Meson exchange between initial and final state and the $R_D$ ratio in the $B \to D\bar{\nu}\ell(\nu_\tau\tau)$ reactions

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(Dated: July 26, 2019)

We perform a calculation of the strong interaction effects between the $B$ and $D$ mesons in the $B \to D\bar{\nu}\ell$ reaction, as a crossing process of reactions with $BD$ in the final state, where the strong interaction between the mesons leads to a bound $BD$ state. We find corrections to the tree level amplitude of the order of $15\% - 25\%$. We further see the effect of the corrections studied in the $R_D$ ratio for the rates of $B \to D\bar{\nu}_\tau\tau$ and $B \to D\bar{\nu}_\ell\ell$ decays and find corrections of the order of $10\%$. Given the claims of $1\%$ precision in this ratio from fits to data within the standard model, any theoretical model aiming at describing this ratio within the same precision must take into account the corrections described in the present work.

PACS numbers:

I. INTRODUCTION

Different experiments [1–11] have reported values for the semileptonic $B$ decay ratios

$$R_D = \frac{BR(B^- \to D\bar{\nu}_\ell\tau)}{BR(B^- \to D\bar{\nu}_e\ell)} \quad \text{(with } \ell = e \text{ or } \mu \text{),}$$

which exceed the values provided by the Standard Model (SM). The amount of theoretical works offering plausible solutions to this puzzle with different extensions of the Standard Model is huge and we refer the reader to recent reviews on this topic [12–14].

In between the recent Belle data [15] have reduced the value of $R_D$ such that the discrepancies with the SM are significantly reduced. Following Ref. [13], the Heavy Flavor Averaging Group (HFLAV) values for 2018 and 2019, the latter one including the recent Belle data, are given in Table I which also shows the SM value for reference.

| TABLE I: HFLAV averages of $R_D$ for 2018 and 2019, together with the SM results. |
|---------------------------------|----------------------------|----------------|
| HFLAV2018 | HFLAV2019 | SM |
| $R_D$ | $0.407(39)(24)$ | $0.340(27)(13)$ | $0.312 (19)$ |

We can observe that the new HFLAV2019 values are already compatible with the SM predictions within errors. The new Belle alone data are [13]

$$R_D^{Belle} = 0.307 \pm 0.037 \pm 0.016,$$

even closer to the SM value.

In the SM, one writes the weak transition amplitudes in terms of form factors, which are conveniently parameterized [16–20]. Input from lattice QCD calculation [21,22] is also often used [13]. In [23] heavy quark effective theory (HQET) [24,25] is used, with corrections of order $\alpha_s$, $\frac{\Lambda_{QCD}}{m_b}$ and partly $\frac{\Lambda_{QCD}}{m_c}$, following [14], and the free parameters are fitted to data. Within this approach the value of $R_D = 0.300_{-0.005}^{+0.002}$ is reported, with errors smaller than the average in Table I and a value of $R_D$ very close to the new Belle data of Eq. (2). Similarly, in [26] a parameterization to data using form factors inspired on the Muskhelishvili-Omnès (MO) dispersion relation is done and the value $R_D = 0.301(5)$ is reported.

While it is unclear which effects from strong interaction are accounted for in parameterized form factors, it is our purpose here to perform explicitly one source of strong corrections, directly related to the final state interaction in semileptonic decays of heavy mesons, which leads to the formation of hadronic resonances in some cases, and are not part of the usual effects considered in some form factor evaluations, in particular quark models.

In [27] the $\bar{B}_s$ and $\bar{B}^0$ semileptonic decays into the $D_s(2317)$ and $D_s^0(2400)$, respectively, are studied from this perspective. The $\bar{B}_s$ decays to $\bar{\nu}\ell$ and a $c\bar{s}$ pair. After hadronization, generating a $q\bar{q}$ pair with the quantum numbers of the vacuum, a $DK$ or $D_s\eta$ pair is created, and these coupled channels interact strongly (final state interaction) to produce the $D_s(2317)$ [28,31]. Similarly, the $\bar{B}^0$ decays primarily into $c\bar{u}$, which after hadronization produces the $D^0\pi^0$, $D^+\pi^-$, $D^0\eta$, $D^+\bar{K}^-$ channels, which undergo final state interaction to produce the $D_s^0(2400)$ resonance [28,31]. Along similar lines, the $D_s$ and $D$ mesons are studied in [32] and their semileptonic decay leads to $\pi\pi$, $\pi\eta$, $\pi K$, $K\bar{K}$ final states, that upon interaction in coupled channels gives rise to the $f_0(500)$, $f_0(980)$ and $a_0(980)$ and $K_0^*(800)$ resonances. These resonances are generated dynamically from the interac-
tion of these channels, which is most effectively handled within the chiral unitary approach [33–36]. Similarly, the $\Lambda_b \rightarrow \bar{\nu}\Lambda_c(2595)$ and $\Lambda_b \rightarrow \bar{\nu}\Lambda_c(2625)$ reactions are investigated in [37] from the perspective that the $\Lambda_c(2595)$ and $\Lambda_c(2625)$ resonances are dynamically generated from the interactions of pseudoscalar-baryon and vector-baryon components [38]. Along the same lines the $\Xi^-_0 \rightarrow \bar{\nu}(\Xi^0_s(2790), \Xi^0(2815))$ reactions are studied in [39] from the perspective that the $\Xi^0_s(2790), \Xi^0(2815)$ are generated dynamically from the pseudoscalar-baryon and vector-baryon interactions [40]. Another example of work along these lines is the semileptonic decay of $B\rightarrow D\ell \nu$ where instead of having $W \rightarrow BD$ decay, the information on the $BD$ interaction is far less known than in the heavy-light sector and the MO approach is less predictive. Yet, it is still possible to use the MO formalism and parameterize the unknown information in terms of a few parameters which are adjusted to data. This is the procedure used in [41] to evaluate the form factors. It is also interesting to mention that $BD$ phase shifts induced in the analysis of [41] hint to the possible existence of a $BD$ bound state.

Our aim is to use the theoretical tools employed in the study of the $BD$ interaction in [41] and use them in the $B \rightarrow D\ell \nu$ reaction, both for light $\nu\ell$ and $\bar{\nu},\tau$, in order to see the effects of this interaction in the $R_D$ ratio.

II. FORMALISM

In this section we connect with the formalism of [41] and of [42] for the $B \rightarrow \bar{\nu}\ell D$, which are compared in [43]. In [44], a good approximation was found that related the different $B(\tau) \rightarrow \bar{\nu}D(\tau)$ processes which we shall find here. To see which process shall we need to consider, let us first proceed to pin down the diagrams that will be needed to account for the $BD$ interaction.

A. Crossing process accounting for the $BD$ interaction

Let us imagine we have a process depicted in Fig. 1(a) where a $W$ produces a $BD$ state. Following [45], the $BD$ state will interact by exchanging mesons, and in the intermediate states one can have other meson pairs that couple to $BD$, essentially, $B, D, s$, although its relevance is diminished by the large energy gap with $BD$. The exchange of the vector mesons is the essential ingredient in [45]. Actually, we could mix channels, $BD \rightarrow B^*D^*$, via pion exchange, which repercute in the $BD$ interaction via $BD \rightarrow B^*D^* \rightarrow BD$, but this was justified to produce small effects in [45] and indeed in [46] no bound $BD$ state was found with pion exchange.

The crossing process to Fig. 1 is given in Fig. 2. While in Fig. 1 one could in principle concentrate in a region close to the $BD$ threshold where the multiple scattering (Figs. (b)(c)... is important and leads to the bound $BD$
state, in Fig. 2 one is very far from this situation and we shall see that the strong interaction corrections are small effects, which justifies that we stop at the one meson exchange level. Taking into account the coupled channels that we have, the relevant diagrams that originate from this strong interaction are given in Fig. 3.

In diagrams (a)(b)(c) of Fig. 3 pseudoscalar meson exchange is not allowed. In diagram (d) η exchange would also be allowed but is suppressed by the large mass of the η. One can also exchange vector mesons in diagrams (d), (e), (f), but this involves anomalous vector–vector–pseudoscalar (VVP) couplings and these terms are suppressed [32]. In any case, we will find out that the terms with vector meson intermediate states give a very small correction, consistently with the findings from different works mentioned above.

We need two ingredients in the theory: The vector–pseudoscalar–pseudoscalar (VPP) couplings and the B → Dπ, B → D′π transitions. Let us first face the first issue.

B. The vector–pseudoscalar–pseudoscalar couplings

In SU(3) the VPP Lagrangian is given by

\[ \mathcal{L} = -ig \langle [P, \partial_\mu P] V^\mu \rangle \] (3)

where \( \langle \rangle \) stands for the trace and \( P \) and \( V^\mu \) are the ordinary SU(3) matrices for pseudoscalar mesons and vector mesons, respectively. The coupling \( g \) is given by

\[ g = \frac{m_V}{2f} \] (4)

with \( m_V \approx 800 \) MeV, a vector meson mass, and \( f = 93 \) MeV the pion decay constant. Since in Fig. 2 we exchange light mesons, the heavy quarks of the B or D mesons act as spectators and we can get the couplings making a mapping from the SU(3) space. In practice it is shown in [53] that the matrix elements needed in these diagrams are easily obtained using the flavor wave functions for the mesons, and equivalently by using the same Lagrangian of Eq. (3) in its SU(4) extension, using for \( P \) and \( V \) the \( q\bar{q} \) matrix elements in the meson basis.

\[ \begin{array}{c|ccc}
D^0 \rho^0 & D^0 & D^0 & D^0 \\
D^0 \rho^- & D^0 & D^0 & D^0 \\
D^+ K^0 & D^0 & D^0 & D^0 \\
D^0 K^+ & D^0 & D^0 & D^0 \\
\end{array} \]

Table II: Coefficients of Eq. (5) for the DDV vertex.

| \( C_i \) | \( \frac{\lambda}{\sqrt{2}} \) | \( \frac{\lambda}{\sqrt{2}} \) | \( \frac{\lambda}{\sqrt{2}} \) |
|-----------|------------------|------------------|------------------|
| \( \frac{\lambda}{\sqrt{2}} \) | 1                | 1                | 1                |

For the DDV vertices we use

\[ \begin{pmatrix}
\pi^0 \frac{\eta}{\sqrt{2}} + \pi^+ \frac{\eta}{\sqrt{6}} + \pi^- \frac{\eta}{\sqrt{3}} & K^+ & \bar{D}^0 \\
\pi^- \frac{\eta}{\sqrt{2}} + \pi^+ \frac{\eta}{\sqrt{6}} + \pi^- \frac{\eta}{\sqrt{3}} & K^+ & D^0 \\
K^- & K^0 & D^- \\
D^0 & \bar{D}^0 & \eta \\
\end{pmatrix} \]

For the BBV vertices we use

\[ \begin{pmatrix}
\pi^0 \frac{\eta}{\sqrt{2}} + \pi^+ \frac{\eta}{\sqrt{6}} + \pi^- \frac{\eta}{\sqrt{3}} & K^+ & B^0 \\
\pi^- \frac{\eta}{\sqrt{2}} + \pi^+ \frac{\eta}{\sqrt{6}} + \pi^- \frac{\eta}{\sqrt{3}} & K^+ & B^0 \\
K^- & K^0 & B^- \\
B^- & \bar{B}^0 & \eta \\
\end{pmatrix} \]

Then we obtain a transition \( t \) matrix for the DDV vertices

\[ t^{(i)}_{DDV} = C_i g(2P' - \eta) \mu \epsilon^\nu, \] (5)

with \( \epsilon_\mu \) the vector polarization and the \( C_i \) coefficients given in Table II. For the BBV vertices we get

\[ t^{(i)}_{BBV} = C_i g(2P - \eta) \mu \epsilon^\nu, \] (6)

with \( C_i \) given in Table III. For the \( D^*D^0 \) vertices we obtain

\[ t^{(i)}_{D^*D^0} = C_i g(P' + \eta) \mu \epsilon^\nu, \] (7)

with \( C_i \) given in Table IV. For the \( B^*B^0 \) vertices we obtain

\[ t^{(i)}_{B^*B^0} = C_i g(P + \eta) \mu \epsilon^\nu, \] (8)

with the \( C_i \) coefficients given in Table V.
and $Q^\alpha$ the quark current

$$Q^\alpha = \langle \bar{u}_c | \gamma^\alpha (1 - \gamma_5) | u_b \rangle$$  \hspace{1cm} (11)

In [61] the evaluation of the matrix elements is done in the $\nu \bar{\nu}$ rest frame where $\tilde{p}_B = \tilde{p}_D = \tilde{p}$ with $p$ given by

$$p = \frac{\lambda^{1/2}(m_{in}^2, m_{fin}^2, M_{inv}^2)}{2M_{inv}}$$  \hspace{1cm} (12)

with $m_{in}$, $m_{fin}$ the initial and final meson masses and $M_{inv}$ the invariant mass of the $\nu \bar{\nu}$ pair.

The quark spinors are written in terms of the momenta of the mesons, rather than the quarks, using the relationship for the four-momenta of the quarks, $b, c$ and the mesons $B, D$,

$$\frac{p_b}{m_b} = \frac{p_B}{m_B} = \frac{p_c}{m_c} = \frac{p_D}{m_D}$$  \hspace{1cm} (13)

This relationship was shown in [62] to be rather accurate, and it is strictly exact in the limit of infinity heavy quark mass. It is not surprising that the final expressions fulfill the heavy quark limit of infinite mass that allows one to relate the amplitudes to the universal Isgur-Wise function [63, 64]. One has there

$$\langle D, P' | Q_{\mu} | B, P \rangle \sqrt{m_B m_D} = (v + v')_\mu h_+(\omega) + (v - v')_\mu h_-(\omega)$$  \hspace{1cm} (14)

where

$$v = \frac{P}{m_B}, \hspace{0.5cm} v' = \frac{P'}{m_B}$$  \hspace{1cm} (15)

and

$$\omega = vv' = \frac{m_B^2 + m_D^2 - M_{inv}^2}{2m_B m_D}$$  \hspace{1cm} (16)
In the heavy quark limit, \( h_+ = \xi(\omega) \), \( h_- = 0 \) with \( \xi(\omega) \) the Isgur-Wise function. The expressions found in the formalism of [61] respect these properties and provide an explicit quantity for the \( \xi(\omega) \) function.

In the formalism of [61], by using the expression of Eq. (11), one writes the spinors as

\[
 u_r = A \left( \frac{\chi_r}{B \sigma \cdot \tilde{B} \chi_r} \right),
\]

\[
 A = \left( \frac{E_B/m_B + 1/2}{m_B \left( 1 + \frac{E_B}{m_B} \right)} \right)^{1/2}, B = \frac{1}{m_B \left( 1 + \frac{E_B}{m_B} \right)}
\]

and similarly for the \( D \) meson, with \( A', B' \) replacing \( A, B \).

For the \( B \to D \) transitions one finds in [61]

\[
\langle D | Q^0 | B \rangle \equiv M_0 = AA'(1 + BB'p^2)
\]

\[
\langle D | Q^\nu | B \rangle = N^\nu = AA'(B + B')p_\delta \omega_\nu;
\]

\[
N^\nu = AA'(B + B')p^\nu
\]

with \( p^\nu \equiv P^\nu \) of Eq. (14), where the index \( \nu \) in Eq. (20) refers to the spatial components of \( Q^\nu \) in spherical basis and the \( z \) axis is chosen along the \( B, D \) momentum \( \vec{P} \). The relationship of \( h_+(\omega) \) to \( A', B, B' \) is found in [62] as

\[
h_+ = \frac{\sqrt{m_B m_D}}{m_B + m_D} AA'(B + B').
\]

In [61] one also finds the expressions for \( Q^\nu \) for the case of \( B \to D^* \bar{\nu} \ell \) and \( B^* \to D^* \bar{\nu} \ell \). One can also write \( Q^\nu \) in terms of \( h_+ \), taking into account the Isgur-Wise scaling for heavy quarks, which is given in [62] for \( B \to D^* \bar{\nu} \ell \). For \( B^* \to D^* \bar{\nu} \ell \) one can also write an expression as in Eq. (14) and one finds

\[
\langle D^*, P | Q^\mu | B^*, P \rangle = h_+(\omega) [\epsilon^\mu_{B^*} (v \cdot \epsilon_D^*), + \epsilon_D^\mu (v' \cdot \epsilon_{B^*})
\]

\[
- \epsilon_D^\mu + \epsilon^\mu_{B^*} (\epsilon_D^* \cdot \epsilon_{B^*}) - i \epsilon_D^{\mu \nu \rho \sigma} \epsilon_{B^*} \cdot \epsilon_{D^*} (v_\sigma + v'_\sigma)]
\]

with \( \epsilon^{0123} = 1 \), with \( h_+ \) given by Eq. (21) using the masses of \( B^* \) and \( D^* \) instead of \( B \) and \( D \), and the same for \( v, v' \) of Eqs. (15).

D. Evaluation of the \( B \to D \bar{\nu} \ell \) correction terms with intermediate \( B, D \) pseudoscalar mesons

If we look at diagram (a) of Fig. 3 and Eqs. (5), (6), we find a vertex contribution of the type

\[
g(2P - q)_\mu \epsilon'_{\nu} g(2P' - q)_{\nu} \epsilon'.
\]

On the other hand for the evaluation of the loop function we shall only consider the positive energy part of the propagator for the heavy \( B \) and \( D \) mesons, that is, the first term of the decomposition

\[
\frac{1}{p^2 - m^2 + i\epsilon} = \frac{1}{2\omega(p)} \left\{ \frac{1}{p^0 - \omega(p) + i\epsilon} - \frac{1}{p^0 + \omega(p) - i\epsilon} \right\}
\]

(24)

with \( \omega(p) = \sqrt{p^2 + m^2} \). Thus, we have the integral

\[
I = i \int \frac{d^4q}{(2\pi)^3} \frac{1}{2\omega_1} \frac{1}{2\omega_2} \frac{1}{2\omega} \left\{ \frac{1}{q^0 - \omega + i\epsilon} - \frac{1}{q^0 + \omega - i\epsilon} \right\}
\]

(25)

with \( \omega_1 = \sqrt{m^2_B + (\vec{P} - \vec{q})^2} \), \( \omega_2 = \sqrt{m^2_B + (\vec{P'}) - \vec{q})^2} \), \( \omega = \sqrt{m^2_{\nu} + \vec{q}^2} \), where \( P^0 = \sqrt{m^2_B + \vec{P}^2} \), \( P'^0 = \sqrt{m^2_{\nu} + \vec{P'}^2} \), where for the light vector we keep the two terms. Note that \( \vec{P} = \vec{P'} \) in the \( \nu \bar{\nu} \) rest frame as we work. One can immediately see that using Cauchy’s integration the negative energy term of the vector propagator does not give a contribution and we readily find

\[
I = \int \frac{d^3q}{(2\pi)^3} \frac{1}{2\omega_1} \frac{1}{2\omega_2}
\]

(26)

and the \( i\epsilon \) can be removed since these denominators cannot vanish. While the particles in the loop cannot be simultaneously placed on shell, we see, however, that in the Cauchy integral we evaluate the residue of the pole of \( q^0 = \omega = \sqrt{q^2 + m^2_{\nu}} \). For practical purposes, the vector meson has on shell kinematics and then \( q_\mu \epsilon'_{\nu} \equiv 0 \) which allows to write the vertex combination of Eq. (23) as

\[
g2\epsilon'_{\mu} g2\epsilon_{\nu} \sum_{pol} \epsilon_{\rho} \epsilon'_{\sigma}
\]

\[
= 4g^2 \epsilon'_{\mu} g_{\nu} - g_{\mu \nu} + \frac{q_\mu q_\nu}{m^2_{\nu}}
\]

\[
= -4g^2 P \cdot P' + 4g^2 \frac{1}{m_{\nu}} (P \cdot q)(P' \cdot q)
\]

\[
= 4g^2 [-E_B E_D + \vec{F}^2 + \frac{1}{m^2_{\nu}} (E_D \omega - \vec{F} \cdot \vec{q})(E_D \omega - \vec{F} \cdot \vec{q})]
\]

(27)

which has to be placed inside the integrand of Eq. (25).

In addition we have to place the \( (Q^0), (Q^\nu) \) matrix elements of Eqs. (19) (20) inside the integral, evaluated for the loop momenta. Hence,

\[
M_0 \to AA'(1 + BB'(\vec{P} - \vec{q})^2),
\]
\( N^\nu \to N^i \to A A'(B + B')(P - q)^i \)

\( \equiv A A'(B + B')P^i \left( 1 - \frac{\vec{P} \cdot \vec{q}}{P^2} \right) \) \hspace{1cm} (29)

where \( A, A', B, B' \) are new functions of \( (\vec{P} - \vec{q})^2 \), and we have taken into account that \( \int d^3q f(\vec{p}, \vec{q})q^i = \alpha p_i = \rho \int d^3q \frac{\vec{p} \cdot \vec{q}}{p^2} f(\vec{p}, \vec{q}) \) with \( f(\vec{p}, \vec{q}) \) a scalar function. Hence we can see that the integral of \( N^i \) is proportional to \( \vec{P} \) which we have taken in the \( z \) direction in the tree level contribution to \( Q^n \), Eq. (20).

With all these ingredients it becomes straightforward to write to corrections to \( M_0 \) and \( N^i \equiv N^3 \) as

\[ T^0(1 + 2) = 2g^2 \int \frac{d^3q}{(2\pi)^3} \frac{1}{2\omega_1} \frac{1}{2\omega_2} \frac{1}{P^0 - \omega - \omega_1} \frac{1}{P^0 - \omega - \omega_2} \cdot AA'(1 + BB'(\vec{P} - \vec{q})^2 \cdot 4) \cdot \left[ -E_B E_D + \vec{P}^2 + (E_B \omega - \vec{P} \cdot \vec{q})(E_D \omega - \vec{P} \cdot \vec{q}) \right] \]

\[ = \frac{E_B}{M_B} + \frac{E_D}{M_D} \]

\[ T^3(1 + 2) = 2g^2 P \int \frac{d^3q}{(2\pi)^3} \frac{1}{2\omega_1} \frac{1}{2\omega_2} \frac{1}{P^0 - \omega - \omega_1} \frac{1}{P^0 - \omega - \omega_2} \cdot \left[ 1 - \frac{\vec{P} \cdot \vec{q}}{P^2} \right] AA'(B + B') \cdot 4 \cdot \left[ -E_B E_D + \vec{P}^2 + (E_B \omega - \vec{P} \cdot \vec{q})(E_D \omega - \vec{P} \cdot \vec{q}) \right] \]

\[ = \frac{E_B}{M_B} + \frac{E_D}{M_D} \] \hspace{1cm} (30)

where 1 and 2 in the parenthesis refer to the first and second diagrams of the first line in Fig. 3. For the third diagram \( T^0(3) \), we have the same expression changing \( 2g^2 \to g^2 \) and \( m^2_p \to m^2_{K^*} \), and the masses of the intermediate \( B, D \) states to those of \( B_2 \) and \( D_3 \). Next we take into account that for \( P = 0 \), corresponding to \( M^{(\text{ext})} \) maximum, \( \omega = 1 \), the Isgur-Wise function has a fixed value, thus, we make subtractions to our evaluated amplitudes to respect this fixed value. We define

\[ \bar{T}_P^0(1 + 2) = \frac{T^0(1 + 2) + \frac{E_B}{M_B} + \frac{E_D}{M_D}}{P} ; \bar{T}_P^0(3) = \frac{T^0(3) + \frac{E_B}{M_B} + \frac{E_D}{M_D}}{P} \] \hspace{1cm} (32)

\[ \bar{T}_P^3(1 + 2) = \frac{T^3(1 + 2)}{P} ; \bar{T}_P^3(3) = \frac{T^3(3)}{P} \] \hspace{1cm} (33)

and

\[ \bar{T}_P^0(1 + 2) = \frac{T^0_0(1 + 2) + \frac{E_B}{M_B} + \frac{E_D}{M_D}}{P} ; \bar{T}_P^0(3) = \frac{T^0_0(3) + \frac{E_B}{M_B} + \frac{E_D}{M_D}}{P} \]

\[ \bar{T}_P^3(1 + 2) = \frac{T^3_0(1 + 2) + \frac{E_B}{M_B} + \frac{E_D}{M_D}}{P} \]

where the subindex \( P \) stands for the intermediate pseudoscalars contributions. Then define the ratios \( R_P^0, R_P^3 \) as

\[ R_P^0 = \frac{\bar{T}_P^0(1 + 2) + \bar{T}_P^0(3)}{\bar{T}_P^0(1 + 2) + \bar{T}_P^0(3)} \left( \frac{E_B}{M_B} + \frac{E_D}{M_D} \right) \]

\[ R_P^3 = \frac{\bar{T}_P^3(1 + 2) + \bar{T}_P^3(3)}{\bar{T}_P^3(1 + 2) + \bar{T}_P^3(3)} \left( \frac{E_B}{M_B} + \frac{E_D}{M_D} \right) \] \hspace{1cm} (35)

The ratios give the relative change with respect to the tree level in the \( M^0 \) and \( N^i \) amplitude of Eqs. (19) (20).

E. Evaluation of the \( B \to D \bar{q} \ell \) correction terms with intermediate \( B^*, D^* \) states

We proceed to evaluate the last three diagrams of Fig. 3. From the structure of the \( Q^\mu \) matrix element in Eq. (22), we shall have three terms (the \( \epsilon^{\mu \nu \alpha \beta} \) term does not contribute when summing over the \( B^*, D^* \) polarization in the diagrams). We use real polarization vectors and have

1) \( \epsilon^{\mu}_{B^*}(P + q)_\alpha \epsilon^{\nu}_{B^*} v_\gamma \epsilon^{\gamma}_{D^*} \epsilon^{\alpha}_{D^*} \gamma(P' + q)_\gamma \)

the sum over polarization gives

\[ \sum_{\text{pol}} \epsilon^{\mu}_{B^*} \epsilon^{\nu}_{B^*} = -g^{\mu \nu} + \frac{(P - q)\alpha(P - q)\mu}{M_{B^*}^2} \] \hspace{1cm} (36)

\[ \sum_{\text{pol}} \epsilon^{\gamma}_{D^*} \epsilon^{\alpha}_{D^*} = -g^{\gamma \alpha} + \frac{(P' - q)\gamma(P' - q)\mu}{M_{D^*}^2} \] \hspace{1cm} (37)

and we obtain

\[ t_1^{\mu} = \left[ -(P + q)\mu + \frac{(P^2 - q^2)(P - q)\mu}{M_{B^*}^2} \right] \frac{-v \cdot (P' + q) + v \cdot (P' - q)(P' - q)\gamma}{M_{D^*}^2} \] \hspace{1cm} (38)

2) Similarly we can proceed with the second term of Eq. (22) and find

\[ t_2^{\mu} = \left[ -(P + q)\mu + \frac{(P^2 - q^2)\mu(P' + q)}{M_{D^*}^2} \right] \frac{-v \cdot (P' - q)(P' - q)\gamma}{M_{D^*}^2} \] \hspace{1cm} (39)
3) We proceed equally with the third term of Eq. (22) and find

$$t_3^0 = \left[-\left(P + q\right) \cdot \left(P' + q\right) - \frac{(P^2 - q^2)(P + q) \cdot (P' - q)}{M_D^2},
- \frac{(P^2 - q^2)(P - q) \cdot (P' + q)}{M_B^2},
+ \frac{(P^2 - q^2)(P^2 - q^2)(P - q) \cdot (P' - q)}{M_B^2 M_D^2}\right] (\nu^\mu + \nu^\nu),$$

(40)

One can further recall that in the \(q^0\) integration in the loop function of Eq. (23), \(q^0\) becomes \(\sqrt{q^2 + m^2}\) with \(m\) the mass of the pseudoscalar meson exchanged, and thus \(q^2 \rightarrow m^2\). We can further evaluate \(t_3^0\) for \(\mu = 0\), \(\mu = i\) explicitly and we find the terms,

$$t_1^0 = \left[-(E_B + \omega) + \frac{(M_B^2 - m^2)(E_B - \omega)}{M_B^2},
- \frac{(E_B - \omega)(E_D + \omega) - \vec{P}^2 + q^2}{M_B^2},
+ \frac{(E_B - \omega)(E_D - \omega) - \left(\vec{P} - \vec{q}\right)^2}{M_B^2 M_D^2}\right] (M_D^2 - m^2),$$

(41)

$$t_2^0 = \left[-(E_D + \omega) + \frac{(E_D - \omega)(M_B^2 - m^2)}{M_B^2},
- \frac{(E_D - \omega)(E_B + \omega) - \vec{P}^2 + q^2}{M_D^2},
+ \frac{(E_D - \omega)(E_B - \omega) - \left(\vec{P} - \vec{q}\right)^2}{M_D^2 M_B^2}\right] (M_B^2 - m^2),$$

(42)

$$t_3^0 = \left[-\left(E_B - \omega + \frac{E_D - \omega}{M_D^2}\right),
\left(E_B + \omega\right)(E_D + \omega) - \left(\vec{P} + \vec{q}\right)^2
- (M_B^2 - m^2)\left(E_B + \omega\right)(E_D - \omega) - \vec{P}^2 + q^2\right] \frac{1}{M_D^2},
- (M_B^2 - m^2)\left(E_B - \omega\right)(E_D + \omega) - \vec{P}^2 + q^2\right] \frac{1}{M_B^2},
+ \frac{1}{M_B^2 M_D^2}\left(M_B^2 - m^2\right)\left(M_B^2 - m^2\right)
\left(E_B - \omega\right)(E_D - \omega) - \left(\vec{P} - \vec{q}\right)^2\right]\right],$$

(43)

\[ t_1^3 = P \left[-\left(1 + \frac{\vec{P} \cdot \vec{q}}{P^2}\right) + \left(1 - \frac{\vec{P} \cdot \vec{q}}{P^2}\right)\right] \left(M_B^2 - m^2\right) \frac{1}{M_B^2},
\left[- \frac{(E_B - \omega)(E_D + \omega) - \vec{P}^2 + q^2}{M_B^2},
+ \frac{(E_B - \omega)(E_D - \omega) - \vec{P}^2 + q^2}{M_B^2 M_D^2}\right] (M_B^2 - m^2),$$

(44)

\[ t_2^3 = P \left[-\left(1 + \frac{\vec{P} \cdot \vec{q}}{P^2}\right) + \left(1 - \frac{\vec{P} \cdot \vec{q}}{P^2}\right)\right] \left(M_B^2 - m^2\right) \frac{1}{M_B^2},
\left[- \frac{(E_B - \omega)(E_D + \omega) - \vec{P}^2 + q^2}{M_D^2},
+ \frac{(E_B - \omega)(E_D - \omega) - \vec{P}^2 + q^2}{M_D^2 M_B^2}\right] (M_B^2 - m^2),$$

(45)

\[ t_3^3 = -P \left[1 + \frac{\vec{P} \cdot \vec{q}}{P^2}\right] \left(\frac{1}{M_B^2} + \frac{1}{M_D^2}\right),
\left[(E_B + \omega)(E_D + \omega) - \vec{P}^2 + q^2\right] \frac{1}{M_B^2},
- (M_B^2 - m^2)\left(E_B + \omega\right)(E_D - \omega) - \vec{P}^2 + q^2\right] \frac{1}{M_D^2},
+ \frac{1}{M_B^2 M_D^2}\left(M_B^2 - m^2\right)\left(M_D^2 - m^2\right)
\left(E_B - \omega\right)(E_D - \omega) - \left(\vec{P} - \vec{q}\right)^2\right]\right],$$

(46)

where \(P = (P^0, \vec{P}), P' = (P'^0, \vec{P}')\).

It is worth noting that, in spite of the apparent extra two powers in \(q\) from Eqs. (40) and (41) from the \(B^*\) to \(D^*\) propagators, the \(t_1^0, t_3^0\) terms are of the same order in \(q\) as the amplitude of Eq. (27) for the case of intermediate \(B, D\) pseudoscalar mesons. This can be seen from a cancellation of the \(O(q^3)\), \(O(q^4)\) terms in \(t_1^0, t_3^0\) of Eqs. (11) and (15).

Together with the integral of Eq. (26) we obtain the terms contributing to the corrections to \(M^0\) and \(N^3\), \(t^0\) and \(t^3\) as

$$T_0^0(1 + 2) = \frac{3}{2} \frac{g^2}{(2\pi)^3} \frac{d^3q}{2\omega^2} \frac{1}{2\omega_1} \frac{1}{2\omega_2} \frac{1}{P^0 - \omega - \omega_1 + i\epsilon} \frac{1}{P^0 - \omega - \omega_2 + i\epsilon} \frac{1}{AA'(B + B')\frac{M_B^2 M_D^2}{M_B^2 + M_D^2}} \cdot (t_1^0 + t_2^0 + t_3^0).$$

(47)
where now \( \omega = \sqrt{q^2 + m^2} \), \( \omega_1 = \sqrt{m_B^2 + (\vec{P} - \vec{q})^2} \), 
\( \omega_2 = \sqrt{m_D^2 + (\vec{P} - \vec{q})^2} \). \( \bar{T}_V^3(3) \) has the same expression but changing \( \frac{3}{2} g^2 \to g^2 \) 
and \( m_x \to m_K \). And we must take into account that, \( A, A', B, B' \) are now functions of \((\vec{P} - \vec{q})^2\).

Similarly
\[
T_V^3(1 + 2) = \frac{3}{2} g^2 \int \frac{d^3q}{(2\pi)^3} \frac{1}{2\omega_1 2\omega_2} \frac{1}{P^0 - \omega - \omega_1 + i\epsilon} P^0 - \omega - \omega_2 + i\epsilon
\]
\[
\cdot AA'(B + B') \frac{M_B M_D}{M_{B'} + M_D},
\]
\[
\cdot (t_1^2 + t_2^2 + t_3^2).
\]

\( \bar{T}_V^3(3) \) has the same expression but changing \( \frac{3}{2} g^2 \to g^2 \) 
and \( m_x \to m_K \) and the masses of the intermediate \( B, D \) 
states to those of \( B_s \) and \( D_s \).

The next step is to subtract the contribution to the 
Isgur-Wise function at \( P = 0 \). For this, we define
\[
\tilde{T}_V^0(1 + 2) = \frac{T_V^0(1 + 2)}{E_B + E_D}, \quad \tilde{T}_V^3(3) = \frac{T_V^3(3)}{M_B + E_D},
\]
\[
\bar{T}_V^3(1 + 2) = \frac{\bar{T}_V^3(1 + 2)}{P}, \quad \bar{T}_V^3(3) = \frac{\bar{T}_V^3(3)}{P}
\]
and define the functions \( \tilde{T}_V^0(1 + 2), \tilde{T}_V^3(3), \bar{T}_V^3(1 + 2), \bar{T}_V^3(3) \), 
as in Eqs. (49) subtracting the values at \( P = 0 \) of the terms of Eq. (49). After that, the relative 
changes for \( M_0 \) and \( N^3 \) of the tree level contributions of 
Eqs. (10) are given respectively by
\[
R_V^0 = \frac{\tilde{T}_V^0(1 + 2) + \tilde{T}_V^3(3)}{AA'(1 + BB'P^2)} \left( \frac{E_B}{M_B} + \frac{E_D}{M_D} \right),
\]
\[
R_V^3 = \frac{\bar{T}_V^3(1 + 2) + \bar{T}_V^3(3)}{AA'(B + B')}.
\]

Finally, let us see how the changes obtained in 
the \( \bar{v} \ell \) invariant mass distribution \( dt/dM_{iv}(\nu\ell) \). In [61] the differential invariant mass distribution was found as
\[
\frac{d\Gamma}{dM_{inv}(\nu\ell)} = \frac{1}{(2\pi)^3} \frac{1}{M_B P_D \hat{p}_\nu} \sum |t|^2
\]
where
\[
\sum \sum |t|^2 = (AA')^2 \left\{ \frac{m_2^2(M_{iv}(\nu\ell)^2 - m_2^2)}{M_{inv}(\nu\ell)} (1 + BB'P^2)^2 \right. 
\]
\[
+ 2\tilde{E}_\nu \tilde{E}_\ell + \frac{1}{3} \tilde{p}_\nu^2 \tilde{p}_\ell^2 \right\}(B + B')^2 P^2
\]
In Eq. (53), the first term comes from \( M_0^2 \) and the second 
term from \((N^3)^2\). It is then clear how this is renormalized 
now. Eq. (52) is the same but \( \sum \sum |t|^2 \) is changed to
\[
\sum \sum |t|^2 = (AA')^2 \left\{ \frac{m_2^2(M_{inv}(\nu\ell)^2 - m_2^2)}{M_{inv}(\nu\ell)} (1 + BB'P^2)^2 \right. 
\]
\[
\left. \cdot (1 + R^0_P + R^0_P)^2 + 2(\tilde{E}_\nu \tilde{E}_\ell + \frac{1}{3} \tilde{p}_\nu^2 \tilde{p}_\ell^2)(B + B')^2 P^2 \right. 
\]
\[
\left. (1 + R^3_P + R^3_P)^2 \right\}.
\]

### III. RESULTS

In the first place we should stress that what we have 
calculated is a part of the form factor and other ingre-
dients would complement what we have done. Indeed, if 
we look at Fig. 1, we would also get a contribution to 
the form factor from the tree level of Fig. 1(a) which is 
factorized in all the terms (b), (c), \( \cdots \). The global 
amplitude is then given, for instance with one intermediate 
channel, by

\[
f(M_{inv}(BD)) \left[ 1 + G(M_{inv}(BD))T_{BD,BD}(M_{inv}(BD)) \right].
\]

Similarly, in the diagram of Fig. 2, and concretely in the 
one of the Fig. 2(b) we have the form factor of the \( WBD \) 
vertex as a function of \( M_{inv}(\nu\ell) \). In the picture of [61] it is included in the expression of \( M_0 \) in Eq. (19) which 
depends on \( p \), given in Eq. (12) as a function of \( M_{inv}(\nu\ell) \). 
As shown in [62], this falls short of the structure of the 
empirical form factor because the form factor coming 
from the intrinsic quark wave functions of the mesons 
is not implemented. This means that to complete a 
microscopical picture of the form factor to be compared 
with the empirical one [60], one should perform a quark 
model calculation of these intrinsic form factors, as done in [50]. Conversely, we could say that a quark model 
calculation of the form factor should be complemented with 
our contribution.

This said, let us show our results. In Fig. 4 we show the results for \( R^0_0, R^0_0, R^0_0, R^0_3 \) as a function of \( M_{inv}(\nu\ell) \) 
for \( \bar{v} \ell \) production. The amplitudes that we have calculated 
\( T^0, T^3 \) are logarithmically divergent. They converge 
after the subtraction in \( p = 0 \), but following the steps in 
the study of meson-meson interaction we regularize the 
loops by means of a cutoff in \( |q^2|, q_{\text{max}}, \) of the order of 
800 MeV. By construction all these factors are zero at 
\( M_{inv}(\nu\ell) \) maximum. As we can see, \( R^0_0, R^0_3 \), reach sizes 
of as much as 25% around \( M_{inv}(\nu\ell) \approx 0 \). This means 
that the corrections that we have evaluated are relevant 
in a microscopical calculation of the form factors. 
The other point worth mentioning is that \( R^0_0, R^0_3 \) are 
comparatively very small and can be neglected. This means 
that the intermediate \( B, D \) pseudoscalar mesons are the 
relevant elements in the corrections that we evaluate.
FIG. 4: Results for $R_0^0, R_3^0, R_0^3, R_3^3$ as a function of $M_{\text{inv}}(\nu\ell)$ for $\bar{B} \to D\bar{\nu}\ell$. A cutoff $q_{\text{max}} = 800$ MeV is taken in the integrals.

In Fig. 5 we show the same results for the reaction $\bar{B} \to D\bar{\nu}\tau$. The results are similar although the range of $M_{\text{inv}}(\nu\ell)$ is now more restricted.

In Fig. 6 we show $\frac{d\Gamma}{dM_{\text{inv}}(\nu\ell)}$ for $\bar{B} \to D\bar{\nu}\ell$ with and without the corrections done here; line a represents tree level; line b with a factor of $(1 + R_0^0 + R_0^3)^2$; line c with a factor of $(1 + R_3^0 + R_3^3)^2$; line d with both factors $(1 + R_0^0 + R_0^3)^2$ and $(1 + R_3^0 + R_3^3)^2$.

Finally, we would like to see which is the effect of the corrections done in the ratio $R_D$ of Eq. (1). We show the branching ratios $R_D$ for different values of $q_{\text{max}}$ in Table VI.

In Table VI we see that we obtain $R_D \simeq 0.23$ from the tree level. This is a bit short of the SM value $R_D \simeq 0.30$ quoted in the Introduction, but a fair result considering that it is a pure theoretical result with no free parameters.

![Graph](image1)

![Graph](image2)

![Graph](image3)

![Graph](image4)

![Graph](image5)

![Graph](image6)

![Graph](image7)

TABLE VI: Branching ratios $R_D$ changing with $q_{\text{max}}$.

| $q_{\text{max}}$ | a | b | c | d |
|------------------|---|---|---|---|
| 0.7 GeV           | 0.228 | 0.240 | 0.194 | 0.204 |
| 0.8 GeV           | 0.228 | 0.243 | 0.185 | 0.196 |
| 1.0 GeV           | 0.228 | 0.250 | 0.166 | 0.181 |
and no fit to data. Taking $q_{\text{max}} \approx 700 \text{ MeV}$, close to values used in \cite{53,67}, we have $R_D \approx 0.204$. What the results of Table \ref{tab:BD} tell us is that the corrections that we have studied here are responsible for a 10\% change of this ratio. This is a moderate effect, which however gains more strength when it is weighed with respect to the 1.5\% error claimed in the SM results in the analyses of \cite{23} and \cite{26}. This means that in a theoretical evaluation aiming at such a precision, the consideration of the effects evaluated here is a must.

IV. CONCLUSIONS

We have performed a theoretical calculation of the strong interaction corrections between the initial and final meson in the $B \to D\bar{\nu}\ell$ decay. This is the analog of the final state interaction in processes where a $B\bar{D}$ pair is produced at the end. The existence of calculations in which the strong interaction between $B$ and $D$ leads to a bound state indicates that the same interaction in the crossed channel $\bar{B} \to D\ell\bar{\nu}$ should be also relevant. We have performed this evaluation using the same ingredients as those used to bind the $B\bar{D}$ states and we obtain corrections to the tree level $B \to D\bar{\nu}\ell$ amplitudes of the order of $15\% - 25\%$, which are relevant in a theoretical calculation. We also explain that the full theoretical evaluation of the form factor in the $\bar{B} \to D\bar{\nu}\ell$ reaction would require the calculation of the $B \to D$ transitions using quark wave functions for the meson states in addition to the strong interaction corrections evaluated here.

We used the results obtained here to see the effects of these strong corrections in the $R_D$ ratio for $\nu_\tau\tau$ and $\nu\ell\bar{\nu}$ production and we found effects of the order of 10\%. This means that if one wishes to do a theoretical calculation of this ratio with the precision of 1.5\% claimed in fits to data within the Standard Model, the effects studied here must be necessarily considered.

Acknowledgments

We thank J. Nieves for useful discussions. N.I. acknowledges the support from JSPS Overseas Research Fellowships and JSPS KAKENHI Grant Number JP19K14709. LRD acknowledges the support from the National Natural Science Foundation of China (Grant No. 11575076). This work is partly supported by the Spanish Ministerio de Economía y Competitividad and European FEDER funds under Contracts No. FIS2017-84038-C2-1-P B and No. FIS2017-84038-C2-2-P B, and the Generalitat Valenciana in the program Prometeo II-2014/068, and the project Severo Ochoa of IFIC, SEV-2014-0398 (EO).

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