A Systematic Study of the Chirality-Mixing Interactions in QCD

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Abstract

We present a study of the QCD interactions which do not conserve the chirality of quarks. These non-perturbative forces are responsible for the violation of the $U_A(1)$ charge conservation and for the breaking of chiral symmetry. From a systematic analysis we argue that the leading sources of chirality flips are the interactions mediated by topological vacuum field fluctuations. We study in detail the contribution of instantons and derive a simple model-independent semi-classical prediction. This result can be used to check on the lattice if instantons are the dynamical mechanism responsible both for chiral symmetry breaking and for the anomalous violation of the $U_A(1)$ charge conservation.

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I. INTRODUCTION AND MOTIVATION

Understanding the structure of the quark-quark interaction at all scales is a fundamental task of nuclear and high-energy physics, which requires the solution of the non-perturbative sector of QCD. Although this goal is not yet been completely achieved, a good deal of information has been gathered in the past decades. In particular, we known that the non-perturbative dynamics mixes quark modes of different chirality. This property of the strong interaction is signaled by the anomalous violation of the \( U_A(1) \) charge conservation and by the breaking of chiral symmetry. These two phenomena have important consequences on the physics of the light hadrons. The anomalous \( U_A(1) \) Breaking (U1B) allows to explain the absence of a ninth Goldstone boson. The Spontaneous Chiral Symmetry Breaking (SCSB) shapes the structure of the spectrum of the lightest hadrons and their interaction, at low momenta. Clearly, identifying the dynamical mechanism responsible for the SCSB and the U1B is a fundamental step toward our comprehension of the quark-quark interaction.

From the dynamical point of view, the structure of the spectrum of the lightest mesons indicates that quark-quark interaction is particularly attractive in the flavor non-singlet \( 0^- \) channel and less attractive (or possibly, at some distance, even repulsive) in the singlet \( 0^- \) channel (\( \eta' \)). Very useful insight in the physics of chiral symmetry breaking has come from the Nambu Jona-Lasinio (NJL) model \[1, 2\]. In this approach, one postulates an effective quark-quark interaction, which breaks \textit{spontaneously} the \( U_V(3) \times U_A(3) \) symmetry down to \( U_V(3) \). Such a symmetry breaking pattern would obviously generate nine Goldstone bosons, being also the \( U_A(1) \) symmetry broken spontaneously. Hence, in order to reproduce the observed \( \eta \) – \( \eta' \) splitting, an additional term is introduced, which breaks \( U_A(1) \) explicitly and simulates the contribution from the ‘t Hooft determinant interaction. Clearly, in such a framework, there are two independent sources of violation of the \( U_A(1) \) charge, because the dynamical origin of the spontaneous \( U_A(3) \) breaking and of the explicit (i.e. anomalous) U1B are distinct. In fact, by suppressing the instanton contribution in the Lagrangian, the \( \eta \) – \( \eta' \) splitting disappears, yet chiral symmetry remains broken, hence the pions remain light.

Conversely, one can imagine the opposite scenario in which \textit{both} chiral symmetry breaking and the \( \eta \) – \( \eta' \) splitting are consequences of the same dynamical mechanism. In other words, one can ask whether the same gauge configurations which generate the quark condensate
also break the $U_A(1)$ symmetry through the axial anomaly. In such a scenario, the $U_A(1)$ charge conservation is violated only through the anomaly. Moreover, there cannot be an $\eta - \eta'$ splitting without the SCSB, and vice-versa.

Instantons provide an example of gauge configurations which solve the $U(1)$ problem [3] and, at the same time, break chiral symmetry [4] (for a review see also [2]). The standing question is whether these semi-classical fields are the dominant configurations in both phenomena, or if their contribution is only sub-leading. Unfortunately, since a systematic semi-classical approach to QCD is not possible, one has to rely on phenomenological models, such as the Instanton Liquid Model (ILM) [8, 9]. This model was shown to quantitatively reproduce the breaking of chiral symmetry in the vacuum, and its restoration at high temperatures. Moreover, it also provides a good description of the spectrum light hadrons [10, 11, 12, 13] and their form factors [14, 15, 16, 17].

On the other hand, a traditional argument against the hypothesis that instantons drive both the U1B and the SCSB is based on the large $N_c$ limit [18]. In this limit, the topological susceptibility disappears, instantons are suppressed by the exponent of the action, yet chiral symmetry remains broken. One is then lead to suppose that, if the large $N_c$ world is at least qualitatively similar to the real one, instantons might not be the leading mechanism for the SCSB and that the $\eta - \eta'$ splitting and the SCSB have a different dynamical origin.

In a recent work, Schäfer studied in detail the instanton content of QCD with many colors [19]. He observed that the large entropy of instantons in $SU(N_c)$ can overcome the exponential suppression due to the action, in such a way that the instanton density can remain finite, at large $N_c$. From an analysis based on numerical simulations and mean-field estimates, he found that in the ILM the quark condensate is of $O(N_c)$, while the $\eta'$ mass is of $O(1/N_c)$. From this facts, he concluded that the ILM is not necessarily in conflict with the large $N_c$ analysis. Although a number of lattice studies seem to confirm a picture in which instantons drive the SCSB (see e.g. [6] and references therein), a general consensus on these issues has not yet been reached [1], and further studies are therefore needed.

In this work, we set-up a framework to study the time-evolution of the chirality of quarks, and we apply it to investigate the dynamics underlying the U1B and the SCSB, in QCD. In the next section, we shall define two functions of the Euclidean time, $R^S(\tau)$ and $R^{NS}(\tau)$, which measure the rate of chirality-flips, in a quark-antiquark system with zero total angular momentum, and total isospin 0 and 1, respectively. We will show that the typical time
between two chirality-flipping interactions scales with the inverse of the $\eta'$ mass, in the $I = 0$ channel, and with the inverse of the mass of the $\delta$ meson (which is the axial partner of the pion) in the $I = 1$ channel. Moreover, by performing a spectral analysis of our probability amplitude ratios, we shall show that chirality flips are dominated by the anomalous breaking of $U_A(1)$, while the contribution of the SBCS is only sub-leading. This result is model independent, because it is based only on the numerical values of the masses of the lowest lying scalar and pseudo-scalar mesons, which are known experimentally. It implies that, in the $|\bar{q}q\rangle$ system under consideration, the leading chirality mixing interaction is mediated by topologically charged gauge fields.

Then, we will use the Operator Product Expansion (OPE) to perform a systematic study of the dynamical origin of the mixing of chirality in QCD. We shall see that, if vacuum field fluctuations were small (vacuum dominance approximation), then the chirality flips would be dominated by the SCSB, and the contribution from the anomalous U1B would be completely negligible. However, this prediction is not supported by phenomenology which, as we have mentioned above, seems to favor the opposite scenario in which the anomalous U1B dominates. Therefore, one is lead to conclude that vacuum fluctuations play a major role in the mixing of chirality. Collecting all these observations, we shall argue that the leading dynamical source of helicity flips is represented by the interaction mediated by some large topological vacuum fluctuations, which badly violate the factorization assumption.

In section III we will address the question of whether such dynamics is driven by instantons. We shall first compute analytically the leading instanton contribution to the helicity-flip probability amplitudes, in the ILM. Then, we shall derive a simple semi-classical prediction, which does not depend on the phenomenological parameters of the ILM. Such a relationship is a model-independent signature of the instanton-induced interaction and can be used to check in an unambiguous way if instantons are responsible both for the SCSB and for the U1B. The feasibility of checking such a signature using lattice simulations will be discussed.

Interestingly, we find that the single-instanton effects generate the same rate of chirality flips, in the singlet and non-singlet channel. The equality $R^S(\tau) = R^{NS}(\tau)$ (which holds for small values of $\tau$, for which many-instanton effects can be neglected) is non-trivial and represents another model independent signature of instanton-induced forces.

Results and conclusions of this work are summarized in section IV.
II. MIXING OF QUARK CHIRALITY IN QCD

For sake of simplicity, let us consider QCD with two flavors. Moreover, since we are interested in the helicity flips generated by the quark-quark interaction, it is convenient to work in the chiral limit. This way, the purely kinematical chirality flips induced by the quark mass are suppressed.

We begin by defining the following combination of gauge invariant correlation functions:

\[
A_{flip}^{NS}(\tau) := \langle 0 | T [\bar{u}(\tau) P_L d(\tau) \bar{d}(0) P_R u(0)] + (P_L \leftrightarrow P_R) \\
+ \langle 0 | T [\bar{d}(\tau) P_L u(\tau) \bar{u}(0) P_R d(0)] + (P_L \leftrightarrow P_R)
\]

(1)

\[
A_{flip}^{S}(\tau) := \frac{1}{2}[\langle 0 | T [\bar{u}(\tau) P_L u(\tau) \bar{d}(0) P_R d(0)] + (P_R \leftrightarrow P_L) \\
+ \langle 0 | T [\bar{d}(\tau) P_L d(\tau) \bar{u}(0) P_R u(0)] + (P_R \leftrightarrow P_L) \\
+ \langle 0 | T [u(\tau) P_L u(\tau) \bar{u}(0) P_R d(0)] + (P_R \leftrightarrow P_L) \\
+ \langle 0 | T [d(\tau) P_L d(\tau) \bar{u}(0) P_R u(0)] + (P_R \leftrightarrow P_L)]
\]

(2)

where,

\[
P_R := \frac{1 + \gamma_5}{2}, \quad P_L := \frac{1 - \gamma_5}{2}.
\]

(3)

\(A_{flip}^{NS}(\tau)\) and \(A_{flip}^{S}(\tau)\) denote the probability amplitude for a flavor singlet (non-singlet) \(|q \bar{q}\) state to be found, after a time interval \(\tau\), in a state in which the chirality of the quark and antiquark is flipped\(^1\). In QCD, even in the case in which \(m_u = m_d = 0\), quark states with different chirality can mix under time evolution. Hence, we expect such matrix elements to be in non-vanishing, in general.

Similarly, we define:

\[
A_{non\text{-}flip}^{NS}(\tau) := \langle 0 | T [\bar{u}(\tau) P_L d(\tau) \bar{d}(0) P_L u(0)] + (P_L \leftrightarrow P_R) \\
+ \langle 0 | T [\bar{d}(\tau) P_L u(\tau) \bar{u}(0) P_L d(0)] + (P_L \leftrightarrow P_R)
\]

(4)

\[
A_{non\text{-}flip}^{S}(\tau) := \frac{1}{2}[\langle 0 | T [\bar{u}(\tau) P_L u(\tau) \bar{d}(0) P_L d(0)] + (P_L \leftrightarrow P_R) \\
+ \langle 0 | T [\bar{d}(\tau) P_L d(\tau) \bar{u}(0) P_L u(0)] + (P_L \leftrightarrow P_R) \\
+ \langle 0 | T [u(\tau) P_L u(\tau) \bar{u}(0) P_L d(0)] + (P_L \leftrightarrow P_R) \\
+ \langle 0 | T [d(\tau) P_L d(\tau) \bar{u}(0) P_L u(0)] + (P_L \leftrightarrow P_R)]
\]

(5)

\(^1\) In all formulas, analytic continuation to Euclidean time is assumed.
These functions measure the probability amplitude for the quark and antiquark not to have exchanged their chirality, after a time interval $\tau$.

It is convenient to rewrite such matrix elements in terms of scalar and pseudo-scalar mesonic two-point correlation functions:

$$A_{NS}^{\text{flip}}(\tau) = \Pi_\pi(\tau) - \Pi_\delta(\tau)$$

$$A_{NS}^{\text{non-flip}}(\tau) = \Pi_\delta(\tau) + \Pi_\pi(\tau),$$

$$A_{S}^{\text{flip}}(\tau) = \Pi_\sigma(\tau) - \Pi_{\eta'}(\tau)$$

$$A_{S}^{\text{non-flip}}(\tau) = \Pi_\sigma(\tau) + \Pi_{\eta'}(\tau),$$

where the correlation functions are defined as:

$$\Pi_\pi(\tau) = \langle 0 | J_\pi(\tau) J_{\pi}^\dagger(0) | 0 \rangle$$

$$\Pi_{\eta'}(\tau) = \langle 0 | J_{\eta'}(\tau) J_{\eta'}^\dagger(0) | 0 \rangle$$

$$\Pi_\delta(\tau) = \langle 0 | J_\delta(\tau) J_\delta^\dagger(0) | 0 \rangle$$

$$\Pi_\sigma(\tau) = \langle 0 | J_\sigma(\tau) J_\sigma^\dagger(0) | 0 \rangle.$$

The interpolating operators, exciting states with given $(I, J^p)$ quantum numbers, are defined as:

$$J_\pi(\tau) := \bar{u}(\tau) i \gamma_5 d(\tau)$$

$$J_\delta(\tau) := \bar{u}(\tau) d(\tau)$$

$$J_{\eta'}(\tau) := \frac{1}{\sqrt{2}} \left( \bar{u}(\tau) i \gamma_5 u(\tau) + \bar{d}(\tau) i \gamma_5 d(\tau) \right)$$

$$J_\sigma(\tau) := \frac{1}{\sqrt{2}} \left( \bar{u}(\tau) u(\tau) + \bar{d}(\tau) d(\tau) \right)$$

At this point it is worth observing that the operators $J_\pi$, $J_\delta$, $J_{\eta'}$, and $J_\sigma$ can be transformed into each other by means of appropriate $U_A(3)$ transformation (see Fig. 1).

With these amplitudes, we can now construct two ratios which represent the probability amplitude to find the quarks in the flipped chirality state, relative to the amplitude for

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2 In this expression, have denoted with $\eta'$ an iso-singlet pseudo-scalar state, i.e. the $SU(N_f = 2)$ correspondent of the $|\eta_0\rangle$ state. Moreover we can, without loss of generality, choose $\tau > 0$ and disregard the Euclidean time-ordering.
In the chiral limit, these ratios carry information about the rate of chirality mixing interactions, in QCD.

Let us study (18) and (19) in the limit of large Euclidean times. In this case, one expects quarks to interact several times. Intuitively, there should exist a characteristic time scale \( \tilde{\tau} \), determining how often chirality mixing interactions occur in the QCD vacuum. After having exchanged their chirality many times, quarks essentially “loose the memory” of what was their initial state, therefore they should be found in either chirality configuration, with equal probability. That is to say:

\[
R^{NS}(\tau), R^{S}(\tau) \xrightarrow{\tau \gg \tilde{\tau}} 1.
\]

In order to derive this result rigorously and to determine \( \tilde{\tau} \), we use the fact that, at large enough \( \tau \), the spectral representation of the correlation functions \( \Pi_{\pi}, \Pi_{\delta}, \Pi_{\sigma}, \Pi_{\eta'} \) is dominated by the contribution of the lowest lying states with the appropriate quantum numbers \( \Pi_{\pi}, \Pi_{\delta}, \Pi_{\sigma}, \Pi_{\eta'} \):

\[
\Pi_{\pi}(\tau) \xrightarrow{\tau \to \infty} \lambda_{\pi}^{2} \frac{m_{\pi}}{4 \pi^{2} \tau} K_{1}(m_{\pi} \tau) \quad (21)
\]

\[
\Pi_{\delta}(\tau) \xrightarrow{\tau \to \infty} \lambda_{\delta}^{2} \frac{m_{\delta}}{4 \pi^{2} \tau} K_{1}(m_{\delta} \tau) \quad (22)
\]

\[
\Pi_{\sigma}(\tau) \xrightarrow{\tau \to \infty} 2 |\langle 0|qq|0 \rangle|^{2} + \lambda_{\sigma}^{2} \frac{m_{\sigma}}{4 \pi^{2} \tau} K_{1}(m_{\sigma} \tau), \quad (23)
\]

\[
\Pi_{\eta'}(\tau) \xrightarrow{\tau \to \infty} \lambda_{\eta'}^{2} \frac{m_{\eta'}}{4 \pi^{2} \tau} K_{1}(m_{\eta'} \tau) \quad (24)
\]
where \( \lambda_{\pi,\delta,\sigma,\eta'} \) are the coupling constants of the densities to the corresponding lowest lying states.

Hence, in the large \( \tau \) limit, we can rewrite the ratio \( R^{NS}(\tau) \) in the following way:

\[
R^{NS}(\tau) = \frac{1 - \xi(\tau)}{1 + \xi(\tau)} \xrightarrow{\tau \to \infty} 1 - 2\xi(\tau) + o(\xi^2),
\]

(25)

where

\[
\xi(\tau) := \frac{\Pi_\delta(\tau)}{\Pi_\pi(\tau)} \xrightarrow{\tau \to \infty} \left( \frac{\lambda_\delta}{\lambda_\pi} \right)^2 \sqrt{\frac{m_\delta}{m_\pi}} e^{-(m_\delta - m_\pi)\tau},
\]

(26)

These relationships show that the information about the initial chirality of the quarks is exponentially destroyed, as the time increases. From the exponent in (26) can immediately read-off the corresponding characteristic time scale:

\[
\tilde{\tau}^{NS} = \frac{1}{(m_\delta - m_\pi)} \sim 0.2 \text{ fm}.
\]

(27)

We shall now prove that the ratio \( R^{NS}(\tau) \) represents the amplitude for the chirality-flips induced by the non-conservation of the \( U_A(1) \) charge. If the \( U_A(1) \) symmetry is not spoiled, then we have \( m_\pi = m_\delta \) and \( \lambda_\pi = \lambda_\delta \). As a consequence \( \xi(t) \to 1 \) and the chirality-flip amplitude ratio tends to zero, at infinity. This condition is sufficient to imply that \( R^{NS}(\tau) \) is identically zero. In fact, let us suppose that this function is not identically zero, yet vanishes at infinity. This corresponds to the unphysical scenario in which quarks can flip their chirality only in some finite time interval, but eventually have to “go back” to their initial chirality state.

The fact that \( \Pi_\pi(\tau) \neq \Pi_\delta(\tau) \) and therefore \( R^{NS}(\tau) \neq 0 \) only implies that the axial charge is not conserved, regardless if this is predominantly because of a spontaneous or anomalous breaking of the \( U_A(1) \) symmetry\(^3\). The combination of correlators that relates directly to the anomaly is the difference between the \( \pi \) and \( \eta' \) two-point functions, \( \Pi_\pi(\tau) - \Pi_{\eta'}(\tau) \). It is therefore convenient to add and subtract the \( \Pi_{\eta'}(\tau) \) correlator, from the numerator in \( R^{NS}(\tau) \):

\[
R^{NS}(\tau) = \frac{\Pi_\pi(\tau) - \Pi_{\eta'}(\tau)}{\Pi_\pi(\tau) + \Pi_\delta(\tau)} + \frac{\Pi_{\eta'}(\tau) - \Pi_\delta(\tau)}{\Pi_\pi(\tau) + \Pi_\delta(\tau)} =: R^{NS}_{\text{anom.}}(\tau) + R^{NS}_{SU_A}(\tau).
\]

(28)

\(^3\) Indeed, in the NJL model, in which the \( U_A(1) \) is broken spontaneously, one has \( \Pi_\pi(\tau) > \Pi_\delta(\tau) \). We thank T.Schäfer for pointing this out.
We are now in condition to disentangle the violation of the axial charge conservation induced by the anomalous and by the spontaneous breaking of $U_A(1)$. Indeed, if there was no anomaly, and the $U_A(1)$ symmetry was just spontaneously broken, then $\Pi_\pi(\tau) = \Pi_{\eta'}(\tau)$ and $R_{\text{anom.}}^{NS}(\tau)$ would vanish. In this case, the contribution to $R^{NS}(\tau)$ would come entirely from $R_{SU_A}^{NS}(\tau)$. Conversely, if the $SU_A(N_f)$ symmetry was not broken, then we would have $\Pi_{\eta'}(\tau) = \Pi_\delta(\tau)$ (see Fig. 1), therefore $R_{SU_A}^{NS}(\tau) = 0$. In the real world we observe a relatively large $\pi - \eta'$ splitting. As a consequence, $R^{NS}(\tau)$ turns out to be dominated by the anomalous term, $R_{\text{anom.}}^{NS}(\tau)$. To see this, we use again the spectral representation, in the large Euclidean time limit:

$$R_{\text{anom.}}^{NS}(\tau) = 1 - C_{\eta'\Pi} e^{-(m_{\eta'} - m_\pi) \tau} - C_{\delta\Pi} e^{-(m_\delta - m_\pi) \tau} + \ldots,$$

$$R_{SU_A}^{NS}(\tau) = C_{\eta'\pi} e^{-(m_{\eta'} - m_\pi) \tau} (1 - C_{\delta\pi} e^{-(m_\delta - m_\pi) \tau} - C_{\eta'\Pi} e^{-(m_{\eta'} - m_\pi) \tau}) + \ldots,$$

where

$$C_{h'h'} := \left(\frac{\lambda_h}{\lambda_{h'}}\right)^2 \frac{m_h}{m_{h'}},$$

and ellipses denote higher order terms in $e^{-(m_\delta - m_\pi) \tau}$. We can see that, in this limit, $R_{\text{anom.}}^{NS}(\tau)$ is of order 1, while $R_{SU_A}^{NS}(\tau)$ is exponentially suppressed by the $\eta' - \pi$ mass difference. We conclude that the violation of the axial charge conservation, parametrized by the function $R^{NS}(\tau)$, is mainly due to the anomalous breaking of the $U_A(1)$ symmetry. This conclusion has quite important implications on quarks and gluons dynamics. It suggests that, in this channel, the leading chirality mixing interaction is mediated by topologically charged gauge fields.

For sake of completeness, let us now consider the helicity-flip amplitude ratio in the singlet channel:

$$R^S(\tau) = \frac{1 - \epsilon(\tau)}{1 + \epsilon(\tau)} \xrightarrow{\tau \to \infty} 1 - 2\epsilon(\tau) + o(\epsilon^2),$$

where,

$$\epsilon(\tau) := \frac{\Pi_{\eta'}(\tau)}{\Pi_{\eta}(\tau)} \xrightarrow{\tau \to \infty} \frac{\lambda_{\eta'}^2 \sqrt{m_{\eta'}}}{\sqrt{2} 8 (0|\bar{q}q|0)^2} \frac{e^{-m_{\eta'} \tau}}{(\pi \tau)^{3/2}},$$

In this channel, we could not find a simple way to single-out the effects due to the anomalous U1B. This is essentially because the scalar density operator $13$ has a non-vanishing
vacuum expectation value (Eq. 23). As a consequence, \( R^S(\tau) \) receives an explicit contribution from the quark condensate\(^4\)(Eq. 33). Notice that the anomalous U1B contribution induce an exponential mixing of chirality, while the SCSB participates only through an additional pre-exponent. The characteristic time, which determines the exponential mixing of chirality is the inverse of the \( \eta' \) mass:

\[
\tilde{\tau}^S = \frac{1}{m_{\eta'}}.
\]  

(34)

Notice that \( \tilde{\tau}^S, \tilde{\tau}^{NS} \ll \frac{1}{\Lambda_{QCD}} \), which suggests that the non-perturbative physics associated with the U1B and SCSB is characterized by an additional scale, significantly larger than \( \Lambda_{QCD} \).

The phenomenological analysis performed so far has indicated that the helicity flips in the non-singlet channel are dominated by the dynamics responsible for the anomalous U1B. Now we want investigate what can be learnt from this result about the non-perturbative quark-quark dynamics. At this purpose, we need to consider \( R^{NS}(\tau) \) in the opposite limit of small Euclidean times, where quark and gluons are the relevant degrees of freedom. After performing Wick contractions, the correlation functions \([10,13]\) read:

\[
\Pi_{\pi}(\tau) = \langle \text{Tr} [S(\tau, 0) \gamma_5 S(0, \tau) \gamma_5] \rangle,
\]

(35)

\[
\Pi_{\delta}(\tau) = -(\text{Tr} [S(\tau, 0) S(0, \tau)]),
\]

(36)

\[
\Pi_{\eta'}(\tau) = [(\text{Tr} [S(\tau, 0) \gamma_5 S(0, \tau) \gamma_5]) - 2 \langle \text{Tr} [S(\tau, \tau) \gamma_5] \text{Tr}[S(0, 0) \gamma_5]\rangle],
\]

(37)

\[
\Pi_{\sigma}(\tau) = [2 \langle \text{Tr} [S(\tau, \tau)] \text{Tr}[S(0, 0)]\rangle - \langle \text{Tr} [S(\tau, 0) S(0, \tau)]\rangle],
\]

(38)

where \( S(\tau, 0) \) is the quark propagator and \( \langle \cdot \rangle \) denotes the average over all gauge configurations. At extremely small distances, asymptotic freedom demands that the correlators \([10,13]\) approach those evaluated in the free theory. One has:

\[
\Pi_{\pi}(\tau), \Pi_{\eta'}(\tau), \Pi_{\delta}(\tau), \Pi_{\sigma}(\tau) \overset{\tau \to 0}{\to} \Pi_0(\tau) := \frac{N_c}{\pi^4 \tau^6}.
\]

(39)

Consequently, we have \( R^S(\tau), R^{NS}(\tau) \overset{\tau \to 0}{\to} 0 \). This is an expected result: in a free theory of massless quarks, chirality is a conserved quantum number, and the rate of quark helicity flips is zero.

\(^4\) It is tempting to simply replace \( \Pi_{\sigma}(\tau) \) with a modified scalar two-point functions, in which the vacuum contribution has been subtracted out, \( \Pi'_{\sigma}(\tau) := \Pi_{\sigma}(\tau) - 2\langle 0 | \bar{q} q | 0 \rangle^2 \). However, the corresponding amplitude ratio, \( R_{\sigma}^S(\tau) := (\Pi'_{\sigma} - \Pi_{\eta'})/(\Pi_{\sigma} + \Pi_{\eta'}) \) would not have a simple probabilistic interpretation in terms of chirality flips.
As $\tau$ gradually increases from zero, perturbative corrections to the correlators (10-13) become more and more important. However, their contribution to (18) and (19) has to vanish exactly, because the perturbative quark-gluon vertex does not mix quark modes with opposite chirality. Hence, any deviation from zero of the ratios (18) and (19) is a signature of non-perturbative physics.

Non-perturbative QCD corrections, in the limit $\tau \to 0$, can be evaluated in a systematic way, using the Operator Product Expansion (OPE). It is known that, in the chiral limit, the quark and gluon condensate singular terms in OPE cannot distinguish between scalar and pseudo-scalar mesonic correlators [22, 23]. Hence, including the contribution up to dimension 4 operators one finds $R^{NS}(\tau) = 0$.

In the 't Hooft picture, in which the anomaly is realized by instantons, the fact that such singular terms cannot reproduce the chirality flipping due to the non-conservation of the axial charge is an expected result. In fact, the leading instanton contributions to the correlators (10,12) tend to a constant, in the $\tau \to 0$ limit. Therefore, one expects some deviation from zero in $R^{NS}(\tau)$ to appear, starting from the regular terms, i.e. from dimension 6 operators. Indeed, we find:

\begin{align}
R^{NS\text{ anom.}}(\tau) &\approx -\frac{2}{N_c} \frac{\pi^4}{\tau^6} \left( \langle 0 | \bar{u} i \gamma_5 u \bar{d} i \gamma_5 d | 0 \rangle \right) + \ldots \\
R^{NS}_{SU_A}(\tau) &\approx \frac{\pi^4}{N_c} \left[ \left( \langle 0 | \bar{u} i \gamma_5 u \bar{u} \gamma_5 u | 0 \rangle - \langle 0 | \bar{u} u \bar{u} u | 0 \rangle \right) \right. \\
&\quad + \left. \left( \langle 0 | \bar{u} i \gamma_5 u \bar{d} i \gamma_5 d | 0 \rangle + \langle 0 | \bar{d} u \bar{d} d | 0 \rangle \right) \right] \tau^6 + \ldots.
\end{align}

Let us now discuss the implications of this result. Quite interestingly, we have found that the quantity:

\[ \Delta = \lim_{\tau \to 0} \left| \frac{R^{NS}_{SU_A}(\tau)}{R^{NS\text{ anom.}}(\tau)} \right| = \frac{\left| \frac{\langle 0 | \bar{u} i \gamma_5 u \bar{u} \gamma_5 u | 0 \rangle + \langle 0 | \bar{u} i \gamma_5 u \bar{d} i \gamma_5 d | 0 \rangle + \langle 0 | \bar{d} u \bar{d} d | 0 \rangle - \langle 0 | \bar{u} u \bar{u} u | 0 \rangle}{2 \langle 0 | \bar{u} i \gamma_5 u \bar{d} i \gamma_5 d | 0 \rangle} \right| \]

parametrizes the relative contribution to quark helicity flips of the spontaneous symmetry breaking with respect to the anomalous symmetry breaking. In particular, if $\Delta \simeq 0$, then the violation of the $U_A(1)$ charge comes almost entirely from the anomaly.

\footnote{In this expression and in (42) we have used isospin symmetry and we have neglected logarithmic perturbative corrections. Notice that the term proportional to the quark condensate and that proportional to the mixed quark condensate $g(0)\bar{q}G_{\mu\nu}\sigma_{\mu\nu}q|0\rangle$ are suppressed, in the chiral limit.}
It is instructive to study the behavior of the chirality flip ratio $R^{NS}(\tau)$, in the large $N_c$ limit. In this limit, the spectral representation of the condensates is dominated by vacuum insertions and one can use the factorization approximation:

$$
\langle 0 | \bar{q} \Gamma_1 q \bar{q} \Gamma_2 q | 0 \rangle \approx \frac{1}{N^2} (\text{Tr}[\Gamma_1] \text{Tr}[\Gamma_2] - \text{Tr}[\Gamma_1 \Gamma_2]) \langle 0 | \bar{q} q | 0 \rangle^2,
$$

where $N := 4 N_f N_c$, (43)

which gives $\langle 0 | \bar{u} i \gamma_5 u \bar{d} i \gamma_5 d | 0 \rangle \approx 0$, hence $\Delta \rightarrow \infty$. This result implies that, in the asymptotically large $N_c$ world, the chirality flips are dominated by the spontaneous chiral symmetry breaking, while the anomalous contribution disappears, in agreement with common wisdom.

In the real world, the accuracy of the factorization approximation is questionable. Indeed, the analysis performed at large distances indicates that the spin flips are actually dominated by the anomaly and not by the SCSB (Eqs. (29) and (30)). In general, the vacuum dominance assumption is expected to fail in the presence of large vacuum field fluctuations. Hence, our analysis suggests that in QCD the helicity flips are mainly induced by some large vacuum gauge field fluctuations. As we mentioned above, such fluctuations have to be topologically charged, as they induce the U1B though the anomaly. The most natural candidates are instantons. Their contribution will be analyzed in the next section.

### III. INSTANTON-INDUCED CHIRALITY FLIPS

In the previous section we have argued that the helicity flips are dominated by some large field fluctuations which have connection with the axial anomaly. Instantons are example of gauge fields which, at the same time, provide a realization of the anomaly and break chiral symmetry. It is therefore interesting to estimate their contribution to the ratio (19). At short distances, the leading instanton contribution to the relevant correlation functions can be computed analytically, using the Single Instanton Approximation (SIA). This is an effective theory of the instanton vacuum (for a detailed description of this approach, see [24]), in which the contribution of the closest instanton is taken into account explicitly, while all other (infra-red) multi-instanton degrees of freedom are integrated out and replaced by a single effective parameter $m^*$, defined as:

$$
m^* = \left[ \left( \int d\rho d(\rho) \frac{\hat{n}}{5 \pi^2 \rho^4} \right) \frac{1}{\left[ \text{Tr} \sum_{I,J} \psi_{0I}(x) (\frac{1}{T})_{I,J} \psi_{0J}^\dagger(x) \right]^2} \right]^{1/2},
$$

(44)
where \(d(\rho)\) is the instanton size distribution, \(\bar{n}\) is the instanton density, \(\psi_{0\ I}(x)\) is the zero mode wave function in the field of the \(I\)-th pseudo-particle and \(T_{I\ J}\) is the overlap matrix, given by:

\[
T_{I\ J} = \int d^4z \psi_{0\ I}^\dagger(z)(i\partial\psi)\psi_{0\ J}(z).
\] (45)

The effective parameter \(m^*\), computed numerically in the ILM from (44), is of the order of 70 MeV. The main advantage of the SIA is that it allows to obtain the leading instanton contributions to Green’s functions by means of simple analytical calculations. The range of applicability of this approach was studied in detail [17, 24]. It was found to be very accurate for correlation functions smaller than the typical distance between two neighbor instantons.

The evaluation of the correlation functions [10, 12] leads to:

\[
\Pi_{SIA} = \Pi_0(\tau) + \Pi_{OI}(\tau) \quad (46)
\]

\[
\Pi_{SIA} = \Pi_0(\tau) - \Pi_{OI}(\tau) \quad (47)
\]

where \(\Pi_0(\tau)\) is the usual free correlator defined in [39] and \(\Pi_{OI}(\tau)\) is the instanton contribution, given by:

\[
\Pi_{OI}(x) = \frac{4\bar{n}}{m^*^2 \pi^4} \int d^4z \int d\rho \ d(\rho) \left( \frac{\rho^4}{[z^2 + \rho^2]^3 ((z - x)^2 + \rho^2)^3} \right). \quad (49)
\]

The corresponding result for the chirality-flip amplitude ratio is:

\[
R_{NS}^{SIA}(\tau) = \frac{\Pi_{OI}(\tau)}{\Pi_0(\tau)}, \quad (50)
\]

In passing, we observe that the leading instanton contribution to \(R^S(\tau)\) is the same as that to \(R^{NS}(\tau)\):

\[
R_{SIA}^S(\tau) = R_{SIA}^{NS}(\tau). \quad (51)
\]

This equality, which is valid only in the region in which the one-instanton effects are dominant, is non-trivial. In fact, we recall that at large distances these quantities have very

---

\[6\] For sake of simplicity, in this calculation we have replaced the non-zero mode part of the quark propagator with the free propagator (zero-mode approximation). It is possible to show that this approximation is very accurate for the particular correlation functions we are considering [25]. The main results of this section (Eq. [52] and Eq. [54] below) do not change, when one correctly accounts for the non zero-mode contributions.
different spectral representations. Hence, in general, we expect them to be different functions of $\tau$.

Let us now establish the connection with the OPE analysis of the previous section. Expanding Eq. (49) for short Euclidean times one finds:

$$R_{NS}^{SIA}(\tau) = \pi^2 \bar{n} \frac{1}{15 m^*} \int d\rho d(\rho) \frac{1}{\rho^4} \tau^6 + ...$$

(52)

where the ellipsis denote terms which are higher orders in $\tau$.

In general, calculations of instantonic effects require the knowledge of two quantities which cannot be obtained in a systematic way in QCD: the instanton density $\bar{n}$, and the instanton size distribution $d(\rho)$. From a phenomenological estimate, Shuryak suggested to use $\bar{n} \approx 1$ fm$^{-4}$, and $d(\rho) = \delta(\rho-1/3\text{fm})$. In Fig. (2) the SIA prediction (50) for $R_{NS}^{SIA}(\tau)$ is compared with its short-time expansion (52), and with the result of numerical simulations in the Random Instanton Liquid Model (RILM)$^8$, in which many-instantons effects are taken explicitly into account. We observe that the very simple SIA analytic prediction agrees with the much more complicated RILM calculation up to quite large times $\tau \approx 0.6$ fm. Moreover, we observe that the short-time expression (52) converges up to distances of the order of $\tau \lesssim 0.25$ fm. This provides an estimate of the radius of convergence of OPE expansions, when the direct-instanton term is included.

The results discussed so far depend on the phenomenological parameters of the ILM. Now we make one more step and show that it is possible to circumvent all the model dependence$^9$, and obtain the semi-classical prediction. At this purpose, we observe that the instanton contribution to the four-quark condensate reads:

$$\langle 0|\bar{q} q \bar{q} q|0\rangle_{SIA} = \frac{2 \bar{n}}{5 \pi^2 m^*} \int d\rho d(\rho) \frac{1}{\rho^4} = \langle 0|\bar{q} \gamma_5 q \bar{q} \gamma_5 q|0\rangle_{SIA}$$

(53)

We notice that leading instanton effects contribute equally to the pseudo-scalar and scalar four-condensates and therefore badly violate the factorization assumption. This implies that, in the instanton vacuum, $\Delta \approx 0$. Therefore, the instanton-induced U1B is channeled

---

7 We recall that the effective parameter $m^*$ can be obtained from $d(\rho)$ and $\bar{n}$ through Eq. (44).
8 For a detailed description of this model and a comparison with other versions of the ILM, see [9].
9 As another example of model-independent instanton prediction in QCD is discussed in [13]. In that paper, we computed the proton and pion dispersion curves and we derived, in a parameter-free way, the instanton contribution to the nucleon mass.
FIG. 2: The chirality flipping amplitude ratio, in the non-singlet channel, $R^{NS}(\tau)$. Points are numerical simulations in the RILM, the solid line denotes the SIA prediction (50), and the dashed line its short-time expression (52). The dot-dashed line represents the explicit breaking contribution (55), with $m = 150$ MeV.

exclusively through the axial anomaly, in agreement with the results of the phenomenological analysis presented in the previous section.

Combining (53) with (52) we can include all unknown quantities appearing in our calculation in the expression for the quark condensate. This way, we obtain the direct-instanton contribution to OPE for (18) and (19):

$$R^{NS}_{D.I.}(\tau) = \frac{\pi^4}{6} \langle 0 | q \bar{q} q \bar{q} | 0 \rangle \tau^6 + ... ,$$

(54)

It is worth emphasizing that this semi-classical calculation requires no assumption on the instanton size and density, yet predicts a particular relationship between different vacuum expectation values. Hence, if the semi-classical contribution is large, so that instantons are the leading configurations responsible for the U1B and the SCSB, then this relationship must be at least approximatively satisfied, also when the quantum average is performed over all configurations (and not only over the semi-classical fluctuations).

Lattice QCD represents the natural framework in which performing such a test. On the practical level, however, up-to-date simulations still need to use quite large values of the current quark masses. Far from the chiral limit, the explicit breaking of chiral symmetry provides a competing source of chirality flips, which has to taken into account. The impor-
tance of this effect can be estimated by evaluating (19) in the free theory - using a massive quark propagator - and then comparing with the ILM prediction (52). After an elementary calculation, we obtain:

$$R_{\text{explicit}}^{NS} (\tau) = \frac{m^2}{4} \tau^2 + ..., \quad (55)$$

where $m$ is the current quark mass. Clearly, at short distances, the contribution to helicity flips coming from the explicit breaking of chiral symmetry (55) will necessarily dominate over the dynamical effects discussed above.

The key question is if there exists a window of Euclidean times in which the instanton effects are dominant over the kinematic ones, and the single-instanton prediction (54) is reliable. In Fig. (2) the contribution of explicit chiral symmetry breaking (55) with a current mass $m = 150$ MeV (of the order of those used in a typical lattice calculation) is compared with the SIA and RILM estimates. We conclude that, in the region $0.1 \, \text{fm} \lesssim \tau \lesssim 0.25 \, \text{fm}$, the instanton contribution should be dominant. Moreover, in this window, the relevant correlation functions (10,12) are of order 1, so it should not be numerically very challenging to measure this signal on the lattice.

IV. CONCLUSIONS

In this paper we have presented a study of the mixing of chirality under time evolution, in QCD. In the chiral limit, quark helicity-flips have a purely dynamical origin, therefore they carry information about the quark-quark interaction.

We have constructed two combinations of correlation functions, $R^{NS}(\tau)$ and $R^{S}(\tau)$, which are related to the rate of chirality flips in a quark-antiquark pair, propagating in the QCD vacuum. Such functions receive no contribution from any number of perturbative gluon exchanges, therefore they represent useful tools to investigate the non-perturbative dynamics.

We have shown that the chirality flipping processes contributing to $R^{NS}(\tau)$ are induced by the same non-perturbative forces responsible for the non-conservation of the axial charge. In fact, the ratio $R^{NS}(\tau)$ would be identically zero, if the degeneracy between the pion and its $U_A(1)$ partner was not lifted. Using the spectral decomposition at large Euclidean times, we have studied the relative contribution to the U1B coming from the axial anomaly and from the possible spontaneous breaking of $U_A(1)$. We found that the U1B comes almost
entirely from the axial anomaly. From a such a spectral analysis, we have also obtained an estimate of the characteristic time between two consecutive spin-flipping interactions in QCD and found $\bar{\tau} \sim 0.2$ fm. This value suggests that the chirality mixing dynamics is characterized by a non-perturbative length scale which is significantly smaller than the typical confinement scale, $1/A_{QCD} \simeq 1$ fm.

We have calculated the first non-perturbative corrections to the amplitude ratio $R^\text{NS}(\tau)$, using OPE. We have found that these are associated to the regular terms, which are commonly related to the so-called “direct instanton” contribution. This is consistent with the 't Hooft solution of the $U(1)$ problem, in which the axial anomaly is realized by instantons. We observed that the fact that $R^\text{NS}(\tau)$ is dominated by the anomaly implies that vacuum field fluctuations are large. In fact, in the vacuum dominance approximation, in which fluctuations are neglected, the chirality flips are dominated by the SCSB, in disagreement with what we found phenomenologically from the spectral analysis. This suggests that the helicity flips are induced by topological gauge field fluctuations.

We calculated analytically the instanton contribution to $R^S(\tau)$ and $R^\text{NS}(\tau)$ in the ILM, using the SIA. Then, we derived a model-independent semi-classical relation, which connects scalar and pseudo-scalar correlators to the four-quark condensates. This prediction represents a clean signature of instanton-induced dynamics. We suggest that it should be checked with an ab initio calculation, on the lattice. We discussed the feasibility of performing such a calculation and concluded that the artifacts related to large quark mass effects should be negligible.

An application of the same framework to the study the quenching effects in lattice QCD is in preparation.

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