Bernstein-Sato Polynomials for Projective Hypersurfaces with Weighted Homogeneous Isolated Singularities

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Abstract. We present an efficient method to calculate the roots of the Bernstein-Sato polynomial $b_f(s)$ for a defining polynomial $f$ of a projective hypersurface $Z \subset \mathbb{P}^{n-1}$ of degree $d$ having only weighted homogeneous isolated singularities. The computation of roots can be reduced to that of the Hilbert series of the Jacobian ring of $f$ except some special case. For this we prove the $E_2$-degeneration of the pole order spectral sequence. Combined with the self-duality of the Koszul complex and also a theorem of Dimca and Popescu on the weak Lefschetz property of the “torsion part” of the Jacobian ring (with respect to a general linear function), it implies in the case $n = 3$ the discrete connectivity of the absolute values of the roots of $b_f(s)$ supported at 0 modulo the roots coming from the singularities of $Z$, except some special case which does not contain any essential indecomposable central hyperplane arrangements in $\mathbb{C}^n$; more precisely, we have $R_f = \frac{1}{\pi}(Z \cap \{3, k'\}) \cup R_Z$, where $R_f, R_Z$ are the roots of Bernstein-Sato polynomials of $f, Z$ up to sign, and $k' = \max(2d-3, k_{\text{max}}+3)$ with $k_{\text{max}}$ the maximal degree of the “torsion part” of the Jacobian ring.

Introduction

Let $Z$ be a projective hypersurface in $\mathbb{P}^{n-1}$ defined by a homogeneous polynomial $f$ of $n$ variables with $n \geq 3$ and $d := \deg f \geq 3$. We assume that $Z$ is reduced and is not a projective cone of a hypersurface (that is, $f$ is not a polynomial of $n-1$ variables). Let $b_f(s)$ be the Bernstein-Sato polynomial of $f$. Set

$$\mathcal{R}_f := \{ \alpha \in \mathbb{Q} \mid b_f(-\alpha) = 0 \} \subset \mathbb{Q}_{>0},$$

and similarly for $\mathcal{R}_{h_z}$ replacing $f$ by a local defining function $h_z$ of $(Z, z) \subset (Y, z)$ ($z \in Z$). Define the set of roots of Bernstein-Sato polynomial supported at the origin (up to sign) by

$$(1) \quad \mathcal{R}_f^0 := \mathcal{R}_f \setminus \mathcal{R}_Z \quad \text{with} \quad \mathcal{R}_Z := \bigcup_{z \in \text{Sing} Z} \mathcal{R}_{h_z} \subset \mathcal{R}_f.$$

Here the last inclusion follows from the equality $b_{h_z}(s) = b_{f,g}(s)$ with $b_{f,g}(s)$ the local Bernstein-Sato polynomial of $f$ at $y \in \mathbb{C}^n \setminus \{0\}$ with $[y] = z$ in $Y = (\mathbb{C}^n \setminus \{0\})/\mathbb{C}^*$. (This follows from the assertion that $b_{f,g}(s)$ depends only on $f^{-1}(0)$, see 4.2 (i) below.)

Set $F_f := f^{-1}(1) \subset X := \mathbb{C}^n$, the Milnor fiber of $f$. We have the pole order filtration $P$ on each monodromy eigenspace $H^j(F_f, \mathbb{C})_\lambda := \text{Ker}(T_\lambda - \lambda) \subset H^j(F_f, \mathbb{C})$ ($\lambda \in \mathbb{C}^*$) with $T = T_s T_u$ the Jordan decomposition of the monodromy. Recall the following.

Theorem 1 ([Sa 07, Theorem 2]). Assume $\alpha \notin \mathcal{R}_Z$. If $\alpha$ satisfies the condition

$$(2) \quad \alpha \notin \mathcal{R}_Z + \mathbb{Z}_{\leq 0},$$

then

$$(3) \quad \alpha \in \mathcal{R}_f^0 \iff \text{Gr}_{p}^{h} H^{n-1}(F_f, \mathbb{C})_\lambda \neq 0 \quad (p = [n - \alpha], \quad \lambda = e^{-2\pi i \alpha}).$$

If condition (2) does not hold, then only the implication $\Longrightarrow$ holds in (3).

Remark 1. There are examples of non-reduced hyperplane arrangements of 3 variables such that condition (2) is unsatisfied and the implication $\Longrightarrow$ in (3) fails, see [Sa 20, Example 4.5]. This problem is related to the asymptotic expansion of $f^s \in D_{X}[s]f^{s}$ via the V-filtration, and is rather complicated, see 4.2 (iii) below.

Theorem 1 gives a partial generalization of a well-known theorem in the isolated singularity case asserting that the Steenbrink spectral numbers coincide with the roots of the microlocal
Bernstein-Sato polynomial \( \tilde{b}_f(s) := b_f(s)/(s+1) \) up to a sign (forgetting the multiplicities) in the case \( f \) is a weighted homogeneous polynomial, see (1.8.2) below.

We have the pole order spectral sequence associated with the pole order filtration on the (algebraic) microlocal Gauss-Manin complex of \( f \). Set

\[
R := \mathbb{C}[x_1, \ldots, x_n],
\]

with \( x_1, \ldots, x_n \) the coordinates of \( \mathbb{C}^n \). This is a graded ring with \( \deg x_i = 1 \). Let \( \Omega^* \) be the complex of the exterior products of the Kähler differentials of \( R \) over \( \mathbb{C} \) so that \( \Omega^p \) is a free \( R \)-module of rank \( \binom{n}{p} \). This complex has anti-commuting two differentials \( d \) and \( df\wedge \) preserving the grading (up to the shift by \( d = \deg f \) in the case of \( df\wedge \) ), where each component of the complex is graded with \( \deg x_i = \deg dx_i = 1 \).

In this paper we assume the isolated singularity condition:

(IS)

\[
\sigma_Z := \dim \Sigma = 0 \quad \text{with} \quad \Sigma := \text{Sing} \ Z.
\]

We have the vanishing

\[
H^j_d(H^*_{df\wedge} \Omega^*) = 0 \quad \text{unless} \quad j = n-1 \ or \ n,
\]

where \( H^*_{df\wedge} \) means that the cohomology with respect to the differential \( df\wedge \) is taken, and similarly for \( H^*_{d} \). These are members of the \( E_2 \)-term of the spectral sequence associated with the double complex with anti-commuting two differentials \( d \) and \( df\wedge \) on \( \Omega^* \), see also (1.1.1) below. The latter has been studied in [Di.92], although the relation to the Gauss-Manin system and the Brieskorn modules was not mentioned there. Note that the usual (that is, non-microlocal) pole order spectral sequence was considered there, but its \( E_2 \)-degeneration is equivalent to that of the microlocal one (see [DiSa.24, Corollary 4.7]), and the latter is related to the pole order filtration as in (1.2.8) below. It has been observed in many examples by A. Dimca and G. Sticlaru [DiSt.19] that the pole order spectral sequence degenerates at \( E_2 \) if the singularities of \( Z \) are isolated and weighted homogeneous. We have the following.

**Proposition 1.** Under the assumption (IS), the \( E_2 \)-degeneration of the pole order spectral sequence is equivalent to each of the following two conditions:

\[
\dim H^p_d(H^*_{df\wedge} \Omega^*) = \dim H^{p-1}(F_j, \mathbb{C})
\]

(5) \[\quad \dim H^{n-1}_d(H^*_{df\wedge} \Omega^*) = \dim H^{n-2}(F_j, \mathbb{C}).\]

If these equivalent conditions are satisfied, then the pole order spectrum \( \text{Sp}_p(f) \) (see [DiSa.24]) is given by the difference of the Hilbert series of the graded \( \mathbb{C} \)-vector spaces

\[
M_+(2) := H^p_d(H^*_{df\wedge} \Omega^*), \quad N_+(2) := H^{p-1}_d(H^*_{df\wedge} \Omega^*)(-d),
\]

with variable \( t \) of the Hilbert series replaced by \( t^{1/d} \) (where \( m \) for \( m \in \mathbb{Z} \) denotes the shift of grading so that \( E(m) := E_{m+k} \) for any graded module \( E \) ), and moreover, in the notation of (3), we have the canonical isomorphisms

\[
M_k^{(2)} = \text{Gr}_p H^{n-1}(F_j, \mathbb{C}) \lambda \quad (\alpha = \frac{t}{d}, \ p = [n - \alpha], \ \lambda = e^{-2\pi i \alpha}).
\]

In this paper we prove the following.

**Theorem 2.** Under the assumption (IS), assume further

(WH) Every singularity of \( Z \) is analytically defined by a weighted homogeneous polynomial with an isolated singularity, see 1.7 below.

Then the pole order spectral sequence degenerates at \( E_2 \) so that (7) holds.

This is quite nontrivial even in the case \( n = 3 \). Without assuming (WH), Theorem 2 never holds, see [DiSa.24, Theorem 5.2]. (Here “weighted homogeneity” cannot be replaced by “semi-weighted-homogeneity”.) The assumption (WH) is equivalent to that every singularity of \( Z \) is quasihomogeneous, that is, \( h_z \in (\partial h_z) \ (\forall \ z \in \Sigma) \), where \( h_z \) is as in (1), and
(\partial h_z) is the Jacobian ideal generated by the partial derivatives of \( h_z \), see [SaK 71] (and 1.7 below). Note that \( \mathcal{R}_{h_z} \) can be determined only by the weights in the weighted homogeneous isolated singularity case, see (1.8.2–3) and A.1 below. This is quite different from the non-quasihomogeneous isolated hypersurface singularity case where we need a computer program to determine the local Bernstein-Sato polynomials of \( Z \) in general.

By [DiSa 24, Corollary 4.7], Theorem 2 implies the following.

**Corollary 1.** Under the hypotheses (IS)-(WH), the Brieskorn module \( H^n A^*_f \) in the notation of [DiSa 24] is torsion-free.

Combining Theorem 2 with [DiSa 24], [Sa 07], we can show the following.

**Corollary 2.** Under the assumptions (IS)-(WH), any \( \alpha \in \mathcal{R}_f^0 \) satisfying condition (2) in Theorem 1 can be detected by using the Hilbert series of the graded \( \mathbb{C} \)-vector space \( M^{(2)} \) in Proposition 1.

Set

\[
M_* := H^n_{df} \Omega^* \quad \text{and} \quad N_* := H^{n-1}_{df} \Omega^*(-d).
\]

These are graded \( \mathbb{C} \)-vector spaces. Let \( y \) be a sufficiently general linear combination of the coordinates \( x_i \) of \( \mathbb{C}^n \), and \( M'_* \subset M_* \) be the \( y \)-torsion part. Set \( M''_* := M_*/M'_* \). Note that \( M_* \) is a finite dimensional graded \( \mathbb{C} \)-vector subspace of \( M_* \), and is equal to the zeroth local cohomology \( H^0_{df} M_\mathfrak{m} \) with \( \mathfrak{m} \subset R \) the maximal ideal at 0, see [DiSa 24].

**Theorem 3.** Under the assumptions (IS)-(WH), we have the injectivity of the composition of canonical morphisms

\[
M'_* \hookrightarrow M_* \twoheadrightarrow M''_*.
\]

Using Theorems 1, 2, 3 together with [DiSa 24, Corollary 2 and Theorem 5.3], we can deduce

**Corollary 3.** Under the hypotheses (IS)-(WH), assume \( M_{k}^{(2)} \neq 0 \) for some \( k \in \mathbb{Z} \). Then \( M_{k}^{(2)} \neq 0 \), hence \( \frac{k}{d} \in \mathcal{R}_f \). The converse holds if \( k > (n-1)d-n \) and moreover \( \frac{k}{d} \notin \mathcal{R}_Z \) in the case \( n \geq 4 \).

These do not hold if the assumption (WH) in Theorem 2 is unsatisfied, see [Sa 15]. Note that the condition \( \frac{k}{d} \notin \mathcal{R}_Z \) follows from the inequality \( k > (n-1)d-n \) if \( n = 3 \), using for instance [dFEM].

Let \( \tau_z := \sum_{z \in \text{Sing}_Z} \tau_{h_z} \) with \( \tau_{h_z} \) the Tjurina number of a local defining function \( h_z \), that is, \( \tau_{h_z} = \dim \mathcal{O}_{Y,z}/((\partial h_z), h_z) \). Under the assumptions (IS)-(WH), this coincides with the Milnor number \( \mu_{h_z} = \dim \mathcal{O}_{Y,z}/(\partial h_z) \). We have the inequalities (see [DiSa 24]):

\[
\mu_k' := \dim M''_k \leq \tau_z, \quad \nu_k := \dim N_k \leq \tau_z \quad (\forall k \in \mathbb{Z}),
\]

where the equalities hold if \( k \geq nd \).

The differential \( d \) of \( \Omega^* \) induces \( d^{(1)} : N_{k+d} \rightarrow M_k \) (\( k \in \mathbb{Z} \)) so that \( M_{k}^{(2)}, N_{k+d}^{(2)} \) are respectively its cokernel and kernel. It is easy to calculate the dimensions of \( M_k, N_{k+d} \) by using computers, and moreover, under the assumptions (IS)-(WH), we have the following:

\[
\text{The composition } N_{k+d} \xrightarrow{d^{(1)}} M_k \xrightarrow{d^{(2)}} M_k'' \text{ is injective if } \frac{k}{d} \notin \mathcal{R}_Z.
\]

This follows from [DiSa 24, Theorem 5.3 and Remark 5.6(i)] together with the assertion that the Hodge and pole order filtrations coincide in the case of weighted homogeneous isolated singularities, see (1.8.2) below.

**Remark 2.** In order to determine \( \mathcal{R}_f^0 \), we do not have to calculate \( d^{(1)} : N_{k+d} \rightarrow M_k \) in (9) for \( \frac{k}{d} \in \mathcal{R}_Z \) because of the last inclusion \( \mathcal{R}_Z \subset \mathcal{R}_f \) in (1). In particular, the information of \( H^{n-2}(F_j, \mathbb{C}) \) (which is given by the kernel of \( d^{(1)} \)) is unnecessary to determine \( \mathcal{R}_f^0 \).
Let \((\partial f) \subset R\) be the Jacobian ideal generated by the partial derivatives of \(f\). Then \(M\) is identified with the Jacobian ring \(R/(\partial f)\) with grading shifted by \(-n\). It is well known that the \(\nu_k = \dim N_k\) are expressed by the \(\mu_k = \dim M_k\); more precisely
\[
\nu_k = \mu_k - \gamma_k \quad (\forall k \in \mathbb{Z}) \quad \text{with} \quad \sum_{k \in \mathbb{Z}} \gamma_k t^k = (t + \cdots + t^{d-1})^n,
\]
see [DiSa 24, Formula (3)]. (This follows from the assertion that the Euler characteristic of a bounded complex of finite dimensional vector spaces is independent of the differential.)

We say that a projective hypersurface \(Z\) or its defining polynomial \(f\) is very special type if for a general \(\mathbb{C}\)-linear combination \(y\) of coordinates \(x_i\) of \(X := \mathbb{C}^n\), there is a nonzero \(\mathbb{C}\)-linear combination \(z\) of coordinates such that \(zh \in (\partial g)\), where \(g\) and \(h\) are the restrictions of \(f\) and \(\xi f\) to the hypersurface \(X'_y := \{y = 0\} \subset X\) with \(\xi\) any constant vector field (that is, a \(\mathbb{C}\)-linear combination of the \(\partial x_i\)) such that \(\xi y \neq 0\), see also 1.9 below. The main theorem of this paper is as follows.

**Theorem 4.** Under the assumptions (IS)-(WH), assume \(f\) is not very special type. Then we have \(\mu_n > \nu_{n+d}\), hence \(\frac{4}{9} \in \mathcal{R}_f\). In the case \(n = 3\), we have \(\mu_k > \nu_{d+k}\), hence \(\frac{2}{9} \in \mathcal{R}_f\), for any \(k \in \mathbb{Z} \cap [3, d-1]\).

This follows from Theorem 2 and [Sa 07, Theorem 2] (that is, Theorem 1 in this paper) together with the non-degeneracy of the Grothendieck residue pairing, see [GrHa 78, p. 659]. Combined with the symmetry of \(\delta_k'' := \mu_k'' - \nu_{k-d}\) with center \(\frac{n-1}{2}\) (where \(\mu'_k := \dim M'_k\), \(\mu_k'' := \dim M''_k\); see [DiSa 24, Corollary 2]) and also the weak Lefschetz property of \(M'_k\) (in particular, \(\mu'_k \leq \mu'_k\) if \(k \leq \frac{2}{3}\), see [DiPo 16]), Theorem 4 and Corollary 3 imply the following.

**Corollary 4.** Under the hypotheses of Theorem 4, assume further \(n = 3\). Then
\[
\mathcal{R}_f = \frac{1}{d}([\mathbb{Z} \cap [3, k'']]) \cup \mathcal{R}_Z,
\]
with \(k'' = \max(2d-3, k'_{\text{max}})\), \(k'_{\text{max}} := \max\{k \mid M'_k \neq 0\}\).

In the case of an essential indecomposable reduced central hyperplane arrangement in \(\mathbb{C}^3\), this is shown in [Ba 24] by a completely different method. Here \(\max\mathcal{R}_f < 2 - \frac{4}{d}\), see [Sa 16, Theorem 1]. In the case \(n = 3\), the latter theorem is reproved in [DiSt 19, Corollary 7.3] using an estimate of Castelnuovo-Mumford regularity in [DIM 20, Corollary 3.5], see also [Ba 24]. We can verify that such an arrangement is not very special type by applying the following.

**Proposition 2.** Under the assumptions (IS)-(WH), assume further \(n = 3\), \(f\) is very special type, and all the singularities of \(Z\) are homogeneous. Then \(h \notin (\partial g)\) and \(\delta_k := \mu_k - \nu_{k+d} = 0\) for \(k \leq \lceil \frac{d}{2} \rceil\) (where \(\lceil \beta \rceil := \min\{k \in \mathbb{Z} \mid k \geq \beta\}\)).

For any essential indecomposable reduced central hyperplane arrangement in \(\mathbb{C}^3\) it is actually proved that \(\mathrm{Gr}^2_p H^2(F_j, \mathbb{C}) \neq 0\) for \(\lambda = e^{-2 \pi i \beta / d}\) in [BSY 11]. One can verify that \(\mathrm{Gr}^1_p H^1(F_j, \mathbb{C}) = 0\) using [DiSa 24, Theorem 5.3] and [BDS 11, (1.5.3)] (see also [Fa 97]). Here \(P\) may be replaced with \(F\) in view of [DiSt 20, Proposition 2.2]. The argument is not trivial if there is a point of \(Z\) with multiplicity \(m\) such that \(\frac{2}{m} = 3 - \frac{3}{d}\). In this case however there are at least two lines not passing through the point of multiplicity \(m\), so the condition in [ESV] is satisfied.

As a conclusion it is recommended to try the code in A.3 below calculating the \(\mu'_k\) and \(\delta_k\) if conditions (IS)-(WH) are satisfied with \(n = 3\). In the case the coefficients of \(T^k\) in \(\delta\) are all positive for \(k \in [3, d-1]\), then \(d\mathcal{R}_f \cap \mathbb{Z} = \{3, \ldots, k''\}\) modulo \(d\mathcal{R}_Z\) where \(k''\) is as in Corollary 4. If the coefficient of \(T^3\) vanishes, then \(f\) must be very special type. For the moment no example of very special type is known other than extremely degenerated ones, see 1.9 and Remark 4.8 below.

We thank A. Dimca for useful discussions on pole order spectral sequences and for making a computer program based on [Sa 20] (which encouraged us to make a less sophisticated one). This work was partially supported by Kakenhi 24540039 and 15K04816.
In Section 1 we review some basics of pole order spectral sequences associated with the algebraic microlocal Gauss-Manin systems, and prove Proposition 1 and also Theorem 1.5 about the compatibility of certain self-duality isomorphisms. In Section 2 we study the fundamental exact sequences of vanishing cycles, and prove Theorem 2.2 which is a key to the proof of Theorem 2. In Section 3 we calculate the filtered twisted de Rham complexes associated with weighted homogeneous polynomials having isolated singularities. In Section 4 we prove Theorems 2–4 and Proposition 2 after reviewing some basics of Bernstein-Sato polynomials. In Section 5 we calculate some examples explicitly. In Appendix we explain how to use the computer programs Macaulay2 and Singular for explicit calculations of roots of Bernstein-Sato polynomials.

1. Microlocal pole order spectral sequences

In this section we review some basics of pole order spectral sequences associated with the algebraic microlocal Gauss-Manin systems, and prove Proposition 1 and also Theorem 1.5 about the compatibility of certain self-duality isomorphisms.

1.1. Algebraic microlocal Gauss-Manin systems. For a homogeneous polynomial \( f \), we have the algebraic microlocal Gauss-Manin complex \( \tilde{C}_j^* \) defined by

\[
\tilde{C}_j^* := \Omega^j[\partial_t, \partial_t^{-1}] \quad \text{with differential } \quad d - \partial_t d f \wedge,
\]

where \( \Omega^* \) is as in the introduction. It is a complex of graded \( \mathbb{C}[t] \langle \partial_t, \partial_t^{-1} \rangle \)-modules with

\[
\deg t = - \deg \partial_t = d,
\]

and the actions of \( t, \partial_t \) are defined by

\[
t(\omega \partial_t^k) = (f \omega) \partial_t^k - k \omega \partial_t^{k-1}, \quad \partial_t(\omega \partial_t^k) = \omega \partial_t^{k+i} \quad (\omega \in \Omega^j, i \in \mathbb{Z}).
\]

The algebraic microlocal Gauss-Manin systems are defined by

\[
\tilde{G}_j^* := H^{n-j} \tilde{C}_j^* \quad (j \in [0, \sigma_Z + 1]),
\]

where \( \sigma_Z := \dim \text{Sing } Z \) as in the introduction. These are free graded \( \mathbb{C}[\partial_t, \partial_t^{-1}] \)-modules of finite type, and there are canonical isomorphisms

\[
\tilde{G}_j^* \cong H^{n-j}(F_j, \mathbb{C}) e^{-k/d} \quad (\forall k \in \mathbb{Z}),
\]

where \( e(\alpha) := \exp(2\pi i \alpha) \) for \( \alpha \in \mathbb{Q} \), \( F_j := f^{-1}(1) \) (which is viewed as the Milnor fiber of \( f \)), and \( E_\lambda := \text{Ker}(T_s - \lambda) \) in \( E := \tilde{H}^*(F_j, \mathbb{C}) \) with \( T_s \) the semisimple part of the monodromy \( T \), see [DiSa 06], [DiSa 24], etc.

1.2. Pole order spectral sequences. In the notation of 1.1, we have the filtration \( P' \) on \( \tilde{C}_j^* \) defined by

\[
P'_i \tilde{C}_j^* := \bigoplus_{k \in i+j} \Omega^j \partial_t^k.
\]

This is an exhaustive increasing filtration, and induces the filtration \( P' \) on the Gauss-Manin systems \( \tilde{G}_j^* \) compatible with the grading. Since \( \deg \partial_t = -d \), we have

\[
\Omega^j \partial_t^k = \Omega^j(kd),
\]

where \( (m) \) for \( m \in \mathbb{Z} \) denotes the shift of grading as in Proposition 1. Set

\[
P'^i = P'_{-i}.
\]

We have

\[
\text{Gr}_{P'} \tilde{C}_j^* = K_j^*(n-i)d).
\]
Here \( K^*_j \) is the Koszul complex defined by

\[
K^j := \Omega^j((j-n)d) \quad \text{with differential} \quad df. 
\]

We have the microlocal pole order spectral sequence

\[
p_rE_r^{j,-i} = H^jGr^i_qC^*_j \implies \tilde{G}^{j+n}_r (= H^j\tilde{C}^*_j) .
\]

In the notation of Proposition 1, we have

\[
p_rE_r^{j,-i} = \begin{cases} 
H^j_{d_f\wedge}(\Omega^r)((j-i)d) = (H^jK^*_j)((n-i)d) & \text{if} \ r = 1, \\
H^j_d(\tilde{H}^r_{d_f\wedge}(\Omega^r))((j-i)d) & \text{if} \ r = 2.
\end{cases}
\]

Set

\[
P := P'[1], \quad \text{that is,} \quad P^i := P'^{i+1} \quad (i \in \mathbb{Z}).
\]

The isomorphisms in (1.1.4) induce the filtered isomorphisms

\[
(\tilde{G}^{j}_{r,k}, P) = \left( \tilde{H}^{r-n-i-j}(F_j, \mathbb{C}_{e(-k/d)}), P \right) \quad (\forall \, k \in [1, d]),
\]

where \( P \) on the right-hand side is the pole order filtration, see [Di 92], [DiSa 06, Section 1.8].

By the definition (1.2.1) we have the filtered graded isomorphisms

\[
\partial^k_t : (\tilde{C}^*_t, P') \sim \to (\tilde{C}^*_{t}(-kd), P'[-k]) \quad (k \in \mathbb{Z}).
\]

Recall that \( (P'[m])^i = P'^{i+m}, \) \( (P'[m])_i = P'^{i-m}, \) \( \) and \( G(m)_k = G_{m+k} \) for any graded module \( G \), \( m \in \mathbb{Z} \) in general.

By (1.2.9) we get the graded isomorphisms of spectral sequences for \( k \in \mathbb{Z} : 
\]

\[
\partial^k_r : p_rE_r^{i,j,-i} \sim \to p_rE_{r-k}^{i-j-k+i+k}(-kd) \quad (r \geq 1),
\]

which are compatible with the differentials \( d_r \) of the spectral sequences.

1.3. Isolated singularity case. In the notation of the introduction, assume \( Z \subset Y \) has only isolated singularities, that is,

\[
\sigma_Z := \dim \Sigma = 0 \quad \text{with} \quad \Sigma := \text{Sing} \, Z.
\]

Set

\[
M := H^nK^*_j, \quad N := H^{n-1}K^*_j,
\]

where \( k^*_j \) is as in (1.2.4). These are graded \( \mathbb{C} \)-vector spaces.

Let \( y \) be a sufficiently general linear combination of coordinates \( x_i \) of \( \mathbb{C}^n \). Then \( M, N \) are finitely generated graded \( \mathbb{C}[y] \)-modules. It is known that \( N \) is \( y \)-torsion-free (see, for instance, [DiSa 24]). Set

\[
M' := M_{\text{tor}}, \quad M'' := M_{\text{free}} = M/M_{\text{tor}},
\]

where \( M_{\text{tor}} \) denotes the \( y \)-torsion part of \( M \). (The latter is independent of \( y \) as long as \( y \) is sufficiently general). These are also finitely generated graded \( \mathbb{C}[y] \)-modules. Set

\[
\tilde{M} := M[y^{-1}] = M \otimes_{\mathbb{C}[y]} \mathbb{C}[y, y^{-1}] \quad (= \tilde{M}'' := M''[y^{-1}]),
\]

\[
\tilde{N} := N[y^{-1}] = N \otimes_{\mathbb{C}[y]} \mathbb{C}[y, y^{-1}].
\]

These are finitely generated free graded \( \mathbb{C}[y, y^{-1}] \)-modules. By the grading, we have the direct sum decompositions

\[
M = \bigoplus_{k \in \mathbb{Z}} M_k, \quad N = \bigoplus_{k \in \mathbb{Z}} N_k, \quad \tilde{M} = \bigoplus_{k \in \mathbb{Z}} \tilde{M}_k, \quad \tilde{N} = \bigoplus_{k \in \mathbb{Z}} \tilde{N}_k, \quad \text{etc.}
\]

By definition there are isomorphisms

\[
y : \tilde{N}_k \sim \to \tilde{N}_{k+1}, \quad y : \tilde{M}_k \sim \to \tilde{M}_{k+1} \quad (k \in \mathbb{Z}),
\]
together with natural inclusions

\begin{equation}
(1.3.6) \quad N_k \hookrightarrow \mathcal{N}_k, \quad M''_k \hookrightarrow \mathcal{M}''_k = \mathcal{M}_k \quad (k \in \mathbb{Z}),
\end{equation}

inducing isomorphisms for \( k \gg 0 \), since \( N \) and \( M'' \) are \( y \)-torsion-free. In the notation of [DiSa 24, 5.1] we have

\begin{equation}
(1.3.7) \quad H^{-1}(sK^*_{f'}) = \mathcal{N}, \quad H^{0}(sK^*_{f'}) = \mathcal{M}.
\end{equation}

In the notation of 1.2, set

\begin{equation}
(1.3.8) \quad M^{(r)} := p^r \mathcal{E}^{n,0}_r, \quad N^{(r)} := p^r \mathcal{E}^{n,-1}_r \quad (r \geq 1).
\end{equation}

Using the differentials \( d_r \) of the microlocal pole order spectral sequence (1.2.5) together with the isomorphisms in (1.2.10), we get the morphisms of graded \( \mathbb{C} \)-vector spaces of degree \(-rd\):

\begin{equation}
(1.3.9) \quad d^{(r)} : N^{(r)} \to M^{(r)} \quad \text{for any } r \geq 1,
\end{equation}

such that \( N^{(r)}, M^{(r)} \) are respectively the kernel and cokernel of \( d^{(r-1)} \) for any \( r \geq 2 \), and are independent of \( r \gg 0 \) (that is, \( d^{(r)} = 0 \) for \( r \gg 0 \)). More precisely, (1.3.9) is given by the composition (using the isomorphisms (1.2.10)):

\begin{equation}
(1.3.10) \quad p^r \mathcal{E}^{n,0}_r \xrightarrow{d_r} p^r \mathcal{E}^{n+r,-r-1}_r \xrightarrow{d^{(r)}_r} p^r \mathcal{E}^{n,-1}_r(-rd).
\end{equation}

By (1.2.3) or (1.2.6), we have for \( r = 1 \)

\begin{equation}
(1.3.11) \quad M^{(1)} = M, \quad N^{(1)} = N,
\end{equation}

and \( d^{(1)} : N \to M \) is induced by the differential \( d \) of the de Rham complex \((\Omega^*, d)\). By (1.2.6) we get moreover

\begin{equation}
(1.3.12) \quad M^{(2)} = H^0_d(H^f_{df} \Omega^*), \quad N^{(2)} = H^{-1}_d(H^f_{df} \Omega^*)(-d).
\end{equation}

These \( M^{(r)}, N^{(r)} \), and \( d^{(r)} \) are equivalent to the microlocal pole order spectral sequence because of the isomorphisms (1.2.9–10). Set

\begin{equation}
(1.3.13) \quad M^{(\infty)} := M^{(r)}, \quad N^{(\infty)} := N^{(r)} \quad (r \gg 0).
\end{equation}

Here \( p^{(r)}_k := \dim M^{(r)}_k, \quad \nu^{(r)}_k := \dim N^{(r)}_k \) are finite and non-increasing for \( r \geq 1 \) with \( k \) fixed. Hence they are stationary for \( r \gg 0 \) with \( k \) fixed.

Note that the shift of the grading by \(-d\) for \( N^{(2)} \) in (1.3.12) comes from the definition of the Koszul complex \( K^*_f \) where the differential \( df\wedge \) preserves the grading.

### 1.4. Weighted homogeneous isolated singularity case.

Assume the isolated singularity condition (1.3.1) together with condition (WH) in Theorem 2, see also 1.7 below. In the notation of 1.3, set as in [DiSa 24, 5.1]:

\begin{equation}
(1.4.1) \quad h := f/y^d \quad \text{on} \quad Y' := Y \setminus \{y = 0\} \cong \mathbb{C}^{n-1}.
\end{equation}

Note that \( h \) is a defining function of \( Z \setminus \{y = 0\} \) in \( Y' \), and we have \( \Sigma = \Sigma Z \subset Y' \) (since \( y \) is sufficiently general).

For \( z \in \Sigma \), set

\begin{equation}
(1.4.2) \quad \Xi_{h_z} := \Omega^{
}_{Y_{df},z}/dh_z \wedge \Omega^{n-1}_{Y_{df},z} \quad (n' := \dim Y' = n-1).
\end{equation}

These are finite dimensional vector spaces of dimension \( \mu_z \), where \( \mu_z \) is the Milnor number of \((h_z, z)\) which coincides with the Tjurina number \( \tau_z \) \((z \in \Sigma)\) under the assumption (WH) in Theorem 2. We have the canonical isomorphism

\begin{equation}
(1.4.3) \quad \Xi^\text{alg}_{h_z} \sim \Xi_{h_z}.
\end{equation}
where \( \Xi_{h_z}^{\text{alg}} \) is defined in the same way as \( \Xi_{h_z} \) with \( \Omega^*_{Y',z} \) replaced by \( \Omega^*_{Y',z} \). (Indeed, \( \mathcal{O}_{Y',z} \) is flat over \( \mathcal{O}_{Y',z} \) so that the tensor product of \( \mathcal{O}_{Y',z} \) over \( \mathcal{O}_{Y',z} \) is an exact functor. Applying this to the exact sequence defining \( \Xi_{h_z}^{\text{alg}} \), that is,

\[
\Omega_{Y',z}^{r-1} \xrightarrow{d\wedge} \Omega_{Y',z}^r \rightarrow \Xi_{h_z}^{\text{alg}} \rightarrow 0,
\]

we get the isomorphism

\[
\Xi_{h_z} = \Xi_{h_z}^{\text{alg}} \otimes_{\mathcal{O}_{Y',z}} \mathcal{O}_{Y',z}.
\]

We have a canonical morphism \( \Xi_{h_z}^{\text{alg}} \rightarrow \Xi_{h_z} \) induced by the inclusion \( \mathcal{O}_{Y',z} \hookrightarrow \mathcal{O}_{Y',z} \). So the assertion follows by suing a finite filtration on \( \Xi_{h_z}^{\text{alg}} \) such that its graded quotients are \( \mathbb{C} \), since \( \mathbb{C} \otimes_{\mathcal{O}_{Y',z}} \mathcal{O}_{Y',z} = \mathbb{C} \).

Fixing \( y \), we have the isomorphisms compatible with (1.3.5)

\[
\tilde{N}_k = \bigoplus_{\Sigma} \Xi_{h_z}, \quad \tilde{M}_k = \bigoplus_{\Sigma} \Xi_{h_z} \quad (\forall k \in \mathbb{Z}).
\]

This follows from the argument in [DiSa 24, 5.1] by using the isomorphisms in (1.3.7), see also the proof of Theorem 1.5 below.

By the theory of Gauss-Manin connections on Brieskorn lattices (see [Br 70]) in the weighted homogeneous polynomial case, we have the finite direct sum decompositions

\[
\Xi_{h_z} = \bigoplus_{\alpha \in \mathbb{Q}} \Xi_{h_z}^\alpha \quad (z \in \Sigma).
\]

defined by

\[
\Xi_{h_z}^\alpha := \text{Ker}(\partial_t - \alpha) \subset \Xi_{h_z} \quad (\alpha \in \mathbb{Q}, z \in \Sigma),
\]

which is independent of the choice of analytic local coordinates. (Note that we have a well-defined action of \( \partial_t \) on \( \Xi_{h_z} \), since \( h_z \) is contained in the ideal generated by its partial derivatives, see (3.2.2) below.)

1.5. Compatibility of duality isomorphisms. In this section we show the following.

**Theorem 1.5.** Under the isomorphisms in (1.4.4), the duality isomorphism between \( \tilde{N}_k \) and \( \tilde{M}_{nd-k} \) induced from the self-duality isomorphism for the Koszul complex in [DiSa 24, Theorem 1] (using the graded local duality as in [DiSa 24, 1.1.4 and 1.7.3]) is identified up to a constant multiple with the direct sum of the canonical self-duality isomorphisms of \( \Xi_{h_z} \) for \( z \in \Sigma \).

**Proof.** The self-duality isomorphism used in the proof of [DiSa 24, Theorem 1] is a canonical one, and is induced by the canonical graded isomorphism

\[
\Omega^r = \text{Hom}_R(\Omega^{r-j}, \Omega^n),
\]

where \( \Omega^n[n] \) is the graded dualizing complex, see also [Ei 05], [Ha 66], etc. (There is no shift of grading here, and it appears in the duality isomorphisms in [DiSa 24, Theorem 1].)

The duality isomorphisms are compatible with the localization by \( y \) in (1.3.4) (see also [DiSa 24, 5.1]). Here we consider the graded modules only over \( \mathbb{C}[y] \) or \( \mathbb{C}[y, y^{-1}] \) as in [DiSa 24, Remark 1.7], and then consider the graded duals also over \( \mathbb{C}[y] \) or \( \mathbb{C}[y, y^{-1}] \). The graded dual over \( \mathbb{C}[y] \) is defined by using the graded dualizing complex \( \mathbb{C}[y] \mathcal{D}[y][1] \), where the degree of complex is shifted by 1, and the degree of grading is also shifted by 1 because of \( dy \). We have the same with \( \mathbb{C}[y] \) replaced by \( \mathbb{C}[y, y^{-1}] \).

As for the relation with the graded local duality as in [DiSa 24, 1.1.4 and 1.7.3], we can take graded free generators \( \{u_i\}, \{v_i\} \) of \( M'' \), \( N \) over \( \mathbb{C}[y] \) such that \( u_i \in M''_i \), \( v_i \in N_{nd+1-k_i} \), with \( k_i \in \mathbb{Z}_{>0} \), and \( \mathbb{C}[y]u_i \) is orthogonal to \( \mathbb{C}[y]v_j \) for \( i \neq j \). (Here the shift of grading by 1 comes from the degree of \( dy \) in the above remark.) We can get the \( u_i \) by using the filtration on \( M_k \) defined by \( M_k \cap y^p M''_p \) \( (p \in \mathbb{Z}) \), and similarly for the \( v_i \). Then \( \{y^{k_i-p} u_i\} \) and \( \{y^{k_i-p-nd-1} u_i\} \) for \( p > 0 \) are \( \mathbb{C} \)-bases of \( \tilde{M}_p = M''_p \) and \( \tilde{N}_p = (\tilde{N}/N)_{-p} \), which are orthogonal to each other under the pairing given by [DiSa 24, 1.1.4 and 1.7.3].
By the above argument, it is enough to consider the graded \(\mathbb{C}[y, y^{-1}]\)-dual of \(\tilde{M}\) instead of the \(\mathbb{C}\)-dual of the \(M_p\) for \(p \gg 0\). In particular, we can neglect the shift of grading in the self-duality isomorphism (by using the isomorphisms in (1.3.5)) for the proof of Theorem 1.5.

Consider the blow-up along the origin

\[ \pi : \tilde{X} \to X := \mathbb{C}^n. \]

(This blow-up is used only for the coordinate change as is explained below. Here we may assume \(y = x_n\) replacing \(x_n\) if necessary.)

Let \(\tilde{X}' \subset \tilde{X}\) be the complement of the total transform of \(\{ y = 0 \} \subset X\). We have the isomorphism

\[ \tilde{X}' = \mathbb{C}^{n'} \times \mathbb{C}^* \quad \text{with} \quad n' = n-1. \]

(This must be distinguished from the isomorphism \(\tilde{X}' = X \setminus \{ y = 0 \} \cong \mathbb{C}^{n'} \times \mathbb{C}^*\), where the effect of the blow-up is neglected, since the coordinates \(x_i\) are used here instead of the \(x_i'\) defined below.)

In the notation of 1.3, set \(x'_i := x_i/y\ (i \in [1, n'])\), and

\[ R' := \mathbb{C}[x'_1, \ldots, x'_n]. \]

Then

\[ (1.5.1) \quad \pi^* f|_{\tilde{X}'} = y^d h \quad \text{with} \quad h := f/y^d \in R', \]

see also (1.4.1). This decomposition is compatible with the decomposition (1.5.1), where the \(x'_i\) and \(y\) are identified with the coordinates of \(\mathbb{C}^{n'}\) and \(\mathbb{C}^*\) respectively. Since

\[ \partial_y(y^d) = dy^{d-1}y, \quad \partial_{x_i}(y^d) = y^d h_i \quad \text{with} \quad h_i := \partial_{x_i} h \in R', \]

the localized Koszul complex \(K_{y^d h}^*\) for the partial derivatives of \(y^d h\) on \(\tilde{X}' = Y' \times \mathbb{C}^*\) can be identified, up to a non-zero constant multiple, with the shifted mapping cone

\[ (1.5.3) \quad C(h : K^*_h[y, y^{-1}] \to K^*_h[y, y^{-1}])[-1], \]

where \(K^*_h[y, y^{-1}]\) is the scalar extension by \(\mathbb{C} \hookrightarrow \mathbb{C}[y, y^{-1}]\) of the Koszul complex \(K^*_h\) defined by

\[ h_i \in \text{End}(R') \quad (i \in [1, n']). \]

Note that \(y^d dy\) and \(y^d\) can be omitted in the above descriptions, since they express only the shift of grading, and can be neglected by using the isomorphisms in (1.3.5) as is explained above. (Indeed, only \(y\) has non-zero degree, since \(\deg y = 1\) and \(\deg h = \deg h_i = \deg x'_i = 0\).)

By (1.4.3) there is a quasi-isomorphism (or an isomorphism in the derived category)

\[ (1.5.4) \quad K^*_h[y, y^{-1}] \xrightarrow{\sim} \bigoplus_{z \in \Sigma} \Xi_{h_z}[y, y^{-1}][n'-1]. \]

Consider the filtration \(G\) on \(K^*_h[y^d h]\) such that

\[ \begin{align*}
\text{Gr}^0_G \tilde{K}^*_h &\cong K_h[y, y^{-1}] \quad \xrightarrow{\sim} \bigoplus_{z \in \Sigma} \Xi_{h_z}[y, y^{-1}][n'-1], \\
\text{Gr}^1_G \tilde{K}^*_h &\cong K_h[y, y^{-1}][-1] \quad \xrightarrow{\sim} \bigoplus_{z \in \Sigma} \Xi_{h_z}[y, y^{-1}][-n'-1],
\end{align*} \]

where \(K^*_h[y^d h]\) is identified with the mapping cone (1.5.3) and (1.5.4) is used for the second isomorphisms in the derived category).

By condition (WH) in Theorem 2, we have the vanishing of the morphisms

\[ h : \Xi_{h_z}[y, y^{-1}] \to \Xi_{h_z}[y, y^{-1}] \quad (\forall \, z \in \Sigma). \]

So the filtration \(G\) on \(\tilde{K}^*_h[y^d h]\) splits in the bounded derived category of graded \(R'[y, y^{-1}]\)-modules \(D^b(R'[y, y^{-1}])_{gr}\), and there is a (non-canonical) isomorphism

\[ (1.5.6) \quad \tilde{K}^*_h[y^d h] \cong \text{Gr}^0_G \tilde{K}^*_h[y^d h] \oplus \text{Gr}^1_G \tilde{K}^*_h[y^d h] \quad \text{in} \quad D^b(R'[y, y^{-1}])_{gr}, \]
A polynomial compatible with the filtration $G$ so that the $\text{Gr}_G^k$ of (1.5.6) are the identity morphisms $(k = 0, 1)$. Here we use the isomorphism

$$\text{Hom}_A(M, M') = \text{Hom}_{D^b(A)}(M, M'),$$

for objects $M, M'$ of an abelian category $A$ in general.

The self-duality isomorphism for $\tilde{K}_y^{\ast}$, is compatible with the filtration $G$, and induces a duality between

$$\text{Gr}_G^0 \tilde{K}_y^{\ast} \text{ and } \text{Gr}_G^1 \tilde{K}_y^{\ast},$$

which can be identified, up to a non-zero constant multiple and also a shift of grading, with the canonical self-duality of $K_y^\ast [y, y^{-1}]$ by the above argument. Moreover the last self-duality is the scalar extension by $\mathbb{C} \to \mathbb{C}[y, y^{-1}]$ of the self-duality of the Koszul complex $K_y^\ast$. So the assertion follows (since the above induced duality between $\text{Gr}_G^0 \tilde{K}_y^{\ast}$ and $\text{Gr}_G^1 \tilde{K}_y^{\ast}$ is sufficient for the proof of Theorem 1.5). This finishes the proof of Theorem 1.5.

**Remark 1.5a** The self-duality of $\Xi_h^\ast$, is given by using the so-called residue pairing (see for instance [Ha 66]) which is compatible with the direct sum decompositions (1.4.5), and implies the duality between $\Xi_h^\ast$ and $\Xi_h^{\ast}$.

**Remark 1.5b** In the notation of the introduction, $N_k$ and $M''_{dn-k}$ are orthogonal subspaces to each other by the duality in Theorem 1.5. This is compatible with Corollary 3.

**1.6. Proof of Proposition 1.** The assertion follows from the filtered isomorphisms (1.2.8) together with the identification of the pole order spectral sequence with the $M^{(r)}$, $N^{(r)}$, $d^{(r)}$ in 1.3. This finishes the proof of Proposition 1.

**1.7. Weighted homogeneous polynomials.** We say that $h$ is a **weighted homogeneous polynomial** with positive weights $(w_i)$ for a coordinate system $(y_i)$ of $Y := \mathbb{C}^n$, if $h$ is a linear combination of $\prod y_i^a_i$ with $\sum_i w_i a_i = 1$, where $w_i \in \mathbb{Q}_{>0}$, $a_i \in \mathbb{N}$. We have

$$\lambda^m h(y_1, \ldots, y_n) = h(\lambda^m y_1, \ldots, \lambda^m y_n) \quad (\lambda \in \mathbb{C}^*),$$

(1.7.1)

where $m_i := mw_i \in \mathbb{N}$ with $m$ the smallest positive integer satisfying $mw_i \in \mathbb{N}$, and (1.7.1) holds with $\lambda$ replaced by $g \in \mathcal{O}_Y$. This implies that condition (WH) in Theorem 2 is independent of the choice of a defining function $h_z$.

We also have

$$h = v(h) \quad \text{with} \quad v := \sum_i w_i y_i \partial y_i \in \mathcal{O}_Y.$$  

This implies that $h$ is quasihomogeneous, that is, $h \in (\partial h)$ with the notation in a remark after Theorem 2. The converse holds in the isolated singularity case, see [SaK 71].

**1.8. Steenbrink spectrum and Bernstein-Sato polynomials.** Let $h$ be a weighted homogeneous polynomial of $n$ variables with weights $w_1, \ldots, w_n$ as above. The **Steenbrink spectrum** $\text{Sp}(h) = \sum_{i=1}^{\mu_h} t^{\alpha_{h,i}}$ and the **spectral numbers** $\alpha_{h,i} \in \mathbb{Q}$ are defined by

$$0 < \alpha_{h,1} \leq \cdots \leq \alpha_{h,\mu_h} < n,$$

(1.8.1)

$$\# \{ i \mid \alpha_{h,i} = \alpha \} = \dim \text{Gr}_F^\mu H^{n-1}(F_h, \mathbb{C}) \lambda = \dim \Xi^\ast_h,$$

where $p := [n-\alpha]$, $\lambda := \exp(-2\pi i \alpha)$, and $\mu_h$ is the Milnor number of $h$, see [St 77b], [ScSt 85]. (Here $\Xi^\ast_h$ is as in 1.4.) We have moreover

$$b_h(s) = (s+1) \left[ \prod_{i=1}^{\mu_h} (s + \alpha_{h,i}) \right]_{\text{red}},$$

(1.8.2)

where $\left[ \prod (s + \beta_j)^{m_j} \right]_{\text{red}} := \prod (s + \beta_j)$ for $\beta_i \neq \beta_j$ $(i \neq j)$ and $m_j \in \mathbb{Z}_{>0}$ in general. This equality can be proved by combining [Ma 75] and [ScSt 85], [Va 82] (which show that the Bernstein-Sato polynomial and the Hodge filtration on the Milnor cohomology can be
obtained by using the Brieskorn lattice [Br 70], see also [Sat 75, St 77a]. It is also well known that
\begin{equation}
(1.8.3) \quad \text{Sp}(h) = \prod_{j=1}^{n} (t^{\nu_j} - t)/(1 - t^{\nu_j}).
\end{equation}
This implies the symmetry of spectral numbers
\begin{equation}
(1.8.4) \quad \alpha_{h,i} = \alpha_{h,j} \quad (i + j = \mu_h + 1).
\end{equation}
Taking the limit of (1.8.3) for \( t \to 1 \), we also get
\begin{equation}
(1.8.5) \quad \mu_h = \prod_{j=1}^{n} \left( \frac{1}{w_j} - 1 \right).
\end{equation}
This is well known in the Brieskorn-Pham type case, that is, if \( w_i = a_i^{-1} \) with \( a_i \in \mathbb{N} \).

**Remark 1.8.** We can prove (1.8.3) by using the Koszul complex \( K_h^n \) as in (1.2.4) with \( f \) replaced by \( h \). Indeed, the denominator of the right hand side of (1.8.3) gives the Hilbert series of the graded ring \( R \) with \( \deg x_i = w_i \), and the numerator corresponds to the shift of the grading of each component of the Koszul complex (since \( \deg \partial_x, h = 1 - w_i \)), where the complex is viewed as the associated single complex of an \( n \)-ple complex associated with the multiplications by partial derivatives \( \partial_x, h (i \in [1, n]) \), see for instance [Se 75, Section IV.2].

We can use (1.8.3) for an explicit computation using Macaulay2, see A.1 below.

### 1.9. Extremely degenerated hypersurfaces.

A reduced hypersurface \( Z \subset \mathbb{P}^{n-1} \) or its defining polynomial \( f \) is called extremely degenerated if for some coordinates \( x_i \) of \( \mathbb{C}^n \), the polynomial \( f \) is a linear combination of monomials \( x^\nu \) with \( \nu = (\nu_1, \ldots, \nu_n) \in \mathbb{N}^n \) satisfying
\begin{equation}
\sum_{i=1}^{n} c_i \nu_i = 0
\end{equation}
for some fixed \( (c_1, \ldots, c_n) \in \mathbb{Q}^n \) with at least two of the \( c_i \) nonzero.

In the case \( n = 3 \) with variables \( x, y, z \), the polynomial \( f \) must contain two monomials \( x^i y^j z^k, x^{i'} y^{j'} z^{k'} \) with \( i, j, k, i', j', k' \in \{0, 1\} \) up to a permutation of variables, since \( Z \) is assumed to be reduced. Indeed, \( f \) must contain certain monomials corresponding to the conditions that \( f \) is not divisible by \( x^2, y^2, z^2 \), and the points of \( \mathbb{N}^3 \) corresponding to the monomials of \( f \) are contained in a line. We can calculate the reduced Bernstein-Sato polynomial \( b_f(s)/(s+1) \) if \( f = x^a y^d - z^d (0 < a < d) \) using the Thom-Sebastiani type theorem (see [Sa 94]), where the \( c_i \) are \( a-d, a, 0 \).

In the extremely degenerated curve case, it seems that the irreducible components of \( Z \) are rational curves and the complement \( U := \mathbb{P}^2 \setminus Z \) has Euler characteristic 0 or 1, see also 2.7 below. It seems to be 0 if \( Z \) contains an ordinary double point. Note that this cannot be determined by the \( c_i \); for instance, \( f = x^3 y^3 + xyz^4 \) and \( f = x^3 y^3 + z^6 \).

**Remark 1.9a.** Assume \( f \) is extremely degenerated or more generally \( \nu_{d+n} \neq 0 \). Then \( f \) is very special type, see Theorem 4 for the definition. Indeed, the assumption means that
\begin{equation}
(1.9.1) \quad \sum_{i=1}^{n} \ell_i \partial_{x_i} f = 0 \quad \text{with} \quad \ell_i \in R_1.
\end{equation}
Note that \( \ell_i = c_i x_i \) in the case \( f \) is extremely degenerated. We may assume \( \ell_n \neq 0 \) replacing the order of coordinates. We have \( y = \sum_{i=0}^{n} a_i x_i \) with \( a_i \in \mathbb{C}^* \), since \( y \) is general. We may then assume \( a_n = 1 \) multiplying \( y \) by a nonzero constant so that \( \partial_{x_n} y = 1 \). The constant vector fields \( \partial_{x_n} - a_i \partial_{x_n} \) are along the hypersurface \( X_{y'} := \{ y = 0 \} \subset \mathbb{C}^n \), and induce vector fields on \( X_{y'} \). So \( f \) is very special type, since
\begin{equation}
(1.9.2) \quad (\sum_{i=1}^{n-1} a_i \ell_i + \ell_n) \partial_{x_n} f = -\sum_{i=1}^{n-1} \ell_i (\partial_{x_i} - a_i \partial_{x_n}) f.
\end{equation}
Here \( \sum_{i=1}^{n-1} a_i \ell_i + \ell_n \neq 0 \), since the \( a_i \) are general.

Proposition 2 implies that the condition \( \nu_{d+3} \neq 0 \) is equivalent to that \( f \) is very special type, assuming that \( n = 3, d \geq 5 \), and all the singularities of \( Z \) are homogeneous.

**Remark 1.9b.** It does not seem easy to construct an example of a polynomial of very special type other than extremely degenerated ones. If one considers a deformation of homogeneous polynomial \( g \) in two variables of degree \( d \) having an isolated singularity at the origin by
adding \( hy \) to \( g \) with \( h \) a homogenous polynomial in two variables of degree \( d-1 \), then \( y \) is not a Lefschetz element in general.

2. Fundamental exact sequences of vanishing cycles

In this section we study the fundamental exact sequences of vanishing cycles, and prove Theorem 2.2 which is a key to the proof of Theorem 2.

2.1. Calculation of the Milnor fiber cohomology. In the notation and assumption of the introduction, set

\[
n' := \dim Y = n-1, \quad U := Y \setminus Z,
\]

with \( j : U \hookrightarrow Y \) the canonical inclusion. Set

\[
(2.1.1) \quad \Lambda_d := \{ \lambda \in \mathbb{C}^* \mid \lambda^d = 1 \}.
\]

Let \( H^*(F_j, \mathbb{C})_\lambda, H^c_*(F_j, \mathbb{C})_\lambda \) be the \( \lambda \)-eigenspaces of the monodromy \( T \). It is well known that \( T^d = id \), and hence \( H^*(F_j, \mathbb{C})_\lambda = 0 \) for \( \lambda \notin \Lambda_d \) (by using the geometric monodromy).

Indeed, we have more generally

\[
(2.1.2) \quad H^*(U, L_\lambda) = H^*(F_j, \mathbb{C})_\lambda, \quad H^c_*(U, L_\lambda) = H^c_*(F_j, \mathbb{C})_\lambda,
\]

see for instance [DiSa 24, 3.1.1]. More precisely, let \( \pi : \tilde{X} \to X \) be the blow-up at the origin, and set \( \tilde{f} := \pi^* f \). The exceptional divisor \( E \) is identified with \( Y \). Define

\[
(2.1.3) \quad L_\lambda := \psi_{\tilde{f}, \lambda} \mathcal{C}_{\tilde{X}}|_U.
\]

This is justified by [BuSa 05, Theorem 4.2], see also [BuSa 10, Section 1.6].

The second isomorphism of (2.1.2) follows from the first isomorphism by using the self-duality of the nearby cycle sheaves \( \psi_{\tilde{f}} \mathcal{C}_{\tilde{X}} \), which implies the isomorphisms

\[
(2.1.4) \quad L_\lambda^\vee = L_{\lambda^{-1}} \quad (\lambda \in \Lambda_d), \quad L_1 = \mathbb{C}_U,
\]

see also [BuSa 10, Sections 1.3–4]. The shifted local system \( \bigoplus_{\lambda \in \Lambda_d} L_\lambda[n'] \) naturally underlie a mixed Hodge module on \( U \) by (2.1.3), and the isomorphisms in (2.1.2) are compatible with mixed Hodge structures.

Note that the rank 1 local systems \( L_\lambda \) are uniquely determined by the local monodromies at smooth points of \( Z \), which are given by the multiplication by \( \lambda^{-1} \). Indeed, in the notation of (1.4.1), \( L_\lambda \) is uniquely determined by its restriction over \( U \cap Y' \) (using the direct image by \( U \cap Y' \hookrightarrow U \)), and we have the isomorphism

\[
(2.1.5) \quad L_\lambda|_{U \cap Y'} = h^* E_{\lambda^{-1}},
\]

where \( E_{\lambda^{-1}} \) is a rank 1 local system on \( \mathbb{C}^* \) with monodromy \( \lambda^{-1} \). (We can verify that \( L_\lambda|_{U \cap Y'} \otimes h^* E_\lambda \) is extendable as a rank 1 local system over \( Y' \setminus \Sigma \), and then over \( Y' = \mathbb{C}^{n'} \) since \( \dim Y' \geq 2 \).

Note that (2.1.5) also implies (2.1.4) as well as the isomorphisms

\[
(2.1.6) \quad \psi_{h, \eta} L_\lambda|_{U \cap Y'} = \psi_{h, \eta} \mathbb{C}_{U \cap Y'} \quad (\forall \eta \in \mathbb{C}^*).
\]

Indeed, we have more generally

\[
(2.1.7) \quad \psi_{h, \lambda} \mathcal{F} = \psi_{h, 0}(\mathcal{F} \otimes h^* E_{\lambda^{-1}}) \quad \text{for any } \mathcal{F} \in D_c^b(\mathbb{C}_{U \cap Y'}).
\]

It is well known (see [Mi 68] and also [DiSa 04, 2.1.2], etc.) that if \( \dim \text{Sing} f^{-1}(0) = 1 \), then

\[
(2.1.8) \quad \tilde{H}^j(U, L_\lambda) = \tilde{H}^*(F_j, \mathbb{C})_\lambda = 0 \quad \text{unless } j \in \{n'-1, n'\},
\]

with \( \tilde{H}^j(U, L_\lambda) := H^j(U, L_\lambda) \) for \( \lambda \neq 1 \). (Recall that \( L_1 = \mathbb{C}_U \), see (2.1.4).)
For \( z \in \Sigma = \text{Sing } Z \) (see (1.3.1)), there are distinguished triangles

\[
\begin{align*}
\mathbf{R} j_* \mathcal{L}_\lambda \to \psi_{h,1}(L\lambda|U \cap \gamma') \to N \psi_{h,1}(L\lambda|U \cap \gamma') \to \mathbf{R} j_* \mathcal{L}_\lambda.
\end{align*}
\]

The associated long exact sequence is strictly compatible with the Hodge filtration \( F \) coming from the theory of mixed Hodge modules [Sa90a] (by using for instance [Sa90b, Proposition 1.3]).

By (2.1.4) for \( \eta = 1 \) together with condition (WH) in Theorem 2, we have the vanishing of \( N \) in the long exact sequences associated with the distinguished triangles in (2.1.9). Using this, we can prove the canonical isomorphisms for \( z \in \Sigma \)

\[
\begin{align*}
(2.1.10) \quad (R^i j_* L_k) \cong \begin{cases} 
H^{n'-1}(F_{h,1})_{\lambda} & \text{if } i = n'-1, \\
H^{n'-1}(F_{h,1})(-1) & \text{if } i = n', \\
0 & \text{if } i \in [2, n'-2] \text{ or } i > n'.
\end{cases}
\end{align*}
\]

where \((p)\) for \( p \in \mathbb{Z} \) denotes the Tate twist in general (see [De71]), \( F_{h,1} \) is the Milnor fiber of \( h_z \), and the coefficient field \( \mathbb{C} \) of the cohomology is omitted to simplify the notation. These isomorphisms are compatible with the Hodge filtration \( F \) and the weight filtration \( W \) of mixed Hodge structures.

Actually we can also construct the isomorphisms in (2.1.10) by using another method. Indeed, let \( \mathbb{Q}_{h,X} := a^*_X \mathbb{Q} \in D^b\text{MHM}(X) \) with \( a_X : X \to pt \) the structure morphism. Let \( i_0 : \{0\} \hookrightarrow X := \mathbb{C}^n, j_0 : X \setminus \{0\} \hookrightarrow X \) be natural inclusions. Using [BuSa05, Theorem 4.2] together with the distinguished triangle

\[
\begin{align*}
(i_0)_* j_0^! \varphi_f \mathbb{Q}_{h,X} \to \varphi_f \mathbb{Q}_{h,X} \to (j_0)_* j_0^* \varphi_f \mathbb{Q}_{h,X} \xrightarrow{+1},
\end{align*}
\]

we can prove Theorem 2.2 below by showing the self-dual exact sequences of mixed Hodge structures (see also [DiSa04, 2.1]):

\[
\begin{align*}
0 \to H^{n'-1}(F_f)_{\not\equiv 1} & \to \bigoplus_{z \in \Sigma} H^{n'-1}(F_{h_z})_{T_1} \xrightarrow{\rho^z} H^{n'}(F_f)_{\not\equiv 1} \\
0 \to H^{n'}(F_f)_{\not\equiv 1} & \xrightarrow{\rho} \bigoplus_{z \in \Sigma} H^{n'-1}(F_{h_z})_{T_1}(-1) \to H^{n'+1}(F_f)_{\not\equiv 1} \to 0,
\end{align*}
\]

\[
\begin{align*}
0 \to H^{n'-1}(F_f)_{1} & \to \bigoplus_{z \in \Sigma} H^{n'-1}(F_{h_z})_{T_1} \xrightarrow{\rho^z} H^{n'}(F_f)_{1}(-1) \\
0 \to H^{n'}(F_f)_{1} & \xrightarrow{\rho} \bigoplus_{z \in \Sigma} H^{n'-1}(F_{h_z})_{T_1}(-1) \to H^{n'+1}(F_f)_{1}(-1) \to 0,
\end{align*}
\]

where \( E_{T_1} \) denotes the invariant part by the action of the local system monodromy \( T_1 \) on \( E := H^{n'-1}(F_{h_z}) \). These are called the fundamental exact sequences of vanishing cycles. Since this proof is rather long, it is omitted in this version. In the proof of Theorem 2.2 in 2.3–4 below, we will construct long exact sequence which should correspond to the above ones (except for the case \( \lambda = 1, n = 3 \)). For the moment it is still unclear whether these two canonical isomorphisms really coincide (even though this is very much expected to hold).

2.2. Key theorem to the proof of Theorem 2. We show the following theorem whose proof will be given in Sections 2.3–6.

**Theorem 2.2.** With the notation and assumption of 2.1 together with the assumption (WH) in Theorem 2, there is a canonical isomorphism compatible with the Hodge filtration \( F \) :

\[
\begin{align*}
(2.2.1) \quad \text{Coker} \left( H^{n'}(F_f)_{\lambda} \xrightarrow{\rho} \bigoplus_{z \in \Sigma} H^{n'-1}(F_{h_z})_{\lambda}(-1) \right) \cong \begin{cases} 
H^{n'-1}(F_f)^{\vee}_{\lambda^{-1}}(-n') & \text{if } \lambda \not= 1, \\
H^{n'-1}(F_f)^{\vee}(-n'-1) & \text{if } \lambda = 1,
\end{cases}
\end{align*}
\]

where \( F_f \) is the Milnor fiber of \( f \) as in the introduction, \( \rho \) is a canonical morphism, and the coefficient field \( \mathbb{C} \) of the cohomology is omitted to simplify the notation. Moreover \( \rho \) can be identified via the isomorphisms as in (2.1.12) and (2.1.10) with the following canonical morphism:

\[
\begin{align*}
(2.2.2) \quad H^{n'}(U, L_\lambda) \to \bigoplus_{z \in \Sigma} (R^n j_* L_k)_{z}.
\end{align*}
\]
**Remark 2.2.** By the first injective morphisms in (2.1.11–12), \( H^{n'-1}(F_j) \neq 1 \) and \( H^{n'-1}(F_j)_1 \) are identified respectively with subspaces of
\[
\bigoplus_{z \in \Sigma} H^{n'-1}(F_{h_z}) \neq 1, \quad \bigoplus_{z \in \Sigma} H^{n'-1}(F_{h_z})_1,
\]
which are pure \( \mathbb{Q} \)-Hodge structures of weight \( n'-1 \) and \( n' \) under the assumption (WH) in Theorem 2, see \([St \, 77a]\). In particular, polarizations of Hodge structures induce self-duality isomorphisms of \( \mathbb{Q} \)-Hodge structures
\[
(2.2.3) \quad H^{n'-1}(F_j) \neq 1 \cong H^{n'-1}(F_j)^{\vee}_{\neq 1}(1-n'), \quad H^{n'-1}(F_j)_1 \cong H^{n'-1}(F_j)^{\vee}_1(-n').
\]
These are closely related to the difference in the Tate twist on the right-hand side of (2.2.1) for \( \lambda \neq 1 \) and \( \lambda = 1 \).

We give a proof of Theorem 2.2 (which is closely related to \([Di \, 92]\), \([Kl \, 14]\)) by dividing it into three cases.

**2.3. Proof of Theorem 2.2 for \( \lambda \neq 1 \).** Set
\[
Y^\circ := Y \setminus \Sigma \text{ with } j^\circ : Y^\circ \hookrightarrow Y \text{ the inclusion.}
\]
Let \( L^\circ_\lambda \) be the zero extension of \( L_\lambda \) over \( Y^\circ \). There is a distinguished triangle
\[
(2.3.1) \quad j^\circ_! L^\circ_\lambda \to \mathbf{R} j^\circ_! L^\circ_\lambda \to \bigoplus_{z \in \Sigma} (\mathbf{R} j_* L_\lambda)_z \xrightarrow{+1},
\]
together with isomorphisms
\[
(2.3.2) \quad j^\circ_! L^\circ_\lambda = j_! L_\lambda, \quad \mathbf{R} j^\circ_! L^\circ_\lambda = \mathbf{R} j_* L_\lambda.
\]
These induce exact sequences
\[
(2.3.3) \quad 0 \to H^{n'-1}(U, L_\lambda) \to \bigoplus_{z \in \Sigma} (R^{n'-1} j_* L_\lambda)_z \to H_c^{n'}(U, L_\lambda)

\to H^{n'}(U, L_\lambda) \to \bigoplus_{z \in \Sigma} (R^{n'} j_* L_\lambda)_z \to H_c^{n'+1}(U, L_\lambda) \to 0,
\]
which are essentially self-dual by replacing \( \lambda \) with \( \lambda^{-1} \).

By (2.1.2), (2.1.10), these give essentially self-dual exact sequences
\[
(2.3.4) \quad 0 \to H^{n'-1}(F_j)_\lambda \to \bigoplus_{z \in \Sigma} H^{n'-1}(F_{h_z})_\lambda \xrightarrow{\rho^\vee} H_c^{n'}(F_j)_\lambda

\to H^{n'}(F_j)_\lambda \xrightarrow{\rho} \bigoplus_{z \in \Sigma} H^{n'-1}(F_{h_z})_\lambda(-1) \to H_c^{n'+1}(F_j)_\lambda \to 0.
\]
It is expected that their direct sum over \( \lambda \in \Lambda_d \setminus \{1\} \) would be identified with (2.1.11). Applying Poincaré duality to the last term of (2.3.4), we get Theorem 2.2 in the case \( \lambda \neq 1 \).

**2.4. Proof of Theorem 2.2 for \( \lambda = 1, \, n' \geq 3 \).** We assume for the moment *only* \( \lambda = 1 \).

Set
\[
Z^\circ := Z \setminus \Sigma \text{ with } j_Z : Z^\circ \hookrightarrow Z \text{ the inclusion.}
\]
We have the distinguished triangle
\[
(2.4.1) \quad \mathbb{Q} Y^\circ \to \mathbf{R} j^\circ_* \mathbb{Q} U \to \mathbb{Q} Z^\circ(-1)[-1] \xrightarrow{+1},
\]
where \( j^\circ : U \hookrightarrow Y^\circ \) is the inclusion, and \( Y^\circ := Y \setminus \Sigma \) with \( j^\circ : Y^\circ \hookrightarrow Y \) as in 2.3. Applying \( \mathbf{R} j^\circ_* \) to (2.4.1), we get the distinguished triangle
\[
(2.4.2) \quad \mathbf{R} j^\circ_* \mathbb{Q} Y^\circ \to \mathbf{R} j_* \mathbb{Q} U \to \mathbf{R} (j_Z)_* \mathbb{Q} Z^\circ(-1)[-1] \xrightarrow{+1}.
\]
These naturally underlie distinguished triangles of complexes of mixed Hodge modules.

Setting \( m := |\Sigma| \), we have
\[
(2.4.3) \quad H^i(Y^\circ) = \begin{cases} 
\mathbb{Q}(-j) & \text{if } i = 2j \text{ with } j \in [0, n'-1], \\
\bigoplus_{m-1} \mathbb{Q}(-n') & \text{if } i = 2n'-1, \\
0 & \text{otherwise}.
\end{cases}
\]
Assume \( n' \geq 3 \) from now on. The distinguished triangle (2.4.2) implies the isomorphisms for \( z \in \Sigma \):

\[
(R^i j_! \mathbb{Q})_z = (R^{i-1}(j_!)_* \mathbb{Q}_{Z^o})_z(-1) \quad (i \in [1, 2n'-3]),
\]
as well as the short exact sequences of mixed Hodge structures

\[
0 \to H^{i+1}(U) \to H^i(Z^o)(-1) \to H^{i+2}(Y^o) \to 0 \quad (i \in \mathbb{N}),
\]
where the coefficient field \( \mathbb{Q} \) of the cohomology is omitted to simplify the notation. Indeed, the restriction morphisms \( H^i(Y^o) \to H^i(U) \) vanish for any \( i > 0 \) (where we use (2.4.4) together with \( 2n'-1 > n' \) in the case \( i = 2n'-1 \) in (2.4.3)).

Define the primitive part by

\[
H^i_{prim}(Z^o) := \ker (H^i(Z^o) \to H^{i+2}(Y^o)(1)) \quad (i \in \{n'-2, n'-1\}).
\]

By (2.4.6) we get the isomorphisms

\[
H^i_{prim}(Z^o) = H^{i+1}(U)(1) \quad (i \in \{n'-2, n'-1\}).
\]

Define the primitive part for the cohomology with compact supports by

\[
H^i_{c,prim}(Z^o) := H^{2n'-2-i}_{prim}(Z^o)(1-n') \quad (i \in \{n'-1, n'\}).
\]

By (2.4.7) this vanishes for \( i = n'-2 \), and we have

\[
H^i_{c,prim}(Z^o) = H^{2n'-i-1}(U)(-n') = H^{i+1}_{c}(U) \quad (i \in \{n'-1, n'\}).
\]

On the other hand, there is a distinguished triangle

\[
(j_!)(j_! \mathbb{Q}_{Z^o}) \to R(j_!)_* \mathbb{Q}_{Z^o} \to (R(j_!)_* \mathbb{Q}_{Z^o})|_{\Sigma^+} \to_{+1},
\]
underlying naturally a distinguished triangle of complexes of mixed Hodge modules. By 2.5 below, this induces a self-dual exact sequence of mixed Hodge structures

\[
0 \to H^{n'-2}_{prim}(Z^o) \to \bigoplus_{z \in \Sigma} (R^{n'-2}(j_!)_* \mathbb{Q}_{Z^o})_z \to H^{n'-1}_{c,prim}(Z^o)
\]

\[
0 \to H^{n'-1}_{prim}(Z^o) \to \bigoplus_{z \in \Sigma} (R^{n'-1}(j_!)_* \mathbb{Q}_{Z^o})_z \to H^{n'}_{c,prim}(Z^o) \to 0,
\]
assuming \( n' \geq 3 \). Combined with (2.1.10), (2.4.5), and (2.4.7–8), this gives a self-dual exact sequence of mixed Hodge structures

\[
0 \to H^{n'-1}(U) \to \bigoplus_{z \in \Sigma} H^{n'-1}(F_{h_0})_1 \to H^{n'}_{c}(U)(-1)
\]

\[
0 \to H^{n'}(U) \to \bigoplus_{z \in \Sigma} H^{n'-1}(F_{h_0})_1(-1) \to H^{n'+1}_{c}(U)(-1) \to 0,
\]
(which is expected to be identified with (2.1.12) in this case). Theorem 2.2 then follows in the case \( \lambda = 1 \) and \( n' \geq 3 \) by using Poincaré duality for \( U \) together with (2.1.2).

2.5. Proof of the exact sequence (2.4.10). Define the non-primitive part by

\[
H^i_{c, np}(Z^o) := \text{Im}(H^i_c(Z^o) \to H^i_c(Y^o)) \quad (i \in \{n'-1, n'\}),
\]

\[
H^i_{np}(Z^o) := \text{Im}(H^i(Y^o) \to H^i(Z^o)) \quad (i \in \{n'-1, n'-2\}).
\]

For \( i = n'-1 \), we have a canonical isomorphism

\[
H^{n'-1}_{c, np}(Z^o) \cong H^{n'-1}_{np}(Z^o),
\]
by using the canonical isomorphism \( H^{n'-1}_{c}(Y^o) \cong H^{n'-1}(Y^o) \).
For \( i = n' \), a similar argument implies the injectivity of the canonical isomorphism

\[
H_{c,np}^{n'}(Z^\circ) \hookrightarrow H^n(Z^\circ).
\]

(2.5.2)

There are moreover canonical isomorphisms

\[
H_{c,prim}^i(Z^\circ) = H_{c}^i(Z^\circ)/H_{c,np}^i(Z^\circ) \quad (i \in \{n'-1, n'\}),
\]

(2.5.3)

\[
H_{prim}^i(Z^\circ) = H^i(Z^\circ)/H_{np}^i(Z^\circ) \quad (i \in \{n'-1, n'-2\}).
\]

Indeed, the first isomorphism easily follows from the definition (since the dual of the kernel of a morphism is the cokernel of the dual morphism in general), and the second follows from the bijectivity of the composition

\[
H^i(Y^\circ) \rightarrow H^i(Z^\circ) \rightarrow H^{i+2}(Y^\circ)(1) \quad (i \in \{n'-1, n'-2\}).
\]

By (2.5.1) and (2.5.3) for \( i = n'-1 \), we get the exactness of the middle part of (2.4.10) using the long exact sequence associated with (2.4.9). By the self-duality of (2.4.10) together with the first isomorphism of (2.5.3) for \( i = n' \) and the injectivity of (2.5.2), it now remains to show the surjectivity of the morphism to \( H_{c,prim}^{n'}(Z^\circ) \) in (2.4.10).

Since \( H_{c,prim}^{n'}(Z^\circ) \) has weights \( \leq n' \) (see [De 71], [Sa 90a]), it is enough to show

\[
H^{n'}(Z^\circ)/H_{c,np}^{n'}(Z^\circ) \quad \text{has weights} \quad > n',
\]

by using the long exact sequence associated with (2.4.9) together with the injectivity of (2.5.2). However, (2.5.4) easily follows from (2.4.4) and (2.4.6). This finishes the proof of the exact sequence (2.4.10).

2.6. **Proof of Theorem 2.2 for \( \lambda = 1, n' = 2 \).** In this case, \( Z \) is a curve on \( Y = \mathbb{P}^2 \). Using the distinguished triangle (2.4.2) together with the snake lemma, we can get the commutative diagram of exact sequences

\[
\begin{array}{ccc}
H^2(U) & \hookrightarrow & H^1(Z^\circ)(-1) & \twoheadrightarrow & H^3(Y^\circ) \\
\downarrow \rho & & \downarrow & & \cap \\
\bigoplus_z (R^2j_*\mathcal{Q}_U)_z & \hookrightarrow & \bigoplus_z (R^1j_*\mathcal{Q}_{Z^\circ}(1))_z & \twoheadrightarrow & \bigoplus_z (R^3j'_*\mathcal{Q}_{Y^\circ})_z \\
\downarrow \text{Coker } \rho & & \downarrow & & \downarrow \\
& & H^2(Z^\circ)(-1) & \twoheadrightarrow & H^4(Y^\circ)
\end{array}
\]

(2.6.1)

where the direct sums are taken over \( z \in \Sigma \), see [Di 92, Ch. 6, 3.14] and Remark 2.6 below. (As for the right vertical exact sequence, note that \( \dim H^3(Y^\circ) = m - 1, \dim H^4(Y^\circ) = 1 \).

We have moreover the canonical isomorphism

\[
\text{Coker } \rho = H^1(U)^{\vee}(-3),
\]

(2.6.2)

using the dual of the isomorphism

\[
H^1(U) = \text{Coker} \left( H^0(Y^\circ) \rightarrow H^0(Z^\circ) \right)(-1).
\]

(2.6.3)

The latter follows from (2.4.6) for \( i = 0 \) by using the bijectivity of the composition

\[
H^0(Y^\circ) \rightarrow H^0(Z^\circ) \rightarrow H^2(Y^\circ)(1).
\]

So (2.2.1) also follows in the case \( \lambda = 1, n' = 2 \). This finishes the proof of Theorem 2.2.

**Remark 2.6.** In the case \( n' = 2 \), let \( r_z \) and \( r_Z \) be respectively the number of local and global irreducible components of \((Z, z)\) and \( Z \). Then

\[
\dim(R^2j_*\mathcal{Q}_U)_z = r_z - 1, \quad \dim(R^1j_*\mathcal{Q}_{Z^\circ})_z = r_z, \quad \dim(R^3j'_*\mathcal{Q}_{Y^\circ})_z = 1.
\]

(2.6.4)

\[
\dim H^1(U) = r_Z - 1, \quad \dim H^0(Z^\circ) = r_Z, \quad \dim H^0(Y^\circ) = \dim H^2(Y^\circ) = 1.
\]

(2.6.5)

**2.7. Relation with the Euler characteristics.** Set \( U := Y \setminus Z \). By (2.1.2) we have

\[
\sum_i (-1)^i \dim H^i(F_j, \mathcal{O}_\lambda) = \chi(U) \quad (\lambda \in \Lambda_d).
\]

(2.7.1)
In the case the pole order spectral sequence degenerates at $E_2$ as in Theorem 2, we then get

\begin{equation}
\sum_{j \in \mathbb{Z}} (\mu_{k+jd} - \nu_{k+jd+d}) = \begin{cases} 
(-1)^n\chi(U) & \text{if } k \in \mathbb{Z} \setminus d\mathbb{Z}, \\
(-1)^n(\chi(U) - 1) & \text{if } k \in d\mathbb{Z}.
\end{cases}
\end{equation}

Here $n' = n-1$. On the other hand, it is well known that $\chi(U) = n' + 1 - \chi(Z)$ (since $\chi(\mathbb{P}^n') = n' + 1$), and for a smooth hypersurface $V \subset \mathbb{P}^n'$ of degree $d$, we have

\begin{equation}
\chi(Z) = \chi(V) - (-1)^{n'-1}\mu_Z \quad \text{with} \quad \mu_Z := \sum_{z \in \Sigma} \mu_z,
\end{equation}

by using a one-parameter family $\{f + sg\}_{s \in \mathbb{C}}$ with $V = \{g = 0\}$, where $\mu_z$ is the Milnor number of $h_z$, see [De 73]. (Here we may assume $V \cap \text{Sing } Z = \emptyset$ since $\text{Sing } Z$ is isolated.) So we get

\begin{equation}
\chi(U) = \chi(\mathbb{P}^n' \setminus V) - (-1)^n\mu_Z.
\end{equation}

In the case $n' = 2$, we have

\begin{equation}
\chi(U) = (d-1)(d-2) + 1 - \mu_Z.
\end{equation}

since a well-known formula for the genus of smooth plane curves implies that

\begin{equation}
\chi(\mathbb{P}^2 \setminus V) = (d-1)(d-2) + 1.
\end{equation}

### 3. Calculation of twisted de Rham complexes

In this section we calculate the filtered twisted de Rham complexes associated with weighted homogeneous polynomials having isolated singularities.

#### 3.1. Twisted de Rham complexes

Let $h$ be a weighted homogeneous polynomial for a local coordinate system $(y_1, \ldots, y_n')$ of $(Y, 0) := (\mathbb{C}^{n'}, 0)$ (as a complex manifold) with positive weights $w_1, \ldots, w_n'$, see 1.7. We assume moreover that $h^{-1}(0)$ has an isolated singularity at $0$. We take and fix

\[ \alpha \in \mathbb{Q} \cap (0, 1). \]

Consider the twisted de Rham complex

\begin{equation}
K^*(h, \alpha) := \Omega^*_{Y,0}[h^{-1}]h^{-\alpha} \cong (\Omega^*_{Y,0}[h^{-1}]; d - \alpha \frac{dh}{h} \wedge),
\end{equation}

where $\Omega^*_{Y}$ is the analytic de Rham complex. Define the pole order filtration $P$ by

\begin{equation}
P^kK^i(h, \alpha) := \Omega^i_{Y,0} h^{-\alpha + k - i} \quad (i, k \in \mathbb{Z}),
\end{equation}

so that

\begin{equation}
\text{Gr}_P^kK^*(h, \alpha) \cong (\Omega^*_{Y,0}/P^k\Omega^*_{Y,0}; dh \wedge).
\end{equation}

Set $P_k = P^{-k}$ as usual. Note that $K^*(h, \alpha)$ is a graded complex with degrees defined by

\begin{equation}
\deg y_i = \deg dy_i = w_i, \quad \deg h = \deg dh = 1, \quad \deg h^{-\alpha} = -\alpha,
\end{equation}

and $P$ is compatible with the grading. Note that the $H^i(K^*(h, \alpha))$ ($i \in \mathbb{Z}$) are graded $\mathbb{C}$-vector spaces, and have the induced filtration $P$ compatible with the grading.

Set $\lambda := \exp(-2\pi i \alpha)$. We have the isomorphism

\begin{equation}
H'^u(K^*(h, \alpha)) = H'^{-1}(F_{h}, \mathbb{C}) \lambda(-1).
\end{equation}

This follows from a generalization of (2.1.10) with $L_\lambda$ replaced by $h^*E_{h-1}$ as in (2.1.5), where $h$ is a weighted homogeneous polynomial for some local coordinates of a complex manifold $Y'$. Indeed, $\mathcal{O}_Y[h^{-1}]h^{-\alpha}$ is a regular holonomic $\mathcal{D}_Y$-module corresponding to $\mathcal{R}_{j,h^*E_{h-1}[n']}$ by the de Rham functor $\text{DR}_Y$ so that we have a canonical isomorphism in the derived category

\begin{equation}
K^*(h, \alpha) = \mathcal{R}_{j,h^*E_{h-1}},
\end{equation}
where $E_{\lambda-1}$ is as in the explanation after (2.1.5), and $j$ is the inclusion of the complement of $h^{-1}(0)$.

### 3.2. Brieskorn modules

In the above notation and assumption, define the Brieskorn module $H''_n$ (see [Br 70]) by

$$
H''_n := \Omega''_{Y,0}/dh \wedge d\Omega''_{Y,0},
$$

where $n' = \dim Y$. This is a graded module with degrees defined by (3.1.4).

The actions of $t$ and $\partial_t$ on $H''_n$ are defined by

$$
t[\omega] := [h\omega], \quad \partial_t[\omega] := d\eta \quad \text{with} \quad dh \wedge \eta = h\omega,
$$

where $\omega \in \Omega''_{Y,0}$, $\eta \in \Omega''_{Y,0}$. It is well known that $H''_n$ is a free $\mathbb{C}\{t\}$-module of rank $\mu$ with $\mu$ the Milnor number of $h$. For the definition of $\partial_t$, we use a property of the weighted homogeneous polynomial that $h$ is contained in the Jacobian ideal $(\partial h) \subset O_{Y,0}$, see 1.7. More precisely, using (1.7.2), it is easy to show that

$$
\partial_t[\omega] = \beta[\omega] \quad \text{for} \quad [\omega] \in H''_{h,\beta},
$$

(see the proof of Proposition 3.3 below), where $H''_{h,\beta} \subset H''_n$ is the degree $\beta$ part of graded module. (It is then easy to show the $t$-torsion-freeness of $H''_n$ by using (3.2.3) in the weighted homogeneous polynomial case, since $H''_{h,\beta} = 0$ for $\beta \leq 0$.)

### 3.3. Relation with Brieskorn modules

In this section we prove the following.

**Proposition 3.3.** With the above notation and assumption (in particular, $\alpha \in (0,1]$), we have

$$
H''(K^*(h, \alpha))_{\beta} = 0 \quad (\forall \beta \neq 0), \quad \text{that is,}
$$

$$
H''(K^*(h, \alpha)) = H''(K^*(h, \alpha))_0,
$$

and there are canonical isomorphisms

$$
t_k : H''_{h,\alpha+k} \overset{\sim}{\longrightarrow} P_{k-n'}H''(K^*(h, \alpha)) \quad (k \in \mathbb{N}),
$$

satisfying

$$
t_{k+i} \circ t_i = t_k \quad \text{in} \quad H''(K^*(h, \alpha)) \quad (k, i \in \mathbb{N}),
$$

where $t_i : H''_{h,\alpha+k} \hookrightarrow H''_{h,\alpha+k+i}$ is induced by the action of $t$ on $H''_n$ defined in (3.2.2).

**Proof.** Set

$$
\omega_0 := dy_1 \wedge \cdots \wedge dy_{n'}, \quad \eta_0 := \sum_i (-1)^{i-1}w_i y_i dy_1 \wedge \cdots \wedge \hat{dy}_i \wedge \cdots \wedge dy_{n'}.
$$

The latter is the inner derivation (or contraction) of $\omega_0$ by the vector field $v$ in (1.7.2) so that

$$
dh \wedge g \eta_0 = hg \omega_0 \quad \text{for} \quad g \in O_{Y,0}.
$$

Set $\beta_0 := \sum_i w_i = \deg \omega_0$. We can easily verify that

$$
d(g \eta_0) = \beta g \omega_0 \quad \text{for} \quad g \in (O_{Y,0})_{\beta-\beta_0} \quad \text{(that is,} \quad g\omega_0 \in (\Omega''_{Y,0})_\beta),
$$

by using the equality $v(g) = \beta g$ for $g \in (O_{Y,0})_\beta$ together with $\sum_i w_i \partial_y y_i = v + \beta_0$.

The assertion (3.3.1) then follows from (3.3.4–5) by the definition of the differential of $K^*(h, \alpha)$ in (3.1.1). (Note that (3.2.3) also follows from (3.3.4–5).)

Define the canonical morphisms $t_k$ in (3.3.2) by

$$
t_k[\omega] = [\omega h^{-\alpha-k}] \in H''(K^*(h, \alpha))_0 \quad \text{for} \quad \omega \in (\Omega''_{Y,0})_{\alpha+k}.
$$

Then (3.3.3) and the surjectivity of $t_k$ in (3.3.2) follow from the definition. For the well-definedness of (3.3.6), we have to show that

$$
 t_k[\omega] = 0 \quad \text{if} \quad \omega = dh \wedge d\xi \quad \text{with} \quad \xi \in (\Omega''_{Y,0})_{\alpha+k-1}.
$$
Set $\eta := d\xi$. Then
\begin{equation}
\tag{3.3.8}
d(\eta h^{-\alpha-k+1}) = -(\alpha + k - 1)dh \wedge \eta = -(\alpha + k - 1)\omega.
\end{equation}
Here we may assume $\alpha + k - 1 \neq 0$, since we have $d\xi = 0$ if $\xi \in (\Theta_{V,0}^{n'-2})_0$. (Note that $\xi$ may be nonzero if $n' = 2$.) So (3.3.7) follows from (3.3.8).

It now remains to show the injectivity of (3.3.2). For $k \gg 0$, this follows from the surjectivity of (3.3.2) by using (3.1.5). Indeed, the pole order filtration $P$ is exhaustive, and we have the isomorphisms
\begin{equation}
\tag{3.3.9}
H''_{\alpha+k} = H''_{\alpha+k-1}(F_h)e(-\alpha) \quad \text{if} \quad k \gg 0,
\end{equation}
compatible with the morphism $t : H''_{\alpha+k} \to H''_{\alpha+k+1}$ (where $e(-\alpha) := \exp(-2\pi i\alpha)$), see [Br70] (and also Remark 3.3 below). This implies the injectivity for any $k$ by using (3.3.3) together with the injectivity of the action of $t$ on $H''_h$, see the remark after (3.2.3). This finishes the proof of Proposition 3.3.

**Remark 3.3.** (i) The action of $\partial_t^{-1}$ on $H''_h$ can be defined in a compatible way with (3.2.2), and the Gauss-Manin system $G_h$ is the localization of the Brieskorn module $H''_h$ by the action of $\partial_t^{-1}$. It is a graded $\mathbb{C}[t]e(t^{-1})$-module, and (3.2.3) holds with $H''_h$ replaced by $G_h$. We have moreover the canonical isomorphisms
\begin{equation}
\tag{3.3.10}
G_{h,\alpha} = H''_{\alpha-1}(F_h)e(-\alpha),
\end{equation}
compatible with the morphism $\partial_t : G_{h,\alpha} \to G_{h,\alpha-1}$. Note that the condition $N = 0$ is equivalent to that $\partial t = \alpha$ on $H''_{h,\alpha}$. So the actions of $t$ and $\partial_t^{-1}$ on $H''_{h,\alpha}$ (and also on $G_{h,\alpha}$ ($\alpha \neq 0$)) can be identified with each other up to a non-zero constant multiple.

(ii) We have the canonical isomorphisms (see [ScSt 85], [Va 82]):
\begin{equation}
\tag{3.3.11}
F^{n'-1-k}H''_{\alpha-1}(F_h)e(-\alpha) = H''_{h,\alpha+k} \quad (\alpha \in (0,1], \ k \in \mathbb{N}),
\end{equation}
using the canonical isomorphism (3.3.9) together with the action of $t$ or $\partial_t^{-1}$ on $H''_h$ (by using $N = 0$ as is explained above).

4. Proof of the main theorems

In this section we prove Theorems 2-4 and Proposition 2 after reviewing some basics of Bernstein-Sato polynomials.

4.1. Bernstein-Sato polynomials. Let $f : X \to \mathbb{C}$ be a holomorphic function on a complex manifold in general. Let $i_f : X \hookrightarrow X \times \mathbb{C}$ be the graph embedding by $f$, and $t$ be the coordinate of $\mathbb{C}$. We have the canonical inclusion
\begin{equation}
\tag{4.1.1}
M := D_X[f^s] \hookrightarrow \mathcal{B}_f := (i_f)^*\mathcal{O}_X,
\end{equation}
where the last term is the direct image of $\mathcal{O}_X$ as a $D$-module, and is freely generated by $\mathcal{O}_X[\partial_s]$ over the canonical generator $\delta(f-t)$. Indeed, $f^s$ and $s$ can be identified respectively with $\delta(t-f)$ and $-\partial_t t$, see [Ma 75], [Ma 83]. We have the filtration $V$ of Kashiwara [Ka 83] and Malgrange [Ma 83] on $\mathcal{B}_f$ along $t = 0$ indexed decreasingly by $\mathbb{Q}$ so that $\partial_t t - \alpha$ is nilpotent on $\text{Gr}_V^\alpha \mathcal{B}_f$. This induces the filtration $V$ on $M$ and on $M/tM$ so that
\begin{equation}
\tag{4.1.2}
b_f(-\alpha) \neq 0 \iff \text{Gr}_V^\alpha(M/tM) \neq 0,
\end{equation}
since $b_f(s)$ is the minimal polynomial of the action of $-\partial_t$ on $M/tM$ by definition. (Here we assume that $b_f(s)$ exists by shrinking $X$ if necessary.) Note that the support of $\text{Gr}_V^\alpha(M/tM)$ is a closed analytic subset, and the above construction is compatible with the pull-back by smooth morphisms. These together with 4.2(i) below imply the last inclusion in (1) in the introduction.

Set
\begin{equation*}
G_i \mathcal{B}_f := t^{-i}M \subset \mathcal{B}_f[t^{-1}] \quad (i \in \mathbb{Z}).
\end{equation*}
We have the canonical inclusion $B_f \hookrightarrow B_f[t^{-1}]$, inducing the isomorphisms  
\[
Gr^\alpha_i B_f \simeq Gr^\alpha_i (B_f[t^{-1}]) \quad (\alpha > 0).
\]
This gives an increasing filtration $G$ on $Gr^\alpha_i B_f$ for $\alpha > 0$ so that  
\[
b_f(-\alpha - i) \neq 0 \iff Gr^\alpha_i Gr^\alpha_i B_f \neq 0 \quad \text{for each } \alpha \in (0, 1], i \in \mathbb{N}.
\]
Indeed, by the isomorphisms  
\[
t^i : Gr^\alpha_i Gr^\alpha_i (B_f[t^{-1}]) \simeq Gr^\alpha_i Gr^\alpha_{i+i}(B_f[t^{-1}]) \quad (\alpha \in (0, 1], i \in \mathbb{Z}),
\]
we get the canonical isomorphisms  
\[
Gr^\alpha_i Gr^\alpha_i B_f = Gr^\alpha_i(M/tM) \quad (\alpha \in (0, 1], i \in \mathbb{Z}).
\]
For $\alpha \in (0, 1]$, the filtration $G$ on $Gr^\alpha_i B_f$ is a finite filtration such that  
\[
Gr^\alpha_i Gr^\alpha_i B_f = 0 \quad \text{unless } i \in \mathbb{N}.
\]
This is closely related to the negativity of the roots of $b_f(s)$, see [Ka 66].

For any $\beta \in \mathbb{Q}$, we have the canonical decomposition  
\[
B_f/V^\beta B_f = \bigoplus_{\alpha < \beta} Gr^\alpha_i B_f.
\]
by using the action of $s = -\partial_t$. This is obtained by taking the inductive limit of the decomposition of $V^\gamma B_f/V^\beta B_f$ for $\gamma \to -\infty$, and is compatible with the canonical surjections  
\[
B_f/V^\beta B_f \to B_f/V^\beta B_f \quad (\beta > \beta').
\]
(This means that (4.1.9) corresponds via (4.1.8) to the canonical surjection associated to shrinking the index set of direct sums.)

We then get the asymptotic expansion for any $\xi \in B_f$ :  
\[
\xi \sim \sum_{\alpha \geq \alpha_\xi} \xi^{(\alpha)} \quad \text{with } \xi^{(\alpha)} \in Gr^\alpha_i B_f, \ \alpha_\xi \in \mathbb{Q}.
\]
This means that the following equality holds for any $\beta > \alpha_\xi$ via the isomorphism (4.1.8):  
\[
\xi \mod V^\beta B_f = \bigoplus_{\alpha_\xi \leq \alpha < \beta} \xi^{(\alpha)}.
\]
We have to control this expansion at least in the case $\xi = f^s$ in order to determine $b_f(s)$.
Indeed, we have the decomposition of $f^s$ modulo $V^\beta B_f$ in a compatible way with the action of $\mathcal{D}_X[s]$ by using the action of $\mathbb{C}[s]$ for any $\beta > \alpha_\xi$ as in (4.1.11), and the direct summands of $f^s$ generate $\mathcal{D}_X[s]f^s$ modulo $V^\beta B_f$ over $\mathcal{D}_X[s]$. So the determination of the leading term of the expansion is not enough to calculate $b_f(s)$. A similar phenomenon is known for the Gauss-Manin systems in the isolated singularity case, and this is the reason for which the determination of $b_f(s)$ is so complicated. In the weighted homogeneous isolated singularity case, this problem does not occur for the Gauss-Manin systems. However, it does occur for $B_f$ in the non-isolated homogeneous singularity case, since the situation is quite different in the case of $B_f$.

4.2. Some remarks. In this section we note some remarks as below.

Remark 4.2a. In the notation of 4.1, it is well known that $b_f(s)$ depends only on the divisor $D := f^{-1}(0)$. Indeed, we have a line bundle $L$ over $X$ corresponding to $D$, and the Euler field corresponding to the natural $\mathbb{C}^*$-action on $L$ and the ideal sheaf of the zero-section of $L$ are sufficient to determine the Bernstein-Sato polynomial by using the filtration $V$ of Kashiwara and Malgrange along the zero-section on the direct image of $O_X$ as a $\mathcal{D}$-module by the canonical section $i_D : X \hookrightarrow L$ defined by $D$ (where $i_D$ is locally identified with the graph embedding if a local defining function of $D$ is chosen).

Remark 4.2b. For any reduced homogeneous polynomial $f$, we have  
\[
(4.2.1) \quad \mathcal{R}^0_f \subset \frac{1}{d}\mathbb{Z}.
\]
where the left-hand side is the roots of \( b_f(s) \) supported at the origin up to sign. Indeed, setting \( X^* := \mathbb{C}^n \setminus \{0\} \) with \( j : X^* \hookrightarrow X := \mathbb{C}^n \) the canonical inclusion, we have the canonical isomorphism

\[
(4.2.2) \quad j_!^P \mathcal{M} \sim j_*^P \mathcal{M}
\]

for any holonomic subquotient \( D \)-module \( \mathcal{M} \) of \( \text{Gr}_\alpha^\circ B_f|_{X^*} \) if \( \alpha \notin \frac{1}{d} \mathbb{Z} \). Here \( j_!^P, j_*^P \) denote the functors between the bounded derived category of \( D \)-modules with holonomic cohomology \( D \)-modules, which can be defined by using \( C \)ech-type complexes, see for instance [Sa 22].

Taking the blow-up \( \pi : \tilde{X} \to X \) at the origin, this isomorphism can be reduced to a similar isomorphism for the inclusion \( \tilde{X} \setminus E \hookrightarrow \tilde{X} \) with \( E \) the exceptional divisor (since \( \pi \) is proper), and follows from the Verdier-type extension theorem (which involves only the unipotent monodromy part, see for instance [Sa 90a, Proposition 2.8]) choosing a local defining function of \( E \subset \tilde{X} \). Indeed, \( X^* \) is a \( \mathbb{C}^* \)-bundle over \( E \cong \mathbb{P}^{n-1} \), and \( \text{Gr}_\alpha^\circ B_f|_{X^*} \) is a monodromical \( D \)-module, whose monodromy around \( E \) coincides with \( T^{-d} \), where \( T \) is the monodromy of the nearby cycle functor for \( \pi^*f \) (since we have locally \( \pi^*f = \frac{d}{n}h \) with \( z_1, \ldots, z_n \) local coordinates of \( \tilde{X} \) and \( h \in \mathbb{C}[z_1, \ldots, z_{n-1}] \)). (Recall that \( \text{Gr}_\alpha^\circ B_f \) corresponds to the \( \lambda \)-eigenspace of the monodromy \( T \) on the nearby cycle complex \( \psi_f \mathcal{C}_X[n-1] \) with \( \lambda = e^{-2\pi i \alpha} \) if \( \alpha \notin \mathbb{Z}_{\leq 0} \).

Using an analogue of the theory of intermediate extensions, the above isomorphism implies the isomorphisms of holonomic \( D_X \)-modules

\[
(4.2.3) \quad j_!^P \mathcal{M} = \mathcal{H}^0 j_!^P \mathcal{M} \sim j_*^P \mathcal{M} \sim \mathcal{H}^0 j_*^P \mathcal{M} = j_*^P \mathcal{M},
\]

where the middle term is the intermediate extension. Note that \( \mathcal{H}^k j_*^P \mathcal{M} = \mathcal{H}^{-k} j_*^P \mathcal{M} = 0 \) for \( k > 0 \) by the construction using \( C \)ech-type complexes.

By the adjunction relations, we have for any \( D_{(0)} \)-module \( \mathcal{M}' \)

\[
(4.2.4) \quad \text{Ext}^1_{D_X}(j_!^P \mathcal{M}, i_*^P \mathcal{M}') = \text{Ext}^1_{D_X}(i_*^P \mathcal{M}', j_*^P \mathcal{M}) = 0,
\]

where \( i : \{0\} \hookrightarrow X \) is the canonical inclusion.

These imply that any holonomic subquotient \( D_X \)-module of \( \text{Gr}_\alpha^\circ B_f \) supported at the origin is a direct factor of \( \text{Gr}_\alpha^\circ B_f \), and is detectable by applying \( i_*^P \) to \( \text{Gr}_\alpha^\circ B_f \) (which corresponds to taking the Milnor fiber cohomology) assuming \( \alpha \notin \frac{1}{d} \mathbb{Z} \). We have however \( H^{n-1}(F_j, \mathcal{C}) = 0 \) for \( \lambda \neq 1 \) using the geometric monodromy, since \( f \) is homogeneous of degree \( d \). We thus conclude that there are no holonomic subquotients of \( \text{Gr}_\alpha^\circ B_f \) supported at the origin if \( \alpha \notin \frac{1}{d} \mathbb{Z} \). So (4.2.1) follows.

**Remark 4.2c.** In the notation of Remark 4.2b above, assume the hypotheses (IS)-(WH) in Theorem 2 and moreover \( \alpha \in (0,1) \cap \frac{1}{d} \mathbb{Z} \). Then

\[
(4.2.5) \quad \text{Gr}_i^W \text{Gr}_\alpha^\circ B_f = 0 \quad \text{for} \quad |i - n'| > 1,
\]

\[
\text{supp} \text{Gr}_i^W \text{Gr}_\alpha^\circ B_f \subset \{0\} \quad \text{for} \quad |i - n'| = 1,
\]

where \( W \) is the weight filtration which has symmetry with center \( n' := n-1 \).

If the asymptotic expansion of some \( \xi \in F_0 B_f = \mathcal{O}_X(t-f) \) in (4.1.10) has a nonzero direct factor satisfying

\[
(4.2.6) \quad \xi^{(\alpha)} \in W_{n'-1} \text{Gr}_\alpha^\circ B_f \setminus D_X[s](F_0 \text{Gr}_\alpha^\circ B_f) \quad (\alpha \in (0,1) \cap \frac{1}{d} \mathbb{Z}),
\]

then this would give an example where the implication \( \Longrightarrow \) in (3) fails with condition (2) unsatisfied in Theorem 1. Note that the direct image of \( \text{Gr}_\alpha^\circ B_f \) by the projection to \( S = \mathbb{C} \) defined by the function \( y \) in 1.3 contains direct factors isomorphic to

\[
E := D_s/D_s(y \partial_y),
\]

where \( y \) is identified with the natural coordinate of \( S = \mathbb{C} \). We have the isomorphism

\[
\text{DR}(E)_0 = C(\text{Gr}_{\nu_y} \partial_y : \psi_{y,1} E := \text{Gr}_{1,y} E \to \varphi_{y,1} E := \text{Gr}_{0,y} E)_0,
\]
with $V_y$ the filtration of Kashiwara and Malgrange along $y = 0$, and the image of $\text{Gr}_{V_y} \partial_y$ coincides with the subspace obtained by taking $\varphi_{y,1}$ of the intersection of $E$ with the direct image of $W_{n'-1} \text{Gr}_{V_y} B_f$.

**Remark 4.2d.** By a calculation of multiplier ideals in [Sa07, Proposition 1] and using [BuSa05], we get

\begin{equation}
\dim \text{Gr}_{V_y}^n \mathcal{O}_X = \binom{k-1}{n-1} \quad \text{for } \alpha = \frac{k}{d} \in \left[ \frac{n}{d}, \alpha_Z \right),
\end{equation}

where $\mathcal{O}_X = \text{Gr}_0^f B_f$. By (1.2.5), (1.2.8), (1.3.8), this implies

\begin{equation}
M_{k}^{(\infty)} \neq 0 \quad \text{for } \frac{k}{d} \in \left[ \frac{n}{d}, \alpha_Z \right).
\end{equation}

Indeed, $\text{Gr}_{V_y}^n \text{Gr}_0^f B_f = \text{Gr}_0^f \text{Gr}_{V_y}^n B_f$ is supported at the origin for $\alpha \in \left[ \frac{n}{d}, \alpha_Z \right)$, and we have the inclusion $F^n P^p \subset P^n \ (\forall p \in \mathbb{Z})$ together with the vanishing $P^n H^{n-1}(F_j, \mathbb{C}) = 0$. (Note that $\alpha_Z = \min(\bar{\alpha}_Z, 1) \leq 1$.)

**Remark 4.2e.** Under the assumption (IS), we can show

\begin{equation}
\alpha_f := \min \mathcal{R}_f = \min \{ \alpha_Z, \frac{n}{d} \}.
\end{equation}

In the notation of 4.1, we index the Hodge filtration $F$ on $B_f$ so that

\begin{equation}
F_0 B_f = \mathcal{O}_X \delta(t-f).
\end{equation}

By [BuSa05], we can essentially identify the induced filtration $V$ on it with the multiplier ideals of $D := f^{-1}(0) \subset X$ (except the difference between left and right continuities at jumping coefficients). This implies that $\alpha_f$ coincides with the log canonical threshold, that is, the minimal jumping number, of $D \subset X$. Indeed, we have

\begin{equation}
\mathcal{D}_X[s] f^s \subset V^\alpha B_f \quad \text{if} \quad \mathcal{O}_X \delta(t-f) \subset V^\alpha B_f.
\end{equation}

So we can replace $\alpha_f$, $\alpha_Z$ in (4.2.9) with the log canonical threshold of $D \subset X$ and that of $D \setminus \{0\} \subset X \setminus \{0\}$. The assertion then follows from [Sa07, Theorem 2.2].

**4.3. First step to the proof of Theorem 2.** In this section we prove the following.

**Theorem 4.3.** In the notation and assumption of 1.4 (including the assumption (WH) in Theorem 2), we have the equality

\begin{equation}
\sum_{k \in \mathbb{N}} \dim \mathbb{C} \text{Coker} \left( r_k : M_k \to \bigoplus_{z \in \Sigma} \mathbb{Z}_{h_z}^{k/d} \right) = \dim \mathbb{C} H^{n-1}(F_j, \mathbb{C}),
\end{equation}

where $r_k$ is the composition of

\begin{equation*}
M_k \to \widetilde{M}_k \quad \text{and} \quad \widetilde{M}_k \to \bigoplus_{z \in \Sigma} \mathbb{Z}_{h_z}^{k/d},
\end{equation*}

induced by (1.3.6) and (1.4.4–5) respectively.

**Proof.** The assertion follows from Theorem 2.2 and Proposition 3.3 together with [Di92], [DiSa06]. In the notation of [DiSa06, Section 1.8], set

\begin{equation*}
\mathcal{L}_\lambda = S^i \quad (i = 0, \ldots, d-1, \ \lambda = e(i/d)),
\end{equation*}

with $e(\alpha) := \exp(2\pi i \alpha)$. This is a locally free $\mathcal{O}_Y$-module of rank 1 having a canonical meromorphic connection so that the localization $\mathcal{L}_\lambda(*Z)$ of $\mathcal{L}_\lambda$ along $Z$ is a regular holonomic $\mathcal{D}_Y$-module with

\[ \text{DR}(\mathcal{L}_\lambda) = R_j L_\lambda. \]

The restriction of $\mathcal{L}_\lambda$ to the smooth points of $Z$ is the Deligne extension with residue $i/d$, see also [BuSa10, 1.4.1]. On $Y' := Y \setminus \{u = 0\}$, there is an isomorphism of regular holonomic $\mathcal{D}_{Y'}$-modules

\[ \mathcal{L}_\lambda(*Z)|_{Y'} \cong \mathcal{O}_{Y'}(*Z') h^{i/d} \quad \text{inducing} \quad \mathcal{L}_\lambda|_{Y'} \cong \mathcal{O}_{Y'} h^{i/d}, \]
with \( h := f/u^d \), \( Z' := Z \cap Y' \). We have the pole order filtration \( P \) on \( \mathcal{L}(\star Z) \) defined by

\[
P_j(\mathcal{L}_\lambda(\star)) = \begin{cases} 
\mathcal{L}_\lambda((j + 1)Z) & \text{if } j \geq 0, \\
0 & \text{if } j < 0,
\end{cases}
\]

which induces the pole order filtration \( P \) on the Milnor cohomology \( H^{n-1}(F_j)_\lambda \) by taking the induced filtration on the de Rham complex (where the shift of filtration by \( n-1 \) occurs), see [DiSa06, Section 1.8], [Di92]. By using the Bott vanishing theorem, the filtered de Rham complex can be described by using the differential \( d_f \) on \( \Omega^* \) defined by

\[
d_f \omega := f d\omega - \frac{k}{d} f d\omega \quad (\omega \in \Omega^*_k),
\]

(see [Do82, 4.1.1] and also [Di92, Ch. 6, 1.18]). We have moreover the surjection

\[
M^{(2)}_{(q+1)d-i} \rightarrow \text{Gr}_{P}^{n-1-q} H^{n-1}(F_j, \mathbb{C})_\lambda,
\]

using the pole order spectral sequence together with [Di92, Ch. 6, Theorem 2.9]. Here \([\omega] \in M^{(2)}_{(q+1)d-i} \) with \( \omega \in \Omega^n_{(q+1)d-i} \) is sent to

\[
[i_E(\omega/f^{q+1}) f^{i/d}] \in \text{Gr}_{P}^{n-1-q} H^{n-1}(U, L_\lambda) \quad (\lambda = e(i/d)),
\]

by using the isomorphism (2.1.2), where \( i_E \) denotes the contraction with the Euler field \( E := \sum y_i \partial_{y_i} \), and \( f^{i/d} \) is a symbol denoting a generator of \( \mathcal{L}_\lambda(i) \cong \mathcal{O}_Y \) (which is unique up to a non-zero constant multiple, and is closely related to the above differential \( d_f \)).

The target of (2.2.2) is calculated by Proposition 3.3. (Note that there is a shift of filtration by 1 coming from (3.1.5).) The graded pieces of the pole order filtration \( P \) of restriction morphism (2.2.2) is then induced up to a non-zero constant multiple by the morphism \( r_k \) in (4.3.1). Here the source of \( \text{Gr}_{P}^{n-1-q} \) of (2.2.2) can be replaced by \( M^{(2)}_{(q+1)d-i} \) as above, and \( r_k \) can be defined by using the contraction with the Euler field together with the substitution \( y = 1 \) after restricting to \( Y' := \{ y \neq 0 \} \subset Y = \mathbb{P}^{n-1} \).

Note that this is compatible with the blow-up construction in the proof of Theorem 1.5, where the Euler field \( E \) becomes the vector field \( y \partial_y \) using the product structure of the affine open piece of the blow-up.

We have the strict compatibility of (2.2.2) with the pole order filtration \( P \) by Remark 4.3 below. (Here \( F \) is the Hodge filtration. We need Proposition 3.3 together with (3.1.5) and (3.3.11) to show \( F = P \) on the target.) So \( \text{Gr}_{P}^{*} \) of the cokernel of (2.2.2) coincides with the cokernel of \( \text{Gr}_{P}^{*} \) of (2.2.2), and the latter is given by the cokernel of \( r_k \) by the above argument. We thus get (4.3.1). This finishes the proof of Theorem 4.3.

**Remark 4.3.** Let \( \phi : (V; F, P) \rightarrow (V'; F, P) \) be a morphism of bifiltered vector spaces with \( F \subset P \) and \( F = P \) on \( V' \). If \( \phi \) is strictly compatible with \( F \), then it is so with \( P \).

Indeed, the strict compatibility for \( F \) means \( \phi(F^p V) = F^p V' \cap \phi(V) \), and

\[
\phi(F^p V) \subset \phi(F^p V') \subset F^p V' \cap \phi(V). \quad (\phi(F^p V) = F^p V' \cap \phi(V), \quad \text{that is, } \phi \text{ is also strictly compatible with } P.
\]

**4.4. Second step to the proof of Theorem 2.** From Theorems 1.5 and 4.3, we can deduce the following.

**Theorem 4.4.** We have

\[
\sum_{q \in \mathbb{N}} \dim N_{q+d} \cap \left( \bigoplus_{z \in \Sigma} \Xi_{z/d}^{q} \right) = \dim H^{n-2}(F_j, \mathbb{C}),
\]

where the intersection on the left-hand side is taken in \( \tilde{N}_{k+d} \) via the isomorphism (1.4.4) using the function \( y \).
Proof. By (1.3.6) and (1.4.4–5), we have the inclusions
\[(4.4.2) \quad V_1 := N_{q+d} \subset V := \widetilde{N}_{q+d} \supset V_2 := \bigoplus_{z \in \Sigma} \mathbb{E}^{q/d},\]
where \(p + q + d = nd\). By the graded local duality as is explained in [DiSa 24, 1.1] together with the compatibility of duality isomorphisms in Theorem 1.5, we have
\[(4.4.3) \quad V^* = \overline{M}_p, \quad V_1^\perp = M_p'', \quad V_2^* = \bigoplus_{z \in \Sigma} \mathbb{E}^{q/d},\]
where \(V^*\) denotes the dual vector space of \(V\). Using the diagram of the nine lemma, we get a surjection of short exact sequences
\[(4.4.4) \quad 0 \to V_1^\perp \to V^* \to V_1^* \to 0\]
which implies the exact sequence
\[(4.4.5) \quad V_1^\perp \to V_2^* \to (V_1 \cap V_2)^* \to 0.\]
By Theorem 1.5 this gives the exact sequence
\[(4.4.6) \quad M_p'' \to \bigoplus_{z \in \Sigma} \mathbb{E}^{q/d} \to (N_{q+d} \cap \left( \bigoplus_{z \in \Sigma} \mathbb{E}^{q/d} \right))^* \to 0.\]
The assertion (4.4.1) then follows from Theorem 4.3. This finishes the proof of Theorem 4.4.

4.5. Proof of Theorem 2. By the pole order spectral sequence, we have the inequality
\[(4.5.1) \quad \sum_{k \in \mathbb{N}} \dim \text{Ker}(d^{(1)} : N_{k+d} \to M_k) \geq \dim H^{n-2}(F_\gamma, \mathbb{C}),\]
where the equality holds if and only if the spectral sequence degenerates at \(E_2\).

On the other hand, the proof of [DiSa 24, Theorem 5.2] actually proves the inequality
\[(4.5.2) \quad \sum_{k \in \mathbb{N}} \dim \text{Ker}(d^{(1)} : N_{k+d} \to M_k) \leq \sum_{k \in \mathbb{N}} \dim N_{k+d} \cap \left( \bigoplus_{z \in \Sigma} \mathbb{E}^{k/d} \right).\]
So the \(E_2\)-degeneration follows from (4.5.1–2) and Theorem 4.4. This finishes the proof of Theorem 2.

4.6. Proof of Theorem 3. The above proof of (4.5.2) shows the inequality
\[(4.6.1) \quad \sum_{k \in \mathbb{N}} \dim \text{Ker}(d^{(1)} : N_{k+d} \to M_k'') \leq \sum_{k \in \mathbb{N}} \dim N_{k+d} \cap \left( \bigoplus_{z \in \Sigma} \mathbb{E}^{k/d} \right).\]
Note that the intersection of \(N_{k+d}\) with the kernel of \(d^{(1)} : \widetilde{N}_{k+d} \to \widetilde{M}_k\) is studied in [DiSa 24, 5.1.2]. (The latter morphism is also constructed there.) This implies the injectivity of the composition in Theorem 3, since the failure of this injectivity implies a strict inequality in (4.5.2) by using (4.6.1). This finishes the proof of Theorem 3.

4.7. Proof of Theorem 4. Set \(R' := \mathbb{C}[x']\), where \(x' = (x_1', \ldots, x_{n-1}')\) is a coordinate system of \(X'_y \cong \mathbb{C}^{n-1}\). For \(j = 0, 1\), let \(H_j^\bullet\) be the \(j\)th cohomology of the shifted mapping cone of
\[(4.7.1) \quad h : (R'/(\partial g))(1-d) \to R'/(\partial g).\]
From the short exact sequence of complexes
\[(4.7.2) \quad 0 \to K_f^\bullet(-1) \to K_f^\bullet / yK_f^\bullet(-1) \to 0,\]
we get a long exact sequence
\[(4.7.3) \quad 0 \to N_\bullet(n-1) \to N_\bullet(n) \to H_f^0 \to M_\bullet(n-1) \to M_\bullet(n) \to H_f^1 \to 0,\]
using the \(n\)-ple complex structure of the Koszul complex. Here \(M_\bullet = H^nK_f^\bullet, \ N_\bullet = H^{n-1}K_f^\bullet,\) and \(K_f^n, K_f^{n-1}\) are direct sums of copies of \(R := \mathbb{C}[x]\) shifted respectively by \(-n\) and \(1-n-d\), see (1.2.4). These shifts are compatible with the shift by the differential of the complex defining the \(H_f^\bullet\) which is defined by multiplication by \(h\) of degree \(d-1\), see (4.7.1).

Since \(f\) is not very special type, we see that
\[(4.7.4) \quad H_k^0 = 0 \quad \text{if} \ k \leq d.\]
Indeed, we have \( zh = 0 \) if \( z \in H^0_d \subset R'_1 \). By the exact sequence (4.7.3), this implies that

\[ N_{d+n} = 0. \]

So the first assertion follows using Theorem 1, since \( M_n = \mathbb{C} \). (Note that the difference between \( \deg h (= d - 1) \) and the degree \( d \) of the differential \( d_{1} \) of the pole order spectral sequence comes from the degree of the derivations \( \partial_{z_i} \) in the exterior derivation \( d \).)

Assume now \( n = 3 \). Since \( y \) is general, the subvariety \( \{ g = 0 \} \subset \mathbb{P}^1 \) is reduced, hence

\[(4.7.6) \quad \dim(R'/((\partial g)))_k = \begin{cases} k+1 & \text{if } 0 \leq k \leq d-2, \\ 2d-3-k & \text{if } d-2 \leq k \leq 2d-4, \end{cases} \]

and 0 otherwise. Using the non-degeneracy of the Grothendieck residue pairing (see [GrHa 78, p. 659], there is a monomial \( m \in R' \) such that \( mh \notin (\partial g) \) for any \( k \in [0, d-3] \). (This can be reduced to the case \( k = d-3 \).) We then get that

\[ \nu_{k+d+2} - \nu_{k+d+1} \leq \dim H^0_{k+d-1} \leq k \quad \text{if } 0 \leq k \leq d-3. \]

(This holds even in the very special case if \( h \notin (\partial g) \).) On the other hand we have

\[ \mu_{k+2} - \mu_{k+1} = k \quad \text{if } 0 \leq k \leq d-1, \]

since \( \deg \partial_{z_i} f = d-1 \). Combined with (4.7.5), these imply that

\[ \mu_{k+2} - \nu_{k+d+2} > 0 \quad \text{if } 1 \leq k \leq d-3. \]

So the assertion follows using Theorem 1. This finishes the proof of Theorem 4.

4.8. Proof of Proposition 2. We first show the first assertion, that is,

\[ (4.8.1) \quad h \notin (\partial g). \]

From the weak Lefschetz property for \( M'_{\bullet} \) (see [DiPo 16]) we get

\[ (4.8.2) \quad \text{Ker}(y : M_k \to M_{k+1}) = 0 \quad \text{if } k < \frac{3d}{2}. \]

Using the exact sequence (4.7.3), this implies that

\[ (4.8.3) \quad \nu_{k+3} - \nu_{k+2} = \dim H^0_k \quad \text{if } k+2 < \frac{3d}{2}. \]

Assume \( h \in (\partial g) \), that is, the morphism (4.7.1) vanishes. For \( k < \frac{3d}{2} - 2 \), the exact sequence (4.7.3) and (4.8.2) then imply

\[ \mu_{k+n} - \mu_{k+n-1} = \max(k+1, 0), \]

\[ \nu_{k+n} - \nu_{k+n-1} = \max(k-d+2, 0). \]

Applying these to \( k \leq -1 \) and \( k \leq d-1 \) respectively, we get by increasing induction on \( k \)

\[ (4.8.5) \quad \delta_{k+n} = \begin{cases} 0 & \text{if } k < -1, \\ -1 & \text{if } k = -1. \end{cases} \]

In view of [DiSa 24, Theorem 5.3], this assertion for \( k = -1 \) implies that \( \frac{2}{m} \) must be a spectral number of a singular point of \( Z \), but this is a contradiction, since \( Z \) has only homogeneous singularities (whose minimal spectral number is \( \frac{2}{m} \) with \( m \) the multiplicity) and \( f \) is not a polynomial of two variables. Note that a projective hypersurface of degree \( d \) is a projective cone if it has a point of multiplicity \( \geq d \) (considering the intersection of \( Z \) with general lines passing through this point). So the first assertion, that is, (4.8.1), is proved.

Using the non-degeneracy of the Grothendieck residue pairing as in 4.7, we get

\[ (4.8.6) \quad \dim H^0_{k+d-1} = \begin{cases} k & \text{if } 0 \leq k \leq d-3, \\ k+1 & \text{if } k = d-2, \end{cases} \]

in the \( f \) very special type case, since \( H^0_{\bullet} = \text{Ker} h \subset (R'/((\partial g)))(1-d) \) is an ideal.
On the other hand we have
\[(4.8.7) \quad \dim H_k^1 = \max(k+1,0) \text{ if } k \leq d-2.\]

Using the exact sequence (4.7.3) and (4.8.2) again, we then get that
\[(4.8.8) \quad \delta_{k+3} = 0 \text{ if } k+2 < \frac{3d}{2} - d,\]

Here the last term of (4.8.4) is replaced by \(\max(k-d+1,0)\). (Note that the condition \(k \leq d-3\) in (4.8.6) is equivalent to \(k-1 \leq d-4\) and also to \((k-1)+3 \leq d-1\).) This finishes the proof of Proposition 2.

**Remark 4.8.** In the very special type case with \(n = 3\), setting \(\tilde{\delta}''_k := \delta''_k + \delta_{k,d} (k \in \mathbb{Z})\) with \(\delta_{k,d}\) Kronecker delta, we have
\[(4.8.9) \quad \mu' = \sum_{a=0}^{e} m_a \mu'_{a,k}, \quad \tilde{\delta}''_{a,k} = \sum_{a=0}^{e} m_a \tilde{\delta}''_{a,k},\]

with \(m_a \in \mathbb{N} (a \in [0,e])\). Here \(e := \left\lfloor \frac{d}{2} \right\rfloor - 1\), and
\[(4.8.10) \quad \tilde{\delta}''_{a,k} := \begin{cases} 1 & \text{if } k \in [d-a,d+a], \\ 0 & \text{otherwise}, \end{cases}\]

This is equivalent to that the \(\tilde{\delta}''_{k}\) are weakly increasing for \(k \leq d\), vanish for \(k \leq \left\lfloor \frac{d}{2} \right\rfloor\), and moreover \(\tilde{\delta}''_{k} + \mu'_{k} = \tilde{\delta}''_{d} = \chi(U)\) for \(d \leq k \leq \left\lfloor \frac{3d}{2} \right\rfloor\) with \(U := \mathbb{P}^2 \setminus Z\) (using the symmetries of the \(\tilde{\delta}''_{k}\) and \(\mu'_{k}\) with centers \(d\) and \(\frac{3d}{2}\) respectively). These are compatible with the exact sequence (4.7.3).

If \(f = x^a y^d-a + z^d (0 < a < d)\), we have \(\tilde{\delta}''_{k} = \tilde{\delta}''_{0,k}\), \(\mu' = \mu'_{0,k}\) (that is, \(m_a = \delta_{0,a}\)) using the Thom-Sebastiani type theorem, see [DiSa 24]. This is expected to hold for any extremely degenerated curves with \(\chi(U) = 1\).

If \(f = x^a y^d-a + x y z^d-2 (0 < a < d)\), it seems that \(\tilde{\delta}''_{k} = \mu'_{k} = 0\) for any \(k\) (that is, \(m_a = 0\) for any \(a\)) as far as computed using Macaulay2. This is expected to hold for any extremely degenerated curves with \(\chi(U) = 0\).

**5. Explicit calculations**

In this section we calculate some examples explicitly.

**5.1. Example I.** Set \(\mu'_k = \dim M'_k, \mu''_k = \dim M''_k\), etc., and
\[
f_1 = x^5 + y^4 z \quad \text{with} \quad h_1 = x^5 + y^4,
\]

where \(n = 3, d = 5, \tau_Z = 12\), and \(\chi(U) = 1\), see (2.7.2). In this case \(\Sigma\) consists of one point \(p := [0 : 0 : 1] \in \mathbb{P}^2\), and \((Z,p)\) is defined by \(h_1\). We have
\[
\begin{align*}
k & : 3 4 5 6 7 8 9 10 11 12 13 \cdots \\
\gamma_k & : 1 2 3 4 5 6 7 8 9 10 11 \cdots \\
\mu_k & : 1 2 3 4 5 6 7 8 9 10 11 \cdots \\
\nu_k & : 1 2 3 4 5 6 7 8 9 10 11 \cdots \\
\mu''_k & : 1 2 3 4 5 6 7 8 9 10 11 \cdots \\
\mu'_k & : 1 2 3 4 5 6 7 8 9 10 11 \cdots \\
\end{align*}
\]

(5.1.1)

and
\[
\begin{align*}
b_{f_1}(s) &= b_{h_1}(s) \prod_{i=6}^{9} (s + \frac{i}{5}), \\
& \quad \text{with} \quad b_{h_1}(s) = (s + 1) \prod_{i=1}^{4} \prod_{j=1}^{3} (s + \frac{i}{5} + \frac{j}{4}).
\end{align*}
\]

(5.1.2)
These are done by using respectively Macaulay2 and RISA/ASIR as in [Sa15]. Note that \( \mu_k^{(2)} = \mu_k - \nu_{k+d} \) by (9) in the introduction, and \( b_h(s) \) can be calculated also by applying (1.8.2). These two calculations both imply
\[
(5.1.3) \quad R^0_{f_1} = \{ \frac{6}{5}, \frac{7}{5}, \frac{8}{5}, \frac{9}{5} \}.
\]
This is an example of an extremely degenerated curve (see 1.9), and is a good example for Theorem 3 and Corollary 3.

5.2. Example II. Let
\[
f_2 = x^5 + x^2y^3 + y^4z \quad \text{with} \quad h_2 = x^5 + x^2y^3 + y^4,
\]
where \( n = 3, d = 5, \tau_Z = 12, \) and \( \chi(U) = 1 \), see (2.7.2). Again \( \Sigma \) consists of one point \( p := [0 : 0 : 1] \in \mathbb{P}^2 \), and \( (Z, p) \) is defined by \( h_2 \). In this case, it is rather surprising that \( h_2 \) is quasihomogeneous with \( \tau_{h_2} = \mu_{h_2} = 12 \), see a remark after Theorem 2. (This cannot be generalized to polynomials of higher degrees as far as tried.) We have
\[
(5.2.1) \quad \begin{align*}
    k : & \quad 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9 \ 10 \ 11 \ 12 \ 13 \ \cdots \\
    \gamma_k : & \quad \begin{array}{ccccccc}
      1 & 3 & 6 & 10 & 12 & 12 & 6 & 3 & 1
    \end{array} \\
    \mu_k : & \quad \begin{array}{ccccccc}
      1 & 3 & 6 & 10 & 12 & 12 & 12 & 12 & 12 & 12 & \cdots \\
    \end{array} \\
    \nu_k : & \quad \begin{array}{ccccccc}
      2 & 6 & 9 & 11 & 12 & 12 & \cdots \\
    \end{array} \\
    \mu_k^{(2)} : & \quad 1 \ 1 \ 1 \ 1 \\
    \mu'_k : & \quad \end{align*}
\]
\[
(5.2.2) \quad \begin{align*}
    b_{f_2}(s) = b_{h_2}(s) \prod_{i=3}^{4}(s + \frac{i}{5}) \cdot \prod_{i=6}^{7}(s + \frac{i}{5}), \\
    \text{with} \quad b_{h_2}(s) = (s + 1) \prod_{i=1}^{4} \prod_{j=1}^{3}(s + \frac{i}{5} + \frac{j}{4}).
\end{align*}
\]
\[
(5.2.3) \quad R^0_{f_2} = \{ \frac{3}{5}, \frac{4}{5}, \frac{6}{5}, \frac{7}{5} \}.
\]
These are compatible with Theorem 3 and Corollary 3, since \( M' = 0 \) in this case.

5.3. Example III. Let
\[
f_3 = x^5 + x^3y^2 + y^4z \quad \text{with} \quad h_3 = x^5 + x^3y^2 + y^4,
\]
where \( n = 3, d = 5, \tau_Z = 11, \) and \( \chi(U) = 1 \), see (2.7.2). As before \( \Sigma \) consists of one point \( p := [0 : 0 : 1] \in \mathbb{P}^2 \), and \( (Z, p) \) is defined by \( h_3 \). However, \( h_3 \) is not quasihomogeneous with \( \mu_{h_3} = 12, \tau_{h_3} = 11 \) in this case. We have by RISA/ASIR
\[
(5.3.1) \quad \begin{align*}
    b_{f_3}(s) = b_{h_3}(s) \begin{pmatrix} \frac{1}{5} \end{pmatrix} \prod_{i=6}^{8}(s + \frac{i}{5}), \\
    b_{h_3}(s) = (s + 1) \prod_{i=1}^{4} \prod_{j=1}^{3}(s + \frac{i}{5} + \frac{j}{4} - \delta_{i,4}\delta_{j,3}),
\end{align*}
\]
where \( \delta_{i,k} = 1 \) if \( i = k \), and \( 0 \) otherwise. So we get
\[
(5.3.3) \quad R^0_{f_3} = \{ \frac{4}{5}, \frac{6}{5}, \frac{7}{5}, \frac{8}{5} \}.
\]
In this case the pole order spectral sequence degenerates at \( E_3 \), and \( \mu_k^{(3)} \neq 0 \) if and only if \( k \in \{ 4, 6, 7, 8 \} \) by calculations using computer programs based on [DiSt19], [Sa20]. We then get \( R^0_{f_3} \) by applying Theorem 1 since \( R_Z \cap \frac{1}{3} \mathbb{Z} = \{ 1 \} \) in this case.

5.4. Example IV. Let
\[
f_4 = x^4y^2z + z^7 \quad \text{with} \quad h_4 = x^4z + z^7, \quad h'_4 = y^2z + z^7,
\]
where \( n = 3, \quad d = 7, \quad \tau_Z = \mu_Z = 22 + 8 = 30, \quad \text{and} \quad \chi(U) = 1, \quad \text{see} \quad (2.7.2-6) \quad \text{and} \quad A.1 \quad \text{below.} \quad \text{We have}

\[
\begin{align*}
    k & : 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10 \quad 11 \quad 12 \quad 13 \quad 14 \quad 15 \quad 16 \quad 17 \quad 18 \quad 19 \\
    \gamma_k & : 1 \quad 3 \quad 6 \quad 10 \quad 15 \quad 21 \quad 25 \quad 27 \quad 25 \quad 21 \quad 15 \quad 10 \quad 6 \quad 3 \quad 1 \\
    \mu_k & : 1 \quad 3 \quad 6 \quad 10 \quad 15 \quad 21 \quad 25 \quad 28 \quad 30 \quad 31 \quad 30 \quad 30 \quad 30 \quad 30 \quad 30 \quad 30 \\
    \nu_k & : 1 \quad 3 \quad 6 \quad 10 \quad 15 \quad 20 \quad 24 \quad 27 \quad 29 \quad 30 \quad 30 \quad 30 \quad 30 \quad 30 \quad 30 \quad 30 \\
    \mu_k^{(2)} & : 1 \quad 1 \quad 1 \quad 2 \quad 2 \quad 2 \quad 2 \quad 1 \quad 1 \quad 1 \\
    \nu_k^{(2)} & : 1 \quad 1 \quad 1 \quad 2 \quad 1 \quad 1 \quad 1 \\
    \delta_k & : 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1
\end{align*}
\]

where we have \( \delta_k := \mu_k - \nu_{k+d} = \mu_k^{(2)} \) in this case. The calculation of \( \mu_k^{(2)}, \nu_k^{(2)} \) is made by computer programs based on [DiSt19], [Sa20].

On the other hand we get by A.1 below and RISA/ASIR

\[
14 \mathcal{R}_h \subset 14 \mathcal{R}_h = 14 \mathcal{R}_Z = \{5\} \cup \{7, \ldots, 21\} \cup \{23\},
\]

\[
14 \mathcal{R}_f = \{5\} \cup \{7, \ldots, 24\} \cup \{26\},
\]

\[
\mathcal{R}_f^0 = \{(\frac{11}{7}, \frac{12}{7}, \frac{13}{7})\}.
\]

Here \( 3/7 \in (\mathcal{R}_Z + \mathbb{Z}_{<0}) \setminus \mathcal{R}_Z \) with \( 3/7 > \alpha_Z = 5/14 \), and \( \mu_3 = \nu_{10} = 1 \). (There is a similar phenomenon in the case \( f = x^6 y^3 - z^9 \).)

**Example V.** It is known that for \( f = x^6 + y^6 + x z^3 + z^2 w \), \( \text{Supp}\{\mu_k^{(2)}\} \) is not discretely connected, according to Aldo Conca, see [Sti15]. Indeed, \( \sum_k \mu_k^{(2)} T^k = T^9 + T^{11} \) in this case (using (A.3) below). We also have \( \sum_k \mu_k^{(2)} T^k = T^{12} + T^{16} \) if \( f = x^7 + y^7 + y z^2 (x^4 + z^3 w) \). Such an example is unknown if condition (WH) is satisfied.

**Appendix: Explicit computations using computers**

In this Appendix we explain how to use the computer programs Macaulay2 and Singular for explicit calculations of roots of Bernstein-Sato polynomials.

**A.1. Calculation of \( \mathcal{R}_Z \).** Using (1.8.2–3) together with the computer program Macaulay2, one can easily calculate the Steenbrink spectrum and the Bernstein-Sato polynomial of a weighted homogeneous polynomial \( h \) with weights \( w_i \) and having an isolated singularity.

To determine the weights of weighted homogeneous polynomials with isolated singularities for \( n = 2 \), it is enough to consider the following cases up to a permutation of variables:

\[
(A.1.1) \quad x^i + y^j, \quad x(x^i + y^j), \quad xy(x^i + y^j),
\]

where the weights \( (w_1, w_2) \) are given respectively by

\[
(A.1.2) \quad (\frac{i}{ij}, \frac{j}{ij}), \quad (\frac{i}{i(j+1)}, \frac{j}{i(j+1)}), \quad (\frac{i}{ij+(j+1)}, \frac{j}{ij+(j+1)}).
\]

Combined with (1.8.5), this implies that the Milnor numbers are respectively

\[
(A.1.3) \quad (i-1)(j-1), \quad (i+1)(j-1) + 1, \quad (i+1)(j+1).
\]

Set \( e = \text{GCD}(i, j), \quad a = j/e, \quad b = i/e, \) and let \( m \) be either

\[
i j/e \quad \text{or} \quad (i+1)j/e \quad \text{or} \quad (ij+i+j)/e,
\]

depending on the above three cases. One can calculate the right-hand side of (1.8.3) with \( T = t^{1/m} \) in the case \( h = xy(x^3 + y^2) \), for instance, by typing in Macaulay2 as follows:

\[
A=\text{QQ[T]; i=3; j=2; e=gcd(i,j); a=j/e; b=i/e; m=(i*j+i+j)/e}
\]

\[
((T^a-T^m)/(1-T^a))\ast((T^b-T^m)/(1-T^b))
\]

(This can be copied from a pdf file usually.) In this case, the output should be

\[
T^{17} + T^{15} + T^{14} + T^{13} + T^{12} + 2T^{11} + T^{10} + T^9 + T^8 + T^7 + T^5
\]

In the other two cases, the definition of \( m \) should be replaced by \( i\ast j/e \) or \( (i+1)\ast j/e \).
These calculations can be used to determine $R_Z$. (For this it would be also possible to use RISA/ASIR if the singularities are not very complicated.)

**A.2. Determination of the singularities of $Z$.** It is sometimes difficult to see whether a semi-weighted-homogeneous polynomial $h_z$ is quasi-homogeneous or not, see 1.7. In our case this can be verified by seeing whether the total Tjurina number $\tau_Z$ coincides with the total Milnor number $\mu_Z$. The latter can be obtained by using the method in A.1, and the former is given by high coefficients of the polynomial $\mu$ in (A.3) below. This can be seen explicitly if the separator “;” at the end of the definition of $\mu$ is removed (where Return must be pressed).

However, it may be possible that $\tau_Z$ and $\mu_Z$ obtained by the above procedure coincide even though some singularity is not quasi-homogeneous. This may occur if there is a hidden singular point of $Z$. It may be verified by typing (or copying) for instance the following in Macaulay2:

```
R=QQ[x,y,z]; f=x^5*y*z+x^4*y^2*z+z^7;
S=R/(f); I=(radical ideal singularLocus S); decompose I
```

The output in this case should be

\{ideal (z, y), ideal (z, x+y), ideal (z, x)\}.

This shows that there is a hidden singular point at [1 : -1 : 0].

There is another method to see whether all the singular points of $Z$ are quasi-homogeneous by using a computer program like Singular as in [DiSt 19]. Setting $h := f|_{z=1}$ after a general coordinate change of $\mathbb{C}^3$ such that $\text{Sing } Z \cap \{z = 0\} = \emptyset$, we can compare

$$
\tau_h := \dim \mathbb{C}[x,y]/(h, h_x, h_y) \quad \text{and} \quad \mu_h := \dim \mathbb{C}[x,y]/(h^2, h_x, h_y),
$$

using Singular (and also [BrSk 74]), see [DiSt 19]. For instance, setting

$$
h = x^5(x+1)z + x^4(x+1)^2z + z^7,
$$

(which is the restriction of $f$ to $y = x+1$, and defines $Z|_{x\neq y} \subset \mathbb{C}^2$), we can calculate the total Tjurina number of $h$ for the singular points of $Z|_{x\neq y}$ by typing (or copying) the following in the computer program Singular [DGPS 20]:

```
ring R = 0, (x,z), dp; poly y=x+1;
poly f=x^5*y*z+x^4*y^2*z+z^7;
ideal J=(jacob(f),f); vdim(groebner(J));
```

We also get the total Milnor number $\mu_h$ of $h$ by replacing $(\text{jacob}(f),f)$ with $(\text{jacob}(f),f^2)$ (where $f^2$ should be $f^{(n-1)}$ in general, see [BrSk 74]). The problem is then whether we have

$$
\text{Sing } Z \cap \{x = y\} = \emptyset.
$$

This can be verified by comparing $\tau_h$ and $\tau_Z$, where $\tau_Z$ can be obtained by $\mu_k$ for $k \gg 0$, see also the calculation of $\mu'_k$ as in (A.3) below (with $f$ replaced by $f$ in this subsection). Note that $\tau_Z$ is also obtained as tau in the Macaulay2 calculation in (A.3) below, which can be seen by removing “;” after the definition of tau.

**A.3. Calculation of $\mu'_k$, $\delta_k$.** We can get $\sum_k \mu'_k T^k$ and $\sum_k \delta_k T^k$ for instance in the case of Example I in 5.1 by typing (or copying) the following in Macaulay2:
\[ R = \mathbb{Q}[x, y, z]; \ n = \dim R; \ f = x^5 + y^4 z; \]
\[ d = \text{first degree } f; \ d_1 = n \cdot d - d + 1; \ d_2 = n \cdot d - n; \ d_3 = n \cdot d - n + d; \]
\[ J = \text{ideal (jacobian ideal}(f)); \ A = \text{frac(QQ[T]);} \]
\[ \mu_1 = \text{sub(hilbertSeries}(R/J, \text{Order} = d_1), A); \]
\[ \mu_2 = \text{sub(hilbertSeries}(R/J, \text{Order} = d_2), A); \]
\[ \mu_3 = \text{sub(hilbertSeries}(R/J, \text{Order} = d_3), A); \]
\[ \nu = \mu_1 T^n - \frac{(T^d - T)}{(T - 1)}^n; \]
\[ \tau = \frac{\mu_1 - \mu_2}{T^{n \cdot d - n}}; \]
\[ \mu_5 = \frac{\tau \cdot (T^{n \cdot d + 1} - 1)}{(T - 1)} - \text{sub}(\nu, \{T = 1/T\}) \cdot T^{n \cdot d}; \]
\[ \mu_5 = \frac{\mu_3 T^n - \frac{(T^d - T)}{(T - 1)}^n}{(T^d - T)/(T - 1)} \cdot n; \]
\[ \delta = \mu_2 T^n - \frac{\nu_2}{T^d}; \]

Here, \( \mu', \mu'' \) mean \( \mu' \cdot, \mu'' \cdot \) respectively. The output should be both \( T^9 + T^8 + T^7 + T^6 \), since this is an example of an extremely degenerated curve, see 1.9. One can see the intermediate results by removing “;” (and pressing Return). In the four variable case, \( \mathbb{Q}[x, y, z] \) should be replaced by \( \mathbb{Q}[x, y, z, w] \).

We can apply the above calculation to Walther’s example [Wa 17] (see also [Sa 16]) where \( f \) is given by
\[
\begin{align*}
&x \cdot y \cdot z \cdot (x + 3z) \cdot (x + y + z) \cdot (2x - 2y + z) \cdot (2x - y - 2z) \cdot (2x + y + z) \cdot (2x - y - z);
\end{align*}
\]
In this case, \( \sum \delta_k T^k \) is a polynomial of degree 16 and 15 respectively. Since \( \frac{16}{9} \not\in \frac{1}{3} \mathbb{Z} \) and \( \mathcal{R}_Z = \{ \frac{1}{3}, 1, \frac{2}{3}, 1 \} \), this implies that \( b_f(s) \) is not a combinatorial invariant of a hyperplane arrangement, see also [Sa 16]. Note that the above first polynomial gives the same \( \{\delta_k\} \) as Ziegler’s (see [Zi 89]):
\[
\begin{align*}
&x \cdot y \cdot z \cdot (x + y - z) \cdot (x - y + z) \cdot (2x - 2y + z) \cdot (2x + y + z) \cdot (2x - y - z);
\end{align*}
\]

References

[Ba 24] Bath, D., Bernstein–Sato polynomials for positively weighted homogeneous locally everywhere divisors, hyperplane arrangements, in \( \mathbb{C}^3 \) (arxiv:2402.08342, 2024).

[BrSk 74] Briançon, J., Skoda, H., Sur la clôture intégrale d’un idéal de germes de fonctions holomorphes en un point de \( \mathbb{C}^n \), C. R. Acad. Sci. Paris, Sér. A, 278 (1974), 949–951.

[Br 70] Brieskorn, E., Die Monodromie der isolierten Singularität en von Hyperflächen, Manuscripta Math. 2 (1970), 103–161.

[BDS 11] Budur, N., Dimca, A., Saito, M., First Milnor cohomology of hyperplane arrangements, Contemp. Math., 538 (2011), 279–292.

[BuSa 05] Budur, N., Saito, M., Multiplier ideals, V- filtration, and spectrum, J. Alg. Geom. 14 (2005), 269–282.

[BuSa 10] Budur, N., Saito, M., Jumping coefficients and spectrum of a hyperplane arrangement, Math. Ann. 347 (2010), 545–579.

[BSY 11] Budur, N., Saito, M., Yuzvinsky, S., On the local zeta functions and the \( b \)-functions of certain hyperplane arrangements. With an appendix by Willem Veys. J., London Math. Soc. (2) 84 (2011), 631–648.

[DGPS 20] Decker, W., Greuel, G.-M., Püister, G. and Schönenmann, H., SINGULAR 4.2.0 — A computer algebra system for polynomial computations, available at http://www.singular.uni-kl.de (2020).

[dFEM] de Fernex, T., Ein, L., Mustaţă, M., Bounds for log canonical thresholds with applications to birational rigidity, Math. Res. Lett. 10 (2003), 219–236.

[De 71] Deligne, P., Théorie de Hodge II, Publ. Math. IHES, 40 (1971), 5–58.

[De 73] Deligne, P., Le formalisme des cycles évanescents, in SGA7 XIII, Lect. Notes in Math. 340, Springer, Berlin (1973), 82–115.

[Di 92] Dimca, A., Singularities and Topology of Hypersurfaces, Universitext, Springer, Berlin, 1992.

[Di 17] Dimca, A., On the syzygies and Hodge theory of nodal hypersurfaces, Ann. Univ. Ferrara 63 (2017), 87–101.

[DIM 20] Dimca, A., Ibadula, D., Mačinč, D.A., Numerical invariants and moduli spaces for line arrangements, Osaka J. Math. 57 (2020), 847–870.

[DiPo 16] Dimca, A., Popescu, D., Hilbert series and Lefschetz properties of dimension one almost complete intersections, Comm. Alg. 44 (2016), 4467–4482.
[St 85] Steenbrink, J.H.M., Semicontinuity of the singularity spectrum, Inv. Math. 79 (1985), 557–565.
[St 89] Steenbrink, J.H.M., The spectrum of hypersurface singularities, Astérisque 179-180 (1989), 163–184.
[Sti 15] Sticlaru, G., Log-concavity of Milnor algebras for projective hypersurfaces, Math. Reports 17(67), 3 (2015), 315–325.
[Va 82] Varchenko, A.N, Asymptotic Hodge structure in the vanishing cohomology, Math. USSR-Izv. 18 (1982), 469–512.
[Va 83] Varchenko, A.N., Semicontinuity of the spectrum and an upper bound for the number of singular points of the projective hypersurface, (Russian) Dokl. Akad. Nauk SSSR 270 (1983), 1294–1297.
[Wa 05] Walther, U., Bernstein-Sato polynomial versus cohomology of the Milnor fiber for generic hyperplane arrangements, Compos. Math. 141 (2005), 121–145.
[Wa 17] Walther, U., The Jacobian module, the Milnor fiber, and the $D$-module generated by $f^*$, Inv. math. 207 (2017), 1239–1287.
[Zi 89] Ziegler, G.M., Combinatorial construction of logarithmic differential forms, Adv. Math. 76 (1989) 116–154.

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