Strong hypercontractivity and strong logarithmic Sobolev inequalities for log-subharmonic functions on stratified Lie groups

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Abstract

On a stratified Lie group $G$ equipped with hypoelliptic heat kernel measure, we study the behavior of the dilation semigroup on $L^p$ spaces of log-subharmonic functions. We consider a notion of strong hypercontractivity and a strong logarithmic Sobolev inequality, and show that these properties are equivalent for any group $G$. Moreover, if $G$ satisfies a classical logarithmic Sobolev inequality, then both properties hold. This extends similar results obtained by Graczyk, Kemp and Loeb in the Euclidean setting.

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1 Introduction

1.1 Background and motivation

The topic of this paper is inspired by two papers of P. Graczyk, T. Kemp, and J.-J. Loeb [14, 15], in which they introduced notions of strong hypercontractivity and a strong logarithmic Sobolev inequality for log-subharmonic functions on real Euclidean space equipped with an appropriate probability measure, and showed the intrinsic equivalence of these two notions. In the present paper, we extend their results to the setting of a stratified real Lie group equipped with hypoelliptic heat kernel measure, which we view in this context as a natural generalization of Euclidean space with Gaussian measure.

As motivation, we begin by recalling the classical notion of hypercontractivity and its relationship to the logarithmic Sobolev inequality. Let \( \mu \) be standard Gaussian measure on \( \mathbb{R}^n \), and let \( A \) be the self-adjoint Ornstein–Uhlenbeck operator on \( L^2(\mu) \) given by \( Af(x) = -\Delta f(x) + x \cdot \nabla f(x) \). (For this introduction, we will work formally and ignore domain considerations.)
Hypercontractivity is the statement that

\[ \| e^{-tA} \|_{L^q(\mu)} \leq \| f \|_{L^p(\mu)}, \quad t \geq t_N(p, q) := \frac{1}{2} \log \left( \frac{q-1}{p-1} \right), \]

\[ f \in L^p(\mu), \quad 1 < p \leq q < \infty. \] (1.1)

This result was proved by E. Nelson [31, 32] with improvements by J. Glimm [13]; see [19] for a broad historical survey of results in this area. Intuitively, (1.1) says that after a certain characteristic time \( t_N \), known as “Nelson’s time,” the Ornstein–Uhlenbeck semigroup \( e^{-tA} \) improves integrability from \( L^p(\mu) \) to \( L^q(\mu) \). The value of \( t_N \) given in (1.1) is sharp.

In the same context, the logarithmic Sobolev inequality, in its “\( L^1 \) form,” is the statement that

\[ \int f \log f \, d\mu \leq \frac{1}{2} \int \frac{|\nabla f|^2}{f} \, d\mu + \| f \|_{L^1(\mu)} \log \| f \|_{L^1(\mu)}, \]

\[ f \in C^1(\mathbb{R}^n), \quad f > 0 \] (1.2)

or equivalently, in the perhaps more familiar “\( L^2 \) form”,

\[ \int |f|^2 \log |f| \, d\mu \leq \int |\nabla f|^2 \, d\mu + \| f \|_{L^2(\mu)}^2 \log \| f \|_{L^2(\mu)}, \quad f \in C^1(\mathbb{R}^n) \] (1.3)

where the equivalence follows by replacing \( f \) by \( |f|^2 \) or vice versa. The earliest known version of this inequality is due to A.J. Stam [35], with another version discovered independently by P. Federbush [12]. The form given here was obtained by L. Gross [16], who coined the name. Gross also showed, at the level of Markovian semigroups, that (1.2) and (1.1) are equivalent. For instance, (1.3) can be formally obtained from (1.1) by setting \( p = 2, \quad q = 1 + e^{2t}, \) so that \( t(p, q) = t, \) and differentiating at \( t = 0. \)

In 1983, S. Janson [22] discovered a fascinating phenomenon of hypercontractivity in a complex setting. Consider (1.1) with \( n = 2 \) and identify \( \mathbb{R}^2 \) with \( \mathbb{C}. \) Janson showed that if we restrict the inequality (1.1) to the space \( \mathcal{H} \) of holomorphic functions, then we obtain the following improvement:

\[ \| e^{-tA} f \|_{L^q(\mu)} \leq \| f \|_{L^p(\mu)}, \quad t \geq t_J(p, q) := \frac{1}{2} \log \left( \frac{q}{p} \right), \]

\[ f \in \mathcal{H} \cap L^p(\mu), \quad 0 < p \leq q < \infty. \] (1.4)

In this result, the critical time \( t_J := \frac{1}{2} \log \left( \frac{q}{p} \right) \) (“Janson’s time”) is strictly smaller than Nelson’s time, so integrability improves faster if the initial
function $f$ is holomorphic. Moreover, Janson’s result has content even if $p = 1$ or $0 < p < 1$. Inequalities of this form have come to be called \textit{(complex) strong hypercontractivity}. For alternate proofs, extensions (including to $\mathbb{C}^n$), and related results, see [5, 23, 40].

Part of the reason for this strengthening in the holomorphic case is that, since holomorphic functions are harmonic, the action of $A$ on holomorphic functions reduces to that of the first-order operator $Ef(x) = x \cdot \nabla f(x)$, which is simply the generator of dilations on $\mathbb{C}^n = \mathbb{R}^{2n}$. This idea was pursued by Graczyk, Kemp, and Loeb in [14, 15], in which they chose to explicitly consider the behavior of the dilation semigroup $e^{-tE}$ on real Euclidean space $\mathbb{R}^n$. In this setting, the holomorphic functions are replaced by the log-subharmonic (LSH) functions; i.e. those nonnegative functions $f$ for which $\log f$ is subharmonic. (This is effectively a generalization: when $n$ is even and $f$ is holomorphic, then $|f|$ is log-subharmonic.) In the case of Gaussian measure $\mu$, they proved the following version of strong hypercontractivity:

$$
\|e^{-tE}f\|_{L^q(\mu)} \leq \|f\|_{L^p(\mu)}, \quad t \geq t_J(p, q),
$$

$$
f \in \text{LSH} \cap L^p(\mu), \quad 0 < p \leq q < \infty. \quad (1.5)
$$

They also obtained a corresponding \textit{strong logarithmic Sobolev inequality}:

$$
\int f \log f \, d\mu \leq \frac{1}{2} \int Ef \, d\mu + \|f\|_{L^1(\mu)} \log \|f\|_{L^1(\mu)}, \quad f \in \text{LSH}. \quad (1.6)
$$

The classical logarithmic Sobolev inequality (1.2) is a key ingredient in their proof; indeed, (1.2) implies (1.6) rather directly. More generally, Graczyk, Kemp and Loeb proved, for a wider class of measures $\mu$, that the statements (1.5) and (1.6)$^1$ are equivalent. For instance, as with (1.1) and (1.2), one may formally obtain (1.6) from (1.5) by taking $p = 1$, $q = e^{2t}$ and differentiating at $t = 0$.

In some cases, the hypothesis $f \in \text{LSH} \cap L^p(\mu)$ in (1.5) must be strengthened to $f \in \text{LSH} \cap L^q(\mu)$; they call this statement \textit{partial strong hypercontractivity}. We discuss this subtle issue in Remark 1.4, later in this paper.

Another line of research inspired by Janson’s strong hypercontractivity (1.4) was to study the phenomenon in non-Euclidean settings. In the papers [17, 18], L. Gross considered the case of a complex Riemannian manifold $M$ equipped with an arbitrary smooth probability measure $\mu$, where the Ornstein–Uhlenbeck operator $A$ is taken to be the generator of the Dirichlet

\[^1\text{Here, and for the rest of this section, the inequalities stated above should be read as including appropriate constants in the obvious places.}\]
form $E(f) = \int_M |\nabla f|^2 d\mu$. Gross showed, under certain assumptions, that if $(M, \mu)$ satisfies the logarithmic Sobolev inequality (1.2) then it satisfies the strong hypercontractivity property (1.4). In this context, it still happens that $A$ reduces, on holomorphic functions, to a first-order vector field, whose geometric and complex-analytic properties become crucially important.

In the paper [11], L. Gross, L. Saloff-Coste and the present author were interested in extending the complex Riemannian results of [17, 18] into a complex sub-Riemannian setting. We replaced the complex Riemannian manifold $M$ with a stratified complex Lie group $G$ equipped with a left-invariant sub-Riemannian geometry, taking the measure $\mu$ to be the hypoelliptic heat kernel associated to this geometry. A relevant feature of stratified Lie groups is that, like Euclidean space, they admit a canonical group of dilations. In this setting, the Ornstein–Uhlenbeck operator $A$ fails to be holomorphic, so we studied instead its $L^2(\mu)$-orthogonal projection $B$ onto the holomorphic functions, which, we showed, coincides with the vector field $E$ generating the dilations. We were able to show that, if the logarithmic Sobolev inequality (1.2) holds, then complex strong hypercontractivity (1.4) holds with the operator $B$ in place of $A$. Of course, in retrospect, this statement is really (1.5) for holomorphic functions.

We remark in passing that the logarithmic Sobolev inequality (1.2) is known to hold for a few stratified complex Lie groups (specifically, the complex Heisenberg–Weyl groups), but it is not currently known whether it holds for all of them.

The aim of the present paper is, in a sense, to unify [11] with [14, 15] by considering statements akin to (1.5) (in its “partial” form) and (1.6), in the setting of a real stratified Lie group $G$, again equipped with a left-invariant sub-Riemannian geometry and the associated hypoelliptic heat kernel measure. Our main Theorem 1.1 is, roughly, that (1.5) and (1.6) are equivalent in any stratified Lie group $G$, and if $G$ satisfies the logarithmic Sobolev inequality (1.2) then (1.5) and (1.6) are both true. Again, we stress that (1.2) is known to hold for some stratified Lie groups (specifically, the H-type groups), but it is not currently known whether it holds for all of them; see Remark 1.2 below.

In our view, stratified Lie groups are a natural setting in which to generalize (1.5), (1.6), since the dilation structure of a stratified Lie group is perhaps the most direct generalization of the dilation structure of Euclidean space. Rather than considering more general measures $\mu$ as in [14, 15], we have chosen to restrict our attention to the canonical hypoelliptic heat kernel measure: partly because it is the natural generalization of Gaussian measure in this setting, and partly because we need to make use of strong
heat kernel estimates from the literature (Theorem 2.14 below).

1.2 Statement of results

We briefly summarize the notation required to state our results. Complete definitions are given in Sections 2 and 3 below.

Let $G$ be a stratified Lie group equipped with a left-invariant sub-Riemannian metric $\langle \cdot, \cdot \rangle$, for which the horizontal space is given by the first layer of the stratification of the Lie algebra of $G$. Let $m$ be some normalization of Haar (Lebesgue) measure on $G$. We denote by $\nabla$ and $\Delta$ the canonical sub-gradient and sub-Laplacian induced by the metric, and by $\rho_s$ the hypoelliptic heat kernel for $\Delta$ at time $s > 0$. A function $f \in C^2(G)$ is said to be log-subharmonic (LSH) if $f > 0$ and $\Delta \log f \geq 0$ (we discuss alternative formulations in Section 8).

We denote by $E$ the vector field which generates the canonical dilations $\delta_r$ of the group $G$, and by $e^{-tE}f = f \circ \delta_{e^{-t}}$ the corresponding operator semigroup.

The $L^p$ norms and spaces in the following statements are taken with respect to the heat kernel probability measure $\rho_s \, dm$ at some fixed time $s$. We let $L^p = \bigcup_{q \geq p} L^q$, and write $f \in W^{1,p}$ if $f, |\nabla f| \in L^p$, where $|\cdot|$ is the norm induced by the metric $\langle \cdot, \cdot \rangle$.

The aim of this paper is to study the relationship between the following three statements, for fixed constants $c, \beta \geq 0$. The time parameter $s > 0$ may be taken as arbitrary; each of the following statements holds for one $s > 0$ iff it holds for all $s > 0$, with the same constants $c, \beta$ (see Remark 2.20 below).

- The classical logarithmic Sobolev inequality:

$$\int_G f \log f \, \rho_s \, dm \leq \frac{cs}{2} \int_G |\nabla f|^2 \rho_s \, dm + \|f\|_{L^1} \log \|f\|_{L^1} + \beta \|f\|_{L^1},$$

$$f \in C^1(G), f \geq 0 \quad \text{(LSI)}$$

We have stated this in its “$L^1$ form”. By replacing $f$ by $f^2$, one can see that (LSI) is equivalent to the “$L^2$ form”:

$$\int_G f^2 \log |f| \, \rho_s \, dm \leq cs \int_G |\nabla f|^2 \rho_s \, dm + \|f\|_{L^2}^2 \log \|f\|_{L^2} + \frac{\beta}{2} \|f\|_{L^2}^2,$$

$$f \in C^1(G) \quad \text{($L^2$-LSI)}$$
The original formulation (1.2) of the logarithmic Sobolev inequality corresponds to taking $\beta = 0$. When $\beta > 0$, (LSI) is sometimes referred to as a “defective logarithmic Sobolev inequality”.

As remarked in Section 1.1, (LSI) is well known to be equivalent to the hypercontractivity of the Ornstein–Uhlenbeck semigroup of $G$.

- The **strong logarithmic Sobolev inequality**:

$$
\int_G f \log f \rho_s \, dm \leq c \int_G E f \rho_s \, dm + \| f \|_{L^1} \log \| f \|_{L^1} + \beta \| f \|_{L^1},
$$

$$
f \in LSH \cap W^{1,1+} \quad (s\text{LSI})
$$

The name “strong logarithmic Sobolev inequality” comes from [14]. However, in our present context, we show in Theorem 7.1 that (LSI) implies (sLSI), so (sLSI) is in fact logically weaker. Of course, (sLSI) applies to a much smaller class of functions.

- (Partial) strong hypercontractivity:

$$
\| e^{-tE} f \|_{L^q} \leq M(p, q) \| f \|_{L^p}, \quad t \geq t_J(p, q),
$$

$$
f \in LSH \cap L^q, \ 0 < p \leq q < \infty \quad (s\text{HC})
$$

where

$$
M(p, q) := \exp(\beta \cdot (p^{-1} - q^{-1})), \quad t_J(p, q) := c \log \left( \frac{q}{p} \right). \quad (1.7)
$$

Here $t_J(p, q)$ is Janson’s time. Note that the word “contractivity” is more apt when $\beta = 0$, since in that case, $M(p, q) = 1$ and (sHC) says that $e^{-tE}$ is a contraction from a subset of $L^p$ into $L^q$. The word “partial” comes from [15] and refers to the hypothesis $f \in LSH \cap L^q$ (rather than $L^p$); see Remark 1.4 below.

In comparing these statements to [14, 15], note that our $c$ is their $\frac{c}{2}$.

The main results of this paper can be summarized as follows:

**Theorem 1.1.** In any stratified Lie group $G$, the statements (sLSI) and (sHC) are equivalent. If $G$ satisfies (LSI), then (sLSI) and (sHC) are both satisfied. That is,

$$
(\text{LSI}) \implies (\text{sLSI}) \iff (\text{sHC})
$$

The implication $(\text{LSI}) \implies (\text{sLSI})$ is Theorem 7.1; $(\text{sHC}) \implies (\text{sLSI})$ is Theorem 7.2; and $(\text{sLSI}) \implies (\text{sHC})$ is Theorem 7.6.
Remark 1.2. It is an open problem to determine which stratified Lie groups satisfy the logarithmic Sobolev inequality (LSI), and we hope this paper may provide additional motivation for further work on this difficult question. The current state of the art, as far as we are aware, is that (LSI) is true for H-type groups ([9, 21]; see Example 4.3 below for definitions and references), and of course in the “step 1” Euclidean case (Example 4.1). In all other stratified Lie groups, including all those of step \( \geq 3 \), it is apparently unknown whether (LSI) holds or not.

Corollary 1.3. If \( G \) is an H-type group, then (sLSI) and (sHC) are both true.

Remark 1.4. In the statement (sHC), the hypothesis \( f \in L^q \) may seem somewhat unnatural, given that the result is to bound the \( L^q \) norm by the \( L^p \) norm. It is reasonable to conjecture that if (sHC) holds for all \( f \in LSH \cap L^q \) then in fact it holds for all \( f \in LSH \cap L^p \). However, the obvious density argument is not available, because we do not know in this setting whether \( LSH \cap L^q(\rho_s) \) is dense in \( LSH \cap L^p(\rho_s) \). (Density arguments in stratified Lie groups can be subtle; for example, it is shown in [28, Proposition 8] that polynomials are dense in \( L^2(\rho_s) \) only for groups of step \( m \leq 4 \).) This issue arose, in the Euclidean setting, in the work of Graczyk, Kemp and Loeb [15]; our (sHC) is the analogue of the statement “partial strong hypercontractivity” appearing in their Theorem 1.17.1(b). The stronger statement, requiring only \( f \in LSH \cap L^p \) (in our notation), is their Theorem 1.17.1(a); but they are able to show this only under significantly stronger assumptions on the measure, one of which is that the measure be \( \alpha \)-subhomogeneous for some \( \alpha \leq 1/c \), where \( c \) is our constant in the strong logarithmic Sobolev inequality (sLSI) (recall that our \( c \) is their \( \frac{c}{2} \)). This approach does not appear to succeed in our stratified Lie group setting, given current technology. For instance, if we consider the Heisenberg group \( \mathbb{H}^3 \), we can use the heat kernel estimates of [8, 26] to see that the heat kernel is \( \alpha \)-subhomogeneous only for \( \alpha \geq 2 \). However, current methods for proving (LSI) in this setting produce a constant \( c > \frac{1}{2} \) (see [3, Sections 1.2 and 6.1] in conjunction with [7, Theorem 1.6]), although we do not know whether this is sharp. The situation for H-type groups is similar [8, 9, 21, 27].
2 Stratified Lie groups and hypoelliptic heat kernels

In this section, we review the standard definitions and properties of stratified Lie groups (also known as Carnot groups), and of the sub-Riemannian geometry and hypoelliptic heat kernels on these groups. The material in this section is adapted from [11]. Some motivating examples are discussed in Section 4.

2.1 Stratified Lie groups

A comprehensive reference on stratified Lie groups is [4].

Definition 2.1. Let \( \mathfrak{g} \) be a finite-dimensional real Lie algebra. We say \( \mathfrak{g} \) is stratified of step \( m \) if it admits a direct sum decomposition

\[
\mathfrak{g} = \bigoplus_{j=1}^{m} V_j
\]

and

\[
[V_1, V_j] = V_{j+1}, \quad [V_1, V_m] = 0.
\]

A finite-dimensional real Lie group \( G \) is stratified if it is connected and simply connected and its Lie algebra \( \mathfrak{g} \) is stratified.

Stratified Lie groups are also known as Carnot groups. Various equivalent definitions can be found in [4, Chapters 1 and 2].

As a trivial example, Euclidean space \( \mathbb{R}^n \) with its usual addition is a (commutative) stratified Lie group of step 1. (Here the Lie bracket is simply 0.) The simplest nontrivial example of a stratified Lie group is the Heisenberg group \( \mathbb{H}_3 \), which has step 2. See Section 4 below for further discussion of these and other examples.

It is easy to check that a stratified Lie group \( G \) is necessarily nilpotent, and thus diffeomorphic to its Lie algebra \( \mathfrak{g} \) via the exponential map. In particular, a stratified Lie group is diffeomorphic to Euclidean space \( \mathbb{R}^n \) as a smooth manifold (though certainly not isomorphic to \( \mathbb{R}^n \) as a Lie group).

Notation 2.2. Let \( L_x, R_x : G \to G \) denote the left and right translation maps \( L_x(y) = xy \) and \( R_x(y) = yx \).

Notation 2.3. Let \( e \) denote the identity element of \( G \). We identify the Lie algebra \( \mathfrak{g} \) with the tangent space \( T_e G \). For \( \xi \in \mathfrak{g} \), let \( \tilde{\xi}, \hat{\xi} \) denote, respectively, the unique left-invariant and right-invariant vector fields on \( G \) with \( \tilde{\xi}(e) = \hat{\xi}(e) = \xi \).
Notation 2.4. Being a connected nilpotent Lie group, $G$ is unimodular, so it has a bi-invariant Haar measure which is unique up to scaling. For our purposes, there is no particular natural choice of scaling, so from now on $m$ will denote some fixed Haar measure on $G$. Integrals like $\int_G f(x) \, dx$ will denote Lebesgue integrals with respect to $m$. It is easy to verify that the Haar measure on $G$ is the push-forward under the exponential map of Lebesgue measure on $g$.

Notation 2.5. Convolution on $G$ is defined by
\[ (f \ast g)(x) = \int_{G'} f(xy^{-1})g(y) \, dy = \int_{G'} f(z)g(z^{-1}x) \, dz \] (2.2)
when the integral exists.

Suppose $\varphi \in C^1_c(G)$, $f \in C^1(G)$ and $\xi \in g$. By using the formulas $\tilde{\xi} g(x) = \frac{d}{dt}|_{t=0} g(xe^{t\xi})$ and $\hat{\xi} g(x) = \frac{d}{dt}|_{t=0} g(e^{t\xi}x)$ and differentiating under the integral sign, we obtain the identities
\[ \tilde{\xi} [\varphi \ast f] = \varphi \ast (\tilde{\xi} f), \quad \hat{\xi} [\varphi \ast f] = (\hat{\xi} \varphi) \ast f. \] (2.3)

2.2 The dilation semigroup

Definition 2.6. For $\lambda \geq 0$, the dilation map $\delta_\lambda$ on $g$ is defined by
\[ \delta_\lambda(v_1 + \cdots + v_m) = \sum_{j=1}^{m} \lambda^j v_j \quad v_j \in V_j \quad j = 1, \ldots, m. \] (2.4)

By an abuse of notation, we will also use $\delta_\lambda$ to denote the corresponding map on $G$ defined by $\delta_\lambda(\exp(v)) = \exp(\delta_\lambda(v))$.

It is straightforward to verify that for each $\lambda > 0$, the dilation $\delta_\lambda$ on $g$ is an automorphism of the Lie algebra, and the dilation $\delta_\lambda$ on $G$ is an automorphism of the Lie group. Also,
\[ \delta_{\lambda \mu} = \delta_\lambda \circ \delta_\mu \quad \lambda, \mu \geq 0. \] (2.5)

Moreover, the derivative at the identity of $\delta_\lambda : G \to G$ is $(\delta_\lambda)_e = \delta_\lambda : g \to g$.

Since $\delta_\lambda$ is a group automorphism, we have the identity
\[ \delta_\lambda \circ L_x = L_{\delta_\lambda(x)} \circ \delta_\lambda. \]

Hence if $\xi \in g = T_e G$ and $\tilde{\xi}$ is the corresponding left-invariant vector field, we have
\[
(\delta_\lambda)_* \tilde{\xi}(x) = (\delta_\lambda)_*(L_x)_* \xi = (\delta_\lambda L_x)_* \xi = (L_{\delta_\lambda(x)} \delta_\lambda)_* \xi = (L_{\delta_\lambda(x)} \delta_\lambda)_* \xi.
\]
In particular, if $\xi \in V_j$, then $(\delta_\lambda)_* \xi = \lambda^j \xi$ and so
\[
(\delta_\lambda)_* \tilde{\xi}(x) = \lambda^j (L_{\delta_\lambda(x)})_* \xi = \lambda^j \tilde{\xi}(\delta_\lambda(x))
\]
(2.6)
or in other words
\[
\tilde{\xi}(f \circ \delta_\lambda) = \lambda^j (\tilde{\xi}f) \circ \delta_\lambda.
\]
(2.7)

The dilation structure is a fundamental property of stratified Lie groups, and since the aim of this paper is to generalize results on the dilation in Euclidean space, stratified Lie groups are a natural setting to consider. Indeed, in a certain sense, we are studying what happens if we are allowed to dilate at different rates in different directions (linearly in $V_1$ directions, quadratically in $V_2$ directions, and so on).

**Definition 2.7.** We define the **dilation vector field** or **Euler vector field** $E$ on $G$ by
\[
(Ex)(x) = \frac{d}{dr} \bigg|_{r=0} f(\delta_{e^r}(x)) \quad f \in C^1(G).
\]
(2.8)

The main object of study in this paper is the one-parameter semigroup of dilation operators
\[
e^{-tE}f = f \circ \delta_{e^{-t}}, \quad t \geq 0.
\]
(2.9)

**Notation 2.8.** Let $\xi_{j,k}$ be a basis for $\mathfrak{g} = G$ adapted to the stratification $\{V_j\}$, so that $\{\xi_{j,k} : 1 \leq k \leq \dim V_j\}$ is a basis for $V_j$. Then each $x \in G$ can be written uniquely as $x = \exp \left( \sum_{j=1}^m \sum_{k=1}^{\dim V_j} x_{j,k} \xi_{j,k} \right)$, so that $x_{j,k}$ is a smooth system of coordinates on $G$.

In this system of coordinates, we have
\[
E = \sum_{j=1}^m \sum_{k=1}^{\dim V_j} jx_{j,k} \frac{\partial}{\partial x_{j,k}}.
\]
(2.10)

In the Euclidean case $m = 1$, we have $E = \sum x_k \frac{\partial}{\partial x_k} = x \cdot \nabla$, a vector field pointing radially away from the origin with magnitude $|x|$.

**Notation 2.9.** The **homogeneous dimension** of $\mathfrak{g}$ or $G$ is
\[
D = \sum_{j=1}^m j \dim V_j.
\]
We note that $\delta_\lambda$ scales the Lebesgue measure $m$ by
\[ m(\delta_\lambda(A)) = \lambda^D m(A). \] (2.11)
Thus for an integrable function $f$, we have
\[ \int_G f \circ \delta_\lambda \, dm = \lambda^{-D} \int_G f \, dm. \] (2.12)

To conclude this subsection, we observe that the vector fields discussed above can be expressed in terms of each other in a well-behaved manner.

Define the adapted basis $\{\xi_{j,k}\}$ for $g$ and the coordinates $\{x_{j,k}\}$ on $G$ as in Notation 2.8. For each $j,k$, the vector fields $\frac{\partial}{\partial x_{j,k}}, \tilde{\xi}_{j,k}, \hat{\xi}_{j,k}$ (see Notation 2.3) coincide at the identity but in general nowhere else.

**Lemma 2.10.** We can write
\[ \tilde{\xi}_{j,k} = \frac{\partial}{\partial x_{j,k}} + \sum_{\alpha=j+1}^m \sum_{\beta=1}^{\dim V_\alpha} a_{j,k}^{\alpha,\beta} \frac{\partial}{\partial x_{\alpha,\beta}} \] (2.13)
and
\[ \frac{\partial}{\partial x_{j,k}} = \tilde{\xi}_{j,k} + \sum_{\alpha=j+1}^m \sum_{\beta=1}^{\dim V_\alpha} b_{j,k}^{\alpha,\beta} \tilde{\xi}_{j,k} \] (2.14)
where the coefficient functions $a_{j,k}^{\alpha,\beta}, b_{j,k}^{\alpha,\beta}$ are polynomials in the coordinates $x_{\alpha,\beta}$.

We can likewise express $\{\hat{\xi}_{j,k}\}$ and $\{\frac{\partial}{\partial x_{j,k}}\}$ in terms of each other, with polynomial coefficients, as well as $\{\tilde{\xi}_{j,k}\}$ and $\{\hat{\xi}_{j,k}\}$.

**Proof.** By the Baker–Campbell–Hausdorff formula, in the coordinates $\{x_{j,k}\}$, the group operation on $G$ has the form
\[ (xy)_{j,k} = x_{j,k} + y_{j,k} + R_{j,k}(x,y) \]
where $R_{j,k}$ is a polynomial which only depends on the coordinates $x_{\alpha,\beta}, y_{\alpha,\beta}$ with $\alpha < j$. See [4, Proposition 2.2.22 (4)] for details. Then (2.13) follows immediately, since $\tilde{\xi}_{j,k}(x) = \left. \frac{d}{dt} \right|_{t=0} x \exp(t \xi_{j,k})$. We then obtain (2.14) by solving the system (2.13) for $\frac{\partial}{\partial x_{j,k}}$. (Note, for instance, that from (2.13) we have $\tilde{\xi}_{m,k} = \frac{\partial}{\partial x_{m,k}}$, so that (2.14) is trivially satisfied when $j = m$. One can then proceed by downward induction on $j$.)
An identical argument applies to \( \{ \widehat{\xi}_{j,k} \} \) and \( \{ \partial_{x_{j,k}} \} \). To write \( \tilde{\xi}_{j,k} \) in terms of \( \{ \widehat{\xi}_{j,k} \} \), first write \( \{ \widehat{\xi}_{j,k} \} \) in terms of \( \{ \partial_{x_{j,k}} \} \) as just noted, and then substitute this into (2.13).

\[ \text{Corollary 2.11} \quad \text{(See also [28, Lemma 4]).} \quad \text{We can write} \]

\[ E = \sum_{j,k} \hat{c}_{j,k} \hat{\xi}_{j,k} = \sum_{j,k} \hat{c}_{j,k} \hat{\xi}_{j,k} \quad (2.15) \]

where the coefficient functions \( \hat{c}_{j,k} \), \( \hat{c}_{j,k} \) are polynomials in the \( x_{\alpha,\beta} \) coordinates.

**Proof.** Substitute (2.14) into (2.10).

\[ \text{\square} \]

### 2.3 Sub-Riemannian geometry on \( G \)

In this subsection, we review some facts about the sub-Riemannian geometry of a stratified Lie group \( G \) and its hypoelliptic sub-Laplacian. For background on the general notions of sub-Riemannian geometry, see [29, 33, 36, 37].

Fix an inner product \( \langle \cdot, \cdot \rangle \) on \( V_1 \subset \mathfrak{g} \). From now on, when we speak of a stratified Lie group \( G \), we really mean a triple \( (G, \langle \cdot, \cdot \rangle, m) \), including a choice of inner product on \( V_1 \) and a choice of normalization for the Haar measure. Objects such as the sub-Laplacian, heat kernel, etc, which we discuss below, are not really intrinsic to the Lie group \( G \), but depend on the choice of \( \langle \cdot, \cdot \rangle \) and \( m \).

The inner product gives rise to a **sub-Riemannian geometry** on the smooth manifold \( G \) in the following way. For \( x \in G \), let \( H_x = (L_x)_* V_1 \subset T_x G \), so that \( H \) is a left-invariant subbundle of the tangent bundle \( TG \). This \( H \) is called the **horizontal bundle** or **horizontal distribution**. Then the inner product \( \langle \cdot, \cdot \rangle \) on \( V_1 \) induces, by left translation, an inner product \( \langle \cdot, \cdot \rangle_x \) on \( H_x \), which is a left-invariant **sub-Riemannian metric** on \( G \). We may drop the subscript \( x \) when no confusion will arise. Since \( V_1 \) generates \( \mathfrak{g} \), the horizontal bundle \( H \) satisfies Hörmander’s bracket generating condition. We will use \( | \cdot | \) to denote the norm on \( H_x \) induced by \( \langle \cdot, \cdot \rangle \).

This metric gives rise to a canonical left-invariant **sub-Laplacian** \( \Delta \) on \( G \), which is easiest to define in terms of a basis. Let \( \xi_1, \ldots, \xi_n \) be an orthonormal basis for \( V_1 \), and let

\[ \Delta = \xi_1^2 + \cdots + \xi_n^2 \quad (2.16) \]
where, as in Notation 2.3, $\tilde{\xi}_i$ is the extension of $\xi_i$ to a left-invariant vector field on $G$. It is easy to check this definition is independent of the basis chosen. Since $H$ satisfies the bracket generating condition, the operator $\Delta$ is hypoelliptic [20]. It is shown in [39] that $\Delta$, with domain $C^\infty_c(G)$, is essentially self-adjoint on $L^2(G,m)$.

As a consequence of (2.7), we have

$$\Delta[f \circ \delta_\lambda] = \lambda^2(\Delta f) \circ \delta_\lambda.$$  \hbox{(2.17)}

Likewise, if $e^{s\Delta/4}$ is the heat semigroup for $\Delta$, we have

$$e^{s\Delta/4}[f \circ \delta_\lambda] = (e^{s\lambda^2\Delta/4} f) \circ \delta_\lambda.$$  \hbox{(2.18)}

Much more information about the sub-Laplacian can be found in [4]. We may also define the sub-gradient $\nabla$ by

$$\nabla f(x) = \sum_{i=1}^n (\tilde{\xi}_i f)(x)\tilde{\xi}_i(x) \in H_x, \quad f \in C^1(G).$$  \hbox{(2.19)}

This too is well-defined independent of the chosen basis.

Finally, let $d$ be the Carnot–Carathéodory distance induced by the sub-Riemannian metric (see [4, Section 5.2]). Intuitively, $d(x,y)$ is the length of the shortest horizontal path joining $x$ and $y$. The Chow–Rashevskii and ball-box theorems [29, 30] imply that $d(x,y)$ is finite and that $d$ is a metric which induces the manifold topology on $G$ (which in turn is just the Euclidean topology on the finite-dimensional vector space $G = g$). A straightforward computation shows that $d$ is left-invariant with respect to the group structure on $G$:

$$d(x,y) = d(zx,zy), \quad x, y, z \in G$$  \hbox{(2.20)}

and invariant with respect to the inverse:

$$d(e,x^{-1}) = d(e,x)$$  \hbox{(2.21)}

and also homogeneous with respect to the dilation $\delta_\lambda$:

$$d(\delta_\lambda(x), \delta_\lambda(y)) = \lambda d(x,y).$$  \hbox{(2.22)}

See [4, Propositions 5.2.4 and 5.2.6] for details.
2.4 Properties of the heat kernel

It is shown in [39] that the Markovian heat semigroup $e^{s \Delta/4}$ admits a right convolution kernel $\rho_s$, i.e. $e^{s \Delta/4} f = f * \rho_s$; it is also shown that $\rho_s$ is $C^\infty$ and strictly positive. This function $\rho_s$ is the (hypoelliptic) heat kernel associated to $(G, \langle \cdot, \cdot \rangle, m)$. Since $e^{s \Delta/4}$ is Markovian, the heat kernel measure $\rho_s \, dm$ is a probability measure. In this subsection, we collect several properties of the heat kernel from the literature.

**Notation 2.12.** For $s > 0$ and $0 < p \leq \infty$, we write $L^p(\rho_s)$ as short for $L^p(G, \rho_s \, dm)$. Let

$$L^p^+(\rho_s) := \bigcup_{q > p} L^q(\rho_s)$$

$$L^p^-(\rho_s) := \bigcap_{q < p} L^q(\rho_s).$$

We will say $f_n \to f$ in $L^p^+(\rho_s)$ (respectively, in $L^p^-(\rho_s)$) if $f_n \to f$ in $L^q(\rho_s)$ for some $q > p$ (respectively, for all $q < p$). Also, $W^{1,p^+}(\rho_s)$ will denote the space of functions $f$ with $f, |\nabla f| \in L^{p^+}(\rho_s)$. We shall only have occasion to deal with $C^1$ functions in $W^{1,p^+}$, so we do not discuss weak derivatives here.

Notice that $L^p^+(\rho_s)$ and $L^p^-(\rho_s)$ are vector spaces, and $L^{\infty^-}(\rho_s)$ is an algebra. By Hölder’s inequality, if $f \in L^p^+(\rho_s)$ and $g \in L^{\infty^-}(\rho_s)$ then $fg \in L^p^+(\rho_s)$. We also note that if $f \in L^1(\rho_s)$ is positive and bounded away from zero then $\log f \in L^{\infty^-}(\rho_s)$.

**Lemma 2.13.** The heat kernel $\rho_s$ is invariant under the group inverse operation: we have $\rho_s(x) = \rho_s(x^{-1})$.

**Proof.** See [34, Theorem III.2.1 (4)] or the discussion in [6, Proposition 3.1 (3)].

We shall need to make use of sharp upper and lower estimates for the heat kernel.

**Theorem 2.14.** For each $0 < \epsilon < 1$ there are constants $C(\epsilon), C'(\epsilon)$ such that for every $x \in G$ and $s > 0$,

$$\frac{C(\epsilon)}{m(B(e, \sqrt{s}))} e^{-d(e,x)^2/(1-\epsilon)s} \leq \rho_s(x) \leq \frac{C'(\epsilon)}{m(B(e, \sqrt{s}))} e^{-d(e,x)^2/(1+\epsilon)s} \quad (2.23)$$

where $m(B(e, \sqrt{s}))$ is the Lebesgue (Haar) measure of the $d$-ball centered at the identity (or any other point of $G$) with radius $\sqrt{s}$. 

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Proof. The upper bound is Theorem IV.4.2 of [39]. The lower bound is Theorem 1 of [38]. Note that our choice to consider the semigroup $e^{s\Delta/4}$ rather than $e^{s\Delta}$ accounts for a missing factor of 4 in the exponents compared to the results stated in [38, 39].

Corollary 2.15. Any polynomial $p$ in the coordinates $x_{j,k}$ (see Notation 2.8) is in $L^{\infty-}(\rho_s)$.

Proof. It suffices to consider $p(x) = x_{j,k}^r$ for some fixed $j, k, r$. Let $S = \{x : d(e, x) = 1\}$ be the unit sphere of $d$, and let $C = \sup_S |p|$ which is finite by the compactness of $S$. Then the inequality $|p(x)| \leq Cd(e, x)^{rj}$ holds trivially on $S$. The scaling relations $p(\delta_\lambda(x)) = \lambda^{rj}p(x)$ and (2.22) now imply that $|p(x)| \leq Cd(e, x)^{rj}$ for all $x \in G$. The result now follows, via massive overkill, from the upper bound in Theorem 2.14. □

Corollary 2.16. If $|\nabla f| \in L^{p+}(\rho_s)$ then $Ef \in L^{p+}(\rho_s)$.

Proof. Combine Corollary 2.11 with Corollary 2.15. □

Lemma 2.17. (Special case of [39, Theorem IV.3.1]) Let $r \geq 0$ and $0 < s < t < \infty$. There is a constant $C$, depending on $r, s, t$, such that for all $y \in G$,

$$\sup_{d(e, x) \leq r} \rho_s(xy) \leq C \rho_t(y).$$

Proof. Replacing $x, y$ by $x^{-1}, y^{-1}$ and using Lemma 2.13 and (2.21), it is enough to show the result for $\rho_s(yx)$ instead of $\rho_s(xy)$.

Let $x \in B(e, r)$ be arbitrary. By the bounds in Theorem 2.14, we have

$$\frac{\rho_s(yx)}{\rho_t(y)} \leq C''(s, t, \epsilon) \exp \left( - \frac{d(e, xy)^2}{(1+\epsilon)s} - \frac{d(e, y)^2}{(1-\epsilon)t} \right).$$

By the left invariance of the distance $d$ and the triangle inequality, we have $d(e, yx) \geq d(e, y) - r$, which yields

$$\frac{\rho_s(yx)}{\rho_t(y)} \leq C''(s, t, \epsilon) \exp \left( -d(e, y)^2 \left( \frac{1}{(1+\epsilon)s} - \frac{1}{(1-\epsilon)t} \right) + \frac{2rd(e, y)}{(1+\epsilon)s} - \frac{r^2}{(1+\epsilon)s} \right) \leq C''(s, t, \epsilon) \exp \left( -d(e, y)^2 \left( \frac{1}{(1+\epsilon)s} - \frac{1}{(1-\epsilon)t} \right) + \frac{2rd(e, y)}{(1+\epsilon)s} \right).$$

Now since $s < t$, we can take $\epsilon$ sufficiently small that $\frac{1}{(1+\epsilon)s} - \frac{1}{(1-\epsilon)t} > 0$. If we now choose some $r_0$ with

$$r_0 > \left( \frac{1}{(1+\epsilon)s} - \frac{1}{(1-\epsilon)t} \right)^{-1} \frac{2r}{(1+\epsilon)s},$$

then...
then for all $y$ with $d(e, y) \geq r_0$, the exponent is negative and we have $\frac{\rho_s(yx)}{\rho(y)} \leq C''(s, t, \epsilon)$. This suffices, since by continuity the supremum over all $y \in B(e, r_0)$ is finite.

\[ \square \]

**Lemma 2.18.** Let $s > 0$, $p > 1$ and $0 < t_0 \leq t_1 < \frac{p}{p-1} s$. Then

$$\sup_{t \in [t_0, t_1]} \frac{\rho_t}{\rho_s} \in L^p(\rho_s).$$

**Proof.** For any $t \in [t_0, t_1]$ we have by Theorem 2.14 that

$$\frac{\rho_t(x)^p}{\rho_s(x)^{p-1}} \leq \frac{C''(\epsilon)^p m(B(e, \sqrt{s}))^{p-1}}{C(\epsilon)^{p-1} m(B(e, \sqrt{t}))^p} \exp \left( - \left( \frac{p}{(1+\epsilon)t} - \frac{p-1}{(1-\epsilon)s} \right) d(e, x)^2 \right).$$

The right side is independent of $t$, and will be integrable on $G$ (with respect to $m$) provided that $\frac{p}{(1+\epsilon)t_1} - \frac{p-1}{(1-\epsilon)s} > 0$. But since

$$\lim_{\epsilon \to 0} \frac{p}{(1+\epsilon)t_1} - \frac{p-1}{(1-\epsilon)s} = \frac{p}{t_1} - \frac{p-1}{s} > 0$$

we can choose $\epsilon$ sufficiently small that this coefficient is indeed positive. \[ \square \]

**Lemma 2.19.** The heat kernel $\rho_s$ obeys the scaling relation

$$\rho_s(\delta_r(y)) = |\lambda|^{-D} \rho_s(\lambda^{-2}(y)). \quad (2.24)$$

**Proof.** This follows from the corresponding scaling properties of the semigroup $e^{s\Delta/4}$ (2.18) and of the Haar measure $m$ (2.11).

**Remark 2.20.** Using (2.7) and (2.24), one may verify, by replacing $f$ by an appropriate dilation $f \circ \delta_r$, that each of the statements (LSI), (sLSI), (sHC) in our main theorem holds for one $s > 0$ iff it holds for all $s > 0$, with the same constants $c, \beta$.

**Lemma 2.21.** Suppose $f \in C^2(G) \cap L^1(\rho_s)$ and $E f, |\nabla f|, \Delta f \in L^1(\rho_s)$. Then

$$\int_G E f \rho_s \, dm = \frac{s}{2} \int_G \Delta f \rho_s \, dm. \quad (2.25)$$

Moreover, the same result holds if we assume $\Delta f \geq 0$ instead of $\Delta f \in L^1(\rho_s)$.
Proof. Suppose first that \( f \in C_c^\infty(G) \). By (2.18) we have
\[
\int_G f \circ \delta_e \rho_s \, dm = \int_G f \rho_s e^x \, dm.
\]
Differentiating under the integral sign at \( r = 0 \), we obtain
\[
\int_G Ef \rho_s \, dm = 2s \int_G f \frac{d}{ds} \rho_s \, dm = \frac{s}{2} \int_G f \Delta \rho_s \, dm = \frac{s}{2} \int_G \Delta f \rho_s \, dm.
\]
Now to show the general case, let \( \phi \in C_c^\infty(G) \) be a cutoff function with \( \phi = 1 \) on a neighborhood of the identity \( e \), and set \( \phi_n = \phi \circ \delta_{1/n} \). Then it is easy to check that
\[
\nabla \phi_n = \frac{1}{n} (\nabla \phi) \circ \delta_{1/n}, \quad \Delta \phi_n = \frac{1}{n^2} (\Delta \phi) \circ \delta_{1/n}, \quad E\phi_n = (E \phi) \circ \delta_{1/n}
\]
Hence we have, pointwise and boundedly,
\[
\phi_n \to 1, \quad |\nabla \phi_n| \to 0, \quad \Delta \phi_n \to 0, \quad E\phi_n \to 0,
\]
the last following from the fact that \( E\phi = 0 \) on a neighborhood of \( e \). Applying our result to \( \phi_n f \), we have
\[
\int_G E\phi_n \cdot f \rho_s \, dm + \int_G \phi_n \cdot Ef \rho_s \, dm
\]
\[
= \int_G \Delta \phi_n \cdot f \rho_s \, dm + \int_G g(\nabla \phi_n, \nabla f) \rho_s \, dm + \int_G \phi_n \cdot \Delta f \rho_s \, dm.
\]
By dominated convergence, using the integrability assumptions on \( f \) and its derivatives, letting \( n \to \infty \) gives the desired identity.

If we only assume \( \Delta f \geq 0 \), then if we choose \( \phi_n \) with a little more care, we can get \( \phi_n \to 1 \) monotonically. Then we can repeat the argument above, in which we have \( \int g(\phi_n, \nabla f) \rho_s \, dm \to \int \Delta f \rho_s \, dm \) by monotone convergence instead of dominated convergence.

3 Log-subharmonic functions

In general, a function \( f : G \to [0, \infty) \) on \( G \) is said to be log-subharmonic (LSH) if \( \log f \) is subharmonic with respect to the sub-Laplacian \( \Delta \). There are many possible notions of subharmonicity in this setting. In this paper, we shall work primarily with a strong "classical" notion of subharmonicity, in order to avoid obscuring the main ideas with technicalities; but see Section 8 below, where we discuss how the results of this paper can be applied to functions which are log-subharmonic in a weaker sense.
Definition 3.1. Suppose \( f \in C^2(G) \). We will say \( f \) is \textbf{subharmonic} if \( \Delta f \geq 0 \). We will say \( f \) is \textbf{log-subharmonic (LSH)} if \( f > 0 \) and \( \Delta \log f \geq 0 \).

Lemma 3.2. If \( f \in C^2(G) \) and \( f > 0 \), then \( f \) is LSH if and only if
\[
\Delta f \geq \frac{|\nabla f|^2}{f}. \tag{3.1}
\]
In particular, LSH functions are subharmonic.

Proof. By the chain and product rules,
\[
\Delta \log(f) = -\frac{|\nabla f|^2}{f^2} + \frac{\Delta f}{f} = \frac{1}{f} \left( -\frac{|\nabla f|^2}{f} + \Delta f \right)
\]
so that \( \Delta \log(f) \geq 0 \) iff \( \Delta f \geq \frac{|\nabla f|^2}{f} \). \( \square \)

Proposition 3.3. Suppose \( f, g \) are LSH. The following functions are LSH:

1. Positive constants
2. \( fg \)
3. \( f^p \) for any \( p > 0 \)
4. \( f + g \)
5. \( f \circ \delta_{\lambda} \) for any \( \lambda > 0 \)

Proof. Items 1–3 are immediate.

For item 4, we use a trick suggested in [14, Proposition 2.2]. We have that \( u = \log f \) and \( v = \log g \) are subharmonic. Fix \( x \in G \) and assume without loss of generality that \( \Delta u(x) \geq \Delta v(x) \). Now
\[
\Delta \log(f + g) = \Delta \log(e^u + e^v) = \Delta [v + \log(e^{u-v} + 1)] \geq \Delta \log(e^{u-v} + 1).
\]
Let \( \psi(t) = \log(e^t + 1) \) and note that \( \psi'(t) = \frac{e^t}{1+e^t} > 0 \) and \( \psi''(t) = \frac{e^t (1+e^t)^2}{(1+e^t)^2} > 0 \). By the chain and product rules, we have
\[
\Delta \log(e^{u-v} + 1) = \Delta \psi(u - v) = \psi''(u - v)|\nabla [u - v]|^2 + \psi'(u - v)\Delta [u - v].
\]
This is nonnegative at \( x \) since by assumption \( \Delta u(x) \geq \Delta v(x) \).

Item 5 is an immediate consequence of (2.17) which implies that
\[
\Delta \log(f \circ \delta_{\lambda}) = \lambda^2 (\Delta \log(f)) \circ \delta_{\lambda}.
\]
\( \square \)
Lemma 3.4. If \( f \) is LSH and \( \varphi \in C^\infty_c(G) \) is nonnegative then \( \varphi \ast f \) is LSH.

Proof. We will show that \( \varphi \ast f \) satisfies (3.1). By rescaling, let us suppose without loss of generality that \( \int_G \varphi \, dm = 1 \). Fix an orthonormal basis \( \xi_1, \ldots, \xi_n \) for \( V_1 \). From (2.16) and (2.3), we have 
\[
\Delta [\varphi \ast f] = \varphi \ast \Delta f \geq \varphi \ast \frac{|\nabla f|^2}{f} = \sum_{i=1}^n \varphi \ast \frac{\tilde{\xi}_i f}{f}.
\]

Applying the multivariate Jensen inequality with the convex function \( \psi(u, v) = u^2/v \) and the probability measure \( \varphi \, dm \), we have 
\[
\left( \frac{\varphi \ast \frac{\tilde{\xi}_i f}{f}}{f} \right)(x) = \int_G \varphi(y) \frac{\tilde{\xi}_i f(y^{-1} x)}{f(y^{-1} x)} \, dy \geq \left( \frac{\int_G \varphi(y) \tilde{\xi}_i f(y^{-1} x) \, dy}{\int_G \varphi(y) f(y^{-1} x) \, dy} \right)^2 \varphi \ast f(x) = \frac{\tilde{\xi}_i [\varphi \ast f](x)}{\varphi \ast f}(x)
\]
using (2.3) again, since \( \tilde{\xi}_i \) is left-invariant. Thus we have 
\[
\Delta [\varphi \ast f] \geq \sum_{i=1}^n \frac{\tilde{\xi}_i [\varphi \ast f]^2}{\varphi \ast f} \geq \frac{|\nabla (\varphi \ast f)|^2}{\varphi \ast f}
\]
and so by Lemma 3.2, \( \varphi \ast f \) is LSH. \( \square \)

4 Examples and special cases

4.1 Euclidean space

Example 4.1. As a trivial example, \( G = \mathbb{R}^n \) with Euclidean addition is an (abelian) stratified Lie group of step 1. (Indeed, these are all the step 1 stratified Lie groups.) Here the Lie bracket is zero and the dilation is \( \delta_\lambda(x) = \lambda x \). If we equip \( V_1 = \mathfrak{g} = \mathbb{R}^n \) with the Euclidean inner product, then the sub-Laplacian and sub-gradient are the usual Euclidean Laplacian and gradient, and the Carnot–Carathéodory distance \( d \) is Euclidean distance.
The heat kernel $\rho_s$ is the Gaussian density, appropriately scaled. (Note that, in our normalization, standard Gaussian density corresponds to $s = 2$.)

As such, the results of this paper include statements about Gaussian measure on Euclidean space, similar to those obtained in [14, 15]. It is well known that (LSI) is true for Gaussian measures [16], with constant $c = \frac{1}{2}$.

### 4.2 The Heisenberg group

**Example 4.2.** The simplest nontrivial example of a stratified Lie group is the 3-dimensional real **Heisenberg group** $G = \mathbb{H}^3$, which we may realize as $\mathbb{R}^3$ equipped with the group operation

\[
(x_1, x_2, x_3)(x'_1, x'_2, x'_3) = \left( x_1 + x'_1, x_2 + x'_2, x_3 + x'_3 + \frac{1}{2}(x_1 x'_2 - x_2 x'_1) \right).
\]

If we let $\xi_i = \frac{\partial}{\partial x_i} \in \mathfrak{g} = T_e G$ for $i = 1, 2, 3$, the Lie bracket is given by

\[
[\xi_1, \xi_2] = \xi_3, \quad [\xi_1, \xi_3] = [\xi_2, \xi_3] = 0
\]

so we have the decomposition $\mathfrak{g} = V_1 \oplus V_2$ where $V_1 = \text{span}\{\xi_1, \xi_2\}$, $V_2 = \text{span}\{\xi_3\}$. Thus the Heisenberg group is stratified of step 2. A natural inner product $\langle \cdot, \cdot \rangle$ on $V_1$ is given by taking $\xi_1, \xi_2$ to be orthonormal.

The corresponding left-invariant vector fields are given by

\[
\tilde{\xi}_1 = \frac{\partial}{\partial x_1} - \frac{1}{2} x_2 \frac{\partial}{\partial x_3}, \quad \tilde{\xi}_2 = \frac{\partial}{\partial x_2} + \frac{1}{2} x_1 \frac{\partial}{\partial x_3}, \quad \tilde{\xi}_3 = \frac{\partial}{\partial x_3}.
\]

The Heisenberg group $\mathbb{H}^3$ was the first nontrivial stratified Lie group that was shown to satisfy the logarithmic Sobolev inequality (LSI). This statement can be found in [3] and follows from heat semigroup gradient bounds previously established in [25], via a variant of a standard $\Gamma_2$-calculus argument from [2] or [1, pp. 69–70]. A key ingredient is sharp upper and lower heat kernel estimates, obtained in [26]. As such, our Theorem 1.1 implies that (sLSI) and (sHC) are satisfied by $\mathbb{H}^3$ as well.

The Heisenberg group construction immediately generalizes to the family of **Heisenberg–Weyl groups** $\mathbb{H}^{2n+1}$, which is realized as $\mathbb{R}^{2n+1}$ with a group operation defined again by (4.1), where now we take $x_1, x_2 \in \mathbb{R}^n$.

The Heisenberg and Heisenberg–Weyl groups are examples of H-type groups, which we discuss next.
4.3 H-type groups

Example 4.3. Suppose that \( G \) is a (real) stratified Lie group of step 2. Let the inner product \( \langle \cdot, \cdot \rangle \) on \( V_1 \) be extended to an inner product on all of \( \mathfrak{g} = V_1 \oplus V_2 \), still denoted by \( \langle \cdot, \cdot \rangle \), for which the decomposition \( \mathfrak{g} = V_1 \oplus V_2 \) is orthogonal. For each \( z \in V_2 \), define the linear map \( J_z : V_1 \to V_1 \) by \( \langle J_z v, w \rangle = \langle z, [v, w] \rangle \). We say that \((G, \langle \cdot, \cdot \rangle)\) is **H-type** if, for each \( z \in V_2 \) with \( \langle z, z \rangle = 1 \), the map \( J_z \) is a partial isometry. H-type groups were introduced in [24]; see [4, Chapter 18] for more background on these groups. The Heisenberg and Heisenberg–Weyl groups discussed in Example 4.2 are H-type (indeed, the H stands for Heisenberg).

H-type groups satisfy the same type of heat semigroup gradient bounds as the Heisenberg group \( \mathbb{H}^3 \). This was shown independently in [9, 21]; for the required heat kernel estimates, see [8, 27]. Thus, such groups satisfy (LSI) as well, by the same general argument given in [3]. As we noted in Corollary 1.3, our Theorem 1.1 then implies that (sLSI) and (sHC) are also true in H-type groups. We do not know of any further examples of stratified Lie groups where (LSI) has been proved.

4.4 Complex stratified Lie groups

Example 4.4. Suppose that \( G \) is a stratified Lie group which is also a complex Lie group, so that the Lie algebra \( \mathfrak{g} \) admits a complex structure \( J : \mathfrak{g} \to \mathfrak{g} \) satisfying \( [Jv, w] = J[v, w] \). Then \( \mathfrak{g} \) is a complex vector space and the subspaces \( V_i \) in the decomposition (2.1) are complex vector spaces as well. The complex structure on \( \mathfrak{g} \) induces a complex manifold structure on \( G \) for which the exponential map is holomorphic. In this setting, it is natural to ask that the inner product \( \langle \cdot, \cdot \rangle \) on \( V_1 \) be compatible with the complex structure, by being Hermitian: \( \langle Jv, w \rangle = -\langle v, Jw \rangle \). We call such \( G \) a **complex stratified Lie group**. A simple example is the complex Heisenberg group \( \mathbb{H}^3_C \), or the complex Heisenberg–Weyl groups \( \mathbb{H}^{2n+1}_C \).

Lemma 4.5. Suppose \( G \) is a complex stratified Lie group. Let \( f : G \to \mathbb{C} \) be holomorphic. Then for any \( \epsilon > 0 \), the function \( g = \sqrt{|f|^2 + \epsilon} \) is LSH.

We cannot say that \( |f| \) itself is LSH by our definition, because \( |f| \) need not be either \( C^2 \) nor strictly positive, but it is weakly LSH in the sense of Section 8; see Proposition 8.4.

Proof. Let \( x \in G \) and suppose for the moment that \( f(x) \neq 0 \). By continuity, there is a disk \( D \subseteq \mathbb{C} \setminus \{0\} \) and an open neighborhood \( U \) of \( x \) such
that \( f(U) \subset D \). Let \( L(z) \) be a branch of the complex logarithm which is holomorphic on \( D \). Then \( L \circ f \) is holomorphic on \( U \). We are assuming that the inner product on \( V_1 \) is Hermitian, so the real and imaginary parts of any holomorphic function are harmonic with respect to the sub-Laplacian \( \Delta \) (see [11] for further details). Thus \( \Delta \log |f| = \Delta \text{Re} L \circ f = 0 \) on \( U \). It follows that \( \Delta \log g \geq 0 \) on \( U \) (see Proposition 3.3 items 3 and 4). So we have shown \( \Delta \log g \geq 0 \) on \( \{f \neq 0\} \). But since \( f \) is holomorphic, \( \{f \neq 0\} \) is dense in \( G \) (unless \( f \equiv 0 \) in which case the statement is trivial). Since \( g \) is strictly positive and \( C^\infty \), \( \Delta \log g \) is continuous. Thus we have \( \Delta \log g \geq 0 \) everywhere.

**Corollary 4.6.** Let \( G \) be a complex stratified Lie group satisfying (sHC). Then (sHC) also holds for all holomorphic \( f \in L^p(\rho_s) \).

**Proof.** Apply (sHC) to \( \sqrt{|f|^2 + \epsilon} \), and let \( \epsilon \downarrow 0 \) using Lemma 4.5 and dominated convergence.

In particular, by Theorem 1.1, this holds whenever \( G \) satisfies (LSI). This implication was one of the main results of [11], which also gave a density argument for holomorphic \( L^p \) that can be used to show that (sHC) also holds for holomorphic \( f \in L^p(\rho_s) \). See the related discussion in Remark 1.4. Unfortunately, we do not know of any similar density results for LSH functions in the real case.

For a complex stratified Lie group \( G \), the vector field \( E \) has an additional significance: as shown in [11], it is the holomorphic projection of the Ornstein–Uhlenbeck operator \( A \). In the special case of \( \mathbb{C}^n \) with the Gaussian heat kernel, if \( f \) is holomorphic then we actually have \( Af = Ef \) because the Laplacian term vanishes. For more general complex stratified groups, this is no longer true because \( Af \) may fail to be holomorphic, but its \( L^2(\rho_s) \) projection onto the holomorphic functions equals \( Ef \).

There is not much overlap between the complex and H-type Lie groups; we showed in [10] that the only complex Lie groups which are also H-type are the complex Heisenberg–Weyl groups \( \mathbb{H}_c^{2n+1} \). As such, these are the only complex stratified Lie groups for which (LSI) is currently known to hold.

## 5 Convolution and approximation

**Lemma 5.1.** If \( f \in L^p(\rho_s) \), \( p \geq 1 \), and \( \varphi \in C_c(G) \) then \( \varphi \ast f \in L^p(\rho_s) \).

**Proof.** By considering positive and negative parts, we can assume without loss of generality that \( \varphi \geq 0 \). Also, by rescaling we can assume \( \int_G \varphi \, dm = 1 \).
Let \( q > p \) and \( r > 1 \) be so small that \( f \in L^{qr}(\rho_s) \). Let \( \frac{1}{r} + \frac{1}{r^*} = 1 \) and choose any \( t \) with \( s < t < rs = \frac{r^*}{r-1}s \). Then by Lemma 2.18 we have \( \hat{\varphi} \in L^{r^*}(\rho_s) \). Next, let \( K \) be the support of \( \varphi \), which is compact, and use Lemma 2.17 to choose \( C \) such that \( \sup_{z \in K} \rho_s(zy) \leq C \rho_t(y) \) for all \( y \in G \).

To start, use Jensen’s inequality with the probability measure \( \varphi \, dm \) to see that

\[
|\varphi * f(x)|^q = \left| \int_G \varphi(z) f(z^{-1}x) \, dz \right|^q \leq \int_G \varphi(z) |f(z^{-1}x)|^q \, dz
\]

so that, by Fubini’s theorem,

\[
\|\varphi * f\|_{L^q(\rho_s)}^q \leq \int_K \int_G \varphi(z) |f(z^{-1}x)|^q \rho_s(x) \, dx \, dz = \int_K \int_G \varphi(z) |f(y)|^q \rho_s(zy) \, dy \, dz
\]

making the change of variables \( y = z^{-1}x \) and using the translation invariance of \( m \). Now for all \( z \) in the support \( K \) of \( \varphi \), we have \( \rho_s(zy) \leq C \rho_t(y) \). Since \( \int_G \varphi(z) \, dz = 1 \) by assumption, we now have

\[
\|\varphi * f\|_{L^q(\rho_s)}^q \leq C \int_G |f(y)|^q \rho_t(y) \, dy
\]

\[
= C \int_G |f(y)|^q \frac{\rho_t(y)}{\rho_s(y)} \rho_s(y) \, dy
\]

\[
\leq C \|f\|_{L^{qr}(\rho_s)}^q \left\| \frac{\rho_t}{\rho_s} \right\|_{L^{r^*}(\rho_s)}
\]

by Hölder’s inequality. By our choices of \( t,q,r \), both norms are finite. \( \square \)

**Lemma 5.2.** Suppose \( f \in L^{p^+}(\rho_s) \), \( p \geq 1 \), and \( \varphi \in C_c^\infty(G) \). Then for any \( \xi \in g \), we have \( \hat{\xi}[\varphi * f] \in L^{p^+}(\rho_s) \). As a consequence, we also have

\[
|\nabla[\varphi * f]|, \Delta[\varphi * f], E[\varphi * f] \in L^{p^+}(\rho_s).
\]

**Proof.** By writing \( \xi \) as a linear combination of an adapted basis \( \{\xi_{j,k}\} \) and using Lemma 2.10, we can write \( \hat{\xi} = \sum_{j,k} c_{j,k} \hat{\xi}_{j,k} \) for some polynomials \( c_{j,k} \). In particular, by Corollary 2.15 we have \( c_{j,k} \in L^{\infty-}(\rho_s) \). Now using (2.3), we have

\[
\hat{\xi} [\varphi * f] = \sum_{j,k} c_{j,k} \hat{\xi}_{j,k} [\varphi * f] = \sum_{j,k} c_{j,k} \cdot [\hat{\xi}_{j,k}[\varphi] * f]
\]

which is in \( L^{p^+}(\rho_s) \) by Lemma 5.1.
The desired statement for $\Delta[\varphi * f]$ follows by applying this twice to get $\tilde{\xi}_2[\varphi * f] \in L^{p+}(\rho_s)$, then summing over an orthonormal basis $\{\xi_i\}$ for $V_1$. For $E[\varphi * f]$, use Corollary 2.16.

**Lemma 5.3.** Suppose $f \in L^{1+}(\rho_s)$. There is a sequence of nonnegative $\varphi_n \in C^\infty_c(G)$ such that $\varphi_n * f \to f$ almost everywhere and in $L^{1+}(\rho_s)$. If moreover $f \in C(G)$ the convergence is uniform on compact sets.

**Proof.** Let $\varphi \in C^\infty_c(G)$ be nonnegative with $\int_G \varphi \, dm = 1$, and let $\varphi_n = n^D \varphi \circ \delta_n$, so that $\varphi_n$ is a sequence of standard mollifiers. It is standard that $\varphi_n * f \to f$ almost everywhere, after passing to a subsequence if necessary, and that the convergence is uniform on compact sets if $f$ is continuous.

Now we note that the $\varphi_n$ are all supported in some compact neighborhood $K$ of the identity. As in the proof of Lemma 5.1, if we choose $q, r > 1$ such that $f \in L^{qr}(\rho_s)$, then we can choose $C, t$, independent of $n$, such that

$$\|\varphi_n * f\|_{L^q(\rho_s)}^q \leq C \|f\|_{L^{qr}(\rho_s)}^q \left|\frac{\rho_t}{\rho_s}\right|_{L^r(\rho_s)}$$

In particular, if $1 < q' < q$ then $\{|\varphi_n * f|^{q'} : n \geq 1\}$ is uniformly integrable with respect to $\rho_s \, dm$, hence $\varphi_n * f$ converges in $L^{q'}(\rho_s)$.

**6 Differentiation under the integral sign**

As mentioned in Section 1, the strong logarithmic Sobolev inequality (sLSI) is essentially an infinitesimal version of the strong hypercontractivity inequality (sHC). Thus, at a purely formal level, the equivalence between them is completely natural, and consists mainly of differentiating under the integral sign. The difficulty is to verify that this is justified.

The following abstract lemma is a general principle for differentiating under the integral sign. We have not seen this particular form in the literature, so we give the proof.

**Lemma 6.1.** Let $(X, \mu)$ be a probability space, and let $F : [0,T] \times X \to \mathbb{R}$ be jointly measurable. Suppose that for each $x \in X$, we have $F(\cdot, x) \in C^1([0,T])$, so that $\partial_t F : [0,T] \to X$ is also jointly measurable. Furthermore, suppose that the family of functions $\{\partial_t F(t, \cdot) : 0 \leq t \leq T\}$ is uniformly integrable on $(X, \mu)$. Set

$$v(t) = \int_X F(t, x) \, \mu(dx)$$

$$w(t) = \int_X \partial_t F(t, x) \, \mu(dx).$$

(6.1)
Then $v \in C^1([0,T])$ and $v'(t) = w(t)$ on $[0,T]$.

We use the term “uniformly integrable” here in the probabilist’s sense: a family of functions $\mathcal{G}$ is uniformly integrable with respect to $\mu$ iff

$$\lim_{M \to \infty} \sup_{g \in \mathcal{G}} \int_{|g| \geq M} |g| \, d\mu = 0.$$  

This is the necessary and sufficient hypothesis for the Vitali convergence theorem. In particular, a uniformly integrable family is bounded in $L^1(\mu)$. We also recall, for later use, the fact that if $\sup_{g \in \mathcal{G}} \|g\|_{L^p(\mu)} < \infty$ for some $p > 1$ then $\mathcal{G}$ is uniformly integrable.

**Proof of Lemma 6.1.** First, by the Vitali convergence theorem, $w$ is continuous on $[0,T]$.

The uniform integrability also implies that $\partial_t F(t,\cdot)$ is uniformly $L^1$ bounded, so we have $\|\partial_t F(t,\cdot)\|_{L^1(\mu)} \leq K$ for some finite $K$. Thus for any $0 \leq \tau \leq T$ we have

$$\int_0^\tau \int_X |\partial_t F(t,x)| \, \mu(dx) \, dt \leq KT < \infty.$$  

So by Fubini’s theorem and the first fundamental theorem of calculus, we have

$$\begin{align*}
\int_0^\tau w(t) \, dt &= \int_0^\tau \int_X \partial_t F(t,x) \, \mu(dx) \, dt \\
&= \int_X \int_0^\tau \partial_t F(t,x) \, dt \, \mu(dx) \\
&= \int_X (F(\tau,x) - F(0,x)) \, \mu(dx) \\
&= v(\tau) - v(0).
\end{align*}$$

Hence by the second fundamental theorem of calculus, $v$ is differentiable and $v' = w$. \qed

The following lemma, which follows from the heat kernel bounds in Section 2.4, will be convenient in verifying the uniform integrability hypothesis for our applications.

**Lemma 6.2.** Suppose $f \in L^p(\rho_s)$. Then for some $q > p$ and any $T < \infty$ we have

$$\sup_{0 \leq t \leq T} \|e^{-tE}f\|_{L^q(\rho_s)} < \infty.$$
In particular, if \( f \in L^{1+}(\rho_s) \), then \( \{e^{-tE}f : 0 \leq t \leq T\} \) is uniformly integrable with respect to \( \rho_s \, dm \).

**Proof.** Choose \( q > p \) and \( r > 1 \) so small that \( f \in L^{qr}(\rho_s) \). Let \( r^* = \frac{r}{r-1} \). By Lemma 2.18, taking \( t_0 = se^{-2T} \) and \( t_1 = s \), we have \( \sup_{0 \leq t \leq T} \rho_se^{-2t}/\rho_s \in L^{r^*}(\rho_s) \). Then for any \( t \in [0,T] \) we have

\[
\|e^{-tE}f\|_{L^q(\rho_s)}^q = \int_G |f \circ \delta_{e^{-t}}|^q \rho_s \, dm
\]

\[
= \int_G |f|^q \rho_{se^{-2t}} \, dm
\]

\[
\leq \int_G |f|^q \left( \sup_{0 \leq t \leq T} \frac{\rho_{se^{-2t}}}{\rho_s} \right) \rho_s \, dm
\]

\[
\leq \|f\|_{L^q(\rho_s)}^q \left( \sup_{0 \leq t \leq T} \frac{\rho_{se^{-2t}}}{\rho_s} \right) \|L^{r^*}(\rho_s)
\]

which is independent of \( t \) and finite. \( \square \)

**Lemma 6.3.** Suppose \( r \in C^1([0,T]) \) with \( 1 \leq r(t) \leq q \), and suppose \( f \in W^{1,q^+}(\rho_s) \) is positive. Set \( f_t = e^{-tE}f^{r(t)} \). Then the functions

\[
f_t, \quad f_t \log f_t, \quad |\nabla f_t|, \quad Ef_t, \quad 0 \leq t \leq T
\]

are all uniformly bounded in \( L^p(\rho_s) \) norm for some \( p > 1 \). In particular, they are uniformly integrable.

We note that the conclusion of this lemma implies \( f_t \in W^{1,1+}(\rho_s) \) for each \( 0 \leq t \leq T \).

**Proof.** For \( f_t \), note that \( 1 + f^q \) is in \( L^{1+} \), so by Lemma 6.2 we have that the family \( \{e^{-tE}[1 + f^q] : 0 \leq t \leq T\} \) is uniformly bounded in \( L^p \) for some \( p > 1 \). But \( f^{r(t)} \leq 1 + f^q \) for each \( t \), so \( f_t \leq e^{-tE}[1 + f^q] \), and \( f_t \) is uniformly bounded in \( L^p \).

For \( f_t \log f_t \), note that since \( f^q \in L^{1+} \), we also have \( (1 + f^q) \log f^q \in L^{1+} \). Then, since

\[
|\log f^{r(t)}| = r(t) |\log f| \leq q |\log f| = |\log f^q|
\]

we have \( |f^{r(t)} \log f^{r(t)}| \leq |(1 + f^q) \log f^q| \). By the same argument as in the previous case, \( f_t \log f_t \) is uniformly bounded in some \( L^p \).
For $|∇f_t|$, note that
$$
|∇f_t| = r(t)e^{-tE[fr(t)-1]}|∇[e^{-tE}f]| = r(t)e^{-tE} \left[fr(t)-1\right]|∇f|
$$
using (2.7). Now $fr(t)-1 \leq 1 + f^{-1}$, and we have $(1 + f^{-1})|∇f| \in L^{1+}$ by Hölder’s inequality. So by Lemma 6.2, $\{e^{-tE \left[(1 + f^{-1})|∇f|]\right]\}$ is uniformly bounded in some $L^p$, $p > 1$, and the same thus holds for $|∇f_t|.$

Since $|∇f| \in L^{q+}$, we have $Ef \in L^{q+}$ as well, by Lemma 2.16. So a similar argument applies for $Ef_t$ as for $|∇f_t|$, noting that $Ef_t = r(t)e^{-tE[fr(t)-1]}E[e^{-tE}f] = r(t)e^{-tE} \left[fr(t)-1\right]Ef.$

\[ \text{Lemma 6.4. Again suppose } r \in C^1([0, T]) \text{ with } 1 \leq r(t) \leq q, \text{ and suppose } f \in C^1(G) \cap W^{1,q+}(\rho_s) \text{ is positive. Set } f_t = e^{-tE}fr(t). \text{ Let}
\]
\[ v(t) = \int_G f_t \rho_s \, dm = \|e^{-tE}f\|_{L^r(t)(\rho_s)}
\]
\[ w(t) = \int_G \partial_t f_t \rho_s \, dm = \int_G \left[-Ef_t + \frac{r'(t)}{r(t)}f_t \log f_t\right] \rho_s \, dm. \quad (6.2)
\]

Then $v \in C^1([0, T])$ and $v'(t) = w(t)$ on $[0, T].$

\[ \text{Proof. Differentiate under the integral sign using Lemma 6.1, with } F(t, x) = f_t(x). \text{ The continuity of } \partial_t f_t \text{ follows from the assumption that } f \in C^1(G), \text{ and the uniform integrability hypothesis is verified by Lemma 6.3.} \]

7 Proofs of the main results

7.1 LSI implies sLSI

\[ \text{Theorem 7.1. In any stratified Lie group } G, \text{ if } (LSI) \text{ holds, then } (sLSI) \text{ holds, with the same constants } c, \beta. \]

\[ \text{Proof. Suppose } f \in LSH \cap W^{1,1+}(\rho_s); \text{ by Corollary 2.16 we have } Ef \in L^{1+}(\rho_s). \text{ Since LSH functions are subharmonic } (\Delta f \geq 0), \text{ we can apply Lemmas 3.2 and 2.21 to obtain}
\]
\[ \int_G \frac{|∇f|^2}{f} \rho_s \, dm \leq \int_G \Delta f \rho_s \, dm = \frac{2}{s} \int_G Ef \rho_s \, dm.
\]

Inserting this inequality into (LSI) yields (sLSI). \]
7.2 sHC implies sLSI

**Theorem 7.2.** In any stratified Lie group $G$, if (sHC) holds, then (sLSI) holds, with the same constants $c, \beta$.

**Proof.** Suppose (sHC) holds with constants $c, \beta$. Fix $f \in LSH \cap W^{1,1+}(\rho_s)$. Set $r(t) = e^{t/c}$ and choose $T > 0$ so small that $f \in W^{1,r(T)+}(\rho_s)$.

For $t \in [0, T]$, applying (sHC) with $p = 1$ and $q = r(T)$ yields

$$
\|e^{-tE}f\|_{L^{r(T)}(\rho_s)} \leq M(t)\|f\|_{L^1(\rho_s)}
$$

where

$$
M(t) := M(1, r(t)) = \exp(\beta \cdot (1 - e^{-t/c})).
$$

Define $v(t), w(t)$ as in (6.2), and set

$$
\alpha(t) = \frac{1}{M(t)}\|e^{-tE}f\|_{L^{r(T)}(\rho_s)} = \frac{1}{M(t)}v(t)^{1/r(t)}.
$$

Note that $\alpha(0) = \|f\|_{L^1(\rho_s)}$, so (7.1) says that $\alpha(t) \leq \alpha(0)$ for all $t \in [0, T]$.

Now applying Lemma 6.4 with $q = r(T)$, we have that $v$ is continuously differentiable on $[0, T]$; hence so is $\alpha(t)$. As such, we must have

$$
0 \geq \alpha'(0) = -\frac{M'(0)}{M(0)^2}v(0)^{1/r(0)}
$$

$$
+ \frac{1}{M(0)}\frac{1}{r(0)}v(0)^{(1/r(0))-1}v'(0)
$$

$$
+ \frac{1}{M(0)}v(0)^{1/r(0)} \log v(0) - \frac{r'(0)}{r(0)^2}.
$$

Observing that

$$
r(0) = 1 \quad r'(0) = \frac{1}{c}
$$

$$
M(0) = 1 \quad M'(0) = \frac{\beta}{c}
$$

and

$$
v(0) = \|f\|_{L^1(\rho_s)}
$$

$$
v'(0) = w(0) = -\int_G Ef \rho_s \, dm + \frac{1}{c} \int_G f \log f \rho_s \, dm
$$

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we see that (7.4) reads

\begin{align*}
0 \geq \frac{\beta}{c} \|f\|_{L^1(\rho_s)} - \int_G E f \rho_s \, dm &+ \frac{1}{e} \int_G f \log f \rho_s \, dm - \frac{1}{c} \|f\|_{L^1(\rho_s)} \log \|f\|_{L^1(\rho_s)} \\
&+ 1 - \hat{G} E f \rho_s \, dm
\end{align*}

which after rearranging is precisely (sLSI).

\begin{remark}
Theorem 7.2 does not rely on any properties of LSH functions, except the assumption that they satisfy (sHC). So more broadly, any appropriate class of functions satisfying (sHC) will also satisfy (sLSI).
\end{remark}

\section{7.3 sLSI implies sHC}

In this section, we show that if the strong logarithmic Sobolev inequality is satisfied for LSH functions, then so is strong hypercontractivity.

We begin by noting that the semigroup $e^{-tE}$ is contractive on log-subharmonic functions. We assume some integrability on $|\nabla f|$ but this assumption will be removed later.

\begin{lemma}
Suppose $f \in LSH \cap W^{1,1+}(\rho_s)$. Then for any $t \geq 0$ we have

\[ \|e^{-tE}f\|_{L^1(\rho_s)} \leq \|f\|_{L^1(\rho_s)}. \]

\begin{proof}
Let $T > 0$ be arbitrary. Applying Lemma 6.4 with $r(t) \equiv 1 = q$, we have

\[ \frac{d}{dt}\|e^{-tE}f\|_{L^1(\rho_s)} = -\int_G E e^{-tE}f \rho_s \, dm. \]

Now from Lemma 6.3, again with $r(t) \equiv 1 = q$, we have in particular that $e^{-tE}f, |\nabla e^{-tE}f|, E e^{-tE}f \in L^1(\rho_s)$. Also, $f$ is subharmonic and hence so is $e^{-tE}f$ by (2.17). So by Lemma 2.21, we have

\[ \int_G E e^{-tE}f \rho_s \, dm = \frac{s}{2} \int_G \Delta e^{-tE}f \rho_s \, dm \geq 0. \]

Hence $\|e^{-tE}f\|_{L^1(\rho_s)}$ is a decreasing function of $t$.
\end{proof}

The next step is to show that (sLSI) implies that (sHC) holds at time $t = t_f$. We take $p = 1$ and again assume, for now, sufficient integrability for $|\nabla f|$.
Lemma 7.5. Suppose that (sLSI) holds. Let $1 \leq q < \infty$, and set

$$t_J = t_J(1, q) = c \log q,$$
$$M = M(1, q) = \exp(\beta \cdot (1 - q^{-1})).$$

Suppose $f \in LSH \cap W^{1,q+}(\rho_s)$. Then

$$\|e^{-tJE}f\|_{L^q(\rho_s)} \leq M\|f\|_{L^1(\rho_s)}. \quad (7.5)$$

Proof. Set $r(t) = e^{t/c}$, so that $r(t_J) = q$, and let $f_t, v(t), w(t)$ be as in Lemma 6.4. The hypotheses of Lemma 6.4 are satisfied, with $T = t_J$, so we have $v \in C^1([0,t_J])$ and $v'(t) = w(t)$.

On the other hand, for each $0 \leq t \leq t_J$, we have $f_t \in LSH$, by Proposition 3.3 items 3 and 5. Moreover, Lemma 6.3 implies $f_t \in W^{1,1+}(\rho_s)$. So (sLSI) applies to $f_t$. In terms of $v(t), w(t)$, this reads

$$cw(t) \leq v(t) \log v(t) + \beta v(t) \quad (7.6)$$

where we note that $\frac{v'(t)}{r(t)} = \frac{1}{c}$. Since $w(t) = v'(t)$, we may rewrite (7.6) as

$$\frac{d}{dt} \log v(t) \leq -\frac{1}{c} \log v(t) + \frac{\beta}{c}. \quad (7.7)$$

Define

$$M(t) = M(1, r(t)) = \exp(\beta \cdot (1 - e^{-t/c}))$$
$$\alpha(t) = \frac{1}{M(t)}\|e^{-tJE}f\|_{L^q(\rho_s)} = \frac{1}{M(t)}v(t)^{1/r(t)}$$

as in the proof of Theorem 7.2. Note that $\alpha(0) = \|f\|_{L^1(\rho_s)}$ and $\alpha(t_J) = M^{-1}\|e^{-tJE}f\|_{L^q(\rho_s)}$. Then we have

$$\frac{d}{dt} \log \alpha(t) = e^{-t/c} \left(-\frac{1}{c} \log v(t) + \frac{d}{dt} \log v(t) - \frac{\beta}{c}\right) \leq 0$$

using (7.7). Hence $\alpha(t)$ is a decreasing function on $[0,t_J]$, so in particular $\alpha(t_J) \leq \alpha(0)$, which is the desired statement.

Theorem 7.6. In any stratified Lie group $G$, if (sLSI) holds, then (sHC) holds, with the same constants $c, \beta$. 

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Proof. Fix \( f \in LSH \cap L^q(\rho_s) \) and \( t \geq t_J(p, q) \).

Suppose first that \( p = 1 \) and \( f \in LSH \cap W^{1,q+}(\rho_s) \). Then Lemma 7.5 gives
\[
\| e^{-tE} f \|_{L^2(\rho_s)} \leq M(1, q) \| f \|_{L^1(\rho_s)}. \tag{7.8}
\]

Set \( \tau = t - t_J(1, q) \), and let \( g = e^{-tE} f^q \in LSH \). We apply Lemma 6.3 with \( T = t_J \) and \( r(t) \equiv q \), so that \( g = f_{t_J} \), to see that \( g \in W^{1,q+}(\rho_s) \). Applying Lemma 7.4 to \( g \), we have
\[
\| e^{-\tau E} g \|_{L^1(\rho_s)} \leq \| g \|_{L^1(\rho_s)}
\]
or in other words
\[
\| e^{-tE} f \|_{L^q(\rho_s)}^q \leq \| e^{-tE} f \|_{L^q(\rho_s)}^q. \tag{7.9}
\]
Combining (7.8) and (7.9) gives (sHC) in this case.

Next, suppose only that \( p = 1 \), \( f \in LSH \cap L^q(\rho_s) \), but make no assumptions about \( \nabla f \). Let \( \varphi_n \) be a sequence of standard mollifiers as in Lemma 5.3, and set \( f_n = \varphi_n * f \), so that \( f_n \to f \) pointwise and in \( L^{1+}(\rho_s) \); then \( e^{-tE} f_n \to e^{-tE} f \) pointwise as well. We have \( f_n \in LSH \) by Lemma 3.4; \( f_n \in L^{q+}(\rho_s) \) by Lemma 5.1; and \( |\nabla f_n| \in L^{q+}(\rho_s) \) by Lemma 5.2. So by the previous case, we have
\[
\| e^{-tE} f_n \|_{L^q(\rho_s)} \leq M(1, q) \| f_n \|_{L^1(\rho_s)} \tag{7.10}
\]
and by Fatou’s lemma, the same holds for \( f \).

Next, suppose \( p = 1 \) and \( f \in LSH \cap L^q(\rho_s) \). Then for any \( 0 < \alpha < 1 \), we have \( f^\alpha \in LSH \cap L^{q+}(\rho_s) \), so that by the previous case,
\[
\| e^{-tE} f^\alpha \|_{L^q(\rho_s)} \leq M(1, q) \| f^\alpha \|_{L^1(\rho_s)}.
\]
Letting \( \alpha \to 1 \), we have \( f^\alpha \to f \) pointwise and in \( L^1(\rho_s) \) (by dominated convergence, using for instance \( 1 + f \) as the dominating function). So by Fatou’s lemma, the result holds for \( f \).

Finally, let \( 0 < p \leq q \) be arbitrary and \( f \in LSH \cap L^q(\rho_s) \). Set \( g = f^p \) and \( r = q/p \). Then we have \( g \in LSH \cap L^r(\rho_s) \), and so by the previous case we have
\[
\| e^{-tE} g \|_{L^r(\rho_s)} \leq M(1, r) \| g \|_{L^1(\rho_s)}, \quad t \geq t_J(1, r).
\]
Noting that \( M(1, r) = M(p, q)^p \) and \( t_J(1, r) = t_J(p, q) \), this reads
\[
\| e^{-tE} f \|_{L^q(\rho_s)}^p \leq M(p, q)^p \| f \|_{L^p(\rho_s)}^p
\]
which is equivalent to (sLSI). \( \square \)
8 Weaker notions of subharmonicity

We have chosen to focus our attention in this paper on log-subharmonic functions which are $C^2$. In this section, we note that our results for strong hypercontractivity can be extended to functions which are LSH in a weaker sense.

A comprehensive discussion of the various possible definitions of subharmonicity on stratified Lie groups is beyond the scope of this paper. We refer the reader to [4], in which the basic definition of subharmonic functions (Definition 7.2.2) is in terms of harmonic measure. Many other equivalent characterizations are given; perhaps the simplest is the following definition in terms of distributional derivatives.

**Definition 8.1.** We say a function $f : G \to [-\infty, \infty)$ is weakly subharmonic if $f \in L^1_{\text{loc}}(G, m)$ and $\Delta f \geq 0$ in the sense of distributions. We say a function $f : G \to [0, \infty)$ is weakly log-subharmonic if either $f \equiv 0$ or $\log f$ is weakly subharmonic, and we write $f \in wLSH$.

Strictly speaking, a function $f$ is weakly subharmonic in this sense iff it has an $m$-version which is subharmonic in the sense of [4, Definition 7.2.2]; see [4, Theorem 8.2.15 and Corollary 8.2.4]. The distinction is irrelevant for our current purposes, since null sets will not concern us.

Let us also call attention to [4, Corollary 8.2.3], where it is shown that a function is (weakly) subharmonic iff it satisfies a sub-averaging property, which is analogous to the definition of subharmonic used in [14, 15].

**Lemma 8.2.** Suppose $f \in wLSH \cap L^{q+}(\rho_s)$, where $q \geq 1$. Then there is a sequence $f_n \in LSH \cap L^{q+}(\rho_s)$ with $f_n \to f$ almost everywhere and in $L^{1+}(\rho_s)$.

**Proof.** If $f \equiv 0$ this is trivial by taking $f_n = 1/n$. Otherwise, $\log f$ is weakly subharmonic. Let $\varphi_n \in C_c^\infty$ be a sequence of nonnegative standard mollifiers with $\int_G \varphi_n \, dm = 1$, as in Lemma 5.3, and set $g_n = \varphi_n \ast \log f$. Then $g_n \to \log f$ almost everywhere. By [4, Theorem 8.1.5 and Corollary 8.2.3], $g_n$ is also weakly subharmonic; moreover, since $g_n \in C^\infty(G)$, we have by [4, Proposition 7.2.5] that $\Delta g_n \geq 0$.

Set $f_n = \exp(g_n)$, so that $f_n \in LSH$ and $f_n \to f$ almost everywhere. Now $f_n \leq \varphi_n \ast f$ by Jensen’s inequality. By Lemma 5.1, we have $\varphi_n \ast f \in L^{q+}(\rho_s)$, so the same is true for $f_n$. And by Lemma 5.3, we have $\varphi_n \ast f \to f$ in $L^{1+}(\rho_s)$, so $f_n \to f$ in $L^{1+}(\rho_s)$ as well. \[\square\]

**Theorem 8.3.** If (sHC) holds for all $f \in LSH \cap L^q(\rho_s)$, then it holds for all $f \in wLSH \cap L^q(\rho_s)$. 

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Proof. As in the proof of Theorem 7.6, it suffices to prove (sHC) with $p = 1$ and for all $f \in wLSH \cap L^{q+}(\rho_s)$. Using Lemma 8.2, choose $f_n \in LSH \cap L^{q+}(\rho_s)$ with $f_n \to f$ almost everywhere and in $L^{1+}(\rho_s)$. Then (sHC) holds for each $f_n$. We have $\|f_n\|_{L^1(\rho_s)} \to \|f\|_{L^1(\rho_s)}$, so by Fatou’s lemma, (sHC) holds for $f$. \qed

In the setting of complex stratified Lie groups (Example 4.4), the modulus of a holomorphic function is weakly LSH.

**Proposition 8.4.** Let $G$ be a complex stratified Lie group, and suppose $f : G \to \mathbb{C}$ is holomorphic. Then $|f| \in wLSH$.

Proof. Let $f_n = \sqrt{|f|^2 + \frac{1}{n}}$. We showed in Lemma 4.5 that $f_n \in LSH$. Now $f_n \downarrow |f|$, and so $\log |f|$ is a decreasing limit of subharmonic functions. By [4, Theorem 8.2.7], $\log |f|$ is therefore weakly subharmonic. \qed

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