Quantum toroidal $\mathfrak{gl}_1$ and Bethe ansatz

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Abstract
We establish the method of Bethe ansatz for the XXZ type model obtained from the $R$ matrix associated to quantum toroidal $\mathfrak{gl}_1$. We do this by using shuffle realizations of the modules and by showing that the Hamiltonian of the model is obtained from a simple multiplication operator by taking an appropriate quotient. We expect this approach to be applicable to a wide variety of models.

Keywords: quantum toroidal algebra, Bethe ansatz, shuffle algebra

1. Introduction

The XXZ type models constitute a well-known large family of integrable quantum models, which was one of the main motivations for the very discovery of quantum groups.

These models arise in the following algebraic setting. We start with a quantum algebra with a triangular decomposition $\mathcal{E} = \mathcal{E}_0 \otimes \mathcal{E}_0 \otimes \mathcal{E}_e$, and an associated $R$ matrix $R$ in a completion of $\mathcal{E}_0 \otimes \mathcal{E}_e$, where $\mathcal{E}_0 = \mathcal{E}_0 \otimes \mathcal{E}_0$ and $\mathcal{E}_e = \mathcal{E}_0 \otimes \mathcal{E}_e$. We also fix a group like element $t$ in (a completion of) $\mathcal{E}$. For an $\mathcal{E}$ module $U$, we have the transfer matrix $T_U(t) = (1 \otimes T_U((1 \otimes t)R))$, provided the trace is well-defined. The assignment $U \mapsto T_U(t)$ gives us a map from the Grothendieck ring of a suitable category of $\mathcal{E}$ modules to a completion of $\mathcal{E}_0$. The standard properties of the trace and the $R$ matrix imply that this map is a ring homomorphism, and that the image is a commutative subalgebra. Given a suitable $\mathcal{E}$
module $V$, the commutative subalgebra of transfer matrices acts in $V$ and produces the XXZ type Hamiltonians associated to $E$ and $V$.

The problem of diagonalizing the action of the XXZ type Hamiltonians has been extensively studied for more than 80 years. It is done almost exclusively by the Bethe ansatz method. The approach is always the same: one writes a candidate for the eigenvector depending on auxiliary parameters in some explicit form—the so-called off-shell Bethe vector. Then one proves that if the parameters satisfy a system of algebraic equations, then the off-shell Bethe vector is indeed an eigenvector with an explicit eigenvalue. The system of equations is called Bethe equations and the corresponding eigenvector is called Bethe vector. Then, in good situations, one proves that the Bethe ansatz is complete, meaning that the Bethe vectors form a basis of the representation $V$-module explicit symmetries if any.

In this paper, we study the case of $E$ being the quantum toroidal algebra of type $gl_1$, also known as elliptic Hall algebra, $(q, \gamma)$ analog of $W_{1+\infty}$, Ding–Iohara algebra, etc. This algebra enjoys a wave of popularity due to its appearance in geometry [BS, FT, S, SV1, SV2] and in integrable systems [FKSW, FKS2, KS].

It appears that the known methods of finding off-shell Bethe vectors are not directly applicable to $E$. We propose an alternative way to obtain the spectrum of the Hamiltonians. The idea is to introduce an appropriate space of functions, and to identify the Hamiltonians with the projection of simple operators of multiplication by symmetric functions. The Bethe equations appear naturally as the condition for describing the kernel of the projection. We expect that this method can be applied to many cases including the ones where the standard Bethe ansatz technique is already established.

Let us describe the logic of our approach in more detail. The quantum toroidal $gl_1$ algebra $E$ depends on complex parameters $q_1, q_2, q_3$ such that $q_1 q_2 q_3 = 1$. The algebras $E_2$, $E_e$ and $E_0$ are generated by currents $e(z)$, $f(z)$ and $\psi^\pm(z)$ (plus central elements and their duals) respectively. The commutation relations are similar to that of the quantum affine $sl_2$ algebra, but they are written in terms of the cubic polynomial $g(z, w) = (z - q_1 w)(z - q_2 w)(z - q_3 w)$, see section 2. There is a projective action of the group $SL(2, \mathbb{Z})$ on $E$ by automorphisms. Along with the initial currents $e(z)$, $f(z)$ and $\psi^\pm(z)$, we also use the currents $e^+ (z)$, $f^+ (z)$ and $\psi^\pm(z)$ obtained by applying the automorphism from $SL(2, \mathbb{Z})$ corresponding to the rotation by 90 degrees [BS, M]. We call them “perpendicular” currents.

We define the coproduct and the $R$ matrix in terms of perpendicular currents. We consider modules which are lowest weight modules with respect to initial currents, namely modules generated by a vector $|\emptyset\rangle$ such that

$$f(z)|\emptyset\rangle = 0, \quad \psi^\pm(z)|\emptyset\rangle = \phi(z)|\emptyset\rangle \quad (1.1)$$

where $\phi(z)$ is a rational function. The most important example is a family of Fock modules $F(u)$ depending on a complex parameter $u$. These modules are irreducible under the Heisenberg subalgebra of $E$ generated by perpendicular currents $\psi^{\pm}(z)$. Other perpendicular currents $e^\pm(z)$, $f^\pm(z)$ act in $F(u)$ by vertex operators [FKSW], while operators $\psi^{\pm}(z)$ can be identified with Macdonald operators [FFJMM2, FHHSSY].

Among the Hamiltonians of the model, the simplest is the first degree term $H_p$ of the transfer matrix $T_{F(u)}(pd^\pm)$, where $p \in \mathbb{C}$ and $d^\pm$ is the degree operator counting $e(z)$ as 1 and $f(z)$ as $-1$, see lemma 5.1. It turns out that $H_p$ coincides with the operator considered in [FKSW, FKS2, KS] in relation to the deformed Virasoro algebra. The operator $H_p$ acting in a generic tensor product of Fock modules for generic $p$ has simple spectrum, and we do not consider other Hamiltonians in the present paper.

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Following the ideas of [FO, Ng], we realize $\mathcal{E}$ algebra in an appropriate space of functions $Sh_0$, see section 3. We also introduce another space of functions $Sh_1(u)$ together with left and right actions of algebra $Sh_0$. Moreover, we extend the left action to the action of $\mathcal{E}$. We denote $J_0$ the image of the right action: $J_0 = Sh_1(u)Sh_0^{\prime}$, where the prime denotes the augmentation ideal. We use certain filtration, see appendix, to prove that the quotient $Sh_1(u)/J_0$ is isomorphic to the Fock module $F(u)$ as $\mathcal{E}$ module. We introduce a subspace $N$ of functions in $Sh_1(u)$ defined by certain regularity conditions and show that $N \oplus J_0 = Sh_1(u)$, see section 3.4. Under a natural embedding of $Sh_1(u)$ to a completion of $\mathcal{E}$, the space $N$ is identified with the space of matrix elements of $L$ operators of the form $L_{\varnothing,v} = (1 \otimes \langle \varnothing | ) R (1 \otimes v )$, $v \in F(u)$. Moreover under the projection $Sh_1(u) \to Sh_1(u)/J_0 = F(u)$, the function corresponding to $L_{\varnothing,v}$ is mapped to $v$.

The coefficients of the series $\psi^p(z)$ act in the space of functions $Sh_1(u)$ by multiplications by symmetric polynomials. It is easy to see that $H_0 = \lim_{p \to 0} H_p$ coincides (up to an explicit constant) with the linear term $h_1$ of $\psi^p(z)$, and in particular that, in the subspace $Sh_{1,n}(u)$ of functions in $n$ variables, $H_0$ acts simply by multiplication by $\sum_{i=1}^{n} x_i$ (up to multiplicative and additive explicit constants), see theorem 5.2, (5.6), (5.2). In the limit $p \to 0$ the algebra of all Hamiltonians of the model coincides with the algebra generated by coefficients of $\psi^p(z)$.

Finally, we define the space of $p$-commutators (see (5.4) below)

$$J_p = \{ G^* F - p^{deg F} F^* G \mid F \in Sh_0, G \in Sh_1(u) \}.$$ 

The multiplication by symmetric polynomials clearly preserves this space, and for generic $p$ we have the direct sum decomposition of vector spaces: $N \oplus J_p = Sh_1(u)$.

Our principal result is: the projection of operator $H_0$ acting in $Sh_1(u)$ to the space of matrix elements of $L$ operators $N$ along space $J_p$ coincides with Hamiltonian $H_p$ acting on $F(u) = N$. In other words, $Pr_{\varnothing,v} H_p v = Pr_{\varnothing,v} H_0 v$ for all $v \in N$, see theorem 5.2.

This identification immediately leads to the Bethe equations and the computation of the spectrum.

Namely, we consider the dual space to $Sh_1(u)$ and evaluation functionals defined as evaluation of functions in $Sh_1(u)$ at fixed complex numbers $\{a_i\}$. Such a functional is obviously an eigenvector with respect to multiplication by a function $f$, with the eigenvalue given by evaluation of $f$ at $\{a_i\}$. Also clearly, the evaluation functional has $J_p$ in the kernel if and only if the evaluation numbers $\{a_i\}$ satisfy the Bethe equations

$$\phi(a_j) \prod_{j \neq i} \frac{\xi(a_j, a_i)}{\xi(a_i, a_j)} = p^{-1} \quad \text{for all} \, i,$$

see (5.7) and theorem 5.4, where $\phi(z)$ is the weight of the module in (1.1). Therefore, we obtain a description of the spectrum of the Hamiltonians in the dual module $V = F(u)^\#$.

We also study the off-shell Bethe vector. The result of [FHSSY] allows us to write the canonical element of $F(u)^\# \otimes F(u)$ in the form $\sum_i \langle a_i \mid \otimes f_j(x) \in F(u)^\# \otimes N$ with explicit functions $f_j(x)$. The latter is the off-shell Bethe vector, from which the Bethe vector is obtained by evaluating the second component at $\{a_i\}$. We give the result in proposition 5.5.

In this paper we consider only tensor products of Fock spaces of $\mathcal{E}$, but we expect such a scheme can be used for many modules over many quantum algebras. Also we skip the question of the completeness of the Bethe ansatz here, but we expect it can be proved for generic $p$ by deforming the $p = 0$ evaluation maps $\rho_p$ described in appendix in a standard way.
We note that operators $\psi^\pm(z)$ acting in $F(u)$ can be identified with operators acting in equivariant $K$-theory of the Hilbert scheme of points on $\mathbb{C}^2$, where $q_1, q_3$ are equivariant parameters. Then the algebra of the XXZ-type Hamiltonians $\{T_{(u)}(p^d)\}$ provides the deformation of these operators which is expected to be related to ‘quantum equivariant $K$-theory’. Such an interpretation was one of motivations for our work.

In the conformal limit, $\mathcal{E}$-algebra becomes the $W$ algebra, and the corresponding integrals of motion in relation with Bethe equations were studied in [AL, L]. Another Hamiltonian of a similar kind was considered in [Sa1, Sa2]. We also feel that there is some connection to the work [NS], where the authors find a connection between Bethe ansatz and supersymmetric gauge theory.

This paper is constructed as follows. In section 2 we describe algebraic properties of quantum toroidal $\mathfrak{gl}_1$ algebra and the Fock module. In section 3 we establish functional realizations of $\mathcal{E}$, and the Fock module. For that we use Gordon filtration established in the appendix. We study matrix elements of $L$ operators and their relation to the shuffle algebras in section 4. In section 5 we describe the XXZ-type Hamiltonians, compute explicitly the first one and diagonalize it. In section 6 we extend our results to the case of tensor product of Fock modules.

2. Quantum toroidal $\mathfrak{gl}_1$

In this section, we introduce our notation concerning the quantum toroidal $\mathfrak{gl}_1$ algebra.

2.1. $\mathcal{E}$-algebra

Fix complex numbers $q, q_1, q_2, q_3$ satisfying $q_2 = q^2$ and $q_1 q_2 q_3 = 1$. We assume further that, for integers $l, m, n \in \mathbb{Z}$, $q_l, q_2^m q_3^n = 1$ holds only if $l = m = n$. We set

$$g(z, w) = (z - q_1 w)(z - q_2 w)(z - q_3 w),$$

$$\kappa_r = \left(1 - q_1^r\right)\left(1 - q_2^r\right)\left(1 - q_3^r\right).$$

The quantum toroidal algebra of type $\mathfrak{gl}_1$, which we denote by $\mathcal{E}$, is a $\mathbb{C}$-algebra generated by elements $e_k, f_k \quad (k \in \mathbb{Z}), \quad h_r \quad (r \in \mathbb{Z} \setminus \{0\})$

and invertible elements $C, C^{\perp}, D, D^{\perp}$, subject to the relations given below. We write them in terms of the generating series

$$e(z) = \sum_{k \in \mathbb{Z}} e_k z^{-k}, \quad f(z) = \sum_{k \in \mathbb{Z}} f_k z^{-k},$$

$$\psi^\pm(z) = \left(C^{\perp}\right)^{\frac{1}{2}} \exp\left(\sum_{r=1}^{\infty} \kappa_r h_r z^{2r}\right).$$

The defining relations of $\mathcal{E}$ read as follows.

$C, C^{\perp}$ are central, $DD^{\perp} = D^{\perp} D,$

$De(z) = e(qz) D,$ $DF(z) = f(qz) D,$ $D\psi^\pm(z) = \psi^\pm(qz) D,$

$D^2e(z) = q e(z) D^{\perp},$ $D^2f(z) = q^{-1} f(z) D^{\perp},$ $D^2\psi^\pm(z) = \psi^\pm(z) D^{\perp},$

$\psi^\pm(z) \psi^\mp(w) = \psi^\mp(w) \psi^\pm(z).$
In particular we have the relations
\[ [h_r, e_n] = -\frac{1}{r} e_{n+r} C^{-r-1}/2, \]  
(2.1)  
\[ [h_r, f_n] = \frac{1}{r} f_{n+r} C^{-r+1}/2, \]  
(2.2)  
\[ [h_r, h_s] = \delta_{r+s,0} \frac{C^{r} - C^{-r}}{r \kappa_r}, \]  
(2.3)
for all \( r, s \in \mathbb{Z} \setminus \{0\} \) and \( n \in \mathbb{Z} \).

The subalgebra of \( \mathcal{E} \) generated by \( e_n, f_n (n \in \mathbb{Z}), h_r (r \in \mathbb{Z} \setminus \{0\}) \) and \( C, C^\perp \) will be denoted by \( \mathcal{E}' \).

\( \mathcal{E} \)-algebra admits an automorphism \( \theta \) of order 4 [BS, M] such that (see figure \( 1 \))
\[ \theta: e_0 \mapsto h_{-1}, \quad h_{-1} \mapsto f_0, \quad f_0 \mapsto h_1, \quad h_1 \mapsto e_0, \]
\[ C^\perp \mapsto C, \quad C \mapsto (C^\perp)^{-1}, \quad D^\perp \mapsto D, \quad D \mapsto (D^\perp)^{-1}. \]  
(2.4)
Quite generally, we write \( x^\perp = \theta^{-1}(x) \) for an element \( x \in \mathcal{E} \). In this notation
\( e_0^\perp = h_1, \quad f_0^\perp = h_{-1}, \quad h_1^\perp = f_0, \quad h_{-1}^\perp = e_0. \)
The relations (2.1)–(2.3) imply further that

\[
\begin{array}{ccc}
\varepsilon_1 & \varepsilon_2 & \varepsilon_3 \\
\varepsilon_4 & \varepsilon_5 & \varepsilon_6 \\
f_{-2} & f_{-1} & f_0, f_1, f_2 \\
\ldots & h_{-2} & h_{-1}, h_0, h_1, h_2 \\
\ldots & e_{-2} & e_{-1}, e_0, e_1, e_2 \\
\end{array}
\]

Figure 2. Subalgebras \( \mathcal{E}_2, \mathcal{E}_3, \mathcal{E}_4, \mathcal{E}_5 \). The elements \( C, C^+, D, D^+ \) placed at the center; \( \bullet \) are common to all these subalgebras.

The relations (2.1)–(2.3) imply further that

\[
\begin{align*}
e_1^+ &= f_1 C^+, \quad e_{-1}^- = e_1 C^{-1}, \\
f_1^- &= f_1 C, \quad f_{-1}^+ = e_{-1} \left( C^+ \right)^{-1}, \\
e_{m+1}^- &= \left[ e_m, f_0 \right] C^-, \quad e_{m-1}^- = \left[ e_0, e_m^- \right]. \\
f_{m+1}^- &= \left[ f_0, f_m \right], \quad f_{m-1}^+ = \left[ f_{-m}, e_0 \right] \left( C^+ \right)^{-1}.
\end{align*}
\]

\( \mathcal{E} \)-algebra is equipped with a \( \mathbb{Z}^2 \) grading defined by the assignment

\[
\text{deg } e_n = (1, n), \quad \text{deg } f_n = (-1, n), \quad \text{deg } h_r = (0, r), \quad (2.5)
\]

\[
\text{deg } x = (0, 0) \quad \left( x = C, C^+, D, D^+ \right). \quad (2.6)
\]

We have

\[
\text{deg } e_n^+ = (-n, 1), \quad \text{deg } f_n^- = (-n, -1), \quad \text{deg } h_r^- = (-r, 0).
\]

For a homogeneous element \( x \in \mathcal{E} \) with \( \text{deg } x = (n_1, n_2) \), we say that \( x \) has a principal degree \( n_1 \) and a homogeneous degree \( n_2 \) and write

\[
pdeg x = n_1, \quad hdeg x = n_2.
\]

Note that \( D^x (D^x)^{-1} = q^{pdeg x} x \) and \( Dx D^{-1} = q^{-hdeg x} x \).
Introduce the following subalgebras:
\[
\begin{align*}
\mathcal{E}_{>} &= \{e_n \ (n \in \mathbb{Z}), \ h_r \ (r > 0), \ C, \ C^\perp, \ D, \ D^\perp\}, \\
\mathcal{E}_{<} &= \{f_n \ (n \in \mathbb{Z}), \ h_{-r} \ (r > 0), \ C, \ C^\perp, \ D, \ D^\perp\}, \\
\mathcal{E}_{>}^\perp &= \{e_n^\perp \ (n \in \mathbb{Z}), \ h_r^\perp \ (r > 0), \ C, \ C^\perp, \ D, \ D^\perp\}, \\
\mathcal{E}_{<}^\perp &= \{f_n^\perp \ (n \in \mathbb{Z}), \ h_{-r}^\perp \ (r > 0), \ C, \ C^\perp, \ D, \ D^\perp\}.
\end{align*}
\]
We picture generators of $\mathcal{E}$-algebra and their perpendicular counterparts on a plane according to their grading. The subalgebras $\mathcal{E}_{>}, \mathcal{E}_{<}, \mathcal{E}_{>}^\perp, \mathcal{E}_{<}^\perp$ are generated by the elements appearing respectively in the lower, upper, right and left half plane, see figure 2.

We set also
\[
\begin{align*}
\mathcal{E}_{>} &= \{e_n \ (n \in \mathbb{Z})\}, \quad \mathcal{E}_{<} = \{f_n \ (n \in \mathbb{Z})\}, \\
\mathcal{E}_{>}^\perp &= \{e_n^\perp \ (n \in \mathbb{Z})\}, \quad \mathcal{E}_{<}^\perp = \{f_n^\perp \ (n \in \mathbb{Z})\}.
\end{align*}
\]
One can easily check that $h_r^\perp, C e_r^\perp \in \mathcal{E}_{>}$ for $r > 0$.

2.2. Bialgebra structure and $R$ matrix

$\mathcal{E}$-algebra is endowed with a topological bialgebra structure. We choose the following coproduct $\Delta$ and counit $\varepsilon$, defined in terms of the perpendicular generators
\[
\begin{align*}
\Delta(e_n^\perp) &= \sum_{j \geq 0} e_{n-j}^\perp \otimes \psi_j^\perp (C^\perp)^n + 1 \otimes e_n^\perp, \\
\Delta(f_n^\perp) &= f_n^\perp \otimes 1 + \sum_{j \geq 0} \psi_{-j}^\perp (C^\perp)^n \otimes f_{n+j}^\perp, \\
\Delta h_r^\perp &= h_r^\perp \otimes 1 + (C^\perp)^{-r} \otimes h_r^\perp, \\
\Delta h_{-r}^\perp &= h_{-r}^\perp \otimes (C^\perp)^r + 1 \otimes h_{-r}^\perp, \\
\Delta x &= x \otimes x \quad (x = C, C^\perp, D, D^\perp).
\end{align*}
\]
\[
\varepsilon(e_n^\perp) = \varepsilon(f_n^\perp) = 0, \quad \varepsilon(h_r^\perp) = 0, \quad \varepsilon(x) = 1 \quad (x = C, C^\perp, D, D^\perp).
\]
for all $n \in \mathbb{Z}$ and $r > 0$. Here we set $\psi^\perp_{-j}(z) = \sum_{j \geq 0} \psi_j^\perp z^{-j}$, $\psi^\perp_0 = C^\perp$.

Quite generally, a bialgebra pairing on a bialgebra $A$ is a symmetric non-degenerate bilinear form $(,): A \times A \to C$ with the properties
\[
\begin{align*}
(a, b_1 b_2) = (\Delta(a), b_1 \otimes b_2), \quad (a, 1) = \varepsilon(a)
\end{align*}
\]
for any $a, b_1, b_2 \in A$. With each such pair $(A, (,))$, there is an associated bialgebra $DA$ called the Drinfeld double of $A$. As a vector space $DA = A \otimes A^\text{op}$, where $A^\text{op}$ is a copy of $A$ endowed with the opposite coalgebra structure. Moreover $A^+ = A \otimes 1$ and $A^- = 1 \otimes A^\text{op}$ are sub bialgebras of $DA$, and the commutation relation
\[
\sum (a_{(2)}, b_{(1)}) a_{(1)}^\perp b_{(2)}^\perp = \sum (b_{(2)}, a_{(1)}) b_{(1)}^\perp a_{(2)}^\perp
\]
is imposed for $a, b \in A$. Here $a^+ = a \otimes 1$, $a^- = 1 \otimes a$, and we use the Sweedler notation $\Delta(a) = \sum a_{(1)} \otimes a_{(2)}$ for the coproduct. The canonical element of $DA = A \otimes A^\text{op}$ considered
as an element of a suitable completion of $A^+ \otimes A^- \subset DA \otimes DA$ is called the universal $R$ matrix and is denoted by $\mathcal{R}$. It has the properties

\[
\mathcal{R} \Delta(x) = \Delta^{op}(x) \mathcal{R} \quad (x \in DA),
\]

\[
(\Delta \otimes \text{id}) \mathcal{R} = \mathcal{R}_{1,3} \mathcal{R}_{2,3},
\]

\[
(\text{id} \otimes \Delta) \mathcal{R} = \mathcal{R}_{1,3} \mathcal{R}_{1,2},
\]

\[
\mathcal{R}_{1,2} \mathcal{R}_{1,3} \mathcal{R}_{2,3} = \mathcal{R}_{2,3} \mathcal{R}_{1,3} \mathcal{R}_{1,2},
\]

where, as usual, the suffixes $i,j$ of $\mathcal{R}_{i,j}$ stand for the tensor components, e.g., $\mathcal{R}_{1,2} = \mathcal{R} \otimes 1$.

The bialgebra $A = E^\perp_\mathbb{Z}$ has a bialgebra pairing such that the non-trivial pairings of the generators are given by

\[
\left( e^1_m, e^1_n \right) = \frac{1}{\kappa_1} \delta_{m,n}, \quad \left( h^1_x, h^1_{x'} \right) = \frac{1}{r\kappa_r},
\]

\[
(C, D) = \left( C^\perp, D^\perp \right) = q^{-1}.
\]

This pairing respects the $\mathbb{Z}^2$ grading in the sense that $(a, b) = 0$ unless $\deg a = \deg b$. We identify $A^{op}$ with $E^\perp_\mathbb{Z}$ through the following isomorphism of algebras, which is also an anti-isomorphism of coalgebras

\[
e^1_n \mapsto f^1_{-n}, \quad h^1_x \mapsto h^1_{-x}, \quad x \mapsto x^{-1} \quad (x = C, C^\perp, D, D^\perp).
\]

The Drinfeld double of $E^\perp_\mathbb{Z}$ is then identified with $E^\perp_\mathbb{Z} \otimes E^\perp_\mathbb{Z}$. Its quotient by the relation $x \otimes 1 = 1 \otimes x$ isomorphic to the algebra $\mathcal{E}$ [BS].

The universal $R$ matrix is an element of a certain completion of $E^\perp_\mathbb{Z} \otimes E^\perp_\mathbb{Z} \subset \mathcal{E} \otimes \mathcal{E}$, with the structure

\[
\mathcal{R} = \mathcal{R}^{(0)} \mathcal{R}^{(1)} \mathcal{R}^{(2)}.
\]

Here

\[
\mathcal{R}^{(1)} = \exp \left( \sum_{r \geq 1} r \kappa_r h^1_{-r} \otimes h^1_r \right),
\]

\[
\mathcal{R}^{(2)} = 1 + \kappa_1 \sum_{i \in \mathbb{Z}} e^1_i \otimes f^1_{-i} + \cdots
\]

is the canonical element of $E^\perp_\mathbb{Z} \otimes E^\perp_\mathbb{Z}$. In (2.18), \cdots stands for terms whose first component has homogeneous degree $\geq 2$. The element $\mathcal{R}^{(0)}$ is formally defined as

\[
\mathcal{R}^{(0)} = q^{-\sum d \otimes d - c \otimes c - d \otimes d - c \otimes c},
\]

\[
C = q^d, \quad C^\perp = q^{-d}, \quad D = q^c, \quad D^\perp = q^{-c}.
\]

The expression $(\mathcal{R}^{(0)})^{-1} \Delta^{op}(x) \mathcal{R}^{(0)}$ has a well-defined meaning, and the intertwining property (2.13) should be understood as

\[
\mathcal{R}^{(1)} \mathcal{R}^{(2)} \Delta(x) = \left( \left( \mathcal{R}^{(0)} \right)^{-1} \Delta^{op}(x) \mathcal{R}^{(0)} \right) \mathcal{R}^{(1)} \mathcal{R}^{(2)} \quad (x \in \mathcal{E}).
\]

The element $\mathcal{R}^{(0)}$ is well defined on tensor products of representations which are principally graded and on which $c$ acts as 0. This is the case for all representations considered in this paper.
2.3. Fock representations

Let $V$ be an $E'$ module, and let $L, K \in \mathbb{C}^\times$. We say that $V$ has level $(L, K)$ if the central element $C$ acts as the scalar $L$ and $C^\perp$ as $K$. In this paper we consider only modules of level $(1, K)$. Then the operators $h_i$ are mutually commutative on $V$. We say that $V$ is quasi-finite if it is graded by the principal degree, $V = \bigoplus_{n \in \mathbb{Z}} V_n$, and $\dim V_n < \infty$ for all $n$. We say it is bounded if $V_n = 0$ for $n \ll 0$. For an $E'$ module $V$ and $u \in \mathbb{C}^\times$, we denote by $V(u)$ the pullback of $V$ by the automorphism

$$s_u : e(z) \mapsto e(z/u), \quad f(z) \mapsto f(z/u), \quad \psi^\pm(z) \mapsto \psi^\pm(z/u), \quad C \mapsto C, \quad C^\perp \mapsto C^\perp.$$  

Let $\phi(z)$ be a rational function such that $\phi(z)$ is regular at $z = 0$, $\phi(0) = 1$. We say that $V$ is a lowest weight module with lowest weight $\phi(z)$ if it is generated by a vector $v$ which satisfies

$$f(z)v = 0, \quad \psi^\pm(z)v = \phi^\pm(z)v.$$  

Here $\phi^\pm(z)$ means the expansion of $\phi(z)$ at $z^\pm 1 = 0$. For each such $\phi(z)$, there exists a unique irreducible lowest weight module $L_{\phi(z)}$ with lowest weight $\phi(z)$. Assigning degree 0 to $v$ we have the principal grading $L_{\phi(z)} = \bigoplus_{n \geq 0} (L_{\phi(z)})_n$, and $L_{\phi(z)}$ is quasi-finite [M].

The most basic lowest weight $E'$ module is the Fock module. For $u \in \mathbb{C}^\times$, the Fock module $\mathcal{F}(u)$ is defined to be the irreducible lowest weight $E'$ module with level $(1, q)$ and lowest weight

$$\phi(u, z) = \frac{q^{-1} - q u/z}{1 - u/z}.$$  

As a vector space, $\mathcal{F}(u)$ has a basis $\{|\lambda\rangle\}_{\lambda \in \mathcal{P}}$ labeled by all partitions.

We use the following convention for partitions. A partition is a sequence of non-negative integers $\lambda = (\lambda_1, \lambda_2, \ldots)$ such that $\lambda_i \geq \lambda_{i+1}$ for all $i \geq 1$, and $\lambda_i = 0$ for $i$ large enough. In particular, we write $\emptyset = (0, 0, 0, \ldots)$. The set of all partitions is denoted by $\mathcal{P}$. The dual partition $\lambda'$ is given by $\lambda'_i = \# \{j \mid \lambda_j \geq i\}$. We set $|\lambda| = \sum_{j \geq 1} \lambda_j$ and $\ell(\lambda) = \lambda'_1$ for $\lambda \in \mathcal{P}$. For $j \geq 1$ and $\lambda \in \mathcal{P}$ we write $\lambda + 1_j = (\lambda_1, \lambda_2, \ldots, \lambda_{j-1}, j, \lambda_{j+1}, \ldots)$. We call a pair of natural numbers $(x, y)$ convex corner of $\lambda$ if $\lambda'_{x+1} < \lambda'_{y+1} = x$, and concave corner of $\lambda$ if $\lambda'_{y+1} = x-1$ and in addition $y = 1$ or $\lambda'_{y-1} > x - 1$. We denote by $CC(\lambda)$ and $CV(\lambda)$ the set of concave and convex corners of $\lambda$, respectively.

Then the action of the generators is given as follows [FT]:

$$\langle \lambda + 1_j | e(z) | \lambda \rangle = \prod_{s=1}^{j-1} \psi \left( q_1^{\lambda - \lambda_{j-1} - 1} q_3^{\lambda_{j-1} - 1} \right) \prod_{s=1}^{j-1} \psi \left( q_1^{\lambda - \lambda_{j} - 1} q_3^{\lambda_j - 1} \right) \cdot \delta \left( q_1^{\lambda_1} q_3^{j-1} u/z \right),$$  

$$\langle \lambda | f(z) | \lambda + 1_j \rangle = \frac{q - q^{-1}}{\kappa_1} \prod_{s=j+1}^{\ell(\lambda)} \psi \left( q_1^{\lambda - \lambda_{j} - 1} q_3^{\lambda_j - 1} \right) \prod_{s=j+1}^{\ell(\lambda)} \psi \left( q_1^{\lambda - \lambda_{j} - 1} q_3^{\lambda_j - 1} \right) \cdot \delta \left( q_1^{\lambda_1} q_3^{j-1} u/z \right),$$  

$$\langle \lambda | \psi^\pm(z) | \lambda \rangle = \prod_{(i,j) \in \text{CC}(\lambda)} \psi \left( q_1^{\lambda_i} q_3^{\lambda_j} u/z \right) \prod_{(i,j) \in \text{CV}(\lambda)} \psi \left( q_1^{\lambda_i} q_3^{\lambda_j} u/z \right)^{-1}.$$  

In the above, we set $\psi(z) = (q - q^{-1}z)/(1 - z)$ and assume that $\lambda, \lambda + 1_j \in \mathcal{P}$. In all other cases the matrix elements are defined to be zero. In terms of the generators $h_i$, we have for $r \in \mathbb{Z} \setminus \{0\}$
\[ h_r | \emptyset \rangle = \gamma_r | \emptyset \rangle, \quad \gamma_r = \frac{1 - q_r^2}{r \kappa_r}. \tag{2.20} \]

The generators \( h_r^\perp \) act as a Heisenberg algebra on \( \mathcal{F}(u) \),

\[
\left[ h_r^\perp, h_{r'}^\perp \right] = \frac{q_r^r - q_{r'}^{-r}}{r \kappa_r} \delta_{r + r', 0} \quad (r, s \in \mathbb{Z} \setminus \{0\}), \tag{2.21}
\]
and \( \mathcal{F}(u) \) is an irreducible module over this Heisenberg algebra. The generators \( e^\perp(z) \), \( f^\perp(z) \) act by vertex operators

\[
e^\perp(z) = \frac{1 - q_2^{-2}}{\kappa_1} u \exp \left( \sum_{r=1}^{\infty} \frac{\kappa_r}{1 - q_2^r} h_{-r}^\perp z^{-r} \right) \exp \left( \sum_{r=1}^{\infty} \frac{q_r^r \kappa_r}{1 - q_2^r} h_r^\perp z^{-r} \right), \tag{2.22}
\]

\[
f^\perp(z) = \frac{1 - q_2^{-1}}{\kappa_1} u^{-1} \exp \left( -\sum_{r=1}^{\infty} \frac{q_r^r \kappa_r}{1 - q_2^r} h_{-r}^\perp z^{-r} \right) \exp \left( -\sum_{r=1}^{\infty} \frac{q_r^2 \kappa_r}{1 - q_2^r} h_r^\perp z^{-r} \right). \tag{2.23}
\]

### 3. Shuffle algebras

It is known that the algebra \( \mathcal{E}_z = \langle e_n \ (n \in \mathbb{Z}) \rangle \) has a presentation in terms of certain algebra of rational functions called the shuffle algebra. In this section we introduce an extension of the shuffle algebra which gives a functional realization of the Fock modules.

#### 3.1. \( S_{0} \)-algebra

First, let us recall the definition of the shuffle algebra

\[ S_{0} = \bigoplus_{n=0}^{\infty} S_{0,n}. \]

We set \( S_{0,0} = \mathbb{C} \), \( S_{0,1} = \mathbb{C}[x^{\pm 1}] \). For \( n \geq 2 \), \( S_{0,n} \) is the space of all symmetric rational functions of the form

\[ F(x_1, \cdots, x_n) = \frac{f(x_1, \cdots, x_n)}{\prod_{1 \leq i < j \leq n} (x_i - x_j)^2}, \quad f(x_1, \cdots, x_n) \in \mathbb{C}\left[x_1^{\pm 1}, \cdots, x_n^{\pm 1}\right]^{\text{Sym}}, \]

satisfying the wheel condition

\[ f(x_1, \cdots, x_n) = 0 \quad \text{if} \quad (x_1, x_2, x_3) = (x, q_1 x, q_2 x) \text{ or } (x, q_2 x, q_1 x). \tag{3.1} \]

Note that since \( f(x_1, \cdots, x_n) \) is symmetric, from (3.1), we also have \( f(x_1, \cdots, x_n) = 0 \) if \( (x_1, x_2, x_3) = (x, q x, q q x) \) or \( (x_1, x_2, x_3) = (x, q q x, q x) \) for \( i, j \in \{1, 2, 3\}, i \neq j \).

We define the shuffle product \( * \) of elements \( F \in S_{0,n} \) and \( G \in S_{0,m} \) by the formula

\[
(F \ast G)(x_1, \cdots, x_{m+n}) = \text{Sym} \left[ F(x_1, \cdots, x_m) G(x_{m+1}, \cdots, x_{m+n}) \prod_{1 \leq i < j \leq m} \omega(x_{m+j}, x_i) \right].
\]
where
\[ \omega(x, y) = \frac{(x - q_1 y)(x - q_2 y)(x - q_3 y)}{(x - y)^3} = \frac{g(x, y)}{(x - y)^3}. \]

Here and after we set
\[ \text{Sym} f(x_1, \ldots, x_n) = \frac{1}{n!} \sum_{\sigma \in S_n} f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}). \]

It is easy to check that the space \( S_0 \) becomes an associative algebra under the product \(*\). The following fact is known (see [FT, Ng, SV2]).

**Proposition 3.1.** \( S_0 \)-algebra is generated by the subspace \( S_{0,1} \). There is an isomorphism of algebras
\[ \sigma : \mathcal{E}_i \rightarrow S_0, \]
where \( c_i = q_i/(1 - q_i)(1 - q_i) \).

Under the isomorphism above, the graded component \( \mathcal{E}_d \) corresponds to the subspace \( S_{0,n} \) consisting of functions of homogeneous degree \( d \in \mathbb{Z} \).

### 3.2. The bimodule \( S_{1,1} \)

We fix \( u \in \mathbb{C} \), and consider a linear space
\[ S_{1,1}(u) = \bigoplus_{n=0}^{\infty} S_{1,1}(u). \]

We set \( S_{1,0}(u) = \mathbb{C}, S_{1,1}(u) = (x - u)^{-1}\mathbb{C}[z^\pm] \). For \( n \geq 2 \), \( S_{1,n}(u) \) is the space of all rational functions of the form
\[
F(x_1, \ldots, x_n) = \frac{f(x_1, \ldots, x_n)}{\prod_{1 \leq i < j \leq n} (x_i - x_j)^2 \prod_{i=1}^n (x_i - u)},
\]
\[ f(x_1, \ldots, x_n) \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]^{\mathbb{Z}_0}, \]
such that they satisfy both the wheel condition (3.1) and an additional wheel condition
\[ f(u, q_2 u, x_3, \ldots, x_n) = 0. \quad (3.2) \]

In what follows, we denote the element \( 1 \in S_{1,0}(u) = \mathbb{C} \) by \( 1 \).

For \( F \in S_{1,m} \) and \( G \in S_{1,n} \), we set
\[
(F^*G)(x_1, \ldots, x_{m+n}) = \text{Sym}[F(x_1, \ldots, x_m)G(x_{m+1}, \ldots, x_{m+n})]
\]
\[ \prod_{1 \leq i \leq m} \omega(x_{m+i}, x_i) \prod_{i=1}^n \phi(u, x_i), \]
\[
(G^*F)(x_1, \ldots, x_{m+n}) = \text{Sym}[G(x_{m+1}, \ldots, x_{m+n})F(x_1, \ldots, x_m)
\]
\[ \prod_{1 \leq i \leq m} \omega(x_i, x_{m+i})], \]
where \( \phi(u, z) \) is given in (2.19). With this definition, by proposition 3.1, \( S_{1,1}(u) \) is an \( \mathcal{E}_i \)-bimodule. Later we will prove that \( S_{1,1}(u) \) is a cyclic bimodule, and \( 1 \) is a cyclic vector (see corollary 3.5).
Clearly, there is an action of multiplication by symmetric Laurent polynomials on both $S_{h_0}$ and $S_{h_0^1}$. In the case of $S_{h_0}$, this action is pulled back by the isomorphism $\sigma: E_\infty \to S_{h_0}$ to the adjoint action $\text{ad}(h_r) = [h_r, \cdot]$ on $E_\infty$ (see (2.1)):

$$\text{ad}(h_r)F = -\left(-\frac{1}{r} \sum_{i=1}^{r} x_i^r\right) F, \quad F \in S_{h_0,n}. \quad (3.3)$$

(In the left-hand side, $\text{ad}(h_r)$ means $\sigma \circ \text{ad}(h_r) \circ \sigma^{-1}$.)

We upgrade the left $E_\infty$ action to make $S_{h_0^1}$ a left $E'$ module of level $(1,q)$.

**Proposition 3.2.** The following formula defines a left action $\varpi$ of $E'$ on $S_{h_0^1}$.

$$\varpi\left(\varepsilon_k\right)F = c_k x^k F, \quad \varpi\left(h_r\right)F = \left(-\frac{1}{r} \sum_{i=1}^{n} x_i^r + \gamma_r\right) F, \quad \varpi\left(\varepsilon_k\right)F = c_2 n (\text{res}_{z=0} + \text{res}_{z=\infty}) \frac{F(x_1, \ldots, x_{n-1}, z) z^k dz}{\prod_{i=1}^{n-1} \omega(z, x_i)} z.$$

Here $F \in S_{h_0^1}(u), k \in \mathbb{Z}, r \in \mathbb{Z}\setminus\{0\}, \gamma_r$ is defined in (2.20), $c_1 = q_1/(1-q_1)(1-q_3)$ and $c_2 = q_3^{-1}/(1-q_2)$. This action commutes with the right action of $E_\infty$.

**Proof.** The proof is done by a direct computation. As an example we sketch the verification of the relation $[\varepsilon(z), f(w)] = (1/k_1) \delta(z/w)(\psi^+(z) - \psi^-(z))$. Let $F \in S_{h_1,\sigma}(u), k, l \in \mathbb{Z}$.

From the above definition we deduce that

$$\varpi\left(\left[\varepsilon_{k}, f_l\right]\right)F = -\frac{1}{k_1} \times \left(\text{res}_{z=0} + \text{res}_{z=\infty}\right) \prod_{i=1}^{n} \frac{\omega(x_i, z)}{\omega(z, x_i)} \cdot \phi(u, z) \cdot z^{k+l} \frac{dz}{z} \times F.$$

Comparing this with the expansions at $z^{\pm 1} \to \infty$

$$\frac{\omega(x, z)}{\omega(z, x)} = \exp\left(-\sum_{2g>0} \frac{1}{\kappa_g} x^g z^{-\gamma}\right), \quad \phi(u, z) = q^{z+1} \exp\left(\sum_{2g>0} \kappa_g \gamma_g z^{-\gamma}\right),$$

we obtain the desired relation. The rest of the relations can be checked similarly. In particular, the cubic Serre relations follow from the identity

$$\text{Sym} \frac{X_2}{x_1 x_2 x_3 x_4} (\omega_{3,1} \omega_{3,2} \omega_{2,1} - \omega_{3,1} \omega_{2,3} \omega_{2,1} - \omega_{1,3} \omega_{1,2} \omega_{3,2} + \omega_{1,2} \omega_{1,3} \omega_{2,3}) = 0,$$

where $\omega_{i,j} = \omega(x_i, x_j)$. \hfill \Box

### 3.3. Functional realization of the Fock module

We consider the following left $E'$ submodule of $S_{h_1}$(u),

$$J_0 = \text{Span}_c \left\{ G^* F \mid G \in S_{h_1}(u), F \in S_{h_0,n}, n \geq 1 \right\} \subset S_{h_1}(u). \quad (3.5)$$

The following gives a realization of the Fock module as a quotient of a space of rational functions.
Proposition 3.3. We have the isomorphism of left $\mathcal{E}'$ modules $Sh_1(u)/J_0 \cong \mathcal{F}(u)$.

Proof. The module $Sh_1(u)/J_0$ contains the lowest weight vector $1$ with the same lowest weight (2.19) as the Fock module. Hence, in order to prove the isomorphism, it is sufficient to show that each of its graded component has the same dimension as that of the Fock module. We show this in the appendix, corollary A2. □

Therefore we have the canonical projection map $\pi$.

Corollary 3.4. There exists a unique surjective homomorphism of left $\mathcal{E}'$ modules

$$\pi: Sh_1(u) \to \mathcal{F}(u), \quad 1 \to |\varnothing\rangle$$

which factorizes through $Sh_1(u)/J_0$.

3.4. The subspace $N$

We have a short exact sequence of left $\mathcal{E}'$ modules

$$0 \to J_0 \to Sh_1(u) \to \mathcal{F}(u) \to 0.$$ 

In this section, we split this sequence in the category of vector spaces. The reason for the choice of this particular splitting will be clarified later (see (4.1)).

Define a linear map $\kappa: \mathcal{F}(u) \to Sh_1(u)$ by the requirements $\kappa(|\varnothing\rangle) = 1$ and

$$\kappa\left(h_{-r}^+(v)\right) = h_{-r}^+\kappa(v) - q'\kappa(v)h_{-r}^+,$$

for all $r > 0$ and $v \in \mathcal{F}(u)$. Here we use the bimodule action of $h_{-r}^+ \in \mathcal{E}_\mathbb{C}^\times$.

Since $\mathcal{F}(u)$ is cyclic with respect to the algebra generated by $h_{-r}^+$, $r > 0$, the map $\kappa$ is uniquely defined. We clearly have $\pi\kappa = \text{id}$ and in particular, $\kappa$ is injective. Let

$$N = \kappa(\mathcal{F}(u)) \subset Sh_1(u).$$

We clearly have a direct sum of vector spaces

$$Sh_1(u) = J_0 \oplus N.$$  

From proposition 3.3 we obtain the following.

Corollary 3.5. The space $Sh_1(u)$ is a cyclic $\mathcal{E}_\mathbb{C}^\times$ bimodule with cyclic vector $1$.

The subspace $N$ has a curious description in terms of regularity conditions.

We call a function $G(x_1,\ldots,x_n) \in Sh_1(u)$ regular at zero if there exists a well-defined limit

$$\lim_{r \to 0} G(t_{y_1},\ldots,t_{y_k}, x_{k+1},\ldots,x_n), \quad k = 1,\ldots,n.$$ 

We call a function $G(x_1,\ldots,x_n) \in Sh_1(u)$ regular at infinity if there exists a well-defined limit

$$\lim_{r \to \infty} G(t_{y_1},\ldots,t_{y_k}, x_{k+1},\ldots,x_n), \quad k = 1,\ldots,n.$$ 

Proposition 3.6. A function $G(x_1,\ldots,x_n) \in Sh_1(u)$ belongs to $N$ if and only if it is regular at zero, regular at infinity and

$$\lim_{r \to 0} G(t_{y_1},\ldots,t_{y_i}, y_{i+1},\ldots,y_n) = 0 \quad \text{for} \quad 1 \leq i \leq n.$$
Proof. It is known [Ng], that a function \( F(x_1, ..., x_m) \in S_{h_{10}, ...} \) belongs to the commutative algebra generated by \( h_{r_1}, ..., h_{r_n} \), if and only if it is regular at zero and at infinity. Then it is easy to check that action (3.7) preserves the regularity and vanishing conditions described in the proposition. It is easy to check that action (3.7) preserves the regularity conditions at zero and at infinity. Noting \( \lim \rightarrow q(u, \tau) = q \) we see further that the vanishing condition is also preserved. Since \( I \) satisfies these conditions, we obtain the only if part. For the if part, we compute the dimension of the space of functions using the same filtration as in [Ng]. □

We remark that if one defined a map \( \tilde{k}: \mathcal{F}(u) \rightarrow S_{h_1}(u) \) by changing \( q \) to \( q^{-1} \) in (3.7) then the image of \( \tilde{k} \) would consist of functions \( G(x_1, ..., x_n) \in S_{h_1}(u) \) which are regular at zero, regular at infinity and satisfy \( \lim_{t \rightarrow 0} G(ty_1, ..., ty_i, y_{i+1}, ..., y_n) = 0 \) for \( 1 \leq i \leq n \).

4. The subspace of matrix elements of \( L \) operators

In this section we construct an inclusion of bimodule \( S_{h_1}(u) \) to a completion of algebra \( \mathcal{E}_G \). Under this inclusion the subspace \( N \subset S_{h_1}(u) \), see (3.8), has a description in terms of matrix elements of \( L \) operators.

From now on we consider only modules of level \( (1, K) \) with some \( K \). We work with the quotient algebra \( \mathcal{E}/(C - 1) \), and denote it by the same letter \( \mathcal{E} \).

4.1. The matrix elements of \( L \) operators

Let \( R \) be the universal \( R \) matrix (2.16). Since we set \( C = 1 \), we have \( R^{(0)} = q^{-c} \otimes q^{c} - q^{c} \otimes q^{-c} \).

For bounded quasi-finite modules \( V, W \), \( R \) gives a well-defined operator on a tensor product \( V(u_1) \otimes W(u_2) \) for generic \( u_1, u_2 \).

For \( v \in \mathcal{F}(u) \) and \( w \in \mathcal{F}(u)^* \), let

\[
L_{w,v} = (1 \otimes w)R(1 \otimes v)
\]

denote the matrix element of \( R \) with respect to the second component. We call elements \( L_{w,v} \) matrix elements of \( L \) operators.

We are mostly concerned with the case \( w = (\emptyset) \). In what follows we abbreviate \( (\emptyset), (\emptyset) \) simply as \( \emptyset \) in the index of the matrix elements of \( L \) operators.

If a coproduct of an element of \( \mathcal{E} \) is known, one can compute its commutation relations with the matrix elements of \( L \) operators. In particular, we have the following commutation relations with perpendicular generators which involve only matrix elements of \( L \) operators with \( w = (\emptyset) \).

Lemma 4.1. For all \( r, n > 0 \) and \( v \in \mathcal{F}(u) \), we have

\[
\left[ h_{r}^+, L_{\emptyset,v} \right]_{q^r} = L_{\emptyset,h_{r}^+,v},
\]

(4.1)

\[
\left[ e_{n}^+, L_{\emptyset,v} \right]_{q^n} = L_{\emptyset,e_{n}^+,v} + q^{-n} \sum_{j \neq 1} L_{\emptyset,e_{n+j}^+,v} \cdot e_{n-j}^+, \quad (4.2)
\]

In addition we have

\[
\left[ e_{0}^+, L_{\emptyset,v} \right] = L_{\emptyset,e_{0}^+,v} - \gamma_1 L_{\emptyset,v} + \sum_{j \neq 1} L_{\emptyset,e_{j}^+,v} \cdot e_{1-j}^+, \quad (4.3)
\]
\[
\left[ f_{j}^{+}, L_{\emptyset, v} \right] = L_{\emptyset, f_{j}^{+} v} - \gamma_{-1} L_{\emptyset, v} + \sum_{j \geq 1} L_{\emptyset, f_{j}^{+} v} \cdot \psi_{j}^{-1}. \tag{4.4}
\]

Here we set \([A, B]_{p} = AB - pBA\).

**Proof.** The element \(R\) has the intertwining property
\[
R\left( h_{r}^{+} \otimes 1 + \left( C^{-1}\right)^{-r} \otimes h_{r}^{+} \right) = \left( q^{-1} h_{r}^{+} \otimes 1 + 1 \otimes h_{r}^{+} \right) R,
\]
\[
R\left( q^{r} h_{-r}^{+} \otimes 1 + 1 \otimes h_{-r}^{+} \right) = \left( h_{-r}^{+} \otimes 1 + \left( C^{-1}\right)^{-r} \otimes h_{-r}^{+} \right) R,
\]
\[
R\left( q^{r} e_{n}^{+} \otimes 1 + 1 \otimes e_{n}^{+} + q^{n} \sum_{j \geq 1} e_{n-j}^{+} \otimes \psi_{j}^{+1} \right)
\]
\[
= \left( e_{n}^{+} \otimes 1 + \left( C^{-1}\right)^{n} \otimes e_{n}^{+} + \sum_{j \geq 1} \left( C^{-1}\right)^{n} \otimes e_{n-j}^{+} \psi_{j}^{+1} \right) R,
\]
\[
R\left( f_{n}^{+} \otimes 1 + \left( C^{-1}\right)^{n} \otimes f_{n}^{+} + \sum_{j \geq 1} \left( C^{-1}\right)^{n} \otimes f_{n+j}^{+} \psi_{j}^{-1} \right)
\]
\[
= \left( q^{n} f_{n}^{+} \otimes 1 + 1 \otimes f_{n}^{+} + q^{n} \sum_{j \geq 1} f_{n+j}^{+} \otimes \psi_{j}^{-1} \right) R,
\]

where \(r > 0\) and \(n \in \mathbb{Z}\). Taking the matrix element between \(\langle \emptyset | \) and \(v\) in the second component, we obtain the lemma. \(\square\)

The above intertwining relations allow us to compute \(L_{\emptyset, \emptyset}\) explicitly.

**Proposition 4.2.** The element \(L_{\emptyset, \emptyset}\) satisfies the commutation relations
\[
(z - u) e(z) L_{\emptyset, \emptyset} = \left( q^{-1} z - qu \right) L_{\emptyset, \emptyset} e(z), \tag{4.5}
\]
\[
\left( q^{-1} z - qu \right) f(z) L_{\emptyset, \emptyset} = (z - u) L_{\emptyset, \emptyset} f(z). \tag{4.6}
\]
\[
\left[ h_{r}, L_{\emptyset, \emptyset} \right] = 0 \quad (\forall r \neq 0). \tag{4.7}
\]

Explicitly we have
\[
L_{\emptyset, \emptyset} = q^{-d_{r}} \exp \left( \sum_{r=1}^{\infty} \left( 1 - q_{z}^{-r} \right) h_{r} u^{-r} \right). \tag{4.8}
\]

**Proof.** In lemma 4.1, we consider the case \(v = \langle \emptyset \rangle\). We need two more formulas derived similarly:
Using

\[ e_0^\perp = h_1, \quad e_i^\perp = C^\perp f_i, \quad e_{-i}^\perp = e_i, \quad h_1^\perp = f_0, \]

\[ f_0^\perp = h_{-1}, \quad f_i^\perp = f_{-i}, \quad f_{-i}^\perp = (C^\perp)^{-1} e_{-i}, \quad h_{-1}^\perp = e_0, \]

and the relations

\[ e_1^\perp \langle \varnothing \rangle = u h^\perp_1 \langle \varnothing \rangle, \quad f_{-1}^\perp \langle \varnothing \rangle = (qu)^{-1} h^\perp_{-1} \langle \varnothing \rangle, \]

\[ \langle \varnothing \rangle e_1^\perp = qu \langle \varnothing \rangle h^\perp_1, \quad \langle \varnothing \rangle f_{-1}^\perp = u^{-1} \langle \varnothing \rangle h^\perp_{-1}, \]

which follow from (2.22)–(2.23), we find

\[
\begin{bmatrix}
  h_1, \quad L_{\varnothing, \varnothing}
\end{bmatrix} = \begin{bmatrix}
  h_{-1}, \quad L_{\varnothing, \varnothing}
\end{bmatrix} = 0, \\
( e_1 - u e_0 ) L_{\varnothing, \varnothing} = L_{\varnothing, \varnothing} \left( q^{-1} e_1 - q u e_0 \right), \\
( q^{-1} f_1 - q u f_0 ) L_{\varnothing, \varnothing} = L_{\varnothing, \varnothing} ( f_1 - u f_0 ).
\]

Taking commutators between the last two lines and \( h_{\pm 1} \), we obtain (4.5) and (4.6). Furthermore, (4.5) and (4.6) imply that

\[ (z - u) \left( q^{-1} z - q u \right) \delta(z/w) \left[ \psi^+(z) - \psi^-(z), \quad L_{\varnothing, \varnothing} \right] = 0. \]

Let \( [\psi^+(z) - \psi^-(z), \quad L_{\varnothing, \varnothing}] = \sum_{j \in \mathbb{Z}} X_j z^{-j}. \) Then

\[ q^{-1} X_{j+1} - \left( q + q^{-1} \right) u X_j + qu^2 X_{j-1} = 0. \]

We have \( X_0 = 0 \), and we already know that \( X_{\pm 1} = 0 \). From this follows (4.7).

The unique element in the completion of \( \mathcal{E}_\perp \) with respect to the homogeneous degree \( \text{hdeg} \) when it becomes large satisfying (4.5)–(4.7) is given by (4.8). The lemma follows.

We denote by \( \hat{\mathcal{E}}_\perp \) the completion of algebra \( \mathcal{E}_\perp \) with respect to the homogeneous degree \( \text{hdeg} \) when it becomes large.

**Corollary 4.3.** We have \( L_{\varnothing, v} \in \hat{\mathcal{E}}_\perp \) for all \( v \in \mathcal{P}(u) \).

**Proof.** We have \( L_{\varnothing, \varnothing} \in \hat{\mathcal{E}}_\perp \) from (4.8). Since \( h_{-r}^\perp \in \mathcal{E}_\perp \) for \( r > 0 \), the corollary follows from (4.1).

We denote by \( \mathcal{N} \) the space of matrix elements of \( L \) operators with the first component \( \langle \varnothing \rangle \):

\[ \mathcal{N} = \text{Span}_C \left\{ L_{\varnothing, v} | v \in \mathcal{P}(u) \right\} \subset \hat{\mathcal{E}}_\perp. \]
4.2. Inclusion of the shuffle algebra to $\hat{E}_\mathfrak{g}$

Consider the $E_\mathfrak{g}$ bimodule

$$S(u) = E_\mathfrak{g} \cdot L_{\mathfrak{g},0} \cdot E_\mathfrak{g} \subset \hat{E}_\mathfrak{g}.$$ 

Lemma 4.4. There exists an isomorphism of $E_\mathfrak{g}$ bimodules

$$\iota: Sh_1(u) \rightarrow S(u), \quad 1 \mapsto L_{\mathfrak{g},0}. \quad (4.9)$$

Proof. Let $G = \sum_j F_j^* \mathbf{1} \ast H_j$ be an element of $Sh_1(u)$, and let $a_j, b_j \in E_\mathfrak{g}$ be the elements corresponding to $F_j, H_j \in Sh_0$, respectively. We set

$$\iota(G) = \sum_j a_j L_{\mathfrak{g},0} b_j.$$ 

We must show that this definition does not depend on the presentation of $G$. To see this, suppose that the right-hand side is 0. In the completion of $E_\mathfrak{g}$ we have the commutation relation following from (4.8),

$$e_n L_{\mathfrak{g},0} = L_{\mathfrak{g},0} e_n, \quad \bar{e}_n = q e_n + (q - q^{-1}) \sum_{j \geq 1} a_j^j e_{n+j}.$$ 

Using this we move the $a_j$’s to the right and obtain

$$0 = \sum_j a_j L_{\mathfrak{g},0} b_j = \sum_j L_{\mathfrak{g},0} \bar{a}_j b_j,$$ 

where $\bar{a}_j$ is obtained from $a_j$ by substituting $e_n$ by $\bar{e}_n$. Since $L_{\mathfrak{g},0}$ is invertible, this implies that $\sum_j \bar{a}_j b_j = 0$. We may assume that $G$ has principal degree, say, $m$. Let $\tilde{G}$ be the element of the completion of $Sh_0$ corresponding to $\sum_j \bar{a}_j b_j$. Then we observe that $\prod_{1 \leq i < j \leq m} (x_i - x_j)^2 \tilde{G}$ is nothing but the expansion of the rational function $\prod_{1 \leq i < j \leq m} (x_i - x_j)^2 G$ at $x_1 = \cdots = x_m = 0$. Hence we obtain $G = 0$, thereby showing that $\iota$ is well defined. The map $\iota$ is clearly surjective, and the above argument shows that it is also injective. \hfill $\Box$

Then we have the identifications of spaces $N, \mathcal{N}$ and $F(u)$.

Lemma 4.5. We have $\iota(N) = \mathcal{N} \subset S(u)$. For any $v \in F(u)$, we have

$$\pi^{-1}(L_{\mathfrak{g},v}) = v \in F(u).$$

Proof. We have $\pi^{-1}(L_{\mathfrak{g},0}) = |\mathfrak{g}| \in F(u)$. The module $F(u)$ is cyclic with respect to the action of the $h^+_{\mathfrak{g}}$’s. Formula (4.1) implies that $h^+_{\mathfrak{g}} L_{\mathfrak{g},v} \equiv L_{\mathfrak{g},h^+_{\mathfrak{g}}} v$ holds modulo right action. Since the map $\iota$ is $E_\mathfrak{g}$ linear and $h^+_{\mathfrak{g}} \in E_\mathfrak{g}$, we obtain the assertion. \hfill $\Box$

We capture various maps on figure 3.
5. Bethe ansatz

5.1. Integrals of motion

Let $W$ be a bounded quasi-finite module. We have $W = \bigoplus_{n \in \mathbb{Z}} W_n$, and $d^2|_{W_n} = n$. Fixing a parameter $p \in \mathbb{C}$, consider the weighted trace

$$T(u; p) = \text{Tr}_{W(u); 2} \left( p^1 \Phi^1 R \right) = \sum_{n \in \mathbb{Z}} \left( pq^{-1} \right)^n \text{Tr}_{W(u); 2} \left( q^{-d} \Phi^1 R^{(1)} \right),$$

where $R$ is defined in subsection 4.1, and $\text{Tr}_{W(u); 2}$ signifies the trace in the second tensor component.

Note that the first tensor component of $\Phi^{(1)}$ (2.17) has the homogeneous degree 0, and that of $R^{(2)}$ (2.18) has non-negative homogeneous degree. Therefore, the operator $T(u; p)$ has the form

$$\sum_{l = 0}^{\infty} T_{W(u); l} \left( u^{-1} \right)^{l},$$

where $T_{W(u); l}$ is an operator on bounded quasi-finite modules and $\text{deg} T_{W(u); l} = (0, l)$.

We introduce the integrals of motion $I_{W; l}(p)$ by

$$\log \left( T_{W,0}(p)^{-1} T_{W(u); 2} \right) = \sum_{n = 1}^{\infty} I_{W; l}(p) u^{-l}.$$

For fixed $p$, the integrals of motions form a commutative family:

$$\left[ T_{W; l_1}(p), T_{W; l_2}(p) \right] = 0,$$

for all $l_1, l_2 \in \mathbb{Z}_{\geq 0}$ and all bounded quasi-finite modules $W_1, W_2$.

When $W(u)$ is the Fock module $F(u)$, we have the following expression for $I_{F; l}(p)$.

**Lemma 5.1.** Set $\tilde{\rho} = pq^{-1}$. Then the operator $I_{F; l}(p)$ is given by the coefficient of $z^0$ of the twisted current

$$-(q_1^{-1} q_3^{-1}, \tilde{\rho}; \tilde{\rho})_{\infty} \prod_{j=1}^{\infty} q^{+j} \left( \tilde{\rho}^{-j} q^{-1} z \right) \cdot e^1(z),$$

where $(a_1, \ldots, a_m; \tilde{\rho})_{\infty} = \prod_{j=1}^{m} (a_j; \tilde{\rho})_{\infty}$, $(a; \tilde{\rho})_{\infty} = \prod_{j=0}^{\infty} (1 - q^j \tilde{\rho}).$
Proof. From the definition of $T_r(u, p)$ along with (2.17), (2.18), we obtain

$$
T_{r,0}(p) = q^{-d^i} \text{Tr}_{F(u),2}\left[\rho^{1\otimes d^i} \mathcal{R}^{(1)}\right],
$$

$$
T_{r,1}(p) = q^{-d^i} \kappa_1 \text{res}_{z=0} \text{Tr}_{F(u),2}\left[\rho^{1\otimes d^i} \mathcal{R}^{(1)} \cdot 1 \otimes f^-(z) \right] e^-(z) \frac{dz}{z}.
$$

We then substitute (2.23) for $f^-(z)$. The trace can be calculated by using

$$
\text{Tr}_{F(u),2}\left[\rho^{1\otimes d^i} \exp\left(\sum_{r=1}^{\infty} A_r 1 \otimes h^+_{-r}\right) \exp\left(\sum_{r=1}^{\infty} B_r 1 \otimes h^+_{-r}\right)\right] = \frac{1}{(\rho; \bar{\rho})_\infty} \exp\left(\sum_{r=1}^{\infty} A_r B_r \frac{\bar{\rho}^r q^r - q^{-r}}{1 - \bar{\rho}^r \rho^r}ight),
$$

where we set

$$
A_r = r \kappa, h^+_{-r} \otimes 1 - \frac{q^r \kappa_r}{1 - q^{2r} z^{-r}}, \quad B_r = -\frac{q^{2r} \kappa_r}{1 - q^{2r} z^{-r}}.
$$

Note that

$$
\exp\left(\sum_{r=1}^{\infty} \kappa_r \frac{\bar{\rho}^r q^r}{1 - \bar{\rho}^r \rho^r} h^+_{-r} z^{-r}\right) = \prod_{j>1} \psi^{-1}\left(\bar{\rho}^{-j} q^{-1} z\right).
$$

After simplification we find

$$
T_{r,0} = q^{-d^i} \frac{1}{(\rho; \bar{\rho})_\infty}, \quad T_{r,1} = u^{-1} q^{-d^i} \left(1 - q_1 q_3\right) \frac{\left(\bar{\rho} q_2; \bar{\rho}\right)_\infty}{\left(\bar{\rho} q_1^{-1}, \bar{\rho} q_3^{-1}; \bar{\rho}\right)_\infty} \tilde{e}_0^+(p).
$$

This completes the proof. \qed

Using the commutation relation

$$
\prod_{j>1} \psi^{-1}\left(\bar{\rho}^{-j} q^{-1} z\right) \cdot e^+(z) = \prod_{i=1}^{3} \frac{\left(\bar{\rho} q_i^{-1} q_1^{-1} z^{-i}; \bar{\rho}\right)_\infty}{\left(\bar{\rho} q_i, \bar{\rho} q_1; \bar{\rho}\right)_\infty} \times e^+(z) \cdot \prod_{j>1} \psi^{-1}\left(\bar{\rho}^{-j} q^{-1} z\right)
$$

we can bring $e^+(z)$ to the left in (5.1). When $c^1 = 1$ we obtain

$$
I_{r,1}(p) = k \cdot \tilde{e}_0(p), \quad k = \left(\frac{q_1 q_3, \bar{\rho}; \bar{\rho}}{\bar{\rho} q_1, \bar{\rho} q_3; \bar{\rho}}_\infty\right), \quad \text{(5.2)}
$$

where $\tilde{e}_0(p)$ is the coefficient of $z^0$ in

$$
e^+(z; p) = e^+(z) \prod_{j=1}^{\infty} \psi^{+1}\left(\bar{\rho}^{-j} q^{-1} z\right). \quad \text{(5.3)}
$$

More generally, if $W$ is a tensor product of several Fock modules, the operators $\{h_{W,n}(p)\}_{n=1}^{\infty}$ are closely related to the commutative family of operators introduced and studied in [FKSW, FKSW2, KS].
5.2. Action of $\tilde{\epsilon}^\perp_0 (p)$ on Fock module as a projection

In this subsection we identify the action of the integral of motion $\tilde{\epsilon}^\perp_0 (p)$ on the Fock module with a projection of operator $h_1$ acting in $Sh_1(u)$ to $N$, along the space $J_p$.

We fix $p \in \mathbb{C}^*$ and consider the subspace of $Sh_1(u)$

$$J_p = \text{Span}_\mathbb{C}\{ G^* F - p^n F^* G | G \in Sh_1(u), F \in Sh_{0,n}, n \geq 1 \} \subset Sh_1(u).$$

Unlike (3.5), it is not an $Sh_0$ submodule. However, it is clearly preserved by the action of $h_r$, see (3.4). From (3.9), we see that for generic $p$, we have a direct sum of vector spaces

$$Sh_1(u) = J_p \oplus N. \quad (5.5)$$

Denote $\pi : Sh_1(u) \rightarrow N$ the projection operator in (5.5) along the first summand.

Recall that for $G(x_1,\ldots,x_n) \in Sh_{1,n}(u)$, the left action of $h_r$ is simply given by

$$\varpi(h_r)G(x_1,\ldots,x_n) = \left(-\frac{1}{r} \sum_{i=1}^n x_i^r + \chi \right) G(x_1,\ldots,x_n), \quad (5.6)$$

see (3.4). Let $L_0 = \sum_j a_j L_0 b_j \ (a_j, b_j \in \mathcal{E}_{\gamma})$ be an element of $S(u)$. Since $[h_r, L_0] = 0$, we have in the completed algebra $\hat{\mathcal{E}}_{\gamma}$

$$\left[ h_r, L_0 \right] = \sum_j \left[ h_r, a_j \right] L_0 b_j + \sum_j a_j L_0 \left[ h_r, b_j \right].$$

In view of (3.3), the right-hand side corresponds to the multiplication by $-1/r \sum_{i=1}^n x_i^r$. Therefore the left action of $h_r$ on $S(u)$ can be written as (suppressing $i$ for simplicity of notation)

$$\varpi(h_r) L_0 = [h_r, L_0] + \gamma L_0.$$

The crucial observation is that the projection of this simple operator to $N$ along $J_p$ produces the desired integral of motion $\tilde{\epsilon}^\perp_0 (p)$.

**Theorem 5.2.** Under the identification of $N$ and $F(u)$, we have $\pi_1 h_1 = \tilde{\epsilon}^\perp_0 (p)$. In other words, for any $v \in F(u)$ we have

$$\tilde{\epsilon}^\perp_0 (p) v = \left( \kappa^{-1} \circ \pi \right) (h_1 \kappa(v)).$$

**Proof.** We use the isomorphism $\iota$, see (4.9) and lemma 4.5. We work in $S(u) \subset \hat{\mathcal{E}}_{\gamma}$ and make use of the matrix elements of $L$ operators to compute the projection.

By (4.2) and (4.3) we have

$$[e_0^\perp, L_0, v] + \gamma L_0, v = L_0, e_0^\perp v + \sum_{j \geq 1} L_0, e_{j+1}^\perp v \cdot e_{-j}^\perp \equiv L_0, e_0^\perp v + \sum_{j \geq 1} p^j L_0, e_{j+1}^\perp v \mod J_p,$$
and for \( n > 0 \)
\[
e_{-n}^+ L_{\varnothing, v} = L_{\varnothing, e_{-n}^+ v} + q^{-n} \sum_{j \geq 0} L_{\varnothing, w_j^{+1} v} \cdot e_{j-n}^+ = L_{\varnothing, e_{-n}^+ v} + \sum_{j \geq 0} q^{-n} p^{j+n} e_{j-n}^+ L_{\varnothing, w_j^{+1} v} \mod J_p.
\]
Iterating the latter, we obtain
\[
\sigma(h) L_{\varnothing, v} = \left[ e_0^+, L_{\varnothing, v} \right] + \eta L_{\varnothing, v} = L_{\varnothing, e_0^+ v} + \sum_{j \geq 1} q^{l_1+\cdots+l_j} \frac{1}{l_1! \cdots l_j!} \left( pq^{-1} \right)^{l_1} \cdots \left( pq^{-1} \right)^{l_j} 
\times L_{\varnothing, e_{l_1}^{+1} \cdots e_{l_j}^{+1} w_j^{+1} w_j^{+1} v} \mod J_p.
\]
where we used definition (5.3).

\[\square\]

### 5.3. Bethe ansatz

Theorem 5.2 immediately leads to Bethe ansatz statements for the dual module.

Given a quasi-finite left \( \mathcal{E} \) module \( V = \bigoplus_{n=0}^{\infty} V_n \), we consider the graded dual space \( \mathcal{V} = \bigoplus_{n=0}^{\infty} \operatorname{Hom}(V_n, \mathbb{C}) \). As usual, \( V^* \) is a right \( \mathcal{E} \) module with action given by \( g f(v) = f(gv) \), \( g \in \mathcal{E}, v \in V, f \in V^* \). Note that the spectrum of any operator and of the dual operator coincide. Note also that the dual to a lowest weight Fock left module is a highest weight Fock right module.

For a point \( a = (a_1, \..., a_n) \in \mathbb{C}^n \) such that \( a_i \neq a_j \) (\( i \neq j \)) and \( a_i \neq u \), we denote by \( e_{ui} \) the evaluation map \( e_{ui} : \operatorname{Sh}_1(u) \to \mathbb{C} \) defined by
\[
e_{ui}(F(x_1, \..., x_n)) = F(a_1, \..., a_n), \quad F(x_1, \..., x_n) \in \operatorname{Sh}_1(u),
\]
and \( e_{ui}(\operatorname{Sh}_{m}(u)) = 0 \) for \( m \neq n \).

**Lemma 5.3.** We have \( e_{ui}(J_p) = 0 \) if and only if \( a = (a_1, \..., a_n) \) satisfies the Bethe equations
\[
1 = q^{-p} \cdot \frac{a_i - q^{-1} a_i}{a_i - u} \prod_{j \neq i} \frac{(a_j - q a_i)(a_j - q_2 a_i)(a_j - q_3 a_i)}{(a_j - q^{-1} a_i)(a_j - q_2^{-1} a_i)(a_j - q_3^{-1} a_i)}, \quad i = 1, \..., n.
\]

**Proof.** Let \( F \in \operatorname{Sh}_{0,m}, G \in \operatorname{Sh}_{1,k}(u) \) and \( m \geq 1, m + k = n \). Then \( p^m F^* G \) and \( G^* F \) by definition are symmetrization and can be compared term-wise so that the substitutions of variables match. These terms become equal for all \( m \) if and only if (5.7) holds.

**Theorem 5.4.** Let \( a = (a_1, \..., a_n) \) be a solution of (5.7) such that \( e_{ui} \) is non-zero. Then the restriction of \( e_{ui} \) to \( N = \kappa(F(u)) \) is an eigenvector in \( F(u)^\kappa \) of the first integral of motion \( \tilde{e}_0^+(p) \) (5.3) with the eigenvalue
\[ E_1(a) = -\sum_{i=1}^{n} a_i + \gamma_i, \]

where we recall that \( \gamma_i = u/(1 - q_1)(1 - q_3) \). For generic \( p, q_1, q_3 \), \( ev_a \) is a joint eigenvector of \( \{ I_{F,u}(p) \}_{p \in \mathbb{C}} \).

**Proof.** When \( p = 0 \), the operator \( I_{F,u}(p) \) on \( F(u) \) has simple spectra if \( q_1, q_3 \) are generic. Hence it is enough to show that \( ev_a \) is an eigenvector of \( \tilde{e}_0^+ (p) \) with eigenvalue \( E_1(a) \).

We simply have for any \( G \in \mathcal{N} \)

\[ \tilde{e}_0^+ (p) ev_a(G) = ev_a \left( \tilde{e}_0^+ (p) G \right) = ev_a \left( h_1 G \right) = E_1(a) ev_a(G), \]

where the second equality follows from theorem 5.2 together with lemma 5.3 and the third equality follows from (5.6).

\[ \square \]

5.4. The off-shell Bethe vector

For us, an off-shell Bethe vector is a vector depending on parameters \( a_i \) such that if \( a_i \) satisfy the Bethe equation, it becomes an eigenvector of the Hamiltonians. However, such a requirement does not determine it uniquely.

An off-shell Bethe vector is obviously given by the formula \((id \otimes ev_a \circ \kappa)K\), where

\[ K = \sum_{\lambda} \frac{\langle \emptyset | h^+_1 \otimes h^+_3 | \emptyset \rangle}{\langle \emptyset | h^+_1 h^+_3 | \emptyset \rangle} \]

is the canonical element of the space \( F^+(u) \otimes F(u) \). Here and after, for a partition \( \lambda = (\lambda_1, \ldots, \lambda_{\ell(\lambda)}) \) we use the notation \( h^+_1 = h^+_2 = \cdots = h^+_\ell, \) etc.

More generally, the off-shell Bethe vector is given by \((id \otimes ev_a)(K_p + (1 \otimes \kappa)K)\), where \( K_p \in F(u)^{\otimes} \otimes J_p \). Our goal is to give an explicit formula in terms of partitions for a suitable choice of \( K_p \).

Let \( \Lambda \) denote the space of symmetric functions in infinitely many variables. For \( \lambda \in \mathcal{P} \), let \( p_1, m_1 \in \Lambda \) be the power sum and the monomial symmetric functions, respectively. Using the rescaled generators

\[ \tilde{h}_r^+ = r \left( 1 - q_1^r \right) q_2^r q_3^r h_r^+, \]

we identify the algebra generated by \( h_r^+, r > 0 \) with \( \Lambda \) by

\[ \nu^+: \Lambda \rightarrow \mathbb{C} h_r^+ \big| r > 0, \quad p_1 \mapsto \tilde{h}_1^+. \]

Introduce further the elements of \( Sh_0 \)

\[ e_\lambda^{(q_i)} = e_\lambda^{(q_1)} \ast \cdots \ast e_\lambda^{(q_i)} , \quad e_\lambda^{(q_i)}(x) = \prod_{i < j} \frac{(x_i - q_3 x_j)(x_i - q_3^{-1} x_j)}{(x_i - x_j)^2}. \]

(5.8)
Since for \( v \in \mathcal{F}(u) \) we have
\[
L_{\emptyset, h^+_v} = \left[ h^+_{-r}, L_{\emptyset, v} \right]_{\text{id}} = (1 - p' q_{2}^{E/2}) h^+_{-r} L_{\emptyset, v} \mod J_p,
\]
we obtain
\[
(id \otimes \kappa) K \simeq \sum_{I} \left< \emptyset | h^+_I \right> \otimes \left( \prod_{i=1}^{(l_i)} (1 - p^i q_{2}^{E/2}) h^+_{-l_i} * 1 \right)
= \sum_{I} \left< \emptyset | a(h^+_I) \otimes \sigma(h^+_I) * 1 \right>.
\]
(5.9)

where \( a \) is an algebra homomorphism given by
\[
a(h^+_I) = (1 - p' q_{2}^{E/2}) h^+.
\]
(5.10)

Recall the isomorphism \( \sigma \simeq \mathcal{S}h_0 \) in proposition 3.1. The following formula is known ([FHSSY], proposition 1.12).
\[
\sum_{I} h^+_I \otimes \sigma(h^+_I) = \sum_{I} \frac{1}{(q_1 - 1)^{\prod_{i=1}^{l_i} l_i!}} \nu^a(m_{\lambda}) \otimes \epsilon^{(q_i)}_{\lambda}.
\]
(5.11)

Combining (5.11) and (5.9), we arrive at the following.

**Theorem 5.5.** An off-shell Bethe vector in \( \mathcal{F}(u)^a \otimes J_p \) is given by
\[
e_{\alpha} = \sum_{I} \frac{1}{(q_1 - 1)^{\prod_{i=1}^{l_i} l_i!}} \nu^a(m_{\lambda}) \otimes \epsilon^{(q_i)}_{\lambda} \times \left< \emptyset | a(\nu^a(m_{\lambda})) \right>.
\]
Here \( \epsilon^{(q_i)}_{\lambda} \) is given by (5.8), \( \nu^a(m_{\lambda}) \) denotes the element of \( \mathcal{F}^a(u) \) corresponding to the monomial symmetric function \( m_{\lambda} \), and \( a \) stands for the substitution (5.10).

### 6. Bethe ansatz in tensor product of Fock modules

The method described above for diagonalizing \( \epsilon^a_n(p) \) in the Fock module works for the case of other highest weight modules with straightforward modifications. Here we give some detail for the case of generic tensor products of Fock modules.

Consider \( V = \mathcal{F}(u_1) \otimes \mathcal{F}(u_2) \otimes \ldots \otimes \mathcal{F}(u_k) \), where \( u_1, \ldots, u_k \in \mathbb{C}^k \) are generic numbers. For generic \( u_1, \ldots, u_k \) this module is well defined and it is a bounded tame irreducible module, see [FFJMM2]. Set \( u = (u_1, \ldots, u_k) \) and denote the lowest weight vector of \( V \) by \( |0\rangle \).

We define the corresponding shuffle algebra \( \mathcal{S}h_1(u) = \bigoplus_{n=0}^{\infty} \mathcal{S}h_{1,n}(u) \). The space \( \mathcal{S}h_{1,n}(u) \) consists of all rational functions of the form
\[
F(x_1, \ldots, x_n) = \frac{f(x_1, \ldots, x_n)}{\prod_{1 \leq i < j \leq n} (x_i - x_j) \prod_{j=1}^{n} (x_i - u_j)}.
\]

where \( f(x_1, \ldots, x_n) \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]^{\mathcal{S}h_0} \).

such that they satisfy both the wheel condition (3.1) and additional wheel conditions

\[ f(u_i, q_2 u_i, x_3, \ldots, x_n) = 0, \quad i = 1, \ldots, k. \]

We denote the element \( 1 \in Sh_{1,0}(u) = C \) by \( 1 \).

Next we set

\[ \phi(u, x) = \prod_{i=1}^k \phi(u_i, x) = \prod_{i=1}^k \frac{q u_i - q^{-1} x}{u_i - x} \]

and define the left and the right action of \( Sh_0 \) on \( Sh_1(u) \) by the same formulas as in the case \( k = 1 \).

Similarly, we extend the left action of \( Sh_0 \) to the left action of \( L' \), define \( \pi \) to be the projection map.

For the definition of \( \kappa \) and the space \( N \), it is not enough to use \( h_\pm \) only since \( V \) is not a cyclic Heisenberg module. So, in addition to (3.7) we impose the condition

\[ \sum_{\kappa} \kappa(v) = -q^n \sum_{j \geq 1} \kappa\left( w_j \right) e^{-n-j}, \]

for all \( v \in V, n > 0 \), cf (4.2). Note that the sum on the right-hand side is finite for any \( v \).

Then \( N \) is well defined and corollary 3.5 still holds.

We define the space of \( p \) commutators \( J_p \) in the same way. And then theorem 5.2 holds just the same. We introduce the Bethe equation

\[ 1 = q^{-p} \prod_{i=1}^k \frac{a_i - q_2 a_j}{a_i - a_j} \cdot \prod_{j \neq i} \frac{(a_j - q_1 a_i)(a_j - q_2 a_i)(a_j - q_3 a_i)}{(a_j - q_1^{-1} a_i)(a_j - q_2^{-1} a_i)(a_j - q_3^{-1} a_i)}, \quad i = 1, \ldots, n. \]

and arrive at the generalization of theorem 5.4.

**Theorem 6.2.** Let \( a = (a_1, \ldots, a_n) \) be a solution of (6.1) such that \( e^\alpha \) is non-zero. Then the restriction of \( e^\alpha \) to \( N = \kappa(V) \) is an eigenvector in \( V^* \) of the first integral of motion \( \tilde{e}_0^\pm (p) \) (5.3) with the eigenvalue
\[ E_i(u) = -\sum_{i=1}^{a} d_i + \frac{\sum_{i=1}^{a} u_i}{(1 - q_1)(1 - q_2)}. \]

For generic \( p \), \( e_{\lambda} \) is a joint eigenvector of \( \{ I_{F,\pi}(p) \}_{\pi=1}^{\infty} \).

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**Appendix. Gordon filtration**

In this section we study the size of the space \( S_{h_1}(u) \) using the technique of the Gordon filtration. Our goal is to prove corollary A2 below.

Let \( n \) be a positive integer, and let \( \lambda \) be a partition such that \( |\lambda| \leq n \). For an element \( F \in S_{h_1,n}(u) \), we introduce an operation of specialization \( \rho_{\lambda}(F) \) as follows.

First we set

\[ \rho_{\lambda}^{(0)}(F)(y_1, \ldots, y_{|\lambda|+1}, \ldots, y_n) = F(x_1, \ldots, x_n) \bigg|_{u_{i+\ldots+i_{|\lambda|}=q_{-i}^{\lambda}y_i}}^{(0)} \]  

The wheel condition (3.1) implies that \( \rho_{\lambda}^{(0)}(F) \) is divisible by the factor

\[ \prod_{1 \leq i < b \leq |\lambda|} \prod_{1 \leq j \leq n-1} \left( q_i^{j-1}y_b - q_i^j y_a \right) \left( q_i^{j-1}y_b - q_i^j y_b \right) \]

\[ \prod_{k=|\lambda|+1}^{n} \prod_{1 \leq i < j \leq |\lambda|-1} \left( x_k - q_i^j y_j \right) \left( x_k - q_i^j y_i \right). \]  \hspace{1cm} (A.1)

Next we set

\[ \rho_{\lambda}^{(1)}(F)(y_2, \ldots, y_{|\lambda|+1}, \ldots, y_n) = \left[ (y_1 - u) \rho_{\lambda}^{(0)}(F)(y_1, \ldots, y_{|\lambda|+1}, \ldots, y_n) \right]_{y_1=u}. \]

Then \( \rho_{\lambda}^{(1)}(F) \) is divisible further by the factor

\[ \prod_{k=|\lambda|+1}^{n} (x_k - q_2^k u) \]

due to the wheel conditions (3.1) and (3.2). For \( i \geq 2 \), we remove a factor contained in (A.1) to define
\[ \rho_1^{(i)}(F) \left( y_{i+1}, \ldots, y_{\ell(i)}, x_{\ell(i)+1}, \ldots, x_n \right) = \left[ (y_i - q_3^{-1}u)^{i+1} \rho_1^{(i-1)}(F) \left( y_i, \ldots, y_{\ell(i)}, x_{\ell(i)+1}, \ldots, x_n \right) \right] \bigg|_{y_i=q_3^{-1}u}. \]

Finally we set \( \rho_1(F) = \rho_1^{(i,j)}(F) \).

At each step, the wheel condition produces further factors. Collecting them together, we find that

\[ \rho_1(F) \in Sh_{0,n-\ell} \times \prod_{k=\ell+1}^n \prod_{(i,j) \in \lambda} \omega(x_k, q_3^{-1}q_1^{-1}u) \times \frac{1}{\prod_{(i,j) \in CC(\lambda)} (x_k - q_3^{-1}q_1^{-1}u)} \quad (A.2) \]

Let \( P_{\lambda,n} \) denote the set of all partitions \( \lambda \) with \( |\lambda| \leq n \). Define a total ordering \( > \) on \( P_{\lambda,n} \) by setting \( \mu > \lambda \) iff there is a \( k \) such that \( \mu_i = \lambda_i, \ldots, \mu_{k-1} = \lambda_{k-1}, \mu_k > \lambda_k \). We introduce a decreasing filtration \( \{ V_{n,\lambda} \}_{\lambda \in P_{\lambda,n}} \) on the space \( V_n = Sh_{1,n}(u) \) by setting

\[ V_{n,\lambda} = \bigcap_{\mu > \lambda} \ker \rho_{\mu} \subset V_n. \]

**Proposition A1.** If \( |\lambda| > m \) then \( \rho_1(Sh_{1,m}(u) \ast Sh_{0,n-m}) = 0. \)

If \( |\lambda| = m \) then \( \rho_1(Sh_{1,m}(u) \ast Sh_{0,n-m}) = \rho_1(V_{n,\lambda}). \)

If \( |\lambda| = n \) then \( \rho_1(V_{n,\lambda}) = C. \)

**Proof.** The first statement is straightforward as all terms in the symmetrization of the product \( Sh_{1,m}(u) \ast Sh_{0,n-m} \) clearly vanish under evaluation \( \rho_1 \) if \( |\lambda| > m \).

Let \( m = |\lambda| \). From the definition of the space \( V_{n,\lambda} \) we see that, if \( F \) is an element of \( V_{n,\lambda} \), then \( \rho_1(F) \) contains an extra factor \( \prod_{k=m+1}^{n} \prod_{(i,j) \in \lambda} (x_k - q_3^{-1}q_1^{-1}u) \), which cancels the denominator of \( (A.2) \). Therefore we have

\[ \rho_1(V_{n,\lambda}) \subset Sh_{0,n-m} \times \prod_{k=m+1}^{n} \prod_{(i,j) \in \lambda} \omega(x_k, q_3^{-1}q_1^{-1}u). \quad (A.3) \]

On the other hand, the following identity holds for elements \( G \in Sh_{1,m}(u) \) and \( H \in Sh_{0,n-m} \):

\[ \rho_1(G \ast H) = \text{const.} \rho_1(G) \cdot H(x_{m+1}, \ldots, x_n) \times \prod_{m+1 \leq k \in \lambda} \omega(x_k, q_3^{-1}q_1^{-1}u), \quad (A.4) \]

where const. is non-zero. We choose

\[ G = e_{\lambda'}(q_1)(x) \times \prod_{i=1}^{m} \frac{x_i - q_2u}{x_i - u} \in Sh_{1,m}(u), \]

where \( \lambda' = (\lambda'_1, \ldots, \lambda'_\ell) \) is the partition dual to \( \lambda \) and \( e_{\lambda'}(q_1)(x) \) is defined in \((5.8)\) where the parameter \( q_3 \) is changed to \( q_1 \). It is easy to see that \( G \in V_{n,\lambda'} \), and that \( \rho_1(G) \) is a non-vanishing complex number. Since \( H \in Sh_{0,n-m} \) is arbitrary in \((A.4)\), we conclude that the inclusion in \((A.3)\) is actually an equality and that \( \rho_1(V_{n,\lambda}) = \rho_1(Sh_{1,m}(u) \ast Sh_{0,n-m}) \).

In particular, if \( m = n \), then \( \rho_1(V_{n,\lambda}) \) is a one-dimensional vector space. \( \square \)
Corollary A2. The space $\sum = \left( \sum_{m=0}^{n-1} \text{Sh}_1(n) \ast \text{Sh}_{0,n-m} \right)$ is a finite dimensional vector space of dimension $p(n)$, the number of partitions of $n$.

Proof. We consider the associated graded space related to the filtration $\{V_{\lambda}\}$ of the space $V_n = \text{Sh}_1(n) \ast \text{Sh}_{0,n-m}$. By the definition, we have $g_{\lambda}(V_n) = \rho_{\lambda}(V_{n,\lambda})$. Now, for the factor space, from proposition A1 we have $g_{\lambda}(\text{Sh}_1(n) \ast \text{Sh}_{0,n-m})$ is zero if $|\lambda| < n$ and one-dimensional if $|\lambda| = n$. The corollary follows. □

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