Freezing sandpiles and Boolean threshold networks: 
equivalence and complexity

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Abstract

The NC versus P-hard classification of the prediction problem for sandpiles on the two dimensional grid with von Neumann neighborhood is a famous open problem. In this paper we make two kinds of progresses, by studying its freezing variant. First, it enables to establish strong connections with other well known prediction problems on networks of threshold Boolean functions such as majority. Second, we can highlight some necessary and sufficient elements to the dynamical complexity of sandpiles, with a surprisingly crucial role of cells with two grains.

1 Introduction

The sand pile model, as well as the Boolean threshold automata, have been studied and applied extensively in various domains [4, 20, 9, 14, 12, 7, 13]. The classical sandpile model on the two dimensional grid with von Neumann neighborhood was introduced in the 1980 by Bak, Tang and Wiesenfeld, as a simple and natural model of some physical phenomena [1]. In [8] Goles and Margenstern showed that in arbitrary graphs, any given Turing machine can be simulated by a configuration of the sandpile model. This means that on an arbitrary topology the dynamic of the sandpile model is Turing-universal. After that, in [21], Moore and Nilsson started the study of how difficult it is to predict the behavior of sandpiles, bringing the question to the formal theory of computational complexity. Sandpile prediction problems are usually solvable in polynomial time by simply running the simulation until the dynamics reaches a stable state. Essentially, the results of Moore and Nilsson say that sandpiles in one dimension are efficiently predictable in parallel (in NC), and that sandpiles in three dimensions or more are intrinsically sequential (P-complete). It leaves open the two-dimensional case, which has not yet been closed despite considerable efforts [6, 5, 13, 24, 3].

Following a trend of research leading to new discoveries around well known open problems on majority dynamical systems (reviewed in Section 3), we introduce in this paper the freezing variant of sandpiles, where each site can be fired at most once.
interestingly, the freezing world breaks a fundamental barrier between majority and sandpiles. Though it is known that two-dimensional majority can simulate two-dimensional sandpiles [13], it is unknown whether the converse is true. Indeed, the main difficulty lies in the so called abelian property of sandpile models (the fact that sand grains may be toppled in any order), which is absent in the majority rule. This later model therefore heavily depends on the parallel schedule of cells update (see [10]), which is not the case in sandpiles and makes a simulation result hard to establish. It turns out that the freezing world breaks this frontier, as majority and other threshold Boolean functions are not sensitive to the order of cells update in this case. Predicting freezing variants of dynamical systems may be thought as a “simplest case” study of their complexity (see [4, Proposition 6] and Remark [1] for a formal discussion).

The complexity classes at stake are AC$^0$ (constant time in parallel), NL (non-deterministic logarithmic space), NC (poly-logarithmic time in parallel), and P (polynomial time), with AC$^0$ ⊆ NL ⊆ NC ⊆ P (see for example [16]).

In Section 2 we define the model and problem under consideration and in Section 3 we review results on the computational complexity of prediction problems in the freezing world. Section 4 establishes in this setting an isomorphism between the dynamics of sandpiles and threshold Boolean functions on a grid layout. Finally, Section 5 studies all possible restrictions for the freezing sandpile prediction problem, consisting in allowing only a subset of sand contents in the configuration given as input. All but two cases are classified as being in NC or as hard as the general (freezing) case, which allows to define even simpler sandpile prediction problems yet preserving the complexity of the general case. The two remaining cases are discussed at the end of the Section; the difficulty to relate them with other models brings novel insights on a possible hierarchy of sandpile prediction problems, between NC and P.

2 Definitions

We consider the freezing variant of the classical sandpile model introduced by Bak, Tang and Wiesenfeld in [1] on the two dimensional grid with von Neumann neighborhood. A configuration $c \in (\mathbb{N} \cup \{-\infty\})^{\mathbb{Z}^2}$ assigns a number of sand grains to each cell of the grid, or $-\infty$ when a cell has already fired. For commodity let $c_v$ denote the sand content at position $v \in \mathbb{Z}^2$ in configuration $c$. When the sand content of a cell exceeds its number of out-neighbors (four in the grid with von Neumann neighborhood), then the cell gives one grain to each of its out-neighbors and enters the state $-\infty$ (freezing state) so that it never gives grains again. Formally, with $\mathcal{N}(i,j) = \{(i, j + 1), (i + 1, j), (i, j - 1), (i - 1, j)\}$ the dynamics is defined by $F : (\mathbb{N} \cup \{-\infty\})^{\mathbb{Z}^2} \to (\mathbb{N} \cup \{-\infty\})^{\mathbb{Z}^2}$ such that for all $v \in \mathbb{Z}^2$,

$$F(c)_v = \begin{cases} -\infty & \text{if } c_v \geq 4 \\ c_v + \sum_{u \in \mathcal{N}(v)} 1_{\mathbb{N}}(c_u - 4) & \text{otherwise} \end{cases}$$

where $1_{\mathbb{N}}(x)$ is the indicator function of $\mathbb{N}$, which equals 1 when $x \geq 0$, and 0 when $x < 0$, for any $x \in \mathbb{Z}$. Remark that this discrete dynamical system is deterministic.
When a cell gives grains to its neighbors we say that it fires, and immediately freezes.

Classically, a configuration $c$ is finite when the number of non-empty cells is finite, that is when $|\{v \mid c_v \neq 0\}| < \infty$. Note that up to translation, all non-empty cells of a finite configuration can always be placed inside a rectangle of (finite) size $n \times m$ with the bottom left corner at the origin, hence going from $(0,0)$ to $(n-1,m-1)$. Such rectangular non-empty parts of finite configurations will be given as inputs to the problem we consider. For the purpose of this article, since we will restrict the allowed values on configurations and sometimes forbid the value 0, we define a finite configuration $c$ as having the freezing state outside the encompassing rectangle, that is with $c_{(v_x,v_y)} = -\infty$ when $v_x < 0$ or $v_x \geq n$ or $v_y < 0$ or $v_y \geq m$. A configuration $c$ is stable when no grain moves, that is when $c_v < 4$ for all cells $v$. Additionally, we say that a finite configuration $c$ is simple when for all cell $v$ inside the rectangle of size $n \times m$ we have $c_v \in \{0, 1, 2, 3, 4\}$.

**Freezing sandpiles prediction problem (FSPP)**

*Input:* a simple finite configuration $c$ and a cell $v$.

*Question:* does there exist $t$ such that $F^t(c)_v \geq 4$?

It is straightforward to notice that the problem is solvable in quadratic time (FSPP $\in \mathbb{P}$) by running the simulation: one step takes a linear time to be computed, and at least one cell freezes at each step or we have reached a stable configuration. Since cells remain frozen, after linearly many steps the configuration is stable, and we can answer.

Let Sandpiles prediction problem (SPP) be the analogous prediction problem on classical (non-freezing) sandpile model, the one not known to be in $\mathbb{NC}$ nor $\mathbb{P}$-hard. We denote $\leq_{\mathbb{NC}}$ the many-one reduction in $\mathbb{NC}$, and $\leq_{\mathbb{AC}^0}$ the many-one reduction in $\mathbb{AC}^0$.

**Remark 1.** As stated in [24] (Lemma 1), when there is only one value 4, and if furthermore this value 4 is placed on the border of the $n \times m$ rectangle containing the finite configuration, then SPP is the same as FSPP because each cell is fired at most once.

### 3 Known results

For the sandpile model [21] and majority cellular automata [20], it has early been proven that the prediction problem is in $\mathbb{NC}$ for dimension one, and $\mathbb{P}$-hard for dimension three and above. Such $\mathbb{P}$-hardness results [18, 22, 23] are commonly proven via reductions from the canonical circuit value problem (CVP) originally proven to be $\mathbb{P}$-complete by Ladner [17], or its monotone variant (MCVP), or its planar variant (PCVP) (see [16]).

It is remarkable that all these reductions employ Bank’s encoding [2]: dynamical chains of reactions implement circuit computations on a quiescent background, as electrons moving along wires. Note that planar monotone circuit value problem (PMCVP), which is easily reducible to sandpiles and majority prediction problems (even freezing), has however been proven to be in $\mathbb{NC}$, i.e. efficiently computable in parallel [27].

Studies of the freezing world have been introduced in cellular automata [15], where the authors prove that Turing universality can be achieved even in one dimension, and
that the prediction problem may be P-complete in two dimensions, though in one dimension it is in NL. On the two-dimensional grid with von Neumann neighborhood, it is proven in [3] that at least two state changes are necessary to be intrinsically universal according to block simulation, and furthermore that two state changes are sufficient. The paper [25] places freezing cellular automata universality and prediction in the broader context of bounded-change and convergent cellular automata.

Previous works on the prediction of threshold (majority-like) functions will be useful in our analysis of sandpile prediction problems complexity. It is known that predicting freezing strict majority is P-complete for undirected graphs of maximum degree at least five (\(\Delta(G) \geq 5\)), and in NC for undirected graphs of maximum degree at most four (\(\Delta(G) \leq 4\)) [14]. Regarding freezing non-strict majority, its prediction is P-complete for undirected graphs of maximum degree at least four (\(\Delta(G) \geq 4\)), and in NC for undirected graphs of maximum degree at most three (\(\Delta(G) \leq 3\)) [14]. This latter has also been proven to be in NC for the two-dimensional grid with von Neumann neighborhood [7].

Remark that, although planarity is known to forbid information crossing on sandpile models [5, 24], it is an obstacle that can be overcome on non-freezing majority [11] (based on a planar traffic light gadget of degree five, exploiting the non-freeziness).

4 Sandpiles as a patchwork of threshold Boolean functions

We begin with a remark linking the dynamics of (freezing) sandpiles to that of an assembly of Boolean functions. These relations will be useful to employ the literature in order to prove that some problems are in NC (Section 5). Indeed, the dynamics of finite freezing sandpiles can be seen as a grid network of freezing threshold Boolean functions.

Let us define finite freezing Boolean networks on the two dimensional grid with von Neumann neighborhood. Let \(G_{n \times m} = (V_{n \times m}, E_{n \times m})\) be the finite undirected graph defined as the subgraph of the two dimensional grid induced by vertices in the rectangle of size \(n \times m\) with the bottom left corner at the origin. Formally,

\[
V_{n \times m} = \{(x, y) \in \mathbb{Z}^2 \mid 0 \leq x \leq n - 1 \text{ and } 0 \leq y \leq n - 1\}
\]

\[
E_{n \times m} = \{(u, v) \in V^2 \mid v \in \mathcal{N}(u)\}
\]

For simplicity, when the dimensions are clear from the context, we will denote \(G = (V, E)\) such a graph. The set of configurations is \(\{0, 1\}^V\), and each vertex \(v\) is equipped with a local Boolean function which is freezing (state 1 is always sent to state 1). We use five such local functions, given a configuration \(c\):

- \(\land\) (and) defined as \(f^\land_v(c) = \begin{cases} 1 & \text{if } c_v = 1 \text{ or } \sum_{u \in \mathcal{N}(v)} c_u = 4 \\ 0 & \text{otherwise} \end{cases}\)

- \(M\) (strict majority) defined as \(f^M_v(c) = \begin{cases} 1 & \text{if } c_v = 1 \text{ or } \sum_{u \in \mathcal{N}(v)} c_u > 2 \\ 0 & \text{otherwise} \end{cases}\)
• \(m\) (non-strict majority) defined as \(f^m_v(c) = \begin{cases} 1 & \text{if } c_v = 1 \text{ or } \sum_{u \in \mathcal{N}(v)} c_u \geq 2 \\ 0 & \text{otherwise} \end{cases}\)

• \(\lor\) (or) defined as \(f^\lor_v(c) = \begin{cases} 1 & \text{if } c_v = 1 \text{ or } \sum_{u \in \mathcal{N}(v)} c_u \geq 1 \\ 0 & \text{otherwise} \end{cases}\)

• \(1\) (constant 1) defined as \(f^1_v(c) = 1\)

Note that each local function only depends on the state of the vertex and its neighbors, and is invariant by permutation of the neighbors, as is the case in freezing sandpiles. Also, the formulation of local functions takes into account the fact that some vertices on the border of the graph \(G\) are missing some neighbors (the number of neighbors, four, is hard-coded in the local functions). Let \(B : \{0, 1\}^V \rightarrow \{0, 1\}^V\) be the dynamics obtained by applying in parallel the local function assigned to each vertex, i.e. such that for all \(v \in V\) we have

\[B(c)_v = f_v(c).\]

Given a finite simple sandpile configuration \(c\) of size \(n \times m\) for the freezing sandpile model, we define the corresponding freezing threshold Boolean network \(B_c\) of size \(n \times m\) (where the local function \(f_v\) at \(v\) depends on the value of \(c_v\)) by

\[
\begin{array}{c|cccc}
  c_v & 0 & 1 & 2 & 3 \\
  f_v & \land & M & m & \lor \\
\end{array}
\]

and a configuration on this network as

\[\phi(c)_v = \begin{cases} 1 & \text{if } c_v = -\infty \\ 0 & \text{otherwise}. \end{cases}\]

Given any finite simple sandpile configuration \(c\) of size \(n \times m\), we obtain the freezing Boolean network \(B_c\) of size \(n \times m\), which dynamics commutes with the transformation \(\phi\) on configurations.

**Proposition 1.** For any finite simple \(c\) and all \(t \in \mathbb{N}\) we have \(B^t_c(\phi(c)) = \phi(F^t(c))\).

**Proof.** Note that initially freezed sandpile cells (outside the rectangle of size \(n \times m\)) are discarded in the corresponding freezing Boolean network. Starting from the initial configuration with all cells in Boolean state 0, this latter will therefore simply transform a grain move from \(u\) to \(v\) into the fact that \(u\) is a neighbor of \(v\) in state 1. At each time step, a cell in the freezing state remains in the freezing state in both dynamics. Regarding other cells, in both dynamics and at each step, they enter the freezing state if and only if at least the same number (in both dynamics) of their neighbors are in the freezing state. And \(\phi\) depends only on the freezing (or not) state of each cell. \(\square\)
5 Computational complexity of FSPP

We study the computational complexity of restrictions on FSPP, depending on the sand contents that each cell of the simple finite configuration given as input can take among \{0, 1, 2, 3, 4\}. It is obvious that forbidding the value 4 leads to answering no to any prediction question, and allowing only the values 3 and 4 (or just 4) leads to always answering yes, therefore we consider only the 14 remaining cases. For any \(A \subseteq \{0, 1, 2, 3, 4\}\) we say that a configuration \(c\) is \(A\)-simple when for all cell \(v\) we have \(c_v \in A\). With this notation, simple means \(\{0, 1, 2, 3, 4\}\)-simple.

\[
\text{A-freezing sandpiles prediction problem (A-FSPP)}
\]

\text{Input: an A-simple finite configuration c and a cell v.}
\text{Question: does there exist \(t\) such that \(F^t(c)_v \geq 4\)?}

Let us underline that the restriction to \(A \subseteq \{0, 1, 2, 3, 4\}\) is sound, as follows.

**Proposition 2.** If we generalize the definition of \(A\)-simple configuration, then we still have \(A\)-FSPP \(\leq^m_{AC^0} \text{FSPP}\) for any finite \(A \subseteq \mathbb{N}\).

**Proof.** Let \((c, v)\) be an instance of \(A\)-FSPP. Since the model is freezing, the cell to cell transformation of \(c\) into \(c'\) defined as \(c_v \mapsto \min\{c_v, 4\}\) (note that the outer part of the finite rectangle is not modified) preserves the answer (after one step we have \(F(c) = F(c')\)), and \((c', v)\) is an instance of FSPP. \(\square\)

Considering \(A\) among

\[
\{0, 4\}, \{1, 4\}, \{0, 1, 4\}, \{2, 4\}, \{0, 2, 4\}, \{1, 2, 4\}, \{0, 1, 2, 4\}, \{0, 3, 4\}, \{1, 3, 4\}, \{0, 1, 3, 4\}, \{2, 3, 4\}, \{0, 2, 3, 4\}, \{1, 2, 3, 4\}, \{0, 1, 2, 3, 4\},
\]

the results are summed-up in Theorems \(1\) and \(2\) plus the Open question \(1\).

**Theorem 1.** \(A\)-FSPP \(\in\text{NC}\) when \(A\) is one of

\[
\{0, 4\}, \{1, 4\}, \{0, 1, 4\}, \{2, 4\}, \{0, 3, 4\}, \{2, 3, 4\}.
\]

**Theorem 2.** FSPP \(\leq^m_{AC^0} A\)-FSPP when \(A\) is one of

\[
\{0, 2, 4\}, \{1, 2, 4\}, \{0, 1, 2, 4\}, \{0, 2, 3, 4\}, \{1, 2, 3, 4\}, \{0, 1, 2, 3, 4\} \text{ and } \{0, 1, 2, 3, 4\}.
\]

**Open question 1.** \(\{0, 1, 3, 4\}\)-FSPP \(\leq^m_{AC^0} \{1, 3, 4\}\)-FSPP, but does \(\{1, 3, 4\}\)-FSPP \(\in\text{NC}\) or FSPP \(\leq^m_{AC^0} \{0, 1, 3, 4\}\)-FSPP?

Subsections \(5.1\) and \(5.2\) will present respectively the results of Theorems \(1\) and \(2\), Subsection \(5.3\) will present some perspectives on Open question \(1\).

5.1 Restrictions efficiently predictable in parallel

This section makes use of the developments presented in Section \(4\), in order to apply results from the literature on problems in NC.
Figure 1: Transformation in $\mathcal{AC}^0$ of a $\{1, 4\}$-simple sandpile configuration to a configuration for the freezing strict majority dynamics on the grid [14].

5.1.1 $\{0, 4\}$-FSPP

When FSPP is restricted to $\{0, 4\}$-simple configurations, according to Proposition 1 it corresponds to a finite freezing Boolean network on the grid with only and and constant 1 local functions, which can be decided in constant parallel time: the instance $(c, v)$ is positive if and only if $\phi(c)_v = 1$ or $\sum_{u \in N(v)} \phi(c)_u = 4$ (with $|N(v)| \leq 4$).

Proposition 3. $\{0, 4\}$-FSPP $\in \mathcal{AC}^0$.

5.1.2 $\{1, 4\}$-FSPP and $\{0, 1, 4\}$-FSPP

When FSPP is restricted to $\{1, 4\}$-simple configurations, according to Proposition 1 it corresponds to a finite freezing Boolean network on the grid with only strict majority and constant 1 local functions. However, constant 1 local functions are the same as strict majority cells initially in state 1 since we are in a freezing world. As a consequence, we are left with only strict majority local functions on a grid with von Neumann neighborhood, which can be predicted in $\mathcal{NC}^2$ according to [14] (to adapt the setting it is sufficient to add a border of cells in state 0). The transformation is easily performed in $\mathcal{AC}^0$, leading to an overall algorithm in $\mathcal{NC}^2$. See Figure 1 for an illustration.

Proposition 4. $\{1, 4\}$-FSPP $\in \mathcal{NC}^2$.

The result of [14] can also be applied to prove that $\{0, 1, 4\}$-FSPP is in $\mathcal{NC}$. The idea is that cells $u$ with $c_u = 0$ are completely passive in the freezing dynamics (they fire if and only if all their four neighbors are already fired and frozen). More precisely, given an instance $(c, v)$ we consider two cases.

1. If $c_v \neq 0$ then we perform in $\mathcal{AC}^0$ the following modification of the grid: each vertex $u = (u_x, u_y)$ such that $c_u = 0$ is replaced with four vertices $u_n, u_e, u_s, u_w$ and the arcs $\{(u_x, u_y + 1), u_n\}, \{(u_x + 1, u_y), u_e\}, \{(u_x, u_y - 1), u_s\}, \{(u_x - 1, u_y), u_w\}$, and $\{u_n, u_e\}, \{u_e, u_s\}, \{u_s, u_w\}, \{u_w, u_n\}$. With state 1 on vertices $u$ such that $c_u = 4$ and state 0 elsewhere, answers to the prediction under freezing strict majority on this graph $G$ and to the freezing sandpiles prediction problem are identical. Indeed, in the freezing strict majority dynamics the newly created vertices corresponding to cells such that $c_u = 0$ will never reach state 1 because they always have two of their three neighbors in state 0. Since cells such that $c_u = 0$ are completely passive
in the sandpile dynamics (i.e. considering that they do not fire leaves the behavior of other cells unchanged), and since the questioned cell $v$ is not one of these, $v$ will fire from $c$ if and only if it reaches state 1 in the freezing strict majority dynamics on $G$. Finally, we have $\Delta(G) \leq 4$, therefore [14] gives an NC$^2$ algorithm to predict the freezing strict majority dynamics. See an example on Figure 2.

2. If $c_v = 0$ then, if furthermore at least one of the four neighbors of $v$ is 0 (or $-\infty$) then $v$ cannot fire and the answer is negative. Otherwise we do the same transformation as in the case $c_v \neq 0$, and ask if each of the four neighbors of $v$ will fire (still in NC$^2$). The answer for $v$ is positive (it will fire) if and only if all its four neighbors will fire.

**Proposition 5.** \{0, 1, 4\}-FSPP $\in$ NC$^2$.

### 5.1.3 \{2, 4\}-FSPP

When FSPP is restricted to \{2, 4\}-simple configurations, according to Proposition [1] it corresponds to a finite freezing Boolean network on the grid with only non-strict majority and constant 1 local functions, which are the same as non-strict majority cells initially in state 1 since we are in a freezing world, and can be decided in NC$^2$ according to [7].

**Proposition 6.** \{2, 4\}-FSPP $\in$ NC$^2$.

### 5.1.4 \{0, 3, 4\}-FSPP

When FSPP is restricted to \{0, 3, 4\}-simple configurations, according to Proposition [1] it corresponds to a finite freezing Boolean network on the grid with only and, or and constant 1 local functions. Given an instance $(c, v)$, we consider three cases.

1. If $c_v \neq 0$, then we can simply remove the vertices $v$ with $c_v = 0$ from the graph supporting the finite freezing Boolean network dynamics since they are completely passive (they freeze to 1 if and only if their four neighbors are already fired and frozen). This construction is done in AC$^0$ and comes down to deciding if there is a path from a cell in state 1 to $v$, which can be done in NL (choose non-deterministically a starting cell in state 1 and travel non-deterministically through a path of length at most $nm$).
Figure 3: Cell to macrocell correspondence in the reduction from \{2, 3, 4\}\text{-FSPP} to \{2, 4\}\text{-FSPP}. After one step the grey cells in the macrocell corresponding to a cell with three grains have three grains (they are neighbor of exactly one cell with four grains).

2. If \(c_v = 0\) and \(c_u \neq 0\) for all \(u \in \mathcal{N}(v)\), then we compute sequentially the answers of the four instances \((c, u)\) for \(u \in \mathcal{N}(v)\) (still in NL), and answer positively if and only if all these four instances are positive.

3. If \(c_v = 0\) and \(c_u = 0\) for at least one \(u \in \mathcal{N}(v)\) then we can answer negatively: \(v\) needs \(u\) to go to state 1 first (strictly before \(v\) does), and conversely.

Deciding in which of these three cases we are and answering it gives an algorithm in NL for \{0, 3, 4\}\text{-FSPP}.

**Proposition 7.** \{0, 3, 4\}\text{-FSPP} \(\in\) NL.

### 5.1.5 \{2, 3, 4\}\text{-FSPP}

The idea is to reduce the question on a \{2, 3, 4\}-simple configuration to a question on a \{2, 4\}-simple configuration, still on the grid. The transformation is presented on Figure 3: each cell at position \((u_x, u_y)\) \(\in\) \(\mathbb{Z}^2\) of the \{2, 3, 4\}-simple configuration is transformed into a macrocell of size 5\(\times\)6 whose bottom left corner is at position \((5u_x, 6u_y)\). The questioned cell is placed on the bottom left corner of the corresponding macrocell (other positions are possible). This reduction can be computed in constant parallel time, i.e. in AC\(^0\).

The correctness of the reduction is easily deduced from the abelian property of sandpiles (the fact that, when the dynamics converges to a stable configuration, it converges to the same stable configuration regardless of the order in which firings are performed, in parallel or sequentially \[6\]). Indeed, if we first consider the firing of values four in the macrocell corresponding to cell with three grains, then a firing can occur on the \{2, 3, 4\}-simple configuration if and only if the whole corresponding macrocell can be fired (otherwise, none of the macrocell’s cells is fired, apart from the initially fired values four in the macrocells corresponding to cells with three grains). The result follows by induction.

**Proposition 8.** \{2, 3, 4\}\text{-FSPP} \(\leq_{\text{AC}^0}^m\) \{2, 4\}\text{-FSPP}, therefore from Proposition 8 we have \{2, 3, 4\}\text{-FSPP} \(\in\) NC\(^2\).
5.2 Restrictions as hard to predict as FSPP

We begin with a trivial remark that $FSPP \leq_{AC^0} \{0, 1, 2, 3, 4\}$-FSPP with the identity function since the two problems are identical. We treat subsequent cases one by one, and always use the same reduction technique: a cell of an input for $FSPP$ is converted to a macrocell (i.e. a fixed size rectangle of cells) of an input for $A$-FSPP, in constant time and in parallel.

5.2.1 $\{1, 2, 3, 4\}$-FSPP

The reduction is defined as follows: given an instance $(c, v)$ of $FSPP$, we replace each vertex $(u_x, u_y) \in \mathbb{Z}^2$ of $c$ with a macrocell of size $5 \times 5$ whose bottom left corner is at position $(5u_x, 5u_y)$. The cell to macrocell correspondence is given on Figure 4. This reduction can be computed in constant parallel time, i.e. in $AC^0$. Let us denote $c'$ the obtained configuration with $v'$ the new questioned cell.

We now argue in details that $(c, v) \in FSPP$ if and only if $(c', v') \in \{1, 2, 3, 4\}$-FSPP, i.e. the reduction is correct. First, $(c', v')$ is a valid instance of $\{1, 2, 3, 4\}$-FSPP since $c'$ is a finite $\{1, 2, 3, 4\}$-simple configuration. Except for the questioned cell and cells without grains (these latter having no influence on the dynamics), there is a strict correspondence between the dynamics of $c$ and $c'$: if vertex $(u_x, u_y)$ of $c$ fires at time $t$, then vertex $(5u_x + 2, 5u_y + 2)$ of $c'$ fires at time $5t$. Indeed, each background value 1 surrounding lines and columns of 3 (which link the centers of macrocells) is neighbor of at most two values 3 (even if we consider the macrocells neighbor to the macrocell corresponding to $v$ when $c_v = 0$), therefore none of them is ever fired and the correspondence is strict. Regarding the new questioned cell $v'$ and its associated macrocell, one can simply remark that for any value of $c_v$ the new cell $v'$ is fired if and only if $4 - c_v$ centers of neighboring macrocells are fired, and that when this is not (yet) the case then no value 1 in this macrocell is fired. As a consequence we get the result.

Figure 4: Cell to macrocell correspondence in the reduction from $FSPP$ to $\{1, 2, 3, 4\}$-FSPP. Top: cells different from $v$. Bottom: $v$, with the new questioned cell highlighted.
\[ \forall a \in \{0, 2, 3, 4\} : a \mapsto \begin{bmatrix} 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 3 & 3 & a & 3 & 3 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \end{bmatrix} \]

\[ 1 \mapsto \begin{bmatrix} 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 3 & 2 & 3 & 2 & 3 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 3 & 3 & 0 & 0 \end{bmatrix} \]

\[ \forall a \in \{0, 2, 3, 4\} : a \mapsto \begin{bmatrix} 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 3 & 3 & a & 3 & 3 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \end{bmatrix} \]

\[ 1 \mapsto \begin{bmatrix} 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 3 & 2 & 3 & 0 & 3 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 3 & 3 & 0 & 0 \end{bmatrix} \]

**Figure 5:** Cell to macrocell correspondence in the reduction from \( \text{FSPP} \) to \( \{0, 2, 3, 4\}-\text{FSPP} \). Top: cells different from \( v \). Bottom: \( v \), with the new questioned cell highlighted.

**Proposition 9.** \( \text{FSPP} \leq_{\text{AC}^0} \{1, 2, 3, 4\}-\text{FSPP} \).  

**5.2.2 \( \{0, 2, 3, 4\}-\text{FSPP} \)**  
The reduction is defined as follows: given an instance \((c, v)\) of \( \text{FSPP} \), we replace each vertex \((u_x, u_y)\) in \( c \) with a macrocell of size \( 5 \times 5 \) whose bottom left corner is at position \((5u_x, 5u_y)\). The cell to macrocell correspondence is given on Figure 5. This reduction can be computed in constant parallel time, i.e. in \( \text{AC}^0 \). Let us denote \( c' \) the obtained configuration with \( v' \) the new questioned cell.

The argumentation regarding the correctness of this reduction is analogous to the case of Proposition 9 except that firings may not be perfectly synchronized because of the macrocell corresponding to the value 1 doing some zigzag, but this has no consequence thanks to the so called *abelian property* of sandpiles which still holds on freezing sandpiles (any sequence of firings in \( c \) is reproduced in \( c' \), and conversely).

**Proposition 10.** \( \text{FSPP} \leq_{\text{AC}^0} \{0, 2, 3, 4\}-\text{FSPP} \).  

**5.2.3 \( \{0, 1, 2, 4\}-\text{FSPP} \)**  
The reduction is defined as follows: given an instance \((c, v)\) of \( \text{FSPP} \), we replace each vertex \((u_x, u_y)\) in \( c \) with a macrocell of size \( 7 \times 7 \) whose bottom left corner is at position \((7u_x, 7u_y)\). The cell to macrocell correspondence is given on Figure 6. This reduction can be computed in constant parallel time, i.e. in \( \text{AC}^0 \). In the constructed macrocells, each value 2 is neighbor of exactly one value 4, and consequently all become value 3. The rest of the argumentation regarding the correctness of this reduction is analogous to the case of Proposition 10.

**Proposition 11.** \( \text{FSPP} \leq_{\text{AC}^0} \{0, 1, 2, 4\}-\text{FSPP} \).
∀ a ∈ \{0, 1, 2, 4\} : a \mapsto

\begin{array}{cccccccc}
0 & 0 & 0 & 2 & 4 & 0 & 0 \\
0 & 0 & 0 & 2 & 4 & 0 & 0 \\
0 & 0 & 0 & 2 & 4 & 4 & 4 \\
2 & 2 & 2 & a & 2 & 2 & 2 \\
4 & 4 & 4 & 2 & 0 & 0 & 0 \\
0 & 0 & 4 & 2 & 0 & 0 & 0 \\
0 & 0 & 4 & 2 & 0 & 0 & 0 \\
\end{array}

3 \mapsto

\begin{array}{cccccccc}
0 & 0 & 0 & 2 & 4 & 0 & 0 \\
0 & 0 & 0 & 2 & 4 & 0 & 0 \\
0 & 0 & 0 & 2 & 4 & 0 & 0 \\
2 & 2 & 2 & 2 & 4 & 2 & 2 \\
4 & 4 & 4 & 2 & 2 & 2 & 0 \\
0 & 0 & 4 & 2 & 0 & 4 & 0 \\
0 & 0 & 4 & 2 & 0 & 0 & 0 \\
\end{array}

Figure 6: Cell to macrocell correspondence in the reduction from **FSPP** to \{0, 1, 2, 4\}-**FSPP**. Macrocells corresponding to the questioned cell \(v\) are identical, with the new questioned cell in the center (relative position (3, 3)).

5.2.4 \{1, 2, 4\}-FSPP

We give a reduction from \{0, 1, 2, 4\}-**FSPP** to \{1, 2, 4\}-**FSPP**, by replacing each vertex \((u_x, u_y) \in \mathbb{Z}^2\) of \(c\) with a macrocell of size \(5 \times 7\) whose bottom left corner is at position \((5u_x, 7u_y)\). The cell to macrocell correspondence is given on Figure 7. This reduction can be computed in constant parallel time, \textit{i.e.} in \(\mathcal{AC}^0\).

The argumentation regarding the correctness of this reduction is analogous to the case of Proposition 11, with the additional remark that some values 1 in the background may fire, without any side effect. Since \(\mathcal{AC}^0\) is closed by composition, Proposition 11 gives the result.

**Proposition 12.** \(\text{FSPP} \leq^m_{\mathcal{AC}^0} \{1, 2, 4\}-\text{FSPP}\).

5.2.5 \{0, 2, 4\}-FSPP

We give a reduction from \{0, 2, 3, 4\}-**FSPP** to \{0, 2, 4\}-**FSPP**, by replacing each vertex \((u_x, u_y) \in \mathbb{Z}^2\) of \(c\) with a macrocell of size \(7 \times 7\) whose bottom left corner is at position \((7u_x, 7u_y)\). The cell to macrocell correspondence is the same as the one given on Figure 6 from **FSPP** to \{0, 1, 2, 4\}-**FSPP**), except that the case \(a = 1\) is removed (indeed, remark that macrocells do not make use of value 1). This reduction can be computed in constant parallel time, \textit{i.e.} in \(\mathcal{AC}^0\).

The argumentation regarding the correctness of this reduction is analogous to the case of Proposition 11. Since \(\mathcal{AC}^0\) is closed by composition, Proposition 11 gives the result.

**Proposition 13.** \(\text{FSPP} \leq^m_{\mathcal{AC}^0} \{0, 2, 4\}-\text{FSPP}\).

5.3 Perspectives on \{1, 3, 4\}-FSPP and \{0, 1, 3, 4\}-FSPP

Let us first notice that the complexity of predicting both models are equivalent for \(\mathcal{AC}^0\) reductions, with the cell to macrocell correspondence given on Figure 8.

**Proposition 14.** \(\text{FSPP} \leq^m_{\mathcal{AC}^0} \{1, 3, 4\}-\text{FSPP}\).
∀ \(a \in \{0, 1, 2, 4\}\) : \(a \mapsto\) 

\[
\begin{array}{ccc}
1 & 1 & 2 \\
1 & 1 & 2 \\
2 & 2 & a \\
4 & 4 & 2 \\
1 & 4 & 2 \\
1 & 4 & 2 \\
\end{array}
\]

\[
\begin{array}{ccc}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
\end{array}
\]

0 \(\mapsto\) 

\[
\begin{array}{ccc}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
\end{array}
\]

\[
\begin{array}{ccc}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
\end{array}
\]

Figure 7: Cell to macrocell correspondence in the reduction from \(\{0, 1, 2, 4\}\)-FSPP to \(\{1, 2, 4\}\)-FSPP. Top: cells different from \(v\). Bottom: \(v\), with the new questioned cell highlighted.

0 \(\mapsto\) 

\[
\begin{array}{ccc}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
\end{array}
\]

\[
\begin{array}{ccc}
1 & 3 & 1 \\
3 & 3 & 3 \\
1 & 3 & 1 \\
1 & 3 & 1 \\
1 & 3 & 1 \\
1 & 3 & 1 \\
\end{array}
\]

\[
\begin{array}{ccc}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
\end{array}
\]

\[
\begin{array}{ccc}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
\end{array}
\]

1 \(\mapsto\) 

\[
\begin{array}{ccc}
1 & 3 & 1 \\
3 & 1 & 3 \\
1 & 3 & 1 \\
1 & 3 & 1 \\
1 & 3 & 1 \\
1 & 3 & 1 \\
\end{array}
\]

\[
\begin{array}{ccc}
1 & 3 & 1 \\
3 & 4 & 3 \\
1 & 3 & 1 \\
1 & 3 & 1 \\
1 & 3 & 1 \\
1 & 3 & 1 \\
\end{array}
\]

\[
\begin{array}{ccc}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
\end{array}
\]

0 \(\mapsto\) 

\[
\begin{array}{ccc}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
\end{array}
\]

\[
\begin{array}{ccc}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
\end{array}
\]

\[
\begin{array}{ccc}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
\end{array}
\]

\[
\begin{array}{ccc}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
\end{array}
\]

Figure 8: Cell to macrocell correspondence in the reduction from \(\{0, 1, 3, 4\}\)-FSPP to \(\{3, 4\}\)-FSPP. Left: correspondence when the questioned cell is not a 0, in this case the new questioned cell is in the center of the corresponding macrocell. Right: if the questioned cell is a 0 then we inflate all macrocells to be \(7 \times 7\), and use the pictured macrocell to replace the questioned cell.
When trying to find an NC algorithm to solve \( \{1, 3, 4\}\)-FSPP, our attempts to adapt the reduction to strict majority employed for \( \{0, 1, 4\}\)-FSPP in Subsection 5.1.2 failed, because gadgets replacing a value 3 seem to require degree five, though they can be made planar (see Figure 9, but the case planar of degree at most five is left open by [14, 7]).

Remark that we can answer efficiently in many cases using previous developments:

- when the questioned cell is a 1 in a cycle of 1, or a 1 on a path whose endpoints are connected to cycles of 1 (decidable in NC\(^2\)), then the answer is negative (as is the case for strict majority [14]);

- when the questioned cell is a 3 connected via values 3 to a 4 (decidable in NL), then the answer is positive (as is the case for \( \{0, 3, 4\}\)-FSPP in Subsection 5.1.4).

We conjecture that the remaining cases are equivalent to planar and-or freezing networks with fan in two and fan out one, but wires are undirected (this is not a circuit) which leads to difficulties analogous to the general case of \( \{0, 1, 2, 3, 4\}\)-FSPP, though interestingly in a seemingly more restrictive setting.

When trying to prove that FSPP reduces to \( \{0, 1, 3, 4\}\)-FSPP, we failed to build a macrocell (with 0, 1, 3, 4) corresponding to a cell with 2 sand grains (other elements are straightforward to design), though we found some close constructions. For example, the construction illustrated on Figure 10 behaves almost as a value 2, except that the combination of west plus east signals does not trigger signals to the north and south (any other combination of at least two signals triggers signals to the remaining sides). We can deduce that if the number of values 2 is upper bounded by a polylogarithmic function of the input’s size, then there is an NC\(^1\) truth-table reduction:

- for each cell with value 2 we try the macrocell of Figure 10 and the same rotated;

- the answer to FSPP will be positive if and only if at least one combination (truth-table) of such macrocells for all values 2 gives a positive answer;

- we need to compute the number \( x \) of values 2 (in NC\(^1\)), and then \( 2^x \) transformations in parallel (a polynomial number, each in AC\(^0\)).

We can also use the planar monotone circuit realizing threshold function \( T_2^{(4)} \) from [19, Figures 1 or 2] in order to create a macrocell corresponding to a cell with 2 sand grains.
Figure 10: White cells have no sand grain (0), and diode mechanisms are highlighted. Macrocell with 0, 1, 3, 4 corresponding to a value 2, except that the west plus east signals do not trigger signals to the north and south sides. Any other combination of at least two signals triggers signals to the remaining sides. The same macrocell rotated by 90 degrees is only missing the north plus south combination.

(using diodes as on Figure [10], but the result given by the last gate is "trapped" inside the macrocell (signals are not sent to the remaining sides). We can nevertheless deduce from such a construction that, if there is only one value 2 and if furthermore this is the questioned cell, then we have a proper AC$^0$ reduction.

6 Conclusion

The freezing world allows to make insightful progresses related to difficult questions. Exploiting the formal connections with threshold Boolean functions established by Proposition 1, Theorems 1 and 2 characterise the computational complexity of all but two restrictions of freezing sandpile prediction problem (FSPP): either the problem is as hard as unrestricted FSPP; or it is proven to be in NC (or below). The results are displayed on Table 1.
The results show interesting fine-grained view on necessary and sufficient conjunctions of elements (values among \{0, 1, 2, 3, 4\}) for the dynamics to be “as expressive as” \textsc{FSPP}. We propose three remarks.

First, in \cite{14} it is proven that the prediction problem on the non-strict majority cellular automata is \textsc{P}-hard for the family of graphs with maximum degree at least 4, and that it is in \textsc{NC} for the family of graphs with maximum degree at most 3. In \cite{7} it is proven that the same problem is in \textsc{NC} when the graph restricted to the two dimensional grid with von Neumann neighborhood (a particular case of regular graph where each vertex has degree 4), which corresponds to \{2, 4\}-\textsc{FSPP}. According to Section \ref{sec:4} the problem \{0, 2, 4\}-\textsc{FSPP} introduces a new refinement: when restricted to the two dimensional grid with von Neumann neighborhood on which some vertices are somehow removed (with sand value 0), the problem becomes as hard as \textsc{FSPP}.

Second, in the reduction of Proposition \ref{prop:11} (and some subsequent ones) it seems important to have many values 4. What is the computational complexity of the weak prediction problem (given a finite stable configuration, plus only one sand grain addition, namely \textsc{1st-col-S-PRED} of the survey \cite{4}) in this case?

Third, the Open question \ref{open:1} puts in light a surprisingly complex refinement, where forbidding only the value 2 seems to decrease the expressiveness of the model, yet not flattening it to another known case. Could it be that, if \textsc{NC} \neq \textsc{P}, then \textsc{FSPP} and \{0, 1, 3, 4\}-\textsc{FSPP} would belong to different intermediate classes strictly between \textsc{NC} and \textsc{P} (which would exist according to an analog of Ladner’s theorem \cite{26})?

The present work circumvents the question of whether \textsc{FSPP} itself is in \textsc{NC} or \textsc{P}-hard. One can implement conjunctions, disjunctions, but the relationship between the impossibility of crossing wires \cite{5} and the possibility of using undirected wires, or even other forms of signal implementation, leaves open its reduction to \textsc{MPCVP} \cite{27} (prediction in \textsc{NC}), or the possibility to implement non-planar or non-monotone gates \cite{17} (\textsc{P}-hard prediction). The general case of \textsc{FSPP} reduces to \{0, 2, 4\}-\textsc{FSPP}, could that help in order to find an efficient algorithm? Advances on \textsc{FSPP} would constitute great insights for the classical sandpile prediction problem (\textsc{SPP}) in two dimensions, left open in the original paper by Moore and Nilsson \cite{21}, even though some relationship between \textsc{FSPP} and \textsc{SPP} is still to be formally established.

Finally, the relationship between threshold functions and cell’s sand content opens perspectives on the prediction of Boolean functions on the grid: in the freezing and

| \textsc{AC}^0 | \textsc{NL} | \textsc{NC}^2 | \textsc{FSPP} \leq \textsc{AC}^0 | \textsc{Open} |
|------------|--------|---------|-----------------|--------|
| \{0, 4\}  | \{0, 3, 4\} | \{1, 4\} | \{1, 2, 3, 4\} | \{0, 1, 3, 4\} |
|           |         | \{0, 1, 4\} | \{0, 2, 3, 4\} | \{0, 1, 3, 4\} |
|           |         | \{2, 4\} | \{0, 1, 2, 4\} | \{0, 1, 3, 4\} |
|           |         | \{2, 3, 4\} | \{1, 2, 4\} | \{0, 1, 3, 4\} |
|           |         |           | \{0, 2, 4\} | \{0, 1, 3, 4\} |

Table 1: Summary of Theorems \ref{thm:1} \ref{thm:2} and Open question \ref{open:1}
non-freezing worlds, what are the necessary and sufficient elements in order to have easy/hard prediction problems?

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