COMPOSITION WITH A TWO VARIABLE FUNCTION

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1. Introduction

In [11] and [12], A. Némethi studied the Milnor fiber and monodromy zeta function of composed functions of the form $f(g_1, g_2)$ with $f$ a two variable polynomial and $g_1$ and $g_2$ polynomials with distinct sets of variables. The present paper addresses the question of proving similar results for the motivic Milnor fiber introduced by Denef and Loeser, cf. [1], [3], [10], [4]. In fact, Némethi later considered in [13] the more general situation of a composition $f \circ g: (X, x) \to (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$, where $g$ has a reasonable discriminant. Still later, Némethi and Steenbrink [14] proved similar results at the level of the Hodge spectrum [16], [17], [18], using the theory of mixed Hodge modules. In particular, they were able to compute, under mild assumptions, the Hodge spectrum of composed functions of the form $f(g_1, g_2)$ without assuming the variables in $g_1$ and $g_2$ are distinct. Their result involves the discriminant of the morphism $g = (g_1, g_2)$. In a previous paper [7], we computed the motivic Milnor fiber for functions of the form $g_1 + g_2^\ell$ when $\ell$ is large without assuming the variables in $g_1$ and $g_2$ are distinct. The corresponding result for the Hodge spectrum goes back to M. Saito [15] and is a special case of the results of Némethi and Steenbrink [14]. So, it seems very natural to search for a full motivic analogue of the results of [14]. At the present time, we are unable to realize this program and we have to limit ourself, as we already mentioned, to the case when $g_1$ and $g_2$ have no variable in common. Already extending our result to the case when one only assumes the discriminant of the morphism $g$ is contained in the coordinate axes seems to require new ideas.

In this paper we consider a polynomial $f$ in $k[x, y]$ and we assume that $f(0, y)$ is non zero of degree $m$. We denote by $i_p$ the closed embedding into $\mathbb{A}^2_k$ of a point $p$ in $F_0 = f^{-1}(0) \cap x^{-1}(0)$. We consider the motivic Milnor fiber $\mathcal{S}_f$ of the function $f: \mathbb{A}^2_k \to \mathbb{A}^1_k$ whose restriction $i_p^*\mathcal{S}_f$ above $p$ is an element of the Grothendieck ring $\mathcal{M}^G_{\mathbb{C}, m}$. We then reformulate Guibert’s computation of the motivic Milnor fiber of germs of plane curve singularities [6] using generalized convolution operators of [8]. More precisely, we express it in terms of the tree $\tau(f, p)$ associated to $f$, depending on the given coordinate system $(x, y)$ on the affine plane $\mathbb{A}^2_k$. Let us recall the tree $\tau(f, p)$ is obtained by considering the Puiseux expansions of the roots of $f$ at $p$, cf. [9], [5]. To any so-called rupture vertex $v$ of this graph, we attach a weighted homogeneous
polynomial $Q_{v,f}$ in $k[c,d]$. We have defined in [5] a generalized convolution by such a polynomial. It is a morphism from $\mathcal{M}_{\mathbb{G}_m}^{G_m}$ to $\mathcal{M}_{\mathbb{G}_m}^{G_m}$, but can be extended to a morphism from $\mathcal{M}_{\mathbb{A}_k^1 \times \mathbb{G}_m}^{G_m}$ to $\mathcal{M}_{\mathbb{A}_k^1 \times \mathbb{G}_m}^{G_m}$. We denote by $\varpi_j$ the morphism $x \mapsto x^j$ from $\mathbb{G}_m$ to $\mathbb{G}_m$ and by $m_p$ the order of $p$ as a root of $f(0,y)$. One can then reformulate Guibert's theorem as

$$ (*) \quad i_p^* \mathcal{S}_f = [\varpi_{m_p} : \mathbb{G}_m \to \mathbb{G}_m] - \sum_v \Psi_Q_{v,f}([\text{Id} : \mathbb{A}_k^1 \times \mathbb{G}_m \to \mathbb{A}_k^1 \times \mathbb{G}_m])$$

where the sum runs over the set of rupture vertices of $\tau(f,p)$.

For $1 \leq j \leq 2$, let $g_j : X_j \to \mathbb{A}_k^1$ be a function on a smooth $k$-variety $X_j$. By composition with the projection, $g_j$ becomes a function on the product $X = X_1 \times X_2$ and we write $g$ for the map $g_1 \times g_2 : X \to \mathbb{A}_k^2$. The main result of this paper, Theorem 4.2, gives a formula for $i_p^* \mathcal{S}_{f,g}$, where $i$ denotes the inclusion of $g_1^{-1}(0) \cap g_2^{-1}(0)$, similar to (3), with $[\text{Id} : \mathbb{A}_k^1 \times \mathbb{G}_m \to \mathbb{A}_k^1 \times \mathbb{G}_m]$ replaced by a virtual object $A_v$. The virtual object $A_v$ is defined inductively in terms of the tree associated to $f$ at the origin of $\mathbb{A}_k^2$ and of $A_{v_0}$, where $v_0$ is the first (extended) rupture vertex of $\tau(f,p)$, and $A_{v_0}$ depends only on $g$.

2. Preliminaries and combinatorial setup

2.1. We fix an algebraically closed field $k$ of characteristic 0. For a variety $X$ over $k$, we denote by $\mathcal{L}(X)$ and $\mathcal{L}_n(X)$ the total spaces of arcs, resp. arcs mod $t^{n+1}$ as defined in [2]. As in [7], we denote by $\mathcal{M}_X$ the localisation of the Grothendieck ring of varieties over $X$ with respect to the class of the relative line. We shall also use the $\mathbb{G}_m$-equivariant variant $\mathcal{M}_{X \times \mathbb{G}_m}^{G_m}$ defined in [3], which is generated by classes of objects $Y \to X \times \mathbb{G}_m$ endowed with a monomial $\mathbb{G}_m$-action.

Also, if $p$ is a closed point of $X$ we denote by $i_p$ the inclusion $i_p : p \to X$ and by $i_p^*$ the corresponding pullback morphism at the level of rings $\mathcal{M}$.

2.2. Let us start by recalling some basic constructions introduced by Denef and Loeser in [1, 4] and [3].

Let $X$ be a smooth variety over $k$ of pure dimension $d$ and $g : X \to \mathbb{A}_k^1$. We set $X_0(g)$ for the zero locus of $g$, and consider, for $n \geq 1$, the variety

$$ (2.2.1) \quad X_n(g) := \{ \varphi \in \mathcal{L}_n(X) \mid \text{ord}_x g(\varphi) = n \}.$$ 

Note that $X_n(g)$ is invariant by the $\mathbb{G}_m$-action on $\mathcal{L}_n(X)$. Furthermore $g$ induces a morphism $g_n : X_n(g) \to \mathbb{G}_m$, assigning to a point $\varphi$ in $\mathcal{L}_n(X)$ the coefficient $ac(g(\varphi))$ of $t^n$ in $g(\varphi)$, which we shall also denote by $ac(g(\varphi))$. This morphism is homogeneous of weight $n$ with respect to the $\mathbb{G}_m$-action on $X_n(g)$ since $g_n(a \cdot \varphi) = a^n g_n(\varphi)$, so we can consider the class $[X_n(g)]$ of $X_n(g)$ in $\mathcal{M}_{X_0(g) \times \mathbb{G}_m}^{G_m}$.

We now consider the motivic zeta function

$$ (2.2.2) \quad Z_{g}(T) := \sum_{n \geq 1} [X_n(g)] \mathbb{L}^{-nd} T^n$$

in $\mathcal{M}_{X_0(g) \times \mathbb{G}_m}^{G_m}[[T]]$. Note that $Z_g = 0$ if $g = 0$ on $X$. 

Denef and Loeser showed in [1] and [3] (see also [4]) that $Z_g(T)$ is a rational series by giving a formula for $Z_g(T)$ in terms of a resolution of $f$. They also showed that one can consider $\lim_{T \to \infty} Z_g(T)$ in $M^{\mathbb{G}_m}_{X_0(y) \times \mathbb{G}_m}$ and they define the motivic Milnor fiber of $g$ as

\begin{equation}
S_g := - \lim_{T \to \infty} Z_g(T).
\end{equation}

2.3. In this subsection we do not assume $X$ to be smooth. For technical reasons we shall use in the present paper the following innocuous variant of $M^{\mathbb{G}_m}_{X \times \mathbb{G}_m}$: replacing everywhere in the definition the first $\mathbb{G}_m$-factor endowed with the $\mathbb{G}_m$-action by multiplicative translation $\lambda \cdot x = \lambda x$ by $\mathbb{A}^1_k$ with “the same” $\mathbb{G}_m$-action one gets a ring $M^{\mathbb{G}_m}_{X \times \mathbb{A}^1_k \times \mathbb{G}_m}$ generated by classes of objects $Y \to X \times \mathbb{A}^1_k \times \mathbb{G}_m^{-1}$ endowed with a monomial $\mathbb{G}_m$-action.

If $Q$ is a quasihomogeneous polynomial in $p$ variables, we defined in [8] a convolution operator

$$\Psi_Q : M^{\mathbb{G}_m}_{X \times \mathbb{G}_m} \to M^{\mathbb{G}_m}_{X \times \mathbb{G}_m}.$$ 

In this paper we shall use the slight variant, still denoted by $\Psi_Q$, which is obtained with the same definition, replacing $\mathbb{G}_m^p$ by $\mathbb{A}^1_k \times \mathbb{G}_m^{-1}$,

$$\Psi_Q : M^{\mathbb{G}_m}_{X \times \mathbb{A}^1_k \times \mathbb{G}_m^{-1}} \to M^{\mathbb{G}_m}_{X \times \mathbb{G}_m}.$$ 

In fact, such constructions carry over for any toric variety with torus $\mathbb{G}_m^p$, not only for $\mathbb{A}^1_k \times \mathbb{G}_m^{-1}$.

2.4. Fix a positive integer $N$ and consider the ring of fractional power series $k[[x^\frac{1}{r}]]$. Given a positive rational number $r$ we denote by $I_{\geq r}$ the ideal of power series of order at least $r$ in $k[[x^\frac{1}{r}]]$. We call the quotient $k[[x^\frac{1}{r}]]/I_{\geq r}$ the ring of $r$-truncated fractional power series.

To a $r$-truncated fractional power series $y$ one assigns a labelled rooted real metric tree $\tau_r(y)$ in the following way. The total space $\tau_r(y)$ is the half-open interval $[0, r)$ and its vertices are the positive exponents with non zero coefficients of the expansion of $y$ in powers of $x$ together with the origin which is the root. We define the height of a vertex to be its distance to the root. We label each vertex by the coefficient of the corresponding term (this coefficient is non-zero for all vertices except maybe for the root). The vertices are ordered by the height. Starting above a vertex there is only one edge. This edge ends with the next vertex if there is one and remains open above the last vertex. We label each edge by 0 and we say $\tau_r(y)$ is of height $r$. We denote by $|\tau_r(y)|$ the underlying unlabelled tree. Notice that we can see the labels as degree 1 polynomials of $k[X]$ (or cycles in $\mathbb{A}_k^1$), that is, $X$ for an edge and $X - a$ for a vertex labelled by $a$.

If now $y$ is a power series in $k[[x^\frac{1}{r}]]$, we denote by $\tau_r(y)$ the height $r$ tree associated its truncation of $y$ at order $r$. Thus, for $r < r'$, $\tau_r(y)$ is obtained from $\tau_{r'}(y)$ by truncating up to height $r$. We denote by $\tau(y)$ the inductive limit of the system $(\tau_r(y))_{r \in \mathbb{Q}}$ and call it the tree associated to the power series $y$. 
2.5. We consider a two variable polynomial $f$ in $k[x, y]$. We assume that $f(0, y)$ is non zero of degree $m$ and we consider the $m$ Newton-Puiseux expansions $y_i$, $1 \leq i \leq m$, associated to $f$ at the points of $f^{-1}(0) \cap x^{-1}(0)$. There exists an integer $N$ such that these roots are elements of the ring of fractional power series $k[[x^{1/N}]]$, namely, they are the roots of the polynomial $f$ in $k[[x^{1/N}]]$.

Fix a positive rational number $r$. We denote by $\bigcup_{i=1}^{m} \tau_r(y_i)$ the labelled rooted real metric tree which is obtained as follows. In the disjoint union of the trees $\bigsqcup_{i=1}^{m} |\tau_r(y_i)|$ we identify two vertices (resp. two edges) if they have same height and same label. If $v$ is a vertex (resp. an edge) shared by trees $\tau_r(y_i)$ for $i \in J$, then its label on the union is the $|J|$-th power of its label on any of the $\tau_r(y_i)$, $i \in J$.

Since the group of $N$-roots of unity acts on the $r$-truncated expansions $(y_1, \ldots, y_m)$, it also acts on $|\bigcup_{i=1}^{m} \tau_r(y_i)|$. We denote by $|\tau_r(f)|$ the separated quotient and by $\pi : |\bigcup_{i=1}^{m} \tau_r(y_i)| \to |\tau_r(f)|$ the quotient morphism. Note that the connected components of $|\tau_r(f)|$ are in natural bijection with points of $f^{-1}(0) \cap x^{-1}(0)$. For any such point $p$, we denote by $|\tau(f, p)|$ the corresponding connected component which is naturally endowed with the structure of a rooted real metric tree. We attach labels to the vertices and edges of $|\tau_r(f)|$ in the following way:

- If $e$ is an edge of $|\tau_r(f)|$, the label attached to $e$ is the label on any element of $\pi^{-1}(e)$. It is a power of $X$ in $k[X]$ and we denote it by $P_{e,f}$. We will call degree of the edge $e$ the degree of $P_{e,f}$.
- If $v$ is a vertex of $|\tau_r(f)|$, the label on $v$ is the product of the labels on $\pi^{-1}(v)$. We denote it by $P_{v,f}$. We will call degree of the vertex $v$ the degree of $P_{v,f}$. Notice that the degree of a vertex $v$ is equal to the degree of the edge $e$ which ends in $v$.

For $r < r'$, the graph $\tau_{r'}(f)$ is the truncation of $\tau_r(f)$ at height $r$. The graph of contacts $\tau(f)$ defined by $f$ along $f^{-1}(0) \cap x^{-1}(0)$ is the inductive limit of the graphs $\tau_r(f)$, $r \in \mathbb{Q}$, cf. [9], [11]. We say that a vertex $v$ of $\tau(f)$ is a rupture vertex if the set of zeroes of $P_{v,f}$ contains at least two points in $\mathbb{A}^1_k$. We define the augmented set of rupture vertices of the tree $\tau(f, p)$ as the set of rupture vertices of $\tau(f, p)$ together with the vertex of minimal non zero height on $\tau(f, p)$.

We fix from now on a point $p$ which will be assumed for simplicity to be the origin in $\mathbb{A}^2_k$. For any arc $\varphi$ in $\mathcal{L}(\mathbb{A}_k^2)$ such that $\varphi(0) = p$ and $x(\varphi) \neq 0$, there exist power series $\omega$ in $k[[t]]$ and $\sum b_j \omega^j$ in $k[[\omega]]$, and an integer $M$ such that $\gcd(M, \{j \mid b_j \neq 0\}) = 1$ and

$$
\begin{align*}
x(\varphi(t)) &= \omega(t)^M \\
y(\varphi(t)) &= \sum_j b_j \omega(t)^j.
\end{align*}
$$

Hence

$$
y(\varphi(t)) = \sum_j b_j (x(\varphi(t)))^{\frac{j}{M}}
$$

is a fractional power series in $x(\varphi(t))$. We consider the tree $\tau(y)$ with

$$
y(x) := \sum_j b_j x^{\frac{j}{M}}
$$
in $k[[x^{1/\ell}]]$.

The power series $\omega$ is defined up to an $M$-root of unity so that the $b_j$'s are defined up to a factor $\zeta^j$ with $\zeta$ an $M$-root of unity. Two different choices lead to trees in the same $\mu_N$-orbit. This orbit is denoted by $\tau(\varphi)$. Notice that $\tau(\varphi)$, as well as $\tau(f)$, depends on the system of coordinates $(x,y)$.

2.6. **Definition.** Consider $\varphi$ in $\mathcal{L}(A_k^2)$ and $f$ in $k[x,y]$ as before. The **order of contact** of $\varphi$ with $f$ is the maximum number $s$ in $\mathbb{Q} \cup \{\infty\}$ such that $\tau_s(\varphi)$ is included in $\tau_s(f)$ (it is infinite if and only if $f(\varphi) = 0$). The **contact** of $\varphi$ with $f$ is the tree $\tau_r(\varphi)$ where $r$ is the order of contact of $\varphi$ with $f$.

2.7. From now on the polynomial $f$ is fixed in $k[x,y]$ and we denote by $m$ the degree of $f(0,y)$. For a positive rational number $r$, by a contact $\tau$ of order $r$, we mean a subtree of $\tau_r(f,p)$ which is isomorphic to $[0,r)$. In particular $\tau$ is rooted at $p$ and its closure in $\tau_r(f,p)$ contains a unique point of height $r$, not necessarily a vertex of $\tau_r(f,p)$, which completely determines $\tau$. To such a contact $\tau$ we assign a polynomial $P_{\tau,f}$ in the following way. The last and (semi)open edge of $\tau$ is contained in a unique edge $e$ of $\tau_r(f,p)$.

- If $e$ ends at a vertex $v$ at height $r$ of $\tau_r(f,p)$ (in this case we say that $\tau$ ends at the vertex $v$), we will set $P_{\tau,f} = P_{\tau,v}$.
- Otherwise, (in that case we say that $\tau$ ends at the edge $e$) we set $P_{\tau,f} = P_{\tau,e}$.

By definition of contact, there is an integer $M$ and a polynomial $y_\tau$ in $k[\omega]$, of degree strictly smaller than $rM$, both depending only on $\tau$, such that for any arc $\varphi$ of contact $\tau$ with $f$, there exists a series $\omega$ in $k[[t]]$, $\text{ord}_t(\omega) = \ell$, such that

\[
\begin{align*}
x(\varphi(t)) &= \omega(t)^M \\
y(\varphi(t)) &= y_\tau(\omega(t)) \quad [\text{mod}(t^{rM\ell})].
\end{align*}
\]

For an arc $\varphi$ of contact $\tau$ with $f$, the quotient $\text{ord}_t(f(\varphi))/\ell$ is an integer and depends only on $\tau$. We denote it by $\nu(\tau)$. One always has the inequality $\nu(\tau) \geq Mr$.

The tree $\tau(f,p)$ is built from the Puiseux expansions of the $m$ roots of $f(x,y)$ in the ring of fractional power series $\bigcup_N k[[x^{1/N}]]$. Conversely, to any finite subtree $\varsigma$ of $\tau(f,p)$, we can associate a polynomial $f_\varsigma$ in $k[x,y]$ which is the minimal polynomial of the $m$ Puiseux expansions restricted to $\varsigma$. Considering the tree associated to the polynomial $f_\varsigma$, we get a tree $\tau(f_\varsigma,p)$ which is an infinite tree with a finite number of vertices. The intersection of $\tau(f_\varsigma,p)$ with $\tau(f,p)$ contains $\varsigma$. As an example, we can consider the tree $\tau_r$ obtained from $\tau(f,p)$ by truncation at height $r$. We will denote by $\overline{\tau_r}$ the tree $\tau(f_{\tau_r},p)$.

3. Guibert’s theorem revisited

3.1. We consider the following set of arcs:

$$X_{\tau,\ell} := \left\{ \varphi \in \mathcal{L}_{\nu(\tau)}(A_k^2) \mid \varphi \text{ has contact } \tau \text{ with } f, \text{ ord}_t x(\varphi) = Mr \right\},$$

where $\nu(\tau)$ is the maximum of the integers $\nu(\tau)$ and $M$. 
We denote by $Q_{\tau,f}$ the function $\omega^{\nu(\tau)}P_{\tau,f}(\omega^{-Mr}c)$. One should note that $Q_{\tau,f}$ is a polynomial in $k[c, \omega]$, even if $Mr$ may not be an integer.

3.2. Lemma. Consider a contact $\tau$ and an integer $\ell$ and denote by $N(\tau, \ell)$ the integer $2\nu(\tau)\ell - Mr - \lfloor Mr\ell \rfloor$. For any arc $\varphi$ in $\mathcal{X}_{\tau,\ell}$, there exist two series $\omega$ and $\varepsilon$ in $k[t]/t^{\nu(\tau)+1}$ such that

(1) $\text{ord}_t(\omega) = \ell$, $x(\varphi) = \omega^M$

(2) $\text{ord}_t(\varepsilon) \geq Mr \ell$ (resp. $= Mr \ell$ if $\tau$ ends in an edge), $y(\varphi) = y_r(\omega) + \varepsilon$.

The mapping $(\varepsilon, \omega) \mapsto (\omega^M, y_r(\omega) + \varepsilon)$ induces an isomorphism

$$
\Phi : (A^1_k \times \mathbb{G}_m) \setminus Q_{\tau,f}^{-1}(0) \times A^N_{k(\tau,\ell)} \rightarrow \mathcal{X}_{\tau,\ell}
$$

given by

$$
(c, \omega_\ell, a) \mapsto (t^{\ell}M(\omega_\ell + \sum_{k=1}^{\ell} a_k \nu^k \omega^k + ct^{Mr\ell} + \sum_{k=0}^{\ell} a_k \nu^k \omega^k)).
$$

Via the isomorphism $\Phi$, the angular coefficient $ac(f(\varphi))$ is given, up to a non-zero constant, by the following formula:

$$
ac(f(\varphi)) \sim \omega^{\nu(\tau)}P_{\tau,f}(\omega^{-Mr}c) = Q_{\tau,f}(c, \omega_\ell).
$$

For the $\mathbb{G}_m$-action $\sigma$ on $(A^1_k \times \mathbb{G}_m)$ given by $\sigma(\lambda) \cdot (c, \omega_\ell) = (\lambda^M c, \lambda^\ell \omega_\ell)$, the polynomial $Q_{\tau,f}$ is homogeneous of degree $\nu(\tau)\ell$.

Proof. We did already notice that the map $\Phi$ is surjective. Conversely $\omega$ is determined by $x(\varphi)$ up to a $M$-th root of unity, and uniquely determined by $x(\varphi)$ and $y(\varphi)$ for the $\text{gcd}$ of $M$ and exponents of non zero terms in $y_r(\omega)$ is equal to 1. \[\square\]

3.3. On the constructible set $\mathcal{X}_{\tau,\ell}$, via the isomorphism $\Phi$, we have a morphism to $A^1_k \times \mathbb{G}_m$ induced by the first projection from $(A^1_k \times \mathbb{G}_m) \times A^N_{k(\tau,\ell)}$. The constructible set $\mathcal{X}_{\tau,\ell}$ defines a class in $\mathcal{M}_{A^1_k \times \mathbb{G}_m}$ we denote by $[\mathcal{X}_{\tau,\ell}]$. On the other hand, the function $ac(f)$ induces a $\mathbb{G}_m$-equivariant morphism from $\mathcal{X}_{\tau,\ell}$ to $\mathbb{G}_m$, hence defines a class in $\mathcal{M}_{\mathbb{G}_m}$ we denote by $[\mathcal{X}_{\tau,\ell}(f)]$. By Lemma 3.2, the morphism $\mathcal{X}_{\tau,\ell} \rightarrow \mathbb{G}_m$ is equal to the composition of the morphism $\mathcal{X}_{\tau,\ell} \rightarrow A^1_k \times \mathbb{G}_m$ with $Q_{\tau,f}$.

3.4. If $v$ is a rupture vertex of height $r$, there is only one contact ending in $v$ that we denote by $\tau_v$. We set $Q_{v,f} := Q_{\tau_v,f}$.

We are now in position to restate Guibert’s theorem [6] in the following form:

3.5. Theorem (Guibert). With the above notation, the following holds:

$$
i^*_p S_f = [\varpi_{m_p} : \mathbb{G}_m \rightarrow \mathbb{G}_m] - \sum_v \Psi_{Q_{v,f}}([\text{Id} : A^1_k \times \mathbb{G}_m \rightarrow A^1_k \times \mathbb{G}_m])
$$

where the second sum runs over the rupture vertices of $\tau(f)$ above $p$. 
Proof. Note that for a two variable quasihomogeneous polynomial $Q$
\[ (3.5.1) \quad \Psi_Q([\text{Id} : \mathbb{A}_k^1 \times \mathbb{G}_m \rightarrow \mathbb{A}_k^1 \times \mathbb{G}_m]) = \]
\[- [Q : (\mathbb{A}_k^1 \times \mathbb{G}_m) \setminus Q^{-1}(0) \rightarrow \mathbb{G}_m] + S_Q([\mathbb{A}_k^1 \times \mathbb{G}_m]) \]
where $S_Q$ is defined as in [7], [8]. We denote by $\pi_E$ the morphism $(a, b) \mapsto a^E$ from
$\mathbb{G}_m \times \mathbb{G}_m$ to $\mathbb{G}_m$. When the zeroes of $Q$ are a disjoint union of one dimensional
$\mathbb{G}_m$-orbits, $S_Q(\mathbb{A}_k^1 \times \mathbb{G}_m)$ decomposes into a sum
\[ S_Q(\mathbb{A}_k^1 \times \mathbb{G}_m) = - \sum_i [\pi_{E_i} : \mathbb{G}_m \times \mathbb{G}_m \rightarrow \mathbb{G}_m] \]
where $E_i$ is the multiplicity of $Q$ along the $i$-th component of $Q^{-1}(0)$. If $v$ is a
rupture vertex we consider the following zeta function:
\[ Z_f^v(T) := \sum_{\ell \geq 1} \sum_{\tau} |\mathcal{X}_{\tau, \ell}(f)| L^{-2\nu(\tau)\ell} T^{\nu(\tau)\ell} \]
where the second sum is extended to the set of contacts $\tau$ which contain $\tau_v$ and do
not contain or end in any successor of $v$ in the set of rupture vertices. From [6] (3.3)
and (5.2), we deduce that $Z_f^v(T)$ has a limit $-S_f^v$ in the Grothendieck ring $\mathcal{M}_{\mathbb{G}_m}$
when $T$ goes to infinity, which is given by the formula
\[ S_f^v = - \Psi_{Q_{v,J}}([\text{Id} : \mathbb{A}_k^1 \times \mathbb{G}_m \rightarrow \mathbb{A}_k^1 \times \mathbb{G}_m]). \]
We consider the series
\[ Z_f^p(T) := \sum_{\ell \geq 1} \sum_{\tau} |\mathcal{X}_{\tau, \ell}(f)| L^{-2\nu(\tau)\ell} T^{\nu(\tau)\ell} \]
where the second sum is extended to the set of contacts $\tau$ starting from the root
(corresponding to $p$, which do not contain or end in any successor of $v$ in the set of
rupture vertices. Again by [6], loc. cit., $Z_f^p(T)$ has limit $-|\mathcal{Z}_{m_p} : \mathbb{G}_m \rightarrow \mathbb{G}_m|$ when
$T$ goes to infinity. The restriction $i_p^*S_f$ is the limit as $T \rightarrow \infty$ of $-i_p^*Z_f(T)$ which
decomposes into
\[ -i_p^*Z_f(T) = -Z_f^p(T) - \sum_v Z_f^v(T) \]
where the sum extends to all the rupture vertices of $\tau(f)$ above $p$. The result
follows. \qed

4. Composition with a morphism

4.1. For $1 \leq j \leq 2$, let $g_j : X_j \rightarrow \mathbb{A}_k^1$ be a function on a smooth $k$-variety $X_j$. By
composition with the projection, $g_j$ becomes a function on the product $X = X_1 \times X_2$.
We write $d_j$ for the dimension of $X_j$, $j$ from 1 to 2, and $d$ for $d_1 + d_2$. Define $g$
as the map $g_1 \times g_2$ on $X$ and $G$ as the product $G = g_1g_2$. For any subvariety $Z$ of the
set $X_0(G)$ containing $X_0(g) := g_1^{-1}(0) \cap g_2^{-1}(0)$, we denote by $i$ the closed immersion
of $X_0(g)$ in $Z$.

As in section 2 we denote by $f$ a two variable polynomial, we assume that
$f(0, y)$ is a nonzero polynomial, we denote by $p$ the origin and will denote by $m_p$ the
order of 0 as a root of $f(0, y)$. We consider the augmented set of rupture vertices of the tree $\tau(f, p)$, namely the set of rupture vertices together with the vertex of minimal non zero height on $\tau(f, p)$. Denote that vertex by $v_0$ and consider its associated polynomial $Q_{v_0}$. We denote by $S'_{g_2}$ the element in $\mathcal{M}^{G_m}_{\mathcal{X}_0(g_2) \times \mathbb{A}_k^1}$ which is the “disjoint sum” of $S_{g_2}$ in $\mathcal{M}^{G_m}_{\mathcal{X}_0(g_2) \times G_m}$ and $X_0(g_2)$ in $\mathcal{M}_{\mathcal{X}_0(g_2)}$. We set $A_{v_0} := S'_{g_2} \boxtimes S_{g_1}$, considered as an element in $\mathcal{M}^{G_m}_{\mathcal{X}_0(g) \times (\mathbb{A}_k^1 \times G_m)}$. For any rupture vertex $v$ of the tree $\tau(f, p)$, we denote by $a(v)$ the predecessor of $v$ in the augmented set of rupture vertices and we define by induction a virtual variety $A_v$ in $\mathcal{M}^{G_m}_{\mathcal{X}_0(g) \times \mathbb{A}_k^1 \times G_m}$. We assume we are given a virtual variety $A_{a(v)}$ in $\mathcal{M}^{G_m}_{\mathcal{X}_0(g) \times \mathbb{A}_k^1 \times G_m}$ whose restriction over $X_0(g) \times G_m \times G_m$ is diagonally monomial in the sense of [7] (2.3), or more precisely whose $G_m$-action is diagonally induced from a diagonally monomial $G_m^2$-action in the sense of [7]. To any successor of $a(v)$ corresponds a factor of the polynomial $Q_{a(v)}$. Denote by $Q'_{a(v)}$ the factor associated to $v$. Notice that $(Q'_{a(v)})^{-1}(0)$ is a smooth subvariety in $G_m \times G_m$, equivariant under a diagonal $G_m$-action and that the second projection $p_2$ of the product $\mathbb{A}_k^1 \times G_m$ induces an homogeneous fibration from $(Q'_{a(v)})^{-1}(0)$ to $G_m$. We denote by $B_v$ the restriction of $A_{a(v)}$ above $(Q'_{a(v)})^{-1}(0)$. The external product of the identity of the affine line $\mathbb{A}_k^1$ by the induced map $B_v : G_m \to G_m$ defines an element $A_v$ in $\mathcal{M}^{G_m}_{\mathcal{X}_0(g) \times \mathbb{A}_k^1 \times G_m}$ which is diagonally monomial when restricted to $X_0(g) \times G_m \times G_m$.

4.2. Theorem. With the previous notations and hypotheses, the following formula holds

\[(4.2.1) \quad i^*S_{f_{\mathbb{X}g}} = S_{(g_2)^{mp}}(X_0(g_1)) - \sum_v \Psi_{Q_v}(A_v),\]

where the sum is runs over the augmented set of rupture vertices of the tree $\tau(f, p)$.

Proof. We first reduce to the case where $\tau(f, p)$ has only a finite number of vertices.

4.3. Lemma. There exists a rational number $\gamma$ such that, for any $r$ greater than $\gamma$,

\[(4.3.1) \quad i^*S_{f_{\mathbb{X}g}} = i^*S_{f_{r, \mathbb{X}g}}.\]

Proof. Consider a rupture vertex $v$. The quotient $\text{ord}_t(f \circ g(\varphi))/\text{ord}_t(g_1(\varphi))$ is an affine function of $r$ whenever $\tau$ contains $\tau_v$ and does not contain or end in any rupture vertex greater than $v$. Hence the quotient $\text{ord}_t(f \circ g(\varphi))/\text{ord}_t(g_1(\varphi))$ is a function on the tree $\tau(f)$, the restriction of which on each semi-open branch joining two consecutive rupture vertices (resp. on each infinite branch above a rupture vertex) is an increasing affine function of the height.

We consider the following set of arcs

\[\mathcal{X}_{n_1, n_2}(x \circ g, f \circ g) := \left\{ \varphi \in \mathcal{L}_{n_1 + n_2}(X) \mid \text{ord}_t x \circ g = n_1, \text{ord}_t f \circ g = n_2 \right\}.\]

For $\gamma$ large enough, the zeta function

\[Z_{\mathbb{X}g, f_{\mathbb{X}g}}(T) = \sum_{n_2 \geq \gamma n_1} [\mathcal{X}_{n_1, n_2}(x \circ g, f \circ g)] T^{-n_2} \]
To prove the theorem, it is enough to consider the case when the tree \( \tau(f, p) \) has a finite number of vertices. The proof goes by induction on the number of vertices of the tree \( \tau(f, p) \). Certainly the result holds if there is no vertex. Assume first the tree has only one vertex \( v_0 \). The formula is then a particular case of the main formula in [5]. Assume now we have at least two vertices. Choose a maximal vertex \( v \) for the height function on \( \tau(f, p) \) and consider the subtree \( \tau^- \) obtained from \( \tau(f, p) \) by deleting the vertex \( v \). Denote by \( a(v) \) the predecessor of \( v \) on \( \tau(f, p) \) and by \( f^- \) the polynomial associated to \( \tau^- \).

Consider an arc \( \varphi \) in \( \mathbb{A}^2_k \) with origin \( p \). Then one of the following two statements holds:

- The contact of \( \varphi \) with \( f \) does not contain \( \tau_v \). Then \( \text{ord}_t(f(\varphi)) = \text{ord}_t(f^-(\varphi)) \) and \( \text{ac}(f(\varphi)) = \text{ac}(f^-(\varphi)) \).
- The contact of \( \varphi \) with \( f \) contains \( \tau_v \).

According to these two different cases, we can split the zeta function \( Z_{f \circ g} \) in two pieces, namely

\[
Z_{f \circ g} = Z_{\leq_v} + Z_{\geq_v},
\]

Similarly, the zeta function \( Z_{f^- \circ g} \) decomposes in

\[
Z_{f^- \circ g} = Z^-_{\leq_v} + Z^-_{\geq_v}.
\]

We noticed that \( Z^-_{\leq_v} = Z^-_{\leq_v} \), hence we get

\[
Z_{f \circ g} - Z_{f^- \circ g} = Z_{\geq_v} - Z^-_{\geq_v}.
\]

In section 3.3 for any contact \( \tau \) and integer \( \ell \), we have considered a set \( \mathcal{X}_{\tau, \ell} \) associated to a polynomial \( f \). Similarly we have a set \( \mathcal{X}^-_{\tau, \ell} \) associated to \( f^- \). These two sets map to \( \mathbb{A}^2_k \times \mathbb{G}_m \). Consider now the inverse image by \( g \) of \( \mathcal{X}_{\tau, \ell} \) (resp. \( \mathcal{X}^-_{\tau, \ell} \)) and denote it by \( \mathcal{X}_{\tau, \ell}(g) \) (resp. \( \mathcal{X}^-_{\tau, \ell}(g) \)). We assume, by induction, that the motivic nearby cycles of \( f^- \) have the given form and that for any contact \( \tau \) greater than \( \tau_a(v) \) the set \( \mathcal{X}_{\tau, \ell}(g) \) is a piecewise affine bundle on \( X_0(g) \times \mathbb{G}_m \times B_{a(v)} \).

An arc \( g \circ \varphi \) in \( \mathcal{L}_{\nu(\tau)\ell}(A^2_k) \) having contact \( \tau \) with \( f^- \) has contact \( \tau_v \) with \( f \) if and only if \( \tau = \tau_v \) and \( Q_{a(v)}^c \) does not vanish on \( \varphi \) or if \( \tau \) contains strictly \( \tau_v \). In that case \( \varphi \) maps to \( \{0\} \times B_{a(v)} \). Hence the set \( \mathcal{X}_{\tau_v, \ell}(g) \) is a disjoint union of piecewise affine bundles on \( X_0(g) \times ((A^2_k \times B_{a(v)}) \setminus B_v) \) and the function \( \text{ac}(f \circ g) \) is given by the composition of the canonical map with \( Q_v \).

An arc \( g \circ \varphi \) in \( \mathcal{L}_{\nu(\tau)\ell}(A^2_k) \) has contact greater than \( \tau_v \) with \( f \) if and only if it has contact \( \tau_v \) with \( f^- \) and \( Q_{a(v)}^c \) vanish on \( \varphi \). Hence, for any contact \( \tau \) greater than \( \tau_v \), the set \( \mathcal{X}_{\tau, \ell}(g) \) is a disjoint union of piecewise affine bundles on \( X_0(g) \times \mathbb{G}_m \times B_v \) and the function \( \text{ac}(f \circ g) \) is given by the composition of the canonical map with the projection \( X_0(g) \times \mathbb{G}_m \times B_v \longrightarrow \mathbb{G}_m \).

We can compute the difference \( Z_{\geq_v} - Z^-_{\geq_v} \) and check that it has limit \( \Psi_{Q_v}(A_v) \) as \( T \) goes to infinity.
It is a consequence of the following lemma, which follows from direct computation, that only extended rupture vertices have a non zero contribution.

4.4. Lemma. Consider a vertex \( v \) of \( \tau(f, p) \) and assume it is not an extended rupture vertex. Then the polynomial \( Q_v \) is of the form:

\[
Q_v(c, \omega) = ((c - \alpha \omega^R)\omega^N)^E
\]

where \( R, N \) and \( E \) are integers and \( \alpha \) a non-zero constant. It defines a map from \( \mathbb{A}^1_k \times \mathbb{G}_m \) to \( \mathbb{A}^1_k \) the zero set of which is isomorphic to \( \mathbb{G}_m \). Then:

- \( \Psi_{Q_v}(A_v) = 0 \) in \( \mathcal{M}^{G_m}_{X_0(g) \times \mathbb{G}_m} \).
- For the unique successor \( s(v) \) of \( v \), the equality \( A_{s(v)} = A_v \) holds in \( \mathcal{M}^{G_m}_{X_0(g) \times \mathbb{A}^1_k \times \mathbb{G}_m} \).

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