Interferometric Quantum Cascade Systems

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In this work we consider quantum cascade networks in which quantum systems are connected through unidirectional channels that can mutually interact giving rise to interference effects. In particular we show how to compute master equations for cascade systems in an arbitrary interferometric configuration by means of a collisional model. We apply our general theory to two specific examples: the first consists in two systems arranged in a Mach-Zender-like configuration; the second is a three system network where it is possible to tune the effective chiral interactions between the nodes exploiting interference effects.

I. INTRODUCTION

Quantum cascade systems (QCSs) describe those physical situations where a first party (the controller) can influence the dynamic of a second party (the idler) without being affected by the latter. The asymmetric character of these couplings originates from the presence of an environmental medium (e.g. an optical isolator or a bosonic chiral channel) which acts as mediator of the interactions and which allows for unidirectional propagation of pulses from a controller to its associated idler.

First interests in these models grew in the 80’s because of the necessity of a formalism able to take into account the reaction of a quantum system (say an atom or an electromagnetic cavity) to the light emitted by another one. In recent years there has been a revival of interest towards QCSs, due to the possibility of creating entangled states and other tasks for quantum computation, chiral optical networks, and in the managing of heat transmission; also several experimental implementations have been proposed, exploiting, for instance, nanophotonic waveguides and spin-orbit coupling.

In the QCS models studied so far, the parties composing the systems are typically assumed to be organized to form an oriented linear chain, each acting as controller for the elements that follow along the line through the mediation of a single environmental channel. Here instead we shall consider more complex configurations where several subsystems interact, unidirectionally via a network of mutually intercepting channels as shown in the left panel of Fig. 1.

In this scenario the QCS couplings while being intrinsically dissipative in nature, can be affected by interference effects which originate from the propagation of the controlling pulses along the network of connections (for instance in the case of figure, the signals from the subsystem $S_1$ split and recombine before reaching subsystem $S_3$). Also, depending on the topology of the scheme, controlling signals from different parties (say the subsystems $S_2$ and $S_3$ of the figure) can merge before reaching a given idler ($S_4$). The study of such architectures is intriguing as it widens the possibility of engineering system-bath coupling in quantum optical systems, which in turn may help in dissipatively preparing quantum many-body states of matter with important consequences in the analysis of non-equilibrium condensed matter physics problems and quantum information. Aim of the present work is to derive a mathematical framework that incorporate these phenomena in a consistent way.

![FIG. 1. Left panel: Pictorial representation of the typical QCS model we are considering here: a collection $S$ of quantum subsystems $S_1, S_2, \ldots, S_M$ (gray circles in the figure) interact unidirectionally by exchanging signals through an oriented network of environmental channels $E^{(1)}, E^{(2)}, \ldots, E^{(K)}$ which may interfere when intercepting (gray/yellow elements). Right panel: collisional model description of the scheme: the propagation of signals along the network is represented in terms of sequence of ordered collisional events involving the quantum subsystem and a collection of quantum information carriers (black circles). Interference among the signals arises from collisions between carriers associated with different connecting paths.](image-url)

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that evolve in time stroboscopically through a series of time-ordered collisions involving the various subsystems — see right panel of Fig. 1. Interference effects are also described in terms of collisions, this time involving carriers associated with different channels (e.g. the red and black carriers of the figure). Similar cascade networks could also be studied in the Heisenberg picture within the so called input-output formalism [6, 7], from which in principle a master equation could be derived using quantum stochastic calculus [6, 8]. The collisional model presented in this work allows to directly obtain the desired master equation and, being based on a simple and operational model of dissipation, naturally generates a Markovian completely positive dynamics without the necessity of introducing further hypothesis and approximations typical of other microscopic derivations.

Here is the outline of the paper: in Sec. II we review briefly the collisional approach to QCS of Ref. [31, 32] and adapt it for writing the master equation of our model. The resulting expression is then cast in standard Gorini-Kossakowski-Sudarshan-Lindblad (GKSL) form [33–36] in Sec. II B. Building from these results, in Sec. III we describe the arising of interference effects in the model, by discussing some specific examples. In particular in Sec. III A we deal with a Mach-Zender-like interferometer, showing how with a phase shift it is possible to modify the effective temperature felt by the second optical cavity. Then in Sec. III B we turn to a configuration of three cavities where we show how, by appropriately exploiting interference effects, it is possible to have a system with only first-neighbor interactions. The paper then ends with Sec. IV where we draw conclusions and give an outlook for future works, and with the Appendices where we present some technical derivations.

II. THE MODEL

In the collisional model approach [37, 38] to open quantum systems dynamics the environment is represented as a large many-body quantum system whose constituents (quantum information carriers or carriers in the following) interact with the system of interest via an ordered sequence of impulsive unitary transformations (collisional events). This yields a, time-discrete, stroboscopic evolution which can then be turned into a continuous time dynamics by properly sending to infinity the number of collisions and to zero the time interval among them while keeping constant their product. By means of collisional models it is possible to derive both Markovian [39, 40] and non-Markovian [41, 42] master equations. In this paper we take our steps from the collisional approach to QCS presented in Refs. [31, 32], generalizing it to include network configurations similar to the one presented in the left panel of Fig. 1.

To this aim we consider a system $S$ made out of $M$ (not necessarily identical) subsystems $S_1$, $S_2$, · · · , $S_M$ (e.g. $M$ optical cavities). Similarly to the scheme of Fig. 1 they are connected via a network of QCS interactions in such a way that for each $m = 1, · · · , M$, the element $S_m$ is capable of controlling all the elements $S_m'$ with $m' > m$ without being affected by their dynamics, the coupling being provided by a collection of unidirectional environmental channels $\mathcal{E}^{(1)}$, $\mathcal{E}^{(2)}$, · · · , $\mathcal{E}^{(K)}$ which intercept to form a graph. In what follows each of these channels are represented in terms of a long, ordered string of quantum carriers which act as mediators of the interactions, propagating along the network and experiencing impulsive interactions (collisional events) with the system elements as sketched on the right panel of Fig. 1. Specifically, for $k = 1, · · · , K$, the $k$-th channel $\mathcal{E}^{(k)}$ is described by the carriers $\{E_n^{(k)}; n = 1, 2, · · · \}$, the subscript $n$ indicating the order with which they start interacting with $S$. Accordingly we find it convenient to regroup these elements into sets which includes those that posses the same value of $n$ independently from the channel they belong to, e.g. the set $\mathcal{E}_1 := \{E_1^{(1)}, E_1^{(2)}, · · · , E_1^{(K)}\}$, the set $\mathcal{E}_2 := \{E_2^{(1)}, E_2^{(2)}, · · · , E_2^{(K)}\}$, and so on and so forth. This way, neglecting the time it takes from one carrier to move from one element of $S$ to the next, we can use
the label \( n \) as the discrete temporal coordinate of the model (more on this on the following paragraphs). In particular, indicating with \( \hat{U}_{S_m,E_n} \) the unitary operator associated with the collisional event that couples \( S_m \) and the carriers which enters at the \( n \)-th temporal step, i.e. the carriers of \( E_n \), the causal structure of the model is enforced by imposing that such operator should precede \( \hat{U}_{S_{m+1},E_n} \) (meaning that \( S_{m+1} \) sees \( E_n \) only after it has interacted with \( S_m \)) and \( \hat{U}_{S_m,E_{n+1}} \) (meaning that the element of \( E_n \) enters the network before those of \( E_{n+1} \)) – the relative ordering of \( \hat{U}_{S_{m+1},E_n} \) and \( \hat{U}_{S_m,E_{n+1}} \), being instead irrelevant as they act on different systems and hence commute. The unitaries \( \hat{U}_{S_m,E_n} \)’s trigger the dissipative evolution of \( S \) which is responsible for the QCS dynamics. In our model they are interweaved with Completely Positive and Trace preserving (CPT) super-operators \([14]\) acting on the quantum carriers only, which describe the propagation of signals along the channels and (possibly) their mutual interactions. In particular, in what follows we shall use the symbol \( \mathcal{M}_{E_n}^{(m)} \) to indicate the CPT map which acts on the carriers of the set \( E_n \) after the collisional event that couples them with \( S_m \) and before the one that instead couples them with \( S_{m+1} \) – see Fig. 2. A convenient way to express the resulting evolution is obtained by introducing the density matrix \( \hat{\rho}(n) \) which describes the joint state of \( S \) and of the first \( n \)-th carriers of all channels (i.e. the carriers belonging to the sets \( E_1, E_2, \ldots, E_n \)) after they have interacted. From the above construction, the relation between such state and its evolved counterpart \( \hat{\rho}(n+1) \) can then be expressed as

\[
\hat{\rho}(n+1) = \mathcal{C}_S \hat{\rho}(n) \otimes \hat{\eta}_{E_{n+1}},
\]

where \( \hat{\eta}_{E_{n+1}} \) indicates the input state of the elements of \( E_{n+1} \) when they enter the network, while \( \mathcal{C}_S \) is the super-operator associated with the collisional events they participate. Explicitly, using the short hand notation

\[
\prod_{m=1}^M \mathcal{A}^{(m)} = \mathcal{A}^{(M)} \mathcal{A}^{(M-1)} \cdots \mathcal{A}^{(2)} \mathcal{A}^{(1)}
\]

(2)

to represent the ordered product of the symbols \( \{ \mathcal{A}^{(1)}, \mathcal{A}^{(2)}, \ldots, \mathcal{A}^{(M)} \} \) this is given by

\[
\mathcal{C}_S \mathcal{E}_n = \prod_{m=1}^M [\mathcal{M}_{E_n}^{(m)} \circ \mathcal{U}_{S_m,E_n}],
\]

(3)

where “\( \circ \)” indicates composition of super-operators and where for each \( m = 1, \ldots, M \) the symbol \( \mathcal{U}_{S_m,E_n} \) indicates the super-operator counterpart of the unitary transformation \( \hat{U}_{S_m,E_n} \), i.e.

\[
\mathcal{U}_{S_m,E_n} = \hat{U}_{S_m,E_n} \mathcal{U}_{S_m,E_n}^\dagger.
\]

(4)

Few remarks are mandatory at this point:

i) the density matrices \( \hat{\rho}(n) \) and \( \hat{\rho}(n+1) \) operate on different spaces (indeed \( \hat{\rho}(n+1) \) operates also to the carriers of the set \( E_{n+1} \) while \( \hat{\rho}(n) \) does not). What is relevant for us is the fact that by taking the partial trace over the carriers they give us the temporal evolution of the system of interest at the various step of the process. In particular

\[
\hat{\rho}(n) := \left\langle \hat{\rho}(n) \right\rangle_{E} = \text{Tr}_E [\hat{\rho}(n)],
\]

(5)

is the joint state of the subsystems \( S \) at the \( n \)-th time step;

ii) as already mentioned in our analysis the time it takes for a carrier to move from one collision to the next is assumed to be negligible, only the causal ordering of these events being preserved. Accordingly in Fig. 2 time flows from left to right for all the \( S_j \) synchronously. This assumption is introduced because, differently to the case of a simple linear chain of cascaded systems \([7]\), when dealing with multiple channels one cannot eliminate the delay time by simply shifting the time origin of each subsystem. Actually, significant delay times can give rise to non-Markovian effects \([42]\), whose study goes beyond the goal of the present work.

iii) in writing Eq. (1) we are implicitly assuming that the input state of the carriers factorizes with respect to the grouping \( E_1, E_2, \ldots, \) i.e. no correlations are admitted among carriers which enters the scheme at different time steps. Yet, at this level, the model still admits the possibility of correlations among carriers of different channels. In what follows we shall however enforce a further constraint that limits the choices of the input \( \hat{\eta}_{E_n} \), see next point and Eq. (12) below.

iv) in the original QCS model of Fig. 1 the unidirectional channels \( E^{(1)}, E^{(2)} \) and \( E^{(M)} \) form a stationary medium which contributes to the dynamics only by allowing signals from one subsystem to propagate to the next one (in other words in the absence of the interactions with the elements of \( S \) they will not present any temporal evolution). To enforce this special character in the collisional model we require it to be translationally invariant with respect to the index \( n \), e.g. imposing that all the input states \( \hat{\eta}_{E_1}, \hat{\eta}_{E_2}, \ldots, \hat{\eta}_{E_n} \) of the carriers sets \( E_1, E_2, \ldots, E_n \) coincide, and that for given \( m \) the unitary couplings \( \mathcal{U}_{S_m,E_n} \) and the maps \( \mathcal{M}_{E_n}^{(m)} \) should be independent from \( n \). This hypothesis can however be relaxed \([32]\) with the condition that the change in the coupling is slow compared to the characteristic time scale of the systems \( S_m \).

A. The continuous time limit

By solving the recursive equation (1) and taking the partial trace as in Eq. (1) one obtains a collection of density matrices \( \hat{\rho}(0), \hat{\rho}(1), \ldots, \hat{\rho}(n) \), which provides an effective description of the temporal evolution of the joint
state of the subsystems $S_1$, $S_2$, $\cdots$, $S_M$ in the presence of a collection of quantum carriers that connects them through a network of unidirectional channels. Such stroboscopic representation of the dynamics can be turned into a continuous time description by taking a proper limit in which the number of collisions per second experienced by the element of $S$ goes to infinity $[31, 32]$. Accordingly we write the interaction unitaries as

$$
\hat{U}_{S_m, E_n} = \exp\left[-ig \sum_{k=1}^{K} \hat{H}_{S_m, E_n} \Delta t\right],
$$

(6)

where $g$ is a coupling constant that we shall use to gauge the intensity of the system-carrier interactions, $\Delta t$ is the duration of a single collisional event, and where

$$
\hat{H}_{S_m, E_n} = \sum_{\ell} \hat{A}^{(\ell, k)}_{S_m} \otimes \hat{B}^{(\ell, m)}_{E_n},
$$

(7)

is the most general Hamiltonian describing the interactions between $S_m$ and $E_n$, with $\hat{A}^{(\ell, k)}_{S_m}$ and $\hat{B}^{(\ell, m)}_{E_n}$ nonzero operators acting locally on such systems respectively $[36]$. Next we take the product $g\Delta t$ to be a small quantity and expand our equations up to the second order in such term. In this regime, upon tracing upon the degree of freedom of the carriers, Eq. (4) yields the identity

$$
\frac{\dot{\rho}(n+1) - \dot{\rho}(n)}{\Delta t} = -ig \sum_{m, k, \ell} \gamma^{(\ell, k)}_{m(k)} \left[\hat{A}^{(\ell, k)}_{S_m}, \hat{\rho}(n)\right] - g^2 \Delta t \left\{ \sum_{m=1}^{M} \mathcal{L}_m(\hat{\rho}(n)) + \sum_{m'=m+1}^{M-1} \mathcal{D}_{m, m'}(\hat{\rho}(n))\right\} + \mathcal{O}(g^3 \Delta t^2),
$$

(8)

with

$$
\mathcal{L}_m(\cdots) = \frac{1}{2} \sum_{k, k'=1}^{K} \sum_{\ell, \ell'} \gamma^{(\ell, \ell')}_{m(kk')} \left\{ 2 \hat{A}^{(\ell, k)}_{S_m} \cdots \hat{A}^{(\ell', k')}_{S_m} - \left[\hat{A}^{(\ell', k')}_{S_m}, \hat{A}^{(\ell, k)}_{S_m}, \cdots\right] \right\},
$$

(9)

and for $m' > m$,

$$
\mathcal{D}_{m \rightarrow m'}(\cdots) = \sum_{k, k'=1}^{K} \sum_{\ell, \ell'} \gamma^{(\ell, \ell')}_{m'(kk')} \left\{ 2 \hat{A}^{(\ell, k)}_{S_m} \cdots \hat{A}^{(\ell', k')}_{S_m} - \left[\hat{A}^{(\ell', k')}_{S_m}, \hat{A}^{(\ell, k)}_{S_m}, \cdots\right] \right\},
$$

(10)

where $\left[\cdots, \cdots, \right]_\pm$ represent the commutator ($-$) and anti-commutator ($+$) brackets respectively. In the above expressions $\gamma^{(\ell, k)}_{m(k)}$, $\gamma^{(\ell, \ell')}_{m(kk')}$, and $\gamma^{(\ell, \ell')}_{m'(kk')}$, are complex coefficients which depend upon correlation term of the input state of the carriers (see Appendix A for the explicit definitions).

The continuous time limit is finally obtained sending to infinity $n$ of collisions while the time interval $\Delta t$ of each collision goes to zero and the coupling constant $g$ explodes in such a way that

$$
\lim_{\Delta t \rightarrow 0^+} n\Delta t = t, \quad \lim_{\Delta t \rightarrow 0^+} g^2 \Delta t = \gamma,
$$

(11)

with $\gamma$ being a positive constant which set the time scale of the model. Notice that the last assumption could lead to problem in the first-order term of the series expansion of Eq. (8), whose contribution to the final expression would explode. Such instability is a typical trait in the derivation of master equations $[36]$ for open quantum systems. It can be solved by imposing a stability condition $[31, 32]$ for the environmental degree of freedom of the system, i.e. by requiring that the input carrier states $\hat{\eta}_{E_n}$ and their evolved counterparts along the network should not be influenced (at first order) by the collisions with the subsystems. This is consistent with the description of the environmental channels as composed by many small sub-environments all in the same reference state that interacts weakly with the subsystems. In the standard derivation of master equations such stability condition is usually assumed as well, and it amounts to the possibility of approximating the joint density matrix as a tensor product between the reduced density matrix of the system and the one of the environment at any time. In our case this corresponds to nullify the coefficients $\gamma^{(\ell, k)}_{m(k)}$ appearing in the rhs of Eq. (8), i.e. by imposing (see Eq. (A10) of the Appendix A)

$$
\left\langle \hat{B}^{(\ell, 1)}_{E_{n(1)}} \hat{\eta}_{E_n}\right\rangle_{\xi} = 0, \quad \left\langle \hat{B}^{(\ell, 2)}_{E_{n(2)}} \mathcal{M}^{(1)}_{n_{E_n}}(\hat{\eta}_{E_n})\right\rangle_{\xi} = 0, \quad \left\langle \hat{B}^{(\ell, 3)}_{E_{n(3)}} \mathcal{M}^{(2)}_{n_{E_n}}(\hat{\eta}_{E_n})\right\rangle_{\xi} = 0, \quad \cdots \quad \left\langle \hat{B}^{(\ell, m)}_{E_{n(m)}} \mathcal{M}^{(m-1)}_{n_{E_n}} \cdots \mathcal{M}^{(1)}_{n_{E_n}}(\hat{\eta}_{E_n})\right\rangle_{\xi} = 0,
$$

(12)

for all $k = 1, \cdots, K$, for all $m = 1, \cdots, M$, and for all $\ell$ ($\hat{B}^{(\ell, k)}_{E_{n(k)}}$ being the carriers operators which participate to the coupling Hamiltonian $[4]$).

By enforcing the condition (12), Eq. (8) finally can be casted in the following differential form

$$
\frac{\partial \hat{\rho}(t)}{\partial t} = \gamma C(\hat{\rho}(t)),
$$

(13)

with $C$ the QCS super-operator

$$
C(\cdots) = \sum_{m=1}^{M} \mathcal{L}_m(\cdots) + \sum_{m'=m+1}^{M-1} \sum_{m=1}^{M-1} \mathcal{D}_{m \rightarrow m'}(\cdots).
$$

(14)

Equation (13) is a Markovian master equation which describes the dynamical evolution of the joint density matrix $\hat{\rho}(t)$ for the system of interest $S$. The term on the rhs is the generator of the dynamics and can be casted in GKSL form $[33, 35]$ by properly reorganizing the various
contributions (see next section). It is however worth analyzing the causal structure of the model a bit further by looking directly at the expression presented in \[14\]. On the one hand we have the terms \(\mathcal{L}_m\) which describe local effects of the interaction between the various element of \(S\) and the environment: they are not capable of creating correlations among the \(S_m\)'s and only account for dissipative behaviors. On the other hand the non-local terms \(\mathcal{D}_{m\rightarrow m'}\) describe the interaction between the \(m\)-th and \(m'-\)th subsystem (with \(m' > m\)) originating by the propagation of the carries from the former to the latter. In principle these are capable of building up correlations among the various elements of \(S\). However, at variance with what would happen with a direct Hamiltonian interaction, such couplings are intrinsically asymmetric in agreement with the cascade structure of the network of connections. In particular one may observe that by tracing over \(S_m\) the term \(\mathcal{D}_{m\rightarrow m'}(\hat{\rho}(t))\) always nullifies, i.e.

\[
\text{Tr}_{S_m}[\mathcal{D}_{m\rightarrow m'}(\hat{\rho}(t))] = 0 ,
\]

while this is not necessarily the case when the same term is traced over \(S_m\). This implies, for instance, that the reduced density matrix \(\hat{\rho}_1(t)\) of the first element of \(S\) (the one which in principle controls all the others without being controlled by them) evolves in time without being affected by the presence of the latter. Similarly the evolution of the first \(m\) elements of \(S\) does not depend upon the remaining ones.

The derivation of Eq. \[13\] we have presented here closely follows the one of Ref. \[31\]. The main difference with the latter is the inner structure of the generators \(\mathcal{L}_m\) and \(\mathcal{D}_{m\rightarrow m'}\) which in our case includes contributions from multiple unidirectional channels as indicated by the sum over the indexes \(k\) and \(k'\) of Eqs. \[9\] and \[10\]. As it will be clear discussing some explicit examples (see next Section) this is what allows us to account for interference effects that originate with the signals propagation through the network.

\[\text{B. Standard GKSL form and effective Hamiltonian couplings}\]

The decomposition of the coupling Hamiltonians presented in Eq. \[7\] is clearly not unique. Alternatives can be obtained by replacing the \(\hat{A}_{S_m}^{(\ell,k)}\)s (resp. the \(\hat{B}_{E_n}^{(\ell,k)}\)s) with proper linear combinations of the same objects for instance by expanding them into an operator basis. The master equation \[12\] clearly does not depend on this choice as it derives from a perturbative expansion on the coupling parameter \(q\) which enters in the model as a multiplicative factor of \(\hat{H}_{S_m}^{(\ell,k)}\), and from the stability conditions \[12\], which are explicitly invariant under linear combinations of the \(\hat{B}_{E_n}^{(\ell,m)}\)s. In this section we shall invoke this freedom assuming the \(\hat{A}_{S_m}^{(\ell,k)}\)’s and the \(\hat{B}_{E_n}^{(\ell,k)}\)’s to be self-adjoint (a possibility which is allowed by the fact that \(\hat{H}_{S_m,E_n}^{(\ell)}\) has to be self-adjoint as well). This working hypothesis is not fundamental but, as pointed out in Refs. \[31, 32\], turns out to be useful as it makes explicit some structural properties of the resulting super-operators, ensuring for instance the identities

\[
\gamma_{m(k,k')}^{(\ell,\ell')} = \begin{pmatrix} \gamma_{m(k,k')}^{(\ell,\ell')} \end{pmatrix}^\dagger , \quad \epsilon_{m(m',kk')}^{(\ell,\ell')} = \begin{pmatrix} \epsilon_{m(m',kk')}^{(\ell,\ell')} \end{pmatrix}^\dagger ,
\]

as evident from Eqs. \[A12-A14\] of the Appendix. Our aim is to exploit these properties to generalize the analysis of Ref. \[12\] by casting the QCS super-operator \[14\] into an explicit standard GKSL form \[36\], i.e. as the sum of an effective Hamiltonian term plus a collection of purely dissipative contributions

\[
\mathcal{C}(\cdots) = -i[\hat{H},(\cdots)] + \sum \tilde{L}^{(i)}(\cdots)\tilde{L}^{(i)\dagger} - \begin{pmatrix} \tilde{L}^{(i)}(\cdots) \end{pmatrix}^\dagger ,
\]

with \(\hat{H}\) being self-adjoint and with the \(\tilde{L}^{(i)}\)’s being a collection of operators acting on \(S\). In Ref. \[12\] this trick was used to show that a collection of two-level atoms coupled in QCS fashion via an unidirectional optical fiber, initialized at zero temperature, can be described as originating from an effective two-body coupling Hamiltonian with chiral symmetry.

We start by focusing on the local contributions of Eq. \[13\]. Indicating with \(j\) the joint index (\(\ell, k\), Eq. \[10\] implies that, for each \(m\) assigned, the matrix \(\Theta_{jj'}\) of elements \(\gamma_{m(k,k')}/2\) is Hermitian, i.e. \(\Theta_{jj'} = \Theta_{jj'}^\dagger\). Furthermore, by direct inspection of Eq. \[A12\] one can easily prove that, being \(\tilde{B}_{E_n}^{(\ell,m)}\) self-adjoint, such matrix is also semi-positive definite. Accordingly Eq. \[9\] can be expressed as a purely dissipative term

\[
\mathcal{L}_m(\cdots) = \sum s \lambda_s \left\{ 2\tilde{A}^{(s)}_{S_m}(\cdots)\tilde{A}^{(s)}_{S_m}\dagger - \left[\tilde{A}^{(s)}_{S_m}\tilde{A}^{(s)}_{S_m}\dagger,\cdots\right]_+ \right\} ,
\]

where \(\{\lambda_s\}_s\) are the eigenvalues of \(\Theta_{jj'}\) and where we have introduced the operators

\[
\tilde{A}^{(s)}_{S_m} = \sum_{k,\ell} v_{(\ell,k),s} \hat{A}^{(\ell,k)}_{S_m} ,
\]

with \(v_{j,s}\) being the unitary matrix which allows us to diagonalize \(\Theta_{jj'}\), i.e. \(\Theta_{jj'} = \sum s v_{j,s} \lambda_s v_{s,j'}\). In the absence of the coupling contributions \(\mathcal{D}_{m\rightarrow m'}\), Eq. \[13\] will hence reduce to the standard form \[18\] with \(\bar{H} = 0\) and with the dissipative operators \(\tilde{L}^{(i)}\) being identified with \(\sqrt{\lambda_s} \tilde{A}^{(s)}_{S_m}\).

Consider next the non-local contributions of Eq. \[13\]. Due to their peculiar structure they cannot directly produce terms as those on the right hand side of Eq. \[18\].
We notice however that for all $m' > m$ one can write
\[
\hat{A}_{S_m}^{(\ell,k)} \left[ \ldots, \hat{A}_{S_{m'}}^{(\ell',k')} \right] = -\frac{1}{2} \hat{A}_{S_m}^{(\ell,k)} \hat{A}_{S_{m'}}^{(\ell',k')} + \frac{1}{2} \hat{A}_{S_{m'}}^{(\ell',k')} \hat{A}_{S_m}^{(\ell,k)},
\]
and
\[
\left[ \ldots, \hat{A}_{S_{m'}}^{(\ell',k')} \right] \hat{A}_{S_m}^{(\ell,k)} = -\frac{1}{2} \hat{A}_{S_{m'}}^{(\ell',k')} \hat{A}_{S_m}^{(\ell,k)} + \frac{1}{2} \hat{A}_{S_m}^{(\ell,k)} \hat{A}_{S_{m'}}^{(\ell',k')} + \ldots,
\]
which simply follow from the fact that $\hat{A}_{S_m}^{(\ell,k)}$, $\hat{A}_{S_{m'}}^{(\ell',k')}$ operate on different quantum systems and hence commute. Replacing these identities into Eq. (10) we can write
\[
D_{m \rightarrow m'}(\cdots) = -i \left[ \hat{H}_{m,m'}, (\cdots) \right] + \Delta \mathcal{L}_{m,m'}(\cdots).
\] (22)

In this expression the first contribution is an effective Hamiltonian term with
\[
\hat{H}_{m,m'} = \sum_{k,k'=1,\ell,\ell'} \xi_{mm'}^{(k,k')}(\ell,\ell') \hat{A}_{S_m}^{(\ell,k)} \otimes \hat{A}_{S_{m'}}^{(\ell',k')} + \sum_{k,k'=1,\ell,\ell'} \operatorname{Im}[\xi_{mm'}^{(k,k')}(\ell,\ell')] \hat{A}_{S_m}^{(\ell,k)} \otimes \hat{A}_{S_{m'}}^{(\ell',k')},
\] (23)
where in the second line we used (16). The second contribution on the right hand side of (22) instead features the super-operator
\[
\Delta \mathcal{L}_{m,m'}(\cdots) = \sum_{m_1,m_2=m,m' \setminus k,k_1,k_2=1,\ell_1,\ell_2} \Delta D_{mn,m_2}(\cdots) \times \left\{ 2 \hat{A}_{S_{m_1}}^{(\ell_1,k_1)}(\cdots) \hat{A}_{S_{m_2}}^{(\ell_2,k_2)} - \left[ \hat{A}_{S_{m_1}}^{(\ell_1,k_1)} \hat{A}_{S_{m_2}}^{(\ell_2,k_2)} + \cdots \right] \right\},
\] (24)
with coefficients
\[
\Delta D_{mn,m_2}(\ell_1,\ell_2) = \frac{1}{2} \begin{cases} \xi_{mm_1}^{(k_1,k_2)}(\ell_1,\ell_2) \text{ for } m_1 < m_2 \\ 0 \text{ for } m_1 = m_2, m_2 > m_1 \end{cases}
\] (25)
\[
\Delta D_{mn,m_2}(\ell_2,\ell_1) = \xi_{mm_2}^{(k_2,k_1)}(\ell_2,\ell_1) \ast \text{ for } m_1 > m_2.
\]

One may notice that indicating with $j$ the joint index $(\ell, k, m)$, then from Eq. (16) it follows that the matrix $\Delta \Omega_{jj'}$ of elements $\Delta D_{mn,m_2}(\ell_1,\ell_2)$ is Hermitian, i.e.
\[
\Delta D_{mn,m_2}(\ell_1,\ell_2) = \left[ \Delta D_{mn,m_2}(\ell_2,\ell_1) \right]^{\ast}.
\]
Yet there is no guarantee that $\Delta \Omega_{jj'}$ is semi-positive definite (an explicit counter-example will be presented in the next section) thus preventing one from directly expressing (24) as a sum of dissipative contributions by diagonalization of $\Delta \Omega_{jj'}$ as we did for the local terms of $\mathcal{L}$. However by replacing Eq. (22) into (14) we arrive to
\[
\mathcal{C}(\cdots) = -i [\hat{H}, (\cdots)] + \sum_{m_1,m_2=1,k_1,k_2=1,\ell_1,\ell_2}^{M} \sum_{K}^{K} \sum_{m=1}^{M} \sum_{m'=m+1}^{M} D_{m_1,m_2}(\ell_1,\ell_2) \times \left\{ 2 \hat{A}_{S_{m_1}}^{(\ell_1,k_1)}(\cdots) \hat{A}_{S_{m_2}}^{(\ell_2,k_2)} - \left[ \hat{A}_{S_{m_1}}^{(\ell_1,k_1)} \hat{A}_{S_{m_2}}^{(\ell_2,k_2)} + \cdots \right] \right\}
\] (26)
where now $\hat{H}$ is the effective Hamiltonian
\[
\hat{H} = \sum_{m'=m+1}^{M} \sum_{m=1}^{M} \hat{H}_{m,m'},
\] (27)
and where the coefficients $D_{m_1,m_2}(\ell_1,\ell_2)$ are obtained from those of Eq. (25) by using the elements $\xi_{mm'}^{(k,k')}$ to fill the zero’s on the $m_1, m_2$ diagonal, i.e.
\[
D_{m_1,m_2}(\ell_1,\ell_2) = \begin{cases} \Delta D_{m_1,m_2}(\ell_1,\ell_2) & \text{for } m_1 \neq m_2 \\ \sqrt{\kappa_i} \xi_{m_1,m_2}(\ell_1,\ell_2) / 2 & \text{for } m_1 = m_2.
\end{cases}
\] (28)

To complete the derivation of Eq. (18) one should prove the non-negativity of the matrix $\Omega_{jj'} = D_{m_1,m_2}(\ell_1,\ell_2)$ (i.e. $\ell$ being once more the joint index $(\ell, k, m)$). This is shown explicitly in App. B. Indicating hence with $\kappa_i(\geq 0)$ the eigenvalues of $\Omega_{jj'}$ and with $w_{j,i}$ the elements of the unitary matrix that diagonalizes it (i.e. $\Omega_{jj'} = \sum_i w_{j,i} \kappa_i w_{j,i}^{\ast}$) we can finally identify the operators $\hat{L}^{(i)}$ of Eq. (18) with
\[
\hat{L}^{(i)} = \sqrt{\kappa_i} \sum_{\ell,k,m} w_{(\ell,k),m} \hat{A}_{S_m}^{(\ell,k)}.
\] (29)

A final remark before concluding the section: as already mentioned in deriving the above results we find it convenient to assume the operators $\hat{A}_{S_m}^{(\ell,k)}$ and $\hat{A}_{E_n}^{(l,m)}$ to be self-adjoint. Yet the analysis presented here is still valid even when this assumption does not hold - simply some of the structural properties of the involved mathematical objects are less explicit. In particular the eigenvalues of the matrices (25) and (28) can be shown to be independent from the decomposition adopted in writing (17) (the associated matrices being related by similarity transformations).

III. INTERFERENCE EFFECTS

Here we present a couple of examples of QCSs which enlighten the arising of interference effects during the propagation of signals on a network of unidirectional connections and how they can be used to externally tune the couplings among the various subsystems.
follow the causal structure depicted on the right panel with optical quantum carriers unidirectional (chiral) optical channels two-level atoms of energy gap pressed as output from one of the two ports is sent to the second first subsystem be in a thermal state of temperature specifically indicating with and the cavities, i.e. the environment is then mixed with the second with the phase shift transform PS acting on the carriers of second only, i.e. while the action of by the identity while the action of by the identity It is worth observing that, in the limit where \( \epsilon_1 = \epsilon_2 = 1 \) (i.e. no mixing between \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \)) and \( T_1 = 0 \), the model just described reproduce the one analyzed in Ref. [12] for \( M = 2 \) two-level atoms. We first observe that with the above choices the stability condition [12] is fulfilled. Indeed from Eq. (32) follows trivially \( \gamma_{m(2)} = 0 \). Instead from Eq. (33) we can take so that

\[
\gamma_{1(1)} = [\gamma_{1(1)}]^* = \left( \hat{b}_{E_n} \right. \hat{\eta}_{E_n} \left. \right|_\varepsilon \right) \left( \hat{b}_{E_n} \hat{\eta}_{E_n} \right|_\varepsilon = 0 ,
\]
which trivially follow from the fact that the annihilation operator admits zero expectation value on Gibbs states. Analogously we have

\[ \gamma_{2(1)}^{(1)} = \left[ \gamma_{2(1)}^{(2)} \right]^* = \left\langle \hat{b}_{E_n} \right| \mathcal{M}_{E_n} \left( \hat{n}_{E_n} \right) \left| \hat{\eta}_{E_n} \right\rangle_{\mathcal{E}} = \left\langle \hat{\eta}_{E_n} \right| \mathcal{M}_{E_n} \left( \hat{b}_{E_n} \right) \left| \hat{n}_{E_n} \right\rangle_{\mathcal{E}} = c(\varphi) \left\langle \hat{b}_{E_n} \right| \hat{n}_{E_n} \rangle_{\mathcal{E}} + s(\varphi) \left\langle \hat{b}_{E_n} \right| \hat{n}_{E_n} \rangle_{\mathcal{E}} = 0, \]  

(42)

where \( \mathcal{M}_{E_n}^{(1)} \) is the complementary counterpart of \( \mathcal{M}_{E_n}^{(1)} \) fulfilling the property

\[ \mathcal{M}_{E_n}^{(1)} \left( \hat{b}_{E_n} \right) := \left( V_{BS_1} \right)^\dagger \left( \hat{V}_{S_2} \right)^\dagger \left( \hat{V}_{BS_1} \right)^\dagger \hat{b}_{E_n} \left( \hat{V}_{BS_1} \right)^\dagger \hat{V}_{S_2} \left( V_{BS_1} \right)^\dagger \]  

\[ = c(\varphi) \hat{b}_{E_n} + s(\varphi) \hat{b}_{E_n}^2, \]  

(43)

with

\[ c(\varphi) = e^{-i \varphi} \sqrt{\epsilon_1 \epsilon_2 \left( 1 - \epsilon_1 \right) \left( 1 - \epsilon_2 \right)} , \]

\[ s(\varphi) = -i e^{-i \varphi} \sqrt{\epsilon_1 \epsilon_2 \left( 1 - \epsilon_1 \right) \left( 1 - \epsilon_2 \right)} . \]  

(44)

In a similar way we can evaluate the coefficients \( \gamma_{m(kk')}^{(1)} \) and \( \gamma_{m(kk')}^{(2)} \) that define the super-operators \( \mathcal{D} \) and \( \mathcal{D}^{(1)} \). First of all we notice that from Eq. (32) it follows that only the terms with \( k = k' = 1 \) can have non vanishing values. Next, indicating with

\[ N_k = (\epsilon^{bk} - 1)^{-1}, \]  

(45)

the mean photon numbers of the \( k \)-th thermal bath, we observe that for the local terms of \( S_1 \) the following identities hold:

\[ \gamma_{1(1)}^{(1)(kk')} = \left[ \gamma_{1(1)}^{(2)(kk')} \right]^* = \delta_{k,k'} \delta_{k',1} \left\langle \hat{b}_{E_n} \right| \hat{n}_{E_n} \rangle_{\mathcal{E}} = 0 , \]

\[ \gamma_{1(2)(kk')} = \delta_{k,k'} \delta_{k',1} \left\langle \hat{b}_{E_n} \right| \hat{n}_{E_n} \rangle_{\mathcal{E}} = \delta_{k,k'} \delta_{k',1} N_1 , \]

\[ \gamma_{1(1)(kk')} = \delta_{k,k'} \delta_{k',1} \left\langle \hat{b}_{E_n} \right| \hat{n}_{E_n} \rangle_{\mathcal{E}} = \delta_{k,k'} \delta_{k',1} (N_1 + 1) , \]

where \( \delta_{k,k'} \) indicates the Kronecker delta and where we used known properties of the second order expectation values of the Gibbs states. Accordingly the associated super-operator \( \mathcal{D}^{(1)} \) becomes

\[ \mathcal{L}_1 (\cdots) = (N_1 + 1) \left( \hat{a}_1 (\cdots) \hat{a}_1^\dagger - \frac{1}{2} \left[ \hat{a}_1 \hat{a}_1, \cdots \right]_+ \right) + N_1 \left( \hat{a}_1 (\cdots) \hat{a}_1 - \frac{1}{2} \left[ \hat{a}_1 \hat{a}_1, \cdots \right]_+ \right) , \]

(46)

which is already in the standard GKS form (19) and which describes a thermalization process where \( S_1 \) absorbs and emits excitations from a thermal bath at temperature \( T_1 \). Similarly the local terms for the \( S_2 \) gives

\[ \gamma_{2(1)}^{(1)(kk')} = \left[ \gamma_{2(1)}^{(2)(kk')} \right]^* = \delta_{k,k'} \delta_{k',1} \left\langle \hat{b}_{E_n} \right| \mathcal{M}_{E_n} \left( \hat{n}_{E_n} \right) \left| \hat{\eta}_{E_n} \right\rangle_{\mathcal{E}} = \delta_{k,k'} \delta_{k',1} \left\langle \left( c(\varphi) \hat{b}_{E_n} + s(\varphi) \hat{b}_{E_n}^2 \right)^2 \hat{\eta}_{E_n} \right\rangle_{\mathcal{E}} = 0 , \]

(47)

and

\[ \gamma_{2(1)}^{(2)(kk')} = \left( \gamma_{2(1)}^{(1)(kk')} \right)^* = \delta_{k,k'} \delta_{k',1} \left\langle \left( c(\varphi) \hat{b}_{E_n} + s(\varphi) \hat{b}_{E_n}^2 \right)^2 \hat{\eta}_{E_n} \right\rangle_{\mathcal{E}} = \delta_{k,k'} \delta_{k',1} N_1 \left( N_1 + 1 \right) , \]

(48)

\[ \gamma_{2(1)}^{(1)(kk')} = \delta_{k,k'} \delta_{k',1} \left\langle \left( c(\varphi) \hat{b}_{E_n} + s(\varphi) \hat{b}_{E_n}^2 \right)^2 \hat{\eta}_{E_n} \right\rangle_{\mathcal{E}} = \delta_{k,k'} \delta_{k',1} N_1 \left( N_1 + 1 \right) . \]

(49)

where we introduced

\[ N_1(\varphi) = |c(\varphi)|^2 N_1 + |s(\varphi)|^2 N_2 = N_2 + (N_1 - N_2) |c(\varphi)|^2 , \]

(50)

Replacing all this into Eq. (9) we hence get the following super-operator

\[ \mathcal{L}_2 (\cdots) = \left( N_1(\varphi) + 1 \right) \left( \hat{a}_2 (\cdots) \hat{a}_2^\dagger - \frac{1}{2} \left[ \hat{a}_2 \hat{a}_2, \cdots \right]_+ \right) + N_1(\varphi) \left( \hat{a}_2 (\cdots) \hat{a}_2 - \frac{1}{2} \left[ \hat{a}_2 \hat{a}_2, \cdots \right]_+ \right) , \]

(51)

which represents a thermalization process induced by an effective bath whose temperature is intermediate between the one of \( E_1 \) and \( E_2 \) and depends by the mixing of the signals induced by their propagation through the Mach-Zehnder.

Consider next the non-local contribution \( \mathcal{D}_{1,2} \) of the master equation. In this case we get

\[ \gamma_{1,2(1)}^{(1)(kk')} = \left[ \gamma_{1,2(2)(kk')} \right]^* = \delta_{k,k'} \delta_{k',1} \left\langle \hat{b}_{E_n} \right| \mathcal{M}_{E_n} \left( \hat{b}_{E_n} \right) \left| \hat{\eta}_{E_n} \right\rangle_{\mathcal{E}} = \delta_{k,k'} \delta_{k',1} N_1 \left( N_1 + 1 \right) , \]

(52)

\[ \gamma_{1,2(2)(kk')} = \left[ \gamma_{1,2(1)(kk')} \right]^* = \delta_{k,k'} \delta_{k',1} \left\langle \left( c(\varphi) \hat{b}_{E_n} + s(\varphi) \hat{b}_{E_n}^2 \right)^2 \hat{\eta}_{E_n} \right\rangle_{\mathcal{E}} = \delta_{k,k'} \delta_{k',1} c(\varphi) N_1 , \]

\[ \gamma_{1,2(1)}^{(2)(kk')} = \left[ \gamma_{1,2(2)(kk')} \right]^* = \delta_{k,k'} \delta_{k',1} \left\langle \left( c(\varphi) \hat{b}_{E_n} + s(\varphi) \hat{b}_{E_n}^2 \right)^2 \hat{\eta}_{E_n} \right\rangle_{\mathcal{E}} = \delta_{k,k'} \delta_{k',1} c(\varphi) \left( N_1 + 1 \right) , \]

(53)

so that

\[ \mathcal{D}_{1 \to 2} (\cdots) = N_1 \left( c(\varphi) \hat{a}_2 \left[ \cdots, \hat{a}_2 \right] - c(\varphi) \left[ \cdots, \hat{a}_2 \right] \hat{a}_1 \right) + (N_1 + 1) \left( c(\varphi) \hat{a}_1 \left[ \cdots, \hat{a}_2 \right] - c^*(\varphi) \left[ \cdots, \hat{a}_2 \right] \hat{a}_1 \right) . \]

(54)
One notices that at variance with the contribution (46) which fully define the dynamics of $S_1$, both the local term (22) of $S_2$ and the coupling super-operator (54) are modulated by the phase $\varphi$. In particular by setting the transmissivities of $BS_1$ and $BS_2$ at 50% (i.e. $\eta_1 = \eta_2 = 0.5$), the coefficient $c(\varphi)$ will acquire an oscillating behavior nullifying for $\varphi = \pm \pi$ (specifically we get $c(\varphi) = -i e^{-i \varphi/2}/\sin(\varphi/2)$). By controlling the parameter $\varphi$ we can hence modify the cascade coupling between $S_1$ and $S_2$.

Following the derivation of Sec. [11B] we can finally write the QCS super-operator in the GKS form (22). In particular in this case the effective Hamiltonian appearing in Eq. (10) is given by

$$H_{1,2} = -\frac{i}{2} \left( c(\varphi) \hat{a}_1^\dagger \hat{a}_1 - c^*(\varphi) \hat{a}_1^\dagger \hat{a}_2 \right)$$

$$= -\frac{i}{2} |c(\varphi)| \left( e^{i \arg[c(\varphi)]} \hat{a}_1^\dagger \hat{a}_1 - e^{-i \arg[c(\varphi)]} \hat{a}_1^\dagger \hat{a}_2 \right),$$

which by absorbing the phase $\arg[c(\varphi)]$ into (say) $\hat{a}_1$ exhibits the same chiral symmetry under exchange of $S_1$ and $S_2$ (i.e. $H_{2,1} = -H_{1,2}$) observed in Ref. [12]. The super-operator $\Delta L_{1,2}$ of Eq. (10) instead in this case is given by

$$\Delta L_{1,2}(\cdots) = N_1 c^*(\varphi) \left( \hat{a}_1^\dagger (\cdots) \hat{a}_2 - \frac{1}{2} \left[ \hat{a}_1^\dagger \hat{a}_2, (\cdots) \right] \right)_+$$

$$+ (N_1 + 1)c(\varphi) \left( \hat{a}_1(\cdots) \hat{a}_2^\dagger - \frac{1}{2} \left[ \hat{a}_2 \hat{a}_1, (\cdots) \right] \right)_+ + h.c.$$  

(56)

which, remembering (39), can be expressed as in (24) with

$$\Delta D_{m_1,m_2}(\ell_1,\ell_2) = \delta_{k_1 k'_1} \delta_{k_2 k'_2} \Delta D_{m_1,m_2(1,1)}(\ell_1,\ell_2),$$

(57)

where $m_1, m_2 = 1, 2$ and $\ell_1, \ell_2 = 1, 2$, $\Delta D_{m_1,m_2(1,1)}(\ell_1,\ell_2)$ is the $4 \times 4$ matrix of elements

$$\begin{bmatrix} 0 & 0 & N_1 c^*(\varphi) & 0 \\ 0 & 0 & 0 & (N_1 + 1)c(\varphi) \\ N_1 c(\varphi) & 0 & 0 & 0 \\ 0 & (N_1 + 1)c^*(\varphi) & 0 & 0 \end{bmatrix}$$

the top-left and bottom right $2 \times 2$ blocks being associated with $m_1 = m_2 = 1$ and $m_1 = m_2 = 2$ respectively. As anticipated in the previous section, while being Hermitian, this is in general not positive semi-definite (indeed it admits eigenvalues $\pm N_1 |c(\varphi)|$ and $\pm N_1 |1 + c(\varphi)|$). On the contrary the matrix (28) which describe the sum of $\Delta L_{1,2}$ with the local terms $L_1$ of Eq. (16) and $L_2$ of Eq. (50) is given by

$$\begin{bmatrix} N_1 & 0 & N_1 c^*(\varphi) & 0 \\ 0 & (N_1 + 1) & 0 & (N_1 + 1)c(\varphi) \\ N_1 c(\varphi) & 0 & N_1 c^*(\varphi) & 0 \\ 0 & (N_1 + 1)c^*(\varphi) & 0 & N_1 c(\varphi) \end{bmatrix}$$

and has eigenvalues

$$\kappa_{1,\pm} = \frac{1}{2} (N_1 + N_1(\varphi) + 2 \pm \sqrt{(N_1 - N_1(\varphi))^2 + 4(N_1 + 1)^2 |c(\varphi)|^2}),$$

$$\kappa_{2,\pm} = \frac{1}{2} (N_1 + N_1(\varphi) \pm \sqrt{(N_1 - N_1(\varphi))^2 + 4N_1^2 |c(\varphi)|^2}),$$

(58)

(59)

which are non-negative for all possible choices of $N_1, N_1 \geq 0$ and $|c(\varphi)| \in [0, 1]$. The associated Lindblad operators (29) can instead be shown to be equal to

$$\hat{L}_{1,\pm} = \sqrt{k_{1,\pm}} \frac{\hat{a}_1 + \hat{a}_2}{\sqrt{1 + |w_{1,\pm}|^2}},$$

$$\hat{L}_{2,\pm} = \sqrt{k_{2,\pm}} \frac{\hat{a}_2 - \hat{a}_1}{\sqrt{1 + |w_{2,\pm}|^2}},$$

(60)

(61)

(62)

(63)

with

$$w_{1,\pm} = \frac{1}{2(N_1 + 1)c^*(\varphi)} \left[ N_1 - N_1(\varphi) \pm \sqrt{(N_1 - N_1(\varphi))^2 + 4(N_1 + 1)^2 |c(\varphi)|^2} \right],$$

$$w_{2,\pm} = \frac{1}{2N_1 c(\varphi)} \left[ N_1 - N_1(\varphi) \pm \sqrt{(N_1 - N_1(\varphi))^2 + 4N_1^2 |c(\varphi)|^2} \right].$$

(64)

(65)

It is worth noticing that if in the already cited limit of $\epsilon_{1,2} = 1$ and $T_1 = 0$ reproducing the model in [12], we have that only the eigenvalue $k_{1,\pm} = 2$ is different from zero, so that one has only one collective jump operator $\hat{L}_{1,\pm} = \hat{a}_1 + \hat{a}_2$.

### B. Example 2: controlling the topology of the network via interference

In this section we discuss how interference can be used to effectively modify the topology of the QCS interaction network by selectively activating/deactivating some of the couplings which enter the scheme. In particular we focus on the case of three quantum systems, dubbed $Q_1$, $Q_2$ and $Q_3$ connected as schematically shown in Fig. 4. This is basically the same configuration discussed in Sec. III A where $Q_1$ and $Q_3$ take the positions of $S_1$ and $S_2$ respectively, while $Q_2$ is placed inside the Mach-Zehnder interferometer. Accordingly the model exhibits direct QCS connections among first neighboring elements (i.e. the couple $Q_1$ and $Q_2$ and the couple $Q_2$ and $Q_3$), while the QCS coupling among $Q_1$ and $Q_3$ is mediated by two
channels which interfere. The dynamics of the model can be derived following the same line of the previous section – see Appendix C for the explicit calculations.

Expressed as in Eq. (14) the resulting master equation exhibits the following local contributions:

$$\mathcal{L}(\cdots) = (N_1 + 1)\left(\hat{a}_1 \cdots \hat{a}^\dagger_1 - \frac{1}{2}[\hat{a}^\dagger_1 \hat{a}_1, \cdots]_+\right)$$

$$+ N_1\left(\hat{a}_1 \cdots \hat{a}^\dagger_1 - \frac{1}{2}[\hat{a}^\dagger_1 \hat{a}_1, \cdots]_+\right),$$

$$\mathcal{L}_2(\cdots) = \left(\bar{N}_{12} + 1\right)\left(\hat{a}_2 \cdots \hat{a}^\dagger_2 - \frac{1}{2}[\hat{a}^\dagger_2 \hat{a}_2, \cdots]_+\right)$$

$$+ \bar{N}_{12}\left(\hat{a}_2 \cdots \hat{a}^\dagger_2 - \frac{1}{2}[\hat{a}^\dagger_2 \hat{a}_2, \cdots]_+\right),$$

$$\mathcal{L}_3(\cdots) = \left(N_{12}(\varphi) + 1\right)\left(\hat{a}_3 \cdots \hat{a}^\dagger_3 - \frac{1}{2}[\hat{a}^\dagger_3 \hat{a}_3, \cdots]_+\right)$$

$$+ N_{12}(\varphi)\left(\hat{a}_3 \cdots \hat{a}^\dagger_3 - \frac{1}{2}[\hat{a}^\dagger_3 \hat{a}_3, \cdots]_+\right),$$

with $N_{12}(\varphi)$ defined as in Eq. (50) and $\bar{N}_{12}$ being the average photon number of the environments perceived by $Q_2$, i.e.

$$\bar{N}_{12} = \epsilon_1 N_1 + (1 - \epsilon_1) N_2 = N_2 + \epsilon_1(N_1 - N_2).$$

Notice that the local terms of $Q_1$ and $Q_3$ coincide respectively with those of $S_1$ and $S_2$ of the previous section and the $\mathcal{L}_2$ doesn’t depend upon the phase $\varphi$.

The non-local contributions of the model are instead given by two first-neighboring elements, connecting the couples $Q_1, Q_2$ and $Q_2Q_3$, plus a second-neighboring contribution, connecting $Q_1$ and $Q_3$. The first two are given by

$$\mathcal{D}_{1\rightarrow 2}(\cdots) = \sqrt{c_1} N_1\left(\hat{a}_1 \cdots \hat{a}^\dagger_1 + [\hat{a}^\dagger_1 \hat{a}_1, \cdots]_+\right) + \sqrt{c_1}(N_1 + 1)\left(\hat{a}_1 \cdots \hat{a}^\dagger_1 + [\hat{a}^\dagger_1 \hat{a}_1, \cdots]_+\right),$$

$$\mathcal{D}_{2\rightarrow 3}(\cdots) = M^*_1(\varphi)\hat{a}^\dagger_1 \cdots \hat{a}^\dagger_3 + M_1(\varphi)\hat{a}_3 \cdots \hat{a}^\dagger_1 + (M_1(\varphi) + \lambda(\varphi))\hat{a}_2 \cdots \hat{a}^\dagger_3 + (M^*_1(\varphi) + \lambda^*(\varphi))\hat{a}_3 \cdots \hat{a}^\dagger_2,$$

where we introduced the functions

$$M_1(\varphi) = \sqrt{c_1} c(\varphi) N_1 + i\sqrt{1 - c_1 s(\varphi)} N_2,$$

$$\lambda(\varphi) = \sqrt{c_1} c(\varphi) + i\sqrt{1 - c_1 s(\varphi)},$$

with $c(\varphi)$ and $s(\varphi)$ as in Eq. (44). The third term instead is given by

$$\mathcal{D}_{1\rightarrow 3}(\cdots) = N_1\left\{c^*(\varphi) \hat{a}^\dagger_1 \cdots \hat{a}^\dagger_3 + c(\varphi) [\hat{a}^\dagger_3, \cdots]_+ \hat{a}_1\right\} + (N_1 + 1)\left\{c(\varphi) [\hat{a}_3, \cdots]_+ + c^*(\varphi) [\hat{a}_3, \cdots]_+ \hat{a}_1\right\},$$

and formally coincides with the element $\mathcal{D}_{1\rightarrow 2}(\cdots)$ of the previous section which connected $S_1$ and $S_2$. The above expressions make it clear that the various coupling terms have different functional dependences upon the phase parameter $\varphi$. To better appreciate this it is useful to focus on the zero temperature regime (i.e. $N_1 = N_2 = 0$), and to assume the beam splitters to have 50% transmissivities (i.e. $\epsilon_1 = \epsilon_2 = 1/2$). Under these assumptions all the local contributions describe a purely dissipative evolution which is independent from $\varphi$, i.e.

$$\mathcal{L}_m(\cdots) = \hat{a}_m \cdots \hat{a}^\dagger_m - \frac{1}{2}[\hat{a}^\dagger_m \hat{a}_m, \cdots]_+,$$

while Eqs. (70) – (73) yield

$$\mathcal{D}_{1\rightarrow 2}(\cdots) = \frac{1}{\sqrt{2}}\left(\hat{a}_1 \cdots \hat{a}^\dagger_1 + \hat{a}_2 \cdots \hat{a}^\dagger_2\right),$$

$$\mathcal{D}_{2\rightarrow 3}(\cdots) = \frac{1}{\sqrt{2}}\left(e^{-i\varphi} \hat{a}_2 \cdots \hat{a}^\dagger_3 + e^{i\varphi} [\hat{a}_3, \cdots]_+ \hat{a}^\dagger_2\right),$$

$$\mathcal{D}_{1\rightarrow 3}(\cdots) = -i \sin \frac{\varphi}{2}\left(e^{-i\varphi} \hat{a}_1 \cdots \hat{a}^\dagger_3 + e^{i\varphi} [\hat{a}_3, \cdots]_+ \hat{a}^\dagger_1\right).$$

The above equations make it explicit that the parameter $\varphi$ contributes to the system dynamics in two different ways. First it introduces a non-trivial relative phase between $Q_1, Q_2$ and $Q_3$ which, at variance with the two body problem of the previous section cannot be removed by simply redefining their corresponding annihilation/creation operators. Second it induces a selective modulation of the intensity of the $Q_1Q_3$ interactions. These facts are reflected into the structure of the effective Hamiltonian $\mathcal{H}$ stemming from the reshaping
of the ME in Lindblad form, i.e.
\[
\hat{H}_{1,2} = -\frac{i}{2\sqrt{2}} \left( \hat{a}_1 \hat{a}_2^\dagger - \hat{a}_2 \hat{a}_1^\dagger \right),
\]
(75)
\[
\hat{H}_{2,3} = -\frac{i}{2\sqrt{2}} \left( e^{i\frac{\varphi}{2}} \hat{a}_2 \hat{a}_3^\dagger - e^{-i\frac{\varphi}{2}} \hat{a}_3 \hat{a}_2^\dagger \right),
\]
(76)
\[
\hat{H}_{1,3} = -\frac{i}{2} \sin \frac{\varphi}{2} \left( e^{i\frac{\varphi+\pi}{2}} \hat{a}_1 \hat{a}_3^\dagger - e^{-i\frac{\varphi+\pi}{2}} \hat{a}_3 \hat{a}_1^\dagger \right),
\]
(77)
see Eqs. (C9) – (C11) of Appendix C. Accordingly we see that acting on \(\varphi\) the topology of the system interactions can be modified, moving from the case where the interactions among \(Q_1\) and \(Q_3\) is null (e.g. \(\varphi = 0\)) or amplified (\(\varphi = \pi\)) with respect to their \(Q_1Q_2\) and \(Q_2Q_3\) counterparts, whose associated intensities are instead independent from \(\varphi\) – see Fig. 5.

IV. CONCLUSIONS

In this work we developed a general theoretical framework for modeling complex networks of quantum systems organized in a cascade fashion, i.e. such that the coupling between the various subsystems is mediated by unidirectional environmental channels. Differently from previous approaches, our framework allows also to consider interactions and interference effects between environmental channels, inducing a rich and complex effective dynamics on the nodes of the network.

The theoretical derivation is based on a collisional model that allows to derive a many-body master equation which preserves the positivity of the density matrix and correctly incorporates the causal structure of the network. Moreover, expressing the master equation in Lindblad form, we obtain an effective Hamiltonian coupling between the systems which is externally tunable by properly modifying the parameters of the network.

We focused on two particular examples: a cascade system in a Mach-Zehnder-like configuration showing dissipative interference effects, and a tripartite cascade network where the topology of the interactions is controllable by means of a simple phase shifter. More generally, the possibility of engineering Hamiltonian and dissipative interactions exploiting interference effects in cascade systems is very intriguing and worth to be further investigated in future works.

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Appendix A: Derivation of the master equation

The second order expansion of Eq. [4] with respect to the product $g\Delta t$ is

$$\mathcal{U}_{Sm,\varepsilon_n} = \mathcal{I}_{Sm,\varepsilon_n} + (g\Delta t) \mathcal{U}'_{Sm,\varepsilon_n} + (g\Delta t)^2 \mathcal{U}''_{Sm,\varepsilon_n} + \mathcal{O}(g\Delta t)^3,$$

(A1)

with $\mathcal{I}_{Sm,\varepsilon_n}$ being the identity super-operator and

$$\mathcal{U}'_{Sm,\varepsilon_n}(\cdots) = -i \sum_{k=1}^{K} \left[ \hat{H}_{Sm,E_n^{(k)}}(\cdots) \right]_{\varepsilon_n},$$

(A2)

$$\mathcal{U}''_{Sm,\varepsilon_n}(\cdots) = \sum_{k,k'=1}^{M} \left\{ H_{Sm,E_n^{(k)}}(\cdots) H_{Sm,E_n^{(k')}}(\cdots) \right\} - \frac{1}{2} \left[ H_{Sm,E_n^{(k)}} H_{Sm,E_n^{(k')}}(\cdots) + (\cdots) H_{Sm,E_n^{(k')}} H_{Sm,E_n^{(k)}}(\cdots) \right].$$

(A3)

By replacing these expressions into Eq. [4] we then obtain the expansion of the super-operator $\mathcal{C}_{S,\varepsilon_n}$, i.e.

$$\mathcal{C}_{S,\varepsilon_n} = \mathcal{C}_{S,\varepsilon_n}^0 + (g\Delta t) \mathcal{C}_{S,\varepsilon_n}' + (g\Delta t)^2 \mathcal{C}_{S,\varepsilon_n}'' + \mathcal{O}(g\Delta t)^3,$$

(A4)

where

$$\mathcal{C}_{S,\varepsilon_n}^0 = \mathcal{M}_{\varepsilon_n}^{(M-1)},$$

$$\mathcal{C}_{S,\varepsilon_n}' = \sum_{m=1}^{M} \mathcal{M}_{\varepsilon_n}^{(M-m)} \circ \mathcal{U}'_{Sm,\varepsilon_n} \circ \mathcal{M}_{\varepsilon_n}^{(m-1)},$$

$$\mathcal{C}_{S,\varepsilon_n}'' = \mathcal{C}_{S,\varepsilon_n}'(a) + \mathcal{C}_{S,\varepsilon_n}'(b),$$

(A5)

and

$$\mathcal{C}_{S,\varepsilon_n}'(a) = \sum_{m=1}^{M} \mathcal{M}_{\varepsilon_n}^{(M-m)} \circ \mathcal{U}'_{Sm,\varepsilon_n} \circ \mathcal{M}_{\varepsilon_n}^{(m-1)},$$

$$\mathcal{C}_{S,\varepsilon_n}'(b) = \sum_{m'=1}^{M-1} \sum_{m=1}^{M} \left\{ \mathcal{M}_{\varepsilon_n}^{(M-m')} \circ \mathcal{U}'_{Sm',\varepsilon_n} \circ \mathcal{M}_{\varepsilon_n}^{(m'-1)} \right\} \circ \mathcal{M}_{\varepsilon_n}^{(m'-1+1)} \circ \mathcal{U}'_{Sm',\varepsilon_n} \circ \mathcal{M}_{\varepsilon_n}^{(m'-1+1)}.$$  

(A6)

where we defined

$$\mathcal{M}_{\varepsilon_n}^{(m_2-m_1)} := \begin{cases} \Pi_{m=m_1}^{m_2} \mathcal{M}_{\varepsilon_n}^{(m)} & \text{for } m_2 \geq m_1, \\ \mathcal{I} & \text{for } m_2 < m_1, \end{cases}$$

(A7)

to indicate the ordered product of the maps $\mathcal{M}_{\varepsilon_n}^{(m_1)}, \mathcal{M}_{\varepsilon_n}^{(m_1+1)}, \ldots, \mathcal{M}_{\varepsilon_n}^{(m_2)}$ – see also definition (2). Inserting all this into Eq. [4] and taking the partial trace with respect to the carriers then allows us to write the following equation

$$\frac{\hat{p}(n+1) - \hat{p}(n)}{\Delta t} = g \left\langle \mathcal{C}'_{S,\varepsilon_{n+1}}(\hat{R}(n) \otimes \hat{\eta}_{n+1}) \right\rangle_{\varepsilon}$$

$$+ g^2 \Delta t \left\langle \mathcal{C}''_{S,\varepsilon_{n+1}}(\hat{R}(n) \otimes \hat{\eta}_{n+1}) \right\rangle_{\varepsilon} + \mathcal{O}(g^3 \Delta t^2),$$

(A8)

which by explicit evaluation of the various terms reduces to Eq. [8] of the main text. Indeed the first order term in $g$ of this expression can be written as

$$\left\langle \mathcal{C}'_{S,\varepsilon_{n+1}}(\hat{R}(n) \otimes \hat{\eta}_{n+1}) \right\rangle_{\varepsilon}$$

(A9)

$$= \sum_{m,k} \left\langle \hat{B}_{E_n^{(k)}}^{(\ell,m)}(m-1) \mathcal{M}_{E_{n+1}^{(k)}}^{(m-1)}(\hat{\eta}_{n+1}) \right\rangle_{\varepsilon} \hat{A}_{Sm}^{(\ell,k)} \hat{p}(n),$$

(A10)

and coincides with the first order contribution of Eq. [8] with

$$\gamma_{m(k)}^{(\ell)} = \left\langle \hat{B}_{E_n^{(k)}}^{(\ell,m)}(m-1) \mathcal{M}_{E_{n+1}^{(k)}}^{(m-1)}(\hat{\eta}_{n+1}) \right\rangle_{\varepsilon}. $$

Similarly the second order term of [A8] is given by two contributions:
\[
\langle C_{S,E_n+1}^m(R(n) \otimes \hat{\eta}_{E_{n+1}}) \rangle = \frac{1}{2} \sum_{m=1}^{M} \sum_{k,k', \ell, \ell'} \gamma_{m(kk')}^{(m,\ell,\ell')} \{ 2 \hat{A}_S^{\ell,\ell'}(\hat{\rho}(n) \hat{A}_S^{\ell,\ell'} - [\hat{A}_S^{\ell,\ell'}, \hat{\rho}(n)]_\epsilon \} 
\]
(A11)

\[
\langle C_{S,E_n+1}^m(R(n) \otimes \hat{\eta}_{E_{n+1}}) \rangle = \sum_{m'=m+1}^{M} \sum_{m=1}^{M-1} \sum_{k,k', \ell, \ell'} \xi_{mm'(kk')}^{(m,\ell,\ell')} \{ \hat{A}_S^{\ell,\ell'}[\hat{\rho}(n), \hat{A}_S^{\ell,\ell'}] - \xi_{mm'(kk')}^{(m,\ell,\ell')} \} 
\]
with coefficients
\[
\gamma_{m(kk')}^{(m,\ell,\ell')} = \langle \hat{B}_S^{(\ell',\ell)}(\hat{\eta}_{E_{n+1}}) \hat{M}_S^{(m-1)}(\hat{\eta}_{E_{n+1}}) \rangle / \epsilon ,
\]
(A12)
\[
\xi_{mm'(kk')}^{(m,\ell,\ell')} = \langle \hat{B}_S^{(\ell',\ell)}(\hat{\eta}_{E_{n+1}}) \hat{M}_S^{(m-1)}(\hat{\eta}_{E_{n+1}}) \rangle / \epsilon ,
\]
(A13)
\[
\xi_{mm'(kk')}^{(m,\ell,\ell')} = \langle \hat{B}_S^{(\ell',\ell)}(\hat{\eta}_{E_{n+1}}) \hat{M}_S^{(m-1)}(\hat{\eta}_{E_{n+1}}) \rangle / \epsilon .
\]
(A14)

Appendix B: Positivity of the matrix \( D_{m_1,m_2}^{(\ell_1,\ell_2)} \)

As anticipated in Sec. [13] one can show that the matrix \( \Omega_{j,j'} = D_{m_1,m_2(k_1,k_2)}^{(\ell_1,\ell_2)} \) (\( j \) being the joint index \((\ell, k, m)\) and \( D_{m_1,m_2}^{(\ell_1,\ell_2)} \) as in Eq. (28) is non-negative, i.e. that for all row vectors \( \vec{q} \) of complex elements \( q_j \) the following inequality applies
\[
\vec{q} \Omega \vec{q}^\dagger := \sum_{j,j'} q_j \Omega_{j,j'} q_{j'}^* \geq 0 .
\]
(B1)

Indeed from Eq. (A13)-(A14) it follows that

\[
2 \vec{q} \Omega \vec{q}^\dagger = \sum_m q_{(\ell,k,m)} \bar{q}_{(\ell,k',m')} q_{m}^{(\ell,k)} + \sum_{m'=m+1}^{M} \sum_{m=1}^{M-1} \sum_{k,k', \ell, \ell'} \xi_{mm'(kk')}^{(m',\ell,\ell')} \{ \hat{A}_S^{\ell,\ell'}[\hat{\rho}(n), \hat{A}_S^{\ell,\ell'}] - \xi_{mm'(kk')}^{(m',\ell,\ell')} \} + h.c.,
\]
(B2)

where in the first line we use Eq. (17) and, for the ease of notation, the convention of sum over repeated indexes, while in the second line we introduce the operators
\[
\hat{Q}_{E_{n+1}}^{(m)} = \sum_{\ell,k} q_{(\ell,k,m)} \hat{B}_S^{(\ell,k)} / \epsilon .
\]
(B3)

To proceed further we invoke the Stinespring decomposition [24] to write
\[
\hat{M}_{E_n}^{(m)}(\cdots) = \text{Tr}_{A_n} [\hat{V}_{E_n}^{(m)}(\cdots \otimes |0\rangle_A(0))] ,
\]
(B4)
\[
\hat{V}_{E_n}^{(m)}(\cdots) := \hat{V}_{E_n}^{(m)}(\cdots) \hat{V}_{E_n}^{(m)} \dagger ,
\]
(B5)

with \(|0\rangle_A\) being a (fixed) reference state of an ancillary system \( A \) and \( \hat{V}_{E_n}^{(m)} \) being a unitary transformation that couples it with \( E_n \). Accordingly from Eq. (17) it follows that for all \( m_2 \geq m_1 \) one has
\[
\hat{M}_{E_n}^{(m_2-m_1)} = \text{Tr}_{A_n} [\hat{V}_{E_n}^{(m_2-m_1)}(\cdots \otimes |0\rangle_A(0))] ,
\]
(B6)
\[
\hat{V}_{E_n}^{(m_2-m_1)}(\cdots) := \hat{V}_{E_n}^{(m_2-m_1)}(\cdots) \hat{V}_{E_n}^{(m_2-m_1)} \dagger ,
\]
(B7)

Hence Eq. (B2) now rewrites as
\[
2 \vec{q} \Omega \vec{q}^\dagger = \sum_m \langle \hat{Q}_{E_{n+1}}^{(m)} / \epsilon \rangle \hat{V}_{E_{n+1}}^{(m)}(\hat{\eta}_{E_{n+1}} \otimes |0\rangle_A(0)) / \epsilon_A 
\]
\[
\langle \hat{Q}_{E_{n+1}}^{(m)} / \epsilon \rangle \hat{V}_{E_{n+1}}^{(m)}(\hat{\eta}_{E_{n+1}} \otimes |0\rangle_A(0)) / \epsilon_A + h.c. 
\]
\[
= \sum_m \langle \hat{V}_{E_{n+1}}^{(m-1)}(\hat{\eta}_{E_{n+1}} \otimes |0\rangle_A(0)) / \epsilon_A 
\]
\[
= \sum_{m'} \langle \hat{V}_{E_{n+1}}^{(m-1)}(\hat{\eta}_{E_{n+1}} \otimes |0\rangle_A(0)) / \epsilon_A 
\]
\[
+ \sum_{m'} \langle \hat{V}_{E_{n+1}}^{(m-1)}(\hat{\eta}_{E_{n+1}} \otimes |0\rangle_A(0)) / \epsilon_A + h.c. 
\]
(B9)
where we used the cyclicity of the trace, where \( \tilde{V}_{e,A}^{(m_2 \leftrightarrow m_1)} \) is the conjugate transformation of \( V_{e,A}^{(m_2 \leftrightarrow m_1)} \), i.e. the mapping
\[
\tilde{V}_{e,A}^{(m_2 \leftrightarrow m_1)}(\cdots) := V_{e,A}^{(m_2 \leftrightarrow m_1)\dagger}(\cdots) V_{e,A}^{(m_2 \leftrightarrow m_1)},
\]
and where we introduced the symbol \( {}^* \) to indicate the regular product between operators whenever needed to avoid possible misinterpretations. Now observe that
\[
\tilde{V}_{e,A}^{(m \leftrightarrow m)}(\hat{Q}_{e,A}^{(m)} \cdot \hat{Q}_{e,A}^{(m)} \dagger) = \tilde{T}_{e,A}^{(m)} \hat{\gamma}(m) \tilde{T}_{e,A}^{(m)\dagger},
\]
where now, for \( 0 \leq m \leq 1 \), integer one has
\[
V_{e,A}^{(m_3 \leftrightarrow m_1)} = V_{e,A}^{(m_3 \leftrightarrow m_2)} V_{e,A}^{(m_2 \leftrightarrow m_1)},
\]
and introduced the operators
\[
\tilde{T}_{e,A}^{(m)} \equiv \tilde{V}_{e,A}^{(m \leftrightarrow m)}(\hat{Q}_{e,A}^{(m)} \dagger).
\]
Replacing all these into Eq. (B9) and re-organizing the various terms finally yields the thesis, i.e.
\[
2 \tilde{q} \cdot \Omega \cdot \tilde{q}^\dagger = \sum_{m,m'=1}^M \left( \hat{T}_{e,A}^{(m')\dagger} \hat{T}_{e,A}^{(m)} \left( \hat{\eta}_{e,A} \otimes |0,0\rangle \right) \right) \geq 0.
\]

Appendix C: Derivation of the three body QCS master equation

Here we report the explicit calculation of the model described in Fig. 4. Following the flowchart representation presented in the right panel of the figure we write the Hamiltonians 7 as
\[
\begin{align*}
\hat{H}_{Q_{m,E_n}^{(1)}} &= \hat{a}^\dagger \hat{b}_{E_n^{(1)}}^m + \hat{a}_m \hat{b}^\dagger_{E_n^{(1)}}, \\
\hat{H}_{Q_{m,E_n}^{(2)}} &= 0,
\end{align*}
\]
where now, for \( m = 1,2,3 \), \( \hat{a}_m \) and \( \hat{a}^\dagger_{E_n^{(1)}} \) are the lowering and raising operators of the system \( Q_m \) while \( \hat{b}_{E_n^{(2)}}^m \) and \( \hat{b}^\dagger_{E_n^{(1)}} \) are the bosonic operators associated with the quantum carriers of the unidirectional channel \( E^{(k)} \) (notice that no direct coupling is assigned between the \( Q_m \)'s and \( E^{(k)} \)). The free dynamics of the environmental elements are instead defined by two distinct maps: the map \( \mathcal{M}_{E_n}^{(1)}(\cdots) \) associated with the beam-splitter \( BS_1 \) that characterizes the evolution of the quantum carriers after the interactions with \( Q_1 \) and before the interactions with \( Q_2 \); and the map \( \mathcal{M}_{E_n}^{(2)}(\cdots) \) associated with the beam-splitter \( BS_2 \) and the phase shift element \( PS \) which instead acts after the collisional events with \( Q_2 \) and before those involving \( Q_3 \). Adopting the same convention used in Eqs. (35)-(38), they can be expressed as
\[
\begin{align*}
\mathcal{M}_{E_n}^{(1)}(\cdots) &= \hat{V}_{BS_1}(\cdots) \hat{V}_{BS_1}^\dagger, \\
\mathcal{M}_{E_n}^{(2)}(\cdots) &= \hat{V}_{BS_2} \hat{V}_{PS}(\cdots) \hat{V}_{PS}^\dagger \hat{V}_{BS_2}^\dagger.
\end{align*}
\]
With this choice and assuming then the same initial conditions of Eq. (11) one can verify that stationary condition still holds for the same reasons of Sec. IIIA so we won’t repeat the calculation of the coefficients \( \gamma_{m,k}^{(r)} \). By the same token it follows that the local term \( \mathcal{E}_1 \) is identical to the one in Eq. (46), because the collisional scheme is identical up to this point. Similarly the computation of the coefficients \( \gamma_{s(k')r}^{(e,f)} \), associated with the local term of \( Q_3 \), and the computation of \( \gamma_{s(k')r}^{(e,f)} \) and \( \gamma_{1,3(k'k')}^{(e,f)} \) associated with the QCS coupling connecting \( Q_1 \) with \( Q_3 \) coincide with the corresponding elements of \( S_2 \) and \( S_1 \) of the model of Sec. IIIBA yielding the expressions reported in Eqs. (69) and (73) of the main text. What is left is hence the computation of the terms associated with \( Q_2 \), i.e. \( \mathcal{E}_2, \mathcal{D}_{1 \rightarrow 2} \) and \( \mathcal{D}_{2 \rightarrow 3} \). Regarding the first we notice that exploiting (65) and invoking the definition of \( \hat{N}_{12} \) prezentet in Eq. (69), the coefficients \( \gamma_{2(k'k')}^{(e,f)} \) can be expressed as
\[
\begin{align*}
\gamma_{2(k'k')}^{(1,1)} &= \left[ \gamma_{2(2k')}^{(2,2)} \right]^* = \delta_{k_1 k_1} \delta_{k_1} \hat{b}_{E_n^{(2)}}^2 \mathcal{M}_{E_n}^{(1)}(\hat{\eta}_{E_n}) = 0, \\
\gamma_{2(k'k')}^{(1,2)} &= \delta_{k_1 k_1} \delta_{k_1} \left( \hat{b}_{E_n^{(2)}}^2 \mathcal{M}_{E_n}^{(1)}(\hat{\eta}_{E_n}) \right) \\
&= \delta_{k_1 k_1} \delta_{k_1} \left( \sqrt{\epsilon} \hat{b}^\dagger_{E_n^{(2)}} + i \sqrt{1 - \epsilon} \hat{b}_{E_n^{(1)}} \right) \\
&\times \left( \sqrt{\epsilon} \hat{b}_{E_n^{(1)}} - i \sqrt{1 - \epsilon} \hat{b}^\dagger_{E_n^{(2)}} \right) \hat{\eta}_{E_n} \\
&= \delta_{k_1 k_1} \delta_{k_1} \hat{N}_{12},
\end{align*}
\]
\[
\begin{align*}
\gamma_{2(k'k')}^{(2,1)} &= \delta_{k_1 k_1} \delta_{k_1} \left( \hat{b}_{E_n^{(2)}}^2 \mathcal{M}_{E_n}^{(1)}(\hat{\eta}_{E_n}) \right) \\
&= \delta_{k_1 k_1} \delta_{k_1} \left( \sqrt{\epsilon} \hat{b}_{E_n^{(1)}} - i \sqrt{1 - \epsilon} \hat{b}^\dagger_{E_n^{(2)}} \right) \\
&\times \left( \sqrt{\epsilon} \hat{b}_{E_n^{(2)}}^2 + i \sqrt{1 - \epsilon} \hat{b}_{E_n^{(1)}} \right) \hat{\eta}_{E_n} \\
&= \delta_{k_1 k_1} \delta_{k_1} (\hat{N}_{12} + 1),
\end{align*}
\]
which give Eq. (67). The expression (70) for $\mathcal{D}_{1\to2}$ instead follows from the identities

\begin{align*}
\gamma_{12}^{(1,1)} &= [\gamma_{12}^{(2,2)}]_k^* = \delta_k \delta_{k'} \left( \hat{b}_{E_n^{(1)}} \mathcal{M}_{E_n^{(1)}} \left( \hat{b}_{E_n^{(1)}} \hat{\eta}_{E_n} \right) \right) = 0, \\
\gamma_{12}^{(2,2)} &= [\gamma_{12}^{(2,2)}]_k^* = \delta_k \delta_{k'} \left( \hat{b}_{E_n^{(2)}} \mathcal{M}_{E_n^{(2)}} \left( \hat{b}_{E_n^{(2)}} \hat{\eta}_{E_n} \right) \right) = 0, \\
\gamma_{12}^{(1,2)} &= [\gamma_{12}^{(1,2)}]_k^* = \delta_k \delta_{k'} \left( \hat{b}_{E_n^{(1)}} \mathcal{M}_{E_n^{(1)}} \left( \hat{b}_{E_n^{(2)}} \hat{\eta}_{E_n} \right) \right) = 0,
\end{align*}

while finally (71) for $\mathcal{D}_{2\to3} \cdots$ follows from

\begin{align*}
\gamma_{23}^{(1,1)} &= [\gamma_{23}^{(2,2)}]_k^* = \delta_k \delta_{k'} \left( \hat{b}_{E_n^{(1)}} \mathcal{M}_{E_n^{(2)}} \left( \hat{b}_{E_n^{(1)}} \hat{\eta}_{E_n} \right) \right) = 0, \\
\gamma_{23}^{(2,2)} &= [\gamma_{23}^{(2,2)}]_k^* = \delta_k \delta_{k'} \left( \hat{b}_{E_n^{(2)}} \mathcal{M}_{E_n^{(2)}} \left( \hat{b}_{E_n^{(2)}} \hat{\eta}_{E_n} \right) \right) = 0,
\end{align*}

where we adopted the definitions (72).

The matrix $B^{(\ell', k')}_{mm'}$ for this system can then be cast in the following form

\[
\begin{bmatrix}
N_1 & 0 & 0 & c*(\varphi)N_1 & 0 \\
0 & N_1 + 1 & \sqrt{\epsilon_1}N_1 & 0 & c(\varphi)(N_1 + 1) \\
\sqrt{\epsilon_1}N_1 & 0 & \epsilon_1N_1 + (1 - \epsilon_1)N_2 & 0 & M_{1,2}(\varphi) \\
0 & \sqrt{\epsilon_1}(N_1 + 1) & \epsilon_1(N_1 + 1) + (1 - \epsilon_1)(N_2 + 1) & 0 & M_{1,2}(\varphi) + \lambda(\varphi) \\
c(\varphi)N_1 & 0 & M_{1,2}(\varphi) & 0 & N_{12}(\varphi) \\
0 & c*(\varphi)(N_1 + 1) & 0 & M_{1,2}(\varphi) + \lambda*(\varphi) & N_{12}(\varphi) + 1
\end{bmatrix},
\]

which upon diagonalization yields the following effective Hamiltonians contributions

\begin{align*}
\hat{H}_{1,2} &= -\frac{i}{2} \sqrt{\epsilon_1} \left( \hat{a}_1 \hat{a}_2 + \hat{a}_2 \hat{a}_1 \right), \quad (C9) \\
\hat{H}_{2,3} &= -\frac{i}{2} \left( \lambda(\varphi) \hat{a}_2 \hat{a}_3 - \lambda(\varphi) \hat{a}_3 \hat{a}_2 \right), \quad (C10) \\
\hat{H}_{1,3} &= -\frac{i}{2} \left( c*(\varphi) \hat{a}_1 \hat{a}_3 - c(\varphi) \hat{a}_3 \hat{a}_1 \right). \quad (C11)
\end{align*}