Small Schwarzschild de Sitter black holes, quantum extremal surfaces and islands

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Abstract: We study 4-dimensional Schwarzschild de Sitter black holes in the regime where the black hole mass is small compared with the de Sitter scale. Then the de Sitter temperature is very low compared with that of the black hole and we study the black hole, approximating the ambient de Sitter space as a frozen classical background. We consider distant observers in the static diamond, far from the black hole but within the cosmological horizon. Using 2-dimensional tools, we find that the entanglement entropy of radiation exhibits linear growth in time, indicative of the information paradox for the black hole. Self-consistently including an appropriate island emerging at late times near the black hole horizon leads to a reasonable Page curve. There are close parallels with flat space Schwarzschild black holes in the regime we consider.

Keywords: AdS-CFT Correspondence, Gauge-Gravity Correspondence

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1 Introduction

The black hole information paradox [1] can be regarded as the tension between the apparent unbounded growth of entanglement entropy of thermal Hawking radiation [2] outside the black hole and the expectation from quantum mechanics that entanglement entropy must become small at late times if purity of the original matter state is to be recovered, i.e. it must follow the Page curve [3, 4]. See e.g. [5, 6] for discussions of various aspects of the information paradox. Recent exciting discoveries unravelled via the study of entanglement and quantum extremal surfaces [7–11] have found that including nontrivial “island” contributions does in fact do this job. Quantum extremal surfaces are extrema of the generalized gravitational entropy [12, 13] obtained from the classical area of the entangling RT/HRT surface [14–17] after incorporating the bulk entanglement entropy of matter, with explicit calculation possible in effective 2-dimensional models where 2-dim CFT techniques enable detailed analysis of the bulk entanglement entropy. The island, arising as a nontrivial solution to extremization (near the black hole horizon, and only at late times), reflects new replica wormhole saddles [10, 11] and serves to purify the early Hawking radiation thereby leading to the entanglement entropy decreasing. There is a large body of literature on various aspects of these issues, reviewed in e.g. [18–20]: see e.g. [21–30, 32–38, 40–58, 60–62, 65] for a partial list of investigations on black holes in this regard. Scrutinies of the island formulation and alternative perspectives appear in e.g. [66–68].

In this paper, we study “small” Schwarzschild de Sitter black holes, i.e. the regime where the black hole mass $m$ is small compared with the de Sitter scale $l$, but large enough
that a quasi-static approximation to the geometry is valid. This translates to saying that the de Sitter temperature is very low compared with that of the black hole. In this regime, we approximate the ambient de Sitter space as a frozen classical background and study the two-sided (eternal) black hole. We can imagine that the black hole has formed from initial matter in a pure state: strictly speaking this can only be an approximation to the bulk CFT at the thermal state at the de Sitter temperature, but it is a reasonable approximation if the de Sitter temperature is very low. We focus on one black hole coordinate patch in the Penrose diagram, figure 1 (which roughly comprises a line of alternating Schwarzschild and de Sitter patches), and consider observers in the static diamond patches far outside the black hole but within the cosmological horizons which bound the black hole patch, figure 2. Then the entanglement entropy of the radiation exhibits an unbounded linear growth in time, which is inconsistent at late times with the entropy of the black hole, and indicative of the information paradox for the black hole. Using the island rule in the extremization of the generalized entropy shows an island emerging at late times a little outside the black hole horizon semiclassically: this then shows finiteness of the entanglement entropy of radiation recovering the expectations on the Page curve. In some essential sense, our analysis (which is purely bulk, with no holography per se) closely mirrors island studies of flat space Schwarzschild black holes in the literature, with the ambient de Sitter space entering only through more complicated coordinates.

In section 2, we review 4-dim Schwarzschild de Sitter black holes as required for our purposes, and in section 3, we describe our setup for the generalized entropy via 2-dim techniques. Section 3.1 discusses the entanglement entropy in the absence of the island (with details in appendix A), while section 4 discusses the island calculation (details in appendix B; see also appendix C for early times). Finally section 5 contains a Discussion of our approximations and open questions.

2 Small Schwarzschild de Sitter black holes → 2-dim

The Schwarzschild de Sitter black hole spacetime in 3 + 1-dimensions has the metric

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2d\Omega_2^2, \quad f(r) = 1 - \frac{2m}{r} - \frac{r^2}{l^2}. \quad (2.1)$$

This is a Schwarzschild black hole in de Sitter space [69] with an “outer” cosmological (de Sitter) horizon and an “inner” Schwarzschild horizon. The surface gravity at both horizons is generically distinct: Euclidean continuations removing a conical singularity can be defined at each horizon separately but not simultaneously at both [70] (see also [71, 72]). The only (degenerate) exception is in an extremal, or Nariai, limit [73] where both periodicities of Euclidean time match: the spacetime develops a nearly $dS_2$ throat in this extremal limit [70]. More on the nearly $dS_2$ limit and the wavefunction of the universe appears in [74] (see also [75]). Related discussions with some relevance to this paper also appear in [76].

The general $d + 1$-dimensional SdS spacetime is of similar form as (2.1) but with $f(r) = 1 - \frac{2m}{l^2}r^{d-2} - \frac{r^2}{l^2}$, and will have qualitative parallels. We will focus on the 4-dim Schwarzschild de Sitter case in what follows. For $SdS_4$, the function $f(r)$ is a cubic and
the zeroes of \( f(r) \), i.e. solutions to \( f(r) = 0 \), give the locations of the horizons. We can parametrize this as
\[
 f(r) = 1 - \frac{2m}{r} - \frac{r^2}{l^2} = \frac{1}{l^2 r} (r_D - r)(r - r_S)(r + r_S + r_D),
\]
\[
 r_D r_S (r_D + r_S) = 2ml^2, \quad r_D^2 + r_D r_S + r_S^2 = l^2; \quad 0 \leq r_S \leq r_D \leq l; \quad \frac{m}{l} \leq \frac{1}{3\sqrt{3}}.
\]

We will take the roots \( r_S \) and \( r_D \) to label the Schwarzschild black hole and de Sitter (cosmological) horizons respectively. (The third zero \(- (r_D + r_S)\) does not correspond to a physical horizon.) The roots \( r_S, r_D \) are constrained as above.

The case with \( m = 0 \), or \( r_S = 0 \), \( r_D = l \), is pure de Sitter space, while the flat space Schwarzschild black hole has \( l = 0 \), or \( r_S = m \), \( r_D = 0 \). The above structure of horizons is valid for \( \frac{m}{l} < \frac{1}{3\sqrt{3}} \), beyond which there are no horizons [69]. The limit \( \frac{m}{l} = \frac{1}{3\sqrt{3}} \) corresponds to the cosmological and Schwarzschild horizon values coinciding: here we have \( r_S = r_D = r_0 = \frac{l}{\sqrt{3}} \) from (2.2). This special value leads to the extremal, or Nariai, limit where the near horizon region (between the horizons) becomes \( dS_2 \times S^2 \). Overall the range of physically interesting \( r_S, r_D \) satisfies \( 0 < r_S < r_0 < r_D \) for generic values. The fact that \( r_S < r_D \) implies that the cosmological horizon is “outside” the Schwarzschild one. The black hole interior has \( r < r_S \) with \( r \to 0 \) the singularity. The region \( r_D < r \leq \infty \) describes the future and past de Sitter universes, with \( r \to \infty \) the future boundary \( I^+ \) (or past, \( I^- \)). The maximally extended Penrose diagram figure 1 shows an infinitely repeating pattern of Schwarzschild coordinate patches or “unit-cells” containing Schwarzschild black hole horizons cloaking interior regions: these patches are bounded by cosmological horizons on the left and right, with future/past universes beyond the cosmological horizons.

The intermediate static diamond region \( D \) is the exterior of the black hole, i.e. the static patch with a timelike Killing vector where physical timelike observers can be stationary:
\[
 D: \quad r_S < r < r_D; \quad 0 < f(r) < 1.
\]

We want to consider the limit of a “small” black hole in de Sitter, i.e.
\[
 m \ll l, \quad l \to \text{large} \quad \Rightarrow \quad r_D \gg r_S.
\]

The horizon locations can then be found perturbatively to be \( r_S \approx 2m \), \( r_D \approx l - m \gg r_S \), from (2.2). In this limit (in a sense opposite to the Nariai limit where \( r_S \sim r_D \)), the black
hole is much smaller than the ambient de Sitter scale, i.e. we have a small black hole in a large accelerating universe. So we expect that the ambient cosmology can be taken as a frozen classical background while the black hole undergoes Hawking evaporation. This is corroborated by the fact that the black hole Hawking temperature is much larger than the Gibbons-Hawking temperature of the ambient de Sitter horizon: i.e. using the surface gravities $\kappa$ [71, 72] (see also [77]) and $T = \frac{\kappa}{2\pi}$ we obtain\(^1\) in the limit (2.4),

$$T_{BH} \sim \frac{1}{8\pi m}, \quad T_{dS} \sim \frac{1}{2\pi l}; \quad T_{dS} \ll T_{BH}. \quad (2.5)$$

Pushing this to the extreme leads to the flat space limit

$$r_D \sim l \to \infty : \quad \frac{r_D}{l} \to 1, \quad r_S \to 2m; \quad T_{dS} \to 0, \quad (2.6)$$

where the ambient de Sitter background acquires a vanishingly small temperature, approaching asymptotically flat space. Our entire analysis will in fact focus on these limits (2.4), (2.5), with the flat limit (2.6) as a special case to corroborate with.

Towards analysing the generalized entropy, we will require recasting the Schwarzschild de Sitter metric (2.1) in Kruskal coordinates which are regular at the black hole horizon. These are not regular in the vicinity of the de Sitter horizon (where a distinct set of Kruskal variables is more appropriate), but we will find that the black hole Kruskal variables suffices for our considerations. This is consistent with the fact the ambient de Sitter space simply serves as a frozen classical background in our regimes of interest. With this in mind, we define the black hole tortoise coordinate following the discussion in [78]:

$$r^* = \int \frac{1}{f(r)} dr = \int \frac{1}{1 - \frac{2m}{r} - \frac{r^2}{l^2}} dr = \int \frac{l^2 r}{(r_D - r)(r - r_S)(r + r_S + r_D)} dr. \quad (2.7)$$

Taking $f(r) > 0$ as pertains to the region $D$ in (2.3), this gives

$$e^{r^*} = (r_D - r)^{-\beta_D}(r - r_S)^{\beta_S}(r + r_D + r_S)^{\beta_M}, \quad (2.8)$$

with the parameters (which simplify $dr^*/dr$ to $1/f(r)$)

$$\beta_D = \frac{l^2 r_D}{(r_D - r_S)(2r_D + r_S)}, \quad \beta_S = \frac{l^2 r_S}{(r_D - r_S)(2r_S + r_D)}, \quad \beta_M = \frac{l^2 (r_D + r_S)}{(2r_D + r_S)(2r_S + r_D)}. \quad (2.9)$$

In the flat limit (2.6), $\beta_S \to r_S$ and $\beta_D \to l$.

The $SdS_4$ metric (2.1) is recast as $ds^2 = f(r)(-dt^2 + dr^*^2) + r^2 d\Omega_2^2$. In the neighborhood of the black hole horizon, the Kruskal coordinates are then defined as $U_S$, $V_S$, and the Schwarzschild de Sitter metric becomes [78]

$$\alpha_S = \frac{1}{2\beta_S}; \quad U_S V_S = -e^{2\alpha_S r^*}, \quad \frac{U_S}{V_S} = -e^{-2\alpha_S t}; \quad ds^2 = -\frac{dU_S dV_S}{W^2} + r^2 d\Omega^2,$$

$$W = \sqrt{r l \alpha_S (r_D - r)^{\frac{(1 + 2\alpha_S \beta_D)}{2}} (r - r_S)^{\frac{2\alpha_S \beta_S - 1}{2}} (r + r_S + r_D)^{\frac{2\alpha_S \beta_M - 1}{2}}}. \quad (2.10)$$

\(^1\kappa_{BH,dS} = \frac{1}{2\sqrt{(m l)^2}} \left[ \frac{2}{\sqrt{r_S r_D}} \right] \) which give $\frac{1}{2\beta_S, D} \frac{1}{\sqrt{1 - \frac{3m l}{2r}}}$, with $\beta_S, D$ in (2.9).
The value of $\alpha_S$ here ensures regularity at the black hole horizon. (noting $\beta_M + \beta_S = \beta_D$ we see that $W$ has dimensions of inverse length.) The Kruskal coordinates cover both the left and right static diamonds $D$ of the black hole patch: in the right side (containing $R_+$ in figure 1) we define

$$U_S = -e^{-\alpha_s(t-r^*)}, \quad V_S = e^{\alpha_s(t+r^*)},$$

while on the left side (with $R_-$) there is a relative minus sign, i.e. $U_S \to -U_S$, $V_S \to -V_S$.

For our purposes, it is a reasonable approximation to look at the s-wave sector of the black hole and consider the bulk matter as a 2-dim CFT: this enables the use of 2-dim CFT tools to study the entanglement entropy of bulk matter. With this in mind, we consider a reduction ansatz of the form

$$ds^2 = g_{\mu \nu} dx^\mu dx^\nu + \lambda^{-2} \phi d\Omega_2^2; \quad g_{\mu \nu} = \phi^{1/2} g_{\mu \nu}^{(2)},$$

(2.12)

to obtain 2-dimensional dilaton gravity [79, 80] (see also [81]; applications to certain families of cosmologies appears in [82]). The lengthscale $\lambda^{-1}$ has been introduced to make the dilaton dimensionless. The dilaton $\phi$ translates to the 4-dim transverse area of 2-spheres $4\pi\phi \lambda^2$. The final term represents a Weyl transformation to absorb the dilaton kinetic term giving $\frac{1}{16\pi G_2} \int d^2x \sqrt{-g} \left( \phi R - \frac{6}{\pi} \phi^{1/2} + 2\lambda^2 \phi^{-1/2} \right)$ as the 2-dim action. The 2-dim metric and dilaton are

$$ds^2 = -\lambda r \frac{dU_S dV_S}{W^2} \equiv - \frac{dU_S dV_S}{(W')^2}, \quad \phi = r^2 \lambda^2,$$

(2.13)

where $W' = \frac{W}{\sqrt{\lambda^2}}$ and $W$ is the conformal factor given in (2.10). For our discussions of entanglement entropy in these 2-dim theories, $\lambda$ will be regarded as some fixed length scale independent of the de Sitter scale $l$ so as to not interfere with the flat limit. With $G_N$ the 4-dim Newton constant, $G_2 = \frac{G_N}{V_2}$ and $V_2 = \frac{4\pi}{\lambda^2}$, the area term in the 2-dim theory is $\frac{\phi}{4G_2} = \frac{4\pi r^2}{4G_N}$ equivalent to the 4-dim one.

3 Black holes in de Sitter and entanglement entropy

In the regimes (2.4), (2.5), that we are considering, we see that there is a version of the information paradox for the black hole that is at play, with some parallels with that for the eternal AdS black hole in [21]. If the black hole forms from initial collapsing matter in a pure state, then information recovery at late times compared with the black hole timescale requires that the generalized gravitational entropy of the radiation obeys the Page curve. Note that in the current situation, this is only approximate since the ambient de Sitter space is only consistent with bulk CFT matter in a thermal state at the de Sitter temperature. However if the ambient de Sitter temperature is very low, then one might imagine that a pure state approximation is reasonable. This is the limit (2.5) we study to understand the black hole evaporation process here, which we find to be consistent. We will find, as in various previous investigations, that a nontrivial island emerges at late times a little outside the horizon as a nontrivial quantum extremal surface solution to the extremization
Figure 2. The Penrose diagram of a Schwarzschild de Sitter black hole, with focus on one black hole “patch” bounded by cosmological horizons on both sides. Depicted are the radiation regions $R_- \equiv [r_{D-}, b_-]$ and $R_+ \equiv [b_+, r_{D+}]$ and the late time island $I \equiv [a_-, a_+]$. 

of the generalized gravitational entropy. Including this and using the island rule shows the late time entropy to be bounded. Our analysis has close parallels with that of flat space Schwarzschild black holes e.g. [25, 26], our expressions showing essential agreement in the flat space limit (2.6).

In our case, we consider distant observers that are stationary, represented by timelike worldlines in the static patch $D$ region (2.3), between the Schwarzschild and de Sitter (cosmological) horizons. Towards simulating the flat space limit, we will consider the far end of the radiation region as asymptoting to the cosmological horizon. The outgoing radiation hits the cosmological horizon and as such is expected to go beyond and eventually hit the future boundary $I^+$ (future timelike infinity). However we assume that the observers propagating in the static patch $D$ collect this outgoing radiation. So we will focus on the Schwarzschild patch bounded by the cosmological horizons on both sides and ignore the regions beyond: see figure 2.

The island proposal [9] for the fine-grained entropy of the Hawking radiation is

$$S(R) = \min \left\{ \text{ext} \left[ \frac{\text{Area}(\partial I)}{4G_N} + S_{\text{matter}}(R \cup I) \right] \right\}$$

(3.1)

where $R$ is a region far from the black hole where the radiation is collected by distant observers and $I$ is a spatially disconnected island around the horizon that is entangled with $R$. The intuition behind this expression is that after about half the black hole has evaporated, the Hawking radiation going out (roughly $I$) begins to purify the radiation that was emitted early on (roughly $R$). This purification of the early radiation by the late Hawking radiation is a reflection of the entanglement between the two parts, and stems from the picture of Hawking radiation as due to production of entangled particle pairs near the horizon (which is taken as vacuum). Thus $R \cup I$ purifies over time and its entanglement thus does not grow: the slowly evaporating black hole has decreasing area, so $S(R)$ decreases in time.
The Hawking process is dominated by s-wave modes. So we calculate the bulk entropy technically using 2-dimensional techniques where we approximate the bulk matter by a 2-dim CFT propagating in the 2-dim background obtained by dimensional reduction. In what follows, we will employ these techniques to calculate the entanglement entropy in the Schwarzschild de Sitter geometry by considering the 2-dim background (2.13) obtained from the reduction of (2.10). There is no holography in our entire discussion here: we are simply calculating the entropy of radiation in the bulk spacetime using the island rule (3.1).

In the absence of the island, the radiation regions are given by the intervals $R_{\pm} \in D$, 

$$R = R_- \cup R_+; \quad R_- \equiv \{r_{D_-}, b_- \}, \quad R_+ \equiv \{b_+, r_{D_+} \},$$

(3.2)

which are the two regions marked in the figure, one in either asymptotic region of the black hole patch. Since the radiation region is far from the black hole horizon, we have $b_- - r_S \gg r_S$. The entropy of the Hawking radiation is

$$S_{\text{matter}} = S(R).$$

(3.3)

In the 2-dim CFT, the matter entanglement entropy for a single interval $A = [x, y]$ is obtained from the replica formulation [83, 84] after also incorporating in $d[x, y]$ the conformal transformation \(^2\) to a curved space [8], stemming from the $W'$-factor in the 2-dim metric (2.13),

$$S_A = \frac{c}{3} \log d[x, y] = \frac{c}{6} \log \left( \frac{-\Delta U_{x} \Delta V_{y}}{W'_{x} W'_{y}} \right).$$

(3.4)

From (2.13), we see that there is one factor of $\lambda$ that arises inside the logarithm in each expression of the last form: in addition there is the ultraviolet cutoff $\epsilon_{UV}$ as in footnote 2. This factor $\frac{\lambda}{\epsilon_{UV}^2}$ in each such term will not play much role and we will ignore this except where necessary.

The entanglement entropy for multiple disjoint intervals

$$A = [x_1, y_1] \cup [x_2, y_2] \cup \ldots \cup [x_N, y_N]$$

(3.5)

is more complicated: this arises from the multi-point correlation functions of twist operators and so it depends on not just the central charge but detailed CFT information. In the limit where the intervals are well-separated, one can expand the twist operator products and obtain [83–86]

$$S_A = \frac{c}{3} \log \frac{\Pi_{i<j} d[x_j, y_i]}{\Pi_{i<j} d[x_j, x_i] \Pi_{i<j} d[y_j, y_i]}$$

(3.6)

$^2$Any 2-dim metric is conformally flat so $ds^2 = e^{\eta_{uv}} dz^a dx^v$. The twist operator 2-point function scales under a conformal transformation as $(\sigma(x_1) \sigma(x_2))_{\epsilon f g} = e^{-\Delta_\alpha f/2}_{\epsilon_{1,1}} e^{-\Delta_\alpha f/2}_{\epsilon_{2,2}} \sigma(x_1) \sigma(x_2))_{\epsilon f g}$ with $\Delta_\alpha = \frac{c}{2} \sum_{a=1}^{2} \frac{1}{\eta_{aa}}$. Since the partition function in the presence of twist operators scales as the twist operator 2-point function, the entanglement entropy becomes $S_{\epsilon f g}^{12} = - \lim_{a \to 1^+} \partial_{a} (\sigma(x_1) \sigma(x_2))_{\epsilon f g} = S^{12} + \frac{c}{6} \sum_{\text{endpoints}} \log \epsilon_{UV} d^{f,2}$, giving for a single interval $S_{\epsilon f g}^{12} = \frac{c}{6} \log \left( \frac{\Delta^{2}_{f} \Delta^{2}_{g}}{\epsilon_{UV} d^{f,2}} \right) \to S_{\epsilon f g}^{12} = \frac{c}{6} \log \left( \frac{\Delta^{2}_{f} d^{f}}{\epsilon_{UV} d^{f,2}} \right)$. 

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For two intervals \([x_1, y_1] \cup [x_2, y_2]\), this is a limit where the cross ratio \(x\) is small, i.e.

\[
x = \frac{d[x_1, y_1]d[x_2, y_2]}{d[x_1, x_2]d[y_1, y_2]} ; \quad x \ll 1,
\]

where we use the Kruskal distances in (3.4) in constructing the cross-ratio. In 2-dim CFTs with a holographic dual, this is the situation where the two intervals \(A, B\) are well-separated and their mutual information exhibits a disentangling transition \([87]\) with \(I[A, B] = S[A] + S[B] - S[A \cup B] \rightarrow 0\), i.e. the disconnected surface \(S_{\text{dis}} = S[A] + S[B]\) has lower area than the connected surface \(S_{\text{conn}} = S[A \cup B]\). It turns out that the cross-ratio for the points in question becomes small at late times, as we will see later, justifying the use of (3.6) for our purposes.

In what follows, we first calculate the entanglement entropy for the configuration without any island, using (3.3): this shows the entropy of radiation as increasing linearly in time at late times, indicative of the information paradox. Towards resolving this we will include a possible island and calculate the entanglement entropy using the island rule (3.1): this results in a late time entropy that is time-independent.

### 3.1 Entanglement entropy: no island

In this section, we will evaluate the entanglement entropy of the radiation at late times in the absence of any island. Then we have only the radiation regions \(R_- \cup R_+\) in figure 2, given by the intervals (3.2), i.e. \(R_- \equiv [r_D, b_-]\) and \(R_+ \equiv [b_+, r_D]\). In the limit (2.4), (2.5), with the de Sitter temperature very low, we can approximate the entire system as a pure state on any slice. Then the bulk matter CFT entropy of \(R\) is the same as that of the complementary region \(R_c = [b_-, b_+]\), so we obtain

\[
S_{\text{matter}} = \frac{c}{3} \log [d(b_+, b_-)].
\]

(3.8)

We label the spacetime coordinates in the left and right asymptotic regions in the Schwarzschild patch as

\[
b_+ : \quad (t, r) = (t_b, b), \quad b_- : \quad (t, r) = (-t_b + \frac{i\beta}{2}, b); \quad \beta = \frac{2\pi}{\alpha_S}.
\]

(3.9)

This choice of \(\beta\) is simply a convenient way of incorporating the relative minus signs in the Kruskal coordinates (2.10), (2.11), in the left and right regions through \(e^{i\beta \alpha_S/2} = e^{i\pi} = -1\). With this parametrization of the left and right time coordinates, we can conveniently use the expressions in (2.11), with \(\beta\) automatically doing the left-right book-keeping.

Then we evaluate the bulk matter entropy in the Schwarzschild de Sitter geometry (2.13) using (3.4) to obtain

\[
S_{\text{matter}} = \frac{c}{6} \cdot \log \left( b \frac{(U_S(b_-) - U_S(b_+))(V_S(b_+) - V_S(b_-))}{W(b_+)W(b_-)} \right).
\]

(3.10)

The total entanglement entropy then becomes

\[
S = \frac{c}{6} \log \left[ 16\beta_S^2 (b - r_S) \left( \frac{(r_D - b)(b + r_D + r_S)}{f^2} \cosh^2 \left( \frac{t_b}{2\beta_S} \right) \right) \right].
\]

(3.11)
The details of the calculation are shown in appendix A. The late time approximation is \( t_b \gg b > r_S, \frac{b}{r_S} \gg 1 \): the above result then approximates as

\[
S \approx \text{const} + \frac{c}{6} \frac{t_b}{\beta_S}.
\]  

(3.12)

Now in the flat space limit (2.6), the total entropy at late times becomes \( S \approx \frac{c}{6} \frac{t_b}{r_S} \), in agreement with [25, 26]. This linear growth in time means that the entropy of the radiation will eventually be infinitely larger than the Bekenstein-Hawking entropy of the black hole.

4 Late time entanglement entropy with island

In this section we will evaluate the entropy of radiation in the presence of an island and show that it saves the entropy bound, recovering the Page curve for the black hole in Schwarzschild de Sitter space at late times \( t_a, t_b \gg r_S \). The island is the region marked \( I \) in figure 2: the intervals in question are

\[
R_- \equiv [r_{D-}, b_-], \quad R_+ \equiv [b_+, r_{D+}], \quad I \equiv [a-, a_+].
\]  

(4.1)

Since we are considering the limit (2.4), (2.5), with the black hole evaporation well-separated from de Sitter physics, we expect the island in question to emerge near the black hole horizon. Then

\[
b - r_S \gg a - r_S \sim 0; \quad b \gg r_S,
\]  

(4.2)

and the last inequality reflects the fact that we are considering distant observers far from the black hole. Since the ambient de Sitter temperature is very low in our considerations as mentioned earlier, we approximate the bulk matter to be in a pure state so the entanglement entropy of \( A \equiv R_- \cup I \cup R_+ \) is equal to that of the complementary intervals \( A_c \equiv [b_-, a_-] \cup [a_+, b_+] \) (see (B.21), for details on the intervals \( A \)). The assumptions (4.2) imply that the intervals are well-separated: then we can express the entanglement entropy for \( A_c \) using (3.6) as

\[
S_{\text{matter}} = \frac{c}{3} \log \frac{d[a_+, a_-] d[b_+, b_-] d[a_+, b_+] d[a_-, b_-]}{d[a_+, b_-] d[a_-, b_+]}.
\]  

(4.3)

For the intervals \([b_-, a_-] \cup [a_+, b_+]\), as the cross-ratio \( x \) in (3.7) becomes small, we have \( 1 - x = \frac{d[a_+, a_-] d[b_+, b_-]}{d[a_+, b_-] d[a_-, b_+]} \to 1 \). Adding the area term \( \frac{\phi}{4G_2} = \frac{4\pi a^2}{4G_N} \) for both left and right sides, the total generalized entropy becomes \( S_{\text{total}} \sim 2 \left( \frac{\pi a^2}{G_N} + \frac{c}{3} \log d[a_+, b_+] \right) \) since the left and right sides give essentially equal contributions. In detail, using the Kruskal
coordinates (2.10), (2.11), (A.1), we evaluate the total generalized entropy (4.3) obtaining

\[
S_{\text{total}} = \frac{2\pi a^2}{G_N} + c \log \left[ \frac{16r_S^4(b-r_S)^2}{(r_D-r_s)^4} \right] \frac{2(\alpha_s + \alpha_s^M)}{r_D + r_s} \frac{(a-r_s)(b-r_s)}{l} \left( \frac{a-r_D}{l} \right) \left( \frac{b-r_D}{l} \right) \cosh \left( \frac{t_a}{2\beta_s} \right) \cosh \left( \frac{t_b}{2\beta_s} \right).
\]

\[
+ \frac{c}{3} \log \left[ 1 - 2 \frac{a-r_s}{b-r_s} \frac{\alpha_s}{\alpha_s^M} C(a) \cosh \left( \alpha_s(t_a - t_b) \right) \right]
\]

\[
- \frac{c}{3} \log \left[ 1 + 2 \frac{a-r_s}{b-r_s} \frac{\alpha_s}{\alpha_s^M} C(a) \cosh \left( \alpha_s(t_a + t_b) \right) \right],
\]

(4.4)

with \(C(a)\) defined as

\[
C(a) = \frac{(r_D - b)^{\alpha_s^{M}}(a + r_s + r_D)^{\alpha_s}}{(r_D - a)^{\alpha_s^{M}}(b + r_s + r_D)^{\alpha_s}}.
\]

(4.5)

See appendix B for details of this calculation. At late times, \(t_a, t_b \gg r_s\), we approximate \(\cosh(\alpha_s(t_a + t_b)) \sim 2 \cosh(\alpha_s t_a) \cosh(\alpha_s t_b)\) which is large. Then taking into account the expectation \(a - r_s \sim 0\) and simplifying, we obtain

\[
S_{\text{total}} \approx \frac{2\pi a^2}{G_N} + \frac{c}{6} \log \left[ \frac{16r_S^4(b-r_S)^2}{(r_D-r_s)^4} \right] \frac{(r_D-a)(r_D-b)(a+r_s+r_D)(b+r_s+r_D)}{l^4 (C(a))^2}
\]

\[
- \frac{2c}{3} \sqrt{\frac{a-r_s}{b-r_s}} C(a) \cosh \left( \alpha_s(t_a-t_b) \right).
\]

(4.6)

In the flat space limit (2.6), we take \(r_D\) large and expand using (2.9) to approximate (4.5) as

\[
C(a) \sim \left( 1 - \frac{b-a}{r_D} \right)^{\alpha_s} \left( 1 - \frac{b-a}{r_D+r_s} \right)^{\alpha_s^M} \sim 1 - \frac{b-a}{2r_s},
\]

(4.7)

which can further approximated as \(e^{-\frac{b-a}{2r_s}}\) in the regime (4.2) to give the total entropy at late times as

\[
S_{\text{total}} \approx \frac{2\pi a^2}{G_N} + \frac{c}{6} \log \left[ \frac{16r_S^4(b-r_S)^2 e^{\frac{b-a}{r_s}}}{(r_D-r_s)^4} \right] - \frac{2c}{3} \sqrt{\frac{a-r_s}{b-r_s}} e^{-\frac{b-a}{2r_s}} \cosh \left( \alpha_s(t_a-t_b) \right).
\]

(4.8)

This is in agreement with e.g. [26] up to a factor of \(ab\) inside the logarithm, stemming from the fact that we are using the strict 2-dim bulk metric (2.13) after reduction, with the additional conformal factor in \(W'\) relative to \(W\) in (2.10). This detailed difference (which also arises in other such expressions) does not affect the qualitative behaviour of the generalized entropy in our regime since the s-wave sector is expected to be dominant in the Hawking process.
Now, extremizing (4.6) with respect to the location of the island boundary $a$ gives
\[
\frac{\partial S_{\text{total}}}{\partial a} = 0 \quad \Rightarrow 
\]
\[
\frac{4\pi a}{G_N} + \frac{c}{6} \left[ \frac{(1 - 2\alpha S\beta_M)(r_D - a) - (1 + 2\alpha S\beta_D)(a + r_S + r_D)}{(a + r_S + r_D)(r_D - a)} \right] 
- \frac{2c}{3} \cosh (\alpha_S(t_a - t_b)) \cdot \sqrt{\frac{a - r_S}{b - r_S}} C(a) \left[ \frac{\alpha_S\beta_M}{a + r_S + r_D} + \frac{\alpha_S\beta_D}{r_D - a} + \frac{1/2}{a - r_S} \right] = 0.
\]

Here, since $r_D$ is large in our considerations, the second term scales as $O(\frac{1}{r_D^3})$ and can thus be ignored: further the $\frac{1}{a + r_S + r_D}$ and $\frac{1}{r_D - a}$ also are suppressed relative to $\frac{1}{a - r_S}$. With these approximations (4.9) becomes
\[
a \simeq \frac{1}{\sqrt{a - r_S}} \frac{G_N c}{12\pi} \frac{1}{\sqrt{b - r_S}} C(a) \cosh (\alpha_S(t_a - t_b)). \quad (4.10)
\]

Now we note that we are in the semiclassical regime where
\[
0 \ll c \ll \frac{1}{G_N}, \quad (4.11)
\]
so that the classical area term in the generalized entropy is dominant but the bulk matter makes nontrivial subleading contributions (which are not so large as to cause significant backreaction on the classical geometry).

We are looking for an island with boundary $a \sim r_S$ near the black hole horizon: this corroborates with the fact that since the entire right hand side is $O(G_N c)$, in the classical limit $c = 0$ we have $a = r_S$. Thus we can solve the above expression in perturbation theory setting $a \sim r_S$ to find the first order correction in $G_N c \ll 1$: then schematically we have
\[
a - r_S \simeq \frac{K}{r_S^2} \frac{1}{b - r_S}, \quad K = \frac{G_N c}{12\pi} \cosh (\alpha_S(t_a - t_b)) C(r_S), \quad (4.12)
\]
and we finally obtain (with $C(r_S)$ from (4.5) with $a = r_S$)
\[
a \simeq r_S + \frac{(G_N c)^2}{144\pi^2 r_S^2(b - r_S)} C(r_S)^2 \cosh^2 (\alpha_S(t_a - t_b)). \quad (4.13)
\]

Using (4.7) and the comments there, we obtain in the flat space limit (2.6)
\[
a \simeq r_S + \frac{(G_N c)^2}{144\pi^2 r_S^2(b - r_S)} e^{\frac{r_S - b}{r_S}} \cosh^2 (\alpha_S(t_a - t_b)), \quad (4.14)
\]
in agreement with [26]. With the value of $a$ in (4.13) the total on-shell entanglement entropy in (4.6) becomes
\[
S_{\text{o.s.}} = \frac{2\pi r_S^2}{G_N} + \frac{c}{6} \log \left[ 16\beta_4^4 \frac{(b - r_S)^2}{l^4} \frac{(b + r_S + r_D)(1 + 2\alpha_S\beta_M)}{(2r_S + r_D)^{2\alpha_S\beta_M - 1}} \frac{(r_D - r_S)^{1 + 2\alpha_S\beta_D}}{(r_D - b)^{2\alpha_S\beta_D - 1}} \right] 
- \frac{c^2 G_N}{36\pi r_S(b - r_S)} C(r_S)^2 \cosh^2 (\alpha_S(t_a - t_b)). \quad (4.15)
\]
Now varying and extremizing the above expression with respect to $t_a$, we obtain $t_a = t_b$. Using this in (4.15) gives the entanglement entropy (keeping only leading terms) to be

$$S_{o.s.} = \frac{2\pi r_S^2}{G_N} + \frac{c}{6} \log \left[ \frac{16\beta_s^4 (b - r_S)^2}{l^4} \frac{(b + r_S + r_D)^{(1+2\alpha S\beta M)}}{(2r_S + r_D)^{2\alpha S\beta M-1}} \frac{(r_D - r_S)^{1+2\alpha S\beta D}}{(r_D - b)^{2\alpha S\beta D-1}} \right],$$

(4.16)

which is time-independent, stemming from the presence of the island. The first term is twice the Bekenstein-Hawking entropy of the black hole and the second term, arising from the bulk entropy of the radiation region purified by the island, is a constant term not growing in time. This recovers the expectations on the Page curve for the evaporating small black hole in de Sitter in the limits we are considering (2.4), (2.5), (2.6). In our discussion which is entirely gravitational, it is natural to take the Planck length as the natural UV scale and set $\lambda^{-1} \sim \epsilon_{UV} \sim l_P$; then putting back the $\frac{c}{\epsilon_{UV}}$ factors gives $S_{o.s.} \sim A_s + \frac{c}{3} \log A_S + \frac{c}{3} \log b l_P$.

Comparison of the late time entanglement entropy without the island (3.12) and that with the island (4.16) enables an estimate of the Page time when the island configuration emerges as the preferred quantum extremal surface with lower area. Making a coarse comparison at the Page time $t_P$,

$$\frac{c}{6} t_P \sim 2 S_{BH} \quad \Rightarrow \quad t_P \sim \frac{12\pi r_S^2 \beta S}{G_N c}.$$

(4.17)

Beyond this time $t_P$, the no-island configuration (3.12) has bigger area and we discard it in favour of the island one (4.16) (which is a new saddle stemming from replica wormholes) which does not grow in time. Of course all our analysis is carried out in a quasi-static approximation with $r_S$ fixed, since $r_S$ is decreasing very slowly in time as the black hole evaporates.

It is interesting to note that no such island configuration near the black hole horizon emerges at early times, $t_a, t_b \ll r_S$, when not much Hawking radiation has gone out: see (C.2) in appendix C, where we consider a coarse approximation with $t_a, t_b \sim 0$. However we might imagine that there arises a “vanishing extremal surface” with the island boundary $a$ far inside the black hole horizon so $a \ll r_S$. Setting up the generalized entropy using Kruskal coordinates in the black hole interior and extremizing in fact reveals such an island in (C.10), with a corresponding generalized entropy that is not significant, vindicating approximate purity in the regimes we are considering.

We have regarded the ambient de Sitter space as a frozen classical background, at very low temperature, with regard to the black hole Hawking process, and found consistency in our island studies and the 2-dim CFT calculations approximating the bulk matter to be in a pure state. The calculation in (B.21) of the entanglement entropy of the complementary intervals alongwith the regularization (B.24) vindicates the approximate purity of the state in this regime. If we keep the de Sitter temperature as nonzero, it would appear that e.g. (B.23) would give rise to a growing entanglement in time; see [88] for some related comments.
5 Discussion

We have studied 4-dim “small” Schwarzschild de Sitter black holes (2.1) with mass \( m \) and de Sitter scale \( l \) in the limit \( m \ll l \) where the de Sitter temperature is very low compared with that of the black hole (2.4), (2.5). In this regime which has qualitative parallels with a flat space limit (2.6), the black hole evaporation process is well-separated from any physics of de Sitter space which can be regarded as a frozen background. Strictly speaking the pure state input can only be approximately true in de Sitter where it is consistent to have bulk matter in a thermal state at the de Sitter temperature. However in the limit of very low de Sitter temperature, the ambient de Sitter space is behaving approximately like a zero temperature bath. In this regime then, we recover the expectations on the Page curve for the Hawking radiation in the black hole evaporation process, incorporating appropriate island contributions, as we have seen. We are simply regarding this as a gravitational system of a black hole in de Sitter space, with no explicit recourse to holography or string theory (although one might generically expect gravity to be intrinsically holographic): this is consistent with the island rule via replica wormholes which does not rely on holography. It is unclear if we can shed light on questions about microstates here: however perhaps embedding into some AdS (via a \( dS \) bubble with a black hole) may be a way to approach this in principle.

We focussed on one Kruskal black hole patch in the Penrose diagram figure 1 of eternal Schwarzschild de Sitter (which in the maximal extension comprises a line of alternating black hole and cosmological patches). This single patch (figure 2) is in some sense equivalent to the Penrose diagram of the flat space Schwarzschild black hole embedded in de Sitter, with the cosmological horizons on both sides serving as asymptotic boundaries (akin to null infinity in flat space). Technically we dimensionally reduce the \( SdS_4 \) spacetime to 2-dimensions and use 2-dim CFT techniques for calculating the entanglement entropy of bulk matter approximated as a 2-dim CFT. We saw in section 3.1 that in the absence of the island the bulk entropy for the radiation region exhibits unbounded linear growth, inconsistent at late times when it exceeds the entropy of the black hole. In section 4 including an appropriate island as in (4.1), upon extremizing the generalized entropy incorporating the island rule, we find an island a little outside the horizon (4.13) semiclassically: the late time entropy (4.6) on-shell becomes (4.15) and is twice the Bekenstein-Hawking entropy of the black hole, plus finite bulk corrections, in the quasi-static approximation. In the flat space limit (2.6), our expressions are in essential agreement with those in [26] (as discussed around (4.8)). This suggests that contributions due to further islands in the other Kruskal patches in our regime are suppressed (if they exist), perhaps consistent with gravity effectively being very weak far from the black hole.

We have restricted to observers propagating within the static patch, with \( f(r) < 1 \), as appropriate for the physics pertaining to the Hawking evaporation of the black hole. Let us now imagine considering observers near the future boundary \( I^+ \) of the de Sitter patch, beyond the cosmological horizon. Various studies on quantum extremal surfaces and cosmologies appear in e.g. [89–107]. In the present context, one can study the generalized entropy for such \( I^+ \) observers as well (see [76] for some discussions on classical RT/HRT
surfaces at $I^+$ in $S_dS$, and \cite{108} and earlier work in ($dS$). This reveals quantum extremal surfaces that are timelike-separated from observers at $I^+$: there are parallels with studies in the Poincare patch of de Sitter \cite{90, 106}. In $S_dS$ however, one might imagine mapping the radiation region $R_+$ to a corresponding interval at the future boundary e.g. by sending out light rays from $R_+$ to $I^+$ (see figure 2). This suggests that intervals at $I^+$ should also be able to access information about black hole evaporation. In particular it might seem possible to find nontrivial island contributions to analyse the generalized entropy for observers at $I^+$ to access black hole physics in de Sitter. It would be interesting to explore these questions further.

More broadly, in our entire analysis, de Sitter space plays very little role, although the black hole Kruskal coordinates we employed do encode detailed aspects of the de Sitter space. Strictly speaking an intermediate regime where the black hole temperature is comparable to the de Sitter temperature will require a more detailed analysis of bulk CFT matter in a mixed state corresponding to the thermal state at the de Sitter temperature, and would be interesting to study as a nontrivial nonequilibrium situation. The regions in Schwarzschild de Sitter parameter space we have explored are very far from the Nariai (extremal) limit where a $dS_2$ throat emerges: see e.g. \cite{109–112} for some recent investigations on the latter. Our island solution (4.13) emerges as a self-consistent solution near the black hole horizon in the regime of very low de Sitter temperature: it would seem that the extremization equation (4.9) has other solutions as well, pertaining to other regimes of more cosmological relevance, worth exploring. These might dovetail with various studies on quantum extremal surfaces and cosmologies e.g. \cite{89–107}, and perhaps broader issues with de Sitter space e.g. \cite{113}.

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A Details: entropy in the no-island case

This section contains some details on the calculations of entanglement entropy in the absence of the island in section 3.1. Using (2.8), (2.10), (2.11), the black hole patch Kruskal coordinates are:

\[ U_S = -e^{-\alpha_S t}(r_D - r)^{-\alpha_S \beta_D}(r - r_S)^{\alpha_S \beta_S}(r + r_D + r_S)^{\alpha_S \beta_M}, \]
\[ V_S = e^{\alpha_S t}(r_D - r)^{-\alpha_S \beta_D}(r - r_S)^{\alpha_S \beta_S}(r + r_D + r_S)^{\alpha_S \beta_M}. \]

Calculating each part of $S_{\text{matter}}$ in equation (3.10) separately gives

\[ U_S(b_--) - U_S(b_+) = (r_D - b)^{-\alpha_S \beta_D}(b - r_S)^{\alpha_S \beta_S}(b + r_D + r_S)^{\alpha_S \beta_M}[e^{-\alpha_S t_b} - e^{\alpha_S (t_b - \frac{i\beta}{2})}], \]

(A.3)
\[ V_S(b_+)-V_S(b_-) = (r_D-b)^{-\alpha_S \beta_D} (b-r_S)^{\alpha_S \beta_S} (b+r_D+r_S)^{\alpha_S \beta_M} \left[ e^{\alpha_S t_b} - e^{-\alpha_S \left( t_b - \frac{i \pi}{2} \right)} \right], \] (A.4)

\[ W(b_+) = W(b_-) = \sqrt{b} \alpha_S (r_D-b)^{-1+2 \alpha_S \beta_D} (b-r_S)^{2 \alpha_S \beta_S} (b+r_D)^{2 \alpha_S \beta_M} \left[ e^{\alpha_S t_b} - e^{-\alpha_S \left( t_b - \frac{i \pi}{2} \right)} \right], \] (A.5)

\[ W(b_+) W(b_-) = b l^2 \alpha_S^2 (r_D-b)^{-1+2 \alpha_S \beta_D} (b+r_S+r_D)^{2 \alpha_S \beta_M-1} (b-r_S)^{2 \alpha_S \beta_S-1}. \] (A.6)

Plugging all these expressions together in (3.10) gives

\[ S_{\text{matter}} = \]

\[ = c \frac{6}{\alpha_S} \log \left[ b (r_D-b)^{-2 \alpha_S \beta_D} (b-r_S)^{2 \alpha_S \beta_S} (b+r_D+r_S)^{2 \alpha_S \beta_M} \left( e^{\alpha_S t_b} - e^{-\alpha_S \left( t_b - \frac{i \pi \beta_S}{2} \right)} \right) \right], \]

\[ \left( e^{\alpha_S t_b} - e^{-\alpha_S \left( t_b - \frac{i \pi \beta_S}{2} \right)} \right) \frac{1}{b \alpha_S^2} (r_D-b)^{1+2 \alpha_S \beta_D} (b-r_S)^{-2 \alpha_S \beta_S} (b+r_S+r_D)^{-2 \alpha_S \beta_M} \] \[ = c \frac{6}{\alpha_S} \log \left[ (r_D-b) (b+r_S+r_D) \frac{1}{b \alpha_S^2} \left( 2 - \left( e^{\alpha_S t_b - \frac{i \pi \beta_S}{2}} + e^{-\alpha_S \left( t_b - \frac{i \pi \beta_S}{2} \right)} \right) \right) \right], \] (A.7)

using \( \beta = \frac{2 \pi}{\alpha_S} \) from (3.9). From (2.10) we have \( \alpha_S = \frac{1}{2 \beta_S} \) so this becomes

\[ S = c \frac{6}{\alpha_S} \log \left[ (b-r_S) 4 \beta_S^2 \left( \frac{r_D}{l} - \frac{b}{l} \right) \left( \frac{b}{l} + \frac{r_S}{l} + \frac{r_D}{l} \right) \right] \cos \left( \frac{2 \pi t_b}{2 \beta_S} \right). \] (A.8)

Thus finally, we obtain (3.11).

**B Details: late-time entropy with island**

Here we give details on section 4. We are looking to calculate (4.3), i.e.

\[ S_{\text{matter}} = c \frac{6}{\alpha_S} \log \left[ \frac{d(a_+, a_-) d(b_+, b_-) d(a_+, b_+) d(a_-, b_-)}{d(a_+, b_-) d(a_-, b_+)} \right]. \] (B.1)

Now calculating each part in \( S_{\text{matter}} \) separately,

\[ \log[d(a_+, a_-)] = \frac{1}{2} \log \left[ \frac{(U_S(a_-) - U_S(a_+)) (V_S(a_+) - V_S(a_-))}{W'(a_+) W'(a_-)} \right]. \] (B.2)

with \( W' \) as in (2.13). Then we have

\[ U_S(a_-) - U_S(a_+) = (r_D-a)^{-\alpha_S \beta_D} (a-r_S)^{\alpha_S} (a+r_S+r_D)^{\alpha_S \beta_M} \left[ e^{\alpha_S t_a} - e^{-\alpha_S \left( t_a - \frac{i \pi \beta_S}{2} \right)} \right], \] (B.3)

\[ V_S(a_+) - V_S(a_-) = (r_D-a)^{-\alpha_S \beta_D} (a-r_S)^{\alpha_S} (a+r_S+r_D)^{\alpha_S \beta_M} \left[ e^{\alpha_S t_a} - e^{-\alpha_S \left( t_a - \frac{i \pi \beta_S}{2} \right)} \right], \] (B.4)

\[ W'(a_+) W'(a_-) = l^2 \alpha_S^2 (r_D-a)^{-1+2 \alpha_S \beta_D} (a+r_S+r_D)^{2 \alpha_S \beta_M-1} (a-r_S)^{2 \alpha_S \beta_S-1}. \] (B.5)
Putting all these expressions together in (B.2) gives
\[
\log[d(a_+,a_-)] = \frac{1}{2} \log \left[ \frac{1}{[r\alpha_S^2] (a-r_D)(a-r_S)(a+r_S+r_D)} \left( e^{(2\alpha_S t_a - \frac{i\alpha_S^2 \beta}{4})} + e^{- (2\alpha_S t_a - \frac{i\alpha_S^2 \beta}{4}) - 2} \right) \right]
\]

Similarly we obtain
\[
\log[d(b_+,b_-)] = \frac{1}{2} \log \left[ \frac{1}{[b\alpha_S^2] (b-r_D)(b-r_S)(b+r_S+r_D)} \left( e^{(2\alpha_S t_a - \frac{i\alpha_S^2 \beta}{4})} + e^{- (2\alpha_S t_a - \frac{i\alpha_S^2 \beta}{4}) - 2} \right) \right]
\]

Now, putting (B.6) and (B.7) together gives, using \(\beta = \frac{2\pi}{\alpha_S}\),
\[
\frac{c}{3} \log[d(a_+,a_-)d(b_+,b_-)] = \frac{c}{6} \log \left[ \frac{2^{8\alpha_S^2}}{(r_D-r_a)^4 (a-r_S)^4 (b-r_S)(b-r_D)^4 (r_D-b)^4)} \left( \frac{r_D-a}{l} \right)^{(r_D-b)} \right.
\]
\[
\left. \left( \frac{a+r_S+r_D}{l} \right)^{(b+r_S+r_D)} \cosh^2 \frac{t_a}{2\beta_S} \cosh^2 \frac{t_b}{2\beta_S} \right] \]

We next calculate other relevant contributions:
\[
d(a_+,b_+) = \frac{1}{W'(a_+)W'(b_+)} \left[ (U_S(b_+) - U_S(a_+))(V_S(a_+) - V_S(b_+)) \right]^{1/2}
\]
\[
= \frac{1}{W'(a_+)W'(b_+)} \left[ 2e^{\alpha_S(r^*(a)+r^*(b))} \right]
\]
\[
\left( \cosh \left( \alpha_S(r^*(a) - r^*(b)) \right) - \cosh \left( \alpha_S(t_a - t_b) \right) \right)^{1/2}
\]

\[
d(a_-,b_-) = \frac{1}{W'(a_-)W'(b_-)} \left[ (U_S(b_-) - U_S(a_-))(V_S(a_-) - V_S(b_-)) \right]^{1/2}
\]
\[
= \frac{1}{W'(a_-)W'(b_-)} \left[ 2e^{\alpha_S(r^*(a)+r^*(b))} \right]
\]
\[
\left( \cosh \left( \alpha_S(r^*(a) - r^*(b)) \right) - \cosh \left( \alpha_S(t_a - t_b) \right) \right)^{1/2}
\]

Now, putting (B.9) and (B.10) together
\[
d(a_+,b_+)d(a_-,b_-) = \frac{1}{W'(a_+)W'(b_+)W'(a_-)W'(b_-)} \left[ 2e^{\alpha_S(r^*(a)+r^*(b))} \right]
\]
\[
\left( \cosh \left( \alpha_S(r^*(a) - r^*(b)) \right) - \cosh \left( \alpha_S(t_a - t_b) \right) \right)^{1/2}
\]

\[
= 16 \quad \text{(B.11)}
\]
Similarly
\[
\begin{align*}
    d(a_+, b_-) &= \frac{1}{W'(a_+)W'(b_-)} \left[ (U_S(b_-) - U_S(a_+))(V_S(a_+) - V_S(b_-)) \right]^{\frac{1}{2}} \\
    &= \frac{1}{W'(a_+)W'(b_-)} \left[ 2e^{\alpha S(r^*(a) + r^*(b))}. \right] \\
    &\left( \cosh (\alpha S(r^*(a) - r^*(b))) - \cosh \left( \alpha S \left( t_a + t_b - \frac{i\beta}{2} \right) \right) \right)^{\frac{1}{2}} \quad (B.12)
\end{align*}
\]

Now, putting (B.12) and (B.13) together
\[
\begin{align*}
    d(a_+, b_-)d(a_-, b_+) &= \frac{1}{W'(a_+)W'(b_-)W'(a_-)W'(b_+)} \left[ 2e^{\alpha S(r^*(a) + r^*(b))}. \right] \\
    &\left( \cosh (\alpha S(r^*(a) - r^*(b))) - \cosh \left( \alpha S \left( t_a + t_b - \frac{i\beta}{2} \right) \right) \right)^{\frac{1}{2}} \quad (B.14)
\end{align*}
\]

Putting (B.11) and (B.14) together we get
\[
\begin{align*}
    \frac{c}{3} \log d(a_+, b_+)d(a_-, b_-) &= \frac{c}{3} \log \left[ \frac{\cosh (\alpha S(r^*(a) - r^*(b))) - \cosh (\alpha S(t_a - t_b))}{\cosh (\alpha S(r^*(a) - r^*(b))) - \cosh (\alpha S(t_a + t_b - \frac{i\beta}{2}))} \right] \quad (B.15)
\end{align*}
\]

Here
\[
\begin{align*}
    \cosh (\alpha S(t_a + t_b - \frac{i\beta}{2})) &= - \cosh (\alpha S(t_a + t_b)) \quad (B.16)
\end{align*}
\]

and
\[
\begin{align*}
    \cosh (\alpha S(r^*(a) - r^*(b))) &= \frac{1}{2} \left[ (a - r_S)\alpha\beta \gamma S \right] \left[ (a + r_D + r_S)\alpha\beta M (r_D - b)\alpha\beta D \right] \\
    &\left( r_D - a \right)\alpha\beta D \left( b + r_D + r_S \right)\alpha\beta M \left( b - r_S \right)\alpha\beta S \\
    &= \frac{1}{2} \left( b - r_S \right)\alpha\beta S \left( a + r_D + r_S \right)\alpha\beta M (r_D - b)\alpha\beta D \left( a - r_S \right)\alpha\beta S \gamma C(a) \quad (B.17)
\end{align*}
\]
using the approximations (4.2), and (4.5). Thus we obtain
\[
\frac{c}{3} \log \frac{d(a_+, b_+)d(a_-, b_-)}{d(a_+, b_-)d(a_-, b_+)} = \frac{c}{3} \log \left[ 1 - 2 \frac{(a - r_S)^{\alpha_S \beta_S}}{(b - r_S)^{\alpha_S \beta_S}} C(a) \cosh \left( \alpha_S (t_a - t_b) \right) \right] \\
- \frac{c}{3} \log \left[ 1 + 2 \frac{(a - r_S)^{\alpha_S \beta_S}}{(b - r_S)^{\alpha_S \beta_S}} C(a) \cosh \left( \alpha_S (t_a + t_b) \right) \right].
\]  
(B.18)

The total bulk matter entanglement entropy thus is (B.8) plus (B.18): along with the area term this gives (4.4). At late times, i.e. \( t_a, t_b \gg r_S \) the total entanglement entropy \( S_{\text{total}} \), after adding the area term, becomes
\[
S_{\text{total}} \sim \frac{2\pi a^2}{G_N} + \frac{2c}{6} \log \left[ \frac{24r_2^2}{l} \frac{(r_D - r_S)^2}{l} \frac{(b - r_S)^2}{l} \sqrt{(a - r_S)(b - r_S)} \right] \notag \\
\frac{\sqrt{(r_D - a)} (r_D - b) (a + r_S + r_D) (b + r_S + r_D)}{l} \cosh \frac{t_a}{2\beta_S} \cosh \frac{t_b}{2\beta_S} \right] \notag \\
+ \frac{c}{3} \log \left[ 1 - 2 \frac{(a - r_S)^{\alpha_S \beta_S}}{(b - r_S)^{\alpha_S \beta_S}} C(a) \cosh \left( \alpha_S (t_a - t_b) \right) \right] \\
- \frac{c}{3} \log \left[ 4 \frac{(a - r_S)^{\alpha_S \beta_S}}{(b - r_S)^{\alpha_S \beta_S}} C(a) \cosh \frac{t_a}{2\beta_S} \cosh \frac{t_b}{2\beta_S} \right].
\]  
(B.19)

which upon simplifying (taking \( a - r_S \sim 0 \) so \( \log(1 - y) \sim -y \)) gives (4.6).

**Entanglement entropy of the intervals \( R_- \cup I \cup R_+ \):** It is instructive to compare the above late time calculation of the entanglement entropy of the intervals \([b_, a_-] \cup [a_+, b_+]\) with that of the original three intervals \( R_- \cup I \cup R_+ \equiv [r_{D_-}, b_-] \cup [a_-, a_+] \cup [b_+, r_{D_+}]\) which are complementary intervals in the black hole Kruskal patch of \( SdS \). Here, \( r_{D_-} \) and \( r_{D_+} \) are the boundaries of the entanglement wedge of the Hawking radiation in the left and right universes respectively. We will take these two points \( r_{D_-} \) and \( r_{D_+} \) to be very close to the corresponding cosmological (de Sitter) horizons which might be approximated as the effective boundaries of the black hole Kruskal patch in the flat space like limit (2.6) of de Sitter that we are considering here: see figure 2. Thus we define the spacetime coordinates of these two points as
\[
r_{D_+} : \quad (t, r) = (t_D, r_D - \delta) ; \quad r_{D_-} : \quad (t, r) = (-t_D + \frac{i\beta}{2}, r_D - \delta). 
\]  
(B.20)

Under the assumptions (4.2), using (3.6), we approximate the entanglement entropy for \( R_- \cup I \cup R_+ \) as
\[
S_{\text{matter}} = \frac{c}{3} \log \left[ d(a_+, a_-) d(b_+, b_-) \right] + \frac{c}{3} \log \left[ \frac{d(a_+, b_+) d(a_-, b_-)}{d(a_+, b_-) d(a_-, b_+)} \right] \\
+ \frac{c}{3} \log \left[ \frac{d(a_+, r_{D_-}) d(a_-, r_{D_-})}{d(a_-, r_{D_-}) d(a_+, r_{D_-})} \right] + \frac{c}{3} \log \left[ \frac{d(b_+, r_{D_+}) d(b_-, r_{D_-})}{d(b_+, r_{D_-}) d(b_-, r_{D_+})} \right] \\
+ \frac{c}{3} \log \left[ d(r_{D_+}, r_{D_-}) \right].
\]  
(B.21)
The expressions in the first line are the same as the matter entanglement entropy for the intervals \([b_-,a_-] \cup [a_+,b_+]\) complementary to \(R_- \cup I \cup R_+\). Simplifying using the various Kruskal variable distances as described earlier, we obtain for the second and third lines

\[
\frac{c}{3} \log \left[ 1 + 2 \frac{\delta \alpha^s \beta^p (a - r_S) \alpha^s \beta^s (a + r_S + r_D) \alpha^s \beta^M}{(r_D - \delta - r_S) \alpha^s \beta^s (r_D - a) \alpha^s \beta^p (2r_D - \delta + r_S) \alpha^s \beta^M} \cosh \left( \alpha_S (t_a + t_D) \right) \right] \\
- \frac{c}{3} \log \left[ 1 - 2 \frac{\delta \alpha^s \beta^p (a - r_S) \alpha^s \beta^s (a + r_S + r_D) \alpha^s \beta^M}{(r_D - \delta - r_S) \alpha^s \beta^s (r_D - a) \alpha^s \beta^p (2r_D - \delta + r_S) \alpha^s \beta^M} \cosh \left( \alpha_S (t_a - t_D) \right) \right] \\
+ \frac{c}{3} \log \left[ 1 - 2 \frac{\delta \alpha^s \beta^p (b - r_S) \alpha^s \beta^s (b + r_S + r_D) \alpha^s \beta^M}{(r_D - \delta - r_S) \alpha^s \beta^s (r_D - b) \alpha^s \beta^p (2r_D - \delta + r_S) \alpha^s \beta^M} \cosh \left( \alpha_S (t_a - t_D) \right) \right] \\
- \frac{c}{3} \log \left[ 1 + 2 \frac{\delta \alpha^s \beta^p (b - r_S) \alpha^s \beta^s (b + r_S + r_D) \alpha^s \beta^M}{(r_D - \delta - r_S) \alpha^s \beta^s (r_D - b) \alpha^s \beta^p (2r_D - \delta + r_S) \alpha^s \beta^M} \cosh \left( \alpha_S (t_a + t_D) \right) \right] \\
+ \frac{c}{6} \log \left[ \frac{\delta (r_D - r_S - \delta) (2r_D + r_S - \delta)}{l} \cosh^2 \frac{t_D}{2\beta S} \right]. \tag{B.22} \]

Since \(\delta \ll r_D\) and the exponent \(\alpha_S \beta^p \sim \frac{r_D}{r_S} \gg 1\), we see that the expressions in the first four lines are of the form \(\log(1 + \cdots)\) and thus vanishingly small. The last term

\[
\frac{c}{3} \log \left[ d(r_{D+},r_{D-}) \right] \sim \frac{c}{6} \log \left[ \frac{(r_D - r_S)(2r_D + r_S)}{l} \left( \frac{\delta}{\alpha^S \beta^P} \cosh^2 \frac{t_D}{2\beta S} \right) \right] \tag{B.23} \]

requires a regularization of the cosmological horizon which is akin to spatial infinity \(\ell^0\) in the flat space limit. Let us define

\[
\frac{(r_D - r_S)(2r_D + r_S)}{l} \frac{\delta \lambda}{\alpha^S \epsilon^2 U^2} \cosh^2 \frac{t_D}{2\beta S} \rightarrow c_D = finite \tag{B.24} \]

where we have reinstituted the various lengthscales, as discussed after (3.4), and using (2.13).

With this regularization of the observer near the cosmological horizon \(r_D\), we see that the entanglement entropy of the intervals \([b_-,a_-] \cup [a_+,b_+]\) used in the text and that of the complementary intervals \(R_- \cup I \cup R_+\) are essentially equivalent, as expected for bulk matter in a pure state on the entire slice \(R_- \cup [b_-,a_-] \cup I \cup [a_+,b_+] \cup R_+\) in the black hole Kruskal patch in figure 2, in the regime of very low de Sitter temperature (2.4), (2.5), (2.6).

### C Entanglement entropy with island at early times

In this section, we study the entanglement entropy looking for an island configuration at early times i.e. at some small \(t_a,t_b\) (with \(t_a,t_b \ll r_S\)). In this case Hawking quanta have not had time to escape out so we do not expect any island near the black hole horizon of the form (4.13). Indeed, simplifying (4.4) at early times with the coarse
approximation $t_a, t_b \sim 0$ gives

$$S_{\text{total}} = \frac{2\pi a^2}{G_N} + \frac{c}{6} \log \left[ \frac{2^8 r_S^4}{(r_D-r_S)^4} \right] \frac{(a-r_S)(b-r_S)}{l^2} \left( \frac{r_D-a}{l} \right) \left( \frac{r_D-b}{l} \right) .$$

$$\left( \frac{a+r_S+r_D}{l} \right) \left( \frac{b+r_S+r_D}{l} \right)$$

$$+ \frac{c}{3} \log \left[ 1 - 2 \frac{(a-r_S)^{\alpha_S \beta_S}}{(b-r_S)^{\alpha_S \beta_S}} C(a) \right] - \frac{c}{3} \log \left[ 1 + 2 \frac{(a-r_S)^{\alpha_S \beta_S}}{(b-r_S)^{\alpha_S \beta_S}} C(a) \right] . \quad (C.1)$$

Extremizing with $a-r_S \sim 0$ and simplifying with $r_D$ large (so $C(a) \sim 1$) gives

$$\frac{4\pi a}{G_N} + \frac{c}{6} \frac{1}{a-r_S} - \frac{c}{3} \frac{1}{(a-r_S)(b-r_S)} \sim 0 , \quad (C.2)$$

keeping only leading terms. So, since $b \gg r_S$, there is no island solution with $a \gtrsim r_S$.

However we might imagine that there arises a vanishing extremal surface with the island boundary $a$ far inside the black hole horizon so $a \ll r_S$. The boundary $b$ of the entanglement wedge of the Hawking radiation continues to be far away from the horizon i.e. $b-r_S \gg r_S-a$. Towards analysing this, we will employ Kruskal coordinates different from those previously used, defined in the black hole interior since $a < r_S$. So we define the tortoise coordinate as

$$r^* = - \int \frac{1}{f(r)} dr = - \int \frac{1}{1 - \frac{2m}{r} - \frac{r^2}{l^2}} dr = \int \frac{l^2 r}{(r_D-r)(r_S-r)(r+r_S+r_D)} dr \quad (C.3)$$

This gives

$$e^{r^*} = (r_D-r)^{-\beta_D}(r_S-r)^{\beta_S}(r+r_D+r_S)^{\beta_M} \quad (C.4)$$

where the $\beta$-parameters are as in (2.9). The radial null coordinates are $U = t-r^*$, $V = t+r^*$. Then the Kruskal coordinates adapted to the interior are

$$U_S = -e^{-\alpha_S(t-r^*)} = -e^{-\alpha_S t}(r_D-r)^{-\alpha_S \beta_D}(r_S-r)^{\alpha_S \beta_S}(r+r_D+r_S)^{\alpha_S \beta_M} ,$$

$$V_S = e^{\alpha_S(t+r^*)} = e^{\alpha_S t}(r_D-r)^{-\alpha_S \beta_D}(r_S-r)^{\alpha_S \beta_S}(r+r_D+r_S)^{\alpha_S \beta_M} , \quad (C.5)$$

with $\alpha_S = \frac{1}{2 \beta_S}$. The reduced two dimensional Schwarzschild de Sitter metric is given by

$$ds^2 = -\lambda r \frac{dU_S dV_S}{W^2} , \quad W = \sqrt{l} \alpha_S (r_D-r)^{-\frac{(1+2\alpha_S \beta_D)}{2}} (r_S-r)^{\frac{2\alpha_S \beta_S-1}{2}} (r+r_S+r_D)^{\frac{2\alpha_S \beta_M-1}{2}} , \quad (C.6)$$

with $W$ the interior conformal factor.
Towards approximating early times, we will set $t_a, t_b \sim 0$: then assuming $b - r_S \gg r_S - a$ as stated above and calculating as earlier reveals the total entanglement entropy to be

$$S_{\text{total}} = \frac{2\pi a^2}{G_N} + \frac{c}{6} \log \left[ \frac{2^8 r_s^4}{(r_D - r_S)^2} \left( \frac{2r_s + r_D}{l} \right)^3 (r_S - a) \left( \frac{r_D - a}{l} \right) \left( \frac{r_D - b}{l} \right) \right].$$

Extremizing (C.7) with respect to the island boundary $a$ gives

$$\frac{4\pi a}{G_N} + \frac{c}{6} \left[ -\frac{1}{r_S - a} - \frac{1}{r_D - a} + \frac{1}{a + r_S + r_D} \right] = \frac{4c}{6} \sqrt{\frac{r_s - a}{r_s}} \frac{C(a)}{(b - r_s)(a + r_s + r_D)} + \frac{1}{r_S - a} + \frac{\alpha_s \beta_M}{a + r_S + r_D} + \frac{\alpha_s \beta_D}{r_D - a}.$$  

We see that there exists a quantum extremal surface with $0 < a \ll r_S$ and low generalized entropy: approximating (C.8) with $a \ll r_S$ in all $O(c)$ terms gives

$$\frac{4\pi a}{G_N} \sim \frac{c}{6} \left( \frac{1}{r_S} + \frac{1}{r_D} - \frac{1}{r_S + r_D} \right) + \frac{c}{3} \sqrt{\frac{r_s - a}{r_s}} \frac{C(0)}{(b - r_s)} \left( \frac{1}{r_S} + \frac{2\alpha_s \beta_M}{r_S + r_D} + \frac{2\alpha_s \beta_D}{r_D} \right).$$  

With $b \gg r_S$, the second set of terms on the right is subleading to the first so we obtain

$$a \sim \frac{G_N c}{24\pi} \left( \frac{1}{r_S} + \frac{1}{r_D} - \frac{1}{r_S + r_D} \right).$$  

The quantity in brackets is positive revealing a small quantum extremal surface $a \sim O(G_N c)$ deep in the interior at early times. It is worth noting that we have made a coarse approximation in setting $t_a, t_b \sim 0$: doing this more carefully requires retaining $t_a$ and extremizing, but we expect similar qualitative behaviour at early times.

For this small QES (C.10), the total on-shell entanglement entropy approximates to

$$S_{\text{O.s.}} = \frac{c^2 G_N}{288\pi} \left( \frac{1}{r_S} + \frac{1}{r_D} - \frac{1}{r_S + r_D} \right)^2 + \frac{c}{3} \log \left[ \frac{2^4 r_s^2 \sqrt{r_s (b - r_s) (r_D - b) (r_D - r_s) (2r_s + r_D) / l}}{(r_D - r_S)^2 \left( \frac{2r_s + r_D}{l} \right)^2} \right].$$  

ignoring terms scaling as $r_D^{-\alpha_s \beta_D}$. Thus at early times when Hawking evaporation has not yet kicked in significantly, the generalized entropy is not significant, in accordance with the approximate purity of the early time state.
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