TWISTORIAL CONSTRUCTION OF MINIMAL
HYPERSURFACES

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ABSTRACT. Every almost Hermitian structure \((g, J)\) on a four-manifold \(M\) determines a hypersurface \(\Sigma_J\) in the (positive) twistor space of \((M, g)\) consisting of the complex structures anti-commuting with \(J\). In this note we find the conditions under which \(\Sigma_J\) is minimal with respect to a natural Riemannian metric on the twistor space in the cases when \(J\) is integrable or symplectic. Several examples illustrating the obtained results are also discussed.

Keywords: Twistor spaces; minimal hypersurfaces.

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1. INTRODUCTION

The twistor space \(Z\) of a Riemannian manifold \((M, g)\) is the bundle on \(M\) parametrizing the complex structures on the tangent spaces of \(M\) compatible with the metric \(g\). Thus the almost Hermitian structures on \((M, g)\) are sections of \(Z\). Given such a structure \(J\), we can consider the hypersurface \(\Sigma_J\) of points of \(Z\) representing complex structures anti-commuting with \(J\). The twistor space admits a 1-parameter family \(h_t\) of Riemannian metrics, the so-called canonical variation of \(g\). Then it is natural to relate geometric properties of the hypersurface \(\Sigma_J\) in the Riemannian manifold \((Z, h_t)\) to properties of the almost Hermitian structure \((g, J)\). In this note we address the problem of when \(\Sigma_J\) is a minimal hypersurface in the twistor space of a manifold of dimension four. In this dimension, there are three basic classes in the Gray-Hervella classification - those of Hermitian, almost Kähler (symplectic) and Kähler manifolds. If \((g, J)\) is Kähler, \(\Sigma_J\) is a totally geodesic submanifold, as one can expect. In the case of an Hermitian manifold, we express the condition for minimality of \(\Sigma_J\) in terms of the Lee form of \((M, g, J)\), while for an almost Kähler manifold we show that \(\Sigma_J\) is minimal if and only if the \(\ast\)-Ricci tensor of \((M, g, J)\) is symmetric. Several example illustrating these results are discussed in the last section of the paper.

2. PRELIMINARIES

Let \((M, g)\) be an oriented Riemannian manifold of dimension four. The metric \(g\) induces a metric on the bundle of two-vectors \(\pi : \Lambda^2 TM \to M\) by the formula

\[
g(v_1 \wedge v_2, v_3 \wedge v_4) = \frac{1}{2} \det[g(v_i, v_j)].
\]
The Levi-Civita connection of \((M, g)\) determines a connection on the bundle \(\Lambda^2 TM\), both denoted by \(\nabla\), and the corresponding curvatures are related by

\[
R(X \wedge Y)(Z \wedge T) = R(X, Y)Z \wedge T + Z \wedge R(X, Y)T
\]

for \(X, Y, Z, T \in TM\). Let us note that we adopt the following definition for the curvature tensor \(R: R(X, Y) = \nabla_{[X, Y]} - [\nabla_X, \nabla_Y]\).

The Hodge star operator defines an endomorphism \(\ast\) of \(\Lambda^2 TM\) with \(s^2 = \text{Id}\). Hence we have the decomposition

\[
\Lambda^2 TM = \Lambda^2_- TM \oplus \Lambda^2_+ TM
\]

where \(\Lambda^2 \pm TM\) are the subbundles of \(\Lambda^2 TM\) corresponding to the \((\pm 1)\)-eigenvalues of the operator \(\ast\).

Let \((E_1, E_2, E_3, E_4)\) be a local oriented orthonormal frame of \(TM\). Set

\[
\begin{align*}
  s^+_1 &= E_1 \wedge E_2 \pm E_3 \wedge E_4, & \quad s^+_2 &= E_1 \wedge E_3 \pm E_4 \wedge E_2, & \quad s^+_3 &= E_1 \wedge E_4 \pm E_2 \wedge E_3.
\end{align*}
\]

Then \((s^+_1, s^+_2, s^+_3)\) is a local orthonormal frame of \(\Lambda^2_+ TM\) defining an orientation on \(\Lambda^2_+ TM\), which does not depend on the choice of the frame \((E_1, E_2, E_3, E_4)\).

For every \(a \in \Lambda^2 TM\), define a skew-symmetric endomorphism of \(T_{\pi(a)} M\) by

\[
g(K_a X, Y) = 2g(a, X \wedge Y), \quad X, Y \in T_{\pi(a)} M.
\]

Note that, denoting by \(G\) the standard metric \(-\frac{1}{2} \text{Trace} PQ\) on the space of skew-symmetric endomorphisms, we have \(G(K_a, K_b) = g(a, b)\) for \(a, b \in \Lambda^2 TM\). If \(\sigma \in \Lambda^2 \pm TM\) is a unit vector, then \(K\sigma\) is a complex structure on the vector space \(T_{\pi(\sigma)} M\) compatible with the metric and the orientation of \(M\). Conversely, the 2-vector \(\sigma\) dual to one half of the Kähler 2-form of such a complex structure is a unit vector in \(\Lambda^2 \pm TM\). Thus the unit sphere subbundle \(Z_+ = Z_+(M)\) of \(\Lambda^2 \pm TM\) parametrizes the complex structures on the tangent spaces of \(M\) compatible with its metric and orientation. This subbundle is called the twistor space of \(M\).

The Levi-Civita connection \(\nabla\) of \(M\) preserves the bundles \(\Lambda^2 \pm TM\), so it induces a metric connection on these bundles denoted again by \(\nabla\). The horizontal distribution of \(\Lambda^2_+ TM\) with respect to \(\nabla\) is tangent to the twistor space \(Z_+\). Thus we have the decomposition \(TZ_+ = H \oplus V\) of the tangent bundle of \(Z_+\) into horizontal and vertical components. The vertical space \(V_\tau = \{ V \in T_\tau Z_+ : \pi_* V = 0 \}\) at a point \(\tau \in Z_+\) is the tangent space to the fibre of \(Z_+\) through \(\tau\). Thus, considering \(T_\tau Z_+\) as a subspace of \(T_\tau (\Lambda^2_+ TM)\) (as we shall always do), \(V_\tau\) is the orthogonal complement of \(\mathbb{R}r\) in \(\Lambda^2_+ T_{\pi(\tau)} M\). The map \(V_\tau \ni V \mapsto K_V\) gives an identification of the vertical space with the space of skew-symmetric endomorphisms of \(T_{\pi(\tau)} M\) that anti-commute with \(K_\tau\). Let \(s\) be a local section of \(Z_+\) such that \(s(p) = \tau\) where \(p = \pi(\tau)\). Considering \(s\) as a section of \(\Lambda^2_+ TM\), we have \(\nabla_X s \in V_\tau\) for every \(X \in T_p M\) since \(s\) has a constant length. Moreover, \(X^i_\tau = s_* X - \nabla_X s\) is the horizontal lift of \(X\) at \(\tau\).

Denote by \(\times\) the usual vector cross product on the oriented 3-dimensional vector space \(\Lambda^2_+ T_p M, p \in M\), endowed with the metric \(g\). Then it is easy to check that

\[
g(R(a)b, c) = g(R(b \times c), a)
\]
for \( a \in \Lambda^2 T_p M, \; b, c \in \Lambda^2_r T_p M \). It is also easy to show that for every \( a, b \in \Lambda^2 T_p M \)
\[ K_a \circ K_b = -g(a, b)Id + K_{a \wedge b}. \tag{4} \]

For every \( t > 0 \), define a Riemannian metric \( h_t \) by
\[ h_t(X^b + V, Y^b + W) = g(X, Y) + tg(V, W) \]
for \( \sigma \in \mathbb{Z}_+, \; X, Y \in T_{\pi(\sigma)}M, \; V, W \in \mathcal{V}_\sigma \).

The twistor space \( \mathcal{Z}_+ \) admits two natural almost complex structures that are compatible with the metrics \( h_t \). One of them has been introduced by Atiyah, Hitchin and Singer who have proved that it is integrable if and only if the base manifold is anti-self-dual [1]. The other one, introduced by Eells and Salamon, although never integrable, plays an important role in harmonic maps theory [10].

The action of \( SO(4) \) on \( \Lambda^2 \mathbb{R}^4 \) preserves the decomposition \( \Lambda^2 \mathbb{R}^4 = \Lambda^2_+ \mathbb{R}^4 \oplus \Lambda^2_- \mathbb{R}^4 \). Thus, considering \( S^2 \) as the unit sphere in \( \Lambda^2_+ \mathbb{R}^4 \), we have an action of the group \( SO(4) \) on \( S^2 \). Then, if \( SO(M) \) denotes the principal bundle of the oriented orthonormal frames on \( M \), the twistor space \( \mathcal{Z}_+ = Z_+(M) \) is the associated bundle \( SO(M) \times_{SO(4)} S^2 \). It follows from the Vilms theorem (see, for example, [3, Theorem 9.59]) that the projection map \( \pi : (\mathcal{Z}_+ , h_t) \rightarrow (M, g) \) is a Riemannian submersion with totally geodesic fibres (this can also be proved by a direct computation).

Denote by \( D \) the Levi-Chivita connection of \((\mathcal{Z}_+, h_t)\).

Let \((N, x_1, ..., x_4)\) be a local coordinate system of \( M \) and let \((E_1, ..., E_4)\) be an oriented orthonormal frame of \( TM \) on \( N \). If \((s_1^+, s_2^+, s_3^+)\) is the local frame of \( \Lambda^2_+ TM \) defined by (1), then \( \bar{x}_a = x_a \circ \pi, \; y_j(\tau) = g(\tau, (s_j^+ \circ \pi)(\tau)), \; 1 \leq a \leq 4, \; 1 \leq j \leq 3, \) are local coordinates of \( \Lambda^2_+ TM \) on \( \pi^{-1}(N) \).

The horizontal lift \( X^h \) on \( \pi^{-1}(N) \) of a vector field
\[ X = \sum_{a=1}^{4} X^a \frac{\partial}{\partial x_a} \]
is given by
\[ X^h = \sum_{a=1}^{4} (X^a \circ \pi) \frac{\partial}{\partial \bar{x}_a} - \sum_{j,k=1}^{3} y_j(g(\nabla X s_j, s_k) \circ \pi) \frac{\partial}{\partial y_k}. \tag{5} \]
Hence
\[ [X^h, Y^h] = [X, Y]^h + \sum_{j,k=1}^{3} y_j(g(R(X \wedge Y)s_j, s_k) \circ \pi) \frac{\partial}{\partial y_k} \tag{6} \]
for every vector fields \( X, Y \) on \( N \). Let \( \tau \in \mathcal{Z}_+ \). Using the standard identification \( T_\tau(\Lambda^2_+ T_p M) \equiv \Lambda^2_+ T_{\pi(\tau)} M \) we obtain from (6) the well-known formula
\[ [X^h, Y^h]_\tau = [X, Y]_\tau^h + R_p(X \wedge Y)_\tau, \; p = \pi(\tau). \tag{7} \]

Then we have the following

**Lemma 1.** ([7]) If \( X, Y \) are (local) vector fields on \( M \) and \( V \) is a vertical vector field on \( \mathcal{Z}_+ \), then
\[ (D_X Y^h)_\tau = (\nabla_X Y)_\tau^h + \frac{1}{2} R_p(X \wedge Y)_\tau, \tag{8} \]
By (4), the points of $\Sigma_J$ of the twistor space defined by $\alpha$ are compatible with the metric and the orientation, and anti-commute with $M$.

Thus, at any point of $\tau \in Z_+$, $\pi(\tau)$, and $\mathcal{H}$ means "the horizontal component".

**Proof.** Identity (8) follows from the Koszul formula for the Levi-Chivita connection and (7).

Let $W$ be a vertical vector field on $Z_+$. Then

$$h_t(D_t X^h W) = -h_t(X^h D_t W) = 0$$

since the fibres are totally geodesic submanifolds, so $D_t W$ is a vertical vector field. Therefore $D_t X^h$ is a horizontal vector field. Moreover, $[V, X^h]$ is a vertical vector field, hence $D_t X^h = \mathcal{H}D_t X^h V$. Thus

$$h_t(D_t X^h Y^h) = h_t(D_t X^h V, Y^h) = -h_t(V, D_t X^h Y^h).$$

Now (9) follows from (8) and (3).

3. A hypersurface in $Z_+$ determined by an almost Hermitian structure on $M$

Let $(g, J)$ be an almost Hermitian structure on a four-manifold $M$. Define a section $\alpha$ of $\Lambda^2 T \pi \mathcal{H}$ by

$$g(\alpha, X \wedge Y) = \frac{1}{2} g(JX, Y), \quad X, Y \in TM.$$ 

Thus, at any point of $M$, $\alpha$ is the dual 2-vector of one half of the Kähler 2-form of the almost Hermitian manifold $(M, g, J)$. Note also that $K_{\alpha_p} = J_p$ for every $p \in M$.

Consider $M$ with the orientation yielded by the almost complex structure $J$. Then $\alpha$ is a section of the twistor bundle $Z_+$. This section determines a hypersurface of the twistor space defined by

$$\Sigma_J = \{ \sigma \in Z_+ : g(\sigma, \alpha_{\pi(\sigma)}) = 0 \}.$$ 

By (4), the points of $\Sigma_J$ are complex structures on the tangent spaces of $M$ that are compatible with the metric and the orientation, and anti-commute with $J$.

Clearly, $\Sigma_J$ is the circle bundle of the rank 2 vector bundle $\Lambda^2_{\pi 0} = \{ \sigma \in \Lambda^2 \pi \mathcal{H} : g(\sigma, \alpha_{\pi(\sigma)}) = 0 \}$.

As is well-known (and easy to see), the complexification of this bundle is the bundle $\Lambda^{2,0}_0 \oplus \Lambda^{0,2}_0$ where $\Lambda^{r,s}$ stands for the bundle of $(r+s)$-vectors of type $(r,s)$ with respect to $J$.

We shall compute the second fundamental form $\Pi$ of the hypersurface $\Sigma_J$ in $(Z_+, h_t)$.

Note that for $\sigma \in \Sigma_J$

$$T_{\sigma} \Sigma_J = \{ E \in T_{\sigma} Z_+ : g(\nabla E, \alpha_{\pi(\sigma)}) = -g(\sigma, \nabla E, \alpha) \}$$

where $\nabla E$ means "the vertical component of $E$". Therefore

$$T_{\sigma} \Sigma_J = \{ X^h + g(\sigma, \nabla X) \alpha_{\pi(\sigma)} : X \in T_{\pi(\sigma)} M \} \oplus \mathbb{R}(\alpha_{\pi(\sigma)} \times \sigma).$$

Given $\tau \in Z$ and $X \in T_{\pi(\tau)} M$, define a vertical vector of $Z_+$ at $\tau$ by

$$X^v = -g(\tau, \nabla X) \alpha_{\pi(\tau)} + g(\tau, \alpha_{\pi(\tau)}) \nabla X \alpha.$$ 

Set

$$\hat{X}_\tau = X^h + X^v.$$
Thus every (local) vector field $X$ on $M$, gives rise to a vector field $\hat{X}$ on $\mathcal{Z}_+$ tangent to $\Sigma_J$.

Let $\rho(\tau) = g(\tau, \alpha_{\pi(\tau)})$, $\tau \in \mathcal{Z}_+$, be the defining function of $\Sigma_J$ and let $\text{grad} \rho$ be the gradient vector field of the function $\rho$ with respect to the metric $h_t$. Fix a point $\tau \in \mathcal{Z}_+$ and take a section $s$ of $\mathcal{Z}_+$ such that $s_{\pi(\tau)} = \tau$, $\nabla s|_{\pi(\tau)} = 0$. Then, for $X \in T_{\pi(\tau)}M$,

$$h_t(X^h, \text{grad} \rho) = s_*(X)(\rho) = X(g(s, \alpha)) = g(\tau, \nabla_X \alpha). \quad (10)$$

Moreover, if $V \in \mathcal{V}_\tau$,

$$h_t(V, \text{grad} \rho) = V(\sum_{k=1}^{3} y_k(g(s_k, \alpha) \circ \pi)) = \sum_{k=1}^{3} V(y_k)g(s_k, \alpha)|_{\pi(\tau)} = g(V, \alpha_{\pi(\tau)}). \quad (11)$$

**Lemma 2.** If $\sigma \in \Sigma_J$ and $X, Y \in T_{\pi(\sigma)}M$, then

$$h_t(\Pi(\hat{X}, \hat{Y}), \text{grad} \rho)_\sigma = \frac{t}{2}[g(\sigma, \nabla_X \alpha)g(\sigma, \nabla_{R(\sigma \times \alpha_{\pi(\sigma)})}Y \cdot \alpha)$$

$$- g(\sigma, \nabla_Y \alpha)g(\sigma, \nabla_{R(\sigma \times \alpha_{\pi(\sigma)})}X \cdot \alpha)]$$

$$- \frac{1}{2}g(\sigma, \nabla_X^2 \alpha) - \frac{1}{2}g(\sigma, \nabla_Y^2 \alpha)$$

where $\nabla_X^2 \alpha = \nabla_X \nabla_Y \alpha - \nabla_{\nabla_X Y} \alpha$ is the second covariant derivative of $\alpha$.

**Proof.** Extend $X$ and $Y$ to vector fields in a neighbourhood of the point $p = \pi(\sigma)$. It follows from (8), (10) and (11) that

$$h_t(D_X h^h Y^h, \text{grad} \rho)_\sigma = g(\nabla_X Y \cdot \alpha, \sigma) + \frac{1}{2}g(R(X \wedge Y)|\sigma, \alpha_p). \quad (12)$$

Identities (9) and (10) imply

$$h_t(D_X h^h Y^h, \text{grad} \rho)_\sigma = \frac{t}{2}g(\sigma, \nabla_X \alpha)g(\sigma, \nabla_{R(\sigma \times \alpha_p)}Y \alpha). \quad (13)$$

Next, note that

$$h_t(D_X h^h Y^h, \text{grad} \rho) = h_t([(X^h, Y^h), \text{grad} \rho]) + h_t(D_Y h^h X^h, \text{grad} \rho).$$

Take an oriented orthonormal frame $(E_1, ..., E_4)$ of $M$ near $p$ such that $\nabla E_a|_p = 0$, $a = 1, ..., 4$. Then $\nabla s_i^+|_p = 0$, $i = 1, 2, 3$, which implies

$$X^h = \sum_{a=1}^{4} X^a(p) \frac{\partial}{\partial x_a}(\sigma), \quad [X^h, \frac{\partial}{\partial y_i}]_\sigma = 0, \quad i = 1, 2, 3. \quad (14)$$

We have

$$Y^v = \sum_{j,k=1}^{3} y_j(g(s_j^+, \alpha)g(\nabla_Y \alpha, s_j^+) - g(s_j^+, \alpha)g(\nabla_Y \alpha, s_j^+)) \circ \pi \frac{\partial}{\partial y_j}. \quad (15)$$

It follows from (14) and (15) that

$$[X^h, Y^v]_\sigma = g(\sigma, \nabla_X \alpha)\nabla_Y \alpha - g(\sigma, \nabla_Y \alpha)\nabla_X \alpha - g(\sigma, \nabla_X \nabla_Y \alpha)\alpha_p.$$

Hence, by (11),

$$h_t([X^h, Y^v], \text{grad} \rho)_\sigma = -g(\sigma, \nabla_X \nabla_Y \alpha).$$
Thus we have
\[ h_t(D_Xv, \nabla Xv) = -g(\alpha, \nabla X\alpha) + t \frac{1}{2} g(\alpha, \nabla^2 X\alpha). \] (16)

The fibres of of \( Z_+ \) are totally geodesic submanifolds, hence \( (D_Xv)_{\alpha} \) is the standard covariant derivative on the unit sphere in the vector space \( \mathbb{R}^2 T_p M \). It follows from (15) that
\[ (D_Xv)_{\alpha} = g(X, \alpha) \nabla Y v - g(\nabla Y v, \alpha) - g(X, \alpha) g(\alpha, \nabla R(\alpha \times \alpha_\rho) X\alpha). \]

Hence
\[ h_t(D_Xv, \nabla Xv) = 0. \] (17)

Now the lemma follows from identities (12), (13), (16), and (17).

If \( \sigma \in \Sigma_J \), the vertical part of \( T_{\alpha}\Sigma_J \) is \( \mathbb{R}(\alpha, \alpha_\rho) \times \sigma \). Define a vertical vector field \( \xi \) on \( Z_+ \) tangent to \( \Sigma_J \) setting
\[ \xi_\tau = \alpha_\tau \times \tau, \quad \tau \in Z_+. \]

**Lemma 3.** If \( \sigma \in \Sigma_J \) and \( X \in T_{\alpha\Sigma_J} M \), then
\[ h_t(\Pi(\xi, X), \nabla X) = -g(\xi_\sigma, \nabla X\alpha) - t \frac{1}{2} g(\alpha, \nabla R(\alpha \times \alpha_\rho) X\alpha). \]
\[ h_t(\Pi(\xi, \xi), \nabla X) = 0. \]

**Proof.** Identity (9) implies
\[ h_t(D_Xh, \nabla X) = -t \frac{1}{2} g(\alpha, \nabla R(\alpha \times \alpha_\rho) X\alpha). \] (18)

A simple computation gives
\[ (D_X \xi)_{\alpha} = -g(\xi_\sigma, \nabla X\alpha) \alpha_\sigma, \quad (D_X \xi)_{\sigma} = 0. \]

Hence
\[ h_t(D_X \xi, \nabla X\alpha), \quad h_t(D_X \xi, \nabla X\alpha) = 0. \] (19)

Thus the result follows from (18) and (19).

**Proposition 1.** Let \( \sigma \in \Sigma_J \) and \( E, F \in T_{\alpha\Sigma_J} M \). Set \( X = \pi_\sigma E, Y = \pi_\sigma F, V = \nabla E, W = \nabla F \). Then
\[ h_t(\Pi(E, F), \nabla X) = \]
\[ + t \frac{1}{2} g(\alpha, \nabla X\alpha) \nabla R(\alpha \times \alpha_\rho) X\alpha + t \frac{1}{2} g(\alpha, \nabla X\alpha) g(\alpha, \nabla R(\alpha \times \alpha_\rho) X\alpha) \]
\[ -t \frac{1}{2} g(\alpha, \nabla^2 X\alpha) - t \frac{1}{2} g(\alpha, \nabla^2 X\alpha) \]
\[ + t \frac{1}{2} g(\alpha_\tau \times \alpha_\rho) \nabla R(\alpha_\sigma X\alpha) + t \frac{1}{2} g(\alpha_\tau \times \alpha_\rho) \nabla R(\alpha_\sigma X\alpha) \]
\[ -g(V, \nabla \alpha) - g(W, \nabla \alpha). \]

**Proof.** This follows from Lemmas 2 and 3 taking into account that \( E = \tilde{X}_\sigma + g(V, \xi_\sigma) \xi_\sigma, F = \tilde{Y}_\sigma + g(W, \xi_\sigma) \xi_\sigma \).

**Corollary 1.** If \( (M, g, J) \) is Kähler, \( \Sigma_J \) is a totally geodesic submanifold of \( Z_+ \).
4. Minimality of the hypersurface \( \Sigma_J \)

Let \( \Omega(X, Y) = g(JX, Y) \) be the fundamental 2-form of the almost Hermitian manifold \( (M, g, J) \). Denote by \( N \) the Nijenhuis tensor of \( J \),

\[
N(Y, Z) = -[Y, Z] + JY \cdot JZ - J[JY, Z].
\]

It is well-known (and easy to check) that

\[
2g((\nabla_X J)(Y), Z) = d\Omega(X, Y, Z) - d\Omega(X, JY, JZ) + g(N(Y, Z), JX).
\]

(20)

4.1. The case of integrable \( J \). Suppose that the almost complex structure \( J \) is integrable. Note that the integrability condition for \( J \) is equivalent to

\[
(\nabla_X J)(Y) = (\nabla_JX)(JY), \quad X, Y \in TM \quad [12, Corollary 4.2].
\]

Let \( B \) be the vector field on \( M \) dual to the Lee form \( \theta = -\delta \Omega \circ J \) with respect to the metric \( g \). Then (20) and the identity \( d\Omega = \Omega \wedge \theta \) imply the following well-known formula

\[
2(\nabla_X J)(Y) = g(JX, Y)B - g(B, Y)JX + g(X, Y)JB - g(JB, Y)X.
\]

(21)

We have

\[
2g((\nabla_X \alpha, Y \wedge Z) = \frac{1}{2}g((\nabla_X J)(Y), Z)
\]

and it follows that

\[
\nabla_X \alpha = \frac{1}{2}(JX \wedge B + X \wedge JB).
\]

(22)

The latter identity implies

\[
\nabla^2_{XY} \alpha = \frac{1}{2}[(\nabla_X J)(Y) \wedge B + Y \wedge (\nabla_X J)(B) + JY \wedge \nabla_X B + Y \wedge J\nabla_X B].
\]

(23)

Let \( \sigma \in \Sigma_J \) and \( X, Y \in T_{\pi(\sigma)}M \). Then a simple computation using identities (2), (4), (21) - (23) gives

\[
g(\sigma, \nabla_X \alpha) = \frac{1}{2}g(X, K_{\xi_\sigma}B),
\]

\[
4g(\sigma, \nabla^2_{XY} \alpha) = -g(K_{\xi_\sigma}B \wedge B, X \wedge Y + JX \wedge JY) - g(JX, B)g(JY, K_{\xi_\sigma}B)
\]

\[
+ \frac{1}{2}|B|^2g(X, K_{\xi_\sigma}Y) - 2g(\nabla_X B, K_{\xi_\sigma}Y)
\]

where, as above, \( \xi_\sigma = \alpha_{\pi(\sigma)} \times \sigma \). Moreover, if \( V \in T_{\sigma} \Sigma_J \) is a vertical vector,

\[
g(\alpha_{\pi(\sigma)} \times V, \nabla_X \alpha) = -\frac{1}{2}g(V, \xi_\sigma)g(\sigma, JX \wedge B + X \wedge JB) = -\frac{1}{2}g(V, \xi_\sigma)g(X, K_{\xi_\sigma}B),
\]

\[
g(V, \nabla_X \alpha) = -\frac{1}{2}g(V, \xi_\sigma)g(X, K_{\xi_\sigma}B).
\]

Now Proposition 1 can be rewritten as

**Proposition 2.** Suppose that the almost complex structure \( J \) is integrable. Let \( \sigma \in \Sigma_J \) and \( E, F \in T_{\sigma} \Sigma_J \). Set \( X = \pi_\ast E, Y = \pi_\ast F, V = \nabla E, W = \nabla F \) and
\[ \xi_\sigma = \alpha_{\pi(\sigma)} \times \sigma. \]

Then

\[ h_t(\Pi(E, F), \text{grad } \rho)_\sigma = \]

\[ -\frac{t}{8} g(X, K_{\xi_\sigma} B) g(Y, R(\xi_\sigma) K_{\xi_\sigma} B) - \frac{t}{8} g(Y, K_{\xi_\sigma} B) g(X, R(\xi_\sigma) K_{\xi_\sigma} B) \]

\[ + \frac{1}{8} g(JX, B) g(JY, K_{\xi_\sigma} B) + \frac{1}{8} g(JY, B) g(JX, K_{\xi_\sigma} B) \]

\[ + \frac{1}{4} g(\nabla X, B, K_{\xi_\sigma} Y) + \frac{1}{4} g(\nabla Y, B, K_{\xi_\sigma} X) \]

\[ - \frac{1}{4} g(V, \xi_\sigma) g(R(\alpha_{\pi(\sigma)}) Y, K_{\xi_\sigma} B) - \frac{1}{4} g(W, \xi_\sigma) g(R(\alpha_{\pi(\sigma)}) X, K_{\xi_\sigma} B) \]

\[ + \frac{1}{4} g(V, \xi_\sigma) g(X, K_{\sigma} B) + \frac{1}{4} g(W, \xi_\sigma) g(Y, K_{\sigma} B) \]

**Corollary 2.** Let \( \sigma \in \Sigma_J \).

Then

\[ h_t(\text{Trace } \Pi, \text{grad } \rho)_\sigma = \frac{1}{2} (d\theta + \theta \wedge d\ln \sqrt{8 + 2t ||\theta_{\pi(\sigma)}||^2}) (\alpha_{\pi(\sigma)} \times \sigma). \]

**Proof.** Set \( p = \pi(\sigma) \). Suppose first that \( B_p \neq 0 \). Then \( E_1 = ||B_p||^{-1} B_p, \ E_2 = K_{\alpha_p} E_1, E_3 = K_j E_1, E_4 = K_{\xi_\sigma} E_1 \) form an oriented orthonormal basis of \( T_p M \) such that \( \alpha_p = (s_1)_p^+, \ \xi_\sigma = (s_3)_p^+ \) where \( s_1^+, s_2^+, s_3^+ \) is the basis of \( \Lambda^2 T_p M \) defined by means of \( E_1, ..., E_4 \) via (1). We have

\[ g(\sigma, \nabla X \alpha) = \frac{1}{2} g(X, K_{\xi_\sigma} B) = \frac{1}{2} ||B_p|| g(X, E_4), \quad X \in T_p M. \]

Hence \( \widetilde{(E_1)}_\sigma = (E_1)_\sigma^b \) for \( i = 1, 2, 3 \) and \( \widetilde{(E_4)}_\sigma = (E_4)_\sigma^b - \frac{1}{2} ||B_p|| \alpha_p \). Thus \( \widetilde{(E_1)}_\sigma^b, \ i = 1, 2, 3, \) \( (1 + \frac{t}{4} ||B_p||^2)^{-\frac{1}{2}} \widetilde{(E_4)}_\sigma^b \), \( \frac{1}{\sqrt{1} \xi_\sigma} \) constitute an orthonormal basis of \( T_\sigma \Sigma_J \).

Note that

\[ g(E_4, R(\xi_\sigma) K_{\xi_\sigma} B) = ||B_p||^{-1} g(K_{\xi_\sigma} B, R(\xi_\sigma) K_{\xi_\sigma} B) = 0 \]

and

\[ g(JE_4, B) = ||B_p||^{-1} g(K_{\alpha_p} \circ K_{\xi_\sigma} B, B) = -||B_p||^{-1} g(K_{\xi_\sigma} B, B) = 0. \]

Then, by Proposition 2,

\[ h_t(\text{Trace } \Pi, \text{grad } \rho)_\sigma = -\frac{1}{4} g(K_{\xi_\sigma} B, R(\xi_\sigma) K_{\xi_\sigma} B) + \frac{1}{4} g(B, K_{\xi_\sigma} B) \]

\[ + \frac{1}{2} \sum_{i=1}^{3} g(\nabla E_i, B, K_{\xi_\sigma} E_i) + \frac{1}{2} (1 + \frac{t}{4} ||B_p||^2)^{-1} g(\nabla E_i, B, K_{\xi_\sigma} E_i) \]

\[ = \frac{1}{2} \sum_{j=1}^{4} g(\nabla E_j, B, K_{\xi_\sigma} E_j) + \frac{t}{8} ||B_p||^2 g(\nabla E_4, B, B) \]

\[ = \frac{1}{2} d\theta(E_1 \wedge E_4 + E_2 \wedge E_3) + \frac{t}{16} + 4t ||B_p||^2 \theta \wedge d(\theta)^2 (E_1 \wedge E_4 + E_2 \wedge E_3) \]

If \( B_p = 0 \), then \( \nabla X \alpha = 0 \) for every \( X \in T_p M \) by (22). Taking a unit vector \( E_1 \in T_p M \) we set \( E_2 = K_{\alpha_p} E_1, E_3 = K_j E_1, E_4 = K_{\xi_\sigma} E_1 \). Then \( \widetilde{(E_j)}_\sigma^b = (E_j)_\sigma^b, \ j = 1, ..., 4, \) \( \frac{1}{\sqrt{1} \xi_\sigma} \) constitute an orthonormal basis of \( T_\sigma \Sigma_J \). It follows from
Proposition 2 that
\[ h_t(\text{Trace } \Pi, \text{grad } \rho) = \frac{1}{2} \sum_{j=1}^{4} g(\nabla E_j B, K_{\xi_j} E_j) = \frac{1}{2} \delta(h_1 E_1 \wedge E_4 + E_2 \wedge E_3). \]

**Proposition 3.** If \( J \) is integrable, the hypersurface \( \Sigma_J \) is a minimal submanifold of \((\mathcal{Z}_+ h_t)\) if and only if the 2-form
\[ d\frac{\theta}{\sqrt{8 + 2t||\theta||^2}} \]
is of type \((1, 1)\) with respect to \( J \).

**Proof.** The condition that \( \Sigma_J \) is a minimal submanifold means that \( h_t(\text{Trace } \Pi, \text{grad } \rho) = 0 \) on \( \Sigma_J \).

Let \( p \in M \) and take an orthonormal basis of \( T_p M \) of the form \( E_1, E_2 = J E_1, E_3, E_4 = J E_3 \). Then \( \alpha_p = (s_1)_p^+ \), so \( (s_2)_p^+, (s_3)_p^+ \in \Sigma_J \).

It is easy to check that, for every \( a \in \Lambda^2 T_p M \) and \( b \in \Lambda^2 T_p M \), the endomorphisms \( K_a \) and \( K_b \) of \( T_p M \) commute. It follows that, for every \( X, Y \in T_p M \), the 2-vector \( X \wedge Y - JX \wedge JY \) is orthogonal to \( \Lambda^2 T_p M \), therefore it lies in \( \Lambda^2 T_p M \). Moreover, \( X \wedge Y - JX \wedge JY \) is orthogonal to \( \alpha_p \), hence is a linear combination of \( (s_2)_p^+ = -\alpha_p \wedge (s_3)_p^+ \) and \( (s_3)_p^+ = \alpha_p \wedge (s_2)_p^+ \). Thus if \( h_t(\text{Trace } \Pi, \text{grad } \rho) = 0 \) on \( \Sigma_J \),
\[
(d\theta + \theta \wedge d \ln \sqrt{8 + 2t||\theta||^2})(X \wedge Y - JX \wedge JY) = 0.
\]
Conversely, if this identity holds, then \( h_t(\text{Trace } \Pi, \text{grad } \rho) = 0 \) at the points \( (s_2)_p^+ \) and \( (s_3)_p^+ \) of \( \Sigma_J \). For every \( \sigma \in \Sigma_J \) with \( \pi(\sigma) = p \), the 2-vector \( \alpha_p \wedge \sigma \) is a linear combination of \( (s_2)_p^+ \) and \( (s_3)_p^+ \), hence \( h_t(\text{Trace } \Pi, \text{grad } \rho) = 0 \) on \( \Sigma_J \).

Thus, \( \Sigma_J \) is minimal if and only if the form \( d\theta + \theta \wedge d \ln \sqrt{8 + 2t||\theta||^2} \) is of type \((1, 1)\). But the this condition is equivalent to the condition that the form
\[ \frac{\theta}{\sqrt{8 + 2t||\theta||^2}} \]
is of type \((1, 1)\).

**4.2. The case of symplectic \( J \).** Now suppose that \( d\Omega = 0 \). Then, by (20),
\[ g(\nabla_X J)(Y), Z) = \frac{1}{2} g(N(Y, Z), JX). \]
The Nijenhuis tensor \( N(Y, Z) \) is skew-symmetric, so it induces a linear map \( \Lambda^2 TM \rightarrow TM \) which we denote again by \( N \). The identity
\[ g(\nabla_X \alpha, Y \wedge Z) = \frac{1}{4} g(N(Y, Z), JX) \]  \hspace{1cm} (24)
implies that, for every \( a \in \Lambda^2 TM \) and \( X \in T_{\pi(a)} M \),
\[ g(\nabla_X \alpha, a) = \frac{1}{4} g(N(a), JX). \]
In particular, if \( V \in \mathcal{V}_\sigma \), then
\[ g(\nabla_X \alpha, V) = \frac{1}{4} g(V, \xi_\sigma) g(N(\xi_\sigma), JX). \]
Proposition 1 implies the following.
Proposition 4. Suppose that $d\Omega = 0$. Let $\sigma \in \Sigma_J$ and $E, F \in T_\sigma \Sigma_J$. Set $X = \pi_* E, Y = \pi_* F, V = \nabla E, W = \nabla F$ and $\xi _\sigma = \alpha_{\pi(\sigma)} \times \sigma$. Then

$$h_t(\Pi(E, F), \nabla \rho)_\sigma = \frac{t}{32} g(JN(\sigma), X)g(JN(\sigma), R(\xi _\sigma)Y) + \frac{t}{32} g(JN(\sigma), Y)g(JN(\sigma), R(\xi _\sigma)X)$$

$$- \frac{1}{2} g(\sigma, \nabla^2_X Y) - \frac{1}{2} g(\sigma, \nabla^2_Y X)$$

$$+ \frac{t}{8} g(N(\alpha_{\pi(\sigma)} \times V), JR(\alpha_{\pi(\sigma)})Y) + \frac{t}{8} g(N(\alpha_{\pi(\sigma)} \times W), JR(\alpha_{\pi(\sigma)})X)$$

$$- \frac{1}{4} g(V, \xi _\sigma)g(N(\xi _\sigma), JX) - \frac{1}{4} g(W, \xi _\sigma)g(N(\xi _\sigma), JY).$$

Corollary 3. Let $\sigma \in \Sigma_J$. Then

$$h_t(\text{Trace } \Pi, \nabla \rho)_\sigma = -g(\text{trace } \nabla^2 \alpha, \sigma).$$

Proof. Suppose first that $N(\sigma) \neq 0$. Take an orthonormal basis of $T_{\pi(\sigma)} M$ of the form $E_1, E_2 = JE_1, E_3 = ||N(\sigma)||^{-1} N(\sigma), E_4 = JE_3$. Then, by (24),

$$\widehat{(E_k)}_{\sigma} = (E_k)^B + \frac{1}{4} ||N(\sigma)|| g(E_k, E_k) \alpha_{\pi(\sigma)},$$

$k = 1, \ldots, 4$. Thus $\widehat{(E_i)}_{\sigma}, i = 1, 2, 3, (1 + \frac{t}{16} ||N(\sigma)||^2)^{-\frac{1}{2}} \widehat{(E_4)}_{\sigma}, \frac{1}{\sqrt{t}} \xi _\sigma$ form an orthonormal basis of $T_\sigma \Sigma_J$. Note also that

$$g(JN(\sigma), R(\xi _\sigma)E_4) = ||N(\sigma)|| g(E_4, R(\xi _\sigma)E_4) = 0.$$

Then Proposition 4 implies

$$h_t(\text{Trace } \Pi, \nabla \rho)_\sigma = -\frac{t}{16} g(JN(\sigma), R(\xi _\sigma)JN(\sigma)) + g(\text{trace } \nabla^2 \alpha, \sigma)$$

$$- g(\text{trace } \nabla^2 \alpha, \sigma).$$

If $N(\sigma) = 0$, then, in view of (24), $g(\nabla_X \alpha, \sigma) = 0$. Thus $\widehat{(E_k)}_{\sigma} = (E_k)^B$ for any orthonormal basis $E_k$ of $T_{\pi(\sigma)} M, k = 1, \ldots, 4$, and the result is a direct consequence of Proposition 4.

Denote by $\rho^*$ the *-Ricci tensor of the almost Hermitian manifold $(M, g, J)$. Recall that it is defined as $\rho^*(X, Y) = \text{trace} \{ Z \rightarrow R(JZ, X)JY \}$. Note that

$$\rho^*(JX, JY) = \rho^*(Y, X),$$

(25)
in particular $\rho^*(X, JX) = 0$.

Proposition 5. If $d\Omega = 0$, then the hypersurface $\Sigma_J$ is a minimal submanifold of $(Z_{+}, h_t)$ if and only if the tensor $\rho^*$ is symmetric.

Proof. The form $\Omega$ is harmonic since $d\Omega = 0$ and $*\Omega = \Omega$. Then, by Corollary 3 and the Weitzenböck formula, $\Sigma_J$ is minimal if and only if, for every 2-form $\tau \in \Lambda^2_+ T^* M$ orthogonal to $\Omega$, $g(S(\Omega), \tau) = 0$ where

$$S(\Omega)(X, Y) = \text{trace} \{ Z \rightarrow (R(Z, Y)\Omega)(Z, X) - (R(Z, X)\Omega)(Z, Y) \}$$

(see, for example, [9]). We have

$$(R(Z, Y)\Omega)(Z, X) = -\Omega(R(Z, Y)Z, X) - \Omega(Z, R(Z, Y)X)$$

$$= g(R(Z, Y)Z, JX) + g(R(Z, X)Y, JZ).$$
Hence
\[ S(\Omega)(X, Y) = \text{Ricci}(Y, JX) - \text{Ricci}(X, JY) + 2\rho^*(X, JY). \]
By (25), in order to show that \( \rho^*(X, Y) = \rho^*(Y, X) \) for every \( X, Y \), it is enough to check that \( \rho^*(X, Y) = \rho^*(Y, X) \) for all unit vectors \( X, Y \in TM \) with \( g(X, Y) = g(X, JY) = 0 \). If \( X, Y \) are such vectors, \( E_1 = X, E_2 = JE_1, E_3 = Y, E_4 = JY \) is an orthonormal basis and the condition \( g(S(\Omega), \tau) = 0 \) is equivalent to
\[ S(\Omega)(E_1, E_3) + S(\Omega)(E_4, E_2) = 0, \quad S(\Omega)(E_1, E_4) + S(\Omega)(E_2, E_3) = 0. \] (26)
These identities are equivalent to \( \rho^*(E_1, E_4) = \rho^*(E_4, E_1) \) and \( \rho^*(E_1, E_3) = \rho^*(E_2, E_4) \) where \( \rho^*(E_2, E_4) = \rho^*(JE_1, JE_3) = \rho^*(E_3, E_1) \). Taking into account (25) we see that (26) is equivalent to \( \rho^*(X, Y) = \rho^*(Y, X) \).

5. Examples

5.1. Generalized Hopf surfaces. Clearly, if \( M \) is locally conformally Kähler \( (d\theta = 0) \) and the Lee form \( \theta \) has constant length, the hypersurface \( \Sigma_J \) is minimal. We have \( ||\theta|| \equiv \text{const} \) on every homogeneous locally conformally Kähler manifold. Also, if \( \theta \) is parallel, then \( d\theta = 0 \) and \( ||\theta|| \equiv \text{const} \). Recall that a Hermitian surface with parallel Lee form is called a generalized Hopf surface [18] (or a Vaisman surface [8]); we refer to [8, 18] for basic properties and examples of such surfaces. The product of a Sasakian 3-manifold with \( \mathbb{R} \) or \( S^3 \) admits a structure of generalized Hopf surface in a natural way. Conversely, every such a surface locally is the product of a Sasakian manifold and \( \mathbb{R} \) [18] (cf. also [11]). A global structure theorem for compact generalized Hopf manifolds is obtained in [16].

As it is shown in [17], certain Inoue surfaces admit locally conformally Kähler structures with \( ||\theta|| \equiv \text{const} \) and non-parallel Lee form \( \theta \).

Fix two complex numbers \( \alpha \) and \( \beta \) such that \( |\alpha| \geq |\beta| > 1 \). Let \( \Gamma_{\alpha,\beta} \) be the group of transformation of \( \mathbb{C}^2 \setminus \{0\} \) generated by the transformation \( (u, v) \to (\alpha u, \beta v) \). Then, by a result of [11], the quotient \( M_{\alpha,\beta} = (\mathbb{C}^2 \setminus \{0\})/\Gamma_{\alpha,\beta} \) admits a structure of a generalized Hopf surface. Note that \( M_{\alpha,\beta} \) (as any primary Hopf complex surface) is diffeomorphic to \( S^3 \times S^1 \).

The hypersurface \( \Sigma_J \) in the twistor space of \( S^3 \times S^1 \)

We shall consider the Hopf surface \( S^3 \times S^1 \) with its standard complex structure \( J \) and the product metric.

According to [5, Example 5], we have
\[ \mathcal{Z}_+\left(S^3 \times S^1\right) \cong \{ [z_1, z_2, z_3, z_4] \in \mathbb{CP}^3 : |z_1| + |z_2| = |z_3| + |z_4| \} \times S^1. \]
In order to give an explicit description of this isomorphism, we first recall that the twistor space of an odd-dimensional oriented Riemannian manifold \((M, g)\) is the bundle \( \mathcal{C}_+(M) \) over \( M \) whose fibre at a point \( p \in M \) consists of all (linear) contact structures on the tangent space \( T_pM \) compatible with the metric and the orientation, i.e. pairs \( (\varphi, \xi) \) of endomorphism \( \varphi \) of \( T_pM \) and a unit vector \( \xi \in T_pM \) such that \( \varphi^2X = -X + g(X, \xi)\xi \), \( g(\varphi X, \varphi Y) = g(X, Y) - g(X, \xi)g(Y, \xi) \) for \( X, Y \in T_pM \), and the orientation of \( T_pM \) is induced by the orthogonal decomposition \( T_pM = Im \varphi \oplus \mathbb{R} \xi \) where the vector space \( Im \varphi \) is oriented by means of the complex structure \( \varphi/Im \varphi \) on it. We refer to [5, 6] and the references therein for more information about the twistor spaces of odd-dimensional manifolds. The twistor
space $\mathcal{C}_+(M)$ admits a 1-parameter family of Riemannian metrics $h^c_t$ defined in a way similar to the definition of the metrics $h_t$ on the twistor space $\mathcal{Z}_+$.

As is well-known, given $(\varphi, \xi)$ and $a \in S^1$, we can define a complex structure $I$ on $T_pM \times T_aS^1$ in the following way. Denote by $\frac{\partial}{\partial t}$ the vector field on $S^1$ determined by the local coordinate $e^{it} \to t$. Then set $I = \varphi$ on $Im \varphi$, $I\xi = (\frac{\partial}{\partial t})(a)$, $I\varphi = -\xi$. The complex structure $I$ is compatible with the product metric of $M \times S^1$ and its orientation, $S^1$ (as well as any other sphere) being oriented by the inward normal vector field. In this way we have a map

$$F: \mathcal{C}_+(M) \times S^1 \to \mathcal{Z}_+(M \times S^1).$$

Endow $\mathcal{C}_+(M) \times S^1$ with the product metric. It is a simple observation that the map $F$ is a bundle isomorphism preserving the metrics (and having other nice properties) [5, Example 4]. Now we apply this observation to the case when $M = S^3$ and shall define an embedding of $\mathcal{C}_+(S^3)$ into $\mathbb{C}P^3$ as in [6, Examples 2 and 3].

Denote the standard basis of $\mathbb{R}^6$ by $a_1, \ldots, a_6$ and consider $\mathbb{R}^6$ and $\mathbb{R}^4$ as the subspaces $span\{a_1, a_2, a_3\}$ and $span\{a_4, \ldots, a_6\}$. Let $(\varphi, \xi) \in \mathcal{C}_+(S^3)$ with $\varphi \in End(T_pS^3)$ and $\xi \in T_pM$, $p \in S^3$. Then we define a complex structure $J$ on $\mathbb{R}^6$ by means of the orthogonal decomposition

$$\mathbb{R}^6 = Im \varphi \oplus \mathbb{R} \xi \oplus \mathbb{R}\{-p\} \oplus Ra_5 \oplus Ra_6$$

setting $J = \varphi$ on $Im \varphi$, $J\xi = -a_5$, $Jp = -a_6$, $Ja_5 = \xi$, $Ja_6 = p$. In this way we obtain an embedding $\kappa$ of $\mathcal{C}_+(S^3)$ into the space $J_+(\mathbb{R}^6)$ of complex structures on $\mathbb{R}^6$ compatible with the metric and the orientation. The tangent space of $J_+(\mathbb{R}^6)$ at any point $I$ consists of skew-symmetric endomorphisms $Q$ of $\mathbb{R}^6$ anti-commuting with $I$. Denote by $G$ the standard metric $-\frac{1}{2}Tr PQ$ on the space of skew-symmetric endomorphisms. Then $\frac{1}{2}\kappa^*G = h^c_1/2$ [6, Example 2]. It is well-known that $J_+(\mathbb{R}^6)$ and $\mathbb{C}P^3$ are both isomorphic to the twistor space of $S^4$ (see, for example, [20]), so $J_+(\mathbb{R}^6) \cong \mathbb{C}P^3$; for a direct proof see [2, 19]). We shall make use of the biholomorphism that sends a point $[z_1, z_2, z_3, z_4] \in \mathbb{C}P^3$ to the complex structure $J$ of $\mathbb{R}^6$ defined as follows: Let $a_1, \ldots, a_6$ be the standard basis of $\mathbb{R}^6$ and set

$$A_k = \frac{1}{\sqrt{2}}(a_{2k-1} - ia_{2k}), k = 1, 2, 3.$$ 

Then the structure $J$ is given by

$$-|z|^2JA_1 = (|z_1|^2 - |z_2|^2 - |z_3|^2 + |z_4|^2)A_1 + 2\overline{z_2}z_2A_2 + 2\overline{z_1}z_3A_3 + 2\overline{z_4}z_3A_4 - 2\overline{z_4}z_2A_3$$

$$-|z|^2JA_2 = 2\overline{z_2}z_1A_1 + (|z_1|^2 + |z_2|^2 - |z_3|^2 + |z_4|^2)A_2 + 2\overline{z_3}z_3A_3 - 2\overline{z_4}z_3A_1 + 2\overline{z_4}z_1A_3$$

$$-|z|^2JA_3 = 2\overline{z_3}z_1A_1 + 2\overline{z_3}z_2A_2 + (|z_1|^2 - |z_2|^2 + |z_3|^2 + |z_4|^2)A_3 + 2\overline{z_4}z_3A_1 - 2\overline{z_4}z_1A_2,$$

where $z = (z_1, z_2, z_3, z_4)$.

For every $p \in S^3$, denote by $\times$ the vector cross-product on the oriented 3-dimensional Euclidean space $T_pS^3$. If $(\varphi, \xi)$ is a linear contact structure on $T_pM$ compatible with the metric and the orientation, then

$$\varphi(v) = \xi \times v, \quad v \in T_pS^3.$$ 

In particular, $(\varphi, \xi)$ is uniquely determined by $\xi$. Define an oriented orthonormal global frame of the bundle $TS^3$ by

$$\xi_1(p) = (-p_2, p_1, -p_4, p_3), \quad \xi_2(p) = (-p_3, p_4, p_1, -p_2), \quad \xi_3(p) = (-p_4, -p_3, p_2, p_1),$$

for $(p_1, p_2, p_3, p_4) \in S^3$. 

Set \( \varphi_1(v) = \xi_1(p) \times v, v \in T_p S^3 \). Then the standard complex structure \( \mathcal{J} \) of \( S^3 \times S^1 \) corresponds to the section \( (\varphi_1, \xi_1) \times \frac{\partial}{\partial t} \) of \( \mathcal{C}_+ (S^3) \times S^1 \) under the isomorphism \( F : \mathcal{C}_+ (S^3) \times S^1 \rightarrow \mathcal{Z}_+ (S^3 \times S^1) \). We note also that if \( J', J'' \) corresponds to \( (\varphi', \xi'), (\varphi'', \xi'') \) under \( F \), then \( G(J', J'') = g(\xi', \xi'') \). In particular, \( J' \) and \( J'' \) are orthogonal if and only if \( \xi' \) and \( \xi'' \) are so. Let \( (\varphi, \xi) \in \mathcal{C}_+ (S^3) \) and \( \xi \perp \xi_1(p) \), thus \( \xi = \lambda_2 \xi_2(p) + \lambda_3 \xi_3(p) \) where \( \lambda_2^2 + \lambda_3^2 = 1 \). Then the point \( [z_1, \ldots, z_4] \in \mathbb{CP}^3 \) corresponding to \( (\varphi, \xi) \) under the embedding \( \mathcal{C}_+ (S^3) \hookrightarrow J_+ \mathbb{R}^6 \cong \mathbb{CP}^3 \) is given by

\[
\begin{align*}
  z_1 &= \frac{1}{2} - (p_1 + ip_2 - (\lambda_3 - i\lambda_2)(p_3 - ip_4)), \\
  z_2 &= \frac{1}{2}[(\lambda_3 - i\lambda_2)(p_1 - ip_2) - (p_3 + ip_4)], \\
  z_3 &= \frac{1}{2}, \\
  z_4 &= -\frac{1}{2}(\lambda_3 - i\lambda_2).
\end{align*}
\]

In particular, we have \( 4|z_3|^2 = 4|z_4|^2 = |z|^2 \). Conversely, let \( [z_1, \ldots, z_4] \in \mathbb{CP}^3 \) be a point for which \( 4|z_3|^2 = |z|^2 \) and \( 4|z_4|^2 = |z|^2 \). Let \( p_1, \ldots, p_4, \lambda_2, \lambda_3 \) be the real numbers determined by the equations

\[
\begin{align*}
p_1 + ip_2 &= -\frac{2(z_1 \bar{z}_3 + \bar{z}_2 z_4)}{|z|^2}, \\
p_3 + ip_4 &= \frac{2(z_1 z_4 - z_2 \bar{z}_3)}{|z|^2}, \\
\lambda_3 + i\lambda_2 &= -\frac{4z_2 \bar{z}_4}{|z|^2}.
\end{align*}
\]

Then \( p = (p_1, \ldots, p_4) \in \mathbb{R}^3 \), \( \lambda_2^2 + \lambda_3^2 = 1 \) and \( [z_1, \ldots, z_4] \) corresponds under the embedding \( \mathcal{C}_+ (S^3) \hookrightarrow J_+ \mathbb{R}^6 \cong \mathbb{CP}^3 \) to \( (\varphi, \xi) \) determined by \( \xi = \lambda_2 \xi_2(p) + \lambda_3 \xi_3(p) \). It follows that

\[
\Sigma_J \cong \{ [z_1, z_2, z_3, z_4] \in \mathbb{CP}^3 : 4|z_3|^2 = 4|z_4|^2 = |z|^2 \} \times S^1.
\]

5.2. Kodaira surfaces. Recall that every primary Kodaira surface \( M \) can be obtained in the following way [14, p.787]. Let \( \varphi_k(z, w) \) be the affine transformations of \( \mathbb{C}^2 \) given by

\[
\varphi_k(z, w) = (z + a_k, w + \bar{a}_k z + b_k),
\]

where \( a_k, b_k, k = 1, 2, 3, 4 \), are complex numbers such that

\[
a_1 = a_2 = 0, \quad Im(a_3 \bar{a}_4) = mb_1 \neq 0, \quad b_2 \neq 0
\]

for some integer \( m > 0 \). They generate a group \( G \) of transformations acting freely and properly discontinuously on \( \mathbb{C}^2 \), and \( M \) is the quotient space \( \mathbb{C}^2 / G \).

It is well-known that \( M \) can also be described as the quotient of \( \mathbb{C}^2 \) endowed with a group structure by a discrete subgroup \( \Gamma \). The multiplication on \( \mathbb{C}^2 \) is defined by

\[
(a, b), (z, w) = (z + a, w + \bar{a} z + b), \quad (a, b), (z, w) \in \mathbb{C}^2,
\]

and \( \Gamma \) is the subgroup generated by \( (a_k, b_k), k = 1, \ldots, 4 \) (see, for example, [4]).

Further we shall consider \( M \) as the quotient of the group \( \mathbb{C}^2 \) by the discrete subgroup \( \Gamma \). Every left-invariant object on \( \mathbb{C}^2 \) descends to a globally defined object on \( M \) and both of them will be denoted by the same symbol.

We identify \( \mathbb{C}^2 \) with \( \mathbb{R}^4 \) by \( (z = x + iy, w = u + iv) \rightarrow (x, y, u, v) \) and set

\[
A_1 = \frac{\partial}{\partial x} - x \frac{\partial}{\partial u} + y \frac{\partial}{\partial v}, \quad A_2 = \frac{\partial}{\partial y} - y \frac{\partial}{\partial u} - x \frac{\partial}{\partial v}, \quad A_3 = \frac{\partial}{\partial u}, \quad A_4 = \frac{\partial}{\partial v}.
\]

These fields form a basis for the space of left-invariant vector fields on \( \mathbb{C}^2 \). We note that the Lie brackets of the vector fields \( A_1, \ldots, A_4 \) are

\[
[A_1, A_2] = -2A_4, \quad [A_i, A_j] = 0 \text{ for all other } i, j, i < j.
\]

It follows that the group \( \mathbb{C}^2 \) defined above is solvable.
Denote by $g$ the left-invariant Riemannian metric on $M$ for which the basis $A_1, \ldots, A_4$ is orthonormal.

We shall consider almost complex structures $J$ on $M$ compatible with the metric $g$ obtained from left-invariant almost complex structures on $\mathbb{C}^2$. Note that by [13] every complex structure on $M$ is induced by a left-invariant complex structure.

I. If $J$ is a left-invariant almost complex structure compatible with $g$, we have $J A_i = \varepsilon_i A_2$, $J A_3 = \varepsilon_2 A_4$, $\varepsilon_1, \varepsilon_2 = \pm 1$.

Since we are dealing with the complex structures orthogonal to $J$, it is enough to consider the two structures $J_\varepsilon$ defined by

$$J A_1 = \varepsilon_1 A_2, \quad J A_3 = \varepsilon_2 A_4, \quad \varepsilon_1, \varepsilon_2 = \pm 1.$$ 

Endow $M$ with the orientation induced by $J_\varepsilon$. Then $\Lambda^2_+ M$ admits a global orthonormal frame defined by $s_1^\varepsilon = \varepsilon A_1 \wedge A_2 + A_3 \wedge A_4$, $s_2^\varepsilon = A_1 \wedge A_3 + \varepsilon A_4 \wedge A_2$, $s_3^\varepsilon = A_1 \wedge A_4 + \varepsilon A_2 \wedge A_3$.

Hence we have a natural diffeomorphism $F^\varepsilon : \mathcal{Z}_+(M) \cong M \times S^2$, $\sum_{k=1}^3 x_k s_k^\varepsilon(p) \rightarrow (p, x_1, x_2, x_3)$, under which

$$\Sigma_\varepsilon \cong \{(p, x) \in M \times S^2 : x_1 = 0\}.$$ 

In order to find an explicit formula for the metrics $h_\varepsilon$ we compute the covariant derivatives of $s_1^\varepsilon, s_2^\varepsilon, s_3^\varepsilon$ with respect to the Levi-Civita connection $\nabla$ of $g$. The non-zero covariant derivatives $\nabla_{A_i} A_j$ are

$$\nabla_{A_1} A_2 = -\nabla_{A_2} A_1 = -A_4, \quad \nabla_{A_3} A_4 = \nabla_{A_4} A_1 = A_2, \quad \nabla_{A_2} A_4 = \nabla_{A_4} A_2 = -A_1.$$ 

Then

$$\nabla_{A_1} s_1^\varepsilon = -\nabla_{A_4} s_2^\varepsilon = -\varepsilon s_3^\varepsilon, \quad \varepsilon \nabla_{A_1} s_3^\varepsilon = -\nabla_{A_2} s_2^\varepsilon = s_1^\varepsilon; \quad \nabla_{A_2} s_1^\varepsilon = -\varepsilon \nabla_{A_4} s_3^\varepsilon = s_2^\varepsilon,$$

and all other covariant derivatives $\nabla_{A_i} s_k^\varepsilon$ are zero. It follows that $F^\varepsilon$ sends the horizontal lifts $A_1^h, \ldots, A_4^h$ at a point $\sigma = \sum_{k=1}^3 x_k s_k^\varepsilon(p) \in \mathcal{Z}_+(M)$ to the vectors $A_1 + \varepsilon(-x_3, 0, x_1), \quad A_2 + (x_2, -x_1, 0), \quad A_3, \quad A_4 + \varepsilon(0, x_3, -x_2)$.

For $x = (x_1, x_2, x_3) \in S^2$, set

$$u_1^\varepsilon(x) = \varepsilon(-x_3, 0, x_1), \quad u_2^\varepsilon(x) = (x_2, -x_1, 0), \quad u_3^\varepsilon(x) = 0, \quad u_4^\varepsilon(x) = \varepsilon(0, x_3, -x_2).$$

These are tangent vectors to $S^2$ at the point $x$. Denote by $h_\varepsilon^P$ the pushforward of the metric $h_\varepsilon$ by $F^\varepsilon$. Then, if $X, Y \in T_p M$ and $P, Q \in T_p S^2$,

$$h_\varepsilon^P(X + P, Y + Q) = g(X, Y) + t < P - \sum_{i=1}^4 g(X, A_i) u_i^\varepsilon(x), Q - \sum_{j=1}^4 g(Y, A_j) u_j^\varepsilon(x) >$$

(27)

where $<, >$ is the standard metric of $\mathbb{R}^3$.

Let $\theta_\varepsilon$ be the Lee form of the Hermitian manifold $(M, g, J_\varepsilon)$. We have $\theta_\varepsilon(X) = -2\varepsilon g(X, A_3)$ which implies $\nabla \theta_\varepsilon = 0$. Therefore, by Proposition 3, the hypersurface
\{(p, x) \in M \times S^2 : x_1 = 0\} in M \times S^2 is minimal with respect to the metrics \(h^*_t\) given by (27).

II. Suppose again that \(J\) is an almost complex structure on \(M\) obtained from a left-invariant almost complex structure on \(G\) and compatible with the metric \(g\). Denote the fundamental 2-form of the almost Hermitian structure \((g, J)\) by \(\Omega\). Set \(JA_i = \sum_{j=1}^4 a_{ij} A_j\). The basis dual to \(A_1, \ldots, A_4\) is \(\alpha_1 = dx, \alpha_2 = dy, \alpha_3 = xdx + ydy + du, \alpha_4 = -ydx + xdy + dv\). We have \(d\alpha_1 = d\alpha_2 = d\alpha_3 = 0, d\alpha_4 = 2dx \wedge dy\). Hence \(d\Omega = d\sum_{i<j} a_{ij} \alpha_i \wedge \alpha_j = -2a_{34} dx \wedge dy \wedge du\). Thus \(d\Omega = 0\) is equivalent to \(a_{34} = 0\). If \(a_{34} = 0\), we have \(a_{1j} = a_{2j} = 0\) for \(j = 1, 2\), \(a_{3k} = a_{4k} = 0\), for \(k = 3, 4\), \(a_{13}^2 + a_{14}^2 = 1, a_{13}a_{23} + a_{14}a_{24} = 0, a_{23}^2 + a_{24}^2 = 1\). It follows that the structure \((g, J)\) is almost Kähler (symplectic) if and only if \(J\) is given by ((15))

\[
\begin{align*}
J A_1 &= -\varepsilon_1 \sin \varphi A_3 + \varepsilon_1 \varepsilon_2 \cos \varphi A_4, \quad J A_2 = -\cos \varphi A_3 - \varepsilon_2 \sin \varphi A_4, \\
J A_3 &= \varepsilon_1 \sin \varphi A_1 + \cos \varphi A_2, \quad J A_4 = -\varepsilon_1 \varepsilon_2 \cos \varphi A_1 + \varepsilon_2 \sin \varphi A_2,
\end{align*}
\]

\(\varepsilon_1, \varepsilon_2 = \pm 1, \varphi \in \mathbb{R}\).

For fixed \(\varepsilon = (\varepsilon_1, \varepsilon_2)\) and \(\varphi\), denote by \(J^{\varepsilon, \varphi}\) the almost complex structure defined by these identities. Set

\[E_1 = A_1, \quad E_2 = -\varepsilon_1 \sin \varphi A_3 + \varepsilon_1 \varepsilon_2 \cos \varphi A_4, \quad E_3 = \cos \varphi A_3 + \varepsilon_2 \sin \varphi A_4, \quad E_4 = A_2.\]

Then \(E_1, \ldots, E_4\) is an orthonormal frame of \(TM\) for which \(J^{\varepsilon, \varphi} E_1 = E_2\) and \(J^{\varepsilon, \varphi} E_3 = E_4\). The only non-zero Lie bracket of these fields is

\[
[E_1, E_4] = -2(\varepsilon_1 \varepsilon_2 \cos \varphi E_2 + \varepsilon_2 \sin \varphi E_3).
\]

The non-zero covariant derivatives \(\nabla_{E_i} E_j\) are

\[
\begin{align*}
\nabla_{E_1} E_2 &= \nabla_{E_2} E_1 = \varepsilon_1 \varepsilon_2 \cos \varphi E_4, \quad \nabla_{E_2} E_3 = \nabla_{E_3} E_1 = \varepsilon_2 \sin \varphi E_4, \\
\nabla_{E_1} E_3 &= -\nabla_{E_3} E_1 = -\varepsilon_1 \varepsilon_2 \cos \varphi E_2 - \varepsilon_2 \sin \varphi E_3, \\
\nabla_{E_2} E_4 &= \nabla_{E_4} E_2 = -\varepsilon_1 \varepsilon_2 \cos \varphi E_1, \quad \nabla_{E_3} E_4 = \nabla_{E_4} E_3 = -\varepsilon_2 \sin \varphi E_1.
\end{align*}
\]

Using \(E_1, \ldots, E_4\), we define a global orthonormal frame \(s_1^+, s_2^+, s_3^+\) of \(A^2 TM\) via (1).

Using the following table for the covariant derivatives of \(s_1^+, s_2^+, s_3^+\):

\[
\begin{align*}
\nabla_{E_1} s_1^+ &= \nabla_{E_4} s_2^+ = \varepsilon_1 \varepsilon_2 \cos \varphi s_3^+, \quad \nabla_{E_2} s_2^+ = -\varepsilon_1 \varepsilon_2 \cos \varphi s_1^+, \\
\nabla_{E_1} s_2^+ &= \varepsilon_1 \varepsilon_2 \cos \varphi s_2^+, \quad \nabla_{E_2} s_3^+ = -\varepsilon_1 \varepsilon_2 \cos \varphi s_1^+, \quad \nabla_{E_3} s_3^+ = 0, \\
\nabla_{E_1} s_3^+ &= \varepsilon_1 \varepsilon_2 \cos \varphi s_3^+, \quad \nabla_{E_2} s_3^+ = -\varepsilon_2 \sin \varphi s_1^+, \quad \nabla_{E_3} s_3^+ = 0, \\
\nabla_{E_1} s_2^+ &= -\varepsilon_1 \varepsilon_2 \cos \varphi s_1^+ - \varepsilon_2 \sin \varphi s_2^+, \quad \nabla_{E_4} s_3^+ = \varepsilon_2 \sin \varphi s_1^+ - \varepsilon_1 \varepsilon_2 \cos \varphi s_2^+.
\end{align*}
\]

The frame \(s_1^+, s_2^+, s_3^+\) gives rise to an obvious diffeomorphism \(F^{\varepsilon, \varphi} : Z_+(M) \cong M \times S^2\) for which \(\Sigma_{F^{\varepsilon, \varphi}} \cong \{(p, x) \in M \times S^2 : x_1 = 0\}\). To describe the pushforward \(h^{\varepsilon, \varphi}_t\) of the metric \(h^*_t\) by \(F^{\varepsilon, \varphi}\), we set

\[
\begin{align*}
u_1^{\varepsilon, \varphi}(x) &= (x_3 \varepsilon_1 \varepsilon_2 \cos \varphi, x_3 \varepsilon_1 \varepsilon_2 \sin \varphi, -x_1 \varepsilon_1 \varepsilon_2 \cos \varphi - x_2 \varepsilon_2 \sin \varphi), \\
u_2^{\varepsilon, \varphi}(x) &= (x_2 \varepsilon_1 \varepsilon_2 \cos \varphi, -x_1 \varepsilon_1 \varepsilon_2 \cos \varphi, 0), \quad \nu_3^{\varepsilon, \varphi}(x) = (x_2 \varepsilon_2 \sin \varphi, -x_1 \varepsilon_2 \sin \varphi, 0), \\
u_4^{\varepsilon, \varphi}(x) &= (-x_3 \varepsilon_2 \sin \varphi, x_3 \varepsilon_1 \varepsilon_2 \cos \varphi, x_1 \varepsilon_2 \sin \varphi - x_2 \varepsilon_1 \varepsilon_2 \cos \varphi).
\end{align*}
\]
for \( x = (x_1, x_2, x_3) \in S^2 \). Then, if \( X, Y \in T_pM \) and \( P, Q \in T_2S^2 \),

\[
h^{\epsilon, \phi}_t(X + P, Y + Q) = g(X, Y) + t < P - \sum_{i=1}^4 g(X; E_i) u^{\epsilon, \phi}_i(x), Q - \sum_{j=1}^4 g(Y; E_j) u^{\epsilon, \phi}_j(x) >.
\]  

(28)

It is easy to compute that

\[
\rho^*(E_1, E_4) = \rho^*(E_4, E_1) = -\epsilon_1 \sin \varphi, \cos \varphi, \quad \rho^*(E_1, E_3) = \rho^*(E_3, E_1) = 0.
\]

It follows from Proposition 5 that \( \{(p, x) \in M \times S^2 : x_1 = 0\} \) is a minimal hypersurface in \( M \times S^2 \), the latter manifold being endowed with the metrics \( h^{\epsilon, \phi}_t \) given by (28).

Secondary Kodaira surfaces are quotients of primary ones by groups of order 2,3,4 or 6. Every secondary Kodaira surface is a homogeneous manifold (in fact a solvmanifold). It admits a basis of left-invariant vector fields \( A_1, ..., A_4 \) such that the complex structure sends \( A_1, A_3 \) to \( A_2, A_4 \) and \( [A_1, A_2] = -2A_4, 2[A_3, A_1] = A_2, 2[A_3, A_2] = -A_1 \) [13]. Computations as above show that if \( J \) is a left-invariant complex or symplectic structure, then \( \Sigma_J \) is a minimal hypersurface.

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