THE OSTROVSKY-HUNTER EQUATION WITH A SPACE DEPENDENT FLUX FUNCTION

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Abstract. We study the periodic Ostrovsky-Hunter equation in the case where the flux function may depend on the spatial variable. Our main results are that if the flux function is twice differentiable, then there exists a unique entropy solution. This entropy solution may be constructed as a limit of approximate solutions generated by a finite volume scheme, and the finite volume approximations converge to the entropy solution at a rate $1/2$.

1. Introduction

To model small-amplitude long waves in a rotating fluid of finite depth, Ostrovsky [1] derived the following non-linear evolution equation
\begin{equation}
\partial_x (\partial_t u + \partial_x f(u) - \beta \partial_{xxx} u) = \gamma u, \tag{1.1}
\end{equation}
where $\beta$ and $\gamma > 0$ are real constants and $f(u) = \frac{u^2}{2}$. Here $u = u(t, x)$ denotes the amplitude of waves, while $x$ and $t$ are the space and time variables respectively. The equation can be formally deduced using two asymptotic expansions of the shallow water equations, once with respect to the rotation frequency and then with respect to the amplitude of the waves, see [2]. Later, in a study of long internal waves in a rotating fluid Hunter [2], investigated the limit of no high-frequency dispersion $\beta \to 0$. This formally reduces (1.1) to the Ostrovsky-Hunter (OH) equation:
\begin{equation}
\partial_x (\partial_t u + \partial_x f(u)) = \gamma u \tag{1.2}
\end{equation}
The OH equation also arises as a model of high frequency waves in a relaxing medium, see [3]. In both cases $f(u) = \frac{u^2}{2}$.

Equation (1.2) can also be derived by including the effects of background rotation in the shallow water equation, and then using singular perturbation methods, see [6, 7]. In this context, it is worth mentioning that equation (1.1) generalizes the KdV equation, which corresponds to $\gamma = 0$. The equation (1.2) is also known as the reduced Ostrovsky equation [1, 8, 10], short wave equation [2], Ostrovsky-Vakhnenko equation [11, 12], or Vakhnenko equation [4, 5, 9]. Also, equation (1.1) is used to model ultra short light pulses in silica optical fibres [13, 14, 15, 20], in which case $f(u) = -\frac{1}{6}u^3$. In this case (1.1) is sometimes referred to as the “short-pulse-equation”.

In this context we note that Hunter established the connection between the KdV equation and short wave equation (1.2), see [2], as the no-rotation and no-long wave dispersion limits of the same equation. But in the case of oceanic waves near the shore, the waves usually propagate on a background whose properties vary. In such a variable medium the linear phase speed of the wave, which is encoded in the flux...
term $f(x, u)$ (instead of $f(u)$) has spatial dependency. To model such scenario the variable coefficient KdV equation was derived by Johnson [23] for water waves and by Grimshaw [24] for internal waves (see also [25] for a review). Motivated by this, in this paper we aim to design and analyze a numerical scheme for the OH equation with spatial dependency in the flux.

As is commonly done, we rewrite the OH equation (1.2) as the following system

$$u_t + f(x, u)_x = \gamma P, \quad P_x = u.$$  

Without loss of generality, we can set $\gamma = 1$, and will in the sequel do so. Since $P$ is defined as any anti-derivative of $u$, we need an additional constraint to close the system. This can be done in different ways, see [18, 19, 21]. We shall adopt the approach in [22], where we study the problem (1.2) in a periodic setting $x \in [0, 1]$. In this case it is natural to redefine the right hand side by subtracting the (constant) term $\int_0^1 P(t, x) \, dx$. This has the attractive side effect of conserving $\int_0^1 u(x) \, dx$, i.e.,

$$\frac{d}{dt} \int_0^1 u(t, x) \, dx = 0,$$

if $u$ satisfies the zero mean condition $\int_0^1 u \, dx = 0$. This condition was also assumed to hold initially in [14, 2]. So the equation we are studying in this paper reads

$$(1.3) \quad u_t + f(x, u)_x = P(t, x) - \int_0^1 P(t, y) \, dy,$$

where $x \mapsto u(t, x)$ is periodic with period 1, and $P(t, x) = \int_0^x u(t, y) \, dy$. Note that since $u$ satisfies the zero mean condition, $x \mapsto P(t, x)$ is also periodic. This is essentially an extension of the system studied in [22] to the case where $f$ is allowed to depend on the spatial variable $x$. As in [22], discontinuities in $u$ will develop independently of the smoothness of the initial data, so that (1.3) must be interpreted in the weak sense. Furthermore, as with scalar conservation laws, in order to show well posedness, we shall consider entropy solutions.

Our main results are as follows. Assuming that the mapping $x \mapsto f(x, v)$ is uniformly Lipschitz continuous locally in $v$, and that the initial data are of bounded variation and satisfy the zero mean condition, we have that

$$\|u(t, \cdot) - v(t, \cdot)\|_{L^1((0, 1))} \leq C e^{2t} \|u_0 - v_0\|_{L^1((0, 1))},$$

where $u$ and $v$ are entropy solutions with initial data $u_0$ and $v_0$ respectively. Furthermore, we establish convergence to the unique entropy solution of approximate solutions generated by an upwind scheme. We also prove a Kuznetsov-type lemma, see [26], satisfied by the entropy solution, and this lemma allows us to conclude that the approximate solutions converge at the rate $1/2$ in $L^1$.

The rest of this paper is organized as follows. In Section 2 we detail precise definitions and assumptions and notation, as well as the definition of the finite volume scheme. In Section 3 we prove the necessary bounds which imply that the approximate solutions form a strongly compact family in $C([0, T]; L^1((0, 1)))$. Furthermore, we prove that the approximate solutions satisfy an entropy inequality, and this is used to show that any limit of the approximate solutions is an entropy solution. In Section 4 we establish a “Kuznetsov type lemma” enabling us to compare exact entropy solutions with arbitrary functions. Then this comparison result is used to show that the approximate solutions “converge at a rate”. Finally, in Sections 5 we exhibit some concrete numerical results.
2. Preliminaries and notation

The problem we study is the following
\begin{equation}
    u_t + f(x, u)_x = \int_0^x u(t, y) \, dy - \int_0^1 \int_0^y u(t, z) \, dz \, dy, \quad \text{for } x \in (0, 1), \ t \in (0, T),
\end{equation}
\begin{align*}
    u(0, x) &= u_0(x), \quad \text{for } x \in (0, 1), \\
    u(t, 0) &= u(t, 1), \quad \text{for } t \in (0, T).
\end{align*}

Regarding the initial data we shall assume that
\[ u_0 \in BV([0, 1]) \quad \text{and} \quad \int_0^1 u_0(x) \, dx = 0. \]

The flux function \( f(x, u) \) is assumed to be in \( C^2_{\text{loc}} \), which in particular implies that it is Lipschitz continuous in \( x \) and locally Lipschitz continuous in \( u \). Since solutions of (2.1) generically develop discontinuities, solutions must be considered in the weak sense. A function in \( C([0, T]; L^1((0, 1))) \) is a weak solution of (2.1) if
\[
    \int_0^T \int_0^1 u \phi_t + f(x, u) \phi_x + P[u] \phi \, dx \, dt \\
    + \int_0^1 u(0, x) \phi(x, 0) \, dx - \int_0^1 u(T, x) \phi(T, x) \, dx = 0,
\]
for all test functions \( \varphi = \varphi(t, x) \) which are 1-periodic in \( x \). Here \( P[u] := \int_0^x u(t, y) \, dy - \int_0^1 \int_0^y u(t, z) \, dz \, dy \). Let \( \Pi_T = [0, T] \times [0, 1] \), following [22] we define entropy solutions as

**Definition 1** (Entropy Solution). A function \( u \in C([0, T]; L^1((0, 1))) \cap L^\infty(\Pi_T) \) is called an entropy solution of the Ostrovsky-Hunter Equation (2.1), if for all constants \( k \) the following inequality holds
\begin{equation}
    \int_{\Pi_T} \eta(u, k) \phi_t + q(x, u, k) \phi_x + \text{sign} (u - k) f_x(x, k) \varphi + \text{sign} (u - k) P[u] \varphi \, dx \, dt \\
    - \int_0^1 \eta(u(t, x), k) \phi(t, x) \, dx \bigg|_{t=0}^{t=T} \geq 0,
\end{equation}
for all non-negative test functions \( \varphi \) which are 1-periodic in the \( x \) variable. Here \( \eta \) and \( q \) are defined as the Kružkov entropy and entropy flux respectively, \( \eta(u, k) = |u - k|, \ q(x, u, k) = \text{sign} (u - k) (f(x, u) - f(x, k)). \)

Throughout this paper we employ the following convention, \( f_x(x, u) \) and \( f_u(x, u) \) denote the partial derivatives of \( f \) with respect to \( x \) and \( u \) respectively. If \( u = u(t, x) \) is differentiable, then we have
\[
    \frac{\partial}{\partial x} f(x, u(t, x)) = f_x(x, u) + f_u(x, u) u_x.
\]
Furthermore, we use the convention that \( C \) denotes a generic positive constant, whose actual value may change from one occurrence to the next.

In order to define the numerical scheme, set
\[
    \Delta x = \frac{1}{N}, \quad \text{and} \quad \Delta t = \frac{T}{M + 1},
\]
where \( N \) and \( M \) are positive integers, and \( T > 0 \). We also define \( x_{j+1/2} = j \Delta x \) for \( j = 0, \ldots, N \), \( x_j = x_{j+1/2} - \Delta x/2 \) for \( j = 1, \ldots, N \) and \( t^n = n \Delta t \) for \( n = 0, 1, 2, \ldots \). We also define the intervals \( I_j = [x_{j-1/2}, x_{j+1/2}) \) for \( j = 1, \ldots, N \).
and $I^n = [t^n, t^{n+1})$. In order to define a piecewise constant approximations, set $I^n_j = I^n \times I_j$.

Next we define the finite volume approximation. Let $F(x, u, v)$ be a numerical flux which is monotone and consistent, i.e.,

$$
u \mapsto F(x, u, v) \text{ is non-decreasing},$$

$$F(x, u, v) \text{ is non-increasing},$$

$$F(x,u,u) = f(x,u).$$

In addition we assume that $F$ is differentiable in $x$ and that both $F_x$ and $F$ are Lipschitz continuous in $u$ and $v$.

We can now define the finite volume scheme. Set $\lambda = \Delta t / \Delta x$, and let $u_{n+1}^j$ be defined by

$$u_{n+1}^j = u_n^j - \lambda \left( F_{n+1/2}^j - F_{n-1/2}^j \right) + \Delta t P_n^j,$$

for $n \geq 0$ and $j = 1, \ldots, N$. Here

$$F_{n+1/2}^j = F(x_{j+1/2}, u_n^j, u_{n+1}^j),$$

where we use periodic boundary conditions $u_{N+1}^n = u_1^n$ and $u_0^n = u_N^n$. The term $P_n^j$ is defined as

$$P_n^j = \Delta x \left( \sum_{i=1}^{j-1} u_i^n + \frac{1}{2} u_j^n \right) - (\Delta x)^2 \sum_{i=1}^{N} \left( \sum_{i=1}^{\ell-1} u_i^n + \frac{1}{2} u_{\ell}^n \right).$$

Finally we define the initial values $u_0^j$ by

$$u_0^j = \frac{1}{\Delta x} \int_{I_j} u_0(x) \, dx, \quad \text{for } j = 1, \ldots, N.$$

Observe that since $u_0$ is assumed to be of bounded variation,

$$|u_0|_{BV} := \sum_{j=1}^N |u_0^j - u_0^{j-1}| \leq |u_0|_{BV([0,1])} < \infty.$$

The spatial discretization $\Delta x$ and the temporal $\Delta t$ are related through a CFL-condition. Consider the map

$$\Psi_j(u,v,w) = u - \lambda (F(x_{j+1/2}, v, w) - F(x_{j-1/2}, u, v)).$$

We choose $\lambda$ so small that for all $j$, $\Psi_j$ is non-decreasing in all its arguments. For this monotonicity to hold it is sufficient to choose

$$\Delta t \leq c_f \Delta x,$$

where $c_f$ is a constant depending on $f$ (through $F$).

It is also useful to define

$$D_a a_j = \frac{a_j - a_{j-1}}{\Delta x}, \quad \text{and} \quad D_a^t a_n = \frac{a_n^{n+1} - a_n^n}{\Delta t},$$

where $a_j$ and $a_n$ are any sequences. With this notation the scheme can be written

$$D_a^t u_0^n + D_a F_{n+1/2}^j = P_n^j.$$

Using (2.4) we see that $\sum_{j=1}^N P_n^j = 0$, and that it is consistent to define $P_0^n = P_N^n$ and $P_{N+1}^n = P_1^n$. With this convention we also have that

$$D_a P_n^j = \frac{1}{2} (u_n^j + u_{n-1}^j).$$
We also observe that
\[ D_+^T \sum_{i=1}^N u^n_i = \sum_{i=1}^N P^n_j = 0. \]
so that if \( \int_0^1 u_0 \, dx = 0 \), then also
\[ \Delta x \sum_{i=1}^N u^n_i = 0 \text{ for } n \geq 0. \]
Defining \( \|u^n\|_\infty := \max_j |u^n_j| \), we can estimate \( P^n_j \) as
\[ |P^n_j| \leq N \Delta x \|u^n\|_\infty + N^2 \Delta x^2 \|u^n\|_\infty = 2 \|u^n\|_\infty. \]
We shall often use the short hand notations \( F_{j+1/2}(u,v) = F(x_{j+1/2}, u, v), \Delta_- a_j = a_j - a_{j-1} = \Delta x D_- a_j \) and \( \|\cdot\|_1 = \|\cdot\|_{L^1((0,1))} \).

3. Discrete estimates and convergence

In this section our aim is to prove the compactness of our scheme using Kolo-
morov’s compactness theorem. To employ this theorem we require a supremum
bound, a \( BV \) bound and an \( L^1 \) continuity-in-time bound on the approximate
solutions, all of which are uniform in the discretization variable \( \Delta x \).

For simplicity of exposition, we shall show such estimates in the case where
\( f_u(x,u) \geq 0 \). This means that \( F(x,u,v) = F(x,u), \) which is non-decreasing in \( u \).
The proof in the general case then follows \textit{mutatis mutandis}.

\textbf{Lemma 2 \((L^\infty\)-bound). } The solution \( u^n \) of the scheme (2.3) satisfies the following bound
\[ \|u^n\|_\infty \leq e^{2 \tau_n} \|u_0^0\|_\infty. \]
\textbf{Proof. } Using the monotonicity of \( \Psi_j \)
\[ u^{n+1}_j = u^n_j - \lambda \Delta_- F^n_{j+1/2} + \Delta t P^n_j = \Psi_j(u^n_{j-1},u^n_j,u^n_{j+1}) + \Delta t P^n_j \]
\[ \leq (1 + 2 \Delta t) \|u^n\|_\infty. \]
The result follows by an application of Gronwall’s inequality. \( \square \)

Now in the next lemma we are going to obtain a uniform bound on total variation
in space for the numerical solutions.

\textbf{Lemma 3 \((BV\)-bound). } The solution \( u^n \) of the scheme (2.3) satisfies the bound
\[ \|u^n\|_{BV((0,1))} \leq e^{C_f \tau_n} \|u^0\|_{BV((0,1))} + C_f \left(e^{C_f \tau_n} - 1\right), \]
where \( C_f \) is a positive constant depending on \( f \) and its first and second derivatives.
\textbf{Proof. } Assume that \( v^n_j \) satisfies
\[ v^{n+1}_j = v^n_j - \lambda \Delta_- G^n_{j+1/2} + \Delta t R^n_j \]
\[ = v^n_j - \lambda \Delta_- F^n_{j+1/2} + \lambda \Delta_- ((F-G)_{j+1/2}(v^n_j)) + \Delta t R^n_j, \]
where \( G(x,u) \) is a given function and \( \Delta_- = \Delta x D_- \). Using the monotonicity of \( \Psi_j \),
we find that
\[ u^n_j \wedge v^n_j - \lambda \Delta_- F^j_{j+1/2} \left( u^n_j \wedge v^n_j \right) + \Delta t P^n_j \]
\[ \leq u^{n+1}_j \wedge v^n_j \leq u^n_j \vee v^n_j - \lambda \Delta_- F^j_{j+1/2} \left( u^n_j \vee v^n_j \right) + \Delta t P^n_j, \]
and similarly
\[ u^n_j \wedge v^n_j - \lambda \Delta_- F^j_{j+1/2} \left( u^n_j \wedge v^n_j \right) + \lambda \Delta_- \left((F-G)_{j+1/2}(v^n_j)\right) + \Delta t R^n_j \]
\[ \leq v^{n+1}_j \wedge u^n_j \leq v^n_j \vee u^n_j - \lambda \Delta_- F^j_{j+1/2} \left( u^n_j \vee v^n_j \right) + \lambda \Delta_- \left((F-G)_{j+1/2}(v^n_j)\right) + \Delta t R^n_j. \]
Subtracting, we find that
\[(3.2) \quad u_j^{n+1} - v_j^{n+1} \leq |u_j^n - v_j^n| - \lambda \Delta_-(Q_{j+1/2}(u^n_j, v^n_j)) - \Delta t (P^n_{j-1} - R^n_j),\]
and
\[(3.3) \quad v_j^{n+1} - u_j^{n+1} \leq |u_j^n - v_j^n| - \lambda \Delta_-(Q_{j+1/2}(u^n_j, v^n_j)) + \Delta t (P^n_{j-1} - R^n_j),\]
with
\[Q_{j+1/2}(u, k) = F_{j+1/2}(u \vee k) - F_{j+1/2}(u \wedge k) = \text{sign}(u - k)(F_{j+1/2}(u) - F_{j+1/2}(k)).\]
This implies
\[(3.4) \quad |u_j^{n+1} - v_j^{n+1}| \leq |u_j^n - v_j^n| - \lambda \Delta_-(Q_{j+1/2}(u^n_j, v^n_j)) - \Delta t |P^n_{j-1} - R^n_j|.|\]
Regarding the “\(F - G\)” term
\[|\Delta_-(Q_{j+1/2}(v^n_j))| \leq |\Delta^v_-(F - G)_{j+1/2}(v^n_j)| + |\Delta^x_-(F - G)_{j+1/2}(v^n_{j-1})|,\]
where
\[\Delta^v_-(F - G)_{j+1/2}(u, w_j) = H(x_{j+1/2}, w_j) - H(x_{j+1/2}, w_{j-1})\]
and
\[\Delta^x_-(F - G)_{j+1/2}(u, w_j) = H(x_{j+1/2}, w_j) - H(x_{j-1/2}, w_j).\]
This gives
\[|\Delta_-(Q_{j+1/2}(v^n_j))| \leq \max_j \|F_{j+1/2} - G_{j+1/2}\|_{\text{Lip}} |u_j^n - v_j^n| + |\Delta^x_-(F - G)_{j+1/2}(v^n_{j-1})|.\]
Now we set \(u_j^n = u^n_{j-1}\), then \(G_{j+1/2}^n = F_{j-1/2}^n\) and \(R^n_j = P^n_{j-1}\) for \(n \geq 1\). Furthermore,
\[|\Delta^v_-(F - G)_{j+1/2}(v^n_{j-1})| \leq \Delta x^2 \max_{x \in [0,1]} |\partial^2_{xx}f(x, v)|.\]
Then we get the \(BV\) bound
\[\sum_{j=1}^{N} |u_j^{n+1} - u_j^{n-1}| \leq \sum_{j=1}^{N} |u_j^n - u_{j-1}^n| + \Delta t \max_j \|F_{j+1/2} - F_{j-1/2}\|_{\text{Lip}} \sum_{j=1}^{N} |u_j^n - u_{j-1}^n| + \Delta t \|\partial^2_{xx}f\|_\infty + \Delta t \|u^n\|_\infty \leq (1 + \Delta t \|\partial^2_{xx}f\|_\infty) \sum_{j=1}^{N} |u_j^n - u_{j-1}^n| + \Delta t \|\partial^2_{xx}f\|_\infty + \Delta t \|u^n\|_\infty.\]
The estimate (3.1) follows after applying Lemma 2 and then Gronwall’s inequality.

Next, we show a so-called “discrete entropy inequality”.

**Lemma 4.** For all \(n \geq 0\) and all constants \(k\), we have
\[D^1_-\eta(u^n_j, k) + D_-Q_{j+1/2}(u^n_j, k) + \text{sign}(u^n_j+1 - k)D_-F_{j+1/2}(k) \leq \text{sign}(u^n_j+1 - k)P^n_j.\]
Proof. Choose $R^0_j = 0$, $G = 0$ and $v^n_j = v^{n+1}_j = k$ in (3.2) and (3.3), the results are 
$c \leq a + b$, and $-c \leq a - b$,
with
$$
d = u^{n+1}_j - k, \quad a = |u^n_j - k| - \lambda \Delta_c Q_{j+1/2}(u^n_j, k)
$$
and
$$
b = -\lambda \Delta_c F_{j+1/2}(k) + \Delta t P^n_j.
$$
Note that $a \geq 0$, multiplying the first inequality with $H(c)$ and the second with $H(-c)$, where $H$ is the Heaviside function, yields
$$
H(c)c \leq H(c)a + H(c)b \quad \text{and} \quad H(-c)c \leq H(-c)a - H(-c)b.
$$
Add these and divide by 2, then rearrange and divide by $\Delta t$ to get (3.5). \qed

Lemma 5 (Time continuity bound). For all $n \geq 0$ we have that

$$
\Delta x \sum_{j=1}^{N} |D^+_x u^n_j| \leq \Delta x \sum_{j=1}^{N} |D^+_x u^0_j| + 2 \left( e^{2^n} - 1 \right) \|u^0\|_{\infty},
$$

where $C_f$ is a constant depending on $f$ and its derivatives.

Proof. Using $v^n_j = u^{n-1}_j$ in (3.4), we find that $G = F$ and $P^n_j = P^{n-1}_j$. Thus

$$
\sum_{j=1}^{N} |u^{n+1}_j - u^n_j| \leq \sum_{j=1}^{N} |u^n_j - u^{n-1}_j| + \Delta t |P^n_j - P^{n-1}_j|.
$$

Multiplying with $\lambda$ and using the bound $|P^n_j| \leq 2 \|u^n\|$ and Lemma 2, we get

$$
\Delta x \sum_{j=1}^{N} |D^+_x u^n_j| \leq \Delta x \sum_{j=1}^{N} |D^+_x u^{n-1}_j| + 4 \Delta t e^{2^n} \|u^0\|_{\infty}.
$$

We use Gronwall’s inequality to conclude the proof. \qed

Note that our assumptions on $f$ and the initial data imply that

$$
\Delta x \sum_{j=1}^{N} |D^+_x u^n_j| \leq C_f \left( |u_0|_{BV([0, 1])} + 1 \right) < \infty,
$$

for some constant depending on $f$.

Next we define the piecewise constant approximation $u_{\Delta x}$ by

$$
u = \sum_{j,n} u^n_j \chi_{r^n_j}(t,x),
$$

With these three bounds, Lemmas 2, 3, 5, we can apply Helly’s theorem, [17, Theorem A.11], to prove that $\{u_{\Delta x}\}_{\Delta x > 0}$ is compact.

Lemma 6 (Compactness lemma). Let $\{u_{\Delta x}\}_{\Delta x > 0}$ be the family obtained from the scheme (2.3) with $\lambda$ chosen such that $\Psi_j(u^{n-1}_j, u^n_j, u^{n+1}_{j+1})$ is monotone for all $j$ and for all $t^n < T$. Then there exists a sequence $\{\Delta x_k\}_{k=1}^{\infty}$ with $\Delta x_k \to 0$ as $k \to \infty$, and a function $u \in C([0, T]; L^1(0, 1))$ such that $u_{\Delta x_k} \to u$ in $C([0, T]; L^1(0, 1))$.

Now we can use the discrete entropy condition (3.5) to show that any limit $u$ satisfies the entropy condition (2.2).

Theorem 7. Assume that the initial data $u_0 \in BV([0, 1])$ satisfies the zero mean condition $\int_0^1 u_0 dx = 0$, and that $\lambda$ satisfies (2.6) (so that $\Psi_j$ is monotone). Then

$$u = \lim_{k \to \infty} u_{\Delta x_k}$$

is an entropy solution according to Definition 1.
Proof. For simplicity we write $\Delta x$ for $\Delta x_k$. Choose a non-negative, $x$-periodic test function $\varphi$ and set $\varphi_0^n = \varphi(t^n, x_j)$. Let $T = t^M$, multiply the discrete entropy inequality (3.5) with $\Delta t \Delta x \varphi^n_j$, and sum by parts in $n$ and $j$ to obtain

\begin{align}
(3.9a) \quad \Delta t \Delta x \sum_{n=1}^{M-1} \sum_{j=1}^{N} \eta_j^n D^t \varphi^n_j + \Delta t \Delta x \sum_{n=0}^{M-1} \sum_{j=1}^{N} Q_{j+1/2}(u_j^n, k) D_- \varphi^n_j \\
(3.9b) \quad + \Delta x \sum_{j=1}^{N} \eta_j^M \varphi_j^{M-1} - \eta_j^0 \varphi_j^0 \\
(3.9c) \quad + \Delta t \Delta x \sum_{n=0}^{M-1} \sum_{j=1}^{N} \eta_j^t \varphi_j^{n+1} D_- F_{j+1/2}(k) \varphi^n_j \\
(3.9d) \quad + \Delta t \Delta x \sum_{n=0}^{M-1} \sum_{j=1}^{N} \eta_j^t \varphi_j^{n+1} P^n_j \varphi^n_j \\
\geq 0,
\end{align}

where

$$\eta_j^t = \text{sign} (u_j^{n+1} - k), \quad \text{and} \quad D^t \varphi^n_j = \frac{\varphi^n_j - \varphi_j^{n-1}}{\Delta t}.$$

Using similar arguments to those that can be found in the proof of the analogous result in [22], it is straightforward to show that we can let $\Delta x \downarrow 0$ in (3.9) to conclude that $\varphi$ satisfies (2.2). The proof of this uses in particular that

$$\varphi \in C^2, \quad u_{\Delta x}(t, \cdot) \in BV,$$

and $\|u_{\Delta x}(t, \cdot) - u_{\Delta x}(s, \cdot)\|_1 \leq O(\max \{|t-s|, \Delta t\})$, and that both $Q$ and $F$ are consistent and Lipschitz continuous in both $x$ and $u$. \hfill \square

4. A Kuznetsov type lemma, stability and convergence rate

As in [22] the similarity of the OH equation to a scalar conservation law allows us to estimate the $L^1$-difference between the an entropy solution and other functions which are not necessarily solutions of (2.1). In this section we establish such a comparison result and use it to prove that the approximations defined by the finite volume scheme (2.3) – (2.5) converge to an entropy solution as $O(\sqrt{\Delta x})$.

4.1. A comparison result. For any function $u \in L^\infty([0, T]; L^1(0, 1))$ define

$$L(u, k, \varphi) = \int_{[0, T]} \left( \eta(u, k) \varphi_t + q(x, u, k) \varphi_x - \eta'(u, k) f_x(x, k) \varphi + \eta'(u, k) P[u][\varphi] \right) dx dt \bigg|_{t=0}^{t=T},$$

where $\eta$ and $q$ are defined in Definition 1 and $\eta'(u, k) = \text{sign} (u - k)$. Choose the test function

$$\varphi_{\epsilon, c_0}(t, x, s, y) = \theta_{\epsilon}(x - y) \omega_{c_0}(t - s),$$

where $\omega_{c_0}$ and $\theta_{\epsilon}$ are standard mollifiers, with $\theta_\epsilon$ being extended periodically outside the interval $(-1/2, 1/2)$. Let $v \in L^\infty([0, T]; L^1(0, 1))$ and put $k = v(s, y)$ in $L$ and integrate in $s$ and $y$. This defines the following functional

$$\Lambda_{\epsilon, c_0}(u, v) = \int_{[0, T]} L \left( u, v(s, y), \varphi_{\epsilon, c_0} \right) dy ds.$$
For any function \( w \in C([0,T]; L^1(0,1)) \), we define the moduli of continuity
\[
\mu(w(t,\cdot),\epsilon) = \sup_{|y| \leq \epsilon} \|w(t,\cdot + y) - w(t,\cdot)\|_1, \\
\nu_\epsilon(w,\epsilon_0) = \sup_{|s| \leq \epsilon_0} \|w(t + s,\cdot) - w(t,\cdot)\|_1, \\
\nu(w,\epsilon_0) = \sup_{0 \leq t \leq T} \nu_\epsilon(w,\epsilon_0).
\]

**Remark 8.** From the proofs of Lemmas 2, 3, and 5 it follows that
\[
\nu(u_{\Delta t,\epsilon_0}) \leq (\epsilon_0 + \Delta t)C_T, \\
\mu(u_{\Delta t}(t,\cdot),\epsilon) \leq \epsilon|u_{\Delta t}(t,\cdot)|_{BV([0,1])}.
\]

In this setting, we have the following version of the Kuznetsov lemma.

**Lemma 9.** Let \( u \) be an entropy solution of the Ostrovsky-Hunter equation with the associated initial data \( u_0 \in BV([0,1]) \). Then for any \( v \in L^\infty([0,T]; L^1(0,1)) \cap L^\infty(\Pi_T) \) the following estimate holds
\[
\|u(T,\cdot) - v(T,\cdot)\|_1 \leq e^{2T} \|u_0(\cdot) - v(0,\cdot)\|_1 \\
+ (e^{2T} - 1) \\
\times \left[ -\Lambda_{\epsilon,\epsilon_0}(v,u) \\
+ \frac{1}{2} (\mu(v(T,\cdot),\epsilon) + \mu(v(T,\cdot),\epsilon) + \mu(v(0,\cdot),\epsilon) + \mu(u_0,\epsilon)) \\
+ C_{f,T}(\epsilon + T \sup_{\varepsilon \in [0,T]} \mu(v(t,\cdot),\epsilon) + \nu(v,\epsilon_0)) \right].
\]

where \( C_{f,T} \) is a constant depending on \( T, u_0, f \) and its first derivatives.

**Proof.** For simplicity we write \( \varphi \) for \( \varphi_{\epsilon,\epsilon_0} \). Using that \( \varphi_x = -\varphi_y, \varphi_t = -\varphi_s \) and that \( \Lambda_{\epsilon,\epsilon_0}(u,v) \geq 0 \), we add \( \Lambda_{\epsilon,\epsilon_0}(u,v) \) and \( \Lambda_{\epsilon,\epsilon_0}(v,u) \) to compute
\[
\Lambda_{\epsilon,\epsilon_0}(v,u) \leq \int_{\Pi_T} \int_{\Pi_T} \text{sign}(u-v) \left[ (f(x,u) - f(x,v))\varphi_x - f_y(x,v)\varphi \right] dx dy ds dt \\
\leq -\int_{\Pi_T} \int_0^1 \varphi(T,x,s,y) |u(T,x) - v(s,y)| dx dy ds dt \\
+ \int_{\Pi_T} \int_0^1 \varphi(0,x,s,y) |u_0(x) - v(s,y)| dx dy ds dt \\
- \int_0^1 \int_{\Pi_T} \varphi(y,x,T,y) |u(x,t) - v(y,T)| dy dt dx \\
+ \int_0^1 \int_{\Pi_T} \varphi(t,x,0,y) |u(t,x) - v_0(y)| dy dt dx \\
+ \int_{\Pi_T} \int \text{sign}(v-u) (P[v] - P[u]) \varphi(t,x,s,y) dx dy ds dt \\
\]

As is standard for scalar conservation laws, see e.g., [17], the terms (4.4) – (4.7) can be estimated to yield the terms containing the initial and final data, plus the term starting with ”\( \frac{1}{2}(\cdot,\cdot) \)” in (4.2). The term (4.8) can be overestimated as in [22] by
\[
2 \int_0^T \|u(t,\cdot) - v(t,\cdot)\|_1 dt + 2T\nu(v,\epsilon_0) + \epsilon T \|v\|_\infty.
\]
The term (4.3) can be estimated as in [16] by
\[ C_f \left( 2\varepsilon + T \sup_{t\in[0,T]} \mu(v(t,\cdot),\varepsilon) + \nu(v,\varepsilon_0) \right). \]

Collecting these bounds
\[ \|u(T,\cdot) - v(T,\cdot)\|_1 \leq \|u_0 - v_0\|_1 - \Lambda_{\varepsilon,\varepsilon_0}(v,u) \]
\[ + \frac{1}{2} \left( \mu(v(T,\cdot),\varepsilon) + \mu(u(T,\cdot),\varepsilon) + \mu(v(0,\cdot),\varepsilon) + \mu(u_0,\varepsilon) \right) \]
\[ + C_f \left( 2\varepsilon + \varepsilon T \|v\|_{\infty} + T \left( \sup_{t\in[0,T]} \mu(v(t,\cdot),\varepsilon) + \nu(v,\varepsilon_0) \right) + \nu(v,\varepsilon_0) \right) \]
\[ + 2 \int_0^T \|u(t,\cdot) - v(t,\cdot)\|_1 \ dt. \]

The proof is concluded by applying Gronwall’s inequality. \( \square \)

If \( v \) is another entropy solution with initial data \( v_0 \) (satisfying the zero mean condition), then \( \Lambda_{\varepsilon,\varepsilon_0}(v,u) \geq 0 \), and we can send \( \varepsilon \) and \( \varepsilon_0 \) to zero in (4.2) to prove the following.

**Theorem 10.** If \( u \) and \( v \) are entropy solutions of the Ostrovsky-Hunter equation (2.1) with initial data \( u_0 \) and \( v_0 \) respectively, then
\[ \|u(T,\cdot) - v(T,\cdot)\|_1 \leq e^{2T} \|u_0 - v_0\|_1. \]

Observe that since the entropy solution is unique, then the whole sequence \( \{u_{\Delta x}\} \), rather than only a subsequence, converges.

### 4.2. Convergence rate.

We can use Lemma 9 with \( v = u_{\Delta x} \) to measure the \( L^1 \) error of the finite volume scheme.

**Lemma 11.** Let \( u \) be the entropy solution to (2.1) and let \( u_{\Delta x} \) be the piecewise constant interpolation defined by (3.8), where \( u^n_j \) is obtained by the scheme (2.3) – (2.5). Assume that \( u_0 \) is in \( L^1((0,1)) \cap BV((0,1)) \), and that \( f \) is locally bounded, \( x \)-periodic and twice continuously differentiable. Then there exists a constant \( C_T \), depending on \( f \), \( u_0 \) and \( T \), but not on \( \varepsilon \), \( \varepsilon_0 \) or \( \Delta x \), such that
\[ (4.9) \quad -\Lambda_{\varepsilon,\varepsilon_0}(u_{\Delta t},u) \leq C_T \left( \Delta t + \Delta x + \frac{\Delta x}{\varepsilon} + \frac{\Delta t}{\varepsilon_0} \right). \]

**Proof.** Again we write \( \varphi = \varphi_{\varepsilon,\varepsilon_0} \), after summation by parts we find that
\[
L(u_{\Delta x},k,\varphi) = -\sum_{n=0}^M \sum_{j=1}^N D^+_x \eta^n_j \int_{I^n_j} \varphi^{n+1} \ dx \ dt - \sum_{n=0}^M \sum_{j=1}^N D^-_x q^n_{j+1/2} \int_{I^n_j} \varphi_{j+1/2} \ dx \ dt \\
+ \sum_{n=0}^M \sum_{j=1}^N \int_{I^n_j} (q(x,u_{\Delta x}) - q(x_{j+1/2},u_{\Delta x})) \varphi_x \ dx \ dt \\
- \sum_{n=0}^M \sum_{j=1}^N \eta^n_j \int_{I^n_j} f_x(x,k) \varphi \ dx \ dt \\
+ \sum_{n=0}^M \sum_{j=1}^N \eta^n_j \int_{I^n_j} P[u_{\Delta x}] \varphi \ dx \ dt,
\]
where \( L \) is defined in (4.1) and
\[ \varphi^{n+1} = \varphi^{(n+1),x}, \quad \varphi_{j+1/2} = \varphi(t,x_{j+1/2}), \quad \eta^n_j = \left| u^n_j - k \right|, \]
\[ q^n_{j+1/2} = \text{sign} \left( u^n_j - k \right) \left( f(x_{j+1/2},u^n_j) - f(x_{j+1/2},k) \right), \quad \text{and} \quad \eta^n_j = \text{sign} \left( u^n_j - k \right). \]
Next, multiply the discrete entropy inequality (3.5) by $\int_{I_T^n} \varphi \, dxdt$ and sum over $n$ and $j$ to get

$$\ell := \sum_{n=0}^M \sum_{j=1}^N D^e_n \eta^i_j \int_{I_T^n} \varphi \, dxdt$$

$$+ \sum_{n=0}^M \sum_{j=1}^N D^- Q^i_{j+1/2} \int_{I_T^n} \varphi \, dxdt$$

$$+ \sum_{n=0}^M \sum_{j=1}^N \eta^{i+n+1}_j D^- F_{j+1/2}(k) \int_{I_T^n} \varphi \, dxdt$$

$$- \sum_{n=0}^M \sum_{j=1}^N \eta^{i+n+1}_j F^m_j \int_{I_T^n} \varphi \, dxdt \leq 0.$$ 

Write $\mathcal{L}(k) = L(u_{\Delta x}, k, \varphi)$, so that $-\mathcal{L} \leq -\mathcal{L} - \ell$. Then

$$\mathcal{L}(k) \leq \sum_{n=0}^M \sum_{j=1}^N D^e_n \eta^i_j \int_{I_T^n} \left( (\varphi^{n+1} - \varphi) \right) \, dxdt$$

$$+ \sum_{n=0}^M \sum_{j=1}^N D^- \left( Q^i_{j+1/2} - Q^i_{j+1/2} \right) \int_{I_T^n} \int_{I_T^n} \varphi \, dxdt$$

$$+ \sum_{n=0}^M \sum_{j=1}^N D^- q^i_{j+1/2} \int_{I_T^n} \left( \varphi_{j+1/2} - \varphi \right) \, dxdt$$

$$+ \sum_{n=0}^M \sum_{j=1}^N \int_{I_T^n} \left( q^i_n f_x(x, k) - \eta^{i+n+1} D^- F_{j+1/2} \right) \varphi \, dxdt$$

$$- \sum_{n=0}^M \sum_{j=1}^N \int_{I_T^n} \left( \eta^{i+n} P[u_j^n] - \eta^{i+n+1} P_j^n \right) \varphi \, dxdt$$

$$- \sum_{n=0}^M \sum_{j=1}^N \int_{I_T^n} \left( q(x, u_{\Delta x}) - q(x_{j+1/2}, u_{\Delta x}) \right) \varphi_x \, dxdt.$$ 

We need to bound $-\int_{\Pi_T} \mathcal{L}(u(s, y)) \, dy$. The integral of the first term on the right can be bounded as follows,

$$\left| \int_{\Pi_T} (4.10) \, dy \right| = \left| \sum_{n=0}^M \sum_{j=1}^N \int_{\Pi_T} D^e_n \eta^i_j \int_{I_T^n} \left( \omega_{\epsilon_0}(t^{n+1} - s) - \omega_{\epsilon_0}(t - s) \right) \theta_{\epsilon}(x - y) \, dx dt dy ds \right|$$

$$\leq \sum_{n=0}^M \sum_{j=1}^N |D^e_n u^i_j| \int_{\Pi_T} \int_{I_T^n} \int_{I_T^n} \left| \omega_{\epsilon_0}(\sigma - s) \right| \, d\sigma \, \theta_{\epsilon}(x - y) \, dx dt dy ds$$

$$\leq \sum_{n=0}^M \sum_{j=1}^N |D^e_n u^i_j| \frac{C}{\epsilon_0} \int_{I_T^n} (t^{n+1} - t) \, dx dt$$

$$\leq \frac{C}{\epsilon_0} \sum_{n=0}^M \Delta t C_T$$

$$\leq C_T \frac{\Delta t}{\epsilon_0},$$
where we have used (3.6) and (3.7). The terms (4.11) and (4.12) can be bounded similarly,
\[ \left| \int_{I_T} (4.11) + (4.12) \, dy ds \right| \leq C_T \frac{\Delta x}{\varepsilon}. \]

Next, we consider (4.13), we split this into a sum of two terms
\[ (4.13)_a = \sum_{n,j} \int_{I_j^n} \left( \eta_j^{n+1} f_x(x, k) - D_- F_{j+1/2}(k) \right) \varphi \, dx dt \]
\[ (4.13)_b = - \sum_{n,j} \left( \eta_j^{n+1} - \eta_j^n \right) f_x(x, k) \, dx dt. \]

The second of these can be bounded by summation by parts,
\[ (4.13)_b = - \sum_{n=1}^{N} \sum_{j=1}^{M} \int_{I_j^M} f_x(x, k)(\varphi(t, x, s, y) - \varphi(t, x - \Delta t, s, y)) \, dx dt. \]

The first integral of the first term of this expression can be estimated as
\[ \left| \int_{\Pi_T} \sum_{j=1}^{N} \sum_{n=1}^{M} \eta_j^{n+1} \int_{I_j^n} f_x(x, k) \varphi \, dx dt ds \right| \]
\[ \leq \| f_x \|_{\infty} \sum_{j=1}^{N} \Delta x \Delta t \]
\[ \leq C \Delta t. \]

The bound on the second term is identical. To bound (4.13)_b we must bound the integral of the last term,
\[ \left| \int_{\Pi_T} \sum_{n=1}^{M} \sum_{j=1}^{N} \eta_j^{n} \int_{I_j^n} f_x(x, k)(\varphi(t, x, s, y) - \varphi(t, x - \Delta t, s, y)) \, dx dt ds \right| \]
\[ \leq \| f_x \|_{\infty} \sum_{n=1}^{M} \sum_{j=1}^{N} \Delta x \Delta t \]
\[ \leq C \frac{\Delta t}{\varepsilon_0}. \]

To bound the integral of (4.13)_a we use the continuity of f_x and the observation that
\[ \Delta_- F_{j+1/2}(k) = \Delta_- F(x_{j+1/2}, k, k) = \Delta_- f(x_{j+1/2}, k), \]
which implies that
\[ D_- F_{j+1/2}(k) = f_x(\xi_j, k) \]
for some \( \xi_j \in I_j \). Therefore
\[ \left| \int_{\Pi_T} (4.13)_a \, dy ds \right| = \left| \int_{\Pi_T} \sum_{n,j} \int_{I_j^n} (f_x(x, k) - f_x(\xi_j)) \varphi \, dx dt ds \right| \]
\[ \leq \Delta x \| f_x \|_{\infty} \sum_{n,j} \Delta x \Delta t \]
\[ \leq C_T \Delta x. \]
Hence
\[ \left| \int_{\Omega_T} \left( \frac{4.13}{4} \right) dy ds \right| \leq C_T \left( \Delta x + \frac{\Delta t}{\varepsilon_0} \right). \]

The term (4.14) is bounded in [22, Section 6.2] as
\[ \left| \int_{\Omega_T} \left( \frac{4.14}{4} \right) dy ds \right| \leq C_T \left( \Delta t + \Delta x + \frac{\Delta t}{\varepsilon_0} \right). \]

The last term, the integral of (4.15), can be bounded using the Lipschitz continuity of \( q \),
\[ \left| \int_{\Omega_T} \left( \frac{4.15}{4} \right) dy ds \right| \leq \| q_x \|_\infty \int_{\Omega_T} \sum_{n,j} \int_{I_{n,j}} \left( x_j + 1/2 - x \right) |q_x'(x - y)| \omega_{n,j}(t - s) dxdtdyds \]
\[ \leq \frac{C}{\varepsilon} \sum_{n,j} \Delta x^2 \Delta t \]
\[ \leq C_T \frac{\Delta x}{\varepsilon}. \]

The proof is concluded by collecting the bounds on the integrals of all the terms (4.10) – (4.15).

From Lemma 11 it easily follows that \( u_{\Delta x} \) converges at a rate 1/2.

**Theorem 12.** Let \( u \) and \( u_{\Delta x} \) be as in Lemma 11. then
\[ \| u(T, \cdot) - u_{\Delta x}(T, \cdot) \|_1 \leq C_T \sqrt{\Delta x}, \]
where \( C_T \) is a constant independent of \( \Delta x \).

**Proof.** This result follows by setting \( \varepsilon = \varepsilon_0 = \sqrt{\Delta t} = C\sqrt{\Delta x} \) in (4.9). We have that \( u_{\Delta x}(0, x) \) is defined in (2.5), therefore \( \| u_{\Delta x}(0, \cdot) - u_0 \|_1 \leq C \Delta x \) since \( u_0 \) is in \( BV \). To conclude the proof apply Lemma 9, and recall that for \( u_{\Delta x} \), all moduli of continuity are uniformly linear in the last argument.

5. **Numerical examples**

In this section we complement our theoretical results by two numerical experiments. Both experiments use the flux function
\[ f(x, u) = \frac{1}{2} u^2 \exp(\sin(2\pi x)). \]

As far as we know, with \( f \) given above, there are no solutions to (2.1) in closed form. When measuring the accuracy of the approximations, we therefore use an approximation generated by the finite volume scheme with a small \( \Delta x \). We used the Engquist-Osher numerical flux
\[ F(x, u, v) = \frac{1}{2} \exp(\sin(2\pi x)) \left( \left( u \lor 0 \right)^2 + \left( v \land 0 \right)^2 \right). \]

Our first example uses initial data that coincides with those of the so-called "corner wave". This is a closed form solution of the OH-equation with \( f = u^2/2 \), but not so in our case. This corner wave initial data is given by

\[ u_0(x) = \begin{cases} \frac{1}{8} \left( x - \frac{1}{2} \right)^2 + \frac{1}{8} \left( x - \frac{1}{2} \right) + \frac{1}{36}, & \text{for } x \in [0, 1/2), \\ \frac{1}{8} \left( x - \frac{1}{2} \right)^2 - \frac{1}{8} \left( x - \frac{1}{2} \right) + \frac{1}{36}, & \text{for } x \in [1/2, 1]. \end{cases} \]

Figure 1 shows the initial data, as well as the approximate solutions at \( t = 36 \), with \( \Delta x = 2^{-8} \) and the reference solution using \( \Delta x = 2^{-13} \) for \( t = 36 \). The second example uses smoother initial data
\[ u_0(x) = -0.05 \cos(2\pi x), \]
Figure 1. The initial data given by (5.1) and the approximate solution with $N = 128$ and the reference solution at $t = 36$.

Figure 2. The initial data given by (5.2) and the approximate solution with $N = 128$ and the reference solution at $t = 36$.

and Figure 2 shows the approximations for $t = 36$, $\Delta x = 2^{-7}$ and $\Delta x = 2^{-13}$. We observe that although the data are smooth, the solution seems to have a discontinuity.

By running the scheme with different $\Delta x$, we can try to estimate the convergence rate numerically. In Table 1 we show the relative $L^1$-errors, defined by

$$E = 100 \frac{\| u_{\Delta x}(t, \cdot) - u_{\text{ref}}(t, \cdot) \|_1}{\| u_{\text{ref}}(t, \cdot) \|_1}.$$  

We have done this for both examples, and as a reference solution, $u_{\text{ref}}$, we used the finite volume approximation with $\Delta x = 2^{-13}$. We observe that the convergence rates are higher than the theoretically proven rate. Although we are measuring “self-convergence”, it may well be the case that when the solution is a smooth as our examples seem to show (continuously differentiable except for a single discontinuity), the actual convergence rate is higher than $1/2$.

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Table 1. Errors and numerical convergence rate in $L^1$ at $t = 36$. 

| $N$  | $E$ (rate) | $E$ (rate) |
|------|------------|------------|
| 32   | 56.4       | 65.3       |
| 64   | 40.9 0.5   | 39.5 0.7   |
| 128  | 26.4 0.6   | 21.8 0.9   |
| 256  | 15.5 0.8   | 11.4 0.9   |
| 512  | 7.8 1.0    | 5.9 1.0    |
| 1024 | 3.7 1.1    | 2.7 1.1    |
| 2048 | 1.6 1.2    | 1.2 1.2    |

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