Bi-criteria multiple knapsack problem with grouped items

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Abstract
The multiple knapsack problem with grouped items aims to maximize rewards by assigning groups of items among multiple knapsacks, without exceeding knapsack capacities. Either all items in a group are assigned or none at all. We study the bi-criteria variation of the problem, where capacities can be exceeded and the second objective is to minimize the maximum exceeded knapsack capacity. We propose approximation algorithms that run in pseudo-polynomial time and guarantee that rewards are not less than the optimal solution of the capacity-feasible problem, with a bound on exceeded knapsack capacities. The algorithms have different approximation factors, where no knapsack capacity is exceeded by more than 2, 1, and 1/2 times the maximum knapsack capacity. The approximation guarantee can be improved to 1/3 when all knapsack capacities are equal. We also prove that for certain cases, solutions obtained by the approximation algorithms are always optimal—they never exceed knapsack capacities. To obtain capacity-feasible solutions, we propose a binary-search heuristic combined with the approximation algorithms. We test the performance of the algorithms and heuristics in an extensive set of experiments on randomly generated instances and show they are efficient and effective, i.e., they run reasonably fast and generate good quality solutions.

Keywords Multiple knapsacks · Approximation algorithms · Heuristics · Bi-criteria optimization · Computational study
1 Introduction

In this paper we study the **Multiple Knapsack Problem with Grouped Items** (GMKP), a generalization of the **Multiple Knapsack Problem** (MKP). MKP assigns a subset of items to multiple knapsacks, aiming to maximize the total reward without exceeding the capacity of any knapsack. In GMKP, items are partitioned into disjoint groups, each group has a reward (if assigned), and either all or none of the items from a group are assigned to knapsacks (not necessarily to the same knapsack).

One motivation for GMKP comes from patient scheduling, where patients may require multiple therapy sessions during their treatment (e.g., for rehabilitation). In this case, each knapsack corresponds to a day (capacities are the time slots available in each day), each group of items corresponds to the therapy sessions that need to be scheduled for a patient, and each item corresponds to a session (weights representing the session duration). A patient is scheduled only if their treatment can be scheduled entirely; i.e., all therapy sessions must be scheduled, or the patient needs to wait and be scheduled in the future.

Knapsack problems have been well-studied in the literature. Multiple variations of the single **0/1 Knapsack Problem** (KP) exist, such as: **multi-dimensional KP** (mKP), a KP with additional capacity constraints; disjunctively constrained KP, where a set of item pairs or edges define items that cannot be assigned together; multiple-choice KP, where each item belongs to a class and exactly one item of each class must be assigned; and bounded/unbounded KP, where assignment variables accept bounded/unbounded positive integers instead of 0/1 values (Wilbaut et al. 2008). There are also several generalizations of MKP, such as MKP with assignment restrictions (Dawande et al. 2000), MKP with color constraints (Forrest et al. 2006), knapsack-of-knapsacks problem (Nip and Wang 2018), and other variants of the generalized assignment problem (Öncan 2007).

MKP and other similar problems are frequently solved with meta-heuristics such as genetic algorithms (Liu and Wang 2015; Khuri et al. 1994), tabu search (Woodcock and Wilson 2010), and swarm intelligence algorithms (Krause et al. 2013; Liu et al. 2014). Other approaches include exact methods such as branch and bound (Posta et al. 2012; Martello and Toth 1981), cutting plane (Avella et al. 2010; Ferreira et al. 1996), and column generation algorithms (Forrest et al. 2006).

The KP is \( \mathcal{NP} \)-hard but solvable in pseudo-polynomial time by dynamic programming (Horowitz and Sahni 1974). Several **Polynomial Time Approximation Schemes** (PTAS) and **Fully PTAS** (FPTAS) exist for KP (Kellerer and Pferschy 1999). MKP is not solvable in pseudo-polynomial time; it is strongly \( \mathcal{NP} \)-hard (Martello and Toth 1990). There is a PTAS, but not an FPTAS for MKP; even for two knapsacks (Chekuri and Khanna 2005). There is no PTAS nor constant-ratio approximation algorithm for GMKP (Chen and Zhang 2018).

Since there are no efficient approximation algorithms for GMKP, we relax capacity constraints to find good solutions (in terms of rewards from assignments), with a bound on how much the capacities are exceeded. Hence, our focus is on the **bi-criteria GMKP** (bi-GMKP), where the goal is to simultaneously maximize the total reward and minimize the maximum exceeded knapsack capacity. The capacity-relaxed bi-GMKP...
is also motivated by patient scheduling, where additional time slots can be added to the schedule by utilizing overtime or temporary personnel.

For a special case of GMKP, when all knapsack capacities are equal and the heaviest group weighs at most $2/3$ of the total capacity, parameterized-approximation algorithms were proposed by Chen and Zhang (2018). Adany et al. (2016) proposed a PTAS for a generalized assignment problem with grouped items that has additional constraints: there is a limit on the number of items per group, and knapsacks can accommodate at most one item from each group. The algorithms proposed by Chen and Zhang (2018) and Adany et al. (2016) guarantee feasibility while sacrificing rewards; the algorithms proposed in this paper sacrifice feasibility (bounded by a maximum exceeded knapsack capacity) while generating solutions achieving the optimal reward (in relation to the original GMKP).

We show the proposed bi-GMKP algorithms can be adapted into GMKP heuristics to generate capacity feasible solutions, and also adapted into bi-GMKP heuristics to generate solutions with different combinations of rewards and maximum exceeded knapsack capacities. In an extensive computational study, algorithms and heuristics exhibit excellent performance overall. In addition, we show that when capacities and weights are powers of the same positive integer, some of the proposed algorithms find optimal solutions.

This paper is organized as follows: In Sect. 2, we define GMKP and present an Integer Programming (IP) formulation. In Sect. 3, we define bi-GMKP, and propose three approximation algorithms in Sects. 4 to 6. Section 7 focuses on the algorithms’ guarantees for some special cases of bi-GMKP. In Sect. 8, we show how the proposed approximation algorithms for bi-GMKP can be used as heuristics for GMKP and bi-GMKP. Finally, we test all algorithms and heuristics in an extensive computational study (Sect. 9), and present the conclusions in Sect. 10.

### 2 GMKP definition and IP model

In GMKP we are given a set of knapsacks $M = \{1, 2, \ldots, m\}$, $m \geq 2$, and a set of items $N = \{1, 2, \ldots, n\}$. Each knapsack $i \in M$ has a capacity $c_i > 0$, and each item $j \in N$ has a weight $0 < w_j \leq \max_{i \in M} c_i$. Items are partitioned into disjoint groups $G_1 \cup \ldots \cup G_k = N$, where $K = \{1, 2, \ldots, k\}$, and each group $G_l, l \in K$, results in a reward $p_l > 0$ if assigned. At least one group must have two or more items, if not, the instance would be an MKP. Table 1 shows a summary of indices, sets, and parameters. In a feasible assignment of items to knapsacks:

- Each item is assigned to at most one knapsack.
- The capacity of a knapsack is not exceeded by the total weight of its assigned items.
- Whenever an item in a group is assigned to a knapsack, then all items in that group must be assigned to knapsacks.

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[1] In this paper, the proposed methods that solve bi-GMKP with performance bounds are referred to as algorithms; proposed methods that solve GMKP (capacity-feasible) and bi-GMKP, both with no performance guarantees, are referred to as heuristics.
Table 1 Indices, sets, and parameters

| Indices & sets |  |
|----------------|----------------|
| $i \in M = \{1, \ldots, m\}$ | Index and set of knapsacks |
| $j \in N = \{1, \ldots, n\}$ | Index and set of items |
| $l \in K = \{1, \ldots, k\}$ | index and set of groups |
| $G_1 \cup \ldots \cup G_k = N$ | Disjoint groups that partition the set of items |

| Parameters |  |
|------------|----------------|
| $c_i$ | Capacity of knapsack $i \in M$ |
| $w_j$ | Weight of item $j \in N$ |
| $p_l$ | Reward of group $l \in K$ |
| $c_{\text{max}} = \max_{i \in M} c_i$ | Capacity of the largest knapsack |
| $w_{\text{max}} = \max_{j \in N} w_j$ | Weight of the heaviest item |

The objective is to find a solution that maximizes the total reward from the groups assigned to knapsacks. Without loss of generality we assume the following for the remainder of the paper:

- \( \exists l \in K \) such that \( \sum_{j \in G_l} w_j > \sum_{i \in M} c_i \). Such groups cannot be feasibly assigned.
- \( \min_{i \in M} c_i \geq \min_{j \in N} w_j \). If not, no item fits into the smallest knapsack and that knapsack can be removed.

The following is an IP formulation for GMKP:

\[
\begin{align*}
    x_{ij} & = \begin{cases} 
    1, & \text{if item } j \in N \text{ is assigned to knapsack } i \in M \\
    0, & \text{otherwise} 
    \end{cases} \\
    z_l & = \begin{cases} 
    1, & \text{if all items in group } l \in K \text{ are assigned to knapsacks} \\
    0, & \text{otherwise} 
    \end{cases}
\end{align*}
\]

**Definition 1** Given a GMKP instance:

\[
\text{IP–GMKP : } v^* = \max_{l \in K} \sum_{l \in K} p_l z_l \\
\text{s.t. } \sum_{j \in N} w_j x_{ij} \leq c_i \quad i \in M \\
\sum_{i \in M} x_{ij} = z_l \quad l \in K, j \in G_l \\
x_{ij} \in \{0, 1\} \quad i \in M, j \in N \\
z_l \in \{0, 1\} \quad l \in K
\]

The objective function (1) maximizes the total reward from the groups assigned to knapsacks. Constraints (2) ensure that no knapsack capacity is exceeded. Con-
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strains (3) guarantee that either all items within a group are assigned, or none are assigned at all; they also ensure that each item is assigned to at most one knapsack.

3 Bi-GMKP definition and approximation algorithms

In bi-GMKP, we relax the capacity constraints (2) and incorporate them as a second objective function to minimize the maximum exceeded knapsack capacity, i.e.,

\[
\min \max_{i \in M} \left\{ \sum_{j \in N} w_j x_{ij} - c_i \right\}
\]

(4)

bi-GMKP is strongly \(\mathcal{NP}\)-hard, since GMKP is (Chen and Zhang 2018).

Definition 2 For \(0 < \alpha \leq 1\) and \(\beta \geq 0\), an algorithm for bi-GMKP is an \((\alpha, \beta)\) bi-criteria approximation algorithm if any solution returned satisfies

\[- \sum_{l \in K} p_l z_l \geq \alpha v^*, \text{ where } v^* \text{ is the optimal objective value (1) of the analogous GMKP.}\]

\[- \max_{i \in M} \left\{ \sum_{j \in N} w_j x_{ij} - c_i \right\} \leq \beta c_{\max}.\]

In an optimal solution of GMKP, \(\alpha = 1\) and \(\beta = 0\). We propose three types of approximation algorithms for bi-GMKP, following a two step approach: group selection and item assignment.

Group selection: Algorithms first select the groups to assign, focusing on maximizing rewards. Each algorithm does this by solving a different relaxation of GMKP, which are:

1. Algorithm 0: LP-GMKP, a Linear Programming (LP) relaxation of IP-GMKP, where all binary variables \(x_{ij}\) and \(z_l\) are replaced with continuous counterparts \(0 \leq x_{ij} \leq 1\) and \(0 \leq z_l \leq 1\) respectively.
2. Algorithm 1: KP-GMKP, a knapsack relaxation constructed by combining all capacity constraints (2).
3. A multi-dimensional KP (mKP) relaxation, which is a KP-GMKP with extra capacity constraints. We define three versions of this algorithm:
   - Algorithm 2: A single extra capacity constraint (2mKP-GMKP).
   - Algorithm 3: Two additional capacity constraints (3mKP-GMKP).
   - Algorithm 6: Multiple extra capacity constraints based on a finite set \(D \subset \mathbb{R}_{>0}\) (mKPD-GMKP); which generalizes 2mKP-GMKP and 3mKP-GMKP. The theoretical results of the generalized version can be found in Appendix 2.

Note that the last two group selection methods sacrifice polynomial time, but are often faster to solve in computational experiments than the original GMKP.

Item assignment: Algorithms assign all items of the selected groups, focusing on minimizing the maximum exceeded knapsack capacity (4). This is done greedily by sequentially assigning items (sorted from heavier to lightest) to the knapsacks with the most free capacity.
### Table 2  Summary of bi-GMKP algorithms’ guarantees, $\alpha=1$ for all algorithms

| Algorithm    | Relaxed Problem Solved | $\beta$ | Note                          |
|--------------|------------------------|---------|-------------------------------|
| Algorithm 0  | LP-GMKP                | 2       | General case                  |
| Algorithm 1  | KP-GMKP                | 1       |                               |
| Algorithm 2  | 2mKP-GMKP              | 1/2     |                               |
| Algorithm 3  | 3mKP-GMKP              | 1/2     |                               |
| Algorithm 6  | mKP$_p$-GMKP, $[c_{\text{max}}/2] \subseteq D$ | 1/2 |                               |
| Algorithm 0  | LP-GMKP                | 2       | Equal knapsack capacities     |
| Algorithm 1  | KP-GMKP                | 1       |                               |
| Algorithm 2  | 2mKP-GMKP              | 1/2     | Equal knapsack capacities and |
|              |                        |         | items heavier than $c_{\text{max}}/2$ |
| Algorithm 3  | 3mKP-GMKP              | 1/3     | Equal knapsack capacities and |
|              |                        |         | capacities/weights are powers |
|              |                        |         | of the same positive integer  |
| Algorithm 6  | mKP$_p$-GMKP, $[c_{\text{max}}/2, c_{\text{max}}/3] \subseteq D$ | 1/3 |                               |

This item assignment sub-problem is equivalent to a parallel machine scheduling problem where each item corresponds to a job (durations are their weights), and each knapsack $i \in M$ corresponds to an identical parallel machine with earliest available time $c_{\text{max}} - c_i$ (i.e., the release time for machine $i$); aiming to minimize the total makespan. This sub-problem is $NP$-hard and has several polynomial time approximation algorithms (Lee 1991; Lee et al. 2000).

Table 2 summarizes the algorithms proposed and their approximation guarantees, for the general case of bi-GMKP and some special cases. Note that all algorithms have $\alpha = 1$, i.e., they achieve the optimal reward of the equivalent GMKP.

### 4 LP based approximation algorithm for bi-GMKP

**Definition 3** In a solution of an LP-GMKP instance, group $l \in K$ is a **partially assigned group** if $0 < z_l < 1$.

Some instances of LP-GMKP can have multiple optimal solutions, and there may be more than one partially assigned group in some of these solutions (Proposition 1 in Appendix 1). To solve LP-GMKP, we propose a polynomial time greedy method which is guaranteed to return an optimal solution with at most one partially assigned group (Proposition 2 and Corollary 4 in Appendix 1); this property is used in proving the approximation guarantee of Algorithm 0.
Algorithm 0: LP based approximation algorithm for bi-GMKP

Input: bi-GMKP instance.
Output:

Group selection:
1. Solve the corresponding LP-GMKP instance greedily (Algorithm 5 in Appendix 1), and get solution $(x, z)$.
2. $z^a_l \leftarrow 1, \forall l \in K$ such that $z_l > 0$.

Item assignment:
3. $x^a_{ij} \leftarrow 0, \forall i \in M, j \in N$.
4. Let $N^s = \bigcup_{l: z^a_l = 1} G_l$, be the set containing all items of the selected groups.
5. for each $j^s \in N^s$ do
   6. $x^a_{ijs} \leftarrow 1$, where $i = \arg \min_{i \in M} \left\{ \sum_{j \in N} w_j x_{ij} - c_i \right\}$.
7. end
8. Return solution $(x^a, z^a)$.

Theorem 1 Algorithm 0 (i) is a $(1,2)$-approximation algorithm, and (ii) runs in polynomial time. (iii) This is a tight approximation.

Proof of Theorem 1 (i) Let $(x^a, z^a)$ be the solution obtained by the algorithm. Guarantee $\alpha \geq 1$ is trivial, since LP-GMKP is a linear relaxation of GMKP.

Now we prove guarantee $\beta \leq 2$. The algorithm solves an LP-GMKP instance and selects all groups that are entirely or partially assigned. Then it greedily assigns the items of the selected group, one by one, to the knapsack with the least exceeded (or most remaining) capacity. Suppose by contradiction that as items are assigned to knapsacks, the capacity is exceeded by more than $2c_{\max}$ after assigning some item $j$. This means that every knapsack has its capacity exceeded by more than $c_{\max}$ (if not, item $j$ would be assigned to the knapsack with the least exceeded capacity, without exceeding any knapsacks capacity by more than $2c_{\max}$), therefore,

$$\sum_{i \in M} \sum_{j \in N} w_j x^a_{ij} > \sum_{i \in M} (c_i + c_{\max}) \geq 2 \sum_{i \in M} c_i \quad (5)$$

On the other hand, the partially assigned group (at most one from Corollary 4 in Appendix 1) cannot weigh more than the sum of all capacities, and the same goes for all other selected groups together. Therefore $\sum_{i \in M} \sum_{j \in N} w_j x^a_{ij} \leq 2 \sum_{i \in M} c_i$, contradicting (5).

(ii) Algorithm 0 runs in polynomial time $O(k \log(k) + n + m \log(m))$; where $k \log(k) + n$ corresponds to the run time of greedily solving LP-GMKP (Proposition 2 in Appendix 1), and $m \log(m)$ corresponds to the run time of the greedy assignment of items (line 6), by sorting and keeping the list updated.

(iii) See Examples for Theorem 1 in Appendix 3. \qed
5 KP based pseudo-polynomial time approximation algorithm for bi-GMKP

Definition 4 Given a GMKP instance:

\[ \text{KP–GMKP:} \quad \max \sum_{l \in K} p_l z_l \]  

s.t. \[ \sum_{l \in K} \sum_{j \in G_l} w_j z_l \leq \sum_{i \in M} c_i \]  

\[ z_l \in \{0, 1\} \quad l \in K \]  

Lemma 1 KP–GMKP is a relaxation of the corresponding GMKP.

Proof of Lemma 1 Consider IP-GMKP. Adding all capacity constraints (2) and replacing \( \sum_{i \in M} x_{ij} \) with \( z_l \) for \( j \in G_l \), we get

\[ \sum_{i \in M} c_i \geq \sum_{i \in M} \sum_{j \in N} w_j x_{ij} = \sum_{j \in N} w_j \left[ \sum_{i \in M} x_{ij} \right] = \sum_{l \in K} \sum_{j \in G_l} w_j \left[ \sum_{i \in M} x_{ij} \right] = \sum_{l \in K} \sum_{j \in G_l} w_j z_l \]  

This is constraint (6). \( \square \)

Algorithm 1: KP based approximation algorithm for bi-GMKP

Input: bi-GMKP instance.  
Output: \( x_{ij}^a \in \{0, 1\}, \forall i \in M, j \in N; z_l^a \in \{0, 1\}, \forall l \in K. \)  
Run Algorithm 0, changing lines 1 and 2 with the following:  
Solve the corresponding KP-GMKP instance, and get solution \( z^a \).

Theorem 2 Algorithm 1 (i) is a \((1,1)\)-approximation algorithm, and (ii) runs in pseudo-polynomial time. (iii) This is a tight approximation.

Proof of Theorem 2 (i) Guarantee \( \alpha \geq 1 \) is trivial since KP-GMKP is a relaxation of GMKP (Lemma 1). Now we prove guarantee \( \beta \leq 1 \). The groups selected by KP-GMKP do not exceed the total knapsack capacity. Therefore, during the greedy item assignment stage, items are always assigned to a knapsack with free capacity (the capacity might be exceeded once the item gets assigned). Thus, the capacity of a knapsack cannot be exceeded by more than \( w_{\text{max}} \leq c_{\text{max}} \).

(ii) The algorithm runs in pseudo-polynomial time

\[ O \left( m \log(m) + n \sum_{i \in M} c_i \right) \]
where \(m \log(m)\) corresponds to the greedy assignment of items, and \(n \sum_{i \in M} c_i\) to the pseudo-polynomial solution time of KP (Horowitz and Sahni 1974).

(iii) See Examples for Theorem 2 in Appendix 3. \qed

### 6 2mKP & 3mKP based pseudo-polynomial time approximation algorithm for bi-GMKP

For \(d > 0\) define the following function \(f_d : \mathbb{R}_{>0} \to \mathbb{Z}_{\geq 0}\)

\[
f_d(y) = \max \left\{ q \in \mathbb{Z} : q < \frac{y}{d} \right\} = \begin{cases} \frac{y}{d} - 1, & \text{if } \frac{y}{d} \in \mathbb{Z}, \\ \left\lfloor \frac{y}{d} \right\rfloor, & \text{if } \frac{y}{d} \notin \mathbb{Z} \end{cases}
\]

Intuitively, function \(f_d(y)\) represents the number of times items slightly larger than \(d\) completely fit into \(y\). \(\lfloor \cdot \rfloor\) denotes the integer part or floor function.

**Definition 5** Given a GMKP instance:

\[
2\text{mKP–GMKP} : \quad \max_{l \in K} \sum_{l \in K} p_l z_l \tag{1}
\]

s.t.

\[
\sum_{l \in K} \sum_{j \in G_l} w_j z_l \leq \sum_{i \in M} c_i \tag{6}
\]

\[
\sum_{l \in K} \sum_{j \in G_l} f_{\max} \left( \frac{w_j}{d} \right) z_l \leq \sum_{i \in M} f_{\max} \left( \frac{c_i}{d} \right) \tag{7}
\]

\(z_l \in \{0, 1\}\)

Constraint (7) is a valid cut for GMKP, i.e., avoids some infeasible GMKP solutions that are feasible in KP-GMKP. For example, consider two knapsacks with equal capacities 1, and only one group with three items that weigh 0.6 each. The group cannot be assigned without exceeding the capacity of some knapsack, but KP-GMKP would still select the group. Adding constraint (7) namely, \(3 f_{\frac{d}{2}} (0.6) z_1 = 3 z_1 \leq 2 f_{\frac{d}{2}} (1) = 2\), avoids selecting the group.

**Lemma 2** A 2mKP-GMKP instance is a relaxation of its corresponding GMKP instance.

**Proof of Lemma 2** KP-GMKP is a relaxation of GMKP (Lemma 1), therefore it suffices to show that constraint (7) is satisfied by all feasible solutions of GMKP. For \(d = c_{\max}/2\)

\[
\sum_{j \in N} f_d(w_j) x_{ij} \leq f_d(c_i) \tag{8}
\]

holds for all GMKP solutions, because capacity constraints (2) are satisfied by GMKP solutions and \(f_d\) is supper-additive (i.e., \(f_d(y_1) + f_d(y_2) \leq f(y_1 + y_2)\), for all
By adding all constraints (8) together we get

\[
\sum_{i \in M} f_d(c_i) \geq \sum_{i \in M} \sum_{j \in N} f_d(w_j) x_{ij} = \sum_{l \in K} \sum_{j \in G_l} f_d(w_j) \left[ \sum_{i \in M} x_{ij} \right] = \sum_{l \in K} \sum_{j \in G_l} f_d(w_j) z_l
\]

This is constraint (7) when we substitute \(d = c_{max}/2\).

**Algorithm 2:** 2mKP based approximation algorithm for bi-GMKP

**Input:** bi-GMKP instance.

**Output:** \(x_{ij}^{a} \in \{0, 1\}, \forall i \in M, j \in N; z_l^{a} \in \{0, 1\}, \forall l \in K\).

Run Algorithm 0, changing lines 1 and 2 with the following:

Solve the corresponding 2mKP-GMKP instance, and get solution \(z^{a}\).

**Theorem 3** Algorithm 2 (i) is a \((1, 1/2)\)-approximation algorithm, and (ii) runs in pseudo-polynomial time. (iii) This is a tight approximation.

**Proof of Theorem 3** (i) Guarantee \(\alpha \geq 1\) is trivial, since 2mKP-GMKP is a relaxation of GMKP (Lemma 2). We now prove guarantee \(\beta \leq 1/2\). The groups selected by 2mKP-GMKP do not exceed the total knapsack capacity. Therefore, during the greedy item assignment stage, items are always assigned to a knapsack with free capacity and thus knapsack capacities can be exceeded by more than \(c_{max}/2\) only when items heavier than \(c_{max}/2\) are assigned. Constraint (7) implies that when an item \(j\) with \(w_j \geq c_{max}/2\) is assigned, there must exist a knapsack with capacity larger than \(c_{max}/2\), and item \(j\) is the first item assigned to this knapsack. Hence, no knapsacks’ capacity can be exceeded by more than \(c_{max}/2\).

(ii) The algorithm runs in pseudo-polynomial time

\[
\mathcal{O} \left( m \log(m) + n \left( \sum_{i \in M} c_i \right) \left( \sum_{i \in M} f_{c_{max}}(c_i) \right) \right)
\]

where \(m \log(m)\) corresponds to the greedy assignment of items, and the second term corresponds to the pseudo-polynomial solution time of mKP (Fréville 2004).

(iii) See Examples for Theorem 3 in Appendix 3. \(\Box\)

**Definition 6** Given a GMKP instance:

**3mKP–GMKP :**

\[
\max_{l \in K} \sum_{l} p_l z_l
\]

s.t.

\[
\sum_{l \in K} \sum_{j \in G_l} w_j z_l \leq \sum_{i \in M} c_i
\]
\[
\sum_{l \in K} \sum_{j \in G_l} f_{\text{max}} (w_j) z_l \leq \sum_{i \in M} f_{\text{max}} (c_i) \tag{7}
\]

\[
\sum_{l \in K} \sum_{j \in G_l} f_{\text{max}} (w_j) z_l \leq \sum_{i \in M} f_{\text{max}} (c_i) \tag{9}
\]

\[z_l \in \{0, 1\} \quad l \in K\]

**Algorithm 3:** 3mKP based approximation algorithm for bi-GMKP

**Input:** bi-GMKP instance.

**Output:** \( x_{ij}^l \in \{0, 1\}, \forall i \in M, j \in N; z_l^a \in \{0, 1\}, \forall l \in K. \)

Run Algorithm 0, changing lines 1 and 2 with the following:

Solve the corresponding 3mKP-GMKP instance, and get solution \( z_l^a \).

Algorithm 3 has the same guarantees as Algorithm 2 for the general case of bi-GMKP; they are both \((1, 1/2)\)-approximation algorithms. Although, Algorithm 3 has a better guarantee for some special case as seen in Sect. 7.1.

A generalization of 2mKP-GMKP and 3mKP-GMKP can be found in Appendix 2. There, Theorem 6 shows that additional constraints in the form of constraints (7) (with different \(d > 0, d \neq c_{\text{max}}/2\) values) do not improve the \(\beta \leq 1/2\) guarantee of Theorem 3, even when all possible constraints of such form are included (Corollary 5 in Appendix 2).

## 7 Special cases of bi-GMKP

In this section we study two special cases of bi-GMKP, namely, equal capacity knapsacks (Sect. 7.1), and when item weights and knapsack capacities are powers of the same positive integer (Sect. 7.2).

### 7.1 Equal capacity knapsacks

**Corollary 1** When all knapsacks have equal capacities and all items are heavier than \(c_{\text{max}}/2\), then Algorithm 2 returns an optimal solution of its corresponding GMKP instance.

**Proof of Corollary 1** To satisfy constraint (7), there exists a knapsack for each item assigned. Hence, no capacity is exceeded. \(\square\)

**Theorem 4** When all knapsacks have equal capacities, Algorithm 3 (i) is a \((1, 1/3)\)-approximation algorithm, and (ii) runs in pseudo-polynomial time. (iii) This is a tight approximation.
Proof of Theorem 4  (i) Guarantee $\alpha \geq 1$ is trivial, since 3mKP-GMKP is a relaxation of GMKP (analogous to Lemma 2). We now prove guarantee $\beta \leq 1/3$. The groups selected by 3mKP-GMKP do not exceed the total knapsack capacity. Therefore, during the greedy item assignment stage, items are always assigned to a knapsack with free capacity, so any item that weighs $c_{\text{max}}/3$ or less cannot exceed the capacity by more than $c_{\text{max}}/3$ when assigned. Constraint (7) ensures that the number of selected items larger than $c_{\text{max}}/2$ cannot exceed the number of knapsacks. Hence, the algorithm always assigns an item heavier than $c_{\text{max}}/2$ to an empty knapsack.

The interesting case is assigning an item $j \in N$ with $c_{\text{max}}/3 < w_j \leq c_{\text{max}}/2$ to a knapsack $i \in M$. Suppose the algorithm assigns such an item $j$ to a knapsack $i$ and exceeds its capacity; we will show that the capacity cannot be exceeded by more than $c_{\text{max}}/3$.

- If knapsack $i$ is empty, its capacity cannot be exceeded.
- If knapsack $i$ has one item assigned previously, such item must weigh more than $5c_{\text{max}}/6$ for the knapsack capacity to be exceeded by more than $c_{\text{max}}/3$ after the assignment of item $j$. This means that, after assigning item $j$, this knapsack’s contribution to the left-hand side (lhs) of constraint (9) would be 3; while the contribution to the right-hand side (rhs) of each knapsack would be 2. Therefore, to satisfy the constraint, there must exist another knapsack whose contribution to the lhs is either 1 or 0. 0 is not possible since the knapsack must have at least one item at least as heavy as item $j$. If the contribution to the lhs is 1, it means the knapsack has only one item which is lighter than $2c_{\text{max}}/3$, but in such a case, the capacity cannot be exceeded by more than $c_{\text{max}}/3$.
- If knapsack $i$ has two or more items, then this knapsack’s contribution to the lhs of constraint (9) would be at least 3 (after assigning item $j$). From the previous part, this contradicts the fact that constraint (9) is satisfied.

(ii) The algorithm runs in pseudo-polynomial time

$$O \left( m \log(m) + n \left( \sum_{i \in M} c_i \right) \left( \sum_{i \in M} f_{\text{max}} \left( c_i \right) \right) \left( \sum_{i \in M} f_{\text{max}} \left( c_i \right) \right) \right)$$

where $m \log(m)$ corresponds to the greedy assignment of items, and the second term corresponds to the pseudo-polynomial solution time of mKP (Fréville 2004).

(iii) See Examples for Theorem 4 in Appendix 3. \(\square\)

Theorem 7 in Appendix 2 shows that if a finite number of constraints in the form of constraint (9) (with different $d > 0$, $d \neq c_{\text{max}}/2$, $d \neq c_{\text{max}}/3$ values) are added to 3mKP-GMKP, this does not improve the $\beta \leq 1/3$ guarantee of Theorem 4.

Corollary 2 When all knapsacks have equal capacities, all approximation guarantees in Theorems 1 to 4 are tight.

Proof of Corollary 2 All tight examples used in the proofs have equal capacities. \(\square\)
7.2 Capacities and weights are powers of integer $a$

**Corollary 3** When all capacities and weights are powers of integer $a > 0$, and knapsacks have equal capacities, Algorithm 1 returns an optimal solution of its corresponding GMKP instance.

**Proof of Corollary 3** Suppose by contradiction that the algorithm assigns some item $j \in N$ to knapsack $i \in M$, exceeding its capacity. The weight and capacity can be expressed, respectively, as $w_j = a^q$ and $c_i = a^r$, for some $q, r \in \mathbb{Z}_{>0}, r \geq q$. The free capacity in knapsack $i$, before assigning item $j$, can be expressed as $a^r - sa^q$ for some $s \in \mathbb{Z}_{>0}$ (recall heavier items are assigned first). To exceed the knapsack’s capacity, $0 < a^r - sa^q < a^q$ must hold, which implies $s < a^{r-q} < s + 1$; not possible for $r \geq q$ since $s$ is integer. □

**Definition 7** Given a GMKP instance:

$$\text{mKP'}-\text{GMKP} : \max \sum_{l \in K} p_l z_l$$

s.t. $$\sum_{l \in K} \sum_{j \in G_l} w_j z_l \leq \sum_{i \in M} c_i$$

$$\sum_{l \in K} \sum_{j \in G_l} \left\lfloor \frac{w_j}{d} \right\rfloor z_l \leq \sum_{i \in M} \left\lfloor \frac{c_i}{d} \right\rfloor \quad d \in D = \left\{ w_j : j \in N, w_j > \min_{i \in M} c_i \right\}$$

$$z_l \in \{0, 1\} \quad l \in K$$

**Algorithm 4:** Alternative mKP based approximation algorithm for bi-GMKP

**Input:** bi-GMKP instance.

**Output:** $x_{ij}^d \in \{0, 1\}, \forall i \in M, j \in N; z_l^d \in \{0, 1\}, \forall l \in K$.

Run Algorithm 0, changing lines 1 and 2 with the following:

- Solve the corresponding mKP'-GMKP instance, and get solution $z^a$.

**Theorem 5** When all capacities and weights are powers of integer $a > 0$, Algorithm 4 returns an optimal solution of its corresponding GMKP instance.

**Proof of Theorem 5** Since mKP'-GMKP is a relaxation of its respective GMKP instance (analogous to Lemma 2), it suffices to show that the solution returned by the algorithm does not exceed any knapsack capacity.

Suppose by contradiction that the algorithm assigns item $j \in N$ to some knapsack, exceeding its capacity. Note that all knapsacks $i \in M$ with $c_i \geq w_j$ are completely
full; if not, item $j$ would fit entirely in some knapsack’s free capacity, since capacities and weights are powers of $a$ (recall heavier items are assigned first). This also implies that $w_j > \min_{i \in M} c_i$ must hold for item $j$ to exceed the capacity of a knapsack (if not, all knapsacks are full which contradicts constraint (6)).

When $w_j > \min_{i \in M} c_i$, consider constraint (10) for $d = w_j \in D$. Each knapsack $i \in M$ such that $c_i \geq w_j$ is full and contributes to the lhs and rhs equally. Other knapsacks with capacities smaller than $w_j$ contribute 0 to the rhs, but item $j$ contributes 1 to the lhs. Thus, constraint (10) for $d = w_j \in D$ would not be satisfied, a contradiction.

8 Heuristics for GMKP and bi-GMKP

Algorithm 0–3 and 6 can exceed knapsack capacities, and their solutions can be improved through local search heuristics, e.g., jump and swap operations: a jump operation consists of moving one item to another knapsack, and a swap operation consists of exchanging the assignments of two different items. A swap-optimal solution is such that there are no swap or jump operations that improve the solution.

This approach is widely used to solve parallel-machine scheduling problems (Schuurman and Vredeveld 2007), and swap-optimal solutions can be found in polynomial time (Finn and Horowitz 1979). Therefore, given a bi-GMKP solution, it is possible to obtain a swap-optimal solution in polynomial time.

Heuristic 1 is a binary-search GMKP heuristic that is trivially guaranteed to stop, and obtains a capacity-feasible solution. In the worst-case scenario, the obtained solution has no groups nor items assigned. The heuristic’s logic is to find a feasible solution utilizing the capacity as much as possible, by exploring the solution space through binary search. Note Heuristic 1 incorporates a swap-optimal improvement, and assumes without loss of generality that all capacities and weights are integer.

A modified versions of Algorithm 0–3 and 6, as seen in Heuristic 2, can run with different $Total\ Capacity$ values to obtain bi-GMKP solutions. If we run Heuristic 2 for several alternative $Total\ Capacity$ values, we no longer maintain the $\alpha, \beta$ guarantees but can identify a set of non-dominated solutions (in terms of bi-criteria: rewards vs. maximum exceeded knapsack capacity).

9 Computational study

We created an extensive set of randomly generated GMKP instances to test the performance of bi-GMKP approximation algorithms (Algorithm 0–3 and 6), GMKP heuristics (Heuristic 1), and bi-GMKP heuristics (Heuristic 2).

The implementation was done on Python 3.6.5, and solver Gurobi 8.0.0 (with default settings) to solve IP models: IP-GMKP, KP-GMKP, 2mKP-GMKP, 3mKP-GMKP, and mKPD-GMKP. Instances used in the experimental evaluation can be found in the online repository Castillo-Zunino (2020). All bi-GMKP algorithms tested are abbreviated as follows
Heuristic 1: Binary-search GMKP heuristic

**Input:** GMKP instance with integer capacities and weights.

**Output:** $x^h_{ij} \in \{0, 1\}, \forall i \in M, j \in N; z^h_l \in \{0, 1\}, \forall l \in K$.

1. $left \leftarrow 0$
2. $right \leftarrow \sum_{i \in M} c_i$
3. $(x^h, z^h) \leftarrow (0, 0)$, i.e., solution with no items nor groups assigned.
4. while $l \leq r$
   5. $TotalCapacity \leftarrow \left\lfloor \frac{left + right}{2} \right\rfloor$
   6. Solve the corresponding bi-GMKP instance with a slightly modified Algorithm 0–3 or 6, and get solution $(x^a, z^a)$. The modified version of each algorithm consists in replacing the rhs of the capacity constraints, $\sum_{i \in M} c_i$, with the current value of $TotalCapacity$.
   7. Do a swap-optimal improvement on solution $(x^a, z^a)$.
   8. if maximum exceeded knapsack capacity of solution $(x^a, z^a)$ is 0 or less then
      9. $left \leftarrow TotalCapacity + 1$
     10. $(x^h, z^h) \leftarrow (x^a, z^a)$
   else
      12. $right \leftarrow TotalCapacity - 1$
13. Return solution $(x^h, z^h)$.

Heuristic 2: bi-GMKP heuristic

**Input:** bi-GMKP instance, $TotalCapacity > 0$.

**Output:** $x^h_{ij} \in \{0, 1\}, \forall i \in M, j \in N; z^h_l \in \{0, 1\}, \forall l \in K$.

1. Solve the corresponding bi-GMKP instance with a slightly modified Algorithm 0–3 or 6, and get solution $(x^h, z^h)$. The modified version of each algorithm consists in replacing the rhs of the capacity constraints, $\sum_{i \in M} c_i$, with the value of $TotalCapacity$.
2. Do a swap-optimal improvement on solution $(x^h, z^h)$.
3. Return solution $(x^h, z^h)$.

- LP: Algorithm 0 (sub-problem LP-GMKP).
- KP: Algorithm 1 (sub-problem KP-GMKP).
- 2mKP: Algorithm 2 (sub-problem 2mKP-GMKP).
- 3mKP: Algorithm 3 (sub-problem 3mKP-GMKP).
- 100mKP: Algorithm 6 for $D = \{100/2, 100/3, \ldots, 100/100\}$ (sub-problem mKP_D-GMKP).
- Best: Selects the best solution between all previous algorithms.
- IP: Solves IP-GMKP with Gurobi, limiting each running time to three hours.

Experiments ran on a computer cluster with over 500 nodes; Appendix 4 summarizes the hardware characteristics of the computer cluster. When solving a specific instance, all algorithms/heuristics solved the instance in the same node to ensure a fair comparison of running time between algorithms.

### 9.1 Instance generation

All instances generated have integer weights and all knapsacks have an equal capacity of 100. We set the reward of each group equal to the total weight of the items in
that group. We focused the computational study on instances with equal knapsack capacities, since Algorithm 0–3 have different guarantees on the maximum exceeded knapsack capacity ($\beta \leq 2, 1, 1/2, 1/3$ respectively). Instances with alternative reward values and unequal knapsack capacities were also tested (see Appendices 6 and 7).

We generated 3000 instances from a 6-dimensional maximum projection Latin hypercube design (Joseph et al. 2015) by using the R package MaxPro; based on a simulated annealing algorithm (Ba and Joseph 2018). A maximum projection Latin hypercube design simultaneously considers space-filling in the full-dimensional space and space-filling on projections to all dimensional subsets—getting the most information from the experimental samples. Appendix 5 shows examples of 1-dimensional and 2-dimensional projections of the resulting space-filling design and gives an overview of the generated instances’ size.

The parameters of each instance are based on six random variables uniformly distributed between 0 and 1, where each random variable is later transformed to:

1. $m$: Uniformly sets the number of knapsacks from 2 to 100.
2. $w_{\text{split}}$: The weight difference between the heaviest and lightest items. Uniformly choose a random integer from 1 to 99.
3. $w_{\text{min}}$: The weight of the lightest item. Uniformly choose a random integer from $1$ to $\min(50, 100 - w_{\text{split}})$. $w_{\text{min}}$ is capped at $c_{\text{max}}/2 = 50$, to avoid instances solved to optimality by Algorithm 2 (Corollary 1).
4. $w_{\text{mode}}$: The desired mode of the item weights. Uniformly choose a random integer from $w_{\text{min}}$ to $w_{\text{max}} = w_{\text{min}} + w_{\text{split}}$.
5. $r_{\text{load}}$: The desired load-ratio $(\sum_{j \in N} w_j)/(\sum_{i \in M} c_i)$ for the knapsacks. Uniformly choose a real number from 1 to 20.
6. $r_{\text{conc}}$: The desired concentration-ratio $1 - k/n$ (correlated with the average number of items in a group, $n/k$). Uniformly choose a real number from 0 to 1.

Once these parameters are defined for an instance, items and groups are created as follows:

- Generate two items of weights $w_{\text{min}}$ and $w_{\text{max}}$.
- Generate items until $(\sum_{j \in N} w_j)/(\sum_{i \in M} c_i)$ exceeds the desired load-ratio. Each weight is randomly chosen from a discretized triangular distribution with parameters $(w_{\text{min}}, w_{\text{mode}}, w_{\text{max}})$.
- Set the number of groups to $k = \lceil n \cdot (1 - r_{\text{conc}}) \rceil$.
- Assign one item to each group.
- For each remaining item, identify the groups that would not exceed the total group weight of $100m$ if this item is assigned to that group. Pick any of those groups randomly with equal probability and assign the item to that group. If there is no such group, then create a new group and assign the item to it.

We used discretized triangular distributions to generate items’ weights to make instances more realistic; most therapies (items in a group) have a minimum, maximum, and most likely durations. Uniform distributions were an obvious alternative but are very unnatural for most real applications. We designed $r_{\text{load}}$ and $r_{\text{conc}}$ such that $r_{\text{load}} \sim \text{Uniform}(1, 20)$ and $r_{\text{conc}} \sim \text{Uniform}(0, 1)$. The instance generation method can slightly exceed the desired $r_{\text{load}}$ and can have a smaller $r_{\text{conc}}$; so we tested the
Bi-criteria multiple knapsack problem with grouped items

9.2 Results for bi-GMKP algorithms

Plots (a) in Figs. 1 and 2 show a box plot of the maximum exceeded knapsack capacities obtained by bi-GMKP algorithms on all instances, without and with swap-optimal improvement, respectively. The box plots mark the 25th, 50th, and 75th percentiles and contain a density plot. Plots (b) of the figures show the respective cumulative density plots of each algorithm, and (c) show a summary table containing some percentile values of the maximum exceeded knapsack capacity. Tables (d) show the share of instances in which each algorithm ranked relative to other algorithms. For example in Fig. 1d, reading row Rank 2 for column 100mKP means that in 14.2% of instances, algorithm 100mKP obtained the second lowest maximum exceeded knapsack capacity (columns add up to 100%, but rows might not because algorithms can tie). Note the complexity and solution quality of each bi-GMKP algorithm increases from LP to 100mKP as ordered in the figures. Results improve significantly after doing a swap-optimal improvement, where 3mKP exceeds the capacity by at most 16 in 99% of instances.

Plot (a) of Fig. 3 shows a logarithmic graph of the computation times of each bi-GMKP algorithm after swap-optimal improvement (times are sorted), and (b) shows a summary table containing some percentile values of the computation times. Over 33%
of instances were not solved by IP in 3 h, while each bi-GMKP algorithm ran for less than 4 min on each instance; 99% of instances were solved by each algorithm in less than 19 s. The 3mKP bi-GMKP algorithm achieves a good balance between computation times and performance (see Figs. 1, 2, 3). Adding some constraints seems to improve computation time, but including too many constraints slows down bi-GMKP algorithm. Although, using many constraints is still very fast and obtained the best results.

Whenever maximizing rewards is a priority and we can slightly exceed capacities, we recommend using 3mKP because it balances performance and computation time, and has the best approximation guarantee of $\beta \leq 1/3$ when knapsack capacities are equal. In most cases, 3mKP runs faster than LP, KP, and 100mKP, while having a similar computation time to 2mKP and a lower maximum exceeded knapsack capacity. We also recommend using 100mKP when computation time is not an issue, since 100mKP obtains slightly better results than 3mKP in most instances. Both 3mKP and 100mKP solve most instances in less than 20 s, so using 100mKP (or other variation with many additional constraints) is encouraged.

### 9.3 Results for GMKP heuristics

All bi-GMKP algorithms, after swap-optimal improvement, were also tested with the binary-search GMKP heuristic (Heuristic 1) to find feasible GMKP solutions. The optimal reward ratio corresponds to the reward obtained by the respective GMKP heuristic, divided by the reward of GMKP’s optimal solution. Since we could not find
the optimal solution of some instances, in such cases we re-ran the IP solver for up to three more hours; warm starting it from the best feasible solution found so far. If after such run Gurobi could not determine an optimal solution, we instead set the denominator of the optimal reward ratio to be the best reward found increased by the duality gap. Note 95% of all 3000 instances had a gap smaller than 4.3%, and 99% of instances a gap smaller than 10.7%.

Results of the optimal reward ratio appear in Fig. 4; (a) shows the box plots and density plots of the optimal reward ratio obtained by GMKP heuristics on all instances, (b) shows the respective cumulative density plots of each heuristic, (c) shows a summary table containing some percentile values of the optimal reward ratio, and (d) shows the share of instances in which each algorithm ranked relative to other algorithms’ optimal reward ratio. No significant improvement is obtained when having additional
constraints beyond 2mKP. Only 5% of instances solved by 2mKP had an optimal reward ratio worse than 0.79, and only 1% worse than 0.69.

Figure 5 shows the computation times of GMKP heuristics; Plot (a) shows a logarithmic graph of the computation time of each GMKP heuristic after swap-optimal improvement, and (b) shows a summary table containing some percentile values of the computation times. Every heuristic ran on any instance in less than 48 min, and on 99% of instances in less than 4 min. Although some instances might take longer, the binary-search GMKP heuristic can stop at any time and return the best feasible solution found.

We recommend using 2mKP based GMKP heuristic when solving GMKP instances, since there is no noticeable improvement when adding additional constraints (such as 3mKP and 100mKP), and because it has the shortest computation time in most instances. 2mKP does not find the optimal solution of most instances, but most of the times it runs in less than 2 min obtaining a reward no more than 20% away from optimal.

9.4 Results for bi-GMKP heuristics

We also tested a bi-GMKP heuristic by using Heuristic 2 with the modified versions of Algorithm 2 (2mKP). We tested 2mKP since the results of GMKP heuristics do not improve after adding additional constraints (see Fig. 4 of Sect. 9.3). In order to generate a non-dominated frontier of instances, we ran Heuristic 2 for several TotalCapacity values; multiplying each instance’s total capacity by factors going from 0.75 to 1.25 (with 0.05 increments; the heuristic ran 11 times per instance).

Plot (a) of Fig. 6 shows an example of a single bi-GMKP instance with several solutions obtained by 2mKP based Heuristic 2, for different TotalCapacity values. Each solution has a combination of optimal reward ratio and maximum exceeded knapsack capacity, and some are dominated by other solutions (i.e., the dominated solution does worse than another solution in both objectives). Note that negative values of the maximum exceeded knapsack capacity can be obtained when no capacity is exceeded, and the negative value represents the free capacity of the knapsack with the least free capacity. (E.g., a value of $-20$ means that the knapsack with least free capacity has 20 unutilized capacity, and therefore all other knapsacks have 20 unutilized capacity.)
or more.) The diagonal line represents the case where changes in the optimal reward ratio generate a proportional change in the maximum exceeded knapsack capacity.

Plot (b) of Fig. 6 shows the contour plot of the non-dominated solutions of all 3000 instances; where darker means that more solutions are in the area. Note 11.3% of the 33,000 solutions were dominated, meaning that the contour plot (b) includes 88.7% of the solutions obtained. Most solutions lie above the diagonal line, showing that the heuristic does a good job in maximizing rewards while slightly exceeding capacities. Therefore, Heuristic 2 can be used to obtain different bi-GMKP solutions to decide among different bi-criteria combinations.

We recommend using Heuristic 2 in practice, since it generates several alternative solutions to pick from, that allow decision makers to evaluate the trade-off between exceeding knapsack capacities and maximizing rewards. Heuristic 2 generates solutions that efficiently maximize rewards while barely exceeding knapsack capacities.

10 Concluding remarks

This paper studies GMKP, a strongly \( \mathcal{NP} \)-hard problem with no polynomial time approximation algorithm. We are not the first studying GMKP (Chen and Zhang 2018), but we are the first studying the bi-criteria version of GMKP and offering a broad computational study of our suggested algorithms and heuristics. We propose several pseudo-polynomial time approximation algorithms for bi-GMKP with tight guarantees, that can be adapted as binary-search heuristics for GMKP, and heuristics for bi-GMKP.

The proposed algorithms for bi-GMKP either solve a KP or an mKP (KP with multiple capacity constraints) to maximize rewards, and then assign all picked items among knapsacks to minimize the maximum exceeded knapsack capacity. bi-GMKP algorithms can be combined with binary-search heuristics to obtain capacity feasible GMKP solutions. In a similar way, modified versions of bi-GMKP algorithms can also be used to obtain different bi-criteria combinations of rewards and maximum exceeded knapsack capacities.

An extensive computational study shows that algorithms and heuristics run fast and obtain good results. We tested a total of 3000 instances that had 2 to 100 knapsacks, where the total item weight can be 1 to 20 times larger than the total knapsack capac-
ities. In 99% of instances, approximation algorithms for bi-GMKP ran in less than 19 s and exceeded the knapsack capacities by at most 16%. In 95% of the cases, GMKP heuristics obtained feasible solutions in less than a minute where the worst reward obtained was 21% below the optimal reward. Running a swap-optimal improvement after running any algorithm/heuristic is greatly encouraged since they run in polynomial time and improve solutions significantly.

Patient scheduling is one application of bi-GMKP, where the suggested algorithms and heuristics give the decision maker a good set of tools to analyze the trade-off between rewards and maximum exceeded knapsack capacities. Motivated by patient scheduling, future research could focus on a multi-period variation of the GMKP with “consistency,” where some items/group have been previously assigned (e.g., therapy sessions scheduled in a previous week) and the goal is to assign new items/groups (schedule new patients and their therapies) while making the least number of changes to the previous schedule to maintain consistency. Future research could also consider an on-line variation of GMKP where items/groups (patients) arrive over time, and we must decide whether and how to assign items/groups (schedule a patient) or save capacity for future arrivals. We hope that this research also motivates further bi-criteria approaches in solving combinatorial optimization problems since most problems in practice have multiple objectives and/or opportunities to relax (possibly with a cost/penalty) some of the constraints.

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Appendices

Greedy LP-GMKP Algorithm

Proposition 1 Optimal extreme points of an LP-GMKP instance can have more than one partially assigned group.

Proof of Proposition 1 Consider the case with two knapsacks of capacities $c_1 = 3$ and $c_2 = 1$, and two groups with rewards $p_1 = p_2 = 3$. The first group has two items that weigh $w_{1a} = 1$ and $w_{1b} = 2$, and the second group has one item with $w_{2} = 3$. Consider the solution $(x, z)$ where:

\[
\begin{align*}
  z &= (z_1, z_2) = (1/2, 5/6) \\
  x &= (x_{1a}, x_{1b}, x_{12}, x_{21a}, x_{21b}, x_{22}) = (1/2, 0, 5/6, 0, 1/2, 0)
\end{align*}
\]

Solution $(x, z)$ is feasible and optimal, with two partially assigned groups. Also, it is an extreme point since it has 8 variables and 8 activate linearly independent constraints that are:

\[
\begin{align*}
  3 &= x_{1a} + 2x_{1b} + 3x_{12}, & \text{(from constraints (2))} \\
  1 &= x_{21a} + 2x_{21b} + 3x_{22} & \text{(from constraints (2))}
\end{align*}
\]
\[ z_1 = x_{11a} + x_{21a} = x_{11b} + x_{21b}, \]  
(from constraints (3))

\[ z_2 = x_{12} + x_{22} \]  
(from constraints (3))

\[ x_{11b} = x_{21a} = x_{22} = 0, \]  
(from constraints \( x_{ij} \geq 0 \))

\[ \square \]

**Algorithm 5:** Greedy LP-GMKP

**Input:** LP-GMKP instance.

**Output:** \( x_{ij} \in [0, 1], \forall i \in M, j \in N; z_l \in [0, 1], \forall l \in K \).

1. Enumerate groups in \( K \) such that
   \[ \frac{p_1}{\sum_{j \in G_1} w_j} \geq \frac{p_2}{\sum_{j \in G_2} w_j} \geq \cdots \geq \frac{p_k}{\sum_{j \in G_k} w_j} \]

2. Initialize: \( x_{ij} \leftarrow 0, \forall i \in M, j \in N; z_l \leftarrow 0, \forall l \in K; l' \leftarrow 1; i' \leftarrow 1; \)

3. while \( l' \leq k \) and \( i' \leq m \) do
   4. \( z_{l'} \leftarrow \min \left( 1, \frac{\sum_{i \in M} c_i - \text{TotalWeight}}{\sum_{j \in G_{l'}} w_j} \right) \)
   5. for each \( j' \in G_{l'} \) do
      6. \( \text{ItemWeight} \leftarrow z_{l'} w_{j'} \)
      7. while \( \text{ItemWeight} > 0 \) do
         8. \( x_{i'j'} \leftarrow \min \left( \text{ItemWeight}, \frac{c_{i'j'} - \text{KnapsackWeight}}{w_{j'}} \right) \)
         9. \( \text{ItemWeight} \leftarrow \text{ItemWeight} - x_{i'j'} w_{j'} \)
         10. \( \text{KnapsackWeight} \leftarrow \text{KnapsackWeight} + x_{i'j'} w_{j'} \)
         11. \( \text{TotalWeight} \leftarrow \text{TotalWeight} + x_{i'j'} w_{j'} \)
         12. if \( \text{KnapsackWeight} = c_{i'j'} \) then
            13. \( \text{KnapsackWeight} \leftarrow 0 \)
            14. \( i' \leftarrow i' + 1 \)
     15. \( l' \leftarrow l' + 1 \)

**Proposition 2** Algorithm 5 (i) generates an optimal solution for any feasible LP-GMKP instance, and (ii) runs in polynomial time.

**Proof of Proposition 2** (i) The algorithm sorts all groups in a non-increasing reward to total weight ratio and then greedily assigns their items into the knapsacks, filling them one by one. Hence, there are no other groups that can fill the knapsacks with higher rewards; i.e., the solution, if feasible, would be optimal.

To guarantee feasibility, the algorithm checks before assigning a new group \( l \in K \) if it fits into the remaining capacity (considering all knapsacks). If it does, it assigns the group \( (z_l = 1, \sum_{i \in M} x_{ij} = 1, \forall j \in G_l) \). If the group does not fit entirely, it will fill up all the remaining capacity \( (z_l < 1 \) and \( \sum_{i \in M} x_{ij} = z_l, \forall j \in G_l) \). In both cases, constraints (3) are satisfied.

\[ \square \]
Finally, capacity constraints (2) are satisfied because the knapsacks are filled up one by one, and whenever an item does not fit the current knapsack, it is fractionally split among the current knapsack and the next. Thus, the solution is feasible.

(ii) The sorting (line 1) runs in polynomial time $O(k \log k)$. The first “while” loop (line 3) iterates no more than $k$ times, the “for each” loop (line 5) iterates no more than $n$ times, and the second “while” loop (line 7) iterates no more than $m$ times. Combining all loops together, in the worst case scenario, the algorithm iterates over $k$ groups to go through all $n$ items, while also iterating through all $m$ knapsacks. Thus the algorithm runs in polynomial time $O(k \log k + n + m)$. 

Corollary 4 Any solution found by Algorithm 5 has at most one partially assigned group.

Proof of Corollary 4 By construction, only the last group assigned can have $0 < z_l < 1$. 

Generalized mKP based pseudo-polynomial time approximation algorithm for bi-GMKP

Definition 8 Given a GMKP instance and a finite set $D \subset \mathbb{R}_{>0}$:

$$\text{mKP}_D-\text{GMKP} : \max \sum_{l \in K} p_l z_l$$

s.t. $\sum_{l \in K} \sum_{j \in G_l} w_j z_l \leq \sum_{i \in M} c_i$ \hfill (6)

$\sum_{l \in K} \sum_{j \in G_l} f_d(w_j) z_l \leq \sum_{i \in M} f_d(c_i) \quad d \in D$ \hfill (11)

$z_l \in \{0, 1\} \quad l \in K$

$mKP_D$-GMKP generalizes KP-GMKP, 2mKP-GMKP and 3mKP-GMKP. KP-GMKP corresponds to $mKP_D$-GMKP for $D = \emptyset$, 2mKP-GMKP corresponds to $mKP_D$-GMKP for $D = \{c_{\text{max}}/2\}$, and 3mKP-GMKP to $mKP_D$-GMKP for $D = \{c_{\text{max}}/2, c_{\text{max}}/3\}$.

Proposition 3 Let $Z(D)$ be the feasible region of an $mKP_D$-GMKP instance, and let

$$D' = (0, w_{\text{max}}) \cap \left\{ \frac{c_i}{q} : i \in M, q \in \mathbb{Z}_{>0} \right\}.$$ \hfill (12)

Then $Z(D') = Z(\mathbb{R}_{>0})$. 

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Proof of Proposition 3 This proposition shows that only finite sets $D \subset D'$ are worth considering when defining an mKP_D-GMKP instance.

We first prove upper bound $w_{\text{max}}$. Whenever $d \geq w_{\text{max}}$ then $f_d(w_j) = 0, \forall j \in N$. This makes the left-hand side (lhs) coefficients of constraints (11) to be 0.

Sort elements of $D'$ increasingly and let $d' > 0$ be such that $d_h < d' < d_{h+1}$, for some consecutive $d_h, d_{h+1} \in D'$. We claim that constraint (11) for such $d'$ is redundant with constraint (11) for $d_h$, i.e.,

$$
\sum_{l \in K} \sum_{j \in G_l} f_{d_h}(w_j)z_l \leq \sum_{i \in M} f_{d_h}(c_i) \implies \sum_{l \in K} \sum_{j \in G_l} f_{d'}(w_j)z_l \leq \sum_{i \in M} f_{d'}(c_i) \quad (13)
$$

If (13) holds, then all such $d'$ can be removed from $D'$, and $Z(D') = Z(\mathbb{R}_{\geq 0})$ still holds. Since $d_h < d'$, when comparing the lhs of constraints (11) for $d'$ and $d_h$ we have

$$
\sum_{l \in K} \sum_{j \in G_l} f_{d'}(w_j)z_l \leq \sum_{l \in K} \sum_{j \in G_l} f_{d_h}(w_j)z_l \quad (14)
$$

As the value of $d$ increases, the right-hand side (rhs) of constraints (11) only change at the points contained in $D'$ by integer amounts. Therefore

$$
\sum_{i \in M} f_{d_h}(c_i) = \sum_{i \in M} f_{d'}(c_i) \quad (15)
$$

Combining constraint (11) for $d_h$, with (14) and (15), we get

$$
\sum_{l \in K} \sum_{j \in G_l} f_{d'}(w_j)z_l \leq \sum_{l \in K} \sum_{j \in G_l} f_{d_h}(w_j)z_l \leq \sum_{i \in M} f_{d_h}(c_i) = \sum_{i \in M} f_{d'}(c_i).
$$

Thus, (13) holds.

Algorithm 6: Generalized mKP based approximation algorithm for bi-GMKP

**Input:** bi-GMKP instance, finite set $D \subset D'$ as defined in (12).

**Output:** $x_{ij}^{d} \in \{0, 1\}, \forall i \in M, j \in N; z_l^{d} \in \{0, 1\}, \forall l \in K$.

Run Algorithm 0, changing line 1 with the following:

Solve the corresponding mKP_D-GMKP instance, and get solution $z^{d}$.

**Theorem 6** Let $D'$ be defined as (12) (in Proposition 3), and let $D \subset D'$ be a finite set containing $c_{\text{max}}/2$. For such $D$, Algorithm 6 (i) is a $(1, 1/2)$-approximation algorithm, and (ii) runs in pseudo-polynomial time. (iii) This is a tight approximation.

**Proof of Theorem 6** This theorem shows that even when set $D$ is very large, the worst case approximation obtained by $D = \{c_{\text{max}}/2\}$ is not improved (equivalent to the worst case approximation of Algorithm 2).
(i,ii) Analogous to Theorem 3 proof, since \( \{c_{\text{max}}/2\} \in D \). The pseudo-polynomial time is
\[
O\left(m \log(m) + n \left( \sum_{i \in M} c_i \right) \prod_{d \in D} \left( \sum_{i \in M} f_d(c_i) \right) \right).
\]

(iii) See Examples for Theorem 6 in Appendix 3. □

**Corollary 5** Even if Algorithm 6 could solve an instance for \( D = \mathbb{R}_{>0} \), the \((1, 1/2)\)-approximation guarantee from Theorem 6 does not improve.

**Proof of Corollary 5** Consider the tight example of Theorem 6 (Fig. 10 in Appendix 3). The groups picked by the algorithm have a feasible assignment in the corresponding GMKP instance (by rearranging items). Since all constraints (11) are valid inequalities for GMKP (proof analogous to Lemma 2), then the solution found by the algorithm is not removed by constraints (11) for any \( d \in D \); thus the tight \( \beta \leq 1/2 \) example works for \( D = \mathbb{R}_{>0} \). □

**Theorem 7** Let \( D' \) be defined as (12) (in Proposition 3), and let \( D \subset D' \) be a finite set such that \( \{c_{\text{max}}/2, c_{\text{max}}/3\} \subseteq D \). For such \( D \), when all knapsacks have equal capacities, Algorithm 6 (i) is a \((1, 1/3)\)-approximation algorithm, and (ii) runs in pseudo-polynomial time. (iii) This is a tight approximation.

**Proof of Theorem 7** This theorem shows that even when set \( D \) is very large, the worst case approximation obtained by \( D = \{c_{\text{max}}/2, c_{\text{max}}/3\} \) (equivalent to the worst case approximation of Algorithm 3) is not improved when all knapsacks have equal capacities.

(i,ii) Analogous to Theorem 4 proof, since \( \{c_{\text{max}}/2, c_{\text{max}}/3\} \in D \). The pseudo-polynomial time is the same as in Theorem 6.

(iii) See Examples for Theorem 7 in Appendix 3. □

**Tight examples**

**Examples for Theorem 1**. The tightness of guarantee \( \alpha \geq 1 \) is trivial; any example where the algorithm gives an optimal solution to GMKP works. Refer to Fig. 7 for the tight \( \beta \leq 2 \) example. Consider \( m \geq 3 \) knapsacks of equal capacities 1, and two groups where

- Group 1 has \( m - 1 \) items that weigh 1 each and one item that weighs \( m/(m + 1) \).
- Group 2 has \( m + 1 \) items that weigh \( m/(m + 1) \).

If all rewards equal total group weights then, given this ordering of groups, Algorithm 0 generates a solution where \( m - 1 \) knapsacks each contain two items of weights 1 and \( m/(m + 1) \). The last knapsack contains three items of weight \( m/(m + 1) \). Therefore, the maximum exceeded knapsack capacity is \( 2 - 3/(m + 1) \), and as \( m \to \infty \) it converges to 2.

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Examples for Theorem 2. The tightness of guarantee $\alpha \geq 1$ is trivial; any example where the algorithm gives an optimal solution to GMKP works. For the tight $\beta \leq 1$ example, refer to the same instance as in Examples for Theorem 1, but only considering group 2 with $m + 1$ items that weigh $m/(m + 1)$. Algorithm 1 generates a solution where $m - 1$ knapsacks have one item assigned, and one knapsack has two items assigned. Therefore, the maximum exceeded knapsack capacity is $1 - 2/(m + 1)$, and as $m \to \infty$ the bound converges to 1. Note constraint (6) of KP-GMKP is satisfied.

Examples for Theorem 3. The tightness of guarantee $\alpha \geq 1$ is trivial; any example where the algorithm gives an optimal solution to GMKP works. Refer to Fig. 8 for the tight $\beta \leq 1/2$ example. Consider $m \geq 3$ knapsacks of equal capacities 1, and one group that has 2$m + 1$ items that weigh $m/(2m + 1)$ each. Algorithm 2 generates a solution where $m - 1$ knapsacks each contain two items and one knapsack contains three items. Therefore, the maximum exceeded knapsack capacity is $1/2 - 3/(4m + 2)$, and as $m \to \infty$ it converges to 1/2. Note constraints (6) and (7) of 2mKP-GMKP are satisfied.
Examples for Theorem 4. The tightness of guarantee \( \alpha \geq 1 \) is trivial; any example where the algorithm gives an optimal solution to GMKP works. Refer to Fig. 9 for the tight \( \beta \leq 1/3 \) example. Consider \( m \geq 3 \) knapsacks of equal capacities 1, and one group with \( m \) items that weigh \( (3m - 1)/(3m) \) and one item of weight 1/3. Algorithm 3 generates a solution where one knapsack has the item of weight 1/3 assigned and one item that weighs \( (3m - 1)/(3m) \). Therefore, the maximum exceeded knapsack capacity is \( 1/3 - 1/\left(3m\right) \), and as \( m \to \infty \) the bound converges to 1/3. Note constraints (6), (7), and (9) of 3mKP-GMKP are satisfied.

Examples for Theorem 6. Refer to Fig. 10 for the tight \( \beta \leq 1/2 \) example.

Consider \( m \geq 3 \) knapsacks were capacities are \( c_i = (2m + 1 - i)/(2m), \forall i \in M \), and consider a single group with \( m + 1 \) items, whose weights are \( w_j = (2m - j)/(2m) = c_j - 1/2m, \forall j \in N \setminus \{n\} \) and \( w_n = 1/2 \). The group is feasible in GMKP instance, since the first \( m \) items \( j \in N \setminus \{n\} \) can be assigned respectively to knapsacks \( j + 1 \) where they fit exactly, and both items with \( w_{n-1} = w_n = 1/2 \) can be assigned to the first knapsack of size 1. On the other hand, the algorithm sequentially assigns each item \( j \in N \setminus \{n\} \) to knapsack \( i = j \). Before assigning the last item \( n \), all knapsacks have \( 1/(2m) \) free capacity, so assigning \( n \) anywhere exceeds the capacity by \( 1/2 - 1/(2m) \). Having \( m \to \infty \) gets bound 1/2. This example is also tight for the \( \alpha \geq 1 \) bound.
Examples for Theorem 7. The tightness of guarantee \( \alpha \geq 1 \) is trivial; any example where the algorithm gives an optimal solution to GMKP works.

Refer to Fig. 11 for the tight \( \beta \leq 1/3 \) example, where all knapsacks have equal capacities of 1. Recall from Proposition 3 that only elements of \( D \) of the form \( c_{\text{max}}/q = 1/q, \quad q \in \mathbb{Z}_{>0} \), are relevant to be considered. Let \( D \) be partitioned into \( D_{\text{odd}} \) and \( D_{\text{even}} \), where for all \( 1/q \in D_{\text{odd}}, \quad q \) is odd; and for all \( 1/q \in D_{\text{even}}, \quad q \) is even. Let \( 1/q_{\text{max}} \in D_{\text{even}} \) be the smallest number in \( D_{\text{even}} \).

For a small \( \epsilon > 0 \), consider

\[
\begin{align*}
\text{Group 1} & \quad \text{has an item that weighs } 1/3 \quad \text{and} \quad \left\lceil \frac{1}{3\epsilon} \right\rceil \text{ items that weigh } 1 - \epsilon. \\
\text{Group 2} & \quad \text{has } 2 \left\lfloor \frac{q_{\text{max}}}{3} \right\rfloor \text{ items that weigh } 1/2. \\
\text{Group 3} & \quad \text{for each } 1/(2p+1) \in D_{\text{odd}}, \quad \text{has } \left\lfloor (2p+1)/3 \right\rfloor \text{ items that weigh } 1/(2p+1), \quad \text{and} \quad \left\lfloor (2p+1)/3 \right\rfloor \text{ items that weigh } p/(2p+1).
\end{align*}
\]

If all rewards equal total group weights then, given this ordering of groups, Algorithm 2 generates a solution where all groups are selected. One knapsack has an item of weight \( 1 - \epsilon \) and another of weight \( 1/3 \). Therefore, the maximum exceeded knapsack capacity is \( 1/3 - \epsilon \), so as \( \epsilon \to 0 \) it converges to bound \( 1/3 \).

We show the solution is feasible in the mKP\(_D\)-GMKP instance. Capacity constraint (6) is satisfied

\[
\text{lhs} = \sum_{i \in K} \sum_{j \in G_i} w_{ij} z_l \\
= \frac{1}{3} + \left\lceil \frac{1}{3\epsilon} \right\rceil (1 - \epsilon) + 2 \left\lceil \frac{q_{\text{max}}}{3} \right\rceil \frac{2}{3} + \sum_{\frac{2p+1}{2p+1} \in D_{\text{odd}}} \left\lfloor \frac{p+1}{2p+1} \right\rfloor \left( \frac{p+1}{2p+1} + \frac{p}{2p+1} \right) \\
\leq \left\lceil \frac{1}{3\epsilon} \right\rceil + \left\lceil \frac{q_{\text{max}}}{3} \right\rceil + \sum_{\frac{1}{q} \in D_{\text{odd}}} \left\lfloor \frac{q}{3} \right\rfloor = m = \sum_{i \in M} c_i = \text{rhs}
\]
Constraints (11) are satisfied for any $1/(2p) \in D_{\text{even}}$

\[
\text{lhs} = \sum_{l \in K} \sum_{j \in G_l} f_{\frac{1}{2p}}(w_j)z_l \\
= f_{\frac{1}{2p}} \left( \frac{1}{3} \right) + \left[ \frac{1}{3e} \right] f_{\frac{1}{2p}}(1 - \epsilon) + 2 \left[ \frac{q_{\text{max}}}{3} \right] f_{\frac{1}{2p}} \left( \frac{1}{2} \right) \\
+ \sum_{\frac{1}{2p+1} \in D_{\text{odd}}} \left[ \frac{2p+1}{3} \right] \left( f_{\frac{1}{2p+1}} \left( \frac{p+1}{2p+1} \right) + f_{\frac{1}{2p+1}} \left( \frac{p'}{2p+1} \right) \right) \\
\leq \left[ \frac{2p+1}{3} \right] + \left[ \frac{1}{3e} \right] 2p + 2 \left[ \frac{q_{\text{max}}}{3} \right] p \\
+ \left[ \frac{2p+1}{3} \right] \left( f_{\frac{1}{2p+1}} \left( \frac{p+1}{2p+1} \right) + f_{\frac{1}{2p+1}} \left( \frac{p'}{2p+1} \right) \right) + \sum_{\frac{1}{q} \in D_{\text{odd}} \setminus \{\frac{1}{2p+1}\}} \left[ \frac{q}{3} \right] f_{\frac{1}{2p+1}}(1) \\
\leq \left[ \frac{2p+1}{3} \right] + \left[ \frac{1}{3e} \right] 2p + 2 \left[ \frac{q_{\text{max}}}{3} \right] p + \left[ \frac{2p+1}{3} \right] (2p - 1) + \sum_{\frac{1}{q} \in D_{\text{odd}} \setminus \{\frac{1}{2p+1}\}} \left[ \frac{q}{3} \right] 2p \\
= \left( \left[ \frac{1}{3e} \right] + \left[ \frac{q_{\text{max}}}{3} \right] + \sum_{\frac{1}{q} \in D_{\text{odd}}} \left[ \frac{q}{3} \right] \right) 2p = m2p \\
= \sum_{i \in M} f_{\frac{1}{2p+1}}(1) = \sum_{i \in M} f_{\frac{1}{2p+1}}(c_i) = \text{rhs}
\]

Constraints (11) are satisfied for any $1/(2p + 1) \in D_{\text{odd}}$

\[
\text{lhs} = \sum_{l \in K} \sum_{j \in G_l} f_{\frac{1}{2p+1}}(w_j)z_l \\
= f_{\frac{1}{2p+1}} \left( \frac{1}{3} \right) + \left[ \frac{1}{3e} \right] f_{\frac{1}{2p+1}}(1 - \epsilon) + 2 \left[ \frac{q_{\text{max}}}{3} \right] f_{\frac{1}{2p+1}} \left( \frac{1}{2} \right) \\
+ \sum_{\frac{1}{2p+1} \in D_{\text{odd}}} \left[ \frac{2p+1}{3} \right] \left( f_{\frac{1}{2p+1}} \left( \frac{p+1}{2p+1} \right) + f_{\frac{1}{2p+1}} \left( \frac{p'}{2p+1} \right) \right) \\
\leq \left[ \frac{2p+1}{3} \right] + \left[ \frac{1}{3e} \right] 2p + 2 \left[ \frac{q_{\text{max}}}{3} \right] p \\
+ \left[ \frac{2p+1}{3} \right] \left( f_{\frac{1}{2p+1}} \left( \frac{p+1}{2p+1} \right) + f_{\frac{1}{2p+1}} \left( \frac{p'}{2p+1} \right) \right) + \sum_{\frac{1}{q} \in D_{\text{odd}} \setminus \{\frac{1}{2p+1}\}} \left[ \frac{q}{3} \right] f_{\frac{1}{2p+1}}(1) \\
\leq \left[ \frac{2p+1}{3} \right] + \left[ \frac{1}{3e} \right] 2p + 2 \left[ \frac{q_{\text{max}}}{3} \right] p + \left[ \frac{2p+1}{3} \right] (2p - 1) + \sum_{\frac{1}{q} \in D_{\text{odd}} \setminus \{\frac{1}{2p+1}\}} \left[ \frac{q}{3} \right] 2p \\
= \left( \left[ \frac{1}{3e} \right] + \left[ \frac{q_{\text{max}}}{3} \right] + \sum_{\frac{1}{q} \in D_{\text{odd}}} \left[ \frac{q}{3} \right] \right) 2p = m2p \\
= \sum_{i \in M} f_{\frac{1}{2p+1}}(1) = \sum_{i \in M} f_{\frac{1}{2p+1}}(c_i) = \text{rhs}
\]
Table 3  CPU and RAM of computer cluster nodes

| Total threads | Threads per server | Servers | CPU       | RAM (GB) |
|---------------|-------------------|---------|-----------|----------|
| 96            | 24                | 4       | E5-2680 v3 | 128      |
| 80            | 20                | 4       | E5-2650 v3 | 256      |
| 32            | 16                | 2       | E5-2630 v3 | 288      |
| 176           | 44                | 4       | Xeon 6152 | 384      |
| 256           | 64                | 4       | EPYC 7502 | 512      |
| 128           | 64                | 2       | EPYC 7542 | 1024     |

**Computer cluster hardware**

We ran all algorithms and heuristics on a computer cluster with CPU and RAM characteristics as described in Table 3. RAM was assigned as needed up to 64 GB; no instances from the main experimental study reached the RAM threshold, only two instances from Appendix 6 reached the RAM threshold and their results were not included. Therefore, our computational study is almost unrestricted by RAM limitations. IP solver running IP-GMKP used the most RAM by far.

To have a fair comparison, when solving a specific instance all algorithms and heuristics solved the instance in the same node. In this way, no algorithm or heuristic is benefited by hardware differences. We also measured processing clock time when calculating computation times—meaning that putting a node to sleep did not affect the computation time of our experiments.

**Characteristics of generated instances**

The outcome of the 3000 generated instances described in Sect. 9 can be visualized in Figs. 12 and 13. Figure 12 shows histograms of the instances’ parameters: \(m, w_{\text{split}}, w_{\text{min}}, w_{\text{mode}}, r_{\text{load}}, r_{\text{conc}}, n,\) and \(k\). The first six listed correspond to the 6-dimensions defined by the maximum projection Latin hypercube design; the last two (number of items \(n\) and groups \(k\)) are a result of the instance generation procedure. Note how \(m, w_{\text{split}}, r_{\text{load}},\) and \(r_{\text{conc}}\) are uniformly distributed as expected by design. \(w_{\text{min}}\) and \(w_{\text{mode}}\) are not uniformly distributed since the former depends on \(w_{\text{split}}\) and the latter depends on both \(w_{\text{split}}\) and \(w_{\text{min}}\). The instance with the most items has \(n = 71, 604\) and the instance with the most groups has \(k = 19, 205\); both histograms were cut-off since the count decreased significantly beyond the 5000 and 2500 threshold, respectively.

Figure 13 shows six 2-dimensional projections of the experimental design to exemplify how the maximum projection Latin hypercube design is space-filling in every dimensional subset.
All instances solved in Sect. 9 had rewards of each group equal to the total weight of items in that group. Here we test the same instances after modifying each reward $p_l, l \in K$, in three different ways:

- **Original Reward R0**: $p_l^0 \leftarrow \sum_{j \in G_l} w_j$
- **Reward R1**: $p_l^1 \leftarrow \lceil 100 \sqrt{p_l^0} \rceil$
- **Reward R2**: $p_l^2 \leftarrow \lfloor p_l^0 / \sqrt{p_l^0} \rfloor$
- **Reward R3**: $p_l^3 \leftarrow \lfloor \text{Random}(1, 10) \cdot p_l^0 \rfloor$

Function $\lfloor \cdot \rfloor$ denotes rounding to the nearest integer, to avoid precision issues with the IP solver. Groups with reward R1 have a reward to weight ratio of approximately $100 / \sqrt{p_l^0}$, giving an incentive to pick lighter groups (we multiplied by 100 to have more precision when rounding). Groups with reward R2 have a reward to weight ratio of approximately $\sqrt{p_l^0}$, giving an incentive to pick heavier groups. Finally, reward R3 consists of multiplying the original rewards with a real random number between 1 and 10.
10 (each $p_j^3$ is multiplied by a different random number); adding noise to the instance while still having rewards proportional to weights in expectation.

We repeated all experiments from Sect. 9 for an additional 9000 instances, given by the combination of the original 3000 instances and the three additional reward structures. Only one instance for reward R1 and one for reward R2 were not solved by Gurobi due to memory limitations, so we removed them from the results.

**Different rewards: results for bi-GMKP algorithms**

In Figs. 14, 15 and 16 we see the details of the maximum exceeded knapsack capacity of bi-GMKP algorithms, after swap-optimal improvement, for different reward structures. Rewards R1 had solutions with lower maximum exceeded knapsack capacity than rewards R2, which makes sense since R1 prioritizes lighter groups while R2 prioritizes heavier groups. Rewards R0 and R3 are somehow similar (see Figs. 2, 16, respectively), showing that having the same reward to weight ratio in expectation seems to obtain similar results. Algorithms 3mKP and 100mKP obtained the least exceeded knapsack capacity independent of reward structure.

Figures 17, 18 and 19 show the computation times of each bi-GMKP algorithm on instances with different reward structures. Algorithms take a longer time solving instances where heavier groups are prioritized (reward R2), even obtaining outliers that take almost an hour to run; although 99% of instances took under a minute. 3mKP persists as the most time-effective alternative independent of the reward structure, while running 100mKP might still be recommended since it obtains better results and computation times remain short. It is interesting to note how for rewards R1, R2 and R3, Gurobi reached the time limit of 3 h in around 67% of instances, while in the
Fig. 14  Maximum exceeded knapsack capacity per bi-GMKP algorithm after swap-optimal improvement for reward R1

Fig. 15  Maximum exceeded knapsack capacity per bi-GMKP algorithm after swap-optimal improvement for reward R2
original reward R0 (Fig. 3) it only reached the time limit in about 33%. It seems that Gurobi works better when total group weights and rewards are equal.

**Different rewards: results for GMKP heuristics**

Figures 20, 21 and 22 show the details of the optimal reward ratio of bi-GMKP heuristics, after swap-optimal improvement, for different reward structures. As in the original reward R0 (Fig. 4), there does not seem to be an improvement after adding constraints beyond 2mKP in any reward structure. GMKP heuristics obtained the best performance when prioritizing smaller groups (R1), and random noise on rewards did not affect the performance significantly (R4).

The performance of the best GMKP heuristic dropped slightly in comparison to the original reward R0; the 5th percentile of optimal reward ratio was 0.83 in R0 (i.e., 95%
Fig. 18 Computation time per bi-GMKP algorithm after swap-optimal improvement for reward R2

Fig. 19 Computation time per bi-GMKP algorithm after swap-optimal improvement for reward R3

Fig. 20 Optimal reward ratio per GMKP heuristic after swap-optimal improvement for reward R1
of instances did better than 0.83) while for rewards R1, R2, and R3 the 5th percentile dropped to 0.79, 0.73, and 0.74 respectively. This difference might not be due to a loss in performance, but because most instances were not solved to optimality by the IP solver in rewards R1, R2, and R3 and their gap obtained was larger; 95% of instances had a gap of 4.3% or less in reward R0, while the 95th percentile of gaps increased to 9.1%, 11.1%, and 8.2% for rewards R1, R2, and R3 respectively.

Figures 23, 24 and 25 show the computation times of each GMKP heuristic on instances with different reward structures. 2mKP runs faster than KP and similar to 3mKP, independent of reward structure. Computation times increased in comparison to the original reward structure (see Fig. 5), where 95% of instances were solved in less than 103 s in reward R0, and in less than 111, 183, and 118 s in reward R1, R2, and R3 respectively. This difference might also be explained by Gurobi having better performance when total group weights equal rewards (recall sub-problems are also solved with Gurobi).

**Different rewards: results for bi-GMKP heuristics**

For different reward structures, Fig. 26 shows the density of the non-dominated solutions obtained by bi-GMKP heuristic for the modified versions of Algorithm 2 (2mKP); as seen in line 6 of Heuristic 1. Analogous to reward R0 (see Fig. 6), most solutions lie slightly above the diagonal line that represents the case where changes in the optimal reward ratio generate a proportional change in the maximum exceeded knapsack capacity. This shows how the proposed bi-GMKP heuristic can be used, independent of the reward structure, to generate different bi-criteria combinations doing a good job in maximizing rewards while slightly exceeding knapsack capacities.
Fig. 22  Optimal reward ratio per GMKP heuristic after swap-optimal improvement for reward R3

Fig. 23  Computation time per GMKP heuristic after swap-optimal improvement for reward R1

Fig. 24  Computation time per GMKP heuristic after swap-optimal improvement for reward R2
Fig. 25 Computation time per GMKP heuristic after swap-optimal improvement for reward R3

Fig. 26 Contour plots of non-dominated solutions of all instances for rewards R1, R2 & R3

Fig. 27 Maximum exceeded knapsack capacity per bi-GMKP algorithm after swap-optimal improvement for unequal knapsack capacities
Results for unequal knapsack capacities

All instances solved in Sect. 9 had identical knapsack capacities. Here we modify all instances by changing each knapsack capacity $c_i$, $i \in M$, through a two-step procedure. The first step consists in multiplying each knapsack capacity of 100 by a real random number between 0.5 and 1.5—note knapsack capacities are 100 in expectation (each knapsack capacity is multiplied by a different random number). Since $w_{\text{min}} \leq 50$ for all instances by design, this procedure does not generate knapsack capacities where no item would fit. The second step of the procedure avoids having groups that exceed the total knapsack capacity. Formally, the steps are the following for each instance:

1. Define new knapsack capacities $c'_i$, $\forall i \in M$, as a function of the original knapsack capacity $c_i = 100$ by doing:
   
   $c'_i \leftarrow \lceil \text{Random}(0.5, 1.5) \cdot 100 \rceil$

2. Let $g_{\text{max}} = \max_{l \in K} \sum_{j \in G_l} w_j$ and $c_{\text{total}} = \sum_{i \in M} c'_i$. If $g_{\text{max}} > c_{\text{total}}$, update all knapsack capacities $c'_i$, $i \in M$, by doing:
   
   $c'_i \leftarrow \lceil c'_i \cdot \frac{g_{\text{max}}}{c_{\text{total}}} \rceil$

We repeated all experiments from Sect. 9 for the 3000 instances with the modified knapsack capacities.

Unequal knapsack capacities: results for bi-GMKP algorithms

In Fig. 27 we see the details of the maximum exceeded knapsack capacity of bi-GMKP algorithms, after swap-optimal improvement, for instances with unequal knapsack capacities. Similar to the original instances with equal knapsack capacities, algorithms 3mKP and 100mKP obtained the least exceeded knapsack capacities. Performance improved in comparison to the original instances, likely because larger knapsacks allowed algorithms to reduce the maximum exceeded knapsack capacity even further.

Figure 28 shows the computation times of each bi-GMKP algorithm on instances with unequal knapsack capacities. 3mKP persists as the most time-effective alternative, while running 100mKP might still be recommended since it obtains better results and computation times remain short. Computation times remained almost identical to instances with equal knapsack capacities (compare Fig. 28 to 3).

Unequal knapsack capacities: results for GMKP heuristics

Figure 29 shows the details of the optimal reward ratio of bi-GMKP heuristics, after swap-optimal improvement, for unequal knapsack capacities. As in the original instances (Fig. 4), there does not seem to be a relevant improvement after adding

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constraints beyond 2mKP. The performance of all GMKP heuristics improved in contrast to the original instances; for example, the 5th percentile of 3mKP improved from 0.78 to 0.85.

Figure 30 shows the computation times of each GMKP heuristic on instances with unequal knapsack capacities. As in instances with equal knapsack capacities, 2mKP runs faster than KP and similar to 3mKP. Computation times remained almost identical to instances with equal knapsack capacities (compare Figs. 30 to 5). Computation times do not seem affected by different knapsack capacities.

**Unequal knapsack capacities: results for bi-GMKP heuristics**

Figure 31 shows the density of the non-dominated solutions obtained by bi-GMKP heuristic for the modified versions of Algorithm 2 (2mKP); as seen in line 6 of Heuris-
Fig. 30 Computation time per GMKP heuristic after swap-optimal improvement for unequal knapsack capacities

Fig. 31 Countour plots of non-dominated solutions of all instances with unequal knapsack capacities

tic 1. Analogous to the original instances with equal knapsack capacities (see Fig. 6), most solutions lie slightly above the diagonal line that represents the case where changes in the optimal reward ratio generate a proportional change in the maximum exceeded knapsack capacity. This shows how the proposed bi-GMKP heuristic can be used, independent of knapsack capacities, to generate different bi-criteria combinations doing a good job in maximizing rewards while slightly exceeding knapsack capacities.

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