A numerical study of the Schrödinger-Newton equation

1: Perturbing the spherically-symmetric stationary states

R Harrison, I Moroz and K P Tod
Mathematical Institute
St Giles
Oxford OX1 3LB

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Abstract

In this article we consider the linear stability of the spherically-symmetric stationary solutions of the Schrödinger-Newton equations. These have been found numerically by Moroz et al [9] and Bernstein et al [3]. The ground state, characterised as the state of lowest energy, turns out to be linearly stable, with only imaginary eigenvalues. The \((n + 1)\)-th state is linearly unstable having \(n\) quadruples of complex eigenvalues (as well as imaginary eigenvalues), where a quadruple consists of \(\{\lambda, \bar{\lambda}, -\lambda, -\bar{\lambda}\}\) for complex \(\lambda\).

1 Introduction

The Schrödinger-Newton equations (hereafter the SN equations) for a quantum-mechanical particle of mass \(m\) moving in its own gravitational potential, are
the pair of coupled non-linear partial differential equations:

\[
\begin{align*}
    i\hbar \frac{\partial \Psi}{\partial t} &= -\frac{\hbar^2}{2m} \nabla^2 \Psi + m\Phi \Psi \\
    \nabla^2 \Phi &= 4\pi G m |\Psi|^2
\end{align*}
\]

where \(\hbar\) is Planck's constant, \(\Psi\) is the wave function, \(\Phi\) is the gravitational potential, \(G\) is Newton's gravitational constant and \(t\) is time.

The SN equations were introduced by Penrose \[10\],\[11\] in his discussion of quantum state reduction by gravity. Penrose suggested that a superposition of two quantum states corresponding to two separated 'lumps' of matter should spontaneously reduce to one or other of the states within a time related to the self-energy of the gravitational field generated by the difference of the densities. This idea requires that there be a preferred basis or special set of quantum states which collapse no further and these, Penrose suggested, should in the non-relativistic limit be the stationary states of the SN equations.

The spherically-symmetric stationary states of the SN equations have been found numerically by a number of authors starting with Ruffini and Bonazzola \[12\] and including Moroz, Penrose and Tod \[9\] and Bernstein, Giladi and Jones \[3\]. Our aim in this article is to test the linear stability of these solutions by linearising the time-dependent SN equations about a spherically-symmetric stationary state. In a later article, we shall present a numerical study of the full non-linear evolution in certain symmetric cases.

The SN equations as introduced by Penrose are closely related to the Schrödinger-Poisson equations which have been studied for much longer (see e.g. \[6\]). They are also the non-relativistic limit of the Einstein equations with a complex Klein-Gordon field as source (see e.g. \[12\]) which is why their solutions are sometimes known as boson stars (see e.g. \[13\]).

The plan of this article will be as follows. In the remainder of this section we review the nondimensionalised SN equations and collect some analytical results on them. In Section 2, we review the spherically-symmetric stationary states and in Section 3 we derive the linearised time-dependent SN equations. In Section 4 we describe the numerical methods which will be used to solve the linearised equations and in Section 5 we present the results.

We shall find that the ground-state solution is linearly stable, having purely imaginary eigenvalues, and the \(n\)-th excited state, equivalently the
(n + 1)-th state, has n unstable modes corresponding to n quadruples of complex eigenvalues, together with purely imaginary eigenvalues. These results are in line with the expectation one might have from similar nonlinear problems like those described in [16] and [4].

Following Moroz et al [9] we begin by introducing dimensionless variables for the system (1). We write the rescaled $\Psi, \Phi$ as $\psi, \phi$, and we rescale space and time but write them using the same variables as before to obtain the nondimensional SN equations:

\[ i\frac{\partial \psi}{\partial t} = -\nabla^2 \psi + \phi \psi, \quad (2) \]
\[ \nabla^2 \phi = |\psi|^2. \]

The most important rescaling, which we note for later use, is that for the time variable, for which $t_{\text{new}} = \gamma t_{\text{old}}$ with

\[ \gamma = \frac{32\pi^2 G^2 m^5}{\bar{\hbar}^3}. \quad (3) \]

Stationary solutions as usual take the form $\psi(x, y, z)e^{-iEt}$ and satisfy the nondimensional time-independent SN equations:

\[ E\psi = -\nabla^2 \psi + \phi \psi, \quad (4) \]
\[ \nabla^2 \phi = |\psi|^2. \quad (5) \]

At this point it is worth collecting some analytical results on the SN equations.

- Existence and uniqueness for solutions of (2) is established by the following simplified version of a theorem of Illner et al [7].

  Given $\chi(x) \in H^2(\mathbb{R}^3)$ with $L_2$-norm equal to 1, the system (2) has a unique, strong solution $\psi(x, t)$, global in time, with $\psi(x, 0) = \chi(x)$ and $\int |\psi|^2 = 1$.

Illner et al [7] give precise regularity properties for the resulting solution.
The equation (4) can be obtained as a variational problem from the Lagrangian:

$$\mathcal{E} = \int (\nabla \psi \cdot \nabla \bar{\psi} + \frac{1}{2} \phi |\psi|^2)$$  \hspace{1cm} (6)$$

where it is understood that $\phi$ is the solution of (5), and the overbar denotes complex conjugation. (All integrals will be over $\mathbb{R}^3$ unless indicated to the contrary.) Here $E$ is a Lagrange multiplier for the normalisation constraint

$$I = \int |\psi|^2 = 1.$$  \hspace{1cm} (7)$$

The quantity $\mathcal{E}$ of (6) is bounded below (14). It is reasonable to suppose, though a proof has not appeared in the literature, that the infimum of $\mathcal{E}$ is attained and that the ground state is the lowest energy spherically-symmetric solution found numerically by the authors mentioned above and proved to exist in [8].

The system (2) preserves the normalisation (7) and the quantity $\mathcal{E}$ of (6) even with time-dependent $\psi$ and $\phi$. We shall therefore call $\mathcal{E}$ the conserved energy. Note that it is different from the energy eigenvalue $E$ for a stationary state appearing in (4).

If we define the kinetic energy $T$ and potential energy $V$ in the obvious way by

$$T = \int |\nabla \psi|^2, \quad V = \int \phi |\psi|^2,$$  \hspace{1cm} (8)$$

then $\mathcal{E} = T + \frac{1}{2} V$. It was shown in [14] that, in a stationary state,

$$T = -\frac{1}{3} E, \quad V = \frac{4}{3} E, \quad \mathcal{E} = \frac{1}{3} E.$$  \hspace{1cm} (9)$$

In particular, as one expects, the energy eigen-values are all negative.

There is an interesting observation on the dispersion of the wavefunction due to Arriola and Solar [1]. Define the second moment $Q$ by

$$Q = \int |x|^2 |\psi|^2$$  \hspace{1cm} (10)$$
assuming that the initial data is such that this integral exists, then it follows from (2) that
\[ \tilde{Q} = \int (8|\nabla \psi|^2 + 2\phi|\psi|^2) = 8\mathcal{E} - 2V. \] (11)

Now recall that, as a consequence of the maximum principle (see e.g. [5]) \( \phi \) is everywhere negative and therefore so is \( V \). Thus if the conserved energy \( \mathcal{E} \) is positive, then the dispersion grows at least quadratically in time. In this sense, a positive energy solution necessarily scatters.

2 The spherically-symmetric stationary states

For a spherically-symmetric stationary state, there is no loss of generality in assuming that the wave-function is real. The time-independent SN equations can therefore be written in the form
\[
\begin{align*}
(r\psi)_{rr} &= -r\psi U, \\
(rU)_{rr} &= -r\psi^2,
\end{align*}
\] (12)

where we have introduced the variable \( U = E - \phi \) and used a subscript \( r \) for \( \frac{d}{dr} \).

The boundary conditions are that \( \psi \to 0 \) as \( r \to \infty \) and \( \psi_r = 0 = \phi_r \) at \( r = 0 \). If \( (\psi, U, r) \) is a solution then so is \( (\lambda^2 \psi, \lambda^2 U, \lambda^{-1} r) \) for any nonzero \( \lambda \). This scaling can be used to impose the normalisation (7) retrospectively. To solve (12), Moroz et al [9] use a shooting method. They set \( U(0) = 1 \), which can be done with a suitable \( \lambda \) leaving the solution not correctly normalised, then they choose values of \( \psi(0) \) and integrate away from \( r = 0 \). The integration continues until the solution for \( \psi \) diverges at some value of \( r \). This value of \( r \) can be pushed out to larger distances and the solution may diverge to large positive or large negative values. By adjusting \( \psi(0) \) it is possible to bracket a value at which \( \psi \) remains finite and then \( E \) can be identified from the limit of \( U \) at large distances.

With better triggering to recognise the diverging solutions, this technique can be refined and it becomes feasible to obtain the first 50 energy levels in a reasonable time. The first 20 eigenvalues calculated with the above routine are given in Table [4]. The first 16 agree to this order with those
| Number of zeros | Energy Eigenvalue |
|-----------------|-------------------|
| 0               | -0.163            |
| 1               | -0.0308           |
| 2               | -0.0125           |
| 3               | -0.00675          |
| 4               | -0.00421          |
| 5               | -0.00287          |
| 6               | -0.00209          |
| 7               | -0.00158          |
| 8               | -0.00124          |
| 9               | -0.00100          |
| 10              | -0.000823         |
| 11              | -0.000689         |
| 12              | -0.000585         |
| 13              | -0.000503         |
| 14              | -0.000437         |
| 15              | -0.000384         |
| 16              | -0.000339         |
| 17              | -0.000302         |
| 18              | -0.000271         |
| 19              | -0.000244         |
| 20              | -0.000221         |

Table 1: The first 20 eigenvalues given by Bernstein et al [3]. A log-log plot of the $n$-th energy $E_n$ against $n$ is shown in figure 1. The slope is asymptotically very close to -2, which would be the exact value for the Hydrogen atom (as was previously noted by Bernstein et al [3]). Once one has the eigenvalues, one can alternatively use a spectral method to find the eigenfunctions and corresponding potentials at the Chebyshev points. These will be needed in Sections 3 and 4 (where the definition of Chebyshev points is also given).
3 The perturbation equations

In this section we shall set up the linear stability problem, ready for numerical solution in Section 4. We look for a solution to (2) which can be expanded in terms of a small parameter $\epsilon$ as

$$\begin{align*}
\psi(x,t) &= \psi_0 + \epsilon \psi_1 + \ldots, \\
\phi(x,t) &= \phi_0 + \epsilon \phi_1 + \ldots,
\end{align*}$$

Substituting (13) into the SN equations (2) and equating powers of $\epsilon$ we obtain

$$\begin{align*}
i \psi_0 + \nabla^2 \psi_0 &= \psi_0 \phi_0 \\
\nabla^2 \phi_0 &= |\psi_0|^2 \\

i \psi_1 + \nabla^2 \psi_1 &= \psi_0 \phi_1 + \psi_1 \phi_0 \\
\nabla^2 \phi_1 &= \psi_0 \tilde{\psi}_1 + \tilde{\psi}_0 \psi_1.
\end{align*}$$

We restrict to spherical symmetry and take

$$\begin{align*}
\psi_0 &= R_0(r)e^{-iEt}, \\
\phi_0 &= E - U_0(r),
\end{align*}$$

Figure 1: Log-log plot of the energy values against $n$
where $R_0$ and $U_0$, both real, are known from the work described in Section 2. We seek solutions of the form

$$
\psi_1 = \frac{1}{r}R(r,t)e^{-iEt},
$$

$$
\phi_1 = \frac{1}{r}\phi(r,t),
$$

(17)

where $R$ is complex and $\phi$ is real. With this choice of variables, (15) becomes

$$
i R_t + R_{rr} = R_0\phi - U_0R,
$$

$$
\phi_{rr} = R_0\bar{R} + \bar{R}_0R.
$$

(18)

We look for a solution of the form

$$
R = (A + B)e^{\lambda t} + (\bar{A} - B)e^{\bar{\lambda} t},
$$

$$
\phi = W e^{\lambda t} + \bar{W} e^{\bar{\lambda} t},
$$

(19)

where we assume for now that $\lambda$ is not real and that $A, B$ and $W$ are functions only of $r$. Now we substitute into (18) and equate coefficients of $e^{\lambda t}$ and $e^{\bar{\lambda} t}$ still assuming that $\lambda$ is not real. This gives

$$
W_{rr} = 2R_0 A,
$$

$$
A_{rr} + U_0 A - R_0 W = -i\lambda B,
$$

$$
B_{rr} + U_0 B = -i\lambda A.
$$

(20)

If $(A, B, W)$ is a solution of (20) with eigenvalue $\lambda$ then $(A, -B, W)$ is a solution with eigenvalue $-\lambda$. Thus eigenvalues come in quadruples $(\lambda, -\lambda, \bar{\lambda}, -\bar{\lambda})$ for complex $\lambda$. If there exists a solution of (20) with real $\lambda$ then the real and imaginary parts of the system decouple. We may assume without loss of generality that $A$ and $W$ are real and $B$ is pure imaginary and then there is once again a solution of the form of (19) to (18). Thus (20) gives the perturbation equations with no restriction on $\lambda$. From the definitions (17) we need $A, B$ and $W$ to vanish at $r = 0$ and to vanish asymptotically as $r$ tends to infinity (or at the outer edge of the grid in a numerical calculation).

We note that $\lambda = 0$ will be an eigenvalue of (20) with $A = W = 0, B = R_0$. This is a trivial ‘zero-mode’ corresponding to an infinitesimal constant rotation in the phase of the unperturbed state. It will be found by the
numerical calculation in Section 5 and we shall discard it by hand.

We can obtain a condition on $\lambda^2$ as follows: multiply the second of (20) by $\bar{A}$ and the third by $\bar{B}$ and integrate to find with the aid of the first that

$$-i\lambda \int_0^\infty \bar{A}Bdr = \int_0^\infty (U_0|A|^2 - |A_r|^2 - \frac{1}{2}|W_r|^2)dr,$$

$$-i\lambda \int_0^\infty \bar{B}Adr = \int_0^\infty (U_0|B|^2 - |B_r|^2)dr.$$

The right-hand-sides in (21) are both real, so that the quotient

$$\frac{-i\lambda \int_0^\infty \bar{A}Bdr}{i\lambda \int_0^\infty \bar{B}Adr},$$

is real provided it is well-defined. Hence we have a dichotomy:

either $\lambda^2$ is real or $\int_0^\infty \bar{A}Bdr = 0.$

This will provide a useful check on the numerics in Section 5.

By a more sophisticated argument, which we shall relegate to an appendix, we may obtain an upper bound on the real part of any eigenvalue in terms of the energy of the state being perturbed. For perturbations of a state with energy-eigenvalue $E$ this is

$$|\Re(\lambda)| \leq \frac{4}{9\pi^2} \sqrt{-E}.$$  

This is an interesting analytic result which limits the rate at which an unstable state can decay. It also will provide a useful check in Section 5. We now turn to the problem of solving the perturbation equations.

4 Solving the perturbation equations

We solve the perturbation equations by a spectral method (see e.g. [3]) which reduces the calculation to that of solving a matrix eigenvalue problem.

We choose a range $(0, L)$ for $r$, regarding $r = L$ as being at infinity for the purpose of setting boundary conditions, and set $x = \frac{2r}{L} - 1$. Next we choose a number $N$ and approximate any function $f(x)$ by the unique polynomial
$p(x)$ of degree $N$ or less such that $p(x_k) = v_k$ for $k = 0, 1 \ldots N$, where $v_k = f(x_k)$ and $x_k = \cos(k\pi/N)$ are the Chebyshev points.

To approximate the derivative we introduce $w_k$ by $w_k = p'(x_k)$ for $k = 0, 1 \ldots N$. Then the differentiation matrix for polynomials of degree $N$, denoted $D_N$, is defined by the equation

$$
\begin{pmatrix}
w_0 \\
w_1 \\
\vdots \\
w_N
\end{pmatrix} = D_N
\begin{pmatrix}
v_0 \\
v_1 \\
\vdots \\
v_N
\end{pmatrix}.
$$

(25)

For an explicit formula for $D_N$, see e.g. [15].

The second-derivative matrix is just $D_N^2$. The requirement that $A$, $B$ and $W$ be zero at the boundary is imposed by deleting the first and last rows and columns of the relevant differentiation matrix, in this case $D_N^2$. This yields an $(N-1) \times (N-1)$ matrix conveniently denoted by $\tilde{D}_N^2$.

For the perturbation equations (21) we therefore obtain the matrix eigenvalue equation

$$
\begin{pmatrix}
-2R_0 & 0 & \tilde{D}_N^2 \\
0 & \tilde{D}_N^2 + U_0 & 0 \\
-\tilde{D}_N^2 - U_0 & 0 & R_0
\end{pmatrix}
\begin{pmatrix}
A \\
B \\
W
\end{pmatrix} = i\lambda
\begin{pmatrix}
0 & 0 & 0 \\
-I & 0 & 0 \\
0 & I & 0
\end{pmatrix}
\begin{pmatrix}
A \\
B \\
W
\end{pmatrix},
$$

(26)

where

$$
\begin{pmatrix}
A(x_1) \\
\vdots \\
A(x_{N-1}) \\
B(x_1) \\
\vdots \\
B(x_{N-1}) \\
W(x_1) \\
\vdots \\
W(x_{N-1})
\end{pmatrix},
$$

(27)

and

$$
R_0 = \text{diag}(R_0(x_1), R_0(x_2), \ldots, R_0(x_{N-1}))
$$

$$
U_0 = \text{diag}(U_0(x_1), U_0(x_2), \ldots, U_0(x_{N-1})).
$$
where \( diag \) indicates a diagonal matrix.

One might worry that since the matrix on the right in (26) is singular, accuracy in the eigenvalues was lost. By solving the equation for \( W \) in terms of \( A \) we can rewrite (26) as

\[
\begin{pmatrix}
0 & \tilde{D}_N^2 + U_0 \\
\tilde{D}_N^2 + U_0 - 2R_0(\tilde{D}_N^2)^{-1}R_0 & 0
\end{pmatrix}
\begin{pmatrix}
A \\
B
\end{pmatrix}
= -i\lambda
\begin{pmatrix}
I & 0 \\
0 & I
\end{pmatrix}
\begin{pmatrix}
A \\
B
\end{pmatrix}
\]

(28)

However the difference between the eigenvalues computed by using (28) and by using (26) turns out to be small and we can just use (26).

5 Numerical results

We consider first the results obtained by solving (26) about the ground state of the system (4),(5). Recall for comparison that the eigenvalue for the ground state in non-dimensionalized units is \(-0.163\). The conversion factor to dimensional units is given by (3). In figure 2 we plot those eigenvalues obtained by solving (26) which are closest to the origin, excluding the near-zero eigenvalue which does not correspond to a non-trivial perturbation. To compute these results, we used \( N = 60 \) and \( L = 150 \). (That the near-zero eigenvalue found does indeed correspond to the trivial zero-mode can be seen by plotting the corresponding eigenfunction. One finds that \( A, W \) are very small (\( O(10^{-7}) \)) as compared to \( B \), and that \( B \) is close to \( R_0 \), so that this solution indeed corresponds to the zero-mode foreseen in Section 3.)

In figure 3 we plot all the eigenvalues obtained by solving (26) (note the difference in vertical scale from figure 2) so that the quadruple of complex eigenvalues appears as a pair) to show that up to these limits there are no eigenvalues other than imaginary ones. The first few eigenvalues obtained, now including the near-zero one, are presented in table 2. The non-trivial eigenvalues are all imaginary and we may conclude that the ground state is linearly stable.

To test convergence of the method we plot graphs of a calculated eigenvalue against increasing \( N \) and \( L \). As an example we show the graph of the fifth eigenvalue \( 0.0765i \) as the value of \( N \) increases in figure 4 and as the
Figure 2: The smallest eigenvalues of the perturbation about the ground state

Figure 3: All the computed eigenvalues of the perturbation about the ground state
Table 2: Eigenvalues of the perturbation about the ground state

| ±0.000000116 |
| ±0.0341i |
| ±0.0603i |
| ±0.0688i |
| ±0.0731i |
| ±0.0765i |
| ±0.0810i |
| ±0.0867i |

value of $L$ increases in figure 5. There is satisfactory convergence.

Using the same method we compute the numerical solutions of the perturbation about the second state. This time we obtain some eigenvalues with nonzero real parts. In figure 6 we plot the eigenvalues nearest to the origin for the case where $N = 60$ and $L = 150$. In figure 7 we plot all the eigenvalues obtained with a different vertical scale (so that the quadruple of complex eigenvalues appears as a pair) to see that up to these limits there are no other complex ones.

From Section 3 we have two analytic conditions on the eigenvalues. The first (23) relates to complex eigenvalues and we do now have some. By the result of (23) we expect that for these $\int_{0}^{\infty} A B d\tau = 0$. To see whether this is the case (up to numerical error) we compute

$$Q = \frac{\left| \int_{0}^{L} A B d\tau \right|}{\left( \int_{0}^{L} |A|^2 d\tau \right)^{1/2} \left( \int_{0}^{L} |B|^2 d\tau \right)^{1/2}}$$

which we want to be much less than one. For $L = 145$ and $N = 60$ we display in table 3 the calculated values of $Q$ with the eigenvalues nearest the origin. As expected, for the eigenvalues with nonzero real part, $Q$ is zero within numerical error and the result of (23) is confirmed.

For the third state, with $N = 60$ and $L = 450$, we present the eigenvalues in figure 8. We have also calculated the eigenvalues for the fourth state and a consistent pattern emerges: there are $n$-quadruples of complex eigenvalues for perturbations about the $(n+1)$-th bound state (equivalently for the $n$-th excited state); the rest of the eigenvalues come in purely imaginary pairs.

With the computed eigenvalues for the higher states available we can check their consistency with the other result, (24), from Section 3. In ta-
Figure 4: The change in the sample eigenvalue with increasing values of $N$ ($L = 150$)

Figure 5: The change in the sample eigenvalue with increasing values of $L$ ($N = 60$)
Figure 6: The lowest eigenvalues of the perturbation about the second bound state

Figure 7: All the computed eigenvalues of the perturbation about the second bound state
Table 3: $Q$ for different eigenvalues of the perturbation about the second state

| $\lambda$                  | $Q$     |
|---------------------------|---------|
| $0.003i$                  | 0.235   |
| $0.00860i$                | 0.368   |
| $-0.00139 - 0.010i$       | $3.53e^{(-14)}$ |
| $0.00139 + 0.010i$        | $3.92e^{(-14)}$ |
| $0.0153i$                 | 0.894   |

Table 4: Bound on the real part of the eigenvalues

| State | Computed maximum $\Re(\lambda)$ | Bound from (24) |
|-------|----------------------------------|-----------------|
| 1     | 0                                | 0.00362         |
| 2     | 0.00139                          | 0.00158         |
| 3     | 0.000520                         | 0.00101         |
| 4     | 0.000225                         | 0.000738        |
| 5     | 0.000114                         | 0.000583        |
| 6     | 0.0000653                        | 0.000482        |

can check the accuracy of the spectral method by solving (20) with each eigenvalue in turn. Using a Runge-Kutta method, we arrive at the same eigenfunctions as by the spectral method.

To summarise, in this paper we have used a spectral method to analyse the linear stability of the stationary solutions of the SN equations in the context of the time-dependent SN equations. We have found that the ground-state solution is linearly stable having purely imaginary eigenvalues and the $n$-th excited state has $n$ unstable modes corresponding to $n$ quadruples of complex eigenvalues, together with purely imaginary eigenvalues. These results are tested for convergence and compliance with two analytical results. They are also in line with the expectation one might have from similar nonlinear problems like those described in [16] and [4]. We shall consider the nonlinear
Figure 8: The eigenvalues of the perturbation about the third bound state

stability in a second paper where this picture is confirmed and one can see
the excited states decaying to the ground state.

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References

[1] E.R. Arriola and J. Soler, Asymptotic behaviour for the 3-D Schrödinger-
Poission system in the attractive case with positive energy, App. Math. Lett. 12 (1999) 1-6

[2] T. Aubin, “Nonlinear analysis on manifolds; Monge-Ampere equations”, Grundlehren der mathematischen Wissenschaften 252 New York: Springer-Verlag (1982)
[3] D.H.Bernstein, E.Giladi and K.R.W.Jones, Eigenstates of the Gravitational Schrödinger Equation, Modern Physics Letters A13 (1998) 2327-2336

[4] P.Bizoń, Equivariant self-similar wave maps from Minkowski spacetime into 3-sphere, math-ph/9910026 to appear in Comm. Math. Phys.

[5] L.C.Evans, “Partial differential equations”, AMS Graduate Studies in Mathematics 19 AMS (1998)

[6] H.Lange, B.Toomire and P.F.Zweifel, An overview of Schrödinger-Poisson Problems, Reports on Mathematical Physics 36 (1995) 331-345

[7] R.Illner, P.F.Zwiefel and H.Lange, Global existence, uniqueness and asymptotic behaviour of solutions of the Wigner-Poisson and Schrödinger-Poisson systems, Math.Meth in App.Sci.17 (1994) 349-376

[8] I.M.Moroz and K.P.Tod, An Analytical Approach to the Schrödinger-Newton equations, Nonlinearity 12 (1999) 201-16

[9] I.M.Moroz, R.Penrose and K.P.Tod, Spherically-symmetric solutions of the Schrödinger-Newton equations, Class Quantum Grav. 15 (1998) 2733-2742

[10] R.Penrose, On gravity’s role in quantum state reduction, Gen.Rel.Grav. 28 (1996) 581-600

[11] R.Penrose, Quantum computation, entanglement and state reduction, Phil.Trans.R.Soc. (Lond) A 356 (1998) 1927

[12] R.Ruffini and S.Bonazzola, Systems of Self-Gravitating Particles in General Relativity and the concept of an Equation of State, Phys.Rev.187 (1969) 1767

[13] B.Schupp, J.J. van der Bij, An axially-symmetric Newtonian boson star, Phys. Lett. B 366 (1996) 85-88

[14] K.P.Tod, The ground state energy of the Schrödinger-Newton equations, Phys.Lett.A 280 (2001) 173-176

[15] L.N.Trefethen, “Spectral Methods in MATLAB”, Philadelphia: SIAM (2000).
Appendix

In this appendix, we shall prove (24) starting from (20). The calculation uses the best constant for the Sobolev inequality in three-dimensions given by Aubin ([2]). To work in three-dimensions, we first define $a = A/r$, $b = B/r$ and $w = W/r$. In these variables we can write (20) as

\begin{align*}
\nabla^2 w &= 2R_0a, \\
\nabla^2 a + U_0 a - R_0 w &= -i\lambda b, \\
\nabla^2 b + U_0 b &= i\lambda a.
\end{align*}  \tag{30}

We multiply the second of (30) by $\bar{a}$, the complex conjugate of the third by $b$, add and integrate by parts to find

\begin{equation}
i \int [\lambda a \bar{a} + \bar{\lambda} b b] = - \int R_0 b \bar{w}, \tag{31}\end{equation}

where, as usual, the integrals are over $\mathbb{R}^3$. Now we make repeated use of the Holder inequality and the Sobolev inequality in 3-dimensions:

\begin{equation}(\int F^6)^\frac{1}{6} \leq K(\int |\nabla F|^2)^\frac{1}{2} \tag{32}\end{equation}

where $K$ is the relevant Sobolev constant, namely

\begin{equation}K = \frac{2^\frac{5}{3}}{3^\frac{2}{3} \pi^\frac{1}{3}}. \tag{33}\end{equation}

From the first of (30) and the Holder inequality we find

\begin{equation}\int |\nabla w|^2 \leq 2 \int |R_0 aw| \leq 2(\int |R_0|^3)^\frac{1}{3}(\int |a|^2)^\frac{1}{2}(\int |w|^6)^\frac{1}{6}\tag{33}\end{equation}

and using the Sobolev inequality in this gives

\begin{equation}(\int |\nabla w|^2)^\frac{1}{2} \leq 2K(\int |R_0|^3)^\frac{1}{3}(\int |a|^2)^\frac{1}{2}. \tag{33}\end{equation}
Next we calculate

$$\int |R_0w| \leq (\int |R_0|^3)^{\frac{1}{2}} (\int |b|^2)^{\frac{1}{2}} (\int |w|^6)^{\frac{1}{6}}$$

$$\leq K (\int |R_0|^3)^{\frac{1}{2}} (\int |b|^2)^{\frac{1}{2}} (\int |w|^6)^{\frac{1}{6}}$$

$$\leq 2K^2 (\int |R_0|^3)^{\frac{1}{2}} (\int |a|^2)^{\frac{1}{2}} (\int |b|^2)^{\frac{1}{2}}$$

where the first step uses the Holder inequality, the second uses the Sobolev inequality and the third uses equation (33). We shall put this together with (31). First we normalise the perturbation so that

$$\int |a|^2 = \cos^2 \theta, \int |b|^2 = \sin^2 \theta,$$

for some $\theta$ and set $\lambda = \mu + i\nu$. Now from (31) and (34) we have

$$|\mu + i\nu \cos 2\theta| \leq K^2 \sin \theta (\int |R_0|^3)^{\frac{1}{2}}.$$

from which it follows that

$$|\mu| \leq K^2 (\int |R_0|^3)^{\frac{1}{2}}.$$

We need an upper limit for the right-hand-side in this, which we obtain as follows:

$$\int |R_0|^3 \leq (\int R_0^2)^{\frac{1}{2}} (\int R_0^6)^{\frac{1}{6}}$$

$$\leq K^{\frac{1}{2}} (\int |\nabla R_0|^2)^{\frac{1}{2}}$$

$$= K^{\frac{1}{2}} T^{\frac{2}{3}}$$

using the Holder and Sobolev inequalities again and the definition (8) of $T$. Using (9) for $T$ in a stationary state and the definition (32) of $K$ we finally arrive at the desired limit

$$|\mu| \leq \frac{4}{9\pi^2} \sqrt{-E}.$$  (35)