Sparse Multivariate ARCH Models: Finite Sample Properties

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Abstract

We provide finite sample properties of sparse multivariate ARCH processes, where the linear representation of ARCH models allows for an ordinary least squares estimation. Under the restricted strong convexity of the unpenalized loss function, regularity conditions on the penalty function, strict stationary and $\beta$-mixing process, we prove non-asymptotic error bounds on the regularized ARCH estimator. Based on the primal-dual witness method of Loh and Wainwright (2017), we establish variable selection consistency, including the case when the penalty function is non-convex. These theoretical results are supported by empirical studies.

Keywords: Multivariate ARCH; non-convex regularizer; statistical consistency; support recovery.

1 Introduction

Modelling the joint behavior of a high-dimensional random vector has become a key challenge for academics and practitioners. In the financial area, it is arduous to build a realistic model that is statistically relevant and consistent with some well-known stylized features of asset returns such as fat tails, volatility clustering, cross-sectional dependency, and the like. In such discrete time multivariate framework, the usual key quantities are the covariance matrices of the current asset returns, given their past values.

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In the literature, many specifications for discrete-time multivariate dynamic models have been proposed. Most of them basically belong to the multivariate GARCH (MGARCH) family or to the multivariate stochastic volatility family: see the surveys of Bauwens, Laurent and Rombouts (2006) and Asai, McAleer and Yu (2006), respectively. These models can be simulated by specifying the dynamics of the first two conditional moments of the underlying distributions on one side, and the law of the innovations on the other side. However, a classical drawback is the so-called “curse of dimensionality” as the specification of a general multivariate dynamic model induces an explosion of the number of parameters to be estimated. This implies practical problems of inference and possibly over-fitting.

The inference for \( p \)-dimensional multivariate GARCH models is usually led by the quasi-maximum likelihood method (see Francq and Zakoïan (2010), e.g.), where the order of the free parameters is \( O(p^4) \) for the most general MGARCH model called vectorial GARCH (VEC-GARCH). The corresponding objective function is highly non-linear (say a multivariate Gaussian QML) and thus fosters the use of reduced versions of such multivariate models: for instance the scalar BEKK of Engle and Kroner (1995). However, when \( p \) is large, capturing heterogeneous patterns will be difficult within scalar models.

Another approach is given by factor modeling, where a small number of factors aims at capturing the cross-sectional information. Among others, Fan, Fan and Lv (2008) emphasized the relevance of factor models for high-dimensional precision matrix estimation. However, this modeling requires the identification of the corresponding factors. An “expert” analysis is based on some priors regarding the leading underlying factors. Otherwise, latent unobserved factors induce particular estimation issues and their number is questionable.

In this paper, to tackle the curse of dimensionality inherent to MGARCH modeling, we propose to use the multivariate ARCH (MARCH) process derived from the VEC-GARCH model and to consider a regularized - or penalized - ordinary least squares (OLS) statistical criterion. MARCH models actually admit a linear representation with respect to the parameters, contrary to GARCH ones. Besides, any “invertible” GARCH process may be written as an infinite order ARCH model, under some conditions on its coefficients. Therefore, we argue that highly parameterized ARCH models (with numerous lags) should behave at least as well as more GARCH models that entail an autoregressive component, in terms of realism and flexibility. Nonetheless, we believe that not all the covariance/variance components have a significant influence on the other components.
Based on this sparsity assumption, we consider a regularization procedure of the multivariate ARCH model to recover the true set of non-zero parameters. This regularized MARCH approach is a major contribution compared to previous studies in the GARCH literature that proposed methods for high-dimensional vectors: for instance Engle, Ledoit and Wolf (2017) propose a composite likelihood approach to estimate the covariance dynamic on pairs of variables, instead of the whole vector at once; Bauwens, Grigoryeva and Ortega (2016) propose an algorithmic method to estimate non-scalar version of large-dimensional GARCH models. In these studies, sparsity is not explicitly considered and no penalization methods have been proposed so far for MGARCH models.

The theoretical properties of the penalized OLS loss function have widely been investigated in numerous studies, which rely on the key assumption that the samples are independent and identically distributed: e.g. Knight and Fu (2000), Fan and Li (2001), Zhang and Zou (2009) for the asymptotic properties; e.g. van de Geer (2008), van de Geer and Bühlmann (2009) for finite sample studies. Recent works have extended these results to the dependent case: asymptotic properties of the Lasso for high-dimensional time series were considered by Loh and Wainwright (2012) or Wu and Wu (2014); Basu and Michailidis (2015) derive finite sample error bounds for Lasso regularized Vector Autoregression models. The convexity of the statistical criterion is a key point to derive such results.

In this paper, we are interested in regularized ARCH models, where the regularizer can potentially be non-convex such as the SCAD or the MCP. We thus rely on the general framework developed by Loh and Wainwright (2014, 2017), which covers a broad range of non-convex objective functions such as the corrected Lasso for error in variables linear models, or simply the scad-penalized OLS loss function. Under the assumption of restricted strong convexity (Negahban, Ravikumar, Wainwright and Yu, 2012) of the unpenalized loss function and suitable regularity conditions on the regularizer, they establish that any stationary points of the penalized function, including both local and global optimum, lies within statistical precision of the true sparse parameter and provide conditions for variable selection consistency, a property also called support recovery. Our study uses this setting to derive finite sample properties of the regularized MARCH model under stationary and $\beta$-mixing process with sub-Gaussian innovations. To do so, we show that the unpenalized loss function satisfies a certain type of restricted strong convexity called restricted eigenvalue condition. Then our main contribution is to quantify the statistical accuracy of the sparse MARCH estimator by deriving bounds on the Frobenius
and entrywise max norm errors between the penalized estimator and the true parameter. Furthermore, we provide sufficient conditions for the sparse MARCH estimator to satisfy the support recovery property. Interestingly, the usual incoherence condition can be dropped under suitable conditions on the regularizers.

The remainder of the paper is organized as follows. In Section 2 we describe the multivariate ARCH framework and the penalized ordinary least squares criterion. Section 3 is devoted to the model assumptions and a preliminary concentration result. In Section 4 we demonstrate that the unpenalized loss function satisfies the restricted strong convexity property and we derive non-asymptotic upper bounds on the estimation error together with the conditions for which the support recovery property is satisfied. Section 5 illustrates these theoretical results through a simulated experiment and the relevance of the proposed regularization method is carried out on real a portfolio analysis.

**Notations.** Throughout this paper, we denote the cardinality of a set $E$ by $|E|$. For a vector $v \in \mathbb{R}^d$, we denote the $l_p$ norm as $\|v\|_p = \left( \sum_{k=1}^{d} |v_k|^p \right)^{1/p}$ for $p > 0$, and $\|v\|_\infty = \max_i |v_i|$. Let the subset $S \subseteq \{1, \ldots, d\}$, then $v_S \in \mathbb{R}^{|S|}$ it the vector $v$ restricted to $S$. For a matrix $A$, we denote $\|A\|_s$, $\|A\|_\infty$ and $\|A\|_F$ the spectral, infimum and Frobenius norms, respectively, and write $\|A\|_{\max} = \max_{i,j} |A_{i,j}|$ the coordinate-wise maximum (in absolute value). We write vec$(A)$ to denote the vectorization operator that stacks the columns of $A$ on top of one another into a vector. For a $p$-square and symmetric matrix $A$, we denote by vech$(A)$ the $p(p+1)/2$ vector that stacks the columns of the lower triangular part of its argument. We denote by $A \succ 0$ (resp $A \succeq 0$) the positive definiteness (resp semi-definiteness) of $A$. The notation $\otimes$ refers to the Kronecker operator. For a function $f : \mathbb{R}^d \to \mathbb{R}$, we denote by $\nabla f$ the gradient or subgradient of $f$ and $\nabla^2 f$ the Hessian of $f$. We denote by $(\nabla^2 f)_{SS}$ the Hessian of $f$ restricted to the block $S$.

## 2 Sparse VEC-ARCH process

### 2.1 VEC-ARCH model

The vector GARCH model is a direct generalization of univariate GARCH models, where the conditional covariance matrix is a function of lagged conditional variances and lagged cross-products of all components. Denoting by $(\epsilon_t)$ a sequence of $p$-dimensional random vectors, whose dynamics is specified by $\theta$, a finite-dimensional parameter, the vector
GARCH model corresponds to

\[
\begin{align*}
\epsilon_t &= H_t^{1/2} \eta_t, \text{ with } H_t := \mathbb{E}[\epsilon_t \epsilon_t^t | \mathcal{F}_{t-1}] > 0 \text{ so that } \\
vech(H_t) &= \omega + \sum_{k=1}^q A_k \vech(\epsilon_{t-k} \epsilon_{t-k}^t) + \sum_{i=1}^r B_i \vech(H_{t-i}), \quad (2.1)
\end{align*}
\]

where \((\eta_t)\) is a sequence of i.i.d. variables with distribution \(\eta\) independent of \(\mathcal{F}_t = \sigma(\epsilon_s, s \geq t)\) the natural filtration, \((A_k)_{1 \leq k \leq q}\) and \((B_i)_{1 \leq i \leq p}\) are \(p(p + 1)/2\) square matrices and \(\omega \in \mathbb{R}^{p(p+1)/2}\).

This VEC-GARCH\((r,q)\) approach definitely hampers any high-dimensional modelling as everything is explained by everything. Since the estimation procedure is based on a quasi-maximum likelihood method, it is challenging to apply a penalization procedure to obtain a parsimonious estimator due to the non-linear objective loss function. Moreover, the theoretical properties of the sparse estimator, such as asymptotic probability bounds or finite sample error bounds, are hard to derive since the penalized objective function would not be convex. In this paper, we thus omit the autoregressive part in \(2.1\), that is \(r = 0\), and using the vectorial specification, we consider the VEC-ARCH\((q)\) process

\[
\begin{align*}
\vech(H_t) &= \omega + \sum_{k=1}^q A_k \vech(\epsilon_{t-k} \epsilon_{t-k}^t) = A_q X_{q,t-1}, \quad (2.2)
\end{align*}
\]

where \(A_q = (\omega, A) \in M_{m \times 1 + qm}(\mathbb{R})\), with \(m = p(p + 1)/2\), so that \(A = (A_1, \ldots, A_q)\) with \(A_k \in M_{m \times m}(\mathbb{R})\) and \(X_{q,t-1} = (1, \vech(\epsilon_{t-1} \epsilon_{t-1}^t), \ldots, \vech(\epsilon_{t-q} \epsilon_{t-q}^t))' \in \mathbb{R}^{1+qm}\). This specification includes the diagonal VEC-ARCH process, where

\[
H_t = \Omega + \sum_{k=1}^q \text{diag}(\epsilon_{t-k}) A_k \text{diag}(\epsilon_{t-k}), \quad (2.3)
\]

so that \(\forall k, A_k = \text{diag}(\vech(A_k))\) and \(\omega = \vech(\Omega)\). The diagonal VEC-ARCH will be of interest in our applications due to the shrinking number of parameters it allows for.

Importantly, the conditional process \((\vech(H_t))\) given in \(2.2\) must be parameterized so that \((H_t)\) is positive-definite. To fix the ideas, we write \(h_t = \vech(H_t)\) and denote by \(a_{ij,i',j'}^k\) the entry of \(A_k\) located in the same row as \(h_{ij,t}\) (where \(i \geq j\)) and belonging to the same column as the element of \(h_{i',j',t}\) of \(h_t'\). Hence, we have

\[
\begin{align*}
h_{ij,t} &= \omega_{ij} + \sum_{k=1}^q \sum_{1 \leq j' \leq p} a_{ij,i',j'}^k \epsilon_{t-k} \epsilon_{t-k}^t \iff h_{ij,t} = \omega_{ij} + \sum_{k=1}^q \epsilon_{t-k} A_{ij,k} \epsilon_{t-k}, \quad (2.4)
\end{align*}
\]
where $A_{ij,k} \in \mathcal{M}_{p \times p}(\mathbb{R})$ and symmetric with $(i', j')$th entries as $a_{ij,i'j'}^k/2$ for $i' \neq j'$ and $a_{ij,i'i'}^k$ on the diagonal. Thus, following the notations of Chrétien and Ortega (2014), we introduce the matrix operator $\Sigma(A)$ defined as

$$
\forall k \geq l, (\Sigma(A)_{kl})_{ij} = \begin{cases} 
\frac{1}{2} A_{\pi(k,l)\pi(i,j)}, & \text{if } i > j \\
A_{\pi(k,l)\pi(i,j)}, & \text{if } i = j \\
\frac{1}{2} A_{\pi(k,l)\pi(j,i)}, & \text{if } i < j,
\end{cases}
$$

$$
\forall k \leq l, \Sigma(A)_{kl} = \Sigma(A)_{lk},
$$

where $\pi : S \to \{1, \cdots, p\}$, with $S = \{(i, j) \in \{1, \cdots, p\} \times \{1, \cdots, p\}, i \geq j\}$ denotes the map that yields the position of component $(i, j), i \geq j$, of any symmetric matrix in its equivalent $vech(.)$ representation. Then relationship (2.4), and thus (2.2), is equivalent to

$$
H_t = \Omega + \sum_{k=1}^q (I_p \otimes \epsilon_{t-k}) \Sigma(A_k) (I_p \otimes \epsilon_{t-k}),
$$

with $\Omega$ a symmetric matrix so that $\omega = vech(\Omega)$. Following Gouriéroux (1997), we have the sufficient condition

$$
(H_t) \text{ is positive definite if } \Sigma(A_k) \succeq 0, \; \Omega \succ 0. \tag{2.5}
$$

There is a one-to-one mapping between the components of $A_k$ and $\Sigma(A_k)$ and the positivity condition on $A_k$ forms a convex set, which will be taken into account in the parameter constraint set.

### 2.2 Regularized statistical criterion

Interestingly, (2.2) can be written as a linear model

$$
vech(\epsilon_t \epsilon_t') = A_q X_{q,t-1} + \zeta_t, \mathbb{E}[\zeta_t | \mathcal{F}_{t-1}] = 0, \tag{2.6}
$$

so that $\zeta_t = vech(\nu_t) = vech(\epsilon_t \epsilon_t') - vech(H_t)$ with $\nu_{ij,t} = \nu_{ji,t}$ for every couple $(i, j)$. Note that $(\zeta_t, \mathcal{F}_t)$ is a martingale difference. The previous linear model (2.6) will be estimated by a penalized least squares procedure. In terms of inference, this is a dramatic advantage w.r.t. the usual quasi-maximum likelihood estimation procedure of GARCH models. Therefore, in practical terms, it will be easier to estimate ARCH-type models with a lot of variables and lags ($p >> 1, q >> 1$) than a GARCH model with the same $p$.
and \( q = 1 \).

In this parameterization, the total number of parameters is \( d = (qp(p+1)/2 + 1)p(p + 1)/2 \). As for the diagonal ARCH, the total number of parameters is \( d = (q+1)p(p+1)/2 \).

Our main assumption concerns the sparsity of the true support, which corresponds to the total number of non-zero elements supposed to be strictly inferior to \( d \) (or \( \tilde{d} \) for the diagonal case). Our objective is to estimate a sparse \((\omega, A) \in \mathbb{R}^m \times \mathcal{M}_{m \times qm}(\mathbb{R})\) under the suitable positive definiteness constraint. To do so, we consider the OLS loss function

\[
G_T(A_q) = \frac{1}{2T} \sum_{t=1}^{T} \| Y_t - A_q X_{q,t-1} \|_2^2, \tag{2.7}
\]

where \( Y_t = \text{vech}(\epsilon_t \epsilon_t') \). The true parameter is denoted by \( A_{0,q} = (\omega_0, A_0) \) and is supposed to uniquely minimize \( \mathbb{E}[G_T(A_q)] \). The sparsity assumption concerns both parameters \( \omega_0 \) and \((A_{0,1}, \ldots, A_{0,q})\). Actually, the proposed framework could also accommodate a variance targeting procedure with respect to \( \omega \) to reduce the number of parameters\(^1\).

To recover the sparse support, we consider the regularized OLS estimator

\[
\hat{A}_q = \arg \min_{\{A_q : g(\text{vec}(A_q)) \leq R, \Omega > 0, \forall i, \Sigma(A_i) \geq 0\}} \{ G_T(A_q) + p(\lambda_T, \text{vec}(A_q)) \},
\]

\( p(\lambda_T, \cdot) : \mathbb{R}^d \rightarrow \mathbb{R} \) is a regularizer applied to each component of \( A_q \); \( \lambda_T \) is the regularization parameter, which depends on the sample size, and enforce a particular type of sparse structure in the solution; \( R > 0 \) is a supplementary regularization parameter. Due to the potential non-convexity of this penalty and following Loh and Wainwright (2014), we include the side condition \( g(\text{vec}(A_q)) \geq \| \text{vec}(A_q) \|_1 \) with \( g : \mathbb{R}^d \rightarrow \mathbb{R} \) a convex function and \( R \) to ensure the existence of local/global optima. We also impose \( \| \text{vec}(A_{0,q}) \|_1 \leq R \), where \( A_{0,q} \) is the true parameter vector of interest. \( A_{0,q} \) is supposed to be sparse so that

\( k = \text{card} (S) \), with \( S = \text{supp}(A_{0,q}) := \{ (i,j) | A_{0,(i,j)} \neq 0 \} \). Using the one-to-one mapping between \( A_k \) and \( \Sigma(A_k) \), \( \Sigma(A_k) \) must belong to the convex set of positive semi-definite matrices, which also contains \( \Sigma(A_{0,k}) \).

\(^1\Omega \) can be parameterized in the cone of non-negative matrices directly. The natural basis is provided by the spectral decomposition of \( \mathbb{E}[\epsilon_t \epsilon_t'] \) (the empirical approximation \( \hat{\text{cov}}_{i,j} \) instead), there exists an orthonormal family \((v_1, \ldots, v_p)\) in \( \mathbb{R}^p \) s.t. \( \mathbb{E}[\epsilon_t \epsilon_t'] \simeq [\hat{\text{cov}}_{i,j}]_{1 \leq i,j \leq p} = \sum_{l=1}^{p} \nu_l v_l v_l' \), where \((\nu_1, \ldots, \nu_p)\) is the associated spectrum, \( \nu_1 \geq \nu_2 \geq \cdots \geq \nu_p \geq 0 \). Then, we assume \( \exists ! \pi_l \geq 0, l = 1, \ldots, p \) s.t. \( \Omega = \sum_{l=1}^{p} \pi_l v_l v_l' \). Alternatively, one can replace \( \omega = (I_m - A_1 - \cdots - A_q) \text{vech}(H) \) with \( H = \mathbb{E}[\epsilon_t \epsilon_t'] \) and consider the linear model \( \text{vech}(\epsilon_t \epsilon_t') = \text{vech}(H) + (A_1, \ldots, A_q) X_{q,t-1} + \zeta_t \), with \( X_{q,t-1} = (\text{vech}(\epsilon_{t-1} \epsilon_{t-1} - H)'), \ldots, \text{vech}(\epsilon_{t-q} \epsilon_{t-q} - H)')' \).
Our proposed framework applies to any estimator \( \hat{A}_q \) that belongs to the interior of the constrained parameter set and satisfies the first order orthogonality conditions, that is
\[
\nabla_{\text{vec}(A_q)} G_T(\hat{A}_q) + \partial_{\text{vec}(A_q)} p(\lambda_T, A_q) = 0,
\]
where \( \partial p(\lambda_T, \cdot) \) is the sub-gradient of the regularizer. Such parameter \( \hat{A}_q \) is called a stationary point.

3 Preliminaries and assumptions

Establishing theoretical properties for penalized multivariate ARCH models requires some probabilistic structure on the process. In particular, Boussama (2006) provides conditions for the existence of a unique stationary and non-anticipative solution of the general vector GARCH model. This solution is unique, geometrically ergodic and \( \beta \)-mixing, which implies that the solution is geometrically \( \beta \)-mixing. Importantly, the \( \beta \)-mixing coefficients of a process are maintained when applying a measurable mapping. For the sequence \( \xi = (\epsilon_1, \cdots, \epsilon_T) \), we define the algebras \( \sigma_l = \sigma(\epsilon_1, \cdots, \epsilon_l), \sigma_{l+k}' = \sigma(\epsilon_{l+k}, \epsilon_{l+k+1}, \cdots) \). The \( \beta \)-mixing (also called completely regular) coefficient of the sequence \( \xi \) is defined as
\[
\beta(\xi)_n = \frac{1}{2} \sup \left\{ \sum_{i=1}^l \sum_{j=1}^j |P(A_i \cap B_j) - P(A_i)P(B_j)| : \right. \\
\left. A_i \text{ any finite partition in } \sigma_l, B_j \text{ any finite partition in } \sigma_{l+n}', l \geq 1 \right\}.
\]

For a measurable function \( f(\cdot) \), the stationary sequence \( (f(\epsilon_t)) \) has its \( \beta \)-mixing rate bounded by the corresponding rate of \( (\epsilon_t) \).

We now provide the conditions of Boussama (2006) adapted to the VEC-ARCH(q) model [2.2].

**Theorem 3.1.** (Boussama, 2006)
There exists a strictly stationary and non-anticipative solution of the VEC-ARCH(q) model [2.2] if
(i) the positivity condition on \( (A_k)_{1 \leq k \leq q} \) and \( \Omega \) given in (2.3) is satisfied.

(ii) the distribution of \( \eta \) has a density, positive on a neighborhood of 0, with respect to the Lebesgue measure on \( \mathbb{R}^p \).

(iii) \( \| \sum_{i=1}^q A_i \|_s < 1 \).
This solution is unique, $\beta$-mixing and ergodic.

In this paper, the conditions obtained by Boussama (2006) are assumed so that $(\epsilon_t)$ is a strictly stationary and $\beta$-mixing process.

**Assumption 1.** Under the conditions of Theorem 3.1, the process $(\epsilon_t)$ is strictly stationary, ergodic, non-anticipative and is geometrically $\beta$-mixing so that it satisfies $\beta(\epsilon)_n \leq \exp(-k_\beta n)$.

The next assumption requires the definition of the sub-Gaussian norm for a random variable $U$, which is
\[
\|U\|_{\psi_2} = \sup_{p \geq 1} \frac{1}{p^{1/2}} (\mathbb{E}[|U|^p])^{1/p},
\]
so that $U$ has a sub-Gaussian constant $M$ if $\|U\|_{\psi_2} \leq M$.

**Assumption 2.** The components of $(\eta_t)$ are sub-Gaussian such that
\[
\max_{t=1,\ldots,T} \|\eta_t\|_{\psi_2} \leq K,
\]
where $K > 0$ and does not depend on $T$.

Assumption 1 is key to derive concentration inequalities. Under $\beta$-mixing processes, the concentration inequalities for dependent variables will be close to the concentration inequalities for i.i.d. variables, the mixing coefficients behaving like a “penalty” in the probability. The sub-Gaussian property on $(\eta_t)$ in assumption 2 is another key ingredient to derive the concentration inequality in Lemma 3.3. This assumption implies that $(\epsilon_t)$ is also a sub-Gaussian process since $(H_t)$ is a $\mathcal{F}_{t-1}$ measurable process and thus entirely depends on past observations.

**Assumption 3.** Sparsity assumption: $\text{card}(S) = k < d$.

**Assumption 4.** We consider regularization functions that are supposed to be amenable regularizers defined as follows. We denote $p(\cdot, \cdot) : \mathbb{R}_+ \times \mathbb{R}^d$ the penalty function - or regularizer - , which is supposed to be coordinate-separable, id est $p(\lambda_T, \theta) = \sum_{k=1}^{d} p(\lambda_T, \theta_i)$. Furthermore, let $\mu \geq 0$, and $p(\lambda_T, \cdot)$ is called $\mu$-amenable if
\begin{enumerate}
\item $(i)$ $\rho \mapsto p(\lambda_T, \rho)$ is symmetric around zero and $p(\lambda_T, 0) = 0$.
\item $(ii)$ $\rho \mapsto p(\lambda_T, \rho)$ is non-decreasing on $\mathbb{R}^+$. \\
\item $(iii)$ $\rho \mapsto \frac{p(\lambda_T, \rho)}{p(\lambda_T, 0)}$ is non-increasing on $\mathbb{R}_+$. \\
\item $(iv)$ $\rho \mapsto p(\lambda_T, \rho)$ is differentiable for any $\rho \neq 0$.
\end{enumerate}
\[ \lim_{\rho \to 0^+} p'(\lambda_T, \rho) = \lambda_T. \]

\( (vi) \) \( \rho \mapsto p(\lambda_T, \rho) + \frac{\mu}{2} \rho^2 \) is convex for some \( \mu \geq 0 \).

The regularizer \( p(\lambda_T, \cdot) \) is \((\mu, \gamma)\)-amenable if in addition

\( (vii) \) There exists \( \gamma \in (0, \infty) \) such that \( p'(\lambda_T, \rho) = 0 \) for \( \rho \geq \lambda_T \gamma \).

We denote by \( q : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R} \) the function \( q(\lambda_T, \rho) = \lambda_T \| \rho \|_1 - p(\lambda_T, \rho) \) so that the function \( \frac{\mu}{2} \| \rho \|_2^2 - q(\lambda_T, \rho) \) is convex.

Assumption 3 implies that the true support (unknown) is sparse, that is the matrix \( A_0, q \) contains zero components. The regularization - or penalization - procedure provides an estimate of \( S \) by discarding past volatilities and covariances. To derive our theoretical properties, assumption 4 provides regularity conditions that potentially encompass non-convex functions. These regularity conditions are the same than Loh and Wainwright (2014, 2017) or Loh (2017). In this paper, we focus on the Lasso, the SCAD due to Fan and Li (2001) and the MCP due to Zhang (2010), given by

\[
\begin{align*}
\text{Lasso} : & \quad p(\lambda_T, \rho) = \lambda_T \| \rho \|_1, \\
\text{MCP} : & \quad p(\lambda_T, \rho) = \text{sign}(\rho) \lambda_T \int_0^{\| \rho \|} (1 - z/(\lambda_T b))_+ dz, \\
\text{SCAD} : & \quad p(\lambda_T, \rho) = \begin{cases} \\
\lambda_T \| \rho \|_1, & \text{for } |\rho| \leq \lambda_T, \\
-(\rho^2 - 2a \lambda_T |\rho| + \lambda_T^2)/(2(a - 1)), & \text{for } \lambda_T \leq |\rho| \leq a \lambda_T, \\
(a + 1) \lambda_T^2/2, & \text{for } |\rho| > a \lambda_T,
\end{cases}
\end{align*}
\]

where \( a > 2 \) and \( b > 0 \) are fixed parameters for the SCAD and MCP respectively. The Lasso is a \( \mu \)-amenable regularizer, whereas the SCAD and the MCP regularizers are \((\mu, \gamma)\)-amenable. More precisely, \( \mu = 0 \) (resp. \( \mu = 1/(a - 1) \), resp. \( \mu = 1/b \)) for the Lasso (resp. SCAD, resp. MCP): see the derivations of Loh and Wainwright (2014).

Contrary to the large sample approach such as the study of Hafner and Preminger (2009), who provide consistency and asymptotic normality results based on the Gaussian quasi-maximum likelihood estimator for the VEC-GARCH process, we are interested in the finite sample analysis. To derive non-asymptotic results that hold with a certain probability, the key tool is the concentration inequality. In this paper and for some reasons that will appear hereafter, we aim at obtaining a concentration inequality for a sum of polynomials of degree two and four of dependent sub-Gaussian variables. To derive such bound, we rely on Theorem 1.4 of Adamczak and Wolff (2015) obtained for the i.i.d.
setting. We first need to introduce the following notations. Let \( d \) and \( p \) be integers and \( C = (c_{i_1, \ldots, i_d})_{i_1, \ldots, i_d=1}. \) For \( \mathbf{i} = (i_1, \ldots, i_d) \in [p]^d \) and \( I \subseteq [p] = \{1, \ldots, p\}, \) let \( \mathbf{i}_I = (i_k)_{k \in I}. \) We denote by \( P_d \) the set of partitions of \([d]\) nonempty and pairwise disjoint sets. Given a partition \( \mathcal{J} = \{J_1, \ldots, J_k\}, \) we define the norm

\[
\|C\|_{\mathcal{J}} = \sup \{ \sum_{\mathbf{i} \in [p]^d} a_{\mathbf{i}} \prod_{l=1}^k x_{i_{l_1}^{(l)}}: \|x_{i_{l_1}^{(l)}}\|_F \leq 1, 1 \leq l \leq k \}, \quad (3.1)
\]

where \( x_{i_{l_1}^{(l)}} \) is a \(|J_l|-indexed\) matrix. We note that taking the union of subsets in the partition increases the norm.

**Theorem 3.2.** (Adamczak and Wolff, 2015)

Let \( f : \mathbb{R}^p \rightarrow \mathbb{R} \) be a polynomial of \( p \) variables of total degrees smaller or equal to \( D. \) For any integer \( d \geq 1, \) let \( D^d \) denote the \( d-th \) derivative of \( f. \) Let \( X = (X_1, \ldots, X_p) \) be a \( p \)-dimensional sub-Gaussian vector so that \( \|X_i\|_{\psi_2} \leq L, i \leq p. \) Let \( C_D \) a positive constant. Then for any \( \eta > 0 \)

\[
\mathbb{P}(\|f(X) - \mathbb{E}[f(X)]\| > \eta) \leq 2 \exp(-\frac{1}{C_D \gamma_f(\eta)}),
\]

with

\[
\gamma_f(\eta) = \min_{1 \leq d \leq D} \min_{\mathcal{J} \in P_d} \left\{ \frac{\eta}{L^d \|\mathbb{E}[D^d f(X)]\|_\mathcal{J}} \right\}^{\frac{2}{d}}.
\]

The \( \beta \)-mixing condition allows us to extend this concentration inequality for dependent variables.

**Lemma 3.3.** Let \( \xi = (\xi_1, \ldots, \xi_T) \) be a strictly stationary and geometrically \( \beta \)-mixing process, centered with \( \xi_t \in \mathbb{R}^p \) and sub-Gaussian so that \( \max_{t} \|\xi_t\|_{\psi_2} \leq L. \) Let \( f(.) \) a polynomial function of degree 4 such that \( f(\xi) = \sum_{\mathbf{i} \in [p]^4} c_{\mathbf{i}} \xi_{i_1} \xi_{i_2} \xi_{i_3} \xi_{i_4}, \) with \( C = (c_{i_1, \ldots, i_4})_{i_1, \ldots, i_4=1}. \) Let \( s_T \) a strictly positive block length and \( \mu_T = \lfloor T/(2s_T) \rfloor. \) Then for any \( \eta > 0, \) we have the concentration bound

\[
\mathbb{P}(\frac{1}{T} f(\xi) - \mathbb{E}[f(\xi)] > \eta) \leq 6 \exp(-\mu_T \gamma_f(\eta)) + 2c(\mu_T - 1) \exp(-k_\beta s_T),
\]

with

\[
\gamma_f(x) = \frac{x^2}{\|C\|_F^2} \wedge \min_{\mathcal{J} \in P_4} \left\{ \wedge_{k=2,3,4} \left( \frac{x^2}{\|C\|_\mathcal{J}^2} \right)^{1/k} \right\}.
\]

**Proof.** To carry out the proof, we use the independent block sequence technique to derive
the concentration inequality for dependent variables, developed e.g. in the proof of Theorem 3.1 in Yu (1994). To do so, we construct an independent block sequence based on the original dependent sequence so that this new sequence is close in terms of distribution to the mixing sequence. And by Corollary 2.7 of Yu (1994), the expected value of a random variable defined over the dependent blocks is close to the one based on these independent blocks under the $\beta$-mixing assumption. The idea is to transfer the dependent problem to the independent block and use standard concentration techniques of the independent case. Following Yu (1994), we divide the stationary process $(\xi_t)$ into $2\mu_T$ independent blocks of equal size $s_T$, where $\mu_T = \lfloor T/2s_T \rfloor$ and the remainder block is of size $T - 2\mu Ts_T$. The partition has $2\mu_T$ blocks so that $T - 2s_T < 2\mu Ts_T \leq T$ and a residual block. We denote the indices in these independent blocks by $H$ (resp. $N$) for the odd (resp. even) blocks and $Re$ the indices in the remainder part. Generally, for $1 \leq j \leq \mu_T$, we have

$$H_j = \{i : 2(j - 1)s_T + 1 \leq i \leq (2j - 1)s_T\},$$

$$N_j = \{i : (2j - 1)s_T + 1 \leq i \leq (2j)s_T\},$$

$$Re = \{2\mu Ts_T + 1, \ldots, T\},$$

so that $H = \bigcup_{j=1}^{\mu_T} H_j$ and $N = \bigcup_{j=1}^{\mu_T} N_j$. We thus denote the random variable set $\xi_h = \{\xi_t, t \in H\}$ (resp. $\xi_n = \{\xi_t, t \in N\}$, resp. $\xi_re = \{\xi_t, t \in Re\}$) the collection of random vectors belonging in the odd blocks (resp. even blocks, resp. remainder block). We take a sequence of identically distributed independent blocks $(\tilde{\xi}_{H_j})_{1 \leq j \leq \mu_T}$ so that $\tilde{\xi}_{H_j}$ is independent of $\xi_t, t = 1 \cdots, T$ and each block has the same distribution as a block from the original sequence $\xi_{H_j}$. We do the same for the even and remainder blocks, and denote these sequences as $(\tilde{\xi}_{H_j})_{1 \leq j \leq \mu_T}$ and $(\tilde{\xi}_{Re})$. We finally denote $\tilde{\xi}_H = \bigcup_{j=1}^{\mu_T} \tilde{\xi}_{H_j}$ (resp. $\tilde{\xi}_N = \bigcup_{j=1}^{\mu_T} \tilde{\xi}_{N_j}$) the union of odd (resp. even) block indices. From this block construction, we have

$$\mathbb{P}(\frac{1}{T} f(\xi) - \mathbb{E}[f(\xi)] > \eta) \leq \mathbb{P}(\frac{2}{T} \sum_{j=1}^{\mu_T} \sum_{i \in H_j} f(\xi_i) - \mu_T \sum_{i \in H_j} \mathbb{E}[f(\xi_i)] > \eta)$$

$$+ \mathbb{P}(\frac{2}{T} \sum_{j=1}^{\mu_T} \sum_{i \in N_j} f(\xi_i) - \mu_T \sum_{i \in N_j} \mathbb{E}[f(\xi_i)] > \eta) + \mathbb{P}(\frac{1}{T} \sum_{i \in Re} f(\xi_i) - |Re| \sum_{i \in Re} \mathbb{E}[f(\xi_i)] > \eta).$$

We first consider a collection of random variables in the even block. Let $\eta > 0$ and $c$ a
positive constant, we have
\[
\mathbb{P}\left(\frac{2}{T}|f(\xi_N) - \mathbb{E}[f(\xi_N)]| > \eta\right) = \mathbb{E}\left[1_{\frac{2}{T}|f(\xi_N) - \mathbb{E}[f(\xi_N)]| > \eta}\right] \\
\leq \mathbb{E}\left[1_{\frac{2}{T}|f(\xi_N) - \mathbb{E}[f(\xi_N)]| > \eta}\right] + c(\mu_T - 1)\beta(s_T) \\
\leq \mathbb{P}\left(\frac{2}{T}|f(\xi_N) - \mathbb{E}[f(\xi_N)]| > \eta\right) + c(\mu_T - 1)\beta(s_T) \\
\leq \mathbb{P}(\forall j = 1, \cdots, \mu_T : \frac{1}{s_T}|f(\xi_{N_j}) - \mathbb{E}[f(\xi_{N_j})]| > \eta) + c(\mu_T - 1)\beta(s_T) \\
\leq 2\exp(-\mu_T\gamma_f(\eta s_T)) + c(\mu_T - 1)\beta(s_T),
\]
where we used Lemma 4.1 of Yu (1994) to relate \(\xi_{s_T}\) and \(\tilde{\xi}_{s_T}\) using the mixing condition to obtain the first inequality and Theorem 1.4 of Adamczak and Wolff (2015) to obtain the last inequality. More precisely, \(f(.)\) corresponds to a linear combination of sub-Gaussian polynomials of degree 4 of the \(s_T\mu_Tp\) blocks of the sub-Gaussian variable \(\tilde{\xi} = (\tilde{\xi}_{ij})_{1 \leq i \leq T, 1 \leq j \leq p}\) with respect to \(\tilde{\xi}\). Since \(f(\tilde{\xi})\) is the sum of an homogeneous degree 4 polynomial, we only have to consider the derivatives of order 4 in (3.2) since the terms of order 1, 2 and 3 are null. For each of the even block, using the four indexed matrix \(C\), the functional form is
\[
\forall N_j, f(\tilde{\xi}_{N_j}) = \sum_{i_1, \cdots, i_4 = 1}^{s_T} \sum_{k_1, \cdots, k_4 = 1}^p C_{(i_1,k_1),(i_2,k_2),(i_3,k_3),(i_4,k_4)} \tilde{\xi}_{i_1,k_1} \tilde{\xi}_{i_2,k_2} \tilde{\xi}_{i_3,k_3} \tilde{\xi}_{i_4,k_4},
\]
with the real symmetric matrix \(C = (C_{ij})_{i,j \leq s_T}\). The expectation of \(\mathbb{E}[D^4 f(X)]\) is obtained by a symmetrization of \(C\). For any index \((j_1, \cdots, j_4) \in ([s_T] \times [p])^4\)
\[
\mathbb{E}[D^4 f(\tilde{\xi}_{N_j})]_{j_1,j_2,j_3,j_4} = \sum_{\pi} C_{j_{\pi(1)}, \cdots, j_{\pi(4)}},
\]
where the sum runs over all permutations of the index set \{1, \cdots, 4\}. We shall obtain an upper bound on \(\|C\|_\mathcal{J}\) to upper bound \(\|\mathbb{E}[D^4 f(X)]\|_\mathcal{J}\). For any partition \(\mathcal{J}\), we hence need to upper bound \(\|C\|^2_\mathcal{J}\). Starting with \(\mathcal{J} = \{1, 2, 3, 4\}\), we have \(\|C\|^2_{1,2,3,4} = \|C\|^2_F\). If we consider a partition \(\mathcal{J} = \{J_1, \cdots, J_r\}\) that has a cardinality larger than 2, we observed in the previous subsection \(\|C\|^2_\mathcal{J} \leq \|C\|^2_{\bigcup J_s, \bigcup J_s J_s}\). Moreover, the sub-Gaussian
property implies that $\|\bar{\xi}_N\|_{\psi_2} \leq s_T L$. Hence
\[
\min_{J \in P_4} \left\{ \frac{\eta s_T}{s_T^{d/2} L^d \| D^4 f(\bar{\xi}_N) \|_J} \right\}^2 \geq \frac{(\eta s_T)^2}{s_T^2 L^d \| C \|_F^2} \wedge \left\{ \wedge_{k=2,3,4} \left( \frac{(\eta s_T)^2}{s_T^2 L^d \| C \|_F^2} \right)^{1/k} \right\}.
\]

We thus obtain
\[
\gamma_f(\eta) \geq \frac{\eta^2}{L^d \| C \|_F^2} \wedge \min_{J \in P_4} \left\{ \wedge_{k=2,3,4} \left( \frac{\eta^2}{L^d \| C \|_F^2} \right)^{1/k} \right\}.
\]

Furthermore, we have $c(\mu_T - 1)\beta(s_T) \leq c(\mu_T - 1) \exp(-k_\beta s_T)$. This implies
\[
\mathbb{P}(\| f(\xi_N) - \mu_T \mathbb{E}[f(\xi_N)] \| > \eta) \leq 2 \exp(-\mu_T \gamma_f(\eta)) + c(\mu_T - 1) \exp(-k_\beta s_T).
\]

The same reasoning holds for the collection in the odd block. Taking the union of the collection of odd and even blocks, we obtain
\[
\mathbb{P}(\| \frac{1}{\mu_T} \sum_{i,j=1}^{\mu_T} f(\xi_{H_i \cup N_j}) - \mu_T \sum_{i,j=1}^{\mu_T} \mathbb{E}[f(\xi_{H_i \cup N_j})] \| > \eta) \leq 4 \exp(-\mu_T \gamma_f(\eta)) + 2c(\mu_T - 1) \exp(-k_\beta s_T).
\]

Let $\eta > 0$, we obtain
\[
\mathbb{P}(\| \frac{1}{\mu_T} \sum_{i,j=1}^{\mu_T} f(\xi_{H_i \cup N_j}) - \mu_T \sum_{i,j=1}^{\mu_T} \mathbb{E}[f(\xi_{H_i \cup N_j})] \| > \eta) \leq 4 \exp(-\mu_T \gamma_f(\eta)) + 2c(\mu_T - 1) \exp(-k_\beta s_T).
\]

The size of the remainder block is $s_T \leq T/(2\mu_T)$. Using Theorem 1.4 of Adamczak and Wolff (2015), we have
\[
\mathbb{P}(\| f(\xi_{Re}) - Re \mathbb{E}[f(\xi_{Re})] \| > \eta) \leq 2 \exp(-\mu_T \gamma_f(\eta)).
\]

We thus obtain by taking the union on all blocks
\[
\mathbb{P}(\| f(\xi) - \mathbb{E}[f(\xi)] \| > \eta) \leq 6 \exp(-\mu_T \gamma_f(\eta)) + 2c(\mu_T - 1) \exp(-k_\beta s_T),
\]

with
\[
\gamma_f(\eta) = \frac{\eta^2}{L^d \| C \|_F^2} \wedge \min_{J \in P_4} \left\{ \wedge_{k=2,3,4} \left( \frac{\eta^2}{L^d \| C \|_F^2} \right)^{1/k} \right\}.
\]

**Remark.** This concentration inequality holds for strictly stationary and $\beta$-mixing sub-
Gaussian processes. In this paper, we would also need a concentration inequality for second degree sub-Gaussian polynomials for \( \beta \)-mixing and strictly stationary processes. It is given as

\[
\mathbb{P}(\frac{1}{T}g(\xi) - \mathbb{E}[g(\xi)] > \eta) \leq 6 \exp(-\mu_T \gamma_g(\eta)) + 2c(\mu_T - 1) \exp(-k_\beta s_T),
\]

where \( g(.) \) is a polynomial function of degree 2 such that \( g(\xi) = \sum_{i \in [pT]^2} b_i \xi_i \xi_{i_2} \), with \( B = (b_{i_1,i_2})_{i_1,i_2=1}^{pT} \), and \( \gamma_g(\eta) = \frac{\sigma^2}{L^4 \|B\|_F} \wedge \frac{\eta}{L^2 \|B\|_2} \). The proof follows the exact same steps as in the proof of Lemma 3.3.

By assumption 3, for any \( t \), conditionally on \( \mathcal{F}_{t-1} \), the random vector \( \epsilon_t \) is actually sub-Gaussian. Indeed, the process \( H_t \) is \( \mathcal{F}_{t-1} \)-measurable. Then for \( (\epsilon_{t-1}, \cdots, \epsilon_{t-q}) = (x_1, \cdots, x_q) \) with each \( x_i \in C \subset \mathbb{R}^p \) any compact set, we have \( \mathrm{vech}(H_t) = \mathbf{A}_q \mathbf{X}_t \), where \( \mathbf{X} = (1, \mathrm{vech}(x_1x_1'), \cdots, \mathrm{vech}(x_qx_q'))' \in \mathbb{R}^{1+qm} \). Thus \( H_t := H(\mathbf{X}) \) depends on the past \( q \) realizations of the random vector \( (\epsilon_{t-1}, \cdots, \epsilon_{t-q}) \). Then under the stationarity condition of Theorem 3.1, let \( L > 0 \), we have

\[
\|H_t^{1/2} \eta_t\|_{\psi_2} = \|H^{1/2}(\mathbf{X}) \eta_t\|_{\psi_2} = \|\mathbf{v}'H^{1/2}(\mathbf{X}) \eta_t\|_{\psi_2} \leq \|\mathbf{v}' \eta_t\|_{\psi_2} = \|H^{1/2}(\mathbf{X})\|_2 \|\eta_t\|_{\psi_2} \leq \|H^{1/2}(\mathbf{X})\|_2 C \left\{ \max_{1 \leq i \leq p} \|\eta_{i,t}\|_{\psi_2} \right\} \leq L.
\]

4 Finite sample results

In this section, we derive finite sample error bounds of the regularized VEC-ARCH(q) estimator based on the linear representation (2.6) and provide the conditions to satisfy the support recovery property. The statistical criterion is

\[
\left\{ \begin{array}{l}
\hat{\mathbf{A}}_q = \arg \min_{\mathbf{A}_q \in \mathcal{K}} \{ \mathcal{G}_T(\mathbf{A}_q) + \mathcal{P}(\lambda_T, \mathrm{vec}(\mathbf{A}_q)) \}, \text{ with }
\mathcal{K} = \{ \mathbf{A}_q : g(\mathrm{vec}(\mathbf{A}_q)) \leq \mathbb{R}; \Omega > 0, \forall l, \Sigma(A_l) \succeq 0 \},
\end{array} \right. \tag{4.1}
\]

where the regularizer \( \mathcal{P}(\lambda_T, .) \) is the Lasso, the SCAD or MCP. The convexity of the parameter set is a key point for the theoretical study. As a consequence, the constraint \( \|\sum_{i=1}^{q} A_i\|_s < 1 \) must be left aside due to its non-convexity. Indeed, the subadditivity cannot be ensured since the \( A_i \) matrices are non-symmetric. It can be checked only ex-post. In a first step, we prove that the unpenalized loss function \( \mathcal{G}_T(.) \) in (4.1) satisfies
a form of restricted strong convexity (RSC) called restricted eigenvalue condition, with high probability. This property is a key ingredient to derive our finite sample statistical analysis.

4.1 Restricted strong convexity

Intuitively, we would like to handle a loss function that locally admits some curvature. To ensure this property, we rely on the strong convexity (local) of the loss function. The strong convexity of a differentiable loss function corresponds to a strictly positive lower bound on the eigenvalues of the Hessian matrix uniformly valid over a local region around the true parameter. This amounts to a curvature condition. More precisely, we are interested only in a particular direction, that is the difference \( \Gamma = \hat{\theta} - \theta_0 \). The notion of restricted strong convexity weakens the (local) strong convexity by adding a tolerance term. A detailed explanation is provided in Negahban, Ravikumar, Wainwright and Yu (2012).

Following the definition of Loh and Wainwright (2017), for \( T \) observations and \( \theta \in \mathbb{R}^d \), an empirical loss function \( G_T(.) \) satisfies the restricted strong convexity condition (RSC), that is \( \exists \alpha_1, \alpha_2 > 0 \) and \( \tau_1, \tau_2 \geq 0 \) given any \( \theta, \Gamma \in \mathbb{R}^p \), such that

\[
\langle \nabla_{\theta} G_T(\theta + \Gamma) - \nabla_{\theta} G_T(\theta), \Gamma \rangle \geq \alpha_1 \|\Gamma\|_2^2 - \tau_1 \log \frac{d}{T} \|\Gamma\|_1, \forall \|\Gamma\|_2 \leq 1, \quad (4.2)
\]

\[
\langle \nabla_{\theta} G_T(\theta + \Gamma) - \nabla_{\theta} G_T(\theta), \Gamma \rangle \geq \alpha_2 \|\Gamma\|_2 - \tau_2 \sqrt{\log \frac{d}{T} \|\Gamma\|_1}, \forall \|\Gamma\|_2 \geq 1. \quad (4.3)
\]

The following proposition provides the conditions for which \( G_T(.) \) satisfies the RSC proposition with respect to \( \text{vec}(A_q) \) with high probability.

**Proposition 4.1.** Let \( a = \min\{\frac{1}{54L^2} \lambda_{\min}(H_X), 1\} \), \( b = \min\{a^2, k_\beta\} \) and \( H_X = \mathbb{E}[X_{q,t-1} X_{q,t-1}' \otimes I_m] \). Under assumptions \( \square \), for a sample size \( T \geq \{\frac{\log d}{b} \}^{\frac{1}{\kappa-1} + \frac{1}{2}} \), where \( d = m(1 + qm) \), with probability at least

\[
1 - 6 \exp(-a^2 T^{1-\kappa}) - 2c(T^{(1-\kappa)} - 1) \exp(-k_\beta T^\kappa),
\]

with \( \kappa \in (0, 1) \), the unpenalized least squares loss function \( (2.7) \) satisfies the restricted
strong convexity condition

\[ \forall \mathbf{v} \in \mathbb{R}^d, \mathbf{v}' \left\{ \frac{1}{T} \sum_{t=1}^{T} (X_{q,t-1}X'_{q,t-1} \otimes I_m) \right\} \mathbf{v} \geq \alpha \| \mathbf{v} \|^2 - \tau \| \mathbf{v} \|^2_1, \]

with parameters \( \alpha = \frac{1}{2} \lambda_{\min}(H_X) \) and \( \tau = \frac{27aL^4 \log d}{bT^{1-\kappa T}}. \)

For \( \kappa \to 0 \), then (4.2) is satisfied with \( \alpha_1 = \alpha \) and \( \tau_1 = \tau \). Thus by Lemma 9 of Loh and Wainwright (2014), (4.3) is also satisfied with \( \alpha_2 = \alpha_1 \) and \( \tau_2 = \tau_1 \).

**Proof.** The loss function is \( G_T(A_q) = \frac{1}{2T} \sum_{t=1}^{T} (Y_t - A_q X_{q,t-1})'(Y_t - A_q X_{q,t-1}) \). Using the matrix derivatives formula of Abadir and Magnus (2005), we obtain for the Jacobian

\[ \nabla_{\text{vec}(A_q)} G_T(A_q) = -\frac{1}{T} \sum_{t=1}^{T} (X_{q,t-1} \otimes Y_t - A_q X_{q,t-1}). \]

As for the Hessian, we aim at extracting the form \( \text{tr}(L(d\Lambda)M(d\Lambda)) \) for \( L \) (resp. \( M \)) any square \( m \times m \) matrix (resp. \( p \times p \)). We thus obtain

\[ \nabla^2_{\text{vec}(A_q)\text{vec}(A_q')} G_T(A_q) = \frac{1}{T} \sum_{t=1}^{T} (X_{q,t-1}X'_{q,t-1} \otimes I_m). \]

As a consequence, for any \( \Gamma \in \mathcal{M}_{m \times 1 + qm}(\mathbb{R}) \) so that \( \| \Gamma \|_F \leq 1 \), with \( \Gamma = (\Gamma_1, \cdots, \Gamma_{q+1}) \), where \( \forall k \geq 2, \Gamma_k \) satisfies the positivity condition (2.5), and \( \Gamma_1 = \text{vech}(\Pi) \), where \( \Pi \in \mathcal{M}_{p \times p}(\mathbb{R}) \) so that \( \Pi \succ 0 \), then for \( \text{vec}(\Gamma) \in \mathbb{R}^d \), we have

\[ \langle \nabla_{\text{vec}(A_q)} G_T(A_{0,q} + \Gamma) - \nabla_{\text{vec}(A_q)} G_T(A_{0,q}), \text{vec}(\Gamma) \rangle = \text{vec}(\Gamma)' \{ \nabla^2_{\text{vec}(A_q)\text{vec}(A_q')} G_T(A_{0,q} + \Gamma) \} \text{vec}(\Gamma). \]

We now shall provide a concentration bound to control for \( \frac{1}{T} \sum_{t=1}^{T} (X_{q,t-1}X'_{q,t-1} \otimes I_m) \).

We denote \( \text{vec}(\Gamma) = \mathbf{v} \). The quantity \( \mathbf{v}'(X_{q,t-1}X'_{q,t-1} \otimes I_m)\mathbf{v}, \) with \( \mathbf{v} \in \mathbb{R}^d \) is a combination of a degree 4 polynomial and degree 2 polynomial of sub-Gaussian variables. Under the strict stationarity condition, they have the same distribution. We thus can apply Lemma 3.3.

To do so, we define the real valued sequence \( f(\epsilon_1, \cdots, \epsilon_T) = \sum_{t=1}^{T} \epsilon_t' Z_t \mathbf{v}, \) with \( Z_t = X_{q,t-1}X'_{q,t-1} \otimes I_m \). It is a strictly stationary sequence and the \( \beta \)-mixing rate of \( (f(\epsilon_1, \cdots, \epsilon_T)) \)
is the same as the process \((\epsilon_t)\). More precisely, this combination of a polynomial of degree 4 and a polynomial of degree 2 corresponds to a sum containing cross products of the form 
\[ \epsilon_{t-k,a}\epsilon_{t-l,b} + \epsilon_{t-m,c}\epsilon_{t-m,d} \]
with \(1 \leq k \leq l \leq q, 1 \leq m \leq n \leq q\) and \(a, b, c, d, e, f \in \{1, \ldots, p\}\). Using the \(Z_{t-1}\) matrix, we have
\[
f(\epsilon_1, \ldots, \epsilon_T) = \sum_{t=1}^{T} \sum_{i,j} v_i Z_{i,j,t} v_j.
\]

Let \(\eta = aL^4\), then by Lemma 3.3 we obtain
\[
\Pr \left( \frac{1}{T} f(\epsilon_1, \ldots, \epsilon_T) - \mathbb{E}[f(\epsilon_1, \ldots, \epsilon_T)] > aL^4 \right) \leq 6 \exp(-\mu_T a^2) + 2c(\mu_T - 1) \exp(-k_T s_T),
\]
where we used the same upper bound for both polynomials since \(\|\Gamma\|_F \leq 1\). Using Lemma F.2 of Basu and Michailidis (2015), we can take the previous concentration inequality over all vectors belonging to \(\mathcal{M}(w)\), the set of \(w\) sparse vectors of unit norm, where
\[
\mathcal{M}(w) = \bigcup_{|V| \leq w} \mathcal{S}_V \text{ with } \mathcal{S}_V = \{ u \in \mathbb{R}^d : \|u\|_2 \leq 1, \text{supp}(u) \subseteq V \} \text{ and } V \subset \{1, \ldots, d\}.
\]

Denoting \(v = \text{vec}(\Gamma)\), we have
\[
\Pr \left( \sup_{v \in \mathcal{M}(w)} \frac{1}{T} \sum_{t=1}^{T} Z_t - \mathbb{E}[Z_t] > aL^4 \right) \leq 6 \exp(-\mu_T a^2 + w \log d) + 2c(\mu_T - 1) \exp(-k_T s_T + w \log d).
\]

This concentration inequality is equivalent to
\[
\Pr \left( \sup_{v \in \mathcal{M}(w)} v' \left( \frac{1}{T} \sum_{t=1}^{T} (X_{q,t-1}X_{q,t-1}' \otimes I_m) - \mathbb{E}[(X_{q,t-1}X_{q,t-1}' \otimes I_m)] \right) v \leq aL^4 \right) \geq 1 - 6 \exp(-\mu_T a^2 + w \log d) + 2c(\mu_T - 1) \exp(-k_T s_T + w \log d).
\]

We shall extend the concentration to all vectors \(v = \text{vec}(\Gamma) \in \mathbb{R}^d\). To do so, by Lemma 12 of Loh and Wainwright (2012), for a parameter \(w \geq 1\), we have
\[
v' \left( \frac{1}{T} \sum_{t=1}^{T} (X_{q,t-1}X_{q,t-1}' \otimes I_m) - \mathbb{E}[(X_{q,t-1}X_{q,t-1}' \otimes I_m)] \right) v \leq 27aL^4[\|v\|_2^2 + \frac{1}{w}\|v\|_2^2],
\]
with probability at least
\[
1 - 6 \exp(-\mu_T a^2 + w \log d) - 2c(\mu_T - 1) \exp(-k_T s_T + w \log d).
\]
Hence, we have
\[
\frac{1}{T} \sum_{t=1}^{T} (X_{q,t-1} X'_{q,t-1} \otimes I_m) v \geq (\lambda_{\min}(H_X) - 27aL^{4}) \|v\|_2^2 - \frac{27aL^{4}}{\log d} \|v\|_1^2.
\]

We now pick \( w = \frac{1}{\log d} \{\mu_T a^2 \land k_\beta s_T \} \). Let \( 0 < \kappa < 1 \) so that we set \( s_T = T^\kappa \) and \( \mu_T = T^{1-\kappa} \). Thus \( w = \frac{b}{\log d} \{T^{(1-\kappa)} \land T^\kappa \} \), with \( b = \min\{a^2, k_\beta\} \) and so that we choose \( T \geq \{ \frac{\log d}{c} \} (\frac{1}{\log d} \wedge \frac{2}{a^2}) \). Then with probability at least \( 1 - 6 \exp(-T^{(1-\kappa)}a^2) - 2c(T^{1-\kappa} - 1) \exp(-k_\beta T^\kappa) \), we obtain
\[
\v' \left\{ \frac{1}{T} \sum_{t=1}^{T} (X_{q,t-1} X'_{q,t-1} \otimes I_m) \right\} v \geq (\lambda_{\min}(H_X) - 27aL^{4}) \|v\|_2^2 - \frac{27aL^{4} \log d}{b \{T^{(1-\kappa)} \land T^\kappa \}} \|v\|_1^2.
\]

Let \( a = \frac{1}{54k} \lambda_{\min}(H_X) \land 1 \) so that \( \lambda_{\min}(H_X) - 27aL^{4} \geq \frac{1}{2} \lambda_{\min}(H_X) \). Hence for a sample size \( T \geq \{ \frac{\log d}{b} \} (\frac{1}{\log d} \wedge \frac{1}{a^2}) \), we obtain
\[
\v' \left\{ \frac{1}{T} \sum_{t=1}^{T} (X_{q,t-1} X'_{q,t-1} \otimes I_m) \right\} v \geq \frac{1}{2} \lambda_{\min}(H_X) \|v\|_2^2 - \frac{27aL^{4} \log d}{b \{T^{(1-\kappa)} \land T^\kappa \}} \|v\|_1^2,
\]

with probability at least \( 1 - 6 \exp(-T^{(1-\kappa)}a^2) - 2c(T^{1-\kappa} - 1) \exp(-k_\beta T^\kappa) \). Since \( \kappa \) can arbitrarily be taken close to 0, this inequality holds for all \( v \in \mathbb{R}^d \) whenever \( T \geq \log d \). Hence by Lemma 9 of Loh and Wainwright (2014), the RSC condition
\[
\v' \left\{ \frac{1}{T} \sum_{t=1}^{T} (X_{q,t-1} X'_{q,t-1} \otimes I_m) \right\} v \geq \frac{1}{2} \lambda_{\min}(H_X) \|v\|_2^2 - 27aL^4 \left( \frac{\log d}{\log T} \right) \|v\|_1^2;
\]
holds with the same probability.

\[\Box\]

### 4.2 Error bound

We first remind of Theorem 1 of Loh and Wainwright (2014) stated for \( n \)-independent observations, we thus keep the same notations as these authors. It provides error bounds over the penalized parameters, assuming that the loss function satisfies the RSC condition and the penalty is \( \mu \)-amenable. To do so, we denote by \( \mathbb{L}_n(.) \) a generic empirical loss function so that the population risk function is defined as \( \mathbb{L}(\theta) = \mathbb{E}[\mathbb{L}_n(\theta)] \), which is uniquely minimized at \( \theta_0 \) and independent of the sample size \( n \). The regularized estimator
thus satisfies
\[
\hat{\theta} = \arg \min_{\theta : g(\theta) \leq R, \theta \in \Theta} \{L_n(\theta) + p(\lambda_n, \theta)\},
\]  
(4.4)

where \( R > 0 \), \( g(\cdot) \) is convex with \( \|\theta_0\| \leq g(\theta_0) \) so that \( \theta_0 \) is a feasible point of the problem and \( \Theta \) is a convex set. Then we have the following Theorem.

**Theorem 4.2.** (Loh and Wainwright, 2014)

Suppose \( \theta \in \mathbb{R}^p \) and the objective function \( G_n(\cdot) : \mathbb{R}^p \rightarrow \mathbb{R} \) satisfies the RSC condition and \( p(\lambda_n, \cdot) \) is \( \mu \)-amenable, with \( \frac{3}{4} \mu < \alpha_1 \). Suppose the choice
\[
4 \max\{\|\nabla_\theta L_n(\theta_0)\|\infty, \sqrt{\frac{\log p}{n}}\} \leq \lambda_n \leq \frac{\alpha_2}{6R},
\]  
(4.5)

and suppose \( n \geq \frac{16R^2 \max\{\tau_1^2, \tau_2^2\}}{\alpha_2^2} \) log \( p \). Let \( \hat{\theta} \) be a stationary point of (4.4). Then \( \hat{\theta} \) satisfies
\[
\|\hat{\theta} - \theta\|_2 \leq \frac{6\lambda_n \sqrt{k}}{4\alpha_1 - 3\mu}, \quad \|\hat{\theta} - \theta\|_1 \leq \frac{24\lambda_n k}{4\alpha_1 - 3\mu}.
\]

The proof of this Theorem can be found in Loh and Wainwright (2014) or Loh (2017). Furthermore, the key quantity for consistency is the boundedness of \( \|\nabla_\theta L_n(\theta_0)\|\infty \). Corollary 4.4 is an application of Theorem 4.2. It requires to upper bound the score of the unpenalized loss function \( G_T(A_{0,q}) \). To do so, as a preliminary result, we need a deviation bound condition.

**Proposition 4.3.** Let \( d = m(1 + qm) \). Under assumptions 2-3, let \( 0 < \kappa < 1 \), for a sample size
\[
T \geq \max\left(\left\{\frac{\|A_{0,q}\|_F^2 \log d}{L^4}, \frac{(L^4 \log d)^{\frac{1}{\alpha_1}}}{\frac{1}{k_\beta} \log d}\right\}\right),
\]

then
\[
\mathbb{P}\left(\frac{1}{T} \max_{1 \leq i \leq T, 1 \leq j \leq m} \left\{\sum_{t=1}^{T} X_{q,t-1,i} \zeta_{t,j}\right\} \leq \Upsilon(\xi, \omega_0, A_{0,q}) \Xi(q, m, T) \right) \geq 1 - 12 \exp(-\log d) - 4c(T^{1-\kappa} - 1) \exp(-k_\beta T^\kappa),
\]

where
\[
\Upsilon(\xi, \omega_0, A_{0,q}) = \sqrt{4L^4 \left\{\|A_{0,q}\|_F^2 \vee 1\right\}}, \quad \Xi(q, m, T) = \sqrt{\frac{\log d}{T^{(1-\kappa)}}}.
\]
Remark. The parameter $\kappa$ can arbitrarily be set close to 0, so that we would obtain a rate similar to $\sqrt{\log \frac{d}{T}}$. In that case, if we consider the diagonal VEC-ARCH process (2.3), the number of parameters is $d = m(1 + q)$, so that the rate is $\sqrt{\log(m(1+q))}$.

Proof. By the orthogonality condition, we have

$$E[\sum_{t=1}^{T} (X_{q,t-1} \otimes Y_t - A_{0,q} X_{q,t-1})] := E[\sum_{t=1}^{T} (X_{q,t-1} \otimes \zeta_t)] = 0.$$ 

By definition, $\|\sum_{t=1}^{T} (X_{q,t-1} \otimes \zeta_t)\|_{\infty} = \max_{1 \leq i \leq d, 1 \leq j \leq m} \{|\sum_{t=1}^{T} X_{q,t-1,1} \zeta_{t,j}|\}$. We note that $\zeta_{t,j}$ corresponds to a degree 2 polynomial of sub-Gaussian components of the form

$$\zeta_{t,j} = (\epsilon_{t,k} \epsilon_{t,\ell}) j - (\sum_{k=1}^{m} A_{0,q,(k)} X_{q,t-1}) j,$$

with $A_{0,q,(k)}$ denoting the $k$-th line of $A_{0,q}$. For any $1 \leq i \leq 1 + qm$ and $1 \leq j \leq m$, using the orthogonality conditions, we have

$$|X_{q,t-1,i} \zeta_{t,j}| = |X_{q,t-1,i} \zeta_{t,j} - E[X_{q,t-1,i} \zeta_{t,j}]|$$

$$\leq |X_{q,t-1,i}(\epsilon_{t,k} \epsilon_{t,\ell})j - E[X_{q,t-1,i}(\epsilon_{t,k} \epsilon_{t,\ell})j]|$$

$$+ |X_{q,t-1,i}(\sum_{k=1}^{m} A_{0,q,(k)} X_{q,t-1}) j - E[X_{q,t-1,i}(\sum_{k=1}^{m} A_{0,q,(k)} X_{q,t-1}) j]|.$$

Thus, for $\eta > 0$, we obtain

$$P\left(\frac{1}{T} \sum_{t=1}^{T} X_{q,t-1,i} \zeta_{t,j} > 2\eta\right) \leq P\left(\frac{1}{T} \sum_{t=1}^{T} X_{q,t-1,i}(\epsilon_{t,k} \epsilon_{t,\ell})j - E[X_{q,t-1,i}(\epsilon_{t,k} \epsilon_{t,\ell})j] > \eta\right)$$

$$+ P\left(\frac{1}{T} \sum_{t=1}^{T} X_{q,t-1,i}(\sum_{k=1}^{m} A_{0,q,(k)} X_{q,t-1}) j - E[X_{q,t-1,i}(\sum_{k=1}^{m} A_{0,q,(k)} X_{q,t-1}) j] > \eta\right).$$

We thus can use Lemma 3.3 for each probability. The second probability entails an average sum of a degree 4 polynomial and a degree 2 polynomial of sub-Gaussian variables. Applying Lemma 3.3 and the remark following this Lemma, we obtain

$$P\left(\frac{1}{T} \sum_{t=1}^{T} X_{q,t-1,i}(\sum_{k=1}^{m} A_{0,q,(k)} X_{q,t-1}) j - E[X_{q,t-1,i}(\sum_{k=1}^{m} A_{0,q,(k)} X_{q,t-1}) j] > \eta\right)$$

$$\leq 6 \exp\left(-\eta^2 T^{(1-\kappa)} \frac{L^4 \|A_{0,q}\|_F^2}{L^4 \|A_{0,q}\|_F^2}ight) + 2c(T^{1-\kappa} - 1) \exp(-k_3 T^\kappa).$$

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We thus obtain

\[
P\left( \frac{1}{T} \sum_{t=1}^T X_{q,t-1} \zeta_{t,j} \mid > 2\eta \right) \leq 6 \exp\left(-\frac{\eta^2 T (1-\kappa)}{L^4} \right) + 6 \exp\left(-\frac{\eta^2 T (1-\kappa)}{L^4 \|A_{0,q}\|_F^2} \right) + 4c(T^{1-\kappa} - 1) \exp(-k_T \kappa)
\]

Taking an union bound over all \(i\), we obtain

\[
P\left( \max_{1 \leq i \leq d, 1 \leq j \leq m} \left\{ \sum_{t=1}^T X_{q,t-1,i} \zeta_{t,j} \right\} \mid > 2\eta \right) \leq d \left\{ 6 \exp\left(-\frac{\eta^2 T (1-\kappa)}{L^4} \right) + 6 \exp\left(-\frac{\eta^2 T (1-\kappa)}{L^4 \|A_{0,q}\|_F^2} \right) + 4c(T^{1-\kappa} - 1) \exp(-k_T \kappa) \right\}
\]

For a sample size \(T \geq \max((\|A_{0,q}\|_F^2 L^4 \log d)^{\frac{1}{1-\kappa}}, (L^4 \log d)^{\frac{1}{1-\kappa}}, (\frac{1}{k^2} \log d)^{\frac{1}{2}})\). If we take \(\eta = \sqrt{L^4 \log d \|A_{0,q}\|_F^2 (1-\kappa)}\), then

\[
P\left( \frac{1}{T} \sum_{t=1}^T X_{q,t-1,i} \zeta_{t,j} \mid > \sqrt{4 L^4 \log d \|A_{0,q}\|_F^2 (1-\kappa}) \right) \leq 12 \exp(- \log d) + 4c(T^{1-\kappa} - 1) \exp(-k_T \kappa).
\]

Equipped with this deviation bound, we can provide a finite sample error bound that holds with high probability for any stationary point \(\hat{A}_q\).

**Corollary 4.4.** Let \(d = m(1 + qm)\). Suppose the regularizer is \(\mu\)-amenable, under the conditions of Proposition 4.1, if the regularization parameter satisfies

\[
C_1 \sqrt{\frac{\log d}{T}} \leq \lambda_T \leq C_2 \frac{R}{T},
\]

with \(C_1, C_2 > 0\) so that \(C_1 > \sqrt{\|A_{0,q}\|_F^2 \sqrt{\log d}}\) \(\mid \sqrt{\frac{\log d}{T}}\); suppose \(\frac{3}{4} \mu < \alpha\) with \(\alpha = \frac{1}{2} \lambda_{\min}(H_X)\) provided in Proposition 4.1, then with a sample size \(T > Ck \log d\) for a sufficiently large \(C > 0\), any local optimum \(\hat{A}_q\) of the nonconvex program (4.1) satisfies

\[
\|\hat{A}_q - A_{0,q}\|_F \leq \frac{c \lambda_T \sqrt{k}}{4\alpha - 3\mu},
\]

(4.6)
with probability at least \(1 - 12 \exp(-\log d) - 4c(T - 1) \exp(-k\beta)\), where \(c > 0\), \(k = |S|\) and \(S = \{x := (i, j) : A_{0,x} \neq 0, 1 \leq i \leq m, 1 \leq j \leq 1 + qm\} \).

**Remark.** Corollary 4.4 is a consequence of Theorem 4.2 where the regularization procedure aims at estimating a sparse vector \(\theta_0\). We note that the squared bound (4.6) depend on the true sparse support \(k\). The regularization rate of \(\lambda_T\) proportional to \(\sqrt{\log d_T}\) implies that the squared norm (4.6) scales at the rate \(k\log d_T\). Moreover, although the unpenalized loss function \(G_T(.)\) is convex with respect to the parameters, the overall criterion might not be convex when the penalty function is the SCAD or MCP.

**Proof.** Under Proposition 4.1, the loss function satisfies the RSC condition. We need to prove that the proposed choice of \(\lambda_T\) satisfies (4.5). To do so, the score is

\[
\nabla_{\text{vec}(A_q)} G_T(A_{0,q}) = -\frac{1}{T} \sum_{t=1}^{T} (X_{q,t-1} \otimes Y_t - A_q X_{q,t-1}).
\]

Taking the infimum norm \(\|\cdot\|_\infty\), we obtain

\[
\|\nabla_{\text{vec}(A_q)} G_T(A_{0,q})\|_\infty = \frac{1}{T} \sum_{t=1}^{T} \|X_{q,t-1} \otimes \zeta_t\|_\infty \leq \frac{1}{T} \max_{1 \leq i \leq d, 1 \leq j \leq m} \{\|\sum_{t=1}^{T} X_{q,t-1,i} \zeta_t,j\| \leq \Upsilon(\xi, A_{0,q})\Xi(q, m, T),
\]

with probability at least \(1 - 12 \exp(-\log d) - 4c(T^{1-\kappa} - 1) \exp(-k\beta T^\kappa)\) by proposition 4.3. Consequently, for \(T\) large enough, for \(\kappa \to 0\), we have

\[
\|\nabla_{\text{vec}(A_q)} G_T(A_{0,q})\|_\infty \leq C \sqrt{\|A_{0,q}\|_F^2} \vee 1 \sqrt{\frac{\log d}{T}}.
\]

\(\square\)

### 4.3 Support recovery

Based on the Karush-Kuhn-Tucker optimality conditions, Wainwright (2009) developed the primal dual witness (PDW) approach to derive selection consistency for convex problems. There exist similar approaches in Zhao and Yu (2006), Candes and Plan (2009). The PDW approach consists in plugging the true subset model \(S\) in the KKT optimality conditions, which are necessary and sufficient if the problem is convex, and checking if they can be satisfied. It means that any solution of the non restricted problem (the original problem providing \(S\)) is also a solution to the restricted problem (the regularized
one). Loh and Wainwright (2017) show that this approach can be extended to nonconvex problems. The PDW construction is reported in Appendix 6.2.

Loh and Wainwright’s main result (2017) concerns the support recovery and \( l_\infty \)-bounds. To obtain the support recovery property, the RSC condition of the loss function with parameters \((\alpha_k, \tau_k)_{k=1,2}\) and the \( \mu \)-amenability of the regularizer are key points. Then, their result ensures the model selection property if two conditions are satisfied: one condition on both regularization parameters \( \lambda_T \) and \( R \); one condition about strict dual feasibility. Both support recovery and error bound results are reported in the appendix.

Based on this framework, we provide \( l_\infty \)-guarantees for the regularized VEC-ARCH(q) estimator together with the support recovery conditions. More precisely, we show that any local/global optimum of (4.1) corresponds to the oracle estimator, which is defined as

\[
\hat{A}^O_{S,q} = \arg \min_{A_{q,S} \in K, \text{ supp}(A_q) = S} \{ G_T((A_{q,S}, 0_{S^c})) \},
\]

with \( \text{vec}(\hat{A}^O_{q}) = (\text{vec}(\hat{A}^O_{S,q}), \text{vec}(0_{S^c})) \), where \( 0 \) is the zero matrix.

**Corollary 4.5.** Let \( d = m(1 + qm) \). Under assumptions 1-3 under the sample size \( T \geq Ck \log d \) with \( C > 0 \) a sufficiently large constant, under the RSC condition provided in Proposition 4.1 for \( \kappa \to 0 \), suppose the regularization parameters \((\lambda_T, R)\) are chosen so that \( \|\text{vec}(A_{0,q})\|_1 \leq \frac{R}{2} \) and \( C(\omega_0, A_{0,q}) \sqrt{\frac{\log d}{T}} \leq \lambda_T \leq \frac{\overline{C}}{R} \), with \( C(\omega_0, A_{0,q}) \) a constant depending on the true parameters and \( \overline{C} \) another constant. Let \( \Upsilon(\epsilon, \omega_0, A_{0,q}) = \sqrt{4L^4 \|A_{0,q}\|_F^2 \vee 1} \). Under the condition

\[
\|\left\{ \frac{1}{T} \sum_{t=1}^{T} (X_{q,t-1}X'_{q,t-1} \otimes I_m) \right\}^{-1}\|_\infty \leq \alpha_\infty,
\]

(i) Suppose \( p(\lambda_T, \cdot) \) is \( \mu \)-amenable with \( \mu \leq \lambda_{\min}(HX) \) and the incoherence condition is satisfied so that

\[
\|\left\{ \frac{1}{T} \sum_{t=1}^{T} (X_{q,t-1}X'_{q,t-1} \otimes I_m) \right\}_{S \cap S^c} \|_{S \cup S^c} \leq \tau < 1.
\]

Then with probability \( 1 - 12 \exp(-\log d) - 4c(T-1) \exp(-k_\beta) \), the objective function (4.1) admits a unique optimum so that \( \hat{S} \subseteq S \) and

\[
\|\hat{A}_q - A_{0,q}\|_{\max} \leq C \Upsilon(\epsilon, \omega_0, A_{0,q}) \sqrt{\frac{\log d}{T}} + \lambda_T \alpha_\infty.
\]
(ii) Suppose \( p(\lambda_T, \ldots) \) is \((\mu, \gamma)\)-amenable with \( \mu \leq \lambda_{\text{min}}(H_X) \) and

\[
\min_{i \in S}|\text{vec}(A_{0,q})_i| \geq \lambda_T(\gamma + \alpha_\infty) + C\gamma(\xi, \omega_0, A_{0,q})\sqrt{\frac{\log d}{T}},
\]

then with probability \( 1 - 12 \exp(-\log d) - 4c(T - 1)\exp(-k_\beta) \), the objective function (4.1) admits a unique minimum so that \( \hat{S} \subseteq S \), \( \hat{\mathcal{A}}_q \) is given by \( \hat{\mathcal{A}}^O \) the oracle estimator defined in (4.7) and

\[
\|\hat{\mathcal{A}}_q - A_{0,q}\|_{\max} \leq C\gamma(\xi, \omega_0, A_{0,q})\sqrt{\frac{\log d}{T}}.
\]

Remark. This result emphasizes the statistical gain when using the SCAD or MCP regularization method compared to the Lasso: the upper bound in (ii) is sharper than the one of part (i), which requires the incoherence condition. The Lasso can recover the true underlying sparse model only if the incoherence condition is satisfied: see Wainwright (2007, 2009). A regularization method based on a non-convex regularizer that is \((\mu, \gamma)\)-amenable ensures the support recovery property even when the incoherence condition is not satisfied.

Proof. Proof of (i). By the zero gradient condition (6.2), we obtain

\[
\nabla_{\text{vec}(A_q)}G_T \hat{\mathcal{A}}_q - \nabla_{\text{vec}(A_q)}G_T(A_{0,q}) + \nabla_{\text{vec}(A_q)}G_T(A_{0,q}) - \nabla q(\lambda_T, \text{vec}(\hat{\mathcal{A}}_q)) + \lambda_T \hat{z} = 0,
\]

where \( q(\lambda_T, \ldots) \) is defined in assumption [4]. This implies

\[
\hat{K} \text{vec}(\hat{\mathcal{A}}_q - A_{0,q}) + \nabla_{\text{vec}(A_q)}G_T(A_{0,q}) - \nabla q(\lambda_T, \text{vec}(\hat{\mathcal{A}}_q)) + \lambda_T \hat{z} = 0,
\]

with \( \hat{K} = \int_0^1 \nabla^2_{\text{vec}(A_q)\text{vec}(A_q)}G_T(A_{0,q} + u(\hat{\mathcal{A}}_q - A_{0,q}))/du \). Equivalently, we have

\[
\begin{pmatrix}
\hat{K}_{SS} & \hat{K}_{SSc} \\
\hat{K}_{ScS} & \hat{K}_{ScSc}
\end{pmatrix}
\begin{pmatrix}
\text{vec}(\hat{\mathcal{A}}_q - A_{0,q})_S \\
0
\end{pmatrix}
+ \begin{pmatrix}
\nabla_{\text{vec}(A_q)}G_T(A_{0,q})_S - \nabla q(\lambda_T, \text{vec}(\hat{\mathcal{A}}_q))_S \\
\nabla_{\text{vec}(A_q)}G_T(A_{0,q})_Sc - \nabla q(\lambda_T, \text{vec}(\hat{\mathcal{A}}_q))_Sc
\end{pmatrix}
+ \lambda_T \begin{pmatrix}
\hat{z}_S \\
\hat{z}_Sc
\end{pmatrix}
= 0.
\]

Consequently, we obtain

\[
\hat{z}_Sc = \frac{1}{\lambda_T} \left\{ \nabla q(\lambda_T, \text{vec}(\hat{\mathcal{A}}_q))_Sc - \nabla_{\text{vec}(A_q)}G_T(A_{0,q})_Sc + \hat{K}_{ScS}^{-1} \hat{K}_{SS} \nabla_{\text{vec}(A_q)}G_T(A_{0,q})_S + \lambda_T \hat{z}_S \right\}.
\]
Using the regularity condition (v), \( \nabla q(\lambda_T, vec(\hat{A}_q))_{\mathcal{S}^c} = \nabla q(\lambda_T, 0)_{\mathcal{S}^c} = 0_{\mathcal{S}^c} \). This implies
\[
\hat{z}_{\mathcal{S}^c} = \frac{1}{\lambda_T} \{- \nabla_{vec(A_q)} G_T(A_{0,q})_{\mathcal{S}^c} + \hat{K}_{s^c \mathcal{S}} \hat{K}_{s^c \mathcal{S}}^{-1} (\nabla_{vec(A_q)} G_T(A_{0,q})_{\mathcal{S}^c} - \nabla q(\lambda_T, vec(\hat{A}_q))_{\mathcal{S}} + \lambda_T \hat{z}_{\mathcal{S}})\}.
\]

More explicitly, with \( \hat{K} = \frac{1}{T} \sum_{t=1}^{T} (X_{q,t-1} X'_{q,t-1} \otimes I_m) \), since the true support is known with respect to \( A_{0,q} \), this relationship corresponds to
\[
\hat{z}_{\mathcal{S}^c} = \frac{1}{\lambda_T} \{- \hat{K}_{s^c \mathcal{S}} vec(A_{0,q})_{\mathcal{S}} + \frac{1}{T} \sum_{t=1}^{T} (X_{q,t-1} \otimes Y_t)_{\mathcal{S}^c} + \hat{K}_{s^c \mathcal{S}} \hat{K}_{s^c \mathcal{S}}^{-1} (\hat{K}_{s^c \mathcal{S}} vec(A_{0,q})_{\mathcal{S}}) - \frac{1}{T} \sum_{t=1}^{T} (X_{q,t-1} \otimes Y_t)_{\mathcal{S}^c} \} \]
\[
\leq \frac{1}{\lambda_T} \{- \hat{K}_{s^c \mathcal{S}} vec(A_{0,q})_{\mathcal{S}} + \frac{1}{T} \sum_{t=1}^{T} (X_{q,t-1} \otimes Y_t)_{\mathcal{S}^c} + \hat{K}_{s^c \mathcal{S}} vec(A_{0,q})_{\mathcal{S}} \}
\]
\[
- \hat{K}_{s^c \mathcal{S}} \hat{K}_{s^c \mathcal{S}}^{-1} \frac{1}{T} \sum_{t=1}^{T} (X_{q,t-1} \otimes Y_t)_{\mathcal{S}^c} \} + \| \hat{K}_{s^c \mathcal{S}} \hat{K}_{s^c \mathcal{S}}^{-1} \|_\infty \]
\[
\leq \frac{1}{\lambda_T} \{- \frac{1}{T} \sum_{t=1}^{T} (X_{q,t-1} \otimes Y_t)_{\mathcal{S}^c} + \hat{K}_{s^c \mathcal{S}} \hat{K}_{s^c \mathcal{S}}^{-1} \frac{1}{T} \sum_{t=1}^{T} (X_{q,t-1} \otimes Y_t)_{\mathcal{S}^c} \} + \| \hat{K}_{s^c \mathcal{S}} \hat{K}_{s^c \mathcal{S}}^{-1} \|_\infty ,
\]
where we used Lemma 8 of Loh and Wainwright (2017) to obtain the upper bound
\[
\| \lambda_T \hat{z}_{\mathcal{S}} - \nabla q(\lambda_T, vec(\hat{A}_q))_{\mathcal{S}} \|_\infty = \| \nabla p(\lambda_T, vec(\hat{A}_q))_{\mathcal{S}} \|_\infty \leq \lambda_T .
\]
Moreover, we have

\[
\frac{1}{T} \sum_{t=1}^{T} \| (X_{q,t-1} \otimes Y_t)_S - \hat{K}_{S'} S \hat{K}_{SS}^{-1} S \sum_{t=1}^{T} (X_{q,t-1} \otimes Y_t) \|_\infty
\]

\[
= \frac{1}{T} \sum_{t=1}^{T} \| (X_{q,t-1} X'_{q,t-1} \otimes I_m)_{S'} S vec(A_{0,q})_S + \frac{1}{T} \sum_{t=1}^{T} (X_{q,t-1} \otimes \zeta_t)_S \|
\]

\[
= \frac{1}{T} \sum_{t=1}^{T} (X_{q,t-1} \otimes \zeta_t)_S - \hat{K}_{S'} S \hat{K}_{SS}^{-1} S \sum_{t=1}^{T} (X_{q,t-1} \otimes \zeta_t)_S \|_\infty
\]

\[
\leq \max_{i \in S' \setminus j} \frac{1}{T} \sum_{t=1}^{T} |X_{q,t-1,i} \zeta_{t,j}| + \| \hat{K}_{S'} S \hat{K}_{SS}^{-1} S \|_\infty \max_{i \in S' \setminus j} \frac{1}{T} \sum_{t=1}^{T} |X_{q,t-1,i} \zeta_{t,j}| 
\]

\[
\leq c_1 \Upsilon(\xi, \omega_0, A_{0,q}) \sqrt{\frac{\log d}{T}} + \tau c_1 \Upsilon(\xi, \omega_0, A_{0,q}) \sqrt{\frac{\log d}{T}},
\]

with probability \( 1 - 12 \exp(-\log d) - 4c(T - 1) \exp(-k\beta) \). Thus

\[
\| \hat{z}_S \|_\infty \leq \frac{1}{\lambda_T} (c_1 \Upsilon(\xi, \omega_0, A_{0,q}) \sqrt{\frac{\log d}{T}} + \tau \{ c_1 \Upsilon(\xi, \omega_0, A_{0,q}) \sqrt{\frac{\log d}{T}} \}) + \tau.
\]

Consequently, strict dual feasibility of Theorem 6.1 of Loh and Wainwright is satisfied when

\[
\lambda_T \geq \frac{1}{1 - \tau} (c_1 \Upsilon(\xi, \omega_0, A_{0,q}) \sqrt{\frac{\log d}{T}} + \tau \{ c_1 \Upsilon(\xi, \omega_0, A_{0,q}) \sqrt{\frac{\log d}{T}} \}).
\]

We can thus select \( \lambda_T \geq C(\omega_0, A_{0,q}) \sqrt{\frac{\log d}{T}} \). Furthermore, we have

\[
\| \hat{K}_{SS}^{-1} \nabla_{vec(A_q) \otimes T} (A_{0,q}) \|_\infty = \| - \hat{K}_{SS}^{-1} \sum_{t=1}^{T} (X_{q,t-1} \otimes \zeta_t)_S \|_\infty
\]

\[
\leq \| \hat{K}_{SS}^{-1} \|_\infty \max_{i \in S' \setminus j} \frac{1}{T} \sum_{t=1}^{T} |X_{q,t-1,i} \zeta_{t,j}|.
\]
Consequently, by part (i) of Theorem 6.2 provided in Appendix 6.2, we obtain
\[ \| \hat{A}_q - A_{0,q} \|_{\text{max}} \leq C \Upsilon(\epsilon,\omega_0, A_{0,q}) \sqrt{\frac{\log d}{T}} + \lambda_T \alpha_\infty. \]

Proof of (ii). The same approach as in the proof of (i) can be applied. Since the regularizer is supposed to be \((\mu, \gamma)\)-amenable, we have by Lemma 5 of Loh and Wainwright (2017) than
\[ \lambda_T \hat{z}_S - \nabla_{\text{vec}(A_q)} q(\lambda_T, \text{vec}(\hat{A}_q)_S) = 0. \]
Hence we have
\[ \| \hat{z}_S \|_{\infty} \leq \frac{1}{\lambda_T} \left( c_1 \Upsilon(\epsilon,\omega_0, A_{0,q}) \sqrt{\frac{\log d}{T}} + \tau \{ c_1 \Upsilon(\epsilon,\omega_0, A_{0,q}) \} \right), \]
with high probability. Then the remainder follows from part (ii) of Theorem 6.2.

5 Empirical applications

In this section, we carry out a simulation study to verify the theoretical results on the regularized VEC-ARCH(q) estimator.

Optimization procedure. We consider the penalized loss function
\[
\begin{cases}
\hat{A}_q = \arg \min_{A_q \in \mathcal{K}} \{ \frac{1}{2T} \sum_{t=1}^{T} \| Y_t - A_q X_{q,t-1} \|_2^2 + p(\lambda_T, \text{vec}(A_q)) \}, \\
\mathcal{K} = \{ A_q : g(\text{vec}(A_q)) \leq R; \Omega > 0, \forall \Sigma(A_l) \succeq 0 \},
\end{cases}
\]
where the non-convexity can potentially come from the regularizer, under the convex constrained parameter set \( \mathcal{K} \). We set \( g(\text{vec}(A_q)) = \| \text{vec}(A_q) \|_1 \). To solve this optimization problem, we follow the composite gradient descent procedure of Loh and Wainwright (see their section 4, 2014), which consists in a three step updating procedure of the optimized parameter value. For the SCAD (resp. MCP), we set \( a = 3.7 \) (resp. \( b = 3.5 \)) following Loh and Wainwright (2014).

Regularization parameters. Following Loh and Wainwright (2014,2017), we select \( R = 1.1 \{ p(\lambda_T, \text{vec}(A_{0,q})) + \mu \| \text{vec}(A_{0,q}) \|_2^2 / 2 \} / \lambda_T \) so that \( A_{0,q} \) is feasible. If the true support were unknown, we would have had to perform a cross-validation method on
Furthermore, we set \( \lambda_T = c \sqrt{\frac{\log d}{T}} \) with \( c \) a grid of ten values within \([1, 2]\) so that we perform a cross-validation procedure to choose the optimal \( \lambda_T \). The standard cross-validation developed for i.i.d. data can not directly be used in our time series framework. To fix this issue, we use the hv-cross-validation procedure devised by Racine (2000), which consists in leaving a gap between the test sample and the training sample, on both sides of the test sample.

**Simulation experiment.** We simulate the stochastic \( p \)-vectorial process \( (\epsilon_t) \) based on the following data generating process

\[
\begin{align*}
\epsilon_t &= H_t^{1/2} \eta_t, \\
vech(H_t) &= \omega + \sum_{k=1}^{q^*} A_k vech(\epsilon_{t-k} \epsilon_{t-k}'),
\end{align*}
\]

where \( \omega = vech(\Omega) \) and the \( A_k \) matrices satisfy the conditions of Theorem 3.1 and the positivity condition through \( \Sigma(A_k) \succeq 0 \) in (2.5). We initialize the observations \((\epsilon_{q^*}, \cdots, \epsilon_1)\) with centered and unit variance multivariate Gaussian distribution. Then conditionally on the past \( q^* \) observations, we generate \((H_t)\) and thus \( \epsilon_t \) according to a centered multivariate Gaussian distribution with variance covariance \( (H_t) \). The matrix \( \mathcal{A}_q = (\omega, A_1, \cdots, A_{q^*}) \) is \( k \)-sparse, where \( k \) is arbitrarily set. As for the non-zero components, we generate the diagonal elements of \( \Sigma(A_k) \) from a uniform distribution \( U([0.05, 0.15]) \) and the off-diagonal ones from \( U([-0.03, 0.03]) \) under the ordering constraint \( \forall k \geq 2, \forall i, j, |\Sigma(A_{k,ij})| \leq |\Sigma(A_{k-1,ij})| \).

As for the matrix \( \Omega \), the diagonal elements are simulated in \( U([0.1; 0.2]) \) and the off-diagonals components in \( U([-0.01, 0.01]) \).

We consider the problem sizes \( p = 4 \) (resp. \( p = 6 \)) and \( q^* = 2 \) so that the total number of parameters is \( d = m(1 + qm) = 210 \) (resp. \( d = 903 \)). The total number of non-zero entries, that is \( |\mathcal{S}| \), is defined as \( k = 104 \) for \( p = 4 \): 100 zeros among \((A_1, \cdots, A_q)\) and 4 zeros among \( \omega \). As for \( p = 6 \), \( k = 450 \): 440 zeros among \((A_1, \cdots, A_q)\) and 10 in \( \omega \). To obtain an estimate of \( \mathcal{S} \), we consider the Lasso, the SCAD and the MCP penalties, which are \( \mu \)-amenable. We report the \( \|\cdot\|_F \)-statistical consistency in Figure 1. The \( \|\cdot\|_\infty \)-consistency is reported in Figure 2 even though the design matrix is not incoherent as in (4.8). For both Figures, each point represents an average of 200 trials. The error curved lines for each regularized estimators decrease to zero as the number of samples increases, which supports that the regularizations are consistent. The Lasso is also reported for the \( \|\cdot\|_\infty \) error when the design matrix is not incoherent: since the Lasso is not a \((\mu, \gamma)\)-amenable regularizer, the estimation error is more significant compared to the SCAD and
**Real data experiment.** To assess the relevance of the proposed penalized method, we propose a real data experiment. To do so, we compare the forecasting performances of the covariance matrices $H_t$ for a portfolio of daily financial returns composed with the FTSE 100, the CAC 40, the DAX 30 and the Nikkei 225. We focus on direct out-of-sample evaluation methods, which allow for pairwise comparisons. They test whether some of the variance covariance models provide better forecasts in terms of portfolio volatility behavior. Following the methodology of Engle and Colacito (2006), we develop a mean-variance portfolio approach to test the $H_t$ forecasts. Intuitively, if a conditional covariance process is misspecified, then the minimum variance portfolio should emphasize such a shortcoming, compared to other models. Then, consider an investor who allocates a fixed amount between $p$ stocks, according to a minimum-variance strategy and independently at each time $t$. At each date $t$, he/she solves

$$
\min_{w_t} w_t' H_t w_t, \quad \text{s.t. } \iota' w_t = 1,
$$

where $w_t$ is the $p \times 1$ vector of portfolio weights chosen at (the end of) time $t-1$, $\iota$ is a $p \times 1$ vector of 1 and $H_t$ is the estimated conditional covariance matrix of the asset returns at time $t$. They are deduced from some dynamics that have been estimated on the sub-sample January 1994 - July 2008. Once the latter process is estimated in-sample, out-of-sample predictions are plugged into the program (5.1) between July 2008 and August 2010. The solution of (5.1) is given by the global minimum variance portfolio $w_t = H_t^{-1} \iota \iota' H_t^{-1} \iota$.

Engle and Colacito (2006) show that the realized portfolio volatility is the smallest one when the variance covariance matrices are correctly specified. As a consequence, if wealth is allocated using two different dynamic models $i$ and $j$, whose predicted covariance matrices are $(H^i_t)$ and $(H^j_t)$, the strategy providing the smallest portfolio variance will be considered as the best one. To do so, we consider a sequence of minimum variance portfolio weights $(w_{i,t})$ and $(w_{j,t})$, depending on the model. Then, we consider a distance based on the difference of the squared returns of the two portfolios, defined as $u_{ij,t} = \{w_{i,t}' \epsilon_t\}^2 - \{w_{j,t}' \epsilon_t\}^2$. The portfolio variances are the same if the predicted covariance matrices are the same. Thus we test the null hypothesis $H_0 : \mathbb{E}[u_{ij,t}] = 0$ by the Diebold and Mariano (1995) test. It consists of a least squares regression using HAC standard errors, given by $u_{ij,t} = \alpha + \epsilon_{u,t}, \mathbb{E}[\epsilon_{u,t}] = 0$, and we test $H_0 : \alpha = 0$. If the mean of $u_{ij,t}$ is significantly positive (resp. negative), then the forecasts given by the covariance matrices
of model $j$ (resp. $i$) are preferred.

We run the latter test to compare the scalar DCC, the scalar BEKK, the factor model O-GARCH(1,1), the MCP VEC-ARCH, the SCAD VEC-ARCH and the Lasso VEC-ARCH. The three first dynamics are described in Appendix 6.1. The three last processes are parameterized as diagonal VEC-ARCH($q$) processes as in (2.3) with $q = 10$ lags and a cross validation procedure is applied to select $\lambda_t$ and $R$. Moreover, the variance targeting procedure $\Omega = (I_p - A_1 - \cdots - A_q) \tilde{H}$ is applied, with $\tilde{H} = E[\epsilon_t \epsilon_t']$, which is a procedure commonly used in practice. Only $(A_1, \cdots, A_q)$ is thus penalized. The matrix forecast comparisons are provided in Table 1. We first note that all regularized VEC-ARCH models outperform both DCC and BEKK models. Moreover, no alternative models outperform the regularized VEC-ARCH. This supports the relevance of the proposed method. But all these results are not sufficiently clear-cut to draw any strong conclusion concerning a potential hierarchy between all these models, at least in terms of a “naive” investment strategy.

Table 1: Diebold Mariano Test of Multivariate GARCH models

|       | DCC   | O-GARCH | BEKK   | SCAD   | MCP    | Lasso   |
|-------|-------|---------|--------|--------|--------|---------|
| DCC   |       | 0.7137  | 1.1484 | 1.8889** | 1.9094** | 1.8211** |
| O-GARCH | -0.7137 |         | 0.1248 | 0.7975 | 0.8200 | 0.7918   |
| BEKK  | -1.1484 | -0.1248 |       | 1.4367* | 1.4212* | 1.4105*  |
| SCAD  | -1.8889** | -0.7975 | -1.4367* |       | 1.0712 | 0.7641   |
| MCP   | -1.9094** | -0.8200 | -1.4212* | -1.0712 |       | -0.0793  |
| Lasso | -1.8211** | -0.7918 | -1.4212* | -0.7641 | -0.0793 |         |

This table reports the out-of-sample t-statistics of the Diebold-Mariano test that checks the equality between covariance matrix forecasts using the loss function $u_{ij,t}$ over the period July 2008 and August 2010. This loss function is defined as the difference of squared realized returns of alternative MGARCH models. When the null hypothesis of equal predictive accuracy is rejected, a positive number is evidence in favor of the model in the column. *, **, ***: rejection of the null hypothesis at 10%, 5% and 1% respectively.

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6 Appendix

6.1 Some competing M-GARCH models

The DCC model specifies dynamics of the covariance matrix of the standardized returns $u_t = (u_{1,t}, \ldots, u_{p,t})$ with $u_{i,t} = \epsilon_{i,t}/\sqrt{h_{ii,t}}$. In its full form, called “Full DCC”, the model belongs to the MARCH family of Ding and Engle (2001) and is specified as

$$
\begin{align*}
\epsilon_t &= H_t^{1/2}\eta_t, \quad \text{with} \quad H_t := \mathbb{E}[\epsilon_t \epsilon_t'|\mathcal{F}_{t-1}] > 0 \text{ so that} \\
H_t &= D_t R_t D_t, \\
Q_t &= (\omega' - A - B) \odot S + A \odot u_{t-1}'u_{t-1} + B \odot Q_{t-1}, \quad R_t = Q_t^{1/2} Q_t^{1/2},
\end{align*}
$$

where $Q_t = [q_{ij,t}]$ and $D_t = \text{diag} \left( \sqrt{h_{11,t}}, \sqrt{h_{22,t}}, \ldots, \sqrt{h_{pp,t}} \right)$, $Q_t^* = \text{diag} \left( q_{11,t}, q_{22,t}, \ldots, q_{pp,t} \right)$. Above, $S$, $A$ and $B$ denote $p \times p$ symmetric matrices of unknown parameters and $\odot$ is the usual Hadamard product of two identically sized matrices. Following Ding and Engle
(2001), if \((u' - A - B) \odot S\), where \(A\) and \(B\) are positive semi-definite, then the matrix \(Q_t\) is positive semi-definite. The significant downside of the full DCC model is its intractability as the \((Q_t)\) process encompasses \(3p(p+1)/2\) coefficients. In our empirical study, the scalar DCC-GARCH is considered instead, where \(A\) and \(B\) are replaced by non negative scalars \(\alpha\) and \(\beta\) times the identity matrix. Moreover, the unknown matrix \(S\) is estimated by the empirical covariance matrix of the standardized returns (the so-called “correlation targeting” technique). A two step Gaussian QMLE is used for estimation, where in a first step the univariate conditional variance \(h_{ii,t}\) are estimated as GARCH(1,1), and conditionally on this first step estimator, the \((Q_t)\) and thus \((R_t)\) processes are estimated.

Instead of focusing on the correlation dynamic, the BEKK model directly generates a variance covariance process. Developed by Baba, Engle, Kraft and Kroner, in a preliminary version of Engle and Kroner (1995), the BEKK is specified as

\[
\left\{
\begin{align*}
\epsilon_t &= H_t^{1/2} \eta_t, \text{ with } H_t := \mathbb{E}[\epsilon_t \epsilon_t' | \mathcal{F}_{t-1}] \succ 0 \text{ so that } \\
H_t &= \Omega + \sum_{k=1}^q \sum_{j=1}^K A_{kj} \epsilon_{t-k} \epsilon_{t-k}' A_{kj}' + \sum_{i=1}^r \sum_{j=1}^K B_{ij} H_{t-i} B_{ij}',
\end{align*}
\right.
\]

where \(K\) is an integer, \(\Omega\), \(A_{kj}\) and \(B_{kj}\) are square \(p \times p\) matrices and \(\Omega \succ 0\). One advantage of the BEKK model is there is no positive semi-definite constraint on the \(A_{kj}\) and \(B_{kj}\) matrices. However, it imposes highly artificial constraints on the volatilities and covariances of the components. As a consequence, the coefficients of a BEKK representation are difficult to interpret. In our application, a scalar BEKK was considered, where \(A_{kj}\) and \(B_{kj}\) are scalar with \(K = 1\), \(q = r = 1\), together with a Gaussian QMLE estimation.

Beside BEKK type dynamics, factor models provide rather natural alternatives. The O-GARCH assumes the decomposition \(H_t = \Lambda_t P'\), where \(\Lambda_t = \text{diag}(\lambda_{1,t}, \cdots, \lambda_{K,t})\), with \(K\) the number of factors. Here, we choose \(K = p\) factors and each \(\lambda_i\) is supposed to follow a univariate GARCH(1,1) process that is estimated by maximum likelihood. The matrix \(P\) is nonsingular and it is estimated by applying a PCA on the empirical variance covariance matrix of \(\epsilon_t\): see Alexander (2001), e.g.

### 6.2 Technical results for support recovery

In this section, we present Loh and Wainwright’s (2017) results. Theorem [6.1] establishes the conditions for support recovery. The primal-dual witness construction is the main tool to prove that the regularized estimator satisfies the support recovery property. Its
argument are defined as:

**Step 1.** We define the estimator

$$
\hat{\theta}_A = \arg \min_{\theta \in \mathbb{R}^{\lvert A \rvert} : g(\theta) \leq R, \Lambda \in \Omega} \{ \mathbb{L}_n(\theta) + p(\lambda_n, \theta) \}. 
$$

(6.1)

We solve problem (6.1), under the constraint \( \hat{A} \subseteq A \) and prove \( g(\hat{\theta}_A) \leq R \).

**Step 2.** Defining \( \hat{z}_A \in \partial \| \hat{\theta}_A \| \), we choose \( \hat{z}_{A^c} \) satisfying the orthogonality condition

$$
\nabla_\theta \mathbb{L}_n(\theta) - \nabla_\theta q(\lambda_n, \hat{\theta}) + \lambda_n \hat{z} = 0, 
$$

(6.2)

with \( \hat{z} = (\hat{z}_A, \hat{z}_{A^c}) \), \( \hat{\theta} = (\hat{\theta}_A, 0_{A^c}) \) and \( q(\lambda_n, \rho) = \lambda_n \rho - p(\lambda_n, \rho) \). We then prove the strict dual feasibility \( \| \hat{z}_{A^c} \|_\infty < 1 \).

**Step 3.** We prove that \( \hat{\theta} \) is a local optimum of (4.4) and that any stationary point of (4.4) satisfies \( \text{supp}(\hat{\theta}) \subseteq A \).

The PDW procedure does not allow for practically solving the regularization problem (4.4) as step 1 requires to know the true subset model \( A \). However, this approach is useful as a proof method to characterize the optimal solution \( \hat{A}_q \).

The next Theorem provides the conditions for which the support recovery property is satisfied by the regularized estimators (4.4).

**Theorem 6.1.** (Loh and Wainwright, 2017)

Suppose \( \mathbb{L}_n(.) \) satisfies the RSC condition with \( (\alpha_k, \tau_k)_{k=1,2} \) parameters and \( p(\lambda_n, .) \) is a \( \mu \)-amenable penalty, with \( 0 \leq \mu < \alpha_1 \). Suppose

(i) The parameters \( (\lambda_n, R) \) satisfy

$$
2 \max \{ \| \nabla_\theta \mathbb{L}_n(\theta_0) \|_\infty, \alpha_2 \sqrt{\frac{\log k}{n}} \} \leq \lambda_n \leq \sqrt{\frac{(2\alpha_1 - \mu)\alpha_2}{56k}}
$$

$$
\max \{ 2\| \theta_0 \|_1, \frac{60k\lambda_n}{2\alpha_1 - \mu} \} \leq R \leq \min \{ \frac{\alpha_2}{8\lambda_n}, \frac{\alpha_2}{\tau_2} \sqrt{\frac{n}{\log p}} \}.
$$

(ii) For some \( \delta \in \left[ \frac{4R\tau_1 \log p}{n\lambda_n}, 1 \right] \), the vector \( \hat{z} \) from the PDW construction satisfies the
strict dual feasibility condition

\[ \| \hat{z}_{AV} \|_\infty \leq 1 - \delta. \]  

(6.3)

Then for any \( k \)-sparse vector \( \theta_0 \), the program \((4.4)\) with a sample size \( n \geq \frac{2\tau_1}{\alpha_1 - \mu} k \log p \) has a unique stationary point given by the primal output \( \hat{\theta} \) of the PDW construction.

Under the condition of Theorem 6.1, Theorem 6.2 provides a \( l_\infty \)-error bound on the optimum of the regularized problem and shows that any local/global optimum of the latter problem corresponds to the oracle estimator, which is defined as

\[ \hat{\theta}_S^O = \arg \min_{\theta_S \in \mathbb{R}^{|S|}} \{ \mathbb{L}_n(\theta_S, 0_S) \}, \]

with \( \hat{\theta}^O := (\hat{\theta}_S^O, 0_{S^c}) \).

**Theorem 6.2.** (Loh and Wainwright, 2017)

Under the conditions of Theorem 6.1, suppose strict dual feasibility \((6.3)\) holds, suppose \( p(\lambda_n, \cdot) \) is \( \mu \)-amenable with \( \mu \in [0, \alpha_1) \). Then the unique stationary solution of \((4.4)\) satisfies

(i) \[ \| \hat{\theta} - \theta_0 \|_\infty \leq \|\hat{K}^{-1}_{SS} \nabla_\theta \mathbb{L}_n(\theta_0)\|_\infty + \lambda T \|\hat{K}^{-1}_{SS}\|_\infty, \]

with \( \hat{K} = \int_0^1 \nabla^2_{\theta\theta} \mathbb{L}_n(\theta_0 + u(\hat{\theta} - \theta_0))du. \)

(ii) If \( p(\lambda_n, \cdot) \) is \( (\mu, \gamma) \)-amenable and if the lower bound \[ \min_{i \in S} |\theta_{0,i}| \geq \lambda T (\gamma + \|\hat{K}^{-1}_{SS}\|_\infty) + \|\hat{K}^{-1}_{SS} \nabla_\theta \mathbb{L}_n(\theta_0)\|_\infty, \]

holds, then \( \hat{\theta} \) agrees with the oracle estimator \( \hat{\theta}^O \) and we have the bound

\[ \| \hat{\theta} - \theta_0 \|_\infty \leq \|\hat{K}^{-1}_{SS} \nabla_\theta \mathbb{L}_n(\theta_0)\|_\infty. \]
Figure 1: Statistical consistency in the Frobenius sense of linear regression for the sparse VEC-ARCH model. SCAD, MCP and Lasso results are represented by solid, dashed and dash-dotted lines, respectively. The $p = 4$ (resp. $p = 6$) case is represented in blue (resp. red). Each point represents an average of 200 trials.

Figure 2: Statistical consistency in the $\|.|_{\infty}$ sense of linear regression for the sparse VEC-ARCH model. SCAD, MCP and Lasso results are represented by solid, dashed and dash-dotted lines, respectively. The $p = 4$ (resp. $p = 6$) case is represented in blue (resp. red). Each point represents an average of 200 trials.