MARTINGALES AND DESCENTSTATISTICS

ALPEREN Y. ÖZDEMIR

Abstract. We provide a martingale proof of the well-known fact that the number of descents in random permutations is asymptotically normal. The same technique is shown to be applicable to other descent and descent-related statistics as they satisfy certain recurrence relation conditions. These statistics include inversions, descents in signed permutations, descents in Stirling permutations, the length of the longest alternating subsequences, descents in matchings and two-sided Eulerian numbers.

1. Introduction

Let $S_n$ be the symmetric group defined over $n$ elements. A permutation $\pi \in S_n$ is said to have a descent at position $i$ if $\pi(i) < \pi(i+1)$. Let $D_n$ be the random variable counting the number of descents of a random permutation in $S_n$.

The procedure which leads to a martingale is as follows. Given a permutation $\pi$ in the usual notation (i.e., $\pi = \pi(1)\pi(2) \cdots \pi(n)$), inserting $(n+1)$ in a random position yields a descent except the terminal position. Yet if it is inserted right next to a position where $\pi$ has a descent, the descent is broken up. Therefore, after inserting $(n+1)$, there are exactly $D_n + 1$ positions that the number of descents stays the same, otherwise it increases by 1. Then for the sigma field $\mathcal{F}_n = \sigma(D_1, \cdots, D_n)$, we have

$$E(D_{n+1}|\mathcal{F}_n) = D_n \frac{D_n + 1}{n+1} + (D_n + 1) \frac{n - D_n}{n+1}$$

$$= \frac{n}{n+1} D_n + \frac{n}{n+1}.$$

It is not hard to see that the mean of $D_n$ is $\frac{n-1}{2}$. Subtracting the mean followed by a proper scaling gives the martingale below.

$$Z_n := n \left(D_n - \frac{n-1}{2}\right).$$

The author would like to thank Jason Fulman for suggesting the problem and possible directions to follow.
Observe that \(E(Z_n) = 0\) and \(E(Z_{n+1} | F_m) = Z_m\) for \(n > m\). Therefore, \(\{Z_n, F_n\}\) is a zero-mean martingale for \(n \geq 1\). We want to show

**Theorem 1.1.** Let \(D_n\) be the number of descents in a uniformly chosen permutation from \(S_n\). Then

\[
\sup_{x \in \mathbb{R}} \left| \mathbb{P} \left( \frac{D_n - \frac{n-1}{2}}{\sqrt{\frac{n+1}{12}}} \leq x \right) - \Phi(x) \right| \leq \frac{C}{\sqrt{n}},
\]

where \(\Phi\) is the standard normal distribution and \(C\) is a constant.

The theorem has been proved by various different methods. We recall that the number of permutations in \(S_n\) with \(k\) descents is the Eulerian number \(A_{n,k}\). The fact that the Eulerian polynomial \(A_n(t) = \sum_k A_{n,k} t^k\) has only real roots, which was known to Frobenius [15], gives asymptotic normality by Harper’s approach [23]. Bender [4] expands the generating function for Eulerian numbers around its singularities to approximate the characteristic function of the distribution of number of descents. Indeed, the explicit formula for \(A(n,k)\) matches with the distribution function of \(n\) uniform random variables whose sum is between \(k\) and \((k+1)\), which is shown by Tanny [38]. Fulman [16] uses Stein’s method of exchangeable pairs to prove the asymptotic normality. Also central limit theorems involving locally dependent random variables can be employed to show the result (See the next section.)

In this paper, we use martingale limit theorem to prove asymptotic normality and apply it to other combinatorial statistics. The paper can be described in two parts. In the first part, which includes the following two sections, we study the moments of \(D_n\) and show the central limit theorem. In the second part, we abstract the martingale formulation from the initial idea by means of recurrence relations to apply it to other descent-related statistics. The last section supplies examples.

2. Moments

We already noted that \(E(D_n) = \frac{n-1}{2}\). The variance of \(D_n\) is also well-known, \(\sigma^2 := \frac{n+1}{12}\). The calculation of variance can be found in Chapter 6 of [6]; we adopt the same technique to obtain the fourth central moment. One can also use method of moments as in [7]. An efficient method is also given by Fulman [16].
First define 2-dependent random variables over $S_n$ as follows:

$$T_i(\pi) = \begin{cases} 1 & \text{if } \pi(i) > \pi(i+1), \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $D_n = \sum_{i=1}^{n-1} T_i$. The third moment can be written as

$$E(D^3_n) = \sum_i E(T^3_i) + 6 \sum_{i<j} E(T^2_i T_j) + 6 \sum_{i<j<k} E(T_i T_j T_k).$$

The first term satisfies

$$\sum_i E(T^3_i) = \sum_i E(T_i) = \frac{n-1}{2}.$$ 

For the second term, $6 \sum_{i<j} E(T^2_i T_j) = 6 \sum_{i<j} E(T_i T_j)$, we use 2-dependence of $T_i$'s. There are $(n-2)$ cases where $j = i+1$, for which $E(T_i T_j) = \frac{1}{6}$. For the remaining $(n-2)^2$ cases, $i$ and $j$ differ by more than 1, so they are independent. The expected value in the latter case is $\frac{1}{4}$. Analyzing the cases in a similar fashion for the last term yields

$$E(D^3_n) = \frac{n-1}{2} + 6 \left( \frac{n-2}{6} + \frac{(n-2)}{4} \right) + 6 \left( \frac{n-3}{24} + 2 \frac{(n-3)}{12} + \frac{(n-3)}{8} \right)$$

$$= \frac{(n^2 - n + 2)(n-1)}{8}.$$ 

The fourth moment is given by

$$E(D^4_n) = \sum_i E(T^4_i) + 8 \sum_{i<j} E(T^3_i T_j) + 6 \sum_{i<j} E(T^2_i T^2_j) +$$

$$36 \sum_{i<j<k} E(T^2_i T_j T_k) + 24 \sum_{i<j<k<l} E(T_i T_j T_k T_l).$$

By the same argument as above,

$$E(D^4_n) = \frac{n^4}{16} + \frac{5n^3}{8} + \frac{13n^2}{48} + \mathcal{O}(n).$$

Finally, we calculate the fourth central moment,

$$E[(D_n - E(D_n))^4] = \frac{3n^2}{144} + \mathcal{O}(n),$$

as expected by normality.

3. Central limit theorem

A comprehensive account on martingale limit theorems and related topics is [22]. The relevant limit theorem therein requires conditions analogous to classical central limit theorems’ for sums of independent random variables. The first one is Lindeberg condition, or asymptotic
negligibility as is put in [22]. The second condition is on the con-
vergence of conditional variance of martingale differences. The limiting
behavior of the martingale is essentially determined by this quantity
or its counterpart \( U^2_n := \sum X_i^2 \) where \( X_i \)'s are martingale differences.
See Chapter 2 of [22] for their relationship.
A martingale array with small deviations for martingale differences
can be obtained as prescribed in [22]. Take \( F_{ni} = F_i \). Then define
\( Z_{ni} = s^{-1}n Z_i \) for \( 1 \leq i \leq n \), where \( s_n \) is the standard deviation of \( Z_n \).
In this way, the expectation of the conditional variance is normalize
d to be 1.
The central limit theorem reads:

**Theorem 3.1.** [22] Let \( \{Z_{ni}, F_{ni}, 1 \leq i \leq k_n, n \geq 1\} \) be a zero-mean,
square integrable martingale array with differences \( X_{ni} \), and \( \eta^2 \) be an
almost surely finite random variable. Suppose that
for all \( \epsilon > 0 \),
\[
\mathbb{E}[X_{ni}^2 I(|X_{ni}| > \epsilon)|F_{n,i-1}] \xrightarrow{p} 0,
\]
and the \( \sigma \)-fields are nested, i.e., \( F_{ni} \subseteq F_{n+1,i} \) for \( 1 \leq i \leq k_n, n \geq 1 \).
Then \( Z_{nk_n} = \sum_{i=1}^{k_n} X_{ni} \xrightarrow{d} Z \), where \( Z \) has characteristic function
\[
\mathbb{E}[\exp(-\frac{1}{2} \eta^2 t^2)].
\]

**Proof of Asymptotics Normality in Theorem 1.1.** We verify the condi-
tions of Theorem 3.1 to conclude asymptotic normality. The row length
\( k_n \) of the array in the theorem is equal to \( n \) in our case. \( \{F_{ni}, 1 \leq i \leq n-1\} \) is a nested sequence of \( \sigma \)-fields by definition. Now we look at
the martingale differences, \( X_{ni} = Z_{ni} - Z_{n,i-1} \). Notice that \( s_n = n \sigma_n \) in
our case. We have,
\[
X_{ni} = \frac{1}{n \sigma_n} (iD_i - (i-1)D_{i-1} - (i-1)).
\]
Let \( W_i := D_i - \frac{i-1}{2} \) be the zero-mean random variable for the number
of descents. Conditioned on \( F_{n,i-1} \), \( X_{ni} \) is
\[
\frac{1}{n \sigma_n} \begin{cases} W_{i-1} - \frac{i}{2}, & \text{with prob. } \frac{1}{2} + \frac{W_{i-1}}{2}, \\ W_{i-1} + \frac{i}{2}, & \text{with prob. } \frac{1}{2} - \frac{W_{i-1}}{2}. \end{cases}
\]
We first verify the Lindeberg condition (2). Since \( \sigma_n = \sqrt{\frac{n+1}{12}} \),
we have \( |X_{ni}| \leq \frac{\sqrt{n+1}}{n \sqrt{n+1}} \). So, for given \( \epsilon > 0 \) and sufficiently large \( n \),
\( P(|X_{ni}| > \epsilon) = 0 \) where \( 1 \leq i \leq n - 1 \). Hence, (2) is satisfied.
Next we verify the convergence of conditional variance (3). By (4),

\[ \mathbb{E}[X_{ni}^2 | F_{n,i-1}] = \frac{1}{n^2 \sigma_n^2} \left( \frac{i^2}{4} - W_{i-1}^2 \right). \]

Then the conditional variance for \( Z_{nn} \) is

\[ V_{nn}^2 := \sum_i \mathbb{E}[X_{ni}^2 | F_{n,i-1}] = \frac{1}{n^2 \sigma_n^2} \left( \frac{\sum i^2}{4} - \sum W_{i-1}^2 \right), \]

(5)

\[ = 1 + \frac{1}{2n} - \frac{12}{n^2(n+1)} \sum W_{i-1}^2. \]

Observe that

\[ \mathbb{E}(V_{nn}^2) = 1 + \frac{1}{2n} - \frac{12}{n^2(n+1)} \sum \mathbb{E}(W_{i-1}^2) \]

\[ = 1 + \frac{1}{2n} - \frac{12}{n^2(n+1)} \sum \frac{i}{12} \]

\[ = 1. \]

Below, we use Cauchy-Schwarz inequality and (1) to show that the variance of \( V_{nn}^2 \) converges to 0.

\[ \mathbb{E}[(V_{nn}^2 - 1)^2] = \frac{1}{n^4 \sigma_n^4} \sum_{1 \leq i, j \leq n-1} \mathbb{E}(W_{i-1}^2 W_{j-1}^2) - \frac{1}{n^3 \sigma_n^4} \]

\[ \leq \frac{1}{n^4 \sigma_n^4} \sum_{1 \leq i, j \leq n-1} \sqrt{\mathbb{E}(W_{i-1}^4) \mathbb{E}(W_{j-1}^4)} - \frac{1}{n^3 \sigma_n^4} \]

\[ \leq \frac{C}{n^4 \sigma_n^4} \sum_{1 \leq i, j \leq n-1} \frac{i j}{n^3 \sigma_n^4} \]

\[ \leq \frac{C(n-1)^2 n^2}{n^4 \sigma_n^4} - \frac{1}{n^3 \sigma_n^4} \to 0. \]

Finally, Chebyschev’s inequality gives that \( V_{nn}^2 = \sum_i \mathbb{E}[X_{ni}^2 | F_{n,i-1}] \) converges to 1 in probability.

Both conditions are verified; the result follows from Theorem 3.1.

\[ \square \]

4. Recurrence relations and martingale formulation

The primary goal of this section is to extend the martingale technique to different permutation statistics. First consider the Eulerian
polynomial
\[ A_n(t) = \sum_{\pi \in S_n} t^{\text{des}(\pi)} = \sum_{k \geq 0} A_{n,k} t^k, \]
where \( \text{des}(\pi) \) denotes the number of descents in \( \pi \in S_n \). Another definition of the Eulerian polynomial is the generating function
\[ \frac{A_n(t)}{(1-t)^{n+1}} = \sum_{k \geq 0} (k+1)^n t^k. \]
The coefficients in \( A_n(t) \) can be expressed recursively by simple manipulations of the sum, (See Foata’s survey [14] for eight different expressions for Eulerian numbers.)
\[ A_{n+1,k} = (k+1)A_{n,k} + (n-k+1)A_{n,k-1}. \]
Dividing both sides by \( A_{n+1}(1) = (n+1)! \),
\[ \frac{A_{n+1,k}}{(n+1)!} = \frac{k+1}{n+1} \frac{A_{n,k}}{n!} + \frac{n-k+1}{n+1} \frac{A_{n,k-1}}{n!}. \]
Equivalently,
\[ \mathbb{P}(D_{n+1} = k) = \frac{k+1}{n+1} \mathbb{P}(D_n = k) + \frac{n-k+1}{n+1} \mathbb{P}(D_n = k-1). \]
In order to relate it to martingales, we write the next equation.
\[ \mathbb{P}(D_{n+1} = k + 1) = \frac{k+2}{n+1} \mathbb{P}(D_n = k + 1) + \frac{n-k}{n+1} \mathbb{P}(D_n = k). \]
Observe that the coefficients of \( \mathbb{P}(D_n = k) \) in subsequent recurrences above agree with the probabilities of the martingale defined in the Introduction.

Next we find conditions for martingale representation in a general setting. Let \( P_{n,k} \) be nonnegative integers for \( n = 1, 2, \ldots \) and \( 0 \leq k \leq n \), which is referred as combinatorial array in [32]. Also define \( P_n(t) = \sum_{k=0}^n P_{n,k} t^k \) as before. Suppose \( \{P_{n,k}\} \) satisfies recurrences of the form
\[ P_{n+1,k} = \alpha_{n+1,k}^{(0)} P_{n,k} + \alpha_{n+1,k}^{(1)} P_{n,k-1} + \cdots + \alpha_{n+1,k}^{(s)} P_{n,k-s}, \]
\[ P_{n+1,k+1} = \alpha_{n+1,k+1}^{(0)} P_{n,k+1} + \alpha_{n+1,k+1}^{(1)} P_{n,k} + \cdots + \alpha_{n+1,k+1}^{(s)} P_{n,k-s+1}, \]
\[ \vdots \]
\[ P_{n+1,k+s} = \alpha_{n+1,k+s}^{(0)} P_{n,k+s} + \alpha_{n+1,k+s}^{(1)} P_{n,k+s-1} + \cdots + \alpha_{n+1,k+s}^{(s)} P_{n,k}, \]
where $\alpha^{(i)}_{n+1,k} \geq 0$ for all $0 \leq i \leq s$. In addition, we want the diagonal coefficients to add up to $\frac{P_{n+1}(1)}{P_n(1)}$, i.e.,

$$
\alpha^{(0)}_{n+1,k} + \alpha^{(1)}_{n+1,k+1} + \cdots + \alpha^{(s)}_{n+1,k+s} = \frac{P_{n+1}(1)}{P_n(1)}.
$$

Let $p_{n+1,i} := \frac{1}{P_{n+1}(1)/P_n(1)} \alpha^{(i)}_{n+1,k+i}$ for ease of notation. Following the argument above, (6) and nonnegativity of the coefficients are sufficient to conclude that

$$
Z_{n+1} = \begin{cases} 
Z_n, & \text{with prob. } p_{n+1,0}, \\
Z_n + 1, & \text{with prob. } p_{n+1,1}, \\
\vdots \\
Z_n + s, & \text{with prob. } p_{n+1,s}
\end{cases}
$$

is a submartingale with respect to $\mathcal{F}_n = \sigma(Z_1, \cdots, Z_n)$. Submartingale is defined in the same way with martingale, the only difference is that we replace $\mathbb{E}(Z_{n+1}|\mathcal{F}_m) = Z_m$ by the weaker condition: $\mathbb{E}(Z_{n+1}|\mathcal{F}_m) \geq Z_m$ for $n > m$.

In Section 5, we give examples of descent-related statistics that have recurrence relations which satisfy those conditions, and admit a martingale formulation. A counterexample is Stirling numbers of second kind, which is recursively defined as

$$
\{ \binom{n+1}{k} \} = k \binom{n}{k} + \binom{n}{k-1}.
$$

It fails the condition (6) in which the right hand side is taken as $\sum_k \binom{n+1}{k+1} / \sum_k \binom{n}{k}$, which indeed has no nice expression.

The next goal is to derive recurrence relations from the rational generating functions of the form

$$
\frac{P_{dn}(t)}{(1-t)^{dn+1}} = \sum_{k \geq 0} f_n(k) t^k,
$$

where $f_n$ has degree $d(n+1)$ if and only if $P(1) \neq 0$ (See Corollary 4.3.1 of [35]). These functions appear in many different contexts. The order polynomials of labelled posets has rational generating functions, [35] where the classical Eulerian polynomial is associated with the special case of an anti-chain. Another example is Ehrhart polynomials of
convex polytopes; if the polytope is taken to be the unit cube, the generating function has Eulerian polynomial in numerator (See Chapter 4 of [35] and Chapter 8 of [31].)

Suppose $f_{n+1}(k) = g_n(k)f_n(k)$ for some degree $d$ polynomial $g_n(k)$ for all $n$. For instance, $d = 1$ and $g_n(k) = k + 1$ for Eulerian numbers. In general, if $g_n(k) = \alpha_n k + \beta_n$, we have

$$
P_n(t) = \alpha_n t \left( \frac{P_{n-1}(t)}{(1-t)^n} \right) + \beta_n \frac{P_{n-1}(t)}{(1-t)^n}.
$$

This implies

$$
P_n(t) = ((\alpha_n n - \beta_n) t + \beta_n) P_{n-1}(t) + \alpha_n t(1-t) P'_{n-1}(t),
$$

and $P_n(1) = n! \prod_{i=1}^{n} \alpha_i$. By Proposition 3.5. in [29], $P_n(t)$ has only real roots as long as the leading coefficient $\alpha_n \geq 0$. In that case, the following theorem applies.

**Theorem 4.1.** [4] Let $P_n(t) = \sum_k P_{n,k} t^k$ is a polynomial with all non-positive real roots, and $X_n$ be a random variable such that $P(X_n = k) = \frac{P_{n,k}}{P_n(t)}$. If $\text{var}(X_n)$ goes to infinity as $n \to \infty$, then $X_n$ is asymptotically normal.

See [32] for a great account of polynomials with only real zeros in this context.

If $g_n(k)$ is not linear in $k$, real-rootedness does not necessarily hold (See Proposition 5.1.) Next, we identify the coefficients of recurrences for an arbitrary degree of $g_n$. An example where $g_n(k)$ is of degree 2 is the number of descents in fixed-point free permutation, which we study in the next section.

It is easy to derive the following from (9).

$$
P_{dn,k} = \sum_{i=0}^{k} (-1)^i \binom{d(n+1)+1}{i} f_n(k-i).
$$

Working the expression above, Koutras [28] obtains the formula

$$
P_{n+1,k} = (k \alpha_n + \beta_n) P_{n,k} + (\alpha_n(n-k+2) - \beta_n) P_{n,k-1}
$$

where $f_n(k) = \alpha_n k + \beta_n$. The technique is to use special cases of the identities below to break (12) into parts. The first one is

$$
(i)_l \binom{d(n+1)+1}{i} = (d(n+1)+d)_l \binom{dn+d-l+1}{i-l},
$$
and the second one is a consequence of Vandermonde’s identity,

\[
\binom{dn + d - l + 1}{i - l} = \sum_{m=l}^{d} \binom{d - l}{m - l} \binom{dn + 1}{i - m}.
\]

We refer to them later in the section.

Now consider the operator \( \Delta = \sum_{k \geq 0} \frac{(-1)^k}{k!} D^k \) (not to be confused with the difference operator), where \( D \) is the usual differentiation. \( \Delta^i \) means applying \( \Delta \) \( i \) times. It acts on monomials by shifting, i.e. \( \Delta^i x^n = (x - i)^n \), which can be shown inductively \cite{14}. At the same time, it can be interpreted as Taylor expansion, i.e.,

\[
\Delta^i f(k) = \sum_{j=0}^k (-1)^i \frac{j^i}{j!} f^{(j)}(k).
\]

Given that \( f_{n+1}(k) = g_n(k)f_n(k) \) where \( g_n(k) \) is a polynomial of degree \( d \), we have

\[
P_{d(n+1),k} = \sum_{i=0}^k (-1)^i \binom{d(n+1) + 1}{i} g_n(k - i)f_n(k - i)
\]

\[
= \sum_{i=0}^k (-1)^i \binom{d(n+1) + 1}{i} \Delta^i g_n(k)f_n(k - i)
\]

by \((12)\). In order to get use of \((14)\), we change the basis as

\[
x^n = \sum_{k=0}^n \binom{n}{k} (x)_k,
\]

where \( \binom{n}{k} \) is Stirling number of second type and \( (x)_k = x(x-1)\cdots(x-k+1) \) is the falling factorial (Section 2.4. of \cite{2}). Then, we have

\[
\Delta^i g_n(k) = \sum_{j=0}^i (-1)^j \frac{j^i}{j!} g_n^{(j)}(k)
\]

\[
= \sum_{j=0}^i \frac{(-1)^j}{j!} g_n^{(j)}(k) \sum_{l=0}^j \binom{j}{l} (i)_l
\]

\[
= \sum_{l=0}^{d} \left( \sum_{j=l}^{d} \frac{(-1)^j}{j!} g_n^{(j)}(k) \binom{j}{l} \right) (i)_l.
\]
Plugging the expression above in (16) and applying (15) and (14) accordingly, we arrive at (17)

\[ P_{d(n+1),k} = \sum_{m=0}^{d} (-1)^m \left( \sum_{l=0}^{m} (d(n+1)+1)l \binom{d-l}{m-l} [(i)_l] \Delta^i g_n(k) \right) P_{d_n,k-m}, \]

where

\[ [(i)_l] \Delta^i g_n(k) = \sum_{j=l}^{d} \frac{(-1)^j}{j!} g^{(j)}_n(k) \left\{ \frac{j}{l} \right\} \]

is the coefficient of \((i)_l\) in \(\Delta^i g_n(k)\). This gives a rather complicated but explicit recurrence relation for combinatorial arrays defined in the rational form (9) for which \(f_{n+1}(k)/f_n(k)\) is a polynomial of \(k\).

5. Applications

The first two examples are simple cases, in the sense that they can be written as sums of independent random variables (which is shown in [12]) by the same formulation for descents.

5.1. Inversions. The number of inversions in a permutation \(\pi\) is defined to be the number of all pairs \(i \leq j\) such that \(\pi(j) < \pi(i)\). Let \(I_n\) be the random variable counting the number of inversions in a randomly chosen permutation from \(S_n\). Inserting \(n+1\) in any possible \(n+1\) positions and counting the probabilities, we obtain

\[ I_{n+1} = \left\{ I_n + i \text{ with probability } \frac{1}{n+1}, 0 \leq i \leq n \right\}. \]

Then subtracting the mean of \(I_n\), we have the zero-mean martingale below.

\[ Z_n := I_n - \binom{n}{2}. \]

The martingale differences \(X_{ni}\) conditioned on \(\mathcal{F}_{n,i-1}\) is

\[ \frac{1}{\sigma_n} \left\{ j - \frac{i-1}{2} \text{ with probability } \frac{1}{i}, 0 \leq j \leq i - 1 \right\}, \]

which are independent random variables for \(1 \leq i \leq n\). Therefore, we can use classical limit theorems for sums of independent random variables. The Lindeberg condition suffices to show the asymptotic normality (See Chapter 10 of [12]).

We first calculate the variance of the differences,

\[ \mathbb{E}[X^2_{ni} | \mathcal{F}_{n,i-1}] = \frac{1}{\sigma_n^2} \sum_{j=0}^{i-1} \frac{(j - \frac{i-1}{2})^2}{i} = \frac{1}{\sigma_n^2} \frac{i^2 - 1}{12}. \]
Since \( V_{n}^{2} := \sum_{i} E[X_{n,i}^{2} | \mathcal{F}_{n,i-1}] \) is deterministic in this case and its expected value is 1, we have
\[
\sigma_{n}^{2} = \sum_{i} i^{2} - \frac{1}{12} = \frac{n(2n + 5)(n - 1)}{72}.
\]
So that \(|X_{n,i}| \leq C \frac{1}{\sqrt{n}}\), which goes to 0 as \( n \to \infty \). The Lindeberg condition is verified, the central limit theorem applies. It can be shown that the absolute third moment is bounded, so Berry-Esseen theorem gives an error term of order \( n^{-1/2} \).

5.2. Cycles. Let \( Q_{n} \) denote the number of cycles of a uniformly chosen random permutation in \( S_{n} \). Goncharov [19] shows a central limit theorem for \( Q_{n} \) considering it asymptotically as sum of Poisson distributions of the number of fixed length cycles. Suppose we insert \( n + 1 \) either in any of the cycles of a given permutation, or place it as a fixed-point. This defines the zero-mean martingale
\[
Z_{n} = Q_{n} - \sum_{k=1}^{n} \frac{1}{k}.
\]
Observe that \( E(Q_{n}) \sim \log n \). The martingale difference \( X_{n,i} = Z_{n,i} - Z_{n,i-1} \) conditioned on \( \mathcal{F}_{n,i-1} \) is
\[
= \frac{1}{\sigma_{n}} \left\{ \begin{array}{ll}
1 - \frac{1}{i} & \text{with prob. } \frac{1}{i}, \\
-\frac{1}{i} & \text{with prob. } \frac{i-1}{i}.
\end{array} \right.
\]
The variance of differences is
\[
E[X_{n,i}^{2} | \mathcal{F}_{n,i+1}] = \frac{1}{\sigma_{n}^{2}} \left( \frac{i-1}{i^{2}} \right),
\]
and
\[
\sigma_{n}^{2} = \sum_{i} \sum_{i=1}^{n} \frac{i-1}{i^{2}} \sim \log n.
\]
The Lindeberg conditioned is easily verified, since \(|X_{n,i}| \leq \frac{1}{\sqrt{\log n}} \). Asymptotic normality follows.

5.3. Signed permutations. The notion of descent in relation with group structure has its generalization in Coxeter groups (See Chapter 11 and 13 of [31].) A particular case other than the symmetric group that we are interested is Coxeter groups of type B, also known as hyperoctahedral groups. We denote them by \( W_{n} \) for \( n = 1, 2, \cdots \). They are isomorphic to the set of signed permutations with usual multiplication in \( S_{n} \). A signed permutation \( \pi \) is defined to be the mapping
\[
\pi : \{-n, \cdots, -1, 0, 1, \cdots, n\} \to \{-n, \cdots, -1, 0, 1, \cdots, n\}
\]
which satisfies $\pi(-i) = -\pi(i)$. The descent set of $\pi$ is defined to be

$$\text{Des}(\pi) = \{0 \leq i \leq n-1 : \pi(i) < \pi(i+1)\}$$

It is clear that the probability that $\pi$ has a descent at position $i$ is $\frac{1}{2}$. Unlike the case in $S_n$, we also take the zeroth position into account. Let $B_n$ count the number of descents of a random signed permutations. Then the linearity of expectation gives $E(B_n) = \frac{n}{2}$. Provided that

$$B_n(t) := \sum_{\pi \in W} t^{\text{des}(\pi)} = \sum_{k=1}^{n} B_{n,k} t^k,$$

we have (Theorem 13.3. of [31])

$$\frac{B_n(t)}{(1-t)^{n+1}} = \sum_{k \geq 0} (2k+1)^n t^k,$$

and also the recurrence relation

$$B_{n+1,k} = (2k+1)B_{n,k} + (2n-2k+3)B_{n,k-1}.$$ 

It satisfies the condition (6), so we have the submartingale below.

$$B_{n+1} = \begin{cases} B_n, & \text{with prob. } \frac{2B_{n+1}}{2n+2}, \\ B_n + 1, & \text{with prob. } \frac{2n-2B_{n+1}}{2n+2}. \end{cases}$$

After normalization, $Z_n = n \left(B_n - \frac{n}{2}\right)$ is a zero-mean martingale. In order to derive the moments, we first use (11) and obtain

$$B_n(t) = (1 + (2n-1)t)B_{n-1}(t) + 2t(1-t)B'_{n-1}(t).$$

Then, we can use method of moments observing that

$$E(B_n) = \frac{B''_n(1)}{B_n(1)}.$$ 

Similarly,

$$E(B^2_n) = \frac{B''_n(1) + B'_n(1)}{B_n(1)}.$$ 

We do not carry out the calculations, but point out that the leading terms of the moments are asymptotically same with the classical Eulerian numbers’. Consider the generating function

$$B(u, t) = \sum_{n \geq 0} B_n(t) \frac{u^n}{n!} = \frac{(t-1)e^{u(t-1)}}{t - e^{2u(t-1)}},$$ 

which can be found in Chapter 13 of [31]. It has a simple pole at $r(t) = \frac{\log t}{2(t-1)}$, which is a constant multiple of the simple pole of $A(u, t)$, the generating function for Eulerian numbers. (See Section 9.6. of [13] for an accessible singularity analysis for combinatorial arrays.) Then,
Theorem 1 in [?] verifies the claim.

In fact, Eulerian polynomials for all Coxeter groups have only real roots, so that Theorem 4.1 applies. The last open case was for Coxeter groups of type D, which is shown in [34]. A simple recurrence relation for Eulerian numbers for Coxeter groups of type D is not available to the best of author’s knowledge. But their relation to the first two types’ is rather simple,

\[ D_{n,k} = B_{n,k} - n2^{n-1}A_{n,k-1}. \]

5.4. Stirling permutations. Gessel and Stanley define Stirling polynomials, \( f_n(k) = \binom{n+k}{k} \), and their generating function

\[
\frac{C_n(t)}{(1-t)^{n+1}} = \sum_{k \geq 0} \left\{ \binom{n+k}{k} \right\} t^k,
\]

in [18], and they provide a combinatorial interpretation for \( C_n(t) \). The coefficient of \( t^k \) in \( C_n(t) \), call it \( C_{n,k} \), is the number of Stirling permutations with exactly \( k \) descents. Stirling permutations are permutations of the multiset \( \{1, 1, 2, 2, \ldots, n, n\} \) such that the numbers between two occurrences of \( i \) are larger than \( i \) for all \( 1 \leq i \leq n \). The numbers \( \{C_{n,k}\} \), known as second-order Eulerian numbers, appear in different branches of combinatorics (See [21].)

By (8), it can be easily shown that

\[
C_n(t) = (2n-1)tC_{n-1}(t) + t(1-t)C'_{n-1}(t).
\]

Let \( C_n \) be the random variable counting descents in Stirling permutations. Then working the coefficients, we obtain the submartingale

\[
C_{n+1} = \begin{cases} 
C_n, & \text{with prob. } \frac{C_{2n+1}}{2n+1}, \\
C_n + 1, & \text{with prob. } \frac{2n-C_n+1}{2n+1}.
\end{cases}
\]

As in the previous example, the moments can be studied by (19). Note that the number of Stirling permutations is \( C_n(1) = (2n - 1)!! \equiv 1 \cdot 3 \cdots (2n - 1) \). For the first moment, we have

\[
\mathbb{E}(C_n) = 1 + \frac{2n - 2}{2n - 1} \mathbb{E}(C_{n-1}),
\]

which does not yield a simple expression. But a purely combinatorial count for the first moment by Bóna [5] leads him to this curious identity

\[
\mathbb{E}(C_n) = \sum_{k=0}^{n-1} \prod_{i=1}^{k} \frac{2n - 2i}{2n - 2i + 1} = \frac{2n + 1}{3}.
\]
It is shown in [5] that \( C_n(t) \) has only real zeros. Also, observe that (19) satisfies (11) with a positive coefficient. Therefore, Theorem 4.1 is applicable to show asymptotic normality.

5.5. **Alternating runs.** For \( \pi \in S_n \), \( \pi \) is said to change direction at position \( i \), if either \( \pi(i − 1) > \pi(i) < \pi(i + 1) \) or \( \pi(i − 1) < \pi(i) > \pi(i + 1) \). Then \( \pi \) has \( k-\)alternating runs if there exist exactly \( k − 1 \) positions that it changes direction. (See Section 1.2 of [6].) Let \( G_{n,k} \) be the number of \( k \)-alternating runs, then it is known to satisfy the recurrence relation

\[
G_{n+1,k} = kG_{n,k} + 2G_{n,k-1} + (n - k + 1)G_{n,k-2},
\]

(20)

\[
G_{n+1} = \begin{cases} 
G_n, & \text{with prob. } \frac{G_n}{n+1}, \\
G_n + 1, & \text{with prob. } \frac{2}{n+1}, \\
G_n + 2, & \text{with prob. } \frac{n-G_n}{n+1}.
\end{cases}
\]

We can calculate the first moment simply by observing that the probability of \( \pi \) to change direction at each position \( i \) is \( \frac{2}{3} \). We have \( \mathbb{E}(G_n) = \frac{2}{3}(n - 2) + 1 \), whence we have the zero-mean martingale

\[
Z_n = \frac{n(n-1)}{2} \left( G_n - \frac{2n - 1}{3} \right).
\]

Another important statistic, which is closely related to alternating runs is the length of the longest alternating subsequences. Let \( L_n \) be the corresponding random variable. We have the simple relation [37],

\[
L_{n,k} = \frac{1}{2}(G_{n,k-1} + G_{n,k}).
\]

Putting (20) and (21) together, we have

\[
L_{n+1,k} = kL_{n,k} + L_{n,k-1} + (n - k + 2)L_{n-1,k-2},
\]

which defines the submartingale

\[
L_{n+1} = \begin{cases} 
L_n, & \text{with prob. } \frac{L_n}{n+1}, \\
L_n + 1, & \text{with prob. } \frac{1}{n+1}, \\
L_n + 2, & \text{with prob. } \frac{n-L_n}{n+1}.
\end{cases}
\]

Similarly, \( Z_n = \frac{n(n-1)}{2} \left( L_n - \frac{4n+1}{6} \right) \) is a zero-mean martingale. The martingale difference \( Z_{n,i} - Z_{n,i-1} = X_{n,i} \) conditioned on \( \mathcal{F}_{i-1} \) is

\[
Z_{n,i} = \frac{(i-1)}{n(n-1)\sigma_n} \begin{cases} 
2L_{i-1} - 2i + 1, & \text{with prob. } \frac{i-1}{i}, \\
2L_{i-1} - i + 1, & \text{with prob. } \frac{1}{i}, \\
2L_{i-1} + 1, & \text{with prob. } \frac{i-L_{i-1}-1}{i}.
\end{cases}
\]
where $\sigma_n = \frac{8n}{45} - \frac{13}{80}$. The higher moments are also given in [37]. Different techniques are used to show central limit theorem [25, 39, 40].

5.6. Matchings. A matching is a fixed-point free involution, a permutation consisting only of transpositions, in $S_{2n}$. The asymptotic normality of number of descents in matchings is addressed by Kim [26]. It is proven by pointwise convergence of the moment generating function. We obtain the same result by martingales. Let

$$J_{2n}(t) := \sum_{\pi \in J_{2n}} t^{des(\pi)} = \sum_{k=0}^{2n-1} J_{2n,k} t^k.$$ 

Désarménien and Foata [10] derived the generating function below using algebraic properties of Schur functions.

$$\sum_{n \geq 0} J_n(t) \frac{u^n}{(1-t)^{n+1}} = \sum_{k \geq 0} \frac{t^k}{(1-u^2)^{(k+1)/2}}.$$ 

Note that $J_n(t) = 0$ if $n$ is odd, and $J_{2n}(1) = (2n-1)!!$. Equating the coefficients of $u^{2n}$ gives

$$\frac{J_{2n}(t)}{(1-t)^{2n+1}} = \sum_{k \geq 0} \binom{(k+1)/2 + n - 1}{n} t^k.$$ 

It is tempting to ask if Theorem 4.1 is applicable in this case. The answer is negative by the following proposition.

**Proposition 5.1.** $J_{2n}(t)$ is not log-concave.

**Proof:** By (12) we have

$$J_{2n}(t) = \sum_{k=0}^{2n-1} J_{2n,k} t^k = \sum_{k=0}^{2n-1} \left( \sum_{i=0}^k (-1)^i \binom{2n+1}{i} \binom{(k-i+1)/2 + n - 1}{n} \right) t^k.$$ 

By inverting the coefficients (See Chapter 3.3 of [33].),

$$a_{n,k} := \binom{(k+1)/2 + n - 1}{n} = \sum_{i=0}^k \binom{2n+i}{i} J_{2n,k-i}.$$ 

Then, we have

$$\sum_{k} a_{n,k} x^k = \left( \sum_{i} \binom{2n+i}{i} x^i \right) \left( \sum_{j} J_{2n,j} x^j \right).$$
The first three terms of \( \{a_{n,k}\} \) reveal below that it is not a log-concave sequence for any \( n \in \mathbb{N} \).

\[
a_{n,2}^2 = \binom{n+2}{2}^2 < \binom{n+6}{2} = a_{n,1}a_{n,3}.
\]

On the other hand, the coefficients of the first sum on the right hand side is a log-concave sequence. But Proposition 1 in [36] says that the product of two log-concave polynomials is also log-concave, which implies \( \{J_{2n,k}\} \) is not log concave. \( \square \)

Since real rootedness implies log-concavity, the roots of \( J_{2n}(t) \) are not real only. So Theorem 4.1 is not applicable in this case. Nevertheless, asymptotic normality is shown by Kim using generating function to determine the asymptotic behaviour of the moments. We give a martingale proof below. The recurrence relation is derived in [20], it can also be obtained from (17).

\[
J_{2n+2,k} = \frac{k(k+1)+2n}{2n+2}J_{2n,k} + \frac{2(k-1)(2n-k+1)+2}{2n+2}J_{2n,k-1} + \frac{(2n-k+2)(2n-k+3)+2n}{2n+2}J_{2n,k-2}.
\]

Let \( M_{2n} \) count the number of descents in random matchings. We can verify (6) and define the submartingale

\[
M_{2n+2} = \begin{cases} 
M_{2n}, & \text{with prob. } \frac{M_{2n}(M_{2n}+1)+2n}{(2n+1)(2n+2)}, \\
M_{2n} + 1, & \text{with prob. } \frac{2M_{2n}(2n-M_{2n})+2}{(2n+1)(2n+2)}, \\
M_{2n} + 2, & \text{with prob. } \frac{(2n-M_{2n})(2n-M_{2n}+1)+2n}{(2n+1)(2n+2)}. 
\end{cases}
\]

We have \( \mathbb{E}(M_{2n+2} | \mathcal{F}_n) = \frac{n}{n+1}M_{2n} + \frac{2n+1}{n+1} \). Note that \( \mathbb{E}(M_{2n}) = n \) (See [26] for its derivation.) Then by properly scaling, we have the zero-mean martingale \( Z_n = n(M_{2n} - n) \). Conditioned on \( \mathcal{F}_{n-1} \), the martingale difference \( X_{ni} \) is

\[
= \frac{1}{n\sigma_n} \begin{cases} 
W_{i-1} - i, & \text{with prob. } \frac{(W_{i+1}+i-1)(W_{i+1}+i)+2i-2}{2i(2i-1)}, \\
W_{i-1}, & \text{with prob. } \frac{2(W_{i+1}+i-1)(i-W_{i+1})+2}{2i(2i-1)}, \\
W_{i-1} + i, & \text{with prob. } \frac{(i-1-W_{i+1})(i-W_{i+1}+2i-2)}{2i(2i-1)}, 
\end{cases}
\]

where \( W_i = M_{2i} - i \). Next we verify the convergence of conditional variance (3). By (4),

\[
\mathbb{E}[X_{ni}^2 | \mathcal{F}_{n-1}] = \frac{(i-1)}{(2i-1)n^2\sigma_n^2} (i^2 + 2i - W_{i-1}^2).
\]
Then, the conditional variance for $Z_{nn}$ is
\[
V_{nn}^2 := \sum_i E[X_{ni}^2 | F_{n,i-1}] = \frac{1}{n^2 \sigma_n^2} \left( \sum_i \frac{i^3 + i^2 - 2i}{2i - 1} - \sum_i \left( \frac{i - 1}{2i - 1} \right) W_{i-1}^2 \right),
\]
\[
= \frac{4n^2 + 15n - 4}{24n^2} - \frac{1}{n^2 \sigma_n^2} \sum_i \left( \frac{i - 1}{2i - 1} \right) W_{i-1}^2 + o(1)
\]
\[
= 1 - \frac{1}{n^2 \sigma_n^2} \sum_i \left( \frac{i - 1}{2i - 1} \right) W_{i-1}^2 + o(1).
\]

Finally, we want to show that $V_{nn} \xrightarrow{p} 1$.
\[
E[(V_{nn}^2 - 1)^2] = \frac{1}{n^4 \sigma_n^4} \sum_{1 \leq i,j \leq n-1} \left( \frac{i - 1}{2i - 1} \right) \left( \frac{j - 1}{2j - 1} \right) E(W_{i-1}^2 W_{j-1}^2) + o(1)
\]
\[
\leq \frac{1}{4n^2 \sigma_n^2} \sum_{1 \leq i,j \leq n-1} \sqrt{E(W_{i-1}^4)E(W_{j-1}^4)} + o(1).
\]

The moments up to fifth degree are calculated in [27]. We can compute the fourth central moment to show that it is of order $n^2$. Then similar to the case in Section 3,
\[
E[(V_{nn}^2 - 1)^2] \leq \frac{C}{n^4 \sigma_n^4} \sum_{1 \leq i,j \leq n-1} ij + o(1)
\]
\[
\leq \frac{C(n - 1)^2 n^2}{n^4 \sigma_n^4} + o(1) \to 0.
\]

By Chebyshev’s inequality, $V_{nn} \xrightarrow{p} 1$. Therefore, Theorem 3.1 applies.

Matchings define a particular conjugacy class in $S_{2n}$, namely the one with $n$ 2-cycles. In general, let $C$ be conjugacy class in $S_n$ with $n_i$ $i$–cycles and $A_C(t) = \sum_{\pi \in C} t^{\text{des}(\pi)}$. It can be found in [17] that
\[
\frac{A_C(t)}{(1 - t)^{n+1}} = \sum_{k \geq 0} t^k \prod_{i=1}^n \binom{f_{ik} + n_i - 1}{n_i},
\]
where $f_{ik} = \frac{1}{t} \sum_{d|i} \mu(d) k^{i/d}$ and $\mu(\cdot)$ is the Möbius function. We are interested in finding recursive formulae for the number of permutations in a fixed conjugacy class with a given number of descents. These numbers arise in riffle shuffles [11]. A recursive formula can be derived from (17) for certain cases. For example, it is possible to find such recurrence relation in the case of $n$ 3–cycles in $S_{3n}$, but it is not possible for involutions with no restriction on the number of fixed-points. But
if the number of fixed-points is fixed at a certain proportion, then a martingale formulation is possible.

5.7. Two-sided Eulerian numbers. The last example is a vector descent statistic which can be studied by multivariate martingale limit theorems. Two-sided Eulerian numbers are introduced in [8] as the coefficients of

$$A_n(t, s) = \sum_{\pi \in S_n} t^{\text{des}(\pi)} s^{\text{des}(\pi^{-1})}.$$ 

The generating function is obtained by counting $n$ unlabelled balls in $kl$ distinct boxes, which is a two-dimensional analogue of barred permutations. (See [30] for the counting argument and a survey on these numbers.) It is given by

$$A_n(t, s) = \sum_{k,l \geq 0} \binom{kl + n - 1}{n} t^k s^l.$$ 

The recurrence relation on coefficients are derived in [8]; it satisfies the two-dimensional generalization of (6). Let $D_n$ be the random variable counting the number of descents of a uniformly chosen permutation $\pi$, while $D'_n$ counts the number of descents in $\pi^{-1}$. Then,

$$(D_{n+1}, D'_{n+1}) = \begin{cases} (D_n, D'_n), & \text{with prob. } \frac{(D_{n+1})(D'_{n+1})+n}{(n+1)^2}, \\ (D_n + 1, D'_n), & \text{with prob. } \frac{(n-D_n)(D'_{n+1})-n}{(n+1)^2}, \\ (D_n, D'_n + 1), & \text{with prob. } \frac{(D_{n+1})(n-D'_n)-n}{(n+1)^2}, \\ (D_n + 1, D'_n + 1), & \text{with prob. } \frac{(n-D_n)(n-D'_n)+n}{(n+1)^2} \end{cases}$$

is a submartingale, and

$$(Z_n, Z'_n) = n \left( D_n - \frac{n-1}{2}, D'_n - \frac{n-1}{2} \right)$$

is a zero-mean martingale. A central limit theorem is recently shown in [9]. We use below a multivariate limit theorem for martingales, whose proof is in [1]. The theorem is in functional form, but we can embed $(Z_n, Z'_n)$ in Brownian motion (See Appendix of [22].), apply the theorem, and the unit time for standard Brownian motion gives standard normal distribution. Later, equivalent conditions for the theorem are given in [24]. They are similar to one-dimensional case. First, we need to verify Lindeberg-condition and the convergence of conditional variance for both coordinates. Since $D_n$ and $D'_n$ are identically distributed, and the case for $D_n$ is already covered in Section 3, we only need to
show the additional condition on covariances,
\[(22)\quad \sum_i E[X_{ni}X'_{ni}|F_{n,i-1}] \xrightarrow{p} 0,\]
where \(X_{ni}\) and \(X'_{ni}\) are defined as in Section 3 to be martingale differences. We also define the central random variables, \(W_i = D_i - \frac{i-1}{2}\) and \(W'_i = D'_i - \frac{i-1}{2}\). It can be calculated from (4) that
\[E[X_{ni}X'_{ni}|F_{n,i-1}] = \frac{36}{n^2(n+1)} W_{i-1}W'_{i-1}.\]

In order to use Chebyshev’s inequality to show (22), we first prove the following lemma.

**Lemma 5.1.** For a uniformly chosen permutation \(\pi \in S_n\), define the random variables, \(D_n(\pi) = \text{des}(\pi)\) and \(D'_n(\pi) = \text{des}(\pi^{-1})\). Then, \(E[(D_n - \frac{n-1}{2})^2 (D'_n - \frac{n-1}{2})^2]\) is of order \(n^2\).

**Proof:** Define \(D_n = \sum_{i=1}^{n-1} T_i\) and \(D'_n = \sum_{i=1}^{n-1} S_i\) as in Section 2, where \(T_i\) is as before and
\[S_i(\pi) = \begin{cases} 1 & \text{if } \pi^{-1}(i) > \pi^{-1}(i+1), \\ 0 & \text{otherwise}. \end{cases}\]

We first evaluate
\[E(D_nD'_n) = \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} E(T_iS_j).\]

As suggested in [9], we define \(K = \{\pi(i), \pi(i+1)\} \cap \{j, j+1\}\), and break the expectation into three terms with respect to the size of \(K\),
\[E(T_iS_j) = P(T_iS_j = 1|K = 0) \cdot P(K = 0) + P(T_iS_j = 1|K = 1) \cdot P(K = 1) + P(T_iS_j = 1|K = 2) \cdot P(K = 2).\]

It is straightforward that \(P(T_i = S_j = 1|K = 0) = \frac{1}{4}\) by independence. The same holds true for \(K = 1\) by symmetry. Suppose \(\pi(i) = j\), which has \(\frac{1}{4}\) probability conditioned on \(K = 1\). Then \(T_iS_j = 1\) if and only if \(\pi(i+1) < j\) and \(\pi^{-1}(j+1) < i\). While if \(\pi(i+1) = j\), \(T_iS_j = 1\) if and only if \(\pi(i) > j\) and \(\pi^{-1}(j+1) < i\). Considering the other two cases as well, we observe that the probabilities of \(T_iS_j = 1\) add up to 1 through four equiprobable cases. Therefore, \(P(T_i = S_j = 1|K = 2) = \frac{1}{2}\). Finally, we deal with the last term. There are four possible pairs, and two of them give descents in both positions, so \(P(T_i = S_j = 1|K = 2) = \frac{1}{2}\).
A simple counting argument shows that \( P(K = 2) = \frac{1}{(\frac{2}{n})^2} = \frac{2}{n(n-1)} \).

Hence,
\[
E(T_i S_j) = \frac{1}{4} + \frac{1}{2n(n-1)},
\]
which eventually shows that
\[
E(D_n D'_n) - E(D_n)E(D'_n) = \frac{n-1}{2n},
\]
which is of constant order. We argue for \( E(D_n^2 D'_n) \) and \( E(D_n D_n^2) \) along the same lines. Consider
\[
E(D_n^2 D'_n) = \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \sum_{k=1}^{n-1} E(T_i T_j S_k),
\]
(We refer to Section 2 for treatment of cases where \( j = i \) and \( j = i + 1 \)) Again, depending on the size of \( \{\pi(i), \pi(i+1), \pi(j), \pi(j+1)\} \cap \{k, k+1\} \), we can perform case-by-case analysis. Given that the size of the intersection is 0 or 1, it can be shown that \( T_i T_j \) is independent of \( S_k \). Since, the probability that the size of the intersection is 2, is of order \( \frac{1}{n^2} \) and there are \((n-1)^3\) terms, we have
\[
E(D_n^2 D'_n) = E(D_n^2)E(D'_n) = O(n).
\]
A similar argument gives
\[
E(D_n^2 D^n) - E(D^2_n)E(D'_n) = O(n^2).
\]
Therefore, we can check term by term that
\[
E(W_n^2 W'^2_n) - E(W_n^2)E(W'^2_n) = O(n^2).
\]
\[\square\]
where the third inequality follows from Lemma 5.1. Therefore, we verify (22) by Chebyshev’s inequality. Then the conditions of Theorem 3.3 in [24] are satisfied. We conclude that

\[
\left( \frac{D_n - \frac{n-1}{2}}{\sqrt{\frac{n+1}{12}}}, \frac{D'_n - \frac{n-1}{2}}{\sqrt{\frac{n+1}{12}}} \right)
\]

is asymptotically bivariate normal with zero mean and unit covariance matrix.

References

[1] Aalen, O.O., *Weak convergence of stochastic integrals related to counting processes*, Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete, **38** No.4, 261-277 (1977)

[2] Aigner, M., *A course in enumeration*, 238, Springer Science & Business Media (2007)

[3] Azuma, K., *Weighted sums of certain dependent random variables*, Tohoku Mathematical Journal, Second Series, **19**, No. 3, 357-367, (1967)

[4] Bender, E. A. *Central and local limit theorems applied to asymptotic enumeration*, Journal of Combinatorial Theory Series A **15** No.1, 91-111 (1973)

[5] Bóna, M., *Real zeros and normal distribution for statistics on Stirling permutations defined by Gessel and Stanley*, SIAM Journal on Discrete Mathematics, **23**, No.1, 401-406, (2009)

[6] Bóna, M., *Combinatorics of permutations* Chapman and Hall/CRC, (2016)

[7] Carlitz, L., Kurtz, D.C., Scoville, R., Stackelberg, O.P., *Asymptotic properties of Eulerian numbers*, Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete, **23**, No.1, 47-54, (1972)

[8] Carlitz, L., Roselle, D.P., Scoville, R.A., *Permutations and sequences with repetitions by number of increases*, Journal of Combinatorial Theory, **1**, No.3, 350-374, (1966)

[9] Chatterjee, S., Diaconis, P., *A central limit theorem for a new statistic on permutations*, Indian Journal of Pure and Applied Mathematics, **48**, No.4, 561-573, (2017)

[10] Désarménien, J., Foata, D., *Fonctions symétriques et séries hypergéométriques basiques multivariées*, Bulletin de la Société Mathématique de France, **113**, 3-22, (1985)

[11] Diaconis, P., McGrath, M., Pitman, J., *Riffle shuffles, cycles, and descents*, Combinatorica, **15**, No.1, 11-29, (1995)

[12] Feller, W., *An introduction to probability theory and its applications Vol.1*, Wiley, New York, (1968)

[13] Flajolet, P., Sedgewick, R., *Analytic Combinatorics*, Cambridge University Press, (2009)

[14] Foata, D., *Eulerian polynomials: from Eulers time to the present*, In The legacy of Alladi Ramakrishnan in the mathematical sciences, Springer, New York, NY, 253-273, (2010)

[15] Frobenius, G., *Über die Bernoullischen und die Eulerschen Polynome*, Sitzungsberichte der Preussische Akademie der Wissenschaften, 809-847, (1910)
[16] Fulman, J., *Stein’s method and non-reversible Markov chains*, Institute of Mathematical Statistics Lecture Notes-Monograph Series, **46**, 66-74 (2004)
[17] Fulman, J., *The distribution of descents in fixed conjugacy classes of the symmetric groups*, Journal of Combinatorial Theory Series A, **84**, No.2, 171-180, (1998)
[18] Gessel, I., Stanley R.P., *Stirling polynomials*, Journal of Combinatorial Theory Series A **24**, No.1, 24-33, (1978)
[19] Goncharov, V., *Du domaine d’analyse combinatoire*, Izv. Akad. Nauk SSSR Ser. Mat., **8**, 3-48, (1944)
[20] Guo, V. J. W., Zeng, J. *The Eulerian distribution on involutions is indeed unimodal*, Journal of Combinatorial Theory Series A 113, no. **6**, 1061-1071, (2006)
[21] Haglund, J., Visontai, M., *Stable multivariate Eulerian polynomials and generalized Stirling permutations*, European Journal of Combinatorics, **33**, No.4, 477-487, (2012)
[22] Hall, P., Heyde, C., *Martingale limit theory and its application*, Academic Press, New York, (2014)
[23] Harper, L.H., *Stirling behavior is asymptotically normal*, The Annals of Mathematical Statistics, **38**, No.2, 410-414 (1967)
[24] Helland, I.S., *Central limit theorems for martingales with discrete or continuous time*, Scandinavian Journal of Statistics, **9**, No.2, 79-94, (1982)
[25] Houdré, C., Restrepo, R., *A probabilistic approach to the asymptotics of the length of the longest alternating subsequence*, The Electronic Journal of Combinatorics, **16**, #R00, (2009)
[26] Kim, G.B., *Distribution of descents in matchings*, arXiv:1710.03896 [math.CO]
[27] Kim, G.B., *Distribution of descents in matchings*, PhD dissertation, University of Southern California, (2017)
[28] Koutras, M.V., *Eulerian numbers associated with sequences of polynomials*, Fibonacci Quart, **32**, No.1, 44-57, (1994)
[29] Liu, L.L., Wang, Y., *A unified approach to polynomial sequences with only real zeros*, Advances in Applied Mathematics, **38**, No.4, 542-560, (2007)
[30] Petersen, T.K., *Two-sided Eulerian numbers via balls in boxes*, Mathematics Magazine, **86**, No.3, 159-176 (2013)
[31] Petersen, T.K., *Eulerian numbers*, Birkhäuser, New York, NY, (2015)
[32] Pitman, J., *Probabilistic bounds on the coefficients of polynomials with only real zeros*, Journal of Combinatorial Theory Series A, **77**, No.2, 279-303, (1997)
[33] Riordan, J., *Combinatorial identities*, **6**, New York, Wiley, (1968)
[34] Savage, C., Visontai, M., *The s-Eulerian polynomials have only real roots*, Transactions of the American Mathematical Society, **367**, No.2, 1441-1466, (2015)
[35] Stanley, R.P., *Enumerative Combinatorics. Vol. I*, The Wadsworth Brooks/Cole Mathematics Series, Wadsworth Brooks, (1986)
[36] Stanley, R.P., *Log-Concave and Unimodal Sequences in Algebra, Combinatorics, and Geometry*, Annals of the New York Academy of Sciences, **576**, No.1, 500-535, (1989)
[37] Stanley, R.P., *Longest alternating subsequences of permutations*, Michigan Mathematical Journal, **57**, No.1, 675-687, (2008)
[38] Tanny, S., *A probabilistic interpretation of Eulerian numbers*, Duke Mathematical Journal **40** No. 4, 717-722 (1973)

[39] Warren, D.I., Seneta, E., *Peaks and Eulerian numbers in a random sequence*, Journal of Applied Probability, **33**, No.1, 101-114 (1996)

[40] Widom, H., *On the limiting distribution for the length of the longest alternating sequence in a random permutation*, Electronic Journal of Combinatorics, **13**, R25, (2006)

University of Southern California

E-mail address: aozdemir@usc.edu