Recoloring bounded treewidth graphs

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Recoloring graphs \implies \textbf{Reconfiguration graphs}

Solutions \parallel Vertices. Adjacent solutions \parallel Neighbors.
Reconfiguration graph

More formally

\textbf{$k$-Reconfiguration graph of $G$}

- Vertices: Proper $k$-colorings of $G$
- Edges between any two $k$-colorings which differ on exactly one vertex.
Reconfiguration graph

More formally

\textit{k-Reconfiguration graph of } G

- Vertices: Proper \( k \)-colorings of \( G \)
- Edges between any two \( k \)-colorings which differ on exactly one vertex.

Remark

Two colorings equivalent up to color permutation are distinct.
Interesting questions

- Two solutions:
  - Are in the same connected component?
  - What distance between them?
Interesting questions

- **Two solutions:**
  - Are in the same connected component?
  - What distance between them?

- **Reconfiguration graphs:**
  - Connex?
  - What diameter?
$k$-mixing graphs

$k$-mixing

A graph is $k$-mixing if its $k$-reconfiguration graph is connected.
$k$-mixing graphs

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A graph is $k$-mixing if its $k$-reconfiguration graph is connected.

Gap

No function $f$ on the chromatic number ensures that $G$ is $k$-mixing if $k \geq f(\chi)$. 
State of the art

**Theorem (Cereceda, van den Heuvel, Johnson ’07)**
Determining if a bipartite graph is 3-mixing is co-NP hard.
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Recoloring diameter
Given a $k$-mixing graph, the recoloring diameter is in $\mathcal{O}(A(n))$ if the diameter of the $k$-reconfiguration graph is bounded by $C \times A(n)$. ($n$ is the number of vertices)
Upper bounds on recoloring

Theorem (Cereceda)
As long as $k \geq n + 1$, the clique $K_n$ is $k$-mixing in $O(n)$. 

Theorem (Bonamy, Johnson, Lignos, Paulusma, Patel '12)
Trees are 3-mixing in $O(n^2)$. 

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\begin{center}
\begin{tikzpicture}
  \node[draw,circle,fill=red] (A) at (0,0) {};
  \node[draw,circle,fill=blue] (B) at (-1,-1) {};
  \node[draw,circle,fill=green] (C) at (1,-1) {};

  \node[draw,circle,fill=red] (D) at (2,-2) {};
  \node[draw,circle,fill=blue] (E) at (1,-3) {};
  \node[draw,circle,fill=green] (F) at (-1,-3) {};

  \draw (A) -- (B);
  \draw (A) -- (C);
  \draw (D) -- (E);
  \draw (D) -- (F);
\end{tikzpicture}
\end{center}
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![Diagram of trees](image-url)
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\begin{center}
\begin{tikzpicture}
\node (A) at (0,0) [circle,fill,blue] {};
\node (B) at (2,0) [circle,fill,red] {};
\node (C) at (4,0) [circle,fill,green] {};
\draw (A) -- (B);
\draw (C) -- (B);
\end{tikzpicture}
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Trees are 3-mixing in $O(n^2)$. 
Chordal graphs

- No induced cycle of length at least 4.
- The graph admits a clique tree.

Theorem (Bonamy, Johnson, Lignos, Paulusma, Patel '12)

The chordal graphs are \((k+1)\)-mixing in \(O(n^2)\) for every \(k \geq \chi + 1\).

Questions

- Does the same hold for bounded treewidth graphs?
- And for perfect graphs?
Chordal graphs

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- Identify it with a vertex of the bag of its parent.

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- Does the same hold for bounded treewidth graphs?
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Perfect graphs: a counter-example
Bounded Treewidth graphs

Definition

\[ tw(G) = \min_H \{ \chi(H) - 1 \mid G \subseteq H, H \text{ chordal} \}. \]
Bounded Treewidth graphs

Definition

- \( tw(G) = \min_H \{ \chi(H) - 1 | G \subseteq H, H \text{ chordal} \} \).
- \( G \) admits a tree decomposition where each bag has size at most \( tw(G) + 1 \) and each edge appears in at least one bag.

\[
\begin{array}{c}
\text{1} & \text{2} & \text{3} & \text{4} & \text{5} & \text{6} & \text{7} & \text{8} & \text{9} \\
\text{1} & \text{2} & \text{3} & \text{4} & \text{5} & \text{6} & \text{7} & \text{8} & \text{9}
\end{array}
\]

Theorem (Cereceda et al.)

Every \( k \)-degenerate graph is \((k + 2)\)-mixing in \( 2^n \).
Bounded Treewidth graphs

Definition

- $tw(G) = \min_H \{ \chi(H) - 1 | G \subseteq H, H \text{ chordal} \}$.
- $G$ admits a tree decomposition where each bag has size at most $tw(G) + 1$ and each edge appears in at least one bag.

Theorem (Cereceda et al.)
Every $k$-degenerate graph is $(k + 2)$-mixing in $2^n$. 
Our main result

Theorem (Bonamy, B.)
Every graph $G$ is $(tw(G) + 2)$-mixing in $O(n^2)$. 
Our main result

Theorem (Bonamy, B.)
Every graph $G$ is $(\text{tw}(G) + 2)$-mixing in $O(n^2)$.

Optimal

- $\text{tw}(K_n) = n - 1$, so $\text{tw}(G) + 2$ colors are necessary.
- Since 3-colorings of paths have a quadratic recoloring diameter, the quadratic bound is necessary.
Sketch of the proof

There exists a tree decomposition such that every bag has size $tw(G) + 2$. 

Objective: identify vertices with their parents.

Waiting for the identification with a not too costful operation.
Sketch of the proof

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There exists a tree decomposition such that every bag has size $\text{tw}(G) + 2$.

Objective: identify vertices with their parents.

Waiting for the identification with a not too costful operation.
Conclusion

Question

- Are $k$-degenerate graphs $(k + 2)$-mixing in $O(n^2)$?

- And planar graphs?

- Other classes of graphs? (cographs and distance hereditary graphs are $\chi + 1$-mixing (Bonamy, B. '13+))

Thanks for your attention!
Conclusion

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- How does the diameter decrease when there are more colors?
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