THE LERCH ZETA FUNCTION AND THE HEISENBERG GROUP

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Abstract. This paper gives a representation-theoretic interpretation of the Lerch zeta function and related Lerch $L$-functions twisted by Dirichlet characters. These functions are associated to a four-dimensional solvable real Lie group $H^J$, called here the sub-Jacobi group, which is a semi-direct product of $GL(1, \mathbb{R})$ with the Heisenberg group $H(\mathbb{R})$. The Heisenberg group action on $L^2$-functions on the Heisenberg nilmanifold $H(\mathbb{Z}) \backslash H(\mathbb{R})$ decomposes as $\bigoplus_{N \in \mathbb{Z}} \mathcal{H}_N$, where each space $\mathcal{H}_N$ ($N \neq 0$) consists of $|N|$ copies of an irreducible infinite-dimensional representation of $H(\mathbb{R})$ with central character $e^{2\pi i Nz}$. The paper shows that these can further decompose $\mathcal{H}_N$ ($N \neq 0$) into irreducible $H(\mathbb{R})$-modules $\mathcal{H}_{N,d}(\chi)$ indexed by Dirichlet characters (mod $d$) for $d \mid N$, each of which carries an irreducible $H^J$-action. On each $\mathcal{H}_{N,d}(\chi)$ there is an action of certain two-variable Hecke operators $\{T_m : m \geq 1\}$; these Hecke operators have a natural global definition on all of $L^2(H(\mathbb{Z}) \backslash H(\mathbb{R}))$, including the space of one-dimensional representations $\mathcal{H}_0$. For $\mathcal{H}_{N,d}(\chi)$ with $N \neq 0$ suitable Lerch $L$-functions on the critical line $1/2 + it$ form a complete family of generalized eigenfunctions (purely continuous spectrum) for a certain linear partial differential operator $\Delta_L$. These Lerch $L$-functions are also simultaneous eigenfunctions for all two-variable Hecke operators $T_m$ and their adjoints $T_m^*$, provided $(m, N/d) = 1$. Lerch $L$-functions are characterized by this Hecke eigenfunction property.

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1. INTRODUCTION

The Lerch zeta function is defined by

\[ \zeta(s, a, c) = \sum_{n=0}^{\infty} e^{2\pi ina(n+c)^{-s}}. \] (1.1)

It is named after M. Lerch \[11\], who in 1887 derived a three term functional equation that it satisfies. The parameter value \( a = 0 \) reduces to the Hurwitz zeta function, and the further specialization to \( (a, c) = (0, 1) \) gives the Riemann zeta function. It is well known that the functional equations for the Hurwitz and Riemann zeta function can be derived by specialization from that of the Lerch zeta function.

This paper addresses the question where the Lerch zeta function fits in the framework of automorphic representation theory. It provides the following answer, as a special case \( (N,d) = (1,1) \) in Theorem \[9.5\].

**Theorem 1.1.** The two symmetrized Lerch-zeta functions

\[ L^\pm(s, a, c) = \zeta(s, a, c) \pm e^{2\pi i a} \zeta(s, 1-a, 1-c) \]

References
are Eisenstein series for the real Heisenberg group $H(\mathbb{R})$ with respect to the discrete subgroup given by the integer Heisenberg group $H(\mathbb{Z})$. They form a continuous family of eigenfunctions in the $s$-parameter with respect to a “Laplacian operator” $\Delta_L = \frac{1}{2\pi i} \frac{\partial}{\partial a} \frac{\partial}{\partial c} + c \frac{\partial^2}{\partial c^2} + \frac{1}{4}$. The operator $\Delta_L$ defines a left-invariant vector field on $H(\mathbb{R})$, and acts on a certain Hilbert space $\mathcal{H}_1$ inside $L^2(H(\mathbb{Z}) \backslash H(\mathbb{R}))$, which is invariant under the right $H(\mathbb{R})$-action, and carries an irreducible action of $H(\mathbb{R})$, the Schrödinger representation, with central character $e^{2\pi i z}$. The operator $\Delta_L$ is specified as an unbounded operator by a dense domain $\mathcal{W}(D_{1,1}) \subset \mathcal{H}_1$ with respect to which it is skew-adjoint. It has pure continuous spectrum with generalized eigenfunctions given by $L^\pm (\frac{1}{2} + it, a, c)$, for $t \in \mathbb{R}$, with a specified spectral measure.

More generally, this paper provides a complete spectral decomposition of $L^2(H(\mathbb{R})/H(\mathbb{Z}))$ with respect to this vector field. The Hilbert space $L^2(H(\mathbb{R})/H(\mathbb{Z}))$ decomposes into a countable direct sum of pieces $\mathcal{H}_N$ indexed by the (integer) value $N$ of the central character. This paper treats the case of nonzero $N$ and shows that all such spaces carry pure continuous spectra of various multiplicities, with generalized eigenfunctions of $\Delta_L$ being symmetrized Lerch functions twisted by (primitive or imprimitive) Dirichlet characters, which we term Lerch $L$-functions. We give a decomposition of $\mathcal{H}_N$ for $N \neq \pm 1$ into smaller pieces, labeled $\mathcal{H}_{N,d}(\chi)$ where $d|n$ and $\chi$ is a (primitive or imprimitive) Dirichlet character (mod $d$), and each piece carries an irreducible representation of $H(\mathbb{R})$ with central character value $N$. A variant of the operator $\Delta_L$ acts on each space $\mathcal{H}_{N,d}(\chi)$ (on a suitable dense domain) and its spectrum is pure continuous, with associated generalized eigenfunctions given by a pair of Lerch $L$-functions on the critical line. The remaining case $N = 0$ will be treated in a sequel paper.

We also introduce two-variable “Hecke operators” $T_m$ acting on these spaces $\mathcal{H}_N$, where for $N = \pm 1$ the functions $L^\pm (s, a, c)$ are simultaneous eigenfunctions of all the $T_m$. For $|N| \geq 2$ all the operators $\{T_m : m \geq 1\}$ form a commuting family acting on $\mathcal{H}_N$, however these operators are normal operators on $\mathcal{H}_N$ exactly when $(m, |N|) = 1$. Smaller spaces $\mathcal{H}_{N,d}(\chi)$ are invariant under the action of the Hecke operators $T_m$ having $(m, \frac{N}{d}) = 1$. The associated pair of Lerch $L$ functions on this space are simultaneous eigenfunctions of all the Hecke operators $T_m$ and their adjoints $T_m$ that have $(m, N) = 1$. These spaces are not invariant under other Hecke operators having $(m, \frac{N}{d}) > 1$, and to restore full Hecke algebra invariance requires combining some of these spaces into larger spaces, as indicated below. One can characterize the Lerch $L$-functions in terms of the Hecke operator action, using results of [38].

The spectral analysis on the Hilbert space $\mathcal{H}_0$, corresponding to $N = 0$ has will be treated in a sequel paper ([34]); it has a different structure. Under the action of $\Delta_L$ alone it splits into one-dimensional eigenspaces. The dilation action of the sub-Jacobi group is broken, but there exist two-variable Hecke operators $T_m$ retaining part of this action. These operators are not normal operators for $m \geq 2$. Their action is more complicated and is dissipative on parts of $\mathcal{H}_0$. 

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1.1. **Previous work.** In four joint papers with Winnie Li ([35]–[38]) the author studied properties of the Lerch zeta function. The paper [35] derived two symmetric four-term functional equations for the Lerch zeta function, valid for values of \((a, c)\) in the closed unit square \(\square := \{(a, c) : 0 \leq a, c \leq 1\}\), following an approach used in Tate’s thesis. They apply to the two functions

\[
L^{\pm}(s, a, c) := \zeta(s, a, c) \pm e^{-2\pi i a} \zeta(s, 1 - a, 1 - c),
\]

which for \(\Re(s) > 1\) are given by the absolutely convergent series

\[
L^{\pm}(s, a, c) = \sum_{n \in \mathbb{Z}} (\text{sgn}(n + c))^{k} e^{2\pi i a |n + c| - s},
\]

with \((-1)^k = \pm\), for \(k = 0\) or \(1\). The four-term functional equations take the form

\[
\hat{L}^{\pm}(s, a, c) = \epsilon(\pm) e^{-2\pi i a} \hat{L}^{\pm}(1 - s, 1 - c, a),
\]

in which \(\epsilon(+) = 1\) and \(\epsilon(-) = i\), and the hat indicates a completion of the function at the archimedean place, which adds a factor \(\pi^{-s/2} \Gamma(s/2)\), resp. \(f((s + 1)/2) \Gamma((s + 1)/2)\) according as sign is + or −. They were first obtained by Weil [62] in 1976. Paper I observed that the Lerch zeta function is discontinuous on the boundary of the square, indicating a singular nature of the values \(a = 0\), resp. \(c = 0\), and we characterized the behavior of the function as the boundary is approached.

The papers [36] and [37] obtained an analytic continuation of \(\zeta(s, a, c)\) in all three complex variables to a (nearly) maximal domain of holomorphy; the functions are multi-valued and their monodromy was explicitly determined. Integer values of \(a\) and nonpositive integer values of \(c\) are singular values omitted from the analytic continuation; this explains some of the discontinuities observed in [35]. It also observed that these functions for fixed \(s\) give eigenfunctions of a linear partial differential differential operator in the \((a, c)\)-variables,

\[
D_L := \frac{1}{2\pi i} \frac{\partial}{\partial a} \frac{\partial}{\partial c} + c \frac{\partial}{\partial c},
\]

which states

\[
D_L \zeta(s, a, c) = -s \zeta(s, a, c).
\]

It also showed that the monodromy functions are also eigenfunctions of this operator. The operator \(\Delta_L = D_L + \frac{1}{2} I\) in the Theorem above is a shifted version of this operator, see Section 9.2 for its general definiton.

The paper [38] introduced certain two-variable operators which were termed “Hecke operators”, and studied their action on the Lerch zeta function. These operators have the form

\[
T_m f(a, c) := \frac{1}{m} \sum_{k=0}^{m-1} f\left(\frac{a + k}{m}, mc\right)
\]

which dilate in one direction and contract and shift in another. When these operators are applied term-by-term to the Dirichlet series representation (1.1), one obtains \(T_m(\zeta)(s, a, c) = m^{-s} \zeta(s, a, c)\), more generally one has

\[
T_m(L^{\pm})(s, a, c) = m^{-s} L^{\pm}(s, a, c).
\]
Thus for fixed $s$ the Lerch zeta function is a simultaneous eigenfunction of all $\{T_m : m \geq 1\}$; conversely, we show for all $s \in \mathbb{C}$ that these simultaneous eigenfunction conditions plus some ”twisted periodicity” conditions and integrability conditions characterize a two-dimensional vector space spanned by $L^\pm(s,a,c)$, which includes the Lerch zeta function, see [38, Theorem 6.2].

1.2. Present paper. In the present paper we view $(a,c)$ as real variables, and characterize the symmetrized Lerch zeta function viewed as related to actions on the Heisenberg group.

The functions themselves can more generally be viewed as associated to certain automorphic representations of a four dimensional solvable real Lie group $H^J(\mathbb{R})$. This group $H^J(\mathbb{R})$ is an extension of the real Heisenberg group $H(\mathbb{R})$ by the multiplicative group $\mathbb{R}^*$. It has faithful matrix representations given in Appendix A. We call $H^J(\mathbb{R})$ the sub-Jacobi group because it is a subgroup of the Jacobi group $Sp(2,\mathbb{R}) \rtimes H(\mathbb{R})$. We show that the Lerch zeta function, along with generalizations of it that involve twisting by Dirichlet characters, called Lerch $L$-functions, are associated to representations of the sub-Jacobi group acting on certain $H(\mathbb{R})$-invariant subspaces of $L^2(H(\mathbb{Z}) \backslash H(\mathbb{R}),d\mu)$, which carry a representation of the larger real Lie group $H^J(\mathbb{R})$. Given a (primitive or imprimitive) Dirichlet character $\chi (\mod d)$, with $d$ dividing $N$, in Section 9 we define the Lerch $L$-functions $L_{N,d}^\pm(\chi,s,a,c)$ by

$$L_{N,d}^\pm(\chi,s,a,c) := \sum_{n \in \mathbb{Z}} \chi\left(\frac{nd}{N}\right)(\text{sgn}(n+Nc))^k e^{2\pi i na} |n+Nc|^{-s}$$

in which $(-1)^k = \pm$ with $k = 0$ or 1. We extend these functions to functions on the Heisenberg group by inserting a central character

$$L_{N,d}^\pm(\chi,s,a,c,z) := e^{2\pi i Nz}L_{N,d}^\pm(s,a,c),$$

and continue to call them Lerch $L$-functions, since $z = 0$ recovers (1.3). The case $N = d = 1$ with $\chi = \chi_0$ the trivial character and $z = 0$ recovers the functions $L^\pm(s,a,c)$ studied in [35]. We show that the functions $L_{N,d}^\pm(\chi,s,a,c,z)$ for fixed $s \in \mathbb{C}$ are well-defined on the associated space $N^3_\chi := \Gamma \backslash H(\mathbb{R})$, which is often called the Heisenberg Nilmanifold.

We give an automorphic interpretation of the Lerch zeta function and, more generally, of Lerch $L$-functions twisted by Dirichlet characters, in terms of representations on Heisenberg modules. Here the Lerch zeta function plays the role of an Eisenstein series, in the sense that it parametrizes on the critical line $s = \frac{1}{2} + i\tau$ the continuous spectrum of the operator $\Delta_L$, acting on the Heisenberg module $\mathcal{H}_1$. The operator $\Delta_L$ can be identified with a certain left-invariant differential operator on the Heisenberg group, which can be expressed in the form $\frac{1}{4\pi i}(XY + YX)$, where $X$ and $Y$ are standard left-invariant vector fields. This operator encodes a “shift by $\frac{1}{2}$” which moves the line of unitarity from the imaginary axis to the critical line, coming from the Heisenberg commutation relations. More generally, Lerch $L$-functions play a similar Eisenstein series role on certain
submodules of the Heisenberg module $\mathcal{H}_N$ for various $N$, denoted $\mathcal{H}_{N,d}(\chi)$ with $d \mid |N|$ and $\chi$ a Dirichlet character (mod $d$).

We also give a generalization of the two-variable Hecke operators studied in [38] to arbitrary Heisenberg modules, given by

$$T_m f(a, c, z) := \frac{1}{m} \sum_{k=0}^{m-1} f \left( \frac{a + k}{m}, mc, z \right)$$

which act as correspondences on certain single-valued functions defined almost everywhere on the real Heisenberg group. We make a detailed study of the action of these two-variable Hecke operators on the different Heisenberg modules. In each case for each $s \in \mathbb{C}$ there is a two-dimensional space $\mathcal{E}_s(\mathcal{H}_{N,d}(\chi))$ of simultaneous eigenfunctions of these operators.

We note that Lerch $L$-functions multiplied by a suitable central character may be lifted to (discontinuous) functions on the real Heisenberg group, with twisted boundary conditions reflecting the Heisenberg group action. That is, they may be viewed as sections of a vector bundle $B_N$ over the manifold $X = H(\mathbb{Z}) \backslash H(\mathbb{R})$, having singularities above certain points.

1.3. Related work. There is a very large literature on the harmonic analysis of the Heisenberg group, see the survey of Howe [28] and the book of Thangavelu [53].

The Heisenberg nilmanifold $N_3 := \Gamma \backslash H(\mathbb{R})$, with $\Gamma = H(\mathbb{Z})$ is known to provide an appropriate setting for the action of Jacobi theta functions $\theta(z, \tau)$, with both variables $(z, \tau) \in \mathbb{C} \times \mathfrak{h}$. The Lerch $L$-functions $L_{N,d}^s(s; a, c, z)$ give well-defined functions on $N_3$. Much work was done in the 1970's relating theta functions, nilmanifolds, and abelian varieties, see for Auslander [3], Auslander and Brezin [4], Auslander and Tolimieri [5], Brezin [12], [13], and Tolimieri [56], [57], [58]. See also the survey of Auslander and Tolimieri [6], and the book of Corwin and Greenleaf [19].

Harmonic analysis on the generalized Heisenberg nilmanifold $N_n = H_n(\mathbb{Z}) \backslash H_n(\mathbb{R})$, a $(2n + 1)$-dimensional manifold, has been studied extensively. This includes work on $H^p$-spaces (Korányi [32]) and expansions taken with respect to various differential and integral operators, including the Radon transform (Strichartz [51]), the Kohn Laplacian (Folland [20]), the horizontal Heisenberg sublaplacian (Thangavelu [54]). A spectral theory for a general class of such operators was given by Ponge [50].

Notation. We write the complex variable $s = \sigma + i\tau$ where $\sigma, \tau$ are real. We reserve $t$ for a real variable, viewed in the multiplicative group $\mathbb{R}^\times$. The additive Fourier transform of a function $f(x) \in L^2(\mathbb{R}, dx)$ is $\mathcal{F}f(y) := \int_{-\infty}^{\infty} f(x)e^{-2\pi ixy}dx$, following Tate [52]. A Hilbert space inner product is denoted $(\cdot, \cdot)$, with $(f, g) = (g, f)$ and $(f, \alpha g) = \bar{\alpha}(f, g)$. We use the convention that divisibility relations $d \mid N$ have $d > 0$ but $N$ may have a sign, and we interpret it as $d$ divides $|N|$.

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2. Main Results

We coordinatize the real Heisenberg group $H(\mathbb{R})$ by $[a, c, z]$ using the (asymmetric) three-dimensional representation

$$H(\mathbb{R}) = \{ [a, c, z] \in \mathbb{R}^3 : a, c, z \in \mathbb{R} \}.$$ 

and has a normalized two-sided Haar measure $d\mu = da \, dc \, dz$. The group law is

$$[a, c, z] \cdot [a', c', z'] = [a + a', c + c', z + z' + ca'].$$

The subgroup $H(\mathbb{Z})$ consists of those $[a, c, z]$ which are all integers, and the left-coset space $H(\mathbb{Z}) \backslash H(\mathbb{R})$ is compact with volume 1.

In Section 3 we recall that the Hilbert space $H = L^2(H(\mathbb{Z}) \backslash H(\mathbb{R}), d\mu)$ with $H(\mathbb{R})$ acting by $\rho_h(F)(g) = F(gh)$ has a canonical “Fourier” decomposition under the unitary characters $e^{2\pi i Nz}$ of the center $Z(H(\mathbb{R}))$ which restrict to the identity on $H(\mathbb{Z})$ as

$$H := L^2(H(\mathbb{Z}) \backslash H(\mathbb{R}), d\mu) = \bigoplus_{N \in \mathbb{Z}} H_N$$

(2.1)

It is known that $N \neq 0$ the space $H_N$ is isotypic, with multiplicity $|N|$ of the irreducible representation with central character $e^{2\pi i Nz}$. We also introduce a Heisenberg-Fourier operator

$$R(F)([a, c, z]) := F(\alpha(a, c, z)) = F(-c, a, z - ac).$$

(2.2)

which is of order 4, and which gives rise to a unitary operator acting on $H$.

In Section 4 we introduce and study basic properties of the two-variable Hecke operators $\{T_m : m \in \mathbb{Z}\backslash\{0\}\}$ given by

$$T_m f(a, c, z) := \frac{1}{m} \sum_{k=0}^{m-1} f \left( \frac{a + k}{m}, mc, z \right)$$

We show that these operators induce bounded operators on the Hilbert space $H = L^2(H(\mathbb{Z}) \backslash H(\mathbb{R}), d\mu)$, and we consider their adjoint operators $T_m^*$ as well. We show that the operators can be viewed as acting on functions on the real Heisenberg group $H(\mathbb{R})$ that are $H(\mathbb{Z})$-invariant, and furthermore they leave each Hilbert space $H_N$ for $N \in \mathbb{Z}$ invariant. (Lemma 4.2). We show the adjoint operators $T_m^*$ are bounded and are conjugate to the operators $T_m$ under the R-operator action, as follows.

**Theorem 4.3.** Let $N \in \mathbb{Z}$, allowing $N = 0$. Then:

(i) The adjoint operator $T_m^*$ of $T_m$ on $H$ leaves each $H_N$ invariant and acts on $H_N$ by

$$T_m^*(F)(a, c, z) = \frac{1}{m} \sum_{k=0}^{m-1} e^{2\pi i k N} F(\frac{ma}{m}, \frac{c+k}{m}, z).$$
(ii) On each $\mathcal{H}_N$ each $T^*_m$ satisfies with respect to the Heisenberg-Fourier operator $R$ the relation

$$T^*_m = R^* \circ T_m \circ R$$

where $R^* = R^3$.

(iii) For $N \in \mathbb{Z}$, and $m \geq 1$, with $d = (m, N)$, then for $F \in \mathcal{H}_N$

$$T^*_m \circ T_m(F)(a, c, z) = \frac{1}{m} \sum_{\ell=0}^{d-1} F(a + \frac{\ell}{d}, c, z)$$

and

$$T_m \circ T^*_m(F)(a, c, z) = \frac{1}{m} \sum_{\ell=0}^{d-1} e^{2\pi i \frac{(N\ell)}{d}} F(a + \frac{\ell}{d}, c, z)$$

The operators $T_m$ and $T^*_m$ commute on $\mathcal{H}_N$ when $d = (m, N) = 1$, and then satisfy

$$T_m \circ T^*_m = T^*_m \circ T_m = \frac{1}{m} I.$$  

They do not commute on $\mathcal{H}_N$ when $d > 1$.

They operators $T_m$ do not commute with the Heisenberg group action on these functions, but transform in a simple way under the a Heisenberg group automorphism

$$\beta(t)[a, c, z] = \left[ \frac{1}{t} a, tc, z \right],$$  \hspace{1cm} (2.3)

as follows.

**Theorem 4.4.** For any $h \in H(\mathbb{R})$ and $F \in \mathcal{H}$ there holds

$$\rho_h \circ T_m(F)(a, c, z) = T_m \circ \rho_{\beta(m)h}(F)(a, c, z).$$

and

$$\rho_h \circ T^*_m(F)(a, c, z) = T^*_m \circ \rho_{\beta(1/m)h}(F)(a, c, z).$$

In Section 5 we establish a decomposition of each of the spaces $\mathcal{H}_N$ for $N \neq 0$ into irreducible Heisenberg modules, compatible with the Hecke operators above. This decomposition is associated to multiplicative (Dirichlet) characters, with the submodules being indexed by a divisor $d | N$ and a Dirichlet character $\chi (\mod d)$, written $\mathcal{H}_{N,d}(\chi)$.

**Theorem 5.5.** (Multiplicative Decomposition) For $N \neq 0$, the Hilbert space $\mathcal{H}_N$ has an orthogonal direct sum decomposition

$$\mathcal{H}_N = \bigoplus_{d | N} \left( \bigoplus_{\chi \in (\mathbb{Z}/d\mathbb{Z})^*} \mathcal{H}_{N,d}(\chi) \right).$$

Here $\chi$ runs over all Dirichlet characters, primitive and imprimitive. Each $\mathcal{H}_{N,d}(\chi)$ is invariant under the $H(\mathbb{R})$-action on $\mathcal{H}_N$ and is an irreducible representation of $H(\mathbb{R})$ with central character $e^{2\pi i N z}$. 

We call this decomposition the multiplicative decomposition of $H_N$. These spaces are studied using twisted Weil-Brezin maps $W_{N,d}(\chi) : L^2(\mathbb{R}, dx) \to H_{N,d}(\chi)$ defined for Schwartz functions $f(x) \in \mathcal{S}(\mathbb{R})$ by

$$W_{N,d}(\chi)(f)(a,c,z) := \sqrt{C_{N,d}} e^{2\pi i N z} \sum_{n \in \mathbb{Z}} \chi(n d N) f(n + N c) e^{2\pi i a}, \quad (2.4)$$

which extends to an isometric isomorphism of Hilbert spaces. (Here $C_{N,d}$ is a normalizing constant.) The two-variable Hecke operators $T_m$ with $(m, \frac{N}{d}) = 1$ leave the spaces $H_{N,d}(\chi)$ invariant, and the twisted Weil-Brezin map shows that they intertwine with operators corresponding to an $\mathbb{R}^*$-action on $L^2(\mathbb{R}, dx)$. (Theorem 5.6). However the two-variable Hecke operators $T_m$ with $(m, \frac{N}{d}) > 1$ do not leave all the spaces $H_{N,d}(\chi)$ invariant. Theorem 5.7 gives a coarse multiplicative decomposition of $H_N (N \neq 0)$ into simultaneous invariant subspaces for all Hecke operators $\{T_m : m \in \mathbb{Z}\setminus\{0\}\}$, as follows.

**Theorem 5.7.** (Coarse Multiplicative Decomposition) Let $N \neq 0$ and to each primitive character $\chi \pmod{f}$ with $f|N$ assign the Hilbert space

$$H_N(\chi; f) := \bigoplus_{d | f} H_{N,d}(\chi|d).$$

Then the Hilbert space $H_N$ has the orthogonal direct sum decomposition

$$H_N = \bigoplus_{\chi, f} H_N(\chi; f),$$

in which $\chi$ runs over all primitive characters (mod $f$) for all $f|N$. Each Hilbert space $H_N(\chi; f)$ is invariant under all two-variable Hecke operators $\{T_m : m \in \mathbb{Z}\setminus\{0\}\}$.

The factors in this decomposition are indexed by the primitive Dirichlet characters $\chi \pmod{f}$ for each $f \mid N$, and are given by

$$H_N(\chi; f) := \bigoplus_{f \mid d | N} H_{N,d}(\chi|d),$$

in which $\chi|d$ denotes the (imprimitive) character (mod $d$) coming from $\chi \pmod{f}$.

In Section 6 we study another decomposition of Heisenberg modules $H_N$ into irreducible submodules, introduced in 1973 by Auslander and Brezin [4]. This decomposition is associated to additive characters rather than multiplicative characters. More generally, their decompositions are based on a choice of “distinguished subgroup.” We determine the action of the two-variable Hecke operators on the resulting modules $H(\psi)$. These modules are usually not invariant under the Hecke operator action. We show that the image of one module $H(\psi)$ under a given $T_m$ is always contained in another module $H(\psi')$ and that if $(m, N) = 1$ this action is a permutation (Theorem 6.5). The results of this section are included for comparison and contrast with the multiplicative decomposition in Section 5.

In Section 7 we study the $\mathbb{R}^*$-action on the multiplicative submodules $H_{N,d}(\chi)$. This action gives a semidirect product with the $H(\mathbb{R})$-action, and defines an action on $H_{N,d}(\chi)$.
of a particular four-dimensional solvable real Lie group $H^J$, which we call the sub-Jacobi group.

**Theorem 7.2.** (Sub-Jacobi group action) For $N \neq 0$, each positive $d|N$ and each Dirichlet character $\chi \pmod d$ the Hilbert space $\mathcal{H}_{N,d}(\chi)$ carries an irreducible unitary representation of a four-dimensional solvable real Lie group $H^J$ with central character $e^{2\pi i N z}$. Two such representations are unitarily equivalent $H^J$-modules if and only if they have the same value of $N$.

This result shows that a large subspace of $L^2(H(\mathbb{Z}) \setminus H(\mathbb{R})), d\mu)$ carries an $H^J$-action, namely the orthogonal complement of $H_0$. The remaining space $H_0$, carrying the action of the one-dimensional representations of $H(\mathbb{R})$ on $L^2(H(\mathbb{Z}) \setminus H(\mathbb{R})), d\mu)$, does not carry an $H^J$-action.

For $N \neq 0$ the individual spaces $\mathcal{H}_{N,d}(\chi)$ in $\mathcal{H}_N$ carry isomorphic $H^J$-actions. The two-variable Hecke operator eigenvalues $T_m$ for $(m,|N|) = 1$ distinguish the spaces $\mathcal{H}_N(\chi;f)$ in the coarse multiplicative decomposition of $\mathcal{H}_N$ given in Theorem 5.7 because they determine the primitive character $(\chi)$, see Theorem 9.9 (1) below. The individual Hilbert spaces $\mathcal{H}_{N,d}(\chi)$ are completely distinguished in terms of their associated spectral measure associated to the operator $\Delta_L$ described in Section 9, which determines a particular pair of Lerch L-function $sL^\pm_{N,d}(\chi,s,a,c,z)$, as defined in 9.

In Section 8 we study the action of the automorphism $\alpha(a,c,z) := (-c,a,z-ac)$ of the Heisenberg group acting as an operator $R$ on the Heisenberg modules $\mathcal{H}_N$. On $\mathcal{H}_1$ this operator intertwines under the Weil-Brezin map with the additive Fourier transform. We show that its action on $\mathcal{H}_N$ intertwines with the dilation $U(N)$ composed with the additive Fourier transform, and that this intertwining action mixes various of the spaces $\mathcal{H}_{N,d}(\chi)$, over all different (imprimitive) $\chi \pmod f$ associated to a fixed primitive character $\chi \pmod f$ with $f|d|N$. We show that the R-operator respects the coarse multiplicative decomposition of $\mathcal{H}_N$ in the following way.

**Theorem 8.3.** For $N \neq 0$, the Heisenberg-Fourier operator $R$ restricted to the invariant subspace $\mathcal{H}_N$ is a unitary operator $R_N$ which acts to permute the Hilbert spaces $\mathcal{H}_N(\chi;f)$ given by the coarse multiplicative decomposition of $\mathcal{H}_N$. It satisfies

$$R_N(\mathcal{H}_N(\chi;f)) = \mathcal{H}_N(\bar{\chi};f).$$

In Section 9 we define Lerch L-functions $L^\pm_{N,d}(\chi,s,a,c,z)$ and show that they are analogous to Eisenstein series in three distinct ways:

(i) as giving a continuous spectral decomposition of a Laplacian-like operator,

(ii) as being a simultaneous eigenfunction of a family $\{T_m : m \geq 1$ with $(m,N) = 1\}$ of Hecke-like operators,

(iii) as satisfying suitable functional equations relating $s$ to $1-s$.

In Section 9.1 we define $L^\pm_{N,d}(\chi,s,a,c,z)$ as functions on the Heisenberg group, for $0 < \Re(s) < 1$, via $L^\pm_{N,d}(\chi,s,a,c,z) = e^{2\pi i N z}L^\pm_{N,d}(\chi,s,a,c)$, via the conditionally convergent
multiplier function on $L$ variant Laplacian $\Delta$ group distribution and observe the distributional variant agrees with Lerch parameter, $-\infty < \tau < \infty$, in which $\hat{F}$ on the Heisenberg nilmanifold.

In Section 9.2 we treat Lerch $L$-functions as simultaneous eigenfunctions of the differential operators

$$\Delta_L = \frac{1}{2\pi i} \frac{\partial}{\partial a} + \frac{\partial}{\partial c} + \frac{N}{2}$$

and $\frac{\partial}{\partial z}$, and show that the two families of functions $L_{N,d}^\pm(\chi, \frac{1}{2} + i\tau, a, c, z)$, viewing $\tau$ as a parameter, $-\infty < \tau < \infty$, are a complete set of generalized eigenfunctions for $\Delta_L$, with

$$\Delta_L L_{N,d}^\pm(\chi, \frac{1}{2} + i\tau, a, c, z) = -i\tau L_{N,d}^\pm(\chi, \frac{1}{2} + i\tau, a, c, z),$$

on a suitable dense domain inside the Hilbert space $\mathcal{H}_{N,d}(\chi)$. A main result of this paper concerns the spectral interpretation of Lerch $L$-functions as Eisenstein series.

**Theorem 9.5.** (Eisenstein Series Interpretation of Lerch $L$-functions)
Let $N \neq 0$ and $d \geq 1$ with $d \mid N$.

1. Consider the unbounded operator $\Delta_L = \frac{1}{2\pi i} \frac{\partial}{\partial a} + \frac{\partial}{\partial c} + \frac{N}{2}$ on the dense domain $\mathcal{D}_{N,d}(\chi) := \mathcal{W}_{N,d}(\chi)(\mathcal{D})$ in the Hilbert space $\mathcal{H}_{N,d}(\chi)$, in which $\mathcal{D}$ denotes the maximal domain for $D = x \frac{\partial}{\partial x} + \frac{1}{2} \mathbb{I}$ on $L^2(\mathbb{R}, dx)$. The operator $(\Delta_L, \mathcal{D}_{N,d}(\chi))$ commutes with all elements of the unitary group $\{V(t) : t \in \mathbb{R}^*\}$.

2. The operator $(\Delta_L, \mathcal{D}_{N,d}(\chi))$ is skew-adjoint on $\mathcal{H}_{N,d}(\chi)$, and its associated spectral multiplier function on $L^2(\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{R}, d\tau)$ is $a_0(\tau) = -i\tau$ and $a_1(\tau_1) = -i\tau_1$.

3. The two families of Lerch $L$-functions $L_{N,d}^\pm(\chi, \frac{1}{2} + i\tau, a, c, z)$, parameterize the (pure) continuous spectrum of $(\Delta_L, \mathcal{D}_{N,d}(\chi))$ on $\mathcal{H}_{N,d}(\chi)$, giving a complete set of generalized eigenfunctions, as $\tau$ varies over $\mathbb{R}$. All functions $F(a, c, z)$ in the dense subspace $\mathcal{S}(\mathcal{H}_{N,d}(\chi))$ have a convergent spectral representation

$$F(a, c, z) = \frac{1}{4\pi} \int_{-\infty}^{\infty} \hat{F}^+(\frac{1}{2} + i\tau)L_{N,d}^+(\chi, \frac{1}{2} - i\tau, a, c, z)d\tau + \frac{1}{4\pi} \int_{-\infty}^{\infty} \hat{F}^-(\frac{1}{2} + i\tau)L_{N,d}^-(\chi, \frac{1}{2} - i\tau, a, c, z)d\tau,$$

in which $\hat{F}^+(s) = \mathcal{M}_0(\mathcal{W}_{N,d}(\chi)^{-1}(F))(s)$ and $\hat{F}^-(s) = \mathcal{M}_1(\mathcal{W}_{N,d}(\chi)^{-1}(F))(s)$.

The operator $\Delta_L$ is the pushforward under the Weil-Brezin map of the dilation invariant Laplacian $D = x \frac{\partial}{\partial x} + \frac{1}{2}$ acting on $L^2(\mathbb{R}_{>0}, dx)$, which itself corresponds to the dilation invariant Laplacian $\tilde{D} = x \frac{\partial}{\partial x}$ acting on $L^2(\mathbb{R}_x > 0: \frac{dx}{2})$.

In Section 9.3 we show these functions can be characterized as simultaneous joint eigenfunctions for the set of two-variable Hecke operators $T_m$, viewing them as tempered distributions using the generalized Weil-Brezin maps. We introduce a notion of Lerch $L$-distribution and observe the distributional variant agrees with Lerch $L$-functions in the
critical strip $0 < \Re(s) < 1$, where these distributions correspond to locally $L^1$-functions.

**Theorem 9.9.** (Hecke operator tempered distribution eigenspace) Let $N$ be a nonzero integer, and $\chi$ be a Dirichlet character (mod $d$), with $d|N$.

1. For each fixed $s \in \mathbb{C}$, let $\mathcal{E}_s(\mathcal{H}_{N,d}(\chi))$ be the vector space of tempered distributions $\Delta \in S'(\mathcal{H}_{N,d}(\chi))$ such that
   \[ T_m(\Delta) = \chi(m)m^{-s}\Delta, \text{ for all } m \geq 1 \text{ with } (m,N) = 1. \]

Then $\mathcal{E}_s(\mathcal{H}_{N,d}(\chi))$ is a two-dimensional vector space, and is spanned by an even homogeneous tempered distribution of homogeneity $|t|^{1-s}$ and an odd homogeneous tempered distribution of homogeneity $\text{sgn}(t)|t|^{1-s}$.

2. For all non-integer $s \in \mathbb{C}$ the two Lerch $L$-distributions $L_{N,d}^\pm(\chi,s,a,c,z)$ are nonzero even and odd homogeneous distributions spanning $\mathcal{E}_s(\mathcal{H}_{N,d}(\chi))$, respectively. For $0 < \Re(s) < 1$ these two distributions are induced by the Lerch $L$-functions $L_{N,d}^\pm(\chi,s,a,c,z)$, which both lie in $L^1(\mathcal{H}_{N,d}(\chi))$.

In Section 9.4 we show these functions satisfy suitable functional equations taking $s \mapsto 1 - s$. These functional equations are more complicated than that for Dirichlet $L$-functions.

**Theorem 9.10.** (Generalized Lerch Functional Equations) Suppose that $N \neq 0$. Let $\chi$ be a primitive character (mod $f$) and suppose that $f|d$ and $d|N$, and let $\chi|d$ denote the (generally imprimitive) character (mod $d$) co-trained with $\chi$. Then for $0 < \Re(s) < 1$ the two Lerch $L$-functions $L_{N,d}^\pm(\chi|d,s,a,c,z)$ associated to $\mathcal{H}_{N,d}(\chi|d)$ satisfy the functional equations

\[
\mathcal{R}(L_{N,d}^\pm)(\chi|d,1-s,a,c,z) = \chi(-1)\tau(\chi)|N|^{s-1}\gamma^\pm(s) \left( \sum_{\bar{d}|N} C_{N,d}^{\pm}(\bar{d},\chi)L_{N,d}^\pm(\chi|d,s,a,c,z) \right),
\]

in which $\mathcal{R}f(a,c,z) = f(-c,a,z-ac)$ and $\gamma^\pm(s)$ are Tate-Gelfand-Graev gamma functions, and the coefficients $C_{N,d}(\bar{d},\chi)$ vanish whenever $\bar{d} \nmid d$.

These functional equations correspond to the action of the $\mathcal{R}$ operator studied in Section 8 which mixes together Lerch $L$-functions involving all imprimitive characters coming from a fixed primitive character $\chi$ (mod $e$) of conductor $e|N$. The Tate-Gelfand-Graev gamma functions are defined in (9.27).

In Section 10 we make concluding and summarizing remarks.

There are two appendices (Sections 11 and 12). Appendix A gives facts about the asymmetric form of the Heisenberg group used in this paper, and its relation to the symmetric Heisenberg group. It also discusses properties of the sub-Jacobi group $H^J$. Appendix B discusses dilation-invariant operators and their spectral theory, particularly the operator $x\frac{d}{dx} + \frac{1}{2}$ on $L^2(\mathbb{R},dx)$, based on work of Burnol.
3. Preliminary Results

We recall basic facts about the Heisenberg nilmanifold \( X = H(\mathbb{Z}) \backslash H(\mathbb{R}) \) and functions on it. It was studied by Brezin [12] and Auslander and Br ezin [4]. Useful references are Auslander and Tolimieri [5] and Thangavelu [54].

3.1. Decomposition of \( H(\mathbb{Z}) \backslash H(\mathbb{R}) \). We may view the Hilbert space \( \mathcal{H} = L^2(H(\mathbb{Z}) \backslash H(\mathbb{R}), d\mu) \) as a space of (equivalence classes of) measurable functions on \( H(\mathbb{R}) \) satisfying the periodicity condition

\[
F(\gamma g) = F(g) \quad \text{for all} \quad \gamma \in H(\mathbb{Z}),
\]

and square-summable on a fundamental domain of \( H(\mathbb{Z}) \backslash H(\mathbb{R}) \). The Hermitian inner product on this Hilbert space is

\[
\langle F_1, F_2 \rangle := \int_{H(\mathbb{Z}) \backslash H(\mathbb{R})} F_1(g) \overline{F_2(g)} \, dg
= \int_0^1 \int_0^1 \int_0^1 F_1(a, c, z) \overline{F_2(a, c, z)} \, dadcdz.
\]

The space \( C_{0, \text{bdd}}(H(\mathbb{R})) \cap L^2(H(\mathbb{Z}) \backslash H(\mathbb{R}), d\mu) \) is dense in \( L^2(H(\mathbb{Z}) \backslash H(\mathbb{R}), d\mu) \).

The group \( H(\mathbb{R}) \) acts on \( L^2(H(\mathbb{Z}) \backslash H(\mathbb{R}), d\mu) \) on the right as

\[
\rho_h(F)(g) = F(gh) \quad g, h \in H(\mathbb{R}).
\]

Indeed \( \rho_{h_1} \circ \rho_{h_2}(F)(g) = \rho_{h_2}(F)(gh_1) = F((gh_1)h_2) = F(g(h_1h_2)) \). Thus if \( g = [a, c, z] \) and \( h = [a', c', z'] \) then

\[
\rho_h(F)(a, c, z) = F(a + a', c + c', z + z' + ca').
\]

All elements of this Hilbert space \( \mathcal{H} \) satisfy the periodicity property

\[
F([a, c, z + 1]) = F([0, 0, 1] \circ [a, c, z]) = F(a, c, z).
\]

in the \( z \)-variable. Thus we can decompose \( \mathcal{H} \) into Fourier eigenspaces in the \( z \)-direction, obtaining the orthogonal decomposition

\[
\mathcal{H} = \bigoplus_{N \in \mathbb{Z}} \mathcal{H}_N,
\]

in which \( \mathcal{H}_N \) consists of \( L^2 \)-functions having \( e^{2\pi i Nz} \) as central character; that is,

\[
F(a, c, z) = e^{2\pi i Nz} F(a, c, 0),
\]

almost everywhere. The conditions for membership \( F(a, c, z) \in \mathcal{H}_N \) can be expressed in terms of values \( z = 0 \), and given in terms of the left action of \( H(\mathbb{Z}) \), as follows. Such a function must satisfy (almost everywhere)

\[
F(a + 1, c, 0) = f([1, 0, 1] \circ [a, c, 0]) = F(a, c, 0),
\]

a condition which is independent of \( N \). Such a function must also satisfy (almost everywhere)

\[
F(a, c + 1, 0) = F([1, 1, 0] \circ [a, c, -a]) = e^{-2\pi i Na} F(a, c, 0),
\]

\[
F(a, c, z + 1) = F([0, 0, 1] \circ [a, c, z]) = F(a, c, z).
\]
a condition which depends on $N$. Since $H(\mathbb{Z})$ is generated by $[0,0,1],[0,1,0],[1,0,0]$ any function on $H(\mathbb{R})$ satisfying properties (3.6), (3.7), (3.8) almost everywhere will be invariant under the left action of $H(\mathbb{Z})$ on $H_N$ almost everywhere.

It is known that for $N \neq 0$ the action of $H(\mathbb{R})$ on the space $H_N$ decomposes into $|N|$ copies of the unique (infinite-dimensional) unitary irreducible representation $\pi_N$ of $H(\mathbb{R})$ having central character $e^{2\pi i Nz}$ (cf. [12], [5]).

For $N = 0$ the central character is trivial and the functions in $H_0$ are constant in the $z$-direction, so the Hilbert space $L^2(\mathbb{R}^2/\mathbb{Z}^2,d\mu)$ can serve as a representation module for $H(\mathbb{R})$.

### 3.2. Heisenberg-Fourier operator

The map $\alpha : H(\mathbb{R}) \to H(\mathbb{R})$ given by

$$\alpha([a,c,z]) := [-c,a,z-ac].$$

is an automorphism of $H(\mathbb{R})$, i.e. $\alpha(g) \circ \alpha(h) = \alpha(gh)$. It is of order 4. The map $\alpha$ induces an operator acting on functions in $H_N$. We let $C^0_{\text{bdd}}(H(\mathbb{R}))$ denote the set of bounded continuous functions on the real Heisenberg group $H(\mathbb{R})$. This space includes all $H(\mathbb{Z})$-periodic continuous functions, which form a dense subset of functions in $H$.

**Definition 3.1.** The Heisenberg-Fourier operator $R : C^0_{\text{bdd}}(H(\mathbb{R})) \to C^0_{\text{bdd}}(H(\mathbb{R}))$ is given by

$$R(F)(a,c,z) := F(\alpha(a,c,z)) = F(-c,a,z-ac).$$

It satisfies $R^4 = I$ on this domain.

**Lemma 3.2.** The operator $R$ restricted to $C^0_{\text{bdd}}(H(\mathbb{R})) \cap L^2(H(\mathbb{Z})\backslash H(\mathbb{R}),d\mu)$ extends uniquely to a unitary operator on $H$, also denoted $R$.

(i) The unitary operator $R$ leaves each space $H_N$ invariant, including $N = 0$. The restriction $R_N$ of this operator to the domain $H_N$ is given by

$$R_N(F)(a,c,z) = e^{-2\pi i Na} F(-c,a,z), \quad F \in H_N.$$  

(ii) The operator $R$ satisfies $R^4 = I$, and has adjoint operator

$$R^*(F)(a,c,z) = R^{-1}(F)(a,c,z) = F(c,-a,z+ac).$$

(iii) The unitary operator $J := R^2$ is an involution, is self-adjoint, and is given by

$$J(F)(a,c,z) = F(-a,-c,z).$$

**Proof.** In proving (i) below we will check that $R$ applied to continuous functions on $H_N$ maps them to $H_N$ and Assuming this, since such functions form a dense subspace of $H_N$, and the map $R$ uniquely extends to an isometry of $H_N$. Consequently it extends uniquely to an isometry of $L^2(H(\mathbb{Z})\backslash H(\mathbb{R})) = \oplus_{N \in \mathbb{Z}} H_N$, which is therefore unitary.
(i) A continuous function $F(a, c, z) \in \mathcal{H}_N$ if and only if $F(a, c, z) = F(a, c, 0)e^{2\pi i Nz}$, together with relations

$$F(a + 1, c, z) = F(a, c, z)$$
$$F(a, c + 1, z) = e^{-2\pi i Na}F(a, c, z).$$

For $F \in \mathcal{H}_N$ these relations yield:

$$R(F)(a, c, z) = F(-c, a, z - ac) = e^{-2\pi i N ac}F(-c, a, z),$$

which is (3.13). We now check $R(F)(a, c, z) \in \mathcal{H}_N$. We have

$$R(F)(a, c, z) = F(-c, a, z - ac) = e^{2\pi i Nz}F(-c, a, -ac) = e^{2\pi i N z}R(F)(a, c, 0).$$

In addition, using these relations, we obtain

$$R(F)(a + 1, c, z) = F(-c, a + 1, z - (a + 1)c)$$
$$= e^{-2\pi i N(-c)}F(-c, a, z - ac - c)$$
$$= e^{2\pi i Nc}e^{2\pi i N(-c)}F(-c, a, z - ac) = R(F)(a, c, z).$$

and

$$R(F)(a, c + 1, z) = F(-c - 1, a, z - a(c + 1))$$
$$= F(-c, a, z - ac - a)$$
$$= e^{2\pi i N(-1)}F(-c, a, z - ac) = e^{-2\pi i Na}R(F)(a, c, z).$$

Thus $R(F) \in \mathcal{H}_N$. Finally recall that the Hilbert space norm of $F(a, c, z)$ is

$$||F||^2 = \int_0^1 \int_0^1 \int_0^1 |F(a, c, z)|^2 dada dz.$$  

Now for $F \in \mathcal{H}_N$, we obtain using the relations

$$||R(F)||^2 = \int_0^1 \int_0^1 \int_0^1 |F(-c, a, z - ac)|^2 dada dz$$
$$= \int_0^1 \int_0^1 \int_0^1 |F(1 - c, a, z)|^2 dada dz = ||F||^2.$$  

It follows that $R$ is an isometry, which is onto since $R$ is invertible.

(ii), (iii) The relation $R^4 = I$ on $L^2(H(\mathbb{Z}) \backslash H(\mathbb{R}), d\mu)$ is inherited from its holding on the dense subspace $C^0_{bdd}(H(\mathbb{R})) \cap L^2(H(\mathbb{Z}) \backslash H(\mathbb{R}), d\mu)$. The adjoint $R^*$ equals $R^{-1}$ since it is unitary, and $R^{-1} = R^3$. The formulas (3.13) and (3.14) follow by explicit calculation. □

Remark. The restriction operator $R_1$ on $\mathcal{H}_1$ is related to the Fourier transform, in the sense that it intertwines with the (additive) Fourier transform $\mathcal{F}$ on $L^2(\mathbb{R}, dx)$ under the Weil-Brezin map $W = W_{1,1}(\chi_0) : L^2(\mathbb{R}, dx) \to \mathcal{H}_1$ defined in Section 5. This explains our nomenclature. The operator $R_N$ on $\mathcal{H}_N$ has a more complicated relation to the Fourier transform, which is computed in Section 7.
4. Two-Variable Hecke Operators

We define and study two-variable Hecke operators on the Heisenberg group. The definition extends the two-variable Hecke operators studied in [37] by inserting the third Heisenberg variable \( z \), which however is not changed under this action.

4.1. Hecke operator definition. We now define Hecke operators on the space of bounded continuous functions \( C_{\text{bdd}}^0(H(\mathbb{R})) \).

**Definition 4.1.** For all nonzero integers \( m \) we define the two-variable Hecke operator

\[
T_m(F)(a,c,z) := \frac{1}{|m|} \sum_{j=0}^{\lfloor |m|/2 \rfloor} F\left(\frac{a+j}{m},mc,z\right), \tag{4.1}
\]

Here \( T_m : C_{\text{bdd}}^0(H(\mathbb{R})) \to C_{\text{bdd}}^0(H(\mathbb{R})) \), and the two variables in the name refer to variables \((a,c)\), noting that the action on the \( z \) variable is trivial.

It is easy to compute that

\[
T_m \circ T_n = T_n \circ T_m = T_{mn}, \tag{4.2}
\]

as operators on \( C_{\text{bdd}}^0(H(\mathbb{R})) \). Indeed setting \( G(a,c,z) = T_m F(a,c,z) \), we have

\[
T_n \circ T_m(F)(a,c,z) = \frac{1}{|n|} \sum_{k=0}^{\lfloor |n|/2 \rfloor} T_m(F)\left(\frac{a+k}{n},nc,z\right) \tag{4.3}
\]

\[
= \frac{1}{|n|} \sum_{k=0}^{\lfloor |n|/2 \rfloor} \frac{1}{|m|} \sum_{j=0}^{\lfloor |m|/2 \rfloor} F\left(\frac{a+k+j}{mn},nmc,z\right)
\]

\[
= \frac{1}{|mn|} \sum_{l=0}^{\lfloor |mn|/2 \rfloor} F\left(\frac{a+l}{mn},mnc,z\right)
\]

\[
= T_{mn}(F)(a,c,z). \tag{4.3}
\]

**Lemma 4.2.** The operators \( \{T_m : m \in \mathbb{Z}\setminus\{0\}\} \) on \( C_{\text{bdd}}^0(H(\mathbb{R})) \cap L^2(H(\mathbb{Z})\setminus H(\mathbb{R}),d\mu) \) extend uniquely to bounded operators \( \{T_m : m \in \mathbb{Z}\setminus\{0\}\} \) on \( L^2(H(\mathbb{Z})\setminus H(\mathbb{R}),d\mu) \).

(i) Each \( T_m \) leaves every space \( H_N \) invariant, including the case \( N = 0 \).

(ii) On \( H_N \) these operators satisfy the relations

\[
T_m \circ T_n = T_n \circ T_m = T_{mn}, \tag{4.4}
\]

for all \( m,n \in \mathbb{Z} \).

(iii) With respect to the involution \( J = R^2 \) the operators \( T_m \) satisfy

\[
T_{-m} = T_m \circ R^2. \tag{4.5}
\]

**Proof.** The space \( C_{\text{bdd}}^0(H(\mathbb{R})) \cap H \) is dense in \( H \) so any continuous extension of \( T_m \) to all of \( H \) is unique. We have \( \|T_m\| \leq 1 \) on \( C_{\text{bdd}}^0(H(\mathbb{R})) \cap H \), so a continuous extension exists.

(i) A continuous function \( F(a,c,z) \in H_N \) if \( F(a,c,z) = F(a,c,0)e^{2\pi i N z} \), with

\[
F(a+1,c,0) = F(a,c,0)
\]

\[
F(a,c+1,0) = e^{-2\pi i N a} F(a,c,0).
\]
For a continuous \( F(a, c, z) \in \mathcal{H}_N \) we have

\[
T_m(F)(a + 1, c, 0) = \frac{1}{|m|} \sum_{j=0}^{|m|-1} F\left(\frac{a + 1 + j}{m}, mc, 0\right)
\]

\[
= \frac{1}{|m|} \left( F\left(\frac{a}{m} + \frac{|m|}{m}\right) + \sum_{j=1}^{|m|-1} F\left(\frac{a + j}{m}, mc, 0\right) \right)
\]

\[
= \frac{1}{|m|} \sum_{j=0}^{|m|-1} F\left(\frac{a + j}{m}, mc, 0\right)
\]

\[
= T_m(F)(a, c, 0),
\]

where (3.7) was used in the second to last line. In addition

\[
T_m(F)(a, c + 1, 0) = \frac{1}{|m|} \sum_{j=0}^{|m|-1} F\left(\frac{a + j}{m}, mc + m, 0\right)
\]

\[
= \frac{1}{|m|} \sum_{j=0}^{|m|-1} e^{-2\pi i m N} F\left(\frac{a + j}{m}, mc, 0\right)
\]

\[
= e^{-2\pi i N a} \frac{1}{|m|} \sum_{j=0}^{|m|-1} e^{-2\pi i j N} F\left(\frac{a + j}{m}, mc, 0\right)
\]

\[
= e^{-2\pi i N a} T_m(F)(a, c, 0),
\]

where (3.8) was used in the second line. We conclude \( T_m(F) \in \mathcal{H}_N \). The continuous functions in \( \mathcal{H}_N \) are dense, so the result holds on all of \( \mathcal{H}_N \) by boundedness of the operator \( T_m \).

(ii) The commutativity relation (4.4) is inherited from (4.2).

(iii) The relation (4.5) is verified using

\[
T_{-m}(F)(a, c, z) = \frac{1}{|m|} \sum_{j=0}^{|m|-1} F\left(\frac{a + j}{-m}, -mc, z\right)
\]

\[
= \frac{1}{|m|} \sum_{j=0}^{|m|-1} R^2(F)\left(\frac{a + j}{m}, mc, z\right)
\]

\[
= T_m \circ R^2(F)(a, c, z)
\]

\[
\square
\]

4.2. Adjoint two-variable Hecke operators. Since the Hecke operators act as bounded operators on each \( \mathcal{H}_N \), they have well defined adjoint Hecke operators \( T_m^* \). In the next result we allow all integers \( N \), allowing \( N = 0 \), using the convention that the g.c.d. \( (m, 0) = m \).
Theorem 4.3. (i) The adjoint operator $T_m^*$ of $T_m$ on $\mathcal{H}$ leaves each space $\mathcal{H}_N$ invariant and acts on $\mathcal{H}_N$ by

$$T_m^*(F)(a, c, z) = \frac{1}{m} \sum_{k=0}^{m-1} e^{2\pi i k N a} F(m a, c + \frac{k}{m}, z). \quad (4.7)$$

(ii) On each $\mathcal{H}_N$ each $T_m^*$ satisfies with respect to the Heisenberg-Fourier operator $R$ the relation

$$T_m^* = R^* \circ T_m \circ R. \quad (4.8)$$

where $R^* = R^3$.

(iii) For $N \in \mathbb{Z}$, and $m \geq 1$, with $d = (m, N)$, then for $F \in \mathcal{H}_N$

$$T_m \circ T_m^*(F)(a, c, z) = \frac{1}{m} \sum_{\ell=0}^{d-1} F(a, c + \frac{\ell}{d}, z) \quad (4.9)$$

$$T_m^* \circ T_m(F)(a, c, z) = \frac{1}{m} \sum_{\ell=0}^{d-1} e^{2\pi i (\frac{N}{N'})^2} F(a + \frac{\ell}{d}, c, z) \quad (4.10)$$

The operators $T_m$ and $T_m^*$ commute on $\mathcal{H}_N$ when $d = (m, N) = 1$, and then satisfy

$$T_m \circ T_m^* = T_m^* \circ T_m = \frac{1}{m} I. \quad (4.11)$$

They do not commute on $\mathcal{H}_N$ when $d > 1$.

Remarks. (1) In the case $N = 0$, we use the convention that $\gcd(m, 0) = m$.

(2) A bounded operator $M$ on a Hilbert space is normal if $M^* M = MM^*$. The result (iii) above implies that $T_m$ is not a normal operator on $\mathcal{H}_N$ when $d = (m, N) \geq 2$.

Proof. (i) We verify (4.7) on $\mathcal{H}_N$, and later use it to prove (4.8). Let $\tilde{T}_m f$ denote the right side of (4.7).

We first show $T_m^*$ leaves each $\mathcal{H}_N$ invariant. Suppose $F \in \mathcal{H}_N$, $g \in \mathcal{H}_N'$. If $N \neq N'$ then

$$\langle F, T_m(G) \rangle = \int_0^1 \int_0^1 F(a, c, 0) \overline{T_m(G)(a, c, 0)} \int_0^1 e^{2\pi i (N-N')z} dz = 0. \quad (4.12)$$

Since

$$\langle T_m^*(F), G \rangle = \langle F, T_m(G) \rangle = 0,$$

it follows that $T_m^* F$ is orthogonal to all $G \in \mathcal{H}_N'$, $N' \neq N$. Thus $T_m^* f \in \mathcal{H}_N$, as required.

It therefore suffices to determine $T_m^*$ on $\mathcal{H}_N$. For $F, G \in \mathcal{H}_N$ we integrate out the $z$-variable to obtain

$$(F, T_m(G)) = \frac{1}{m} \int_0^1 \int_0^1 F(a, c, 0) \sum_{j=0}^{m-1} G \left( a + \frac{k}{m}, mc, 0 \right) dadc.$$

For $\frac{k}{m} \leq c < \frac{k+1}{m}$ set $c = \frac{\tilde{c} + k}{m}$, and by change of variables

$$(F, T_m(G)) = \frac{1}{m^2} \int_0^1 \sum_{k=0}^{m-1} \sum_{j=0}^{m-1} F \left( a, \frac{\tilde{c} + k}{m}, 0 \right) G \left( \frac{a + j}{m}, \tilde{c} + k, 0 \right) d\tilde{c}. \quad (4.13)$$
Next, for each \( j \), set \( \tilde{a} = \frac{a + j}{m} \), so \( a = m\tilde{a} - j \) with \( \frac{j}{m} < \tilde{a} < \frac{j+1}{m} \). We obtain

\[
(F, T_m(G)) = \frac{1}{m} \sum_{k=0}^{m-1} \sum_{j=0}^{m-1} \int_0^1 \int_{\frac{j}{m}}^{\frac{j+1}{m}} F \left( m\tilde{a} - j, \frac{\tilde{c} + k}{m}, 0 \right) G(\tilde{a}, \tilde{c} + k, 0) d\tilde{a} d\tilde{c}.
\]

Using \( F(a - j, c, 0) = F(a, c, 0) \) and \( G(a, c + k, 0) = e^{-2\pi i N a} g(a, c, 0) \) leads to

\[
(F, T_m(G)) = \frac{1}{m} \sum_{k=0}^{m-1} \sum_{j=0}^{m-1} \int_0^1 \int_{\frac{j}{m}}^{\frac{j+1}{m}} e^{2\pi i N a} F \left( m\tilde{a} - j, \frac{\tilde{c} + k}{m}, 0 \right) G(\tilde{a}, \tilde{c} + k, 0) d\tilde{a} d\tilde{c}
\]

\[
= \int_0^1 \int_0^1 \left( \frac{1}{m} \sum_{k=0}^{m-1} e^{2\pi i N a} F \left( m\tilde{a} - j, \frac{\tilde{c} + k}{m}, 0 \right) G(\tilde{a}, \tilde{c}, 0) d\tilde{a} d\tilde{c} \right)
\]

\[
= \langle \tilde{T}_m f, g \rangle,
\]

where \( \tilde{T}_m \) denotes the right side of (4.7). Since \( (T^*_m(F), G) \) for all \( G \in \mathcal{H}_N \) determines the element \( T^*_m(F) \in \mathcal{H}_N \), we conclude that (4.7) holds.

(ii) To prove (4.8) it suffices to show that it holds for \( F \in \mathcal{H}_N \) for all \( N \in \mathbb{Z} \). It then follows by linearity for all \( F \in \mathcal{H} \), since \( R \) and \( R^* \) leave all \( \mathcal{H}_N \) invariant. We verify it by direct computation on \( \mathcal{H}_N \); it suffices to check that \( R_N \circ T^*_m = T_m \circ R_N \) and \( R_N^* \circ T^*_m = T_m \circ R_N^* \) on \( \mathcal{H}_N \). We compute

\[
R_N \circ T^*_m(F)(a, c, z) = T^*_m(F)(-c, a, z - ac)
\]

\[
= \frac{1}{m} \sum_{k=0}^{m-1} e^{-2\pi i N c} F(-mc, \frac{a + k}{m}, z - ac)
\]

\[
= \frac{1}{m} \sum_{k=0}^{m-1} e^{-2\pi i N c - 2\pi i N c} F(-mc, \frac{a + k}{m}, z)
\]

while

\[
T_m \circ R_N(F)(a, c, z) = \frac{1}{m} \sum_{k=0}^{m-1} R_N(F)(\frac{a + k}{m}, mc, z)
\]

\[
= \frac{1}{m} \sum_{k=0}^{m-1} F(-mc, \frac{a + k}{m}, z - (\frac{a + k}{m})mc)
\]

\[
= \frac{1}{m} \sum_{k=0}^{m-1} e^{-2\pi i N c} e^{-2\pi i N c} F(-mc, \frac{a + k}{m}, z).
\]
(iii) We verify (4.9). Using the formula (4.7) we have

\[ T_m^* \circ T_m(F)(a, c, z) = \frac{1}{m} \sum_{j=0}^{m-1} e^{2\pi ij Na} T_m(F)(ma, \frac{c+j}{m}, z) \]

\[ = \frac{1}{m} \sum_{j=0}^{m-1} \sum_{k=0}^{m-1} e^{2\pi iNja} F(ma+k, \frac{m(c+j)}{m}, z) \]

\[ = \frac{1}{m} \sum_{j=0}^{m-1} \sum_{k=0}^{m-1} e^{2\pi iNja} F(a+\frac{k}{m}, c+j, z) \]

\[ = \frac{1}{m} \sum_{j=0}^{m-1} \sum_{k=0}^{m-1} e^{2\pi iNja} e^{-2\pi iNj(a+\frac{k}{m})} F(a+\frac{k}{m}, c, z) \]

\[ = \frac{1}{m} \sum_{k=0}^{m-1} \left( \frac{1}{m} \sum_{j=0}^{m-1} e^{-2\pi i \frac{Nj}{m}} \right) F(a+\frac{k}{m}, c, z) \]

If \( N \neq 0 \) the condition \( d = (m, N) \) the condition \( m|Nk \) requires that \( k = \frac{N}{d} \ell \) for some integer \( 0 \leq \ell < d - 1 \), which yields (4.9) in this case. In case \( N = 0 \) set \((m,0) = m\) and (4.9) still holds. The derivation of (4.10) is similar:

\[ T_m \circ T_m^*(F)(a, c, z) = \frac{1}{m} \sum_{k=0}^{m-1} T_m^*(F)(\frac{a+k}{m}, mc, z) \]

\[ = \frac{1}{m} \sum_{k=0}^{m-1} \sum_{j=0}^{m-1} e^{2\pi iNj(\frac{a+k}{m})} F(m(a+k), \frac{mc+j}{m}, z) \]

\[ = \frac{1}{m} \sum_{k=0}^{m-1} \sum_{j=0}^{m-1} e^{2\pi iNja} e^{2\pi i \frac{Nj}{m}} F(a, c+j, z) \]

\[ = \frac{1}{m} \sum_{j=0}^{m-1} e^{2\pi iNja} \left( \frac{1}{m} \sum_{k=0}^{m-1} e^{-2\pi i \frac{Nj}{m}} \right) F(a, c+j, z) \]

\[ = \frac{1}{m} \sum_{0 \leq j < m} e^{2\pi i \frac{Nj}{m}} F(a, c+j, z) \]

\[ = \frac{1}{m} \sum_{\ell=0}^{d-1} e^{2\pi i \frac{N\ell}{d}} F(a, c+\frac{\ell}{d}, z) \]

To derive the last line, for \( N \neq 0 \) we used the fact that \( m|jN \) makes \( j = \frac{N}{d} \ell \) for some \( 0 \leq \ell < d \). The case \( N = 0 \) follows by inspection.

The relations (4.9) and (4.10) imply that \( T_m \) and \( T_m^* \) do not commute on \( \mathcal{H}_N \) for \( N \neq 0 \) when \( d = (m,|N|) \geq 2 \). When \( (m,N) = 1 \), only the \( \ell = 0 \) term occurs in the sums above and \( T_m^* \) and \( T_m \) commute, with \( T_m^* \circ T_m(F)(a, c, z) = \frac{1}{m} F(a, c, z) \) and \( T_m \circ T_m^*(F)(a, c, z) = \frac{1}{m} F(a, c, z) \). □
Remark. Theorem 4.3 (ii) shows that $T_m$ and $T_m^*$ are unitarily equivalent operators on each $\mathcal{H}_N$ (including $N = 0$), so they have equal operator norms

$$\|T_m\| = \|T_m^*\|,$$  \hspace{1cm} (4.14)

(Here $\|T_m\| := \sup_{x \in \mathcal{H}} \{x : \|x\| = 1\} \|T_m x\|$.) Theorem 4.3 (iii) implies that the operator $T_m$ on $\mathcal{H}_N$ is invertible when $(m, N) = 1$ and has operator norm

$$\|T_m\| = \frac{1}{\sqrt{|m|}}.$$ \hspace{1cm} (4.15)

In part II we explicitly compute their action on $\mathcal{H}_0$ using the basis $\{\phi_{jk} : (j, k) \in \mathbb{Z}^2\}$, and show that for $|m| \geq 2$ these operators are not invertible and have infinite-dimensional kernels, and we show that

$$\|T_m\| = \|T_m^*\| = 1,$$ \hspace{1cm} (4.16)

holds for all $m \neq 0$.

4.3. Heisenberg group action on $T_m$ and $T_m^*$. The operators $T_m$ and $T_m^*$ do not commute with the Heisenberg action on the right, but transform in a simple way under this action. For $t \in \mathbb{R}^*$ the maps $\beta(t) : H(\mathbb{R}) \rightarrow H(\mathbb{R})$ given by

$$\beta(t)[a, c, z] = \left[ \frac{1}{t}a, tc, z \right],$$ \hspace{1cm} (4.17)

form a one-parameter group of automorphisms of $H(\mathbb{R})$, i.e. $\beta(t)g \cdot \beta(t)h = \beta(t)(gh)$ and $\beta(t) \circ \beta(t') = \beta(tt')$.

**Theorem 4.4.** For any $h \in H(\mathbb{R})$ and $F \in L^2(H(\mathbb{Z}) \setminus H(\mathbb{R}), d\mu)$ there holds

$$\rho_h \circ T_m(F)(a, c, z) = T_m \circ \rho_{\beta(m)h}(F)(a, c, z).$$ \hspace{1cm} (4.18)

and

$$\rho_h \circ T_m^*(F)(a, c, z) = T_m^* \circ \rho_{\beta(1/m)h}(F)(a, c, z).$$ \hspace{1cm} (4.19)

**Proof.** Let $h = [a', c', z'] \in H(\mathbb{R})$, so that $\beta(m)h = [a' m^{-1}, mc, z]$. Then

$$\rho_h \circ T_m(F)(a, c, z) = T_m(F)(a + a', c + c', z + z' + ca')$$

$$= \frac{1}{|m|} \sum_{k=0}^{[m]-1} F\left(\frac{a + a' + k}{m}, m(c + c'), z + z' + ca' \right)$$

and

$$T_m \circ \rho_{\beta(m)h}(F)(a, c, z) = \frac{1}{|m|} \sum_{k=0}^{[m]-1} \pi_{\beta(m)h}(F)\left(\frac{a + k}{m}, mc, z \right)$$

$$= \frac{1}{|m|} \sum_{k=0}^{[m]-1} F\left(\frac{a + k}{m}, \frac{a' m + mc, z + z' + (mc a')}{m} \right)$$

which yields (4.18).
For the adjoint Hecke operator, by linearity it suffices to consider \( F \in H_N \) for each \( N \) separately. Then

\[
\rho_h \circ T_m^\ast(F)(a, c, z) = T_m^\ast(F)(a + a', c + c', z + z' + ca')
\]

\[
= \frac{1}{|m|} \sum_{k=0}^{|m|-1} e^{2\pi i N(a+a')} F(m(a + a'), \frac{c + c' + k}{m}, z + z' + ca').
\]

Next, using the fact that \( F(a, c, z + w) = e^{2\pi i N w} F(a, c, z) \) for \( f \in H_N \), we have

\[
T_m^\ast \circ \rho_{\beta(1/m)h}(F)(a, c, z) = \frac{1}{|m|} \sum_{k=0}^{|m|-1} e^{2\pi i N a} \rho_{\beta(1/m)h}(F)(ma, \frac{c + k}{m}, z)
\]

\[
= \frac{1}{|m|} \sum_{k=0}^{|m|-1} e^{2\pi i k N a} F(ma + ma', \frac{c + k}{m} + \frac{c'}{m}, z + z' + \left( \frac{c + k}{m} \right) ma')
\]

\[
= \frac{1}{|m|} \sum_{k=0}^{|m|-1} e^{2\pi i k N(a+a')} F(m(a + a'), \frac{c + c' + k}{m}, z + z' + ca'), (4.20)
\]

which yields (4.19).

\[ \square \]

5. Multiplicative Character Decomposition of \( H_N \)

In this section we construct a decomposition of \( H_N \) for \( N \neq 0 \) into irreducible \( H(\mathbb{R}) \) modules, which is associated to Dirichlet characters \( \chi \in (\mathbb{Z}/d\mathbb{Z})^* \) for all \( d|N \). We call this the multiplicative character decomposition of \( H_N \). Note that

\[
\sum_{d|N} \phi(d) = N , \tag{5.1}
\]

so this decomposes \( H_N \) into \(|N|\) subspaces, labelled \( H_{N,d}(\chi) \).

5.1. Schrödinger representations of \( H(\mathbb{R}) \). The Stone-von Neumann theorem asserts: that for each real \( \lambda \neq 0 \) there is (up to unitary isomorphism) a unique an irreducible (infinite-dimensional) unitary representation \( \pi_\lambda \) for the real Heisenberg group \( H(\mathbb{R}) \) having central character \( e^{2\pi i \lambda z} \), which is unique up to unitary isomorphism. The Schrödinger representation provides such a representation on the Hilbert space \( L^2(\mathbb{R}, dx) \), constructed using the operations of modulation and translation, defined here by

\[
\pi_\lambda([a, c, z]) f(x) := e^{2\pi i a x} f(x + \lambda c) e^{2\pi i \lambda z} .
\]  

(5.2)

Here modulation action is

\[
\pi_\lambda([a, 0, 0]) f(x) = e^{2\pi i a x} f(x)
\]

and the translation action is

\[
\pi_\lambda([0, c, 0]) f(x) = f(x + \lambda c).
\]

The central character multiplies by \( e^{2\pi i \lambda z} \) and describes a phase shift action

\[
\pi_\lambda([0, 0, z]) f(x) = e^{2\pi i \lambda z} f(x).
\]

All these operators leave the Schwartz class \( S(\mathbb{R}) \) in \( L^2(\mathbb{R}) \) invariant.
The Schrödinger representation $\pi_\lambda$ for arbitrary $\lambda \neq 0$ can be obtained on $L^2(\mathbb{R}, dx)$ from the Schrödinger representation $\pi_1$, by rescaling under an automorphism of $H(\mathbb{R})$ given by

$$\gamma_\lambda([a, c, z]) = [a, \lambda c, \lambda z]. \quad (5.3)$$

We have

$$\pi_\lambda([a, c, z]) := \pi_1(\gamma_\lambda[a, c, z]).$$

5.2. Twisted Weil-Brezin maps. We recall first the Weil-Brezin map, named after Weil [59] and Brezin [13, Sect. 4]. The key property is that the Weil-Brezin map intertwines the Schrödinger representation $\pi_1$ of $H(\mathbb{R})$ on $L^2(\mathbb{R}, dx)$ with the Heisenberg action on $H_1$.

**Definition 5.1.** (1) The Weil-Brezin map $\mathcal{W} : L^2(\mathbb{R}, dx) \to H_1$ is defined for Schwartz functions $f \in S(\mathbb{R})$ by

$$\mathcal{W}(f)(a, c, z) := e^{2\pi iz} \left( \sum_{n \in \mathbb{Z}} f(n + c) e^{2\pi ina} \right). \quad (5.4)$$

Under Hilbert space completion this map extends to an isometry of these Hilbert spaces.

(2) The inverse Weil-Brezin map is

$$\mathcal{W}^{-1}(g)(x) = \int_{-1}^1 g(a, x - n, 0) e^{-2\pi ina} da \text{ for } n < x < n + 1. \quad (5.5)$$

This map was independently discovered in time-frequency signal analysis, with $f(x)$ being the time-domain signal, by Zak [65], [66], where it is now called the Zak transform, cf. Janssen [29], [30].

As an example, the image under the Weil-Brezin map of the Gaussian $\phi(x) = e^{-\pi tx^2}$ is

$$\mathcal{W}(f)(a, c, z) = e^{2\pi iz} \sum_{n \in \mathbb{Z}} e^{-\pi t(n+c)^2} e^{2\pi ina}$$

$$= e^{2\pi iz} e^{-\pi tc^2} \vartheta_3(it, a + ict) \quad (5.6)$$

where $\vartheta_3(\tau, z)$ is the Jacobi theta function

$$\vartheta_3(\tau, z) := \sum_{n \in \mathbb{Z}} e^{\pi in^2 \tau} e^{2\piinz}, \quad (5.7)$$

where $\tau \in \mathbb{H}_C$ and $z \in \mathbb{C}$. L. Auslander [3] introduced a class of $C^\infty$ functions on $H(\mathbb{R})$ which he called nil-theta functions, generalizing (5.6).

We now generalize this map to $N \neq 0$, inserting a multiplicative character. Given a (primitive or imprimitive) Dirichlet character $\chi(\text{mod } d)$ with $d$ dividing $|N|$, we define a notion of twisted Weil-Brezin map, and introduce a Hilbert space $H_{N,d}(\chi)$ as its image. This map will intertwine a copy of the Schrödinger representation $\pi_{N,d}$ on $L^2(\mathbb{R}, dx)$ with the Heisenberg action on its image space.
Definition 5.2. Given a (primitive or imprimitive) Dirichlet character \(\chi(\text{mod } d)\), and an integer \(N\) with \(d|N\), the twisted Weil-Brezin map \(W_{N,d}(\chi) : L^2(\mathbb{R}, dx) \to \mathcal{H}_{N,d}(\chi)\) is defined for Schwartz functions \(f \in \mathcal{S}(\mathbb{R})\) by

\[
W_{N,d}(\chi)(f)(a, c, z) := \sqrt{C_{N,d}}e^{2\pi i N z} \sum_{n \in \mathbb{Z}} \chi\left(\frac{nd}{N}\right) f(n + N c)e^{2\pi i a} \quad (5.8)
\]

in which we set \(\chi(r) := 0\) if \(r \notin \mathbb{Z}\), and

\[
C_{N,d} := \frac{N}{\phi(d)} \quad (5.9)
\]

is a normalizing factor. (Note also that \(\chi(r) = 0\) for those \(r \in \mathbb{Z}\) having \((r, d) > 1\).)

Lemma 5.3. The twisted Weil-Brezin map \(W_{N,d}(\chi) : \mathcal{S}(\mathbb{R}) \to C_\infty(\mathcal{H}_N)\) extends to a Hilbert space isometry

\[
W_{N,d}(\chi) : L^2(\mathbb{R}, dx) \to \mathcal{H}_{N,d}(\chi) \subseteq \mathcal{H}_N
\]

whose range \(\mathcal{H}_{N,d}(\chi)\) is a closed subspace of \(\mathcal{H}_N\). The Hilbert space \(\mathcal{H}_{N,d}(\chi)\) is invariant under the action of \(H(\mathbb{R})\), and the map \(W_{N,d}(\chi)\) intertwines the Schrödinger representation \(\pi_N\) on \(L^2(\mathbb{R}, dx)\) with this action.

Proof. We check that for \(f, g \in \mathcal{S}(\mathbb{R})\),

\[
(f, g)_{L^2(\mathbb{R}, dx)} = (W_{N,d}(\chi)(f), W_{N,d}(\chi)(g))_{\mathcal{H}_N}. \quad (5.11)
\]

Since \(W_{N,d}(\chi)(f), W_{N,d}(\chi)g \in \mathcal{H}_N\) we have

\[
(W_{N,d}(\chi)(f), W_{N,d}(\chi)(g)) = C_{N,d} \int_0^1 \int_0^1 W_{N,d}(\chi)(f)(a, c, 0) \overline{W_{N,d}(\chi)(g)(a, c, 0)} dadc = C_{N,d} \int_0^1 \int_0^1 \sum_{n_1 \in \mathbb{Z}} \sum_{n_2 \in \mathbb{Z}} \chi\left(\frac{n_1 d}{N}\right) \chi\left(\frac{n_2 d}{N}\right) f(n_1 + N c)g(n_2 + N c)e^{2\pi i (n_1 - n_2)a} dadc = C_{N,d} \int_0^1 \sum_{n_1 \in \mathbb{Z}} \chi\left(\frac{n_1 d}{N}\right) \chi\left(\frac{n_1 d}{N}\right) f(n_1 + N c)g(n_1 + N c) dc.
\]

Now introduce the new summation variable \(n = \frac{nd}{N} \in \mathbb{Z}\). On noting that \(\chi(\cdot)\) is a character (mod \(d\)) so that \(\chi(n)\overline{\chi}(n) = 1\) if \((n, d) = 1\) and is 0 otherwise, we obtain

\[
(W_{N,d}(\chi)(f), W_{N,d}(\chi)(g)) = \frac{N}{\phi(d)} \int_0^1 \sum_{n \in \mathbb{Z}} \frac{n N}{d} f\left(\frac{n N}{d} + N c\right)g\left(\frac{n N}{d} + N c\right) dc = N \int_{-\infty}^{\infty} f(N c)g(N c) dc = (f, g)_{L^2(\mathbb{R}, dx)}. \quad (5.11)
\]

We now use the general fact that an isometry \(f : \mathcal{D} \to V\) defined on a dense subspace \(\mathcal{D}\) of a separable Hilbert space \(\mathcal{D} \subseteq \mathcal{H}_1\) into a subspace \(V \subseteq \mathcal{H}_2\) extends to an isometry \(f : \mathcal{H}_1 \to \mathcal{H}_2\) whose range is a closed subspace of \(\mathcal{H}_2\).

It remains to show that \(\mathcal{H}_{N,d}(\chi)\) is invariant under the action of \(H(\mathbb{R})\) acting on \(\mathcal{H}_N\). Given \(h = (a', c', z') \in H(\mathbb{R})\) and \(F \in \mathcal{H}_{N,d}(\chi)\), we have

\[
\rho_h(F)((a, c, z)) = F((a, c, z) \circ (a', c', z')) = F(a + a', c + c', z + z' + ca').
\]
Now $F(a, c, z) = W_{N,d}(\chi)(f)(a, c, z)$ and one checks that
\[
\rho_h(F)(a, c, z) = e^{2\pi i \frac{Nz + \phi'}{d}} e^{2\pi i N\frac{c'}{d}} \sum_{n \in \mathbb{Z}} \chi(\frac{nd}{N}) f(n + n(\frac{c + c'}{d}))) e^{2\pi i (a + a')} \\
= W_{N,d}(\chi)(\tilde{f})(a, c, z),
\]
where
\[
\tilde{f}(x) := e^{2\pi i x a'} f(x + Nc') e^{2\pi i Nz'} \in L^2(\mathbb{R}, dx).
\] (5.12)
Thus $\rho_h(F) \in \mathcal{H}_{N,d}(\chi)$. (Strictly speaking this is verified on the dense set of Schwartz functions on $L^2(\mathbb{R}, dx)$ and extended to the whole space by completion.) Note that $\tilde{f}(x) = \pi_N(f)(x)$ gives the Schrödinger representation action of $[a, c, z]$ of $H(\mathbb{R})$ with central character $e^{2\pi i N z}$ on $L^2(\mathbb{R}, dx)$. This establishes the intertwining, and shows that the $H(\mathbb{R})$-action on $\mathcal{H}_{N,d}(\chi)$ is irreducible with central character $e^{2\pi i N z}$.

Note that every function $F(a, c, z) \in \mathcal{H}_N$ has a unique Fourier expansion
\[
F(a, c, z) = e^{2\pi i N z} \sum_{m \in \mathbb{Z}} h_m(c) e^{2\pi i m a}
\]
and that those $F \in \mathcal{H}_{N,d}(\chi)$ have Fourier coefficients $h_m(c)$ supported on those $m$ with $(m, N) = \frac{N}{d}$, as is evident from (5.8).

The next result shows that the twisted Weil-Brezin map $W_{N,d}(\chi)$ for $\chi$ a Dirichlet character (mod $d$) is just a rescaling of the twisted Weil-Brezin map $W_{d,d}(\chi)$.

**Lemma 5.4.** Given a Dirichlet character $\chi \pmod{d}$, for any $f \in L^2(\mathbb{R}, dx)$ there holds
\[
W_{N,d}(\chi)(f)(a, c, z) = W_{d,d}(\chi)(U(\frac{N}{d}))(f) \left( \frac{Na}{d}, dc, \frac{N}{d} \right).
\] (5.13)
in which $U(t) : L^2(\mathbb{R}, dx) \rightarrow L^2(\mathbb{R}, dx)$ is the unitary transformation
\[
(U(t)(f))(x) = |t|^{1/2} f(tx).
\] (5.14)

**Proof.** Let $f \in S(\mathbb{R})$. Starting from (5.8) only terms $n \equiv 0 \pmod{\frac{N}{d}}$ give nonzero contributions on the right side, and setting $n = \frac{N}{d} \tilde{\eta}$
\[
W_{N,d}(\chi)(f)(a, c, z) = \sqrt{C_{N,d}} e^{2\pi i N z} \left( \sum_{\tilde{l} \in \mathbb{Z}} \chi(\tilde{l}) f \left( \frac{N}{d} \tilde{l} + Nc \right) e^{2\pi i \frac{N\tilde{\eta}}{d}} \right) \\
= \left( \frac{C_{N,d}}{\frac{N}{d}} \right)^{1/2} e^{2\pi i N z} \left( \sum_{\tilde{l} \in \mathbb{Z}} \chi(\tilde{l})(U(\frac{N}{d}))(\tilde{l} + dc) e^{2\pi i \frac{N\tilde{\eta}}{d}} \right) \\
= \left( \frac{C_{N,d}}{C_{d,d} \cdot \frac{N}{d}} \right)^{1/2} W_{d,d}(\chi)(U(\frac{N}{d}))(f) \left( \frac{Na}{d}, dc, \frac{N}{d} \right).
\]
Since
\[
\left( \frac{C_{N,d}}{C_{d,d} \cdot \frac{N}{d}} \right)^{1/2} \left( \frac{N}{\phi(d)} \cdot \frac{N}{d} \right) = 1,
\]
the result follows. \qed
5.3. Multiplicative character decomposition.

**Theorem 5.5.** (Multiplicative Decomposition) For \( N \neq 0 \), the Hilbert space \( \mathcal{H}_N \) has an orthogonal direct sum decomposition

\[
\mathcal{H}_N = \bigoplus_{d|N} \left( \bigoplus_{\chi \in \left( \mathbb{Z}/d\mathbb{Z} \right)^*} \mathcal{H}_{N,d}(\chi) \right). \tag{5.15}
\]

Here \( \chi \) runs over all Dirichlet characters, primitive and imprimitive. Each \( \mathcal{H}_{N,d}(\chi) \) is invariant under the \( H(\mathbb{R}) \)-action on \( \mathcal{H}_N \) and is an irreducible representation of \( H(\mathbb{R}) \) with central character \( e^{2\pi i Nz} \).

**Proof.** To prove the theorem it suffices to show that the spaces \( \mathcal{H}_{N,d}(\chi) \) are mutually orthogonal. Lemma 5.3 showed that each \( \mathcal{H}_{N,d}(\chi) \) carries a representation of \( H(\mathbb{R}) \) with central character \( e^{2\pi i Nz} \) of multiplicity at least 1, while \( \mathcal{H}_N \) carries such a representation of multiplicity \( |N| \). Since we have \( |N| = \sum_{d|N} \phi(d) \) summands on the right side, we conclude that each multiplicity is 1. Since the representation on \( \mathcal{H}_N \) is completely reducible, the orthogonal complement \( \mathcal{H}^\perp \) of the right side of (5.15) carries a representation of \( \mathcal{H}_N \), with central character \( e^{2\pi i Nz} \). Its multiplicity must be 0, so \( \mathcal{H}^\perp = \{0\} \) and (5.15) follows.

To verify pairwise orthogonality of the \( \mathcal{H}_{N,d}(\chi) \), it suffices to check it on the dense subspace of functions

\[
\mathcal{S}_{N,d}(\chi) := \{ \mathcal{W}_{N,d}(\chi)(f) : f \in \mathcal{S}(\mathbb{R}) \}. \tag{5.16}
\]

Given \( f_1, f_2 \in \mathcal{S}(\mathbb{R}) \) we have

\[
( \mathcal{W}_{N,d_1}(\chi_1)(f_1), \mathcal{W}_{N,d_2}(\chi_2)(f_2) )_{\mathcal{H}_N} = \sqrt{C_{N,d_1}C_{N,d_2}} \int_0^1 \int_{n_1,n_2 \in \mathbb{Z}} \chi_1 \left( \frac{n_1d_1}{N} \right) \chi_2 \left( \frac{n_2d_2}{N} \right) f_1(n_1 + Nc) f_2(n_2 + Nc) e^{2\pi i (n-n_c)a} \, da \, dc
\]

\[
= \sqrt{C_{N,d_1}C_{N,d_2}} \int_0^1 \sum_{n_1 \in \mathbb{Z}} \chi_1 \left( \frac{n_1d_1}{N} \right) \chi_2 \left( \frac{n_1d_2}{N} \right) f_1(n_1 + Nc) f_2(n_1 + Nc) dc.
\]

If \( d_1 \neq d_2 \) then

\[
\chi_1 \left( \frac{nd_1}{N} \right) \chi_2 \left( \frac{nd_2}{N} \right) = 0, \quad \text{for all} \quad n \in \mathbb{Z}, \tag{5.17}
\]

i.e. they have disjoint support. Thus

\[
( \mathcal{W}_{N,d_1}(\chi_1)(f), \mathcal{W}_{N,d_2}(\chi_2)(g) ) = 0, \quad d_1 \neq d_2.
\]

If \( d_1 = d_2 \), set \( n = \frac{nd_1}{N} \) and \( \chi = \chi_1 \chi_2 \), to obtain

\[
( \mathcal{W}_{N,d_1}(\chi_1)(f_1), \mathcal{W}_{N,d_1}(\chi_2)(f_2) )_{\mathcal{H}_N} = \frac{N}{\phi(d_1)} \int_0^1 \sum_{n \in \mathbb{Z}} \chi(n) f_1 \left( \frac{Nn}{d_1} + Nc \right) f_2 \left( \frac{Nn}{d_1} + Nc \right) dc
\]

\[
= \frac{N}{\phi(d_1)} \left( \sum_{k=0}^{d_1-1} \chi(k) \right) f_1(Nc) f_2(Nc) dc
\]

\[
= 0, \tag{5.18}
\]

because \( \sum_{k=0}^{d_1-1} \chi(k) = 0 \) since \( \chi \) is a nontrivial Dirichlet character \( \mod d_1 \). \( \square \)
5.4. **Hecke operator action on** \( \mathcal{H}_{N,d}(\chi) \). We next consider the action of the Hecke operators on the spaces \( \mathcal{H}_{N,d}(\chi) \). For \( m \in \mathbb{Z}_{>0} \) and a Dirichlet character \( \chi \), define \( T_{N}^{\chi} : L^{2}(\mathbb{R}, dx) \to L^{2}(\mathbb{R}, dx) \) by

\[
T_{N}^{\chi}(f)(x) := \chi(m)f(mx) .
\]  

(5.19)

If \( \chi \) is a character \( (\mod d) \) and \( d|d' \) then we let \( \chi|_{d'} \) denote the (imprimitive) character \( (\mod d') \) which equals \( \chi(n) \) if \( (n, d') = 1 \) and 0 otherwise.

**Theorem 5.6.** (Hecke-invariant Hilbert Subspaces) *Let \( N \neq 0 \) and suppose that \( \chi \) is a Dirichlet character \( (\mod d) \) with \( d|N \). Given \( m \), and letting \( d' = (m, \frac{N}{d}) \), the two-variable Hecke operator \( T_{N}^{\chi} \) leaves invariant the Hilbert space

\[
\mathcal{H}_{N,d}(\chi; d') := \bigoplus_{e|d'} \mathcal{H}_{N,de}(\chi|_{de}) ,
\]

and for \( f \in \mathcal{H}_{N,d}(\chi) \),

\[
T_{N}^{\chi} \circ W_{N,d}(\chi) = \sum_{e|d'} \sqrt{\frac{\phi(de)}{\phi(d')}} \chi\left(\frac{m}{e}\right) W_{N,de}(\chi|_{de}) \circ T_{N}^{\chi} .
\]

(5.21)

In particular, if \( (m, \frac{N}{d}) = 1 \) then \( T_{N}^{\chi} \) leaves \( \mathcal{H}_{N,d}(\chi) \) invariant, and

\[
T_{N}^{\chi} \circ W_{N,d}(\chi) = W_{N,d}(\chi) \circ T_{N}^{\chi} = \chi(m) W_{N,d}(\chi) \circ T_{N}^{\chi} .
\]

(5.22)

**Proof.** Given \( W_{N,d}(\chi)(f) \in \mathcal{H}_{N} \) with \( f \in \mathcal{S}(\mathbb{R}) \), we have

\[
T_{N}^{\chi} \circ W_{N,d}(\chi)(f)(a, c, z) = \frac{1}{m} \sum_{k=1}^{m} W_{N,d}(\chi)(f)\left(\frac{a+k}{m}, mc, z\right)
\]

\[
= \sqrt{C_{N,de}} e^{2\pi i N z} \sum_{n \in \mathbb{Z}} \chi\left(\frac{nd}{N}\right) f(n + Nmc) \left(\frac{1}{m} e^{2\pi i \frac{mk}{m}} \sum_{k=0}^{m-1} e^{2\pi i \frac{mk}{m}}\right)
\]

\[
= \sqrt{C_{N,de}} e^{2\pi i N z} \sum_{n = lm + mc \atop n = lm} \chi\left(\frac{ldm'}{N}\right) f(lm + Nmc)e^{2\pi ila} .
\]

Writing \( m = d'm' \) with \( d' = (m, \frac{N}{d}) \), we obtain

\[
T_{N}^{\chi} \circ W_{N,d}(\chi)(f)(a, c, z) = \sqrt{C_{N,de}} e^{2\pi i N z} \sum_{l \in \mathbb{Z}} \chi\left(\frac{ldm'}{N}\right) T_{N}^{\chi}(f)(l + Nc)e^{2\pi ila} .
\]

(5.23)

We subdivide \( l \in \mathbb{Z} \) into arithmetic progressions \( (\mod N) \). Now \( \chi\left(\frac{ld'}{N}\right) = 0 \) unless \( \frac{N}{d'} \big| l \).

Suppose this holds, and define \( e \) by

\[
(l, N) = \frac{N}{de} ,
\]

with \( e|d' \). For fixed \( e \) there are \( \phi(de) \) such arithmetic progressions \( (\mod N) \). On such a progression, writing \( l = \frac{N}{de} \cdot l' \), one has \( (l', de) = 1 \), so that

\[
\chi(l \frac{dd'm'}{N}) = \chi(l' \frac{d'm'}{e}) = \chi\left(\frac{d'm'}{e}\right) \chi(l') = \chi\left(\frac{d'm'}{e}\right) \chi\left|_{de}\right( l' \right) .
\]

(5.24)
Now (5.23) yields for $f \in S(\mathbb{R})$ that
\[
T_m \circ W_{N,d}(\chi)(f)(a,c,z) = \sqrt{C_{N,d}e^{2\pi i N z}} \sum_{d | d'} \chi \left( \frac{d'}{e} \right) \left( \sum_{\substack{l,N \mid (l') \equiv \frac{N}{(l')}, \ N \equiv \frac{N}{(l')}}} \chi \left( \frac{d'}{e} \right) \chi(l') T_{m'}^\chi(f)(l + N c)e^{2\pi ila} \right).
\]
This relation holds for all $f \in H_{N,d}(\chi)$ by Hilbert space completion, and (5.20) follows since
\[
\frac{C_{N,d}}{C_{N,de}} = \varphi(\frac{d}{e}) \frac{\varphi(d)}{\varphi(d)}.
\]

The relation (5.20) shows that $T_m(H_{N,d}(\chi)) \subseteq \bigoplus_{e | d'} H_{N,de}(\chi \mid d_e)$.

(5.25)

Applying this to all $H_{N,de}(\chi \mid d_e)$ on the left establishes that $\bigoplus_{e | N/d} H_{N,de}(\chi \mid d_e)$ is an invariant subspace for $T_m$ in $H_N$.

5.5. **Coarse multiplicative character decomposition.** Theorem 5.6 yields a coarser orthogonal decomposition of $H_N$, labeled by primitive Dirichlet characters $\chi \pmod{f}$ with $f \mid N$, whose summands are left invariant by all two-variable Hecke operators. We call the resulting decomposition (5.28) below the **coarse multiplicative decomposition** of $H_N$.

**Theorem 5.7.** (Coarse Multiplicative Decomposition) Let $N \neq 0$ and to each primitive character $\chi \pmod{f}$ with $f | N$ assign the Hilbert space
\[
H_N(\chi; f) := \bigoplus_{d \mid f} H_{N,d}(\chi \mid d).
\]
(5.27)

Then the Hilbert space $H_N$ has the orthogonal direct sum decomposition
\[
H_N = \bigoplus_{\chi, f} H_N(\chi; f),
\]
(5.28)
in which $\chi$ runs over all primitive characters $\pmod{f}$ for all $f | N$. Each Hilbert space $H_N(\chi; f)$ is invariant under all two-variable Hecke operators $\{T_m : m \in \mathbb{Z} \setminus \{0\}\}$.

**Proof.** The orthogonality of the decomposition follows from Theorem 5.5. The invariance under all the two-variable Hecke operators $T_m$ follows from the Hecke operator action given in Theorem 5.6.

The formulas in Theorem 5.6 indicate that the mutually commuting action of the two-variable Hecke operators on $H_N(\chi; f)$ is not semisimple, except in the case that $f = N$. Their action is simultaneously triangularizable, but not always diagonalizable.
6. Additive character decomposition of $\mathcal{H}_N$

In this section, for comparative purposes, we describe another orthogonal decomposition of $\mathcal{H}_N$ for $N \neq 0$ into irreducible $H(\mathbb{R})$-submodules, due to Brezin [12]. This decomposition, which we term the *additive character decomposition*, was generalized in Auslander and Brezin [3] to other nilpotent Lie groups, including the $(2n+1)$-dimensional Heisenberg group $H_n(\mathbb{R})$. In order to state results in parallel with the multiplicative decomposition given in Section 5, our notation and formulation differs slightly from that in Auslander and Brezin [14] and Auslander [3]. We then determine the action of the two-variable Hecke operators on the additive character decomposition modules.

### 6.1. Distinguished subgroups and additive Brezin maps.

These decompositions are associated to additive characters of certain “distinguished subgroups” of a given discrete subgroup of $H_n(\mathbb{R})$. We define for $N \geq 1$ the discrete subgroups

$$\Gamma^U \left( \frac{1}{N} \right) := \begin{bmatrix} 1 & \frac{1}{N} \mathbb{Z} & \frac{1}{N} \mathbb{Z} \\ 0 & 1 & \mathbb{Z} \\ 0 & 0 & 1 \end{bmatrix}$$

(6.1)

of $H(\mathbb{R})$. Note that $H(\mathbb{Z}) = \Gamma^U(1) \subseteq \Gamma^U \left( \frac{1}{N} \right)$ and $\Gamma^U(1)$ is a normal subgroup of $\Gamma^U \left( \frac{1}{N} \right)$.

**Definition 6.1.** For each additive character $\psi_k$ on $\mathbb{Z}/N\mathbb{Z}$, with

$$\psi_k(m) := e^{2\pi i \frac{km}{N}}, \quad 0 \leq k \leq N - 1,$$

(6.2)

we define the *additive Brezin map* $W_N(\psi_k) : L^2(\mathbb{R}, dx) \to \mathcal{H}_N$ on Schwartz function $f \in S(\mathbb{R})$ by

$$W_N(\psi_k)(f)(a, c, z) := e^{2\pi i N z} \sum_{n \in \mathbb{Z}} e^{\frac{2\pi i kn}{N}} f(n + Nc) e^{2\pi i na}.$$  

(6.3)

**Lemma 6.2.** (1) The additive Brezin map $W_N(\psi) : S(\mathbb{R}) \to C_\infty(\mathcal{H}_N)$ extends to a Hilbert space isometry

$$W_N^d(\psi_k) : L^2(\mathbb{R}, dx) \to \mathcal{H}_N$$

whose image $\mathcal{H}_N(\psi_k)$ is a closed subspace of $\mathcal{H}_N$.

(2) The space $\mathcal{H}_N(\psi_k)$ consists of those elements $F \in \mathcal{H}_N$ that satisfy for $0 \leq j \leq N - 1$ the relations

$$F \left( a + \frac{j}{N}, c, z \right) = \psi_k(j) F(a, c, z),$$  

(6.4)

almost everywhere.

**Proof.** (1) As in Lemma 5.3 one computes that for all $f, g \in S(\mathbb{R})$,

$$(f, g)_{L^2(\mathbb{R}, dx)} = \langle W_N(\psi_k)(f), W_N(\psi_k)(g) \rangle_{\mathcal{H}_N}.$$  

The map therefore extends under Hilbert space completion to an isometry.

(2) If $F = W_N(\psi)(f) \in \mathcal{H}_N(\psi_k)$ then the relation (6.4) hold by inspection of terms in the formula (6.3). It can be checked that, conversely, the functional equations (6.4) for $0 \leq j \leq N - 1$ suffice to determine the Fourier coefficients on the right side of (6.3).
The group $\Gamma_U(\frac{1}{N})$ is associated to these Weil-Brezin maps in that the transformation (6.4) involves a translation in the group $\Gamma_U(\frac{1}{N})$.

The next result shows that the additive Weil-Brezin maps $W_N(\psi_k)$ arise from the original Weil-Brezin map by a rescaling of variables.

**Lemma 6.3.** For $f \in L^2(\mathbb{R}, dx)$ and the additive character $\psi_k(n) = e^{\frac{2\pi i k n}{N}}$ on $\mathbb{Z}/NZ$, the additive Brezin map is given by

$$W_N(\psi_k)(f)(a, c, z) = W(f)(\frac{a + k}{N}, Nc, Nz)$$

(6.5)

where $W(f) = W_1(\psi_0)(f) = W_{1,1}(\chi_0)(f)$ is the Weil-Brezin map.

**Proof.** This formula holds for Schwartz functions by (6.3) and carries over to all $f \in L^2(\mathbb{R}, dx)$ under Hilbert space completion. Note that the right side of (6.5) is $\Phi_k^N \circ W(f)$ where

$$\Phi_k^N(F)(a, c, z) := F(\frac{a + k}{N}, Nc, Nz).$$

(6.6)

In particular (6.5) shows that $\Phi_k^N : H_1 \rightarrow H_N(\psi_k)$ is a Hilbert space isometry mapping $H_1$ onto $H_N(\psi_k)$.

6.2. **Additive Character Decomposition.** The following result is a special case of a result of Auslander and Brezin [4].

**Theorem 6.4.** (Auslander-Brezin) For $N \neq 0$ the Hilbert space $H_N$ has an orthogonal direct sum decomposition

$$H_N = \bigoplus_{k=0}^{N-1} H_N(\psi_k).$$

(6.7)

Each $H_N(\psi_k)$ is invariant under the $H(\mathbb{R})$-action on $H_N$ and is an irreducible representation of $H(\mathbb{R})$ with central character $e^{2\pi i N z}$.

**Proof.** This is Theorem 2 (iii) of Auslander and Brezin [4]. It can also be proved by similar orthogonality calculations to those in Theorem 5.5. □

The Hecke operators $T_m$ act on the additive character decomposition (6.2) of $H_N$ by mapping each subspace $H(\psi)$ into another subspace, and when $(m, N) = 1$ this action is a permutation.

**Theorem 6.5.** For each $N \neq 0$ and each $m \geq 1$ the Hecke operator $T_m : H_N \rightarrow H_N$ restricts to a map

$$T_m : H_N(\psi_k) \rightarrow H_N(\psi_{km})$$

(6.8)

for $0 \leq k \leq N - 1$. In particular $H_N(\psi_k)$ is invariant under $T_m$ if and only if $k = 0$ or $k \equiv 0 \pmod{N}$ and $m \equiv 1 \pmod{N/(k, N)}$. If $(m, N) = 1$ the action of $T_m$ permutes the $H_N(\chi_k)$. 

Proof. Let $\mathcal{W}_N(\psi_k)(f) \in \mathcal{H}_N(\psi_k)$, and let $\tilde{T}_m : L^2(\mathbb{R}, dx) \rightarrow L^2(\mathbb{R}, dx)$ by
\[
\tilde{T}_m(f)(x) := f(mx).
\] (6.9)

Then
\[
T_m \circ \mathcal{W}_N(\psi_k)(f)(a,c) = \frac{1}{m} \sum_{j=0}^{m-1} \mathcal{W}_N(\psi_k)(f)\left(\frac{a+j}{m},mc,z\right)
\]
\[
= \frac{1}{m} \sum_{j=0}^{m-1} \sum_{n \in \mathbb{Z}} e^{\frac{2\pi i k n}{N}} f(n+Nm) e^{2\pi i \left(\frac{2\pi j}{m}\right)}
\]
\[
= \sum_{n \in \mathbb{Z}} e^{\frac{2\pi k n}{N}} f(n+Nm) e^{2\pi i \left(\frac{m-1}{m}\sum_{j=0}^{m-1} e^{\frac{2\pi i j}{m}}\right)}
\]
\[
= \sum_{n \in \mathbb{Z}} e^{\frac{2\pi k n}{N}} f(m(l+Mc)) e^{2\pi i la}
\]
\[
= \mathcal{W}_N(\psi_{km}) \circ \tilde{T}_m(f).
\] (6.10)

Now (6.10) shows that $T_m \circ \mathcal{W}(\psi)(f) \in \mathcal{H}_N(\psi_{km})$, as asserted.

Next $\psi_{km} \equiv \psi_k$ if and only if either $k \equiv 0 \pmod{N}$ or $k \not\equiv 0 \pmod{N}$ and $m \equiv 1 \pmod{N/(k,N)}$. If $(m,N) = 1$ the map on additive characters $\psi_k$ to $\psi_{km}$ is bijective. \qed

Remarks. (1) There are similar orthogonal direct sum decompositions associated to additive characters of other discrete subgroups. For example, there is an orthogonal direct sum decomposition of $\mathcal{H}_N$ associated to the characters of
\[
\Gamma_L \left(\frac{1}{N}\right) := \left[\begin{array}{ccc}1 & \frac{1}{N}Z & \frac{1}{N}Z \\ 0 & 1 & \frac{1}{N}Z \\ 0 & 0 & 1 \end{array}\right].
\] (6.11)

that leave the normal subgroup $\Gamma_L(1)$ fixed.

(2) For $N = M^2$ there is also an orthogonal direct sum decomposition
\[
\mathcal{H}_N = \bigoplus_{\psi_1,\psi_2 \in (\mathbb{Z}/M\mathbb{Z})} \mathcal{H}(\psi_1, \psi_2)
\] (6.12)

associated to the discrete subgroup
\[
\Gamma_L^U \left(\frac{1}{N^2}\right) := \left[\begin{array}{ccc}1 & \frac{1}{N}Z & \frac{1}{N}Z \\ 0 & 1 & \frac{1}{N}Z \\ 0 & 0 & 1 \end{array}\right].
\] (6.13)

of $H(\mathbb{R})$.

(3) This additive character decomposition is associated to the finite Fourier transform is studied by Auslander and Tolimieri [3].

7. Dilation Action on $\mathcal{H}_N$ and the Sub-Jacobi Group

The group $GL(1,\mathbb{R}) := \mathbb{R}^*$ has a unitary action on $L^2(\mathbb{R}, dx)$ by dilations
\[
U(t)(f)(x) := |t|^{1/2} f(tx), \ t \neq 0.
\] (7.14)
Using the Weil-Brezin maps $W_{N,d}(\chi)$, we show this action carries over to all Heisenberg modules $\mathcal{H}_N$, for $N \neq 0$, and that it behaves nicely with respect to all the two-variable Hecke operators.

7.1. Dilations and Two-Variable Hecke Operators. The dilation operators give an action of $\mathbb{R}^*$ on each Heisenberg module $\mathcal{H}_{N,d}(\chi)$ using the intertwining map $W_{N,d}(\chi)$. Set $V(t) : \mathcal{H}_{N,d}(\chi) \rightarrow \mathcal{H}_{N,d}(\chi)$ by

$$V(t) := W_{N,d}(\chi) \circ U(t) \circ W_{N,d}(\chi)^{-1} \text{ for } t \neq 0,$$

(7.1)

Writing $F = W_{N,d}(\chi)(f) \in \mathcal{H}_{N,d}(\chi)$ as (5.8), we have

$$V(t)(F)(a,c,z) = |t|^{1/2} \sqrt{C_{N,d}} e^{2\pi i N z} \sum_{n \in \mathbb{Z}} \chi \left( \frac{nd}{N} \right) f(t(n + Nc)) e^{2\pi i a}.$$  

(7.2)

Now $V(t)$ is a unitary operator on $W_{N,d}(\chi)$, and taking direct sums defines the (diagonal) dilation action $V(t) : \mathcal{H}_N \rightarrow \mathcal{H}_N$, for all $N \neq 0$.

**Theorem 7.1.** For each $N \neq 0$, the $\mathbb{R}^*$-action $\{V(t) : t \in \mathbb{R}^*\}$ on $\mathcal{H}_N$ commutes with all two-variable Hecke operators. That is, for each $m \neq 0$ and $t \in \mathbb{R}^*$,

$$V(t) \circ T_m = T_m \circ V(t).$$

(7.3)

**Proof.** This is a consequence of Theorem 5.6. It suffices to check it on each $\mathcal{H}_{N,d}(\chi)$ separately. For $m \geq 1$, on setting $d' = \gcd(m, \frac{N}{d})$ it yields

$$T_m \circ W_{N,d}(\chi)(f) = \frac{1}{\sqrt{|m|}} \sum_{e \mid d'} \chi \left( \frac{m}{e} \right) \sqrt{\phi(d)\phi(e)} W_{N,de}(\chi|de)(U(m)(f)).$$

(7.4)

Then

$$V(t) \circ T_m \left( W_{N,d}(\chi)(f) \right) = \frac{1}{\sqrt{|m|}} \sum_{e \mid d'} \chi \left( \frac{m}{e} \right) \sqrt{\phi(d)\phi(e)} V(t) \circ W_{N,de}(\chi|de)(U(m)(f))$$

$$= \frac{1}{\sqrt{|m|}} \sum_{e \mid d'} \chi \left( \frac{m}{e} \right) W_{N,de}(\chi|de) \sqrt{\phi(d)\phi(e)} U(t) \circ U(m)(f)$$

$$= \frac{1}{\sqrt{|m|}} \sum_{e \mid d'} \chi \left( \frac{m}{e} \right) \left( W_{N,de}(\chi|de) \circ U(m) \right)(U(t)(f))$$

$$= T_m \circ W_{N,d}(\chi)(U(t)(f))$$

$$= T_m \circ V(t) \left( W_{N,d}(\chi)(f) \right),$$

as asserted.

Finally $T_{-m} = R^2 \circ T_m$ and $R^2 = V(-1)$ on $\mathcal{H}_N(\chi)$ so the result follows for negative $m$. \hfill \box

Note that (7.4) gives for $(m, N) = 1$ that on $\mathcal{H}_{N,d}(\chi)$ there holds $T_m = |m|^{-1/2} \chi(m)V(m)$. 

7.2. Sub-Jacobi Group Action on $\mathcal{H}_N$. The $\mathbb{R}^*$-action on each $\mathcal{H}_{N,d}(\chi)$ ($N \neq 0$) combines with the Heisenberg $H(\mathbb{R})$-action to give an irreducible unitary representation of a certain four-dimensional real Lie group $H^J$, defined below, which we call the sub-Jacobi group.

Let $N \neq 0$ and the Dirichlet character $\chi \pmod{d}$ be given. The $H(\mathbb{R})$-action on $F(a,c,z) \in \mathcal{H}_{N,d}(\chi)$ is

$$\rho([a',c',z'])(F)(a,c,z) := F((a,c,z) \circ (a',c',z')) = F(a + a', c + c', z + z' + ca'). \quad (7.5)$$

**Definition 7.2.** We define for $t \in \mathbb{R}$ and $[a',c',z'] \in H(\mathbb{R})$ the (unitary) operators $\rho_J([t,a',c',z'])$ acting on $\mathcal{H}_{N,d}(\chi)$ by

$$\rho_J([t,a',c',z'])(F)(a,c,z) := V(t) \circ \rho([a',c',z'])(F)(a,c,z).$$

Here $V(t)$ acts on $F = W_{N,d}(\chi)(f)$ for suitable $f \in L^2(\mathbb{R}, dx)$ via (7.2).

We obtain the following result.

**Theorem 7.3.** (Sub-Jacobi group action)

For $N \neq 0$, each positive divisor $d|N$ and each Dirichlet character $\chi \pmod{d}$ the Hilbert space $\mathcal{H}_{N,d}(\chi)$ carries an irreducible unitary representation of a four-dimensional solvable real Lie group $H^J$ with central character $e^{2\pi i Nz}$. Two such representations are unitarily equivalent $H^J$-modules if and only if they have the same value of $N$.

**Proof.** To show that the set of all operators $\rho_J([t,a',c',z'])$ forms a (unitary) representation on $\mathcal{H}_{N,d}(\chi)$ of a four-dimensional real Lie group, we pull the action back to $L^2(\mathbb{R}, dx)$ using the inverse modified Weil-Brezin map $W_{N,d}(\chi)^{-1}$. The $H(\mathbb{R})$-representation pulls back as follows. If $f(x) = W_{N,d}(\chi)^{-1}(F)$ then we have

$$\tilde{\rho}_N([a',c',z'])(f)(x) = e^{2\pi i Nz'} f(x + c') e^{2\pi i Na'}, \quad (7.6)$$

which depends only on $N \neq 0$ and is independent of $d$ and the character $\chi$. Note here that for $N = 0$ the formula (7.6) makes sense and defines an $H(\mathbb{R})$-action with trivial central character viewed with image in $L^2(\mathbb{R}, dx)$, but there is no corresponding Weil-Brezin map.

The $\mathbb{R}^*$-action $V(t)$ pulls back to $U(t)$, and we set

$$\tilde{\rho}_N([t,a,c,z])(f)(x) := U(t) \circ \tilde{\rho}_N(a,c,z)(f)(x) \quad (7.7)$$

Using (7.6) we compute

$$U(t) \circ \tilde{\rho}_N([a',c',z'])(f)(x) = e^{2\pi i Nz'} f(tx + c') e^{2\pi i Na'}, \quad (7.8)$$

$$\tilde{\rho}_N([a',c',z'] \circ U(t))(f)(x) = e^{2\pi i Nz'} f(tx + c') e^{2\pi i Na'}. \quad (7.9)$$

These formulas combine to give

$$U(t) \circ \tilde{\rho}_N([a,c,z]) \equiv \tilde{\rho}_N([a, \frac{1}{t} c, z]) \circ U(t), \quad (7.10)$$
whose functional form is independent of $N \in \mathbb{Z}$. Thus we have a representation of a four-dimensional real Lie group $H^J$ whose general element is

$$[t, a, c, z] \in \mathbb{R}^* \times \mathbb{R}^3,$$

having multiplication law

$$[t, a, c, z] \circ [t', a', c', z'] = [tt', \frac{1}{t}a + a', t'c + c', z + z' + t'ca']. \quad (7.11)$$

The group $H^J$ has a faithful $4 \times 4$ matrix representation

$$\rho([t, a, c, z]) = \begin{pmatrix}
1 & c & a & z \\
0 & t & 0 & ta \\
0 & 0 & \frac{1}{t} & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}. \quad (7.12)$$

It is a semi-direct product $\mathbb{R}^* \ltimes H(\mathbb{R})$, and is a solvable Lie group.

The action $\tilde{\rho}_N([t, a, c, z])$ in (7.11) defines a unitary representation of $H^J$ on $L^2(\mathbb{R}, dx)$, where unitarity follows from (7.10). For each $N \neq 0$ this $H^J$-representation is irreducible because it is already irreducible as an $H(\mathbb{R})$-representation. This property then holds for the representation $\rho_J$ acting on $H_N(d\chi)$ using the intertwining map $W_{N,d}(\chi)$.

Two such representations viewed as $H(\mathbb{R})$-representations are unitarily equivalent if and only if they have the same value of $N$. Viewed as $H^J$-representations, they are unitarily equivalent if they have the same value of $N$, by inspection of the action (7.8) on $L^2(\mathbb{R}, dx)$.

We term the group $H^J$ the sub-Jacobi group because it can be identified as a subgroup of the Jacobi group $\text{Aut}(H(\mathbb{R})) \ltimes H(\mathbb{R})$, see Appendix A. This group $H^J$ has been called the extended $(1+1)$-dimensional Poincaré group in the physics literature, see de Mello and Rivelles [43].

The action of $H^J$ on each $H_N(d\chi)$ extends to an action of $H^J$ on $H_N$ for each $N \neq 0$, by taking the direct sum. It is an interesting question whether this action has a natural definition directly on $H_N$ without invoking the Weil-Brezin maps. A crucial feature of this action is that it commutes with the two-variable Hecke operators as given in Theorem 7.1.

### 7.3. Compatiblity of Additive and Multiplicative character $\mathbb{R}^*$-actions

One can define in a similar fashion an (possibly different) $H^J$-action on $H_N$ for each $N \neq 0$ by using the additive character decomposition of $H_N$ given in Section 6. That is, one pushes forward the action (7.8) to each $H_N(\psi)$ by the appropriate Weil-Brezin map $W_N(\psi)$, and then takes a direct sum. We show that this action $\tilde{V}(t)$ coincides with the $H^J$-action above.

**Theorem 7.4.** For each $N \neq 0$ the $\mathbb{R}^*$-action $V(t)$ on $H_N$ induced from the multiplicative character decomposition coincides with the $\mathbb{R}^*$-action $\tilde{V}(t)$ induced from the additive character decomposition of $H_N$. 
**Proof.** The action \( \tilde{V}(t) \) for a function \( F = \mathcal{H}_N(\psi)(f) \in \mathcal{W}_N(\psi) \) is given by

\[
\tilde{V}(t)(F)(a,c,z) = |t|^{1/2} e^{2\pi i N z} \sum_{n \in \mathbb{Z}} \psi(n) f(t(n + Nc)) e^{2\pi ina}.
\]

By linearity it suffices to check the equivalence for any function \( G = \mathcal{W}_{N,d}(\chi)(g) \in \mathcal{H}_{N,d}(\chi) \). We use the fact that the function \( h(n) = \chi(\frac{nd}{N}) \) is periodic with period \( N \) and therefore can be expressed as a linear combination \( h(n) = \sum_{k=0}^{N-1} a(k) \psi_k(n) \), for certain (complex) coefficients \( a(k) \). It follows that for all \( g(x) \in L^2(\mathbb{R}, dx) \) there holds

\[
\mathcal{W}_{N,d}(\chi)(g) = \sqrt{C_{N,d}} \sum_{k=0}^{N-1} a(k) \mathcal{W}_N(\psi_k)(g).
\]

We now obtain

\[
V(t)(G)(a,c,z) = |t|^{1/2} \sqrt{C_{N,d}} e^{2\pi i N z} \sum_{n \in \mathbb{Z}} \chi(\frac{nd}{N}) g(t(n + Nc)) e^{2\pi ina}
\]

\[
= |t|^{1/2} \sqrt{C_{N,d}} e^{2\pi i N z} \sum_{k=0}^{N-1} a(k) \sum_{n \in \mathbb{Z}} \psi_k(n) g(t(n + Nc)) e^{2\pi ina}
\]

\[
= \sqrt{C_{N,d}} \sum_{k=0}^{N-1} a(k) \tilde{V}(t)(\mathcal{W}_N(\psi_k)(g))
\]

\[
= \tilde{V}(t)(\mathcal{W}_{N,d}(\chi)(g)) (a,c,z) = \tilde{V}(t)(G)(a,c,z),
\]

(7.1)

as asserted. \( \square \)

### 7.4. Lerch L-Functions as Mellin Transforms

The two functions \( L^\pm(s,a,c) \) studied in [35] and given in (1.2) can be interpreted as arising from a multiplicative Fourier transform (Mellin transform) associated to the \( \mathbb{R}^* \)-action \( \{ V(t) \mid t \in \mathbb{R}^* \} \). These two functions can be identified with the value \( z = 0 \) of Lerch \( L \)-functions \( L^\pm_{1,1}(\chi_0, s, a, c, z) \) with character \( \chi_0 \) on the Heisenberg group defined in Section 9.1 following, in which \( \chi_0 \) is the principal character (mod 1).

Recall that the two-sided Mellin transforms \( \mathcal{M}_k(f) \) for \( k = 0, 1 \) were defined by

\[
\mathcal{M}_k(f)(s) := \int_{-\infty}^{\infty} f(x)(\text{sgn}(x))^k |x|^s \frac{dx}{|x|}.
\]

(7.2)

The one-sided Mellin transform is

\[
\mathcal{M}(f)(s) := \int_0^{\infty} f(x)x^s \frac{dx}{x},
\]

(7.3)

which satisfies, formally,

\[
\mathcal{M}(f)(s) = \frac{1}{2}(\mathcal{M}_0(f)(s) + \mathcal{M}_1(f)(s)).
\]

The function \( f(x) \) must have some growth restrictions as \( x \to 0^+ \) and \( x \to \infty \) in order for these integrals to converge for some \( s \in \mathbb{C} \). The multiplicative averaging operator \( A_{a,c}[f](t) \) introduced in [35] is given, for \( f(x) \in \mathcal{S}(\mathbb{R}) \) and \( t \in \mathbb{R}^* \), by

\[
A_{a,c}[f](t) = \sum_{n \in \mathbb{Z}} f((n + c)t) e^{2\pi ina} = \mathcal{W}(U(t)(f))(a,c,0) = [V(t) \circ \mathcal{W}(f)](a,c,0),
\]

(7.4)

where \( \mathcal{W} = \mathcal{W}_{1,1}(\chi_0) \) is the Weil-Brezin map. We obtain a Mellin transform
Proposition 7.5. (Mellin Integral Representation) For any test function $f(x) \in S(\mathbb{R})$, on the half-plane $\Re(s) > 1$ there holds

$$
\frac{1}{2} \mathcal{M}_k(f)(s) L_{1,1}^\pm(x_0, s, a, c) = \int_0^\infty \left[ \mathcal{W}_{1,1}(U(t)(f))(a, c, 0) + (-1)^k \mathcal{W}_{1,1}(U(t)(f)(-a, -c, 0) \right] t^{s-1} dt. \tag{7.5}
$$

Remark. The formula (7.5) exhibits the $\mathbb{R}^*$-action $U(t)$ inside the Mellin transform. The combination of two terms on the right side of (7.5) was needed to get a Mellin transform having a nonempty half-plane of absolute convergence. This combination of terms creates a function invariant under the reflection automorphism $a^2(a, c, z) = (-a, -c, z)$ of $H(\mathbb{R})$.

Alternatively, since $V(t)(\mathcal{W}_{1,1}(f)) = \mathcal{W}_{1,1}(U(t)f)$ the right side of (7.5) equals

$$
\int_0^\infty \left[ V(t)\mathcal{W}_{1,1}(f)(a, c, 0) + (-1)^k V(t)\mathcal{W}_{1,1}(f)(-a, -c, 0) \right] t^{s-1} dt.
$$

Proof. In [35, Lemma 2.1] it was shown that for $k = 0, 1$ the function given by the operator

$$
B_k^{a,c}[f](t) := A^{a,c}[f](t) + (-1)^k e^{-2\pi i a} A^{1-a,1-c}[f](t), \tag{7.6}
$$

with $0 < a, c < 1$ applied to a Schwartz function $f(s)$, has one-sided Mellin transforms $\mathcal{M}(B_k^{a,c}[f])(s)$ which are absolutely convergent in the half-plane $\Re(s) > 1$, with

$$
\mathcal{M}(B_k^{a,c}[f])(s) = \frac{1}{2} \mathcal{M}_k(f)(s) L_{1,1}^\pm(x_0, s, a, c) \text{ with } (-1)^k = \pm. \tag{7.7}
$$

The formula (7.5) is derived using (7.6) together with the identity

$$
e^{-2\pi i a} \mathcal{W}(U(t)(f))(1-a, 1-c, 0) = \mathcal{W}(U(t)(f))(a, c, 0).
$$

Given a primitive Dirichlet character $\chi \pmod{N}$, one may derive formulas similar to (7.5) which replace the Weil-Brezin map with an appropriate modified Weil-Brezin map $\mathcal{W}_{N,d}(\chi)$. In Section 9.1 we extend the definition of Lerch $L$-functions to all $\mathcal{W}_{N,d}(\chi)$ and to all (primitive or imprimitive) Dirichlet characters $\chi$, specifying $L_{N,d}(\chi, s, a, c, z)$. In Section 9.4 we show that these $L$-functions satisfy suitable functional equations. The integral formulas above specialize to the value $z = 0$.

8. R-Operator Action and Additive Fourier Transform

In this section we determine the action of the Heisenberg-Fourier operator $R(F)(a, c, z) = F(-c, a, z - ac)$ on the spaces $\mathcal{H}_{N,d}(\chi)$, where $d$ divides $N$. Recall from Section 3.2 that on the invariant subspace $\mathcal{H}_{N}$ we denote this operator by $R_N$, and it is given by

$$
R_N(F)(a, c, z) = e^{-2\pi i cN} F(-c, a, z), \quad F \in \mathcal{H}_{N}.
$$

A. Weil [59] observed in 1964 that $R_1(F)(a, c, z)$ intertwines with the additive Fourier transform $\mathcal{F}$ under the Weil-Brezin map on $\mathcal{H}_1 = \mathcal{H}_{1,1}(\chi_0)$. That is,

$$
R_1(\mathcal{W}_{1,1}(\chi_0)(f)) = \mathcal{W}_{1,1}(\chi_0)(\mathcal{F}(f)), \tag{8.1}
$$
where the (normalized) Fourier transform is given by
\[
\mathcal{F}(f)(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{2\pi i xy} dx.
\] (8.2)

We show that for general \( N \neq 0 \) the action of \( R_N(F)(a, c, z) \) has a more complicated intertwining with the additive Fourier transform. In particular it is necessary to distinguish between primitive and imprimitive Dirichlet characters, and the dilation operator \( U(N) \) is needed to describe the action.

8.1. Gauss sums for imprimitive characters. We need to make use of known formulae for Gauss sums of imprimitive characters. Let \( \chi \) be a primitive Dirichlet character \((\text{mod } f)\) and if \( f \mid d \) let \( \chi|_d \) be the (imprimitive) Dirichlet character \((\text{mod } d)\) defined by \( \chi|_d(m) = \chi(m) \) if \((m, d) = 1\), and 0 otherwise. For any integer \( m \) the Gauss sum \( G(m, \chi|_d) \) is given by
\[
G(m, \chi|_d) := \sum_{k \pmod{d}} \chi|_d(k) e^{2\pi i km/d}.
\]
The standard Gauss sum \( \tau(\chi) \) of a primitive character \( \chi \) is
\[
\tau(\chi) := G(1, \chi),
\]
and it satisfies \(|\tau(\chi)|^2 = f\).

**Proposition 8.1.** Let \( \chi \) be a primitive Dirichlet character \((\text{mod } f)\) and suppose \( f \nmid d \). For any integer \( m \) set \( m' = m(m, d) \) and \( d' = d(m, d) \). Then
\[
(i) \text{ If } f \nmid d', \text{ then } G(m, \chi|_d) = 0.
\] (8.3)
\[
(ii) \text{ If } f \mid d', \text{ then } G(m, \chi|_d) = \left( \frac{\phi(d')}{\phi(d)} \frac{d'}{f} \mu \left( \frac{d'}{f} \right) \chi \left( \frac{d'}{f} \right) \bar{\chi}(m') \right) \tau(\chi).
\] (8.4)

**Proof.** This is derived in Hasse \[26\] pp. 444–450 and in Joris \[31\] Theorem A. A generalization appears in Nemchenok \[48\]. \(\square\)

More general formulas for Gauss sums on finite (commutative or noncommutative) rings were derived by Lamprecht \[39\], \[40\]. We note that in all cases the modulus squared of such a Gauss sum is an integer.

For primitive or imprimitive characters, if \((m, d) = 1\) then one has \( G(m, \chi|_d) = \bar{\chi}|_d(m) G(1, \chi) \). However only for primitive characters is it true that \( G(m, \chi) = 0 \) otherwise.

8.2. Fourier transform intertwining with R-operator under Weil-Brezin maps. The nonvanishing of \( G(m, \chi|_d) \) for some \((m, d) > 1\) for imprimitive characters leads to complications in the formulas for the R-operator action given in the following result.

**Theorem 8.2.** Let \( N \neq 0 \), and let \( d \) be a positive divisor of \(|N|\). Suppose that \( \chi \) is a primitive Dirichlet character \((\text{mod } f)\) with \( f \mid d \). Then for \( f(x) \in L^2(\mathbb{R}, dx) \), there holds
\[
R_N(\mathcal{W}_N,d(\chi|_d)(f)) = \chi(-1) \frac{\tau(\chi)}{|N|^2} \sum_{d \mid |N|} C_{N,d}(\bar{d}, \chi) \mathcal{W}_N,d(\bar{\chi}|_d)(\mathcal{F} \circ U(N)(f)),
\] (8.5)
for certain coefficients $C_{N,d}(\tilde{d}, \chi)$. These coefficients are given in terms of $d' := \frac{d}{(|N|/d, d)}$ as follows. If $\not| d'$ then

$$C_{N,d}(\tilde{d}, \chi) := \sqrt{\frac{\phi(d)}{\phi(d')}} \left( \frac{\phi(d)}{\phi(d')} \mu\left( \frac{d'}{d} \right) \chi\left( \frac{d'}{d} \right) \chi\left( \frac{|N|d'}{|Nd|} \right) \right),$$  \hspace{1cm} (8.6)

and if $\not| \not| d'$ then $C_{N,d}(\tilde{d}, \chi) = 0$.

**Remark.** The proof shows that $\not| \not| \tilde{d}$ implies $\not| \not| d'$, so that the coefficients $C_{N,d}(\tilde{d}, \chi) = 0$ whenever $\not| \not| \tilde{d}$. However it can happen that $C_{N,d}(\tilde{d}, \chi) \neq 0$ for some $\tilde{d}$ a strict divisor of $d$, e.g. when $N = d = 3$ and $\tilde{d} = 1$, where $d' = 1$.

**Proof.** Set

$$F(a, c, z) = \mathcal{W}_{N,d}(\chi|d)(f)(a, c, z) = \sqrt{C_{N,d}} e^{2\pi i Nz} \sum_{N \in \mathbb{Z}} \chi|d(\frac{nd}{N}) f\left( n + Nc e^{2\pi i Na} \right),$$

in which $C_{N,d} = \frac{N}{\varphi(d)}$. Then, taking $\tilde{n} = \frac{nd}{N}$, we have

$$R(F)(a, c, z) = \sqrt{C_{N,d}} e^{2\pi i Nz} \sum_{n \in \mathbb{Z}} \chi|d(\frac{nd}{N}) f\left( n + Nd \right) e^{-2\pi i \frac{Na}{d}} e^{-2\pi i \frac{Na}{d}}. \hspace{1cm} (8.7)$$

We compute the partial Fourier expansion

$$R(F)(a, c, z) = e^{2\pi i Nz} \sum_{m \in \mathbb{Z}} h_m(c) e^{2\pi i ma}, \hspace{1cm} (8.8)$$

with Fourier coefficients

$$h_m(c) := \int_{a=0}^{1} R(F)(a, c, 0) e^{-2\pi i ma} da.$$

We obtain

$$h_m(c) = \sum_{\tilde{n} \in \mathbb{Z}} \chi|d(\tilde{n}) e^{-2\pi i \frac{\tilde{n}c}{d}} \int_{0}^{1} f\left( N(a + \frac{\tilde{n}}{d}) \right) e^{-2\pi i Na(c + \frac{2m}{d})} e^{-2\pi i Na} da$$

$$= \sum_{\tilde{n} \in \mathbb{Z}} \chi|d(\tilde{n}) e^{-2\pi i \frac{\tilde{n}c}{d}} \int_{0}^{1} f\left( N(a + \frac{\tilde{n}}{d}) \right) e^{-2\pi i Na(c + \frac{2m}{d})} e^{-2\pi i Na} da. \hspace{1cm} (8.9)$$

Splitting the sum on the right into residue classes (mod $d$) gives

$$h_m(c) = \sqrt{C_{N,d}} \sum_{k=1}^{d} \chi|d(k) e^{-2\pi i \frac{km}{d}} \int_{-\infty}^{\infty} f\left( N(a + \frac{k}{d}) \right) e^{-2\pi i Na(c + \frac{2m}{d})} e^{-2\pi i Na} d\left( \frac{|N| \tilde{n}}{|N|} \right) da$$

$$= \sqrt{C_{N,d}} \sum_{k=1}^{d} \chi|d(k) e^{-2\pi i \frac{km}{d}} \int_{-\infty}^{\infty} f(N\tilde{a}) e^{-2\pi i N\tilde{a}(c + \frac{2m}{d})} d\left( \frac{|N| \tilde{a}}{|N|} \right)$$

$$= \sqrt{C_{N,d}} G(-m, \chi|d) \cdot \frac{1}{|N|} \mathcal{F}(f)(c + \frac{m}{N})$$

$$= \sqrt{C_{N,d}} G(-m, \chi|d) \cdot \frac{1}{|N|} \mathcal{F}(f)(c + \frac{m}{N}).$$
Now $\mathcal{F}(f)(c + \frac{m}{d}) = |N|^{1/2}U(\frac{1}{N})(\mathcal{F}(f))(c + m)$ and, using the fact that for all $t \in \mathbb{R}^*$, $U(\frac{1}{t}) \circ \mathcal{F} = \mathcal{F} \circ U(t)$ on $L^2(\mathbb{R}, dx)$, we obtain

$$h_m(c) = \sqrt{C_{N,d}} \frac{G(-m, \chi|d)}{|N|^{1/2}} \left(U\left(\frac{1}{N}\right) \circ \mathcal{F}\right)(f)(m + Nc)$$

$$= \sqrt{C_{N,d}} \frac{G(-m, \chi|d)}{|N|^{1/2}} \left(\mathcal{F} \circ U(N)\right)(f)(m + Nc). \quad (8.10)$$

Now we group terms in the Fourier expansion $(8.7)$ according to the value of $\tilde{d} = \frac{N}{(m,N)}$, noting that $(m, N) = \frac{N}{d}$. We will apply Proposition $8.1$ to evaluate the Gaussian sums, and for this we set

$$d' = \frac{d}{(m, d)} = \frac{d}{(|N|/d, d)} \quad \text{and} \quad m' = \frac{m}{(m, d)} = \frac{m}{(|N|/d, d)},$$

Proposition $8.1$ then states, if $f|d'$, that

$$G(m, \chi|d) = \frac{\phi(d)}{\phi(d')} \mu\left(\frac{d'}{d}\right) \chi\left(\frac{d'}{d}\right) \cdot \bar{\chi}(m'), \quad (8.11)$$

while $G(m, \chi|d) = 0$ if $f \nmid d'$. We will show that, when $f|d'$, that

$$G(-m, \chi|d) = \chi(-1)C_{N,d}(\tilde{d}, \chi) \cdot \bar{\chi}(\frac{md}{N}). \quad (8.12)$$

with

$$C_{N,d}(\tilde{d}, \chi) := \frac{\phi(d)}{\phi(d')} \mu\left(\frac{d'}{d}\right) \chi\left(\frac{d'}{d}\right) \bar{\chi}(\frac{N}{dd}). \quad (8.13)$$

and that $G(-m, \chi|d) = 0$ when $f \nmid d'$.

We have

$$G(-m, \chi|d) = \bar{\chi}(\frac{md}{N}) \cdot \bar{\chi}(\frac{N}{d}), \quad (8.14)$$

so this gives $G(-m, \chi|d) = 0$ if $f \nmid d'$, and it remains to consider the case $f|d'$.

We will need to relate $\tilde{d}$ and $d'$, and now show that $f|d'$ implies that $f|\tilde{d}$, or equivalently, that $f \nmid \tilde{d}$ implies $f \nmid d'$, as remarked after the theorem statement. So suppose $f|d'$, and let a prime $p|f$, and set $p^{\nu} || f, p^{\nu} || f, p^{\nu} || N$, so that $1 \leq f_p \leq f_p \leq h_p$. Now $f|d'$ gives $ord_p((\frac{N}{d}, d)) \leq f_p - f_p < f_p$ so that $ord_p((\frac{N}{d}, d)) = ord_p((\frac{N}{d}, d))$, which yields $ord_p(\tilde{d}) \geq h_p - (f_p + f_p) = f_p + (h_p - f_p) \geq f_p$. Since this holds for all primes dividing $f$, we have $f|\tilde{d}$.

Next, define $\tilde{g} := \frac{N}{dd}$ and note that the definition of $d'$ yields $\frac{N}{d} := (\frac{N}{d}, d)$, which shows that $\tilde{g}$ is an integer. We have $m' = \frac{md}{N} \cdot \tilde{g}$, and since $\frac{md}{N}$ is an integer,

$$\bar{\chi}(m') = \bar{\chi}(\frac{md}{N}) \cdot \bar{\chi}(\tilde{g}).$$

Whenever $f|\tilde{d}$, the relation $(\frac{md}{N}, \tilde{d}) = 1$ yields

$$\bar{\chi}(m') = \bar{\chi}(\frac{md}{N}) \cdot \bar{\chi}(\tilde{g}). \quad (8.15)$$

We showed above that $f|d'$ implies $f|\tilde{d}$, so we can combine this with $(8.11)$, and $(8.14)$ to deduce $(8.12)$. 
We next note that
\[ \mathcal{W}_{N,d}(\bar{\chi}|_{d})(\mathcal{F} \circ U(N)(f)) = \sqrt{C_{N,d}} \sum_{m \in \mathbb{Z}} \bar{\chi}|_{d}(\frac{md}{N})(\mathcal{F} \circ U(N))(f)(m + Nc)e^{2\pi ima}. \quad (8.16) \]

The nonzero terms in the sum on the right all have \((m, N) = \frac{|N|}{d}\). (Indeed, we must have \(|N|(n, N)\) for \(\frac{md}{N}\) to be an integer, and if any further factor of \(d\) divides \((m, N)\) then the imprimitive character \(\chi|_{d}\) vanishes.) Comparing this formula with (8.10) on those terms with \((m, N) = \frac{|N|}{d}\) and using (8.12), we find that they agree up to a multiplicative scale factor, given by multiplying (8.16) by
\[ C_{N,d}(\tilde{d}, \chi) := \sqrt{C_{N,d}} C_{N,d}^*(\tilde{d}, \chi) = \sqrt{\frac{\phi(d)}{\phi(\tilde{d})}} C_{N,d}^*(\tilde{d}, \chi). \]

Combining this with (8.13) yields (8.16) and completes the proof. \(\Box\)

8.3. Action of R-operator on coarse multiplicative decomposition. The simplest case of Theorem 8.2 is the case of a primitive character \(\chi \pmod{|N|}\) in which case \(\mathcal{R}(\mathcal{H}_{N,|N|}(\chi)) = \mathcal{H}_{N,|N|}(\chi)\), and (8.3) simplifies to
\[ \mathcal{R}(\mathcal{W}_{N,|N|}(\chi)(f)) = \epsilon(\chi)\mathcal{W}_{N,|N|}(\bar{\chi})(\mathcal{F} \circ U(N)(f)), \quad (8.17) \]
where \(\epsilon(\chi)\) is given by
\[ \epsilon(\chi) := \chi(-1) \frac{\tau(\chi)}{|N|^\frac{1}{2}}, \]
and satisfies \(|\epsilon(\chi)| = 1\) and \(\epsilon(\chi)\epsilon(\bar{\chi}) = 1\). More generally, we obtain the following result.

**Theorem 8.3.** For \(N \neq 0\), the Heisenberg-Fourier operator \(\mathcal{R}\) restricted to the invariant subspace \(\mathcal{H}_N\) is a unitary operator \(\mathcal{R}_N\) which acts to permute the Hilbert spaces \(\mathcal{H}_N(\chi; \bar{f})\) given by the coarse multiplicative decomposition of \(\mathcal{H}_N\). It satisfies
\[ \mathcal{R}_N(\mathcal{H}_N(\chi; \bar{f})) = \mathcal{H}_N(\bar{\chi}; \bar{f}). \quad (8.18) \]

**Proof.** The coarse multiplicative decomposition of \(\mathcal{H}_N\) was given in Theorem 5.7. The map \(\mathcal{R}_N\) is the restriction of \(\mathcal{R}\) to \(\mathcal{H}_N\). and map \(\mathcal{R}\) is a unitary transformation of \(\mathcal{H}_N\) into itself, which is onto since \(\mathcal{R}^4 = \mathbf{I}\). Theorem 8.2 and the remark following its statement show that when \(d|f\) the image of \(\mathcal{H}_{N,d}(\chi|_{d})\) falls in \(\mathcal{H}_N(\chi; \bar{f})\) so we conclude that \(\mathcal{R}(\mathcal{H}_N(\chi; \bar{f})) \subseteq \mathcal{H}_N(\chi; \bar{f})\). Since the image \(\mathcal{R}(\mathcal{H}_N(\chi'; \bar{f}'))\) falls in \(\mathcal{H}_N(\chi'; \bar{f}')\), which is orthogonal to \(\mathcal{H}_N(\chi; \bar{f})\) inside \(\mathcal{H}_N\), we conclude that the map \(\mathcal{R}_N\) on \(\mathcal{H}_N(\chi; \bar{f})\) must be an isometry onto \(\mathcal{H}_N(\chi; \bar{f})\). \(\Box\)

9. **Lerch L-Functions viewed as Eisenstein Series**

The classical Eisenstein series \(E(z, s)\) associated to \(SL(2, \mathbb{Z}) \setminus SL(2, \mathbb{R})\) has three characteristic properties. First, for each fixed \(s \in \mathbb{C}\) it is a (generalized) eigenfunction in the \(z = x + iy\) variable of the non-Euclidean Laplacian \(\Delta_{\mathbb{H}} = y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)\) (with eigenvalue \(s(s - 1)\)), and on the critical line \(s = \frac{1}{2} + it\) these eigenfunctions comprise the continuous spectrum of \(\Delta_{\mathbb{H}}\) on \(SL(2, \mathbb{Z}) \setminus SL(2, \mathbb{R})\). Second, for each fixed \(s \in \mathbb{C}\) it is a simultaneous eigenfunction of a commutative algebra of Hecke operators acting on the
z-variable. Third, it has a functional equation in the spectral variable \( s \), relating \( E(z, s) \) and \( E(z, 1-s) \). In this section we define Lerch \( L \)-functions \( L_{N,d}^{\pm}(\chi, s, a, c, z) \) and their tempered distribution analogues, and show they possess analogues of all three properties.

9.1. Lerch \( L \)-functions. Let \( d \) divide \( |N| \) and suppose that \( \chi \) is a (primitive or imprimitive) Dirichlet character \((\text{mod } d)\). We apply the intertwining map \( W_{N,d}(\chi) \) to functions and operators on \( L^2(\mathbb{R}, dx) \), carrying them to functions and operators associated to \( \mathcal{H}_{N,d}(\chi) \). In Appendix B we give the resulting correspondence for \( \mathcal{H}_1 = \mathcal{H}_{1,1}(\chi_0) \), for the operators discussed in Section 8, plus the two variable Hecke operators and their adjoints. The dilation group \( U(t) \) is carried to a group \( V(t) := W_{N,d}(\chi) \circ U(t) \circ W_{N,d}(\chi)^{-1} \) of unitary operators on \( \mathcal{H}_{N,d}(\chi) \); we therefore call operators on \( \mathcal{H}_{N,d}(\chi) \) dilation-invariant if they commute with all \( V(t) \). The action of the additive Fourier transform \( \mathcal{F} \) on \( \mathcal{H}_1 \), given in Weil [39], was derived in Section 6.

Now apply the intertwining operator \( W_{N,d}(\chi) \) to the generalized eigenfunctions \((\text{sgn}(x))^k|x|^{-1/2+it}\) of \( x \frac{d}{dx} + \frac{1}{2} \) to obtain, formally,

\[
W_{N,d}(\chi)((\text{sgn}(x))^k|x|^{-1/2+it})(a, c, z) = \sqrt{C_{N,d}e^{2\pi inz}} \sum_{n \in \mathbb{Z}} \chi(n) e^{2\pi ina} |n+ac|^{-\frac{1}{2}+it},
\]

in which \( \pm = (-1)^k \chi(-1) \), and \( C_{N,d} = N/\phi(d) \). This series converges conditionally in the critical strip \( 0 < \Re(s) < 1 \), for non-integer values of \( a \) and \( c \). (One may split the sum into \( n \geq 0 \) and \( n < 0 \) and get absolute convergence in the half-plane \( \Re(s) > 1 \), resp. \( \Re(s) < 0 \).)

We use it to make the following definitions of Lerch \( L \)-function attached to \( \mathcal{H}_{N,d}(\chi) \) in the critical strip, as analytic functions in the \( s \)-variable.

**Definition 9.1.** For fixed \( s \) in \( 0 < \Re(s) < 1 \) and non-integer \( a \) and \( c \) the Lerch \( L \)-function \( L_{N,d}^{\pm}(\chi, s, a, c, z) \) is given by

\[
L_{N,d}^{\pm}(\chi, s, a, c, z) := e^{2\pi iz} L_{N,d}^{\pm}(\chi, s, a, c),
\]

in which the last term is given by the conditionally convergent series,

\[
L_{N,d}^{\pm}(\chi, s, a, c) := \sum_{n \in \mathbb{Z}} \chi(n) e^{2\pi ina} |n+ac|^{-s}.
\]

Here \( \pm = (-1)^k \), with \( k = 0, 1 \). We also call \( L_{N,d}^{\pm}(\chi, s, a, c) \) a Lerch \( L \)-function; it corresponds to setting \( z = 0 \) in (9.1).

Now consider the special case \( N = 1 \), where necessarily \( d = 1 \) and \( \chi = \chi_0 \) is the trivial character. We have

\[
L_{1,1}^{\pm}(\chi_0, s, a, c, z) = e^{2\pi iz} \left( \sum_{n \in \mathbb{Z}} (\text{sgn}(n+c))^k e^{2\pi ina} |n+c|^{-s} \right) = e^{2\pi iz} L^{\pm}(s, a, c),
\]

where \( L^{\pm}(s, a, c) \) are the Lerch functions studied in Lagarias and Li [35, Theorem 2.2], which for \( \Re(s) > 0 \) and \( (a, c) \in \mathbb{R} \times \mathbb{R} \) are given by

\[
L^{\pm}(s, a, c) = \sum_{n \in \mathbb{Z}} (\text{sgn}(n+c))^k e^{2\pi ina} |n+c|^{-s}, \text{ with } (-1)^k = \pm.
\]
All Lerch $L$-functions can be expressed in terms of the functions $L^\pm(s, a, c)$ treated in Lagarias and Li [35], as follows.

**Lemma 9.2.** Let $\chi$ be a (primitive or imprimitive) Dirichlet character (mod $d$), with $d$ dividing $N$. Then for fixed $(a, c) \in \mathbb{R} \times \mathbb{R}$, and $\Re(s) > 1$ there holds

$$
L_{N, d}^\pm(\chi, s, a, c, z) = e^{2\pi i N z} N^{-s} \left( \sum_{m=0}^{d-1} \chi(m) e^{2\pi i (\frac{N}{d}) m a} L^\pm(s, Na, c + \frac{m}{d}) \right). \quad (9.4)
$$

Here we use the convention $\chi(r) = 0$ if $r$ is not an integer.

**Proof.** We have, for $\Re(s) > 1$, writing $n = N j$, with $j \in \mathbb{Z}$ and $j = ld + m$, with $l \in \mathbb{Z}$ and $0 \leq m \leq d - 1$, that

\[
L_{N, d}^\pm(\chi, s, a, c) := \sum_{n \in \mathbb{Z}} \chi(\frac{nd}{N}) (\text{sgn}(n + N c))^{k e^{2\pi i a} |n + N c| - s}
\]

\[
= N^{-s} \left( \sum_{j \in \mathbb{Z}} \chi(j) (\text{sgn}(\frac{Nj}{d} + N c))^{k e^{2\pi i (\frac{N}{d}) a} j |d + c|^{-s}} \right)
\]

\[
= N^{-s} \left( \sum_{m=0}^{d-1} \chi(m) \left( \sum_{l \in \mathbb{Z}} (\text{sgn}(l + (\frac{m}{d} + c))^{k e^{2\pi i (\frac{N}{d}) a} |l + (\frac{m}{d} + c)|^{-s}} \right) \right)
\]

\[
= N^{-s} \left( \sum_{m=0}^{d-1} \chi(m) e^{2\pi i (\frac{m}{d}) (Na)} L^\pm(s, Na, c + \frac{m}{d}) \right),
\]

giving the result. \qed

From this lemma we deduce an analytic continuation of $L_{N, d}^\pm(\chi, s, a, c)$ in the $s$-variable, and that these functions satisfy the "twisted periodicity" conditions needed to belong to $H_N$.

**Theorem 9.3.** Let $N \neq 0$ with $d \mid N$ and let $\chi$ (mod $d$) be a Dirichlet character. Then for fixed $(a, c) \in \mathbb{R} \times \mathbb{R}$ the function $L_{N, d}^\pm(\chi, s, a, c)$ analytically continues to a meromorphic function of $s$, whose only singularities are a possible simple pole at $s = 0$, which may only occur if $a$ is an integer, and a possible simple pole at $s = 1$, which may only occur if $c$ is an integer. It satisfies the “twisted periodicity” conditions

$$
L_{N, d}^\pm(\chi, s, a + \frac{d}{N}, c, z) = L_{N, d}^\pm(\chi, s, a, c, z) \quad (9.5)
$$

$$
L_{N, d}^\pm(\chi, s, a, c + 1, z) = e^{-2\pi i Na} L_{N, d}^\pm(\chi, s, a, c, z). \quad (9.6)
$$

The first of these implies that

$$
L_{N, d}^\pm(\chi, s, a + 1, c, z) = L_{N, d}^\pm(\chi, s, a, c, z). \quad (9.7)
$$

Remark. These twisted periodicity relations imply $L_{N, d}^\pm(\chi, s, a+1, c, z) = L_{N, d}^\pm(\chi, s, a, c, z)$, so are sufficient to make $L_{N, d}^\pm(\chi, s, a, c, z)$ a well-defined function on the Heisenberg nilmanifold $N_3 = H(\mathbb{Z}) \backslash H(\mathbb{R})$. 
Proof. The \( z \)-variable plays no role, so it suffices to prove the result for \( L_{N,d}^\pm(\chi, s, a, c) \). The meromorphic continuation follows from the right side of (9.4) in Lemma 9.2 using Lagarias and Li [35, Theorem 2.2]. In fact a stronger result holds: the meromorphic continuation as stated holds for the completed functions

\[
\hat{L}_{N,d}^\pm(\chi, s, a, c) := \gamma^\pm(s)L_{N,d}^\pm(\chi, s, a, c),
\]

in which \( \gamma^\pm(s) \) is the Tate gamma function, which is meromorphic and has no zeros. This implies that \( L_{N,d}^\pm(\chi, s, a, c) \) must have "trivial zeros" at the appropriate set of negative integers.

The "twisted periodicity conditions" (9.5) and (9.6) now follow from the "twisted periodicity conditions" for \( L^\pm(s, a, c) \) in [35, Theorem 2.2],

\[
L^\pm(s, a + 1, c) = e^{-2\pi ia}s^\pm L^\pm(s, a, c) \quad \text{and} \quad L^\pm(s, a, c + 1) = e^{-2\pi ia}s^\pm L^\pm(s, a, c)
\]

substituted into (9.4). \( \square \)

For fixed non-integer \( s \in \mathbb{C} \) the Lerch \( L \)-functions \( L_{N,d}^\pm(\chi, s, a, c, z) \) are continuous and real-analytic on \( H(\mathbb{R}) \) except possibly at values \((a, c, z) = \frac{l}{N} \) and \( c = \frac{m}{N} \) for integer \( l, m \). This follows from the right side of (9.4), using the fact that \( L^\pm(s, a, c) \) has discontinuities only at integer values of \( a \) and \( c \), see [35, Theorem 2.3]. In addition, for \( 0 < \Re(s) < 1 \) the Lerch \( L \)-functions \( L_{N,d}^\pm(\chi, s, a, c, z) \) are locally \( L^1 \)-functions on \( H(\mathbb{R}) \), as a consequence of the same result for \( L^\pm(s, a, c) \), in [35, Theorem 2.4]. However for fixed \( s \) outside of \( 0 \leq \Re(s) \leq 1 \) this function is not locally \( L^1 \) on \( H(\mathbb{R}) \). For this paper it is sufficient to know these functions are analytic in the (open) critical strip \( 0 < \Re(s) < 1 \), which is derivable from the Dirichlet series representation above.

Note that each Lerch \( L \)-function possess a reflection symmetry

\[
L_{N,d}^\pm(\chi, s, -a, -c, z) = \pm \chi(-1)L_{N,d}^\pm(\chi, s, a, c, z).
\]

This follows directly from the definition (9.1), using a change of the summation variable from \( n \) to \(-n\).

There are also simple rescaling relations relating Lerch \( L \)-functions at different levels having the same Dirichlet character. The following result relates functions associated to the Heisenberg modules \( \mathcal{H}_N \) and \( \mathcal{H}_{N'} \), when \( |N'| \mid N \).

**Theorem 9.4.** Let \( N \neq 0 \) with \( d \mid N \). Suppose that \( N' \) is a (positive or negative) integer satisfying \( d \mid N' \) and \( |N'| \mid |N| \), Then for any Dirichlet character \( \chi(\mod d) \), there holds

\[
L_{N,d}^\pm(\chi, s, a, c, z) = (\text{sgn}(\frac{N}{N'}))^k|\frac{N}{N'}|^{-s}L_{N',d}^\pm(\chi, s, \frac{N}{N'}a, c, \frac{N}{N'}z),
\]

where \( \pm = (-1)^k \). In particular,

\[
L_{-N,d}^\pm(\chi, s, a, c, z) = \pm L_{N,d}^\pm(\chi, s, -a, c, -z).
\]
Proof. This result parallels Lemma 5.4. For $0 < \Re(s) < 1$ the definition of Lerch $L$-function, taking $\tilde{n} = \frac{Na}{N}$, yields

$$L_{N,d}^\pm(\chi, s, a, c, z) = |N|^{-s} e^{2\pi i N z} \sum_{\tilde{n} \in \mathbb{Z}} \chi(\tilde{n})(sgn(N)) \frac{k}{d}(\tilde{n} + c) e^{2\pi i \frac{N}{d} \tilde{n}} |\frac{\tilde{n}}{d} + c|^{-s}.$$ 

On the other hand,

$$L_{N',d}^\pm(\chi, s, \frac{Na}{N'}, c, \frac{N}{N'}z) := e^{2\pi i N' \frac{N}{N'} z} \sum_{n \in \mathbb{Z}} \chi(\frac{N}{N'}(n + N'c)) e^{2\pi i \frac{N}{N'} \frac{N}{N'} \tilde{n}} |n + N'c|^{-s} = |N'|^{-s} e^{2\pi i N z} \sum_{\tilde{n} \in \mathbb{Z}} \chi(\tilde{n})(sgn(N')) \frac{k}{d}(\tilde{n} + c) e^{2\pi i \frac{N}{d} \tilde{n}} |\frac{n}{d} + c|^{-s}.$$ 

The theorem follows for $0 < \Re(s) < 1$ on comparing these two formulae. It then holds for all complex $s$ by analytic continuation. □

9.2. Generalized Eigenfunctions of $\Delta_L$. The “Laplacian operator” $\Delta_L$ is a left-invariant differential operator on the Heisenberg group, given in (9.12) below. It acts on all Heisenberg modules $\mathcal{H}_N$, including $\mathcal{H}_0$. On $\mathcal{H}_{N,d}(\chi)$ it can be identified with the image of the dilation-invariant operator $D = x \frac{d}{dx} + \frac{1}{2}$ under the twisted Weil-Brezin map $W_{N,d}(\chi)$, and this permits the results of Section 5 to be applied.

For $\mathcal{H}_{N,d}(\chi)$ the dilation-invariant differential operator $x \frac{d}{dx} + \frac{1}{2}$ is carried to the differential operator

$$\Delta_L := W_{N,d}(\chi) \circ (x \frac{d}{dx} + \frac{1}{2}) \circ W_{N,d}(\chi)^{-1} = \frac{1}{2\pi i} \frac{\partial}{\partial a} + \frac{Nc}{\partial c} + \frac{N}{2}.$$ (9.11)

This operator coincides with

$$\Delta_L = \frac{1}{4\pi i} (XY + YX)$$ (9.12)

in which

$$X := \frac{\partial}{\partial a} + c \frac{\partial}{\partial z}, \quad Y := \frac{\partial}{\partial c}$$

are left-invariant differential operators on the (non-symmetric) Heisenberg group. This definition of $\Delta_L$ is intrinsic on the Heisenberg group $\mathcal{H}(\mathbb{R})$, and it also makes sense on the “degenerate” Heisenberg module $\mathcal{H}_0$.

From the definition (9.11) one checks that for any fixed value of $s$ with $0 < \Re(s) < 1$ the Lerch $L$-functions $L_{N,d}^\pm(\chi, s, a, c, z)$ formally are generalized eigenfunctions of $\Delta_L$ acting on $\mathcal{H}(\mathbb{R})$, with eigenvalue $-(s - \frac{1}{2})$. The eigenfunction property obviously holds for each term separately in the expansion (9.11). This can be rigorously justified, and extended to all $s \in \mathbb{C}$ if one uses tempered distributions, using the notion of Lerch $L$-distribution defined in Section 9.3. (Here we note that the analytically continued version of the Lerch $L$-function for a fixed $s \in \mathbb{C}$ is differentiable in $a$ and $c$ away from the singular set where $a$ or $c$ are integers, and there satisfies the eigenfunction equation with eigenvalue $-(s - \frac{1}{2})$. To get the generalized eigenfunction property at the singular set, one must use tempered distributions.)
The linear partial differential operator $\Delta_L$ has several features of a classical Laplacian operator, but also has some differences. A classical Laplacian is a differential operator in the center of the universal enveloping algebra of some real Lie algebra. Here the operator $\Delta_L$ a left-invariant operator in the Heisenberg Lie algebra, but is not right-invariant, so it is not in the center of the universal enveloping algebra.

In the classical $SL(2)$ case Eisenstein series on the critical line are generalized eigenfunctions of the non-Euclidean Laplacian operator. The following result interprets Lerch $L$-functions as an “Eisenstein series” in a similar sense.

**Theorem 9.5.** (Eisenstein Series Interpretation of Lerch $L$-functions)

Let $N \neq 0$ and $d \geq 1$ with $d \mid N$.

1. Consider the unbounded operator $\Delta_L = \frac{1}{2\pi i} \frac{\partial}{\partial x} + Nc \frac{\partial}{\partial c} + \frac{N}{2}$ on the dense domain $\mathcal{D}_{N,d}(\chi) := W_{N,d}(\chi)(\mathcal{D})$ in the Hilbert space $\mathcal{H}_{N,d}(\chi)$, in which $\mathcal{D}$ denotes the maximal domain for $D = x \frac{\partial}{\partial x} + \frac{1}{2} I$ on $L^2(\mathbb{R}, dx)$. The operator $(\Delta_L, \mathcal{D}_{N,d}(\chi))$ commutes with all elements of the unitary group $\{V(t) : t \in \mathbb{R}^*\}$.

2. The operator $(\Delta_L, \mathcal{D}_{N,d}(\chi))$ is skew-adjoint on $\mathcal{H}_{N,d}(\chi)$, and its associated spectral multiplier function on $L^2(\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{R}, d\tau)$ is $a_0(\tau) = -i\tau$ and $a_1(\tau_1) = -i\tau_1$.

3. The two families of Lerch $L$-functions $L^\pm_{N,d}(\chi, \frac{1}{2} + i\tau, a, c, z)$, parameterize the (pure) continuous spectrum of $(\Delta_L, \mathcal{D}_{N,d}(\chi))$ on $\mathcal{H}_{N,d}(\chi)$, giving a complete set of generalized eigenfunctions, as $\tau$ varies over $\mathbb{R}$. All functions $F(a, c, z)$ in the dense subspace $\mathcal{S}(\mathcal{H}_{N,d}(\chi))$ have a convergent spectral representation

$$F(a, c, z) = \frac{1}{4\pi} \int_{-\infty}^{\infty} \hat{F}^+(\frac{1}{2} + i\tau)L^+_{N,d}(\chi, \frac{1}{2} - i\tau, a, c, z)d\tau + \frac{1}{4\pi} \int_{-\infty}^{\infty} \hat{F}^-(\frac{1}{2} + i\tau)L^-_{N,d}(\chi, \frac{1}{2} - i\tau, a, c, z)d\tau,$$

in which $\hat{F}^+(s) = \mathcal{M}_0(W_{N,d}(\chi)^{-1}(F))(s)$ and $\hat{F}^-(s) = \mathcal{M}_1(W_{N,d}(\chi)^{-1}(F))(s)$.

**Proof.** The main idea of the proof is that the nice properties of the operators follow from the property of commuting with the group of unitary dilations on a suitable dense domain in the Hilbert space which is invariant under all the operators involved. It transforms the problem to that of a dilation-invariant operator on $L^2(\mathbb{R}, dx)$, viewed as a rigged Hilbert space, with Schwartz functions $\mathcal{S}(\mathbb{R})$ as the smallest class in the triple.

1. Using the Weil-Brezin transform we pull back the problem from each Hilbert space $\mathcal{H}_{N,d}(\chi)$ to a problem on $L^2(\mathbb{R}, dx)$, in which the operator $\Delta_L$ is transformed to $x \frac{d}{dx} + \frac{1}{2}$. In $L^2(\mathbb{R}, dx)$ we take the Schwartz space $\mathcal{S}(\mathbb{R})$ as our dense domain, since the domain $\mathcal{S}(\mathcal{H}_{N,d}(\chi))$ is the push-forward under the Weil-Brezin map $W_{N,d}(\chi)$ of $\mathcal{S}(\mathbb{R})$. The group of dilation operators $V(t)$ is pulled back to the group of dilations $U(t)$ on $L^2(\mathbb{R}, dx)$. The Schwartz space $\mathcal{S}(\mathbb{R})$ is invariant under dilations, the operator $D = x \frac{d}{dx} + \frac{1}{2}$ is dilation-invariant and preserves this space.

2. (2), (3) We apply results of Burnol [17] Theorems 2.2 and 2.4 concerning dilation invariant operators on $L^2[\mathbb{R}_{\geq 0}, \frac{dx}{x}]$ (the group $G = \mathbb{R}_{>0}$), which can be transferred to
$L^2(\mathbb{R}, dx)$ by an inverse Mellin transform identity. Applied to the operator $iD$ it gives a unique self-adjoint extension domain and an absolutely continuous spectral measure as specified. These issues are discussed in Appendix B, where Proposition [12.5] supplies an answer to both (2) and (3) for $x \frac{d}{dx} + \frac{1}{2}$.

9.3. Hecke Operator Eigenfunctions. We now characterize Lerch $L$-functions for fixed $s \in \mathbb{C}$ as simultaneous generalized eigenfunctions of the set of two-variable Hecke operators $T_m, m \geq 1$. This parallels the second property of classical Eisenstein series mentioned above. We give complete details for $L^\pm(\chi_0, s, a, c, z)$ and then sketch the result for general Lerch $L$-functions.

We obtain the result using a form of the inverse Weil-Brezin map to pull back the question from the Heisenberg group to the real line, and formalize our results in terms of tempered distributions. The basic observation is that $L^\pm(\chi_0, s, a, c, z)$ is the formal image under the Weil-Brezin map of the quasicharacter $\text{sgn}(x)^k|x|^{-s}$; we justify this for tempered distributions when $0 < \Re(s) < 1$. Let $S'(\mathbb{R})$ denote the space of tempered distributions, the dual space to Schwarz functions $S(\mathbb{R})$, characterized as continuous linear functionals $F : S(\mathbb{R}) \to \mathbb{C}$. We write $\langle F, \varphi(x) \rangle$ for the value of the linear functional on $\varphi(x) \in S(\mathbb{R})$. (Note that this scalar product is not a Hilbert space inner product; it is linear in both arguments for complex-valued $\varphi(x)$, and in particular is not conjugate-linear in the second argument.) If the distribution corresponds to an $L^1$ function then we have

$$\langle F, \varphi(x) \rangle := \int_{-\infty}^{\infty} F(x)\varphi(x)dx.$$  

For $t \in \mathbb{R}^*$ we define a dilation action $U(t) : S'(\mathbb{R}) \to S'(\mathbb{R})$ by defining for a distribution $F$ its image $U(t)(F)$ by

$$\langle U(t)(F), \varphi(x) \rangle := \langle F, U(\frac{1}{t})\varphi(x) \rangle = \langle F, |t|^{-1/2}\varphi(\frac{x}{t}) \rangle.$$  

(9.14)

These operators satisfy $U(t_1t_2) = U(t_1) \circ U(t_2)$ and $U(0) = 1$.

A multiplicative quasicharacter on $\mathbb{R}^*$ has the form $\chi(t) = (\text{sgn}(t)^k|t|^s$ with $s \in \mathbb{C}$, $k = 0, 1$. We write $\chi_+(t) := 1$ for the trivial character and $\chi_-(t) = \text{sgn}(t)$ for the sign character. A tempered distribution $\Delta$ is said to be homogeneous with quasicharacter $\chi$ if

$$U(t)(\Delta) = \chi(t)|t|^{-1/2}\Delta \text{ for all } t \in \mathbb{R}^*.$$  

(9.15)

A tempered distribution $\Delta$ is even if $U(-1)(\Delta) = \Delta$ and is odd if $U(-1)\Delta = -\Delta$.

Proposition 9.6. (Weil) For each quasicharacter $\chi(t) = (\text{sgn}(t)^k|t|^s$ on $\mathbb{R}^*$ there exists a tempered distribution $\Delta(x)$ on $\mathbb{R}$ having homogeneity $\chi$. This distribution on $\mathbb{R}$ is unique up to multiplication by a nonzero constant.

\[\text{Homogeneous distributions were treated by Gel'fand and Sapiro [23] and Gårding [21]. Burnol [16, p. 16] states that a tempered distribution $\Delta$ has homogeneity $\chi$ if $\Delta(x) = \chi(x)|x|^{-1}\Delta(t)$, i.e. if for all Schwartz functions $\int_{-\infty}^{\infty} \Delta(t)\phi(t)dt = \chi(x)\int_{-\infty}^{\infty} \Delta(t)\phi(x)dt$.}\]
Proof. This appears in Weil [60]. Weil’s paper treats more general situations, including homogeneous distributions on all local fields, and on adeles.

The homogeneous distributions, excluding a countable number of values of \( s \), can be grouped into two families \( \Delta^+_s \) and \( \Delta^-_s \) associated to \( |t|^s \) and \( \text{sgn}(t)|t|^s \), respectively, which are meromorphic in the parameter \( s \in \mathbb{C} \), in the sense that for each test function \( \varphi \in \mathcal{S}(\mathbb{R}) \), the function

\[
f^\pm_\varphi(s) := \langle \Delta^\pm_\varphi, \varphi(x) \rangle
\]

is a meromorphic function of \( s \). Furthermore the functions \( f^\pm_\varphi(s) \) have at most simple poles, which can occur only at the quasicharacters where the associated homogeneous distribution is local, which means supported at the point \( \{0\} \). Some cases of local distributions are that of homogeneity \( \chi_0 \), where the distribution is the Dirac delta function \( \delta_0 \) at \( x = 0 \), and those of homogeneity \( |t|^{-2n} \) for \( n \geq 1 \), where the distribution is the \( n \)-th derivative of the Dirac delta function \( \delta_0^{(n)} \) at \( x = 0 \).

For \( \Re(s) > 0 \) the function \( x \mapsto |x|^{s-1} \) is locally integrable, and we take

\[
\Delta^\pm_s := (\text{sgn}(x))^k|x|^{s-1} \text{ with } (-1)^k = \pm,
\]

where the distribution is defined by

\[
\langle \Delta^\pm_s, \varphi(x) \rangle := \int_{-\infty}^{\infty} \varphi(x)(\text{sgn}(x))^k|x|^{s-1}dx.
\]

The distributions \( \Delta^\pm_s \) form an analytic family in the region \( \Re(s) > 1 \), and Weil showed they analytically continue to a meromorphic family on \( s \in \mathbb{C} \). The polar divisors are exactly the values of \( s \) where the homogeneous distribution is local, and are a subset of \( \mathbb{Z} \).

The Hecke operator \( T_m^\chi_0(f)(x) := f(mx) \) when regarded as a linear operator acting on locally integrable functions of moderate growth at \( \pm \infty \) has a unique extension to a (continuous) operator, also denoted \( T_m^\chi_0 \), acting on tempered distributions \( \mathcal{S}'(\mathbb{R}) \), defined by

\[
\langle T_m^\chi_0(F), \varphi(x) \rangle := \langle F, (T_m^\chi_0)^*(\varphi)(x) \rangle, \quad \phi(x) \in \mathcal{S}(\mathbb{R}),
\]

in which

\[
(T_m^\chi_0)^*(\varphi)(x) := \frac{1}{m} \sum_{j=0}^{m-1} \varphi\left(x + \frac{j}{m}\right).
\]

Using Theorem 5.6 one deduces that \( T_m^\chi_0 = |m|^{-1/2}U(m) \), where \( U(m) \) is the \( \mathbb{R}^* \)-action on tempered distributions given above.

**Theorem 9.7.** Let \( d \geq 1 \) be an integer, and for each \( s \in \mathbb{C} \) fixed, let \( E_s \) be the vector space consisting of those tempered distributions \( \Delta \) on \( \mathbb{R} \) such that

\[
T_m^\chi_0(\Delta) = m^{-s}\Delta, \text{ for all } m \equiv 1 \pmod{d} \text{ with } m \geq 1.
\]

Then \( E_s \) is two-dimensional, and is independent of \( d \). It is spanned by an even homogeneous tempered distribution of homogeneity \( |t|^{1-s} \) and an odd homogeneous tempered
distribution of homogeneity $\text{sgn}(t)|t|^{1-s}$. For $s \notin \mathbb{Z}$, these distributions can be taken to be $\Delta_1^{\pm}$ and $\Delta_{-1}^{\pm}$, respectively.

Proof. The condition (9.21) is satisfied by homogeneous distributions of the two types, as is evident from the relation of $T_{m}^{\chi}$ to $U(m)$. By Proposition 9.6 these generate a two-dimensional vector space $\tilde{E}_s$ of tempered distributions, independent of $d$, with $\tilde{E}_s \subseteq E_s$. It remains to show that $\tilde{E}_s = E_s$.

Given $\Delta \in E_s$, we must show that $\Delta \in \tilde{E}_s$. We first show that (9.21) implies that, for all $t > 0$,

$$U(t)(\Delta) = |t|^{-(s-\frac{1}{2})}\Delta.$$  \hfill (9.22)

Since $T_{m}^{\chi} = m^{-1/2}U(m)$, (9.21) gives

$$U(m)(\Delta) = |m|^{-(s-\frac{1}{2})}\Delta$$

for $m \equiv 1 \pmod{d}$, $m > 0$. Letting $U(\frac{1}{m})$ operate on both sides of this equation yields

$$U(\frac{1}{m})(\Delta) = m^{s-\frac{1}{2}}\Delta.$$

Now

$$U(m_1) \circ U(\frac{1}{m_2})(\Delta) = U(m_1)(|m_2|^{s-\frac{1}{2}}\Delta) = \frac{m_1}{m_2}^{-(s-\frac{1}{2})}\Delta.$$

Thus (9.22) holds for the values $\{t = \frac{m_1}{m_2}: m_1 \equiv m_2 \equiv 1 \pmod{d}\}$ which are dense in $\mathbb{R}_{>0}$. The relation (9.22) now holds by a limiting process. Take a sequene $t_k = \frac{m_1}{m_2,k}$ with $t_k \to t$ and one has

$$\langle U(t_k)(\Delta), \varphi(x) \rangle = \left\langle |t_k|^{-(s-\frac{1}{2})}\Delta, \varphi(x) \right\rangle \to \left\langle t^{-(s-\frac{1}{2})}\Delta, \varphi(x) \right\rangle,$$

while also

$$\langle U(t_k)(\Delta), \varphi(x) \rangle = \langle \Delta, U(\frac{1}{t_k})\varphi(x) \rangle = \langle \Delta, |t_k|^{\frac{1}{2}}\varphi(\frac{x}{t_k}) \rangle \quad \xrightarrow{k \to \infty} \quad \langle \Delta, |t|^{\frac{1}{2}}\varphi(\frac{x}{t}) \rangle = \langle \Delta, U(\frac{1}{t})\varphi(x) \rangle = \langle U(t)(\Delta), \varphi(x) \rangle.$$

This verifies (9.22).

To complete the proof we use $U(-1)$, which commutes with $U(t)(t > 0)$ so that

$$U(t) \circ U(-1)\Delta = |t|^{-(s-\frac{1}{2})}U(-1)\Delta.$$  

It follows that the tempered distribution

$$\Delta^{+} := \frac{1}{2}(\Delta + U(-1)\Delta)$$

satisfies $U(-1)(\Delta^{+}) = \Delta^{+}$ and, using (9.22),

$$U(t)\Delta^{+} = |t|^{-(s-\frac{1}{2})}\Delta^{+} \quad \text{all } t > 0.$$

Thus $\Delta^{+}$ is a homogeneous tempered distribution with homogeneity $\chi(x) = |x|^{1-s}$, hence it lies in $\tilde{E}_s$. Similarly we find that the tempered distribution

$$\Delta^{-} := \frac{1}{2}(\Delta - U(-1)\Delta)$$
is homogeneous with homogeneity \( \text{sgn}(x)|x|^{1-s} \), so also lies in \( \mathcal{E}_s \). We conclude that \( \Delta = \frac{i}{2}(\Delta^+ + \Delta^-) \in \mathcal{E}_s \).

Finally we note that \( \Delta_{-s}^\pm \) are even (odd) homogeneous distributions in the appropriate spaces, as required, as long as \( s \not\in \mathbb{Z} \), or more generally whenever \( s \) is not a polar value of the meromorphic family. \( \square \)

We transport the notion of tempered distribution to \( \mathcal{H}_{N,d}(\chi) \) using the twisted Weil-Brezin map \( W_{N,d}(\chi) \). We let \( \mathcal{S}(\mathcal{H}_{N,d}(\chi)) \) denote the image of the Schwartz space under \( W_{N,d}(\chi) \). This map is one-to-one and we put the induced topology on \( \mathcal{S}(\mathcal{H}_{N,d}(\chi)) \). We call the topological dual space \( \mathcal{S}'(\mathcal{H}_{N,d}(\chi)) \) the space of tempered distributions on \( \mathcal{H}_{N,d}(\chi) \). We obtain a twisted Weil-Brezin map on tempered distributions \( W_{N,d}(\chi) : \mathcal{S}'(\mathbb{R}) \rightarrow \mathcal{S}'(\mathcal{H}_{N,d}(\chi)) \) which associates to a tempered distribution \( \Delta \in \mathcal{S}(\mathbb{R}) \) a distribution \( W_{N,d}(\chi)^\sharp(\Delta) \in \mathcal{S}'(\mathcal{H}_{N,d}(\chi)) \) by the linear form

\[
\left\langle W_{N,d}(\chi)^\sharp(\Delta), W_{N,d}(\chi)(\phi) \right\rangle := \langle \Delta, \phi \rangle, \text{ for all } \varphi \in \mathcal{S}(\mathbb{R}).
\]

This map is an isomorphism onto. We use the “sharp” notation to distinguish the distribution \( W_{N,d}(\chi)^\sharp(\varphi) \) associated to the function \( \varphi \) from the function \( W_{N,d}(\chi)(\varphi) \), because for \( f \in L^1(\mathbb{R}, dx) \) one computes the linear form as

\[
\left\langle W_{N,d}(\chi)^\sharp(f), W_{N,d}(\chi)(\phi) \right\rangle := \int_0^1 \int_0^1 \int_0^1 W_{N,d}(\chi)^\sharp(f) \cdot \overline{W_{N,d}(\chi)(\varphi)} \, d\alpha \, d\beta \, dz,
\]

Two complex conjugations appear in the last term in this integral because \( W_{N,d}(\chi) \) respects the Hilbert space inner product, which is conjugate-linear in its second argument. We define the space \( L^1(\mathcal{H}_{N,d}(\chi)) \) to be the space of locally \( L^1 \)-functions on \( H(\mathbb{R}) \) which transform according to the relations in \( \mathcal{H}_{N,d}(\chi) \). To each function \( F \in L^1(\mathcal{H}_{N,d}(\chi)) \) there is naturally associated a tempered distribution in \( \mathcal{S}'(\mathcal{H}_{N,d}(\chi)) \).

**Definition 9.8.** For arbitrary \( s \in \mathbb{C} \) the Lerch \( L \)-distribution \( L^\pm_{N,d}(\chi, s, a, c, z) \) in \( \mathcal{H}_{N,d}(\chi) \) is the tempered distribution \( W_{N,d}(\chi)(\Delta^\pm_{1-s}) \) in \( \mathcal{S}'(\mathcal{H}_{N,d}(\chi)) \).

For fixed \( s \) in \( 0 < \Re(s) < 1 \) the Lerch \( L \)-distribution can be identified with the Lerch \( L \)-function \( L^\pm_{N,d}(\chi, s, a, c, z) \), viewed as a locally \( L^1 \)-function on \( H(\mathbb{R}) \). We therefore use the same notation for it, in an abuse of language. Note that outside \( 0 \leq \Re(s) \leq 1 \) one cannot necessarily identify the Lerch \( L \)-distribution with the (analytically continued) Lerch \( L \)-function, since this function is not locally \( L^1 \) on \( H(\mathbb{R}) \).

**Theorem 9.9.** (Hecke operator tempered distribution eigenspace) Let \( N \) be a nonzero integer, and \( \chi \) be a Dirichlet character (mod \( d \)), with \( d|N \).

1. For each fixed \( s \in \mathbb{C} \), let \( \mathcal{E}_s(\mathcal{H}_{N,d}(\chi)) \) be the vector space of tempered distributions \( \Delta \in \mathcal{S}'(\mathcal{H}_{N,d}(\chi)) \) such that

\[
T_m(\Delta) = \chi(m)m^{-s}\Delta, \text{ for all } m \geq 1 \text{ with } (m, N) = 1.
\]

\[ (9.23) \]
Then $E_s(\mathcal{H}_{N,d}(\chi))$ is a two-dimensional vector space, and is spanned by an even homogeneous tempered distribution of homogeneity $|t|^{1-s}$ and an odd homogeneous tempered distribution of homogeneity $\text{sgn}(t)|t|^{1-s}$.

(2) For all non-integer $s \in \mathbb{C}$ the two Lerch $L$-distributions $L^\pm_{N,d}(\chi, s, a, c, z)$ are nonzero even and odd homogeneous distributions spanning $E_s(\mathcal{H}_{N,d}(\chi))$, respectively. For $0 < \Re(s) < 1$ these two distributions are induced by the Lerch $L$-functions $L^\pm_{N,d}(\chi, s, a, c, z)$, which both lie in $L^1(\mathcal{H}_{N,d}(\chi))$.

Proof. We pull the condition (9.23) back to $L^2(\mathbb{R}, dx)$ using the map $\mathcal{W}_{N,d}(\chi)^{-1}$ acting on tempered distributions. Any tempered distribution $\Delta = \mathcal{W}_{N,d}(\chi)(\Delta')$ for a (unique) tempered distribution $\Delta'$, and the analogue of Theorem 5.6 on tempered distributions asserts that

$$T_m \circ \mathcal{W}_{N,d}(\chi)(\Delta') = \chi(m)\mathcal{W}_{N,d}(\chi)(T_m(\Delta')),$$

provided $(m, \frac{N}{d}) = 1$. This implies that all elements of $\mathcal{W}_{N,d}(\chi)(E_s)$ satisfy (9.23). In the converse direction, if $\Delta \in E_s(\mathcal{H}_{N,d}(\chi))$ then applying $\mathcal{W}_{N,d}(\chi)^{-1}$ to (9.23) using (9.24) yields

$$\chi(m)T_m(\Delta') = \chi(m)m^{-s}\Delta'.$$

when $(m, \frac{N}{d}) = 1$. For $m \equiv 1 \pmod{N}$ this gives

$$T_m(\Delta') = \chi(m)m^{-s}\Delta',$$

and Theorem 9.7 applied with $d = N$ establishes that $\Delta' \in E_s$. We conclude that $E_s(\mathcal{H}_{N,d}(\chi)) = \mathcal{W}_{N,d}(\chi)(E_s)$. It follows from Theorem 9.7 that $E_s(\mathcal{H}_{N,d}(\chi))$ has dimension 2, and the rest of (1) follows because $\mathcal{W}_{N,d}(\chi)$ preserves evenness and oddness of distributions.

This argument also establishes (2), aside from the identification of the tempered distribution with the corresponding Lerch $L$-function when $0 < \Re(s) < 1$. The identification follows from the property that these functions are locally $L^1$ on the Heisenberg group. This local $L^1$ property can be established by extending the proof of the local $L^1$-property in [35, Theorem 2.4] to general Lerch $L$-functions.

Remarks. (1) One can prove a similar result for the adjoint Hecke operators $(T_m)^*$. The vector space $E_s^*(\mathcal{H}_{N,d}(\chi))$ of tempered distributions in $\mathcal{W}_{N,d}(\chi)$ such that

$$(T_m)^*(\Delta) = \bar{\chi}(m)m^{1-s}\Delta, \text{ for all } m \geq 1 \text{ with } (m, N) = 1,$$

is two-dimensional, with $E_s^*(\mathcal{H}_{N,d}(\chi)) = E_s(\mathcal{H}_{N,d}(\chi))$. To prove this one can use the formula $(T_m)^* = R^* \circ T_m \circ R$ together with Theorem 5.2. We use the fact that the action of $T_m$ is constant on $\mathcal{H}_N(\hat{\chi}; c)$ as long as $(m, N) = 1$.

(2) Theorem 9.9 is analogous to a result of Milnor [46, Theorem 1], see also Lagarias and Li [38, Theorem 5.5]. In fact Milnor’s result can be interpreted as describing simultaneous continuous eigenfunctions of a two-variable Hecke operator on a certain vector subspace of the Hilbert space $\mathcal{H}_0$, as shown in part II.
9.4. Generalized Lerch Functional Equations. The original Lerch zeta function satisfies two symmetrized four-term functional equations relating $s$ to $1 - s$, given in Weil [62] p. 57 and in Lagarias and Li [35]. Each functional equation encodes the action of the additive Fourier transform. It can be derived from that of the homogeneous distributions $\Delta_{R}^\pm$ on the real line, using the Weil-Brezin maps $\mathcal{W}_{N,d}(\chi)$ on tempered distributions.

The additive Fourier transform acts on tempered distributions by $\langle \mathcal{F}(\Delta), \varphi \rangle = \langle \Delta, \mathcal{F}(\varphi) \rangle$. Weil [60] observed that the additive Fourier transform takes homogeneous distributions $\Delta_{R}^\pm$ to homogeneous distributions $\Delta_{1-s}^\pm$, with

$$\mathcal{F}(\Delta_{R}^\pm) = \gamma^\pm(s)\Delta_{1-s}^\pm,$$  \hspace{1cm} (9.25)

in which $\gamma^\pm(s)$ is a certain meromorphic function of $s$, with the data $(\chi^\pm, s)$ specifying the homogeneity type of the homogeneous distribution on the left side of the equation. This follows for $0 < \Re(s) < 1$ from the identity valid for Schwartz functions $\varphi \in \mathcal{S}(\mathbb{R})$,

$$\int_{-\infty}^{\infty} \mathcal{F}(\varphi)(x)(\text{sgn}(x))^{k}\text{sgn}^{-1}(x) = \pi^{-s/2}\Gamma(\frac{s}{2})\int_{-\infty}^{\infty} \varphi(y)(\text{sgn}(y))^{k}\text{sgn}^{-1}(y) \, dy. \hspace{1cm} (9.26)$$

and extends by analytic continuation to $s \in \mathbb{C}$. We call the functions $\gamma^\pm(s)$ Tate-Gelfand-Graev gamma functions, following the terminology of Burnol [14], [15], who named them after Tate [52] and Gelfand and Graev, cf. [22]. Recall that they are

$$\gamma^+(s) = \frac{\pi^{-s/2}\Gamma(\frac{s}{2})}{\pi^{-2s/2}\Gamma(\frac{s+1}{2})}, \quad \gamma^-(s) = -i \frac{\pi^{-(s+1)/2}\Gamma(\frac{s+1}{2})}{\pi^{-2s/2}\Gamma(\frac{s+1}{2})}. \hspace{1cm} (9.27)$$

These functions provide the necessary correction factor in a nonsymmetric form of the local functional equation at the real place. Recall that the local Euler factor at the real place is $\Gamma_{R}(s) := \pi^{-s/2}\Gamma(\frac{s}{2})$, and the local functional equation can be written

$$\Gamma_{R}(s) = \gamma^+(s)\Gamma(1 - s).$$

Note that $\gamma^\pm(s)\gamma^\pm(1 - s) = \pm 1$.

We next state functional equations for the Lerch $L$-functions expressed using the R-operator, to relate functions at value $s$ with functions at value $1 - s$. These functional equations are obtained by pushing the relation (9.25) forward through the Weil-Brezin map $\mathcal{W}_{N,d}(\chi)$.

**Theorem 9.10.** (Generalized Lerch Functional Equations) Suppose that $N \neq 0$. Let $\chi$ be a primitive character $\pmod{f}$ and suppose that $f|d$ and $d|N$, and let $\chi|_{d}$ denote the (generally imprimitive) character $\pmod{d}$. Then for $0 < \Re(s) < 1$ the two Lerch $L$-functions $L_{N,d}^\pm(\chi|_{d}, s, a, c, z)$ associated to $\mathcal{H}_{N,d}(\chi|_{d})$ satisfy the functional equations

$$R(L_{N,d}^\pm(\chi|_{d}, 1 - s, a, c, z)) = \chi(-1)^{s}\tau(\chi)|N|^{s-1}\gamma^\pm(s) \left( \sum_{d|N} C_{N,d}(\tilde{d}, \chi)L_{N,d}^\pm(\chi|_{d}, s, a, c, z) \right), \hspace{1cm} (9.28)$$

in which $Rf(a, c, z) = f(-c, a, z - ac)$ and $\gamma^\pm(s)$ are Tate-Gelfand-Graev gamma functions, and the coefficients $C_{N,d}(\tilde{d}, \chi)$ vanish whenever $\tilde{f} \nmid \tilde{d}$.  

Remarks. (1) The coefficients $C_{N,d}(\tilde{d}, \chi)$ in \(9.28\) are those in Theorem 8.2.

(2) The restriction to $0 < \Re(s) < 1$ allows both sides of the functional equation to be viewed as functions of four variables that are locally $L^1$ functions on the Heisenberg group in the $(a, c, z)$ variables, which are continuous off the set where $a$ or $c$ take integer values. Assuming the values of $a$ and $c$ are fixed, and are not integers, the associated function of the variable $s$ can be meromorphically continued from $0 < \Re(s) < 1$ to $s \in \mathbb{C}$, with polar divisor set contained in $\mathbb{Z}$. This can be proved in a standard fashion, similar to that in [35].

(3) The left side of \(9.28\) satisfies the identity

$$R(L_{N,d}^\pm(\chi|d, 1-s, a, c, z)) = L_{N,d}^\pm(\chi|d, 1-s, -c, a, z - ac) = e^{-\pi i N ac} L_{N,d}^\pm(\chi|d, 1-s, 1-c, a, z). \tag{9.29}$$

This identity facilitates comparison with the functional equations given in [35], for $N = 1, \chi = \chi_0$.

Proof. We assume $0 < \Re(s) < 1$ to that we can identify the homogeneous distribution $\Delta_s^\pm = (\text{sgn}(x))^k|x|^{s-1}$ with the corresponding locally $L^1$-function. The calculations of the functional equation will be at the level of locally $L^1$-functions, thus avoiding the question of identifying distributions on different Hilbert spaces.

We push forward the local functional equation \(9.25\) for the homogeneous distribution $\Delta_s^\pm$ through the Weil-Brezin map $W_{N,d}(\chi|d)$ acting on tempered distributions $S(W_{N,d}(\chi|d))$. We have, formally, that the tempered distribution $W_{N,d}(\chi|d)(\Delta_s^\pm)$ is given by

$$W_{N,d}(\chi|d)(\Delta_s^\pm) = e^{2\pi i N z} \left( \sum_{n \in \mathbb{Z}} \chi|d\left(\frac{nd}{N}\right)(\text{sgn}(n + N c))^k|n + N c|^{-(1-s)} e^{2\pi i n a} \right) = L_{N,d}^\pm(\chi|d, 1-s, a, c, z).$$

The sum on the right is conditionally convergent for $0 < \Re(s) < 1$ but gives absolutely convergent sums when evaluated against any test function. Now the formula of Theorem 8.2 asserts

$$R(W_{N,d}(\chi|d)(f)) = \chi(-1) \frac{\tau(\chi)}{|N|^2} \sum_{d | |N|} C_{N,d}(\tilde{d}, \chi) W_{N,d}(\tilde{\chi}|d)(\mathcal{F} \circ U(N)(f)), \tag{9.30}$$

which we can apply for any Schwartz function $f$. This yields

$$R(L_{N,d}^\pm(\chi|d, 1-s, a, c, z) = \chi(-1) \frac{\tau(\chi)}{|N|^2} \sum_{d | |N|} C_{N,d}(\tilde{d}, \chi) W_{N,d}(\tilde{\chi}|d)(\mathcal{F} \circ U(N)(\Delta_s^\pm)). \tag{9.30}$$
Next, using (9.14) we obtain
\[ U(N)(\Delta^\pm_s) = |N|^{s-\frac{1}{2}}\Delta^\pm_s. \]
We deduce the equality of tempered distributions
\[ \mathcal{W}_{N,d}(\check{\chi}|d)(\mathcal{F} \circ U(N)(\Delta^\pm_s)) = \mathcal{W}_{N,d}(\check{\chi}|d)(|N|^{s-\frac{1}{2}}\gamma^\pm(s)\Delta^\pm_{-s}) \]
\[ = |N|^{s-\frac{1}{2}}\gamma^\pm(s)e^{2\pi i N \frac{n d}{N}}(\sum_{n \in \mathbb{Z}} \check{\chi}(\frac{n d}{N})(\text{sgn}(n + Nc))k |n + Nc|^{-s}e^{2\pi i a n}) \]
\[ = |N|^{s-\frac{1}{2}}\gamma^\pm(s)L^\pm_{N,d}(\check{\chi}|d, s, a, c, z). \]
Substituting this in the right side of (9.30) yields the functional equation (9.28).

The functional equations given in Theorem 9.10 typically involve several different Lerch L-functions on their right side. They can however be reformulated as a vector-valued functional equations for each Hilbert space \( \mathcal{H}_N(\chi; f) \) in Theorem 8.3. One uses a vector of the Lerch L-functions \( L_{N,d}(\chi|d, s, a, c, z) \) with a fixed sign \( \pm \) associated to characters \( \chi|d \) indexed by the set
\[ \Sigma(f, N) := \{ d \geq 1 : f|d \text{ and } d|N \}, \]
and relates it to Lerch L-functions associated to \( \mathcal{H}_N(\check{\chi}; f) \). The functional equation involves a matrix \( M(\chi) \) whose entries are the \( C_{N,d}(d, \chi) \) with rows and columns indexed by \( d \) (resp. \( \check{d} \)), both drawn from \( \Sigma(f, N) \).

**Remark.** The Lerch L-function (with a proper choice of sign \( \pm \)) recovers the corresponding Dirichlet L-function by taking a limit as \( (a, c) \rightarrow (0, 0) \). The functional equation for the Dirichlet L-function with a primitive character \( \chi \mod N \) is recoverable from the functional equation of the Lerch L-function \( L_{N,N}(\chi, s, a, c, z) \) with \( \epsilon = \chi(-1) \) under this limiting process, taking \( z = 0 \).

### 10. Concluding Remarks

#### 10.1. *Is the Lerch zeta function a global or a local object?*

The Lerch zeta function has some unusual features. One may view it as a “global” zeta function attached to the rational field \( \mathbb{Q} \), in the sense that, when specialized the corners of the unit square \( \square \), it yields formally the Riemann zeta function. The Lerch L-functions in this paper, correspondingly specialize at the corners to Dirichlet L-functions, and sometimes specialize to be identically zero. When these are nonzero, these specializations are global L-functions.

On the other hand, the Lerch L-functions \( L_{N,d}^\pm(\chi, s, a, c, z) \) appear to behave like a kind of local L-function at the real place, in that Theorem 9.5 exhibits them as the image under a Weil-Brezin map of a local homogeneous distribution at the real place. This interpretation leads to the question whether there exist analogous constructions of “Lerch L-functions” at other local places. The thesis of Ngo [49] gives an adelic construction, in terms of certain zeta integrals, for local fields and globally for number fields and function fields. In the globalization the archimedean places play a special role,
leading to the global functions not having an Euler product. The Lerch $L$-functions in (1.2) appear as global zeta integrals for particular adelic test functions.

10.2. Heisenberg modules invariant under all Hecke operators and $R$-operator.
One may consider the algebra $A_N$ of operators acting on $H_N$ having $\mathbb{C}$-coefficients generated by the set of all two-variable Hecke operators $\{T_m : m \geq 1\}$ together with the $R$-operator. This algebra is a $\ast$-algebra (by Theorem 4.3), which has an interesting non-commutative structure, particularly for $|N| \geq 2$. To formulate a decomposition into subspaces of $H_N$ that are invariant under the action of the whole algebra $A_N$ we find (on combining Theorem 5.7 with Theorem 8.3) that one must use a coarser decomposition than any so far, indexed by a primitive Dirichlet character $\chi \pmod{f}$ together with its contragredient $\bar{\chi}$, which is given by

$$H_N^{\text{prim}}(\chi, \bar{\chi}) := \bigoplus_{d \mid N} \left( H_{N,d}(\chi | d) \oplus H_{N,d}(\bar{\chi} | d) \right).$$

It might prove worthwhile to study the action of these operators on the full Hilbert space associated to a single primitive character $\chi \pmod{f}$ and its contragredient, where the level $N$ may vary, as

$$H^{\text{prim}}(\chi, \bar{\chi}) := \bigoplus_{N \in \mathbb{Z} \setminus \{0\}} \left( \bigoplus_{N \mid |N|} H_N^{\text{prim}}(\chi, \bar{\chi}) \right).$$

Here one allows positive and negative $N$ in the sum. Note that $H_N$ for positive $N$ are associated to holomorphic functions (theta functions) while those for negative $N$ are associated to anti-holomorphic functions.

10.3. $xp$ operator and Riemann hypothesis. This paper showed that the Lerch $L$-functions on the critical line are generalized eigenfunctions for a spectral decomposition associated to the action of the dilation group $V(t)$. In particular they are generalized eigenfunctions for a differential operator having the form “$xp$” noted by Berry and Keating [8], explicitly exhibited in (9.12).

At certain limiting values of their domain variables, such as $(a, c, z) = (1, 1, 0)$, they yield Dirichlet $L$-functions, generalizing the case of the Riemann zeta function treated in [35]. What seems noteworthy is that Dirichlet $L$-functions are associated with a multiplicative structure, while Lerch $L$-functions embody an additive structure, coming from the group law on the Heisenberg group. The Heisenberg group structure brings together both the additive and multiplicative structures via a limit process.

This structure might conceivably be relevant to understanding the Riemann hypothesis. Berry and Keating ([8], [9]) have suggested that suitable operators of “$xp$” form might be involved in a spectral interpretation of the Riemann hypothesis. They studied $\frac{1}{2}(xp + px)$ which is a (formally) self-adjoint operator. The operator $\Delta_L$ is of an analogous form, although in our framework it is a (formally) skew-adjoint operator, cf Theorem 9.5.
10.4. Further work. A sequel ([34]), will address further points in understanding the two-variable Hecke operators acting on the Heisenberg module $L^2(H(\mathbb{Z})\backslash H(\mathbb{R}))$. First it shows that all the $GL(1)$ $L$-functions associated to $\mathbb{Q}$ (Dirichlet $L$-functions) do appear directly in their traditional multiplicative guise as generalized eigenvalues rather than as generalized eigenfunctions. Their values on the critical line then give the spectral multiplier function for pure continuous spectra of a “zeta operator”

$$Z := \frac{1}{2} \left( \sum_{m \in \mathbb{Z}\setminus\{0\}} T_m \right),$$

acting as an unbounded operator on suitable subspaces of $L^2(H(\mathbb{Z})\backslash H(\mathbb{R}))$, namely on $H_{N,N}(\chi)$. Second, it treats the Hecke operator action on the “degenerate” module $H_0$, which completes the study of their action on $H_N (N \neq 0)$ given here. Third, it observes that the results of Milnor [46] and of Bost and Connes [11] have an interpretation in terms of this action restricted to certain subspaces of $H_0$.

The results of this paper motivate further study of automorphic representations and automorphic forms on the sub-Jacobi group $H^J$ and related groups. Automorphic representations over adelic nilpotent groups, including the Heisenberg group, were worked out in 1965 by C. C. Moore [47]. There has been extensive study of automorphic forms on the full Jacobi group, see Berndt and Schmidt [7]. However the structures considered here have not been studied in the context of the Jacobi group. Some other points relevant to an adelic treatment for the Heisenberg group are given in Haran [25, Chap. 12].

11. Appendix A. Heisenberg and Sub-Jacobi Groups

We describe various matrix realizations of the (real) Heisenberg and sub-Jacobi groups.

We consider the one-parameter family of real Lie groups $G_\lambda$ specified by the real parameter $\lambda$, whose underlying elements $(x, y, z) \in \mathbb{R}^3$ with group law given by

$$[x_1, y_1, z_1]_\lambda \circ [x_2, y_2, z_2]_\lambda = [x_1 + x_2, y_1 + y_2, z_1 + z_2 + \lambda x_1 y_2 + (1 - \lambda) y_1 x_2]_\lambda. \quad (11.1)$$

The group law $\lambda = 0$ is the nonsymmetric Heisenberg group considered in this paper, and the case $\lambda = \frac{1}{2}$ is the symmetric Heisenberg group. The groups $G_\lambda$ are isomorphic as real Lie groups. An explicit isomorphism $\alpha_\lambda : G_1 \rightarrow G_\lambda$ is given by

$$\alpha_\lambda([x, y, z])_1 = [x, y, z - (1 - \lambda)xy]_\lambda \quad (11.2)$$

with

$$\alpha_\lambda^{-1}([\tilde{x}, \tilde{y}, \tilde{z}])_\lambda = [\tilde{x}, \tilde{y}, \tilde{z} + (1 - \lambda)\tilde{x}\tilde{y}]_1. \quad (11.3)$$

These groups have finite-dimensional matrix representations. In terms of the parameter $\lambda$, the case $\lambda = 0$ has the three-dimensional (real) matrix representation

$$[x, y, z]_0 = \begin{bmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix}. $$
and the case $\lambda = 1$ has the three-dimensional (real) matrix representation

$$[x, y, z]_1 = \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}.$$  

For general $\lambda \in \mathbb{R}$ the groups $G_\lambda$ have a $4 \times 4$ (real) linear representation

$$[x, y, z]_\lambda = \begin{bmatrix} 1 & x & y & z \\ 0 & 1 & 0 & \lambda y \\ 0 & 0 & 1 & -(1 - \lambda) x \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (11.4)$$

The symmetric Heisenberg group $G_{1/2}$ has an alternative $4 \times 4$ matrix representation in terms of variables $(p, q, z)$, as

$$[p, q, z]_{1/2} =: \begin{bmatrix} 1 & p & q & 2z \\ 0 & 1 & 0 & q \\ 0 & 0 & 1 & -p \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (11.5)$$

Certain properties of the Heisenberg group are more easily visible using the $4 \times 4$ matrix representation. Setting $G_1 = H(\mathbb{R})$, $G_{1/2} = H_{\text{sym}}(\mathbb{R})$ and define

$$\Gamma_{\text{sym}}(1) := \alpha_{1/2}(H(\mathbb{Z})). \quad (11.6)$$

Then

$$\Gamma_{\text{sym}}(1) := \left\{ (p, q, z) : p, q \in \mathbb{Z} \text{ and } z \in \frac{1}{2} \mathbb{Z} \setminus \mathbb{Z} \right\} \quad (11.7)$$

Let $F(p, q, z) : H_{\text{sym}}(\mathbb{R}) \to \mathbb{C}$ be a function in $L^2(\Gamma_{\text{sym}}(1) \setminus H_{\text{sym}}(\mathbb{R}))$. We have the Hilbert space decomposition

$$L^2(\Gamma_{\text{sym}}(1) \setminus H_{\text{sym}}(\mathbb{R})) = \bigoplus_{N \in \mathbb{Z}} \mathcal{H}_{N}^{\text{sym}}. \quad (11.8)$$

The smooth functions $F \in \mathcal{H}_{N}^{\text{sym}}$ satisfy

$$F(p, q, z) = e^{2\pi i N z} F(p, q, 0)$$
$$F(p + 1, q, 0) = e^{\pi i q} F(p, q, 0)$$
$$F(p, q + 1, 0) = e^{-\pi i p} F(p, q, 0). \quad (11.9)$$

The Hilbert space inner product on $\mathcal{H}_{N}^{\text{sym}}$ is

$$\langle F, G \rangle = \int_0^1 \int_0^1 \int_0^1 \overline{F(p, q, z)} \overline{G(p, q, z)} dpdqdz. \quad (11.10)$$

There is a Hilbert space isomorphism $\alpha_{1/2}^* : \mathcal{H}_N \to \mathcal{H}_N^{\text{sym}}$ which sends a smooth function $F(a, c, w)$ in $\mathcal{H}_N$ to the smooth function

$$\tilde{F}(p, q, z) = F \left( p, q, z - \frac{1}{2} pq \right) = e^{\pi i N pq} F(p, q, z). \quad (11.11)$$

We now consider matrix versions the sub-Jacobi group $H^J(\mathbb{R})$ treated in Section 7. This is an extension $H^J(\mathbb{R})$ of the Heisenberg group $G_1(\mathbb{R})$ by $\mathbb{R}_{>0}^*$. It is a four-dimensional
exponential solvable Lie group, and is unimodular. The Haar measure is \( d\mu = \frac{dt}{t} da dc dz \).

In Section 7 we observed that it has a \( 4 \times 4 \) matrix representation, given by, for \( t > 0 \),

\[
[c, a, z, t] = \begin{bmatrix}
1 & c & a & z \\
0 & t & 0 & ta \\
0 & 0 & \frac{1}{t} & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]  

(11.12)

A symmetrized form of this group action is

\[
[c, a, z, t] = \begin{bmatrix}
1 & c & a & z \\
0 & t & 0 & ita \\
0 & 0 & \frac{1}{t} & -\frac{1}{t} c \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]  

(11.13)

The latter group multiplication does not coincide with (11.12), but the resulting groups are isomorphic as Lie groups. Using the change of variable \( t = e^u \), we can view this solvable group as being homeomorphic to \( \mathbb{R}^4 \), with variables \( (u, a, c, z) \) and Haar measure \( d\mu' := du da dc dz \).

The universal enveloping algebra of the sub-Jacobi group has a two-dimensional center. One generator is the vector field \( \frac{\partial}{\partial z} \) associated with the center of the Heisenberg Lie algebra. The other generator is a second order differential operator which is a “lift” of the Heisenberg operator

\[
\Delta_L := \frac{1}{2\pi i} \frac{\partial}{\partial a} \frac{\partial}{\partial c} + c \frac{\partial}{\partial c} \frac{\partial}{\partial z} + \frac{1}{2} \frac{\partial}{\partial z}.
\]

The sub-Jacobi group is a subgroup of the Jacobi group, viewed in its \( 4 \times 4 \) matrix representation, contained inside \( SL(4, \mathbb{R}) \).

12. Appendix B. The Dilation-Invariant Operator \( x \frac{d}{dx} + \frac{1}{2} \)

This appendix gives results on (unbounded) operators on a Hilbert space that commute with the action of a locally compact Lie group \( G \) due to Burnol [17], [18]. We apply it to the group of dilations \( G = \mathbb{R}^* \) and the operator \( x \frac{d}{dx} + \frac{1}{2} \) acting inside \( L^2(\mathbb{R}, dx) \). Burnol’s work was originally done to understand invariances of the “explicit formula” in prime number theory ([14], [15], [16]), which includes this case.

12.1. Operators Commuting with a Locally Compact Group Action. Let \( G \) be a locally compact abelian group and \( \hat{G} \) be its dual group of unitary characters. There is a two-sided Haar measure \( dg \) on \( G \) unique up to a multiplicative constant, and there is a dual Haar measure \( dx \) on \( \hat{G} \) such that the \( G \)-Fourier transform

\[
\mathcal{F}_G(\varphi)(\chi) := \int_G \varphi(g) \chi(g) dg
\]

is an isometry from \( L^2(G, dg) \) to \( L^2(\hat{G}, dx) \). The group action \( U(g) f(x) = f(xg) \) on \( L^2(G, dg) \) consists of unitary operators. We consider (possibly unbounded) operators that respect the group action.
Definition 12.1. (1) A (possibly unbounded) operator $M$ with dense domain $D$ on a separable Hilbert space $H$ is said to commute with a bounded operator $A$ on $H$ if $A$ maps $D$ into $D$ and

\[ M(Av) = A(Mv), \quad \text{for all } v \in D. \quad (12.1) \]

(2) Let $G$ be a locally compact group, with (left) Haar measure $dg$. A (possibly unbounded) operator $M$ with dense domain on $L^2(G, dg)$ is said to commute with $G$ if it commutes with all unitary operators $\{U(g) : g \in G\}$, given by $U(g)(f)(h) := f(hg)$.

If $G$ is a locally compact abelian group, then the closed operators that commute with $G$ can be characterized as multiplication operators on $\hat{G}$, as follows. Let $a(\chi)$ denote a Borel-measurable function on $\hat{G}$, not necessarily bounded. Let $D_a \subset L^2(G, dg)$ be the domain of (equivalence-classes of) square-integrable functions $\varphi(g)$ on $G$ such that $a(\chi)\mathcal{F}_G(\varphi)(\chi)$ belongs to $L^2(\hat{G}, d\chi)$. Let $(\hat{M}_a, D_a)$ denote the (possibly unbounded) closed operator on $L^2(G, dg)$ with domain $D_a$ acting by $\hat{M}_a(\varphi) := \mathcal{F}_G^{-1} \circ M_a \circ \mathcal{F}_G(\varphi)$, where $M_a$ is the multiplication operator by $a(\chi)$ on $L^2(\hat{G}, d\chi)$. We call $a(\chi)$ its associated spectral multiplier function. In the case $G = \mathbb{R}^*$ for a closed dilation-invariant operator $T$ we use the alternate notations $a_0(\tau, T), a_1(\tau, T) (-\infty < \tau < \infty)$ or just $a_0(\tau), a_1(\tau)$ to denote the spectral multiplier functions associated to the characters $|y|^{i\tau}$ and $\text{sgn}(y)|y|^{i\tau}$, respectively.

Proposition 12.2. (Burnol) Let $G$ be a locally compact abelian group with two-sided Haar measure $dg$.

(1) For each measurable function $a(\chi)$ on $\hat{G}$ the (spectral) multiplication operator $(\hat{M}_a, D_a)$ on $L^2(G, dg)$ is closed, has $D_a$ as a dense domain, and commutes with $G$. If $(\hat{M}_b, D_b)$ extends $(\hat{M}_a, D_a)$, then $a(\chi) = b(\chi)$ (up to sets of measure zero) and $(\hat{M}_b, D_b) = (\hat{M}_a, D_a)$. The adjoint of $(\hat{M}_a, D_a)$ is $(\hat{N}_a, D_a)$, with $D_a = D_a$.

(2) Suppose that $(M, D)$ is a (possibly unbounded) operator with dense domain in $L^2(G, dg)$ which is closed and commutes with the elements of $G$. Then $(M, D) \equiv (\hat{M}_a, D_a)$ for a measurable function $a(\chi)$ on $\hat{G}$, which is unique up to a set of measure zero.

(3) The operator $(\hat{M}_a, D_a)$ is bounded if and only if $a(\chi)$ is essentially bounded.

(4) The adjoint of the operator $(\hat{M}_a, D_a)$ is $(\hat{N}_a, D_a)$, with $D_a = D_a$. It is self-adjoint if and only if $a(\chi)$ is (essentially) real-valued.

Proof. (1) This is Lemma 2.3 in Burnol [17].

(2) This is Theorem 2.4 in Burnol [17].

(3) This follows from the definition of the domain $D_a$.

(4) This is part of Lemma 2.3 in Burnol [17]. \qed

We next recall another result of Burnol in which the domain $D$ is not maximal.

Proposition 12.3. (Burnol) Let $G$ be a locally compact abelian group with two-sided Haar measure $dg$. Suppose that $(M, D)$ is a (possibly unbounded) operator with a dense domain in $L^2(G, dg)$ which is symmetric and commutes with $G$. Then $(M, D)$ is essentially self-adjoint, and if it is closed then it is self-adjoint.
Proof. This is Corollary 2.6 in Burnol [17]. The closure of such an operator is a spectral multiplication operator, with a real-valued multiplier function \( a(\chi) \), by Proposition 12.2.

\[ \hat{G} := \mathbb{R}^* \] consists of two one-parameter families of unitary characters, \( \chi_+^\pm(t) = |t|^\pm i\tau \) and \( \chi_-^\pm(t) = \text{sgn}(t)|t|^i\tau \), with \( \tau \in \mathbb{R} \), and can be identified with \( \hat{G} = \mathbb{R} \oplus \mathbb{Z}/2\mathbb{Z} \). The corresponding Fourier transform \( \mathcal{F}_G \) is a variant of the Mellin transform, which uses the two-sided Mellin transforms, understood, associated to the characters \( \chi_\pm \). The transform is as follows.

\[
\mathcal{M}_k(f)(s) := \int_{\mathbb{R}^*} f(t)(\text{sgn}(t))^k |t|^s \frac{dt}{|t|}, \quad (12.2)
\]

The transform is as follows.

**Lemma 12.4.** The Mellin transform integrals \( \mathcal{M}_k(f)(s) \) are well-defined for Schwartz functions \( f(t) \) on \( \mathbb{R}^* \) on the imaginary axis \( s \in i\mathbb{R} \) and the map

\[
f(t) \in S(\mathbb{R}^*) \leftrightarrow \mathcal{F}_G(\tau) := (\mathcal{M}_0(f)(i\tau), \mathcal{M}_1(f)(i\tau))\). 
\]

extends to an isometry \( \mathcal{F}_G : L^2(\mathbb{R}^*, \frac{dy}{|y|}) \to L^2(\mathbb{R}^*, d\chi) \) where

\[
L^2(\mathbb{R}^*, d\chi) \equiv L^2(\mathbb{R} \oplus \mathbb{Z}/2\mathbb{Z}, \frac{d\tau}{2\sqrt{2}\pi})
\]
corresponds to the action of the Mellin transform on the imaginary axis.

Now consider a closed dilation-invariant operator \( M \) on \( L^2(\mathbb{R}^*, \frac{dy}{|y|}) \). We use a special notation for the spectral multiplier functions \( a(\chi) \) above of such a function. We write them as two families \( a_0(\tau, M), a_1(\tau, M) \) \((-\infty < \tau < \infty)\), or just \( a_0(\tau), a_1(\tau) \) when \( M \) is understood, associated to the characters \( |y|^\tau \) and \( \text{sgn}(y)|y|^\tau \), respectively.

These results for \( L^2(\mathbb{R}^*, \frac{dy}{|y|}) \) carry over to \( L^2(\mathbb{R}, dx) \) with the action \( U(t)(f)(x) = |t|^2 f(tx) \), using the (inverse of the) isometry \( \varphi : L^2(\mathbb{R}, dx) \to L^2(\mathbb{R}^*, \frac{dy}{|y|}) \) given by \( f(x) \leftrightarrow \varphi(f)(y) := |y|^{1/2} f(y) \). Operators on \( L^2(\mathbb{R}, dx) \) that commute with the unitary operators \( U(t) \) push forward to operators on \( L^2(\mathbb{R}^*, \frac{dy}{|y|}) \) that commute with the \( \mathbb{R}^* \)-action.

The isometry

\[
\mathcal{F}_G \circ \varphi : L^2(\mathbb{R}, dx) \to L^2(\mathbb{R} \oplus \mathbb{Z}/2\mathbb{Z}, \frac{d\tau}{2\sqrt{2}\pi})
\]

has an inverse given by an integral formula (inverse Mellin transform) for all sufficiently nice functions \( f(x) \in L^2(\mathbb{R}, dx) \). For Schwartz functions \( f(x) \in S(\mathbb{R}) \) the inversion formula is

\[
f(x) = \int_{-\infty}^{\infty} \mathcal{M}_0(f)(\frac{1}{2} + i\tau)|x|^{-\frac{1}{2} - i\tau} \frac{d\tau}{2\sqrt{2}\pi} + \int_{-\infty}^{\infty} \mathcal{M}_1(f)(\frac{1}{2} + i\tau)\text{sgn}(x)|x|^{-\frac{1}{2} - i\tau} \frac{d\tau}{2\sqrt{2}\pi}.
\]

In part II we will tabulate spectral multiplier functions for various dilation-invariant operators on \( \mathcal{H}_{N,d}(\chi) \). The dilation operators on \( \mathcal{H}_{N,d}(\chi) \) are \( V(t) := W_{N,d}(\chi) \circ U(t) \circ W_{N,d}(\chi)^{-1} \). Note that the reflection operator

\[
\text{R}^2(F)(a, c, z) := F(-a, -c, z)
\]
has $\mathbb{R}^2 = V(-1) = \chi(-1)T_{-1}$, and its spectral multiplier function is $a_0(\tau) \equiv 1$, $a_1(\tau) \equiv -1$.

### 12.3. Continuous spectrum for $x \frac{d}{dx} + \frac{1}{2}$ on $L^2(\mathbb{R}, dx)$. Proposition 12.2 specifies for each unbounded operator above a (maximal) domain on which it is closed and commutes with $\mathbb{R}^*$. For some purposes it is useful to have a smaller dense domain $\Phi$ which is left invariant by the operator, with the maximal domain being recoverable by taking the closure of this operator. This is relevant in describing continuous spectra, which fall outside the Hilbert space. Continuous spectra can sometimes be described as generalized functions (distributions), using the dense domain $\Phi$ as a space of allowed test functions.

We recall that a rigged Hilbert space (or Gelfand triple) consists of $(\Phi, \mathcal{H}, \Phi')$ in which $\Phi$ is a dense vector subspace of a separable Hilbert space $\mathcal{H}$ endowed with a Fréchet topology finer than the Hilbert space topology, and $\Phi'$ is the dual space to $\Phi$, viewed as a space of generalized functions so that one has the inclusions $\Phi \subset \mathcal{H} \subset \Phi'$. The original definition requires that $\Phi$ be a nuclear space in the sense of Grothendieck and we impose this requirement, although some authors do not, e.g. Wickramasehara and Bohm [63], [64]. (In the physics literature, such as [63] the space $\Phi'$ is a space of conjugate-linear functionals, rather than linear functionals as in our definition.)

For the dilation invariant operator $D = x \frac{d}{dx} + \frac{1}{2}$ on $L^2(\mathbb{R}, dx)$ we take $\Phi = \mathcal{S}(\mathbb{R})$, one natural domain is the Schwartz space with its Fréchet topology, in which case $\Phi' = \mathcal{S}'(\mathbb{R})$ is the space of tempered distributions. The Schwartz space is relevant because it is the space of smooth vectors for the Schrödinger representation of the Heisenberg group acting on $L^2(\mathbb{R}, dx)$, see Howe [27], [28, p. 827]. The Schwartz space is invariant under all dilations $\{U(t) : t \in \mathbb{R}\}$, under the Fourier transform $\mathcal{F}$ and under additive translations $T(t)f(x) = f(x + t)$. However it is not invariant under the inversion $I f(x) = \frac{1}{\sqrt{\pi}} f(\frac{1}{x})$, or under the modified Poisson operator $P$ or the co-Poisson operator $P'$. For the dilation-invariant differential operator $D = x \frac{d}{dx} + \frac{1}{2}$ with this domain we have the following facts.

**Proposition 12.5.** (1) The operator $D = x \frac{d}{dx} + \frac{1}{2}$ leaves the domain $\mathcal{S}(\mathbb{R})$ invariant, and on this domain it is essentially skew-adjoint.

(2) It has a purely absolutely continuous spectrum, with generalized eigenfunctions $f_{k,\tau}(x)$ parametrized by $(k, \tau) \in \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{R}$, as

\[
\begin{align*}
  f_{0,\tau}(x) &= |x|^{-\frac{1}{2} + i\tau}, \\
  f_{1,\tau}^1(x) &= \text{sgn}(x)|x|^{-\frac{1}{2} + i\tau},
\end{align*}
\]

viewed as tempered distributions. The spectral measure is $\frac{d\tau}{2\sqrt{2\pi}}$ on both real components of the continuous spectrum, with spectral multiplier functions

\[
a_0(\tau) = -i\tau \quad \text{and} \quad a_1(\tau) = -i\tau.
\]

(3) For all elements $f(x) \in \mathcal{S}(\mathbb{R})$ the following two formulae converge absolutely:

\[
f(x) = \int_{-\infty}^{\infty} \mathcal{M}_0(f)(\frac{1}{2} + i\tau)|x|^{-\frac{1}{2} - i\tau} \frac{d\tau}{2\sqrt{2\pi}} + \int_{-\infty}^{\infty} \mathcal{M}_1(f)(\frac{1}{2} + i\tau)\text{sgn}(x)|x|^{-\frac{1}{2} - i\tau} \frac{d\tau}{2\sqrt{2\pi}}.
\]
and
\[
Df(x) = -\int_{-\infty}^{\infty} i\tau M_0(f)(\frac{1}{2}+i\tau)|x|^{-\frac{1}{2}-i\tau} \frac{d\tau}{\sqrt{2\pi}} - \int_{-\infty}^{\infty} i\tau M_1(f)(\frac{1}{2}+i\tau)|x|^{-\frac{1}{2}-i\tau} \frac{d\tau}{\sqrt{2\pi}}.
\]

Proof. The Schwartz space is a nuclear Fréchet space with its usual topology using seminorms, and the result on self-adjointness and continuous spectrum follows from the Nuclear Spectral theorem of Gel’fand-Maurin (Gelfand and Vilenkin [24], Maurin [42], see Bohm and Gadella [10, p. 25]) applied to the operator \( iD \).

The first formula is the inverse Mellin transform, separated into even and odd function parts. (The scaling factor in the measure divides by an extra factor of 2 coming from the definition of \( M_j \) integrating over the whole real line.) The second formula identifies the spectral multipliers and is obtained by differentiating the first formula under the integral sign.

\[ \square \]

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