Abstract  The main purpose of this paper is study and investigate a skew-commuting and skew-centralizing $d$ and $g$ be a derivations on noncommutative prime ring and semiprime ring $R$, we obtain the derivation $d(R)=0$ (resp. $g(R)=0$).

Keywords  Skew-commuting, Derivation, Noncommutative Prime Ring, Semiprime Ring

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1. Introduction

Derivations on rings help us to understand rings better and also derivations on rings can tell us about the structure of the rings. For instance a ring is commutative if and only if the only inner derivation on the ring is zero. Also derivations can be helpful for relating a ring with the set of matrices with entries in the ring (see, [5]). Derivations play a significant role in determining whether a ring is commutative, see ( [1],[3],[4],[18],[19] and [20]). Derivations can also be useful in other fields. For example, derivations play a role in the calculation of the eigenvalues of matrices (see, [2]) which is important in mathematics and other sciences, business and engineering. Derivations also are used in quantum physics (see, [18]). Derivations can be added and subtracted and we still get a derivation, but when we compose a derivation with itself we do not necessarily get a derivation. The history of commuting and centralizing mappings goes back to (1955) when Divinsky [6] proved that a simple Artinian ring is commutative if it has a commuting nontrivial automorphism. Two years later, Posner[7] has proved that the existence of a non-zero centralizing derivation on prime ring forces the ring to be commutative (Posner's second theorem). Luch [8] generalized the Divinsky result, we have just mentioned above, to arbitrary prime ring. In[9] M.N.Daif, proved that, let $R$ be a semiprime ring and $d$ a derivation of $R$ with $d^2\neq 0$.If $[d(x),d(y)]=0$ for all $x,y \in R$, then $R$ contains a non-zero central ideal. M.N.Daif and H.E. Bell [10] proved that, let $R$ be a semiprime ring admitting a derivation $d$ for which either $xy+d(xy)=yx+d(yx)$ for all $x,y \in R$ or $xy-d(xy)=yx-d(yx)$ for all $x,y \in R$, then $R$ is commutative. V.DeFilippis [11] proved that, when $R$ be a prime ring let $d$ a non-zero derivation of $R$, $U\neq(0)$ a two-sided ideal of $R$, such that $d([x,y])=[x,y]$for all $x,y \in U$, then $R$ is commutative. Recently A.H. Majeed and Mehsin Jabel [12], give some results as , let $R$ be a 2-torsion free semiprime ring and $U$ a non-zero ideal of $R$.R admitting a non-zero derivation $d$ satisfying $d([d(x),d(y)])=[x,y]$ for all $x,y \in U$. If $d$ acts as a homomorphism, then $R$ contains a non-zero central ideal. Our aim in this paper is to investigate skew-commuting $d$ and $g$ be derivations on noncommutative prime ring and semiprime ring $R$.

2. Preliminaries

Throughout $R$ will represent an associative ring with identity, $Z(R)$ denoted to the center of $R$, $R$ is said to be $n$-torsion free, where $n \neq 0$ is an integer, if whenever $nx=0$, with $x \in R$, then $x = 0$. We recall that $R$ is semiprime if $xRx = (0)$ implies $x = 0$ and it is prime if $xRy = (0)$ implies $x = 0$ or $y = 0$. A prime ring is semiprime but the converse is not true in general. An additive mapping $d: R \rightarrow R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x,y \in R$, and is said to be $n$-centralizing on $U$ (resp. $n$-commuting on $U$), if $[x^n,d(x)] \in Z(R)$ holds for all $x \in U$ (resp. $[x^n,d(x)]=0$ holds for all $x \in U$, where $n$ be a positive integer. Also is called skew-centralizing on subset $U$ of $R$ (resp. skew-commuting on subset $U$ of $R$) if $d(x)x+xd(x) \in Z(R)$ holds for all $x \in U$ (resp. $d(x)x+xd(x)=0$ holds for all $x \in U$), and $d$ acts as a homomorphism on $U$ (resp. anti-homomorphism on $U$) if $d(xy)-d(x)d(y)$ holds for all $x,y \in U$ (resp. if $d(xy)=d(y)d(x)$ holds for all $x,y \in U$). We write $[x,y]$ for $xy-yx$ and make extensive use of basic commutator identities $[x,y]z=x[y,z]+[x,z]y$ and $[x,yz]=[x,y]z+x[y,z]$. In some parts of the proof our theorems(3.1 and 3.2), we using same technique in [21].

First we list the lemmas which will be needed in the sequel.

Lemma1[7]
If d is commuting derivation on noncommutative prime ring, then d=0.

Lemma 2 [13: Theorem 1.2]
Let S be a set and R a semiprime ring. If functions d and g of S into R satisfy \( d(s)xg(t) = g(s)dx(t) \) for all \( s,t \in S \), \( x \in R \), then there exists idempotents \( a_i, \ alpha_1, \ alpha_2 \in C \) such that \( \alpha_1 \alpha_2 + \alpha_2 \alpha_1 = 1 \), and \( \alpha_i g(s) = \alpha_i g(s) = 0 \), \( \alpha_3 d(s) = 0 \) hold for all \( s \in S \).

Lemma 3 [14: Theorem 2]
Let R be a 2-torsion free semiprime ring. If functions d and g be a derivations of R and w a non-zero element of \( \text{Z}(R) \), we obtain
\[
\begin{align*}
\text{d}(x)w + d(w)x + xg(w) + wg(x) &= 0 \\
\text{for all } x \in R.
\end{align*}
\]
Let R be a semiprimering and U a non-zero ideal of R. Then \( \text{d}(s)xg(t) = g(s)xd(t) \) for all \( s,t \in U \), then d is commuting derivation on R, then d=0.

Lemma 4 [15: Lemma 4]
Let R be a semiprimering and U a non-zero ideal of R. If d is a derivation of R which is centralizing on U, then d is commuting on U.

3. The Main Results

Theorem 3.1
Let R be a noncommutative prime ring, \( d \) and \( g \) be a derivations of R. If R admits to satisfy \( \text{d}(x)x + xg(x) \in \text{Z}(R) \) for all \( x \in R \), then \( d(R) = 0 \) (resp. \( g(R) = 0 \)) or \( wd \) (resp. \( wg \)) is central and \( wR \) is a prime ring, thus \( w = 0 \).

Proof: At first we suppose there exists an element say \( w \in R \), such that \( w \in \text{Z}(R) \). Let \( w \) be a non-zero element of \( \text{Z}(R) \). By linearizing our relation
\[
\text{d}(x)x + xg(x) \in \text{Z}(R) \quad \text{for all } x,y \in R.
\]
Taking \( y = w \) in (1), we get
\[
\text{d}(x)w + d(w)x + xg(w) + wg(x) \in \text{Z}(R) \quad \text{for all } x \in R.
\]
Again in (1) replacing \( y \) by \( w^2 \), we obtain
\[
\text{d}(x)w^2 + d(w^2)x + xg(w^2) + w^2g(x) \in \text{Z}(R) \quad \text{for all } x \in R,
\]
which implies
\[
\text{d}(x)w^2 + d(w^2)x + xg(w^2) + w^2g(x) = 0.
\]
According to (2) the relation (4) gives
\[
\text{w}(d(x)x + xg(w)) + d(w)x + xg(w) + wg(x) \in \text{Z}(R) \quad \text{for all } x \in R.
\]
Subtracting (13) and (12), we obtain
\[
\text{sg}(w,u) + \text{ug}(w,u) = 0 \quad \text{for all } u,v \in R.
\]
Since \( w \in \text{Z}(R) \), then (5) gives
\[
\text{d}(w)x + xg(w) = 0 \quad \text{for all } x,y \in R.
\]
Also from (2), we obtain
\[
\text{d}(w)x + xg(w) + wg(x) = 0 \quad \text{for all } x,y \in R. \quad \text{for all } x,y \in R.
\]
Now from (6) and (7), we obtain
\[
\text{w}(d(x)x + xg(w),y) = 0 \quad \text{for all } x,y \in R.
\]
Replacing y by \( zy \), with using (8), we get
\[
\text{w}(d(x)x + xg(w),y) = 0 \quad \text{for all } x,y \in R,
\]
which implies
\[
\text{w}(d(x)x + xg(w),y) = 0 \quad \text{for all } x,y \in R.
\]
Replacing z by \( d(x)x + xg(w) \) and since R is prime ring, which implies
\[
\text{w}(d(x)x + xg(w),y) = 0 \quad \text{for all } x,y \in R.
\]
If \( w = 0 \), then obviously, we obtain \( d(R) \) (resp. \( g(R) \)) is
central for all w ∈ Z(R).

**Theorem 3.2**

Let R be a noncommutative prime ring, d be a skew-centralizing derivation of R (resp. g be a skew-centralizing derivation of R), if R admits to satisfy d(x)x+xg(x) ∈ Z(R) for all x ∈ R. Then d(R)=0 (resp. g(R)=0).

**Proof:** Let xo ∈ R and c=d(xo)xo+xog(xo). Thus, according to our hypothesis, we obtain c ∈ Z(R). Then by Theorem 3.1, we get cd and cg are commuting, then [cd(x),y]=0 for all x,y ∈ R. Then
cd(x)y=ycd(x) for all x,y ∈ R.

Since c ∈ Z(R), then above relation become
d(x)y=ycd(x) for all x,y ∈ R. (14)

Now taking S=R, g(x)=c with applying Lemma 2 to (14), we obtain that there exist idempotents α₁, α₂, α₃ ∈ C and an invertible element λ ∈ C such that

a₁a₂=0 for i≠j, a₁+a₂+α₃=1, and α₃d(x)=λα₁c, α₂c=0, a₃d(x)=0 for all x ∈ R. (15)

For the first identity of (15) replacing x by xy and using it again, we obtain

λα₁c=α₂d(xy)=α₁d(y)+α₂d(y)=λα₁c for all x,y ∈ R. Then

λα₁c=λα₁c+yλα₁c for all x,y ∈ R. (16)

Replacing y by −x in (16), we obtain

λα₁c=λα₁c+xλα₁c for all x,y ∈ R. (17)

Thus, we get

λα₁c=0. Therefore, the first identity of (15) become

λα₁c=α₁d(x) for all x ∈ R. Hence, using (15), we obtain
d(x)=(α₁+,α₂+α₃)d(x)=α₂d(x) for all x ∈ R. Then
cd(x)=α₂d(x)=α₂cd(x) for all x ∈ R. Then, from second identity in (15), we obtain cd(x)=0 for all x ∈ R. Since cg is commuting, then cg is central, therefore, analogously, it follows that cg(x)=0 for all x ∈ R. Hence

cd(x)x=0 and xcd(x)=cxg(x)=0 for all x ∈ R. Thus from these relations, we obtain cd(x)x+g(x)=0 for all x ∈ R.

In particular, c(d(xo)xo+xog(xo))=c²=0. Since a semiprime ring has no nonzero central nilpotent, therefore, we get c=0, which implies d(xo)xo+xog(xo)=0. Since xo is an arbitrary element of R, therefore
d(x)x+xg(x)=0 for all x ∈ R. (18)

If we taking d(x)=g(x), then
d(x)x+xd(x)=0 for all x ∈ R. Then by using Lemma 3, we obtain d(R)=0 (resp. g(R)=0).

If d(x)≠g(x), this case lead to d(x)x+xg(x) ∈ Z(R) for all x ∈ R. By Theorem 3.1, we complete our proof.

**Theorem 3.3**

Let R be a 2-torsion free semiprime ring with cancellation property. If R admits a derivation d to satisfy

(i) d acts as a skew-commuting on R.

(ii) d acts as a skew-centralizing on R. Then d(R) is commuting of R.

**Proof:** (i) Since d is skew-commuting, then
d(x)x+xd(x)=0 for all x ∈ R. (18)

Left –multiplying (18) by x, we obtain

xd(x)x+2xd(x)=0 for all x ∈ R. (19)

From (18), we get

d(x)x=0 for all x ∈ R. (20)

In (20) replacing x by x+y, we obtain
d(x+y)d(x+y)+d(y)x+yd(x)=o for all x,y ∈ R. According to (20), a above equation become
d(x+y)d(x+y)=0 for all x,y ∈ R. Then
d(x+y)d(y)+yd(x)=0 for all x,y ∈ R. (21)

Replacing y by x² and according to (20), we arrived to
d(x)x²+2xd(x)=0 for all x ∈ R. (22)

Then x²d(x)=−d(x)x² for all x ∈ R.

By substituting (21) in (19), we get

xd(x)x²=0 for all x ∈ R. Then

[x,R]=0 for all x ∈ R. Then apply the cancellation property on x,

we get, we obtain

[x,d(x)]=0 for all x ∈ R. Then d(R) is commuting of R.

(ii) We will discuss, when d acts as a skew- centralizing on R.

Then we have d(x)x+xd(x) ∈ Z(R) for all x ∈ R.

d(x²) ∈ Z(R) for all x ∈ R. i.e.

d(x²)=0 for all x, r ∈ R. (22)

Also, by replacing r by x in (22), we obtain
d(x)r =0 for all x,y ∈ R.

Then

d(x)=0 for all x,y ∈ R. Then

d(x)x²−x²d(x)=0 for all x ∈ R. Then

[d(x),x²]=0 for all x ∈ R. (23)

In (22) , replacing x by x+y, we obtain

[d(x²)+d(x²)+d(y²)+d(x²)r] for all x,y,r ∈ R. According to (22), we obtain

[d(y)]=0 for all x,y,r ∈ R. (24)

Replacing y by x², we obtain

[d(x²)+d(x²)+d(x²)+d(x²)r] for all x,y,r ∈ R. According to (22) and (23) , we get

[x²d(x)+d(x²)]=0 for all x,y,r ∈ R. Then

2[x²d(x)d(x²)+d(x²)r]=0 for all x,y,r ∈ R.

Since R is 2-torsion free, we obtain

[x²d(x)+d(x²)]r=0 for all x,r ∈ R.

According to (22), we have

[x²d(x)+d(x²)+d(x²)+d(x²)r] for all x,y,r ∈ R. Then

[x²d(x)+d(x²)+d(x²)+d(x²)r] for all x,r ∈ R. Then

2[x²d(x)+d(x²)]r=0 for all x,y,r ∈ R.

Replacing r by x, we obtain

x²d(x)x+dx(x²)r=0 for all x ∈ R. Then

Then x²d(x)x=0 for all x ∈ R. Apply the cancellation property on x², we get

[d(x),x]=0 for all x ∈ R. We complete the proof of theorem.
Theorem 3.4

Let \( R \) be a 2-torsion free noncommutative prime ring. If \( R \) admits a derivation \( d \) to satisfy one of following
(i) \( d \) acts as a homomorphism on \( R \). Then \( d(R)=0 \).
(ii) \( d \) acts as an anti-homomorphism on \( R \). Then \( d(R)=0 \).

Proof: (i) \( d \) acts as a homomorphism on \( R \). We have \( d \) is a derivation, then
\[
d(xy)=d(x)y+xd(y) \quad \text{for all } x, y \in R.
\]
Then \( [d(xy),r]=[d(x)y]+[xd(y),r] \) for all \( x, y, r \in R \). Since \( d \) acts as a homomorphism, then \( [d(x)y],r=[d(xy),r]+[xd(y),r] \) for all \( x, y, r \in R \). Replacing \( r \) by \( d(y) \), we obtain
\[
[d(x),d(y)]d(y)=[d(x)y,d(y)]+[xd(y),d(y)] \quad \text{for all } x, y \in R.
\]
Replacing \( y \) by \( x \), we obtain
\[
d(x)[x,d(x)]+[x,d(x)]d(x)=0.
\]
Since \( \text{char. } R=2 \), then
\[
d(x)x+xd(x)=0.
\]
We complete the proof of (ii) as it is shown in the following example.

Example 3.6

Let \( R \) be the ring of all \( 2 \times 2 \) matrices over a field \( F \) with char. \( R=2 \), let \( a=\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \), \( R=\{a,b \in F \} \). Let \( d \) be the inner derivation given by:
\[
d(x)=\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} x \quad \text{when } x \in R \text{ then } x=\begin{pmatrix} a \\ b \end{pmatrix},
\]
therefore, \( d(x)=\begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} \).

Then \( d(x)x+xd(x)=\begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} \begin{pmatrix} a & b \\ b & a \end{pmatrix} + \begin{pmatrix} a & b \\ b & a \end{pmatrix} \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} \)
\[
=\begin{pmatrix} b^2 & ab \\ -ba & b^2 \end{pmatrix} + \begin{pmatrix} -b^2 & ab \\ -ba & b^2 \end{pmatrix}
=\begin{pmatrix} 0 & 2ba \\ -2ba & 0 \end{pmatrix} .\text{Since char. } R=2, \text{ then}
\]
\[
=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} .\text{Then } d \text{ is skew-centralizing and}
\]
skew-commuting on \( R \), i.e. \( d(R)=0 \).

Also when we have \( d \) acts as homomorphism (resp. acts as an anti-homomorphism).

\[
d(x)d(x)=d\left(\begin{pmatrix} a & b \\ b & a \end{pmatrix}\right)d\left(\begin{pmatrix} a & b \\ b & a \end{pmatrix}\right)
=\begin{pmatrix} a^2 + b^2 & ab + ba \\ ba + ba & a^2 + b^2 \end{pmatrix}
=\begin{pmatrix} 0 & 2ba \\ -2ba & 0 \end{pmatrix} .\text{Since char. } R=2, \text{ we obtain}
\]
\[
=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} .\text{Thus } d \text{ is skew-centralizing and}
\]
skew-commuting on \( R \), i.e. \( d(R)=0 \).

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