I. INTRODUCTION

Properly functioning supply and transport networks essentially underlie our everyday lives. They enable the transport of nutrients and fluids [1] in biological organisms such as our body, the supply of electric energy in power grids [2], the transport of people and goods in traffic networks [3] as well as the flow of information through communication networks such as the internet. Structural changes of supply networks impact their core functionality. Increasing the strength of an edge or adding a new edge to a supply network constitutes a common strategy for improving its overall transport performance and to adapt it to current or future needs [4]. However, not every enhanced edge actually improves performance.

Indeed, already in 1968, Braess [5] highlighted an intriguing phenomenon in traffic networks: that opening a new street can worsen overall traffic as each individual tries to selfishly optimize their own travel time. Thus, adding certain edges may decrease the transport performance of supply and transport networks, a collective phenomenon today known as Braess’ paradox.

In its extreme form, Braess’ paradox may induce a complete loss of operating state of the supply network and thus a total collapse of its functionality, see e.g. [6]. Weaker forms of Braess’ paradox reduce system performance and causes higher stress in the network, e.g. reduce overall traffic flow in a street network [7] or reduce system stability [8]. Since its first identification, Braess’ paradox has been shown to prevail across many networked systems, including traffic networks, DC electrical circuits, AC electricity grids and other oscillatory networks, linear supply networks, discrete message passing systems, and two-dimensional electron gases [6, 9–12]. Notable theoretical results achieved over the decades include necessary and sufficient conditions for the occurrence of Braess’ Paradox [13–18], its prevalence even if individual agents behave non-selfishly [19], and the likelihood of occurrence in large random networks [20]. An algorithmic heuristic of identifying Braussian edges in traffic networks have been proposed by [21]. Coletta and Jacquod [22] recently showed how to predict which edges, if enhanced, cause Braess’ paradox, i.e. which edges are “Braessian” for heterogeneous one-dimensional chain topologies Yet, a general theory to better understand and to predict which individual edges are Braessian in a network is still missing to date.

To intuitively understand where and why Braess’ paradox occurs, we here take a new direction towards such a theory of predicting Braessian edges in networks with a wide class of dynamics. We propose an alternative perspective and consider Braess’ paradox in terms of increasing maximal flows and inducing potential overloads of edges due to differential changes of the network structure. Accordingly, we call an edge Braessian if infinitesimally increasing its strength yields an increase in the maximum flow in the network.

The problem of identifying such differentially Braessian edges maps exactly to a dual problem from electrostatic theory: that of identifying the direction of network currents induced by one dipole current on the maximum-flow carrying edge. Guided by the intuition resulting from this mapping, we propose an intuitive approximate graph theoretic predictor of Braessian edges based on the direction
of rerouted flows if the maximum flow carrying edge is removed. We illustrate the inverse consequence of these results to indicate ways to intentionally reduce the strength of a Brassian edge to recover an operating state of originally overloaded networks. The insights thus not only further our theoretical understanding of Braess’ paradox by providing intuitive insights about where to expect Braessian edges and provide drastic computational simplifications, they also offer practical advice on how to keep supply and transport networks functional.

II. GUIDING BACKGROUND

For the theory we develop below, we consider a broad class of supply and transport networks as occurring in natural and engineered systems. Before we provide more details, let us first mention some key properties of the networks constituting that class. By supply networks we refer to graphs \( G(\mathcal{V}, \mathcal{E}) \) of vertices (nodes) \( i \in \{1, \ldots, N\} =: \mathcal{V} \) and edges (lines) \((i, j) \in \mathcal{V} \times \mathcal{V}\) having the following additional vertex and edge properties:

1. flows \( F_{ij} \in \mathbb{R} \) across an edge \((i, j)\) quantify the amount of material or energy transported across that edge per unit time;

2. scalar vertex variables \( \varphi : \mathcal{V} \to \mathbb{R}, \, i \mapsto \varphi_i \) define potential functions in the sense of physical potentials such that these scalars \( \varphi_i \) constitute state variables making the resulting flows conservative and

3. edge strengths \( K_{ij} \in \mathbb{R} \) are the inverse of edge resistances that oppose flows, edge strengths are thus generalized susceptances known from DC electric networks.

These quantities are related by

\[
F_{ij} = K_{ij} f(\varphi_j - \varphi_i)
\]

where \( f : \mathbb{R} \to \mathbb{R} \) is a differentiable, strictly monotonic and odd function of the (potential) difference \((\varphi_j - \varphi_i)\) of the state variables at the vertices \(i\) and \(j\) the edge is incident to.

A. Linear supply networks

For linear \( f \) and thus \( f(x) = x \), without loss of generality, we obtain the most basic model setting where the flows from \( j \) to \( i \) are given by

\[
F_{ij} = K_{ij} (\varphi_j - \varphi_i).
\]

Such networks provide suitable approximate models for electric circuits, water, gas and heat supply networks as well as biological systems such as plant venation networks supplying, e.g. plant leaves [1, 23].

For electric circuits, \((2)\) represents Ohm’s law, approximating the current \( F_{ij} \) flowing through a conductor for a given potential difference \( \varphi_j - \varphi_i \) between its end nodes by a linear function with the current proportional to the conductance, i.e. the inverse resistance, \( K_{ij} := 1/R_{ij} \). In reality, the conductance depends on several factors, including the temperature of the conductor which itself depends on the current flowing through it. Thus, a linear relation approximates the actual nonlinear relation between voltage (potential difference) and current (flow) at a given operating point.

B. Nonlinear supply networks

Any nonlinearity of \( f \) characterizes system-specific details beyond the linearization of a network near a given operating point. For instance, coupled swing equations exhibit sinusoidal \( f \) and model lossless electric AC transmission grids. In that model class, each power generator and each consumer is modeled as a synchronous machine, and thus assigned a phase \( \theta_j \), a moment of inertia \( M_j \), a damping constant \( D_j \) as well as the power produced (for generators) or consumed (for consumers) \( P_j \) [24, 25].

The equation of motion at each node is given by

\[
M_j \frac{d^2 \theta_j}{dt^2} + D_j \frac{d\theta_j}{dt} = P_j + \sum_{k=1}^{N} K_{jk} \sin(\theta_k - \theta_j), \tag{3}
\]

where the edge strengths \( K_{ij} \) depend on the grid voltage approximated to be constant in time and the admittance of the transmission line.

During the steady operation of the power grid, the flow of electrical power from node \( j \) to node \( i \) is given by

\[
F_{ij} = K_{ij} \sin(\varphi_j - \varphi_i), \tag{4}
\]

identifying \( \theta_i = \varphi_i \).

Model flows with nonlinear \( f \) equally represent flows that can be assigned to dynamical systems that originally do not model real supply or transport networks. For instance, equations \((4)\) define abstract flows for the Kuramoto model [26, 27] with variables \( \varphi_i(t) \) satisfying

\[
\frac{d\varphi_i}{dt} = \omega_i + \sum_{j=1}^{N} K_{ij} \sin(\varphi_j - \varphi_i), \tag{5}
\]

where \( \omega_j \) are the natural frequencies of each node. The Kuramoto model constitutes a paradigmatic model of weakly coupled, strongly attracting limit cycle oscillators and does not include any types of material, energy or other flows. We may thus assign flows to systems that are not models of supply or transport networks to uncover system properties employing approaches for supply networks, for instance the approximate prediction scheme for Braessian edges presented below.

III. BRAESS’ PARADOX IN SUPPLY NETWORKS

Now, equipped with basic ideas about the system class considered, let us define conservative supply networks and
introduce an \textit{infinitesimal} perspective onto Braess’ paradox, on which we base the core results of this article.

\textit{a. Conservative supply networks.} As sketched above, supply networks are graphs whose edges model the transport of a certain quantity – which can be matter, energy or information. This quantity enters the system through a subset of \textit{source} nodes and exits it through another subset of \textit{sink} nodes. We formalize this in the following definition.

\textbf{Definition 1} (Supply network). Let \(G(\mathcal{V}, \mathcal{E})\) be a graph with the vertex set \(\mathcal{V} = (v_1, v_2, \cdots, v_N)\) and the edge set \(\mathcal{E} = (e_1, e_2, \cdots, e_M)\) and let us denote the input (current) at each node \(j \in \mathcal{V}\) as \(I_j\) and the flow from \(j\) to \(i\) across each edge \(e = (i, j) \in \mathcal{E}\) as \(F_{ij}\). Moreover, let \(\overline{\mathcal{T}} := (I_1, I_2, \cdots, I_N)\) and \(\overline{F} = (F_1, F_2, \cdots, F_M)\). Then the tuple \((G, \overline{T}, \overline{F})\) is called a supply network.

Let us focus one supply networks where the flow is conserved such that the continuity equation
\[
I_j = \sum_{(i,j) \in \mathcal{E}} F_{ij}, \text{ for all } j \in \mathcal{V} \tag{6}
\]
holds. It means that the input at each node equals the total outward flow through all the edges that node is part of. We remark that both, the \(I_j\) and the \(F_{ij}\) may be positive, negative or zero.

\textbf{Definition 2} (Conservative supply network). A supply network \((G, \overline{T}, \overline{F}^{con})\) is called a conservative supply network if the continuity equation \((6)\) is satisfied and the flow \(F_{ij}\) across any edge \((i, j)\) is a monotonically increasing, continuous, differentiable and odd function of the difference between a certain vertex property \textit{across the edge} \((i, j)\),
\[
F_{ij} = K_{ij}f(\phi_{ij} - \phi_{ij}) \tag{7}
\]
\[
f(y) > f(x) \iff y > x \tag{8}
\]
\[
f(-x) = -f(x) \tag{9}
\]
\[
K_{ij} = K_{ji} \tag{10}
\]
for all \(i, j \in \mathcal{V}\) and all \(x, y\) in the domain of \(f\).

We note that the oddness of \(f\) together with the symmetry of the \(K_{ij}\) implies the flow directionality condition \(F_{ij} = -F_{ji}\) that makes the flows well-defined.

\textit{b. Differential Braess’ paradox.} Most works on Braess’ paradox, including the first work by Braess [5] define it as a some form of “decrease in performance” of a supply network upon adding an edge. We here broaden this perspective by considering the \textit{infinitesimal strengthening} of an edge, a continuous procedure, instead of adding an edge, a discrete procedure.

\textbf{Definition 3} (Braessian edge). In a supply network \(\mathcal{F} = (G, \overline{T}, \overline{F})\), let the maximum flow be across the edge \((s, t)\),
\[
\max_{(i,j) \in \mathcal{E}} |F_{ij}| = |F_{s,t}|. \text{ After increasing the strength of only one edge } (a, b) \text{ by a small amount, } K'_{ab} = K_{ab} + \kappa,
\]
let the new flows across the edges be \(F'_{ij} = F_{ij} + \delta F_{ij}\). The edge \((a, b)\) is called Braessian if and only if
\[
|F'_{s,t}| > |F_{s,t}| \tag{11}
\]
as \(\kappa \to 0\).

We note that the condition \((11)\) is equivalent to \(F_{s,t}\delta F_{s,t} > 0\). We illustrate this definition in FIG 1 by means of a simple four-node supply network. The maximum flow is across the left edge \((1, 4)\). Upon increasing the strength of the top edge infinitesimally (panel b), the resulting incremental flow change at the maximum flow edge is \textit{anti-aligned} with (i.e. not in the same direction as) its original flow. Thus the maximum flow increases upon increasing the strength of the top edge, which makes it non-Braessian. However, the top edge is Braessian, because upon increasing its strength (panel c),

![FIG. 1. Braessian edge that aligns with differential flow change.](image)

(a) Simple supply network with four nodes (gray disks) with heterogeneous inputs \(I_i\) (not illustrated), four edges (blue), and identical edge strengths \(K_{ij} = K_{ji}\). (b) Maximum flow is across the edge at the left. (c) Top edge slightly strengthened. (d) In response, the incremental flow changes along a cycle, (dashed red lines) that disaligns with and thus decreases maximum flow. Hence, the top edge is not Braessian. (e) Bottom edge is slightly strengthened instead and (f) maximum flow increases. Hence, the bottom edge is Braessian. (g) Bar chart showing, for each edge, the original flows (red), flows due to strengthening top edge (light red), and due to strengthening bottom edge (dark red).
the incremental flow change at the left edge is aligned to the original (maximum) flow.

IV. ELECTROSTATIC ANALOG AND KEY SYMMETRY

Intriguingly, the problem of determining Braessian edges has a simple electrostatic analog that is crucial for our core result of understanding Braess’ Paradox (BP) based on the network topology. If a single edge of a conservative supply network is strengthened, the resulting flow changes are equal to the currents in a specifically constructed resistor network, where a constant current source is placed across the edge with the maximum flow. We will now present this equivalence in detail.

**FIG. 2.** Duality between flow changes and currents in a resistor network utilized to identify Braessian edges. (a) Flows in a four-node network (solid red edges) and incremental flow changes (dashed green edges) on strengthening the edge on the right. (b) A resistor network with resistances given by (12) and a constant current source connected across the right edge. The resulting currents in (b) equal the incremental flow changes in (a) at all edges except the right edge. The current at the left edge is aligned to the maximum flow, hence Braess’ paradox occurs. (c) - (d) The same as (a)-(b), but with the top edge strengthened. The current at the left edge is anti-aligned to the maximum flow, hence there is no Braess’ paradox.

**Claim 1 (Duality).** Let $F = (G, \vec{F}, \vec{I})$ be a conservative supply network with flows $F_{ij} = K_{ij} f(\varphi_j - \varphi_i)$ across each edge as per (7). Suppose the flow across the edge $(a, b)$ is positive from $a$ to $b$, $F_{ba} > 0$, and let its strength be increased as per Definition 3. Let the resulting flow changes across each edge be $F'_{ij} = F_{ij} + \delta F_{ij}$. Now consider a resistor network $G$ that has the same vertex and edge sets as $G$, and each edge $(i, j) \in \mathbb{E}$ has resistance $1/K_{ij} = 1/K_{ij}'(\varphi_j - \varphi_i)$.

**FIG. 3.** Detecting all Braessian edges in a network using the resistor equivalence. The edges for which the original flow and the current flow in the resistor network are in the same direction are Braessian. The edges for which these two are in opposite directions are not Braessian.

**Lemma 1.** Consider a supply network $F = (G, \vec{F}, \vec{I})$ with maximum flow across edge $(s, t)$, directed from $s$ to $t$. Consider a resistor network $G$ with a constant current source connected with the positive terminal attached to $t$ and the negative terminal attached to $s$, resulting in currents $I_{ab}$ for each edge $(a, b) \in G$. If $I_{ab}$ is directed identically as the flow $F_{ab}$ in the original flow network $F$, i.e. $I_{ab} K_{ab} > 0$, then, and only then, $(a, b)$ is a Braessian edge.
As a consequence, exploiting the symmetry of the resistor currents, determining the Braessianess of all the edges requires just one step: place a dipole across the maximum flow, and compute the currents. The brute-force method would be to strengthen each edge one by one and compute the new steady flows. We illustrate this in FIG 3.

We now demonstrate that the resistor network analog, beyond a major speedup in numerical identification, enables us to gain an intuitive topological understanding about which edges are likely to be Braessian.

V. TOPOLOGICAL UNDERSTANDING OF BRAESS PARADOX

If we had a graph theoretical quantity – preferably easy to compute – that predicts the direction of current in a resistor network due to a constant current source across one of its edges, our problem of predicting Braessian edges based on topology would be completely solved, thanks to Lemma 1.

As it happens, there exists such a quantity, presented by Shapiro [28], which we will paraphrase here.

Lemma 2. (Based on [28, Lemma 1]) Consider a resistor network with 1 unit of [29] current across an edge \( (s, t) \), directed from \( s \) to \( t \). We are interested in finding out if the current across an arbitrary edge \( (a, b) \) is directed from \( a \) to \( b \) or from \( b \) to \( a \). Let \( N(s, a \rightarrow b, t) \) be the set of spanning trees containing a path \( s, v_2, \ldots, a, b, \ldots, v_m-1, t \). Let \( N(s, b \rightarrow a, t) \) be defined in an analogous manner. Then the current across \( (a, b) \) is directed from \( a \) to \( b \) (\( b \) to \( a \)) if

\[
\sum_{T \in N(s,a \rightarrow b,t)} \sum_{(i,j) \in T} R_{ij} \gtrless \sum_{T \in N(s,b \rightarrow a,t)} \sum_{(i,j) \in T} R_{ij}, \tag{13}
\]

where \( R_{ij} \) is the resistance of the edge \( (i,j) \) and the sums run over the spanning trees \( T \).

Unfortunately the double sum in (13) is complex to compute, hence not useful in our quest of predicting Braessian edges. We will thus now present a simple topological concept we call rerouting alignment, inspired by (13). It is easy to compute, intuitive to understand, and frequently agrees with (13) to act as an approximate predictor of Braessian edges.

Definition 4 (Rerouting alignment). Consider the same resistor network as in Lemma 2. Let \( P_{s,a,b,t} \) be the shortest simple path that starts at \( s \), ends at \( t \) and contains the edge \( (a, b) \). If \( a \) precedes \( b \) in \( P_{s,a,b,t} \), then we say \( (a, b) \) is aligned by rerouting to \( (s, t) \). Otherwise, we say \( (a, b) \) is anti-aligned by rerouting to \( (s, t) \).

Now state a Heuristic that in the setup described in Lemma 2, the current across the edge \( (a, b) \) will be directed from \( a \) to \( b \) (from \( b \) to \( a \)) if \( (a, b) \) is aligned (anti-aligned) by rerouting to \( (s, t) \). Whenever this Heuristic holds, Lemma 1 yields the following predictor for Braessian edges in a network:

**Heuristic 1.** Suppose the maximum flow is across the edge \( (s, t) \), directed from \( s \) to \( t \). Given any other edge \( (a, b) \), carrying flow from \( a \) to \( b \), is Braessian if and only if it is aligned by rerouting to the edge \( (s, t) \).
stinate this claim, we analyzed its performance in three classes of drastically different network topologies (FIG 5): a 15 × 15 square lattice, a Voronoi tessellation of 20 uniformly randomly drawn points from a unit square and the IEEE 300 bus test case. In each topology, 1/4th of the nodes were chosen to have inputs 1 and an equal number to have inputs −1. The remaining nodes have inputs 0. For all three topologies, we generated and analyzed 200 independent realizations. We find that the classifier based on the above Heuristic performs reasonably well. The exact implementation of the predictor is described in Appendix C.

VI. HEURISTICS FOR MITIGATING NETWORK OVERLOAD

Braessian edges, by their very definition, increase the maximum flow in the network when strengthened. Vice versa, they decrease the maximum flow when weakened. Utilizing this property, we will now show how to mitigate overload in a network caused by damage at an edge by damaging a second, Braessian edge. We note that a similar phenomenon was reported in [30], where intentionally removing certain nodes and edges were shown to reduce the extent of cascading failures in a network. In Figure 6, we illustrate this for a 5 × 5 square lattice, with each edge having the same weight of unity, $K_{ij} = 1$. Reducing the strength of one edge (colored red) edge by 0.1 causes an overload in the maximum flow-carrying edge (colored sky blue). Among the Braessian edges, many mitigated the overload, when damaged to a suitable degree (by reducing their strengths). The colormap in the figure illustrates the amount by which the weight of an edge must be reduced to bring the maximum flow in the network back to its original value. Not coincidentally, the non-Braessian edges were incapable of mitigating the overload by this strategy of weight reduction. However, some Braessian edges cannot mitigate the overload, no matter how much they are managed. According to our systematic observations, this was due to one of two reasons. First, there were edges that, even when damaged to the maximum degree (i.e. completely taken out), could not completely reverse the overload. Secondly, there were some edges, which when damaged suitably, although reversed the overload in the previously maximally loaded edge, ended up overloading another edge so much that the maximum flow in the network increased. The flow across another edge became the new maximum flow.

VII. CONCLUSION

In this article we have presented an intuitive and topological way of classifying which edges in a supply network exhibit Braess’ paradox such that increasing their strength increases the maximum flow. In real world networks that often are capacity constrained, such increased maximum flows may easily induce overloads and system dysfunction.

Many supply networks crucial for our society need upgrading single edges from time to time. We thus believe that an improved intuitive understanding of the consequences of upgrading infrastructures may help planning such upgrades. We have shown that the incremental flow changes upon an infinitesimal strength increase of an edge are equivalent to the currents in a suitably constructed resistor network. This equivalence may be exploited beyond predicting Braess’ paradox, because it contributes an intuitive understanding of how the flow across any edge of choice would be affected if any other edge strength is changed.

Moreover, the resistor network analog may be extended to understand the effect of changes at multiple edges at once: It is equal to currents due to multiple dipole current sources in the resistor network. The latter follows from the resulting linearity of the differential approach and therefore the superposition principle underlying the problem.

We have concentrated in this article on infinitesimal increases in edge strengths, and it remains to investigate how our results translate to settings with non-infinitesimal changes in edge strengths, including a newly added or entirely removed edge, using, e.g., line outage
Acknowledgements. We thank Rainer Kree, Franziska Wegner, Benjamin Schäfer, and Malte Schröder for valuable discussions and hints about the manuscript presentation. We gratefully acknowledge support from the Federal Ministry of Education and Research (BMBF grant no. 03EK3055A-F), the International Max Planck Research School for the Physics of Biological and Complex Systems (to DM), the German Science Foundation (DFG) by a grant toward the Cluster of Excellence ‘Center for Advancing Electronics Dresden’ (cfaed).

Code availability. Code to reproduce key results is available at [32].

Appendix A: Flows in a resistor network

We demonstrate here how incremental flow changes upon strengthening an edge in a conservative supply network are equivalent to the electrical currents in a suitably constructed DC resistor network. Suppose a resistor network is described by a graph $G(V, E)$, with each edge $(i, j)$ having resistance $1/K_{ij}$. Let the input of electrical current at each node $j$ be $P_j$. Then the input at each node $j$ must equal the total outwards current from $j$ to all its neighbours, i.e.

$$P_j = \sum_{(i, j) \in E} F_{ij}, \text{ for all } j \in V, \quad (A1)$$

meaning the continuity equation (6) is satisfied. In addition, Ohm’s law gives

$$F_{ij} = K_{ij}(V_j - V_i), \quad (A2)$$

where $V_j$ is the voltage at node $j$. 

[1] E. Katifori, G. J. Szöllősi, and M. O. Magnasco, Phys. Rev. Lett. 104, 048704 (2010).
[2] P. Kundur, N. J. Balu, and M. G. Lauby, Power system stability and control, vol. 7 (McGraw-hill New York, 1994).
[3] A. Nagurney et al., Books (2000).
[4] M. Amin, in [in](2000).
[5] D. Braess, Unternehmensforschung 12, 258 (1968).
[6] D. Witthaut and M. Timme, The European Physical Journal B 86, 377 (2013), ISSN 1434-6028, 1434-6036, URL https://link.springer.com/article/10.1140/epjb/e2013-40469-4.
[7] D. Braess, A. Nagurney, and T. Wakolbinger, Transportation Science 39, 446 (2005).
[8] M. Colombo and H. Holden, Journal of Optimization Theory and Applications 168, 216 (2016).
[9] J. E. Cohen and P. Horowitz, Nature 352, 699 (1991).
[10] D. Witthaut and M. Timme, New journal of physics 14, 083036 (2016).
[11] L. S. Nagurney and A. Nagurney, EPL (Europhysics Letters) 115, 28004 (2016).
[12] S. Toussaint, D. Logoteta, M. Pala, V. Bayot, B. Hackens, et al., Bulletin of the American Physical Society 61 (2016).
[13] M. Frank, Mathematical Programming 20, 283 (1981).
[14] R. Steinberg and W. I. Zangwill, Transportation Science 17, 301 (1983), URL http://pubsonline.informs.org/doi/abs/10.1287/trsc.17.3.301.
[15] S. Dafermos and A. Nagurney, Transportation Research Part B: Methodological 18, 101 (1984).
[16] Y. A. Korilis, A. A. Lazar, and A. Orda, Journal of Applied Probability 36, 211 (1999).
[17] R. Azouzi, E. Altman, and O. Pourtallier (IEEE, 2002), vol. 4, pp. 3646–3651.
[18] A. Nagurney, EPL (Europhysics Letters) 91, 48002 (2010).
[19] E. I. Pas and S. L. Principio, Transportation Research Part B: Methodological 31, 265 (1997).
[20] G. Valiant and T. Roughgarden, Random Structures & Algorithms 37, 495 (2010).
[21] S. A. Baglee, A. A. Ceder, M. Tavana, and C. Bozic, Transportmetrica A: Transport Science 10, 437 (2014), URL https://doi.org/10.1080/23249935.2013.787557.
[22] T. Coletta and P. Jacquod, Physical Review E 93, 032222 (2016).
[23] B. Stott, J. Jardim, and O. Alsac, IEEE Transactions on Power Systems 24, 1290 (2009).
[24] D. Witthaut, F. Hellmann, J. Kurths, S. Kettemann, H. Meyer-Ortmanns, and M. Timme, Reviews of Mod-
Flows in a resistor network due to a single constant current source

Suppose in a resistor network, a constant current source with current $I_0$ is placed across edge $(a, b)$ so that $P_j = I_0(\delta_{jb} - \delta_{ja})$.

Now combining (A1) and (A2), we can see that

$$F_{ij} = K_{ij}(V_j - V_i)$$
$$P_j = \sum_{(i,j) \in E} K_{ij}(V_j - V_i) = I_0(\delta_{jb} - \delta_{ja}).$$  \hspace{1cm} (A3)

Relation with flow changes on infinitesimal strengthening of an edge

Now we justify our claim 1 that in any conservative supply network, if the strength of an edge is infinitesimally increased as per Definition 3, the resulting flow changes across any edge will be equal to currents in a suitably constructed resistor network.

In a supply network $\mathcal{F} = (G, \tilde{T}, \tilde{F})$, combining (6) and (7), we see that the flows will satisfy

$$I_j = \sum_{(i,j) \in E} F_{ij} = \sum_{(i,j) \in E} K_{ij}f(\phi_j - \phi_i).$$  \hspace{1cm} (A4)

Now if the strength of a single edge $(a, b)$ is increased from $K_{ab}$ to $K_{ab} + \kappa$, let the $\phi_j$ at each node $j$ be changed to $\phi_j + \xi_j$. Then the new flows will be

$$F_{ij} + \delta F_{ij} = (K_{ij} + \kappa \delta_{ai}\delta_{bj} + \kappa \delta_{aj}\delta_{bi}) f(\phi_j + \xi_j - \phi_i - \xi_i).$$

Defining

$$\tilde{K}_{ij} = K_{ij}f' (\phi_j - \phi_i),$$

we see that the flow changes at all edges $(i, j) \neq (a, b)$ will be

$$\delta F_{ij} = \tilde{K}_{ij} (\xi_j - \xi_i).$$  \hspace{1cm} (A7)

With the new flows (A5), (A4) will become

$$I_j = \sum_i (K_{ij} + \kappa \delta_{ai}\delta_{bj} + \kappa \delta_{aj}\delta_{bi}) f(\phi_i - \phi_j + \xi_i - \xi_j)$$
$$= I_j + \kappa \delta_{bj} f(\phi_a - \phi_j) + \kappa \delta_{aj} f(\phi_b - \phi_j) + \sum_i [K_{ij}] f(\phi_i - \phi_j) + f'(\phi_i - \phi_j)(\xi_i - \xi_j)] + O((\xi_i - \xi_j)^2).$$

Subtracting Eq. (A4):

$$0 = \kappa \delta_{bj} f(\phi_a - \phi_j) + \kappa \delta_{aj} f(\phi_b - \phi_j) + \sum_i \tilde{K}_{ij} (\xi_i - \xi_j)$$
$$+ O((\xi_i - \xi_j)^2).$$

Rearranging and putting together with (A7) we see

$$\delta F_{ij} = \tilde{K}_{ij} (\xi_j - \xi_i)$$
$$\sum_i \tilde{K}_{ij} (\xi_j - \xi_i) = \kappa f(\phi_a - \phi_b)(\delta_{jb} - \delta_{ja})$$
$$= \frac{F_{ab} \kappa}{K_{ab}} (\delta_{jb} - \delta_{ja}).$$  \hspace{1cm} (A8)

Comparing the flow changes (A8) and the currents in a resistor network (A3) yields Claim 1. That is, we see that the flow changes upon increasing the strength of edge $(a, b)$ with original flow $F_{ab}$ directed from $a$ to $b$; the resulting flow changes $\delta F_{ij}$ across all edges $(i, j) \neq (a, b)$ is given by the electrical currents in a resistor network with the same topology, and each edge having resistance $1/K_{ij}$, and a single constant current source with current $I = \frac{F_{ab} \kappa}{K_{ab}}$ placed across the enhanced edge $(a, b)$, such that $b$ is the current source and $a$ is the sink. This equivalence is illustrated in FIG 2 for a simple example.

Appendix B: A symmetry of resistor currents

In a resistor network, let a constant current source (CCS) be placed across the edge $(a, b)$ (+ input at $b$, − input at $a$) and the resulting current at edge $(i, j)$ from $i$ to $j$ be $F_{ji}^{b+a}$. We show that if we swapped $(a, b)$ and $(i, j)$ simultaneously, i.e. placed a constant current source across $(i, j)$ with a suitably chosen current, the current across $(a, b)$ satisfies

$$F_{ji}^{b+a} = F_{ab}^{i+j}.$$

To demonstrate this, we will go back to the definition
equations for currents in a resistor network (A3)

\[ F_{ij} = K_{ij} (V_j - V_i) \]

\[ P_j = \sum_{(i,j) \in E} K_{ij} (V_j - V_i) = I(\delta_{jb} - \delta_{ja}). \]  

(B2)

It is beneficial to recast this equation in matrix form. To this end, we will introduce two vectors in \( \mathbb{R}^n \) (\( n \) being the number of nodes in \( G \))

\[ \vec{1} = (1, 1, \cdots, 1) \]  

\[ \vec{e}^i = \begin{cases} n^{-1} \sum_{(i,k) \in E} K_{ik} & \text{if } i = j, \\ -K_{ij} & \text{if } i \neq j \text{ and } (i,j) \in E, \\ 0 & \text{otherwise}. \end{cases} \]  

(B3)

(B4)

and a matrix \( L \in \mathbb{R}^{n \times n} \), the weighted Laplacian matrix \([33, \text{p. 286}]\) of the graph \( G \), the edge weights being the inverse resistances

\[ L_{i,j} = \begin{cases} \sum_{(i,k) \in E} K_{ik} & \text{if } i = j, \\ -K_{ij} & \text{if } i \neq j \text{ and } (i,j) \in E, \\ 0 & \text{otherwise}. \end{cases} \]  

(B5)

Then (B2) becomes

\[ \vec{P} = IL^+ \vec{e}^b - I\vec{e}^a = L\vec{V}. \]  

(B6)

Now, (B6) does not have an unique solution for \( \vec{V} \) because \( L \) is singular, with a null space of dimension 1 spanned by the vector \( \vec{1} = (1, 1, \cdots, 1) \). This is no surprise, since the voltages in a DC resistor network are defined up to an arbitrary additive constant. Following \([34, 35]\), the node voltages are given by

\[ \vec{V} = L^+ \vec{P} + c\vec{1}, \]  

(B7)

where \( L^+ \) is the Moore-Penrose pseudoinverse of \( L \) and \( c \) is any real number. Then substituting (B6) into (B7), we obtain the voltage at any node \( m \).

\[ \vec{V} = IL^+ (\vec{e}^b - \vec{e}^a) \]

\[ V_{ma} = I \sum_{k=1}^{n} \left( L_{mk}^+ \vec{e}_{k} - L_{mk}^- \vec{e}_{k}^+ \right) + c \]

\[ = I \left( L_{mb}^+ - L_{ma}^- \right) + c. \]

Then the current \( F_{ji}^{b+a} \) from \( i \) to \( j \), following (B2), is

\[ F_{ji}^{b+a} = K_{ij} (V_i - V_j) \]

\[ = K_{ij} I \left[ \left( L_{ib}^+ - L_{ia}^- \right) - \left( L_{jb}^+ - L_{ja}^- \right) \right] \]

\[ = K_{ij} I \left( L_{ib}^+ - L_{ia}^- - L_{jb}^+ + L_{ja}^- \right). \]

Now, analogously, if a constant current source with is placed across \( (i,j) \) with \(+I \) input at \( i \) and \(-I \) input at \( j \), then the current from \( b \) to \( a \) will be

\[ F_{ab}^{i+j} = K_{ab} I \left( L_{bi}^+ - L_{bj}^+ - L_{ai}^- + L_{aj}^- \right). \]

Since \( L^+ \) is a symmetric matrix (for proof, see \([36, \text{Lemma 6.A.1}]\)), we have

\[ F_{ab}^{i+j} = K_{ab} \frac{K_{ji}}{K_{ij}} F_{ji}^{b+a}. \]  

(B8)

The prefactor \( \frac{K_{ji}}{K_{ij}} \) is a positive constant, and equals zero if the strength of the constant current source placed across \( (i,j) \) is \( I = \frac{K_{ji}}{K_{ij}} \).

More importantly, (B8) states the following regarding the direction of currents in resistor networks: If due to a constant current source across \((a,b)\) with positive input at \( b \) and negative input at \( a \), the resulting flow across edge \((i,j)\) is directed from \( i \) to \( j \), then a constant current source with positive input at \( i \) and negative input at \( j \) yields a current across \((a,b)\) directed from \( b \) to \( a \). This is illustrated in FIG 7, and provides Lemma 1.

Appendix C: Implementation of the predictor (heuristic for rerouting alignment)

The Heuristic 1 introduced above not only helps understanding the origin of Braessian edges (and non-Braessian ones) but also enables us to develop an algorithm for (approximately) predicting Braessian edges in any conservative supply network. The crucial part of this algorithm is determining if in a supply network an edge \((u,v)\) is aligned by rerouting to the maximum flow at edge \((s,t)\), directed from \( s \) to \( t \).

We note that the concept of an edge \((u,v)\) being aligned by rerouting to another edge \((s,t)\) is undefined if either of these two edges is a bridge. We call an edge \((i,j)\) a bridge if an originally connected network becomes disconnected by removing that edge. Such edges by definition are not part of any cycle, and therefore, do not support any rerouting flow. Indeed, strengthening them has no impact on the flows across other edges. Since they are therefore not interesting for the present article, such edges have been excluded from all analyses in this article. While generating random networks, we also have ignored realizations where the maximum flow itself is across a bridge.

Algorithm 1 describes the predictor in detail. However, the step of determining the shortest path in a graph that traverses the nodes \(s, i, j, t\) in that order proved too computationally expensive to solve exactly. Therefore we have developed a heuristic for determining approximations for such paths. As illustrated in FIG 5, that heuristic works reasonably well. The implementation of
this heuristic is available in [32].

**Algorithm 1: Topological predictor for Braessian edges**

**Data:** $(G(V, E), \overline{T}, \overline{F}^{\text{con}})$, a conservative supply network.

The maximum flow $F_{\text{max}}$ is across edge $(s, t)$, from $s$ to $t$.

**Result:** Set of predicted Braessian edges $E_{\text{be}}$, set of predicted non-Braessian edges $E_{\text{nbe}}$.

```plaintext
/* algorithm */
E_{\text{be}} \leftarrow \{\};
E_{\text{nbe}} \leftarrow \{\};
H \leftarrow G;
for (i, j) \in E do
    // set edge weights of $H$
    Weight of $(i, j) = K'_{ij} \leftarrow K_{ij} f'(\phi_j - \phi_i)$, as per (12);
end
for (i, j) \in E do
    // flow is from $i$ to $j$
    if there exists any simple path in $H$ containing the nodes $s, i, j, t$ in that order then
        $l_{al} \leftarrow$ (heuristically) the length of the shortest such path;
    else
        $l_{al} \leftarrow \infty$
    end
    if there exists any simple path in $H$ containing the nodes $s, j, i, t$ in that order then
        $l_{\text{nal}} \leftarrow$ (heuristically) the length of the shortest such path;
    else
        $l_{\text{nal}} \leftarrow \infty$
    end
    if $l_{al} < l_{\text{nal}}$ then
        $E_{\text{be}} \leftarrow E_{\text{be}} \cup \{(i, j)\};$
    else
        if $l_{al} > l_{\text{nal}}$ then
            $E_{\text{nbe}} \leftarrow E_{\text{be}} \cup \{(i, j)\};$
        else
            Braessianness of the edge cannot be predicted.
        end
    end
end
```