Signatures of Fractional Exclusion Statistics in the Spectroscopy of Quantum Hall Droplets

Nigel R. Cooper
T.C.M. Group, Cavendish Laboratory, University of Cambridge,
J. J. Thomson Ave., Cambridge CB3 0HE, United Kingdom

Steven H. Simon
Rudolf Peierls Centre for Theoretical Physics, University of Oxford,
1 Keble Road, Oxford OX1 3NP, United Kingdom

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We show how spectroscopic experiments on a small Laughlin droplet of rotating bosons can directly demonstrate Haldane fractional exclusion statistics of quasihole excitations. The characteristic signatures appear in the single-particle excitation spectrum. We show that the transitions are governed by a “many-body selection rule” which allows one to relate the number of allowed transitions to the number of quasihole states on a finite geometry. We illustrate the theory with numerically exact simulations of small numbers of particles.

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One of the most dramatic features of strongly correlated phases is the emergence of quasiparticle excitations with unconventional quantum statistics. The archetypal example is the fractional, “anyonic”, quantum statistics predicted for the quasiparticles of the fractional quantum Hall phases [1, 2]. While experiments on semiconductor devices have shown that these quasiparticles have fractional charges [3–5], a direct observation of the fractional statistics has remained lacking.

In this Letter we show how precision spectroscopy measurements of rotating droplets of ultracold atoms could be used to demonstrate the Haldane fractional exclusion statistics [6] of quasiholes in the Laughlin state of bosons. By involving only spectroscopic signatures of the rotating droplet, our proposal plays to the strengths of atomic physics experiments. We show that evidence of the fractional exclusion statistics appears in counting the numbers of lines in the radio-frequency (RF) absorption spectrum. In this sense, the method is conceptually similar to classic evidence of quantum statistics, as appearing in rotational levels of homonuclear diatomic molecules (e.g. the Fermi statistics of the proton causing the rotational levels of H₂ to depend on whether the spins of the nuclei are in singlet or triplet state). Our method differs substantially from proposals to measure the fractional braiding statistics of quasiholes [7–9], notably by not requiring local time-dependent potentials for the adiabatic braiding statistics of quasiholes, in which a “many-body selection rule” which allows one to relate the number of allowed transitions to the number of quasihole states on a finite geometry.

In addition, we shall consider a weak quartic potential — weak compared to both \( \hbar \omega_0 \) and \( V_0 \) — for reasons to be described below. Specifically, we shall consider an initial state of \( N_i \) atoms which has been spun up to the angular momentum \( L_i = N_i(N_i - 1) \). Then, for the case of contact repulsive interactions relevant in typical cold gas experiments, the groundstate is the (exact) \( \nu = 1/2 \) Laughlin state. Furthermore, for the case of contact interactions, the quasihole excitations of these Laughlin droplets are non-interacting: this will allow us to find evidence of the fractional exclusion statistics even in small systems of \( N_i \lesssim 10 \) atoms. Experimental protocols to generate this initial state for small numbers of atoms have been identified [10, 11], and experimental work on driven lattices [12] has investigated the properties of rapid rotation on multi-droplet systems. We shall focus on the properties of a single droplet, and the spectroscopic signatures we seek shall require single-atom imaging; such conditions are likely to require technologies developed in ultracold gas microscopes [13, 14].

Now consider making an RF excitation of a single atom from internal state \( \uparrow \) into an internal state \( \downarrow \) which does not interact with the initial \( \uparrow \) atoms. For hyperfine states, such situations can be found by tuning to the zero of a Feshbach resonance. The promoted atom can carry away an angular momentum, \( m_{i\uparrow} \), in the range \( 0 \leq m_{i\uparrow} \leq m_{\text{max}} \), leaving the \( N_i = N_i - 1 \) atoms with angular momentum \( L_i = L_i - m_{i\uparrow} \) in the range \( L_i - m_{\text{max}} \leq L_i \leq L_{i\downarrow} \). (The upper limit \( m_{\text{max}} = 2(N_i - 1) = 2N_i \) is the highest angular momentum carried by any one particle in the initial Laughlin state of \( N_i \) particles.) In the following, we shall imagine that the transition spectrum can be resolved into components labelled by this final angular momentum \( L_i \). In principle this could be done by measuring the final angular momentum of the excited \( \downarrow \) atom. However, note that, in general, the change in in-
The counting of the zero interaction energy states can be obtained from a simple picture based on the generalized clustering principle\cite{18}, which identifies the “root states” of the exact quantum states. These root states are single Fock states, with definite particle number \( n_m \) in orbitals \( m = 0, 1, 2 \ldots \). For the \( \nu = 1/2 \) Laughlin state, the clustering principle is that no two particles can be in orbitals, \( m \) and \( m' \), with \( |m - m'| < 2 \). The resulting root configurations for \( N_f = 5 \) are shown in Fig. 2 for small total angular momentum. Note that (nonorthogonal) basis vectors of the zero interaction energy space can be made from the root Fock state superposed with daughter Fock states that do not generally obey the clustering principle. The daughter states have the interesting feature that they are always “squeezed” from the root state\cite{13}, meaning that if we write out a string to represent the occupancies of single particle orbitals \( m \), squeezing always numerically reduces the value of this string while preserving the total number of particles as well as the total angular momentum, \( L \). For example if we write the root state string \( 101001 \) to mean we have filled the orbitals \( m = 5, m = 3, m = 0 \) each once, a daughter state squeezed from this would be the string \( 100110 \) which is numerically less than \( 101001 \).

\[
L_f \quad \# N_f, L_f \quad \# \text{allowed} \quad L_f^2 = L_f - N_f^2
\]

| \( L_f \) | \( \# N_f, L_f \) | \( \# \text{allowed} \) | \( L_f^2 = L_f - N_f^2 \) |
|---|---|---|---|
| 20 | 1, 1 | -5 |
| 21 | 1, 1 | -4 |
| 22 | 2, 2 | -3 |
| 23 | 3, 2 | -2 |
| 24 | 5, 3 | -1 |
| 25 | 7, 3 | 0 |
| 26 | 10, 3 | 1 |
| 27 | 13, 2 | 2 |
| 28 | 18, 2 | 3 |
| 29 | 23, 1 | 4 |
| 30 | 30, 1 | 5 |
| 31 | 37, 0 | |
| \vdots | \vdots | \vdots | \vdots |

TABLE I: The number of zero-energy final states of \( N_f = 5 \) contact interacting bosons \( \# N_f = 5, L_f \), and the number of these states with allowed transitions, \( \# \text{allowed} \), following the many-body selection rule described in the text. The final column gives the \( z \)-component of the angular momentum of the effective spherical system described in the text. No states are allowed for \( L_f > 30 \) and here there is no meaningful value for \( L_f^2 \).

If there were no quartic potential, all of the final states with zero interaction energy at a given \( L_f \) would be at exactly the same energy (since interaction energy is zero). The quartic potential splits these degenerate states, allowing separate transitions to be resolved. Fig. 3 shows the splitting for \( N_f = 5 \). For weak quartic potential (compared to \( V_0 \)) this splitting does not obscure the many-body gap separating these zero-interaction energy states from the high energy states. Therefore, by detecting the number of low-energy spectral lines (i.e. those below the gap, arising from many-body repulsion, in Fig. 3) as a...
are very significant restrictions on such matrix elements. This leads to a strong “many-body selection rule” on the RF transitions in Laughlin clusters. Specifically, we find that strong transitions — which we shall refer to as “allowed” transitions — exist only to those states whose root configurations have all particles within the $m = 0 \rightarrow 2N_f$ orbitals.

The origin of the selection rule lies in the fact that the initial Laughlin state, with $N_i = N_f + 1$ particles, is a state in which all particles are in these $m = 0 \ldots 2N_f$ orbitals (see Fig. 2). Hence, removal of a particle from the Laughlin state cannot produce a Fock state with a particle in orbital $m > 2N_f$. Since we find, from numerical diagonalizations, that the root configuration has a very large weight in the exact eigenstates, matrix elements are large only with those states whose root configurations have all particles within $0 \leq m_f \leq 2N_f$.

The allowed states are a well-defined subset of the zero-interaction energy states. This subset can be readily found from Fig. 2 by retaining only those root states for which all particles lie to the left of the red dashed line. For this case of $N_f = 5$ particles, the number of states that have allowed transitions is shown in Table I.

The set of allowed states has a simple interpretation: it is the set of zero-energy states that can be formed for contact interacting bosons on a system of fixed area with 2 quasiholes (e.g., a sphere with flux $N_{\phi} = 2N_f$). This follows from the fact that the number of orbitals over which the $N_i$ particles can be distributed is $2N_f + 1$, while the (unique) Laughlin groundstate is formed when the number of states is $2N_f - 1$. These $(2N_f + 1) - (2N_f - 1) = 2$ excess orbitals can be viewed as two quasiholes in the Laughlin groundstate.

Each can be placed in $d_{\text{qh}} = N_f + 1$
possible locations, with respect to the $N_f$ particles. For two quasiholes, the total number of states is simply given by the number of ways to put two identical quasiholes in $d_{\text{qh}} = N_f + 1$ states: $\# \text{allowed} = \sum_{L_f} \# \text{allowed} = \frac{1}{2}(N_f + 1)(N_f + 2)$. This can be resolved into states of fixed angular momentum $L_f$. Again we exploit the equivalence to the states of $N_i$ particles on a sphere of flux $N_{\phi} = 2N_i$: with $L^*_z = -N_i \ldots N_i$ replacing $m = 0 \ldots 2N_i$, such that the $z$-projection of total angular momentum on the sphere is $L^*_z = L_f - N^*_f$. Since the Laughlin groundstate for $N_f$ particles is at $N_{\phi} = 2(N_f - 1)$, the flux $N_{\phi} = 2N_f$ corresponds to the addition of $n = 2$ quasiholes. It is known that the total number of zero energy states for $n$ quasiholes is given by the binomial coefficient $C(N_f + n, n)$ (i.e., choose $n$ from $N_f + n$). For $n = 2$ these states can be indexed as states of total angular momentum $L^*_z = 1, 3, 5, \ldots N_i$ ($N_i$ odd), or $L^*_z = 0, 2, 4, \ldots N_i$ ($N_i$ even), which is consistent with the counting in Table I.

The relation (1), together with the fact that for a system of fixed area, the removal of a single particle
works as well as it does (with the matrix elements for
experimental goal will be to count the transitions with
the counting rules described above (see Table I). The
follows the pattern (1,1,2,2,3,3,3,2,2,1,1) expected from
of the allowed states (marked by red circles in Fig. 4)
imperceptible on the linear scale of Fig. 4. The counting
ement smaller by at least two orders of magnitude, and
between allowed states, with matrix element squared of
values of
agonalization for the case of
we show the matrix elements (squared) for all excited
ments numerically for
have computed the many-body states and matrix ele-
describe the many-body selection rule. How well does this many-
detection of these exclusion statistics.
\[ \Delta N = -1 \] corresponds to the creation of two quasi-
holes \( \Delta n_{\text{qh}} = -2\Delta N \) fixes the Haldane exclusion statis-
tics of the quasiholes. These generalized exclusion statistics relate the change in dimension of the Hilbert space \( \Delta d \) available to a quasiparticle to the change in number of quasiparticles \( \Delta n \) via \( \Delta d = -g\Delta n \) with \(-g\) being the exclusion statistic parameter. For bosons or fermions, \( g = 0 \) or 1 respectively. Here we have \( \Delta d = -\frac{1}{2}\Delta n \) showing that \( g = \frac{1}{2} \), indicative of “semionic” statistics of the quasiholes. Thus, by counting the number of allowed transitions in the RF spectra, one obtains direct evidence for the counting formula \([1]\). As described above, the dependence of the counting formulas on \( N_f \) lies at the heart of the fractional exclusion statistics for the quasiholes: thus, by detecting the multiplicities for different \( N_f \) (i.e. different initial \( N_f = N_f + 1 \)), amounts to a direct detection of these exclusion statistics.

The preceding discussion is based on the existence of the many-body selection rule. How well does this many-body selection rule apply in practice? To test this, we have computed the many-body states and matrix elements numerically for \( N_f = 2 \ldots 10 \) particles. In Fig. 4 we show the matrix elements (squared) for all excited states at each final momentum \( L_f \), computed by exact diagonalization for the case of \( N_f = 5 \). (Results for other values of \( N_f \) are consistent.) There is a clear separation between allowed states, with matrix element squared of order one, and forbidden states, with square matrix element smaller by at least two orders of magnitude, and imperceptible on the linear scale of Fig. 4. The counting of the allowed states (marked by red circles in Fig. 4) follows the pattern (1,1,2,2,3,3,3,2,2,1,1) expected from the counting rules described above (see Table I). The experimental goal will be to count the transitions with nonnegligible weights in each angular momentum sector.

One might wonder why the many-body selection rule works as well as it does (with the matrix elements for
states that violate the selection rule being suppressed by factors of 100 or more). Our above argument that each eigenstate contains a large component of its root Fock state turns out not to be a sufficient explanation since there is substantial mixing with daughter states\([15]\). Let us examine a more general potential \( V \sim r^3 \) instead of the quartic potential \( \gamma = 4 \) (while keeping the potential the smallest energy scale of the system). In terms of orbital occupations \( n_m \), this yields potential energies \( \sim \sum_m m^{\gamma/2} n_m \). In the limit \( \gamma \to \infty \) this is dominated by the occupied orbitals with the largest \( m \), and the potential energies of Fock states are then ordered with the same “squeezing” relationship as described above. Beginning with the basis of zero interaction energy states described by root states and their corresponding daughters, in the \( \gamma \to \infty \) limit the exact energy eigenstates can then be found by successively orthogonalizing these wavefunctions, starting with the root state with lowest potential energy and continuing to successively higher root states. Since removing a particle from the Laughlin state must generate a superposition of wavefunctions in this zero-interaction energy space which also contains no occupied orbitals with \( m > N_f \) the selection rule becomes exact. We find numerically that this “orthogonalized root state basis” is an extremely accurate representation of the exact eigenstates even when \( \gamma = 4 \), providing some justification for why the selection rule works so well.

In summary, the spectroscopic probe removing one atom creates two quasiholes in the Laughlin cluster. Measurements of the number of allowed transitions would determine the number of these quasihole states, testing theories of the properties (degeneracy and mutual statistics) of this pair of fractionalized particles. While experiments of this kind are certainly technically challenging — requiring control of small numbers of atoms with high fidelity, and the detection of single atoms in spectroscopic probes — these are within reach of new technologies of quantum gas microscopes. Our work provides a very appealing direct link between multiplicities in RF spectra, and counting formulas for fractionalized excitations in strongly correlated many-body systems.

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