Quenched mean-field theory for the majority-vote model on complex networks

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Abstract – The majority-vote (MV) model is one of the simplest nonequilibrium Ising-like model that exhibits a continuous order-disorder phase transition at a critical noise. In this paper, we present a quenched mean-field theory for the dynamics of the MV model on networks. We analytically derive the critical noise on arbitrary quenched unweighted networks, which is determined by the largest eigenvalue of a modified network adjacency matrix. By performing extensive Monte Carlo simulations on synthetic and real networks, we find that the performance of the quenched mean-field theory is superior to a heterogeneous mean-field theory proposed in a previous paper (CHEN H. et al., Phys. Rev. E, 91 (2015) 022816), especially for directed networks.

Introduction. – Dynamical processes taking place on complex networks are often used to model a wide variety of phenomena [1–5]. Examples include spreading of diseases or opinions through a population [6,7], neural activity in the brain [8], and cascading failure on power grid [9]. Owing to the inherent randomness and heterogeneity in the interacting patterns, it has lead to dynamics on complex networks drastically different from those on regular lattices in Euclidean space [10]. So far, unveiling the relationship between the topologies of the underlying networks and the dynamics on them is still a hot topic of considerable attention.

Mean-field theory (MFT) is the most commonly used to obtain (relatively) low-dimensional descriptions for the systems under study. For homogeneous networks, MFT is quite accurate. However, for heterogeneous networks such as scale-free networks (SFN), ordinary MFT fails since the degrees of nodes are largely different from each other. An improved MFT, termed heterogeneous MFT (HMFT) was proposed for the analysis of general dynamical processes on heterogeneous networks. By assuming that nodes with the same degree are statistically equivalent, the HMFT writes down a set of degree-based equations for governing the evolutions of dynamical processes on complex networks [6,10]. Some well-known conclusions of the HMFT include the absence of epidemic threshold in susceptible-infected-susceptible (SIS) model [11,12] and the anomalous behavior of Ising model [13–17] in SFN with degree exponent less than 3. It is widely accepted that the HMFT theory is exact only when the underlying networks are annealed where dynamical correlations among neighboring nodes are absent [6,10]. An annealed network requires that the time scale of network evolution is much faster than that of dynamics on it. In other words, an annealed network is far from static, in contrasts with a network where links are fixed permanently in time. While the latter network is referred to as a quenched network. Almost concurrently, a quenched MFT (QMFT) was proposed to study the dynamics on arbitrary quenched networks. The basic idea of QMFT is to derive evolution equations for each node by assuming that the dynamic state of each node is statistically independent of its nearest neighbors. For example, the QMFT predicts that the epidemic threshold of SIS model equals the inverse of the largest eigenvalue of the adjacency matrix of the underlying network [18–20]. Furthermore, several extensions of HMFT-based [21–26] and QMFT-based [27–29] theories have been developed that take into account the role of dynamical correlations.
The majority-vote (MV) model is a simple nonequilibrium Ising-like system with up-down symmetry [30]. In the model, each individual is assigned to a binary spin variable ±1. In each time step, each spin tends to align with the local neighborhood majority but with a noise parameter \( f \) giving the probability of misalignment. As \( f \) increases, the model presents a continuous order–disorder phase transition at a critical value \( f_c \). The MV model has been extensively studied for various interacting substrates, such as regular lattices [31–34], random graphs [35,36], small-world networks [37–39], scale-free networks [40,41], and modular networks [42].

In a recent work [43], we have used the HMFT to derive the critical noise \( f_c \), which is determined by

\[
(1 - 2f_c) \sum_k k^2 P(k) \langle k \rangle^{2 - k} C_{k-1}^{(k-1)/2} = 1,
\]

where \( P(k) \) is the degree distribution defined as the probability that a node chosen at random has degree \( k \), and \( \langle k \rangle \) is the average degree. \( C_{k-1}^{(k-1)/2} \approx 2^{k-1}/\sqrt{k\pi/2} \), and \( f_c \) is thus given explicitly,

\[
f_c = \frac{1}{2} - \frac{1}{2} \sqrt{\frac{\pi}{2} \langle k \rangle / \langle k^2 \rangle},
\]

where \( \langle k^2 \rangle = \sum_k k^2 P(k) \) is the \( n \)-th moment of degree distribution.

As mentioned before, HMFT is only exact for annealed networks. While for quenched networks, QMFT is expected to be more accurate than HMFT. To fill this gap, here we develop a QMFT for the dynamics of the MV model. We show that the critical noise can be determined by the leading eigenvalue of a modified adjacency matrix of the underlying network. By performing extensive Monte Carlo (MC) simulations on diverse types of networks, we find that the QMFT is superior to the HMFT.

**Model.** -- Let us first define the MV model on an unweighted network with size \( N \), where the network is described by an adjacency matrix \( \mathbf{A} \). The elements of \( \mathbf{A} \) are defined as \( A_{ij} = 1 \) if there exists an edge from node \( i \) to node \( j \), and \( A_{ij} = 0 \) otherwise. The spin of each node is assigned to a binary variable \( \sigma_i \in \{-1, +1\} \) \( \{i = 1, \ldots, N\} \). The system evolves as follows: for each node \( i \), we determine the majority spin of its neighborhood. With probability \( f \) the node \( i \) takes the opposite sign of the majority, otherwise it takes the same sign. \( f \) is called the noise parameter. In this way, the spin flipping probability of node \( i \) is given by

\[
w_i = \frac{1}{2} \left[ 1 - (1 - 2f) \sigma_i S(\Theta_i) \right],
\]

with

\[
\Theta_i = \sum_j A_{ij} \sigma_j,
\]

where \( S(x) = \text{sgn}(x) \) if \( x \neq 0 \) and \( S(0) = 0 \). In the latter case the spin \( \sigma_i \) is flipped to \( \pm 1 \) with equal probabilities \( 1/2 \). For \( f = 0 \), the MV model is equivalent to the zero-temperature Ising model with Glauber dynamics [44,45].

The MV model does not only play an important role in the study of nonequilibrium phase transitions, but also helps to understand opinion dynamics in social networks [46]. In this model, binary spins can represent two opposite opinions, and the noise parameter \( f \) plays the role of the temperature in equilibrium systems.

**Results.** -- To proceed the QMFT, let us define \( p_i \) as the probability that the spin of node \( i \) takes \( \sigma_i = 1 \). The dynamical evolution of \( p_i \) is governed by the rate equation,

\[
\frac{dp_i}{dt} = -p_i w_i^+ + (1 - p_i) w_i^-,
\]

where \( w_i^+ \) (\( w_i^- \)) is the flipping probability of node \( i \) with spin +1 (−1). According to the dynamics of the MV model, \( w_i^+ \) can be written as the sum of two parts,

\[
w_i^+ = f \varphi_i + (1 - f) (1 - \varphi_i),
\]

where the first part is the product of the probability \( \varphi_i \) of the majority spin among the neighborhood of the node \( i \) being +1 and the probability \( f \) of the minority rule being applied, and the second part is the product of the probability \( 1 - \varphi_i \) of the minority spin among the neighborhood of the node \( i \) being +1 and the probability \( 1 - f \) of the majority rule being applied. Likewise, \( w_i^- \) is expressed as

\[
w_i^- = (1 - f) \varphi_i + f (1 - \varphi_i).
\]

Considering \( w_i^+ + w_i^- = 1 \), eq. (5) reduces to

\[
\frac{dp_i}{dt} = -p_i + w_i^-.
\]

The probability \( \varphi_i \) can be written as

\[
\varphi_i = \sum_{n=0}^{k_{\text{in}}^\text{out}} \left( 1 - \frac{1}{2} \delta_{n,k_{\text{in}}^\text{out}/2} \right) \xi_n k_{\text{in}}^\text{out},
\]

where \( k_{\text{in}}^\text{out} \) is the outdegree of node \( i \), and \( \delta \) is the Kronecker delta function. \( \xi_n k_{\text{in}}^\text{out} \) is the probability that there are \( n \) up spins among the neighborhood \( \mathcal{N}(i) \) of node \( i \), which can be calculated by

\[
\xi_n k_{\text{in}}^\text{out} = \sum_{\mathcal{U}(i) \subseteq \mathcal{N}(i)} \prod_{j \in \mathcal{U}(i)} p_j \prod_{j \notin \mathcal{U}(i)} (1 - p_j),
\]

where \( \mathcal{U}(i) \) are all the subsets of \( \mathcal{N}(i) \) that contains \( n \) neighbor(s) of node \( i \), \( |\mathcal{U}(i)| = n \in [0, k_{\text{in}}^\text{out}] \). \( \mathcal{U}(i) \) is the complement of the subset \( \mathcal{U}(i) \), i.e., \( \mathcal{U}(i) \cup \mathcal{U}(i) = \mathcal{N}(i) \) and \( \mathcal{U}(i) \cap \mathcal{U}(i) = \emptyset \). Obviously, there are \( C_n^{k_{\text{in}}^\text{out}} \) possibilities for the subsets \( \mathcal{U}(i) \).

Since \( \varphi_i = 1/2 \) at \( p_i = 1/2 \) for all \( i \), one can easily check that \( p_i = 1/2 \) is always a stationary solution of eq. (8).
For convenience, let us note this trivial solution as $\mathbf{p}^*$. Such a trivial solution corresponds to a disorder phase. An ordered phase may emerge when the noise parameter $f$ is lower than the so-called critical value, $f_c$, below which the trivial solution $p_i = 1/2$ loses its stability. Thus, $f_c$ is determined by the maximal eigenvalue of the Jacobian matrix that is null, $\Lambda_{\text{max}}(\mathbf{J}) = 0$, where the elements of the matrix $\mathbf{J}$ are given by

$$J_{ij} = -\delta_{ij} + (1 - 2f)\frac{\partial \varphi_i}{\partial p_j}|_{\mathbf{p}^*}. \quad (11)$$

If $j \notin \mathcal{N}(i)$, $\varphi_i$ is considered to be independent of $p_j$, so that $\frac{\partial \varphi_i}{\partial p_j} = 0$. If $j \in \mathcal{N}(i)$, $\frac{\partial \varphi_i}{\partial p_j}$ at $\mathbf{p}^*$ can be derived as follows. For $j \in \mathcal{U}(i)$ and $|\mathcal{U}(i)| = n$, the remaining $n - 1$ nodes different from node $j$ are included in the set $\mathcal{U}(i)$. There exist $C_{n-1}^{k_{i \text{out}}-1}$ possibilities for $\mathcal{U}(i)$, and its contribution to the partial derivation $\frac{\partial \varphi_i}{\partial p_j}$ is positive in terms of eq. (10). For $j \in \mathcal{U}(i)$, there exist $C_{n-1}^{k_{i \text{out}}-1}$ possibilities for $\mathcal{U}(i)$, and its contribution to the partial derivation is negative. $C_m^{k_{i \text{out}}} = C_{m-1}^{k_{i \text{out}}-1} + C_{m-1}^{n-1}$ holds for any $n$. Therefore, we have

$$\frac{\partial \varphi_i}{\partial p_j}|_{\mathbf{p}^*} = A_{ij} \sum_{n=|k_{i \text{out}}/2|}^{k_{i \text{out}}} \left(1 - \frac{1}{2} \delta_{n, k_{i \text{out}}/2}\right) \times \left(C_{n-1}^{k_{i \text{out}}-1} - C_{n}^{k_{i \text{out}}-1}\right) \left(\frac{1}{2}\right)^{-k_{i \text{out}}-1}$$

$$= A_{ij} C_{k_{i \text{out}}-1}^{n-1} 2^{1-k_{i \text{out}}}. \quad (12)$$

Substituting eq. (12) into eq. (11), we obtain

$$J_{ij} = -\delta_{ij} + (1 - 2f) A_{ij} C_{k_{i \text{out}}-1}^{n-1} 2^{1-k_{i \text{out}}}. \quad (13)$$

Again, $f_c$ is obtained when the largest eigenvalue of $\mathbf{J}$ is null, yielding

$$f_c = \frac{1}{2} \left(1 - \frac{1}{\Lambda_{\text{max}}(\mathbf{A})}\right), \quad (14)$$

where the elements of the matrix $\mathbf{A}$ are

$$\tilde{A}_{ij} = A_{ij} C_{k_{i \text{out}}-1}^{n-1} 2^{1-k_{i \text{out}}}. \quad (15)$$

Equation (14) is the central result of the present work. From the Perron-Frobenius theorem, since $\tilde{A}_{ij}$ is non-negative, and assuming that it is irreducible, its largest eigenvalue is real and positive.

For $f < f_c$, analytically deriving nontrivial solution of $p_i$ is generally impossible. To this end, one can numerically iterate $N$-intertwined eq. (8) at stationary, $p_i = \varphi_i$. However, direct calculation of $\varphi_i$ is not practical, since the combination number of the subsets $\mathcal{U}(i)$ is tremendously large, especially for nodes with high outdegrees. To overcome this difficulty, we adopt a recursive formulae developed in ref. [47]. The probability $\xi_{n,k_{i \text{out}}}$ in eq. (10) can be calculated by

$$\xi_{k,j} = (1 - p_j)\xi_{k,j-1} + p_j \xi_{k-1,j-1}, 0 \leq k \leq j \leq k_{i \text{out}}. \quad (16)$$

The boundary conditions for eq. (16) are $\xi_{-1,j} = \xi_{j+1,j} = 0 \ (j = 0, \ldots, k_{i \text{out}})$, and $\xi_{0,0} = 1$. Once $p_i$ is obtained, average magnetization of node $i$ can be calculated as $m_i = 2p_i - 1$, and average magnetization per node as $m = \sum_{i=1}^{N} m_i / N$.

To numerically determine the critical noise $f_c$ from MC simulations, we calculate the magnetization $m$ and the susceptibility $\chi$, defined as

$$m = \left\langle \frac{1}{N} \sum_{i=1}^{N} \sigma_i \right\rangle, \quad (17)$$

$$\chi = N \left(\langle m^2 \rangle - \langle m \rangle^2 \right), \quad (18)$$

where $\langle \cdots \rangle$ denotes time averages taken in the stationary regime.

Let us now start the validation of the QMFT for a simple homogeneous network, the random $k$-regular networks (RkRN). For RkRN, all nodes have the same degree $k$ and the degree distribution follows $P(k) = \delta(k)$, while the edges among nodes are linked at random. From $\Lambda_{\text{max}}(\tilde{A}) = C_{k-1}^{(k/2)-1}2^{1-k} \Lambda_{\text{max}}(\mathbf{A})$ and $\Lambda_{\text{max}}(\mathbf{A}) = k$, we arrive at the critical noise of RkRN,

$$f_{c_{\text{RkRN}}} = \frac{1}{2} - \frac{1}{k C_{k-1}^{(k/2)-1} 2^{2-k}}. \quad (19)$$

For large $k$, $C_{k-1}^{(k/2)-1} \approx 2^{k-1}/\sqrt{(k-1)\pi/2}$ by Stirling’s approximation, and thus

$$f_{c_{\text{RkRN}}} \approx \frac{1}{2} - \frac{1}{2k} \sqrt{\frac{(k-1)\pi}{2}} \approx \frac{1}{2} - \frac{1}{2} \sqrt{\frac{\pi}{2k}}. \quad (20)$$

which is consistent with the result of HMFT, eq. (2). However, for any randomly heterogeneous networks, on the one hand, it is generally impossible to analytically obtain $\Lambda_{\text{max}}(\tilde{A})$, which can be instead calculated numerically. It should be noted that there were some attempts to examine the spectral properties of network adjacency matrices with given degree distributions [48], which might be a heuristic effect on the present work. On the other hand, the result of QMFT is different from that of HMFT.

Figure 1(a) shows the magnetization $m$ as a function of noise intensity $f$ in RkRN with $N = 10000$ and three different average degrees $\langle k \rangle = 6, 10, 20$. The networks are generated according to the Molloy-Reed model [49]; each node is assigned a random number of stubs $k$ that is drawn from a given degree distribution. Pairs of unlinked stubs are then randomly joined. The lines and symbols in fig. 1(a) indicate the QMFT and simulation results, respectively. Figure 1(b) shows the susceptibility $\chi$ as a
function of $f$. $\chi$ exhibits a sharp peak at the critical noise $f_c$. MC simulation gives $f_c = 0.212$, 0.284, and 0.350 for ($k$) = 6, 10, and 20, respectively. The QMFT predicts $f_c = 0.233$, 0.297, and 0.358 for ($k$) = 6, 10, and 20, respectively. The discrepancy between them becomes more apparent for smaller ($k$). The better accuracy of the QMFT for larger ($k$) is intuitive since topological distance among nodes decreases as the average degree increases making the mean-field premise a more credible hypothesis.

We next demonstrate the results on directed networks. A directed network is constructed as follows. We start with an undirected network, and then turn each undirected edge, $i \rightarrow j$, into a directed edge, $i \rightarrow j$ or $i \leftarrow j$ with equal probabilities 1/2. The average outdegree and the average indegree of the resulting directed network are ($k^{out}$) = ($k$)/2, where ($k$) is the average degree of the original undirected network. Unlike HMFT, the QMFT does not require that the underlying networks are undirected. Figure 2 shows $m$ (left axis) and $\chi$ (right axis) as a function of $f$ on a directed network, generated by a RkRN with $N = 10000$ and ($k$) = 10 (named as D-RkRN for shorts). MC simulation and QMFT show the critical noises are respective $f_c = 0.196$ and 0.200, as shown by the symbols and lines (left axis). There are in excellent agreement between them. Moreover, we substitute $k$ in eq. (1) and eq. (2) with $k^{out}$, which enables HMFT to be applied to directed networks. It is found that the HMFT result is $f_c = 0.230$ (see table 1 for comparison). As expected, QMFT is obviously superior to HMFT in predicting critical point on directed networks.

To further test the potential of our QMFT on other networks, we consider synthetic networks: Erdős-Rényi (ER) random networks (undirected) [50] Barabási-Albert (BA) scale-free networks (undirected) [51], directed ER (D-ER) networks and directed BA (D-BA) networks, as well as real networks: the Email network at Rovira i Virgili (containing 1133 nodes and 5451 directed edges) [52] and Wikivote network (containing 7115 nodes and 103689 directed edges) [53]. The sizes and the average degrees of the ER network and BA network we use are both $N = 10000$ and ($k$) = 6. The results for the critical noise are summarized in table 1. One can see that the performances of the QMFT on all the listed networks are better than HMFT, especially for directed networks. The reason why HMFT does not work well in directed networks maybe lies in an assumption of degree uncorrelated networks used in HMFT to obtain explicitly the expression of $f_c$. In general, such an assumption is no longer valid for directed networks. While for QMFT, it is only based on the network adjacency matrix that attains the full information of the network topology, whether the network is undirected or directed.

Recently, an inertial effect was added to the dynamics of the MV model, such that the spin-flip probability of each node depends not only on the its neighboring spins, but also on its own spin [54]. In the inertial MV model, eq. (5) is rewritten as

$$
\Theta_i = (1 - \theta) \sum_j A_{ij} \sigma_j/k^{out}_i + \theta \sigma_i,
$$

where $\theta \in (0, 0.5]$ controls the strength of the inertia. Note that for $\theta = 0$ one recovers the original MV model.

Interestingly, the order-disorder phase transition is changed from a usual continuous or second-order type to a discontinuous or first-order one when $\theta$ is larger than a critical value $\theta_c$ [54,55]. If the inertia is placed to partial nodes, a mixture of discontinuous and continuous phase

| Network Type | MC | QMFT | HMFT |
|--------------|----|------|------|
| D-RkRN directed | 0.196 | 0.200 | 0.230 |
| ER undirected | 0.272 | 0.310 | 0.321 |
| BA undirected | 0.236 | 0.261 | 0.265 |
| D-ER directed | 0.116 | 0.121 | 0.180 |
| D-BA directed | 0.190 | 0.207 | 0.259 |
| Email directed | 0.073 | 0.102 | 0.268 |
| Wikivote directed | 0.312 | 0.366 | 0.437 |
transitions from fully ordered phase to partially ordered phase and then to disordered phase was observed [56]. To derive QMFT in the inertial MV model, the probability \( \varphi \), in eq. (7) and eq. (8) should be replaced by \( \varphi^+ \) and \( \varphi^- \), respectively. Here \( \varphi^+ \) (\( \varphi^- \)) is the conditional probability that the majority spin among the neighborhood of node \( i \) being +1, providing that the spin of node \( i \) is up (down). They can be written as

\[
\varphi^\pm_i = \sum_{n=\pm 1} k_{n,i}^{\text{out}} \left( 1 - \frac{1}{2} \delta_{n,n_i} \right) \xi_{n,k_{n,i}^{\text{out}}},
\]

where \( n_i = k_{n,i}^{\text{out}} (1 - 2\theta) /[2(1 - \theta)] \) and \( n_i^* = k_{n,i}^{\text{out}} /[2(1 - \theta)] \) satisfying \( \Theta_i = 0 \) in eq. (21). In the steady state, one has that

\[
p_i = \frac{w_i^-}{w_i^+ + w_i^-}.
\]

Figure 3 shows \( m \) as a function of \( f \) for two different values of \( \theta \) on a ER network with \( N = 10^4 \) and \( \langle k \rangle = 20 \). The simulation results (symbols) are obtained by performing forward and backward simulations, respectively. The former is done by calculating the stationary value of \( m \) as \( f \) increases from 0 to 0.5 in steps of 0.01 and using the final configuration of the last simulation run as the initial condition of the next run, while the latter is performed by decreasing \( f \) from 0.5 to 0 with the same step. For \( \theta = 0.1 \), forward and backward simulations coincide indicating that the order-disorder transition is continuous. For \( \theta = 0.3 \), forward and backward simulations form a hysteresis loop, a feature of a discontinuous phase transition. The lines show the theoretical results obtained by numerically solving eq. (23). Although the theory agrees qualitatively with MC simulation, there exist obvious disagreements between them in evaluating the critical point of the discontinuous phase transition.

Conclusions. – In conclusion, we have developed a QMFT for studying the MV model on complex networks. The QMFT cannot only predict the phase transition behavior of the MV model, but also can analytically derive the critical point of the transition. Our proposed theory has shown that the critical point of the MV model is determined by the leading eigenvalue of a modified adjacency matrix of the underlying network. This result is similar to but different from the results of QMFT in the SIS model and in the Ising model. In the latter two models, the critical points are directly determined by the leading eigenvalue of the adjacency matrix. By extensive MC simulations on various networks, we found that the QMFT is better than the HMF in predicting the critical point, especially for directed networks. However, for networks with low average degrees, both of them overestimate the critical point. Therefore, in the future it will be desirable to develop higher-order theories (such as pair approximation) to obtain a more accurate estimation of the critical point of the networked MV model.

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