Robust chaos with prescribed natural invariant measure and Lyapunov exponent

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Abstract

We extend in several ways a recently proposed method to construct one-dimensional chaotic maps with exactly known natural invariant measure \cite{1,2}. First, we assume that the given invariant measure depends on a continuous parameter and show how to construct maps with robust chaos —i.e., chaos that is not destroyed by arbitrarily small changes of the parameter— and prescribed invariant measure and constant Lyapunov exponent. Then, by relaxing one condition in the approach of Refs. \cite{1,2}, we describe a method to construct robust chaos with prescribed constant invariant measure and varying Lyapunov exponent. Another extension of a condition in Refs. \cite{1,2} provides a new method to get robust chaos with known varying Lyapunov exponent. In this third approach the invariant measure can be computed exactly in many particular cases. Finally we discuss how to use diffeomorphisms to construct maps with robust chaos, any number of parameters and prescribed invariant measure and Lyapunov exponent.

Key words: nonlinear dynamical system, deterministic chaos, robust chaos, natural invariant measure

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1. Introduction

The inverse problem for chaotic one-dimensional maps has been recently proposed and solved in some cases by Sogo \cite{1,2}. Starting from a given invariant measure, the method allows to construct maps with that measure and Lyapunov

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exponents in the form $\lambda = \ln m$, with $m = 2, 3, \ldots$. This method can be used to construct exact examples of chaotic one-dimensional maps depending on the discrete parameter $m$ and having exactly known invariant measure and Lyapunov exponent. The first goal of this work is to extend this method to families of maps that depend on a continuous parameter, display robust chaos—i.e., they have a chaotic attractor which is not destroyed by arbitrarily small changes of the parameter—and have known invariant measure and Lyapunov exponent.

Piecewise smooth maps may show robust chaos and have been used to describe robust chaos in circuits [3]. On the other hand, many families of smooth maps have fragile chaos. For instance, the logistic map, $x_{n+1} = 4rx_n(1 - x_n)$, is chaotic for $r = 1$, but the attractor is periodic for a set of values of the parameter $r$ that is dense in the interval $0 \leq r \leq 1$ [4]. If such a dynamical system describes a real device, it may be impossible to know in advance whether the behavior of the system will in fact be chaotic or periodic for some parameter value, which is always known with finite precision. However, robust chaotic behavior is required in many applications, including encrypting messages [5, 6], random number generators [7], and engineering applications [8]. Robust chaos also arises in neural networks [9, 10] as well as in the study of brain and population dynamics [11, 12]. Thus, simple and easy ways of constructing robust chaotic attractors with exactly known properties, as those explored in this work, can be of interest in different fields of science and engineering.

Andrecut and Ali first found a smooth map [13] and later a method of generating smooth maps [14] whose evolution is chaotic for whole intervals of the parameter. Another case is discussed in Ref. [15] and we have recently explored several new ways to construct smooth one-dimensional maps with robust chaos [16].

The purpose of this work is to present other ways of constructing maps with robust chaos. The advantage of these new methods is that the Lyapunov exponent is known exactly and the invariant measure is known always with the first, second and fourth methods and in many particular cases in the third approach. The first three methods will be extensions of Sogo’s inverse problem [1, 2]. In Sect. 2 we will show how to construct families of chaotic maps with a constant Lyapunov exponent and a given invariant measure depending continuously on a parameter. In Sect. 3 we present a method to construct families of chaotic maps sharing the same prescribed invariant measure and having a known Lyapunov exponent depending on the parameter. In Sect. 4 we will explore a method to construct robust chaos with a known Lyapunov exponent varying continuously with the parameter and a natural measure that can be computed exactly in many particular cases. Fi-
nally, in Sect. 5 and 6 we will discuss several ways to construct robust chaos with prescribed Lyapunov exponent and natural invariant measure by using diffeomorphisms.

We will consider one-dimensional maps on a finite interval $[a, b]$, which for commodity will be reduced to $[0, 1]$ by means of a linear transformation.

2. Robust chaos with prescribed invariant measure

We will use an easy extension of Sogo’s method [1, 2] to construct families of dynamical systems, $x_{n+1} = f_r(x_n)$, that are chaotic for a full range of the parameter $r$ and have exactly known invariant measure and Lyapunov exponent. Let us assume that the maps $f_r$ are $m$-to-1 so that each value $x \in [0, 1]$ has $m$ preimages $y_k$, such that $f_r(y_k) = x$, for $k = 1, 2, \ldots, m$. The natural invariant measure $d\mu_r = \rho_r(x) \, dx$ satisfies the Frobenius-Perron equation [4]:

$$\rho_r(x) = \int_0^1 \rho_r(y) \delta(x - f_r(y)) \, dy = \sum_{k=1}^m \frac{\rho_r(y_k)}{|f'_r(y_k)|}. \quad (1)$$

To find solutions of this equation we will simplify it by further assuming that all the terms in the sum make the same contribution, so that the substitution $y_1 \rightarrow x$ gives the condition

$$\rho_r(f_r(x)) \left| f'_r(x) \right| = m \rho_r(x). \quad (2)$$

Eq. (2) provides a useful practical way to construct a family of maps $f_r$ with a prescribed invariant density $\rho_r$: just choose, for a given integer value of $m \geq 2$, a family of solutions of the differential equation (2) that map the phase space $[0, 1]$ onto itself and are $m$-to-1. This will be a family of maps with robust chaos. By extending the argument in Ref. [2] one can show that all the maps in the family will have the same Lyapunov exponent:

$$\lambda_r = \int_0^1 \rho_r(x) \ln |f'_r(x)| \, dx$$

$$= \ln m + \int_0^1 \rho_r(x) \ln \rho_r(x) \, dx$$

$$- \int_0^1 \rho_r(x) \ln \rho_r(f_r(x)) \, dx = \ln m. \quad (3)$$
The two last integrals cancels each other because of (1):

\[ \int_0^1 \rho_r(x) \ln \rho_r(x) \, dx \]
\[ = \int_0^1 \int_0^1 \rho_r(y) \, \delta(x - f_r(y)) \ln \rho_r(x) \, dy \]
\[ = \int_0^1 \rho_r(y) \ln \rho_r(f_r(y)) \, dy. \tag{4} \]

2.1. Some solutions with \( m=2 \)

To give some examples, let us further assume \( m = 2 \) and that the invariant densities \( \rho_r \in C^0(0,1) \) are strictly positive, \( \rho_r(x) > 0 \) for \( 0 < x < 1 \), so that

\[ \mu_r(x) \equiv \int_0^x \rho_r(y) \, dy \]  

monotonously increases from \( \mu_r(0) = 0 \) to \( \mu_r(1) = 1 \) and has a unique inverse \( \mu^{-1}_r(x) \) in \([0,1]\). If we choose the boundary conditions \( f_r(0) = f_r(1) = 0 \), the solution of (2) is

\[ f_r(x) = \mu^{-1}_r(1 - |1 - 2\mu_r(x)|). \tag{6} \]

Each map will increase from \( f_r(0) = 0 \) to the maximum

\[ f_r(\alpha_r) = 1, \quad \alpha_r \equiv \mu^{-1}_r \left( \frac{1}{2} \right) \tag{7} \]

and then decrease to \( f_r(1) = 0 \). The Lyapunov exponent will be \( \lambda_r = \ln 2 \).

2.1.1. A family of piecewise smooth chaotic maps

If one chooses the invariant densities

\[ \rho_r(x) = rx^{r-1}, \quad (r > 0), \tag{8} \]

the solution (6) will be

\[ f_r(x) = (1 - |1 - 2x^r|)^{1/r}, \quad (r > 0). \tag{9} \]

Three members of this family of chaotic maps are displayed in Fig. 1. The full tent map \( T(x) = 1 - |1 - 2x| \) is recovered with \( r = 1 \).

4
2.1.2. A family of smooth chaotic maps

The solutions (9) are not differentiable at $x = \alpha_r$. In fact, since

$$\lim_{x \to \alpha_r^\pm} f'_r(x) = \pm \frac{2\rho_r(\alpha_r)}{\rho_r(1)},$$

the condition for $f'_r$ to be continuous at the maximum is $\lim_{x \to 1} \rho_r(x) = \infty$.

We can use this condition to find families of chaotic maps that are smooth along the full interval $[0, 1]$. To provide an example, let us consider the family of densities given for every $r > 0$ by

$$\rho_r(x) = \frac{2^{-1} r \arcsin^{-1} 1/r}{\pi^r \sqrt{x(1-x)}}.$$  \hspace{1cm} (11)

Then the solution (6) is

$$f_r(x) = \begin{cases} 
\sin^2 \left( \frac{2^{1/r} \arcsin \sqrt{x}}{\pi} \right), & 0 \leq x \leq \alpha_r; \\
\sin^2 \left( 2^{1/r} \left( \frac{x}{2} - \arcsin^r \sqrt{x} \right)^{1/r} \right), & \alpha_r \leq x \leq 1,
\end{cases}$$  \hspace{1cm} (12)

with $\alpha_r \equiv \sin^2 \left( 2^{-1} r \pi \right)$.

Some members of the family (12) are displayed in Fig. 2 where one recognizes the full logistic map $f_1(x) = 4x(1-x)$. We have $f_r \in C^1[0,1]$ for $0 < r \leq 1$, but $f'_r(x) \to -\infty$ as $x \to 1$ for $r > 1$. 

Figure 1: Three members of the family of maps (9).
The boundary conditions for $f_r(x)$ can be chosen in many other ways. For instance, with $f_r(0) = f_r(1) = 1$ the maps will have a single minimum, instead of a maximum, and the substitute for (6) is

$$f_r(x) = \mu_r^{-1} (|1 - 2\mu_r(x)|).$$

(13)

The single minimum is $f_r(\alpha_r) = 0$ with $\alpha_r \equiv \mu_r^{-1}(1/2)$.

With the densities (8) one gets

$$f_r(x) = |1 - 2x^r|^{1/r}, \quad (r > 0),$$

(14)

which has a smooth minimum only for $0 < r < 1$, since now the condition is $\lim_{x \to 0} \rho_r(x) = \infty$.

It is also easy to extend the results of this section for values of $m \neq 2$. But let us present another way to construct robust chaos.

3. Robust chaos with constant invariant measure

Following Refs. [1, 2], to get Eq. (2) we assumed that all terms in sum (1) had the same value. It is obvious that this simplifying condition can be relaxed in many ways. Let us explore a simple one. One could construct a family of 2-to-1 maps, with the same prescribed invariant density $\rho(x)$ but with a generalized
relation between the terms in sum (1),

\[
\frac{\rho (y_1)}{|f'_r (y_1)|} = r \frac{\rho (y_2)}{|f'_r (y_2)|}, \quad (r > 0),
\]

for instance. With this assumption, one would replace the constant multiplicity \( m \) of Eq. (2) by

\[
m_r (x) = \begin{cases} 
\frac{1 + r}{r}, & 0 \leq x < \alpha_r; \\
1 + r, & \alpha_r < x \leq 1;
\end{cases}
\]

(16)

where \( x = \alpha_r \) is the value where the two preimages coincide: \( y_1 (\alpha_r) = y_2 (\alpha_r) \).

The family \( f_r \) would satisfy the following equation:

\[
\rho (f_r (x)) |f'_r (x)| = m_r (x) \rho (x), \quad (r > 0).
\]

For example, for any \( \rho \in C^0 (0, 1) \) with \( \rho (x) > 0 \) and \( \mu (x) \equiv \int_0^x \rho (y) \, dy \), the solution of (17) with \( f_r (0) = f_r (1) = 0 \) is

\[
f_r (x) = \begin{cases} 
\mu^{-1} \left( \frac{1 + r}{r} \mu (x) \right), & 0 \leq x \leq \alpha_r; \\
\mu^{-1} \left( (1 + r) (1 - \mu (x)) \right), & \alpha_r \leq x \leq 1;
\end{cases}
\]

(18)

with

\[
\alpha_r \equiv \mu^{-1} \left( \frac{r}{1 + r} \right).
\]

(19)

By using (17) and (19) the varying Lyapunov exponent is readily computed:

\[
\lambda_r = \int_0^1 \rho (x) \ln |f'_r (x)| \, dx = \ln (1 + r) - \frac{r}{1 + r} \ln r.
\]

(20)

This value starts from \( \lim_{r \to 0} \lambda_r = 0 \), increases until its maximum \( \lambda_1 = \ln 2 \) and then decreases towards \( \lim_{r \to \infty} \lambda_r = 0 \). The condition for the maximum at \( x = \alpha_r \) to be smooth is again \( \lim_{x \to 1} \rho (x) = \infty \).

### 3.1. Another family of piecewise-smooth chaotic maps

For example, if one selects for all \( r > 0 \) the same constant invariant density \( \rho_r (x) = \rho (x) = 1 \) the solution of Eq. (18) is the family

\[
f_r (x) = (1 + r) \begin{cases} 
x/r, & 0 \leq x \leq \alpha_r; \\
1 - x, & \alpha_r \leq x \leq 1;
\end{cases}
\]

(21)

with \( \alpha_r = r/(1 + r) \) and the Lyapunov exponent of Eq. (20). \( f_1 \) is the full tent map \( T(x) = 1 - |1 - 2x| \).
3.2. A second family of smooth chaotic maps

If the starting point is the invariant density of the full logistic map, \( \rho_r(x) = \rho(x) = [\pi^2 x(1 - x)]^{-1/2} \), we get, for every \( r > 0 \),

\[
f_r(x) = \begin{cases} 
\sin^2 \left( \frac{1 + r}{r} \arcsin \sqrt{x} \right), & 0 \leq x \leq \alpha_r; \\
\sin^2 \left( (1 + r) \arccos \sqrt{x} \right), & \alpha_r \leq x \leq 1;
\end{cases}
\]

with \( \alpha_r = \cos^2 \frac{\pi}{2(1 + r)} \) \hspace{1cm} (23)

and the Lyapunov exponent (20). The full logistic map is recovered with \( r = 1 \):
\[ f_1(x) = 4x(1 - x). \]

It is straightforward to extend the method in this section for \( m = 3, 4, \ldots \)

4. Robust chaos with varying Lyapunov exponent

In Eq. (2) it was assumed that \( m \) is an integer larger than one. But let us substitute for it a real number \( r > 1 \):

\[
\rho \left( f_r(x) \right) |f_r'(x)| = r \rho(x), \quad (r > 1).
\]

(24)

Notice that we are using again the same invariant density \( \rho(x) \) for all values of the continuous parameter \( r \), which now replaces the constant multiplicity \( m \). If we get a solution of Eq. (24) for a given \( \rho(x) \), the latter will not satisfy the Frobenius-Perron equation for non-integer values of \( r \), but this does not prevent the solution \( f_r \) from being chaotic for all \( r > 1 \), as we will show in the following. In fact, by using an argument very similar to the one leading to (3), one can show that the Lyapunov exponent of the family satisfying (24) will be \( \lambda_r = \ln r \), for all \( r > 1 \).

The solution of Eq. (26) for \( f_r(0) = 0, f'_r(0) > 0 \) and \( 1 < r \leq 2 \) can be written as

\[
f_r(x) = \begin{cases} 
\mu^{-1} \left( r \mu(x) \right), & 0 \leq x \leq \alpha_r \equiv \mu^{-1}(1/r); \\
\mu^{-1} \left( 2 - r \mu(x) \right), & \alpha_r \leq x \leq 1,
\end{cases}
\]

with \( \mu(x) \equiv \int_0^x \rho(y) \, dy \). It is also easy to write down the solution for other initial conditions or other ranges of the parameter \( r \).
4.1. A third family of smooth chaotic maps

If we choose again the invariant density of the full logistic map, \( \rho(x) = [\pi^2 x(1-x)]^{-1/2} \), a solution of Eq. (24) for all \( r > 1 \) is

\[
f_r(x) = \sin^2 \left( r \arcsin \sqrt{x} \right).
\] (26)

For \( r = 2, 3, \ldots \) the maps \( f_r(x) \) reduce to polynomials and we recover the ‘Chebyshev hierarchy’ of Refs. [1, 2]:

\[
f_n(x) = \frac{1}{2} - \frac{(-1)^n}{2} T_n(2x - 1), \quad (n = 2, 3, \ldots),
\] (27)

where \( T_n \) are the Chebyshev polynomials of the first kind. All these maps are chaotic, with Lyapunov exponent \( \lambda_n = \ln n \) and the same invariant invariant density as the full logistic map \( f_2(x) = 4x(1-x) \). The question is what happens for non-integer values of \( r \)?

Their Lyapunov exponent is \( \lambda_r = \ln r \) and we can see in the bifurcation diagram of Fig. 3 that the attractor only fills the phase space \([0, 1]\) for \( r \geq 2 \).

But, what about the invariant measure? The point is that now all points \( x \) do not have the same number \( m \) of preimages \( y_k \), such that \( f_r(y_k) = x \), as assumed in Refs. [1, 2] and Sect. 2 and 3. In consequence, the starting invariant density will not satisfy the Frobenius-Perron equation (1) for non-integer values of \( r \). Obviously, the actual invariant measure will satisfy that equation with a value of \( m \) depending on the point \( x \). It is clear from Fig. 4, which display in solid line the graphs of two members of family (26), that the value \( m(x) \) will change in this example at point \( x = \alpha \equiv f_r(1) \) and the same will happen at all its images, so
that we can expect the invariant measure to be discontinuous at every point in the orbit of \( f_r(1) \):

\[ O_r \equiv \{ f_r^k(1) : k = 1, 2, \ldots \} . \] (28)

The task of finding the invariant measure can thus be very difficult unless the orbit \( O_r \) is simple enough (a cycle), as happens in the ‘Chebyshev hierarchy’ (where \( r = 2, 3, \ldots \) and \( O_{2n} = \{ 0, 0, \ldots \} \) and \( O_{2n-1} = \{ 1, 1, \ldots \} \), so that the invariant density is continuous for \( 0 < x < 1 \)), but also for many non-integer values of \( r \) as we shall see now.

For instance, in the left graph of Fig. 4 the parameter equals the golden ratio, \( r = \phi \equiv (1 + \sqrt{5})/2 \), and then the orbit of \( f_\phi(1) \) is a 3-cycle:

\[ O_\phi = \left\{ \alpha \equiv \sin^2 \frac{\phi \pi}{2}, 1 - \alpha, \alpha, 1 - \alpha, 1, \ldots \right\} . \] (29)

Since one can multiply a solution of Eqs. (1) and (24) with any constant, one may suspect that the actual invariant measure of the maps (26) is that of the full logistic map multiplied by a function which is constant except at the points lying on the orbit of \( f_\phi(1) \). In fact, we can find the exact expression of the invariant density for \( r = \phi \) by using the normalization condition \( \int_0^1 \rho_\phi(x) \, dx = 1 \) and trying in the Frobenius-Perron equation (1) a density in the form

\[
\rho_\phi(x) = \frac{1}{\pi \sqrt{x(1-x)}} \begin{cases} 
  a, & 0 < x < \alpha; \\
  b, & \alpha < x < 1 - \alpha; \\
  c, & 1 - \alpha < x < 1;
\end{cases}
\] (30)

with constant \( a, b \) and \( c \) and the following multiplicity:

\[
m(x) = \begin{cases} 
  1, & 0 < x < \alpha; \\
  2, & \alpha < x < 1.
\end{cases}
\] (31)
One finally finds
\[ a = 0, \quad b = \frac{c}{\phi} = \frac{1 + 3\phi}{5}. \] (32)

The result can be easily checked by means of numerical simulations [17].

Even simpler is the orbit of \( f_r(1) \) for \( r = 1 + \sqrt{2}: \)
\[ O_{1+\sqrt{2}} = \left\{ \alpha = \cos^2 \frac{\pi}{\sqrt{2}}, 1, \alpha, 1, \ldots \right\}. \] (33)

From this 2-cycle and the Frobenius-Perron equation (1) it is easy to find the invariant density:
\[ \rho_{1+\sqrt{2}} = \frac{2 + \sqrt{2}}{4\pi \sqrt{x(1-x)}} \left\{ \begin{array}{ll} \sqrt{2}, & 0 < x < \alpha; \\ 1, & \alpha < x < 1. \end{array} \right. \] (34)

Other cases with piecewise continuous invariant measure can be found in a similar way.

It should be noticed that maps (26) are a nice example of solvable robust chaos, since the solution can be explicitly written in terms of the initial condition \( x_0 \) as \( x_n = \sin^2 \left( r^n \arcsin \sqrt{x_0} \right) \).

5. Robust chaos through diffeomorphisms

Another way to construct examples of robust chaos takes advantage of the invariance of Lyapunov exponents through diffeomorphisms on the interval \([0, 1]\).

We are going to consider three different approaches.

5.1. A single diffeomorphism

If we know a family of chaotic maps \( f_r(x) \) with invariant densities \( \rho_r(x) \) and choose a diffeomorphism \( \varphi(x) \), the topologically conjugated family \( f_r = \varphi \circ f_r \circ \varphi^{-1} \) will have the same Lyapunov exponent and the following invariant density [4]:
\[ \tilde{\rho}_r(x) = \rho_r \left( \varphi^{-1}(x) \right) \left| \frac{d}{dx} \varphi^{-1}(x) \right|. \] (35)

For example if \( f_r \) is the family (21) and the diffeomorphism
\[ \varphi(x) = \sin^2 \frac{\pi x}{2}, \] (36)
the topologically conjugated family is (22).
On the other hand, if \( f_r \) is the family \((26)\) and the diffeomorphism
\[
\varphi(x) = \frac{2}{\pi} \arcsin \sqrt{x},
\]
the topologically conjugated family is
\[
\tilde{f}_r(x) = 1 - |1 - r x \mod 2|.
\]
(38)

Maps (38) are piecewise linear and two particular cases are depicted in dashed line in Fig. 4. Moreover, one recovers the full tent map for \( r = 2 \). It is obvious that since the slope of the graph of \( \tilde{f}_r(x) \) is nearly everywhere \( \pm r \) the maps are chaotic for all \( r > 1 \) and that the Lyapunov exponent is \( \lambda_r = \ln r \). This in turn proves again that the topologically conjugated family \((12)\) is chaotic and has the same Lyapunov coefficient. The invariant density is \( \tilde{\rho}_n(x) = 1 \) for \( n = 2, 3, \ldots \) and can be computed for other values of \( r \) by using \((35)\) or the method discussed in the last section. For example, the measure corresponding to \((34)\) is piecewise constant:
\[
\rho_{1+\sqrt{2}}(x) = \frac{2 + \sqrt{2}}{4} \begin{cases} 
\sqrt{2}, & 0 \leq x < \sqrt{2} - 1; \\
1, & \sqrt{2} - 1 < x \leq 1.
\end{cases}
\]
(39)

5.2. A family of diffeomorphisms

A second possibility is to choose a single chaotic map \( f(x) \) with known invariant density \( \rho(x) \) and Lyapunov exponent \( \lambda \) and a family of diffeomorphisms depending on a continuous parameter: \( \varphi_r(x) \). Then the maps \( \tilde{f}_r = \varphi_r \circ f \circ \varphi_r^{-1} \) will have the same Lyapunov exponent and the following natural density:
\[
\tilde{\rho}_r(x) = \frac{\rho(\varphi_r^{-1}(x))}{|\varphi_r'(\varphi_r^{-1}(x))|}.
\]
(40)

For instance, if the starting map is the full tent map, \( f(x) = T(x) = 1 - |1 - 2x| \), with \( \rho(x) = 1 \) and \( \lambda = \ln 2 \) and the diffeomorphism \( \varphi_r(x) = x^{1/r} \), with \( r > 0 \), the family \( \tilde{f}_r \) is precisely \((9)\) with the same Lyapunov exponent and the invariant density \((8)\). (Notice that in this example \( f = \tilde{f}_1 \).)

On the other hand, given the full tent map and the diffeomorphisms \( \varphi_r(x) = \sin^{1/r} \frac{\pi x}{2} \), with \( r > 0 \), the topologically conjugated smooth maps
\[
\tilde{f}_r(x) = \sin^{1/r} (2 \arcsin x^r) = x \left( 4 - 4x^{2r} \right)^{1/2r}
\]
(41)
have the same Lyapunov exponent and the natural densities

\[ \tilde{\rho}_r(x) = \frac{2r \cdot x^{r-1}}{\pi \sqrt{1 - x^{2r}}}. \]  

(42)

This map, which is also obtained from the full logistic map \( f(x) = 4x(1 - x) \) by means of the diffeomorphism \( \varphi_r(x) = x^{1/2r} \), is a simple example of exactly solvable robust chaos, in which everything is known, including the general solution \( x_n = \sin^{1/r} (2^n \arcsin x_0^r) \). We recover the full logistic map with \( r = 1/2 \).

5.3. Prescribed invariant density

Finally, one can choose a chaotic map \( f(x) \) with known invariant density \( \rho(x) \) and Lyapunov exponent \( \lambda \) and a prescribed family of natural densities \( \tilde{\rho}_r(x) \) depending on the parameter \( r \). Then each family of diffeomorphisms \( \varphi_r \) satisfying the differential equation

\[ \tilde{\rho}_r(\varphi_r(x)) |\varphi_r'(x)| = \rho(x) \]  

(43)

will provide a family of maps \( \tilde{f}_r = \varphi_r \circ f \circ \varphi_r^{-1} \) with the same Lyapunov exponent and the desired natural measure.

If we want \( \varphi_r(x) \) to be increasing, the solution of (43) is

\[ \varphi_r(x) = \tilde{\mu}_r^{-1} (\mu(x)), \]  

(44)

with

\[ \tilde{\mu}_r(x) \equiv \int_0^x \tilde{\rho}_r(y) \, dy, \quad \mu(x) \equiv \int_0^x \rho(y) \, dy. \]  

(45)

For example if \( f(x) \) is the full tent map and \( \tilde{\rho}_r(x) = r \cdot x^{r-1} \), the solution (44) is \( \varphi_r(x) = x^{1/r} \) for \( r > 0 \), and we recover once more the family (9), which in turn shows again that the latter is chaotic for all positive real values of \( r \) and that its invariant density is (8).

If we choose \( \varphi_r(x) \) to be decreasing, the solution of (43) is \( \varphi_r(x) = \tilde{\mu}_r^{-1} (1 - \mu(x)) \) and for \( \tilde{\rho}_r(x) = r \cdot x^{r-1} \) and \( r > 0 \) we have \( \varphi_r(x) = (1 - x)^{1/r} \), which transforms the full tent map into the family (14). Since families (14) and (9) are topologically conjugated to the full tent map, they are also conjugated to each other. In fact, the diffeomorphism conjugating them is \( \varphi_r(x) = \varphi_r^{-1}(x) = (1 - x^r)^{1/r} \). This a nice example of two topologically conjugated maps sharing not only the Lyapunov exponent but also the natural invariant measure.

This simple example shows that the maps found in previous sections are not necessarily the only solution to the corresponding problem. It was pointed out in
Refs. [1, 2] that in general the inverse problem does not have a unique solution and several examples were described with different maps sharing a given invariant density while having different Lyapunov exponents in the form \( \lambda_m = \ln m \). We see here that it is also possible to have different maps with the same Lyapunov exponent and invariant measure.

An obvious variant of the method in this section is a starting family of densities \( \rho_r(x) \) depending on the parameter \( r \) to construct a family of chaotic maps with a constant density \( \tilde{\rho}(x) \) by solving
\[
\tilde{\rho} (\varphi_r(x)) |\varphi'_r(x)| = \rho_r(x).
\] (46)

6. The inverse problem

Obviously, one can combine the methods in the previous sections to get families of maps depending on several parameters which display robust chaos. Instead, we will discuss a more general and easier method to accomplish the same.

All the maps considered before can be written in the form \( f = \mu^{-1} \circ g \circ \mu \), for some appropriate \( g \) and \( \mu \). In fact, for the particular case of smooth unimodal families with robust chaos it has been proven that all maps within the family are topologically conjugate [13]. This suggests a general way to construct a map \( f \) (smooth or not, unimodal or not) with prescribed Lyapunov exponent \( \lambda \) and natural invariant density \( \rho(x) \), such that \( \mu(x) \equiv \int_0^x \rho(y) \, dy \) is a diffeomorphism, even in the case in which \( f, \lambda \) and \( \rho \) depend on one or several parameters, which will not be written explicitly.

The starting point is a set of \( m \) values \( 0 < a_k < 1 \) such that
\[
b_m \equiv \sum_{k=1}^{m} a_i = 1.
\] (47)

If \( b_0 \equiv 0 \) and \( b_k \equiv b_{k-1} + a_k \), for \( k = 1, 2, \ldots, m \), the piecewise linear map
\[
g(x) \equiv \begin{cases} 
\frac{x - b_{2i-2}}{a_{2i-1}}, & b_{2i-2} \leq x \leq b_{2i-1}; \\
\frac{b_{2i} - x}{a_{2i}}, & b_{2i-1} \leq x \leq b_{2i};
\end{cases}
\] (with \( i = 1, 2, \ldots \)) has the invariant density \( \rho_0(x) = 1 \), because of (47), and the Lyapunov exponent
\[
0 < \lambda_0 = - \sum_{k=1}^{m} a_k \ln a_k \leq \ln m.
\] (49)
The maximum Lyapunov exponent $\lambda_0 = \ln m$ is obtained with $a_1 = a_2 = \cdots = a_m = 1/m$.

Now, if one chooses a value of $m$ large enough (so that $\lambda \leq \ln m$) and a set of values $a_k$ (depending on the parameters of the desired map) such that $\lambda = \lambda_0$, then the map $f = \mu^{-1} \circ g \circ \mu$ will have the prescribed Lyapunov exponent $\lambda$ and natural invariant density $\rho(x) = \mu'(x)$. The derivative $f'$ will be continuous at the maxima (minima) if $\lim_{x \to 1} = \infty$ ($\lim_{x \to 0} = \infty$). The map $f$ will not be unique, since many others can be constructed in the same way with higher values of $m$ and, even for the same $m$, there will be in general many (infinite) ways of choosing the values $a_k$. For instance, this method reduces to (13) for $m = 2$, $a_1 = a_2 = 1/2$ and $g(x) = T(x) = 1 - |1 - 2x|$. On the other hand, solution (18) is recovered with $m = 2$, $a_2 = 1 - a_1 = 1/(1 + r)$ and $g$ given by (21).

It is also easy to change slightly this method to construct maps with known Lyapunov exponent and piecewise natural density. Let us consider a single case. Instead of (48) the starting point will be

$$\hat{g}(x) = g(x) \begin{cases} 1, & 0 \leq x \leq b_{m-1}; \\ a_1, & b_{m-1} \leq x \leq 1, \end{cases}$$

for $m = 3, 5, \ldots$ Since the orbit of $\hat{g}(1)$ is $\mathcal{O} = \{a_1, 1, a_1, \ldots\}$ there will be a single discontinuity point in the natural invariant density, which in fact is

$$\hat{\rho}_0(x) = \frac{1}{a_1(1 + a_m)} \begin{cases} a_1 + a_m, & 0 < x < a_1; \\ a_1, & a_1 < x < 1, \end{cases}$$

while the Lyapunov exponent is

$$0 < \hat{\lambda}_0 = -\sum_{k=1}^{m} a_k \ln a_k \leq \arcsinh n, \quad n \equiv (m - 1)/2.$$  

The maximum value is reached with $a_1 = \cdots = a_{m-1} = \sqrt{a_m} = \sqrt{1 + n^2} - n$. Then for each choice of the values $a_k$, the map $\hat{f} = \mu^{-1} \circ \hat{g} \circ \mu$ will have the Lyapunov exponent $\hat{\lambda} = \hat{\lambda}_0$ and the natural invariant density

$$\hat{\rho}(x) = \frac{\rho(x)}{a_1(1 + a_m)} \begin{cases} a_1 + a_m, & 0 < x < \mu^{-1}(a_1); \\ a_1, & \mu^{-1}(a_1) < x < 1. \end{cases}$$  


For instance, taking \( m = 3 \) and \( a_1 = a_2 = \sqrt{2} - 1 \), one recovers the map (38) for \( r = \sqrt{2} + 1 \), which can be extended to a piecewise linear family with known Lyapunov exponent and invariant measure by letting \( a_2 \) run from 0 to \( 2 - \sqrt{2} \). In turn, applying to that family the diffeomorphism \( \mu(x) = \frac{x}{2} \arcsin \sqrt{x} \) one gets a family of smooth maps with known Lyapunov exponent and piecewise continuous natural invariant density, which coincides with (26) when \( a_2 = 1/r = \sqrt{2} - 1 \).

It is easy to construct other examples starting from a piecewise linear map with simple orbits of \( \tilde{g}(1) \) (or \( \tilde{g}(0) \)).

7. Final comments

We have presented some new easy methods to construct families of one-dimensional maps with robust chaos and exactly known Lyapunov exponent and natural invariant measure. As far as we know, this is the first time general methods to do that are discussed.

It should be stressed that the explicit examples with robust chaos discussed above have been selected for simplicity, but many other can be easily constructed with the ideas discussed in this work.

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References

[1] K. Sogo, J. Phys. Soc. Japan 68 (1999) 3469–3472.
[2] K. Sogo, Chaos, Solitons & Fractals 41 (2009) 1817–1822.
[3] S. Banerjee, J.A. Yorke, C. Grebogi, Phys. Rev. Lett. 80 (1998) 3049–3052.
[4] E. Ott, Chaos in Dynamical Systems, second ed., Cambridge, Cambridge, 2002, Chap. 2.
[5] S. Hayes, C. Grebogi, E. Ott, Phys. Rev. Lett. 70 (1993) 3031–3034.
[6] N. Nagaraj, M.C. Shastry, P.G. Vaidya, Eur. Phys. J. Special Topics 165 (2008) 73–83.
[7] M. Drutarovský, P. Galajda, Radioengineering 16 (2007) 120–127.

[8] See the references in Z. Elhadj, J.C. Sprott, Frontiers of Physics in China, 3 (2008) 195–204.

[9] A. Priel, I. Kanter, Europhysics Lett. 51 (2000) 230–236.

[10] A. Potapov, M.K. Ali, Phys. Lett. A 277 (2000) 310–322.

[11] M.P. Dafilis, D.T.J. Liley, P.J. Cadusch, Chaos 11 (2001) 474–478.

[12] V. Botella-Soler, J.A. Oteo, J. Ros, “Dynamics of a map with power-law tail”, arXiv:0812.4551 (2008).

[13] M. Andrecut, M.K. Ali, Europhys. Lett. 54 (2001) 300–305.

[14] M. Andrecut, M.K. Ali, Phys. Rev. E 64 (2001) 025203-1–025203-3.

[15] M.C. Shastry, N. Nagaraj, P.G. Vaidya, “The B-Exponential Map: A Generalization of the Logistic Map and its Applications in Generating Pseudo-Random Numbers”, arXiv:cs.CR/0607069 (2006).

[16] J.M. Aguirregabiria, Chaos, Solitons & Fractals 42 (2009) 2531–2539.

[17] J.M. Aguirregabiria, Dynamics Solver. Free program to simulate continuous and discrete dynamical systems available from [http://tp.lc.ehu.es/jma/ds/ds.html](http://tp.lc.ehu.es/jma/ds/ds.html).

[18] S. van Strien, “One-parameter families of smooth interval density of hyperbolicity and robust chaos”, arXiv:0912.0656v1 (2009).