THE PROTON DISTRIBUTION FUNCTION IN WEAKLY MAGNETIZED TURBULENT PLASMAS

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ABSTRACT

We calculate the proton distribution function due to heating by subsonic, Alfvénic turbulence in a weakly magnetized collisionless plasma. The distribution function is nonthermal. For nonrelativistic energies it is an exponential of the magnitude of the proton velocity. For ultrarelativistic energies, it can be characterized as a power law with a momentum-dependent slope. We briefly discuss the implications of this work for gamma-ray emission (via pion decay) from hot accretion flows.

Subject headings: MHD — plasmas — turbulence

1. INTRODUCTION

The proton distribution function (PDF) in a collisionless plasma depends on the history of the plasma. If, for example, the plasma is heated by shocks, the PDF is believed to be a power law at high energies, while for the thermal particles, the form of the PDF is not thought to be universal (e.g., Blandford & Eichler 1987). In this paper we calculate the PDF for a weakly magnetized plasma (the Alfvén speed \(v_A\) is less than the ion thermal speed \(v_i\)) heated by subsonic, Alfvénic turbulence.

Although we do not have a fully quantitative theory for the turbulence (§ 2), we are still able to calculate the distribution function (§ 3). This is because of a special feature of particle heating by low-frequency, subsonic waves (§ 3.1). In § 4 we mention a possible application of our results to accreting black holes. We summarize our results in § 5.

2. PROTON HEATING BY SUBSONIC TURBULENCE

The nature of subsonic turbulence in a collisionless weakly magnetized plasma is not fully understood. A plausible scenario, based on Goldreich & Sridhar (1995), is as follows. Large-scale motions (larger than the ion Larmor radius \(\rho\)) are nearly dissipationless and incompressible. If the characteristic velocity of the energy containing eddies (eddies of the largest scale \(L\)) is small, \(V \ll v_A \ll v_i\), the dynamics is adequately described by reduced MHD (incompressible MHD without slow modes, e.g., Strauss 1976; Ng & Bhattacharjee 1996). As in hydrodynamics, the turbulent energy cascades to small scales where it is dissipated. The cascade is described by a Kolmogorov 5/3 law, but it is strongly anisotropic. Typical wavevectors \(k\) are nearly perpendicular to the local magnetic field, so that perturbations are strongly elongated along the local magnetic field. The energy per unit mass in eddies of perpendicular wavenumber \(\sim k_{\perp}\) is \(\sim \epsilon^{2/3} k_{\perp}^{-2/3}\), where \(\epsilon \sim V^3/L\) is the net energy dissipation rate per unit mass. The degree of anisotropy is determined by the requirement that the Alfvén frequency not exceed the nonlinear frequency, which yields \(k_{\perp} \lesssim k_A^{2/3} L^{-1/3}\).

“Viscous” heating occurs when the perpendicular size of the eddy is \(\sim \rho\) (Quataert 1998; Gruzinov 1998; Quataert & Gruzinov 1999; see also Blackman 1998). The small-scale eddies are essentially Alfvén waves. Alfvén waves with a perpendicular wavenumber \(\sim \rho^{-1}\) have a nonzero parallel magnetic field perturbation (in contrast to long-wavelength perturbations). As a result, particles interact with waves through “magnetic mirror” forces, i.e., through the coupling of the magnetic moment of the particle and the parallel gradient of the magnetic field. A charged particle moving along the field will be randomly accelerated and decelerated along the field and will therefore diffuse in parallel velocity space.

Strong parallel diffusion requires \(|v_\parallel| \leq v_A\), where \(v_\parallel\) is the component of the particle’s velocity along the magnetic field. This can be understood as follows. A parallel magnetic field perturbation of frequency \(\omega\) and parallel wavenumber \(k_\parallel\) will accelerate particles that satisfy the resonance condition \(v_\parallel = \omega/k_\parallel\). If the turbulence were strictly a superposition of linear Alfvén waves, the resonance condition would imply that only particles with \(|v_\parallel| = v_A\) are accelerated (because Alfvén waves have the dispersion relation \(\omega = |k_\parallel| v_A\)).

It is likely, however, that the strong turbulence of interest is not fully describable as linear Alfvén waves. In this case, at the smallest scales \((k_\perp \sim \rho^{-1})\), perturbations of all frequencies \(\omega \lesssim \omega_{\text{Alf}}\) and wavenumbers \(k_\parallel \lesssim \omega_{\text{Alf}}/v_A\) will exist, where \(\omega_{\text{Alf}} \sim \epsilon^{1/3} \rho^{-2/3}\) is the nonlinear frequency on scales \(\sim \rho\). The quantity \(\omega/k_\parallel\) then takes on values between 0 and \(\infty\). This does not, however, mean that a particle with an arbitrary parallel velocity will be accelerated. Acceleration of high parallel velocity particles, \(|v_\parallel| > v_A\), will be insignificant because perturbations resonating with a high parallel velocity particle must have small \(k_\parallel\). These perturbations will exert negligibly small magnetic mirror forces for two reasons. The parallel force is proportional to the parallel gradient of the parallel magnetic field perturbation (see eq. [4]). In addition, the parallel magnetic field perturbation is itself proportional to the parallel gradient of the electric field perturbation, because an incompressible two-dimensional plasma flow with \(k_\parallel = 0\) does not perturb the parallel magnetic field. Magnetic mirror forces are therefore \(\sim k_{\perp}^2\); perturbations with small \(k_\parallel\) are consequently negligible.

3. THE EVOLUTION OF THE PROTON DISTRIBUTION FUNCTION

Diffusion in parallel velocity due to turbulent heating can

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\(^3\) Formally, this follows from the small \(k_\parallel\) limit of the plasma dielectric tensor.
be described by the equation
\[ \frac{\partial f}{\partial t} = \frac{\partial}{\partial p_{||}} \left( D_{||} \frac{\partial f}{\partial p_{||}} \right), \tag{1} \]
where \( D_{||} \) is the parallel momentum diffusivity. Anisotropies in momentum space are quickly erased by high-frequency small-scale electromagnetic instabilities. Parallel momentum diffusion is then equivalent to an isotropic diffusion in momentum space,
\[ \frac{\partial f}{\partial t} = \frac{1}{p^2} \frac{\partial}{\partial p} \left( p^2 D \frac{\partial f}{\partial p} \right), \tag{2} \]
where
\[ D = \frac{1}{2} \int_{-1}^{1} d\mu \mu^2 D_{||} \tag{3} \]
and \( \mu \) is the cosine of the particle’s pitch angle (the angle between the velocity vector and the local magnetic field).

Suppose that protons are strongly heated, so that their final thermal energy is much larger than their initial thermal energy. In this case, the final distribution function depends primarily on the properties of the diffusion under consideration, but only weakly on the initial distribution function. If \( D \) does not depend on \( p \), the final distribution function will be Maxwellian. As we show below, for our problem \( D \propto p \gamma \), where \( \gamma \) is the particle’s Lorentz factor. Since \( D \) is not constant, diffusion establishes a non-Maxwellian PDF.

3.1. The Diffusion Coefficient

We first calculate the parallel diffusivity, \( D_{||} \). The parallel motion of a particle interacting with low-frequency perturbations is described by
\[ \frac{dv_{||}}{dt} = \frac{v_{||}^2}{2} \frac{\nabla_{||} B}{B}, \tag{4} \]
where we have assumed \(|v_{||}| \ll v_{\perp}\). Since \( p_{||} = \gamma m v_{||} \), the parallel momentum diffusivity is
\[ D_{||} \propto \gamma^2 v_{\perp}^4 \Phi(v_{||}/v_{\perp}), \tag{5} \]
where \( \Phi(x) \) is a positive dimensionless function, which is \( \sim 1 \) for \(|x| \leq 1\) and rapidly approaches zero at larger \(|x|\). This is because only particles with sub-Alfvénic parallel velocities efficiently interact with the turbulence (see §2). The value of \( \Phi(x) \) cannot be calculated in detail because the spectrum of turbulence at small scales, where nonlinear and kinetic effects are both important, is unknown. Because we are interested in \( v_{\perp} \ll v_{\perp} \), however, our final answer will be independent of \( \Phi \).

The isotropic diffusivity, \( D \), can be obtained from \( D_{||} \) using equation (3), which yields
\[ D \propto \gamma^2 v_{\perp}^4 \int d\mu \mu^2(1 - \mu^2)^2 \Phi(\mu v/v_{\perp}) \propto \gamma^2 v_{\perp}. \tag{6} \]

3.2. The Distribution Function

For nonrelativistic protons, \( D \propto p \). Solving equation (2) for an initial PDF of the form \( f_0(p) \propto \delta(p) \) gives (Appendix A)
\[ f(p) = \frac{1}{\pi T_{\text{m}}} \exp\left( -\frac{2p}{\sqrt{T_{\text{m}}}} \right), \tag{7} \]
where the “temperature” is defined by the usual expression \( \langle p^2 \rangle = 3 T_{\text{m}} \). Although the PDF is nonthermal, the number of suprathermal protons is exponentially small. Subsonic turbulence is therefore intrinsically inefficient at accelerating particles in the nonrelativistic limit.

At ultrarelativistic energies, \( D \propto p^2 \). Solving equation (2) for \( f_0(p) \propto \delta(p - p_{\text{f}}) \) gives (Appendix B)
\[ f(p) \propto \left( \frac{p}{p_{\text{f}}} \right)^{-a(p)}, \tag{8} \]
where
\[ a(p) = \frac{7}{2} + \frac{\log(p/p_{\text{f}})}{\log(p/p_{\text{f}})} \tag{9} \]
and \( p_{\text{f}} \) is the mean momentum of the final PDF. At relativistic energies, there is a significant population of particles above the mean energy of the plasma. Realizing this efficient acceleration, however, requires that the bulk of the plasma be heated to at least mildly relativistic energies, which limits its practical significance.

Using equation (2) and equation (6), we numerically calculated the distribution function for a plasma that was heated from nonrelativistic to mildly relativistic energies.

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\( ^4 \) The Green’s function of eq. (2) with \( D = \text{const} \) is a Gaussian curve.

\( ^5 \) For linear MHD Alfvén waves, \( \Phi(x) \propto \delta(|x| - 1) \).
Figure 1 shows the calculated distribution in energy space, \( dN/dE \), where \( E = (\gamma - 1)mc^2 \). Also shown is a Maxwellian with a temperature of 0.2 GeV, which has the same mean energy per particle. The viscously heated PDF has a noticeable nonthermal tail of relativistic particles.

4. APPLICATION TO ACCRETION FLOWS

A possible application of our results is to hot, quasi-spherical, collisionless plasmas which have been proposed to exist near accreting black holes. By spherical accretion we mean any of the proposed accretion models, Bondi (Bondi 1952), ion tori (Rees et al. 1982), or advection-dominated accretion flows (Narayan & Yi 1995). Our calculated PDF may be relevant to these gravitationally confined plasmas (which must have \( v_A \ll v_i \)). In such hot plasmas, even a Maxwellian PDF leads to interesting levels of gamma-ray emission because of the decay of neutral pions produced in proton-proton collisions (Mahadevan, Narayan, & Krolik 1998; Ginzburg & Syrovatskii 1964 give a clear description of how to estimate the rates).

For protons with roughly virial energies near the Schwarzchild radius (\( \approx R_s \)) of a black hole, the PDF calculated in the previous section is noticeably non-Maxwellian, with an excess of high-energy protons. It should, in principle, produce distinct observable features in the gamma-ray spectrum. As with a thermal plasma, however, relativistic protons capable of producing pions only exist within \( \sim 10R_s \) of the black hole (because the bulk of the PDF must itself be relativistic to have any noticeable population of relativistic protons; in the nonrelativistic limit, the PDF still cuts off exponentially with proton momentum). This implies that detailed predictions of the gamma-ray flux from pion decay are very sensitive to the poorly understood details of the flow structure near \( R_s \) (cf. Mahadevan et al. 1998).

One relatively robust and testable conclusion which can be drawn from our analysis is that heating by subsonic, Alfvénic, turbulence is unable to give rise to a substantial gamma-ray flux above \( \sim 1 \) GeV. There are simply no protons with sufficiently high Lorentz factors.

5. SUMMARY

We have calculated the PDF due to heating by subsonic Alfvénic turbulence in a weakly magnetized collisionless plasma. For nonrelativistic energies, the PDF is nonthermal only to the extent that it cuts off exponentially, rather than as a Gaussian, with particle momenta. By contrast, at mildly relativistic or relativistic energies, a potentially observable "suprathermal tail" develops.

Our results seem to be relatively robust. For plasmas heated to well above their initial thermal energy, the distribution function is determined solely by the momentum space diffusivity. The diffusivity we have calculated (eq. [6]) should be valid on quite general grounds, provided the protons are indeed predominantly heated by low-frequency, subsonic, waves.

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APPENDIX A

NONRELATIVISTIC DIFFUSION

Here we solve the diffusion equation

\[
\frac{\partial f}{\partial t} = \frac{1}{p^2} \frac{\partial}{\partial p} \left( p^2 D \frac{\partial f}{\partial p} \right),
\]

assuming the nonrelativistic diffusivity \( D(p) = A(t)p \), where \( A(t) \) is a function of time only. We assume that at \( t = 0 \) the protons were cold, \( f_0(p) \propto \delta(p) \).

The change of variables \( p = q^2 \) and \( dt = A(t)dt/4 \) gives

\[
\frac{\partial f}{\partial \tau} = \frac{1}{q^2} \frac{\partial}{\partial q} \left( q^5 \frac{\partial f}{\partial q} \right).
\]

This is an ordinary diffusion equation in six dimensions. The solution is

\[
f \propto \exp \left( -q^2/4\tau \right) = \exp \left( -2p/\sqrt{Tm} \right),
\]

where we replace the time variable, \( \tau \), with the more physically relevant temperature variable, \( T = 64\pi^2/m \). The temperature, \( T \), of the plasma increases with time so long as the heating continues [i.e., so long as \( A(t) \neq 0 \)]. At each moment of time, the PDF is given by equation (7).

APPENDIX B

ULTRARELATIVISTIC DIFFUSION

In the ultrarelativistic case, \( D \propto p^2 \), and we make the change of variables \( p = \exp q \) and \( dt = A(t)dt \). Then

\[
\frac{\partial f}{\partial \tau} = \frac{\partial^2 f}{\partial q^2} + 3 \frac{\partial f}{\partial q}.
\]
Another change, $y = q + 3\tau$, gives an ordinary diffusion equation in one dimension:

$$\frac{\partial f}{\partial \tau} = \frac{\partial^2 f}{\partial y^2}. \quad (B2)$$

Equation (8) can now be derived along standard lines.

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