Vector breathers in the Manakov system

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\textbf{Abstract}

We study theoretically the nonlinear interactions of vector breathers propagating on an unstable wavefield background. As a model, we use the two-component extension of the one-dimensional focusing nonlinear Schrödinger equation—the Manakov system. With the dressing method, we generate the multibreather solutions to the Manakov model. As shown previously in [D. Kraus, G. Biondini, and G. Kovačič, Nonlinearity 28(9), 3101, (2015)], the class of vector breathers is presented by three fundamental types I, II, and III. Their interactions produce a broad family of the two-component (polarized) nonlinear wave patterns. First, we demonstrate that the type I and the types II and III correspond to two different branches of the dispersion law of the Manakov system in the presence of the unstable background. Then, we investigate the key interaction scenarios, including collisions of standing and moving breathers and resonance breather transformations. Analysis of the two-breather solution allows us to derive general formulas describing phase and space shifts acquired by breathers in mutual collisions. The found expressions enable us to describe the asymptotic states of the breather interactions and interpret the resonance fusion and decay of breathers as a limiting case of infinite space shift in the case of merging breather eigenvalues. Finally, we demonstrate that only type I breathers participate in the development of modulation...
instability from small-amplitude perturbations within the superregular scenario, while the breathers of types II and III, belonging to the stable branch of the dispersion law, are not involved in this process.

KEYWORDS
breathers, integrable systems, modulation instability, rogue waves, solitons

1 | INTRODUCTION

Breathers are coherent nonlinear pulsating wave structures living on an unstable background, which theoretical description represents a generalization of the soliton theory.\textsuperscript{1–3} The interest in studies breathers is both theoretical and practical. On one side, these nonlinear wave groups can be described by exactly solvable, that is, integrable models, for example, by the one-dimensional focusing nonlinear Schrödinger equation (NLSE).\textsuperscript{1,4–7} As such, the class of breather solutions describes an essential part of the integrable system dynamics. On the other side, the breathers model is applicable in a wide range of physical systems as diverse as light in optical fibers, fluids, plasma, and Bose–Einstein condensates.\textsuperscript{8–10} Many experiments have confirmed the existence of breathers in nature, encouraging theoreticians to predict novel scenarios of breather propagation and interactions.

The scalar NLSE breathers have been the focus of the studies for the past decades, revealing such fundamental building blocs of the breather dynamics as Kuznetsov, Akhmediev, Peregrine, and Tajiri-Watanabe solutions;\textsuperscript{5,11–13} as well as superregular and ghost interaction patterns,\textsuperscript{14–17} and breather wave molecules.\textsuperscript{17} All these scenarios of nonlinear wavefield evolution have been confirmed experimentally with optical, hydrodynamical, and plasma setups.\textsuperscript{14,17–24} In addition, the breathers play an essential role in the formation of rational rogue waves,\textsuperscript{25,26} modulation instability (MI) development,\textsuperscript{11,27} and in the dynamics and statistics of complex nonlinear random wave states.\textsuperscript{28–31}

In this work, we consider the vector two-component extension of the NLSE—the Manakov system.\textsuperscript{32} In the presence of a constant background fields having amplitudes $A_{1,2} = \text{const}$, $A = \sqrt{A_{1}^{2} + A_{2}^{2}}$, the Manakov system can be written as follows:

\begin{align*}
    i\psi_{1t} + \frac{1}{2}\psi_{1xx} + (|\psi_{1}|^{2} + |\psi_{2}|^{2} - A^{2})\psi_{1} &= 0, \\
    i\psi_{2t} + \frac{1}{2}\psi_{2xx} + (|\psi_{1}|^{2} + |\psi_{2}|^{2} - A^{2})\psi_{2} &= 0, \quad (1)
\end{align*}

where $t$ is time, $x$ is spatial coordinate, and $\psi_{1,2}$ is a two-component complex wave field. The presence of the constant background, which often refers as condensate, means the following boundary conditions: $|\psi_{1,2}|^{2} \to |A_{1,2}|^{2}$ at $x \to \pm\infty$.

In the case of small-amplitude condensate perturbations, the linear analysis of the system (1) reveals two branches of the dispersion law $\omega_{I}(k)$ and $\omega_{II}(k)$, see Appendix Section A.1 for the
derivation details,

$$\omega_1(k) = \pm k \sqrt{k^2/4 - A^2}, \quad \omega_{II}(k) = \pm k^2/2. \tag{2}$$

The first branch $\omega_1(k)$ is the same as in the scalar NLSE with one-component condensate of amplitude $A$, and leads to the long-wave MI in the spectral region $k \in (-2A, 2A)$. The second one $\omega_{II}(k)$ is the same as in the scalar NLSE on zero background, and corresponds to stable small-amplitude linear waves.

The system (1) first considered by S.V. Manakov in Ref. 32 is now widely used in nonlinear optics as a model of optical pulse propagation in a birefringent optical fiber.10,33 The two components $\psi_1$ and $\psi_2$ describe different light polarizations, which interact nonlinearly with each other producing a broad family of complex nonlinear phenomena. Remarkably, the Manakov system, like its scalar counterpart NLSE, belongs to the class of equations integrable using the inverse scattering transform (IST) technique.32 The IST allows finding exact multi-soliton solutions and asymptotic description of an arbitrary pulse evolution.1,2 The key role in the IST construction for the Manakov system plays an auxiliary linear system of $3 \times 3$ matrix wave functions (Jost functions), see, for example, Ref. 10. In the case of zero background, each complex eigenvalue of the auxiliary system corresponds to a vector (polarized) soliton in the wavefield. S.V. Manakov presented the first study of vector soliton dynamics and demonstrated that they can change polarization as a result of mutual collisions.32 By now, such solitons are studied in detail, see, for example, Ref. 34 and the monograph.10

In the presence of a condensate, vector solitons transform into vector breathers, characterized by discrete eigenvalues of the same $3 \times 3$ auxiliary system. The vector breathers are in the recent trends of nonlinear studies focused on increasing the level of systems complexity. The first results on vector generalizations of Kuznetsov, Akhmediev, and Peregrine breathers have been presented within the past decade in Refs. 35–37, see also the recent work.37 The vector rogue waves have been studied theoretically, numerically, and experimentally in Refs. 38–42. In addition, the work43 provided a detailed study of the initial value problem for the system (1) based on the IST Riemann approach and suggested a general classification of the types of vector breathers based on analytical properties of the wavefield Jost functions. According to this classification, the vector breathers of the Manakov system represent three fundamental types—type I, II, and III. Type I is a direct analog of the scalar NLSE breathers, while types II and III exhibit fundamentally different dynamics specific to the vector case. Recently, we have found that the vector breathers can participate in a resonance interaction, see our Letter.44 The resonance represents a three-breather process, that is, a fusion of two breathers of the type I and II into one breather of the type III, which we denote schematically as $I + II \rightarrow III$; or the opposite, that is, the decay $III \rightarrow I + II$.

In this work, we study interactions of the vector breathers, including the resonance situations. With the vector variant of the dressing method scheme, see Ref. 44, we obtain a general multibreather solution of Equation (1). First, we analyze single breathers and show that the type I breathers correspond to the first branch $\omega_1(k)$ of the dispersion laws (2), while the types II and III are linked to the second branch $\omega_{II}(k)$. This correspondence takes place for the decaying tails of the breathers. Then, we study the resonant breather interactions and two-breathers solutions. We derive asymptotic expressions for the position and phase shifts acquired by breathers after mutual collisions. The found formulas allow us to describe the asymptotic states of the multibreather ensembles and reveal the mathematical nature of the resonance interactions. More precisely, we find that similar to the three-wave system case,45 the resonance interaction of vector breathers can be explained as a limiting case of infinite space shift acquired by one of the breathers. Finally, we
demonstrate that only type I breathers participate in the development of MI from small-amplitude perturbations within the superregular scenario, see Refs. 15, 27, while the breathers of types II and III, belonging to the stable branch of the dispersion law, are not involved in this process.

The paper is organized as follows. In the next Section 2, we construct the general scheme of the vector dressing method and find the $N$-breather solution of the Manakov system in the presence of the condensate. In Section 3, we analyze the single-breather solutions of the Manakov system and their relations to the branches of the dispersion law. Then, in Section 4, we consider the case of resonant interactions. In Section 5, we consider the general two-breather solution of the Manakov system (1), which describes elastic collisions of breathers. We end up Section 5 with finding the shifts of the positions and phases acquired by the breathers after their interaction. Finally, in Section 6, we investigate important particular cases of vector two-breather solution. The last Section 7 presents discussions and conclusions. The Appendix Section A provides additional computational details, a complete table of the positions, and phases shifts expressions and additional illustrations.

## 2 | DRESSING METHOD FOR VECTOR BREATHERS

In this section, we build a dressing method scheme for constructing multibreather solutions to the Manakov system. Previously in the Letter, 44 we presented a shorter version of this scheme limited to single-eigenvalue solutions. Note, that the dressing method, also known as the Darboux dressing scheme, being a popular tool for constructing exact solutions to integrable nonlinear PDE, has many variations, see Refs. 1, 46–48. We use a vector analog of the scalar dressing scheme for the NLSE developed in Ref. 15.

The dressing method starts from introducing the auxiliary linear system for the $3 \times 3$ matrix wave function $\Phi$ depending on $x$, $t$, and the complex spectral parameter $\lambda$:

$$
\Phi_x = U\Phi,
$$

$$
\Phi_t = V\Phi = -(\lambda U + iW/2)\Phi.
$$

Here, $U$ and $W$ are the following $3 \times 3$ matrixes:

$$
U = \begin{pmatrix}
-i\lambda & \psi_1 & \psi_2 \\
-\psi_1^{\ast} & i\lambda & 0 \\
-\psi_2^{\ast} & 0 & i\lambda
\end{pmatrix},
$$

$$
W = \begin{pmatrix}
|\psi_1|^2 + |\psi_2|^2 - A^2 & \psi_1x & \psi_2x \\
\psi_1^{\ast}x & -|\psi_1|^2 + A^2 & -\psi_1^{\ast}\psi_2 \\
\psi_2^{\ast}x & -\psi_1\psi_2^{\ast} & -|\psi_2|^2 + A^2
\end{pmatrix}.
$$

The Manakov system (1) represents the compatibility condition of Equations (3) and (4) written as:

$$
\Phi_{xt} = \Phi_{tx}.
$$
From Equations (3) and (4), we find the following auxiliary equations for $\Phi^{-1}$ and $\Phi^\dagger$:

$$\Phi^{-1}_x = -\Phi^{-1}_U, \quad \Phi^{-1}_t = -\Phi^{-1}_V, \quad (7)$$

$$\Phi^\dagger_x = \Phi^\dagger U^\dagger, \quad \Phi^\dagger_t = \Phi^\dagger V^\dagger. \quad (8)$$

Here, the sign $\dagger$ means Hermitian conjugation. Comparing formulas (7) and (8), and using the symmetry properties $U^\dagger(\lambda^\ast) = -U(\lambda)$, $V^\dagger(\lambda^\ast) = -V(\lambda)$, we find that $\Phi$ satisfies the following reduction:

$$\Phi^\dagger(\lambda^\ast) = \Phi^{-1}(\lambda). \quad (9)$$

At the first step of the dressing procedure, we find the solution $\Phi_c$ of the system (3), (4) for the condensate background $\psi_{1,2} = A_{1,2}$:

$$\Phi_c(x, t, \lambda) = \left[ (1 + r^2)e^{-\varphi_0} \right]^{-1/3} \begin{pmatrix} 0, & e^{\varphi}, & -i re^{-\varphi} \\ -\frac{A_2}{A} e^{-\varphi_0}, & -\frac{A_1}{A} i re^{\varphi}, & \frac{A_1}{A} e^{-\varphi} \\ \frac{A_1}{A} e^{-\varphi_0}, & -\frac{A_2}{A} i re^{\varphi}, & \frac{A_2}{A} e^{-\varphi} \end{pmatrix}, \quad (10)$$

where,

$$r = A/(\lambda + \zeta), \quad \zeta = \sqrt{\lambda^2 + A^2}, \quad (11)$$

and the functions $\varphi_0$ and $\varphi$ are,

$$\varphi_0 = -i \lambda x + \frac{i}{2} (\lambda^2 + \zeta^2) t, \quad \varphi = -i \zeta x + i \lambda \zeta t. \quad (12)$$

We imply that the function $\zeta(\lambda)$ has the branchcut on the interval $[-iA, iA]$, which differs from the automatic choice $\{-i\infty, -iA\} \cup \{iA, \infty i\}$ implied in software packets, such as Wolfram Mathematica. As we see later, the choice of the branchcut is essential for constructing breather solutions.

The solution $\Phi_c(x, t, \lambda)$ satisfies the auxiliary system:

$$\Phi_{c,x} = U_c \Phi, \quad \Phi_{c,t} = V_c \Phi, \quad (13)$$

where $V_c = -(\dot{\lambda} U_c + i W_c)$, and

$$U_c = \begin{pmatrix} -i\lambda & A_1 & A_2 \\ -A_1 & i\lambda & 0 \\ -A_2 & 0 & i\lambda \end{pmatrix},$$

$$W_c = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & A_2^2 & -A_1 A_2 \\ 0 & -A_1 A_2 & A_1^2 \end{pmatrix}.$$
In accordance with Ref. 15, we introduce the dressing function $\chi$ as:

$$\chi = \Phi \Phi_0^{-1}. \quad (14)$$

The dressing function satisfies the asymptotic condition:

$$\chi(\lambda) \to E + \frac{N}{\lambda} + O(\lambda^{-2}), \quad \text{at} \quad |\lambda| \to \infty, \quad (15)$$

where $N$ is a constant matrix and $E$ is the unit matrix. From Equation (9), we also find that the function $\chi(x, t, \lambda)$ satisfies the following reduction:

$$\chi^*(\lambda^*) = \chi^{-1}(\lambda). \quad (16)$$

Then, using Equations (3), (4), (7), and (8), we obtain the system of linear equations for the inverse dressing function:

$$\chi_x^{-1} = -\chi^{-1}U + U \chi^{-1},$$

$$\chi_t^{-1} = -\chi^{-1}V + V \chi^{-1}. \quad (17)$$

Now, choosing matrix $\chi$ so that the matrices $U$ and $V$ are regular in the $\lambda$-plane, we obtain a new solution of Equations (3) and (4). Substituting the expansion (15) into Equation (17), we find out the final formulas to calculate the components $\psi_{1,2}$:

$$\psi_1 = A_1 + 2i N_{12}, \quad \psi_2 = A_2 + 2i N_{13}. \quad (18)$$

First, we propose that the function $\chi$ has only one pole at $\lambda = \lambda_1$, so that it can be written as:

$$\chi = E + \frac{N}{\lambda - \lambda_1}. \quad (19)$$

The constant $\lambda_1$ represents discrete eigenvalue of the system (3, 4), which means that the corresponding wave function is bounded from both sides in space, see, for example, Ref. 43. Then, from (16), we obtain that:

$$\chi^{-1} = E + \frac{N^\dagger}{\lambda - \lambda^*_1}. \quad (20)$$

From the condition $\chi \chi^{-1} = E$, we find:

$$\chi(\lambda_1) N^\dagger(\lambda_1) = 0. \quad (21)$$

From this, it follows that the matrices $N$, $N^\dagger$ are degenerated and can be expressed via three-component vectors $p$ and $q$ as follows:

$$N_{\alpha\beta} = p_{\alpha} q_{\beta}, \quad N^\dagger = q_{\alpha}^* p_{\beta}^*, \quad \alpha, \beta = 1, 2, 3. \quad (22)$$
To eliminate an extra pole at the point $\lambda = \lambda_1^*$ in the expression (17), we impose on the vector $\mathbf{q}(x, t)$ the condition:

$$\partial_x \mathbf{q}^* - \mathbf{U}_c(x, t, \lambda_1^*) \mathbf{q}^* = 0.$$  \hspace{1cm} (23)

From this, we find the vector $\mathbf{q}$ in the form,

$$\mathbf{q}(x, t) = \Phi_c^*(x, t, \lambda_1^*) \mathbf{C},$$  \hspace{1cm} (24)

where the vector of integration constants,

$$\mathbf{C} = (C_0, C_1, C_2)^T,$$  \hspace{1cm} (25)

is an arbitrary three-component complex vector with the superscript $T$ meaning transposing. Finally, from (21), we obtain the vector $\mathbf{p}$ in the form:

$$\mathbf{p} = \frac{\mathbf{q}^*}{|\mathbf{q}|^2} (\lambda_1 - \lambda_1^*).$$  \hspace{1cm} (26)

Thereby, the one-pole function $\chi(x, t, \lambda)$ from Equation (19) is completely defined. Now, using the formula (18), we obtain the components $\psi_{1,2}$ of the single-eigenvalue solution of the Manakov system (1) in the presence of the condensate background:

$$\psi_1 = A_1 + \frac{2i(\lambda_1 - \lambda_1^*) q_1^* q_2}{|\mathbf{q}|^2},$$

$$\psi_2 = A_2 + \frac{2i(\lambda_1 - \lambda_1^*) q_1^* q_3}{|\mathbf{q}|^2}.$$  \hspace{1cm} (27)

Following the analogy with Ref. 15, we obtain that if the dressing matrix $\chi(x, t, \lambda)$ have $N$ poles, $\lambda = \lambda_j, j = 1, \ldots N$, then the corresponding $N$-eigenvalue solution of the Manakov system can be found by means of the Cramer’s rule as:

$$\psi_1 = A_1 + 2 \tilde{M}_{12}/M,$$

$$\psi_2 = A_2 + 2 \tilde{M}_{13}/M.$$  \hspace{1cm} (28)

Here, $\tilde{M}_{\alpha, \beta}$ and $M$ are the following determinants:

$$\tilde{M}_{\alpha\beta} = \begin{vmatrix} 0 & q_{1,\beta} & \cdots & q_{n,\beta} \\ q_{1,\alpha} & \ddots & \vdots & \vdots \\ \vdots & \ddots & 0 & q_{n,\beta} \\ q_{N,\alpha} & \cdots & q_{n,\beta} & 0 \end{vmatrix} M^T; \quad M = \det(M); \quad M_{nm} = \frac{i(q_n \cdot q_m^*)}{\lambda_n - \lambda_m^*},$$  \hspace{1cm} (29)

with $\alpha, \beta = 1, 2, 3$ and $n, m = 1, 2, \ldots N$. Note that it is sufficient to consider the poles of the dressing function located only in the upper half of the $\lambda$-plane, that is:

$$\text{Im}[\lambda_j] > 0, \quad j = 1, \ldots N,$$  \hspace{1cm} (30)
because the choices $\text{Im}[\lambda_j] < 0$ lead to the same class of multibreather solutions. Here and below, $\text{Re}$ and $\text{Im}$ mean real and imaginary parts of a complex number. Recall that the eigenvalue set $\{\lambda_j\}$ represents discrete spectrum of the system (3, 4). Meanwhile, the real $\lambda$-axes region belongs to the continuous part of the system (3, 4), which we do not consider here.

From Equation (24), we find the vectors $q_n$ as:

$$q_n = \begin{pmatrix}
0, & e^{-\varphi_n}, & i r_n e^{\varphi_n} \\
-\frac{A_2}{A} e^{\varphi_0 n}, & \frac{A_1}{A} i r_n e^{-\varphi_n}, & \frac{A_1}{A} e^{\varphi_n} \\
\frac{A_1}{A} e^{\varphi_0 n}, & \frac{A_2}{A} i r_n e^{-\varphi_n}, & \frac{A_2}{A} e^{\varphi_n}
\end{pmatrix}
\begin{pmatrix}
C_{n0} \\
C_{n1} \\
C_{n2}
\end{pmatrix},$$

(31)

or, in component-wisely form,

$$q_{n1} = e^{-\varphi_n} C_{n1} + i r_n e^{\varphi_n} C_{n2}$$

$$q_{n2} = \frac{1}{A} \left[-A_2 e^{\varphi_0 n} C_{n0} + A_1 (i r_n e^{-\varphi_n} C_{n1} + e^{\varphi_n} C_{n2})\right],$$

$$q_{n3} = \frac{1}{A} \left[A_1 e^{\varphi_0 n} C_{n0} + A_2 (i r_n e^{-\varphi_n} C_{n1} + e^{\varphi_n} C_{n2})\right].$$

(32)

Here, the functions $\varphi_{0n}$ and $\varphi_n$ are the following, see Equation (12):

$$\varphi_{0n} = -i \lambda_n x + \frac{i}{2} (\lambda_n^2 + \zeta_n^2) t = u_{0n} - i v_{0n},$$

$$\varphi_n = -i \zeta_n x + i \lambda_n \zeta_n t = u_n - i v_n.$$  

(33)

The functions $u_n, v_n, u_{0n}, v_{0n}$ distinguish real and imaginary parts of the functions $\varphi_{0n}$ and $\varphi_n$ as:

$$u_{0n} = \text{Im}[\lambda_n] x - \frac{1}{2} \text{Im}[\lambda_n^2 + \zeta_n^2] t,$$

$$v_{0n} = \text{Re}[\lambda_n] x - \frac{1}{2} \text{Re}[\lambda_n^2 + \zeta_n^2] t,$$

$$u_n = \text{Im}[\zeta_n] x - \text{Im}[\lambda_n \zeta_n] t,$$

$$v_n = \text{Re}[\zeta_n] x - \text{Re}[\lambda_n \zeta_n] t.$$  

(34)

In further calculations, we will use the square of the modulus $q_n$, which can be written as:

$$|q_n|^2 = |e^{-\varphi_n} C_{n1} + i r_n e^{\varphi_n} C_{n2}|^2 + |i r_n e^{-\varphi_n} C_{n1} + e^{\varphi_n} C_{n2}|^2 + |e^{\varphi_0 n} C_{n0}|^2.$$  

(35)

Note that the transformation,

$$q_n \rightarrow \kappa q_n,$$

(36)

where $\kappa$ is an arbitrary complex constant, does not change the solution (27). The latter means that an arbitrary choice of the vector $C$ corresponds to four real-valued solution parameters. The nontrivial solutions of the Manakov system appear when the vector $C$ has at least two nonzero components. We discuss the physical meaning of the breather parameters and different choices of the vector $C$ in the following paragraphs.
In this section, we describe the elementary building blocks of the vector breather dynamics—the single breathers of the fundamental types I, II, and III. This classification based on analytical properties of the wavefield Jost functions was proposed in the work. Type I coincides with the breathers solutions to the scalar NLSE, while types II and III exhibit fundamentally different nonlinear wave dynamics specific to the vector (polarized) case. Previously in Ref. 44, we have established that on the language of the dressing method, see also Section 2, the types I, II, and III correspond to the subsequent setting to zero one of the components of the vector \( \mathbf{C} \), see Equation (25). Here, following Ref. 44, we present an analytical description for all three types of vector breathers and consider important particular cases which have not been touched in Ref. 44. Also, we emphasize that type II and type III solutions can be transformed into each other by changing the Riemann surface sheets of the spectral parameter plane. Finally, we demonstrate that type I corresponds to the first branch of the dispersion law \( \omega_I(k) \), while types II and III to the second branch \( \omega_{II}(k) \).

To avoid the sign issues of square root function \( \zeta(\lambda) \) and also to simplify computations, we used the following parameterizations for the spectral parameter and associated with it functions,

\[
\lambda = A \sinh(\xi + i \alpha), \\
\zeta = A \cosh(\xi + i \alpha), \\
r = e^{-\xi - i\alpha}.
\]

This transformation of the two-sheeted Riemann surface of \( \zeta(\lambda) \) into one-sheeted plane with coordinates \( (\xi, \alpha) \) is called uniformization and often used in the breathers studies, see, for example, Refs. 15, 43. According to (30), we consider only the regions \( \xi \in (0, \infty) \) and \( \alpha \in (0, \pi) \) for the breather parameters.

We use the general single-eigenvalue solution (27) with \( N = 1 \), and start with the case \( C_0 = 0 \) and \( C_{1,2} \neq 0 \) corresponding to the breather of type I. When dealing with single breathers, we omit the subscripts \( n \) for the formulas of the previous Section 2. Substituting \( C_1 = 0 \) into (27), we find that the breather represents a simple vector generalization of the solution of the scalar NLSE, when the two components of the Manakov system do not interact, satisfying the relation:

\[
\psi_2 = (A_2/A_1)\psi_1.
\]

The latter means that each wavefield component represents a well-known breather solution of the scalar NLSE, see, for example, Ref. 49. When the scalar NLSE for one-component wavefield \( \psi \) is written in the form:

\[
i\psi_t + \frac{1}{2}\psi_{xx} + (|\psi|^2 + |\psi|^2 - A_0^2)\psi = 0,
\]

the vector breathers of type I can be obtained from the known breather solutions of Equation (39) using the following transformation:

\[
\psi_1(x, t) = \frac{A_1}{A_0}\psi\left(A_0^2t, \frac{A}{A_0}x\right),
\]
From Equation (36), we see that the vector $\mathbf{C}$ has only one independent complex parameter, which allows us to parameterize its components as follows:

$$C_0 = 0, \quad C_1 = C_2^{-1} = e^{i\theta/2},$$

where $\delta$ and $\theta$ are real valued parameters controlling the space position of the breather and its phase.

From the general solution (27), the real and imaginary wavefield components for type I breather can be written as:

$$\text{Re} \psi_{1,2}^I = A_{1,2} - \frac{2A_{1,2} \sin \alpha \cosh \xi [\cos(2v_I) \cosh \xi + \cosh(2u_I) \sin \alpha]}{\cosh \xi \cosh(2u_I) + \sin \alpha \cos(2v_I)},$$

$$\text{Im} \psi_{1,2}^I = \frac{2A_{1,2} \sin \alpha \cosh \xi [\sinh(2u_I) \cos \alpha + \sin(2v_I) \sinh \xi]}{\cosh \xi \cosh(2u_I) + \sin \alpha \cos(2v_I)},$$

where

$$2u_I = l_I^{-1}(x - V_I t - \delta), \quad 2v_I = k_I x - \omega_I t + \theta,$$

are expressed via the breather characteristic length $l_I$, group velocity $V_I$, characteristic wave vector $k_I$, and characteristic frequency $\omega_I$:

$$l_I = (2\text{Im}[\xi])^{-1} = (2A \sin \alpha \sinh \xi)^{-1},$$

$$V_I = \text{Im}[\lambda \xi] = \frac{A \cos \alpha \cosh 2\xi}{\sinh \xi},$$

$$k_I = 2\text{Re}[\xi] = 2A \cos \alpha \cosh \xi,$$

$$\omega_I = 2\text{Re}[\lambda \xi] = A^2 \cos 2\alpha \sinh 2\xi.$$  

(45)

The type I breather has the following spatial asymptotics,

$$\psi_{1,2}^I \to A_{1,2} e^{\pm 2i\alpha}; \quad x \to \pm \infty,$$

so that the total phase shift of the background field caused by the presence of the breather is $4\alpha$.

Figure 1 demonstrates a general case, when the breather is localized in space and moves with a nonzero group velocity. In the figure, we indicate the characteristic $l_I$ and $k_I$, and the asymptotic values (46). We choose the following set of breather parameters:

$$A_1 = 1, \quad A_2 = 1;$$

$$\alpha = \pi/5, \quad \xi = 1/4, \quad \theta = 0, \quad \delta = 0,$$

(47)
Figure 1: Vector breather of type I, see Equation (43), which can be obtained from scalar breather solution using the transformation (40). The solution parameters are defined in (47). (A) $|\psi_1|$ at $t = 0$ with indicated asymptotic values, see Equation (46), characteristic size and wavelength computed according to Equation (45). (B) Arg[$\psi_1$] at $t = 0$ with indicated asymptotic values, where Arg means complex phase. (C) Spatio-temporal evolution of $|\psi_1|$. The wavefield component $\psi_2$ is not shown because it coincides with the first one after rescaling the amplitude, see Equation (38). Here and in next figures, the function Arg is defined in the cyclic interval $[-\pi, \pi)$, so that the function value can exhibit a jump as in panel (B).

which later we also use to show examples of type II and type III breathers. The choice of $\alpha$ and $\xi$ in (47) corresponds to a moving spatially localized breather, see (45).

When $C_1 = 0$, we obtain another nontrivial solution of the Manakov system, which we call type II breather, again referring to the classification from Ref. 43. Writing the components of the vector $C$ as follows:

$$C_0 = e^{-\text{Im}[\lambda] \delta - i \theta / 2}, \quad C_1 = 0, \quad C_2 = e^{-\text{Im}[\xi] \delta + i \theta / 2},$$

from (27), we obtain:

$$\begin{align*}
\psi_{1I} & = A_1 + \frac{4xe^{i\lambda x} \sin \alpha \cosh \xi(A_1 - A_2 e^{\mu t - \nu})}{e^{2\mu + \xi} + 2 \cosh \xi}, \\
\psi_{2I} & = A_2 + \frac{4xe^{i\lambda x} \sin \alpha \cosh \xi(A_2 + A_1 e^{\mu t - \nu})}{e^{2\mu + \xi} + 2 \cosh \xi},
\end{align*}$$

(49)

where

$$2\mu = l_1^{-1}(x - V t - \delta), \quad \nu = k x - \omega t + \theta,$$

(50)

are expressed via the physical characteristics of type II breather,

$$l_1 = (2(\text{Im}[\lambda] - \text{Im}[\xi]))^{-1} = (2A e^{-\xi} \sin \alpha)^{-1}, \quad V_{1I} = \frac{\text{Im}[\lambda^2 + \xi^2] / 2 - \text{Im}[\lambda \xi]}{\text{Im}[\lambda] - \text{Im}[\xi]} = -A \cos \alpha e^{-\xi},$$

$$k_{1I} = \text{Re}[\lambda] - \text{Re}[\xi] = -A \cos \alpha e^{-\xi}, \quad \omega_{1I} = \frac{1}{2} \text{Re}[\lambda^2 + \xi^2] - \text{Re}[\lambda \xi] = \frac{A^2}{2} e^{-2\xi} \cos 2\alpha.$$
FIGURE 2  Vector breather of type II. The solution parameters are defined in (47). (a1,a2) $|\psi_{1,2}|$ at $t = 0$ with indicated asymptotic values, see Equation (52), and characteristic size and wavelength computed according to Equation (51). (b1,b2) Arg[$\psi_{1,2}$] at $t = 0$ with indicated asymptotic values. (c1,c2) Spatio-temporal evolution of $|\psi_{1,2}|$.

Figure 2 shows an example of type II localized breather having parameters (47), which moves with a nonzero group velocity.

The type II breather has the following asymptotics:

$$\psi_{1,2}^\Pi \rightarrow A_{1,2} e^{2i\alpha}; \quad x \rightarrow -\infty$$

$$\psi_{1,2}^\Pi \rightarrow A_{1,2}; \quad x \rightarrow +\infty.$$

(52)

Finally, for $C_2 = 0$, we obtain type III breather. Writing components $C$ in the form:

$$C_0 = e^{-i\text{Im}[\lambda] \delta - i\beta/2},$$

$$i r C_1 = e^{i\text{Im}[\xi] \delta + i\beta/2}, \quad C_2 = 0,$$

(53)

from (27), we obtain:

$$\psi_1^\Pi = A_1 - 4i \sin \alpha e^{-i\xi} \cosh \xi \frac{A_1 - A_2 e^{iu_{\Pi} - i\nu_{\Pi}}}{e^{2iu_{\Pi} - \xi} + 2 \cosh \xi},$$

(54)

$$\psi_2^\Pi = A_2 - 4i \sin \alpha e^{-i\xi} \cosh \xi \frac{A_2 + A_1 e^{iu_{\Pi} - i\nu_{\Pi}}}{e^{2iu_{\Pi} - \xi} + 2 \cosh \xi},$$

(55)

where

$$u_{\Pi} = \frac{l_{\Pi}^{-1}(x - V_{\Pi} t - \delta)}{2}, \quad \nu_{\Pi} = k_{\Pi} x - \omega_{\Pi} t + \vartheta.$$

(56)
Physical characteristics of type III breather are the following:

\[ l_{III} = (2(\text{Im}[\lambda] + \text{Im}[\zeta]))^{-1} = (2Ae^\xi \sin \alpha)^{-1}, \]
\[ V_{III} = \frac{\text{Im}[\lambda^2 + \zeta^2]/2 + \text{Im}[\lambda \zeta]}{\text{Im}[\lambda] + \text{Im}[\zeta]} = Ae^\xi \cos \alpha, \]
\[ k_{III} = \text{Re}[\lambda] + \text{Re}[\zeta] = Ae^\xi \cos \alpha, \]
\[ \omega_{III} = \frac{1}{2} \text{Re}[\lambda^2 + \zeta^2] + \text{Re}[\lambda \zeta] = \frac{A^2}{2} e^{2\xi} \cos 2\alpha. \] (57)

The type III breather has the following asymptotics:

\[ \psi_{1,2}^{III} \rightarrow A_{1,2} e^{-2i\alpha}; \quad x \rightarrow -\infty \]
\[ \psi_{1,2}^{III} \rightarrow A_{1,2}; \quad x \rightarrow +\infty. \] (58)

Figure 3 shows an example of the type III breather, moving with a nonzero group velocity.

The solutions of type II and III have a similar structure, however, differ in asymptotic behavior and characteristic parameters. For the same eigenvalue, these breathers always propagate in opposite directions and the breather II has larger size and characteristic wavelength according to the inequalities \(|k_{II}| < |k_{III}|\) and \(|l_{II}| > |l_{III}|\), see Equations (51) and (57). The following change of the spectral parameter:

\[ \xi \rightarrow -\xi, \quad \alpha \rightarrow \pi - \alpha, \] (59)
transforms type II solution (49) into type III solution (55). In terms of $\lambda$ spectral variable, the transformation (59) means that we change the Riemann sheets of the function $\zeta(\lambda)$. This situation is not typical in the IST theory, where usually different Riemann sheets correspond to the same class of solutions. For example, in the scalar NLSE model, the jump to another Riemann sheet only changes the breather phase, leaving the solution the same. One can check that type I solution (43) is invariant to the transformation (59), when the additional replacement $\theta \to -\theta$ is applied.

Figure 4 briefly reminds the key properties of these nonlinear structures. The Kuznetsov breather is a standing one-humped wave group oscillating on the condensate background with a finite time period $T = 4\pi/(A^2 \sinh 2\xi)$, see Figure 4A. The Peregrine breather is a degenerate limit of the Kuznetsov solution appearing in Equation (43) at the spectral parameter (62). It can be found by resolving an uncertainty of the type 0/0 in solution (43) that leads to the following
rational solution:

\[ \psi_{1,2} = -A_{1,2} + 4A_{1,2} \frac{1 - 2iA^2 t}{1 + 4A^2 x^2 + 4A^4 t^2}. \]  

(64)

The Peregrine breather (64) emerges from a small amplitude spatially localized condensate perturbation and then disappears, that is, in other words, it is localized both in space and time, see Figure 4B. This property makes the Peregrine solution a popular elementary model of extreme waves formation, see Refs. 9, 49–51. The Peregrine and the high-order Peregrine solutions of the NLSE also often referred to as rational rogue waves.50 Finally, the Akhmediev breather is a periodic solution with spatial period \( L = 2\pi/(A \sin \alpha) \), which, similar to the Peregrine breather, emerges only once in time, see Figure 4C. The Akhmediev breather describes an important scenario of the MI development of a periodically perturbed condensate.9,49,50

Similar to as it is typically done in the linear theory of polarized light, see, for example, Ref. 52, the wavefield components can be considered as vector \((\psi_1, \psi_2)^T\), which can be rotated by a rotation matrix \( T \) providing the same solution of the Manakov system written in a new basis. In particular, one can switch between solutions of the Manakov system \((\psi_1, \psi_2)^T\) and \((\tilde{\psi}_1, \tilde{\psi}_2)^T\) having the asymptotics,

\[ \psi_{1,2} \to \begin{pmatrix} A_1 e^{i\phi} \\ A_2 e^{i\phi} \end{pmatrix}, \quad \tilde{\psi}_{1,2} \to \begin{pmatrix} e^{i\phi} \\ 0 \end{pmatrix}; \quad x \to \pm \infty, \]  

(65)

using the rotation matrix \( T \) and its inverse counterpart \( T^{-1} \) as follows:

\[
\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = T \begin{pmatrix} \tilde{\psi}_1 \\ \tilde{\psi}_2 \end{pmatrix}, \quad \begin{pmatrix} \tilde{\psi}_1 \\ \tilde{\psi}_2 \end{pmatrix} = T^{-1} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}; \quad T = \frac{1}{A^2} \begin{pmatrix} A_1 & -A_2 \\ A_2 & A_1 \end{pmatrix}, \quad T^{-1} = \frac{1}{A^2} \begin{pmatrix} A_1 & A_2 \\ -A_2 & A_1 \end{pmatrix}.
\]

(66)

Our solutions for the breathers of types I, II, and III with asymptotics \( A_{1,2} e^{i\phi} \) defined by Equations (46), (52), and (58) can be transformed using the inverse matrix \( T^{-1} \) into solutions having zero condensate level in the second component, see Equations (65) and (66). Most interestingly, for type I breathers, the second component in the new bases is exactly canceled, that is, \( \tilde{\psi}_2(x, t) \equiv 0 \), due to the symmetry (38). In other words, type I solutions are flat in the sense of polarization, meanwhile, type II and type III breathers always have both wavefield components different from zero. In particular, the work43 uses the polarization basis corresponding to the case \((\tilde{\psi}_1, \tilde{\psi}_2)^T\), see the illustrations for the second wavefield component in Ref. 43. One can check that the solutions (43), (49), and (55) boil down to those presented in Ref. 43 after the transformation with the matrix \( T^{-1} \).

Mathematically, the diversity of scalar breathers emerges due to the nontrivial spectral parameter plane geometry produced by the function \( \zeta(\lambda) \) branchcut. Indeed, the Kuznetsov, Akhmediev, and Peregrine breathers correspond to the eigenvalue location at the imaginary \( \lambda \)-axis, respectively, outside/inside the branchcut and precisely at the branch point of the function \( \zeta(\lambda) \), see Equations (61–63). Meanwhile, the general Tajiri-Watanabe breather emerges when the eigenvalue is located outside the imaginary \( \lambda \)-axis. As we already discussed, the general type II and type III breathers exhibit in principal a similar to the scalar case behavior (in the sense that they are localized moving pulsating breathers). The choices of the eigenvalue locations (61–63) lead to
FIGURE 5  Vector breathers of the type II with spectral parameters defined by Equations (61–63).
(A) Spectral parameter belongs to the Kuznetsov region, see Equation (61). The corresponding solution is a standing dark-bright oscillating breather. (B) Spectral parameter belongs to the Peregrine region, see Equation (61). The corresponding solution exhibits a particular case of type II solution (no degeneration of the solution occurs). (C) Spectral parameter belongs to the Akhmediev region, see Equation (61). The corresponding solution exhibits a particular case of type II general dynamics (no spatially periodic dynamic occurs).

fundamentally different wavefield dynamics, which was recently studied in Ref. 37. We illustrate the all three cases in Figure 5 (type II) and Figure 6 (type III). For the spectral parameter choice (61), the type II and III breathers represent a dark-bright standing wave group oscillating on the condensate background, see Figures 5A and 6A. Unlike their scalar counterpart, these breathers change the condensate phase to $\pi$, according to the asymptotics (52) and (58). Meanwhile, for the set of parameters (62), no degeneration in the solutions (49) and (55) occurs. The wavefield dynamic is similar to the type II and III cases, see Figures 5B and 6B, meaning that there are no nontrivial vector analogs of the rational rogue waves in the sense that the vector counterparts of the Peregrine breather are not localized simultaneously in space and time. Finally, when the spectral parameter belongs to the set (63), the solutions of the types II and III are moving localized breathers, see Figures 5C and 6C. Accordingly, these solutions are a particular case of the general type II and III breathers, and there are no nontrivial vector analogs of the periodic Akhmediev breather dynamic. In addition, we note that in the case of Akhmediev type eigenvalues (63), the transformation (59) boils down to a change of parameter $\alpha$ only, leaving the eigenvalue on the branchcut in the upper half of the $\lambda$-plane. In other words, for the eigenvalues (63), the type II and type III solutions merge into one class.

Each breather type corresponds to a certain branch of the dispersion law (2). To establish this connection, we consider the breathers’ tails as condensate perturbations and study them asymptotically. The role of small parameter plays the value $e^{-L}$, where $L$ is the characteristic distance from the breather center to the point where we study the breather tail. The latter can be done in the case of finite characteristic size, while for periodic Akhmediev breathers, one can similarly consider asymptotics at large times. We choose $L \gg 1$, so that we are far away from breather center and perform asymptotic expansion of the solutions (43), (49), and (55),
FIGURE 6  Vector breathers of type III with spectral parameters defined by Equations (61–63). (A) Spectral parameter belongs to the Kuznetsov region, see Equation (61). The corresponding solution is a standing dark-bright oscillating breather. (B) Spectral parameter belongs to the Peregrine region, see Equation (61). The corresponding solution exhibits a particular case of type III solution (no degeneration of the solution occurs). (C) Spectral parameter belongs to the Akhmediev region, see Equation (61). The corresponding solution exhibits a particular case of type III Tajiri-Watanabe dynamics (no spatially periodic dynamic occurs).

see Appendix Section A.1 for the computational details. For type I solution (43), the first-order terms represent the main (zero-order) asymptotic (46) plus a linear combinations of the first-order terms having structure \( p e^{2 \varphi} \) and \( \bar{p} e^{-2 \varphi} \), where \( p \) and \( \bar{p} \) are constants, while the functions \( \varphi \) are defined by (33). We write these exponents in the form \( e^{ikx+i\omega t} \). Considering, for example, \( e^{2\varphi} \), we obtain:

\[
e^{2\varphi} = e^{ikx+i\omega t}, \quad k = -2\zeta, \quad \omega = 2\lambda \zeta.
\]

Now using that \( \zeta(\lambda) = \sqrt{\lambda^2 + A^2} \), see Equation (11), we find \( \omega(k) = \pm ik \sqrt{A^2 - k^2/4} \). Thereby, the breather tails obey the first branch of the dispersion law \( \omega_I(k) \) with complex \( k \) and \( \omega \). The complexity of \( k \) in (67) means exponential decay of the breather tail. The same result can be obtained for the terms \( \bar{p} e^{2 \varphi} \).

For breather types II and III, a similar analysis gives as zero-order terms the asymptotic values (52) and (58) plus the first-order terms having structure \( p e^{\varphi_0 - \varphi} \) and \( \bar{p} e^{-\varphi_0 - \varphi^*} \) (for type II) and the terms of the structure \( p e^{-\varphi_0 - \varphi} \) and \( \bar{p} e^{-\varphi_0 - \varphi^*} \) (for type III). For all the listed exponents, one gets the second branch of the dispersion law \( \omega_{II}(k) \). For instance, in the case \( e^{\varphi_0 - \varphi} \), we obtain:

\[
e^{\varphi_0 - \varphi} = e^{ikx+i\omega t}, \quad k = \zeta - \lambda, \quad \omega = -\frac{1}{2}(\zeta - \lambda)^2,
\]

and finally easily retrieve the second branch \( \omega(k) = -k^2/2 \). Again, as in the type I case, the complexity of \( k \) means exponential decay of the breather tails.
4 | Resonance Interactions of Breathers

In this section, we study resonant interactions of the vector breathers, that is, a fusion of two breathers into one or decay of one breather into two, such that the characteristic wave vectors and frequencies of the breathers satisfy resonance conditions. The phenomena of inelastic mutual coherent structure transformations have been known for integrable systems since the work devoted to solitons in the three-wave model, see also Ref. 53. Another examples represent two-dimensional field theory, chiral field models, and relativistic \( O(1, 1) \) sine-Gordon model. In the case of zero background, such nontrivial interactions are possible for solitons in the three (or more than three) component systems, such as the mentioned three-wave model. However, in the presence of a nontrivial background, the constrain on the number of dimensions can be relaxed and nontrivial interactions can be observed already in two-component systems. Mathematically speaking, the nontrivial interactions are possible when the IST auxiliary problem admits different types of the eigenvalues, which can be merged into one point without solution degeneration, see Ref. 45. Recently, we have observed resonance interactions for vector breathers in the Manakov system (1), see our Letter and the recent paper. The resonance represents a three-breather process of a fusion or decay, where each of the three participating breathers has a different type either I, II, or III. Here, we present the theory of these nontrivial interactions in more detail than in Refs. 44, 59.

The resonant interaction of three vector breathers is described by the one-pole solution (27), when all the integration constants \( C_0, C_1, \) and \( C_2 \) are nonzero. First, we consider the situation when the eigenvalue is of general type, see Equation (60), and after that, we switch to the particular choices (61–63). In the general case, the solution has the following asymptotics:

\[
\psi_{1,2} \to A_{1,2} e^{-2i\alpha}; \quad x \to -\infty,
\]

\[
\psi_{1,2} \to A_{1,2}; \quad x \to +\infty,
\]

(69)

that on one side coincides with the asymptotic III (58), and on the other side can be obtained by linear superposition of the asymptotics I and II, see Equations (52) and (58). For definiteness, we consider the case \( \pi/2 > \alpha > 0 \). To investigate the asymptotic states of the solution with nonzero integration constants, we move at \( t \to -\infty \) to the reference frame,

\[
u = \text{const},
\]

(70)

\[
u_0 - \nu = \text{const},
\]

(71)

while at \( t \to \infty \), we move to the reference frame,

\[
u_0 + \nu = \text{const}.
\]

(72)

Recall that \( \nu \) and \( \nu_0 \) are defined in (34). Then, the conditions (70–72) lead to that in the expressions (32), \( e^{\varphi_0} \to 0, e^{-\varphi} \to 0, \) and \( e^{\varphi} \to 0 \) correspondingly, and for each of the reference frames, one can obtain exactly the single breather solutions (43), (49), or (55). The latter means that the asymptotic state of the resonance process represents single breathers of the types I, II, and III with the following integration constants \( C \): (I) \{0, \( C_1, C_2 \), (II) \{\( C_0, 0, C_2 \), and (III) \{\( C_0, C_1, 0 \). The interaction itself
Resonance interaction $I + II \rightarrow III$ of vector breathers described by single-pole solution (27) with $\mathbf{C} = \{1, 1, 1\}$. Blue lines in (a1,a2) and (b1,b2) show $|\psi_{1,2}|$ before ($t = -7.0$) and after ($t = 7.0$) the resonance interaction. The dotted green and red lines show local approximation of the breathers of the types I, II, and III (see the corresponding notations in the figures) by solution (27) with: (I) $\mathbf{C} = \{0, 1, 1\}$, green line in (a1,a2), (II) $\mathbf{C} = \{1, 0, 1\}$, red line in (a1,a2), and (III) $\mathbf{C} = \{1, 1, 0\}$, red line in (b1,b2). Panels (c1,c2) show spatio-temporal evolution of $|\psi_{1,2}|$ for the whole resonance interaction and its asymptotic state.

represents fusion of the breathers I and II into the breather III, what we denote as $I + II \rightarrow III$. We show the full resonance process and the single-breather approximation in Figure 7. Note that for $\pi > \alpha > \pi/2$, the resonance represents an opposite process—the decay of the breather III into the breathers I and II, that is, $III \rightarrow I + II$.

From the expressions (45), (51), and (57) describing breather characteristics, we find that the resonance process satisfies the standard resonant conditions:

$$k_I + k_{II} = k_{III},$$

(73)

$$\omega_I + \omega_{II} = \omega_{III}.$$  

(74)

Note that the resonance conditions (73) and (74) cannot be derived from the dispersion laws (2). Indeed, the characteristic breather wave vectors and frequencies follow from the fully nonlinear solutions (43), (49), and (55), meanwhile, the dispersion laws describe only the breather tails.

The resonance is always represented by either the process $I + II \rightarrow III$ or the process $III \rightarrow I + II$. Other configurations, such as a fusion of breather I cannot exist, what can be seen from the structure of the solution asymptotics, as soon as the resonance asymptotic (69) coincides with the asymptotic (58). In addition, such process as $I \rightarrow II + III$ is prohibited by the resonant conditions (73) and (74). Indeed, let us consider the region of spectral parameter, where $\pi/4 > \alpha > 0$. Then, according to Equations (45), (51), and (57), we find that $k_I > 0$, $k_{II} < 0$, and $k_{III} > 0$. In addition for the whole range of spectral parameter, $k_I > k_{III}$, so that $k_I$ cannot be represent as a sum of $k_{II}$ and $k_{III}$. 
To conclude this section, we consider the particular cases of the resonance interactions corresponding to the choices of the eigenvalues (61–63). For the eigenvalue of the Kuznetsov type (61), we observe a standing wave group exhibiting complex oscillations, see Figure 8A. In the limit (62) solution (27) with nonzero, $C_0$, $C_1$, and $C_2$ degenerate, leading, after resolving the uncertainty of the type $0/0$, to the following rational formula:

$$\psi_1 = A_1 + \frac{4\left[1 - 2A(x - x_1) - 2iA^2t\right]}{(e^{2A(x-x_0)} + 2 + 8A^2(x - x_1)^2 + 8A^4t^2)} \times$$

$$\times (A_2e^{A(x-x_0)-iA^2(t-t_0)/2} + A_1[1 + 2A(x - x_1) - 2iA^2t]),$$

$$\psi_2 = A_2 + \frac{4\left[1 - 2A(x - x_1) - 2iA^2t\right]}{(e^{2A(x-x_0)} + 2 + 8A^2(x - x_1)^2 + 8A^4t^2)} \times$$

$$\times (-A_1e^{A(x-x_0)-iA^2(t-t_0)/2} + A_2[1 + 2A(x - x_1) - 2iA^2t]),$$

(75)

where $x_0$, $x_1$, and $t_0$ are real valued parameters. The semirational solution (75) represents a localized wave group, decaying with time as $t^{-2}$. At certain coordinates, it exhibits a Perigine-type bump coexisting with the rest of the solution, see Figure 8B, that was previously studied in Ref. 38 in the context of vector rogue waves formation. Finally, for the Akhmediev type eigenvalues (63), we observe a moving type II breather which at some point decays into a Akhmediev type I wave excitation in one-half of space, plus a type III breather moving in another half of space, see Figure 8C. As we noted in the previous section, for the eigenvalues (63), the class of types II and III solution merges into one, which explains why the moving breathers before and after the resonance interaction in Figure 8C are similar. Note that Figure 8 demonstrates a particular case of the general scenario shown in Figure 7C when one uses the eigenvalues (63) for each of the three asymptotic breather states.
The two-eigenvalue solution of the model (1), see Equation (28) with $N = 2$, has the following general form:

$$
\psi_1 = A_1 + 2\tilde{M}_{12}/M,
$$

$$
\psi_2 = A_2 + 2\tilde{M}_{13}/M,
$$

where

$$
\tilde{M}_{12} = i\left[ -m_2 q_{12}^* q_{12} |q_2|^2 + n_2 q_{12}^* q_{21}(q_2, q_1^*) + n_1 q_{22}^* q_{11}(q_1, q_2^*) - m_1 q_{22}^* |q_1|^2 \right],
$$

$$
\tilde{M}_{13} = i\left[ -m_2 q_{13}^* q_{13} |q_2|^2 + n_2 q_{13}^* q_{21}(q_2, q_1^*) + n_1 q_{23}^* q_{11}(q_1, q_2^*) - m_1 q_{23}^* |q_1|^2 \right],
$$

$$
M = -[m |q_1|^2 |q_2|^2 - n (q_1, q_2^*)(q_2, q_1^*)].
$$

Here, the coefficients $m_{1,2} \equiv 1/\lambda_{1,2} - \lambda_{1,2}^*$, $n_{1,2} \equiv 1/\lambda_{1,2} - \lambda_{1,2}^*$, $m = m_1 m_2$, $n = n_1 n_2 \leq 0$. The vectors $q_{1,2}$ are defined in (32). In the parameterization (37), the coefficients can be written as:

$$
m_i = \frac{2|r_i|^2}{A(r_i^* - r_i)(1 + |r_i|^2)}, \quad n_i = \frac{2r_i r_j^*}{A(1 + r_i r_j^*)(r_j^* - r_i)}, \quad i = 1, 2, \quad j = 3 - i;
$$

$$
m = \frac{4|r_1 r_2|^2}{A^2(1 + |r_1|^2)(1 + |r_2|^2)(r_1 - r_1^*)(r_2 - r_2^*)} = \frac{4A^2 \sin \alpha_1 \sin \alpha_2 \cosh \xi_1 \cosh \xi_2}{4 \sin \alpha_1 \sin \alpha_2 \cosh \xi_1 \cosh \xi_2}^{-1},
$$

$$
n = \frac{4|r_1 r_2|^2}{A^2(1 + r_1 r_2^*)(1 + |r_2|^2)(r_1 - r_2^*)}.
$$

The solution (76) describes a wide family of vector breather interactions. It is characterized by two eigenvalues and six integration constants $C_{ij}$, $i = 1, 2, \quad j = 1, 2, 3$. Depending on their choice, the solution (76) represents either two elastically colliding breathers, a breather plus fusion/decay resonance wave pattern, or a combination of two resonance wave patterns. Taking into account three fundamental types of breathers, we obtain more than 10 scenarios of the breather interactions described via Equation (76). Here, we do not consider all of them, instead focus on fundamental aspects of the vector breather interactions, such as collision wavefield profiles and asymptotic states of the breathers at large times. For the latter question, we compute exact formulas describing the shifts of the positions and phases acquired by the breathers after their collision.

We start with elastic collisions, that is, we put zero one of the components in vectors $C_1$ and $C_2$. For each breather, we chose parameterization of its integration constant according to the breather type. We begin with the case of type I breathers collision, that is, $I + I \rightarrow I + I$. First, we choose the parameterization for the vectors $C_1$ and $C_2$ according to Equation (42):

$$
C_{i,0} = 0, \quad C_{i,1} = C_{i,2}^{-1} = e^{i \text{Im}[\xi_i] \delta_i + i \delta_i^* / 2}, \quad i = 1, 2.
$$
Each of the two breathers changes the phase of the condensate according to (46), so that the asymptotic of the I+I solution reads as:

$$\psi_{1,2}^{I+I} \rightarrow A_{1,2} e^{\pm 2i(\alpha_1 + \alpha_2)}; \quad x \rightarrow \pm \infty. \quad (80)$$

The asymptotic states of the scalar two-breather NLSE solution, which is linked to the vector case through the transformation (40), have been found in Ref. 60, see also Ref. 61 for additional details. Here, we reobtain this result. We consider the two-breather solution in the reference frame moving with the group velocity $V_i$ of the breather $i$ ($i = 1, 2$), which collides with the breather $j$ ($j = 2, 1$). Then, we analyze solution (76) at large times, see computational details in Ref. 61, and also in Appendix Section A.3, and find the asymptotic state for each of the breathers. The full asymptotic state of the solution (76) at $t \rightarrow \pm \infty$ represents single breathers with shifted position $\delta$ and phase $\theta$ parameters, as well as shifted general phase,

$$\psi_{1,2}^{I+I}(\lambda_1, \delta_1, \theta_1; \lambda_2, \delta_2, \theta_2) \rightarrow \begin{cases} e^{\mp 2s_1 \alpha_2} \psi_{1,2}^I(\lambda_1, \delta_1 + \delta_{0,1}^\pm, \theta_1 + \theta_{0,1}^\pm), & \text{at } x \sim V_1 t, \\ e^{\mp 2s_2 \alpha_1} \psi_{1,2}^I(\lambda_2, \delta_2 + \delta_{0,2}^\pm, \theta_2 + \theta_{0,2}^\pm), & \text{at } x \sim V_2 t, \end{cases} \quad (81)$$

where the sign $s_i = \pm 1$, and the position shift $\delta_{0,i}^\pm$, and the phase shift $\theta_{0,j}^\pm$ are defined at $t \rightarrow \pm \infty$ by the following expressions:

$$s_i \equiv \text{sign}(V_j - V_i),$$

$$\delta_{0,i}^\pm \equiv \mp s_i \frac{l_i(\lambda_i)}{2} \log \left| \frac{(r_i - r_j^*) (1 + r_i r_j^*)}{(r_i - r_j)(1 + r_i r_j^*)} \right|^2,$$

$$\theta_{0,j}^\pm \equiv \mp s_i \text{Arg} \left[ \frac{(r_i^* - r_j^*)(1 + r_i r_j^*)}{(r_i^* - r_j)(1 + r_i r_j^*) \sin \alpha_j} \right]. \quad (82)$$

Figure 9 shows an example of two breathers of types I elastic collision and also illustrates the asymptotic formula (81). One can see the change of the breathers’ phases and positions by comparing the final wavefield state with the situation when one of the breathers travels alone, that is, without collision. For this and some of the subsequent examples of elastic two-breather interactions, we use the following set of parameters:

$$A_1 = 1, \quad A_2 = 1;$$

$$\alpha_1 = \pi/5, \quad \xi_1 = 1/4, \quad \theta_1 = 0, \quad \delta_1 = 0,$$

$$\alpha_2 = 4\pi/5, \quad \xi_2 = 1/2, \quad \theta_2 = 0, \quad \delta_2 = 0. \quad (83)$$

In general case, there are six possible combinations of the elastic two-breather interactions $B_i + \bar{B}_j \rightarrow B_i + \bar{B}_j$. Here, $B$ and $\bar{B}$ stand for one of the three breather types, while the subscripts $i = 1, 2$ and $j = 2, 1$ indicate the breather index number. Note that the indexes are not related to the breather type and can be freely chosen, that is, they indicate which breather we call the first and which the second. The asymptotic state of this interaction represents the following generalization
Elastic collision $I + I \rightarrow I + I$ of vector breathers with spectral parameters defined by Equation (83). (A,B) $|\psi_j|$ and Arg[$\psi_j$], where Arg means complex phase, after the breathers collision at $t = 4.5$. (C) Spatio-temporal plot of the wavefield evolution. The dotted green and red lines in (A,B) show a local approximation of the breathers after collision by single-breather solutions from the asymptotic (81). Thin black dashed line in (A) shows how the first breather would have been if it had been traveling alone. Both wavefield components shown on the same plot because they coincide after the rescaling of the amplitude, see Equation (38).

of Equation (81):

$$
\psi_{1,2}^{B_i + \tilde{B}_j} (\lambda_i, \delta_i, \theta_i; \lambda_j, \delta_j, \theta_j) \rightarrow \begin{cases} 
e^{2i\delta^+_{i,j}} \psi_{1,2}^B (\lambda_i, \delta_i + \delta^+_{0,i} \theta_i + \theta^+_{0,i}), & \text{at } x \sim V_{B_i} t, \\ e^{2i\delta^+_{j,i}} \psi_{1,2}^\tilde{B} (\lambda_i, \theta_j + \delta^+_{0,j} \theta_j + \theta^+_{0,j}), & \text{at } x \sim V_{\tilde{B}_j} t, \\ \end{cases}
$$

(84)

where the shifts of positions $\delta$, phases $\theta$, and general phases are defined as follows:

$$
\{\delta^-_{0,i}, \delta^+_{0,i}, \theta^-_{0,i}, \theta^+_{0,i}, \beta^-_{i}, \beta^+_{i}\} = \begin{cases} \{a_i, b_i, 0, 0, 0, 0\}, & \text{at } s_i = 1 \\ \{b_i, a_i, 0, 0, 0, 0\}, & \text{at } s_i = -1. \\ \end{cases}
$$

(85)

As before, in Equations (84) and (85), the subscript indexes can be freely chosen to distinguish breathers, that is, $i = 1, 2$ and $j = 2, 1$. Meanwhile, $B$ and $\tilde{B}$ indicate the breather type. The lower and the first upper indexes of the coefficients $a$, $b$, and so on from (85) indicate the breather for which the shift is presented, while the second upper index means the breather with which the studied one interacts. The lower index shows only what we call the studied breather, that is, either the first or the second one. Meanwhile, the upper indexes also indicate the type of interacting breathers. For example, $b_i^{B_i, \tilde{B}_j}$ at $s_i = 1$ ($a_i^{B_i, \tilde{B}_j}$ at $s_i = -1$) represents the correction $\delta^+_{0,i}$ to the position of the $i$-th breather of type $B$ at large time after the collision with the breather of type $\tilde{B}$.

Similar to the formulas (83), we find the rest of the coefficients in (85) by asymptotic analysis of the solution (76) at large times, see details in the Appendix Section A.3. We summarize these results in Table A.1, which we present in Appendix Section A.5. In addition, these tables provide asymptotic wavefield values at $x \to \pm \infty$ for each of the two-breather configuration. Consider a concrete example of how to use the notations (85). Suppose the process is $I + II \rightarrow I + II$ and we need to know the asymptotic shifts of positions at large positive time. We say, for instance, that $B_1 = I$ and $\tilde{B}_2 = II$. For the first breather, we find $\delta^+_{0,1} = b_1^{II}$ if $s_1 = 1$ and $\delta^+_{0,1} = a_1^{II}$ if $s_1 = -1$, while for the second one, $\delta^+_{0,2} = b_2^{II}$ if $s_2 = -s_1 = 1$ and $\delta^+_{0,2} = a_2^{II}$ if $s_2 = -1$. Then, we go to Table A.1 and find the corresponding values of the coefficients $b$ or $a$ in its fourth row.
Elastic collision $\text{II} + \text{II} \rightarrow \text{II} + \text{II}$ of vector breathers with spectral parameters defined by Equation (83). (A,B) $|\psi_1|$ and Arg [$\psi_1$], where Arg means complex phase, after the breathers collision at $t = 25.0$. (C) Spatio-temporal plot of the wavefield evolution. The dotted green and red lines in (A,B) show a local approximation of the breathers after collision by single-breather solutions from the asymptotic (84). Thin black dashed line in (A) shows how the second breather would have been if it had been traveling alone.

By analogy with Figure 9, Figure 10 shows one example of elastic collisions of two equal-type breathers $\text{II} + \text{II} \rightarrow \text{II} + \text{II}$, while Figure 11 shows one mixed case $\text{I} + \text{III} \rightarrow \text{I} + \text{III}$. More examples are presented in Appendix Section A.4. The asymptotic result (84) is illustrated by approximation of the two breathers after mutual collision by single breathers with appropriately shifted phases and position. Note that in addition to (83), we also use for the illustrations the following set of breather parameters:

$$A_1 = 1, \quad A_2 = 1;$$
$$\alpha_1 = \pi/4, \quad \xi_1 = 1/4, \quad \theta_1 = 0, \quad \delta_1 = 0,$$
$$\alpha_2 = \pi/5, \quad \xi_2 = 1/2, \quad \theta_2 = 0, \quad \delta_2 = 0. \quad (86)$$

While the formulas (84) and (85) allow to describe the asymptotic state of the two-breather interaction mathematically, the physical meaning plays the total values of the position and phase shifts $\Delta \delta_{\text{II},\text{II}}$ and $\Delta \theta_{\text{II},\text{II}}$ acquired by the breather $\text{III}_\text{i}$ as a result of collision with the breather $\text{III}_\text{j},$

$$\Delta \delta_{\text{II},\text{II}} = \delta_{0,i}^+ - \delta_{0,i}^-,$$  
$$\Delta \theta_{\text{II},\text{II}} = \theta_{0,i}^+ - \theta_{0,i}^-.$$  

The total shift values (87) and (88) can be found using the coefficients listed in Table A.1 of the Appendix Section A.5. They represent a broad family of expressions. Here, we focus on
FIGURE 11 Elastic collision $I + III \rightarrow I + III$ of vector breathers with spectral parameters defined by (86). (A,B) $|\psi_1|$ and $\text{Arg}[\psi_1]$, where $\text{Arg}$ means complex phase, after the breathers collision at $t = 12.0$. (C) Spatio-temporal plot of the wavefield evolution. The dotted green and red lines in (A,B) show a local approximation of the breathers after collision by single-breather solutions from the asymptotic (84). Thin black dashed line in (A) shows how breathers would have been if they had been traveling alone.

particular features of the space shifts which are responsible for qualitative changes of the breather collision behavior.

In Table 1, we summarize the total spatial shifts in all possible breather interaction scenarios. As was shown in Ref. 61, for scalar NLSE breathers participating in the head-on collisions, the value of the spatial shift in the direction of the breather propagation can be positive, negative, and even zero depending on the breather parameters, which means that breather can move forward or backward relative to its initial trajectory or remain on it. In the vector case, we have a similar situation for those interactions where the type I breathers participate. More precisely, the sign of the spatial shift depends on which breather is faster, what is controlled by the value of $s$. However, the fastest breather not necessarily moves forward, and in addition in some cases, the sign of $\Delta \theta_{B_i,B_j}$ can be switched by changing breather parameters keeping $s_i$ the same. Note that this situation is unusual taking into account that for NLSE solitons, the fastest one always moves forward with respect to its propagation direction, while another one moves backward.

To illustrate the described above behavior of the vector breathers, we consider the following particular case of parameters:

$$\alpha_1 = \alpha, \quad \alpha_2 = \pi - \alpha, \quad \xi_1 = \xi \quad \xi_2 = \xi,$$

and assume that the first breather is always of type I, that is, $B_1 = I_1$. When $B_2 = I_2$ or $B_2 = III_2$, the breathers collide in the head-on manner, while for $B_2 = II_2$, the collision is overtaking. In addition, the absolute value of the breather I velocity is always bigger than for the breathers II and III, see Equations (45), (51), and (57). The latter means that the sign of $s_i$ does not change when changing $\alpha$ or $\xi$. Meanwhile, the sign of the total spatial shift can be changed by changing $\xi$ at a fixed value of $\alpha$. Indeed, the logarithm in the corresponding shift expressions, see Table 1,
| Process | Total space shift |
|---------|------------------|
| $I_i + I_j \rightarrow I_i + I_j$ | $\Delta \delta^{I_i,I_j}_{I_i} = -s_i l_i(\lambda_i) \log \left| \frac{(r_i - r_j')(1 + r_i r_j)}{(r_i - r_j)(1 + r_i r_j')} \right|^2$ |
| $II_i + II_j \rightarrow II_i + II_j$ | $\Delta \delta^{II_i,II_j}_{II_i} = s_i l_I(\lambda_i) \left( \log \left[ 1 - \frac{n[1 + r_j' r_j]}{m(1 + |r_j'|^2)(1 + |r_j|^2)} \right] + \log \left[ 1 - \frac{n}{m} \right] \right)$ |
| $III_i + III_j \rightarrow III_i + III_j$ | $\Delta \delta^{III_i,III_j}_{III_i} = s_i l_{III}(\lambda_i) \left( \log \left[ 1 - \frac{n[1 + r_j' r_j]}{m(1 + |r_j'|^2)(1 + |r_j|^2)} \right] + \log \left[ 1 - \frac{n}{m} \right] \right)$ |
| $II_i + I_j \rightarrow II_i + I_j$ | $\Delta \delta^{II_i,I_j}_{I_i} = s_j l_{II}(\lambda_j) \log \left| \frac{(r_j - r_i')(1 + r_i r_i')}{(r_j - r_i)(1 + r_j r_i')} \right|^2$ |
| $I_i + III_j \rightarrow I_i + III_j$ | $\Delta \delta^{I_i,III_j}_{I_i} = -s_i l_{III}(\lambda_i) \left( \log \left[ 1 - \frac{n[r_i - r_i']^2}{m(1 + |r_i|^2)(1 + |r_i'|^2)} \right] + \log \left[ 1 - \frac{n}{m} \right] \right)$ |

Left column indicates the collision process type in the form $B_i + \tilde{B}_j \rightarrow B_i + \tilde{B}_j$, where $i = 1 \text{ or } 2$, while $j = 2 \text{ or } 1$, respectively. Right column presents the corresponding values of $\Delta \delta^{B_i,\tilde{B}_j}_{B_i}$ and $\Delta \delta^{\tilde{B}_j,B_i}_{\tilde{B}_j}$ for each of the two breathers participating in the interaction. When $B$ has the same type as $\tilde{B}$, we leave only $\Delta \delta^{B_i,\tilde{B}_j}_{B_i}$ value. The meaning of the indexes has been explained in the main text. For example, $\Delta \delta^{I_i,III_j}_{I_i}$ describes the total space shift acquired by $i$th breather of type I as a result of a collision with the breather of type III.

can be simplified under the constrain (89) as follows:

$$\log \left| \frac{(r_1 - r_2')(1 + r_1 r_2)}{(r_1 - r_2)(1 + r_1 r_2')} \right|^2 = \log \left( \frac{\sinh^2 \xi}{\cos^2 \alpha (\cos^2 \alpha \sinh^2 \xi + \sin^2 \alpha \cosh^2 \xi)} \right).$$

(90)

Now, one sees that for a fixed $\alpha \neq \pi/2$, the argument of the logarithm (90) changes from zero to $\cos^{-2} \alpha$, when $\xi$ changes from zero to infinity. At the point $\xi_0$ satisfying transcendental condition $\sinh^2 \xi_0 = \cos^2 \alpha (\cos^2 \alpha \sinh^2 \xi_0 + \sin^2 \alpha \cosh^2 \xi_0)$, the shift changes sign, what we illustrate in Figure 12A. In addition, Figure 12B,C shows examples of the overtaking collision $I + II \rightarrow I + II$, where I overtakes II, for $\xi < \xi_0$ (b) and $\xi > \xi_0$ (c). One can see that in the case (b), the first breather I shifts forward along its trajectory, while the second breather II shifts backward. In the case (c), the geometry of collision is the same, that is, the first breather being faster overtakes the second one; however, the signs of the shift are opposite. The latter can be easily seen for the breather II and poorly pronounced for the breather I due to a relatively small absolute value of the shift, see also Figure 12A.
FIGURE 12 Illustration of the total space shifts behavior in the collisions $I_1 + B_2 \rightarrow I_1 + B_2$, where $B_2$ is either type I, II, or III breather. (A) Dependence of the total space shift for the breathers with parameters defined by (89) as a function on $\xi$ and fixed $\alpha_1 = 6\pi/16$. Black line corresponds to the case $B_2 = I_2$, blue lines to the process $B_2 = I_I$ (solid blue line for the breather $I_1$ and dashed one for the breather $II_2$), and red lines to the case $B_2 = III_2$ (solid red line for the breather $I_1$ and dashed one for the breather $III_2$). The shift curves change sign at the point $\xi_0$, where the argument of the logarithm (90) turns unity. (B,C) Spatio-temporal plots for collisions $I_1 + II_2$ in the cases $\alpha_1 = 9\pi/16$, $\xi = 0.05 < \xi_0$ (B), and $\xi = 0.8 > \xi_0$ (C).

In addition, we note that the sign of the logarithmic expressions in the total space shift formulas of the processes $II + II \rightarrow II + II$ and $III + III \rightarrow III + III$ is always negative, that is,

$$
\log \left[ 1 - \frac{n|1 + r_i^* r_j|^2}{m(1 + |r_i|^2)(1 + |r_j|^2)} \right] + \log \left[ 1 - \frac{n}{m} \right] < 0,
$$

$$
\log \left[ 1 - \frac{n|1 + r_i^* r_j|^2}{m(1 + |r_i|^2)(1 + |r_j|^2)} \right] + \log \left[ 1 - \frac{n}{m} \right] < 0. \tag{91}
$$

The inequalities (91) mean that the sign of the total shifts $\Delta \delta_I^{II,II}$ and $\Delta \delta_I^{III,III}$ is determined only by the sign of $s_i$. The latter is different for the processes $II + II \rightarrow II + II$ and $III + III \rightarrow III + III$, because the signs of $V_{II}$ and $V_{III}$ are opposite, see Equations (51) and (57). Thus, for the same set of eigenvalues, the signs of $\Delta \delta_I^{II,II}$ and $\Delta \delta_I^{III,III}$ are also opposite.

6 PARTICULAR CASES OF VECTOR TWO-BREATHER SOLUTION

The resonant fusion and decay of breathers presented in Section 4 is based on the single-eigenvalue solution with nonzero integration constants $C_0$, $C_1$, and $C_2$. However, the same expression can be also obtained from the two-eigenvalue solution (76) in the case of merging eigenvalues, that is, when $\lambda_1 = \lambda_2 = \lambda$. More specifically, one must choose the integration constants in (76) so that each breather has a different type, either I, II, or III. In other words, each of the vectors $C_1$ and $C_2$ has one zero in its components, and the positions of these zeros are different. For example, consider the case I+II, that is, $C_{0,1} = 0$ and $C_{1,2} = 0$. We substitute $\lambda_1 = \lambda_2 = \lambda$ in (76), so that $r_1 = r_2 = r$. In addition, we change $\alpha \rightarrow -\alpha$, and end up with the resonant solution described in Section 4, characterized by the following three nonzero integration constants:

$$
C_0 = \left[C_{1,1} C_{2,2} (1 + r^2)/C_{0,2} \right]^*, \quad C_1 = C_{2,1}^*, \quad C_2 = -C_{1,1}^*. \tag{92}
$$
Zakharov and Manakov proposed the interpretation of the resonant interaction as a result of merging eigenvalues in Ref. 45. They found that when \( \lambda_1 \to \lambda_2 \), the three-wave system soliton acquires infinite space shifts due to collision. The same situation takes place for vector breathers. Indeed, assuming \( \lambda_1 \to \lambda_2 \) in (87), see also Table 1, for the processes \( \text{I+II} \to \text{I+II} \) or \( \text{I+III} \to \text{I+III} \), we obtain an infinite value of the space shift. To get a feeling of this limit, we plot the spatio-temporal diagrams for the breather collisions corresponding to a set of small differences \( \varepsilon \) between the breather eigenvalues, defined as:

\[
\alpha_1 = \alpha_2 = \alpha, \quad \xi_1 = \xi, \quad \xi_2 = \xi_1 + \varepsilon,
\]

(93)

see Figure 13. At small \( \varepsilon \), the point where the breathers collide transforms into an increasing straight junction, which begins at the point of the breathers association and later ends at the point of the breathers separation. The length of the junction is of order of the total shift \( \Delta \delta^{\text{I,II}} \), whose dependence on \( \varepsilon \) in case of eigenvalues (93) is defined by the following logarithm, see Table 1,

\[
\ln \left( \frac{(r_i - r_j)(1 + r_i r_j)}{(r_i - r_j)(1 + r_i r_j)} \right)^2 = \ln(1 + X), \quad X = \frac{4a b \sin^2 \alpha [(a^2 - 1)(b^2 - 1) + 4ab \cos^2 \alpha]}{(a - b)^2(1 + a b)^2},
\]

(94)

where \( a \equiv e^{-\xi_i}, b \equiv e^{-\xi_j} \). In case of small \( \varepsilon \), one can see that \( a - b \sim \varepsilon \) and thus \( \Delta \delta^{\text{I,II}} \sim \ln(1/\varepsilon) \); that is, the junction logarithmically increases with decreasing \( \varepsilon \). The logarithmic behavior of the junction length can be also seen in Figure 13, where we plot three collision portraits (a), (b), and (c) with \( \varepsilon = 10^{-2}, 10^{-4}, \) and \( 10^{-6} \) correspondingly. The beginning of the junction remains in the same position on the spatio-temporal diagram, while its end goes to infinity. The junction itself became the breather of the type different from the types of the colliding breathers. Finally, at \( \varepsilon = 0 \), that is, when the eigenvalues merge precisely, the two-breather solution transforms into the resonance solution. As we noted above, a similar transformation of the two-soliton collision into the resonance pattern is described in Ref. 45 for the three-wave system.

Now, let us consider the particular case of vector breathers, which emerges when the breathers are placed close to each other, and their group velocities coincide. Such nonlinear wave complex, also called breather molecule, has been studied theoretically in the scalar case, see Refs. 15, 17, 62, 63, and recently reproduced experimentally in a nearly conservative optical fiber system. Here,
we briefly consider its vector generalization. The group velocity condition of the two-breather is as follows:

\[ V_{B_1} = V_{B_2}. \]  

(95)

For example, in the case II + II, and fixed parameters \( \xi_1, \xi_2, \alpha_1 \), the condition (95) results in \( \alpha_2 = \arccos(e^{-\xi_1+\xi_2 \cos \alpha_2}) \). Figure 14 shows the typical two-breather molecules I + I, II + II, and III + III. In the general case, the breather molecule exhibits quasi-periodic oscillations because the breathers’ individual oscillation frequencies are not commensurate, see Figure 14. As was shown in Ref. 17, the commensurate condition for the breather oscillation frequencies in the scalar case leads to a high-order polynomial equation. In cases when the equation has solutions in the region of breather parameters validity, then the corresponding breather molecule is periodic, which was observed experimentally.17 We leave the question of the construction of periodic vector breather molecules to further studies.

Finally, we discuss one more important case—the so-called superregular scenario of breather interactions.15,27,60 The superregular interactions represent a near annihilation into a small-amplitude localized condensate perturbation of a pair of scalar NLSE breathers resulting from their collision. A reverse process—the emergence of breathers—is also possible, that is, the formation of breathers due to the perturbation growth and evolution. The latter makes superregular breathers important exactly solvable scenarios of the nonlinear stage of modulation instability development, see also, Refs. 16, 64, 65. Is it natural to ask whether finding nontrivial generalizations of the scalar superregular breathers in the vector case is possible? The recent study66 has not found nontrivial vector analogs of the superregular breathers emerging from small-amplitude condensate perturbations. Here, we present our analysis of the question.

We use the mathematical interpretation of the breathers annihilation provided in Ref. 15, according to which the folding of two breathers into one small localized condensate perturbation emerges due to the exact cancellation of the numerator of the two-breather solution in the case when,

\[ \xi_1 = \xi_2 = 0, \quad \alpha_1 = -\alpha_2 = \alpha, \]  

(96)

so that the NLSE solution represents a pure unperturbed condensate. Then, in case when \( \xi_1 = \xi_2 = \epsilon \ll 1 \), the solution at the moment of collision has to be a small localized condensate
Collision profiles of vector two-breather solutions of types (A) I + I, (B) II + II, (C) III + III at \( t = 0 \) and parameters \( \xi_1 = \xi_2 = \epsilon, \alpha_1 = -\alpha_2 = \pi/3, \theta_{1,2} = \pi/2, \delta_{1,2} = 0 \). The value of parameter \( \epsilon \) is 0.4 (blue curves), 0.2 (red curves), and 0.1 (green curves). Panel (B) shows a trivial vector analog of the superregular folding of a pair of scalar NLSE breathers into small-amplitude condensate perturbations. Panels (B) and (C) illustrate that there is no such folding for the breathers of types II and III. Only the first wavefield component is shown; the behavior of \( \psi_2 \) is analogous.

perturbation because the breathers having opposite group velocities collide in the head-on manner so that no other continuous limit to the condensate solution at \( \epsilon \rightarrow 0 \) is possible, see details in Ref. 15.

In the vector case, the first numerator \( \tilde{M}_{12} \) of the two-breather solution (76) under the constrain (96) simplifies as follows (for the second numerator \( \tilde{M}_{13} \) the derivations are analogous):

\[
\tilde{M}_{12} = i\tilde{m}(q_{12}q_{23} - q_{13}q_{22})(q_{11}^*q_{23}^* - q_{13}^*q_{21}^*),
\]

(97)

where \( \tilde{m} = m_1 = m_2 \). One can see that the numerator (97) is exactly canceled at any \( x \) and \( t \), when \( q_{12} = hq_{13}, \ q_{22} = hq_{23} \), where \( h \) is an arbitrary constant. The latter happens only when \( C_{0,1} = 0, C_{0,2} = 0 \), so that \( h = A_2/A_1 \), that is, when both breathers are of type I and the vector two-breather solution is the trivial generalization of the scalar one, see transformation (40).

In the case when one or both breathers are of type II or III, the numerator (97), can be canceled only together with denominator \( M \), see (76), so that the vector two-breather solution transforms into a degenerate one (one needs to resolve the indeterminate form \( 0/0 \); see examples of degenerate scalar breathers in Refs. 15, 67. Here, we do not study the degenerate limit and instead focus on the behavior of vector two-breather solution at \( \xi_1 = \xi_2 = \epsilon \ll 1 \). Figure 15 shows two-breather solutions of different types at \( t = 0, \alpha = \pi/3 \) and different values of \( \epsilon \). In addition, we choose breather phases and positions as \( \theta_{1,2} = \pi/2 \) and \( \delta_{1,2} = 0 \), which corresponds to the most efficient folding of the breathers in the scalar case, see details in Ref. 15. One can see the trivial vector analog (I + I) of the superregular folding in Figure 15A, which shows that the amplitude of the condensate perturbation (produced by the breather collision) decreases when decreasing \( \epsilon \). At the same time, in the cases of collisions II + II and III + III shown in Figure 15B,C, the amplitude of the wavefield remains large even at very small \( \epsilon \). The latter means that instead of the superregular folding, the vector breathers at \( \epsilon \rightarrow 0 \) tend to a degenerate limit, as we discussed above. We conclude that the vector breathers of types II and III do not participate in the MI development from small-amplitude perturbations, which is consistent with Ref. 66 and also with the correspondence of these types of vector breathers to the stable branch of the dispersion law \( \omega_{II} \), see Section 3.
7 CONCLUSIONS AND DISCUSSIONS

In this work, we have studied theoretically the vector breathers and their interactions in the framework of the two-component NLSE—the Manakov system. Our model implies the focusing-type nonlinearity in both system components, see Equation (1), and the presence of a nonzero constant background; see also Ref. 68 for the defocusing case and gray vector solitons. As a starting point, we take the vector variant of the dressing method,44 the studies on the three fundamental breather types I, II, III,43 and on the resonance vector breather interactions.44 Then, we reveal the connection between breather type and the branches of the dispersion law, analyze important particular cases of the breather solutions, and describe the asymptotic state of the breather interactions by computing spatial and phase shifts acquiring by breathers as a result of collisions. The three types of breathers generate a family of nine different shift expressions, which we summarize using Equation (85) and the corresponding to it Table A.1. We find that the spatial shifts of the vector breathers can change sign depending on the spectral parameters without changing the sign of the difference between the breather velocities. Finally, the obtained shift expressions allowed us to interpret the resonance fusion and decay of breathers as a limiting case of infinite space shift in the case of merging breather eigenvalues. In the future, the shift expressions can be used to build a spectral theory of vector breather gases, similar to the recent scalar case studies, see Ref. 69.

The breathers of types II and III exhibit fundamentally different wavefield dynamics when compared to the breathers of type I. The latter, as a trivial vector generalization of the scalar NLSE breathers, see transformation (40), describes particular scenarios of the MI and formation of rogue waves. In contrast, the breathers of types II and III, belonging to the stable branch of the dispersion law, do not participate in the development of MI from small-amplitude perturbations. Indeed, for the Akhmediev-type eigenvalues, see Equation (63), as well as for superregular-type eigenvalues, see Section 6, the breathers of types II and III exhibit a localized condensate pulsations which are never small. On the other side, the breathers of types II and III represent an important class of localized pulsating exact solutions and, together with the type I breathers, are responsible for inelastic resonance interactions, see Section 4.

A fundamental question that needs further study is the eigenvalues portraits of localized small-amplitude arbitrary-shaped perturbations of the vector condensate. As was shown in Ref. 65 for the scalar case, superregular eigenvalue pairs can be embedded into small-amplitude and arbitrary-shaped condensate perturbations under certain conditions. Meanwhile, our present studies show that the II and III breathers cannot be folded in such a way, at least within the standard superregular scenario. All this leads us to a conjecture that only type I superregular breathers exist in the locally perturbed vector condensate. Our conjecture can be tested in the future using numerical computation of the eigenvalue spectrum of the auxiliary system (3). Moreover, we think that the vector arbitrary-shaped perturbation evolution is driven by the interaction between type I breathers and the unstable continuous spectrum; see the works65,70,71 explaining how it happens in the scalar case.

We believe that our study will also benefit to the rapidly developing area of statistical description of nonlinear waves in integrable systems—the so-called integrable turbulence, see Refs. 29, 72–77. The first studies on the random polarized nonlinear waves have been recently obtained in Ref. 41 and we think that our analysis of the vector breather interactions will provide new insights into this complex subject. Meanwhile, experimental observation of various aspects of the integrable scalar NLSE dynamics and statistics has been successfully performed in many different works, see, for example, Refs. 14, 17–24, 78, 79. In addition, the development of vector MI and vector dark
rogue waves has been studied experimentally in a Manakov fiber system.\textsuperscript{40,80} At the same time, the experimental observation of vector breathers represents a challenging task for further studies, see also, Ref. 80 where experimental conditions for experimental observation of vector breathers have been discussed.

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**DATA AVAILABILITY STATEMENT**

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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APPENDIX A

In the Appendix section, we provide details of theoretical derivations, additional illustrations, and full expressions for the shift coefficients.

A.1 | Linearization of the Manakov system for small-amplitude perturbations

In this section, we derive the two branches of the dispersion law for the Manakov system, see Equation (2) in the main part of the work. We consider small-amplitude perturbations of the condensate solution \( \delta \psi_1 \) and \( \delta \psi_2 \), so that the wavefield components are,

\[
\psi_1 = A_1 + \delta \psi_1, \\
\psi_2 = A_2 + \delta \psi_2.
\]

(A.1)

Then, we substitute (A.1) into the Manakov system (1) and linearize it by leaving only the first-order terms proportional to \( \delta \psi_1 \) or \( \delta \psi_2 \). It is convenient to consider the following new variables,

\[
\delta \psi_j + \mathbf\delta \psi_j^* = f_j, \\
\delta \psi_j - \mathbf\delta \psi_j^* = g_j,
\]

(A.2)

where \( j = 1 \) or \( 2 \). Using (A.2), the linearized Manakov system can be obtained in the following form:

\[
ig_{jt} + \frac{1}{2}f_{jxx} + 2A_j(A_1f'_1 + A_2f'_2) = 0, \\
if_{jt} + \frac{1}{2}g_{jxx} = 0.
\]

(A.3)

We assume that the initial perturbations are simple linear harmonics, that is, \( \delta \psi_j \sim e^{i(kx-\omega t)} \), and, accordingly, choose,

\[
f_j = f^0_j \text{Re}[e^{i(kx-\omega t)}] = f^0_j \cos(kx - \omega t), \\
g_j = ig^0_j \text{Im}[e^{i(kx-\omega t)}] = ig^0_j \sin(kx - \omega t),
\]

(A.4)
where \( f_j^0 \) and \( g_j^0 \) are arbitrary real-valued constants. Substituting expressions (A.4) into Equation (A.3), we obtain the two branches of the dispersion law \( \omega_1(k) = \pm k \sqrt{k^2/4 - A^2} \) and \( \omega_{II}(k) = \pm k^2/2 \), see Equation (2), as the consistency condition of the linearized system (A.3).

**A.2 Linearization of the breathers’ tails**

Now, we show how to derive the connection between the breather type and the branch of the dispersion law, which was discussed in Section 3. We consider the small-amplitude breathers’ tails as condensate perturbations and study them in a linear approximation. Let us for definiteness assume that we are in the region of space far away from the breather center where \( \text{Im}[\zeta]x > 0 \). Then, for the breather of type I, we have the following small parameter,

\[
|e^{-\varphi}| \sim e^{-\text{Im}[\zeta]x} = \varepsilon \ll 1,
\]

so that \( |e^\varphi| \sim 1/\varepsilon \gg 1 \). Using the smallest of \( \varepsilon \), we present the first component (the computations are similar for the first and second components, so we present only the first ones) of the type I single-breather solution as:

\[
\psi_1 = A_1 + \frac{2i(\lambda_1 - \lambda_1^*)q_1^*q_2}{|q|^2} \approx A_1 + \frac{F + \delta f}{G + \delta g} \approx A_1 + \frac{F}{G} + \frac{\delta f}{G} - \frac{\delta g}{G^2},
\]

where the leading order terms \( F, G \sim 2/\varepsilon \), and the terms \( \delta f, \delta g \sim 1 \) can be written as follows:

\[
F = p_1 e^{\varphi + \varphi^*}, \quad \delta f = p_2 e^{\varphi - \varphi^*} + p_3 e^{\varphi - \varphi^*},
\]

\[
G = q_1 e^{\varphi + \varphi^*}, \quad \delta g = q_2 e^{\varphi - \varphi^*} + q_3 e^{\varphi - \varphi^*},
\]

and \( p_i \) and \( q_i \) are coefficients of the order of unity, where particular composition is not important for this consideration. Note that to obtain the final result on the right-hand side of (A.6), we neglected the terms \( \delta f \delta g \sim \varepsilon^2 \).

The leading term in (A.6) \( A_1 + F/G \) gives the asymptotic (46), that is, the background condensate with a changed phase. Meanwhile, to obtain the connection with the dispersion relation, we need the second-order terms, that is,

\[
\frac{\delta f}{G} = \frac{p_2}{s_1} e^{-2\varphi} + \frac{p_3}{s_1} e^{-2\varphi^*}, \quad -\frac{F}{G^2} \delta g = -\frac{p_1}{s_1} \left( \frac{s_2}{s_1} e^{-2\varphi} + \frac{s_3}{s_1} e^{-2\varphi^*} \right).
\]

From formulas (A.8), we find that the linearization of the breather tail with respect to the background condensate gives a linear combinations of the first-order terms proportional to either \( e^{2\varphi} \) or \( e^{2\varphi^*} \). Now, to retrieve the dispersion law, we rewrite these exponents in the form \( e^{ikx+i\omega t} \), see also Section 3. Considering, for example, the first exponent \( e^{2\varphi} \), we obtain that \( k = -2\zeta \) and \( \omega = 2\lambda \xi \). Now, using \( \zeta(\lambda) = \sqrt{\lambda^2 + A^2} \), see Equation (11), we find \( \omega(k) = \pm ik \sqrt{A^2 - k^2/4} \). Thereby, the breather tails in this case obey the first branch of the dispersion law \( \omega_1(k) \) with complex \( k \) and \( \omega \). The complexity of \( k \) in (67) means exponential decay of the breather tail. The same result can be obtained for the terms \( p e^{2\varphi^*} \). One can repeat the presented procedure for types II and III breathers and in similar way find that they correspond to the second branch of the dispersion law \( \omega_{II}(k) \).
A.3 Computation of the phase and position shifts

Details of the computations of the phase and position shifts for the process $I + I \rightarrow I + I$ are similar to those for the scalar NLSE breathers presented in Ref. 61. The main idea boils down to asymptotic expansion of the two-breather solution (76) at the moments of time far before/after the collision and regions of space where each of the breathers is located. Here, we briefly consider the computational details for the full spatial shift in the process $II + II \rightarrow II + II$, while the other answers listed in Tables 1 and A.1 can be obtained in a similar way. We choose the vectors $\mathbf{C}_1$ and $\mathbf{C}_2$ according to Equation (A.9):

$$
C_{n,0} = e^{-1 \text{Im}[\lambda_n]t_n - i \delta n}/2, \quad C_{n,1} = 0, \quad C_{n,2} = e^{-1 \text{Im}[\xi_n]t_n + i \delta n}/2, \quad n = 1, 2.
$$

(A.9)

Then, in addition, we renormalize the vectors $\mathbf{q}_1$ and $\mathbf{q}_2$ using the property (36), so that they can be written in the following form:

$$
\begin{align*}
q_{11} &= i r_1, \quad q_{12} = \frac{1}{A} (A_1 - A_2 \exp[u_{II,1} - i v_{II,1}]), \quad q_{13} = \frac{1}{A} (A_2 + A_1 \exp[u_{II,1} - i v_{II,1}]), \\
q_{21} &= i r_2, \quad q_{22} = \frac{1}{A} (A_1 - A_2 \exp[u_{II,2} - i v_{II,2}]), \quad q_{23} = \frac{1}{A} (A_2 + A_1 \exp[u_{II,2} - i v_{II,2}]),
\end{align*}
$$

(A.10)

(A.11)

where, in accordance with (50), $u_{II,n} = (x - V_{II,n} t - \delta_n)/(2 l_{II,n})$, $v_{II,n} = p_{II,n} x - \omega_{II,n} t + \theta_n$, $n = 1, 2$. One can derive from (A.10) the following useful equality:

$$
|q_{1,2}|^2 = 1 + |r_{1,2}|^2 + \exp(2 u_{II}^{(1,2)}),
$$

$$
(q_1, q_2^*) = r_1 r_2^* + 1 + \exp[u_{II}^{(1)} + u_{II}^{(2)} - i (v_{II}^{(1)} - v_{II}^{(2)})].
$$

(A.12)

Let us for definiteness consider the breather with index 1 and assume that $0 \leq \alpha_{1,2} \leq \pi$. We move to the reference frame associated with the first breather, where $u_{II,1} = \text{const}$. According to (83), we define for the first breather $s_1 = \text{sign}(V_{II,2} - V_{II,1})$. Then, in the limit $t \rightarrow \pm \infty$, we obtain:

$$
u_{II,2} = (2 l_{II,2})^{-1} [2 l_{II,2} u_{II,1} + (V_{II,2} - V_{II,1}) t + \delta_1 - \delta_2] \rightarrow \pm s_1 \infty.
$$

(A.13)

In the case $u_{II,2} \rightarrow \infty$, the denominator $M$ of the two-breather solution (76) can be simplified as:

$$
M = -[m |q_1|^2 |q_2|^2 - n (q_1^*, q_2^*) (q_1, q_2^*)] = -e^{2 u_{II,2}} m \left[1 + |r_1|^2 + e^{2 u_{II,1}} \left(1 - \frac{n}{m}\right)\right],
$$

(A.14)

meanwhile, in the case $u_{II,2} \rightarrow \infty$ as,

$$
M = -e^{-2 u_{II,2}} (1 + |r_1|^2)^{-1} [m (1 + |r_1|^2)(1 + |r_2|^2) - n |1 + r_1 r_2|^2] \times
$$

$$
\times \left[1 + |r_1|^2 + e^{2 u_{II,1}} \left(\frac{m (1 + |r_1|^2)(1 + |r_2|^2)}{m (1 + |r_1|^2)(1 + |r_2|^2) - n |1 + r_1 r_2|^2}\right)\right].
$$

(A.15)
| Process | Integration constants and wavefield asymptotics | Space-phase shifts coefficients |
|---------|------------------------------------------------|------------------------------|
| $I_i + I_j \rightarrow I_i + I_j$ | $C_{i,0} = 0$, $C_{i,1} = e^{\text{Im}[\zeta_i] \delta_i + i\phi_i/2}$, $C_{i,2} = C_{1,1}^{-1}$, $\psi_{1,2}^{\leq \infty} = A_{1,2} e^{2i(\alpha_i + \sigma_i)}$, $\psi_{1,2}^{\geq \infty} = A_{1,2} e^{2i(\alpha_i + \sigma_i)}$ | $a_{i,j}^{1,1} = l_{i}(\lambda_i) \log \left[ \frac{(r_j - r_j^*)(1 + r_i r_j)}{(r_j - r_j^*)(1 + r_i r_j^*)} \right]$, $b_{i,j}^{1,1} = -a_{i,j}^{1,1}$, $c_{i,j}^{1,1} = \text{Arg} \left[ \frac{(r_j^* - r_j^*)(1 + r_i r_j)}{(r_j^* - r_j)^*(1 + r_i r_j^*) \sin \alpha_j} \right]$, $d_{i,j}^{1,1} = -c_{i,j}^{1,1}$, $e_{i,j}^{1,1} = \alpha_j$, $f_{i,j}^{1,1} = -\alpha_j$. |
| $I_i + I_j \rightarrow I_i + I_j$ | $C_{i,0} = 0$, $C_{i,1} = 0$, $C_{i,2} = e^{\text{Im}[\zeta_i] \delta_i + i\phi_i/2}$, $\psi_{1,2}^{\leq \infty} = A_{1,2} e^{2i(\alpha_i + \sigma_i)}$, $\psi_{1,2}^{\geq \infty} = A_{1,2}$ | $a_{i,j}^{1,2} = l_{i}(\lambda_i) \log \left[ \frac{1}{m(1 + |r_i|^2)} \right]$, $b_{i,j}^{1,2} = l_{i}(\lambda_i) \log \left[ \frac{1}{m(1 + |r_j|^2)(1 + |r_j|^2)} \right]$, $c_{i,j}^{1,2} = -\text{Arg}[1 - \frac{n_i}{m_j}]$, $d_{i,j}^{1,2} = \text{Arg} \left[ \frac{n_i(1 + r_i r_j^*)}{m_j(1 + |r_j|^2)} \right]$, $e_{i,j}^{1,2} = 0$, $f_{i,j}^{1,2} = \alpha_j$. |
| $II_i + II_j \rightarrow II_i + II_j$ | $C_{i,0} = e^{-\text{Im}[\lambda_i] \delta_i - i\phi_i/2}$, $C_{i,1} = 0$, $C_{i,2} = e^{-\text{Im}[\zeta_i] \delta_i + i\phi_i/2}$, $\psi_{1,2}^{\leq \infty} = A_{1,2} e^{2i(\alpha_i + \sigma_i)}$, $\psi_{1,2}^{\geq \infty} = A_{1,2}$ | $a_{i,j}^{2,2} = -l_{i}(\lambda_i) \log \left[ \frac{1 - \frac{n_i}{m_j}}{1} \right]$, $b_{i,j}^{2,2} = l_{i}(\lambda_i) \log \left[ \frac{1}{m(1 + |r_i|^2)} \right]$, $c_{i,j}^{2,2} = \text{Arg} \left[ \frac{n_i(1 + r_i r_j^*)}{m_j(1 + |r_j|^2)} \right]$, $d_{i,j}^{2,2} = \text{Arg} \left[ \frac{n_i r_i(1 + r_i r_j^*)}{m_j r_i(1 + |r_j|^2)} \right]$, $e_{i,j}^{2,2} = 0$, $f_{i,j}^{2,2} = -\alpha_j$. |
| $III_i + III_j \rightarrow III_i + III_j$ | $C_{i,0} = e^{-\text{Im}[\lambda_i] \delta_i - i\phi_i/2}$, $C_{i,1} = e^{\text{Im}[\zeta_i] \delta_i + i\phi_i/2}$, $C_{i,2} = 0$, $\psi_{1,2}^{\leq \infty} = A_{1,2} e^{2i(\alpha_i + \sigma_i)}$, $\psi_{1,2}^{\geq \infty} = A_{1,2}$ | $a_{i,j}^{3,3} = -l_{i}(\lambda_i) \log \left[ \frac{1 - \frac{n_i}{m_j}}{1} \right]$, $b_{i,j}^{3,3} = l_{i}(\lambda_i) \log \left[ \frac{1}{m(1 + |r_i|^2)} \right]$, $c_{i,j}^{3,3} = c_{i,j}^{2,2}$, $d_{i,j}^{3,3} = \text{Arg} \left[ \frac{n_i r_i(1 + r_i r_j^*)}{m_j r_i(1 + |r_j|^2)} \right]$, $e_{i,j}^{3,3} = 0$, $f_{i,j}^{3,3} = -\alpha_j$. |

(Continues)
| Process | Integration constants and wavefield asymptotics | Space-phase shifts coefficients |
|---------|-----------------------------------------------|---------------------------------|
| $I_i + II_j \rightarrow$ | $C_{i,0} = 0, \quad C_{i,1} = e^{\Im[\zeta_i \delta_i + i\theta_i/2]}, \quad C_{i,2} = C_{i,1}^{-1}$ | $a_{i,II_j}^I = 0, \quad b_{i,II_j}^I = a_{i,II_j}^I, \quad c_{i,II_j}^I = 0, \quad d_{i,II_j}^I = c_{i,II_j}^I, \quad e_{i,II_j}^I = 0, \quad f_{i,II_j}^I = \alpha_j$ |
| $\rightarrow I_i + II_j$ | $C_{j,0} = e^{-\Im[\zeta_j \delta_j - i\theta_j/2]}, \quad C_{j,1} = 0, \quad C_{j,2} = e^{-\Im[\zeta_j \delta_j + i\theta_j/2]}$ | $a_{j,II_j}^{II} = b_{j,II_j}^{II}, \quad b_{j,II_j}^{II} = l_{II}(\lambda_j) \log \left(1 - \frac{n|r_j - r_j^*|^2}{m(1 + |r_j|^2)(1 + |r_j^*|^2)}\right), \quad c_{j,II_j}^{II} = d_{j,II_j}^{II}, \quad d_{j,II_j}^{II} = \text{Arg} \left(1 - \frac{n_j(r_j^* - r_j^*)}{m_j(1 + |r_j^*|^2)}\right), \quad e_{j,II_j}^{II} = \alpha_j, \quad f_{j,II_j}^{II} = -\alpha_j$ |
| $I_i + III_j \rightarrow$ | $C_{i,0} = e^{\Im[\zeta_i \delta_i - i\theta_i/2]}, \quad C_{i,1} = 0, \quad C_{i,2} = C_{i,1}^{-1}$ | $a_{i,III_j}^I = 0, \quad b_{i,III_j}^I = -a_{i,II_j}^I, \quad c_{i,III_j}^I = 0, \quad d_{i,III_j}^I = -c_{i,II_j}^I, \quad e_{i,III_j}^I = 0, \quad f_{i,III_j}^I = -\alpha_j$ |
| $\rightarrow I_i + III_j$ | $C_{j,0} = e^{-\Im[\zeta_j \delta_j - i\theta_j/2]}, \quad C_{j,1} = 0, \quad C_{j,2} = e^{-\Im[\zeta_j \delta_j + i\theta_j/2]}$ | $a_{j,III_j}^{III} = l_{III}(\lambda_j) \log \left(1 - \frac{n|r_j - r_j^*|^2}{m(1 + |r_j|^2)(1 + |r_j^*|^2)}\right), \quad b_{j,III_j}^{III} = b_{j,II_j}^{III}, \quad c_{j,III_j}^{III} = \text{Arg} \left(1 - \frac{n_j(r_j - r_j^*)}{m_j r_j(1 + |r_j|^2)}\right), \quad d_{j,III_j}^{III} = d_{j,III_j}^{III}, \quad e_{j,III_j}^{III} = \alpha_j, \quad f_{j,III_j}^{III} = -\alpha_j$ |
| $II_i + III_j \rightarrow$ | $C_{i,0} = e^{-\Im[\zeta_i \delta_i - i\theta_i/2]}, \quad C_{i,1} = 0, \quad C_{i,2} = e^{-\Im[\zeta_i \delta_i + i\theta_i/2]}$ | $a_{i,III_j}^{II} = a_{i,II_j}^{II}, \quad b_{i,III_j}^{II} = b_{i,II_j}^{II}, \quad c_{i,III_j}^{II} = c_{i,II_j}^{II}, \quad d_{i,III_j}^{II} = d_{i,II_j}^{II}, \quad e_{i,III_j}^{II} = 0, \quad f_{i,III_j}^{II} = -\alpha_j$ |
| $\rightarrow II_i + III_j$ | $C_{j,0} = e^{\Im[\zeta_j \delta_j - i\theta_j/2]}, \quad C_{j,1} = 0, \quad C_{j,2} = e^{\Im[\zeta_j \delta_j + i\theta_j/2]}$ | $a_{j,III_j}^{III} = a_{j,II_j}^{III}, \quad b_{j,III_j}^{III} = a_{j,II_j}^{III}, \quad c_{j,III_j}^{III} = c_{j,II_j}^{III}, \quad d_{j,III_j}^{III} = c_{j,II_j}^{III}, \quad e_{j,III_j}^{III} = 0, \quad f_{j,III_j}^{III} = \alpha_j$ |

0Left column indicates the collision process type in the form $B_i + \bar{B}_j \rightarrow B_i + \bar{B}_j$, where $i = 1$ or $2$ while $j = 2$ or $1$ respectively. The middle column shows the corresponding parametrization of the integration constants $C_i$ and asymptotic values of the wavefield at $x \to \pm \infty$. Right column presents the values of the space-phase shifts coefficients defined in Eq. (85). When $B$ and $\bar{B}$ represent the same type we leave only $\Delta \delta^{II,III}$ value in the table. The meaning of the indexes has been explained in the main text. For example, $b_{j,II_j}^{III}$ at $s_j = 1$ ($a_{j,II_j}^{III}$ at $s_j = -1$) represents the correction to the position of the $i$-th breather of type I at large time after the collision with the breather of type II
FIGURE A.1  Elastic collision III + III → III + III of vector breathers with spectral parameters defined by Equation (83). (A,B) $|\psi_1|$ and Arg$[\psi_1]$, where Arg means complex phase, after the breathers collision at $t = 10.0$. (C) Spatio-temporal plot of the wavefield evolution. The dotted green and red lines in (A,B) show a local approximation of the breathers after collision by single-breather solutions from the asymptotic (84). Thin black dashed line in (A) shows how the first breather would have been if it had been traveling alone.

Finally, comparing (A.14) and (A.15) with the denominator of the type II single-breather solution (49), we find that the shift of the center of the first breather after collision with the second breather is equal to

$$\Delta \delta_{\text{I}, \text{II}} = \frac{l_{\text{II}}}{2} \left( \log \left[ 1 - \frac{n}{m} \right] + \log \left[ 1 - \frac{n(1 + r_2^* r_2^2)}{m(1 + |r_1|^2)(1 + |r_2|^2)} \right] \right)s_1, \quad (A.16)$$

which is one of the answers presented in Table 1.

**A.4  Additional illustrations**

In this section, we provide additional to the shown in Section 5 illustrations of the breather collisions and approximations of their asymptotic states using single breathers with shifted according to (84) parameters. Figures A.1–A.3 show the processes III + III → III + III, I + II → I + II, and II + III → II + III, respectively.

**A.5  Tables with coefficients for the asymptotic state of elastic breather collision**

In this section, we complete the information on the asymptotic state of the two-breather collision, see Equation (84), providing the values of the coefficients $a_{B_1 B_2}^{B_1 B_2}$, $b_{B_1 B_2}^{B_1 B_2}$, and so on from Equation (85) and the wavefield asymptotics at $x \to \pm \infty$. Table A.1 summarizes the results of our derivations. Note that the wavefield asymptotics of the two-breather solution represent a linear sum of the single-breather wavefield asymptotics derived in the main part of the paper, see Equations (46), (52), and (58). We also remind that the lower and the first upper indexes of the coefficients $a_{B_1 B_2}^{B_1 B_2}$, $b_{B_1 B_2}^{B_1 B_2}$, and so on indicate the breather for which the shift is presented, while the
FIGURE A.2  Elastic collision $I + II \rightarrow I + II$ of vector breathers with spectral parameters defined by Equation (86). (A,B) $|\psi_1|$ and $\text{Arg}[\psi_1]$, where Arg means complex phase, after the breathers collision at $t = 7.5$. (C) Spatio-temporal plot of the wavefield evolution. The dotted green and red lines in (A,B) show a local approximation of the breathers after collision by single-breather solutions from the asymptotic (84). Thin black dashed line in (A) shows how the first breather would have been if it had been traveling alone.

FIGURE A.3  Elastic collision $II + III \rightarrow II + III$ of vector breathers with spectral parameters defined by Equation (86). (A,B) $|\psi_1|$ and $\text{Arg}[\psi_1]$, where Arg means complex phase, after the breathers collision at $t = 12.0$. (C) Spatio-temporal plot of the wavefield evolution. The dotted green and red lines in (A,B) show a local approximation of the breathers after collision by single-breather solutions from the asymptotic (84). Thin black dashed line in (A) shows how the first breather would have been if it had been traveling alone.
second upper index means the breather with which the studied one interacts. The lower index shows only what we call the studied breather, that is, either the first or the second one. Meanwhile, the upper indexes also indicate the type of interacting breathers. For example, $b^{B_i,\tilde{B}_j}_i$ at $s_i = 1$ ($a^{B_i,\tilde{B}_j}_i$ at $s_i = -1$) represents the correction $\delta_{0,i}^+$ for the position of the $i$-th breather of type $B$ after the collision with the breather of type $\tilde{B}$.

We find that the coefficients $a^{B_i,\tilde{B}_j}_i$, $b^{B_i,\tilde{B}_j}_i$, and so on for the processes with different types of breathers often coincide with some of the coefficients computed for the process with the same types of breathers. For example, $b^{I_i,\Pi_j}_i = a^{I_i,\Pi_j}_i$, while $a^{\Pi_i,\Pi_j}_i = a^{\Pi_i,\Pi_j}_i$, see Table A.1. In addition, in the expressions for $b^{\Pi_i,\Pi_j}_i$ and $b^{\Pi_i,\Pi_j}_j$ appears the same logarithm. Nevertheless, we do not identify any symmetry properties of the coefficients and consider all of them as independent.