 PIECEWISE EQUIDISTANT MESHES FOR QUASILINEAR TURNING POINT PROBLEMS: TECHNICAL REPORT

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Abstract

A class of quasilinear singularly perturbed boundary value problems with a turning point of attractive type is considered. The problems are solved numerically by a finite–difference scheme on a special discretization mesh which is dense near the turning point. The scheme is a combination of the standard central and midpoint schemes and is practically second–order accurate. Pointwise accuracy is uniform in the perturbation parameter, ε, and, moreover, $L^1$ errors decrease when $ε \to 0$. This is achieved by the use of meshes which generalize the piecewise equidistant Shishkin mesh. Two particular types of meshes are considered and compared.

Keywords: singular perturbation, boundary value problem, finite differences, Shishkin mesh.

1 Introduction

The following singularly perturbed boundary value problem was considered in Vulanović and Lin [1]:

$$ -εu'' - xb(x, u)u' + c(x, u) = 0, \quad x ∈ [-1, 1], \quad u(±1) = U_±, $$

where $ε$ is a small positive parameter, $b > 0$ and $c = O(|x| + ε)$ are sufficiently smooth functions, and $U_±$ are two given numbers. Some additional

*This manuscript was prepared in February, 2000, but has remained unpublished and I decided to make it accessible through arXiv.
conditions were assumed as well, but we shall not list them here. It was shown that any solution of problem (1) has an interior layer of exponential type at the turning point $x = 0$. An appropriate numerical method was proposed, based on finite differences on a special discretization mesh dense in the layer. The error of this method was estimated in a discrete $L^1$ norm by

$$M(\sqrt{\varepsilon} + e^{-N})N^{-1}. \quad (2)$$

(In (2) and throughout the present paper, $N$ denotes the number of mesh steps and $M$ stands for any, in the sense of $O(1)$, positive constant independent of $\varepsilon$ and $N$.)

In the present paper, we are interested in improving the result from [1]. This is achieved by applying a different kind of discretization mesh and a higher–order scheme. To simplify the presentation, we shall only consider a special case of problem (1), viz. the case $b = b(u)$ and $c \equiv 0$. However, the same theoretical results hold for the general problem if the method and some conditions are modified appropriately.

Singly perturbed boundary value problems arise in various applications, see Chang and Howes [2] for instance. Two recent books discuss numerical methods for these problems: Miller et al. [3] and Roos et al. [4]. Numerical methods for different types of turning point problems have attracted a considerable attention, let us only mention the papers by Berger et al. [5], Lin [6], Vulanović and Farrell [7], Clavero and Lisbona [8], and Sun and Stynes [9]. Prior to [1], some weaker versions of problem (1) were considered in Vulanović [10], [11], and [12], where special discretization meshes were also used. The meshes belong to the class of explicitly constructed meshes, which means that they are formed before the discrete problem is solved. This requires sharp derivative estimates of the continuous solution. The same approach will be applied here.

In general, the explicitly constructed meshes can be divided in two main classes, meshes of Bakhvalov type (B meshes) and meshes of Shishkin type (S meshes). The former were introduced in Bakhvalov [13] and later on generalized and simplified in Vulanović [14]. Many different types of singular perturbation problems have been successfully solved numerically on B meshes. A B mesh is constructed by a smooth mesh generating function which maps equidistant points into mesh points which are dense in the layer(s). In each layer, the mesh generating function corresponds to the inverse of the function describing how the solution behaves there. On the other hand, S meshes (Shishkin [15]) are piecewise equidistant, and therefore much
simpler. An S mesh for a problem like (1) would typically consist of three equidistant parts: a fine part around \( x = 0 \) and two coarse parts of the mesh outside the layer, with the transition points between these parts located at \( \pm \alpha \sqrt{\varepsilon \ln N} \), where \( \alpha > 0 \) is independent of \( \varepsilon \) and of \( N \). However, S meshes produce less accurate numerical results (see Vulanović [16] for a comparison of B and S meshes). There are two possibilities for improving numerical results obtained on S meshes. One of them is to use more accurate schemes that may be more complicated but still easier to analyze on S meshes than on B meshes, and another one is to improve the mesh itself. This paper uses both approaches: a scheme which is practically second–order accurate and a modified and improved S mesh.

Different improvements of S meshes have been considered so far. That by Linß [17] uses a modification which makes the mesh more similar to a B mesh. The resulting mesh is not piecewise equidistant any more. The approach in Vulanović [18] improves the S mesh while keeping it piecewise equidistant (the mesh has more equidistant parts which are constructed in a special way). A recent paper by Roos and Linß [19] (see Linß et al. [20] as well) provides for a unified theory which covers both S and B meshes (the latter only slightly different from those in [14]) and their generalizations. All these papers deal with non–turning point problems.

We are not going to consider all the different types of meshes here. We shall only analyze the mesh from [18] and its slight modification. In this way, we show that meshes of this type can be applied also to turning point problems. The modification of the mesh from [18] consists of replacing \( N \) in the transition point formulas by \( 1/\varepsilon \). Such transition points in S meshes were briefly discussed in [19] in the non–turning point case. They are closer to the points marking the beginning and the end of the layer and they improve both theoretical and numerical results for problems of type (1). The two kinds of meshes give quite satisfactory results and the discretization scheme is easier to discuss on these piecewise equidistant meshes. This is why we are not going to consider B meshes here. Besides, the particular B mesh applied to (1) in [1] is even more complicated than some other, more standard, B meshes.

The scheme which we shall use in this paper belongs to the class of hybrid (or switching) finite–difference schemes. The standard central scheme is used inside the layer, where the mesh is fine and the scheme is unconditionally stable, whereas a midpoint upwind scheme is used outside the layer, on the coarse part of the mesh. For such schemes, see Vulanović [21] and [22], and more recently, Stynes and Ross [23], and Linß [24]. Of these papers, only [22]
and [24] deal with hybrid schemes for quasilinear problems. The scheme in [24] is less general, since it is constructed for a non-turning point problem. However, it is simpler and that is why we are going to use a very similar approach here.

The paper is organized as follows. Precise assumptions on the continuous problem and properties of its solution are given in section 2. In addition to the interior turning point case, a boundary turning point is also considered. In that case, the turning point is still \( x = 0 \), but the interval \([0, 1]\) is considered instead of \([-1, 1]\). The numerical method is easier to describe for the boundary turning point problem, thus this case also serves the purpose of simplifying the presentation in section 3. Subsection 3.1 introduces the discretization scheme and analyzes its stability. The special mesh is described and the main result is stated and proved in subsection 3.2. The necessary changes for the interior turning point case are explained in section 4. Some additional remarks are also given there. This is followed by numerical results in section 5.

To illustrate our main result, let us state it for the interior turning point case when the mesh consists of five equidistant parts (the central one around \( x = 0 \) being the finest). Then, an error estimate of the form

\[ M \sqrt{\varepsilon} \left[ \ln \left( \ln \left( \frac{1}{\varepsilon} \right) \right) \right]^2 N^{-2} \]

can be proved in a discrete \( L^1 \) norm. This is an improvement over (2).

Let us finally mention that [9] is to our knowledge the only other paper which uses a piecewise equidistant mesh to solve a turning point problem numerically. However, the problem considered there is different from (1) and requires a different, more complicated mesh. In particular, the number of equidistant parts of the mesh depends on \( N \), which is not the case here.

## 2 The Continuous Problem

For simplicity, we are going to use \( \varepsilon^2 \) instead of \( \varepsilon \) in the rest of the paper. We consider the problem

\[ \varepsilon^2 u'' + a(x, u) u' = 0, \quad x \in I = [\nu, 1], \quad u(\nu) = U_-, \quad u(1) = U_+, \quad (3) \]

where \( \varepsilon \) is a perturbation parameter, \( 0 < \varepsilon << 1 \), and where \( \nu \) stands for either 0 or \(-1\). We assume that

\[ a(x, u) = xb(u) \quad (4) \]
with \( b \in C(\mathbb{R}) \), and that \( U_\pm \) are two different constants (otherwise \( U_+ = U_- \) solves (3)). The case \( \nu = 0 \) describes a boundary turning point problem, whereas if \( \nu = -1 \), we have an interior turning point problem.

We can assume without loss of generality that \( U_- < U_+ \). Then \( U_- \) and \( U_+ \) are respectively the lower and upper solutions of (3), and therefore, the problem has a unique solution, \( u_\varepsilon \in C^2(I) \), satisfying

\[
u
\]

\[ u_\varepsilon(x) \in U := [U_-, U_+], \quad x \in I \]

(see Lorenz [25]). Note that \( U_\pm \) are also the solutions of the reduced equation

\[
u
\]

\[ a(x, u)u' = 0, \quad x \in I, \]

subject to only one of the original boundary conditions.

Throughout the paper, we shall assume that \( b \in C^3(U) \), so that \( u_\varepsilon \in C^5(I) \). Another assumption that will be needed in this paper is

\[
u
\]

\[ b^* \geq b(u) \geq b_* > 0, \quad u \in U, \quad (5) \]

(of course, the upper bound on \( b \) is not a restriction here). Then the solution \( u_\varepsilon \) of the problem (3)–(5) satisfies the following estimate:

\[
u
\]

\[ |u_\varepsilon^{(k)}(x)| \leq M\varepsilon^{-k}y(x) \leq M\varepsilon^{-k}z(x), \quad k = 0, 1, 2, 3, 4, \quad x \in I, \quad (6) \]

where

\[
u
\]

\[ y(x) = e^{-b_{**}x^2/2\varepsilon^2} \]

and

\[
u
\]

\[ z(x) = e^{-m|x|/\varepsilon}. \]

The above constants \( b_{**} \) and \( m \) are positive and independent of \( \varepsilon \). \( b_{**} \) satisfies \( b_{**} < b_* \), whereas \( m \) is arbitrary. As the estimate (6) is sharp, it clearly shows that \( u_\varepsilon \) has a layer of exponential type at \( x = 0 \). Moreover,

\[
u
\]

\[ |u_\varepsilon(x) - U_+| \leq My(x) \leq Mz(x), \quad x \in [0, 1], \quad (7) \]

and similarly

\[
u
\]

\[ |u_\varepsilon(x) - U_-| \leq My(x) \leq Mz(x), \quad x \in [-1, 0], \quad \text{if } \nu = -1. \]

How to prove (6) and (7), can be found in [1] and [10], cf. [11] as well. Note that the proof of (6) requires \( b \in C^3(U) \).

It is another novelty of this paper, as compared to [1], [10], and [12], that the condition (5) is given locally, i.e. for \( u \in U \), and not for \( u \in \mathbb{R} \).
3 The Case $\nu = 0$

3.1 The Discretization

Let $I^h$ denote the discretization mesh with points $x_i, i = 0, 1, \ldots, N$, $0 = x_0 < x_1 < \cdots < x_N = 1$, and let $h_i = x_i - x_{i-1}, i = 1, 2, \ldots, N$. The only assumption on $I^h$ needed here is

$$h_i \leq h_{i+1}, \quad i = 1, 2, \ldots, N-1,$$

(8)

(the special mesh will be introduced in the next subsection). Also, let $h_i = (h_i + h_{i+1})/2, i = 1, 2, \ldots, N-1$ and $x_{i+\frac{1}{2}} = (x_i + x_{i+1})/2$. Let $w^h$ be an arbitrary mesh function on $I^h \setminus \{0,1\}$, which is identified with a column vector in $\mathbb{R}^{N-1}$,

$$w^h = [w_1, w_2, \ldots, w_{N-1}]^T,$$

where for simplicity $w_i = w_i^h$. For any mesh function, we shall formally set $w_0 = U_-$ and $w_N = U_+$. The restriction of the continuous solution $u_{\varepsilon}$ on $I^h \setminus \{0,1\}$ will be denoted by $u_{\varepsilon}^h$. Let $W = \{w^h \in \mathbb{R}^{N-1} \mid w_i \in U, \quad i = 1, 2, \ldots, N-1\}$.

The following quantity is used to define the discretization:

$$\rho_i = \frac{b^* x_{i-1} h_i}{2\varepsilon^2},$$

where $b^*$ is given in (5). Let us consider the set of indices

$$J = \{i \mid \rho_i \leq 1\} \subseteq \{1, 2, \ldots, N-1\}.$$

Note that $1 \in J$ and let $n = \max J$. Because of (8),

$$\rho_i \leq 1, \quad i = 1, 2, \ldots, n.$$  

(9)

If $1 < n < N-1$, we define

$$\chi_i = \begin{cases} 
    h_i & \text{if } 1 \leq i \leq n-1, \\
    \frac{h_i}{2} + h_{i+1} & \text{if } i = n, \\
    h_{i+1} & \text{if } n+1 \leq i \leq N-1.
\end{cases}$$

If $n = 1$, $\chi_i = h_{i+1}$ and if $n = N-1$, $\chi_i = h_i$, in both cases for all $i = 1, 2, \ldots, N-1$. 

6
We can now introduce the finite–difference operators

\[ D'' \varepsilon u_i = 1 \chi_i \left( \frac{w_{i-1} - w_i}{h_i} + \frac{w_{i+1} - w_i}{h_{i+1}} \right), \]

\[ D' \varepsilon u_i = \frac{w_{i+1} - w_{i-1}}{2h_i}, \]

\[ D' + \varepsilon u_i = \frac{w_{i+1} - w_i}{h_{i+1}}, \]

\[ D' t \varepsilon u_i = 2 \frac{w_{i+1} - w_i - w_i}{2 \chi_i}, \]

\[ D^0 \varepsilon u_i = \frac{w_{i+1} + w_i}{2}. \]

The differential equation of problem (3) is discretized in the following form:

\[-\varepsilon^2 u'' - f(x, u)' + f_x(x, u) = 0, \] (10)

where

\[ f(x, u) = \int_{U^+} a(x, t) dt. \]

When discretizing \( f \) on \( I^h \), we use the notation \( f_i = f(x_i, w_i) \). The notation \( f_{x,i}, a_i, b_i \), etc. has an analogous meaning. Then the following schemes are used to discretize (10):

\[ T_c w_i = -\varepsilon^2 D'' w_i - D' f_i + f_{x,i}, \]

\[ T_+ w_i = -\varepsilon^2 D'' w_i - D'_+ f_i + D^0 f_{x,i}, \]

\[ T_t w_i = -\varepsilon^2 D'' w_i - D'_t f_i + f_{x,i}. \]

By combining those schemes, we obtain the discretization of (10) on \( I^h \),

\[ T w^h = 0, \] (11)

where, if \( 1 < n < N - 1 \),

\[ T w_i = \begin{cases} 
T_c w_i \text{ if } 1 \leq i \leq n - 1, \\
T_t w_i \text{ if } i = n, \\
T_+ w_i \text{ if } n + 1 \leq i \leq N - 1.
\end{cases} \]

\( T_c \) is the standard central scheme, \( T_+ \) is the midpoint scheme used to discretize (10) at the point \( x_{i+1} \), and \( T_t \) is a transition scheme between \( T_c \).
and $T_+$. If $n = 1$, $T_+$ is not used and there is no need for the transition scheme. In that case, $T \equiv T_+$. Likewise, if $n = N - 1$, $T_+$ is not used and we set $T \equiv T_c$. For simplicity, in what follows, we shall only consider $1 < n < N - 1$.

The central and midpoint schemes are combined above in the same way as in [21], but the transition scheme was not required for the type of problems considered there. The present transition scheme is a little simpler than the one used in [24]. Our stability analysis needs such a transition, but this may be just a technical requirement. Note that $\chi_i = h_{i+1}$ when $T_+$ is used. This gives a nonstandard scheme $D''$ for discretizing $u''$. However, such schemes have been used earlier, see [12] and [20].

Let us finally introduce some vector and matrix norms. By $\| \cdot \|_\infty$ and $\| \cdot \|_1$ we denote the vector norms

$$
\|w^h\|_\infty = \max_{1 \leq i \leq N-1} |w_i^h|, \quad \|w^h\|_1 = \sum_{i=1}^{N-1} |w_i|
$$

and, at the same time, their subordinate matrix norms. The diagonal matrix

$$
H = \text{diag}(\chi_1, \chi_2, \ldots, \chi_{N-1})
$$

is used to define the following discrete $L^1$ norm:

$$
\|w^h\|_H = \|Hw^h\|_1,
$$

and its subordinate matrix norm

$$
\|A\|_H = \|HAH^{-1}\|_1,
$$

where $A$ is an arbitrary $(N - 1) \times (N - 1)$ matrix. The vector norm

$$
\sum_{i=1}^{N-1} h_i |w_i|
$$

and the corresponding matrix norm are usually used for nonequidistant discretizations of quasilinear problems, see [1] for instance. For the modified norms like $\| \cdot \|_H$ above, cf. [12] and [24].

Let $G = [g_{ij}] = T'(w^h)$ be the Fréchet derivative of the discrete operator $T$ on mesh $I^h$ at some $w^h \in W^h$.

**Lemma 1** Let (5) and (8) hold. Then $G$ is an $L$–matrix.
Proof. Since $G$ is a tridiagonal matrix, we have to show that

$$g_{ii} > 0 \quad \text{and} \quad g_{i,i \pm 1} \leq 0.$$ 

It is easy to see that $g_{ii} > 0$ for all the schemes used. When $T_c$ or $T_t$ are applied, (9) guarantees that $g_{i,i \pm 1} \leq 0$. For $T_+$, $g_{i,i-1} \leq 0$ is immediate, and

$$g_{i,i+1} = -\frac{\varepsilon^2}{h_{i+1} h_i} - \frac{a_{i+1}}{h_{i+1}} + \frac{b_{i+1}}{2} \leq b_{i+1} \left( \frac{1}{2} - \frac{x_{i+1}}{h_{i+1}} \right) < 0,$$

since $h_{i+1} < x_{i+1}$. \qed

**Theorem 1** Let (5) and (8) hold. Then the discrete problem (11) has a unique solution, $w^h$, which belongs to $W$. Moreover, the following stability inequality holds for any two mesh functions $w^h$, $v^h \in W$:

$$\|w^h - v^h\|_H \leq \frac{2}{b_*} \|Tw^h - Tv^h\|_H. \quad (12)$$

Proof. Let $e^h = [1, 1, \ldots, 1]^T \in \mathbb{R}^{N-1}$. It can be shown that

$$s^h := \left(HGH^{-1}\right)^T e^h \geq \frac{b_*}{2} e^h,$$ 

where the inequality should be understood componentwise. The proof of (13) is elementary and it requires the transition scheme. In fact, if $i \neq n+1$,

$$s_i = \frac{X_{i-1}}{X_i} g_{i-1,i} + g_{ii} + \frac{X_{i+1}}{X_i} g_{i+1,i} \geq b_i \geq b_*,$$

where we formally set $g_{01} = g_{N,N-1} = 0$. If $i = n+1$, then from $T_t$ we get

$$g_{i+1,i} = \frac{\varepsilon^2}{h_i h_{i+1}^2} + \frac{a_i}{h_{i+1}} + \frac{b_i}{2},$$

and $T_+$ gives

$$g_{ii} = \frac{\varepsilon^2}{h_i h_{i+1}^2} + \frac{a_i}{h_{i+1}} + \frac{b_i}{2},$$

and

$$g_{i+1,i} = \frac{\varepsilon^2}{h_{i+1}^2},$$

so that

$$s_i = \frac{b_i}{2} \geq \frac{b_*}{2}.$$
The discussion above also illustrates how the transition scheme enables the proof of (13).

The inequality (13) implies that $G$ is an inverse–monotone matrix (and therefore an M–matrix) and also that

$$
\|G^{-1}\|_H \leq \frac{b_*}{2}.
$$

This result can be applied immediately to the matrix

$$
A = \int_0^1 T'(v^h + s(w^h - v^h))ds
$$

in

$$
Tw^h - Tv^h = A(w^h - v^h),
$$

and (12) follows.

That (11) has a solution in $W$ can be proved by showing that

$$
T(U_+ e^h) \leq 0 \leq T(U_- e^h).
$$

The second inequality is immediate because of the way $f$ is defined. To illustrate the proof of the first inequality, let us consider

$$
[T(U_- e^h)]_n = [T(U_- e^h)]_n = \int_{U_+}^{U_-} b(t) \left( 1 - \frac{2x_{n+1} - x_n - x_{n-1}}{h_n + 2h_{n+1}} \right) dt = 0,
$$

and

$$
[T(U_- e^h)]_{N-1} = [T(U_+ e^h)]_{N-1} = -\varepsilon^2 \frac{U_+ - U_-}{h_N^2} + \int_{U_+}^{U_-} b(t) \left( \frac{x_{N-1}}{h_N} + \frac{1}{2} \right) < 0
$$

(recall that $w_N$ is replaced by $U_+$).

The solution is unique because of (12).

\[ \square \]

### 3.2 The Main Result

We shall now define the special discretization mesh. Let $\lambda$ be either $1/\varepsilon$ or $N$ and let

$$
\ln^0 \lambda = \lambda, \quad \ln^k \lambda = \ln(\ln^{k-1} \lambda), \quad k = 1, 2, \ldots, K,
$$

10
where \( K = K(\lambda) \) is a positive integer such that \( 0 < \ln^K \lambda < 1 \). Let \( \ell, 1 \leq \ell \leq K \), denote a fixed integer independent of \( \varepsilon \). Also, let \( \alpha \) be a positive constant independent of \( \varepsilon \) and \( N \). Then we define the transition points
\[
\tau_k = \alpha \varepsilon \ln^{\ell-k+1} \lambda, \quad k = 1, 2, \ldots, \ell.
\]
We shall assume that \( \tau_\ell < 1 \), since \( \varepsilon << 1 \). By formally setting \( \tau_0 = 0 \) and \( \tau_{\ell+1} = 1 \), we can split up the interval \([0, 1]\) into \( \ell + 1 \) subintervals,
\[
[0, 1] = \bigcup_{k=1}^{\ell+1} I_k, \quad I_k = [\tau_{k-1}, \tau_k].
\]
Each interval \( I_k \) is then divided into \( N_k \geq 2 \) equidistant subintervals, so that
\[
N_1 + N_2 + \ldots + N_{\ell+1} = N
\]
and
\[
N \leq MN_k, \quad k = 1, 2, \ldots, \ell + 1.
\]
The points obtained in this way form the mesh on \([0, 1]\), which we shall refer to as the \( S(\ell) \) mesh (Shishkin mesh with \( \ell \) transition points). The standard \( S \) mesh is \( S(1) \) with \( \lambda = N \). \( S(\ell) \) satisfies (8).

Note that if (9) holds with \( n = N - 1 \) on the \( S(\ell) \) mesh, this practically means that \( 1/N \leq M \varepsilon^2 \), whereas usually \( \varepsilon \leq 1/N \). Thus it is not realistic to expect that the discrete operator \( T \) be identical to the central scheme \( T_c \).

On the other hand, it is possible that \( n = 1 \) and \( T \equiv T_+ \) if \( \varepsilon \) is sufficiently small, \( \varepsilon^2 \leq MN^{-2}(\ln N)^2 \). Even though we are only showing details for \( 1 < n < N - 1 \), all our results are true for \( n = 1 \) or \( n = N - 1 \) as well.

Let
\[
\delta = \varepsilon \left( \frac{\ln \lambda}{N} \right)^2
\]
and
\[
v(x) = e^{-\beta x/\varepsilon}
\]
with a positive constant \( \beta \).

We first consider the case \( \lambda = 1/\varepsilon \).

**Lemma 2** On the \( S(\ell) \) mesh with \( \lambda = 1/\varepsilon \) it holds that
\[
\frac{h_i^2}{\varepsilon} v_{i-1} \leq M \delta,
\]
where \( i = 1, 2, \ldots, N - 1 \) and \( v_i = v(x_i) \) with a sufficiently large constant \( \beta \) independent of \( \varepsilon \) and \( N \).
Proof. We shall consider several cases. If \( x_i \in (0, \tau_1) \), then \( h_{i+1} \leq M\varepsilon N^{-1} \ln^\ell \lambda \) and (15) follows immediately.

In all other cases (15) can be proved with \( \varepsilon N^{-2} \) instead of \( \delta \). If \( x_i \in (\tau_k, \tau_{k+1}) \) for \( 1 \leq k \leq \ell \), then

\[
\frac{h_{i+1}^2}{\varepsilon} v_{i-1} \leq M\varepsilon \left( \frac{\ln^{\ell-k} \lambda}{N} \right)^2 e^{-\alpha\beta \ln^{\ell-k+1} \lambda} = M\varepsilon N^{-2} (\ln^{\ell-k} \lambda)^{2-\alpha\beta} \leq M\varepsilon N^{-2},
\]

where \( \beta \) is chosen so that \( \alpha\beta \geq 2 \).

If \( x_i = \tau_k \), \( k = 1, 2, \ldots, \ell \), then

\[
x_{i-1} \geq \tau_k \left( 1 - \frac{1}{N_k} \right) \geq \frac{\tau_k}{2}
\]
and (15) follows like in the previous case but with \( \alpha\beta \geq 4 \). \( \square \)

Let us introduce some more notation. By \( \mathcal{I} \) we denote

\[
\mathcal{I} = \bigcup_{k=0}^{\ell} \mathcal{I}_k,
\]

where

\[
\mathcal{I}_k = \{ i \mid x_i \in (\tau_k, \tau_{k+1}) \}.
\]

Lemma 3 On the \( S(\ell) \) mesh with \( \lambda = 1/\varepsilon \) it holds that

\[
\sigma_k := \sum_{i \in \mathcal{I}_k} \left( \frac{h_{i+1}}{\varepsilon} \right)^2 \int_{x_{i-1}}^{x_{i+1}} v(x) dx \leq M\delta,
\]

where \( k = 0, 1, \ldots, \ell \) and where \( v(x) \) is given in (14) with a sufficiently large constant \( \beta \) independent of \( \varepsilon \) and \( N \).

Proof. We have

\[
\sigma_k \leq M \left( \frac{\tau_{k+1}}{\varepsilon N} \right)^2 \int_{\tau_k}^{\tau_{k+1}} v(x) dx
\]

\[
\leq M \left( \frac{\tau_{k+1}}{\varepsilon N} \right)^2 \varepsilon v(\tau_k) \leq M\delta.
\]

The last inequality follows similarly to the proof of Lemma 2. \( \square \)

We can now prove the main result of the paper.
Theorem 2 Let (5) hold and let $w^h_\varepsilon$ be the solution of the discrete problem (11) on the $S(\ell)$ mesh with $\lambda = 1/\varepsilon$. Then the following error estimate holds:

$$\|w^h_\varepsilon - u^h_\varepsilon\|_H \leq M\delta.$$ 

Proof. Using (12) with $w^h = w^h_c$ and $v^h = u^h_c$, we see that in order to prove the theorem, it suffices to show that the consistency error $r^h = Tu^h_\varepsilon$ satisfies

$$\|r^h\|_H \leq M\delta. \tag{16}$$ 

Let

$$r^h_c = T_cu^h_\varepsilon \quad \text{and} \quad r^h_+ = T_+u^h_\varepsilon.$$ 

Since the transition between $T_c$ and $T_+$ can occur at any mesh point, we can consider separately $\|r^h_c\|_H$ and $\|r^h_+\|_H$. As for $T_t$, it is only used at $x_n$ and therefore, it is sufficient to estimate $\chi_n|r_n|$. 

When the central scheme is used, Taylor’s expansion of $r_i = r_{c,i}$ about $x_i$ gives

$$\chi_i|r_i| \leq M \left\{ \varepsilon^2 \left[ h_{i+1}(h_{i+1} - h_i)|u^{(3)}_\varepsilon(x_i)| + h_{i+1}^2 \int_{x_{i-1}}^{x_{i+1}} |u^{(4)}_\varepsilon(x)|dx \right] \\
+ h_{i+1}(h_{i+1} - h_i)|p''(x_i)| + h_{i+1}^2 \int_{x_{i-1}}^{x_{i+1}} |p^{(3)}(x)|dx \right\},$$

where $p(x) = f(x, u_\varepsilon(x))$ (for the integral form of the consistency error, cf. Kellogg and Tsan [26] for instance). Then (6) implies

$$\chi_i|r_i| \leq M(Q_i + R_i),$$

where

$$Q_i = h_{i+1}(h_{i+1} - h_i)\varepsilon^{-1}v_{i-1}$$

and

$$R_i = \left( \frac{h_{i+1}}{\varepsilon} \right)^2 \int_{x_{i-1}}^{x_{i+1}} v(x)dx.$$ 

Here, $v(x)$ is like in (14), with a constant $\beta$ satisfying $0 < \beta < m$. $\beta$ is independent of $\varepsilon$ and $N$, it can be chosen arbitrarily close to $m$, and therefore it can be made sufficiently large. On writing

$$\|r^h\|_H = \sum_{i \not\in I} \chi_i|r_i| + \sum_{i \in I} \chi_i|r_i|,$$
from Lemma 3 we immediately get
\[ \sum_{i \in I} \chi_i |r_i| \leq M \sum_{i \in I} R_i \leq M \delta. \]
On the other hand, if \( i \notin I \), i.e. if \( x_i = \tau_k \), it holds that
\[ Q_i \leq \frac{h_{i+1}}{\varepsilon} v_{i-1} \leq M \delta, \]
because of Lemma 2. Since the number \( \ell \) of transition points does not
depend on \( N \), this proves (16) for \( r^h = r^h_c \).

Let us now consider the consistency error when \( T = T^+ \). Then, \( r_i = r_{+,i} \)
is expanded about \( x_{i+1/2} \). The errors due to \( D'_+ \) and \( D^0 \) can be treated as
above, but the one due to \( D'' \) requires a closer attention. Let
\[ r''_i = \varepsilon^2 [D'' u_\varepsilon(x_i) - u''_\varepsilon(x_{i+1/2})]. \]
We have
\[ \chi_i |r''_i| \leq M \varepsilon^2 \left[ (h_{i+1} - h_i)|u''_\varepsilon(x_{i+1/2})| + h_{i+1} \int_{x_i}^{x_{i+1}} |u''_\varepsilon(x)| dx \right]. \]
Since in this case \( \rho_i > 1 \), it holds that
\[ \varepsilon^2 \leq M x_{i-1} h_i. \]
This and (6) imply
\[ \chi_i |r''_i| \leq M (Q''_i + R''_i), \]
\[ Q''_i = h_i (h_{i+1} - h_i) \varepsilon^{-1} v_{i-1}, \]
\[ R''_i = \frac{h_{i+1}^2}{\varepsilon^3} \int_{x_{i-1}}^{x_{i+1}} z(x) dx. \]
We can handle \( Q''_i \) in the same way as \( Q_i \) above, and
\[ R''_i \leq \left( \frac{h_{i+1}}{\varepsilon} \right)^2 \frac{x_{i-1}}{\varepsilon} e^{-mx_{i-1}/2\varepsilon} \int_{x_{i-1}}^{x_{i+1}} e^{-mx/2\varepsilon} dx \]
\[ \leq M \left( \frac{h_{i+1}}{\varepsilon} \right)^2 \int_{x_{i-1}}^{x_{i+1}} e^{-mx/2\varepsilon} dx, \]
so that Lemma 3 can be applied. Thus, (16) holds with \( r^h = r^h_c \).
Finally, let us consider the transition scheme at $x_n$. We again expand $r_n$ about $x_n$ to get

$$
\chi_n |r_n| \leq M \left\{ \varepsilon^2 \left[ h_{n+1} |u''_\varepsilon(x_n)| + h_{n+1}^2 \max_{x_{i-1} \leq x \leq x_{i+1}} |u^{(3)}_\varepsilon(x)| \right] + h_{n+1}^2 |p''(\theta)| \right\}
$$

(17)

with some $\theta \in (x_{n-1}, x_{n+1})$. The fact that in this case $\rho_{n+1} > 1$ implies

$$
\varepsilon^2 \leq M x_n h_{n+1}.
$$

We need this inequality and (6) in the estimate

$$
\varepsilon^2 h_{n+1} |u''_\varepsilon(x_n)| \leq M h_{n+1}^2 \frac{x_n}{\varepsilon^2} z(x_n) \leq M \frac{h_{n+1}^2}{\varepsilon} v_n.
$$

The other terms on the right–hand side of (17) can be estimated directly using (6). Thus, it follows that

$$
\chi_n |r_n| \leq \frac{h_{n+1}^2}{\varepsilon} v_{n-1} \leq M \delta,
$$

where the last inequality follows from Lemma 2. 

Let us now turn to the case $\lambda = N$.

**Theorem 3** Let (5) hold and let $w^h_\varepsilon$ be the solution of the discrete problem (11) on the $S(\ell)$ mesh with $\lambda = N$. Then the following error estimate holds:

$$
\|w^h_\varepsilon - u^h_\varepsilon\|_H \leq M \left( \delta + N^{-(1+\eta \ln N)} \right),
$$

where $\eta$ is some positive constant independent of $\varepsilon$ and $N$.

**Proof.** The result can be proved analogously to the proof of Theorem 2, cf. [18] as well. The only case which produces the $N^{-(1+\eta \ln N)}$ term is when $x_i = \tau_\ell, \tau_\ell + \kappa$, where $\kappa$ is the mesh step in $[\tau_\ell, 1]$. For instance, when estimating expressions like the left–hand side of (15) (which is also needed in the proof of Lemma 3), the best we can get at those points is

$$
\frac{h_{i+1}^2}{\varepsilon} v_{i-1} \leq M \frac{1}{N^2 \varepsilon} e^{-\alpha \beta \ln N} = M \frac{1}{\varepsilon N^{2+\alpha \beta}}.
$$
The case \( x_i \in (\tau_\ell + \kappa, 1) \) still gives the same result as before, but requires a somewhat different technique:

\[
\frac{h_{i+1}^2}{\varepsilon} v_{i-1} \leq M \frac{\varepsilon}{N^2} \frac{1}{N\alpha\beta\varepsilon^2} e^{-\gamma/N\varepsilon} \leq M \frac{\varepsilon}{N^2},
\]

where \( \gamma \) is a positive constant independent of \( \varepsilon \) and \( N \), and where \( \alpha\beta \geq 2 \).

Therefore, when \( x_i = \tau_\ell, \tau_\ell + \kappa \), we use a different estimate of the consistency error:

\[
\chi_i |r_i| = \chi_i |Tu_\varepsilon(x_i)| \leq M h_{i+1} \left\{ \varepsilon^2 \max_{x_{i-1} \leq x \leq x_{i+1}} \left[ |u''_\varepsilon(x)| + |p'(x)| + |q(x)| \right] \right\},
\]

where \( q(x) = f_x(x, u_\varepsilon(x)) \). We use here estimates (6) and (7) with \( y(x) \) to get

\[
\chi_i |r_i| \leq M h_{i+1} y(x_{i-1})
\]

and

\[
\chi_i |r_i| \leq M N^{-1} e^{-\eta (\ln N)^2} \leq M N^{-(1+\eta \ln N)},
\]

on setting \( \eta = b_\star\alpha^2/2 \).

The error estimate of Theorem 2 is better than that of Theorem 3 when \( \varepsilon \to 0 \). It may look like the term \( N^{-(1+\eta \ln N)} \) in the estimate above arises for purely technical reasons. However, the numerical experiments in section 5 will show that when that term dominates in the error, the error does not decrease together with \( \varepsilon \).

4 The Case \( \nu = -1 \) and Other Remarks

Theorem 2 holds also for the case \( \nu = -1 \), when the problem (3) is considered on \( I = [-1, 1] \), if the numerical method is modified appropriately. As for the mesh, the simplest thing to do is to extend \( S(\ell) \) to \([-1, 0] \) symmetrically to \( x_0 = 0 \):

\[
x_{i} = -x_{i}, \quad i = 1, 2, \ldots, N,
\]

\[
h_{-i} = x_{-i} - x_{-i-1} = h_{i+1}, \quad i = 0, 1, \ldots, N - 1.
\]

This is accompanied with other symmetrically changed definitions, like

\[
\chi_i = \begin{cases}
    h_i & \text{if } -N + 1 \leq i \leq -n - 1, \\
    h_i + \frac{h_{i+1}}{2} & \text{if } i = -n, \\
    h_i & \text{if } -n + 1 \leq i \leq n - 1, \\
    \frac{h_i}{2} + h_{i+1} & \text{if } i = n, \\
    h_{i+1} & \text{if } n + 1 \leq i \leq N - 1,
\end{cases}
\]
and

\[ D'_{-} w_{i} = \frac{w_{i} - w_{i-1}}{h_{i}}, \]
\[ D'_{+} w_{i} = \frac{w_{i+1} + w_{i} - 2w_{i-1}}{2\chi_{i}}, \]
\[ D_{-} w_{i} = \frac{w_{i} + w_{i-1}}{2}. \]

The scheme is also symmetrical:

\[ T w_{i} = \begin{cases} T_{-} w_{i} & \text{if } -N + 1 \leq i \leq -n - 1 \\ T_{t-} w_{i} & \text{if } i = -n, \\ T_{c} w_{i} & \text{if } -n + 1 \leq i \leq n - 1, \\ T_{t} w_{i} & \text{if } i = n, \\ T_{+} w_{i} & \text{if } n + 1 \leq i \leq N - 1, \end{cases} \]

where

\[ T_{-} w_{i} = -\varepsilon^{2}D''w_{i} - D'_{-}f_{i} + D_{-}f_{x,i}, \]

and

\[ T_{t-} w_{i} = -\varepsilon^{2}D''w_{i} - D'_{t-}f_{i} + f_{x,i}. \]

Note that \( T_{c} \) is always used at least at \( x_{0} = 0 \). To make the discretization even more symmetric, we also replace \( U_{+} \) in \( f \) with \( U_{-} \) when discretizing (10) at \( x_{i} \in (-1, 0) \), and with \((U_{+} + U_{-})/2\) at \( x_{0} = 0 \).

Under the conditions of Theorem 2, the following estimate holds:

\[ \|w_{h}^{\varepsilon} - u_{h}^{\varepsilon}\|_{\infty} \leq M\frac{\ln^{\ell}(1/\varepsilon)}{N}. \quad (18) \]

This is because the smallest mesh step is \( h_{1} = \tau_{1}/N \). The above result does not mean \( \varepsilon \)-uniform pointwise convergence, but \( \ln^{\ell}(1/\varepsilon) \) grows very slowly when \( \varepsilon \rightarrow 0 \). For instance, if \( \ell = 3 \) and \( \varepsilon = 10^{-12} \), \( \ln^{3}(1/\varepsilon) \approx 1.2 \). The numerical results of the next section show that pointwise accuracy is even better than what (18) indicates. A similar discussion is true for \( S(\ell) \) with \( \lambda = N \).

It is possible to adjust the present method to the most general problem (1) under the conditions given in [1]. The reduced solutions are not \( U_{-} \) and \( U_{+} \) in that case, but some more complicated functions. They have to be incorporated in the function \( f \), like it was done in [1]. In order to prove the stability inequality corresponding to (12), the conditions of type (5) on \( b \) have to hold for \( u \in \mathbb{R} \). The same has to be assumed of all conditions on
c. This is because in this case it is generally difficult to find the upper and lower solutions of (1) and of the discrete problem. For Theorems 2 and (3), more complicated estimates corresponding to (6) and (7) have to be used, see [1].

5 Numerical Results

Our test problem is more general than (3) but less general than (1),

\(-\varepsilon^2 u'' - xuu' + c(x) = 0, \quad u(\pm 1) = U_{\pm},\)

where \(c(x), U_- \approx 1,\) and \(U_+ \approx 3\) are determined by the exact solution being

\(u_\varepsilon(x) = 2 + \tanh \frac{x}{\varepsilon}.\)

We have tested the S(\(\ell\)) mesh with both \(\lambda = 1/\varepsilon\) and \(\lambda = N,\) and with values of \(\ell = 1, 2, 3.\) Let

\(q_k = \frac{N_k}{N}, \quad k = 1, 2, \ldots, \ell + 1,\)

where \(N\) is the number of mesh steps in \([0, 1].\) Table 1 shows the values of the ratios \(q_k\) that are used in the meshes below. They were kept fixed for each \(\ell\) regardless of other mesh parameters, including \(N.\)

| \(\ell\) | 1 | 2 | 3 |
|---|---|---|---|
| \(q_1\) | 3/4 | 1/4 | 1/8 |
| \(q_2\) | - | 1/2 | 1/8 |
| \(q_3\) | - | - | 1/2 |

The choice of the ratios \(q_k\) may influence the errors significantly. Other ratios have been also tested and the results for this test problem were the best when there were around 75\% of the mesh points in the layer. The question of the optimal choice of the ratios is problem–dependent and seems to be difficult to solve in general.

The tables below show the errors

\(E = E(N) = \|w^h_\varepsilon - u^h_\varepsilon\|_\infty\) and \(E_1 = E_1(N) = \|w^h_\varepsilon - u^h_\varepsilon\|_H,\)

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where the dependence on $N$ indicates that the mesh is used with $2N$ mesh steps in $[-1, 1]$. The corresponding numerical orders of convergence, \( \text{Ord} = \text{Ord}(N) \) and \( \text{Ord}_1 = \text{Ord}_1(N) \), are also listed, where

$$\text{Ord}(N) = \frac{\ln E(N) - \ln E(N/2)}{\ln 2},$$

and \( \text{Ord}_1(N) \) is defined analogously. The results in Tables 2–4 are given for the transition point coefficient \( \alpha = 1 \), whereas Table 5 presents some results for \( \alpha = 2 \).

Table 2 illustrates that the errors are smaller if \( \ell \) is larger, that is, if the meshes have more subintervals within the layer. This is to be expected. The case \( \lambda = 1/\varepsilon \) is shown. If \( \lambda = N \), the errors behave analogously with respect to the change in \( \ell \).

| \( \ell \) | \( \varepsilon = 2^{-14} \) | \( \varepsilon = 2^{-18} \) | \( \varepsilon = 2^{-22} \) | \( E \) | \( E_1 \) | \text{Ord} | \text{Ord}_1 |
|---|---|---|---|---|---|---|
| 1 | | | | | | |
| 2 | | | | | | |
| 3 | | | | | | |

In the remaining tables, only \( \ell = 3 \) is considered. Table 3 gives more details of what can already be observed in Table 2, viz. errors \( E_1 \) decreasing together with \( \varepsilon \), while at the same time, errors \( E \) slightly increase, still preserving a high accuracy. The increase of \( E \) is what we can expect from (18), but the accuracy and its order are higher than what (18) indicates. Both \( \text{Ord} \) and \( \text{Ord}_1 \) are around 2, with \( \text{Ord}_1 \) being somewhat smaller for larger \( \varepsilon \) values.
Table 3. S(3) mesh with $\lambda = 1/\varepsilon$, $\alpha = 1$

| $N$ | $\varepsilon = 2^{-14}$ | $\varepsilon = 2^{-18}$ | $\varepsilon = 2^{-22}$ | $E$ | $E_1$ |
|-----|----------------|----------------|----------------|-----|-----|
| 64  | 1.18–03 1.88–07 | 1.57–03 1.73–08 | 1.73–03 1.27–09 | 1.20 1.89 |
|     | 2.01 2.06       | 1.51 2.00       | 1.48 1.89       |
| 128 | 3.13–04 4.80–08 | 3.63–04 2.97–09 | 3.94–04 2.45–10 | 1.91 2.62 |
|     | 1.92 1.97       | 2.12 2.54       | 2.13 2.37       |
| 256 | 7.83–05 1.38–08 | 9.16–05 7.51–10 | 1.04–04 5.33–11 | 1.90 2.70 |
|     | 2.00 1.80       | 1.99 1.98       | 1.92 2.20       |
| 512 | 1.96–05 4.29–09 | 2.29–05 1.93–10 | 2.60–05 1.33–11 | 2.00 2.00 |
|     | 2.00 1.68       | 2.00 1.96       | 2.00 2.00       |

When $\lambda = N$, both $E$ and $E_1$ errors are more uniform in $\varepsilon$ but the accuracy is worse than for $\lambda = 1/\varepsilon$. This is shown in Table 4, where we can see that Ord is less than 2 as can be expected from the error estimate of Theorem 3. Also, $E_1$ is much worse than in Table 3 even though Table 4 shows Ord$_1$ significantly higher than 2. In fact, in Table 4, $E_1$ does not decrease together with $\varepsilon$, which is possible according to the error estimate of Theorem 3. This can be improved by increasing the value of $\alpha$ so that the $\varepsilon$-independent term of the error estimate gets negligible. Table 5 shows that $E_1$ errors decrease for $\alpha = 2$ when $\varepsilon \to 0$ and that they become almost as accurate as in Table 3, but at the same time, the larger $\alpha$ spoils $E$ a little.

Table 4. S(3) mesh with $\lambda = N$, $\alpha = 1$

| $N$ | $\varepsilon = 2^{-14}$ | $\varepsilon = 2^{-18}$, $2^{-22}$ | $E$ | $E_1$ |
|-----|----------------|-------------------------------|-----|-----|
| 64  | 1.89–03 9.21–05 | 1.90–03 9.21–05               | 1.20 1.89 |
|     | 1.91 2.62       | 1.91 2.62                     |
| 128 | 5.07–04 1.41–05 | 5.10–04 1.42–05               | 1.90 2.70 |
|     | 1.90 2.70       | 1.89 2.70                     |
| 256 | 1.37–04 2.10–06 | 1.39–04 2.10–06               | 1.89 2.75 |
|     | 1.89 2.76       | 1.88 2.75                     |
| 512 | 3.64–05 3.02–07 | 3.76–05 3.04–07               | 1.91 2.79 |
|     | 1.91 2.79       | 1.88 2.79                     |
Table 5. S(3) mesh with $\lambda = N$, $\alpha = 2$

| $N$ | $\varepsilon = 2^{-14}$ | $\varepsilon = 2^{-18}$ | $\varepsilon = 2^{-22}$ |
|-----|-----------------|-----------------|-----------------|
| 64  | 1.59–03 2.63–07 | 1.59–03 3.02–08 | 1.59–03 1.56–08 |
|     | 3.09 3.15       | .09 4.07        | 3.09 4.66       |
| 128 | 4.44–04 4.81–08 | 4.44–04 3.51–09 | 4.44–04 7.24–10 |
|     | 1.84 2.45       | 1.84 3.10       | 1.84 4.43       |
| 256 | 1.40–04 1.67–08 | 1.40–04 1.07–09 | 1.40–04 8.44–11 |
|     | 1.66 1.53       | 1.66 1.73       | 1.66 3.10       |
| 512 | 4.34–05 5.53–09 | 4.34–05 3.46–10 | 4.34–05 2.23–11 |
|     | 1.69 1.59       | 1.69 1.61       | 1.69 1.92       |

How fast do the $E_1$ errors in Tables 3 and 5 decrease as $\varepsilon \to 0$? This $\varepsilon$–order can also be measured numerically, analogously to $\text{Ord}_1$:

$$\text{Ord}_\varepsilon = \frac{E_1(4\varepsilon) - E_1(\varepsilon)}{\ln 4},$$

where $N$ is kept fixed and the dependence of $E_1$ on the value of $\varepsilon$ is expressed. The results are given in Table 6. We can see that they confirm the expected value of 1 for $\lambda = N$ and $\alpha = 2$, whereas for $\lambda = 1/\varepsilon$, they are better than what Theorem 2 indicates, particularly for larger $\varepsilon$.

Table 6. $\text{Ord}_\varepsilon$ on S(3) mesh with $N = 512$

| $\varepsilon$ | $\lambda = 1/\varepsilon$, $\alpha = 1$ | $\lambda = N$, $\alpha = 2$ |
|--------------|---------------------------------|-----------------|
| $2^{-14}$    | 1.45                            | 1.00 |
| $2^{-16}$    | 1.21                            | 1.00 |
| $2^{-18}$    | 1.03                            | 1.00 |
| $2^{-20}$    | 0.97                            | 1.00 |
| $2^{-22}$    | 0.96                            | 0.98 |

Let us finally mention that we have also tested the discretization in which the central scheme is used instead of the transition schemes $T_t$ and $T_{t-}$. The errors are somewhat worse, the difference being greater when $\lambda = 1/\varepsilon$ than when $\lambda = N$. Thus, it may be possible that the use of the transition schemes is not entirely for technical reasons.

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