Graph Cohomologies from Arbitrary Algebras

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Abstract

For each commutative, graded algebra with finite dimension in each degree, we construct a graded cohomology theory for graphs whose graded Euler characteristic is the chromatic polynomial of the graph. This extends our previous work which was based on the algebra $\mathbb{Z}[x]/(x^2)$.

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1 Introduction

In [7], Khovanov introduced a graded cohomology theory for classical links and showed it yields the Jones polynomial by taking the graded Euler characteristic. This construction has sparked a good deal of interests in recent years. In [5], a graded cohomology theory for graphs was constructed. The graded Euler characteristic of the cohomology groups is the chromatic polynomial of the graph.

Both constructions in [7] and in [5] depend on a given graded algebra (more precisely a Frobenius algebra in the case of knots and links) which is the building block for the chain groups in the chain complex. However, the amounts of choices of algebras are quite different. In the case of links, the choices are quite limited due to invariance of Reidemeister moves. For the case of graphs, the choices are abundant. The algebra used in [5], $\mathbb{Z}[x]/(x^2)$, is the simplest natural choice. The purpose of this note is to show the construction in [5] can be made for any graded $\mathcal{R}$-algebra that is finite dimensional in each degree, commutative, and whose product is degree preserving, i.e. $\deg(yz) = \deg y + \deg z$ for all homogeneous elements $y$ and $z$, where $\mathcal{R}$ is an integral domain.

In section 2, we explain the definition of the chain complex, and show that the Euler characteristic of the cohomology groups is equal to the chromatic polynomial of the graph evaluated at $\lambda = q \dim \mathcal{A}$ where $q \dim \mathcal{A}$ is the graded dimension of the algebra $\mathcal{A}$. In section 3, we discuss some basic properties of our cohomology groups. In particular, we construct a long exact sequence which can be considered as a categorification of the deletion-contraction rule of the chromatic polynomial. In section 4, we show some computational examples. When the algebra is $\mathbb{Z}[x]/(x^3)$, the cohomology groups can have a torsion of order three. This is contrary to the cohomology groups in [5] and in [7], where the computations suggest no odd torsion can occur [10]. We also show some computations when the algebra has no grading, in which case $q \dim \mathcal{A} = \dim \mathcal{A}$ is an integer. The Euler characteristic of the cohomology groups is therefore an integer $P_G(\dim \mathcal{A})$. This should probably be compared to the work of Eastwood and Huggett [2], who constructed, for each positive integer $\lambda$ and each graph $G$, a topological space $M$ whose Euler characteristic is the integer $P_G(\lambda)$. However, we don’t know if there is any connections between our work and theirs. We also make some comments on the strength of the cohomology groups. In particular, we show that (a twisted version) of our cohomology groups can be stronger than the chromatic polynomial. The last section is an appendix in which we classify rings whose additive group is $\mathbb{Z} \oplus \mathbb{Z}$.

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2 The Construction

2.1 The chromatic polynomial

We recall some basic properties for the chromatic polynomial. Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. For each positive integer $\lambda$, let $\{1, 2, \ldots, \lambda\}$ be the set of $\lambda$-colors. A $\lambda$-coloring of $G$ is an assignment of a $\lambda$-color to each vertex of $G$ such that vertices that are connected by an edge in $G$ always have different colors. Let $P_G(\lambda)$ be the number of $\lambda$-colorings of $G$. It is well-known that $P_G(\lambda)$ satisfies the deletion-contraction relation

$$P_G(\lambda) = P_{G-e}(\lambda) - P_{G/e}(\lambda)$$

Furthermore, it is obvious that

$$P_{N_n}(\lambda) = \lambda^n$$

where $N_n$ is the graph with $n$ vertices and no edges.

These two equations uniquely determines $P_G(\lambda)$. They also imply that $P_G(\lambda)$ is always a polynomial of $\lambda$, known as the chromatic polynomial.

There is another formula for $P_G(\lambda)$ that is useful for us. For each $s \subseteq E(G)$, let $[G : s]$ be the graph whose vertex set is $V(G)$ and whose edge set is $s$, let $k(s)$ be the number of connected components of $[G : s]$. We have

$$P_G(\lambda) = \sum_{s \subseteq E(G)} (-1)^{|s|} \lambda^{k(s)}.$$  \hfill (1)

Equivalently, grouping the terms $s$ with the same number of edges yields the following state sum formula

$$P_G(\lambda) = \sum_{i \geq 0} (-1)^i \sum_{s \subseteq E(G), |s| = i} \lambda^{k(s)}.$$  \hfill (2)

2.2 Graded algebras

We recall some definitions and specify what kind of algebras we will work with.

**Definition 2.1** Let $R$ be a commutative ring with identity. An algebra over $R$ is a ring $\mathcal{A}$ that is simultaneously a $R$-module and such that $r(ab) = (ra)b = a(rb)$ for all $r \in R$ and $a, b \in \mathcal{A}$. 
Definition 2.2 A graded $R$-algebra $\mathcal{A}$ is an algebra with direct sum decomposition $\mathcal{A} = \bigoplus_{i=0}^{\infty} A_i$ into $R$-submodules such that $a_i a_j \in A_{i+j}$ for all $a_i \in A_i$ and $a_j \in A_j$. The elements of $A_j$ are called homogeneous elements of degree $j$.

Most of our results can be stated for any integral domain $R$ (see section 2.4). However, for simplicity, we will assume that $R = \mathbb{Z}$, i.e. $\mathcal{A}$ is a ring itself. Furthermore, we want each $A_i$ to have finite free rank, where the free rank of a $\mathbb{Z}$-module $M$ is defined to be $\dim_{\mathbb{Q}}(M \otimes_{\mathbb{Z}} \mathbb{Q})$ and is denoted by $\text{rank}(M)$.

Thus, from now on, we will work with algebras that satisfy the following conditions:

Assumptions 2.3 $\mathcal{A} = \bigoplus_{i=0}^{\infty} A_i$ is a commutative, graded $\mathbb{Z}$-algebra with 1 such that each $A_i$ is free of finite rank.

Note that these assumptions can sometimes be relaxed. For instance, the construction can still be made even if there is no identity or if the $A_i$’s are not free.

Since $\mathcal{A}$ can be consider as a $\mathbb{Z}$-module, the graded dimension of $\mathcal{A}$ is defined using the definition below.

Definition 2.4 Let $M = \bigoplus_{j=0}^{\infty} M_j$ be a graded $\mathbb{Z}$-module where $M_j$ denotes the set of homogeneous elements of degree $j$ of $M$. Assume that $\text{rank}(M_j) < \infty$ for each $j$. The graded dimension of $M$ is the power series

$$q \dim M := \sum_{j=0}^{\infty} q^j \text{rank}(M_j).$$

2.3 The chain complex

Figure 1 shows what the chain complex will look like and the details can be found after the figure. The diagram comes first because we thought it might be helpful to have a picture of what is going on while reading the formal definitions.

Let $G$ be a graph and $E = E(G)$ be the edge set of $G$. Let $n = |E|$ be the cardinality of $E$. We fix an ordering on $E$ and denote the edges by $e_1, \ldots, e_n$. Consider the $n$-dimensional cube $\{0,1\}^E = \{0,1\}^n$. Each vertex $\alpha$ of this cube corresponds to a subset $s = s_\alpha$ of $E$, where $e_i \in s_\alpha$ if and only if $\alpha_i = 1$. The height $|\alpha|$ of $\alpha$, is defined by $|\alpha| = \sum \alpha_i$, which is also equal to the number of edges in $s_\alpha$.

For each vertex $\alpha$ of the cube, we associate the graded $\mathbb{Z}$-module $C^\alpha(G)$ as follows. Consider $[G : s]$, the graph with vertex set $V(G)$ and edge set $s$. We assign a copy of $\mathcal{A}$ to each component of $[G : s]$ and then taking tensor product over the components. Let
Figure 1: The chain complex when $G = P_3$

$C^\alpha(G)$ be the resulting graded $\mathbb{Z}$-module, with the induced grading from $\mathcal{A}$. Therefore, $C^\alpha(G) \cong \mathcal{A}^\otimes k$ where $k$ is the number of components of $[G : s]$. We define the $i^{th}$ chain group of our complex to be $C^i(G) := \bigoplus_{|\alpha|=i} C^\alpha(G)$. Keep in mind that $C^i(G)$ depends on the algebra $\mathcal{A}$. Thus one may want to denote it by $C^i_{\mathcal{A}}(G)$. However, we will omit the letter $\mathcal{A}$ unless there is an ambiguity. Also, we sometimes interchange the notions $\alpha$ and $s$. Thus $C^\alpha(G)$ is sometimes denoted by $C^\alpha_{\mathcal{A}}(G)$. This should not cause any confusion.

To define the differential maps $d^i$, we need to make use of the edges of the cube $\{0, 1\}^E$. Each edge $\xi$ of $\{0, 1\}^E$ can be labeled by a sequence in $\{0, 1, *\}^E$ with exactly one *. The tail of the edge is obtained by setting $* = 0$ and the head is obtained by setting $* = 1$. The height $|\xi|$ is defined to be the height of its tail, which is also equal to the number of 1’s in $\xi$.

Given an edge $\xi$ of the cube, let $\alpha_1$ be its tail and $\alpha_2$ be its head. The per-edge map $d_{\xi} : C^{\alpha_1}(G) \to C^{\alpha_2}(G)$ is defined as follows. For $j = 1$ and $2$, the $\mathbb{Z}$-module $C^{\alpha_j}(G)$ is $\mathcal{A}^\otimes k_j$ where $k_j$ is the number of connected components of $[G : s_j]$ (here $s_j$ stands for $s_{\alpha_j}$).

Let $e$ be the edge with $s_2 = s_1 \cup \{e\}$.

If $e$ joins a component of $[G : s_1]$ to itself, then $k_1 = k_2$ and the components of $[G : s_1]$
and the components of \([G : s_2]\) naturally correspond to each other. We let \(d_\xi\) to be the identity map.

If \(e\) joins two different components of \([G : s_1]\), say \(E_1\) to \(E_2\) where \(E_1, E_2, \ldots, E_{k_1}\) are the components of \([G : s_1]\), then \(k_2 = k_1 - 1\) and the components of \([G : s_2]\) are \(E_1 \cup E_2 \cup \{e\}, E_3, \ldots, E_{k_1}\). We define \(d_\xi\) to be the identity map on the tensor factors coming from \(E_3, \ldots, E_{k_1}\), and \(d_\xi\) on the remaining tensor factors to be the multiplication map \(A \otimes A \to A\) sending \(x \otimes y\) to \(xy\).

Now, we define the differential \(d^i : C^i(G) \to C^{i+1}(G)\) by \(d^i = \sum_{|\xi|=i} (-1)^\xi d_\xi\), where \((-1)^\xi = -1\) (resp. 1) if the number of 1’s in \(\xi\) before * is odd (resp. even).

We have

**Theorem 2.5**  
(a) \(0 \to C^0(G) \overset{d^0}{\to} C^1(G) \overset{d^1}{\to} \cdots \overset{d^{n-1}}{\to} C^n(G) \to 0\) is a graded chain complex whose differential is degree preserving.  
(b) The cohomology groups \(H^i(G) = H^i_A(G)\) are independent of the ordering of the edges of \(G\), and therefore are invariants of the graph \(G\).  
(c) The graded Euler characteristic of the chain complex is equal to the chromatic polynomial of the graph \(G\) evaluated at \(\lambda = q \dim A\), i.e. \(\chi_q(C) = \sum_{0 \leq i \leq n} (-1)^i q \dim(H^i) = \sum_{0 \leq i \leq n} (-1)^i q \dim(C^i) = P_G(q \dim A)\)

**Proof.** The proof is rather standard. We sketch the ideas here.

(a). To prove this defines a chain complex, we need to show that \(d\) is a differential. That is, \(d \circ d = 0\). This is done in two steps. First, we verify that the maps \(d_\xi\) makes the cube commutative, a fact follows from the associativity of the algebra. Second, the signs \((-1)^\xi\) in \(d\) allow us to cancel out all terms in \(d \circ d\). Thus \(d \circ d = 0\).

To show that \(d\) is degree preserving, we note that the multiplication map on \(A\) is always degree preserving, which then implies each map \(d_\xi\) is degree preserving, and therefore so is \(d\).

(b). The proof is similar to the one for \(\mathbb{Z}[x]/(x^2)\) in [3]. Each permutation of the edges of \(G\) is a product of transpositions of the form \((k, k+1)\). An explicit isomorphism can be constructed for each such transposition. In fact, this shows that the isomorphism class of the chain complex is an invariant of the graph.

(c). First, a standard homological algebra argument shows that \(\sum_{0 \leq i \leq n} (-1)^i q \dim(H^i) = \sum_{0 \leq i \leq n} (-1)^i q \dim(C^i)\). Next, we note that \(q \dim(C^\alpha(G)) = q \dim(A^{\otimes k}) = (q \dim(A))^k\) where \(k\) is the number of connected components of \([G : s]\). Taking direct sum over \(s \subseteq E(G), |s| = i\), and then taking alternating sum over \(i\), we obtain the equation \(\sum_{0 \leq i \leq n} (-1)^i q \dim(C^i) = P_G(q \dim A)\).
Remark 2.6  (a) The above graded chain complex can be turned into a bi-graded chain complex. Let $C^{i,j}(G)$ be the subgroup of $C^i(G)$ consisting of homogeneous elements with degree $j$. For each $j$ we have a chain complex

$$0 \to C^{0,j}(G) \to C^{1,j}(G) \to \cdots \to C^{n,j}(G) \to 0$$

The direct sum of these chain complexes, with the obvious gradings, is equal to the chain complex in Theorem 2.5.

(b) Our chain complexes can also be described in terms of enhanced states. One defines an enhanced state $S$ of $G$ to be a pair $(s,c)$ where $s \subseteq E(G)$ and $c$ is an assignment of an element of $A$ to each connected component of $[G:s]$. Identify $S$ with the element $c(E_1) \otimes \cdots \otimes c(E_k)$ of $C^*(G) = \mathcal{A}^\otimes k$, where $E_1, \ldots, E_k$ are components of $[G:s]$. Thus $C^i(G)$ is generated by states with $|s| = i$. When each $c(E_i)$ is a homogeneous element of $A$, we say the coloring $c$ and the enhanced state $S$ are homogeneous, and we define its degree to be $j(S) = \sum_i \deg c(E_i)$. It is easy to see that $C^{i,j}$ above is generated by all homogeneous enhanced states $S$ with $i(S) = i, j(S) = j$. The differential of each enhanced state is then defined to be the operation of adding each edge not in $s$, adjusting the coloring $c$ using the multiplication on $A$ or the identity map, and then taking the summation over the edges in $E(G) - s$ with appropriate $\pm 1$ signs in front of each term.

2.4 Some variations in the construction

Some variations can be introduced in our construction.

(a) As we discussed in 2.2, our coefficient ring $\mathbb{Z}$ can be replaced by any integral domain $R$, i.e. a commutative ring with $1 \neq 0$ and no zero divisors. By analogy with the abelian group case, we define the free rank of a module $M$ over an integral domain $R$ to be $\text{rank}(M) := \dim_M M \otimes_R Q$ where $Q$ is the field of fractions of $R$. Note that $M \otimes_R Q$ is a vector space over $Q$ so its dimension is well defined.

Let $A = \oplus_{i=0}^{\infty} A_i$ be a graded algebra over an integral domain $R$ such that each $A_i$ has finite free rank. Using $A$ as the building block for our chain $R$-modules (the counterpart of the chain groups in the $R$-module case), the same construction can still be made and

$$\sum_{0 \leq i \leq n} (-1)^i q \dim(C^i) = P_G(q \dim A)$$

where the graded dimension of $A$ is obviously defined to be the power series $q \dim A = \sum_i q^i \dim A$.

The proof that $\sum_{0 \leq i \leq n} (-1)^i q \dim(H^i) = \sum_{0 \leq i \leq n} (-1)^i q \dim(C^i)$ is the same as in the abelian case, we just need to prove the counterpart in the $R$-module category of a well known result for abelian groups. Namely, it is enough to show that if $0 \to L \to M \to N \to 0$ is an exact sequence of $R$-modules, then $\text{rank } M = \text{rank } L + \text{rank } N$. This is true because the quotient field of an integral domain $R$ is a flat $R$-module ([9], p86). Hence
taking the tensor product of the exact sequence $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ by $Q$ yields an exact sequence of $R$-modules that turns out to be an exact sequence of $Q$-vector spaces and the result follows.

Some properties, e.g. the result (3.3) on pendant edge in the next section, can be extended to the case when $R$ is a principal idea domain.

(b) We can define a twisted version of the differential map $d$ as follows. Let $f : A \rightarrow A$ be a degree preserving homomorphism of the algebra $A$. That is, $f$ is linear, $f(yz) = f(y)f(z)$ for all $y, z$, and $\deg(f(y)) = \deg(y)$ for all homogeneous elements $y$. Let $\xi$ be an edge of the cube we had before. Let $\alpha_1$ be its tail, $\alpha_2$ be its head, and $e$ be the edge of $G$ corresponding to $\xi$ so that $s_2 = s_1 \cup \{e\}$. As before, the $\mathbb{Z}$-module $C^{\alpha_j}(G) = \mathcal{A}^{\otimes k_j}$ where $j = 1, 2$ and $k_j$ is the number of connected components of $[G : s_j]$ (here $s_j = s_{\alpha_j}$).

If $e$ joins a component of $[G : s_1]$ to itself, then $k_1 = k_2$ and the components of $[G : s_1]$ and the components of $[G : s_2]$ naturally correspond to each other. We let \( d_\xi = f^{\otimes k_1} : \mathcal{A}^{\otimes k_1} \rightarrow \mathcal{A}^{\otimes k_2} \) (note that $k_1 = k_2$).

If $e$ joins two different components of $[G : s_1]$, we define $d_\xi$ the same way as before. This yields the differential $d^i : C^i(G) \rightarrow C^{i+1}(G)$ using the same formula: $d^i = \sum_{|\xi|=i} (-1)^{i \xi} d_\xi$.

If $f$ is the identity map, this is the same as our construction above. Otherwise, the constructions can be different. For example, if $f = 0$ and $G$ is the graph with one vertex and $n$ loops, then the differential $d = 0$. Thus $H^i(G) \cong C^i(G)$ which are nontrivial for $i = 0, 1, \cdots, n$. On the other hand, the (untwisted) cohomology groups all vanishes when $G$ has a loop (see Corollary 3.2 in the next section).

3 Some Properties

In this section, we discuss some basic properties of the cohomology groups. Most of these are parallel to the case when the algebra is $\mathbb{Z}[x]/(x^2)$ whose discussion was given in [5]. The most interesting property is a long exact sequence, which can be considered as a categorification for the deletion-contraction rule for the chromatic polynomial.

3.1 An exact sequence

The long exact sequence comes from a short exact sequence of graded chain homomorphisms

$$0 \rightarrow C^{i-1}(G/e) \xrightarrow{\alpha} C^i(G) \xrightarrow{\beta} C^i(G - e) \rightarrow 0$$
which we explain here. Basically \( \alpha \) is the map that recovers the edge \( e \), and \( \beta \) is the projection map that kills every state containing \( e \). A more precise description is given below. First, we order the edges of \( G \) so that \( e \) is the last edge. This induces natural orderings on the edge sets of \( G/e \) and \( G - e \) by deleting \( e \) from the list. For each \( s \subseteq E(G/e) \), let \( \tilde{s} = s \cup \{e\} \). Then \( \tilde{s} \subseteq E(G) \). Recall that \( C^s(G/e) \) (resp. \( C^\tilde{s}(G) \)) is the tensor product of \( A \) taken over components of \( [G/e : s] \) (resp. \( [G : \tilde{s}] \)). The components of \( [G/e : s] \) and the components of \( [G : \tilde{s}] \) are the same except for the one involving \( e \) in which case they are related by a contraction of \( e \). Thus we have \( C^s(G/e) \cong C^\tilde{s}(G) \) via a natural isomorphism, since the tensor factors naturally correspond to each other. Let \( \alpha|_{C^i(G/e)} : C^s(G/e) \to C^\tilde{s}(G) \) be this isomorphism. Taking direct sum over \( s \), we obtain the homomorphism \( \alpha : C^{i-1}(G/e) \to C^i(G) \). Next, we explain the map \( \beta : C^i(G) \to C^i(G - e) \). We have \( C^i(G) = \oplus_{s \in E(G), |s| = i} C^s(G) \). If \( e \notin s \), \( s \) is automatically a subset of \( E(G - e) \). We have \( C^s(G) = C^\tilde{s}(G - e) \) since the graphs \( [G : s] \) and \( [G - e : s] \) are identical. The map \( \beta \) is the identity map from \( C^s(G) \) to \( C^s(G - e) \). If \( e \in s \), we let \( \beta|_{C^s(G)} \) be the zero map. Taking direct sum over \( s \) with \(|s| = i\), we obtain the map \( \beta : C^i(G) \to C^i(G - e) \). A standard diagram chasing argument shows that this defines a short exact sequence of chain complexes. Thus we have

**Theorem 3.1** Let \( G \) be a graph, and \( e \) be an edge of \( G \).

(a) For each \( i \), there is a short exact sequence of graded chain homomorphisms: \( 0 \to C^{i-1}(G/e) \xrightarrow{\alpha} C^i(G) \xrightarrow{\beta} C^i(G - e) \to 0 \), and therefore by the zig-zag lemma,

(b) it induces a long exact sequence of cohomology groups: \( 0 \to H^0(G) \xrightarrow{\alpha^*} H^0(G/e) \xrightarrow{\beta^*} \ldots \to H^i(G/e) \xrightarrow{\gamma^*} H^i(G) \xrightarrow{\beta^*} H^i(G - e) \xrightarrow{\gamma^*} \ldots \)

Taking the alternating sum of the graded dimensions in the above long exact sequence, we obtain the deletion-contraction rule. It is in this sense that the long exact sequence is considered as a categorification of the deletion-contraction rule.

It is useful to understand the following geometric description of the maps \( \alpha^* \), \( \beta^* \), and \( \gamma^* \): \( \alpha^* \) expands the edge \( e \) and keeps in the same coloring, \( \beta^* \) is the projection map that kills every state containing \( e \). For \( \gamma^* \), we add the edge \( e \), contract it to a point, assign a natural coloring described below and then multiply by \((-1)^i\). The corresponding coloring is the same if \( e \) connects a component to itself. If \( e \) connects two separate components, say \( E_1 \) and \( E_2 \), then new coloring on the component \( E_1 \cup E_2 \cup \{e\} \) is the product \( c(E_1)c(E_2) \) and is the same as \( c(E_i) \) for all other components \( E_i \).

Other basic properties can follow either from this exact sequence or from the definition. For example, we have
3.2 Graphs with loops or multiple edges

Corollary 3.2 (a) If a graph has a loop then all the cohomology groups are trivial.
(b) The cohomology groups are unchanged if you replace all the multiple edges of a graph by single edges.

Proof. (a). In the long exact sequence

\[ \ldots \rightarrow H^{i-1}(G - e) \xrightarrow{\alpha^*} H^i(G/e) \xrightarrow{\beta^*} H^i(G) \xrightarrow{\gamma^*} H^i(G/e) \xrightarrow{\alpha^*} \ldots \]

we have \( G/e = G - e \) and the map \( \gamma^* \) is the identity map multiplied by \((-1)^i\). It follows that \( H^i(G) = 0 \) for each \( i \).

(b) Let \( e_1 \) and \( e_2 \) be two edges connecting the same pair of vertices in \( G \). In the exact sequence

\[ \rightarrow H^{i-1}(G/e_2) \xrightarrow{\alpha^*} H^i(G) \xrightarrow{\beta^*} H^i(G - e_2) \xrightarrow{\gamma^*} H^i(G/e_2) \rightarrow \]

the graph \( G/e_2 \) contains a loop coming from \( e_1 \). Therefore \( H^{i-1}(G/e_2) = H^i(G/e_2) = 0 \). It follows that \( H^i(G) \cong H^i(G - e_2) \). One can repeat this process until all redundant edges in \( G \) are removed. 

3.3 Adding a pendant edge.

Next, we consider the effect of adding a pendant edge to a graph. Recall that a pendant vertex in a graph is a vertex of degree one, and a pendant edge is an edge connecting a pendant vertex to another vertex. Let \( e \) be a pendant edge in a graph \( G \), then \( P_G(\lambda) = (\lambda - 1)P_{G/e}(\lambda) \). An analogous equation on the cohomology level is given below, as long as our algebra \( A \) has an identity. First we prove an algebraic lemma.

Lemma 3.3 Let \( A \) be an algebra over \( \mathbb{Z} \) that with identity \( 1_A \). Assume that the additive group \((A, +)\) is a free abelian group of finite rank. Then \( 1_A \) generates a direct summand of the abelian group \((A, +)\).

Proof. Let \( d \) be a positive integer such that \( 1_A = de_1 \) and \( \{e_1, \ldots, e_k\} \) is a basis of the abelian group \((A, +)\) (see [6] p73). It enough to show that \( d = 1 \). We have \( 1_A = de_1 = d(1_Ae_1) = d^2(e_1^2) \). We write \( e_1^2 \) as a linear combination of the basis elements: \( e_1^2 = b_1e_1 + \cdots + b_ke_k \). This implies \( 1_A = d^2b_1e_1 + \cdots d^2b_ke_k \). On the other hand, we have \( 1_A = de_1 \). The uniqueness of coefficients implies that \( d^2b_1 = d \), which implies \( d = 1 \).

By the lemma, if our graded algebra \( A \) has an identity, then \( A = \mathbb{Z}1_A \oplus A' \) as a \( \mathbb{Z} \)-module, where \( \mathbb{Z}1_A \) is generated by the identity of \( A \) and \( A' \) is a submodule of \( A \). We
simply apply the lemma to our graded algebra \( A \). If \( A \) has infinite dimension, we apply the lemma to the subalgebra \( A_0 \) instead which is free of finite rank by assumptions (2.3).

**Proposition 3.4** Let \( A \) and \( A' \) be as above. Then \( H^i(G) \cong H^i(G/e) \otimes A' \) where \( e \) is a pendant edge of \( G \).

**Proof.** Consider the operations of contracting and deleting \( e \) in \( G \). Denote the graph \( G/e \) by \( G_1 \). We have \( G/e = G_1 \cup \{v\} \), where \( v \) is the end point of \( e \) with \( \deg v = 1 \). Consider the exact sequence

\[
\cdots \to H^{i-1}(G_1 \cup \{v\}) \xrightarrow{\gamma^*} H^{i-1}(G_1) \xrightarrow{\alpha^*} H^i(G) \xrightarrow{\beta^*} H^i(G_1 \cup \{v\}) \xrightarrow{\gamma^*} H^i(G_1) \to \cdots
\]

We need to understand the map

\[
H^i(G_1 \cup \{v\}) \xrightarrow{\gamma^*} H^i(G_1)
\]

It is easy to compute the cohomology groups for the one point graph \( \{v\} \) (see the first example in the next section). We have \( H^0(\{v\}) \cong A \) and \( H^i(\{v\}) = 0 \) for all \( i > 0 \). Thus, the K"unneth type formula (3.6) below implies

\[
H^i(G_1 \cup \{v\}) \cong H^i(G_1) \otimes A
\]

by a natural isomorphism \( h_* \). Since, \( A = Z_1 A \oplus A' \), we identify \( H^i(G_1 \cup \{v\}) \) with \( H^i(G_1) \otimes (Z \oplus A') \). The map \( \gamma^* : H^i(G_1 \cup \{v\}) \to H^i(G_1) \) sends \( x \otimes 1 \) to \((-1)^i x \). In particular, \( \gamma^* \) is onto. Therefore, the above long exact sequence becomes a collection of short exact sequences

\[
0 \to H^i(G) \xrightarrow{\beta^*} H^i(G_1 \cup \{v\}) \xrightarrow{\gamma^*} H^i(G_1) \to 0 \tag{3}
\]

Hence, \( H^i(G) \cong \ker \gamma^* \). We define a homomorphism:

\[
f : H^i(G_1) \otimes A' \to \ker \gamma^* \text{ by } f(x \otimes a') = x \otimes a' - (-1)^i \gamma^*(x \otimes a') \otimes 1
\]

One checks that \( f \) is an isomorphism of \( \mathbb{Z} \)-modules. Therefore, \( H^i(G) \cong \ker \gamma^* \cong H^i(G_1) \otimes A' \).

**Remark 3.5** The isomorphism \( f : H^i(G_1) \otimes A' \to H^i(G) \) defined above can be visualized as follows. Each cycle in \( C^i(G_1) \) is a linear combination of terms of the form \( u \), where the vertex to which \( e \) collapses is labelled with \( u \in A \). Such a term becomes \( u \) under the action of \( f \), where the element of \( A \) written close to a vertex indicates the label that has been assigned to this component.
3.4 Disjoint union

Finally, we state a Künneth theorem type formula for our cohomology groups under disjoint union. It will be used in our computation in the next section.

**Proposition 3.6** Let \( \mathcal{A} \) be an algebra satisfying the assumptions \([2.3]\). For each \( i \in \mathbb{N} \), we have:

\[
H^i_{\mathcal{A}}(G_1 \sqcup G_2) \cong \bigoplus_{p+q=i} \left[ H^p_{\mathcal{A}}(G_1) \otimes H^q_{\mathcal{A}}(G_2) \right] \oplus \bigoplus_{p+q=i+1} \left[ H^p_{\mathcal{A}}(G_1) \ast H^q_{\mathcal{A}}(G_2) \right]
\]

where \( \ast \) denotes the torsion product of two abelian groups.

**Proof.** This is a corollary of Künneth theorem \([3]\), since the chains complexes \( \mathcal{C}(G_1) \) and \( \mathcal{C}(G_2) \) are free. \( \blacksquare \)

4 Some Computations

We show some computational examples. The examples for null graphs and trees follow easily from the previous section. The results are needed for other computations. The examples for the graph \( P_3 \), “the triangle”, with different algebras show that there can be a variety of torsions. Our final example shows the (twisted) cohomology theory (see section 2.4) can be stronger than the chromatic polynomial, by varying the algebra \( \mathcal{A} \) and the homomorphism \( f \).

**Example 4.1** Let \( G = N_n \) be the order \( n \) null graph. That is, the graph with \( n \) vertices and no edges. Then \( C^0(G) \cong \mathcal{A}^\otimes n \) and \( C^i(G) = 0 \) for \( i > 0 \). It follows that \( H^0(G) \cong \mathcal{A}^\otimes n \) and \( H^i(G) = 0 \) for all \( i > 0 \).

**Example 4.2** Given any graph \( G \), let \( G \sqcup N_n \) be the graph obtained by adding \( n \) isolated vertices to \( G \). By Proposition \([3.6]\) and Example \([4.1]\), \( H^i(G \sqcup N_n) \cong H^i(G) \otimes \mathcal{A}^\otimes n \) for all \( i \geq 0 \).

**Example 4.3** Let \( G = T_n \) be a tree with \( n+1 \) vertices and \( n \) edges. We assume \( \mathcal{A} \) has an identity and therefore \( \mathcal{A} \cong \mathbb{Z} \oplus \mathcal{A}' \) for some \( \mathbb{Z} \)-submodule \( \mathcal{A}' \). By Proposition \([3.4]\), \( H^0(T_n) \cong \mathcal{A} \otimes \mathcal{A}'^{\otimes n} \) and \( H^i(T_n) = 0 \) for all \( i > 0 \).

Recall that \( \cdot \{\ell\} \) is the degree shift operation on graded \( \mathbb{Z} \)-modules that increases the degree of all homogeneous elements by \( \ell \). For instance, in the example below, \( \mathbb{Z}^3\{4\} \) denotes the graded \( \mathbb{Z} \)-module \( \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \) where the three generators have degree 4.
Example 4.4 Let $G = P_3$, the polygon graph on 3 vertices (see Figure [4]). Let $A = \mathbb{Z}[x]/(x^3)$, we will show below that

$$H^0(P_3) \cong \mathbb{Z}\{3\} \oplus \mathbb{Z}^3\{4\} \oplus \mathbb{Z}^3\{5\} \oplus \mathbb{Z}\{6\}$$

and $H^i(G) = 0$ for $i > 1$.

The computation is done using the exact sequence. We start with $N_1$, and then add edges until we have $P_3$.

If $G = N_1$, $H^0(G) \cong \mathcal{A}$ with generators being $\{x^r | r = 0, 1, 2\}$.

If $G = T_1$, the tree with one edge, Proposition 3.4 and Remark 3.5 imply that $H^0(G) \cong \mathcal{A} \otimes \mathcal{A}'$ with generators $\{e_{rs} | r = 0, 1, 2, s = 1, 2\}$, where $e_{rs} = x^r \otimes x^s - x^{r+s} \otimes 1$ in $H^0(G)$. For simplicity, we drop the tensor notation and denote $e_{rs}$ by $e_{rs} = x^r x^s - x^{r+s} 1$. A picture of $e_{rs}$ is $e_{rs} = x^r \otimes x^s - x^{r+s} 1$.

If $G = T_2$, the tree with two edges, the same argument shows that a set of generators of $H^0(G) \cong H^0(T_1) \otimes \mathcal{A}'$ is $\{e_{rst} | r = 0, 1, 2, s = 1, 2, t = 1, 2\}$ where $e_{rst} = (x^r x^s x^t - x^r x^{s+t} 1) - (x^{r+s} 1 x^t - x^{r+s} x^t 1) = x^r x^s x^t - x^r x^{s+t} 1 - x^{r+s} x^t 1 + x^r x^{s+t} 1$. A picture of $e_{rst}$ is $e_{rst} = x^r \otimes x^s \otimes x^t - x^{r+s+t} 1 - x^{r+s} x^t 1 + x^{r+s+t} 1$.

Now, let $G = P_3$. Let $e$ be an edge of $P_3$. The exact sequence on $(P_3, e)$ gives

$$0 \rightarrow H^0(P_3) \xrightarrow{\partial^*} H^0(P_3 - e) \xrightarrow{\gamma^*} H^0(P_3/e) \xrightarrow{\partial^*} H^1(P_3) \rightarrow 0$$

where $H^0(P_3 - e) = H^0(T_2)$ is freely generated by $\{e_{011}, e_{012}, e_{021}, e_{111}, e_{022}, e_{112}, e_{121}, e_{211}, e_{122}, e_{021}, e_{122}, e_{222}\}$, and $H^0(P_3/e) \cong H^0(T_1)$ is freely generated by $\{e_{01}, e_{02}, e_{11}, e_{12}, e_{21}, e_{22}\}$. The map $\gamma^*$ sends $a \otimes b \otimes c$ to $ac \otimes b$. Thus $\gamma^*(e_{rst}) = x^{r+t} x^s - x^{r+s+t} - x^{r+s+t} 1 + x^{r+s} x^t = (x^{r+t} x^s - x^{r+s+t} 1) - (x^r x^{s+t} - x^{r+s+t} 1) + (x^{r+s} x^t - x^{r+s+t} 1) = e_{r+t,s} - e_{r,s+t} + e_{r+s,t}$.

This gives

\[
\begin{align*}
\gamma^*(e_{011}) &= e_{11} - e_{02} + e_{11} = 2e_{11} - e_{02} \\
\gamma^*(e_{012}) &= e_{21} - e_{03} + e_{12} + e_{21} \quad \text{(here $e_{03} = 0$ since $x^3 = 0$, same for $e_{13}$ etc)} \\
\gamma^*(e_{021}) &= e_{12} - e_{03} + e_{21} = e_{12} + e_{21} \\
\gamma^*(e_{111}) &= e_{21} - e_{12} + e_{21} = -e_{12} + 2e_{21} \\
\gamma^*(e_{022}) &= e_{22} - e_{04} + e_{22} = 2e_{22} \\
\gamma^*(e_{121}) &= e_{31} - e_{13} + e_{22} = e_{31} \\
\gamma^*(e_{122}) &= e_{22} - e_{13} + e_{31} = e_{22} \\
\gamma^*(e_{211}) &= e_{31} - e_{22} + e_{31} = -e_{22}
\end{align*}
\]
Here, we group the basis elements into five groups according to their degrees (i.e. $r + s + t$ in the domain). This breaks the $12 \times 6$ presentation matrix into five blocks of submatrices. From there, we obtain the cohomology groups of $P_3$ using $H^0(P_3) \cong \ker \gamma^*$ and $H^1(P_3) \cong H^0(P_3/e)/\text{Im} \gamma^*$.

In the next two examples, let us consider the special case when $A = A_0$, i.e., every element in $A$ has degree 0. The graded dimension of $A$ is then an integer, namely $\dim A$. Thus the Euler characteristic of our cohomology groups is the integer $P_G(\lambda)$ where $\lambda = \dim A$. We study two examples, the first one is when $\dim A = 1$, the second is when $\dim A = 2$.

**Example 4.5** Let $A = \mathbb{Z}$ with the usual ring structure. We have $q \dim A = 1$, $P(G, 1) = 1$ if $G$ has no edge and 0 otherwise. For the cohomology groups, we have

$$H^i(G) \cong \begin{cases} 
\mathbb{Z} & \text{if } i = 0 \text{ and } G \text{ has no edge}, \\
0 & \text{otherwise}.
\end{cases}$$

This can be easily proved by inducting on the number of edges and using the long exact sequence.

Next, we consider the case when $\dim A = 2$. We assume that $A$ has an identity. By Lemma 3.3, the abelian group $(A, +)$ is generated by 1 and $x$ where 1 is the identity of $A$. Let us consider various ring structures on $A$. The multiplication $*$ satisfies $1 * 1 = 1, 1 * x = x * 1 = x,$ and $x * x = a1 + bx$ where $a, b$ are two fixed integers. In the appendix, we will show that the isomorphism type of such a ring depends on $b \pmod{2}$ and $b^2 + 4a$.

**Example 4.6** Let $A$ be the ring above. Then

$$H^0(P_3) \cong \begin{cases} 
\mathbb{Z} & \text{if } b^2 + 4a = 0, \\
0 & \text{otherwise}.
\end{cases}$$

$$H^1(P_3) \cong \begin{cases} 
\mathbb{Z}_2 \oplus \mathbb{Z} & \text{if } b^2 + 4a = 0, \\
\mathbb{Z}_{|b^2+4a|} & \text{if } b^2 + 4a \neq 0 \text{ and } b \text{ is odd} \\
\mathbb{Z}_2 \oplus \mathbb{Z}_{\frac{|b^2+4a|}{2}} & \text{if } b^2 + 4a \neq 0 \text{ and } b \text{ is even}
\end{cases}$$

$H^i(P_3) = 0$ for all $i > 1$. 

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This example shows that all torsions that are not of the form \(3 + 4k\) can occur.

Again, the computation is based on the exact sequence.

If \(G = N_1\), the graph with one vertex and no edge, then \(H^0(G) \cong \mathcal{A}\) with generators being 1 and \(x\).

If \(G = T_1\), the tree with one edge, Proposition 3.4 implies that \(H^0(G) \cong \mathcal{A} \otimes Zx \cong \mathcal{A}\). By Remark 3.5, \(H^1\) has the following basis: \(e_1 = 1 \otimes x - x \otimes 1 = 1x - x1, e_2 = x \otimes x - x \otimes x = (a1 + bx) \otimes 1 = x \otimes x - a1 \otimes 1 - bx \otimes 1 = xx - a11 - bx1\). Here, for simplicity of notation, we drop the tensor product symbol \(\otimes\) again. Thus \(1x\) stands for \(1 \otimes x\).

If \(G = T_2\), the tree with two edges, the same argument shows that \(H^0(T_2) \cong \mathcal{A}\). To describe the basis, we denote the three vertices of \(T_2\) by 1, 2, and 3 with 2 being the middle vertex with degree two. A basis of \(H^0(T_2)\) is:

\[
f_1 = (1x) - (a1 + bx)1 - (xx1) = (1xx) - a(111) - b(1x1) - (x1x) + (xx1),
\]

\[
f_2 = (xx) - [x(x \otimes 1) - a(11x - 1x1) - b(x1x - xx1)
\]

\[
= (xx) - a1 + bx)x - a(11x) + a(1x) - b(x1x) + b(xx1)
\]

\[
= (xx) - a(111) - b(xx1) - a11 + a1x - b(x1x) + b(xx1).
\]

Now, let \(G = P_3\). Let \(e\) be an edge of \(P_3\). The exact sequence on \((P_3, e)\) gives

\[
0 \to H^0(P_3) \xrightarrow{\beta} H^0(P_3 - e) \xrightarrow{\gamma^*} H^0(P_3/e) \xrightarrow{\alpha^*} H^1(P_3) \to 0
\]

we have \(H^0(P_3 - e) = H^0(T_2) \cong \mathbb{Z} \oplus \mathbb{Z}\) with \(\{f_1, f_2\}\) being a basis, and \(H^0(P_3/e) \cong H^0(T_1) \cong \mathbb{Z} \oplus \mathbb{Z}\) with \(\{e_1, e_2\}\) being a basis. The map \(\gamma^*\) sends \(a \otimes b \otimes c\) to \(ac \otimes b\). Thus

\[
\gamma^*(f_1) = \gamma^*(1xx) - a(111) - b(1x1) - (x1x) + (xx1))
\]

\[
= xx - a1 + bx - (xx)1 + xx = 2(xx) - a(11) - b(1x) - (a1 + bx)1
\]

\[
= 2[(xx) - a(11) - b(x1)] - [b(1x) - b(x1)]
\]

\[
= -be_1 + 2e_2.
\]

\[
\gamma^*(f_2) = \gamma^*(xx) - a(11) - b(xx1) - a(11x) + a(1x) - b(x1x) + b(xx1)
\]

\[
= (x \otimes x)x - a(x1) - b(xx) - a(x1) + a(1x) - b((x \otimes x)1) + b(xx)
\]

\[
= (a1 + bx)x - 2a(x1) - b(xx) + a(1x) - b((a1 + bx)1) + b(xx)
\]

\[
= a(1x) + b(xx) - 2a(x1) - b(xx) + a(1x) - ab(11) - b^2(x1) + b(xx)
\]

\[
= b(xx) - 2a(x1) - b^2(x1) + 2a(1x) - ab(11)
\]

\[
= 2a(1x - x1) + b(xx) - ab(11) - b^2(x1)
\]

\[
= 2ae_1 + be_2.
\]

Therefore, the linear map \(\gamma^*\) is given by the matrix

\[
\begin{pmatrix}
-b & 2 \\
2a & b
\end{pmatrix}
\]

Standard linear algebra then implies the computation results.
In our final example, we consider the twisted cohomology groups described in section 2.4, and show that they can be stronger than the chromatic polynomial. Of course, as we noted earlier in section 2.4, graphs with one vertex and \( n \) loops provide such examples. They all have zero chromatic polynomials. The example below contains no loops, and their chromatic polynomials are nonzero.

**Example 4.7** Let \( G_1 \) be two copies of \( P_3 \) glued together at one vertex, \( G_2 \) be two copies of \( P_3 \) glued together at one edge, plus a pendant edge (here \( P_3 \) is the polygon graph on 3 vertices, see Figure (7)). It is easy to check that \( G_1 \) and \( G_2 \) share the same chromatic polynomial:

\[
P_{G_1} = P_{G_2} = \lambda(\lambda - 1)^2(\lambda - 2)^2
\]

On the other hand, they often have non-isomorphic chain complexes, and sometimes non-isomorphic twisted cohomology groups. For a concrete example, let our algebra \( A \) be \( \mathbb{Z}x \) with \( \deg x = 1 \) and \( x \ast x = 0 \). Let \( f = 0 \) be the homomorphism from \( A \) to \( A \). Then the differential \( d = 0 \). Therefore \( \text{H}^i(G) \cong \text{C}^i(G) \) for all \( i \) and \( G = G_1, G = G_2 \). The chain groups \( \text{C}^i \) are computed below. We have

For \( G_1 \), \( \text{C}^0 \cong \mathbb{Z}\{5\}, \text{C}^1 \cong \mathbb{Z}^6\{4\}, \text{C}^2 \cong \mathbb{Z}^{15}\{3\}, \text{C}^3 \cong \mathbb{Z}^2\{3\} \oplus \mathbb{Z}^{18}\{2\}, \text{C}^4 \cong \mathbb{Z}^7\{2\} \oplus \mathbb{Z}^8\{1\}, \text{C}^5 = \mathbb{Z}\{2\} \oplus \mathbb{Z}^5\{1\}, \text{C}^6 = \mathbb{Z}\{1\}, \text{C}^7 = \mathbb{Z}\{2\} \oplus \mathbb{Z}^9\{1\}, \text{C}^8 = \mathbb{Z}\{1\}. \) Here \( \mathbb{Z}^2\{3\} \) denotes the graded \( \mathbb{Z} \)-module \( \mathbb{Z} \oplus \mathbb{Z} \) with both generators having degree 3.

This shows that \( \text{C}^i(G_1) \not\cong \text{C}^i(G_2) \) for \( i = 4, 5 \), and therefore their cohomology groups \( \text{H}^i \) are different for \( i = 4 \) and 5.

### 5 Appendix - Ring structures on \( \mathbb{Z} \oplus \mathbb{Z} \)

Given an abelian group, it is generally difficult to classify all rings whose additive group is the group. However, when the abelian group is small, one can work out of the classification by hand. Here, we classify all commutative \( \mathbb{Z} \)-algebras (i.e. rings) with identity whose additive group is the free abelian group of rank two. The result is used in Example 4.6. Let \( A \) be such a ring. Its additive group \( (A, +) \) is generated by 1 and \( x \). We have \( 1 \ast 1 = 1, 1 \ast x = x \ast 1 = x, \) and \( x \ast x = a1 + bx \) where \( a, b \) are arbitrary integers. Obviously the ring structure of \( A \) is completely determined by \( (a, b) \). Let \( A' \) be another ring whose additive group is generated by \( 1' \) and \( x' \) with \( x' \ast x' = a'1' + b'x' \). We have

**Proposition 5.1** The two rings \( A \) and \( A' \) are isomorphic if and only if \( b^2 + 4a = b'^2 + 4a' \) and \( b \equiv b' \pmod{2} \). In other words, the isomorphism type of \( A \) is completely determined by \( (b^2 + 4a, b \pmod{2}) \in (\mathbb{Z}, \mathbb{Z}_2) \).
Proof. Suppose \( A \cong A' \) as modules, and let \( f : A \to A' \) be an isomorphism. Then \( f(1) = 1', f(x) = k1' + lx' \) where \( k, l \) are integers. Since \( \{ f(1), f(x) \} \) spans the \( A' \) as an abelian group, we have \( l = \pm 1 \).

Since \( f \) preserves the multiplication, \( f(x^2) = (k1' + lx')^2 = k^21' + 2klx' + l^2x^2 = k^21' + 2klx' + l^2(a'1' + b'x') = (k^2 + l^2a')1' + (2kl + l^2b')x' \). We also have \( f(a1 + bx) = a1' + b(k1' + lx') = (a + kb)1' + lx' \). Since \( x^2 = a1 + bx \), we have

\[
\begin{align*}
\begin{cases}
    k^2 + l^2a' = a + kb \\
    2kl + l^2b' = lb
\end{cases}
\end{align*}
\]

The above two equations are equivalent to the following, where (3) is obtained by taking \( 1 - \frac{k}{l} \ast (2) \), and (4) is \( \frac{1}{l} \ast (2) \):

\[
\begin{align*}
\begin{cases}
    l^2a' - klb' - k^2 = a \\
    2k + lb' = b
\end{cases}
\end{align*}
\]

Take \( 4 \ast (3) + (4)^2 \), we obtain:

\[
l^2(b^2 + 4a') = b^2 + 4a
\]

This is equivalent to \( b^2 + 4a' = b^2 + 4a \) since \( l = \pm 1 \).

The relation \( b \equiv b' \mod 2 \) follows from equation (4) by taking mod 2 in both sides.

Conversely, if \( (a, b) \) and \( (a', b') \) satisfy the two relations, we let \( l = 1 \), and \( k = \frac{1}{2}(b - b') \). A straight forward computation shows that equations (1) and (2) hold. This implies that the map \( f : A \to A' \) defined by \( f(1) = 1', f(x) = k1' + x' \) is a ring isomorphism.

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