A NEW GOODNESS OF FIT TEST FOR GAMMA DISTRIBUTION WITH CENSORED OBSERVATIONS

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\textbf{Abstract.} In the present paper, we develop a new goodness fit test for gamma distribution using the fixed point characterization. U-Statistic theory is employed to derive the test statistic. We discuss how the right censored observations are incorporated in the test developed here. The asymptotic properties of the test statistic in both censored and uncensored cases are studied in detail. Extensive Monte Carlo simulation studies are carried out to validate the performance of the proposed tests. We also illustrate the test procedure using several real data sets.

\textit{Keywords:} Gamma distribution; Right censoring; Stein’s identity; U-statistics.

1. Introduction

Lifetime data analysis involve the modeling of time to event data. Several parametric distributions are used to model lifetime data. Exponential, gamma and Weibull distributions are some commonly used lifetime models. In this context, it is important to check the validity of an assumed parametric model. The goodness of fit tests are employed to validate the assumption that the lifetime data follows a particular distribution. A detailed study of goodness of fit tests for lifetime data is given in Lawless (2011). Gamma distribution has a great significance in lifetime distributions, due to its ability in modelling different ageing patterns. Gamma distribution generalises exponential, $\chi^2$ and Erlang distributions.

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For applications of gamma distribution in lifetime data analysis one can refer Barlow and Proschan (1996) and Deshpande and Purohit (2015) among many others. Widespread applications of gamma distribution include modeling of rainfall data in Africa (Husak et al., 2007) and vinyl chloride data from an environmental study (Bhaumik et al., 2009) along with others.

A number of goodness of fit tests are available for gamma distribution in literature. Kallioras et al. (2006) proposed a method using the empirical moment generating function to develop goodness of fit for gamma distribution. Henze et al. (2012) developed a goodness of fit test for gamma distribution based on empirical Laplace transform. Villaseñor and González-Estrada (2015) suggested a variance ratio test for testing gamma distribution. Baringhaus et al. (2017) proposed tests based on some independent properties of the gamma distribution. Recently, Betsch and Ebner (2019) developed a new characterisation for gamma distribution and an associated goodness of fit test. Note that all these tests are developed for complete data.

Stein (1972) introduced a natural identity for a random variable whose distribution belongs to an exponential family. Stein’s identity and its role in inference procedures have been discussed widely in the literature. For a detailed discussion on Stein’s type identity for a general class of probability distributions and related characterizations, one can refer to Sudheesh (2009) and Sudheesh and Dewan (2016) and the references therein. Using Stein’s type identity, Betsch and Ebner (2019) developed a fixed point characterization for gamma distribution. Making use of this characterization, we develop a U-statistic based goodness fit test for gamma distribution for complete data. We also develop a new goodness fit test for gamma distribution with censored observations.

The rest of the paper is organized as follows. In Section 2, we develop a new non-parametric test for gamma distribution for complete data. In Section 3, we discuss how to incorporate right-censored observations in the testing procedure. We obtain the asymptotic
distribution of these test statistics separately. The result of Monte Carlo simulation studies are reported in Section 4 to evaluate the finite sample performance of the proposed tests. The procedures are illustrated with real data sets in Section 5. Finally, Section 6 concludes the study with a discussion on future works.

2. Test statistics: Uncensored case

In this section, we develop a new goodness of fit test for gamma distribution with complete data. Let \( X \) be non-negative random variable having distribution function \( F \). Then \( X \) has gamma distribution with parameter \( k \) and \( \lambda \) (denoted as \( \Gamma(k, \lambda) \)), if its probability density function is given by

\[
f(x) = \frac{\lambda^{-k}}{\Gamma(k)} x^{k-1} e^{-x/\lambda}, \quad x > 0.
\]

We use the fixed point characterization based on Steins’s type identity for gamma distribution to develop the test.

**Theorem 1** (Betsch and Ebner, 2019). *The random variable \( X \) has gamma distribution with parameter \( k \) and \( \lambda \) if and only if*

\[
F(t) = E \left[ \left( \frac{1-k}{X} + \frac{1}{\lambda} \right) \min(X, t) \right].
\]

Based on a random sample \( X_1, ..., X_n \) from \( F \), we are interested in testing the null hypothesis

\[
H_0 : F \in \Gamma(k, \lambda)
\]

against

\[
H_1 : F \notin \Gamma(k, \lambda).
\]
For testing the above hypothesis, we define a departure measure which discriminate between null and alternative hypothesis. Consider \( \Delta(F) \) given by

\[
\Delta(F) = \int_0^\infty \left( E \left[ \left( \frac{1-k}{X} + \frac{1}{\lambda} \right) \min(X,t) \right] - F(t) \right) dF(t).
\]  

(1)

In view of Theorem 1, \( \Delta(F) \) is zero under \( H_0 \) and non zero under \( H_1 \). Hence \( \Delta(F) \) can be considered as a measure of departure from the null hypothesis \( H_0 \) towards the alternative hypothesis \( H_1 \). As we propose the test using the theory of U-statistics, first we simplify \( \Delta(F) \) in terms of expectation of the function of random variables. Consider

\[
\Delta(F) = \int_0^\infty E \left[ \left( \frac{1-k}{X} + \frac{1}{\lambda} \right) \min(x,t) \right] dF(t) - \int_0^\infty F(t)dF(t).
\]

\[
= \int_0^\infty \int_0^\infty \left( \frac{1-k}{x} \min(x,t) + \frac{1}{\lambda} \min(x,t) \right) dF(x)dF(t) - 1/2
\]

\[
= \Delta_1 + \frac{1}{\lambda} E(\min(X_1, X_2)) - \frac{1}{2},
\]  

(2)

where \( \Delta_1 = \int_0^\infty \int_0^\infty \frac{1-k}{x} \min(x,t) dF(x) dF(t) \). Now

\[
\Delta_1 = \int_0^\infty \int_0^\infty \frac{1-k}{x} \min(x,t) dF(x) dF(t).
\]

\[
= (1-k) \int_0^\infty \int_0^\infty \frac{1}{x} \min(x,t) dF(x) dF(t).
\]

\[
= (1-k) \int_0^\infty \left( \int_0^\infty I(x < t) + \frac{t}{x} I(t < x) \right) dF(x) dF(t).
\]

\[
= (1-k) P(X_1 < X_2) + (1-k) \int_0^\infty \int_t^\infty \frac{t}{x} dF(x) dF(t).
\]

\[
= (1-k) \frac{1}{2} + (1-k) E \left[ \frac{X_1}{X_2} I(X_1 < X_2) \right],
\]  

(3)
where $I(A)$ denote the indicator function of a set $A$. Substitute equation (3) in equation (2) we obtain

$$\Delta(F) = \frac{1 - k}{2} - \frac{1}{2} + (1 - k)E \left[ \frac{X_1}{X_2} I(X_1 < X_2) \right] + \frac{1}{\lambda} E(\min(X_1, X_2))$$

$$= \frac{1}{\lambda} E(\min(X_1, X_2)) + (1 - k)E \left( \frac{X_1}{X_2} I(X_1 < X_2) \right) - \frac{k}{2}. \quad (4)$$

We find test statistic using U-statistics theory. Define $h_1(X_1, X_2) = \min(X_1, X_2)$. Then a U-statistic defined by

$$U_1 = \frac{2}{n(n-1)} \sum_{i=1}^{n} \sum_{j=1, j<i}^{n} h_1(X_i, X_j),$$

is an unbiased estimator of $E(\min(X_1, X_2))$. Defined a symmetric kernel

$$h_2(X_1, X_2) = \frac{1}{2} \left( \frac{X_1}{X_2} I(X_1 < X_2) + \frac{X_2}{X_1} I(X_2 < X_1) \right).$$

Then a U-statistic defined by

$$U_2 = \frac{2}{n(n-1)} \sum_{i=1}^{n} \sum_{j=1, j<i}^{n} h_2(X_i, X_j),$$

is an unbiased estimator of $E \left( \frac{X_1}{X_2} I(X_1 < X_2) \right)$. Let $\hat{\lambda}$ and $\hat{k}$ be the consistent estimators of $\lambda$ and $k$, respectively. Hence the test statistic is given by

$$\hat{\Delta} = \frac{U_1}{\lambda} + (1 - \hat{k})U_2 - \frac{\hat{k}}{2}. \quad (5)$$

We reject the null hypothesis $H_0$ against the alternative $H_1$ for large value of $\hat{\Delta}$.

Next we study the asymptotic properties of the test statistic. Since $U_1$ and $U_2$ are U-statistics they are consistent estimators of $E(\min(X_1, X_2))$ and $E \left( \frac{X_1}{X_2} I(X_1 < X_2) \right)$, respectively (Lehmann, 1951). Hence the following result is straightforward.

**Theorem 2.** Under $H_1$, as $n \to \infty$, $\hat{\Delta}$ converges in probability to $\Delta$. 


Theorem 3. As $n \to \infty$, $\sqrt{n}(\widetilde{\Delta} - \Delta)$ converges in distribution to normal random variable with mean zero and variance $\sigma^2$, where $\sigma^2$ is given by

$$\sigma^2 = \text{Var} \left( \frac{2X \bar{F}(X)}{X} \right) + \frac{2}{X} \int_0^X ydF(y) + (1-k)X \int_X^\infty \frac{1}{y}dF(y) + \frac{(1-k)}{X} \int_0^X ydF(y).$$  \hspace{1cm} (6)

Proof: Define

$$\widehat{\Delta}^* = \frac{U_1}{\lambda} + (1-k)U_2.$$

Since $\widehat{\lambda}$ and $\widehat{k}$ are consistent estimators of $\lambda$ and $k$, respectively, by Slutsky’s theorem, the asymptotic distribution of $\sqrt{n}(\widetilde{\Delta} - \Delta)$ and $\sqrt{n}(\widehat{\Delta}^* - E(\widehat{\Delta}^*))$ are same. Now we observe that $\widehat{\Delta}^*$ is a U statistic with symmetric kernel,

$$h(X_1, X_2) = \frac{1}{2} \left[ \left( \frac{2\min(X_1, X_2)}{\lambda} \right) + (1-k) \left( \frac{X_1}{X_2}I(X_1 < X_2) + \frac{X_2}{X_1}I(X_2 < X_1) \right) \right].$$

Hence using the central limit theorem for U-statistics we have the asymptotic normality of $\widehat{\Delta}^*$. The asymptotic variance is $4\sigma_1^2$ where $\sigma_1^2$ is given by (Lee, 2019)

$$\sigma_1^2 = \text{Var} \left[ E \left( h(X_1, X_2) | X_1 \right) \right].$$  \hspace{1cm} (7)

Consider

$$E[2 \min(x, X_2)] = 2E[xI(x < X_2) + X_2I(X_2 < x)]$$

$$= 2xP(x < X_2) + 2 \int_0^\infty yI(y < x) dF(y)$$

$$= 2x\bar{F}(x) + 2 \int_0^x ydF(y).$$  \hspace{1cm} (8)

Also

$$E\left[ \frac{x}{X_2} I(x < X_2) + \frac{X_2}{x} I(X_2 < x) \right] = xE\left[ \frac{1}{X_2} I(x < X_2) + \frac{1}{x} E[X_2 I(X_2 < x)] \right]$$

$$= x \int_0^\infty \frac{1}{y}I(x < y)dF(y) + \frac{1}{x} \int_0^x ydF(y)$$

$$= x \int_x^\infty \frac{1}{y}dF(y) + \frac{1}{x} \int_0^x ydF(y).$$  \hspace{1cm} (9)
Substituting equations (8) and (9) in equation (7) we obtain the variance expression as specified in the theorem.

Under the null hypothesis $H_0$, $\Delta(F) = 0$. Hence we have the following corollary.

**Corollary 1.** Under $H_0$, as $n \rightarrow \infty$, $\sqrt{n} \hat{\Delta}$ converges in distribution to normal with mean zero and variance $\sigma_0^2$, where $\sigma_0^2$ is the value of $\sigma^2$ evaluated under $H_0$.

An asymptotic critical region of the test can be obtained using Corollary 1. Let $\hat{\sigma}_0^2$ be a consistent estimator of $\sigma_0^2$. We reject the null hypothesis $H_0$ against the alternative hypothesis $H_1$ at a significance level $\alpha$, if

$$\frac{\sqrt{n}|\hat{\Delta}|}{\hat{\sigma}_0} > Z_{\alpha/2},$$

where $Z_\alpha$ is the upper $\alpha$-percentile point of the standard normal distribution. Since the distribution function $F$ has no closed form for the gamma distribution, it is difficult to evaluate the null variance $\hat{\sigma}_0^2$ be a consistent estimator of $\sigma_0^2$. Hence we find the critical region of the test based on Monte Carlo simulation. We determine lower ($c_1$) and upper ($c_2$) quantiles in such a way that $P(\hat{\Delta} < c_1) = P(\hat{\Delta} > c_2) = \alpha/2$. Finite sample performance of the test is evaluated through Monte Carlo simulation study and the results are reported in Section 4.

3. **Test for right censored case**

Next we discuss how the censored observations can be incorporated in the proposed testing procedure. Consider the right-censored data $(Y, \delta)$, with $Y = \min(X, C)$ and $\delta = I(X \leq C)$, where $C$ is the censoring time. We assume censoring times and lifetimes are independent. Now we develop test discussed in Section 2 based on $n$ independent and identical observation $\{(Y_i, \delta_i), 1 \leq i \leq n\}$. Since we consider U-statistics theory for right censored
data, we use the same departure measure $\Delta(F)$ given in (11). We rewrite (11) as

$$
\Delta(F) = \frac{1}{\lambda} E(\min(X_1, X_2)) + (1 - k) E\left(\frac{X_1}{X_2} I(X_1 < X_2)\right) - k P(X_1 < X_2). \quad (10)
$$

To develop the test statistic for right censored case, we estimate each quantity in (10) using U-statistics for right censored data (Datta et al., 2010). An estimator of $E(\min(X_1, X_2))$ is given by

$$
\hat{\Delta}_{1c} = \frac{2}{n(n-1)} \sum_{i=1}^{n} \sum_{j<i, j=1}^{n} \frac{\min(Y_1, Y_2) \delta_i \delta_j}{\hat{K}_c(Y_i) \hat{K}_c(Y_j)}, \quad (11)
$$

provided $\hat{K}_c(Y_i) > 0$ and $\hat{K}_c(Y_j) > 0$, with probability 1 and $\hat{K}_c$ is the Kaplan-Meier estimator of $K_c$, the survival function of $C$. Again, an estimator of $E\left(\frac{X_1}{X_2} I(X_1 < X_2)\right)$ is given by

$$
\hat{\Delta}_{2c} = \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j<i, j=1}^{n} \frac{\left(\frac{X_i}{X_j} I(X_i < X_j) + \frac{X_j}{X_i} I(X_j < X_i)\right) \delta_i \delta_j}{\hat{K}_c(Y_i) \hat{K}_c(Y_j)}, \quad (12)
$$

An estimator of $P(X_1 < X_2)$ is given by

$$
\hat{\Delta}_{3c} = \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j<i, j=1}^{n} \frac{(I(X_i < X_j) + I(X_j < X_i)) \delta_i \delta_j}{\hat{K}_c(Y_i) \hat{K}_c(Y_j)}, \quad (13)
$$

Similarly, the estimators of $E(X)$ and $E(X^2)$ are given by (Datta, 2005)

$$
\bar{X}_c = \frac{1}{n} \sum_{i=1}^{n} \frac{Y_i \delta_i}{\hat{K}_c(Y_i)},
$$

and

$$
\bar{X}_c^2 = \frac{1}{n} \sum_{i=1}^{n} \frac{Y_i^2 \delta_i}{\hat{K}_c(Y_i)}.
$$

Using $\bar{X}_c$ and $\bar{X}_c^2$ we obtain the estimators of $k$ and $\lambda$. Moment estimators of $k$ and $\lambda$ are given by

$$
\hat{k}_m = \frac{E^2(X)}{\text{Var}(\bar{X})}.
$$
and

\[ \hat{\lambda}_m = \frac{Var(X)}{E(X)}. \]

Hence under right censored case we estimate these quantities by

\[ \hat{k}_c = \frac{(\bar{X}_c)^2}{\bar{X}^2_c - (X_c)^2}. \] (14)

and

\[ \hat{\lambda}_c = \frac{\bar{X}^2_c - (X_c)^2}{X_c}. \] (15)

Since \( \bar{X}_c \) and \( \bar{X}^2_c \) are consistent estimators of \( E(X) \) and \( E(X^2) \), we can easily verify that \( \hat{k}_c \) and \( \hat{\lambda}_c \) are consistent estimators of \( k \) and \( \lambda \), respectively. Using the estimators given in equations (14-15), we obtain the test statistic in right censored case as

\[ \hat{\Delta}_c = \frac{\hat{\Delta}_{1c}}{\hat{\lambda}_c} + (1 - \hat{k}_c)\hat{\Delta}_{2c} - \hat{k}_c\hat{\Delta}_{3c}. \] (16)

We reject \( H_0 \) in favour of \( H_1 \) for large values of \( \hat{\Delta}_c \).

Next we obtain the limiting distribution of \( \hat{\Delta}_c \). Let \( N^c_i(t) = I(Y_i \leq t, \delta_i = 0) \) be the counting process corresponds to the censoring variable \( C_i \), \( R_i(t) = I(Y_i \geq t) \). Also let \( \lambda_c \) be the hazard rate of \( C \). The martingale associated with this counting process \( N^c_i(t) \) is given by

\[ M^c_i(t) = N^c_i(t) - \int_0^t R_i(u)\lambda_c(u)du. \]

Let \( G(x, y) = P(X_1 \leq x, Y_1 \leq y, \delta = 1), x \in \mathcal{X}, \tilde{H}(t) = P(Y_1 > t) \) and

\[ w(t) = \frac{1}{H(t)} \int_{\mathcal{X} \times [0, \infty)} \frac{h_1(x)}{K_c(y -)} I(y > t) dG(x, y), \]

where \( h_1(x) = E(h(X_1, X_2|X_1 = x)) \). The proof of next result follows from Theorem 1 of Datta et al. (2010) for a particular choice of the kernel.
Theorem 4. Let

\[ h_1(x) = \frac{1}{2} E \left[ \left( \frac{2 \min(x, Y_2)}{\lambda} \right) + (1 - k) \left( \frac{x}{Y_2} I(x < Y_2) + \frac{Y_2}{x} I(Y_2 < x) \right) - k \left( I(x < Y_2) + I(Y_2 < x) \right) \right]. \]

Suppose the conditions

\[ E \left[ \frac{1}{2} \left( \frac{2 \min(Y_1, Y_2)}{\lambda} \right) + (1 - k) \left( \frac{Y_1}{Y_2} I(Y_1 < Y_2) + \frac{Y_2}{Y_1} I(Y_2 < Y_1) \right) - k \right]^2 < \infty, \]

\[ \int_{X \times [0, \infty)} \frac{h_2(x)}{K_c(t)} dG(x, y) < \infty \quad \text{and} \quad \int_0^\infty w^2(t) \lambda_c(t) dt < \infty \]

holds. As \( n \to \infty \), \( \sqrt{n} (\hat{\Delta}_c - \Delta) \) converges in distribution to Gaussian random variable with mean zero and variance \( 4\sigma_c^2 \), where \( \sigma^2_c \) is given by

\[ \sigma^2_c = \text{Var} \left( \frac{h_1(X) \delta_1}{K_c(Y_1 -)} + \int w(t) dM_v(t) \right). \]

Next we find an estimator of \( \sigma^2_c \) using the reweighed techniques. An estimator of \( \sigma^2_c \) is given by

\[ \hat{\sigma}^2_c = \frac{4}{(n - 1)} \sum_{i=1}^n (V_i - \bar{V})^2, \]

where

\[ V_i = \frac{\hat{h}_1(X_i) \delta_i}{K_c(Y_i)} + \hat{w}(X_i)(1 - \delta_i) - \sum_{j=1}^n \frac{\hat{w}(X_i)I(X_i > X_j)(1 - \delta_i)}{\sum_{i=1}^n I(X_i > X_j)}, \]

\[ \bar{V} = \frac{1}{n} \sum_{i=1}^n V_i, \quad \hat{h}_1(X) = \frac{1}{n} \sum_{i=1}^n \frac{h(X, Y_i) \delta_i}{K_c(Y_i -)}, \quad R(t) = \frac{1}{n} \sum_{i=1}^n I(Y_i > t) \]

and

\[ \hat{w}(t) = \frac{1}{R(t)} \sum_{i=1}^n \frac{\hat{h}_1(X_t) \delta_i}{K_c(Y_i)} I(X_t > t). \]

Let \( \hat{\sigma}^2_{0c} \) be the value of \( \hat{\sigma}^2_c \) evaluated under \( H_0 \). Under right censored situation, we reject the null hypothesis \( H_0 \) against the alternative hypothesis \( H_1 \) at a significance level \( \alpha \), if

\[ \frac{\sqrt{n} \left| \hat{\Delta}_c \right|}{\hat{\sigma}_{0c}} > Z_{\alpha/2}. \]

The results of the Monte Carlo simulation which assess the finite sample performance of the test is also reported in Section 4.
4. Empirical evidence

To evaluate the finite sample performance of the proposed test procedure, we conduct a Monte Carlo simulation study using R software. To show the competitiveness of our test with the existing test procedures for complete data, we compare the empirical powers of the same. In censored case we evaluate the power of our test against different alternatives.

4.1. Uncensored case. We find the type I error and empirical power of the proposed test and other acknowledged tests. The algorithm used to find the empirical power can be summarised as follows.

1. Generate lifetime data from the desired alternative and calculate the test statistic.
2. Estimate the parameters $k$ and $\lambda$ from the data obtained in Step 1.
3. Generate a sample of size $n$ from $\Gamma(k, \lambda)$.
4. Obtain the bootstrap distribution of the test statistic with 10000 bootstrap samples obtained from the data generated in Step 3 and determine the critical point.
5. Repeat Steps 1-4 10000 times and calculate empirical power as the proportion of significant test statistics.

First we find empirical type I error of the test. We generate lifetimes from gamma distribution with different samples sizes $n = 25, 50, 75, 100, 200$ to calculate the empirical type I error. To find the empirical power, lifetime random variables are generated from different choices of alternative including Weibull, lognormal and Pareto distributions where the distribution functions are;

Weibull distribution: $F(x) = 1 - e^{(-x/\lambda)^k}$, $x > 0$, $k, \lambda > 0$,
Pareto distributions: $F(x) = (\lambda/x)^\alpha$ $x > 0$, $\alpha, \lambda > 0$,
Lognormal distribution: $F(x) = \Phi(\frac{\ln x - \mu}{\sigma})$, $x > 0$, $-\infty < \mu < \infty$, $\sigma^2 > 0$,

where $\Phi(x)$ is the cumulative distribution function of the standard normal random variable.

We compare the performance of our test with goodness of fit test for gamma distribution
proposed by Henze et al. (2012) and Betsch and Ebner (2019) and also with the well-known Kolmogorov Smirnov (KS) test and Cramer von Mises (CvM) test. The test statistic by Henze et al. (2012) is given by

\[ HME = \int_0^{\infty} Z^2_{1n}(t)w(t)dt, \]

where \( Z_{1n}(t) = \sqrt{n}\left[(1 + t)L_n'(t) + \hat{k}L_n(t)\right] \) with \( L_n(t) = \frac{1}{n} \sum_{i=1}^{n} \exp(-tY_j) \), \( w(t) \) is a weight function satisfying some predefined conditions and \( Y_i = \frac{X_i}{\lambda}, i = 1, 2, \ldots, n \). Test statistic proposed by Betsch and Ebner (2019) based on fixed point characterization is given by

\[ BE = \int_0^{\infty} Z^2_{2n}(t)w(t)dt, \]

where \( Z_{2n}(t) = \sqrt{n}\left[\frac{1}{n} \sum_{i=1}^{n} \left(\frac{1-k}{Y_i} + 1\right) \min(Y_i, t) - \frac{1}{n} \sum_{i=1}^{n} I(Y_i \leq t)\right] \). The value of the Henze test and Betsch and Ebner test are obtained using the R-Package ‘gofgamma’. The Kolmogorov-Smirnov test statistic is given by \( KS = \max\{D^+, D^-\} \) where

\[ D^+ = \max_{i = 1, 2, \ldots, n} \left(\frac{i}{n} - \hat{F}(X(i))\right) \quad \text{and} \quad D^- = \max_{i = 1, 2, \ldots, n} \left(\hat{F}(X(i)) - \frac{i - 1}{n}\right) \]

and Cramer von Mises statistics is given by

\[ CM = \frac{1}{12n} + \sum_{i=1}^{n} \left(\hat{F}(X(i)) - \frac{2i - 1}{2n}\right)^2, \]

where \( \hat{F}(.) \) is the empirical distribution function. The results of the simulation study are given in Tables 1-4. In Table 1, we report the empirical type I error of the proposed test and in all other tables, we give the empirical power against different alternatives. From Table 1, we observe that the size of the test attains chosen level of significance. The test has good power against all choices of alternatives which increases with sample size. We can see that the newly proposed test performs better than other tests in most of the cases.
we considered. When the alternative is Pareto distribution, we note other test perform better than our test for small sample sizes. The proposed test has high power even for small sample size, which affirms the efficiency of the test.

Table 1. Empirical type I error

| n  | α = 0.01 | α = 0.05 | α = 0.01 | α = 0.05 | α = 0.01 | α = 0.05 | α = 0.01 | α = 0.05 | α = 0.01 | α = 0.05 |
|----|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|
| 25 | 0.0104   | 0.0494   | 0.0013   | 0.0595   | 0.0130   | 0.0603   | 0.0095   | 0.0472   | 0.0104   | 0.0470   |
| 50 | 0.0089   | 0.0491   | 0.0113   | 0.0542   | 0.0116   | 0.0552   | 0.0115   | 0.0510   | 0.0101   | 0.0530   |
| 75 | 0.0089   | 0.0484   | 0.0119   | 0.0546   | 0.0117   | 0.0533   | 0.0092   | 0.0507   | 0.0099   | 0.0495   |
| 100| 0.0097   | 0.0487   | 0.0114   | 0.0520   | 0.0113   | 0.0528   | 0.0102   | 0.0530   | 0.0097   | 0.0534   |
| 200| 0.0095   | 0.0498   | 0.0115   | 0.0541   | 0.0113   | 0.0529   | 0.0118   | 0.0512   | 0.0112   | 0.0507   |

Table 2. Empirical power: Log normal distribution ($\mu = 2, \sigma = 1$)

| n  | α = 0.01 | α = 0.05 | α = 0.01 | α = 0.05 | α = 0.01 | α = 0.05 | α = 0.01 | α = 0.05 | α = 0.01 | α = 0.05 |
|----|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|
| 25 | 1.0000   | 1.0000   | 1.1442   | 0.3335   | 0.1583   | 0.3322   | 1.0000   | 1.0000   | 1.0000   | 1.0000   |
| 50 | 1.0000   | 1.0000   | 0.4079   | 0.6463   | 0.4019   | 0.6218   | 1.0000   | 1.0000   | 1.0000   | 1.0000   |
| 75 | 1.0000   | 1.0000   | 0.5738   | 0.7991   | 0.5697   | 0.7620   | 1.0000   | 1.0000   | 1.0000   | 1.0000   |
| 100| 1.0000   | 1.0000   | 0.9951   | 1.0000   | 0.9930   | 0.9989   | 1.0000   | 1.0000   | 1.0000   | 1.0000   |
| 200| 1.0000   | 1.0000   | 1.0000   | 1.0000   | 1.0000   | 1.0000   | 1.0000   | 1.0000   | 1.0000   | 1.0000   |

Table 3. Empirical power: Pareto distribution ($\alpha = 2, \lambda = 1$)

| n  | α = 0.01 | α = 0.05 | α = 0.01 | α = 0.05 | α = 0.01 | α = 0.05 | α = 0.01 | α = 0.05 | α = 0.01 | α = 0.05 |
|----|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|
| 25 | 0.8105   | 0.8493   | 0.7132   | 0.9166   | 0.5918   | 0.8463   | 1.0000   | 1.0000   | 1.0000   | 1.0000   |
| 50 | 0.8587   | 0.9052   | 0.9811   | 0.9981   | 0.9607   | 0.9974   | 1.0000   | 1.0000   | 1.0000   | 1.0000   |
| 75 | 0.8802   | 0.9884   | 1.0000   | 1.0000   | 1.0000   | 1.0000   | 1.0000   | 1.0000   | 1.0000   | 1.0000   |
| 100| 0.9871   | 1.0000   | 1.0000   | 1.0000   | 1.0000   | 1.0000   | 1.0000   | 1.0000   | 1.0000   | 1.0000   |
| 200| 1.0000   | 1.0000   | 1.0000   | 1.0000   | 1.0000   | 1.0000   | 1.0000   | 1.0000   | 1.0000   | 1.0000   |

Table 4. Empirical power: Weibull distribution ($k = 2, \lambda = 1$)

| n  | α = 0.01 | α = 0.05 | α = 0.01 | α = 0.05 | α = 0.01 | α = 0.05 | α = 0.01 | α = 0.05 | α = 0.01 | α = 0.05 |
|----|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|
| 25 | 0.8424   | 0.9836   | 0.0426   | 0.1344   | 0.1130   | 0.1139   | 0.1147   | 0.3836   | 0.0780   | 0.4192   |
| 50 | 1.0000   | 1.0000   | 0.0862   | 0.2352   | 0.1972   | 0.1972   | 0.4962   | 0.8252   | 0.5622   | 0.9236   |
| 75 | 1.0000   | 1.0000   | 0.1391   | 0.3234   | 0.2625   | 0.2621   | 0.8188   | 0.9768   | 0.9028   | 0.9967   |
| 100| 1.0000   | 1.0000   | 0.1836   | 0.3772   | 0.3188   | 0.3188   | 0.9539   | 0.9980   | 0.9873   | 1.0000   |
| 200| 1.0000   | 1.0000   | 0.4204   | 0.6608   | 0.5742   | 0.5744   | 1.0000   | 1.0000   | 1.0000   | 1.0000   |
4.2. Censored case. We calculate empirical type I error and power of the test statistic proposed for right censored data using Monte Carlo simulation studies. To calculate the empirical type I error, lifetimes are generated from gamma distribution. We considered the same alternatives as in uncensored case for finding the empirical power. Here, the censoring percentages are chosen to be 20% and 40%. In all cases, the censoring random variable $C$ is generated from exponential distribution with parameter $b$, where $b$ is chosen such that $P(T > C) = 0.2(0.4)$. Re-weighting techniques explained in Section 3 is used to estimate the variance of $\hat{\Delta}_c$. Results of the simulation study are presented in Tables 5 and 6.

Table 5. Empirical type I error and power of the test when 20% of lifetimes are censored.

| $n$ | Gamma (1,1) | Lognormal (2,1) | Weibull (2,1) | Pareto (2,1) |
|-----|-------------|-----------------|---------------|--------------|
|     | $\alpha = 0.01$ | $\alpha = 0.05$ | $\alpha = 0.01$ | $\alpha = 0.05$ | $\alpha = 0.01$ | $\alpha = 0.05$ | $\alpha = 0.01$ | $\alpha = 0.05$ |
| 50  | 0.0132 | 0.0461 | 0.9863 | 0.9916 | 0.9891 | 0.9948 | 0.8381 | 0.8892 |
| 75  | 0.0121 | 0.0489 | 0.9992 | 1.0000 | 1.0000 | 1.0000 | 0.8675 | 0.9543 |
| 100 | 0.0118 | 0.0508 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9782 | 0.9999 |
| 200 | 0.0104 | 0.0497 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |

Table 6. Empirical type I error and power of the test when 40% of lifetimes are censored.

| $n$ | Gamma(1,1) | Lognormal (2,1) | Weibull (2,1) | Pareto (2,1) |
|-----|-------------|-----------------|---------------|--------------|
|     | $\alpha = 0.01$ | $\alpha = 0.05$ | $\alpha = 0.01$ | $\alpha = 0.05$ | $\alpha = 0.01$ | $\alpha = 0.05$ | $\alpha = 0.01$ | $\alpha = 0.05$ |
| 50  | 0.0140 | 0.0514 | 0.5248 | 0.6069 | 0.9594 | 0.9847 | 0.7893 | 0.8456 |
| 75  | 0.0112 | 0.0489 | 0.6785 | 0.7494 | 0.9976 | 0.9995 | 0.8175 | 0.8893 |
| 100 | 0.0108 | 0.0507 | 0.8044 | 0.8798 | 1.0000 | 1.0000 | 0.9274 | 0.9632 |
| 200 | 0.0106 | 0.0505 | 0.9627 | 0.9778 | 1.0000 | 1.0000 | 0.9461 | 0.9874 |

We can see that the empirical power of the test approaches the chosen level significance as $n$ increases. The performance of the test is good in terms of empirical power. From Tables 5 and 6, we observe that the power of the test increases with sample size and decreases with censoring percentage.
5. Data analysis

The proposed test procedures are illustrated using several real data sets.

5.1. Complete data. We consider two data sets for the analysis. To find the critical region, we use the following algorithm.

1. Estimate the parameters of gamma distribution $k$ and $\lambda$ from the observed data.
2. Generate a random sample from gamma distribution using the estimated parameters in Step 1.
3. Obtain the bootstrap distribution of the test statistic with 10000 bootstrap samples from the data generated in Step 2 and determine the critical points.

Illustration 1: We consider the data on survival times in weeks for 20 male rats that were exposed to a high level of radiation. The data is discussed in Lawless (2011) in Example 4.2.1. We choose a data with small sample size to make sure that our test is suitable for any sample size. The test statistic is obtained as 0.0964 where the 5% level critical values are 0.0456 and 1.2664 and 1% level critical values are -0.0055 and 2.1042 respectively. Hence we accept $H_0$ that the data on survival times of male rats follows gamma distribution.

Illustration 2: We also examine the lifetime data on number of millions of revolutions before failure for 23 ball bearings. The data is studied in Lawless (2011) in context of goodness of fit tests for parametric models and given in Example 3.3.1. We obtain the test statistic as -0.0379 where the 5% level critical values are 0.0956 and 1.2978 and 1% level critical value are 0.0385 and 2.0577. Hence we reject the null hypothesis that the data follows gamma distribution.

5.2. Censored data. Two real data sets are considered for illustrating the proposed test procedure. We use the normal based critical region given in Section 3 to make a decision. The asymptotic null variance of $\hat{\Delta}_c$ is estimated using the re-weighting techniques explained in Section 3.
Illustration 1: We analyse stanford heart transplant data available in R software named ‘stanford2’ to test for gamma assumption. The data consist of 184 lifetimes where 72 of them are censored lifetimes. The censoring percentage of data is 38.5%. The test statistic is calculated as 0.3503. So we accept the null hypothesis that the data follows gamma distribution at both 1% and 5% level of significance.

Illustration 2: We examine the data on lifetimes of disk break pads on 40 cars studied in Lawless (2011). Complete data set is given in Table 6.11, Page 337 of Lawless (2011). Out of the 40 observed lifetimes, 9 are censoring times, hence data contains 22.5% of censored observations. The test statistic is obtained as 3.5142. Hence we reject the hypothesis of gamma distribution assumption for this data at both 1% and 5% level of significance.

6. Concluding Remarks

Based on fixed point characterization arising from Stein’s type identity, we developed new goodness of fit test for gamma distribution. We studied the asymptotic properties of the proposed test statistic. The proposed test has well controlled error rate. The power of the test is compared with recently developed test for gamma distribution. The proposed test is illustrated using two real data sets.

Even though several tests are available for gamma distribution in literature, as our knowledge, all of these test are developed for complete data. Motivated by this we develop a new goodness of fit test for gamma distribution with right censored data. We prove that the asymptotic distribution of the proposed test statistic is normal. We also find a consistent estimator of the asymptotic variance. The finite sample performance of the test is evaluated through Monte Carlo simulation study. Apart from right censoring, truncation and other types of censoring are common in lifetime data analysis. The proposed test can be modified to incorporates these situations. The similar goodness of fit test can be developed for other lifetime distribution using Stein’s type characterization.
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