SOLVING GAUGE-FIELD THEORY
BY DISCRETIZED LIGHT-CONE QUANTIZATION

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The canonical front form Hamiltonian for non-Abelian SU(N) gauge theory in 3+1 dimensions is mapped non-perturbatively on an effective Hamiltonian which acts only in the Fock space of a quark and an antiquark. The approach is based on the novel method of iterated resolvents and on discretized light-cone quantization, driven to the continuum limit. It is free of the usual Tamm-Dancoff truncations of the Fock space, rather the perturbative series are consistently resumed to all orders in the coupling constant. Emphasis is put on dealing with the many-body aspects of gauge field theory. The effective interaction turns out to be the kernel of an integral equation in the momentum space of a single quark, which is frame-independent and solvable on comparatively small computers. Important is that the higher Fock-space amplitudes can be retrieved self-consistently from these solutions.

1 The Hamiltonian matrix

In canonical field theory the four components of the energy-momentum vector \( P^\mu \) commute and are constants of the motion. In the front form of Hamiltonian dynamics they are denoted by \( P^\mu = (P^+, P_\perp, P^-) \). Its spatial components \( P_\perp \) and \( P^+ \) are independent of the interaction and diagonal in momentum representation. Their eigenvalues are the sums of the single particle momenta, \( P^+ = \sum p^+ = \frac{2\pi}{L} K \) and \( P_\perp = \sum p_\perp \). Each single particle has four-momentum \( p^\mu = (p^+, p_\perp, p^-) \) and sits on its mass-shell \( p^\mu p_\mu = m^2 \). Each particle state \( "q" \) is then characterized by six quantum numbers \( q = (p^+, p_\perp, \lambda, c, f) \): the three spatial momenta, helicity, color and flavor. The temporal component \( P^- = 2P_0 \) depends on the interaction and is a very complicated non-diagonal operator. It propagates the system in the light-cone time \( x^+ = x^0 + x^3 \). The contraction of \( P^\mu \) is the operator of invariant mass-squared,

\[
P^\mu P_\mu = P^+P^- - \vec{P}_\perp^2 \equiv H_{LC} \equiv H.
\]

It is Lorentz invariant and referred to somewhat improperly but conveniently as the ‘light-cone Hamiltonian’ \( H_{LC} \), or shortly \( H \). One seeks a representation in which \( H \) is diagonal \( H |\Psi\rangle = E |\Psi\rangle \). Introductory texts are available.

By convenience, the (light-cone) Hamiltonian is split into four terms: \( H = T + V + F + S \). The kinetic energy \( T \) survives the limit of the coupling constant \( g \)

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Figure 1: The Hamiltonian matrix for a SU(N)-meson and a harmonic resolution $K = 4$. Only vertex diagrams are included. Zero matrices are marked by (·).

\[\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
4 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
5 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
6 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
7 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
8 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
9 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
10 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
11 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
12 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
13 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{pmatrix}\]

Going to zero. Since it is diagonal in Fock-space representation, its eigenvalue is the free invariant mass squared of the particular Fock state. The vertex interaction $V$ is the relativistic interaction per se. It is linear in $g$ and changes the particle number by 1. Those matrix elements of $V$ which change the particle number by 3 are present in the instant form, but vanish for light-cone kinematics: The vacuum does not fluctuate. The instantaneous interactions $F$ and $S$ are consequences of working in the light-cone gauge $A^+ = 0$. They are proportional to $g^2$. By definition, the seagull interaction $S$ conserves the particle number and the fork interaction $F$ changes it by 2.

The eigenfunctions $|\Psi\rangle$ are complicated superpositions of Fock-space projections with contributions like $\langle q\bar{q}|\Psi\rangle$ or $\langle q\bar{q}g|\Psi\rangle$. The eigenvalue equation stands thus for an infinite set of coupled integral equations which are even difficult to write down. However, if one works with periodic boundary conditions, with discretized light-cone quantization (DLCQ), the infinite set of coupled integral equations is mapped onto a finite set of coupled matrix equations, as illustrated in Fig. 1. As consequence of discretization, the Fock states are denumerable and orthonormal. The momentum fraction carried by the $n$-th
particle is $x_n = p_n^+ / P^+$. The sum of all momentum fractions and the sum of all transversal momenta is constrained to $\sum_n x_n = 1$ and $\sum_n \vec{k}_{n_x} = 0$, with the intrinsic transversal momenta $\vec{k}_1$ defined by $\vec{p}_{n_x} = x_n \vec{P}_1 + \vec{k}_{n_x}$. Since $P^+$ has only positive eigenvalues and since each particle has a lowest possible value of $p^+$, the number of particles in a Fock state is limited for any fixed value of the harmonic resolution $\frac{4}{K}$.

Here then is the problem, the bottle neck of any Hamiltonian approach in field theory: The dimension of the Hamiltonian matrix increases exponentially fast, even if one regulates the transversal momenta by some suitable cut-off $\Lambda$, like the invariant-mass cut-off of Lepage and Brodsky or similar constructs. Suppose, the regularization procedure allows for 10 discrete momentum states in each direction. A single particle has then about $10^3$ degrees of freedom. A Fock-space sector with $n$ particles has then roughly $10^{n-1}$ different Fock states. Sector 13 with its 8 particles has thus about $10^{21}$ and sector 1 ($q\bar{q}$) about $10^3$ Fock states. Deriving an effective interaction can be understood as reducing the dimension in a matrix diagonalization problem from $10^{21}$ to $10^3$!

2 The effective interaction

Effective interactions are a well known tool in many-body physics. In field theory the method is known as the Tamm-Dancoff-approach. The Fock-space sectors $|i\rangle$ appear in a most natural way and the Hamiltonian matrix can be understood as a matrix of block matrices, whose rows and columns are enumerated by $i = 1, 2, \ldots N$ like in Fig. 1. The eigenvalue equation can then be written as a coupled block matrix equation:

$$\sum_{j=1}^{N} \langle i| H |j \rangle \langle i| \Psi \rangle = E \langle n| \Psi \rangle \quad \text{for all } i = 1, 2, \ldots, N . \quad (2)$$

The rows and columns can always be split into the $P$-space and the rest, the $Q$-space. The block matrix equation becomes

$$\langle P| H |P \rangle \langle P| \Psi \rangle + \langle P| H |Q \rangle \langle Q| \Psi \rangle = E \langle P| \Psi \rangle , \quad (3)$$

and

$$\langle Q| H |P \rangle \langle P| \Psi \rangle + \langle Q| H |Q \rangle \langle Q| \Psi \rangle = E \langle Q| \Psi \rangle . \quad (4)$$

Rewrite the second equation as $\langle Q| E - H |Q \rangle \langle Q| \Psi \rangle = \langle Q| H |P \rangle \langle P| \Psi \rangle$, and observe that the quadratic matrix $\langle Q| E - H |Q \rangle$ could be inverted to express the $Q$-space wave-function $\langle Q| \Psi \rangle$ in terms of $\langle P| \Psi \rangle$. The eigenvalue $E$ is unknown at this point. One introduces therefore a redundant parameter $\omega$, and defines $G_Q(\omega) = [\langle Q| \omega - H |Q \rangle]^{-1}$. The so obtained effective interaction

$$\langle P| H_{\text{eff}}(\omega) |P \rangle = \langle P| H |P \rangle + \langle P| H |Q \rangle G_Q(\omega) \langle Q| H |P \rangle . \quad (5)$$
acts only in the $P$-space: $H_{\text{eff}}(\omega)\ket{\Phi_k(\omega)} = E_k(\omega)\ket{\Phi_k(\omega)}$. Varying $\omega$ one generates a set of energy functions $E_k(\omega)$. Whenever one finds a solution to the fix-point equation $E_k(\omega) = \omega$, one has found one of the true eigenvalues and eigenfunctions. The advantage of working with an effective interaction is of analytical nature to the extent that resolvents can be approximated systematically. The two resolvents

$$G_Q(\omega) = \frac{1}{\langle Q|\omega - T - U|Q \rangle} \quad \text{and} \quad G_0(\omega) = \frac{1}{\langle Q|\omega - T|Q \rangle},$$

defined once with and once without the non-diagonal interaction in the Hamiltonian $H = T + U$, respectively, are identically related by $G_Q(\omega) = G_0(\omega) + G_0(\omega) U G_Q(\omega)$, or by the infinite series of perturbation theory $G_Q(\omega) = G_0(\omega) + G_0(\omega) U G_0(\omega) + G_0(\omega) U G_0(\omega) U G_0(\omega) + \ldots$. The point is, of course, that the kinetic energy $T$ is a diagonal operator which can be trivially inverted to get the unperturbed resolvent $G_0(\omega)$.

Albeit exact in principle, the Tamm-Dancoff-approach (TDA) suffers in practice from two serious defects: (1) The approach is technically useful only, if one truncates the perturbative series to its very first term. This destroys Lorentz and gauge invariance. (2) If one identifies $\omega$ with the eigenvalue, as one should, the effective interaction has a non-integrable singularity, both in the original instant form\(^6\),\(^7\) and in the front form\(^8\),\(^9\).

The above can be understood as to reduce the block matrix dimension from 2 to 1. But there is no need to identify the $P$-space with the lowest sector. One also can choose the $Q$-space identical with last sector and the $P$-space with the rest, $P = 1 - Q$: The same steps as above reduce then the block matrix dimension from $N$ to $N - 1$. The effective interaction acts in the now smaller space. This procedure can of course be iterated, but then one has to deal with ‘resolvents of resolvents’, or with iterated resolvents\(^{10}\). Ultimately, one arrives at block matrix dimension 1 where the procedure stops: The effective interaction in the Fock-space sector with only one quark and one antiquark is defined unambiguously.

Suppose, in the course of this reduction, one has arrived at block matrix dimension $n$, with $1 \leq n \leq N$. Denote the corresponding effective interaction $H_n(\omega)$. The eigenvalue problem reads then

$$\sum_{j=1}^{n} \langle i|H_n(\omega)|j \rangle \langle j|\Psi(\omega) \rangle = E(\omega) \langle i|\Psi(\omega) \rangle, \quad \text{for } i = 1, 2, \ldots, n.$$  \(7\)

Observe that $i$ and $j$ refer here to sector numbers. Now, like in the above,
define the resolvent of the sector Hamiltonian $H_n(\omega)$ by

$$G_n(\omega) = \frac{1}{\langle n|\omega - H_n(\omega)|n \rangle}$$

thus

$$\langle n|\Psi(\omega)\rangle = G_n(\omega) \sum_{j=1}^{n-1} \langle n|H_n(\omega)|j\rangle \langle j|\Psi(\omega)\rangle.$$  

The effective interaction in the $(n-1)$-space becomes then

$$H_{n-1}(\omega) = H_n(\omega) + H_n(\omega)G_n(\omega)H_n(\omega)$$

for every block matrix element $\langle i|H_{n-1}(\omega)|j\rangle$. Everything proceeds like in above, including the fixed point equation $E(\omega) = \omega$. But one has achieved much more: Eq.(10) is a recursion relation which holds for all $1 < n < N$! Since one has started from the bare Hamiltonian in the last sector, one has to convene that $H_N = H$. The rest is algebra and interpretation.

Applying the method to the block matrix structure of QCD, as displayed in Fig. 1, is particularly easy and transparent. By definition, the last sector contains only the diagonal kinetic energy, thus $H_{13} = T_{13}$. Its resolvent is calculated trivially. Then $H_{12}$ can be constructed unambiguously, followed by $H_{11}$, and so on, until one arrives at sector 1. Grouping the so obtained results in a different order, one finds for the sectors with one $q\bar{q}$-pair:

$$H_{q\bar{q}} = H_1 = T_1 + VG_3V + VG_3VG_2V G_3V,$$  

$$H_{q\bar{q}g} = H_3 = T_3 + VG_6V + VG_6VG_5V G_6V + VG_4V,$$  

$$H_{q\bar{q}gg} = H_6 = T_6 + VG_{10}V + VG_{10}VG_9V G_{10}V + VG_7V.$$

The quark-gluon content of the respective sectors is added here for an easier identification. Correspondingly, one obtains for the sectors with two $q\bar{q}$-pairs

$$H_{q\bar{q}q\bar{q}} = H_4 = T_4 + VG_7V + VG_7VG_6VG_7V,$$  

$$H_{q\bar{q}q\bar{q}g} = H_7 = T_7 + VG_{11}V + VG_{11}VG_{10}V G_{11}V + VG_9V.$$

In the pure glue sectors, the structure is even simpler:

$$H_{gg} = H_2 = T_2 + VG_5V + VG_5V,$$  

$$H_{ggg} = H_5 = T_5 + VG_6V + VG_9V.$$

Note that the above relations are exact. They hold for an arbitrarily large $K$ and thus in the continuum limit. Note also that the vertex interaction appears only in even pairs, typically in the combination $VGV$. This is the deeper
reason why the instantaneous interactions can be implemented \( \textit{ex post} \). The effective Hamiltonian \( H_{\text{eff}} = H_{\bar{q}q} \) as given by Eq.\((11)\) is illustrated in Fig.\(2\).

As a net result the interaction scatters a quark with helicity \( \lambda_q \) and four-momentum \( p = (xP^+, x\vec{P}_\perp + \vec{k}_\perp, p^-) \) into a state with \( \lambda'_q \) and four-momentum \( p' = (x'P^+, x'\vec{P}_\perp + \vec{k}'_\perp, p'^-) \).

In the continuum limit, the resolvents are replaced by propagators and the eigenvalue problem \( H_{\text{eff}} \langle \psi_b \rangle = M^2 \langle \psi_b \rangle \) becomes an integral equation

\[
M^2_b \langle x, \vec{k}_\perp; \lambda_q, \lambda_{\bar{q}} | \psi_b \rangle = \left[ \frac{\vec{P}^2}{x} + \frac{\vec{k}_\perp^2}{1-x} \right] \langle x, \vec{k}_\perp; \lambda_q, \lambda_{\bar{q}} | \psi_b \rangle + \sum_{\lambda'_{q}, \lambda'_{\bar{q}}} \int_D dx' d^2\vec{k}'_\perp \langle x, \vec{k}'_\perp; \lambda_q, \lambda_{\bar{q}} | U + U_{a}| x', \vec{k}'_\perp; \lambda'_{q}, \lambda'_{\bar{q}} \rangle \langle x', \vec{k}'_\perp; \lambda'_{q}, \lambda'_{\bar{q}} | \psi_b \rangle
\] (18)

The domain \( D \) restricts integration in line with regularization. The effective potential \( U \) is diagrammatically defined in diagram \( U_{1,1} \) of Fig.\(2\) and cannot change the flavor of the quark. It includes all fine and hyperfine interactions. Diagram \( U_{1,2} \) contributes to the effective quark mass \( \tilde{m} \). The flavor-changing annihilation interaction \( U_{a} = U_{1,0} \) is probably of less importance in a first assault. The eigenvalues \( M^2_b \) are the invariant mass\(^2 \) of a physical particle and the corresponding wave-function \( \langle x, \vec{k}_\perp; \lambda_q, \lambda_{\bar{q}} | \psi_b \rangle \) gives the probability amplitudes for finding a flavored quark with momentum fraction \( x \), intrinsic transverse momentum \( \vec{k}_\perp \) and helicity \( \lambda_q \). These are boost-invariant quantities. The eigenfunctions represent the normalized projections of the full eigenfunction \( |\Psi\rangle \) onto the Fock states \( |q; \bar{q}\rangle = b_q^\dagger d_{\bar{q}}^\dagger |\text{vac}\rangle \).

The knowledge of \( \psi_b \) is sufficient to retrieve all desired Fock-space components of the total wave-function. The key is the upwards recursion relation Eq.\((9)\). Obviously, one can express the higher Fock-space components \( \langle n | \Psi \rangle \) as functionals of \( \psi_{q\bar{q}} \) by a finite series of quadratures, \textit{i.e.} matrix multiplications or of momentum-space integrations. One need not solve another
eigenvalue problem. I show this by calculating the probability amplitude for a \( |gg\rangle \)- or a \( |q\bar{q} g\rangle \)-state in an eigenstate of the full Hamiltonian. The first two equations of the recursive set in Eq.(9) are
\[
\langle 2|\Psi \rangle = G_2 \langle 2|H_2|1\rangle \langle 1|\Psi \rangle , \quad (19)
\]
and
\[
\langle 3|\Psi \rangle = G_3 \langle 3|H_3|1\rangle \langle 1|\Psi \rangle + G_3 \langle 3|H_3|2\rangle \langle 2|\Psi \rangle . \quad (20)
\]
The sector Hamiltonians \( H_n \) have to be substituted from Eqs.(12) and (16). In taking block matrix elements of them, the formal expressions are simplified considerably since many of the Hamiltonian blocks in Fig. 1 are zero. One thus gets simply \( \langle 2|H_2|1\rangle = \langle 2|VG_3V|1\rangle \) and therefore \( \langle 2|\Psi \rangle = G_2 VG_3V \langle 1|\Psi \rangle \). Substituting this into Eq.(19) gives \( \langle 3|\Psi \rangle = G_3 V \langle 1|\Psi \rangle + G_3 VG_2 VG_3V \langle 1|\Psi \rangle \). These findings can be summarized more succinctly as
\[
|\psi gg\rangle = G_{gg} VG_{q\bar{q}} \langle 1|\Psi \rangle , \quad (21)
\]
and
\[
|\psi q\bar{q} g\rangle = G_{q\bar{q} g} V \langle 1|\Psi \rangle + G_{q\bar{q} g} VG_{gg} VG_{q\bar{q} g} V \langle 1|\Psi \rangle . \quad (22)
\]
Note that the above relations are exact. The finite number of terms is in strong contrast to the infinite number of terms in perturbative series. Iterated resolvents sum the perturbative series to all orders in closed form.

3 The mean field and the vertex functions

Let us discuss in some greater detail the structure of the sector Hamiltonians, particularly of \( H_{q\bar{q} g} \) in Eq.(12). The corresponding graphs are displayed diagrammatically in Figs. 3 and 4. Those in Fig. 3 differ from those in Fig. 2 only by an additional gluon. The gluon does not change quantum numbers under impact of the interaction and acts like a spectator. The graphs in Fig. 3 will be referred to as the ‘spectator interaction’ \( \mathcal{U}_3 \). Correspondingly, the graphs of Fig. 4 will be referred to as the ‘participant interaction’ \( \mathcal{U}_3 \). The gluon quantum numbers are changed and the gluon participates in the interaction. The analogous separation into spectator and participant interactions can be made in all quark-pair-gluon sector Hamiltonians:
\[
H_n = T_n + \mathcal{U}_n + \mathcal{U}_n , \quad \text{for} \quad n = 3, 6, 10, 15, \ldots . \quad (23)
\]
Note that the spectator interaction has the same diagrams as in Fig. 2, except for the additional free and non-interacting gluons.

The essence is that the spectator interactions \( \mathcal{U}_n \) define mean fields, and that they can be associated with their own resolvents \( \mathcal{G}_n \):
\[
\mathcal{G}_n = \frac{1}{\omega - T_n - \mathcal{U}_n} , \quad (24)
\]
while $G_n$ was

$$G_n = \frac{1}{\omega - T_n - U_n - \bar{U}_n}. \tag{25}$$

Both are related by $G_n = \overline{G}_n + \overline{G}_n \bar{U}_n G_n$. Equivalently, they can be written as an infinite series $G_n = \overline{G}_n + \overline{G}_n \bar{U}_n \overline{G}_n \bar{U}_n \overline{G}_n \bar{U}_n \overline{G}_n \bar{U}_n \cdots$, as usual. The main difference to the above conventional series is, that there the ‘unperturbed propagator’ $G_0(\omega)$ refers to the system without interactions while here the ‘unperturbed propagators’ $\overline{G}_n$ include the mean fields $\bar{U}_n$. One therefore deals here with ‘perturbation theory in medium’. Note that the present series differ from the above also with respect to the physics: The system stays in sector $n$. This allows for the identical rearrangement

$$G_n = R_n \overline{G}_n R_n, \tag{26}$$

with

$$R_n = 1 + \frac{1}{2} \overline{G}_n \bar{U}_n + \frac{3}{8} \overline{G}_n \bar{U}_n \overline{G}_n \bar{U}_n + \cdots = \frac{1}{\sqrt{1 - G_n \bar{U}_n}}. \tag{27}$$

This can be verified order by order and has, to our recollection, not been given before. The square matrices $R_n$ are always sandwiched between a quark-pair-glue resolvent $\overline{G}$ and the vertex $V$. It is therefore meaningful to introduce the effective vertex $\overline{V} = R_n V$ or $\overline{V} = R_n V$, thus

$$V G_n V = \overline{V} \overline{G}_n \overline{V}. \tag{28}$$
The vertex operators $R_n$ come with same power as the coupling constant. They are a genuine part of the interaction and accumulate the many-body aspects manifesting themselves as non-point-like vertices. Finally, rewriting systematically Eqs. (11)-(13), one gets

$$H_6 = T_6 + V G_{10} V G_9 V G_{10} V ,$$
(29)

$$H_3 = T_3 + V G_6 V G_5 V G_6 V ,$$
(30)

$$H_1 = T_1 + V G_3 V G_2 V G_3 V .$$
(31)

Instead of being similar, the quark-pair-glue sector Hamiltonians $H_n = T_n + U_n$ now apparently all look the same. Note that the resolvents $G_2$, $G_5$ and $G_9$ carry no bar. They correspond to the pure glue sectors. The distinction between participants and spectators makes no sense there.

4 The approximations

Thus far the approach is formally exact. For calculating the effective interaction the two resolvents $G_3$ and $G_2$ and the vertex function $R_3$ are needed. How can they be approximated? Suppose to have solved the eigenvalue problem in the $q\bar{q}$-space,

$$\sum_{q',\bar{q}'} \langle q; \bar{q} | H_{q\bar{q}}(\omega) | q'; \bar{q}' \rangle \psi_b(\omega) \rangle = M^2_b(\omega) \langle q; \bar{q} | \psi_b(\omega) \rangle .$$
(32)

The eigenvalues are enumerated by $b = 1, 2, \ldots$. The corresponding eigenfunctions $\langle q; \bar{q} | \psi_b(\omega) \rangle$ are a complete set. Despite working in the continuum limit, we continue to use summation symbols for the sake of a more compact notation. Suppose further that an $\omega$ was found which has the same value as the lowest eigenvalue $M^2 = M^2_1$. The substitution $\omega = M^2$ will hence forward be done without explicitly mentioning. Next, ask for the eigenvalues and eigenfunctions in the $q\bar{q}g$-space. One need not to solve another eigenvalue problem, since the result is known ahead of time! By construction, the gluon moves relative to the meson subject to momentum conservation. The eigenfunction is a thus a product state $|\psi_{b,s}⟩ = |\psi_b⟩ \otimes |\phi_s⟩$. Parameterizing the gluons four-momentum as $p_\mu_g = (y P^+, y \vec{P}_\perp + \vec{q}_\perp, p_g^-)\rangle$, the eigenvalues are

$$M^2_{b,s} = \frac{M^2_b + \vec{q}_\perp^2}{1 - y} + \frac{\vec{q}_\perp^2}{y} .$$
(33)

Knowing the eigenvalues and eigenfunctions, one can calculate the exact resolvent. After a few identical rewritings one gets

$$\langle q; \bar{q} | G_3 | q'; \bar{q}' \rangle = G_3(q; \bar{q} | g [q; \bar{q}; g] [q; \bar{q}; g | q'; \bar{q}' \rangle - \langle q; \bar{q}; g | A | q'; \bar{q}; g' \rangle .$$
(34)
The operator $A$ can never become a Dirac-$\delta$ function $\langle q; \bar{q} ; g | q' ; \bar{q}' ; g' \rangle$, since

$$
A = \sum_{b,s} \langle \psi_{b,s} | \frac{y(M_b^2 - M^2)}{Q^2 + y(M_b^2 - M^2)} | \psi_{b,s} \rangle.
$$

(35)

It is therefore dropped, as an approximation. The proportionality coefficient is

$$
\mathcal{G}_3(q; \bar{q}; g) = -\frac{y(1 - y)}{Q^2}, \quad \text{with} \quad Q^2 = y^2 M^2 + \vec{q}^2.
$$

(36)

The same approximation yields for the two gluon propagator

$$
\tilde{G}_2(g_1; g_2) = \frac{1}{M^2 - M_{gb}^2}.
$$

(37)

It is parameterized in terms of the glue ball mass $M_{gb}$.

For calculating the operator $R_3^2$, one addresses first to calculate the operator $B := \mathcal{G}_3(V \mathcal{G}_4 V + V \mathcal{G}_6 V)$. An inspection of Fig. 3 yields that the loop diagrams (34) and (36) are the leading terms, by far. They have been calculated long ago by Thorn for $n_c = 3$ colors and massless quarks, using the perturbative propagators. Repeating this calculations with the non-perturbative propagators $\mathcal{G}_4$ and $\mathcal{G}_6$, which can be approximated in line with Eq.(34), one gets a diagonal $B$ with

$$
B = -\frac{1}{4} \alpha_s c_0 \ln \left[ 1 + \frac{1}{(1 - y) Q^2} \right],
$$

(38)

with $\alpha_s = g^2/4\pi$ and $c_0 \equiv (33 - 2N_f)/6\pi$. The latter number is not unfamiliar from the work of Gross and Wilczek. Details will be given elsewhere. Applying the same procedure to even higher spaces gives again and again the same type of vacuum polarization diagrams, and thus a diagonal

$$
R_3^2 = \frac{1}{1 + \frac{B}{1 + \frac{B}{1 + \ldots}}}.
$$

(39)

This continued fraction can be resumed to all orders by $R_3^2 = 1/(1 + BR_3^2)$ and solved

$$
R_3^2 = \frac{2}{1 + \sqrt{1 + \alpha_s c_0 \ln \left[ 1 + \frac{1}{(1 - y) Q^2} \right]}}.
$$

(40)

Now, one can calculate analytically the effective interaction!
5 Outlook

The strength of the interaction \( R_3 \) tends to zero for increasing cut-off \( \Lambda \rightarrow \infty \). One may (or may not) interpret this result as asymptotic freedom. But this would be premature. The above is only the regulated effective interaction, albeit being well defined for any fixed value of the cut-off \( \Lambda \). The dependence on \( \Lambda \) must be removed by a future renormalization group analysis. It looks as if the present explicit form is almost ideally suited for an application of the Hamiltonian flow equations. Finally, one should emphasize that the above approach is based on comparatively weak approximations being thoroughly non-perturbative: No smallness assumption on the coupling constant was ever needed, nor was the Fock space ever truncated. Gauge invariance is maintained, and Lorentz covariance (including rotations) is strictly observed in the continuum limit.

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