Frame Spectral Pairs and Exponential Bases

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Abstract

Given a domain $\Omega \subset \mathbb{R}^d$ with positive and finite Lebesgue measure and a discrete set $\Lambda \subset \mathbb{R}^d$, we say that $(\Omega, \Lambda)$ is a frame spectral pair if the set of exponential functions $E(\Lambda) := \{e^{2\pi i \lambda \cdot x} : \lambda \in \Lambda\}$ is a frame for $L^2(\Omega)$. Special cases of frames include Riesz bases and orthogonal bases. In the finite setting $\mathbb{Z}^d_N$, $d, N \geq 1$, a frame spectral pair can be similarly defined. In this paper we show how to construct and obtain new classes of frame spectral pairs in $\mathbb{R}^d$ by “adding” a frame spectral pair in $\mathbb{R}^d$ to a frame spectral pair in $\mathbb{Z}^d_N$. Our construction unifies the well-known examples of exponential frames for the union of cubes with equal volumes. We also remark on the link between the spectral property of a domain and sampling theory.

Keywords  Frames · Riesz bases · Exponential bases and sampling

1 Introduction

Let $\Omega \subset \mathbb{R}^d$ be a set with positive and finite Lebesgue measure, $0 < |\Omega| < \infty$, and let $\Lambda \subset \mathbb{R}^d$ be a discrete and countable set. Let $E_\Omega(\Lambda)$ denote the set of exponentials, $E_\Omega(\Lambda) = \{e_\lambda : \Omega \to S^1 | e_\lambda(x) := e^{2\pi i x \cdot \lambda}\}$. For simplicity, we will drop the subscript $\Omega$ in the sequel and write $E(\Lambda)$. The set of exponentials $E(\Lambda)$ is a frame for $L^2(\Omega)$ when there exist finite constants $c > 0$ and $C > 0$ such that for any $u \in L^2(\Omega)$
\[ \|u\|_{L^2(\Omega)}^2 \leq \sum_{\lambda \in \Lambda} |\langle u, e_\lambda \rangle|_{L^2(\Omega)}^2 \leq C \|u\|_{L^2(\Omega)}^2. \] (1)

The constants \(c\) and \(C\) are called lower and upper frame constants (see the classical references \([4,5]\) for the basic properties of frames). The frame is called tight if \(C = c\).

The set of exponentials \(E(\Lambda)\) is a Riesz sequence in \(L^2(\Omega)\), if there are positive constants \(c > 0\) and \(C > 0\) such that for any finite set of scalars \(\{c_\lambda\}_{\lambda \in F} (F \subset \Lambda\) finite), the following inequalities hold:

\[ c \sum_{\lambda \in F} |c_\lambda|^2 \leq \left\| \sum_{\lambda \in F} c_\lambda e_\lambda \right\|_{L^2(\Omega)}^2 \leq C \sum_{\lambda \in F} |c_\lambda|^2. \] (2)

When only the right inequality in (2) holds, then \(E(\Lambda)\) is called a Bessel sequence.

A Riesz sequence is called a Riesz basis if it is complete. Any Riesz basis is a frame with unified frame constants, while the converse is not always true; see e.g. \([5]\).

**Definition 1** Given a set \(\Omega \subset \mathbb{R}^d\) and discrete and countable set \(\Lambda \subset \mathbb{R}^d\), we term \((\Omega, \Lambda)\) a frame spectral pair when the system of exponentials \(E(\Lambda)\) is a frame for \(L^2(\Omega)\).

For a given frame pair, when the frame is a Riesz basis, we shall call the pair a Riesz spectral pair. When it is an orthogonal basis, we shall simply call the pair a spectral pair. An interesting question in the frame theory is the following:

**Question A** How can a given frame spectral be modified to become a Riesz basis or an orthogonal basis? This question is more general, and in its light in this paper we provide an answer to the following question:

**Question B** How can current frame spectral pairs, Riesz spectral pairs, or orthogonal bases be combined to form new ones?

The study of frame spectral pairs in this paper is motivated by the interpolation formula for bandlimited signals with structured spectra and the link between spectral properties of a domain to its sampling properties.

### 1.1 Main Contributions

To address Question B., the simplest and perhaps the most natural way of creating a new frame spectral pair in \(\mathbb{R}^d\) is by appropriately “adding” two frame spectral pairs in both finite and continuous settings. In this paper, we will investigate whether for a given pair \((\Omega, \Lambda)\) in \(\mathbb{R}^d\), the pair \((\Omega + A, \Lambda + J / N)\), \(A, J \subset \mathbb{Z}^d_N\), will inherit the analytical property of the pair \((\Omega, \Lambda)\).

Our main findings are as follows.

**Theorem 1** Suppose that \(\Omega_1 \subset \mathbb{R}^d\) is a set with positive and finite Lebesgue measure, \(\Lambda_1 \subset \mathbb{R}^d\) is a discrete and countable set, and \(N \geq 1\) is an integer. Let \(A\) be any subset of \(\mathbb{Z}^d_N\) for which the translates of \(\Omega_1\) by \(A\) are disjoint, that is,

\[ |\Omega_1 + a \cap \Omega_1 + a'| = 0, \quad \forall a, a' \in A, \quad a \neq a'. \] (3)
Let \( J \subset \mathbb{Z}^d \), and define
\[
\Omega := \Omega_1 + A, \quad \Lambda := \Lambda_1 + J/N,
\]
where the sum is taken to be Minkowski addition. If \((\Omega_1, \Lambda_1)\) is a frame spectral pair in \( \mathbb{R}^d \) with frame constants \( \alpha, \beta \) and \((A, J)\) is a frame spectral pair in \( \mathbb{Z}^d \) with frame constants \( c, C \), then \((\Omega, \Lambda)\) is also a frame spectral pair in \( \mathbb{R}^d \) with frame constants \( \alpha c, \beta C \) if for all \( \lambda \in \Lambda_1 \) we have
\[
\hat{\delta}_\lambda \equiv 1 \text{ on } A,
\]
where \( \delta_\lambda \) is the Kronecker delta function at \( \lambda \).

**Theorem 2** With the assumptions of Theorem 1 and condition (5), \( E(\Lambda) \) is a Riesz basis for \( L^2(\Omega) \) if \( E(\Lambda_1) \) is a Riesz basis for \( L^2(\Omega_1) \) and \( E(J) \) is a basis for \( \ell^2(A) \).

In this case, the frame constants are given as in Theorem 1.

Moreover, when \( \Lambda_1 \) is a lattice, the dual Riesz basis \( \{g_\lambda\}_{\lambda \in \Lambda} \) is given by
\[
g_{\lambda_1 + j/N}(x) = \left( G_j, E_{j} \chi_{\Pi \Lambda_1}(x - \cdot) \right)_{\ell^2(A)} e_{\lambda_1 + j/N}(x) \quad \text{a.e. } x \in \Omega.
\]
Here, \( \Pi \Lambda_1 \) is the fundamental domain of the lattice \( \Lambda_1 \), \( \chi_{\Pi \Lambda_1} \) denotes the indicator function of the domain \( \Pi \Lambda_1 \), and \( E_j(z) = e^{2\pi i \frac{z}{|z|}} \) and \( G_j \) are given in (17).

**Theorem 3** With the assumptions of Theorem 1, \( E(\Lambda) \) is an orthogonal basis for \( L^2(\Omega) \) if \( E(\Lambda_1) \) is an orthogonal basis for \( L^2(\Omega_1) \) and \( E(J) \) is an orthogonal basis for \( \ell^2(A) \) and (5) holds.

As an indirect application of Theorem 3, we obtain the following result, which connects a domain’s spectral property to sampling and reconstruction.

**Proposition 1** Assume that \( N > 0 \), \( A, J \subseteq \mathbb{Z}^N \). Let \( \Omega := [0, 1] + A \) and \( \Lambda := \mathbb{Z} + J/N \). Assume that \( f \in PW_{\Omega} \) such that \( \{f(\lambda)\}_\Lambda \) are predetermined. If \((A, J)\) is a spectral pair in \( \mathbb{Z}^N \), then \( f \) can be completely recovered and
\[
\hat{f}(\xi) = (\#J)^{-1} \sum_{\lambda \in \Lambda} f(\lambda) e^{-2\pi i \lambda \cdot \xi} \quad \text{a.e. } \xi \in \Omega.
\]

### 1.2 Comparison with Existing Work

Regarding Theorem 1, any bounded domain \( \Omega \) with Lebesgue measure \( 0 < |\Omega| < \infty \) admits an exponential tight frame \( E(\Lambda) \) by projecting an exponential orthonormal basis of a cube (containing \( \Omega \)) onto \( \Omega \). In this method, the frame spectrum \( \Lambda \) is necessarily a lattice. In Theorem 1, we provide an alternative method for the construction of exponential frames (not necessarily tight) for bounded domains where the frame spectrum is not necessarily a lattice.
Notice that Theorem 2 illustrates how to construct a Riesz spectrum for the union of domains with equal measure. There are many cases where it is known that a set $\Omega$ admits a Riesz basis of exponential functions, such as multi-tiling (bounded and unbounded) domains in $\mathbb{R}^d$ [1,3,17,26]. Recently, it was established in [7] that any convex polytope which is centrally symmetric and whose faces of all dimensions are also centrally symmetric, admits a Riesz basis of exponentials. For the existence of exponential Riesz bases in other special cases see, e.g., [6,29] and the references therein. While there are known cases where $\Omega$ does not admit an orthogonal basis of exponentials, such as the unit Ball in $\mathbb{R}^d$ ($d \geq 2$) (for other cases see e.g. [15,19]), less is known about Riesz bases of exponentials. Finding a domain $\Omega$ that does not admit any Riesz basis of exponentials is still an unsolved problem.

Below, we point out the difference between our results and some well-known results for the construction of exponential Riesz bases. Constructive proofs of the existence of exponential Riesz bases can be found in [17], and later in [26] with fewer assumptions and a simpler proof. For example, in [26], the author considers a multi-tiling domain $\Omega$ in $\mathbb{R}^d$ which multi-tiles the space by a lattice $\Lambda$ and proves that the domain admits an exponential Riesz basis. More precisely, he obtains a Riesz spectrum for the domain $\Omega$ using a finite union of translates of the dual lattice $\Lambda$, i.e., $\Lambda + J$. Here, we choose a Riesz spectral pair $(\Omega, \Lambda)$ (in $\mathbb{R}^d$) and a basis pair $(A, J)$ (in $\mathbb{Z}^d_N$) and show that $(\Omega + A, \Lambda + J/N)$ is a Riesz spectral pair in $\mathbb{R}^d$ if (5) is satisfied for $A$ and $\Lambda$.

Another well-known example of exponential Riesz bases for a union of co-measurable cubes has been constructed by DeCarli [6] for the union of unit cubes. In this paper, the author takes a finite set of vectors in $\mathbb{R}^d$ with an arithmetic progression and proves that the union of shifts of $\mathbb{Z}^d$ by these vectors is a Riesz spectrum for the union of cubes if and only if the evaluation matrix is invertible. Equivalently, for a given finite number of vectors in $\mathbb{R}^d$, the union of translations of $\mathbb{Z}^d$ by the vectors is a Riesz spectrum if the matrix is an invertible Vandermonde matrix. Theorem 2 requires fewer assumptions to establish the existence of Riesz bases for a finite union of unit cubes. More precisely, when $\Omega_1$ is a $d$-dimensional cube, we construct a Riesz spectrum by taking the disjoint union of multi-rational shifts of $\mathbb{Z}^d$ with the sufficient condition that the matrix is invertible.

In Theorem 2, when $\Omega_1$ is a cube in $\mathbb{R}^d$, the structure of the Riesz spectrum for the union of cubes is similar to the well-known example of sampling and interpolation sequences constructed by Lyubarskii and Seip [31] for the union of intervals of equal length in $\mathbb{R}$, and by Marzo in higher dimensions [32]. In [31], the authors consider a union of $p$ disjoint intervals with equal size and construct a sampling and interpolation sequence for the set using the $p$ shifts of the spectrum of a single interval. Like in the current paper, the sufficient condition is the invertibility of an associated matrix (7). Our result in Theorem 2 provides machinery for such constructions with more general domains beyond intervals (or cubes).

Regarding Theorem 3, in recent years, research on orthogonal bases of exponentials has flourished in response to the development of exponential bases in Banach spaces, and in particular, to the growing interest in the Fuglede Conjecture [14]. The Fuglede Conjecture asserts that every domain of $\mathbb{R}^d$ with positive finite Lebesgue measure admits an orthogonal basis of exponentials if and only if it tiles $\mathbb{R}^d$ by translation.
Although the Fuglede Conjecture is, in general, false, as shown by Tao in one direction [35] and by Kolountzakis and Matolcsi in the other [27,28,33], it has given rise to active investigations of the connections between orthogonal bases of exponentials and tilings in Euclidean space. The conjecture has been proved affirmative in special cases in various settings (continuous and discrete). See, for example, [2,10,11,16,20,21], and the references contained therein. There are many cases where it is known that a domain admits no orthogonal exponential bases. See, for example, [15,18,19,25] and the references contained therein. The conjecture is still open in dimensions $d = 1, 2$.

However, there are special cases in these dimensions where the conjecture has been proved affirmative (see e.g. [19,30]).

With regard to Theorem 3, we shall point it out that when $\Omega_1$ is a $d$-dimensional cube, the new spectrum set that we construct in Theorem 3 is different from the well-known spectra in the literature, namely, the one for the union of cubes in $\mathbb{R}^d$ ($d \geq 5$) that was presented by Tao in [35]. The example is used to disprove the direction “spectral $\not\rightarrow$ tiling” of the Fuglede Conjecture. In the construction, Tao considers a union of translations of a spectral set in the finite domain $\mathbb{Z}_p^d$ and lifts it to a higher dimension. Our construction of a new spectral set is the result of adding a spectral set in $\mathbb{Z}_p^d$ to a spectral set in the continuous domain $\mathbb{R}^d$.

Another well-known example of a spectral set was constructed by Łaba in dimension $d = 1$ [30]. In her paper, Łaba characterizes the spectrum of the union of two co-measurable intervals. When the left endpoints of the intervals are integers, the spectrum of the union coincides with the construction of the spectrum we present in Theorem 3. Here, we construct a spectral set for any ($\geq 2$) union of co-measurable intervals.

Outline. This paper is organized as follows. After introducing the notations and preliminaries in Sect. 2, we prove Theorem 1 in Sect. 3. In Sect. 4 we prove Theorem 2 and illustrate the explicit structure of biorthogonal dual Riesz bases along with some examples for a special subclass of Riesz spectral pairs and Riesz bases, using the techniques that were developed earlier by the first listed author and Okoudjou in [13]. Results and examples of dual bases in both the continuous and finite settings appear in Sects. 4.1 and 4.2. In Sect. 5, we prove Theorem 3 followed by examples of spectral pairs. In Sect. 5.2 we prove Proposition 1.

2 Notations and Preliminaries

Throughout this paper, $\Omega \subset \mathbb{R}^d$ is a Lebesgue measurable set with $0 < |\Omega| < \infty$, and $\Lambda \subset \mathbb{R}^d$ is countable and discrete. The inner product of $u, v \in L^2(\Omega)$ is defined by $\langle u, v \rangle_{L^2(\Omega)} = \int_\Omega u(x)v(x)dx$. In the sequel, we shall drop the subscripts and simply write $\langle f, g \rangle$ when the underlying Hilbert space is clear from the context.

For $f \in L^2(\mathbb{R}^d)$, we denote by $\hat{f}$ or $\mathcal{F}(f)$ the Fourier transform of $f$, and it is defined by $\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x)e^{-2\pi i x \cdot \xi}dx$, $\xi \in \mathbb{R}^d$, where, $x \cdot \xi = \sum_{i=1}^d x_i \xi_i$ is the scalar products of two vectors. The inverse Fourier transform, denoted by $\mathcal{F}^{-1}(\hat{f}) = f$, is then given by $f(x) = \int_{\mathbb{R}^d} \hat{f}(\xi)e^{2\pi i x \cdot \xi}dx$, $x \in \mathbb{R}^d$.

Here, and in the sequel, we denote the cardinality of a finite set $F$ by $|F|$. For $d, N \geq 1$, $\mathbb{Z}_N^d$ denotes the $d$-dimensional vector space over the cyclic group $\mathbb{Z}_N$. 

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For any functions \( f, g \in \ell^2(\mathbb{Z}_N^d) \), the inner product is defined by \( \langle f, g \rangle_{\ell^2(\mathbb{Z}_N^d)} = N^{-d} \sum_{x \in \mathbb{Z}_N^d} f(x)g(x) \). In general, for a function \( f : \mathbb{Z}_N^d \to \mathbb{C}, d \geq 1 \), the cyclic Fourier transform is given by \( \hat{f}(x) = N^{-d} \sum_{y \in \mathbb{Z}_N^d} e^{2\pi i y \cdot x/N} \).

A frame spectral pair in the finite setting can be defined in a fashion that is similar to the continuous setting. Definition 1 can also be expressed in terms of matrices, as follows: Let \( A, J \subseteq \mathbb{Z}_N^d \), and assume that \( A, J \) induce a Riesz basis (or an orthogonal basis) for \( \ell^2(A) \). The submatrix \( \mathcal{F}_{A,J} \) induces a Riesz basis (an orthogonal basis) for \( \ell^2(A) \) if \( \mathcal{F}_{A,J} \) is an invertible (unitary) matrix. Since the transpose matrix preserves the invertibility and unitary property, the notion of symmetry discussed above loses its meaning in the infinite or continuous setting, e.g., \( \mathbb{Z}_N^d \) or \( \mathbb{R}^d \). These settings require a well-defined notion of symmetry, which is a challenging problem.

**Notation:** In the sequel, we use \( e_\lambda(x) = e^{2\pi i x \cdot \lambda} \) for exponential functions in \( \mathbb{R}^d \), and \( E_j(z) = e^{2\pi i z \cdot J/N} \) in the finite setting \( \mathbb{Z}_N^d \).

### 3 Proof of Theorem 1

**Proof** Assume that \( \Omega \) and \( A \) are given as in (4)–(3), and also assume that (5) holds.

Let \( u \in L^2(\Omega) \). Then

\[
\sum_{(\lambda, j) \in A_1 \times J} \left| \langle u, e_{\lambda+j/N} \rangle_{L^2(\Omega)} \right|^2
= \sum_{(\lambda, j) \in A_1 \times J} \left| \sum_{a \in A} \langle u, e_{\lambda+j/N} \rangle_{L^2(\Omega_1+a)} \right|^2
= \sum_{(\lambda, j) \in A_1 \times J} \left| \sum_{a \in A} \int_{\Omega_1} u(x+a) e_{\lambda+j/N}(x+a) \, dx \right|^2
\]

\[\Box \]
Proposition 2. Suppose that $E \subseteq \mathbb{R}^d$. Assume that the exponentials $e^{i x N}$ of $Z$ is a frame spectral pair and $u_j(x) := \sum_{a \in A} u(x + a) e^{i x N} = \text{in} L^2(\Omega_1)$, there exists an upper frame constant $\beta > 0$ for which

$$
\sum_{(\lambda, j) \in \Lambda_1 \times J} \left| \int_{\Omega_1} \left( \sum_{a \in A} u(x + a) e^{i x N} \right)^2 e^{i \lambda x} dx \right|^2.
$$

(8)

The last line follows from (5). For any fixed $j \in J$, since $(\Omega_1, A_1)$ is a frame spectral pair and $u_j(x) := \sum_{a \in A} u(x + a) e^{i x N} = \text{is in} L^2(\Omega_1)$, there exists an upper frame constant $\beta > 0$ for which

$$
\sum_{j \in J} \int_{\Omega_1} \left| \sum_{a \in A} u(x + a) e^{i x N} \right|^2 dx = \beta \int_{\Omega_1} \left| u(x + \cdot) \right|^2 \ell^2(A) dx = C \beta \|u\|_{L^2(\Omega)}^2,
$$

where $C$ is the upper frame constant for the frame spectral pair $(A, J)$ in $\mathbb{Z}_N^d$. This completes the proof of the upper frame estimate for the pair $(\Omega, A)$ with the upper bound constant $C \beta$. The lower bound estimate is obtained similarly. The completeness of the frame sequence is obtained by the frame lower bound property. \hfill \Box

We conclude this section with some examples.

**Examples** 1. For any given $A, J \subseteq \mathbb{Z}_N^d$ and any frame pair $(\Omega_1, A_1)$ in $\mathbb{R}^d$, the set of exponentials $E(A)$ is a Bessel sequence in $L^2(\Omega)$ with the Bessel constant $\|A\|_J$. When $\# A = 1 = \# J$, the Bessel sequence is a tight frame, defined as a sequence of functions in $L^2(\Omega)$ satisfying (1) with equal frame constants, i.e., $c = C$.

2. Assume that the exponentials $E(A_1)$ have a lower frame bound in $L^2(\Omega_1)$. Then for any $A \subseteq \mathbb{Z}_N^d$ and $J \subseteq \mathbb{Z}_N^d$, the exponentials $E(A)$ also have a lower frame bound in $L^2(\Omega)$, where $\Lambda = A_1 + J$ and $\Omega = \cup_{a \in A} \Omega_1 + a$.

The following result is of independent interest.

**Proposition 2**. Suppose that $\Omega_1 \subset \mathbb{R}^d$ is a set with positive and finite Lebesgue measure, $A_1 \subset \mathbb{R}^d$ is a discrete and countable set, and $N \geq 1$ is an integer. Let $A$ be any subset of $\mathbb{Z}_N^d$ with $|\Omega_1 + a \cap \Omega_1 + a'| = 0$, $\forall a' \in A, a' \neq a$, and let $J$ be any subset of $\mathbb{Z}_N^d$. Define the pair $(\Omega, \Lambda)$ by

$$
\Omega := \Omega_1 + A, \quad \Lambda := A_1 + J / N.
$$

(9)

If $E(A_1)$ is a complete set in $L^2(\Omega_1)$, then the family $E(A)$ is also a complete set in $L^2(\Omega)$ provided that the condition (5) holds.

**Proof**. Let $u \in L^2(\Omega)$ and $\lambda = \lambda_1 + j / N \in \Lambda$. Then

$$
\langle u, e_\lambda \rangle_{L^2(\Omega)} = \sum_{a \in A} \langle u, e_{\lambda_1 + j / N} \rangle_{L^2(\Omega_1 + a)}
$$


\[
\sum_{a \in A} \langle u(\cdot + a), e_{\lambda_1+j/N}(\cdot + a) \rangle_{L^2(\Omega_1)} = \sum_{a \in A} u(\cdot + a) e_{j/N}(\cdot + a), e_{\lambda_1} \rangle_{L^2(\Omega_1)}. \tag{10}
\]

Assume that \( \langle u, e_{\lambda} \rangle_{L^2(\Omega)} = 0 \) for all \( \lambda = \lambda_1 + j/N \in \Lambda, \) (i.e. for all \( j \in J \) and \( \lambda_1 \in \Lambda_1 \)). By the completeness of the pair \((\Omega_1, \Lambda_1)\), by (10) for all \( j \in J \) we obtain

\[
\sum_{a \in A} u(x + a) e_{j/N}(x + a) = 0 \quad a.e. \ x \in \Omega_1.
\]

Or,

\[
\sum_{a \in A} u(x + a) e_{j/N}(a) = 0 \quad a.e. \ x \in \Omega_1.
\]

By the completeness of the pair \((A, J)\) in the setting of \( \mathbb{Z}_N^d \), the preceding equality implies that for all \( a \in A, \)

\[
u(x + a) = 0 \quad a.e. \ x \in \Omega_1,
\]

or equivalently,

\[
u(x) = 0 \quad a.e. \ x \in \Omega_1 + a.
\]

This proves that \( u = 0 \) in \( L^2(\Omega_1 + a) \) for all \( a \in A \) and therefore \( u = 0 \) in \( L^2(\Omega) \). \( \square \)

### 4 Proof of Theorem 2

First we have a motivational result.

**Proposition 3** Assume that \( \mathcal{E}(\Lambda_1) \) is a Bessel sequence in \( L^2(\Omega_1) \) with a Bessel constant \( C > 0 \), and let \( A, J \subset \mathbb{R}^d \) be any finite sets. Define

\[
\Omega := \Omega_1 + A, \quad \Lambda := \Lambda_1 + J.
\]

Then for the family of exponentials \( \mathcal{E}(\Lambda) \) the Bessel inequality holds, provided that (5) holds. Indeed, for any finite set of scalars \( \{c(\lambda, j)\}_{(\lambda, j) \in F}, \ F = \Gamma \times I \subset \Lambda_1 \times J, \) we have

\[
\left\| \sum_{(\lambda, j) \in F} c(\lambda, j)e_{\lambda + j} \right\|_{L^2(\Omega)}^2 \leq B \sum_{(\lambda, j) \in F} |c(\lambda, j)|^2, \tag{11}
\]

where \( B = (\sharp A \sharp J)C. \)
**Proof** The proof is obtained using the Cauchy-Bunyakovsky-Schwarz inequality:

\[
\left\| \sum_{(\lambda, j) \in F} c_{(\lambda, j)} e_{\lambda+j} \right\|_{L^2(\Omega)}^2 = \left\| \sum_{j \in I} \left( \sum_{\lambda \in \Gamma} c_{(\lambda, j)} e_\lambda \right) e_j \right\|_{L^2(\Omega)}^2 \\
\leq \sum_{a \in A} \left\| \sum_{j \in I} \left( \sum_{\lambda \in \Gamma} c_{(\lambda, j)} e_\lambda \right) e_j \right\|_{L^2(\Omega_1+a)}^2.
\]

(12)

For \( a \in A \),

\[
\int_{\Omega_1+a} \left| \sum_{j \in I} \left( \sum_{\lambda \in \Gamma} c_{(\lambda, j)} e_\lambda(x) \right) e_j(x) \right|^2 dx \leq (\# I) \int_{\Omega_1+a} \left| \sum_{\lambda \in \Gamma} c_{(\lambda, j)} e_\lambda(x) \right|^2 dx
\\
\leq (\# J) \sum_{j \in I} \left\| \sum_{\lambda \in \Gamma} c_{(\lambda, j)} e_\lambda(x) \right\|_{L^2(\Omega_1+a)}^2
\\
= (\# J) \sum_{j \in I} \left\| \sum_{\lambda \in \Gamma} c_{(\lambda, j)} e_\lambda(x) \right\|_{L^2(\Omega_1+a)}^2
\\
\leq C(\# J) \sum_{(\lambda, j) \in F} |c_{(\lambda, j)}|^2.
\]

The last inequality is obtained using the Bessel property of \( E(\Lambda_1) \) in \( L^2(\Omega_1) \) and the property (5). Now, using the inequality in (12) and after the summing over \( A \), the Bessel inequality (11) holds for \( E(\Lambda) \) in \( L^2(\Omega) \) with a Bessel constant \( B = (\# A \# J)C \).

Theorem 2 improves the result of Proposition 3 in the sense that a Bessel sequence is a Riesz sequence or a Riesz basis when additional assumptions on \( A \), \( J \) and \( E(\Lambda_1) \) are satisfied.

**Proof of Theorem 2** The completeness of the exponentials \( E(\Lambda) \) in \( L^2(\Omega) \) is due to Theorem 1. Indeed, \( E(\Lambda) \) is a frame for \( L^2(\Omega) \) and is therefore complete. It is then sufficient to prove

the Riesz inequalities (2) for \( E(\Lambda) \). For this, let \( F \) be a finite index set as in Proposition 3 and \( \{c_{(\lambda, j)}\}_{(\lambda, j) \in F} \) be any finite set of scalars. Using the equality in (12), we have

\[
\left\| \sum_{(\lambda, j) \in F} c_{(\lambda, j)} e_{\lambda+j/N} \right\|_{L^2(\Omega)}^2 = \sum_{a \in A} \left\| \sum_{j \in I} \left( \sum_{\lambda \in \Gamma} c_{(\lambda, j)} e_\lambda \right) e_j/N \right\|_{L^2(\Omega_1+a)}^2
\]
\[ = \sum_{a \in A} \int_{\Omega_1 + a} \left| \sum_{j \in I} \left( \sum_{\lambda \in \Gamma} c(\lambda, j) e_\lambda(x) \right) e_{j/N}(x) \right|^2 dx \]

\[ = \int_{\Omega_1} \sum_{a \in A} \left| \sum_{j \in I} \left( \sum_{\lambda \in \Gamma} c(\lambda, j) e_{\lambda + j/N}(x) \right) e_{j/N}(a) \right|^2 dx. \]

(13)

The last equality is obtained after applying the assumption that \( e_\lambda(a) = 1 \) for all \( \lambda \in \Lambda_1 \) and \( a \in A \).

For \( j \in I \), define \( d_j(x) := e_{j/N}(x) \sum_{\lambda \in \Gamma} c(\lambda, j) e_\lambda(x) \) a.e. \( x \in \Omega_1 \). Then,

\[ (13) = \int_{\Omega_1} \sum_{a \in A} \left| \sum_{j \in I} d_j(x) e_{j/N}(a) \right|^2 dx. \]

(14)

Since \( \mathcal{E}(J) \) is a basis for \( \ell^2(A) \), it is a Riesz basis by a result in [12]. Assume that \( \beta > 0 \) is the upper frame constant for the Riesz basis. Then we have

\[ (14) \leq \beta \int_{\Omega_1} \sum_{j \in I} |d_j(x)|^2 dx \]
\[ = \beta \sum_{j \in I} \int_{\Omega_1} \left| \sum_{\lambda \in \Gamma} c(\lambda, j) e_\lambda(x) \right|^2 dx \]
\[ = \beta \sum_{j \in I} \left\| \sum_{\lambda \in \Gamma} c(\lambda, j) e_\lambda \right\|^2_{L^2(\Omega_1)} \]
\[ \leq \beta C \sum_{j \in I} \sum_{\lambda \in \Gamma} |c(\lambda, j)|^2. \]

Notice that we obtained the last inequality by the Riesz basis property of \( \mathcal{E}(\Lambda_1) \) in \( L^2(\Omega_1) \). A lower Riesz bound can be obtained with a similar calculation. This completes the proof for the first part of the theorem. The proof of the second part is illustrated in Sect. 4.1.2.

4.1 Explicit Form of Biorthogonal Dual Riesz Bases

It is known that any Riesz basis in a Hilbert space has a biorthogonal dual Riesz basis [5]. In the setting of exponential Riesz bases, this statement reads as follows.
Proposition 4  Given any Riesz basis $E(\Lambda)$ for $L^2(\Omega)$, there is a unique collection of functions $\{h_\lambda\}_{\lambda \in \Lambda}$ in $L^2(\Omega)$ such that the biorthogonality condition holds:

$$\langle h_\lambda, e_{\lambda'} \rangle = |\Omega| \delta_\lambda(\lambda') \quad \forall \lambda, \lambda' \in \Lambda.$$ 

Moreover $\{h_\lambda\}_{\lambda \in \Lambda}$ is a Riesz basis for $L^2(\Omega)$ and any $u \in L^2(\Omega)$ can be represented uniquely as

$$u = |\Omega|^{-1} \sum_{\lambda \in \Lambda} \langle u, h_\lambda \rangle e_\lambda = |\Omega|^{-1} \sum_{\lambda \in \Lambda} \langle u, e_\lambda \rangle h_\lambda.$$ 

Here, $\delta_t$ is the Dirac function given by

$$\delta_t(x) = \begin{cases} 1 & \text{if } x = t \\ 0 & \text{otherwise}. \end{cases}$$ (15)

The basis $\{h_\lambda\}$ is called biorthogonal dual Riesz basis for $E(\Lambda)$. When $E(\Lambda)$ is an orthogonal basis for $L^2(\Omega)$, the system of exponentials is self-dual, i.e. $e_\lambda = h_\lambda$, and in (2) we have $c = C = |\Omega|^{-1}$.

4.1.1 Finite Case: $\mathbb{Z}_N^d$

Let $(A, J)$ be a Riesz spectral pair in $\mathbb{Z}_N^d$, and suppose that $k = \#A$, that is, $A = \{a_r\}_{r=1}^k$, and $J = \{j_s\}_{s=1}^k$. This means that the $k \times k$ matrix $F_{A,J}$ in (7) is invertible.

For $F \in \ell^2(A)$,

$$\langle F, E_{j_s} \rangle = \sum_{r=1}^k F(a_r) \omega^{a_r \cdot j_s},$$

where $\omega = \exp(-2\pi i/N)$. The reconstruction of $F$ in $\ell^2(A)$ using the Riesz basis $E(J)$ is accomplished using the linear system

$$\begin{pmatrix} \langle F, E_{j_1} \rangle \\ \langle F, E_{j_2} \rangle \\ \vdots \\ \langle F, E_{j_k} \rangle \end{pmatrix} = F_{A,J} \begin{pmatrix} F(a_1) \\ F(a_2) \\ \vdots \\ F(a_k) \end{pmatrix}$$ (16)

Assume that $\{G_{j_s}\}_{1 \leq s \leq k}$ is the dual in $\ell^2(A)$. Then, we see that by biorthogonality $\langle G_{j_s}, E_{j_{s'}} \rangle = \#A \delta_s(s')$, we can recover the dual functions $G_{j_s}$. More precisely, for
\[ F = G_{j_s} \] in (16) we get
\[
\mathcal{F}_{A,J}^{-1}(k e_s) = \begin{pmatrix} G_{j_s}(a_1) \\ G_{j_s}(a_2) \\ \vdots \\ G_{j_s}(a_k) \end{pmatrix},
\]
where \( e_s \) is a \( k \)-dimensional vector with \((e_s)_t = 1\) if \( t = s \) and 0 elsewhere. In summary, the biorthogonal dual Riesz basis can be obtained using the \( s \)th column of \( \mathcal{F}_{A,J}^{-1} \):
\[
G_{j_s}(a_r) = k(\mathcal{F}_{A,J}^{-1})_{r,s} \quad r = 1, \ldots, k.
\] (17)

Notice that by (17), the dual basis is also a set of exponentials only when \( \mathcal{F}_{A,J} \) is a unitary matrix.

### 4.1.2 Continuous Case: Multi-tiling Domains and Proof of (6)

In this section, we shall illustrate an explicit form of the dual Riesz basis for a class of exponential Riesz bases for a bounded domain \( \Omega \) given as in Theorem 2 (although the result extends to a class of unbounded domains). First let us recall a definition. For a lattice \( \Lambda = M(\mathbb{Z}^d) \) given by an invertible \( d \times d \) matrix \( M \), the volume of \( \Lambda \) is defined by \( \text{vol}(\Lambda) = \det(M) \).

Let \( \Lambda_1^* \) again be a full lattice with a fundamental domain \( \Pi_{\Lambda_1} \) and dual lattice \( \Lambda_1 \). Then, let \( A = \{a_r\}_{r=1}^k \subset \mathbb{Z}_N^d \), and let \( J = \{j_s\}_{s=1}^k \subset \mathbb{Z}_N^d \) be a finite set of vectors such that the \( k \times k \) matrix \( \mathcal{F}_{A,J} \) in (7) is invertible. Then, \((A,J)\) is a Riesz spectral pair. Assume that \((\Omega_1, \Lambda_1)\) is also a Riesz spectral pair in \( \mathbb{R}^d \), such that the assumptions (5) and (3) hold. Then \( \mathcal{E}(A) \) is a Riesz basis for \( L^2(\Omega) \) as given in Theorem 2.

In the given situation, the associated biorthogonal Riesz basis to \( \mathcal{E}(A) \) with \( \Lambda = \Lambda_1 + \frac{1}{N} J \) has been recently illustrated explicitly and constructively by the first author and Okoudjou in [13]. The result is presented in [13] for a more general class of multi-tiling sets, but here we present the special case where \( \Omega \) is a multi-rectangle in \( \mathbb{R}^d \). In this case,

for \( \lambda \in \Lambda_1 \) and \( 1 \leq s \leq k \), the dual basis functions in \( L^2(\Omega) \) are given by
\[
g_{\lambda + j_s/N}(x) = e^{2\pi i a_r \cdot j_s/N} \chi_{\Pi_{\Lambda_1}}(x - \cdot) \chi_{A}(x) \quad \text{a.e. } x \in \Omega.
\]

Here, \( \chi_{\Pi_{\Lambda_1}} \) denotes the indicator function of the domain \( \Pi_{\Lambda_1} \). For an illustration of a multi-tiling set in dimension \( d = 2 \) see Fig. 1.

**Remark 1** If the system is self-dual, then this formula implies that \( (\mathcal{F}_{A,J})_{r,s}^{-1} = \frac{1}{k} e^{2\pi i a_r \cdot j_s/N} = \frac{1}{k} \delta_{a_r,j_s} \). (Recall that \( k = \sharp A \).) In this case, \((\Omega, \Lambda)\) is a spectral pair if and only if \( (\mathcal{F}_{A,J})^*(\mathcal{F}_{A,J}) = kI \) meaning that \( \mathcal{F}_{A,J} \) is a (log) Hadamard matrix [28].

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Fig. 1 An example of a multi-tiling set of level $k = 2$ in dimension $d = 2$ and a corresponding Riesz spectrum

**Remark 2** A special case is a 1-tiling with respect to the full lattice $\Lambda_1^*$ with a fundamental domain $\Omega_1 = \Pi \Lambda_1$ and dual lattice $\Lambda_1$. In this case, for any $f \in L^2(\Omega_1)$ we have

$$\sum_{\lambda \in \Lambda_1} \langle f, e_{\lambda} \rangle e_{\lambda}(x) = \text{vol}(\Lambda_1) \sum_{\lambda^* \in \Lambda_1^*} f(x - \lambda^*) = \text{vol}(\Lambda_1) f(x).$$

Here we used the Poisson summation formula and the fact that $\Omega_1$ is a fundamental domain of $\Lambda_1^*$ (the only value of $\Lambda^*$ that gives a nonzero summand is $\lambda^* = 0$). This means that for $f \in L^2(\Omega_1)$,

$$f(x) = \frac{1}{\text{vol}(\Lambda_1)} \sum_{\lambda \in \Lambda_1} \langle f, e_{\lambda} \rangle e_{\lambda}(x) = \sum_{\lambda \in \Lambda_1} \langle f, e_{\lambda} \rangle g_{\lambda}(x).$$

The biorthogonal dual Riesz basis is then given by

$$g_{\lambda}(x) = \frac{1}{\text{vol}(\Lambda_1)} e_{\lambda}(x).$$

**4.2 Examples**

In this section, we provide few examples of a Riesz spectral pair as well as the biorthogonal dual basis (6) in dimensions $d = 1, 2$. Let $\Lambda_1 = \mathbb{Z}_2$, and let $\Pi \Lambda_1 = \mathcal{Q}_2$ be the unit square. For a pair of distinct multi-integers $a_1, a_2 \in \Lambda_1$, let

$$\Omega := (\mathcal{Q}_2 + a_1) \cup (\mathcal{Q}_2 + a_2).$$

The set $\Omega$ multi-tiles $\mathbb{R}^2$ with the lattice $\mathbb{Z}_2$ at level $k = 2$. Let $N > 2$ be an integer and $j_1, j_2$ be any two vectors in $\mathbb{Z}_2$ such that the following matrix is invertible:

$$V = \mathcal{F}_{A,J} := \begin{pmatrix} \omega^{a_1,j_1} & \omega^{a_2,j_1} \\ \omega^{a_1,j_2} & \omega^{a_2,j_2} \end{pmatrix}.$$ (18)
Take

\[ W = (F_{A,J})^{-1} = \frac{1}{\text{det}(V)} \begin{pmatrix} \omega^{a_2\cdot j_2} & -\omega^{a_2\cdot j_1} \\ -\omega^{a_1\cdot j_2} & \omega^{a_1\cdot j_1} \end{pmatrix}. \]

Since \( \text{det}(F_{A,J}) \neq 0 \), then the following system of exponential functions is a Riesz basis for \( L^2(\Omega) \):

\[ \{e_{n+j_1/N}(x)\}_{n \in \mathbb{Z}^d} \cup \{e_{n+j_2/N}(x)\}_{n \in \mathbb{Z}^d}, \]

and by (6) the dual basis functions are given by

\[ \{g_{n+j_1/N}(x)\}_{n \in \mathbb{Z}^d} \cup \{g_{n+j_2/N}(x)\}_{n \in \mathbb{Z}^d}, \]

where

\[ g_{n+j_1/N}(x) = 2e_{n+j_1/N}(x) \left( v_{11}w_{11} \chi_{Q_2}(x-a_1) + v_{12}w_{21} \chi_{Q_2}(x-a_2) \right) \quad n \in \mathbb{Z}^2, \]

and

\[ g_{n+j_2/N}(x) = 2e_{n+j_2/N}(x) \left( v_{21}w_{12} \chi_{Q_2}(x-a_1) + v_{22}w_{22} \chi_{Q_2}(x-a_2) \right) \quad n \in \mathbb{Z}^2. \]

**Example 1** (case \( d = 1 \)) Following the notation in Sect. 4.2, let \( a_1 = 0, a_2 = 2 \), let \( \{j_1 = 0, j_2 = 1\} \), and let \( N > 2 \) be any integer. Then the matrix \( V \) in (18) is invertible, so \( (\Omega, \Lambda) \) is a Riesz spectral pair, where

\[ \Omega = [0, 1] \cup [2, 3], \quad \Lambda = \mathbb{Z} \cup \mathbb{Z} + 1/N. \]

Then the exponential Riesz basis constructed above coincides with the exponential Riesz basis given as in Theorem 2, when \( \Omega_1 = [0, 1], \Lambda_1 = \mathbb{Z}, \Lambda = [0, 2], J = [0, 1], \) and \( N > 2 \). By the symmetry property, we obtain that \( \Omega = [0, 2] \) and \( \Lambda = \mathbb{Z} \cup \mathbb{Z} + 2/N \) is also a Riesz spectral pair. It can be readily verified that the system is an orthogonal basis only for \( N = 4 \); see Theorem 3. In this case, \( V \) is a unitary matrix with \( V^*V = 2I \) and the basis is self-dual.

**Example 2** (case \( d = 2 \)) As a concrete example in \( d = 2 \), let \( N = 4, a_1 = (0, 0), a_2 = (2, 0), j_1 = (0, 0), \) and \( j_2 = (1, 0) \). For this choice, the matrix \( V \) in (18) is invertible, so the following pair \((\Omega, \Lambda)\) builds a Riesz spectral pair:

\[ \Omega = Q_2 \cup Q_2 + (2, 0), \quad \Lambda = \mathbb{Z}^2 \cup \mathbb{Z}^2 + (1/4, 0). \]

By the symmetry property, for \( \Omega = [0, 2] \times [0, 1] \) and \( \Lambda = \mathbb{Z}^2 \cup \mathbb{Z}^2 + (1/2, 0) = 2^{-1}\mathbb{Z} \times \mathbb{Z} \), the pair \((\Omega, \Lambda)\) is also a Riesz spectral pair, as we expected.
Example 3 (case \(d > 1\)) Let \(\Omega\) be the convex hull of a finite number of points in \(\mathbb{R}^d\) which is centrally symmetric. Moreover, assume that all faces of \(\Omega\) in all dimensions are also centrally symmetric. Then by Theorem 1.1 in [8], \(L^2(\Omega)\) has a Riesz basis of exponentials \(\mathcal{E}(\Lambda)\). Assume that \((\Lambda, J)\) is a pair in \(\mathbb{Z}_N^d\) whose evaluation matrix is invertible. Then the pair \((\Omega + A, A + J)\) is a Riesz spectral pair in \(\mathbb{R}^d\) if (5) holds.

5 Proof of Theorem 3 and Proposition 1

Given positive integers \(N, d \geq 1\), and sets \(A, J \subseteq \mathbb{Z}_N^d\) of the same cardinality, we say that the family of exponentials \(\mathcal{E}(J) = \{ E_j(z) = e^{2\pi i j \cdot z/N} \}_{j \in J} \) is an orthogonal basis for \(\ell^2(A)\) if the elements of \(\mathcal{E}(J)\) are mutually orthogonal on \(A\); that is

\[
\langle E_j, E_{j'} \rangle_{\ell^2(A)} = \sum_{a \in A} e^{2\pi i (j - j') \cdot a/N} = 0 \quad \forall j, j' \in J, j \neq j'.
\] (19)

Notice that the orthogonality relation (19) can also be expressed as

\[
\hat{\chi}_A(j - j') = \sum_{a \in A} e^{2\pi i (j - j') \cdot a/N} = 0, \quad \forall j \neq j'
\]

where \(\hat{\chi}_A\) is the discrete Fourier transform of the indicator function \(\chi_A\).

Proof of Theorem 3 To prove mutual orthogonality of the exponentials \(\mathcal{E}(\Lambda)\) in \(L^2(\Omega)\), let \(\lambda_1 + j_1/N\) and \(\lambda_2 + j_2/N\) be two distinct vectors in \(\Lambda\). We have the following:

\[
\langle e_{\lambda_1+j_1/N}, e_{\lambda_2+j_2/N} \rangle_{L^2(\Omega)} = \sum_{a \in A} \langle e_{\lambda_1+j_1/N}, e_{\lambda_2+j_2/N} \rangle_{L^2(\Omega_1+a)}
\]

\[
= \left( \sum_{a \in A} e_{\lambda_1+j_1/N}(a) e_{\lambda_2+j_2/N}(-a) \right) \langle e_{\lambda_1+j_1/N}, e_{\lambda_2+j_2/N} \rangle_{L^2(\Omega_1)}
\]

\[
= \left( \sum_{a \in A} e_{j_1/N}(a) e_{j_2/N}(-a) \right) \langle e_{\lambda_1+j_1/N}, e_{\lambda_2+j_2/N} \rangle_{L^2(\Omega_1)}
\] (20)

\[
= \langle E_{j_1}, E_{j_2} \rangle_{\ell^2(A)} \langle e_{\lambda_1+j_1/N}, e_{\lambda_2+j_2/N} \rangle_{L^2(\Omega_1)}.
\] (21)

To pass from (20) to (21) we used the assumption that \(\hat{\delta}_a(a) = 1\) for all \(a \in A\) and \(\lambda \in \Lambda_1\). If \(j_1 \neq j_2\), by the orthogonality of \(\mathcal{E}(J)\), the first inner product in (22) is zero. If \(j_1 = j_2\), then by the assumption we must have \(\lambda_1 \neq \lambda_2\), and by the orthogonality of \(\mathcal{E}(A_1)\) we get \(\langle e_{\lambda_1+j_1/N}, e_{\lambda_2+j_2/N} \rangle_{L^2(\Omega_1)} = \langle e_{\lambda_1}, e_{\lambda_2} \rangle_{L^2(\Omega_1)} = 0\). The completeness of \(\mathcal{E}(\Lambda)\) in \(L^2(\Omega)\) is obtained by a density condition (see [9]), or directly by Theorem 1 since \(\mathcal{E}(\Lambda)\) is a frame for \(L^2(\Omega)\).
Remark 3  Note due to the symmetry property of spectral pairs in finite settings, by the assumption of Theorem 3, the domain $\Omega = \Omega_1 + J$ admits an orthogonal basis of exponentials $\mathcal{E}(A)$ with $A = A_1 + A/N$, provided that (3) and (5) are satisfied.

Remark 4  Let $(\Omega_1, A_1)$ be a spectral pair. Assume that $A \subset \mathbb{Z}^d$ is finite such that (3) and (5) are satisfied, and $B \subset \mathbb{R}^d$ is a finite set of rational numbers. Assume that $\sharp A = \sharp B = d$. Then there exists an integer number $N \geq 1$ such that $NB \subset \mathbb{Z}^d$. Consider the $d \times d$ matrix $V = (v_{a,b})_{a \in A, b \in B}$ with entries given by

$$v_{a,b} = e^{-2\pi i a \cdot b}.$$ 

If $V$ is a unitary (or an invertible) matrix, then $\mathcal{E}(A)$ is an orthogonal basis (or a Riesz basis) for $L^2(\Omega)$, where

$$\Omega = \Omega_1 + A, \ \Lambda = \Lambda_1 + B.$$ 

This can be readily obtained by Theorem 3 since $(A, NB)$ is a spectral pair in $\mathbb{Z}^d_N$. The proof of the Riesz basis is due to Theorem 2 when $V$ is an invertible matrix.

5.1 Examples of Spectral Pairs

The first example shows that the sufficient condition (5) is also a necessary condition in Theorem 3.

Example 4  Let $\Omega_1 = [0, 2]$. Take $N \in 6\mathbb{Z}, N > 0$, and $A = \{0, 3\}, J = \{0, \alpha = N/6\}$. Then $(A, J)$ is a spectral pair in $\mathbb{Z}^d_N$, while $(\Omega, \Lambda)$ fails to be a spectral pair in $\mathbb{R}^d$. Indeed, for $\Lambda = 2^{-1}\mathbb{Z} \cup 2^{-1}\mathbb{Z} + 6^{-1}$, the exponentials $\mathcal{E}(A)$ are not mutually orthogonal on $\Omega = [0, 2] \cup [3, 5]$. Otherwise we would have had $\Lambda \subseteq \Lambda - \Lambda \subseteq Z_\Omega$, where

$$Z_\Omega = \{0\} \cup \{\xi \in \mathbb{R} : \hat{\chi}_\Omega(\xi) = 0\}.$$ 

On the other hand, $Z_\Omega = 2^{-1}\mathbb{Z} \cup 3^{-1}\mathbb{Z}$, and $\Lambda$ is not contained in $Z_\Omega$. This is a contradiction, thus the functions in $\mathcal{E}(A)$ are not orthogonal on $\Omega$.

Example 5  Let $N \geq 1$, and assume that $(A, J)$ is a spectral pair in $\mathbb{Z}^d_N$. Take $\Omega := \cup_{a \in A}[a, a+1)$. Then, by Theorem 3, the pair $(\Omega, A)$ is spectral in $\mathbb{R}$ for $A = \mathbb{Z} + J/N$.

Example 6  [$d = 1$] Let $N = 4, A = \{0, 2\}$ and $J = \{0, 1\}$. Define the $2 \times 2$ evaluation matrix

$$H = \frac{1}{\sqrt{2}} (\omega^{a \cdot j})_{a \in A, j \in J}.$$ 

Then, $H$ is a unitary matrix, that is, $H^* H = I$. This proves that $(A, J)$ is a spectral pair in $\mathbb{Z}_4$. By Theorem 3, for $\Omega = [0, 1] \cup [2, 3]$ and $\Lambda = \mathbb{Z} \cup \mathbb{Z} + 1/4$, the pair $(\Omega, \Lambda)$ is a spectral pair $\mathbb{R}$. This example is illustrated in Fig. 2.
Fig. 2 The figure illustrates $\Omega = [0, 1] \cup [2, 3]$ and its spectral set $\Lambda = \mathbb{Z} \cup \mathbb{Z} + 1/4$ in $d = 1$.

Fig. 3 An example of a spectral set with its spectrum in $d = 2$.

By the symmetry property, the pair $(J, A)$ is also a spectral pair. Therefore, using the result of Theorem 3 we obtain the well-known spectral pair $(\tilde{\Omega} = [0, 2], \tilde{\Lambda} = 2^{-1} \mathbb{Z})$.

**Example 7** [$d = 2$] To construct a spectral pair in higher dimensions, one possible technique is using the Cartesian product as in the results of Jorgensen and Pedersen in [24, Section 2]. See also [22, 23] for more on the construction and ‘closeness’ of spectral pairs in Cartesian settings. An example of a spectral pair in dimension $d = 2$ is depicted in Fig. 3. This example is constructed from the self Cartesian product of the spectral pair given in Example 6.

### 5.2 Spectral Set and Function Recovery

In this section, we remark on the link between spectral pairs and recovery problem, known as Shannon Sampling theorem. First, we need the following general result for any pair in $\mathbb{Z}_N$.

**Lemma 1** Define $\Omega = [0, 1] + A$ and $\Lambda = \mathbb{Z} + J/\mathbb{N}$ for any sets $A, J \subset \mathbb{Z}_N$, and suppose that $f$ is a function in the Paley-Wiener space $PW_\Omega = \{f \in L^2(\mathbb{R}) \mid \hat{f}(\xi) = 0 \text{ a.e. } \xi /\in \Omega\}$. Then, the distribution

$$F_s(t) = \sum_{\lambda \in \Lambda} f(\lambda) \delta_{\lambda}(t), \ t \in \mathbb{R}$$

(23)
has a Fourier transform given by
\[ \hat{F}_s(\xi) = \sum_{k \in \mathbb{Z}} \hat{\chi}_J(k) \hat{f}(\xi - k) \ \text{a.e. } \xi \in \Omega. \] (24)

**Proof** We express the distribution (general function) \( F_s \) as
\[ F_s(t) = f(t)P(t) \] (25)
where \( P \in \mathcal{S}'(\mathbb{R}) \) is a distribution, known as a \((\Lambda\text{-sampling})\) pattern function, given by
\[ P(t) := \sum_{\lambda \in \Lambda} \delta_\lambda(t) = \sum_{n \in \mathbb{Z}} \delta_n(t) \ast \sum_{j \in J} \delta_{j/N}(t), \ t \in \mathbb{R}. \]

An example of such pattern is illustrated in Fig. 4. By applying the Fourier transform to (25) we obtain
\[ \hat{F}_s = \hat{f} \ast \hat{P}, \] (26)
where
\[ \hat{P}(\xi) = \mathcal{F}(\sum_{n \in \mathbb{Z}} \delta_n(\xi))\mathcal{F}(\sum_{j \in J} \delta_{j/N}(\xi)), \ \xi \in \mathbb{R}. \] (27)

By applying the Poisson summation formula to the first term of (27), we get
\[ \mathcal{F}\left(\sum_{n \in \mathbb{Z}} \delta_n \right)(\xi) = \sum_{n \in \mathbb{Z}} e^{-2\pi in\xi} = \sum_{k \in \mathbb{Z}} \delta_k(\xi), \ \xi \in \mathbb{R} \] (28)

For the second term in (27), in terms of distributions we have
\[ \mathcal{F}\left(\sum_{j \in J} \delta_{j/N} \right)(\xi) = \sum_{j \in J} e^{-2\pi ij\xi/N}, \ \xi \in \mathbb{R}. \] (29)
Define the function \( \hat{\chi}_J(\xi) := \sum_{j \in J} e^{-2\pi i j \xi / N} \). By (28) and plugging (29) in (27), we obtain

\[
\hat{P}(\xi) = \hat{\chi}_J(\xi) \sum_{k \in \mathbb{Z}} \delta_k(\xi) = \sum_{k \in \mathbb{Z}} \hat{\chi}_J(k) \delta_k(\xi).
\]

By substituting this expression in (26) we obtain

\[
\hat{F}_s(\xi) = \hat{f} \ast \hat{P}(\xi) = \left( \hat{f} \ast \sum_{k \in \mathbb{Z}} \hat{\chi}_J(k) \delta_k \right)(\xi) = \sum_{k \in \mathbb{Z}} \hat{\chi}_J(k) \left( \hat{f} \ast \delta_k \right)(\xi) = \sum_{k \in \mathbb{Z}} \hat{\chi}_J(k) \hat{f}(\xi - k) \quad \forall \xi \in \Omega.
\]

This completes the proof\(^2\).

Note that Lemma 1 holds for any domain \( \Omega \subset \mathbb{R}^d, d \geq 1 \), with finite and positive measure and any \( \Lambda = \Lambda_1 + J/N, J \subset \mathbb{Z}_N^d \), such that the Poisson summation holds for \( \Lambda_1 \).

We are now ready to prove Proposition 1.

**Proof (Proof of Proposition 1)** Let \( F_s \) be as in (24). Then

\[
\hat{F}_s(\xi) = (\sharp J) \hat{f}(\xi), \text{ a.e. } \xi \in \Omega.
\]

Indeed, we need to show that all the terms in (30d) are zero except for \( k = 0 \). We do this by considering different cases for the values of \( k \).

**I:** If \( k = 0 \), we have \( \hat{\chi}_J(k) \hat{f}(\xi - k) = \sharp J \hat{f}(\xi) \).

**II:** If \( k \neq 0 \), the summation is known as **aliasing term** and we consider two cases as well:

(a) \( k \in A - A, k \neq 0 \). Then, \( k = a - a' \) for some distinct \( a, a' \in A \). By the symmetry property of the spectral pairs in \( \mathbb{Z}_N \) (and more generally in \( \mathbb{Z}_N^d \), \( A \) is a spectral set for the set \( J \) and we obtain \( \hat{\chi}_J(k) = \hat{\chi}_J(a - a') = 0 \).

---

1. Note that the definition of this general function over \( \mathbb{R} \), also called the **symbol** of \( J \), coincides with the Fourier transform of the characteristic function \( \chi_J \) over the cyclic group when the domain is restricted to \( \mathbb{Z}_N \) up to a constant.
2. By (30d), the Fourier transform of \( F_s \) is equal to a sum of translated copies of the Fourier transform of \( f \) on \( k \)-shifts of \( \Omega \) multiplied with coefficients \( \hat{\chi}_J(k) \). The theorem proves that the Fourier transform of \( F_s \) over \( \Omega \) is the exact Fourier transform of \( f \) up to some constant, and the translations do not overlap. In the language of signal processing, this means that the **aliasing term** is zero.
(b) \( k \in \mathbb{Z}\backslash\{A - A\}, k \neq 0 \). In this case, we claim that \(|\Omega \cap \Omega + k| = 0\). If not, then there must be \( a, a' \in A \) such that \(|\{a + k, a + k + 1\} \cap \{a', a' + 1\}| \neq 0\). Since \( a \in \mathbb{Z} \) and each interval has length unit 1, then we must have \( k = a' - a \), which is a contradiction. Thus the sum over all \( k \notin A - A \) is equal to zero.

In summary, we obtain
\[
\hat{F}_s(\xi) = (\sharp J) \hat{f}(\xi) \quad \text{for a.e. } \xi \in \Omega,
\]
\[
\hat{f}(\xi) = (\sharp J)^{-1} \hat{F}_s(\xi).
\]

(31)

Note that by the definition of \( F_s \) in (24), we obtain
\[
\hat{F}_s(\xi) = \int_{\mathbb{R}} \left( \sum_{\lambda \in \Lambda} f(\lambda) e^{-2\pi i t \lambda} \right) e^{-2\pi i t \xi} dt = \sum_{\lambda \in \Lambda} f(\lambda) e^{-2\pi i \lambda \xi} \quad \text{a.e. } \xi \in \Omega.
\]

Substituting this in (31) we complete the proof:
\[
\hat{f}(\xi) = (\sharp J)^{-1} \sum_{\lambda \in \Lambda} f(\lambda) e^{-2\pi i \lambda \xi} \quad \text{a.e. } \xi \in \Omega.
\]

\( \square \)

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