Technical Details of the Proof of the Sine Inequality

\[ \sum_{k=1}^{n-1} \left( \frac{n}{k} - \frac{k}{n} \right)^\beta \sin(kx) \geq 0 \]

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Abstract

In a recent study, H. Alzer and the author showed that the sine polynomial

\[ \sum_{k=1}^{n-1} \left( \frac{n}{k} - \frac{k}{n} \right)^\beta \sin(kx) > 0 \]

is nonnegative for \( x \in [0, \pi] \), \( n \geq 2 \), \( \beta \geq \beta_1 := \frac{\log(2)}{\log(16/5)} \). This result, among others, will be presented in a forthcoming article. The proof relies on quite a number of technical Lemmas and inequalities. We have decided to delegate all the tedious details of the proofs of these Lemmas in a separate article, namely, the current one. Some of the proofs require brute-force numerical computation, performed with the help of the computer software MAPLE.

A few of the Lemmas included here are of independent interest.

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1 Introduction

As stated in the Abstract, this article is meant to be a supplement to a hopefully forthcoming paper on a recent project done in collaboration with H. Alzer. The results presented here are proved mainly by elementary techniques, sometimes with the assistance of the
computer for carrying out tedious computations. There is no intention to seek formal publication in an official research journal. For that reason, often more detailed arguments as well as heuristic discussions may be included.

The result to be established is

**Theorem AK.** The inequality

\[ 0 \leq \sum_{k=1}^{n-1} \left( \frac{n}{k} - \frac{k}{n} \right) \beta \sin(kx) \quad (\beta \in \mathbb{R}) \quad (1.1) \]

holds for all integers \( n \geq 2 \) and real numbers \( x \in [0, \pi] \) if and only if \( \beta \geq \log(2)/\log(16/5) = 0.59592... \).

Throughout the paper, \( a_k, b_k, c_k, d_k \) denote NN (nonnegative) numbers, and \( m, n, k \) positive integers. For convenience, we adopt the notations:

\[ s(k) = \sin(kx), \quad c(k) = \cos(kx), \]

\[ [a_1, a_2, ..., a_n] = \sum_{k=1}^{n} a_k s(k), \]

\[ [a_1, a_2, ..., a_n]^- = \sum_{k=1}^{n} (-1)^{k+1} a_k s(k). \]

The following identities are well-known:

\[ \tau_k = [1, ..., 1] = \frac{c(\frac{1}{2}) - c(k + \frac{1}{2})}{2s(\frac{1}{2})}. \quad (1.2) \]

\[ \tau_k^- = [1, ..., 1]^- = \frac{s(\frac{1}{2}) - (-1)^k s(k + \frac{1}{2})}{2c(\frac{1}{2})}. \quad (1.3) \]

A tool used frequently below is explained in detail in [3]. Let \( f(x) = g_1(x) - g_2(x) \) be the difference of two specific increasing (or decreasing) functions in a given bounded interval \([\alpha, \beta]\) and they do not involve any further variable parameters. The tool, called \texttt{dif}, is a numerical algorithm, implemented as a MAPLE procedure, can be applied to rigorously prove \( f(x) \geq 0 \) (if that is true). In the sequel, if an inequality is proved using this technique, we will attach the computer output.

The author would like to thank Horst Alzer for an enjoyable long-time collaboration in the study of classical inequalities. The current work is a part of one of our projects. Horst should be a coauthor, but he insisted otherwise.
2 Proof of Theorem AK

The result for small $n = 2, \cdots, 6$ was proved in [1]. Thus our starting point is $n \geq 7$.

The proof, in its entirety, is quite long. In order not to mask the main ideas, we postpone the proofs, some rather technical, of quite a number of Lemmas and inequalities to Section 3. In addition, we divide the proof into small chunks in order to highlight the different ideas involved.

1. Instead of studying the sine polynomial in (1.1) directly, we consider two equivalent ones. First, we divide it by the coefficient of the first term to get

$$S_{n, \beta}(x) := [a_{n,k}] = \sum_{k=1}^{n-1} \left( \frac{n^2 - k^2}{(n^2 - 1)k} \right)^\beta s(k)$$

(2.1)

with leading coefficient $a_{n,1} = 1$. For convenience, the dependence of $a_{n,k}$ on $\beta$ is not explicit in the notation. In addition, we often suppress even $n$ and/or $x$ in the notation. For example, we write $S$ and sometimes $S_n$ instead of $S_{n, \beta}(x)$ and $a_k$ instead of $a_{n,k}$.

We also denote the sequence of second differences of the coefficients by

$$\Box_k = a_{k-1} - 2a_k + a_{k+1} \quad (k = 2, \cdots, n-1).$$

(2.2)

Next, by reflection, we see that $[a_k]$ is NN if and only if $S\neg \neg = [a_k]^{-}$ is NN. (Caution: In [1], the superscript * is used instead of $\neg \neg$. ) The $\neg \neg$ operator can be extended to general sine polynomials. Using $S\neg \neg$ in placed of $S$ simplifies some intermediate inequalities.

2. Our first step is to show that for $n \geq 7$, $S\neg \neg$ is NN on $[0.75, \pi]$.

For all $n$, $a_2 \leq 2^{-\beta_1} < 0.662$. Using (3.2) of Lemma 3, we have

$$[a_k]^{-} \geq \frac{s\left(\frac{3}{2}\right) + s\left(\frac{1}{2}\right)}{2c\left(\frac{1}{2}\right)} - 0.662 \left[1 + s\left(\frac{3}{2}\right)\right] = s\left(\frac{3}{2}\right) + 0.338s\left(\frac{3}{2}\right) - 0.662.$$

The numerator of the RHS is NN in $[0.75, \pi]$ ([3, Lemma 7]), so is $[a_k]^{-}$.

3. A sequence $\{c_k\}_{k=1}^{m}$ is said to be convex (concave) if

$$c_{k-1} - 2c_k + c_{k+1} \geq (\leq) 0 \quad \text{for } k = 2, \cdots, m-1.$$

Lemma 1(ii) assets that when the coefficient sequence of a sine polynomial, appended with a 0, is convex, the polynomial is NN. For certain combinations of $n$ and $\beta$ (for instance, when $n = 7$ and $\beta > 0.8714$), the coefficient sequence $\{a_k\}_{k=1}^{n}$ (Note that the last member of the sequence is $a_n = 0$.) of (2.1) is convex, and hence $S$ is NN, and Theorem AK hold for those combinations.

However, in general, $\{a_k\}$ is not convex. In such cases, we invoke Lemma 5 to find a positive integer $m < n$ such that the sub-sequence $\{a_1, \cdots, a_m\}$ is convex while
the sub-sequence \( \{a_m, a_{m+1}, \ldots, a_n\} \) has an odd number of terms and is concave.

After subtracting \( a_m \) from each of the terms of the first sub-sequence, we see that \( \{a_1 - a_m, \ldots, a_{m-1} - a_m, 0\} \) is convex. By Lemma 1 (ii),

\[
S_1 := [a_1 - a_m, a_2 - a_m, \ldots, a_{m-1} - a_m]
\]

is NN. Note that \( S \) has the decomposition

\[
S = S_1 + T
= S_1 + [a_m, \ldots, a_m, a_{m+1}, \ldots, a_{n-1}].
\]

It is easy to verify that \( T \) has the alternative representation

\[
T = \sum_{k=m}^{n-1} d_k \tau_k,
\]

with an even number of summands on the RHS and

\[
d_k = a_k - a_{k+1} \uparrow \text{ in } k.
\]

4. By Lemma 3 (ii), \( S_1 \) has at least five terms. We decompose it further as follows

\[
S_1 = H + K
\]

where

\[
H = [a_1 - 4a_4 + 3a_5, a_2 - 3a_4 + 2a_5, a_3 - 2a_4 + a_5].
\]

The convexity of \( \{a_1, \ldots, a_m\} \) implies that the coefficients of \( H \) and \( K \) are positive and satisfy the hypotheses of Lemma 1 (ii), respectively. Hence, both \( H \) and \( K \) are NN. It follows that

\[
S = H + K + T \geq H + T.
\]

In view of this, an appropriate tactic to prove Theorem AK is to find good lower bounds for \( H \) and \( T \).

5. For convenience, we have suppressed the explicit dependence of \( H \) and \( T \) on \( n \) and \( \beta \).

The more precise notations ought to be \( H(n, \beta) \) and \( T(n, \beta) \).

After the substitution \( y = \frac{1}{n^2-1} \in (0, \frac{1}{38}) \), the coefficients of \( H \) are given by \( h_i(y; \beta), i = 1, 2, 3 \) while the two second differences are \( h_i(y; \beta), = 4, 5 \) as given before Lemma 8.

For fixed \( n \) and \( \beta > \beta_1 \), Lemma 8 (ii) asserts that \( H(n, \beta) - H(n, \beta_1) \) satisfies Lemma 1 (ii) and so is NN, implying that \( H(n, \beta) \geq H(n, \beta_1) \).

Likewise, with \( \beta = \beta_1 \) fixed, and \( n > n_1 \), Lemma 8 (i) asserts that \( H(n, \beta_1) \geq H(n_1, \beta_1) \).

It follows that for \( n \geq n_1 \) and \( \beta \geq \beta_1 \),

\[
H(n, \beta) \geq H(n, \beta_1) \geq H(n_1, \beta_1).
\]

(2.6)
6. In particular, for $n > 7$ and $\beta > \beta_1$,

$$H^-(n, \beta) \geq H^-(7, \beta_1)$$

$$= \left(1 - 4 \left(\frac{11}{64}\right)^{\beta_1} + 3 \left(\frac{1}{10}\right)^{\beta_1}\right) \sin(x) - \left(\frac{15}{32}\right)^{\beta_1} - 3 \left(\frac{11}{64}\right)^{\beta_1} + 2 \left(\frac{1}{10}\right)^{\beta_1}\right) \sin(2x)$$

$$+ \left(\frac{5}{18}\right)^{\beta_1} - 2 \left(\frac{11}{64}\right)^{\beta_1} + \left(\frac{1}{10}\right)^{\beta_1}\right) \sin(3x).$$

By Lemma 10, this sine polynomial is concave in $[0, 0.75]$, and so $H^-(7, \beta_1)/x$ is decreasing, implying that

$$\frac{H^-}{x} \geq \frac{H^-(7, \beta_1)}{x} \bigg|_{x=0.75} = 0.2232352723... \quad (2.7)$$

7. To get a lower bound for $T^-$ we make use of (2.3) and Lemma 2 (iii). By definition,

$$d_{n-1} = a_{n-1} = \left(\frac{2n - 1}{(n^2 - 1)(n - 1)}\right)^{\beta_1} \leq \left(\frac{2n - 1}{(n^2 - 1)(n - 1)}\right)^{\beta_1} \leq \frac{1.105}{n}.$$  

The last inequality holds because $n \left(\frac{2n - 1}{(n^2 - 1)(n - 1)}\right)^{\beta_1}$ is a decreasing function of $n$ (Lemma 13) and the upper bound $1.105$ is obtained by letting $n = 7$. Then Lemma 2 (iii) gives

$$T^- \geq \frac{1.105}{n} \min\{\tau_{n-1}, 0\}$$

$$\geq \frac{1.105}{n} \left(\frac{s\left(\frac{1}{2}\right) - 1}{2c\left(\frac{1}{2}\right)}\right)$$

$$\geq -\frac{1.105}{2n \cos(0.375)}.$$  

(2.8)

8. From (2.5), (2.7), and (2.8), we get

$$S^- \geq H^- + T^- \geq 0.2232x - \frac{1.105}{2n \cos(0.375)}.$$  

The righthand side is NN when

$$x \geq \frac{1.105}{2(0.2232) \cos(0.375)} \frac{1}{n}$$

$$> \frac{2.660223693}{n}. \quad (2.9)$$

This greatly improves ¶2.
9. Now, (2.7) can be improved, by using the shorter interval $[0, \frac{2.67}{7}]$ instead of $[0, 0.75]$, to give
\[
\frac{H^-}{x} \geq \left. \frac{H^-(7, \beta_1)}{x} \right|_{x=2.67/7} > 0.2285. \tag{2.10}
\]
This can be used to replace $0.2232$ in the first line of (2.9).

Then, the factor $\cos(0.375)$ in the denominator of the same expression can be replaced by the larger number $\cos(\frac{2.67}{7})$. After that, we conclude that $S^-$ is NN when
\[
x \geq \frac{1.105}{2(0.2285) \cos(\frac{2.67}{7})} \frac{1}{n} > \frac{2.4602482}{n}.
\]
This proves Proposition 1 of [1].

Although the same arguments can be used iteratively to bootstrap the assertion further, but the improvements gained this way are very slight.

10. When $n$ is even, (1.3) gives
\[
\tau^-_{n-1} = \frac{s(\frac{1}{2}) + s(n - \frac{1}{2})}{2c(\frac{1}{2})} = \frac{s(\frac{3}{2})c(\frac{n-1}{2})}{c(\frac{1}{2})} \geq 0, \quad x \in [0, \frac{\pi}{n}].
\]

By Lemma 2(iii), $T^-$ is NN in $[0, \frac{\pi}{n}] \supset [0, \frac{2.5}{n}]$. By (2.5), $S^-$ is thus also NN in $[0, \frac{\pi}{n}]$. Combining with ¶9 yields Theorem AK. This is Proposition 2 of [1].

It now remains to show that $S^-$ is NN in $[0, \frac{2.5}{n}]$ for odd $n \geq 7$.

11. Let us determine $T$ for $n = 7$. The last second difference
\[
\Box_6 = \left( \frac{1}{10} \right)^\beta - 2 \left( \frac{13}{288} \right)^\beta
\]
is nonnegative for $\beta \geq \beta_2 := \frac{\ln(2)}{\ln(288) - \ln(130)} \approx 0.8714162659$. For such values of $\beta$, $T = 0$ and Theorem AK holds. On the other hand,\[
\Box_5 = \left( \frac{11}{64} \right)^\beta - 2 \left( \frac{1}{10} \right)^\beta + \left( \frac{13}{288} \right)^\beta > 0
\]
for all $\beta > \beta_1$. The construction presented in ¶2 yields $T$ with two summands in the form of $Tdk$.

12. For general $n$, we can find an upper bound of the number of summands of $T$ as follows. We first compute $\Box_k$ for $S$ when $\beta = \beta_1$. Suppose that the last $N$ differences are negative, then $T$ is constructed using the last $N$ or $N - 1$ coefficients, whichever is even. By Remark 2 for $\beta > \beta_1$, $T$ has fewer or equal number of summands than that for $\beta_1$.
Following this scheme, we find that for $n \leq 13$, $T$ has no more than 2 summands while for $n \leq 43$, there are no more than 10.

Recall that in all cases, $T$ has an even number of summands in (2.3) and (3.1) of Lemma 2 (iv) is applicable to yield a lower bound.

In (3.1),

\[ d_{2j} - d_{2j-1} = 2a_{2j} - a_{2j-1} - a_{2j+1} = -\Box_{2j}. \]

By defining $\delta_k := -\Box_{n-k}$, (3.1) becomes

\[ \frac{2|T^-|}{x} \leq (m + 1)\delta_{n-m-1} + ... + (n - 3)\delta_3 + (n - 1)\delta_1. \quad (2.11) \]

We only need to worry about $x \in [0, \frac{2.5}{n}]$ in which $T^- < 0$; this explains the use of $|T^-|$ on the LHS. Note that only odd subscripts of $\delta$ are involved. If $T$ has $2j$ summands, the RHS of (2.11) has $j$ terms involving $\delta_1, \delta_3, \ldots, \delta_{2j-1}$. The necessary lower bounds for $\delta_k$ are derived in Lemmas 12 and 13.

13. Let us first treat the simplest case when $T$ has 2 summands. This is true for $n \leq 13$.

Applying (3.14) of Lemma 12 to (2.11) containing only one summand, leads to

\[ \frac{|T^-|}{x} \leq 0.196. \]

With (2.10), this implies $S^- \geq H^- + T^- \geq 0$ in $[0, \frac{2.5}{n}]$ and Theorem AK is proved.

14. Next, suppose $T$ has 10 or less summands. This is true for $n \leq 43$.

For the rest of the proof we can assume $n \geq 15$ odd. By (2.6), we have $H(n, \beta) \geq H(15, \beta_1)$. In addition, we use the shorter interval $[0, \frac{2.5}{15}]$ to improve (2.10) to

\[ \frac{H^-}{x} \geq \left. \frac{H^-}{x} \right|_{x=2.5/15} > 0.248. \quad (2.12) \]

Using (2.11) with $\delta_k$ up to $\delta_9$ and the estimates (3.21) and (2.12), we obtain

\[ \frac{|T^-|}{x} \leq 0.196 + 0.0206 + 0.009 + 0.005171 + 0.003451 = 0.234222 < \frac{H^-}{x}, \]

proving the Theorem.

15. Finally we consider the case when $T$ has more than 10 summands.

Using $H(n, \beta) \geq H(45, \beta_1)$ and the interval $[0, \frac{2.5}{45}]$, we improve (2.12) further to

\[ \left. \frac{H^-}{x} \right|_{x=2.5/45} > 0.250772629. \quad (2.13) \]

From (2.11), we obtain

\[ \frac{2|T^-|}{x} \leq (n - 11)[\delta_{n-m-1} + ... + \delta_{11}] + (n - 9)\delta_9 + ... + (n - 1)\delta_1. \quad (2.14) \]
The monotonicity of $\square_k$ implies
\[ \delta_{n-m-1} + \ldots + \delta_{11} \leq \delta_{n-m} + \ldots + \delta_{10}. \]

Note that the subscripts of $\delta$ on the RHS are even; those on the LHS are odd. It follows from (2.14) that
\[
\frac{2|T^-|}{x} \leq \frac{(n-11)}{2} \left[ \delta_{n-m-1} + \ldots + \delta_{10} \right] + (n-10)\delta_9 + \ldots + (n-1)\delta_1. \tag{2.14}
\]

The subscripts of $\delta$ in the sum $[\ldots]$ are now consecutive. Fortunately, this sum telescopes:
\[
\delta_{n-m-1} + \ldots + \delta_{10} = (-a_m + 2a_{m+1} - a_{m+2}) + \ldots + (-a_{n-11} + 2a_{n-10} - a_{n-9}) \\
= -a_m + a_{m+1} + a_{n-10} - a_{n-9} \\
\leq a_{n-10} - a_{n-9}.
\]

Hence,
\[
\frac{2|T^-|}{x} \leq \frac{(n-11)}{2} \left( a_{n-10} - a_{n-9} \right) + (n-9)\delta_9 + \ldots + (n-1)\delta_1. \tag{2.15}
\]

The RHS, using (3.21) and (3.18), adds up to less than
\[
\frac{0.1636}{2} + 0.006902 + 0.010342 + 0.0326 + 0.3428 = 0.492444. \tag{2.16}
\]

This together with (2.15), (2.16), and (2.13) implies that $S^-$ is NN, as desired.

The proof of Theorem AK is thus complete.

### 3 Lemmas

**Lemma 1** (Fejér [2, Satz XXVII]).

(i) If $\{c_1, c_2, \ldots, c_m, c_{m+1}\}$ is convex, then $[c_1, \ldots, c_m, \frac{c_{m+1}}{2}]$ is NN in $[0, \pi]$.

(ii) In particular, if $\{c_1, c_2, \ldots, c_m, 0\}$ is convex, then $[c_1, \ldots, c_m]$ is NN in $[0, \pi]$.

**Remark 1.** In (ii), assuming only the convexity of $\{c_1, c_2, \ldots, c_m\}$ is not enough to guarantee NN. We need to augment the coefficient sequence with a 0 at the end. This is equivalent to requiring the convexity of $\{c_1, c_2, \ldots, c_m\}$ plus $a_{m-1} \geq 2a_m$.

**Remark 2.** It follows from the convexity of the power function $x^\alpha$ ($\alpha > 1, x \geq 0$) that if $\{c_k^\gamma\}, \beta > 0$, is a convex sequence, so is $\{c_k^\gamma\}$, for $\gamma > \beta$.

**Lemma 2.** (i) For any integer $k > 0$, $\tau_{k-1} + \tau_k$ is NN in $[0, \pi]$.

(ii) Let $0 < A < B$. Then
\[
A\tau_{2j-1}^- + B\tau_{2j}^- \geq -j(B-A)x \quad (x \geq 0).
\]
(iii) Let $0 < m < m^*$ be integers, and $d_k > 0$, $k = m, \ldots, m^*$. Then
\[
\sum_{k=m}^{m^*} d_k \tau_k \geq d_{m^*} \min \{\tau_{m^*}, 0\}.
\]

(iv) In addition to the hypotheses of (iii), assume that $m$ is odd and $m^*$ even. Then
\[
\sum_{k=m}^{m^*} d_k \tau_k \geq -\left[ \sum_{j=(m+1)/2}^{m^*/2} j (d_{2j} - d_{2j-1}) \right] x.
\]

\[\text{(3.1)}\]

**Proof.** (i) This is a corollary of Lemma 1 (i), when $c_k = 2$, $k = 1, \ldots, m + 1$.

(ii) By (1.3),
\[
\tau_{2j}^+ \geq \frac{s\left(\frac{1}{2}\right) - s(2j + \frac{1}{2})}{2c(\frac{1}{2})} = \frac{-c(j + \frac{1}{2}) s(j)}{c(\frac{1}{2})}.
\]

In $[0, \frac{\pi}{2j}]$, $c(j + \frac{1}{2})/c(\frac{1}{2})$ is a decreasing function. Hence, it is less than its value at $x = 0$, which is 1. It follows that $\tau_{2j}^+ \geq -s(j) \geq -jx$. In $[\frac{\pi}{2j}, \pi]$, we obtain a lower bound of $\tau_{2j}^-$ by replacing $s(2j + \frac{1}{2})$ in the middle expression above by $-1$ to obtain
\[
\tau_{2j}^- \geq \frac{s\left(\frac{1}{2}\right) - 1}{2c(\frac{1}{2})}.
\]

The RHS is negative but it is an increasing function of $x$. At $x = \frac{\pi}{2j}$, the RHS is greater than $-jx$, implying that the same is true for all $x \in [\frac{\pi}{2j}, \pi]$. Thus we have shown that $\tau_{2j}^+ \geq -jx$ for all $x \in [0, \pi]$.

By (i),
\[
A\tau_{2j-1}^- + B\tau_{2j}^- = A(\tau_{2j-1}^- + \tau_{2j}^-) + (B - A)\tau_{2j}^- \geq (B - A)\tau_{2j}^- \geq -j(B - A)x.
\]

(iii) Rearrange the sum in question, in reverse order of the terms, as
\[
d_{m^*} \tau_{m^*} + d_{m^*-1} \tau_{m^*-1} + \ldots + d_m \tau_m
\]

We compare this with the following sum, in which the coefficients are all the same:
\[
d_{m^*} (\tau_{m^*} + \tau_{m^*-1} + \ldots + \tau_m).
\]

Let us look at the partial sums of the latter. Using (i), we see that if the partial sum has an even number of terms, then it is NN. If there are an odd number of terms, the sum from the second term on is NN, implying that the whole sum is not less than the first term. In all cases, the partial sums $\geq \min \{d_{m^*} \tau_{m^*}, 0\}$. Applying the Comparison Principle then yields the conclusion. The Comparison Principle is a well-known result. An explanation can be found in [4, See Lemma 2].
Lemma 3. Suppose $c_k > 0$, $\forall k$. Then

$$[c_1, c_2, \ldots, c_n]^- \geq \frac{c_1 \left[s\left(\frac{1}{2}\right) + s\left(\frac{3}{2}\right)\right] - c_2 \left[1 + s\left(\frac{3}{2}\right)\right]}{2c\left(\frac{1}{2}\right)}.$$  \hfill (3.2)

Proof. The special case $c_1 = c_2 = \cdots = 1$ corresponds to $\tau_n^-$. By (1.3), its partial sums satisfy

$$\tau_m^- \geq \frac{s\left(\frac{1}{2}\right) - 1}{2c\left(\frac{1}{2}\right)} \quad (1 < m < n).$$

Applying the Comparison principle yields

$$[1, 1, c_2, \cdots]^- \geq \frac{s\left(\frac{1}{2}\right) - 1}{2c\left(\frac{1}{2}\right)},$$

which is (3.2) when $c_1 = c_2$. The general case follows from the relation

$$[c_1, c_2, \cdots, c_n]^- = (c_1 - c_2)s(1) + [c_2, c_2, \cdots, c_n].$$

Lemma 4. For $x \in [0.75, \pi]$,

$$s\left(\frac{1}{2}\right) + 0.338 s\left(\frac{3}{2}\right) - 0.662 \geq 0$$

Proof. Using the transformation $x = \pi - 2t$, we see that the desired inequality is equivalent to

$$\cos(t) - 0.338 \cos(3t) - 0.662 \geq 0, \quad t \in \left[0, \frac{\pi - 0.75}{2}\right].$$

With a further substitution $X = \cos(t)$, the inequality becomes

$$\frac{1}{500}(1 - X)(676X^2 + 676X - 331) \geq 0, \quad X \in \left[\cos\left(\frac{\pi - 0.75}{2}\right), 1\right] \subset [0.4, 1],$$

which is true, since all the roots of the LHS lie outside $[0.366, 1]$. \hfill \blacksquare

Monotonicity properties of a sequence can often be deduced from corresponding properties of its continuous analog. For instance, if $\phi(x)$ is a differentiable function in $(0, \infty)$, then $\phi(k), k = 1, 2, \cdots$ is a decreasing sequence if $\phi'(x) \leq 0$. Likewise,

$$\phi'(x) \nearrow (\searrow) \implies \phi(k + 1) - \phi(k) \nearrow (\searrow)$$

$$\phi''(x) \nearrow (\searrow) \implies \phi(k - 1) - 2\phi(k) + \phi(k + 1) \nearrow (\searrow).$$ \hfill (3.3)

Lemma 5. (i) For $\beta \in [0, 1]$. \square_k \searrow in $k$.

(ii) Either $\{a_k\}_{k=1}^n$ is convex, or there is an $m < n$ such that the sub-sequence $\{a_1, \cdots, a_m\}$ is convex and the sub-sequence $\{a_m, a_{m+1}, \cdots, a_n\}$ has an odd number of terms and is concave.
(iii) When $n \geq 7$ and $\beta \geq \beta_1$, $\square_2, \square_3, \square_4 \geq 0$.

Proof. (i) Note that

$$a_k = \left( \frac{n}{n^2 - 1} \right)^\beta f \left( \frac{k}{n} \right),$$  

(3.4)

where

$$f(x) := \left( \frac{1}{x} - x \right)^\beta, \quad x \in [0, 1].$$

In view of (3.3), the monotonicity of $\square_k$ will follow if we can show that for a fixed $\beta \in [0, 1]$, $f''(x) < 0$ for $x \in (0, 1]$. Direct computation gives

$$f''(x) = -\left( \frac{f(x)}{x^3(1-x^2)^3} \right) g(x),$$

where

$$g(x; \beta) = (x^6 + 3x^4 + 3x^2 + 1) \beta^2 - (3x^6 + 15x^4 + 9x^2 - 3) \beta + (2x^6 + 18x^4 - 6x^2 + 2).$$

The conclusion follows if we can show that $g(x) > 0$ for $(x, \beta) \in [0, 1] \times [0, 1]$. For each fixed $x \in [0, 1]$, $g(x, \beta)$, when extended to $\beta \in (-\infty, \infty)$, attains its minimum at

$$\sigma = \frac{3(x^4 + 4x^2 - 1)}{2(x^4 + 2x^2 + 1)},$$

which falls inside $[0, 1]$ only when $x \in [x_1, x_2]$, where $x_1 = \sqrt{5/2} - 2, \ x_2 = \sqrt{21/4} - 4$.

**Case 1:** $x \in [0, x_1]$. Minimum of $g(x, \beta)$ for $\beta \in [0, 1]$ is attained when $\beta = 0$. Hence,

$$g(x; \beta) \geq g(x; 0) = 2x^6 + 18x^4 - 6x^2 + 2 > 0,$$

The positivity of the polynomial is verified using Sturm’s procedure.

**Case 2:** $x \in [x_2, 1]$. Minimum of $g(x, \beta)$ for $\beta \in [0, 1]$ is attained when $\beta = 1$. Hence,

$$g(x; \beta) \geq g(x; 1) = 6x^4 - 12x^2 + 6 > 0.$$

**Case 3:** $x \in (x_1, x_2)$. Minimum of $g(x, \beta)$ for $\beta \in [0, 1]$ is attained when $\beta = \sigma$. Hence,

$$g(x; \beta) \geq g(x; \sigma) = \frac{-x^8 + 8x^6 - 78x^4 + 56x^2 - 1}{4(x^2 + 1)} > 0.$$

Again positivity (for $x \in \{0.4, 0.8\} \cap [x_1, x_2]$) is checked with the Sturm procedure.

In all three cases, $g(x; \beta) \geq 0$; the first assertion of the Lemma is proved.
(ii) It is easy to verify $\Box_2 \geq 0$. If it happens that $\Box_k \geq 0$ for all $k = 2, \ldots, n-1$, then the sequence is convex. This situation prevails when $\beta$ is very close to 1.

In general, when $n$ is large and $\beta$ is close to $\beta_1$, $f(x)$ is not convex in $[0, 1]$, as exemplified by the red curve in Figure 1. Recall that after an appropriate scaling, see (3.4), $\hat{a}_n = f\left(\frac{k}{n}\right)$ are discrete points on the graph of $f(x)$. The curve starts out being convex and becomes concave after the point of inflection, marked as a blue dot on the curve, between $\hat{a}_{12}$ and $\hat{a}_{13}$. The two sub-sequences stipulated in the Lemma are $\{a_1, \ldots, a_{12}\}$ and $\{a_{13}, \ldots, a_{16}, 0\}$, with the latter having five terms.

By (i), $\Box$ will transition from $\geq 0$ to $< 0$ somewhere, as depicted below:

\[
\begin{align*}
\underbrace{a_1, \ldots, a_{\kappa-1}, a_{\kappa}, a_{\kappa+1}, a_{\kappa+2}, \ldots, a_n = 0}_{\Box \geq 0} & \quad \underbrace{\quad \Box < 0}_{} \\
\end{align*}
\]

If we split the sequence right before $a_{\kappa}$ or right after it, or after the next one, we get, in all three cases, a convex sub-sequence followed by a concave one. We can always choose one of these cases so that the second sub-sequence (including $a_n = 0$) has an odd number of terms. Letting $a_m$ be the last term of the first sub-sequence, we construct the two polynomials represented, respectively, by the RHS of (??) and (??).

(iii) The monotonicity of $\Box_k$ implies that, for a fixed $\beta$, there is a unique $x_*$ at which
\[ f''(x) = 0, \] and \[ f''(x) > 0 \] for \( x \in (0, x_*) \). Direct computation gives

\[
f''(x) = -\left(\frac{\beta f(x)}{x^2(1-x^2)^2}\right) ((\beta - 1)x^4 + (2\beta - 4)x^2 + 1),
\]

and

\[
x_* = \sqrt{(\sqrt{5} - 4\beta + \beta - 2)/(1 - \beta)}.
\]

As a function of \( \beta \), \( x_* \) is increasing in \( \beta \). Thus \( x_* \geq x_*(\beta_1) = 0.5281747 \cdots \). In terms of \( a_k \), this means that \( 2k \geq 0 \) for \( k = 2, \cdots, \kappa \), where \( \kappa \) is the largest integer less than or equal to \( 0.5281747n - 1 \). For \( n \geq 10 \), \( \kappa \geq 4 \) and the second assertion of the Lemma is proved. For \( 5 \leq n \leq 9 \), the assertion can be verified directly by computation.

**Lemma 6.** Let \( \lambda, \mu \geq 0 \), and \( 0 < C < B < A \leq 1 \). The function

\[
\xi(\beta) = \lambda A^\beta - (\lambda + \mu) B^\beta + \mu C
\]

is \( \geq 0 \) at some point \( \beta_0 > 0 \) \( \Rightarrow \) \( \xi(\beta) \) cannot have a local minimum in \( (\beta_0, \infty) \).

**Proof.** \( \xi(\beta_0) \geq 0 \) \( \Rightarrow \) \( \xi(\beta) > 0 \) for \( \beta > \beta_0 \) (due to the convexity of \( x^\beta, \beta > 1 \)).

A local minimum must be a critical point. Our tactic is to show that at a critical point \( \sigma \) (i.e. when \( \xi'(\sigma) = 0 \), \( \xi''(\sigma) < 0 \), which implies that the critical point cannot be a local minimum.

We can assume, without loss of generality, that \( \sigma = 1 \) (use \( A^\sigma \) and \( B^\sigma \) as the new \( A \) and \( B \), respectively). That 1 is a critical point gives

\[
\xi'(1) = \lambda A \ln(A) - (\lambda + \mu) B \ln(B) + C \ln(C) = 0
\]
and we need to show

\[
\xi''(1) = \lambda A \ln^2(A) - (\lambda + \mu) B \ln^2(B) + C \ln^2(C) < 0.
\]

The point \( \sigma = e^{-1} \approx 0.3678794412 \) divides \([0, 1]\) into two sub-intervals: the function \( f(t) = |t \ln(t)| \) is \( \nearrow \) in \([0, \sigma]\), but \( \searrow \) in \([\sigma, 1]\). In each sub-interval, \( f(t) \) is invertible. Let \( g(s) = f^{-1}(s) \) be the inverse of \( f(t) \) in \([0, \sigma]\), i.e. \( |g(s)\ln(g(s))| = s \).

We divide the proof of (3.7) into three cases.

**Case 1:** \( A \in [0, \sigma] \). Then \( B \) and \( C \) are also in \([0, \sigma]\).

That (3.7) follows from (3.6) is a consequence of the concavity of \( t \ln^2(t) = g(s) \ln^2(g(s)) \) as a function of \( s = t \ln(t) \). Using the chain rule, one can verify that

\[
\frac{d}{ds} (t \ln^2(t)) = \frac{d}{dt} (t \ln^2(t)) \frac{dt}{ds} = \frac{\ln^2(t) + 2 \ln(t)}{\ln(t) + 1}
\]
and
\[ \frac{d^2}{ds^2}(t \ln^2(t)) = \frac{d}{dt} \left( \frac{\ln^2(t) + 2 \ln(t)}{\ln(t) + 1} \right) \frac{dt}{ds} = \frac{\ln^2(t) + 2 \ln(t) + 2}{t(\ln(t) + 1)}. \]
The last expression is $-$ for $t \in [0, \sigma]$ (the numerator is $+$, but the denominator is $-$), proving the claim.

![Figure 2: Graph of $f(t) = |t \ln(t)|$. Case 1, sub-case 1.](image)

**Case 2:** $A \in [\sigma, 1]$ and $C \in [0, \sigma]$. There are two sub-cases. The first is $f(A) \geq f(C)$. One such example is depicted in Figure 2. There exists $A_1 \in [C, \sigma]$ such that $f(A_1) = f(A)$. Then, with $A_1$ replacing $A$, we have the same situation as Case 1. Thus, (3.7), with $A_1$ in place of $A$, i.e.

\[ \lambda A_1 \ln^2(A_1) - (\lambda + \mu)B \ln^2(B) + C \ln^2(C) < 0. \]

(3.8)

Since $\ln(A_1) < \ln(A) < 0$,

\[ A \ln^2(A) = (A \ln(A)) \ln(A) = (A_1 \ln(A_1)) \ln(A) < (A_1 \ln(A_1)) \ln(A_1). \]

(3.9)

Now (3.8) and (3.9) imply (3.7).

The second sub-case is when $f(A) < f(C)$, as depicted in Figure 3. Let $D = (\lambda A + \mu C) / (\lambda + \mu)$; it lies between $C$ and $A$. The assumption $\xi(1) \geq 0 \implies B \in [C, D]$.

The dash green line joins $(C, f(C))$ and $(A, f(A))$. Concavity of $f(t) \implies$ dash green line $\leq$ red curve. Let the vertical line through $D$ cut the dash green line at $E$. The length $DE$ is $(\lambda f(A) + \mu f(C)) / (\lambda + \mu) = f(B)$, by (3.6). However, this contradicts the earlier assertion that $B$ lies between $C$ and $D$ because for all $B \in [C, D]$, $B \ln(B)$ (the red curve) is obviously larger than $DE$ (the red curve is above $E$). This contradiction means that (3.6) cannot hold. In other words, 1 cannot be a critical point of $\xi(b)$.

**Case 3:** $C \in [\sigma, 1]$. Then $f(A) < f(C) \implies$ contradiction just as in the last case.

A simple corollary is
Lemma 7. Let $\xi(\beta)$ be defined as in Lemma 6 and $\xi(\beta_1) \geq 0$.

(i) If $\xi'(1) \geq 0$, then $\xi(\beta) \nearrow$ in $[\beta_1, 1]$.

(ii) If $\xi'(\beta_1) \leq 0$, then $\xi(\beta) \searrow$ in $[\beta_1, 1]$.

Proof. (i) Suppose the contrary. Then $\xi'(\beta)$ must be negative at some point $\beta_3 \in [\beta_1, 1)$, and there must exist a local minimum between $\beta_3$ and 1, contradicting the Lemma.

(ii) In a similar way, suppose the contrary. Then $\xi'(\beta)$ must be positive at some point $\beta_3 \in [\beta_1, 1)$, and there must exist a local minimum between $\beta_3$ and 1, a contradiction.

The coefficients of $H$, as defined in (2.4), written in terms of $y \in \left[0, \frac{1}{15}\right]$ and $\beta \in [\beta_1, 1]$
are given by \( h_i(y; \beta) \), \( i = 1, 2, 3 \) below, while the second differences are given by \( i = 4, 5 \).

\[
\begin{align*}
    h_1(y; \beta) & := 1 - 4 \left( \frac{1 - 15y}{4} \right)^\beta + 3 \left( \frac{1 - 24y}{5} \right)^\beta \\
    h_2(y; \beta) & := \left( \frac{1 - 3y}{2} \right)^\beta - 3 \left( \frac{1 - 15y}{4} \right)^\beta + 2 \left( \frac{1 - 24y}{5} \right)^\beta \\
    h_3(y; \beta) & := \left( \frac{1 - 8y}{3} \right)^\beta - 2 \left( \frac{1 - 15y}{4} \right)^\beta + \left( \frac{1 - 24y}{5} \right)^\beta \\
    h_4(y; \beta) & := h_1(y; \beta) - 2h_2(y; \beta) + h_3(y; \beta) \\
    & = 1 - 2 \left( \frac{1 - 3y}{2} \right)^\beta + \left( \frac{1 - 8y}{3} \right)^\beta \\
    h_5(y; \beta) & := h_2(y; \beta) - 2h_3(y; \beta) \\
    & = \left( \frac{1 - 3y}{2} \right)^\beta - 2 \left( \frac{1 - 8y}{3} \right)^\beta + \left( \frac{1 - 15y}{4} \right)^\beta .
\end{align*}
\]

Lemma 8.  
(i) With \( \beta = \beta_1 \), all five functions \( h_i(y; \beta_1) \) \( (i = 1, \cdots, 5) \), are \( + \searrow \) in \( y \).

(ii) For fixed \( y \), all five functions \( h_i(y; \beta) \) \( (i = 1, \cdots, 5) \), are \( \nearrow \) in \( \beta \).

Proof. (i) All conclusions are proved using the MAPLE procedure \texttt{dif}, see [3].

\( h_1(y; \beta_1) \): We want to show that

\[
- \frac{5h_1'(y; \beta_1)}{\beta_1} = 72 \left( \frac{1 - 24y}{5} \right)^{\beta_1-1} - 75 \left( \frac{1 - 15y}{4} \right)^{\beta_1-1}
\]

is positive for \( y \in [0, \frac{1}{48}] \). The MAPLE display is shown in Figure ??.
The first command defines $b_1(\beta_1)$. The next two commands defines $g_1$ and $g_2$. The fourth invokes `dif` in the verbose format (by adding the option `long=1`) to generate four points (the second column of the displayed matrix):

\[
\begin{array}{cccccc}
1 & 0 & 137.5658221 & 137.81116865 & 0.1544356 \\
2 & 0.0075 & 149.4847220 & 149.25523566 & 0.2294654 \\
3 & 0.0181 & 173.6906278 & 152.7900123 & 20.9065155 \\
4 & 0.0203333333 & 182.5620384 & 152.7900123 & 29.7720261
\end{array}
\]

\[\tau_1 = 0, \quad \tau_2 = 0.0075, \quad \tau_3 = 0.0181, \quad \tau_4 = \frac{1}{38}.\]

The third column of the matrix lists $g_1(\tau_i), \ (i = 1, 2, 3, 4)$ the fourth lists $g_2(\tau_i), \ (i = 2, 3, 4, 4)$, while the fifth column is the third column minus the fourth and must be positive. The successful generation of the matrix proves that $g_1 - g_2$ is positive in the interval under study. The graphs are plot as a visual check that $g_1$ (the red curve) and $g_2$ (the blue curve) are monotone.
\( h_2(y; \beta_1) \):  
We have

\[
- \frac{h_2'(y; \beta_1)}{\beta_1} = \left[ \frac{3}{2} \left( \frac{1 - 3y}{2} \right)^{\beta_1 - 1} + \frac{48}{5} \left( \frac{1 - 24y}{5} \right)^{\beta_1 - 1} \right] - \frac{45}{4} \left( \frac{1 - 15y}{4} \right)^{\beta_1 - 1}.
\]

This time, we invoke \texttt{dif} without the \texttt{long=1} option. The output then only shows the list of \( \tau_i \), not the verbose matrix or the graphs.

\begin{verbatim}
> g1 := 3/2 * (1 - 3*y)^beta1 + 48/5 * (1 - 24*y)^beta1
> g2 := 45/4 * (1 - 15*y)^beta1 - 1
> g3 := 8/3 * (1 - 8*y)^beta1 + 24/5 * (1 - 24*y)^beta1

> evalf([g1, g2, g3], 10)

[0.000000000, 0.000000000, 0.000000000]
\end{verbatim}

Figure 5. MAPLE output for \( h_2(y; \beta_1) \).

\( h_3(y; \beta_1) \):

\[
- \frac{h_3'(y; \beta_1)}{\beta_1} = \left[ \frac{8}{3} \left( \frac{1 - 8y}{3} \right)^{\beta_1 - 1} + \frac{24}{5} \left( \frac{1 - 24y}{5} \right)^{\beta_1 - 1} \right] - \frac{15}{2} \left( \frac{1 - 15y}{4} \right)^{\beta_1 - 1}.
\]

\begin{verbatim}
> g1 := 8/3 * (1 - 8*y)^beta1 + 24/5 * (1 - 24*y)^beta1
> g2 := 15/2 * (1 - 15*y)^beta1 - 1
> g3 := 8/3 * (1 - 8*y)^beta1 + 24/5 * (1 - 24*y)^beta1

> evalf([g1, g2, g3], 10)

[0.000000000, 0.000000000, 0.000000000]
\end{verbatim}

Figure 6. MAPLE output for \( h_3(y; \beta_1) \).
(ii) Although we assume that $y$ is fixed, it is not given a specific value. In other words, it is used as a parameter, and for that reason, dif cannot be applied directly. Instead, we make use of Lemma \[7\] noting that $h_i(y; \beta)$ (with fixed $y$) has the same form as $\xi(\beta)$ in (3.5). By Lemma \[7\] (i), it suffices to show that

$$\frac{\partial}{\partial \beta} h_i(y; 1) \geq 0, \quad i = 1, \ldots, 5.$$ 

Now we can apply dif to each $\partial h_i(y; 1)/\partial \beta$.  

$h_4(y; \beta_1)$:

$$\begin{align*}
g_1 &= 4 \left( \frac{1}{3} - \frac{1}{3} y \right)^{6/3 - 1} \\
g_2 &= 3 \left( \frac{1}{2} - \frac{3}{2} y \right)^{6/3 - 1} \\
\text{eval}(g_1, g_2, \left[ 0, \frac{1}{48} \right])
\end{align*}$$

Figure 7. MAPLE output for $h_4(y; \beta_1)$.

$h_5(y; \beta_1)$:

$$\begin{align*}
g_1 &= \frac{3}{2} \left( \frac{1}{2} - \frac{3}{2} y \right)^{6/3 - 1} + \frac{15}{4} \left( \frac{1}{4} - \frac{15}{4} y \right)^{6/3 - 1} \\
g_2 &= \frac{16}{3} \left( \frac{1}{3} - \frac{8}{3} y \right)^{6/3 - 1} \\
\text{eval}(g_1, g_2, \left[ 0, \frac{1}{48} \right])
\end{align*}$$

Figure 8. MAPLE output for $h_5(y; \beta_1)$.  

\[19\]
\[
\frac{\partial h_1(y; 1)}{\partial \beta} = \frac{3}{5} \left( \frac{1 - 24y}{5} \right) \ln \left( \frac{1 - 24y}{5} \right) - \frac{4}{5} \left( \frac{1 - 15}{4} \right) \ln \left( \frac{1 - 15}{4} \right)
\]

Figure 9. MAPLE output for \(\frac{\partial h_1(y; 1)}{\partial \beta}\).

\[
\frac{\partial h_2(y; 1)}{\partial \beta}:
\]

\[
\geq 1 = \left( \frac{1}{2} - \frac{3}{4} \right) \ln \left( \frac{1}{2} - \frac{3}{4} \right) + 2 \left( \frac{1}{5} - \frac{24}{5} \right) \ln \left( \frac{1}{5} - \frac{24}{5} \right)
\]

\[
\geq 2 = 3 \left( \frac{1}{4} - \frac{15}{4} \right) \ln \left( \frac{1}{4} - \frac{15}{4} \right)
\]

\[
\partial \beta \geq [\geq 1, \geq 2, 0, \frac{1}{48}]
\]

Figure 10. MAPLE output for \(\frac{\partial h_2(y; 1)}{\partial \beta}\).

\[
\frac{\partial h_3(y; 1)}{\partial \beta}:
\]

\[
\geq 1 = \left( \frac{1}{3} - \frac{8}{3} \right) \ln \left( \frac{1}{3} - \frac{8}{3} \right) + \left( \frac{1}{5} - \frac{24}{5} \right) \ln \left( \frac{1}{5} - \frac{24}{5} \right)
\]

\[
\geq 2 = 2 \left( \frac{1}{4} - \frac{15}{4} \right) \ln \left( \frac{1}{4} - \frac{15}{4} \right)
\]

\[
\partial \beta \geq [\geq 1, \geq 2, 0, \frac{1}{48}]
\]

Figure 11. MAPLE output for \(\frac{\partial h_3(y; 1)}{\partial \beta}\).
Lemma 9. For any $0 < B < 1$, $\beta \in [\frac{1}{2}, 1]$, $(1 + B)^\beta + (1 - B)^\beta \nearrow$ in $\beta$.

**Proof.** The conclusion is true if we can show that
\[
\frac{\partial}{\partial \beta} \left[ (1 + B)^\beta + (1 - B)^\beta \right] \\
= (1 + B)^\beta \ln(1 + B) + (1 - B)^\beta \ln(1 - B) \\
\geq 0.
\] (3.10)

Since $\beta > \frac{1}{2}$, (3.10) follows from the stronger assertion
\[
(1 + B)^{1/2} \ln(1 + B) + (1 - B)^{1/2} \ln(1 - B) \geq 0,
\]
which is equivalent to
\[ \theta_1(B) - \theta_2(B) := (1 + B) \ln^2(1 + B) - (1 - B) \ln^2(1 - B) \geq 0. \]

We divide the proof into three cases:

1. \( \theta_1(B) \) is increasing in \( B \in [0, 1] \). \( \theta_2(B) \) is increasing in \( [0, \tau] \) and decreasing in \( [\tau, 1] \), where \( \tau = 1 - e^{-2} \). Since \( \theta_1(\tau) > \theta_2(\tau) \), we have \( \theta_1(B) > \theta_2(B) \) in \( [\tau, 1] \).

2. In \( [0, 0.4] \), we can use the DIF technique to confirm that \( \theta_1(B) > \theta_2(B) \).

3. The Taylor series of \( \ln(1 + x) \) is an alternating (i.e. + and −) series. Hence,
\[
\ln(1 + B) \geq B - \frac{B^2}{2} - \frac{B^3}{3} - \frac{B^4}{4}.
\]

\[
\implies \theta_1(B) \geq (1 + B) \left( \frac{B^2}{2} + \frac{B^3}{3} - \frac{B^4}{4} \right)^2. \tag{3.11}
\]

On the other hand, the Taylor series of \( \theta_2(B) \) is
\[
B^2 - \frac{1}{12} B^4 - \frac{1}{12} B^5 - \frac{13}{180} B^6 - \frac{11}{180} B^7 - \frac{29}{560} B^8 - \frac{223}{5040} B^9 + ..., \]
suggesting that
\[
B^2 - \frac{1}{12} B^4 - \frac{1}{12} B^5 \geq \theta_2(B). \tag{3.12}
\]

This is confirmed because, for \( B \in [0, 0.4] \),
\[
\frac{d^6}{dB^6} \left( B^2 - \frac{1}{12} B^4 - \frac{1}{12} B^5 - \theta_2(B) \right) = \frac{52 - 48 \ln(1 - B)}{(1 - B)^5} \geq 0.
\]

The Sturm procedure \( \implies \) for \( B \in [0, 0.4] \),
\[
(1 + B) \left( \frac{B^2}{2} + \frac{B^3}{3} - \frac{B^4}{4} \right)^2 \geq B^2 - \frac{1}{12} B^4 - \frac{1}{12} B^5. \tag{3.13}
\]

\((3.11), (3.12), + (3.13) \implies \theta_1(B) \geq \theta_2(B) \) in \( [0, 0.4] \).

This completes the proof of the Lemma. \( \blacksquare \)

**Lemma 10.** The function
\[
\left( 1 - 4 \left( \frac{11}{64} \right)^{\beta_1} + 3 \left( \frac{1}{10} \right)^{\beta_1} \right) \sin(x) - \left( \left( \frac{15}{32} \right)^{\beta_1} - 3 \left( \frac{11}{64} \right)^{\beta_1} + 2 \left( \frac{1}{10} \right)^{\beta_1} \right) \sin(2x)
\]
\[+ \left( \left( \frac{5}{18} \right)^{\beta_1} - 2 \left( \frac{11}{64} \right)^{\beta_1} + \left( \frac{1}{10} \right)^{\beta_1} \right) \sin(3x)\]
is concave in \( [0, 0.75] \).
Proof. Denote the function by $\varphi(x)$. We need to show that $\varphi''(x) < 0$ in the given interval. In terms of $X = \cos(x) \in [\cos(0.75), 1]$,}

$$
\frac{\varphi''(x)}{\sin(x)} = \left(72 \left(\frac{11}{64}\right)^{\beta_1} - 36 \left(\frac{1}{10}\right)^{\beta_1} - 36 \left(\frac{5}{18}\right)^{\beta_1}\right)X^2 + \left(8 \left(\frac{15}{62}\right)^{\beta_1} - 24 \left(\frac{11}{64}\right)^{\beta_1} + 16 \left(\frac{1}{10}\right)^{\beta_1}\right)X \\
+ \left(9 \left(\frac{5}{18}\right)^{\beta_1} - 14 \left(\frac{11}{64}\right)^{\beta_1} + 6 \left(\frac{1}{10}\right)^{\beta_1} - 1\right).
$$

It is straightforward to check that the roots of the RHS are

$$
0.39281956258689586, \quad 0.67755077339437549,
$$

computed using MAPLE with an accuracy of 40 digits, both of which lie outside the given interval. Hence, $\varphi''(x)$ is of one sign in the interval, and the sign is negative, as desired.

Lemma 11. For $n \geq 7$, $n \left(\frac{2n-1}{(n^2-1)(n-1)}\right)^{\beta_1}$ is a decreasing function of $n$.

Proof. Since $\beta_1 > \frac{1}{2}$, the conclusion follows if we can show that

$$
n \left(\frac{2n-1}{(n^2-1)(n-1)}\right)^{1/2}
$$

is decreasing in $n$, or equivalently, if

$$
\frac{n^2(2n-1)}{(n^2-1)(n-1)}
$$

is decreasing in $n$. This is a routine exercise in calculus.

Let $\delta_k := -\Omega_{n-k}$ be as defined in Section 2 §12. Then,

$$
\delta_1 = \left(\frac{1}{n^2-1}\right)^{\beta} \left[2 \left(\frac{2n-1}{n-1}\right)^{\beta} - \left(\frac{4n-4}{n-2}\right)^{\beta}\right] \\
= \left(\frac{2n-1}{(n^2-1)(n-1)}\right)^{\beta} \left[2 - \left(\frac{4(n-1)^2}{(2n-1)(n-2)}\right)^{\beta}\right].
$$

Lemma 12.

$$(n-1)\delta_1 \leq \theta(12) < 0.3921 \quad \text{for } n \geq 7. \quad \text{(3.14)}$$

$$(n-1)\delta_1 \leq \theta(45) < 0.3428 \quad \text{for } n \geq 45. \quad \text{(3.15)}$$

Proof. Since $\frac{2n-1}{(n^2-1)(n-1)} < 1$, the first factor $\gamma$ in $\beta$. On the other hand, $\frac{(4n-4)(n-1)}{(2n-1)(n-2)} > 1$, implying that the second factor is also $\gamma$ in $\beta$. It follows that

$$(n-1)\delta_1 \leq (n-1) \left(\frac{2n-1}{(n^2-1)(n-1)}\right)^{\beta_1} \left[2 - \left(\frac{4(n-1)^2}{(2n-1)(n-2)}\right)^{\beta_1}\right]. \quad \text{(3.16)}$$
Denote by $\theta(n)$ the RHS of (3.16). Numerics gives

$$\theta(7) < \theta(8) < \theta(9) < \theta(10) < \theta(11) < \theta(12) > \theta(13) > \theta(14) > ...$$

We rewrite

$$\theta(n) = \left(\frac{2n - 1}{(n + 1)(n - 1)^{0.1}}\right)^{\beta_1} \left[2 - \left(\frac{4(n-1)^2}{(2n-1)(n-2)}\right)^{\beta_1} \right].$$  \hspace{1cm} (3.17)

Direct computation gives

$$\frac{d}{dn} \left(\frac{2n - 1}{(n + 1)(n - 1)^{0.1}}\right) = -\frac{2n^2 - 29n + 29}{10(n+1)^2(n-1)^{1.1}} < 0 \quad \text{when } n > 14.$$

It follows that the first factor in (3.17) is $\searrow (n > 14)$. Rewrite the second factor as $\frac{\eta(y)}{y^\alpha}$, where $y = n - 1$, $\eta(y) = 2 - \left(\frac{4y^2}{(2y+1)(y-1)}\right)^{\beta_1}$, $\alpha = 1.9\beta_1 - 1$. Obviously, $\eta(y)$ is $\nearrow$ in $y > 12$.

Direct computation gives

$$y\eta'(y) = \beta_1 \left(\frac{y+2}{2y+1}\right) \left(\frac{4y^2}{(2y+1)(y-1)^{1+1/\beta_1}}\right)^{\beta_1}.$$

Both fractions are $\searrow$ in $y \Rightarrow y\eta'(y) \searrow$ in $y$.

$$\left(\frac{\eta(y)}{y^\alpha}\right)' = \frac{y\eta'(y) - \alpha\eta(y)}{y^{\alpha+1}}.$$

The numerator, being a $\searrow$ function minus an $\nearrow$ function, is $\searrow$ in $y$. Substituting $y = 10$ shows that the numerator is negative. Hence, the numerator is negative for all larger $y$. Thus the second factor in (3.17) is $\searrow$, implying that $\theta(n)$ is $\searrow$ in $n > 14$. Direct computation shows $\theta(12) > \theta(13) > \theta(14)$. Thus $\theta(n)$ is $\searrow$ for $n \geq 12$. \hfill \qed

**Lemma 13.** For odd $n \geq 15$,

$$\begin{align*}
(n - 3)\delta_3 &< 0.0412 \quad \text{(3.18)} \\
(n - 5)\delta_5 &< 0.018 \quad \text{(3.19)} \\
(n - 7)\delta_7 &< 0.010342 \quad \text{(3.20)} \\
(n - 9)\delta_9 &< 0.006902. \quad \text{(3.21)}
\end{align*}$$

For $n \geq 45$

$$\begin{align*}
(n - 3)\delta_3 &< 0.0326. \quad \text{(3.22)}
\end{align*}$$

**Proof.** For (3.18)-(3.21), we only give the proof of (3.18). The other inequalities can be established in exactly the same way.
\[ \delta_3 = \left( \frac{6n - 9}{(n^2 - 1)(n - 3)} \right)^\beta \left[ 2 - \left( \frac{(8n - 16)(n - 3)}{(6n - 9)(n - 4)} \right) - \left( \frac{(4n - 4)(n - 3)}{(6n - 9)(n - 2)} \right) \right]. \quad (3.23) \]

The first factor is \( \searrow \) in \( \beta \). We claim that the second factor is also \( \searrow \), which is equivalent to

\[ \left( \frac{(8n - 16)(n - 3)}{(6n - 9)(n - 4)} \right)^\beta + \left( \frac{(4n - 4)(n - 3)}{(6n - 9)(n - 2)} \right)^\beta = (1 + A)^\beta + (1 - B)^\beta \]

being \( \nearrow \) in \( \beta \). It is easy to verify that

\[ 0 < B = \frac{2n^2 - 5n + 6}{3(2n - 3)(n - 2)} < \frac{2n^2 - 7n + 12}{3(2n - 3)(n - 4)} = A < 1. \]

Obviously, \((1 + A)^\beta - (1 + B)^\beta \nearrow ic in \beta \). The claim then follows from this and Lemma 9.

It then follows that

\[ \delta_3 \leq \left( \frac{6n - 9}{(n^2 - 1)(n - 3)} \right)^\beta_1 \left[ 2 - \left( \frac{(8n - 16)(n - 3)}{(6n - 9)(n - 4)} \right)^{\beta_1} - \left( \frac{(4n - 4)(n - 3)}{(6n - 9)(n - 2)} \right)^{\beta_1} \right] \]

\[ = F(n)G(n), \quad (3.24) \]

where \( F(n) \) and \( G(n) \) are the first and second factor of (3.23), respectively. Obviously, \( F(n) \) is \( \searrow \) in \( n \).

Our next claim is that \( G(n) \) is \( \nearrow \) in \( n \geq 15 \). This is equivalent to

\[ \left( \frac{(8n - 16)(n - 3)}{(6n - 9)(n - 4)} \right)^{\beta_1} + \left( \frac{(4n - 4)(n - 3)}{(6n - 9)(n - 2)} \right)^{\beta_1} \searrow . \]

After differentiating this expression and canceling some common factors, we see that this is equivalent to

\[ -2 \left( \frac{n^2 - 6}{(n - 4)^2} \right) \left( \frac{8n - 16}{n - 4} \right)^{\beta_1 - 1} + \left( \frac{n^2 - 3}{(n - 2)^2} \right) \left( \frac{4n - 4}{n - 2} \right)^{\beta_1 - 1} \leq 0 \]

which is, in turn, equivalent to

\[ \left( \frac{(8n - 16)(n - 2)}{(4n - 4)(n - 4)} \right)^{1 - \beta_1} \leq \frac{2(n^2 - 6)(n - 2)^2}{(n^2 - 3)(n - 4)^2}. \quad (3.25) \]

Since the fraction inside the parentheses on the LHS is \( > 1 \), (3.25) follows from

\[ \frac{(8n - 16)(n - 2)}{(4n - 4)(n - 4)} \leq \frac{2(n^2 - 6)(n - 2)^2}{(n^2 - 3)(n - 4)^2}, \]

which is easy to verify. The proof of the claim is complete.

Then (3.24) leads to

\[ \delta_3 \leq F(n)G(\infty), \quad (3.26) \]
where
\[ G(\infty) = \left[ 2 - \left( \frac{4}{3} \right)^{\beta_1} - \left( \frac{2}{3} \right)^{\beta_1} \right]. \]

For later use, we define
\[ C_k = \left[ 2 - \left( \frac{k+1}{k} \right)^{\beta_1} - \left( \frac{k-1}{k} \right)^{\beta_1} \right]. \]

Thus
\[
(n-3)\delta_3 \leq C_3(n-3) \left( \frac{6n-9}{(n^2-1)(n-3)} \right)^{\beta_1} \\
= C_3 \left( \frac{(6n-9)(n-3)^{1/\beta_1-1}}{n^2-1} \right)^{\beta_1} \\
\leq C_3 \left( \frac{(6n-9)(n-3)^{1/\beta_1-1}}{n^2-1} \right)^{\beta_1} \bigg|_{n=15} \\
< 0.0412.
\]

It is easy to verify that the expression in the second line is decreasing in \( n \) for \( n \geq 15 \), justifying the inequality given by third line.

In the final step of the proof of Theorem 1, we need the sharper bound (3.22) on \( (n-3)\delta_3 \). In the above proof, there is room for improvement because we have used a fairly crude bound \( C_3 \) for the last factor in (3.24).

Using (3.26) we see that (3.22) is true for \( n \geq 99 \). It remains to verify (3.22) for the finite set \( n = 45, 47, \cdots, 97 \). That can be achieved by brute force, simply by computing each of these values numerically.

**Lemma 14.** For fixed \( n \geq 45 \),
\[ a_{n-10} - a_{n-9} < 0.1636. \]

**Proof.**
\[ a_{n-10} - a_{n-9} = \left( \frac{20n-100}{(n^2-1)(n-10)} \right)^{\beta} - \left( \frac{18n-91}{(n^2-1)(n-9)} \right)^{\beta} \]
has the form of \( \xi(\beta) \) in (3.5), with \( \lambda = 1, \mu = 0 \), and satisfies Lemma 7 (ii). This means that we only need to find an upper bound when \( \beta = \beta_1 \). Thus
\[ (n-11)(a_{n-10} - a_{n-9}) \leq \frac{(n-11)}{(n^2-1)^{\beta_1}} \left[ \left( \frac{20n-100}{n-10} \right)^{\beta_1} - \left( \frac{18n-81}{n-9} \right)^{\beta_1} \right]. \]
The first factor is $\downarrow$ in $n \geq 45$. The second factor is a difference of two convex functions. The DIF technique proves that it is $\downarrow$. Therefore,

$$(n - 11)(a_{n-10} - a_{n-9}) \leq \frac{(n - 11)}{(n^2 - 1)^{\beta_1}} \left[ \left( \frac{20n - 100}{n - 10} \right)^{\beta_1} - \left( \frac{18n - 81}{n - 9} \right)^{\beta_1} \right] \bigg|_{n=45} < 0.1636.$$

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