Asymmetric simple exclusion process on a Cayley tree

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Abstract. We study the asymmetric exclusion process on a regular Cayley tree with arbitrary co-ordination number. In this model particles can enter the system only at the parent site and exit from any of the sites at the last level. In the bulk they move downward to one of the unoccupied neighbours chosen randomly. We show that the steady state current that flows from one level to the next is independent of the exit rate, and increases monotonically with the entry rate and the co-ordination number. Unlike the TASEP, the model has only one phase and the density profile shows no boundary layers.

Keywords: driven diffusive systems (theory), stochastic particle dynamics (theory), random graphs, networks
Exclusion processes [1] have been studied extensively as the paradigm models of non-equilibrium phase transitions. They exhibit a rich variety of phases, phase-coexistence, shock profiles and non-trivial boundary layers. Some variations of these models are exactly solvable [2] on a one dimensional lattice, which provides deep understanding of non-equilibrium transport, traffic and jamming. However, very little is known about the systems beyond one dimension. In particular, transport in irregular structures, like networks, has been a recent topic of interest [3]. In generic undirected networks particles can enter or exit at any arbitrary sites. Again, the presence of loops in these networks make the study of particle transport difficult. On a Cayley tree, however, both the entry and exit sites and the direction of transport are well defined. The absence of loops makes the study relatively simpler. Again, several physical systems like water transport in trees, transport of nutrients in the blood-circulation system [4], transport of antibodies in idiotypic networks in the immune system [5], air circulation in the lungs [6], and flow on disordered networks [7] are strikingly similar to this model system of Cayley trees.

In this paper we study transport in a Cayley tree with arbitrary co-ordination number. Particles can enter the Cayley tree only at the parent site with rate $\alpha$ and are allowed to leave from any of the sites at the last level with rate $\beta$. In the bulk, if allowed by hard-core interaction, the particle can move to one of its downward neighbours chosen randomly. The resulting current that flows between neighbouring levels is found to be independent of the exit rate $\beta$ when $q \geq 2$. Further, the current is found to be larger than that of the TASEP in one dimension for any value of $(\alpha, \beta)$. The average density at each level, except the last one, does not show any finite size correction and decays exponentially as one moves away from the parent site. In the last level, the density depends both on $\alpha$ and $\beta$ and decays exponentially with the system size. We did a mean-field analysis, which correctly reproduces all these features.

The model is defined on a connected cycle-free graph, namely the Cayley tree. An $N$-level Cayley tree, labelled as $i = 0, \ldots, N-1$, with co-ordination number $z = q+1$ can be generated as follows. Starting from the parent level $i = 0$ with one single site $j = 0$ (called root) the next generation of sites is created iteratively by connecting each site in level $i$ with $q$ new sites in the next level $i+1$. Thus level $i$ has $q^i$ sites and the resulting graph has a total of $M = (q^N - 1)/(q-1)$ sites, labelled by $j = 0, \ldots, M$.

Note that the Cayley tree with $q = 1$ is a one dimensional lattice with $N$ sites. The first non-trivial structure can be obtained for $q = 2$, which is described in figure 1. Here, in every level, each site is connected to two downward neighbours in the next level. There are $M = (2^N - 1)$ sites which are labelled by $j$ increasing from top to bottom and left to right. Further, each site of a Cayley tree can either be vacant or occupied by at most one particle. Correspondingly we define a variable $s_j = 1, 0$ at site $j$, which represents the presence or absence of a particle respectively. These hard-core particles flow downwards (from level $i$ to level $i+1$) with the following particle conserving rules. A particle present in a level $i$, at the site say $j$, will move to its neighbour $j'$ in the next lower level $i+1$ only when $s_{j'} = 0$. If more than one neighbour of $j$ is vacant, the particle is transferred to one of the sites chosen randomly. In the following, the bulk dynamics for $q = 2$ is described.

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Asymmetric simple exclusion process on a Cayley tree

Figure 1. Rooted Cayley tree with $q = 2$. There are $N$ levels and $M = (2^N - 1)$ sites, labelled by $i = 0, 1, \ldots, N - 1$ and $j = 0, 1, \ldots, M$ respectively. Particles enter at the parent site $j = 0$ with rate $\alpha$ and exit from any of the sites at the last level $N - 1$ with rate $\beta$. Schematically:

The system is driven by the in- and out-fluxes of particles at boundaries. A particle can enter the system with rate $\alpha$ if the parent site $j = 0$ is unoccupied ($s_j = 0$). Any particle from the $(q^N - 1)$ sites at the boundary level $N - 1$ can leave the system with rate $\beta$. Note that for $q = 1$, each site has only one downward neighbour and the dynamics is simply $10 \rightarrow 01$. Such an exclusion process, namely the totally asymmetric simple exclusion process (TASEP) [8] on a one dimensional lattice, is studied extensively for its application in vehicular traffic and cellular transport by motor-proteins. Exact solution of the TASEP [9, 10] reveals novel macroscopic properties of the steady state, like boundary driven phase transition, shock formation and propagation, condensation, and jamming.

In the steady state of the system, a constant average current $I_q$ flows between any two neighbouring levels $i$ and $i + 1$, which are connected by $q^{i+1}$ links. Hence the current

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on each link is
\[ J_{i \rightarrow i+1} = J_i = \frac{I_q}{q^{i+1}}. \] (1)

For convenience, let us define a level density \( \phi_i \) as the sum of the densities of all those sites belonging to the level \( i \),
\[ \phi_i = \sum_{j \in i} \langle s_j \rangle. \] (2)

In the following we use the mean-field theory (MFT) where both the fluctuations of particle density at the individual sites and variation of densities among sites in the same level are neglected. At the level \( i \), thus, every site is assumed to have an average density
\[ \rho_i = \frac{\phi_i}{q^i}. \] (3)

Since a site of level \( i \) is occupied with probability \( \rho_i \) (and is vacant with probability \( 1 - \rho_i \)), the average current through the bonds is given by
\[ qJ_i = \rho_i (1 - \rho_i^q), \] (4)
where \( (1 - \rho_i^q) \) is the mean-field probability that at least one of the \( q^{i+1} \) sites at the \((i + 1)\)th level is unoccupied. The factor \( q \) in front of \( J_i \) ensures that the outgoing current flows through \( q \) bonds.

The conservation of particle density in the bulk of the system leads to a continuity equation
\[ \frac{d\rho_i}{dt} = J_{(i-1) \rightarrow i} - qJ_{i \rightarrow (i+1)} = J_{i-1} - qJ_i = \frac{1}{q} \rho_{i-1}(1 - \rho_i^q) - \rho_i(1 - \rho_i^q) \quad \text{for } 1 \leq i \leq N - 2. \] (5)

These equations must be supplemented by suitable boundary conditions. At the parent level \( i = 0 \) (where particles enter) we have
\[ \frac{d\rho_0}{dt} = \alpha(1 - \rho_0) - qJ_0 = \alpha(1 - \rho_0) - \rho_0(1 - \rho_1^q) \] (6)
and at the last level \( i = N - 1 \) (where particles exit)
\[ \frac{d\rho_{N-1}}{dt} = J_{N-2} - \beta \rho_{N-1} = \frac{1}{q} \rho_{N-2}(1 - \rho_{N-1}^q) - \beta \rho_{N-1}. \] (7)

The mean-field current and density profile can be obtained as a solution of equations (5)–(7) as follows. Clearly from equation (5), the bulk current \( J_i = J_{i-1}/q \) results in \( J_i = J_0/q^i \), where \( J_0 = I_q/q \) (obtained using equation (1)). Again, equations (4) and (1) provide an iterative equation for the density profile,
\[ \rho_{i+1} = \sqrt{1 - \frac{I_q}{q^i \rho_i}}. \] (8)

It is evident from equation (8) that the \( q = 1 \) case (i.e. the TASEP) is conceptually different from \( q \geq 2 \) as the right hand side of the above map for \( q = 1 \) does not depend
Asymmetric simple exclusion process on a Cayley tree

Figure 2. Solution of equation (10) with $\phi_{i+1} = \phi_i = \phi_i^*$ for different $I_q$ and $i$. As $i$ increases, $\phi_i^*$ approaches the fixed point $I_q$ exponentially.

explicitly on $i$. Correspondingly $\rho_{i+1} = 1 - I_i/\rho_i$ is identical to the mean-field equation for the TASEP [11], which shows macroscopic changes in both the density profile and the current as one varies the boundary rates $\alpha$ and $\beta$. The mean-field theory [11] predicts three different phases: (i) the high density phase ($\alpha > \beta, \beta < \frac{1}{2}$), (ii) the low density phase ($\alpha < \frac{1}{2}, \beta > \alpha$), and (iii) the maximal current phase ($\alpha > \frac{1}{2}, \beta > \frac{1}{2}$). The steady state currents and the density profiles in all these phases agree well with exact results [9].

One must treat the $q \geq 2$ cases separately as the function in the right hand side of equation (8) changes in each iteration. It is not obvious whether such a map iteratively approaches any fixed point. Let us re-write equation (8) in terms of $\phi_i = q^i \rho_i$, which is an $O(1)$ quantity;

$$\frac{\phi_{i+1}}{q^{i+1}} = \sqrt{1 - \frac{I_q}{\phi_i}} \quad \text{for } q \geq 2 \quad \text{and} \quad i \leq N - 2. \quad (9)$$

To find the fixed point of the above map, let us use $\phi_{i+1} = \phi_i = \phi_i^*$ in equation (9) and solve for $\phi_i^*$ for any given $i$ and $I_q$. When $i$ is increased, keeping $I_q$ fixed, $\phi_i^*$ approaches $I_q$ exponentially (shown in figure 2 for $q = 2$ and $I_q = 0.15, 0.3$ and 0.45). This is expected, as the left hand side of equation (9) vanishes for large $i$ resulting in $\phi_i = I_q$. Thus in the bulk of the system, i.e. for large $i$, the density at any level is $I_q$. Even at the level $i = 1$, the density $\phi_1$ is quite close to this bulk value $I_q$ (evident from figure 2); thus one can use $\phi_1 = I_q$ as a good approximation. Since $\phi_1 = q \sqrt{1 - I_q/\phi_0}$ (from equation (9)) we have

$$\left( \frac{I_q}{q} \right)^q + \frac{I_q}{\phi_0} = 1. \quad (10)$$

Note that $\phi_0$ is the same as $\rho_0$ defined in equation (3). Again, $I_q$ can be obtained as a function of $\rho_0$ from the boundary condition (6) as

$$I_q = \alpha(1 - \rho_0). \quad (11)$$

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Therefore, equation (10) becomes a \((q+1)\)th order equation in \(\rho_0\)
\[
\left[\frac{\alpha(1-\rho_0)}{q}\right]^q + \frac{\alpha(1-\rho_0)}{\rho_0} = 1, \quad (12)
\]
which can be solved for \(\rho_0\) to obtain the mean-field current (given by equation (11)) and
the density profile (using equation (3) and \(\phi_i = I_q\))
\[
\rho_i = \frac{\alpha(1-\rho_0)}{q^i}. \quad (13)
\]

Numerical solution of equation (12) can be obtained with high accuracy for any given
\(\alpha\) and \(q\). A closed form solution is possible only for \(q = 2, 3\). For \(q = 2\) we have
\[
\rho_0 = \frac{2}{3} \left[1 - \frac{\sqrt{\alpha^2 + 12\alpha + 12}}{\alpha} \cos \left(\frac{\pi + \theta}{3}\right)\right],
\]
where \(\tan \theta = \frac{6\sqrt{6} \sqrt{\alpha^4 + 18\alpha^3 + 20\alpha^2 + 24\alpha + 8}}{36 - 18\alpha - \alpha^2}. \quad (14)\]

Thus, equations (11) and (13) together with the solution of \(\rho_0\) from equation (14)
provide the complete mean-field solution of the asymmetric exclusion process on the
Cayley tree. To check the validity of the MF theory, we have carried out Monte Carlo
simulation of this exclusion process with \(\alpha = 0.7\) and \(\beta = 0.2\) on a Cayley tree \((q = 2)\)
with \(N = 6, 8, 10, 12\). Starting from a random initial condition, the system is relaxed for
\(10^5\) MCS and the expectation values of the current and the density profile are calculated
over \(10^7\) realizations. It has been checked (data not shown) that the current reaches its
stationary value \(I_q\) in about \(10^3\) MCS. Clearly the densities \(\rho_i\), except \(\rho_{N-1}\), show an
exponential decay which agrees quite well with the MF results (13) and suggest that the
finite size corrections are absent here; i.e., the densities at all levels except \(N - 1\) are
independent of the system size. For example in this figure \(\rho_3\) is same for \(N = 6, 8, 10, 12\)
and \(\rho_5\) is same for \(N = 10, 12\).

It also appears from figure 3 that \(\rho_{N-1}\) varies exponentially with the system size \(N\) as
\(\rho_{N-1} = c/2^{N-1}\), with \(c > \rho_0\). The value of \(c\) can be calculated using the second boundary
condition (7); in steady state \(J_{N-2} = \beta \rho_{N-1}\). Since \(J_{N-2} = I_q/q^{N-1}\) (from equation (1)),
we have
\[
\rho_{N-1} = \frac{c}{q^{N-1}} \quad \text{with} \quad c = \frac{I_q}{\beta} = \frac{\alpha(1-\rho_0)}{\beta}. \quad (15)
\]
In the inset of figure 3, values of \(c\) versus \(\beta\) obtained from numerical simulations for two
different values of \(\alpha = 0.4, 0.7\) are compared with corresponding mean-field values given
by equation (15).

Both from Monte Carlo simulation and from the MF analysis it is evident that the
average density \(\phi_i = \alpha(1-\rho_0)\) in this model is independent of \(i\), except for \(i = N - 1\).
This is similar to the case \(q = 1\) (TASEP) where the bulk density does not vary with \(i\).
However, the density profile in the TASEP shows a boundary layer at both ends which is
absent here. Another difference with the TASEP is that the exit rate \(\beta\) controls only the
average density of the last level \(\phi_{N-1} = c\) as shown in equation (15).

For a generic Cayley tree with \(q \geq 2\), the current \(I_q = \alpha(1-\rho_0)\), where \(\rho_0\) is given by
the solution of equation (12). Note that \(I_q\) is independent of the exit rate \(\beta\) and increases
The density profiles $\rho_i$ for $q = 2$ obtained from Monte Carlo simulations (symbols) for different $N$ are compared with the mean-field results equation (13) (dashed line). Here $\alpha = 0.7$, $\beta = 0.2$, and the corresponding $\rho_0 = 0.422$ is calculated using equation (14). Clearly $\rho_i$, except for $\rho_{N-1}$, is independent of $N$. $\rho_{N-1}$ also follows the mean-field equation (15) (dotted line). The inset shows $c$ as a function of $\beta$, where the simulation results (symbols) for $\alpha = 0.4, 0.7$ are compared with equation (15).

Figure 3. The density profiles $\rho_i$ for $q = 2$ obtained from Monte Carlo simulations (symbols) for different $N$ are compared with the mean-field results equation (13) (dashed line). Here $\alpha = 0.7$, $\beta = 0.2$, and the corresponding $\rho_0 = 0.422$ is calculated using equation (14). Clearly $\rho_i$, except for $\rho_{N-1}$, is independent of $N$. $\rho_{N-1}$ also follows the mean-field equation (15) (dotted line). The inset shows $c$ as a function of $\beta$, where the simulation results (symbols) for $\alpha = 0.4, 0.7$ are compared with equation (15).

monotonically with $\alpha$ (shown in figure 4). To verify this we calculate $I_q$ for a Cayley tree with $q = 2$ using Monte Carlo simulations for two different values of $\beta = 0.2, 0.6$ and plot them against $\alpha$ (inset of figure 4). The mean-field current (11) for $q = 2$, drawn as a solid line, shows an excellent agreement.

Figure 4 compares $I_q$ for $q = 2, 3, 4$. As expected, $I_q$ is a strictly non-decreasing function of the entry rate $\alpha$. Thus, the maximum achievable current on a Cayley tree is $I_{q}^\text{max} = \lim_{\alpha \to \infty} I_q$, which can be calculated analytically for $q = 2$ and 3,

\[
I_{2}^\text{max} = (2\sqrt{2} - 2); \quad I_{3}^\text{max} = \left[\frac{27}{4}\right]^{1/3} \left[(\sqrt{5} + 1)^{1/3} - (\sqrt{5} - 1)^{1/3}\right].
\]

From figure 4, it appears that $\lim_{q \to \infty} I_{q}^\text{max} = 1$. This can be understood from the fact that when $q \to \infty$, $I_q = \phi_0$ for any finite value of $\alpha$ (from equation (10)). Now, in the $\alpha \to \infty$ limit, the parent site $j = 0$ remains occupied with probability 1, resulting in $\phi_0 = 1$. Thus $I_{\infty}^\text{max} = 1$. Thus the maximum achievable current for any $q \geq 2$ is much larger than that of the TASEP ($q = 1$).\(^1\) Another interesting fact about $I_q$ is the following inequality:

\[
I_1(\alpha, \beta) < I_2(\alpha) < \cdots < I_\infty(\alpha),
\]

which holds for any arbitrary value of $(\alpha, \beta)$.

In conclusion, we have studied the asymmetric exclusion process on a regular Cayley tree with arbitrary co-ordination number $z = q + 1$, where particles enter the system only

\(^1\) In the TASEP, the largest current $J = \frac{1}{2}$ flows only in the maximal current phase ($\alpha > \frac{1}{2}$ and $\beta > \frac{1}{2}$).
Figure 4. \( I_q = \alpha(1 - \rho_0) \), where \( \rho_0 \) is a solution of equation (12), is shown as a function of \( \alpha \) for different \( q = 2, 3, 4 \). In the inset we compare this mean-field result for \( q = 2 \) with Monte Carlo simulations (symbols). As expected, the current \( I_q \) is the same for the two different \( \beta = 0.2, 0.6 \).

at the parent site with rate \( \alpha \), and exit from one of the sites at the last level with rate \( \beta \). In the bulk they move from occupied sites to one of their downward unoccupied neighbours chosen randomly. The TASEP is a special case of this model for \( q = 1 \) which exhibits boundary driven phase transitions. In this case the steady state current \( I_q \) is different in different phases and depends both on \( \alpha \) and \( \beta \). Surprisingly, for \( q \geq 2 \), there is only one phase and the average current \( I_q \) is independent of \( \beta \) and increases monotonically with \( \alpha \). Again, the density profiles do not show any boundary layers or finite size corrections. It is like a free flow of particles from one end to the other; only the last level is affected by the exit rate. The model could find application in nutrient transport in blood-circulation systems, air circulation in the lungs or in antibody transport in immune systems or water transport in trees. Possibly, branching is essential in all these systems (say trees) as the flow (of water at different levels) needs to be maintained independent of the exit rates (weather conditions) at the last level (leaves).

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Asymmetric simple exclusion process on a Cayley tree

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