SINGULAR MEASURES ON THE LIMIT SET OF A KLEINIAN GROUP

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Abstract. We consider a finitely generated torsion free Kleinian group $H$ and a random walk on $H$ with respect to a symmetric nondegenerate probability measure $\mu$ with finite support. When $H$ is geometrically infinite without parabolics or when $H$ is Gromov hyperbolic with parabolics, we prove that the Patterson-Sullivan measure is singular with respect to the harmonic measure coming from $\mu$.

1. Introduction

Let $H$ be a finitely generated Kleinian group, i.e., a discrete subgroup of $\text{PSL}(2, \mathbb{C})$. We assume that $H$ has no elliptic elements. Since $\text{PSL}(2, \mathbb{C})$ is the group of orientation preserving isometries of the hyperbolic 3-space $\mathbb{H}^3$, we can consider the action of $H$ on $\mathbb{H}^3$ which can be continuously extended to the ideal boundary $S^2_\infty$ of $\mathbb{H}^3$. The limit set $\Lambda_H$ of $H$ is the set of limit points of an orbit $H \cdot o$ where $o \in \mathbb{H}^3$. This definition does not depend on the choice of $o$ and $\Lambda_H$ is contained in $S^2_\infty$.

We consider two measures on $\Lambda_H$. One is the Patterson-Sullivan measure $\rho_x$ based at $x \in \mathbb{H}^3$ and the other is the harmonic measure $\nu_H$ coming from a random walk on $H$. Given a probability measure $\mu$ on $H$ and the corresponding random walk $\{Y_n\}$ on $H$, we can describe $\nu_H$ as follows. For $A \subset S^2_\infty$, $\nu_H(A)$ is the probability for $Y_n \cdot o$ to converge to a point in $A \subset S^2_\infty$. When $H$ is a Fuchsian group i.e., a discrete subgroup of $\text{PSL}(2, \mathbb{R})$, $H$ acts isometrically on $\mathbb{H}^2$ and Y. Guivarc’h and Y. Le Jan proved the following.

Theorem 1.1. (Y. Guivarc’h and Y. Le Jan [16]) Let $H$ be a discrete subgroup of $\text{PSL}(2, \mathbb{R})$ such that $\mathbb{H}^2/H$ is a noncompact surface with finite area. Let $\nu_H$ is the harmonic measure on $S^1_\infty$ coming from a symmetric nondegenerate probability measure $\mu$ with finite support on $H$. Then $\nu_H$ is singular with respect to the Lebesgue measure on $S^2_\infty$.

In this paper, we generalize Theorem 1.1 to Kleinian groups as follows.
Theorem 1.2. Suppose that $H$ is a finitely generated Kleinian group such that

1. $H$ is Gromov hyperbolic.
2. $H$ is not convex-cocompact.
3. The orbit map $\tau_o : H \to \mathbb{H}^3$ sending $h$ to $h \cdot o$ extends continuously to $\bar{\tau}_o : \partial H \to S_\infty^2$.

Here $\partial H$ is the Gromov boundary of $H$. Let $\nu_H$ be the harmonic measure on $\Lambda_H$ coming from a symmetric nondegenerate probability measure $\mu$ with finite support on $H$. Then $\nu_H$ is singular with respect to the Patterson-Sullivan measure on $\Lambda_H$.

We remark that when $\Lambda_H = S_\infty^2$, the Patterson-Sullivan measure $\rho_o$ where $o$ is the origin of the Poincaré ball model is equal to the Lebesgue measure $\text{Leb}_{S_\infty^2}$ on $S_\infty^2$ up to constant multiple. Thus every Patterson-Sullivan measure $\rho_x$ is equal to $\text{Leb}_{S_\infty^2}$ up to homothety. The continuous extension $\bar{\tau}_o$ of the orbit map $\tau_o$ for $H$ is called the Cannon-Thurston map of $H$ and its existence has been verified recently for surface Kleinian groups by [24]. The paper [25] dealt with the general case. For the proof of Theorem 1.2, we modify and extend the argument used in the proof of [3, Prop 5.4, Prop 5.5]. The assumption of the existence of the Cannon-Thurston map is crucial.

2. Preliminaries

2.1. Hyperbolic spaces. For $\delta \geq 0$, a geodesic metric space $(X, d)$ is called $\delta$-hyperbolic if for any geodesic triangle in $X$, each side of the triangle is contained in the $\delta$-neighborhood of the union of the other two sides. We call a geodesic metric space a hyperbolic space (in the sense of Gromov) if it is $\delta$-hyperbolic for some $\delta \geq 0$. A hyperbolic metric space $(X, d)$ has a boundary at infinity $\partial X$ called the Gromov boundary which can be defined as follows. We say a sequence $\{x_n\}$ in $(X, d)$ converges to infinity if

$$\liminf_{i,j \to \infty} (x_i|y_j)_x = \infty$$

for some (hence every) basepoint $x$, where

$$(y|z)_x := \frac{1}{2}(d(x,y) + d(x,z) - d(y,z))$$

for $x, y, z \in X$. This product $(y|z)_x$ is called the Gromov product of $y$ and $z$ with respect to $x$ and approximates within $2\delta$ the distance from $x$ to any geodesic $[y,z]$ joining $y, z$. Two sequences $\{x_n\}, \{y_n\} \subset X$ converging to infinity are said to be equivalent if

$$\liminf_{i,j \to \infty} (x_i|y_j)_x = \infty$$

for some $x \in X$. The Gromov boundary $\partial X$ is defined as the set of the equivalence classes of sequences converging to infinity in $X$. This definition is independent of the choice of the base point $x \in X$. The Gromov product
can be extended to \( \partial X \). We can define a natural metric \( d_\epsilon(\cdot, \cdot) \) on \( \partial X \) such that \( C^{-1}e^{-\epsilon(g,h)} \leq d_\epsilon(p,q) \leq C e^{-\epsilon(g,h)} \), where \( (p,q) \in \partial X \times \partial X \).

Fixing a finite generating set \( S \) of \( H \), the word metric \( d_w(\cdot, \cdot) \) on \( H \) is defined by setting \( d_w(g,h) = |g^{-1}h|_S \) where \( |g|_S \) is the minimum of the number of elements of \( S \) whose product is \( g \). When we consider the right Cayley graph \( \Gamma_H \) of \( H \) with the length of every edge being 1, \( d_w(g,h) \) is the minimum of the lengths of paths joining \( g \) and \( h \) in \( \Gamma_H \). A finitely generated group \( H \) is called a \textit{hyperbolic group} if its Cayley graph \( \Gamma_H \) is hyperbolic with respect to the word metric for a finite generating set \( S \) of \( H \). We simply denote the word length of \( g \) as \( |g| \) by fixing a finite generating set. We denote the Gromov boundary of \( \Gamma_H \) by \( \partial H \). For basic properties of Gromov hyperbolic spaces, hyperbolic groups and their Gromov boundaries, we refer the reader to [15] [13] and [20].

### 2.2. Kleinian groups.

Let \( H \) be a finitely generated \textit{Kleinian group}. We assume that \( H \) has no elliptic elements. The \textit{limit set} \( \Lambda_H \) of \( H \) is the set of limit points of an orbit \( H \cdot o \) where \( o \in \mathbb{H}^3 \). The complementary open set \( \Omega_H = S^2_\infty \setminus \Lambda_H \) becomes the domain of discontinuity of the action of \( H \) and by Ahlfors’ finiteness theorem, \( \Omega_H/H \) consists of finitely many Riemann surfaces which are called as the \textit{conformal boundaries at infinity} of \( H \).

The \textit{convex hull} of \( H \) is defined to be the smallest convex set in \( \mathbb{H}^3 \) whose closure in \( \mathbb{H}^3 \cup S^2_\infty \) contains \( \Lambda_H \). The \textit{convex core} of \( H \) is the quotient of its convex hull by \( H \) itself. The Kleinian group \( H \) is called \textit{geometrically finite}(resp. \textit{convex cocompact}) if its convex core has finite volume(resp. if its convex core is compact). Every convex cocompact Kleinian group is a hyperbolic group by the Schwarz-Milnor lemma(See [5] for example). By Thurston’s uniformization theorem in [28], every geometrically infinite Kleinian group without parabolics has a convex cocompact representation which is faithful. Thus if \( H \) is geometrically infinite without parabolics, then it is hyperbolic.

If the orbit map \( \tau_o : H \to \mathbb{H}^3 \) sending \( h \to h \cdot o \) can be extended continuously to \( \tilde{\tau}_o : \partial H \to S^2_\infty \), we call \( \tilde{\tau}_o \) as the \textit{Cannon-Thurston map} of \( H \) following [9] [4] [22] [23] [24] [25] [27]. If \( H \) is convex cocompact, then \( \tau_o \) is a quasi-isometric embedding and we have a natural homeomorphism \( \tilde{\tau}_o \) from the Gromov boundary \( \partial H \) to the limit set \( \Lambda_H \). When \( H \) is a geometrically finite Kleinian group with parabolics, we still have a continuous extension of \( \tau_o \) from the Floyd boundary (See [12] [14]) of \( H \) to \( \Lambda_H \). If we further assume \( H \) is hyperbolic, then the Floyd boundary of \( H \) is equal to the Gromov boundary \( \partial H \) and a parabolic element \( h \) gives two points \( \{ h_\infty, h_{-\infty}\} \) in \( \partial H \) such that \( \tilde{\tau}_o(h_\infty) = \tilde{\tau}_o(h_{-\infty}) \) where \( \{ h_\infty, h_{-\infty}\} \) are the accumulation points of \( \{ h^i \mid i \in \mathbb{Z}\} \) in \( \Gamma_H \).

Now we consider the case when \( H \) is a geometrically infinite hyperbolic Kleinian group and we assume the Cannon-Thurston map \( \tilde{\tau}_o \) exists. In this case, we can find an exiting sequence of closed geodesics \( \{ c_n \} \) in \( \mathbb{H}^3/H \). The exiting property of \( \{ c_n \} \) gives a bi-infinite quasigeodesic \( l \) in \( \Gamma_H \) such that
the two endpoints of $l$ are identified by $\hat{\tau}_o$. See [26] for details. We need the following lemma for the proof of Theorem 1.2.

**Lemma 2.1.** Let $H$ be a finitely generated Kleinian group with $\mathbb{H}^3/H$ not being convex cocompact and such that $H$ is Gromov hyperbolic. We assume that the Cannon-Thurston map $\hat{\tau}_o : \partial H \to S^2_\infty$ exists. Then there exists $\{h_n\} \subset H$ such that for any constant $D > 0$, $|h_n| - Dd_{H^3}(o, h_n \cdot o) \to \infty$.

**Proof.** We fix a finite generating set $S$ of $H$. If $H$ has a parabolic element $g$, then we can just take $h_n$ as $g^n$. If not, then $H$ is geometrically infinite and there exists a bi-infinite quasigeodesic $l$ in $\Gamma_H$ joining two points $p, q$ in $\partial H$ such that $\hat{\tau}_o(p) = \hat{\tau}_o(q)$. We represent $l$ as a bi-infinite sequence of vertices in $\Gamma_H$

$$\{\cdots g_{-n}, g_{-(n-1)}, \cdots, \text{id}, \cdots, g_n, g_{n+1}, \cdots\}$$

such that $|g^{-1}_i g_{i+1}| = 1$. Then since $\hat{\tau}_o(p) = \hat{\tau}_o(q)$ and $\hat{\tau}_o$ is a continuous extension of $\tau_o$, we have $(g_{-n} \cdot o) | g_n \cdot o) \to \infty$. If $\tau_o|_l$ is a quasi-isometric embedding then by the Morse-Mostow lemma (see [5] for example), the image of $\tau_o|_l$ is contained in a uniform neighborhood of a bi-infinite geodesic in $\Gamma_H$. Since $(g_{-n} \cdot o) | g_n \cdot o) \to \infty$, this is not possible and thus $\tau_o|_l$ cannot be a quasi-isometric embedding. We note that $d_{H^3}(g, h) \leq C|g^{-1}h|$ by the triangle inequality where $C := \max_{s \in S} d_{H^3}(o, s \cdot o)$. We can also see the following. There exists a sequence of integers $i(n)$ and $j(i) > i$ such that for any $D > 0$, $|g_{-i(n)}^{-1}g_{j(i)}| - Dd_{H^3}(g_i \cdot o, g_j \cdot o) \to \infty$ as $n \to \infty$. We take $h_n$ as $g_{i(n)}^{-1} g_{j(i)}$.

$\square$

2.3. **Poisson boundary.** Let $\mu$ be a probability measure on a group $H$.

- $\mu$ is called *nondegenerate* if the support of $\mu$ generates $H$ as a semi-group.
- $\mu$ is called *symmetric* if $\mu(A) = \mu(A^{-1})$ where $A \subset H$ and $A^{-1}$ is the set of the inverses of elements of $A$.

The word length $|h|$ of an element $h$ in $H$ is with respect to a fixed finite generating set of $H$. The *first moment* of $(H, \mu)$ is defined to be \( \sum_{h \in H} |h| \mu(h) \).

The *entropy* of $(H, \mu)$ is \( \lim_{n \to \infty} - \sum_{h \in H} \mu(h) \log \mu(h) \). When $\mu$ has a finite support, both of its entropy and its first moment are finite.

The *random walk* on $H$ with respect to $\mu$ is a Markov chain $\{Y_n\}$ with the transition probabilities given by $p_{h_1, h_2} = \mu(h_1^{-1} h_2)$. We denote $\{Y_n\}$ by $Y$. A random walk $Y$ can also be described as a sequence of independent random variables $\{Z_n\}$ with values in the probability space $(H, \mu)$ such that $Y_n = Z_0 Z_1 \cdots Z_n$. When $Y_0 = \text{id}$, the random walk starts from the identity element $\text{id}$ of $H$. We regard $Z = \{Z_n\}$ as an element of the product probability space $(H^\mathbb{N}, \mu^\mathbb{N})$ and $Y$ as an element of the Kolmogorov representation space $(H^\mathbb{N}, \mathbb{P}_{\text{id}})$ where

$$\mathbb{P}_{\text{id}}(\{Y_n\} | Y_0 = \text{id}, Y_1 = h_1, Y_2 = h_2, \cdots, Y_n = h_n) = p_{\text{id}, h_1, h_2, \cdots, h_{n-1}, h_n}$$
and $\mathbb{P}_{id}(\{Y_n\}|Y_0 \neq id)$ is defined to be zero. We denote $\mathbb{P}_{id}$ by $\mathbb{P}$.

The time shift operator $T$ acts on $(H^\mathbb{N}, \mathbb{P})$ by $(TY)_n = Y_{n+1}$ and defines an equivalence relation $\sim$ by saying $Y \sim Y'$ if and only if there exist positive integers $k, k'$ such that $T^kY = T^{k'}Y'$.

**Definition 2.2.** The Poisson boundary of $H$ with respect to $\mu$ is the quotient space of $(H^\mathbb{N}, \mathbb{P})$ by the smallest measurable equivalence relation generated by the equivalence relation $\sim$.

The smallest measurable equivalence relation generated by $\sim$ is called as the measurable envelope [31]. When $H$ is a hyperbolic group, we can consider the hitting measure $\nu$ of a random walk $Y$ on $\partial H$ and the Poisson boundary of $H$ can be identified with $(\partial H, \nu)$ by [17, 18]. More precisely, it can be shown that $\{Y_n\}$ converges to a point in $\partial H$ for almost every sample path $\{Y_n\}$ and $\nu(A)$ can be defined as the probability for a sample path $\{Y_n\}$ converges to a point in $A \subset \partial H$. Thus $\nu$ can also be called as the law of $Y_\infty$. When $H$ is a hyperbolic Kleinian group, $H$ acts on $\mathbb{H}^3$ and we can consider the law of $Y_\infty \cdot o$ if $\{Y_n \cdot o\}$ converges to a point in $S^2_\infty$ for almost every sample path $\{Y_n\}$. In fact, the law of $Y_\infty \cdot o$ is the push forward of $\nu$ by $\hat{\tau}_o$ by the following Theorem 2.3 which summarizes [21, Corollary 6.2].

**Theorem 2.3.** (Karlsson and Margulis [21]) Let $H$ be a countable group isometrically acting on a uniformly convex, complete metric space $(X, d)$ which is nonpositively curved in the sense of Busemann. Let $\mu$ be a probability measure on $H$ with $\sum_{h \in H} d(o, h \cdot o) \mu(h) < \infty$. Let $\partial X$ be the ideal boundary of $X$ consisting of asymptotic classes of geodesic rays. Then

- Almost every sample path $\{Z_n\}$ in $H^\mathbb{N}$ with respect to $\mu^\mathbb{N}$ converges to a geodesic ray in $X$.

Thus we have a map $\xi : (H^\mathbb{N}, \mu^\mathbb{N}) \to \partial X$ and we give $\partial X$ the push-forward measure $\xi_* (\mu^\mathbb{N})$ so that $\xi$ is measurable.

- $(\partial X, \xi_* (\mu^\mathbb{N}))$ becomes the Poisson boundary of $(H, \mu)$ if the lattice counting function is subexponential, i.e., if

$$\# \{h \in H | d(o, h \cdot o) < r\} \leq e^{Cr}$$

for some constant $C > 0$.

A similar form of Theorem 2.3 was also mentioned in Remark 3 following [18, Theorem 7.7]. It is known that the lattice counting function is subexponential for a large class of discrete subgroups of the isometry groups of Cartan-Hadamard manifolds. In particular, for pinched negatively curved case, it is proven in [34, Theorem 3.6.1]. Thus every Kleinian group has a subexponential lattice counting function. Moreover, since we are assuming $H$ is finitely generated, we have $d_{\mathbb{H}^3}(o, h \cdot o) \leq D \cdot |h|$ by the triangle inequality.

Therefore we can apply Theorem 2.3 to a Kleinian group $H$ acting on $\mathbb{H}^3$. We assume $\mu$ is a symmetric, nondegenerate probability measure on $H$ with
Lemma 2.4. \( \alpha(0) = \) convergent or divergent.

Hence it becomes an \( H \), \( \mu \) Poisson boundary of \((H, \mu)\) and thus \( \tau_\mu : (\partial H, \nu) \to (S^2_\infty, \nu_H) \) is a measurable isomorphism.

2.4. Conformal density. For \( x \in \mathbb{H}^3 \), the Busemann function \( b_{x, \eta}(\cdot) \) at \( \eta \) with \( b_{x, \eta}(x) = 0 \) can be defined by choosing a geodesic ray \( \alpha(t) \) from \( \alpha(0) = x \in \mathbb{H}^3 \) toward \( \eta \in S^2_\infty \) as follows.

\[
b_{x, \eta}(y) = \lim_{t \to \infty} (d_{\mathbb{H}^3}(y, \alpha(t)) - d_{\mathbb{H}^3}(x, \alpha(t)))
\]

Lemma 2.4. ([29] Lemma 3.2.1) Consider the Poincaré ball model \( \{ x \in \mathbb{R}^3 : |x| < 1 \} \) of \( \mathbb{H}^3 \) where \( |x| \) is the usual Euclidean norm of \( x \). Then

\[
e^{b_{x, \eta}(y)} = \frac{P(x, \eta)}{P(y, \eta)}
\]

where \( P(x, \eta) \) is the Poisson kernel \( (1 - |x|^2)/(|x - \eta|^2) \). Thus \( b_{x, \eta}(y) \) is a continuous function of \( x, y \in \mathbb{H}^3 \) and \( \eta \in S^2_\infty \).

Fixing \( x, y \) in \( \mathbb{H}^3 \), the Poincaré series for \( H \) is

\[
g_s(x, y) = \sum_{h \in H} e^{-s d_{\mathbb{H}^3}(x, h \cdot y)}
\]

The critical exponent \( \delta_H \) of \( H \) is defined as

\[
\delta_H = \limsup_{r \to \infty} \frac{1}{r} \log(\#\{ h \in H | d^3_{\mathbb{H}}(o, h \cdot o) \leq r \})
\]

Equivalently, \( \delta_H \) is the infimum of the set of \( s \) such that \( g_s(x, y) \) is finite. We call \( H \) is divergent if \( g_s(x, y) \) diverges at \( s = \delta_H \). Otherwise \( H \) is called convergent. These definitions are independent of the choices of \( x, y \in \mathbb{H}^3 \).

Consider a family of measures \( \{ \rho^s_x \} \) defined by

\[
\rho^s_x = \frac{1}{g_s(y, y)} \sum_{h \in H} e^{-s d_{\mathbb{H}^3}(x, h \cdot y)} \delta_{h \cdot y}
\]

where \( \delta_{h \cdot y} \) is the Dirac measure at \( h \cdot y \). Then we can find a sequence \( \{ s_i \} \) with \( s_i \to \delta_H^* \) such that \( \rho^s_x \) weakly converges to a finite measure \( \rho_x \) on the compact space \( \mathbb{H}^3 \cup S^2_\infty \). When \( H \) is divergent, \( \rho_x \) has its support on the limit set \( \Lambda_H \). Since \( \{ \rho_x \} \) satisfies

\[
\frac{d \rho_x}{d \rho_y}(\eta) = e^{-\delta_H b_{x, \eta}(y)}, \ h^* \rho_x = \rho_{h^{-1} \cdot x}
\]

it becomes an \( H \)-invariant conformal density of dimension \( \delta_H \). Here the pull-back measure \( h^* \rho_x \) is defined by setting \( (h^* \rho_x)(A) := \rho_x(h \cdot A) \) for \( A \subset \Lambda_H \). Even when \( H \) is convergent, we still can construct a conformal density by increasing the Dirac mass on each orbit point suitably [30, 32] and we denote a resulting conformal density also by \( \rho_x \). We call \( \rho_x \) as the Patterson-Sullivan measure with base point \( x \in \mathbb{H}^3 \) in either case of \( H \) being convergent or divergent.
As an application of the Tameness theorem (see section 9 in [6] and [11, Prop. 3.9]), a Kleinian group $H$ is divergent if and only if either $H$ is geometrically finite or $\Lambda_H = S_\infty^2$. When $\Lambda_H = S_\infty^2$, $\rho_x$ is equal to $\text{Leb}_{S_\infty^2}$ up to homothety and the diagonal action of $H$ on $S_\infty^2 \times S_\infty^2 \setminus \Delta$ is ergodic with respect to $\text{Leb}_{S_\infty^2} \otimes \text{Leb}_{S_\infty^2}$. Here $\Delta$ means the diagonal set.

2.5. Harmonic Density. The Green function on a group $H$ with a probability measure $\mu$ is defined by

$$G(g, h) = \sum_{n=0}^{\infty} \mu^n(g^{-1} h)$$

where $\mu^n$ is the $n$-th convolution power of $\mu$. Then $F(g, h) = G(g, h)/G(h, h)$ is the probability that there exists $n \in \mathbb{N}$ such that $gY_n = h$. The Green metric $d_G$ is defined by

$$d_G(g, h) = -\log F(g, h)$$

If $H$ is a hyperbolic group and if $\mu$ is finitely supported nondegenerate symmetric probability measure on $H$, then $d_G$ is a left invariant hyperbolic metric quasi-isometric to the word metric on $H$ by [3, Corollary 1.2].

The Martin kernel $K : H \times H \rightarrow \mathbb{R}$ is defined by

$$K(g, h) = \frac{F(g, h)}{F(id, h)}$$

There exist constants $\{C_g\}$ such that for all $h \in H$, $K(g, h) \leq C_g$ for each $g \in H$. Let $H \cup \partial_M H$ be the metric completion of $H$ with respect to the metric $d_M$ defined by

$$d_M(h_1, h_2) = \sum_{h \in H} D_h \frac{|K(h, h_1) - K(h, h_2)| + |\delta_{h, h_1} - \delta_{h, h_2}|}{C_h + 1}$$

where $\{D_h\}$ is chosen so that $\sum_{h \in H} D_h < \infty$. $\delta_{h, g}$ is the Kronecker delta.

Then it can be shown that $H \cup \partial_M H$ is in fact a compactification of $H$ and the Martin kernel can be continuously extended to $H \times (H \cup \partial_M H)$. We call $\partial_M H$ as the Martin boundary of $H$. When $H$ is a hyperbolic group, $\partial_M H$ is homeomorphic to the Gromov boundary of $H$. For this, see [2, 19] or [3, Corollary 1.8].

A positive function $u$ on $H$ is called harmonic if $u(h) = \sum_{g \in H} \mu(h^{-1} g)u(g)$ for all $h \in H$ and the Martin representation theorem says for any harmonic function $u$ on $H$, there exists a measure $\nu_u$ on $\partial_M H$ such that

$$u(h) = \int_{\partial_M H} K(h, \xi) d\nu_u(\xi)$$

When we take $u$ as a constant function $u \equiv 1$, the support of $\nu_u$ with the measure $\nu_u = \nu_1$ becomes the Poisson boundary of $H$. It is known that almost every sample path $\{Y_n\}$ converges to a point on $\partial_M H$. Thus for
a hyperbolic group $H$, we can identify $\nu$ with $\nu_1$. We have the following change of variable formula.

$$
\frac{d(h^*\nu)}{d\nu}(\xi) = K(h^{-1}, \xi)
$$

For a hyperbolic group $H$, we can define the Busemann function on $H$ with respect to the Green metric $d_G$ by

$$
b_G^{\text{id,}\xi}(h) = \sup_{\{h_n\}} \limsup_{n \to \infty} (d_G(h_n, \text{id}) - d_G(h_n, h))
$$

where $\sup$ is taken over all sequences $\{h_n\}$ converging to $\xi$. If we choose $\{h_n\}$ along a quasigeodesic ray toward $\xi$ in $\Gamma_H$, then we can just take the usual limit as $h_n \to \xi$ to define $b_G^{\text{id,}\xi}(h)$. By the definition of the Martin kernel, we get

$$
\frac{d(h^*\nu)}{d\nu}(\xi) = e^{b_G^{\text{id,}\xi}(h^{-1})}
$$

We have the same formula for the push-forward measure $\nu_H$ on $S^2_\infty$ which was used in the proof of [3, Prop. 5.5]

Lemma 2.5. Let $H$ be a Kleinian group which is hyperbolic and let $\nu_H$ be the push-forward measure $\hat{\tau}_o(\nu)$ on $S^2_\infty$ as before. Then

$$
\frac{d(h^*\nu_H)}{d\nu_H}(\hat{\tau}_o(\xi)) = e^{b_G^{\text{id,}\xi}(h^{-1})}
$$

where $\xi \in \partial H$ and $(h^*\nu_H)(A) = \nu_H(h \cdot A)$ for $A \subset S^2_\infty$ for any $\nu_H$-measurable set $A \subset S^2_\infty$.

Now we apply [19, Theorem 3.3] to the Cayley graph of the hyperbolic group $H$. Since the Gromov boundary $\partial H$ itself is the conical limit set of $H$, we can see that the diagonal action of $H$ on $\partial H \times \partial H \setminus \Delta$ with respect to the measure class of $\nu \otimes \nu$ is ergodic.

Lemma 2.6. Let $H$ be a Kleinian group which is hyperbolic. Let $\mu$ be a symmetric nondegenerate probability measure on $H$ with finite support. Then the diagonal action of $H$ on $S^2_\infty \times S^2_\infty \setminus \Delta$ is ergodic with respect to $\nu_H \otimes \nu_H$.

Proof. We know $(\partial H, \nu)$ and $(S^2_\infty, \nu_H)$ are measurably isomorphic by the equivariant map $\hat{\tau}_o$ by Theorem 2.3. □

3. Singularity of measures on $\Lambda_H$

In this section, we prove Theorem 1.2. Recall that for $\eta_1, \eta_2 \in S^2_\infty$, the Busemann cocycle $B_o(\eta_1, \eta_2)$ is defined as $b_{o,\eta_1}(y) + b_{o,\eta_2}(y)$ where $y$ is any point in the bi-infinite geodesic $l$ joining $\eta_1$ and $\eta_2$. Geometrically $B_o(\eta_1, \eta_2)$ is the length of the geodesic subsegment of $l$ contained in the intersection of the horoballs passing through $o$ and centered at $\eta_1, \eta_2$. Thus it is nonnegative for any $(\eta_1, \eta_2) \in S^2_\infty \times S^2_\infty \setminus \Delta$ and it is zero if and only if $o$ is contained in $l$. 


Assume, seeking a contradiction, $J$ is a $H$-invariant measure although it may not be an ergodic measure with respect to the $H$-action. We define the measure $\rho_o$ on $S^2_\infty \times S^2_\infty \setminus \Delta$ by
$$d\rho_o(\eta_1, \eta_2) = e^{2\delta_H B_o(\eta_1, \eta_2)}d\rho_o(\eta_1)d\rho_o(\eta_2)$$
Then $\rho_o$ is a $H$-invariant measure.

Thus we have for $\bar{\nu}_H$ the measure $\nu_H$ on $S^2_\infty \times S^2_\infty \setminus \Delta$ as
$$d\bar{\nu}_H(\tilde{o}_\xi_1, \tilde{o}_\xi_2) = e^{2(\xi_1, \xi_2)_G}d\nu_H(\tilde{o}_\xi_1)d\nu_H(\tilde{o}_\xi_2)$$
Then $\bar{\nu}_H$ is invariant under the action of $H$ by \[2] Prop. 2.2. Furthermore it is an ergodic measure by Corollary \[2,6] Now we suppose $\nu_H$ is equivalent to $\rho_o$ and we claim that the Radon-Nikodym derivative $d\rho_o/d\nu_H$ is $\nu_H$-essentially upper and lower bounded by positive constants.

Proof of the claim: We have a $\nu_H$-integrable function $J$ and a $\bar{\nu}_H$-measurable function $\tilde{J}$ defined by $d\nu_H = Jd\rho_o$ and $d\bar{\nu}_H = \tilde{J}d\rho_o$. Since $\tilde{J}$ is positive almost everywhere, there exists a constant $C > 0$ such that the set $A := \{(\eta_1, \eta_2) \in S^2_\infty \times S^2_\infty \setminus \Delta| \tilde{J}(\eta_1, \eta_2) \leq C\}$ has positive $\nu_H$-measure.

Since $\nu_H$ is ergodic, there exists $h \in H$ such that $(h\eta_1, h\eta_2) \in A$ for $\nu_H$-almost every $(\eta_1, \eta_2)$. Since $\rho_o$ is also $H$-invariant, we have $\tilde{J}(\eta_1, \eta_2) = \tilde{J}(h\eta_1, h\eta_2)$ and thus $A$ has the full $\tilde{\nu}_H$-measure. When $\eta_i = \tilde{o}_\xi_i$ for $i = 1, 2$, we have
$$\tilde{J}(\eta_1, \eta_2) = J(\eta_1)J(\eta_2)\frac{e^{2(\xi_1, \xi_2)_G}}{e^{2\delta_H B_o(\eta_1, \eta_2)}}$$
Thus we have for $\tilde{\nu}_H$-almost every $(\eta_1, \eta_2)$,
$$C^{-1}\frac{e^{2\delta_H B_o(\eta_1, \eta_2)}}{e^{2(\xi_1, \xi_2)_G}} \leq J(\eta_1)J(\eta_2) \leq C\frac{e^{2\delta_H B_o(\eta_1, \eta_2)}}{e^{2(\xi_1, \xi_2)_G}}$$
Assume, seeking a contradiction, $J$ is unbounded in $B_s \cap \Lambda_H \subset S^2_\infty$ where $B_s$ is a small spherical open ball in $S^2_\infty$. We choose another small spherical open ball $B_s'$ far from $B_s$ so that for all $(\eta, \eta') \in B_s \times B_s'$, $D^{-1} < e^{B_o(\eta, \eta')} < D$ for some constant $D > 0$. Since $\tilde{o}_\xi$ is a continuous map, the distance in $\partial H$ from the open set $\tilde{o}_\xi^{-1}(B_s')$ to the open set $\tilde{o}_\xi^{-1}(B_s)$ has a positive lower bound. If we let $\tilde{o}_\xi = \eta$ and $\tilde{o}_\xi' = \eta'$, this means $(\xi'|\xi)_G$ has an upper bound and the same is true for $(\xi'|\xi)_G$ from this, we get that $J(\eta)J(\eta')$ is positively upper and lower bounded for $\tilde{\nu}_H$-almost all $(\eta, \eta') \in B_s \times B_s'$. But there is a constant $D_1 > 0$ such that $B := \{\eta' \in B_s'|D_1^{-1} \leq J(\eta') \leq D_1\}$ has a positive measure so that $B_s \times B$ has a positive $\tilde{\nu}_H$-measure. Since $J$ is unbounded on $B_s$, $J(\eta)J(\eta')$ cannot be essentially bounded in $B_s \times B$. This is a contradiction and we have proved our original claim.

Now we recall the change of variable formulas for conformal and harmonic measures.
$$\frac{d(h^s\rho_o)}{d\rho_o}(\eta) = e^{\delta_H b_o, s(h^{-1}.o)}$$
$$\frac{d(h^s\nu_H)}{d\nu_H}(\tilde{o}_\xi(\xi)) = e^{G_i, s(h^{-1})}$$
Since the density of $\rho_o$ with respect to $\nu_H$ is uniformly bounded away from zero, there exists $C_1 > 0$
\[ |\text{ess sup}_{\eta \in \Lambda_H} \delta_H b_{o,\eta}(h^{-1} \cdot o) - \text{ess sup}_{\xi \in \partial H} b_{id,\xi}^G(h^{-1})| < C_1 \]
By [3, Lemma 2.5], there exists a constant $C_2 > 0$ such that
\[ |d_G(id, h^{-1}) - \text{ess sup}_{\xi \in \partial H} b_{id,\xi}^G(h^{-1})| < C_2 \]
Here we can replace ‘esssup’ by ‘sup’ because every open set in $\partial H$ has a positive $\nu$-measure and $b_{id,\xi}^G(h^{-1})$ is a continuous function with respect to $\xi$ by Lemma 2.4. We also have
\[ \sup_{\eta \in \Lambda_H} b_{o,\eta}(h^{-1} \cdot o) \leq d_{\mathbb{H}^3}(o, h^{-1} \cdot o) \]
by the triangle inequality. Therefore there exist $C_3$ (which may not be a positive number) such that for all $h \in H$,
\[ \delta_H d_{\mathbb{H}^3}(o, h^{-1} \cdot o) - d_G(id, h^{-1}) > C_3 \]
But by Lemma 2.1 there exists a sequence $\{h_n\} \subset H$ such that $|h_n^{-1}| - \delta_H d_{\mathbb{H}^3}(o, h_n^{-1} \cdot o)$ goes to infinity. Since the word metric is quasi-isometric to the Green metric, we have a contradiction. □

Note that even in the case of $\Lambda_H = S^2_{\infty}$, Theorem 1.2 is not a direct consequence of [3, Prop. 4.5]. In fact, [3, Prop. 4.5] is using [3, Prop. 4.4] which is valid for the case that $\tau_o$ is a quasi-isometry and $\hat{\tau}_o$ is the natural homeomorphism between $\partial H$ and $S^2_{\infty}$. For the case that $\tau_o$ is not a quasi-isometry, we need to assume the existence of the continuous boundary extension $\hat{\tau}_o$ of $\tau_o$.

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