Steady dynamos in finite domains: an integral equation approach

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Abstract

The paper deals with the integral equation approach to steady kinematic dynamo models in finite domains based on Biot-Savart’s law. The role of the electric potential at the boundary is worked out explicitly. As an example, a modified version of the simple spherical α-effect dynamo model proposed by Krause and Steenbeck is considered in which the α-coefficient is no longer constant but may vary with the radial coordinate. In particular, the results for the original model are re-derived. Possible applications of this integral equation approach for numerical simulations of dynamos in arbitrary geometry and for an "inverse dynamo theory" are sketched.

I. INTRODUCTION

For decades, theoretical and numerical work on dynamos has been done for the main part in terms of differential equation systems. Countless computer simulations, in particular for spherical geometry, have led to a good understanding of dynamos, at least on the kinematic level.
In a few papers the steady state of dynamos in an infinitely extended fluid with constant conductivity has been investigated on the basis of integral equations; see Gailitis (1967), Gailitis (1970), Gailitis and Freiberg (1974), Dobler and Rädler (1998). In the case of a finite fluid body surrounded by free space the electric potential at the boundary has to be taken into account. This was already pointed out by Roberts (1967) but not really utilized for investigations of specific models. In a recent paper by Dobler and Rädler (1998) spatial variations of the electrical conductivity were also allowed to occur. However, the case of surrounding vacuum had to be excluded in this formulation.

It is the aim of this paper to re-consider the integral equation formulation of steady kinematic dynamo models with finite fluid bodies. This approach may be interesting for numerical simulations of dynamos in arbitrary geometry. E.g., realistic simulations of some recent laboratory dynamo experiments are still rare due to the fact that for general geometry the handling of boundary conditions for the induction equation is not as easy as in the spherical case. In calculations concerning the Karlsruhe dynamo experiment it was still appropriate to circumvent this problem by virtually embedding the non-spherical (but relatively compact) dynamo module into a spherically shaped surrounding of low conductivity (Rädler et al. 1998). For the Riga dynamo facility with its large aspect ratio another method was used (Stefani, Gerbeth and Gailitis 1999). The time-dependent induction equation was treated for the dynamo domain with solving, at every time-step, a Laplace equation in the outer part with appropriate matching conditions to the inner part. Using this time-consuming procedure, a more efficient numerical method which is restricted to the very dynamo domain and its boundary seemed highly desirable. In summary, the first reason for our interest in the integral equation approach is connected with the search for alternative schemes for the efficient and stable numerical treatment of dynamo problems in arbitrary geometry.

The second reason is connected with some new developments concerning inverse problems of magnetohydrodynamics (MHD). In two recent papers (Stefani and Gerbeth 1999; Stefani and Gerbeth 2000) the problem of reconstructing the velocity field of an electrically
conducting fluid from measurements of induced magnetic fields outside the fluid and measurements of electric potentials at the fluid boundary was addressed. Up to now, this work is restricted to small magnetic Reynolds numbers $R_m$, hence an external magnetic field has to be applied. The long-term objective is to generalize this inverse problem approach to large $R_m$, in particular with regard to laboratory dynamos. Of course, there is a long tradition in geophysics concerning inverse problems and (concerning the geodynamo) there is a wide literature on reconstructing the tangential velocity at the core-mantle boundary from magnetic field observations (see Bloxham 1989 and references therein). However, the frozen-flux approximation, which is crucial for that kind of velocity reconstruction, was seriously put into question recently (Gubbins and Kelly 1996; Love 1999). For obvious reasons, geophysicists never expected that the electric potential at the fluid boundary could also be known from measurement. But this situation will be different for laboratory dynamos. Considering the huge technical problems for velocity measurements in liquid sodium, an inverse dynamo theory for velocity determination from measured magnetic fields and electric potentials seems to be attractive all the more. For those applications, the integral equation approach with its explicit use of the electric potential at the boundary seems to be an appropriate starting point.

The general scheme of the integral equation approach will be represented in section 2. The integral equation for the magnetic field is derived from Biot-Savart’s law and contains a boundary integral over the electric potential. This electric potential, in turn, has to fulfill an integral equation over the boundary. It should be pointed out that the incorporation of the electric potential at the boundary is well elaborated in the quite different context of electrocardiology, electroencephalography, and magnetoencephalography. For dynamos in arbitrary geometry, in section 2 also some hints are given concerning the numerical implementation of the general scheme.

In section 3, the general scheme is applied to the case of a spherically symmetric mean-field dynamo with an arbitrary radial dependence of $\alpha$. For that case, we find three coupled integral equations for the two defining scalars of the magnetic field and for the electric
potential at the boundary. For the special case that $\alpha$ is constant inside the volume we re-derive the well-known analytical result obtained by Krause and Steenbeck (1967).

In section 4 some remarks are made on the generalization of the method to the non-steady case and to the case of varying electrical conductivity, and on possible applications to inverse dynamo theory.

II. THE INTEGRAL EQUATION APPROACH

A. General formulation

Let us consider a dynamo acting in an electrically conducting non-magnetic fluid which occupies a finite domain $D$ with boundary $S$ surrounded by non-conducting space. We restrict all the following considerations to the steady case. Then the magnetic field $B$ and the electric current density $j$ have to satisfy the Maxwell equations

$$\nabla \times B = \mu_0 j, \quad \nabla \cdot B = 0$$

(1)

everywhere with $\mu_0$ being the magnetic permeability of the free space. The electric field $E$ has to be irrotational, $E = -\nabla \varphi$, with some electric potential $\varphi$. The current density $j$ is assumed to satisfy Ohm’s law in the form

$$j = \sigma (F - \nabla \varphi)$$

(2)

inside $D$, and it vanishes outside. Here $\sigma$ is the electrical conductivity assumed to be constant. $F$ denotes the electromotive force $u \times B$ where $u$ is the velocity of the fluid motion. In the framework of mean-field electrodynamics (see, e.g., Krause and Rädler 1980) $B$, $j$, $\varphi$, and $F$ can be interpreted as mean fields. Then $F$ can be fixed, e.g., to the form

$$F = u \times B + \alpha B - \beta \nabla \times B,$$

(3)

where $u$ denotes now the mean velocity, the term $\alpha B$ describes the $\alpha$-effect and the term $\beta \nabla \times B$ another effect which can be interpreted by introducing a mean-field conductivity
different from $\sigma$. With $\alpha = \beta = 0$ we formally return to the case considered before. The considerations of this section apply for all $\mathbf{F}$ being homogeneous linear functions of $\mathbf{B}$ and its derivatives independent of their precise form.

The equations given so far define a problem for $\mathbf{B}$. Together with the requirements that there are no surface currents on $S$ and that $\mathbf{B}$ vanishes at infinity they allow to determine $\mathbf{B}$ (apart from a constant factor) if the dependence of $\mathbf{F}$ on $\mathbf{B}$ is fixed.

As it is well known equations (1) are equivalent to Biot-Savart’s law

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_D \frac{\mathbf{j}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \, dV'. \quad (4)$$

With $\mathbf{j}$ according to (2) we obtain

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 \sigma}{4\pi} \int_D \frac{\mathbf{F}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \, dV' - \frac{\mu_0 \sigma}{4\pi} \int_S \varphi(\mathbf{s}') \mathbf{n}(\mathbf{s}') \times \frac{\mathbf{r} - \mathbf{s}'}{|\mathbf{r} - \mathbf{s}'|^3} \, dS' \quad (5)$$

with $\mathbf{n}(\mathbf{s}')$ denoting the outward directed unit vector at the boundary point $\mathbf{s}'$ and $dS'$ denoting an area element at this point.

In contrast to the differential equation approach which usually deals with the magnetic field only, we have now to deal with both $\mathbf{B}$ and the electric potential $\varphi$ which is, however, needed only at the very boundary.

According to (1) the current density $\mathbf{j}$ is source-free everywhere. Therefore we may conclude from (2) that

$$\Delta \varphi(\mathbf{r}) = \nabla \cdot \mathbf{F}(\mathbf{r}) \quad (6)$$

in $D$. Since there are no surface currents on $S$ we have there $\mathbf{j} \cdot \mathbf{n} = 0$, i.e, the normal derivative of the potential on the inner side of $S$ has to satisfy

$$\frac{\partial \varphi}{\partial n}|_S = \mathbf{F}(\mathbf{s}) \cdot \mathbf{n}(\mathbf{s}). \quad (7)$$

Using Green’s theorem, it can be shown (Courant and Hilbert 1962; Barnard et al. 1967a) that
\begin{equation}
p \varphi(r) = -\frac{1}{4\pi} \int_{D} \frac{\nabla r' \cdot F(r')}{|r - r'|} \, dV' + \frac{1}{4\pi} \int_{S} n(s') \cdot \frac{F(s')}{|r - s'|} \, dS' \\
- \frac{1}{4\pi} \int_{S} \varphi(s') n(s') \cdot \frac{r - s'}{|r - s'|^3} \, dS'
\end{equation}

where \( p = 1 \) for points \( r \) inside \( D \), \( p = 1/2 \) for points \( r = s \) on \( S \) and \( p = 0 \) for points \( r \) outside \( D \). A solution for \( \varphi \) on \( S \) can be found by either taking the limit \( r \to s \) from outside or inside (for the latter case it is important to note that \( \varphi \) is continuous in \( D + S \)) or by solving the version of (8) for \( r = s \),

\begin{equation}
\varphi(s) = -\frac{1}{2\pi} \int_{D} \frac{\nabla r' \cdot F(r')}{|s - r'|} \, dV' + \frac{1}{2\pi} \int_{S} n(s') \cdot \frac{F(s')}{|s - s'|} \, dS' \\
- \frac{1}{2\pi} \int_{S} \varphi(s') n(s') \cdot \frac{s - s'}{|s - s'|^3} \, dS'.
\end{equation}

In this context it is of importance that

\begin{equation}
\lim_{r \to s} \int_{S} \varphi(s') n(s') \cdot \frac{r - s'}{|r - s'|^3} \, dS' = \mp 2\pi \varphi(s) + \int_{S} \varphi(s') n(s') \cdot \frac{s - s'}{|s - s'|^3} \, dS'
\end{equation}

where the upper and lower signs correspond to the approaches to the point \( s \) from inside or outside \( D \), respectively. The last integrals in (9) and (10) have to be understood in the sense of principal values, which are obtained by first integrating over an area \( \tilde{S} \) which is \( S \) diminished by a small tangential disk of radius \( \epsilon \) around \( s \) and taking than the limit \( \epsilon \to 0 \).

The two integral equations (5) and (9) provide another complete formulation of the problem for \( B \) as it was defined above on the basis of differential equations.

**B. Some numerical aspects**

Let us make some formal remarks concerning the numerical implementation of the coupled system of equations (5) and (9) for dynamo problems in arbitrary geometry. Assume certain discretizations of integrals and let us denote all components of \( B \) at the grid points \( i \) by \( B_i \) as well as all \( \varphi \) at the boundary grid points \( m \) by \( \varphi_m \). The strength of the induction effects incorporated in \( F \) is scaled by a factor \( \lambda \). Using Einstein’s summation convention, equation (5) can formally be written as
\begin{equation}
B_i = \lambda M_{ik} B_k + N_{im} \varphi_m .
\end{equation}

For any chosen discretization, the precise form of the matrices $M$ and $N$ can easily be derived from equation (5). We want to point out that $M$ depends on $F$ but $N$ depends only on the geometry of the boundary.

The integral equation (9) for the electric potential can as well be written in matrix notation as

\begin{equation}
\varphi_I + E_{lm} \varphi_m = \lambda H_{ln} B_n
\end{equation}

or

\begin{equation}
G_{lm} \varphi_m = \lambda H_{ln} B_n
\end{equation}

with

\begin{equation}
G_{lm} = \delta_{lm} + E_{lm} .
\end{equation}

The matrix $G$ depends only on the geometry of the boundary. Concerning the solution of (13) some care is needed as $G$ is singular. This is connected with the fact that $\varphi$ is only determined up to a constant. However, there exist methods to circumvent this problem, one of them being the deflation method (see, e.g., Barnard et al. 1967b) where the singular matrix $G$ is replaced by an appropriate non-singular matrix $\tilde{G}$ giving a unique inversion. Rather than going into the details of the deflation method, let us assume for the moment that we have found the inverse of $\tilde{G}$. Then $\varphi$ can formally be written as

\begin{equation}
\varphi_m = \lambda (\tilde{G}^{-1})_{ml} H_{ln} B_n .
\end{equation}

In a last step this can be inserted into equation (11) to give the matrix eigenvalue equation

\begin{equation}
B_i = \lambda (M_{ik} B_k + N_{im} (\tilde{G}^{-1})_{ml} H_{ln}) B_n .
\end{equation}

Note again that $N$ and $\tilde{G}^{-1}$ depend only on the geometry of the boundary. In principle, when dealing with various $u$, $\alpha$ and $\beta$ the product matrix $N \cdot \tilde{G}^{-1}$ must be computed only
once for a given geometry.
The preceding considerations provide the framework for numerical computations for finite domains of arbitrary shape.

III. A SPHERICAL MEAN-FIELD DYNAMO MODEL

We apply our integral equation approach now to a simple spherical mean-field dynamo model which is a slightly modified version of a model proposed by Krause and Steenbeck (1967) (see also Krause and Rädler 1980). To define this model we specify $D$ to be a spherical region and put $F = \alpha B$, i.e., $\mathbf{u} = 0$ and $\beta = 0$. In contrast to the original model, whose advantage is the possibility of an analytical treatment, we consider $\alpha$ no longer as necessarily constant but admit it to vary with the radial coordinate $r$. Starting from equations (5) and (8) we will derive three integral equations for three functions of the radial coordinate $r$, two of which define $B$ and the third one $\varphi$. In particular, we will re-derive the known analytical result for the original model within this new approach.

We want to point out that our model involves quite a few simplifications, and therefore the results have to be considered with care. In particular, an ideal $\alpha$-effect as given by $F = \alpha B$ occurs only with a homogeneous isotropic turbulence and then $\alpha$ has to be constant. A spatial variation of $\alpha$ requires deviations from this assumption which necessarily leads to other contributions to $F$, which are ignored here. Those deviations from homogeneous isotropic turbulence and additional contributions to $F$ must of course occur near the boundary of the fluid body. In the original model of Krause and Steenbeck this neglect led to a conflict between the results obtained for the high conductivity limit and a statement by Bondi and Gold (1950) according to which in this limit, roughly speaking, any growth of the magnetic field in outer space has to be excluded. Actually, this conflict was resolved by a consequent treatment of a model taking into account additional contributions to $F$ (Rädler 1982), and it was shown that nevertheless a dynamo is well possible even in the high-conductivity limit but the magnetic field is then completely confined inside the fluid.
A. Mathematical preliminaries

We start with splitting the magnetic field $B$ into poloidal and toroidal parts, $B_P$ and $B_T$, and representing them by defining scalars $S$ and $T$,

$$
B_P = \nabla \times \nabla \left( \frac{S}{r} r \right), \quad B_T = \nabla \left( \frac{T}{r} r \right).
$$  \hfill (17)

We refer here to spherical coordinates $r, \theta, \phi$ and denote the radius vector by $r$. The defining scalars and the electric potential are expanded in series of spherical harmonics $Y_{lm}$,

$$
S(r, \theta, \phi) = \sum_{l,m} s_{lm}(r) Y_{lm}(\theta, \phi), \quad T(r, \theta, \phi) = \sum_{l,m} t_{lm}(r) Y_{lm}(\theta, \phi),
$$

$$
\varphi(r, \theta, \phi) = \sum_{l,m} \varphi_{lm}(r) Y_{lm}(\theta, \phi).
$$  \hfill (18)

The $Y_{lm}(\theta, \phi)$ are defined as

$$
Y_{lm}(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P_{lm}(\cos \theta) e^{im\phi},
$$  \hfill (19)

with $P_{lm}$ being associated Legendre Polynomials. The summation in (18) is over all $l$ and $m$ satisfying $l \geq 0$ and $|m| \leq l$. Since, however, terms with $l = 0$ are without interest in the following we restrict all discussions to $l \geq 1$. Since $S, T$ and $\varphi$ are real we have $s_{l-m} = s_{lm}^*$ and analogous relations for $t_{lm}$ and $\varphi_{lm}$. The definition (19) implies

$$
\int_0^{2\pi} d\phi \int_0^\pi \sin \theta d\theta Y_{lm}^* \left( \theta, \phi \right) Y_{lm} \left( \theta, \phi \right) = \delta_{l\nu} \delta_{mm'} .
$$  \hfill (20)

In addition we have

$$
\Omega Y_{lm} = -l(l+1)Y_{lm}
$$  \hfill (21)

where the operator $\Omega$ is defined by

$$
\Omega f = \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 f}{\partial \phi^2}.
$$  \hfill (22)
From (17) and (18) we obtain with the help of (21) the components of $B$

\[ B_r(r, \theta, \phi) = \sum_{l,m} \frac{l(l+1)}{r^2} s_{lm}(r) Y_{lm}(\theta, \phi) \]  
\[ B_\theta(r, \theta, \phi) = \sum_{l,m} \left( \frac{t_{lm}(r)}{r \sin \theta} \frac{\partial Y_{lm}(\theta, \phi)}{\partial \phi} + \frac{1}{r} \frac{d s_{lm}(r)}{dr} \frac{\partial Y_{lm}(\theta, \phi)}{\partial \theta} \right) \]  
\[ B_\phi(r, \theta, \phi) = \sum_{l,m} \left( -\frac{t_{lm}(r)}{r} \frac{\partial Y_{lm}(\theta, \phi)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{d s_{lm}(r)}{dr} \frac{\partial Y_{lm}(\theta, \phi)}{\partial \phi} \right) \] .

Finally we recall the expression for the inverse distance between two points $r$ and $r'$,

\[ \frac{1}{|r - r'|} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \frac{r_{>l}^l}{r_{>l}^{l+1}} Y_{lm}^*(\theta', \phi') Y_{lm}(\theta, \phi) \]  

where $r_>$ denotes the larger of the values $r$ and $r'$, and $r_<$ the smaller one.

In the following we will derive integral equations for the functions $s_{lm}(r)$ and $t_{lm}(r)$, and relations for the coefficients $\varphi_{lm}(R)$.

**B. The defining scalar of the poloidal part of the magnetic field**

Taking the scalar product of both sides of (5) with the unity vector $e_r$ we obtain

\[ B(r) \cdot e_r = \frac{\mu_0 \sigma}{4\pi} \int_D \frac{\alpha(r') B(r') \times (r - r')}{|r - r'|^3} \cdot e_r \, dV' - \frac{\mu_0 \sigma}{4\pi} \int_S \varphi(s') n(s') \times \frac{r - s'}{|r - s'|^3} \cdot e_r \, dS' \]  
\[ = \frac{\mu_0 \sigma}{4\pi} \int_D \frac{\nabla_r \times (\alpha(r') B(r'))}{|r - r'|} \cdot e_r \frac{r'}{r} \, dV' . \]  

In the derivation of the second line of (27) we have expressed $e_r$ under the integrals by $(r - r')/r + (r'/r)e_{r'}$ and used the fact that $n(s')$ and $e_{r'}$ coincide for $r' = s'$. Considering now

\[ \nabla_r \times (\alpha(r') B(r')) = -B(r') \times \nabla_r \alpha(r') + \alpha(r') \nabla_r \times B(r') \]  

in (27) we see that the scalar product of the first term on the right hand side with $e_{r'}$ vanishes as the gradient of $\alpha(r')$ points in $r'$-direction, too. From (21), (24) and (25) we obtain

\[ (\nabla_r \times B(r')) \cdot e_{r'} = \sum_{l', m'} \frac{l'(l' + 1)}{r'^2} t_{l'm'}(r') Y_{l'm'}(\theta', \phi') . \]
Taking (27), (28) and (29) together we find
\[
\sum_{l,m} l(l+1) \frac{s_{lm}(r)Y_{lm}(\theta, \phi)}{r^2} = \int_D \alpha(r') \sum_{l' m'} \frac{l'(l'+1)}{r'^2} t_{l'm'}(r') \times Y_{l'm'}(\theta', \phi') \frac{r'}{|r - r'|} \, dV'.
\] (30)

Expressing the inverse distance according to equation (26) we have to distinguish between the cases \( r > r' \) and \( r < r' \). After integrating on the right-hand side of (30) over the primed angles, multiplying then both sides of (30) with \( Y_{lm}^* (\theta, \phi) \) and integrating over the non-primed angles we obtain the first integral equation of our problem in the form
\[
s_{lm}(r) = \mu_0 \sigma \frac{1}{2l + 1} \left[ \int_0^r \frac{r'^{l+1}}{r'} \alpha(r') t_{lm}(r') \, dr' + \int_r^R \frac{r'^{l+1}}{r'^2} \alpha(r') t_{lm}(r') \, dr' \right].
\] (31)

C. The electric potential at the boundary

For the determination of the potential at the boundary we start from (8) for points \( r \) outside \( D \). As for the last boundary integral we have
\[
\frac{1}{4\pi} \int_S \varphi(s') \mathbf{n}(s') \cdot \frac{\mathbf{r} - s'}{|\mathbf{r} - s'|^3} \, dS' = \frac{1}{4\pi} \int_S \varphi(s') \frac{\partial}{\partial s'} \frac{1}{|\mathbf{r} - s'|} \, dS'
\]
\[
= \sum_{l,m} \varphi_{lm}(R)Y_{lm}(\theta', \phi') \sum_{l' m'} \frac{1}{2l' + 1} \frac{\partial}{\partial s'} \frac{s'^{l'}}{r'^{l'+1}} Y_{l'm'}^*(\theta', \phi') Y_{lm'}(\theta, \phi) \, dS'.
\] (32)

and thus
\[
\lim_{\mathbf{r} \to \mathbf{s}} \frac{1}{4\pi} \int_S \varphi(s') \mathbf{n}(s') \cdot \frac{\mathbf{r} - s'}{|\mathbf{r} - s'|^3} \, dS' = \sum_{l,m} \frac{l}{2l + 1} \varphi_{lm}(R)Y_{lm}(\theta, \phi).
\] (33)

For the evaluation of the volume integral in (8) we use
\[
\nabla_{r'} \cdot \left( \alpha(r') \mathbf{B}(r') \right) = \left( \nabla_{r'} \alpha(r') \right) \cdot \mathbf{B}(r') + \alpha(r') \nabla_{r'} \cdot \mathbf{B}(r')
\]
\[
= \frac{d\alpha(r')}{dr'} \mathbf{B}_r(r')
\] (34)

and take \( B_r \) from (23). In this way we find
\[
\frac{1}{4\pi} \int_D \frac{\nabla_{r'} \cdot \left( \alpha(r') \mathbf{B}(r') \right)}{|\mathbf{r} - \mathbf{r}'|} \, dV' = \frac{1}{4\pi} \int_D \frac{d\alpha(r')}{dr'} \sum_{l' m'} \frac{l'(l'+1)}{r'^2} s_{l'm'}(r') Y_{l'm'}(\theta', \phi') \frac{1}{|\mathbf{r} - \mathbf{r}'|} \, dV'.
\] (35)
and thus
\[
\lim_{r \to s} \frac{1}{4\pi} \int_D \frac{\nabla_{r'} \cdot (\alpha(r') B(r'))}{|r - r'|} dV' = \sum_{lm} \frac{l(l+1)}{2l+1} Y_{lm}(\theta, \phi) \int_0^R \frac{r'^l}{R^{l+1}} \frac{d\alpha(r')}{dr'} s_{lm}(r') \, dr'.
\]  
(36)

Analogously, we obtain for the remaining boundary integral in (8)
\[
\lim_{r \to s} \frac{1}{4\pi} \int \mathbf{n}(s') \cdot \frac{\alpha(s') B(s')}{|r - s'|} dS' = \sum_{lm} \frac{l(l+1)}{2l+1} \frac{1}{R} \alpha(R) s_{lm}(R) Y_{lm}(\theta, \phi).
\]  
(37)

Evaluating now (8) for \( r \to s \) with the help of (33), (36) and (37) we find
\[
\varphi_{lm}(R) = -(l+1) \int_0^R \frac{r'^l}{R^{l+1}} \frac{d\alpha(r')}{dr'} s_{lm}(r') \, dr' + \frac{l+1}{R} \alpha(R) s_{lm}(R).
\]  
(38)

D. The defining scalar of the toroidal part of the magnetic field

We take now the curl of both sides of (5) thus obtaining
\[
\nabla \times B(r) = \frac{\mu_0 \sigma}{4\pi} \left[ \nabla \times \left( \nabla \times \int_D \frac{\alpha(r') B(r')}{|r - r'|} dV' \right) - \nabla \times \int_S \varphi(s') \mathbf{n}(s') \times \frac{r - s'}{|r - s'|^3} dS' \right].
\]  
(39)

Considering first the case \( r \leq R \) we further form on both sides of (39) the scalar product with \( \mathbf{e}_r \). We note that
\[
\mathbf{e}_r \cdot \left( \nabla \times \left( \nabla \times \int \frac{\alpha(r') B(r')}{|r - r'|} dV' \right) \right) = \mathbf{e}_r \cdot \left( \nabla, \nabla \cdot -\Delta_r \right) \int \frac{\alpha(r') B(r')}{|r - r'|} dV'
\] 
\[
= \frac{\partial}{\partial r} \int \frac{\nabla_{r'} \cdot (\alpha(r') B(r'))}{|r - r'|} dV' - \frac{\partial}{\partial r} \int_S \mathbf{n}(s') \cdot \frac{\alpha(s') B(s')}{|r - s'|^3} dS'
\] 
\[
+ 4\pi \alpha(r) B_r(r)
\]  
(40)

where we have used the identity \( \Delta_r |r - r'|^{-1} = -4\pi \delta(r - r') \). The two integrals on the second line of (40) were already treated in the last subsection. Concerning the boundary integral in (39) over the electric potential we note that
\[
\mathbf{e}_r \cdot \left( \nabla \times \left( \mathbf{n}(s') \times \frac{r - s'}{|r - s'|^3} \right) \right) = -\frac{\partial^2}{\partial r \partial s'} \frac{1}{|r - s'|}.
\]  
(41)

Putting everything together we have
\[
(\nabla_r \times B(r)) \cdot e_r = \mu_0 \sigma \alpha(r) B_r(r) + \frac{\mu_0 \sigma}{4\pi} \left[ \frac{\partial}{\partial r} \int_D \frac{d\alpha(r')}{dr'} B_r(r') \frac{1}{|r - r'|} dV' - \frac{\partial}{\partial r} \int_S \frac{\alpha(s') B_r(s')}{|r - s'|} dS' \right] + \int_S \varphi(s') \frac{\partial^2}{\partial r \partial s'} \frac{1}{|r - s'|} dS'.
\] (42)

Representing the right-hand side according to (29), expressing \( \varphi \) according to (18) and (38), using (26) and integrating both sides over the angles we obtain
\[
t_{lm}(r) = \mu_0 \sigma \left[ \alpha(r) s_{lm}(r) - \frac{l + 1}{2l + 1} \int_0^r r'^{l} \frac{d\alpha(r')}{dr'} s_{lm}(r') dr' + \frac{l}{2l + 1} \int_r^R \frac{r'^{l+1}}{r'^{l+1}} \frac{d\alpha(r')}{dr'} s_{lm}(r') dr' + \frac{l + 1}{2l + 1} \int_0^R \frac{r'^{l}}{R^{2l+1}} s_{lm}(r') dr' \right] \quad \text{for } r \leq R. \quad (43)
\]

From (39) we can also conclude that \( \nabla \times B = 0 \) for \( r > R \). Without going into the details of the proof we note only the consequence
\[
t_{lm}(r) = 0 \quad \text{for } r > R. \quad (44)
\]

E. Connection with the differential equation approach

Notwithstanding the fact that the differential and the integral equation approach are equivalent in a general sense it might be instructive to show this equivalence for our special problem. Differentiating equations (31) and (43) two times with respect to the radial component we obtain the equations
\[
\frac{d^2 s_{lm}(r)}{dr^2} - \frac{l(l + 1)}{r^2} s_{lm}(r) + \mu_0 \sigma \alpha(r) t_{lm}(r) = 0 \quad (45)
\]
\[
\frac{d^2 t_{lm}(r)}{dr^2} - \mu_0 \sigma \frac{d\alpha(r)}{dr} \frac{ds_{lm}(r)}{dr} - \mu_0 \sigma \alpha(r) \frac{d^2 s_{lm}(r)}{dr^2} - \frac{l(l + 1)}{r^2} \left( t_{lm}(r) - \mu \alpha(r) s_{lm}(r) \right) = 0. \quad (46)
\]

These are (apart from a factor \( r \) in the definitions of \( s_{lm} \) and \( t_{lm} \)) the same differential equations for the considered problem of radially varying \( \alpha \) for the steady case as they were already derived by Rädler (1986).

The boundary conditions which are used in the differential equation approach can as well be derived from (43) and (31). For the case \( r = R \) we see that the third term on the
right hand side of (43) vanishes identically and that the second and fourth term as well as the first and the fifth term are canceling each other. Thus we arrive in a natural way at

\[ t_{lm}(R) = 0 \] (47)

which is one of the boundary conditions. The second boundary condition

\[ \frac{ds_{lm}(r)}{dr} \bigg|_{r=R} + \frac{l}{r} s_{lm}(R) = 0. \] (48)

can be derived by differentiating equation (31) with respect to the radius and using (47).

**F. Result for the dynamo model of Krause and Steenbeck**

Let us now specify our results to the original model by Krause and Steenbeck, i.e., to the case of constant \( \alpha \). From equations (31), (38), and (43) we obtain

\[
s_{lm}(r) = \mu_0 \sigma \alpha \frac{1}{2l+1} \left[ \int_0^r \frac{r^{d+1}}{r^{l}} t_{lm}(r') \, dr' + \int_r^R \frac{r^{l+1}}{r^{d}} t_{lm}(r') \, dr' \right] \]

(49)

\[
\varphi_{lm}(R) = \alpha \frac{l+1}{R} s_{lm}(R) \]

(50)

\[
t_{lm}(r) = \begin{cases} 
\mu_0 \sigma \alpha \left( s_{lm}(r) - \frac{x^{l+1}}{R^{l+1}} s_{lm}(R) \right) & \text{for } r \leq R \\
0 & \text{for } r \geq R
\end{cases}
\]

(51)

The equation (50) is already incorporated in (43) or (51) and is not needed for the determination of the magnetic field but allows to calculate the electric potential afterwards. Introducing the dimensionless variable \( x = r/R \) we can rewrite now the two equations (49) and (51) for the expansion coefficients of the defining scalars in the form

\[
s_{lm}(x) = \mu_0 \sigma \alpha \frac{R^2}{2l+1} \left[ \int_0^x \frac{x^{d+1}}{x^{l}} t_{lm}(x') \, dx' + \int_x^1 \frac{x^{l+1}}{x^{d}} t_{lm}(x') \, dx' \right] \]

(52)

\[
t_{lm}(x) = \begin{cases} 
\mu_0 \sigma \alpha \left( s_{lm}(x) - \frac{x^{l+1}}{R^{l+1}} s_{lm}(R) \right) & \text{for } r \leq R \\
0 & \text{for } r \geq R
\end{cases}
\]

(53)

These equations are solved by

\[
s_{lm}(x) = \begin{cases} 
c_{lm} \left( \frac{1}{\mu_0 \sigma_0} x^{1/2} J_{l+1/2}(\mu_0 \sigma_0 R x) - \frac{R}{2^{l+1}} x^{l+1} J_{l-1/2}(\mu_0 \sigma_0 R) \right) & \text{for } r \leq R \\
-c_{lm} R \frac{R}{2^{l+1}} x^{-l} J_{l-1/2}(\mu_0 \sigma_0 R) & \text{for } r \geq R
\end{cases}
\]

(54)
\[ t_{lm}(x) = \begin{cases} c_{lm} x^{1/2} J_{l+1/2}(\mu_0 \sigma_0 \alpha R) & \text{for } r \leq R \\ 0 & \text{for } r \geq R \end{cases} \] (55)

if the condition
\[ J_{l+1/2}(\mu_0 \sigma_0 \alpha R) = 0 \] (56)

is satisfied. Here \( c_{lm} \) means an arbitrary constant and the \( J_n \) are Bessel functions of the first kind with the order \( n \). This result can be proved with the help of the known relations
\[
\int x^{n+1} J_n(x) \, dx = x^{n+1} J_{n+1}(x), \quad \int x^{-n+1} J_n(x) \, dx = -x^{-n+1} J_{n-1}(x),
\] (57)

\[ J_{n+1}(x) + J_{n-1}(x) = 2nx^{-1} J_n(x). \] (58)

Apart from a slight difference in the definitions of the defining scalars (by a factor \( r \)) the result (54)-(56) coincides with that reported by Krause and Rädler (1980).

**IV. GENERALIZATIONS AND PROSPECTS**

We have restricted our considerations to the steady case and to a constant electrical conductivity of the fluid. It is easy to modify them in such a way that they apply to a spatially varying conductivity. Then the electric potential is not only needed at the boundary of the fluid but at all places with a non-zero gradient of the conductivity. As for the non-steady case, it should be noted that for an infinitely extended fluid Dobler and Rädler (1998) were able to formulate an integral equation approach under the assumption that the time dependence of the magnetic field has the form \( \mathbf{B}(r, t) = \exp(\gamma t) \tilde{\mathbf{B}}(r) \). A similar procedure should also be possible for time-dependent dynamos in finite domains.

As already mentioned in the introduction, one of the long-term goals is to develop an inverse dynamo theory. For small \( R_m \), using an applied external magnetic field, it was shown (Stefani and Gerbeth 2000) that the velocity field can be reconstructed from the measurements of the induced magnetic field and the induced electric potential at the boundary if some regularization of the inverse problem is used. The presented integral formulation is
intended also as a first step to generalize this method to large $R_m$. Still needed is an appropriate formulation of the inverse dynamo theory in terms of an eigenvalue equation combined with a least-square adjustment calculus giving an optimal fit of the model to the measured quantities. Surely, some kind of regularization (e.g. with some quadratic functionals of the velocity or the magnetic field) will be needed for that purpose. Ideas pointing in a similar direction are discussed by Love and Gubbins (1996). For those investigations in inverse dynamo theory the presented integral formulation of the forward dynamo problem might be useful.
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