Discrete Wiener algebra in the bicomplex setting, spectral factorization with symmetry, and superoscillations

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Abstract

In this paper we present parallel theories on constructing Wiener algebras in the bicomplex setting. With the appropriate symmetry condition, the bicomplex matrix valued case can be seen as a complex valued case and, in this matrix valued case, we make the necessary connection between bicomplex analysis and complex analysis with symmetry. We also write an application to superoscillations in this case.

Keywords Bicomplex analysis · Wiener algebra · Rational functions · Spectral factorization · Superoscillations

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1 Introduction

1.1 Prologue

In the present work we define and study the properties of the Wiener algebra in the framework of bicomplex numbers, i.e. $\mathbb{BC}$. Here we address $\mathbb{BC}$ questions in two ways and each, with the added the expense of doubling the dimension, reduces the problem to the complex setting. One way is through the introduction of symmetry and the other goes along idempotent decomposition. In the end, we also write an application to superoscillations in this case.

To set the framework we first recall the definition of the matrix-valued Wiener algebra.

**Definition 1.1** The complex Wiener algebra $\mathcal{W}^{p \times p}$ consists of the matrix-valued functions of the form

$$f(e^{it}) = \sum_{n \in \mathbb{Z}} f_ne^{int}$$

with $f_n \in \mathbb{C}^{p \times p}$ and $\sum_{n \in \mathbb{Z}} \|f_n\| < \infty$ (where $\|f_n\|$ denotes the operator norm of the matrix $f_n$) endowed with pointwise multiplication and norm

$$\|f\| = \sum_{n \in \mathbb{Z}} \|f_n\|.$$  \hspace{1cm} (1.2)

We note that in analysis quite often the Wiener algebra setting provides a convenient intermediate step, between the general $L_\infty$ case and the rational case (see Theorem 2.22 below for the latter). The algebra $\mathcal{W}^{p \times p}$ is an important example of Banach algebra. When $p = 1$ this algebra is commutative, and will be denoted by $\mathcal{W}$. Counterparts and extensions of the Wiener algebra have been studied in various other settings; see e.g. [4, 8, 27], [22, §II.1].

In the scalar case (that is, $p = 1$), the celebrated Wiener-Lévy theorem asserts that point invertibility of an element of $\mathcal{W}$ implies invertibility in the algebra itself; see [29, Lemma IIe p. 14]. We also note that in this case ($p = 1$), the same theorem can be proved using Gelfand’s theory of commutative Banach algebras and we write the statement as follows:

**Theorem 1.2** Suppose that: $f(e^{i\theta}) = \sum_{n=-\infty}^{\infty} c_n e^{in\theta}$, with $\sum_{n=-\infty}^{\infty} |c_n| < \infty$, and $f(e^{i\theta}) \neq 0$ for any $\theta \in \mathbb{R}$, then there exists a sequence $\{\gamma_n\}_{n \in \mathbb{Z}}$ such that:

$$\frac{1}{f(e^{i\theta})} = \sum_{n=-\infty}^{\infty} \gamma_n e^{in\theta},$$

and $\sum_{n=-\infty}^{\infty} |\gamma_n| < \infty$.

We already note at this point that the above theorem yields a direct proof in the bicomplex case, which we present in Sect. 4.1. The result still holds in the matrix-valued case by considering the determinant of the function.
There exist two important subalgebras of $\mathcal{W}^{p \times p}$ used in spectral factorizations, namely $\mathcal{W}^{p \times p}_+$ and $\mathcal{W}^{p \times p}_-$, defined as follows:

**Definition 1.3** The subalgebra $\mathcal{W}^{p \times p}_+$ consists of the elements $f \in \mathcal{W}^{p \times p}$ for which $f_n = 0$ for $n < 0$ in the representation (1.1). Similarly, the subalgebra $\mathcal{W}^{p \times p}_-$ consists of the elements $f \in \mathcal{W}^{p \times p}$ for which $f_n = 0$ for $n > 0$ in the representation (1.1).

A right (resp. left) spectral factorization of $f \in \mathcal{W}^{p \times p}$ is a representation of the form $f = f_- f_+$ (resp. $f_+ f_-$) where $f^\pm_+$ (resp. $f^\pm_-$) belongs to $\mathcal{W}^{p \times p}_+$ (resp. $\mathcal{W}^{p \times p}_-$).

We recall:

**Theorem 1.4** Let $f \in \mathcal{W}^{p \times q}$ be a function with a non-identically vanishing determinant. If $f(e^{it}) > 0$ (pointwise, in the sense of positive matrices), then it admits both right and left spectral factorizations in $\mathcal{W}^{p \times q}$.

We note that, while $\mathcal{W}^{p \times p}$ is not commutative, factorization and inversion results are readily deduced from the scalar case using determinant; see e.g. [18, §2]. For completeness we also mention the sources [15, 20, 21] for more information on matrix-valued and operator-valued Wiener algebras. See in particular [21, XXIX].

We define and study the Wiener algebra in the bicomplex ($\mathbb{BC}$) setting in two different ways, both reducing the problem to the complex setting. One way, as in our previous paper [9], is through the introduction of a specific symmetry to complex matrices of twice the size, which allows us to construct a bridge with the bicomplex setting. The other way goes along the idempotent decomposition of bicomplex numbers. In both cases, one doubles the dimension of the underlying framework, but in a different way. The first approach, restricting a realization to be symmetric, comes to the expense of minimality, however we can write significant results in this direction as well, as this second approach is classical in $\mathbb{BC}$ analysis.

In this paper, our research focuses on three types of results in the theory of matrix Wiener algebra on the bicomplex space:

(i) Inversion Theorems,
(ii) Factorization Theorems,
(iii) Applications to Superoscillations.

We present our results in Sects. 1.2 and 1.3, with complete proofs written in the subsequent parts of the paper. In order to do so, we first recall some properties of $\mathbb{BC}$, the algebra of bicomplex numbers.

### 1.2 $\mathbb{BC}$ and the scalar bicomplex case

We set the stage through the introduction of $\mathbb{BC}$, in the same fashion as [17, 24, 25], the key definitions and results for the case of holomorphic functions of a bicomplex variable, with a description of the bicomplex matrix setting following in Sect. 3.

The algebra of bicomplex numbers, $\mathbb{BC}$, is generated by $i$ and $j$, two commuting imaginary units. The product of the two commuting units $i$ and $j$ is denoted by $k := ij$ and we note that $k$ is a hyperbolic unit, i.e. it is a unit which squares to 1. Because of these various units in $\mathbb{BC}$, there are several different conjugations that can be defined.
naturally. We will make use of these appropriate conjugations in this paper, and we refer the reader to [24, 28] for more information on bicomplex and multicomplex analysis.

\( \mathbb{BC} \) is not a division algebra: it has two distinguished zero divisors, \( e_1 \) and \( e_2 \),

\[
\begin{align*}
e_1 &:= \frac{1+k}{2}, & e_2 &:= \frac{1-k}{2},
\end{align*}
\]

which are idempotent, linearly independent over the reals, and mutually annihilating with respect to the bicomplex multiplication:

\[
\begin{align*}
e_1 \cdot e_2 &= 0, & e_1^2 &= e_1, & e_2^2 &= e_2, \\
e_1 + e_2 &= 1, & e_1 - e_2 &= k.
\end{align*}
\]

Just like \( \{1, j\} \), they form a basis of the complex algebra \( \mathbb{BC} \), which is called the idempotent basis. If we define the following complex variables in \( \mathbb{C}(i) \):

\[
\begin{align*}
\lambda_1 &= z_1 - iz_2, & \lambda_2 &= z_1 + iz_2,
\end{align*}
\]

the \( \mathbb{C}(i) \)--idempotent representation for \( Z = z_1 + jz_2 \) is given by

\[
Z = \lambda_1 e_1 + \lambda_2 e_2.
\]

A straightforward computation yields:

\[
\begin{align*}
z_1 &= \frac{\lambda_1 + \lambda_2}{2}, \\
z_2 &= \frac{i(\lambda_1 - \lambda_2)}{2}.
\end{align*}
\]

The \( \mathbb{C}(i) \)--idempotent representation is the only one for which multiplication is component-wise, as shown in the next lemma.

**Lemma 1.5** The addition and multiplication of bicomplex numbers can be realized component-wise in the idempotent representation above. Specifically, if \( Z = \lambda_1 e_1 + \lambda_2 e_2 \) and \( W = \mu_1 e_1 + \mu_2 e_2 \) are two bicomplex numbers, where \( a_1, a_2, b_1, b_2 \in \mathbb{C}(i) \), then

\[
\begin{align*}
Z + W &= (\lambda_1 + \mu_1) e_1 + (\lambda_2 + \mu_2) e_2, \\
Z \cdot W &= (\lambda_1 \mu_1) e_1 + (\lambda_2 \mu_2) e_2, \\
Z^n &= \lambda_1^n e_1 + \lambda_2^n e_2.
\end{align*}
\]

Moreover, the inverse of an invertible bicomplex number \( Z = \lambda_1 e_1 + \lambda_2 e_2 \) (in this case \( \lambda_1 \cdot \lambda_2 \neq 0 \)) is given by

\[
Z^{-1} = \lambda_1^{-1} e_1 + \lambda_2^{-1} e_2.
\]
where \( \lambda_1^{-1} \) and \( \lambda_2^{-1} \) are the complex multiplicative inverses of \( \lambda_1 \) and \( \lambda_2 \), respectively.

One can see this also by computing directly which product on the bicomplex numbers of the form

\[
x_1 + ix_2 + jx_3 + kx_4, \quad x_1, x_2, x_3, x_4 \in \mathbb{R}
\]

is component wise, and one finds that the only one with this property is given by the mapping:

\[
x_1 + ix_2 + jx_3 + kx_4 \mapsto ((x_1 + x_4) + i(x_2 - x_3), (x_1 - x_4) + i(x_2 + x_3)),
\]

which corresponds to the idempotent decomposition

\[
Z = z_1 + jz_2 = (z_1 - iz_2)e_1 + (z_1 + iz_2)e_2,
\]

where \( z_1 = x_1 + ix_2 \) and \( z_2 = x_3 + ix_4 \).

A special subset of the bicomplex space is defined here:

**Definition 1.6** The space of hyperbolic numbers \( \mathbb{D} \) is defined as the subset of the bicomplex space \( \mathbb{BC} \) of elements the form \( Z = a + bk \), with \( a \) and \( b \) in \( \mathbb{R} \).

**Definition 1.7** The bicomplex number \( Z \) will be said to be positive if both \( \lambda_1 \) and \( \lambda_2 \) in (1.3) are positive real numbers in the idempotent representation, and we write \( Z \in \mathbb{D}_+ \). The space \( \mathbb{D}_+ \) is called the cone of hyperbolic positive numbers.

This generalizes to a partial order on the bicomplex space that is fundamental in defining notions of positivity, as well as a hyperbolic valued norm which we use throughout our paper. Using the idempotent representation one can define three norms associated with it, see [26]. We will make use of all of these norms in our work, according to the types of properties that we wish to analyze. The first two definitions represent the Lie norm and the dual Lie norm as follows:

**Definition 1.8** The dual Lie norm is, up to a factor of 2:

\[
\|Z\| = |\lambda_1| + |\lambda_2| = 2\mathcal{L}^*(Z),
\]

and

**Definition 1.9** The Lie norm is:

\[
\mathcal{L}(Z) = \max\{|\lambda_1|, |\lambda_2|\}.
\]

A slightly non-standard norm is:

**Definition 1.10** The hyperbolic valued norm: \( \|Z\|_{\mathbb{D}_+} = |\lambda_1|e_1 + |\lambda_2|e_2 \).
It is easy to see that \( \|Z\|_{\mathbb{D}_+} \in \mathbb{D}_+ \).

**Remark 1.11** We will note that if \( \|Z\| \leq M \) then \( \mathcal{L}(Z) \leq M \) and \( \|Z\|_{\mathbb{D}_+} \leq M \) and we will use this throughout our paper.

We will need the following results regarding the various norms:

**Proposition 1.12** The norm in Definition 1.8 is sub-multiplicative, i.e.

\[
\|ZW\| \leq \|Z\| \cdot ||W||. \tag{1.6}
\]

**Proof**

\[
Z = \lambda_1 e_1 + \lambda_2 e_2 \quad \text{and} \quad Z = \mu_1 e_1 + \mu_2 e_2.
\]

Then, \( ZW = \lambda_1 \mu_1 e_1 + \lambda_2 \mu_2 e_2 \) and

\[
\|ZW\| = |\lambda_1 \mu_1| + |\lambda_2 \mu_2| \leq (|\lambda_1| + |\lambda_2|)(|\mu_1| + |\mu_2|) = \|Z\| \cdot ||W||.
\]

\( \square \)

We will now revise notions of the unit bicomplex bi-disk and throughout this discussion, we will be associating to the bicomplex number \( Z = z_1 + jz_2 \) the complex pair \((z_1, z_2)\) and denote them interchangeably. We will not do so for the idempotent representation.

For \( \mathbb{B}\mathbb{C} \)-valued functions, the natural counterpart of the bi-disk in \( \mathbb{C}^2 \) is:

\[
\mathbb{K} = \left\{ (z_1, z_2) \in \mathbb{C}^2; |z_1 \pm iz_2| < 1 \right\}. \tag{1.7}
\]

We note that \( \mathbb{K} \) contains \( B(0, 1) \times \{0\} \), and so the single variable theory is obtained as a special case. Here we discuss the boundaries and properties of this bicomplex domain \( \mathbb{K} \).

**Lemma 1.13** The distinguished boundary \( |\lambda_1| = |\lambda_2| = 1 \) of the bidisk in the decomposition along the idempotents corresponds to

\[
\left\{ e^{it}(\cos s, \sin s) ; t, s \in [0, 2\pi] \right\}. \tag{1.8}
\]

i.e.

\[
Z = e^{it} e^{js} \tag{1.9}
\]

**Proof** Since

\[
|z_1 \pm i z_2|^2 = |z_1|^2 + |z_2|^2 - 2 \text{Re}(iz_1 \overline{z_2})
\]

...
the conditions $|z_1 \pm iz_2| = 1$ are equivalent to the two conditions

$$ |z_1|^2 + |z_2|^2 = 1 $$

$$ \text{Im}(z_1 \bar{z}_2) = 0. $$

Setting $z_1 = uz_2$ when $z_2 \neq 0$, $u \in \mathbb{R}$, we get

$$ \partial K = \left\{ (e^{i\theta}, 0), \theta \in [0, 2\pi] \right\} \bigcup \left\{ \bigcup_{u \in \mathbb{R}} \left\{ \left( \frac{ue^{i\theta}}{\sqrt{u^2 + 1}}, \frac{e^{i\theta}}{\sqrt{u^2 + 1}} \right), \theta \in [0, 2\pi] \right\} \right\}, $$

which is rewritten as (1.8). To conclude, note that $\partial K$

$$ z_1 + jz_2 = e^{it}(\cos s + j \sin s) = e^{it}e^{js}. $$

We will call (1.8) the distinguished boundary of $\mathbb{K}$ and denote it by $\partial \mathbb{K}$. The definition is well posed, since this constitutes the distinguished boundary of $\mathbb{K}$ in $\mathbb{C}^2$ as well.

The counterpart of the unit circle can now be defined:

**Definition 1.14** The bicomplex unit circle $\partial \mathbb{K}$ is:

$$ \partial \mathbb{K} = \left\{ e^{it}e^{js}, t, s \in [0, 2\pi] \right\} $$

the following two lemmas show us the structure of $\partial \mathbb{K}$:

**Lemma 1.15** In the idempotent representation, the bicomplex unit circle becomes:

$$ \partial \mathbb{K} = \left\{ e^{i(t+s)}e_1 + e^{i(t-s)}e_2, t, s \in [0, 2\pi] \right\} $$

**Proof** Since:

$$ z_1 = e^{it} \cos s \quad \text{and} \quad z_2 = e^{it} \sin s, $$

we have that $\lambda_1$ and $\lambda_2$, defined by (1.3), are equal to

$$ \lambda_1 = e^{i(t-s)} \quad \text{and} \quad \lambda_2 = e^{i(t+s)} $$

and

$$ z_1 + jz_2 = e^{i(t-s)}e_1 e^{i(t+s)}e_2 $$
This bicomplex unit circle has also been defined \[ 10, 24 \] in terms of hyperbolic curves, here we provide an alternate definition.

**Lemma 1.16** Elements of \( \partial \mathbb{K} \) are invertible in \( \mathbb{B} \mathbb{C} \).

**Proof** This follows from (1.10). \( \square \)

**Definition 1.17** Let the bicomplex Wiener algebra, denoted \( \mathcal{W} \) be the set of functions \( f : \partial \mathbb{K} \to \mathbb{B} \mathbb{C} \), such that:

\[
f(Z) = \sum_{n \in \mathbb{Z}} f_n Z^n, \quad Z \in \partial \mathbb{K},
\]

such that \( \sum_{n \in \mathbb{Z}} \| f_n \| < \infty \), where \( \| f_n \| \) is the dual Lie norm in Definition 1.8.

We can also define the counterparts of \( \mathcal{W}_+ \) and \( \mathcal{W}_- \) in the usual way:

**Definition 1.18** Let \( \mathcal{W}_+ \) be the set of functions \( f : \partial \mathbb{K} \to \mathbb{B} \mathbb{C} \), such that:

\[
f(Z) = \sum_{n \geq 0} f_n Z^n, \quad Z \in \partial \mathbb{K},
\]

such that \( \sum_{n \geq 0} \| f_n \| < \infty \).

Then, let \( \mathcal{W}_- \) be the set of functions \( f : \partial \mathbb{K} \to \mathbb{B} \mathbb{C} \), such that:

\[
f(Z) = \sum_{n \geq 0} f_{-n} Z^{-n}, \quad Z \in \partial \mathbb{K},
\]

such that \( \sum_{n \geq 0} \| f_{-n} \| < \infty \).

At the end of Sect. 4 we write two corollaries that extend the Wiener algebra results in the scalar case. These theorems (Corollaries 4.6, 4.7) are a direct result of the main theorems proven in the bicomplex matrix case.

**Theorem 1.19** Assume that \( f(Z) \) invertible on \( \partial \mathbb{K} \). Then \( f \) is invertible in the bicomplex Wiener algebra.

**Theorem 1.20** Assume that \( f(Z) > 0 \) invertible on \( \partial \mathbb{K} \). Then \( f \) admits left and right spectral factorizations.

### 1.3 Outline of the matrix valued case

In Sect. 4 we define the Wiener algebra in the bicomplex matrix valued case as:

**Definition 1.21** The bicomplex Wiener algebra \( \mathcal{W}^{p \times p} \) consists of the matrix-valued functions of the form

\[
f(Z) = \sum_{n \in \mathbb{Z}} f_n Z^n
\]

with \( Z \in \partial \mathbb{K} \), \( f_n \in \mathbb{B} \mathbb{C}^{p \times p} \) and \( \sum_{n \in \mathbb{Z}} \| f_n \| < \infty \).
**Remark 1.22** Here \(||f_n||\) denotes the following extension of the dual Lie norm in Definition 1.8 to the space of bicomplex matrices \(f_n = f_{n,1}e_1 + f_{n,2}e_2\):

\[
||f_n|| = ||f_{n,1}|| + ||f_{n,2}||,
\]

where \(||f_{n,1}||\) and \(||f_{n,2}||\) are the operator norms as complex matrices.

This Wiener algebra is endowed with point-wise multiplication and norm

\[
\|f\| = \sum_{n \in \mathbb{Z}} \|f_n\|. \tag{1.17}
\]

The elements of \(\mathcal{W}^{p \times p}_+\) are of the form

\[
f(Z) = \sum_{n \geq 0} f_n Z^n, \tag{1.18}
\]

while elements of \(\mathcal{W}^{p \times p}_-\) are of the form

\[
f(Z) = \sum_{n \geq 0} f_{-n} Z^{-n}. \tag{1.19}
\]

In Sect. 4 we prove an invertibility theorem in the bicomplex case (Theorem 4.2), namely if \(f \in \mathcal{W}^{p \times p}\) be such that \(f(Z)\) is invertible for \(Z \in \partial \mathbb{K}\). Then \(f\) is invertible in \(\mathcal{W}^{p \times p}\). We also prove a spectral factorization in this case (Theorem 4.4), i.e. for \(f \in \mathcal{W}^{p \times p}\) be such that \(f(Z) > 0\) for \(Z \in \partial \mathbb{K}\) we have that \(f\) admits a spectral factorization \(f(Z) = f_+(Z) f_+(Z)^*\) where \(f_+^{\pm 1} \in \mathcal{W}^{p \times p}_+\).

In Sect. 4.2 we also prove an equivalence between this bicomplex Wiener algebra and one induced by a special type of symmetry, called \(\#\) symmetry in the complex matrix domain. We will have the same type of factorization result in this case, see Theorem 4.10. For an equivalence theorem between the two algebras see Theorem 4.11. In the same Section, in Sect. 4.3, we address the rational bicomplex case with \(\#\) symmetry as well.

Last but not least, as applications of these results, in Sect. 5 we introduce a theory of superoscillations in the bicomplex case.

## 2 Matrix-valued complex Wiener algebras

### 2.1 Wiener algebra and matrix symmetries

We first describe relations in the Wiener algebras induced by symmetries with respect to invertible matrices. The Wiener Algebra in the complex matrix case see [20] and [21], here we only extract the definitions and results that will be useful in the bicomplex case.
Definition 2.1 Let $Y \in GL_n(\mathbb{C})$ and we define the following map on the space of square complex matrices:

$$M_Y = YMY^{-1}, \quad M \in \mathbb{C}^{n \times n}.$$ (2.1)

Remark 2.2 It is immediate that, for $M, N \in \mathbb{C}^{n \times n}$ and $\lambda \in \mathbb{C},$

$$(MN)_Y = MYN_Y$$

$$(M + N)_Y = M_Y + N_Y$$

$$(\lambda I_n)_Y = \lambda I_n.$$ We now have, using the map in Definition 2.1:

Lemma 2.3 Let now $f \in \mathcal{W}_p^{p \times p}$ be of the form (1.1). If we define $f_Y(e^{it}) = \sum_{n \in \mathbb{Z}} (f_n)_Y e^{int}$ for $Y \in GL_n(\mathbb{C}),$ it follows that

$$f_Y(e^{it}) = (f(e^{it}))_Y.$$ From the uniqueness of the Fourier coefficients and Theorem 1.4 one obtains the following:

Proposition 2.4 Let $f, g \in \mathcal{W}_p^{p \times p}$ and $Y \in GL_n(\mathbb{C}).$ We have:

$$(fg)_Y(e^{it}) = f_Y(e^{it})g_Y(e^{it})$$

$$f_Y(e^{it}) = (f(e^{it}))_Y$$

$$(f(e^{it}))_Y^* = ((f(e^{it}))_Y)^*.$$ Furthermore, $f \in \mathcal{W}_+^{p \times p}$ if and only if $f_Y \in \mathcal{W}_+^{p \times p}.$

Definition 2.5 The function $f \in \mathcal{W}_p^{p \times p}$ is $Y$-symmetric if $f(e^{it}) = f_Y(e^{it}),$ $t \in [0, 2\pi].$

From uniqueness of the Fourier coefficients we obtain:

Proposition 2.6 The function $f \in \mathcal{W}_p^{p \times p}$ is $Y$-symmetric if and only if its Fourier coefficients are $Y$-symmetric:

$$f_n = (f_n)_Y, \quad n \in \mathbb{Z}.$$ Theorem 2.7 Let $f \in \mathcal{W}_p^{p \times p}$ be $Y$-symmetric for some $Y \in \mathbb{C}^{p \times p}$ and such that $f(e^{it}) > 0$ for $t \in [0, 2\pi].$ Then, there is a uniquely determined spectral factorization $f = f_+ f_+^*$ where $f_+ \in \mathcal{W}_+^{p \times p}$ is $Y$-symmetric and satisfies $f_+(1) = I_p$ and invertible and any other spectral factorization differs by an $Y$-symmetric unitary multiplicative constant on the right from $f_+.$

Proof We leave the proof to the reader, since it is very similar to the proof of theorem 2.16 and uses parallel techniques. This above $Y$ symmetry does not apply for the more general, non-square case. We introduce another symmetry that will overcome this limitation.
2.2 Wiener algebra with special symmetry

We now turn towards a type of symmetry that will be useful in the bicomplex case and could be applied to the non-square case as well.

Let \( J \in \mathbb{C}^{2 \times 2} \) be a matrix such that \( JJ^* = I_2 \).

**Definition 2.8** Let \( M \in \mathbb{C}^{2 \times 2} \). We define the \( \# \)-conjugate with respect to \( J \) to be:

\[
M^\# = JMJ^*.
\]

More generally, given \( M \in \mathbb{C}^{2a \times 2b} \) a block matrix, we define

\[
M^\# = J_a M J_b^*,
\]

with

\[
J_a = J \otimes I_a, \quad J_a^* = J^* \otimes I_a
\]

and where \( \otimes \) denotes the Kronecker (tensor) product of matrices.

**Proposition 2.9** Let \( M \in \mathbb{C}^{2p \times 2q} \) and \( N \in \mathbb{C}^{2q \times 2r} \) be matrices of possibly different sizes. Then

\[
(MN)^\# = M^\# N^\#.
\]

**Definition 2.10** For \( f \in \mathcal{W}^{2p \times 2p} \) of the form (1.1), we define its \( \# \)-symmetric function to be:

\[
f^\# (e^{it}) = \sum_{n \in \mathbb{Z}} (f_n)^\# e^{int}.
\]

**Remark 2.11** It is easy to see that:

\[
f^\# (e^{it}) = (f(e^{it}))^\#,
\]

therefore \( f^\# \in \mathcal{W}^{2p \times 2p} \) of the form (1.1).

The counterpart of Proposition 2.4 is:

**Proposition 2.12** Let \( p, q, s \in \mathbb{N} \) and let \( f \in \mathcal{W}^{2p \times 2q} \) and \( g \in \mathcal{W}^{2q \times 2s} \). We have:

\[
(fg)^\# (e^{it}) = f^\# (e^{it}) g^\# (e^{it})
\]

\[
f^\# (e^{it}) = (f(e^{it}))^\#
\]

\[
((f(e^{it}))^\#)^* = ((f(e^{it}))^*)^\#
\]
Furthermore, \( f \in \mathcal{W}_+^{2p \times 2q} \) if and only if \( f^\sharp \in \mathcal{W}_+^{2p \times 2q} \).

**Definition 2.13** The matrix \( M \in \mathbb{C}^{2 \times 2} \) is called \( \sharp \)-symmetric with respect to \( J \) if and only if it satisfies \( M = M^\sharp \). More generally the block matrix \( M \in \mathbb{C}^{2a \times 2b} \) is called \( \sharp \)-symmetric with respect to \( J \) if \( M = M^\sharp \). This will hold if and only if all its \( \mathbb{C}^{2 \times 2} \) block entries are \( \sharp \)-symmetric with respect to \( J \).

**Definition 2.14** The function \( f \in \mathcal{W}_+^{2p \times 2p} \) is called \( \sharp \)-symmetric if \( f(e^{it}) = f^\sharp(e^{it}), \quad t \in [0, 2\pi] \).

From the uniqueness of the Fourier coefficients we obtain:

**Proposition 2.15** The function \( f \in \mathcal{W}_+^{2p \times 2p} \) is \( \sharp \)-symmetric if and only if its Fourier coefficients are \( \sharp \)-symmetric:

\[
fn = (fn)^\sharp, \quad n \in \mathbb{Z}.
\]

**Theorem 2.16** Let \( f \in \mathcal{W}_+^{2p \times 2p} \) be \( \sharp \)-symmetric and such that \( f(e^{it}) > 0 \) for \( t \in [0, 2\pi] \). Then, there is a uniquely determined spectral factorization \( f = f_+ f_+^\ast \) where \( f_+ \in \mathcal{W}_+^{2p \times 2p} \) is \( \sharp \)-symmetric and satisfies \( f_+(1) = I_p \) and invertible and any other spectral factorization differs by an \( \sharp \)-symmetric unitary multiplicative constant on the right from \( f_+ \).

**Proof** Using Proposition 2.4 we can write:

\[
f(e^{it}) = f_+(e^{it}) f_+(e^{it})^\ast
\]

and

\[
f(e^{it}) = (f(e^{it}))^\sharp
\]

\[
= (f_+(e^{it}) f_+(e^{it})^\ast)^\sharp
\]

\[
= (f_+(e^{it}))^\sharp (f_+(e^{it})^\ast)^\sharp
\]

\[
= (f_+(e^{it}))^\sharp ((f_+(e^{it}))^\sharp)^\ast.
\]

By uniqueness up to left multiplicative factor of the spectral factor we have

\[
f_+(e^{it}) = (f_+(e^{it}))^\sharp \cdot U
\]

and so

\[
f_+(z) = (f_+(z))^\sharp \cdot U, \quad |z| < 1. \tag{2.9}
\]

for some \( U \in GL_p(\mathbb{C}) \) which is moreover unitary. To show that \( U = I_p \) remark that \( f_+(0) \) is invertible and so, with \( f_+(z) = \sum_{n=0}^\infty a_n z^n \) we have \( a_0 \in GL_p(\mathbb{C}) \). Setting \( z = 0 \) in (2.9) leads to \( a_0 = a_0 U \) and so \( U = I_p \) since \( a_0 \) is invertible. \( \square \)
In the case of the bicomplex algebra, as seen in [9] we have an important structural matrix:

\[ J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \]  

(2.10)

which has the obvious property that \( J J^* = I_2 \).

**Definition 2.17** The matrix \( M \in \mathbb{C}^{2 \times 2} \) is called bicomplex \( \sharp \)-symmetric if it satisfies \( M = M^\sharp \), with respect to \( J \) defined in (2.10) above.

As in [9] we have:

**Remark 2.18** For \( J \) as in (2.10), we have that \( M \) is \( J \)-symmetric if and only if it is of the form

\[ M = \begin{pmatrix} z_1 & -z_2 \\ z_2 & z_1 \end{pmatrix}; \quad z_1, z_2 \in \mathbb{C}, \]  

(2.11)

which, in Sect. 4.2 will be discussed at length in relation with bicomplex analysis. In this decomposition, the matrix \( J \) becomes the matricial counterpart of the bicomplex unit \( j \) and this symmetry is fundamental to the definition of bicomplex numbers.

**Remark 2.19** As we will see in Sect. 3.3, a bicomplex number can be viewed as a matrix of the form (2.11) (this gives in fact an explicit construction of the bicomplex numbers in terms of complex matrices). Bicomplex matrices will be identified with \( \sharp \)-symmetric complex ones of the appropriate dimension. This allows us to follow the strategy introduced in our previous work [9], where various results are reduced to ones in a complex Wiener algebra with symmetry. Note that we will also use the second approach, splitting the problem into two classical complex ones via the idempotent representation.

### 2.3 Rational functions and the Wiener algebra

In the general theory of rational matrix-valued functions, we can speak about several types of realizations. We recall results that pertains to our work in the bicomplex case.

**Theorem 2.20** Every matrix-valued (say \( \mathbb{C}^{a \times b} \)-valued) rational function which is analytic at infinity admits a realization centered at infinity, i.e. can be written in the form:

\[ f(z) = D + C(zI_N - A)^{-1}B. \]

**Remark 2.21** If \( f \) is not analytic at infinity, then it has an additional polynomial part \( P(z) \) which will not be of this form, i.e.:

\[ f(z) = P(z) + D + C(zI_N - A)^{-1}B. \]
The realization is called minimal if $N$ is minimal. We write a proof of Theorem 2.20 which gives the existence of a (possibly non minimal) realization; this sequence of arguments will allow us to sketch similar proofs in the subsequent parts of the paper, where we need specific symmetries arising in the bicomplex case. Theorems 2.23 and 2.24 make use of these arguments, which in turn prove the bicomplex rational case in Theorems 4.12 and 4.13.

**Proof of Theorem 2.20** Let $p_1, \ldots, p_M$ be the poles of $f$ (in the most elementary sense, meaning points where any entry of $f$ has a non-removable singularity). At each of these points $f$ has a finite Laurent expansion and we can write (since $f$ is assumed analytic at infinity)

$$f(z) = D + \sum_{m=1}^{M} \left( \sum_{k=1}^{k_m} \frac{H_{k,m}}{(z - p_m)^k} \right), \quad z \notin \{p_1, \ldots, p_M\},$$

where $D$ and the coefficients $H_{k,m}$ belong to $\mathbb{C}^{a \times b}$. We now write

$$f_m(z) = C_m(zI_{N_m} - A_m)^{-1}B_m \quad (2.12)$$

where

$$C_m = \left( \begin{array} {ccc} I_a & 0_a & \cdots & 0_a \\ 0_a & I_a & \cdots & 0_a \\ \vdots & \vdots & \ddots & \vdots \\ 0_a & \cdots & \cdots & I_a \end{array} \right)_{k_m \times a \times a \text{ blocks}}$$

$$B_m = \left( \begin{array} {c} H_1 \\ \vdots \\ H_{k_m} \end{array} \right) \quad (2.13)$$

and

$$A_m = \left( \begin{array} {cccc} p_mI_a & I_a & 0_a & \cdots & 0 \\ 0_a & p_mI_a & I_a & \cdots & 0_a \\ 0_a & \cdots & p_mI_a & I_a \\ 0_a & \cdots & 0_a & p_mI_a \end{array} \right), \quad (2.14)$$

where $A_m \in \mathbb{C}^{k_m a \times k_m a}$. Thus neither the realization of $f_m$ nor the realization of $f$ (given by $f(z) = D + C(I_N - A)^{-1}B$) are necessarily minimal, where

$$C = \left( \begin{array} {c} C_1 \\ \vdots \\ C_M \end{array} \right), \quad A = \text{diag}(A_1, \ldots, A_M), \quad B = \left( \begin{array} {c} B_1 \\ \vdots \\ B_M \end{array} \right). \quad (2.15)$$

$\square$
We recall the following result (see [19] for more details and information). Note that we do not require the realization to be minimal as the proof in [19] still holds since $A$ is assumed to have its spectrum off the unit circle. The realization is then called regular on the unit circle.

**Theorem 2.22** Let $f$ be a $\mathbb{C}^{p\times p}$-valued rational function. If $f$ has no singularities on the unit circle, then $f \in \mathcal{W}^{p\times p}$. Let $f(z) = D + C(zI_N - A)^{-1}B$ be a realization of $f$, possibly not minimal but with no spectrum on the unit circle, and let $P$ denote the Riesz projection corresponding to the spectrum of $A$ outside the closed unit disk:

$$P = I_N - \frac{1}{2\pi i} \int_{|z|=1} (zI_N - A)^{-1} dz. \quad (2.17)$$

Then $f(z) = \sum_{n \in \mathbb{Z}} f_n z^n$ with

$$f_n = \begin{cases} C A^{n-1}(I_N - P)B, & n > 0, \\ D \delta_{n0} - C A^{n-1}PB, & n \leq 0, \end{cases} \quad (2.18)$$

and $\sum_{\mathbb{Z}} \|f_n\| < \infty$.

We now turn to the $\sharp$ symmetric case. Let us consider $J \in \mathbb{C}^{2 \times 2}$ as above, with $JJ^* = I_2$.

**Theorem 2.23** Let $f(z)$ be a $\mathbb{C}^{2p\times 2p}$-valued rational function, analytic at infinity, $\sharp$-symmetric with respect to $J$, and regular on the unit circle. Then $f$ admits a $\sharp$-symmetric realization, analytic at infinity, and regular on the unit circle.

**Proof** Going back to the construction in the proof of Theorem 2.20 we see that the matrix coefficients $H_{k,m}$ are $\sharp$-symmetric, and so is the corresponding realization as written in Theorem 2.20. We note that the Riesz projection $P$ obtained in Theorem 2.22 is also $\sharp$-symmetric. $\square$

**Theorem 2.24** Let $f \in \mathcal{W}^{2p\times 2p}$ rational and taking strictly positive values and $\sharp$ symmetric. Then it admits rational $\sharp$-symmetric left and right factorizations.

**Proof** This follows from the fact that the $a$ $\sharp$-symmetric rational function admits a sharp-symmetric realization. $\square$

Assuming $f(e^{it}) > 0$, one can write $f(z) = w(z)(w(1/\overline{z}))^*$ where the spectral factor $w(z)$ is rational. For the following, see also [5, Lemma 1.2, p. 145] where the realization is centered at the origin. The computations appear also in [6].

**Proposition 2.25** Let

$$d + c(zI_m - a)^{-1}b$$

be a realization of the left spectral factor of $f(z)$ with both $\sigma(a)$ and $\sigma(a - bd^{-1}c)$ inside the open unit disk. Then,

$$f(z) = D + C(zI_{2m} - A)^{-1}B,$$
where

\[ A = \begin{pmatrix} a & -bb^*a^{-*} \\ 0 & a^{-*} \end{pmatrix}, \]
\[ B = \begin{pmatrix} b(d^* - b^*a^{-*}c^*) \\ a^{-*}c^* \end{pmatrix}, \]
\[ C = \begin{pmatrix} c & -db^*a^{-*} \end{pmatrix}, \]
\[ D = d(d^* - b^*a^{-*}c^*). \]

The Riesz projection \( P \) from (2.17) becomes:

\[
\begin{pmatrix} I_m & X \\ 0 & 0 \end{pmatrix},
\]

(2.19)

where \( X \) is the unique solution to the Stein equation:

\[ X - aXa^* = bb^*. \]

(2.20)

We now give formulas for the Fourier coefficients in terms of \( a, b, c \) and \( d \).

**Proposition 2.26** The Fourier coefficients of the spectral function are given by

\[ f_0 = dd^* + cXc^*, \]
\[ f_k = (db^* + cXa^*)a^*(k-1)c^*, \quad k = 1, 2, \ldots \]
\[ f_{-k} = f_k^*, \quad k = 1, 2, \ldots, \]

in terms of a realization (2.25) of the spectral factor.

Using Definition 2.8, one can re-write Propositions 2.25 and 2.26 in terms of the \( \# \) conjugate as well.

**Remark 2.27** Let us assume that \( f(e^{i\theta}) > 0 \) is also \( \# \)-symmetric and the dimensions are even. It follows that the spectral factor \( w(z) \), which yields the decomposition of \( f \) i.e. \( f(z) = w(z)(w(1/\overline{z}))^* \), is rational and \( \# \) symmetric. In this context, in Propositions 2.25 and 2.26 all factors \( a, b, c, d \) as well as \( A, B, C, D \) will be \( \# \) symmetric. The Riesz Projector \( P \) will be \( \# \)-symmetric as well.

The results in this section will be interpreted in Sect. 4 in the setting of the \( BC \) Wiener algebra.

### 2.4 Wiener algebras and classical Superoscillations

The notion of superoscillations originate with the works of Aharonov and Berry and the notion of weak measurements. We refer to the papers [1–3, 13–15]. We refer to [11, 12, 16] for recent developments and to the paper [7] for recent connections with Schur analysis.
Definition 2.28 A superoscillatory sequence is a sequence of complex-valued functions \( \{F_m(t, a)\}_{m \geq 0} \) defined on \( \mathbb{R} \) as follows:

\[
F_m(t, a) = \left( \cos\left(\frac{t}{m}\right) + i \ a \sin\left(\frac{t}{m}\right) \right)^m = \sum_{k=0}^{m} c_k(m, a)e^{i\frac{t(1-2k/m)}{m}}, \tag{2.21}
\]

where:

\[
c_m(n, a) = \binom{m}{k} \left( \frac{1 + a}{2} \right)^{m-k} \left( \frac{1 - a}{2} \right)^k,
\]

with \( a > 1 \).

One can then see that for fixed \( t \in \mathbb{R} \) we have:

\[
\lim_{m \to \infty} F_m(t, a) = e^{iat} \tag{2.23}
\]

and convergence is uniform on compact subsets of the real line. We note that:

Lemma 2.29 Let \( z = \rho e^{i\theta} \) with \( \theta \in (-\pi, \pi) \), i.e. \( z \in \Omega = \mathbb{C} \setminus (-\infty, 0] \), and define for \( c \in \mathbb{C} \)

\[
z^c = e^{c\ln\rho} e^{ic\theta}.
\]

The function

\[
G_m(z, a) = \sum_{k=0}^{m} c_k(m, a)z^{1-2k/m}, \quad z \in \Omega,
\]

coincides with \( F_m(t, a) \) for \( t \in (-\pi, \pi) \), and \( z^m G_m(z^m, a) \in \mathcal{W}_+ \).

Proof The first claim is clear and the second follows from the equality

\[
G_m(z^m, a) = \sum_{k=0}^{m} c_k(m, a)z^{m(1-2k/m)} = \sum_{k=0}^{m} c_k(m, a)z^{m-2k}.
\]

The sequence \( F_m(t, a) \) plays an important role in the respective Wiener algebra setting. Let us now turn to the matrix-valued Wiener algebra \( \mathcal{W}_{p \times p} \) case. We will show that the superoscillatory sequence \( F_m(t, a) \) provides a good approximation in this case as well. We have the following result which will expand to the \( \mathbb{B} \mathbb{C} \) as well.
Theorem 2.30 Let $f \in \mathcal{W}^{p \times p}$, then $f$ can be uniformly approximated on compact subsets of $\mathbb{R}$ by sums of the form

$$f_{-1}e^{-it} + f_0 + f_1e^{it} + \sum_{|n| \leq N, n \notin \{-1,0,1\}} f_n F_m(t, n).$$

Proof Let $\epsilon > 0$ and $f \in \mathcal{W}^{p \times p}$. If we write

$$f(e^{it}) = f_0 + f_{-1}e^{-it} + f_1e^{it} + g(e^{it}),$$

there exists $N \in \mathbb{N}$ such that

$$\sum_{|n| > N} \|f_n\| < \frac{\epsilon}{2},$$

and by (2.23) there exists $M$ such that

$$\forall n \in \{-N, \ldots, N\} \setminus \{-1,0,1\}, \quad m \geq M \implies \|F_m(t, n) - e^{int}\| \leq \frac{\epsilon}{2N - 2}.$$

Thus for $m \geq M$

$$\|g(e^{it}) - \sum_{n \in \mathbb{Z}, n \notin \{-1,0,1\}} f_n F_m(t, n)\| \leq \epsilon,$$

and so

$$\left\|f(e^{it}) - \left(f_{-1}e^{-it} + f_0 + f_1e^{it} + \sum_{|n| \leq N, n \notin \{-1,0,1\}} f_n F_m(t, n)\right)\right\| \leq \epsilon.$$

\[\square\]

In preparation of the following two results, recall that a (say $\mathbb{C}^{p \times q}$-function analytic in the open unit disk and with a real positive part there is called Carathéodory function, and that a $\mathbb{C}^{p \times q}$-valued function analytic and contractive in the open unit disk is called a Schur function. When $p = q$, and given a Schur function $S$, the function $I_p - S$ is a Carathéodory function.

Theorem 2.31 Let $\varphi$ be a $\mathbb{C}^{p \times p}$-valued Carathéodory function belonging to the Wiener algebra and such that $\text{Re} \varphi(e^{it}) \geq \epsilon_0 I_p$ for some $\epsilon_0 > 0$. Then, $\varphi$ can be approximated uniformly on compact subsets of the real line by functions $\varphi_N$ which are finite linear combinations of functions of the form $F_m(t, n)$ with the following property: these functions are not in the Wiener algebra, but for some integer $m_0$, the functions $z^{m_0} \varphi_N(z^{m_0})$ are in $\mathcal{W}_+^{p \times p}$.
Proof The proof is a slight adaptation of the proof of Theorem 2.30. We set
\[ \varphi(e^{it}) = f_0 + 2f_1e^{it} + 2 \sum_{n=2}^{\infty} f_n e^{in}, \]
where the \( f_n \in \mathbb{C}^{p \times p} \) satisfy \( \sum_{n=2}^{\infty} \| f_n \| < \infty \). We first choose \( N \) such that
\[ \sum_{n=N+1}^{\infty} \| f_n \| < \frac{\varepsilon_0}{16}, \]
and fix a compact subset, say \( K \), of the real line. For such a \( N \) we choose \( m_0 \) such that
\[ \max_{t \in K} |F_m(t, n) - e^{i t n}| < \frac{\varepsilon_0}{8 \left( \sum_{\ell=2}^{N} \| f_{\ell} \| \right)}. \tag{2.25} \]
With \( K = [-\pi, \pi] \) we define:
\[ \psi_N(t) = f_0 + 2f_1e^{it} + \sum_{n=2}^{N} f_n F_m(t, n). \]
With \( G_m \) defined by \( 2.24 \) and for every \( t \in (-\pi, \pi) \) we have
\[ \psi_N(t) = \varphi_N(e^{it}) = f_0 + 2f_1e^{it} + \sum_{n=2}^{N} f_n G_m(e^{it}, n). \]
We have for \( t \in (-\pi, \pi) \)
\[ \| \varphi(e^{it}) - \varphi_N(e^{it}) \| \leq \sum_{n=2}^{N} \| f_n \| \cdot \| G_m(e^{it}, n) - e^{i t n} \| + 2 \sum_{n=N+1}^{\infty} \| f_n \| \leq \frac{\varepsilon_0}{4}. \]
It follows that, still for \( t \in (-\pi, \pi) \),
\[ \| \text{Re} \varphi(e^{it}) - \text{Re} \varphi_N(e^{it}) \| \leq \frac{1}{2} \left( \| \varphi(e^{it}) - \varphi_N(e^{it}) \| + \| (\varphi(e^{it}))^* - (\varphi_N(e^{it}))^* \| \right) \leq \frac{\varepsilon_0}{2}. \tag{2.26} \]
When \( p = 1 \) the triangle inequality implies that
\[ \text{Re} \varphi(e^{it}) - \frac{\varepsilon_0}{2} \leq \text{Re} \varphi_N(e^{it}) \leq \text{Re} \varphi(e^{it}) + \frac{\varepsilon_0}{2}, \quad t \in (-\pi, \pi). \]
Thus, with $m_0$ defined by (2.25) (for $K = [-\pi, \pi]$), we have that $z^{m_0} \varphi_N(z^{m_0})$ is in $\mathcal{W}_+$. The case of matrix-valued function is adapted as follows. For every $c \in \mathbb{C}^p$ we have

$$\langle \Re \varphi(e^{it}) - \Re \varphi_N(e^{it})c, c \rangle \leq \frac{c^*c \epsilon_0}{2}.$$  

Hence

$$c^* \Re \varphi(e^{it})c - \frac{c^*c \epsilon_0}{2} \leq \Re \varphi_N(e^{it}) \leq c^* \Re \varphi(e^{it})c + \frac{c^*c \epsilon_0}{2} \leq \varphi_N(e^{it}).$$  

The proof follows since $\frac{cc^* \epsilon_0}{4} \leq c^* \Re \varphi(e^{it})c$.  

**Remark 2.32** A similar result holds for Schur functions belonging to the Wiener algebra and strictly contractive on the unit circle. The proof reduces to the case of Carathéodory functions in two steps. The function is then $\mathbb{C}^{r \times r}$-valued with $r = \max\{p, q\}$, and first we can assume that the Schur function $S$ is square (by adding a number of columns or rows with entries equal to 0). One can then consider the function $\varphi(e^{it}) = I_r - S(e^{it})$. These ideas will be discussed in future works.

In what follows, we extend these results to the bicomplex case using the intrinsic structure of its algebra and analysis.

### 3 Bicomplex analysis and symmetry domains

In this section we revisit concepts of bicomplex analysis and write the appropriate symmetry domains that will allow us to introduce a bicomplex Wiener algebra in this case.

#### 3.1 Bicomplex algebra as $2 \times 2$ complex matrices

As described in [9], in a way similar to the space of complex numbers and the Pauli model, one can re-write the space of bicomplex numbers as a subspace of complex matrices $\mathbb{C}^{2 \times 2}$ as:

$$\mathbb{BC} = \left\{ \begin{pmatrix} z_1 & -z_2 \\ z_2 & z_1 \end{pmatrix} ; z_1, z_2 \in \mathbb{C} \right\},$$  

(3.1)

where the complex unit in $\mathbb{C}$ is denoted by $i$. We write

$$\begin{pmatrix} z_1 & -z_2 \\ z_2 & z_1 \end{pmatrix} = z_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + z_2 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = z_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + Jz_2,$$

with $J$ as in (2.10). We will use the shorter notation $Z = z_1 + jz_2$ to denote a bicomplex number. For example, in this writing, we have that the original complex
unit $i$ becomes:

\[
i I = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}.
\]

**Remark 3.1** In the same notation as in Sect. 1.2 one obtains the corresponding matrix to the *hyperbolic unit* $k = i j$, (i.e. $k^2 = 1$) as $K = iJ$.

In the matrix notation $k$ is represented by:

\[
i J = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},
\]

and one can easily check that the square of this matrix is the identity matrix.

In the matricial writing of a bicomplex number $Z \in \mathbb{B}\mathbb{C}$, we make use of the following primary decomposition and the fact that a normal matrix is unitary diagonalizable to obtain:

\[
\begin{pmatrix} z_1 & -z_2 \\ z_2 & z_1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} z_1 - iz_2 & 0 \\ 0 & z_1 + iz_2 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ i & -1 \end{pmatrix}.
\]

The unitary and hermitian diagonalization matrix is:

\[
U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ i & -1 \end{pmatrix},
\]

and the eigenvalues are:

\[
\lambda_1 = z_1 - iz_2 \\
\lambda_2 = z_1 + iz_2.
\]

**Remark 3.2** In this commutative setting we have the same zero divisors $e_1$, $e_2$ as in Sect. 1.2, given by the diagonalization. As expected, these split the bicomplex space in $\mathbb{B}\mathbb{C} = \mathbb{C}e_1 \oplus \mathbb{C}e_2$, as:

\[
Z = z_1 + jz_2 = (z_1 - iz_2)e_1 + (z_1 + iz_2)e_2.
\]

Using normality again, the corresponding matrix form of a bicomplex number can be written as the associated sum of two weighted orthogonal projections

\[
\begin{pmatrix} z_1 & -z_2 \\ z_2 & z_1 \end{pmatrix} = \lambda_1 \cdot E_1 + \lambda_2 \cdot E_2,
\]

where the eigenvalues $\lambda_{1,2}$ are given in (3.3) and the orthogonal projections are

\[
E_1 = \frac{1}{2} \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \quad \text{and} \quad E_2 = \frac{1}{2} \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}.
\]
It is easily seen that $E_1$ and $E_2$ are the corresponding matrix forms of $e_1$ and $e_2$. We will now define the conjugates in the bicomplex setting, as in [17, 24].

**Definition 3.3** For any $Z \in \mathbb{BC}$ we have the following three conjugates:

\[
Z = z_1 + jz_2, \\
Z^\dagger = z_1 - jz_2, \\
Z^* = \overline{Z}^\dagger = \overline{z}_1 - j\overline{z}_2.
\]

**Remark 3.4** In the matrix form the space of hyperbolic numbers $\mathbb{D}$ is realized as the subset of matrices (3.1) with:

\[
\begin{pmatrix}
a & -ib \\
ib & a
\end{pmatrix}.
\]

As in [10], one can define the corresponding notions of positivity:

**Definition 3.5** A matrix $M \in \mathbb{BC}^{n \times n}$ is called Hermitian if $M = M^*$ and positive if $c^*Mc \geq 0$ for all $c \in \mathbb{BC}^n$. Here $M^*$ is the $^*$--conjugate of the matrix $M^t$, namely, for $M = (Z_{ab})_{1 \leq a, b \leq n}$, we have $M = (Z_{ba}^*)_{1 \leq a, b \leq n}$, in the sense of Definition 3.3.

In [9] we proved the following equivalent definition, in terms of our matricial form:

**Proposition 3.6** Let $M = M_1 + jM_2 \in \mathbb{BC}^{n \times n}$ with $M_1, M_2 \in \mathbb{C}^{n \times n}$. Then $M$ is Hermitian, (resp. positive), in the sense of Definition 3.5 if and only if

\[
\tilde{M} = \begin{pmatrix}
M_1 & -M_2 \\
M_2 & M_1
\end{pmatrix}
\]

is Hermitian (resp. a positive matrix, in the sense of positive matrices with complex entries). If we write

\[
M_1 + jM_2 = P_1e_1 + P_2e_2,
\]

with $P_1 = M_1 - iM_2$ and $P_2 = M_1 + iM_2$. Then, $M$ is positive in the sense of Definition 3.5 if and only if both $P_1$ and $P_2$ are positive elements of $\mathbb{C}^{n \times n}$.

**3.2 $\mathbb{BC}$ analyticity**

As seen in [17, 24] and [28], the function $F = F_1 + jF_2$ is $\mathbb{BC}$--analytic if and only if the functions $F_1$ and $F_2$ are complex holomorphic in $z_1$ and $z_2$ i.e.

\[
\frac{\partial F_1}{\partial z_1} = \frac{\partial F_1}{\partial z_2} = \frac{\partial F_2}{\partial z_1} = \frac{\partial F_2}{\partial z_2} = 0,
\]

(3.5)
and if the following Cauchy-Riemann like equations hold:

\[
\begin{align*}
\frac{\partial F_1}{\partial z_1} &= \frac{\partial F_2}{\partial z_2}, \\
\frac{\partial F_1}{\partial z_2} &= -\frac{\partial F_2}{\partial z_1}.
\end{align*}
\] (3.6)

(3.7)

**Remark 3.7** The identity function $Z = z_1 + jz_2$ is $\mathbb{BC}$-analytic, but the $\mathbb{C}$-valued functions $Z \mapsto z_1$ and $Z \mapsto z_2$ are not.

### 3.3 Bicomplex matrices as complex matrices with symmetry

We will now view bicomplex matrices in $\mathbb{BC}^{p \times p}$ as matrices in $\mathbb{C}^{2p \times 2p}$ with the bicomplex $\sharp$-symmetry defined below. We then can reduce the various results to ones in a complex Wiener algebra with symmetry. Note that we will also use the second approach, splitting the problem into two classical complex ones via the idempotent representation.

**Remark 3.8** This type of symmetry can be viewed as a complexification of the geometry of complex numbers as well. While in the complex framework a rotation matrix takes the Pauli form, in the bicomplex case it becomes a “double rotation”.

We recall that in the case the bicomplex algebra, as seen is [9] and in (2.10), we have an important structural matrix:

\[ J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \]

which has the obvious property that $JJ^* = I_2$. In this context, we recall Definition 2.17 and we have that the matrix $M \in \mathbb{C}^{2 \times 2}$ is called bicomplex $\sharp$-symmetric if it satisfies $M = M^\sharp$, with respect to this $J$. This matrix gives us the structure of bicomplex numbers as $M$ is $J$-symmetric if and only if it is of the form (see 2.11).

\[ M = \begin{pmatrix} z_1 & -z_2 \\ z_2 & z_1 \end{pmatrix}; \quad z_1, z_2 \in \mathbb{C}. \]

We now extend this definition to the general matrix case, by first defining

\[ J_a = J \otimes I_a = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes I_a. \]

In the same way we define $J_a^* = J^* \otimes I_a$, where $\otimes$ denotes the Kronecker (tensor) product of matrices. Then we have:

**Definition 3.9** The block matrix $M \in \mathbb{C}^{2a \times 2b}$ is called bicomplex $\sharp$-symmetric if and only if it satisfies $M = M^\sharp$, where

\[ M^\sharp = J_a M J_b^*. \]
We will make use of all of these notions in the following Sect. 4, specifically in Sect. 4.2.

The following lemma is an immediate result:

**Lemma 3.10** Let \( M \in \mathbb{BC}^{p \times p} \), with \( M = (Z_{ab})_{1 \leq a, b \leq p} \) and \( Z_{ab} = z_1^{ab} + jz_2^{ab} \in \mathbb{BC} \). We can decompose \( M \) as a matrix in \( \mathbb{C}^{2p \times 2p} \) first as \( \tilde{M} \) in Proposition 3.6 and second as a matrix where each \( Z_{ab} = z_1^{ab} + jz_2^{ab} \) is written as the \( 2 \times 2 \) block

\[
\begin{pmatrix}
z_1^{ab} & -z_2^{ab} \\
z_2^{ab} & z_1^{ab}
\end{pmatrix},
\]

namely as \( Z_1 \otimes I_2 + Z_2 \otimes J \), where \( Z_1 = (z_1^{ab})_{1 \leq a, b \leq p} \) and \( Z_2 = (z_2^{ab})_{1 \leq a, b \leq p} \). The matrix from the second decomposition is bicomplex \( \# \)-symmetric and the definitions of positivity coincide.

We now turn our attention to building the various bicomplex Wiener algebras.

### 4 The scalar and matrix-valued bicomplex Wiener algebra

We extend the Wiener algebra setting to the set of bicomplex numbers in two ways, as follows. The bicomplex Wiener algebra can be defined in a direct way using the idempotent split, or as a symmetry based Wiener algebra in the complex domain, using a block matrix that fixes the bicomplex structure.

#### 4.1 Wiener bicomplex algebra using bicomplex analyticity

In this section we describe the general matrix-valued bicomplex Wiener algebra, induced by the analytic and algebraic structures of the set of bicomplex numbers, via the idempotent split.

We first recall the definition of matrix-valued bicomplex Wiener algebras, as in Definition 1.16.

**Definition 4.1** The bicomplex Wiener algebra \( \mathbb{M}^{p \times p} \) consists of the matrix-valued functions of the form

\[
f(Z) = \sum_{n \in \mathbb{Z}} f_n Z^n
\]

with \( Z \in \partial \mathbb{K} \), \( f_n \in \mathbb{BC}^{p \times p} \) and \( \sum_{n \in \mathbb{Z}} ||f_n|| < \infty \).

Here \( ||f_n|| \) denotes the dual Lie norm extended to set of bicomplex matrices \( f_n = f_n,1e_1 + f_n,2e_2 : ||f_n|| = ||f_n,1|| + ||f_n,2|| \), where \( ||f_n,1|| \) and \( ||f_n,2|| \) are the operator norms as complex matrices.

This Wiener algebra is endowed with point-wise multiplication and norm

\[
||f|| = \sum_{n \in \mathbb{Z}} ||f_n||.
\]

(4.2)
The elements of $\mathcal{W}^{p \times p}_+$ are of the form
\[ f(Z) = \sum_{n \geq 0} f_n Z^n, \quad (4.3) \]
while elements of $\mathcal{W}^{p \times p}_-$ are of the form
\[ f(Z) = \sum_{n \geq 0} f_{-n} Z^{-n}. \quad (4.4) \]

We have the following theorems that describe the structure of the Wiener algebras in this setting.

**Theorem 4.2** Let $f \in \mathcal{W}^{p \times p}$ be such that $f(Z)$ is invertible for $Z \in \partial \mathbb{K}$. Then $f$ is invertible in $\mathcal{W}^{p \times p}$.

**Proof** We write $f(Z) = \sum_{n \in \mathbb{Z}} f_n Z^n$ with $f_n \in \mathbb{B}\mathbb{C}^{p \times p}$, and $f_n = f_{1n}e_1 + f_{2n}e_2$ where $f_{1n}$ and $f_{2n}$ belong to $\mathbb{C}^{p \times p}$.

As $Z \in \partial \mathbb{K}$ is given by corresponding $\lambda_1 = e^{i(t+s)}$ and $\lambda_2 = e^{i(t-s)}$ given by Lemma 1.15
\[ \sum_{n \in \mathbb{Z}} f_{1n} e^{in(b-a)} > 0 \quad \text{and} \quad \sum_{n \in \mathbb{Z}} f_{2n} e^{in(b+a)} > 0, \quad a, b \in [0, 2\pi]. \]

Setting $s = 0$, the classical inversion result leads to
\[ \left( \sum_{n \in \mathbb{Z}} f_{\ell n} e^{i\ell t} \right)^{-1} = g_{\ell}(e^{i\ell t}) \in \mathcal{W}^{p \times p}_+, \quad \ell = 1, 2. \]

we set
\[ g(Z) = \left( \sum_{n \in \mathbb{Z}} g_{1n} \lambda_1^n \right) e_1 + \left( \sum_{n \in \mathbb{Z}} g_{2n} \lambda_2^n \right) e_2, \]
then $g \in (\mathcal{W}(\mathbb{B}\mathbb{C}))^{p \times p}$ and $f(Z)g(Z) = I_p$ for $Z \in \partial \mathbb{K}$ and this completes the proof. \(\Box\)

**Corollary 4.3** In the previous theorem it is enough to require invertibility on $e^{it}$, $t \in [0, 2\pi]$ to ensure invertibility in $\mathcal{W}^{p \times p}$.

**Theorem 4.4** Let $f \in \mathcal{W}^{p \times p}$ be such that $f(Z) > 0$ for $Z \in \partial \mathbb{K}$. Then $f$ admits a spectral factorization $f(Z) = f_+(Z)f_+(Z)^*$ where $f^+ \in \mathcal{W}^{p \times p}_+$.

**Proof** In the same notation as in the proof of the previous theorem, and with $Z \in \partial \mathbb{K}$ given by (1.10) and corresponding $\lambda_1$ and $\lambda_2$ given by (5.2)
\[ \sum_{n \in \mathbb{Z}} f_{1n} e^{in(b-a)} > 0 \quad \text{and} \quad \sum_{n \in \mathbb{Z}} f_{2n} e^{in(b+a)} > 0, \quad a, b \in [0, 2\pi]. \]
Setting $a = 0$, the classical factorization result leads to the spectral factorization

$$\sum_{n \in \mathbb{Z}} f_{\ell n} e^{inb} = f_{\ell+}(e^{ib})(f_{\ell+}(e^{ib}))^*, \quad \ell = 1, 2.$$ 

with $f_{\ell+} \in \mathcal{W}_+^{p \times p}$ and invertible in $\mathcal{W}_+^{p \times p}$. Let

$$f_{\ell+}(e^{ib}) = \sum_{n=0}^{\infty} g_{\ell,n} e^{inb}, \quad \ell = 1, 2,$$

with $\sum_{n=0}^{\infty} \|g_{\ell,n}\| < \infty$ for $\ell = 1, 2$, and set

$$f_+(Z) = \left(\sum_{n=0}^{\infty} g_{1n} e^{in(b-a)}\right) e_1 + \left(\sum_{n=0}^{\infty} g_{2n} e^{in(b+a)}\right) e_2, \quad Z = e^{ib} + ja \in \partial \mathbb{K}.$$ 

Then, $f(Z) = f_+(Z)f_+(Z)^*$. To conclude we need to verify that $f_+$ is invertible in $\mathcal{M}_+^{p \times p}$. Let

$$(f_{\ell+}(e^{ib}))^{-1} = \sum_{n=0}^{\infty} h_{\ell,n} e^{inb}, \quad \ell = 1, 2,$$

with $\sum_{n=0}^{\infty} |h_{\ell,n}| < \infty$ for $\ell = 1, 2$. We have

$$(f_+(Z))^{-1} = \left(\sum_{n=0}^{\infty} h_{1n} e^{in(b-a)}\right) e_1 + \left(\sum_{n=0}^{\infty} h_{2n} e^{in(b+a)}\right) e_2, \quad Z = e^{ib} + ja \in \partial \mathbb{K},$$

so that $f_+(Z)$ is invertible in $\mathcal{M}_+^{p \times p}$.

**Corollary 4.5** In the previous theorem it is enough to assume that $f(e^{ib}) > 0$ for $b \in [0, 2\pi]$.

When $p = 1$ the above results reduce to the bicomplex scalar case as shown in the following two corollaries:

**Corollary 4.6** Assume that $f(Z)$ invertible on $\partial \mathbb{K}$. Then $f$ is invertible in the bicomplex Wiener algebra $\mathcal{W}$.

**Corollary 4.7** Assume that $f(Z) > 0$ invertible on $\partial \mathbb{K}$. Then $f$ admits left and right spectral factorizations in $\mathcal{W}_+$ and $\mathcal{W}_-$. 

The proofs of these theorems follow directly from Theorems 4.2 and 2.7, in the case when $p = 1$. 
4.2 Bicomplex Wiener algebras using $\#$ symmetries

Let us return to bicomplex matrices with $\#$ symmetry as in Definition 3.9.

**Proposition 4.8** Using our specific bicomplex $\#$ definition (with $J$ defined in (2.10)) and setting $p, q, s \in \mathbb{N}$ and $f \in \mathcal{W}^{2p \times 2q}$ and $g \in \mathcal{W}^{2q \times 2s}$, we have:

\[
(fg)^\#(e^{it}) = f^\#(e^{it})g^\#(e^{it}) \quad (4.5)
\]
\[
f^\#(e^{it}) = (f(e^{it}))^\# \quad (4.6)
\]
\[
((f(e^{it}))^\#)^* = ((f(e^{it}))^\#)^\#. \quad (4.7)
\]

Furthermore, $f \in \mathcal{W}^{2p \times 2q}_+$ if and only if $f^\# \in \mathcal{W}^{2p \times 2q}_+$.

**Proposition 4.9** The function $f \in \mathcal{W}^{2p \times 2p}$ is bicomplex $\#$-symmetric if and only if its Fourier coefficients are bicomplex $\#$-symmetric:

\[
f_n = (f_n)^\#, \quad n \in \mathbb{Z}.
\]

**Theorem 4.10** Let $f \in \mathcal{W}^{2p \times 2p}$ be bicomplex $\#$-symmetric and such that $f(e^{it}) > 0$ for $t \in [0, 2\pi]$. Then, there is a uniquely determined spectral factorization $f = f_+^\#f_+^\#$ where $f_+ \in \mathcal{W}^{2p \times 2p}_+$ is bicomplex $\#$-symmetric and satisfies $f_+(1) = I_{2p}$ and invertible and any other spectral factorization differs by a bicomplex $\#$-symmetric unitary multiplicative constant on the right from $f_+$.

Using Lemma 3.10, we can now establish the equivalence between the two types of Wiener algebras in the bicomplex case.

**Theorem 4.11** $\mathcal{W}^{p \times p}$ is isomorphic to the space of $\mathcal{W}^{2p \times 2p}$ with bicomplex $\#$ symmetry.

### 4.3 Rational bicomplex functions and the bicomplex Wiener algebra

We now re-write Theorems 2.23 and 2.24 in the bicomplex case. The proofs are left to the reader.

**Theorem 4.12** Let $f(z)$ be a $\mathbb{C}^{2p \times 2p}$-valued rational function, analytic at infinity, bicomplex $\#$-symmetric, and regular on the unit circle. Then $f$ admits a bicomplex $\#$-symmetric realization, analytic at infinity, and regular on the unit circle.

**Proof** Follows from proof of Theorem 2.23.

**Theorem 4.13** Let $f \in \mathcal{W}^{2p \times 2p}$ rational and taking strictly positive values and bicomplex $\#$ symmetric. Then it admits rational bicomplex $\#$-symmetric left and right factorizations.

**Proof** This follows from the fact that the a bicomplex $\#$-symmetric rational function admits a bicomplex $\#$-symmetric realization.
**Remark 4.14** Using this isomorphism between the bicomplex Wiener algebra and the complex Wiener algebra of double dimension with specific symmetries, our proof can avoid the issue of singularities which are not isolated points, just as in the case of realization of rational functions of several complex variables. We point the reader to [23] for a thorough discussion of Laurent expansions of bicomplex functions where singularities are not isolated.

### 5 Bicomplex superoscillations

In this Section we apply the previous matrix decomposition of the bicomplex Wiener algebra to write a result on bicomplex superoscillations. We start with the distinguished boundary $|\lambda_1| = |\lambda_2| = 1$ of the bicomplex bidisk. We use Lemma 1.13, and formulae (1.8) and (1.10), to show that the decomposition along the idempotents corresponds to

$$\left\{ e^{it}(\cos s, \sin s); t, s \in [0, 2\pi] \right\},$$

i.e.

$$Z = e^{it} e^{js}.$$ (5.1)

We have for $z_1 = u z_2$ when $z_2 \neq 0$:

$$\partial K = \left\{ (e^{i\theta}, 0), \theta \in [0, 2\pi] \right\} \bigcup \left\{ \left. \left( \frac{ue^{i\theta}}{\sqrt{u^2+1}}, \frac{e^{i\theta}}{\sqrt{u^2+1}} \right) \right| \theta \in [0, 2\pi] \right\},$$

which is rewritten as (1.8). To conclude, note that

$$z_1 + jz_2 = e^{it}(\cos s + j \sin s) = e^{it} e^{js},$$

and that from Lemma 1.15 we have

$$\lambda_1 = e^{i(t-s)} \quad \text{and} \quad \lambda_2 = e^{i(t+s)}$$ (5.2)

and

$$z_1 + jz_2 = e^{i(t-s)} e_1 + e^{i(t+s)} e_2.$$ 

Following Definition 2.28 in the classical case, we now write a superoscillatory sequence in the bicomplex case. For given $x, y, a, b \in \mathbb{R}$ and $m \geq 0$ we take specific bicomplex numbers $Z = \lambda_1 e_1 + \lambda_2 e_2$, where

$$\lambda_1 = \cos\left(\frac{x}{m}\right) + i a \sin\left(\frac{x}{m}\right)$$
and

\[ \lambda_2 = \cos \left( \frac{y}{m} \right) + i b \sin \left( \frac{y}{m} \right). \]

**Definition 5.1** With the notation above, we have the following bicomplex superoscillatory sequence

\[ F_m(x, y, a, b) = (\lambda_1 e_1 + \lambda_2 e_2)^m = \lambda_1^m e_1 + \lambda_2^m e_2, \]

namely

\[ F_m(x, y, a, b) = \left( \cos \left( \frac{x}{m} \right) + i a \sin \left( \frac{x}{m} \right) \right)^m e_1 + \left( \cos \left( \frac{y}{m} \right) + i b \sin \left( \frac{y}{m} \right) \right)^m e_2. \]

Using Lemma 2.29 in the complex case we have the following structure of bicomplex superoscillations:

**Lemma 5.2** For the bicomplex superoscillatory sequence in Definition 5.1 we have that:

\[ F_m(x, y, a, b) = \sum_{k=0}^{n} c_k(m, a) e^{i x (1 - 2k/m)} e_1 + c_k(m, b) e^{i y (1 - 2k/m)} e_2, \]

where

\[ c_k(m, a) = \binom{m}{k} \left( \frac{1 + a}{2} \right)^{m-k} \left( \frac{1 - a}{2} \right)^k, \quad c_k(m, b) = \binom{m}{k} \left( \frac{1 + b}{2} \right)^{m-k} \left( \frac{1 - b}{2} \right)^k. \]

(5.3)

We also have the same type of limiting behavior:

**Lemma 5.3** We have that:

\[ \lim_{m \to \infty} F_m(x, y, a, b) = e^{iax} e_1 + e^{iby} e_2, \]

and

\[ F_m(x, y, a, b) = F_m(x, a) e_1 + F_m(y, b) e_2. \]

**Theorem 5.4** Let \( f \in W^{p \times p} \), then \( f(Z) = \sum_{n \in \mathbb{Z}} f_n Z^n \), where \( Z = e^{it} e^{js} \) can be uniformly approximated on compact subsets of \( \mathbb{R} \) by a superoscillatory sequence defined by:

\[ g_m(Z) = f_{-1} e^{-it} e^{-js} + f_0 + f_1 e^{it} e^{js} + \sum_{\substack{n \in \mathbb{Z} \backslash \{-1, 0, 1\}}} f_n F_m(t - s, t + s, n, n). \]

**Proof** We have that

\[ f(Z) = \sum_{n \in \mathbb{Z}} f_n (e^{it} e^{js})^n = \sum_{n \in \mathbb{Z}} f_n (e^{i(t-s)} e_1 + e^{i(t+s)} e_2)^n. \]
Therefore
\[ f(Z) = \sum_{n \in \mathbb{Z}} (A_n e_1 + B_n e_2)(e^{i(t-s)} e_1 + e^{i(t+s)} e_2)^n. \]
\[ f(Z) = \sum_{n \in \mathbb{Z}} A_n e^{i(t-s)} e_1 + B_n e^{i(t+s)} e_2 =: f_1(e^{i(t-s)}) e_1 + f_2(e^{i(t+s)}) e_1. \]

Using Theorem 2.30 we have that \( f_1(e^{i(t-s)}) \) is approximated by:
\[ \mathcal{E}_1 = f_1^1 e^{-i(t-s)} + f_0^1 + f_1^1 e^{i(t-s)} + \sum_{n \in \mathbb{Z}, n \notin \{-1,0,1\}} f_n^1 F_m(t - s, n). \]

Similarly, \( f_2(e^{i(t+s)}) \) is approximated by:
\[ \mathcal{E}_2 = f_2^2 e^{-i(t+s)} + f_0^2 + f_1^2 e^{i(t+s)} + \sum_{n \in \mathbb{Z}, n \notin \{-1,0,1\}} f_n^2 F_m(t + s, n). \]

Using the Lie norm induced on the space of bicomplex matrices we can write an approximation for \( f(Z) \) to be \( \mathcal{E}_1 e_1 + \mathcal{E}_2 e_2 \):
\[ f(Z) = (f_1^1 e^{-i(t-s)} + f_0^1 + f_1^1 e^{i(t-s)}) e_1 + (f_2^2 e^{-i(t+s)} + f_0^2 + f_1^2 e^{i(t+s)}) e_2 + \sum_{n \in \mathbb{Z}, n \notin \{-1,0,1\}} f_n^1 F_m(t - s, n) e_1 + \sum_{n \in \mathbb{Z}, n \notin \{-1,0,1\}} f_n^2 F_m(t + s, n) e_2. \]

which yields:
\[ f(Z) = f_1 e^{-it} e^{-js} + f_0 + f_1 e^{it} e^{js} + \sum_{n \in \mathbb{Z}, n \notin \{-1,0,1\}} f_n(F_m(t - s, n) e_1 + F_m(t + s, n) e_2), \]

namely:
\[ f(Z) = f_1 e^{-it} e^{-js} + f_0 + f_1 e^{it} e^{js} + \sum_{n \in \mathbb{Z}, n \notin \{-1,0,1\}} f_n F_m(t - s, t + s, n, n). \]

This completes the proof. \( \square \)

This application concludes the paper and we expect to be able to apply this framework to other settings as well. Wiener Algebras have many applications in the theory of Linear Systems in the classical case, so we expect results in this direction as well.

We are also investigating whether the setting of bicomplex Wiener algebras is useful in other applications of Quantum Mechanics.

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Declarations

Conflict of interest  The authors (Alpay, Lewkowicz, and Vajiac) declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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