Periodic three-body orbits in the Coulomb potential

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(Dated: December 21, 2018)

We numerically discovered around 100 distinct non-relativistic collisionless periodic three-body orbits in the Coulomb potential in vacuo, with vanishing angular momentum, for equal-mass ions with equal absolute values of charges. These orbits are classified according to their symmetry and topology, and a linear relation is established between the periods, at equal energy, and the topologies of orbits. Coulombic three-body orbits can be formed in ion traps, such as the Paul, or the Penning one, where one can test the period vs. topology prediction.

PACS numbers:
Keywords: three-body problem, Coulomb potential

The Newtonian three-body problem is one of the outstanding classical open questions in science. After more than 300 years of observation, only two topologically distinct types of periodic three-body systems, or orbits, have been observed in the skies [1]: 1) the so-called hierarchical systems, such as the Sun-Earth-Moon one, to which type belong more than 99% of all observed three-body systems; 2) Lagrangian three-body systems, such as Jupiter’s Trojan satellites, to which the remaining ≤ 1% belong.

There has been some significant theoretical progress on the subject over the past few years: several hundred new, topologically distinct families of periodic solutions have been found by way of numerical simulations [2–16], and unexpected regularities have been observed among them [9, 13, 15, 16] relating the periods, topologies and linear stability of orbits.

Of course, one would like to observe at least some of the new orbits and test their properties in an experiment, but such a test would be impeded by a number of obstacles: (1) only stable orbits have a chance of actually existing for a sufficiently long time to be observed; (2) stability depends on the ratio(s) of masses, and on the value of angular momentum, neither of which can be controlled in astronomical settings; (3) even if an orbit is stable in a wide range of mass ratios and angular momenta, there is no guarantee that such a system will have been formed sufficiently frequently and sufficiently close to Earth, that it may be observed by our present-day instruments.

All of the above prompted us to look for alternative three-body systems that share (at least) some of the same properties with Newtonian three-body systems. The Coulombic potential shares one basic similarity with the Newtonian gravity – its characteristic $1/r$ (homogeneous) spatial dependence – as well as several important differences: (1) the (much) larger coupling constant; (2) both attractive and repulsive nature; (3) naturally identical (quantized) electric charge(s); (4) ions with opposite charges may have masses equal to one part in a few thousand; (5) ions have a finite probability of elastic scattering in head-on collisions; and (6) Coulombic bound states can be formed in tabletop ion-trap experiments [17]. For these reasons we turn to the study of periodic three-body orbits bound by Coulombic potential. The application of only the Coulomb interaction amounts to a non-relativistic approximation, which is good only in the low-velocity limit [18].

In this Rapid Communication we present the results of a search that led to around 100 distinct collisionless or-

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bits, only four of which are stable, and around 80 isosceles quasi-colliding (free-fall, or “brake”) ones. We use the collisionless orbits to display a new regularity, akin to Kepler’s third law, in the form of a linear dependence

$$T |E|^{3/2} \sim N,$$

(1)

between the scale-invariant period $T |E|^{3/2}$, where $T$ is the period, and $E$ is the energy of an orbit, on one hand, and the orbit’s topological complexity $N$, expressed as the number of collinear configurations (“syzygies”) encountered during one cycle, see the text below, on the other. This prediction ought to be tested in ion trap experiments.

We used the same search method as in the Newtonian gravity three-body problem [5]. There are 12 independent variables that define the initial state of this system, for each body there are the $x$ and $y$ coordinates of the body, and the $v_x$ and $v_y$ components of their velocity. Adopting the center-of-mass reference frame reduces this number (12) to eight. Fixing the value of angular momentum ($L = 0$) reduces this further to six. Using the scaling rules [12] for the solutions and the fact that periodic solution must pass through at least one syzygy (collinear configuration) during one period, yields a four-dimensional search space for all zero-angular-momentum periodic solutions. We search for solutions in the two-dimensional subspace of orbits that pass through the Euler configuration, defined as the symmetric collinear configuration wherein the positively charged particle with velocity $(-2v_x, -2v_y)$ passes through the origin $(0, 0)$, i.e., exactly between the two negatively charged particles, as follows. For both class A and class B, sequence (I): $w_{n,k}^{(I)} = [(ab)^n(ab)^m]^k$ with integers $n, k = 1, 2, \ldots$; and for class A only, sequence (II): $w_{m,n,k}^{(II)} = [(ab)^n(ab)^m]^kA[(ab)^n(ab)^m]^kB$, with $m, n, k = 1, 2, 3, \ldots$; and sequence (III): $w_{n}^{(III)} = [(ab)^nAB(ab)^2Bb]^n$, with $n = 1, 2, 3, \ldots$.

Note that the 100-odd collisionless Coulombic orbits are substantially fewer than roughly 200 collisionless Newtonian orbits with similar search parameters, and that there are only four linearly stable solutions in contrast to more than 20 in the Newtonian case.

All of this is a consequence of just one sign change in the potential: one pair of charged particles must experience repulsion, contrary to Newtonian gravity, where all pairs are attractive. Therefore, no choreographic solution, i.e., permutationally symmetric solution with all three particles following the same trajectory, such as the famous “figure-8” orbit, may exist in the Coulombic case. Moreover, at least one orbit, similar to Olrov’s [4] colliding “S-orbit” (in Newtonian gravity) still exists in the Coulombic case, but it is not stable any more, and consequently does not produce an infinite sequence of periodic orbits, see [13].

The initial conditions of all 100-odd orbits and their corresponding topological and kinematical properties can be found in [24]; in Fig. 2 and Table I we have shown six representative solutions.

Next we show that Eq. (1), the (striking) property of orbits that was first observed in Newtonian three-body systems [4], also features in the Coulombic three-body systems. This relation between topological and kinematical properties of Newtonian three-body systems was first reported in [4] and later studied in more detail in Refs. [11, 13, 15, 16]. Equation (1) is a (simple) linear dependence of the scale-invariant period $T |E|^{3/2}$ on the topological complexity $N$. The topological complexity $N$ can be measured in at least two different ways: (1) we used the length $N_w$ of the free-group element (word) describing the orbit’s topology, which, due to symmetry in our case, is equal to the number of asymmetric syzygies, i.e., collinear configurations wherein the two equal-charge particles are next to each other, over one period; (2) the number $N_e$ of all syzygies (collinear configurations) was considered in Refs. [6, 12] as the measure of topological complexity $N$ of Newtonian orbits.

In Fig. 3 one can see that Eq. (1) holds for three-body orbits in the Coulomb potential: (1) with $N = N_e$, a linear fit yields a slope equal to 1.8252, with asymptotic standard error of 0.08% and an average relative deviation of points from fit values that equals 0.63%; (2) with $N = N_w$ the number of all syzygies (collinear configu-
TABLE I: Initial conditions of six orbits, depicted in Fig. 2, that belong to the sequence described by the free-group elements [(AB)\(^2\)(ab)]\(^k\), with \(k = 1, 2, 3, \ldots\), and four linearly stable orbits, Table II. The columns correspond to: solution label, name of the sequence that the solution belongs to, initial velocities \([\dot{x}_1(0), \dot{y}_1(0)]\), period, negative energy, scaled period, free group element, and the total number of syzygies over a period. For initial conditions of all other found solutions, see [20].

| Label | Seq. | \(x_1(0)\) | \(y_1(0)\) | \(T\) | \(-E\) | \(T/|E|^{1/2}\) | Free group element | \(N_w\) | \(N_s\) |
|-------|------|-------------|-------------|-------|--------|----------------|-------------------|--------|--------|
| A.4   | I    | 0.191764    | 0.339958    | 13.4332| 1.06108| 14.6826       | (AB)^2(ab)^2     |        |        |
| A.12.a| I    | 0.147917    | 0.323693    | 37.1599| 1.12003| 44.0473       | (AB)^2(ab)^3     | 24     | 42     |
| A.12.b| I    | 0.246251    | 0.335527    | 45.3784| 0.98035| 44.0472       | (AB)^2(ab)^3     | 24     | 42     |
| B.4   | I    | 0.111427    | 0.305087    | 11.3981| 1.18352| 14.6755       | (AB)^2(ab)^2     | 8      | 14     |
| B.12.a| I    | 0.327539    | 0.337033    | 57.4554| 0.83738| 44.0266       | (AB)^2(ab)^3     | 24     | 42     |
| B.12.b| I    | 0.345214    | 0.344247    | 63.0644| 0.78062| 44.0266       | (AB)^2(ab)^3     | 24     | 42     |
| A.15.b| II   | 0.108065    | 0.323579    | 44.7536| 1.15086| 55.2534       | (ab)^2ABA(ba)^2babA | 30     | 52     |
| A.18  | III  | 0.105224    | 0.336995    | 55.6513| 1.12609| 66.5019       | (ab)^2ABA(ba)^2babA |        |        |
| A.20.b| II   | 0.126494    | 0.315968    | 59.3293| 1.15249| 73.4049       | (ab)^2ABA(ba)^2babA | 40     | 70     |
| A.24.a| II   | 0.249577    | 0.291337    | 80.2223| 1.0585 | 87.364        | (ab)^2ABA(ba)^2babA | 48     | 86     |

FIG. 2: Trajectories of orbits A.4, with topology \((AB)^2(ab)^2\); A.12.a and A.12.b, both with topology \[((AB)^2(ab)^2)^3\), in class A (upper row); and orbits B.4, with topology \((AB)^2(ab)^3\), and B.12.a and B.12.b, both with topology \[((AB)^2(ab)^3)^3\), in class B (lower row), respectively. Note the independent symmetries of the class A (upper row) trajectories with respect to the reflections about the horizontal and the vertical axis, whereas the class B (lower row) trajectories have only this symmetry under combined reflections. Black lines correspond to the positively charged particle while red and blue lines correspond to the negatively charged ones.
in Coulombic three-body orbits, and are not features of Newtonian gravity alone. The homogeneity of the Coulombic, Newtonian, and the strong Jacobi-Poincaré potentials is common to all three known cases of manifestation of this regularity. This supports indirectly the explanation offered in Refs. [13, 24].

Our next concern ought to be the observation of some of these orbits in an experiment. The trajectories of a number (ranging between 1 and 32) of positively charged macroscopic particles in an ion trap has remained unanswered to the present day, to our knowledge. It is well known that Paul and/or Penning traps can lead to binding of pairs of identical ions, including periodic orbits as well as their chaotic motions, when the circumstances (such as the frequency and amplitudes of the applied electric and/or magnetic fields) are right. Such periodic orbits are impossible in free space, however, as there the identical ions experience only Coulomb repulsion. So, before one observes any periodic three-body orbits in an ion trap, and declares them genuine Coulomb orbits, one must know which periodic three-body orbits exist in free space — information that we have provided here. With the present work we have prepared the terrain for future numerical, and we hope also experimental studies of three-ion motions in traps.

FIG. 3: Dependence of the scale-invariant period $T|E|^{3/2}$ on the number of asymmetric syzygies $N_w$ (collinear configurations with two particles of the same charge on one side) during an orbit. Inset: dependence of the scale-invariant period $T|E|^{3/2}$ on the number of all syzygies $N_e$.

There are no records, to our knowledge, of searches for periodic Coulombic three-body systems with equal masses and equal charges, which are the closest to the equal-mass Newtonian system that was studied in Refs. [4–12, 14–16]. As we wished to compare the closest analogs of the Coulombic and Newtonian three-body systems, we had to repeat a search for periodic collisionless orbits, of which we have provided around 100, that ought to suffice for a starting point.

At any rate, trap-induced corrections will have to be calculated for each three-ion orbit in any trap where experiments are conducted, before an interpretation is given. With this Rapid Communication we hope to start a discussion of trap-induced corrections for periodic three-ion orbits: in order to calculate such corrections, one needs the (initial conditions of) free-space periodic orbits, of which we have provided around 100, that ought to suffice for a starting point.

To be sure, we are not the first ones who have studied Coulombic periodic three-body motion: the subject has a long history, see e.g. Refs. [29, 30], with a revival in the 1980s, since when a number of studies have been published. Numerical discovery of more than 8000 collinear colliding periodic orbits with He atom mass ratios was reported in Ref. [36], and of somewhat fewer collisionless ones in Ref. [37]. The initial conditions were not published, so one could not simply retrieve these previously discovered orbits and use them here.

With this Rapid Communication we also hope to induce practitioners to consider experimental searches, particularly in view of the fact that, at least in the case of past periodic-orbit discoveries, the theory did not precede experiment. We thank Marija Janković, Ana Hudomal and Srđjan Marjanović for their help in the early stages of the work of V. D. and M. Šuvakov was supported by the Serbian Ministry of Education, Science and Technological Development under Grants No. OI 171037 and No. III 41011. M. Šindik conducted her work during the summer breaks of 2016 and 2017, when she was supported by a Student scholarship from the Serbian Ministry of Education, Science and Technological Development. All numerical work was done on the Zebram cluster, Laboratory for gaseous electronics, Center for nonequilibrium processes, at the Institute of Physics, Belgrade.

| name | Re($\lambda_j$) | Im($\lambda_j$) | $|\lambda_j|^2$ | $\nu_j$ | $N_e$ |
|------|----------------|----------------|----------------|-------|-------|
| A.15.b | 0.510145 | 0.860102 | 1.000023 | 0.164797 | 30 |
|      | -0.11507 | 0.993357 | 0.999999 | 0.268355 |   |
| A.18 | -0.002025 | 0.999961 | 0.999926 | 0.250322 | 36 |
|      | -0.820340 | 0.571882 | 1.000007 | 0.403107 |   |
| A.20.b | 0.009875 | 0.998966 | 0.998931 | 0.248427 | 40 |
|      | 0.94189 | 0.339728 | 1.002572 | 0.055094 |   |
| A.24.a | -0.988601 | 0.174067 | 1.007631 | 0.472261 | 48 |
|      | 0.993975 | 0.116863 | 1.001643 | 0.018627 |   |
this work.

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[20] See Supplemental Material at [URL will be inserted by publisher] for Periodic three-body orbits in the Coulomb potential, which includes, but is not limited to: (a) tables of initial conditions, topologies of orbits; (b) figures of orbits’ trajectories; (c) description of the search method; (d) stability analysis.

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[42] from a linearly stable orbit, in agreement with the Birkhoff-Lewis theorem.

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[44] We put the word colliding into quotation marks here, because only zero-angular momentum orbits experience actual collisions, whereas ion traps generally impart angular momentum to ions.