Orthogonal run-and-tumble walks

L. Angelani$^{1,2}$

$^1$ ISC-CNR, Institute for Complex Systems, P.le A. Moro 2, 00185 Rome, Italy and
$^2$ Dipartimento di Fisica, Sapienza Università di Roma, P.le A. Moro 2, 00185 Rome, Italy

Abstract

Planar run-and-tumble walks with orthogonal directions of motion are considered. After formulating the problem with generic transition probabilities among the orientational states, we focus on the symmetric case, giving general expressions of the probability distribution function (in the Laplace-Fourier domain), the mean-square displacement and the effective diffusion constant in terms of transition rate parameters. As case studies we treat and discuss two classes of motion, alternate/forward and isotropic/backward, obtaining, when possible, analytic expressions of probability distribution functions in the space-time domain. We discuss at the end also the case of cyclic motion. Reduced (enhanced) effective diffusivity, with respect to the standard 2D active motion, is observed in the cyclic and backward (forward) cases.
I. INTRODUCTION

Random walk models with finite velocity describe many different physical and biological phenomena [1–3], from the motion of electrons in metals [4] to the swimming of motile bacteria, such as *E. coli* [5, 6]. In many cases the particle motion can be described by the so-called run-and-tumble models, in which the particle trajectory is a straight line interrupted by abrupt changes of motion direction. Analytical results of run-and-tumble equations describing the time evolution of probability densities can be obtained in different simplified situations in one-dimensional space. Higher dimensions are in general quite harder to treat. Focusing on the two-dimensional case, an explicit expression of the probability density function exists in the case of uniform turning-angle distribution [7], while some other analytical results can be achieved under some approximations in the case of general turning-angle distribution [8] or for Active Brownian particles [9, 10]. Among planar motions, of some interest is the case of discrete turning-angle [11], and, in particular, of orthogonal directions of motion [12, 14].

We investigate here the planar random motion of a particle which moves at constant speed along four different orthogonal directions of motion, switching between them at given rates. The switching process is described by a transition probability matrix, whose elements are, in general, different one from each other. While the problem has been previously treated in some special cases (see Ref. [14] and references within), we give here a very general and unified formulation, allowing us to obtain expressions valid for generic transition probabilities among the different orientational states of the particle. We are then able to specialize the general formulae to various interesting case studies.

The paper is organized as follows. In Sec. II, we introduce the orthogonal run-and-tumble model, giving the formal general solution of the dynamical equations for the probability distribution functions. In Sec. III, we analyze the symmetric case, reporting explicit expressions of the probability distribution function (in the Laplace-Fourier domain) and the mean square displacement as a function of the transition rate parameters. We then specialize to some interesting case studies, such as the alternate motion, with orthogonal switch of the direction of motion at each tumble event (as interesting byproduct we obtain the expression for the case of a 1D run-and-tumble particle with finite tumbling time), the isotropic motion, characterized by equal transition rates, and the cases of forward and backward motion,
which turn out to be equivalent to the previous two cases with rescaled parameters. In Sec. IV we consider instead the case of a cyclic motion, where the particle’s direction of motion, after a tumble, rotates 90 degrees counterclockwise. Conclusions are drawn in Sec. V.

II. ORTHOGONAL RUN-AND-TUMBLE MODEL

We consider a run-and-tumble particle in a plane which can move along two orthogonal directions of motion, parallel to \( \hat{x} \) and \( \hat{y} \) axes. Therefore there are only four possible self-propelling orientations for the particle, i.e., \(+\hat{x} (R, \text{Right}), -\hat{x} (L, \text{Left}), +\hat{y} (U, \text{Up}), -\hat{y} (D, \text{Down})\). We denote with \( P_\mu(x,t) \) – with \( \mu \in \{R, L, U, D\} \) – the probability density functions (PDF) for the \( \mu \)-oriented particles. Reorientation of the particle motion is described by a Poisson process with rate \( \alpha \) and we denote with \( \gamma_{\mu\nu} \) the transition probability from state \( \nu \) to state \( \mu: \nu \to \mu \). We note that in general \( \gamma_{\mu\mu} \) can be different from zero, i.e., after a tumble the new direction chosen could be the same as the previous one. The orthogonal run-and-tumble motion is described by the following general equations for the PDFs

\[
\frac{\partial P_R}{\partial t} = - v \frac{\partial P_R}{\partial x} - \alpha P_R + \alpha \sum_{\mu} \gamma_{R\mu} P_\mu \tag{1}
\]

\[
\frac{\partial P_L}{\partial t} = v \frac{\partial P_L}{\partial x} - \alpha P_L + \alpha \sum_{\mu} \gamma_{L\mu} P_\mu \tag{2}
\]

\[
\frac{\partial P_U}{\partial t} = - v \frac{\partial P_U}{\partial y} - \alpha P_U + \alpha \sum_{\mu} \gamma_{U\mu} P_\mu \tag{3}
\]

\[
\frac{\partial P_D}{\partial t} = v \frac{\partial P_D}{\partial y} - \alpha P_D + \alpha \sum_{\mu} \gamma_{D\mu} P_\mu \tag{4}
\]

We define the vector \( \mathbf{P} \)

\[
\mathbf{P} = \begin{pmatrix} P_R \\ P_L \\ P_U \\ P_D \end{pmatrix}
\]

the derivatives matrix

\[
\mathbf{D} = \begin{pmatrix} v \partial_x & 0 & 0 & 0 \\ 0 & -v \partial_x & 0 & 0 \\ 0 & 0 & v \partial_y & 0 \\ 0 & 0 & 0 & -v \partial_y \end{pmatrix}
\]

3
and the transition matrix
\[ \Gamma = \begin{pmatrix}
\gamma_{RR} & \gamma_{RL} & \gamma_{RU} & \gamma_{RD} \\
\gamma_{LR} & \gamma_{LL} & \gamma_{LU} & \gamma_{LD} \\
\gamma_{UR} & \gamma_{UL} & \gamma_{UU} & \gamma_{UD} \\
\gamma_{DR} & \gamma_{DL} & \gamma_{DU} & \gamma_{DD}
\end{pmatrix} \] (7)
with the constraint (probability conservation)
\[ \sum_{\mu} \gamma_{\mu\nu} = 1 \] (8)

We can write the Eq.s (1-4) in a concise form
\[ \frac{\partial P}{\partial t} = -[D + \alpha(1 - \Gamma)]P \] (9)

where 1 is the identity matrix. It is more convenient to work in the Laplace-Fourier domain
\[ \tilde{\tilde{P}}(k, s) = \int_0^\infty dt \, e^{-st} \int dr \, e^{ik \cdot r} \, P(r, t) \] (10)

where the symbols \( \tilde{\cdot} \) and \( \hat{\cdot} \) denote, respectively, Laplace and Fourier transforms. Eq.(9) becomes
\[ [(s + \alpha)1 + D' - \alpha \Gamma] \tilde{\tilde{P}} = \hat{P}_0 \] (11)

where the RHS is the Fourier transform of the initial distribution \( P_0(r) = P(r, t=0) \) and the matrix \( D' \) is
\[ D' = \begin{pmatrix}
-ivk_x & 0 & 0 & 0 \\
0 & ivk_x & 0 & 0 \\
0 & 0 & -ivk_y & 0 \\
0 & 0 & 0 & ivk_y
\end{pmatrix} \] (12)

By defining the matrix \( A \)
\[ A = (s + \alpha)1 + D' - \alpha \Gamma \] (13)

the Eq.(11) can be concisely written as
\[ A \tilde{\tilde{P}} = \hat{P}_0 \] (14)

Then, the formal expression of the Laplace-Fourier transformed PDF, for generic initial conditions, can be written as
\[ \hat{P} = A^{-1} \hat{P}_0 \] (15)
Let us now specialize to the case of isotropic initial conditions

\[ P_\mu(r, t = 0) = \frac{1}{4} \delta(r) \quad \forall \mu \in \{R, L, U, D\} \] (16)

corresponding in the Fourier space to

\[ (\hat{P}_0)_\mu = \frac{1}{4} \quad \forall \mu \in \{R, L, U, D\} \] (17)

We are interested in the total distribution function, independent of particle orientation

\[ P = P_R + P_L + P_U + P_D \] (18)

which can be then written, from Eq. (15)

\[ \hat{P} = \frac{1}{4} \sum_{\mu, \nu} (A^{-1})_{\mu \nu} \] (19)

The above expression allows us to obtain the probability distribution function as a sum of the elements of the inverse of the matrix \( A \) defined in (13). The obtained expression is very general, valid for generic transition probabilities \( \gamma_{\mu \nu} \).

In the following section we give generic explicit solutions for the symmetric case, considering rotational symmetry and equivalence among the orientational states \( R, L, U, D \). The general obtained expressions will allow us to specialize to few interesting case studies. We analyze the case of isotropic transition probabilities, i.e., after a tumble the particle can assume with equal probability each one of the four possible propelling directions. Another case we consider is the one with right reorientational angles, i.e., the particle orientation switches between the two orthogonal directions \( \hat{x} \) and \( \hat{y} \). An interesting byproduct of this planar motion is obtained projecting the solution onto the \( x \) axis, resulting in a one dimensional motion with 3 states, considering a finite rest time during tumble events. We also analyze the problems of forward and backward moving particle, which, after a tumble, can only move forward/backward or orthogonal to the previous direction of motion.

In a final section we analyze the case of cyclic motion, considering unidirectional rotational motion of the self-propelled direction.

### III. SYMMETRIC CASE

We consider here the symmetric case, where all the orientational states are equivalent and, moreover, symmetric rotational symmetry is assumed, considering equal transition
probabilities from a given state towards the two perpendicular directions. We can write the transition matrix as follow:

\[
\Gamma = \begin{pmatrix}
\gamma_F & \gamma_B & \gamma_P & \gamma_P \\
\gamma_B & \gamma_F & \gamma_P & \gamma_P \\
\gamma_P & \gamma_P & \gamma_F & \gamma_B \\
\gamma_P & \gamma_P & \gamma_B & \gamma_F
\end{pmatrix}
\]  (20)

where \(\gamma_F\), \(\gamma_B\) and \(\gamma_P\) are, respectively, forward, backward and perpendicular transition probabilities after a tumble event, satisfying the constraint

\[\gamma_F + \gamma_B + 2\gamma_P = 1\]  (21)

While in principle one can treat the symmetric cases by reabsorbing the diagonal terms of the transition matrix in a rescaled tumbling rate, we prefer to maintain the original formulation with the presence of forward transition terms, allowing for generalisations to non-symmetric cases, as for example in the presence of orientational dependent forward transition rates.

The simplified transition matrix with a reduced number of independent elements, with respect to the general case in Eq. (7), allows us to obtain explicit solutions of dynamical equations in a simple form. By solving the linear equations (14) for \(P_\mu\) – or inverting the A matrix (13) and using (15) – we can obtain, after some algebra, the general expression of the PDF in the Laplace-Fourier space \((s, k)\) as a function of transition probability parameters:

\[
\hat{P}(s) = \frac{(s + \alpha_1)[(s + \alpha_1)(s + 2\alpha_2) + k^2v^2/2]}{[k_x^2v^2 + (s + \alpha_1)(s + \alpha_2)][k_y^2v^2 + (s + \alpha_1)(s + \alpha_2)] - \alpha_2^2(s + \alpha_1)^2}
\]  (22)

where \(k^2 = k_x^2 + k_y^2\) and

\[
\alpha_1 = \alpha(1 + \gamma_B - \gamma_F) = 2\alpha(\gamma_B + \gamma_F)
\]  (23)

\[
\alpha_2 = \alpha(1 - \gamma_B - \gamma_F) = 2\alpha\gamma_F
\]  (24)

An interesting quantity characterizing the motion is the mean square displacement (MSD), obtained through the relation [15]

\[
\langle r^2 \rangle = -\nabla_k^2 \hat{P} \bigg|_{k=0}
\]  (25)

By deriving Eq. (22) we obtain the Laplace-transformed MSD

\[
\tilde{r^2}(s) = \frac{2v^2}{s^2} \frac{1}{s + \alpha_1} = \frac{2v^2}{\alpha_1^2} \left[ \frac{\alpha_1}{s^2} - \frac{1}{s} + \frac{1}{s + \alpha_1} \right]
\]  (26)
corresponding, in the time domain, to

\[ r^2(t) = \frac{2v^2}{\alpha_1} \left[ \alpha_1 t - 1 + e^{-\alpha_1 t} \right] \]  \hspace{1cm} (27)

We note that this expression corresponds to the usual MSD for active particles with rescaled tumbling rate \( \alpha_1 \). It is worth also noting that a similar expression as (26) can be obtained as special case of a more general form with continuous distribution of tumbling angles (see Eq. 45 of Ref. [16] specialized to Poissonian tumbling). The diffusive limit is obtained for \( v, \alpha \to \infty \) with \( v^2/\alpha \) constant. In this limit the PDF (22) reduces to the well known expression for the Brownian motion

\[ \tilde{P}_{Diff} = \frac{1}{s + Dk^2} \]  \hspace{1cm} (28)

corresponding to the time dependence

\[ \hat{P}_{Diff} = \exp \left( -Dk^2 t \right) \]  \hspace{1cm} (29)

with \( D \) the effective diffusion constant of the run-and-tumble particle

\[ D = \frac{v^2}{2\alpha_1} = \frac{v^2}{2\alpha} \frac{1}{2(\gamma_B + \gamma_P)} \]  \hspace{1cm} (30)

Then, in the diffusive limit, the MSD reduces to the usual linear form

\[ r^2(t) = 4Dt \]  \hspace{1cm} (31)

In the following subsections we specialize to some interesting case studies, reporting expressions of the PDFs and mean-square displacements and summarizing the values of effective diffusivity in Table I.

A. Alternate and forward motions

The first class of models we analyze is that including alternate and forward motions. We first consider the alternate motion, which has been quite extensively investigated in the past and then it serves as a benchmark of our results.

1. Alternate motion

We consider the case of a particle performing alternate motion along \( \hat{x} \) and \( \hat{y} \) axes. At each tumbling event the particle can switch between the two orthogonal directions of motion, as described in the following picture
TABLE I: Long time effective diffusivity (in unit of standard run-and-tumble diffusivity in two dimensions $D_0 = v^2/2\alpha$) for the different analyzed run-and-tumble models.

| Model     | Effective Diffusivity |
|-----------|-----------------------|
| Isotropic | 1                     |
| Alternate | 1                     |
| Backward  | $3/4$                 |
| Forward   | $3/2$                 |
| Cyclic    | $1/2$                 |

The transition matrix now reads

$$
\Gamma = \frac{1}{2} \begin{pmatrix}
0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 \\
\end{pmatrix}
$$

(32)

corresponding to $\gamma_F = \gamma_B = 0$ and $\gamma_P = 1/2$. The PDF in this case is given by the following expression ($\alpha_1 = \alpha_2 = \alpha$)

$$
\tilde{P} = \frac{(s+\alpha)[(s+\alpha)(s+2\alpha)+k^2v^2/2]}{[k^2v^2+(s+\alpha)^2][k^2v^2+(s+\alpha)^2]-\alpha^2(s+\alpha)^2}
$$

(33)

The mean square displacement is given by the usual form

$$
r^2(t) = \frac{2v^2}{\alpha^2} \left[ \alpha t - 1 + e^{-\alpha t} \right]
$$

(34)

and the effective diffusivity in the diffusive limit reads

$$
D = \frac{v^2}{2\alpha}
$$

(35)
The present case is of particular relevance, as we can write an explicit expression of the PDF in the real space. Indeed, we observe that the orthogonal motion in the \((x, y)\) coordinates reference corresponds to a sum of two independent run-and-tumble motions (with rescaled velocity \(v/\sqrt{2}\) and tumbling rate \(\alpha\)) in the \(\pi/4\) rotated coordinates reference \((x', y')\), as also observed in Refs. [14, 17]). We can then write the explicit solution as a product

\[
P(x, y, t; \alpha, v) = P_{1d}^0 \left( \frac{x + y}{\sqrt{2}}, t; \alpha, \frac{v}{\sqrt{2}} \right) P_{1d}^0 \left( \frac{x - y}{\sqrt{2}}, t; \alpha, \frac{v}{\sqrt{2}} \right)
\]

(36)

where \(P_{1d}^0\) is the PDF of the 1D standard run-and-tumble motion [1, 7].

\[
P_{1d}^0(x, t; \alpha, v) = e^{-\alpha t/2} \left\{ \delta(x - vt) + \delta(x + vt) + \left[ \frac{\alpha}{2v} I_0 \left( \frac{\alpha \Delta(x, t)}{2v} \right) + \frac{\alpha t}{2\Delta(x, t)} I_1 \left( \frac{\alpha \Delta(x, t)}{2v} \right) \right] \theta(vt - |x|) \right\}
\]

(37)

where we have explicitly indicated the parametric dependence on tumbling rate \(\alpha\) and velocity \(v\). \(I_0, I_1\) are the modified Bessel functions of zero and first order and

\[
\Delta = \sqrt{v^2 t^2 - x^2}
\]

(38)

The reported results for the case of alternate motion can be used to obtain the exact solution of the one-dimensional run-and-tumble motion with finite values of the tumbling times. We provide this derivation in Appendix A.

2. Forward motion

Strictly related to the previous case is that of forward motion. In this case a particle, after a tumble, cannot move backward, but only forward or orthogonal to the previous direction of motion, as shown in the following diagram
The transition matrix is then of the form
\[
\Gamma = \frac{1}{3} \begin{pmatrix}
1 & 0 & 1 & 1 \\
0 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 \\
1 & 1 & 0 & 1
\end{pmatrix}
\] (39)
corresponding to \(\gamma_F = \gamma_P = 1/3\) and \(\gamma_B = 0\). The PDF can be written as
\[
\tilde{P} = \frac{(s + \alpha_1)[(s + \alpha_1)(s + 2\alpha_1) + k^2v^2/2]}{[k_x^2v^2 + (s + \alpha_1)^2][k_y^2v^2 + (s + \alpha_1)^2] - \alpha_1^2(s + \alpha_1)^2}
\] (40)
where the effective tumbling rate is
\[
\alpha_1 = \frac{2}{3}\alpha
\] (41)
In other words, the motion is the same of the alternate orthogonal case Eq.(33), with the effective reduced tumbling rate \(\alpha_1\). The mean square displacement has the usual form, with the rescaled tumbling parameter \(\alpha_1\)
\[
r^2(t) = \frac{2v^2}{\alpha_1^2 \alpha_1} \left[ \alpha_1 t - 1 + e^{-\alpha_1 t} \right]
\] (42)
In the diffusive limit, contrary to the previous case, we observe an enhanced diffusion of a factor \(3/2\)
\[
D = \frac{3}{2} \frac{v^2}{2\alpha}
\] (43)

B. Isotropic and backward motions

The second class of motions we consider is that including isotropic and backward cases.
1. Isotropic motion

The isotropic motion refers to a particle that, after a tumble, chooses the new direction of motion among the four allowed ones in an isotropic way. The allowed transitions among states can be represented by the following schematic picture:

![Isotropic Motion Diagram]

The transition probabilities are all equal

$$\gamma_{\mu\nu} = \frac{1}{4} \quad \forall \mu, \nu \in \{R, L, U, D\}$$  \hfill (44)

and the transition matrix reads

$$\Gamma = \frac{1}{4} \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
\end{pmatrix}$$  \hfill (45)

This case is then obtained from the previously derived formulae by setting $$\gamma_F = \gamma_B = \gamma_P = \frac{1}{4}$$. From the general expression (22), with $$\alpha_1 = \alpha$$ and $$\alpha_2 = \alpha/2$$, we have

$$\tilde{P} = \frac{(s + \alpha)[(s + \alpha)^2 + k^2v^2/2]}{[k_x^2v^2 + (s + \alpha)(s + \alpha/2)][k_y^2v^2 + (s + \alpha)(s + \alpha/2)] - \alpha^2(s + \alpha)^2/4}$$  \hfill (46)

where $$k^2 = k_x^2 + k_y^2$$.

The mean square displacement now reads

$$r^2(t) = \frac{2v^2}{2\alpha} \left[ \alpha t - 1 + e^{-\alpha t} \right]$$  \hfill (47)

and the diffusion constant is

$$D = \frac{v^2}{2\alpha}$$  \hfill (48)
2. Backward motion

The backward motion belongs to the same class of the previous case. Here the particle, after a tumble, can only move backward or orthogonal to the previous direction of motion [18].

The transition matrix reads

$$\Gamma = \frac{1}{3}\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$  \hspace{1cm} (49)

corresponding to $\gamma_f = 0$ and $\gamma_B = \gamma_P = 1/3$. The PDF can be then written as

$$\tilde{P} = \frac{(s + \alpha_1)((s + \alpha_1)^2 + k^2v^2/2)}{[k_x^2v^2 + (s + \alpha_1)(s + \alpha_1/2)][k_y^2v^2 + (s + \alpha_1)(s + \alpha_1/2)] - \alpha_1^2(s + \alpha_1)^2/4}$$  \hspace{1cm} (50)

where $\alpha_1$ is the effective tumbling rate

$$\alpha_1 = \frac{4}{3}\alpha$$  \hspace{1cm} (51)

The above expression for the PDF is similar to that of the isotropic case Eq.(46), indicating that the effect of not-forward motion is simply encoded in the rescaled tumbling rate $\alpha_1$.

The mean square displacement is then

$$r^2(t) = \frac{2v^2}{\alpha_1^2} \left[ \alpha_1 t - 1 + e^{-\alpha_1 t} \right]$$  \hspace{1cm} (52)

In the diffusive limit, we observe a reduced diffusivity with respect to the isotropic case, with a diffusion constant reduced by a factor 3/4:

$$D = \frac{3}{4} \frac{v^2}{2\alpha}$$  \hspace{1cm} (53)
We mention that in a recent work the backward model has been studied with generic transition rates, and the above expression for the MSD is recovered in the limit of equal rates [19].

IV. CYCLIC CASE

Many interesting cases in nature show circular motion. This is for example the case of circular trajectories of bacteria close to surfaces [20, 21]. In our discrete orthogonal model this corresponds to consider a rotational cyclic motion [22, 23], with the following sequence of orientation switches

Without loss of generality we are considering here the case of anti-clockwise motion. The transition matrix is not expressed by the symmetric form (20), and it now reads

\[
\Gamma = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}
\]

(54)

Also in this case it is possible to give an explicit expression of the PDF in the Laplace-Fourier space. Proceeding as before, we can solve the linear equations (14) for \( P_\mu \) using the above expression of transition matrix (54). We finally obtain:

\[
\tilde{P} = \frac{(s + 2\alpha)[(s + \alpha)^2 + \alpha^2] + (s + \alpha)k^2v^2/2}{[k_x^2v^2 + (s + \alpha)^2][k_y^2v^2 + (s + \alpha)^2] - \alpha^4}
\]

(55)

From (25) we can obtain the mean square displacement in the Laplace domain

\[
\tilde{r}^2(s) = \frac{2v^2}{s^2} \frac{s + \alpha}{(s + \alpha)^2 + \alpha^2} = \frac{v^2}{\alpha} \left[ \frac{1}{s^2} - \frac{1}{(s + \alpha)^2 + \alpha^2} \right]
\]

(56)
corresponding, in the time domain, to

\[ r^2(t) = \frac{v^2}{\alpha^2} \left[ \alpha t - e^{-\alpha t}\sin(\alpha t) \right] \]  \hspace{1cm} (57)

This expression differs from the previous one \((27)\). However, it has similar asymptotic behaviors: ballistic at short time \( r^2 \simeq v^2t^2 \), and diffusive at long time, \( r^2 \simeq v^2t/\alpha \). In the diffusive limit, \( v, \alpha \to \infty \) with \( v^2/\alpha \) constant, the PDF \((55)\) reduces again to the well known expression

\[ \hat{P}_{\text{Diff}} = \frac{1}{s + Dk^2} \]  \hspace{1cm} (58)

with

\[ D = \frac{1}{2} \frac{v^2}{2\alpha} \]  \hspace{1cm} (59)

The cyclic rotational motion results in a slower diffusion of the particle and the effective diffusivity is reduced by a factor two with respect to the standard active motion (see Table I).

V. CONCLUSIONS

In this work we have treated the planar run-and-tumble walk with orthogonal directions of motion. After formulating the general problem with generic transition probabilities matrix, we focused on symmetric cases, giving analytic expressions of PDF, see Eq.\((22)\), mean-square displacements \((27)\) and effective diffusivity \((30)\) in terms of rescaled tumbling rates. The obtained general formulas have been then specialized to some interesting cases previously investigated in the literature with different approaches (alternate/forward and isotropic/backward motions). We finally discussed the case of circular cyclic motion, reporting expressions of PDF \((55)\) and MSD \((57)\). The reported formulation allows us to treat in a simple way discrete orientational motions, making possible to extend the analysis to different and more complex situations, such as, for example, the cases of run-and-tumble walks with orientational dependent motilities or drift terms, with non-instantaneous tumble events, with non-orthogonal directions of motion, or also extending the analysis of some orthogonal models to higher dimensions or to stochastic resetting processes \[17,24,25\].

14
Acknowledgements

I acknowledge financial support from the MUR PRIN2020 project 2020PFCXPE. I thank Roberto Garra for useful discussions and comments.

Appendix A. 1D Run-and-Tumble with finite tumbling time

An interesting byproduct of the results reported in Sec.3A is obtained considering the marginal distribution, i.e., projecting the solution onto the $\hat{x}$ axis. It is easy to recognize that this corresponds to a one-dimensional run-and-tumble motion with exponentially distributed rest time during tumbling (with the same mean value $1/\alpha$ of the run-time). In other words we have a 1D three-states model in which the particle alternates run motion and rest periods switching between them at the rate $\alpha$ \cite{11,26,27}. By denoting with $P_{1d}^{(3s)}$ the PDF of this one-dimensional three-states motion, we have

$$
\hat{P}_{1d}^{(3s)} (k, s) = \hat{P}(k_x = k, k_y = 0, s)
$$

where the RHS is the previously obtained quantity \cite{33}. We finally obtain

$$
\hat{P}_{1d}^{(3s)} (k, s) = \frac{1}{2(s + \alpha)} \left[ \frac{(s + 2\alpha)^2}{s(s + 2\alpha) + k^2 v^2} + 1 \right]
$$

in agreement with Eq.(17) of Ref. \cite{26} – by setting in that reference $\psi(t) = e^{-\alpha t}$, $\psi(s) = \alpha/(s + \alpha)$, $\tau_T = 1/\alpha$, $P_0 = (s + \alpha)/[(s + \alpha)^2 + k^2 v^2]$. The explicit expression of the PDF in the $(x,t)$ domain can be obtained by performing the inverse Laplace-Fourier transform of the above expression, giving rise, after some algebra

$$
P_{1d}^{(3s)}(x,t) = \frac{e^{-\alpha t}}{4} \left\{ 2\delta(x) + \delta(x - vt) + \delta(x + vt) + \int_{|x|/v}^{t} dt' I_0 \left( \frac{\alpha \Delta(x,t')}{v} \right) \theta(vt - |x|) \right\}
$$

where $\Delta = \sqrt{v^2 t^2 - x^2}$ and $I_0$, $I_1$ are the modified Bessel functions of zero and first order. We note that the above expression is in agreement with that reported in \cite{9} (expressed in a different form) and also with the expression reported by Kolensik \cite{28} considering the
sum of two independent telegraph processes on a line. One can recognize, indeed, that the three-states run-and-tumble process with tumbling rate $\alpha$ and velocity $v$ (run-right, run-left, tumble-rest) can be mapped into a process which is the sum of two independent two-states processes, each one with tumbling rate $\alpha/2$ and velocity $v/2$. Indeed, when the two processes correspond to run motions in the same direction one has twice the velocity of the single process, while when they correspond to two motions in opposite directions one has a rest situation. Moreover, the rate at which happens a tumble of one of the two processes is twice the single rate.

References

[1] Weiss G H 2002, Phys. A (Amsterdam, Neth.) 311, 381
[2] Goldstein S 1951 Q. J. Mech. Appl. Math. 4, 129
[3] Kac M 1974 Rocky Mt. J. Math. 4, 497
[4] Lorentz H A 1905 Arch. N’eerl. 10, 336
[5] Berg H C, E. Coli In Motion (Springer, New York, 2004)
[6] Schnitzer M J 1993 Phys. Rev. E 48, 2553
[7] Martens K, Angelani L, Di Leonardo R and Bocquet L 2012 Eur. Phys. J. E 35, 84
[8] Sevilla F J 2020 Phys. Rev. E 101, 022608
[9] Basu U, Majumdar S N, Rosso A, and Schehr G 2018 Phys. Rev. E 98 062121
[10] Malakar K, Das A, Kundu A, Vijay Kumar K, Dhar A 2020 Phys. Rev. E 101 022610
[11] Santra I , Basu U and Sabhapandit S 2020 Phys. Rev. E 101, 062120
[12] Godoy S and García-Colín 1997 Phys. Rev. E 55, 2127
[13] Orsingher E 2000 Stochastics and Stochastics Reports 69, 1
[14] Cinque F and Orsingher E 2021 [arXiv:2108.10027]
[15] Klafter J and Sokolov, I M First Steps in Random Walks: From Tools to Applications, Oxford University Press, New York (2011).
[16] Detcheverry F 2017 Phys. Rev. E 96 012415
[17] Smith N R, Le Doussal P, Majumdar S N and Schehr G 2022 [arXiv:2207.10445]
[18] Kolesnik A D and Orsingher E 1999 *Theory Probab. Appl.* **46**, 132

[19] Mallikarjun R and Pal A 2022 *arXiv:2209.05912*

[20] Lauga E, DiLuzio W R, Whitesides G M, and Stone H A 2006 *Biophys. J.* **90**, 400

[21] Di Leonardo R, Dell’Arciprete D, Angelani L, and Iebba V 2011 *Phys. Rev. Lett.* **106**, 038101

[22] Kolesnik A D 2004 *Bul. Acad. Științe Republ. Mold. Mat.* **2**, 27

[23] Orsingher E, Garra R and Zeifman A I 2020 *Markov Processes Relat. Fields* **26**, 381

[24] Evans M R and Majumdar S N 2018 *J. Phys. A: Math. Theor.* **51** 475003

[25] Masoliver J 2019 *Phys. Rev E* **99**, 012121

[26] Angelani L 2013 *EPL* **102** 20004

[27] Basu U, Majumdar S N, Rosso A, Sabhapandit S, and Schehr G 2020 *Phys. A: Math. Theor.* **53** 09LT01

[28] Kolesnik A D 2015 *Stochastic and Dynamics* **15**(2) 1550013