Research Article

Initial Bounds for Certain Classes of Bi-Univalent Functions Defined by Horadam Polynomials

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1. Introduction

Let $\mathbb{R}$ be the set of real numbers, $\mathbb{C}$ be the set of complex numbers and

\[ \mathbb{N} := \{1, 2, 3, \ldots \} \]

be the set of positive integers. Let $\mathcal{A}$ denote the class of functions of the form

\[ f(z) = z + \sum_{n=2}^{\infty} a_n z^n \]

which are analytic in the open unit disk $\Delta = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Further, by $\mathcal{S}$ we shall denote the class of all functions in $\mathcal{A}$ which are univalent in $\Delta$.

It is well known that every function $f \in \mathcal{S}$ has an inverse $f^{-1}$, defined by

\[ f^{-1}(f(z)) = z \quad (z \in \Delta) \]

and

\[ f(f^{-1}(w)) = w \quad (|w| < r_0(f); \quad r_0(f) \geq \frac{1}{4}) \]

where

\[ f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2a_3 + a_4\right) w^4 + \cdots \]

\[ f(z) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2a_3 + a_4\right) w^4 + \cdots \]

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\Delta$ if both the function $f$ and its inverse $f^{-1}$ are univalent in $\Delta$. Let $\sigma$ denote the class of bi-univalent functions in $\Delta$ given by (2).

In 2010, Srivastava et al. [1] revived the study of bi-univalent functions by their pioneering work on the study of coefficient problems. Various subclasses of the bi-univalent function class $\sigma$ were introduced and nonsharp estimates on the first two coefficients $|a_2|$ and $|a_3|$ in the Taylor–Maclaurin series expansion (2) were found in the recent investigations (see, for example, [2–23]) and including the references therein. The afore-cited all these papers on the subject were actually motivated by the work of Srivastava et al. [1]. However, the problem to find the coefficient bounds on $|a_n|$ ($n = 3, 4, \ldots$) for functions $f \in \sigma$ is still open problem.

For analytic functions $f$ and $g$ in $\Delta$, $f$ is said to be subordinate to $g$ if there exists an analytic function $w$ such that $w(0) = 0$, $|w(z)| < 1$ and $f(z) = g(w(z))$ $(z \in \Delta)$. This subordination will be denoted here by $f < g \quad (z \in \Delta)$.
or, conventionally, by
\[ f(z) < g(z) \quad (z \in \Delta). \quad (8) \]

In particular, when \( g \) is univalent in \( \Delta \)
\[ f < g \quad (z \in \Delta) \iff f(0) = g(0) \text{ and } f(\Delta) \subset g(\Delta). \quad (9) \]

The Horadam polynomials \( h_n(x, a, b; p, q) \), or briefly \( h_n(x) \)
are given by the following recurrence relation (see [22, 23]):
\[
\begin{align*}
h_0(x) &= a, & h_1(x) &= bx, & h_2(x) &= pxh_{n-1}(x) + qh_{n-2}(x) & (n \geq 3)
\end{align*}
\]
for some real constants \( a, b, p, \) and \( q \).

The generating function of the Horadam polynomials \( h_n(x) \)
(see [23]) is given by
\[ \Pi(x, z) := \sum_{n=0}^{\infty} h_n(x)z^{n-1} = \frac{a + (b - ap)xz}{1 - pzx - qz^2}. \quad (11) \]

Here, and in what follows, the argument \( x \in \mathbb{R} \) is independent
of the argument \( z \in C \); that is, \( x \neq \mathbb{R}(z) \).

Note that for particular values of \( a, b, p, \) and \( q \), the Horadam
polynomial \( h_n(x) \) leads to various polynomials, among those,
we list a few cases here (see [22, 23] for more details):

1. For \( a = b = p = q = 1 \), we have the Fibonacci polynomials \( F_n(x) \).
2. For \( a = 2 \) and \( b = p = q = 1 \), we obtain the Lucas
   polynomials \( L_n(x) \).
3. For \( a = q = 1 \) and \( b = p = 2 \), we get the Pell
   polynomials \( P_n(x) \).
4. For \( a = b = p = 2 \) and \( q = 1 \), we attain the Pell-Lucas
   polynomials \( Q_n(x) \).
5. For \( a = b = 1, p = 2 \) and \( q = -1 \), we have the
   Chebyshev polynomials \( T_n(x) \) of the first kind.
6. For \( a = 1, b = p = 2 \) and \( q = -1 \), we obtain the
   Chebyshev polynomials \( U_n(x) \) of the second kind.

Recently, in literature, the coefficient estimates are found
for functions in the class of univalent and bi-univalent functions
associated with certain polynomials such as the Faber
polynomial [8], the Chebyshev polynomials [6], and
the Horadam polynomial [15]. Motivated in these lines, estimates
on initial coefficients of the Taylor–Maclaurin series expansion
(2) and Fekete–Szegö inequalities for certain classes of bi-univalent
functions defined by means of Horadam polynomials

\[
|a_3 - va_2^2| \leq \frac{|b|}{2 + 6\lambda} \frac{|b|x|v - 1|}{\left[(1 + 4\lambda)b - p(1 + 2\lambda)^2|bx^2 - qa(1 + 2\lambda)^2\right]} \quad \text{and for } v \in \mathbb{R}
\]

Proof. Let \( f \in S^*_+(\lambda, x) \) be given by the Taylor–Maclaurin
expansion (2). Then, there are analytic functions \( u \) and \( v \) such that
\[ u(0) = 0; \quad v(0) = 0; \quad |u(z)| < 1 \text{ and } |v(z)| < 1 \quad (\forall z, w \in \Delta), \]

are obtained. The classes introduced in this paper are motivated
by the corresponding classes investigated in [2, 10, 14, 15].

2. Coefficient Estimates and Fekete–Szegö

Inequalities

A function \( f \in A \) of the form (2) belongs to the class \( S^*_+(\lambda, x) \)
for \( \lambda \geq 0 \) and \( z, w \in \Delta \), if the following conditions are satisfied:
\[ \frac{zf'(z)}{f(z)} + \lambda z^2f''(z) < \Pi(x, z) + 1 - a \quad (12) \]

and for \( g(w) = f^{-1}(w) \)
\[ \frac{wg'(w)}{g(w)} + \lambda \frac{w^2g''(w)}{g(w)} < \Pi(x, w) + 1 - a \quad (13) \]

where the real constant \( a \) is as in (10).

Note that \( S^*_+(0, x) = S^*_+(0, 0) \) was introduced and studied by
Srivastava et al. [15].

Remark 1. When \( a = 1, b = p = 2, q = -1 \) and \( x = t \), the
function in (11) reduces to that of the Chebyshev
polynomial \( U_n(t) \) of the second kind, which is given explicitly by
\[ U_n(t) = (n + 1)_2F_1\left(-n, n + 2; \frac{3}{2}, \frac{1 - t}{2} \right) \]
\[ = \frac{\sin(n + 1)\phi}{\sin \phi}, \quad (t = \cos \phi) \quad (14) \]
in terms of the hypergeometric function \(_2F_1_\).

In view of Remark 1, the bi-univalent function class \( S^*_+(\lambda, x) \)
reduces to \( S^*_+(t) \) and this class was studied earlier in [3, 12].
For functions in the class \( S^*_+(\lambda, x) \), the following coefficient
estimates and Fekete–Szegö inequality are obtained.

Theorem 1. Let \( f(z) = z + \sum_{n=2}^{\infty} a_nz^n \) be in the class \( S^*_+(\lambda, x) \).
Then
\[ |a_3| \leq \frac{|b|}{2 + 6\lambda} \frac{|b|x|v - 1|}{\left[(1 + 4\lambda)b - p(1 + 2\lambda)^2|bx^2 - qa(1 + 2\lambda)^2\right]} \quad \text{and for } v \in \mathbb{R}
\]

\[ |a_4| \leq \frac{|b|}{2 + 6\lambda} \frac{b^2x^2}{(1 + 2\lambda)^2} \quad (15) \]

and for \( v \in \mathbb{R} \)
\[ if \quad |v - 1| \leq \frac{[(1 + 4\lambda)b - p(1 + 2\lambda)^2|bx^2 - qa(1 + 2\lambda)^2]}{2b^2x^2(1 + 3\lambda)}, \]
\[ if \quad |v - 1| \geq \frac{[(1 + 4\lambda)b - p(1 + 2\lambda)^2|bx^2 - qa(1 + 2\lambda)^2]}{2b^2x^2(1 + 3\lambda)} \quad (16) \]

and we can write
\[ \frac{zf'(z)}{f(z)} + \lambda z^2f''(z) = \Pi(x, u(z)) + 1 - a \quad (18) \]

and
\[
\frac{w^3'(w)}{g(w)} + \lambda \frac{w^2g''(w)}{g(w)} = \Pi(x, v(w)) + 1 - a.
\]

Equivalently,
\[
\frac{zf'(z)}{f(z)} + \lambda \frac{z^2f''(z)}{f(z)} = 1 + h_1(x) - a + h_2(x)u(z) + h_3(x)[u(z)]^2 + \cdots
\]

and
\[
\frac{w^3'(w)}{g(w)} + \lambda \frac{w^2g''(w)}{g(w)} = 1 + h_1(x) - a + h_2(x)v(w) + h_3(x)[v(w)]^2 + \cdots.
\]

From (20) and (21) and in view of (11), we obtain
\[
\frac{zf'(z)}{f(z)} + \lambda \frac{z^2f''(z)}{f(z)} = 1 + h_2(x)u_z + \left[h_2(x)u_2 + h_3(x)u_1\right] z^2 + \cdots
\]

and
\[
\frac{w^3'(w)}{g(w)} + \lambda \frac{w^2g''(w)}{g(w)} = 1 + h_2(x)v_z + \left[h_2(x)v_2 + h_3(x)v_1\right] w^2 + \cdots.
\]

If
\[
u(z) = \sum_{n=1}^{\infty} u_n z^n \quad \text{and} \quad v(z) = \sum_{n=1}^{\infty} v_n w^n,
\]

then it is well known that
\[
|u_n| \leq 1 \quad \text{and} \quad |v_n| \leq 1 \quad (n \in \mathbb{N}).
\]

Thus upon comparing the corresponding coefficients in (22) and (23), we have
\[
(1 + 2\lambda)a_2 = h_2(x)u_1,
\]
\[
2(1 + 3\lambda)a_3 - (1 + 2\lambda)a_2^2 = h_2(x)u_2 + h_3(x)u_1^2,
\]
\[
- (1 + 2\lambda)a_2 = h_2(x)v_1
\]

and
\[
3 + 10\lambda)a_3^2 - 2(1 + 3\lambda)a_3 = h_2(x)v_2 + h_3(x)v_1^2.
\]

From (26) and (28), we can easily see that
\[
u_1 = -v_1, \quad \text{provided} \quad h_2(x) = bx \neq 0
\]

and
\[
2(1 + 2\lambda)^2a_2^3 = h_2(x)u_1^2 + v_1^2
\]
\[
a_2^2 = \frac{h_2(x)u_1^2 + v_1^2}{2(1 + 2\lambda)^2}.
\]

If we add (27) to (29), we get
\[
2(1 + 4\lambda)a_3^2 = h_2(x)(u_2 + v_2) + h_3(x)(u_1^2 + v_1^2).
\]

By substituting (31) into (32), we obtain
\[
a_2^2 = \frac{h_2(x)u_1^2 + v_1^2}{2(1 + 4\lambda)[h_2(x)]^2 - 2h_2(x)(1 + 2\lambda)^2}.
\]

and by taking \(h_2(x) = bx\) and \(h_3(x) = bpx^2 + qa\) in (33), it further yields
\[
|a_2| \leq \frac{|b|x \sqrt{|b|x}}{\sqrt{\left|1 + 2\lambda\right| b - p(1 + 2\lambda)^2} |b|x^2 - qa(1 + 2\lambda)^2}}.
\]

By subtracting (29) from (27) and in view of (30), we obtain
\[
4(1 + 3\lambda)a_1 - 4(1 + 3\lambda)a_2^2 = h_2(x)(u_2 - v_2) + h_3(x)(u_1^2 - v_1^2)
\]
\[
a_1 = \frac{h_2(x)(u_2 - v_2)}{4(1 + 3\lambda)} + a_2^2.
\]

Then in view of (31), (35) becomes
\[
a_3 = \frac{h_2(x)(u_2 - v_2)}{4(1 + 3\lambda)} + \frac{[h_2(x)]^2(u_1^2 + v_1^2)}{2(1 + 2\lambda)^2}.
\]

Applying (10), we deduce that
\[
|a_3| \leq \frac{|b|x}{2 + 6\lambda} + \frac{b^2x^2}{(1 + 2\lambda)^2}.
\]

From (35), for \(v \in \mathbb{R}\), we write
\[
a_3 - va_2^2 = \frac{h_2(x)(u_2 - v_2)}{4(1 + 3\lambda)} + (1 - v)a_1.
\]

By substituting (33) in (38), we have
\[
a_3 - va_2^2 = \frac{h_2(x)(u_2 - v_2)}{4(1 + 3\lambda)} + \left(1 - v\right)\left(h_2(x)(u_1^2 + v_1^2) + \frac{(1 - v)[h_2(x)]^2(u_1^2 + v_1^2)}{2(1 + 2\lambda)(h_2(x)^2 - h_3(x)(1 + 2\lambda)^2)}\right)
\]
\[
= h_2(x)\left[\left(\Omega(v, x) + \frac{1}{4(1 + 3\lambda)}\right)u_1 + \left(\Omega(v, x) - \frac{1}{4(1 + 3\lambda)}\right)v_1\right].
\]

where
\[
\Omega(v, x) = \frac{(1 - v)[h_2(x)]^2}{2(1 + 4\lambda)[h_2(x)]^2 - 2h_2(x)(1 + 2\lambda)^2}.
\]

Hence, we conclude that
\[
|a_3 - va_2^2| \leq \left\{ \begin{array}{ll}
\frac{|h_2(x)|}{2 + 6\lambda} & \text{if} \quad |v - 1| \leq \frac{1}{2(1 + 3\lambda)} \\
\frac{|h_2(x)|}{2(1 + 4\lambda)} |\Omega(v, x)| & \text{if} \quad |\Omega(v, x)| \geq \frac{1}{2(1 + 3\lambda)}
\end{array} \right.
\]

and in view of (10), it evidently completes the proof of Theorem 1. \(\square\)

For \(\lambda = 0\), Theorem 1 readily yields the following coefficient estimates for \(S''_q(x)\).

**Corollary 1.** Let \(f(z) = z + \sum_{a=1}^{\infty} a_3^a z^n\) be in the class \(S''_q(x)\). Then
\[
|a_2| \leq \frac{|b|x \sqrt{|b|x}}{|b|x^2 - qa}, \quad \text{and} \quad |a_3| \leq \frac{|b|x}{2} + b^2x^2.
\]

and for \(v \in \mathbb{R}\)
\[
|a_3 - va_2^2| \leq \left\{ \begin{array}{ll}
\frac{|b|x}{2} & \text{if} \quad |v - 1| \leq \frac{|b|x^2 - qa}{2b^2x^2} \\
\frac{|b|x^2|v - 1|}{|b|x^2 - qa} & \text{if} \quad |v - 1| \geq \frac{|b|x^2 - qa}{2b^2x^2}.
\end{array} \right.
\]
In view of Remark 1, Theorem 1 can be shown to yield the following result.

**Corollary 2.** Let \( f(z) = z + \sum_{n=0}^{\infty} a_n z^n \) be in the class \( S'_v(\lambda, t) \). Then

\[
|a_2| \leq \frac{|2t| \sqrt{2t}}{\sqrt{(1 + 2\lambda)^2 - 16\lambda^2 t}}, \quad \text{and} \quad |a_3| \leq \frac{|t|}{1 + 3\lambda} + \frac{4t^2}{(1 + 2\lambda)^2}
\]

and for \( v \in \mathbb{R} \)

\[
|a_3 - va_2^2| \leq \left\{ \begin{array}{ll}
\frac{|b|}{1 + 3\lambda} & \text{if } |v - 1| \leq \frac{(1 + 2\lambda)^2 - 16\lambda^2 t}{8t^2(1 + 3\lambda)}, \\
\frac{|b|}{(1 + 2\lambda)^2 - 16\lambda^2 t} & \text{if } |v + 1| \geq \frac{(1 + 2\lambda)^2 - 16\lambda^2 t}{8t^2(1 + 3\lambda)}
\end{array} \right.
\]

**Remark 2.** Results obtained in Corollary 1 coincide with results obtained in [15]. For \( \lambda = 0 \), Corollary 2 reduces to the results discussed in [3, 12].

Next, a function \( f \in A \) belonging to the class \( M_\nu(\alpha, x) \) for \( 0 \leq \alpha \leq 1 \) and \( z, w \in \Delta \), if the following conditions are satisfied:

\[
(1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) < \Pi(x, z) + 1 - a \tag{46}
\]

and for \( g(w) = f^{-1}(w) \)

\[
(1 - \alpha) \frac{wg'(w)}{g(w)} + \alpha \left( 1 + \frac{wg''(w)}{g'(w)} \right) < \Pi(x, w) + 1 - a,
\]

where the real constant \( a \) is as in (10).

Note that the class \( M_\nu(\alpha, x) \) reduces to the classes \( S'_v(x) \) and \( K_\nu(x) \) as \( M_\nu(0, x) = S'_v(x) \) and \( M_\nu(1, x) = K_\nu(x) \). In view of Remark 1, the bi-univalent function classes \( M_\nu(\alpha, x) \) would become the class \( M_\nu^*(\alpha, t) \) introduced and studied by Altmkaya and Yalçın [4]. For functions in the class \( M_\nu(\alpha, x) \), the following coefficient estimates and Fekete–Szegő inequality are obtained.

**Theorem 2.** Let \( f(z) = z + \sum_{n=0}^{\infty} a_n z^n \) be in the class \( M_\nu(\alpha, x) \). Then

\[
|a_2| \leq \frac{|bx| \sqrt{|bx|}}{\sqrt{[(1 + \alpha)b - p(1 + \alpha)^2]bx^2 - qa(1 + \alpha)^3}}, \quad \text{and}
\]

\[
|a_3| \leq \frac{|b|}{2 + 4\alpha} + \frac{b^2 x^2}{(1 + \alpha)^2}
\]

and for \( v \in \mathbb{R} \)

\[
if \ |v - 1| \leq \frac{[(1 + \alpha)b - p(1 + \alpha)^2]bx^2 - qa(1 + \alpha)^3}{b^2 x^2(2 + 4\alpha)},
\]

\[
if \ |v + 1| \geq \frac{[(1 + \alpha)b - p(1 + \alpha)^2]bx^2 - qa(1 + \alpha)^3}{b^2 x^2(2 + 4\alpha)}.
\]

From (53), (54) and in view of (11), we obtain

\[
(1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) = 1 + h_1(x)u_1 z + [h_2(x)u_2 + h_3(x)u_3] z^2 + \cdots
\]

and

\[
(1 - \alpha) \frac{wg'(w)}{g(w)} + \alpha \left( 1 + \frac{wg''(w)}{g'(w)} \right) = 1 + h_1(x)v_1 w + [h_2(x)v_2 + h_3(x)v_3] w^2 + \cdots
\]

If

\[
u(z) = \sum_{n=1}^{\infty} u_n z^n \quad \text{and} \quad v(z) = \sum_{n=1}^{\infty} v_n w^n, \tag{57}
\]

then it is well known that

\[
|u_n| \leq 1 \quad \text{and} \quad |v_n| \leq 1 \quad (n \in \mathbb{N}). \tag{58}
\]

Thus upon comparing the corresponding coefficients in (55) and (56), we have

\[
(1 + \alpha) a_2 = h_2(x)u_1 \tag{59}
\]
\begin{align*}
2(1 + 2\alpha) a_3 - (1 + 3\alpha) a_2^2 &= h_2(x) u_2 + h_3(x) u_1^2 \quad (60) \\
-(1 + \alpha) a_2 &= h_2(x) v_1 \quad (61)
\end{align*}

and
\begin{align*}
(3 + 5\alpha) a_2^2 - 2(1 + 2\alpha) a_3 &= h_2(x) v_2 + h_3(x) v_1^2. \quad (62)
\end{align*}

From (59) and (61), we can easily see that
\begin{align*}
u_i = -v_i, \quad \text{provided} \quad h_1(x) = bx \neq 0 \quad (63)
\end{align*}

and
\begin{align*}
2(1 + \alpha) a_2^2 &= [h_2(x)]^2 \left(u_1^2 + v_1^2\right) \quad (64)
\end{align*}

If we add (60) to (62), we get
\begin{align*}
2(1 + \alpha) a_2^2 &= h_2(x) (u_2 + v_2) + h_3(x) \left(u_1^2 + v_1^2\right). \quad (65)
\end{align*}

By substituting (64) in (65), we obtain
\begin{align*}
a_2^2 &= \frac{[h_2(x)]^2 (u_2 + v_2)}{2(1 + \alpha) [h_2(x)]^2 - 2h_3(x)(1 + \alpha)^2}. \quad (66)
\end{align*}

and by taking \( h_2(x) = bx \) and \( h_3(x) = bpx^2 + qa \) in (66), it further yields
\begin{equation}
|a_2| \leq \frac{|bx| \sqrt{|bx|}}{\sqrt{|(1 + \alpha)b - p + (1 + \alpha)^2|}}. \quad (67)
\end{equation}

By subtracting (62) from (60) and in view of (63), we obtain
\begin{align*}
4(1 + 2\alpha) a_3 - 4(1 + 2\alpha) a_2^2 &= h_2(x) (u_2 - v_2) + h_3(x) \left(u_1^2 - v_1^2\right) \\
a_3 &= \frac{h_2(x) (u_2 - v_2)}{4(1 + 2\alpha)} + a_2^2. \quad (68)
\end{align*}

Then in view of (64), (68) becomes
\begin{align*}
a_3 &= \frac{h_2(x) (u_2 - v_2)}{4(1 + 2\alpha)} + \frac{[h_2(x)]^2 (u_1^2 + v_1^2)}{2(1 + \alpha)^2}. \quad (69)
\end{align*}

Applying (10), we deduce that
\begin{align*}
|a_3| \leq \frac{|bx|}{2 + 4\alpha} + \frac{b^2 x^2}{(1 + \alpha)^2}. \quad (70)
\end{align*}

From (68), for \( v \in \mathbb{R} \), we write
\begin{align*}
a_3 - va_2^2 &= \frac{h_2(x) (u_2 - v_2)}{4(1 + 2\alpha)} + (1 - v)a_2^2. \quad (71)
\end{align*}

By substituting (66) in (71), we have
\begin{align*}
a_3 - va_2^2 &= \frac{h_2(x) (u_2 - v_2)}{4(1 + 2\alpha)} + (1 - v) \frac{(1 - \alpha) [h_2(x)]^2 (u_1^2 + v_1^2)}{2(1 + \alpha) [h_2(x)]^2 - 2h_3(x)(1 + \alpha)^2} \\
&= h_2(x) \left(\Omega(v, x) + \frac{1}{4(1 + 2\alpha)} \left(\Omega(v, x) - \frac{1}{4(1 + 2\alpha)}\right) v_2\right), \quad (72)
\end{align*}

where
\begin{equation}
\Omega(v, x) = \frac{(1 - v) [h_2(x)]^2}{2(1 + \alpha) [h_2(x)]^2 - 2h_3(x)(1 + \alpha)^2}. \quad (73)
\end{equation}

Hence, we conclude that
\begin{align*}
|a_3 - va_2^2| &\leq \begin{cases} 
\frac{|h_2(x)|}{2 + 4\alpha} & 0 \leq |\Omega(v, x)| \leq \frac{1}{4(1 + 2\alpha)} \\
2|h_2(x)||\Omega(v, x)| & |\Omega(v, x)| \geq \frac{1}{4(1 + 2\alpha)}.
\end{cases} \quad (74)
\end{align*}

which in view of (10), evidently completes the proof of Theorem 2.

\[\square\]

For \( \alpha = 1 \), Theorem 2 readily yields the following coefficient estimates for \( K_a(x) \).

\begin{corollary}
Let \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) be in the class \( K_a(x) \). Then
\begin{align*}
|a_3| &\leq \frac{|bx| \sqrt{|bx|}}{\sqrt{|(2b - 4p)bx^2 - 4qa|}}, \quad \text{and} \quad |a_3| \leq \frac{|bx|}{6} + \frac{b^2 x^2}{4}. \quad (75)
\end{align*}

and for \( v \in \mathbb{R} \)
\begin{align*}
|a_3 - va_2^2| &\leq \begin{cases} 
\frac{|bx|}{6} & |v - 1| \leq \frac{|b - 2p| |bx^2 - 4qa|}{8b^2 (1 + 2\alpha)} \\
\frac{|bx|}{2b - 4p} & |v - 1| \geq \frac{|b - 2p| |bx^2 - 4qa|}{8b^2 (1 + 2\alpha)}.
\end{cases} \quad (76)
\end{align*}

In view of Remark 1, Theorem 2 yields the following result.

\begin{corollary}
Let \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) be in the class \( M_a(x, t) \). Then
\begin{align*}
|a_3| &\leq \frac{2|t| \sqrt{|t|}}{\sqrt{|(1 + \alpha)^2 - 4\alpha(1 + \alpha)^2|}}, \quad \text{and} \quad |a_3| \leq \frac{|t|}{1 + 2\alpha} + \frac{4t^2}{(1 + \alpha)^2}. \quad (77)
\end{align*}

and for \( v \in \mathbb{R} \)
\begin{align*}
|a_3 - va_2^2| &\leq \begin{cases} 
\frac{|t|}{1 + 2\alpha} & |v - 1| \leq \frac{|(1 + \alpha)^2 - 4\alpha(1 + \alpha)^2|}{8\alpha(1 + 2\alpha)} \\
\frac{8|t| |v - 1|}{|1 + \alpha|^2 - 4\alpha(1 + \alpha)^2} & |v - 1| \geq \frac{|(1 + \alpha)^2 - 4\alpha(1 + \alpha)^2|}{8\alpha(1 + 2\alpha)}.
\end{cases} \quad (78)
\end{align*}

In view of Remark 1, Corollary 3 yields the following result.

\begin{corollary}
Let \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) be in the class \( K_a(t) \). Then
\begin{align*}
|a_3| &\leq \frac{|t| \sqrt{|t|}}{\sqrt{|1 - 2t^2|}}, \quad \text{and} \quad |a_3| \leq \frac{|t|}{3} + t^2. \quad (79)
\end{align*}

and for \( v \in \mathbb{R} \)
\begin{align*}
|a_3 - va_2^2| &\leq \begin{cases} 
\frac{|t|}{3} & |v - 1| \leq \frac{|1 - 2t^2|}{6t} \\
\frac{2|t| |v - 1|}{|1 - 2t^2|} & |v - 1| \geq \frac{|1 - 2t^2|}{6t}.
\end{cases} \quad (80)
\end{align*}

Remark 3. The results obtained in Corollary 4 and 5 coincide with results of Altinkaya and Yal\c{c}in [4].
Next, a function \( f \in \sigma \) of the form (2) belongs to the class \( \mathcal{L}_\omega(\mu, x) \) for \( 0 \leq \mu \leq 1 \) and \( z, w \in \Delta \) if the following conditions are satisfied:

\[
\left( \frac{zf'(z)}{f(z)} \right)^\mu \left( 1 + \frac{zf''(z)}{f'(z)} \right)^{1-\mu} < \Pi(x, z) + 1 - a \tag{81}
\]

and for \( g(w) = f^{-1}(w) \)

\[
\left( \frac{wg'(w)}{g(w)} \right)^\mu \left( 1 + \frac{wg''(w)}{g'(w)} \right)^{1-\mu} < \Pi(x, w) + 1 - a \tag{82}
\]

where the real constants \( a \) is as in (10).

This class also reduces to \( \mathcal{S}_\omega(x) \) and \( \mathcal{K}_\omega(x) \). In view of Remark 1, the bi-univalent function class \( \mathcal{L}_\omega(\mu, x) \) would become the class \( \mathcal{L}_\omega^*(\mu, t) \). For functions in the class \( \mathcal{L}_\omega(\mu, x) \), the following coefficient estimates are obtained.

**Theorem 3.** Let \( f(z) = z + \sum_{n=0}^{\infty} a_n z^n \) be in the class \( \mathcal{L}_\omega(\mu, x) \). Then

\[
|a_3| \leq \frac{|bx| \sqrt{2|x|}}{6 - 4\mu + \frac{b^2 \cdot (2-\mu)^2}{(2-\mu)^2}} \tag{83}
\]

and for \( v \in \mathbb{R} \)

\[
|a_n| \leq |v| - 1 
\]

\[
= \frac{1 + h_2(x)v_1 + h_3(x)v_1^2}{|v| - 1} \tag{84}
\]

where the real constants \( a_n \) and for \( \mathcal{L}_\omega(\mu, x) \) be given by the Taylor–Maclaurin expansion (2). Then, there are analytic functions \( u \) and \( v \) such that

\[
u(0) = 0, \quad v(0) = 0, \quad |u(z)| < 1 \quad \text{and} \quad |v(z)| < 1 \quad (\forall z, w \in \Delta), \tag{85}
\]

and

\[
(zf'(z))^{\mu} \left( 1 + zf''(z) \right)^{1-\mu} = \Pi(x, u(z)) + 1 - a \tag{86}
\]

and

\[
(wg'(w))^{\mu} \left( 1 + wg''(w) \right)^{1-\mu} = \Pi(x, v(w)) + 1 - a \tag{87}
\]

Equivalently,

\[
(zf'(z))^{\mu} \left( 1 + zf''(z) \right)^{1-\mu} = 1 + h_1(x) - a + h_2(x)u(z) + h_3(x)u(z)^2 + \cdots \tag{88}
\]

and

\[
(wg'(w))^{\mu} \left( 1 + wg''(w) \right)^{1-\mu} = 1 + h_1(x) - a + h_2(x)v(w) + h_3(x)v(w)^2 + \cdots \tag{89}
\]

From (88) and (89) and in view of (11), we obtain

\[
(zf'(z))^{\mu} \left( 1 + zf''(z) \right)^{1-\mu} = 1 + h_1(x)u_1 z + [h_2(x)u_2 + h_3(x)u_2^2] z^2 + \cdots \tag{90}
\]

If

\[
u(z) = \sum_{n=1}^{\infty} u_n z^n \quad \text{and} \quad v(z) = \sum_{n=1}^{\infty} v_n z^n, \tag{92}
\]

then it is well known that

\[
|u_n| \leq 1 \quad \text{and} \quad |v_n| \leq 1 \quad (n \in \mathbb{N}). \tag{93}
\]

Thus upon comparing the corresponding coefficients in (90) and (91), we have

\[
(2-\mu)a_2 = h_2(x)u_1 \tag{94}
\]

\[
2(3-2\mu)a_3 + (\mu^2 + 5\mu - 8) \frac{a_1^2}{2} = h_2(x)u_2 + h_3(x)u_2^2 \tag{95}
\]

\[
-2(3-2\mu)a_2 = h_2(x)v_1 \tag{96}
\]

and

\[
(\mu^2 - 12\mu + 16) \frac{a_1^2}{2} - 2(3-2\mu)a_3 = h_2(x)v_2 + h_3(x)v_2^2. \tag{97}
\]

From (94) and (96), we can easily see that

\[
u_1 = -v_1, \quad \text{provided} \quad h_2(x) = bx \neq 0 \tag{98}
\]

and

\[
2(3-2\mu)^2 a_2 = [h_2(x)]^2 (u_1^2 + v_1^2) \tag{99}
\]

\[
a_2^2 = \frac{[h_2(x)]^2 (u_1^2 + v_1^2)}{2(3-2\mu)^2}. \tag{99}
\]
If we add (95) to (97), we get
\[(\mu^2 - 3\mu + 4)a_2^2 = h_2(x)(u_2 + v_2) + h_3(x)(u_1^2 + v_1^2). \tag{100}\]

By substituting (99) in (100), we obtain
\[a_2^2 = \frac{[h_2(x)]^2(u_2 + v_2)}{(\mu^2 - 3\mu + 4)[h_2(x)]^2 - 2h_3(x)(2 - \mu)^2} \tag{101}\]

and by taking \(h_2(x) = bx\) and \(h_3(x) = bpx^2 + qa\) in (101), it further yields
\[|a_2| \leq \frac{|bx| \sqrt{2|bx|}}{\sqrt{|(\mu^2 - 3\mu + 4)b - 2p(2 - \mu)^2|bx^2 - 2qa(2 - \mu)^2|}}. \tag{102}\]

By subtracting (97) from (95) and in view of (98), we obtain
\[4(3 - 2\mu)a_3 - 4(3 - 2\mu)a_2^2 = h_2(x)(u_2 - v_2) + h_3(x)(u_1^2 - v_1^2)\]
\[a_3 = \frac{h_2(x)(u_2 - v_2)}{4(3 - 2\mu)} + a_2^2. \tag{103}\]

Then in view of (99), (103) becomes
\[a_3 = \frac{h_2(x)(u_2 - v_2)}{4(3 - 2\mu)} + \frac{[h_2(x)]^2(u_1^2 + v_1^2)}{2(2 - \mu)^2}. \tag{104}\]

Applying (10), we deduce that
\[|a_3| \leq \frac{|bx|}{6 - 4\mu} + \frac{b^2x^2}{(2 - \mu)^2}. \tag{105}\]

From (103), for \(v \in \mathbb{R}\), we write
\[a_3 - va_2^2 = \frac{h_2(x)(u_2 - v_2)}{4(3 - 2\mu)} + (1 - v)a_2^2. \tag{106}\]

By substituting (101) in (106), we have
\[a_3 - va_2^2 = \frac{h_2(x)(u_2 - v_2)}{4(3 - 2\mu)} + \frac{(1 - v)[h_2(x)]^2(u_2 + v_2)}{(\mu^2 - 3\mu + 4)[h_2(x)]^2 - 2h_3(x)(2 - \mu)^2}\]
\[= h_2(x)\left\{\Omega(v, x) + \frac{1}{(4 - 2\mu)}u_2 + \frac{\Omega(v, x) - \frac{1}{4(3 - 2\mu)}}{4(3 - 2\mu)}v_2\right\}, \tag{107}\]

where
\[\Omega(v, x) = \frac{(1 - v)[h_2(x)]^2}{(\mu^2 - 3\mu + 4)[h_2(x)]^2 - 2h_3(x)(2 - \mu)^2}. \tag{108}\]

Hence, we conclude that
\[|a_3 - va_2^2| \leq \left\{\frac{|h_2(x)|}{5 - 4\mu} 0 \leq |\Omega(v, x)| \leq \frac{1}{4(3 - 2\mu)} \right\} \tag{109}\]

which in view of (10) evidently completes the proof of Theorem 2.

In view of Remark 1, Theorem 3 yields.

**Corollary 6.** Let \(f(z) = z + \sum_{n=2}^{\infty}a_nz^n\) be in the class \(L_{\infty}(\mu, t)\). Then
\[|a_2| \leq \frac{2|\mu| \sqrt{2|\mu|}}{\sqrt{(2 - \mu)^2 - 2(\mu^2 - 5\mu + 12)|t|}}\tag{110}\]
and for \(v \in \mathbb{R}\)
\[|a_3 - va_2^2| \leq \left\{\frac{|h_2(x)|}{5 - 4\mu} \right\} \tag{111}\]

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that there have no conflicts of interest.

**Authors’ Contributions**

All authors contributed equally towards writing, reading, and approval of this manuscript.

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