Technique for Separating Velocity and Density Contributions in Spectroscopic Data and Its Application to Studying Turbulence and Magnetic Fields

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Abstract

Based on the theoretical description of position–position–velocity (PPV) statistics in Lazarian & Pogosyan, we introduce a new technique called the velocity decomposition algorithm (VDA) for separating the PPV fluctuations arising from velocity and density fluctuations. Using MHD turbulence simulations, we demonstrate its promise in retrieving the velocity fluctuations from the PPV cube in various physical conditions and its prospects in accurately tracing the magnetic field. We find that for localized clouds, the velocity fluctuations are most prominent in the wing part of the spectral line, and they dominate the density fluctuations. The same velocity dominance applies to extended HI regions undergoing galactic rotation. Our numerical experiment demonstrates that velocity channels arising from the cold phase of atomic hydrogen (H I) are still affected by velocity fluctuations at small scales. We apply the VDA to HI GALFA-DR2 data corresponding to the high-velocity cloud HVC186+19-114 and high-latitude galactic diffuse HI data. Our study confirms the crucial role of velocity fluctuations in explaining why linear structures are observed within PPV cubes. We discuss the implications of VDA for both magnetic field studies and predicting polarized galactic emission that acts as the foreground for cosmic microwave background studies. Additionally, we address the controversy related to the filamentary nature of the HI channel maps and explain the importance of velocity fluctuations in the formation of structures in PPV data cubes. VDA will allow astronomers to obtain velocity fluctuations from almost every piece of spectroscopic PPV data and allow direct investigations of the turbulent velocity field in observations.

Unified Astronomy Thesaurus concepts: Interstellar magnetic fields (845); Interstellar medium (847); Interstellar dynamics (839); Interstellar atomic gas (833); Interstellar filaments (842)

1. Introduction

Spectroscopic Doppler-shifted lines carry information about astrophysical turbulence in interstellar media, in particular, neutral hydrogen and ionized and molecular species. The corresponding data for H I and molecular lines, for example, CO, are stored in position–position–velocity (PPV) data cubes that contain a wealth of information for diagnosis of the properties of turbulence in interstellar media. It is thus essential to understand what is contained in the velocity channel data, which is the v-slice of data from the PPV cube, that is, a slice with a certain velocity \( v_0 \) and channel width \( \Delta v \).

The theory that relates the statistics of spectroscopic intensity fluctuations in the PPV space to the underlying statistical fluctuations of turbulent velocities and density was developed in Lazarian & Pogosyan (2000, henceforth LP00) and extended in subsequent theoretical studies (Lazarian & Pogosyan 2004, 2006, 2008; Kandel et al. 2016, 2017a, 2017b). The aforementioned theory describes how the statistics of intensity fluctuations in PPV arise from velocity crowding. The latter we will call velocity caustics, although previously in LP00 this term is applicable only in the case of thermal broadening. Such structures can be seen in PPV cubes obtained with spectroscopic data in turbulent regions. The formation of caustics does not require shocks but is a natural result of turbulent motions.

LP00 for the first time formulated the statistical description of the PPV statistics and stimulated many further developments both in terms of theory and observational studies (see Lazarian 2009). In particular, it predicted the change in the spectral index of the spectrum of the channel maps’ emissivity fluctuations with the change in the thickness of the velocity slice and related this change to the change in the relative contributions of density and velocity fluctuations. The velocity fluctuations are actually the velocity caustics we refer to above and are the center of interest in the series of papers by Lazarian & Pogosyan. The quantitative relations derived in LP00 provided the theoretical foundations of the velocity channel analysis (VCA) technique that proved to be a useful tool for extracting the velocity statistics from turbulent interstellar media. In particular, the VCA predictions were successfully tested numerically (Chepurnov & Lazarian 2009) and were applied to numerous sets of observational HI and CO data (Stanimirović & Lazarian 2001; Padoan et al. 2006; Chepurnov & Lazarian 2010; see also a summary in Yuen et al. 2019; see Section 2 for a comprehensive review).³ The LP00 approach was generalized to address studies of turbulence anisotropies in Kandel et al. (2016). The corresponding study made use of the description of different modes of anisotropic turbulence in Lazarian & Pogosyan (2012). The aforementioned works open an opportunity in isolating the contributions of fast, slow, and Alfvén basic MHD modes in observations (Zhang et al. 2020).

It is very important that the aforementioned understanding of the mapping of the anisotropic velocity and density fluctuation into PPV cubes, together with the nature of this anisotropy that follows from the theory of MHD turbulence (Goldreich & Sridhar 1995) and turbulent reconnection (Lazarian & Vishniac 1999), resulted in a radically new technique of tracing magnetic fields using PPV cube data. This technique,³

³ The publicly available VCA software is available at https://github.com/Arstoya/TurboStat.
termed the velocity gradient technique (VGT), showed promising results in mapping magnetic fields using both H\textsc{i} (Yuen & Lazarian 2017a, 2017b; Lazarian & Yuen 2018a) and CO and other molecules (Lazarian & Yuen 2018a; Hu et al. 2019a, 2019b; Hsieh et al. 2019). The VGT does use the information about the velocity caustics in order to trace the magnetic field. The density fluctuations are sensitive to shocks (Hu et al. 2019c; Yuen & Lazarian 2017b). Therefore, it is very advantageous to separate velocity and density contributions in the PPV cube.

While providing a technique for the statistical separation of the velocity and density contribution in PPV cubes, LP00 did not directly address the actual practical separations of velocity and density contributions to the individual observed fluctuations. The separation of velocity and density fluctuation is fundamentally important for studying turbulence statistics in PPV cubes for the following reasons: (1) It enables the study of true velocity statistics from observational data. (2) It allows one to cross-check with the theoretical framework of LP00. (3) The original development of LP00 dealt with the statistics of the central channels of the spectral line. In this paper, employing the statistical approach in Lazarian & Pogosyan (2004), we deal with the statistics of the channels that are far away from the spectral line peak, that is, wing channels, and find the differences from those of the central channels. The separation of velocity fluctuations in PPV cubes will help us to identify the channels that carry the largest portion of the velocity information. (4) Based on the theory of PPV statistics, we would like to examine the influence of thermal broadening on turbulent statistics. The thermal broadening is an essential factor that affects the velocity caustics in the PPV cubes. LP00 showed that the thermal broadening effect acts to increase the effective thickness of velocity channels. The thermal broadening effect also blurs velocity caustics, decreasing the contribution of the latter to the intensity fluctuations observed within velocity channels. Earlier, a number of approaches for dealing with the problem were discussed: the effective kernel as discussed in LP00, thermal kernel modeling (Dib et al. 2006), the use of centroids (Esquivel & Lazarian 2005; Kandel et al. 2017a), and the thermal deconvolution method (Yuen & Lazarian 2020a). However, it is hard to connect the LP00 theory to the product of these approaches since fundamentally we do not have access to true velocity caustics nor to their turbulence statistics in observations.

In this paper, we examine a broad range of interstellar conditions to which the LP00 theory is applicable, from molecular species like CO that are considered isothermal to a more complex multiphase gas like H\textsc{i}, which contains cold, warm, and unstable phases. While the isothermal media are straightforward to analyze, the multiphase gas requires a more complicated numerical setting and an additional analysis. For instance, in LP00 it was noted that, due to thermal instability, cold clumps are expected to form within the eddies of turbulent warm H\textsc{i} and to carry the momentum of the warm gas. The nonthermal turbulent velocities of cold H\textsc{i} atoms are larger than the intrinsic thermal velocities of cold H\textsc{i} atoms, and, as a result, one can assume that the caustics of cold clumps in sufficiently thin channel maps are marginally affected by the thermal broadening according to LP00. Another scenario is the formation of caustics due to the presence of passive tracers in turbulent molecular hydrogen gas H$_2$, for example, CO, in which the thermal width of the passive tracers is significantly narrower than that of H$_2$ because the former has a larger molecular weight.

In this paper, we do not appeal to passive tracer arguments but address the thermal broadening problem in its most complicated form, that is, assuming a single-phase or multiphase fluid with the thermal line widths larger than the turbulent line width. We show that even in this case, our approach to the velocity–density separation provides a way for us to both study velocity turbulence and accurately trace the magnetic field through the caustics structures. In particular, to understand the nature of fluctuations in H\textsc{i} velocity channels, we need to build an appropriate model for cold neutral media (CNM) and warm neutral media (WNM) caustics and cross-check with observations. In particular, we would like to answer the following questions:

1. Is the concept of density/velocity fluctuations pixel-based, or is it only valid in a statistical sense?
2. What is the role of velocity caustics in channel maps when the CNM dominates the emission?
3. What is the relative importance of velocity and density fluctuations in a spectral line’s central and wing channels?

We expect our present study to resolve several controversial points regarding the caustics in the literature. For instance, the importance of velocity caustics in creating fluctuations of 21 cm intensity fluctuations in thin channels was questioned in Clark et al. (2019). With our new procedure, termed the velocity decomposition algorithm (VDA), we can extract velocity caustics in observations and discuss their importance relative to density fluctuations in H\textsc{i} emissions. Furthermore, discussed in later sections of the paper, the availability of velocity caustics will have a far-reaching impact that allows new, unexplored sets of data to be easily extracted from nearly every spectroscopic and interferometric data cube.

This paper is structured as follows. In Section 2 we first review how velocity caustics are defined theoretically and what we expect in the presence of multiphase media. In Section 3 we discuss a new algorithm for obtaining the velocity caustics from any PPV cube based on the statistical principle. In Section 4 we discuss our numerical setup. In Section 5 we test the method of VDA in both isothermal and multiphase simulations. In Section 6 we discuss a unique property of velocity caustics fluctuations as a function of line-of-sight velocity named the “1σ criterion” based on the LP00 framework. In Section 7 we examine an important observation example, the high-velocity cloud (HVC), where the velocity caustics fluctuations are dominant in the HVC H\textsc{i} velocity channels. In Section 8 we discuss the implication of VDA for the VCA method. In Section 9 we discuss the implications of VDA for a wide range of studies related to the physics of spectroscopic data and its underlying velocity statistics. In particular, we discuss the potential impact of this paper and compare our results to a recent series of papers (Clark et al. 2019; Peek & Clark 2019; Kalberla & Haud 2019, 2020; Kalberla et al. 2020) that challenge the concept of velocity caustics. In Section 10 we discuss the prospects for VDA for future studies. In Section 11 we conclude our paper. The important discoveries of the current paper are listed in the flowchart in Figure 1.

We also include some essential supplementary materials in the appendices. In Appendix A we discuss how the correlation of density and velocity would impact our result. In Appendix B
we discuss the differences between the thermally broadened channels and the intensity maps. In Appendix C we discuss the theoretical meaning of “velocity channel gradients” based on the formulation of PPV statistics. In Appendix D we discuss how to use VDA in obtaining the constant-density velocity centroid. The velocity centroid in constant-density form was
2. Theoretical Considerations of the PPV Statistical Theory

2.1. Correlation and Structure Function of Turbulent Media

In what follows, we consider turbulent gas with temperature $T$ and mean mass of the turbulent fluid $m$. The latter is usually written as $m = m_H$, where $m_H$ is the mean molecular weight, and $m$ is the mass of the hydrogen. The thermal broadening of such gas is given by $\beta_T = k_B T / m$. The turbulent velocities of gas $v$ in our model can be larger or smaller than $\beta_T$. The intensity fluctuations in the PPV arise from both density and velocity fluctuations, where the statistics of the latter are characterized by the line-of-sight (LOS) component of the velocity structure function $D(r)$ (hereafter denoted as the $z$-direction):

$$D_z(r = (X, z)) = \langle (v_z(r') - v_z(r + r'))^2 \rangle$$

where $v_z$ is the $z$-direction velocity and $D_z$ averages over $r'$. Similarly, the correlation function $\zeta$ of turbulent density is given by

$$\zeta(r) = \langle \rho(r) \rho(r + r') \rangle$$

where $\rho$ is the density. From the theory of MHD turbulence (see Beresnyak & Lazarian 2019), it follows that the fluctuations of velocity are generally larger as the scale increases. Notice that the correlation/structure functions are connected to the power spectrum under a Fourier transform. For example, one can show that a one-dimensional velocity spectrum $E(k) \sim k^{-\alpha}$ is equivalent to a velocity structure function as a scalar distance of $r$, $D_z(r) \sim r^\alpha$ (see Lazarian & Pogosyan 2006). The value of $\alpha$ would allow one to classify the turbulence system into two regimes. The $\alpha > 1$ case is termed the “steep spectrum” in LP00; the other is termed the “shallow spectrum.” For instance, the Kolmogorov turbulence has $\alpha = 5/3$, which is a steep spectrum. The density spectrum can have both a steep and shallow spectrum ($\alpha < 1$). Notice that the shallow density spectrum emerges from either high sonic Mach number turbulence (Kowal & Lazarian 2010) or self-gravity (see Li 2018). We note that the sonic Mach number is defined as $M_s = V / c_s$, where $V$ is the turbulent velocity at the injection scale and $c_s$ is the sound speed.

It is worth noting that both the velocity structure function and the density correlation functions are mostly anisotropic (Goldreich & Sridhar 1995; see also Beresnyak & Lazarian 2019 for the condition of anisotropy), with the local magnetic field playing the dominant role in determining the anisotropy of velocities. The latter property is essential for using PPV information in determining the magnetic field direction with the VGT.

2.2. Thin and Thick Velocity Channels

For our study, we have to define thin and thick velocity channels. Importantly, the channel that is thin or thick is given by the following criterion:

$$\text{thin} : \Delta v^2 + 2\beta_T D_z(X, z = 0) \approx D_z(X, z = 0)$$

$$\text{thick} : \Delta v^2 + 2\beta_T \gg D_z(X, z = 0),$$

where the right-hand side of the equation is $D_z(X, z = 0)$, the $z$-component velocity structure function, depending on the plane-of-sky (POS) separation $X$. In other words, in thick channels, the velocity effects are integrated out in velocity channels, while in thin channels, they are retained. It was shown in LP00 that the relative contributions of velocity and density change as the velocity channel goes from the thin to thick regime. Also, note that the definition of the thin and thick channels is position-dependent, that is, it depends on the scale of the fluctuations that we measure. Therefore, the channels can be thin for small-scale fluctuations and remain thick for large-scale fluctuations or otherwise. Notice that LP00 demonstrated that the effect of the thermal broadening is similar to the increases in the effective width of the velocity channel $\Delta v_{\text{effective}}$ following Equation (3):

$$\Delta v_{\text{effective}} \approx \Delta v^2 + \frac{2k_B T}{m}.$$ 

We emphasize that whether a channel is thin or not depends on the planar position $X$. Checking the thin criterion in multiphase turbulent environments with a varying temperature requires extra caution because both $D_z$ and $T$ are functions of spatial distance $r = (X, z)$.

2.3. Velocity Caustics from Thin Channel Maps

The concept of velocity caustics in PPV is a theoretical object used in LP00 to denote the effect of velocity crowding that is due to the turbulent velocities along the line of sight (see Figure 2 for a cartoon explaining the reasoning of velocity caustics). Because of this effect, the atoms that are at different physical positions along the same line of sight happen to overlap when viewed in the PPV space. A more detailed discussion of the mapping from real to PPV space is provided in Lazarian (2009). In the following, we shall first discuss pictorially what is meant by velocity caustics and how it is defined analytically based on LP00.

2.3.1. Physical Picture

Suppose we have five fluid elements A–E along the line of sight associated with some specific density and velocity values associated with interstellar turbulence in panel (1) of Figure 2. We further assume that this system’s thermal width is 1 km s$^{-1}$, which is considered strong here because $\Delta v \sim c_s$. In the absence of thermal broadening, the spectral line is simply the histogram of velocities weighted by the mean density $\langle \rho \rangle$. In our example, $\langle \rho \rangle = 8.6$ g cm$^{-3}$, as we show in panel (2) of Figure 2. In the presence of thermal broadening, the contribution of each fluid element to PPV will be Gaussians instead of discrete numbers.

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Notice that in the case of no thermal broadening, the thin criterion of Equation (3) collapses to $\Delta v^2 \ll D_z(X, z = 0)$, which is a comparison between the channel width and the turbulence dispersion along the line of sight.
For our example here, the caustics density is \( \rho = 1 g cm^{-3} \) at \( v = 1 km/s \), like what we show in panel (3) of Figure 2. The density profile in the PPV will then be a Gaussian with height \( 2(\rho) \sqrt{2\pi} c_s \) (see Equation (4)) and width \( \sigma = 1 km/s \). These Gaussians (the blue, red, and green curves in the lower part of panel (3) in Figure 2) at different velocities will sum up, and the resultant profile (the purple dashed curve in panel (3) of Figure 2) corresponds to the velocity fluctuations, that is, the velocity caustics, of this turbulence system. Notice that the caustics profile is independent of the individual density values of the fluid elements. Therefore, this velocity caustics map is uncorrelated to the density field. The true PPV cube has both density and velocity effects, but the construction of the PPV cube from individual fluid elements is
similar to the case of velocity caustics, as we show in panel (4) of Figure 2. The only difference between panels (3) and (4) in Figure 2 is that the heights of the Gaussians are related to the true densities of the fluid elements. For example, the total density of the fluid elements at 1 km s$^{-1}$ (i.e., A and C) is just 3 g cm$^{-3}$, while those for 2 km s$^{-1}$ and 3 km s$^{-1}$ are both 20 g cm$^{-3}$. The resultant profile generated from the Gaussians will then be the black curve in panel (4) of Figure 2. One can see from panel (5) of Figure 2 that the density-weighted profiles and the caustics profiles are very different. In particular, the differences between the two profiles correspond to the contributions of the density fluctuations to PPV. In our example here, we see that the velocity contribution is significantly larger at 1 km s$^{-1}$ relative to the total PPV cube but smaller at v = 3 km s$^{-1}$. Notice that the density contribution can be negative but not for the velocity contribution. This simple example illustrates how velocity caustics and density fluctuations are defined in the channel maps.

One of the most important consequences for the effect of velocity caustics is the creation of PPV intensity structures that are independent of the true density structures in 3D. As a result, there are nonzero PPV intensity fluctuations in the spectroscopic data, even from incompressible turbulence or in observational maps that are unrelated to velocity fluctuations (e.g., synchrotron intensity fluctuations that contain density and magnetic field fluctuations). Even for the case of incompressible turbulence, the thin channel maps still contain intensity fluctuations, due to the nonlinear mapping of velocity turbulence structures along the line of sight (see panels (4) and (5) of Figure 2). This explains why the velocity channel has to be thin in observing the velocity fluctuations in PPV data.

2.3.2. Mathematical Formulation

To formulate the statistical behavior of velocity channels in a nonisothermal environment mathematically, we need to rewrite the velocity channel formula from Lazarian & Pogosyan (2004) with explicit consideration of thermal broadening. The density in PPV space of emitters with temperature $T$ moving along the line of sight with stochastic turbulent velocity $v_{\text{turb}}(x)$ and regular coherent velocity $v_g(x)$ is (see Lazarian & Pogosyan 2004)

$$p(X, v_0, \Delta v) = \int dz \rho(X, z) \left( \frac{m}{2\pi k_B T} \right)^{1/2} \times \int_{v_0-\Delta v/2}^{v_0+\Delta v/2} dv W(v) e^{-\frac{m(v-v_{\text{turb}}(X,z))^2}{2k_B T(x,z)}}, \quad (5)$$

where the sky position is described by the 2D vector $X = (x, y)$, $z$ is the line-of-sight coordinate, and $W(v)$ is a window function given by the instrument. Equation (5) is exact, including the case when the temperature of emitters varies in space, $T = T(X)$. One can see that the three quantities, namely $\rho$, $v_{\text{turb}}$, and $T$, enter differently into Equation (5). This provides the physical basis for separating these different contributions to the velocity channel maps.

To facilitate the discussions of what is contained in velocity channels, we would use the PPV density:

$$\tilde{\rho}(X, v) = \int dz \rho(X, z) f(v(X, z)), \quad (6)$$

where $f$ is the distribution function of the turbulent velocities along the $z$-axis. Note that $\tilde{\rho}(X, v)$ is the measure directly available from observations. Then Equation (5) can also be rewritten as

$$p(X, v_0, \Delta v) = \int_{v_0-\Delta v/2}^{v_0+\Delta v/2} dv \tilde{\rho}(X, v) \times W(v) \left( \frac{m}{2\pi k_B T(X, v)} \right)^{1/2} e^{-\frac{m(v-v_0)^2}{2k_B T(X, v)}}, \quad (7)$$
LP00 quantifies the statistics of intensities observed in velocity channels through correlation functions.\(^5\) There, the two-point correlation function of velocity channel intensity is given by

\[
\zeta_c(X, v_0, \Delta v) \propto \int_0^S dz \frac{\zeta'_\rho}{(D_z(r) + 2/\beta r)^{3/2}} \times \int_{v_0 - \Delta v/2}^{v_0 + \Delta v/2} dv W(v) e^{-\frac{(v-v_0)^2}{D_z(r) + 2/\beta r}},
\]

(8)

where \(\zeta'_\rho = \zeta_\rho/\rho^2\) (see Equation (2)) is the normalized three-dimensional overdensity function, \(D_z(r)\) is the velocity structure function projected along the line of sight (Equation (1)), \(\beta r\) is the thermal contribution,\(^6\) and \(S\) is the line-of-sight depth of the turbulent cloud.

For any density spectral index, the overdensity correlation function can be modeled as (see appendix of Lazarian & Pogosyan 2006 or Kandel et al. 2017a, and a numerical test in Kowal & Lazarian 2010)

\[
\zeta'_\rho \propto 1 - \text{sign}(\gamma + 3) \left(\frac{r_0}{r}\right)^{\gamma + 3},
\]

(9)

where \(r_0\) is a characteristic length scale signifying the turbulent correlation length,\(^7\) and \(\gamma\) is a modeled parameter denoting the density dependencies on \(r\). By inserting Equation (9) back into Equation (8), we get two terms, one arising from the first term (i.e., the 1 in Equation (9)), which provides the pure velocity contribution effect:

\[
I_0(X) = \int_0^S dz \frac{1}{(D_z(r) + 2/\beta r)^{3/2}} \times \int_{v_0 - \Delta v/2}^{v_0 + \Delta v/2} dv W(v) e^{-\frac{(v-v_0)^2}{D_z(r) + 2/\beta r}},
\]

(10)

and the second part of Equation (9) contains mainly density fluctuations:\(^8\)

\[
I_2(X) = \int_0^S dz \frac{-\text{sign}(\gamma + 3) \left(\frac{r_0}{r}\right)^{\gamma + 3}}{(D_z(r) + 2/\beta r)^{3/2}} \times \int_{v_0 - \Delta v/2}^{v_0 + \Delta v/2} dv W(v) e^{-\frac{(v-v_0)^2}{D_z(r) + 2/\beta r}}.
\]

(11)

We note that \(D_z(r) \to 0\) as \(r \to r_0\). Therefore, at small scales \(r \sim r_0\), the integral is affected by the density structures weighted by the combined effect from turbulent and thermal velocities. While dealing with the multiphase H1 case, whose temperature is not constant, we shall assume the H1 environment is in local thermal equilibrium (LTE). This simplifies our theoretical treatment.

It is important that while the \(I_0\) presents the pure velocity contribution, the density fluctuations dominate \(I_2\). The relative importance of \(I_0\) and \(I_2\) depends on the length scale \(r\). It is useful to compare the Fourier transforms of \(I_0\) and \(I_2\), namely \(P_r(K)\) and \(P_\rho(K)\), respectively, as LP00 did in characterizing the relative importance of velocity and density fluctuations as a function of wavenumber \(K = \sqrt{X}\).\(^9\) In fact, as long as the density is not constant everywhere, the relative importance of velocity and density fluctuations is always finite (i.e., the quantities \(P_r\) and \(P_\rho\) do not diverge).

Moreover, it is evident to see that the effective width of the velocity channel is critical in determining the relative importance of velocity and density fluctuations. For example, in the case of extremely thin channels (\(\Delta v \to 0\) in Equation (3)), the velocity integral can be approximated as a constant integral:

\[
\int_{v_0 - \Delta v/2}^{v_0 + \Delta v/2} dv W(v) e^{-\frac{(v-v_0)^2}{D_z(r) + 2/\beta r}} \approx \text{const}.
\]

(12)

In this scenario, both Equations (10) and (11) will be left with

\[
I_0 \approx \int_0^S dz \frac{1}{(D_z(r) + 2/\beta r)^{3/2}},
\]

(13a)

\[
I_2 \approx \int_0^S dz \frac{-\text{sign}(\gamma + 3) \left(\frac{r_0}{r}\right)^{\gamma + 3}}{(D_z(r) + 2/\beta r)^{3/2}}.
\]

(13b)

Notice that \(I_0\) contains no density-related terms. We can see from here that the factor \(-\text{sign}(\gamma + 3) \left(\frac{r_0}{r}\right)^{\gamma + 3}\) is critical in determining the relative importance of velocity and density fluctuations.

The LP00 study results in the development of the VCA that employs the difference in how \(I_0\) and \(I_2\) depend on the thickness of the channel map\(^10\) for studying the velocity and density statistics of the interstellar medium. VCA turned out to be a reliable tool in probing the velocity spectral index from observational spectroscopic maps. Table 1 lists the predicted spectral dependencies of VCA according to Lazarian & Pogosyan (2000, 2004), in which the 3D velocity spectral index \(m\) is involved in the discussion of VCA. The velocity and density contributions in VCA were separated by changing the thickness of the velocity channels. The technique was successfully tested numerically in Chepurnov & Lazarian (2009) and was applied to both H1 and CO data (see Stanimirović & Lazarian 2001; Padoan et al. 2006; Yuen et al. 2019).

2.4. The Need for Exact Velocity Caustics in Real Observational Data

The issue of thermal broadening is not that severe in the case of molecular tracers like \(^{12/13}\)CO, or in molecular hydrogen H2 according to LP00 since the turbulent environment in such

\(^{9}\) We use the capital \(K\) to refer to the 2D wavenumber, and \(K = |K|\) as the wavenumber of the 2D vector. Similarly, \(k\) is the 3D wavenumber, and \(k = |k|\) is the 3D wavenumber.

\(^{10}\) The spectrum of fluctuations along with the velocity coordinate is also affected in different ways by the velocity and density contributions. The latter will bring changes to the velocity coordinate spectrum (VCS) technique (Lazarian & Pogosyan 2000, 2006, 2008; Chepurnov & Lazarian 2009; Chepurnov et al. 2015) that we do not consider in this paper.
instance is usually isothermal and supersonic.\footnote{Notice that the turbulence in molecular gas can be subsonic, but the thermal broadening \( \sim 2k_B T / m_{\text{H}_2} \) may not be important. Naturally, the thin slice regime is available for CO and heavier molecules even though the turbulent velocities in \( \text{H}_2 \) are still subsonic (see Appendix F). As a result, it is rather easy to see a velocity-dominant channel, that is, velocity caustics, in observation.} As a result, the thin channel criterion (see Equation (3)) can be easily fulfilled.\footnote{From our numerical experiment, \( \langle p_d p_v \rangle \) is not always zero for supersonic turbulence (Appendix A). \textit{LP00} discussed this situation in their Appendix A.} 

\textit{LP00} applied similar considerations to neutral hydrogen media consisting of warm (\( \sim 10^4 \) K) and cold (\( \sim 10^2 \) K) components (McKee 1990; Dib & Burkert 2005; Draine 2011). The turbulence in galactic HI is generally subsonic (at high latitudes) to transsonic (near the inner regions of galactic disks; see Dib et al. 2006) in terms of its warm component and often supersonic for the colder one. However, the two components are not independent but interconnected. First of all, they are connected by the magnetic field. Second, the two components are subjected to thermal instability, with a significant fraction of matter being in the intermediate, formally unstable state (McKee 1990; Dib & Burkert 2005; Draine 2011). In this situation, it is natural to consider that the condensation of HI into the cold phase is happening within the flow of the warm turbulent phase, which means the two phases share the same turbulent velocity. This was the model adopted in \textit{LP00} to represent galactic HI. In terms of coupling of motions of the warm and cold phases, other authors adopted similar models concerned with aspects of multiphase HI dynamics not related to the applicability of the VCA. For instance, Inoue & Inutsuka (2009) discussed the condition of CNM formation in WNM collision flow. Inoue & Inutsuka (2016) further characterize the conditions for the filamentary structures to be perpendicular to the magnetic field in multiphase media (see also Seifried et al. 2020 for the molecular cloud variant).

The concept of comoving phases has far-reaching implications for the determination of thin and thick channels in multiphase neutral hydrogen observations since the comoving phase argument is an assumption well taken by theorists (Lazarian & Pogosyan 2000; Inoue & Inutsuka 2009) but has never been characterized in observation as far as we know. If the two phases have similar velocities, their velocity structure functions \( D_x \) are the same. As a result, for a given velocity channel width \( \Delta v \), it is entirely possible that the channel itself is thin for CNM but thick for WNM, due to the dramatic differences in their temperature by Equation (3). The resultant velocity channel can be composed of velocity caustics from the colder component with the density fluctuations from the warmer component in multiphase neutral hydrogen media. This coupling of density and velocity structures between different phases was never considered adequately in \textit{LP00}. Nevertheless, it was previously impossible to disentangle the density and velocity fluctuations from different phases to our knowledge, making the determination of velocity turbulence statistics very difficult, as reported by some authors in the community.

In this study, we, however, go beyond the particular model adopted in \textit{LP00} and formulate a recipe for decoupling the density and velocity fluctuations from observational multiphase data. Our current paper addresses a general and more complex problem of the effect of thermal broadening on the channel map statistics. In other words, we consider the case where the intensity fluctuation in PPV contains significant thermal broadening and without heavy tracers, which is the worst case from the point of VCA study if turbulence is subsonic. This paper will show how our new procedure of separating velocity and density contributions works in this case.

3. Construction of the Velocity Decomposition Algorithm

To proceed with the analysis, we describe a mathematical procedure that allows us to separate the velocity and density contributions in velocity channels. In what follows, we will term this procedure the VDA. This algorithm uses the statistical properties of fluctuations induced by MHD turbulence within thin velocity channel maps. Here we mostly focus on the subsonic regime, which was the most challenging case within the VCA as it is formulated in \textit{LP00} (see Section 2.4). We also test numerically the same approach for the case of supersonic turbulence. Below we lay out the formal derivation and discuss the possible improvements in Appendix A for readers to understand the theoretical foundations of the algorithm. Moreover, we also discuss another method of velocity decomposition in Appendix E.

\textit{LP00} analytically described how the density and velocity fluctuations contribute to the fluctuation of velocity channels (see Section 6 for the formal expressions in terms of their respective correlation functions). Formally we can always write the fluctuation of velocity channel intensity as

\begin{equation}
\rho(X, v_0, \Delta v) - \langle \rho |_{X \in A} = \rho_d(X, v_0, \Delta v) + \rho_v(X, v_0, \Delta v)
\end{equation}

where \( \langle \rho |_{X \in A} \) represents the velocity channel averaged over a certain spatial area \( A \). The subtraction of the mean value in Equation (14) is required as we deal with the fluctuations arising from turbulence. Notice that \( \rho_d \) and \( \rho_v \) are functions of the POS two-dimensional vector \( X \), as well as the velocity channel position \( v_0 \) and channel width \( \Delta v \). In what follows, we shall refer to \( \rho_v \) that is, the velocity contribution to velocity channels, as the velocity caustics contribution.

There are a few properties of the density (\( \rho_d \)) and velocity (\( \rho_v \)) contributions that we will use below:

1. Orthogonality of \( \rho_d \) and \( \rho_v \) when \( M_s \ll 1 \): \textit{LP00} postulated that the density and the velocity parts are statistically uncorrelated for the case of MHD turbulence. That means that the average \( \langle \rho_d \rho_v \rangle_A \) should be zero in the case of subsonic turbulence.\footnote{From our numerical experiment, \( \langle p_d p_v \rangle \) is not always zero for supersonic turbulence (Appendix A). \textit{LP00} discussed this situation in their Appendix A, and we provide a corresponding derivation in the case of nonorthogonality in Appendix E.}
2. \( p_v = 0 \) when \( \Delta v \to \infty \): The caustics should not be observed when the channel width is large. This is a property that naturally follows from LP00 theory for velocity caustics. The increase of the number of emitters in one channel means the removal of emitters from other channels. Thus the caustics fluctuations must decrease when the channel width increases.

3. \( p_d \propto I \) when \( M_s \ll 1 \): When the sonic Mach number is low, LP00 predicts that the thermal broadening will significantly increase the effective channel width of the velocity channel. When we deal with subsonic turbulence, the emission from the entire volume arrives at every channel map (if we do not consider the galactic rotation curve). This emission is proportional to the total column density. In the case of thick channels, the integration kernel will cover the whole velocity axis. As a result, the thick velocity channel would then look like the intensity map (see Appendix B for the technical reason and the differences in the integration between a thick channel and the intensity map, and Appendix E for a method that is independent of \( M_s \)). Moreover, this construction guarantees that \( \langle p_d \rangle = \langle p_v \rangle = 0 \).

In what follows, we shall restrict our discussion to subsonic turbulence and leave the supersonic counterpart to Appendix E as the latter case is less influenced by thermal broadening. Using Property 3, we can then write the form \( p_d \) for each single channel:

\[
p_d(\Delta v = \infty) \propto I - \langle I \rangle \times A
\]

where \( I \) is the intensity map. From Property 1 we have \( \langle p(p) \rangle_A = 0 \). That means for any velocity channel \( p \) that can be expressed by Equation (14), we can multiply it by \( p_d \) and take the areal average:

\[
\langle (p - \langle p \rangle)p_d \rangle = \langle p dp_d \rangle + \langle p_p \rangle
\]

where the second term is zero according to Property 1.\(^{13}\) With Properties 1 and 3, we can then formally write \( p_v \) as

\[
p_v = p - \langle p \rangle - C(I - \langle I \rangle)
\]

for some constant \( C \) that is a function of the line-of-sight velocity \( v \). We first multiply the two sides of Equation (17) by \( I - \langle I \rangle \) and then take the areal average operator, and using Property 2 we get

\[
C = \frac{1}{\sigma_I} \left( \langle p - \langle p \rangle \rangle \left( \frac{I - \langle I \rangle}{\sigma_I} \right) \right)
\]

\(^{13}\) In fact, the areal average operator is an inner product in linear algebra, and here \( p_d \) and \( p_v \) “span” the velocity channel space (as we see in Equation (14)). That means if we can write out the “unit vector” of \( p_d \) and \( p_v \), namely \( \tilde{p}_d \) and \( \tilde{p}_v \), we can simply write

\[
p = A\tilde{p}_d + B\tilde{p}_v
\]

for some \( A, B \). It is straightforward to derive that \( \tilde{p}_v = p_{uv}/\sigma_{p_{uv}} \). The space spanned by \( (\tilde{p}_d, \tilde{p}_v) \) is a complete vector space with the inner product operator being the areal average operator (\( \langle \ldots \rangle \)).

where \( \sigma_I^2 = \langle (I - \langle I \rangle)^2 \rangle \) is the dispersion of \( I \) within the area \( A \). Then we can write \( p_v \) as

\[
p_v(X, v_0, \Delta v) = (p - \langle p \rangle) \left( \frac{I - \langle I \rangle}{\sigma_I} \right)
\]

\[
\times \frac{I - \langle I \rangle}{\sigma_I}.
\]

Equation (19) gives an expression of velocity caustics in the case of subsonic media. Notice that Equation (19) fulfills the following properties:

1. The sum of \( p_v \) across all channels is zero, which we can see from the definition that \( \sum_v p(v) = I \sum_v \langle p(v) = \langle I \rangle \) (since the averaging operator and the summation direction are commutative), and \( \langle (I - \langle I \rangle)^2 \rangle = \sigma_I^2 \). This is exactly Property 2 we listed before.

2. If the area of averaging \( A \) is not statistically large enough (meaning it does not recover the MHD statistics in this area), then \( p_v \to 0 \). This property is very important since velocity caustics can only be measured statistically (see Section 6). We shall explore numerically in Section 5 the size of \( A \) required in the simulation.

With these properties in mind, we simplify Equation (19) to

\[
p_v = p - \langle p \rangle \langle I \rangle \left( \frac{I - \langle I \rangle}{\sigma_I^2} \right)
\]

\[
p_d = p - p_v = \langle p \rangle \langle I \rangle \left( \frac{I - \langle I \rangle}{\sigma_I^2} \right).
\]

This constitutes the foundation of the velocity decomposition algorithm, that is, the VDA. The suggested procedure is entirely new, and it is not developed based on LP00 theory, but the underlying idea was based on the properties of velocity caustics described in LP00. We expect the VDA to work in the case of significant thermal broadening, that is, when the performance of the traditional procedures described in LP00 drops. In particular, for such a significant thermal broadening, both the VCA and the velocity channel gradients (VChGs) introduced in Lazarian & Yuen (2018a) are not expected to work. We shall augment both the VCA (Section 8) and VChGs (Section 9.3) with the VDA and test numerically the new techniques below.

4. Numerical Simulations

4.1. Isothermal Simulations

The isothermal simulations are set up similar to the simulations in Lazarian & Yuen (2018a). The 3D simulation cubes are obtained from a single-fluid, operator-split, staggered-grid MHD Eulerian code ZEUS-MP/3D (Hayes et al. 2006) to set up a three-dimensional, isothermal \((T = 10 \text{ K})\), saturated turbulent medium under triply periodic conditions. The turbulence is injected solenoidally by Fourier-space forced driving at \( k = 2 \) (see, e.g., Stone et al. 1998) and used by several authors (Cho & Lazarian 2003; Dib et al. 2007, 2008). The choice of forced stirring over the other popular choice of decaying turbulence is made because only the former will exhibit the full characteristics of turbulence statistics (e.g., power law, turbulence anisotropy) extended from \( k = 2 \) to a dissipation scale of 12 pixels in a simulation, and it matches with what we see in observations (e.g., Armstrong et al. 1995; Chepurnov & Lazarian 2010).
For the particular application of the VDA, we consider two extreme cases that correspond to the highly subsonic and supersonic turbulence, with both being sub-Alfvénic, $M_A < 1$. The conditions of these simulations are listed in Table 2. Due to the spatial resolution requirements on both thermal broadening and the VCA method (see Chepurnov & Lazarian 2009), the two simulations are both 12003, such that the inertial range extends to close to two orders of magnitude ($k = 2$ to $k = 100$). The two limiting cases will correspond to the strong thermal limit and the supersonic density crowding limits.

### 4.2. Multiphase Simulation

The multiphase simulations are set up with the modern MHD code ATHENA++14 (White et al. 2016; Stone et al. 2020). To solve the radiative heating and cooling, we adopt the simplified generalized heat-loss function proposed by Koyama & Inutsuka (2000).15 We neglect explicit thermal conduction as it cannot be resolved in the resolution used in this simulation. For our purposes, we only provide one simulation that has a condition similar to the realistic multiphase neutral hydrogen gases with mass/volume fractions being consistent with observations. For the initial state, we set up a 3D periodic cube with length 200 pc, and we are assuming the fluid represents the bulk neutral hydrogen in the interstellar media. The mean number density for the neutral hydrogen is set as $n_h = 3 \text{ cm}^{-3}$, and the magnetic field strength is $1 \mu G$. The turbulence is driven at every iteration step to generate a saturated turbulent medium. We take a snapshot at about 110 Myr. We obtain a classical three-phase turbulence system under this setup, namely cold, unstable, and warm phases (McKee & Ostriker 1977; Hopkins et al. 2012). For the convenience of the discussion, we define two temperature thresholds to differentiate the three phases. We shall define the gas as the cold phase when the temperature of the gas is below 200 K, while that above 5250 K is the warm phase, and the gas in between is the unstable phase. Figure 3 shows a volume-weighted phase diagram. We also computed the volume-filling fraction and mass-filling fraction of each phase; see Table 3. Those fractions are consistent with the previous study of the multiphase simulation (Heiles & Troland 2003; Kritsuk et al. 2017). In Figure 4, we show the channel map at the peak of the spectral line for each phase, which are cold ($T < 200$ K, left), unstable ($200 < T < 5250$ K, middle), and warm neutral media ($T > 5250$ K, right). We can see from Figure 4 that structurally the cold and unstable phases are more filamentary than are the warm phases. More details of the numerical method and the simulations’ physical picture are provided in K. W. Ho et al. (2021, in preparation).

### 5. Numerical Testing of the VDA

In this section, we numerically test the VDA method using the simulations from Section 4. In particular, we test the accuracy of VDA in Section 5.1, using VDA to discuss the important question of whether velocity or density fluctuations are dominating the channel fluctuations in Section 5.2, and we discuss the effects of noise in VDA in Section 5.3.

#### 5.1. Accuracy of the VDA in Numerical Simulations

The parameter that influences the thermal broadening strength is the ratio between the line-of-sight velocity and that of the sonic speed $v_{\text{LOS}}/c_s$, which we will refer to as the line-of-sight sonic Mach number $M_{\text{LOS}}$. If the gas is isothermal and isotropic, we expect $M_{\text{LOS}} = \delta v_{\text{LOS}}/c_s \sim M_s/\sqrt{3}$. In the case of anisotropic turbulence, a higher or lower ratio between $M_{\text{LOS}}$ and $M_s$ could be possible depending on the cloud’s inclination and also the beam size. The ratio between $M_{\text{LOS}}$ and $M_s$ is expected to be smaller in the case when the turbulence is magnetized, with the mean field being perpendicular to the line of sight. Readers should be careful that it is possible for a supersonic cube ($M_s > 1$) to experience strong thermal broadening, since the decisive parameter in determining the strength of the thermal broadening is $M_{\text{LOS}}$, not $M_s$.

We would first like to visualize the power of VDA using numerical simulations. We compute the realistic thermal

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14 https://github.com/PrincetonUniversity/athena-public-version/wiki

15 The volume- and mass-filling fractions depend also on the other sources of heating, such as the local star formation rate, which is not accounted for in Koyama & Inutsuka (2000).
broadening synthesis (PPV) based on Yuen et al. (2019; see Section 6), which requires three inputs: the density \( \rho \), the line-of-sight velocity \( v_{\text{LOS}} \), and the sonic Mach number \( c_s \). We compute both the density-weighted PPV cube (i.e., the velocity channels) \( p(X, v) = ppv(\rho, v, c_s) \) and the constant-density PPV cube (i.e., the velocity caustics) \( n(X, v) = ppv(\langle \rho \rangle, v, c_s) \). For the density-weighted PPV cube, we compute the \( p_r \) term based on Equation (20). Notice that in the isothermal simulations that we employ in this experiment, the sonic speed can be rescaled with other simulation variables because periodic saturated, magnetized turbulence simulations are scale-free (see Yuen et al. 2018). That means we can freely adjust the broadening strength by varying \( c_s \) in the simulation. We shall utilize this fact in later sections, but here we prepare two simulations with different sonic Mach numbers and perform synthetic observations by using their intrinsic \( \delta v_{\text{LOS}}/c_s \).

Figure 5 illustrates how the algorithm (Equation (20)) works pictorially (see Appendix A for the discussion on density and velocity correlation). For visualization purposes, we only display the relative enhancement/discrepancy of local structures in the channel map according to Equation (20). Therefore, the color bar of Figure 5 has been adjusted to be \( \langle (p) \rangle, \langle p \rangle + 3\sigma_p \rangle \), where \( \langle p \rangle \) and \( \sigma_p \) are the mean and standard deviation of each image’s intensity. We can see visually from Figure 5 that in both the subsymmetric and supersonic cases, their respective decomposed \( p_r \) are highly similar to those of the constant-density velocity channel (i.e., velocity caustics) proposed in LP00. The result from Figure 5 is auspicious in terms of estimating the statistics of velocity caustics using observationally available parameters \( p_r \).

From here we define the normalized covariance coefficient (NCC; Yuen et al. 2019), which is defined as

\[
\text{NCC}(A, B) = \frac{\langle (A - \langle A \rangle)(B - \langle B \rangle) \rangle}{\sigma_A \sigma_B} \tag{21}
\]

where \( A \) and \( B \) are generic terms that refer to two distinct images. Note that NCC \( (A, B) \in [-1, 1] \). The use of NCC would allow us to determine whether the two 2D maps are correlated. If NCC = 1, then the two maps \( A \) and \( B \) are statistically perfectly correlated. If NCC = -1, then the two maps are statistically anticorrelated. If NCC = 0, then the two maps are statistically uncorrelated. For our purpose, we would like to see whether NCC\((n, p_r)\) would be close to 1. Since we are unable to obtain the true velocity caustics map \( n(X, v) \) from observations, if we can obtain NCC\((n, p_r) \sim 1 \) for most of the velocity values, then we have a reliable way to reconstruct \( n(X, v) \) from observations.

We are going to apply VDA with simulations (see Table 2) to see if the channels after VDA do indeed return the correct caustics map. We use a simulation “e5r3” that has \( M_e \sim 0.61 \). On the top left of Figure 6, we show the results of the NCC correlation when applied to the numerical data from “e5r3” using the simulation’s intrinsic thermal broadening strength. To quantify the effectiveness of VDA, we compute the NCC between the true caustics map \( n \) and the VDA-decomposed caustics map \( p_r \). Our results demonstrate that the decomposition works well for channels in different velocity positions \( v \). We see that these nine randomly drawn velocity channels have NCC values close to 1. This indicates that the decomposition algorithm (Equation (20)) works very well numerically for channels in different velocity positions. Moreover, we compute the power spectral slopes for both \( p_r \) (red text in Figure 6) and \( n \) for each \( v \) (blue text in Figure 6). We can see that the computed spectral slopes are very similar, with a deviation of less than 10% observed. We can conclude that the decomposed \( p_r \) captures most of the statistical features of the velocity caustics for every velocity channel in the case of subsymmetric turbulence.

The excellent performance of VDA suggests that the velocity caustics\(^{16} \) can be obtained from observations, which is important for both the statistical and structural studies in the PPV cube. In particular, the agreement of spectral slopes between \( n \) and \( p_r \) is also very important as the VCA (LP00, Chepurnov & Lazarian 2009) and VCS techniques (Lazarian & Pogosyan 2000, 2006, 2008; Chepurnov & Lazarian 2009; Chepurnov et al. 2015) can be applied much more easily in the case where we can extract the caustics. Also, with the caustics, we can perform the real velocity gradients according to the recipe of Yuen & Lazarian (2017a; see also Lazarian & Yuen 2018a) as the formulation of VGT is based on the

\(^{16}\) Notice that while the density term \( p_d \) does still contain a small contribution of velocity, the \( p_d \) term in subsymmetric turbulence is structurally similar to the intensity map. Yet we are aware that in supersonic turbulence the claim that \( p_d \) contains purely a density contribution does not hold, so we address in Appendix E how to tackle this problem.
dominance of statistics of velocity fluctuations in velocity channels.\textsuperscript{17}

\textsuperscript{17}The magnetic field tracing is also possible if density fluctuations are passively induced by velocity fluctuations. This is the case of the passive scalar approximation for the description of density statistics (see Beresnyak & Lazarian 2019). In this situation, the intensity gradient tracing is feasible (Yuen & Lazarian 2017b; Hu et al. 2019c). However, the accuracy of the passive density approximation fails as shocks are formed in supersonic turbulence.

We can also apply the same method to supersonic simulations. On the right of Figure 6 we show the structure of velocity channels for the case when we have supersonic simulations, or “huge 0,” which has an intrinsic $M_s \sim 6.36$. We can see that comparatively the VDA performs not as well in the supersonic cases as in the subsonic cases, particularly at the center channel $v = -1.14 \text{ km s}^{-1}$, where the spectral peak lies. While wing channels generally have a better NCC than the center channels, the spectral slopes of $p_c$ in the wings are
the numerical value of NCC is close to 1 for all nine cases. Notably, the decomposed power spectral slope for each as that of the constant-density PPV channel.

Figure 6. Left: illustration of the decomposition algorithm (Equation (20)) on the different density-weighted velocity channels synthesized from the numerical simulation “e5r3” \((M_\alpha \sim 0.61)\). We compute the NCC between \(p_i\) (from Equation (20)) and the real constant-density PPV channel \(n\), that is, NCC\((n, p_i)\). We see that the numerical value of NCC is close to 1 for all nine cases. Notably, the decomposed power spectral slope for each \(p_i\) based on Equation (20) (red) is almost the same as that of the constant-density PPV channel (blue). The color-bar scaling is adjusted such that the minimum scale corresponds to the figure’s mean value (white), while the maximum scale corresponds to the mean value plus three standard deviations. Right: a similar figure with the same method applied to the supersonic simulation “huge 0” \((M_\alpha \sim 0.36)\).

Generally steeper. The reason why VDA works for supersonic turbulence is rather simple: the \(p_i\) term now does not collect as much information because it is in the subsonic counterpart (see, e.g., Appendix B), due to thermal broadening. Interestingly enough, when the VDA algorithm applies to supersonic simulations, we see from the right of Figure 6 that VDA can recover the caustics reasonably well in terms of NCC, in particular in the wing channels. We have to emphasize that the drop of the NCC would not be a problem for VDA because (1) highly supersonic simulations do not suffer from the thermal problem, as indicated by the success of the VCA method in supersonic molecular clouds (see Stanimirović & Lazarian 2001; Padoan et al. 2006; see also Appendix E); and (2) in multiphase media, the supersonic regions, likely to be cold neutral media, occupy only a few percent in terms of volume fraction (see Table 3). By performing proper Gaussian decomposition along the line of sight, we can ignore those regions and focus on the underlying warm neutral media statistics.

One might wonder about the sensitivity of the VDA algorithm to the size of the area that we are taking the average with, since the algorithm we proposed in Equation (20) depends on the statistical area that is used for the VDA computation. Therefore, we perform a simple test to see what is the smallest area required for the VDA to work. To start, we first subsample the PPV cube with size \(N \times N \times N\), where \(N\) is the block size and \(N_0\) is the number of channels along the line of sight. We prepared eight samples with \(N = 20, 40, 60, 80, 100, 120, 140, 160\). Note that the original PPV cube has \(N_0 = 1200\). For each \(N\) and also \(N_0\), we extract the PPV cube \(N \times N \times N\), and compute VDA on this smaller cube. After VDA, we extract the same region and plot their structure as in Figure 7. To compare, we also show the same area when we have \(N_0 = 1200\) and compute the NCC (Equation (21)) between the sample map and the original map. We can see visually from Figure 7 that only for the case of \(N = 20\) was the structure of the \(p_i\) map behaving differently from the other maps. This is reflected in the drop of the NCC value in the top left panel of Figure 7. As for other panels, the NCC values are almost equal to 1, indicating that the VDA technique is robust even when one samples a tiny area.\(^{18}\)

Readers might also wonder how the same experiments we made above will behave for multiphase media. The multiphase media is a more complicated case since it involves both the subsonic warm phases and the supersonic cold phases. The former will perform precisely the same as the subsonic limit in this section, while the latter will be similar to this section’s supersonic counterpart. We show the visual decomposition results in the middle column of Figure 5. The NCC would therefore be a function of the relative fractions between the cold and warm parts. However, since we learned from Figure 6 that the NCC and both the subsonic and supersonic cases are relatively high (>0.5), we expect that the performance of VDA in multiphase media will perform reasonably well both in the center channel and in the wing channels. More precisely, we expect that the center channel, which is more CNM-crowded, has a lower NCC, while the wing channel has a higher NCC. Indeed, the NCC across synthetic channels from multiphase media is relatively high, as we can see from Figure 8, suggesting that our method also works for multiphase media.

\(^{18}\) Another test that might influence the performance of VDA is to change the telescope beam size of the synthetic map. We report that the VDA method under the change of beam size has no change in terms of its performance even if we degrade the resolution by 60 times.
5.2. Relative Contributions of Density and Velocity Fluctuations

5.2.1. Isothermal Simulation Test

With the decomposition algorithm, it is much easier to compare the relative contributions of density and velocities in each velocity channel in observations. The LP00 study contains a description of how the spectra of density and velocity affect the spectrum of the velocity channels and specifically discusses the case of a shallow density spectrum (see Section 6 and Table 1). It was noted that in the latter case, it is unavoidable to have both density and velocity contributions at small-scale intensities in thin velocity channels, while velocity fluctuations dominate over that of density in the case of a steep density spectrum (see Table 1). Therefore, we develop a systematic method of quantifying the relative contributions of density and velocity fluctuations in observation using Equation (20).

To start, we prepare two sets of simulations that are supersonic and subsonic based on Table 2. With the simulations, we produce the thermally broadened velocity channels and proceed with the decomposition method as we suggested in Section 3. We then compute the power spectra for density-only $p_d$ and velocity-only channel $p_v$ for different channel widths $\Delta v$ and different $v$ and plot their ratio as a function of wavenumber $K \sim 1/R$ (see Figure 9). Here we normalize the velocity channel width with the injection velocity $v_{inj}$, the rms velocity of injected turbulent motions. Note that the channel width that we measure here should be checked with

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$^{19}$ It is well known (see, e.g., Kowal et al. 2007) that the density spectrum becomes shallow in systems with high Mach number.
Figure 9. Set of figures showing how the spectral ratio \( P_d / P_v \) behaves as a function of 2D wavenumber \( K \) in our two simulations. We first plot the channel dispersion \( \sigma \)-velocity position \( v \) diagram (center column) for both density-only channel \( P_d \) and velocity-only channel \( P_v \), and we notice that the maximum dispersions of \( p_v \) for both cases are not at the center of the spectral line. We then compute the \( P_d / P_v \) for the center (left column) and wing channel (right column) with maximum \( \sigma \) in the \( \sigma - v \) diagram for both subsonic (top row) and supersonic cases (lower row) as a function of the normalized channel width \( \Delta v / v_{\text{maj}} \). The horizontal dashed line indicates that the relative contributions of density and velocity in the scale \( K \) are equal to each other.

Equation (3). We can see from Figure 9 that the features predicted by LP00 are observed in the case of thermal broadening.

Before proceeding with the analysis of the power spectral ratio \( P_d(K) / P_v(K) \) in Figure 9, we would first like to see what the simple dispersions of \( p_d \) and \( p_v \) look like as a function of \( v \) for both cases. In the center column of Figure 9, we plot the channel dispersion \( \sigma \)-velocity position \( v \) diagram for both the density-only channel \( P_d \) and velocity-only channel \( P_v \). If \( p_d \) and \( p_v \) truly represent the fluctuations of density and velocity in velocity channels, the \( \sigma - v \) diagram will tell us whether a particular channel is velocity or density dominant. While it is expected that the dispersion of \( p_d \) would be largest near the peak of the spectral line, it is surprising for us to see that there is a two-peak distribution for the dispersion of \( p_v \) as a function of \( v \). The maximum dispersion of \( p_v \) for both the subsonic and supersonic cases is not at the center of the spectral line. Moreover, the local minimum of the dispersion of \( p_v \) is located at the center of the spectral line. More importantly, if we consider the ratio of dispersions between \( p_d \) and \( p_v \), this value will be smaller than one in the case of supersonic media. For the subsonic case, the ratio of dispersion will only be smaller than one where we are inspecting the wing channels.

Similar conclusions can be drawn when one computes the power spectral ratio \( P_d(K) / P_v(K) \). In the case of high sonic Mach number (lower row of Figure 9), we can see that in the case of thin velocity channels both at the center (lower left of Figure 9) and the wings (lower right of Figure 9) the relative contribution \( P_d(K) / P_v(K) \) is generally smaller than one when the normalized channel width is smaller than 0.4 at small scales \( (k > 30) \). In the other case \( (\Delta v / v_{\text{maj}} > 0.4) \), we can see that the small-scale relative contribution \( P_d(K) / P_v(K) \) is larger than one. This experiment indicates that in the case of supersonic turbulence, we can achieve velocity dominance by (1) reducing the width of the velocity channels and (2) filtering large-scale contributions away to retain small scales that have velocity contributions dominant over that of densities. These effects correspond to the predictions in LP00, where the dominance of velocity fluctuations was predicted for thin velocity channels for asymptotically large \( k \). This effect of decreasing the fluctuations of channel intensity arising from velocity fluctuations with the decrease in \( \Delta v \) corresponds to the prediction from LP00, where it was shown that thermal velocity acts to increase the effective width of the channel maps. This effect naturally decreases the effects of velocity caustics induced by turbulence.

A more nontrivial case is the subsonic limit, for which we are showing the behavior of \( P_d(K) / P_v(K) \) as a function of \( K \) and \( \Delta v / v_{\text{maj}} \) in the top row of Figure 9. We can see that the relative ratio of the power spectra has a significantly different behavior than the supersonic case. The most important thing to notice is that, when we are looking at the center channel (top left of Figure 9), the ratio \( P_d(K) / P_v(K) \) stays larger than 1 for almost all length scales for all possible choices of channel width that we selected. This agrees with the top central panel of Figure 9 in that the velocity caustics fluctuations attain a minimum at the center channel. However, as we move from the center to the wings, we can see that velocity fluctuations start to dominate over density fluctuations. The top right of Figure 9 shows the spectral ratio \( P_d(K) / P_v(K) \) at one of the local maxima in the \( \sigma - v \) diagram. We can see that the ratio is smaller than 1, indicating that the velocity caustics, which are nicely traced by \( p_v \), as indicated by Figure 6 by the NCC value, actually dominate over the density counterpart in the wing.
channel. As a side note, we do not observe any changes in the spectral ratio \( P_d(K)/P_v(K) \) as a function of the channel width, as opposed to LP00’s point of view. We also report that the spectral ratio \( P_d(K)/P_v(K) \) will decrease further if we move away from the spectral peaks.

For techniques like VCA, VCS, VGT, or RHT, it is not necessary to discuss the relative contributions of density and velocity in \( p_\nu \) in every channel in detail even though we now can trace back the caustics in every channel (see Equation (20)). However, some modifications have to be done to these techniques for them to be applicable in subsonic media (see Sections 8, 9.3) as the statistics of velocity caustics is far more complicated than what LP00 studied. Also, in realistic turbulence systems, the injection scale is separated from the smallest observable scale much further than in numerical simulations. Therefore, we do not expect that the first few points with small \( K \) in both panels of Figure 9 will play an important role in discussing the dominance of either density or velocity fluctuations in velocity channels.

5.2.2. Multiphase Media Test

The same analysis framework can be extended to cases in a multiphase simulation. Here we use our multiphase simulation (see Section 4) and perform the same analysis as in the previous section. For our current purpose, this simulation is statistically supersonic and contains three phases with the correct volume and mass fractions consistent with observation. Here we define the cold phase as \( T < 200 \text{K} \) and the warm phase as \( T > 5250 \text{K} \).

The result is shown in Figure 10. We can see that the spectral ratio \( P_d/P_v \) is sensitive to the channel width, indicating the supersonic nature of the channel map in multiple areas. Moreover, we observe that the spectral ratio is significantly below 1 (notice that the y-axis of Figure 10 is in log scale), which indicates that in the case of sub-Alfvénic multiphase media, it is possible to have velocity fluctuations be dominant even though the cold neutral media in this simulation has a higher mass fraction (see Table 3) and is highly supersonic.

We can further analyze the multiphase data by separating the phases using the two density thresholds as defined above. Figure 11 shows the \( \sigma-v \) diagram and the spectral ratio \( P_d/P_v \) figures for both the center channel and wing channel for the cold, warm, and unstable phase gas.

5.2.3. Similarities of the Results of the Isothermal and Multiphase Media Cases

In Sections 5.2.1 and 5.2.2, we see some consistent behavior on both the \( \sigma-v \) diagram and the power spectral ratio \( P_d/P_v \). We can see from Figures 9 and 11 that (1) there is always a clear double peak seen for the \( \sigma-v \) diagram for \( P_v \), which is true for both isothermal and multiphase simulations.

(2) The power spectral ratio \( P_d/P_v \) is generally smaller than 1 when \( \delta v/c_s > 1 \). This is true also for CNM since \( c_{s,\text{CNM}} \) is usually very small. (3) The power spectral ratio \( P_d/P_v \) varies much more for supersonic isothermal turbulence/CNM than for subsonic isothermal turbulence/WNM. For instance, one can see in the top row of Figure 9 that the \( P_d/P_v \) simply does not change as a function of \( \Delta v \). That happens similarly for the WNM in the lower row of Figure 9. Conversely, the trend of \( P_d/P_v \) as a function of \( \Delta v \) is basically the same for the supersonic isothermal turbulence (lower row of Figure 9) and both CMW and WNM (top and middle rows of Figure 11). From here we see that the phenomena (double peak, spectral ratio) that occur in the isothermal case are also occurring in the multiphase limit.

The similarities between the statistical properties of velocity caustics in isothermal and multiphase media make the study of the statistical properties of velocity caustics in multiphase media much easier since we can reduce the multiphase simulations to a superposition of isothermal simulations with different \( M_c \). In particular, we can mimic the properties of velocity caustics by using supersonic isothermal simulations as the CNM and that of the subsonic one as the WNM.

These similarities can be utilized to analyze the dynamics of CNM and WNM in observations. Notice that cold neutral media could be formed within the collision flow of warm neutral media (Inoue & Inutsuka 2009); that is, any compression occurring in the WNM will lead to the formation of a CNM phase. It is rather natural to guess that the ambient WNM is moving the resultant CNM. If such a guess is correct, then the velocity caustics of CNM should also be embedded inside the WNM, which can be easily checked from observation using multi-Gaussian decomposition and VDA. Here we shall postpone the full study of multiphase media and its statistical analysis to elsewhere (K. W. Ho et al. 2021, in preparation).
5.3. Effects of Noise on VDA

The VDA is affected by the observational noise, and therefore we would like to see how accurate our prediction of the caustics using VDA would be when we add white noise to the channels. To start, we select two channels from the subsonic simulation “e5r3,” one of which is the center channel \( p(v = v_{\text{peak}}) \), while the other is the wing channel at \( 0.5 \Delta v_{\text{effective}} \) away from the peak position \( p(v = v_{\text{peak}} - \Delta v_{\text{effective}}) \). We add white noise with certain amplitudes to these channels and then perform the decomposition algorithm based on the noise-added map with respect to the noise-added intensity map. We then compare the decomposed \( p_v \) map from these two cases to the true velocity caustics map obtained by setting \( p = \text{const} \) in Equation (5). We use the NCC function (Equation (21)) to characterize how similar the decomposed map \( p_v \) and the true caustics map \( n \) are as a function of the noise-to-signal ratio. Figure 12 shows the results for both center and wing channels. We can see that the \( p_v \) from the wing channels is less influenced by noise than center channels compared to true caustics. In particular, when the noise-to-signal ratio is 0.5, the wing channel \( p_v \) still has an NCC value of \( \approx 0.8 \), while that of the center channel drops below 0.5. Therefore, practically it is easier to extract the wing channel caustics in observations in thermally subdominant media.

![Figure 11](image_url)

**Figure 11.** Set of figures showing how the spectral ratio \( P_d/P_v \) behaves as a function of 2D wavenumber \( K \) in the multiphase simulation for each phase similar to what we did in Figures 9 and 10. Top: cold; middle: unstable; bottom: warm neutral media. The choice of the channel widths is the same for all six cases (see the legend of the middle row).

![Figure 12](image_url)

**Figure 12.** Behavior of the NCC when we compare the realistic caustics \( n \) to the VDA-decomposed \( p_v \) from the center (blue) and wing channels (red) as a function of the noise-to-signal level \( (N/S) \) in the simulation e5r3. Here we select the wing channel position to be \( v = v_{\text{peak}} - 0.5 \Delta v_{\text{effective}} \) (see Section 6).
6. Standard Deviations of Velocity Caustics as a Function of Velocity: The 1σ Criterion

As we can see from the previous section (Section 5, Figures 9–11), the dispersion of velocity caustics as a function of $v$ exhibits a double-peak shape, but that does not happen for the density fluctuations, which was not expected in the original theory in LP00. This suggests that the maximal velocity fluctuations do not occur at the center of the spectral line, but rather at some velocity position away from the spectral peak. The locations of the double peaks are not random and can be explained by the theory of Lazarian & Pogosyan (2004; also D. Pogosyan, private communication). Here we will show where the peaks are located for GS95-type turbulence using numerical simulations, but we shall defer the rigorous theoretical treatment that applies to a general turbulence system to our forthcoming paper. We shall analyze the unique phenomenon of velocity caustics numerically, but the formal analytical studies will be presented elsewhere.

In the following, we shall separate the discussion of two of the most important cases in observations: localized emission regions that correspond to molecular clouds (Section 6.1), and regions with the line-of-sight differential velocity that corresponds to general H I emissions (Section 6.2). In particular, we shall derive a relation called the 1σ criterion that allows us to find the maximal velocity caustics fluctuations in observations in a spectral-line PPV cube.

6.1. The 1σ Criterion in Localized Emission Regions

From our previous discussion, we see that in terms of the behavior of velocity caustics, the multiphase simulations can be treated as a linear combination of supersonic and subsonic isothermal simulations. To simplify the analysis, we can analyze the behavior of the $\sigma$–$v$ curve using our isothermal simulations as a function of $v_{\text{los}}/c_s$. To test how $\sigma$–$v$ changes as a function of $v_{\text{los}}/c_s$, we change the value of $c_s$ in our simulation “c5r3” ($M_c = 0.63$; see Table 2). Figure 13 shows that the peak difference $|v_{\text{left peak}} - v_{\text{right peak}}|$ is a clear function of $v_{\text{los}}/c_s$. The double-peak feature of the $\sigma$–$v$ diagram is shifted outward as the thermal broadening strength increases (i.e., $v_{\text{los}}/c_s$ decreases).

Based on the observations from Figure 13, we can measure how the distance of the two peaks $|v_{\text{left peak}} - v_{\text{right peak}}|$ in the $\sigma$–$v$ diagram for $p_r$ varies as a function of effective channel width $\Delta v_{\text{effective}} \sim \Delta v \sqrt{1 + (c_r/\Delta v)^2}$ in Figure 14. We see that for subthermal cases there is a rather simple linear relation:

$$|v_{\text{left peak}} - v_{\text{right peak}}| \approx (1.09 \pm 0.02) \Delta v_{\text{effective}} + (-0.013 \pm 0.002)$$  \hspace{1cm} (22)

with 95% confidence. Notice that the intercept is almost negligible compared to the choice of values we used for the $\Delta v_{\text{eff}}$. Also, one can estimate the relation $|v_{\text{left peak}} - v_{\text{right peak}}|/\Delta v_{\text{effective}}$ using the theory of LP00. Theoretically, Lazarian & Pogosyan (2004; also see D. Pogosyan, private communication) suggest that $|v_{\text{left peak}} - v_{\text{right peak}}| \sim 0.95$. Since these special velocity locations (Equation (22)) are exactly the locations where we have the maximal velocity caustics fluctuations, we would then refer to Equation (22) as the 1σ criterion.

As we emphasized earlier, we do not see a similar phenomenon in the $\sigma$–$v$ diagram for $p_d$, which we will discuss in a forthcoming paper. The dramatic differences for the $\sigma$–$v$ diagrams $p_r$ and $p_d$ have very important consequences in discussing the dominance of the velocity/density contribution in velocity channels, as raised in the recent debate by Clark et al. (2019), Peek & Clark (2019), Kalberla & Haud (2019, 2020), and Kalberla et al. (2020). Notice that we do not consider only the warm phase, but the two phases with the cold phase that is moved together with the warm one in the galactic disk, at least. We consider in this paper the most challenging setting for the VCA situation (see Section 8.1 for a further discussion).

In the series of papers, it is claimed that the density contribution generally dominates velocity in every velocity channel. However, as we can see in Figure 13, the velocity fluctuations are maximal from the center channel (i.e., $v = 0$), especially when the thermal broadening strength is large. In the latter case, it is expected according to LP00 to have the...
suppression of velocity fluctuation at the center channels, but not at wing channels. Therefore, even in this extreme case, the density contributions are not dominant for every channel (see an example in Figure 24). Naturally, in the case of multiphase media, the thermal broadening effects present a more complex pattern than that found in Figure 13. Nevertheless, the double-peak pattern of the velocity contribution is a substantial effect that is important to consider when any judgment on the contributions to the PPV intensities is made.

The current study combined with the previous work on thermal deconvolution (Yuen & Lazarian 2020a) also sheds light on applying the VCA techniques to velocity channels heavily dominated by density and affected by thermal effects. We shall discuss this possibility in Section 8.

6.2. The 1σ Criterion in the Presence of Galactic Rotation Curve

For observations of galactic-disk neutral hydrogen data, the presence of galactic rotation cannot be neglected. LP00 showed how the galactic rotation affects the velocity channel/caustics map. In terms of the present study, this will impact the double-peak feature that we discussed above. In this subsection, we discuss this effect using synthetic observations.

Here we consider a linear shear along the line of sight. For given 3D numerical data of line-of-sight depth $L$, we add a linear velocity field on top of the original $z$-component velocity field $v_z$ with turbulent dispersion $\delta v_{\text{los}}$ in the simulation:

$$v_{\text{new}}(z) = v_z(z) + C \delta v_{\text{los}} \frac{z}{L}$$  \hspace{1cm} (23)

for some constant $C$. Here we shall write $v_{\text{shear}} = C \delta v_{\text{los}}$. We then synthesize the velocity channel map using $\rho(X, v) =ppv(\rho, v_{\text{new}}, c_i)$. The caustics map can be obtained using our recipe in Section 3.

Figure 15 shows how the addition of the line-of-sight velocity shear will change the $\sigma$-$v$ diagram for velocity caustics. For illustrative purposes, we keep only three curves that correspond to $C = 0, 1, 2$ in Figure 15 to illustrate the effects: (1) The whole double-peak structure is shifted to the right according to the magnitude of $C$. (2) The peak on the right-hand side of the $\sigma$-$v$ curve is increased with respect to the value of $C$, but that change is minimal for the peak’s left-hand side. (3) Most importantly, we do not notice any statistically significant changes to the value of the velocity position difference $|v_{\text{left peak}} - v_{\text{right peak}}|$ as a function of $C$.

These results are of vital importance. First of all, in disk H I emission, both turbulent and shear velocity contributions are present. While the shear contributions are discussed in LP00, the impact of the shear is not discussed in the context of the velocity caustics dispersion. We can see here that in the presence of velocity shear along the line of sight, the heights of the peaks are different. Therefore, we expect to see unbalanced peaks in the $\sigma$-$v$ diagram for velocity caustics in H I observations. More importantly, by measuring the differences in the height of the peaks, we can measure the local velocity shear strength. Having the latter value for H I clouds would be advantageous for decoupling the turbulent and shear contributions in observations.

Readers should notice that the notation of the wings and channels is ambiguous in the presence of galactic rotation. Moreover, in the presence of the galactic rotation curve, the introduction of galactic shear $v_{\text{shear}}$ is usually smaller than the intrinsic turbulent velocity value (i.e., $C < 1$). Under this condition, the double-peak feature would not be distorted very much. Very importantly, the VDA technique will still work regardless of the shape of the 1σ double-peak feature. The VDA and the unique double-peak feature will allow us to determine the statistics of the velocity field in observations (see Section 8).

7. Observational Application to HVC

With what we established in the previous sections, we can apply our technique to observations. We select a unique object, the high-velocity cloud HVC186+19–114, which is available in the Galactic Arecibo L-band Feed Array–HI (GALFA-HI) survey (Peek et al. 2018) and carries a relatively simple phase structure: a cold core plus a warm envelope (Stanimirović et al. 2006). This simple structure allows us to easily discuss CNM and WNM behavior without worrying about much of the phase exchanges in general H I emissions.
The observational data are obtained from the GALFA-DR2 (Peek et al. 2018) survey, which deals with a wide range of observational data of neutral hydrogen 21 cm emission with full ranges of R.A. and 0°–34° for decl. The pixelized resolution is 1′ (FWHM \(\sim4′\)), and the pixelized velocity channel resolution is 0.18 km s\(^{-1}\), which is more than enough for resolving CNM with temperature on the order of 1 K. In that extreme of temperature, the CNM is believed to be absorption-dominant. We would only expect the coldest neutral media observable to have their temperature at \(\sim100\) K, which corresponds to a sound speed of \(\sim0.6\) km s\(^{-1}\). The high-velocity cloud HVC186+19–114 is located at R.A. = 108°6.6–109°.8 and decl. 31°0.0–32°.4, spanning a total angular area of 1°2 \times 1°4.

7.1. Visualizing the Caustics from HVC

To study the velocity structure of this HVC, we follow Section 3 and apply the VDA to the observational data. This allows us to explore how the properties of velocity caustics arise from the HVC. Figure 16 shows the velocity caustics fluctuations we extracted following Section 3 with a velocity channel width of \(\Delta v=1\) km s\(^{-1}\), which extends from \(v=-125\) km s\(^{-1}\) to \(v=-108\) km s\(^{-1}\). We include the spectral slopes of each caustics channel \(p_v\) in red in Figure 16, and we write NCC\((p_v, I)\) in blue in Figure 16. We can see that the velocity caustics are indeed uncorrelated to the column density map. Moreover, the velocity caustics fluctuations can be spatially different from the channel intensity structure (see Stanimirović et al. 2006). The rich velocity caustics information that is decomposed by the algorithm in Section 3 provides an additional tool for observers to determine the turbulence properties in neutral hydrogen structures (see Section 10). Stanimirović et al. (2006) suggests the maximum temperature of CNM is about 350–1000 K. If taking the lower limit, that suggests that the CNM in HVC might be supersonic.

7.2. Analysis under VDA

We can perform a multi-Gaussian decomposition (Haud & Kalberla 2007) that allows one to separate the cold and warm parts of neutral media in GALFA data in the area of interest. Figure 17 shows an illustration of the two-Gaussian fitting of a single velocity spectral line. We can see that the fitting algorithm can recognize two components. This feature is generally true for the spectral lines within the area of interest. Therefore, we can group the fitting Gaussians that have narrower width to be the cold neutral media and those with wider width to be the warm neutral media in our study, that is, if the velocity channel \(p(x, y, v)\) could be fitted with two Gaussian profiles similar to Stanimirović et al. (2006). The choice of Gaussian over a more theoretically found Voigt profile is made for multiple reasons. First of all, the majority of the profile decomposition algorithms (e.g., Haud & Kalberla 2007) are based on Gaussian models. Second, our numerical testing shows that the composite Voigt profile fits the profile as well as that of a Gaussian. Last, there are no profile dependencies of the VDA method as long as the thermal broadening effect is active and the three conditions in Section 3 are fulfilled. The resultant spectral line for the whole region is shown in the top left of Figure 18. We can see that the cold neutral media contribute approximately twice the emission...
strength compared to the warm neutral media at the center of the spectral line.

Since this HVC contains a relatively simple physical structure, it would be interesting to see how the CNM and WNM velocity caustics behave statistically. We first plot the $\sigma-\nu$ diagram that corresponds to the PPV cube of HVC in the top left of Figure 18 as a function of the line-of-sight velocity. After we distinguish the CNM and WNM using Gaussian decomposition, we plot the $\sigma-\nu$ diagrams for CNM and WNM in the lower left and lower right panels of Figure 18. Here, we remove the $k = 1, 2$ contributions from the decomposed $p_d, p_v$ map as they usually do not give insight into the relative dominance of density and velocity fluctuations. Moreover, we zoom into the velocity ranges where we see the whole CNM profile. We can see that, while in the case of total PPV (top left of Figure 18) the density contribution is comparable to that of the velocity contribution in the core part of the spectral line, the latter is not negligible. In particular, in the case of the WNM (lower right of Figure 18), the velocity caustics fluctuations totally dominate over the density fluctuations. Even for the case of CNM (lower left of Figure 18), the velocity fluctuations for a large part of the spectral lines are significantly higher than that of the densities, for example, at $v \sim -110$ km s$^{-1}$. This shows that velocity caustics can be the dominant fluctuations of intensity observed in velocity channels, for example, in the wing part of the velocity spectral line or warm neutral media. Even in the case of CNM, the velocity fluctuations are not negligible. More importantly, we see the double-peak feature as predicted from Section 6. Notice that the skewness of the $\sigma-\nu$ diagram suggests that the HVC should have a nonzero line-of-sight bulk velocity component.

We can address our findings from the perspective of the theory of intensity fluctuation in PPV space from Lazarian & Pogosyan (2000, 2004). First of all, Figure 18 tells us that the turbulent motions associated with the CNM are velocity dominant, and the CNM emission dominates the intensity of emission in the velocity range $\approx 123$ to $110$ km s$^{-1}$. This means that for the corresponding velocity channels, the traditional VCA should be applicable. This also invalidates the arguments in Clark et al. (2019) that the VCA cannot apply to the multiphase H I due to strong thermal broadening. We see that this type of problem can occur only beyond the velocity range as mentioned before. However, we should mention that the spatial resolution of the HVC data is relatively poor, such that it is likely that the spectral slope estimation will have a large error.

We can further analyze the relative contributions of the density and velocity fluctuations in velocity channels using the $P_d(K)/P_v(K)$ parameter discussed in Section 3. Figure 19 shows $P_d(K)/P_v(K)$ for both CNM and WNM with variations of the channel widths. We can see very clearly from Figure 19 that the small-scale structures of both CNM and WNM have $P_d(K)/P_v(K) < 1$, meaning that velocity fluctuations are dominant at small scales. In particular, for CNM, we see an average $P_d(K)/P_v(K) \approx 1/2$ at small scale, while for WNM, this ratio can extend to the order of $10^{-2}$. This shows a clear example that, even though CNM exists extensively in velocity channels in terms of mass fraction, the caustics fluctuations associated with CNM are still stronger than its density fluctuation. The variation of the power spectral ratio $P_d(K)/P_v(K)$ for HVC’s CNM is consistent with that of our simulation result (Figure 11) that (1) the relative dominance of density and velocity is indeed a function of $\Delta v$ and (2) in small $\Delta v$, velocity fluctuations dominate over the density fluctuations. For HVC’s WNM, we observe that its power spectral ratio $P_d(K)/P_v(K)$ is smaller than 1 at almost all scales. This can be seen usually in channels that are very far away from the spectral peak (see Figures 9, 10), or the density fluctuations are simply too small to contribute to the $P_d(K)/P_v(K)$, that is, the incompressible limit. Nevertheless, for HVC we see an observational example where both CNM and WNM are velocity dominant in small scales at the center velocity channel in the thin channel limit. We now have a detailed caustics map for each channel of a physical object to which the prediction of the PPV statistical theory (LP00) can be applied readily without worrying about the density contamination in channel maps.

We can further examine the changes in spectral slopes as a function of the line-of-sight velocity for the decomposed channels, which are shown in Figure 20. We can see that the total velocity channel power spectral slope (black line of Figure 20) shares the core part of that of the CNM $p_d$ (blue of Figure 20) and the wing part of the WNM $p_v$ (red line of Figure 20). This shows that velocity caustics do indeed dominate the intensity fluctuation in thin velocity channels.

8. Synergy of VCA and VDA

With the availability of velocity caustics from VDA, it is natural to ask what are the implications for the techniques that depend on the theoretical formulation of LP00. In particular, several questions can be asked. (1) With the knowledge that the velocity caustics are dominant on wing channels, should we give less importance to the centers of the velocity channel? (2) How do we use VCA in heavily thermally broadened channels?

8.1. Necessary Changes to the VCA Method

The VCA method relies on the difference of the spectral slopes in thin and thick channels to predict the 3D velocity spectral index (see Lazarian 2009, Table 1). The success of applying VCA in observation of the SMC
and Perseus (Padoan et al. 2006) does make VCA seem like a universal method for obtaining the 3D velocity spectral index. However, there is no change of spectral index between the thin and thick velocity channels in the case of strong thermal broadening. Hence the VCA method is not applicable in subsonic media. The new developments presented in this paper allow one to extract the caustics from subsonic media, which allows one to compare the spectral index between the thin channel caustics and the intensity map to obtain the 3D velocity spectral index. Below we show two examples of how we can estimate the velocity spectral index for a case with the spectral ratio $P_d/P_v < 1$ and $P_d/P_v > 1$.

First, we have to summarize what changes we need for the VCA method to work in subsonic media. First of all, the caustics statistics should be used in substituting the velocity channel in computing the thin channel spectral slope (Table 1). The value of $m$ as required by the VCA method would be the difference in the spectral index between the thin channel caustics and the thick velocity channel. It is evident that we only have the contribution from velocity fluctuations in the case of constant density. Using the VCA method’s formula becomes straightforward without any worry about the spectral distortions arising from the density fluctuations. One can also compare the results of the 3D velocity spectral index with that of the constant-density velocity centroid (see Appendix D) as the two numbers should be the same in the ideal case. Second, since the peak fluctuations of caustics are not at the center of the spectral line, but $\sim \pm 0.5 \Delta \nu_{\text{effective}}$ away from it because the

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21 We also show numerically with the high-resolution data that the SMC has $P_d/P_v \sim 1/5$ for most $K$ using the newest observational data (K. H. Yuen et al. 2021, in preparation).
peaks are symmetric about the center of the spectral line (Section 6), all channels should be computed with the center channel put at \(|v - v_{\text{peak}}| \sim \pm 0.5 \Delta v_{\text{effective}}\), where \(v_{\text{peak}}\) is the spectral peak. In other words, we would consider the spectral index of the velocity channel, for example, take \(v = 0.5 \Delta v_{\text{effective}}\)

\[
p(X, v_0 \sim 0.5 \Delta v_{\text{effective}}; \Delta v) \\
\propto \int_{0.5 \Delta v_{\text{effective}} - \Delta v/2}^{0.5 \Delta v_{\text{effective}} + \Delta v/2} \rho(v) W(v)\exp\left(-\frac{(v - v_{\text{central}})^2}{2\sigma^2}\right) \quad (24)
\]

as a function of velocity channel width \(\Delta v\). In simple words, we are going to consider the differences of the spectral slopes between the column density map and the velocity caustics from thin wing channels at \(|v - v_{\text{peak}}| \sim \pm 0.5 \Delta v_{\text{effective}}\) for the application of VCA.

Below, we show two GALFA examples from Yuen et al. (2019): in one we knew that VCA works in its original form, while the other example was identified as a heavily thermally broadened map. We shall illustrate how the above new procedure combined with VDA works for these channels.

8.2. Applying VDA to GALFA Data

8.2.1. Subdominant Thermal Broadening

We shall first use the GALFA data to which we successfully applied VCA in Yuen et al. (2019). The region centered at R.A. = 4° and decl. = 10°35', spanning an 8° \times 8° region, is a representative region that has some VCA statistics as discussed in Yuen et al. (2019). We would first like to see how VDA works for the channels. Figure 21 shows the channels’ structures and their decomposition under VDA for two selected velocities. The two velocities are selected because they are the velocities that produce peaks of the \(\sigma_p\) and \(\sigma_x\). In particular, the velocity position \(-9.29\) km s\(^{-1}\) fulfills the 1\(\sigma\) criterion.

We can quantify the results using the \(P_d/P_v\) diagram for easier visualization. On the left of Figure 22 we see the variation of \(P_d/P_v\) as a function of \(k\) for both velocities. We see that for the case of \(v = -3.77\) km s\(^{-1}\), the \(P_d/P_v\) factor does not deviate much from unity, which signifies that the density and velocity contributions are in “equipartition” in this selected velocity channel. However, for the case of \(v = -9.29\) km s\(^{-1}\), we see that the \(P_d/P_v\) factor is mostly smaller than 1. Notice that \(v = -9.29\) km s\(^{-1}\) is the location where \(\sigma_p\) attains its maximum. Therefore, according to Section 8.1 we should use it as the center channel in computing VCA. On the right of Figure 22 we see the variations of the spectral indices for both \(p_x\) and \(p = p_d + p_x\) as a function of \(\Delta v\) by putting \(v_0 = -9.29\) km s\(^{-1}\). We estimate \(m \sim 3.4 - 2.9 \sim 0.5\) for this region, indicating that the 3D velocity spectral index is \(E(k) \sim k^{-3.5}\).

8.2.2. Regions with Dominant Thermal Broadening

It is interesting to see how we could apply VCA to regions that suffer from strong thermal broadening. Note that those regions correspond to the N/A entry in Yuen et al. (2019). For these regions, the change of the channel width does not change the spectral slope. The region centered at R.A. = 228° and decl. = 18°35', spanning an 8° \times 8° region, is a good candidate...
since we can see visually from the top of Figure 23 that the $p_v$ map does not look to be very different from that of the $p_d$ map, which is a bad sign of equipartition of velocity and density contributions in this particular velocity channel. We can see from the $P_d/P_v$ curve in the bottom left of Figure 23 that it is unity for most scales. As for the VCA method, while we see that there is no variation of the spectral slope as a function of $\Delta v$ for the total velocity channel (blue curve in the bottom right

![Figure 21](https://example.com/figure21.png)

**Figure 21.** Structures of raw and decomposed velocity channels at two selected velocities $-3.77$ km s$^{-1}$ (upper row) and $-9.29$ km s$^{-1}$ (lower row) for the GALFA data centered at R.A. = 4.00 and decl. = 10.35. The latter is one of the positions fulfilling the 1σ condition for this piece of data. Left: total intensities of velocity channel; middle: $p_v$; right: $p_d$. The color bars are set to be $[\langle p \rangle, \langle p \rangle + 3\sigma_p]$.

![Figure 22](https://example.com/figure22.png)

**Figure 22.** Left: variation of $P_d/P_v$ as a function of $k$ for both selected velocities for the GALFA data centered at R.A. = 4° and decl. = 10°35. Right: variation of $\gamma$ as a function of $\Delta v$ when we set $v_0 = -9.29$ km s$^{-1}$. The dashed line on the right hand figure indicates the spectral index for the column intensity map.
of Figure 23), there is a significant variation for that of $p_v$. Using the spectral index values of thin channel $p_v$ and thick channel $p$, we estimated that $m \sim 3.1 - 2.7 \sim 0.4$, which corresponds to a 3D velocity spectrum of $E(k) \sim k^{-3.4}$. We note that this value of $m$ cannot be found if one only has the information on the blue curve in the bottom right of Figure 23. Only in the case when we remove the density contamination can we study the channel’s velocity statistics. Formally, the spectrum is a little bit shallower than the Kolmogorov one ($2/3$), which is likely because we did not perform a thermal deconvolution in the decomposed velocity channel, or there is a possible self-absorption in these data. Not to mention we did not estimate the noise here since we want to illustrate the power of the VDA for VCA studies. However, the result of this analysis should not be underestimated since it is simply impossible to apply VCA to regions with severe thermal broadening. Indeed, the VDA amplifies the capabilities of the VCA significantly. The contribution of velocity caustics arising from the WNM is suppressed in the traditional VCA approach, due to the thermal broadening. Combining the VCA and the VDA, we can study velocity turbulence in both the CMN and in the WNM, and, therefore, we can compare the properties of turbulence in different phases of HI. This is very advantageous for understanding the dynamics of the multiphase medium. We will provide the corresponding analysis in our later publications.

9. Implications for Previous Studies

With our results in the previous sections, it is worth discussing how important velocity caustics are for some previous research. We discuss some of the important research directions and discuss how the caustics extracted by VDA would change these directions. We shall first discuss the importance of caustics to channel studies in Section 9.1. Studies related to the rolling Hough transform will be discussed in Section 9.2. We shall discuss how the VDA fundamentally changes the current recipe of VGT in Section 9.3. Last, we shall discuss the nature of HI filaments under the analysis framework of VDA in Section 9.4.

9.1. Importance of Velocity Caustics in Velocity Channels

The theory of PPV statistics (LP00) has been verified by numerical simulations and observations. One of the most

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22 Notice that according to our discussion in Section 3, the sum of $p_v$ across channels is zero. Since the $\gamma$ for a nearly constant map is zero, the thick channel limit of $p_v$ must have a flatter slope than that of the thin channel $p_v$. As a result, the trend of $\gamma$ for $p_v$ might seem different to some readers compared to what LP00 predicted: in thin channels, the spectral slope of $p_v$ is steeper, but that should eventually decrease as the channel width increases. Yet LP00 was actually comparing the spectral slope between the thin channel $p_v$ and the thick channel $p = p_d + p_v$. Therefore, we perform the same in this section.
important predictions in LP00 is the change in the velocity channel power spectral slope as a function of velocity channel width, which has been confirmed observationally in the SMC (Stanimirović & Lazarian 2001) and also numerically in MHD simulations (Lazarian & Esquivel 2003). The method of obtaining the velocity spectral index from the difference of power spectral slopes of thin and thick velocity channels is later tailored as VCA in Lazarian & Pogosyan (2004). Since then, the development of VCA has been extended to emission and absorption lines for different tracers (Lazarian & Pogosyan 2004, 2006, 2008). More importantly, the VCA method was further improved numerically in Chepurnov & Lazarian (2009), and the absorption case was tested by Burkhart et al. (2013). The VCA is primarily tested observationally for molecular tracers (Padoan et al. 2006) and multiple H I cubes that are not severely affected by thermal broadening (Yuen et al. 2019).

Aside from the spectral slope differences as a function of channel width, LP00 and subsequent works (Lazarian & Pogosyan 2004, 2006, 2008; Chepurnov & Lazarian 2009) also predict that the anisotropy of velocity channels is associated with the velocity caustics fluctuations. The velocity caustics are both filamentary (see Sections 9.2, 9.4)\(^{23}\) and independent of the density fluctuations (see Figure 2). In this paper, we clarify the concept of velocity caustics and show that it can be observed and fulfills the properties that are predicted by LP00.

Nevertheless, the importance of velocity caustics was challenged in Clark et al. (2019), where it was claimed that LP00 is not applicable to multiphase H I, and the filaments in channel maps were mostly density filaments aligned with the magnetic field. Our initial response to Clark et al. (2019) was made public in Yuen et al. (2019). With the multiphase simulations and the tools developed in this paper, we are in a position to provide a quantitative response to the critiques by Clark et al. (2019) and subsequent publications supporting the pure density explanation of 21 cm emission filaments (see Kalberla & Haud 2019). A key criticism of LP00 from Clark et al. (2019) is related to the increased density contribution that is due to thermal effects. To address criticisms such as these, a tool that allows us to disassociate and quantify the relative density and velocity fluctuations in velocity channels is required.

It is for this purpose that we present the VDA method. The results provided in this paper demonstrate that VDA can answer many of the concerns from the critics of the LP00 approach to H I. First of all, the caustics are the most important theoretical case that is studied in LP00 and subsequent works and has been known to follow Table 1 nicely. The extraction of velocity caustics would allow one to determine the 3D velocity spectral slopes based on the caustics map derived from Equation (20). Second, obtaining the exact contribution of caustic structures enables us to settle the recent debate on whether the velocity channels are dominated by densities or velocities (Clark et al. 2019; Peek & Clark 2019; Yuen et al. 2019; Kalberla & Haud 2019, 2020; Kalberla et al. 2020). From the examples in Sections 7, 8, and 9.2, we see that the statement that “H I velocity channels are exclusively dominated by CNM density fluctuations” is incorrect. Note that, even in the regions that we identified as “density dominated,” ignoring the velocity fluctuations would be inaccurate. While the total intensity fluctuations can arise from densities on large scales, the small-scale intensity fluctuations are still dominated by velocity caustics (see Figures 9–19), as opposed to the empirical claim by Clark et al. (2019). Furthermore, we can see from the upper row of Figure 11 that only when one considers a rather thick channel limit (\(\Delta v/\langle\nu_{th}\rangle > 1.3\)) will the spectral ratio \((P_d/P_v)\) in Figure 11 stay above unity, which suggests that LP00 is applicable to H I media.

9.2. Implication for Rolling Hough Transform and Tracing of Cold Neutral Media Based on Linear Structures

A series of papers (Kalberla & Haud 2019, 2020; Peek & Clark 2019; Kalberla et al. 2020) studied the correlation of the enhancement of intensities in H I velocity channels with different observables in order to infer the distribution of cold neutral media on the sky. These works are based on the argument from Clark et al. (2019) that the features extracted by the unsharp mask (USM), which is the first step of the rolling Hough transform (RHT; Clark et al. 2014), are mostly dominated by density fluctuations and reflected cold neutral media structures. In particular, they used a nonnormalized version of the NCC (see Equation (21) and Yuen et al. 2019) and compared the value of NCC between channel maps and the dust column density map to a curve erroneously attributed to LP00 to claim that the velocity channels are density dominated regardless of the channel width. Based on these sorts of arguments, Clark et al. (2019) concluded with their Section 3.3 title that “Thin-channel H I Intensity Structures Are Not Velocity Caustics,” and also in their main text “Evidently, not only are thin-channel H I intensity structures not dominated by velocity caustics, but there is no measurable contribution to the H I intensity from velocity caustics at all,”\(^{24}\) suggesting that velocity caustics are completely absent in H I observation.

The above conclusion contradicts our testing in this paper. In fact, our analysis in this paper reveals that each channel contains both density and velocity fluctuations for each phase (see Figure 11). In particular, previous sections show that (1) velocity fluctuations tend to contribute more when we are observing the wing channels, and (2) the dominance of density fluctuations in velocity channels occurs mostly in the subsonic environment when \(|v - v_{\text{peak}}| < 0.5\Delta v_{\text{effective}}\) (see Section 6). It is not hard to imagine that the USM or RHT of the \(p_v\) map, which is purely velocity features, would still exhibit a correlation to cold neutral media. Moreover, it is also possible that the structure of the cold neutral media in channel maps is inherited from the velocity fluctuations. We want to illustrate this effect using the region employed in Clark et al. (2015) to show that the velocity channels do contain velocity caustics according to our analysis method in Section 3 and further show that caustics are dominant in a number of channels.

To start with, we select the regions that have the most “RHT fibers” (the linear structures that are detected according to RHT) in the region used in Clark et al. (2015; see top panel of Figure 24). We then perform VDA in the region and compute RHT for both \(p_d\) and \(p_v\). The RHT output indicates the location of the linear structures in the \(p_d\) maps with the intensity map being rescaled to \([0, 1]\). The intensity value of the RHT output indicates the pixel’s probability of being a part of the linear feature in the neighboring region. We can see visually from the

\(^{23}\) The formation of “fibrous filaments” arising from velocity fluctuations, that is, from velocity caustics in PPV space, was reported in simulations by Clarke et al. (2018), in agreement with the LP00 predictions.

\(^{24}\) Clark et al. (2019), Section 3.3, p. 9, last line.
middle panels of Figure 24 that both \( p_d \) and \( p_v \) maps exhibit a variety of linear structures. To compare their location, we can multiply the RHT output of \( p_d \) and \( p_v \), whose intensity value is also in the range [0, 1]. The lower panel of Figure 24 shows the multiplied output. We can see that the linear features from \( p_d \) and \( p_v \) are uncorrelated to each other, which is consistent with the NCC value (NCC \( \approx 0.087 \)); i.e., the two maps are statistically not correlated; see Equation (21)) of the RHT output of these two maps. More importantly, it is clear that velocity caustics contain linear features that are identified as filaments by RHT. The appearance of these features follows from MHD turbulence theory and the theory of space-velocity mapping in LP00. Therefore, it is wrong to disregard the effects of velocity fluctuations in those fibers identified by the RHT.

It is then important to see how the spectral ratio \( (P_d/P_v) \); see Section 5) would behave as a function of the channel width. The channel width in this data is considered to be “thin” according to LP00 (see Equation (3)). Figure 25 shows that the small-scale \( P_d/P_v \) ratio is indeed smaller than one when the channel is thin. This shows that velocity fluctuations dominate over that of density at small scales for this particular channel. This result can be easily generalized to different regions. In fact, from our analysis, 34 channels out of 41 are velocity dominant. Furthermore, we would like to estimate the importance of velocity caustics in all channels by computing the weighted percentage along the line of sight. We first compute the importance of velocity fluctuations for each velocity channel by \( \sigma_{p_v}/(\sigma_{p_d} + \sigma_{p_v}) \) and then multiply this quantity to the spectral line \( N(v) \). That is to say, the weighted percentage is given by

\[
\text{Weighted Percentage} = \frac{1}{\sum_v N(v)} \sum_v N(v) \sigma_{p_v}(v) / (\sigma_{p_d}(v) + \sigma_{p_v}(v)).
\]

From the above equation, we see that more than 50% of the total pixels in all of the channels in the regions are velocity fluctuations even with the density effects fully accounted for. This shows that disregarding the effect of velocity caustics for the formation of the structures in thin channels is incorrect. The structures of velocity caustics are linear, filamentary, and expected to align with the magnetic field. We also note that the degree of alignment of the velocity filaments arising from velocity fluctuations with the polarization is higher than for those from the density filaments. This corresponds well to the theory’s expectation that the velocity fluctuations in MHD turbulence trace the magnetic field direction better than those of density (see Section 9.3). However, we believe that the VGT, especially under the modification of VDA (see Section 9.3), is a more reliable way to trace magnetic fields than the currently available methods that utilize the filamentary nature of velocity channel maps.

9.3. Implications for the Velocity Gradient Technique

The velocity gradient technique, which is based on the modern MHD theory (Goldreich & Sridhar 1995; Lazarian & Vishniac 1999) and the statistics of spectroscopic channel maps (Lazarian & Pogosyan 2000, 2004), can successfully trace magnetic fields in various astrophysical environments. However, the velocity channel gradients (VChGs) are founded on the assumption that velocity caustics dominate the fluctuations of the velocity channels in MHD turbulence (Lazarian & Yuen 2018a). In this paper, we show that this argument is not entirely correct. In fact, from our previous sections (e.g., Section 5), we see that there can be velocity channels that are density dominated. Despite that fact, Lazarian & Yuen (2018a) showed in a variety of MHD simulations with different sonic and Alfvénic Mach numbers that the gradient technique is
shown to be applicable to H I media. It is then natural to ask: Why do the channel gradients in Lazarian & Yuen (2018a) correctly trace the magnetic field directions? How would the VDA developed in this paper improve the accuracy of the velocity channel gradient in observation? Very importantly, we will have a new quantity under the framework of VGT called the velocity caustics gradient that would follow the assumptions of Lazarian & Yuen (2018a) nicely, and we expect that to be more reliable and accurate in tracing the magnetic field.

9.3.1. Improving Magnetic Field Tracing with Velocity Channel Maps

The numerical tests (Yuen & Lazarian 2017a, 2017b; Lazarian & Yuen 2018a) of the VGT cover a large parameter space with various $M_\text{s}$ and $M_\text{c}$ and are performed with various synthetic observations (see Hsieh et al. 2019). It has been shown repeatedly with different numerical setups that the gradients of different observables (e.g., column density, velocity centroids, velocity channels) are tracing the block-averaged magnetic field directions as long the Yuen & Lazarian (2017a) Gaussianity is satisfied. That means one could fit a Gaussian function to the gradient orientation histogram. In this scenario, the peak of the Gaussian fitting function returns the local magnetic field directions. This procedure is termed block-averaging in Yuen & Lazarian (2017a) and has been the core of the gradient technique even as other procedures or improvements have been introduced (e.g., Hu et al. 2018; Lu et al. 2020, Ho & Lazarian 2021). Aside from some special situations, for example, regions of gravitational collapse (see, e.g., Yuen & Lazarian 2017b; Hu et al. 2019c), velocity gradient directions after block-averaging are statistically perpendicular to the local block-averaged magnetic field directions. The block-averaging criterion has been extended by Lu et al. (2020) under theoretical considerations such that it should be a special Lorentzian-sinusoidal function that fits the gradient orientation histogram. The success of the VGT is further confirmed by the numerous application examples available in different astrophysical environments (see, e.g., Yuen & Lazarian 2017a; Lazarian & Yuen 2018a; Hu et al. 2019a, 2019b). On the contrary, while the density field can mimic the statistics of velocities for low $M_\text{s}$, it significantly deviates from the velocity statistics in high $M_\text{s}$ flows or in more complicated physical settings (Yuen & Lazarian 2017b; Lazarian & Yuen 2018a). Those include, for instance, multiphase fluids with thermal instability. However, previous work shows that even with the density contamination in velocity channels, the VChGs still trace the magnetic field reliably (Lazarian & Yuen 2018a).

Simultaneously, the thermal line width of turbulent gas presents a prominent complication in terms of the tracing of the magnetic field as the channel map is affected by the morphological changes of density structure. According to the LP00 theory, an increase in the thermal line width will also increase the effective thickness of a velocity channel (see Equation (4)). Moreover, as we see in this paper (see, e.g., Figure 9), the contribution of density fluctuations increases in proportion to the thermal line width. Under a strong broadening scenario, the velocity channel will be effectively thicker, which complicates the interpretation of VChGs in terms of velocity gradients using the Lazarian & Yuen (2018a) recipe. Earlier, this problem was circumvented by using heavier tracers, for example, CO, with reduced thermal width, or appealing to the physical model that cold H I clumps move together with the warm H I. With VDA, the problem of density interference due to thermal broadening is solved as we can compute the gradients of the decomposed caustics $p_\nu$ (see Equation (20)) from observational data.

We want to stress that, while some channels are density dominated, the VDA technique provides the pure caustics structures for each velocity channel, and thus the underlying structures will reflect the velocity statistics as predicted by LP00 and numerically tested in Lazarian & Yuen (2018a).

Our exploration of the velocity channel based on VDA shows that even in the case of subsonic turbulence, the velocity caustics contribute to a significant amount of fluctuations of intensities in velocity channels (Section 5), which is confirmed by both numerical (Section 5) and observational data (Sections 7, 8). Elsewhere, we shall discuss how to use the velocity channel data more productively by combining the VGT with the VDA approach.

9.3.2. Dependencies of Caustics Gradients on Physical Conditions and MHD Modes

In fact, when turbulence is highly subsonic, the following is expected by the MHD theory: (1) Alfvén-mode velocity fluctuations are anisotropic along with the local magnetic field directions (Goldreich & Sridhar 1995); (2) the slow-mode velocity fluctuations are moved by Alfvénic modes as a passive scalar and, therefore, mimic the statistics of Alfvén velocity fluctuations; (3) the fast-mode velocity fluctuations are isotropic (see Cho & Lazarian 2003) with the relative energy in the fast mode being small compared to the Alfvén and slow modes combined (Cho & Lazarian 2002, 2003). Notice that the slow-mode velocity statistics are slaved by the Alfvén mode. Under this condition, the velocity fluctuations in channel maps, which mostly come from Alfvén and slow modes, are expected to be anisotropic to the local magnetic field.

The statistics of density fluctuations are far more complicated than that of velocity fluctuations (Beresnyak et al. 2005; Kowal et al. 2007). In subsonic turbulence, the Alfvén mode does not contribute to density fluctuations. The majority of the density fluctuations arise from slow modes, and they follow the same Alfvénic turbulence scaling as the velocity fluctuations. This also follows from our previous numerical experiments in Lazarian & Yuen (2018a), where the orientations of the density and velocity gradients are shown to be similar, that is, perpendicular to the local magnetic field in the case of a mildly subsonic environment. In fact, it is shown numerically (K. W. Ho & A. Lazarian 2021, in preparation) that in the subsonic limit, the velocity channel gradients perform exceptionally well regardless of the relative contributions of density and velocity fluctuations in the channel maps. Moreover, both of these fluctuations are anisotropic along with the local magnetic field directions. Yet in supersonic cases, the density

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25 For large $M_\text{s}$ cases, the fitting should be a von Mise function.

26 The importance of such structures becomes more prominent for high sonic Mach numbers $M_\text{s}$. However, for $M_\text{s} \gg 1$, the importance of thermal broadening decreases.

27 It is worth noting that the gradients of a certain observable will only be able to trace the magnetic field when (1) the power spectral slope of that observable is steeper than $-1$ and (2) is anisotropic along the magnetic field (see Lazarian & Yuen 2018b and Appendix C). The MHD turbulence (Goldreich & Sridhar 1995; see also Beresnyak & Lazarian 2019) falls into the requirement, but the gradient technique will also apply to other kinds of cascade as long as the Lazarian & Yuen (2018b) criterion is satisfied.
fluctuations show a rather nontrivial behavior. It was shown in Beresnyak et al. (2005) that, while the density field below a certain threshold obeys the Alfvenic turbulence statistics, a small fraction of volume filled by high-density structures will have a shallow spectrum and very different statistical properties arising from shocks. Those high-density structures tend to align perpendicularly to the magnetic field (Yuen & Lazarian 2017b), which results in a decrease of the accuracy for the channel gradients approach (Lazarian & Yuen 2018a). Under the framework of VDA, observers can deal with the real velocity fluctuations from spectroscopic data, allowing the observers to not consider the statistical behavior of the density field in MHD turbulence.

9.3.3. Caustics Gradient Dispersion as a Probe of Media Magnetization

The VGT in its present formulation goes beyond tracing the magnetic field. For example, the dispersion of the channel gradients can be used to study magnetization of the media (Lazarian et al. 2018). Notice that the gradient dispersion statistics of density and velocity in MHD turbulence could be different in different modes even if their gradient orientations are statistically the same. This leads to an essential question on whether the magnetization estimation technique (Lazarian et al. 2018) is applicable if we have a different weight of density and velocity contributions in velocity channels, as there might be concerns on whether the fitting result from Lazarian et al. (2018) will be applicable to the caustics map decomposed from VDA.

Nevertheless, as the orientation of the velocity caustics gradients is defined purely from velocity statistics (see Section C), we do not expect that the orientation-related statistics (e.g., Lazarian et al. 2018; Yuen & Lazarian 2020b) would change dramatically. We also reported numerically that the dispersion of density-weighted channel gradients is not much different from that of the caustics gradients. Thus we believe that the fitting result from Lazarian et al. (2018) should be applicable, but a more careful study should be performed in the near future. Similarly, the improvement techniques (moving average from Lazarian & Yuen 2018a; PCA from Hu et al. 2018; spectral filtering from Lazarian et al. 2017) will all have to be tested accordingly in the case of the caustics map. It would also be interesting to see how the mode decomposition technique can be synergetically used with the gradients of velocity caustics in observations (see Cho & Lazarian 2003; Zhang et al. 2020).

9.3.4. Shock Identification with the VDA

The introduction of the VDA technique also allows us to proceed with an accurate shock-detection algorithm based on the properties of the velocity caustics gradient. It is discussed in Yuen & Lazarian (2017b) and subsequently in later works (Lazarian & Yuen 2018a; Hu et al. 2019c) that the gradients of densities will be perpendicular to that of velocities in the region of shocks. While in the raw velocity channels we cannot separate the density and velocity channels, it was argued in Yuen & Lazarian (2017b) that thin channel gradients would be a good estimate of the caustics map. Therefore, by comparing the gradients of the thin and thick channels, one might locate shocks. However, as we see from this paper, the aforementioned method may sometimes be problematic since the gradients of the velocity channel are linear combinations of density and velocity fluctuations. Moreover, the density gradients are now parallel to the magnetic field statistically. A numerical example can be given using the results from Section 5: even for the thinnest channel we have at the spectral peak in the lower left panel of Figure 9 ($\Delta v/v_{\text{ms}} \sim 0.07$), the relative contribution of density to velocity in the center channel is still $\sim 0.7$. That means the gradients of the center channel in the vicinity of the shocks in Figure 9 in the thin channel limit would be composed of $\sim 41\%$ of the gradients that are density-like being parallel to the magnetic field, while $\sim 57\%$ of the gradients that are velocity-like will then be perpendicular to the magnetic field. With VDA, we can separate the channel maps into $p_d$ and $p_v$ and consider their gradients’ relative orientations in the vicinity of shock regions. We shall refer to the formulation in the forthcoming paper.

9.3.5. Caustics Gradients under Self-gravity

A similar argument also applies when we are using gradients to probe the self-gravitating regions. It was suggested by Yuen & Lazarian (2017b) that the gradients of both density-like and velocity-like features will gradually rotate 90°. With VDA, we can see how the velocity caustics behave in the presence of gravity. According to the description in Yuen & Lazarian (2017b), we believe that the caustics gradients should be least affected by gravity. However, when the gravitational force takes over, the caustics gradients will eventually turn 90°. This change happens as the acceleration induced by gravity gets larger than the acceleration induced by turbulence. Both the relative orientation between the intensity gradient and $p_v$ gradient and that between $p_d$ and polarization will change. This is similar to the use of the velocity gradients (see Yuen & Lazarian 2017b), but using $p_v$, we expect to determine the collapse regions significantly better. We shall discuss the corresponding algorithm in the forthcoming paper.

9.3.6. Caustics Gradients in Multiphase HI Emissions

The ability to separate the gradients arising from velocities and densities would change the ways to apply VGT in multiphase media. It was shown in Figure 11 that all of the channels for the cold and unstable neutral media are velocity dominant, while for the warm neutral media, the velocity channels are velocity dominant only in the wings. When they are combined and observed spectroscopically, we see that only the supersonic, cold/unstable media features are displayed in both the $\sigma - v$ and spectral ratio $P_d/P_v$ diagrams. We can see that the peak location of $\sigma_p - v$ in Figure 10 is consistent with that of cold neutral media in Figure 11. Similarly, the $P_d/P_v$ diagrams in both Figures 10 and 11 are sensitive to the channel gradient vectors to flip their directions. Instead, as discussed in Yuen & Lazarian (2017b), the relative orientations of both intensity and velocity centroid gradients will change from perpendicular to parallel according to the stage of collapse, with the latter being slower process.

28 We note that the response of density fluctuations to the presence of the magnetic field can be used to get properties of magnetized media (see Hu et al. 2019c). Using the approaches developed with the velocity gradients, we can also trace magnetic fields (Yuen & Lazarian 2017b; Hu et al. 2019c). Notice that density gradients are usually not reliable tracers of the magnetic field in supersonic interstellar media (Yuen & Lazarian 2017b). They, however, can be used in combination with velocity gradients to study both magnetic fields and shocks (see Hu et al. 2019c; also see Section 9.3.4).

29 The word “gradually” is important here since it is not a one-time process for the gradient vectors to flip their directions. Instead, as discussed in Yuen & Lazarian (2017b), the relative orientations of both intensity and velocity centroid gradients will change from perpendicular to parallel according to the stage of collapse, with the latter being a slower process.
width, which is a signature of a supersonic system. As we show in Figure 11, the numerical simulations suggest that the caustics arising from the CNM clumps dominate that from the other phases in multiphase media. It is worth noting that, due to the large temperature differences and the definition of the sonic Mach number, the turbulence flow can be supersonic for CNM but subsonic for WNM. Again, we emphasize that the thermal problem depends on the parameter $\nu_{\text{los}}/c_s$, not the true sonic Mach number. For a two-phase model, there will be at least two numbers $\nu_{\text{los}}/c_{\text{sonic}}$ and $\nu_{\text{los}}/c_{\text{wmm}}$ that will decide whether the velocity channel will be density-like or velocity-like, where $c_{\text{sonic}}$ and $c_{\text{wmm}}$ are the localized sonic Mach number for CNM and WNM. Readers should be cautious of the dramatic differences of the thermal properties for CNM and WNM and the respective impact on the channel maps (see Section 5.2.2).

The VDA approach brings the gradient technique to a new level in applications to H I media. This paper shows that the VDA resolves the problem of separating the velocity and density contributions in the situation where the thermal broadening dominates the turbulent contribution. As a result, the VDA makes the caustics gradients applicable in the most unfavorable scenario described in Clark et al. (2019). In particular, our practical applications of the VDA to the H I data in Sections 7 and 8 demonstrate that, using caustics gradients in multiphase H I, we can (1) prove the applicability of caustics gradients in their application to most of the galactic H I, (2) increase the accuracy of both techniques in probing turbulence and the magnetic field, and (3) obtain reliable results for the regions where application of the original versions of techniques may be problematic. (4) The fundamental reason why small-scale gradients are more accurate than that at large scales due to the spectral behavior we show numerically in Figures 9, 11 and observationally in Figure 19.

The channel gradients are being developed based on the better understanding of the position–velocity mapping of turbulent motions (LP00). We expect that the VDA would be a way to significantly improve their performance, due to the removal of density contamination in channel maps. The synergy with other approaches presents a plausible way to further improve the methods of studying turbulence and magnetic fields. We discussed above the application of the phase decomposition technique (Kalberla & Haud 2019) in combination with the VCA. We believe that this approach could be promising together with the caustics gradient and expect that this synergistic approach can provide insight into how different phases interact with the magnetic field. Besides, while the classical VCA (Section 8) focuses on obtaining turbulent properties of the cold medium, with the VDA, turbulence in both cold and warm media can be studied, and its properties can be compared. This is very important for understanding the dynamics of the multiphase media.

9.3.7. Use of Caustics Gradients for Predicting CMB Polarized Foreground

The VDA, together with VGT, also enables us to study the foreground magnetic field, which would be beneficial in cosmological studies. The VDA is a technique applicable to any spectroscopic data. The application of it to H I will have significant consequences for foreground polarization studies. Indeed, dust-polarized emission is an essential component to be removed in CMB polarization studies aimed at detecting the enigmatic gravitational waves in the early universe.

As we see from Section 9.2, the H I fibers that are extracted by RHT and related techniques could be either CNM density structures or velocity caustics. While Clark et al. (2015) did show that the fibers traced by RHT have a statistical correlation to the local magnetic field on the plane of the sky, whether the correlation is related to density, velocity, or both is open to question. As we discussed in the paper, the VDA, with its ability to identify velocity fluctuations, mitigates the effects of density fluctuations on the VGT. As we also discussed earlier, the latter fluctuations are more poorly aligned with respect to the magnetic field compared to the velocity fluctuations. In other words, on the basis of our study, we conclude that the H I density elongated structures, the orientation of which was viewed in Clark et al. (2015) as a way to trace the magnetic field and the foreground polarization, are in fact an impediment to accurately predicting the foreground CMB polarization. A more precise prediction of the CMB foreground polarization is expected when the contribution of these density filaments is filtered out with the VDA. The corresponding study will be presented elsewhere.

9.4. Implications for Studying the Nature of the H I Velocity Channel Structures

The current study of the properties of the PPV statistics is not limited to specific astrophysical settings. At the same time, the present study is timely for a number of reasons. First of all, the intensity fluctuations within H I channel maps were an issue of recent intense debates. Clark et al. (2019) suggested that the structures of thin channels, both at the spectral peak and in wing channels, are mostly associated with the density enhancement of cold neutral media. Their argument is based on the following. (1) In cases with low sonic Mach number, thermal broadening would suppress the contribution of velocity fluctuations in velocity channels. (2) In cases with high sonic Mach number, the high-density enhancements due to compression dominate the fluctuations of velocity channels. (3) When using high-pass filters on velocity channels, one will likely detect channel fluctuations that are associated with the density enhancements. (4) Those channel fluctuations are shown to be correlated with a number of alternative observational signatures that are directly or indirectly related to cold neutral media. Several follow-up works support the argument from Clark et al. (2019) and claim that the observed velocity channel intensity arises exclusively from cold neutral media density fluctuations (Clark et al. 2019; Peek & Clark 2019; Kalberla & Haud 2019, 2020; Kalberla et al. 2020).

These debates have far-reaching implications because the existence of caustics is the foundation of many statistical techniques that are shown to be applicable in observation. For instance, some of the papers (see Kalberla & Haud 2020) question the applicability of the VCA to H I and claim that the whole crop of VCA results obtained in this direction is “fake.” The velocity gradient technique (VGT) is also challenged by the papers that claim that the H I channel gradients are density gradients rather than velocity gradients. Note that VCA provides a unique insight into the velocity statistics, and VGT is a very promising tool to, for example, map galactic magnetic fields both at high latitudes (see, e.g. Yuen & Lazarian 2017a; Lazarian & Yuen 2018a) and within the galactic disk (González-Casanova & Lazarian 2019). Nevertheless, it can be argued that even though VCA or VGT shows correspondences to the theoretical expectation, it is entirely
possible that the velocity channel is density dominated as both VCA and VGT rely on the fundamental properties of velocity channels as formulated by LP00. This paper is timed perfectly to give a formal response to those papers (Clark et al. 2019; Peek & Clark 2019; Kalberla & Haud 2019, 2020; Kalberla et al. 2020) and to discuss how the two methods (VCA: Section 8; VGT: Section 9.3) will be impacted in the extreme cases when $P_d/P_v > 1$.

The points in Clark et al. (2019) were addressed in Yuen et al. (2019), and this paper also provides the further theoretical, numerical, and observational foundations for the theory describing the statistics of the velocity caustics. As we understand the current situation, the debates are focused on whether LP00 theory and the VCA technique adequately reflect the statistics of the multiphase neutral hydrogen. The nature of the neutral hydrogen velocity channel is highly related to the underlying model of turbulent HI. It is a well-known fact that neutral hydrogen has at least two stable, pressure-balanced phases (see Draine 2011; McKee & Ostriker 2007), which are called cold neutral media (CNM; $T \sim 200$ K) and warm neutral media (WNM; $T > 5250$ K; see also Kritsuk et al. 2017). In between these temperature ranges are the thermodynamically unstable phases. Under this differentiation of phases, WNM is believed to be subsonic, while that of CNM is believed to have sonic Mach number $M_s = v/c_s \sim 2 – 3$. Moreover, both phases are believed to be sub-Alfvénic (Crutcher et al. 2010). In the theory of interstellar turbulence with the Alfvénic Mach number small ($M_A < 1$), the statistical properties of supersonic turbulence are very different from that of subsonic turbulence (see Goldreich & Sridhar 1995; Kowal et al. 2007). Nevertheless, observationally, supersonic turbulence velocity channels display different properties from that of subsonic turbulence. How supersonic CNM and subsonic WNM behave geometrically under the framework of interstellar turbulence and their respective spatial behavior in velocity channels becomes a crucial question that we would like to understand.

For example, the application of VDA to HVC (Section 7) allows observers to extract the velocity caustics with high accuracy in observed neutral hydrogen velocity channels and further enables observers to characterize the importance of turbulence velocity statistics in observations. Notice that HVC has a well-studied core–envelope structure. We can see that the density structure does indeed dominate the center part of the velocity spectral line, which we found to originate in CNM density fluctuations. Yet as we move to the wing channels or focus only on small-scale structures, the velocity caustics dominate over density fluctuations. Nevertheless, the WNM caustics dominate over density fluctuations for nearly all channels, which means the contributions of velocity caustics cannot be ignored for every channel.

A similar situation also happens for the GALFA data that we test in Section 9.2. By analyzing the GALFA data, we show regions where the intensity fluctuations are clearly dominated by velocity fluctuations (see left panel of Figure 22), and we also showed that even using the so-called “cold neutral media tracing tools” like USM or RHT, we can obtain more than 50% filamentary structures that are caustics (see Figure 25). In fact, there is no channel that has nonzero intensity or contains zero caustics contributions. Even at the peak of the velocity spectral line, the caustics contribution is still about 33% of the total fluctuations of the velocity channel when including the large-scale contribution. If we discuss only small scales ($k > 5$) at $\Delta v = 2.94 \text{ km s}^{-1}$, the percentage should be raised to 87%. The velocity caustics map contains a unique spatial structure that reflects the properties of turbulence (see Figure 16). The rich physics that is contained in velocity caustics should therefore be studied extensively.

Readers should keep in mind that the existence of velocity caustics in a specific place in the velocity channel does not exclude the possibility of cold neutral media in the same location, and vice versa. In other words, cold neutral media as a supersonic object displays both density enhancements and velocity caustics in approximately the same spatial location of the map, and the fluctuations that we see from velocity channels are the sum of these two (see, e.g., Figure 11). The dominance of density fluctuations in H I channel maps was first proposed in Clark et al. (2019). In fact, as we noted before, Clark et al. (2019) believes that there is no measurable contribution to the H I intensity from velocity caustics at all. This argument was adopted in some of the follow-up works that correlate the locations of unsharp-masked channel structures to some other CNM diagnostics (e.g., probing using NaI equivalent width, Peek & Clark 2019; or CNM probed by multi-Gaussian decomposition, Kalberla & Haud 2019). Nevertheless, as we see from Section 9.2, more than half of the “filamentary structure found by unsharp masks” should be associated with velocity caustics. Moreover, those “strong density fluctuations” originate from the few smallest $K$ from the $P_d/P_v$ diagram. Most of the small-scale $P_d/P_v$ is smaller than 1, suggesting a velocity dominance in the observed H I 21 cm channel maps. The works that are based on the unsharp-mask algorithm should be seriously revisited.

Finally, we should mention that some authors were so worried about the applicability of the VCA to H I that, in an arXiv preprint with a telling title “Are observed H I filaments turbulent fraud or density structures? Velocity caustics, facts and fakes” (Kalberla & Haud 2020), they give the reader the impression that some published results analyzing H I and supporting the predictions of LP00 theory were fake ones and no one could reproduce them, apart from the authors of the LP00 technique. We believe that the analysis provided in the present paper will help the reader to make a fact-based judgment whether the confirmations of the LP00 predictions were real or “fake.”

10. Impact on Various Fields of Astrophysical Study

Aside from the previous studies, the development of the VDA is expected to impact general spectroscopic and polarimetric studies. On one hand, the VDA method allows simple yet efficient ways to extract the caustics map from spectroscopic PPV data, meaning that one can now study the velocity statistics and dynamics based on the currently available PPV data using the framework developed in this paper, which we shall discuss in Section 10.1. On the other hand, the method can be easily migrated to other studies like polarimetric maps, which also suffer from density contamination. We shall discuss the possible ways to recover the density-free physical quantities in Section 10.2. We shall also discuss how the current paper could be used in conjunction with the differential measure analysis method (Lazarian et al. 2020) to find the magnetic field strength on the sky; see Section 10.3. We also answer the questions that we asked in Section 1 in Section 10.4.
10.1. Importance to General Spectroscopic Studies in Interstellar Media

Because turbulence is ubiquitous in numerous astrophysical processes, the development of the current paper suggests that every piece of spectroscopic PPV data could be revisited by extracting the corresponding velocity caustics structure, practically giving a second life to all spectroscopic data for astronomers to explore. This paper demonstrates not only the key role that velocity caustics play in the formation of channel intensity fluctuations within the spectroscopic PPV cubes, but, most importantly, it provides the algorithm for extracting velocity caustics from observation (Sections 3, 7, 8). These caustics structures are expected to carry different morphologies compared to the channel map itself, following the theory of PPV statistics (LP00) and also the properties that we discuss in the current paper (Sections 5, 6; see also Appendices E, F).

The velocity caustics is the central quantity in studying the turbulent fluctuations in the quantitative theory of PPV statistics established in LP00 (see also Kandel et al. 2016). This theory formulated how different statistical theories of MHD turbulence (e.g., Goldreich & Sridhar 1995; Lazarian & Vishniac 1999; Beresnyak & Lazarian 2019, and the references therein) could impact the spectrum and anisotropy revealed through spectroscopic observations. Nevertheless, since the density field can be non-Kolmogorov in terms of the power spectrum in a number of physical conditions (see Beresnyak et al. 2005; Kowal et al. 2007), the interpretation of the PPV data based on the prediction derived from a pure velocity statistics assumption could be ambiguous. LP00 provided a way to statistically determine the spectral indexes of the velocity and density spectra, but this study provides a way to separate the actual PPV contributions arising from density and velocity fluctuations. Our present work effectively opens a new direction in studying turbulence by applying the VCA (LP00) and the VCS (Lazarian & Pogosyan 2006), as well as the LP00-based methods of applying turbulence anisotropy studies (see Lazarian & Pogosyan 2012; Kandel et al. 2016, 2017a) to the PPV data with velocity caustics extracted (see Section 8 for our suggested procedure).

The velocity caustics map also has great importance for techniques that are based on the statistics of velocity channels, as velocity caustics statistics are connected to a number of studies that allow us to infer the turbulence properties (Section 9.1), the direction and magnitude of the magnetic field (Section 9.3), and the nature of the filamentary structures in observational data (Sections 9.2, 9.4). The new set of tools that we developed in this paper allows astronomers to explore the velocity dynamics directly without density contamination. Similarly, studies of turbulence anisotropy (Lazarian & Esquivel 2003; Esquivel & Lazarian 2005) and the decomposition of observational data into the contributions arising from Alfvén, fast, and slow modes (Zhang et al. 2020, see also Kandel et al. 2016, 2017a) can be done much more precisely and reliably when the PPV data is not contaminated by the density fluctuations. We note that the VDA provides a good synergy with the techniques that use the channel map data, different moments of channel maps (e.g., see our construction of the constant-velocity centroid in Appendix D that shows significant promise compared to the traditional velocity centroids; see Lazarian & Esquivel 2003; Esquivel & Lazarian 2005), and more sophisticated constructions. The caustics map can be combined with other tools developed by the community that apply to channel maps, for example, wavelet transforms (Kowal & Lazarian 2010) or the principal component analysis (Heyer et al. 2008) with new insight as the caustics are density-independent and follow PPV statistics from LP00. Naturally, the VDA also opens new possibilities for developing new tools to directly study the properties of the velocity field of astrophysical turbulence.

10.2. Implications of the VDA for Polarization Studies

The separation of the contributions arising from velocity and density fluctuations is important for other branches of research. Take the problem of ground-state atomic alignment (GSA) as an example (Yan & Lazarian 2006, 2007, 2008). The observational output of the ground-state alignment would be the Stokes parameter maps at a range of velocities $v$ with velocity width $\Delta v$ (hereafter GSA-PPV). The mathematical structures of the Stokes parameter in ground-state alignment problems are rather similar to that of spectroscopic maps that we study in the current paper, in which the resultant observational parameter is a density-weighted average of velocity (for Doppler-shifted lines) or magnetic field information (for ground-state alignments along the line of sight). As a result, the GSA-PPV would contain “magnetic field caustics” that arise from the fluctuations of the magnetic field lines sampled in a narrow velocity range. However, Stokes parameters are produced with a weighting related to the tracers’ density. For the case of ground-state alignment, that would be the atoms like [C II]. Therefore, a method similar to VDA is required in analyzing the fluctuations of GSA-PPV if we need to study only the statistics of the magnetic field in atomic alignment measurements.

A similar technique can also be used for a dust and synchrotron polarization map where both suffer from density contamination. Using dust polarization as an example, the Stokes parameters (in the complex form $P = Q + iU$) are a density-weighted sum of the double angle along the line of sight:

$$P(X) \propto \epsilon \int dz \rho_{dust} e^{2i \theta (X,z)} \sin^2 \gamma$$

(25)

where $\rho_{dust}$ is the dust density, $\theta$ is the 3D planar angle, and $\gamma$ is the inclination angle. Observationally we obtain both $P$ and column dust intensity $I = \int dz \rho_{dust}$. Similar to the spectroscopic counterpart, the function $P$ contains both density and magnetic field fluctuations, as discussed in Lazarian & Pogosyan (2012, under a more general framework, though) as it is rather natural to consider the linear combinations of $P$ and $I$ to extract both density and magnetic fluctuations from dust emission maps. We shall discuss the fundamentals of the method and how to extend the decomposition to synchrotron polarization with strong Faraday rotation in the upcoming paper.

10.3. Studying Magnetic Field Strength

Aside from tracing the magnetic field using VGT (see Section 9.3 for a dedicated discussion on how VDA could fundamentally change VGT), the VDA technique can also help to estimate the magnetic field strength based on either the Davis–Chandrasekhar–Fermi (DCF) technique (Davis 1951; Chandrasekhar & Fermi 1953) or the recent synergy of gradient dispersion (Lazarian et al. 2018) and the differential measure approach (DMA; Lazarian et al. 2020).
In Lazarian et al. (2020), we derived the relation between the mean squared magnetic field strength and the structure functions of the velocity centroid and polarization angles. The mean squared planar magnetic field strength, which is based on the MHD theory of turbulence, is given by

\[
\overline{B}_c^2 = f^2 \frac{4\pi \rho}{D_n^c} \frac{\overline{v}_c^2}{D_n^v}
\]

where \(\rho\) is the mean density of the system, and \(D_n^c, D_n^v\) are the \(n\)th-order structure functions for velocity centroid and polarization angle, respectively. The \(f^2\) factor is theoretically deduced and numerically computed (Lazarian et al. 2020) to be \(\sim 1\)–2 in most cases. The method of Equation (26) is shown to have high accuracy even in supersonic and mildly super-Alfvénic cases in Lazarian et al. (2020), which allows observers to estimate the magnetic field strength readily with theoretical support.

However, the method of Equation (26) has a pronounced deficiency: both \(V\) and \(\phi\) are implicit functions of density, even though they do not carry any units of density. For instance, in observations, we can only observe the normalized velocity centroid \(\overline{V}_c \sim 1\) and the Stokes polarization angle \(\phi \sim \tan^{-1} \left( \frac{\int \rho \cos 2\theta}{\int \rho \sin 2\theta} \right)\) where \(\rho\) is the density. Therefore, it is obvious to ask whether we have ways to remove the contribution of density fluctuations in observations for both centroid and polarization angles. In Section D we discuss a method of obtaining the constant-density centroid based on VDA, while the constant-density Stokes parameter could be possibly extracted using the idea from Section 10.2. The ability of the VDA to separate the velocity and density information opens new avenues for precision magnetic field studies using the spectroscopic information. We intend to explore those in future papers.

10.4. Our Questions in the Introduction That Are Answered in the Present Study

Based on this VDA algorithm and the multi-Gaussian decomposition algorithm (Haud & Kalberla 2007), we answered the questions that are asked in Section 1:

1. Is the concept of density/velocity fluctuations pixel-based, or is it only valid in a statistical sense?  
   Answer: The concept of density/velocity fluctuation is a statistical concept. We cannot see any velocity statistics if we do not consider the statistical effect of caustics (see Section 3, Figure 6).

2. What is the role of velocity caustics in channel maps when the CNM dominates the emission?  
   Answer: The relative fluctuations from velocity caustics are not insignificant (Figures 16, 18, 22).

3. What is the relative importance of velocity and density fluctuations in a spectral line’s central and wing channels?  
   Answer: Even in the case of subsonic media, most of the channels are actually velocity-dominated at small scales. An example would be the high-velocity cloud (Section 7), where its CNM density contribution is approximately the same as of the velocity counterpart (Figure 18). However, at small scales, the density fluctuations in velocity channels are approximately one-half that of velocity (Figure 19).

11. Summary

This paper develops a set of self-consistent, comprehensive tools for extracting the velocity caustics from PPV cubes by combining analytical and numerical approaches, and the tools have been thoroughly tested in observations. Our numerical part includes isothermal and multiphase numerical simulations. We developed a new algorithm for isolating the velocity caustics, explored the difference in the PPV statistics of the central and wing velocity channels, and applied our new approaches to H1 21 cm GALFA data. Our results are summarized as follows:

1. We developed a new algorithm termed the velocity decomposition algorithm (VDA) for extracting the velocity information from the observational spectroscopic PPV cube. The algorithm provides an excellent decomposition of velocity and density contributions for subsonic turbulence, and it also shows significant promise for supersonic turbulence (see also Appendix E).

2. Our numerical study demonstrates that in thin velocity channels the contributions from velocity caustics dominate at small scales, and the density fluctuations dominate in the intensity fluctuations of the wing channel at small scales.

3. We also applied our approach to multiphase simulations with realistic CNM and WNM mass fractions and demonstrated that the ratio of the spectral energies associated with density and velocity fluctuations, that is, \(P_d/P_v\), is less than unity when the channel width is thin regardless of whether the channel is at the center or at the wing (Figure 10). In particular, the cold phase media that weighs the most in terms of spectral studies exhibit velocity-dominant features at small scales, as predicted in LP00.

4. We demonstrated that the contribution of velocity caustics fluctuations is maximized in the wings, and they exhibit a double-peak pattern for isolated emission regions. The distance of the peaks as a function of velocity is approximately \(\Delta_{\text{effective}}\) (see Figures 13, 14). This distance is unchanged in the presence of the galactic rotation curve (Figure 15).

5. We test our method with a selected HVC (Section 7). This high-velocity, low-metallicity cloud is isolated and has simple core–envelope geometry, making it a good testing ground for our approach. We determine that the small-scale fluctuations in the HVC PPV data are dominated by velocity caustics regardless of the cloud’s multiphase nature (Figure 11). Moreover, we see the double-peak behavior as we predicted from Section 14.

6. Using the VDA and the 1\(\sigma\) (Section 6) condition, we propose a modification of the VCA technique that is advantageous when the observational data are significantly thermally broadened (Section 8). The modification above extends the applicability of the VCA technique and increases its accuracy in recovering the 3D velocity spectral indices. With VDA, we can focus our analysis on the statistics of pure velocity caustics. That means we can employ LP00’s theory directly without worrying about the density contributions.

7. We show that the possible “cold neutral media” location traced by RHT/USM can be associated with the velocity caustics of cold neutral media (Sections 9.1, 9.2). More
importantly, our study confirms that the velocity caustics are also filamentary and ubiquitous under the RHT algorithm in HI channel maps (Figure 24) and establishes the foundations for further advancing the methods of studying turbulence and magnetic fields using spectroscopic data.

8. The VDA method has significant implications for both the gradient technique (GT; Section 9.3) and other structure identification algorithms like RHT (Section 9.2). Since the velocity caustics have well-predicted properties from the theory of MHD turbulence (Goldreich & Sridhar 1995) and the PPV statistics (LP00), the gradients or “fibers” of the caustics are expected to trace the magnetic field and shocks much better than any density-weighted variants can.

9. Most importantly, the VDA could derive a completely new set of unexplored velocity caustics data (Section 10.1) from every spectroscopic data set. Moreover, the VDA allows one to check with a well-established theory of PPV statistics (LP00) and apply the methods that are derived from LP00. The potential of the VDA in studying MHD turbulence in observations should not be underestimated.

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Appendix A
Correlation of Density and Velocity in the Case of Subsonic Media

To derive the orthogonality condition of VDA (see Section 3 and also Equation (E4)) from LP00’s point of view, we need to start with the basic thermal broadening equation in subsonic turbulence in the velocity space form (Equation (7)) and perform some simple analysis in the case of very thin channels ($\Delta \nu < \delta \nu_R = D(X, 0) < c_s$). Then from Equation (5) we can simply write

$$p_{\delta} \propto \delta \rho e^{-\zeta_1/2c_s^2}$$
$$p_{\nu} \propto \langle \rho \rangle e^{-\zeta_2/2c_s^2}$$

where $\zeta_{1,2}$ are two values in ($v_0 - \Delta \nu/2, v_0 + \Delta \nu/2$) selected according to the mean value theorem. We can already make some observations here. First of all, if $c_s$ is large and we know that from MHD theory $\langle \delta \rho \cdot v \rangle = 0$, $\langle \delta \rho \rangle = 0$, the correlation term can be separated and proved to be zero:

$$\langle p_{\rho} p_{\nu} \rangle \sim \langle \rho \rangle \langle \delta \rho \rangle \langle e^{-\zeta_1/2c_s^2} e^{-\zeta_2/2c_s^2} \rangle = 0.$$  

(A2)

The problem here is that, the orthogonality of the inner product above relies on two factors: (1) $\langle \delta \rho \cdot v \rangle = 0$ and (2) $\langle \delta \rho \rangle = 0$. The first one is not true when we have large $M_s$ (see the correction of VDA in Appendix E), and the second one is not true if we have insufficient statistical sampling (see the discussion of statistical sampling in Yuen & Lazarian 2017a).

To proceed, we have to employ the numerical simulations and see how $\langle p_{\rho} p_{\nu} \rangle$ varies as a function of $M_s$. To start, we have to compute the channel map $p$ from Equation (5) and the true velocity caustics map $n$ by setting $\rho = \text{const}$ in Equation (5). Then we compute the term $\text{NCC}(p - n, n) = \langle (p - n) n \rangle$ as from definition in Section 3 $p_d = p - n$ and $p_n = n$, and NCC is a normalized correlation (see Equation (21)). Figure 26 shows the relation of $\langle p_d p_{\nu} \rangle$ as a function of the line-of-sight Mach number $M_{s,\text{LOS}} = v_{\text{LOS}}/c_s$. We compute $\langle p_d p_{\nu} \rangle$ for both the center and the wing channels. From Figure 26 we see that the term $\langle p_d p_{\nu} \rangle$ is generally nonzero especially when $M_s \gg 1$, regardless of whether one picks the center or the wing. However, it is worth noting that even though the term $\langle p_d p_{\nu} \rangle$ is nonzero in supersonic turbulence, we still have a surprisingly good extraction result in the main text (see Figure 5) for the velocity caustics.

Appendix B
Kernel of Velocity Channels

A fundamental concept throughout this paper is the effective shape of the observational kernel in the velocity space (see Figure 27). This discussion of the observational kernel shape is

![Figure 26. Correlation of $\langle p_d p_{\nu} \rangle$ as a function of $M_{s,\text{LOS}} = v_{\text{LOS}}/c_s$ for the center channel ($\nu = \nu_{\text{peak}}$) and wing channel ($\nu = \nu_{\text{peak}} - \delta \nu_{\text{peak}}$) using the simulation set “beta” from Lazarian & Yuen (2018a).](image-url)
crucial in understanding the differences between the physical scenarios in LP00 and Clark et al. (2019) and the physical reality. In simple words, the effective kernel shape describes the relation of the observed maps and the unbroadened velocity channels, according to Equation (7). For instance, the kernel to produce the column density map is a squared top hat over all channels, while the kernel to produce the thick velocity channel map is a Gaussian function with large upper and lower limits. Considering the kernel’s shape is crucial in determining what is contained in a certain observed map (see also Figure 2).

Figure 27. Illustration of how the intensity map kernel should be different from that of the channel map kernel in a realistic MHD simulation. Left column: the intensity and the thin channel map from the simulation. Middle column: the respective shape of the kernel. We can see that the thin channel kernel is intrinsically different from the full intensity kernel. Right column: ratio of intensities between the observed map and the total intensity map.

Appendix C
What Gradients Are We Exactly Computing in Velocity Channels?

There were a number of concerns on whether the gradients of velocity channels are actually “velocity gradients” in some of the recent publications (see, e.g., Clark et al. 2019 for their discussion section). We shall make a simple derivation in this section to show that the gradients of velocities purely define the gradient directions of velocity channels within the velocity channel width as long as the channel is thin. We shall start with Equation (7) and explicitly write out the spatial and spectral dependencies of the channel map by assuming the x-axis is the LOS direction:

\[
p(X = (y, z), v_v(X); \Delta \nu) \propto \int_{v_v - \Delta \nu/2}^{v_v + \Delta \nu/2} dv f(v) W(v) \exp \left( -\frac{(v - v_{\text{center}})^2}{2c_s^2} \right) \tag{C1}
\]

where \( f(v) \) is the probability distribution function (PDF) for velocity \( v \), and \( W(v) \) is the thermal kernel (see Appendix B).

We consider the isothermal case, but the result can be easily generalized to the nonisothermal cases. The spatial gradient can be derived by the fundamental theorem of calculus, setting \( v_{\text{center}} = 0 \) without loss of generality, and we will put \( W(v) = 1 \) for simplicity:

\[
\nabla_X p(X, v_v(X), \Delta \nu) \propto \nabla_X v_v \left[ f(v_v(X) + \Delta \nu/2) \right. \\
\times \exp \left( -\frac{(v_v(X) + \Delta \nu/2)^2}{2c_s^2} \right) \\
- f(v_v(X) - \Delta \nu/2) \times \exp \left( -\frac{(v_v(X) - \Delta \nu/2)^2}{2c_s^2} \right) \right] \tag{C2}
\]

in which we can see that the gradients of a velocity channel at velocity \( v_v \) are actually given by the gradients of the velocity itself with a complicated weighting parameter. There is an ambiguity in the above equation since the \( \nabla v_v \) here actually represents the gradients of velocity within the velocity range \( v_v \pm \Delta \nu/2 \) only. To proceed, we assume \( \Delta \nu \ll c_s \) (which is an extra condition for the “thin” channel for this paper, which we shall call thermally thin in the moment), and thus an expansion is doable:

\[
\nabla_X p(X, v_v(X), \Delta \nu) \propto \nabla_X v_v \bigg|_{v_v \in [v_v - \Delta \nu/2, v_v + \Delta \nu/2]} \\
\times \exp \left( -\frac{v_v^2(X)}{2c_s^2} \right) \left[ \frac{df}{dv} \bigg|_{v = v_v(X)} \right. \\
\times \frac{\Delta \nu}{c_s} - f(v_v(X)) \left( -\frac{v_v \Delta \nu}{c_s^2} \right) \bigg]. \tag{C3}
\]
One can see that the formulation above actually works even for \( f(v) \) being the true density PDF. As a well-known fact if we assume \( f \) to be a Gaussian \( f \sim \exp(-v^2/2\delta^2) \) with \( \delta \) the line width (usually \( \delta \sim \sqrt{\delta v^2 + 2c_s^2} \)), then the above expression gives

\[
\nabla_X p(X, v(X), \Delta v) \propto v(X) \Delta v \nabla_X v(X) \bigg|_{v(X) \in [v_0-\Delta v/2, v_0+\Delta v/2]} \\
\times \exp \left(-\frac{v(X)^2}{2c_s^2}\right) f(v(X)) \\
\times \left[ \frac{1}{c_s^2} + \frac{1}{\delta^2} \right]. \tag{C4}
\]

The key here is that the gradients of velocity channels are actually defined by the map that contains the velocity pixel values in \([v_0-\Delta v/2, v_0+\Delta v/2]\) multiplied by two exponent factors. Observe that the gradient operator can actually be regrouped:

\[
\nabla_X p(X, v(X), \Delta v) \propto \Delta v \left(1 + \frac{c_s^2}{\delta v^2 + 2c_s^2}\right) \\
\times f(v(X)) \nabla_X \exp \left(-\frac{v(X)^2}{2c_s^2}\right). \tag{C5}
\]

The important thing is that it is the last factor \( \nabla_X \exp \left(-\frac{v(X)^2}{2c_s^2}\right) \) that gives the direction of the gradients, and Equation (C5) works for all \( v \). The result here indicates that the orientation of gradients in thermally thin velocity channels is purely defined by the gradients of the velocity pixel values in \([v_0-\Delta v/2, v_0+\Delta v/2]\), with the amplitude of the gradients being related to the PDF \( f \) and also a number of extra constant factors.

**Appendix D**

**Constant-density Velocity Centroid**

Lazarian & Esquivel (2003; see also Esquivel & Lazarian 2005; Esquivel et al. 2015) discussed the properties of the velocity centroid and the gradients of which that are extensively applied in observations (see Yuen & Lazarian 2017a and works from the same authors). However, the realistic velocity centroid computed by the formula

\[
C(X) = \frac{\int dz \rho(X, z)v(X, z)}{\int dz \rho(X, z)} \tag{D1}
\]

has a density contribution both in the numerator and the denominator. Statistical analysis on the velocity centroid (for example, Kandel et al. 2016) often needs to assume that the density is constant in order to study the properties of the velocity field in such observables. The weighting of density in the velocity centroid also makes it a less favorable observable than thin velocity channels in terms of tracing magnetic fields.
with gradients, since the latter contains a higher portion of velocity contributions. With the availability of velocity caustics $p_v$, using VDA, the current study allows one to construct the so-called velocity-weighted caustics in subsonic turbulence by the following formula:

$$V_n(X) = \int dv p_v(X, \nu).$$

(D2)

Notice that in the case of constant density, the centroid is simply the sum of the LOS velocity:

$$V = \frac{1}{L_z} \int dz v(X, z)$$

(D3)

where $L_z$ is the line-of-sight depth. We show the structures of the two quantities from Equations (D2) and (D3) in the top row of Figure 28, and it is obvious that they look almost exactly the same. To further test whether their statistical properties are the same, we plot the spectra of the two maps in the lower left panel of Figure 28 and their ratio of correlation functions in the lower right panel of Figure 28, computed by

$$R(K) = \frac{|\mathcal{F}(V)|^2}{|\mathcal{F}(V_n)|^2}.$$  

(D4)

The expression $R$ is a function of position and will only be a constant if the two maps are exactly the same. It is obvious from the lower left corner of Figure 28 that the spectra of both Equations (D2) and (D3) have similar structures, with their inertial ranges (the dashed lines) carrying the same slope. As for the correlation function ratio (lower right corner of Figure 28), the value is basically a constant, indicating that the structures of the $V$ and $V_n$ maps are basically the same.

Appendix E

Supersonic Velocity Decomposition Algorithm

To obtain a unified decomposition method for both subsonic and supersonic turbulence, one has to make changes to the properties that we listed in Section 3 and proceed with the supersonic turbulence. Notice that the modification in this section can also be used in some other cases where turbulence is involved, for example, in situations where we have strong shocks or self-gravity.

Before that, let us first recall the three properties (Section 3):

1. $\langle p_v p_v \rangle = 0$;
2. $p_v = 0$ when $\Delta \nu \to \infty$;
3. $p_v \sim 1$ when $c_\nu > \delta_{\nu,\text{LOS}}$.

Even with the simple recipe in Section 3, we can see from Figure 29 that the decomposed $p_v$ correctly traces the true velocity caustics $n$ produced by synthesizing the velocity channel with a constant-density term in terms of the NCC value (Equation (21)). In fact, even when $M_{\nu,\text{LOS}} = \delta_{\nu,\text{LOS}}/c_\nu > 5$, the NCC value between $p_v$ and $n$ is still above 0.5 for both the center and the wing channels, meaning our approach in Section 3 actually works very well even in supersonic media. To further improve the method, we have to recall the properties of supersonic turbulence.

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30 This statement is given quantitatively from our knowledge according to the alignment measure of the gradients of the centroid and channel when compared to the local B-field; see Lazarian & Yuen (2018a).
In matrix form, we can write

\[ p(v) = \begin{bmatrix} U & 0 \\ V & 1 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = U\alpha_{v} + (V + B)\hat{\rho}_{v}. \] (E3)

Then with some algebra we can obtain the exact form of \( p_v \) and \( p_d \) for the case when \( \langle p, p_d \rangle = 0 \):

\[ p_d = \left( p - \langle p \rangle \right) \left( \frac{V}{U} \right) \hat{\rho}_v \]
\[ p_v = \left( p - \langle p \rangle \right) \left( \frac{V}{U} \right) \hat{\rho}_v \] (E4)

where the value of \( V \) is given by Figure 29.

**E.3. Can We Modify Property 2?**

Readers might wonder whether we can modify Property 2 in our derivation. Unfortunately, Property 2 is subject to the approximation and the knowledge that \( \langle \rho \rangle = 0 \) in the supersonic channel. Moreover, the method is more accurate in supersonic turbulence, which is surprising good in the wing channel. We shall not discuss the possible modification of Property 2 in the current paper.

**E.4. Modification of Property 3: The Velocity Channel Differential**

Here we describe a brand new method based on VDA in supersonic turbulence, which is surprisingly good in the wing channel. Moreover, the method is more accurate than the method that we developed in the main text (Section 3), with a cost that we do not observe the velocity caustics directly from observations, and it is more noise dependent. To start, we shall consider the observationally available function:

\[ V(X, v, \Delta v) = -c_t^2 \frac{\partial \log p_v(X, v, \Delta v)}{\partial v}. \] (E5)

Notice that when \( \Delta v \to \infty \), from Equation (7), we have

\[ V(X, v, \Delta v \to \infty) = -c_t^2 \left( \int \frac{d'v'}{p_v(v')} \left( \frac{v' - v}{c_t^2} \right) \exp \left( \frac{(v' - v)^2}{2c_t^2} \right) \right) \]
\[ = C(X) \] (E6)

which is basically the normalized velocity centroid (Equation (D1)). We shall name Equation (E5) the partial velocity centroid (see also the discussion of the centroid in Appendix D) at velocity \( v \) and velocity channel width \( \Delta v \). Readers should notice that Equation (E5) is simply the centroid function that integrates within the range \( [v - \Delta v/2, v + \Delta v/2] \).

This special \( V \) function has a unique behavior in the tracing of \( p_v \) in supersonic media. Notice that if we assume \( \Delta v \ll c_t \ll \delta v \) (the latter inequality comes from \( M_t > 1 \)), then the partial centroid (Equation (E5)) is simply the partial velocity projection along the line of sight regardless of what density weights we use:

\[ V(X, v_0, \Delta v \ll c_t \ll \delta v) \sim \sum_{v \in [v_0 - \Delta v/2, v_0 + \Delta v/2]} v. \] (E7)

Therefore, we can compute \( V \) both with density weighting (denoted as \( V_p \)) and without density weighting (denoted as \( V_n \)) and use NCC (Equation (21)) to see whether they are alike.

Figure 30 shows how \( V_p \) and \( V_n \) look in the wing and the center channel for the simulation h0-1200 (see Table 2), which has a total \( M_t = 6.36 \) and \( \delta v_{LOS}/c_t = 3.49 \). We can see that they look pretty much alike. Notice that \( V_p \) is purely velocity dominant. This means the \( V_p \) term is also very much dominated by velocity fluctuations. Furthermore, we can see from the left of Figure 31 that the NCC of the partial centroids to the caustics counterpart is generally higher than that of the \( p_v \) to the caustics. Moreover, at \( v = v_{peak} \pm \sigma \) (\( \sigma = \delta v_{LOS} \)), the difference of the NCC between the partial centroid and \( p_v \) is the largest. This shows that the partial centroid method is a more robust method than the VDA method that we developed in the main text in supersonic turbulence. However, note that the supersonic variant that we developed here is very sensitive to noise. The observation application of the supersonic method will be tested elsewhere.

How can we understand the results physically? Notice that technically we can always write the partial centroid as the linear combination of \( p_d \) and \( p_v \) in different \( v \) using the differential approximation and the knowledge that \( V_p \approx V_n \):

\[ V_p(X, v) \approx V_n(X, v) \approx \frac{c_t^2}{p_v(X, v)\Delta v} \left[ p_v(X, v) - p_v(X, v + \Delta v) \right] \]
\[ \approx p_v(v + \Delta v) \approx p_v(X, v) \left[ 1 - \frac{\Delta v}{c_t^2} V_p(X, v) \right]. \] (E8)

We can see that as long as we have an initial guess of \( p_v \) at some velocity, we can extrapolate \( p_v \) for different velocities. Notice that in the extreme wing channels, the logarithm of velocity fluctuations is purely velocity-like. Then we can successively produce the eigenmaps of \( p_d(v) \) and use Equation (E4) to also compute \( p_d \). Readers might wonder whether the parameter \( V_p \) could be used in subsonic turbulence.

We show the same NCC dependence curve for subsonic turbulence on the right in Figure 31. One can see that the performance of \( V_p \) is comparable to that of \( p_v \) in tracing the corresponding velocity caustics structures. This indicates that the partial centroid \( V \) is a versatile parameter in studying caustics in both subsonic and supersonic turbulence.

As a remark, incidentally, we also report that the logarithm of the channel map \( \log(p) \) itself also performs very well in the wings in tracing its velocity caustics equivalent \( \log(n) \). There is a theoretical and numerical finding (e.g., Kowal et al. 2007) that using the logarithm of channel maps suppresses the high-density fluctuations, and \( \log(p) \) is also a natural parameter in analyzing the linear waves in MHD turbulence (see Biskamp 2003). However, we found that a good correlation usually only occurs when \( |v - v_{peak}| > 1.5\sigma \). Nevertheless, while we can postulate that the channel map or the \( V \) parameter (Equation (E5)) is the linear combination of density and velocity contributions, the logarithm of velocity channels is not. Therefore, based on the formulation of this work, there still needs to be a way to understand why the logarithm of channels could trace those far wing caustics in observations. Incidentally, the \( V \sim -d(\log(p))/dv \) parameter
Equation (E5) traces the caustics counterpart \( V_n \propto -d(\log n)/dv \) very well in the range of velocities \(|v - v_{\text{peak}}| < 1.5\sigma\) (see Figure 31), exactly the opposite of the ranges of applicability of the \( \log p \) method, which means they can be complementary to each other in terms of observational applications.
In the text, we assume that the observed velocity channel’s thermal kernel is related to the system’s intrinsic temperature. This is true for neutral hydrogen, since the emission agent is the major species of the turbulence system. However, in realistic observation, the thermal speed of the observed species is different from the ambient thermal speed of the turbulence system. For example, suppose we have observed turbulent, partially ionized gases in channel maps, where we recognize that the neutrals are lighter (e.g., H$_2$) while the ions are heavier (HCO$^+$). Under these circumstances, the effective sonic speed for the emission lines of these two species would be inversely related to the mean molecular weight of the observed species since $c_s^2 = dP/d\rho = \gamma P/\rho \propto \mu^{-1}$ where $\mu$ is the mean molecular weight. For example, if the line-of-sight sonic Mach number of a certain turbulence system is $v_{\text{los}}/c_s \sim 0.5$ and we have the mean molecular weight of ions as 10 times that of neutrals, then the effective line-of-sight sonic Mach number of the ions would be $v_{\text{los}}/c_s \sim 5.0$. This means that while the neutrals’ velocity channels suffer from strong thermal broadening, the thermal broadening effect can be mostly ignored in the ions’ channel map.

From our discussion in the main text, it is rather evident that the line-of-sight sonic Mach number $v_{\text{los}}/c_s$ (not the intrinsic 3D Mach number; see Section 2) is a critical parameter affecting the performance of VDA. In this section, we would like to briefly study how the change of the effective thermal kernel width, which is inversely proportional to the molecular weight, would affect the performance of VDA. Figure 32 shows how the change of the effective thermal kernel width will alter the performance of VDA in the center channel of “e5r3” in the main text.

For the sake of completeness, we consider both heavier and lighter tracers, which correspond to a decrease and increase in $c_s$, respectively. We can see from Figure 32 that when the molecular tracers are heavier than the fluid in bulk (i.e., $c_s/c_{s,0} < 1$), the structure of the raw velocity channel will be more fragmented, and vice versa for the lighter tracers.

In Figure 32 we clearly see that the structure of the observed maps gets closer and closer to the maps of pure velocity caustics as the thermal broadening of the species decreases. At the same time, it resembles more the density map as the related thermal broadening increases. This is a direct consequence of the LP00 theory, which predicts that the thermal width is similar to the width of the velocity channels. The theory also predicts that the effect of velocity caustics decreases with the increase in the velocity channel width (see Equation (4)). We note that the role of the heavy species with lower thermal broadening can also be played by clumps of the colder gas that are being moved by the hotter gas in the multiphase medium. This was the model of the multiphase H I that was adopted in LP00 to justify the application of the theory to the cold fraction of interstellar H I. Our multiphase numerical simulations presented in the main body of the paper support this model of H I.

One might wonder why there are still density fluctuations in the heavy tracer limits as the latter should correspond to the limiting case discussed in LP00. It is true that in the case of $c_s \rightarrow 0$, there should be only velocity caustics fluctuations in velocity channels, but since realistic observations always have $c_s > 0$, the density fluctuations would still exist despite being small in magnitude. Not to mention, the relative fluctuation of density and velocity is always a function of velocity channel position (see Sections 3, 5). Neither velocity nor density fluctuations are negligible in real spectroscopic data. Yet, with our development of the VDA, it is always possible to separate the density and velocity fluctuations in observations and thus quantify their relative importance. To better quantify the performance of VDA under this scenario, we use the NCC (Equation (21)) again as in the main text.

The corresponding value of NCC$(n, p)$ for each row is listed on the left-hand side of the row. We observe that while the raw velocity channel structure is changing as a function of $c_s$, the decomposed $p_v$ is almost an exact match with the real velocity caustics map. From previous sections, we know that the intrinsic line-of-sight sonic Mach number is an essential parameter affecting the performance of VDA. However, there is no drop in performance from the change in the effective thermal kernel width that is due to the change in molecular species. Therefore, we conclude that the VDA technique can apply to molecular tracer emission lines.

It is also worth noting that the caustics map structure is still a function of thermal kernel width, as we discussed in the main text. Readers might wonder what physically the caustics represent when $c_s$ changes. Indeed, they are still pure velocity structures, but integrated for a different effective channel width $\Delta v_{\text{eff}}^2 \sim \delta v^2 + 2c_s^2$ (see Equation (3)). A larger $c_s$ would represent a larger collection of velocity structures along the line of sight.
Figure 32. Set of figures showing how the structure of the channel maps from the v = 0 channel of the simulation “e5r5” (M_t = 0.61, v_los/c_s,0) will change as a function of the effective thermal speed c_s. Here we represent c_s as multiples of c_s,0, the intrinsic thermal speed of the simulation we use in the main text. Columns from left to right: raw velocity channel p, decomposed \rho_d, decomposed \rho_v, and the real velocity channel map n. The corresponding NCC value between \rho_d and n for each row is labeled on the left. The color bar is drawn for \[ p, p + 3\sigma_p. \]

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