A note about fractional Stefan problem

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Abstract We derive the fractional version of one-phase one-dimensional Stefan model. We assume that the diffusive flux is given by the time-fractional Riemann-Liouville derivative, i.e. we impose the memory effect in the examined model.

Key words: fractional derivatives, Stefan problem.

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1 Introduction

The purpose of this paper is to derive the one-dimensional one-phase Stefan model. We are motivated by the paper [1], where the authors represent the nonlocal in time effects by assuming that the diffusive flux is given in the form of time-fractional Riemann-Liouville derivative of temperature gradient. The similar result has been already obtained in [2].

2 Formulation of the problem

We assume that \( \Omega = (0, L) \) for a positive \( L \). Further, we assume that at the initial time \( t = 0 \) the domain \( \Omega \) is divided onto two parts: \((0, x_0)\) - “liquid” and \((x_0, L)\) - “solid”. In particular, we admit the case where \( x_0 = 0 \). Following [1] we define the enthalpy function by \( E = T + \phi \), where \( T(x, t) \) is the temperature at point \( x \in \Omega \) at time \( t \) and \( \phi \) represents the latent heat. We consider the sharp-interface model, hence we assume that \( \phi \) is given in the following form

\[
\phi = \begin{cases} 
1 & \text{in liquid} \\
0 & \text{in solid.}
\end{cases}
\]
We shall consider the one-phase model, i.e. we assume that \( T \equiv 0 \) in “solid” part. We denote by \( q^*(x,t) \) the flux at \( x \in \Omega \) at time \( t \). The main principle, which we assume is the conservation law which here has the following form: for each \( V = (a,b) \subseteq \Omega \)

\[
\frac{d}{dt} \int_V E(x,t)dx = q^*(a,t) - q^*(b,t).
\]  

(2)

We may easily see that if the model does not exhibit memory effects then identity (2) leads to classical one-phase Stefan problem. We state this result in the remark.

**Remark 1.** If the flux is defined by the Fourier law

\[
q^*(x,t) = -T_x(x,t),
\]

then (2) leads to the classical Stefan problem

\[
\frac{d}{dt}T(x,t) - T_{xx}(x,t) = 0 \quad \text{for} \quad t > 0 \quad \text{and} \quad x \in (0,L) \setminus \{s(t)\},
\]

(3)

\[
\dot{s}(t) = -T_x^-(s(t),t) \quad \text{for} \quad t > 0,
\]

(4)

where \( s(t) \) is a interface and

\[
T_x^-(s(t),t) = \lim_{\varepsilon \to 0^+} T_x(s(t) - \varepsilon, t).
\]

We assume that the flux is given by the Riemann-Liouville fractional derivative with respect to the time variable, i.e.

\[
q^*(x,t) = -\text{RL}_0D_t^{1-\beta}T_x(x,t),
\]

(5)

where

\[
\text{RL}_0D_t^{1-\beta}T_x(x,t) = \frac{1}{\Gamma(\beta)} \frac{d}{dt} \int_0^t (t-\tau)^{\beta-1}T_x(x,\tau)d\tau, \quad \beta \in (0,1).
\]

In the next remark we give a formal idea why such form of the flux seems to be reasonable in the model exhibiting memory effects.

**Remark 2.** Let us denote by \( s(t) \) the phase interface. We decompose the domain \( \Omega \) on the solid and liquid parts.

\[
\Omega_l(t) = (0,s(t)) \quad \text{liquid}, \quad \Omega_s(t) = (s(t),L) \quad \text{solid}.
\]

Let \( V \subseteq \Omega \) be arbitrary. Then, if we assume that \( V = (a,b) \) and denote

\[
V_l(t) = \Omega_l(t) \cap V, \quad V_s(t) = \Omega_s(t) \cap V
\]

then, (2) has a form

\[
\frac{d}{dt} \left[ \int_{V_l(t)} (T(x,t) + 1)dx \right] + \frac{d}{dt} \left[ \int_{V_s(t)} T(x,t)dx \right] = \text{RL}_0D_t^{1-\beta}T_x(b,t) - \text{RL}_0D_t^{1-\beta}T_x(a,t).
\]

(6)
Assuming that the temperature gradient is bounded with respect to time variable, after integrating with respect to time we get

\[
\int_{V_l(t)} (T(x,t) + 1) dx + \int_{V_s(t)} T(x,t) dx = \int_{V_l(0)} (T(x,0) + 1) dx + \int_{V_s(0)} T(x,0) dx \\
+ \frac{1}{\Gamma(\beta)} \int_{t_0}^{t} (t - \tau)^{\beta - 1} [T_x(b, \tau) - T_x(a, \tau)] d\tau,
\]  
(7)
i.e. the total enthalpy in \( V \) at time \( t \) is a sum of the initial enthalpy and the time-average of differences of local fluxes at the endpoints of \( V \).

3 Derivation of the model

In this section we derive (see (22)-(25)) the fractional Stefan model from the balance law (2) with the diffusive flux given by (5). In order to do it rigorously we have to impose some regularity conditions on the phase interface \( s \) and the temperature function \( T \). At first, we assume that \( t^* \) is positive and

\[
s(t) \in AC[0, t^*], \quad T_x(x, \cdot) \in L^\infty(U^x) \quad \text{for every } x \in \Omega,
\]

\[
T_x(\cdot, t) \in AC[0, s(t) - \varepsilon] \quad \text{for every } \varepsilon > 0 \quad \text{and every } t \in (0, t^*), \tag{A1}
\]

\[
T_t(\cdot, t) \in L^1(0, s(t)) \quad \text{for each } t \in (0, t^*),
\]

where we denote

\[
Q_{s,t^*} = \{(x, t) : 0 < x < s(t), \quad t \in (0, t^*)\},
\]

\[
U^x = \{t : (x, t) \in Q_{s,t^*}\}
\]

and \( AC \) denotes the space of absolutely continuous functions.

We equip our model with an initial condition:

\[
T(x,0) = T_0(x) \geq 0
\]

and the Dirichlet or Neumann boundary condition

\[
T(0,t) = T_D(t) \geq 0 \quad \text{or} \quad T_x(0,t) = T_N(t) \leq 0.
\]

If \( T_0, T_D \equiv 0 \) or \( T_0, T_N \equiv 0 \), then we expect that \( T \equiv 0 \). Otherwise, we expect

\[
\dot{s}(t) > 0, \quad \tag{A2}
\]
i.e. we observe melting of solid. Since we consider one-phase problem, we have \( T(x,t) = 0 \) for \( x \in \Omega_s(t) \). Therefore, the flux is nonzero only in liquid part of the domain, i.e. in \( Q_{s,t^*} \) and it is given by the formula

\[
q^*(x,t) = \begin{cases} 
-\frac{R L}{s(t)} D_t^{1-\beta} T_x(x,t) & \text{for } (x,t) \in Q_{s,t^*}, \\
0 & \text{for } (x,t) \notin Q_{s,t^*},
\end{cases}
\]

(8)
where
\[
RL_{s(t)} \, D_t^{1-\beta} T(x, t) = \begin{cases} 
\frac{1}{\Gamma(\beta)} \frac{d}{dt} \int_0^t (t - \tau)^{\beta-1} T_x(x, \tau) d\tau & \text{for } x \leq s(0) \\
\frac{1}{\Gamma(\beta)} \frac{d}{dt} \int_{s-1(x)}^t (t - \tau)^{\beta-1} T_x(x, \tau) d\tau & \text{for } x > s(0).
\end{cases}
\] (9)

We recall the definitions of fractional integral \(0_I^\beta f\) and fractional Caputo derivative \(C_0^\beta D_t^\beta\) for a later use
\[
0_I^\beta f(x, t) = \frac{1}{\Gamma(\beta)} \int_0^t (t - \tau)^{\beta-1} f(x, \tau) d\tau, \quad C_0^\beta D_t^\beta f(x, t) = \frac{1}{\Gamma(1 - \beta)} \int_0^t (t - \tau)^{-\beta} f_t(x, \tau) d\tau.
\]

We note that since we consider one-phase Stefan problem the temperature in the solid vanishes. This together with (1) leads to the following form of equality (2)
\[
\frac{d}{dt} \left[ \int_{V(t)} T(x, t) + 1 dx \right] = -q^*(b, t) + q^*(a, t).
\] (10)

In order to derive the system of equations from (10) we fix a positive time \(t \in (0, t^*)\) and we consider the conservation of energy on a subset \(V\) of the domain at time \(t\). We will consider two cases.

- If \(V = (a, b) \subseteq (0, s(0))\), then from (A2) we have \(V \subseteq (0, s(t))\) for each \(t \in (0, t^*)\) and (10) gives
\[
\frac{d}{dt} \left[ \int_V T(x, t) + 1 dx \right] = RL_{s(t)} \, D_t^{1-\beta} T_x(b, t) - RL_{s(t)} \, D_t^{1-\beta} T_x(a, t)
\]
hence,
\[
\int_V \frac{d}{dt} T(x, t) dx = RL_{s(t)} \, D_t^{1-\beta} T_x(b, t) - RL_{s(t)} \, D_t^{1-\beta} T_x(a, t).
\]

We apply the fractional integral \(0_I^{1-\beta}\) with respect to the time variable to both sides of the identity and with a use of assumption (A1) we arrive at
\[
\int_V C_0^\beta D_t^\beta T(x, t) dx = T_x(b, t) - T_x(a, t)
\]
and after using the fundamental theorem of calculus we obtain
\[
\int_V \left[ C_0^\beta D_t^\beta T(x, t) - T_{xx}(x, t) \right] dx = 0.
\]

Since \(V \subseteq (0, s(0))\) is arbitrary, we get
\[
C_0^\beta D_t^\beta T(x, t) - T_{xx}(x, t) = 0 \quad \text{for } (x, t) \in (0, s(0)) \times (0, t^*).
\] (11)
• If $V = (a, b)$, where $s(0) < a < s(t) < b$, then (10) has a form

$$\frac{d}{dt} \left[ \int_a^{s(t)} T(x, t) + 1dx \right] = q^*(a, t) = -\frac{1}{\Gamma(\beta)} \frac{d}{dt} \int_{s^{-1}(a)}^{t} (t - \tau)^{\beta-1}T_x(a, \tau)d\tau.$$

Differentiating the integral on the left hand side leads to

$$\int_a^{s(t)} \frac{d}{dt} T(x, t)dx + s(t)[T(s(t), t) + 1] = -\frac{1}{\Gamma(\beta)} \frac{d}{dt} \int_{s^{-1}(a)}^{t} (t - \tau)^{\beta-1}T_x(a, \tau)d\tau.$$

Using $T(s(t), t) = 0$ and applying operator $\frac{1}{\Gamma(1-\beta)} \int_{s^{-1}(a)}^{t} (t - \tau)^{-\beta} \cdot d\tau$ we obtain

$$\frac{1}{\Gamma(1-\beta)} \int_{s^{-1}(a)}^{t} (t - \tau)^{-\beta} \int_{a}^{s(\tau)} \frac{d}{d\tau} T(x, \tau)dx d\tau + \frac{1}{\Gamma(1-\beta)} \int_{s^{-1}(a)}^{t} (t - \tau)^{-\beta} \cdot s(\tau)d\tau$$

$$= -\frac{1}{\Gamma(\beta)} \frac{1}{\Gamma(1-\beta)} \int_{s^{-1}(a)}^{t} (t - \tau)^{-\beta} \frac{d}{d\tau} \int_{s^{-1}(a)}^{\tau} (\tau - \beta)T_x(a, \beta)d\tau (12).$$

Let us denote

$$C_s(t)D^\beta_t T(x, t) = \begin{cases} \frac{1}{\Gamma(1-\beta)} \int_{0}^{t} (t - \tau)^{-\beta} \frac{d}{d\tau} T(x, \tau)d\tau & \text{for } x \leq s(0) \\ \frac{1}{\Gamma(1-\beta)} \int_{s^{-1}(a)}^{t} (t - \tau)^{-\beta} \frac{d}{d\tau} T(x, \tau)d\tau & \text{for } x > s(0). \end{cases} (13)$$

If we apply the Fubini theorem to the first and third integral in (12) and make use of the assumption (A1) we arrive at the identity

$$\int_{a}^{s(t)} C_{s(t)}D^\beta_t T(x, t)dx + \frac{1}{\Gamma(1-\beta)} \int_{s^{-1}(a)}^{t} (t - \tau)^{-\beta} s(\tau)d\tau = -T_x(a, t). (14)$$

Applying the substitution $\tau = s^{-1}(x)$ we get

$$\frac{1}{\Gamma(1-\beta)} \int_{s^{-1}(a)}^{t} (t - \tau)^{-\beta} s(\tau)d\tau = \frac{1}{\Gamma(1-\beta)} \int_{a}^{s(t)} (t - s^{-1}(x))^{-\beta}dx.$$

We expect that $T_x$ may admit singular behaviour near the phase change point. Thus, we proceed very carefully and fix $\varepsilon > 0$ such that $a < s(t) - \varepsilon$. Then, by (A1) we have

$$-T_x(a, t) = \int_{a}^{s(t)-\varepsilon} T_{xx}(x, t)dx - T_x(s(t) - \varepsilon, t).$$

Making use of this results in (14) we obtain

$$\int_{a}^{s(t)-\varepsilon} \left[ C_{s(t)}D^\beta_t T(x, t)dx - T_{xx}(x, t) + \frac{1}{\Gamma(1-\beta)}(t - s^{-1}(x))^{-\beta} \right]dx$$

$$= -\int_{s(t)-\varepsilon}^{s(t)} \left[ C_{s(t)}D^\beta_t T(x, t)dx + \frac{1}{\Gamma(1-\beta)}(t - s^{-1}(x))^{-\beta} \right]dx - T_x(s(t) - \varepsilon, t). (15)$$
Let us choose arbitrary $\tilde{a}$ such that $s(0) < \tilde{a} < a$, then we get
\[
\int_{\tilde{a}}^{s(t)-\varepsilon} \left[ \frac{C}{s(t)} D_t^\beta T(x,t)dx - T_{xx}(x,t) + \frac{1}{\Gamma(1-\beta)}(t-s^{-1}(x))^{-\beta} \right] dx
\]
\[
= - \int_{s(t)-\varepsilon}^{s(t)} \left[ \frac{C}{s(t)} D_t^\beta T(x,t)dx + \frac{1}{\Gamma(1-\beta)}(t-s^{-1}(x))^{-\beta} \right] dx - T_x(s(t)-\varepsilon,t). \tag{16}
\]
Subtracting the sides of (15) and (16) we arrive at
\[
\int_{\tilde{a}}^a \left[ \frac{C}{s(t)} D_t^\beta T(x,t)dx - T_{xx}(x,t) + \frac{1}{\Gamma(1-\beta)}(t-s^{-1}(x))^{-\beta} \right] dx = 0 \tag{17}
\]
for arbitrary $a, \tilde{a} \in (s(0), s(t) - \varepsilon)$ hence, we may deduce that
\[
\frac{C}{s(t)} D_t^\beta T(x,t)dx - T_{xx}(x,t) + \frac{1}{\Gamma(1-\beta)}(t-s^{-1}(x))^{-\beta} = 0 \quad \text{for} \quad x \in (0, s(t)). \tag{18}
\]
We impose another regularity assumption concerning $s$. Namely,
\[
t^{1-\beta} \dot{s}(t) \in L^\infty(0, t^*). \tag{A3}
\]
Then
\[
\lim_{\varepsilon \to 0^+} \int_{s(t)-\varepsilon}^{s(t)} (t-s^{-1}(x))^{-\beta}dx = 0. \tag{19}
\]
Next, we assume that
\[
\forall t \in (0, t^*) \exists \varepsilon_0 > 0, \exists a > \frac{1}{1-\beta} \quad \text{such that} \quad \int_{s(t)-\varepsilon_0}^{s(t)} \int_{s^{-1}(x)}^t |T_t(x,\tau)|^a dx d\tau < \infty. \tag{A4}
\]
Then, from the above assumption, for $\varepsilon \in (0, \varepsilon_0)$ we have
\[
\left| \int_{s(t)-\varepsilon}^{s(t)} \frac{C}{s(t)} D_t^\beta T(x,t)dx \right|
\]
\[
\leq \frac{1}{\Gamma(1-\beta)} \left( \int_{s(t)-\varepsilon_0}^{s(t)} \int_{s^{-1}(x)}^t |T_t(x,\tau)|^a dx d\tau \right)^{1/a} \left| \int_{s(t)-\varepsilon}^{s(t)} \int_{s^{-1}(x)}^t (t-\tau)^{-\frac{a\beta}{a-1}} d\tau dx \right|^{\frac{a-1}{a}}
\]
hence, we get
\[
\left| \int_{s(t)-\varepsilon}^{s(t)} \frac{C}{s(t)} D_t^\beta T(x,t)dx \right| \to 0, \quad \text{if} \quad \varepsilon \to 0^+. \tag{20}
\]
We note that (16) and (18) lead to
\[
0 = - \int_{s(t)-\varepsilon}^{s(t)} \left[ \frac{C}{s(t)} D_t^\beta T(x,t)dx + \frac{1}{\Gamma(1-\beta)}(t-s^{-1}(x))^{-\beta} \right] dx - T_x(s(t)-\varepsilon,t).
\]
Making use of (19) and (20) we obtain

\[
\lim_{\varepsilon \to 0^+} T_x(s(t) - \varepsilon, t) = 0.
\]  

(21)

Finally, we have obtained the following system

\[
C \frac{D^\beta}{s(t)} T(x, t) dx - T_{xx}(x, t) = \begin{cases} 
0 & \text{for } x < s(0) \\
-\frac{1}{\Gamma(1-\beta)}(t - s^{-1}(x))^{-\beta} & \text{for } x \in (s(0), s(t))
\end{cases}
\]

(22)

with the boundary conditions

\[
T(s(t), t) = 0, \quad T_x(s(t), t) = 0,
\]

(23)

\[
T(0, t) = T_D(t) \geq 0 \quad \text{or} \quad T_x(0, t) = T_N(t) \leq 0,
\]

(24)

and the initial condition

\[
T(x, 0) = T_0(x) \geq 0.
\]

(25)

References

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