QUANTUM GRAVITY ON POLYgons AND $\mathbb{R} \times \mathbb{Z}_n$ FLRW MODEL

J. N. ARGOTA QUIROZ AND S. MAJID

Abstract. We fully solve the quantum geometry of $\mathbb{Z}_n$ as a polygon graph with arbitrary metric lengths on the edges, finding a $\ast$-preserving quantum Levi-Civita connection which is unique for $n \neq 4$. As a first application, we numerically compute correlation functions for Euclideanised quantum gravity on $\mathbb{Z}_n$ for small $n$. We then study an FLRW model on $\mathbb{R} \times \mathbb{Z}_n$, finding the same expansion rate as for the classical flat FLRW model in 1+2 dimensions. We also look at particle creation on $\mathbb{R} \times \mathbb{Z}_n$ and find an additional $m = 0$ adiabatic no particle creation expansion as well as the particle creation spectrum for a smoothed step expansion.

1. Introduction

Quantum spacetime or the idea that space and time coordinates are noncommutative or ‘quantum’ has been speculated on since the early days of quantum theory but has also emerged by now as a better-than-classical effective theory that includes some quantum gravity effects. This was first discussed in modern times in [16] in the context of non-commutativity of phase space and quantum born reciprocity or observable-state duality, where it led to the bicrossproduct class of quantum group (rather differently from the other main class, the q-deformation ones, arising from integrable systems). Bicrossproduct quantum groups act canonically on the dual of one of their factors and later on provided natural models of quantum spacetime with quantum group Poincaré symmetry, notably the bicrossproduct Minkowski spacetime $[x_i, t] = i\lambda p x_i$ in [17]. This class of models also relates via semidualisation and quantum Born reciprocity to quantum gravity with point sources[22]. Other early works were [31] which did not itself propose a closed spacetime algebra, its adaptation [10] with classical (not quantum) symmetry and the proposal [13] of the angular momentum algebra as a quantum spacetime. We refer to [19] for more details and literature.

Also argued back in [10] was that a true test of quantum gravity model building would be models where the spacetime (and indeed the entire phase space, both position and momentum) was both curved and quantum. For this one would need a mathematical framework for spacetimes with curvature. One approach, which already goes back to the 1980s, is ‘noncommutative geometry à la Connes’[8] coming

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out of cyclic cohomology, K-theory and an abstract notion of ‘spectral triples’ modelling a Dirac operator using methods of operator theory. Another, which is the one we will use, is a constructive ‘quantum groups approach’ motivated by quantum groups and their homogeneous spaces as examples but ultimately working for any algebra $A$ equipped with differential structure, over any field. The starting point here is to specify the latter as a bimodule $\Omega^1$ of ‘1-forms’ (this means we can multiply them from either side by elements of $A$) equipped with an exterior derivative $d : A \to \Omega^1$ obeying the Leibniz rule. We then define a metric as an invertible element $g \in \Omega^1 \otimes_A \Omega^1$ with some kind of symmetry condition and a quantum Levi-Civita connection (QLC) in these terms is a bimodule connection $\nabla : \Omega^1 \to \Omega^1 \otimes_A \Omega^1$ which is metric compatible and torsion free.

We provide a short introduction to the formalism in the preliminaries Section 2 with full details in [5]. For each quantum Riemannian geometry one can compute a Laplacian $\Delta = (\ , \ )\nabla d : A \to A$ and, with a little more ‘lifting’ data, a Ricci tensor in $\Omega^1 \otimes_A \Omega^1$ and a Ricci scalar $S \in A$. The above bicrossproduct model spacetime in 1+1 dimensions turned out [3, 23] from this point of view to admit two main classes of translation invariant 2D differential structures and each of these to admit a unique form of quantum metric up to a parameter and associated QLCs. The bicrossproduct model coordinate algebra here also has a flat Minkowski-type geometry with quantum Poincaré symmetry, but for this one needs an extra cotangent dimension in $\Omega^1$. There are also ‘noncommutative algebraic geometry’ approaches in the mathematics literature different from both Connes’ approach and ours.

Until recently, however, few models were known where metrics could be arbitrary and the QLC still found across the whole moduli of metrics, a necessary prerequisite for quantum gravity in this approach. Namely, $\mathbb{Z}_2 \times \mathbb{Z}_2$ as a square graph and $\mathbb{Z}$ as a lattice line graph were solved for arbitrary metrics (in the form of a real number of length squared dimension associated to each edge) in [19, 20] respectively. Quantum gravity on the square was studied in [19], while [20] focussed on cosmological particle creation for 1+0 QFT on $\mathbb{Z}$ as a ‘discrete line’ approximation of $\mathbb{R}$ with varying metric. In the present paper, we extend this small body of discrete quantum Riemannian geometries to $n$-gons $\mathbb{Z}_n$ with $n \geq 3$ and arbitrary metric on the edges, as a discrete version of a circle $S^1$. We find a natural $*$-preserving (or ‘real’) QLC for any prescribed metric which turns out to be a periodic subset of the solutions on $\mathbb{Z}$ and to be unique for $n \neq 4$. This is the main result of Section 3.1. Note that the algebra $A = C(\mathbb{Z}_n)$ of functions on $\mathbb{Z}_n$ is commutative, but differentials as spanned by the graph directed edges, do not commute with functions. Also, in all these cases, $\Omega^1$ is translation invariant with respect to the stated group and this also determines $\Omega^2$ by stating that the translation-invariant 1-forms $\{e^a\}$ form a Grassmann algebra. Note that the quantum geometry of $\mathbb{Z}_4$ here is not the same as $\mathbb{Z}_2 \times \mathbb{Z}_2$ due to the different $\Omega^2$, albeit we do find somewhat similar results. The role of the group structure here is broadly analogous to the use of local $\mathbb{R}^n$ coordinates for a classical manifold and does not force us to fix the metric. Section 3.2 studies Euclideanised quantum gravity on $\mathbb{Z}_n$ as a first application of our results. Although not regular quantum gravity, the Euclidean version is still of interest for classical compact Riemannian manifolds without boundary, see [15].
Section 4 turns to quantum metrics and QLCs on $\mathbb{R} \times \mathbb{Z}_n$, where $\mathbb{R}$ is classical. We find in Section 4.1 that quantum metrics are forced to have the block form $g = \mu dt \otimes dt + h_{ab} e^a \otimes e^b$ and moreover that $h_{ab}$ has to have a specific form where the time dependence enters only in the overall scale of the spatial metric. Section 4.2 focusses on the FLRW cosmology case of uniform ‘circle’ metric on $\mathbb{Z}_n$ expanded by a time-dependent factor, so

$$g = -dt \otimes dt - a(t)(e^+ \otimes e^- + e^- \otimes e^+).$$

The negative sign in the second term is needed to match the comparable classical cases and relates to the interpretation of $e^\pm$. We find that the Friedmann equations for $a(t)$ in our discrete case then actually come out the same as for the usual flat FLRW model in two spatial dimensions, which is in line with our $\Omega^1$ being 2-dimensional. Section 4.3 provides some elementary checks for QFT in the constant spatial dimensions, which is in line with our $\Omega^1$ of complex functions on a manifold as well as the discrete case that we are interested in. This ensures that out constructions are not ad-hoc to the discrete case and also provides a common setting within which one can hope to take a continuum limit as well as consider the mixed case of $\mathbb{R} \times \mathbb{Z}_n$. For each layer of the theory we recall the general set up over a unital algebra $A$ as in [5], for orientation purposes, then given details for the discrete graph case, which is the same setting as in [19, 20].

Section 5 concludes with some directions for further work. We work in units with $\hbar = c = 1$.

2. Preliminaries

It is important that the formalism of quantum Riemannian geometry that we use is functorial across a wide range of algebras, including the classical case of functions on a manifold as well as the discrete case that we are interested in. This ensures that out constructions are not ad-hoc to the discrete case and also provides a common setting within which one can hope to take a continuum limit as well as consider the mixed case of $\mathbb{R} \times \mathbb{Z}_n$. For each layer of the theory we recall the general set up over a unital algebra $A$ as in [5], for orientation purposes, then given details for the discrete graph case, which is the same setting as in [19, 20].

2.1. Differentials and metrics. As explained in the introduction, the first step is a graded exterior algebra $(\Omega, d)$ where $\Omega^0 = A$ is the algebra of ‘functions’ and $d$ increases the differential form degree by 1, obeys a graded-Leibniz rule and $d^2 = 0$. We also require $\Omega$ to be generated by $A, dA$. If one fixes $\Omega^1$ first then there is a unique ‘maximal’ $\Omega^2$ of which one can chose a quotient for on we want. In our case, will be interested in the commutative algebra $A = C(X)$ of complex functions on a discrete set $X$ with pointwise product. Then choosing $\Omega^1$ is equivalent to assigning arrows to make a graph with vertex set $X$. Denoting a linear basis of $\Omega^2$ by $\{\omega_{x \rightarrow y}\}$ labelled by arrows $x \rightarrow y$, the bimodule products and exterior derivative are

$$f \cdot \omega_{x \rightarrow y} = f(x) \omega_{x \rightarrow y}, \quad \omega_{x \rightarrow y} \cdot f = f(y) \omega_{x \rightarrow y}, \quad df = \sum_{x \rightarrow y} (f(y) - f(x)) \omega_{x \rightarrow y}.$$
We will be interested in the case where the graph is bidirected i.e., for every arrow $x \rightarrow y$ there is an arrow $y \rightarrow x$. In other words, the data is just a usual undirected graph which we understand as arrows both ways in the above formulae. A metric as a tensor $g \in \Omega^1 \otimes_A \Omega^1$ then has the form
\[
g = \sum_{x \rightarrow y} g_{x \rightarrow y} \omega_{x \rightarrow y} \otimes \omega_{y \rightarrow x} \in \Omega^1 \otimes_{C(X)} \Omega^1
\]
for nonzero weights $g_{x \rightarrow y}$ for every edge. Canonically, a metric is ‘quantum symmetric’ if $\wedge(g) = 0$ for the wedge product of $\Omega$. Specific to graphs, we also have a slightly different notion that $g$ is edge-symmetric if $g_{x \rightarrow y} = g_{y \rightarrow x}$ for all $x \rightarrow y$, i.e., does not depend on the direction of travel. As in [19, 20] for the line graph, we will see that this variant also works better when we apply it to the polygon.

Next, it is useful to endow $X$ with a group structure and look for $\Omega^1$ which is left and right translation invariant. These will be the Cayley graph for an $\text{Ad}$-stable set of generators $C \subseteq G \setminus \{e\}$ (where $e$ is the group identity), with arrows of the form $x \rightarrow xa$ for $a \in C$. In this case one has a basis of invariant 1-forms $e^a = \sum_{x \rightarrow xa} \omega_{x \rightarrow xa}$ with $\Omega^1 = A \{e^a\}$ with bimodule relations and derivative
\[
e^a f = R_a(f)e^a, \quad df = \sum_a (\partial_a f) e^a, \quad \partial_a = R_a - \text{id}, \quad R_a(f)(x) = f(xa)
\]
defined by the right translation operators $R_a$ as stated. These formulae now makes sense even when $X$ is infinite as long as $C$ is finite. Moreover $\Omega$ is canonically generated by functions and basic 1-forms with the above as well as certain ‘braided-anticommutation relations’ between the $\{e^a\}$. In the case of an Abelian group (which is all we will need) this is just the usual Grassmann algebra on the $e^a$, i.e., they anticommute and we also have $de_a = 0$ in this case.

2.2. Connections. A connection in quantum Riemannian geometry is a map $\nabla : \Omega^1 \rightarrow \Omega^1 \otimes_A \Omega^1$. If given a quantum vector field in the form of a right module map $X : \Omega^1 \rightarrow A$ then we can evaluate this against the first output to obtain a covariant derivative $\nabla_X : \Omega^1 \rightarrow \Omega^1$, but the connection itself is defined independently of any vector field. Rather, it obeys two Leibniz rules as follows. From the left we ask for
\[
\nabla(\omega \alpha) = d\alpha \otimes \omega + \alpha \nabla \omega
\]
for all $\alpha \in A, \omega \in \Omega^1$. From the right we similarly ask for [12, 23]
\[
\nabla(\omega \alpha) = (\nabla \omega) \alpha + \sigma(\omega \otimes d\alpha); \quad \sigma : \Omega^1 \otimes_A \Omega^1 \rightarrow \Omega^1 \otimes_A \Omega^1
\]
for some ‘bimodule map’ $\sigma$ (i.e., commuting with the action of $A$ from either side, i.e. ‘tensorial’ in a strong sense.)

In the case we need of a Cayley graph calculus on a finite group, we see that $\nabla$ just needs to be specified on the $e^a$ provided this is consistent with is extension to $\Omega^1$ by the two Leibniz rules. We write
\[
\nabla e^a = \Gamma^a_{bc} e^b \otimes e^c, \quad \sigma(e^a \otimes e^b) = \sigma e^a_{mn} e^m \otimes e^n
\]
for coefficients in $A$ with a certain compatibility between these tensors for a bimodule connection. In general, torsion-free amounts to $\wedge \nabla - d = 0$ as maps from $\Omega^1 \rightarrow \Omega^2$ and needs in the case of an abelian group the additional relations
\[
\Gamma^a_{bc} = \Gamma^a_{cb}, \quad \sigma e^a_{mn} e^m \wedge e^n + e^a \wedge e^b = 0.
\]
Next, any bimodule connection extends canonically to a connection on tensor products. This implies a meaning to $\nabla g = 0$, namely if $g = g^1 \otimes g^2$ say then this is
\[
\nabla g^1 \otimes g^2 + (\sigma(g^1 \otimes (\cdot)) \otimes \text{id})\nabla g^2 = 0.
\]
In the discrete Cayley graph setting we write $g = h_{ab} e^a \otimes e^b$ where centrality needs
\[
h_{ab} = \delta_{a^{-1}, b}h_a
\]
for some functions $h_a$. In these terms (i) edge symmetry and, in the case of the Grassmann algebra, quantum symmetry (ii) appear as
\[
(i) \quad h_a = R_a(h_{a^{-1}}), \quad (ii) \quad h_a = h_{a^{-1}}.
\]

2.3. $*$-structures, inner calculi and structure constants. For physics, there should also be a $*$-involution on $A$ which in our examples is just complex conjugation, and everything should be unitary or ‘real’ in the sense of $*$-preserving. We require this to extend to $\Omega$ with an extra minus signs for swapping two odd elements and to commute with $d$. For the metric and connection ‘reality’ means
\[
(2.1) \quad g^\dagger = g, \quad \nabla \circ \cdot = \sigma \circ \cdot \circ \nabla,
\]
which also implies $\dagger \circ \cdot = \sigma^{-1} \circ \dagger \circ \alpha$. A Cayley graph calculus is inner with $\theta = \sum_a e^a$. In this case, since to be bimodule maps, we need $\sigma^{ab}_{mn} = 0$ unless $ab = mn$ in the group and $\alpha(e^a) = \alpha^{a}_{mn} e^m \otimes e^n$ needs $\alpha^{a}_{mn} = 0$ unless $a = mn$ in the group, see [18] 5. The indices here range over elements of the generating set $C$ of the calculus and are not being multiplied in the $4$-index and $3$-index tensors $\sigma^{ab}_{mn}, \alpha^{a}_{mn}$. We will need this a little more explicitly than currently in the literature.

**Lemma 2.1.** Let $\Omega(G)$ be a Cayley graph calculus and cf. [18] 5, write a bimodule connection on $\Omega^1$ in the form
\[
\sigma^{ab}_{mn} = \delta^{a}_{n}\delta^{b}_{m} + \delta^{a}_{a^{-1}mn}\tau^{a}_{mn}, \quad \Gamma^{a}_{bc} = \tau^{a}_{bc} - \delta^{a}_{bc}\alpha_{bc}
\]
for coefficient functions $\tau^{a}_{bc} = 0$ unless $a^{-1}bc \in C$ and $\alpha_{bc} = 0$ unless $bc \in C$.

1. For $G$ abelian, the condition for torsion freeness is $\tau^{a}_{bc}, \alpha_{bc}$ symmetric in $b,c$.

2. The condition for ‘reality’ of the connection (to be $*$-preserving) is
\[
\alpha_{bc} + R_{bc}(\alpha_{c^{-1}b^{-1}}) + \sum_n R_{nbcn^{-1}}(\alpha_{c^{-1}b^{-1}n^{-1}, n})\tau^{n^{-1}}_{bc} = 0,
\]
$$\tau^{a^{-1}}_{\, c d} + R_{c d}(\tau^{a^{-1}}_{\, c^{-1}d^{-1}}) + \sum_{n} R_{c d}(\tau^{a^{-1}}_{\, c^{-1}d^{-1}n,n^{-1}})\tau^{n}_{\, c d} = 0$$

for all $a, b, c, d$.

(3) The condition for metric compatibility with an edge-symmetric metric is

$$h_{m n} \alpha_{m n} + R_n(h_{n^{-1} \alpha_{m,n^{-1}m^{-1}}}) - \sum_{a} R_{a^{-1}}(h_{a \alpha_{a m n, n^{-1}m^{-1}}}) - R_n(h_{n^{-1} \tau^{n^{-1}}_{\, m,n^{-1}m^{-1}}}) = 0$$

$$\delta^{p}_{\, n^{-1}} \partial_m h_n = h_{p^{-1} \tau^{p^{-1}}_{\, m n}} - \sum_{a} R_{a^{-1}}(h_{a \tau^{a}_{\, amn,p}})\tau^{a^{-1}}_{\, mn}$$

for all $m, n, p$.

Proof. (1) The first formula displayed is basically [18] (or see [5]) in the inner case with $\theta = \sum_n e^a$, merely put in terms of the components of $\Gamma$ and after subtracting off the flip map from $\sigma$ and imposing the bimodule properties of the maps $\alpha, \sigma$ (hence $\tau$). It is easy to see that $\wedge \alpha = 0$ and $\wedge(id + \sigma) = 0$ for the Grassmann algebra case reduce to symmetry in the lower indices (this technique is used in [5] but is in any case straightforward). Note that $e \notin \mathcal{C}$ so $\Gamma^a_{\, b c}$ has value $-\alpha_{b c} := -\alpha_{b,c}$ when $a = b c$ and $\tau^a_{\, b c} := \tau^a_{\, b,c}$ otherwise, where we omit the commas when there are only two elements not being multiplied.

(2) The condition for $\alpha$ is immediate from $\sigma \circ \dagger \circ \alpha = \alpha \circ \ast$ evaluated on $e^a$ with $e^{a \ast} = -e^{a^{-1}}$. The condition $\sigma \circ \dagger \circ \sigma = \dagger$ is easily seen (as in the proof of [5, Lemma 8.17] for $\alpha = 0$) to be

$$\sum_{m,n} R_{m^{-1}n^{-1} \alpha_{mn}}(\tau^{ab}_{\, mn})\sigma^{n^{-1}m^{-1}}_{\, cd} = \delta^{b}_{\, a^{-1}} \cdot \delta^{a^{-1}}_{\, d},$$

which we now evaluate for the stated form of $\sigma$.

(3) Metric compatibility is

$$\nabla(h_{ab}e^a) \otimes e^b - \sigma(h_{ab}e^a \otimes \Gamma^b_{\, cd}e^c) \otimes e^d = 0$$

which expands out using the Leibniz rules and the form of the metric to

$$\delta_{p,n^{-1}} \partial_m h_n - h_{p^{-1} \Gamma^{p^{-1}}_{\, mn}} - h_a R_a(\Gamma^{a^{-1}}_{\, bp}) \sigma^{ab}_{\, mn} = 0$$

In the edge-symmetric case this becomes

$$\delta_{p,n^{-1}} \partial_m h_n - h_{p^{-1} \Gamma^{p^{-1}}_{\, mn}} - R_a(h_{a^{-1} \Gamma^{a^{-1}}_{\, bp}}) \sigma^{ab}_{\, mn} = 0.$$
2.4. Curvature. Given a left connection $\nabla$ on an algebra with differential calculus (it does not even need to be a bimodule one) we have Riemann curvature

$$R_\nabla : \Omega^1 \to \Omega^2 \otimes_A \Omega^1,$$

$$R_\nabla = (d \otimes 1 - 1 \otimes \nabla)(\alpha - \sigma_0)(\alpha - \sigma_0) \omega.$$

For example, in the inner case of a connection defined by maps $\sigma, \alpha$ as above, this is

$$R_\nabla \omega = \theta \otimes \theta \otimes \omega - (\otimes \otimes 1)(1 \otimes (\alpha - \sigma_0)(\alpha - \sigma_0)) \omega.$$

Next, given a bimodule ‘lift’ map $i : \Omega^2 \to \Omega^1 \otimes A \Omega^1$ such that $\otimes \circ i = 1$, we define Ricci and Ricci scalar $S$ relative to it as

$$\text{Ricci} = ((1 \otimes 1)(1 \otimes i) \otimes 1)(1 \otimes R_\nabla)g,$$

$$S = (1, \text{Ricci}).$$

This is a ‘working definition’ rather than part of a fully developed theory (for which in understanding of conservation laws and the stress-energy tensor would be needed). In the Cayley graph case of Lemma 2.1 there is a canonical $\Omega$ and with it a canonical $i$ which for an abelian group is just

$$i(e^a \wedge e^b) = \frac{1}{2}(e^a \otimes e^b - e^b \otimes e^a)$$

on the Grassmann algebra generators (extended as bimodule map). Thus, once we have found a QLC for our quantum metric, the route to the scalar curvature needed for the Einstein-Hilbert action is canonical at least for Abelian groups such as $\mathbb{Z}_n$.

3. Quantization of $\mathbb{Z}_n$

Here we consider the general theory above for the case of an $n$-gon for $n \geq 3$. A metric is a free assignment of a ‘square-length’ to each edge and Section 3.1 solves the quantum Riemannian geometry to find the quantum Levi-Civita connection. Section 3.2 then constructs Euclidean quantum gravity on the polygon.

3.1. Quantum Riemannian geometry on $\mathbb{Z}_n$. Just as it is useful in classical geometry to use local coordinates where the differential structure is the standard one for $\mathbb{R}^n$, it is similarly useful to regard the $n$-gon as the group $G = \mathbb{Z}_n$ for its differential structure as explained in Section 2. Here the calculus $\Omega^1(\mathbb{Z}_n)$ with generators $\mathcal{C} = \{1, -1\}$ and corresponding left-invariant basis $\{e^+, e^\}$, where

$$e^+ = \sum_{i=0}^{n-1} \omega_{i+i+1}; \quad e^- = \sum_{i=0}^{n-1} \omega_{i+i-1}.$$

The $n = 2$ case is different and was already solved for its quantum Riemannian geometry in [19].

Since the $e^\pm$ are a basis over the algebra, a bimodule invertible quantum metric must take the central form

$$g = ae^+ \otimes e^- + be^- \otimes e^+$$

for non-vanishing functions $a, b \in \mathbb{R}(\mathbb{Z}_n)$ and the inverse metric

$$(e^+, e^+) = (e^-, e^-) = 0, \quad (e^+, e^-) = 1/R_+(a), \quad (e^-, e^+) = 1/R_-(b).$$

Besides we have the inner element $\theta = e^+ + e^-$ and the canonical $\star$-structure $(e^\star)^* = -e^\star; (e^\star)^* = -e^\star$. We also write $R_{e^\pm} = R_{e^\pm 1}$ for the shift operators. On the other hand, from the graph perspective, the relevant Cayley graph of $\mathbb{Z}_n$ with the above
The restriction to periodic metrics mod $n$ for Proposition 3.1.

A quantum metric on $\mathbb{Z}_n$ is given by metric coefficient functions $a, b$ or equivalently by directed edge weights $g_{i \to i+1}$.

Proposition 3.1. For $n \geq 3$, there is a $*$-preserving QLC for any given edge-symmetric metric (3.1) on $\Omega^1(\mathbb{Z}_n)$. This is the unique for $n \neq 4$ and coincides with the restriction to periodic metrics mod $n$ of the unique such connection on $\mathbb{Z}$ in [20], namely

$$
\sigma(e^+ \otimes e^+) = \rho e^+ \otimes e^+, \quad \sigma(e^+ \otimes e^-) = e^- \otimes e^+, \\
\sigma(e^- \otimes e^+) = e^+ \otimes e^-, \quad \sigma(e^- \otimes e^-) = R^2 \rho^{-1} e^- \otimes e^-
$$

with the geometric structures

$$
\nabla e^+ = (1 - \rho)e^+ \otimes e^+, \quad \nabla e^- = (1 - R\rho^{-1}) e^- \otimes e^-, \\
R e^+ = \partial_- \rho e^+ \otimes e^- \otimes e^+, \quad R e^- = -\partial_+(R^2 \rho^{-1}) e^+ \otimes e^- \otimes e^-,
$$

$$
\text{Ricci} = \frac{1}{2} \left( -\partial_+ R \rho^{-1} e^+ \otimes e^- - \partial_- R \rho^{-1} e^- \otimes e^+ \right), \\
S = \frac{1}{2} \left( -\partial_+ \rho^{-1} a + \partial_-(R \rho) R \rho \right), \quad \Delta f = -\frac{R - \rho + 1}{a} (\partial_+ + \partial_-) f.
$$

(For $n = 4$, there is a second $*$-preserving QLC given below.)

Proof. Due to the grading restrictions for a bimodule map, the most general $\sigma$ for $n \neq 4$ has the form

$$
\sigma(e^+ \otimes e^+) = \sigma_0 e^+ \otimes e^+, \quad \sigma(e^+ \otimes e^-) = \sigma_1 e^+ \otimes e^- + \sigma_2 e^- \otimes e^+, \\
\sigma(e^- \otimes e^+) = \sigma_3 e^+ \otimes e^- + \sigma_4 e^- \otimes e^+, \quad \sigma(e^- \otimes e^-) = \sigma_5 e^- \otimes e^-.
$$
where the $\sigma_i$ are functional parameters) while for $n = 4$ we can have additional terms leading to another solution (given below). Similarly, for $n \neq 3$ we can only have the map $\alpha = 0$ while for $n = 3$ we may have additional terms leading to non $\ast$-preserving solutions in the Appendix. Taking the displayed main form of $\sigma$ and $\alpha = 0$, torsion freeness $\wedge (\id + \sigma) = 0$ amounts to
\[
\sigma_2 = \sigma_1 + 1, \quad \sigma_3 = \sigma_4 + 1,
\]
while metric compatibility is
\[
R_+ (a) = a R_+ (\sigma_3) \sigma_0, \quad a = a R_+ (\sigma_4) \sigma_1 + R_- (a) R_- (\sigma_0) \sigma_3, \\
R_- (a) = a R_+ (\sigma_3) \sigma_2 + R_- (a) R_- (\sigma_1) \sigma_4, \quad R_-^2 (a) = R_- (a) R_- (\sigma_2) \sigma_5, \\
0 = a R_1 (\sigma_3) \sigma_1 + R_- (a) R_- (\sigma_1) \sigma_3, \quad 0 = a R_+ (\sigma_4) \sigma_2 + R_- (a) R_- (\sigma_0) \sigma_4.
\]
It is then a matter of solving these, which was done using SAGE. Among the solutions, we find a unique one that is $\ast$-preserving. The others are described for completeness in the Appendix.

That the restriction of the unique $\ast$-preserving QLC on $\mathbb{Z}$ in [20] to periodic metrics gives a $\ast$-preserving QLC is not surprising, but that this gives all $\ast$-preserving solutions for $n \neq 4$ is a nontrivial result of solving the equations as described. For $n = 4$, similar methods lead to a further 2-parameter moduli of $\ast$-preserving connections of the form
\[
\sigma (e^+ \otimes e^+) = \gamma e^- \otimes e^-; \quad \sigma (e^+ \otimes e^-) = e^+ \otimes e^-,
\]
\[
\sigma (e^- \otimes e^+) = e^- \otimes e^+; \quad \sigma (e^- \otimes e^-) = \frac{R_+ a}{R_- (a) \gamma} e^+ \otimes e^+,
\]
where $\gamma = (\gamma_0, \gamma_1, \bar{\gamma}_0^{-1}, \bar{\gamma}_1^{-1})$ specifies a function on the four points of $\mathbb{Z}_4$ (in order) in terms of two complex parameters $\gamma_0, \gamma_1$, such that $R_-^2 \gamma = \bar{\gamma}^{-1}$. The associated quantum geometric structures are
\[
\nabla e^+ = e^+ \otimes e^+ + e^- \otimes e^+ - e^+ \otimes e^- - \gamma e^- \otimes e^-;
\]
\[
\nabla e^- = e^- \otimes e^- + e^+ \otimes e^- - e^- \otimes e^+ - r e^+ \otimes e^+,
\]
\[
R_\gamma e^+ = (R_- r - 1) e^+ \otimes e^- \wedge e^+, \quad R_\gamma e^- = (1 - r) e^+ \wedge e^- \otimes e^-,
\]
\[
\text{Ricci} = \frac{1}{2} (R_- r - 1) e^+ \otimes e^- + \frac{1}{2} (R_-^2 r - 1) e^- \otimes e^+,
\]
\[
S = \frac{1}{2a} \left( (R_- \rho) (R_-^2 r - 1) + R_- r - 1 \right),
\]
\[
\Delta f = - \frac{2}{a} (\partial_+ f + (R_- \rho) \partial_- f),
\]
where we use the shorthand
\[
r := \frac{R_+ (a)}{R_- (a)} |\gamma|^2.
\]
This is the $\ast$-preserving case of the general $n = 4$ solution (i) in the Appendix.

3.2. Euclideanised quantum gravity on $\mathbb{Z}_n$. As for the integer line graph [20], the two-dimensional cotangent bundle on $\mathbb{Z}_n$ represents a kind of fattening of a circle in the discrete case, which then admits the possibility of curvature due to the 2-dimensional cotangent bundle. We envision that there could be various applications of such curved discrete ‘tori’, but here we focus on just one, namely Euclideanised
quantum gravity on $\mathbb{Z}_n$. For integration on $\mathbb{Z}_n$ needed in the action, we take a sum over $\mathbb{Z}_n$ with a weight $a$ (in the commutative case, this would be $\sqrt{|\det g|}$), which has the merit that then the action is

$$S_g = \frac{1}{2} \sum_{\mathbb{Z}_n} (R_\rho \partial_\rho R_\rho) = \frac{1}{2} \sum_{\mathbb{Z}_n} \rho \partial_\rho = \frac{1}{2} \sum_{\mathbb{Z}_n} \rho \partial_x \rho = \frac{1}{4} \sum_{\mathbb{Z}_n} \rho (\partial_x + \partial_\rho) \rho,$$

where $\partial_x + \partial_\rho$ is the usual lattice double-differential on $\mathbb{Z}_n$. This has the same form as for a scalar field except that $\rho$ is a positive function, as already observed for $\mathbb{Z}$ in (20). We consider two approaches, depending on what we regard as our underlying field, and in both cases maintaining $\mathbb{Z}_n$ symmetry in the result.

(i) As suggested by the form of the action, we can thus of

$$\rho_0 = \frac{a(1)}{a(0)}, \quad \ldots, \quad \rho_{n-2} = \frac{a(n-1)}{a(n-2)}, \quad \rho_{n-1} = \frac{a(0)}{a(n-1)}$$

as $n$ dynamical variables subject to the constraint $\rho_0 \cdots \rho_{n-1} = 1$. We think of the constraint as a hypersurface in $\mathbb{R}^{n\rho}_0$ which induces a metric $g_\rho$ on the hypersurface, and use the Riemannian measure in this. Thus, we can take $\rho_0, \ldots, \rho_{n-2}$ as local coordinates and measure $D\rho = (\prod_{i=0}^{n-2} d\rho_i) \sqrt{|\det(g_\rho)|}$. The measure here maintains the $\mathbb{Z}_n$ symmetry as ultimately independent of the choice of coordinates.

Explicitly, for $n = 3$, we take $\rho_0, \rho_1$ as coordinates and the constrained surface in $\mathbb{R}^3_{n=0}$ is $\rho_2 = 1/(\rho_0 \rho_1)$. The coordinate tangent vectors and induced metric are

$$v_0 = (1, 0, -\frac{1}{\rho_0 \rho_1}), \quad v_1 = (0, 1, -\frac{1}{\rho_0 \rho_1});$$

$$g_\rho = (v_i, v_j) = \left(1 + \frac{1}{\rho_0^2 \rho_1}, \frac{1}{\rho_0^2 \rho_1^2}, \frac{1}{\rho_0^2 \rho_1^2}, 1 + \frac{1}{\rho_0^2 \rho_1^2}, \frac{1}{\rho_0^2 \rho_1^2}\right), \quad \det(g_\rho) = 1 + \frac{1}{\rho_0^2 \rho_1^2} + \frac{1}{\rho_0^2 \rho_1^2}.$$ 

Hence the partition function is

$$Z = \int_0^\infty d\rho_0 \int_0^\infty d\rho_1 \sqrt{|\det(g_\rho)|} e^{-\frac{1}{\rho_0^2 \rho_1^2}} (\rho_0^2 + \rho_1^2 - \rho_0 \rho_1 - \rho_0 \rho_1), \quad \rho_2 \approx \frac{1}{\rho_1 \rho_2}.$$

These integrals can be done numerically and appear to converge for all values $G > 0$ of the coupling constant (the numerical results need $G$ not too small for working precision but this case can be analysed separately). We are interested in expectation values $\langle \rho_i \cdots \rho_m \rangle$ where we insert $\rho_i \cdots \rho_m$ in the integrand and take the ratio with $Z$.

Some results obtained from this theory for $n = 3$ are plotted in Figure 2. Numerical evidence is limited due to convergence accuracy issues, but it seems clear that expectation values of products of $\rho_i$ tend to 1 as $G \to 0$, as might be expected. As in (19), this should be thought of as the weak gravity limit given that fluctuations expressed in $\rho$ enter the action relative to $G$. Meanwhile, it appears that $\frac{\Delta \rho_i}{\langle \rho_i \rangle} \sim 1.1$ as $G \to \infty$ (at least for the limited range of $G$ accessible numerically), which would be a similar phenomenon for the relative metric uncertainty in (19) in the ‘strong gravity’ limit. By contrast, it would appear that $\frac{\langle \rho_i \rho_j \rangle}{\rho_i \rho_j}$ for $i \neq j$ has a minimum of around 0.808 for $G \approx 6.55$.

(ii) We can take (as more usual) the metric coefficients as the underlying field, so in our case the edge lengths $a = (a_0, \ldots, a_n) \in \mathbb{R}^n_{\geq 0}$. Assuming Lebesgue measure,
the partition function is

$$Z = \int_0^\infty (\prod_i d\alpha_i) e^{\frac{S_g}{\pi}} = \int_0^L (\prod_i d\alpha_i) e^{\frac{1}{\pi} \Sigma_{\alpha_i} \rho^2 \alpha_i}$$

and we introduce a field strength upper bound $L$ to control divergences as in [19]. One can then look at ratios independent of $L$ or indeed consider a formal renormalisation process.

On the other hand, the divergences come from the global scaling symmetry $a_i \mapsto \lambda a_i$ for $\lambda \in \mathbb{R}_{>0}$ of the action (since this depends only on the ratios $\rho$) and therefore another approach would be to ‘factor out’ the overall value and not do its integral. This is again in the spirit of [19], except that we proceed multiplicatively. Thus we let $A = (\prod_i a_i)^\frac{1}{n}$ be the geometric mean and $b_i = a_i/A$, which by construction obey $b_0 \cdots b_{n-1} = 1$. These are similar to the $\rho_i$ variables in forming the corresponding hypersurface in $\mathbb{R}^n_{>0}$ but the action is different and the measure is also different since it is inherited from the Lebesgue measure on the $a_i$.

Again, we will look at this explicitly for $n = 3$. Then the action is

$$S_g = \frac{1}{2} \left( \frac{b_0}{b_1} + \frac{b_1}{b_2} + \frac{b_0}{b_2} \right) - \left( \frac{b_1}{b_0} \right)^2 - \left( \frac{b_2}{b_1} \right)^2 - \left( \frac{b_0}{b_2} \right)^2; \quad b_2 = \frac{1}{b_0 b_1},$$

while the Jacobean for the change of variables from $a_0, a_1, a_2$ to $b_0, b_1, A$ gives us

$$da_0 da_1 da_2 = \frac{3A^2}{b_0 b_1} db_0 db_1 dA.$$ 

Omitting the now decoupled integration over $A$ as an (infinite) constant, we have effectively

$$Z = \int_0^b db_0 \int_0^b db_1 \frac{1}{b_0 b_1} e^{\frac{1}{2} \lambda (-1 + 1 + b_0^3)(-1 + b_0^3)(-1 + b_0^3)(-1 + b_0^3)(b_0^6))}. $$

The graphical expectation values against $G$ look qualitatively similar to those of $\rho_i$ in Figure 2 but one also has $\langle b_i \rangle = \langle b_i b_j \rangle$ for $i \neq j$, but this is specific to $n = 3$. 

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{Euclidean quantum gravity vevs on $\mathbb{Z}_3$ for gauge invariant variables $\rho_i$}
\end{figure}
Larger \( n > 3 \) can proceed entirely similarly and one has \( 1 < \langle b_i \rangle < \langle b_{i+1} \rangle \). One can also then see that the \( i \)-step correlations \( \langle b_0 b_i \rangle \) (or between any two points different by \( i \)) decrease as \( i \) increases from \( i = 0 \) to reach a minimum (as expected) halfway around the polygon. This is based on numerical data for small \( n \) as shown in Figure 3. The data for \( n = 6 \) are noisy due to numerical convergence issues.

4. Quantum geometric cosmological models on \( \mathbb{R} \times \mathbb{Z}_n \)

In this section, we first start with an analysis of quantum metrics and QLCs on \( \mathbb{R} \times \mathbb{Z}_n \), where \( \mathbb{R} \) is a classical time and \( \mathbb{Z}_n \) is discrete. We find that the full ‘strongly tensorial’ bimodule properties for an invertible quantum metric force us to the block diagonal case, without taking this as an assumption. Existence of a QLC further dictates its form, again without taking this as an assumption, and we then find a unique \( * \)-preserving one. We then focus on the case where the \( \mathbb{Z}_n \) geometry is flat (modelling an actual geometric circle) but possibly time-dependent as in FLRW cosmology.

4.1. Quantum metric and QLC on \( \mathbb{R} \times \mathbb{Z}_n \). We consider a general metric on the product \( \mathbb{R} \times \mathbb{G} \) where \( \mathbb{R} \) has ‘time’ variable \( t \) and we are interested in the finite group \( \mathbb{G} = \mathbb{Z}_n \) and \( e^a = e^a \), but we do not need to specialise at this stage. We consider metrics of the form

\[
g = \mu dt \otimes dt + h_{ab} e^a \otimes e^b + \beta_a (e^a \otimes dt + dt \otimes e^a)
\]

for \( \mu, h_{ab}, \beta_a \) in \( A = C^\infty (\mathbb{R}) \otimes C(\mathbb{G}) \) but note right away that if we take the tensor product calculus where the time variable and its differential \( t, dt \) graded commute with functions and forms on \( \mathbb{G} \) then centrality of the metric needed for a bimodule inverse dictates that \( \beta_a = 0 \). We therefore proceed in this case.

Similarly, we look for general QLCs of the form

\[
\nabla dt = -\Gamma a dt \otimes dt + c_a (e^a \otimes dt + dt \otimes e^a) + d_{ab} e^a \otimes e^b,
\]

\[
\nabla e^a = -\Gamma a_{bc} e^b \otimes e^c + \gamma a (e^b \otimes dt + dt \otimes e^b) + f^a dt \otimes dt
\]
and note that for the tensor form of calculus along with the natural choice where \( \sigma(\text{d}t \otimes e^a) \), \( \sigma(\text{d} e^a) \) are the flip on the basic 1-forms \( dt, e^a \), requiring the above to be a bimodule connection compatible with the relations of each algebra forces us to

\[
e_a = 0, \quad f^a = 0, \quad \gamma_a^b = \gamma_a \delta_{a,b}, \quad d_a = d_a \delta_{a,b^{-1}}
\]

for some functions \( \gamma_a \). We therefore proceed in this case.

Next, for zero torsion, we need that

\[
d_{ab} = d_{ba}, \quad \Gamma^a_{bc} = \Gamma^a_{cb}, \quad \therefore (\text{id} + \sigma)(e^a \otimes e^b) = 0
\]

(which means \( \sigma \) restricted to the \( \{ e^a \} \) has the form studied before for a torsion free bimodule connection on an inner calculus, but note the calculus as a whole is not inner). And for \( \nabla g = 0 \), we obtain 8 equations which we compute under our assumptions above for a central metric and bimodule connection, with \( \mu = \frac{d}{\partial t} \mu \),

\[
d \otimes \otimes e^a : \quad \hat{\mu} = -\mu \Gamma,
\]

\[
d \otimes e^a \otimes e^a : \quad 0 = 0,
\]

\[
e^a \otimes e^a \otimes e^a : \quad \partial_a \mu = 0,
\]

\[
d \otimes e^a \otimes e^b : \quad h_{cb} \gamma^c_a + h_{ac} \Gamma^c_a + \dot{h}_{ab} = 0,
\]

\[
e^a \otimes e^a \otimes e^b : \quad h_{cb} \gamma^c_a + \mu d_{ab} = 0,
\]

\[
e^a \otimes e^b \otimes e^b : \quad \mu d_{ab} + h_{mp} R_{mn} (\gamma_{p}^n) \sigma_{mn} = 0
\]

\[
e^m \otimes e^a \otimes e^p : \quad \partial_m h_{np} - h_{ap} \Gamma^a_{mn} - h_{ac} \Gamma^c_a \sigma_{mn} = 0.
\]

The first and last of the 8 equations are just that \( \Gamma \) is a QLC on the line and \( \sigma, \Gamma^a_{bc} \) a QLC on \( G \). The 4th tells us that \( \mu \) is constant on \( G \). If we write the metric as \( h_{ab} = h_a \delta_{a,b^{-1}} \) for functions \( h_a \) etc., then the 6th equation tells us

\[
d_a = -\frac{h_a \gamma_a}{\mu}
\]

and the 5th and 7th reduce to

\[
\dot{h}_a + h_a \gamma_a + R_a (h_{a^{-1}} \gamma_{a^{-1}}) = 0, \quad \sum_p \Gamma_{p^{-1}} (h_p \gamma_p) \sigma_{p^{-1}} = h_a \gamma_a \delta_{a,b^{-1}}.
\]

Finally, we impose *-structure \( d^* = dt \) and suppose that the connection on \( G \) is also *-preserving for \( e^{a*} = -e^{a^{-1}} \) as usual. The extended metric is then obeys the quantum reality condition if \( \mu \) is real, which we suppose henceforth, and the metric on \( G \) is ‘real’ in the required sense (which amounts to \( h_a \) real-valued). Then the additional condition for our extended \( \nabla \) to be *-preserving comes down to \( \Gamma \) real and

\[
\overline{\gamma}_a = R_a \gamma_a^{-1}, \quad \sum a \overline{d}_a \sigma(e^a \otimes e^{a^{-1}}) = \sum a d_{a^{-1}} e^{a^{-1}} \otimes e^a,
\]

where the 1st part comes from \( \nabla e^{a*} \) and the 2nd from \( \nabla dt^* \). Next we use \((4.1)\) and that \( h_a \) are real and edge-symmetric to deduce from the 1st part that \( \overline{d}_a = R_a d_{a^{-1}} \). Then since \( d_a \) are constant on \( G \), we have \( \overline{d}_a = d_{a^{-1}} \) and our condition to be *-preserving is

\[
\overline{\gamma}_a = R_a \gamma_a^{-1}, \quad \sum a d_{a^{-1}} (\sigma(e^a \otimes e^{a^{-1}}) - e^{a^{-1}} \otimes e^a) = 0.
\]
Since \( \mu \) has to be a constant on \( G \), it is some function of \( t \) alone. Generically, we can absorb this in a change of the variable \( t \), so we proceed for simplicity with \( \mu = -1 \) for a cosmological type solution.

**Theorem 4.1.** For \( \sigma, \nabla^Z \) the *-preserving QLC on \( \mathbb{Z}_n \) in Proposition 3.1, a quantum metric on \( \mathbb{R} \times \mathbb{Z}_n \) admitting a *-preserving QLC has the form

\[
g = -dt \otimes dt - ae^+ \otimes e^- - R_a e^- \otimes e^+
\]

up to choice of the \( t \) parameterization, such that \( \partial_\ast \dot{a} = 0 \), i.e., \( a \) has the form

\[
a(t, i) = \alpha(t) + \beta(i)
\]

for some functions \( \alpha, \beta \) with \( \sum_i \beta(i) = 0 \). In these terms, there is a unique *-preserving QLC with scalar curvature and Laplacian

\[
2S = -\ddot{a} \left( \frac{1}{\alpha + \beta} + \frac{1}{\alpha + R \beta} \right) + \frac{\dot{a}^2}{4} \left( \frac{1}{(\alpha + \beta)^2} + \frac{1}{(\alpha + R \beta)^2} \right) + \frac{s}{(\alpha + \beta)^2(\alpha + R \beta)} + R_\ast \left( \frac{s}{(\alpha + \beta)^2(\alpha + R \beta)} \right),
\]

\[
\Delta f = -\partial_\ast^2 + \left( \frac{1}{\alpha + \beta} + \frac{1}{\alpha + R \beta} \right) \left( -\frac{\dot{a}}{2} \partial_t f + \Delta_{Z_n} f \right),
\]

where

\[
s := (\alpha + R_\ast \beta)(\alpha + R \beta) - (\alpha + \beta)^2 = (\alpha + \beta)(\Delta_{Z_n} \beta) + (\partial_\ast \beta) \partial_\ast \beta
\]

in terms of the usual Laplacian \( \Delta_{Z_n} \beta = (\partial_\ast + \partial_\ast) \beta = R_\ast \beta + R \beta - 2 \beta \) on \( \mathbb{Z}_n \).

**Proof.** We use the general analysis above applied in the specific case of \( \mathbb{Z}_n \). Also, for the purpose of the proof, it is convenient to have a shorthand notation \( a_\ast = a \) and \( a_- = R \ast a \), so that \( h_\ast = a_\ast \) for our particular metric. Then the 2nd of \((4.2)\) holds automatically as \( \sigma(e^\ast \otimes e^\ast) = e^\ast \otimes e^\ast \) and \( a_\ast \gamma_\ast = d_\ast(t) \) are constants on \( \mathbb{Z}_n \) for a solution, while the 1st of \((4.2)\) is that \( \dot{a}_\ast = -\dot{a}_- - d_\ast \), which requires \( \partial_\ast \dot{a} = 0 \) as stated. We assume the QLC on \( \mathbb{Z}_n \) at each \( t \) for the metric functions \( a = a(t, i) \) at each \( t \). The flip form of \( \sigma(e^\ast \otimes e^\ast) \) for this also means that the 2nd part of \((4.3)\) is automatic and we just need \( \gamma_\ast = R_\ast \gamma_\ast \) or equivalently \( \dot{d}_\ast = d_\ast \) for a *-preserving connection. This means that

\[
d_\ast = \dot{a}_\ast \frac{\dot{a}_\ast}{2} + db, \quad d_- = \ddot{a}_\ast = -\dot{a}_\ast - db; \quad \gamma_\ast = -\dot{a}_\ast \frac{\dot{a}_\ast}{2a_\ast} + \frac{\dot{a}_\ast}{a_\ast}
\]

for any real-valued function \( b(t) \). The unique solution with real coefficients for \( \nabla \) in our basis is \( b = 0 \) and gives the *-preserving QLC

\[
(4.4) \quad \nabla dt = \dot{a}_\ast e^\ast \otimes e^- + e^- \otimes e^+, \quad \nabla e^\ast = \nabla^{\mathbb{Z}_n} e^\ast - \frac{\dot{a}_\ast}{2a_\ast} (e^\ast \otimes dt + dt \otimes e^\ast).
\]
The $\sigma$ for this when one argument is $dt$ is the flip. We then proceed to compute the curvature of this QLC,

$$R_{\mathcal{NC}} \epsilon^\pm = R_{\mathcal{NC}}^2 \epsilon^\pm - \left( \tilde{T}_{\pm ab} \Gamma_{\pm ab} \left( \frac{\dot{a}}{2a_\pm} \right) + \frac{\dot{a}}{2a_\pm} \Gamma_{\pm ab} \right) dt \wedge \epsilon^a \otimes \epsilon^b - \Gamma_{\pm ab} \left( \frac{\dot{a}}{2a_\pm} \right) \epsilon^a \wedge \epsilon^b \otimes dt$$

$$\pm \left( \frac{\dot{a}}{2a_\pm} \right)^2 a_\pm \epsilon^+ \wedge \epsilon^- \wedge \epsilon^+ + \frac{\dot{a}}{2} \partial_\pm \left( \frac{1}{a_\pm} \right) \epsilon^b \wedge \epsilon^+ \otimes dt + \frac{\dot{a}}{2} \partial_\pm \left( \frac{1}{a_\pm} \right) dt \wedge \epsilon^b \otimes \epsilon^+$$

$$- \dot{a} \left( \frac{\dot{a}}{2a_\pm} \right) + \left( \frac{\dot{a}}{2a_\pm} \right)^2 dt \wedge \epsilon^\pm \otimes dt,$$

$$R_{\mathcal{NC}} dt = \frac{\dot{a}}{2} dt \wedge \left( \epsilon^+ \otimes \epsilon^- + \epsilon^- \otimes \epsilon^+ \right) + \frac{\dot{a}}{2} \epsilon^+ \wedge \Gamma^- \wedge e^- \otimes \epsilon^b + \frac{\dot{a}}{2} \epsilon^- \wedge \Gamma^+ \wedge e^+ \otimes \epsilon^b$$

$$+ \sum \left( \frac{\dot{a}}{2a_\pm} \right)^2 a_\pm \epsilon^+ \wedge \left( \epsilon^\pm \otimes dt + dt \otimes \epsilon^\pm \right),$$

in terms of the Christoffel symbols on $\mathbb{Z}_n$. The Ricci tensor and the Ricci scalar $S$ are then

$$Ricci = Ricci^{\pm} + \frac{\dot{a}}{4} \left( \epsilon^+ \otimes \epsilon^- + \epsilon^- \otimes \epsilon^+ \right) + \frac{1}{2} \left( R_{\mathcal{NC}} \left( \tilde{T}^{\pm} \right) - \frac{\dot{a}}{2} \left( R_{\mathcal{NC}} \left( \tilde{T}^{\pm} \right) + 1 \right) \partial_\pm \left( \frac{1}{a_\pm} \right) \right) dt \otimes \epsilon^-$$

$$+ \frac{1}{2} \left( R_{\mathcal{NC}} \left( \tilde{T}^{\pm} \right) - \frac{\dot{a}}{2} \left( R_{\mathcal{NC}} \left( \tilde{T}^{\pm} \right) + 1 \right) \partial_\pm \left( \frac{1}{a_\pm} \right) \right) dt \otimes \epsilon^+ + \frac{\dot{a}}{4} \left( \left( R_{\mathcal{NC}} \left( \tilde{T}^{\pm} \right) - \frac{\dot{a}}{2} \right) \partial_\pm \left( \frac{1}{a_\pm} \right) \right) \epsilon^- \otimes dt$$

$$- \frac{\dot{a}}{4} \left( \left( R_{\mathcal{NC}} \left( \tilde{T}^{\pm} \right) - \frac{\dot{a}}{2} \right) \partial_\pm \left( \frac{1}{a_\pm} \right) \right) \epsilon^+ \otimes dt + \frac{1}{2} \left( \partial_\pm \left( \frac{\dot{a}}{2a_\pm} \right) + \frac{\dot{a}^2}{2a_\pm} \right) + \frac{\dot{a}^2}{2a_\pm} \right) \epsilon^+ \otimes dt \otimes dt,$$

$$S = \frac{\dot{a}}{2} \left( \frac{1}{a_+} + \frac{1}{a_-} \right) + \frac{1}{2} \left( \frac{\dot{a}}{2a_+} \right)^2 + \frac{1}{2} \left( \frac{\dot{a}}{2a_-} \right)^2$$

(where we have used that $\Gamma^+ = \Gamma^-$. We now insert values for the QLC in Proposition 3.1 to obtain

$$R_{\mathcal{NC}} \epsilon^\pm = \pm \left( \partial_\pm \left( \frac{a_\pm}{a_\pm} \right) + \frac{\dot{a}}{2a_\pm} \right)^2 a_\pm \epsilon^+ \wedge \epsilon^- \otimes \epsilon^+ + \frac{\dot{a}}{2a_\pm} \partial_\pm \left( a_\pm \right) dt \wedge \epsilon^+ \otimes \epsilon^+$$

$$+ \frac{\dot{a}}{2} \partial_\pm \left( \frac{1}{a_\pm} \right) \left( \epsilon^+ \wedge \epsilon^+ \otimes dt + dt \wedge \epsilon^\pm \otimes \epsilon^\pm \right) + \left( - \frac{\dot{a}}{2a_\pm} + \frac{\dot{a}^2}{2a_\pm} \right) \epsilon^+ \wedge \epsilon^- \otimes dt,$$

$$R_{\mathcal{NC}} dt = \sum \left( \frac{\dot{a}}{2a_\pm} \right) a_\pm dt \wedge \epsilon^\pm \otimes \epsilon^+ + \sum \left( \frac{\dot{a}}{2a_\pm} \right) \partial_\pm \left( a_\pm \right) \epsilon^+ \wedge \epsilon^- \otimes \epsilon^+ + \frac{\dot{a}^2}{2} \partial_\pm \left( \frac{1}{a_\pm} \right) \epsilon^+ \wedge \epsilon^- \otimes dt$$

and

$$Ricci = \frac{1}{2} \sum \left( \left( \frac{\dot{a}}{2} + \partial_\pm \left( \frac{a_\pm}{a_\pm} \right) \right) \epsilon^+ \wedge \epsilon^+ \otimes dt + \frac{\dot{a}}{2} \partial_\pm \left( \frac{1}{a_\pm} \right) \epsilon^+ \otimes dt \right)$$

$$- \frac{1}{2} \left( \frac{\dot{a}}{2} \right) \frac{1}{a_+} + \frac{1}{a_-} \right) + \left( \frac{\dot{a}}{2a_-} \right)^2 \right) \epsilon^+ \wedge \epsilon^- \otimes dt,$$

$$S = \frac{1}{2} \left( \frac{\dot{a}}{2} \right) \frac{1}{a_+} \left( \frac{\dot{a}}{2a_-} \right) \left( 1 \right) \frac{1}{a_-} \right) \right) \epsilon^+ \wedge \epsilon^- \otimes dt,$$

We now note that the requirement $\partial_\pm \dot{a} = 0$ is equivalent to $\dot{a}$ being of the form stated. Clearly such a form obeys this condition as $\dot{a} = \alpha$ is constant on $\mathbb{Z}_n$. Conversely, given $a(t, i)$ obeying the condition we let $\alpha(t) = \frac{1}{n} \sum a(t, i)$ be the average value and $\beta = a - \alpha$. The latter averages to zero and has zero time derivative by the
assumption on $a$, hence depends only on $i$. We now insert this specific form into the curvature calculations to obtain

$$\text{Ricci} = \left(\frac{\dot{\alpha}}{4} - \frac{s}{(\alpha + \beta)(\alpha + R_\beta)}\right) e^+ \otimes e^- + \left(\frac{\dot{\alpha} - R_\alpha}{4} - \frac{s}{(\alpha + \beta)(\alpha + R_\beta)}\right) e^- \otimes e^+$$

$$- \frac{\dot{\alpha} R_\alpha}{4} \frac{\partial_\beta}{(\alpha + \beta)^2} dt \otimes e^+ - \frac{\partial_\beta}{(\alpha + \beta)(\alpha + R_\beta)} e^+ \otimes dt$$

$$- \frac{\dot{\alpha}}{4} \frac{\partial_\beta}{(\alpha + \beta)^2} dt \otimes e^- - R_\alpha \frac{\partial_\beta}{(\alpha + \beta)(\alpha + R_\beta)} e^- \otimes dt$$

$$+ \left(\frac{\dot{\alpha}}{4} - \frac{2\alpha + \beta + R_\beta}{(\alpha + \beta)(\alpha + R_\beta)}\right) e^+ \otimes e^-$$

and the scalar curvature as stated. Without loss of generality, we have fixed $\sum_i \beta(i) = 0$ since this could be shifted into the value of $\alpha$. We also have geometric Laplacian

$$\Delta f = -\Delta^{Z_n} f - \left(\frac{1}{a} + \frac{1}{a_{\infty}}\right) \frac{\dot{a}}{2} \partial_\beta f - \partial_\beta^2 f = -\left(\frac{1}{a} + \frac{1}{a_{\infty}}\right) \left(\frac{\dot{a}}{2} \partial_\beta f - \Delta^{Z_n} f\right)$$

which simplifies as stated. We are using $\Delta^{Z_n}$ for the Laplacian in Proposition 3.1 and $\Delta_{Z_n}$ with lower label for the standard finite difference Laplacian.

In this theorem, $\alpha(t) > 0$ is the average ‘radius’ of the $Z_n$ geometry, evolving with time, while $\beta(i)$ as a fluctuation as we go around $Z_n$ and we see that this has to be ‘frozen’ (does not depend on time) in order for the metric to admit a quantum geometry. It is striking that this includes the FLRW-type models studied in the remaining section in the class forced by the quantum geometry. Note that we also need to restrict to

$$(4.5) \quad \min_i \beta(i) > -\inf_t \alpha(t)$$

so that $\alpha(t, i)$ is everywhere positive.

Although we will not study it here, we now in position to start thinking about quantum gravity on $\mathbb{R} \times Z_n$ in a functional integral approach. This would presumably have the form of a partition function

$$(4.6) \quad Z = \int D\alpha \prod_{t=0}^{n-2} d\beta(i) \ J_\beta e^{\int_0^\infty \int_{Z_n} dt \Sigma_{Z_n} \mu S[\alpha, \beta]}$$

for some measure $\mu(t, i)$. Classically, this would come from the metric coefficients and, for example, we might take something of the form $\mu = \sqrt{(\alpha + \beta)(\alpha + R_\beta)}$ in line with the case of $Z_n$ alone in Section 3.2. It is not clear what would be the right choice, however. For the integral over functions $\{\alpha(t)\}$ there would be usual issues to make this rigorous (as some kind of continuous product of integrals). The new feature is that these should be restricted to values $\alpha(t) > 0$ and for a given configuration $\{\alpha(t)\}$, we should limit the lower bound on the $\int d\beta(i)$ integrations according to $4.5)$. Finally, we presumably would want, to maintain the $Z_n$ symmetry, a Jacobian which we have denoted $J_\beta$ to reflect the geometry of the constraint $\Sigma \beta(i) = 0$. The choice of $\mu$ and the constrained integration are both issues that we already encountered for $Z_n$ in Section 3.2 but are now significantly more complicated. We also should now aim for a physical theory given the Lorentzian signature, hence the $r$ in the action.
4.2. Equations of state in FLRW model on $\mathbb{R} \times \mathbb{Z}_n$. We focus on this cosmological FLRW model case where $a = a(t)$ with no fluctuation $\beta(i)$ over $\mathbb{Z}_n$ and

\begin{equation}
\tag{4.7}
g = -dt \otimes dt - ae^+ \otimes_s e^-,
\end{equation}

where $e^+ \otimes_s e^- = e^+ \otimes e^- + e^- \otimes e^+$. In this case, the results above simplify to

\begin{equation}
\nabla dt = \frac{\dot{a}}{2} e^+ \otimes_s e^-,
\end{equation}

\begin{equation}
\nabla e^\pm = -\frac{\dot{a}}{2a} e^\pm \otimes_s dt,
\end{equation}

\begin{equation}
R e^\pm = -\dot{r} dt \wedge e^\pm \otimes dt \pm \left(\frac{\dot{a}}{2a}\right)^2 ae^+ \wedge e^- \otimes e^\pm, \quad R e_t dt = \ddot{r} dt \wedge e^+ \otimes_s e^-, 
\end{equation}

\begin{equation}
\text{Ricci} = \dot{r} dt \otimes dt + \frac{\ddot{a}}{4} e^+ \otimes_s e^-, \quad S = -\frac{\dot{a}}{a} + \left(\frac{\dot{a}}{2a}\right)^2,
\end{equation}

where

\[ r = \frac{\dot{a}}{2a} - \left(\frac{\dot{a}}{2a}\right)^2. \]

Although a general scheme for a noncommutative Einstein tensor is not known, in the present model it seems sufficient to define it in the usual way, in which case

\begin{equation}
\tag{4.11}
\text{Eins} = \text{Ricci} - \frac{1}{2} S g = -\frac{1}{8} \left(\frac{\dot{a}}{a}\right)^2 dt \otimes dt - \frac{ra}{2} e^+ \otimes_s e^-.
\end{equation}

**Lemma 4.2.** The divergence $\nabla = ((\cdot ,\cdot ) \otimes \text{id}) \nabla$ of a 1-1 tensor of the form

\[ T = f dt \otimes dt - pac^+ \otimes_s e^- \]

defined by functions $f, p$ on $\mathbb{R} \times \mathbb{Z}_n$, and for metric defined as above by $a(t)$, is

\[ \nabla \cdot T = -\left(\dot{f} + \frac{\dot{a}}{a}(f + p)\right) dt + \partial b pe^b. \]

In particular, the Einstein tensor (4.11) is conserved in the sense $\nabla \cdot \text{Eins} = 0$.

**Proof.** The Leibniz rule for the action of the connection produces

\begin{align*}
\nabla (f dt \otimes dt - pac^+ \otimes_s e^-) &= df \otimes dt \otimes dt - dp \otimes ae^+ \otimes_s e^- + f \nabla (dt \otimes dt) - p \nabla (ae^+ \otimes_s e^-) \\
&= df \otimes dt \otimes dt - dp \otimes ae^+ \otimes_s e^- + f \partial b pe^b \otimes dt \otimes dt + \partial b pe^b \otimes ae^+ \otimes_s e^- \\
&\quad + \frac{\ddot{a}}{2}(f + p) (e^+ \otimes_s e^- \otimes dt + e^- \otimes dt \otimes e^+ + e^+ \otimes dt \otimes e^-) \\
\end{align*}

on using metric compatibility so that $\nabla (dt \otimes dt) = -\nabla (ae^+ \otimes_s e^-)$ and then evaluating the former. Now applying the operator $(\cdot ,\cdot ) \otimes \text{id}$ with the inverse metric, we arrive at the stated result for the divergence.

For Eins in (4.11), the coefficients are constant so there is no $e^+$ term in $\nabla \cdot \text{Eins}$. For the $dt$ term it is easy to verify that $\dot{f} + \frac{\dot{a}}{a}(f + p) = 0$ automatically for the effective values of the specific coefficients $f, p$ in (4.11) defined by $a(t)$. \qed

Next, recall from Section 2.4 that our formulation of Ricci is $-1/2$ of the usual value, hence Einstein’s equation for us should be written as

\begin{equation}
\tag{4.12}
\text{Eins} + 4\pi GT = 0
\end{equation}
and from (4.11) we see that this holds if $T$ has the form for dust of pressure $p$ and density $f$, namely

$$T = pg + (f + p)dt \otimes dt = f dt \otimes dt - pae_+ \otimes e_-$$

for pressure and density

$$p = -\frac{1}{8\pi G} \frac{\dot{a}}{2a} - \left( \frac{\dot{a}}{2a} \right)^2, \quad f = \frac{1}{32\pi G} \left( \frac{\dot{a}}{a} \right)^2.$$ 

Note that $T$ is automatically conserved by the same calculation as for the Einstein tensor and this does not give any constraint on $a(t)$. Setting

$$H := \frac{\dot{a}}{a},$$

conservation is equivalent to the continuity equation

$$\dot{f} = -H(f + p),$$

which also holds automatically. The standard consideration in cosmology at this point is to assume an equation of state $p = \omega f$ for a real parameter $\omega$, in which case the continuity equation becomes $\frac{df}{dt} = -f (1 + \omega)$ so that $f \propto a^{-(1+\omega)}$. Given this form of the density $f$, our assumption $p = \omega f$ can be solved for $\omega = -1$ to give

$$a(t) = a_0 (1 + \sqrt{8\pi G f_0 (1 + \omega) t})^{\frac{1}{1+\omega}}$$

for initial radius and pressure $a_0, f_0$. Here $\omega > -1$ leads to an expanding universe. Recall that one usually takes $\omega = 0, 1/3$ for cold dust and radiation respectively.

If we add a cosmological constant so that Eins $-\frac{1}{2}g\Lambda + 4\pi GT = 0$, this is equivalent to a modified stress energy tensor given as before but with modified

$$f^\Lambda = f + \frac{\Lambda}{8\pi G}, \quad p^\Lambda = p - \frac{\Lambda}{8\pi G}, \quad p^\Lambda = \omega f^\Lambda - \frac{1 + \omega}{8\pi G} \Lambda.$$ 

The effective equation of state now leads to

$$a(t) = a_0 \left( \frac{\cosh(\text{arccosh}(\sqrt{-\frac{\Lambda}{8\pi G f_0}})) + \sqrt{\Lambda(1 + \omega) t}}{\sqrt{-\frac{\Lambda}{8\pi G f_0}}} \right)^{\frac{1}{1+\omega}}$$

with reasonable behaviour for $f_0 > 0$ (with $f$ remaining positive) and real $\Lambda$ but a limited range of $t$ when $\Lambda < 0$.

For comparison, note that the classical Einstein tensor on $\mathbb{R} \times S^1$ with $g = -dt \otimes dt + adx \otimes dx$ vanishes as for any 2-manifold and $T = f dt \otimes dt + padx \otimes dx = pg + (f + p) dt \otimes dt$ admits only zero pressure and density if we want Einstein’s equation. One can also add a cosmological constant, in which case we need $p = -\frac{\Lambda}{8\pi G}$ and $f = \frac{\Lambda}{8\pi G}$ and $\omega = -1$.

**Proposition 4.3.** The results (4.14)-(4.15) for $a(t)$ (as well as for $f(t)$) for the FLRW model on $\mathbb{R} \times S^1$ are the same as for the classical flat FLRW-model on $\mathbb{R} \times \mathbb{R}^2$.

**Proof.** The flat FLRW model in 1+2 dimensions is an easy exercise starting with the metric $g = -dt \otimes dt + a(t)(dx \otimes dx + dy \otimes dy)$ to compute the Ricci tensor (in our conventions, which is $-\frac{1}{2}$ of the usual values) as

$$\text{Ricci} = r dt \otimes dt - \frac{\dot{a}}{4}(dx \otimes dx + dy \otimes dy)$$
and the same scalar curvature $S$ as in (4.10). The Einstein tensor is therefore

$$Eins = \frac{1}{8} \left( \frac{\dot{a}}{a} \right)^2 dt \otimes dt + \frac{ra}{2} (dx \otimes dx + dy \otimes dy)$$

by a similar calculation as for (4.11). The stress tensor for dust being similarly $f dt \otimes dt + p(dx \otimes dx + dy \otimes dy)$ means that the Einstein equation give $p, f$ by the same expressions (4.13) as before. The Friedmann equations are therefore the same as we solved.

This is perhaps not too surprising given that $\Omega^1$ on $\mathbb{Z}_n$ is 2-dimensional, indeed $-e^+ \otimes e^-$ plays the same role as the classical spatial metric $dx \otimes dx + dy \otimes dy$. We also recall by way of comparison that the standard $k = 0$ Friedmann equations for the FLRW model $\mathbb{R} \times \mathbb{R}^3$ has the well-known solution (after noting that in our conventions the metric has $a$ not $a^2$),

$$a(t) = a_0(1 + \sqrt{6\pi G f_0(w + 1)t})^{\frac{4}{3(w+1)}}$$

without cosmological constant and can also be solved with it, as

$$a(t) = a_0 \left( \cosh\left( \text{arccosh}\left( \sqrt{\frac{\lambda}{8\pi G f_0}} \right) + \sqrt{\frac{3\Lambda}{4}} (w + 1)t \right) \right)^{\frac{4}{3(w+1)}}.$$

As usual, the case of $a(t)$ independent of time is a solution for the Einstein vacuum equation with $\text{Ricci} = S = 0$. It is easy to see that there are no other solutions of interest with $\text{Ricci} \propto g$ or $\text{Eins} \propto g$. On the other hand, we do have the following.

**Proposition 4.4.** The equation $\text{Ricci} - \lambda S g = 0$ with time-varying $a(t)$ and constant $\lambda$ has a unique solution of the form

$$\lambda = \frac{1}{3}, \quad a(t) = a_0 e^{\mu t}$$

for some growth constant $\mu \neq 0$ and initial $a_0 > 0$.

**Proof.** Considering the equation $\text{Ricci} = \lambda g S$, where $\lambda$ is an arbitrary real constant, we have two equations, one related to $e^+ \otimes e^+$ is

$$\frac{\ddot{a}}{a} + \frac{\lambda}{1 - 4\lambda} \left( \frac{\dot{a}}{a} \right)^2 = 0$$

and other related to $dt \otimes dt$ is

$$\frac{\ddot{a}}{a} + \frac{\lambda - 1}{2 - 4\lambda} \left( \frac{\dot{a}}{a} \right)^2 = 0.$$

This requires $\lambda = \frac{1}{4}$ and $\frac{\dot{a}}{a} = \left( \frac{\dot{a}}{a} \right)^2$, which has the solution claimed. □
4.3. Quantum field theory on $\mathbb{R} \times \mathbb{Z}_n$. Here we consider quantum field theory in the flat case where $a$ is a constant. The corresponding Laplacian operator and the Klein-Gordon equation are
\[
\Delta = \frac{2}{a} (\partial_+ + \partial_-) - \partial_+^2; \quad (-\Delta + m^2)\phi = 0.
\]
Writing $q = e^{\frac{2\pi i}{a}}$ and Fourier transforming on $\mathbb{Z}_n$ by considering solutions of the form $\phi(t, k) = q^{ik}e^{-iw_k t}$ labelled by $k = 0, \ldots, n-1$, we obtain the ‘mass on-shell’ expression
\[
w_k^2 = \frac{2}{a} \sin^2 \left( \frac{\pi k}{n} \right) + m^2.
\]
We then consider the corresponding operator-valued fields starting with
\[
\phi_i = \sum_{k=0}^{n-1} \frac{1}{\sqrt{2w_k}} (q^{ik}a_k + q^{-ik}a_k^\dagger),
\]
where now $a_k, a_k^\dagger$ are self-adjoint operators and $a_k |0\rangle = 0$, where $|n\rangle$ are the eigenvectors of the corresponding Hamiltonian
\[
H = \sum_{k=0}^{n-1} w_k (a_k a_k^\dagger + \frac{n}{2}).
\]
From the commutators $[H, a_k] = -w_k a_k$ and $[H, a_k^\dagger] = w_k a_k^\dagger$, and using the Heisenberg representation for the time evolution of the field, we obtain
\[
\phi_i(t) = e^{iHt} \phi_i e^{-iHt} = \sum_{k=0}^{n-1} \frac{1}{\sqrt{2w_k}} (q^{ik}e^{-iw_k t} a_k + q^{-ik}e^{iw_k t} a_k^\dagger)
\]
with the time-ordered correlation function
\[
\langle 0 | \mathcal{T} [\phi_i(t_a) \phi_j(t_b)] | 0 \rangle = \sum_{k=0}^{n-1} \frac{1}{w_k} \cos \left( \frac{2\pi}{n} k (i-j) \right) e^{-iw_k |t_a-t_b|}.
\]
Next we check that we obtain the same correlation function via a formal path integral approach with the $\hbar$-prescription. The partition functional integral $Z[J]$ is defined as
\[
Z[J] = \int \mathcal{D}\phi e^{\frac{i}{\hbar} S[\phi]} \int dt \sum_{k=0}^{n-1} J_i(t) \phi_i(t)
\]
\[
\int \mathcal{D}\phi e^{\frac{i}{\hbar} S[\phi]} \int dt \sum_{k=0}^{n-1} (\phi_i(t)(\Delta-m^2+\epsilon)\phi_i(t)+2J_i(t)\phi_i(t))
\]
where $\beta$ is a dimensionless coupling constant. We diagonalize the action $S[\phi]$ using Fourier transform to write
\[
\phi_i(t) = \sum_{k=0}^{n-1} \int_{-\infty}^{\infty} \frac{dw}{2\pi} \phi_k(w)q^{ik}e^{iwt}; \quad \tilde{J}_i(t) = \sum_{k=0}^{n-1} \int_{-\infty}^{\infty} \frac{dw}{2\pi} \tilde{J}_k(w)q^{ik}e^{iwt},
\]
which produces the action
\[
S[\tilde{\phi}] = \int_{-\infty}^{\infty} \frac{dw}{2\pi} \frac{1}{2\beta} \sum_{k=0}^{n-1} \left( \tilde{\phi}'_{-k}(-w)(-w^2 + w_k^2)\tilde{\phi}_{k}^\dagger(w) + \tilde{J}_{-k}(-w) \frac{1}{-w^2 + w_k^2} \tilde{J}_{k}(w) \right),
\]
where $\tilde{\phi}'_{k}(w) = \tilde{\phi}_{k}(w) - (-w^2 + w_k^2)^{-1}\tilde{J}_{k}(w)$. The first term in terms of the new variables gives a Gaussian integral which we ignore as an overall factor independent.
of the source. Using

$$\tilde{J}_k(w) = \frac{1}{n} \int dt \sum_{i=0}^{n-1} J_i(t) e^{-i w t},$$

doing the functional integral becomes

$$Z[J] = e^{\frac{1}{2} \int dt' dt'' J_i(t') \Delta_f(i, i'; j, j') J_j(t'')} 
Z[J] = e^{\frac{1}{2} \int dt' dt'' J_i(t') \Delta_f(i, i'; j, j') J_j(t'')},$$

where the Feynman propagator is

$$\Delta_f(i, i'; j, j') = \sum_{k=0}^{n-1} \frac{1}{w_k} \cos \left( \frac{2\pi}{n} k i - j \right) e^{-i w_k |t_a - t_b|}.$$  

Finally, by construction, we have

$$\langle 0 | T[\phi_i(t_a) \phi_j(t_b)] | 0 \rangle = \frac{\beta^2}{i^2} \frac{\partial}{\partial J_i(t_a)} \frac{\partial}{\partial J_j(t_b)} Z[J] = \Delta_f(i, i'; j, j'),$$

which therefore gives the same result as obtained by Hamiltonian quantisation. This is as expected, but provides a useful check that our methodology makes sense at least in the flat case of constant $a$.

4.4. **Particle creation in FLRW model on $\mathbb{R} \times \mathbb{Z}_n$.** Here we follow the procedure developed by Parker [27, 28, 29, 30] to study cosmological particle, adapted now to an FLRW model on $\mathbb{R} \times \mathbb{Z}_n$ with an expanding quantum metric (4.7).

4.4.1. **Model case of $\mathbb{R} \times S^1$.** We start with the classical background geometry case of $\mathbb{R} \times S^1$, which is presumably known but sets up the procedure and our notations. Here the metric has the usual 2D FLRW form

$$g = -dt \otimes dt + adx \otimes dx,$$

where $a(t)$ is an arbitrary function. Thus the Klein-Gordon equation for the field $\phi$ is

$$\left( g^{\mu \nu} \nabla_\mu \nabla_\nu - m^2 \right) \phi = 0$$

or in explicit form

$$\ddot{\phi} + \frac{\dot{a}}{2a} \dot{\phi} - \frac{1}{a} \partial_x^2 \phi + m^2 \phi = 0.\tag{4.19}$$

We impose the periodic boundary condition $\phi(t, x + L) = \phi(t, x)$, where $L$ has units of length. We then expand the field in terms of a Fourier series

$$\phi(t, x) = \sum_k (A_k f_k(t, x) + A_k^* f_k^*(t, x)), \tag{4.20}$$

where

$$f_k(t, x) = \frac{1}{\sqrt{|L a|^{1/4}}} e^{i x k h_k(t)} \tag{4.21}$$
and $k = 2l\pi/L$ for $l$ an integer. Here $k$ is the physical momentum and $l$ the corresponding ‘integer momentum’ on a circle. Here $\phi$ obeys \[4.19\] provided
\[
\dot{h}_k(t) + \left(\frac{k^2}{a} + m^2\right)h_k(t) + \left(\frac{3}{16} \left(\frac{\dot{a}}{a}\right)^2 - \frac{1}{4} \frac{\ddot{a}}{a}\right)h_k(t) = 0
\]
for each momentum mode. We will be particularly interested in the so called ‘adiabatic limit’, where $a$ varies slowly with respect to the time in such way that $\dot{a}/a \to 0, \ddot{a}/a \to 0$. The solutions to \[4.22\] in this approximation are
\[
h_k(t) \sim (w_k)^{-\frac{1}{2}} \left(\alpha_k e^{i\int w_k(t')dt'} + \beta_k e^{-i\int w_k(t')dt'}\right)
\]
where $\alpha_k$ and $\beta_k$ are complex constant that satisfy
\[
|\alpha_k|^2 - |\beta_k|^2 = 1
\]
and
\[
w_k(t) = \sqrt{m^2 + \frac{k^2}{a(t)}}.
\]
In order to have an exact solution, we now let $\alpha_k$ and $\beta_k$ be functions of time such that
\[
h_k(t) = (w_k(t))^{-\frac{1}{2}} \left(\alpha_k(t) e^{i\int w_k(t')dt'} + \beta_k(t) e^{-i\int w_k(t')dt'}\right)
\]
and
\[
|\alpha_k(t)|^2 - |\beta_k(t)|^2 = 1
\]
for all $t$. Equivalently, we can rewrite the expansion of the field as
\[
\phi(t, x) = \sum_k \left(\alpha_k(t) g_k(t, x) + \alpha_k^*(t) g_k^*(t, x)\right),
\]
where now
\[
g_k(t, x) = \frac{\alpha^{-\frac{1}{2}}}{\sqrt{Lw_k}} e^{i(xk - \int w_k(t')dt')}e^{i\frac{1}{2}\int w_k(t')dt'}
\]
and
\[
\alpha_k(t) = \alpha_k(t)^* A_k + \beta_k(t) A_k^*.
\]
In order to follow the usual procedure of canonical quantisation, we next define the conjugate momentum as
\[
\pi(t, x) = a\dot{\phi}(t, x),
\]
promote the field $\phi(t, x)$ and the momentum $\pi(t, x)$ to operators $\hat{\phi}(t, x), \hat{\pi}(t, x)$ respectively, and impose the commutators relations
\[
[\hat{\phi}(t, x), \hat{\phi}(t, x')] = [\hat{\pi}(t, x), \hat{\pi}(t, x')] = 0, \quad [\hat{\phi}(t, x), \hat{\pi}(t, x')] = i\delta(x - x').
\]
This requires that $A_k$ and $A_k^*$ in \[4.29\] are promoted to operators $A_k$ and $A_k^*$ with the usual commutation relations
\[
[A_k, A_k^*] = \delta_{kk'}, \quad [A_k, A_k^*] = 0.
\]
It then follows from these and a conserved quantity (see \[27\]), that the operator versions of \[4.29\] obey
\[
[a_k(t), a_{k'}(t)] = [a_k^*(t), a_{k'}^*(t)] = 0, \quad [a_k(t), a_{k'}^*(t)] = \delta_{kk'}.
\]
Now note that for any function $W_k(t)$ with at least derivatives to second order, the function
\begin{equation}
(4.33) \quad H(t) = W_k(t)^{-1/(2)}(\alpha_k e^{f} df' W_k(t') + \beta_k e^{-f} df' W_k(t'))
\end{equation}
for any constants $\alpha_k, \beta_k$ is an exact solution of the equation
\[ \dot{H}(t) + \left[ W_k^2 - W_k \frac{d^2}{dt^2} W_k^{-\frac{1}{2}} \right] H(t) = 0. \]
Hence if we can solve for $W_k(t)$ such that
\begin{equation}
(4.34) \quad W_k^2 = W_k \frac{d^2}{dt^2} W_k^{-\frac{1}{2}} + w_k^2 + \sigma
\end{equation}
holds, where
\[ \sigma = \frac{3}{16} \left( \frac{\dot{a}}{a} \right)^2 - \frac{1}{4} \frac{\dot{a}}{a}, \]
then $H(t)$ provides exact solutions $h_k(t)$ of (4.22) for each $k$.

We can then expand $W_k$ as a sum of terms
\begin{equation}
(4.35) \quad W_k = w^{(0)} + w^{(1)} + w^{(2)} + \ldots,
\end{equation}
where the superfix denotes the adiabatic order. Putting this into (4.34) and just keeping the elements of order zero, we have $w^{(0)} = w_k$. Just keeping the elements of first order tell us that $w^{(1)} = 0$, while for elements of second adiabatic order we require
\[ w^{(2)} = \frac{(w^{(0)})^{-\frac{1}{2}}}{2} \frac{d^2}{dt^2} \left( (w^{(0)})^{-\frac{1}{2}} \right) + \frac{\sigma}{2w^{(0)}}, \]
We can continue this procedure to any desired order to find odd $w^{(i)} = 0$ and even $w^{(i)}$ determined from lower even ones. The form of the functions $\alpha_k(t)$ and $\beta_k(t)$ can be obtained when we impose (4.27). From its temporal derivative, one is led to the ansatz
\begin{equation}
(4.36) \quad \alpha_k(t) = -\dot{\beta}_k(t) e^{-2{f} df' W_k(t')}, \quad \beta_k(t) = -\dot{\alpha}_k(t) e^{2{f} df' W_k(t')}
\end{equation}
as justified by consistency with (4.22) given (4.34). For an explicit form of these coefficients, see [26].

A special case of interest here is when the $w^{(i)}$ vanish for all the orders bigger that zero (and all $k$). In this case, the operator $a_k(t)$ defined in (4.29) is independent of time, the number of particles is constant and there is no particle creation. From the above remarks, it is sufficient that $w_k^{(2)} = 0$, which amounts to
\begin{equation}
(4.37) \quad \frac{m^2 \left(-3m^2 + 2k^2 \right)}{16 \left( k^2 + m^2 \right)^2} \left( \frac{\dot{a}}{a} \right)^2 + \frac{1}{4} \frac{m^2}{( \frac{k^2}{a} + m^2)} \frac{\dot{a}}{a} = 0.
\end{equation}
The only way that this can hold for all time and $k$ is in the infinite mass limit $m \to \infty$ (cf. [27]), where it reduces to an FLRW-like equation
\[ \frac{\dot{a}}{a} = \frac{3}{4} \left( \frac{\dot{a}}{a} \right)^2 \]
with solution $a \propto t^4$. As well as the obvious flat Minkowski case of constant $a$, this represents a further possibility for no particle creation.
For an actual particle creation computation, it is convenient to move to a new time variable \( \eta \) such that

\[
\frac{d\eta}{dt} = \frac{1}{\sqrt{a(t)}}
\]

in which case our metric becomes conformally flat,

\[
g = C(\eta)(-d\eta \otimes d\eta + dx \otimes dx),
\]

where \( C(\eta) = a(t) \), i.e. \( a \) regarded as a function of \( \eta \). Following the same steps as before but using this metric puts the wave equation \( (4.22) \) on spatial momentum modes in the simpler form

\[
\frac{d^2 h_k(\eta)}{d\eta^2} + w_k(\eta) h_k(\eta) = 0,
\]

where

\[
w_k(\eta) = \sqrt{C(\eta)m^2 + k^2}
\]

as a modification of \( (4.25) \).

We now consider particle creation under the assumption that \( a \) and hence \( C \) has a constant constant value \( C(\eta) = a_{in} \) for early times \( \eta < \eta_{in} \), say, and a constant value \( C(\eta) = a_{out} \) for late times \( \eta > \eta_{out} \), with \( \eta_{in} < \eta_{out} \). For these early and late times, we let

\[
w_k^{in} = \sqrt{a_{in}m^2 + k^2}; \quad w_k^{out} = \sqrt{a_{out}m^2 + k^2}
\]

as functions of \( k \). The fields at early and late times behave exactly as flat Minkowski space-time with the corresponding frequency or effective mass, with solutions of \( (4.40) \) at early and late times provided by

\[
h_k^{in}(\eta) = (w_k^{in})^{-\frac{1}{2}} e^{i w_k^{in}\eta}, \quad h_k^{out}(\eta) = (w_k^{out})^{-\frac{1}{2}} e^{i w_k^{out}\eta}.
\]

Now suppose that we start with \( h_k^{in}(\eta) \) at early times, i.e. \( h_k(\eta) \) for \( \alpha_k(\eta_{in}) = 1 \) and \( \beta_k(\eta_{in}) = 0 \) in the analogue of \( (4.26) \), and extend this by solving \( (4.40) \) to late times. There we expand it as the Bogolyubov transformation

\[
h_k^{in} = \alpha_k h_k^{out} + \beta_k h_k^{out*}
\]

valid at late times and for some complex constants \( \alpha_k, \beta_k \). Comparing with the analogue of \( (4.26) \) at late times, these constants up to phases are just the evolved values \( \alpha_k(\eta_{out}), \beta_k(\eta_{out}) \) in the general scheme. (The phases come from \( e^{i f_{out} \int \omega_k(\eta) d\eta} \) and are not relevant in what follows.)

Finally, we fix a vacuum \( \ket{0} \) as characterised by \( A_k \ket{0} = 0 \) and consider the number operator \( N_k(\eta) = a_k^\dagger(\eta)a_k(\eta) \) it evolves in time, where we use the analogue of \( (4.29) \) as our solution evolves. Starting now with \( \alpha_k(\eta_{in}) = 1, \beta_k(\eta_{in}) = 0 \) in defining \( a_k, a_k^\dagger \), we have of course

\[
\bra{0} N_k(\eta_{in}) \ket{0} = 0
\]

at early times, but in this same state at late times we have the possibility of particle creation according to

\[
\bra{N_k} := \bra{0} N_k(\eta_{out}) \ket{0} = |\beta_k(\eta_{out})|^2 = |\beta_k|^2.
\]

This completes the general scheme, which is also well-known from several other points of view. To proceed further we need to fix on a particular \( C(\eta) \), and the
standard choice for purposes of calculation is to interpolate the initial and final values as

$$C(\eta) = \frac{a_{in} + a_{out}}{2} + \frac{a_{out} - a_{in}}{2} \tanh(\mu \eta),$$

where \(\mu\) is a positive constant parameter. Equation (4.40) can then be solved with hypergeometric functions that have the correct asymptotic limit for late and early times. Using their linear properties and comparison with (4.44) gives (see [7]),

$$\alpha_k = \left(\frac{w_k^{out}}{w_k^{in}}\right)^{1/2} \frac{\Gamma(1 - i w_k^{in} / \mu)}{\Gamma(-i w_k^{in} / \mu)} \frac{\Gamma(1 - w_k^{in} / \mu)}{\Gamma(1 + w_k^{in} / \mu)}, \quad \beta_k = \left(\frac{w_k^{out}}{w_k^{in}}\right)^{1/2} \frac{\Gamma(1 - i w_k^{in} / \mu)}{\Gamma(-i w_k^{in} / \mu)} \frac{\Gamma(1 - w_k^{in} / \mu)}{\Gamma(1 + w_k^{in} / \mu)},$$

where

$$w_k^\pm = \frac{1}{2} (w_k^{out} \pm w_k^{in}).$$

These values result in

$$|\alpha_k|^2 = \frac{\sinh^2\left(\frac{\pi w_k^{in}}{\mu}\right)}{\sinh\left(\frac{\pi w_k^{in}}{\mu}\right) \sinh\left(\frac{\pi w_k^{out}}{\mu}\right)}, \quad |\beta_k|^2 = \frac{\sinh^2\left(\frac{\pi w_k^{in}}{\mu}\right)}{\sinh\left(\frac{\pi w_k^{in}}{\mu}\right) \sinh\left(\frac{\pi w_k^{out}}{\mu}\right)},$$

which one can check obeys the unitarity condition (4.27). Figure 4 includes a plot of \(|\beta_k|^2\) as a function of \(k\) or rather of the associated integer momentum \(l\).

4.4.2. Adaptation to \(\mathbb{R} \times \mathbb{Z}_n\). We now repeat the previous analysis for the polygon case with \(n\) sides and time-varying metric (4.7). Also we have the Laplacian

$$\Delta = -\partial^2_t - \frac{\dot{a}}{a} \partial_a + \frac{2}{a} (\partial_a + \partial_\perp)$$

from Theorem 4.1 with \(\beta = 0\). The Klein-Gordon equation \((-\Delta + m^2)\phi = 0\) is

$$\left(-\frac{2}{a} (\partial_a + \partial_\perp) + \frac{1}{a} \partial_a (a \partial_a) + m^2\right) \phi = 0.$$

Now, we expand the field in terms of a Fourier series

$$\phi(t, i) = \sum_k \left(A_k f_k(t, i) + A_k^* f^*_k(t, i)\right)$$

in place of (4.20), where now

$$f_k(t, i) = \frac{1}{\sqrt{a(t)}} q^{ik} h_k(t)$$

and \(k\) is an integer mod \(n\). For the modes \(f_k\) to obey (4.51), the \(h_k\) have to solve

$$\ddot{h}_k(t) + \left(m^2 + \frac{8}{a(t)} \sin^2\left(\frac{\pi k}{n}\right)\right) h_k(t) + \left(\frac{1}{4} \left(\frac{\dot{a}}{a} - \frac{1}{2} \frac{\dot{a}}{a}\right)^2 - \frac{1}{2} a\right) h_k(t) = 0.$$

The corresponding on shell frequency is therefore

$$w_k(t) = \sqrt{m^2 + \frac{8}{a(t)} \sin^2\left(\frac{\pi k}{n}\right)}$$

instead of (4.25). We again consider an exact solution of the form

$$h_k(t) = (w_k(t))^{-\frac{1}{2}} \left(\alpha_k(t) e^{i \int t' w_k(t') dt'} + \beta_k(t) e^{-i \int t' w_k(t') dt'}\right).$$
Analogously to the previous case, we can re-write the expansion of the field as
\[ \phi(t, i) = \sum_k (a_k(t)g_k(t, i) + a_k^*(t)g_k^*(t, i)), \]
where
\[ g_k(t, i) = (w_k a)^{-\frac{1}{2}} q^{ik} e^{-i \int w_k(t') dt'} \]
and the operator \( a_k(t) \) has the same form as (4.29). The quantisation procedure and analysis then proceeds as before. Our previous expressions for \( W_k(t), \alpha_k(t), \alpha_k \) are still valid, but we have to take into account that the zero adiabatic order term \( w_k \) is different and that now
\[ \sigma = \frac{1}{4} \left( \frac{\dot{a}}{a} \right)^2 - \frac{1}{2} \frac{\ddot{a}}{a} \]
as the factor in (4.54).

For our first result, we look at when the \( w_k^{(2)} \) correction vanishes so that there is no particle creation. In place of (4.37), we now require
\[ -\frac{1}{4} m^4 + \frac{4d}{a} \sin^2 \left( \frac{\pi k}{n} \right) \left( \frac{\ddot{a}}{a} \right)^2 + \frac{1}{2} \frac{\ddot{a}}{a} \left( 2 \frac{\sin^2 \left( \frac{\pi k}{n} \right) + m^2}{a} \right) = 0. \]
This can happen for all time and all \( k \) in the infinite mass limit \( m \to \infty \) if
\[ \frac{\ddot{a}}{a} = \frac{1}{2} \left( \frac{\dot{a}}{a} \right)^2 \]
with solution \( a \propto t^2 \). However, we also have a new possibility when \( m \to 0 \), if
\[ \frac{\ddot{a}}{a} = \frac{1}{4} \left( \frac{\dot{a}}{a} \right)^2 \]
with solution \( a \propto t^\frac{3}{2} \). Thus we have two, not one, additional possibility for no particle creation beyond the constant Minkowski metric case.

For our second result, we want analyses the particle creation in an analog way to the case when the space is a circle, then we make the same change of variable (4.38) in the metric (4.7) to write
\[ g = C(\eta)(-d\eta \otimes d\eta - e^+ \otimes_s e^-), \]
where \( C(\eta) = a(\eta) \), and the corresponding connection is
\[ \nabla d\eta = \frac{\dot{a}}{2a} (-d\eta \otimes d\eta + e^+ \otimes_s e^-), \quad \nabla e^+ = -\frac{\dot{a}}{2a} e^+ \otimes_s d\eta. \]

Using the quantum geometric Laplacian for this connection, we require
\[ \frac{d^2 h_k(\eta)}{d\eta^2} + \left( C(\eta)m^2 + 8 \sin^2 \left( \frac{\pi k}{n} \right) \right) h_k(\eta) = 0 \]
analogously to (4.40), but now in place (4.41) we have
\[ w_k(\eta) = \sqrt{C(\eta)m^2 + 8 \sin^2 \left( \frac{\pi k}{n} \right)}. \]
The rest of the procedure follows in the same way with the same considerations, and in particular (4.49) is still valid but with (4.62) instead of (4.41). Figure 4 shows the expected value of the number operator \( \{N_k\} \) as a function of \( k \) as well as
Figure 4. Number operator for $\mathbb{Z}_{100}$ against $k$ compared to $S^1$ with length scale $L = 100/\sqrt{2}$ plotted against integer momentum $l$ where $k = 2\pi l/L$. In both cases, $m = 0.1$ and $\mu = 100$ for the interpolation parameter.

comparing to the circle case. The big difference of course is that the $\mathbb{Z}_n$ has to be periodic in $k$ since the physical momentum is only defined mod $n$.

5. Concluding Remarks

In Section 3.1 we completely solved the quantum Riemannian geometry on a polygon in $\mathbb{Z}_n$ in the sense of arbitrary lengths $a(i)$ on the edges. As is typical for discrete calculi the increasing and decreasing derivatives are closely related but nevertheless linearly independent so that $\Omega^1$ is 2-dimensional – in effect the polygon is ‘thickened’ to something more like a tube and admits curvature. Clearly one could look beyond to discrete tori $\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_m}$ and as well as to electromagnetism both in flat and curved metrics on the $\mathbb{Z}_n$ factors. Also interesting could be quantum geodesics even on one copy $\mathbb{Z}_n$, using the new formalism of [4].

We then, in Section 3.2, computed Euclideanised quantum gravity expectation values for small $n$. In the spirit of $\mathbb{Z}_2 \times \mathbb{Z}_2$ in [19], we did this in two versions: the full quantisation and one for only fluctuations relative in an average field value. The polygon case is in fact very different and this time the full quantisation in terms of the ratios $\rho_i = a(i+1)/a(i)$ that enter into the action appears to be finite. We found that $\Delta \rho_i \sim 1.1 \langle \rho_i \rangle$ from the numerical work which is, however, a similar phenomenon to results in [19]. In the relative theory we found it useful to work with $b_i = a(i)/A$ where $A$ is the geometric mean of the $a$ field values rather than the additive one as in [19]. The correlation functions $\langle b_i b_j \rangle$ are now more interesting and were plotted up to $n = 6$. Again, it would be useful to continue this programme of baby quantum gravity models to get a feel for what the general theory looks like and what quantities it would be interesting to compute. As discussed in [19], our approach is not immediately comparable with other computable approaches such as [1, 2, 11, 14].

We also looked in Section 4.1 at the quantum geometry on $\mathbb{R} \times \mathbb{Z}_n$ with the main result that consistency with the quantum geometry forces the metric to be block
diagonal with the metric on $Z_n$ to have an a specific form with average value which can depend on time and fluctuations $\beta(i)$ frozen in time, see Theorem 4.1. We identified some issues for quantum gravity in this case and taking this further could be another direction for further work.

Instead, we focussed for the rest of the paper on the special case of the $\mathbb{R} \times Z_n$ FLRW cosmological model, where the metric on $Z_n$ is constant (the $\beta(i) = 0$) but there could be an overall varying $a(t)$ factor. The Friedmann equations for $a(t)$ turned out to be the same as those for the standard flat 1+2 dimensional FLRW model, which is perhaps not too surprising in retrospect. For spatial curvature one could next take a non-Abelian group such as $S_3$ or a fuzzy sphere $\mathbb{C}_\lambda[S^2]$ in place of $Z_n$. This will be considered elsewhere. In the spirit of Connes’ approach to internal symmetries of particle physics by tensoring a classical spacetime by a finite-dimensional algebra such as matrices or quaternions [9], one could also consider one of these in place of $Z_n$ now from an FLRW perspective.

We then turned to quantum field theory and in particular to particle creation in the $\mathbb{R} \times Z_n$ FLRW model taking as model the set up of Parker [26, 27, 28, 29, 30], applied to $\mathbb{R} \times S^1$. The main difference compared to the circle case is that we found adiabatic no particle creation solutions for $a(t)$ at $m = 0$, not only $m = \infty$. Another difference of course is that the particle creation $\langle N_k \rangle$ from constant in to constant out metrics is periodic in the frequency $k$ rather than decaying as $k \to \infty$ as it would on $S^1$. The latter is not surprising since the momentum on $Z_n$ differs fundamentally in being periodic mod $n$. In principle one could consider particle creation between the new $m = 0$ solutions, but this would need new ideas than the ones used (we would not be able to just adapt the circle case).

Also, for the equations of state for the FLRW model on $\mathbb{R} \times Z_n$ in Section 4.2, we considered only the standard form of stress energy tensor for an incompressible fluid. Stress tensors in quantum geometry remain poorly understood with no general theory, and in particular one can check that the obvious choice

$$ T = d\phi \otimes d\phi - \frac{1}{2}((d\phi, d\phi) + m^2 \phi^2)g $$

is not conserved for a free scalar field obeying the Klein-Gordon equation for the geometric Laplacian (4.50). One could still consider further what would be natural as stress tensor for a scalar field, at least in this FLRW background.

Finally, while we have focussed on the quantum field theory, one could consider the quantum mechanics limit. In the flat warm up case of Section 4.3 and following the usual steps of factoring into a wave in the time direction and a slowly varying factor, and adding a potential $V(t, i)$, gives the Schrödinger-like equation

$$ i\partial_t \psi(t, i) = -\frac{1}{am}(\partial_+ + \partial_-) \psi(t, i) + V(t, i)\psi(t, i). $$

The free particle plane-waves are clearly $\psi_k(t, i) = e^{-iE_k t}e^{\frac{2\pi i}{n}ik}$ with energy spectrum $E_k = \frac{1}{ma} \sin^2 \left( \frac{2\pi k}{n} \right)$, for $k = 0, \ldots, n-1$ so that the trace of the free Hamiltonian is $\sum_{k=0}^{n-1} E_k = \frac{2n}{ma}$, compared to the circle case where the trace diverges. Clearly, the discrete-space quantum mechanics could be studied further with specific potentials $V(t, i)$. 
This indicates several directions for further work building on the results in the present paper. Stepping back, the machinery of quantum Riemannian geometry [3] can be applied to almost any unital algebra in a step by step fashion and hence explored in a similar way for other algebras. We refer to the conclusions of [19] for wide-ranging discussion of some directions that could be interesting here.

APPENDIX A. NON *-PRESERVING SOLUTIONS

We have rightly focussed in the text on the unitary or *-preserving quantum geometries over $\mathbb{C}$ on $\mathbb{Z}_n$. However, the underlying classification was done by computer algebra and works over any field of characteristic zero. For completeness, we list the remaining solutions which over $\mathbb{C}$ would not obey the unitarity or ‘reality’ condition (2.1). These could be useful in other contexts over $\mathbb{R}$ or applied to other fields, for example to obtain ‘digital’ quantum geometries over $\mathbb{F}_2$ in the setting of [21] (in this case there could be other solutions also, as the field has non-zero characteristic).

For $n \geq 3$ odd there are two further independent solutions:

(i) $$\begin{align*}
\sigma(e^+ \otimes e^+) &= -\rho e^+ \otimes e^+, \\
\sigma(e^- \otimes e^+) &= -e^+ \otimes e^- - 2e^- \otimes e^+, \\
\sigma(e^+ \otimes e^-) &= e^- \otimes e^+, \\
\sigma(e^- \otimes e^-) &= R^2(\rho^{-1})e^- \otimes e^-,
\end{align*}$$

giving the geometric structures

$$\begin{align*}
\nabla e^+ &= (1 + \rho)e^+ \otimes e^+, \\
\nabla e^- &= (1 - R^2_{\rho^{-1}})e^- \otimes e^- + 2(e^+ \otimes e^- + e^- \otimes e^+), \\
R\nabla e^+ &= -\partial_\rho(e^+ \otimes e^-), \\
R\nabla e^- &= -\partial_\rho(R_{\rho^{-1}}e^+ \otimes e^-) - 2(1 - R_{\rho})e^- \otimes e^-, \\
\text{Ricci} &= \frac{1}{2}\left(-\partial_\rho(R_{\rho})e^+ \otimes e^- + 2(1 - \rho)e^+ \otimes e^+ + \partial_\rho(R_{\rho}^{-1})e^+ \otimes e^\right), \\
S &= \frac{1}{2}\left(\partial_\rho(R_{\rho}^{-1}) - \frac{\partial_\rho(R_{\rho})}{R_{\rho}}\right), \\
\Delta f &= \frac{1}{a}(R_{\rho}f - R_{\rho}(f))(R_{\rho} + 1).
\end{align*}$$

For $n = 3$ we may freely add a map $\alpha$ given by $\alpha(e^-) = \lambda R_{\lambda}(a)e^+ \otimes e^+$ to $\nabla e^-$ for a free parameter $\lambda$, and $\alpha(e^+) = 0$ so no change to $\nabla e^+$. This agrees with the triangle analysis in [3] aside from a different definition of $\rho$.

(ii) $$\begin{align*}
\sigma(e^+ \otimes e^+) &= \rho e^+ \otimes e^+, \\
\sigma(e^+ \otimes e^-) &= -2e^+ \otimes e^- - e^- \otimes e^+, \\
\sigma(e^- \otimes e^+) &= e^+ \otimes e^-, \\
\sigma(e^- \otimes e^-) &= -R^2(\rho^{-1})e^\otimes e^-,
\end{align*}$$
giving the geometric structures

\[ \nabla e^+ = (1 - \rho)e^+ \otimes e^+ + 2(e^+ \otimes e^- + e^- \otimes e^+), \quad \nabla e^- = (1 + \rho^2(\rho^-))e^- \otimes e^- \]

\[ R_e^+ = -\partial_e^+ e^+ \otimes e^+ + 2(1 - R_\pm(\rho^-))e^+ \otimes e^- - e^- \otimes e^+, \]

\[ R_e^- = -\partial_e^- (R_\pm(\rho^-))e^+ \otimes e^- + 2(1 - R_\pm(\rho^-))e^+ \otimes e^- + e^- \otimes e^-, \]

\[ \text{Ricci} = \frac{1}{2} \left( -\partial_e^- (R_\pm(\rho))e^- \otimes e^+ + 2(1 - R_\pm(\rho^-))e^+ \otimes e^- + \partial_e^+ (\rho^-)e^+ \otimes e^- \right), \]

\[ S = \frac{1}{2} \left( \frac{\partial_e^+ (\rho^-)}{a} - \frac{\partial_e^- (R_\pm(\rho))}{R_\pm a} \right), \]

\[ \Delta f = \frac{1}{a} (R_\pm(f) - R_\pm (f)(R_\pm(\rho) + 1). \]

For \( n = 3 \) we may freely add a map \( \alpha \) given by \( \alpha(e^+) = \lambda R_\pm(\alpha)e^- \otimes e^- \) to \( \nabla e^+ \) for a free parameter \( \lambda \), and \( \alpha(e^-) = 0 \) so no change to \( \nabla e^- \). This again agrees with the triangle analysis in [3] aside from a different definition of \( \rho \).

For \( n \geq 4 \) even there are two further independent solutions each with a free nonzero parameter \( q \), from which we define a function

\[ Q = q^{-1} = \begin{pmatrix} q^{-1} \\ q^{-1} \\ \vdots \end{pmatrix}. \]

Then

(i) \[ \sigma(e^+ \otimes e^+) = \rho e^+ \otimes e^+, \quad \sigma(e^+ \otimes e^-) = (Q - 1)e^+ \otimes e^- + Qe^- \otimes e^+, \]

\[ \sigma(e^- \otimes e^+) = e^+ \otimes e^-, \quad \sigma(e^- \otimes e^-) = R_\pm(\rho^-)Qe^- \otimes e^-, \]

giving the geometric structures

\[ \nabla e^+ = (1 - \rho)e^+ \otimes e^+ + (1 - Q)(e^- \otimes e^+ + e^+ \otimes e^-), \quad \nabla e^- = (1 - R_\pm(\rho^-))e^- \otimes e^- \]

\[ R_e^+ = \partial_e^+ (R_\pm(\rho^+)(Q^+))e^+ \otimes e^+ + \partial_e^+ (R_\pm(Q^+ - 1)R_\pm(\rho^-) - (Q - 1))e^- \otimes e^-, \]

\[ R_e^- = \partial_e^- (R_\pm(\rho^-)R_\pm(\rho^+))e^+ \otimes e^- + \partial_e^- (R_\pm(Q^+ - 1)R_\pm(\rho^-) - (Q - 1))e^- \otimes e^-, \]

\[ \text{Ricci} = \frac{1}{2} \left( \partial_e^+ (R_\pm(Q^+)R_\pm(\rho^-))e^+ \otimes e^+ + \partial_e^+ (R_\pm(Q^+ - 1)R_\pm(\rho^-) - (Q - 1))e^- \otimes e^- \right), \]

\[ S = \frac{1}{2a} \left( \partial_e^+ (R_\pm(Q^+)R_\pm(\rho^-)) - R_\pm(\rho)\partial_e^- (R_\pm(\rho^-)Q^+)) \right), \]

\[ \Delta f = -\left( \frac{1}{R_\pm(a)} + \frac{1}{a} \right) (\partial_e f + Q \partial_e f). \]

(ii) \[ \sigma(e^+ \otimes e^+) = \rho Q e^+ \otimes e^+, \quad \sigma(e^+ \otimes e^-) = R_\pm(\rho^-)Qe^- \otimes e^-, \]

\[ \sigma(e^- \otimes e^+) = e^+ \otimes e^+, \quad \sigma(e^- \otimes e^-) = Qe^- \otimes e^- + (Q - 1)e^- \otimes e^+, \]
giving the geometric structures
\[ \nabla e^+ = (1 - \rho Q)e^+ \otimes e^+ , \quad \nabla e^- = (1 - R^2(\rho^{-1}))e^- \otimes e^- + (1 - Q)(e^+ \otimes e^- + e^- \otimes e^+) , \]
\[ R_\nabla e^+ = \partial_\rho Q e^+ \wedge e^- \otimes e^- , \]
\[ R_\nabla e^- = (-R_\gamma (Q-1) R_\gamma (\rho) + Q - 1)e^+ \wedge e^- \otimes e^+ + \partial_\rho (Q R_\gamma (\rho^{-1})) e^+ \wedge e^- \otimes e^- , \]
\[ \text{Ricci} = \frac{1}{2} \left( \partial_\rho (R_\gamma (\rho Q)) e^+ \otimes e^- - (\partial_\rho (R_\gamma (Q) \rho^{-1})) e^+ \otimes e^- + (\rho(Q-1) - R_\gamma (Q-1)) e^+ \otimes e^+ \right) , \]
\[ S = -\frac{1}{2a} \partial_\rho (R_\gamma (Q) \rho^{-1}) , \]
\[ \Delta f = -\left( \frac{1}{R_\gamma (a)} + \frac{1}{a} \right) (Q \partial f + \partial f) . \]

For \( n = 4 \) we have a further more general form for the generalised braiding
\[ \sigma(e^+ \otimes e^+) = \sigma_0 e^+ \otimes e^+ + \sigma_6 e^- \otimes e^- , \quad \sigma(e^+ \otimes e^-) = \sigma_1 e^+ \otimes e^- + \sigma_2 e^- \otimes e^+ , \]
\[ \sigma(e^- \otimes e^+) = \sigma_3 e^+ \otimes e^- + \sigma_4 e^- \otimes e^+ , \quad \sigma(e^- \otimes e^-) = \sigma_5 e^- \otimes e^- + \sigma_7 e^+ \otimes e^+ , \]
for which the conditions for zero torsion are the same as before but metric compatibility now has a more complicated form due to the two extra parameters \( \sigma_6, \sigma_7 \). The QLCs turn out to fall into 10 families of which 3 are the ones with \( \sigma_6 = \sigma_7 = 0 \) already covered above. In addition we have

(i) a 4-parameter solution with a free nonzero function \( \gamma = (\gamma_0, \gamma_1, \gamma_2, \gamma_3) \) and
\[ \sigma(e^+ \otimes e^+) = \gamma e^- \otimes e^- , \quad \sigma(e^+ \otimes e^-) = e^+ \otimes e^- , \]
\[ \sigma(e^- \otimes e^+) = e^- \otimes e^+ , \quad \sigma(e^- \otimes e^-) = R_\gamma (\gamma^{-1}) R_\gamma (\rho') e^+ \otimes e^+ , \]
\[ \nabla e^+ = e^+ \otimes e^+ + e^- \otimes e^- - e^+ \otimes e^- - \gamma e^- \otimes e^- , \]
\[ \nabla e^- = e^- \otimes e^- + e^+ \otimes e^- - e^+ \otimes e^+ - R_\gamma (\gamma^{-1}) R_\gamma (\rho') e^+ \otimes e^+ , \]
where
\[ \rho' = \frac{1}{\rho R_\gamma \rho} . \]

Moreover, this is \( \ast \)-preserving if and only if \( \gamma \) has the 2-parameter form such that \( R_\gamma^2(\gamma) = \gamma^{-1} \) as in the main text.

(ii) a 3-parameter solution with parameter \( \beta \) and functions
\[ \gamma = (p,q,p,q) , \quad \delta = \frac{pq - 1}{R_\gamma (\gamma) - 1} = (pq - 1) \left( \frac{1}{q - 1} , \frac{1}{q - 1} , \frac{1}{p - 1} , \frac{1}{p - 1} \right) , \]
\[ \sigma(e^+ \otimes e^+) = \rho (1 - \delta) e^+ \otimes e^+ + \beta (\gamma - 1) \rho' e^- \otimes e^- , \quad \sigma(e^+ \otimes e^-) = (\gamma - 1) e^+ \otimes e^- + \gamma e^- \otimes e^+ , \]
\[ \sigma(e^- \otimes e^+) = (1 - \delta) e^+ \otimes e^- - \delta e^- \otimes e^+ , \quad \sigma(e^- \otimes e^-) = -\frac{\delta}{\beta R_\gamma^2 \rho} e^+ \otimes e^+ + \frac{\gamma}{R_\gamma^2 \rho} e^- \otimes e^- , \]
where
\[ \rho' = \left( \frac{\rho_0}{\rho_2}, \rho_0 \rho_1, 1, \rho_0 \rho_3 \right) . \]
giving the QLC
\[ \nabla e^+ = (1 - \rho(1 - \delta))e^+ \otimes e^+ + (1 - \gamma)(e^- \otimes e^+ + e^+ \otimes e^-) + \beta \rho'(1 - \gamma)e^- \otimes e^-, \]
\[ \nabla e^- = (1 - \frac{\gamma}{R^2_+ \rho})e^- \otimes e^- + \delta(e^+ \otimes e^- + e^- \otimes e^+) + \frac{\delta}{\beta R^2_+ \rho'}e^+ \otimes e^+. \]

(iii) a 3-parameter solution with parameters \( \beta \) and functions
\[ \gamma = (p, 0, q, 0), \quad \delta = (1, \frac{q}{p}, 1, \frac{p}{q}), \]
\[ \sigma(e^+ \otimes e^+) = R_-(\frac{\gamma}{1 - \gamma})e^+ \otimes e^+ + \frac{\beta \rho'}{1 - R_-(\gamma)}e^- \otimes e^-, \]
\[ \sigma(e^+ \otimes e^-) = (\gamma - 1)e^+ \otimes e^- + \gamma e^- \otimes e^+, \]
\[ \sigma(e^- \otimes e^+) = R_+(\frac{\gamma}{1 - \gamma})e^+ \otimes e^- + \frac{1}{1 - R_+(\gamma)}e^- \otimes e^+, \]
\[ \sigma(e^- \otimes e^-) = \frac{R_-(\delta)}{\beta R^2_+ (\rho')} (1 - \gamma)e^+ \otimes e^+ + R^2_+(\frac{\gamma}{\rho'})e^- \otimes e^- , \]

where
\[ \rho' = (\frac{\rho_0}{\rho_2}, \rho_0 \rho_1, 1, \rho_0 \rho_3), \]
giving the QLC
\[ \nabla e^+ = (1 + R_-(\frac{\gamma}{1 - \gamma})\rho)e^+ \otimes e^+ + (1 - \gamma)(e^- \otimes e^+ + e^+ \otimes e^-) - \frac{\beta \rho'}{1 - R_-(\gamma)}e^- \otimes e^-, \]
\[ \nabla e^- = (1 - R^2_-(\frac{\gamma}{\rho})e^- \otimes e^- + (1 + R_+\frac{\gamma}{1 - \gamma})e^+ \otimes e^- + \frac{1}{1 - R_+(\gamma)}e^- \otimes e^+ - \frac{R_-(\delta)}{\beta R^2_+ (\rho')} (1 - \gamma)e^+ \otimes e^+. \]

(iv) a 3-parameter solution with parameters \( \beta \) and the functions
\[ \gamma = (0, p, 0, q), \quad \delta = (\frac{p}{q}, 1, \frac{q}{p}, 1), \]
\[ \sigma(e^+ \otimes e^+) = \rho R_-(\frac{\gamma}{1 - \gamma})e^+ \otimes e^+ + \frac{\beta \rho'}{1 - R_-(\gamma)}e^- \otimes e^-, \]
\[ \sigma(e^+ \otimes e^-) = (\gamma - 1)e^+ \otimes e^- + \gamma e^- \otimes e^+, \]
\[ \sigma(e^- \otimes e^+) = R_+(\frac{\gamma}{1 - \gamma})e^+ \otimes e^- + \frac{1}{1 - R_+(\gamma)}e^- \otimes e^+, \]
\[ \sigma(e^- \otimes e^-) = \frac{R_-(\delta)}{\beta R^2_+ (\rho')} (1 - \gamma)e^+ \otimes e^+ + R^2_+(\frac{\gamma}{\rho'})e^- \otimes e^- , \]

where
\[ \rho' = (\frac{\rho_0}{\rho_2}, \rho_0 \rho_1, 1, \rho_0 \rho_3), \]
giving the QLC
\[ \nabla e^+ = (1 + R_-(\frac{\gamma}{1 - \gamma})\rho)e^+ \otimes e^+ + (1 - \gamma)(e^- \otimes e^+ + e^+ \otimes e^-) - \frac{\beta \rho'}{1 - R_-(\gamma)}e^- \otimes e^-, \]
\[ \nabla e^- = (1 - R^2_-(\frac{\gamma}{\rho})e^- \otimes e^- + (1 + R_+\frac{\gamma}{1 - \gamma})e^+ \otimes e^- + \frac{1}{1 - R_+(\gamma)}e^- \otimes e^+ - \frac{R_-(\delta)}{\beta R^2_+ (\rho')} (1 - \gamma)e^+ \otimes e^+. \]
(v) a 2-parameter solution with parameter $\beta$ and $Q = (q,q^{-1},q,q^{-1})$ as usual,
\[
\sigma(e^+ \otimes e^+) = \rho e^+ \otimes e^+, \quad \sigma(e^+ \otimes e^-) = (Q-1)e^+ \otimes e^- + Qe^- \otimes e^+,
\]
\[
\sigma(e^- \otimes e^+), \quad \sigma(e^- \otimes e^-) = \beta \rho' e^+ \otimes e^+ + R^2_2(\rho^{-1})Qe^- \otimes e^-,
\]
where
\[
\rho' = (1, -\frac{\rho_1 \rho_2}{q}, \frac{\rho_2}{\rho_0}, -\frac{\rho_2 \rho_3}{q}),
\]
giving the QLC
\[
\nabla e^+ = (1 - \rho) e^+ \otimes e^+ + (1 - Q)(e^+ \otimes e^- + e^- \otimes e^+),
\]
\[
\nabla e^- = (1 - R^2_2(\rho^{-1})Q)e^- \otimes e^+ - \beta \rho' e^+ \otimes e^+.
\]

(vi) a 2-parameter solution with parameter $\beta$ and $Q = (q,q^{-1},q,q^{-1})$ as usual,
\[
\sigma(e^+ \otimes e^-) = e^+ \otimes e^+, \quad \sigma(e^- \otimes e^+) = Qe^+ \otimes e^- + (Q-1)e^- \otimes e^+,
\]
\[
\sigma(e^- \otimes e^-) = \beta \rho' e^+ \otimes e^+ + R^2_2(\rho^{-1})e^+ \otimes e^+,
\]
where
\[
\rho' = (1, -\frac{\rho_1 \rho_2}{q}, \frac{\rho_2}{\rho_0}, -\frac{\rho_2 \rho_3}{q}),
\]
giving the QLC
\[
\nabla e^+ = (1 - \rho Q)e^+ \otimes e^+ + (1 - Q)(e^+ \otimes e^- + e^- \otimes e^+) - \beta \rho' e^+ \otimes e^+.
\]

(vii) a 2-parameter solution with parameter $\beta$ and $Q = (q,q^{-1},q,q^{-1})$ as usual,
\[
\sigma(e^+ \otimes e^-) = -\rho' \rho Qe^+ \otimes e^+ + \beta \rho'' e^- \otimes e^-, \quad \sigma(e^+ \otimes e^-) = e^- \otimes e^+,
\]
\[
\sigma(e^- \otimes e^-) = -\rho' \rho Qe^+ \otimes e^- - (\rho' Q + 1)e^- \otimes e^- + \rho Qe^- \otimes e^- + R^2_2(\rho^{-1})e^+ \otimes e^+,
\]
where
\[
\rho' = (\rho_1 \rho_0, \rho_0^{-1} \rho_1^{-1}, \rho_1 \rho_0, \rho_0^{-1} \rho_1^{-1}), \quad \rho'' = \left(\frac{\rho_0}{\rho_2}, q, 1, q, \frac{\rho_3}{\rho_1}\right),
\]
giving the QLC
\[
\nabla e^+ = (1 + \rho' \rho Q)e^+ \otimes e^+ - \beta \rho'' e^- \otimes e^-, \quad \nabla e^- = (1 - R^2_2(\rho^{-1})e^- \otimes e^+ + (1 + \rho' Q)(e^+ \otimes e^- + e^- \otimes e^+))
\]
Note that $\mathbb{Z}_4$ here is a different group from $\mathbb{Z}_2 \times \mathbb{Z}_2$ treated in [5, 19], even though in both cases the graph is a square. This is because, although $\Omega^1$ and the metric can be made to match up and hence the metric compatibility part of the QLC condition is the same, $\Omega^2$ and hence the condition for torsion freeness are different. This work [19] also treats the $\mathbb{Z}_2$ case.

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Queen Mary, University of London, School of Mathematics, Mile End Rd, London E1 4NS, UK

E-mail address: j.n.argotaquiroz@qmul.ac.uk, s.majid@qmul.ac.uk