The Modeling a Cellular Operator Profit as a Solution of the Optimization Problem in Applied Mathematics

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Abstract. In this paper, we solved the problem of applied mathematics connecting with maximizing the profit of a cellular operator under the assumption of a price duopoly with nonlinear demand functions. The conditions for the existence of an equilibrium in the value-added services market are considered. Response curves of duopolies were found explicitly, their properties were investigated. The results of the practical implementation of the developed theoretical models were presented on the example of the interactive voice services taking into account the obtained estimates of the coefficients for the empirical demand functions for the services of two leading cellular operators.

1. Introduction

In today's world, digital mobile services are developing quite dynamically. Mobile operators compete with each other by providing its subscribers with various new services, the prices of which are set by operators based on its assessment of the market situation. This study is about analysis the interactive voice service (IVR) [1]. IVR is a system of pre-recorded voice messages that performs the function of routing calls within the call center using the information entered by the client on the telephone keypad in tone mode.

Routing performed with the IVR system ensures that the caller is correctly provided with information in the automatic mode. Based on IVR, auto-information systems are built with a view to providing information about the balance of the subscriber, for instance. Other examples of IVR services may encompass legal advice or reading of literary works.

2. Methodology

2.1. The oligopoly models

For the first time, the oligopoly model for the simplest case of two firms was developed by Antoine – Augustin Cournot [2] and modified by Joseph Bertrand [3]. Cournot showed that in the state of equilibrium, each of the duopolists covers one-third of the market demand with their products at a single price. Covering together two-thirds of the market demand, each duopolist ensures the maximum of its own profit, but not that of the national industry.

This conclusion is very important for the theory of the industry organization. Contrary to the long-time dominating notion that the structure of the industry determines behavior, and the latter, in turn, determines the result [4-7]. It follows from the extended Cournot model that the structure of the industry and the results of its operation (price structure) are determined simultaneously.
These findings led economists (Ferguson P. and Ferguson G. [8]) to change the perception of internal relationships within the paradigm of structure - behavior - a result, but they did not solve the problem of oligopolists’ “naivety” when they are unable to draw certain conclusions from their own experience.

Chamberlain attempted to solve the problem by assuming that oligopolists, in particular, would not adhere to the assumption that each other’s output was predetermined if they saw that the opponent’s output changed in response to their own decisions. As a result, they would understand that it was in the interests of each of them to act in such a way that their joint profits would have been maximum. Thus, without colluding, they would come to the desirability of establishing a monopoly price for their (homogeneous) products [9].

2.2. Bertrand model
Joseph Bertrand as an alternative to the Cournot model proposed the classical model of price oligopoly. In the Bertrand model, each of the oligopolists takes the price level of competitors as given and makes a decision about the level of their price independently of all the others.

Oligopolists make decisions irrespectively of one other, take competitors’ prices for granted and choose their price level. At the same time, all consumers purchase products from an oligopolist who has set the minimum price. In the case of equal prices, the market is divided completely.

Assuming all the demand goes to the seller, who sets the minimum price, and the average production costs ci are the same for all, the only Nash equilibrium will be the total sale of products at costs and, as a result, the economic profit will be zero. Actually, at all prices of the competitors, exceeding the cost, the best strategy is to reduce production costs in order to corner the market. Obviously, competitors will not want to put up with such a situation. They also have the opportunity to lower prices, entice customers and ensure maximum profit for themselves under these conditions.

Work [8] shows that demand is much more dependent on the product price set by a cheaper company in the market. Thus, it can be assumed that the industry demand depends precisely on the minimum price \( p_{min} \) prevailing in the market.

If all firms set the same prices, the industry demand is equally divided between them. Industry demand has the form of a linear function in the classical Bertrand model. The average costs of oligopolists do not change with time.

If all firms set the same prices, the industry demand is equally divided between them. The result of the prerequisites of the classical Bertrand model with constant and equal average costs is the Bertrand paradox. It means firms alternately reduce prices to the cost level, and zero profits are obtained at the equilibrium point, which is completely equivalent to a perfect competition situation.

3. Developed math models
The demand of the mobile operators’ subscribers for additional services is considered from the perspective of a potential and actual one [10, 11].

The potential demand is determined by the number of services in quantitative terms which is desirable for an individual to consume at the stated service price level [12,13].

The amount of the potential demand was determined on the basis of the survey data collected by analytical agencies. It was also based on the data from billing systems (Telecommunications billing) of operators and service providers on the results of targeted interaction with consumers of various groups via mobile polling tools. The awareness and demand for various mobile services, the general and current structure of demand for VAS services were studied on the basis of data from the ROMIR Monitoring research holding.

Actual demand is equal to the factual volume of services rendered [14-16]. The value of the actual demand for IVR services is equal to the volume of services provided. This value was determined on the basis of commercial performance reports of cellular operators and service providers.

Potential industry demand for IVR services can be represented by a hyperbolic function:
Parameter $a$ in the denominator is introduced to limit the demand function at $p = 0$. The demand level at the zero price of the service is set by parameters alpha and $a$. As mentioned above, we will assume that a consumer’s choice is limited to the services of only two operators. The actual demand function $D_1(p)$ is the number of services rendered by the first operator provided that subscribers who are able to use the services of both the first and second operators consume IVR services of only the first operator during the given period of time. The demand value depends on the number of subscribers ($N_1$) and the average number of services per subscriber for a unit of time, for example, for one month.

$$D_1(p) = N_1 \bar{q}_1(p); \quad i = 1, 2.$$  

We assume that initially the prices for services of both operators are the same and equal to the minimum ($p_{min}$). The potential demand of each operator in this case can be described by an empirical function of the actual demand, the analytical form of which coincides with the function of industry demand and takes the form of a hyperbola. Estimates of coefficients $\alpha$ and $a$ can be obtained by processing statistical data using the method of the least squares. Let us consider two models of operators’ behavior.

### 3.1. Model 1

The first operator once raises the price of services to the level of $p_1$ and does not respond to price changes made by the second operator. In this case, the number of services consumed by subscribers falls only at the expense of reducing the number of its subscribers, who will switch on to the second operator with lower prices for services. The remaining subscribers will consume the same volume of services per subscriber, so that the analytical view and parameters of the empirical demand function as a whole will not change.

The number of consumed services will be determined by the demand levels of a function having the form (1) with parameters $\alpha_1$ and $a_1$ at the new price $p_1$.

We assume that subscribers have some inertia, i.e. they can move from operator to operator only in the next time interval, for example, next month. This assumption is true, since, most often, consumers use IVR prepaid services. Thus, the demand function of operator 1 can be written as $\Delta N_{12}$

$$D_1(p_1) = \frac{\alpha_1}{a_1 + p_1},$$

where $p_1 = p_{min} + \Delta p$;

$$\Delta D(\Delta p) = \Delta N_{12} \bar{q}_1(p_{min}) = N_0 \Delta p \bar{q}_1(p_{min}).$$

$\Delta N_{12}$ is the number of subscribers who have switched from operator 1 to operator 2; $p_1$ is the new price set by operator 1.

By virtue of the consumer habits invariance, we assume that the subscribers who have made the switch will consume the services of the second operator, according to their demand function with the first operator. At the same time, the greater the difference between the price of operator 2 and that of operator 1, the higher the number of the switched subscribers is. The demand of the second operator will increase due to the influx of new consumers by the value of $\Delta D(\Delta p)$ and it will be equal to

$$D_2(p_2) = \frac{\alpha_2}{a_2 + p_2} + N_0 \Delta p \frac{\alpha_1}{a_1 + p_2}.$$  

We have taken into account that before the price increase by the first operator, the service cost for both operators was the same and equal to the minimum price $p_1 = p_2 = p_{min}$. 

3
In this case, the problem of operators’ profit maximization will have the form of:

$$\pi_2 = D_2(p_2)p_2 - c_2p_2 = \left(\frac{\alpha_2}{a_2 + p_2} + \frac{\alpha_1}{a_1 + p_2}\right)p_2 - c_2p_2 \rightarrow \max_{p_1} (5)$$

In the given formulation of the problem, we will be interested in the reaction of the second operator to the price increase of the first operator.

To simplify calculations and avoid computations with integer variables, we assume that the number of subscribers and the number of services consumed during a month for each operator is a continuous variable. This is possible if the number of subscribers and consumed services change in thousands.

The solution to the problem (5) will be the prices of services maximizing the profits of operators. In this case, the reaction of operator 2 to the price change made by operator 1 by value $\Delta p = p_1 - p_2$ is given by the expression

$$\frac{\alpha_2a_2}{(a_2 + p_2)^2} + \frac{\alpha_1N_0a_1(p_1 - p_2) - \alpha_1N_0(p_2^2 + a_1)}{(a_1 + p_2)^2} = c_2$$

The price maximizing the profit of the first operator, provided that there is no response to the actions of the second operator, depends only on the parameters of the empirical demand function of operator 1 and, assuming unit costs, it can be found from the relation

$$p_1 = \sqrt{a_1a_1} - a_1; \quad a_1, a_2 \geq 0, p_1 > 0.$$  

Since the price of the service cannot be negative, the profit earning by operator 1 is possible provided the inequality is achieved.

$$0 < a_1 < a_1$$

3.2. Model 2

When making a decision to establish new levels of their prices for services, operators may be guided by competitors. In this case the demand function of operator 1 that is the first to increase the price from the $p_{min}$ level to $p_1$ will be

$$D_1(p_1) = D_1(p_{min}) - \Delta D(\Delta p),$$

where

$$\Delta D(\Delta p) = N_{12}\bar{a}_1 = N_0(p_1 - p_{min})\frac{\alpha_1}{a_1 + p_{min}}$$

Formula (9) takes into account the fact that the first operator knows part of its subscribers $N_{12}$ will switch on to the second operator. Provided that at the first step, the second operator does not increase its price simultaneously with the first operator ($p_2 = p_{min}$), the problem of the first operator’s profit maximization will be

$$\pi_1 = \left(\frac{\alpha_1}{a_1 + p_2} - \frac{\alpha_2N_0(p_1 - p_2)}{a_1 + p_2}\right)p_1 - p_1c_1 \rightarrow \max_{p_1} (10)$$

The problem (10) solution is the function of

$$p_1 = \frac{\alpha_1 - \alpha_1c_1}{2\alpha_1N_0} + \frac{\alpha_2N_0 - c_1}{2\alpha_1N_0}p_2$$

The conditions for maximizing the second operator’s profits and the equation of the second operator’s reaction remain unchanged (10), (11).
4. Results

Let us consider the response curves of operators in model 1 and model 2 in more detail. The estimation of the demand functions parameters was carried out by the least squares method [10] in the econometric package of GRETL. It was based on the previously linearized statistical data of the federal cellular operators’ billing systems. Coefficients rounded to integers are shown in Table 1.

The reaction of the second operator to the price increase by the first operator, under two different assumptions about the behavior of the first operator, is shown in Figure 3. We assumed that the number of subscribers who switched from one operator to another is $N_0 = 20$. In the future, this assumption can be replaced by a detailed theoretical or empirical analysis of the disloyal subscribers’ number effect on operator profits and market equilibrium.

Figures 1 - 5 show graphs of the profit maximizing price of the second operator, depending on the value of the price premium set by the first operator. We take into account different parameter values of the empirical demand functions of operator 1 ($a_1, \alpha_1$) and operator 2 ($a_2, \alpha_2$) as well as the number of subscribers who switched from the first operator to the second one ($N_0$). Table 1 gives the values of the equilibrium prices with two options of operator costs.

Table 1. Equilibrium price levels for different duopolist response models. $N_0 = 20$.

| Profit Functions | Costs  | $p_1$ | $p_2$ | $p_1'$ | $p_2'$ |
|------------------|--------|-------|-------|--------|--------|
| $\alpha_1 = 90, a_1 = 5$ | $c_1 = 1.5; c_2 = 1.5$ | 2,697 | 1.37 | 7,126 | 3,583 | 12,321 | 5,432 |
| $\alpha_2 = 70, a_2 = 2$ | $c_1 = 1.5; c_2 = 3.0$ | - | - | 7,247 | 3,639 | 7,247 | 3,639 |

Operator 1 is the first to increase the price. Costs for the content production and the provision of one service by the operators are the same and equal to 1.5. Operator 1 is the first to start increasing the price. The parameters of its demand function are $\alpha_1 = 90; a_1 = 5$. The response curve is depicted with the blue line in Figure 1. Following the increase in price by operator 1, the second operator can also change the price for its services. Depending on the reaction of the first operator, its response curve is shown in Figure 1. The green straight line shows the case when the first operator does not respond to the actions of the second one (Model 1), the red line means that it reacts (Model 2).

If the reaction of the first operator to the actions of the second one is absent (Model 1), there is a single Bertrand-Nash equilibrium at point $p^*$ in the market. This point lies below the straight line $p_1 = p_2$ (the purple line in Figure 1). Therefore, if the initial prices for IVR services were, for example, equal to $p_0 = 6.0$ and the first operator increased its price to $p_1^*$, then to maximize its profit...
and to achieve market equilibrium, the second operator can reduce its price to the level of \( p_2^* = 5.432 \), thereby further increasing its profit by attracting an even greater number of subscribers.

In case the first operator increases its price with a focus on the actions of the second operator, there are two equilibrium points \( p' \) and \( p'' \) in the market. Equilibrium at point \( p' \) can be achieved if operator 2 reduces the price of services below the level of costs. This situation seems unlikely. When the first operator raises the price, the second operator’s gains increase due to the influx of new subscribers. The equilibrium at point \( p'' \) is achieved at price levels lower than those existing under the conditions where the first operator does not respond to the second operator’s actions.

If the costs of the first operator are two times higher than the costs of the second operator \( c_1 = 2 \cdot c_2 = 3.0 \), the response curve in model 1 shifts to the left (Fig. 2) and the equilibrium in both models is achieved at the same price level.

5. Conclusion

From our analysis, we strongly conclude that depending on the duopoly behavior model, there may be one or two Bertrand-Nash equilibrium points in the market. The influence of the value of the cost on the equilibrium shift in the market for IVR services was investigated. The results can be useful for mobile operators to develop strategies of behavior in setting prices for VAS services.

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