Quantum alpha-determinants and $q$-deformed hypergeometric polynomials

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Abstract

The quantum $\alpha$-determinant is defined as a parametric deformation of the quantum determinant. We investigate the cyclic $\mathcal{U}_q(\mathfrak{sl}_2)$-submodules of the quantum matrix algebra $A_q(\text{Mat}_2)$ generated by the powers of the quantum $\alpha$-determinant. For such a cyclic module, there exists a collection of polynomials which describe the irreducible decomposition of it in the following manner: (i) each polynomial corresponds to a certain irreducible $\mathcal{U}_q(\mathfrak{sl}_2)$-module, (ii) the cyclic module contains an irreducible submodule if the parameter is a root of the corresponding polynomial. These polynomials are given as a $q$-deformation of the hypergeometric polynomials. This is a quantum analogue of the result obtained in our previous work [K. Kimoto, S. Matsumoto and M. Wakayama, Alpha-determinant cyclic modules and Jacobi polynomials, to appear in Trans. Amer. Math. Soc.].

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1 Introduction

The $\alpha$-determinant is a common generalization of the determinant and permanent defined by

$$\det^{(\alpha)}(X) = \sum_{\sigma \in S_n} \alpha^{\nu(\sigma)} x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(n)} n,$$

where $\alpha$ is a complex parameter and $\nu(\sigma) = n - (m_1 + m_2 + \cdots + m_n)$ if the cycle type of $\sigma \in S_n$ is $1^{m_1} 2^{m_2} \cdots n^{m_n}$ [10]. By definition, the $\alpha$-determinant $\det^{(\alpha)}(X)$ agrees with the determinant $\det(X)$ when $\alpha = -1$ and with the permanent $\text{per}(X)$ when $\alpha = 1$. In other words, the $\alpha$-determinant interpolates these two.

We recall an invariant property of the determinant and permanent in the following representation-theoretical context. Let $A(\text{Mat}_n)$ be the $\mathbb{C}$-algebra of polynomials in the $n^2$ commuting variables $\{x_{ij}\}_{1 \leq i,j \leq n}$, and $\mathcal{U}(\mathfrak{gl}_n)$ the universal enveloping algebra of the Lie algebra $\mathfrak{gl}_n = \mathfrak{gl}_n(\mathbb{C})$. By defining

$$E_{ij} \cdot f = \sum_{r=1}^{n} x_{ir} \frac{\partial f}{\partial x_{jr}} \quad (f \in A(\text{Mat}_n)),$$

$A(\text{Mat}_n)$ becomes a $\mathcal{U}(\mathfrak{gl}_n)$-module. Here $\{E_{ij}\}_{1 \leq i,j \leq n}$ is the standard basis of $\mathfrak{gl}_n$. Then, both of the determinant $\det(X)$ and the permanent $\text{per}(X)$ generate irreducible $\mathcal{U}(\mathfrak{gl}_n)$-submodules of $A(\text{Mat}_n)$. In fact, the cyclic submodules $\mathcal{U}(\mathfrak{gl}_n) \cdot \det(X)$ and $\mathcal{U}(\mathfrak{gl}_n) \cdot \text{per}(X)$ are equivalent to the skew-symmetric tensor product $\wedge^n(\mathbb{C}^n)$ and symmetric tensor product $\text{Sym}^n(\mathbb{C}^n)$ of the natural representation $\mathbb{C}^n$ respectively, which are irreducible.

In view of this fact, it is natural and interesting to study the irreducible decomposition of the cyclic submodule $\mathcal{U}(\mathfrak{gl}_n) \cdot \det^{(\alpha)}(X)$, or more generally $\mathcal{U}(\mathfrak{gl}_n) \cdot \det^{(\alpha)}(X)^m$. Matsumoto and Wakayama tackled this problem first and obtained explicit irreducible decomposition of $\mathcal{U}(\mathfrak{gl}_n) \cdot \det^{(\alpha)}(X)$, and recently Matsumoto, Wakayama...
and the author investigated the general case \( \mathcal{U}(\mathfrak{gl}_n) \cdot \det^{(\alpha)}(X)^m \) [3]: It is proved that

\[
\mathcal{U}(\mathfrak{gl}_n) \cdot \det^{(\alpha)}(X)^m \cong \bigoplus_{\lambda \vdash mn} \mathcal{M}_n^\lambda \cong \text{rk} F_{n,m}^\lambda(\alpha)
\]

holds for certain square matrices \( F_{n,m}^\lambda(\alpha) \) whose entries are polynomials in \( \alpha \). In this direct sum, \( \lambda \) runs over the partitions of \( mn \) whose length is at most \( n \). Here we identify the dominant integral weights and partitions, and denote by \( \mathcal{M}_n^\lambda \) the irreducible representation of \( \mathcal{U}(\mathfrak{gl}_n) \) with highest weight \( \lambda \). Remark that the matrices \( F_{n,m}^\lambda(\alpha) \) are determined up to conjugacy and non-zero scalar factor. In the particular case where \( m = 1 \), we explicitly have \( F_{n,1}^\lambda(\alpha) = f_\lambda(\alpha)I \), where \( I \) is the identity matrix and \( f_\lambda(\alpha) \) is the (modified) content polynomial for \( \lambda \) [7]. It seems quite difficult to obtain an explicit expression for \( F_{n,m}^\lambda(\alpha) \) in general. However, when \( n = 2 \), all the matrices \( F_{2,m}^\lambda(\alpha) \) are \textit{one by one}, and they are explicitly given by

\[
F_{2,m}^{(m+s,m-s)}(\alpha) = (1 + \alpha)^s 2F_1 \left( \begin{array}{c} s - m, s + 1 \\ -m \end{array} ; -\alpha \right) \quad (s = 0, 1, \ldots, m),
\]

where \( 2F_1(a; b; c; x) \) is the \textit{Gaussian hypergeometric function} [3].

These problems can be also formulated in the framework of \textit{quantum groups}. Namely, we define a quantum counterpart of the \( \alpha \)-determinant, which we call \textit{quantum \( \alpha \)-determinant}, by

\[
\det_q^{(\alpha)} = \sum_{\sigma \in \mathfrak{S}_n} \alpha^{\ell(\sigma)} q^{\ell(\sigma)x_{\sigma(1)}x_{\sigma(2)} \cdots x_{\sigma(n)}}
\]

in the \textit{quantum matrix algebra} \( \mathcal{A}_q(\text{Mat}_n) \) [9]. Here \( \ell(\sigma) \) denotes the inversion number of a permutation \( \sigma \). This agrees with the quantum determinant when \( \alpha = -1 \). We then introduce a \( \mathcal{U}_q(\mathfrak{gl}_n) \)-module structure on it, where \( \mathcal{U}_q(\mathfrak{gl}_n) \) is the quantum enveloping algebra of \( \mathfrak{gl}_n \) [1], and consider the cyclic module \( \mathcal{U}_q(\mathfrak{gl}_n) \cdot (\det_q^{(\alpha)})^m \). In [3], we study the case where \( m = 1 \). In contrast to the classical case, the structure of the cyclic module is much more complicated, so that we have only obtained several less explicit results.

Nevertheless, we can establish a quantum version of the result [1.1] \textit{rather explicitly}, and this is the aim of the present article. We investigate the cyclic \( \mathcal{U}_q(\mathfrak{sl}_2) \)-submodule (instead of \( \mathcal{U}_q(\mathfrak{gl}_2) \)-submodule just for simplicity of the description) of \( \mathcal{A}_q(\text{Mat}_2) \) defined by

\[
V_q^m(\alpha) = \mathcal{U}_q(\mathfrak{sl}_2) \cdot (\det_q^{(\alpha)})^m.
\]

We prove that there exists a collection of polynomials \( F_{m,j}(\alpha) \) \( (j = 0, 1, \ldots, m) \) such that

\[
V_q^m(\alpha) \cong \bigoplus_{0 \leq j \leq m} \mathcal{M}_q(2j + 1),
\]

where \( \mathcal{M}_q(d) \) is the \( d \)-dimensional irreducible representation of \( \mathcal{U}_q(\mathfrak{sl}_2) \) (Theorem [3.3]), and show that the polynomials \( F_{m,j}(\alpha) \) are written in terms of a certain \( q \)-deformation of the hypergeometric polynomials (Theorem [3.4]). Taking a limit \( q \to 1 \), we also obtain the formula [1.1] again (Corollary [3.7]).

2 Preliminaries

We first fix the convention on quantum groups. We basically follow to [1, 8] and [9], but modify slightly.

Let \( q \) be an indeterminate. We always discuss over the rational function field \( \mathbb{C}(q) \). The quantum enveloping algebra \( \mathcal{U}_q(\mathfrak{sl}_2) \) is an associative algebra generated by \( k, k^{-1}, e, f \) with the fundamental relations

\[
kk^{-1} = k^{-1}k = 1, \quad kek^{-1} = q^2 e, \quad kfk^{-1} = q^{-2} f, \quad ef - fe = \frac{k - k^{-1}}{q - q^{-1}}.
\]
We recall two well-known identities involved with $q$-deformed hypergeometric polynomials:

The quantum matrix algebra $A_q$ is an associative algebra generated by $x_{11}, x_{12}, x_{21}, x_{22}$ with the fundamental relations

\begin{align*}
x_{11}x_{12} &= qx_{12}x_{11}, & x_{21}x_{22} &= qx_{22}x_{21}, \\
x_{11}x_{21} &= qx_{21}x_{11}, & x_{12}x_{22} &= qx_{22}x_{12}, \\
x_{12}x_{21} &= x_{21}x_{12}, & x_{11}x_{22} - x_{22}x_{11} &= (q - q^{-1})x_{12}x_{21}.
\end{align*}

(2.1)

For convenience, we put

\[ z_1 = x_{11}x_{22}, \quad z_2 = x_{12}x_{21}. \]

(2.2)

The point is that they **commute**:

\[ z_1z_2 = z_2z_1. \]

The quantum $\alpha$-determinant of size two is then

\[ \det_q^{(\alpha)} = x_{11}x_{22} + \alpha qx_{12}x_{21} = z_1 + \alpha z_2. \]

(2.3)

**Remark 2.1.** The quantum $\alpha$-determinant of size two interpolates the quantum counterparts of the determinant and permanent:

\[ \det_q = x_{11}x_{22} - qx_{12}x_{21} = \det_q^{(-1)}, \quad \text{per}_q = x_{11}x_{22} + q^{-1}x_{12}x_{21} = \det_q^{(q^{-2})}. \]

However, the quantum $\alpha$-determinant of size $n$ does not coincide with the quantum permanent for any $\alpha$ if $n \geq 3$. This is because $\nu(\cdot)$ is a class function on $\mathfrak{S}_n$ in general, whereas the inversion number $\ell(\cdot)$ is not if $n \geq 3$.

The algebra $A_q$ becomes a $U_q(sl_2)$-module by

\begin{align*}
k_{1,1}x_{11} &= q^{\pm 1}x_{11}, & e \cdot x_{11} &= 0, & f \cdot x_{11} &= x_{12}, \\
k_{1,2}x_{12} &= q^{\pm 1}x_{12}, & e \cdot x_{12} &= x_{11}, & f \cdot x_{12} &= 0 \quad (i = 1, 2).
\end{align*}

(2.4)

These are compatible with the fundamental relations (2.1) above. Our main object is the cyclic submodule of $A_q$ given by

\[ V_q^{m}(\alpha) = U_q(sl_2) \cdot \left( \det_q^{(\alpha)} \right)^m. \]

(2.5)

We denote by $M_q(d)$ the $d$-dimensional irreducible representation of $U_q(sl_2)$. Notice that

\[ U_q(sl_2) \cdot (x_{11}x_{21})^s \det_q^{m-s} \cong U_q(sl_2) \cdot (x_{11}x_{21})^s \cong M_q(2s+1). \]

(2.6)

Define $q$-analogues of numbers, factorials and binomial coefficients by

\[ [n]_q := \frac{q^n - q^{-n}}{q - q^{-1}}, \quad [n]_q! := \prod_{i=1}^{n} [i]_q, \quad [n]_q! := \frac{[n]!}{[k]! [n-k]!}. \]

We recall two well-known identities involved with $q$-binomial coefficients which we will use later.

- **$q$-binomial theorem:**

\[ \prod_{i=1}^{n} (x + yq^{2i}) = \sum_{r=0}^{n} q^{(n-r)(r+1)} \binom{n}{r}_q x^r y^{n-r}. \]

(2.7)

- **$q$-Chu-Vandermonde formula:**

\[ \sum_{r=0}^{n} q^{-r(x+y)} \binom{x}{r}_q \binom{y}{r}_q = q^{-ny} \binom{x+y}{n}_q. \]

(2.8)
3 Cyclic modules generated by the quantum alpha-determinant

3.1 Some lemmas

Lemma 3.1.

\[ f^j \cdot (x_{11}x_{21})^j = q^{-j(j-1)/2} |j|_q^j \sum_{r=0}^{j} q^{-r^2} [j^r]_q^2 x_{11}^{j-r} x_{22}^{j-r} (x_{12}x_{21})^r. \]  \hspace{1cm} (3.1)

Proof. For \( 1 \leq i \leq 2j \), put

\[ f_j(i) = 1 \otimes \cdots \otimes 1 \otimes f \otimes k^{-1} \otimes \cdots \otimes k^{-1} \in \mathcal{U}_q(\mathfrak{sl}_2)^{\otimes 2j}. \]

Then

\[ \Delta^{2j-1}(f) = \sum_{i=1}^{2j} f_j(i), \]

so that

\[ \Delta^{2j-1}(f)^j = \sum_{1 \leq n_1, \ldots, n_j \leq 2j} f_j(n_1) \cdots f_j(n_j). \]

Since \( f_j(m)f_j(n) = q^{-2}f_j(n)f_j(m) \) if \( m > n \) and \( f^2 \cdot x_{11} = f^2 \cdot x_{21} = 0 \), we have

\[ \Delta^{2j-1}(f)^j = q^{-j(j-1)/2} |j|_q^j \sum_{1 \leq n_1, \ldots, n_j \leq 2j} f_j(n_1) \cdots f_j(n_j) + R, \]

where \( R \) is a certain element in \( \mathcal{U}_q(\mathfrak{sl}_2)^{\otimes 2j} \) such that \( R \cdot (x_{11}x_{21})^j = 0 \). Here we also use the well-known identity

\[ \sum_{\sigma \in \mathcal{S}_j} x^{j(\sigma)} = (1 + x)(1 + x + x^2) \cdots (1 + x + \cdots + x^{j-1}) \]

with \( x = q^{-2} \). Now we consider

\[ f_j(n_1) \cdots f_j(n_j) \cdot (x_{11}^jx_{21}^j) \]

for given \( n_1, \ldots, n_j \) \( (1 \leq n_1 < \cdots < n_j \leq 2j) \). Suppose that

\[ n_1 < \cdots < n_r < j < n_{r+1} < \cdots < n_j, \]

for some \( r \) and define \( m_1, \ldots, m_r \) \( (1 \leq m_1 < \cdots < m_r \leq j) \) by the condition

\[ \{n_{r+1}, n_{r+2}, \ldots, n_j\} \cup \{j + m_1, j + m_2, \ldots, j + m_r\} = \{j + 1, j + 2, \ldots, 2j\}. \]

Then we have

\[ f_j(n_1) \cdots f_j(n_j) \cdot (x_{11}^jx_{21}^j) = q^\beta \cdot x_{11}^{n_1} \cdots x_{12}^{n_r} \cdots x_{12}^{j-r} \cdots x_{21}^{m_1} \cdots x_{22}^{m_r} \]

\[ = q^\beta + r \cdot x_{11}^{j-r} x_{12}^{j-r} \cdots x_{21}^{j-r} x_{22}^{j-r} \]

\[ = q^\beta + r \cdot x_{11}^{j-r} x_{22}^{j-r} (x_{12}x_{21})^r, \]
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where $\beta$ and $\gamma$ are calculated as

$$
\beta = -\{(2j - n_1) + (2j - n_2) + \cdots + (2j - n_j) - (1 + 2 + \cdots + j - 1)\} + (1 + 2 + \cdots + j - 1) \\
= \frac{j(j - 1)}{2} - rj + (n_1 + \cdots + n_r) - (m_1 + \cdots + m_r),
$$

$$
\gamma = \{(j - m_r) + (j - 1 - m_{r-1}) + \cdots + (j - r + 1 - m_1)\} \\
- \{(j - n_r) + (j - 1 - n_{r-1}) + \cdots + (j - r + 1 - n_1)\} \\
= (n_1 + \cdots + n_r) - (m_1 + \cdots + m_r).
$$

Thus we get

$$
f_j(n_1) \cdots f_j(n_j) \cdot (x_{11}^j x_{22}^j) = q^{-r^2 + j(j-1)/2 + 2(n_1 + \cdots + n_r) - 2(m_1 + \cdots + m_r)} x_{11}^{j-r} x_{22}^{j-r} (x_{12} x_{21})^r.
$$

Using this, we have

$$
f^r \cdot (x_{11}^j x_{22}^j) = q^{-j(j-1)/2} [j]_q! \sum_{1 \leq n_1 < \cdots < n_{2j} \leq 2j} f_j(n_1) \cdots f_j(n_j) \cdot (x_{11}^j x_{22}^j) \\
= [j]_q! \sum_{r=0}^{j} q^{-r^2} \sum_{1 \leq n_1 < \cdots < n_r \leq j, 1 \leq m_1 < \cdots < m_r \leq j} q^{2(n_1 + \cdots + n_r) - 2(m_1 + \cdots + m_r)} x_{11}^{j-r} x_{22}^{j-r} (x_{12} x_{21})^r \\
= [j]_q! \sum_{r=0}^{j} q^{-r^2} e_r(1, q^2, \ldots, q^{2(j-1)}) e_r(1, q^{-2}, \ldots, q^{-2(j-1)}) x_{11}^{j-r} x_{22}^{j-r} (x_{12} x_{21})^r,
$$

where $e_r(x_1, x_2, \ldots, x_j)$ denotes the $r$-th elementary symmetric polynomial in $x_1, x_2, \ldots, x_j$. Using the identity (see, e.g. [6])

$$
e_r(1, q^2, \ldots, q^{2j-2}) = q^{r(j-1)} \left[ \begin{array}{c} j \\ r \end{array} \right]_q
$$

together with the symmetry $\left[ \begin{array}{c} j \\ r \end{array} \right]_q = \left[ \begin{array}{c} j \\ q-1 \end{array} \right]_q$, we obtain

$$
f^r \cdot (x_{11}^j x_{22}^j) = [j]_q! \sum_{r=0}^{j} q^{-r^2} \left[ \begin{array}{c} j^2 \\ r \end{array} \right]_q x_{11}^{j-r} x_{22}^{j-r} (x_{12} x_{21})^r.
$$

Since $(x_{11} x_{21})^j = q^{-j(j-1)/2} x_{11}^j x_{21}^j$, we have the desired conclusion. \hfill \Box

It is straightforward to verify the relations

$$
z_1 \cdot x_{22} = x_{22} \cdot (z_1 + (q^3 - q) z_2),
$$

$$
z_2 \cdot x_{22} = q^2 x_{22} \cdot z_2.
$$

Using this, we get the

**Lemma 3.2.**

$$
x_{11}^l x_{22}^l = \prod_{r=1}^{l} (z_1 + (q^{2r-1} - q) z_2).
$$

**Proof.** By (3.2), it follows that

$$
(z_1 + (q^{2r-1} - q) z_2) \cdot x_{22} = x_{22} \cdot (z_1 + (q^{2r+1} - q) z_2),
$$

by which the lemma is proved by induction on $l$. \hfill \Box
3.2 Irreducible decomposition

**Theorem 3.3.** There exists a collection of \( \mathbb{C}(q) \)-valued functions \( F_{m,j}(\alpha) \) \((j = 0, 1, \ldots, m)\) such that

\[
V_q^m(\alpha) \cong \bigoplus_{\substack{0 \leq j \leq m \\ \text{F}_{m,j}(\alpha) \neq 0}} \mathcal{M}_q(2j + 1).
\]

**Proof.** Notice that

\[
\left( \det_q(\alpha) \right)^m = \sum_{j=0}^{m} \binom{m}{j} (aq)^j z_1^{m-j} z_2^j
\]

is a homogeneous polynomial of degree \( m \) in the commuting variables \( z_1 \) and \( z_2 \). On the other hand, it is also clear that the vectors

\[
v_{m,j} = (f^j \cdot (x_{11} x_{21})^j) \det_q^{m-j} \quad (j = 0, 1, \ldots, m)
\]

are linearly independent (since \( e^j \cdot v_{m,j} \neq 0 \) and \( e^{j+1} \cdot v_{m,j} = 0 \)), and they are homogeneous polynomials of degree \( m \) in \( z_1 \) and \( z_2 \) by Lemma \([3.1]\). Therefore, \( \{v_{m,j}\}_{j=0}^{m} \) form a basis of the space consisting of the homogeneous polynomials of degree \( m \) in \( z_1 \) and \( z_2 \), so that there exist \( \mathbb{C}(q) \)-valued functions \( F_{m,j}(\alpha) \) such that

\[
\left( \det_q(\alpha) \right)^m = \sum_{j=0}^{m} F_{m,j}(\alpha)v_{m,j}.
\]

Since \( \mathcal{U}_q(\mathfrak{sl}_2) \cdot v_{m,j} \cong \mathcal{M}_q(2j + 1) \), this proves the theorem. \( \square \)

The conditions for the functions \( F_{m,j}(\alpha) \) are described in terms of polynomials in \( \mathbb{C}(q)[z_1, z_2] \) by virtue of Lemmas \([3.1, 3.2]\). Since \( z_1 \) and \( z_2 \) commute, it is meaningful to consider the specialization \( z_1 = z, \ z_2 = 1 \), where \( z \) is a new indeterminate. Put

\[
g_j(z) = \prod_{i=1}^{j}(z + q^{2i-1} - q),
\]

\[
v_j(z) = q^{-j(j-1)/2} [j]_q! \sum_{r=0}^{j} q^{-r^2} \left(\begin{atms} j \end{atms} r\right)_q g_{j-r}(z).
\]

Then \([3.6]\) together with Lemmas \([3.1, 3.2]\) yields

\[
(z + qa)^m = \sum_{j=0}^{m} F_{m,j}(\alpha)v_{j}(z-q)^{m-j}.
\]

If we take the \( l \)-th derivative of this formula with respect to \( z \) \((l = 0, 1, \ldots, m)\), then we have

\[
\frac{m!}{(m-l)!}(z + qa)^{m-l} = \sum_{j=0}^{m} F_{m,j}(\alpha) \sum_{t=0}^{l} \binom{l}{t} v_{j}^{(l-t)}(z) \frac{(m-j)!}{(m-j-t)!}(z-q)^{m-j-t}.
\]

Letting \( z = q \), we get the relation

\[
\binom{m}{l} q^{m-l}(1 + \alpha)^{m-l} = \sum_{j=m-l}^{m} F_{m,j}(\alpha) v_{j}^{(l-m+j)}(q) = \sum_{s=0}^{l} F_{m,m-s}(\alpha) v_{m-s}^{(l-s)}(q).
\]
Now we calculate \( v_j^{(i)}(q)/i! \). By the \( q \)-binomial theorem \( (2.7) \), we have

\[
g_j(z) = \sum_{i=0}^{j} q^{j-i} {j \choose i}_q (z - q)^i.
\]

Hence it follows that

\[
v_j(z) = q^{-j(j-1)/2} [j]_q! \sum_{r=0}^{j} q^{-r^2} {j \choose r}_q g_{j-r}(z) = q^{j(j+1)/2} [j]_q! \sum_{r=0}^{j} q^{r(i-2j)} \left\{ \sum_{r=0}^{j-i} q^{r(i-2j)} {j \choose r}_q \right\} (z - q)^i.
\]

Using the \( q \)-Chu-Vandermonde formula \( (2.8) \), we get

\[
\sum_{r=0}^{j-i} q^{r(i-2j)} {j \choose r}_q = q^{j-i} {2j-i \choose j}_q.
\]

Thus provides

\[
v_j(z) = q^{-j(j-1)/2} \sum_{i=0}^{j} \frac{[j]_q! [2j-i]_q!}{[i]_q! [j-i]_q q^{2i}} (z - q)^i; \quad (3.11)
\]

or

\[
\frac{v_j^{(i)}(q)}{i!} = q^{-j} \left( \frac{[j]_q! [2j-i]_q!}{[i]_q! [j-i]_q q^{2i}} \right). \quad (3.12)
\]

Thus the formula \( (3.10) \) is rewritten more explicitly as

\[
[m - l]_q! 2^l \left( \frac{m}{l} \right)_q m^{-l} (1 + \alpha)^{m-l} = \sum_{s=0}^{l} q^{-\left( \begin{array}{c} m-s \no \no z \end{array} \right)_q} \frac{[m-s]_q! [2m-l-s]_q!}{[l-s]_q!} F_{m,m-s}(\alpha). \quad (3.13)
\]

### 3.3 Expression of \( F_{m,j}(\alpha) \) in terms of mixed hypergeometric polynomials

From \( (3.10) \) (or \( (3.13) \)), we can conclude that \( F_{m,j}(\alpha) \) is a polynomial function in \( \alpha \) which is divisible by \( (1 + \alpha)^j \), that is

\[
F_{m,j}(\alpha) = (1 + \alpha)^j Q_{m,j}(\alpha) \quad (3.14)
\]

for some \( Q_{m,j}(\alpha) \in \mathbb{C}(q)[\alpha] \). By \( (3.14) \) and \( (3.13) \), we have

\[
\left[ \frac{2m-2i}{m-i} \right]_q^{-1} \left( \frac{m}{i} \right)_q m^{-i} = \sum_{j=0}^{i} \left[ \frac{2i-2m+1}{i-j} \right]_q (-1)^{i-j} (1 + \alpha)^{i-j} q^{-\left( \begin{array}{c} m-j \no \no z \end{array} \right)_q} [m-j]_q Q_{m,m-j}(\alpha). \quad (3.15)
\]

To solve this, we need the following lemma.

**Lemma 3.4.**

\[
\left( \left[ \frac{2i-2m+1}{i-j} \right]_q \right)_{0 \leq i,j \leq m}^{-1} = \left( \frac{[2m-2i+1]_q}{[2m-2j+1]_q} \right)_{0 \leq i,j \leq m}. \quad (3.16)
\]
for $0 \leq j \leq i \leq m$ since the matrices in (3.10) are lower triangular. The case where $i = j$ is clear. Assume that $i > j$. By putting $d = i - j$, $n = m - j$ and changing the running index by $r = k - j$, (3.17) is reduced to

$$
\sum_{r=0}^{d} \binom{2n+1-2r}{d-r} q^{2d-(2n+1)} \binom{2n+1}{r} = 0 \quad (0 < d \leq n).
$$

(3.18)

To prove this, it suffices to show that the function

$$
f(x) = \sum_{r=0}^{d} \binom{x-2r}{d-r} q^{2d-x} \binom{x}{r} q
$$

is constant, which is readily seen to be zero. Notice that $f(x)$ is a rational function in $z = q^x$, and its numerator is a polynomial in $z$ of degree at most $2d$. Hence it is enough to verify that $f(l) = 0$ for $l = 0, 1, \ldots, 2d$. However, since we easily see that $f(x) + f(2d-x) = 0$, which also implies $f(2d) = 0$, we have only to check $f(l) = 0$ for $l = 0, 1, \ldots, d - 1$.

Let $l \in \mathbb{Z}$ such that $0 \leq l < d$. Then we have

$$
f(l) = \sum_{r=0}^{l} \binom{l-2r}{d-r} q^{2d-l} \binom{l}{r} = 0 \quad (l < r \leq d)
$$

$$
= \sum_{s=0}^{l} \binom{l-2(l-s)}{d-(l-s)} q^{2d-l} \binom{l}{l-s} = -f(l),
$$

which means $f(l) = 0$. This completes the proof of the lemma.

Now we define the mixed hypergeometric series by

$$
\Phi \left( \begin{array}{c}
\alpha_1, \ldots, \alpha_k \\
\beta_1, \ldots, \beta_l,
\end{array} ; q \right) := \sum_{i=0}^{\infty} \frac{\prod_{j=1}^{k} \left( a_j ; q^i \right)}{\prod_{j=1}^{l} \left( b_j ; q^i \right)} x^i,
$$

(3.19)

where $(a; q) = \prod_{i=1}^{\infty} \left( 1 - a q^i \right)$ and $(a; q)_i = \prod_{j=0}^{i-1} \left( 1 - a q^j \right)$ (cf. [2]).

**Theorem 3.5.** For $s = 0, 1, \ldots, m$,

$$
F_{m,s}(\alpha) = q^{(s+1)/2} \binom{m}{s} \frac{[s]_q!}{[2s]_q!} (1 + \alpha)^s \Phi \left( \begin{array}{c}
s-m \\
2s+1
\end{array} ; \frac{s+1}{2s+2}, q(1+\alpha) \right)
$$

(3.20)

holds.

**Proof.** By (3.15) and (3.16), we have

$$
Q_{m,m-i}(\alpha) = q^{(m-1)/2} \binom{m}{m-i} \prod_{j=0}^{i} \left( -1 \right)^{j} q^{m-j} \binom{2m-2i+1}{2m-2j+1} \binom{2m-2j+1}{m-j} (1+\alpha)^{m-j}
$$

$$
= q^{(m-i)/2} \binom{m-i}{m-i} \frac{[m-i]_q!}{[2m-2i+1]_q!} \prod_{r=0}^{i} \left( -q \right)^r \frac{[m-i+r]_q!}{[m-i+r]_q! (m-i+r)!} \frac{[2m-2i+r+1]_q!}{[m-i+r]_q!} (1+\alpha)^r.
$$
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Since

\[(i-r)! = (-1)^r \frac{i!}{(-i-r)!}, \quad (n+r)! = n!(n+1;r), \quad [n+r]_q! = [n]_q!(n+1;r)_q,\]

we have

\[
Q_{m,m-i}(\alpha) = \frac{q^{\binom{m-i+1}{2}} m! [m-i]_q!}{i!(m-i)! [2m-2i]_q!} \sum_{r=0}^{i} \frac{(-i;r) (m-i+1;r)_q^2}{(m-i+1;r) (2m-2i+2;r)_q^2} (q(1+\alpha))^r
\]

\[
= q^{\binom{m-i+1}{2}} \binom{m}{i} \frac{[m-i]_q!}{[2m-2i]_q!} \Phi \left( m-i+1; \frac{m-i+1, m-i+1}{2m-2i+2}; q; q(1+\alpha) \right).
\]

(3.21)

If we substitute this into (3.14) and replace $m-i$ by $s$, then we have the conclusion.

Remark 3.6. The function $\Phi$ given by (3.21) satisfies the difference-differential equation

\[
\left\{ -(E + a_1) \cdots (E + a_k) [E + c_1]_q \cdots [E + c_m]_q
\right.

\[
+ \partial_q(E + b_1 - 1) \cdots (E + b_t - 1) [E + d_1 - 1]_q \cdots [E + d_n - 1]_q \left\} \Phi = 0,
\]

where we put

\[
E = \frac{d}{dx}, \quad [E + a]_q = \frac{q^{E+a} - q^{-E-a}}{q - q^{-1}}, \quad \partial_q f(x) = \frac{f(qx) - f(q^{-1}x)}{qx - q^{-1}x}.
\]

If we take a limit $q \to 1$, then the difference-differential equation above becomes a hypergeometric differential equation for $k+m F_{t+n}(a_1, \ldots, c_m; b_1, \ldots, d_n; x)$.

3.4 Classical case

All the discussion above also work in the classical case (i.e. the case where $q = 1$). Thus, by taking a limit $q \to 1$ in Theorem 3.3, we will obtain Theorem 4.1 in [3] again. We abuse the same notations used in the discussion of quantum case above to indicate the classical counterparts. From (3.21), we have

\[
Q_{m,s}(\alpha) = \frac{m!}{(m-s)!(2s)!} \left[ \begin{array}{c} 2F_1 \left( \begin{array}{c} s-m, s+1 + s+1 \n 1+\alpha \end{array} \right) \frac{s}{2} + 2s+2 \end{array} \right]
\]

\[
= \frac{m!}{(m-s)!(2s)!} \left[ \begin{array}{c} 2F_1 \left( \begin{array}{c} s-m, s+1 + s+1 \n 2s+1 \end{array} \right) \frac{1}{1+\alpha} \end{array} \right]
\]

(3.22)

Notice that

\[
2F_1 \left( \begin{array}{c} s-m, s+1 + s+1 \n 2s+2 \end{array} \right) = \frac{m!(2s+1)!}{s!(m+s+1)!} 2F_1 \left( \begin{array}{c} s-m, s+1 + s+1 \n -m \end{array} \right).
\]

(3.23)

Thus we also get

\[
Q_{m,s}(\alpha) = \frac{m!^2 (2s+1)}{(m-s)!(m+s+1)!} 2F_1 \left( \begin{array}{c} s-m, s+1 + s+1 \n -m \end{array} \right) (s = 0, 1, \ldots, m).
\]

(3.24)

Summarizing these, we have the

Corollary 3.7 (Classical case).

\[
F_{m,s}(\alpha) = \frac{m!}{(m-s)!(2s)!} (1+\alpha)^s 2F_1 \left( \begin{array}{c} s-m, s+1 + s+1 \n 2s+1 \end{array} \right) \frac{1}{1+\alpha}
\]

\[
= \frac{(2m-s) - (2m-s-1)}{(2m-s)!} (1+\alpha)^s 2F_1 \left( \begin{array}{c} s-m, s+1 + s+1 \n -m \end{array} \right) (s = 0, 1, \ldots, m).
\]

(3.25)

for $s = 0, 1, \ldots, m$. 

\[\blacksquare\]
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