Lifespan estimates for local solutions to the semilinear wave equation in Einstein–de Sitter spacetime

Alessandro Palmieri
Department of Mathematics, University of Pisa, Pisa, Italy

ABSTRACT
In this paper, we prove some blow-up results for the semilinear wave equation in generalized Einstein-de Sitter spacetime by using an iteration argument, and we derive upper bound estimates for the lifespan. In particular, we will focus on the critical cases which require the employment of a slicing procedure in the iterative mechanism. Furthermore, in order to deal with the main critical case, we will introduce a non-autonomous and parameter-dependent Cauchy problem for a linear ODE of second-order, whose explicit solution will be determined by applying the theory of special functions.

1. Introduction
In the last four decades, the proof of the Strauss conjecture concerning the critical exponent of the initial value problem for the semilinear wave equation with power nonlinearity required the effort of many mathematicians worldwide. Nowadays, we know that the critical exponent for the Cauchy problem

\[
\begin{cases}
    v_{tt} - \Delta v = |v|^p & x \in \mathbb{R}^n, \ t > 0, \\
    v(0,x) = \varepsilon v_0(x) & x \in \mathbb{R}^n, \\
    v_t(0,x) = \varepsilon v_1(x) & x \in \mathbb{R}^n,
\end{cases}
\]

is the so-called Strauss exponent \( p_{Str}(n) \) (cf. \([1\text{–}11]\)), that is, the positive root of the quadratic equation

\[(n - 1)p^2 - (n + 1)p - 2 = 0.\]

We are also interested not only in the critical exponent but also in lifespan estimates (the lifespan is the maximal existence time of the solution) when the global in time existence cannot be expected. See the introduction of \([12]\) for the complete picture of the lifespan estimates for the classical semilinear wave equation with power nonlinearity.

While the situation is completely understood in the Euclidean case with flat metric on \(\mathbb{R}^n\), in the last years several papers have been devoted to study the semilinear wave equation in the spacetime \(\mathbb{R}^{1+n}_+\) equipped with different Lorentzian metrics. The semilinear wave equation in Schwarzschild has
been investigated in [13–16] in the 1 + 3-dimensional case. Moreover, the wave (or Klein–Gordon) equation in de Sitter and anti-de Sitter spacetimes have been investigated in the linear and semilinear case in [17–22] and [23–26], respectively. Finally, the wave equation in Einstein–de Sitter (EdeS) spacetime has been considered in [27–29]. In this paper, we shall examine the semilinear wave equation with power nonlinearity in a generalized EdeS spacetime. More precisely, let us consider the semilinear equation with singular coefficients

$$
\phi_{tt} - t^{-2k} \Delta \phi + 2t^{-1} \phi_t = |\phi|^p,
$$

(1)

where \( k \in [0, 1) \) and \( p > 1 \). We call this model the semilinear wave equation in a generalized EdeS spacetime since for \( k = 2/3 \) and \( n = 3 \) Equation (1) is the semilinear wave equation in EdeS spacetime with power nonlinearity.

In [29, Theorem 1.3], the authors proved that for

$$
1 < p < \max\{p_0(n, k), p_1(n, k)\}
$$

a local in time solution to the corresponding Cauchy problem (with initial data prescribed at the initial time \( t = 1 \)) blows up in finite time, provided that the initial data fulfil certain integral sign conditions. Here, \( p_0(n, k) \) is the positive root of the quadratic equation

$$
\left( \frac{n-1}{2} + \frac{2-k}{2(1-k)} \right) p^2 - \left( \frac{n+1}{2} + \frac{2+3k}{2(1-k)} \right) p - 1 = 0.
$$

(2)

while

$$
p_1(n, k) = 1 + \frac{2}{(1-k)n}.
$$

(3)

Furthermore, in [29], it is also shown that, for the semilinear wave equation in EdeS spacetime, the blow-up is the effect of the semilinear term. For this reason, we shall focus our analysis on the effect of the nonlinear term, prescribing the Cauchy data at the initial time \( t = 1 \). All results that we are going to prove in the next sections are still true if the Cauchy data are prescribed at the initial time \( t = t_0 > 0 \) (the choice \( t_0 = 1 \) is done for the sake of simplicity, but it has no special meaning).

Performing the transformation \( u = t \phi \), (1) becomes equivalent to the following semilinear equation for \( u \)

$$
\phi_{tt} - t^{-2k} \Delta \phi = t^{1-p} |u|^p.
$$

(4)

In this paper, we investigate the blow-up dynamic for (4) and, in particular, we will focus on the upper bound estimates for the lifespan and on the treatment of the critical case \( p = \max\{p_0(n, k), p_1(n, k)\} \). More precisely, in the next sections, we are going to provide a complete picture of the upper bound estimates for the lifespan of local in-time solutions to (4) when \( 1 < p \leq \max\{p_0(n, k), p_1(n, k)\} \).

In the subcritical case, we employ a Kato-type lemma on the blow-up dynamic for a second-order ordinary differential inequality (ODI). On the other hand, in the critical case, an iteration argument combined with a slicing procedure is applied. More particularly, for \( p = p_0(n, k) \), we adapt the approach from [30,31] to the time-dependent semilinear model (4).

After the completion of the final version of this work, we found out the existence of the paper [32], where a more general model is considered. In Section 5, we explain detailedly the differences between our results and the ones in [32].

1.1. Notations

Throughout the paper, we will employ the following notations: \( \phi_k(t) = \frac{t^{1-k}}{1-k} \) denotes a distance function produced by the speed of propagation \( a_k(t) = t^{-k} \), while the amplitude of the light cone is given
by the function

\[ A_k(t) \doteq \int_1^t \tau^{-k} \, d\tau = \phi_k(t) - \phi_k(1); \]  

(5)

the ball with radius \( R \) around the origin is denoted \( \mathcal{B}_R \); \( f \lesssim g \) means that there exists a positive constant \( C \) such that \( f \leq C g \) and, similarly, for \( f \gtrsim g \); \( I_\nu \) and \( K_\nu \) denote the modified Bessel functions of first and second kind of order \( \nu \), respectively; finally,

\[ N(k) = \frac{1 - 2k + \sqrt{4k^2 - 4k + 9}}{2(1 - k)} \]  

(6)

denotes the threshold for the spatial dimension in determining the dominant exponent between \( p_0(n,k) \) and \( p_1(n,k) \) (more specifically, \( p_0(n,k) > p_1(n,k) \) if and only if \( n > N(k) \), while \( p_0(n,k) \leq p_1(n,k) \) for \( n \leq N(k) \)).

### 1.2. Main results

The main results of this work are the following blow-up results that combined together provide a full picture of the critical case \( p = \max \{ p_0(n,k), p_1(n,k) \} \) for the Cauchy problem

\[
\begin{cases}
    u_{tt} - t^{-2k} \Delta u = t^{1-p} |u|^p & x \in \mathbb{R}^n, \ t \in (1, T), \\
    u(1, x) = \varepsilon u_0(x) & x \in \mathbb{R}^n, \\
    u_t(1, x) = \varepsilon u_1(x) & x \in \mathbb{R}^n,
\end{cases}
\]  

(7)

where \( p > 1, \varepsilon > 0 \) is a parameter describing the size of initial data and \( k \in [0, 1) \).

Before stating the main results, let us introduce the notion of energy solution to the semilinear Cauchy problem (7).

**Definition 1.1:** Let \( u_0 \in H^1(\mathbb{R}^n) \) and \( u_1 \in L^2(\mathbb{R}^n) \). We say that

\[ u \in \mathcal{C}(\{1, T\}, H^1(\mathbb{R}^n)) \cap \mathcal{C}^1(\{1, T\}, L^2(\mathbb{R}^n)) \cap L^p_{\text{loc}}([1, T] \times \mathbb{R}^n) \]

is an energy solution to (7) on \([1, T] \) if \( u \) fulfils \( u(1, \cdot) = \varepsilon u_0 \) in \( H^1(\mathbb{R}^n) \) and the integral relation

\[
\int_{\mathbb{R}^n} \partial_t u(t, x) \psi(t, x) \, dx - \int_1^t \int_{\mathbb{R}^n} (\partial_t u(s, x) \psi_\delta(s, x) - s^{-2k} \nabla u(s, x) \cdot \nabla \psi(s, x)) \, dx \, ds \\
= \varepsilon \int_{\mathbb{R}^n} u_1(x) \psi(1, x) \, dx + \int_1^t \int_{\mathbb{R}^n} s^{1-p} |u(s, x)|^p \psi(s, x) \, dx \, ds
\]  

(8)

for any \( \psi \in \mathcal{C}_0^\infty([1, T] \times \mathbb{R}^n) \) and any \( t \in (1, T) \).

We point out that performing a further step of integration by parts in (8), we find the integral relation

\[
\int_{\mathbb{R}^n} (\partial_t u(t, x) \psi(t, x) - u(t, x) \psi_\delta(t, x)) \, dx - \varepsilon \int_{\mathbb{R}^n} (u_1(x) \psi(1, x) - u_0(x) \psi_\delta(1, x)) \, dx \\
+ \int_1^t \int_{\mathbb{R}^n} u(s, x) (\psi_{\delta s}(s, x) - s^{-2k} \Delta \psi(s, x)) \, dx \, ds = \int_1^t \int_{\mathbb{R}^n} s^{1-p} |u(s, x)|^p \psi(s, x) \, dx \, ds
\]  

(9)

for any \( \psi \in \mathcal{C}_0^\infty([1, T] \times \mathbb{R}^n) \) and any \( t \in (1, T) \).
Remark 1.1: Let us stress that if the Cauchy data are compactly supported, say $\text{supp} \ u_j \subset B_R$ for $j = 0, 1$ and for some $R > 0$, then, for any $t \in (1, T)$, a local solution $u$ to (7) satisfies the support condition

$$\text{supp} \ u(t, \cdot) \subset B_{R+A_k(t)},$$

where $A_k$ is defined by (5). Therefore, in Definition 1.1, we may also consider test functions which are not compactly supported, namely $\psi \in C^\infty (\mathbb{R}^n)$. 

Theorem 1.2: Let $n \in \mathbb{N}^+$ such that $n > N(k)$ and $p = p_0(n, k)$. Let us assume that $u_0 \in H^1 (\mathbb{R}^n)$ and $u_1 \in L^2 (\mathbb{R}^n)$ are non-negative, non-trivial and compactly supported functions with supports contained in $B_R$ for some $R > 0$. Let

$$u \in \mathcal{C} ([1, T), H^1 (\mathbb{R}^n)) \cap \mathcal{C}^1 ([1, T), L^2 (\mathbb{R}^n)) \cap L^p_{\text{loc}} ([1, T) \times \mathbb{R}^n)$$

be an energy solution to (7) according to Definition 1.1 with lifespan $T = T(\varepsilon)$ and satisfying the support condition $\text{supp} \ u(t, \cdot) \subset B_{A_k(t)+R}$ for any $t \in (1, T)$.

Then, there exists a positive constant $\varepsilon_0 = \varepsilon_0(u_0, u_1, n, p, k, R)$ such that for any $\varepsilon \in (0, \varepsilon_0]$ the energy solution $u$ blows up in finite time. Moreover, the upper bound estimate for the lifespan

$$T(\varepsilon) \leq \exp (C_0 \varepsilon)$$

holds, where the constant $C_0 > 0$ is independent of $\varepsilon$.

Theorem 1.3: Let $n \in \mathbb{N}^+$ such that $n \leq N(k)$ and $p = p_1(n, k)$. Let us assume that $u_0 \in H^1 (\mathbb{R}^n)$ and $u_1 \in L^2 (\mathbb{R}^n)$ are non-negative, non-trivial and compactly supported functions with supports contained in $B_R$ for some $R > 0$. Let

$$u \in \mathcal{C} ([1, T), H^1 (\mathbb{R}^n)) \cap \mathcal{C}^1 ([1, T), L^2 (\mathbb{R}^n)) \cap L^p_{\text{loc}} ([1, T) \times \mathbb{R}^n)$$

be an energy solution to (7) according to Definition 1.1 with lifespan $T = T(\varepsilon)$ and satisfying the support condition $\text{supp} \ u(t, \cdot) \subset B_{A_k(t)+R}$ for any $t \in (1, T)$.

Then, there exists a positive constant $\varepsilon_0 = \varepsilon_0(u_0, u_1, n, p, k, R)$ such that for any $\varepsilon \in (0, \varepsilon_0]$ the energy solution $u$ blows up in finite time. Moreover, the upper bound estimate for the lifespan

$$T(\varepsilon) \leq \exp (C_0 \varepsilon^{-p})$$

holds, where the constant $C_0 > 0$ is independent of $\varepsilon$.

The remaining part of the paper is organized as follows: in Section 2, we prove Theorem 1.2 by using the approach introduced in [30]; then, in Section 3, we provide a complete overview on upper bound estimates for the subcritical case (cf. Proposition 3.2), while, in Section 4, we show the proof of Theorem 1.3; finally, in Appendix, we provide a different proof of Proposition 2.1 in the special case of EdeS spacetime.

2. Semilinear wave equation in EdeS spacetime: first critical case

Our goal is to prove a blow-up result in the critical case $p = p_0(n, k)$, where $p_0(n, k)$ is the greatest root of the quadratic Equation (2).

The approach that we will follow is based on the technique introduced in [30] and subsequently applied to different wave models (cf. [31,33–37]).

We are going to introduce a time-dependent functional that depends on a local in time solution to (7) and to study its blow-up dynamic. In particular, the blow-up result will be obtained by applying
the so-called *slicing procedure* in an iteration argument to show a sequence of lower bound estimates for the above-mentioned functional.

The section is organized as follows: in Section 2.1, we determine a pair of auxiliary functions which have a fundamental role in the definition of the time-dependent functional and in the determination of the iteration frame, while, in Section 2.2, we establish some fundamental estimates for these functions; then, in Section 2.3, we establish the iteration frame for the functional and, finally, in Section 2.4, we prove the blow-up result by using an iteration procedure.

### 2.1. Auxiliary functions

In this section, we are going to introduce two auxiliary functions (see $\xi_q$ and $\eta_q$ below) analogously to the corresponding functions introduced in [30], which represent, in turn, a generalization of the solution to the classical free wave equation given in [11]. Those auxiliary functions are defined by using the remarkable function $\phi(x)$ introduced in [9]. Let us recall briefly the main properties of this function: $\phi$ is a positive and smooth function that satisfies $\Delta \phi = \phi$ and asymptotically behaves like $|x|^{-\frac{n-1}{2}} e^{|x|}$ as $|x| \to \infty$ up to a positive multiplicative constant.

In order to introduce the definition of the auxiliary functions, let us begin by determining the solutions $y_j = y_j(t; s, \lambda, k)$, $j \in \{0, 1\}$, of the non-autonomous, parameter-dependent, ordinary Cauchy problems

\[
\begin{aligned}
&\frac{\partial^2 y_j}{\partial t^2}(t; s, \lambda, k) - \lambda^2 t^{-2k} y_j(t; s, \lambda, k) = 0, \quad t > s, \\
y_j(s; s, \lambda, k) = \delta_{0j}, \\
\frac{\partial y_j}{\partial t}(s; s, \lambda, k) = \delta_{1j},
\end{aligned}
\tag{11}
\]

where $\delta_{ij}$ denotes the Kronecker delta, $s \geq 1$ is the initial time and $\lambda > 0$ is a real parameter.

Let us recall that we denote by $\phi_k(t) = \frac{1}{t^{1-k}}$ a primitive of the speed of propagation $a(t) = t^{-k}$ for the wave equation in (7). In order to find a system of independent solutions to

\[
\frac{d^2 y}{dt^2} - \lambda^2 t^{-2k} y = 0,
\tag{12}
\]

we perform first a change of variables. Let $\tau = \tau(t; \lambda, k) = \lambda \phi_k(t)$. Since

\[
\begin{aligned}
\frac{dy}{dt} &= \lambda t^{-k} \frac{dy}{d\tau}, \\
\frac{d^2 y}{dt^2} &= \lambda^2 t^{-2k} \frac{d^2 y}{d\tau^2} - \lambda k t^{-k-1} \frac{dy}{d\tau},
\end{aligned}
\]

then, $y$ solves (12) if and only if it solves

\[
\tau \frac{d^2 y}{d\tau^2} - \frac{k}{1-k} \frac{dy}{d\tau} - \tau y = 0.
\tag{13}
\]

Next, we carry out the transformation $y(\tau) = \tau^\nu w(\tau)$ with

\[
\nu = \frac{1}{2(1-k)}.
\]
Proof: We have seen that

\[ \frac{\tau^2}{d\tau^2} w + \frac{\tau}{d\tau} w - (\nu^2 + \tau^2) w = 0, \]  

(14)

where we applied the straightforward relations

\[ \frac{dy}{d\tau} = \nu \tau^{\nu-1} w(\tau) + \tau^\nu \frac{dw}{d\tau}, \quad \frac{d^2y}{d\tau^2} = \nu(\nu - 1) \tau^{\nu-2} w + 2\nu \tau^{\nu-1} \frac{dw}{d\tau} + \tau^\nu \frac{d^2w}{d\tau^2}. \]

If we employ independent solutions to (14) the modified Bessel function of first and second kind of order \( \nu \), denoted, respectively, by \( I_\nu(\tau) \) and \( K_\nu(\tau) \), then, the pair of functions

\[ V_0(t; \lambda, k) = \tau^\nu I_\nu(\lambda \phi_k(t)), \]
\[ V_1(t; \lambda, k) = \tau^\nu K_\nu(\lambda \phi_k(t)) \]

is a basis of the space of solutions to (12).

**Proposition 2.1:** The functions

\[ y_0(t, s; \lambda, k) \equiv \lambda(t/s)^{1/2} \phi_k(s) [I_{\nu-1}(\lambda \phi_k(s)) K_\nu(\lambda \phi_k(t)) + K_{\nu-1}(\lambda \phi_k(s)) I_\nu(\lambda \phi_k(t))], \]
\[ y_1(t, s; \lambda, k) \equiv (1 - k)^{-1/2} [K_\nu(\lambda \phi_k(s)) I_\nu(\lambda \phi_k(t)) - I_\nu(\lambda \phi_k(s)) K_\nu(\lambda \phi_k(t))], \]

(15)

(16)
solve the Cauchy problems (11) for \( j = 0 \) and \( j = 1 \), respectively, where \( I_\nu, K_\nu \) denote the modified Bessel function of order \( \nu \) of the first and second kind, respectively.

**Proof:** We have seen that \( V_0, V_1 \) form a system of independent solutions to (12). Therefore, we may express the solutions of (11) as linear combinations of \( V_0, V_1 \) as follows:

\[ y_j(t, s; \lambda, k) = a_j(s; \lambda, k) V_0(t; \lambda, k) + b_j(s; \lambda, k) V_1(t; \lambda, k) \]  

(17)

for suitable coefficients \( a_j(s; \lambda, k), b_j(s; \lambda, k), j \in \{0, 1\} \). Using the initial conditions \( \partial_t y_j(s; \lambda, k) = \delta_{ij} \), we find the system

\[
\begin{pmatrix}
V_0(s; \lambda, k) & V_1(s; \lambda, k) \\
\partial_t V_0(s; \lambda, k) & \partial_t V_1(s; \lambda, k)
\end{pmatrix}
\begin{pmatrix}
a_0(s; \lambda, k) & a_1(s; \lambda, k) \\
b_0(s; \lambda, k) & b_1(s; \lambda, k)
\end{pmatrix}
= I,
\]

where \( I \) denotes the identity matrix. So, in order to determine the coefficients in (17), we have to calculate explicitly the inverse matrix

\[
\begin{pmatrix}
V_0(s; \lambda, k) & V_1(s; \lambda, k) \\
\partial_t V_0(s; \lambda, k) & \partial_t V_1(s; \lambda, k)
\end{pmatrix}^{-1} = (\psi(V_0, V_1)(s; \lambda, k))^{-1}
\begin{pmatrix}
\partial_t V_1(s; \lambda, k) & -V_1(s; \lambda, k) \\
-\partial_t V_0(s; \lambda, k) & V_0(s; \lambda, k)
\end{pmatrix},
\]

(18)

where \( \psi(V_0, V_1) \) is the Wronskian of \( V_0, V_1 \). Clearly, we need to express in a more suitable way \( \psi(V_0, V_1) \). Let us calculate the \( t \)-derivative of \( V_0, V_1 \). Recalling that \( \phi_k(t) = t^{1-k}/(1-k) \) and
\[ \nu = 1/(2(1 - k)), \] results
\[ \partial_t V_0(t; \lambda, k) = \nu(\lambda \phi_k(t))^{\nu-1} \lambda \phi_k'(t) I_\nu(\lambda \phi_k(t)) + (\lambda \phi_k(t))^\nu I'_\nu(\lambda \phi_k(t)) \nu \phi_k(t) \]
\[ = \frac{1}{2t} (\lambda \phi_k(t))^\nu I_\nu(\lambda \phi_k(t)) + (\lambda \phi_k(t))^\nu (\lambda \phi_k'(t)) I'_\nu(\lambda \phi_k(t)) \]
and, analogously,
\[ \partial_t V_1(t; \lambda, k) = \frac{1}{2t} (\lambda \phi_k(t))^\nu K_\nu(\lambda \phi_k(t)) + (\lambda \phi_k(t))^\nu (\lambda \phi_k'(t)) K'_\nu(\lambda \phi_k(t)) \]
Consequently, we can express \( \mathcal{W}(V_0, V_1) \) as follows:
\[ \mathcal{W}(V_0, V_1)(t; \lambda, k) = (\lambda \phi_k(t))^{2\nu} (\lambda \phi_k'(t)) [K'_\nu(\lambda \phi_k(t)) I_\nu(\lambda \phi_k(t)) - I'_\nu(\lambda \phi_k(t)) K_\nu(\lambda \phi_k(t))] \]
\[ = (\lambda \phi_k(t))^{2\nu} (\lambda \phi_k'(t)) \mathcal{W}(I_\nu, K_\nu)(\lambda \phi_k(t)) = -(\lambda \phi_k(t))^{2\nu-1} (\lambda \phi_k'(t)) \]
\[ = -\lambda^{2\nu} (\phi_k(t))^{2\nu-1} \phi_k'(t) = -c_k^{-1} \nu, \]
where \( c_k \equiv (1 - k)^{k/(1 - k)} \) and in the third equality we used the value of the Wronskian of \( I_\nu, K_\nu \)
\[ \mathcal{W}(I_\nu, K_\nu)(z) = I_\nu(z) \partial_z K_\nu(z) - K_\nu(z) I_\nu(z) = -\frac{1}{z}. \]
Let us underline that \( \mathcal{W}(V_0, V_1)(t; \lambda, k) \) does not actually depend on \( t \), due to the absence of the first-order term in (12).
Plugging the previous representation of \( \mathcal{W}(V_0, V_1) \) in (18), we obtain
\[ \left( \begin{array}{cc} a_0(s; \lambda, k) & a_1(s; \lambda, k) \\ b_0(s; \lambda, k) & b_1(s; \lambda, k) \end{array} \right) = -c_k \lambda^{-2\nu} \left( \begin{array}{cc} \partial_t V_1(s; \lambda, k) & -V_1(s; \lambda, k) \\ -\partial_t V_0(s; \lambda, k) & V_0(s; \lambda, k) \end{array} \right). \]
Let us begin by proving (15). Using the above representation of \( a_0(s; \lambda, k), b_0(s; \lambda, k) \) in (17), we obtain
\[ y_0(t, s; \lambda, k) \]
\[ = c_k \lambda^{-2\nu} \{ \partial_t V_0(s; \lambda, k) V_1(t; \lambda, k) - \partial_t V_1(s; \lambda, k) V_0(t; \lambda, k) \} \]
\[ = c_k \lambda^{-2\nu} (\lambda \phi_k(s))^{\nu} (\lambda \phi_k(t))^{\nu} \left\{ \frac{1}{2s} I_\nu(\lambda \phi_k(s)) + (\lambda \phi_k'(s)) I'_\nu(\lambda \phi_k(s)) \right\} K_\nu(\lambda \phi_k(t)) \]
\[ - \left\{ \frac{1}{2s} K_\nu(\lambda \phi_k(s)) + (\lambda \phi_k'(s)) K'_\nu(\lambda \phi_k(s)) \right\} I_\nu(\lambda \phi_k(t)) \]
\[ = c_k (\lambda \phi_k(s) \phi_k(t))^{\nu} (2s)^{-1} \left\{ I_\nu(\lambda \phi_k(s) K_\nu(\lambda \phi_k(t)) - K_\nu(\lambda \phi_k(s)) I_\nu(\lambda \phi_k(t)) \right\} \]
\[ + c_k \lambda (\phi_k(s) \phi_k(t))^{\nu} \phi_k'(s) \left\{ I'_\nu(\lambda \phi_k(s) K_\nu(\lambda \phi_k(t)) - K'_\nu(\lambda \phi_k(s)) I_\nu(\lambda \phi_k(t)) \right\}. \]
Applying the recursive relations for the derivatives of the modified Bessel functions
\[ \frac{\partial I_\nu(z)}{\partial z} = -\frac{\nu}{z} I_\nu(z) + I_{\nu-1}(z), \]
\[ \frac{\partial K_\nu(z)}{\partial z} = -\frac{\nu}{z} K_\nu(z) - K_{\nu-1}(z), \]
to the last relation, we arrive at
wecanderivethefollowingrepresentations:

Lemma 2.2: Let \( y \) be the functions defined in (15) and (16), respectively. Then, the following identities are satisfied for any \( t \geq s \geq 1 \)

\[
\frac{\partial y_1}{\partial s}(t, s; \lambda, k) = -y_0(t, s; \lambda, k),
\]

Since \( c_k(\phi_k(s)\phi_k(t)) = (1 - k)^{-1}(st)^{1/2}φ_k(s) \), (19) yields immediately (15). Let us prove now the representation (16). Plugging the above determined expressions for \( a_1(s; λ, k) \), \( b_1(s; λ, k) \) in (17), we have

\[
y_1(t, s; \lambda, k) = c_k(r^{-2i}(V_1(s; λ, k) V_0(t; λ, k) - V_0(s; λ, k) V_1(t; λ, k))
\]

\[
= c_k(r^{-2i}(λφ_k(s))) [K_v(λφ_k(s)) I_v(λφ_k(t)) - I_v(λφ_k(s)) K_v(λφ_k(t))]
\]

\[
= c_k(λφ_k(s)φ_k(t))^v [K_v(λφ_k(s)) I_v(λφ_k(t)) - I_v(λφ_k(s)) K_v(λφ_k(t))].
\]

Thus, using \( c_k(φ_k(s)φ_k(t))^v = (st)^{1/2}(1 - k) \), from (20) follows (16). This concludes the proof.

Remark 2.1: In the special case \( k = 2/3 \), \( y_0(t, s; λ, k) \) and \( y_1(t, s; λ, k) \) can be expressed in terms of elementary functions. Indeed, by using the explicit representations

\[
\begin{align*}
I_{1/2}(z) &= \sqrt{\frac{2}{\pi}} \frac{\sinh z}{z^{1/2}}, & I_{3/2}(z) &= \sqrt{\frac{2}{\pi}} \frac{\cosh z - \sinh z}{z^{3/2}}, \\
K_{1/2}(z) &= \sqrt{\frac{\pi}{2}} \frac{e^{-z}}{z^{1/2}}, & K_{3/2}(z) &= \sqrt{\frac{\pi}{2}} \frac{e^{-(z + 1)}}{z^{3/2}},
\end{align*}
\]

we can derive the following representations:

\[
y_0(t, s; \lambda, 2/3) = \left(\frac{t}{s}\right)^{1/3} \cosh(3λ(t^{1/3} - s^{1/3})) - \frac{1}{3λs^{1/3}} \sinh(3λ(t^{1/3} - s^{1/3})),
\]

\[
y_1(t, s; \lambda, 2/3) = \left[\frac{(st)^{1/3}}{λ} - \frac{1}{9λ^2}\right] \cosh(3λ(t^{1/3} - s^{1/3}))
\]

\[
+ \frac{1}{3λ^2}(t^{1/3} - s^{1/3}) \sinh(3λ(t^{1/3} - s^{1/3})).
\]

Actually, in this case, it is possible to derive the representations of \( y_0(t, s; λ, 2/3) \) and \( y_1(t, s; λ, 2/3) \) by reducing (12) to a confluent hypergeometric equation instead of a modified Bessel equation. For detailed proof, see Appendix.

We stress that we are particularly interested to the case \( k = 2/3 \), since this case has a physical relevance.

Indeed, as we have already pointed out in the introduction, the operator \( \partial_t^2 - t^{-4} Δ + 2t^{-1} ∂_t \) represents the d’Alembertian in EdeS Lorentzian metric.

Lemma 2.2: Let \( y_0, y_1 \) be the functions defined in (15) and (16), respectively. Then, the following identities are satisfied for any \( t \geq s \geq 1 \)

\[
\frac{\partial y_1}{\partial s}(t, s; \lambda, k) = -y_0(t, s; \lambda, k),
\]
\[
\frac{\partial^2 y_1}{\partial s^2}(t, s; \lambda, k) - \lambda^2 s^{-2k} y_1(t, s; \lambda, k) = 0.
\] (24)

**Remark 2.2:** As the operator \((\partial^2_t - \lambda^2 t^{-2k})\) is formally self-adjoint, in particular, (23) and (24) tell us that \(y_1\) solves also the adjoint problem to (12) with final conditions \((0, -1)\).

**Proof:** Let us introduce the pair of independent solutions to (12)

\[
\begin{align*}
z_0(t; \lambda, k) &= y_0(t, 1; \lambda, k), \\
z_1(t; \lambda, k) &= y_1(t, 1; \lambda, k).
\end{align*}
\]

By standard computations, we may show the representations

\[
\begin{align*}
y_0(t, s; \lambda, k) &= z'_1(s; \lambda, k)z_0(t, \lambda, k) - z'_0(s; \lambda, k)z_1(t, \lambda, k), \\
y_1(t, s; \lambda, k) &= z_0(s; \lambda, k)z_1(t, \lambda, k) - z_1(s; \lambda, k)z_0(t, \lambda, k).
\end{align*}
\]

In particular, we used that the Wronskian of \(z_0, z_1\) is identically 1. First, we prove (23). Differentiating the second one of the previous representations with respect to \(s\) and then using the first one, we get immediately

\[
\frac{\partial y_1}{\partial s}(t, s; \lambda, k) = z'_0(s; \lambda, k)z_1(t, \lambda, k) - z'_1(s; \lambda, k)z_0(t, \lambda, k) = -y_0(t, s; \lambda, k).
\]

Since \(z_0, z_1\) are solutions of (12), then,

\[
\begin{align*}
\frac{\partial^2 y_1}{\partial s^2}(t, s; \lambda, k) &= z''_0(s; \lambda, k)z_1(t, \lambda, k) - z''_1(s; \lambda, k)z_0(t, \lambda, k) \\
&= \lambda^2 s^{-2k}z_0(s; \lambda, k)z_1(t, \lambda, k) - \lambda^2 s^{-2k}z_1(s; \lambda, k)z_0(t, \lambda, k) \\
&= \lambda^2 s^{-2k}y_1(t, s; \lambda, k).
\end{align*}
\]

So, we prove (24) as well.

**Proposition 2.3:** Let \(u_0 \in H^1(\mathbb{R}^n)\) and \(u_1 \in L^2(\mathbb{R}^n)\) be functions such that \(\text{supp } u_j \subset B_R\) for \(j = 0, 1\) and for some \(R > 0\) and let \(\lambda > 0\) be a parameter. Let \(u\) be a local in time energy solution to (7) on \([1, T)\) according to Definition 1.1. Then, the following integral identity is satisfied for any \(t \in [1, T)\)

\[
\int_{\mathbb{R}^n} u(t, x)\varphi_\lambda(x) \, dx = \varepsilon y_0(t, 1; \lambda, k) \int_{\mathbb{R}^n} u_0(x)\varphi_\lambda(x) \, dx + \varepsilon y_1(t, 1; \lambda, k) \int_{\mathbb{R}^n} u_1(x)\varphi_\lambda(x) \, dx \\
+ \int_1^t \int_{\mathbb{R}^n} s^{1-p} y_1(t, s; \lambda, k) |u(s, x)|^p \varphi_\lambda(x) \, dx \, ds,
\] (25)

where \(\varphi_\lambda(x) \doteq \varphi(\lambda x)\) and \(\varphi\) is defined by (10).

**Proof:** Since we assumed \(u_0, u_1\) compactly supported, we may consider a test function \(\psi \in C^\infty_0((1, T) \times \mathbb{R}^n)\) in Definition 1.1 according to Remark 1.1.

Therefore, we consider \(\psi(t, x) = y_1(t, s; \lambda, k)\varphi_\lambda(x)\) (here, \(t, \lambda\) can be considered fixed parameters). Hence, \(\psi\) satisfies

\[
\psi(t, x) = y_1(t, t; \lambda, k)\varphi_\lambda(x) = 0,
\]
\[
\psi(1, x) = y_1(t, 1; \lambda, k)\varphi_\lambda(x),
\]
By contradiction, one can prove easily that
\[
\psi_s(t, x) = \partial_y y_1(t, \lambda, k) \varphi_\lambda(x) = -y_0(t, \lambda, k) \varphi_\lambda(x) = -\varphi_\lambda(x),
\]
\[
\psi_s(1, x) = \partial_y y_1(t, 1, \lambda, k) \varphi_\lambda(x) = -y_0(t, 1, \lambda, k) \varphi_\lambda(x)
\]
and
\[
\psi_{ss}(s, x) - s^{-2k} \Delta \psi(s, x) = \left( \partial_x^2 y_1(t, s, \lambda, k) - \lambda^2 s^{-2k} y_1(t, s, \lambda, k) \right) \varphi_\lambda(x) = 0,
\]
where we used (23), (24) and the property \( \Delta \varphi = \varphi \).

Hence, employing this \( \psi \) in (9), we find immediately (25).

**Proposition 2.4**: Let \( y_0, y_1 \) be the functions defined in (15) and (16), respectively. Then, the following estimates are satisfied for any \( t \geq s \geq 1 \)
\[
y_0(t, s; \lambda, k) \geq \cosh(\lambda(\phi(t) - \phi(s))),
\]
\[
y_1(t, s; \lambda, k) \geq (st)^{\frac{k}{2}} \sinh(\lambda(\phi(t) - \phi(s))) \frac{1}{\lambda}.
\]

**Proof**: The proof of the inequalities (26) and (27) is based on the following minimum type principle:
let \( w = w(t, s; \lambda, k) \) be a solution of the Cauchy problem
\[
\left\{
\begin{array}{ll}
\partial_t^2 w - \lambda^2 t^{-2k}w = h, & \text{for } t > s \geq 1, \\
w(s) = w_0, & \partial_t w(s) = w_1,
\end{array}
\right.
\]
where \( h = h(t, s; \lambda, k) \) is a continuous function; if \( h \geq 0 \) and \( w_0 = w_1 = 0 \) (i.e. \( w \) is a supersolution of the homogeneous problem with trivial initial conditions), then, \( w(t, s; \lambda, k) \geq 0 \) for any \( t > s \).

In order to prove this minimum principle, we apply the continuous dependence on initial data, letting \( \epsilon \to 0 \), we find that \( w(t, s; \lambda, k) \geq 0 \) for any \( t > s \).

Note that if \( w_0, w_1 \geq 0 \) and \( w_0 + w_1 > 0 \), then, the positivity of \( w \) follows straightforwardly from the corresponding integral equation via a contradiction argument, rather than working with the family \( \{w_\epsilon\}_{\epsilon > 0} \). Nevertheless, in what follows, we consider exactly the limit case \( w_0 = w_1 = 0 \), for this reason, the previous digression was necessary.

Let us prove the validity of (27). We denote by \( w_1 = w_1(t, s; \lambda, k) \) the function on the right-hand side of (27). It is easy to see that \( w_1(s, s; \lambda, k) = 0 \) and \( \partial_t w_1(s, s; \lambda, k) = 1 \). Moreover,
\[
\partial_t^2 w_1(t, s; \lambda, k) \leq \lambda^{-1}(s)^{\frac{k}{2}} \sinh(\lambda(\phi(t) - \phi(s)))(\lambda t^{-k})^2 = \lambda^2 t^{-2k} w_1(t, s; \lambda, k).
\]
Therefore, \( y_1 - w_1 \) is a supersolution of (28) with \( h = 0 \) and \( w_0 = w_1 = 0 \). Thus, applying the minimum principle we have that \( (y_1 - w_1)(t, s; \lambda, k) \geq 0 \) for any \( t > s \), that is, we showed (27).

In a completely analogous way, one can prove (26), repeating the previous argument based on the minimum principle with \( w_0(t, s; \lambda, k) \equiv \cosh(\lambda(\phi(t) - \phi(s))) \) in place of \( w_1(t, s; \lambda, k) \) and \( y_0 \) in place of \( y_1 \), respectively.

After the preliminary results that we have proved so far in this section, we can now introduce the definition of the following auxiliary function:
\[
\xi_q(t, s; x; k) \equiv \int_0^{\lambda_0} e^{-\lambda(A_k(t) + R)} \cosh(\lambda(\phi_k(t) - \phi_k(s))) \varphi_\lambda(x) \lambda^q \, d\lambda, \tag{29}
\]
Combining (25) and the lower bound estimates (26), (27), we find
\begin{equation}
\eta_q(t, s, x; k) \equiv (st)^{k/2} \int_0^{\lambda_0} e^{-\lambda(A_k(t)+R)} \frac{\sinh(\lambda(\phi_k(t) - \phi_k(s)))}{\lambda(\phi_k(t) - \phi_k(s))} \varphi_\lambda(x) \lambda^q d\lambda,
\end{equation}
where \( q > -1, \lambda_0 > 0 \) is a fixed parameter and \( A_k \) is defined by (5).

**Remark 2.3:** For \( k = 0 \), the functions \( \xi_q \) and \( \eta_q \) coincide with the corresponding ones given in [30], provided, of course, that we shift the initial time in the Cauchy problem from 0 to 1.

Combining the results from Propositions 2.3 and 2.4, we may finally derive a fundamental inequality, whose role will be crucial in the next sections in order to prove the blow-up result.

**Corollary 2.5:** Let \( u_0 \in H^1(\mathbb{R}^n) \) and \( u_1 \in L^2(\mathbb{R}^n) \) such that \( \text{supp} \ u_j \subset B_R \) for \( j = 0, 1 \) and for some \( R > 0 \). Let \( u \) be a local in time energy solution to (7) on \([1, T]\) according to Definition 1.1. Let \( q > -1 \) and let \( \xi_q(t, s; x; k), \eta_q(t, s; x; k) \) be the functions defined by (29) and (30), respectively. Then,
\begin{equation}
\int_{\mathbb{R}^n} u(t, x) \xi_q(t, t, x; k) \, dx \geq \varepsilon \int_{\mathbb{R}^n} u_0(x) \xi_q(t, 1, x; k) \, dx \\
+ \varepsilon (\phi_k(t) - \phi_k(1)) \int_{\mathbb{R}^n} u_1(x) \eta_q(t, s, x; k) \, dx \\
+ \int_1^t (\phi_k(t) - \phi_k(s)) s^{1-p} \int_{\mathbb{R}^n} |u(s, x)|^p \eta_q(t, s, x; k) \, dx \, ds
\end{equation}
for any \( t \in [1, T] \).

**Proof:** Combining (25) and the lower bound estimates (26), (27), we find
\begin{equation}
\int_{\mathbb{R}^n} u(t, x) \varphi_\lambda(x) \, dx \geq \varepsilon \cosh(\lambda(\phi_k(t) - \phi_k(1))) \int_{\mathbb{R}^n} u_0(x) \varphi_\lambda(x) \, dx \\
+ \varepsilon t^k \lambda^{-1} \sinh(\lambda(\phi_k(t) - \phi_k(1))) \int_{\mathbb{R}^n} u_1(x) \varphi_\lambda(x) \, dx \\
+ \int_1^t s^{1-p}(st)^{k} \lambda^{-1} \sinh(\lambda(\phi_k(t) - \phi_k(s))) \int_{\mathbb{R}^n} |u(s, x)|^p \varphi_\lambda(x) \, dx \, ds.
\end{equation}
Multiplying both sides of the previous identity by \( e^{-\lambda(A_k(t)+R)} \lambda^q \), integrating with respect to \( \lambda \) over \([0, \lambda_0]\) and applying Fubini’s theorem, we get (31).

**2.2. Properties of the auxiliary functions**

In this section, we determine some lower- and upper-bound estimates for the auxiliary functions \( \xi_q, \eta_q \) under suitable assumptions on \( q \).

Let us begin with the lower bound estimates

**Lemma 2.6:** Let \( n \geq 1 \) and \( \lambda_0 > 0 \). If we assume \( q > -1, \) then, for \( t \geq s \geq 1 \) and \( |x| \leq A_k(s) + R \), the following lower bound estimates hold:
\begin{align}
\xi_q(t, s, x; k) & \geq B_0 (A_k(s))^{-q-1}; \\
\eta_q(t, s, x; k) & \geq B_1 (st)^{k} (A_k(t))^{-1} (A_k(s))^{-q}.
\end{align}
Here, \( B_0, B_1 \) are positive constants depending only on \( \lambda_0, q, R, k \) and we employ the notation \( \langle y \rangle \equiv 3 + |s| \).
Proof: We follow the main ideas of the proof of Lemma 3.1 in [30]. Since
\[ |x|^{-\frac{n+1}{2}} e^{|x|} \lesssim \varphi(x) \lesssim |x|^{-\frac{n+1}{2}} e^{|x|} \quad (34) \]
holds for any \( x \in \mathbb{R}^n \), we can find a constant \( B = B(\lambda_0, R, k) > 0 \) independent of \( \lambda \) and \( s \) such that
\[ B \leq \inf_{\lambda \in [\lambda_0, 2\lambda_0/A_k(s) \cap \lambda_0]} \inf_{|x| \leq A_k(s) + R} e^{-\lambda(A_k(s) + R)} \varphi_\lambda(x). \]
Let us begin with (32). Shrinking the domain of integration in (29) to \( \left[ \frac{\lambda_0}{A_k(s)}, \frac{2\lambda_0}{A_k(s)} \right] \) and applying the previous inequality, we obtain
\[
\xi_q(t, s, x; k) \\
\geq \int_{\lambda_0/A_k(s)}^{2\lambda_0/A_k(s)} e^{-\lambda(A_k(t) - A_k(s))} \cosh(\lambda(\phi_k(t) - \phi_k(s))) e^{-\lambda(A_k(s) + R)} \varphi_\lambda(x) \lambda^q d\lambda \\
\geq B \int_{\lambda_0/A_k(s)}^{2\lambda_0/A_k(s)} e^{-\lambda(A_k(t) - A_k(s))} \cosh(\lambda(\phi_k(t) - \phi_k(s))) \lambda^q d\lambda \\
= B/2 \int_{\lambda_0/A_k(s)}^{2\lambda_0/A_k(s)} (1 + e^{-2\lambda(\phi_k(t) - \phi_k(s)))}) \lambda^q d\lambda \\
\geq B/2 \int_{\lambda_0/A_k(s)}^{2\lambda_0/A_k(s)} \lambda^q d\lambda = \frac{B(2q+1)\lambda_0^{q+1}}{2(q+1)} A_k(s)^{-q-1}.
\]
We prove now (33). Repeating similar steps as before, we arrive at
\[
\eta_q(t, s, x; k) \\
\geq (st)^{\frac{k}{2}} \int_{\lambda_0/A_k(s)}^{2\lambda_0/A_k(s)} e^{-\lambda(A_k(t) - A_k(s))} \sinh(\lambda(\phi_k(t) - \phi_k(s))) e^{-\lambda(A_k(s) + R)} \varphi_\lambda(x) \lambda^q d\lambda \\
\geq B/2 (st)^{\frac{k}{2}} \int_{\lambda_0/A_k(s)}^{2\lambda_0/A_k(s)} \frac{1 - e^{-2\lambda(\phi_k(t) - \phi_k(s)))}}{\phi_k(t) - \phi_k(s)} \lambda^q d\lambda \\
\geq B/2 (st)^{\frac{k}{2}} \frac{1 - e^{-2\lambda_0 \phi_k(t) - \phi_k(s)}}{\phi_k(t) - \phi_k(s)} \int_{\lambda_0/A_k(s)}^{2\lambda_0/A_k(s)} \lambda^q d\lambda \\
= \frac{B(2q+1)\lambda_0^{q+1}}{2q} (st)^{\frac{k}{2}} A_k(s)^{-q} \frac{1 - e^{-2\lambda_0 \phi_k(t) - \phi_k(s)}}{\phi_k(t) - \phi_k(s)}.
\]
The previous inequality implies (33), provided that
\[ 1 - e^{-2\lambda_0 \phi_k(t) - \phi_k(s)} \gtrsim \langle A_k(t) \rangle^{-1} \]
holds. Let us prove this last inequality. For \( \phi_k(t) - \phi_k(s) \gtrsim \frac{1}{2\lambda_0} \langle A_k(s) \rangle \), we have
\[ 1 - e^{-2\lambda_0 \phi_k(t) - \phi_k(s)} \geq 1 - e^{-1} \]
and, consequently,
\[ 1 - e^{-2\lambda_0 \phi_k(t) - \phi_k(s)} \gtrsim (\phi_k(t) - \phi_k(s))^{-1} \gtrsim A_k(t)^{-1} \gtrsim \langle A_k(t) \rangle^{-1}. \]
On the other hand, in the case \( \phi_k(t) - \phi_k(s) \leq \frac{1}{2\lambda_0} \langle A_k(s) \rangle \), employing the inequality \( 1 - e^{-\sigma} \geq \sigma/2 \) for \( \sigma \in [0, 1] \), we find immediately

\[
1 - e^{-2\lambda_0 \frac{\phi_k(t) - \phi_k(s)}{\langle A_k(s) \rangle}} \geq \frac{\lambda_0}{\langle A_k(s) \rangle} \geq \frac{\lambda_0}{\langle A_k(t) \rangle}.
\]

So, the proof of (33) is completed.

Next, we prove an upper bound estimate in the special case \( s = t \).

**Lemma 2.7:** Let \( n \geq 1 \) and \( \lambda_0 > 0 \). If we assume \( q > (n - 3)/2 \), then, for \( t \geq 1 \) and \( |x| \leq A_k(t) + R \), the following upper bound estimate holds:

\[
\xi_{q}(t, t; x, k) \leq B_2 \langle A_k(t) \rangle^{-\frac{n+1}{2}} \langle A_k(t) - |x| \rangle^{-\frac{n-3}{2} - q}.
\]

Here, \( B_2 \) is a positive constant depending only on \( \lambda_0, q, R, k \) and \( \langle y \rangle \) denotes the same function as in the statement of Lemma 2.6.

**Proof:** We follow the proof of Lemma 3.1(iii) in [30]. Applying (34), we obtain

\[
\xi_{q}(t, t; x, k) = \int_0^{\lambda_0} e^{-\lambda(A_k(t) + R)} \varphi_{\lambda}(x) \lambda^q d\lambda \lesssim \int_0^{\lambda_0} \langle |x| \rangle^{-\frac{n-1}{2}} e^{-\lambda(A_k(t) - |x|)} \lambda^q d\lambda.
\]

Let us consider separately two different cases. If \( |x| \leq (A_k(t) + R)/2 \), then

\[
\xi_{q}(t, t; x, k) \lesssim \int_0^{\lambda_0} e^{-\lambda(A_k(t) + R - |x|)} \lambda^q d\lambda \lesssim \int_0^{\lambda_0} e^{-\lambda(A_k(t) + R)/2} \lambda^q d\lambda
\]

\[
\lesssim (A_k(t) + R)^{-q-1} \int_0^{\infty} e^{-\mu/2} \mu^q d\mu \lesssim (A_k(t) + R)^{-q-1} \lesssim (A_k(t))^{-q-1}
\]

\[
\lesssim (A_k(t))^{-\frac{n+1}{2}} (A_k(t) - |x|)^{-\frac{n-3}{2} - q}.
\]

In particular, in the last estimate we used the inequality \( \langle A_k(t) - |x| \rangle \lesssim \langle A_k(t) \rangle \), which follows trivially from \( |A_k(t) - |x|| \leq A_k(t) \) for \( |x| \leq A_k(t) \) and from \( \langle A_k(t) - |x| \rangle \lesssim 1 \) for \( A_k(t) \leq |x| \leq (A_k(t) + R)/2 \).

On the other hand, for \( |x| \geq (A_k(t) + R)/2 \), we may estimate

\[
\xi_{q}(t, t; x, k) \lesssim (A_k(t) + R)^{-\frac{n-1}{2}} \int_0^{\lambda_0} e^{-\lambda(A_k(t) + R - |x|)} \lambda^q^{-\frac{n-1}{2}} d\lambda
\]

\[
\lesssim (A_k(t))^{-\frac{n-1}{2}} (A_k(t) + R - |x|)^{-q + \frac{n-3}{2}} \int_0^{\infty} e^{-\mu/2} \mu^q^{-\frac{n-1}{2}} d\mu
\]

\[
\lesssim (A_k(t))^{-\frac{n-1}{2}} (A_k(t) + R - |x|)^{-q + \frac{n-3}{2}}.
\]

When \( (A_k(t) + R)/2 \leq |x| \leq A_k(t) \), thanks to the inequality \( A_k(t) + R - |x| \gtrsim \langle A_k(t) - |x| \rangle \), from (36) it follows easily (35); while for \( A_k(t) \leq |x| \leq A_k(t) + R \), as \( \langle A_k(t) - |x| \rangle \approx 1 \), the estimate

\[
\xi_{q}(t, t; x, k) \lesssim (A_k(t) + R)^{-\frac{n-1}{2}} \int_0^{\lambda_0} e^{-\lambda(A_k(t) + R - |x|)} \lambda^q^{-\frac{n-1}{2}} d\lambda
\]

\[
\lesssim (A_k(t))^{-\frac{n-1}{2}} \int_0^{\lambda_0} \lambda^q^{-\frac{n-1}{2}} d\lambda \lesssim (A_k(t))^{-\frac{n-1}{2}}
\]

is sufficient to conclude (35). This completes the proof.
2.3. Derivation of the iteration frame

In this section, we introduce the time-dependent functional whose dynamic is studied in order to prove the blow-up result. Hence, we derive the iteration frame for this functional and a first lower bound estimate of logarithmic type.

Let us introduce the functional

$$\mathcal{U}(t) = t^{-\frac{k}{2}} \int_{\mathbb{R}^n} u(t,x) \xi_q(t,t,x;k) \, dx$$

(37)

for \( t \geq 1 \) and for some \( q > (n-3)/2 \). From (31), (32) and (33), it follows

$$\mathcal{U}(t) \gtrsim B_0 t^{-\frac{k}{2}} \int_{\mathbb{R}^n} u_0(x) \, dx + B_1 t \int_{\mathbb{R}^n} u_1(x) \, dx.$$  

If we assume that \( u_0, u_1 \) are both non-negative and non-trivial, then we find that

$$\mathcal{U}(t) \gtrsim \epsilon$$

(38)

for any \( t \in [1, T) \), where the unexpressed multiplicative constant depends on \( u_0, u_1 \).

In the next proposition, we derive the iteration frame for the functional \( \mathcal{U} \).

**Proposition 2.8:** Suppose that the assumptions in Corollary 2.5 are satisfied and let \( q = \frac{n-1}{2} - \frac{1}{p} \). If \( \mathcal{U} \) is defined by (37), then there exists a positive constant \( C = C(n,p,R,k) \) such that

$$\mathcal{U}(t) \geq C(A_k(t))^{-1} \int_1^t \frac{\phi_k(t) - \phi_k(s)}{s} (\log(A_k(s)))^{-(p-1)} (\mathcal{U}(s))^p \, ds$$

(39)

for any \( t \in (1, T) \).

**Proof:** By the definition of the functional (37), applying Hölder’s inequality we obtain

$$s^\frac{k}{2} \mathcal{U}(s) \leq \left( \int_{\mathbb{R}^n} |u(s,x)|^p \eta_q(t,s,x;k) \, dx \right)^{1/p} \left( \int_{B_{R+A_k(s)}} \frac{\xi_q(s,s,x;k)}{\eta_q(t,s,x;k)} \, dx \right)^{1/p'},$$

where \( 1/p + 1/p' = 1 \). Therefore,

$$\int_{\mathbb{R}^n} |u(s,x)|^p \eta_q(t,s,x;k) \, dx \geq (s^\frac{k}{2} \mathcal{U}(s))^p \left( \int_{B_{R+A_k(s)}} \frac{\eta_q(t,s,x;k)}{\xi_q(s,s,x;k)} \right)^{(p-1)/p}.$$

(40)

Let us determine now an upper bound estimates for the integral on the right-hand side of (40). By using (35) and (33), we obtain

$$\int_{B_{R+A_k(s)}} \frac{\eta_q(t,s,x;k)}{\xi_q(s,s,x;k)} \, dx \leq B(A_k(s))^{-\frac{n-1}{2}} (st)^{-\frac{k}{2p'-1}} (A_k(t))^{1/p-1} (A_k(s))^\frac{q}{p} \int_{B_{R+A_k(s)}} (A_k(s) - |x|)^{\frac{n-3}{2} - q} \, dx$$
\[
\leq \tilde{B}(st) - \frac{k}{2^{p+1}} \langle A_k(t) \rangle^\frac{1}{p+1} \langle A_k(s) \rangle^\frac{1}{p+1} (-\frac{n-1}{2}p + \frac{n-1}{2}) \int_{B_R+A_k(s)} (A_k(s) - |x|)^{-1} \, dx \\
\leq \tilde{B}(st) - \frac{k}{2^{p+1}} \langle A_k(t) \rangle^\frac{1}{p+1} \langle A_k(s) \rangle^\frac{1}{p+1} (-\frac{n-1}{2}p + \frac{n-1}{2}) + n-1 \log(A_k(s)),
\]

where \( \tilde{B} = B_1^\frac{1}{p+1} B_2^\frac{p}{p+1} \) and in the second step we used \( q = (n-1)/2 - 1/p \) to get exactly \(-1\) as power of the function in the integral. Hence, from (40), we obtain

\[
\int_{\mathbb{R}^n} |u(s, x)|^p \eta_q(t, s, x; k) \, dx \\
\geq (s^\frac{k}{2} U(s))^p (st)^\frac{k}{2} \langle A_k(t) \rangle^{-1} \langle A_k(s) \rangle^{-\frac{n-1}{2}(p-1) + \frac{k}{p} -(n-1)(p-1) \log(A_k(s))}^{-(p-1)} \\
\geq t^2 \langle A_k(t) \rangle^{-1} s^\frac{k}{2} (p-1) \langle A_k(s) \rangle^{-\frac{n-1}{2}(p-1) + \frac{k}{p} \log(A_k(s))}^{-(p-1)} \langle U(s) \rangle^p.
\]

If we combine the previous lower bound estimate and (31), we have

\[
\mathcal{U}(t) \geq t^{-\frac{k}{2}} \int_1^t \left( \phi_k(t) - \phi_k(s) \right) s^{1-p} \int_{\mathbb{R}^n} |u(s, x)|^p \eta_q(t, s, x; k) \, dx \, ds \\
\geq \langle A_k(t) \rangle^{-1} \int_1^t \left( \phi_k(t) - \phi_k(s) \right) s^{1-p + \frac{k}{2}(p-1)} \langle A_k(s) \rangle^{-\frac{n-1}{2}(p-1) + \frac{k}{2}} \langle U(s) \rangle^p \, ds \\
\geq \langle A_k(t) \rangle^{-1} \int_1^t \left( \phi_k(t) - \phi_k(s) \right) \langle A_k(s) \rangle^{-\frac{n-1}{2}(p-1) + \frac{k}{2}} \langle U(s) \rangle^p \, ds \\
\geq \int_1^t \frac{\phi_k(t) - \phi_k(s)}{\langle A_k(t) \rangle} \langle A_k(s) \rangle^{-\frac{n-1}{2}(p-1) + \frac{k}{2}} \langle U(s) \rangle^p \, ds,
\]

where in third step we used \( s = (1 - k) s^{1-k} (A_k(s) + \phi_k(1))^{1-k} \approx \langle A_k(s) \rangle^{1-k} \) for \( s \geq 1 \).

Since \( p = p_0(n, k) \) from (2) it follows:

\[
- \left( \frac{n-1}{2} + \frac{2 - k}{2(1-k)} \right) p + \left( \frac{n-1}{2} + \frac{2 + k}{2(1-k)} \right) + \frac{1}{p} = -1 - \frac{k}{1-k} = -\frac{1}{1-k}, \tag{41}
\]

then, plugging (41) in the last lower bound estimate for \( \mathcal{U}(t) \) we find

\[
\mathcal{U}(t) \geq \langle A_k(t) \rangle^{-1} \int_1^t \left( \phi_k(t) - \phi_k(s) \right) \langle A_k(s) \rangle^{-\frac{1}{1-k}} \langle \log(A_k(s)) \rangle^{-p-1} \langle U(s) \rangle^p \, ds \\
\geq \langle A_k(t) \rangle^{-1} \int_1^t \frac{\phi_k(t) - \phi_k(s)}{s} \langle \log(A_k(s)) \rangle^{-p-1} \langle U(s) \rangle^p \, ds,
\]

which is precisely (39). This completes the proof.

\[\blacksquare\]

**Lemma 2.9:** Suppose that the assumptions in Corollary 2.5 are satisfied. Then, there exists a positive constant \( K = K(u_0, u_1, n, p, R, k) \) such that the lower bound estimate

\[
\int_{\mathbb{R}^n} |u(t, x)|^p \, dx \geq K e^{p} \langle A_k(t) \rangle^{(n-1)(1-\frac{p}{2}) + \frac{k p}{2(1-k)}}
\]

holds for any \( t \in (1, T) \).

**Proof:** We adapt the proof of Lemma 5.1 in [30] to our model.
Let us fix \( q > (n - 3)/2 + 1/p' \). Combining (37), (38) and Hölder’s inequality, it results

\[
\varepsilon t^{k/2} \lesssim t^{k/2} \mathcal{U}(t) = \int_{\mathbb{R}^n} u(t, x) \xi_q(t, t, x; k) \, dx
\]

\[
\lesssim \left( \int_{\mathbb{R}^n} |u(t, x)|^p \, dx \right)^{1/p} \left( \int_{B_{R+A_k(t)}} (\xi_q(t, t, x; k)^{p'}) \, dx \right)^{1/p'}.
\]

Hence,

\[
\int_{\mathbb{R}^n} |u(t, x)|^p \, dx \gtrsim \varepsilon^p t^{kp/2} \left( \int_{B_{R+A_k(t)}} (\xi_q(t, t, x; k)^{p'}) \, dx \right)^{-(p-1)}. \tag{43}
\]

Let us determine an upper bound estimates for the integral of \( \xi_q(t, t, x; k)^{p'} \). By using (35), we have

\[
\int_{B_{R+A_k(t)}} (\xi_q(t, t, x; k)^{p'}) \, dx \lesssim \langle A_k(t) \rangle^{-n+1/p'} \int_{B_{R+A_k(t)}} (A_k(t) - |x|)^{(n-3)p'/2 - p'q} \, dx
\]

\[
\lesssim \langle A_k(t) \rangle^{-n+1/p'} \int_0^{R+A_k(t)} r^{n-1} (A_k(t) - r)^{(n-3)p'/2 - p'q} \, dr
\]

\[
\lesssim \langle A_k(t) \rangle^{-n+1/p' + n-1} \int_0^{R+A_k(t)} (A_k(t) - r)^{(n-3)p'/2 - p'q} \, dr.
\]

Performing the change of variable \( A_k(t) - r = \rho \), one obtains

\[
\int_{B_{R+A_k(t)}} (\xi_q(t, t, x; k)^{p'}) \, dx \lesssim \langle A_k(t) \rangle^{-n+1/p' + n-1} \int_{-R}^{A_k(t)} (3 + |\rho|)^{(n-3)p'/2 - p'q} \, d\rho
\]

because of \((n - 3)p'/2 - p'q < -1\).

Combining this upper bound estimates for the integral of \( \xi_q(t, t, x; k)^{p'} \), (43) and using \( t \approx (A_k(t))^{1/2} \) for \( t \gg 1 \), we arrive at (42). The proof is over.

In Proposition 2.8, we derive the iteration frame for \( \mathcal{U} \). In the next result, we shall prove a first lower bound estimate for \( \mathcal{U} \), which shall be the base case of the inductive argument in Section 2.4.

**Proposition 2.10:** Suppose that the assumptions in Corollary 2.5 are satisfied and let \( q = n - 1/2 - 1/p' \). Let \( \mathcal{U} \) be defined by (37). Then, for \( t \gg 3/2 \), the functional \( \mathcal{U}(t) \) fulfils

\[
\mathcal{U}(t) \geq M \varepsilon^p \log \left( \frac{2t}{3} \right), \tag{44}
\]

where the positive constant \( M \) depends on \( u_0, u_1, n, p, R, k \).

**Proof:** From (31), we know that

\[
\mathcal{U}(t) \geq t^{-k/2} \int_1^t (\phi_k(t) - \phi_k(s)) s^{-p} \int_{\mathbb{R}^n} |u(s, x)|^p \eta_q(t, s, x; k) \, dx \, ds.
\]

Consequently, applying (33) first and then (42), we find

\[
\mathcal{U}(t) \geq B_1 \langle A_k(t) \rangle^{-1} \int_1^t (\phi_k(t) - \phi_k(s)) s^{-p + \frac{k}{2}} \langle A_k(s) \rangle^{-q} \int_{\mathbb{R}^n} |u(s, x)|^p \, dx \, ds
\]
\[
\geq B_1 K \varepsilon^p (A_k(t))^{-1} \int_1^t (\phi_k(t) - \phi_k(s)) s^{-p + \frac{k}{2}} (A_k(s))^{-q + (n-1)(1 - \frac{p}{2}) + \frac{kp}{2(n-1)}} ds
\]
\[
\geq \varepsilon^p (A_k(t))^{-1} \int_1^t (\phi_k(t) - \phi_k(s)) (1 - p + \frac{k}{2})^{-1 - \frac{1}{n - 1} + \frac{1}{2} + (n-1)(1 - \frac{p}{2}) + \frac{kp}{2(n-1)}} ds
\]
\[
\geq \varepsilon^p (A_k(t))^{-1} \int_1^t (\phi_k(t) - \phi_k(s))^{-\left(\frac{n-1}{2} + \frac{2}{2(n-1)}\right) + \frac{1}{2} (n-1)(1 - \frac{p}{2}) + \frac{kp}{2(n-1)}} ds
\]
\[
\geq \varepsilon^p (A_k(t))^{-1} \int_1^t (\phi_k(t) - \phi_k(s))^{-\frac{1}{n}} ds
\]
\[
\geq \varepsilon^p (A_k(t))^{-1} \int_1^t \frac{\phi_k(t) - \phi_k(s)}{s} ds.
\]

We estimate now the integral in the right-hand side of the previous chain of inequalities. Integration by parts leads to
\[
\int_1^t \frac{\phi_k(t) - \phi_k(s)}{s} ds = (\phi_k(t) - \phi_k(s)) \log s \big|_{s=1}^{s=t} + \int_1^t \phi_k'(s) \log s ds
\]
\[
= \int_1^t s^{-k} \log s ds \geq t^{-k} \int_1^t \log s ds.
\]

Therefore, for \( t \geq 3/2 \),
\[
\mathcal{U}(t) \geq \varepsilon^p (A_k(t))^{-1} t^{-k} \int_1^t \log s ds \geq \varepsilon^p (A_k(t))^{-1} t^{-k} \int_{2t/3}^t \log s ds
\]
\[
\geq (1/3) \varepsilon^p (A_k(t))^{-1} t^{1-k} \log(2t/3) \geq \varepsilon^p \log(2t/3),
\]

where in the last line we employed \( t \approx (A_k(t))^{-\frac{1}{n}} \) for \( t \geq 1 \). Also, the proof is complete. \(\square\)

2.4. Iteration argument

In this section, we prove the blow-up result. More specifically, we are going to show a sequence of lower bound estimates for \( \mathcal{U} \) and from these lower bound estimates, we conclude that for \( t \) over a certain \( \varepsilon \)-dependent threshold, the functional \( \mathcal{U}(t) \) may not be finite.

Our goal is to show the validity of the sequence of lower bound estimates
\[
\mathcal{U}(t) \geq C_j (\log(A_k(t)))^{-\beta_j} \left( \log \left( \frac{t}{\ell_j} \right) \right)^{\alpha_j} \text{ for } t \geq \ell_j
\]
for any \( j \in \mathbb{N} \), where the bounded sequence of parameters characterizing the slicing procedure is \( \{\ell_j\}_{j=1}^\infty \) with \( \ell_j \doteq 2 - 2^{-j+1} \) and \( \{C_j\}_{j=1}^\infty \), \( \{\alpha_j\}_{j=1}^\infty \), \( \{\beta_j\}_{j=1}^\infty \) are sequences of positive numbers that we will determine throughout the iteration argument.

In order to show (45), we apply an inductive argument. As we have already said, the crucial idea here is to apply a slicing procedure for the domain of integration in the iteration frame (39), in order to increase the power of the second logarithmic term in (45) step by step. This idea was introduced for the first time in [38], and since then it has been applied successfully to study the blow-up dynamic of semilinear wave models in critical cases, overcoming the difficulties in the application of Kato’s lemma for critical cases.

Since (45) is true in the case \( j = 0 \), provided that \( C_0 \doteq M \varepsilon^p \) and \( \alpha_0 = 1, \beta_0 = 0 \) (cf. Proposition 2.10), it remains to prove the inductive step. We assume (45) true for \( j \geq 0 \) and we have to prove...
Therefore, for $t \geq \ell_{j+1}$, we obtain

\[
\mathcal{W}(t) \geq CC_j^p (A_k(t))^{-1} \int_{\ell_j}^t \frac{\phi_k(t) - \phi_k(s)}{s} (\log(A_k(s)))^{-(p-1) - \beta_p} \left( \log \left( \frac{s}{\ell_j} \right) \right)^{\alpha_j p} ds \\
\geq CC_j^p (A_k(t))^{-1} (\log(A_k(t)))^{-(p-1) - \beta_p} \int_{\ell_j}^t \frac{\phi_k(t) - \phi_k(s)}{s} \left( \log \left( \frac{s}{\ell_j} \right) \right)^{\alpha_j p} ds.
\]

Using integration by parts, we find

\[
\int_{\ell_j}^t \frac{\phi_k(t) - \phi_k(s)}{s} \left( \log \left( \frac{s}{\ell_j} \right) \right)^{\alpha_j p} ds = (\phi_k(t) - \phi_k(s))(\alpha_j p + 1)^{-1} \left( \log \left( \frac{s}{\ell_j} \right) \right)^{\alpha_j p + 1} \bigg|_{s=\ell_j}^{s=t} \\
+ (\alpha_j p + 1)^{-1} \int_{\ell_j}^t \phi_k'(s) \left( \log \left( \frac{s}{\ell_j} \right) \right)^{\alpha_j p + 1} ds \\
= (\alpha_j p + 1)^{-1} \int_{\ell_j}^t s^{-k} \left( \log \left( \frac{s}{\ell_j} \right) \right)^{\alpha_j p + 1} ds \\
\geq (\alpha_j p + 1)^{-1} t^{-k} \int_{\ell_j}^t \left( \log \left( \frac{s}{\ell_j} \right) \right)^{\alpha_j p + 1} ds \\
\geq (\alpha_j p + 1)^{-1} t^{-k} \int_{\ell_j}^{\ell_{j+1}} \left( \log \left( \frac{t}{\ell_{j+1}} \right) \right)^{\alpha_j p + 1} ds \\
\geq (\alpha_j p + 1)^{-1} 2^{-(j+3)} \gamma_k (A_k(t)) \left( \log \left( \frac{t}{\ell_{j+1}} \right) \right)^{\alpha_j p + 1},
\]

where in the last step we applied $1 - \ell_j/\ell_{j+1} > 2^{-(j+3)}$ and $t^{1-k} \geq \gamma_k (A_k(t))$ for $t \geq 1$ with

\[
\gamma_k = \begin{cases} 
1/3 & \text{if } k \in [0, 2/3], \\
(1-k) & \text{if } k \in [2/3, 1).
\end{cases}
\]

Therefore,

\[
\mathcal{W}(t) \geq C_{\gamma_k} 2^{-(j+3)} (\alpha_j p + 1)^{-1} C_j^p (\log(A_k(t)))^{-(p-1) - \beta_p} \left( \log \left( \frac{t}{\ell_{j+1}} \right) \right)^{\alpha_j p + 1}
\]

for $t \geq \ell_{j+1}$, that is, we proved (45) for $j+1$, provided that

\[
C_{j+1} = C_{\gamma_k} 2^{-(j+3)} (\alpha_j p + 1)^{-1} C_j^p, \quad \alpha_{j+1} = 1 + p\alpha_j, \quad \beta_{j+1} = p - 1 + p\beta_j.
\]

Next, we establish a lower bound estimate for $C_j$. For this purpose, we provide first an explicit representation of the exponents $\alpha_j$ and $\beta_j$. Employing recursively the relations $\alpha_j = 1 + p\alpha_{j-1}$ and
\( \beta_j = (p - 1) + p \beta_{j-1} \) and the initial exponents \( \alpha_0 = 1, \beta_0 = 0 \), we obtain

\[
\alpha_j = \alpha_0 p^j + \sum_{k=0}^{j-1} p^k = \frac{p^{j+1} - 1}{p - 1} \quad \text{and} \quad \beta_j = p^j \beta_0 + (p - 1) \sum_{k=0}^{j-1} p^k = p^j - 1. \tag{46}
\]

In particular, \( \alpha_{j-1} p + 1 = \alpha_j \leq p^{j+1}/(p - 1) \) implies that

\[
C_j \geq D (2p)^{-j} C_{j-1} \tag{47}
\]

for any \( j \geq 1 \), where \( D = 2^{-2} C_{\gamma_k (p - 1)/p} \). Applying the logarithmic function to both sides of (47) and using iteratively the resulting inequality, we find

\[
\log C_j \geq p \log C_{j-1} - j \log (2p) + \log D
\]

\[
\geq \cdots \geq p^j \log C_0 - \left( \sum_{k=0}^{j-1} (j - k) p^k \right) \log (2p) + \left( \sum_{k=0}^{j-1} p^k \right) \log D
\]

\[
= p^j \left( \log M \varepsilon^p - \frac{p \log (2p)}{(p - 1)^2} + \frac{\log D}{p - 1} \right) + \left( \frac{j}{p - 1} + \frac{p}{(p - 1)^2} \right) \log (2p) - \frac{\log D}{p - 1},
\]

where we used the identity

\[
\sum_{k=0}^{j-1} (j - k) p^k = \frac{1}{p - 1} \left( \frac{p^{j+1} - p}{p - 1} - j \right). \tag{48}
\]

Let us define \( j_0 = j_0(n, p, k) \) as the smallest non-negative integer such that

\[ j_0 \geq \frac{\log D}{\log (2p)} - \frac{p}{p - 1}. \]

Hence, for any \( j \geq j_0 \), we have the estimate

\[
\log C_j \geq p^j \left( \log M \varepsilon^p - \frac{p \log (2p)}{(p - 1)^2} + \frac{\log D}{p - 1} \right) = p^j \log (E \varepsilon^p), \tag{49}
\]

where \( E = M (2p)^{-p/((p - 1)^2)} D^{1/(p - 1)} \).

Combining (45), (46) and (49), we arrive at

\[
\mathcal{U} (t) \geq \exp \left( p^j \log (E \varepsilon^p) \right) \left( \log \langle A_k (t) \rangle \right)^{-\beta_j} \left( \log \left( \frac{t}{2} \right) \right)^{\alpha_j}
\]

\[
= \exp \left( p^j \log (E \varepsilon^p) \right) \left( \log \langle A_k (t) \rangle \right)^{-\beta_j + 1} \left( \log \left( \frac{t}{2} \right) \right)^{\alpha_j (p^{j+1} - 1)/(p - 1)}
\]

\[
= \exp \left( p^j \log \left( E \varepsilon^p \left( \log \langle A_k (t) \rangle \right)^{-1} \left( \log \left( \frac{t}{2} \right) \right)^{\alpha_j (p^{j+1} - 1)/(p - 1)} \right) \right) \log \langle A_k (t) \rangle \left( \log \left( \frac{t}{2} \right) \right)^{-1/(p - 1)}
\]
for \( t \geq 2 \) and any \( j \geq j_0 \). Since for \( t \geq t_0(k) \doteq \max\{4, \gamma_k^{-1/k}\} \) the inequalities

\[
\log(A_k(t)) \leq (1 - k) \log t - \log \gamma_k \leq \log t \quad \text{and} \quad \log \left(\frac{t}{2}\right) \geq 2^{-1} \log t
\]

hold true, then

\[
U(t) \geq \exp(p \log(2^{-p/(p-1)} E \varepsilon^p (\log t)^{1/(p-1)})) \log(A_k(t)) \left(\log \left(\frac{t}{2}\right)\right)^{-1/(p-1)}
\]

for \( t \geq t_0 \) and any \( j \geq j_0 \). Let us denote \( J(t, \varepsilon) \doteq 2^{-p/(p-1)} E \varepsilon^p (\log t)^{1/(p-1)} \).

If we choose \( \varepsilon_0 = \varepsilon_0(n, p, k, \lambda_0, R, u_0, u_1) \) sufficiently small so that

\[
\exp(2P E_1^{-p} \varepsilon_0^{-p(p-1)}) \geq t_0,
\]

then, for any \( \varepsilon \in (0, \varepsilon_0] \) and for \( t > \exp(2P E_1^{-p} \varepsilon_0^{-p(p-1)}) \), we get \( t \geq t_0 \) and \( J(t, \varepsilon) > 1 \). Consequently, for any \( \varepsilon \in (0, \varepsilon_0] \) and for \( t > \exp(2P E_1^{-p} \varepsilon_0^{-p(p-1)}) \) letting \( j \to \infty \) in (50) it results that the lower bound for \( U(t) \) blows up; hence, \( U(t) \) is not finite as well. Also, we showed that \( U \) blows up in finite time and, moreover, we proved the upper bound estimate for the lifespan

\[
T(\varepsilon) \leq \exp(2P E_1^{-p} \varepsilon_0^{-p(p-1)})
\]

Therefore, we completed the proof of Theorem 1.2.

### 3. Semilinear wave equation in generalized EdeS spacetime: subcritical case

As a by-product of the approach developed in Section 2, we derive in this section the upper bound estimates for the lifespan of local in time solutions in the subcritical case \( 1 < p < \max\{p_0(n, k), p_1(n, k)\} \). Our main tool will be the generalization of Kato’s lemma containing the upper bound estimates for the lifespan proved in [39], whose statement is recalled below for the ease of the reader.

**Lemma 3.1:** Let \( p > 1, a > 0, q > 0 \) satisfy

\[
M = \frac{p-1}{2} a - \frac{q}{2} + 1 > 0.
\]

Assume that \( F \in \mathcal{C}^2([\tau, T]) \) satisfies

\[
F(t) \geq A t^a \quad \text{for} \ t \geq T_0 \geq \tau,
\]

\[
F''(t) \geq B(t + R)^{-q} |F(t)|^p \quad \text{for} \ t \geq \tau,
\]

\[
F(\tau) \geq 0, \quad F'(\tau) > 0,
\]

where \( A, B, R, T_0 \) are positive constants. Then, there exists a positive constant \( C_0 = C_0(p, a, q, B, \tau) \) such that

\[
T < 2^{\frac{q}{M}} T_1
\]

holds, provided that

\[
T_1 \doteq \max \left\{ T_0, \frac{F(\tau)}{F'(\tau)} R \right\} \geq C_0 A^{-\frac{p-1}{2M}}.
\]

As we are going to apply this generalization of Kato’s lemma, we will find some estimates already obtained in [29] in the treatment of the subcritical case, although the proofs that lead to these estimates are different.
Let us assume that $u_0, u_1$ are compactly supported with supports in $B_R$ for some $R > 0$, non-negative and non-trivial functions. Let $u$ be a solution on $[1, T)$ of (7) according to Definition 1.1 such that

$$\text{supp}(u, \cdot) \subset B_{R + A_k(t)}$$

for any $t \in (1, T)$, where $T = T(\varepsilon)$ is the lifespan of $u$.

Hence, we introduce as time-dependent functional the spatial average of $u$

$$U(t) \doteq \int_{\mathbb{R}^n} u(t, x) \, dx. \quad (56)$$

Choosing a test function $\psi$ such that $\psi = 1$ on $\{(s, x) \in [1, t] \times \mathbb{R}^n : |x| \leq R + A_k(s)\}$ in (8), we obtain

$$U'(t) = U'(1) + \int_1^t \int_{\mathbb{R}^n} s^{1-p}|u(s, x)|^p \, dx \, ds.$$ 

Also, differentiating the previous identity with respect to $t$, it results

$$U''(t) = t^{1-p} \int_{\mathbb{R}^n} |u(t, x)|^p \, dx. \quad (57)$$

By using the support condition for $u$ and Hölder’s inequality, from the above inequality, we obtain

$$U''(t) \gtrsim t^{1-p}(R + A_k(t))^{-n(p-1)}|U(t)|^p \geq (R + t)^{-(1-k)n+1(p-1)}|U(t)|^p \quad (58)$$

for any $t \in (1, T)$.

Let us derive now two estimates from below for $U$. On the one hand, thanks to the convexity of $U$, we have immediately

$$U(t) \geq U(1) + (t - 1)U'(1) \gtrsim \varepsilon t \quad (59)$$

for any $t \in (1, T)$, where we used that $u_0, u_1$ are non-negative and non-trivial in the unexpressed multiplicative constant. Plugging (59) in (58) and integrating twice, we obtain

$$U(t) \gtrsim \varepsilon^p t^{-(1-k)n+1(p-1)+p+2} \quad (60)$$

for any $t \in [T_0, T)$, where $T_0 > 1$. The first lower bound estimate for $U$ in (60) has been obtained from the convexity of $U$. On the other hand, from Lemma 2.9 and (57), integrating twice, we find a second lower bound estimate for $U$, that is,

$$U(t) \gtrsim \varepsilon^p t^{(1-k)(n-1)(1-\frac{p}{2}) + \frac{kp}{2} + 1 - p + 2} \quad (61)$$

for any $t \in [T_0, T)$.

Next, we apply Lemma 3.1 to the functional $U$. Since $u_0, u_1$ are non-negative and non-trivial we have $U(1), U'(1) > 0$, so (53) is fulfilled.

Moreover, (58) corresponds to (52) with $q \doteq ((1-k)n+1)(p-1)$. Finally, combining (60) and (61) we have (51) with $a = \max\{a_1, a_2\}$, where

$$a_1 \doteq -((1-k)n+1)(p-1) + p + 2,$$

$$a_2 \doteq (1-k)(n-1)\left(1 - \frac{p}{2}\right) + \frac{kp}{2} + 1 - p + 2$$
and $A \approx e^p$. According to this choice, we have two possible values for the quantity $M$ in Lemma 3.1: either we use (60), that is, $a = a_1$ and, consequently,

$$M_1 \doteq \frac{p - 1}{2} a_1 - \frac{q}{2} + 1 = \frac{p}{2} \left[-(1 - k)n(p - 1) + 2 \right]$$

or we use (61), that is, $a = a_2$ and, then,

$$M_2 \doteq \frac{p - 1}{2} a_2 - \frac{q}{2} + 1$$

$$= \frac{1}{2} \left\{ - \left[ (1 - k) \frac{n - 1}{2} + 1 - \frac{k}{2} \right] p^2 + \left[ (1 - k) \frac{n + 1}{2} + 1 + \frac{3k}{2} \right] p + 1 - k \right\}.$$  

Therefore, for $M = \max\{M_1, M_2\} > 0$, Lemma 3.1 provides a blow-up result and the upper bound estimate for the lifespan

$$T \lesssim e^{-\frac{p(p-1)}{2M}}.$$  

Let us make the condition $M > 0$ more explicit. The condition $M_1 > 0$ is equivalent to $p < p_1(n, k)$, while the condition $M_2 > 0$ is equivalent to $p < p_0(n, k)$. Hence, Lemma 3.1 implies the validity of a blow-up result for (7) in the subcritical case $1 < p < \max\{p_0(n, k), p_1(n, k)\}$ (exactly as in [29]) and the upper bound estimates for the lifespan

$$T(e) \lesssim \begin{cases} e^{-\frac{\theta(p, n, k) - (1 - k)n}{p^2}} & \text{if } p < p_1(n, k), \\ e^{-\frac{p(p-1)}{M(p, n, k)}} & \text{if } p < p_0(n, k), \end{cases}$$  

where

$$\theta(p, n, k) \doteq 1 - k + \left[ (1 - k) \frac{n + 1}{2} + 1 + \frac{3k}{2} \right] p - \left[ (1 - k) \frac{n - 1}{2} + 1 - \frac{k}{2} \right] p^2.$$  

Furthermore, we point out that $a > 1$ (so, in particular, $a > 0$ as it is required in the assumptions of Lemma 3.1) if and only if $1 < p < \max\{p_1(n, k), p_2(n, k)\}$, where

$$p_2(n, k) \doteq 2 + \frac{2k}{(1 - k)n + 1}.$$  

We want to show now that the condition $a > 1$ is always fulfilled whenever $M > 0$ holds. For this purpose, we shall determine how to order the exponents $p_0, p_1, p_2$. Since $p_0(n, k)$ is defined through (2), the inequality $p_0(n, k) > p_1(n, k)$ holds if and only if

$$((1 - k)n + 1) p_1(n, k)^2 - ((1 - k)n + 3 + 2k) p_1(n, k) - 2(1 - k) < 0.$$  

By straightforward computations it follows that the last inequality is fulfilled if and only if $n > N(k)$, where $N(k)$ is defined in (6). Similarly, $p_0(n, k) > p_2(n, k)$ if and only if $n < N(k)$. Summarizing,

$$p_2(n, k) < p_0(n, k) < p_1(n, k) \quad \text{if } n < N(k),$$

$$p_0(n, k) = p_1(n, k) = p_2(n, k) \quad \text{if } n = N(k),$$

$$p_1(n, k) < p_0(n, k) < p_2(n, k) \quad \text{if } n > N(k).$$  

Consequently, for $n > N(k)$, the critical condition is $p = p_0(n, k)$, so if $p < p_0(n, k)$, in particular, the condition $p < p_2(n, k)$ is fulfilled (that is, $M_2 > 0$ implies $a_2 > 1$). On the other hand, for $n < N(k)$, it holds $p_2(n, k) < p_1(n, k)$ and the condition $M_1 > 0$ and $a_1 > 1$ are both equivalent to $p < p_1(n, k)$.
(the critical condition is \( p = p_1(n, k) \) in this case). Therefore, we actually proved that \( M > 0 \) implies \( a > 1 \).

**Remark 3.1:** In [29] the condition in the subcritical case on \( p \) under which a blow-up result holds for (7) is written in a slightly different but equivalent way. Indeed, combining [29, Equation (1.9)] with (64), we see immediately that the condition for \( p \) in [29, Theorem 1.3] is satisfied if and only if \( 1 < p < \max\{p_1(n, k), p_0(n, k)\} \).

Finally, we want to compare the upper bound estimates for the lifespan in (62). Clearly, the estimates

\[
T(\varepsilon) \lesssim \begin{cases} 
\varepsilon^{-(\frac{2}{p-1}-(1-k)n)^{-1}} & \text{if } n < N(k) \text{ and } p \in [p_0(n, k), p_1(n, k)), \\
\varepsilon^{-\frac{p(p-1)}{m(p,n,k)}} & \text{if } n > N(k) \text{ and } p \in [p_1(n, k), p_0(n, k)), 
\end{cases}
\]

cannot be improved because it holds either \( p \geq p_0(n, k) \) or \( p \geq p_1(n, k) \). Note that \( p_2(n, k) \) plays no role in the determination of the upper bound estimate for the lifespan.

However, in the case \( 1 < p < \min\{p_0(n, k), p_1(n, k)\} \) it is not clear which of the upper bounds in (62) is better. Of course, in this case we have to compare \( a_1 \) and \( a_2 \). A straightforward computation shows that \( a_1 \geq a_2 \) if and only if

\[
((1 - k)n - 1)p \leq 2(1 - k).
\]

If \( n \leq \tilde{N}(k) = 1/(1 - k) \), then the previous inequality is always true. On the other hand, for \( n > \tilde{N}(k) \), we may introduce the further exponent

\[
p_3(n, k) \doteq \frac{2(1 - k)}{(1 - k)n - 1}.
\]

It turns out that \( p_3(n, k) > 1 \) if and only if \( \tilde{N}(k) < n < \tilde{N}(k) = 2 + 1/(1 - k) \). Moreover, for \( n > \tilde{N}(k) \), the inequalities \( p_1(n, k) < p_3(n, k) \) and \( p_0(n, k) < p_3(n, k) \) are both satisfied if and only if \( n < N(k) \).

In order to clarify the upper bound estimates in (62), we shall consider five different subcases depending on the range for the spatial dimension \( n \).

**Case** \( n \leq \tilde{N}(k) \). In this case, (65) is always satisfied as the left-hand side is non-positive. So, \( a_1 \geq a_2 \). Therefore, for any \( 1 < p < p_1(n, k) \), the following upper bound estimate holds:

\[
T(\varepsilon) \lesssim \varepsilon^{-(\frac{2}{p-1}-(1-k)n)^{-1}}.
\]

**Case** \( \tilde{N}(k) < n < N(k) \). In this case, (65) is satisfied for \( p \leq p_3 \). Hence, by the ordering \( 1 < p_0(n, k) < p_1(n, k) < p_3(n, k) \), we get that \( a_1 > a_2 \) for exponents satisfying \( 1 < p < p_1(n, k) \). Also, even in this case (66) is a better estimates than

\[
T(\varepsilon) \lesssim \varepsilon^{-\frac{p(p-1)}{m(p,n,k)}}.
\]

**Case** \( n = N(k) \). In this limit case, \( p_0(n, k) = p_1(n, k) = p_3(n, k) \).

So, for \( 1 < p < p_1(n, k) = p_3(n, k) \), it holds \( a_1 > a_2 \) and as in the previous case (66) is the best estimate.

**Case** \( N(k) < n < \tilde{N}(k) \). In this case, it results \( 1 < p_3(n, k) < p_1(n, k) < p_0(n, k) \).

So, for \( 1 < p \leq p_3(n, k) \), it holds \( a = a_1 \), while for \( p_3(n, k) < p < p_0(n, k) \) we have \( a = a_2 \). Therefore,

\[
T(\varepsilon) \lesssim \begin{cases} 
\varepsilon^{-(\frac{2}{p-1}-(1-k)n)^{-1}} & \text{if } p \in (1, p_3(n, k)], \\
\varepsilon^{-\frac{p(p-1)}{m(p,n,k)}} & \text{if } p \in (p_3(n, k), p_0(n, k)). 
\end{cases}
\]
In this case, \( p_3(n,k) \leq 1 \) and \( 1 < p_1(n,k) < p_0(n,k) \) so (65) is never satisfied for \( p > 1 \). Hence, \( a_2 > a_1 \) for any \( 1 < p < p_0(n,k) \), that is,

\[
T(\varepsilon) \lesssim \varepsilon^{-\frac{p(p-1)}{2(p,p,k)}}
\]

is a better estimate than (66).

### 3.1. Lifespan estimates in the subcritical case

In summarizing, what we established in the above subcases, we proved the following proposition, that completes [29, Theorem 1.3] with the estimate for the lifespan while Theorems 1.2 and 1.3 deals with the critical case that was not discussed in [29].

**Proposition 3.2:** Let \( n \geq 1 \) and \( 1 < p \leq \max\{p_0(n,k), p_1(n,k)\} \). Let us assume that \( u_0 \in H^1(\mathbb{R}^n) \) and \( u_1 \in L^2(\mathbb{R}^n) \) are non-negative, non-trivial and compactly supported functions with supports contained in \( B_{R} \) for some \( R > 0 \). Let

\[
u \in C^k([1,T), H^1(\mathbb{R}^n)) \cap C^1([1,T), L^2(\mathbb{R}^n)) \cap L^p_{\text{loc}}([1,T) \times \mathbb{R}^n)
\]

be an energy solution to (7) according to Definition 1.1 with lifespan \( T = T(\varepsilon) \) and fulfilling the support condition \( \text{supp}(u(t,\cdot)) \subset B_{A_k(t)+R} \) for any \( t \in (1,T) \). Then, there exists a positive constant \( \varepsilon_0 = \varepsilon_0(u_0, u_1, n, p, k, R) \) such that for any \( \varepsilon \in (0, \varepsilon_0] \) the energy solution \( u \) blows up in finite time. Furthermore, the upper bound estimates for the lifespan

\[
T(\varepsilon) \leq \begin{cases} 
C\varepsilon_{\frac{2}{p-1}-(1-k)n}^{-1} & \text{if } n \leq N(k) \text{ and } p \in (1, p_1(n,k)), \\
C\varepsilon_{\frac{2}{p-1}-(1-k)n}^{-1} & \text{if } n \in (N(k), \hat{N}(k)) \text{ and } p \in (1, p_3(n,k)], \\
C\varepsilon_{\frac{p(p-1)}{\theta(p,n,k)}} & \text{if } n \in (N(k), \hat{N}(k)) \text{ and } p \in (p_3(n,k), p_0(n,k)), \\
C\varepsilon_{\frac{p(p-1)}{\theta(p,n,k)}} & \text{if } n \geq \hat{N}(k) \text{ and } p \in (1, p_0(n,k)), 
\end{cases}
\]

hold, where the constant \( C > 0 \) is independent of \( \varepsilon \) and \( \theta(p,n,k) \) is defined by (63).

### 4. Semilinear wave equation in EdeS spacetime: second critical case

In Section 3, we derived the upper bound for the lifespan in the subcritical case, while, in Section 2, we studied the critical case \( p = p_0(n,k) \). We have already remarked that \( p = p_0(n,k) \) is the critical case when \( n > N(k) \). Therefore, it remains to consider the critical case \( p = p_1(n,k) \) when \( n \leq N(k) \).

In this section, we are going to prove a blow-up result even in this critical case \( p = p_1(n,k) \) and derive the corresponding upper bound estimate for the lifespan. Even in this critical case, our approach will be based on a basic iteration argument combined with the slicing procedure we already applied in Section 2.

As time-depending functional we will use the same one employed in Section 3, namely the function \( U \) defined in (56). Then, since \( p = p_1(n,k) \) is equivalent to the condition

\[
((1-k)n + 1) (p-1) = p+1,
\]

we may rewrite (58) as

\[
U(t) \geq C \int_{1}^{t} \int_{1}^{s} (R + \tau)^{-(p+1)} (U(\tau))^p \, d\tau \, ds
\]

for any \( t \in (1,T) \) and for a suitable positive constant \( C > 0 \). Let us point out that (68) will be the iteration frame in the iteration procedure for the critical case \( p = p_1(n,k) \).
We know that \( U(t) \geq K e^t \) for any \( t \in (1, T) \), where \( K \) is a suitable positive constant, provided that \( u_0, u_1 \) are non-negative, non-trivial and compactly supported (cf. the estimate in (59)). Therefore,

\[
U(t) \geq C K_p e^p \int_1^t \int_1^s (R + \tau)^{-(p+1)} \tau^p \, d\tau \, ds \geq C K_p (R + 1)^{-(p+1)} e^p \int_1^t \int_1^s \tau^{-1} \, d\tau \, ds
\]

\[
= C K_p (R + 1)^{-(p+1)} e^p \int_1^t \log s \, ds \geq C K_p (R + 1)^{-(p+1)} e^p \int_{2t/3}^t \log s \, ds
\]

\[
\geq 3^{-1} C K_p (R + 1)^{-(p+1)} e^p t \log \left( \frac{2t}{3} \right)
\]

(69)

for \( t \geq \ell_0 = 3/2 \), where we used \( R + \tau \leq (R + 1)\tau \) for \( \tau \geq 1 \).

Hence, by using recursively (68), we are going to prove now the sequence of lower bound estimates

\[
U(t) \geq K_j t \left( \log \left( \frac{t}{\ell_j} \right) \right)^{\sigma_j} \text{ for } t \geq \ell_j
\]

(70)

for any \( j \in \mathbb{N} \), where the sequence \( \{\ell_j\}_{j \in \mathbb{N}} \) is defined as in Section 2.3, i.e. \( \ell_j = 2 - (j+1) \), and \( \{K_j\}_{j \in \mathbb{N}}, \{\sigma_j\}_{j \in \mathbb{N}} \) are sequences of positive reals that we shall determine afterwards.

We remark that for \( j = 0 \) (70) holds true, provided that \( K_0 = (C K_p (R + 1)^{-(p+1)} e^p)/3 \) and \( \sigma_0 = 1 \). Next, we are going to prove (70) by using an inductive argument. Assumed the validity of (70) for some \( j \geq 0 \) we have to prove (70) for \( j + 1 \). For this purpose, we plug (70) in (68), thus, after shrinking the domain of integration, we have

\[
U(t) \geq C (R + 1)^{-(p+1)} K_j^p (\sigma_j p + 1)^{-1} \int_{\ell_j}^t \left( \log \left( \frac{s}{\ell_j} \right) \right)^{\sigma_j p + 1} \, ds
\]

for \( t \geq \ell_{j+1} \).

If we shrink the domain of integration to \([((\ell_j/\ell_{j+1})t, t)\) in the last \( s \)-integral, we obtain

\[
U(t) \geq C (R + 1)^{-(p+1)} K_j^p (\sigma_j p + 1)^{-1} \int_{\ell_j}^t \left( \log \left( \frac{s}{\ell_j} \right) \right)^{\sigma_j p + 1} \, ds
\]

\[
= C (R + 1)^{-(p+1)} K_j^p (\sigma_j p + 1)^{-1} \left( 1 - \frac{\ell_j}{\ell_{j+1}} \right) t \left( \log \left( \frac{t}{\ell_{j+1}} \right) \right)^{\sigma_j p + 1}
\]

\[
\geq C (R + 1)^{-(p+1)} 2^{-(j+3)} K_j^p (\sigma_j p + 1)^{-1} t \left( \log \left( \frac{t}{\ell_{j+1}} \right) \right)^{\sigma_j p + 1}
\]

for \( t \geq \ell_{j+1} \), where in the last step we applied the inequality \( 1 - \ell_j/\ell_{j+1} > 2^{-j-3} \). Also, we proved (70) for \( j + 1 \) provided that

\[
K_{j+1} = C (R + 1)^{-(p+1)} 2^{-(j+3)} (\sigma_j p + 1)^{-1} K_j^p \text{ and } \sigma_{j+1} \leq \sigma_j p + 1.
\]
Next, we determine a lower bound estimate for $K_j$. First, we find the value of the exponent $\sigma_j$. Applying iteratively the relation $\sigma_j = 1 + p\sigma_{j-1}$ and the initial exponent $\sigma_0 = 1$, we obtain
\[
\sigma_j = \sigma_0 p^j + \sum_{k=0}^{j-1} p^k = \frac{p^{j+1} - 1}{p - 1}.
\] (71)

In particular, $\sigma_{j-1} p + 1 = \sigma_j \leq p^{j+1} / (p - 1)$ implies that
\[
K_j \geq L (2p)^{-j} K_{j-1}^p
\] (72)
for any $j \geq 1$, where $L \doteq 2^{-2} C (R + 1)^{-1} (p+1) (p - 1) / p$. Applying the logarithmic function to both sides of (72) and reusing the resulting inequality in an iterative way, we arrive at
\[
\log K_j \geq p \log K_{j-1} - j \log (2p) + \log L \\
\geq \cdots \geq p^j \log K_0 - \left( \sum_{k=0}^{j-1} (j - k) p^k \right) \log (2p) + \left( \sum_{k=0}^{j-1} p^k \right) \log L \\
= p^j \left( \log (3^{-1} C K^p (R + 1)^{-1} (p+1) \varepsilon^p) - \frac{p \log (2p)}{(p - 1)^2} + \frac{\log L}{p - 1} \right) \\
+ \left( \frac{j}{p - 1} + \frac{p}{(p - 1)^2} \right) \log (2p) - \frac{\log L}{p - 1},
\]
where we applied again the identity (48). Let us define $j_1 = j_1 (n, p, k)$ as the smallest non-negative integer such that
\[
j_1 \geq \frac{\log L}{\log (2p)} - \frac{p}{p - 1}.
\]
Hence, for any $j \geq j_1$ the estimate
\[
\log K_j \geq p^j \left( \log (3^{-1} C K^p (R + 1)^{-1} (p+1) \varepsilon^p) - \frac{p \log (2p)}{(p - 1)^2} + \frac{\log L}{p - 1} \right) = p^j \log (N \varepsilon^p)
\] (73)
holds, where $N \doteq 3^{-1} C K^p (R + 1)^{-1} (p+1) (2p)^{-p/(p-1)^2} L^{1/(p-1)}$.

Combining (70), (71) and (73), we arrive at
\[
U(t) \geq \exp (p^j \log (N \varepsilon^p)) t \left( \log \left( \frac{t}{\ell_j} \right) \right)^{\sigma_j} \\
\geq \exp (p^j \log (N \varepsilon^p)) t \left( \frac{1}{2} \log t \right)^{(p^{j+1} - 1) / (p - 1)} \\
= \exp (p^j \log (2^{-p/(p-1)} N \varepsilon^p (\log t)^{p/(p-1)})) t \left( \frac{1}{2} \log t \right)^{-1/(p-1)}
\]
for $t \geq 4$ and for any $j \geq j_1$, where we applied the inequality $\log (t / \ell_j) \geq \log (t/2) \geq (1/2) \log t$ for all $t \geq 4$. If we denote $H(t, \varepsilon) \doteq 2^{-p/(p-1)} N \varepsilon^p (\log t)^{p/(p-1)}$, the last estimate may be rewritten as
\[
U(t) \geq \exp (p^j \log H(t, \varepsilon)) t \left( \frac{1}{2} \log t \right)^{-1/(p-1)}
\] (74)
for $t \geq 4$ and any $j \geq j_1$. 
Let us fix \( \varepsilon_0 = \varepsilon_0(n, p, k, R, u_0, u_1) \) so that

\[
\exp \left( 2N^{-(1-p)/p} \varepsilon_0^{-(p-1)} \right) \geq 4.
\]

Then, for any \( \varepsilon \in (0, \varepsilon_0] \) and for \( t > \exp(2N^{-(1-p)/p} \varepsilon^{-(p-1)}) \), we get \( t \geq 4 \) and \( H(t, \varepsilon) > 1 \). Thus, for any \( \varepsilon \in (0, \varepsilon_0] \) and for \( t > \exp(2N^{-(1-p)/p} \varepsilon^{-(p-1)}) \) as \( j \to \infty \) in (74), we find that the lower bound for \( U(t) \) blows up and, consequently, \( U(t) \) cannot be finite too. Summarizing, we proved that \( U \) blows up in finite time and, besides, we showed the upper bound estimate for the lifespan

\[
T(\varepsilon) \leq \exp \left( 2N^{-(1-p)/p} \varepsilon^{-(p-1)} \right).
\]

Altogether, we established Theorem 1.3 in the critical case \( p = p_1(n, k) \).

**Remark 4.1:** Combining the results from Theorems 1.2 and 1.3 and Proposition 3.2, we a full picture of the upper bound estimates for the lifespan of local in time solutions to (7) whenever \( 1 < p \leq \max\{p_0(n, k), p_1(n, k)\} \), of course, under suitable sign, size and support assumptions for the initial data.

5. **Final remarks**

Let us compare our results with the corresponding ones for the semilinear wave equation in the flat case. First, we point out that due to the presence of the term \( t^{1-p} \) in the semilinear term in (4), we have a competition between the two exponents \( p_0, p_1 \) to be the critical exponent. This, for the classical semilinear wave equation with power nonlinearity, does not happen since \( p_{Str}(n) \geq \frac{n+1}{n-1} \) for any \( n \geq 2 \). However, a similar situation, it has been observed when lower-order terms with time-dependent coefficients in the *scale-invariant case* are present, with a competition between a shift of Fujita exponent and a shift Strauss exponent (cf. [40–46]). On the other hand, the presence of the exponent \( p_3 \) for dimensions \( n \in (N(k), \widehat{N}(k)) \) to distinguish among two different upper bounds for the lifespan depending on the range for \( p \) is exactly what happens for the semilinear wave equation in spatial dimensions \( n = 2 \) (see [12,39]). Moreover, the situation for (7) when \( n \leq N(k) \) is completely analogous to what happens for the semilinear wave equation when \( n = 1 \), see [47] for the Euclidean case.

Finally, as anticipated in the introduction, we point out the main differences between our results and the ones in [32]. The approach we used in the critical case is completely different, and we slightly improved their result in the special case of the semilinear wave equation in the generalized EdeS spacetime, by removing the assumption on the size of the support of the Cauchy data (cf. [32, Theorem 2.3]). More specifically, when the coefficient of the damping term is 2 as in (1) differently from Theorem 2.3 in [32], where the condition \( R \leq (2(1-k))^{-1} \) is required on the radius of the ball containing the supports of the Cauchy data, in the present paper, no restriction on \( R \) is requested. Beyond this small improvement on the size of \( B_R \), it is interesting to explain how the approach in the present paper differs from the one in [32] for the treatment of the critical case. Indeed, while in [32] a comparison argument for an ODI is employed to prove the blow-up in finite time of a local solution, in the proof of Theorem 1.2 an iteration argument for a suitable weighted average of a local solution is used. Of course, a comparison argument for an ODI and the derivation of a sequence of lower bound estimates by means of an iteration frame are two different sides of the same coin. Nevertheless, in the personal opinion of the author, the latter treatment seems to be more flexible in regards to generalizations or when adapting it to different semilinear hyperbolic models.

**Acknowledgments**

The author thanks Karen Yagdjian (UTRGV) and Hiroyuki Takamura (Tohoku Univ.) for valuable discussions on the model considered in this work.
Disclosure statement
No potential conflict of interest was reported by the author(s).

Funding
A. Palmieri is supported by the Japan Society for the Promotion of Science (JSPS) – JSPS Postdoctoral Fellowship for Research in Japan (PE20003) and by the GNAMPA project 'Problemi stazionari e di evoluzione nelle equazioni di campo nonlineari dispersive'.

ORCID
Alessandro Palmieri http://orcid.org/0000-0001-8552-517X

References
[1] Georgiev V, Lindblad H, Sogge CD. Weighted Strichartz estimates and global existence for semilinear wave equations. Am J Math. 1997;119(6):1291–1319.
[2] Glassey RT. Existence in the large for □u = F(u) in two space dimensions. Math Z. 1981;178(2):233–261.
[3] Glassey RT. Finite-time blow-up for solutions of nonlinear wave equations. Math Z. 1981;177(3):323–340.
[4] John F. Blow-up of solutions of nonlinear wave equations in three space dimensions. Manuscr Math. 1979;28(1–3):235–268.
[5] Kato T. Blow-up of solutions of some nonlinear hyperbolic equations. Commun Pure Appl Math. 1980;33(4):501–505.
[6] Lindblad H, Sogge CD. Long-time existence for small amplitude semilinear wave equations. Am J Math. 1996;118(5):1047–1135.
[7] Schaeffer J. The equation utt − Δu = |u|^p for the critical value of p. Proc R Soc Edinb A. 1985;101(1–2):31–44.
[8] Sideris TC. Nonexistence of global solutions to semilinear wave equations in high dimensions. J Differ Equ. 1984;52(3):378–406.
[9] Yordanov BT, Zhang QS. Finite time blow up for critical wave equations in high dimensions. J Funct Anal. 2006;231(2):361–374.
[10] Zhou Y. Cauchy problem for semilinear wave equations in four space dimensions with small initial data. J Partial Differ Equ. 1995;8(2):135–144.
[11] Zhou Y. Blow up of solutions to semilinear wave equations with critical exponent in high dimensions. Chin Ann Math Ser B. 2007;28:205–212.
[12] Imai T, Kato M, Takamura H, et al. The sharp lower bound of the lifespan of solutions to semilinear wave equations with low powers in two space dimensions. Asymptotic analysis for nonlinear dispersive and wave equations. Tokyo (Japan): Mathematical Society of Japan; 2019. p. 31–53. doi:10.2969/aspm/08110031
[13] Catania D, Georgiev V. Blow-up for the semilinear wave equation in the Schwarzschild metric. Differ Integral Equ. 2006;19(7):799–830.
[14] Lin Y, Lai N-A, Ming S. Lifespan estimate for semilinear wave equation in Schwarzschild spacetime. Appl Math Lett. 2020;99:105997, 4 pp.
[15] Lindblad H, Metcalfe J, Sogge CD, et al. The Strauss conjecture on Kerr black hole backgrounds. Math Ann. 2014;359(3–4):637–661.
[16] Marzuola J, Metcalfe J, Tataru D, et al. Strichartz estimates on Schwarzschild black hole backgrounds. Math Ann. 2010;293(1):37–83.
[17] Ebert MR, Reissig M. Regularity theory and global existence of small data solutions to semi-linear de Sitter models with power non-linearity. Nonlinear Anal Real World Appl. 2018;40:14–54.
[18] Galstian A. Semilinear shifted wave equation in the de Sitter spacetime with hyperbolic spatial part. Theory, numerics and applications of hyperbolic problems. I. Cham: Springer; 2018. p. 577–587. (Springer proceedings in mathematics & statistics; 236).
[19] Galstian A, Yagdjian K. Global in time existence of self-interacting scalar field in de Sitter spacetimes. Nonlinear Anal Real World Appl. 2017;34:110–139.
[20] Yagdjian K. The semilinear Klein–Gordon equation in de Sitter spacetime. Discrete Contin Dyn Syst S. 2009;2(3):679–696.
[21] Yagdjian K. Global existence of the scalar field in de Sitter spacetime. J Math Anal Appl. 2012;396(1):323–344.
[22] Yagdjian K, Galstian A. Fundamental solutions for the Klein–Gordon equation in de Sitter spacetime. Commun Math Phys. 2009;285:293–344.
[23] Galstian A. U^α − L^β decay estimates for the wave equations with exponentially growing speed of propagation. Appl Anal. 2003;82(3):197–214.
Appendix. Alternative proof of Proposition 2.1 in the special case $k = 2/3$

In this appendix, we determine the representation of the solutions $\{y_j(t, s; \lambda)\}_{j \in \{0,1\}}$ to the Cauchy problems

\[
\begin{align*}
\partial_t^2 y_j(t, s; \lambda) - \lambda^2 t^{-\frac{2}{3}} y_j(t, s; \lambda) &= 0, \\
y_j(s, s; \lambda) &= \delta_{0j}, \\
\partial_t y_j(s, s; \lambda) &= \delta_{1j},
\end{align*}
\]

\[(A1)\]
where $\lambda > 0$ is a parameter and $\delta_j$ denotes the Kronecker delta. We underline that \((A1)\) is related to the d’Alembertian operator in Einstein–de Sitter spacetime (namely, to special value $\kappa = 2/3$), so this case has a privileged role from a physical viewpoint. In particular, in this appendix, we want to emphasize how the hyperbolic trigonometric functions come into play in the description of the solutions to \((A1)\) in this significant case.

Let us introduce the change of variables $z = z(t; \lambda) \equiv -2\lambda \phi(t)$, where for the sake of brevity we denote simply $\phi(t) \equiv \phi^3(t) = 3t^{1/3}$. Furthermore, we perform the transformation $y(t, \lambda) = w(z)e^{-z}$. A straightforward computation shows that

$$
\partial_t y(t, \lambda) = \left[ w'(z) - \frac{1}{2} w(z) \right] e^{-z} \partial_z \frac{\partial z}{\partial t},
$$

$$
\partial_z^2 y(t, \lambda) = \left[ w'(z) - w'(z) + \frac{1}{4} w(z) \right] e^{-z} \left( \frac{\partial \partial z}{\partial t} \right)^2 + \left[ w'(z) - \frac{1}{2} w(z) \right] e^{-\frac{z}{2}} \partial^2 \partial z.
$$

Consequently, $y$ solves the equation

$$
\frac{d^2 y}{dt^2} - \lambda^2 e^{-\frac{z}{2}} y = 0 \tag{A2}
$$

if and only if $z$ is a solution of the confluent hypergeometric equation

$$
z w''(z) - (z + 2) w'(z) + w(z) = 0, \tag{A3}
$$

where we used $\frac{\partial^2 z}{\partial t^2} = 4\lambda t^{-4/3} (\phi(t))^{-1}$ and $(\frac{\partial z}{\partial t})^2 = 4\lambda^2 t^{-4/3}$. According to \cite{48, Equation 13.2.32, p.324}, a fundamental pair of solutions to \((A3)\) is given by $z^3 M(z; 2, 4)$ and $z + 2$. Here, $M(z; a, c)$ denotes Kummer’s function

$$
M(z; a, c) = \sum_{h=0}^{\infty} \frac{(a)_h}{(c)_h h!} z^h,
$$

where $(b)_h$ denotes the Pochhammer symbol (rising factorial) and is defined by $(b)_h = 1$ for $h = 0$ and $(b)_h = b(b + 1) \cdots (b + h - 1)$ for $h \geq 1$.

**Lemma A1**: For any $z \in \mathbb{R}$, the following identity holds:

$$
z^3 M(z; 2, 4) = 6(e^{z}(z - 2) + z + 2). \tag{A4}
$$

**Proof**: In order to prove \((A4)\), we are going to consider the corresponding Taylor series expansions. Let us denote $f(z) \equiv 6(e^{z}(z - 2) + z + 2)$. Since

$$
f'(z) = 6(e^{z}(z - 1) + 1), \quad f''(z) = 6 e^{z},
$$

then, $f(0) = f'(0) = f''(0) = 0$.

Moreover, one can prove recursively that $f^{(2+h)}(z) = 6 e^{z}(z + h)$ for any $h \geq 0$. Therefore,

$$
f(z) = \sum_{h=0}^{\infty} \frac{f^{(h)}(0)}{h!} z^h = z^3 \sum_{h=0}^{\infty} \frac{f^{(h+3)}(0)}{(h+3)!} z^h = z^3 \sum_{h=0}^{\infty} \frac{6(h+1)}{(h+3)!} z^h.
$$

We remark that

$$
\frac{(2)_{h}}{(4)_{h} h!} \equiv \frac{(h+1)!}{(1/6)(h+3)! h!} = \frac{6(h+1)}{(h+3)!}
$$

for any $h \in \mathbb{N}$, because of $(2)_h = (h+1)!$ and $(4)_h = (1/6)(h+3)!$. Hence,

$$
f(z) = z^3 \sum_{h=0}^{\infty} \frac{6(h+1)}{(h+3)!} z^h = z^3 \sum_{h=0}^{\infty} \frac{(2)_{h}}{(4)_{h} h!} z^h = z^3 M(z; 2, 4),
$$

that is, we proved \((A4)\). \textbf{∎}

According to our previous remark, by \((A4)\) it follows that $6(e^{z}(z - 2) + z + 2)$ and $z + 2$ are a fundamental system of solutions for \((A3)\). For the sake of simplicity, we may consider $\{g_1, g_2\}$, where $g_1(z) \equiv e^{z}(z - 2)$ and $g_2(z) \equiv z + 2$ as a basis of the solution space for \((A2)\). We point out that $\{g_1, g_2\}$ is clearly a fundamental system of solutions as

$$
W(g_1, g_2) = g_1(z)g_2'(z) - g_2(z)g_1'(z) = -z^2 e^{z}.
$$

Thus, the pair of functions

$$
\tilde{V}_0(t, \lambda) \equiv -\frac{1}{2} e^{-\frac{z}{2}} g_1(z) = e^{z} \left( \frac{z}{2} + 1 \right) = e^{-\lambda \phi(t)} (\lambda \phi(t) + 1),
$$
\[
\tilde{V}_1(t, \lambda) = -\frac{1}{2} e^{-\tilde{z}} g_2(z) = e^{-\tilde{z}} \left( -\frac{z}{2} - 1 \right) = e^{\lambda \phi(t)} \left( \lambda \phi(t) - 1 \right)
\]
form a system of fundamental solutions to (A2).

Finally, we prove the representations (21) and (22) by using \{\tilde{V}_0, \tilde{V}_1\} as fundamental system of solutions to (A2).

**Proposition A.2:** Let \(y_0(t; s; \lambda, 2/3)\) and \(y_1(t; s; \lambda, 2/3)\) be the functions defined in (21) and (22), respectively. Then, \(y_0(t; s; \lambda, 2/3)\) and \(y_1(t; s; \lambda, 2/3)\) solve the Cauchy problem (A1) for \(j = 0\) and \(j = 1\), respectively.

**Proof:** We know that \(\tilde{V}_0, \tilde{V}_1\) form a system of independent solutions to (A2). Also, we can write the solutions \(y_j(t; s; \lambda)\), \(j = 0, 1\) of (A1) as linear combinations of \(\tilde{V}_0, \tilde{V}_1\) in the following way:

\[
y_j(t; s; \lambda) = a_j(s; \lambda) \tilde{V}_0(t; \lambda) + b_j(s; \lambda) \tilde{V}_1(t; \lambda)
\]
for suitable coefficients \(a_j(s; \lambda)\) and \(b_j(s; \lambda), j = 0, 1\).

The application of the initial conditions \(\partial_t^j y_j(s; s; \lambda) = \delta_j\) yields the system

\[
\begin{pmatrix}
\tilde{V}_0(s; \lambda) & \tilde{V}_1(s; \lambda) \\
\partial_t \tilde{V}_0(s; \lambda) & \partial_t \tilde{V}_1(s; \lambda)
\end{pmatrix}
\begin{pmatrix}
a_0(s; \lambda) & a_1(s; \lambda) \\
b_0(s; \lambda) & b_1(s; \lambda)
\end{pmatrix}
= I,
\]
where \(I\) denotes the identity matrix. Therefore,

\[
\begin{pmatrix}
a_0(s; \lambda) & a_1(s; \lambda) \\
b_0(s; \lambda) & b_1(s; \lambda)
\end{pmatrix}
= (W(\tilde{V}_0, \tilde{V}_1)(s; \lambda))^{-1}
\begin{pmatrix}
\partial_t \tilde{V}_0(s; \lambda) & -\tilde{V}_1(s; \lambda) \\
-\partial_t \tilde{V}_0(s; \lambda) & \tilde{V}_0(s; \lambda)
\end{pmatrix}.
\]

The Wronskian \(W(\tilde{V}_0, \tilde{V}_1)\) is given by

\[
W(\tilde{V}_0, \tilde{V}_1)(t; \lambda) = \tilde{V}_0(t; \lambda) \partial_t \tilde{V}_1(t; \lambda) - \tilde{V}_1(t; \lambda) \partial_t \tilde{V}_0(t; \lambda) = 2\lambda^3 (\phi(t))^2 \phi'(t) = 18\lambda^3,
\]
where we employed

\[
\partial_t \tilde{V}_0(t; \lambda) = -\lambda^2 \phi(t)\phi'(t) e^{-\lambda \phi(t)},
\]
\[
\partial_t \tilde{V}_1(t; \lambda) = \lambda^2 \phi(t)\phi'(t) e^{\lambda \phi(t)}.
\]

Plugging the previous representation of \(W(\tilde{V}_0, \tilde{V}_1)\) in (A6), we find

\[
\begin{pmatrix}
a_0(s; \lambda) & a_1(s; \lambda) \\
b_0(s; \lambda) & b_1(s; \lambda)
\end{pmatrix}
= \frac{1}{18\lambda^3}
\begin{pmatrix}
\partial_t \tilde{V}_1(s; \lambda) & -\tilde{V}_1(s; \lambda) \\
-\partial_t \tilde{V}_0(s; \lambda) & \tilde{V}_0(s; \lambda)
\end{pmatrix}.
\]

Let us begin by proving that \(y_0(t; s; \lambda) = y_0(t; s; \lambda, 2/3)\). Employing the above representation of the coefficients \(a_0(s; \lambda), b_0(s; \lambda)\) in (A5), we have

\[
y_0(t; s; \lambda) = (18\lambda^3)^{-1} [\partial_t \tilde{V}_1(s; \lambda) \tilde{V}_0(t; \lambda) - \partial_t \tilde{V}_0(s; \lambda) \tilde{V}_1(t; \lambda)]
= (18\lambda^3)^{-1} \lambda^2 \phi(s)\phi'(s) \left\{ e^{-\lambda(\phi(t) - \phi(s))} (\lambda \phi(t) + 1) + e^{\lambda(\phi(t) - \phi(s))} (\lambda \phi(t) - 1) \right\}
= 3^{-2} \phi(s) \phi'(s) (\phi(t) - \phi(s))
- 3^{-2} \lambda^{-1} \phi(s)\phi'(s) \sinh(\lambda(\phi(t) - \phi(s)))
= (t/s)^{1/3} \cosh(\lambda(\phi(t) - \phi(s))) - 1/(3\lambda s^{1/3}) \sinh(\lambda(\phi(t) - \phi(s)))
= y_0(t; s; \lambda, 2/3).
\]

Analogously, plugging the previously determined expressions for \(a_1(s; \lambda), b_1(s; \lambda)\) in (A5), we have

\[
y_1(t; s; \lambda) = (18\lambda^3)^{-1} [\tilde{V}_0(s; \lambda) \tilde{V}_1(t; \lambda) - \tilde{V}_1(s; \lambda) \tilde{V}_0(t; \lambda)]
= (18\lambda^3)^{-1} (\lambda \phi(s) + 1)(\lambda \phi(t) - 1) e^{\lambda(\phi(t) - \phi(s))}
- (18\lambda^3)^{-1} (\lambda \phi(s) - 1)(\lambda \phi(t) + 1) e^{\lambda(\phi(t) - \phi(s))}
= (18\lambda^3)^{-1} (\lambda^2 \phi(t)\phi(s) - 1) (e^{\lambda(\phi(t) - \phi(s))} - e^{-\lambda(\phi(t) - \phi(s)))}
\]
\[ + (18\lambda^3)^{-1}\lambda(\phi(t) - \phi(s))(e^{\lambda(\phi(t) - \phi(s))} + e^{-\lambda(\phi(t) - \phi(s))}) \]
\[ = ((st)^{1/3}/\lambda - 1/(9\lambda^3)) \sinh(\lambda(\phi(t) - \phi(s))) \]
\[ + (1/9\lambda^2)(\phi(t) - \phi(s)) \cosh(\lambda(\phi(t) - \phi(s))) = y_1(t, s; \lambda, 2/3). \]

The proof is complete.