Multifractal Features in the Foreign Exchange and Stock Markets

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Abstract

The multifractal behavior for tick data of prices is investigated in Korean financial market. Using the rescaled range analysis (R/S analysis), we show the multifractal nature of returns for the won-dollar exchange rate and the KOSPI. We also estimate the Hurst exponent and the generalized $q$th-order Hurst exponent in the universal multifractal framework. Particularly, our financial market is a persistent process with long-run memory effects, and the statistical value of the Hurst exponents occurs the crossovers at characteristic time scales. It is found that the probability distribution of returns is well consistent with a Lorentz distribution, significantly different from fat-tailed properties.

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In recent years, there has considerable interest in various applications of physical and economic methods in natural and social sciences [1]. The representative topics in econophysics have mainly included the price changes in open market [2], the distribution of income of companies, the scaling relation of size fluctuations of companies, the financial analysis of foreign exchange rates[3], the tick data analysis of bond futures [4], the herd behavior of financial markets [5], and the self-organized segregation [6]. The essential issues with fluctuations have particularly led to a better understanding for the scaling properties based on methods and approaches in scientific fields. It was remarkably argued in the previous work [3] that the price fluctuations follow the anomalous power law from the stochastic time evolution equation, which is clearly represented in terms of the Langevin-type equation. In view of some studies, the power law distribution, the stretched exponential distribution, and the fat-tailed distribution have generally elucidated the functional properties from the numerical results obtained in diverse econophysical models.

On the other hand, in order to measure the multifractals of dynamical dissipative systems, the generalized dimension and the spectrum have effectively used to calculate the trajectory of chaotic attractors that may be classified by the type and number of the unstable periodic orbits. Several attempts [7 − 9] to compute these quantities have primarily been presented from the box-counting method. Recently, we have usually used the box-counting method to analyze precisely generalized dimensions and scaling exponents for mountain heights and sea-bottom depths [10]. For the standard analysis, since there exists notably no statistical correlations between observations, the R/S analysis has extended to distinguish the random time series from correlated ones. The recent work[11] on Norwegian and US stock markets has showed that there exists the notable persistence caused by long-memory in the time series.

For the volume of bond futures, Scalas et al. [4] have studied the correlation function for
bond walks from the time series of BTP (Buoni del tesoro Poliennali) futures exchanged at the London International Financial Futures and options Exchange (LIFFE). They have discussed that the continuous-time random walk theory, formerly introduced by Montroll and Weiss [12], is successfully applied to the dynamical behavior of empirical scaling laws by a set of tick-by-tick data in financial markets. Mainardi et al. [13] have also dealt with the waiting-time distribution for bond futures traded at LIFFE. The theoretical and numerical arguments for the volume of bond futures traded at Korean Futures Exchange market (KOFEX) of Pusan were presented in our previous work [14]. To our knowledge, it is of fundamental importance to treat with the multifractal nature of prices for the won-dollar exchange rate and the KOSPI (Korean stock price index). The studies of multifractals in Korean financial market have not explored up to now. The purpose of this letter is to investigate mainly the generic multifractal behavior for tick data of prices using the R/S analysis for the won-dollar exchange rate and the KOSPI. Particularly, the multifractal Hurst exponents, the height-height correlation function, and the probability distribution of returns are also discussed with long-run memory effects.

To quantify the Hurst exponents, we employ the R/S analysis that is generally contributed to estimate the multifractals of a time series [15, 16]. First of all, we let consider a price time series of length \( n \) given by \( \{ p(t_1), p(t_2), ..., p(t_n) \} \), and the price \( \tau \)-returns \( r(\tau) \) having time scale \( \tau \) and length \( n \) that is represented in terms of \( r(\tau) = \{ r_1(\tau), r_2(\tau), ..., r_n(\tau) \} \), with \( r_i(\tau) = \ln p(t_i + \tau) - \ln p(t_i) \). After dividing the time series or returns into \( N \) subseries of length \( M \), we label each subseries \( E_{M,d}(\tau) = \{ r_{1,d}(\tau), r_{2,d}(\tau), ..., r_{M,d}(\tau) \} \), with \( d = 1, 2, ..., N \).

Then, the deviation \( D_{M,d}(\tau) \) can be defined directly from the mean of returns \( \bar{r}_{M,d}(\tau) \) as

\[
D_{M,d}(\tau) = \sum_{k=1}^{M} (r_{k,d}(\tau) - \bar{r}_{M,d}(\tau)).
\]

(1)

The hierarchical average value \((R/S)_M(\tau)\) that stands for the rescaled/normalized relation between \( R_{M,d}(\tau) \) and \( S_{M,d}(\tau) \) becomes

\[
(R/S)_M(\tau) = \frac{1}{N} \sum_{d=1}^{N} \frac{R_{M,d}(\tau)}{S_{M,d}(\tau)} \propto M^{H(\tau)},
\]

(2)
where the subseries \( R_{M,d}(\tau) \) and the standard deviation \( S_{M,d}(\tau) \) are, respectively, given by

\[
R_{M,d}(\tau) = \max \{ D_{1,d}(\tau), D_{2,d}(\tau), ..., D_{M,d}(\tau) \} - \min \{ D_{1,d}(\tau), D_{2,d}(\tau), ..., D_{M,d}(\tau) \}
\]  

and

\[
S_{M,d}(\tau) = \left[ \frac{1}{M} \sum_{k=1}^{M} (r_{k,d}(\tau) - \bar{r}_{M,d}(\tau))^2 \right]^\frac{1}{2}.
\]

Here \( H(\tau) \) is called the Hurst exponent and the relationship between the fractal dimension \( D_f \) and the Hurst exponent \( H(\tau) \) can be written as \( D_f = 2 - H(\tau) \). As is well known, we can dynamically evolve the Hurst exponent in the following way: (1) The time series is persistent if \( H(\tau) \in (0.5, 1.0] \). It means that this is characterized by long-run memory affecting all time scales. One has increasing persistence as \( H(\tau) \) approaches 1.0. The persistence process means that the chances will continue to be up or down in the future, if the price is up or down. (2) When \( H(\tau) = 0.5 \), the time series is uncorrelated, and this case is really included to Gaussian or gamma white-noise process. The stochastic process with \( H(\tau) \neq 0.5 \) are also referred to as fractional Brownian motions. (3) One has antipersistence if \( H(\tau) \in (0, 0.5] \). Hence, it is important to note that the persistent process has little noises, while it shows high-frequency noise in the antipersistent process.

To investigate the multifractal properties systematically, several methods have been suggested for more than one decade. In particular, Barabási et al. [8] have recently reported the multifractality of self-affine fractals and have also studied the multi-affine function and the multifractal spectrum. For simplicity, the \( q \)-th height-height correlation function \( F_q(\tau) \) that depends only on the time lag \( \tau \) takes the form

\[
F_q(\tau) = \langle |p(t + \tau) - p(t)|^q \rangle \sim \tau^{qH_q},
\]

where \( H_q \) is the generalized \( q \)th-order Hurst exponent and the angular brackets denote a statistical average over time. It would be in reality expected that a nontrivial multi-affine spectrum can be obtained as \( H_q \) varies with \( q \). This has exploited in the multifractal method
and the large fluctuation effects in the dynamical behavior of the price can be explored from Eq.(5). In our scheme, we will make use of Eqs.(2) and (5) to compute the multifractal features of prices. The mathematical techniques discussed in this letter lead us to more general results.

For characteristic analysis of the won-dollar exchange rate and the KOSPI in Korean financial market, we will present more detailed numerical data of Hurst exponents from the results of R/S analysis. The generalized $q$th-order Hurst exponents in the height-height correlation function are further estimated, and the form of the probability distribution of returns is discussed. In this letter, the tick data for the won-dollar exchange rate were taken from April 1981 to December 2002, while we used the tick data of the KOSPI transacted for 23 years from April 1981. In Fig.1 we show the price time series for the the won-dollar exchange rate, in which the time step between ticks is evolved for one day. For the measure of the Hurst exponent from Eq.(2), we restrict ourselves to three cases of $\tau = 1$, 5, and 24, although the time interval can be extended to large number in our simulation. The Hurst exponents for the won-dollar exchange rate and the KOSPI was obtained numerically from the results of R/S analysis, as summarized in Tables 1 and 2. The Hurst exponents for the KOSPI are $H(\tau = 1)=0.6886$ and $H(\tau = 30)=0.7332$, as plotted in Fig.2, and it is in fact found that our Hurst values are significantly different from a well-studied random walk with $H = 0.5$. These are located in the persistence region similar to those of the crude oil prices [16]. Especially, it may be expected that the Hurst exponent is taken anomalously to be near 1 as the time series proceeds with long-run memory effects. Since the crossovers in the function $H(\tau)$ are existed in recent studies, $H(\tau)$ from our tick data is similarly found to have the existence of crossovers at characteristic time $\tau = 9(\tau = 7$ and 35) for the won-dollar exchange rate(the KOSPI).

Next we perform the numerical study of Eq.(5) in order to analyze the generalized $q$th-order Hurst exponents in the height-height correlation function $F_q(\tau)$. Tables 1 and 2 include the values of the generalized $q$th-order Hurst exponent $H_q$ in height-height correlation func-
tion for the won-dollar exchange rate and the KOSPI. Especially, the values $\log(F_q/q)$ for $q = 1, 2, ..., 6$ are plotted in Fig.3 for the won-dollar exchange rate, and the generalized Hurst exponent is taken to be near 0.65 as $q \to 1$. The probability distribution of returns is well consistent with a Lorentz distribution different from fat-tailed properties, as shown in Figs.4 and 5.

In conclusion, we have presented the multifractal measures from the dynamical behavior of prices using the R/S analysis for the won-dollar exchange rate and the KOSPI. The multifractal Hurst exponents, the generalized $q$th-order Hurst exponent, and the form of the probability distribution have discussed with long-run memory effects. Since our Hurst exponents are larger than 0.5 through R/S analysis, the time series of prices is meant to be persistent. Particularly, it is apparent from our data of the Hurst exponent $H(\tau)$ that the existence of crossovers is similar to that of other result [16]. Moreover, it is found that the probability distribution for all returns is well consistent with a Lorentz distribution.

Since a plethora of tick data support to carry out the dynamical behavior in our stock and foreign exchange markets, our analysis would assure that it is in fact able to capture the essential multifractal properties in our present result. In future, we will study extensions of the financial analysis for the exchange rates transacted in financial markets. We expect that our key result will be effectively applied to investigate the other tick data in Korean financial markets and compared with other calculations transacted in other nations in detail.

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FIGURE CAPTIONS

Fig. 1. Plot of the tick data for the won-dollar exchange rate, where one time step is the transaction time evolved for one day. This continuous tick data were taken from April 1981 to December 2002.

Fig. 2. Log-log plot of $R/S(\tau)$ at $\tau = 1$(circle) and 24(triangle) for the won-dollar exchange rate.

Fig. 3. Plot of The $q$-th height-height correlation function $F_q(\tau)$ of the time interval $\tau$ for the KOSPI, where the value of slopes is summarized in Table 2.

Fig. 4. The probability distribution of returns for the KOSPI. the dot line is represented in terms of a Lorentz distribution, i.e. $P(r) = \frac{2b}{\pi} \frac{a}{r^2 + a^2}$, where $a = 3.0 \times 10^{-4}(5.0 \times 10^{-3})$ and $b = 9.4 \times 10^{-5}(1.1 \times 10^{-4})$ for the KOSPI(the won-dollar exchange rate).

Fig. 5. The probability distribution of all returns for the won-dollar exchange rate. The dashed and solid lines show the Gaussian and Lorentz distributions, respectively.

TABLE CAPTIONS
Table 1. Summary of values of the Hurst exponent $H(\tau)$ and the generalized $q$th-order Hurst exponent $H_q$ for the won-dollar exchange rate.

| $H(\tau)$ | $H_q$ |
|-----------|-------|
| $H(\tau = 1) = 0.6886$ | $H_1 = 0.6535$ |
| $H(\tau = 5) = 0.7283$ | $H_2 = 0.5614$ |
| $H(\tau = 24) = 0.7332$ | $H_3 = 0.4859$ |

Table 2. Summary of values of the Hurst exponent $H(\tau)$ and the generalized $q$th-order Hurst exponent $H_q$ for the KOSPI.

| $H(\tau)$ | $H_q$ |
|-----------|-------|
| $H(\tau = 1) = 0.6238$ | $H_1 = 0.7791$ |
| $H(\tau = 5) = 0.6575$ | $H_2 = 0.5426$ |
| $H(\tau = 24) = 0.7278$ | $H_3 = 0.5215$ |
\[ \log R/S(\tau) \]

\[ \log M \]

\[ H(1) = 0.6886 \]

\[ H(24) = 0.7332 \]
\[
\frac{\log(F_q)}{q}
\]
$$P(\|r\|)$$

- Gaussian
- Lorentz
$H(1) = 0.6886$

$H(24) = 0.7332$
\[ \frac{\log F_q}{q} \]
Gaussian
Lorentz
\[ P(\| r \|) \]

- Gaussian
- Lorentz

\[
\begin{align*}
10^{-3} & \quad 10^{-2} \\
10^{-1} & \quad 10^0 \\
10^1 & \quad 10^2 \\
10^3 & \quad 10^4
\end{align*}
\]