INVARIANT LAMINATIONS FOR IRREDUCIBLE AUTOMORPHISMS OF FREE GROUPS

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Abstract. For every atoroidal iwip automorphism \( \varphi \) of \( F_N \) (i.e. the analogue of a pseudo-Anosov mapping class) it is shown that the algebraic lamination dual to the forward limit tree \( T_+(\varphi) \) is obtained as “diagonal closure” of the support of the backward limit current \( \mu^- (\varphi) \). This diagonal closure is obtained through a finite procedure in analogy to adding diagonal leaves from the complementary components to the stable lamination of a pseudo-Anosov homeomorphism. We also give several new characterizations as well as a structure theorem for the dual lamination of \( T_+(\varphi) \), in terms of Bestvina-Feighn-Handel’s “stable lamination” associated to \( \varphi \).

1. Introduction

For a closed surface \( \Sigma \) with Euler characteristic \( \chi(\Sigma) < 0 \) geodesic laminations play an important theoretical role, for many purposes. In particular, if such a lamination \( \mathcal{L} \) is equipped with a transverse measure \( \mu \), it becomes a powerful tool, for example in the analysis of the mapping class \( [h] \) of a homeomorphism \( h : \Sigma \to \Sigma \). The set of projective classes \([\mathcal{L}, \mu]\) of such measured laminations carries a natural topology, and the issuing space \( \mathcal{PML}(\Sigma) \) of projective measured laminations is homeomorphic to a high dimensional sphere: One of Thurston’s fundamental results shows that \( \mathcal{PML}(\Sigma) \) serves naturally as boundary to the Teichmüller space \( \mathcal{T}(\Sigma) \) of \( \Sigma \), and the action of the mapping class group \( \text{Mod}(\Sigma) \) extends canonically to the compact union \( \mathcal{T}(\Sigma) \cup \mathcal{PML}(\Sigma) \).

In this paper we are interested in the “cousin world”, where \( \Sigma \) (or rather \( \pi_1(\Sigma) \)) is replaced by a free group \( F_N \) of finite rank \( N \geq 2 \), the mapping class group \( \text{Mod}(\Sigma) \) is replaced by the outer automorphism group \( \text{Out}(F_N) \), and the role of Teichmüller space \( \mathcal{T}(\Sigma) \) is (usually) taken on by Outer space \( CV_N \). These and the other terms used in this introduction will be explained in section 2.

In the \( \text{Out}(F_N) \)-setting, both, topological laminations \( \mathcal{L} \) and measured laminations \( (\mathcal{L}, \mu) \) have natural analogues, but the situation is quite a bit

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more intricate: There are two competing analogues of measured lamination, namely $\mathbb{R}$-trees $T$ with isometric $F_N$-action, and currents $\mu$ over $F_N$. Both, the space of such $\mathbb{R}$-trees as well as the space of such currents have projectivizations which can be used to compactify $CV_N$ (in two essentially different ways, see [11]), and both are used as important tools to analyze single automorphisms of $F_N$. Also, both, an $\mathbb{R}$-tree $T$ and a current $\mu$, determine an algebraic lamination, denoted $L^2(T)$ (the dual lamination of $T$) and $L^2(\mu) = \text{supp}(\mu)$ (the support of $\mu$) respectively. Algebraic laminations $L^2$ for a free group $F_N$ are the natural analogues of (non-measured) geodesic laminations $\mathcal{L}$ on a surface $\Sigma$ (or rather, of their lift $\hat{\mathcal{L}} \subset \hat{\Sigma}$ to the universal covering $\hat{\Sigma}$ of $\Sigma$: the set $L^2$ consists of $F_N$-orbits of pairs $(X,Y) \in \partial^2 F_N := \partial F_N \times \partial F_N \setminus \{(Z,Z) \mid Z \in \partial F_N\}$.

The two spaces which serve (after projectivization) as analogues of $T(\Sigma) \cup \mathcal{PML}(\Sigma)$, namely the space $\overline{\text{TEML}}(\Sigma)$ of very small $\mathbb{R}$-tree actions of $F_N$, and the space $\text{Curf}(F_N)$ of currents over $F_N$, are related naturally by an $\text{Out}(F_N)$-equivariant continuous intersection form $\langle \cdot, \cdot \rangle : \overline{\text{TEML}}(F_N) \rightarrow \mathbb{R}_{\geq 0}$ (see [12]). In [13] it has been shown that $\langle T, \mu \rangle = 0$ if and only if $\text{supp}(\mu) \subset L^2(T)$. Sometimes this inclusion is actually an equality: for example, if $(\mathcal{L},\mu)$ is a measured lamination of a surface $\Sigma$ with boundary, and $T_\mu$ is the $\mathbb{R}$-tree transverse to the lift of $(\mathcal{L},\mu)$ to the universal covering of $\Sigma$, and if $\mathcal{L}$ is maximal (i.e. every complementary component has either 3 cusps, or 1 boundary component plus 1 cusp), then the current $\hat{\mu}$ defined by $(\mathcal{L},\mu)$ satisfies $\text{supp}(\hat{\mu}) = L^2(T_\mu)$.

However, in general one can not deduce from $\langle T, \mu \rangle = 0$ the equality $\text{supp}(\mu) = L^2(T)$, since the dual laminations of $\mathbb{R}$-trees are by nature always diagonally closed, i.e. the diagonal closure $\overline{\text{diag}}(L^2(T))$ is equal to $L^2(T)$ for all $T \in \overline{\text{TEML}}(\Sigma)$. On the other hand, the support of a current may well be minimal and thus typically a proper subset of its diagonal closure. Moreover, it is easy to find perpendicular pairs $(T,\mu)$ (i.e. $T$ and $\mu$ satisfy $\langle T, \mu \rangle = 0$) where even the diagonal closure $\overline{\text{diag}}(\text{supp}(\mu))$ is only a proper subset of $L^2(T)$. A class of examples with even stronger properties is described in section 3 alternatively, any measured lamination $(\mathcal{L},\mu)$ on a surface which has a complementary component with non-abelian $\pi_1$ defines a perpendicular pair $(T_\mu,\hat{\mu})$ as above, with $\overline{\text{diag}}(\text{supp}(\hat{\mu})) \neq L^2(T_\mu)$.

For pseudo-Anosov mapping classes $[h]$ of $\Sigma$ it is well known that the induced action on $T(\Sigma) \cup \mathcal{PML}(\Sigma)$ has uniform North-South dynamics. In particular, $[h]$ has precisely one attracting and one repelling fixed point on $\mathcal{PML}(\Sigma)$. These very statements are also true for the induced actions of atoroidal iwip automorphisms of $F_N$ on both “cousin spaces”, $\overline{CV}_N$ and $\mathcal{P}\text{Curf}(F_N)$, so that the (much studied) class of atoroidal iwip automorphisms should be considered as strict analogue of the class of pseudo-Anosov mapping classes. In [13] it has been shown that for any atoroidal iwip $\varphi \in \text{Out}(F_N)$ the forward limit tree $T_+ = T_+(\varphi) \in \overline{\text{TEML}}(\Sigma)$, which defines the attracting fixed point $[T_+] \in \overline{CV}_N$, is perpendicular to the backward
limit current $\mu_\ominus = \mu_\ominus(\varphi) \in \text{Curr}(F_N)$, which gives the repelling fixed point $[\mu_\ominus] \in \overline{\text{Curr}}(F_N)$:

$$\langle T_+(\varphi), \mu_\ominus(\varphi) \rangle = 0 \quad \text{for any atoroidal iwip } \varphi \in \text{Out}(F_N)$$

The main result of this paper can now be stated as follows:

**Theorem 1.1.** Let $\varphi$ be an atoroidal iwip automorphism of $F_N$. Let $T_+$ be its forward limit tree, and let $\mu_\ominus$ be its backward limit current. Then the dual lamination $L^2(T_+) = \text{diag}(\text{supp}(\mu_\ominus)) = \text{diag}(\text{supp}(\mu_\ominus))$ (where the “diagonal extension” $\text{diag}(\text{supp}(\mu_\ominus))$ is a specification of the diagonal closure $\text{diag}(\text{supp}(\mu_\ominus))$ defined in Definition-Remark 2.5(a)).

This result raises the question, what the precise conditions are, which allow one to deduce for perpendicular pairs $(T, \mu) \in \overline{\text{cv}}(F_N) \times \text{Curr}(F_N)$ the analogous conclusion, i.e. that $L^2(T) = \text{diag}(\text{supp}(\mu))$. A necessary condition for such a conclusion is that $\mu$ “fills out” enough of the available room in $F_N$ which is “potentially dual” to $T$. Making such an indication precise takes more room than available here in the introduction. We will discuss those matters below in section 8.

A second result of this paper, which is stronger than Theorem 1.1, concerns the structure of the lamination $L^2(T_+)$. The proof of this result is really the main purpose of this paper; we show below how to derive Theorem 1.1 from the following:

**Theorem 1.2.** Let $\varphi$ be an atoroidal iwip automorphism of $F_N$, and let $T_+$ be its forward limit tree. Then there exists a sublamination $L^2_{\text{BFH}} \subset L^2(T_+)$ which satisfies:

1. $L^2_{\text{BFH}}$ is the “stable lamination” exhibited by Bestvina-Feighn-Handel in [2] for $\varphi^{-1}$. In particular, $L^2_{\text{BFH}}$ is minimal and non-empty.
2. $L^2_{\text{BFH}}$ is the only minimal and non-empty sublamination of $L^2(T_+)$. 
3. $L^2(T_+) = \text{diag}(L^2_{\text{BFH}})$ 
4. $L^2(T_+) \setminus L^2_{\text{BFH}}$ is a finite union of $F_N$-orbits of pairs $(X,Y) \in \partial^2 F_N$.

The arguments given below prove also that $L^2_{\text{BFH}}$ is contained in $\text{supp}(\mu_\ominus)$. Indeed, it can be shown directly that the two laminations are equal.

In the course of our proof we also give several alternative characterizations of the dual lamination $L^2(T_+)$, based on (absolute) train track representatives of $\varphi$ or of $\varphi^{-1}$. The precise terminology of the terms used is given in sections 4 and 5 below.

**Proposition 1.3.** Let $\varphi$ be an atoroidal iwip automorphism of $F_N$, and let $T_+$ be its forward limit tree. Let $f_+: \tau_+ \to \tau_+$ and $f_-: \tau_- \to \tau_-$ be stable train track maps that represent $\varphi$ and $\varphi^{-1}$ respectively.

1. The dual lamination $L^2(T_+)$ consists precisely of all pairs $(X,Y) \in \partial^2 F_N$ such that for some $C > 0$ the whole $\varphi$-orbit $\varphi^i(X,Y)$ is totally $C$-illegal with respect to $f_+$. 

The dual lamination $L^2(T_+) \subset \partial^2 F_N$ consists precisely of all pairs $(X,Y) \in \partial^2 F_N$ such that the whole $\varphi$-orbit is uniformly $\varphi$-contracting (or, equivalently, uniformly $\varphi^{-1}$-expanding).

(3) The dual lamination $L^2(T_+) \subset \partial^2 F_N$ consists precisely of all pairs $(X,Y) \in \partial^2 F_N$ such that the whole $\varphi$-orbit is used legal with at most one singularity, with respect to $f_-$.

This proposition is a direct consequence of the fact that in the course of our proof of Theorem 1.2 we show first that $L^2(T_+)$ is included in the set of pairs from $\partial^2 F_N$ defined by the the property stated in part (1) of Theorem 1.2. One then shows that this set of pairs is contained in the set defined by part (2), and that the latter is contained in the set defined by part (3). In a final step one shows the set of $(X,Y)$ defined in part (3) is contained in the diagonal closure of $L^2_{FBH}$. But by part (3) of Theorem 1.2 the lamination $L^2(T_+)$ is equal to this diagonal closure, so that all of the previous inclusions must be in fact equalities.

Using the same type of arguments one deduces Theorem 1.1 from Theorem 1.2. From the perpendicularity of $T_+$ with $\mu_-$ we derived above that $\text{supp}(\mu_-)$ is a subset of $L^2(T_+)$, which implies (compare Definition-Remark 2.5) $\overline{\text{diag}(\text{supp}(\mu_-))} \subset \overline{\text{diag}(L^2(T_+))}$. Since $L^2(T_+)$ is diagonally closed (see Proposition 2.10), we obtain:

$$\overline{\text{diag}(\text{supp}(\mu_-))} \subset L^2(T_+)$$

On the other hand, from the uniqueness property given by part (2) of Theorem 1.2 one deduces that $L^2_{FBH}$ is contained in $\text{supp}(\mu_-)$. Hence (again by Definition-Remark 2.5) $\overline{\text{diag}(L^2_{FBH})}$ must be contained in $\overline{\text{diag}(\text{supp}(\mu_-))}$, so that part (3) of Theorem 1.2 yields

$$L^2(T_+) \subset \overline{\text{diag}(\text{supp}(\mu_-))}.$$

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Personal comment: The main result of this paper was originally known under the name of “Seven steps to happiness”, due to the very origin of this paper, where the more optimistic of the two authors proposed a proof-scheme in seven steps, and the other, the more skeptical between us, decided that it reminded him of one of these advertised programs which guarantee personal fulfillment, wealth, or religious enlightenment. Since then, the result has been internally referred to by that title, and in particular the headers of the later chapters are still a reminiscence to the beginning of this particular collaboration.
Organisation of the paper:

- In §2 we recall the basic facts and definitions about \( \mathbb{R} \)-trees, Outer space and iwip automorphisms. We also review some of the basics of algebraic laminations, define the diagonal closure of a lamination, and recall some of the known facts about the dual lamination of an \( \mathbb{R} \)-tree.

- In §3 we recall the basic facts and definitions of train track maps. We purposefully employ space and care to make this section accessible for, say, a graduate student who is only partially an expert of the subject.

- In §4 we show that the dual lamination \( L^2(T^+ ) \) of the forward limit tree \( T^+ \) of an atoroidal iwip automorphism \( \varphi \) has a strong contraction property with respect to the action of \( \varphi \). This implies that \( L^2(T^+ ) \) has a strong expansion property with respect to the action of \( \varphi^{-1} \).

- In §5 we will show that any lamination with a strong expansion property with respect to \( \varphi^{-1} \), when realized as geodesic paths on a train track representative \( f_- : \tau_- \rightarrow \tau_- \) of \( \varphi^{-1} \), consists only of paths which are entirely “used legal”, or else have precisely one “singularity” (the terms will be defined there). It will also be shown that the sublamination of \( L^2(T^+ ) \) which is realized by used legal geodesics on \( \tau_- \) is contained in Bestvina-Feighn-Handel’s “attracting” lamination \( L_{BFH}^2(f_- ) \) associated to the train track map \( f_- \).

- In §6 we consider elements \((X, Y) \in L^2(T^+ )\) which are realized on \( \tau_- \) by used legal geodesics with precisely one singularity. We show that such geodesics have a rather special form: they are (essentially) the concatenation of two eigenrays of the map \( f_- \). As a consequence, we show that such pairs \((X, Y)\) must lie in the diagonal closure of \( L_{BFH}^2(f_- ) \). Also, the finiteness of such pairs (up to the \( F_N \)-action) as well as the fact that they are non-rational is a direct consequence of this characterization.

- In §7 we determine, as an application of our previous results, the precise relationship between “singular leaves” of \( L^2(T^+ ) \) and branch points (and the eigenrays emanating from them) of the backward limit tree \( T_- \) of \( \varphi \).

- In section 8 we discuss some question issuing from our main result and the surrounding facts.

2. Definition of terms and background

The purpose of this section is to properly define the terms used in the introduction, and to give some background with references about them.

Throughout the paper we fix a “model” free group \( F_N \) of finite rank \( N \geq 2 \). A finite connected graph \( \tau \) is called a marked graph, if it is equipped with a marking isomorphism \( \theta : F_N \xrightarrow{\cong} \pi_1(\tau) \). Here we purposefully suppress
the issue of choosing a basepoint of \( \tau \), as in this paper we are interested in automorphisms of \( F_N \) only up to inner automorphisms.

2.1. Outer space.

We give here only a brief overview of basic facts related to Outer space. We refer the reader to [9] for more detailed background information.

The unprojectivized Outer space \( \text{cv}_N \) consists of all minimal free and discrete isometric actions on \( F_N \) on \( \mathbb{R} \)-trees (where two such actions are considered equal if there exists an \( F_N \)-equivariant isometry between the corresponding trees). There are several different topologies on \( \text{cv}_N \) that are known to coincide, in particular the equivariant Gromov-Hausdorff convergence topology and the so-called length function topology. Every \( T \in \text{cv}_N \) is uniquely determined by its translation length function \( \| \cdot \|_T : F_N \to \mathbb{R} \), where \( \| g \|_T := \min \{ d(gx, x) \mid x \in T \} \) is the translation length of \( g \) on \( T \).

Two trees \( T_1, T_2 \in \text{cv}_N \) are close if the functions \( \| \cdot \|_{T_1} \) and \( \| \cdot \|_{T_2} \) are close pointwise on a large ball in \( F_N \). The closure \( \overline{\text{cv}}_N \) of \( \text{cv}_N \) in either of these two topologies is well-understood and known to consists precisely of all the so-called very small minimal isometric actions of \( F_N \) on \( \mathbb{R} \)-trees, see [1] and [4].

The outer automorphism group \( \text{Out}(F_N) \) has a natural continuous right action on \( \text{cv}_N \) (that leaves \( \text{cv}_N \) invariant) given at the level of length functions as follows: for \( T \in \text{cv}_N \) and \( \varphi \in \text{Out}(F_N) \) we have \( \| g \|_{T \varphi} = \| \Phi(g) \|_T \), with \( g \in F_N \) and \( \Phi \in \text{Aut}(F_N) \) representing \( \varphi \in \text{Out}(F_N) \).

The projectivized Outer space \( \overline{\text{CV}}_N = \mathbb{P}\overline{\text{cv}}_N \) is defined as the quotient \( \overline{\text{cv}}_N / \sim \) where for \( T_1 \sim T_2 \) whenever \( T_2 = cT_1 \) for some \( c > 0 \). One similarly defines the projectivization \( \overline{\text{cv}}_N = \mathbb{P}\overline{\text{cv}}_N \) of \( \overline{\text{cv}}_N \) as \( \overline{\text{cv}}_N / \sim \) where \( \sim \) is the same as above. The space \( \overline{\text{CV}}_N \) is compact and contains \( \overline{\text{cv}}_N \) as a dense \( \text{Out}(F_N) \)-invariant subset. The compactification \( \overline{\text{CV}}_N \) of \( \overline{\text{cv}}_N \) is a free group analogue of the Thurston compactification of the Teichmüller space. For \( T \in \overline{\text{cv}}_N \) its \( \sim \)-equivalence class is denoted by \( [T] \), so that \( [T] \) is the image of \( T \) in \( \overline{\text{CV}}_N \).

2.2. Laminations and the diagonal closure.

For the free group \( F_N \) we define the double boundary

\[ \partial^2 F_N := \{(X,Y) \in \partial F_N \times \partial F_N \mid X \neq Y\}. \]

The set \( \partial^2 F_N \) comes equipped with a natural topology, inherited from \( \partial F_N \times \partial F_N \), and with a natural translation action of \( F_N \) by homeomorphisms. There is also a natural flip map \( \partial^2 F_N \to \partial^2 F_N \), \( (X, Y) \mapsto (Y, X) \), interchanging the two coordinates on \( \partial^2 F_N \).

The following definition has been introduced and systematically studied in [6], where also further background material concerning this subsection can be found.
Definition 2.1. An algebraic lamination on $F_N$ is a non-empty subset $L^2 \subseteq \partial^2 F_N$ which is (i) closed, (ii) $F_N$-invariant, and (iii) flip-invariant.

In analogy to laminations on surfaces, the elements $(X,Y) \in L^2$ are sometimes also referred to as the leaves of the algebraic lamination $L^2$.

In some circumstances it is useful to admit the empty set as algebraic lamination. We will formally stick to the classical non-empty notion, but occasionally informally include the empty set in our discussions about algebraic lamination.

Example 2.2. Let $\Omega$ be a set of conjugacy classes $[w]$ with $w \in F_N$. Denote by $w^\infty$ and $w^-\infty$ the attracting and the repelling fixed points respectively of the action of $w$ on $\partial F_N$. Then

$$L^2(\Omega) := \{(v^w\infty,v^w^-\infty) \mid v \in F_N, [w] \in \Omega \cup \Omega^{-1}\}$$

is an algebraic lamination.

Remark 2.3. (a) Note that, in the above Example 2.2, for finite $\Omega$ taking the closure on the right hand side of the displayed equality can be omitted without changing the set $L^2(\Omega)$.

(b) Thus for finite $\Omega$ one obtains a lamination $L^2(\Omega)$ which consists only of finitely many $F_N$-orbits of leaves. Indeed, it is an easy exercise to show that any such finite lamination occurs precisely in this way. (For the less experienced reader we recommend for this exercise the transition to symbolic laminations, as described in detail in [6].)

Definition 2.4. An algebraic lamination $L^2 \neq \emptyset$ is called minimal if it doesn’t contain any proper sublamination (other than $\emptyset$). This is equivalent to requiring that for any $(X,Y) \in L^2$ the set $F_N \cdot (X,Y) \cup F_N \cdot (Y,X)$ is dense in $L^2$.

The following terminology is inspired by geodesic laminations on surfaces, and isolated leaves which cross diagonally through one of the complementary components.

Definition-Remark 2.5. Let $S$ be a subset of $\partial^2 F_N$.

(a) The diagonal extension $\text{diag}(S)$ of $S$ is the set of all $(X,Y) \in \partial^2 F_N$ such that for some integer $m \geq 1$ there exist elements $X = Z_0, Z_1, \ldots, Z_m = Y$ in $\partial F_N$ such that $(Z_{i-1}, Z_i) \in S$ for $i = 1, \ldots, m$.

(b) It is easy to see that $S \subseteq \text{diag}(S)$ and that $\text{diag}(\text{diag}(S)) = \text{diag}(S)$.

(c) A subset $S \subseteq \partial^2 F_N$ is said to be diagonally closed if $S = \text{diag}(S)$.

(d) If $S' \subseteq S \subseteq \partial^2 F_N$ then $\text{diag}(S') \subseteq \text{diag}(S)$.

(e) If $S$ is $F_N$-invariant, then so is $\text{diag}(S)$. Similarly, if $S$ is flip-invariant then so is $\text{diag}(S)$. However, if $S$ is a closed subset of $\partial^2 S$, then we can deduce that $\text{diag}(S)$ is closed only if $\text{diag}(S) \setminus S$ consists of finitely many $F_N$-orbits.
(f) We denote by \( \text{diag}(S) \) the closure of \( \text{diag}(S) \) in \( \partial^2 F_N \). For any algebraic lamination \( L^2 \) the diagonal closure \( \text{diag}(L^2) \) is again an algebraic lamination. If \( \text{diag}(L^2) \setminus L^2 \) consists of finitely many \( F_N \)-orbits, then \( \text{diag}(L^2) = \text{diag}(L^2) \).

Remark 2.6. Examples of laminations where the diagonal extension is not closed are easy to come by. For example, if \( L^2 \) is given by a geodesic lamination on a surface, then \( \text{diag}(L^2) \) is closed if and only if all complementary components are simply connected.

If one picks a basis \( \mathcal{A} \) for \( F_N \), one can associate naturally to any pair \( (X,Y) \in F_N \) a biinfinite reduced word \( Z = X^{-1} \cdot Y \) in \( \mathcal{A} \cup \mathcal{A}^{-1} \), and to \( Z \) the set \( L \) of all finite subwords. In this way one can associate to every algebraic lamination \( L^2 \) a symbolic lamination \( L_A \) and a laminary language \( L_A \), and both of these translations are inversible. For details see [6].

Interpreting words in \( \mathcal{A} \pm 1 \) as paths in the associated rose \( \rho_A \), or in its universal covering, the Cayley graph \( \Gamma(F_N, \mathcal{A}) \), one can easily generalize the concepts of a “symbolic lamination” or a “laminary language” to more general graphs \( \tau \) with an identification \( F_N = \pi(\tau) \), or to their universal coverings. For the purposes of this paper, however, the following suffices:

**Definition 2.7.** Let \( \tau \) be a graph with a marking \( \theta : F_N \xrightarrow{\sim} \pi_1 \tau \), and let \( \partial \theta : \partial F_N \to \partial \tau \) be the induced homeomorphisms on the Gromov boundaries.

Let \( (X,Y) \in \partial^2 F_N \), and let \( \gamma \) be the biinfinite reduced path (the “geodesic”) in the universal covering \( \tilde{\tau} \) of \( \tau \) which joins the boundary point \( \partial \theta(X) \) to the boundary point \( \partial \theta(Y) \).

The reduced biinfinite path \( \gamma \) in \( \tau \) which is the image of \( \gamma \), under the universal covering map \( \tilde{\tau} \to \tau \), is called the geodesic realization (in \( \tau \)) of the pair \( (X,Y) \), and is denoted by \( \gamma_{\tau}(X,Y) \).

Assume now that the marked graph \( \tau \) comes with a homotopy equivalence \( f : \tau \to \tau \) which represents \( \varphi \in \text{Aut}(F_N) \), i.e. \( \varphi \circ \theta = f_* \circ \varphi \) (up to inner automorphisms of \( \pi_1(\tau) \)). In this case it follows directly from the above definition that, for any integer \( t \geq 0 \) and any \( (X,Y) \in \partial^2 F_N \), the geodesic realizations satisfy the equality

\[
[f^t(\gamma_{\tau}(X,Y))] = \gamma_{\tau}(\partial \varphi^t(X), \partial \varphi^t(Y)),
\]

where for any (possibly non-reduced) path \( \eta \) we denote by \([\eta]\) the reduced path obtained from \( \eta \) by reduction.

We say that a lamination \( L^2 \) is generated by a set \( P \) of edge paths \( \gamma_i \) in a marked graph \( \tau \), if a pair \( (X,Y) \in \partial^2 F_N \) belongs to \( L^2 \) if and only if any finite subpath of the geometric realization \( \gamma_{\tau}(X,Y) \) is also a subpath of some \( \gamma_i \in P \) or \( \overline{\gamma}_i \in P \). In this case we write:

\[
L^2 = L^2(P)
\]

Here typically the \( \gamma_i \) are finite paths, and \( P \) is infinite, but we also include the case of closed, infinite or bi-infinite \( \gamma_i \).
It is easy to see (compare [6]) that any infinite set $P$ of finite (possibly closed) edge paths in $\tau$ generates an algebraic lamination $L^2(P)$. If all $\gamma_i \in P$ are closed paths, then $L^2(P)$ only depends on the set $\Omega$ of conjugacy classes in $F_N$ defined by the $\gamma_i \in P$ (and not by the particular choice of the marked graph $\tau$ and the geodesic realization of the conjugacy classes in $\tau$); in this case we write $L^2(\Omega) := L^2(P)$.

2.3. The algebraic lamination dual to an $\mathbb{R}$-tree.

To any tree $T \in \overline{\mathbb{W}}_N$ in [7] there has been naturally associated a dual algebraic lamination (also called the zero lamination) $L^2(T)$ of $T$:

Definition 2.8. Let $T$ be an $\mathbb{R}$-tree from $\overline{\mathbb{W}}_N$.

(1) For any $\varepsilon > 0$ let $\Omega_\varepsilon(T)$ be the set of conjugacy classes $[w] \subset F_N$ with translation length $||w||_T \leq \varepsilon$, and let $L^2_\varepsilon(T) =: L^2(\Omega_\varepsilon(T))$.

(2) Define $L^2(T) := \bigcap_{\varepsilon > 0} L^2_\varepsilon(T)$.

In [7] it has been shown that set $L^2(T)$ is an algebraic lamination on $F_N$, if $T \in \partial \mathbb{W}_N$, and $L^2(T) = \emptyset$ otherwise (i.e. $T \in \mathbb{W}_N$). It has also been shown in [7], for any basis $A$ of $F_N$, that $L^2(T)$ is precisely the set of all $(X,Y) \in \partial^2 F_N$ such that the associated reduced biinfinite word $Z$ in $A^{\pm 1}$ has the property that for every finite subword $w$ of $Z$ and any $\varepsilon > 0$ there is a cyclically reduced word $v$ which contains $w$ as subword and satisfies $||v||_T \leq \varepsilon$.

As a slight extension of the latter, we can formulate the criterion which alternatively could serve as definition for this paper:

Remark 2.9. For any marked graph $\tau$ the following characterization of $L^2(T)$ holds:

A finite geodesic path $\gamma'$ is a subpath of the geodesic realization $\gamma_T(X,Y)$ for some $(X,Y) \in L^2(T)$ if and only if for any $\varepsilon > 0$ there is an element $w \in F_N$ with translation length on $T$ of seize $||w||_T \leq \varepsilon$, such that the conjugacy class of $w$ in $F_N$ is represented by a geodesic loop $\hat{\gamma}$ which contains $\gamma'$ as subpath.

It is well known that every $T \in \partial \mathbb{W}_N$ either contains points $x \in T$ with non-trivial stabilizer $\text{stab}(x) \subset F_N$, or else $T$ has dense orbits: This means that every orbit $F_N x$ of a point $x \in T$ is a dense subset of $T$. Note that this means that in particular the set of branch points (i.e. points with 3 or more complementary components) is dense in $T$. Of course, there are also trees $T \in \partial \mathbb{W}_N$ which have both, non-trivial point stabilizers, and dense orbits.

For $T \in \overline{\mathbb{W}}_N$ with dense $F_N$-orbits one can give an equivalent characterization of $L^2(T)$ to that given in Definition 2.8 above. For such a tree $T$ there is a canonical $F_N$-equivariant map $Q : \partial F_N \rightarrow \hat{T} = \overline{T} \cup \partial T$, constructed in [7], where $\partial T$ is the metric completion of $T$. The precise nature of how the map $Q$ is defined is not relevant for the present paper. However, it is
proved in [4] that in this case for \((X, Y) \in \partial^2 F_N\) we have \((X, Y) \in L^2(T)\) if and only if \(Q(X) = Q(Y)\). This fact immediately implies:

**Proposition 2.10.** Let \(T \in \overline{\partial N}\) have dense \(F_N\)-orbits. Then \(L^2(T)\) is diagonally closed, that is \(\text{diag}(L^2(T)) = L^2(T)\).

2.4. **Iwip automorphisms.**

**Definition 2.11.** (a) An outer automorphism \(\varphi \in \text{Out}(F_N)\) is called **irreducible with irreducible powers** (iwip) if no positive power of \(\varphi\) preserves the conjugacy class of a proper free factor of \(F_N\).

(b) An outer automorphism \(\varphi \in \text{Out}(F_N)\) is called **atoroidal** if it has no non-trivial periodic conjugacy classes, i.e. if there do not exist \(t \geq 1\) and \(w \in F_N - \{1\}\) such that \(\varphi^t\) fixes the conjugacy class \([w]\) of \(w\) in \(F_N\).

It was proved in [3] that for \(N \geq 2\) an iwip automorphisms \(\varphi \in \text{Out}(F_N)\) is non-atoroidal if and only if \(\varphi\) is induced, via an identification of \(F_N\) with the fundamental group of a compact surface \(S\) with a single boundary component, by a pseudo-Anosov homeomorphisms \(h : S \to S\).

**Remark 2.12.** The terminology “iwip” derives from the groundbreaking paper [3]: Bestvina-Handel call an element \(\varphi \in \text{Out}(F_N)\) is **reducible** if there exists a free product decomposition \(F_N = C_1 \ast \ldots \ast C_k \ast F',\) where \(k \geq 1\) and \(C_i \neq \{1\}\), such that \(\varphi\) permutes the conjugacy classes of subgroups \(C_1, \ldots, C_k\) in \(F_N\). An element \(\varphi \in \text{Out}(F_N)\) is called **irreducible** if it is not reducible.

It is not hard to see that an element \(\varphi \in \text{Out}(F_N)\) is an iwip if and only if for every \(n \geq 1\) the power \(\varphi^n\) is irreducible (sometimes such automorphisms are also called **fully irreducible**).

It is known by a result of Levitt and Lustig [17] that iwips have a simple “North-South” dynamics on the compactified Outer space \(\overline{CV}_N\):

**Proposition 2.13.** [17] Let \(\varphi \in \text{Out}(F_N)\) be an iwip. Then there exist unique \([T_+] = [T_+(\varphi)], [T_-] = [T_-(\varphi)] \in \overline{CV}_N\) with the following properties:

1. The elements \([T_+], [T_-] \in \overline{CV}_N\) are the only fixed points of \(\varphi\) in \(\overline{CV}_N\).

2. For any \([T] \in \overline{CV}_N, [T] \neq [T_-]\) we have \(\lim_{n \to \infty} [T \varphi^n] = [T_+]\) and for any \([T] \in \overline{CV}_N, [T] \neq [T_+]\) we have \(\lim_{n \to \infty} [T \varphi^{-n}] = [T_-]\).

3. We have \(T_+ \varphi = \lambda_+ T\) and \(T_- \varphi^{-1} = \lambda_- T_\) where \(\lambda_+ > 1\) and \(\lambda_- > 1\).

4. Both, \(T_+\) and \(T_-\) are “intrinsically non-simplicial” \(\mathbb{R}\)-trees: Every \(F_N\)-orbit of points is dense in the tree.

In [17] it is also proved that convergence in (2) is locally uniform and hence uniform on compact subsets. For more information see [15]. A description of \(T_+\) in terms of train tracks is given below in section 3.7.
3. Train track technology

This section will be a reference section, organized by subheaders as a kind of glossary. Almost everything in this section is known, or within $\varepsilon$-neighborhood of known facts. It is only for the convenience of the reader that we assemble in this section what is needed later from basic train track theory.

The expert reader is encouraged to completely skip this section and only go back to it if he or she needs an additional explanation in the course of reading the later sections.

Notation 3.1. In order to avoid confusion, in this section the automorphisms in question will be denoted by $\alpha$, as in the later sections we will have to consider both cases, $\alpha = \varphi$ and $\alpha = \varphi^{-1}$. Similarly, a train track map will be called $f : \tau \to \tau$ (to be specified later to $f_+ : \tau_+ \to \tau_+$ or $f_- : \tau_- \to \tau_-$, the forward limit tree will be simply called $T$ (rather than $T_+$ or $T_-$), etc.

3.1. Graphs, paths and graph maps.

In this paper $\tau$ will always denote a graph, specified as follows:

Convention 3.2. By a graph $\tau$ we mean a non-empty topological space which is equipped with a cell structure, consisting of vertices and edges. Furthermore, $\tau$ satisfies the following conditions, unless explicitly otherwise stated:

- $\tau$ is connected.
- $\tau$ is a finite graph, i.e. it consists of finitely many vertices and edges.
  (Of course, the universal covering $\tilde{\tau}$ of a finite graph $\tau$ is in general not finite, but this counts as “explicitly stated exception”.)
- There are no vertices of valence 1 in $\tau$ (but we do admit vertices of valence 2).

We systematically denote vertices of $\tau$ by $v, v'$, and edges by $e, e', e_j$, etc. A point which can be a vertex or an interior point of an edge is usually denoted by $P$ or $Q$.

An edge $e$ of $\tau$ is always oriented, and we denote by $\overline{e}$ the conversely oriented edge. In an edge path $\ldots e_k e_{k+1} e_{k+2} \ldots$ we always write the edges $e_i$ in the direction in which the path runs. We occasionally use $\text{Edges}(\tau)$ to denote the edge set of $\tau$. We use the convention that its elements are, as before, oriented edges, so that the edge set $\text{Edges}(\tau)$ contains for each pair $e, \overline{e}$ only the element $e$. If need be, we use the notation $\overline{\text{Edges}(\tau)} := \{\overline{e} \mid e \in \text{Edges}(\tau)\}$. Note that one is free, at any convenient time, to reorient edges: this operation does not change the graph, in our understanding, only the extra information about notation of edges.

However: to be specific: if we say “an edge of $\tau$”, this may well be the edge $\overline{e}$, for $e \in \text{Edges}(\tau)$.
We will denote the simplicial length of an edge path $\gamma$, i.e. the number of edges traversed by $\gamma$, with $L(\gamma)$. For the edge path $\overline{\gamma}$, by which we mean the path $\gamma$ with reversed orientation, one has of course $L(\overline{\gamma}) = L(\gamma)$. In particular, for any single edge $e$ one has $L(e) = L(\overline{e}) = 1$.

Convention-Definition 3.3. (a) If we use the terminology path without further specification, we always mean a finite path. The reader is free to formalize paths according to his or her own preferences; in most cases we recommend the viewpoint where a path $\gamma$ means an edge path $\gamma = e_1 \circ e_2 \circ \ldots \circ e_q$, where the $e_i$ are edges of $\tau$ such that the terminal vertex of any $e_i$ coincides with the initial vertex of $e_{i+1}$. According to circumstances, we generalize this concept sometimes slightly by allowing that $e_1$ (or $e_q$) is a non-degenerate terminal (or initial) segment of an edge (or, if $q = 1$, any segment of a single edge).

A path $\gamma$ is called trivial or degenerate if it consists of a single point. Otherwise, $\gamma$ is said to be non-trivial or non-degenerate.

We will also sometimes suppress the distinction between an edge (or edge segment) $e$ and a path which traverses precisely $e$.

(b) A path $\gamma'$ is called a backtracking path if its endpoints coincide, and if the issuing loop is contractible in $\tau$. If $\gamma'$ occurs as subpath of a path $\gamma$, then $\gamma'$ is called a backtracking subpath of $\gamma$.

(c) A (possibly infinite or biinfinite) path $\gamma$ is reduced if it doesn’t contain any non-degenerate backtracking subpath. We denote by $[\gamma]$ the reduced path obtained from a possibly non-reduced path $\gamma$ by iteratively contracting all non-degenerate backtracking subpaths. (It is well known that $[\gamma]$ depends only on $\gamma$ and not on the sequence of iteratively contracted backtracking subpaths).

(d) Two subpaths $\gamma_1$ and $\gamma_2$ of a path $\gamma$ are disjoint if in the course of traversing $\gamma$ one first traverses $\gamma_1$ completely without starting $\gamma_2$, or conversely. Similarly we define the overlap of the two subpaths as the maximal subpath of $\gamma$ which is also a subpath of both $\gamma_i$.

Recall from the beginning of section 2 that a finite connected graph $\tau$ is called a marked graph, if it is equipped with a marking isomorphism $\theta : F_N \xrightarrow{\cong} \pi_1(\tau)$. Recall also that we purposefully suppress the issue of choosing a basepoint of $\tau$, as we are only interested in automorphisms of $F_N$ up to inner automorphisms.

Convection-Definition 3.4. (a) A map $f : \tau \to \tau'$ between graphs $\tau$ and $\tau'$ will always map vertices to vertices and edges to edge paths, which a priori may be non-reduced.

(b) A self-map $f : \tau \to \tau$ of a graph $\tau$, provided with a marking isomorphism $\theta : F_N \xrightarrow{\cong} \pi_1(\tau)$, represents an automorphism $\alpha$ of $F_N$ if the induced automorphisms $f_* : \pi_1(\tau) \to \pi_1(\tau)$ satisfies $\theta \circ \alpha = f_* \circ \theta$ up to inner automorphisms.
Remark 3.5. It is well known [5] that for any map \( f : \tau \to \tau' \) between graphs \( \tau \) and \( \tau' \), which induces an isomorphism on \( \pi_1(\tau) \) (or, for the matter, a monomorphism), there is an upper bound to the length of any backtracking path \( \gamma' \) which is contained as subpath in the (non-reduced) image \( f(\gamma) \) of a reduced path \( \gamma \) in \( \tau \).

Definition 3.6. (a) A self-map \( f : \tau \to \tau \) is called expanding if for every edge \( e \) of \( \tau \) there is an exponent \( t \geq 1 \) such that the edge path \( f^t(e) \) has length \( L(f^t(e)) \geq 2 \).

(b) If \( f : \tau \to \tau \) is expanding, then there is a well defined self-map \( Df \) on the set \( \text{Edges}(\tau) \cup \text{Edges}(\tau) \) which associates to every edge \( e \) the initial edge of the edge path \( f(e) \).

Remark 3.7. If a self-map \( f : \tau \to \tau \) represents an iwip automorphism of \( F_N \), then the hypothesis that \( f \) be expanding is always easy to satisfy: It suffices to contract all edges which are not expanded by any iterate \( f^t \) to an edge path of length \( \geq 2 \): The issuing contracted subgraph must be a forest, as otherwise some \( f^t \) will fix (up to conjugacy) a non-trivial proper free factor of \( \pi_1(\tau) \).

Alternatively, one can reparametrize the map \( f \) along each edge so that it becomes a true homothety with respect to the Perron-Frobenius metric introduced in subsection 3.7 below.

Since in this paper we will concentrate on self-maps of graphs that represent iwip automorphisms, we will almost exclusively consider self-maps \( f : \tau \to \tau \) that are expanding.

Lemma 3.8. Let the self-map \( f : \tau \to \tau \) of the graph \( \tau \) be expanding, and assume that \( f \) represents an iwip automorphism of \( F_N \). Then we have:

1. Every \( f \)-invariant subgraph \( \tau_0 \subset \tau \) is either a single vertex (or a collection of vertices, if one relaxes the convention that a graph is connected), or equal to all of \( \tau \).
2. For every edge \( e \) there exists an exponent \( t \geq 0 \) such that the (possibly non-reduced) edge path \( f^t(e) \) crosses three or more times over \( e \) or \( \overline{e} \).

Proof. (1) Assume that \( \tau_0 \subset \tau \) contains at least one edge. From the assumptions that \( f \) is expanding, and that \( f(\tau_0) \subset \tau_0 \), it follows that \( \pi_1(\tau_0) \) is non-trivial. From the iwip hypothesis it follows that the inclusion \( \tau_0 \subset \tau \) induces an isomorphisms on \( \pi_1 \). Hence our convention on graphs (see subsection 3.1), that \( \tau \) is finite and does not contain vertices of valence 1, implies \( \tau_0 = \tau \).

(2) By the expansiveness of \( f \) there is at least one edge \( e \) with the property described in statement (2). We consider a subgraph \( \tau_0 \) which (i) contains at least one such edge, (ii) is \( f \)-invariant, and (iii) is minimal with respect to the properties (i) and (ii). If \( \tau_0 \) contains edges which do not satisfy statement (2), then the union of those must be a non-empty subcomplex
which is \( f \)-invariant (since any edge \( e \), which lies outside of this union but is contained in its image, must have all image paths \( f^t(e) \) disjoint from this union, thus contradicting the above minimality assumption (iii)). But this contradicts the expansiveness of \( f \). Thus \( \tau_0 \) consists only of edges satisfying statement (2). From statement (1) it follows \( \tau_0 = \tau \).

\[
3.2. \text{Gates, turns, train tracks.}
\]

**Definition 3.9.** (a) For any expanding self-map of graphs \( f : \tau \to \tau \) and any vertex \( v \) of \( \tau \) one partitions the edges with initial vertex \( v \) into equivalence classes, called gates \( g_i \), by the following rule: Two edges \( e_1 \) and \( e_2 \), both with initial vertex \( v \), belong to a common gate \( g_i \) if and only if there is an exponent \( t \geq 1 \), such that \( Df^t(e_1) = Df^t(e_2) \). To be specific, in case of a loop edge \( e \) at \( v \) (i.e. the initial and the terminal vertex of \( e \) both coincide with \( v \)) the edges \( e \) and \( \overline{e} \) count as distinct edges with initial vertex \( v \), which can or cannot belong to the same gate at \( v \).

(b) This definition is sometimes extended to points \( P \) in the interior of an edge \( e \), at which there are precisely two gates, one containing precisely the terminal segment \( e'' \) of \( e \) which starts at \( P \), and the other one containing precisely the edge segment \( \overline{e}' \), where \( e' \) is the initial segment of \( e \) which terminates at \( P \).

**Remark 3.10.** For any vertex (or point \( P \)) of \( \tau \) the map \( f \) induces via the map \( Df \) a map \( f^P \) from the gates at \( P \) to the gates at \( f(P) \). It follows directly from Definition 3.9 (a) that this map \( f^P \) is injective, and hence, if \( P \) is a periodic point, that \( f^P \) is bijective.

**Definition 3.11.** (a) A pair of edges of \( \tau \) forms a turn \((e, e')\) at a vertex \( v \) if and only if both, \( e \) and \( e' \), have \( v \) as initial vertex. The turn is said to be degenerated if \( e = e' \). Otherwise it is called non-degenerate.

(b) A path \( \gamma \) crosses over a turn \((e, e')\) (or contains the turn \((e, e')\)) if \( \gamma \) contains the edge path \( \overline{e} \circ e' \) as subpath (or a path \( \overline{e}_0 \circ e'_0 \) for non-degenerate initial segments \( e_0 \) of \( e \) and \( e'_0 \) of \( e' \)).

Note that a path \( \gamma \) is reduced if and only if it doesn’t cross over any degenerate turn.

**Definition 3.12.** Let \( f : \tau \to \tau \) be an expanding self-map of a graph \( \tau \).

1. The map \( f \) induces canonically a map \( D^2f \) on turns, by setting \( D^2((e, e')) = (Df(e), Df(e')) \).
2. A turn \((e, e')\) is called legal if for all \( t \geq 0 \) the image turn \( D^2f^t((e, e')) \) is non-degenerate. Otherwise \((e, e')\) is called illegal. In particular, all degenerate turns are illegal.

A path (or loop) \( \gamma \) in \( \tau \) is called legal if it crosses only over legal turns. Otherwise it is called illegal.
(From Definition 3.9 it follows directly that the turn \((e, e')\) is legal if and only if \(e\) and \(e'\) belong to distinct gates at their common initial vertex.)

(3) The map \(f\) is called a train track map if for every edge \(e\) of \(\tau\) the edge path \(f(e)\) is legal. (It is easy to see that this condition is equivalent to the requirement that \(f\) maps any legal path \(\gamma\) to a legal path \(f(\gamma)\).)

**Remark 3.13.** (a) It follows directly from the Definition 3.12 (2) that \(D^2f\) maps legal turns to legal turns, and illegal turns to illegal turns.

(b) If \(f : \tau \to \tau\) is not expanding, then one can use the following definition of legal paths, which in the expanding case is equivalent to the one given in Definition 3.12 (2):

A path \(\gamma\) in \(\tau\) is said to be legal if the maps \(f^t\) are locally injective along \(\gamma\), for all \(t \geq 1\).

(c) In some circumstances it can be useful to consider more general “train track maps”, i.e. self-maps of graphs which have the train track property that legal maps are mapped to legal paths (where “legal paths” are defined as in part (b) above). However, in this paper we insist on the assumption that a train track map \(f : \tau \to \tau\) is always expanding!

We now state a crucial property of train track maps, which follows directly from the Definition 3.12.

**Remark 3.14.** For any train track map \(f : \tau \to \tau\) and any path \(\gamma\) in \(\tau\) the number of illegal turns which are crossed over by \(\gamma\), denoted by \(\text{ILT}(\gamma)\), satisfies:

\[\text{ILT}([f(\gamma)]) \leq \text{ILT}(f(\gamma)) = \text{ILT}(\gamma)\]

Of course, a path \(\gamma\) is legal if and only if \(\text{ILT}(\gamma) = 0\).

**Lemma 3.15.** If \(f : \tau \to \tau\) is a train track map which represents an iwip automorphism of \(F_N\), then at every vertex of \(\tau\) there are at least 2 gates.

**Proof.** Let \(v\) be any vertex of \(\tau\), and let \(e\) be an edge with initial vertex \(v\). Since \(f\) is a train track map, the path \(f^t(e)\) must be legal, for any \(t \geq 0\). By Lemma 3.8 (2) for some value of \(t\) the path \(f^t(e)\) contains \(e\) (or \(\overline{e}\)) as interior edge. Let \(e'\) be the edge adjacent to this occurrence of \(e\) (or \(\overline{e}\)) on \(f^t(e)\), so that \(e'e\) or \(\overline{e'e}\) is a subpath of \(f^t(e)\). By Definition 3.12 (2) the edges \(\overline{e'}\) and \(e\) belong to distinct gates (at the vertex \(v\)).

The following is one of the fundamental results for automorphisms of free groups (slightly adapted to the language specified in this section):

**Theorem 3.16 (3).** For every iwip automorphism \(\alpha\) of \(F_N\) there exists a train track map \(f : \tau \to \tau\) that represents \(\alpha\).
3.3. Eigenrays.

Throughout this subsection we assume that $f : \tau \to \tau$ is a train track map.

**Definition 3.17.** (a) A ray $\rho$ is an infinite path in $\tau$ which doesn’t necessarily start at a vertex. The path $\rho$ can alternatively be thought of as locally injective map $\mathbb{R}_{\geq 0} \to \tau$ which crosses infinitely often over vertices, or as reduced infinite edge path $e'_1e_2e_3\ldots$, where the $e_i$ with $i \geq 2$ are edges of $\tau$, and $e'_1$ is a non-degenerate terminal segment of some edge $e_1$ of $\tau$ (which includes the possibility $e'_1 = e_1$). The point $P \in \tau$ (not necessarily a vertex!) which is the initial point of $e'_1$ (from the second viewpoint) or the image of $0 \in \mathbb{R}_{\geq 0}$ (from the first viewpoint) is called the starting point of $\rho$. If $P$ is a vertex, then we also call it the initial vertex of $\rho$.

(b) A ray $\rho$ is called an eigenray if one has $f^t(\rho) = \rho$ for some integer $t \geq 1$.

(Note that the condition $f^t(\rho) = \rho$ is stronger than requiring just $[f^t(\rho)] = \rho$ !) In particular, it follows that the starting point $P$ of $\rho$ is $f$-periodic, and that $\rho$ is a legal path.

**Lemma 3.18.** Let $\gamma$ be a non-trivial finite path in $\tau$, and assume that for some $t \geq 1$ the map $f^t$ maps $\gamma$ to a path $f^t(\gamma)$ which contains $\gamma$ as initial subpath. Then there is precisely one eigenray $\rho$ in $\tau$ which has $\gamma$ as initial subpath.

**Proof.** It is easy to see that the union of the $f^{kt}(\gamma)$ for all $k \geq 1$ form an eigenray. Conversely, any eigenray $\rho$ which contains $\gamma$ as initial subpath must also contain any $f^{kt}(\gamma)$ as initial subpath. \qed

We see from Lemma 3.18 that an eigenray can never bifurcate in the forward direction to give rise two eigenrays with same initial point. However, we will see later that, in the negative direction, an eigenray may well bifurcate, giving rise to an INP (compare Lemma 3.26).

**Proposition 3.19.** Let $f : \tau \to \tau$ be a train track map, and let $P$ be a periodic vertex (or interior point) of $\tau$. Then every gate $g_i$ at $P$ is mapped by $f$ periodically. A gate $g_i$ contains precisely one edge $e_i$ on which $Df$ acts periodically (an “eigen edge”), and there is precisely one eigenray $\rho$ which starts “from $g_i$”, i.e. which starts with an edge (or edge segment) that belongs to $g$. This edge is precisely the eigen edge $e_i$.

**Proof.** By Definition 3.9 for every gate $g$ there is an exponent $t \geq 0$ such that $Df^t$ maps every edge of $g$ to a single edge. Recall from Remark 3.10 that for each periodic point $P$ of $\tau$ the map $f$ induces a periodic map $f^*_P$ on the gates at $P$. Hence each gate $g_i$ at $P$ contains precisely one edge $e_i$ (the “eigen edge”) which is periodic under the induced map $Df$, say with period $k(i) \in \mathbb{N}$.

Thus $f^{k(i)}(e_i)$ is an edge path with initial subpath $e_i$. Hence Lemma 3.18 gives us a well defined eigenray which starts from the gate $g_i$, with initial edge $e_i$. 
Since the initial edge (or edge segment) of an eigenray is necessarily periodic under the map \( Df \), and two eigenrays with same initial edge (or edge segment) must agree (see Lemma 3.18), it follows that from every gate only one eigenray can start.

From Proposition 3.19 we obtain directly the following:

**Corollary 3.20.** For every periodic point \( P \) of \( \tau \) there is a canonical bijection between (i) the gates at \( P \), (ii) the eigen edges with initial vertex \( P \), and (iii) the eigenrays with starting point \( P \). 

We also need later on the following property:

**Lemma 3.21.** Let \( f : \tau \to \tau \) be a train track map which represents an automorphism of \( F_N \), and let \( \rho \) be an eigenray in \( \tau \). Then \( \rho \) can not be an eventually periodic path, i.e. a path of the form \( \gamma_0 \circ \gamma_1 \circ \gamma_1 \circ \gamma_1 \circ \ldots \).

**Proof.** By our convention from Definition 3.12 (3) the map \( f \) is expanding, which contradicts the fact that the loop \( \gamma_1 \) would have to be mapped to itself, given that by hypothesis \( f \) represents an automorphisms of \( F_N \). 

### 3.4. INPs.

As before, assume that throughout this subsection \( f : \tau \to \tau \) is a train track map. In addition, we assume in this section that \( f \) induces an automorphism on \( \pi_1(\tau) \).

**Definition 3.22.** A path \( \eta \) in \( \tau \) which crosses over precisely one illegal turn is called a *periodic indivisible Nielsen path* (or INP, for short), if for some exponent \( t \geq 1 \) one has \( [f^t(\eta)] = \eta \), (where \([\gamma]\) denotes as before the path obtained via reduction from a possibly unreduced path \( \gamma \)).

The illegal turn on \( \eta = \gamma' \circ \gamma \) is called the *tip* of \( \eta \), while the two maximal initial legal subpaths \( \gamma' \) and \( \gamma \), of \( \eta \) and \( \gamma \) respectively, are called the *branches* of \( \eta \).

**Remark 3.23.** Let \( \eta = \gamma' \circ \gamma \) be an INP of the train track map \( f : \tau \to \tau \).

1. The two endpoints of \( \eta \) are fixed points (but not necessarily vertices !) of \( f^t \) for some \( t \geq 1 \). The branches \( \gamma \) and \( \gamma' \) of \( \eta \) are initial segments of eigenrays \( \rho \) and \( \rho' \) respectively, defined as unions of the nested sequences of paths \( (f^{kt}(\gamma))_{k \in \mathbb{N}} \) and \( (f^{kt}(\gamma'))_{k \in \mathbb{N}} \) (compare Lemma 3.13). The rays \( \rho \) and \( \rho' \) coincide up to the initial segments \( \gamma \) and \( \gamma' \) respectively: indeed the common terminal segment of \( \rho \) and \( \rho' \) is precisely the union of all the paths crossed back and forth by the backtracking subpaths (see Convention-Definition 3.13 (b)) of the unreduced paths \( f^{kt}(\eta) \), at the tip of the illegal turn of \( \eta \), for all \( k \geq 1 \).

2. Two eigenrays \( \rho \) and \( \rho' \), which coincide up to initial segments \( \gamma \) and \( \gamma' \) respectively, define always an INP \( \eta = \gamma' \circ \gamma \) as above in (1). This follows directly from the definitions.
Remark 3.24. For every train track map $f : \tau \rightarrow \tau$ there are only finitely many INPs in $\tau$, if one assumes (as done throughout this section) that (i) $f$ is expanding, (ii) $\tau$ is finite, and (iii) $f$ induces an automorphism on $\pi_1(\tau)$.

This is a consequence of the fact that $f$ induces a quasi-isometry on the universal covering space (with respect to the simplicial metric $L$), so that for any geodesic path in $\tau$ the length of any backtracking subpath in the image path is bounded above by a constant depending only on $f$. Since $f$ is expanding, this gives a bound to the maximal length of the legal branches $\gamma$ and $\gamma'$ of any INP $\eta = \gamma' \circ \gamma$ in $\tau$, since the only backtracking subpath in $f(\eta)$ is the subpath cancelled at the tip when $f(\eta)$ is reduced to $[f(\eta)]$.

(If the reader wants to fill in the details, we recommend restricting to the case where $f$ induces an iwip automorphism on $\pi_1(\tau)$ and working with the PF-metric $L^{PF}$ introduced in subsection 3.7 rather than with the simplicial metric $L$; by Remark 3.39 the two metrics define a quasi-isometry for $\bar{\tau}$.)

Convention 3.25. As pointed out in Remark 3.23 (1) the endpoints of an INP may a priori well lie in the interior of an edge. However, since these points are $f$-periodic, and since by Remark 3.24 there are only finitely many INPs in $\tau$, we can assume from now on that the edges of $\tau$ have been subdivided accordingly, so that all endpoints of INPs are vertices.

(Of course, such a subdivision must be followed potentially by the procedure lined out in Remark 3.7 in order to make sure that after subdivision the train track map $f$ is still expanding.)

Lemma 3.26. There exists an exponent $r_1 \geq 1$ with the following property: Let $\gamma$ be a path in $\tau$, and assume that it contains precisely two illegal turns, each being the tip of an INP-subpath $\eta_1$ and $\eta_2$ of $\gamma$. Assume that $\eta_1$ and $\eta_2$ overlap in a non-degenerate subpath. Then $f^{r_1}(\gamma)$ reduces to a path $[f^{r_1}(\gamma)]$ which is legal.

Proof. By hypothesis, $\eta_1 = \gamma'_1 \circ \gamma_1$ and $\eta_2 = \gamma'_2 \circ \gamma_2$ intersect in a non-degenerate path, which by Convention 3.25 must be an edge path $\gamma'$ with vertices as initial and terminal point.

Each INP contains only one illegal turn, and by assumption $\gamma$ contains two. Hence the path $\gamma'$ cannot contain either, so that it must be a legal subpath of both, the branch $\gamma'_2$ of $\eta_2$, and the inverse of the branch $\gamma_1$ of $\eta_1$.

Since by assumption $\gamma'$ is non-degenerate, it is expanded under iteration of $f$ to a legal edge path of arbitrary big length.

Assume that an exponent $r \geq 0$ is chosen such that $L(f^r(\gamma'))$ is strictly bigger than the sum $L(\gamma_1) + L(\gamma_2)$. In this case it follows directly that, in reducing $f^r(\gamma)$, both illegal turns disappear into the backtracking subpaths, so that the resulting path $[f^r(\gamma)]$ is legal.

Since $f$ is expanding and since (see Remark 3.24) there are only finitely many INPs in $\tau$, it is easy to find a bound $r_1$ as required in the claim. \(\Box\)
The following is one of the crucial properties of train track maps of graphs. It goes back to the first paper [3] on the subject. An alternative proof is given in [18].

**Proposition 3.27.** For every train track map \( f : \tau \to \tau \) there exists a constant \( r_2 = r_2(\tau) \geq 0 \) such that every path \( \gamma \) with precisely 1 illegal turn satisfies:

Either \( \gamma \) contains an INP as subpath, or else \( [f^{r_2}(\gamma)] \) is legal. \( \square \)

Recall from Remark 3.14 that \( \text{ILT}(\gamma) \) denotes the number of illegal turns in a path \( \gamma \).

**Proposition 3.28.** There exists an exponent \( r = r(f) \geq 0 \) such that every finite path \( \gamma \) in \( \tau \) with \( \text{ILT}(\gamma) \geq 2 \) satisfies

\[
\text{ILT}([f^{r}(\gamma)]) < \text{ILT}(\gamma),
\]

unless every illegal turn on \( \gamma \) is the tip of an INP-subpath \( \eta_i \) of \( \gamma \), where any two \( \eta_i \) are either disjoint subpaths on \( \gamma \), or they overlap precisely in a common endpoint (for this terminology compare Convention-Definition 3.3 (d)).

**Proof.** It suffices to take \( r \) to be the maximum of \( r_1 \) from Lemma 3.26 and \( r_2 \) from Proposition 3.27. We then apply the latter to any maximal subpath with precisely one illegal turn. If each such subpath contains an INP-subpath, we apply Lemma 3.26. \( \square \)

Bestvina-Handel [3] proved the following result which is very useful in many contexts:

**Proposition 3.29.** Every iwip automorphism \( \varphi \) is represented by a train track map \( f : \tau \to \tau \) such that \( \tau \) contains at most one INP \( \eta \). The path \( \eta \) is closed if and only if \( \varphi \) is induced by a homeomorphism of some surface with at least one boundary component or puncture.

Such special train track maps have been termed stable in [3].

### 3.5. Used turns.

As before, let \( f : \tau \to \tau \) be a train track map which is fixed throughout this subsection.

**Definition 3.30.** A turn \((e,e')\) in \( \tau \) is called used if there is an edge \( e'' \) in \( \tau \) with the property that for some \( t \geq 1 \) the image path \( f^t(e'') \) crosses over this turn. Otherwise the turn \((e,e')\) is called unused.

**Remark 3.31.** From Definition 3.30 we derive directly the following facts:

(1) Every used turn is legal. The converse is in general wrong.

(2) The image turn (under the map \( D^2 f \)) of any used turn is also used.
The image of an unused turn \((e, e')\) may well be used. There is, however, a constant \(s \geq 0\) (which only depends on the total number of turns in \(\tau\)) such that either \(D^2 f^s(e, e')\) is used, or else all forward iterates of \((e, e')\) under \(D^2 f\) are unused.

**Lemma 3.32.** If the train track map \(f : \tau \to \tau\) represents an iwip automorphism, then for every edge \(e\) of \(\tau\) there is an edge \(e'\) which has the same initial vertex as \(e\), such that \(ee'\) is a used legal turn.

**Proof.** This is a direct consequence of Lemma 3.8 (2). \[\Box\]

**Lemma 3.33.** Let \(g_1\) and \(g_2\) be two gates at the same periodic vertex \(v\) of \(\tau\), and assume that some turn \((e'_1, e'_2)\) with \(e'_i \in g_i\) is used. Then the turn \((e_1, e_2)\) is also used, where \(e_i\) is the initial edge of the eigenray \(\rho_i\) that starts from the gate \(g_i\), for \(i = 1\) and \(i = 2\). (By Proposition 3.19 this is equivalent to stating that each edge \(e_i\) is the eigen edge of the gate \(g_i\).)

**Proof.** From the definition of a gate it follows directly that for every periodic gate \(g_i\) there is an exponent \(t_i \geq 1\) such that \(Df^{t_i}\) maps every edge \(e'\) in \(g_i\) to the eigen edge \(e_i\) of \(g_i\).

Now, if the turn \((e'_1, e'_2)\) is used, then for some edge \(e\) and some \(t \geq 1\) the edge path \(f^t(e)\) crosses over the turn \((e'_1, e'_2)\). It follows that the path \(f^{tt_1t_2}(e)\) crosses over the turn \((e_1, e_2)\). \[\Box\]

### 3.6. BFH’s “stable” lamination.

The initials BFH in the following definition refer to Bestvina-Feighn-Handel, who introduced and studied the following lamination in [2]; in particular they showed a special attraction property of this lamination, for train track maps which represent iwip automorphisms.

**Definition 3.34.** For any train track map \(f : \tau \to \tau\) we define the **BFH-attracting lamination** \(L^2_{BFH}(f)\) as the lamination which is generated by the family of paths \(f^t(e)\), for any edge \(e\) of \(\tau\) and any exponent \(t \geq 0\).

It is easily seen that, if \(f\) represents an automorphism \(\alpha\) of \(F_N\), that \(L^2_{BFH}\) is invariant under the natural map induced by \(\alpha\) on \(\partial^2 F_N\).

The following is an easy consequence of the definition of “used turn” in the previous section, and of Lemma 3.32.

**Lemma 3.35.** Let \(f : \tau \to \tau\) be a train track map which represents an iwip automorphism. Then any leaf \((X, Y)\) of the lamination \(L^2_{BFH}(f)\) is represented by biinfinite path \(\gamma\) in \(\tau\) which only crosses over used turns. \[\Box\]

The following proposition is due to Bestvina-Feighn-Handel. We note here that an alternative proof of part (1) follows as direct consequence of our Theorem 1.2 since \(L^2(T_+)\) depends only on \(\alpha\) and not on the particular train track representative \(f : \tau \to \tau\).

We would also recommend to the reader to try proving part (2) as an exercise, as it is quite doable and will enhance the reader’s understanding of the main subjects treated in this paper.
Proposition 3.36 (2). Let \( f : \tau \to \tau \) be a train track representative of an iwip automorphism \( \alpha \) of \( F_N \).

(1) The lamination \( L_{BFH}^2(f) \) depends only on \( \alpha \in \text{Out}(F_N) \) and not on the particular choice of the train track representative \( f \).

(2) The lamination \( L_{BFH}^2(f) \) is minimal (see Definition 2.4). \( \square \)

3.7. The limit tree and BBT.

In this subsection we freely use the terminology and notation about \( \mathbb{R} \)-trees introduced previously in section 2.

Let \( T \) be the forward limit tree of the iwip automorphism \( \alpha \) (i.e. \( T\alpha = \lambda T \) for \( \lambda := \lambda_+(\alpha) > 1 \)). Let \( f : \tau \to \tau \) be any train track representative of \( \alpha \).

It is well known (see [10]) that for the universal covering \( \tilde{\tau} \) of \( \tau \) there exists an \( F_N \)-equivariant map \( i : \tilde{\tau} \to T \) which is injective on legal paths (where a turn in \( \tilde{\tau} \) is legal if and only if its image turn in \( \tau \) is legal). If one uses this map to pull back the metric on \( T \) to define an edge length \( L_{PF}(e) \) for every edge \( e \) of \( \tilde{\tau} \) (an thus, by the \( F_N \)-equivariance of \( i \) also for the edges of \( \tau \), then one obtains a Perron-Frobenius column eigenvector \( \vec{v} = (L_{PF}(e))_{e \in \text{Edges}(\tau)} \) (with eigenvalue \( \lambda \)) of the non-negative primitive transition matrix \( M(f) \), see for example [19]. One obtains:

Remark 3.37. The map \( i : \tilde{\tau} \to T \) has the property that, with respect to the PF-metric \( L_{PF} \), every finite legal path \( \gamma \) in \( \tilde{\tau} \) is mapped isometrically to the geodesic segment \( i(\gamma) \).

The map \( i \) satisfies the bounded backtracking property (BBT): There is a constant \( BBT(i) \geq 0 \) such that for any two points \( x, y \in \tilde{\tau} \) the geodesic segment \( [x, y] \subset \tilde{\tau} \) is mapped by \( i \) into the BBT(\( i \))-neighborhood of the geodesic segment \( [i(x), i(y)] \):

\[
i([x, y]) \subset N_{BBT(i)}([i(x), i(y)])
\]

Based on results of [2] it has been shown in [10] that:

\[
BBT(i) \leq \text{vol}^{PF}(\tau) := \sum_{e \in \text{Edges}(\tau)} L_{PF}(e)
\]

Lemma 3.38. Consider any element \( w \in F_N \), with translation length \( 0 \leq ||w||_T < c \) on \( T \), where \( c \) is smallest PF-length of any loop in \( \tau \), and let \( \hat{\gamma} \) be a reduced loop in \( \tau \) which represents the conjugacy class \([w] \subset F_N \). Let \( \gamma' \) be a legal subpath of \( \hat{\gamma} \). The PF-length of \( \gamma' \) satisfies:

\[
L_{PF}(\gamma') \leq ||w||_T + 4BBT(i)
\]

(Indeed, one can also show the stronger inequality \( L_{PF}(\gamma') \leq ||w||_T + 2BBT(i) \))

Proof. We can lift \( \hat{\gamma} \) to a biinfinite geodesic \( \tilde{\gamma} \) in the universal covering \( \tilde{\tau} \), which is mapped by \( i : \tilde{\tau} \to T \) to a (possibly non-reduced) path \( i(\tilde{\gamma}) \) in \( T \)
that covers the axis \( \text{Ax}(w) \) in \( T \). The action of \( w \) on \( \tilde{\tau} \) and on \( T \) fixes \( \tilde{\gamma} \) and \( \text{Ax}(w) \) and translates each of them (by the amount of \( ||w||_T \), for \( \text{Ax}(w) \)).

From the assumption \( ||w||_T < c \) it follows (by Remark 3.37) that \( \tilde{\gamma} \) is not a legal loop, so that in particular the legal subpath \( \gamma' \) cannot wrap around all of \( \tilde{\gamma} \). Let \( P \) be a point on \( \tilde{\gamma} \) which is not contained in \( \gamma' \), and let \( \tilde{P} \) be a lift of \( P \) to \( \tilde{\gamma} \). Hence the point \( P_T := i(\tilde{P}) \) lies in the BBT \( i \)-neighborhood of \( \text{Ax}(w) \). In particular it follows that \( d(P_T, wP_T) \leq ||w||_T + 2 \text{BBT}(i) \).

We can now lift \( \gamma' \) to a legal path \( \tilde{\gamma}' \) which is contained in the geodesic segment \( [\tilde{P}, w\tilde{P}] \subset \tilde{\gamma} \). Hence the endpoints of \( i(\tilde{\gamma}') \) must lie in the BBT \( i \)-neighborhood of \( [P_T, wP_T] \). Thus they have distance at most \( ||w||_T + 4 \text{BBT}(i) \). But \( \tilde{\gamma}' \) is legal and hence (Lemma 3.37) mapped isometrically to its \( i \)-image. Thus \( \tilde{\gamma}' \) and hence also \( \gamma' \) have PF-length bounded above by \( ||w||_T + 4 \text{BBT}(i) \). \( \square \)

Remark 3.39. (1) Note that with respect to the Perron-Frobenius length \( L^{PF} \) any legal path \( \gamma \) satisfies \( L^{PF}(f(\gamma)) = \lambda L^{PF}(\gamma) \). In particular, it follows that no edge \( e \) can have \( L^{PF}(e) \) equal to 0, since otherwise the subgraph \( \tau_0 \subset \tau \) defined by all such edges would contradict Lemma 3.8 (1).

(2) As a consequence we observe that the simplicial edge length \( L \) and the Perron-Frobenius edge length \( L^{PF} \) give rise to quasi-isometric metrics on the universal covering \( \tilde{\tau} \).

4. Steps 1 and 2

The material in this section is reminiscent to some of the techniques used previously by the second author in [20].

Let \( T_+ \) be the forward limit tree of the atoroidal iwip automorphism \( \varphi \) (i.e. \( T_+ \varphi = \lambda_+ T \) for \( \lambda_+ > 1 \)). Let \( f_+ : \tau_+ \to \tau_+ \) be a train track representative of \( \varphi \). By Proposition 3.29 we can assume the following:

**Hypothesis 4.1.** There is at most one INP \( \eta \) in \( \tau_+ \), and the two endpoints of \( \eta \) are distinct.

**Definition 4.2.** For any constant \( C \geq 1 \) a (possibly infinite or bi-infinite) path \( \gamma \) in \( \tau_+ \) is called **totally \( C \)-illegal** if every legal subpath \( \gamma' \) of \( \gamma \) has length

\[
L(\gamma') \leq C.
\]

**Remark 4.3.** The Definition 4.2 is motivated by Lemma 4.4 below. Another, rather useful property of any finite totally \( C \)-illegal path \( \gamma \), which follows directly from the definition, is given by the second of following inequalities (while the first one doesn’t require any hypotheses):

\[
\text{ILT}(\gamma) + 1 \leq L(\gamma) \leq C(\text{ILT}(\gamma) + 1)
\]

**Lemma 4.4.** There exists a constant \( C \geq 0 \) such that for every pair \((X,Y)\) in the dual lamination \( L^2(T_+) \subset \partial^2 F_N \) the reduced biinfinite path \( \gamma = \gamma_{\tau_+}(X,Y) \) in \( \tau_+ \) (the “geodesic realization” of the pair \((X,Y)\), see Definition 2.7) is totally \( C \)-illegal.
Proof. By Remark 3.29 a finite geodesic path \( \gamma' \) is a subpath of the geodesic realization \( \gamma_{\tau_+}(X,Y) \) for some \( (X,Y) \in L^2(T_+) \) if and only if for every \( \varepsilon > 0 \) there is an element \( w \in F_N \) with translation length on \( T_+ \) of size \( ||w||_{T_+} \leq \varepsilon \), such that the conjugacy class of \( w \) in \( F_N \) is represented by a geodesic loop \( \hat{\gamma} \) which contains \( \gamma' \) as subpath. Hence the desired inequality is a direct consequence of Lemma 3.38 where the constant \( C \) can be calculated from the value 4BBT(i) for the map \( \tau_+ \to T_+ \) explained in subsection 3.7 and the quasi-isometry constants between the length functions \( L \) and \( L^{PF} \), see Remark 3.39 (2).

Below we use the following terminology: If \( \gamma \) is a (possibly infinite or biinfinite) path in \( \tau_+ \), and \( \gamma_1 \) a subpath of \( \gamma \), then a boundary subpath \( \gamma' \) of \( [f^t_+(\gamma_1)] \) (for some \( t \geq 1 \)) is cancelled by the reduction of \( f^t_+(\gamma) \) to \( [f^t_+(\gamma)] \) if, for \( \gamma = \gamma_0 \circ \gamma_1 \circ \gamma_2 \), when \( [f^t_+(\gamma_0)] \circ [f^t_+(\gamma_1)] \circ [f^t_+(\gamma_2)] \) is reduced to give \( [f^t_+(\gamma)] \), then \( \gamma' \) is entirely contained in one of backtracking subpaths at the concatenation points of the product path \( [f^t_+(\gamma_0)] \circ [f^t_+(\gamma_1)] \circ [f^t_+(\gamma_2)] \).

Lemma 4.5. Let \( C \geq 1 \) be any constant, and let \( (X,Y) \in \partial^2 F_N \) be such that, for every integer \( t \geq 0 \), the geodesic realization in \( \tau_+ \) of the pair \((\partial \varphi^t(X), \partial \varphi^t(Y))\) by a reduced biinfinite path \( \gamma_t = \gamma_{\tau_+}(\partial \varphi^t(X), \partial \varphi^t(Y)) \) is totally \( C \)-illegal. Then there exists an integer \( s \geq 0 \) which has the property that for any finite subpath \( \gamma' \) of \( \gamma_0 \) with \( ILT(\gamma') \geq 2 \) one has either

\[
ILT([f^s_+(\gamma')]) < ILT(\gamma'),
\]

or else a boundary subpath of \( [f^s_+(\gamma')] \) which crosses over at least one illegal turn is completely cancelled by the reduction of \( f^s_+(\gamma) \) to \( [f^s_+(\gamma)] = \gamma_s \).

Proof. From Proposition 3.28 we know that for any subpath \( \gamma' \) of \( \gamma_0 \) with \( 2 \) or more illegal turns one has \( ILT([f^s_+(\gamma')]) < ILT(\gamma') \) (for \( r \) as given in Proposition 3.28), unless every illegal turn of \( \gamma' \) is the tip of an INP entirely contained as subpath in \( \gamma' \), such that any two such INP-subpaths which are adjacent on \( \gamma' \) either (i) are disjoint, or (ii) intersect precisely in a common endpoint.

The second case (ii) is ruled out by our Hypothesis 4.1 since \( \gamma' \) is a geodesic and the only INP \( \eta \) contained in \( \tau_+ \) is not closed.

In order to rule out the first case (i), we observe that the subpath \( \gamma'' \) of \( \gamma' \), which connects the two endpoints of the adjacent INP-subpaths of \( \gamma' \), has to run over at least one edge, and that \( \gamma'' \) is legal. Hence, by the expansiveness of \( f_+ \), for some \( t \geq 1 \) the path \( [f^t_+(\gamma')] \) contains a legal subpath \( f^t_+(\gamma'') \) of length bigger than the constant \( C \). This contradicts the hypothesis that \( [f^t_+(\gamma)] = \gamma_t \) is totally \( C \)-illegal, unless one of the two INP-subpaths of \( [f^t_+(\gamma')] \) which are adjacent to the subpath \( f^t_+(\gamma'') \) is completely contained in a boundary subpath of \( [f^t_+(\gamma')] \) which is cancelled by the reduction of \( f^t_+(\gamma) \) to \( [f^t_+(\gamma)] = \gamma_t \).

Hence it suffices to take \( s \geq r \) big enough so that \( L(f^s(e)) > C \) for any edge \( e \) of \( \tau_+ \).
Corollary 4.6. Let $C \geq 0, (X, Y) \in \partial^2 F_N, \gamma'$ and $s \geq 1$ be as in Lemma 4.5. Let us furthermore assume that $\text{ILT}(\gamma') \geq 5$. Then one has

$$\text{ILT}([f^+_s(\gamma')]) \leq \frac{1}{4} \text{ILT}(\gamma')$$

Proof. By the assumption $\text{ILT}(\gamma') \geq 5$ we can subdivide $\gamma' = \gamma_1 \circ \ldots \circ \gamma_q$ as concatenation of subpaths $\gamma_i$, where each $\gamma_i$ satisfies $2 \leq \text{ILT}(\gamma_i) \leq 3$, and where at each concatenation vertex the path $\gamma'$ crosses over an illegal turn. We now apply Lemma 4.5 to each of the subpaths $\gamma_i$ and obtain, for $s \geq 1$ as specified there, that either $\text{ILT}([f^+_s(\gamma_i)]) < \text{ILT}(\gamma_i)$, or else one of the illegal turns in $[f^+_s(\gamma_i)]$ is cancelled when $f^+_s(\gamma')$ is reduced to $[f^+_s(\gamma')]$. Thus one obtains, for

$$[f^+_s(\gamma')] = \gamma'_1 \circ \ldots \circ \gamma'_q,$$

where each $\gamma'_i$ is the (possibly trivial) subpath of $[f^+_s(\gamma_i)]$ which is left-over when the concatenation $[f^+_s(\gamma_1)] \circ \ldots \circ [f^+_s(\gamma_q)]$ is reduced, that each of the paths $\gamma'_i$ satisfies $2 \leq \text{ILT}(\gamma'_i) \leq \text{ILT}(\gamma_i) - 1$. This gives $\text{ILT}([f^+_s(\gamma')]) < \frac{1}{4} \text{ILT}(\gamma')$, as claimed. \qed

Proposition 4.7. For any $\lambda > 1$ there is a constant $c > 0$ and an exponent $t \geq 0$ with the following property:

Consider $(X, Y) \in L^2(T_+^\ast) \subset \partial^2 F_N$, and let $\gamma = \gamma_{\tau_+}(X, Y)$ be a geodesic realization as reduced biinfinite path in $\tau_+$. Let $\gamma'$ be a finite subpath of $\gamma$ of simplicial length $L(\gamma') \geq c$. Then the reduced image path $[f^+_s(\gamma')]$ satisfies:

$$L([f^+_s(\gamma')]) \leq \frac{1}{\lambda} L(\gamma')$$

Proof. By Lemma 4.4 there exists a constant $C \geq 0$ such that for any $t \in \mathbb{Z}$ the geodesic realization in $\tau_+$ of the pair $(\partial \varphi^t(X), \partial \varphi^t(Y)) \in L^2(T_+^\ast)$ by a reduced path $\gamma_t = \gamma_{\tau_+}((\partial \varphi^t(X), \partial \varphi^t(Y)))$ is totally $C$-illegal. Hence any finite subpath $\gamma'$ of $\gamma_0$ with sufficiently large $\text{ILT}(\gamma')$ satisfies by Corollary 4.6 the inequality

$$\text{ILT}([f^k_s(\gamma')]) \leq \frac{1}{4k} \text{ILT}(\gamma')$$

for some large $k \geq 1$.

Now, one can decompose $[f^k_s(\gamma')]$ as concatenation $[f^k_s(\gamma')] = \delta_0 \circ [f^k_s(\gamma')] \circ \delta_1$, where $[f^k_s(\gamma')]_{\gamma_k}$ is the maximal subpath of $[f^k_s(\gamma')]$ which is also subpath of $\gamma_k$, while $\delta_0$ and $\delta_1$ are boundary subpaths that are cancelled when $f^k_s(\gamma_0)$ is reduced to $[f^k_s(\gamma_0)] = \gamma_k$. In particular, it follows that for each $\delta_i$ there is a backtracking subpath $\delta'_i$ of $f^k_s(\gamma_0)$, such that $\delta_i$ is obtained from a subpath of $\delta'_i$ by canceling (in the process of reducing the subpath $f^k_s(\gamma')$ of $f^k_s(\gamma_0)$ to $[f^k_s(\gamma')]$) certain further backtracking subpaths. Hence (compare Remark 4.5) $L(\delta'_i)$ and thus $L(\delta_i)$ is bounded above by a constant $K \geq 0$ which depends only on $f^k_s$. 


As subpath of the totally \(C\)-illegal path \(\gamma_{ks}\) the path \([f^k_{+s}(\gamma')]_{\gamma_{ks}}\) must be itself totally \(C\)-illegal. Hence we deduce, using Remark 4.3, that

\[
L([f^k_{+s}(\gamma')]) \leq L([f^k_{+s}(\gamma')]_{\gamma_{ks}} + 2K \\
\leq C(\text{ILT}([f^k_{+s}(\gamma')]_{\gamma_{ks}}) + 1) + 2K \\
\leq C(\frac{1}{4}\text{ILT}(\gamma')) + 1) + 2K \\
\leq C(\frac{1}{4}L(\gamma)) + 1) + 2K
\]

It is now easy to determine first \(k\) and thus \(t\) large enough (in dependence of the given value of \(\lambda\)) and then (using again the inequality \(L(\gamma) \leq C(\text{ILT}(\gamma) + 1)\) from Lemma 4.3) a sufficiently large constant \(c\) which give the desired inequality. \(\Box\)

Remark 4.8. It follows easily from standard arguments of geometric group theory that the strong uniform contraction property of the automorphism \(\varphi\) along the leaves of the dual lamination \(L^2(T_+)\) of its forward limit tree \(T_+\), which is stated in Proposition 4.7 above in terms of the train track representative \(f_+\) of \(\varphi\), is in fact an intrinsic property, which can be similarly expressed with respect to any representative of \(\varphi\) as a self-map of some marked graph, or, more algebraically, directly for the automorphisms \(\varphi\) expressed as traditionally by means of a basis \(A\) of \(F_N\) and the image words of the generators \(a_i \in A\). Any \(\varphi\)-invariant subset of \(\partial^2 F_N\) which satisfies this property is called uniformly expanding.

It is easy to see that, passing over to \(\varphi^{-1}\), there is a similar way to define the same sets by an analogous property as uniformly \(\varphi^{-1}\)-expanding subsets of \(\partial^2 F_N\).

5. STEPS 3 AND 4

Throughout this section we assume that \(f_- : \tau_- \to \tau_-\) is a train track map that represents the automorphism \(\varphi^{-1}\) of \(F_N\). Since \(\varphi\) is assumed to be iwip, it follows (as immediate consequence of Definition 2.11) that \(\varphi^{-1}\) is also iwip. Just as in the previous section for \(f_+ : \tau_+ \to \tau_+\), we can assume here:

**Hypothesis 5.1.** There is at most one INP \(\eta\) in \(\tau_-,\) and the two endpoints of \(\eta\) are distinct.

We consider the lamination \(L^2(T_+)\) from the previous section, and we shorten here the notation to \(L^2 := L^2(T_+)\). From the previous section (see Remark 4.8) we deduce:

**Lemma 5.2.** For any \(\lambda > 1\) there is a constant \(c > 0\) and an exponent \(t \geq 1\) such that for any pair \((X,Y) \in L^2_-\) and any subpath \(\gamma'\) of the geodesic realization \(\gamma = \gamma_{\tau_-}(X,Y)\) of \((X,Y)\) in \(\tau_-\), with simplicial length \(L(\gamma') \geq c\), one has:

\[
L(f^t_-(\gamma')) \geq \lambda L(\gamma')
\]

\(\Box\)
For any $K \geq 0$ we define the set $L_K = L_K(L_2^-,\tau_-)$ to be the set of subpaths $\gamma$ of length $L(\gamma) = K$ of any biinfinite path $\gamma_{\tau_-}(X,Y)$ in $\tau_-$ which is a geodesic realization any pair $(X,Y) \in L_2^-$.

For any constant $C \geq 0$ and any path $\gamma$ in $\tau_-$ we denote by $\gamma_{C}$ the (possibly trivial) subpath of $\gamma$ which is left after erasing from $\gamma$ the two boundary subpaths of length $C$.

Recall from Remark 3.5 that there is a constant $C(f_-) \geq 0$ which bounds the length of any backtracking subpath in the (unreduced) image $f_-(\gamma)$ of any reduced path $\gamma$ in $\tau_-$. 

Remark 5.3. It follows that for any reduced path $\gamma$ in $\tau_-$ and any subpath $\gamma'$ of $\gamma$ the path $[f_-(\gamma')]_{C(f_-)}$ is a subpath of the reduced path $[f_- (\gamma)]$.

Note that this statement is not necessarily true if $[f_-(\gamma')]_{C(f_-)}$ is replaced by the path $[f_-(\gamma')]$.

Lemma 5.4. There exists a constant $C_0 > 0$ such that, for any $C \geq C_0$ and for $t \geq 1$ as in Lemma 5.2, the following holds:

For any path $\gamma \in L_C$ there exists a path $\gamma' \in L_C$ with the property that $\gamma$ is a subpath of $[f_-(\gamma')]_{C(f_-)}$.

Proof. This is a direct consequence of the above Lemma 5.2.

Recall from Convention 3.25 that we can assume without loss of generality that every endpoint of an INP in $\tau_-$ is a vertex of $\tau_-$. 

Lemma 5.5. (a) For $C$ as in Lemma 5.4 there is at most one illegal turn on any path $\gamma \in L_C$. This illegal turn must be the tip of some INP which is contained as subpath in $\gamma$.

(b) Furthermore, if $\gamma$ is legal, then there is at most one unused turn on any path $\gamma \in L_C$. If $\gamma$ contains an INP, then all turns outside the INP are used turns. (To be specific: At the initial and terminal point of the INP the path $\gamma$ may well cross over an unused (but legal) turn.)

Proof. (a) For any path $\gamma =: \gamma_0 \in L_C$ we can assume by Lemma 5.4 that (after possibly replacing $f$ by a positive power $f^t$) there is an infinite family of paths $\gamma_n \in L_C$, for any integer $n \leq 0$, such that each $\gamma_n$ is a subpath of $[f_-(\gamma_{n-1})]_{C(f_-)}$. We note (compare Remark 3.14) that $\text{ILT}(\gamma_n) \geq \text{ILT}(\gamma_m)$, for any $n \leq m \leq 0$.

Since all $\gamma_n$ have length bounded by $C$, it follows from Remark 3.14 that for sufficiently negative $n \leq m$ one has $\text{ILT}(\gamma_n) = \text{ILT}(\gamma_m)$. Thus it follows from Proposition 3.28 for $-m$ sufficiently large, that every illegal turn in $\gamma_{m-r}$ is the tip of some INP which is entirely contained as subpath in $\gamma_{m-r}$, and adjacent such INP-subpaths can only overlap in a common endpoint. The same statement must be true for all $f^k(\gamma_{m-r})$ and hence for all $\gamma_k$ with $k \geq m - r$, and thus also for $\gamma = \gamma_0$.

But the same argument applies to any of the $\gamma_n$. It follows that any maximal legal subpath $\gamma'$ of $\gamma_n$, between two adjacent INP-subpaths, is the $f_-$-image of the maximal legal subpath between adjacent INP-subpaths in
\(\gamma_{n-1}\). Since the endpoints of INPs are vertices, so that such non-degenerate maximal legal subpaths can not become arbitrary small. It follows from the assumption that \(f_-\) is expanding that each of the legal paths \(\gamma'\) between two adjacent INP-subpaths of any \(\gamma_n\) must has length 0. But by Hypotheses 5.1 there is only one INP in \(\tau_-\), and its endpoints are distinct. Thus no reduced path \(\gamma\) in \(\tau_-\) can contain two subsequent INPs which have a common endpoint. This shows statement (a).

(b) Recall from Remark 3.31 (2) that used turns are mapped by \(D^2 f_-\) to used turns. However, by Remark 3.31 (3), any unused legal turn must be mapped similarly to an unused legal turn, as one can take its (necessarily unused legal) preimage arbitrarily far back.

We first assume that \(\gamma\) (and hence any of the \(\gamma_n\) as in part (a)) is legal. From the last paragraph it follows that any maximal legal subpath \(\gamma'\) of \(\gamma_n\), between two adjacent unused turns, is the \(f'\)-image of the maximal legal subpath between adjacent unused turns in \(\gamma_{n-1}\). But the unused turns do only occur at vertices, so that such non-degenerate maximal legal subpaths can not become arbitrary small. It follows from the assumption that \(f'\) is expanding that each of the legal paths \(\gamma'\) between two adjacent unused turns of any \(\gamma_n\) must have length 0; in other words: there is only one such unused turn on any of the \(\gamma_n\).

The same argument applies to the maximal legal subpath between an unused turn and the boundary point of an INP, in case some \(\gamma_n\) would contain both. It follows again that this maximal legal subpath must have length 0, so that the only unused turns, on any \(\gamma_n\) which contains an INP, can occur at the two endpoints of the INP. This proves claim (b). \(\square\)

As a direct consequence of Lemma 5.5 we obtain:

**Proposition 5.6.** (a) For any \((X,Y)\) \(\in L^2_-\) there is on any of the geodesic realizations \(\gamma := \gamma_{\tau_-}(X,Y)\) in \(\tau_-\) at most one “singularity”: This can either be an unused legal turn, or an illegal turn at the tip of an INP, or an INP with one or two unused legal turns at its boundary points. The remainder of \(\gamma\) is legal and crosses only over used turns.

(b) If \(\gamma = \gamma_{\tau_-}(X,Y)\) contains a non-used legal turn, then so does every \(\gamma_t = \gamma_{\tau_-}(\partial \varphi^t(X), \partial \varphi^t(Y)) \in L^2_-,\) for any \(t \in \mathbb{Z}\). The analogous statement holds if \(\gamma\) crosses over an INP with \(k\) unused legal turns at its boundary points, for \(k \in \{0, 1, 2\}\).

**Proof.** (a) It suffices to choose \(C\) in Lemma 5.5 large enough, so that the two singularities would be both contained in a path of \(L_C\), in contradiction to this lemma.

(b) This is a direct consequence of part (a), of the fact that the image of an INP is (after reduction) again an INP, and of the fact (explained in the first paragraph of the proof of part (b) of Lemma 5.5) that under the map \(D^2 f_-\) used turns are mapped to used turns, and any unused legal turn on \(\gamma_t\) is mapped to an unused legal turn on \(\gamma_{t+1}\). \(\square\)
In the following we call a (possibly infinite or biinfinite) path $\gamma$ used legal if it is legal and if it only crosses over used turns.

**Corollary 5.7.** (a) For any $(X,Y) \in L_2^-$ every finite used legal subpath $\gamma'$ of the geodesic realization $\gamma_{\tau_-}(X,Y)$ in $\tau_-$ is also contained as subpath in $f_\tau^t(e)$, for some $t \geq 0$ and some edge $e$ of $\tau_-$.

(b) Assume that for $(X,Y) \in L_2^-$ the geodesic realization $\gamma_{\tau_+}(X,Y)$ is a used legal path. Then $(X,Y)$ belongs to the BFH-attracting lamination $L^2_{BFH}(f_-)$ of $f_- : \tau_- \to \tau_-$ (see Definition 7.3).

**Proof.** (a) From the expansiveness of $f_-$ and the legality of $\gamma'$ we can use Proposition 5.6 (b) to “iterate” $f_-$ backwards until we find a path $\gamma''$ which runs over at most two adjacent edges $e, e'$ in $\tau_-$ such that $f_\tau^t(\gamma'')$ contains $\gamma'$ as subpath, for some $t \geq 0$. But since all turns of $\gamma'$ are used, the same must be true for all of its preimages, so that the turn between $e$ and $e'$ must be used. From Definition 3.3 it follows directly that this proves the claim (a).

(b) This is a direct consequence of part (a), by the definition of $L^2_{BFH}(f_-)$ in Definition 3.3.

$\square$

6. Steps 5 - 7

Throughout this section we use the same conventions as in section 5, in particular Hypothesis 5.1 and the notation from the paragraph preceding it.

**Lemma 6.1.** Let $(\rho_t)_{t \in \mathbb{Z}}$ be a family of infinite geodesic rays in $\tau_-$, and assume that $[f_- (\rho_t)] = \rho_{t+1}$ for all $t \in \mathbb{Z}$. Assume furthermore that each $\rho_t$ is legal, and that each $\rho_t$ starts at a vertex $v_t$ of $\tau_-$. Then each $\rho_t$ is an eigenray for the train track map $f_- : \tau_- \to \tau_-$.  

**Proof.** Since each $\rho_t$ is legal, one has always $[f_- (\rho)] = f_- (\rho) = \rho_{t+1}$. It follows that $f_-$ maps the starting point $v_t$ of $\rho_t$ to the starting point $v_{t+1}$ of $\rho_{t+1}$. Since the starting point of each $\rho_t$ is (by hypothesis) a vertex, it follows from the finiteness of the vertex set of $\tau_-$ that each $v_t$ is a periodic vertex of $\tau_-).

Similarly, since all $\rho_t$ are legal, the initial edge $e_{t+1}$ of $\rho_{t+1}$ must always be equal to the initial edge of the (legal) edge path $f_- (e_t)$, where $e_t$ denotes the initial edge of $\rho_t$. Again by the finiteness of the graph $\tau_-$, we see that for any $t \in \mathbb{Z}$ the initial edge $e_t$ of $\rho_t$ lies on a periodic orbit, with respect to the map $Df_-$ (see Definition 3.6 (b)). In other words: each $e_t$ is the eigen edge of the gate to which it belongs (compare subsection 3.3).

We now argue by induction: Suppose that all $\rho_{kt}$ for some fixed integer $k \neq 0$ start with the same initial subpath $\gamma_n$ of simplicial length $n \geq 1$. Then it follows from the expansiveness of $f_-$ (and from the fact that all paths are legal) that their images have a common initial subpath of simplicial length $\geq n + 1$. Thus the assumption $f_\tau^k(\rho_t) = \rho_{t+k}$ implies that the maximal common initial subpath of the $\rho_t$ must have infinite length, so that we obtain $\rho_t = \rho_{t+k}$ for all $t \in \mathbb{Z}$. But this is the defining equality for eigenrays.  

$\square$
Lemma 6.2. (a) For each eigenray $\rho$ which starts at a vertex of $\tau_-$ there is an eigenray $\rho'$, with same initial vertex, such that $\overline{\rho} \circ \rho'$ is a biinfinite used legal path.

(b) Any path of type $\overline{\rho} \circ \rho'$, with used legal turn at the concatenation point, realizes a pair $(X, Y) \in \partial^2 F_N$ which belongs to the BFH-attracting lamination $L^2_{BFH}(f_-)$.

Proof. (a) We first derive from Lemma 3.32 that there is an edge $e$ with same initial vertex $v$ as $\rho$ such that $\overline{\rho} \circ \rho$ is used legal. In particular $e$ doesn’t lie in the gate from which $\rho$ starts. By Proposition 3.19 there is an eigenray $\rho'$ which starts from the gate that contains $e$, and by Lemma 3.33 the turn from $\rho'$ to $\rho$ is used. Furthermore, every eigenray is used legal, since every finite subpath if contained in some $f_-$-iterate of its initial edge (compare Proposition 3.19 and its proof). This proves the claim.

(b) By part (a) the turn at the concatenation point between $\overline{\rho}$ and $\rho'$ is used: There is an edge $e$ of $\tau_-$ and an exponent $t \geq 1$ such that the edge path $f^t_-(e)$ contains a subpath which consists of non-trivial initial segments of both, $\overline{\rho}$ and $\rho'$. It follows from the expansiveness of $f_-$ and the definition of an eigenray that every path which consists of non-trivial initial segments of both, $\overline{\rho}$ and $\rho'$, is a subpath of $f^t_-(e)$ for some integer $n \geq 0$. By Definition 3.34 it follows that $(X, Y)$ is an element $L^2_{BFH}(f_-)$.

Recall from subsection 2.2 that for any $(X, Y) \in \partial^2 F_N$ the geodesic realization in $\tau_-$ satisfies $[f_-(\gamma_{\tau_-}(X, Y))] = \gamma_-(\partial \varphi^{-1}(X), \partial \varphi^{-1}(Y))$. As in section 5 we use the for the lamination dual to the forward limit tree $T_+$ the abbreviation $L^2_1 = L^2(T_+)$.

Proposition 6.3. Let $(X, Y) \in \partial^2 F_N$, and let $(\gamma_t)_{t \in \mathbb{Z}}$ be a family of biinfinite geodesics $\gamma_t = \gamma_{\tau_-}((\partial \varphi^{-1}(X), \partial \varphi^{-1}(Y)))$ which realize the pair $(\partial \varphi^{-1}(X), \partial \varphi^{-1}(Y)) \in \partial^2 F_N$ in $\tau_-$. In particular we assume that $[f_-(\gamma_t)] = \gamma_{t+1}$ for all $t \in \mathbb{Z}$.

(a) Assume that each $\gamma_t$ is legal, and that on each $\gamma_t$ there is precisely one unused legal turn. Then there are eigenrays $\rho$ and $\rho'$, such that $\gamma_t = \overline{\rho} \circ \rho'$, where the unused turn is situated at the concatenation point.

(b) Assume that on each $\gamma_t$ there is precisely one INP subpath $\eta_t$, and that every turn of $\gamma_t$ is legal except for the turn at the tip of $\eta_t$. Then there are eigenrays $\rho$ and $\rho'$, such that $\gamma_t = \overline{\rho} \circ \eta \circ \rho'$.

Proof. Note that in case (a) each unused turn in $\gamma_t$ takes place at a vertex $v^0_t$ of $\tau_-$. Similarly, in case (b) the endpoints $v^1_t$ and $v^2_t$ of any INP in $\gamma_t$ are vertices, by Convention 3.25. Notice also that in case (a) one has $[f_-(\gamma_t)] = f_-(\gamma_t)$, and in case (b) the only backtracking subpath of $f_-(\gamma_t)$ is situated at the tip of $\eta_{t+1}$. Hence $f_-$ maps the “singular” vertices $v^1_t$ in $\gamma_t$ to the singular vertices $v^1_{t+1}$ in $\gamma_{t+1}$. Thus we obtain the statements of (a) and (b) is a direct consequence of Lemma 6.1. □
Definition 6.4. Let $v$ be a periodic vertex $v$ of $\tau_\cdot$. For any two eigenrays $\rho, \rho'$ with initial vertex $v$ we write

$$\rho \sim \rho'$$

if in the biinfinite path $\mathfrak{p} \circ \rho'$ the turn at the concatenation point is used. We consider the equivalence relation which is generated by this relation, which we also denote by $\sim$.

It follows directly from the definitions that the equivalence relation $\sim$ is modeled precisely after the definition of the diagonal closure, so that one observes:

Remark 6.5. Let $v$ be a periodic vertex of $\tau_\cdot$, and assume that the BFH-attracting lamination $L^2_{BFH}(f_-)$ contains pairs $(X_1, Y_1)$ and $(X_2, Y_2)$. Assume that the geodesic realizations $\gamma_1 = \gamma_{\tau_\cdot}(X_1, Y_1)$ and $\gamma_2 = \gamma_{\tau_\cdot}(X_2, Y_2)$ are both concatenations of eigenrays, $\gamma_1 = \mathfrak{p}_1 \circ \rho'_1$ and $\gamma_2 = \mathfrak{p}_2 \circ \rho'_2$, with concatenation vertex equal to $v$. Then $\rho_1 \sim \rho_2$ implies that $(X_1, X_2)$ is contained in the diagonal closure of $L^2_{BFH}(f_-)$.

Note that the converse implication is also true, but it is a little bit less immediate: one needs to employ Lemma 6.1 and Lemma 6.2 (b).

We will now use the equivalence relation $\sim$ to derive from $\tau_\cdot$ and $f_-$ a new graph $\tau^*_\cdot$ and a new self-map $f^*_\cdot$. This will be defined in several steps as follows:

We assume that at some periodic vertex $v$ of $\tau_\cdot$ there is more than one $\sim$-equivalence class of eigenrays. We first assume that $v$ is fixed by $f_-$. We then split the vertex into several new vertices, called subvertices of $v$, precisely one for each equivalence class, such that all eigenrays starting at the same subvertex are equivalent, and conversely.

We now attach any other edge $e$ of $\tau_\cdot$ with initial vertex $v$ to one of the new subvertices, according to which eigenray started in $\tau_\cdot$ from the same gate as $e$. Recall from Proposition 3.19 that for each gate $g_i$ at a periodic vertex there is precisely one eigenray starting from $g_i$.

We connect the $k \geq 2$ subvertices of $v$ by a $k$-pod $\mathcal{P}$ to obtain a new graph which transforms back to $\tau_\cdot$, if $\mathcal{P}$ is contracted to a single point.

We now consider any vertex $v'$ of $\tau_\cdot$ which is mapped eventually by $f_-$ to $v$, and we do the analogous construction there, where the subdivision of $v'$ into subvertices is simply lifted from $v$ to $v'$. (Indeed, since $v'$ is not periodic, we can not use again the relation $\sim$ for the subdivision of $v'$.) The graph obtained by this blow-up of vertices is denoted by $\tau^*_\cdot$.

We observe that the train track map $f_-$ induces naturally a map $f^*_\cdot : \tau^*_\cdot \to \tau^*_\cdot$ which maps $\mathcal{P}$ homeomorphically onto itself and leaves also $\tau^*_\cdot \setminus \hat{\mathcal{P}}$ invariant. Since $f_-$ is expanding, it follows that $\tau^*_\cdot \setminus \hat{\mathcal{P}}$ has non-trivial fundamental group. Hence $\pi_1(\tau^*_\cdot \setminus \hat{\mathcal{P}})$ is a non-trivial proper $\varphi$-invariant free factor of $F_N$, contradicting the assumption that $f_-$ represents an iwip automorphisms.
Now, if $v$ is not fixed by $f_-$, one has to do the same blow-up procedure at the whole $f_-$-orbit of $v$, but the argument remains in this case precisely the same as in the simpler case just considered.

This shows:

**Lemma 6.6.** At every periodic vertex $v$ of $\tau_-$ there is only one $\sim$-equivalence class of eigenrays. \qed

**Proposition 6.7.** Let $(X,Y) \in L^2_-$ be such that its geodesic realization $\gamma_{\tau_-(X,Y)}$ in $\tau_-$ is not used legal. Then $(X,Y)$ belongs to the diagonal closure of the BFH-attracting sublamination $L^2_{BFH}(f_-)$ of the train track map $f_-$. 

**Proof.** By Proposition 6.6 we know that $\gamma := \gamma_{\tau_-(X,Y)}$ (and hence any $\gamma_t$ which realizes $(\partial \varphi^t(X), \partial \varphi^t(Y))$, for arbitrary $t \in \mathbb{Z}$) contains either precisely one non-used legal turn, or else it runs precisely once over an INP-subpath $\eta$ of $\tau_-$. Thus one deduces from Proposition 6.3 that $\gamma = \rho \circ \rho'$ or $\gamma = \rho \circ \eta \circ \rho'$, for eigenrays $\rho$ and $\rho'$.

In the first case the unused turn occurs at the concatenation vertex $v$ of the two eigenrays, and it follows directly from Lemma 6.2 that there are eigenrays $\rho_1$ and $\rho_2$ at $v$ such that the biinfinite paths $\gamma_1 = \rho \circ \rho_1$ and $\gamma_2 = \rho' \circ \rho_2$ are used legal, and that they are geodesic realizations $\gamma_1 = \gamma_{\tau_-(X,Z_1)}$ and $\gamma_2 = \gamma_{\tau_-(Y,Z_2)}$ of pairs $(X,Z_1)$ and $(Y,Z_2)$ that both belong to the BFH-attracting lamination $L^2_{BFH}(f_-)$. Hence it follows directly from Lemma 6.6 and Remark 6.5 that $(X,Y)$ belongs to the diagonal closure of $L^2_{BFH}(f_-)$.

In the second case, where $\gamma = \rho \circ \eta \circ \rho'$, we consider the eigenrays $\rho_1$ and $\rho_2$ which have the two legal branches of $\eta$ as initial subpaths and agree along an infinite legal subray, see Remark 3.23. By the case treated before, the biinfinite paths $\rho \circ \rho_1$ and $\rho_2 \circ \rho'$ realize elements $(X,Z)$ and $(Z,Y)$ which belong to the diagonal closure of $L^2_{BFH}$. Thus $(X,Y)$ also belongs to this diagonal closure. \qed

We have now assembled all ingredients necessary to prove the main result of this paper:

**Proof of Theorem 1.2.** In Corollary 5.7 (b) together with Proposition 6.7 it is shown that $L^2(T_+)$ is contained in $\text{diag}(L^2_{BFH}(f_-))$. More precisely, a pair $(X,Y) \in L^2(T_+)$ is contained in $\text{diag}(L^2_{BFH}(f_-)) \setminus L^2_{BFH}(f_-)$ if and only if its geodesic realization in $\tau_-$ is a concatenation of eigenrays at either an unused legal turn, or else at an INP. But since eigenrays in $\tau_-$ are in 1-1 relation with periodic gates of train track map $f_- : \tau_- \to \tau_-$ (see Corollary 3.20), there exist only finitely many distinct eigenrays. Since also the number of unused legal turns is finite, as well as the number of INPs in $\tau_-$, it follows that $L^2(T_+)$ contains only finitely many $F_N$-orbits of pairs $(X,Y)$ which are contained in $\text{diag}(L^2_{BFH}(f_-)) \setminus L^2_{BFH}(f_-)$. This establishes claim (4) of Theorem 1.2.
Furthermore, eigenrays of expanding maps are never periodic or eventually periodic (see Lemma 3.21), so that by Remark 2.3(b) any minimal sublamination (see Definition 2.4) of $L^2(T_\pm)$ must intersect $L^2_{BFH}(f_-)$ and hence be contained in $L^2_{BFH}(f_-)$. Note that by Lemma 4.2 of [5] every algebraic lamination contains a minimal sublamination. But $L^2_{BFH}(f_-)$ is itself minimal (see Proposition 3.36), so that it must be contained in $L^2(T_\pm)$, as unique minimal sublamination. This proves parts (1) and (2) of Theorem 1.2.

By Proposition 2.10 the lamination $L^2(T_\pm)$ is diagonally closed, so that it contains with $L^2_{BFH}(f_-)$ also its diagonal closure. This inclusion, together with the converse inclusion stated in the first sentence of this proof, gives the equality claimed in part (3) of Theorem 1.2. \(\square\)

7. EIGENRAYs OF $T_-$ AND SINGULAR EQUIVALENCE CLASSES OF LEAVes OF $L^2(T_\pm)$

Any lamination $L^2 \subset \partial^2 F_N$ can be understood as relation on $\partial F_N$, and one can consider the equivalence relation $\sim_{L^2}$ on $\partial F_N$ generated by $L^2$. For $(X,Y) \in L^2$ we denote by $[X]_{L^2} = [Y]_{L^2}$ the corresponding equivalence class.

It follows directly from the definition of the diagonal extension (see Definition 2.5(a)) that $L^2$ and diag($L^2$) generate the same equivalence relation.

For the special case of the lamination $L^2_\pm = L^2(T_\pm)$ we will abbreviate the above notations to $\sim_-$ and $[X]_-$. As explained in subsection 2.3 it follows from the results of [7] that the classes $[X]_-$ are precisely the fibers of the map $Q_- : \partial F_N \to T_- \cup \partial T_-$. We then obtain the following as a direct consequence of Proposition 6.7.

**Proposition 7.1.** The the equivalence relation $\sim_-$ is generated by the lamination $L^2_{BFH}(f_-)$. \(\square\)

It is well known (e.g. see §4 of [19]) that the trees $T_+$ and $T_-$ are equipped with homotheties that realize the given iwip automorphism $\varphi \in \text{Out}(F_N)$, or rather, any of its representatives $\Phi \in \text{Aut}(F_N)$.

The goal of this section is to prove the following statement:

**Theorem 7.2.** Let $T_-$ be the backward limit tree of an atoroidal iwip $\varphi \in \text{Out}(F_N)$, and denote by $L_-$ the zero-lamination of the forward limit tree $T_+$. Let $\Phi^{-1} \in \text{Aut}(F_N)$ be a representative of $\varphi^{-1}$, and let $H_-$ be the homothety of $T_-$ which realizes $\Phi^{-1}$, i.e. $H_-w = \Phi^{-1}(w)H_- : T_- \to T_-$ for any $w \in F_N$.

(a) For every branchpoint $P$ of $T_-$ there is a class $[X]_- \subset \partial F_N$ and a bijection $\chi_P : \rho_i \mapsto X_i$ between the eigenrays $\rho_i$ starting at $P$ and the elements $X_i \in [X]_-$, which has the property that $Q_-(\chi_P(\rho_i))$ is precisely the endpoint of $\rho_i$ in $\partial T_-$. Furthermore, $P$ is the fixed point of $wH_\pm^t$, for some $w \in F_N$ and $t \geq 1$, if and only if $[X]_-$ is fixed by $w\Phi^{-t}$. (Note that such $w$ and $t$ exist for every branch point of $T_-$.)


(b) Conversely, if \([X]_-\) is an equivalence class of cardinality \(\geq 3\), then there exists a unique branch point \(P = P_{[X]} \in T_-\) with a bijection \(\chi_P\) as specified in part (a), with target set \(\text{Im}(\chi_P) = [X]_-\).

The notion of an eigenray in an \(\mathbb{R}\)-tree \(T\), provided with a homothety \(H : T \to H\) with stretching factor \(\lambda > 1\), has been used frequently before, e.g., see [10, 15, 19]. In the context of this paper and the previous section, we note that the definition given in Definition 3.17 for eigenrays in \(\tau\) carries over directly to the universal covering \(\tilde{\tau}\) and to the \(\mathbb{R}\)-tree \(T_-\). To be specific, we would like to emphasize that by an eigenray in \(\tilde{\tau}\) we mean an eigenray of the map \(w\tilde{\tau}_t\), for any \(w \in F_N\) and any integer \(t \geq 1\). Here \(\tilde{\tau}_-\) is the lift of \(\tau_-\) which realizes the automorphism \(\Phi^{-1}\) specified in the above theorem, i.e., one has \(\tilde{\tau}_-w = \Phi^{-1}(w)\tilde{\tau}_- : T_- \to T_-\) for any \(w \in F_N\). We note the following direct consequence of this set up:

**Remark 7.3.** Every lift to \(\tilde{\tau}_-\) of an eigenray in \(\tau_-\) is itself an eigenray in \(\tilde{\tau}_-\).

Similarly, an eigenray in \(T_-\) is an eigenray of some of the maps \(wH_-^t\), for any \(w \in F_N\) and any integer \(t \geq 1\), and for \(H_-\) as specified above in Theorem 7.2. We note that in particular the origin (= initial point) of any eigenray must be a fixed point of the corresponding map, in \(\tilde{\tau}_-\) as well as in \(T_-\). Since \(H_-\) is a homothety with stretching factor \(\lambda_- > 1\) (see e.g. [19], §4), there is always a unique such fixed point \(P\), for any \(wH_-^t\) as before. However, unless one knows that \(P\) is a branch point, it may well happen that \(P\) is contained in the metric completion \(\overline{T}_-\) of \(T_-\) but not in \(T_-\) itself.

Since every eigenray in \(\tilde{\tau}_-\) quotients to an eigenray in \(\tau_-\), it is a legal path, see Definition 3.17 (b). Hence it follows directly from Remark 3.37 that every \(\tilde{\tau}_-\)-eigenray in \(\tilde{\tau}_-\) is mapped by the canonical edge-isometric surjection \(i_- : \tilde{\tau}_- \to H_-\) (see subsection 3.7) bijectively to an \(H_-\)-eigenray in \(T_-\). In particular, we obtain:

**Lemma 7.4.** For any branch point \(P \in T_-\) the map \(\chi_P\) which associates to every eigenray \(\rho\) originating from \(P\) its endpoint \(\chi_P(\rho) \in \partial T_-\) gives rise to a well defined injective map \(\chi_P : \rho \mapsto X \in \partial F_N\) with \(Q_-(\chi_P(\rho)) = \chi_P(\rho)\).

**Proof.** This is an immediate consequence of the definition of the map \(Q_-\) and the well known fact (see [7]) that this map, restricted to the preimage of \(\partial T_-\), induces a bijection onto \(\partial T_-\). \(\square\)

We consider now any branchpoint \(P \in T_-\), two distinct eigenrays \(\rho\) and \(\rho'\) originating from \(P\) which are fixed by some \(H'_\rho = wH_-^t\) as before, and the biinfinite geodesic line \(\rho^{-1} \circ \rho'\) in \(T_-\) which results from the concatenation of the two eigenrays. We denote the \(Q_-\)-preimages of the endpoints of the two eigenrays by \(X = \chi_P(\rho), Y = \chi_P(\rho') \in \partial F_N\).

Let \(\gamma = \gamma_\tilde{\tau}_-(X,Y)\) be the geodesic realization of \((X,Y)\) in \(\tilde{\tau}_-\). The biinfinite geodesic \(\gamma\) is mapped by the map \(i : \tilde{\tau}_- \to T_-\) to a (possibly non-geodesic) biinfinite path in \(T_-\) which is contained in the BBT\((i)\)-neighborhood of \(\rho^{-1} \circ \rho'\) in \(T_-\) and covers all of \(\rho^{-1} \circ \rho'\).
The representative \( \Phi' = i_w \Phi^{-t} \in \text{Aut}(F_N) \) (where \( i_w : v \mapsto wvw^{-1} \)) of the points \( X, Y \in \partial F_N \). Hence the realization \( \tilde{f}_- = w \tilde{f}'_- \) of \( \Phi' \) on \( \tilde{\gamma}_- \) maps \( \gamma \) to a (potentially non-reduced) biinfinite path \( \tilde{f}_-(\gamma) \) which reduces to \( \tilde{\gamma}_-(\gamma) = \gamma \).

We recall that \( H_- \) is a homothety with stretching factor \( \lambda_- > 1 \): In particular \( H'_- = wH'_l \) leaves \( \rho^{-1} \circ \rho' \) as set invariant but stretches every finite segment by the factor \( \lambda'_- > 1 \). Hence it follows from the fact that \( \tilde{f}'_- \) commutes with \( H'_- \) via \( i \), from the bounded backtracking property of \( i \) (see subsection 3.7), and from the fact that the simplicial length of legal paths in \( \tilde{\gamma}_- \) and the length of their \( i \)-images \( T_- \) are comparable (see Remark 3.39), that \( \gamma \) satisfies the expansion property specified in Lemma 5.2.

But this is all we need to deduce the following, since in sections 5 and 6 we have proved that any such \( f_- \)-invariant expansive path is the geodesic realization of a pair \((X, Y)\) in \( \partial^2 F_N \) which is contained in the diagonal extension of \( L^2_{BFH} \), and thus in \( L^2_- \).

We obtain the following statement, which also proves a major part of statement (a) of Theorem 7.2.

**Proposition 7.5.** (a) For every branchpoint \( P \) of \( T_- \) and any eigenray \( \rho \) in \( T_- \) ranging out from \( P \) there is a leaf \((X, Y)\) in \( L^2_- \) with \( \chi_P(\rho) = X \).

(b) For any two distinct eigenrays \( \rho \) and \( \rho' \) with origin \( P \) their images \( X = \chi_P(\rho) \) and \( X' = \chi_P(\rho') \) are \( L_- \)-equivalent. \( \square \)

We now consider a triple \( X, Y, Z \in \partial F_N \) of three distinct boundary points of \( F_N \), such that the two pairs \((X, Y)\) and \((X, Z)\) belong \( L^2_{BFH} \). Hence (see Lemma 3.35) their geodesic realizations \( \gamma = \gamma_- (X, Y) \) and \( \gamma' = \gamma_- (X, Z) \) are used legal paths, and they define a center in \( \tilde{\gamma}_- \) as follows:

If the issuing the third pair, \((Y, Z)\) in \( L^2_- \), has as geodesic realization \( \gamma'' = \gamma_- (Y, Z) \) also a used legal path, then there is a unique center vertex \( v_1 \) in \( \tilde{\gamma}_- \) which lies on all three geodesics.

The same is true if there is a singularity at the bifurcation locus of \( \gamma \) and \( \gamma' \), with the condition that the path \( \gamma'' \) is legal but it runs over precisely one unused turn, namely at the center vertex \( v_1 \).

In the remaining case it has been shown in Proposition 5.6 (a) that at the bifurcation locus of \( \gamma \) and \( \gamma' \) the issuing singularity is an INP, with one of its branches on each of these two geodesics, and \( \gamma'' \) contains the INP as subpath. In this case there are two center vertices \( v_1 \) and \( v_2 \), namely the two endpoints of the INP, located on \( \gamma \) and on \( \gamma' \) respectively.

In order to simplify the presentation we will below simply denote the center vertex by \( v_i \) without specifying which of the three cases is meant.

We now observe that, since by Proposition 5.6 (b) the singularities and their type are preserved under iteration of \( \tilde{f}_- \), and since \( \tilde{f}_- \) realizes the representative \( \Phi_- \in \text{Aut}(F_N) \) of \( \varphi^{-1} \), the image \( \tilde{f}_-(v_i) \) of a center vertex \( v_i \) will
be a center vertex of the triple of image boundary points \( \Phi^{-1}(X), \Phi^{-1}(Y) \) and \( \Phi^{-1}(Z) \).

But the statement of the last paragraph is true not just for the triple \((X, Y, Z)\), but for any triple \((\Phi^{-1}(X), \Phi^{-1}(Y), \Phi^{-1}(Z))\), for every integer \( t \in \mathbb{Z} \). Hence we can pass to the images in \( \tau_- \) of the paths \( \gamma \) and \( \gamma' \) and apply Lemma 6.1. Via Remark 7.3 this lemma proves directly that each of the three boundary points \( X, Y \) or \( Z \) is the endpoint of an eigenray in \( \tau_- \) which has its origin in a center point \( v \), and which runs along “half” of one of \( \gamma \) or \( \gamma' \) (or of both). Let \( w \in F_N \) and \( t \geq 1 \) be such that \( w^{-t} \) fixes each center vertex \( v \) and each of the above three eigenrays originating at one of \( v \).

We now pass via the map \( i \) from \( \tau_- \) to \( T_- \) and, since the map \( i \) folds any \( \text{INP} \) to a single segment (see e.g. [10]), each center vertex \( v \) is mapped to the point \( P \in T_- \) which is the (unique) fixed point of the homothety \( H'_- = wH'_- \), because of the commutativity of \( f_- \) and \( H_- \) via \( i \) (see subsection 3.7).

Let now \( W \in [X]_- \) be a fourth point assumed to be \( L_- \)-equivalent to the other three, with \((X, W)\) or \((W, Y)\) in \( L^2_{F, BH} \). We consider the same construction as above, and obtain a center vertex \( v'_i \in \tau_- \) for the triple \((X, Y, W)\), and an image fixed point \( P' \in T_- \) of some \( H'' = w'H'' \). We now iterate \( H'' \) backwards to obtain image triples such that the distance between the two fixed points \( P \) and \( P' \) becomes small. But both, \( P \) and \( P' \) are by the above construction images of vertices on the used legal path \( \gamma \). Since the distance of distinct such vertices is bounded below by the minimal edge length in \( \tau_- \), it follows that \( P' = P \).

**Proposition 7.6.** Every \( L^2 \)-equivalence class \([X] \subset \partial F_N\) of cardinality \( \geq 3 \) determines a branch point \( P = P_{[X]} \in T_- \), and every point \( Y \in [X] \) determines an eigenray \( \rho \) in \( T_- \) which originates from \( P \) and has as endpoint the point \( Q_-(Y) \in \partial T_- : \)

\[ \chi_\rho(X) = Y \]

**Proof.** If \([X]\) has cardinality \( \geq 3 \) it follows directly from Proposition 7.1 that there must exist points \( X, Y, Z \in [X] \) which (up to possibly exchanging the role of \( X \) and \( Y \)) have the properties as assumed above. Any further point \( W \in [X] \) is related iteratively via finitely many points \( W_k \in [X] \) to the triple \((X, Y, Z)\) in the same way as the point \( W \) considered above. Hence the above given argument shows that the center vertices of these finitely many triples are all mapped to the same branchpoint \( P \in T_- \), which is then the origin of all of the above constructed eigenrays \( \rho_i \) in \( T_- \), one for each \( X_i \in [X] \). The fact that each \( \rho_i \) has endpoint in \( \partial T_- \) which coincides with \( Q_-(X_i) \) is directly due to the above construction. \( \square \)

The arguments given above and the fact that our constructions are all natural and hence commute with the action of \( F_N \) on both, \( \tau_- \) and on \( T_- \), and also with the action of \( \Phi^{-1} \) realized on \( \tau_- \) by \( f_- \) and on \( T_- \) by \( H_- \), yield directly the following:
Proposition 7.7. (a) The map \([X] \mapsto P[X]\) defines a bijection between the \(L^2\)-equivalence classes in \(\partial F_N\) of cardinality \(\geq 3\) and the branch points of \(T\).

(b) For any integer \(t \geq 1\) and any \(w \in F_N\) the class \([X]_w\) is \(i_w \Phi^{-t}\)-invariant if and only if the associated branchpoint \(P[X]\) is the fixed point of \(wH^1\).

(c) The maps \([X] \mapsto P[X]\) and \(\rho \mapsto \chi_P(\rho) \in [X]\) are \(F_N\)-equivariant. Furthermore one has \(H_-(P[X]) = P[\Phi^{-1}(X)]\) and \(H_-(\chi_P(\rho)) = \chi_{H_-(\rho)}(H_-(\rho)) = \Phi^{-1}(\chi_P(\rho))\).

(d) Every \(L^2\)-equivalence class \([X]_w \subset \partial F_N\) of cardinality \(\geq 3\) has \(F_N\)-orbit that is \(\varphi\)-periodic. \(\square\)

It remains to observe that the last proposition, together with the previous two, completes the proof of Theorem 7.2.

8. Discussion

Throughout this section \(T\) denotes always an \(\mathbb{R}\)-tree from \(\mathbb{C}v_N\), and \(\mu\) a current from \(\text{Curr}(F_N)\). Recall that such a pair \((T, \mu)\) is called perpendicular if \(\langle T, \mu \rangle = 0\). Recall also that following [13] this is equivalent to the statement \(\text{supp}(\mu) \subset L^2(T)\), which in turn is equivalent to \(\text{diag}(\text{supp}(\mu)) \subset L^2(T)\). The purpose of this section is to give (partial) answers to the following question:

Under which circumstances does this last inclusion actually improve to \(\text{diag}(\text{supp}(\mu)) = L^2(T)\)?

To shape the discussion a bit, we propose:

Definition 8.1. (a) A pair \((T, \mu) \in \mathbb{C}v_N \times \text{Curr}(F_N)\) is said to be diagonally equal if \(L^2(T) = \text{diag}(\text{supp}(\mu))\). In this case we also say that \(T\) is diagonally equal to \(\mu\), or \(\mu\) is diagonally equal to \(T\).

(b) A tree \(T\) is called diagonally equalizable (DE) if there exists a current \(\mu\) such that \((T, \mu)\) is diagonally equal. Similarly, a current \(\mu\) is called diagonally equalizable (DE) if there exists a tree \(T\) such that \((T, \mu)\) is diagonally equal.

(c) A tree \(T\) is called totally diagonally equalizable (TDE) if every perpendicular current \(\mu\) is diagonally equal to \(T\). Similarly, a current \(\mu\) is called totally diagonally equalizable (TDE) if every perpendicular tree \(T\) is diagonally equal to \(\mu\).

(d) A tree \(T\) is called uniquely diagonally equalizable (UDE) if among all perpendicular currents there is (up to scalar multiples) precisely one current \(\mu\) which is diagonally equal to \(T\). Similarly, a current \(\mu\) is called uniquely diagonally equalizable (UDE) if among all perpendicular trees there is (up to scalar multiples) precisely one tree \(T\) which is diagonally equal to \(\mu\).

All of these definitions descend directly to the projectivized objects, so that one can speak for example of a “uniquely diagonally equalizable” \([T] \in \mathbb{C}v_N\). The result of Theorem [13] can be rephrased by stating that for any
atoroidal iwip $\varphi \in \Out(F_N)$ the attracting fixed point $[T_+] \in \overline{CV}_N$ and the repelling fixed point $[\mu_-] \in \mathbb{P} \Cur(F_N)$ constitute a pair which is diagonally equal.

By the above stated result of [13] we know that every diagonally equal pair $([T],[\mu])$ must be perpendicular. The set $I_0$ of such perpendicular pairs contains the uniquely determined minimal set $\mathcal{M}^2 \subset \overline{CV}_N \times \mathbb{P} \Cur(F_N)$ with respect to the $\Out(F_N)$-action (see [11]), which is also contained in the cartesian product $\mathcal{M}^\text{cv} \times \mathcal{M}^\text{curr}$ of the (much better understood) minimal sets $\mathcal{M}^\text{cv} \subset \overline{CV}_N$ and $\mathcal{M}^\text{curr} \subset \mathbb{P} \Cur(F_N)$. However, we only know these inclusions; it is open whether $\mathcal{M}^2$ is equal to $I_0 \cap (\mathcal{M}^\text{cv} \times \mathcal{M}^\text{curr})$.

**Remark 8.2.** We will give below examples of perpendicular pairs which are not diagonally equal. The set $\mathcal{DE}$ of diagonally equal pairs $([T],[\mu])$ is by definition $\Out(F_N)$-invariant and non-empty, but it will follow from the results presented below that $\mathcal{DE}$ is not closed in $\overline{CV}_N \times \mathbb{P} \Cur(F_N)$.

**Remark 8.3.** The set $\mathcal{DE}$ defines also an $\Out(F_N)$-invariant subgraph of the intersection graph $\mathcal{I}(F_N)$ defined in [12], and it seems worthwhile to think which kind of a subgraph this is. Since all “limit pairs” $([T_+],[\mu_-])$ for atoroidal iwips are contained in $\mathcal{DE}$, it must consist of many distinct connected components. It seems likely that even the subgraph $\mathcal{DE}_0$ defined by all $\mathcal{DE}$-pairs which are contained in the main component $I_0(F_N)$ of $\mathcal{I}(F_N)$ is non-connected.

We will now turn to the question of necessary and sufficient conditions for a pair $(T,\mu)$ to be diagonally equal, and for $T$ or $\mu$ to be DE, TDE or UDE. We first recall some facts:

**Facts 8.4.** (1) For every algebraic lamination $L^2$ over $F_N$ there exist a current $\mu \neq 0$ with $\supp(\mu) \subset L^2$ (see [8]). In particular, for every tree $T \in \partial CV_N := CV_N \setminus cv_N$ there exists a current $\mu \in \Cur(F_N)$ which is perpendicular to $T$.

The “dual” statement is wrong, if we accept the subspace $\Cur(F_N)_{+} \subset \Cur(F_N)$ of currents $\mu \in \Cur(F_N)$ with full support (i.e. $\supp(\mu) = \partial^2 F_N$) as “dual” of $cv_N$. (Note that $cv_N$ can be characterized also as the subspace in $\overline{CV}_N$ that consists of all $\mathbb{R}$-trees $T$ with $L^2(T) = \emptyset$). There exist currents $\mu \in \Cur(F_N) \setminus \Cur(F_N)_+$ which are not perpendicular to any tree in $\overline{CV}_N$. Examples of such filling currents are given in [14].

(2) If $T, T' \in \partial CV_N$, and if there is a length decreasing $F_N$-equivariant map $T \to T'$, then one has $L^2(T) \subset L^2(T')$. This applies in particular to the case where $T'$ results from $T$ by contracting an $F_N$-invariant forest $T_0$: Here each connected component of $T_0$ is mapped to a distinct point in $T'$. If $T_0$ contains connected components with infinite diameter, then the inclusion $L^2(T) \subset L^2(T')$ is proper: one has $L^2(T) \neq L^2(T')$.

(3) Let $f : \tau \to \tau$ is an expanding train track map which represents some $\psi \in \Aut(F_N)$, but contrary to the case considered in the main body of this
paper, we assume that the non-negative transition matrix \( M(f) \) (assumed to be in normal form for non-negative matrices) has precisely two primitive diagonal submatrices, with eigenvalues \( \lambda_{\text{top}} \) and \( \lambda_{\text{bottom}} \) respectively, and that the upper right off-diagonal block is non-zero (while, according to the normal form, the lower left off-diagonal block must be the zero matrix). For such exponential two-strata train track maps one has to distinguish two cases:

If \( \lambda_{\text{top}} < \lambda_{\text{bottom}} \), then both eigenvalues possess non-negative row eigenvectors \( \vec{v}^*_{\text{top}} \) and \( \vec{v}^*_{\text{bottom}} \) respectively, and each of them determines a Perron-Frobenius tree \( T_{\text{top}} \in \partial \CV_N \) and \( T_{\text{bottom}} \in \partial \CV_N \) respectively. Both projective classes, \([T_{\text{top}}]\) \( \in \partial \CV_N \) and \([T_{\text{bottom}}]\) \( \in \partial \CV_N \) are fixed points of the induced action of the automorphism \( \psi \in \Out(F_N) \) which is represented by this train track map \( f \).

If \( \lambda_{\text{top}} \geq \lambda_{\text{bottom}} \), then only \( \vec{v}^*_{\text{top}} \) is non-negative, so that only \( T_{\text{top}} \in \partial \CV_N \) exists (and is projectively fixed by \( \psi \)), but no \( T_{\text{bottom}} \) as in the other case.

On the other hand, if \( \lambda_{\text{top}} > \lambda_{\text{bottom}} \), then there are two non-negative column eigenvectors \( \vec{v}_{\text{top}} \) and \( \vec{v}_{\text{bottom}} \) respectively, and both determine projectively \( \psi \)-invariant currents \( \mu_{\text{top}}, \mu_{\text{bottom}} \in \Curr(F_N) \). If \( \lambda_{\text{top}} \leq \lambda_{\text{bottom}} \), then there is only one non-negative column eigenvector \( \vec{v}_{\text{bottom}} \) and only one projectively \( \psi \)-invariant current \( \mu_{\text{bottom}} \in \Curr(F_N) \).

It turns out that it is easier to prove results about trees:

**Proposition 8.5.** Let \( T \in \partial \CV_N = \CV_N \setminus \CV_N \).

1. Not every \( T \in \partial \CV_N \) is DE. A counterexample is given, for any 2-strata exponential train track map (which represents some \( \psi \in \Aut(F_N) \)) with \( \lambda_{\text{top}} < \lambda_{\text{bottom}} \), by the Perron-Frobenius tree \( T_{\text{bottom}} \).
2. The tree \( T \) is TDE if and only if the dual lamination \( L^2(T) \) is the diagonal closure of a (uniquely determined) minimal sublamination \( L^2_0(T) \).
3. Every \( \RR \)-tree \( T \) which is UDE must also TDE. A TDE-tree \( T \) is UDE if and only if the minimal sublamination \( L^2_0(T) \subset L^2(T) \) is uniquely ergodic.

**Proof.**

1. The dual lamination of \( T_{\text{bottom}} \) consists, from the bottom up, of a minimal sublamination \( L^2_{\min} \) which is carried by the bottom stratum and has geodesic realizations by used legal paths. This lamination can be completed (by finitely many additional leaves) to \( \text{diag}(L^2_{\min}) \), which is still carried by the bottom stratum. There is another (non-minimal) lamination \( L^2_{\text{used}} \) which is also carried by used legal paths, but those may run through both strata of \( \tau \). The lamination \( L^2(T_{\text{bottom}}) \) is equal to the diagonal closure of \( L^2_{\text{used}} \).

   The current \( \mu_{\text{bottom}} \) has as support the lamination \( L^2_{\min} \). Any other current \( \mu' \) with support in \( L^2(T_{\text{bottom}}) \) can be decomposed as convex linear combination of \( \mu_{\text{bottom}} \) and a current \( \mu'_0 \) with support in \( L^2(T_{\text{bottom}}) \) where \( \mu'_0 \) is extremal (i.e. no non-zero multiple of \( \mu_{\text{bottom}} \) can be subtracted). The
set of such extremal currents (all carried by $L^2(T_{bottom})$) is $\psi$-invariant, and by construction does not contain $\mu_{bottom}$.

We consider the column vectors defined by these extremal currents and consider the action of $f$, which commutes with the action of $\psi$ on the currents. But the vector space of these column vectors is finite dimensional (which isn’t necessarily true for the space of currents carried by $L^2(T_{bottom})$, as $\#\text{Curr}(F_N)$ is infinite dimensional !), so that any invariant proper subspace would lead to a second eigenvector, which we know does not exist. Thus there must be an extremal current which projects to the same vector as $\mu_{bottom}$. This contradicts our results from [16].

For the dual setting we get analogous, but in part weaker results:

**Proposition 8.6.** Let $\mu \in \text{Curr}(F_N) \setminus \text{Curr}(F_N)_+$, and assume that $\mu$ is not filling.

1. Not every non-filling $\mu \in \text{Curr}(F_N) \setminus \text{Curr}(F_N)_+$ is DE. Counterexamples are given, for any 2-strata exponential train track map with $\lambda_{top} > \lambda_{bottom}$, by the Perron-Frobenius current $\mu_{top}$.

2. A current $\mu$ is TDE if its support $\text{supp}(\mu)$ has diagonal closure which is a maximal element (with respect to the inclusion) of $\Lambda^2(F_N) \setminus \{\partial^2 F_N\}$ (diag($\text{supp}(\mu)$) is “maximal”).

3. A TDE-current $\mu$ with maximal diagonal closure of $\text{supp}(\mu)$ is UDE if and only if diag($\text{supp}(\mu)$) is uniquely ergometric.

**Proof.** The proof is given by a dualization of the above proof of the corresponding parts of Proposition [8.5]. The proof of part (1) becomes easier, since $\text{CV}_N$ is finite dimensional. □

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