SET PARTITIONS WITHOUT BLOCKS OF CERTAIN SIZES

JOSHUA CULVER AND ANDREAS WEINGARTNER

Abstract. We give an asymptotic estimate for the number of partitions of a set of \(n\) elements, whose block sizes avoid a given set \(S\) of natural numbers. As an application, we derive an estimate for the number of partitions of a set with \(n\) elements, which have the property that its blocks can be combined to form subsets of any size between 1 and \(n\).

1. Introduction

Let \(B_n\) be the \(n\)-th Bell number, that is the number of set-partitions of a set with \(n\) distinct elements. For example, \(B_3 = 5\) because there are five set-partitions of \(\{a, b, c\}\):

\[
\{\{a, b, c\}\}, \{\{a\}\}, \{\{b, c\}\}, \{\{b\}, \{a, c\}\}, \{\{a\}, \{b\}, \{c\}\}.
\]

We say that the partition \(\{\{a\}\}, \{\{b, c\}\}\) has the block \(\{a\}\) of size 1 and the block \(\{b, c\}\) of size 2. Blocks of size zero (empty blocks) are not allowed.

Let \(S \subset \mathbb{N}\) and let \(B_{n, S}\) denote the number of partitions of a set with \(n\) elements, whose block sizes are not in \(S\). If \(S = \{1, 2, \ldots, m\}\), we write \(B_{n, m} = B_{n, S}\), the number of partitions of a set with \(n\) elements, all of whose block sizes are greater than \(m\). We shall call such partitions \(m\)-rough. For example, \(B_{3, 1} = B_{3, 2} = 1\). We clearly have \(B_{n, 0} = B_n\). We define \(B_{0, S} = 1\) for all \(S\).

Throughout this paper, we write \(r = r(n)\) to denote the solution of

\[ re^r = n. \tag{1} \]

The function \(r\) is called Lambert-W function or product logarithm, and it can be approximated with the asymptotic formula (see [1])

\[ r = \log n - \log \log n + \frac{\log \log n}{\log n} + O \left( \frac{(\log \log n)^2}{\log n} \right). \tag{2} \]

Let

\[ \alpha(z) = \alpha_S(z) = \sum_{k \in S} \frac{z^k}{k!}. \]

Our first result is an asymptotic estimate for \(B_{n, S}\), derived from Cauchy’s residue theorem and the saddle point method.

2010 Mathematics Subject Classification. Primary 05A18; Secondary 05A16.
Theorem 1. Let $\eta_1 = 0.1866823\ldots$, $\eta_2 = 2.1555352\ldots$ be the two real solutions of $\eta(1 - \log \eta) = 1/2$ and let $0 < \delta_1 < \eta_1 < \eta_2 < \delta_2$. We have

$$B_{n,S} = \frac{n^l \exp(e^r - 1 - \alpha(r))}{r^n \sqrt{2 \pi r (r+1)e^r}} \left(1 + O\left(\frac{(\alpha'(r))^2}{e^r}\right)\right),$$

uniformly for sets $S$ with $S \cap [\delta_1 r, \delta_2 r] = \emptyset$.

Lemma 3 shows that the relative error term in Theorem 1 approaches zero as $n \to \infty$, uniformly for sets $S$ with $S \cap [\delta_1 r, \delta_2 r] = \emptyset$. The constant factor implied in the big-O notation depends only on the choice of the constants $\delta_1, \delta_2$, but does not depend on $n$ or $S$. On the other hand, if $S \cap [\gamma_1 r, \gamma_2 r] \neq \emptyset$, for constants $\gamma_1 < \eta_1 < \gamma_2 < \eta_2$, then the error term in Theorem 1 grows unbounded as $n \to \infty$. In this case, it seems that a different method is needed to determine the asymptotic behavior of $B_{n,S}$.

The contribution to $\alpha'(r)$ from $k \geq (e + \epsilon)r$ is $o(1)$ as $n \to \infty$, if $\epsilon > 0$. Thus, $B_{n,S} \sim B_{n,S'}$, if $S$ and $S'$ avoid $[\delta_1 r, \delta_2 r]$ and differ only on $((e+\epsilon)r, \infty)$.

If $\max S \leq r$, the occurrence of $\alpha'(r)$ in the error term of Theorem 1 can be estimated as in equation (3), with $i = 1$, to obtain $\alpha'(r) < (r/m)^{m-1}e^m$, where $m = \max S$.

Corollary 1. With $\delta_1$ as in Theorem 1 and $m = \max S$, we have

$$B_{n,S} = \frac{n^l \exp(e^r - 1 - \alpha(r))}{r^n \sqrt{2 \pi r (r+1)e^r}} \left(1 + O\left(\frac{(er/m)^{2m-2}}{e^r}\right)\right),$$

uniformly for sets $S$ with $1 \leq m \leq \delta_1 r$.

If $S = \emptyset$, then $\alpha(r) = 0$ and Theorem 1 simplifies to the known asymptotic estimate for the Bell numbers $B_n$ (see Moser and Wyman [6]):

Corollary 2. We have

$$B_n = \frac{n^l \exp(e^r - 1)}{r^n \sqrt{2 \pi r (r+1)e^r}} \left(1 + O\left(e^{-r}\right)\right).$$

Dividing the estimate in Theorem 1 by the one in Corollary 2 leads to the following result.

Corollary 3. Let $\delta_1, \delta_2$ be as in Theorem 1. The proportion of set partitions of $n$ objects, whose block sizes are not in $S$, is

$$\frac{B_{n,S}}{B_n} = \exp(-\alpha(r)) \left(1 + O\left(\frac{(\alpha'(r))^2}{e^r}\right)\right) = \exp(-\alpha(r))(1 + o(1)),$$

as $n \to \infty$, uniformly for sets $S$ with $S \cap [\delta_1 r, \delta_2 r] = \emptyset$.

With 1-rough partitions, $\alpha(r) = r$, so that $e^{-r}$ is the main term as well as the relative error term in Corollary 3. This is consistent with Table 1, where the relative errors are $< e^{-r}$. Similarly, for 2-rough partitions, the relative errors in Table 2 are $< (1 + r)^2 e^{-r}$, the relative error term in Corollary 3.
Table 1. Proportion of 1-rough set partitions: numerical examples of Corollary 3 for $S = \{1\}$ and $\alpha(r) = r$, showing the ratio $B_{n,1}/B_n$, the approximation $\exp(-\alpha(r))$, and the relative error $\exp(-\alpha(r))/(B_{n,1}/B_n) - 1$.

| $n$  | $B_{n,1}/B_n$ | $\exp(-r)$ | Rel. Error |
|------|---------------|-------------|------------|
| $2^2$ | 0.266667      | 0.300542    | 0.127032   |
| $2^4$ | 0.116036      | 0.128325    | 0.105906   |
| $2^6$ | 0.045716      | 0.047583    | 0.040834   |
| $2^8$ | 0.015896      | 0.016123    | 0.014298   |
| $2^{10}$ | 0.005122 | 0.005146    | 0.004675   |
| $2^{12}$ | 0.001573 | 0.001575    | 0.001456   |
| $2^{14}$ | 0.000468 | 0.000468    | 0.000438   |

Table 2. Proportion of 2-rough set partitions: numerical examples of Corollary 3 for $S = \{1, 2\}$ and $\alpha(r) = r + r^2/2$, showing the ratio $B_{n,2}/B_n$, the approximation $\exp(-\alpha(r))$, the relative error $\exp(-\alpha(r))/(B_{n,2}/B_n) - 1$, and $(\alpha'(r))^2e^{-r}$, the relative error term in Corollary 3.

| $n$  | $B_{n,2}/B_n$ | $\exp(-r - \frac{r^2}{2})$ | Rel. Error | $(1 + r)^2e^{-r}$ |
|------|---------------|----------------------------|------------|-------------------|
| $2^2$ | $6.667 \cdot 10^{-2}$ | $1.459 \cdot 10^{-1}$   | 1.1886     | 1.4575           |
| $2^4$ | $8.772 \cdot 10^{-3}$ | $1.559 \cdot 10^{-2}$   | 0.7776     | 1.1962           |
| $2^6$ | $3.185 \cdot 10^{-4}$ | $4.610 \cdot 10^{-4}$   | 0.4474     | 0.7787           |
| $2^8$ | $2.628 \cdot 10^{-6}$ | $3.222 \cdot 10^{-6}$   | 0.2257     | 0.4239           |
| $2^{10}$ | $4.356 \cdot 10^{-9}$ | $4.805 \cdot 10^{-9}$   | 0.1033     | 0.2023           |
| $2^{12}$ | $1.368 \cdot 10^{-12}$ | $1.428 \cdot 10^{-12}$ | 0.0438     | 0.0875           |
| $2^{14}$ | $7.902 \cdot 10^{-17}$ | $8.040 \cdot 10^{-17}$ | 0.0175     | 0.0352           |

Since $e^{-r} = r/n$ and $r \sim \log n$ by equation (2), the proportion of 1-rough set partitions (i.e. partitions with no singletons) is asymptotic to $(\log n)/n$. Corollary 4 makes that more precise.

**Corollary 4.** The proportion of 1-rough set partitions of $n$ objects is

$$\frac{B_{n,1}}{B_n} = \frac{r}{n} \left(1 + O\left(\frac{\log n}{n}\right)\right) = \frac{\log(n/\log n)}{n} \left(1 + O\left(\frac{\log \log n}{(\log n)^2}\right)\right).$$

The quantity $B_{n,m}$ appears in [10]. However, [10, Prop. 2] claims that $B_{n,1}/B_n \sim (\log n)/(ne)$, which is false in light of Corollary 4. Moreover, the asymptotic estimate for $B_{n,m}$ in [10, Prop. 4] is not correct, because $\exp(e^r) \sim \exp(n/\log n)$ by (2), even though $e^r = n/r \sim n/\log n$.

We now turn to an application of Theorem 1. In analogy with practical numbers [8][11] and practical integer partitions [2][3], we say that a partition of a set of $n$ objects is practical if its blocks can be combined to form subsets...
of any size between 1 and \( n \). Thus, if the partition has \( l \) blocks of sizes \( a_1, a_2, \ldots, a_l \), the partition is practical if and only if

\[
\left\{ \sum_{i=1}^{l} \varepsilon_i a_i : \varepsilon_i \in \{0, 1\} \right\} = \{0, 1, 2, 3, \ldots, n\}.
\]

For example, when \( n = 7 \), the set partition \( \{\{a\}, \{b, c\}, \{d, e, f, g\}\} \) is practical, but \( \{\{a\}, \{b, c, d\}, \{e, f, g\}\} \) is not, because the blocks cannot be combined to form a set of size 2 or 5. Let \( P_n \) denote the number of practical set partitions and let \( I_n = B_n - P_n \), the number of impractical set partitions, of a set of \( n \) elements. Define \( P_0 = B_0 = 1 \). Partitions which are 1-rough are clearly impractical, since the blocks can not be combined to form a set of size 1. Theorem 2 shows that, as \( n \) grows, almost all impractical set partitions are 1-rough.

**Theorem 2.** We have

\[
I_n = B_{n,1} \left( 1 + O \left( \exp \left( -\frac{(\log n)^2}{3} \right) \right) \right).
\]

Combining Theorem 2 with Corollary 4 yields an estimate for \( I_n / B_n \):

**Corollary 5.** The proportion of impractical set partitions of \( n \) objects is

\[
\frac{I_n}{B_n} = \frac{r}{n} \left( 1 + O \left( \frac{\log n}{n} \right) \right) = \frac{\log(n/\log n)}{n} \left( 1 + O \left( \frac{\log \log n}{\log n} \right) \right).
\]

| \( n \) | \( I_n/B_n \) | \( r/n \) | Relative Error |
|--------|----------------|----------|----------------|
| \( 2^2 \) | 0.533333 | 0.300542 | -0.436484 |
| \( 2^4 \) | 0.141507 | 0.128325 | -0.093156 |
| \( 2^8 \) | 0.046743 | 0.047583 | 0.017954 |
| \( 2^{10} \) | 0.015907 | 0.016123 | 0.013594 |
| \( 2^{10} \) | 0.005122 | 0.005146 | 0.004670 |

Table 3. Proportion of impractical set partitions: numerical examples of Corollary 5 showing the ratio \( I_n/B_n \), the approximation \( r/n \) and the relative error \( (r/n)/(I_n/B_n) - 1 \).

Note that the relative errors in Table 3 are of a similar size as the main term \( r/n \), consistent with the first equation in Corollary 5.

Since \( P_n = B_n - I_n \), we find that almost all set partitions are practical, as in the case of integer partitions (see [2, 3]).

**Corollary 6.** The proportion of practical set partitions of \( n \) objects is

\[
\frac{P_n}{B_n} = 1 - \frac{r}{n} + O \left( \frac{(\log n)^2}{n^2} \right) = 1 - \frac{\log(n/\log n)}{n} + O \left( \frac{\log \log n}{n \log n} \right).
\]
2. Proof of Theorem 1

The following lemma gives a recursive formula for the sequence $B_{n,S}$, which we used to generate the numerical examples in the tables. It is also the basis for deriving the exponential generating function in Lemma 2.

**Lemma 1.** For $n \geq 1$,

$$B_{n,S} = \sum_{0 \leq k \leq n-1 \atop n-k \notin S} \binom{n-1}{k} B_{k,S}.$$ 

**Proof.** We count the number of partitions of the set $\{1, 2, \ldots, n\}$, with no block sizes in $S$. For such a partition, let $k$ be the number of all elements of $\{1, 2, \ldots, n\}$ which are not in the block that contains 1. There are $\binom{n-1}{k}$ ways of selecting those elements from $\{2, 3, \ldots, n\}$, and for each such selection there are $B_{k,S}$ set partitions of those elements, with no block size in $S$. Note that the block containing 1 has $n-k$ elements, so $n-k \notin S$. □

Let $G_S(z) = \sum_{n=0}^{\infty} \frac{B_{n,S}}{n!} z^n$, the exponential generating function for the sequence $B_{n,S}$.

**Lemma 2.** We have

$$G_S(z) = \exp(e^z - 1 - \alpha(z)).$$

**Proof.** It is a standard exercise to derive the differential equation

$$G_S'(z) = G_S(z)(e^z - \alpha'(z))$$

from the recursive formula in Lemma 1. Solving that equation for $G_S(z)$ yields the desired result. Alternatively, the result follows from the general principle in [4, Proposition II.2]. □

We will need the following upper bound for the $i$-th derivative $\alpha^{(i)}(r)$:

**Lemma 3.** Let $\delta_1, \delta_2$ be as in Theorem 1 and let $I, J \geq 0$ be fixed integers. Uniformly for sets $S$ with $S \cap [\delta_1 r, \delta_2 r] = \emptyset$, we have

$$\alpha^{(i)}(r) \ll_{I,J} \frac{r^{i/2}}{r^J} \quad (0 \leq i \leq I).$$

**Proof.** Write $m = \delta_1 r$ and $M = \delta_2 r$. Then

$$\alpha^{(i)}(r) \leq \sum_{i \leq k \leq m} \frac{r^{k-i}}{(k-i)!} + \sum_{k > M} \frac{r^{k-i}}{(k-i)!} = s_1 + s_2,$$

say. We have

$$s_1 = \sum_{0 \leq k \leq m-i} \frac{m^k}{k!} \left( \frac{r}{m} \right)^k \leq \left( \frac{r}{m} \right)^{m-i} \sum_{0 \leq k \leq m-i} \frac{m^k}{k!} < \left( \frac{r}{m} \right)^{m-i} e^m,$$
hence
\[ \log s_1 < m(1 - \log(m/r)). \]
Since \( m/r = \delta_1 < \eta_1 \), the definition of \( \eta_1 \) in Theorem [1] implies
\( (m/r)(1 - \log(m/r)) < 1/2 - \varepsilon_1, \)
for some \( \varepsilon_1 > 0 \). Combining the last two inequalities shows that
\[ s_1 < \exp((1/2 - \varepsilon_1)r) \ll_J \exp(r/2)/r^J. \]
Similarly,
\[ s_2 = \sum_{k > M-i} \frac{M^k}{k!} \left( \frac{r}{M} \right)^k \leq \left( \frac{r}{M} \right)^{M-i} \sum_{k > M-i} \frac{M^k}{k!} < \delta_2^i \left( \frac{r e}{M} \right)^M, \]
hence
\[ \log s_2 < M(1 - \log(M/r)) + i \log \delta_2. \]
Since \( M/r = \delta_2 > \eta_2 \), the definition of \( \eta_2 \) in Theorem [1] implies
\( (M/r)(1 - \log(M/r)) < 1/2 - \varepsilon_2, \)
for some \( \varepsilon_2 > 0 \). With \( i \leq I \), we get
\[ s_2 \ll_I \exp((1/2 - \varepsilon_2)r) \ll_I \exp(r/2)/r^J. \]

\[ \square \]

Proof of Theorem [1]. Let \( r \) be given by (1). Cauchy’s residue theorem yields
\[ (4) \quad \frac{B_{n,S}}{n!} = \frac{1}{2\pi i} \int_{|z|=r} \frac{G_S(z)}{z^n+1} \, dz = \frac{1}{2\pi i} \int_{|z|=r} \frac{\exp(e^z - 1 - \alpha(z))}{z^n+1} \, dz. \]
Writing \( z = re^{i\theta} \), we obtain
\[ (5) \quad \frac{B_{n,S}}{n!} = \frac{\exp(e^r - 1 - \alpha(r))}{2\pi r^n} \int_{-\pi}^{\pi} \exp(h(\theta)) \, d\theta, \]
where
\[ h(\theta) = e^{re^{i\theta}} - e^r + \alpha(r) - \alpha(re^{i\theta}) - i\theta n. \]

Our first task is to show that the contribution to the last integral from \( \delta \leq |\theta| \leq \pi \) is negligible, where
\[ \delta = \sqrt{12(1 + \alpha(r))e^{-r}}. \]
We have
\[ |\exp(h(\theta))| = \exp(\text{Re}(h(\theta))) \leq \exp(\text{Re} e^{re^{i\theta}} - e^r + 2\alpha(r)). \]
Since \( \text{Re} e^z = e^{\text{Re} z} \cos(\text{Im} z), \)
\[ \text{Re} e^{re^{i\theta}-1} = e^{r(\cos \theta - 1)} \cos(r \sin \theta) \leq e^{r(\cos \theta - 1)} \leq e^{r(\cos \delta - 1)}, \]
for \( \delta \leq |\theta| \leq \pi \). Now \( \cos \delta \leq 1 - \delta^2/3 \) for \( 0 \leq \delta \leq 2, \) and \( e^x \leq 1 + x/2 \) for \( -1 \leq x \leq 0 \). Thus
\[ \text{Re} e^{re^{i\theta}-1} \leq 1 + r(\cos \delta - 1)/2 \leq 1 - \frac{r \delta^2}{6}, \]
The even central moments of a normal distribution with variance 1 are given by (see [5, p. 25])

\[
\int e^{r e^{i\theta}} - e^r = e^r \left( \int e^{r(e^{i\theta} - 1)} - 1 \right) \leq -\frac{e^r r^2 \delta^2}{6} = -2r(1 + \alpha(r)),
\]

for \( \delta \leq |\theta| \leq \pi \). Hence

\[
|\exp(h(\theta))| \leq \exp(-2r(1 + \alpha(r)) + 2\alpha(r)) \ll \exp(-2r).
\]

The contribution to the integral in (5) from \( \delta \leq |\theta| \leq \pi \) is thus

\[
\int_{\delta \leq |\theta| \leq \pi} |\exp(h(\theta))| d\theta \ll e^{-2r},
\]

which is acceptable.

The second task is to approximate \( h(\theta) \) for \( |\theta| \leq \delta \) by a Taylor polynomial and show that the error term is negligible. We have

\[
h(\theta) = -i\alpha'(r)r\theta - A \frac{\theta^2}{2} - iB \frac{\theta^3}{6} + O(r^4 e^r \theta^4),
\]

where

\[
a = r^2 e^r (1 - \alpha''(r)e^{-r}) + re^r (1 - \alpha'(r)e^{-r}),
\]

and

\[
b = e^r (r^3 + 3r^2 + r) - (r^3 \alpha'''(r) + 3r^2 \alpha''(r) + r\alpha'(r)) = O(e^r r^3).
\]

Since \( e^{-it} = 1 - it + O(t^2) \) for all real \( t \), we can write \( \exp(h(\theta)) \) as

\[
e^{-A\theta^2/2} \left( 1 - i\alpha'(r)r\theta - iB \frac{\theta^3}{6} + O((r\alpha'(r))^2\theta^2 + B^2 \theta^6) \right) (1 + O(r^4 e^r \theta^4)),
\]

where the last error term is justified since \( r^4 e^r \theta^4 = O(1) \) for \( |\theta| \leq \delta \), by Lemma [3]. Multiplying the two factors, and appealing to \( r^4 e^r \theta^4 = O(1) \), shows that \( \exp(h(\theta)) \) equals

\[
e^{-A\theta^2/2} \left( 1 - i\alpha'(r)r\theta - iB \frac{\theta^3}{6} + O((r\alpha'(r))^2\theta^2 + B^2 \theta^6 + r^4 e^r \theta^4) \right)
\]

and

\[
\int_{-\delta}^{\delta} \exp(h(\theta)) d\theta = \int_{-\delta}^{\delta} e^{-A\theta^2/2} d\theta + 0 + 0 + E,
\]

where

\[
E \ll \int_{-\delta}^{\delta} e^{-A\theta^2/2} ((r\alpha'(r))^2\theta^2 + B^2 \theta^6 + r^4 e^r \theta^4) d\theta.
\]

The even central moments of a normal distribution with variance \( 1/A \) are given by (see [5, p. 25])

\[
\sqrt{\frac{A}{2\pi}} \int_{-\infty}^{\infty} e^{-A\theta^2/2} \theta^{2k} d\theta = \frac{(2k)!}{(2A)^k k!} (k \geq 0).
\]

Since \( A \gg e^r r^2 \), by Lemma [3] and \( B \ll e^r r^3 \), we obtain

\[
E\sqrt{A} \ll \frac{(r\alpha'(r))^2}{A} + \frac{B^2}{A^3} + \frac{r^4 e^r}{A^2} \ll (\alpha'(r))^2 e^{-r} + e^{-r} + e^{-r}.
\]
\[ \int_{-\delta}^{\delta} \exp \left( h(\theta) \right) d\theta = \int_{-\delta}^{\delta} e^{-A\theta^2/2} d\theta + O \left( \frac{1 + (\alpha'(r))^2}{\sqrt{Ae^r}} \right). \]

Our third task is to extend the last integral to \((-\infty, \infty)\). We have
\[ \int_{-\delta}^{\infty} e^{-A\theta^2/2} d\theta \leq \int_{1}^{\infty} e^{-A\theta^2/2} d\theta + \int_{1}^{\infty} e^{-A\delta^2/2} d\theta \leq e^{-A\delta^2/2} + \frac{e^{-A/2}}{A/2} \ll e^{-2r}. \]

Thus,
\[ \int_{-\delta}^{\delta} \exp \left( h(\theta) \right) d\theta = \int_{-\infty}^{\infty} e^{-A\theta^2/2} d\theta + O \left( \frac{1 + (\alpha'(r))^2}{\sqrt{Ae^r}} \right) = \sqrt{\frac{2\pi}{A}} \left( 1 + O \left( \frac{1 + (\alpha'(r))^2}{e^r} \right) \right). \]

To approximate the quantity \(A\) by \(r(r+1)e^r\), we need to estimate \(\alpha''(r)\) in terms of \(\alpha'(r)\). We have
\[ \sum_{1 \leq k-1 \leq 3r, k \in S} \frac{r^{k-2}}{(k-2)!} \leq \sum_{1 \leq k-1 \leq 3r, k \in S} \frac{r^{k-2}}{(k-2)!} \cdot \frac{3r}{k-1} \leq 3\alpha'(r) \]
and
\[ \sum_{k-1 > 3r, k \in S} \frac{r^{k-2}}{(k-2)!} \leq \sum_{k-1 > 3r, k \in S} \frac{(3r)^{k-2}}{(k-2)!} \left( \frac{1}{3} \right)^{3r-1} < e^{3r} \left( \frac{1}{3} \right)^{3r-1} \leq 3. \]

Thus, \(\alpha''(r) \leq 3 + 3\alpha'(r)\), and
\[ A = r(r+1)e^r \left( 1 + O \left( \frac{1 + \alpha'(r)}{e^r} \right) \right). \]

Theorem 4 now follows from combining the estimates (6), (7) and (8) with equation (5). \(\square\)

3. Proof of Theorem 2

Sierpinski [7] and Stewart [9] independently gave the characterization of practical numbers in terms of their prime factors. The following analogue characterizes practical set partitions in terms of their block sizes.

**Lemma 4.** A set partition with \(l\) blocks of sizes \(a_1 \leq a_2 \leq \ldots \leq a_l\) is practical if and only if
\[ a_i \leq 1 + \sum_{1 \leq j < i} a_j \quad (1 \leq i \leq l). \]

**Proof.** Condition (1) is clearly necessary: if \(a_i > 1 + \sum_{1 \leq j < i} a_j\) for some \(1 \leq i \leq l\), then there is no set of size \(1 + \sum_{1 \leq j < i} a_j\) which is the union of different blocks.

To show that (1) is sufficient, we proceed by induction on \(l\). The case \(l = 1\) is obvious. Assume that (1) implies that the corresponding set partition with
block sizes $a_1 \leq \ldots \leq a_l$ is practical for some $l \geq 1$. Assume a set partition with block sizes $a_1 \leq \ldots \leq a_l \leq a_{l+1}$ satisfies (9), with $l$ replaced by $l+1$. The set of sizes of subsets obtained from combining different blocks is

$$A := \left\{ \sum_{i=1}^{l+1} \varepsilon_i a_i : \varepsilon_i \in \{0, 1\} \right\} = \left\{ \sum_{i=1}^{l} \varepsilon_i a_i : \varepsilon_i \in \{0, 1\} \right\} + \{0, a_{l+1}\}$$

By the inductive hypothesis, $A = \{1, 2, 3, \ldots, \sum_{i=1}^{l} a_i\} + \{0, a_{l+1}\}$, and since $a_{l+1} \leq 1 + \sum_{1 \leq j < l+1} a_j$, we have $A = \{1, 2, 3, \ldots, \sum_{i=1}^{l+1} a_i\}$.

The following functional equation is the analogue of [11, Lemma 2.3] for practical numbers and of [2, Lemma 5] for practical integer partitions. Other analogues include polynomials over finite fields [12, Lemma 5] and permutations [12, Lemma 11].

**Lemma 5.** For $n \geq 0$,

$$B_n = \sum_{k=0}^{n} {n \choose k} P_k B_{n-k,k+1}.$$ 

**Proof.** Given any partition of $n$ objects with block sizes $a_1 \leq a_2 \leq \ldots \leq a_l$, let $l_0$ be the largest index such that

$$a_i \leq 1 + \sum_{1 \leq j < i} a_j \quad (1 \leq i \leq l_0),$$

and let $l_0 = 0$ if $a_1 > 1$. By Lemma 4, the blocks of sizes $a_1 \leq \ldots \leq a_{l_0}$ form a practical set partition of a set with $k := \sum_{1 \leq j \leq l_0} a_j$ elements. Since $l_0$ was maximal, the remaining blocks have sizes $k + 1 < a_{l_0+1} \leq \ldots \leq a_l$ and form a $(k+1)$-rough set partition of a set with $n - k$ elements. The lemma now follows since $0 \leq k \leq n$ and there are $\binom{n}{k}$ ways to choose the $k$ elements that belong to the practical set partition. \qed

**Lemma 6.** For $n \geq 1$,

$$I_n = B_{n,1} + \sum_{k=1}^{\lfloor \frac{n-2}{2} \rfloor} {n \choose k} P_k B_{n-k,k+1}.$$ 

**Proof.** In Lemma 5, the term corresponding to $k = 0$ is $B_{n,1}$, since $P_0 = 1$. The term corresponding to $k = n$ is $P_n$, since $B_{0,n+1} = 1$. If $(n-2)/2 < k < n$, then $0 < n-k < k+2$, so $B_{n-k,k+1} = 0$. The result now follows since $I_n = B_n - P_n$. \qed

For the remainder of this section, we write

$$\beta_m(z) = \sum_{j=0}^{m} \frac{z^j}{j!}.$$
Lemma 7. For \( n \geq 1, \ m \geq 0 \), we have
\[
B_{n,m} \leq \frac{n! \exp(e^r - \beta_m(r))}{r^n}
\]

Proof. With \( S = \{1, 2, 3, \ldots, m\} \), the integrand in equation (4) satisfies
\[
\left| \exp \left( e^z - \beta_m(z) \right) \right| = \frac{\exp \left( \text{Re} \sum_{k=m+1}^{\infty} \frac{k^k}{k!} \right)}{|z|^{n+1}} \leq \frac{\exp \left( \sum_{k=m+1}^{\infty} \frac{k^k}{k!} \right)}{r^{n+1}},
\]
and therefore
\[
\frac{B_{n,m}}{n!} \leq \frac{1}{2\pi} \frac{2\pi r}{2\pi} \frac{\exp \left( \sum_{k=m+1}^{\infty} \frac{k^k}{k!} \right)}{r^{n+1}} = \frac{\exp(e^r - \beta_m(r))}{r^n}.
\]

\[\square\]

Proof of Theorem 2. We write \( b_{n,k} = B_{n,k}/n! \), \( p_n = P_n/n! \) and \( i_n = I_n/n! \). Lemma 6 says that
\[
0 \leq i_n - b_{n,1} = \sum_{k=1}^{N} p_k b_{n,k-1} + \sum_{k=1}^{N} b_k b_{n,k-1},
\]
where \( N = \lfloor \frac{n-2}{2} \rfloor \). To establish Theorem 2 we need to show that the last sum satisfies
\[
\sum_{k=1}^{N} b_k b_{n,k-1} \ll b_{n,1} \exp(-\log(n)^2/3).
\]

We write
\[
\sum_{k=1}^{N} b_k b_{n,k-1} = \sum_{1 \leq k \leq L} + \sum_{L < k \leq M} + \sum_{M < k \leq N} = \sum_{1 \leq k \leq L} + \sum_{L < k \leq M} + \sum_{M < k \leq N} = S_1 + S_2 + S_3,
\]
say, where \( L = (\log n)^{3/2} \) and \( M = \lfloor \frac{n-4}{3} \rfloor \). Let
\[
g(n) = n \log(r(n)) - \frac{n}{r(n)} \sim n \log \log n,
\]
by (2). It is easy to verify that
\[
g'(n) = \log r(n) \sim \log \log n.
\]

Theorem 1 shows that
\[
b_{n,1} = \exp(e^r - n \log r + O(r)) = \exp(-g(n) + O(\log n)),
\]
since \( e^r = n/r \). Similarly, Lemma 7 shows that, for \( n \geq 1 \) and \( m \geq 0 \),
\[
b_{n,m} \leq \exp(-g(n) - \beta_m(r(n))).
\]

It follows that
\[
\log(b_k b_{n-k-1}) \leq -g(k) - g(n - k) - \beta_{k+1}(r(n - k)).
\]
For $S_1$, we have $1 \leq k \leq L = (\log n)^{3/2}$, so (12) and (11) yield
\[
\log(b_kb_{n-k,k+1}) \leq 0 - g(n - L) - \beta_2(r(n/2)) \\
\leq -g(n) + O(L \log \log n) - (\log n)^2/(2 + \varepsilon) \\
\leq -g(n) - (\log n)^2/(2 + 2\varepsilon),
\]
for any $\varepsilon > 0$ and $n \geq n_0(\varepsilon)$. Hence
\[
S_1 \leq L \exp(-g(n) - (\log n)^2/(2 + 2\varepsilon)) \\
\leq b_{n,1} \exp(-g(n) - (\log n)^2/(2 + 3\varepsilon)). 
\]
(13)

When $L < k \leq N$, we have
\[
0 \leq e^{r(n-k)} - \beta_{k+1}(r(n-k)) = \sum_{j=k+2}^{\infty} \frac{(r(n-k))^j}{j!} \leq \sum_{j=k+2}^{\infty} \frac{k^j}{j!} \left( \frac{r(n)}{k} \right)^j \\
\leq \left( \frac{r(n)}{L} \right)^{k+2} e^k = o(1),
\]
as $n \to \infty$. Since $e^{r(n-k)} = (n-k)/r(n-k)$, (12) yields
\[
\log(b_kb_{n-k,k+1}) \leq -g(k) + g(n - k) - \frac{n-k}{r(n-k)} + o(1) \\
= -f(n,k) + o(1),
\]
say. We claim that $f(n,k)$ is decreasing (and hence $f(n,k)$ is increasing) in $k$, for $1 \leq k \leq n/2$. Indeed, since $r(n)$ is increasing, (11) shows that $g(k) + g(n - k)$ is decreasing in $k$, for $1 \leq k \leq n/2$. Moreover, $e^{r(n-k)}$ is clearly decreasing in $k$. Thus,
\[
S_2 \leq M \exp(-f(n,M) + o(1)) \\
\leq b_{n,1} \exp(g(n) - f(n, n/3) + O(\log n)) \\
\leq b_{n,1} \exp(-0.03n/\log n),
\]
for $n$ sufficiently large. The last inequality, whose derivation is not difficult but somewhat tedious, follows from (2).

Finally, when $M < k \leq N$, we have $n-k \geq k+2 > (n-k)/2$. Thus, $B_{n-k,k+1} = 1$ and $b_{n-k,k+1} = 1/(n-k)!$. Stirling’s approximation, in the form $\log(n!) = n(\log n - 1) + O(\log n)$, yields
\[
\log(b_kb_{n-k,k+1}) \leq -g(k) - \log((n-k)!)) \\
\leq 0 - (n-k)(\log(n-k) - 1) + O(\log n) \\
\leq -(n/2)(\log(n/2) - 1) + O(\log n).
\]
Hence
\[
S_3 \leq N \exp(-(n/2)(\log(n/2) - 1) + O(\log n)) \\
\leq b_{n,1} \exp(g(n) - (n/2)(\log(n/2) - 1) + O(\log n)) \\
\leq b_{n,1} \exp \left( -\frac{n \log n}{3} \right),
\]
(15)
for $n$ sufficiently large.

The estimates (13), (14) and (15) show that (10) holds, which completes the proof of Theorem 2. □

References

[1] R. M. Corless, G. H. Gonnet, D. E. G. Hare, D. J. Jeffrey, D. E. Knuth, On the Lambert $W$ function, Adv. Comput. Math. 5 (1996), no. 4, 329–359.
[2] J. Dixmier and J.-L. Nicolas, Partitions without small parts. Number theory, Vol. I (Budapest, 1987), 9–33, Colloq. Math. Soc. János Bolyai, 51, North-Holland, Amsterdam, 1990.
[3] P. Erdős and M. Szalay, On some problems of J. Dénes and P. Turán, Studies in pure mathematics, 187–212, Birkhäuser, Basel, 1983.
[4] P. Flajolet and R. Sedgewick, Analytic Combinatorics, Cambridge Univ. Press, 2009.
[5] J. Patel and C. Read, Handbook of the Normal Distribution, Second Edition, CRC Press, 1996.
[6] L. Moser and M. Wyman, An asymptotic formula for the Bell numbers, Transactions of the Royal Society of Canada 49 (1955) 49-54.
[7] W. Sierpinski, Sur une propriété des nombres naturels, Ann. Mat. Pura Appl. (4) 39 (1955), 69–74.
[8] A. K. Srinivasan, Practical numbers, Current Sci. 17 (1948), 179–180.
[9] B. M. Stewart, Sums of distinct divisors, Amer. J. Math. 76 (1954), 779–785.
[10] C. Wang and I. Mező, Some limit theorems with respect to constrained permutations and partitions, Monatsh. Math. 182 (2017), no. 1, 155–164.
[11] A. Weingartner, Practical numbers and the distribution of divisors, Q. J. Math. 66 (2015), 743–758.
[12] A. Weingartner, On the degrees of polynomial divisors over finite fields, Math. Proc. Cambridge Philos. Soc. 161 (2016), no. 3, 469–487.