What is Stochastic Independence?

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Abstract
The notion of a tensor product with projections or with inclusions is defined. It is shown that the definition of stochastic independence relies on such a structure and that independence can be defined in an arbitrary category with a tensor product with inclusions or projections. In this context, the classifications of quantum stochastic independence by Muraki, Ben Ghorbal, and Schürmann become classifications of the tensor products with inclusions for the categories of algebraic probability spaces and non-unital algebraic probability spaces. The notion of a reduction of one independence to another is also introduced. As examples the reductions of Fermi independence and boolean, monotone, and anti-monotone independence to tensor independence are presented.

1 Introduction
In this paper we will deal with the question, what stochastic independence is. Since the work of Speicher\cite{Speicher} and Schürmann\cite{Schürmann1, Schürmann2, Schürmann3} we know that a 'universal' notion of independence should come with a product that allows to construct the joint distribution of two independent random variables from their marginal distributions. It turned out that in classical probability there exists only one such product satisfying a natural set of axioms. But there are several different good notions of independence in non-commutative probability. The most important ones were classified in the work of Speicher\cite{Speicher}, Ben Ghorbal and Schürmann\cite{Ghorbal1, Ghorbal2}, and Muraki\cite{Muraki1, Muraki2}. They are tensor independence, free independence, boolean independence, monotone independence and anti-monotone independence.

We present and motivate here the axiomatic framework used in these articles. We show that the classical notion of stochastic independence is based on a kind of product in the category of probability spaces, which is intermediate to the notion of a (universal) product in category theory - which does not exist in this category - and the notion of a tensor product. Furthermore, we show that the classification of stochastic independence by Ben Ghorbal and Schürmann\cite{Ghorbal1, Ghorbal2} and by Muraki\cite{Muraki1, Muraki2} is also based on such a product, which we call a tensor product with projections or inclusions, cf. Definition \ref{def:tensor_product}. This notion allows to define independence for arbitrary categories, see Definition \ref{def:independence}. If independence is something that depends on a tensor product and projections or inclusions between the original objects and their tensor product, then it is clear that a map between categories that preserves independence should be tensor functor with an additional structure that takes care of
the projections or inclusions. This is formalized in Definition 2.3. We show in several examples that these notions are really the correct one; see Subsection 3.2, 3.3, and 4.1, and Section 7.

But let us first look at the notion of independence in classical probability.

2 Independence for Classical Random Variables

Two random variables \(X_1 : (\Omega, \mathcal{F}, P) \to (E_1, \mathcal{E}_1)\) and \(X_2 : (\Omega, \mathcal{F}, P) \to (E_2, \mathcal{E}_2)\), defined on the same probability space \((\Omega, \mathcal{F}, P)\) and with values in two possibly distinct measurable spaces \((E_1, \mathcal{E}_1)\) and \((E_2, \mathcal{E}_2)\), are called \textit{stochastically independent} (or simply \textit{independent}) w.r.t. \(P\), if the \(\sigma\)-algebras \(\mathcal{F}_1^{-1}(\mathcal{E}_1)\) and \(\mathcal{F}_2^{-1}(\mathcal{E}_2)\) are independent w.r.t. \(P\), i.e. if

\[
P\left(\bigcap_{j=1}^n (X_j^{-1}(M_j))\right) = \prod_{k=1}^n P\left(X_k^{-1}(M_k)\right)
\]

holds for all \(n \in \mathbb{N}\) and all choices of indices \(k_1, \ldots, k_n \in J\) with \(j_k \neq j_\ell\) for \(j \neq \ell\), and all choices of measurable sets \(M_j \in \mathcal{E}_j\).

There are many equivalent formulations for independence, consider, e.g., the following proposition.

\begin{proposition}
Let \(X_1\) and \(X_2\) be two real-valued random variables. The following are equivalent.

\(\text{i) } \) \(X_1\) and \(X_2\) are independent.

\(\text{ii) } \) For all bounded measurable functions \(f_1, f_2\) on \(\mathbb{R}\) we have

\[
\mathbb{E}(f_1(X_1)f_2(X_2)) = \mathbb{E}(f_1(X_1))\mathbb{E}(f_2(X_2)).
\]

\(\text{iii) } \) The probability space \((\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2), P_{(X_1,X_2)})\) is the product of the probability spaces \((\mathbb{R}, \mathcal{B}(\mathbb{R}), P_{X_1})\) and \((\mathbb{R}, \mathcal{B}(\mathbb{R}), P_{X_2})\), i.e.

\[
P_{(X_1,X_2)} = P_{X_1} \otimes P_{X_2}.
\]

We see that stochastic independence can be reinterpreted as a rule to compute the joint distribution of two random variables from their marginal distribution. More precisely, their joint distribution can be computed as a product of their marginal distributions. This product is associative and can also be iterated to compute the joint distribution of more than two independent random variables.

The classifications of independence for non-commutative probability \(\mathbb{R}\) that we are interested in are based on redefining independence as a product satisfying certain natural axioms.

3 Tensor Categories and Independence

We will now define the notion of independence in the language of category theory. The usual notion of independence for classical probability theory and the independences classified in 1, 2, 3, 4, 5, 6, 7, 8 will then be instances of this
general notion obtained by considering the category of classical probability spaces or the category of algebraic probability spaces.

First we recall the definitions of a product, coproduct and a tensor product, see also MacLane[7] for a more detailed introduction. Then we introduce tensor categories with inclusions or projections. This notion is weaker than that of a product or coproduct, but stronger than that of a tensor category. It is exactly what we need to get an interesting notion of independence.

**Definition 3.1** (See, e.g., Maclane[7]) A tuple $(B_1 \coprod B_2, \pi_1, \pi_2)$ is called a product or universal product of the objects $B_1$ and $B_2$ in the category $C$, if for any object $A \in \text{Ob} \ C$ and any morphisms $f_1 : A \to B_1$ and $f_2 : A \to B_2$ there exists a unique morphism $h$ such that the following diagram commutes,

$$
\begin{array}{ccc}
A & & \to \\
\downarrow & & \downarrow \\
B_1 & \xleftarrow{\pi_1} & B_1 \coprod B_2 & \xrightarrow{\pi_2} & B_2.
\end{array}
$$

An object $K$ is called terminal, if for all objects $A \in \text{Ob} \ C$ there exists exactly one morphism from $A$ to $K$.

The product of two objects is unique up to isomorphism, if it exists. Furthermore, the operation of taking products is commutative and associative up to isomorphism and therefore, if a category has a terminal object and a product for any two objects, then one can also define a product for any finite set of objects.

The notion of coproduct is dual to that of a product, i.e., its defining property can be obtained from that of the product by ‘reverting the arrows’. The notion dual to terminal object is an initial object, i.e. an object $K$ such that for any object $A$ of $C$ there exists a unique morphism from $K$ to $A$.

Let us now recall the definition of a tensor category.

**Definition 3.2** A category $(C, \boxtimes)$ equipped with a bifunctor $\boxtimes : C \times C \to C$, called tensor product, that is associative up to a natural isomorphism

$$
\alpha_{A,B,C} : A \boxtimes (B \boxtimes C) \xrightarrow{\cong} (A \boxtimes B) \boxtimes C,
$$

for all $A, B, C \in \text{Ob} \ C$, and an element $E$ that is, up to isomorphisms

$$
\lambda_A : E \boxtimes A \xrightarrow{\cong} A, \quad \text{and} \quad \rho_A : A \boxtimes E \xrightarrow{\cong} A,
$$

for all $A \in \text{Ob} \ C$.

A unit for $\boxtimes$, is called a tensor category or monoidal category, if the pentagon axiom

$$
\begin{array}{ccc}
(A \boxtimes B) \boxtimes (C \boxtimes D) & \xrightarrow{\alpha_{A,B,C,D}} & (A \boxtimes B) \boxtimes (C \boxtimes D) \\
A \boxtimes (B \boxtimes (C \boxtimes D)) & \xrightarrow{\alpha_{A,B,C,D}} & (A \boxtimes (B \boxtimes C)) \boxtimes D
\end{array}
$$

and the triangle axiom

$$
\begin{array}{ccc}
A \boxtimes (E \boxtimes C) & \xrightarrow{\alpha_{A,E,C}} & (A \boxtimes E) \boxtimes C \\
& \xrightarrow{id_A \boxtimes \lambda_C} & A \boxtimes C
\end{array}
$$

$$
\begin{array}{ccc}
& \xrightarrow{\rho_A \boxtimes id_C} & A \boxtimes C
\end{array}
$$

are satisfied for all objects $A, B, C, D$ of $C$.
If a category has products or coproducts for all finite sets of objects, then
the universal property guarantees the existence of the isomorphisms \( \alpha, \lambda, \) and \( \rho \) that turn it into a tensor category.

In order to define a notion of independence we need less than a (co-)
product, but a little bit more than a tensor product. What we need are
inclusions or projections that allow us to view the objects \( A, B \) as sub-
sets of their product \( A \square B \).

**Definition 3.3** A tensor category with projections \((C, \square, \pi)\) is a tensor
category \((C, \square)\) equipped with two natural transformations \( \pi_1 : \square \rightarrow P_1 \) and
\( \pi_2 : \square \rightarrow P_2 \), where the bifunctors \( P_1, P_2 : C \times C \rightarrow C \) are defined by
\( P_1(B_1, B_2) = B_1, \ P_2(B_1, B_2) = B_2 \), on pairs of objects \( B_1, B_2 \) of \( C \), and similarly on pairs of morphisms. In other words, for any pair of objects \( B_1, B_2 \)
there exist two morphisms \( \pi_{B_1} : B_1 \square B_2 \rightarrow B_1, \ \pi_{B_2} : B_1 \square B_2 \rightarrow B_2 \), such that
for any pair of morphisms \( f_1 : A_1 \rightarrow B_1, \ f_2 : A_2 \rightarrow B_2 \), the following diagram commutes,

\[
\begin{array}{ccc}
A_1 & \xrightarrow{\pi_{A_1}} & A_1 \square A_2 \xrightarrow{\pi_{A_2}} A_2 \\
\downarrow{f_1} & & \downarrow{f_2} \\
B_1 & \xrightarrow{\pi_{B_1}} & B_1 \square B_2 \xrightarrow{\pi_{B_2}} B_2.
\end{array}
\]

Similarly, a tensor product with inclusions \((C, \square, i)\) is a tensor category
\((C, \square)\) equipped with two natural transformations \( i_1 : \square \rightarrow \square \) and \( i_2 : \square \rightarrow \square \),
i.e. for any pair of objects \( B_1, B_2 \) there exist two morphisms \( i_{B_1} : B_1 \rightarrow B_1 \square B_2, \ i_{B_2} : B_2 \rightarrow B_1 \square B_2 \), such that for any pair of morphisms \( f_1 : A_1 \rightarrow B_1, \ f_2 : A_2 \rightarrow B_2 \), the following diagram commutes,

\[
\begin{array}{ccc}
A_1 & \xrightarrow{i_{A_1}} & A_1 \square A_2 \xleftarrow{i_{A_2}} A_2 \\
\downarrow{f_1} & & \downarrow{f_2} \\
B_1 & \xrightarrow{i_{B_1}} & B_1 \square B_2 \xleftarrow{i_{B_2}} B_2.
\end{array}
\]

In a tensor category with projections or with inclusions we can define a
notion of independence for morphisms.

**Definition 3.4** Let \((C, \square, \pi)\) be a tensor category with projections. Two mor-
phisms \( f_1 : A \rightarrow B_1 \) and \( f_2 : A \rightarrow B_2 \) with the same source \( A \) are called inde-
pendent (with respect to \( \square \)), if there exists a morphism \( h : A \rightarrow B_1 \square B_2 \) such that the diagram

\[
\begin{array}{ccc}
& A & \\
\downarrow{f_1} & \searrow & \downarrow{f_2} \\
B_1 & \xleftarrow{i_{B_1}} & B_1 \square B_2 \xrightarrow{i_{B_2}} B_2.
\end{array}
\]

commutes.

In a tensor category with inclusions \((C, \square, i)\), two morphisms \( f_1 : A_1 \rightarrow B \)
and \( f_2 : A_2 \rightarrow B \) with the same target \( B \) are called independent, if there exists
a morphism \( h : A_1 \square A_2 \rightarrow B \) such that the diagram

\[
\begin{array}{ccc}
& A & \\
\downarrow{f_1} & \searrow & \downarrow{f_2} \\
B_1 & \xleftarrow{i_{B_1}} & B_1 \square B_2 \xrightarrow{i_{B_2}} B_2.
\end{array}
\]

commutes.
This definition can be extended in the obvious way to arbitrary sets of morphisms.

If □ is actually a product (or coproduct, resp.), then the universal property in Definition 3.4 implies that for all pairs of morphisms with the same source (or target, resp.) there exists even a unique morphism that makes diagram □ (or □, resp.) commuting. Therefore in that case all pairs of morphisms with the same source (or target, resp.) are independent.

We will now consider several examples. We will show that for the category of classical probability spaces we recover usual stochastic independence, if we take the product of probability spaces, cf. Proposition 3.4.

3.1 Example: Independence in the Category of Classical Probability Spaces

The category of measurable spaces consists of pairs (Ω, F), where Ω is a set and F ⊆ P(Ω) a σ-algebra. The morphisms are the measurable maps. This category has a product,

(Ω1, F1) × (Ω2, F2) = (Ω1 × Ω2, F1 ⊗ F2)

where Ω1 × Ω2 is the Cartesian product of Ω1 and Ω2, and F1 ⊗ F2 is the smallest σ-algebra on Ω1 × Ω2 such that the canonical projections p1 : Ω1 × Ω2 → Ω1 and p2 : Ω1 × Ω2 → Ω2 are measurable.

The category of probability spaces Prob has as objects triples (Ω, F, P) where (Ω, F) is a measurable space and P a probability measure on (Ω, F). A morphism X : (Ω1, F1, P1) → (Ω2, F2, P2) is a measurable map X : (Ω1, F1) → (Ω2, F2) such that

P1 ◦ X⁻¹ = P2.

This means that a random variable X : (Ω, F) → (E, E) automatically becomes a morphism, if we equip (E, E) with the measure

PX = P ◦ X⁻¹

induced by X.

This category does not have universal products. But one can check that the product of measures turns Prob into a tensor category,

(Ω1, F1, P1) ⊗ (Ω2, F2, P2) = (Ω1 × Ω2, F1 ⊗ F2, P1 ⊗ P2),

where P1 ⊗ P2 is determined by

(M1 × M2) = P1(M1)P2(M2),

for all M1 ∈ F1, M2 ∈ F2. It is even a tensor category with projections in the sense of Definition 3.3 with the canonical projections p1 : (Ω1 × Ω2, F1 ⊗ F2, P1 ⊗ P2) → (Ω1, F1, P1), p2 : (Ω1 × Ω2, F1 ⊗ F2, P1 ⊗ P2) → (Ω2, F2, P2) given by p1(ω1, ω2) = ω1, p2(ω1, ω2) = ω2 for ω1 ∈ Ω1, ω2 ∈ Ω2.

The notion of independence associated to this tensor product with projections is exactly the one used in probability.

Proposition 3.5 Two random variables X1 : (Ω, F, P) → (E, E) and X2 : (Ω, F, P) → (E, E), defined on the same probability space (Ω, F, P) and with values in measurable spaces (E1, E1) and (E2, E2), are stochastically independent, if and only if they are independent in the sense of Definition 3.4 as morphisms X1 : (Ω, F, P) → (E1, E1, P1) and X2 : (Ω, F, P) → (E2, E2, P2) of the tensor category with projections (Prob, ⊗, p).
Proof. Assume that $X_1$ and $X_2$ are stochastically independent. We have to find a morphism $h : (\Omega, \mathcal{F}, P) \to (E_1 \times E_2, \mathcal{E}_1 \otimes \mathcal{E}_2, P_{X_1} \otimes P_{X_2})$ such that the diagram

\[
\begin{array}{ccc}
(\Omega, \mathcal{F}, P) & \xrightarrow{X_1} & (E_1, \mathcal{E}_1, P_{X_1}) \\
\downarrow & & \downarrow p_1 \\
(E_1 \times E_2, \mathcal{E}_1 \otimes \mathcal{E}_2, P_{X_1} \otimes P_{X_2}) & \xrightarrow{h} & (E_2, \mathcal{E}_2, P_{X_2}) \\
\downarrow & & \downarrow p_2 \\
(\Omega, \mathcal{F}, P) & \xrightarrow{X_2} & (E_1, \mathcal{E}_1, P_{X_1})
\end{array}
\]

commutes. The only possible candidate is $h(\omega) = (X_1(\omega), X_2(\omega))$ for all $\omega \in \Omega$, the unique map that completes this diagram in the category of measurable spaces and that exists due to the universal property of the product of measurable spaces. This is a morphism in $\text{AlgProb}$, because we have

\[
P(h^{-1}(M_1 \times M_2)) = P(X_1^{-1}(M_1) \cap X_2^{-1}(M_2)) = P(X_1^{-1}(M_1))P(X_2^{-1}(M_2)) = P_{X_1}(M_1)P_{X_2}(M_2) = (P_{X_1} \otimes P_{X_2})(M_1 \times M_2)
\]

for all $M_1 \in \mathcal{E}_1, M_2 \in \mathcal{E}_2$, and therefore

\[
P \circ h^{-1} = P_{X_1} \otimes P_{X_2}.
\]

Conversely, if $X_1$ and $X_2$ are independent in the sense of Definition 3.4, then the morphism that makes the diagram commuting has to be again $h : \omega \mapsto (X_1(\omega), X_2(\omega))$. This implies

\[
P(X_1 \times X_2) = P \circ h^{-1} = P_{X_1} \otimes P_{X_2}
\]

and therefore

\[
P(X_1^{-1}(M_1) \cap X_2^{-1}(M_2)) = P(X_1^{-1}(M_1))P(X_2^{-1}(M_2))
\]

for all $M_1 \in \mathcal{E}_1, M_2 \in \mathcal{E}_2$. \hfill \qed

3.2 Example: Tensor Independence in the Category of Algebraic Probability Spaces

By the category of algebraic probability $\text{AlgAlgProb}$ spaces we denote the category of associative unital algebras over $\mathbb{C}$ equipped with a unital linear functional. A morphism $j : (A_1, \varphi_1) \to (A_2, \varphi_2)$ is a quantum random variable, i.e. an algebra homomorphism $j : A_1 \to A_2$ that preserves the unit and the functional, i.e. $j(1_{A_1}) = 1_{A_2}$ and $\varphi_2 \circ j = \varphi_1$.

The tensor product we will consider on this category is just the usual tensor product $(A_1 \otimes A_2, \varphi_1 \otimes \varphi_2)$, i.e. the algebra structure of $A_1 \otimes A_2$ is defined by

\[
1_{A_1} \otimes 1_{A_2} = 1_{A_1} \otimes 1_{A_2},
\]

\[
(a_1 \otimes a_2)(b_1 \otimes b_2) = a_1 b_1 \otimes a_2 b_2,
\]

and the new functional is defined by

\[
(\varphi_1 \otimes \varphi_2)(a_1 \otimes a_2) = \varphi_1(a_1)\varphi_2(a_2),
\]

for all $a_1, b_1 \in A_1, a_2, b_2 \in A_2$.

This becomes a tensor category with inclusions with the inclusions defined by

\[
i_{A_1}(a_1) = a_1 \otimes 1_{A_2},
\]

\[
i_{A_2}(a_2) = 1_{A_1} \otimes a_2,
\]

for $a_1 \in A_1, a_2 \in A_2$. 

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One gets the category of $*$-algebraic probability spaces, if one assumes that the underlying algebras have an involution and the functional are states, i.e. are also positive. Then an involution is defined on $A_1 \otimes A_2$ by $(a_1 \otimes a_2)^* = a_1^* \otimes a_2^*$ and $\varphi_1 \otimes \varphi_2$ is again a state.

The notion of independence associated to this tensor product with inclusions by Definition 3.4 is the usual notion of Bose or tensor independence used in quantum probability, e.g., by Hudson and Parthasarathy.

**Proposition 3.6** Two quantum random variables $j_1 : (B_1, \psi_1) \rightarrow (A, \varphi)$ and $j_2 : (B_2, \psi_2) \rightarrow (A, \varphi)$, defined on algebraic probability spaces $(B_1, \psi_1), (B_2, \psi_2)$ and with values in the same algebraic probability space $(A, \varphi)$ are independent if and only if the following two conditions are satisfied.

(i) The images of $j_1$ and $j_2$ commute, i.e.

$$[j_1(a_1), j_2(a_2)] = 0,$$

for all $a_1 \in A_1, a_2 \in A_2$.

(ii) $\varphi$ satisfies the factorization property

$$\varphi(j_1(a_1)j_2(a_2)) = \varphi(j_1(a_1))\varphi(j_2(a_2)),$$

for all $a_1 \in A_1, a_2 \in A_2$.

We will not prove this Proposition since it can be obtained as a special case of Proposition 3.4, if we equip the algebras with the trivial $\mathbb{Z}_2$-grading $A^{(0)} = A$, $A^{(1)} = \{0\}$.

### 3.3 Example: Fermi Independence

Let us now consider the category of $\mathbb{Z}_2$-graded algebraic probability spaces $\mathbb{Z}_2\text{-AlgProb}$. The objects are pairs $(A, \varphi)$ consisting of a $\mathbb{Z}_2$-graded unital algebra $A = A^{(0)} \oplus A^{(1)}$ and an even unital functional $\varphi$, i.e. $\varphi|_{A^{(1)}} = 0$.

The morphisms are random variables that don’t change the degree, i.e., for $j : (A_1, \varphi_1) \rightarrow (A_2, \varphi_2)$, we have

$$j(A^{(0)}_1) \subseteq A^{(0)}_2 \quad \text{and} \quad j(A^{(1)}_1) \subseteq A^{(1)}_2.$$

The tensor product $(A_1 \otimes_{\mathbb{Z}_2} A_2, \varphi_1 \otimes \varphi_2) = (A_1, \varphi_1) \otimes_{\mathbb{Z}_2} (A_2, \varphi_2)$ is defined as follows. The algebra $A_1 \otimes_{\mathbb{Z}_2} A_2$ is the graded tensor product of $A_1$ and $A_2$, i.e. $(A_1 \otimes_{\mathbb{Z}_2} A_2)^{(0)} = A_1^{(0)} \otimes A_2^{(0)}$, $(A_1 \otimes_{\mathbb{Z}_2} A_2)^{(1)} = A_1^{(1)} \otimes A_2^{(0)} \oplus A_1^{(0)} \otimes A_2^{(1)}$, with the algebra structure given by

$$1_{A_1 \otimes_{\mathbb{Z}_2} A_2} = 1_{A_1} \otimes 1_{A_2},$$

$$(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) = (-1)^{\deg b_1 \deg a_2} a_1 a_2 b_1 \otimes b_2,$$

for all homogeneous elements $a_1, b_1 \in A_1, a_2, b_2 \in A_2$. The functional $\varphi_1 \otimes \varphi_2$ is simply the tensor product, i.e. $(\varphi_1 \otimes \varphi_2)(a_1 \otimes a_2) = \varphi_1(a_1) \otimes \varphi_2(a_2)$ for all $a_1 \in A_1, a_2 \in A_2$. It is easy to see that $\varphi_1 \otimes \varphi_2$ is again even, if $\varphi_1$ and $\varphi_2$ are even. The inclusions $i_1 : (A_1, \varphi_1) \rightarrow (A_1 \otimes_{\mathbb{Z}_2} A_2, \varphi_1 \otimes \varphi_2)$ and $i_2 : (A_2, \varphi_2) \rightarrow (A_1 \otimes_{\mathbb{Z}_2} A_2, \varphi_1 \otimes \varphi_2)$ are defined by

$$i_1(a_1) = a_1 \otimes 1_{A_2} \quad \text{and} \quad i_2(a_2) = 1_{A_1} \otimes a_2,$$

for $a_1 \in A_1, a_2 \in A_2$.

If the underlying algebras are assumed to have an involution and the functionals to be states, then the involution on the $\mathbb{Z}_2$-graded tensor product is defined by $(a_1 \otimes a_2)^* = (-1)^{\deg a_1 \deg a_2} a_1^* \otimes a_2^*$, this gives the category of $\mathbb{Z}_2$-graded $*$-algebraic probability spaces.

The notion of independence associated to this tensor category with inclusions is called Fermi independence or anti-symmetric independence.
Proposition 3.7 Two random variables \( j_1 : (B_1, \psi_1) \to (A, \varphi) \) and \( j_2 : (B_2, \psi_2) \to (A, \varphi) \), defined on two \( \mathbb{Z}_2 \)-graded algebraic probability spaces \((B_1, \psi_1), (B_2, \psi_2))\) and with values in the same \( \mathbb{Z}_2 \)-algebraic probability space \((A, \varphi)\) are independent if and only if the following two conditions are satisfied.

(i) The images of \( j_1 \) and \( j_2 \) satisfy the commutation relations

\[
j_2(a_2)j_1(a_1) = (-1)^{\deg a_1 \deg a_2} j_1(a_1)j_2(a_2)
\]

for all homogeneous elements \( a_1 \in A_1 \), \( a_2 \in A_2 \).

(ii) \( \varphi \) satisfies the factorization property

\[
\varphi(j_1(a_1)j_2(a_2)) = \varphi(j_1(a_1))\varphi(j_2(a_2)),
\]

for all \( a_1 \in A_1 \), \( a_2 \in A_2 \).

Proof. The proof is similar to that of Proposition 3.5, we will only outline it. It is clear that the morphism \( h : (B_1, \psi_1) \otimes_{\mathbb{Z}_2} (B_2, \psi_2) \to (A, \varphi) \) that makes the diagram in Definition 3.4 commuting, has to act on elements of \( B_1 \otimes 1_{B_2} \) and \( 1_{B_1} \otimes B_2 \) as

\[
h(b_1 \otimes 1_{B_2}) = j_1(b_1) \quad \text{and} \quad h(1_{B_1} \otimes b_2) = j_2(b_2).
\]

This extends to a homomorphism from \((B_1, \psi_1) \otimes_{\mathbb{Z}_2} (B_2, \psi_2)\) to \((A, \varphi)\), if and only if the commutation relations are satisfied. And the resulting homomorphism is a quantum random variable, i.e. satisfies \( \varphi \circ h = \psi_1 \otimes \psi_2 \), if and only if the factorization property is satisfied.

4 Reduction of Independences

In this Section we will study the relations between different notions of independence. Let us first recall the definition of a tensor functor.

Definition 4.1 (see, e.g., Section XI.2 in MacLane) Let \((C, \square)\) and \((C', \square')\) be two tensor categories. A cotensor functor or coenoidal functor \( F : (C, \square) \to (C', \square') \) is an ordinary functor \( F : C \to C' \) equipped with a morphism \( F_0 : F(E_C) \to E_{C'} \) and a natural transformation \( F_2 : F(\cdot \square \cdot) \to F(\cdot \square' \cdot) \), i.e. morphisms \( F_2(A, B) : F(A \square B) \to F(A) \square' F(B) \) for all \( A, B \in \text{Ob} C \) that are natural in \( A \) and \( B \), such that the diagrams

\[
\begin{array}{c}
\begin{array}{c}
F(A \square (B \square C)) \\
F_2(A, B \square C) \\
\uparrow \uparrow \uparrow \uparrow \uparrow
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
F(A) \square' F(B \square C) \\
F_2(A \square B, C) \\
\uparrow \uparrow \uparrow \uparrow \uparrow
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
F(A \square' (F(B) \square F(C))) \\
\uparrow \uparrow \uparrow \uparrow \uparrow
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
F(B \square E_{C'}) \\
F_2(B, E_C) \\
\uparrow \uparrow \uparrow \uparrow \uparrow
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
F(B) \square' F(E_C) \\
\uparrow \uparrow \uparrow \uparrow \uparrow
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
F(B \square E_{C'}) \\
\uparrow \uparrow \uparrow \uparrow \uparrow
\end{array}
\end{array}
\]

\[
\text{diagram (3)}
\]

\[
\begin{array}{c}
\begin{array}{c}
F(A \square (B \square C)) \\
F_2(A, B \square C) \\
\uparrow \uparrow \uparrow \uparrow \uparrow
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
F(A) \square' F(B \square C) \\
F_2(A \square B, C) \\
\uparrow \uparrow \uparrow \uparrow \uparrow
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
F(A \square' (F(B) \square F(C))) \\
\uparrow \uparrow \uparrow \uparrow \uparrow
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
F(B \square E_{C'}) \\
F_2(B, E_C) \\
\uparrow \uparrow \uparrow \uparrow \uparrow
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
F(B) \square' F(E_C) \\
\uparrow \uparrow \uparrow \uparrow \uparrow
\end{array}
\end{array}
\]

\[
\text{diagram (4)}
\]
If we denote the even element of \( \mathbb{Z} \) from (\[\text{We will now define the bosonization of Fermi independence as a reduction}\]\), 4.1 Example: Bosonization of Fermi Independence:

\[
\begin{array}{c}
F(E_C \Box B) \xrightarrow{F_2(E_C,B)} F(E_C \Box' F(B) \\
\downarrow F(\lambda_B) \quad \downarrow \lambda_{F(B)} \\
F(B) \xrightarrow{F_0 \Box' \text{id}_B} E_C \Box' F(B)
\end{array}
\]

\( \text{commute for all } A, B, C \in \text{Ob} \mathcal{C}. \)

We have reversed the direction of \( F_0 \) and \( F_2 \) in our definition. In the case of a strong tensor functor, i.e. when all the morphisms are isomorphisms, our definition of a cotensor functor is equivalent to the usual definition of a tensor functor as, e.g., in MacLane [7].

The conditions are exactly what we need to get morphisms \( F_n(A_1, \ldots, A_n) : F(A_1 \Box \cdots \Box A_n) \to F(A_1) \Box' \cdots \Box' F(A_n) \) for all finite sets \( \{A_1, \ldots, A_n\} \) of objects of \( \mathcal{C} \) such that, up to these morphisms, the functor \( F : (\mathcal{C}, \Box) \to (\mathcal{C}', \Box') \) is a homomorphism.

For a reduction of independences we need a little bit more than a cotensor functor.

**Definition 4.2** Let \( (\mathcal{C}, \Box, i) \) and \( (\mathcal{C}', \Box', i') \) be two tensor categories with inclusions and assume that \( \mathcal{C} \) is a subcategory of \( \mathcal{C}' \). A reduction \( (F, J) \) of the tensor product \( \Box \) to the tensor product \( \Box' \) is a cotensor functor \( F : (\mathcal{C}, \Box) \to (\mathcal{C}', \Box') \) and a natural transformation \( J : \text{id}_\mathcal{C} \to F \), i.e. morphisms \( j_A : A \to F(A) \) in \( \mathcal{C}' \) for all objects \( A \in \text{Ob} \mathcal{C} \) such that the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{j_A} & F(A) \\
\downarrow f & & \downarrow F(f) \\
B & \xrightarrow{j_B} & F(B)
\end{array}
\]

commutes for all morphisms \( f : A \to B \) in \( \mathcal{C} \).

Such a reduction provides us with a system of inclusions \( J_n(A_1, \ldots, A_n) = F_n(A_1, \ldots, A_n) \circ J_{A_1} \Box \cdots \Box A_n : A_1 \Box \cdots \Box A_n \to F(A_1) \Box' \cdots \Box' F(A_n) \) with \( J_1(A) = j_A \) that satisfies, e.g., \( J_m(A_1, \ldots, A_{n+m}) = F_2(F(A_1) \Box' \cdots \Box' F(A_n), F(A_{n+1}) \Box' \cdots \Box' F(A_{n+m})) \circ (J_n(A_1, \ldots, A_n) \Box J_m(A_{n+1}, \ldots, A_{n+m})) \) for all \( n, m \in \mathbb{N} \) and \( A_1, \ldots, A_{n+m} \in \text{Ob} \mathcal{C} \).

A reduction between two tensor categories with projections would consist of a tensor functor \( F \) and a natural transformation \( P : F \to \text{id} \).

We have to extend our definition slightly. In our applications \( \mathcal{C} \) will often not be a subcategory of \( \mathcal{C}' \), but we have, e.g., a forgetful functor \( U \) from \( \mathcal{C} \) to \( \mathcal{C}' \) that “forgets” an additional structure that \( \mathcal{C} \) has. An example for this situation is the reduction of Fermi independence to tensor independence in following subsection. Here we have to forget the \( \mathbb{Z}_2 \)-grading of the objects of \( \mathbb{Z}_2 \text{-AlgProb} \) to get objects of \( \text{AlgProb} \). In this situation a reduction of the tensor product with inclusions \( \Box \) to the tensor product with inclusions \( \Box' \) is a tensor function \( F \) from \( (\mathcal{C}, \Box) \to (\mathcal{C}', \Box') \) and a natural transformation \( J : U \to F \).

### 4.1 Example: Bosonization of Fermi Independence

We will now define the bosonization of Fermi independence as a reduction from \( \text{(AlgProb}, \otimes i) \) to \( \text{(Z}_2 \text{-AlgProb}, \otimes \mathbb{Z}_2, i) \). We will need the group algebra \( \mathbb{C}\mathbb{Z}_2 \) of \( \mathbb{Z}_2 \) and the linear functional \( \varepsilon : \mathbb{C}\mathbb{Z}_2 \to \mathbb{C} \) that arises as the linear extension of the trivial representation of \( \mathbb{Z}_2 \), i.e.

\[
\varepsilon(1) = \varepsilon(g) = 1,
\]

if we denote the even element of \( \mathbb{Z}_2 \) by \( 1 \) and the odd element by \( g \).
The underlying functor $F : \mathbb{Z}_2\text{-AlgProb} \to \text{AlgProb}$ is given by

$F : \begin{align*}
\text{Ob } \mathbb{Z}_2\text{-AlgProb} & \ni (A, \varphi) \mapsto (A \otimes_{\mathbb{Z}_2} \mathbb{C}Z_2, \varphi \otimes \varepsilon) \in \text{Ob } \text{AlgProb}, \\
\text{Mor } \mathbb{Z}_2\text{-AlgProb} & \ni f \mapsto f \otimes \text{id}_{\mathbb{C}Z_2} \in \text{Mor } \text{AlgProb}.
\end{align*}$

The unit element in both tensor categories is the one-dimensional unital algebra $\mathbb{C}1$ with the unique unital functional on it. Therefore $F_0$ has to be a morphism from $F(\mathbb{C}1) \cong \mathbb{C}Z_2$ to $\mathbb{C}1$. It is defined by $F_0(1) = F_0(g) = 1$.

The morphism $F_2(A_1, A_2)$ has to go from $F(A \otimes_{\mathbb{Z}_2} B) = (A \otimes_{\mathbb{Z}_2} B) \otimes \mathbb{C}Z_2$ to $F(A) \otimes F(B) = (A \otimes_{\mathbb{Z}_2} \mathbb{C}Z_2) \otimes (B \otimes_{\mathbb{Z}_2} \mathbb{C}Z_2)$. It is defined by

$$a \otimes b \otimes 1 \mapsto \begin{cases} (a \otimes 1) \otimes (b \otimes 1) & \text{if } b \text{ is even}, \\ (a \otimes g) \otimes (b \otimes 1) & \text{if } b \text{ is odd}, \end{cases}$$

and

$$a \otimes b \otimes g \mapsto \begin{cases} (a \otimes g) \otimes (b \otimes g) & \text{if } b \text{ is even}, \\ (a \otimes 1) \otimes (b \otimes g) & \text{if } b \text{ is odd}, \end{cases}$$

for $a \in A$ and homogeneous $b \in B$.

Finally, the inclusion $J_A : A \to A \otimes_{\mathbb{Z}_2} \mathbb{C}Z_2$ is defined by

$$J_A(a) = a \otimes 1$$

for all $a \in A$.

In this way we get inclusions $J_n = J_n(A_1, \ldots , A_n) = F_n(A_1, \ldots , A_n) \circ J(A_1 \otimes_{\mathbb{Z}_2} \cdots \otimes_{\mathbb{Z}_2} A_n)$ of the graded tensor product $A_1 \otimes_{\mathbb{Z}_2} \cdots \otimes_{\mathbb{Z}_2} A_n$ into the usual tensor product $(A_1 \otimes_{\mathbb{Z}_2} \mathbb{C}Z_2) \otimes \cdots \otimes (A_n \otimes_{\mathbb{Z}_2} \mathbb{C}Z_2)$ which respect the states and allow to reduce all calculations involving the graded tensor product to calculations involving the usual tensor product on the bigger algebras $F(A_1) = A_1 \otimes_{\mathbb{Z}_2} \mathbb{C}Z_2, \ldots , F(A_n) = A_n \otimes_{\mathbb{Z}_2} \mathbb{C}Z_2$. These inclusions are determined by

$$J_n(1 \otimes \cdots \otimes 1 \otimes a \otimes 1 \otimes \cdots \otimes 1) = \hat{g} \otimes \cdots \otimes \hat{g} \otimes \hat{a} \otimes 1 \otimes \cdots \otimes 1,$$

$k - 1$ times \hspace{1cm} $n - k$ times \hspace{1cm} $k - 1$ times \hspace{1cm} $n - k$ times

for $a \in A_k, 1 \leq k \leq n$, where we used the abbreviations

$$\hat{g} = 1 \otimes g, \quad \hat{a} = a \otimes 1, \quad \hat{1} = 1 \otimes 1.$$

## 5 Forgetful Functors, Coproducts, and Semi-universal Products

We are mainly interested in different categories of algebraic probability spaces. There objects are pairs consisting of an algebra $A$ and a linear functional $\varphi$ on $A$. Typically, the algebra has some additional structure, e.g., an involution, a unit, a grading, or a topology (it can be, e.g., a von Neumann algebra or a $C^*$-algebra), and the functional behaves nicely with respect to this additional structure, i.e., it is positive, unital, respects the grading, continuous, or normal. The morphisms are algebra homomorphisms, which leave the linear functional invariant, i.e., $j : (A, \varphi) \to (B, \psi)$ satisfies

$$\varphi = \psi \circ j$$

and behave also nicely w.r.t. to additional structure, i.e., they can be required to be *-algebra homomorphisms, map the unit of $A$ to the unit of $B$, respect the grading, etc. We have already seen one example in Subsection 3.3.

The tensor product then has to specify a new algebra with a linear functional and inclusions for every pair of of algebraic probability spaces. If the category of algebras obtained from our algebraic probability space by forgetting the linear functional has a coproduct, then it is sufficient to consider the case where the new algebra is the coproduct of the two algebras.
Proposition 5.1 Let \((\mathcal{C}, \boxtimes, i)\) be a tensor category with inclusions and \(F : \mathcal{C} \to \mathcal{D}\) a functor from \(\mathcal{C}\) into another category \(\mathcal{D}\) which has a coproduct \(\coprod\) and an initial object \(E_{\mathcal{D}}\). Then \(F\) is a tensor functor. The morphisms \(F_2(A, B) : F(A) \coprod F(B) \to F(A \boxtimes B)\) and \(F_0 : E_{\mathcal{D}} \to F(E)\) are those guaranteed by the universal property of the coproduct and the initial object, i.e. \(F_0 : E_{\mathcal{D}} \to F(E)\) is the unique morphism from \(E_{\mathcal{D}}\) to \(F(E)\) and \(F_2(A, B)\) is the unique morphism that makes the diagram

\[
\begin{array}{ccc}
F(A) & \xrightarrow{F_2(A, B)} & F(A \boxtimes B) \\
\downarrow & & \downarrow \\
F(A) \coprod F(B) & \xrightarrow{F_0} & F(B)
\end{array}
\]

commuting.

Proof. Using the universal property of the coproduct and the definition of \(F_2\), one shows that the triangles containing the \(F(A)\) in the center of the diagram

\[
\begin{array}{ccc}
F(A) \coprod (F(B) \coprod F(C)) & \xrightarrow{\alpha_{F(A), F(B), F(C)}} & (F(A) \coprod F(B)) \coprod F(C) \\
\downarrow & & \downarrow \\
F(A) \coprod F(B \boxtimes C) & \xrightarrow{\iota_{F(A)}} & F(A) \coprod F(A \boxtimes B) \coprod F(C) \\
\downarrow & & \downarrow \\
F(A \boxtimes (B \boxtimes C)) & \xrightarrow{F_2(A, B, C)} & F((A \boxtimes B) \boxtimes C)
\end{array}
\]

commute (where the morphism from \(F(A)\) to \(F(A \boxtimes B) \coprod F(C)\) is \(F_2(A, B) \coprod \id_{F(C)}\)), and therefore that the morphisms corresponding to all the different paths form \(F(A)\) to \(F((A \boxtimes B) \boxtimes C)\) coincide. Since we can get similar diagrams with \(F(B)\) and \(F(C)\), it follows from the universal property of the triple coproduct \(F(A) \coprod (F(B) \coprod F(C))\) that there exists only a unique morphism from \(F(A) \coprod (F(B) \coprod F(C))\) to \(F((A \boxtimes B) \boxtimes C)\) and therefore that the whole diagram commutes.

The commutativity of the two diagrams involving the unit elements can be shown similarly.

Let \(\mathcal{C}\) now be a category of algebraic probability spaces and \(F\) the functor that maps a pair \((A, \varphi)\) to the algebra \(A\), i.e., that “forgets” the linear functional \(\varphi\). Suppose that \(\mathcal{C}\) is equipped with a tensor product \(\boxtimes\) with inclusions and that \(F(\mathcal{C})\) has a coproduct \(\coprod\). Let \((A, \varphi), (B, \psi)\) be two algebraic probability spaces in \(\mathcal{C}\), we will denote the pair \((A, \varphi) \boxtimes (B, \psi)\) also by \((A \boxtimes B, \varphi \boxtimes \psi)\). By Proposition 5.1 we have morphisms \(F_2(A, B) : A \coprod B \to A \boxtimes B\) that define a natural transformation from the bifunctor \(\coprod\) to the bifunctor \(\boxtimes\). With these morphisms we can define a new tensor product \(\boxtimes\) with inclusions by

\[
(A, \varphi) \boxtimes (B, \psi) = \left(A \coprod B, (\varphi \boxtimes \psi) \circ F_2(A, B)\right).
\]

The inclusions are those defined by the coproduct.

Proposition 5.2 If two random variables \(f_1 : (A_1, \varphi_1) \to (B, \psi)\) and \(f_2 : (A_1, \varphi_1) \to (B, \psi)\) are independent with respect to \(\boxtimes\), then they are also independent with respect to \(\boxtimes\).

Proof. If \(f_1\) and \(f_2\) are independent with respect to \(\boxtimes\), then there exists a random variable \(h : (A_1 \boxtimes A_2, \varphi_1 \boxtimes \varphi_2) \to (B, \psi)\) that makes diagram in Definition 5.1 commuting. Then \(h \circ F_2(A_1, A_2) : (A_1 \coprod A_2, \varphi_1 \boxtimes \varphi_2) \to (B, \psi)\) makes the corresponding diagram for \(\boxtimes\) commuting.
The converse is not true. Consider the category of algebraic probability spaces with the tensor product, see Subsection §4, and take $B = A_1 \coprod A_2$ and $\psi = (\varphi_1 \otimes \varphi_2) \circ F_2(A_1, A_2)$. The canonical inclusions $i_{A_1} : (A_1, \varphi_1) \to (B, \psi)$ and $i_{A_2} : (A_2, \varphi_2) \to (B, \psi)$ are independent w.r.t. $\otimes$, but not with respect to the tensor product itself, because their images do not commute in $B = A_1 \coprod A_2$.

We will call a tensor product with inclusions in a category of quantum probability spaces semi-universal, if it is equal to the coproduct of the corresponding category of algebras on the algebras. The preceding discussion shows that every tensor product on the category of algebraic quantum probability spaces $\text{AlgProb}$ has a quasi-universal version.

6 The Classification of Independences in the Category of Algebraic Probability Spaces

We will now reformulate the classification by Muraki [10] and by Ben Ghorbal and Schürmann [2, 1] in terms of semi-universal tensor products with inclusions for the category of algebraic probability spaces $\text{AlgProb}$.

In order to define a semi-universal tensor product with inclusions on $\text{AlgProb}$ one needs a map that associates to a pair of unital functionals $(\varphi_1, \varphi_2)$ on two algebras $A_1$ and $A_2$ a unital functional $\varphi_1 \cdot \varphi_2$ on the free product $A_1 \coprod A_2$ (with identification of the units) of $A_1$ and $A_2$ in such a way that the bifunctor

$$\Box : (A_1, \varphi_1) \times (A_2, \varphi_2) \mapsto (A_1 \coprod A_2, \varphi_1 \cdot \varphi_2)$$

satisfies all the necessary axioms. Since $\Box$ is equal to the coproduct $\coprod$ on the algebras, we don’t have a choice for the isomorphisms $\alpha, \lambda, \rho$ implementing the associativity and the left and right unit property, we have to take the ones following from the universal property of the coproduct. The inclusions and the action of $\Box$ on the morphisms also have to be the ones given by the coproduct.

The associativity gives us the condition

$$((\varphi_1 \cdot \varphi_2) \cdot \varphi_3) \circ \alpha_{A_1,A_2,A_3} = \varphi_1 \cdot (\varphi_2 \cdot \varphi_3),$$ (6)

for all $(A_1, \varphi_1), (A_2, \varphi_2), (A_3, \varphi_3)$ in $\text{AlgProb}$. Denote the unique unital functional on $\mathbb{C}1$ by $\delta$, then the unit properties are equivalent to

$$(\varphi \cdot \delta) \circ \rho_A = \varphi \quad \text{and} \quad (\delta \cdot \varphi) \circ \lambda_A = \varphi,$$

for all $(A, \varphi)$ in $\text{AlgProb}$. The inclusions are random variables, if and only if

$$(\varphi_1 \cdot \varphi_2) \circ i_{A_1} = \varphi_1 \quad \text{and} \quad (\varphi_1 \cdot \varphi_2) \circ i_{A_2} = \varphi_2$$ (7)

for all $(A_1, \varphi_1), (A_2, \varphi_2)$ in $\text{AlgProb}$. Finally, from the functoriality of $\Box$ we get the condition

$$(\varphi_1 \cdot \varphi_2) \circ (j_1 \coprod j_2) = (\varphi_1 \circ j_1) \cdot (\varphi_2 \circ j_2)$$ (8)

for all pairs of morphisms $j_1 : (B_1, \psi_1) \to (A_1, \varphi_1), j_2 : (B_2, \psi_2) \to (A_2, \varphi_2)$ in $\text{AlgProb}$.

Our Conditions (6), (7), and (8) are exactly the axioms (P2), (P3), and (P4) in Ben Ghorbal and Schürmann [2], or the axioms (U2), the first part of (U4), and (U3) in Muraki [10].

**Theorem 6.1** (Muraki [10], Ben Ghorbal and Schürmann [2]) There exist exactly two semi-universal tensor products with inclusions on the category of algebraic probability spaces $\text{AlgProb}$, namely the semi-universal version $\otimes$ of the tensor product defined in Section §4 and the one associated to the free product $\ast$ of states.
Voiculescu’s free product \( \varphi_1 \ast \varphi_2 \) of two unital functionals can be defined recursively by

\[
(\varphi_1 \ast \varphi_2)(a_1 a_2 \cdots a_m) = \sum_{I \subseteq \{1, \ldots, m\}} (-1)^{m-|I|+1} (\varphi_1 \ast \varphi_2) \left( \prod_{k \in I} a_k \right) \prod_{k \notin I} \varphi_{k,k}(a_k)
\]

for a typical element \( a_1 a_2 \cdots a_m \in A_1 \coprod A_2 \), with \( a_k \in A_{e_k}, \epsilon_1 \neq \epsilon_2 \neq \cdots \neq \epsilon_m \), i.e. neighboring \( a \)'s don’t belong to the same algebra. \( \coprod \) denotes the number of elements of \( I \) and \( \prod_{k \notin I} a_k \) means that the \( a \)'s are to be multiplied in the same order in which they appear on the left-hand-side. We use the convention \( \varphi_1 \ast \varphi_2 \left( \prod_{k \in I} a_k \right) = 1 \).

Ben Ghorbal and Schürmann\cite{2, 1} and Muraki\cite{10} have also considered the category of non-unital algebraic probability \( \mathfrak{nuAlgProb} \) consisting of pairs \((A, \varphi)\) of a not necessarily unital algebra \( A \) and a linear functional \( \varphi \). The morphisms in this category are algebra homomorphisms that leave the functional invariant. On this category we can define three more tensor products with inclusions corresponding to the boolean product \( \lhd \), the monotone product \( \vartriangleright \) and the anti-monotone product \( \triangleleft \) of states. They can be defined by

\[
\begin{align*}
\varphi_1 \circ \varphi_2(a_1 a_2 \cdots a_m) &= \prod_{k=1}^{m} \varphi_{k,k}(a_k), \\
\varphi_1 \triangleright \varphi_2(a_1 a_2 \cdots a_m) &= \varphi_1 \left( \prod_{k:k_{k,k}=1} a_k \right) \prod_{k:k_{k,k}=2} \varphi_2(a_k), \\
\varphi_1 \triangleleft \varphi_2(a_1 a_2 \cdots a_m) &= \prod_{k:k_{k,k}=1} \varphi_1(a_k) \varphi_2 \left( \prod_{k:k_{k,k}=2} a_k \right),
\end{align*}
\]

for \( a_1 a_2 \cdots a_m \in A_1 \coprod A_2, a_k \in A_{e_k}, \epsilon_1 \neq \epsilon_2 \neq \cdots \neq \epsilon_m \), i.e. neighboring \( a \)'s don’t belong to the same algebra. Note that \( \coprod \) denotes here the free product without units, the coproduct in the category of not necessarily unital algebras.

For the classification in the non-unital case, Muraki imposes the additional condition

\[
(\varphi_1 \circ \varphi_2)(a_1 a_2) = \varphi_{e_1}(a_1) \varphi_{e_2}(a_2)
\]

(9)

for all \((e_1, e_2) \in \{(1, 2), (2, 1)\}, a_1 \in A_{e_1}, a_2 \in A_{e_2}\).

Theorem 6.2 (Muraki\cite{[10]}) There exist exactly five semi-universal tensor products with inclusions satisfying \( \text{(9)} \) on the category of non-unital algebraic probability spaces \( \mathfrak{nuAlgProb} \), namely the semi-universal version \( \varrho \) of the tensor product defined in Section 2.4 and the ones associated to the free product \( ' \), the boolean product \( \circ \), the monotone product \( \triangleright \) and the anti-monotone product \( \triangleleft \).

The monotone and the anti-monotone product are not symmetric, i.e. \((A_1 \coprod A_2, \varphi_1 \triangleright \varphi_2)\) and \((A_2 \coprod A_2, \varphi_2 \triangleright \varphi_1)\) are not isomorphic in general. Actually, the anti-monotone product is simply the mirror image of the monotone product,

\[
(A_1 \coprod A_2, \varphi_1 \triangleright \varphi_2) \cong (A_2 \coprod A_1, \varphi_2 \triangleleft \varphi_1)
\]

for all \((A_1, \varphi_1), (A_2, \varphi_2)\) in the category of non-unital algebraic probability spaces. The other three products are symmetric.

At least in the symmetric setting of Ben Ghorbal and Schürmann, Condition \( \text{(9)} \) is not essential. If one drops it and adds symmetry, one finds in addition the degenerate product

\[
(\varphi_1 \circ \varphi_2)(a_1 a_2 \cdots a_m) = \begin{cases} 
\varphi_{e_1}(a_1) & \text{if } m = 1, \\
0 & \text{if } m > 1.
\end{cases}
\]
and a whole family
\[ \varphi_1 \otimes \varphi_2 = q((q^{-1} \varphi_1) \otimes (q^{-1} \varphi_2)), \]
parametrized by a complex number \( q \in \mathbb{C} \backslash \{0\} \), for each of the three symmetric products, \( \otimes \in \{ \otimes, *, \diamond \} \).

7 The Reduction of Boolean, Monotone, and Anti-Monotone Independence to Tensor Independence

We will now present the unification of tensor, monotone, anti-monotone, and boolean independence of Franz in our category theoretic framework. It resembles closely the bosonization of Fermi independence in Subsection 4.1, but the group \( \mathbb{Z}_2 \) has to be replaced by the semigroup \( M = \{1, p\} \) with two elements, \( 1 \cdot 1 = 1, \ 1 \cdot p = p \cdot 1 = p \cdot p = p \). We will need the linear functional \( \varepsilon: \mathbb{C} M \to \mathbb{C} \) with \( \varepsilon(1) = \varepsilon(p) = 1 \).

The underlying functor and the inclusions are the same for the reduction of the boolean, the monotone and the anti-monotone product. They map the algebra \( \mathcal{A} \) of \((\mathcal{A}, \varphi)\) to the free product \( F(\mathcal{A}) = \mathcal{A} \prod \mathbb{C} M \) of the unitalization \( \mathcal{A} \) of \( \mathcal{A} \) and the group algebra \( \mathbb{C} M \) of \( M \). For the unital functional \( F(\varphi) \) we take the boolean product \( \tilde{\varphi} \otimes \varepsilon \) of the unital extension \( \tilde{\varphi} \) of \( \varphi \) with \( \varepsilon \). The elements of \( F(\mathcal{A}) \) can be written as linear combinations of terms of the form
\[ p^\alpha a_1 \cdots p^\omega a_m \]
with \( m \in \mathbb{N}, \alpha, \omega \in \{0, 1\}, \ a_1, \ldots, a_m \in \mathcal{A} \), and \( F(\varphi) \) acts on them as
\[ F(\varphi)(p^\alpha a_1 \cdots p^\omega a_m) = \prod_{k=1}^{m} \varphi(a_k). \]

The inclusion is simply
\[ J_\mathcal{A}: \mathcal{A} \ni a \mapsto a \in F(\mathcal{A}). \]

The morphism \( F_0: F(\mathbb{C}1) = \mathbb{C} M \to \mathbb{C}1 \) is given by the trivial representation of \( M \), \( F_0(1) = F_0(p) = 1 \).

The only part of the reduction that is different for the three cases are the morphisms
\[ F_2(\mathcal{A}_1, \mathcal{A}_2): \mathcal{A}_1 \prod \mathcal{A}_2 \to F(\mathcal{A}_1) \otimes F(\mathcal{A}_2) = (\mathcal{A} \prod \mathbb{C} M) \otimes (\mathcal{A} \prod \mathbb{C} M). \]

We set
\[ F_2^B(\mathcal{A}_1, \mathcal{A}_2)(a) = \begin{cases} a \otimes p & \text{if } a \in \mathcal{A}_1, \\ p \otimes a & \text{if } a \in \mathcal{A}_2, \end{cases} \]
for the boolean case,
\[ F_2^M(\mathcal{A}_1, \mathcal{A}_2)(a) = \begin{cases} a \otimes p & \text{if } a \in \mathcal{A}_1, \\ 1 \otimes a & \text{if } a \in \mathcal{A}_2, \end{cases} \]
for the monotone case, and
\[ F_2^{AM}(\mathcal{A}_1, \mathcal{A}_2)(a) = \begin{cases} a \otimes 1 & \text{if } a \in \mathcal{A}_1, \\ p \otimes a & \text{if } a \in \mathcal{A}_2, \end{cases} \]
for the anti-monotone case.
For the higher order inclusions $J_n^\ast = F_n^\ast(A_1, \ldots, A_n) \circ J_{A_1 \cdot \cdots \cdot A_n}$, one gets

\[ \begin{align*}  
J_B^n(a) &= p^{\otimes (k-1)} \otimes a \otimes p^{\otimes (n-k)}, \\
J_M^n(a) &= 1^{\otimes (k-1)} \otimes a \otimes p^{\otimes (n-k)}, \\
J_{AM}^n(a) &= p^{\otimes (k-1)} \otimes a \otimes 1^{\otimes (n-k)},
\end{align*} \]

if $a \in A_k$.

One can verify that this indeed defines reductions $(F_B, J)$, $(F_M, J)$, and $(F_{AM}, J)$ from the categories $(\nuAlgProb, \cdot, i)$, $(\nuAlgProb, \triangleright, i)$, and $(\nuAlgProb, \triangleleft, i)$ to $(\AlgProb, \otimes, i)$. The functor $U : \nuAlgProb \to \AlgProb$ mentioned at the end of Section 3 is the unitization of the algebra and the unital extension of the functional and the morphisms.

This reduces all calculations involving the boolean, monotone or anti-monotone product to the tensor product. These constructions can also be applied to reduce the quantum stochastic calculus on the boolean, monotone, and anti-monotone Fock space to the boson Fock space. Furthermore, they allow to reduce the theories of boolean, monotone, and anti-monotone Lévy processes to Schürmann’s theory of Lévy processes on involutive bialgebras, see Franz.

8 Conclusion

We have seen that the notion of independence in classical and in quantum probability depends on a product structure which is weaker than a universal product and stronger than a tensor product. We gave an abstract definition of this kind of product, which we named tensor product with projections or inclusions, and defined the notion of reduction between these products. We showed how the bosonization of Fermi independence and the reduction of the boolean, monotone, and anti-monotone independence to tensor independence fit into this framework.

We also recalled the classifications of independence by Ben Ghorbal and Schürmann and Muraki and showed that their results classify in a sense all tensor products with inclusions on the categories of algebraic probability spaces and non-unital algebraic probability spaces, or at least their semi-universal versions.

There are two ways to get more than the five universal independences. Either one can consider categories of algebraic probability spaces with additional structure, like for Fermi independence, cf. Subsection 3.3, and braided independence, cf. Franz, Schott, and Schürmann, or one can weaken the assumptions, drop, e.g., associativity, see Mlotkowski and the references therein. Romuald Lenczewski has given a tensor construction for a family of new products called $m$-free that are not associative, see also Franz and Lenczewski. His construction is particularly interesting, because in the limit $m \to \infty$ it approximates the free product. But it is not known, if a reduction of the free product to the tensor product in the sense of Definition 4.2 exists.

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