1. Introduction and definitions

In Theorem 3 below, we show that for fractional Calderón-Zygmund operators, the $\kappa$-Cube Testing conditions over polynomials of degree less than $\kappa$ times indicators of cubes, are ‘essentially’ controlled by the familiar 1-Cube Testing conditions over indicators of cubes. This is then applied in Theorem 8 at the end of the paper, to obtain an extension of Stein’s characterization [Ste2, Theorem 4 page 306] of optimal cancellation conditions for Calderón-Zygmund operators via the $T1$ theorem of David and Journé. The extension is to more general pairs of doubling measures, with one weight in $A_\infty$, in place of Lebesgue measure. We now recall the definitions needed to formulate and prove these theorems.

Let $\sigma$ and $\omega$ be locally finite positive Borel measures on $\mathbb{R}^n$, and denote by $\mathcal{P}^n$ the collection of all cubes in $\mathbb{R}^n$ with sides parallel to the coordinate axes. For $0 \leq \alpha < n$, the classical $\alpha$-fractional Muckenhoupt condition for the measure pair $(\sigma, \omega)$ is given by

\begin{equation}
A^\alpha_2 (\sigma, \omega) \equiv \sup_{Q \in \mathcal{P}^n} \frac{|Q|_\sigma}{|Q|^{1-\frac{\alpha}{n}}} \frac{|Q|_\omega}{|Q|^{1-\frac{\alpha}{n}}} < \infty, \tag{1.1}
\end{equation}

and the one-tailed conditions by

\begin{align*}
A^\alpha_2 (\sigma, \omega) &\equiv \sup_{Q \in \mathcal{Q}^n} \mathcal{P}^\alpha (Q, \sigma) \frac{|Q|_\omega}{|Q|^{1-\frac{\alpha}{n}}} < \infty, \tag{1.2} \\
A^{\alpha, *}_2 (\sigma, \omega) &\equiv \sup_{Q \in \mathcal{Q}^n} \frac{|Q|_\sigma}{|Q|^{1-\frac{\alpha}{n}}} \mathcal{P}^\alpha (Q, \omega) < \infty,
\end{align*}

\[1\text{'actually a rather direct consequence of the } T1 \text{ theorem' in the words of Stein [Ste2 page 306].} \]
where the reproducing Poisson integral $P^\alpha$ is given by
\[
P^\alpha (Q, \mu) = \int_{\mathbb{R}^n} \left( \frac{|Q|^\frac{n}{n-\alpha}}{|Q|^\frac{n}{n-\alpha} + |x - x_Q|^2} \right)^n d\mu(x).
\]
The measure $\sigma$ is said to be doubling if there is a positive constant $C_{\text{doub}}$, called the doubling constant, such that
\[
|2Q|^n \leq C_{\text{doub}} |Q|^n , \quad \text{for all cubes } Q \in P^n.
\]
The absolutely continuous measure $d\omega(x) = w(x) \, dx$ is said to be an $A_\infty$ weight if there are constants $0 < \varepsilon, \eta < 1$, called $A_\infty$ parameters, such that
\[
\frac{|E|}{|Q|} < \eta \quad \text{whenever } E \text{ compact } \subset Q \text{ a cube with } \frac{|E|}{|Q|} < \varepsilon.
\]
Let $0 \leq \alpha < n$. For $\kappa_1, \kappa_2 \in \mathbb{N}$ and $\delta > 0$, we say that $K^\alpha(x, y)$ is a standard $(\kappa_1 + \delta, \kappa_2 + \delta)$-smooth $\alpha$-fractional kernel if for $x \neq y$, and with $\nabla_1$ denoting gradient in the first variable, and $\nabla_2$ denoting gradient in the second variable,
\[
\left| \nabla^j K^\alpha(x, y) \right| \leq C_{\text{CZ}} |x - y|^{\alpha - j - n - 1}, \quad 0 \leq j \leq \kappa_1,
\]
\[
\left| \nabla_i K^\alpha(x, y) - \nabla_i K^\alpha(x', y) \right| \leq C_{\text{CZ}} \left( \frac{|x - x'|}{|x - y|} \right)^\delta |x - y|^{\alpha - \kappa_1 - n - 1}, \quad \frac{|x - x'|}{|x - y|} \leq \frac{1}{2},
\]
and where the same inequalities hold for the adjoint kernel $K^{\alpha,*}(x, y) \equiv K^\alpha(y, x)$, in which $x$ and $y$ are interchanged, and where $\kappa_1$ is replaced by $\kappa_2$, and $\nabla_1$ by $\nabla_2$. We also consider vector kernels $K^\alpha = \{K_i^\alpha\}$ where each $K_i^\alpha$ is as above, often without explicit mention. This includes for example the vector Riesz transform in higher dimensions.

Given a standard $\alpha$-fractional CZ kernel $K^\alpha$, we consider truncated kernels $K_{\delta, R}^\alpha(x, y) = \eta_{\delta, R}^\alpha \left( |x - y| \right) K^\alpha(x, y)$ which uniformly satisfy (1.4). Then the truncated operator $T^\alpha_{\delta, R}$ with kernel $K_{\delta, R}^\alpha$ is pointwise well-defined, and we will refer to the pair $T^\alpha = \left( K^\alpha, \left\{ \eta_{\delta, R}^\alpha \right\}_{0 < \delta < R < \infty} \right)$ as an $\alpha$-fractional singular integral operator.

**Definition 1.** We say that an $\alpha$-fractional singular integral operator $T^\alpha = \left( K^\alpha, \left\{ \eta_{\delta, R}^\alpha \right\}_{0 < \delta < R < \infty} \right)$ satisfies the norm inequality
\[
\| T^\alpha_{\delta, R} f \|_{L^2(\omega)} \leq \mathcal{N}_{\tau^2} (\sigma, \omega) \| f \|_{L^2(\sigma)}, \quad f \in L^2(\sigma),
\]
provided
\[
\| T^\alpha_{\delta, R} f \|_{L^2(\omega)} \leq \mathcal{N}_{\tau^\alpha} (\sigma, \omega) \| f \|_{L^2(\sigma)}, \quad f \in L^2(\sigma), 0 < \delta < R < \infty.
\]

In the presence of the classical Muckenhoupt condition $A_2^\alpha$, the norm inequality (1.5) is essentially independent of the choice of truncations used, including nonsmooth truncations as well - see e.g. [LaSaShUr3].

The $\kappa$-cube testing conditions associated with an $\alpha$-fractional singular integral operator $T^\alpha$ introduced by Rahm, Sawyer and Wick in [RaSaWi] are given, with a slight modification, by
\[
\left( \mathcal{T}^{(\kappa)}_{T^\alpha} (\sigma, \omega) \right)^2 = \sup_{Q \in \mathcal{P}^\alpha} \max_{0 \leq |\beta| < \kappa} \frac{1}{|Q|_\sigma} \int_Q \| T^\alpha_{\delta, R} \left( 1_Q m^\beta_Q \right) \|^2 \omega < \infty,
\]
\[
\left( \mathcal{T}^{(\kappa)}_{T^\alpha} (\omega, \sigma) \right)^2 = \sup_{Q \in \mathcal{P}^\alpha} \max_{0 \leq |\beta| < \kappa} \frac{1}{|Q|_\omega} \int_Q \| (T^\alpha)^{\delta, R}_{\delta, R} \left( 1_Q m^\beta_Q \right) \|^2 \sigma < \infty,
\]
with $m^\beta_Q(x) \equiv (x - c_Q)^\beta$ for any cube $Q$ and multiindex $\beta$, where $c_Q$ is the center of the cube $Q$, and where as usual we interpret the right hand sides as holding uniformly over all sufficiently smooth truncations of $T^\alpha$. The more familiar cube testing conditions, as found in $T1$ theorems, are the case $\kappa = 1$ of (1.6) and $m^\beta_Q = 1$.  

We also use the larger full \( \kappa \)-cube testing conditions in which the integrals over \( Q \) are extended to the whole space \( \mathbb{R}^n \): 

\[
\begin{align*}
\left( \mathfrak{F}^n_{T^\circ} (\sigma, \omega) \right)^2 &= \sup_{Q \in \mathcal{P}^n} \max_{0 \leq |\beta| < \kappa} \frac{1}{|Q|_\omega} \int_{\mathbb{R}^n} \left| T^\alpha_\omega \left( 1_{Q} m^\beta_{Q} \right) \right|^2 \omega < \infty, \\
\left( \mathfrak{F}^n_{(T^\circ)^*} \right)^2 &= \sup_{Q \in \mathcal{P}^n} \max_{0 \leq |\beta| < \kappa} \frac{1}{|Q|_\omega} \int_{\mathbb{R}^n} \left| (T^\alpha_\omega)^* \left( 1_{Q} m^\beta_{Q} \right) \right|^2 \sigma < \infty.
\end{align*}
\]

Finally, as in \([SaShUr7]\), an \( \alpha \)-fractional vector Calderón-Zygmund kernel \( K^\alpha = (K^\alpha_j) \) is said to be \textit{elliptic} if there is \( c > 0 \) such that for each unit vector \( u \in \mathbb{R}^n \) there is \( j \) satisfying

\[
|K^\alpha_j(x, xu)| \geq ct^{\alpha-n}, \quad \text{for all } t > 0;
\]

and \( K^\alpha = (K^\alpha_j) \) is said to be \textit{strongly elliptic} if for each \( m \in \{1, -1\}^n \), there is a sequence of coefficients \( \{\lambda^m_j\}_{j=1}^J \) such that

\[
\sum_{j=1}^J \lambda^m_j K^\alpha_j(x, xu) \geq ct^{\alpha-n}, \quad t \in \mathbb{R}.
\]

holds for all unit vectors \( u \) in the \( n \)-ant

\[
V_m = \{x \in \mathbb{R}^n : m_i x_i > 0 \text{ for } 1 \leq i \leq n\}, \quad m \in \{1, -1\}^n.
\]

For example, the vector Riesz transform kernel is strongly elliptic \((SaShUr7)\).

### 1.1. Controlling polynomial testing conditions - main theorems

We begin in dimension \( n = 1 \) with the elementary formula for recovering a linear function from indicators of intervals,

\[
1_{[a,b)}(y) \left( \frac{y-a}{b-a} \right) = \int_a^b 1_{[r,b)}(y) \frac{dr}{b-a}, \quad \text{for all } y \in \mathbb{R},
\]

from which we conclude that for any locally finite positive Borel measure \( \sigma \), and any operator \( T \) bounded from \( L^2(\sigma) \) to \( L^2(\omega) \),

\[
T^\sigma_\omega \left( 1_{[a,b)}(y) \left( \frac{y-a}{b-a} \right) \right)(x) = \int_a^b T^\sigma_\omega \left( 1_{[r,b)}(y) \frac{dr}{b-a} \right)(x) = \int_a^b (T^\sigma_\omega 1_{[r,b)}(x) \frac{dr}{b-a}).
\]

We then use the testing estimate \( \|T^\sigma_\omega 1_{[r,b)} \|_{L^2(\omega)} \leq \left( \mathfrak{F}T \right)^2 \|\cdot\|_{\sigma} \), together with Minkowski’s inequality, to obtain

\[
\begin{align*}
\left\| T^\sigma_\omega \left[ 1_{(a,b)}(y) \left( \frac{y-a}{b-a} \right) \right] \right\|_{L^2(\omega)} &= \left\| T^\sigma_\omega \left[ \int_a^b 1_{[r,b)}(y) \frac{dr}{b-a} \right] \right\|_{L^2(\omega)} \\
&\leq \int_a^b \left\| T^\sigma_\omega \left[ 1_{[r,b)}(y) \right] \right\|_{L^2(\omega)} \frac{dr}{b-a} \leq \int_a^b \mathfrak{F}T \sqrt{\|\cdot\|_{\sigma}} \frac{dr}{b-a} \\
&\leq \mathfrak{F}T \sqrt{\int_a^b \|\cdot\|_{\sigma}} \frac{dr}{b-a} = \mathfrak{F}T \sqrt{\int_a^b \left( \int_{[r,b)} d\sigma(y) \right) \frac{dr}{b-a}} \\
&= \mathfrak{F}T \sqrt{\int_{[a,b)} \left( \int_a^y \frac{dr}{b-a} \right) d\sigma(y)} = \mathfrak{F}T \sqrt{\int_{[a,b)} \frac{y-a}{b-a} d\sigma(y)} \leq \mathfrak{F}T \sqrt{\|\cdot\|_{\sigma}},
\end{align*}
\]

and hence \( \mathfrak{F}T^{(1)} \leq \mathfrak{F}T^{(0)} \equiv \mathfrak{F}T. \) Similarly, the identity

\[
1_{[a,b)}(y) \left( \frac{y-a}{b-a} \right)^2 = \int_a^b 1_{[r,b)}(y) 2 \left( \frac{y-r}{b-a} \right) \frac{dr}{b-a}, \quad \text{for all } y \in \mathbb{R},
\]
shows that
\[
\left\| T \left[ 1_{[a,b]} (y) \left( \frac{y-a}{b-a} \right)^2 \right] \right\|_{L^2(\omega)} = T \left[ \int_a^b 1_{[r,b]} (y) 2 \left( \frac{y-r}{b-a} \right) dr \right] \right\|_{L^2(\omega)} 
\leq 2 \int_a^b \left\| T \left[ 1_{[r,b]} (y) \left( \frac{y-r}{b-a} \right) \right] \right\|_{L^2(\omega)} \frac{dr}{b-a} \leq 2 \delta \Sigma_T^{(1)} \sqrt{|[a,b]|_\sigma},
\]
and hence \( \delta \Sigma_T^{(2)} \leq 2 \delta \Sigma_T^{(1)} \). Continuing in this manner we obtain
\[
\delta \Sigma_T^{(\kappa)} \leq \kappa ! \delta \Sigma_T (\sigma, \omega),
\]
which when iterated gives
\[
\delta \Sigma_T^{(\kappa)} \leq \kappa ! \delta \Sigma_T (\sigma, \omega), \quad \kappa \geq 1,
\]

By a result of Hytönen [Hyt2], see also [SaShUr12] for the straightforward extension to fractional singular integrals, the full testing constant \( \delta \Sigma_T (\sigma, \omega) \) in dimension \( n = 1 \), is controlled by the usual testing constant \( \Sigma_T (\sigma, \omega) \) and the one-tailed Muckenhoupt condition \( A_2^\sigma \). Thus we have proved the following lemma for the case when \( T = T^\alpha \) is a fractional CZ operator.

**Lemma 2.** Suppose that \( \sigma \) and \( \omega \) are locally finite positive Borel measures on \( \mathbb{R} \) and \( \kappa \in \mathbb{N} \). If \( T^\alpha \) is a bounded \( \alpha \)-fractional CZ operator from \( L^2 (\sigma) \) to \( L^2 (\omega) \), then we have
\[
\Sigma_T^{(\kappa)} (\sigma, \omega) \leq \kappa ! \Sigma_T^{(\kappa)} (\sigma, \omega) + C_\kappa A_2^\sigma (\sigma, \omega), \quad \kappa \geq 1,
\]
where the constant \( C_\kappa \) depends on the kernel constant \( C_{\text{CZ}} \) in (1.4), but is independent of the operator norm \( \mathcal{M}_T (\sigma, \omega) \).

The higher dimensional version of this lemma will include a small multiple of the operator norm \( \mathcal{M}_T (\sigma, \omega) \) in place of the one-tailed Muckenhoupt constant \( A_2^\sigma (\sigma, \omega) \) on the right hand side, since we no longer have available an analogue of Hytönen’s result. Nevertheless, we show below that for doubling measures, the two testing conditions are equivalent in the presence of one-tailed Muckenhoupt conditions (1.2) in all dimensions, and so we will be able to prove a \( T1 \) theorem in higher dimensions in certain cases.

**Theorem 3.** Suppose that \( \sigma \) and \( \omega \) are locally finite positive Borel measures on \( \mathbb{R}^n \), and let \( \kappa \in \mathbb{N} \). If \( T \) is a bounded operator from \( L^2 (\sigma) \) to \( L^2 (\omega) \), then for every \( 0 < \varepsilon < 1 \), there is a positive constant \( C (\kappa, \varepsilon) \) such that
\[
\delta \Sigma_T^{(\kappa)} (\sigma, \omega) \leq C (\kappa, \varepsilon) \delta \Sigma_T (\sigma, \omega) + \varepsilon \mathcal{M}_T (\sigma, \omega), \quad \kappa \geq 1,
\]
and the constants \( C (\kappa, \varepsilon) \) depend only on \( \kappa \) and \( \varepsilon \), and not on the operator norm \( \mathcal{M}_T (\sigma, \omega) \).

**Proof.** We begin with the following geometric observation, similar to a construction used in the recursive control of the nearby form in [SaShUr12]. Let \( R = [0,1]^{n-1} \times [0,t) \) be a rectangle in \( \mathbb{R}^n \) with \( 0 < t < 1 \). Then given \( 0 < \varepsilon < 1 \), there is a positive integer \( m \in \mathbb{N} \) and a dyadic number \( t^* \equiv \frac{b}{2^m} \) with \( 0 \leq b < 2^m \), so that
\[
(1.9) \quad R = E \cup \left\{ \bigcup_{i=1}^B K_i \right\};
\]
\[
E = [0,1)^{n-1} \times \left[ t^*, t \right) \text{ with } |t - t^*| < \varepsilon,
\]
\[
B \leq 2^{nm-2m+2},
\]
and where the \( K_i \) are pairwise disjoint cubes inside \( R \). To see (1.9) we choose \( m \in \mathbb{N} \) so that \( \frac{1}{2m} < \varepsilon \) and then let \( b \in \mathbb{N} \) satisfy \( 2^m t - 1 \leq b < 2^m t \). Then with \( t^* = \frac{b}{2^m} \) we have \( |t - t^*| < \frac{1}{2m} \). Next expand \( t^* \) in binary form,
\[
t^* = b_1 \frac{1}{2} + b_2 \frac{1}{4} + \ldots + b_{m-1} \frac{1}{2^{m-1}}, \quad b_k \in \{0,1\}.
\]
Then for each \( k \) with \( b_k = 1 \) we decompose the rectangle
\[
R_k \equiv [0,1]^{n-1} \times \left[ b_1 \frac{1}{2} + b_2 \frac{1}{4} + \ldots + b_k - \frac{1}{2^{k-1}} b_1 \frac{1}{2} + b_2 \frac{1}{4} + \ldots + b_k - \frac{1}{2^{k-1}} + \frac{1}{2^k} \right).
\]
into $2^{(n-1)k}$ pairwise disjoint dyadic cubes of side length $\frac{1}{2^k}$. Then we take the collection of all such cubes, noting that the number $B$ of such cubes is at most

$$\sum_{k=1}^{m-1} 2^{(n-1)k} \leq 2 \cdot 2^{(n-1)(m-1)} = 2^{nm-n-m+2},$$

and label them as $\{K_i\}_{i=1}^B$ with $B \leq 2^{nm-n-m+2}$. Finally we note that

$$\bigcup_{i=1}^B K_i = \bigcup_{k: b_k = 1} R_k = [0, 1)^{n-1} \times [0, t^*).$$

This completes the proof of (1.9). Note that we may arrange to have $m \approx \ln \frac{1}{T}$. We also have the same result for the complementary rectangle $R = [0, 1)^{n-1} \times [r, 1)$ by simply reflecting about the plane $y_n = \frac{1}{2}$ and taking $r = 1 - t$. It is in this complementary form that we will use (1.9).

Again we start by considering the full testing condition $\mathfrak{F}_T$ over linear functions, and we begin by estimating

$$|| T_\sigma (\chi_Q (y) y_j)||^2_{L^2(\omega)}, \quad Q \in \mathcal{P}^n, 1 \leq j \leq n.$$ 

In order to reduce notational clutter in appealing to the complementary form of the geometric observation above, we will suppose - without loss of generality - that $Q = [0, 1)^n$ is the unit cube in $\mathbb{R}^n$, and that $j = n$. Then we have

$$\chi_{[0,1)^n} (y) y_n = \int_0^1 \chi_{[0,1)^{n-1} \times [r,1)} (y) \, dr, \quad \text{for all } y \in \mathbb{R}^n,$$

and

$$T_\sigma \left( \chi_{[0,1)^n} (y) y_n \right) (x) = T_\sigma \left( \int_0^1 \chi_{[0,1)^{n-1} \times [r,1)} (y) \, dr \right) (x) = \int_0^1 \left( T_\sigma \chi_{[0,1)^{n-1} \times [r,1)} \right) (x) \, dr.$$

The norm estimate is complicated by the lack of Hytönen’s result in higher dimensions, and we compensate by using the complementary form of the geometric observation (1.9), together with a simple probability argument. Let $[r, 1) = [r, r^*) \cup [r^*, 1)$ and write

$$\left| T_\sigma \chi_{[0,1)^{n-1} \times [r,1)} \right|_{L^2(\omega)}^2 = \left| \left( T_\sigma \chi_{[0,1)^{n-1} \times [r, r^*)} + \chi_{[0,1)^{n-1} \times [r^*, 1)} \right) \right|_{L^2(\omega)}^2$$

$$\leq \int \left| T_\sigma \chi_{[0,1)^{n-1} \times [r, r^*)} \right|^2 d\omega (x) + \sum_{i=1}^{B} \left| K_i \right|_{\mathcal{P}^n}.$$ 

First, we apply a simple probability argument to the integral over $r$ of the last integral above by pigeonholing the values taken by $r^* \in \left\{ \frac{k}{2^m} \right\}_{0 \leq k \leq 2m}:

$$\int_0^1 \left| T_\sigma \chi_{[0,1)^{n-1} \times [r, r^*)} \right|^2 d\omega (x) \, dr \leq \mathcal{M}_T (\sigma, \omega)^2 \int_0^1 \left\{ \int_{[0,1)^{n-1} \times [r, r^*)} \, d\sigma \right\} 

\leq \mathcal{M}_T (\sigma, \omega)^2 \sum_{0 \leq k \leq 2m} \int_{[0,1)^n} \left\{ \int_{[0,1)^{n-1} \times [r, r^*)} \, d\sigma \right\} 

\leq \mathcal{M}_T (\sigma, \omega)^2 \int_{[0,1)^n} \varepsilon \omega (y_1, \ldots, y_n) = \varepsilon \mathcal{M}_T (\sigma, \omega)^2 || [0, 1)^n ||_{\sigma},$$

since $\frac{b-1}{2m} \leq r \leq y_n < \frac{b}{2m}$ implies $y_n - \varepsilon < y_n - \frac{1}{2m} \leq r \leq y_n$.\]
Combining estimates, and setting \( R_y = [0, 1)^n \times [r, 1) \) for convenience, we obtain
\[
\| T_\sigma [1_{R_y}(y) y_n] \|_{L^2(\omega)} = \left\| T_\sigma \left[ \int_0^1 1_{R_y}(y) \, dr \right] \right\|_{L^2(\omega)}
\leq \int_0^1 \| T_\sigma [1_{R_y}(y)] \|_{L^2(\omega)} \, dr \leq \frac{\mathcal{F} T_\sigma}{\| \mathcal{F} T_\sigma \|} \int_0^1 \sqrt{\| R_y \|} \, dr + \varepsilon \mathcal{N}_T(\sigma, \omega) \| \{0, 1\}^n \|_\sigma,
\]
where
\[
\int_0^1 \sqrt{\| R_y \|} \, dr \leq \int_0^1 \| [r, b] \|_{\sigma} \frac{dr}{b-a} = \sqrt{\int_0^1 \| (0, 1)^n \times [r, 1) \|} \, dr \leq \sqrt{\int_0^1 \| (0, 1)^n \|} \, n \sigma \, dr = \sqrt{\int_0^1 \| (0, 1)^n \| \, y_n \, dr}.
\]
Noting that \( \sqrt{\int_0^1 y_n \, dr} \leq \| (0, 1)^n \|_\sigma \), that the same estimates hold for \( y_i \) in place of \( y_n \), and finally that there are appropriate analogues of these estimates for all cubes \( Q \in \mathcal{P}^n \) in place of \( [0, 1)^n \), we see that
\[
\mathcal{F} T_\sigma^{(1)}(\sigma, \omega) \leq C_m \mathcal{F} T_\sigma^{(1)}(\sigma, \omega) + \varepsilon \mathcal{N}_T(\sigma, \omega).
\]
Similarly, for each \( i < n \) we can consider the monomial \( y_i y_n \), and obtain from the above argument with \( y_i \) included in the integrant, that
\[
\| T_\sigma [1_{R_y}(y) y_i y_n] \|_{L^2(\omega)} \leq \sqrt{\int \| T_\sigma \left( \mathbf{1}_{(0, 1)^n \times [r, r^+]}(y) y_i \right) \right\|^2 \, d\omega(x) + \mathcal{F} T_\sigma^{(1)} \| \{0, 1\}^n \|_\sigma.
\]
For the monomial \( y_n^2 \) we use the identity
\[
\mathbf{1}_{(0, 1)^n}(y) y_n^2 = \int_0^1 \mathbf{1}_{(0, 1)^n \times [r, 1)}(y) 2 (y_n - y) \, dr, \quad \text{for all } y \in \mathbb{R}^n,
\]
to obtain
\[
\| T_\sigma [1_{R_y}(y) y_n^2] \|_{L^2(\omega)} \leq \sqrt{\int \| T_\sigma \left( \mathbf{1}_{(0, 1)^n \times [r, r^+]}(y) y_n^2 \right) \right\|^2 \, d\omega(x) + \mathcal{F} T_\sigma^{(1)} \| \{0, 1\}^n \|_\sigma.
\]
Then in either case, integrating in \( r \), using the simple probability argument above, and finally using the appropriate analogues of these estimates for all cubes \( Q \in \mathcal{P}^n \) in place of \( [0, 1)^n \), we obtain
\[
\mathcal{F} T_\sigma^{(2)}(\sigma, \omega) \leq C_m \mathcal{F} T_\sigma^{(1)}(\sigma, \omega) + \varepsilon \mathcal{N}_T(\sigma, \omega).
\]
Continuing in this way, using the identity
\[
\mathbf{1}_{(0, 1)^n}(y) y^\beta = \int_0^1 \mathbf{1}_{(0, 1)^n \times [r, 1)}(y) \left( y_{\beta_1} \ldots y_{\beta_{n-1}}\right) \left( 2 \beta_n (y_n - y)^{\beta_n-1} \right) \, dr, \quad \text{for all } y \in \mathbb{R}^n,
\]
yields the inequality
\[
\mathcal{F} T_\sigma^{(\kappa)}(\sigma, \omega) \leq C_{m, \kappa-1} \mathcal{F} T_\sigma^{(\kappa-1)}(\sigma, \omega) + \varepsilon \mathcal{N}_T(\sigma, \omega), \quad \kappa \in \mathbb{N}.
\]
Iteration then gives
\[
\mathcal{F} T_\sigma^{(\kappa)}(\sigma, \omega) \leq \varepsilon \mathcal{N}_T(\sigma, \omega) + C_{m, \kappa-1} \mathcal{F} T_\sigma^{(\kappa-1)}(\sigma, \omega) + \varepsilon \mathcal{N}_T(\sigma, \omega) + C_{m, \kappa-2} \mathcal{F} T_\sigma^{(\kappa-2)}(\sigma, \omega) + \varepsilon \mathcal{N}_T(\sigma, \omega) + \ldots + \varepsilon \mathcal{N}_T(\sigma, \omega) + \varepsilon \mathcal{N}_T(\sigma, \omega) + B(\kappa, \varepsilon) \mathcal{F} T_\sigma^{(\kappa)}(\sigma, \omega),
\]
where the constants \( A(\kappa, \varepsilon) \) and \( B(\kappa, \varepsilon) \) are independent of the operator norm \( \mathcal{N}_T(\sigma, \omega) \). Here we have taken \( m \approx \log_2 \frac{1}{\varepsilon} \). This completes the proof of Theorem \( \square \).
We have already pointed out in dimension \( n = 1 \), the equivalence of full testing with the usual 1-testing in the presence of one-tailed Muckenhoupt conditions. In higher dimensions the same is true for at least doubling measures. For this we use a quantitative expression of the fact that doubling measures don’t charge the boundaries of cubes \([\text{Ste2}, \text{see e.g. 8.6 (b) on page 40}]\).

**Lemma 4.** Suppose \( \sigma \) is a doubling measure on \( \mathbb{R}^n \) and that \( Q \in \mathcal{P}^n \). Then for \( 0 < \delta < 1 \) we have

\[
|Q \setminus (1 - \delta) Q|_\sigma \leq \frac{C}{\ln \frac{1}{\delta}} |Q|_\sigma .
\]

**Proof.** Let \( \delta = 2^{-m} \). Denote by \( \mathcal{C}^{(m)}(Q) \) the set of \( m \)-th generation dyadic children of \( Q \), so that each \( I \in \mathcal{C}^{(m)}(Q) \) has side length \( \ell(I) = 2^{-m} \ell(Q) \), and define the collections

\[
\mathcal{G}^{(m)}(Q) \equiv \left\{ I \in \mathcal{C}^{(m)}(Q) : I \subset Q \text{ and } \partial I \cap \partial Q \neq \emptyset \right\},
\]

\[
\mathcal{S}^{(m)}(Q) \equiv \left\{ I \in \mathcal{C}^{(m)}(Q) : 3I \subset Q \text{ and } \partial (3I) \cap \partial Q \neq \emptyset \right\}.
\]

Then

\[
Q \setminus (1 - \delta) Q = \mathcal{G}^{(m)}(Q) \text{ and } (1 - \delta) Q = \bigcup_{k=2}^{m} \mathcal{S}^{(k)}(Q).
\]

From the doubling condition we have \( |3I|_\sigma \leq D |I|_\sigma \) for all cubes \( I \), and so

\[
\left| \mathcal{S}^{(k)}(Q) \right|_\sigma = \sum_{I \in \mathcal{S}^{(k)}(Q)} |I|_\sigma \geq \frac{1}{D} \sum_{I \in \mathcal{S}^{(k)}(Q)} |3I|_\sigma = \frac{1}{D} \int \left( \sum_{I \in \mathcal{S}^{(k)}(Q)} 1_I \right) d\sigma
\]

\[
\geq \frac{1}{D} \int \left( \sum_{I \in \mathcal{G}^{(k)}(Q)} 1_I \right) d\sigma = \frac{1}{D} \left| \mathcal{G}^{(k)}(Q) \right|_\sigma \geq \frac{1}{D} \left| \mathcal{G}^{(m)}(Q) \right|_\sigma = \frac{1}{D} |Q \setminus (1 - \delta) Q|_\sigma .
\]

Thus we have

\[
|Q|_\sigma \geq \sum_{k=2}^{m} \left| \mathcal{S}^{(k)}(Q) \right|_\sigma \geq \frac{m-1}{D} |Q \setminus (1 - \delta) Q|_\sigma ,
\]

which proves the lemma. \( \square \)

**Proposition 5.** Suppose that \( \sigma \) and \( \omega \) are locally finite positive Borel measures on \( \mathbb{R}^n \), and that \( \sigma \) is doubling. Then for \( 0 < \varepsilon < 1 \) there is a positive constant \( C(\varepsilon) \) such that

\[
\mathcal{F} T_{\sigma}(\sigma, \omega) \leq \mathcal{F} T_{\sigma}(\sigma, \omega) + C(\varepsilon) A^2_{\nu}(\sigma, \omega) + \varepsilon \mathcal{N}_T(\sigma, \omega) .
\]

**Proof.** Let \( \delta > 0 \) be defined by the equation \( \varepsilon = \frac{C}{\ln \frac{1}{\delta}} \), i.e. \( \delta = e^{-\frac{C}{\ln \frac{1}{\delta}}} \). Then we write

\[
\int_{\mathbb{R}^n} |T_{\sigma}1_Q|^2 d\omega = \int_{Q} |T_{\sigma}1_Q|^2 d\omega + \int_{\mathbb{R}^n \setminus Q} |T_{\sigma}1_{(1-\delta)Q} + T_{\sigma}1_{Q \setminus (1-\delta)Q}|^2 d\omega
\]

\[
\leq \mathcal{F} T_{\sigma}(\sigma, \omega)^2 |Q|_\sigma + 2 \int_{\mathbb{R}^n \setminus Q} |T_{\sigma}1_{(1-\delta)Q}|^2 d\omega + 2 \int_{\mathbb{R}^n \setminus Q} |T_{\sigma}1_{Q \setminus (1-\delta)Q}|^2 d\omega
\]

\[
\leq \mathcal{F} T_{\sigma}(\sigma, \omega)^2 |Q|_\sigma + C \delta^{-1} A^2_{\nu}(\sigma, \omega) |Q|_\sigma + C \frac{1}{\delta} \mathcal{N}^2_{\nu}(\sigma, \omega) |Q \setminus (1 - \delta) Q|_\sigma .
\]

Now invoke Lemma \( \text{[4]} \) to obtain

\[
\int_{\mathbb{R}^n} |T_{\sigma}1_Q|^2 d\omega \leq \mathcal{F} T_{\sigma}(\sigma, \omega)^2 |Q|_\sigma + C \frac{1}{\delta} A^2_{\nu}(\sigma, \omega) |Q|_\sigma + \varepsilon \mathcal{N}_{\nu}^2(\sigma, \omega) |Q|_\sigma ,
\]

with \( \varepsilon = \frac{2C}{\ln \frac{1}{\delta}} \). \( \square \)
2. A T1 THEOREM FOR DOUBLING WEIGHTS WHEN ONE WEIGHT IS $A_\infty$

The following T1 theorem provides a Cube Testing extension of the T1 theorem of David and Journé [DaJo] to a pair of weights with one doubling and the other $A_\infty$ (and provided the operator is bounded on unweighted $L^2(\mathbb{R}^n)$ when $\alpha = 0$).

**Theorem 6.** Suppose $0 \leq \alpha < n$, and $\kappa_1, \kappa_2 \in \mathbb{N}$ and $0 < \delta < 1$. Let $T^\alpha$ be an $\alpha$-fractional Calderón-Zygmund singular integral operator on $\mathbb{R}^n$ with a standard $(\kappa_1 + \delta, \kappa_2 + \delta)$-smooth $\alpha$-fractional kernel $K^\alpha$, and when $\alpha = 0$, suppose that $T^0$ is bounded on unweighted $L^2(\mathbb{R}^n)$. Assume that $\sigma$ and $\omega$ are locally finite positive Borel doubling measures on $\mathbb{R}^n$ with doubling exponents $\theta_1$ and $\theta_2$ respectively satisfying $\kappa_1 > \theta_1 + \alpha - n$ and $\kappa_2 > \theta_2 + \alpha - n$, and that one of the measures is an $A_\infty$ weight. Set

$$T^\alpha_f = T^\alpha (f \sigma)$$

for any smooth truncation of $T^\alpha$.

Then the operator $T^\alpha_f$ is bounded from $L^2(\sigma)$ to $L^2(\omega)$, i.e.

$$\|T^\alpha_f\|_{L^2(\omega)} \leq \mathcal{N}_{T^\alpha} (\sigma, \omega) \|f\|_{L^2(\sigma)},$$

uniformly in smooth truncations of $T^\alpha$, provided that the one-tailed fractional condition (1.7) of Muckenhoupt holds, and the two dual Cube Testing conditions (1.7) hold with $\kappa = 1$. Moreover we have

$$\mathcal{N}_{T^\alpha} (\sigma, \omega) \leq C \left( \sqrt{A^2_\alpha (\sigma, \omega)} + \sqrt{A^{\alpha,*}_\alpha (\sigma, \omega)} + \mathcal{F}_{T^\alpha} (\sigma, \omega) + \mathcal{F}_{(T^\alpha)^*} (\omega, \sigma) \right),$$

where the constant $C$ depends on $C_{CZ}$ in (1.4) and the doubling parameters $(\beta_1, \gamma_1), (\beta_2, \gamma_2)$ of the weights $\sigma$ and $\omega$, as well as on the $A_\infty$ parameters of one of the weights. If $T^\alpha$ is elliptic, and strongly elliptic if $\frac{\alpha}{2} \leq \alpha < n$, the inequality can be reversed.

**Proof.** For convenience we take $\kappa = \kappa_1 = \kappa_2$. From Theorem 4 of [Saw] we have the inequality

$$\mathcal{N}_{T^\alpha} (\sigma, \omega) \leq C \left( \sqrt{A^2_\alpha (\sigma, \omega)} + \mathcal{F}_{T^\alpha} (\sigma, \omega) + \mathcal{F}_{(T^\alpha)^*} (\omega, \sigma) \right),$$

where the constant $C$ depends on $C_{CZ}$ in (1.4) and the doubling parameters $(\beta_1, \gamma_1), (\beta_2, \gamma_2)$ of the weights $\sigma$ and $\omega$, as well as on the $A_\infty$ parameters of one of the weights. From Theorem 3 above, we obtain that for every $0 < \varepsilon_1 < 1$, there is a positive constant $C (\kappa, \varepsilon_1)$ such that

$$\mathcal{F}_{T^\alpha} (\sigma, \omega) \leq C (\kappa, \varepsilon_1) \mathcal{F}_{T^\alpha} (\sigma, \omega) + \varepsilon_1 \mathcal{N}_{T^\alpha} (\sigma, \omega).$$

Finally from Proposition 3 we obtain that for every $0 < \varepsilon_2 < 1$, there is a positive constant $C (\varepsilon_2)$ such that

$$\mathcal{F}_{(T^\alpha)^*} (\omega, \sigma) \leq \mathcal{F}_{(T^\alpha)^*} (\omega, \sigma) + \varepsilon_2 \mathcal{N}_{T^\alpha} (\sigma, \omega).$$

Now we drop dependence on $(\sigma, \omega)$ to reduce clutter of notation, and combining inequalities we obtain

$$\mathcal{N}_{T^\alpha} \leq C \left( \sqrt{A^2_\alpha} + \mathcal{F}_{T^\alpha} + \mathcal{F}_{(T^\alpha)^*} \right) \leq C \left( \sqrt{A^2_\alpha} + \mathcal{F}_{T^\alpha} + \mathcal{F}_{(T^\alpha)^*} \right) \leq C \left( \sqrt{A^2_\alpha} + C (\kappa, \varepsilon_1) \mathcal{F}_{T^\alpha} + \varepsilon_1 \mathcal{N}_{T^\alpha} + C (\kappa, \varepsilon_1) \mathcal{F}_{(T^\alpha)^*} + \varepsilon_1 \mathcal{N}_{T^\alpha} \right) \leq C \sqrt{A^2_\alpha} + CC (\kappa, \varepsilon_1) \left( \mathcal{F}_{T^\alpha} + \mathcal{F}_{(T^\alpha)^*} + C (\varepsilon_2) \left( \sqrt{A^2_\alpha} + \sqrt{A^{\alpha,*}_\alpha} \right) + 2 \varepsilon_2 \mathcal{N}_{T^\alpha} \right) + 2 C \varepsilon_1 \mathcal{N}_{T^\alpha} \leq CC (\kappa, \varepsilon_1) \left( \mathcal{F}_{T^\alpha} + \mathcal{F}_{(T^\alpha)^*} \right) + CC (\kappa, \varepsilon_1) C (\varepsilon_2) \left( \sqrt{A^2_\alpha} + \sqrt{A^{\alpha,*}_\alpha} \right) + \{ 2CC (\kappa, \varepsilon_1) \varepsilon_2 + 2C \varepsilon_1 \} \mathcal{N}_{T^\alpha}.\leq C \mathcal{N}_{T^\alpha} + \mathcal{F}_{T^\alpha} + \mathcal{F}_{(T^\alpha)^*} + C \mathcal{N}_{T^\alpha}$$

Choose first $\varepsilon_1 > 0$ so that $2C \varepsilon_1 < \frac{1}{2}$, and then choose $\varepsilon_2 > 0$ so that $2CC (\kappa, \varepsilon_1) \varepsilon_2 < \frac{1}{2}$. We can then absorb the final term on the right into the left hand side to obtain

$$\mathcal{N}_{T^\alpha} (\sigma, \omega) \leq C_\kappa \left( \mathcal{F}_{T^\alpha} (\sigma, \omega) + \mathcal{F}_{(T^\alpha)^*} (\omega, \sigma) + \sqrt{A^2_\alpha (\sigma, \omega)} + \sqrt{A^{\alpha,*}_\alpha (\sigma, \omega)} \right),$$

for all suitable truncations of $T^\alpha$, and where the constant $C_\kappa$ depends on the doubling constants of the weights, and on the $A_\infty$ parameters of one of the weights. If $T^\alpha$ is elliptic, and strongly elliptic if $\frac{\alpha}{2} \leq \alpha < n$, then

$$\sqrt{A^2_\alpha (\sigma, \omega)} + \sqrt{A^{\alpha,*}_\alpha (\sigma, \omega)} \leq \mathcal{N}_{T^\alpha} (\sigma, \omega).$$
by [SaShUr7] Lemma 4.1 on page 92.\] \[□\]

Remark 7. If we drop the assumption that one of the weights is \( A_\infty \), inequality (2.3) remains true if we include on the right hand side the Bilinear Indicator Cube Testing constant \( \text{BICT}_{T^\alpha} (\sigma, \omega) \) from Saw2:

\[
\text{BICT}_{T^\alpha} (\sigma, \omega) \equiv \sup_{Q \in \mathcal{P}_n} \sup_{E \subseteq Q} \frac{1}{\sqrt{\langle Q \rangle_\sigma, \langle Q \rangle_\omega}} \left| \int_F T^\alpha_\sigma (1_E) \omega \right| < \infty,
\]
where the second supremum is taken over all compact sets \( E \) and \( F \) contained in a cube \( Q \).

2.1. Optimal cancellation conditions. Using Theorem 6 we can now obtain a \( T1 \) version of Theorem 5 in Saw3. For \( 0 \leq \alpha < n \), let \( T^\alpha \) be a continuous linear map from rapidly decreasing smooth test functions \( \mathcal{S}' \), to which is associated a kernel \( K^\alpha (x, y) \), defined when \( x \neq y \), that satisfies the inequalities,

\[
|\partial_\beta^\alpha \partial_\gamma^\alpha K^\alpha (x, y)| \leq A_{\alpha,\beta,\gamma,n} |x - y|^{n-|\beta|-|\gamma|}, \quad \text{for all multiindices } \beta, \gamma;
\]
such kernels are called smooth \( \alpha \)-fractional Calderón-Zygmund kernels on \( \mathbb{R}^n \). An operator \( T^\alpha \) is associated with a kernel \( K^\alpha \) if, whenever \( f \in \mathcal{S} \) has compact support, the tempered distribution \( T^\alpha f \) can be identified, in the complement of the support, with the function obtained by integration with respect to the kernel, i.e.

\[
T^\alpha f (x) \equiv \int K^\alpha (x, y) f (y) \, d\sigma (y), \quad \text{for } x \in \mathbb{R}^n \setminus \text{Supp} \, f.
\]

The characterization in terms of (2.5) in the next theorem is identical to that in Stein [Ste2, Theorem 4 on page 306], except that the doubling measures \( \sigma \) and \( \omega \) appear here in place of Lebesgue measure in [Ste2].

**Theorem 8.** Let \( 0 \leq \alpha < n \). Suppose that \( \sigma \) and \( \omega \) are locally finite positive Borel doubling measures on \( \mathbb{R}^n \). Suppose also that the measure pair \((\sigma, \omega)\) satisfies the classical \( A_2^\alpha \) condition in (1.1), and that in addition, one of the measures is an \( A_\infty \) weight. Suppose finally that \( K^\alpha (x, y) \) is a smooth \( \alpha \)-fractional Calderón-Zygmund kernel on \( \mathbb{R}^n \). In the case \( \alpha = 0 \), we also assume there is \( T^0 \) associated with the kernel \( K^0 \) that is bounded on unweighted \( L^2 (\mathbb{R}^n) \).

Then there exists a bounded operator \( T^\alpha : L^2 (\sigma) \to L^2 (\omega) \), that is associated with the kernel \( K^\alpha \) in the sense that (2.3) holds, if and only if there is a positive constant \( A_{K^\alpha} (\sigma, \omega) \) so that

\[
\int_{|x-x_0| < N} \int_{|x-y| < N} K^\alpha (x, y) \, d\sigma (y) \, d\omega (x) \leq A_{K^\alpha} (\sigma, \omega) \int_{|x_0-y| < N} d\sigma (y),
\]

for all \( 0 < \varepsilon < N \) and \( x_0 \in \mathbb{R}^n \),

\[
\text{along with a similar inequality with constant } A_{K^{\alpha,+}} (\omega, \sigma), \text{ in which the measures } \sigma \text{ and } \omega \text{ are interchanged and } K^\alpha (x, y) \text{ is replaced by } K^{\alpha,+} (x, y) = K^\alpha (y, x). \text{ Moreover, if such } T^\alpha \text{ has minimal norm, then}
\]

\[
\|T^\alpha\|_{L^2 (\sigma) \to L^2 (\omega)} \leq A_{K^\alpha} (\sigma, \omega) + A_{K^{\alpha,+}} (\omega, \sigma) + \sqrt{A_2^\alpha (\sigma, \omega)} + \sqrt{A_2^\alpha (\sigma, \omega))},
\]

with implied constant depending on \( C_{\text{CZ}} \), the doubling constants of the weights, and the \( A_\infty \) parameters of the \( A_\infty \) weight. If \( T^\alpha \) is elliptic, and strongly elliptic if \( \frac{n}{2} \leq \alpha < n \), the inequality can be reversed.

**Proof.** The proof follows almost verbatim that of Theorem 5 in Saw2, but using Theorem 6 above instead of the \( Tp \) theorem with Bilinear Indicator/Cube Testing in Saw2, and which thus eliminates both the polynomials and the indicator of a compact set \( E \) that appear in the characterization in Theorem 5 of Saw2. The straightforward verification of the details is left to the reader. \[\square\]

**Concluding comments:** The \( T1 \) theorem here is proved for general CZ operators, and thus in the absence of any special positivity properties of the CZ kernels \( K^\alpha \). As a consequence there is no catalyst available to enable control of the difficult ‘far below’ and ‘stopping’ terms by ‘goodness’ of cubes in the NTV bilinear Haar decomposition (see e.g. [NTV4]). In the case of the Hilbert transform, the positivity of the derivative of the convolution kernel \( \frac{1}{x} \) permits the derivation of a strong catalyst, namely the energy condition, from the testing and Muckenhoupt conditions (see e.g. [LaSaShUr3]). But the lack of a suitable catalyst for general CZ operators, see [SaShUr11] and Saw1 for negative results, limits us to using the weighted Alpert wavelets in [RaSaWi] with doubling measures having one weight in \( A_\infty \). It is an intriguing open question whether or not these
restrictions on the weight pair can be removed. There is no known example of the failure of a $T1$ theorem for fractional CZ operators.

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