The extended future tube conjecture for SO(1,n)

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ABSTRACT. Let $C$ be the open upper light cone in $\mathbb{R}^{1+n}$ with respect to the Lorentz product. The connected linear Lorentz group $SO_R(1,n)^0$ acts on $C$ and therefore diagonally on the $N$-fold product $T^N$ where $T = \mathbb{R}^{1+n} + iC \subset \mathbb{C}^{1+n}$. We prove that the extended future tube $SO_C(1,n) \cdot T^N$ is a domain of holomorphy.

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For $K \in \{R, C\}$ let $K^{1+n}$ denote the $(1+n)$-dimensional Minkowski space, i.e., on $K^{1+n}$ we have given the bilinear form

$$(x, y) \mapsto x \cdot y := x_0y_0 - x_1y_1 - \cdots - x_ny_n$$

where $x_j$ are the components of $x$ respectively $y$ in $K^{1+n}$. The group $O_K(1,n) = \{g \in GL_K(1+n); gx \cdot gy = x \cdot y \text{ for all } x, y \in K^{1+n}\}$ is called the linear Lorentz group. For $n \geq 2$ the group $O_K(1,n)$ has four connected components and $O_C(1,n)$ has two connected components. The connected component of the identity $O_K(1,n)^0$ of $O_K(1,n)$ will be called the connected linear Lorentz group. Note that $SO_R(1,n) = \{g \in O_R(1,n); \det(g) = 1\}$ has two connected components and $O_R(1,n)^0 = SO_R(1,n)^0$. In the complex case we have $SO_C(1,n)^0 = O_C(1,n)^0$.

The forward cone $C$ is by definition the set $C := \{y \in \mathbb{R}^{1+n}; y \cdot y > 0 \text{ and } y_0 > 0\}$ and the future tube $T$ is the tube domain over $C$ in $\mathbb{C}^{1+n}$, i.e., $T = \mathbb{R}^{1+n} + iC \subset \mathbb{C}^{1+n}$. Note that $T^N = T \times \cdots \times T$ is the tube domain in the space of complex $(1+n) \times N$-matrices $\mathbb{C}^{(1+n)\times N}$ over $C^N = C \times \cdots \times C \subset \mathbb{R}^{(1+n)\times N}$. The group $SO_C(1,n)$ acts by matrix multiplication on $\mathbb{C}^{(1+n)\times N}$ and the subgroup $SO_R(1,n)^0$ stabilizes $T^N$. In this note we prove the

Extended future tube conjecture:

$$SO_C(1,n) \cdot T^N = \bigcup_{g \in SO_C(1,n)} g \cdot T^N \text{ is a domain of holomorphy.}$$

This conjecture arose in the theory of quantized fields for about 50 years. We refer the interested reader to the literature ([HW], [H], [SV], [SW], [W]). There is a proof of this conjecture in the case where $n = 3$ ([He2], [Z]). The proof there uses essentially that $T$ can be realized as the set $\{Z \in \mathbb{C}^{2 \times 2}; \frac{1}{2i}(Z - Z^\dagger) \text{ is positive definite}\}$. Moreover the proof for $n = 3$ is unsatisfactory. It does not give much information about $SO_C(1,n) \cdot T^N$ except for holomorphic convexity.

Here we prove that more is true. Roughly speaking, we show that the basic Geometric Invariant Theory results known for compact groups (see [He1]) also holds for $X := T^N$ and

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the non compact group $\text{SO}_\mathbb{R}(1, n)^0$. More precisely this means $\text{SO}_\mathbb{C}(1, n) : X = Z$ is a universal complexification of the $G$-space $X$, $G = \text{SO}_\mathbb{R}(1, n)^0$, in the sense of [He1]. There exists complex analytic quotients $X//G$ and $Z//G^C$, $G^C = \text{SO}_\mathbb{C}(1, n)$, given by the algebra of invariant holomorphic functions and there is a $G$-invariant strictly plurisubharmonic function $\rho : X \to \mathbb{R}$, which is an exhaustion on $X/G$. Let

$$\mu : X \to \mathfrak{g}^*,$$ $\mu(z)(\xi) = \frac{d}{dt} \bigg|_{t=0} (t \to \rho(\exp it\xi \cdot z)),$$

be the corresponding moment map. Then the diagram

$$\mu^{-1}(0) \hookrightarrow X \hookrightarrow Z$$

$$\downarrow \quad \downarrow \pi \quad \downarrow \pi^C$$

$$\mu^{-1}(0)/G \equiv X//G \equiv Z//G^C$$

where all maps are induced by inclusion is commutative, $X//G$, $X$, $Z$ and $Z//G^C$ are Stein spaces and $\rho|\mu^{-1}(0)$ induces a strictly plurisubharmonic exhaustion on $\mu^{-1}(0)/G = X//G = Z//G^C$. Moreover the same statement holds if we replace $X = T^N$ with a closed $G$-stable analytic subset $A$ of $X$.

## 1 Geometric Invariant Theory of Stein spaces

Let $Z$ be a Stein space and $G$ a real Lie group acting as a group of holomorphic transformations on $Z$. A complex space $Z//G$ is said to be an analytic Hilbert quotient of $Z$ by the given $G$-action if there is a $G$-invariant surjective holomorphic map $\pi : Z \to Z//G$, such that for every open Stein subspace $Q \subset Z//G$

i. its inverse image $\pi^{-1}(Q)$ is an open Stein subspace of $Z$ and

ii. $\pi^* \mathcal{O}_{Z//G}(Q) = \mathcal{O}(\pi^{-1}(Q))^G$, where $\mathcal{O}(\pi^{-1}(Q))^G$ denotes the algebra of $G$-invariant holomorphic functions on $\pi^{-1}(Q)$ and $\pi^*$ is the pull back map.

Now let $G^C$ be a linearly reductive complex Lie group. A complex space $Z$ endowed with a holomorphic action of $G^C$ is called a holomorphic $G^C$-space.

**Theorem 1.1.** Let $Z$ be a holomorphic $G^C$-space, where $G^C$ is a linearly reductive complex Lie group.

i. If $Z$ is a Stein space, then the analytic Hilbert quotient $Z//G^C$ exists and is a Stein space.

ii. If $Z//G^C$ exists and is a Stein space, then $Z$ is a Stein space.

**Proof.** Part i. is proven in [He1] and part ii. in [HeMP]. □

**Remark 1.1.**

i. If the analytic Hilbert quotient $\pi : Z \to Z//G^C$ exists, then every fiber $\pi^{-1}(q)$ of $\pi$ contains a unique $G^C$-orbit $E_q$ of minimal dimension. Moreover, $E_q$ is closed and $\pi^{-1}(q) = \{ z \in Z; E_q \subset \overline{G^C \cdot z} \}$. Here $\overline{\cdot}$ denotes the topological closure.

ii. Let $X$ be a subset of $Z$, such that $G^C \cdot X := \bigcup_{g \in G} g \cdot X = Z$ and assume that $Z//G^C$ exists. Then $G^C \cdot X$ is a Stein space if and only if $Z//G^C = \pi(X)$ is a Stein space.
iii. Let $V^c$ be a finite dimensional complex vector space with a holomorphic linear action of $G^c$. Then the algebra $\mathbb{C}[V^c]^{G^c}$ of invariant polynomials is finitely generated (see e.g. [K17]).

In particular, the inclusion $\mathbb{C}[V^c]^{G^c} \hookrightarrow \mathbb{C}[V^c]$ defines an affine variety $V^c//G^c$ and an affine morphism $\pi^c : V^c \rightarrow V^c//G^c$. If we regard $V^c//G^c$ as a complex space, then $\pi^c : V^c \rightarrow V^c//G^c$ gives the analytic Hilbert quotient of $V^c$ (see e.g. [He1]).

**Remark 1.2.** For a non-connected linearly reductive complex group $G$ let $G^0$ denote the connected component of the identity and let $Z$ be a holomorphic $G$-space. The analytic Hilbert quotient $Z//G$ exists if and only if the quotient $Z//G^0$ exists. Moreover, the quotient map $\pi_G : Z \rightarrow Z//G$ induces a map $\pi_{G/G^0} : Z//G^0 \rightarrow Z//G$ which is finite. In fact the diagram

$$
\begin{array}{ccc}
Z \\
\pi_{G^0} \searrow \pi_G \\
Z//G^0 & \rightarrow & Z//G \\
\end{array}
$$

commutes and $\pi_{G/G^0}$ is the quotient map for the induced action of the finite group $G/G^0$ on $Z//G^0$.

2 The geometry of the Minkowski space

Let $\mathbb{K}$ denote either the field $\mathbb{R}$ or $\mathbb{C}$ and $(e_0, \ldots, e_n)$ the standard orthonormal basis for $\mathbb{K}^{1+n}$. The space $\mathbb{R}^{1+n}$ together with the quadratic form $\eta(z) = z_0^2 - z_1^2 - \cdots - z_n^2$, where $z_j$ are the components of $z$, is called the $(1+n)$-dimensional linear Minkowski space. Let $<,>_L$ denote the symmetric non-degenerated bilinear form which corresponds to $\eta$, i.e., $z \cdot w := < z, w >_L = z^t J w$ where $z^t$ denotes the transpose of $z$ and $J = (e_0, -e_1, \ldots, -e_n)$ or equivalently $z \cdot w = < z, J w >_E$ where $<,>_E$ denotes the standard Euclidean product on $\mathbb{R}^{1+n}$, respectively its $\mathbb{C}$-linear extension to $\mathbb{C}^{1+n}$.

Let $O_E(1, n)$ denote the subgroup of $GL_E(1+n)$ which leave $\eta$ fixed, i.e., $O_E(1, n) = \{ g \in GL_E(1+n) ; gz \cdot gw = z \cdot w \text{ for all } z, w \in \mathbb{K}^{1+n} \}$. Note that $SO_E(1, n) = \{ g \in O_E(1, n) ; \det g = 1 \}$ is an open subgroup of $O_E(1, n)$. For $\mathbb{K} = \mathbb{C}$, $SO_C(1, n)$ is connected. But in the real case $SO_R(1, n)$ consists of two connected components ($n \geq 2$). The connected component $SO_R(1, n)^0 = O_E(1, n)^0$ of the identity is called the connected linear Lorentz group. Note that $SO_R(1, n)^0$ is not an algebraic subgroup of $SO_R(1, n)$ but is Zariski dense in $SO_R(1, n)$. We have $\mathbb{K}[\eta] = \mathbb{K}[\mathbb{K}^{1+n}]^{SO_E(1, n)} = \mathbb{K}[\mathbb{K}^{1+n}]^{O_E(1, n)}$.

Now let $\mathbb{C}^{(1+n) \times N} = \mathbb{C}^{1+n} \times \cdots \times \mathbb{C}^{1+n}$ be the $N$-fold product of $\mathbb{C}^{1+n}$, i.e., the space of complex $(1+n) \times N$-matrices. The group $O_C(1, n)$ acts on $\mathbb{C}^{(1+n) \times N}$ by left multiplication. A classical result in Invariant Theory says that $\mathbb{C}[\mathbb{C}^{(1+n) \times N}]^{O_C(1, n)}$ is generated by the polynomials $p_k(z_1, \ldots, z_N) = z_k \cdot z_j$ where $z = (z_1, \ldots, z_N) \in \mathbb{C}^{(1+n) \times N}$.

**Remark 2.1.** The (algebraic) Hilbert quotient $\mathbb{C}^{(1+n) \times N}/O_C(1, n)$ can be identified with the space $\Sym_N(\min \{ 1+n, N \})$ of symmetric $N \times N$-matrices of rank smaller or equal $\min \{ 1+n, N \}$.

With this identification the quotient map $\pi_C : \mathbb{C}^{(1+n) \times N} \rightarrow \mathbb{C}^{(1+n) \times N}/O_C(1, n)$ is given by $\pi_C(Z) = ^t Z J Z$ where $^t Z$ denotes the transpose of $Z$ and $J$ is as above. For the group $SO_C(1, n)$ the situation is slightly more complicated. If $N \geq 1+n$ additional invariants appear, but they are not relevant for our considerations, since the induced map $\mathbb{C}^{(1+n) \times N}/SO_C(1, n) \rightarrow \mathbb{C}^{(1+n) \times N}/O_C(1, n)$ is finite.
There is a well known characterization of closed $O_C(1, n)$-orbits in $\mathbb{C}^{(1+n)\times N}$. In order to formulate this we need more notations. Let $z = (z_1, \ldots, z_N) \in \mathbb{C}^{(1+n)\times N}$ and $L(z) := \mathbb{C}z_1 + \cdots + \mathbb{C}z_N$ be the subspace of $\mathbb{C}^{1+n}$ spanned by $z_1, \ldots, z_N$. The Lorentz product $\langle \cdot, \cdot \rangle_L$ restricted to $L(z)$ is in general degenerated. Thus let $L(z)^0 = \{ w \in L(z); <w, v>_L = 0 \text{ for all } v \in L(z) \}$. It follows that $\dim L(z) / L(z)^0 = \rank(z J z) = \rank \pi_C(z)$. Elementary consideration show the following.

**Lemma 2.2.** The orbit $O_C(1, n) \cdot z$ through $z \in \mathbb{C}^{(1+n)\times N}$ is closed if and only if the orbit $SO_C(1, n) \cdot z$ is closed and this is the case if and only if $L(z)^0 = \{ 0 \}$, i.e., $\dim L(z) = \rank \pi_C(z)$. □

The light cone $N := \{ y \in \mathbb{R}^{1+n}; \eta(y) = 0 \}$ is of codimension one and its complement $\mathbb{R}^{1+n} \setminus N$ consists of three connected components (here of course we assume $n \geq 2$). By the forward cone $C$ we mean the connected component which contains $e_0$. It is easy to see that $C = \{ y \in \mathbb{R}^{1+n}; y \cdot e_0 > 0 \text{ and } \eta(y) > 0 \} = \{ y \in \mathbb{R}^{1+n}; y \cdot x > 0 \text{ for all } x \in N^+ \}$ where $N^+ = \{ x \in N; x \cdot e_0 > 0 \}$. In particular, $C$ is an open convex cone in $\mathbb{R}^{1+n}$. Since $J$ has only one positive Eigenvalue, the following version of the Cauchy-Schwarz inequality holds.

**Lemma 2.2.** If $\eta(y) > 0$, then $\tilde{x} \cdot y \leq 0$ for $\tilde{x} := x - \frac{\eta(y)}{\eta(y)} y$ and all $x \in \mathbb{R}^{1+n}$. In particular

$$\eta(x) \cdot \eta(y) \leq (x \cdot y)^2$$

and equality holds if and only if $x$ and $y$ are linearly dependent. □

The elementary Lemma has several consequences which are used later on. For example,

- if $y_1, y_2 \in C^\pm := C \cup (-C) = \{ y \in \mathbb{R}^{1+n}; \eta(y) > 0 \}$, then $y_1 \cdot y_2 \neq 0$. Moreover,

- if $y_1, y_2 \in N = \{ y \in \mathbb{R}^{1+n}; \eta(y) = 0 \}$, and $y_1 \cdot y_2 = 0$, then $y_1$ and $y_2$ are linearly dependent.

The tube domain $T = \mathbb{R}^{1+n} + iC \subset \mathbb{C}^{1+n}$ over $C$ is called the future tube. Note that $SO_\mathbb{R}(1, n)^0$ acts on $T$ by $g \cdot (x + iy) = gx + igy$ and therefore on the $N$-fold product $T^N = T \times \cdots \times T \subset \mathbb{C}^{(1+n)\times N}$ by matrix multiplication.

**Remark 2.2.** It is easy to show that the $SO_\mathbb{R}(1, n)^0$-action on $C$ and consequently also on $T^N$ is proper. In particular $T^N / SO_\mathbb{R}(1, n)^0$ is a Hausdorff space.

The complexified group $SO_C(1, n)$ does not stabilize $T^N$. The domain

$$SO_C(1, n) \cdot T^N = \bigcup_{g \in SO_C(1, n)} g \cdot T^N$$

is called the extended future tube.

## 3 Orbit connectedness of the future tube

Let $G$ be a Lie group acting on $Z$. A subset $X \subset Z$ is called orbit connected with respect to the $G$-action on $Z$ if $\Sigma(z) = \{ g \in G; g \cdot z \in X \}$ is connected for all $z \in X$.

In this section we prove the following

**Theorem 3.1.** The $N$-fold product $T^N$ of the future tube is orbit connected with respect to the $SO_C(1, n)$-action on $\mathbb{C}^{(1+n)\times N}$.
We first reduce the proof of this Theorem for the SO_C(1, n)-action to the proof of the related statement about the Cartan subgroups of SO_C(1, n). For this we use the results of Bremigan in [13]. For the convenience of the reader we briefly recall those parts, which are relevant for the proof of Theorem 3.1.

Starting with a simply connected complex semisimple Lie group G_C with a given real form G defined by an anti-holomorphic group involution, g \mapsto \bar{g}$, there is a subset S of G_C such that GSG contains an open $G \times G$-invariant dense subset of G_C. The set S is given as follows.

Let $Car(G_C) = \{H_1, \ldots, H_l\}$ be a complete set of representatives of the Cartan subgroups of G_C, which are defined over \( \mathbb{R} \). Associated to each $H \in Car(G_C)$ are the Weyl group $W(H) := N_{G_C}(H)/(H)$, the real Weyl group $W_R(H) := \{gH \in W(H); \bar{g}H = gH\}$ and the totally real Weyl group $W_{\mathbb{R}}(H) := \{gH \in W(R); \bar{g} = g\}$. Here $N_{G_C}(H)$ denotes the normalizer of $H$ in $G_C$.

For $H \in Car(G_C)$ let $R(H)$ be a complete set of representatives of the double coset space $W_{\mathbb{R}}(H) \backslash W_R(H)/W_{\mathbb{R}}(H)$ chosen in such a way that $\bar{e} = e^{-1}$ holds for all $e \in Car(G_C)$. Then $S := \cup H_e$ has the claimed properties.

Although SO_C(1, n) is not simply connected, the results above remain true for $G := SO_R(1, n)^0$ and $G_C := SO_C(1, n)$, as one can see by going over to the universal covering.

**Remark 3.1.** Using the classification of the $SO_R(1, n)^0 \times SO_R(1, n)^0$-orbits in SO_C(1, n) as presented in [17], the same result can be obtained for $G_C = SO_C(1, n)$.

Since $T^N$ is $SO_R(1, n)^0$-stable, $SO_R(1, n)^0$ is connected and $SO_R(1, n)^0 \cdot S \cdot SO_R(1, n)^0$ is dense in $SO_C(1, n)$, Theorem 3.1 follows from

**Proposition 3.1.** The set $\Sigma_S(w) := \{g \in S; g \cdot w \in T^N\}$ is connected for all $w \in T^N$.

In the case $n = 2m - 1$ we may choose $Car(SO_C(1, n)) = \{H_0\}$ where

$$H_0 = \left\{ \begin{pmatrix} \sigma & 0 & \cdots & 0 \\ 0 & \tau_1 & \cdots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \tau_{m-1} \end{pmatrix} ; \sigma \in SO_C(1, 1), \tau_j \in SO_C(2) \right\} \text{ and } R(H_0) = \{\text{Id}\}.$$

In the even case $n = 2m$ we make the choice $Car(SO_C(1, n)) = \{H_1, H_2\}$ where

$$H_1 = \left\{ \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} ; h \in H_0 \right\}, H_2 = \left\{ \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \tau_1 & \cdots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \tau_{m} \end{pmatrix} ; \tau_j \in SO_C(2) \right\},$$

$$R(H_1) = \{\text{Id}\} \text{ and } R(H_2) = \{\text{Id}, \epsilon\} \text{ with } \epsilon = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \text{Id}_{2m-3} \end{pmatrix}.$$

Observe that in the case $H_2$, where $\epsilon$ is present, $S$ is not connected. But the “$\epsilon$-part” of $S$ is not relevant, since any $h \in H_2$ does not change the sign of the first component of the imaginary part of $z_j \in T$ and therefore $\Sigma_{H_2, \epsilon}(z)$ is empty for all $z \in T^N$. Thus it is sufficient to prove the following.

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Proposition 3.2. For every possible $H \in \{H_0, H_1, H_2\}$ and every $w \in T^N$ the set $\Sigma_H(w) = \{h \in H; h \cdot w \in T^N\}$ is connected.

Proof. We will carry out the proof in the case where $n = 2m - 1$ and $H = H_0$. The proof in the other cases is analogous. Note that $H$ splits into its real and imaginary part, i.e., $H = H_R \cdot H_I \cong H_R \times H_I$ where $H_R$ denotes the connected component of the identity of $\text{SO}_R(1,n)^0 \cap H = \{h \in H; h = h\}$ and $H_I = \exp itR$. Thus the $2 \times 2$ blocks appearing for $h \in H_1$ are given by

$$\sigma = \begin{pmatrix} a & ib \\ ib & a \end{pmatrix} \text{ where } a^2 + b^2 = 1 \quad \text{and} \quad \tau_j = \begin{pmatrix} c_j & -id_j \\ id_j & c_j \end{pmatrix} \text{ where } c_j^2 - d_j^2 = 1, c_j > 0.$$

Let $S^1 := \{(x,y) \in \mathbb{R}^2; x^2 + y^2 = 1\}$, $\mathcal{H} := \{(x,y) \in \mathbb{R}^2; x^2 - y^2 = 1 \text{ and } x > 0\}$, identify $H_I$ with $S^1 \times \mathcal{H} \times \cdots \times \mathcal{H} \subset \mathbb{R}^2 \times \cdots \times \mathbb{R}^2 = \mathbb{R}^{2m}$ and let

$$\tilde{\psi} : \mathbb{R}^{2m} \to \mathbb{R}^{(1+n) \times (1+n)}, \tilde{\psi}(a,b,c_1,d_1,\ldots,c_{m-1},d_{m-1}) = \begin{pmatrix} \sigma & 0 & \cdots & 0 \\ 0 & \tau_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & \cdots & 0 & \tau_{m-1} \end{pmatrix}$$

where $\sigma = \begin{pmatrix} a & ib \\ ib & a \end{pmatrix}$ and $\tau_j = \begin{pmatrix} c_j & -id_j \\ id_j & c_j \end{pmatrix}$. The restriction $\psi$ of $\tilde{\psi}$ to $S^1 \times \mathcal{H} \times \cdots \times \mathcal{H}$ is a diffeomorphism onto its image $H_I$.

For every $w_k \in T$, $k = 1,\ldots,N$ we get the linear map $\tilde{\varphi}_k : \mathbb{R}^{2m} \to \mathbb{R}^{1+n}$, $p \mapsto \text{Im}(\tilde{\psi}(p) \cdot w_k)$. Note that

- If $p = (p_1,\ldots,p_m) \in \tilde{\varphi}_k^{-1}(C)$, then $(p_1,\ldots,rp_j,\ldots,p_m) \in \tilde{\varphi}_k^{-1}(C)$ for all $0 < r \leq 1$ and $j = 2,\ldots,m$.
- If $p = (p_1,\ldots,p_m), p_j \in \tilde{\varphi}_k^{-1}(C)$, then $(s \cdot p_1,p_2,\ldots,p_m) \in \tilde{\varphi}_k^{-1}(C)$ for all $s > 1$.

where $p_1 = (a,b), p_j = (c_j,d_j) \in \mathbb{R}^2, j = 2,\ldots,m$.

It remains to show that $\Sigma_{H_1}(w)$ is connected for all $w \in T^N$.

Let $e := ((1,0),(1,0),\ldots,(1,0)) = \psi^{-1}(\text{Id}) \in \psi^{-1}(\Sigma_{H_1}(w))$ and $p = (p_1,\ldots,p_m) := \psi^{-1}(e) \in \psi^{-1}(\Sigma_{H_1}(w))$. From the convexity of $C$ and the linearity of $\varphi_k$ it follows that $q(t) = (q_1(t),\ldots,q_m(t)) = e + t(p - e)$ is contained in $\bigcap_{k=1}^N \varphi_k^{-1}(C)$ for $t \in [0,1]$. Thus

$$\tilde{\gamma}_p(t) := \left( \begin{array}{c} \frac{q_1(t)}{\|q_1(t)\|_E} \\ \frac{q_2(t)}{\sqrt{q_2(t)}} \\ \vdots \\ \frac{q_m(t)}{\sqrt{q_m(t)}} \end{array} \right) \in \psi^{-1}(\Sigma_H(w))$$

for $t \in [0,1]$. Here $\| \cdot \|_E$ denotes the standard Euclidean norm. Thus $\gamma_h(t) := \psi(\tilde{\gamma}_p(t))$ gives a curve which connects $\text{Id}$ with $h$.

Since $\text{SO}_R(1,n)^0$ is a real form of $\text{SO}_C(1,n)$, orbit connectivity implies the following (see [He1]).

Corollary 3.1. Let $Y$ be a complex space with a holomorphic $\text{SO}_C(1,n)$-action. Then every holomorphic $\text{SO}_R(1,n)^0$-equivariant map $\varphi : T^N \to Y$ extends to a holomorphic $\text{SO}_C(1,n)$-equivariant map $\Phi : \text{SO}_C(1,n) \cdot T^N \to Y$.

In the terminology of [He1] Corollary 3.1 means that $\text{SO}_C(1,n) \cdot T^N$ is the universal complexification of the $\text{SO}_R(1,n)^0$-space $T^N$. 

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4 The strictly plurisubharmonic exhaustion of the tube

Let $X, Q, P$ be topological spaces, $q : X \to Q$ and $p : X \to P$ continuous maps. A function $f : X \to \mathbb{R}$ is said to be an exhaustion of $X$ mod $p$ along $q$ if for every compact subset $K$ of $Q$ and $r \in \mathbb{R}$ the set $p(q^{-1}(K) \cap f^{-1}((−\infty, r]))$ is compact.

The characteristic function of the forward cone $C$ is up to a constant given by the function $\hat{\rho} : C \to \mathbb{R}, \hat{\rho}(y) = \eta(y) - \eta(x)$ . It follows from the construction of the characteristic function, that log $\hat{\rho}$ is a $\text{SO}_\mathbb{R}(1, n)^0$-invariant strictly convex function on $C$ (see [FK] for details).

In particular

$$\rho : T^N \to \mathbb{R}, \quad (x_1 + iy_1, \ldots, x_N + iy_N) \mapsto \frac{1}{\eta(y_1)} + \cdots + \frac{1}{\eta(y_N)}$$

is a $\text{SO}_\mathbb{R}(1, n)^0$-invariant strictly plurisubharmonic function on $T^N$. Of course this may also be checked by direct computation.

Let $\pi_C : \mathbb{C}^{(1+n)\times N} \to \mathbb{C}^{(1+n)\times N // \text{SO}_\mathbb{C}(1, n)}$ be the analytic Hilbert quotient and $\pi_R : T^N \to T^N / \text{SO}_\mathbb{R}(1, n)^0$ the quotient by the $\text{SO}_\mathbb{R}(1, n)^0$-action. In the following we always write $z = x + iy$, i.e., $z_j = x_j + iy_j$ where $x_j$ denote the real and $y_j$ the imaginary part of $z_j$. For example $z_j \cdot z_k = x_j \cdot x_k - y_j \cdot y_k + i(x_j \cdot y_k - x_k \cdot y_j)$.

The main result of this section is the following

**Theorem 4.1.** The function $\rho : T^N \to \mathbb{R}$, is an exhaustion of $T^N$ mod $\pi_R$ along $\pi_C$.

We do the case of one copy first.

**Lemma 4.1.** Let $D_1 \subset T$ and assume that $\pi_C(D_1) \subset \mathbb{C}$ is bounded. Then $\{(x \cdot y, \eta(x), \eta(y)) \in \mathbb{R}^3; z = x + iy \in D_1\}$ is bounded.

**Proof.** The condition on $D_1$ means, that there is a $M \geq 0$ such that

$$|\eta(x) - \eta(y)| \leq M \quad \text{and} \quad |x \cdot y| \leq M$$

for all $z = x + iy \in D_1$. Since $\eta(x)\eta(y) \leq (x \cdot y)^2$ and $\eta(y) \geq 0$, this implies that $\{(x \cdot y, \eta(x), \eta(y)) \in \mathbb{R}^3; z \in D_1\}$ is bounded. \hfill \Box

**Lemma 4.2.** Let $D_2 \subset T \times T$ be such that $\pi_C(D_2)$ is bounded. Then $\{(\eta(x_1), \eta(y_1), \eta(x_2), \eta(y_2), x_1 \cdot x_2, y_1 \cdot y_2) \in \mathbb{R}^6; (z_1, z_2) \in D_2\}$ is bounded.

**Proof.** Lemma 4.1 implies that there is a $M_1 \geq 0$ such that $|\eta(x_j)| \leq M_1$, $|\eta(y_j)| \leq M_1$ and $|x_j \cdot y_j| \leq M_1$, $j = 1, 2$, for all $(z_1, z_2) \in D_2$. Now $\eta(z_1 + z_2) = \eta(z_1) + \eta(z_2) + 2 \cdot z_1 \cdot z_2$ shows that $\{(\eta(z_1 + z_2) \in \mathbb{R}; (z_1, z_2) \in D_2\}$ is bounded. But $z_1 + z_2 \in T$, thus Lemma 4.1 implies $|\eta(x_1 + x_2)| \leq M_2$ and $|\eta(y_1 + y_2)| \leq M_2$ for some $M_2 \geq 0$ and all $(z_1, z_2) \in D_2$. This gives

$$|x_1 \cdot x_2| \leq \frac{3}{2} \max \{M_1, M_2\} \quad \text{and} \quad |y_1 \cdot y_2| \leq \frac{3}{2} \max \{M_1, M_2\}.$$ 

\hfill \Box

**Remark 4.1.** Based on the following we only need, that the set $\{(\eta(y_1), \eta(y_2), y_1 \cdot y_2) \in \mathbb{R}^3; (z_1, z_2) \in D_2\}$ is bounded. We apply this to points $y_j + iy_1$ where $\pi_C(y_j + iy_1) = \eta(y_j) - \eta(y_1) + 2iy_1 \cdot y_1$.

**Remark 4.2.** For every subset $X$ of $T$, we have

$$X \subset \text{SO}_\mathbb{R}(1, n)^0 \cdot (X \cap (\mathbb{R}^{1+n} + i(\mathbb{R}^{>0} \cdot e_0))),$$

where $\mathbb{R}^{>0} \cdot e_0 = \{te_0; t > 0\} \subset \mathbb{R}^{1+n}$.
Lemma 4.3. For every compact sets $B \subset C$ and $K \subset \mathbb{C}$ the set
\[ M(B, K) := \{ x \in \mathbb{R}^{1+n}; \pi_C(x + iy) \in K \text{ for some } y \in B \} \]
is compact.

Proof. Since $B$ and $K$ are compact, $M(B, K)$ is closed. We have to show that it is bounded. First note that $B_1 \subset B_2$ implies $M(B_1, K) \subset M(B_2, K)$. Using the properness of the $\text{SO}_R(1, n)^0$-action on $C$, we see, that there is an interval $I = \{ t \cdot e_0; a \leq t \leq b \}$, $a > 0$ in $\mathbb{R} \cdot e_0$ and a compact subset $N$ in $\text{SO}_R(1, n)^0$, such that $N \cdot I := \bigcup_{g \in N} g \cdot I \supset B$. Thus $M(B, K) \subset M(N \cdot I, K) = N \cdot M(I, K) := \bigcup_{g \in N} g \cdot M(I, K)$.

It remains to show that $M(I, K)$ is bounded. For $x \in M(I, K)$, $x = \begin{pmatrix} x_0 \\ \vdots \\ x_n \end{pmatrix}$, there exists a $M_1 \geq 0$ such that $|x \bullet (y_0 \cdot e_0)| = |x_0 \cdot y_0| \leq M_1$ for all $y_0 \cdot e_0 \in I$. Since $a \leq y_0 \leq b$ and $a > 0$, this implies $|x_0^2| \leq \frac{M_1^2}{a^2} \leq \frac{M_2^2}{a^2}$. There also exists a $M_2 \geq 0$ such that $|\eta(x)| = |x_0^2 - x_1^2 - \cdots - x_n^2| \leq M_2$, so we get $x_1^2 + \cdots + x_n^2 \leq \frac{M_2^2}{a^2} + M_2$.

Corollary 4.1. For every $r > 0$ the set $M(B, K) \cap \{ y \in \mathbb{R}^{1+n}; r \leq \eta(y) \}$ is compact.

Proof of Theorem 4.1. Using Remark 4.2, it is sufficient to prove that the set
\[ S := (\pi_C^{-1}(K) \cap \{ \rho \leq r \}) \cap ((\mathbb{R}^{1+n} + i(\mathbb{R}^{>0} \cdot e_0)) \times T^{N-1}) \]
is compact. For $z = (z_1, \ldots, z_N) \in S$ let $z_j = x_j + iy_j$, where $x_j$ denotes the real part and $y_j$ the imaginary part of $z_j$. By the definition of $S$ we have $y_1 = y_{10} \cdot e_0$ where $y_{10} = y_1 \cdot e_0$. Moreover, we get $\frac{1}{r} \leq \eta(y_1) = (y_{10})^2 \leq M$. Therefore the set $\{ y_1 \in \mathbb{R}^{1+n}; (z_1, \ldots, z_N) \in S \} = \{ t \cdot e_0; t^2 \in \left[ \frac{1}{r}, M \right], t > 0 \}$ is compact.

By Remark 4.1 we get that the sets $\{ (\eta(y_1), \eta(y_j), y_1 \bullet y_j) \in \mathbb{R}^3; (z_1, \ldots, z_N) \in S \}$ are bounded for $j = 2, \ldots, N$. Therefore we get the boundedness of $\{ \pi_C(y_j + iy_1) \in \mathbb{C} ; (z_1, \ldots, z_N) \in S \}$. Thus the $y_1, y_2, \ldots, y_N$ are lying in the sets $M(I, B_j) \cap \{ y \in \mathbb{R}^{1+n}; r \leq \eta(y) \}$, where $I := \{ t \cdot e_0; t^2 \in \left[ \frac{1}{r}, M \right], t > 0 \}$ and $B_j$ are compact subsets of $\mathbb{C}$, containing $\{ \pi_C(y_j + iy_1) \in \mathbb{C} ; (z_1, \ldots, z_N) \in S \}$. By Corollary 4.1 these sets are compact, which implies that the set $\{ (y_1, \ldots, y_N) \in (\mathbb{R}^{1+n})^N; (z_1, \ldots, z_N) \in S \}$ is compact. Hence using Lemma 4.3 it follows that $\{ (x_1, \ldots, x_N) \in (\mathbb{R}^{1+n})^N; (z_1, \ldots, z_N) \in S \}$ is bounded. Thus $S$ is bounded and therefore compact.

5 Saturation of the extended future tube

We call $A \subset X$ saturated with respect to a map $\rho : X \to Y$ if $A$ is the inverse image of a subset $Y$.

Let $\pi_C : C^{(1+n) \times N} \to C^{(1+n) \times N} \sslash \text{SO}_C(1, n)$ be the analytic Hilbert quotient, which is given by the algebra of $\text{SO}_C(1, n)$-invariant polynomials functions on $C^{(1+n) \times N}$ (see section 4) and let $U_r$ denote the set $\{ z \in T^N; \rho(z) < r \}$ for some $r \in \mathbb{R} \cup \{ +\infty \}$, where $\rho$ is the strictly plurisubharmonic exhaustion function, which we defined in section 4.

Theorem 5.1. The set $\text{SO}_C(1, n) \cdot U_r = \text{SO}_C(1, n) \cdot \{ z \in T^N; \rho(z) < r \}$ is saturated with respect to $\pi_C$.

It is well known, that each fiber of $\pi_C$ contains exactly one closed orbit of $\text{SO}_C(1, n)$ (see section 4). Moreover, every orbit contains a closed orbit in its closure. Therefore it is sufficient to prove
Proposition 5.1. If \( z \in U_r \) and \( \text{SO}_C(1,n) \cdot u \) is the closed orbit in \( \text{SO}_C(1,n) \cdot z \), then \( \text{SO}_C(1,n) \cdot u \cap U_r \neq \emptyset \).

The idea of proof is to construct a one-parameter group \( \gamma \) of \( \text{SO}_C(1,n) \), such that \( \gamma(t)z \in U_r \) for \( |t| \leq 1 \) and \( \lim_{t\to0} \gamma(t)z \in \text{SO}_C(1,n) \cdot u \).

In the following, let \( z = (z_1, \ldots, z_N) \in U_r \) and denote by \( L(z) = Cz_1 + \cdots + Cz_N \) the \( C \)-linear subspace of \( C^{1+n} \) spanned by \( z_1, \ldots, z_N \). The subspace of isotropic vectors in \( L(z) \) with respect to the Lorentz product is denoted by \( L(z)^0 \), i.e., \( L(z)^0 = \{ w \in L(z); w \cdot v = 0 \text{ for all } v \in L(z) \} \). Let \( L(z)^0 \) be its conjugate, i.e., \( L(z)^0 = \{ \overline{v}; v \in L(z)^0 \} \).

Lemma 5.1. For all \( \omega \neq 0 \), \( \omega \in L(z)^0 \) we have \( \eta(\text{Im}(\omega)) < 0 \).

Proof. Let \( \omega = \omega_1 + i\omega_2 \) with \( \omega_1 = \text{Re}(\omega), \omega_2 = \text{Im}(\omega) \). Assume that \( \eta(\text{Im}(\omega)) = \eta(\omega_2) \geq 0 \). Since \( \omega \in L(z)^0 \), we have \( 0 = \eta(\omega) = \eta(\omega_1) = \eta(\omega_2) + 2i\omega_1 \cdot \omega_2 \).

If \( \eta(\omega_2) > 0 \), i.e., \( \omega_2 \in C \) or \( \omega_2 \in -C \), then \( \omega_1 \cdot \omega_2 = 0 \) contradicts \( \eta(\omega_1) = \eta(\omega_2) > 0 \). Thus assume \( \eta(\omega_1) = \eta(\omega_2) = 0 \) and \( \omega_1 \cdot \omega_2 = 0 \). Hence \( \omega_1 \) and \( \omega_2 \) are \( \mathbb{R} \)-linearly dependent and therefore there is a \( \lambda \in \mathbb{C}, \omega_3 \in \mathbb{R}^{1+n} \) such that \( \omega = \lambda \omega_3 \) and \( \omega_3 \cdot e_0 \geq 0 \).

We have \( \eta(\omega_3) = 0 \) and, since \( \omega_3 \in L(z)^0, e_0 \cdot \omega_3 \geq 0 \) and \( z_1 \in T \), we also have \( 0 = \omega_3 \cdot \text{Im}(z_1) \). This implies by the definition of \( C \) that \( \omega_3 = 0 \).

Corollary 5.1. For \( \omega \in L(z)^0, \omega \neq 0 \), we have \( \omega \cdot \overline{\omega} < 0 \). In particular, \( L(z)^0 \cap \overline{L(z)^0} = \{0\} \) and the complex Lorentz product is non-degenerate on \( L(z)^0 \oplus \overline{L(z)^0} \).

Corollary 5.2. Let \( W := (L(z) \oplus \overline{L(z)})^\perp := \{ v \in C^{1+n}; v \cdot u = 0 \text{ for all } u \in L(z)^0 \oplus \overline{L(z)^0} \}. \) Then \( L(z) = L(z)^0 \oplus (L(z) \cap W) \).

Proof of Proposition 5.1. Let \( z \in U_r \). We use the notation of Corollary 5.2. Define

\[
\gamma : \mathbb{C}^* \to \text{SO}_C(1,n) \quad \text{by} \quad \gamma(t) = \begin{cases} \left\{ \begin{array}{l} tv \\ t^{-1}v \\ v \end{array} \right. & \text{for } v \in L(z)^0 \\ & \text{for } v \in \overline{L(z)^0} \end{cases}.
\]

Every component \( z_j \) of \( z \) is of the form \( z_j = u_j + \omega_j \) where \( u_j \in W \) and \( \omega_j \in L(z)^0 \) are uniquely determined by \( z_j \). Recall that \( W \) is the set \( \{ v \in C^{1+n}; v \cdot u = 0 \text{ for all } u \in L(z)^0 \oplus \overline{L(z)^0} \} \). Since \( \lim_{t\to0} \gamma(t)z_j = u_j \) and \( L(u)^0 = \{0\} \) for \( u = (u_1, \ldots, u_N) \), \( u \) lies in the unique closed orbit in \( \text{SO}_C(1,n) \cdot z \) (see Lemma 5.1). It remains to show that \( u \in U_r \). For every \( t \in \mathbb{C} \) we have

\[
\eta(\text{Im}(u_j + t\omega_j)) = \eta(\text{Im}(u_j)) + |t|^2 \eta(\text{Im}(\omega_j)).
\]

Since \( \eta(\text{Im}(u_j + \omega_j)) > 0 \) and \( \eta(\text{Im}(\omega_j)) \leq 0 \), this implies \( \eta(\text{Im}(u_j + t\omega_j)) \in C^\pm \) for all \( t \in [0,1] \). Moreover, \( \eta(\text{Im}(z_j)) < \eta(\text{Im}(u_j)) \), for every \( j \). Thus \( \rho(z) > \rho(u) \) and therefore \( u \in U_r \).

Corollary 5.3. The extended future tube is saturated with respect to \( \pi_C \).

Remark 5.1. The function \( f : \mathbb{R} \to \mathbb{R}, t \mapsto \eta(\text{Im}(u_j + t\omega_j)) \), is strictly concave if \( \omega_j \neq 0 \). The proof shows \( u_j + t\omega_j \in T \) for all \( t \in \mathbb{R} \).
The Kählerian reduction of the extended future tube

If one is only interested in the statement of the future tube conjecture, one can simply apply the main result in [He2] (Theorem 1 in §2). Our goal here is to show that much more is true.

For \( z \in \mathbb{C}^{(1+n)\times N} \) let \( x = \frac{1}{2}(z + \bar{z}) \) be the real and \( y = \frac{1}{2i}(z - \bar{z}) \) the imaginary part of \( z \), i.e., \( z = (z_1, \ldots, z_N) = (x_1, \ldots, x_N) + i(y_1, \ldots, y_N) \) in the obvious sense. The strictly plurisubharmonic function \( \rho : T^N \to \mathbb{R} \), \( \rho(z) = \frac{1}{\eta(y_1)} + \cdots + \frac{1}{\eta(y_N)} \) defines for every \( \xi \in \mathfrak{so}(1, n) = \mathfrak{o}(1, n) \) the function

\[
\mu_\xi(z) = d\rho(z)(i\xi z) = \frac{d}{dt}
\bigg|_{t=0} \rho(\exp it \cdot \xi \cdot z).
\]

Here of course \( \mathfrak{so}(1, n) = \mathfrak{o}(1, n) \) denotes the Lie algebra of \( \mathbb{O}_+(1, n) \). The real group \( \text{SO}_R(1, n)^0 \) acts by conjugation on \( \mathfrak{so}(1, n) \) and therefore by duality on the dual vector space \( \mathfrak{so}(1, n)^* \). It is easy to check that the map \( \xi \to \mu_\xi \) depends linearly on \( \xi \). Thus

\[
\mu : T^N \to \mathfrak{so}(1, n)^*, \quad \mu(z)(\xi) := \mu_\xi(z),
\]

is a well defined \( \mathbb{SO}_R(1, n)^0 \)-equivariant map. In fact \( \mu \) is a moment map with respect to the Kähler form \( \omega = 2i\partial\bar{\partial}\rho \).

In order to emphasize the general ideas, we set \( G := \text{SO}_R(1, n)^0 \), \( G^C := \text{SO}_C(1, n) \), \( X := T^N \) and \( Z := G^C \cdot X \). The corresponding analytic Hilbert quotient, induced by \( \pi_C : \mathbb{C}^{(1+n)\times N} \to \mathbb{C}^{(1+n)\times N}/\text{SO}_C(1, n) \) are denoted by \( \pi_X : X \to X/G, \pi_Z : Z \to Z/G^C \). Note that, by what we proved, we have \( X/G = Z/G^C \).

**Proposition 6.1.**

i. For every \( q \in Z//G^C \) we have \( (\pi_C)^{-1}(q) \cap \mu^{-1}(0) = G \cdot x_0 \) for some \( x_0 \in \mu^{-1}(0) \) and \( G^C \cdot x_0 \) is a closed orbit in \( Z \).

ii. The inclusion \( \mu^{-1}(0) \hookrightarrow X \subset Z \) induces a homeomorphism \( \mu^{-1}(0)/G \to \tilde{Z}//G^C \).

**Proof.** A simple calculation shows that the set of critical points of \( \rho|G^C \cdot x \cap X \), i.e., \( \mu^{-1}(0) \cap G^C \cdot x \), consists of a discrete set of \( G \)-orbits. Moreover, every critical point is a local minimum (see [He2], Proof of Lemma 2 in §2).

On the other hand Remark 5.1 of section 6 says that if \( \rho|G^C \cdot x \cap X \) has a local minimum in \( x_0 \in G^C \cdot x \cap X \), then \( G^C \cdot x_0 = G^C \cdot x \) is necessarily closed in \( Z \). Moreover, \( \rho|G^C \cdot x \cap X \) is then an exhaustion and therefore \( \mu^{-1}(0) \cap (G^C \cdot x_0 \cap X) = G \cdot x_0 \) (see [He2], Lemma 2 in §2). This proves the first part.

The statement i. implies that \( \iota : \mu^{-1}(0) \hookrightarrow X \subset Z \) induces a bijective continuous map \( \tilde{\iota} : \mu^{-1}(0)/G \to \tilde{Z}//G^C \). Since the \( G \)-action on \( X \) is proper and \( \mu^{-1}(0) \) is closed, the action on \( \mu^{-1}(0) \) is proper. In particular \( \mu^{-1}(0)/G \) is a Hausdorff topological space.

Theorem 5.1 implies that \( \tilde{\iota} \) is a homeomorphism, since for every sequence \( q_\alpha \to q_0 \) in \( Z//G^C \) we find a sequence \( (x_\alpha) \) such that \( x_\alpha \) are contained in a compact subset of \( \mu^{-1}(0) \) and \( \pi_C(x_\alpha) = q_\alpha \). Thus every convergent subsequence of \( (x_\alpha) \) has a limit point in \( G \cdot x_0 \) where \( \pi_C(x_0) = q_0 \).

**Proposition 6.2.** The restriction \( \rho|\mu^{-1}(0) : \mu^{-1}(0) \to \mathbb{R} \) induces a strictly plurisubharmonic continuous exhaustion \( \bar{\rho} : Z//G^C \to \mathbb{R} \).

**Proof.** The exhaustion property for \( \bar{\rho} \) follows from Theorem 6.1. The argument that \( \bar{\rho} \) is strictly plurisubharmonic is the same as in [HeHuL].
Theorem 6.1. The extended future tube $Z$ is a domain of holomorphy.

Proof. Proposition 6.2 implies that $Z/G$ is a Stein space (see [N] Theorem II). Hence $Z$ is a Stein space. □

In fact, much more has been proved here. We would like to comment on this. By definition, an analytic subset of a complex manifold is closed. For the following recall that orbit-connectedness is a condition on the $G_C$-orbits.

Proposition 6.3. Every analytic $G$-invariant subset $A$ of $X$ is orbit connected in $Z$ and $G_C \cdot A$ is an analytic subset of $Z$. In particular, $G_C \cdot A$ is a Stein space. Moreover the restriction maps

$$O(Z)^{G_C} \to O(G_C \cdot A)^{G_C} \to O(A)^G$$

are surjective.

Proof. If $b \in G_C \cdot A \cap X$, then $b = g \cdot a$ for some $g \in G_C$ and $a \in A$. Hence $g \in \Sigma_{G_C}(a) = \{ g \in G_C; g \cdot a \in X \}$. The identity principle for holomorphic functions shows that $\Sigma_{G_C}(a) \cdot a \in A$. Thus $b \in A$. This shows $G_C \cdot A \cap X = A$. But $\{ g \cdot X; g \in G_C \}$ is an open covering of $X$ such that $G_C \cdot A \cap g \cdot X = g \cdot A$. This shows that $G_C \cdot A$ is an analytic subset of $Z$. In particular, it is a Stein space. The last statement follows from orbit connectedness (see [He1]). □

Proposition 6.4. For every $G$-invariant analytic subset $A$, its saturation $\hat{A} = \pi^{-1}_X(\pi_X(A))$ is an analytic subset of $X$. Moreover, $\hat{A}/G$ is canonically isomorphic to $A/G$ and $\pi_{\hat{A}} : \hat{A} \to \hat{A}/G \subset X/G$ is the Hilbert quotient of $\hat{A}$ whose restriction to $A$ gives the analytic Hilbert quotient of $A$.

Proof. We already know that $A^c = G_C \cdot A$ is an analytic subset of $Z$. Its saturation $\hat{A}^c = \pi^{-1}_Z(\pi_Z(A^c)) = \pi^{-1}_Z(\pi_Z(A))$ is an analytic subset of $Z$ and it is easily checked that $\hat{A} = \hat{A}^c \cap X = \pi^{-1}_X(\pi_X(A))$ has the desired properties. □

References

[B] R. Bremigan, Invariant analytic domains in complex semisimple groups, Transformation Groups 1 (1996), 279–305

[FK] J. Faraut, A. Koranyi, Analysis on Symmetric Cones, Oxford Press, Oxford 1994

[HW] D. Hall, A. D. Wightman, A theorem on invariant analytic functions with applications to relativistic quantum field theory, Kgl. Danske Videnskaps. Selkap, Mat.-Fys. Medd 31 (1965) 1–14

[He1] P. Heinzner, Geometric invariant theory on Stein spaces, Math. Ann. 289 No. 4 (1991), 631–662

[He2] P. Heinzner, The minimum principle from a Hamiltonian point of view, Doc. Math. J. 3 (1998), 1–14

[HeHuL] P. Heinzner, A. T. Huckleberry, F. Loose, Kählerian extensions of the symplectic reduction, J. reine angew. Math. 455 (1994), 123–140

[HeMP] P. Heinzner, L. Migliorini, M. Polito, Semistable quotients, Ann. Scuola Norm. Sup. Pisa 26 (1998), 233–248
[J] **R. Jost**, The general theory of quantized fields, In: Lectures in applied mathematics vol. IV, 1965

[Kr] **H. Kraft**, Geometrische Methoden in der Invariantentheorie, In: Aspects of Mathematics, Vieweg Verlag 1984

[N] **R. Narasimhan**, The Levi Problem for Complex Spaces II, *Math. Ann.* **146** (1962), 195–216

[SV] **A. G. Sergeev, V. S. Vladimirov**, Complex analysis in the future tube, In: Encyclopaedia of mathematical sciences (Several complex variables II) vol. 8 (1994), 179–253

[StW] **R. F. Streater, A. S. Wightman**, PCT spin statistics, and all that, W. A. Benjamin, INC. 1964

[W] **A. S. Wightman**, Quantum field theory and analytic functions of several complex variables, *J. Indian Math. Soc.* **24** (1960), 625–677

[Z] **X. Y. Zhou**, A proof of the extended future tube conjecture, *Izv. Math.* **62** (1998), 201–213

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