BREAKING THE $2^n$-BARRIER FOR IRREDUNDANCE: 
A PARAMETERIZED ROUTE TO SOLVING EXACT PUZZLES 
(EXTENDED ABSTRACT)

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Abstract. The lower and the upper irredundance numbers of a graph $G$, denoted $ir(G)$ and $IR(G)$ respectively, are conceptually linked to domination and independence numbers and have numerous relations to other graph parameters. It is a long-standing open question whether determining these numbers for a graph $G$ on $n$ vertices admits exact algorithms running in time less than the trivial $\Omega(2^n)$ enumeration barrier. We solve these open problems by devising parameterized algorithms for the dual of the natural parameterizations of the problems with running times faster than $O^*(4^k)$. For example, we present an algorithm running in time $O^*(3.069^k)$ for determining whether $IR(G)$ is at least $n - k$. Although the corresponding problem has been known to be in FPT by kernelization techniques, this paper offers the first parameterized algorithms with an exponential dependency on the parameter in the running time. Additionally, our work also appears to be the first example of a parameterized approach leading to a solution to a problem in exponential time algorithmics where the natural interpretation as an exact exponential-time algorithm fails.

Key words and phrases: Parameterized Algorithms, Exact Exponential-Time Algorithms, Graphs, Irredundant Set.

The first author gratefully acknowledges the support given by The University of Newcastle for RGC CEF grant number G0189479 which supported her work on this project.
1. Introduction

A set $I \subseteq V$ is called an irredundant set of a graph $G = (V, E)$ if each $v \in I$ is either isolated in $G[I]$, the subgraph induced by $I$, or there is at least one vertex $u \in V \setminus I$ with $N(u) \cap I = \{v\}$, called a private neighbor of $v$. An irredundant set $I$ is maximal if no proper superset of $I$ is an irredundant set. The lower irredundance number $\text{ir}(G)$ equals the minimum cardinality taken over all maximal irredundant sets of $G$; similarly, the upper irredundance number $\text{IR}(G)$ equals the maximum cardinality taken over all such sets.

In graph theory, the irredundance numbers have been extensively studied due to their relation to numerous other graph parameters. An estimated 100 research papers [13] have been published on the properties of irredundant sets in graphs, e.g., [1, 3, 4, 6, 8, 11, 12, 15, 19, 20]. For example, if $D \subseteq V$ is an (inclusion-wise) minimal dominating set, then for every $v \in D$ there is some minimality witness, i.e., a vertex that is only dominated by $v$. In fact, a set is minimal dominating if and only if it is irredundant and dominating [7].

Since each independent set is also an irredundant set, the well-known domination chain $\text{ir}(G) \leq \gamma(G) \leq \alpha(G) \leq \text{IR}(G)$ is a simple observation. Here, as usual, $\gamma(G)$ denotes the size of a minimum dominating set, and $\alpha(G)$ denotes the size of a maximum independent set in $G$. It is known that $\gamma(G)/2 < \text{ir}(G) \leq \gamma(G) \leq 2 \cdot \text{ir}(G) - 1$, see [18].

Determining the irredundance numbers is NP-hard even for bipartite graphs [19]. They can be computed in linear time on graphs of bounded treewidth [2], but the fastest currently known exact algorithm for general graphs is the simple $O^*(2^n)$ brute-force approach enumerating all subsets.\footnote{The $O^*$-notation hides polynomial factors, e.g., $f(n, k) \cdot \text{poly}(n, k) = O^*(f(n, k))$.}

Since there has been no progress in the exact exponential time area, it is tempting to study these problems from a parameterized complexity viewpoint (for an introduction, see, e.g., [9]). The hope is that the additional notion of a parameter, e.g., the size $k$ of the irredundant set, allows for a more fine-grained analysis of the running time, maybe even a running time polynomial in $n$ and exponential only in $k$: It has been known for a while (see, e.g., [24]) that it is possible to break the so-called $2^n$-barrier for (some) vertex-selection problems by designing parameterized algorithms that run in time $O^*(c^k)$ for some $c < 4$ by a “win-win” approach: either the parameter is “small” ($k < n/2 + \epsilon$ for an appropriate $\epsilon > 0$) and we use the parameterized algorithm, or we enumerate all $(\binom{n}{\lfloor n/2+\epsilon \rfloor}) < 2^n$ subsets.

Unfortunately, the problem of finding an irredundant set of size $k$ is $W[1]$-complete when parameterized in $k$ as shown by Downey et al. [10], which implies that algorithms with a running time of $O(f(k)\text{poly}(n))$ are unlikely. However, they also proved that the parameterized dual, where the parameter is $k' := n - k$, admits a problem kernel of size $3k^2$ and is therefore in FPT (but the running time has a superexponential dependency on the parameter). What’s more, in order to break the $2^n$-barrier for the unparameterized problems, we can also use the dual parameter. Therefore in this paper we study the parameterized problems (following the notation of [10]) Co-Maximum Irredundant Set (Co-MaxIR) and Co-Minimum Maximal Irredundant Set (Co-MinMaxIR), which given a graph $G = (V, E)$ and positive integer parameter $k$ are to decide whether, respectively, $\text{IR}(G) \geq n - k$ and $\text{ir}(G) \leq n - k$. We also consider the variant exact Co-Minimum Maximal Irredundant Set (exact Co-MinMaxIR), which given a graph $G = (V, E)$ and positive integer parameter $k$, asks to decide whether $\text{ir}(G) = n - k$.\footnote{The $O^*$-notation hides polynomial factors, e.g., $f(n, k) \cdot \text{poly}(n, k) = O^*(f(n, k))$.}
Our contribution. Our first contribution are linear problem kernels with $2k - 1$ vertices for the Co-MinMaxIR problem and $3k$ vertices for Co-MaxIR, which already shows that both problems can be solved with a running time of $O^*(c^k)$, $c \leq 8$. In particular, this improves the kernel with $3k^2$ vertices and the corresponding running time of $O^*(8^{k^2})$ of [10].

Secondly, we present a simple algorithm with a running time of $O^*(3.841^k)$ which solves both Co-MAXIR and exact Co-MinMaxIR simultaneously. The price we pay for this generality is that the running time is only slightly better than $O^*(4^k)$, since we cannot exploit any special properties of Co-MAXIR that do not hold for exact Co-MinMaxIR and vice versa.

Thirdly, we present one modification of the above algorithm, which trades the generality for improved running time and solves Co-MAXIR in time $O^*(3.069^k)$. Although all the algorithms are surprisingly simple, a major effort is required to prove their running time using a non-standard measure and a Measure & Conquer (M&C) approach. While nowadays M&C is a standard technique for the analysis of moderately exponential time algorithms (see, e.g., [16]), it is still seldomly used in parameterized algorithmics.

Finally, as a direct consequence of the above algorithms, we obtain the first exact exponential time algorithm breaking the $2^n$-barrier for computing the irredundance numbers on arbitrary graphs with $n$ vertices, a well-known open question (see, e.g., [17]).

Due to the lack of space, most proofs or parts thereof have been moved to an appendix.

2. Preliminaries and Linear Kernels

The following alternative definition of irredundance is more descriptive and eases understanding the results in this paper: The vertices in an irredundant set can be thought of as kings, where each such king ought to have his very own private garden that no other king can see (where “seeing” means adjacency). Each king has exactly one cultivated garden, and all the other private neighbors degenerate to wilderness. It is also possible that the garden is already built into the king’s own castle. One can easily verify that this alternate definition is equivalent to the formal one given above.

Definition 2.1. Let $G = (V, E)$ be a graph and $I \subseteq V$ an irredundant set. We call the vertices in $I$ kings, the set consisting of exactly one private neighbour for each king we call gardens, and all remaining vertices wilderness. If a king has more than one private neighbor, we fix one of these vertices as a unique garden and the other vertices as wilderness. If a vertex $v \in I$ has no neighbors in $I$, we (w.l.o.g.) say $v$ has an internal garden, otherwise the garden is external. We denote the corresponding sets as $K, G, W$. Note that $K$ and $G$ are not necessarily disjoint, since there might be kings with internal gardens. Kings with external gardens are denoted by $K_e$ and kings with internal garden by $K_i$. Similarly, the set of external gardens is $G_e := G \setminus K$. In what follows these sets are also referred to as “labels”.

The following theorem makes use of the inequality $\text{ir}(G) \leq \gamma(G) \leq n/2$ in graphs without isolated vertices, and improves on the known kernel for Co-MinMaxIR, while the subsequent theorem uses crown reductions and improves over the previously known kernel with a quadratic number of vertices for Co-MAXIR [10].

Theorem 2.2. The Co-MinMaxIR problem admits a kernel with at most $2k - 1$ vertices.

Remark 2.3. By results of Blank and McCuaig/Shepherd, see [22], we know that $\gamma(G) \leq \frac{n}{2}$ for any graph $G$ of minimum degree two, apart from some small exceptional graphs. If
we could design reduction rules to cope with degree-one vertices in a given Co-MinMaxIR instance \((G, k)\), we might be able to show that Co-MinMaxIR admits a kernel with at most \(3k\) vertices, starting with a graph of order \(n\). This is currently an open question.

By using a crown reduction, see [5, 14], we can show:

**Theorem 2.4.** Co-MaxIR admits a kernel with at most \(3k\) vertices.

The above two theorems already show that Co-MinMaxIR and Co-MAXIR allow fixed-parameter tractable algorithms with a running time exponential in \(k\), a new contribution. The status of determining the irredundance numbers when parameterized by their natural parameters is different. While it was shown in [10] that computing \(\text{IR}(G)\) is \(W[1]\)-complete, we have no such result for the lower irredundance number (but membership in \(W[2]\) is easy to see). However, the relation of \(\text{ir}(G)\) and \(\gamma(G)\) yields an interesting link to another (open) problem: we observe that if computing \(\text{ir}(G)\) was in FPT, we could approximate \(\gamma(G)\) up to a factor of two in FPT-time. More generally, one could approximate \(\text{ir}(G)\) up to a constant factor in FPT-time if and only if \(\gamma(G)\) can be approximated up to a constant factor in FPT-time. The latter question is still open, see [21] for a recent survey on FPT approximation.

3. A Simple Algorithm For Computing The Irredundance Numbers

Our algorithm for the irredundance numbers recursively branches on the vertices of the graph and assigns each vertex one of the four possible labels \(K_i, K_e, G_e, W\), until a labeling that forms a solution has been found (if one exists). If \(I\) is an irredundant set of size at least \(n - k\), then is is easy to see that \(|K_e| = |K \setminus G| \leq k\) and \(|G \setminus K| + |W| \leq k\), which indicates a first termination condition. Furthermore, one can easily observe that for any irredundant set \(I \subseteq V\) the following simple properties hold for all \(v \in V\): (1) if \(|N(v) \cap K| \geq 2\) then \(v \in K \cup W\); (2) if \(|N(v) \cap G| \geq 2\) then \(v \in G \cup W\); (3) if \(|N(v) \cap K| \geq 2\) and \(|N(v) \cap G| \geq 2\) then \(v \in W\). Additionally, for all \(v \in K_i\), we have \(N(v) \subseteq W\).

This gives us a couple of conditions the labeling has to satisfy in order to yield an irredundant set: each external garden is connected to exactly one external king and vice versa. Once the algorithm constructs a labeling that cannot yield an irredundant set anymore the current branch can be terminated.

**Definition 3.1.** Let \(G = (V, E)\) be a graph and let \(K_i, K_e, G_e, W \subseteq V\) be a labeling of \(V\). Let \(\overline{V} = V \setminus (K_i \cup K_e \cup G_e \cup W)\). We call \((K_i, K_e, G_e, W)\) valid if the following conditions hold, and invalid otherwise.

- \(K_i, K_e, G_e, W\) are pairwise disjoint,
- for each \(v \in K_i\), \(N(v) \subseteq W\),
- for each \(v \in K_e\), \(N(v) \cap G_e \neq \emptyset\),
- for each \(v \in K_e\), \(|N(v) \cap G_e| \leq 1\),
- for each \(v \in G_e\), \(N(v) \cap (K_i \cup \overline{V}) \neq \emptyset\),
- for each \(v \in G_e\), \(|N(v) \cap K_i| \leq 1\).

As a direct consequence, we can define a set of vertices that can no longer become external gardens or kings without invalidating the current labeling:

\[
\text{Not}G := \{ v \in \overline{V} \mid \text{the labeling } (K_i, K_e, G_e \cup \{v\}, W) \text{ is invalid} \}
\]

\[
\text{Not}K := \{ v \in \overline{V} \mid \text{the labeling } (K_i, K_e \cup \{v\}, G_e, W) \text{ is invalid} \}
\]
Algorithm 1 A fast yet simple algorithm for CO-MaxIR.

Algorithm CO-IR\((G, k, \mathcal{K}_e, \mathcal{K}_i, \mathcal{G}_e, \mathcal{W})\):

Input: Graph \(G = (V, E)\), \(k \in \mathbb{N}\), labels \(\mathcal{K}_e, \mathcal{K}_i, \mathcal{G}_e, \mathcal{W} \subseteq V\)

01: Compute the sets Not\(G\), Not\(K\).
02: Apply the reduction rules exhaustively, updating Not\(G\) and Not\(K\).
03: if the current labeling is invalid then return NO.
04: if \(\varphi(k, \mathcal{K}_e, \mathcal{G}_e, \mathcal{W}) < 0\) then return NO.
05: if \(|\mathcal{K}_e| + |\mathcal{W}| = k\) or \(|\mathcal{G}_e| + |\mathcal{W}| = k\) or all vertices are labeled then
06: return whether \(V \setminus (\mathcal{W} \cup \mathcal{G}_e)\) is a solution.
07: if Not\(G \neq \emptyset\) (or analogously, Not\(K \neq \emptyset\)) then
08: choose \(v \in \text{Not}\(G\);)
09: \return{CO-IR\((G, k, \mathcal{K}_e, \mathcal{K}_i, \mathcal{G}_e, \mathcal{W})\) or CO-IR\((G, k, \mathcal{K}_e, \mathcal{K}_i, \mathcal{G}_e, \mathcal{W} \cup \{v\})\);} \(\text{or CO-IR\((G, k, \mathcal{K}_e, \mathcal{K}_i, \mathcal{G}_e, \mathcal{W} \cup \{v\})\)}\)
10: Choose (in this preferred order) unlabeled \(v \in V\) of degree one, of maximum degree
11: with \(N(v) \cap (\mathcal{G}_e \cup \mathcal{K}_e) \neq \emptyset\) or any unlabeled \(v\) with maximum degree.
\(\text{or} \exists u \in N(v) \setminus (\mathcal{K}_e \cup \mathcal{K}_i \cup W): \text{CO-IR\((G, k, \mathcal{K}_e, \mathcal{G}_e, \mathcal{W} \cup \{u\}, W)\)}\)
\(\text{or} \exists u \in N(v) \setminus (\mathcal{G}_e \cup \mathcal{K}_i \cup W): \text{CO-IR\((G, k, \mathcal{K}_e, \mathcal{G}_e, \mathcal{W} \cup \{u\}, W)\)}\)

It is easy to see that Not\(K\) and Not\(G\) can be computed in polynomial time, and since vertices in Not\(G \cap\) Not\(K\) can only be wilderness, we can also assume that Not\(G \cap\) Not\(K\) = \(\emptyset\) once the following reduction rules have been applied.

Let \(G = (V, E)\) be a graph and let \(\mathcal{K}_i, \mathcal{K}_e, \mathcal{G}_e, \mathcal{W} \subseteq V\) be a valid labeling of \(V\). Let \(\overline{V} = V \setminus (\mathcal{K}_i \cup \mathcal{K}_e \cup \mathcal{G}_e \cup \mathcal{W})\). We define the following reduction rules, to be applied in this order, one at a time:

- \(\text{R}_1\) If there is some \(v \in \mathcal{W}\), remove all edges incident to \(v\).
- \(\text{R}_2\) If there is some \(v \in \overline{V}\) with \(\deg(v) = 0\), then set \(\mathcal{K}_i = \mathcal{K}_i \cup \{v\}\).
- \(\text{R}_3\) If there is \(v \in \mathcal{K}_e\) with \(N(v) \cap \mathcal{G}_e = \emptyset\) and \(N(v) \cap \overline{V} = \{w\}\), then set \(\mathcal{G}_e := \mathcal{G}_e \cup \{w\}\).
  - If there is \(v \in \mathcal{G}_e\) with \(N(v) \cap \mathcal{K}_e = \emptyset\) and \(N(v) \cap \overline{V} = \{w\}\), then set \(\mathcal{K}_e := \mathcal{K}_e \cup \{w\}\).
- \(\text{R}_4\) For every \(v \in \text{Not}\(G\) set \(\mathcal{W} := \mathcal{W} \cup \{v\}\).

A graph and a labeling of its vertices as above is called reduced if no further reduction rules can be applied.

Since the algorithm uses exhaustive branching, we easily obtain:

**Lemma 3.2.** Algorithm 1 correctly solves CO-MaxIR.

**Remark 3.3.** Algorithm 1 can be also used, with slight modifications, to answer the question if a graph \(G\) has an inclusion-minimal co-irredundant set of size exactly \(k\). Namely, if the potential dropped to zero, then either the current labeling corresponds to a valid co-irredundant set of size \(k\) that is inclusion-minimal or not; this has to be tested in addition.

Let \(T(k, \mathcal{K}_e, \mathcal{G}_e, \mathcal{W})\) be the number of recursive calls that reach Line 5 where none of the (possibly zero) following recursive calls (in Lines 9 and 11) reach this line. Since all recursive calls only require polynomial time, the running time of Algorithm 1 is bounded by \(O^*(T(k, \mathcal{K}_e, \mathcal{G}_e, \mathcal{W}))\). Let our measure be:

\[ \varphi(k, \mathcal{K}_e, \mathcal{G}_e, \mathcal{W}) = k - |\mathcal{W}| - 0.5|\mathcal{K}_e| - 0.5|\mathcal{G}_e| \]

**Lemma 3.4.** \(T(k, \mathcal{K}_e, \mathcal{G}_e, \mathcal{W}) \leq \alpha^{\varphi(k, \mathcal{K}_e, \mathcal{G}_e, \mathcal{W})}\) with \(\alpha \leq 3.841\).
Theorem 3.5. Co-MaxIR and exact Co-MinMaxIR can be solved in time $O^*(3.841^k)$. 

Corollary 3.6. The irredundance numbers of a graph $G$ with $n$ vertices can be computed in time $O^*(1.99914^n)$. 

4. Measure & Conquer Tailored To The Problems

In this section, we tailor the general Algorithm 1 to the needs of the Co-MAXIR problem. To this end, we use a more precise annotation of vertices: In the course of the algorithm, they will be either unlabeled $U$, kings with internal gardens $K_i$, kings with external gardens $K_e$, (external) gardens $G_e$, wilderness $W$, not being kings Not$K$, or not being gardens Not$G$. We furthermore partition the set of vertices $V$ into active vertices $V_a = U \cup \text{Not}G \cup \text{Not}K \cup \{v \in K_e \mid N(v) \cap G_e = \emptyset\} \cup \{v \in G_e \mid N(v) \cap K_e = \emptyset\}$ that have to be reconsidered, and inactive vertices $V_i = V \setminus V_a$. This means that the inactive vertices are either from $W$, $K_i$ or paired-up external kings and gardens. Define $K_{ea} = K_e \cap V_a$ and $K_{ei} = K_e \cap V_i$ (and analogously $G_{ea}$, $G_{ei}$).

We use a new measure $\varphi(k, K_i, K_e, G_e, \text{Not}G, \text{Not}K, W, V_a) = k - |W| - |G_e| - \omega_e(|K_{ea}| + |G_{ea}|) - \omega_n(|\text{Not}G| + |\text{Not}K|)$, where Not$G$ and Not$K$ are taken into account. We will later determine the weights $\omega_e$ and $\omega_n$ to optimize the analysis, where $0 \leq \omega_n \leq 0.5 \leq \omega_e \leq 1$ and $\omega_n + \omega_e \leq 1$. We will describe in words how the measure changes in each case, leaving most of the analysis to the appendix.

Let us first present the reduction rules that we employ in Table 1.

Lemma 4.1. The rules listed in Table 1 are sound and do not increase the measure.

Lemma 4.2. In a reduced instance, a vertex $v \in \text{Not}K \cup \text{Not}G$ may have at most one neighbor $u \in G_e \cup K_e$; more precisely, if such $u$ exists, then $u \in G_e$ if and only if $v \in \text{Not}G$. Moreover, $\deg(v) \geq 2$, so $v$ must have a neighbor $z$ that is not in $G_e \cup K_e$.

Proof. Consider, w.l.o.g., $v \in \text{Not}G$. Assume that $N(v) \cap (G_e \cup K_e) = \emptyset$. Then, Reduction Rules 11 and 3 ensure that $\deg(v) \geq 2$.

Assume now that $u \in N(v) \cap (G_e \cup K_e)$ exists. Note that the alternative $u \in K_e$ is resolved by Reduction Rule 6. Hence, $u \in G_e$. If $v$ had no other neighbor but $u$, then Reduction Rule 11 would have triggered. So, $\deg(v) \geq 2$. Let $z \in N(v) \setminus \{u\}$. If the claim were false, then $z \in G_e \cup K_e$. The case $z \in K_e$ is ruled out by Reduction Rule 6. The case $z \in G_e$ is dealt with by Reduction Rule 12. Hence, $z \notin G_e \cup K_e$. 

Lemma 4.3. In each labeled graph which is input of a recursive call of CO-IR there are no two neighbors $u,v$ such that $u \in K_{ea}$ and $v \in G_{ea}$.

Lemma 4.4. Whenever our algorithm encounters a reduced instance, a vertex $v \in G_e$ obeys $N(v) \subseteq U \cup G_e \cup \text{Not}G$. Symmetrically, if $v \in K_e$, then $N(v) \subseteq U \cup K_e \cup \text{Not}K$.

Note that the irredundance numbers can be computed in polynomial time on graphs of bounded treewidth, see [25, Page 75f.], and the corresponding dynamic programming easily extends also to labeled graphs, since the labels basically correspond to the states of the dynamic programming process.
(1) If \( V \) contains a vertex \( x \) with two neighbors \( u, v \) where \( x \in K_1 \cup K_e \) and \( u, v \in G_e \), then return \( \text{NO} \). Exchanging the roles of kings and gardens, we obtain a symmetric rule.

(2) If \( V \) contains an isolated vertex \( v \in (G_i \cup K_e) \), then return \( \text{NO} \).

(3) If \( V \) contains an isolated vertex \( v \in (\text{Not}K \cup \text{Not}G) \), then put \( v \) into \( W \), decreasing the measure by \( 1 - \omega_n \).

(4) If \( V \) contains an isolated vertex \( u \in U \), then put \( u \) into \( K_i \) and set \( V_a = V_a \setminus \{u\} \).

(5) Delete an edge between two external kings or two external gardens.

(6) Delete an edge between a \( K_e \)- and a \( \text{Not}G \)-vertex. Exchanging the roles of kings and gardens, we obtain a symmetric rule.

(7) Remove any edges incident to vertices in \( W \).

(8) Delete an edge between two \( K_i \)-vertices. Delete an edge between two \( \text{Not}G \)-vertices.

(9) If \( u \in U \) such that \( N(u) = \{v\} \) for some \( v \in U \), then put \( u \) into \( K_i \) and set \( V_a = V_a \setminus \{u\} \).

(10) If \( u \in K_i \), then put its neighbors \( N(u) \) into \( W \) and set \( V_a = V_a \setminus N(u) \); this decreases the measure by \( |N(u)| \).

(11) If \( V \) contains two neighbors \( u, v \) such that \( u \in \text{G} \) and \( v \in U \cup \text{Not}G \) with either \( \deg(u) = 1 \) or \( \deg(v) = 1 \), then put \( v \) into \( K_e \), and make \( u, v \) inactive; this decreases the measure by \( 1 - \omega_t \) (if \( u \in U \)) and \( 1 - \omega_t - \omega_n \), respectively. Exchanging the roles of kings and gardens, we obtain a symmetric rule.

(12) If \( V \) contains a vertex \( v \) with two neighboring gardens such that \( v \in U \), then set \( v \in \text{Not}K \); if \( v \in \text{Not}G \), then set \( v \in W \). This decreases the measure by \( \omega_n \) or \( (1 - \omega_n) \), respectively. Exchanging the roles of kings and gardens, we obtain a symmetric rule.

(13) Assume that \( V \) contains two inactive neighbors \( u, v \) where \( u \in K_e \) and \( v \in G_e \), then put all \( x \in (N(u) \cap U) \) into \( \text{Not}G \), all \( x \in (N(u) \cap \text{Not}K) \) into \( W \), all \( x \in (N(v) \cap U) \) into \( \text{Not}K \) and all \( x \in (N(v) \cap \text{Not}G) \) into \( W \).

Table 1: Extensive list of reduction rules.

Although we are looking for a maximal irredundant set, we can likewise look for a complete labeling \( L = (K_1^L, G_e^L, K_e^L, W^L) \) that partitions the whole vertex set \( V = K_i^L \cup G_e^L \cup K_e^L \cup W^L \) into internal kings, external kings and gardens, as well as wilderness. Having determined \( L_i \), \( I_L = K_i^L \cup K_e^L \) should be an irredundant set, an conversely, to a given irredundant set \( I \), one can compute in polynomial time a corresponding complete labeling. However, during the course of the algorithm, we deal with (incomplete) labelings \( L = (K_i, G_e, K_e, \text{Not}G, \text{Not}K, W, V_a) \), a tuple of subsets of \( V \) that also serve as input to our algorithm, preserving the invariant that \( V = K_i \cup G_e \cup K_e \cup \text{Not}G \cup \text{Not}K \cup W \cup U \). A complete labeling corresponds to a labeling with \( \text{Not}G = \text{Not}K = U = V_a = \emptyset \). Since \( (\text{Not}K \cup \text{Not}G) \subseteq V_a \), we have obtained a complete labeling once we leave our algorithm in Line 4, returning \( \text{YES} \). We say that a labeling \( L' = (K_i', G_e', K_e', \text{Not}G', \text{Not}K', W', V_a') \) extends the labeling \( L = (K_i, G_e, K_e, \text{Not}G, \text{Not}K, W, V_a) \) if \( K_i \subseteq K_i' \), \( G_e \subseteq G_e' \), \( K_e \subseteq K_e' \), \( \text{Not}G \subseteq W \cup G_e' \), \( \text{Not}K \subseteq W \cup G_e' \), \( W \subseteq W' \), \( V_a \subseteq V_a' \). We also write \( L \prec_G L' \) if \( L' \) extends \( L \). We can also speak of a complete labeling extending a labeling in the sense described above. Notice that reduction rules and recursive calls only extend labelings (further).

Notice that \( \prec_G \) is a partial order on the set of labelings of a graph \( G = (V, E) \). The maximal elements in this order are precisely the complete labelings. Hence, the labeling \( L_I \) corresponding to a maximal irredundant set \( I \) is maximal, with \( \varphi(k, L_I) \leq 0 \) iff \( |I| \geq |V| - k \). Conversely, given a graph \( G = (V, E) \), the labeling \( L_G = (\emptyset, \emptyset, \emptyset, \emptyset, \emptyset, V) \) is the smallest element of \( \prec_G \); this is also the initial labeling that we start off with when first calling Algorithm 2. If \( L, L' \) are labelings corresponding to the parameter lists of nodes \( n, n' \) in the search tree such that \( n \) is ancestor of \( n' \) in the search tree, then \( L \prec_G L' \). The basic strategy
Algorithm 2 A faster algorithm for CO-MaXIR.

Algorithm CO-IR(G, k, K, K, G, G, NotG, NotK, W, V_a):
Input: Graph G = (V, E), k ∈ N, labels K, K, G, G, NotG, NotK, W, V_a ⊆ V
01: Consecutively apply the procedure CO-IR to components containing V_a-vertices.
02: Apply all the reduction rules exhaustively.
03: if ϕ(k, K, K, G, G, W, NotG, NotK, V_a) < 0 then return NO.
04: if V_a = 0 then return YES.
05: if maxdegree(G[V_a]) ≤ 2 then solve remaining instance by
dynamic programming.
06: if NotG = 0 (and analogously, NotK = 0) then
07: choose v ∈ NotG; if ∃z ∈ N(v) ∩ G then I := {v, z} else I := 0.
08: if there is an unlabeled v ∈ V with exactly two neighbors u, w in G[V_a],
where u ∈ G and w ∈ K then
10: return CO-IR(G, k, K, K, G, G, G, NotG, NotK, W, V_a \ {v, w}) or
CO-IR(G, k, K, K, G, G, G, NotG, NotK, W, V_a \ {v, u}) or
CO-IR(G, k, K, K, G, G, G, NotG, NotK, W, V_a \ {v, u})
11: if K ∩ K ∩ G ∩ G = 0 then
12: Choose some v ∈ K ∩ K ∩ G ∩ G of maximum degree.
13: if v ∈ K ∩ K ∩ G ∩ G (and analogously, v ∈ K ∩ K ∩ G ∩ G) then
14: return ∃u ∈ N(v) : CO-IR(G, k, K, K, G, G, G, NotG, NotK, W, V_a \ {u, v})
15: Choose v ∈ U of maximum degree, preferring v with some u ∈ N(v) of degree two.
16: return CO-IR(G, k, K, K, G, G, NotG, NotK, W ∪ {v}, V_a \ {v}) or
CO-IR(G, k, K, K, G, G, G, NotG, NotK, W ∪ {v}) or
CO-IR(G, k, K, K, G, G, G, NotG, NotK, W ∪ {v}) or
CO-IR(G, k, K, K, G, G, G, NotG, NotK, W ∪ {v})

of Algorithm 2 is to exhaustively consider all complete labelings (only neglecting cases that
cannot be optimal). This way, also all important maximal irredundant sets are considered.

Lemma 4.5. If ϕ(k, K, K, G, G, W, NotG, NotK, V_a) < 0, then for weights 0 ≤ ω ≤ 0.5 ≤
ω ≤ 1 with ω_n + ω ≤ 1, for any complete labeling L = (K, G, G, K, K, L, W, L) extending the
labeling Λ := (K, K, G, G, NotG, NotK, W, V_a) we have ϕ(k, K, K, G, G, K, K, L, 0, 0, W, 0) < 0.

Proof.

We give a table for every possible label transition from Λ to
its extension L. Note that Algorithm CO-IR only computes
such solutions. All entries except two cause a non-increase of
ϕ. The entries number 1 and 3 expose an increase in ϕ. By
the problem definition, there exists a bijection f : K → G, L.
So for a vertex v in K ∩ K we must have f(v) ∈ U ∩ NotK.
By Lemma 4.3 f(v) /∈ G. Taking now into account the label
transition of f(v) which must be of the form U → G or
NotK → G, we see that a total decrease with respect to
v and f(v) of at least 1 − ω_n − ω ≥ 0 can be claimed. If
v ∈ NotG ∩ K then by arguing analogously we get a total
decrease of at least 1 − 2 · ω_n > 0.
Lemma 4.6. Assume that all active vertices are in \( \mathcal{U} \cup \mathcal{G} \cup \mathcal{K} \), with \( \mathcal{G}_a \cup \mathcal{K}_a \neq \emptyset \) and that there is an unlabeled vertex \( v \), which has exactly two neighbors \( v_G \in \mathcal{G} \) and \( v_K \in \mathcal{K} \). In the corresponding branching process, we may then omit the case \( v \in \mathcal{W} \).

**Proof.** We are looking for an inclusion-maximal irredundant set. Hence, only the positions of the kings matter, not the positions of the gardens. So, in particular we cannot insist on the garden of \( v_K \) being placed on some neighbor \( u \) of \( v_K \) different from \( v \). In this sense, any solution that uses \( v \) as wilderness can be transformed into a no worse solution with \( v \in \mathcal{G} \): Simply pair up \( v \) and \( v_K \), turning the hitherto garden of \( v_K \) into wilderness. So, no optimum solution is lost by omitting the case \( v \in \mathcal{W} \) in the branching. 

---

| Weight | 1 | 0 | 1 | \( \omega_v \) | \( \omega_e \) | \( \omega_a \) | 0 | \( \cdots \) |
|--------|---|---|---|------------|------------|------------|---|--------|
| **Case** | **W** | \( u_v \) | \( u_e \) | \( \mathcal{K}_a \) | \( \mathcal{G}_a \) | **Not** \( W \) | **Not** \( K \) | **(M)** | potential. diff. \( \geq \) |
| (1a) | 1+ | +u | u | u | u | \( x \) | x | -- | -- |
| (1b) | 1+ | +u | u | u | u | \( x \) | x | -- | -- |
| (2a) | 1+ | +u | u | u | u | \( x \) | x | -- | -- |
| (2b) | 1+ | +u | u | u | u | \( x \) | x | -- | -- |
| (3a) | 1+ | +u | u | u | u | \( x \) | x | -- | -- |
| (3b) | 1+ | +u | u | u | u | \( x \) | x | -- | -- |
| (4a) | 1+ | +u | u | u | u | \( x \) | x | -- | -- |

Table 2: Overview over different branchings; symmetric branchings due to exchanging roles of kings and gardens are not displayed. Neither are possibly better branches listed.

**Theorem 4.7.** **Co-MaxIR** can be solved in time \( \mathcal{O}^*(3.069^k) \).

**Proof.** The correctness of the algorithm has been reasoned above already. In particular, notice Lemma 4.5 concerning the correctness of the abort.

For the running time, we now provide a partial analysis leading to recurrences that estimate an upper bound on the search tree size \( T_\nu(\mu, h) \), where \( \mu \) denotes the measure and \( h \) the height of the search tree. More details can be found in the appendix. The claimed running time would then formally follow by an induction over \( h \).

1. Assume that the algorithm branches on some vertex \( v \in \text{Not} \mathcal{G} \), the case \( v \in \text{Not} \mathcal{K} \) being completely analogous. By reduction rules, \( N(v) \subseteq \mathcal{U} \cup \mathcal{G}_a \cup \mathcal{K} \).
   - (a) If \( N(v) \subseteq \mathcal{U} \cup \mathcal{K} \), we derive the following branch in the worst case:
     \[
     T_\nu(\mu, h) \leq T_\nu(\mu - (1 - \omega_n), h - 1) + T_\nu(\mu - (\omega_t - \omega_n), h - 1).
     \]
     This follows from a simple branching analysis considering the cases that \( v \) becomes wilderness or that \( v \) becomes a king.
(b) Assume now that \( N(v) \cap (G_{ea}) \neq \emptyset \) and let \( u \in N(v) \cap G_{ea} \). Lemma 4.2 ensures that there can be at most one element in \( N(v) \cap G_e \). Due to Reduction Rule 11, \( \deg(u) \geq 2 \) and \( \deg(v) \geq 2 \) thanks to Lemma 4.2. First assume that \( \deg(u) = 2 \), i.e., \( N(u) = \{v, x\} \). Then, we arrive at the following recursion:

\[
T_\varphi(\mu, h) \leq T_\varphi(\mu - (2 - \omega_x - \omega_n), h - 1) + T_\varphi(\mu - (1 - \omega_x + \omega_n), h - 1). 
\]

This is seen as follows. By setting \( v \in W \), due to Reduction Rule 7, \( u \) will be of degree one and hence will be paired with its neighbor \( x \) due to Reduction Rule 11. If \( x \in U \), the measure decreases by \( 2 - \omega_x - \omega_n \). If \( x \in \text{Not}G \), it decreases by \( 2 - \omega_x - 2\omega_n \). But then by Lemma 4.2 there is \( y \in N(x) \setminus \{u\} \) such that \( y \in \text{Not}K \cup U \). Then by Reduction Rule 11 \( y \) is moved to \( W \cup \text{Not}G \) giving some additional amount of at least \( \omega_n \). Note that \( y \neq v \). If we set \( v \in K_e \), then \( u \) and \( v \) will be paired by Reduction Rule 13. Thereafter, the other neighbor \( x \) of \( u \) will become a member of \( \text{Not}G \) or of \( W \), depending on its previous status. Moreover, there must be a further neighbor \( z \in U \) of \( v \) (by Lemma 4.2 and the fact that \( u \) is the unique \( G_{ea} \) neighbor) that will become member of \( \text{Not}G \). This yields the claimed measure change if \( z \neq x \). If \( z = x \), then \( z \) is in \( U \) and the vertex will be put into \( W \). Thus we get \( T_\varphi(\mu - (2 - \omega_x - \omega_n), h - 1) \leq T_\varphi(\mu - (1 - \omega_x + \omega_n), h - 1) \).

(c) Secondly, assume that \( \deg(u) \geq 3 \) (keeping the previous scenario otherwise).

This yields the following worst-case branch:

\[
T_\varphi(\mu, h) \leq T_\varphi(\mu - (1 - \omega_n), h - 1) + T_\varphi(\mu - (1 - \omega_x + 2\omega_n), h - 1). 
\]

This is seen by a similar (even simpler) analysis. Note that all \( z \in N(v) \cap N(u) \subseteq U \) get labeled \( W \) in the second branch.

We will henceforth not present the recurrences for the search tree size in this explicit form, but rather point to Table 2 that contains the same information. There, cases are differentiated by writing \( B_j \) for the \( j \)th branch.

(2) Assume that all active vertices are in \( U \cup G_e \cup K_e \), with \( G_{ea} \cup K_{ea} \neq \emptyset \). Then, the algorithm would pair up some \( v \in G_{ea} \cup K_{ea} \). Assume that there is an unlabeled vertex \( v \) that has exactly two neighbors \( v_G \in G_e \) and \( v_K \in K_e \). Observe that we may skip the possibility that \( v \in W \) due to Lemma 4.6. Details of the analysis are contained in the appendix and in Table 2.

(3) Assume that all active vertices are in \( U \cup G_e \cup K_e \), with \( G_{ea} \cup K_{ea} \neq \emptyset \). Then, the algorithm tries to pair up some \( v \in G_{ea} \cup K_{ea} \) of maximum degree. There are \( \deg(v) \) branches for the cases labeled (3) \( \# j \). Since the two possibilities arising from \( v \in G_{ea} \cup K_{ea} \) are completely symmetric, we focus on \( v \in G_{ea} \). Exactly one neighbor \( u \) of \( v \in G_{ea} \) will be paired with \( v \) in each step, i.e., we set \( u \in K_e \). Pairing the king on \( u \) with the garden from \( v \) will inactivate both \( u \) and \( v \). Then, reduction rules will label all other neighbors of \( v \) with \( \text{Not}K \) (they can no longer be kings), and symmetrically all other neighbors of \( u \) with \( \text{Not}G \). Note that \( N(u) \setminus (K_e \cup \{v\}) \neq \emptyset \), since otherwise a previous branching case or Reduction Rules 12 or 9 would have triggered. Thus, there must be some \( q \in N(u) \cap U \). From \( q \), we obtain at least a measure decrease of \( \omega_n \), even if \( q \in N(v) \). This results in a set of recursions depending on the degree of \( v \) as given in Table 2.

(4) Finally, assume \( V_a = U \). Since an instance consisting of paths and cycles can be easily seen to be optimally solvable in polynomial time, we can assume that we can always find a vertex \( v \) of degree at least three to branch at. Details of the analysis
are contained in the appendix and in Table 2. There are $\deg(v)$ branches for each of the cases $(4x)\#j$, where $x \in \{a, b\}$.

Finally, to show the claimed running time, we set $\omega_{\ell} = 0.7455$ and $\omega_n = 0.2455$ in the recurrences. If the measure drops below zero, then we argue that we can safely answer NO, as shown in Lemma 4.5.

**Corollary 4.8.** \(\text{IR}(G)\) can be computed in time $O^*(\ell^{96})$.

This can be seen by the balancing “win-win” approach described above, exhaustively testing irredudant candidate sets up to size $\approx 0.4 \cdot n$.

5. Conclusions

We presented a parameterized route to the solution of yet unsolved questions in exact algorithms. More specifically, we obtained algorithms for computing the irredundance numbers running in time less than $O^*(2^n)$ by devising appropriate parameterized algorithms (where the parameterization is via a bound $k$ on the co-irredundant set) running in time less than $O^*(4^k)$.

The natural question arises if one can avoid this detour to parameterized algorithmics to solve such a puzzle from exact exponential-time algorithmics. A possible non-parameterized attack on the problem is to adapt the measure $\varphi$ to a run time less then $O^*(2^n)$ was not possible to achieve. For example, the recurrences under 1a) and 1c) translate to $T_{\varphi}(\mu, h) \leq T_{\varphi}(\mu - (1 - \tilde{\omega}_n), h - 1) + T_{\varphi}(\mu - (\tilde{\omega}_{\ell} - \tilde{\omega}_n), h - 1)$ and $T_{\varphi}(\mu, h) \leq T_{\varphi}(\mu - (1 - \tilde{\omega}_n), h - 1) + T_{\varphi}(\mu - (2 - \tilde{\omega}_{\ell} - 2\tilde{\omega}_n), h - 1)$. Now optimizing over $\tilde{\omega}_{\ell}$, $\tilde{\omega}_n$ and the maximum over the two branching numbers alone we already arrive at a run time bound of $O^*(2.036^n)$ (whereas $\tilde{\omega}_{\ell} = 1.13$ and $\tilde{\omega}_n = 0.08$). Thus, the parameterized approach was crucial for obtaining a run time upper bound better than the trivial enumeration barrier $O^*(2^n)$. Observe that for these particular problems, allowing a weight of $\tilde{\omega}_{\ell} \in [0, 2]$ is valid, while usually only weights in $[0, 1]$ should be considered.

It would be interesting to see this approach used for other problems, as well. Some of the vertex partitioning parameters discussed in [25] seem to be appropriate.

We believe that the M&C approach could also be useful to find better algorithms for computing the lower irredundance number. We are currently working on the details which should yield in running times similar to the computations for the upper irredundance number.

More broadly speaking, we think that an extended exchange of ideas between the field of Exact Exponential-Time Algorithms, in particular the M&C approach, and that of Parameterized Algorithms, could be beneficial for both areas. In our case, we would not have found the good parameterized search tree algorithms if we had not been used to the M&C approach, and conversely only via this route and the corresponding way of thinking we could break the $2^n$-barrier for computing irredundance numbers.

A few days before, we became aware that a group consisting of M. Cygan, M. Pilipczuk and J. O. Wojtaszczyk from Warsaw independently found $c^n$ algorithms, $c < 2$, for computing the irredundance numbers. Unfortunately, we do not know any details.
References

[1] R. B. Allan and R. Laskar. On domination and independent domination numbers of a graph. *Discrete Mathematics*, 23(2):73–76, 1978.

[2] M. W. Bern, E. L. Lawler, and A. L. Wong. Linear-time computation of optimal subgraphs of decomposable graphs. *Journal of Algorithms*, 8(2):216–235, 1987.

[3] B. Bollobás and E. J. Cockayne. Graph-theoretic parameters concerning domination, independence, and irredundance. *J. Graph Theory*, 3:241–250, 1979.

[4] B. Bollobás and E. J. Cockayne. On the irredundance number and maximum degree of a graph. *Discrete Mathematics*, 49:197–199, 1984.

[5] B. Chor, M. Fellows, and D. Juedes. Linear kernels in linear time, or how to save $k$ colors in $O(n^2)$ steps. In J. Hromkovic et al., editors, *30th International Workshop on Graph-Theoretic Concepts in Computer Science WG 2004*, volume 3353 of *LNCS*, pages 257–269. Springer, 2004.

[6] E. J. Cockayne, P. J. P. Grobler, S. T. Hedetniemi, and A. A. McRae. What makes an irredundant set maximal? *J. Combin. Math. Combin. Comput.*, 25:213–224, 1997.

[7] E. J. Cockayne, S. T. Hedetniemi, and D. J. Miller. Properties of hereditary hypergraphs and middle graphs. *Canad. Math. Bull.*, 21(4):461–468, 1978.

[8] E. J. Cockayne and C. M. Mynhardt. Irredundance and maximum degree in graphs. *Combin. Proc. Comput.*, 6:153–157, 1997.

[9] R. G. Downey and M. R. Fellows. *Parameterized Complexity*. Springer-Verlag, 1999.

[10] R. G. Downey, M. R. Fellows, and V. Raman. The complexity of irredundant sets parameterized by size. *Discrete Applied Mathematics*, 100:155–167, 2000.

[11] O. Favaron. Two relations between the parameters of independence and irredundance. *Discrete Mathematics*, 70(1):17–20, 1988.

[12] O. Favaron. A note on the irredundance number after vertex deletion. *Discrete Mathematics*, 121(1-3):51–54, 1993.

[13] O. Favaron, T. W. Haynes, S. T. Hedetniemi, M. A. Henning, and D. J. Knisley. Total irredundance in graphs. *Discrete Mathematics*, 256(1-2):115–127, 2002.

[14] M. R. Fellows. Blow-ups, win/win’s, and crown rules: Some new directions in fpt. In *Graph-Theoretic Concepts in Computer Science, 29th International Workshop (WG)*, volume 2880 of *LNCS*, pages 1–12. Springer, 2003.

[15] M. R. Fellows, G. Fricke, S. T. Hedetniemi, and D. P. Jacobs. The private neighbor cube. *SIAM J. Discrete Math.*, 7(1):41–47, 1994.

[16] F. V. Fomin, F. Grandoni, and D. Kratsch. A measure & conquer approach for the analysis of exact algorithms. *Journal of the ACM*, 56(5), 2009.

[17] F. V. Fomin, K. Iwama, D. Kratsch, P. Kaski, M. Koivisto, L. Kowalik, Y. Okamoto, J. van Rooij, and R. Williams. 08431 open problems – moderately exponential time algorithms. In F. V. Fomin, K. Iwama, and D. Kratsch, editors, *Moderately Exponential Time Algorithms*, number 08431 in Dagstuhl Seminar Proceedings, Dagstuhl, Germany, 2008. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, Germany.

[18] T. W. Haynes, S. T. Hedetniemi, and P. J. Slater. *Fundamentals of Domination in Graphs*, volume 208 of *Monographs and Textbooks in Pure and Applied Mathematics*. Marcel Dekker, 1998.

[19] S. T. Hedetniemi, R. Laskar, and J. Pfaff. Irredundance in graphs: a survey. *Congr. Numer.*, 48:183–193, 1985.

[20] R. Laskar and J. Pfaff. Domination and irredundance in graphs. Technical Report Techn. Rep. 434, Clemson Univ., Dept. of Math. SC., 1983.

[21] D. Marx. Parameterized complexity and approximation algorithms. *The Computer Journal*, 51(1):60–78, 2008.

[22] B. McCuaig and B. Shepherd. Domination in graphs of minimum degree two. *Journal of Graph Theory*, 13:749–762, 1989.

[23] O. Ore. *Theory of Graphs*, volume XXXVIII of *Colloquium Publications*. American Mathematical Society, 1962.

[24] V. Raman and S. Saurabh. Parameterized algorithms for feedback set problems and their duals in tournaments. *Theoretical Computer Science*, 351(3):446–458, 2006.

[25] J. A. Telle. *Vertex Partitioning Problems: Characterization, Complexity and Algorithms on Partial $k$-Trees*. PhD thesis, Department of Computer Science, University of Oregon, USA, 1994.
6. Appendix

Proof of Theorem 2.2

Proof. It is known that $\text{irr}(G)$ is upper bounded by the domination number $\gamma(G)$ of any graph $G$ of minimum degree one, see above. Since $\gamma(G) \leq n/2$ for any graph of minimum degree one (see, e.g., [23]), we can derive that $\text{irr}(G) \leq n/2$. So, in the given CO-MinMaxIR instance $(G, k)$ we can first delete all isolated vertices (they will be in any maximal irredundant set), without changing the parameter, and then kernelize as follows: if $k \leq n/2$, then we are looking for a maximal irredundant set of size at most $n - k \geq n/2 \geq \text{irr}(G)$, so that we can immediately return YES. If $k > n/2$, then we have obtained the desired kernel, which is just the current graph, with the claimed bound.

Proof of Theorem 2.4

Proof. Let $G = (V, E)$ be a graph and let $I$ be an irredundant set of size at least $n - k$ in a graph $G$.

We use a crown reduction, see [5, 14]. A crown is a subgraph $G' = (C, H, E') = G[C \cup H]$ of $G$ such that $C$ is an independent set in $G$, $H$ are all neighbors of $C$ in $G$ (i.e., $H$ separates $C$ from $V \setminus (H \cup C)$, and such that there is a matching $M$ of size $|H|$ between $C$ and $H$.

We first show that if $G$ contains a crown $(C, H, E')$, then $G$ contains a maximum irredundant set $I$ such that $I \supseteq C$ and $H \subseteq V \setminus I$. Assume that this is wrong. So, we have a solution $I$ and let $I = K_i \cup K_e$ be an arbitrary partition of $I$ into internal and external kings. Let $G_e \in V \setminus I$ be an arbitrary set that can serve as a set of gardens for $K_e$. Let $W = V \setminus (I \cup G_e)$. Let $K_i^H = K_i \cap C$. We find partners of $K_i^H$ in $H$ by the matching $M$, formally by considering $K_i^H = K_i^H \cap V(M)$, to which the matching associates a set $H_i^M \subseteq H$ with $|K_i^H| = |H_i^M|$. Let $H = H \cap I = H \cap (K_i \cup K_e)$. These are the kings in the so-called head $H$. Let $G = H \cap G_e$. These are the external gardens in the head. The corresponding kings (which are in $N[H]$) are denoted by $K_H$.

Clearly, $(I \setminus G) \cap H_i^M = \emptyset$, as well as $I \cap G = \emptyset$. $K(H) := K_i^H \cup I \cup K_H$ comprise the kings that interfere with $H \cup C$. Interfere means that either a king is a vertex in the head, has its garden (private neighbor) in the head or the internal kings situated in the crown $C$.

Moreover, $|K(H)| = |K_i^H \cup I \cup K_H| = |K_i^H \cup I \cup K_H| + |K_i^H \setminus K_i^H| \leq |H_i^M| + |I \cup K_H| + |K_i^H \setminus K_i^H| \leq |H| + |K_i^H \setminus K_i^H| \leq |C|$. Observe that every vertex in $v \in K_i^H$ has its distinct partner in $u \in H$ such that also $u \in W$. The partner $u$ can be found via the matching $M$. Now note that $(I \setminus K(H)) \cup C$ gives another irredundant set not smaller than $I$.

It remains to show that we can always find a large crown in $G = (V, E)$ if $|V| > 3k$.

Let $L$ be a maximal matching in $G$. We claim that if $|L| > k$, then we can safely answer NO. Assume that $I$ is a maximum irredundant set in $G$. Let $I = K_i \cup K_e$ be an arbitrary partition of $I$ into internal and external kings. Let $G_e \in V \setminus I$ be an arbitrary set that can serve as a set of gardens for $K_e$. Let $W = V \setminus (I \cup G_e)$. In general, $W \cap V(L)$ cause no trouble for the following counting argument. More formally, let us assign a weight $w(x) = 1$ to such wilderness vertices $x \in W$. 

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If \( x \in K_i \), then let \( w(x) := 0 \). Observe that the vertex \( y \) matched to \( x \) by \( L \) lies in \( W \). So, consider \( x \in (K_e \cup G_e) \). Let us assign a weight of \( \frac{1}{4} \) to each such vertex. Notice that \( |V| - |I| = \sum_{x \in V} w(x) \). Moreover, \( |L| \leq \sum_{x \in V(L)} w(x) \) according to the fact for every \( \{u, v\} \in L \) we have \( w(u) + w(v) \geq 1 \). Hence, if \( |L| > k \), then \( |V| - |I| > k \), so that we can answer \( \text{NO} \) as claimed.

Hence, \( L \) contains at most \( k \) edges if \( G \) contains an irredundant set of size at least \( n - k \). This reasoning also holds for maximal matchings that contain no \( L \)-augmenting path of length three, a technical notion introduced in [5]. The demonstration given in [5, Theorem 3] shows the claimed kernel bound.

Lemma 6.1. Let \( G = (V, E) \) and \( v \in V \) with \( \text{deg}(v) = 1 \). Then there is a maximum irredundant set \( I \) for \( G \) with \( v \in I \).

This follows immediately from the proof of Theorem 2.4, since a vertex \( v \) with only one neighbor \( u \) induces a crown \( \{(v), \{u\}, \{e\}\} \) with \( e = \{u, v\} \).

Proof of Lemma 6.2

Lemma 6.2. Let \( G = (V, E) \) be a graph and let \( K_i, K_e, G_e, W \subseteq V \) be a valid labeling of \( V \) and let \( I \subseteq V \) be an irredundant set of size \( k \) that respects \( (K_i, K_e, G_e, W) \). Then there is also an irredundant set of size \( k \) that respects \( R(K_i, K_e, G_e, W) \), where \( R \) is any of the reduction rules.

Proof. This is obviously true for \( R_1 \) and \( R_2 \), since isolated vertices can always be added to \( K_i \) without decreasing the size of an solution, and edges incident to vertices in wilderness cannot result in an invalid labeling.

\( R_3 \) is also straightforward, since there the corresponding \( v \in K_e \) needs a vertex in \( G_e \) as a neighbor, but there is only one possibility left. (Likewise for \( v \in G_e \)).

\( R_4 \) is obvious.

Proof of Lemma 3.2

Proof. The algorithm uses exhaustive branching and enumerates all possible solutions up to isomorphism, i.e., up to different colorings of connected components.

Note that in each recursive call at least one vertex is added to \( K_e \cup G_e \cup W \). Thus, this algorithm terminates after at most \( 2k \) recursive calls (see also the runtime analysis). Moreover, it can never falsely output \( \text{YES} \), as solutions are verified in Line 5.

Thus, we can assume that there is some solution irredundant set \( I \), with a corresponding set of gardens \( G \). But then, \( I \) and \( G \) imply a labeling \( (\overline{K}_i, \overline{K}_e, \overline{G}_e, \overline{W}) \) by setting \( \overline{K}_i := I \cap G \), \( \overline{K}_e := I \setminus G \), adding a unique external garden in \( N(v) \cap G \) for each \( v \in \overline{K}_e \) into \( G_e \) and setting \( \overline{W} = V \setminus (\overline{K}_i \cup \overline{K}_e \cup \overline{G}_e) \).

It remains to show inductively that Algorithm 1 returns \( \text{YES} \) if called on a labeling \( (K_i, K_e, G_e, W) \) such that \( \overline{K}_i \supseteq K_i \), \( \overline{K}_e \supseteq K_e \), \( \overline{G}_e \supseteq G_e \), and \( \overline{W} \supseteq W \).

Note that \( (\overline{K}_i, \overline{K}_e, \overline{G}_e, \overline{W}) \) can only be valid, if \( (K_i, K_e, G_e, W) \) is valid as well. Moreover, \( (K_i, K_e, \overline{K}_i, \overline{W}) \) must obviously be valid. Since \( |I| \geq |V| - k \), we have \( |K_e| + |W| \leq k \) and \( |G_e| + |W| \leq k \).

If we have \( |K_e| + |W| = k \) and \( |G_e| + |W| = k \), the algorithm obviously outputs \( \text{YES} \), if we have found \( I \). If \( |K_e| + |W| < k \) or \( |G_e| + |W| < k \), there are two possibilities:
• All vertices are labeled, and the algorithm has found $I$, which is checked in Line 5.
• Some vertex $v \in V$ is not labeled yet. If $v \in \text{Not}G$, it cannot be in $G_e$, as this would imply that $(K_e, K_e, K_e, W)$ is invalid, since $(K_e, K_e, G_e) \cup \{v\}, W)$ is invalid. Thus branching whether $v \in K_e$ or $v \in W$ yields the correct solution (and similar for $v \in \text{Not}K$).

If $v \in V$, Algorithm 1 exhaustively branches on $v$ (including the choice which vertex acts as external garden or external king), which obviously yields the correct solution.

Comments on our measure

Let $T(k, K_e, G_e, W)$ be the number of recursive calls that reach Line 5 where none of the (possibly zero) following recursive calls (in Lines 9 and 11) reach this line. This way, we do not count recursive calls that fail immediately in the first four lines. This allows us to ignore the up to $2\deg(v) + 2$ failing calls (Line 11), that only contribute a polynomial runtime factor (since they do not trigger further recursive calls).

We claim that $T(k, K_e, G_e, W) \leq \alpha^{\varphi(k, K_e, G_e, W)}$, for any $k$, $K_e$, $G_e$, $W$ and $\alpha \leq 3.841$. Then in particular, Co-MaxIR and Co-MinMaxIR can be solved in time $O^*(\alpha^{\varphi(k, 0, 0, 0)}) \leq O^*(3.841^k)$.

We prove the claim by induction over search trees for arguments $(G, k, K_e, G_e, W)$. In the following, we analyze each of the many possible cases how the algorithm branches. To get a better bound, we sometimes include subsequent calls in the estimation.

We first show that $\varphi(k, K_e, G_e, W)$ is indeed a correct measure. If $\varphi(k, K_e, G_e, W) < 0$ no recursive calls will be triggered.

Lemma 6.3. If $\varphi(k, K_e, G_e, W) < 0$ then the algorithm correctly outputs NO.

Proof. If $\varphi(k, K_e, G_e, W) < 0$ then we claim that $|K_e| + |W| + |\text{Not}G| > k$ or $|G_e| + |W| + |\text{Not}K| > k$. Assume the contrary then $|K_e| + 2|W| + |G_e| + |\text{Not}G| + |\text{Not}K| \leq 2k$. Therefore we can deduce $0.5(|K_e| + |G_e|) + |W| \leq k$ which contradicts the fact that $\varphi(k, K_e, G_e, W) < 0$.

Proof of Lemma 3.4

Proof. Let $\beta := \frac{1 + \alpha^{0.5}}{\alpha} = \alpha^{-1} + \alpha^{-0.5} < 1$. Note that the first case is used later on to overcome some bad cases. We are thus forced to analyze it very closely. More precisely, we need to guarantee that a certain number of “good” branches is executed.

Case 1. $\text{Not}G \cup \text{Not}K \neq \emptyset$.

Proof. We can assume that both $|K_e| + |W| + |\text{Not}G| \leq k$ and $|G_e| + |W| + |\text{Not}K| \leq k$ hold, because otherwise this branch fails immediately. Let $d = |\text{Not}G \cup \text{Not}K| = |\text{Not}G| + |\text{Not}K|$, where the last equivalence follows from $R_5$. Note that this implies that $\varphi(k, K_e, G_e, W) \geq (|\text{Not}G| + |\text{Not}K|)/2$.

We show $T(k, K_e, G_e, W) \leq \alpha^{\varphi(k, K_e, G_e, W)} \beta^d$ by induction over $d$. Let $v \in \text{Not}G \cup \text{Not}K$ and assume, w.l.o.g., that $v \in \text{Not}G$. 


We can assume that both recursive calls on \( v \) do reach Line 5, because otherwise we have either \( T(k, K_e, G_e, W) = T(k, K_e \cup \{ v \}, G_e, W) \leq \alpha^{\varphi(k, K_e, G_e, W) - 0.5} \) or \( T(k, K_e, G_e, W) = T(k, K_e, G_e, W \cup \{ v \}) \leq \alpha^{\varphi(k, K_e, G_e, W) - 1} \), and \( \alpha^{-0.5} \leq \beta \) as well as \( \alpha^{-1} \leq \beta \).

For \( d = 1 \), we therefore obtain by our overall induction over \( \varphi(k, K_e, G_e, W) \)
\[
T(k, K_e, G_e, W) \leq T(k, K_e \cup \{ v \}, G_e, W) + T(k, K_e, G_e, W \cup \{ v \}) \leq \alpha^{\varphi(k, K_e, G_e, W)} (\alpha^{-0.5} + \alpha^{-1}) = \alpha^{\varphi(k, K_e, G_e, W) \beta}.
\]

We can hence assume \( d > 1 \).

If both recursive calls reach Line 5, we have \( \varphi(k, K_e \cup \{ v \}, G_e, W) \geq |\text{Not} G \setminus \{ v \} \cup \text{Not} K|/2 = (d - 1)/2 \) and \( \varphi(k, K_e, G_e, W \cup \{ v \}) \geq |\text{Not} G \setminus \{ v \} \cup \text{Not} K|/2 = (d - 1)/2 \), because otherwise the condition in Line 4 is already true. If the call where \( v \in K_e \) satisfies one of the conditions in Line 5, we therefore have
\[
T(k, K_e \cup \{ v \}, G_e, W) = 1 \leq \alpha^{\varphi(k, K_e, G_e, W) \alpha^{-(d-1)/2} \alpha^{-0.5}}
\]
and
\[
T(k, K_e, G_e, W \cup \{ v \}) = 1 \leq \alpha^{\varphi(k, K_e, G_e, W) \alpha^{-(d-1)/2} \alpha^{-1}}
\]
in the call where \( v \in W \). Thus, the two recursive calls by induction over \( d \) yield the bounds
\[
T(k, K_e \cup \{ v \}, G_e, W) \leq \alpha^{\varphi(k, K_e, G_e, W) \alpha^{-0.5} \max \left\{ \alpha^{-(d-1)/2}, \beta^{-1} \right\}},
\]
and
\[
T(k, K_e, G_e, W \cup \{ v \}) \leq \alpha^{\varphi(k, K_e, G_e, W) \alpha^{-1} \max \left\{ \alpha^{-(d-1)/2}, \beta^{-d} \right\}}.
\]
But since
\[
\alpha^{-(d-1)/2} \leq \sum_{i=0}^{d-1} \left( \frac{d - 1}{i} \right) \alpha^{-(d-1)+0.5i} \leq \left( \frac{1 + \alpha^{0.5}}{\alpha} \right)^{d-1} = \beta^{-d},
\]
we obtain
\[
T(k, K_e, G_e, W) \leq T(k, K_e \cup \{ v \}, G_e, W) + (k, K_e, G_e, W \{ v \}) \leq \alpha^{\varphi(k, K_e, G_e, W)} \left( \beta^{d-1} \alpha^{-0.5} + \beta^{d-1} \alpha^{-1} \right)
= \alpha^{\varphi(k, K_e, G_e, W)} \beta^{-d-1} (\alpha^{-1} + \alpha^{-0.5}) = \alpha^{\varphi(k, K_e, G_e, W) \beta^d}.
\]

In the following, we now assume \( \text{Not} G = \text{Not} K = \emptyset \) and let \( \overrightarrow{V} = V \setminus (K_e \cup K_e \cup G_e \cup W) \) the set of yet unlabeled vertices. Also note that for all \( v \in \overrightarrow{V} \) we have that \( |N(v) \cap K_e| \leq 1 \) and \( |N(v) \cap G_e| \leq 1 \).

**Case 2.** \( v \in \overrightarrow{V} \) such that \( N(v) = \{ u \} \).

**Proof.** Let \( v \in \overrightarrow{V} \) be a vertex of degree one and let \( \{ u \} = N(v) \). By the reduction rules (removal of edges), \( u \notin W \) and by the preferred branching for Not-K and Not-G vertices, \( u \in \overrightarrow{V} \). W.l.o.g., we can assume \( N := N(u) \setminus \{ v \} \neq \emptyset \). Distinguish the following cases:
If there is $z \in N$ with $z \in \nabla$, then the $v \in G_e$ and $v \in K_e$ branches restrict $z$ and we gain (after inserting Case 1 above once):

$$T(k, K_e, G_e, W) \leq T(k, K_e, G_e, W \cup \{v\}) + T(k, K_e, G_e, W \cup \{u\}) + T(k, K_e \cup \{v\}, G_e \cup \{u\}, W) + T(k, K_e \cup \{u\}, G_e \cup \{v\}, W)$$

$$\leq \alpha^{\varphi(k, K_e, G_e, W)} (\alpha^{-1} + \alpha^{-1} + 2\alpha^{-1} \cdot \beta)$$

$$\leq \alpha^{\varphi(k, K_e, G_e, W)}.$$

If there is $z \in N \cap \text{Not} G$ (or analogously, $z \in N \cap \text{Not} K$), then in the branch where $v \in K_e$ and $u \in G_e$ we immediately obtain $z \in W$ and therefore

$$T(k, K_e, G_e, W) \leq T(k, K_e, G_e, W \cup \{v\}) + T(k, K_e, G_e, W \cup \{u\}) + T(k, K_e \cup \{v\}, G_e \cup \{u\}, W \cup \{z\}) + T(k, K_e \cup \{u\}, G_e \cup \{v\}, W)$$

$$\leq \alpha^{\varphi(k, K_e, G_e, W)} (\alpha^{-1} + \alpha^{-1} + \alpha^{-1} + \alpha^{-1})$$

$$\leq \alpha^{\varphi(k, K_e, G_e, W)}.$$

Finally, if there is $z \in N \cap K_e$ (and similarly, $z \in N \cap G_e$), then the branch $v \in K_e$ immediately fails. Therefore,

$$T(k, K_e, G_e, W) \leq T(k, K_e, G_e, W \cup \{v\}) + T(k, K_e, G_e, W \cup \{u\}) + T(k, K_e \cup \{v\}, G_e \cup \{u\}, W)$$

$$\leq \alpha^{\varphi(k, K_e, G_e, W)} (\alpha^{-1} + \alpha^{-1} + \alpha^{-1}) \leq \alpha^{\varphi(k, K_e, G_e, W)}.$$

**Case 3.** $v \in \nabla$ such that $N(v) \cap K_e = \{v_K\}$ and $N(v) \cap G_e = \{v_G\}$ for some vertices $v_K$ and $v_G$.

**Proof.** Note that the branch $v \in K_i$ does not reach Line 5, as this imposes an invalid coloring. The same holds for each recursive call where $v \in K_e$ and $u \in N(v) \setminus (K_e \cup K_i \cup W)$ except for the case where $u = v_G$ due to invalid labeling ($v$ is a $K_e$ king with two external gardens $G_e$). Similarly, the branches where $u \neq v_K$ do not reach Line 5 for the cases we set $v \in G_e$.

Furthermore, $N(v_K) \cap G_e = \emptyset$, because this would imply that $v \in \text{Not} G$. Similarly, $N(v_G) \cap K_e = \emptyset$. In particular, $v_K$ and $v_G$ are not adjacent. Moreover, $v_K$ and $v_G$ must have at least another unlabelled neighbor (maybe the same) since the instance is reduced.

If $|N(v_K) \setminus K_e| = 2$, setting $K_e = K_e \cup \{v\}$ implies that the remaining unlabeled neighbor $u_K$ of $v_K$ must be added to $G_e$ (and analogously $u_G$ in case we consider $v_G$).

Thus, if both $|N(v_K) \setminus K_e| = 2$ and $|N(v_G) \setminus G_e| = 2$, we obtain

$$T(k, K_e, G_e, W) \leq T(k, K_e \cup \{v\}, G_e \cup \{u_K\}, W) + T(k, K_e \cup \{u_G\}, G_e \cup \{v\}, W) + T(k, K_e, G_e, W \cup \{v\})$$

$$\leq \alpha^{\varphi(k, K_e, G_e, W)} (\alpha^{-1} + \alpha^{-0.5} - 0.5 + \alpha^{-0.5} - 0.5)$$

$$\leq \alpha^{\varphi(k, K_e, G_e, W)}.$$

If, w.l.o.g., $|N(v_K) \setminus K_e| = 2$ and $|N(v_G) \setminus G_e| > 2$, we only obtain

$$T(k, K_e, G_e, W) \leq \alpha^{\varphi(k, K_e, G_e, W)} (\alpha^{-1} + \alpha^{-0.5} - 0.5 + \alpha^{-0.5}).$$
However, after setting $K_e = K_e \cup \{v\}$, all other neighbors of $v_G$ cannot be kings. Since at least one of these vertices $u_1$ is not the neighbor of $v_k$, at least one vertex is added to Not$K$ in this case. The remaining unlabeled neighbor $u_2$ of $v_k$ must be added to Not$G$.

After setting $G_e = G_e \cup \{v\}$, the unique neighbor $u_2$ of $v_k$ cannot be a garden, and is thus added to Not$G$.

Combining the branch on $v$ with the branches on Not$K$ and Not$G$ in the very next recursive calls yields

$$T(k, K_e, G_e, W) \leq \alpha^\varphi(k, K_e \cup \{v\}, G_e \cup \{u_1, u_2\}, W) + \alpha^\varphi(k, K_e \cup \{u_2\}, G_e \cup \{v\}, W) + \alpha^\varphi(k, K_e, G_e \cup \{v\}, W)$$

$$\leq \alpha^\varphi(k, K_e, G_e, W) (\alpha^{-1} + \alpha^{-0.5} \beta^{-1} + (d-1)\alpha^{-1} \beta^{d-2})$$

Hence, we can assume $|N(v_k) \setminus K_e| > 2$ and $|N(v_G) \setminus G_e| > 2$. But then, we gain at least two vertices in Not$G$ or Not$K$ whenever $v \notin W$. This implies (analogously to Case 1)

$$T(k, K_e, G_e, W) \leq \alpha^\varphi(k, K_e, G_e, W) (\alpha^{-1} + \frac{2}{i} \alpha^{-0.5 - 0.5 - (2-1)} + \frac{2}{i} \alpha^{-0.5 - 0.5 - (2-1)})$$

Proof. Let $d := \deg(v)$. Again, note that the branch $v \in K_i$ does not reach Line 5, and whenever we branch $v \in G_e$, the same holds unless the respective $u = v_k$.

For the $v \in K_e$ branch, however, we need to test all possible external gardens, which are $d-1$ branches. Whenever we branch $v \in K_e$ (or $v \in G_e$), the $d-1$ vertices in $N(v) \setminus \{v_k, u\}$ (the $d-1$ vertices in $N(v) \setminus \{v_k\}$ become Not$G$ (Not$K$) vertices in the very next branch. Branching on these vertices will give us a bonus to overcome the poor branching on $v$. It should be noted that none of these branches reaches Line 5, if $|\text{Not}G \cup \text{Not}K|$ becomes large, i.e., $|\text{Not}G \cup \text{Not}K|/2 > \varphi(k, K_e, G_e, W)$. Using the bound of Case 1, we thus obtain

$$T(k, K_e, G_e, W) \leq T(k, K_e, G_e, W \cup \{v\}) + T(k, K_e \cup \{v\}, G_e \cup \{v\}, W) \leq \alpha^\varphi(k, K_e, G_e, W) (\alpha^{-1} + \alpha^{-0.5} \beta^{-1} + (d-1)\alpha^{-1} \beta^{d-2})$$

for any $d \geq 2$. Since

$$f(d) := \alpha^{-1} + \alpha^{-0.5} \beta^{-1} + (d-1)\alpha^{-1} \beta^{d-2}$$

is strictly decreasing for $d \geq 4$, and

$$f(d) = \begin{cases} 0.9138880316045346 & d = 2 \\ 0.9645844017875586 & d = 3 \\ 0.9576263068932915 & d = 4 \end{cases}$$
we obtain
\[ T(k, \mathcal{K}_e, G_e, \mathcal{W}) \leq \alpha^{\varphi(k, \mathcal{K}_e, G_e, \mathcal{W})}. \]

**Case 5.** The case \( N(v) \cap G_e = \{v_G\} \) and \( N(v) \cap \mathcal{K}_e = \emptyset \) for some \( v_G \) is analogous to Case 4.

**Case 6.** \( v \in \mathcal{V} \) of maximum degree, such that \( N(v) \subseteq \mathcal{V} \).

**Proof.** Here, the \( \mathcal{K}_i \) branch can reach Line 5, but enforces \( N(v) \subseteq \mathcal{W} \).

Just as in the previous case, if \( v \) becomes a king with external garden, the branching “guesses” where the corresponding garden is (the same holds for the garden branch). Note that \( N(u) \subseteq \mathcal{V} \) as well, since otherwise the algorithm would branch on \( u \) instead.

We obtain the general recurrence
\[
T(k, \mathcal{K}_e, G_e, \mathcal{W}) \leq T(k, \mathcal{K}_e, G_e, \mathcal{W} \cup \{v\}) + T(k, \mathcal{K}_e, G_e, \mathcal{W} \cup N(v)) + \sum_{u \in N(v)} T(k, \mathcal{K}_e \cup \{u\}, G_e \cup \{u\}, \mathcal{W}) + \sum_{u \in N(v)} T(k, \mathcal{K}_e \cup \{u\}, G_e \cup \{v\}, \mathcal{W})
\]

In the branches where \( v \in \mathcal{K}_e \) and some \( u \in N(v) \) becomes its unique external garden, we also restrict the further possibilities of all vertices in \( N\{v, u\} \): Nodes in \( N(v) \setminus N[u] \) cannot become external gardens, vertices in \( N(u) \setminus N[v] \) cannot become kings, and thus in particular all the vertices in \( N(v) \cap N(u) \) must become wilderness. For each \( u \in N(v) \), we let \( S_u := N(v) \cap N(u) \) and \( T_{u} := N\{v, u\} \setminus (N(v) \cap N(u)) \). We thus obtain by the induction hypothesis (and inserting Case 1),
\[
T(k, \mathcal{K}_e \cup \{v\}, G_e \cup \{u\}, \mathcal{W}) \leq \alpha^{\varphi(k, \mathcal{K}_e, G_e, \mathcal{W})} \cdot \alpha^{-|S_u|} \beta^{|T_u|}
\]

Note that \(|T_u| + 2|S_u| = \deg(v) + \deg(u) - 2\) and \( \alpha^{-|S_u|} \leq \beta^{|T_u|} \) implies
\[
\alpha^{-1} \cdot \beta^{|T_u|} \leq \alpha^{-1} \cdot \beta^{\deg(u) + \deg(v) - 2}.
\]

Since the case \( v \in G_e \) is similar and since \( \deg(u) \geq 2 \), we obtain
\[
T(k, \mathcal{K}_e, G_e, \mathcal{W}) \leq T(k, \mathcal{K}_e, G_e, \mathcal{W} \cup \{v\}) + T(k, \mathcal{K}_e, G_e, \mathcal{W} \cup N(v)) + \sum_{u \in N(v)} T(k, \mathcal{K}_e \cup \{v\}, G_e \cup \{u\}, \mathcal{W}) + \sum_{u \in N(v)} T(k, \mathcal{K}_e \cup \{u\}, G_e \cup \{v\}, \mathcal{W})
\]
\[
\leq \alpha^{\varphi(k, \mathcal{K}_e, G_e, \mathcal{W})} \left( \alpha^{-1} + \alpha^{-d} + 2 \sum_{u \in N(v)} \alpha^{-1} \cdot \beta^{\deg(u) + \deg(v) - 2} \right)
\]
\[
\leq \alpha^{\varphi(k, \mathcal{K}_e, G_e, \mathcal{W})} \left( \alpha^{-1} + \alpha^{-\deg(v)} + 2 \cdot \deg(v) \cdot \alpha^{-1} \cdot \beta^{\deg(v)} \right).
\]

Finally, we have
\[
1 \geq \alpha^{-1} + \alpha^{-\deg(v)} + 2 \cdot \deg(v) \cdot \alpha^{-1} \cdot \beta^{\deg(v)}
\]
because \( f(d) := \alpha^{-1} + \alpha^{-d} + 2 \cdot d \cdot \alpha^{-1} \cdot \beta^{d} \) is strictly decreasing for \( d \geq 4 \) and \( f(2) \leq 0.947 \), \( f(3) \leq 0.993 \), and \( f(4) \leq 0.9994 \). We therefore obtain the desired bound
\[
\alpha^{\varphi(k, \mathcal{K}_e, G_e, \mathcal{W})} \left( \alpha^{-1} + \alpha^{-\deg(v)} + 2 \cdot \deg(v) \cdot \alpha^{-1} \cdot \beta^{\deg(v)} \right) \leq \alpha^{\varphi(k, \mathcal{K}_e, G_e, \mathcal{W})}
\]
for \( \alpha \geq 3.841 \).

This finishes the proof.
Proof of Corollary 3.6

Proof. First enumerate all vertex subsets of maximum size $0.485252n$. Then for all $1 \leq k \leq 0.514748n$ invoke the algorithm for Co-MaxIR.

Proof of Lemma 4.1

Proof. The correctness should be clear for most of the rules; many can be seen as reformulations of those for the simple first algorithm. As an example, we discuss a reasoning for Rule 8: Discuss two neighbors $u,v \in \text{Not}K$. If later on $u$ is put into $G_e$ (with still $v \in \text{Not}K$), then Rule 6 would have triggered. If $u$ is put into $W$, then Rule 7 would have deleted the edge. Hence, we can delete it right away.

Since $0 \leq \omega_n \leq 0.5$ and $\omega_n + \omega_l \leq 1$, the reduction rules do not increase the measure.

Proof of Lemma 4.3

Proof. Clearly the statement holds for the initial input graph. The only reduction rule which could create such a situation is Rule 11. But here the two vertices $u,v$ will be immediately inactivated. In each line where recursive calls are made it is easily checked if such a situation as described could occur. If so, the affected pair $u,v$ is removed from $V_a$.

Proof of Lemma 4.4

Proof. Consider $z \in N(v)$, where $v \in G_e$. $N(v) \not\subseteq U \cup K_e \cup \text{Not}K$ is ruled out by reduction rules and the previous lemma: (a) $z \in G_e$ violates Rule 5. (b) $z \in K_e$ violates the invariant shown in Lemma 4.3. (c) $z \in \text{Not}K$ violates Rule 6. (d) $z \in W$ violates Rule 7.

Details of Theorem 4.7 when all active vertices are in $U \cup G_e \cup K_e$, with $G_{va} \cup K_{va} \neq \emptyset$.

(a) Assume $\deg(v_G) \geq 3$ and $\deg(v_K) \geq 3$. Then, the worst-case recursion given in Table 2 arises. The two branches $v \in K_e$ and $v \in G_e$ are completely symmetric: e.g., if $v \in K_e$, then it will be paired with $v_G$ (by inactivating both of them), and Reduction Rule 13 will put all (at least two) neighbors of $N(v_G) \setminus G_e$ into $\text{Not}G$, and symmetrically all neighbors of $N(v_K) \setminus K_e$ into $\text{Not}K$ in the other branch.

(b) Assume that $v_G$ and $v_K$ satisfy $\deg(v_G) = 2$ and $\deg(v_K) = 2$. Then, the recursion given in Table 2 arises. Assume first that $v_G$ and $v_K$ do not share another neighbor. Then, when $v$ is put into $K_e$, then $v$ is paired up with $v_G$. Since the degree of $v_K$ will then drop to one by Reduction Rule 5, $v_K$ must have its garden on the only remaining neighbor. This will be achieved with Reduction Rule 11. Therefore, all in all the measure decreases by $2 \cdot (1 - \omega_l)$; moreover, we turn at least one neighbor of $v_G$ into a NotK-vertex.

Secondly, it could be that $v_G$ and $v_K$ share one more neighbor, i.e., $N(v_G) = N(v_K) = \{q,v\}$. If $q$ has any further neighbor $x$, then $x \in U$ (confer Reduction Rule 12), or these vertices form a small component that is handled, since it has maximum degree two. The reasoning for the measure having guaranteed the existence of $x \in U$ is similar to the first case.
(c) We now assume that \( N(v) = \{v_K, v_G\} \), \( \deg(v_G) = 2 \), and \( \deg(v_K) \geq 3 \). Then, the worst-case recursion given in Table 2 arises. This can be seen by combining the arguments given for the preceding two cases.

Details when all active vertices are in \( U = V_a \).

There are two cases to be considered: (a) either \( v \) has a neighbor \( u \) of degree two or (b) this is not the case.

The analysis of (a) yields the recursion given in Table 2. The first term can be explained by considering the case \( v \in W \). In that case, Reduction Rules 7 and 9 trigger and yield the required measure change. Notice that all vertices have minimum degree of two at this stage due to Reduction Rule 11. In the case where \( v \in K_i \), the neighbors of \( v \) are added to \( W \), yielding the second term.

The last two terms are explained as follows: We simply consider all possibilities of putting \( v \in K_e \cup G_e \) and looking for its partner in the neighborhood of \( v \). Once paired up with some \( u \in N(v) \), the other neighbors of \( \{u, v\} \) will be placed into Not\( K \) or Not\( G \), respectively.

In case (b), we obtain the recursion given in Table 2.

This is seen by a slightly simplified but similar argument to what is written above. Notice that we can assume that \( \deg(v) \geq 4 \), since the case when \( \deg(v) = 3 \) (excluding degree-1 and degree-2 vertices in \( N(v) \) that are handled either by Reduction Rule 9 or by the previous case) will imply that the graph \( G[V_a] \) is 3-regular due to our preference to branching on high-degree vertices. However, this can happen at most once in each path of the recursion, so that we can neglect it.

Further Discussion of the Branching Cases

We shall now show that out of the infinite number of recurrences that we derived for Co-MAXIR, actually only finitely many need to be considered.

Case (3)#j: For this case we derived a recurrence of the following form where \( i := \deg(v) \).

\[
T_{\varphi}(\mu, h) \leq i \cdot T_{\varphi}(\mu - ((1 - \omega_\ell) + (i + 1) \cdot \omega_n), h - 1) \\
\leq i \cdot 3.069^{k - ((1 - \omega_\ell) + (i + 1) \cdot \omega_n)} \\
= 3.069^k \cdot f(i)
\]

The first inequality should follow by induction on the height \( h \) of the search tree, while this entails \( T_{\varphi}(\mu, h) \leq 3.069^k \) if \( f(i) \leq 1 \) for all \( i \). We now discuss \( f(i) = i \cdot 3.069^{-(1 - \omega_\ell) + (i + 1) \cdot \omega_n} \).

We had a closer look at the behavior of this function \( f(i) \). Its derivative with respect to \( i \) is:

\[
(3.069)^{(-0.5199 - i \cdot 0.2455)} - 0.2455 \cdot i \cdot (3.069)^{(-0.5199 - i \cdot 0.2455) \log(3.069)}
\]

The zero of this expression is at

\[ z = 2000/491/ \log(3.069) \approx 3.625 \]
We validated that this is indeed a saddle point of $f$, and $f$ is strictly decreasing from there on, yielding a value $f(z) < 0.8$. Hence, it is enough to look into all recursions up to $i = 4$ in this case as also $f(3) < 0.8$.

**Case 4a**: We further have to look into:

$$T_\varphi(\mu, h) \leq T_\varphi(\mu - i, h - 1) + 2 \cdot i \cdot T_\varphi(\mu - (1 + i \cdot \omega_n), h - 1) + T_\varphi(\mu - 2, h - 1)$$

Hence, we have to discuss for $i \geq 3$

$$f(i) = 3.069^{-i} + 2 \cdot i \cdot 3.069^{-(1+i\cdot\omega_n)} + 3.069^{-2}.$$

We find that $f(3) < 1$ and $f(4) < 1$ and that a saddle-point of $f(i)$ is between 3 and 4 (namely at 3.2) and $f$ is strictly decreasing from 4 on. So, from $i = 4$ on, all values are strictly below one.

**Case 4b**: Finally, we investigate

$$T_\varphi(\mu, h) \leq T_\varphi(\mu - i, h - 1) + 2 \cdot i \cdot T_\varphi(\mu - (1 + (i + 1) \cdot \omega_n), h - 1) + T_\varphi(\mu - 1, h - 1)$$

So, investigate for $i \geq 4$ the function

$$f(i) = 3.069^{-i} + 2 \cdot i \cdot 3.069^{-(1+(i+1)\cdot\omega_n)} + 3.069^{-1}.$$

Again, we found that the saddle-point of $f$ is between 3 and 4 and that $f$ is strictly decreasing from 4 on, with $f(4) < 0.998$. 

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