A two-component generalization of the reduced Ostrovsky equation and its integrable semi-discrete analogue

Bao-Feng Feng¹, Ken-ichi Maruno² and Yasuhiro Ohta³

¹ School of Mathematical and Statistical Sciences, The University of Texas Rio Grande Valley, Edinburg, TX 78539, USA
² Department of Applied Mathematics, Waseda University, Tokyo 169-8050, Japan
³ Department of Mathematics, Kobe University, Rokko, Kobe 657-8501, Japan

E-mail: baofeng.feng@utrgv.edu, kmaruno@waseda.jp and ohta@math.kobe-u.ac.jp

Received 3 September 2016, revised 8 November 2016
Accepted for publication 11 November 2016
Published 4 January 2017

Abstract
In the present paper, we propose a two-component generalization of the reduced Ostrovsky (Vakhnenko) equation, whose differential form can be viewed as the short-wave limit of a two-component Degasperis–Procesi (DP) equation. They are integrable due to the existence of Lax pairs. Moreover, we have shown that the two-component reduced Ostrovsky equation can be reduced from an extended BKP hierarchy with negative flow through a pseudo 3-reduction and a hodograph (reciprocal) transform. As a by-product, its bilinear form and N-soliton solution in terms of pfaffians are presented. One- and two-soliton solutions are provided and analyzed. In the second part of the paper, we start with a modified BKP hierarchy, which is a Bäcklund transformation of the above extended BKP hierarchy, an integrable semi-discrete analogue of the two-component reduced Ostrovsky equation is constructed by defining an appropriate discrete hodograph transform and dependent variable transformations. In particular, the backward difference form of above semi-discrete two-component reduced Ostrovsky equation gives rise to the integrable semi-discretization of the short wave limit of a two-component DP equation. Their N-soliton solutions in terms of pfaffians are also provided.

Keywords: BKP and modified BKP hierarchy, pseudo 3-reduction, hodograph and discrete hodograph transform, two-component reduced Ostrovsky equation, short wave model of two-component Degasperis–Procesi (DP) equation, integrable discretization

(Some figures may appear in colour only in the online journal)
1. Introduction

The partial differential equation

\[ (u_t + c_0 u_x + uu_x)_x = \gamma u, \tag{1.1} \]

is a special case \((\beta = 0)\) of the Ostrovsky equation

\[ (u_t + c_0 u_x + uu_x + \beta u_{xxx})_x = \gamma u, \tag{1.2} \]

which was originally derived as a model for weakly nonlinear surface and internal waves in a rotating ocean [1, 2]. As pointed out in [3], equation (1.1) is invariant under the transformation

\[ u \rightarrow \mu^2 u, \quad x \rightarrow \mu x, \quad t \rightarrow \mu^{-1} t, \quad c_0 \rightarrow \mu^2 c_0, \tag{1.3} \]

and under the transformation

\[ u \rightarrow -u, \quad t \rightarrow -t, \quad \gamma \rightarrow -\gamma. \tag{1.4} \]

Moreover, the linear term \(c_0 u_t\) can be eliminated by a Galilean transformation. Therefore, without loss of generality, we can assume \(\gamma = 3\) and consider specifically the following equation

\[ (u_t + uu_x)_x - 3u = 0, \tag{1.5} \]

which is called the reduced Ostrovsky equation hereafter. Several authors derived basically the same model equation from different physical situations [4–6]. In particular, it appears as a model for high-frequency waves in a relaxing medium [5, 6]. Therefore the reduced Ostrovsky equation (1.5) is sometimes called the Vakhnenko equation [7–9], the Ostrovsky–Hunter equation [10], or the Ostrovsky–Vakhnenko equation [11, 12].

Differentiating the reduced Ostrovsky equation (1.5) with respect to \(x\), we obtain

\[ u_{txx} + 3u_x u_{xx} + uu_{xxx} - 3u_x = 0, \tag{1.6} \]

or in an alternative form

\[ m_t + um_x + 3mu_x = 0, \quad m = 1 - u_{xx}. \tag{1.7} \]

Equation (1.6) or equation (1.7) is known as the short wave limit of the Degasperis–Procesi (DP) equation [13, 14]. The reason for this lies in the fact that equation (1.6) can be derived from the DP equation [15]

\[ U_T + 3U_X - U_{XXX} + 4UU_X = 3U_X U_{XX} + U_{XXX}, \tag{1.8} \]

by taking a short wave limit \(\epsilon \rightarrow 0\) with \(U = \epsilon^2 (u + c_0 + \cdots), T = \epsilon^1 t, X = \epsilon^{-1} x\). Based on this connection, Matsuno [14] constructed the \(N\)-soliton solution of the short wave model of the DP equation from the \(N\)-soliton solution of the DP equation [16, 17]. By using the reciprocal link between the reduced Ostrovsky equation and 3-reduction of the B-type or C-type two-dimensional Toda lattice, i.e. the \(A_2^{(2)}\) 2D-Toda lattice, multi-soliton solutions to both the reduced Ostrovsky equation (1.5) and its differentiation form were constructed by the authors in [18]. Furthermore, we constructed an integrable semi-discrete reduced Ostrovsky equation [19] from a modified BKP hierarchy based on Hirota’s bilinear approach [20]. The integrability and wave-breaking phenomenon were studied in [21]. Interestingly, the short wave limit of the DP equation (1.6) also serves as an asymptotic model for propagation of surface waves in deep water under the condition of small-aspect-ratio [22]. Most recently, the inverse scattering transform (IST) problem for the short wave limit of the DP equation (1.6) was solved by a Riemann–Hilbert approach [12].
In the present paper, we propose and study a two-component generalization of the reduced Ostrovsky equation

\[ \rho_t + (\rho u)_x = 0, \]  
\[ (u_t + uu)_x = 3u + c(1 - \rho), \]  
\[ (\rho + \rho u)_x = 0, \]

which is shown to be integrable by finding its Lax pair and multi-soliton solution in subsequent sections. Differentiating equation (1.9) with respect to \( x \), we obtain

\[ m_t + m u + 3mu_x = c\rho_x, \quad m = 1 - u_{xx}. \]  

The system (1.10) and (1.11) is also integrable, which can be viewed as the short wave limit of a two-component Degasperis–Procesi equation.

The remainder of the present paper is organized as follows. In section 2, we find Lax pairs for two-component reduced Ostrovsky equation and its differential form. Then in section 3, starting from an extended BKP hierarchy with negative flow and its tau functions, we derive a two-component reduced Ostrovsky equation by a pseudo-3 reduction and an appropriate hodograph transform. Its bilinear form and \( N \)-soliton solution in parametric form are also given. In section 4, starting from a modified BKP hierarchy, which can be viewed as the Bäcklund transformation of the above extended BKP hierarchy, we construct integrable semi-discrete analogues of two-component reduced Ostrovsky equation and of the short wave limit of a two-component DP equation. We conclude our paper by some comments and further topics in section 5.

2. The Lax pairs

Equation (1.10) represents a conservation law, which can be used to define a hodograph (reciprocal) transformation \((x, t) \rightarrow (y, s)\) by

\[ dy = \rho dx - \rho ud\tau, \quad ds = d\tau, \]

then we have

\[ \partial_y = \rho^{-1}\partial_x, \quad \partial_s = \partial_t + u\partial_x. \]

By using above conversion formulas, we have the new conservative law

\[ (\rho^{-1})_t = u_s, \]

or

\[ \phi_t = u_s, \]

by defining \( \phi = \rho^{-1} \). Note that equation (1.9) can be rewritten as

\[ \rho u_{ys} - 3u + c(\rho - 1) = 0, \]

which, in turn, becomes

\[ \phi_{ys} - (3u + c)\phi + c = 0. \]

As shown in \([13, 23]\), equations (2.4) and (2.6) belong to the first negative flow in the Sawada–Kotera hierarchy. The corresponding Lax pair for \( c = 1 \) is of third order, which can be expressed as
\[ \Psi_{ss} - (3u + 1)\Psi = \frac{1}{\lambda}\Psi, \quad (2.7) \]
\[ \Psi_{s} - \lambda(\phi\Psi_{s} - \phi\Psi_{s}) = 0. \quad (2.8) \]

The above Lax pair can be rewritten in a matrix form
\[ \Phi_{s} = U\Phi, \quad \Phi_{s} = V\Phi, \quad (2.9) \]

with
\[ U = \begin{pmatrix} 0 & -\lambda\phi \lambda \phi \\ \phi & \lambda & 0 \\ \phi & \lambda & 0 \end{pmatrix}, \quad (2.10) \]
\[ V = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 3u + 1 & 0 \end{pmatrix}. \quad (2.11) \]

Applying the hodograph (reciprocal) transformation (2.2)–(2.9), we find
\[ \Phi_{s} = U\Phi, \quad \Phi_{s} = V\Phi, \quad (2.12) \]

with
\[ U = \begin{pmatrix} 0 & -\lambda u & \lambda \\ 1 & \lambda \rho & 0 \\ u & 1 & \lambda \rho \end{pmatrix}, \quad (2.13) \]
\[ V = \begin{pmatrix} 0 & 1 & 1 + \lambda uu & -\lambda u \\ -u & -\lambda uu & 1 \\ 1 & -uu & 2u + 1 & -\lambda uu \end{pmatrix}. \quad (2.14) \]

It is easy to find that the zero-curvature condition for (2.12) yields the two-component reduced Ostrovsky equations (1.9) and (1.10).

As the link of Lax pairs found by Hone and Wang in [13] is between the reduced Ostrovsky equation and the short wave limit of the DP equation, by considering the second component in above matrix form (2.12), we have
\[ \psi_{xxx} = 2\lambda \rho \psi_{xx} + (2\lambda \rho_{x} - \lambda^{2} \rho^{2})\psi_{x} + \left(\lambda m + \lambda \rho_{xx} - \lambda^{2} \rho \rho_{x}\right)\psi, \quad (2.15) \]
\[ \psi_{s} = \frac{1}{\lambda} \psi_{s} - (u + \rho)\psi_{s} + (u_{s} - \rho_{s})\psi. \quad (2.16) \]

The compatibility condition \( \psi_{xxxx} = \psi_{xxxx} \) gives the short wave limit of the two-component DP equations (1.10) and (1.11). Therefore, the integrability of the system (1.10) and (1.11) is confirmed by its Lax pair (2.15) and (2.16).
3. Bilinear equation and N-soliton solution for the two-component reduced Ostrovsky equation

3.1. Bilinear equation

The bilinear equation

\[ [(D_{x,-3} - D_{x,-1})D_{x} + 3D_{x,-1}^{2}] \tau \cdot \tau = 0, \quad (3.1) \]

is a dual bilinear equation

\[ [(D_{x} - D_{x,-1})D_{x,-1} + 3D_{x,-1}^{2}] \tau \cdot \tau = 0, \quad (3.2) \]

which belongs to the extended BKP hierarchy \([20, 24, 25]\). It has been shown in \([18]\) that this bilinear equation yields the reduced Ostrovsky equation \((1.5)\) through a hodograph transformation. Based on this finding, an integrable discretization of the reduced Ostrovsky equation \((1.5)\) was constructed in \([19]\).

Impose a pseudo-3 reduction by requesting \(D_{x,-5} = cD_{x,-1}\), and assume \(y = x_1, s = x_{-1}\), equation \((3.1)\) is reduced to

\[ (D_{x}D_{s} - cD_{x}s + 3D_{s}^{2}) \tau \cdot \tau = 0. \quad (3.3) \]

By using the relations

\[ \frac{D_{x}D_{s}^{2} \tau \cdot \tau}{\tau^2} = 2(ln \tau)_{xs} + 12(ln \tau)_{xs}(ln \tau)_{ys}, \]
\[ \frac{D_{x}D_{s} \tau \cdot \tau}{\tau^2} = 2(ln \tau)_{ys}, \quad \frac{D_{s}^{2} \tau \cdot \tau}{\tau^2} = 2(ln \tau)_{ys}, \]

Equation \((3.3)\) is converted to

\[ 2(ln \tau)_{ys} = 6(ln \tau)_{ys}(1 - 2(ln \tau)_{y}) + 2c(ln \tau)_{ys}. \quad (3.4) \]

Introducing a dependent variable transformation

\[ u = -2(ln \tau)_{ys}, \quad (3.5) \]

and a hodograph transformation

\[ x = y - 2(ln \tau)_{y}, \quad t = s, \quad (3.6) \]

we then have

\[ \frac{\partial x}{\partial y} = \rho^{-1}, \quad \frac{\partial x}{\partial s} = u \quad (3.7) \]

by defining \(\rho^{-1} = 1 - 2(ln \tau)_{y}\). Obviously, we have

\[ (\rho^{-1})_{s} = -2(ln \tau)_{ys} = u_{y}. \quad (3.8) \]

which, in turn, becomes

\[ \rho_{s} = -\rho^{2}u_{y} = -\rho u_{s}. \quad (3.9) \]

Furthermore, referring to the hodograph transformation and the resulting conversion formula \((2.2)\), we obtain

\[ \rho_{s} + u\rho_{s} = -\rho u_{s}, \quad (3.10) \]
which is exactly equation (1.10). On the other hand, the dependent variable transformation (3.5) converts equation (3.4) into
\[ \rho u_\tau = 3u + c(1 - \rho). \]  
(3.11)
With the use of the conversion formula (2.2) by hodograph transformation, we have
\[ (u_\tau + uu_\tau)_s = 3u + c(1 - \rho), \]  
(3.12)
which is exactly equation (1.9). In summary, the bilinear equation (3.3) derives the two-component reduced Ostrovsky equations (1.9) and (1.10) through the transformations (3.5) and (3.6).

**Remark 3.1.** A similar pseudo 3-reduction \( D_s = D_{s1} \) acting on the bilinear equation (3.2) leads to the shallow water waves [26]
\[ a_t - u u_x + 3uu_\tau + 3ux_1 u_\tau + 3ux_2 = 0, \]  
(3.13)
through variable transformations \( x = x_1, t = x_2, u = 2(\ln \tau)_x. \)

**Remark 3.2.** If \( c = 0 \), the reduction becomes a 3-reduction satisfying \( D_{s1} = 0 \), and the resulting bilinear equation gives the reduced Ostrovsky equation (1.5). Therefore, the reduced Ostrovsky equation can be viewed as a limiting case of the two-component reduced Ostrovsky equation as \( c \to 0 \). When \( c = 0 \), even if the variable \( \rho \) does not occur in the reduced Ostrovsky equation, it actually exists implicitly, which is embedded in the hodograph transformation.

3.2. N-soliton solution of the two-component reduced Ostrovsky equations (1.9) and (1.10)

It is known that both the bilinear equations (3.1) and (3.2) admit a pfaffian-type solution [18–20]
\[ \tau = \text{Pf}(a_1, a_2, \cdots, a_{2N}), \]  
(3.14)
where the elements of pfaffian are defined by
\[ \text{Pf}(a_i, a_j) = c_{ij} + \frac{p_i - p_j}{p_i + p_j} \varphi_{ij}, \]  
(3.15)
with
\[ c_{ij} = -c_{ji}, \quad \varphi = \exp(p_1^{-3}x_3 + p_1^{-1}x_1 + p_1 x_1 + p_1^3 x_3 + \xi_0). \]
Similar to the 3-reduction of the BKP hierarchy, to realize the pseudo 3-reduction \( D_{s1} = cD_{s1} \), we need to impose a constraint on the parameters of the general pfaffian solution, i.e.
\[ c_{ij} = \delta_{j,2N+1-i} c_{ii}, \quad c_{2N+1-i} = -c_{ii}, \]  
(3.16)
and
\[ p_i^{-3} + p_{2N+1-i}^{-1} = c(p_i^{-1} + p_{2N+1-i}^{-1}). \]  
(3.17)
Note that the pfaffian \( \tau \) can be rewritten as
\[ \tau = \left( \prod_{i=1}^{2N} c_i \right) \text{Pf} \left( \frac{\delta_{j,2N+1-i} c_{ii} + p_i - p_j}{p_i + p_j} \varphi_{ij} \varphi_{2N+1-i} \right). \]
it can be easily shown that \( \tau \) satisfies
\[
\partial_{x_3} \tau = c \partial_{x_1} \tau.
\] (3.18)

Under this reduction, the variable \( x_{-3} \) becomes a dummy variable, which can be viewed as a constant. Summarizing the results in question, we can present the \( N \)-soliton solution by the following theorem.

**Theorem 3.3.** The two-component reduced Ostrovsky equations (1.9) and (1.10) admits the following \( N \)-soliton solution in parametric form
\[
u = -2(\ln \tau)_3, \quad \rho = (1 - 2(\ln \tau)_3)^{-1}, \quad x = y - 2(\ln \tau)_3, \quad t = s,
\] (3.19)
inwhere \( \tau \) is a pfaffian
\[
\tau = \text{Pf}(a_1, a_2, \cdots, a_{2N}),
\] (3.20)
whose elements are defined by
\[
\text{Pf}(a_i, a_j) = \delta_{ij}2\nu + \frac{p_i - p_j e^{\xi_j + \xi_i}}{p_i + p_j e^{\xi_i + \xi_j}}.
\] (3.21)

Here \( \xi_j = p_j y + p_j^{-1} s + \xi_{j0} \) with the wave numbers \( p_j (j = 1, \cdots, 2N) \) satisfy a condition
\[
p_{3j} + p_{3N+1-j}^{-1} = c(p_{1j}^{-1} + p_{1N+1-j}^{-1}).
\] (3.22)

### 3.3. One- and two-soliton solutions

In this section, we provide one- and two-soliton for the two-component reduced Ostrovsky equations (1.9) and (1.10) and give a detailed analysis for their properties.

#### 3.3.1. One-soliton.
For \( N = 1 \), we have
\[
\tau = \text{Pf}(1, 2) = c_1 + \frac{p_1 - p_2 e^{\xi_1 + \xi_2}}{p_1 + p_2},
\] (3.23)
Let \( c_1 = 1, \quad \eta = \xi_1 + \xi_2 + \ln(p_1 - p_2) - \ln(p_1 + p_2), \quad p_{1j}^{-1} + p_{1N+1-j}^{-1} = k_j, \) we then have \( p_1 p_2 = (k_1^2 - c)/3 \) since \( p_{1j}^{-1} + p_{1N+1-j}^{-1} = c(p_{1j}^{-1} + p_{1N+1-j}^{-1}) \). \( \tau \) can be rewritten as
\[
\tau = 1 + e^{\eta} = 1 + e^{k_1^2 + \frac{3k_1^2}{2} y + \eta n}.
\] (3.24)
Therefore, we have the parametric form of the one-soliton solution
\[
u = \frac{k_1^2}{2} \text{sech}^2 \frac{\eta}{2}, \quad \rho = \left(1 - \frac{3k_1^2}{2(k_1^2 - c)} \text{sech}^2 \frac{\eta}{2}\right)^{-1},
\] (3.25)
\[
x = y - \frac{2k_1 e^{\eta}}{1 + e^{\eta}}, \quad t = s.
\] (3.27)
Equation (3.25) represents a soliton of amplitude $k_1^2/2$ with velocity $-(k_1^2 - c)/3$ for $u$-field. The regularity of the solution depends on equation (3.26). Notice that $\rho \to \pm \infty$ as $y \to \pm \infty$, and it attains an extreme value of $-2(k_1^2 - c)/(k_1^2 + 2c)$ at the peak point of the soliton when $\eta_k = 0$. It is not difficult to find that (a): if $c > 0$ and $k_1^2 - c < 0$, or (b): if $c < 0$ and $k_1^2 + 2c < 0$, the solution is regular. Two examples for case (a) $k_1 = 1.0, c = 2.0$ and case (b) $k_1 = 1.0, c = -2.0$ are illustrated in figures 1 and 2, respectively. Even though the $u$-field has the same amplitude for both cases, the $\rho$-field is quite different. The amplitude of $\rho$ is smaller that the asymptotic value of 1 at $\pm \infty$ for case (a), while it is larger than 1 for case (b). Moreover, the soliton moves to the right with velocity $1/3$ for case (a) and to the left with velocity $-1$ for case (b).

**Remark 3.4.** When $c = 0$, the two-component reduced Ostrovsky equation becomes simply the reduced Ostrovsky equation, and the one-soliton solution is always of loop type since $\rho^{-1}$ always has two zeros. Whereas the two-component reduced Ostrovsky equation has the regular solution depending on the values of $c$ and wave number $\kappa_1$.

**Remark 3.5.** Compared with the reduced Ostrovsky equation which only admits the left-moving soliton solution, the two-component reduced Ostrovsky equation may have both the left-moving and right-moving soliton solutions. To be more specific, if $k_1^2 - c > 0$, it has a left-moving soliton, whereas, if $k_1^2 - c < 0$, it has a right-moving soliton. However, the soliton solution does not exist when $k_1^2 - c = 0$.

### 3.3.2. Two-soliton.

By choosing $c_1 = c_2 = 1$, we have the tau function for two-soliton solution ($N = 2$)

$$
\tau = \text{Pf}(1, 2, 3, 4) = \text{Pf}(1, 2)\text{Pf}(3, 4) - \text{Pf}(1, 3)\text{Pf}(2, 4) + \text{Pf}(1, 4)\text{Pf}(2, 3)
$$

$$
\times \frac{p_1 - p_2 e^{\xi_3 + \xi_4}}{p_1 + p_2} \times \frac{p_3 - p_4 e^{\xi_3 + \xi_4}}{p_3 + p_4} \times \frac{p_5 - p_6 e^{\xi_3 + \xi_4}}{p_5 + p_6} \times \frac{p_7 - p_8 e^{\xi_3 + \xi_4}}{p_7 + p_8} + \left(1 + \frac{p_1 - p_2 e^{\xi_3 + \xi_4}}{p_1 + p_2} \right) \left(1 + \frac{p_3 - p_4 e^{\xi_3 + \xi_4}}{p_3 + p_4} \right) \left(1 + \frac{p_5 - p_6 e^{\xi_3 + \xi_4}}{p_5 + p_6} \right) \left(1 + \frac{p_7 - p_8 e^{\xi_3 + \xi_4}}{p_7 + p_8} \right),
$$

under the condition

$$
p_1^3 + p_4^3 = c(p_1^{-1} + p_4^{-1}), \quad p_2^3 + p_3^3 = c(p_2^{-1} + p_3^{-1}).
$$

Similarly, the above $\tau$-function can be rewritten as
\[ \tau = 1 + e^{\eta_1} + e^{\eta_2} + b_{12} e^{\eta_1 + \eta_2}, \]  
(3.29)

with
\[ b_{12} = \frac{(p_1 - p_2)(p_1 - p_3)(p_4 - p_2)(p_4 - p_3)}{(p_1 + p_2)(p_1 + p_3)(p_4 + p_2)(p_4 + p_3)}. \]  
(3.30)

by having \( \eta_1 = \zeta_1 + \xi_3 + \ln(p_1 - p_3) - \ln(p_1 + p_3), \eta_2 = \zeta_2 + \xi_4 + \ln(p_2 - p_3) - \ln(p_2 + p_3) \).

Furthermore, if we let \( p_i^{-1} + p_4^{-1} = k_1, p_2^{-1} + p_3^{-1} = k_2 \), we then have
\[ \eta_i = k_i s + \frac{3 k_i}{k_i^2 - c} y + \eta_{0i}. \]  
(3.31)

for \( i = 1, 2 \) and
\[ b_{12} = \frac{(k_1 - k_2)^2(k_1^2 - k_2 k_1 + k_2^2 - 3c)}{(k_1 + k_2)^2(k_1^2 + k_2 k_1 + k_2^2 - 3c)}. \]  
(3.32)

To avoid the singularity of the soliton solution, the condition for regularity of each soliton needs to be satisfied. With regards to the interactions of two solitons, there are either catch-up collision or head-on collision depending on the values of parameters discussed previously. Furthermore, the collision is always elastic, and there is no change in shape and amplitude of solitons except a phase shift. In figure 3, we illustrate the contour plot for the collision of two solitons, and in figure 4, the profiles before and after the collision. The parameters are taken as \( c = -2.0, k_1 = 1.0 \) and \( k_2 = 1.6 \).

### 4. Integrable semi-discretization of the two-component reduced Ostrovsky equation

We could construct a semi-discrete analogue of the two-component reduced Ostrovsky equation based on the Bäcklund transformation of the extended BKP hierarchy. For the sake of simplicity, here we take \( c = 1 \) without loss of generality. The starting point is a bilinear equation associated with the modified BKP hierarchy
\[ ((D_x - b)^\tau - (D_y - b^\tau))\eta_{n+1} \cdot \eta = 0. \]  
(4.1)
This bilinear equation can be viewed as a Bäcklund transformation of the extended BKP hierarchy. It admits a pfaffian type solution of the form
\[ \tau = \text{Pf}\left(\ldots, 2N\right) \]
whose elements are determined by
\[ \phi = -c_{ij}, \]
where \( c_{ij} = -c_{ji} \) and
\[ \phi_{l}^{(0)}(t) = p_{l}^{1} \left( 1 + b_{p} \right) e^{\xi_{l}}, \quad \xi_{l} = p_{l}^{-1} x + p_{l}^{-3} t + \xi_{l0}. \]

Note that if we take \( c_{ij} \) as in equation (3.16), \( \eta \) is rewritten
\[ \eta = \prod_{l=1}^{2N} \phi_{l}^{(0)}(t) \left( \phi_{l}^{(0)}(t) \phi_{2N+1-l}^{(0)}(t) \right) \left( \frac{p_{l} - p_{l+1}}{p_{l} + p_{l+1}} \right), \]
so by imposing a reduction condition
\[ \frac{1}{p_{l}^{3}} + \frac{1}{p_{2N+1-l}^{3}} = \frac{1}{p_{l}} + \frac{1}{p_{2N+1-l}}, \]
we can easily show that the pfaffian \( \eta \) satisfies
\[ \partial_{x} \eta = \partial_{t} \eta. \]

Therefore equation (4.1) is reduced into
\[ (D_{l}^{3} - 3b D_{l}^{2} + (3b^{2} - 1)D_{l}) \eta_{l} \cdot \eta = 0, \]

based on which we will derive the integrable semi-discretization. First, we introduce a discrete hodograph transformation
\[ x_{l} = 2lb - 2\left(\ln \eta_{l}\right), \quad t = s, \]
and a dependent variable transformation
\[ u_{l} = -2\left(\ln \eta_{l}\right), \]
it then follows that the nonuniform mesh, which is defined by $\delta_l = x_{l+1} - x_l$, can be expressed as

$$\delta_l = 2b - 2\left(\ln \frac{\eta_{l+1}}{\eta_l}\right)_s,$$

(4.8)

which is related to $\rho_l$ by

$$\rho_l = \frac{2b}{\delta_l}.$$

(4.9)

Differentiating equation (4.8) with respect to $s$, one obtains

$$\frac{d\delta_l}{ds} = -2\left(\ln \frac{\eta_{l+1}}{\eta_l}\right)_{ss} = u_{l+1} - u_l.$$

(4.10)

which is equivalent to

$$\frac{d\rho^{-1}_l}{ds} = \frac{u_{l+1} - u_l}{2b}.$$

(4.11)

Dividing $\eta_{l+1}/\eta_l$ on both sides of equation (4.4) and using the following relations

$$\frac{D^2\eta_{l+1}/\eta_l}{\eta_{l+1}/\eta_l} = \left(\ln \frac{\eta_{l+1}}{\eta_l}\right)_s,$$

$$\frac{D^2\eta_{l+1}/\eta_l}{\eta_{l+1}/\eta_l} = \left(\ln(\eta_{l+1}/\eta_l)\right)_{ss} + \left(\ln \frac{\eta_{l+1}}{\eta_l}\right)^2_s,$$

$$\frac{D^3\eta_{l+1}/\eta_l}{\eta_{l+1}/\eta_l} = \left(\ln \frac{\eta_{l+1}}{\eta_l}\right)_{sss} + 3\left(\ln \frac{\eta_{l+1}}{\eta_l}\right)_{ss} \left(\ln(\eta_{l+1}/\eta_l)\right)_s + \left(\ln \frac{\eta_{l+1}}{\eta_l}\right)^3_s,$$
one obtains
\[
\left( \ln \frac{\tau_{i+1}}{\tau_i} \right)_{xx} = (1 - b^2) \left( \ln \frac{\tau_{i+1}}{\tau_i} \right)_x + \left( b - \left( \ln \frac{\tau_{i+1}}{\tau_i} \right)_x \right)
\]
\[
+ \left[ 3(\ln(\tau_{i+1}))_{xx} - \left( \ln \frac{\tau_{i+1}}{\tau_i} \right)_x \right] 2b - \left( \ln \frac{\tau_{i+1}}{\tau_i} \right)_x, \tag{4.12}
\]
which is converted into
\[
\frac{d}{ds}(u_{i+1} - u_i) = \frac{3}{2} \delta_t(u_{i+1} - u_i) - \frac{1}{4} \delta_t(\delta_t^2 - 4) - 2b^3 - 2b, \tag{4.13}
\]
by equations (4.6) and (4.8). In summary we have the following theorem

**Theorem 4.1.** The bilinear equation
\[
(D_t^3 - 3bD_x^2 + (3b^2 - 1)D_x)\tau_{i+1} \cdot \tau_i = 0
\]
determines a semi-discrete analogue of the two-component reduced Ostrovsky equations (1.9) and (1.10)
\[
\frac{d}{ds}(u_{i+1} - u_i) = \frac{3}{2} \delta_t(u_{i+1} - u_i) - \frac{1}{4} \delta_t(\delta_t^2 - 4) - 2b^3 - 2b, \tag{4.14}
\]
\[
\frac{d\rho_t^{-1}}{ds} = \frac{u_{i+1} - u_i}{2b} \tag{4.15}
\]
by dependent variable transformations
\[
u_t = -2(\ln \tau)_x, \quad \rho_t = \left( 1 - b^{-1} \left( \ln \frac{\tau_{i+1}}{\tau_i} \right)_x \right)^{-1}, \tag{4.16}
\]
and a discrete hodograph transformation
\[
x_t = 2b - 2(\ln \tau)_x, \quad t = s. \tag{4.17}
\]

The nonuniform mesh, which is defined by \( \delta_t = x_{i+1} - x_i \), is related to \( \rho_t \) by \( \rho_t \delta_t = 2b \).
Next, we show the continuous limit of semi-discrete two-component reduced Ostrovsky equations (4.14) and (4.15). Since
\[
\frac{\partial x}{\partial s} = \frac{\partial x_0}{\partial s} + \sum_{j=0}^{i-1} \frac{\partial \delta_t}{\partial s} = \frac{\partial x_0}{\partial s} + \sum_{j=0}^{i-1} (u_{j+1} - u_j) \to u,
\]
we then have
\[
\partial_t \tau \equiv \partial_t + \frac{\partial x}{\partial s} \partial_s \to \partial_t + u \partial_s.
\]
Then equation (4.15) is converted into
\[
\partial_s (\partial_t + u \partial_s) \frac{\partial \tau}{\partial s} = u_t,
\]
which, in turn, becomes equation (1.10). By dividing \( \delta_t \) on both sides of equation (4.13), we have
\[ \frac{1}{\delta_t} \frac{d}{ds} (u_{t+1} - u_t) = \frac{3}{2} (u_t + u_{t+1}) - \frac{1}{4} \delta_t^2 + 1 - (1 - b^2) \rho, \]  

(4.18)

Obviously, in the continuous limit, \( b \to 0 (\delta_t \to 0) \), it converges to

\[(\partial_t + u \partial_x)u = 3u + 1 - \rho,\]

which is exactly equation (1.9) with \( c = 1 \). It is interesting to note that we have

\[ \frac{1}{\delta_t} \frac{d}{ds} (u_{t+1} - u_t) = \frac{3}{2} (u_t + u_{t+1}) - \frac{1}{4} (\delta_t^2 - \delta_{t-1}^2) - (1 - b^2)(\rho_t - \rho_{t-1}), \]  

(4.19)

by taking a backward difference of equation (4.18). Furthermore, by defining

\[ m_t = 1 - \frac{2}{\delta_t + \delta_{t-1}} \left( \frac{u_{t+1} - u_t}{\delta_t} - \frac{u_t - u_{t-1}}{\delta_{t-1}} \right), \]

by defining a forward difference operator and an average operator

\[ \Delta f_t = \frac{f_{t+1} - f_t}{\delta_t}, \quad Mu_t = \frac{f_t + f_{t-1}}{2}, \]

we can claim an integrable semi-discrete analogue of equations (4.14) and (4.15) as follows.

**Theorem 4.2.** A semi-discrete analogue for the short wave limit of a two-component DP equations (1.10) and (1.11) is of the form

\[ \frac{d}{ds} m_t = m_t \left( -2M \Delta u_t - \frac{M(\delta_t \Delta u_t)}{M\delta_t} + \frac{1}{2} \left( \delta_t - \delta_{t-1} \right) \right) + \left( 1 - b^2 \right) \frac{\rho_t - \rho_{t-1}}{M\delta_t}, \]

(4.20)

\[ \frac{d}{ds} \rho_{t-1} = \frac{u_{t+1} - u_t}{2b}, \]

(4.21)

\[ m_t = 1 - \frac{2}{\delta_t + \delta_{t-1}} \left( \frac{u_{t+1} - u_t}{\delta_t} - \frac{u_t - u_{t-1}}{\delta_{t-1}} \right). \]

(4.22)

Its \( N \)-soliton solution is the same as the one of the two-component reduced Ostrovsky equation. In the continuous limit, \( b \to 0 (\delta_t \to 0) \), we have

\[ 2M \Delta u_t \to 2u_s, \quad \frac{M(\delta_t \Delta u_t)}{M\delta_t} \to u_s, \quad \frac{\rho_t - \rho_{t-1}}{M\delta_t} \to \rho_s, \]

then equation (4.22) converges to

\[ m_t \to m = 1 - u_s, \]

while equations (4.20) and (4.21) converge to

\[(\partial_t + u \partial_x)m = -3mu_s - \rho_s,\]

and

\[(\partial_t + u \partial_x)\rho = -\rho u_s,\]

which are exactly the short wave limit of a two-component DP equations (1.10) and (1.11).
5. Conclusion and further topics

In the present paper, we proposed a two-component generalization of the reduced Ostrovsky equation and its differential form, which can be viewed a short wave limit of a two-component DP equation. The integrability for both equations is assured by finding their Lax pairs. Moreover, we have shown that the proposed two-component reduced Ostrovsky equation can be reduced from an extended BKP hierarchy through a hodograph transformation under a pseudo 3-reduction. Based on this fact, its bilinear equation, as well as its $N$-soliton solution, is found. One- and two-soliton solutions are analyzed in detail. We should emphasize that, in comparison with the reduced Ostrovsky equation which only admits a multi-valued (loop) soliton solution, the two-component reduced Ostrovsky equation, as well as its differential form, can have regular solutions depending on the spatial wave number and the value of $c$.

The integrable semi-discrete analogues for the two-component generalization of the reduced Ostrovsky equation and its differential form are constructed based on a Bäcklund transform of the extended BKP hierarchy by defining a discrete hodograph transform and mimicking pseudo 3-reduction in continuous case. The $N$-soliton solutions are also provided in terms of pfaffians. It would be interesting to apply integrable semi-discretizations as integrable self-adaptive moving mesh methods [27–29] for numerical simulations of the two-component reduced Ostrovsky equation.

A two-component Camassa–Holm (2-CH) equation [31–33] and its short wave limit, also called a two-component Hunter–Saxton (2-HS) equation [34–37], have been known for while and have drawn some attention in mathematical physics. Both equations can be expressed by the same form

\begin{align}
    m_t + um_x + 2mu_x - \sigma \rho \rho_x &= 0, \\
    \rho_t + (\rho u)_x &= 0,
\end{align}

(5.1)

(5.2)

except for the 2-CH equation $m = \kappa + u - u_{xx}$ and for the 2-HS equation $m = \kappa - u_{xx}$. A similar two-component DP equation has been proposed in [30] but it seems not to be integrable. Does an integrable two-component DP equation share the same form as equations (1.10) and (1.11) except $m = 1 + u - u_{xx}$. If this is true, then what is the Lax pair? We expect that the answers to these questions can be made clear in the near future.

Acknowledgment

BF appreciates the comments and discussions with Professor Youjin Zhang and Professor Qingping Liu and the partial support by the National Natural Science Foundation of China (No. 11428102). The work of KM is partially supported by JSPS Grant-in-Aid for Scientific Research (C-15K04909) and CREST, JST. The work of YO is partly supported by JSPS Grant-in-Aid for Scientific Research (B-24340029, C-15K04909) and for Challenging Exploratory Research (26610029).

References

[1] Ostrovsky L A 1978 Oceanology 18 119–25
[2] Stepanyants Y A 2006 Chaos Solitons Fractals 28 193–204
[3] Parkes E J 2007 Chaos Solitons Fractals 31 602–10
[4] Hunter J 1990 Lect. Appl. Math. 26 301–16
[5] Vakhnenko V O 1992 J. Phys. A: Math. Gen. 25 4181–7
[6] Vakhnenko V O 1999 J. Math. Phys. 40 2011–20
[7] Vakhnenko V O and Parkes E J 1999 Nonlinearity 11 1457–64
[8] Morrison A J, Vakhnenko V O and Parkes E J 1999 Nonlinearity 12 1427–37
[9] Vakhnenko V O 1992 J. Phys. A: Math. Gen. 25 4181–7
[10] Liu Y, Pelinovsky D and Sakovich A 2010 SIAM J. Math. Anal. 42 1967–85
[11] Brunelli J C and Sakovich S 2013 Commun. Nonlinear Sci. Numer. Simul. 18 56–62
[12] Bouet de Monvel A and Shepelsky D 2015 J. Phys. A: Math. Theor. 48 035204
[13] Hone A N W and Wang J P 2003 Inverse Problems 19 129–45
[14] Matsuno Y 2006 Phys. Lett. A 359 451–7
[15] Degasperis A and Procesi M 1999 Asymptotic integrability Symmetry and Perturbation Theory ed A Degasperis and G Gaeta (Singapore: World Scientific) pp 23–37
[16] Matsuno Y 2005 Inverse Problems 21 1553–70
[17] Matsuno Y 2005 Inverse Problems 21 2085–101
[18] Feng B-F, Maruno K and Ohta Y 2012 J. Phys. A: Math. Theor. 45 355203
[19] Feng B F, Maruno K and Ohta Y 2015 J. Phys. A: Math. Theor. 48 135203
[20] Hirota R 2004 The Direct Method in Soliton Theory (Cambridge: Cambridge University Press)
[21] Grimshaw R H J, Helfrich K and Johnson E R 2012 Stud. Appl. Math. 129 414–36
[22] Kanna J, Leblond H and Manna M A 2014 J. Phys. A: Math. Theor. 47 025208
[23] Gordoa P R and Pickering A 1999 J. Math. Phys. 28 2871–88
[24] Jimbo M and Miwa T 1983 Publ. RIMS. Kyoto Univ. 19 943–1001
[25] Hirota R 1989 J. Phys. Soc. Japan 58 2285–96
[26] Hirota R and Satsuma J 1976 J. Phys. Soc. Japan 40 611–2
[27] Ohta Y, Maruno K and Feng B F 2008 J. Phys. A: Math. Theor. 41 355205
[28] Feng B F, Maruno K and Ohta Y 2010 J. Comput. Appl. Math 235 229–43
[29] Feng B F, Maruno K and Ohta Y 2010 J. Phys. A: Math. Theor. 43 085203
[30] Popowicz Z 2006 J. Phys. A: Math. Gen. 39 13717
[31] Chen M, Liu S and Zhang Y 2006 Lett. Math. Phys. 75 1–15
[32] Aratyn H, Gomes J F and Zimerman A H 2006 J. Phys. A: Math. Gen. 39 1099–114
[33] Constantin A, Ivanov R I 2008 Phys. Lett. A 372 1129–32
[34] Wunsch M 2009 Discrete Contin. Dyn. Syst. 12 647–56
[35] Lenells J and Lechtenfeld O 2009 J. Math. Phys. 50 4012704
[36] Moon B and Liu Y 2012 J. Differ. Eq. 253 319–55
[37] Lou S Y, Feng B F and Yao R X 2016 Wave Motion 65 17–28