Zero mode of the Fourier series of some modular graphs from Poincare series

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Abstract

We consider specific linear combinations of two loop modular graph functions on the toroidal worldsheet with $2s$ links for $s = 2, 3$ and $4$. In each case, it satisfies an eigenvalue equation with source terms involving $E_{2s}$ and $E_{s}^{2}$ only. On removing certain combinations of $E_{2s}$ and $E_{s}^{2}$ from it, we express the resulting expression as an absolutely convergent Poincare series. This is used to calculate the power behaved terms in the asymptotic expansion of the zero mode of the Fourier expansion of these graphs in a simple manner.

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1 Introduction

Modular graph functions that arise in the low momentum expansion of multi–graviton amplitudes in type II string theory at genus one are $SL(2,\mathbb{Z})$ invariant. Integrating products of them over the truncated fundamental domain of $SL(2,\mathbb{Z})$ and keeping the finite contributions [1,2] yields terms in the effective action that are analytic in the external momenta. Hence it is useful to understand detailed properties of these modular graphs. Consider modular graphs on the toroidal worldsheet whose links are given by the scalar Green function connecting the vertices which are integrated over the worldsheet. Non–trivial graphs are one particle irreducible and also do not have any vertex with only one link ending on it, which follows from the properties of the Green function. These graphs satisfy eigenvalue equations on moduli space which have proved extremely useful in understanding their various properties as well as performing the integral over the truncated fundamental domain [1–16].

One loop modular graphs are given by the non-holomorphic Eisenstein series whose Fourier and Poincare series are simple and well known. These graphs satisfy Laplace equation on moduli space. However, this analysis gets involved beyond one loop. Generically such graphs satisfy Poisson equations on moduli space, where the source terms involve modular graphs such that every term in the eigenvalue equation preserves the number of links. At two loops, the eigenvalue equation satisfied by the graphs is known [3, 14], while explicit expressions for the Fourier and Poincare series are also known [17–19]. The eigenvalue equation has been derived either by manipulating the expression for the graphs given as lattice sums after acting on them with the Laplacian operator, or by analyzing the variations of the graphs on varying the Beltrami differentials. The Fourier and Poincare series have been derived using the lattice sum representations of these graphs.

In this paper, we reconsider these issues for some two loop modular graphs from a different viewpoint. If the graph has the Fourier series given by

$$ F(\tau) = \sum_{n \in \mathbb{Z}} F_n(\tau_2)e^{2\pi in\tau_1}, $$

where $\tau = \tau_1 + i\tau_2$ is the complex structure of the torus, the zero mode of the Fourier series is simply given by $F_0(\tau_2)$ (which is also referred to as the constant term). This (as well as every other $F_n(\tau_2)$) has terms which are power behaved in $\tau_2$ as well as terms that are exponentially suppressed in $\tau_2$ in the large $\tau_2$ expansion. We want to derive the terms that are power behaved in $\tau_2$ not by directly solving the eigenvalue equation it satisfies but from the Poincare series the graph satisfies. Thus in general while it is interesting to obtain the Fourier series directly from the Poincare series, this is the simplest exercise one can think of carrying out. Though here we shall reproduce known results for the Fourier series, the method is general enough to be possibly extended to other cases.

We now briefly outline the strategy we follow. We consider (linear combinations of) modular graphs $\Psi_{2s}$ with $2s$ links for $s = 2, 3$ and 4. Each of them satisfies an eigenvalue equation with source terms that involve only $E_{2s}$ and $E_{s}^2$, where $E_p$ is the non–holomorphic Eisenstein series. As stressed above, we want to understand the Fourier modes from the Poincare series rather than from the solutions of the eigenvalue equation that $\Psi_{2s}$ satisfies.
In order to do so, we first obtain an eigenvalue equation the seed function of the Poincare series for \( \Psi_{2s} \) satisfies, which we solve with a specific choice of boundary conditions. For the cases we consider, the seed function contains a term that is linear in \( \tau_2 \) and hence the Poincare series is not absolutely convergent. To remedy this, we remove certain combinations of \( E_{2s} \) and \( E_s^2 \) from \( \Psi_{2s} \) such that the resulting expression is given by an absolutely convergent Poincare series. Finally this is used to obtain the terms that are power behaved in the large \( \tau_2 \) expansion of the zero mode \( F_0(\tau_2) \) of the Fourier series of the modular graph. Part of this work generalizes some of the results in [20,21] and offers a somewhat different viewpoint. We end with a discussion on possibly generalizing our analysis to a class of modular graphs and also discuss certain limitations.

2 Two loop modular graph functions on the toroidal worldsheet

We now define the two loop modular graphs. To begin with, note that the coordinate on the toroidal worldsheet \( \Sigma \) is given by

\[
-\frac{1}{2} \leq \text{Re} z \leq \frac{1}{2}, \quad 0 \leq \text{Im} z \leq \tau_2.
\]  

(2.2)

The Green function representing the link connecting the vertices of the graph on the toroidal worldsheet is given by [1,22]

\[
G(z_i, z_j) = \frac{1}{\pi} \sum_{(m,n) \neq (0,0)} \frac{\tau_2}{|m\tau + n|^2} e^{\pi i z_{ij}(m\tau + n) - z_{ij}(m\tau + n)} / \tau_2
\]  

(2.3)

where \( z_{ij} = z_i - z_j \).

We now define the product of a adjoining links by

\[
\mathcal{G}(z_1, z_{a+1}; a) \equiv \int_{\Sigma^a-1} \prod_{i=2}^{a} \frac{d^2 z_i}{\tau_2} G(z_1, z_2) G(z_3, z_4) \ldots G(z_a, z_{a+1}),
\]  

(2.4)

where the first and the last insertion points are not integrated over.

Then the two loop modular graph \( C_{a,b,c} \) which is symmetric under interchange of \( a, b \) and \( c \), is given by

\[
C_{a,b,c} = \int_{\Sigma^2} \prod_{i=1}^{2} \frac{d^2 z_i}{\tau_2} \mathcal{G}(z_1, z_2; a) \mathcal{G}(z_1, z_2; b) \mathcal{G}(z_1, z_2; c).
\]  

(2.5)

In this paper, we shall be interested in some two loop modular graphs with even number of links. The simplest graph \( C_{1,1,2} \) which we consider has four links. With six links, we shall consider the graphs \( C_{1,2,3} \) and \( C_{2,2,2} \). Finally, with eight links, we shall consider a specific linear combination of the graphs \( C_{1,3,4}, C_{2,2,4} \) and \( C_{3,3,3} \).

\[^2\text{Here } \Sigma^a \text{ means } a \text{ copies of } \Sigma.\]
3 The analysis of the modular graph $C_{1,1,2}$ with four links

3.1 The absolutely convergent Poincare series

The modular graph $C_{1,1,2}$ satisfies the eigenvalue equation \[\Delta - 2\] $C_{1,1,2} = 9E_4 - E_2^2, \] where the $SL(2,\mathbb{Z})$ invariant Laplacian is defined by
\[
\Delta = 4\tau_2^2 \frac{\partial^2}{\partial \tau \partial \bar{\tau}},
\]
and the non-holomorphic Eisenstein series that arise as the source terms is defined in appendix A. This leads to the Poincare series representation
\[
C_{1,1,2} - \frac{2}{3}E_4 = \sum_{\gamma \in \Gamma \setminus SL(2,\mathbb{Z})} \Lambda_{1,1,2}(\gamma(\tau)),
\]
where the seed function is given by
\[
\Lambda_{1,1,2}(\tau) = \frac{\pi \zeta(3)}{90} \tau_2 + \frac{\pi \tau_2}{90} \sum_{n=1}^{\infty} \frac{\sigma_3(n)}{n^3}(e^{2\pi in\tau} + e^{-2\pi in\tau}).
\]
Note that the Poincare series is not for the modular graph $C_{1,1,2}$ but a modular invariant combination involving a shift by $-2E_4/3$.

The Poincare series in (3.9) is not absolutely convergent because of the term linear in $\tau_2$ in $\Lambda_{1,1,2}$. This is problematic for our purposes of calculating the zero mode of the Fourier series as we shall soon see, as the calculation involves performing an integral involving each individual term in the seed function. To remedy this, we consider
\[
E_2^2 - \frac{7}{3}E_4
\]
which using (A.113), yields the Poincare series
\[
E_2^2 - \frac{7}{3}E_4 = \sum_{\gamma \in \Gamma \setminus SL(2,\mathbb{Z})} \Upsilon_1(\gamma(\tau)),
\]
where the seed function is given by
\[
\Upsilon_1(\tau) = 2\Lambda_{1,1,2} + \frac{4\zeta(4)\tau_2}{\pi^2} \sum_{n=1}^{\infty} \frac{\sigma_3(n)}{n^2}(e^{2\pi in\tau} + e^{-2\pi in\tau}).
\]

\(^3\)Issues regarding absolute convergence of Poincare series that arise in integrals at genus one in string theory have been considered in a different context in \[23, 24\].

\(^4\)This is exactly for the same reason that the Poincare series (A.113) is absolutely convergent only for $\text{Re} s > 1$. In fact, the case for $s = 1$ needs to be regularized and the $\tau_1$ independent contributions have $\tau_2$ dependence given by $\tau_2$ and $\ln \tau_2$. 

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Thus we have that

$$C_{1,1} + \frac{1}{2} \left( E_4 - E_2^2 \right) = \sum_{\gamma \in \Gamma_{\infty} \backslash SL(2,\mathbb{Z})} \Omega_{1,1,2}(\gamma(\tau)), \quad (3.13)$$

where the seed function is given by

$$\Omega_{1,1,2}(\tau) = -\frac{2\zeta(4)\tau_2^2}{\pi^2} \sum_{n=1}^{\infty} \frac{\sigma_3(n)}{n^2} \left( e^{2\pi in\tau} + e^{-2\pi in\tau} \right). \quad (3.14)$$

Hence we see that $C_{1,1} + (E_4 - E_2^2)/2$ is given by a Poincare series that is absolutely convergent. In fact each term in the Fourier expansion of the seed function is exponentially suppressed at the cusp. We now calculate the zero mode of the Fourier series of $C_{1,1,2}$.

### 3.2 Zero mode of the Fourier series

To calculate the zero mode of the Fourier series of $C_{1,1,2}$, we use the results given in appendix B. Thus in (B.116), substituting

$$F = C_{1,1,2} + \frac{1}{2} \left( E_4 - E_2^2 \right), \quad (3.15)$$

from (3.13) and (B.118), we have that

$$\Omega_0 = 0, \quad \Omega_m = -\frac{2\zeta(4)\tau_2^2\sigma_3(m)}{\pi^2m^2} e^{-2\pi |m|\tau_2} \quad (m \neq 0). \quad (3.16)$$

This yields the expression for the zero mode on using (B.119), leading to

$$F_0 = -\frac{2\zeta(4)}{\pi^2\tau_2} \sum_{n>0} \sum_{m \neq 0} \frac{S(m, 0; n)\sigma_3(m)}{m^2n^4} \int_{-\infty}^{\infty} \frac{dx}{1 + x^2} e^{-2\pi(x+im\tau_2)/m^2}, \quad (3.17)$$

We now analyze this expression along the lines of [20]. We first expand the exponential in an infinite series and perform the sum over $n$ using (B.122). We next perform the $x$ integral using the relation

$$\int_{-\infty}^{\infty} \frac{dx}{(1 + ix)^a(1 - ix)^b} = 2^{2-(a+b)} \pi \frac{\Gamma(a + b - 1)}{\Gamma(a)\Gamma(b)}, \quad (3.18)$$

for Re$(a + b) > 1$. This gives us that

$$F_0 = -\frac{\zeta(4)}{\pi\tau_2} \sum_{k=0}^{\infty} \frac{(k + 2)}{k!\zeta(2k + 4)} \left( -\frac{\pi}{\tau_2} \right)^k \sum_{m>0} \frac{\sigma_3(m)\sigma_{3-2k}(m)}{m^{2-k}}. \quad (3.19)$$

In this expression, the sum over $k$ arises from expanding the exponential in the integrand in (3.17), and we have interchanged the sums over $m$ and $k$.

5 The sum over $k$ is formally divergent since it grows as $m^k$ for fixed $m$, and hence interchanging the two sums is actually not allowed. However, we still proceed with the calculation and see what it gives us.
over \( m \) using the identity

\[
\sum_{m>0} \frac{\sigma_p(m)\sigma_q(m)}{m^r} = \frac{\zeta(r)\zeta(r-p)\zeta(r-q)\zeta(r-p-q)}{\zeta(2r-p-q)} \tag{3.20}
\]

which we analytically continue to all integral values of \( r \). This gives us

\[
F_0 = -\frac{1}{\pi \tau_2} \sum_{k=0}^{\infty} \frac{(k + 2)\zeta(-1-k)\zeta(2-k)\zeta(2+k)\zeta(5+k)}{k!\zeta(4+2k)} \left(-\frac{\pi}{\tau_2}\right)^k. \tag{3.21}
\]

In this infinite sum only the \( k = 0, 1 \) and 2 terms contribute, while the rest vanish. This gives us

\[
F_0 = \frac{5\zeta(5)}{12\pi \tau_2} - \frac{3\zeta(3)^2}{4\pi^2 \tau_2^2} + \frac{7\zeta(7)}{8\pi^3 \tau_2^3}, \tag{3.22}
\]

where the \( k = 1 \) contribution has to be regularized.

Thus having obtained the power behaved terms in the zero mode in the Fourier expansion of (3.15) it is straightforward to obtain those in the Fourier expansion of \( C_{1,1,2} \) using (A.111). This gives us

\[
C_{1,1,2} = \frac{4\zeta(8)\tau_2^4}{3\pi^4} + \frac{\zeta(3)\pi \tau_2}{45} + \frac{5\zeta(5)}{12\pi \tau_2} - \frac{3\zeta(3)^2}{4\pi^2 \tau_2^2} + \frac{9\zeta(7)}{16\pi^3 \tau_2^3}, \tag{3.25}
\]

which precisely agrees with the expression in [3].

4 The analysis of the modular graphs \( C_{1,2,3} \) and \( C_{2,2,2} \) with six links

4.1 The absolutely convergent Poincare series

The modular graphs \( C_{1,2,3} \) and \( C_{2,2,2} \) satisfy the coupled eigenvalue equations [3][9]

\[
\begin{align*}
\left(\Delta - 2\right)\left(4C_{1,2,3} + C_{2,2,2}\right) &= 52E_6 - 4E_3^2, \\
\left(\Delta - 12\right)\left(6C_{1,2,3} - C_{2,2,2}\right) &= 108E_6 - 36E_3^2. \tag{4.26}
\end{align*}
\]

We now obtain absolutely convergent Poincare series for modular invariant expressions involving these graphs.

\footnote{This is done using

\[
\zeta(-1-k) = \frac{\zeta(2+k)\Gamma(1+k/2)}{\pi^{k+3/2}\Gamma(-1/2-k/2)}, \tag{3.23}
\]

which leads to (as \( k \to 1 \))

\[
\zeta(-1-k)\zeta(2-k) = -\frac{\zeta(3)\zeta(1+\epsilon)}{2\pi^2\Gamma(\epsilon/2)} \to \frac{\zeta(3)}{4\pi^2}, \tag{3.24}
\]

as \( \epsilon = 1-k \to 0 \).}
Using the relation \((A.110)\) we rewrite the first equation in \((4.26)\) as
\[
(\Delta - 2)\Psi_1 = \frac{5720}{691}E_6 - 4E_3^2, \tag{4.27}
\]
where
\[
\Psi_1 = 4C_{1,2,3} + C_{2,2,2} - \frac{1079}{691}E_6. \tag{4.28}
\]
The motivation for writing it in this way will be clear shortly.

We now want to obtain the Poincare series for \(\Psi_1\). Letting
\[
\Psi_1 = \sum_{\gamma \in \Gamma \setminus SL(2,\mathbb{Z})} \Phi_1(\gamma(\tau)), \tag{4.29}
\]
we see that the seed function \(\Phi_1\) satisfies the eigenvalue equation
\[
(\Delta - 2)\Phi_1 = \frac{16\pi^6}{893025}\tau_2^6 - \frac{8\pi^3}{945}\tau_2^3E_3 \tag{4.30}
\]
where we have used \((A.113)\). Now from \((A.111)\) we see that the right hand side of \((4.30)\) is \(O(\tau_2)\) for large \(\tau_2\). We shall now use this fact to choose boundary conditions to uniquely solve for the seed function, thus justifying the choice of \((4.28)\).

To understand the choice of boundary conditions we take to uniquely solve for the seed function \(\Phi(\tau)\), consider the generic eigenvalue equation\(^7\)
\[
(\Delta - \lambda)\Phi = f(\tau) \tag{4.31}
\]
it satisfies. Suppose \(f(\tau)\) is \(O(\tau_2^s)\) for large \(\tau_2\). Let the Fourier series of \(\Phi\) be given by
\[
\Phi(\tau) = \sum_{n \in \mathbb{Z}} \Phi_n(\tau_2)e^{2\pi in\tau_1}. \tag{4.32}
\]
Then for \(\Phi_0(\tau_2)\), we choose the boundary condition that it is \(O(\tau_2^s)\) for large \(\tau_2\), and it is \(O(\tau_2^{1-s})\) for small \(\tau_2\). For \(\Phi_n(\tau_2)(n \neq 0)\) we choose the boundary condition that it is \(O(\tau_2^{1-s})\) for small \(\tau_2\)^8. This non–trivial choice of boundary conditions is motivated by \(25\). However, the analysis there is done for a modular invariant eigenfunction, which is not the case for the seed function\(^9\). Hence we simply take it as the definition of the choice of boundary conditions for calculating the seed function. We shall see this yields the correct answer for the Fourier mode of the modular graph in all the cases we consider (similar issues have also been discussed in \(20\)). It would be interesting to understand the issue of boundary conditions to be imposed on the Fourier modes of the seed function in detail.

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\(^7\)For all the cases we consider, \(\lambda = \mu(\mu - 1)\) where \(\mu\) is a non–zero integer.

\(^8\)This mode is exponentially suppressed for large \(\tau_2\), and hence we do not need to impose any boundary condition at large \(\tau_2\) to solve for it.

\(^9\)In fact, for \(s = 1\), the analysis of \(25\) leads to \(O(\ln \tau_2)\) behavior for small \(\tau_2\), rather than \(\tau_2^0\).
Thus for the case at hand, we impose the boundary condition that the zero mode of the Fourier series of the seed function is $O(\tau_2^2)$ for large $\tau_2$, while all the modes are $O(\ln \tau_2)$ for small $\tau_2$ (whether it is $O(\ln \tau_2)$ or $O(\tau_2^0)$ for small $\tau_2$ will be irrelevant for our purposes). In fact, this will be the boundary conditions we shall impose for all the cases we consider because $f(\tau)$ is always $O(\tau_2)$ for large $\tau_2$, as we shall see.

Thus setting

\[ \Phi_1(\tau) = \Phi_{1,0}(\tau_2) + \sum_{n \neq 0} \Phi_{1,n}(\tau_2)e^{2\pi in\tau_1}, \]  

we see that the zero mode $\Phi_{1,0}(\tau_2)$ satisfies

\[ \left( \frac{\tau_2^2}{\tau_2^2} \frac{d^2}{d\tau_2^2} - 2 \right) \Phi_{1,0} = -\frac{2\pi \zeta(5)}{315} \tau_2, \]  

which is solved by

\[ \Phi_{1,0}(\tau_2) = \frac{\pi \zeta(5)}{315} \tau_2 \]  

on using the boundary conditions at large and small $\tau_2$ which remove the homogeneous solution.

The non–zero mode $\Phi_{1,n}(n \neq 0)$ satisfies the equation

\[ \left( \frac{\tau_2^2}{\tau_2^2} \frac{d^2}{d\tau_2^2} - 2 - 4\pi^2 n^2 \tau_2^2 \right) \Phi_{1,n} = -\frac{8\pi^3 \tau_2^3 \sigma_5(n)}{945|n|^3} e^{-2\pi|n|\tau_2} \left(1 + \frac{3}{2\pi|n|\tau_2} + \frac{3}{4\pi^2 n^2 \tau_2^2}\right). \]  

We express the solution as

\[ \Phi_{1,n} = \Phi_{1,n}^h + \Phi_{1,n}^p, \]  

where $\Phi_{1,n}^h$ and $\Phi_{1,n}^p$ are the solutions to the homogeneous equation and the particular solution respectively. While the solution to the homogeneous equation is given by

\[ \Phi_{1,n}^h = a_{1,n} \sqrt{\tau_2} K_{3/2}(2\pi|n|\tau_2), \]  

where $a_{1,n}$ is an arbitrary constant, the particular solution is given by

\[ \Phi_{1,n}^p = \frac{\sigma_5(n)}{1890n^6} \left(\frac{15}{8\pi|n|\tau_2^2} + \frac{15}{4} + 6\pi|n|\tau_2 + 2\pi^2 n^2 \tau_2^2\right) e^{-2\pi|n|\tau_2}. \]  

Thus the boundary condition at small $\tau_2$ which demands cancellation of the $O(\tau_2^{-1})$ contribution yields

\[ a_{1,n} = -\frac{\sigma_5(n)}{252|n|^{11/2}}, \]  

leading to the solution

\[ \Phi_{1,n}(\tau_2) = \frac{\pi \sigma_5(n) \tau_2}{945|n|^5} \left(3 + \pi|n|\tau_2\right) e^{-2\pi|n|\tau_2}. \]  

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10Here and in the cases to be considered later where the analysis is similar, we keep the solution involving $\sqrt{\tau_2} K_{3/2}(2\pi|n|\tau_2)$ and ignore the linearly independent solution involving $\sqrt{\tau_2} I_{3/2}(2\pi|n|\tau_2)$ on physical grounds, as it diverges exponentially for large $\tau_2$. 

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Thus we see that
\[ 4C_{1,2,3} + C_{2,2,2} - \frac{1079}{691} E_6 = \sum_{\gamma \in \Gamma \setminus SL(2,\mathbb{Z})} \Lambda_1(\gamma(\tau)), \]  
(4.42)
where the seed function is given by
\[ \Lambda_1(\tau) = \frac{\pi \zeta(5)}{315} \tau_2 + \frac{\pi \tau}{945} \sum_{n=1}^{\infty} \frac{\sigma_5(n)}{n^5} \left(3 + \pi n \tau_2\right) \left(e^{2\pi i n \tau} + e^{-2\pi i n \tau}\right). \]  
(4.43)

Proceeding similarly, we rewrite the second equation in (4.26) as
\[ (\Delta - 12) \Psi_2 = \frac{51480}{691} E_6 - 36 E_3^2, \]  
(4.44)
where
\[ \Psi_2 = 6C_{1,2,3} - C_{2,2,2} - \frac{1286}{691} E_6. \]  
(4.45)
The Poincare series for \( \Psi_2 \) is given by
\[ \Psi_2 = \sum_{\gamma \in \Gamma \setminus SL(2,\mathbb{Z})} \Phi_2(\gamma(\tau)), \]  
(4.46)
where the seed function satisfies
\[ (\Delta - 12) \Phi_2 = \frac{16\pi^6}{99225} \tau_2^6 - \frac{8\pi^3}{105} \tau_2^3 E_3. \]  
(4.47)
Once again, the right hand side of (4.47) is \( O(\tau_2) \) for large \( \tau_2 \).

Thus setting
\[ \Phi_2(\tau) = \Phi_{2,0}(\tau_2) + \sum_{n \neq 0} \Phi_{2,n}(\tau_2) e^{2\pi i n \tau_2}, \]  
(4.48)
we see that \( \Phi_{2,0}(\tau_2) \) satisfies
\[ \left(\tau_2^2 \frac{d^2}{d\tau_2^2} - 12\right) \Phi_{2,0} = -\frac{2\pi \zeta(5)}{35} \tau_2, \]  
(4.49)
which is solved by
\[ \Phi_{2,0}(\tau_2) = \frac{\pi \zeta(5)}{210} \tau_2. \]  
(4.50)
The non–zero mode \( \Phi_{2,n}(n \neq 0) \) satisfies
\[ \left(\tau_2^2 \frac{d^2}{d\tau_2^2} - 12 - 4\pi^2 n^2 \tau_2^2\right) \Phi_{2,n} = -\frac{8\pi^3 \tau_2^3 \sigma_5(n)}{105 |n|^3} e^{-2\pi |n| \tau_2} \left(1 + \frac{3}{2\pi |n| \tau_2} + \frac{3}{4\pi^2 n^2 \tau_2^2}\right). \]  
(4.51)
Once again, expressing the solution as
\[ \Phi_{2,n} = \Phi_{2,n}^h + \Phi_{2,n}^p, \]  
(4.52)
we see that the homogeneous equation is solved by
\[ \Phi_{h,2,n} = a_{2,n} \sqrt{\tau_2} K_{7/2}(2\pi |n| \tau_2) \] (4.53)
for an arbitrary constant \( a_{2,n} \), while the particular solution is given by
\[ \Phi_{p,2,n} = -\frac{\sigma_5(n)}{840n^6} \left( \frac{525}{8\pi^3 |n|^3 \tau_2^3} + \frac{525}{4\pi^2 n^2 \tau_2^2} + \frac{105}{\pi |n| \tau_2^2} + 35 - 4\pi |n| \tau_2 - 8\pi^2 n^2 \tau_2^2 \right) e^{-2\pi |n| \tau_2}. \] (4.54)

The boundary condition at small \( \tau_2 \) leads to the cancellation of \( \tau_2^{-3} \) and \( \tau_2^{-1} \) terms and yields
\[ a_{2,n} = \frac{\sigma_5(n)}{12 |n|^{11/2}}, \] (4.55)
leading to the expression
\[ \Phi_{2,n}(\tau_2) = \frac{\pi \sigma_5(n) \tau_2}{210 |n|^5} \left( 1 + 2\pi |n| \tau_2 \right) e^{-2\pi |n| \tau_2} \] (4.56)
for the non–zero mode.
This leads to
\[ 6C_{1,2,3} - C_{2,2,2} - \frac{1286}{691} E_6 = \sum_{\gamma \in \Gamma_\infty \setminus SL(2,\mathbb{Z})} \Lambda_2(\gamma(\tau)), \] (4.57)
where the seed function is
\[ \Lambda_2(\tau) = \frac{\pi \zeta(5)}{210} \tau_2 + \frac{\pi \tau_2}{210} \sum_{n=1}^\infty \frac{\sigma_5(n)}{n^5} \left( 1 + 2\pi n \tau_2 \right) \left( e^{2\pi in \tau} + e^{-2\pi in \tau} \right). \] (4.58)

Now the Poincare series in (4.42) and (4.57) are not absolutely convergent because of the term linear in \( \tau_2 \) in \( \Lambda_i \) \( (i = 1, 2) \). To remedy this, we consider
\[ E_3^2 - \frac{1430}{691} E_6 = \sum_{\gamma \in \Gamma_\infty \setminus SL(2,\mathbb{Z})} \Upsilon_2(\gamma(\tau)), \] (4.59)
where the seed function is given by
\[ \Upsilon_2(\tau) = \Lambda_U(\tau) + \frac{2\zeta(6) \tau_2^3}{\pi^3} \sum_{n=1}^\infty \frac{\sigma_5(n)}{n^3} \left( 1 + \frac{3}{2\pi n \tau_2} \right) \left( e^{2\pi in \tau} + e^{-2\pi in \tau} \right). \] (4.60)
In (4.60), the expression for \( \Lambda_U(\tau) \) is given by
\[ \Lambda_U(\tau) = \frac{\pi \zeta(5)}{630} \tau_2 + \frac{\pi \tau_2}{630} \sum_{n=1}^\infty \frac{\sigma_5(n)}{n^5} \left( e^{2\pi in \tau} + e^{-2\pi in \tau} \right), \] (4.61)
which appears in (4.43) as well as in (4.58).

This immediately gives us that

\[
4C_{1,2,3} + C_{2,2,2} + \frac{1781}{691}E_6 - 2E_3^2 = \sum_{\gamma \in \Gamma_{\infty} \setminus SL(2,\mathbb{Z})} \chi_1(\gamma(\tau)),
\]

\[
6C_{1,2,3} - C_{2,2,2} + \frac{3004}{691}E_6 - 3E_3^2 = \sum_{\gamma \in \Gamma_{\infty} \setminus SL(2,\mathbb{Z})} \chi_2(\gamma(\tau)),
\]

with the seed functions

\[
\chi_1(\tau) = -\frac{4\zeta(6)\tau_2^3}{\pi^3} \sum_{n=1}^{\infty} \frac{\sigma_5(n)}{n^3} \left(1 + \frac{5}{4\pi n\tau_2}\right) \left(e^{2\pi in\tau} + e^{-2\pi in\tau}\right),
\]

\[
\chi_2(\tau) = -\frac{6\zeta(6)\tau_2^3}{\pi^3} \sum_{n=1}^{\infty} \frac{\sigma_5(n)}{n^3} \left(e^{2\pi in\tau} + e^{-2\pi in\tau}\right).
\]

Thus in (4.62) we obtain expressions for the modular invariant quantities on the left hand side in terms of absolutely convergent Poincare series, in which every term in the Fourier expansion of the seed function is exponentially suppressed at the cusp.

This allows us to obtain expressions involving the graphs \(C_{1,2,3}\) and \(C_{2,2,2}\) separately which are given by

\[
C_{1,2,3} + \frac{957}{1382}E_6 - \frac{1}{2}E_3^2 = \sum_{\gamma \in \Gamma_{\infty} \setminus SL(2,\mathbb{Z})} \Omega_{1,2,3}(\gamma(\tau)),
\]

\[
C_{2,2,2} - \frac{133}{691}E_6 = \sum_{\gamma \in \Gamma_{\infty} \setminus SL(2,\mathbb{Z})} \Omega_{2,2,2}(\gamma(\tau)),
\]

where the seed functions are given by

\[
\Omega_{1,2,3}(\tau) = -\frac{\zeta(6)\tau_2^3}{\pi^3} \sum_{n=1}^{\infty} \frac{\sigma_5(n)}{n^3} \left(1 + \frac{1}{2\pi n\tau_2}\right) \left(e^{2\pi in\tau} + e^{-2\pi in\tau}\right),
\]

\[
\Omega_{2,2,2}(\tau) = -\frac{3\zeta(6)\tau_2^3}{\pi^4} \sum_{n=1}^{\infty} \frac{\sigma_5(n)}{n^4} \left(e^{2\pi in\tau} + e^{-2\pi in\tau}\right).
\]

### 4.2 Zero mode of the Fourier series

We now calculate the zero mode in the Fourier series expansion of the graphs \(C_{1,2,3}\) and \(C_{2,2,2}\) using results in appendix B. Since the steps involved in the analysis are the same as discussed previously, we skip some intermediate steps.

We first consider \(C_{2,2,2}\). In (B.116), for

\[
F = C_{2,2,2} - \frac{133}{691}E_6,
\]

(4.66)
in \((B.118)\) we have that
\[
\Omega_0 = 0, \quad \Omega_m = - \frac{3\zeta(6)\tau_2^2\sigma_5(m)}{\pi^4 m^4} e^{-2\pi |m| \tau_2} \quad (m \neq 0).
\] (4.67)

This gives us the expression
\[
F_0 = - \frac{3\zeta(6)}{2\pi^3 \tau_2} \sum_{k=0}^{\infty} \frac{(k+2)}{k! \zeta(2k+4)} \left( - \frac{\pi}{\tau_2} \right)^k \sum_{m>0} \frac{\sigma_5(m)\sigma_{3-2k}(m)}{m^{4-k}}
\]
\[
= - \frac{3}{2\pi^3 \tau_2} \sum_{k=0}^{\infty} \frac{(k+2)\zeta(-1-k)\zeta(4-k)\zeta(2+k)\zeta(7+k)}{k! \zeta(4+2k)} \left( - \frac{\pi}{\tau_2} \right)^k.
\] (4.68)

Here only \(k = 0, 2, 3\) and \(4\) contributes in the infinite sum, leading to
\[
F_0 = \frac{\zeta(7)}{24\pi \tau_2} - \frac{7\zeta(9)}{16\pi^3 \tau_2^3} + \frac{15\zeta(5)^2}{16\pi^4 \tau_2^4} - \frac{32175\zeta(11)}{44224\pi^5 \tau_2^5},
\] (4.69)

where the \(k = 3\) term has to be regularized. This leads to
\[
C_{2,2,2} = \frac{266\zeta(12)\tau_2^6}{691\pi^6} + \frac{\zeta(7)}{24\pi \tau_2} - \frac{7\zeta(9)}{16\pi^3 \tau_2^3} + \frac{15\zeta(5)^2}{16\pi^4 \tau_2^4} - \frac{81\zeta(11)}{128\pi^5 \tau_2^5}
\] (4.70)

which exactly agrees with the known result \([10]\).

Now from \((4.64), (4.65)\) and \((B.116)\), for
\[
F = C_{1,2,3} - \frac{1}{6} C_{2,2,2} + \frac{1502}{2073} E_6 - \frac{1}{2} E_3^2,
\] (4.71)

from \((B.118)\), we have that
\[
\Omega_0 = 0, \quad \Omega_m = - \frac{\zeta(6)\tau_2^3\sigma_5(m)}{\pi^3 |m|^3} e^{-2\pi |m| \tau_2} \quad (m \neq 0).
\] (4.72)

This leads to
\[
F_0 = - \frac{\zeta(6)}{16\pi^2 \tau_2^2} \sum_{k=0}^{\infty} \frac{(k+3)(k+4)}{k! \zeta(2k+6)} \left( - \frac{\pi}{\tau_2} \right)^k \sum_{m>0} \frac{\sigma_5(m)\sigma_{5-2k}(m)}{m^{3-k}}
\]
\[
= - \frac{1}{16\pi^2 \tau_2^2} \sum_{k=0}^{\infty} \frac{(k+3)(k+4)\zeta(-2-k)\zeta(3-k)\zeta(3+k)\zeta(8+k)}{k! \zeta(6+2k)} \left( - \frac{\pi}{\tau_2} \right)^k.
\] (4.73)

Here \(k = 1, 2\) and \(3\) contributes to the infinite sum, leading to
\[
F_0 = \frac{35\zeta(9)}{192\pi^3 \tau_2^3} - \frac{45\zeta(5)^2}{64\pi^4 \tau_2^4} + \frac{75075\zeta(11)}{88448\pi^5 \tau_2^5},
\] (4.74)

where the \(k = 2\) term needs regularization. Thus using the expression for the zero mode for \(C_{2,2,2}\) in \((4.70)\), we get that
\[
C_{1,2,3} = \frac{473\zeta(12)\tau_2^6}{691\pi^6} + \frac{\zeta(5)\pi \tau_2}{630} + \frac{\zeta(7)}{144\pi \tau_2} + \frac{7\zeta(9)}{64\pi^3 \tau_2^3} - \frac{17\zeta(5)^2}{64\pi^4 \tau_2^4} + \frac{99\zeta(11)}{256\pi^5 \tau_2^5},
\] (4.75)
in perfect agreement with the expression in \([10]\).
5 The analysis for a linear combination of modular graphs with eight links

5.1 The absolutely convergent Poincare series

As a final example, we consider the linear combination of modular graphs with eight links given by

$$6C_{1,3,4} + 3C_{2,2,4} + 5C_{2,3,3}$$

which satisfies the eigenvalue equation $[3]$

$$\left( \Delta - 2 \right) \left( 6C_{1,3,4} + 3C_{2,2,4} + 5C_{2,3,3} \right) = 153E_8 - 9E_4^2. \tag{5.77}$$

We rewrite this as

$$\left( \Delta - 2 \right) \Psi = \frac{65637}{3617} E_8 - 9E_4^2 \tag{5.78}$$

where

$$\Psi = 6C_{1,3,4} + 3C_{2,2,4} + 5C_{2,3,3} - \frac{27098}{10851} E_8. \tag{5.79}$$

Thus the Poincare series for $\Psi$ defined by

$$\Psi = \sum_{\gamma \in \Gamma_{\infty} \setminus SL(2,\mathbb{Z})} \Phi(\gamma(\tau)) \tag{5.80}$$

has the seed function which satisfies the equation

$$\left( \Delta - 2 \right) \Phi = \frac{\pi^8}{2480625} \tau_2^8 - \frac{\pi^4}{525} \tau_2^4 E_4 \tag{5.81}$$

the right hand side of which is $O(\tau_2^2)$ for large $\tau_2$.

Hence defining the Fourier series of the seed function as

$$\Phi(\tau) = \Phi_0(\tau_2) + \sum_{n \neq 0} \Phi_n(\tau_2)e^{2\pi i n \tau_1}. \tag{5.82}$$

we see that the zero mode $\Phi_0(\tau_2)$ satisfies

$$\left( \tau_2^2 \frac{d^2}{d\tau_2^2} - 2 \right) \Phi_0 = -\frac{\pi \zeta(7)}{840} \tau_2, \tag{5.83}$$

which is solved by

$$\Phi_0(\tau_2) = \frac{\pi \zeta(7)}{1680} \tau_2. \tag{5.84}$$

The non–zero mode $\Phi_{1,n}(n \neq 0)$ satisfies the differential equation

$$\left( \tau_2^2 \frac{d^2}{d\tau_2^2} - 2 - 4\pi^2 n^2 \tau_2^2 \right) \Phi_{1,n} = -\frac{\pi^4 \tau_2^4 \sigma_7(n)}{1575 n^4} e^{-2\pi|n|\tau_2} \left( 1 + \frac{3}{\pi |n| \tau_2} + \frac{15}{4\pi^2 n^2 \tau_2^2} + \frac{15}{8\pi^3 |n|^3 \tau_2^3} \right). \tag{5.85}$$
As before, expressing
\[ \Phi_n = \Phi_h + \Phi_p, \]
we see that the solution to the homogeneous equation is given by
\[ \Phi_h = a_n \sqrt{\tau_2} K_{3/2}(2\pi|n|\tau_2) \]
where \(a_n\) is an arbitrary constant. Now the solution to the particular equation is given by
\[ \Phi_p = \frac{\sigma_\tau(n)}{302400 n^8} \left( \frac{105}{2\pi|n|\tau_2} + 105 + 180\pi|n|\tau_2 + 80\pi^2 n^2 \tau_2^2 + 16\pi^3|n|^3 \tau_2^3 \right) e^{-2\pi|n|\tau_2}, \]
and thus the boundary condition for small \(\tau_2\) demanding the cancellation of the \(\tau_2^{-1}\) term yields
\[ a_n = -\frac{\sigma_\tau(n)}{1440|n|^{15/2}}, \]
leading to the expression
\[ \Phi_n(\tau_2) = \frac{\pi \sigma_\tau(n) \tau_2}{75600 |n|^7} \left( 45 + 20\pi|n|\tau_2 + 4\pi^2 n^2 \tau_2^2 \right) e^{-2\pi|n|\tau_2} \]
for the non–zero mode.

Thus we have that
\[ 6C_{1,3,4} + 3C_{2,2,4} + 5C_{2,3,3} - \frac{27098}{10851} E_6 = \sum_{\gamma \in \Gamma_\infty \setminus SL(2, \mathbb{Z})} \Lambda(\gamma(\tau)), \]
where the seed function is given by
\[ \Lambda(\tau) = \frac{\pi \zeta(7)}{1680} \tau_2 + \frac{\pi \tau_2}{75600} \sum_{n=1}^{\infty} \frac{\sigma_\tau(n)}{n^7} \left( 45 + 20\pi n \tau_2 + 4\pi^2 n^2 \tau_2^2 \right) \left( e^{2\pi i n \tau} + e^{-2\pi i n \tau} \right). \]

To remedy the lack of absolute convergence due to the presence of the term linear in \(\tau_2\) in \(\Lambda(\tau)\), we consider
\[ E_4^2 - \frac{7293}{3617} E_8 = \sum_{\gamma \in \Gamma_\infty \setminus SL(2, \mathbb{Z})} \Upsilon_4(\gamma(\tau)), \]
with the seed function
\[ \Upsilon_4(\tau) = \Lambda_8(\tau) + \frac{2\zeta(8) \tau_2^3}{3\pi^4} \sum_{n=1}^{\infty} \frac{\sigma_\tau(n)}{n^4} \left( 1 + \frac{3}{\pi n \tau_2} + \frac{15}{4\pi^2 n^2 \tau_2^2} \right) \left( e^{2\pi i n \tau} + e^{-2\pi i n \tau} \right). \]

In (5.94) we have that
\[ \Lambda_8(\tau) = \frac{\pi \zeta(7)}{7560} \tau_2 + \frac{\pi \tau_2}{7560} \sum_{n=1}^{\infty} \frac{\sigma_\tau(n)}{n^7} \left( e^{2\pi i n \tau} + e^{-2\pi i n \tau} \right) \]
which arises in (5.92). This yields that

$$6C_{1,3,4} + 3C_{2,2,4} + 5C_{2,3,3} + \frac{142715}{21702}E_8 - \frac{9}{2}E_4^2 = \sum_{\gamma \in \Gamma_\infty \setminus SL(2,\mathbb{Z})} \Omega_8(\gamma(\tau)), \quad (5.96)$$

where the seed function is

$$\Omega_8(\tau) = -\frac{\zeta(8)\tau_2^4}{\pi^4} \sum_{n=1}^{\infty} \frac{\sigma_7(n)}{n^4} \left(3 + \frac{17}{2\pi n\tau_2} + \frac{35}{4\pi^2 n^2\tau_2^2}\right) \left(e^{2\pi i n\tau} + e^{-2\pi i n\tau}\right) \quad (5.97)$$

leading to an absolutely convergent Poincare series.

### 5.2 Zero mode of the Fourier series

We now consider the contribution to the zero mode of the Fourier series that follows from the above analysis. For

$$F = 6C_{1,3,4} + 3C_{2,2,4} + 5C_{2,3,3} + \frac{142715}{21702}E_8 - \frac{9}{2}E_4^2 \quad (5.98)$$

in \((B.116)\), from \((5.97)\) and \((B.118)\) we thus have that

$$\Omega_0 = 0, \quad \Omega_m = -\frac{\zeta(8)\tau_2^4}{m^4\pi^4} (3 + \frac{17}{2\pi |m|\tau_2} + \frac{35}{4\pi^2 m^2\tau_2^2}) e^{-2\pi |m|\tau_2} (m \neq 0). \quad (5.99)$$

The contributions to $F_0$ from the three terms in \((5.99)\) that are $O(\tau_2^0)$, $O(1/\tau_2)$ and $O(1/\tau_2^2)$ in the parentheses are given by

$$F_0^{(1)} = -\frac{\zeta(8)}{64\pi^3\tau_2^3} \sum_{k=0}^{\infty} \frac{(k+6)(k+5)(k+4)}{k!\zeta(2k+8)} \left(3 + \frac{17}{2\pi n\tau_2} + \frac{35}{4\pi^2 n^2\tau_2^2}\right) \left(-\frac{\pi}{\tau_2}\right)^k,$$

$$F_0^{(2)} = -\frac{17\zeta(8)}{32\pi^4\tau_2^4} \sum_{k=0}^{\infty} \frac{(k+4)(k+3)}{k!\zeta(2k+6)} \left(-\frac{\pi}{\tau_2}\right)^k \sum_{m>0} \frac{\sigma_7(m)\sigma_{-5-2k}(m)}{m^5-k},$$

$$F_0^{(3)} = -\frac{35\zeta(8)}{8\pi^5\tau_2^5} \sum_{k=0}^{\infty} \frac{(k+2)}{k!\zeta(2k+4)} \left(-\frac{\pi}{\tau_2}\right)^k \sum_{m>0} \frac{\sigma_7(m)\sigma_{-3-2k}(m)}{m^6-k},$$

$$= -\frac{35}{8\pi^5\tau_2^5} \sum_{k=0}^{\infty} \frac{(k+2)\zeta(-1-k)\zeta(6-k)\zeta(2+k)\zeta(9+k)}{k!\zeta(4+2k)} \left(-\frac{\pi}{\tau_2}\right)^k$$

$$\quad (5.100)$$
respectively.

Adding the three contributions, we get that

\[
F_0 = \sum_{i=1}^{3} F_0^{(i)} = \frac{5\zeta(9)}{432\pi \tau_2} + \frac{1}{64\tau_2^6} \sum_{k=0}^{\infty} g(k) \zeta(-6-k) \zeta(1-k) \zeta(7+k) \zeta(14+k) \left( -\frac{\pi}{\tau_2} \right)^k,
\]

where the factor \( g(k) \) is given by

\[
g(k) = \frac{(k+1)(k+3)(k+7)(k+10)(k+12)}{(k+5)!}.
\]

In (5.101), only \( k = 0 \) and 1 contributes to the sum leading to

\[
F_0 = \frac{5\zeta(9)}{432\pi \tau_2} - \frac{945\zeta(7)^2}{512\pi^6 \tau_2^6} + \frac{7300293\zeta(15)}{1851904\pi^7 \tau_2^7}
\]

where the \( k = 0 \) term requires regularization.

Thus we get that

\[
6C_{1,3,4} + 3C_{2,2,4} + 5C_{2,3,3} = \frac{54196\zeta(16)\tau_2^8}{10851\pi^8} + \frac{\zeta(7)\pi \tau_2}{840} + \frac{5\zeta(9)}{432\pi \tau_2} - \frac{45\zeta(7)^2}{512\pi^6 \tau_2^6} + \frac{2431\zeta(15)}{2048\pi^7 \tau_2^7},
\]

We have checked this exactly matches the expression one gets by directly analyzing the eigenvalue equation this combination of modular graphs satisfies, as well as with the structure one obtains from [18].

6 Discussion

In this paper, we have looked at several examples of (linear combinations of) modular graphs where we have calculated the power behaved terms in the zero mode of their Fourier expansion, starting from an expression for the absolutely convergent Poincare series involving them. There are several issues we would like to mention:

(i) We have only looked at graphs with an even number of links, which satisfy eigenvalue equations with source terms involving \( E_2 s \) and \( E_2^2 \). What about the other cases? For example, one can consider graphs with an odd number of links like \( C_{1,1,3} \) which satisfies the eigenvalue equation

\[
\left( \Delta - 6 \right) \left( C_{1,1,3} + \frac{\zeta(5)}{60} \right) = \frac{86}{5} E_5 - 4E_2 E_3,
\]

or graphs with an even number of links like \( C_{1,1,4} \) which satisfies the eigenvalue equation

\[
\left( \Delta - 12 \right) \left( 6C_{1,1,4} + C_{2,2,2} \right) = 120 E_6 + 12E_3^2 - 36E_2 E_4.
\]
In either case, we have performed the analysis and found that it yields the wrong coefficient of only one term in the zero mode of the Fourier series\(^\text{11}\), even though we use an absolutely convergent Poincare series in \((\text{B.119})\). This seems to be the issue whenever the eigenvalue equation contains a source term that is a product of modular graphs that are not identical (for example, \(E_{s_1}E_{s_2}\) for \(s_1 \neq s_2\) in the examples just mentioned).

(ii) This issue naturally raises the question–why did our analysis work for the examples we considered where the source terms only involve \(E_{2s}\) and \(E_{2s}^2\)? While this is perhaps somewhat surprising, we do not have a good understanding of this issue. In fact, this has been considered from a different point of view in [21], where every case considered needs to be regularized in an involved way to get the correct answer. However for the cases we considered, none of the intricacies of the regularization matter and we obtained the answer in a straightforward way. It would be fascinating to understand why this happens, and what is the crucial difference that makes regularization essential for the cases mentioned in (i). This should be related to the issue of performing the sum in calculating \(F_0\) where we neglected an issue of convergence.

(iii) We only focussed on the power behaved terms in the Fourier expansion of the zero modes, and neglected the contributions that are exponentially suppressed in the large \(\tau_2\) limit which did not arise in our analysis. In fact, in the final expression for \(F_0\) only a few terms contributed to the infinite sum. In [21] it has been suggested that the exponentially suppressed terms can be obtained precisely from these terms that vanish, based on certain assumptions and using the regularization mentioned in (ii). Once again, it would be interesting to understand these issues for our analysis.

(iv) Thus we see that the results we have obtained for several examples yield the correct power behaved terms in the zero mode of the Fourier series, which is really an experimental observation. Our results suggest an obvious generalization which we briefly mention. Consider a linear combination of modular graphs \(\Psi_{2s}\) with \(2s\) links which satisfies the eigenvalue equation\(^\text{12}\)

\[
\left(\Delta - s'(s' - 1)\right)\Psi_{2s} = E_{2s} - \frac{\zeta(4s)}{2\zeta(2s)^2}E_s^2,
\]

where \(s'\) is an integer greater than 1. Thus defining the Poincare series

\[
\Psi_{2s}(\tau) = \sum_{\gamma \in \Gamma_{\infty}\setminus SL(2,\mathbb{Z})} \Phi_{2s}(\gamma(\tau)),
\]

we see that the seed function satisfies the equation

\[
\left(\Delta - s'(s' - 1)\right)\Phi_{2s} = \frac{2\zeta(4s)}{\pi^{2s}}E_{2s} - \frac{\zeta(4s)\tau_2^{2s}}{\zeta(2s)\pi^s}E_s,
\]

where the right hand side is \(O(\tau_2)\) for large \(\tau_2\). Then one can proceed as we did with the choice of boundary conditions for the seed function to obtain the power behaved terms in

\(^{11}\)For \(C_{1,1,3}\) this has been observed before in [20].

\(^{12}\)If the source terms in the eigenvalue equation involve only \(E_{2s}\) and \(E_{2s}^2\), we can always bring it into this form by rescaling \(\Psi_{2s}\) and then shifting \(E_{2s}\) with an appropriate coefficient into the definition of \(\Psi_{2s}\) on using \((\text{A.110})\).
the zero mode of the Fourier series. It is an interesting exercise to see if this yields the correct answer, hence circumventing the need for regularization.

A The $SL(2,\mathbb{Z})$ invariant non–holomorphic Eisenstein series

The non–holomorphic Eisenstein series $E_s(\tau)$ satisfies the eigenvalue equation

$$\Delta E_s = s(s - 1)E_s,$$  \hspace{1cm} (A.110)

and has the Fourier expansion

$$E_s(\tau) = \frac{2\zeta(2s)}{\pi^s}\tau_2^s + \frac{2\Gamma(s - 1/2)}{\pi^{s-1/2}\Gamma(s)}\zeta(2s - 1)\tau_2^{1-s}$$

$$+ \frac{4\sqrt{\tau_2}}{\Gamma(s)}\sum_{n\neq 0} \sigma_{2s-1}(n) \frac{\zeta(s-1/2)}{n^{s-1/2}} K_{s-1/2}(2\pi|n|\tau_2)e^{2\pi in\tau_1},$$  \hspace{1cm} (A.111)

where the divisor function $\sigma_m(n)$ is defined by

$$\sigma_m(n) = \sum_{d|n, d>0} d^m,$$  \hspace{1cm} (A.112)

where the sum is over the positive divisors of $n$.

The Poincare series representation for the Eisenstein series $E_s(\tau)$ which is absolutely convergent for $Re s > 1$ is given by

$$E_s(\tau) = \frac{2\zeta(2s)}{\pi^s} \sum_{\gamma \in \Gamma_{\infty}\setminus SL(2,\mathbb{Z})} (\text{Im} \gamma(\tau))^s,$$  \hspace{1cm} (A.113)

where

$$\gamma(\tau) = \frac{a\tau + b}{c\tau + d}$$  \hspace{1cm} (A.114)

is an $SL(2,\mathbb{Z})$ transformation, under the identification by

$$\Gamma_{\infty} = \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix},$$  \hspace{1cm} (A.115)

for $n \in \mathbb{Z}$ which stabilizes the cusp at $\tau_2 \to \infty$.

B The Fourier series from the Poincare series

Suppose $F(\tau)$, which is modular invariant, has the Fourier expansion

$$F(\tau) = \sum_{n \in \mathbb{Z}} F_n(\tau_2)e^{2\pi in\tau_1}.$$  \hspace{1cm} (B.116)
Also suppose $F(\tau)$ is given by the absolutely convergent Poincare series

$$F(\tau) = \sum_{\gamma \in \Gamma \setminus \text{SL}(2,\mathbb{Z})} \Omega(\gamma(\tau)), \quad \text{(B.117)}$$

where the seed function $\Omega(\tau)$ has the Fourier expansion

$$\Omega(\tau) = \sum_{n \in \mathbb{Z}} \Omega_n(\tau_2)e^{2\pi in\tau_1}. \quad \text{(B.118)}$$

Then the Fourier modes $F_n(\tau_2)$ of $F(\tau)$ can be obtained from the Fourier modes $\Omega_n(\tau_2)$ of the seed function $\Omega(\tau)$.

Focussing only on $F_0(\tau_2)$, we have that (see [26,27], for example)

$$F_0(\tau_2) = \Omega_0(\tau_2) + \sum_{n>0} \sum_{m \in \mathbb{Z}} S(m,0;n) \int_{-\infty}^{\infty} dx \Omega_m \left( \frac{1}{n^2 \tau_2 (1+x^2)} \right) e^{-2\pi imx/[n^2 \tau_2 (1+x^2)]}, \quad \text{(B.119)}$$

where $S(m,0;n)$ is obtained from the Kloosterman sum

$$S(m,0;n) = \sum_{a \in \mathbb{Z}} e^{2\pi i (am+bn)/p}. \quad \text{(B.120)}$$

In (B.120), the restricted sum is over the integers

$$0 < a < p, \quad (a,p) = 1, \quad ab = 1 \pmod{p}. \quad \text{(B.121)}$$

In our analysis, we need the expression for the Kloosterman–Selberg zeta function given by

$$\sum_{n>0} \frac{S(\pm m,0;n)}{n^{2s}} = \frac{\sigma_{1-2s}(m)}{\zeta(2s)}, \quad (m \neq 0). \quad \text{(B.122)}$$

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