Neutral Kaons without Weisskopf–Wigner Approximation

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Abstract

The model-independent formalism is constructed to describe decays of mixed particles without using the Weisskopf-Wigner approximation. Limitations due to various symmetries are traced for neutral $K^-$ mesons. As an application we show that effects of $CPT$-violation and going beyond WWA may be separated and studied independently.

1 Introduction

Neutral kaons present the best-known example of rather rare phenomenon, quantum interference of massive particles. The corresponding interference effects allowed to do unique measurements. We mean, first of all, discovery and investigation of the $CP$-violation.

New data on neutral kaons, with higher statistics and better precision, should appear in the near future. The main goals of their using seem to be searches for the $CPT$-violation and the so called direct $CP$-violation (see, e.g.,[1–3]). To reach those goals one needs to measure very small quantities (present experimental bounds for the direct $CP$-violation see in [4]; estimates for possible $CPT$-violation may be found, e.g., in [5]). Such tiny effects may be influenced by various approximations (or going beyond them). In particular, corrections to the conventional Weisskopf-Wigner approximation (WWA) [6,7] might imitate some properties of neutral kaons which are
usually related to CPT-violation [8] (compare to the final state interaction
imitating T-violation in decays).

Deviations from WWA (i.e., in essence, from the exponential t-dependence)
could be interesting by themselves. Remember here, that the familiar exponen-
tial behavior of decaying unstable systems can be realized in quantum
theory only approximately. It should inevitably be violated for very small
or very large times [9]. Related theoretical problems have been repeatedly
discussed in literature (see, e.g., recent papers [10,11] and references therein
to earlier works). But experiments came to success [12] only for small times
(the quantum Zeno effect), effects at large times stay unobserved. Neutral
kaons, which decays produce oscillations sensitive to various small effects,
could appear useful here as well.

The present paper studies evolution of neutral kaons without using WWA.
The problem has been investigated in [13,14], and later in [10]. But the
approach of Refs. [13,14] is saturated by mathematics, that screens the
underlying physical meaning. Ref.[10] uses a particular toy model. Here we
assume only the many-channel character of the problem and existence of the
Hamiltonian. The necessary formalism is first described for decays of usual
one-particle states (Sections 2,3). Then it is generalized to decays of mixed
states (Section 4). Section 5 concerns with specific features of \( K_0 \) and \( \bar{K}_0 \),
especially with various possible symmetries. As application, we consider in
Section 6 the possible effects of CPT-violation and going beyond WWA for
neutral kaon decays.

\section{Evolution and survival of states}

The Schrödinger equation implies that the wave function evolves in time as

\[ \Psi(t) = e^{-iHt}\Psi(0), \]  

where \( H \) is the total Hamiltonian of the system, \( \Psi(0) \) is its wave function at
t = 0. Useful for description of decays is the survival amplitude [11,15] equal
to

\[ A(t) = \langle \Psi(0) | \Psi(t) \rangle = \langle \Psi(0) | e^{-iHt} | \Psi(0) \rangle. \]  

If we are interested in the evolution only for \( t > 0 \), we can rewrite it as

\[ e^{-iHt} \cdot \Theta(t) = \frac{i}{2\pi} \int_{-\infty}^{\infty} dE e^{-iEt} \frac{1}{E - H + i\epsilon}. \]
\begin{equation}
A(t) = \frac{i}{2\pi} \int_{-\infty}^{\infty} dE e^{-iEt} \langle \Psi^{(0)} | \frac{1}{E - H + i\epsilon} | \Psi^{(0)} \rangle. \tag{3}
\end{equation}

A stationary state having
\[ H \Psi^{(0)} = E^{(0)} \Psi^{(0)} \]
gives to \( A(t) \) the simple time dependence of \( \exp(-iE^{(0)}t) \) with real \( E^{(0)} \) and constant absolute value. Exponential decay would require complex-valued \( E^{(0)} \). It cannot be an eigen-value of any hermitian Hamiltonian, but the corresponding state can be defined in a special sense [11]. However, one is unable to present such a state as a combination of usual physical states. Therefore, we prefer another approach.

Note, first of all, that description of unstable particles uses, as a rule, several kinds of interaction. These may be interactions of quite different nature (e.g., weak decays of hadrons), or various manifestations of the same interactions (as for light emission by the excited atom). Hence, for simplicity we begin with a system of two coupled channels having interactions both inside and between the channels.

The wave function is now a two-component column containing wave functions of those two channels; Hamiltonian is a 2 \( \times \) 2 matrix
\[ H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}. \tag{4} \]

Hermitian character of \( H \) means
\[ H_{ij}^\dagger = H_{ji}, \quad H_{22}^\dagger = H_{22}, \quad H_{12}^\dagger = H_{21}. \tag{5} \]

We use here the symbol of hermitian, not complex, conjugation because of the operator nature of the matrix elements \( H_{jk} \). Each of the indices, 1 and/or 2, may be considered actually as related to a set of channels. Expressions (1)–(3) stay applicable to the two-component wave function \( \Psi(t) \) if one takes in them the matrix (4) for \( H \).

Consider now a more special case where \( \Psi^{(0)} \) has only one nonvanishing component \( \Psi^{(0)}_1 \) corresponding to channel(s) 1. Then we can simplify the survival amplitude. Indeed, only the operator
\[ R(E) = \left( \frac{1}{E - H} \right)_{11} \tag{6} \]
without additional matrix structure is essential here. From the matrix identity
\[ (E - H) \cdot \left( \frac{1}{E - H} \right) = 1 \]  
(7)
we obtain:
\[ R(E) = \frac{1}{E - \tilde{H}(E)}, \]  
(8)
where
\[ \tilde{H}(E) = H_{11} + H_{12} \cdot \frac{1}{E - H_{22}} \cdot H_{21}. \]  
(9)

Expression (8) exactly corresponds to expression (3) for the pure one-channel problem. So \( \tilde{H} \) appears as an effective one-channel Hamiltonian. But the influence of other channels generates two important effects.

1) Unlike the true Hamiltonian \( H \), the effective Hamiltonian \( \tilde{H} \) depends on \( E \), thus revealing time ”nonlocality” of interaction in one channel because of transition to other channel(s) and return. In fact, we can obtain \( A(t) \) by using an effective one-component wave function \( \Psi_1(t) \) such that
\[ A(t) = \langle \Psi_1^{(0)} \mid \Psi_1(t) \rangle, \]
\[ \Psi_1(t) \cdot \Theta(t) = \frac{i}{2\pi} \int_{-\infty}^{\infty} dE e^{-iEt} R(E + i\epsilon) \Psi_1^{(0)}. \]  
(10)

The so defined function \( \Psi_1(t) \) is the first component of the two-component wave function \( \Psi(t) \) if
\[ \Psi(0) = \begin{pmatrix} \Psi_1^{(0)} \\ 0 \end{pmatrix}. \]

It satisfies the integro-differential equation
\[ i \frac{d}{dt} \Psi_1(t) = \int_0^t dt' H_1(t - t') \Psi_1(t'), \]  
(11)
where the lower limit appears because we study the evolution only for \( t > 0 \), the upper limit expresses causality. Here
\[ H_1(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dE e^{-iEt} \tilde{H}(E + i\epsilon). \]  
(12)
Equations (11),(12) demonstrate nonlocal character of time evolution of \( \Psi_1(t) \). Were \( \tilde{H} \) independent of \( E \), equation (11) would transform into the familiar Schrödinger equation, local in time.

2) Even if the total Hamiltonian \( H \) is hermitian, the effective Hamiltonian \( \tilde{H} \) may be nonhermitian. Due to relations (5), the definition (9) implies

\[
\tilde{H}(E) = \tilde{H}(E^*),
\]

(13)

Hence, \( \tilde{H} \) becomes nonhermitian both for complex values of \( E \) and for real values lying at the spectrum, point-like or continuous, of the operator \( H_{22} \).

The above presentation shows that expressions (8),(9) are quite exact. However, they coincide with formal summation of perturbation series (i.e. expansion in \( H_{12} \) and \( H_{21} \)). In particular, they allow to obtain the standard perturbative expressions [16] for energy eigen-values. It is interesting to note that due to the coupling of channels one can use the same expression (9) as the basis to find energies of states which, in the decoupling limit, would refer to both channel 1 and channel 2.

Consider briefly eigen-states of the total Hamiltonian. Each of them has the corresponding two-component wave function solving the familiar eigen-value equation

\[
H \Psi^{(n)} = E^{(n)} \Psi^{(n)},
\]

(14)

\( H \) being the total Hamiltonian. One can easily check that its channel 1 component \( \Psi_1^{(n)} \), by itself, solves an unusual eigen-value equation:

\[
\tilde{H}(E^{(n)}) \Psi_1^{(n)} = E^{(n)} \Psi_1^{(n)},
\]

(15)

with the same value \( E^{(n)} \) as in Eq.(14). \( E^{(n)} \) is, surely, real because of hermiticity of \( H \). The fact that the set of \( \Psi^{(n)} \) is complete and orthonormalized implies completeness of the set of \( \Psi_1^{(n)} \). Note, however, that \( \Psi_1^{(n)} \) are not orthonormalized. Equation (15) gives one more illustration of \( \tilde{H}(E) \) as an effective Hamiltonian in channel 1.

3 Decays of a separate state

Initial state \( \Psi_1^{(0)} \) has not been fixed till now. Let us take it to be an eigen-state of the operator \( H_{11} \) having energy \( E_1^{(0)} \) and wave function \( \psi_1^{(0)} \):

\[
H_{11} \psi_1^{(0)} = E_1^{(0)} \psi_1^{(0)}.
\]

(16)
Such a state would be stationary if transitions to channel(s) 2 were absent. Due to the transitions it will spread. According to the previous Section the survival amplitude of that state is related to the matrix element

$$a(E) = \langle \psi_1^{(0)} | \frac{1}{E - \tilde{H}(E)} | \psi_1^{(0)} \rangle. \quad (17)$$

It can be presented as

$$a(E) = \frac{1}{E - h(E)}. \quad (18)$$

$h(E)$ is a numerical function defined, in essence, by Eqs.(17),(18). It can be written also in the form

$$h(E) = E_1^{(0)} + \frac{1}{a(E)} \langle \psi_1^{(0)} | H_{12}(E - H_{22})^{-1} H_{21}(E - \tilde{H})^{-1} | \psi_1^{(0)} \rangle \quad (19)$$

convenient for perturbative expansion.

The function $h(E)$ has the same meaning for the state $\psi_1^{(0)}$ as $\tilde{H}(E)$ for the whole channel 1. So it can be considered as an effective Hamiltonian of the state $\psi_1^{(0)}$, again nonlocal in time because of $E$-dependence. Applying complex conjugation to Eqs.(17),(18) we obtain relation

$$h^*(E) = h(E^*) \quad (20)$$

directly analogous to (13). Similar to $\tilde{H}(E)$, the function $h(E)$ is complex even at real values of $E$ if they correspond to the spectrum (point-like or continuous) of $H_{22}$.

Solution of the equation

$$E_1 - h(E_1) = 0 \quad (21)$$

produces the pole of $a(E)$ at $E = E_1$. If the solution is real, it gives the eigen-value of the total Hamiltonian taking into account interaction between channels 1 and 2. If $H_{12}$ and $H_{21} = H_{12}^\dagger$ contain a small parameter $\delta$ one can use Eq.(21) to reconstruct standard perturbative formulas [16] for the energy shift of the channel 1 state under influence of channel 2. But if the value $E_1^{(0)}$ lies at the continuous spectrum of channel 2 (this is just decay case) then $h(E_1^{(0)})$ and, therefore, solution of Eq.(21) near $E_1^{(0)}$ are complex.
The survival amplitude of the state $\Psi^{(0)}_1$ equals

$$A(t) = \frac{i}{2\pi} \int_{-\infty}^{\infty} dE e^{-iEt} a(E + i\epsilon).$$  \hspace{1cm} (22)

For small $\delta$ the pole of integrand (22) appears at

$$E_1 \approx h(E^{(0)}_1 + i\epsilon).$$  \hspace{1cm} (23)

Then the structure of $\tilde{H}(E)$ leads to $\text{Im} E_1 < 0$, so the pole produces exponential decrease of $A(t)$.

Now one can easily justify the necessity [9] of deviation of $A(t)$ from pure exponential behaviour at very small and very large $t$. For small $t$ we can directly use Eq.(2) and have

$$A(0) = 1, \quad A(t) \approx 1 - it\langle \Psi^{(0)} | H | \Psi^{(0)} \rangle.$$  \hspace{1cm} (24)

Hermitian total Hamiltonian $H$ gives purely real value of $\langle \Psi^{(0)} | H | \Psi^{(0)} \rangle$. Hence, the survival probability, equal to $|A(t)|^2$, does not contain any term linear in $t$. Its decrease from the initial unit value goes slower than prescribed by the exponential.

More convenient for large $t$ are Eqs. (18),(22) which present $A(t)$ as the sum of contributions from singularities of $a(E)$. The closer to the real axis is the singularity, the slower is exponential decrease of its contribution. But $h(E)$ and $a(E)$ always contain real singularities due to thresholds in various channels. It is just the threshold contributions that give the slowest $t$ decrease. Each of them goes as power law with the power exponent depending on behavior of $h(E)$ near the threshold.

Return now to the small parameter $\delta$ in $H_{12}$ and $H_{21}$. If perturbation expansion in $\delta$ is applicable, then only one singularity (it is just the pole (21),(23)) produces contribution without parametric smallness. All other contributions have order $\sim \delta^2$ or higher. Thus, survival probability is very close to the exponential and deviates only at very small or very large $t$. Note that the true decrease in both cases is slower than exponential.

More diverse time behavior is possible if the perturbation expansion in $\delta$ is inapplicable. E.g., two singularities can produce contributions which are not small, but have opposite signs. At $t = 0$ they may essentially subtract each other. If one of them decreases with $t$ faster than another, the survival probability may even increase in some interval of $t$. 

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Generally, if various singularities give contributions of comparable value
then survival probability inevitably contains oscillating terms due to inter-
ference of various contributions.

Now we are ready to formulate WWA for decay of a separate state [6]. To have it one should change

\[ h(E) \rightarrow h(E_1^{(0)} + i\epsilon) \] (25)

under the integral (22). Eq.(23) for the pole position becomes exact; integral
(22) has only the pole contribution, and \( A(t) \) appears to be pure exponential
of \( t \). At first sight WWA means rejection of contributions from all singular-
ities of \( a(E) \) except the pole. But actually it changes normalization of the
pole term as well, so to reserve the initial condition \( A(0) = 1 \). Both effects
have the same order of smallness if perturbation theory is applicable.

To complete the Section we make one more note. The structure of ex-
pression (18), with function of \( E \) appearing instead of the eigen-state energy,
reminds the exact structure of one-particle propagator in the quan-
tum field theory where the mass operator depending on the momentum replaces the
mass value. One can easily understand that the similarity is not acciden-
tal. Both phenomena have the same reason. It is the influence of virtual
transitions to other channels.

## 4 Decays of coupled states

The above formalism is not convenient for neutral kaons and other mixed
systems, since their description requires to consider both coupled channels
at the same basis. That is why we somewhat modify the approach.

Let us consider three-channel situation. Keeping kaons in mind we can
consider three-channel situation. Keeping kaons in mind we can take channels
1 and 2 as having strangeness \( S = \pm 1 \), while index 3 should correspond to
the totality of channels with all other values of \( S \). Wave function has now
3 components, Hamiltonian \( H \) and resolvent \( (E - H)^{-1} \) are 3 × 3 matrices.
Matrix elements of hermitian total Hamiltonian satisfy relations

\[ H_{jk}^\dagger = H_{kj} \] \((j, k = 1, 2, 3)\). (26)

Exclude channel 3 from explicit consideration, just as was done earlier
for channel 2. From the resolvent we separate its part \( \tilde{R}(E) \) that describes
evolution and mutual transitions of channels 1 and 2. It can be expressed as
\[ \tilde{R}(E) = (E - \tilde{H})^{-1}, \] (27)
similar to Eqs. (6),(8). But now the effective Hamiltonian \( \tilde{H} \) is two-channel; it is \( 2 \times 2 \) matrix with elements \( (j, k = 1, 2) \)
\[ \tilde{H}_{jk}(E) = H_{jk} + H_{j3}(E - H_{33})^{-1}H_{3k}. \] (28)
The two-channel effective Hamiltonian, just as for one-channel case, depends on \( E \), i.e. interaction is nonlocal in time. Hermitian total Hamiltonian \( H \) produces relations for \( \tilde{H} \) (compare to one-channel relation (13)):
\[ \tilde{H}^\dagger(E) = \tilde{H}(E^*). \] (29)
Hence, the matrix operator \( \tilde{H}(E) \) is hermitian only at real \( E \) outside the spectral region of \( H_{33} \).

We were specially interested earlier in evolution of one particular state \( \psi_{1}^{(0)} \). Now we will follow for two states, \( \psi_{1}^{(0)} \) in channel 1 and \( \psi_{2}^{(0)} \) in channel 2. Moreover, we will study both their survival amplitudes and their mutual transitions. In other words, we consider now \( A(t) \) as \( 2 \times 2 \) matrix with elements
\[ A_{jk}(t) = \langle \psi_j^{(0)} | \Psi_k(t) \rangle, \quad j, k = 1, 2. \] (30)
Here \( \Psi_k(t) \) is the 3-component wave function with the initial condition
\[ \Psi_k(0) = \psi_k^{(0)}, \] (31)
while \( \psi_j^{(0)} \) may be viewed as having only one nonvanishing component which corresponds to eigen-state of
\[ H_{jj}\psi_j^{(0)} = E_j^{(0)}\psi_j^{(0)}. \] (32)
Then we obtain
\[ A(t) = \frac{i}{2\pi} \int_{-\infty}^{\infty} dE e^{-iEt} a(E + i\epsilon), \] (33)
where \( a(E) \) generalizes the quantity (17). It is now \( 2 \times 2 \) matrix with elements
\[ a_{jk}(E) = \langle \psi_j^{(0)} | \frac{1}{E - \tilde{H}(E)} | \psi_k^{(0)} \rangle \]
\[ = \langle \psi_j^{(0)} | \frac{1}{E - H} | \psi_k^{(0)} \rangle. \] (34)
Similar to (18), the matrix $a(E)$ may be presented as

$$a(E) = (E - h(E))^{-1},$$

(35)

where $2 \times 2$ matrix $h(E)$ may be defined by the matrix relation

$$h(E) \cdot a(E) = \langle \tilde{H}(E) \tilde{R}(E) \rangle.$$  

(36)

The symbol $\langle ... \rangle$ means "projecting" the product of operators to 2-dimensional space spanned over the two eigen-states. Eq.(36) is, surely, an analog of Eq.(19). And this is the end of simple analogies. E.g., Eq.(21) cannot be readily rewritten in the matrix form (one term would be multiple of the unit matrix, another would not).

Nevertheless, one may consider $h(E)$ as an effective Hamiltonian of two states, $\psi_1^{(0)}$ and $\psi_2^{(0)}$, just as earlier it was an effective one-state Hamiltonian. By tracing relations (27),(28),(34–(36), that connect $h(E)$ to the total Hamiltonian $H$, one can verify that hermitian $H$ (see (26)) produces matrix relation (compare to (29)):

$$h^\dagger(E) = h(E^*).$$

(37)

Consider in more detail $a(E)$ as an effective resolvent for two coupled states. It is convenient to expand $h(E)$ over linearly independent matrices:

$$h(E) = h_0(E) + \sum_{k=1}^3 b_k(E) \sigma_k,$$

(38)

where $\sigma_k$ are the Pauli matrices. New numerical functions $h_0(E)$ and $b_k(E)$ are simply related to components of $h(E)$:

$$h_0 = \frac{1}{2}(h_{11} + h_{22}), \quad b_1 = \frac{1}{2}(h_{12} + h_{21}),$$

$$b_2 = \frac{i}{2}(h_{12} - h_{21}), \quad b_3 = \frac{1}{2}(h_{11} - h_{22}).$$

(39)

If we denote eigen-values of $h(E)$ as $\lambda^{(1)}(E)$ and $\lambda^{(2)}(E)$, then

$$\lambda^{(1),(2)} = h_0 \pm b,$$

$$b = \sqrt{b_1^2 + b_2^2 + b_3^2}.$$
Note that both \( b_k(E) \) and \( b(E) \) are complex. Therefore, the problem arises how to formulate some additional condition so to fix branch of the root in Eq.\((40)\). We discuss it later.

Now we can present \( h(E) \) as
\[
h(E) = \lambda^{(1)}(E) \cdot P_+(E) + \lambda^{(2)}(E) \cdot P_-(E),
\]
where the projecting operators \( P_{\pm} \) equal
\[
P_{\pm}(E) = \frac{1 \pm \vec{n}\vec{\sigma}}{2}, \quad \vec{n}(E) = \frac{\vec{b}(E)}{b(E)}.
\]
Generally, the vector \( \vec{n}(E) \) has complex-valued components. But it is a unit vector in the sense
\[
\vec{n}^2 = n_1^2 + n_2^2 + n_3^2 = 1.
\]
Respectively,
\[
P_+P_- = P_-P_+ = 0, \quad P_+ + P_- = 1, \quad (P_{\pm})^2 = P_{\pm},
\]
but \( P_{\pm}^\dagger \) may not coincide with \( P_{\pm} \).

Eq.\((41)\) separates \( h(E) \) to two parts and allows to do the same for \( a(E) \):
\[
a(E) = \frac{1}{E - \lambda^{(1)}(E)} P_+(E) + \frac{1}{E - \lambda^{(2)}(E)} P_-(E).
\]
Each of the two parts in \((45)\) reminds the case of one-state decay (see Eq.\((18)\)). Two-state character, i.e. mixing and mutual transitions of two decaying states, reveals itself "only" in multiplication by projecting operators. As a result, the quantity \( a(E) \), being described in one-state case by a single complex numerical function \( h(E) \), requires for two-state case to specify four, not two complex functions. They may be either 4 elements of \( 2 \times 2 \) matrix \( h(E) \) (see Eq.\((35)\)), or 2 functions \( \lambda^{(1)}(E) \), \( \lambda^{(2)}(E) \) and 2 independent components of the complex unit vector \( \vec{n}(E) \) (see Eqs. \((42),(43)\)) contained in \( P_{\pm} \) (see Eqs.\((45),(42)\)). Surely, such complication is directly related to the fact that for the two-state case we are interested not only in survival amplitudes of the two states, but also in amplitude of their mutual transitions.

Operators \( P_{\pm} \) are orthogonal to each other (in the sense of the two first Eqs.\((44)\)) and, at first sight, should select two chosen combinations in the 2-dimensional set of states. However, \( E \)-dependence of \( P_{\pm} \) makes the selection nonlocal in time and physically inefficient.
Note one more feature of the two-state problem. If transition elements of \( H \) can be considered as perturbation then \( \lambda^{(1)} \) and \( \lambda^{(2)} \) are expandable. But components of \( \vec{n} \) become nonexpandable if nonperturbative spectra of channels 1 and 2 coincide. It is manifestation of the well-known property of quantum theory [16]: if a system contains energy-degenerate states then perturbation theory can be applicable only when the initial states have been changed by specially (nonperturbatively) selected combinations of them. It is just this nonperturbativeness that gives possibility for survival amplitudes to be oscillating (see discussion in the preceding Section).

Let us discuss some details of the structure of \( A(t) \) for two-state case. Its large \( t \) behavior is determined now by two sets of singularities in accordance with two-term structure (45). Among them there are two poles, at \( E^{(1)} \) and \( E^{(2)} \), solving equations

\[
E^{(1)} - \lambda^{(1)}(E^{(1)}) = 0, \quad E^{(2)} - \lambda^{(2)}(E^{(2)}) = 0. \tag{46}
\]

They generate two different exponentials in \( A(t) \) and, generally, oscillations in survival probabilities. There exist, however, two such combinations of initial states \( \psi^{(0)}_1 \) and \( \psi^{(0)}_2 \) which evolution does not reveal one or another of the exponentials. If the initial two-component function \( \psi^{(2)}(0) \) satisfies the condition

\[
P_+(E^{(1)})\psi^{(2)}(0) = 0, \tag{47}
\]

then it will evolve without the exponential generated by the pole of the first term of (45). Another initial state, \( \psi^{(1)}(0) \) with condition

\[
P_-(E^{(2)})\psi^{(1)}(0) = 0, \tag{48}
\]

will evolve without contribution from the pole of the second term of (45).

Retaining in the integral (33) only pole contributions from both terms of (45) one obtains the pole approximation suggested in [8]. The standard form of generalized WWA suggested in [7] to describe neutral kaons appears if one retains the pole contributions and neglects difference between \( \vec{n}(E^{(1)}) \) and \( \vec{n}(E^{(2)}) \). Taking this difference into account is of higher order smallness if one applies the perturbative expansion with respect to transition elements of \( H \).
5 Decays of kaons

Since $K^0$ and $\bar{K}^0$ can turn to each other their description without WWA should use formalism of the preceding Section. But some problems need to be considered in more detail.

We identify channel 1 with the set of states having strangeness $S = +1$ while channel 2 corresponds to the set of states with $S = -1$. Channel 3 includes all states with $S \neq \pm 1$; the most important in the framework of perturbation theory are states with $S = 0$. Thus, we may enumerate Hamiltonian matrix elements by the corresponding strangeness values ($H_{1,-1}$ instead of $H_{12}$, $H_{10}$ instead $H_{13}$ and so on). Further, we use $K^0$ and $\bar{K}^0$ as $\psi_1^{(0)}$ and $\psi_2^{(0)}$. Channels of various strangeness are coupled by weak interaction which strongly violates space reflection $P$ and charge conjugation $C$. Therefore, we do not consider them separately, but only their combination $CP$. In the limit of $CP$-invariance we can use $CP$-transformation to relate states of $S = +1$ and $S = -1$. In particular, we define phases of one-particle states so that

$$\tilde{K}^0 = (CP)K^0.$$  \hfill (49)

The set of states with $S \neq \pm 1$ goes to itself under $CP$-transformation.

With these conventions we rewrite relations (26) for hermitian Hamiltonian as

$$H_{jk}^\dagger = H_{kj} \quad (j, k = -1, 0, +1).$$  \hfill (50)

Various symmetries produce additional relations between elements of the Hamiltonian. E.g., $CP$-invariance gives

$$H_{11} = H_{-1,-1}; \quad H_{1,-1} = H_{-1,1};$$

$$H_{10} = H_{-1,0}; \quad H_{0,1} = H_{0,-1}.$$  \hfill (51)

We can add time inversion $T$ and consider $CPT$-transformation. Assumption of $CPT$-invariance gives smaller number of relations:

$$H_{11} = H_{-1,-1}; \quad H_{10} = H_{-1,0}^\dagger = H_{0,-1}.$$  \hfill (52)

$CPT$-invariance does not produce any relations for $H_{1,-1}$ and $H_{-1,1}$ additional to (50). Hermitian conjugation arises in (52) due to antiunitary nature of transformations $T$ and $CPT$. 

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Consider now the effective two-channel Hamiltonian \( \tilde{H}(E) \). Hermitian \( H \) leads to relations (29) that connect matrix elements of \( \tilde{H} \) at different values of \( E \). Unlike those, symmetry relations connect various elements of \( \tilde{H} \) with the same \( E \). So, \( CPT \)-invariance gives:

\[
\tilde{H}_{11}(E) = \tilde{H}_{-1,-1}(E).
\]  

(53)

The same relation is true for \( CP \)-invariance which adds one more relation:

\[
\tilde{H}_{1,-1}(E) = \tilde{H}_{-1,1}(E).
\]  

(54)

Note that \( T \)-invariance by itself also leads to Eq.(54) but does not require Eq.(53).

Tracing connections of \( h(E) \) and \( \tilde{H}(E) \) we find that \( CPT \)-invariance leads to

\[
h_{KK}(E) = h_{\bar{K}\bar{K}}(E)
\]  

(55)

in analogy with (53). Similarly, \( CP \)-invariance gives Eq.(55) with addition

\[
h_{K\bar{K}}(E) = h_{\bar{K}K}(E),
\]  

(56)

which is an evident analog of Eq.(54).

These relations take quite simple form for vectors \( \vec{b}(E) \) and \( \vec{n}(E) \) (see Eqs. (39)–(43)). From \( CPT \)-invariance (55) we obtain

\[
n_3(E) = 0,
\]  

(57)

while \( CP \)-invariance relations (55),(56) give

\[
n_2(E) = n_3(E) = 0.
\]  

(58)

\( T \)-invariance requires only vanishing \( n_2(E) \).

Note that all the symmetries do not influence \( \lambda^{(1)}(E) \) and \( \lambda^{(2)}(E) \) but essentially change properties of \( \vec{n}(E) \). For \( CP \)-invariant case there is only one non-zero component of \( \vec{n} \) which looses any \( E \)-dependence due to normalization (43). \( CPT \)-invariance does not eliminate \( E \)-dependence but implies only one independent component of \( \vec{n} \) (two non-vanishing components are related to each other by normalization).
Existing data imply that $CP$-violation is small while $CPT$-violation has not been observed at all. This means

$$|n_3(E)| \ll |n_2(E)| \ll |n_1(E)|$$  \hspace{1cm} (59)

(note that we really know it only for $E \approx m_K$). Therefore, we can choose the branch of root in Eq.(40) so as

$$b(E) \approx b_1(E), \quad n_1(E) \approx 1 - \frac{1}{2} n_3^2(E).$$  \hspace{1cm} (60)

Further, when applying to kaons we denote $\lambda_S$ and $\lambda_L$ instead of $\lambda^{(1)}$ and $\lambda^{(2)}$ respectively.

Solutions of the equations

$$M_S = \lambda_S(M_S), \quad M_L = \lambda_L(M_L)$$  \hspace{1cm} (61)

give positions of two poles in $a(E)$ that generate exponential contributions to $A(t)$. In analogy with $\psi^{(1)}$ and $\psi^{(2)}$ (see Eqs. (47),(48)), there exist two initial states, $K_S$ and $K_L$, which evolutions reveal only one exponential from two possible ones. If we describe their (anti)kaon content by the standard parameters $\epsilon_S$ and $\epsilon_L$ and apply relations corresponding to Eqs. (47) and (48) then

$$\frac{1 - \epsilon_L}{1 + \epsilon_L} = \left. \frac{n_1 + in_2}{1 - n_3} \right|_{E=M_S};$$

$$\frac{1 - \epsilon_S}{1 + \epsilon_S} = \left. \frac{n_1 + in_2}{1 + n_3} \right|_{E=M_L}. \hspace{1cm} (62)$$

We see that without WWA

$$\epsilon_S \neq \epsilon_L$$  \hspace{1cm} (63)

even in the case of $CPT$-invariance (i.e. at $n_3 = 0$) [8]. Meanwhile, it is just the relation (63) which is usually considered to be manifestation of $CPT$-violation.

Note, however, that the states $K_S$ and $K_L$ as defined above appear to have unfamiliar properties. Evolution of each of them reveals only one exponential term indeed. But there are power law terms as well. As a result, the kaon
content of the states changes with time. In this sense one may say that the states \( K_S \) and \( K_L \) are not independent beyond WWA and regenerate each other \([13,14]\).

Surely, all the construction becomes quite standard in the framework of WWA when one discards nonexponential terms and \( E \)-dependence of \( \vec{n} \). Note, however, that this dependence becomes eliminated also by CP-invariance when

\[
    n_2 = n_3 = 0, \quad n_1 = 1.
\]

In such a case the states \( K_S = K_1 \) and \( K_L = K_2 \) appear to be totally decoupled, so their content does not change with time even without WWA.

### 6 Effects of violation of CPT and WWA

Possibility to imitate CPT-violation beyond WWA requires to study in more detail the corresponding effects and their mutual influence. Simultaneously we can have a simple illustration of how the above formalism works.

We begin with the matrix \( A(t) \). Its elements describe two kinds of processes. One of them is survival of (anti)kaons, i.e. two transitions: \( K^0 \rightarrow K^0 \), \( \bar{K}^0 \rightarrow \bar{K}^0 \). Their amplitudes, according to Eqs.\((33),(42),(45)\), are equal

\[
A_{KK}(t) = \frac{i}{4\pi} \int_{-\infty}^{\infty} dE e^{-iEt} \left[ f_+(E + i\epsilon) + n_3(E)f_-(E + i\epsilon) \right],
\]

\[
A_{\bar{K}\bar{K}}(t) = \frac{i}{4\pi} \int_{-\infty}^{\infty} dE e^{-iEt} \left[ f_+(E + i\epsilon) - n_3(E)f_-(E + i\epsilon) \right],
\]

where

\[
f_{\pm}(E) = \frac{1}{E - \lambda_S(E)} \pm \frac{1}{E - \lambda_L(E)}.
\]

Other processes may be called inversion of kaons. They are transitions \( K^0 \rightarrow \bar{K}^0 \) and \( \bar{K}^0 \rightarrow \bar{K}^0 \) with amplitudes

\[
A_{K\bar{K}}(t) = \frac{i}{4\pi} \int_{-\infty}^{\infty} dE e^{-iEt} \left[ n_1(E) + in_2(E) \right] f_-(E + i\epsilon),
\]
A_K\bar{K}(t) = \frac{i}{4\pi} \int_{-\infty}^{\infty} dE e^{-iEt} [n_1(E) - in_2(E)] f_-(E + i\epsilon),

Amplitudes (65) contain $n_3(E)$ and are directly related to the problem of CPT-invariance. The ratio of the same-species amplitudes

$$R_{\text{same}} = \frac{A_{KK}(t)}{A_{\bar{K}\bar{K}}(t)}$$

deviates from unity if CPT is violated. At the same time it gains $t$-dependence. These properties are valid both in WWA and beyond it. The amplitudes (65) and their ratio (68) are insensitive to violation or conservation of CP if CPT is conserved.

Inversion amplitudes (67) have different properties. They are influenced by conservation or violation of CPT since $n_1$ and $n_2$ are related to $n_3$ by normalization (43); but this effect is of higher order smallness. However, it is just their ratio

$$R_{\text{opposite}} = \frac{A_{\bar{K}K}(t)}{A_{K\bar{K}}(t)}$$

that demonstrates the so called CP-violation in kaon mixing. In the framework of WWA

$$(R_{\text{opposite}})_{\text{WWA}} = \frac{n_1 + in_2}{n_1 - in_2} = \frac{1 - \epsilon_S}{1 + \epsilon_S} \cdot \frac{1 - \epsilon_L}{1 + \epsilon_L}.$$  

If CP is violated the ratio (69) becomes $t$-dependent beyond WWA independently of conservation or violation of CPT. Note that $t$-dependence of $R_{\text{opposite}}$ is directly related to $E$-dependence of $\vec{n}$. Therefore, this effect of going beyond WWA has no analogs in decays of uncoupled states. We emphasize, however, that it can be revealed only if CP-invariance (or, more exactly, T-invariance) is violated.

Thus, $R_{\text{same}}$ and $R_{\text{opposite}}$ have essentially different properties: $R_{\text{same}}$ can depend on $t$ only at violated CPT; on the other hand, $R_{\text{opposite}}$ can depend on $t$ only beyond WWA. These examples demonstrate that effects of violation of CPT or WWA can be discriminated and studied separately. Unfortunately, quantitative theoretical estimates of both effects can be made at present only in model-dependent ways.
7 Conclusion

Here we briefly summarize the above results. To describe kaon evolution without Weisskopf-Wigner approximation we suggest formalism which reminds modified one-particle propagator where the mass is changed by the mass operator. Such approach makes more transparent many results obtained earlier [13,14] in a highly mathematical way. Symmetry considerations show, in particular, that violation of $CPT$ and deviation from WWA can be studied independently of each other.

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