Consistently melting crystals

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Abstract

Recently Ooguri and Yamazaki proposed a statistical model of melting crystals to count BPS bound states of certain D-brane configurations on toric Calabi–Yau manifolds [arXiv:0811.2801]. This construction relied on a set of consistency conditions on the corresponding brane tiling, and in this note I show that these conditions are satisfied for any physical brane tiling; they follow from the conformality of the low energy field theory on the D-branes. As a byproduct I also provide a simple direct proof that any physical brane tiling has a perfect matching.

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1 Introduction

Understanding the gauge theory on a stack of D-branes probing a Calabi–Yau singularity is an important problem with many applications. While the case of a general Calabi–Yau manifold remains poorly understood, in the past few years considerable progress has been made in the case of toric Calabi–Yaus. For these, the low energy field theory on the D-branes is given by a quiver gauge theory, and can be analyzed using powerful brane tiling techniques initiated and developed in [1, 2, 3, 4, 5].

A different way of understanding such configurations of D-branes is through the derived category approach proposed in [6, 7, 8]; see [9] for an excellent review. While working with the derived category of coherent sheaves, $D(Coh X)$, is again difficult for a general Calabi–Yau $X$, the situation again simplifies when $X$ is toric and tools like exceptional collections can be used to analyze the system [10, 11, 12, 13].

In [14], based on the mathematical work of Mozgovoy and Reineke [15], Ooguri and Yamazaki proposed a statistical model of crystal melting that counts bound states of D0 and D2 branes in the background of a single D6 brane wrapping the whole toric Calabi–Yau. This model utilizes both approaches mentioned above, and can be used for instance to compute Donaldson–Thomas invariants for an arbitrary toric Calabi–Yau. This crystal is built upon the planar quiver describing the theory, and the construction relies on a set of consistency conditions on the corresponding brane tiling. In this note I show that these consistency conditions are indeed satisfied for any quiver theory arising from a configuration of D-branes; they are shown to follow from the conformality of the low energy field theory.

2 Toric quiver theories and brane tilings

In this section I briefly review quiver gauge theories arising as low energy field theories on the world volume of a stack of D-branes probing the conical singularity of a toric Calabi–Yau; for excellent extended reviews on this topic the reader is referred to [16, 17].
**Toric quiver theories and brane tilings**

**Quivers:** Quiver gauge theories are specified by a set $Q_0$ of gauge groups $SU(N_i)$, and a set $Q_1$ of matter fields transforming in the bifundamental representation $(N_i, \bar{N}_j)$ under two of the gauge groups. In addition there is a superpotential, which for a toric Calabi–Yau is always a sum of monomials, such that each matter field appears in exactly two terms with opposite signs. Figure 1 shows the quiver and the superpotential corresponding to $dP_1$, where $dP_1$ denotes the toric Calabi–Yau that is given by a complex cone over the first del Pezzo surface.

It is well known that any such quiver can be ‘opened up’ and placed on a torus in such a way that the faces of the new graph correspond to the superpotential terms; a clockwise orientation of a face corresponds to a negative term and vice versa [2]. This is known as the planar (or periodic) quiver, and is also pictured in figure 1. We will mostly work with the universal covering of the planar quiver, which is an infinite, periodic graph on $\mathbb{R}^2$. We denote the quiver by $Q = (Q_0, Q_1, Q_2)$, where $Q_2$ contains the faces, i.e. superpotential terms.

**The brane tiling and perfect matchings:** The dual graph of the planar quiver is particularly useful. It is generated by replacing the faces of the
planar quiver by vertices, replacing vertices by faces, and replacing the arrows with perpendicular edges. See figure 2 for an illustration for the case of $dP_1$. Since the faces of the planar quiver correspond to either positive or negative superpotential terms, we can color the vertices of the brane tiling to reflect this sign; we choose black vertices for negative terms and white for positive ones. Thus we see that the brane tiling is a bipartite graph; white vertices are only connected to black vertices and vice versa. Therefore, a brane tiling $G = (G_0^+, G_0^-, G_1, G_2)$ consists of two sets of vertices, $G_0^+$ and $G_0^-$, corresponding to positive and negative superpotential terms, a set of edges $G_1 \subseteq G_0^+ \times G_0^-$ corresponding to the bifundamental matter fields, and a set of faces corresponding to the gauge groups.

A perfect matching of a tiling $G$ is a subset of the edges such that each vertex in $G_0$ is touched by exactly one edge in the perfect matching. Figure 2 shows one of the perfect matchings of $dP_1$. A given bipartite graph need not have a perfect matching; however, I will show in section 3 that for a physical brane tiling, that is a bipartite graph arising from a configuration of D-branes, a perfect matching always exists.

The isoradial embedding and rhombus lattice: To prove the consistency conditions of the crystal model [14, 15] we need to take into account the fact that the low energy field theory on the world-volume of the branes is conformal. This means that the NSVZ beta functions for all gauge groups

\[1\text{Here I follow the notation of [15].}\]
must vanish, which implies (see \cite{18, 16})

\[ \sum_{i \in a} (1 - R(X_i)) = 2, \quad \text{for all gauge groups } a, \quad (1) \]

where the sum is over all the matter fields transforming under that gauge group \( a \), and \( R(X_i) \) denotes the R-charge of the corresponding field. Further, the superpotential has to be marginal, so each term in the superpotential must have R-charge 2. Recalling that in the planar quiver the superpotential terms correspond to faces, this implies

\[ \sum_{i \in F} R(X_i) = 2, \quad \text{for all superpotential terms } F, \quad (2) \]

where \( F \) is a face in the planar quiver, and the sum is over all the edges surrounding that face. The relations (1) and (2) can be written in terms of the brane tiling as

\[ \sum_{e \in V} R_e = 2, \quad (3) \]

\[ \sum_{e \in F} R_e = \# \text{edges} - 2, \quad (4) \]
Figure 4: Left : the rhombus lattice. The dashed black diagonals give the brane tiling, and the dash-dotted magenta diagonals give the planar quiver. Right : One of the rhombi characterized by the angle $\theta$.

where $V$ is any vertex on the brane tiling and the first sum is over all edges connected to that vertex, and $F$ is a face on the tiling and the second sum is over all the edges surrounding that face.

These relations have a very nice geometric interpretation. Consider an embedding of the brane tiling where one draws a unit circle for each of the faces of the tiling, and places the vertices on the circumference of the circle. This is known as the isoradial embedding, and is shown in figure 3 for $dP_1$. Next draw lines from the centers of the circles to the vertices on the circumference; this yields a lattice consisting of rhombi and is shown in figures 3 and 4. Half of the vertices of the rhombi are on the vertices of the brane tiling, i.e. the superpotential terms, while the other half of the vertices are on the faces of the tiling, i.e. the gauge groups. Thus, depending on which diagonals we focus on we get the brane tiling or the planar quiver; the rhombus lattice contains the data of both graphs.

Each rhombus is characterized by an angle $\theta_e$, which we choose to be opposite the diagonal corresponding to an arrow in the quiver, and we associate the angle $\theta_e$ to the R-charge $R_e$ of the corresponding field by

$$ R_e \equiv \frac{\theta_e}{\pi}, $$

We have suppressed the black and white vertices of the brane tiling, or equivalently the direction of the arrows of the quiver. This data needs to be included in the rhombus lattice for it to contain the same data as the tiling or quiver.
we see that (3) and (4) are automatically satisfied; the first equation translates to the angles summing to $2\pi$ around a superpotential vertex, while the second translates to the sum of internal angles in a polygon being $(\#\text{edges} - 2)\pi$. This is shown in figure 5. Note that for the rhombus lattice to be non-degenerate we need all R-charges to lie in the interval $(0, 1)$; this holds for any physical tiling [4].

**Train tracks and zig-zag paths:** The rhombus lattice has a set of special paths, known as rhombus paths or train tracks. These are constructed by joining together rhombi sharing parallel edges, as in figure 6. Each such path defines a path in the planar quiver that turns alternately maximally left and right, known as a zig-zag path. It is known that for a tiling arising from a system of branes, i.e. one having an isoradial embedding, a zig-zag path never intersects itself, and two zig-zag paths intersect at most at one edge [19, 20, 4].

**The path algebra:** Consider the algebra $\mathbb{C}Q$, which consists of all paths in the quiver. The multiplication in $\mathbb{C}Q$ is defined by joining two paths when the first ends where the second begins, otherwise the product is zero. Since the arrows on the quiver correspond to bifundamental fields, the F-term relations give relations in the path algebra. Figure 7 shows three paths in the path algebra of $dP_1$ which are seen to be identical by using the F-terms relations. If we denote the ideal generated by the F-term conditions by $\mathcal{F}$, the independent paths are given by the factor algebra $A = \mathbb{C}Q/\mathcal{F}$. 

Figure 5: Left: Rhombi sharing a superpotential vertex. Right: Rhombi sharing a gauge group vertex.
Let us denote the shortest path between vertices $i$ and $j$ by $v_{ij}$. Also, denote the loop around a face in the quiver by $w$; all such loops are equivalent by the F-term relations. It was shown in [15] that the elements in the factor algebra $A$ can be written in the form $v_{ij}w^n$, where $n \geq 0$ is a natural number.

3 Proofs of the consistency conditions

The crystal model [14] is built on the mathematical work [15], who place three consistency conditions for the brane tiling: (3.5), (4.12) and (5.3) in [15]. The first two of these are fairly trivial, though we prove (3.5) in detail since as a consequence we can also prove that any physical brane tiling has a perfect matching. Although perfect matchings are widely used in the physics literature, to my best knowledge this is the first direct proof that they exist for any physical tiling. Most of this section is devoted proving condition (5.3) of [15].

**Condition 3.5:** The first consistency condition of [15] is that the tiling should be non-degenerate, meaning that every edge in the brane tiling belongs to some perfect matching. We will proceed to show that this holds for physical tilings using the following lemma:

\[\sum_{i,j} c_{ij} z^i w^j,\]

3. Shortest’ meaning of smallest R-charge.

4. The Kasteleyn matrix and its determinant, the characteristic polynomial \(\sum_{i,j} c_{ij} z^i w^j\), are central to the dimer literature, and perfect matchings appear in the coefficients $c_{ij}$. If there were no perfect matchings the coefficients would vanish. In [5] it was argued via mirror symmetry that these coefficients are generally nonzero; the proof given here verifies...
Figure 7: Three equivalent paths in the planar quiver of $dP_1$. To see the equivalence of the red and blue paths, use $\partial W/\partial X_{34} = 0$, and to show that the magenta path is equivalent to these two use $\partial W/\partial X_{13} = 0$ and $\partial W/\partial X_{42} = 0$. (Refer to figure 1 for the superpotential.)

**Lemma 1.** Any physical brane tiling $G = (G^+_0, G^-_0, G_1, G_2)$ has a perfect matching.

**Proof:** For any subset $A \subseteq G^+_0$ of white vertices, we define the set of neighbors $N(A) \subseteq G^-_0$ to consist of all the black vertices that are connected to vertices in $A$ by an edge in $G_1$. Then Hall’s theorem\(^5\) states that

A bipartite graph $G$ has a perfect matching if and only if $|A| \leq |N(A)|$ for any subset $A \subseteq G^+_0$.

Now choose a subset $A \subseteq G^+_0$. We need to show $|N(A)| \geq |A|$. This follows from marginality of the superpotential; since each superpotential term has R-charge 2, the edges connected to the vertices in $A$ have total R-charge $2|A|$. Since each of the edges is connected to a vertex in $N(A)$, this charge is divided among the neighboring vertices $N(A)$. Since each vertex can have only two units of R-charge, the pigeonhole principle states that $|N(A)| \geq |A|$. This is illustrated in figure 8. This proves that $G$ has a perfect matching. \(\square\)

We can go a bit further and show that $|A| < |N(A)|$ when $A \neq \emptyset, G^+_0$. First assume $|A| = |N(A)|$. Then by the previous argument each vertex in $|N(A)|$ must already have R-charge 2, and therefore can’t be connected to any new vertices, i.e. $N(N(A)) = A$. But this implies that either $A = G^+_0$,

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\(^5\) Also known as the ‘marriage theorem’. This result directly without requiring mirror symmetry.
Figure 8: Part of an inconsistent tiling: since the white vertices have R-charge 2, we have \(a + e = b + f = c + g = d + h = 2\), which implies one of the black vertices must have R-charge greater than two, which conflicts with the marginality of \(W\). Similar arguments were used in a different setting recently in \cite{21}.

\(A = \emptyset\), or the tiling consists of two or more disconnected pieces, which is impossible. Thus either \(A = \emptyset\) or \(A = G_0^+\).

**Theorem 1.** Physical brane tilings are non-degenerate, i.e. each edge belongs to a perfect matching.

**Proof:** Pick an edge \(e \in G_1\). Let us construct a new bipartite graph \(G'\) by removing from \(G\) the two vertices that are connected by \(e\), and all the edges that are connected to either of those vertices. Note that if \(G\) only had two vertices, each edge is already a perfect matching; we can thus assume \(G' \neq \emptyset\). Pick any subset of white vertices \(A \subset G_0^+\). Then \(G'\) has a perfect matching iff \(|N'(A)| \geq |A|\)

\[6\] This follows since \(G'\) is a subtiling of \(G\): \(A\) is also a subset of white vertices in \(G\). Furthermore, since \(A \neq G_0^+\) we have \(|A| < |N(A)|\), where the neighbors are taken in \(G\). Restricting to \(G'\) we see that \(|N'(A)| = |N(A)|\) or \(|N'(A)| = |N(A)| - 1\), depending on whether the removed vertex was a neighbor to \(A\). Thus \(|A| \leq |N'(A)|\), and therefore by Hall’s theorem \(G'\) has a perfect matching \(M'\). This proves the theorem, since we can define \(M = M' \cup e\), which is a perfect matching of the tiling \(G\), and contains the edge \(e\).

\(\square\)

**Condition 5.3:** Having proved condition 3.5, we turn our attention to the second consistency condition:

\[^6\]\(N'(A)\) denotes the neighbors in the tiling \(G'\), and \(N(A)\) denotes the neighbors in the tiling \(G\).
Figure 9: A zig-zag path from $i$ to $j$ (black), and the ‘neighboring’ path back to $i$ (blue).

**Theorem 2.** For any two vertices $i, j$ in the planar quiver, there exists an arrow $a : j \rightarrow k$ such that $v_{ik} \sim av_{ij}$.

To prove this, we will need a few useful lemmas:

**Lemma 2.** If two vertices $i$ and $j$ are along the same zig-zag path, the zig-zag path provides $v_{ij}$, the shortest path between $i$ and $j$.

**Proof:** Consider the path from $i$ to $j$ and back again to $i$ portrayed in figure 9. This path is equivalent to $w^n$, where $n$ is the number of faces circled by the path. Due to the structure of the rhombus lattice, $n$ is the minimal number of faces that will be circled by any path $i \rightarrow j \rightarrow i$, and thus the path is equivalent to $v_{iji}$, the path of minimal length from $i$ to $i$ passing through $j$. But clearly $v_{iji} \sim v_{ji}v_{ij}$, which proves that the zig-zag path is equivalent to $v_{ij}$.

**Lemma 3.** For any two vertices $i$ and $j$, if there is a zig-zag path $Z_1$ passing through $j$ such that $v_{ik} \sim av_{ij}$ and $v_{il} \sim bav_{ij}$, then $v_{im} \sim cbav_{ij}$. Here $a, b$ and $c$ are the next three arrows in the zig-zag path starting from $j$, and $k, l$ and $m$ are the end points of $a, b$ and $c$; see figure 10.

**Proof:** Each arrow in the quiver belongs to exactly two zig-zag paths. Thus, there are unique zig-zag paths different from $Z_1$ that pass through $b$ and $c$, let us denote these by $Z_2$ and $Z_3$ respectively. These are also shown in figure 10 however the reader should note that one shouldn’t rely too much on the figure: the way the figure has been drawn it appears that $Z_2$ and $Z_3$ intersect
in the rhombus whose diagonal arrow is denoted $d$. This need not be the case; $d$ is meant to represent the path that completes the face on which the vertices $k, l, m$ lie; if the face is a triangle as drawn then $Z_2$ and $Z_3$ intersect there as shown in the figure, if the face is a polygon with more than three edges, then $d$ is not an edge but rather a path of edges completing the face, in which case $Z_2$ and $Z_3$ do not intersect in that rhombus. The reader can verify that the proof presented here does not rely on such aspects of the graph.

The three zig-zag paths divide the plane into regions denoted I, II and III in the figure. I have drawn vertex $i$ to reside in region I; the reader can verify that if $i$ was in region II the conditions of the lemma would not be satisfied, and if it resided in region III an argument similar to the one given below will show the lemma to be true.

Now let us assume the lemma is not true: $cbav_{ij} \sim uv_{im}$, i.e. there exists a path $v_{im} : i \rightarrow m$ that is shorter than $cbav_{ij}$. This path is shown in figure 10 as a red path. Denote by $x$ the vertex where $v_{im}$ hits the zig-zag path $Z_3$.  

\[\text{Figure 10: The set-up for lemma (This is for a generic tiling, not } dP_1.)\]

\[\text{We have drawn } v_{im} \text{ to cross } Z_3; \text{ if this is not the case, i.e. the path } v_{im} \text{ is directed towards left in the figure, then it must cross the original zig-zag path } Z_1 \text{ and a similar}\]
Now, we can take the path $wv_{im}$ to be given by travelling first along $v_{im}$ from $i$ to $m$, and then around the face given by $cbd$. Then we have $cba v_{ij} \sim cbdv_{im}$, and we can cancel $cb$ to yield $dv_{im} \sim av_{ij} \sim v_{ik}$, i.e. $dv_{im}$ is the shortest path $i \rightarrow k$. Now consider path $V$, which is defined by travelling on $v_{im}$ until $x$, and then taking the magenta path $e$ to $k$. Here $e$ is a neighboring path to $Z_3$ in the sense of the proof of lemma 2; it need not be a single arrow as drawn in the figure. It was shown in the proof of lemma 2 that such a neighboring path is the shortest path between its endpoints. It is also easily shown that the red path $x \rightarrow k$ is longer than the path $e$; thus we have constructed a path $V : i \rightarrow k$ that is shorter than $dv_{im}$. This is a contradiction since our assumption that the lemma is false yielded $dv_{im} \sim v_{ik}$; thus our assumption must be wrong and the lemma holds.

The readers can easily convince themselves that if $i$ was located in region $III$ or if the path $v_{im}$ crossed $Z_1$ instead of $Z_3$ similar arguments can be used to prove the lemma.

\textit{Proof of theorem 2}: Let $i$ and $j$ be two vertices, and $v_{ij}$ the shortest path between them. We need to show that there exists an arrow $a : j \rightarrow k$ such that $av_{ij} \sim v_{ik}$. Consider the last arrow on path $v_{ij}$, which we denote by $b$. It belongs to exactly two zig-zag paths; one turning maximally right and one turning maximally left at $j$; let us denote the next arrows on these paths by $a_R$ and $a_L$ respectively; see figure 11.

\textsuperscript{8}Recall the previous comment that $d$ need not be a single arrow, it represents the rest of the arrows around this face.
We claim that either $a_L$ or $a_R$ can be identified with $a$ so that the theorem holds. We prove this by assuming that one of them does not have the property of $a$, and then showing that the other one must have it. Assume $a_L$ does not have the property, i.e. $a_Lv_{ij} \sim wv_{ikL}$. We can take $wv_{ikL}$ to be first $v_{ikL}: i \rightarrow k_L$, and then a loop around the face indicated in figure 10 which is surrounded by $a_L$, $b$, and the magenta path $m$. Denote by $x$ the last vertex before the magenta path hits $b$; if the face is a triangle then $x = k_L$. Since this path shares the same last arrow with $a_Lv_{ij}$, we can cancel $a_L$ to yield $v_{ij} \sim bmv_{ikL}$. Thus the path constructed is also the shortest path $i \rightarrow j$, and trivially also the shortest path to any vertex it passes. Following this path to $x$ then gives the shortest path $v_{ix}$. But now note that the indicated path from $x$ to $k_R$ is a zig-zag path: one first follows the last arrow of the magenta path, turns maximally left, follows $b$ and then turns maximally right. But we also know that this constructed path is the shortest path $i \rightarrow j$, so the conditions of lemma 3 are satisfied, and if we take another step on the zig-zag path, by lemma 3 we again have a shortest path, and thus $a_Rv_{ij} \sim v_{ikR}$, and the theorem is proved.

\[ \square \]

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