Remarks on boundary layers in singularly perturbed Caputo fractional boundary value problems

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Abstract
Almost nothing is known about the layer structure of solutions to singularly perturbed Caputo fractional boundary value problems. We discuss simple convection-diffusion and reaction-diffusion problems.

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1 Introduction
Fractional differential problems became more and more popular in the last years. For an introduction into fractional differential equations see [3], for instance.

But in the singularly perturbed case we know not much about the existence of layers. In the case of Caputo fractional boundary value problems we discuss two simple examples which show that fractional problems are different from standard boundary value problems.

2 Convection-diffusion
Consider the singularly perturbed boundary value problem

\begin{equation}
-\varepsilon D^{\alpha}_{x} u - u' = f, \quad u(0) = u(1) = 0
\end{equation}

with \(0 < \varepsilon << 1\), \(\alpha \in (0,1)\) and the Caputo derivative \(D^{\alpha}_{x}\). For simplicity we take \(f = -1\) and often choose \(\alpha = 1/2\) to study

\begin{equation}
-\varepsilon D^{3/2}_{x} u - u' = -1, \quad u(0) = u(1) = 0.
\end{equation}

In the classical case \(\alpha = 0\) we can decompose \(u\) into \(u = U + V\), where \(U\) solves the reduced problem

\begin{equation}
-U' = -1, \quad U(1) = 0,
\end{equation}

while the layer component \(V(\xi)\) with \(\xi = x/\varepsilon\) solves

\begin{equation}
D^{2}V + DV = 0, \quad V(0) = -U(0), \quad V(1/\varepsilon) = 0.
\end{equation}
Thus we get the exponentially decaying layer component
\[ V = -U(0) e^{-x/\varepsilon} - \frac{e^{-1/\varepsilon}}{1 - e^{-1/\varepsilon}} \approx -U(0) e^{-x/\varepsilon}. \]

Surprisingly, the case \( \alpha \in (0, 1) \) is very different from the case \( \alpha = 0 \).
Let us in this case again decompose
\[ u = U + V_\alpha. \]

With \( \xi = \frac{x}{\varepsilon^{1/(1-\alpha)}} \) we get
\[ (2.5) \quad D^{2-\alpha}_x V_\alpha + D V_\alpha = 0, \quad V_\alpha(0) = -U(0), \quad V_\alpha(1/\varepsilon^{1/(1-\alpha)}) = 0. \]

In our special case we have \( U(0) = -1 \).
Next we solve instead the initial value problem
\[ (2.6) \quad D^{2-\alpha}_x V_\alpha + D V_\alpha = 0, \quad V_\alpha(0) = 1, \quad D V_\alpha(0) = \theta \]

and try to determine \( \theta \) in such a way that the solution of (2.6) solves the boundary value problem (2.5) as well. One possibility for solving (2.6) consists in the application of the Laplace transform. We get
\[ \hat{V}_\alpha(s) = \frac{s^\alpha + s + \theta}{s^{1+\alpha}(1 + s^{1-\alpha})}. \]

Now it is possible [2] from the asymptotic behavior of the function \( \hat{V}_\alpha \) to conclude the asymptotic behavior of the original function. We observe:
\( V_\alpha \) is exponentially decaying if and only if \( \alpha = 0 \) and \( \theta = -1 \).
Otherwise we have \( V_\alpha \sim \xi^\alpha \) if \( \xi \mapsto \infty \).

Alternatively we solve (2.6) for \( \alpha = 1/2 \) by power series. Setting
\[ V_{1/2} = 1 + \theta V^*_1 \]
we obtain
\[ V^*_1 = \sum_{k=1}^{\infty} \frac{\xi^k}{k!} = \sum_{k=1}^{\infty} \frac{\xi^{k+1/2}}{\Gamma(k + 3/2)}. \]

Using Mittag-Leffler functions [4] one can write
\[ (2.7) \quad V_{1/2}(\xi) = 1 + \theta \xi E_{1/2,2}(-\xi^{1/2}) \]

(this also follows from the Laplace-transform of \( V_{1/2} \)).
Consequently, we obtain for the layer correction with \( \theta = -1/V^*_\alpha(1/\varepsilon^2) \) in the case \( \alpha = 1/2 \)
\[ (2.8) \quad V_{1/2}(x) = 1 - \frac{V^*_1(x/\varepsilon^2)}{V^*_1(1/\varepsilon^2)}. \]
From the asymptotic behavior of the Mittag-Leffler functions we conclude

\[ V_{1/2}(1/\varepsilon^2) = O(1/\varepsilon) \quad \text{and} \quad \theta = O(\varepsilon). \]

For fixed \( x_0 \) we obtain

\[
\lim_{\varepsilon \to 0} V_{1/2}(x_0) = 1 - \lim_{\varepsilon \to 0} \frac{V_{1/2}(x_0/\varepsilon^2)}{V_{1/2}(1/\varepsilon^2)} = 1 - \frac{\sqrt{x_0}/\varepsilon}{1/\varepsilon} = 1 - \sqrt{x_0}.
\]

Thus, \( V_{1/2} \) has nothing to do with a classical boundary layer function. See Figure 2: left the solution, right the layer correction.

### 3 Reaction-diffusion

Consider the boundary value problem

\[
(3.1) \quad -\varepsilon D_{*}^{2-\alpha} u + u = f, \quad u(0) = u(1) = 0
\]

with \( 0 < \varepsilon << 1, \alpha \in (0, 1) \) and the Caputo derivative \( D_{*}^{2-\alpha} \). For simplicity we choose \( f = -1 \) and mainly discuss the case \( \alpha = 1/2 \).

We decompose the solution into \( u = -1 + V_\alpha^0 + V_\alpha^1 \). Then \( V_\alpha^0 \) and \( V_\alpha^1 \) satisfy the homogeneous equation, moreover

\[
V_\alpha^0(0) = 1, \quad V_\alpha^0(1) = \mu; \quad V_\alpha^1(0) = 0, \quad V_\alpha^1(1) = 1 - \mu.
\]

Later we will fix \( \mu \) and show that it is small.

Are \( V_\alpha^0 \) and \( V_\alpha^1 \) boundary layer functions?

At \( x = 0 \) we introduce the local variable \( \xi = x/\varepsilon^{1/(2-\alpha)} \). Thus, for instance for \( \alpha = 1/2 \) we obtain for \( V_\alpha^0 \)

\[
-\varepsilon D_{*}^{3/2} V + V = 0, \quad V(0) = 1, \quad V(1/\varepsilon^{2/3}) = 0.
\]

Again we replace the boundary value problem by an initial value problem with \( DV(0) = \theta \). Its Laplace transform yields

\[
\hat{V}_\alpha^0(s) = \frac{s + \theta}{s^\alpha(s^{2-\alpha} - 1)}.
\]

Consequently, for \( \theta = -1 \) there exists a solution \( V_\alpha^0(\xi) \) that decays as \( \xi^{\alpha-1} \). We set \( \mu = V_\alpha^0(\xi)|_{x=1} \) and get \( \mu = O(\varepsilon^{(1-\alpha)/(2-\alpha)}) \).

For \( V_\alpha^1 \) we solve the correspondent initial value problem with \( DV(0) = \theta \) using the Laplace transform. We get

\[
\hat{V}_\alpha^1(\theta) = \frac{\varepsilon \theta}{s^\alpha(s^{2-\alpha} - 1)}.
\]
For instance for $\alpha = 1/2$, the nominator has a zero at $s = 1/\varepsilon^{2/3}$, thus the original function is characterized by $V \sim \exp(x/\varepsilon^{2/3})$. Choosing $\varepsilon \theta = \exp(-1/\varepsilon^{2/3})$, we observe that $V_{\alpha}^1$ behaves like a typical exponentially decaying boundary layer function, i.e.,

$$V_{\alpha}^1(x) \sim (1 - \mu) \exp(-(1 - x)/\varepsilon^{2/3}).$$

See Figure 1: left the solution, right the layer correction with different layers at $x = 0$ and $x = 1$.

Figure 1: Reaction-Diffusion

Figure 2: Convection-Diffusion

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