Amalgamation, interpolation and congruence extension properties in topological cylindric algebras

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Abstract

We study various amalgamation properties in topological cylindric algebras of all dimensions.

1 Amalgamation; positive results

We turn to investigating amalgamation properties for various subclasses of TCA$_\alpha$, most, but not all, consisting solely of representable algebras. We shall address all dimensions. All our positive results in the infinite dimensional case will follow from the interpolation result proved in part 1, which we recall:

**Theorem 1.1.** Let $\alpha$ be an infinite ordinal. let $\beta$ be a cardinal. Let $\rho : \beta \rightarrow \wp(\alpha)$ such that $\alpha \sim \rho(i)$ is infinite for all $i \in \beta$. Then $\mathfrak{t}_\beta^\alpha$TCA$_\alpha$ has the interpolation property.

This theorem will lead to theorem ?? showing that a certain class of algebras has the super amalgamation property, which is the main source for all positive results obtained.

All negative results (for both finite and infinite dimension) are obtained by bouncing them to the CA case, using the earlier observation made, namely, that any CA$_\alpha$ can be expanded to a TCA$_\alpha$ such that the latter is representable if and only if the former is. We start with the relevant definitions:

**Definition 1.2.** Let $L$ be a class of algebras.

1Topological logic, Chang modal logic, cylindric algebras, representation theory, amalgamation, congruence extension, interpolation Mathematics subject classification: 03B50, 03B52, 03G15.
We say that \( D \) is a super amalgam over \( A_0 \), via \( m_1 \) and \( m_2 \), or simply an amalgam.

(iii) Let everything be as in (i) and assume that the algebras considered are endowed with a partial order. If in addition, \((\forall x \in A_j)(\forall y \in A_k)(m_j(x) \leq m_k(y) \implies (\exists z \in A_0))(x \leq i_j(z) \land i_k(z) \leq y))\) where \( \{j, k\} = \{1, 2\} \), then we say that \( A_0 \) lies in the super amalgamation base of \( L \), and \( D \) is called a super amalgam.

(iv) \( L \) has the amalgamation property, if the amalgam base of \( L \) coincides with \( L \). Same for strong amalgamation and super amalgamation.

We write \( AP \), \( SAP \) and \( SUPAP \) for the amalgamation, strong amalgamation and super amalgamation properties, respectively. We write \( APbase(K) \), \( SAPbase(K) \), and \( SUPAPbase(K) \), for the amalgamation, strong amalgamation, and super amalgamation base of the class \( K \), respectively. Notice that \( SUPAP \) also implies \( SAP \) by writing the extra condition for \( SAP \) as follows:

\[(\forall x \in A_1)(\forall y \in A_2)[m(x) = n(y) \implies (\exists z \in A_0)(x = f(z) \land y = h(z))].\]

We will sometimes seek amalgams, and for that matter strong or super amalgams, for algebras in a certain class in a possibly bigger one.

**Definition 1.3.** Let \( K \subseteq L \). We say that \( K \) has \( AP \) with respect to \( L \) if amalgams can always be found in \( L \). More precisely, for any \( A, B, C \in K \) and any injective homomorphisms \( f : A \to B \) and \( g : A \to C \) then there exist \( D \in L \) and injective homomorphisms \( m : B \to D \) and \( n : C \to D \) such that \( m \circ f = n \circ g \). The analogous definition applies equally well when we replace \( AP \) by \( SAP \) or \( SUPAP \).

The next property is different than the amalgamation property; we do not require that both homomorphisms from the base algebra are injective; only one of them is. The definition is taken from [16].

More precisely:

**Definition 1.4.** A class \( L \) has the transferable injections property, or \( TIP \) for short, if for all \( A_0, A_1, A_2 \in L \) and injective homomorphism \( i_1 : A_0 \to A_1 \), and homomorphism \( i_2 : A_0 \to A_2 \) there exist \( D \in L \) and an injective homomorphism \( m_1 : A_1 \to D \) and a homomorphism \( m_2 : A_2 \to D \) such that \( m_1 \circ i_1 = m_2 \circ i_2 \). We say that \( D \) is a \( TI \) amalgam.
From now on $\alpha$ is an arbitrary ordinal $> 0$.

**Definition 1.5.** Let $\mathfrak{A} \in \text{TCA}_\alpha$. Then a filter $F$ of $\mathfrak{A}$ is a Boolean filter, that satisfies in addition that $q_i x = x$ for every $i < \alpha$ and every $x \in A$.

The following lemma is crucial for our later algebraic manipulations. We show that filters so defined correspond to congruences, thus filters and congruences can be treated equally giving quotient algebras.

**Theorem 1.6.** Let $\mathfrak{A} \in \text{TCA}_\alpha$. Let $\text{Filt}(\mathfrak{A})$ be the lattice of filters (with inclusion) on $\mathfrak{A}$, and $\text{Co}(\mathfrak{A})$ be the lattice of congruences on $\mathfrak{A}$. Then $\text{Filt}(\mathfrak{A}) \cong \text{Co}(\mathfrak{A})$. Furthermore, $\Theta$ restricted to maximal filters is an isomorphism into the set of maximal congruences.

**Proof.** The map $\Theta : \text{Co}(\mathfrak{A}) \to \text{Filt}(\mathfrak{A})$, defined via

$$\cong \mapsto \{x \in \mathfrak{A} : x \cong 1\},$$

is an isomorphism, with inverse $\Theta^{-1} : \text{Filt}(\mathfrak{A}) \to \text{Co}(\mathfrak{A})$ defined by

$$F \mapsto R = \{(a, b) \in A \times A : a \oplus b \in F\},$$

where $\oplus$ as before denotes the symmetric difference.

Indeed, let $\cong$ be a congruence on $\mathfrak{A}$. Then we show that $F = \{a \in A : a \cong 1\}$ is a filter. Let $a, b \in F$. Then $a \cong 1$ and $b \cong 1$, hence $a \cdot b \cong 1$, so that $a \cdot b \in F$. Let $a \in F$ and $a \leq b$. Then $a + b = b$ and we obtain $b = a + b \cong 1 + b = 1$. Hence $b \in F$. Now finally, assume that $a \in F$ and $i < \alpha$. Then $a \cong 1$ so $q_i a \cong q_i 1 = 1$, and we are done.

Conversely, let $F$ be a filter and let $\cong_F$ be given by $a \oplus b \in F$. Then it is straightforward to see that $\cong_F$ is a congruence with respect to the Boolean operations, cylindrifiers and substitutions. It remains to check that $\cong_F$ is a congruence with respect to the interior operators. Let $i < \alpha$. If $a \cong_F b$ then, by definition, $a \oplus b \in F$, hence $q_i(a \oplus b) \in F$ be the definition of $F$. But $q_i(a \oplus b) = q_i(I_i(a) \oplus I_i(b)) \leq I_i(a) \oplus I_i(b) \in F$ by properties of filters, and the interior operator.

Now it remains to show that $F_{\cong_F} = F$ and $\cong_{F_{\cong_F}} = \cong$. We prove only the former. Let $a \in F$. Then $a = a \oplus 1$, that is $a \cong_F 1$, and so $a \in F_{\cong_F}$. Conversely, if $a \in F_{\cong_F}$, then $a \cong_F 1$, that is $a \cong 1 \in F$, hence $a \in F$ and we are done.

Finally, if $R$ is maximal, and $\Theta(R) = F$ is not a maximal filter, then there is a proper filter $J$ extending $F$ properly. Let $x \in J \sim F$. Then $(x, 1) \notin \Theta^{-1} F = R$ and $(x, 1) \in R_f$, so that $R$ is properly contained in the proper congruence $R_f$ which is impossible. \hfill $\square$

If $\mathfrak{A}, \mathfrak{B} \in \text{TCA}_\alpha$, and $F$ is a filter of $\mathfrak{A}$ then $\mathfrak{A}/F$ denotes the quotient algebra $\mathfrak{A}/\cong_F$ which is a homomorphic image of $\mathfrak{A}$. If $h : \mathfrak{A} \to \mathfrak{B}$ is a homomorphism, then $\text{ker} h$ is the filter $\{a \in \mathfrak{A} : h(a) = 1\}$; we have $\mathfrak{A}/\text{ker} h \cong h(\mathfrak{A})$. 

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Theorem 1.8. Let \( \mathfrak{A} \) be a variety. If \( \mathfrak{A} \in \text{TCA}_\alpha \) and \( \Gamma \subseteq \alpha \), \( \Gamma = \{ i_0, \ldots, i_{n-1} \} \) say, then \( q_\Gamma \cdot x = q_{i_0} \cdot \ldots \cdot q_{i_{n-1}} \cdot x \). This does not depend on the order of the \( i_j \)'s because the \( q_i \)'s \( (i < \alpha) \) commute, and so is well defined.

Lemma 1.7. Let \( \mathfrak{A}, \mathfrak{B} \in \text{TCA}_\alpha \) with \( \mathfrak{B} \subseteq \mathfrak{A} \). Let \( X \subseteq \mathfrak{A} \) and \( F \) be a filter of \( \mathfrak{B} \). We then have:

1. \( \mathfrak{A}^\mathfrak{A} X = \{ a \in A : \exists n \in \omega, x_0, \ldots, x_n \in X, \text{and} \in \subseteq \omega \alpha, \mathfrak{A}^\mathfrak{A}(x_0 \cdot x_1 \cdot \ldots \cdot x_n) \leq a \} \).
2. \( \mathfrak{A}^\mathfrak{A} M = \{ x \in A : x \geq b \text{ for some } b \in M \} \).
3. \( M = \mathfrak{A}^\mathfrak{A} M \cap \mathfrak{B} \).
4. If \( \mathfrak{C} \subseteq \mathfrak{A} \) and \( N \) is a filter of \( \mathfrak{C} \), then \( \mathfrak{A}^\mathfrak{A}(M \cup N) = \{ x \in A : b \cdot c \leq x \text{ for some } b \in M \text{ and } c \in N \} \).
5. For every filter \( N \) of \( \mathfrak{A} \) such that \( N \cap B \subseteq M \), there is a filter \( N' \) in \( \mathfrak{A} \) such that \( N \subseteq N' \) and \( N' \cap B = M \). Furthermore, if \( M \) is a maximal filter of \( \mathfrak{B} \), then \( N' \) can be taken to be a maximal filter of \( \mathfrak{A} \).

Proof. Only (iv) might deserve attention. The special case when \( N = \{ 1 \} \) is straightforward. The general case follows from this one, by considering \( \mathfrak{A}/N, \mathfrak{B}/(N \cap \mathfrak{B}) \) and \( M/(N \cap \mathfrak{B}) \), in place of \( \mathfrak{A}, \mathfrak{B} \) and \( M \) respectively.

The next theorem investigates the relationship of TIP and AP.

Theorem 1.8. Let \( K \subseteq \text{TCA}_\alpha \). If \( K \) has AP and \( HK = \mathfrak{A} \) then \( K \) has TIP. If \( PK = \mathfrak{A} \) and \( K \) has TIP then it has AP. In particular, if \( K \) is a variety, then TIP is equivalent to AP.

Proof. Assume \( K \) has TIP. Then for all \( \mathfrak{A}, \mathfrak{B}, \mathfrak{C} \in K, i : \mathfrak{A} \to \mathfrak{B}, j : \mathfrak{A} \to \mathfrak{C} \) and \( x \neq y \) in \( \mathfrak{B} \) (respectively, \( x \neq y \in C \)), there exist \( \mathfrak{D}_{xy} \in K \) and homomorphisms \( h_{xy} : \mathfrak{B} \to \mathfrak{D}_{xy} \) and \( k_{xy} : \mathfrak{C} \to \mathfrak{D}_{xy} \) such that \( h_{xy} \circ i = k_{xy} \circ j \) and \( h_{xy}(x) \neq h_{xy}(y) \) (respectively, \( k_{xy}(x) \neq k_{xy}(y) \)). Let \( \mathfrak{D} \) be the direct product of all algebras \( \mathfrak{D}_{xy} \) for all two element sets \( x, y \). By co-universality of \( \mathfrak{D} \) the homomorphisms \( h_{xy} : \mathfrak{B} \to \mathfrak{D}_{xy} \) and \( k_{xy} : \mathfrak{C} \to \mathfrak{D}_{xy} \) induce injective homomorphisms \( h : \mathfrak{B} \to \mathfrak{D} \) and \( k : \mathfrak{C} \to \mathfrak{D} \), as required.

Let \( \mathfrak{A}, \mathfrak{B}, \mathfrak{C} \in K, m : \mathfrak{C} \to \mathfrak{A} \) be an embedding \( n : \mathfrak{C} \to \mathfrak{B} \) be a homomorphism. Then \( \mathfrak{C}/\text{Kern} \cong n(\mathfrak{C}) \). Let \( M \) be a filter of \( \mathfrak{A} \) such that \( M \cap \mathfrak{C} = \text{Kern} \). Since \( HK = \mathfrak{A} \), then \( \mathfrak{B}/M \) is in \( K \). Now \( \mathfrak{C}/\text{kern} \) embeds into \( \mathfrak{B}/M \), but it also embeds into \( \mathfrak{C} \). By AP in \( K \) there is a \( \mathfrak{D} \in K \) and \( f : \mathfrak{B}/M \to \mathfrak{D} \) and \( g : \mathfrak{C} \to \mathfrak{D} \) such that \( f \circ n = g \circ m \). Then \( f^* : \mathfrak{B} \to \mathfrak{D} \) defined via \( f^*(b) = f(b) \)
and \( g \) is as required; that is \( f^* \circ n = g \circ m \), \( f^* \) is a homomorphism and \( g \) is an embedding.

Then the following can be proved. The references \([17, 13, 29, 24, 27, 26]\) would help a lot. For undefined notions the reader is referred to \([14, 17]\).

**Theorem 1.9.** Let \( \alpha \) be an infinite ordinal.

1. \( \text{TDC}_\alpha \) has the super amalgamation property with respect to \( \text{RTCA}_\alpha \). In other words, \( \text{TDC}_\alpha \) is contained in the super amalgamation base of \( \text{RTCA}_\alpha \). However, it does not have even AP.

2. The classes of semisimple algebras and diagonal algebras (as defined in \([17]\)) of dimension \( \alpha \) have the amalgamation property but not the strong amalgamation property, a fortiori they fail the super amalgamation property.

3. The free algebra \( \mathfrak{A} \) on any number of free generators \( \beta > 1 \) of any variety between \( \text{TeCA}_\alpha \) and \( \text{RTCA}_\alpha \) has a weak form of interpolation, namely, if \( X_1, X_2 \subseteq \mathfrak{A} \) and \( a, b \in \mathfrak{S}^g X_1 \) with \( a \leq b \), then there exists \( c \in \mathfrak{S}^g (X_1 \cap X_2) \), and a finite \( \Gamma \subseteq \alpha \) such that \( q(\Gamma) a \leq c \leq c(\Gamma) b \), but it does not have the usual interpolation property when the number of free generators are \( \geq 4 \).

4. Furthermore, the existence condition on the finite set \( \Gamma \) in the previous item which makes it easier to find an interpolation cannot be omitted in a very strong sense. For every finite \( n \geq 0 \), there is an inequality \( a \leq b \) such that the interpolant can be found using more than \( n \) quantified indices of \( \alpha + \omega \sim \alpha \). In particular for such \( n \), and such inequality the \( \Gamma \) that provides an interpolant has to satisfy that \( |\Gamma| > n \).

5. The former result of weak interpolation is equivalent to the fact that the class of simple algebras (which is a proper class of the class of representable algebras) have the amalgamation property.

6. The free representable algebras on any number of generators have the strong restricted interpolation property, but the free \( \text{CA}_\alpha \) on \( \omega \) free generators does not have the weak interpolation property.

7. The variety \( \text{RTCA}_\alpha \) has the strong embedding property, but the variety \( \text{TCA}_\omega \) does not have the embedding property. The (strong) embedding property is a restricted form of the (strong) amalgamation property, namely, when the base common subagebra is required to be minimal.

8. \( \mathfrak{A} \) is in the amalgamation base of \( \text{RTCA}_\alpha \) iff it has the UNEP, and it is in the super amalgamation base iff it has both \( \text{NS} \) and \( \text{SUPAP} \). In particular, there are representable algebras that do not have UNEP.
(9) $\mathcal{A} \in \text{RTCA}_\alpha$ has UNEP iff $\mathcal{A}$ has universal maps with respect to the neat reduct functor, in particular $\mathcal{N}_\mathcal{R}$ does not have a right adjoint, hence it is not invertible.

(10) In any class between simple algebras and representable algebras of dimension $\alpha$ ES fails, where ES abbreviates that epimorphisms in the categorical are surjective.

Then the following can be proved. The references [17, 13, 29, 24, 27, 26] would help a lot. For undefined notions the reader is referred to [14, 17].

**Theorem 1.10.** Let $\alpha$ be an infinite ordinal.

(1) $\text{TDc}_\alpha$ has the super amalgamation property with respect to $\text{RTCA}_\alpha$. In other words, $\text{TDc}_\alpha$ is contained in the super amalgamation base of $\text{RTCA}_\alpha$. However, it does not have even AP.

(2) The classes of semisimple algebras and diagonal algebras (as defined in [17]) of dimension $\alpha$ have the amalgamation property but not the strong amalgamation property, a fortiori they fail the super amalgamation property.

(3) The free algebra $\mathcal{A}$ on any number of free generators $\beta > 1$ of any variety between $\text{TeCA}_\alpha$ and $\text{RTCA}_\alpha$ has a weak form of interpolation, namely, if $X_1, X_2 \subseteq \mathcal{A}$ and $a, b \in Sg^3X_1$ with $a \leq b$, then there exist $c \in Sg^3(X_1 \cap X_2)$, and a finite $\Gamma \subseteq \alpha$ such that $q(\Gamma)a \leq c \leq c(\Gamma)b$, but it does not have the usual interpolation property when the number of free generators are $\geq 4$.

(4) Furthermore, the existence condition on the finite set $\Gamma$ in the previous item which makes it easier to find an interpolation cannot be omitted in a very strong sense. For every finite $n \geq 0$, there is an inequality $a \leq b$ such that the interpolant can be found using more than $n$ quantified indices of $\alpha + \omega \sim \alpha$. In particular for such $n$, and such inequality the $\Gamma$ that provides an interpolant has to satisfy that $|\Gamma| > n$.

(5) The former result of weak interpolation is equivalent to the fact that the class of simple algebras (which is a proper class of the class of representable algebras) have the amalgamation property.

(6) The free representable algebras on any number of generators have the strong restricted interpolation property, but the free $\mathcal{CA}_\alpha$ on $\omega$ free generators does not have the weak interpolation property.
We can assume, without loss, that $S$.

**Theorem 1.14.** Let $f : C \to A$ and $g : C \to B$ be injective homomorphisms. Then there exist $A^+, B^+, C^+ \in \text{TCA}_\beta$, $e_A : A \to \text{Nr}_\alpha A^+$ and $e_B : B \to \text{Nr}_\alpha B^+$ such that $e_A e_B (A) = B$. Then there exist $A^+, B^+, C^+ \in \text{TCA}_\beta$.

$\Box$

**Lemma 1.13.** If $\alpha \geq \omega$ and $\mathfrak{A} \in \text{TDc}_\alpha$ then $\mathfrak{A}$ has $\text{NS}$ and $\text{UNEP}$.

**Proof.** [7] Theorem 2.6.67-71-72.

**Theorem 1.14.** Let $\alpha$ be an infinite ordinal.

(1) $\text{TDc}_\alpha \subseteq \text{SUPAPbase}(\text{RTCA}_\alpha)$.

(2) $\text{Tlf}_\alpha$ has $\text{SUPAP}$.

**Proof.** For the first part. Let $\beta = \alpha + \omega$. Let $\mathfrak{C} \in \text{Dc}_\beta$, let $\mathfrak{A}, \mathfrak{B} \in \text{RTCA}_\beta$, and let $f : \mathfrak{C} \to \mathfrak{A}$ and $g : \mathfrak{C} \to \mathfrak{B}$ be injective homomorphisms. Then there exist $A^+, B^+, C^+ \in \text{TCA}_\beta$, $e_A : A \to \text{Nr}_\alpha A^+$ and $e_B : B \to \text{Nr}_\alpha B^+$ such that $e_A e_B (A) = B$. Then there exist $A^+, B^+, C^+ \in \text{TCA}_\beta$.

We can assume, without loss, that $\mathfrak{C} = \mathfrak{A}$ and similarly for $\mathfrak{B}$ and $\mathfrak{C}$. Let $f(C)^+ = g(C)^+ = f(g(C))^+$ such that $(e_A \upharpoonright f(C)) \circ f = f \circ e_C$ and $(e_B \upharpoonright g(C)) \circ g = g \circ e_C$. 

$\Box$
Both $\tilde{f}$ and $\tilde{g}$ are injective homomorphisms and $K$ has \textit{SUPAP}, hence there is a $D^+$ in $K$ and $k : A^+ \to D^+$ and $h : B^+ \to D^+$ such that $k \circ \tilde{f} = h \circ \tilde{g}$. Also $k$ and $h$ are injective homomorphisms. Then $k \circ e_A : A \to Nr_a D^+$ and $h \circ e_B : B \to Nr_a D^+$ are one to one and $k \circ e_A \circ f = h \circ e_B \circ g$. Let $D = Nr_a D^+$. Then we have obtained $D \in Nr_a TRC_{\alpha, +\omega}$ and $m : A \to D$, $n : B \to D$ such that $m \circ f = n \circ g$. Here $m = k \circ e_A$ and $n = h \circ e_B$. We have proved \textit{AP}.

Denote $k$ by $m^+$ and $h$ by $n^+$. We further want to show that if $m(a) \leq n(b)$, for $a \in A$ and $b \in B$, then there exists $t \in C$ such that $a \leq f(t)$ and $g(t) \leq b$. So let $a$ and $b$ be as indicated. We have $(m^+ \circ e_A)(a) \leq (n^+ \circ e_B)(b)$, so $m^+(e_A(a)) \leq n^+(e_B(b))$. Since $K$ has \textit{SUPAP}, there exist $z \in C^+$ such that $[e_A(a)] \leq \tilde{f}(z)$ and $\tilde{g}(z) \leq [e_B(b)]$. Let $\Gamma = \Delta z \sim \alpha$ and $z' = c(\Gamma)z$. (Note that $\Gamma$ is finite.) So, we obtain that $e_A(c(\Gamma)a) \leq \tilde{f}(c(\Gamma)z)$ and $\tilde{g}(c(\Gamma)z) \leq e_B(c(\Gamma)b)$. It follows that $e_A(a) \leq \tilde{f}(z')$ and $\tilde{g}(z') \leq e_B(b)$. Now by the $NS$ property for $TDc$, we have $z' \in Nr_a C^+ = \mathbb{G}^{\text{max}}(C) = e_C(C)$. So, there exists $t \in C$ with $z' = c_C(t)$. Then we get $e_A(a) \leq \tilde{f}(e_C(t))$ and $\tilde{g}(e_C(t)) \leq e_B(b)$. It follows that $e_A(a) \leq (e_A \circ f)(t)$ and $(e_B \circ g)(t) \leq e_B(b)$. Hence, $a \leq f(t)$ and $g(t) \leq b$. We are done. For the locally finite case, one takes the subalgebra of $Nr_a D$ (as constructed above) generated by the images of $A$ and $B$ which are now locally finite, as an amalgam, and hence as a super amalgam. This algebra is necessarily locally finite.

In the above theorem, we do not guarantee that the super amalgam is found inside $TDc_\alpha$, for the subalgebra of the amalgam as formed for the locally finite case may not be in $TDc_\alpha$. Indeed, we have:

\textbf{Theorem 1.15.} \textit{For $\alpha \geq \omega$, $TDc_\alpha$ does not have AP.}

\textbf{Proof.} Let $A, B \in TDc_\alpha$, such that their minimal subalgebras are isomorphic and for which there exist $x \in A$ and $y \in B$, such that $\Delta x \cup \Delta y = \alpha$. Clearly such algebras cannot be amalgamated by a $Dc_\alpha$ over the common minimal subalgebra $M$ say of $A$ and $B$, embedded into each by the inclusion map $i$. For if $C \in TDc_\alpha$ and $m : A \to C$ and $n : B \to C$, such that $m \circ i = n \circ i$, then $\Delta(m(x) + n(y)) = \alpha$ which is not possible because the amalgam is assumed to be dimension complemented.

Using the same argument as in theorem \textit{[8]} it can be shown:

\textbf{Theorem 1.16.} \textit{If $C \in TDc_\alpha$, $A, B \in TRCA_{\alpha}$, $n : C \to A$ an embedding and $m : C \to B$ a homomorphism, then there exists $D \in TRCA_{\alpha}$ a homomorphism $f : A \to D$ and an embedding $g : B \to D$ such that $f \circ n = g \circ m$.}

Now we deal with concepts that are localizations of the amalgamation property; in the sense that various amalgamation properties will be proved to hold for a class of algebras if the free algebras of such classes enjoy such local properties, typically \textit{interpolation} properties. We have already dealt with one such property; we introduce weaker ones.
Definition 1.17. Let $\alpha$ be any ordinal (finite included) and $\mathfrak{A} \in \text{TCA}_\alpha$. Then

1. $\mathfrak{A}$ has the *weak interpolation property*, WIP for short, if for all $X_1, X_2 \subseteq \mathfrak{A}$, for all $x \in \mathcal{S}^\mathfrak{A} X_1$, $z \in \mathcal{S}^\mathfrak{A} X_2$ if $x \leq z$, then there exist $\Gamma \subseteq \omega$ and $y \in \mathcal{S}^\mathfrak{A} (X_1 \cap X_2)$ such that

$$q(\Gamma)x \leq y \leq c(\Gamma)z.$$ 

2. $\mathfrak{A}$ has the *universal interpolation property*, UIP for short, if for all $X_1, X_2 \subseteq \mathfrak{A}$, for all $x \in \mathcal{S}^\mathfrak{A} X_1$, $z \in \mathcal{S}^\mathfrak{A} X_2$ if $x \leq z$, then there exist $\Gamma \subseteq \omega$ and $y \in \mathcal{S}^\mathfrak{A} (X_1 \cap X_2)$ such that

$$q(\Gamma)x \leq y \leq z.$$ 

3. $\mathfrak{A}$ has the *existential interpolation property*, EIP for short, if for all $X_1, X_2 \subseteq \mathfrak{A}$, for all $x \in \mathcal{S}^\mathfrak{A} X_1$, $z \in \mathcal{S}^\mathfrak{A} X_2$ if $x \leq z$, then there exist $\Gamma \subseteq \omega$ and $y \in \mathcal{S}^\mathfrak{A} (X_1 \cap X_2)$ such that

$$x \leq y \leq c(\Gamma)z.$$ 

The following theorem is proved by a *compactness argument* taken from [25]. The proof works only for infinite dimension. We shall see that the theorem fails when the dimension is finite. It can be used in even a much wider context, saying that if the *dimension restricted free* algebras in $\omega$ extra dimensions have the interpolation property, then the free algebras *without any restrictions* have a natural weak form of interpolation. The idea is that an interpolant can always be found if we allow infinitely many more dimensions (variables), though only finitely many are used in the interpolant. Then quantifiers may be used to get rid of the extra variables bouncing the interpolant back to using only the number of available variables. For a term $\sigma$ in the language of TCA$_\alpha$, $\text{Var}(\sigma)$ denotes the set of variables occurring in $\sigma$.

**Theorem 1.18.** Let $\alpha \geq \omega$. Let $K$ be a class of algebras such that RTCA$_\alpha \subseteq \mathfrak{K} \subseteq \text{TCA}_\alpha$. Then for any terms of the language of TCA$_\alpha$, $\sigma, \tau$ say, if $K \models \sigma \leq \tau$, then there exist a term $\pi$ with $\text{Var}(\pi) \subseteq \text{Var}(\sigma) \cap \text{Var}(\tau)$ and a finite $\Delta \subseteq \alpha$ such that

$$K \models q(\Delta)\sigma \leq \pi \leq c(\Delta)\tau.$$ 

In particular, for any non-zero cardinal $\beta$, $\mathfrak{F}_\beta K$, has the WIP.

**Proof.** The same argument in [25] but now using theorem 1.1. \qed

We will show that the above form of interpolation is the best possible for such classes $K$, witness theorem 3.3. Furthermore it fails for finite dimensions, theorem 3.3.
We have proved that for $\alpha \geq \omega$, $TDc_{\alpha}$ lies in the super amalgamation base of $TRCA_{\alpha}$. In what follows we define larger classes of algebras, still retaining an amalgamation property. We lose the strong amalgamation property, but in return in such cases the amalgam is always found inside the class in question.

The following class was introduced by Monk for cylindric algebras under the name of \textit{Diagonal cylindric algebras}, and was denoted by $Di_{\alpha}$ in [17]. In the last reference Pigozzi proved that this class has AP. We obtain an analogous result, but we define the class differently, allowing generalization to diagonal free algebras. We use the term definable substitutions corresponding to replacements. In what follows $\alpha$, unless indicated otherwise, is infinite.

\textbf{Definition 1.19.} $\mathfrak{A} \in TCA_{\alpha}$ is called a substitution algebra of dimension $\alpha$, if for all non-zero $x$ in $\mathfrak{A}$, for all finite $\Gamma \subseteq \alpha$, there exist distinct $i,j \in \alpha \sim \Gamma$, such that $s_j^i x \neq 0$. Let $TSc_{\alpha}$ denote the class of substitution algebras.

We know that any simple algebra is in $Sc_{\alpha}$ and semi-simple algebras are subdirect products of simple ones. This does not guarantee that the class of semi-simple algebras is contained in $Sc_{\alpha}$ because we cannot assume a priori that the latter class is closed under products. However, as it happens, we have:

\textbf{Theorem 1.20.} If $\mathfrak{A}$ is semi-simple or $\mathfrak{A} \in TDc_{\alpha}$, then $\mathfrak{A} \in Sc_{\alpha}$.

\textit{Proof.} Let $\mathfrak{A} \in TDc_{\alpha}$ $a \in \mathfrak{A}$ be non zero, and $\Gamma \subseteq \omega \alpha$. Choose $i,j \in \alpha \sim \Delta x$. Then $s_j^i x = x \neq 0$. Let $\mathfrak{A} \in Sc_{\alpha}$, $\Gamma$ be a finite subset of $\alpha$ and $x \in \mathfrak{A} \sim \{0\}$. Using Zorn’s lemma one can find a maximal filter $F$ of $\mathfrak{A}$ such that $x \notin F$. Since $F$ is maximal then $\mathfrak{A}/F$ is simple. But $x \notin F$, hence there exists a finite $\Delta \subseteq \alpha$, such that $c(\Delta)(x/F) = c(\Delta)x/F = 1$.

Let $i,j \in \alpha \sim (\Gamma \cup \Delta)$, then we claim that $s_j^i x \neq 0$. If not, then

$$0 = (c(\Delta)s_j^i x)/F = (s_j^i c(\Delta)x)/F = c(\Delta)x/F = 1,$$

which is impossible.

\textbf{Theorem 1.21.} $Sc_{\alpha} \subseteq RTCA_{\alpha}$. Furthermore, the class $Sc_{\alpha}$ has AP and TIP. If the algebras to be amalgamated are semi-simple then the amalgam can be chosen to be semisimple, too. The same holds when we replace semi-simple by simple.

\textit{Proof.} The same argument in [17, Theorem 2.2.24] using theorem ??.

\textbf{Theorem 1.22.} $SUpTSc_{\alpha} = RTCA_{\alpha}$.

\textit{Proof.} From theorem [1.21] since $TDc_{\alpha} \subseteq Sc_{\alpha}$. 

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The following class is the TCA analogue of the class of cylindric algebras introduced in item (iii) of theorem 2.6.50. Using the neat embedding theorem \[31, \text{Theorem 4.2(2)}\] yet again, together with ultraproduc\textit{t}, we show that such a class consists only of representable TCA\textsubscript{α}s, furthermore, it is easy to see that TSc\textsubscript{α} is contained in it, and as we shall see in a minute properly.

**Definition 1.23.** \(A \in \text{TCA}_\alpha\) is called a weak substitution algebra, a TWSc\textsubscript{α} for short, if for every finite injective map \(\rho\) into \(\alpha\), and for every \(x \in A, x \neq 0\), there is a function \(h\) and \(k < \alpha\) such that \(h\) is an endomorphism of \(\mathcal{R}_{\rho}^A\), \(k \in \alpha \sim \text{rng}(\rho), c_k \circ h = h\) and \(h(x) \neq 0\).

The following theorem can be proved exactly like in \[7, \text{Theorem 2.6.50 (iii)}\] using the hitherto established neat embedding theorem \[31, \text{Theorem 4.2 (2)}\].

**Theorem 1.24.** For any infinite ordinal \(\alpha\), TWSc\textsubscript{α} ⊆ TRCA\textsubscript{α}.

**Theorem 1.25.** The following conditions are equivalent:

1. TWSc\textsubscript{α} is elementary.
2. TWSc\textsubscript{α} is closed under ultraproducts.
3. TWSc\textsubscript{α} = TRCA\textsubscript{α}.
4. TWSc\textsubscript{α} is a variety.

**Proof.**

(i) (1) \(\rightarrow\) (2) is trivial.

(ii) (2) \(\rightarrow\) (3) If TWSc\textsubscript{α} is closed under ultraproducts, then because it is closed under forming subalgebras, we have TWSc\textsubscript{α} = \(\text{SUP}\) TWSc\textsubscript{α} = TRCA\textsubscript{α}.

(iii) (3) \(\rightarrow\) (4) Since RTCA\textsubscript{α} is a variety.

(iv) (4) \(\rightarrow\) (1) Trivial.

We let TSs\textsubscript{α} stand for semisimple algebras. In the next example we show that the inclusions

\[\text{TDC}_\alpha \subseteq \text{TSs}_\alpha \subseteq \text{TSc}_\alpha \subseteq \text{TWSs}_\alpha \subseteq \text{RTCA}_\alpha\]

are all proper for \(\alpha \geq \omega\), for the CA case witness \[7, \text{Remark 2.6.51}\].

But first we need a definition.

**Definition 1.26.** Let \(\alpha\) be any ordinal \(> 1\) and \(A \in \text{TCA}_\alpha\). Then \(I\) is an ideal in \(A\) iff \(I\) is an ideal in \(\mathcal{R}_{\rho}^A\).
It can be easily checked that this definition is sound in the sense that ideals so defined correspond to congruences (hence to filters) via $\cong \mapsto \{a \in A : a \cong 0\}$.

**Example 1.27.**  
(1) For the first inclusion. Let $m \geq 2$ be a finite ordinal. Take $\mathfrak{A} = \wp(\alpha \setminus m)$, it is easy to see that $\mathfrak{A} \in \mathbb{TSc}_\alpha$. However, $\mathfrak{A}$ is not in $\mathbb{TDC}_\alpha$, because for every $s \in \wp m$, we have $\Delta(\{s\}) = \alpha$. We show that $\mathfrak{A}$ is not even semi-simple by showing that for any constant map $f : \alpha \to m$, the singleton $\{f\}$ is in all the maximal proper ideals. Let $f$ be such a map. Let $X = \{f\}$. Let $J$ be a maximal proper ideal. Assume for contradiction that $X \notin J$. Then $X/J \neq 0$ in $\mathfrak{A}/J$. Since $\mathfrak{A}/J$ is simple, then the ideal generated by $X/J$ coincides with $\mathfrak{A}/J$, so there exists a finite $\Gamma \subseteq \alpha$ such that $c_{\Gamma}(X/J) = c_{\Gamma}X/J = 1$. This means that $c_{\Gamma}X \notin J$, but $J$ is maximal, hence $-c_{\Gamma}X \in J$. Now let $k \in \alpha \sim \Gamma$, and let $t$ be the sequence that agrees with $f$ everywhere except at $k$, where its value is $\neq f(k)$. Then $t \in -c_{\Gamma}X$, so $\{t\} \in J$ by maximality of $J$. But $X \subseteq c_{k}\{t\}$, so $X \subseteq J$ which is impossible.

(2) let $\mathfrak{A} = \wp(\alpha \setminus \alpha)$; then of course $\mathfrak{A} \in \mathbb{TRCA}_\alpha$. We show that $\mathfrak{A} \notin \mathbb{TSc}_\alpha$.

Let $\Theta$ be a bijection from $\alpha$ to $\alpha$ and consider the element $x = \{\Theta\} \in \mathfrak{A}$. Then for any distinct $i, j \in \alpha$, $s^i_j x = 0$, because $\sigma \in s^i_j X$ iff $\sigma \circ [i/j] = \Theta$ which is impossible, because $\sigma \circ [i/j](i) = \sigma \circ [i/j](j)$.

We now show that $\mathfrak{A} \in \mathbb{TWSc}_\alpha$. Let $x \in \mathfrak{A}$ be non-zero. Let $\rho$ be a one to one finite function with $\text{rng}(\rho) \subseteq \alpha$. We want to find $H$ as in the conclusion of the definition of a $\mathbb{TWSc}_\alpha$. Let $\tau \in \alpha \setminus \alpha$ such that $k \notin \text{rng}(\tau)$, $\tau \upharpoonright \text{rng}(\rho) \subseteq Id$ and $\tau$ is one to one. Let $H : \mathfrak{A} \to \mathfrak{A}$ by $H(Y) = \{\phi \in \wp(\alpha \setminus \alpha) : \phi \circ \tau \in Y\}$. Then $H$ is as required.

**Corollary 1.28.** Any class $K$, such that $\mathbb{TLf}_\alpha \subseteq K \subseteq \mathbb{TSS}_\alpha$ is not closed under ultraproducts, hence is not elementary. The class of semisimple algebras is not closed under $H$.

*Proof.* Let $\mathfrak{A}$ be a simple, locally finite, non-discrete cylindric algebra of dimension $\alpha$. Here non-discrete means that there is an $i \in \alpha$ such that $c_i \neq Id$ expanded by $I_i = Id$ for all $i < \alpha$.

Let $I$ be an infinite set and $J = \{\Gamma : \Gamma \subseteq I, |\Gamma| < \omega\}$. For $\Gamma \in J$, let $M_\Gamma = \{\Delta \in J : \Gamma \subseteq \Delta\}$. Let $F$ be an ultrafilter on $J$ that contains $M_\Gamma$ for every $\Gamma \in J$; clearly exists for $M_{\Gamma_1} \cap M_{\Gamma_2} = M_{\Gamma_1 \cup \Gamma_2}$.

Let $\mathfrak{B}$ be the following ultrapower of $\mathfrak{A}$, $\mathfrak{B} = J^{\mathfrak{A}}/F$. Then it is proved in [7] remark 2.4.59 that $\mathfrak{Rd}_\alpha \mathfrak{B}$ is not semi-simple and not dimension complemented, hence $\mathfrak{B}$ is not semi-simple, because $I$ is an ideal in $\mathfrak{B}$ if and only if it is an ideal in $\mathfrak{Rd}_\alpha \mathfrak{B}$. Obviously $\mathfrak{B}$ is also not dimension complemented and we are done.

For the last part since $\mathbb{SPS}_\alpha = \mathbb{S}_\alpha$ and the latter is not a variety, hence $\mathbb{HS}_\alpha \neq \mathbb{S}_\alpha$. 

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Corollary 1.29. \( T\mathcal{S}_\alpha \) is not closed under ultraproducts hence it is not first order axiomatizable, least a variety.

Proof. \( T\mathcal{S}_\alpha \) is clearly closed under forming subalgebras, hence by theorem 1.22 and example 1.27 it is not closed under ultraproducts. \( \square \)

2 Interpolation and amalgamation

If we do not have an order (as we do now) what corresponds locally to amalgamation properties, are congruence extension properties. So let us see how these relate to the interpolation properties defined earlier. Recall that for an algebra \( \mathfrak{A} \), \( \text{Co}\mathfrak{A} \) stands for the set of all congruence relations on \( \mathfrak{A} \).

Definition 2.1. Let \( \mathfrak{A} \) be an algebra, and \( C \subseteq \bigcup_{\mathfrak{A} \leq \mathfrak{A}} \text{Co}\mathfrak{A} \). \( \mathfrak{A} \) is said to have the congruence extension property relative to \( C \) if for any \( X_1, X_2 \subseteq \mathfrak{A} \) such that \( X_1 \cap X_2 = X \), if \( R \in \text{Co}(\mathcal{G}^\alpha X_1) \cap C \) and \( S \in \text{Co}(\mathcal{G}^\alpha X_2) \cap C \), such that \( R \cap ^2\mathcal{G}^\alpha (X_1 \cap X_2) = S \cap ^2\mathcal{G}^\alpha (X_1 \cap X_2) \), then there is a \( T \in \text{Co}(\mathfrak{A}) \cap C \) such that \( T \cap ^2\mathcal{G}^\alpha X_1 = R \) and \( T \cap ^2\mathcal{G}^\alpha X_2 = S \). If \( C = \bigcup_{\mathfrak{A} \subseteq \mathfrak{A}} \text{Co}\mathfrak{A} \), we say that \( \mathfrak{A} \) has the congruence extension property, or \( \text{CP} \) for short.

From now on \( \alpha \) is an arbitrary ordinal \( > 0 \). We next formulate the above property for free algebras in various forms. We will see that properties of the free algebras of a variety may be reflected in properties of the corresponding equational consequence relations of the variety, in particular we may focus on properties of the equational consequence relation for a countable list of variables, and this enables us to restrict our attention to countable free algebras only as shown by G. Metcalfe et al. [16]. The Pigozzi property \( \text{PP} \), the Robinson property \( \text{RP} \), the Machura interpolation property \( \text{MIP} \), the deductive interpolation property \( \text{DIP} \) are defined in [16]. Countable \( \text{MIP} \), countable \( \text{DIP} \) and countable \( \text{RP} \), are the restriction of such properties when the variables available are countable.

We note that \( \text{MIP} \) is the interpolation property corresponding to \( \text{TIP} \) which in turn is equivalent in varieties in which congruences of subalgebras of an algebra lift to congruences of the algebra (which is our case\(^2 \) to \( \text{AP} \) [16].

Theorem 2.2. Let \( \alpha \) be an ordinal \( > 0 \). Let \( V \) be a subvariety of \( T\mathcal{C}_{\alpha} \). Then the following conditions are equivalent:

(1) \( V \) has the \( \text{TIP} \).

\(^2\)This property is referred to in the literature as the congruence extension property, but we do not use this term here for we have reserved the term congruence extension property for a different property.
(2) $V$ has the (countable) MIP.

(3) $V$ has the (countable) DIP.

(4) $V$ has AP.

(5) Finitely generated algebras in $V$ has AP

(6) The free algebras have the CP.

(7) The (countable) free algebras have the UIP.

(8) The (countable) free algebras have EIP.

(9) $V$ has PP.

(10) $V$ has the (countable) RP.

Proof. It is known that for any variety $V$ [16] all of (1)-(6) and (9)-(10) are equivalent to each other.

We prove (4) $\implies$ (6) in a form to be used later on. Assume that $V$ has AP and let $A$ be the free algebra on a non empty set of generators. For $R \in \text{Co} A$ and $X \subseteq A$, by $(A/R)^{(X)}$ we understand the subalgebra of $A/R$ generated by $\{x/R : x \in X\}$. We want to show that $A$ has CP. Let $A, X_1, X_2, R$ and $S$ be as specified in the definition of CP. Define

$\theta : \mathfrak{g}^A(X_1 \cap X_2) \rightarrow \mathfrak{g}^A X_1/R$

by

$a \mapsto a/R.$

Then $\ker \theta = R \cap \mathfrak{g}^A(X_1 \cap X_2)$ and $\text{Im} \theta = (\mathfrak{g}^A X_1/R)^{(X_1 \cap X_2)}$. It follows that

$\bar{\theta} : \mathfrak{g}^A(X_1 \cap X_2)/R \cap \mathfrak{g}^A(X_1 \cap X_2) \rightarrow (\mathfrak{g}^A X_1/R)^{(X_1 \cap X_2)}$

defined by

$a/R \cap \mathfrak{g}^A(X_1 \cap X_2) \mapsto a/R$

is a well defined isomorphism. Similarly

$\bar{\psi} : \mathfrak{g}^A(X_1 \cap X_2)/S \cap \mathfrak{g}^A(X_1 \cap X_2) \rightarrow (\mathfrak{g}^A X_2/S)^{(X_1 \cap X_2)}$

defined by

$a/S \cap \mathfrak{g}^A(X_1 \cap X_2) \mapsto a/S$

is also a well defined isomorphism. But

$R \cap \mathfrak{g}^A(X_1 \cap X_2) = S \cap \mathfrak{g}^A(X_1 \cap X_2)$,
Hence
\[ \phi : \left( \mathfrak{S}g^a X_1 / R \right)^{(X_1 \cap X_2)} \to \left( \mathfrak{S}g^a X_2 / S \right)^{(X_1 \cap X_2)} \]
defined by
\[ a / R \mapsto a / S \]
is a well defined isomorphism. Now \( \left( \mathfrak{S}g^a X_1 / R \right)^{(X_1 \cap X_2)} \) embeds into \( \mathfrak{S}g^a X_1 / R \) via the inclusion map; it also embeds in \( \mathfrak{S}g^a X_2 / S \) via \( i \circ \phi \) where \( i \) is also the inclusion map. For brevity let \( A_0 = \left( \mathfrak{S}g^a X_1 / R \right)^{(X_1 \cap X_2)} \), \( A_1 = \mathfrak{S}g^a X_1 / R \) and \( A_2 = \mathfrak{S}g^a X_2 / S \) and \( j = i \circ \phi \). Then \( A_0 \) embeds in \( A_1 \) and \( A_2 \) via \( i \) and \( j \) respectively. Then there exists \( B \in V \) and injective homomorphisms \( f \) and \( g \) from \( A_1 \) and \( A_2 \) respectively to \( B \) such that \( f \circ i = g \circ j \). Let
\[ \bar{f} : \mathfrak{S}g^a X_1 \to B \]
be defined by
\[ a \mapsto f(a / R) \]
and
\[ \bar{g} : \mathfrak{S}g^a X_2 \to B \]
be defined by
\[ a \mapsto g(a / R) . \]
Let \( B' \) be the algebra generated by \( \text{rng} f \cup \text{rng} g \). Then \( \bar{f} \cup \bar{g} \downarrow X_1 \cup X_2 \to B' \) is a function since \( \bar{f} \) and \( \bar{g} \) coincide on \( X_1 \cap X_2 \). By freeness of \( A \), there exists \( h : A \to B' \) such that \( h \downarrow X_1 \cup X_2 = \bar{f} \cup \bar{g} \). Let \( T = \text{ker} h \). Then it is not hard to check that
\[ T \cap \mathfrak{S}g^a X_1 = R \] and \( T \cap \mathfrak{S}g^a X_2 = S \). \( T \) induces the required congruence.

\((6) \implies (7)\). Let \( x \in \mathfrak{S}g^a X_1 \), \( z \in \mathfrak{S}g^a X_2 \) and assume that \( x \leq z \). Then
\[ z \in (\mathfrak{F}X_1 \{ x \} ) \cap \mathfrak{S}g^a X_1 \).

Let
\[ M = \mathfrak{F}X_1 \{ x \} \] and \( N = \mathfrak{F}(M \cap \mathfrak{S}g^a(X_1 \cap X_2)) \).

Then
\[ M \cap \mathfrak{S}g^a(X_1 \cap X_2) = N \cap \mathfrak{S}g^a(X_1 \cap X_2) \]
By identifying ideals with congruences, and using the congruence extension property, there is a filter \( P \) of \( A \) such that
\[ P \cap \mathfrak{S}g^a X_1 = N \text{ and } P \cap \mathfrak{S}g^a X_2 = M \).

It follows that
\[ \mathfrak{F}(N \cup M) \cap \mathfrak{S}g^a X_1 \subseteq P \cap \mathfrak{S}g^a X_1 = N \].
Hence
\[(\mathcal{F}(x) \{z\}) \cap S g^a X_1 \subseteq N.\]
and we have
\[z \in \mathcal{F}(x) \cap S g^a (X_1 \cap X_2).\]
This implies that there is an element \(y\) such that
\[z \geq y \in S g^a (X_1 \cap X_2),\]
and \(y \in \mathcal{F}(x).\)
Hence, there exists a finite \(\Gamma \subseteq \alpha\) such that \(y \geq q_{(\Gamma)} x\),
so we get
\[q_{(\Gamma)} x \leq y \leq z.\]

(7) \(\implies\) (1). Let \(\mathfrak{A}, \mathfrak{B}, \mathfrak{C} \in V\), with inclusions \(m : \mathfrak{C} \to \mathfrak{A}, n : \mathfrak{C} \to \mathfrak{B}\). We want to find an amalgam. Let \(\mathfrak{D}\) be the free algebra on \(|D|\) generators, where
\[|D| > \max |B|, |C|\]
for all \(i < |D|\). Let \(h : \mathfrak{D} \to \mathfrak{C}, h_1 : \mathfrak{D} \to \mathfrak{A}, h_2 : \mathfrak{D} \to \mathfrak{B}\) be homomorphisms such that for \(x \in h^{-1}(\mathfrak{C})\),
\[h_1(x) = m \circ h(x) = n \circ h(x) = h_2(x).\]
Such homomorphisms clearly exist by the freeness, cardinality of \(\mathfrak{D}\), and the fact that \(\mathfrak{A}, \mathfrak{B}, \mathfrak{C} \in V\), Let \(\mathfrak{D}_1 = h^{-1}_1(\mathfrak{A})\) and \(\mathfrak{D}_2 = h^{-1}_2(\mathfrak{B})\). Then \(h_1 : \mathfrak{D}_1 \to \mathfrak{A}\),
and \(h_2 : \mathfrak{D}_2 \to \mathfrak{B}\). Let \(M = \ker h_1\) and \(N = \ker h_2\), and let \(\bar{h}_1 : \mathfrak{D}_1/M \to \mathfrak{A}, \bar{h}_2 : \mathfrak{D}_2/N \to \mathfrak{B}\) be the induced isomorphisms. Let \(l_1 : h^{-1}(\mathfrak{C})/h^{-1}(\mathfrak{C}) \cap M \to \mathfrak{E}\) be defined via \(\bar{x} \to h(x)\), and \(l_2 : h^{-1}(\mathfrak{C})/h^{-1}(\mathfrak{C}) \cap N \to \mathfrak{E}\) be defined via \(\bar{x} \to h(x)\). Then those are well defined, and hence \(k^{-1}(\mathfrak{E}) \cap M = h^{-1}(\mathfrak{E}) \cap N\).
We show that \(\mathfrak{F}(P(M \cup N)\) is a proper filter of \(\mathfrak{D}\) and that \(\mathfrak{D}/P\) is the required amalgam. Let \(x \in \mathfrak{F}(P(M \cup N)\) such that \(b \cdot c \leq x\).
Thus \(c \leq x + b\). But \(x + b \in \mathfrak{D}_1\) and \(c \in \mathfrak{D}_2\), it follows by assumption that there exist \(d \in \mathfrak{D}_1 \cap \mathfrak{D}_2\) such that \(q_{(\Gamma)} c \leq d \leq x + b\).
Notice that \(c \in N\) so \(q_{(\Gamma)} c \in N\), hence \(d \in N\), so \(d \in M\), because \(M \cap \mathfrak{D}_1 \cap \mathfrak{D}_2 = N \cap \mathfrak{D}_1 \cap \mathfrak{D}_2\). Hence \(x \in M\). Thus \(P = \mathfrak{F}(P(M \cup N)\) is proper, and \(\mathfrak{D}/P\) is the required amalgam.

Countable free algebras have \(UIP\) implies \(AP\), because we can restrict our attention only to countable algebras being amalgamated. \(AP\) equivalent to countable \(EIP\) is exactly like above by working with ideals instead of filters.

\[\square\]

Since \(UIP\) and \(EIP\) are equivalent in the case of varieties we call the (one) property they express the almost interpolation property, briefly \(AIP\).

Before our next theorem which provides infinitely many varieties satisfying the conditions of theorem 2.2. For \(R \subseteq \omega \alpha \times \alpha, d(R) = \prod_{i,j \in R} -d_{ij}\). The next result is the topological analogue of a result of Comer [3] proved for cylindric algebras, but we present a different proof depending on theorem 1.2.1.
Theorem 2.3. If $\alpha \geq \omega$, then there are infinitely many subvarieties of $\text{RTCA}_\alpha$ that has $\text{AP}$, hence satisfy all conditions in theorem 2.2.

Proof. For each finite $k \geq 1$, let $V_k$ be the variety consisting of algebras having characteristic $k$ endowed with interior operators. $\mathfrak{A} \in V_k$ iff $\mathfrak{A} \in \text{TCA}_\alpha$ and $\mathfrak{A}$ satisfies the equations $c(k)\bar{d}(k \times k) = 1$ and $c((k+1)\times (k+1)) = 0$. Then $V_k \subseteq \text{TSc}_\alpha$ and amalgams found by theorem 1.21 are necessarily of characteristic $k$ since embeddings preserve equations.

Below in corollary 3.6 we give infinitely many varieties that do not have $\text{AP}$ but their simple algebras do, hence, in view of the coming theorem 2.6, their free algebras have $\text{WIP}$.

Definition 2.4. [16] A variety $V$ has the maximal Pigozzi property, $\text{PP}_m$ for short, if for any sets $Y$ and $Z$ whenever

(i) $Y \cap Z \neq \emptyset,$

(ii) $\Theta_Y \in \text{Co}(\mathfrak{F}_Y(V))$ and $\Theta_Z \in \text{Co}(\mathfrak{F}_Z(V))$; are maximal congruences

(iii) $\Theta_Y \cap \mathfrak{F}_{Y\cap Z}(V)^2 = \Theta_Z \cap \mathfrak{F}_{Y\cap Z}(V)^2,$ then $\Theta_Y$ and $\Theta_Z$ have a common extension to $\mathfrak{F}_{Y\cup Z}(V)$.

In some cases, properties of free algebras may be expressed as properties of the equational consequence relations of the variety, as we proceed to show:

Definition 2.5. [16] A variety $V$ has the maximal Robinson property $\text{RP}$ if for each set of variables $Y$, whenever
(i) \( \Sigma \cup \Pi \cup \{\epsilon\} \subseteq \text{Eq}(Y) \) and \( \text{Var}(\Sigma) \cap \text{Var}(\Pi \cup \{\epsilon\}) \neq \emptyset \);

(ii) \( \Sigma \models_{V} \delta \iff \Pi \models_{V} \delta \), for all \( \delta \in \text{Eq}(Y) \) satisfying \( \text{Var}(\delta) \subseteq \text{Var}(\Pi) \subseteq \text{Var}(\Sigma) \);

(iii) \( \text{Var}(\epsilon) \subseteq \text{Var}(\Sigma) \);

(iv) \( \Sigma \cup \Pi \models_{V} \epsilon \);

(v) The congruences generated by \( \bar{\Sigma} \) and \( \bar{\Pi} \) are maximal, then \( \Sigma \models \epsilon \).

**Theorem 2.6.** The following conditions are equivalent for a subvariety \( V \) of \( \text{TCA}_{\alpha} \), \( \alpha > 1 \):

1. Semisimple algebras in \( V \) have AP.
2. Simple algebras have in \( V \) AP.
3. Free \( V \) algebras have CP with respect to maximal congruences.
4. Free \( V \) algebras have WIP.
5. \( V \) has PP\(_{m} \).
6. \( V \) has RP\(_{m} \).

**Proof.** (1) implies (2) is trivial. (2) implies (1) by the argument used in the last part of theorem \( [1.21] \). (2) equivalent to (3) can be proved by using exactly the above argument using maximal congruences in place of congruences, hence amalgams will be simple algebras in place of algebras.

Now we prove that (3) is equivalent to (4). This is proved for cylindric algebras in \( [34] \). Assume CP relative to \( U \), where \( U \) is the set of proper maximal filters in subalgebras of \( \mathfrak{A} \). Let \( X_{1}, X_{2} \subseteq \mathfrak{A} \), and \( x \in \mathfrak{S}^{\mathfrak{A}}X_{1} \) and \( z \in \mathfrak{S}^{\mathfrak{A}}X_{2} \), such that \( x \leq z \) and assume for contradiction that there is no \( y \) and no finite \( \Gamma \subseteq \alpha \), such that \( \psi_{(\Gamma)}x \leq y \leq \psi_{(\Gamma)}z \).

Then \( \psi_{(\Gamma)}x - y \geq 0 \) or \( \psi_{(\Gamma)}[-z] \cdot y > 0 \) whenever \( y \in \mathfrak{S}^{\mathfrak{A}}(X_{1} \cap X_{2}) \).

Hence for any finite subsets \( \Delta, \theta \) of \( \alpha \), we have \( u \cdot w > 0 \) for all \( u, w \in \mathfrak{S}^{\mathfrak{A}}(X_{1} \cap X_{2}) \) such that \( u \geq \psi_{(\Delta)}x \) and \( w \geq \psi_{(\theta)}[-z] \). Let

\[
P = \mathfrak{S}^{\mathfrak{S}^{\mathfrak{A}}(X_{1} \cap X_{2})}[[\mathfrak{S}^{\mathfrak{S}^{\mathfrak{A}}X_{1}}\{x\} \cap \mathfrak{S}^{\mathfrak{S}^{\mathfrak{A}}(X_{1} \cap X_{2})}] \cup \mathfrak{S}^{\mathfrak{S}^{\mathfrak{A}}X_{2}}\{-z\} \cap \mathfrak{S}^{\mathfrak{S}^{\mathfrak{A}}(X_{1} \cap X_{2})}].
\]

Then \( P \) is proper, so let \( P' \) be a maximal proper filter in \( \mathfrak{S}^{\mathfrak{A}}(X_{1} \cap X_{2}) \) containing \( P \). Then there are maximal filters \( M \) of \( \mathfrak{S}^{\mathfrak{A}}X_{1} \) and \( N \) of \( \mathfrak{S}^{\mathfrak{A}}X_{2} \) such that (*)

\[
\mathfrak{S}^{\mathfrak{S}^{\mathfrak{A}}X_{1}}\{x\} \subseteq M \text{ and } \mathfrak{S}^{\mathfrak{S}^{\mathfrak{A}}X_{2}}\{-z\} \subseteq N,
\]
and \( M \cap \mathfrak{S}^\alpha(X_1 \cap X_2) = P' = N \cap \mathfrak{S}^\alpha(X_1 \cap X_2) \). By assumption, we have \( \mathfrak{F}^\alpha(M \cup N) \) is proper, and so it is not the case that \( x \leq z \), for if \( x \leq z \), then \( x \cdot (-z) = 0 \) and so by (*) we get \( 0 \in \mathfrak{F}^\alpha(M \cup N) \) and so \( \mathfrak{F}^\alpha(M \cup N) = \mathfrak{A} \). This is a contradiction and we are done.

For the converse. Assume that \( \mathfrak{A} \) has WIP. Let \( M \) be a filter of \( \mathfrak{S}^\alpha(X_1 \cap X_2) \) and \( N \) be a filter of \( \mathfrak{S}^\alpha(X_1 \cap X_2) \), both maximal, such that \( M \cap \mathfrak{S}^\alpha(X_1 \cap X_2) = N \cap \mathfrak{S}^\alpha(X_1 \cap X_2) \).

Assume for contradiction that \( \mathfrak{S}^\alpha(M \cup N) = \mathfrak{A} \). Then there exist \( x \in M, z \in N \) such that \( x \cdot z = 0 \). By assumption there is an element \( y \in \mathfrak{S}^\alpha(X_1 \cap X_2) \) and a finite \( \Gamma \subseteq \alpha \) such that \( q_{(\Gamma)}x \leq y \leq c_{(\Gamma)}z \), hence \( y \in \mathfrak{F}^{\mathfrak{S}^\alpha X_1}(\{x\}) \) and \( -y \in \mathfrak{F}^{\mathfrak{S}^\alpha X_2}(\{z\}) \), and so \( -y \in M \cap \mathfrak{S}^\alpha(X_1 \cap X_2) \) and \( y \in N \cap \mathfrak{S}^\alpha(X_1 \cap X_2) \). Hence \( 0 = -y \cdot y \in M \) which is impossible. We conclude that \( \mathfrak{F}^\alpha(M \cup N) \) is proper and maximal, and it induces the required maximal congruence.

It is clear that (5) and (3) are equivalent. We lastly prove the equivalence of (5) and (6). Suppose \( V \) has PP and that conditions (i) (ii) (iii), and (iv) are satisfied for the RP. Let \( Y = \text{Var}(\Sigma) \) and \( Z = \text{Var}(\Pi) \). Let \( \Theta_Y \) be the congruence generated by \( \Sigma \) in \( \mathfrak{F}(Y) \) and \( \Theta_Z \) be the congruence generated by \( \Pi \) in \( \mathfrak{F}(Z) \). Then both are maximal congruences and \( \Theta_Y \cap \mathfrak{F}(Y \cap Z)^2 = \Theta_Z \cap \mathfrak{F}(Y \cap Z)^2 \). Hence by PP there exists \( \Theta \in \text{Co}(\mathfrak{F}(Y \cup Z)) \) such that \( \Theta_Y = \Theta \cap \mathfrak{F}(Y)^2 \) and \( \Theta_Z = \Theta \cap \mathfrak{F}(Z)^2 \). We may assume that \( \Theta \) is the congruence generated by \( \Theta_Y \cup \Theta_Z \) in \( \mathfrak{F}(Y \cup Z) \). By (iv) we have \( \bar{\epsilon} \in \Theta \). But \( \text{Var}(\bar{\epsilon}) \subseteq Y \), we have \( \bar{\epsilon} \in \Theta_Y \) and \( \Sigma \models \bar{\epsilon} \).

Conversely, assume \( V \) has RP and that conditions (i), (ii), (iii) are satisfied for the PP. Choose \( \Sigma \) and \( \Pi \) such that \( \Theta_Y \) is the congruence generated by \( \Sigma \) in \( \mathfrak{F}(Y) \) and \( \Theta_Z \) is the congruence generated by \( \Pi \) in \( \mathfrak{F}(Z) \). Then (i) and (ii) of the RP hold. Let \( \Theta \) be the congruence generated by \( \Theta_Y \cup \Theta_Z \) in \( \mathfrak{F}(Y \cup Z) \) as is required.

The natural question at this point is. What does the usual interpolation property correspond to. On the global level it corresponds to the super amalgamation property. One implication can be distilled without much effort from the proof theorem ???. The other direction, that is SUPAP implies IP in free algebras is proved by Madarasz and Maksimova [11, 15], in a more general setting, of which Boolean algebras with operators, and cylindric algebras with interior operators, are a special case.

**Theorem 2.7.** Let \( \alpha \) be an ordinal \( > 0 \). Let \( V \) be a variety of TCA\( _\alpha \). Then the following conditions are equivalent:

1. \( V \) has SUPAP
2. The free algebras have IP.
If $\alpha < \omega$, then $\text{TCA}_\alpha$ is a discriminator variety, with discriminator term $c_{(\alpha)}$; in particular, every subdirectly indecomposable algebra is simple and hence every algebra is semisimple. This gives:

**Theorem 2.8.** If $\alpha < \omega$ then for any subvariety $V$ of $\text{TCA}_\alpha$ all conditions in theorems 2.2 and 2.6 are equivalent:

**Proof.** It suffices to show that free algebras have $WIP$ implies free algebras have $AIP$. Let $\mathfrak{A} = \mathfrak{F}_{\mathfrak{B}} V X_1, X_2 \subseteq \mathfrak{A}$, $a \in \mathfrak{S}g^{\mathfrak{A}} X_1$ and $b \in \mathfrak{S}g^{\mathfrak{A}} X_2$ such that $a \leq b$. Then $a \leq c_{(\alpha)} a \leq c_{(\alpha)} b$. Hence by $WIP$ there exists $d \in \mathfrak{S}g^{\mathfrak{A}} (X_1 \cap X_2)$ and $\Gamma \subseteq \alpha$ such that $a \leq c_{(\Gamma)} c_{(\alpha)} a \leq d \leq c_{(\Gamma)} c_{(\alpha)} b = c_{(\alpha)} b$, hence $\mathfrak{A}$ has $AIP$ and we are done. Notice that if $\alpha$ is the dimension $< \omega$ then both $AIP$ and $WIP$ are equivalent to; with $\mathfrak{A}$, $a, b$ as above; that there exists $d \in \mathfrak{S}g^{\mathfrak{A}} (X_1 \cap X_2)$ such that $a \leq d \leq c_{(\alpha)} b$. \qed

We will see in corollary 3.3 that the above theorem is not true for the infinite dimensional case. Indeed for $\alpha \geq \omega$, $\text{TCA}_\alpha$ is not a discriminator variety; subdirectly indecomposable algebras that are not simple can be easily constructed. We know from theorem 1.18 that the free algebras in any variety $V$ containing the representable algebras have $WIP$, but we will see in theorem 3.1 that $\text{RTCA}_\alpha$ does not have $AP$, hence by theorem 2.2 the free algebras do not have $AIP$.

Now we define yet other **restricted** interpolation properties, that are the adaptation of Pigozzi’s restricted forms of interpolation defined for cylindric algebras, to our present context. Here we look for the interpolant in the minimal subalgebra of the free algebra. From the logical point of view the formulas to be interpolated contain only the equality symbol as a common symbol, so we are looking for an interpolant that contains no other symbols, we are looking for a formula built up only of equations, that is their atomic subformulas are of the form $x_i = x_j \ (i, j \in \omega)$, where $x_i$ and $x_j$ are variables; reflected algebraically by the diagonal element $d_{ij}$.

**Definition 2.9.** Let $\mathfrak{A} \in \text{TCA}_\alpha$.

1. $\mathfrak{A}$ has the **restricted interpolation property** if whenever $x \leq z$, $x \in \mathfrak{S}g^{\mathfrak{A}} Y$ and $z \in \mathfrak{S}g^{\mathfrak{A}} Z$, with $Y \cap Z = \emptyset$, then there exists $y \in \mathfrak{S}g^{\mathfrak{A}} (Y \cap Z)$ such that either $x \leq y \leq z$.

2. $\mathfrak{A}$ has the **almost restricted interpolation property** if whenever $x \leq z$, as in the previous item, then there exist $y \in \mathfrak{S}g^{\mathfrak{A}} (Y \cap Z)$ and a finite $\Gamma \subseteq \alpha$, such that $q_{(\Gamma)} x \leq y \leq z$.

3. $\mathfrak{A}$ the **weak restricted interpolation property** if whenever $x \leq z$ as in the previous item, then there exist $y \in \mathfrak{S}g^{\mathfrak{A}} (Y \cap Z)$ and a finite $\Gamma \subseteq \alpha$ such that $q_{(\Gamma)} x \leq y \leq c_{(\Gamma)} z = 1$. 

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In the above definition the algebra $S_g(Y \cap Z)$ is a minimal algebra, it is generated by the diagonal elements, and it has no proper subalgebras. Let $TM_n\alpha$ denote the class of such minimal algebras, namely, algebras with no proper subalgebras. Clearly $TM_n\alpha \subseteq TDc_\alpha$ for infinite $\alpha$. This simple observation will be used in the coming proof.

**Theorem 2.10.** Let $\alpha$ be an infinite ordinal. Let $\beta$ be any cardinal $> 0$. Then $Fr\beta RTCA_\alpha$ has the strong restricted interpolation property.

**Proof.** Let $A = Fr\beta TRCA_\alpha$ and let $X_1, X_2 \subseteq \beta$ be disjoint sets. We can assume without loss of generality that $X_1 \cup X_2 = \beta$. Assume that $a \in A_1 = S_g^\alpha X_1$ and $b \in A_2 = S_g^\alpha X_2$ such that $a \leq b$. Since $X_1 \cap X_2 = \emptyset$, we have $A_0 = S_g^\alpha (X_1 \cap X_2) = 2$ embeds in $S_g^\alpha X_1$ and $S_g^\alpha X_2$, respectively via the inclusion maps $i_0$ and $i_1$ say. From theorem 1.14 that there is a $D \in RTCA_\alpha$, a monomorphism $m_1$ from $A_1$ into $D$ and a monomorphism $m_2$ from $A_2$ into $D$ such that $m_1 \circ i_1 = m_2 \circ i_2$, and $(\forall x \in A_j)(\forall y \in A_k)(m_j(x) \leq m_k(y) \implies (\exists z \in A_0)(x \leq i_j(z) \land i_k(z) \leq y))$ where $\{j, k\} = \{1, 2\}$. Now since $A$ is free, there exists a homomorphism $f : A \rightarrow D$ such that $f \upharpoonright A_1 = m_1$ and $f \upharpoonright A_2 = m_2$. Since $f(a) \leq f(b)$ it follows that $m_1(a) \leq m_2(b)$. Hence there exists $z \in A_0$ such that $a \leq z \leq b$.

Before stating our next result, we need:

**Definition 2.11.** $K$ has the (strong) embedding property if it has the (strong) amalgamation property when the base algebra is is minimal.

**Corollary 2.12.** Let $\alpha$ be an infinite ordinal. Then $RTCA_\alpha$ has the strong embedding property. Furthermore, if the algebras $A$ and $B$ to be amalgamated (agreeing on their minimal subalgebras) are simple, semi-simple or in $Sc_\alpha$ then so is the amalgam.

**Proof.** The first part is from the proof of theorem 2.10 using theorem 1.14. The second part follows from the arguments used in theorem 1.21.

We will see in a while that for finite $\alpha > 1$, the situation is different. The class $RTCA_\alpha$ does not have the embedding property and $Fr\beta K$ for any class $K$ between $RTCA_\alpha$ and $TCA_\alpha$ does not have the weakest restricted interpolation property.

### 3 Negative results

Now that we have obtained such equivalences, the natural question is how far can we get as far as interpolation is concerned with the free representable algebras. We know that they enjoy the weak interpolation property. In the next theorem we show that the representable algebras does not have $AP$. It
is known \cite{17} that the class RCA\(_\alpha\) for infinite \(\alpha\) does not have AP, so obtain an analogous result by bouncing it to the cylindric case.

**Theorem 3.1.** Let \(\alpha \geq \omega\). Then any class \(K\) such that \(TCA\alpha \subseteq K \subseteq TRCA\alpha\) does not have AP.

**Proof.** Take \(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2\) in RCA\(_\alpha\) and \(f: \mathfrak{A}_0 \to \mathfrak{A}_1\), and \(g: \mathfrak{A}_0 \to \mathfrak{A}_1\) injective homomorphisms for which there are no \(\mathfrak{D} \in RCA\alpha\) and injective homomorphisms \(m: \mathfrak{A}_1 \to \mathfrak{D}\) \(n: \mathfrak{A}_2 \to \mathfrak{D}\) such that \(m \circ f = n \circ g\), endow the base of each the interior topology stimulating the interior operators as identity functions, and so finding an amalgam for the resulting RTCA\(_\alpha\)s (with same embedding maps), will give an amalgam to the original RCA\(_\alpha\)s, by taking its CA reduct \cite{17,22} which is a contradiction. We explicitly describe such algebras. Let \(\mathfrak{A} = \mathfrak{Fr}_{4}CA\alpha\) with \(\{x, y, z, w\}\) its free generators. Let \(X_1 = \{x, y\}\) and \(X_2 = \{x, z, w\}\). Let \(r, s\) and \(t\) be defined as follows:

\[
\begin{align*}
r &= c_0(x \cdot c_1y) \cdot c_0(x \cdot -c_1y), \\
s &= c_0c_1(c_1z \cdot s_0^1c_1z \cdot -d_{01}) + c_0(x \cdot -c_1z), \\
t &= c_0c_1(c_1w \cdot s_0^1c_1w \cdot -d_{01}) + c_0(x \cdot -c_1w),
\end{align*}
\]

where \(x, y, z, w\) are the first four free generators of \(\mathfrak{A}\). Then \(r \leq s \cdot t\). Let \(\mathfrak{D} = \mathfrak{Fr}_{4}RCA\alpha\) with free generators \(\{x', y', z', w'\}\). Let \(\psi: \mathfrak{A} \to \mathfrak{D}\) be defined by the extension of the map \(t \mapsto t'\), for \(t \in \{x, y, x, w\}\). For \(i \in \mathfrak{A}\), we denote \(\psi(i) \in \mathfrak{D}\) by \(i'\). Let \(I = J_{\mathfrak{g}_{\mathfrak{D}(X_1)}}\{r'\}\) and \(J = J_{\mathfrak{g}_{\mathfrak{D}(X_2)}}\{s' \cdot t'\}\), and let

\[
L = I \cap \mathfrak{D}^{(X_1 \cap X_2)}\text{ and } K = J \cap \mathfrak{D}^{(X_1 \cap X_2)}.
\]

Then \(L = K\), and \(\mathfrak{A}_0 = \mathfrak{D}^{(X_1 \cap X_2)}/L\) can be embedded into \(\mathfrak{A}_1 = \mathfrak{D}^{(X_1)}/I\) and \(\mathfrak{A}_2 = \mathfrak{D}^{(X_2)}/J\), but there is no amalgam even in \(CA\omega\) \cite{27,Theorem 4.1]. \(\Box\)

We have a stronger result for finite dimensions:

**Theorem 3.2.** Let \(n\) be finite \(> 1\). Then any class \(K\) between TRCA\(_n\) and TCA\(_n\) does not have EP. Furthermore, the algebras witnessing failure of EP can be chosen to be set algebras hence are simple.

**Proof.** Assume first that \(n > 1\). Let \(1 < n \leq |U_0| < |U_1|\) and \(|U_0| < \omega\). Let \(\mathfrak{A}_i = \mathcal{A}(n, U_i)\) be the set algebra with unit \(^nU_0\) and universe \(\varphi(nU_0)\) where \(U_i\) has the discrete topology. Let \(D_i\) be the principal diagonal in \(\mathfrak{A}_i\). That is \(D_i = \prod_{k,l<n}d_{kl}\). Let \(\mathfrak{M}\) be the minimal subalgebra of \(\mathfrak{A}_0\). Then \(\mathfrak{M}\) is embeddable in \(\mathfrak{A}_0\) and \(\mathfrak{A}_1\) via \(g_0\) and \(g_1\) such that \(g_0 \circ g^{-1}\) is an isomorphism from \(g_0\mathfrak{M}\) and \(g_1\mathfrak{M}\) and \(g_1g_0^{-1}D_0 = D_1\). Here we are using that the minimal subalgebras of \(\mathfrak{A}_0\) and \(\mathfrak{A}_1\) are isomorphic, and they remain so after endowing their bases with the discrete topology. This follows from the fact that they
are both simple and have characteristic zero \(2.5.30\). Then, as proved in \[3\] there can be no \(B \in TCA_n\), \(f_0 : A_0 \to B'\) and \(f_1 : A_1 \to B'\) injective homomorphisms such that \(f_1 \circ g_1 = f_0 \circ g_0\).

Note that if \(n = 1\) and we drop the cross axiom \(s^i_j I(i) = I(j)s^i_j\), then the same set algebras can be viewed as one dimensional algebras, with \(I_0\) interpreted like the second cylindrifier, which means that \(EP\) fails for this class of one dimensional algebras.

An algebra in \(A \in TCS_\alpha\) is called a full set algebra if the universe of \(A\) is \(\wp(\alpha U)\) for some set \(U\), that is, it consists of all subsets of \(\alpha U\).

**Corollary 3.3.** For \(\alpha \geq \omega\) any subvariety of \(TCA_\alpha\) containing the class of all full set algebras of dimension \(\alpha\) fails all the conditions of theorem 2.2, but satisfies all conditions of theorem 2.6. If \(\alpha\) is finite \(> 1\), any variety containing the class of all full set algebras of dimension \(\alpha\) fails all conditions of theorem 2.6, hence also those in 2.2.

**Proof.** From theorems 3.1 and 3.2.

Next we give a categorical formulation to the \(UNEP\). We review some categorical concepts from \[9\]. For a category \(L\), \(\text{Ob}(L)\) denotes the class of objects of the category and \(\text{Mor}(L)\) denotes the corresponding class of morphisms.

**Definition 3.4.** Let \(L\) and \(K\) be two categories. Let \(G : K \to L\) be a functor and let \(B \in \text{Ob}(L)\). A pair \((u_B, A_B)\) with \(A_B \in \text{Ob}(K)\) and \(u_B : B \to G(A_B)\) is called a universal map with respect to \(G\) (or a \(G\) universal map) provided that for each \(A' \in \text{Ob}(K)\) and each \(f : B \to G(A')\) there exists a unique \(K\) morphism \(\bar{f} : A_B \to A'\) such that

\[
G(\bar{f}) \circ u_B = f.
\]

\[
\begin{array}{ccc}
\mathcal{B} & \xrightarrow{u_B} & G(A_B) \\
\downarrow f & & \downarrow G(f) \\
A_B & \xrightarrow{j} & G(A') \\
\end{array}
\]

Viewing the neat reduct operator as a functor introduced next, is a theme initiated in \[27\], wrapping up deep results on the amalgamation property for both cylindric and polyadic algebras in the language of arrows.

**Definition 3.5.** The neat reduct functor, \(\mathfrak{N}\) for short, is defined from \(K = \{A \in TCA_{\alpha+\omega} : \exists g^a_{\mathfrak{N}_a A} A = A\}\) to \(RTCA_\alpha\) by sending every object \(A \in K\), to \(\mathfrak{N}_a A\), and sending injective homomorphisms to their restrictions, that is for \(A, B \in K\) and \(f : A \to B\), an injective homomorphism, \(\mathfrak{N}_a(f) = f \upharpoonright \mathfrak{N}_a A\).
It is clear that $f(\mathfrak{M}_\alpha, \mathfrak{A}) \subseteq \mathfrak{M}_\beta, \mathfrak{B}$, hence this functor is well defined. Here we are restricting morphisms to only injective homomorphisms.

For a class $K$, we let $\text{Sim}(K)$ denotes the class of simple algebras in $K$.

**Corollary 3.6.** Let $\alpha$ be an infinite ordinal.

(1) For any $k \geq 4$, and any class $K$ such that $\text{RTCA}_\alpha \subseteq K \subseteq \text{TCA}_\alpha$, $\text{Fr}_k K$ does not have AIP but has WIP.

(2) $\text{RTCA}_\alpha$ does not have UNEP.

(3) For any $k \geq 0$, the variety $\mathfrak{N}_\alpha \text{TCA}_{\alpha+k}$ does not have AP, hence it does not have MIP, indeed it does not have any of the properties in theorem 2.2 but it has WIP, hence satisfy all properties in theorem 2.6.

(4) There is an $\mathfrak{A} \in \text{TCA}_\alpha$ that does not have a universal map with respect to the functor $\mathfrak{M}$.

(5) $\mathfrak{M}$ does not have a right adjoint.

**Proof.**

(1) By theorem 1.18 and 3.1.

(2) Let $\mathfrak{A}_0$ be as in the proof of theorem 3.1 so that $\mathfrak{A}_0$ is not in the amalgamation base of $\text{RTCA}_\alpha$. If $\mathfrak{A}_0$ has UNEP then using the same reasoning in theorem 1.14 by appeal to theorem ??, $\mathfrak{A}_0$ would be in the amalgamation base of $\text{RTCA}_\alpha$ which is impossible.

(3) The algebras $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_1$ in the proof of theorem 3.1 are representable, hence they are in $\mathfrak{N}_\alpha \text{TCA}_{\alpha+\omega}$ but they do not have an amalgam in $\text{TCA}_\alpha$, a fortiori in $\mathfrak{N}_\alpha \text{TCA}_{\alpha+k}$. WIP follows from theorem 1.18 or by noting that $\text{Sim}(\text{TCA}_\alpha) = \text{Sim}(\text{RTCA}_\alpha) = \text{Sim}(\mathfrak{N}_\alpha \text{TCA}_{\alpha+k})$ and then using theorem 2.6.

(4) Let $\mathfrak{A}_0$ be as in the proof of theorem 3.1. Then $\mathfrak{A}_0$ does not have UNEP. So $\mathfrak{A}_0$ generates non-isomorphic algebras in extra dimensions, the existence of a universal map for it, will force that these algebras are actually isomorphic, fixing it pointwise, and this cannot happen.

In more detail, assume that $\mathfrak{A}_0$ neatly embeds into $\mathfrak{B}$ via $e_B$ and into $\mathfrak{B}'$ via $e_{B'}$. Let $(e, \mathfrak{C})$ be a universal map for $\mathfrak{A}_0$, so that $\mathfrak{A}$ neatly embeds into $\mathfrak{C}$ via $e$. (See the above diagram). By universality, there exists isomorphisms $f : \mathfrak{C} \to \mathfrak{B}$ and $k : \mathfrak{C} \to \mathfrak{B}'$ such that $f \circ e = e_B$ and $k \circ e = e_{B'}$. The maps are injective by definition, they are surjective, because $\mathfrak{A}_0$ is contained in $\mathfrak{C}$ and it generates both $\mathfrak{B}$ and $\mathfrak{B}'$. We infer that $\mathfrak{B}$ and $\mathfrak{B}'$ are isomorphic, but we want more. We want to exclude special isomorphisms (in principal, isomorphisms can exist as long as
they do not fix $\mathfrak{A}$ pointwise). We have $h = k \circ f^{-1} : \mathfrak{B} \to \mathfrak{B}'$ is an isomorphism such that $h \circ e_B = e_B'$, and this isomorphism is as required, leading to a contradiction.

(5) It is known [9, Theorem 27.3 on p. 196] that if $G : K \to L$ is a functor such that each $\mathfrak{B} \in Ob(K)$ has a $G$ universal map $(\mu_B, \mathfrak{A}_B)$, then there exists a unique adjoint situation $(\mu, \epsilon) : F \to G$ such that $\mu = (\mu_B)$ and for each $\mathfrak{B} \in Ob(L)$, $F(\mathfrak{B}) = \mathfrak{A}_B$. Conversely, if we have an adjoint situation $(\mu, \epsilon) : F \to G$ then for each $\mathfrak{B} \in Ob(L)$ $(\mu_B, F(\mathfrak{B}))$ have a $G$ universal map.

\[ \square \]

**Theorem 3.7.** For any $\alpha > 0$, the following conditions are equivalent for $\mathfrak{A} \in TCA_\alpha$.

1. $\mathfrak{A}$ has the UNEP.
2. $\mathfrak{A} \in \text{APbase}(RTCA_\alpha)$.
3. $\mathfrak{A}$ has a universal map with respect to the functor $\text{Nr}$.

**Proof.** This is proved for cylindric algebras in [34]. The proof lifts with no modifications. \[ \square \]

We need another algebraic counterpart of a yet another definability property, namely, Beth definability. A class $\mathcal{K}$ has ES if epimorphisms in $\mathcal{K}$ (in the categorical sense) are surjective.

The next theorem is folklore [12, 13, 14].

**Theorem 3.8.** Let $\mathcal{K}$ be a class of algebras. Then

1. If $\mathcal{K}$ has $\text{SUPAP}$ then it has $\text{SAP}$, $\text{AP}$ and $\text{ES}$.
2. $\mathcal{S}\mathcal{P}\mathcal{K}$ has $\text{SAP}$ if and only if it has $\text{AP}$ and $\text{ES}$.

Our next ES result can also be easily distilled from the cylindric counterpart, as follows.

**Theorem 3.9.** Let $\alpha > 1$. Then for any class $\mathcal{K} \subseteq TCA_\alpha$, that contains the class of simple representable algebras, ES, hence SAP fail. In particular, for $\alpha \geq \omega$ and $k \geq 1$, ES fails for $S\mathcal{N}\mathcal{R}_\alpha TCA_{\alpha+k}$.

**Proof.** Assume that $\alpha \geq \omega$. Notice that in this case all simple algebras are representable. Let $\mathfrak{A}, \mathfrak{B}$ be the algebras such that $\mathfrak{A} \subseteq \mathfrak{B}$ is an epimorphism that is not surjective. Such algebras are constructed in [13]; they are weak set algebras, $\mathfrak{A}$ is generated by a single element $R$ and $\mathfrak{B}$ is generated by two elements $R$ and $X$ hence $X \notin \mathfrak{A}$. Also $\mathfrak{A}$ and $\mathfrak{B}$ have the same unit $V$, with
common base $U$ that can be endowed with the discrete topology. So we can assume that $\mathfrak{A}, \mathfrak{B} \in \text{RTCA}_\alpha$ and still we have $\mathfrak{A} \subseteq \mathfrak{B}$. Assume that $\mathfrak{D} \in \text{TCA}_\alpha$ and homomorphisms $f, g : \mathfrak{B} \to \mathfrak{D}$ such that $f(X) \neq g(X)$, this is impossible because then $\mathfrak{D}$ witnesses that $\mathfrak{A} \subseteq \mathfrak{B}$ is not an epimorphism. Hence there is $\mathfrak{D} \in \text{TCA}_\alpha$ and homomorphisms $f, g : \mathfrak{B} \to \mathfrak{D}$ such that $f(X) \neq g(X)$, this is impossible because then $\mathfrak{D}$ witnesses that $\mathfrak{A} \subseteq \mathfrak{B}$ is not an epimorphism.

In [13] it is shown that $\mathfrak{A}$ and $\mathfrak{B}$ can be chosen to be semi-simple and in [26] it is shown that they can further be chosen to be simple. For finite $\alpha > 1$ the required follows easily from the construction in [1] expanding, like we often did before, the constructed algebras by the identity interior operators corresponding to the discrete topology. The last part follows from the fact that $\text{RTCA}_\alpha \subseteq \text{Sn}_{\mathfrak{r}_\alpha} \text{CA}_{\alpha+k} \subseteq \text{TCA}_\alpha$ for all $\alpha$.

Theorem 3.10. (1) Let $\mathfrak{D}$ the full set algebra with unit $\omega$; with $\omega$ having the discrete topology. Let $\mathfrak{M}$ be its minimal subalgebra. Then $\mathfrak{M} \in \text{SUPAPbase}(\text{RTCA}_\omega) \sim \text{APbase}(\text{TCA}_\omega)$. Furthermore, $\text{TCA}_\omega$ does not have $\text{EP}$, and $\text{Fr}_\omega \text{CA}_\omega$ does not have weak restricted $\text{IP}$.

(2) If $1 < \alpha < \omega$, then for any $K$ such that $\text{RTCA}_\alpha \subseteq K \subseteq \text{TCA}_\alpha$, $\text{Fr}_\omega K$ does not have weak restricted $\text{IP}$.

Proof. For the infinite dimensional case witness [29, 35]. The second part follows from the reasoning in theorem 2.6. The second item concerning finite dimensional algebras follows from theorem 3.2.

4 Representability and amalgamation for various topological polyadic algebras

The class $\text{TPCA}_\alpha$, to be dealt with next, is defined by restricting the signature and axiomatization of Halmos’ polyadic algebras to finite cylindrifiers, so that we have all substitution operators but only $c_i$ for $i \in \alpha$, and interior operators $I_i$. Here we do not have diagonal elements and we consider only infinite dimensions. In more detail: The dimension set of $x$, in symbols $\Delta x$, is defined exactly as in the TCA case; that is $\Delta x = \{i \in \alpha : c_i x \neq x\}$. If $\mathfrak{A} \in \text{TPCA}_\alpha$, then $\mathfrak{A}_\text{pa} \mathfrak{A}$ denotes its reduct obtained by discarding all interior operators. Henkin ultrafilters are defined exactly like before; they are the ultrafilters that eliminate cylindrifiers. That is for $\mathfrak{A} \in \text{TPCA}_\alpha$ a Boolean ultrafilter $F$ is Henkin if for all $x \in \mathfrak{A}$ for all $k < \alpha$, if $c_k x \in F$, then there exists $l \notin \Delta x$, such that $s^l_k x \in F$.

Theorem 4.1. Let $\alpha < \beta$ be infinite ordinals. Then for every $\mathfrak{A} \in \text{TPCA}_\alpha$ there is a unique $\mathfrak{B} \in \text{TPCA}_\beta$ up to isomorphism that fixes $\mathfrak{A}$ such that $\mathfrak{A} \subseteq \mathfrak{N}_{\mathfrak{r}_\alpha} \mathfrak{B}$ and for all $X \subseteq A$, $\mathfrak{S}g^\mathfrak{A} X = \mathfrak{N}_{\mathfrak{r}_\alpha} \mathfrak{S}g^\mathfrak{B} X$. In particular, $\text{TPCA}_\alpha$ has $\text{NS}$ and $\text{UNEP}$. Furthermore, if $F$ is a Henkin ultrafilter of $\mathfrak{B}$ and $a \in F$, then
there exists a topology on $\beta$ and a homomorphism $f : \mathfrak{A} \to (\varphi(\alpha \beta), J_i)_{i < \alpha}$ with $f(a) \neq 0$ where for $i < \alpha$ and $X \subseteq ^\alpha \beta$, $J_iX = \{s \in ^\alpha \beta : s_i \in \text{int}\{u \in U : s_i^u \in X\}\}$.

**Proof.** Dilations are proved to exist similar to the arguments used in [5][27]; the rest follows using analogues of [31] Theorems 2.9, 3.6. Nevertheless forming dilations here is more involved. We extensively use the techniques in [5], but we have to watch out, for we only have finite cylindrifications. Let $(\mathfrak{A}, \alpha, S)$ be a transformation system in the sense of [5]. Substitutions in $\mathfrak{A}$, induce a homomorphism of semigroups $S : ^\alpha \alpha \to \text{End}(\mathfrak{A})$, via $\tau \mapsto s_\tau$. The operation on both semigroups is composition of maps; the latter is the semigroup of endomorphisms on $\mathfrak{A}$. For any set $X$, let $F(^\alpha X, \mathfrak{A})$ be the set of all functions from $^\alpha X$ to $\mathfrak{A}$ endowed with Boolean operations defined pointwise and for $\tau \in ^\alpha \alpha$ and $f \in F(^\alpha X, \mathfrak{A})$, put $s_\tau f(x) = f(x \circ \tau)$. This turns $F(^\alpha X, \mathfrak{A})$ to a transformation system as well. The map $H : \mathfrak{A} \to F(^\alpha \alpha, \mathfrak{A})$ defined by $H(p)(x) = s_x p$ is easily checked to be an embedding. Assume that $\beta \supseteq \alpha$. Then $K : F(^\alpha \alpha, \mathfrak{A}) \to F(^\beta \alpha, \mathfrak{A})$ defined by $K(f)x = f(x \uparrow \alpha)$ is an embedding, too. These facts are straightforward to establish, cf. [3] Theorems 3.1, 3.2. Call $F(^\beta \alpha, \mathfrak{A})$ a minimal functional dilation of $F(^\alpha \alpha, \mathfrak{A})$. Elements of the big algebra, or the (cylindrifier free) functional dilation, are of form $s_\sigma p$, $p \in F(^\beta \alpha, \mathfrak{A})$ where $\sigma$ is one to one on $\alpha$, cf. [3] Theorems 4.3-4.4.

We can assume that $|\alpha| < |\beta|$. Let $\mathfrak{B}$ be the algebra obtained from $\mathfrak{A}$, by discarding its cylindrifiers, then dilating it to $\beta$ dimensions, that is, taking a minimal functional dilation in $\beta$ dimensions, and then re-defining cylindrifiers in the bigger algebra, on the big algebra, so that they agree with their values in $\mathfrak{A}$ as follows (*):

$$c_\xi s^{\mathfrak{B}}_\sigma p = s^{\mathfrak{B}}_{\sigma, i} c_{\rho \rho^{-1} \cap \sigma (x) \cap \sigma (\alpha)} s^{\mathfrak{A}}_{\rho \sigma (|\alpha|)} p.$$  

$$I(k) s^{\mathfrak{B}}_\sigma p = s^{\mathfrak{B}}_{\sigma, i} I(\rho \{k\} \cap \sigma (\alpha)) s^{\mathfrak{A}}_{\rho \sigma (|\alpha|)} p.$$  

Here $\rho$ is a any permutation such that $\rho \circ \sigma (|\alpha|) \subseteq \sigma (\alpha)$. It can be checked by a somewhat tedious computation [5] that the definition is sound; in other words it is independent of $\rho, \sigma, p$, and it defines the required dilation.

To prove UNEP let $\mathfrak{A}, \mathfrak{A}' \in \text{TPCA}_{\alpha}$ and $\beta > \alpha$. Let $\mathfrak{B}, \mathfrak{B}' \in \text{TPCA}_{\beta}$ and assume that $e_A, e_{A'}$ are embeddings from $\mathfrak{A}, \mathfrak{A}'$ into $\mathfrak{N} \mathfrak{R}_{\alpha} \mathfrak{B}, \mathfrak{N} \mathfrak{R}_{\alpha} \mathfrak{B}'$, respectively, such that $\mathfrak{D} g^{\mathfrak{B}}(e_A(A)) = \mathfrak{B}$ and $\mathfrak{D} g^{\mathfrak{B}'}(e_{A'}(A')) = \mathfrak{B}'$, and let $i : \mathfrak{A} \to \mathfrak{A}'$ be an isomorphism. We need to “lift” $i$ to $\beta$ dimensions. Let $\mu = |A|$. Let $x$ be a bijection from $\mu$ onto $A$. Let $y$ be a bijection from $\mu$ onto $A'$, such that $i(x_j) = y_j$ for all $j < \mu$. Let $\mathfrak{D} = \mathfrak{N} \mathfrak{R}_{\mu} \text{TPCA}_{\beta}$ with generators $(\xi_i : i < \mu)$. Let $\mathfrak{C} = \mathfrak{D} g^{\mathfrak{D}}(i) \{\xi_i : i < \mu\}$. Then $\mathfrak{C} \subseteq \mathfrak{N} \mathfrak{R}_{\alpha} \mathfrak{D}$, $C$ generates $\mathfrak{D}$ and so by the previous lemma $\mathfrak{C} = \mathfrak{N} \mathfrak{R}_{\alpha} \mathfrak{D}$. There exist homomorphisms $f : \mathfrak{D} \to \mathfrak{B}$ and $f' : \mathfrak{D} \to \mathfrak{B}'$ such that $f(\xi) = e_A(x)$ and $f'(\xi) = e_{A'}(y)$ for all $\xi < \mu$. Note that $f$ and $f'$ are both surjective.
have \( e_A \circ i^{-1} \circ e^{-1}_A \circ (f' \restriction \mathcal{C}) = f \restriction \mathcal{C} \). Therefore \( \ker f' \cap \mathcal{C} = \ker f \cap \mathcal{C} \). Hence by \( NS \) we have \( \ker f' = \mathfrak{F}(\ker f' \cap \mathcal{C}) = \mathfrak{F}(\ker f \cap \mathcal{C}) = \ker f \).

Let \( y \in B \), then there exists \( x \in D \) such that \( y = f(x) \). Define \( \hat{i}(y) = f'(x) \). The map is well defined and is as required.

For the last part, assume that a Henkin ultrafilter \( F \) is given and define \( f : \mathfrak{A} \rightarrow \wp(\alpha \beta) \) via
\[
p \mapsto \{ \tau \in \alpha \beta : s_{\tau \cup I_d_{\beta \alpha}}p \in F \}.
\]
Then like the proof of [31, Lemma 3.4] \( f \) preserves the polyadic operations. Handling interior operators here is easier, for we do not have diagonal elements, and hence we are not forced to define a congruence on the base of the representation as done before in case of cylindric topological algebras. For \( i \in \alpha \) and \( p \in \mathfrak{A} \), let
\[
O_{p,i} = \{ k \in \beta : s^k_i I(i)p \in F \}
\]
Let
\[
\mathcal{B} = \{ O_{p,i} : i \in \alpha, p \in A \}.
\]
Then it is easy to check that \( \mathcal{B} \) is the base for a topology on \( \beta \). Let \( W = \alpha \beta \).

For each \( i < \alpha \)
\[
J_i : \wp(W) \rightarrow \wp(W)
\]
by
\[
x \in J_i X \iff \exists U \in \mathcal{B}(x_i \in U \subseteq \{ u \in \alpha : x^i_u \in X \}),
\]
where \( X \subseteq W \). We now check that \( f \) preserves the interior operators \( J(i) \) (\( i < \alpha \)), too. We need to show
\[
\psi(I_i p) = J_i(\psi(p)).
\]
Let \( x \) be in \( \psi(I_i p) \). Then \( x_i \in \{ u : s^k_i I(i)s_{\gamma}p \in F \} \in q \) where \( y : \alpha \rightarrow \beta \), \( y \uparrow \alpha \sim \{ i \} \), "(i)" = \( x \uparrow \alpha \sim \{ i \} \) and \( y(i) = i \). But \( I_s s_{\gamma}p \leq s_{\gamma}p \), hence
\[
U = \{ u : s^k_i I_i s_{\gamma}p \in F \} \subseteq \{ u : s^k_i s_{\gamma}p \in F \}.
\]
It follows that \( x_i \in U \subseteq \{ u : x^i_u \in \Psi(p) \} \). Thus \( x \in J_i \psi(p) \). The other direction is the same as in the proof of [31, Lemma 3.6].

\[\square\]

**Corollary 4.2.** For any ordinal \( \beta > \alpha \) \( \text{TPCA}_\alpha = \text{S}\alpha \text{TPCA}_\beta \).

**Theorem 4.3.**

(1) \( \text{S}\beta \text{TPCA}_\alpha \) has the interpolation property.

(2) \( \text{TPCA}_\alpha \) has \( \text{SUPAP} \).
Proof. The proof of the first item is like the proof of theorem 1.1 undergoing the obvious modifications, and the second item follows from the first using the reasoning in theorem ??.

Note that the representability of any $A \in \text{TPCA}_{\alpha}$ can be easily proved using a simpler version of the above technique, where only one Henkin ultrafilter is needed to establish representability.

Now do we have an omitting types theorem in the context of $\text{TPCA}_{\alpha}$? The question itself is problematic because the signature of $\text{TPCA}_{\alpha}$ is necessarily uncountable even if the dimension is countable because in this case we have continuum many substitution operators, and it is known that for ‘omitting types theorems’ tied so much to the Baire category theorem, countability is essential. But here we have the related notion to omitting types that can be approached in our new context, namely, that of complete representability, summarized in the following question: If $A$ is in $\text{TPCA}_{\alpha}$, is there a representation of $A$ that preserves infinite meets and joins, whenever they exist?

We make the notion of representation precise in our new context. $\mathfrak{B}(X)$ denotes the Boolean set algebra $(\wp(X), \cup, \cap, \sim, \emptyset)$.

Definition 4.4. A representation of $A$ is a pair $(f, V)$ such that $V = \bigcup_{i \in I} \alpha U_i$ for some indexing set $I$, and a family $U_i : i \in I$, of non-empty sets that are pairwise disjoint, that is, $U_i \cap U_j = \emptyset$ for distinct $i \neq j$, and $f : A \to \langle \mathfrak{B}(V), c_i, I_i, s_i \rangle \subseteq \alpha \cup \circ \alpha$ is an injective homomorphism, where $I_i$ as usual is defined by $I_i(X) = \{s \in V : s_i \in \text{int} \{u \in \bigcup_{s \in V} \text{rng}s : s_i \in X\}\}, X \subseteq V$.

In what follows we may drop the operations when talking about $\alpha$ dimensional set algebras (whose top elements consists of $\alpha$-ary sequences) identifying notionally the algebra with its universe. This does not cause any harm since set algebras are uniquely defined by their top element.

A completely additive Boolean algebra with operators is one for which all extra non-Boolean operations preserve arbitrary joins.

Lemma 4.5. Let $A \in \text{TPCA}_{\alpha}$. A representation $f$ of $A$ is atomic if and only if it is complete. If $A$ has a complete representation, then $\mathfrak{R}_{pa}A$ is completely additive.

Proof. The first part is like 10. For the second part replace $A$ by its complete representation where $\sum$ is $\bigcup$. It is clear that in such an algebra the operations of substitutions and cylindrifiers are completely additive.

By Lemma 4.5 a necessary condition for the existence of complete representations is the condition of atomicity and complete additivity of its CPA reduct. We now prove the harder converse to this result, namely, that when $A$ is atomic and $\mathfrak{R}_{pa}A$ is completely additive, then $A$ is completely representable. We note that cylindrifiers are in all cases completely additive.
Theorem 4.6. Any atomic algebra in \( \mathfrak{A} \in \text{TPCA}_\alpha \) such that \( \mathfrak{R}_{\rho_\alpha} \mathfrak{A} \) is completely additive, is completely representable.

Proof. Argument used is like the argument in [27, Theorem 3.10] using a Henkin construction, expressed algebraically by dilating the algebra to large enough dimensions and then forming a Henkin ultrafilter (defined as before) in the dilation, with a very simple topological fact, namely, that in the Stone space of an atomic Boolean algebra principal ultrafilters lie outside sets of the first category; these are countable unions of nowhere dense sets; so we could always find a principal Henkin ultrafilter from which we build the complete representation.

Let \( c \in \mathfrak{A} \) be non-zero. We will find a set algebra \( \mathfrak{B} \in \text{TPCA}_\alpha \) and a homomorphism from \( f : \mathfrak{A} \to \mathfrak{B} \) that preserves arbitrary suprema whenever they exist and also satisfies that \( f(c) \neq 0 \). Now there exists \( \mathfrak{B} \in \text{TPCA}_\alpha \), \( n \) a regular cardinal, such that \( \mathfrak{A} \subseteq \mathfrak{R}_\alpha \mathfrak{B} \) and \( \mathfrak{A} \) generates \( \mathfrak{B} \) and we can assume that \( |n \sim \alpha| = |n| \). We also have for all \( Y \subseteq \mathfrak{A} \), we have \( \mathcal{G}_\mathfrak{A}(Y) = \mathfrak{R}_\alpha(\mathcal{G}_\mathfrak{B}(Y)) \). This dilation also has Boolean reduct isomorphic to \( F(\alpha, \mathfrak{A}) \), in particular, it is atomic because \( \mathfrak{A} \) is atomic. Cylindrifiers are defined on this minimal functional dilation exactly like in theorem 4.1 by restricting to singletons. For all \( i < n \), we have

\[
c_i p = \sum s_i^j p
\]  

This last supremum can be proved to hold using the same reasoning in [5, Theorem 1.6]. Let \( X \) be the set of atoms of \( \mathfrak{A} \). Since \( \mathfrak{A} \) is atomic, then \( \sum^\mathfrak{A} X = 1 \). By \( \mathfrak{A} = \mathfrak{R}_\alpha \mathfrak{B} \), we also have \( \sum^\mathfrak{B} X = 1 \). Because substitutions are completely additive, by assumption, we have for all \( \tau \in ^\alpha n \)

\[
\sum^\mathfrak{B} X = 1.
\]  

Let \( S \) be the Stone space of \( \mathfrak{B} \), whose underlying set consists of all Boolean ultrafilters of \( \mathfrak{B} \). Let \( X^* \) be the set of principal ultrafilters of \( \mathfrak{B} \) (those generated by the atoms). These are isolated points in the Stone topology, and they form a dense set in the Stone topology since \( \mathfrak{B} \) is atomic. So we have \( X^* \cap T = \emptyset \) for every nowhere dense set \( T \) (since principal ultrafilters, which are isolated points in the Stone topology, lie outside nowhere dense sets). For \( a \in \mathfrak{B} \), let \( N_a \) denote the set of all Boolean ultrafilters containing \( a \). Now for all \( \Gamma \subseteq \alpha, \ p \in B \) and \( \tau \in ^\alpha n \), we have, by the suprema, evaluated in (1) and (2):

\[
G_{i,p} = N_{c,p} \sim \bigcup_{\tau \in ^\alpha n} N_{s\tau p}
\]  

and

\[
G_{X,\tau} = S \sim \bigcup_{x \in X} N_{s\tau x}.
\]
are nowhere dense. Let $F$ be a principal ultrafilter of $S$ containing $c$. This is possible since $\mathcal{B}$ is atomic, so there is an atom $x$ below $c$; just take the ultrafilter generated by $x$. Then $F \in X^*$, so $F \notin G_{i,p}$, $F \notin G_{X,\tau}$, for every $i \in \alpha$, $p \in A$ and $\tau \in ^{\alpha}n$. Now define for $a \in A$

$$f(a) = \{\tau \in ^{\alpha}n : s_\tau^a \in F\}.$$ 

Then $f$ is a homomorphism from $\mathfrak{A}$ to the full set algebra with unit $^{\alpha}n$, with interior operators $J(i) i < \alpha$ defined exactly as in theorem 1.1 using all substitutions instead of only finite ones. We have $f(c) \neq 0$ because $Id \in f(c)$. Moreover $f$ is an atomic representation since $F \notin G_{X,\tau}$ for every $\tau \in ^{\alpha}n$, which means that for every $\tau \in ^{\alpha}n$, there exists $x \in X$, such that $s_\tau^a x \in F$, and so $\bigcup_{x \in X} f(x) = ^{\alpha}n$. We conclude that $f$ is a complete representation.

Now let $\text{CTPCA}_\alpha$ be the class of completely representable $\text{TPCA}_\alpha$s.

**Theorem 4.7.** (1) The class $\text{CTPCA}_\alpha$ is elementary, and it is axiomatizable by a finite schema in first order logic.

(2) Let $\mathfrak{Rr} : K \to \text{TPCA}_\alpha$ be the neat reduct functor. Then $\mathfrak{Rr}$ is strongly invertible, namely, there is a functor $G : \text{TPCA}_\alpha \to K$ and natural isomorphisms $\mu : 1_K \to G \circ \mathfrak{Rr}$ and $\epsilon : \mathfrak{Rr} \circ G \to 1_{\text{TPCA}_\alpha}$.

**Proof.** Atomicity can be expressed by a first order sentence, and complete additivity can be captured by the following continuum many formulas, that form a single schema. Let $\text{At}(x)$ be the first order formula expressing that $x$ is an atom. That is $\text{At}(x)$ is the formula $x \neq 0 \land (\forall y)(y \leq x \to y = 0 \lor y = x)$.

For $\tau \in ^{\alpha}n$, let $\psi_\tau$ be the formula:

$$y \neq 0 \to \exists x(\text{At}(x) \land s_\tau x \neq 0 \land s_\tau x \leq y).$$

Let $\Sigma$ be the set of first order formulas obtained by adding all formulas $\psi_\tau$ ($\tau \in ^{\alpha}n$) to the polyadic schema. Then it is easy to show that $\text{CTPCA}_\alpha = \text{Mod}(\Sigma)$. The second part follows by using exactly the same reasoning in [27, Theorem 3.4].

We can also expand the signature of the $\omega$ dimensional algebras studied in [18, 19], whose signature is countable having substitutions coming from a countable rich semigroup $G$, by interior operators with the same equations postulated for $\text{TCA}_\alpha$. Denote the resulting variety by $\text{TPA}_G$. Also usual Halmos polyadic algebras can be enriched with such modalities, call the resulting variety $\text{TPA}_\alpha$. We get all positive results obtained for $\text{TPCA}_\alpha$ with almost the same proofs using the techniques in [19, 27, 21, 18]. In particular we have:

**Theorem 4.8.** (1) $\text{TPA}_G$ and $\text{TPA}_\alpha$ have the super amalgamation property.
(2) In each such variety $\mathfrak{A}$ is completely representable if and only if the reduct obtained by discarding interior operators is completely additive.

(3) For such varieties the functor $\mathfrak{M}_\tau$ (defined like before adapted to the present context) is strongly invertible.

We can add diagonal elements and relativize semantics of topological polyadic algebras, getting the variety of topological cylindric polyadic algebras of dimension $\alpha$ whose signature is like $\text{TPCA}_\alpha$, axiomatized by the set of equations postulated in [3] definition 6.3.7 together with the schema of equations for the interior operators.

Denote this abstract class by $\text{TPCEA}_\alpha$ and the concrete class of representable algebras by $\text{TGp}_\alpha$. Then using the same methods adopted her replacing Henkin ultrafilters by what Ferenzci calls perfect ultrafilters we get:

**Theorem 4.9.** Let $\alpha \geq \omega$. Then the following hold:

1. $\text{TGp}_\alpha = \text{TPCEA}_\alpha$; hence $\text{TGp}_\alpha$ is a variety that can be axiomatizable by a finite schema of Sahlqvist equations. Furthermore, it is canonical and atom-canonical.

2. $\text{TGp}_\alpha$ has the superamalgamation property

3. Any atomic algebra in $\text{TGp}_\alpha$ has a complete representation. In particular, the class of completely representable $\text{TCPEA}_\alpha$s is elementary.

**Sketch.** Suppose $\mathfrak{A}$ is such an algebra. Then a dilation can be formed using all available substitutions, so we get $\mathfrak{A} = \mathfrak{M}_\alpha \mathfrak{B}$ where $\mathfrak{B} \in \text{TCPEA}_\beta$. However in the process of representation only admissible substitution on $\beta$ are used. A substitution $\tau \in \beta$ is such if $\text{dom}\tau \subseteq \alpha$ and $\text{rng}\tau \cap \alpha = \emptyset$. Call the set of all admissible substitutions $\text{adm}$. Henkin ultrafilters can always be found, but they are modified to give perfect ultrafilters in the sense of [29, p.128].

To preserve diagonal elements one factors out the set $\Gamma = \{i \in \beta : \exists j \in \alpha : c_i d_{ij} \in G\}$ by the congruence relation $k \sim l$ iff $d_{kl} \in G$. Then $\Gamma \subseteq \beta$ and the required representation with base $\Gamma / \sim$ is defined via

$$f(a) = \{\bar{\tau} \in ^{\alpha}[\Gamma / \sim] : \tau \in \text{adm}, s^{\beta}_{\tau} a \in G\},$$

where for each $i \in \alpha$ and $\tau \in \text{adm}$ $\tau(i) = \tau(i) / \sim$, witness [29, p. 128]. Next one defines for $p \in \mathfrak{A}$ and $i \in \alpha$ the sets $O_{p,i}$ and the interior operators $J_i$ on the representation as before.
5 Summary of results on amalgamation and interpolation

In the next table we summarize our results on classes of algebras in tabular form. This task was done for different classes of cylindric algebras in the recent [14].

In Table 2, we summarize the results we obtained on the interpolation property for the free algebras corresponding to the classes of algebras dealt with in Table 1.

In the top row of Table 1, we find a list of nine different amalgamation and embedding properties (AP and EP for short), together with the definability property ES and at the leftmost column we find a comprehensive list of classes of algebras occupying six rows. At the top of the third column ‘strong AP w.r.t rep.’ means ‘the strong amalgamation property with respect to the class of representable algebras in question’, while at the top of the fifth column ‘AP w.r.t abs.’ means the amalgamation property with respect to the class of abstract algebras. For example ‘strong AP w.r.t to rep’ is ‘strong AP with respect to RTCAα’ and ‘AP w.r.t abs.’ is ‘AP with respect to TCAα.’

Table 2 contains a summary of the results involving interpolation of the dimension-restricted free algebras. The rows addressing semisimple, substitution, representable cylindric algebras in Table 1 collapse to just one row in Table 2, since the free algebra coincide for all these classes (they all generate the same variety, namely, RTCAα).

In more general contexts than topological predicate logic addressed here, the Craig interpolation property ramifies into several different interpolation properties (IP for short). These properties are summarized in the six columns of the uppermost row of Table 2.

Only the first row in the next table deals with finite dimensional algebras of dimension n > 1. K denotes any subclass of TCA_n containing the variety of representable algebras. All the no’s in this row follow readily from theorem 3.2. Now we clarify the results collected above, stating where can they be found in the text. The last four rows in Table 1 and Table 2 follow from theorems 4.3, 4.8, 4.9. Next we have;

(1) Row two: Here we are dealing with ordinary predicate topological logic which has IP as proved in theorem 1.1. The rest now follow from theorems 1.14 and 3.8.

(2) Row three: All the no’s in second row follows from example 1.15. For simple algebras the algebras $A$ and $B$ taken in example 1.15 can be easily chosen to be simple. Though there is a simple amalgam, it is not dimension complemented. As illustrated in example 1.15, there could not be one.
Table 1:

|                      | strong AP | strong AP w.r.t rep. | AP w.r.t abs. | AP for simple | strong EP | EP for simple algebras | SUP AP | ES |
|----------------------|-----------|----------------------|---------------|---------------|-----------|------------------------|--------|----|
| TRCA<sub>α</sub> ⊆ K | no        | no                   | no            | no            | no        | no                     | no     | no |
| TLF<sub>α</sub>      | yes       | yes                  | yes           | yes           | yes       | yes                    | yes    | yes |
| TDC<sub>α</sub>      | no        | yes                  | no            | yes           | no        | no                     | no     | yes |
| TSc<sub>α</sub>      | no        | no                   | yes           | yes           | yes       | yes                    | no     | no |
| RTCA<sub>α</sub>     | no        | no                   | no            | no            | yes       | yes                    | no     | no |
| S<sub>1</sub><sub>α</sub><sub>TCA</sub><sup><sub>α</sub>k</sup> | no        | no                   | no            | yes           | ?         | ?                      | yes    | no |
| TCA<sub>α</sub>      | no        | no                   | no            | yes           | no        | no                     | yes    | no |
| TP<sub>α</sub>C<sub>α</sub> | yes   | yes                  | yes           | yes           | yes       | yes                    | no     | no |
| TPA<sub>G</sub>      | yes       | yes                  | yes           | yes           | yes       | yes                    | yes    | yes |
| TPA<sub>α</sub>      | yes       | yes                  | yes           | yes           | yes       | yes                    | yes    | yes |
| TGp<sub>α</sub>      | yes       | yes                  | yes           | yes           | yes       | yes                    | yes    | yes |

The first and second yes follow from theorem 1.14 and the third yes is due to the fact that ES follows from the fact that DC<sub>α</sub> has SUP AP w.r.t RTCA<sub>α</sub>, resorting to theorem 3.8.

(3) Row four: TS<sub>α</sub> does not have SAP because it does not have ES, by theorem 3.9. In fact, this last theorem takes care of all the no's. The remaining yes's follow from theorem 1.21 and corollary 2.12.

(4) Row five: The results follow like in the previous item. In particular, the no's follow from theorems 3.9 using theorem 3.8.

(5) Row six: The no's except for the last follow from theorem 3.1; the last no follows from theorem 3.9. The yes's follow from theorem 1.21 and corollary 2.12.

(6) Row seven. Like row six, except that the various forms of EP for algebras that are not simple remains unsettled. In theorem 3.10 the base algebra is the minimal subalgebra of an algebra obtained by twisting a representable algebra, and the other algebra is representable. But twisted algebras do not satisfy the so-called merry go round identities, which algebras in $\sum_{\lambda} CA_{\alpha+2}$ do. So this technique does not work for $\sum_{\lambda} TCA_{\alpha+k}$ when $k \geq 2$.

(7) Row eight. Note that TCA<sub>α</sub> does not have AP with respect to TRCA<sub>α</sub> is trivial. One just takes a non-representable algebra $\mathfrak{A}$ and considers the
inclusion maps \( i : A \to A \) twice, so that we are required to amalgamate \( A \) over \( A \) by a representable algebra, which is impossible for the amalgam necessarily contains an isomorphic copy of \( A \), while any subalgebra of a representable algebra is representable. The \textbf{no}'s follow from theorems 3.1, 3.9 and 3.10, and the only \textbf{yes} from theorem 1.21.

For \( TWSc_\alpha \) all questions involving \( AP \) remains unsettled. If any of the conditions in theorem 1.25 hold, then we get a \textbf{no} for all such questions, for in this case we get that \( TWSc_\alpha = RTCA_\alpha \).

In the following table \( IP \) is short for interpolation property. The top row addresses all interpolation properties introduced and investigated throughout this paper. \( IP \) is the interpolation property, \( WIP \) is weak \( IP \), \( AIP \) is almost \( IP \), \ldots etc. The first column addresses \( K \) where \( K \) is any class between \( TRCA_n \) and \( TCA_n \) \( n \) is finite \( > 1 \). The \textbf{no}'s in this row follows from theorem 3.2 and corollary 3.3. Without loss of generality, we consider (countable) free algebras on \( \omega \) generators.

Notice that \( IP \) implies \( AIP \) implies \( WIP \), and strong restricted \( IP \) implies all other restricted versions of \( IP \).

| \( Fr_\omega K \) | \( IP \) | \( AIP \) | \( WIP \) | \( \text{restricted} \) \( IP \) | \( \text{almost restricted} \) \( IP \) | \( \text{weak} \) \( \text{restricted} \) \( IP \) |
|------------------|--------|--------|--------|------------------|------------------|------------------|
| \( Fr_\omega TCA_\alpha \) which is in \( TLF_\alpha \) | yes | yes | yes | yes | yes | yes |
| \( Fr_\omega TCA_\alpha \) which is in \( TDC_\alpha \) | yes | yes | yes | yes | yes | yes |
| \( Fr_\omega RTCA_\alpha \) | no | no | yes | yes | yes | yes |
| \( Fr_\omega SFr_\alpha TCA_{\alpha+k} \) | no | no | yes | ? | ? | ? |
| \( Fr_\omega TCA_\alpha \) | no | yes | no | no | no | no |
| \( Fr_\omega TPCA_\alpha \) | yes | yes | yes | yes | yes | yes |
| \( Fr_\omega TPG \) | yes | yes | yes | yes | yes | yes |
| \( Fr_\omega TPA_\alpha \) | yes | yes | yes | yes | yes | yes |
| \( Fr_\omega TGp_\alpha \) | yes | yes | yes | yes | yes | yes |

Table 2:
All positive theorems on the free algebras, addressing cases other than \( \text{RTCA}_\alpha \) follow from theorems 1.1 and 4.3.

Concerning the \( \text{RTCA}_\alpha \) case, the positive results follow from theorems 2.12, 2.10 and the negative results follow from theorems 3.1 and 3.10. The various forms of restricted \( IP \) remain unsettled for \( \mathfrak{fr}_\alpha \mathfrak{nr}_\alpha \text{RTCA}_{\alpha + k} \).

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