A Criterion for Dualizing Modules

Kamran DIVAANI-AAZAR, Massoumeh NIKKHAH BABAEI and Massoud TOUSI

Alzahra University and *Shahid Beheshti University
(Communicated by K. Onishi)

Abstract. We establish a characterization of dualizing modules among semidualizing modules. Let $R$ be a finite dimensional commutative Noetherian ring with identity and $C$ a semidualizing $R$-module. We show that $C$ is a dualizing $R$-module if and only if $\text{Tor}_i^R(E, E')$ is $C$-injective for all $C$-injective $R$-modules $E$ and $E'$ and all $i \geq 0$.

1. Introduction

Throughout this paper, $R$ will denote a commutative Noetherian ring with non-zero identity. The injective envelope of an $R$-module $M$ is denoted by $E_R(M)$.

A finitely generated $R$-module $C$ is called semidualizing if the homothety map $R \rightarrow \text{Hom}_R(C, C)$ is an isomorphism and $\text{Ext}_R^i(C, C) = 0$ for all $i > 0$. Immediate examples of such modules are free $R$-modules of rank one. A semidualizing $R$-module $C$ with finite injective dimension is called dualizing. Although $R$ always possesses a semidualizing module, it does not possess a dualizing module in general. Keeping [BH, Theorem 3.3.6] in mind, it is straightforward to see that the ring $R$ possesses a dualizing module if and only if it is Cohen-Macaulay and it is homomorphic image of a finite dimensional Gorenstein ring.

Let $(R, m, k)$ be a local ring. There are several characterizations in the literature for a semidualizing $R$-module $C$ to be dualizing. For instance, Christensen [C, Proposition 8.4] has shown that a semidualizing $R$-module $C$ is dualizing if and only if the Gorenstein dimension of $k$ with respect to $C$ is finite. Also, Takahashi et al. [TYY, Theorem 1.3] proved that a semidualizing $R$-module $C$ is dualizing if and only if every finitely generated $R$-module can be embedded in an $R$-module of finite $C$-dimension. Our aim in this paper is to give a new characterization for a semidualizing $R$-module $C$ to be dualizing.

Let $C$ be a semidualizing $R$-module. An $R$-module $M$ is said to be $C$-projective (respectively $C$-flat) if it has the form $C \otimes_R U$ for some projective (respectively flat) $R$-module $U$. Also, a $C$-free $R$-module is defined as a direct sum of copies of $C$. We can see that every...
$C$-projective $R$-module is a direct summand of a $C$-free $R$-module and over a local ring every finitely generated $C$-flat $R$-module is $C$-free. Also, an $R$-module $M$ is said to be $C$-injective if it has the form $\text{Hom}_R(C, I)$ for some injective $R$-module $I$.

Yoneda raised a question of whether the tensor product of injective modules is injective. Ishikawa in [I, Theorem 2.4] showed that if $E_R(R)$ is flat, then $E \otimes_R E'$ is injective for all injective $R$-modules $E$ and $E'$. Further, Enochs and Jenda [EJ, Theorem 4.1] proved that $R$ is Gorenstein if and only if for every injective $R$-modules $E$ and $E'$ and any $i \geq 0$, $\text{Tor}_i^R(E, E')$ is injective. We extend this result in terms of a semidualizing $R$-module. More precisely, for a semidualizing $R$-module $C$, we show that the following are equivalent (see Theorem 2.7):

(i) $C_p$ is a dualizing $R_p$-module for all $p \in \text{Spec } R$.

(ii) For any prime ideal $p$ of $R$ and any $i \geq 0$,

$$\text{Tor}_i^R(E_C(R/p), E_C(R/p)) = \begin{cases} 0 & \text{if } i \neq \dim_{R_p} C_p \\ E_C(R/p) & \text{if } i = \dim_{R_p} C_p \end{cases},$$

where $E_C(R/p) := \text{Hom}_R(C, E_R(R/p))$.

(iii) For any $C$-injective $R$-modules $E$ and $E'$ and any $i \geq 0$, $\text{Tor}_i^R(E, E')$ is $C$-injective.

2. The Results

Let $p$ be a prime ideal of $R$. Recall that an $R$-module $M$ is said to have property $t(p)$ if for each $r \in R - p$, the map $M \to M$ is an isomorphism and if for each $x \in M$ we have $p^m x = 0$ for some $m \geq 1$. If an $R$-module $M$ has $t(p)$-property, then it has the structure as an $R_p$-module. It is known that $E_R(R/p)$ has $t(p)$-property.

To prove Theorem 2.7, which is our main result, we shall need the following five preliminary lemmas.

**Lemma 2.1.** Let $C$ be a semidualizing $R$-module. Then the following statements hold true.

(i) $E_C(R/p) := \text{Hom}_R(C, E_R(R/p))$ has $t(p)$-property for each $p \in \text{Spec } R$.

(ii) If $p$ and $q$ are two distinct prime ideals of $R$, then $\text{Tor}_i^R(E_C(R/p), E_C(R/q)) = 0$ for all $i \geq 0$.

**Proof.** (i) As $E_R(R/p)$ has $t(p)$-property, one can easily check that for any finitely generated $R$-module $M$, the $R$-module $\text{Hom}_R(M, E_R(R/p))$ has $t(p)$-property.

(ii) By (i) $E_C(R/p)$ has $t(p)$-property and $E_C(R/q)$ has $t(q)$-property. So, [EH, 5] implies that

$$\text{Tor}_i^R(E_C(R/p), E_C(R/q)) = 0$$

for all $i \geq 0$. \qed
LEMMA 2.2. Let \((R, m, k)\) be a local ring, \(C\) a semidualizing \(R\)-module and \(I\) an Artinian \(C\)-injective \(R\)-module. Then \(\text{Hom}_R(I, E_R(k))\) is a finitely generated \(\hat{C}\)-free \(\hat{R}\)-module.

PROOF. Denote the functor \(\text{Hom}_R(-, E_R(k))\) by \((-)^\vee\). We have \(I = \text{Hom}_R(C, I')\) for some injective \(R\)-module \(I'\). Clearly, \(C \otimes_R I\) is also an Artinian \(R\)-module. Since

\[ C \otimes_R I \cong C \otimes_R \text{Hom}_R(C, I') \cong \text{Hom}_R(\text{Hom}_R(C, C), I') \cong I', \]

we deduce that \(I'\) is also Artinian. So, \(I' \cong n \oplus E_R(k)\) for some nonnegative integer \(n\).

Now, one has

\[ I^\vee = \text{Hom}_R(C, I')^\vee \cong C \otimes_R I'^{\vee} \cong n \oplus \hat{C}, \]

and so \(I^\vee\) is a finitely generated \(\hat{C}\)-free \(\hat{R}\)-module. \(\square\)

In the next result, we collect some useful known properties of semidualizing modules. We may use them without any further comments.

LEMMA 2.3. Let \(C\) be a semidualizing \(R\)-module and \(\underline{r} := r_1, \ldots, r_n\) a sequence of elements of \(R\). The following statements hold.

(i) \(\text{Supp}_R C = \text{Spec} R\), and so \(\text{dim}_R C = \text{dim} R\).

(ii) If \(R\) is local, then \(\hat{C}\) is a semidualizing \(\hat{R}\)-module.

(iii) \(\underline{r}\) is a regular \(R\)-sequence if and only if \(\underline{r}\) is a regular \(C\)-sequence.

(iv) If \(\underline{r}\) is a regular \(R\)-sequence, then \(C/(\underline{r})C\) is a semidualizing \(R/(\underline{r})\)-module.

(v) If \(R\) is local and \(\underline{r}\) is a regular \(R\)-sequence, then \(C\) is a dualizing \(R\)-module if and only if \(C/(\underline{r})C\) is a dualizing \(R/(\underline{r})\)-module.

PROOF. (i) and (ii) follow easily by the definition of a semidualizing module.

(iii) and (iv) are hold by [S, Corollary 3.3.3].

(v) Assume that \(R\) is local and \(\underline{r}\) is a regular \(R\)-sequence. Then by (iv), \(C/(\underline{r})C\) is a semidualizing \(R/(\underline{r})\)-module. On the other hand, [BH, Corollary 3.1.15] yields that

\[ \text{id}_R C \underline{r} (\underline{r})C = \text{id}_R C - n. \]

This implies the conclusion. \(\square\)

In the proof of the following result, \(R \times C\) will denote the trivial extension of \(R\) by \(C\). For any \(R \times C\)-module \(X\), its Gorenstein injective dimension will be denoted by \(\text{Gid}_{R \times C} X\). Also, we recall that for a module \(M\) over a local ring \((R, m, k)\), the width of \(M\) is defined by \(\text{width}_R M := \inf \{ i \in \mathbb{N}_0 | \text{Tor}_i^R(k, M) \neq 0 \}\).

LEMMA 2.4. Let \((R, m, k)\) be a local ring and \(C\) a semidualizing \(R\)-module. Then \(E_C(k) \otimes_R E_C(k)\) is a non-zero \(C\)-injective \(R\)-module if and only if \(C\) is a dualizing \(R\)-module of dimension 0.
Suppose that $E_C(k) \otimes_R E_C(k)$ is a non-zero $C$-injective $R$-module. As $E_C(k)$ is Artinian, by [KLS, Corollary 3.9] the length of $E_C(k) \otimes_R E_C(k)$ is finite. So, also, $(E_C(k) \otimes_R E_C(k))^\vee$ has finite length. Since

$$\text{Hom}_R(E_C(k), \hat{C}) \cong (E_C(k) \otimes_R E_C(k))^\vee,$$

by Lemma 2.2, we deduce that $\text{Hom}_R(E_C(k), \hat{C})$ is isomorphic to a direct sum of finitely many copies of $\hat{C}$. This, in particular, implies that $\hat{C}$ has finite length. Thus Lemma 2.3 yields that

$$\dim R = \dim R C = \dim \hat{R} \hat{C} = 0,$$

and so, in particular, $R$ is complete. Next, one has

$$\text{Hom}_R(E_C(k), R) \cong \text{Hom}_R(E_C(k), \text{Hom}_R(C, C))$$

$$\cong \text{Hom}_R(C, \text{Hom}_R(E_C(k), C))$$

$$\cong \bigoplus \text{Hom}_R(C, C)$$

$$\cong R^n$$

for some $n > 0$. This, in particular, implies that

$$\text{Ann}_R(\text{Hom}_R(E_C(k), R)) = \text{Ann}_R R.$$

Since $R$ is Artinian, $m' = 0$ and $m'^{-1} \neq 0$ for some $t > 0$. If for every $f \in \text{Hom}_R(E_C(k), R)$, $\text{im} f \subseteq m$, then $m'^{-1} f = 0$ so $m'^{-1} \text{Hom}_R(E_C(k), R) = 0$ a contradiction. Thus there is an epimorphism $E_C(k) \to R \to 0$, and so $R$ is a direct summand of $E_C(k)$. Next, [HJ1, Lemma 2.6] implies that $R$ is a Gorenstein injective $R \times C$-module. This yields that $C$ is a dualizing $R$-module, because by [HJ2, Proposition 4.5], one has

$$\text{id}_R C \leq \text{Gid}_{R \times C} R + \text{width}_R R.$$

Conversely, if $C$ is a dualizing $R$-module of dimension 0, then $\dim R = 0$ by Lemma 2.3 (i). Hence, $E_R(k)$ is a dualizing $R$-module, and then by [BH, Theorem 3.3.4 (b)] we have $C \cong E_R(k)$. Thus

$$E_C(k) \otimes_R E_C(k) \cong \text{Hom}_R(E_R(k), E_R(k)) \otimes_R \text{Hom}_R(E_R(k), E_R(k))$$

$$\cong R \otimes_R R$$

$$\cong R$$

$$\cong \text{Hom}_R(C, E_R(k)),$$

which is a non-zero $C$-injective $R$-module. \qed

**Remark 2.5** (See [B, (2.5)]). Let $M$ be an $R$-module and let $r \in R$ be a non-unit which is a non-zero divisor of both $R$ and $M$. Let $0 \to M \to I^0 \xrightarrow{d^0} I^1 \to \cdots$ be a
minimal injective resolution of $M$. Then there is a natural $R/(r)$-isomorphism $M/(r)M \cong \text{Hom}_R(R/(r), \text{im } d^0)$ and

$$0 \to \text{Hom}_R(R/(r), I^1) \to \text{Hom}_R(R/(r), I^2) \to \cdots$$

is a minimal injective resolution of the $R/(r)$-module $M/(r)M$.

Next, we recall the definition of the notion of co-regular sequences. Let $X$ be an $R$-module. An element $r$ of $R$ is said to be co-regular on $X$ if the map $X \to X$ is surjective. A sequence $r_1, \ldots, r_n$ of elements of $R$ is said to be a co-regular sequence on $X$ if $r_i$ is co-regular on $(0 :_M (r_1, \ldots, r_{i-1}))$ for all $i = 1, \ldots, n$.

The following result plays a crucial role in the proof of Theorem 2.7.

**Lemma 2.6.** Let $(R, \mathfrak{m}, k)$ be a local ring and $C$ a semidualizing $R$-module. Let $r \in \mathfrak{m}$ be a non-zero divisor of $R$. Assume that $r$ is co-regular on $\text{Tor}_i^R(E_C(k), E_C(k))$ for all $i$. Then for any $i \geq 0$, we have a natural $\bar{R}$-isomorphism

$$\text{Tor}_{i-1}^\bar{R}(E_{\bar{C}}(k), E_{\bar{C}}(k)) \cong \text{Hom}_R(\bar{R}, \text{Tor}_i^R(E_C(k), E_C(k))),$$

where $\bar{R} := R/(r)$, $\bar{C} := C/(r)C$, $E_C(k) := \text{Hom}_R(C, E_R(k))$ and $E_{\bar{C}}(k) := \text{Hom}_\bar{R}(\bar{C}, E_\bar{R}(k))$.

**Proof.** Let $0 \to I^0 \to I^1 \to \cdots$ be a minimal injective resolution of $C$. Then

$$\cdots \to \text{Hom}_R(I^1, E_R(k)) \to \text{Hom}_R(I^0, E_R(k)) \to 0$$

is a flat resolution of $E_C(k)$. Applying $E_C(k) \otimes_R -$, we get the complex

$$\cdots \to E_C(k) \otimes_R \text{Hom}_R(I^1, E_R(k)) \to E_C(k) \otimes_R \text{Hom}_R(I^0, E_R(k)) \to 0.$$ 

We will denote $E_C(k) \otimes_R \text{Hom}_R(I^i, E_R(k))$ by $X_i$ and set

$$X_* := \cdots \to X_i \to \cdots \to X_1 \to X_0 \to 0.$$ 

Then for each $i \geq 0$, we have $H_i(X_*) = \text{Tor}_{i}^R(E_{\bar{C}}(k), E_{\bar{C}}(k))$.

By Remark 2.5,

$$0 \to \text{Hom}_\bar{R}(\bar{R}, I^1) \to \text{Hom}_\bar{R}(\bar{R}, I^2) \to \cdots$$

is a minimal injective resolution of $\bar{C}$ as an $\bar{R}$-module. So,

$$\cdots \to \text{Hom}_\bar{R}(\text{Hom}_\bar{R}(\bar{R}, I^2), E_{\bar{R}}(k)) \to \text{Hom}_\bar{R}(\text{Hom}_\bar{R}(\bar{R}, I^1), E_{\bar{R}}(k)) \to 0$$

is a flat resolution of $E_{\bar{C}}(k)$ as an $\bar{R}$-module. Thus for each $i \geq 1$, the $\bar{R}$-module $\text{Tor}_{i-1}^\bar{R}(E_{\bar{C}}(k), E_{\bar{C}}(k))$ is isomorphic to the $i$th homology of the following complex

$$(\ast) \cdots \to E_{\bar{C}}(k) \otimes_\bar{R} \text{Hom}_\bar{R}(\text{Hom}_\bar{R}(\bar{R}, I^2), E_{\bar{R}}(k))$$
We shall show that the later complex is isomorphic to the complex $Y_\bullet := \text{Hom}_R(\bar{R}, X_\bullet)$.

Noting that $E_{\bar{R}}(k) \cong \text{Hom}_R(\bar{R}, E_{\bar{R}}(k))$ and using Adjointness yields that

$$E_{\bar{C}}(k) = \text{Hom}_{\bar{R}}(\bar{C}, E_{\bar{R}}(k)) \cong \text{Hom}_R(\bar{R}, E_{\bar{C}}(k)).$$

Hence for each $i \geq 0$, by using Adjointness, Hom-evaluation and Tensor-evaluation, one has the following natural $\bar{R}$-isomorphisms:

$$E_{\bar{C}}(k) \otimes \bar{R} \text{Hom}_{\bar{R}}(\text{Hom}_R(\bar{R}, I^i), E_{\bar{R}}(k)) \cong E_{\bar{C}}(k) \otimes \bar{R} \text{Hom}_R(I^i, E_{\bar{R}}(k)) \cong E_{\bar{C}}(k) \otimes \bar{R} \text{Hom}_R(I^i, E_{\bar{R}}(k)) \cong Y_i.$$

Note that $\text{Hom}_R(I^i, E_{\bar{R}}(k))$ is a flat $R$-module. As $r$ is a non-zero divisor of $R$, it is also a non-zero divisor of $C$. This implies that $r$ is a non-zero divisor of $I^0$, and so $\text{Hom}_R(\bar{R}, I^0) = 0$. Thus

$$Y_0 \cong E_{\bar{C}}(k) \otimes \bar{R} \text{Hom}_R(\bar{R}^0, E_{\bar{R}}(k)) = 0.$$

Therefore, the two complexes $(\ast)$ and $Y_\bullet$ are isomorphic, and so we deduce that $\text{Tor}^R_{i-1}(E_{\bar{C}}(k), E_{\bar{C}}(k)) = H_i(Y_\bullet)$ for all $i \geq 0$.

Since $r$ is a non-zero divisor of $C$, it is co-regular on $E_{\bar{C}}(k)$, and so it is co-regular on $X_i$ for all $i$. Thus, we can deduce the following exact sequence of complexes

$$0 \longrightarrow Y_\bullet \longrightarrow X_\bullet \overset{r}{\longrightarrow} X_\bullet \longrightarrow 0.$$

It yields the following exact sequences of modules

$$\cdots \longrightarrow \text{Tor}^R_{i+1}(E_{\bar{C}}(k), E_{\bar{C}}(k)) \overset{r}{\longrightarrow} \text{Tor}^R_{i+1}(E_{\bar{C}}(k), E_{\bar{C}}(k)) \longrightarrow \text{Tor}^R_{i-1}(E_{\bar{C}}(k), E_{\bar{C}}(k)) \overset{f_i}{\longrightarrow} \text{Tor}^R_i(E_{\bar{C}}(k), E_{\bar{C}}(k)) \longrightarrow \cdots.$$

As $r$ is a co-regular element on $\text{Tor}^R_i(E_{\bar{C}}(k), E_{\bar{C}}(k))$ for all $i$, we deduce that $f_i$ is a monomorphism for all $i$. This implies our desired isomorphisms. $\square$

**Theorem 2.7.** Let $C$ be a semidualizing $R$-module. The following are equivalent:

(i) $C_p$ is a dualizing $R_p$-module for all $p \in \text{Spec } R$.

(ii) For any prime ideal $p$ of $R$ and any $i \geq 0$,

$$\text{Tor}^R_i(E_{C}(R/p), E_{C}(R/p)) = \begin{cases} 0 & \text{if } i \neq \dim_{R_p} C_p \\ E_{C}(R/p) & \text{if } i = \dim_{R_p} C_p. \end{cases}$$
where \( E_C(R/p) := \text{Hom}_R(C, E_R(R/p)) \).

(iii) For any \( C \)-injective \( R \)-modules \( E \) and \( E' \) and any \( i \geq 0 \), \( \text{Tor}_i^R(E, E') \) is \( C \)-injective.

**Proof.** (i) \( \Rightarrow \) (ii) Let \( p \) be a prime ideal of \( R \). There are natural \( R_p \)-isomorphisms
\[
E_C(R/p) \cong E(R/p/R_p) \quad \text{and} \quad \text{Tor}_i^R(E_C(R/p), E_C(R/p)) \cong \text{Tor}_i^{R_p}(E_C(p, R_p), E_C(R/p))
\]
for all \( i \geq 0 \). Hence, we can complete the proof of this part by showing that if \( C \) is a dualizing module of a local ring \((R, m, k)\), then
\[
\text{Tor}_i^R(E(k), E(k)) = \begin{cases} 0 & \text{if } i \neq \dim_R C, \\ E(k) & \text{if } i = \dim_R C. \end{cases}
\]

Set \( d := \dim_R C \). As \( C \) is a dualizing \( R \)-module, [BH, Theorem 3.3.10] implies that for any prime ideal \( p \), one has
\[
\mu^i(p, C) = \begin{cases} 0 & \text{if } i \neq \text{ht } p, \\ 1 & \text{if } i = \text{ht } p. \end{cases}
\]

So, if \( I^\bullet = 0 \rightarrow I^0 \rightarrow I^1 \rightarrow \cdots \) is a minimal injective resolution of \( C \), then \( I^d \cong E_R(k) \) and for any \( i \neq d \), \( E_R(k) \) is not a direct summand of \( I^i \). In particular, \( \text{Hom}_R(R/m, I^i) = 0 \) for all \( i \neq d \). Now, \( \text{Hom}_R(I^\bullet, E_R(k)) \) is a flat resolution of \( E_C(k) \). Clearly, one has
\[
E_C(k) \otimes_R \text{Hom}_R(I^d, E_R(k)) \cong E_C(k) \otimes_R \widehat{R} \cong E_C(k).
\]

Next, let \( i \neq d \). Since \( \text{Hom}_R(I^i, E_R(k)) \) is a flat \( R \)-module, [M, Theorem 23.2 (ii)] implies that
\[
\text{Ass}_R(E_C(k) \otimes_R \text{Hom}_R(I^i, E_R(k))) = \text{Ass}_R(R/m \otimes_R \text{Hom}_R(I^i, E_R(k))).
\]

But,
\[
R/m \otimes_R \text{Hom}_R(I^i, E_R(k)) \cong \text{Hom}_R(\text{Hom}_R(R/m, I^i), E_R(k)) = 0,
\]
and so \( E_C(k) \otimes_R \text{Hom}_R(I^i, E_R(k)) = 0 \). Therefore, it follows that the complex \( E_C(k) \otimes_R \text{Hom}_R(I^\bullet, E_R(k)) \) has \( E_C(k) \) in its \( d \)-place and 0 in its other places. Thus, we deduce that
\[
\text{Tor}_i^R(E_C(k), E_C(k)) = H_i(E_C(k) \otimes_R \text{Hom}_R(I^\bullet, E(k))) = \begin{cases} 0 & \text{if } i \neq d, \\ E_C(k) & \text{if } i = d. \end{cases}
\]

(ii) \( \Rightarrow \) (iii) Let \( E \) be an injective \( R \)-module. Since \( E \cong \bigoplus_{p \in \text{Spec } R} E_R(R/p) \mu^0(p, E) \) and \( C \...
is finitely generated, we have
\[ \text{Hom}_R(C, E) \cong \bigoplus_{p \in \text{Spec } R} E \mu_0(p, E). \]

As \( R \) is Noetherian, clearly any direct sum of \( C \)-injective \( R \)-modules is again \( C \)-injective, and so (ii) yields (iii) by Lemma 2.1 (ii).

(iii) \( \Rightarrow \) (i) It is easy to check that a given \( R_p \)-module is \( C_p \)-injective if and only if it is the localization at \( p \) of a \( C \)-injective \( R \)-module. Thus, it is enough to show that if \( C \) is a semidualizing module of a local ring \((R, m, k)\) such that \( \text{Tor}_i^R(E, E') \) is \( C \)-injective for all \( C \)-injective \( R \)-modules \( E \) and \( E' \) and all \( i \geq 0 \), then \( C \) is dualizing.

Let \( \underline{\mathfrak{r}} = r_1, \ldots, r_d \in m \) be a maximal regular \( R \)-sequence. Then \( \underline{\mathfrak{r}} \) is also a regular \( C \)-sequence. It is easy to verify that \( \underline{\mathfrak{r}} \) is a co-regular sequence on any \( C \)-injective \( R \)-module, and consequently \( \underline{\mathfrak{r}} \) is a co-regular sequence on \( \text{Tor}_i^R(E_C(k), E_C(k)) \) for all \( i \geq 0 \). Letting \( \tilde{\mathfrak{r}} : = R/(\underline{\mathfrak{r}}) \) and \( \tilde{C} : = C/(\underline{\mathfrak{r}})C \), by Lemma 2.3 (iv), it turns out that \( \tilde{C} \) is a semidualizing \( \tilde{R} \)-module. Making repeated use of Lemma 2.6, we can establish the following natural \( \tilde{R} \)-isomorphism
\[ E_{\tilde{C}}(k) \otimes_{\tilde{R}} E_{\tilde{C}}(k) \cong \text{Hom}_{\tilde{R}}(\tilde{R}, \text{Tor}_d^R(E_C(k), E_C(k))). \]

So, \( E_{\tilde{C}}(k) \otimes_{\tilde{R}} E_{\tilde{C}}(k) \) is a \( \tilde{C} \)-injective \( \tilde{R} \)-module. Lemma 2.3 implies that
\[ \text{depth}_{\tilde{R}} \tilde{C} = \text{depth}_{\tilde{R}} \tilde{C} = \text{depth}_{\tilde{R}} \tilde{R} = 0, \]
and so there are natural inclusion maps \( k \xhookrightarrow{i} \tilde{C} \) and \( k \xhookrightarrow{j} \hat{\tilde{C}} \). By applying the functor \( \text{Hom}_{\tilde{R}}(\cdot, E_{\tilde{R}}(k)) \) on \( i \), we get an epimorphism \( E_{\tilde{C}}(k) \twoheadrightarrow k \). Next, by applying the functor \( \text{Hom}_{\tilde{R}}(\cdot, \hat{\tilde{C}}) \) on the later map, we see that
\[ \text{Hom}_{\tilde{R}}(E_{\tilde{C}}(k) \otimes_{\tilde{R}} E_{\tilde{C}}(k), E_{\tilde{R}}(k)) \cong \text{Hom}_{\tilde{R}}(E_{\tilde{C}}(k), \hat{\tilde{C}}) \neq 0. \]

Hence, \( E_{\tilde{C}}(k) \otimes_{\tilde{R}} E_{\tilde{C}}(k) \) is a non-zero \( \tilde{C} \)-injective \( \tilde{R} \)-module, and so Lemma 2.4 yields that \( \hat{\tilde{C}} \) is a dualizing \( \tilde{R} \)-module. Now, by Lemma 2.3 (v), we deduce that \( C \) is a dualizing \( R \)-module. \( \square \)

We end the paper with the following immediate corollary.

**Corollary 2.8.** Let \( R \) be a finite dimensional ring and \( C \) a semidualizing \( R \)-module. Then \( C \) is a dualizing \( R \)-module if and only if \( \text{Tor}_i^R(E, E') \) is \( C \)-injective for all \( C \)-injective \( R \)-modules \( E \) and \( E' \) and all \( i \geq 0 \).

**References**

[B] H. Bass, On the ubiquity of Gorenstein rings, Math. Z. **82** (1963), 8–28.
CRITERION FOR DUALIZING MODULES

[ BH ] W. BRUNS and J. HERZOG, Cohen-Macaulay rings, Cambridge Studies in Advanced Mathematics 39, Cambridge University Press, Cambridge, 1998.

[ C ] L. W. CHRISTENSEN, Semi-dualizing complexes and their Auslander categories, Trans. Amer. Math. Soc. 353(5), (2001), 1839–1883.

[ EH ] E. E. ENOCHS and Z. HUANG, Canonical filtrations of Gorenstein injective modules, Proc. Amer. Math. Soc. 139(7), (2011), 2415–2421.

[ EJ ] E. E. ENOCHS and O. M. G. JENDA, Tensor and torsion products of injective modules, J. Pure Appl. Algebra 76(2), (1991), 143–149.

[ HJ1 ] H. HOLM and P. JØRGENSEN, Semi-dualizing modules and related Gorenstein homological dimensions, J. Pure Appl. Algebra 205(2), (2006), 423–445.

[ HJ2 ] H. HOLM and P. JØRGENSEN, Cohen-Macaulay homological dimensions, Rend. Semin. Mat. Univ. Padova 117 (2007), 87–112.

[ I ] T. ISHIKAWA, On injective modules and flat modules, J. Math. Soc. Japan 17(3), (1965), 291–292.

[ KLS ] B. KUBIK, M. J. LEAMER and S. SATHER-WAGSTAFF, Homology of Artinian and Matlis reflexive modules, I, J. Pure Appl. Algebra 215(10), (2011), 2486–2503.

[ M ] H. MATSUMURA, Commutative ring theory, Cambridge Studies in Advanced Mathematics 8, Cambridge University Press, Cambridge, 1986.

[ S ] S. SATHER-WAGSTAFF, Semidualizing modules, 2009, preprint.

[ TYY ] R. TAKAHASHI, S. YASSEMI and Y. YOSHINO, On the existence of embeddings into modules of finite homological dimensions, Proc. Amer. Math. Soc. 138(7), (2010), 2265–2268.

Present Addresses:

K. DIVAANI-AAZAR
DEPARTMENT OF MATHEMATICS,
ALZAHRA UNIVERSITY,
VANAK, POST CODE 19834, TEHRAN, IRAN.

M. NIKKHAIH BAABEI
DEPARTMENT OF MATHEMATICS,
ALZAHRA UNIVERSITY,
VANAK, POST CODE 19834, TEHRAN, IRAN.
e-mail: massnikkhah@yahoo.com

M. TOUSI
DEPARTMENT OF MATHEMATICS,
SHAHID BEHESHITI UNIVERSITY, G.C.,
TEHRAN, IRAN.

School of Mathematics,
Institute for Research in Fundamental Sciences (IPM),
P.O. Box 19395–5746, Tehran, Iran.
e-mail: mtousi@ipm.ir