APPROXIMATION OF A NONLINEAR FRACTAL ENERGY FUNCTIONAL ON VARYING HILBERT SPACES

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(Communicated by Martino Bardi)

Abstract. We study a quasi-linear evolution equation with nonlinear dynamical boundary conditions in a two dimensional domain with Koch-type fractal boundary. We consider suitable approximating pre-fractal problems in the corresponding pre-fractal varying domains. After proving existence and uniqueness results via standard semigroup approach, we prove that the pre-fractal solutions converge in a suitable sense to the limit fractal one via the Mosco convergence of the energy functionals adapted by Tölle to the nonlinear framework in varying Hilbert spaces.

1. Introduction. In the latest years there has been an increasing interest towards the study of boundary value problems in irregular - fractal domains due to the several problems related to model industry and real world applications. A crucial point is to understand whether the solutions of such problems exist and in which sense they can be approximated by the solutions of analogous problems in smoother approximating domains. This could be a step toward numerical approximation, which is not tackled in the present paper.

In the framework of linear and semilinear boundary value problems in irregular domains of fractal type which model heat diffusion, results have been obtained in [26, 28, 29], while for the case of quasi-linear boundary value problems in irregular domains the reader is referred to [27, 38, 39, 42]. For the numerical approximation (in the linear case) see [34, 11].

In this paper we consider a quasi-linear evolution Cauchy problem with nonlinear dynamical boundary conditions in a two-dimensional domain Ω with fractal boundary. Problems of this type are also known as nonlinear Venttsel’ problems [40], [2]. After stating existence and uniqueness results for such problem, we consider

2000 Mathematics Subject Classification. Primary: 35K, 28A80; Secondary: 37L05, 31C25.
Key words and phrases. Asymptotic behavior, Venttsel’ problems, nonlinear energy forms, fractal domains, trace theorems, varying Hilbert spaces, p-Laplacian, nonlinear semigroups.

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the corresponding approximating “pre-fractal” problems. The aim of this paper is to prove the convergence of the solutions of the pre-fractal problems to the limit fractal one in a suitable sense.

More precisely, in the domain $\Omega$ with Koch-type fractal boundary (see Section 2 for more details on this domain), we consider the abstract Cauchy problem

\[ (P) \left\{ \begin{array}{l}
\frac{du}{dt} + Au \ni 0, \quad t \in [0, T] \\
u(0) = u_0,
\end{array} \right. \]

where $T$ is a fixed positive number, $u_0$ is a given function and $A$ is the subdifferential of a suitable proper, convex, lower semicontinuous energy functional $\Phi_p$ (see 3.8). The energy functional associated with the nonlinear Venttsel’ problem is the sum of a nonlinear energy in the bulk and a nonlinear fractal energy form on the boundary plus lower order terms.

We stress the fact that the Koch snowflake domain is a prototype of a domain with irregular boundary for which it is possible to construct a $p$-energy form. The construction of a $p$-energy form as limit of suitable discrete energy forms (defined on the pre-fractal sets) is possible only for the Koch curve and its variants. In the other cases, such as the Sierpinski Gasket, it is not possible to have a formula like 3.3 due to the lack of an explicit value of the renormalization factor in the energy, which for the Koch curve is $\frac{4(p-1)}{p} h^p$. Thus 3.3 is a crucial tool when studying the M-convergence of the approximating energy functionals to a limit one which involves a fractal $p$-energy form.

For each $h \in \mathbb{N}$, we consider the corresponding approximating problems $(P_h)$, with nonlinear dynamical boundary conditions, in the natural approximating pre-fractal domains $\Omega_h$ which are an increasing sequence exhausting $\Omega$. Existence and uniqueness results of the solution are proved via standard semigroup theory in $L^2(\Omega, m)$ and in $L^2(\Omega, m_h)$, respectively, where $m$ and $m_h$ are suitable measures and the latter one changes accordingly to $h$ (see 2.13 and 2.14). Thus, the natural setting to study the convergence of the solutions is that of varying Hilbert spaces and the semigroup approach will turn essential to this end.

In fact, we use the notion of Mosco convergence (M-convergence) [32, 33] of the energy functionals adapted by Tölle to the nonlinear framework in varying Hilbert spaces [36]. The M-convergence of the functionals is equivalent to the G-convergence of their subdifferentials $A_h$ which in turn is equivalent to the convergence of the nonlinear semigroup generated by $-A_h$ (the reader is referred to the pioneering papers [3] and [7] for the case of a fixed Banach space). The proof of the M-convergence requires sophisticated tools typical of fractal analysis such as Decimation techniques in order to prove the liminf and limsup properties of the nonlinear fractal energy forms (see proofs of conditions a) and b) in Theorem 3.5) and delicate trace estimates (see Theorem 2.3). Moreover we prove that the solutions of problems $(P)$ and $(P_h)$ respectively solve in a suitable sense a homogeneous parabolic equation for the $p$-Laplace operator with nonlinear Venttsel’ boundary conditions (see problems ($\tilde{P}$) and $(\tilde{P}_h)$).

The study of linear Venttsel’ problems in fractal domains and their approximation is recent, see [30]. The first existence and regularity results for a quasi-linear nonlocal evolution Cauchy problem with nonlinear dynamical boundary conditions in the fractal domain $\Omega$ have been obtained in [27], where a fractal $p$-Laplace Beltrami-type operator appears in the boundary condition. Problems of this type
appear e.g. in some climate models or in non-isothermal phase separation in a confined container (see [13] and [16] and the references listed in).

To our knowledge, results on the asymptotic behavior of the solutions of nonlinear pre-fractal problems, in the setting of both fixed and varying Hilbert spaces, are new and are the main objective of this paper.

The plan of the paper is the following. In Section 2 we introduce the Koch snowflake, its main properties, the main functional spaces and the definition of varying Hilbert spaces.

In Section 3 we introduce the nonlinear energy forms and the functionals in the fractal and pre-fractal case and we prove the M-Convergence in Theorem 3.5.

In Section 4 we consider the abstract Cauchy problems both in the fractal and pre-fractal case. After stating existence and uniqueness results of a “strong” solution, by using the characterization of the subdifferentials (of the energy functionals) we prove that the solutions of the abstract Cauchy problems solve (in a suitable sense) a nonlinear Venttsel’ boundary value problem. From Theorem 3.5 we deduce the convergence of the approximating solutions via the convergence of the subdifferentials and hence of the nonlinear semigroups.

2. Preliminaries.

2.1. The Koch snowflake. In the paper we denote by $P = (x_1, x_2)$ points in $\mathbb{R}^2$, by $|P - P_0|$ the Euclidean distance and by $B(P_0, r) = \{ P \in \mathbb{R}^2 : |P - P_0| < r \}$, $P_0 \in \mathbb{R}^2$, $r > 0$ the Euclidean ball. By the Koch snowflake $F$, we will denote the union of three com-planar Koch curves (see [14]) $K_1$, $K_2$ and $K_3$. We assume that the junction points $A_1$, $A_3$ and $A_5$ are the vertices of a regular triangle with unit side length, i.e. $|A_1 - A_3| = |A_1 - A_5| = |A_3 - A_5| = 1$. $K_1$ is the uniquely determined self-similar set with respect to a family $\Psi$ of four suitable contractions $\psi_1^{(1)}$, $\ldots$, $\psi_4^{(1)}$, with respect to the same ratio $\frac{1}{3}$ (see [15]). Let $V_0^{(1)} := \{ A_1, A_3 \}$, $\psi_{i_1 \ldots i_h} := \psi_{i_1} \circ \cdots \circ \psi_{i_h}$, $V_{i_1 \ldots i_h}^{(1)} := \psi_{i_1 \ldots i_h}^{(1)}(V_0^{(1)})$ and

$$V_h^{(1)} := \bigcup_{i_1 \ldots i_h = 1}^4 V_{i_1 \ldots i_h}^{(1)}.$$ 

We set $i|h = (i_1, i_2, \ldots, i_h)$, $V_i^{(1)} := \bigcup_{h \geq 0} V_h^{(1)}$. It holds that $K_1 = \overline{V_1^{(1)}}$. Now let $K_0$ denote the unit segment whose endpoints are $A_1$ and $A_3$. We set $K_{i_1 \ldots i_h} = \psi_{i_1 \ldots i_h}(K_0)$ and $V(K_{i_1 \ldots i_h}) = V_{i_1 \ldots i_h}$.

In a similar way, it is possible to approximate $K_2$, $K_3$ by the sequences $(V_h^{(2)})_{h \geq 0}$, $(V_h^{(3)})_{h \geq 0}$, and denote their limits by $V_2^*$, $V_3^*$.

In order to approximate $F$, we define the increasing sequence of finite sets of points $V_h := \bigcup_{i=1}^{3} V_h^{(i)}$, $h \geq 1$ and $V_* := \bigcup_{h \geq 1} V_h$. It holds that $V_* = \bigcup_{i=1}^{3} V_i^*$.

The Hausdorff dimension of the Koch snowflake is given by $d_f = \frac{\ln 4}{\ln 3}$.

One can define, in a natural way, a finite Borel measure $\mu$ supported on $F$ by

$$\mu := \mu_1 + \mu_2 + \mu_3,$$  \hspace{1cm} (2.1)

where $\mu_i$ denotes the normalized $d_f$-dimensional Hausdorff measure, restricted to $K_i$, $i = 1, 2, 3$. Further, for any $h \geq 1$, we define a discrete measure on $V_h^{(i)}$ by:

$$\mu_h^{(i)} := \frac{1}{4^h} \sum_{p \in V_h^{(i)}} \delta_{\{p\}},$$ \hspace{1cm} (2.2)
where \( \delta_{\{p\}} \) denotes the Dirac measure at the point \( p \). In [15], the following result is shown.

**Proposition 2.1.** The sequence \((\mu^i_h)_{h \geq 1}\) is weakly convergent (i.e. in the dual space of \( C(K_i) \)) to the measure \( \mu_i \).

In the following we denote by

\[
F_{h+1} = \bigcup_{i=1}^{3} K_i^{(h+1)}
\]  

(2.3)

the closed polygonal curve approximating \( F \) at the \((h+1)\)-th step. Here we denote by \( K_i^{(h+1)} \) the so-called pre-fractal (polygonal) curve approximating \( K_i \).

The measure \( \mu \) has the property that there exist two positive constants \( c_1, c_2 \) such that

\[
c_1 r^d \leq \mu(B(P,r) \cap F) \leq c_2 r^d, \quad \forall P \in F,
\]  

(2.4)

where \( d = d_f = \frac{\log 4}{\log 3} \). As \( \mu \) is supported on \( F \), it is not ambiguous to write in 2.4 \( \mu(B(P,r)) \) in place of \( \mu(B(P,r) \cap F) \).

We note that, in the terminology of [20], from 2.4 it follows that \( F \) is a \( d \)-set with \( d = d_f \) and the measure \( \mu \) is a \( d \)-measure (see [41]).

**Remark 1** (see [15]). The Koch snowflake can be also regarded as a fractal manifold.

Let \( \Omega \) denote the (open) snowflake domain, and for every integer \( h \geq 1 \) let \( \Omega_h \) be the pre-fractal polygonal domains approximating \( \Omega \) and let \( F_h = \partial \Omega_h \) be the corresponding closed polygonal curves approximating \( F \); we denote by \( M \) every side of the polygonal curve, by \( \circ M \) the corresponding open segment and by \( V(M) \) its vertices. We remark that the sequence \( \{\Omega_h\}_{h \in \mathbb{N}} \) is an increasing sequence of sets exhausting \( \Omega \). We denote by \( T \) the open equilateral triangle whose midpoints are the vertices \( A_1, A_3 \) and \( A_5 \) of \( F \) (see Figure 1).

\[ \text{Figure 1. The Koch snowflake.} \]

**2.2. Sobolev spaces.** By \( L^p(\cdot) \) we denote the Lebesgue space with respect to the Lebesgue measure \( d\mathcal{L}_2 \) on subsets of \( \mathbb{R}^2 \), which will be left to the context whenever that does not create ambiguity. By \( L^p(F) \) we denote the Banach space of \( p \)-summable functions on \( F \) with respect to the invariant measure \( \mu \). By \( \ell \) we denote the natural arc length coordinate on each edge of \( F_h \) and we introduce the
coordinates \( x_1 = x_1(\ell), x_2 = x_2(\ell), \) on every segment \( M_h^{(j)} \) of \( F_h \). By \( d\ell \) we denote the one-dimensional measure given by the arclength \( \ell \).

Given \( S \) a closed set of \( \mathbb{R}^2 \), by \( C(S) \) we denote the space of continuous functions on \( S \), by \( C_0(S) \) we denote the space of continuous functions vanishing on \( \partial S \). Let \( \mathcal{G} \) be an open set of \( \mathbb{R}^2 \), by \( W^{s,p}(\mathcal{G}) \), where \( s \in \mathbb{R}^+ \) we denote the usual (possibly fractional) Sobolev spaces (see [35]); \( W_0^{s,p}(\mathcal{G}) \) is the closure of \( \mathcal{D}(\mathcal{G}) \), (the smooth functions with compact support on \( \mathcal{G} \)), with respect to the \( \| \cdot \|_{W^{s,p}} \)-norm.

In the following, we will make use of trace spaces on boundaries of polygonal domains of \( \mathbb{R}^2 \). By \( W^{1,p}(F_h) \) we denote (see [8]) the set
\[
\{ u \in C(F_h) : u|_{F_h} \in W^{1,p}(M) \}.
\]

In the sequel, we consider \( W^{1,p}(F_h) \) with the norm
\[
\| u \|_{W^{1,p}(F_h)} = \left( \| u \|_{L^p(F_h)}^p + \| Du \|_{L^p(F_h)}^p \right)^{\frac{1}{p}}.
\]

By \( W^{r,p}(F_h), 0 < r \leq 1 \) we denote the Sobolev space on \( F_h \), defined by local Lipschitz charts as in [35].

We now recall two trace theorems. In the following we will denote by \( |A| \) the Lebesgue measure of a subset \( A \subset \mathbb{R}^n \). For \( f \) in \( W^{s,p}(\mathcal{G}) \), we put
\[
\gamma_0 f(P) = \lim_{r \to 0} \frac{1}{|B(P, r) \cap \mathcal{G}|} \int_{B(P, r) \cap \mathcal{G}} f(Q) \, d\mathcal{L}_2
\]
at every point \( P \in \mathcal{G} \) where the limits exist. It is known that the limit 2.5 exists at quasi every \( P \in \mathcal{G} \) with respect to the \((s,p)\)-capacity [1].

We now recall the results of Theorem 3.1 in [18] specialized to our case, referring to [17] for a more general discussion.

**Proposition 2.2.** Let \( \Omega_h \) and \( F_h \) be as above and let \( \frac{1}{p} < s < 1 + \frac{1}{p} \). Then \( W^{s-\frac{1}{p},p}(F_h) \) is the trace space to \( F_h \) of \( W^{s,p}(\Omega_h) \) in the following sense:

(i) \( \gamma_0 \) is a continuous and linear operator from \( W^{s,p}(\Omega_h) \) to \( W^{s-\frac{1}{p},p}(F_h) \),

(ii) there is a continuous linear operator \( \text{Ext} \) from \( W^{s-\frac{1}{p},p}(F_h) \) to \( W^{s,p}(\Omega_h) \), such that \( \gamma_0 \circ \text{Ext} \) is the identity operator in \( W^{s-\frac{1}{p},p}(F_h) \).

In the sequel we denote by the symbol \( f|_{F_h} \) the trace \( \gamma_0 f \) to \( F_h \). Sometimes we will omit the trace subscript and the interpretation will be left to the context.

The following theorem characterizes the trace on the polygonal \( F_h \) of a function belonging to Sobolev space \( W^{\beta,p}(\mathbb{R}^2) \) (for the definitions and the main properties of Sobolev spaces, see [1]).

**Theorem 2.3.** Let \( F_h \) denote \( \partial \Omega_h \). Let \( u \in W^{\beta,p}(\mathbb{R}^2) \) and \( \delta_h = (\frac{3}{4})^h = (3^{1-d})^h \).

Then, for \( \frac{1}{p} < \beta \leq \frac{2}{p} \),
\[
\| u \|_{L^p(F_h)}^p \leq C_{\beta} \| u \|_{W^{\beta,p}(\mathbb{R}^2)}^p,
\]
where \( C_{\beta} \) is independent of \( h \).

**Proof.** We point out that every \( u \in W^{\beta,p}(\mathbb{R}^2) \) can be expressed in the following way:
\[
u = G_{\beta} * g, \quad g \in L^p(\mathbb{R}^2),
\]
Theorem 2.4. Let \( F \) have a \( \beta \)-set (see [20]). Then by Hölder inequality we have

\[
\|u\|_{L^p(F_h)}^p = \int_{F_h} |u|^p \, d\ell = \int_{F_h} \left( \int_{\mathbb{R}^2} G_\beta(x-y) g(y) \, dy \right)^p \, d\ell
\]

\[
\leq \int_{F_h} \left( \int_{\mathbb{R}^2} |G_\beta(x-y)|^{\alpha p} |g(y)|^p \, dy \right)^p \, d\ell
\]

where \( 0 < \alpha < 1 \) will be chosen later. Now, by using Lemma 1 on page 104 in [20], we get

\[
\int_{\mathbb{R}^2} |G_\beta(x-y)|^{(1-\alpha)p'} \, dy \leq C_1,
\]

with \( C_1 \) independent of \( h \), if

\[
(2 - \beta)(1-\alpha)p' < 2.
\]  \hspace{1cm} (2.8)

Moreover, since \( F_h \) is a \( 1 \)-set with constant \( c_2 = C_3 \delta_h^{-1} \) (see 2.4), again from Lemma 1 on page 104 in [20] we get

\[
\int_{F_h} |G_\beta(x-y)|^{\alpha p} \, d\ell \leq C_4 \delta_h^{-1},
\]

with \( C_4 \) again independent of \( h \), if

\[
(2 - \beta)\alpha p < 1.
\]  \hspace{1cm} (2.9)

Hence, by choosing \( \alpha \) in order to satisfy 2.8 and 2.9, by using Fubini’s Theorem we get

\[
\|u\|_{L^p(F_h)}^p \leq C_1 \int_{F_h} \left( \int_{\mathbb{R}^2} |G_\beta(x-y)|^{\alpha p} |g(y)|^p \, dy \right)^p \, d\ell
\]

\[
= C_1 \int_{\mathbb{R}^2} \left( \int_{F_h} |G(x-y)|^{\alpha p} \, d\ell \right) |g(y)|^p \, dy
\]

\[
\leq C_1 C_4 \delta_h^{-1} \|g\|_{L^p(\mathbb{R}^2)}^p = C_\beta \delta_h^{-1} \|u\|_{W^{\alpha,p}(\mathbb{R}^2)}^p,
\]

where \( C_\beta \) is a constant independent of \( h \). \hfill \Box

The following theorem is a consequence of Theorem 1 in Chapter V of [20] as the fractal \( F \) is a \( d \)-set (see Section 2).

**Theorem 2.4.** Let \( u \in W^{\alpha,p}(\mathbb{R}^2) \). Then, for \( \frac{2-d}{p} < \beta \),

\[
\|u\|_{L^p(F)}^p \leq C_\beta \|u\|_{W^{\alpha,p}(\mathbb{R}^2)}^p.
\]  \hspace{1cm} (2.10)

It is possible to prove that the domains \( \Omega_h \) are \( (\varepsilon, \delta) \) domains with parameters \( \varepsilon \) and \( \delta \) independent of the (increasing) number of sides of \( F_h \). Thus by the extension theorem for \( (\varepsilon, \delta) \) domains due to Jones (Theorem 1 in [19]) we obtain the following Theorem 2.5, which provides an extension operator from \( W^{1,p}(\Omega_h) \) to the space \( W^{1,p}(\mathbb{R}^2) \) whose norm is independent of \( h \).

**Theorem 2.5.** There exists a bounded linear extension operator \( \text{Ext}_J : W^{1,p}(\Omega_h) \rightarrow W^{1,p}(\mathbb{R}^2) \), such that

\[
\|\text{Ext}_J v\|_{W^{1,p}(\mathbb{R}^2)}^p \leq C_J \|v\|_{W^{1,p}(\Omega_h)}^p
\]  \hspace{1cm} (2.11)

where \( C_J \) independent of \( h \).
We conclude this part with an extension theorem for fractional Sobolev spaces $W^{s,p}(\Omega)$. We observe that if $0 < \beta < 1$, estimate 2.12 can be deduced from Theorem 1 on page 103 in [20] (see also Theorem 3 on page 155 in [20]) since the domain $\Omega$ is 2-set.

**Theorem 2.6.** There exists a linear extension operator $\text{Ext} : W^{\beta,p}(\Omega) \rightarrow W^{\beta,p}(\mathbb{R}^2)$,

$$\|\text{Ext} f\|_{W^{\beta,p}(\mathbb{R}^2)} \leq C_\beta \|f\|_{W^{\beta,p}(\Omega)} \quad (2.12)$$

with $C_\beta$ depending on $\beta$.

2.3. **Besov spaces.** We now come to the definition of the Besov spaces $B^{s,p}_\alpha$ with $\alpha$ positive and non-integer (see [37] and [20]). Let $\mathcal{S}$ be a $d$-set in $\mathbb{R}^D$, $\alpha > 0$ non integer, $k = [\alpha]$ the integer part of $\alpha$, $j$ a $D$-dimensional multi-index of length $|j| \leq k$.

If $f$ and $\{f^{(j)}\}$ are functions defined $\tilde{\mu}$-a.e. on $\mathcal{S}$, we set

$$R_j(P,P') = f^{(j)}(P) - \sum_{|j+i| \leq k} \frac{f^{(j+i)}(P')}{i!}(P - P')^i,$$

where $f^{(0)} = f$ and $i$ denotes a $D$-dimensional multi-index.

**Definition 2.7.** We say that $f \in B^{s,p}_\alpha(\mathcal{S})$ if there exists a family $\{f^{(j)}\}$ with $|j| \leq k$, as above, such that $f^{(j)} \in L^p(\mathcal{S},\tilde{\mu})$ and $\|\{a_n\}\|_{L^p} < \infty$ where $a_n$ is the smallest number such that

$$\left(3^{nd} \int_{|P-P'| < 3^{-n}} |R_j(P,P')|^p d\tilde{\mu}(P) d\tilde{\mu}(P')\right)^{1/p} \leq 3^{-n(\alpha - |j|)}a_n.$$

The norm of $f$ in $B^{s,p}_\alpha(\mathcal{S})$ is

$$\|f\|_{B^{s,p}_\alpha(\mathcal{S})} = \|f\|_{L^p(\mathcal{S},\tilde{\mu})} + \|\{a_n\}\|_{L^p}.$$

The family $\{f^{(j)}\}$ in the previous definition is uniquely determined by $f$, as shown in [20], for $d$-sets with $d > D - 1$.

Let us note that for $0 < \alpha < 1$ the norm $\|f\|_{B^{s,p}_\alpha(\mathcal{S})}$ can be written as

$$\|f\|_{L^p(\mathcal{S},\tilde{\mu})} + \left(\sum_{n=0}^{\infty} 3^n d^{(d+\alpha)} \int_{|P-P'| < 3^{-n}} |f(P) - f(P')|^p d\tilde{\mu}(P) d\tilde{\mu}(P')\right)^{1/p}.$$

We now state the trace theorem specialized to our case.

**Proposition 2.8.** Let $\mathcal{S} = F$ be the Koch snowflake. Let $s > \frac{(2-d)}{p}, \left(s - \frac{(2-d)}{p}\right) \notin \mathbb{N}$, then $B^{s-p}_{s-\frac{(2-d)}{p}}(F)$ is the trace space to $F$ of $W^{s,p}(\Omega)$ in the following sense:

(i) $\gamma_0$ is a continuous linear operator from $W^{s,p}(\Omega)$ to $B^{s-p}_{s-\frac{(2-d)}{p}}(F)$,

(ii) there is a continuous linear operator $\text{Ext}$ from $B^{s,p}_{s-\frac{(2-d)}{p}}(F)$ to $W^{s,p}(\Omega)$ such that $\gamma_0 \circ \text{Ext}$ is the identity operator in $B^{s-p}_{s-\frac{(2-d)}{p}}(F)$.

For the proof we refer to Theorem 1 of Chapter VII in [20], see also [37].

From Proposition 2.8 it follows that when $s = 1$ the trace space of $W^{1,p}(\Omega)$ is $B^{p-p}_{1-\frac{(2-d)}{p}}(F)$.
In the sequel we denote by the symbol $f|_F$ the trace $\gamma_0 f$ to $F$. For the sake of simplicity we will omit the subscript.

In the following, we also make use of the dual of Besov spaces on $F$. These spaces as shown in [21] coincide with a subspace of Schwartz distributions $D'(\mathbb{R}^2)$, which are supported on $F$. They are built by means of atomic decomposition. Actually, Jonsson and Wallin proved this result in the general framework of $d$-sets; we refer to [21] for a complete discussion.

Throughout the paper $c$ will denote possibly different constants.

2.4. Convergence of Hilbert spaces. We introduce the notion of convergent Hilbert spaces that we will use in the next sections. For further details and proofs of the theorems see [25] and [22].

The Hilbert spaces we consider are real and separable.

**Definition 2.9.** A sequence of Hilbert spaces $\{H_h\}_{h \in \mathbb{N}}$ converges to a Hilbert space $H$ if there exists a dense subspace $C \subset H$ and a sequence $\{Z_h\}_{h \in \mathbb{N}}$ of linear operators $Z_h: C \subset H \rightarrow H_h$ such that

$$\lim_{h \rightarrow \infty} \|Z_h u\|_{H_h} = \|u\|_H$$

for any $u \in C$.

We define the space $H = \bigcup_{h \in \mathbb{N}} H_h$ and define strong and weak convergence in $H$. From now on we assume $\{H_h\}_{h \in \mathbb{N}}$, $H$ and $\{Z_h\}_{h \in \mathbb{N}}$ are as in Definition 2.9.

**Definition 2.10** (Strong convergence in $H$). A sequence of vectors $\{u_h\}_{h \in \mathbb{N}}$ strongly converges to $u$ in $H$ if $u_h \in H_h$, $u \in H$ and there exists a sequence $\{\tilde{u}_m\}_{m \in \mathbb{N}} \subset C$ tending to $u$ in $H$ such that

$$\lim_{m \rightarrow \infty} \lim_{h \rightarrow \infty} \|Z_h \tilde{u}_m - u_h\|_{H_h} = 0$$

**Definition 2.11** (Weak convergence in $H$). A sequence of vectors $\{u_h\}_{h \in \mathbb{N}}$ weakly converges to $u$ in $H$ if $u_h \in H_h$, $u \in H$ and

$$(u_h, v_h)_{H_h} \rightarrow (u, v)_H$$

for every sequence $\{v_h\}_{h \in \mathbb{N}}$ strongly tending to $v$ in $H$.

**Remark 2.** We note that the strong convergence implies the weak convergence (see [25]).

**Lemma 2.12.** Let $\{u_h\}_{h \in \mathbb{N}}$ be a sequence weakly converging to $u$ in $H$. Then

$$\sup_{h \rightarrow \infty} \|u_h\|_{H_h} < \infty, \quad \|u\|_H \leq \lim_{h \rightarrow \infty} \|u_h\|_{H_h}.$$  

Moreover, $u_h \rightarrow u$ strongly if and only if $\|u\|_H = \lim_{h \rightarrow \infty} \|u_h\|_{H_h}$.

Let us recall some characterizations of the strong convergence of a sequence of vectors $\{u_h\}_{h \in \mathbb{N}}$ in $H$.

**Lemma 2.13.** Let $u \in H$ and let $\{u_h\}_{h \in \mathbb{N}}$ be a sequence of vectors $u_h \in H_h$. Then $\{u_h\}_{h \in \mathbb{N}}$ strongly converges to $u$ in $H$ if and only if

$$(u_h, v_h)_{H_h} \rightarrow (u, v)_H$$

for every sequence $\{v_h\}_{h \in \mathbb{N}}$ with $v_h \in H_h$ weakly converging to $v$ in $H$.  


Lemma 2.14. A sequence of vectors \( \{u_h\}_{h \in \mathbb{N}} \) with \( u_h \in H_h \) strongly converges to a vector \( u \) in \( H \) if and only if

\[
\|u_h\|_{H_h} \to \|u\|_H \quad \text{and} \quad \langle u_h, Z_h(\varphi) \rangle_{H_h} \to \langle u, \varphi \rangle_H \quad \text{for every} \quad \varphi \in C.
\]

Lemma 2.15. Let \( \{u_h\}_{h \in \mathbb{N}} \) be a sequence with \( u_h \in H_h \). If \( \|u_h\|_{H_h} \) is uniformly bounded, then there exists a subsequence of \( \{u_h\}_{h \in \mathbb{N}} \) which weakly converges in \( H \).

Lemma 2.16. For every \( u \in H \) there exists a sequence \( \{u_h\}_{h \in \mathbb{N}}, u_h \in H_h \) strongly converging to \( u \) in \( H \).

We now define the G-convergence of operators (see Definition 7.20 in [36]).

Definition 2.17. Let \( n \in \mathbb{N}, A_n: H_n \to 2^{H_n}, A: H \to 2^H \) be multivalued operators. We say that \( A_n \) G-converges to \( A \), \( A_n \rightharpoonup^G A \), if for every \([x, y] \in A \) (i.e. \( x \in D(A) \) and \( y \in A(x) \)) there exists \([x_n, y_n] \in A_n, n \in \mathbb{N} \) such that \( x_n \to x \) and \( y_n \to y \) strongly in \( H \).

In the following we denote by \( L^2(\Omega, m) \) the Lebesgue space with respect to the measure \( m \) with

\[
dm = dL_2 + d\mu,
\]

and by the space \( L^2(\Omega, m_h) \) the Lebesgue space with respect to the measure \( m_h \) with

\[
dm_h = \chi_{\Omega_h} dL_2 + \chi_{F_h} \delta_h d\ell,
\]

where \( \chi_{\Omega_h} \) and \( \chi_{F_h} \) denote the characteristic function of \( \Omega_h \) and \( F_h \) respectively.

Throughout the paper we consider \( H = L^2(\Omega, m) \) where \( m \) is the measure in (2.13), and the sequence \( \{H_h\}_{h \in \mathbb{N}} \) with \( H_h = \{L^2(\Omega) \cap L^2(\Omega, m_h)\} \) where \( m_h \) is the measure in 2.14 with norms

\[
\|u\|_{H}^2 = \|u\|^2_{L^2(\Omega)} + \|u|_x\|^2_{L^2(F, \mu)}, \quad \|u\|_{H_h}^2 = \|u\|^2_{L^2(\Omega_h)} + \|u|_{F_h}\|^2_{L^2(F_h, \delta_h \ell)}.
\]

Proposition 2.18. Let \( \delta_h = \left(\frac{3}{4}\right)^h \). Then the sequence \( \{H_h\}_{h \in \mathbb{N}} \) converges in the sense of Definition 2.9 to \( H \).

For the proof, see Proposition 4.1 in [30], where \( C \) and \( Z_h \) in Definition 2.9 are respectively \( C(\overline{\Omega}) \) and the restriction operator to \( \overline{\Omega_h} \).

3. The M-convergence of energy functionals and of their subdifferentials.

3.1. Nonlinear fractal energy forms. For \( f : V_{\ast}^{(i)} \to \mathbb{R}, i = 1, 2, 3 \), we define for \( 1 < p < \infty \) and \( h \in \mathbb{N}, \)

\[
\mathcal{E}^{(h)}_{p, i}[f] = \frac{1}{p} 4^{(p-1)h} \sum_{i_1, \ldots, i_h = 1}^{4} \sum_{\xi, \eta \in V_0^{(i)}} |f(\psi_{i_1 \ldots i_h}(\xi)) - f(\psi_{i_1 \ldots i_h}(\eta))|^p,
\]

and

\[
\mathcal{E}^{(h)}_p[f] = \sum_{i=1}^{3} \mathcal{E}^{(h)}_{p, i}[f].
\]

We note that the form \( \mathcal{E}^{(h)}_p \) in 3.2 can be also written as

\[
\mathcal{E}^{(h)}_p[f] = \frac{4^{(p-1)h}}{p} \sum_{M \in F_h} \sum_{r,s \in V(M)} |f(r) - f(s)|^p.
\]
It has been shown in [10] that the sequence $E_p^{(h)}[f]$ is non-decreasing; by defining for $f : V^i \rightarrow \mathbb{R}$

$$E_p^{(i)}[f] = \lim_{h \rightarrow \infty} E_p^{(h)}[f],$$

the set

$$\mathcal{F}^{(p)} = \{ f : V_i \rightarrow \mathbb{R} : E_p[f] < \infty \}$$

does not degenerate to a space containing only constant functions.

Each $f \in \mathcal{F}^{(p)}$ can be uniquely extended in $C(K_i)$. We denote this extension on $K_i$ still by $f$ and we define the space

$$D(\mathcal{E}_p^{(i)}) = \{ f \in C(K_i) : \mathcal{E}_p^{(i)}[f] < \infty \},$$

where $\mathcal{E}_p^{(i)}[f] := \mathcal{E}_p^{(i)}[f] |_{V_i}$ [10]. Hence $D(\mathcal{E}_p^{(i)}) \subset C(K_i) \subset L^p(K_i, \mu)$. Moreover, $(\mathcal{E}_p^{(i)}, D(\mathcal{E}_p^{(i)}))$ is a non-negative energy functional in $L^p(K_i, \mu_i)$ and the following result holds (see [10]).

**Theorem 3.1.** The following properties hold.

- **i)** $D(\mathcal{E}_p^{(i)})$ is complete in the norm $\|f\|_{D(\mathcal{E}_p^{(i)})} : = \|f\|_{L^p(K_i, \mu_i)} + (\mathcal{E}_p^{(i)}[f])^{1/p}$.
- **ii)** $D(\mathcal{E}_p^{(i)})$ is dense in $L^p(K_i, \mu_i)$.
- **iii)** $D(\mathcal{E}_p^{(i)}) \subset D(\mathcal{E}_p^{(i)})$, for $1 < p \leq q < \infty$.

By proceeding as in Section 4.1 and 4.2 in [15] one can define on $F$ a $p$-energy form $(\mathcal{E}_p, D(\mathcal{E}_p))$

$$\mathcal{E}_p[u] = \sum_{i=1}^{3} \mathcal{E}_p^{(i)}[u] \|_{K_i}$$

for every $u \in D(\mathcal{E}_p)$, where $D(\mathcal{E}_p) = \{ u \in C(F) : u|_{K_i} \in D(\mathcal{E}_p^{(i)}) \text{ for } i = 1, 2, 3 \}$.

### 3.2. The energy functionals

We now introduce the energy functionals for the fractal and pre-fractal problem respectively. Let be $p \geq 2$ and a strictly positive continuous function in $\Omega$. We set

$$\Phi_p[u] := \begin{cases} \frac{1}{p} \int_{\Omega} |Du|^p \, d\mathcal{L} + \frac{1}{p} \int_{F_h} b|u|^p \, d\mu + \mathcal{E}_p[u] & \text{if } u \in D(\Phi_p), \\ +\infty & \text{if } u \in H \setminus D(\Phi_p), \end{cases}$$

with domain

$$D(\Phi_p) := \{ u \in W^{1,p}(\Omega) : u|_{F} \in D(\mathcal{E}_p) \}.$$

**Proposition 3.2.** $\Phi_p$ is a weakly lower semicontinuous, proper and convex functional in $H$.

For the proof see [27, Proposition 2.3].

We define

$$E_p^{(h)}[u] = \frac{\delta_{h}^{-1-p}}{p} \int_{F_h} |Du|^p \, df,$$

with domain

$$D(E_p^{(h)}) = W^{1,p}(F_h).$$

We now introduce the energy functional on the pre-fractal domain:

$$\Phi_p^{(h)}[u] := \begin{cases} \frac{1}{p} \int_{\Omega} \chi_{\Omega} |Du|^p \, d\mathcal{L} + \frac{\delta_{h}}{p} \int_{F_h} b|u|^p \, d\ell + E_p^{(h)}[u] & \text{if } u \in D(\Phi_p^{(h)}), \\ +\infty & \text{if } u \in H_h \setminus D(\Phi_p^{(h)}), \end{cases}$$

where $\delta_{h}$ is a positive parameter. For the proof see [27, Proposition 2.3].

We define

$$E_p^{(h)}[u] = \frac{\delta_{h}^{-1-p}}{p} \int_{F_h} |Du|^p \, df,$$

with domain

$$D(E_p^{(h)}) = W^{1,p}(F_h).$$

We now introduce the energy functional on the pre-fractal domain:
where
\[ D(\Phi^{(h)}_p) := \left\{ u \in W^{1,p}(\Omega) : u|_{F_h} \in D(E^{(h)}_p) \right\}. \]

By proceeding as in [27, Proposition 2.3], we can prove the following result.

**Proposition 3.3.** \( \Phi^{(h)}_p \) is a weakly lower semicontinuous, proper and convex functional in \( H_h \).

### 3.3. M-convergence.

We recall that the definition of M-convergence of quadratic energy forms was introduced by Mosco in [32] for a fixed Hilbert space and adapted to the case of varying Hilbert spaces by Kuwae and Shioya, see Definition 2.11 in [25]. This notion has been extended to the case of proper convex functionals in Banach spaces by Tölle (see Section 7.5, Definition 7.26 in [36]) as stated in Definition 3.4 below.

Let \( H_h \) be a sequence of Hilbert spaces converging to a Hilbert space \( H \) in the sense of Definition 2.9.

**Definition 3.4.** A sequence of proper and convex functionals \( \{ \Phi^{(h)}_p \} \) defined in \( H_h \) M-converges to a functional \( \Phi_p \) defined in \( H \) if the following hold:

a) for every \( \{ v_h \} \in H_h \) weakly converging to \( u \in H \) in \( H \),
\[ \lim_{h \to \infty} \Phi^{(h)}_p[v_h] \geq \Phi_p[u], \]

b) for every \( u \in H \) there exists \( \{ w_h \} \), with \( w_h \in H_h \) strongly converging to \( u \) in \( H \) such that
\[ \lim_{h \to \infty} \Phi^{(h)}_p[w_h] \leq \Phi_p[u]. \]

We now state the main theorem of this section.

**Theorem 3.5.** Let \( \delta_h = (3^{1-d_f})^h = (\frac{3}{4})^h \). Let \( \Phi_p \) and \( \Phi^{(h)}_p \) be defined as in 3.8 and 3.10 respectively. Then \( \Phi^{(h)}_p \) M-converges to the functional \( \Phi_p \).

We preliminary state the following propositions.

**Proposition 3.6.** If \( \{ v_h \}_{h \in \mathbb{N}} \) weakly converges to a vector \( u \) in \( H \), then \( \{ v_h \}_{h \in \mathbb{N}} \) weakly converges to \( u \) in \( L^2(\Omega) \) and \( \lim_{h \to \infty} \delta_h \int_{F_h} \varphi v_h \, d\ell = \int_{F} \varphi u \, d\mu \) for every \( \varphi \in C(\Omega) \).

For the proof see [30, Proposition 4.4].

**Proposition 3.7.** Let \( v_h \to u \) in \( W^{1,p}(\Omega) \), \( b \in C(\Omega) \). Then
\[ \delta_h \int_{F_h} b|v_h|^p \, d\ell \to \int_F b|u|^p \, d\mu. \]

**Proof.** We first note that
\[ \left| \delta_h \int_{F_h} b|v_h|^p \, d\ell - \int_F b|u|^p \, d\mu \right| \leq \left| \delta_h \int_{F_h} b|v_h|^p \, d\ell - \delta_h \int_{F_h} b|u|^p \, d\ell \right| + \left| \delta_h \int_{F_h} b|u|^p \, d\ell - \int_F b|u|^p \, d\mu \right|. \]

We set
\[ A_h = \left| \delta_h \int_{F_h} b|v_h|^p \, d\ell - \delta_h \int_{F_h} b|u|^p \, d\ell \right|. \]
and

\[ B_h = \left| \delta_h \int_{F_h} b |u|^p \, d\ell - \int_F b |u|^p \, d\mu \right| \]

and we study these two terms separately.

For the first term it holds that

\[ A_h \leq c_1 \delta_h \|b\|_{C(\overline{\Omega})} \left( \|v_h - u\|_{L^p(F_h)} \right) \left( \|v_h\|_{L^p(F_h)} + \|u\|_{L^p(F_h)} \right)^{p-1} \]

Since \( v_h \) weakly converges to \( u \) in \( W^{1,p}(\Omega) \), it follows that \( v_h \) strongly converges to \( u \) in \( W^{\alpha,p}(\Omega) \) for every \( \alpha \in (0,1) \).

If we consider the extension of \((v_h - u)\) to \( W^{\alpha,p}(\mathbb{R}^2) \) we have from Theorems 2.3 and 2.6

\[ \delta_h \left( \|v_h - u\|_{L^p(F_h)} \right) \leq C_\alpha \|\operatorname{Ext}(v_h - u)\|_{W^{\alpha,p}(\mathbb{R}^2)} \leq c_2 \|v_h - u\|_{W^{\alpha,p}(\Omega)} . \]

Hence \( A_h \) goes to 0 when \( h \) tends to \( \infty \).

We now prove that also \( B_h \) goes to 0. Since \( u \) belongs to \( W^{1,p}(\Omega) \) there exists a sequence \( \{g_m\} \in C(\overline{\Omega}) \cap W^{1,p}(\Omega) \) such that \( \|g_m - u\|_{W^{1,p}(\Omega)} \to 0 \) as \( m \) goes to \( \infty \) (see Proposition 4.4 in [19]). Then we have

\[
\begin{align*}
B_h &\leq \left| \delta_h \int_{F_h} b |u|^p \, d\ell - \delta_h \int_{F_h} b |g_m|^p \, d\ell \right| + \left| \delta_h \int_{F_h} b |g_m|^p \, d\ell - \int_F b |g_m|^p \, d\mu \right| \\
&\quad + \left| \int_F b |g_m|^p \, d\mu - \int_F b |u|^p \, d\mu \right|.
\end{align*}
\]

Proceeding as in the case \( A_h \) we can estimate the first and the third term in the right hand side with \( \|u - g_m\|_{W^{1,p}(\Omega)} \) and hence we conclude that for every \( \varepsilon > 0 \) there exists \( m_\varepsilon \in \mathbb{N} \) such that these two terms are less than \( c \varepsilon \). Then, if we choose \( m > m_\varepsilon \) the second term in the right hand side goes to 0 for \( h \) tending to \( \infty \) for Proposition 2.18 (since \( b \) \( g_m \) belongs to \( C(\overline{\Omega}) \)).

We are now ready to prove Theorem 3.5.

**Proof.** We show that both of the conditions of Definition 3.4 hold for the corresponding functionals \( \Phi_p^{(h)} \).

**Proof of condition a.** Let \( v_h \in H_h \) be a weakly converging sequence in \( H \) to \( u \in H \). We can suppose that \( v_h \in D(\Phi_p^{(h)}) \) and

\[
\lim_{h \to \infty} \Phi_p^{(h)}[v_h] < \infty
\]

(otherwise the thesis follows trivially). Then there exists a \( c \) independent of \( h \) such that

\[
\frac{1}{p} \int_{\Omega_h} \chi_{\Omega_h} |Dv_h|^p \, d\mathcal{L}_2 + \frac{\delta_h}{p} \int_{F_h} b |v_h|^p \, d\ell + \frac{\delta_h^{1-p}}{p} \int_{F_h} |Dv_h|^p \, d\ell \leq c \quad (3.11)
\]

In particular we have that \( \|v_h\|_{W^{1,p}(\Omega_h)} < c \). For every \( h \in \mathbb{N} \) from Theorem 2.5 there exists a bounded linear operator \( \operatorname{Ext}: W^{1,p}(\Omega_h) \to W^{1,p}(\mathbb{R}^2) \) such that

\[
\|\operatorname{Ext} v_h\|_{W^{1,p}(\mathbb{R}^2)} \leq C \|v_h\|_{W^{1,p}(\Omega_h)} \leq cC ,
\]

with \( C \) independent of \( h \).

Now we denote by \( \hat{v}_h = \operatorname{Ext} v_h |_{\overline{\Omega}} \). Then \( \hat{v}_h \in W^{1,p}(\Omega) \) and \( \|\hat{v}_h\|_{W^{1,p}(\Omega)} \leq cC \), hence there exists a subsequence, still denoted by \( \hat{v}_h \), weakly converging to \( \hat{v} \) in \( W^{1,p}(\Omega) \). We point out that \( \hat{v}_h \) strongly converges to \( \hat{v} \) in \( L^p(\Omega) \) and also in \( L^2(\Omega) \).
since $p \geq 2$. From Proposition 3.6, $v_h$ weakly converges to $u$ in $L^2(\Omega)$. We prove that $\hat{v} = u$ $L^2$-a.e., that is
\[
\int_\Omega (\hat{v} - u)\varphi \, d\mathcal{L}_2 = 0
\]
for each $\varphi \in L^2(\Omega)$. Indeed, we can write
\[
\int_\Omega (\hat{v} - u)\varphi \, d\mathcal{L}_2 = \int_\Omega (\hat{v} - \hat{v}_h + \hat{v}_h - u)\varphi \, d\mathcal{L}_2 = \int_\Omega (\hat{v} - \hat{v}_h)\varphi \, d\mathcal{L}_2 + \int_{\Omega_h} (\hat{v}_h - u)\varphi \, d\mathcal{L}_2 + \int_{\Omega \setminus \Omega_h} (\hat{v}_h - u)\varphi \, d\mathcal{L}_2.
\]
(3.12)

For every $\epsilon > 0$ there exists $h \in \mathbb{N}$ such that each term in the sum of the right-hand side of 3.12 is less than $\epsilon/3$. Since $\hat{v}_h \rightharpoonup \hat{v}$ in $L^2(\Omega)$ and $v_h \rightharpoonup u$ in $L^2(\Omega)$ we deduce our claim for the first two terms. As to $\int_{\Omega \setminus \Omega_h} (\hat{v}_h - u)\varphi \, d\mathcal{L}_2$, from Hölder inequality we deduce that
\[
\int_{\Omega \setminus \Omega_h} |(\hat{v}_h - u)\varphi| \, d\mathcal{L}_2 \leq \|\varphi\|_{L^2(\Omega \setminus \Omega_h)} (\|\hat{v}_h\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}) \leq \epsilon/3,
\]
since $|\Omega \setminus \Omega_h| \to 0$ as $h \to \infty$.

We now prove that
\[
\lim_{h \to \infty} \int_\Omega \chi_{\Omega_h} |Dv_h|^p \, d\mathcal{L}_2 \geq \int_\Omega |Du|^p \, d\mathcal{L}_2.
\]
(3.13)

It is enough to prove that $\chi_{\Omega_h} Dv_h \rightharpoonup Du$ in $L^p(\Omega)$, from here the claim will follow from the semicontinuity of the norm. Since $\chi_{\Omega_h} Dv_h = \chi_{\Omega_h} D\hat{v}_h$, this amounts to prove that $\int_\Omega \chi_{\Omega_h} D\hat{v}_h \varphi \, d\mathcal{L}_2 \to \int_\Omega Du \varphi \, d\mathcal{L}_2$ for every $\varphi \in L^{p'}(\Omega)$.

It holds that
\[
\int_\Omega Du \varphi \, d\mathcal{L}_2 - \int_{\Omega_h} D\hat{v}_h \varphi \, d\mathcal{L}_2 = \int_\Omega (Du - D\hat{v}_h) \varphi \, d\mathcal{L}_2 - \int_{\Omega \setminus \Omega_h} D\hat{v}_h \varphi \, d\mathcal{L}_2.
\]
The first term vanishes as $h \to \infty$ since $D\hat{v}_h \rightharpoonup Du$ in $L^p(\Omega)$. Now we estimate the second term $\int_{\Omega \setminus \Omega_h} |D\hat{v}_h \varphi| \, d\mathcal{L}_2$. We have
\[
\int_{\Omega \setminus \Omega_h} D\hat{v}_h \varphi \, d\mathcal{L}_2 \leq \|\varphi\|_{L^{p'}(\Omega \setminus \Omega_h)} \|D\hat{v}_h\|_{L^p(\Omega)} \to 0.
\]

Hence 3.13 holds. Now we prove that
\[
\lim_{h \to \infty} \delta_{\Omega_h}^{1-p} \int_{F_h} |Dv_h|^p \, d\ell \geq E_p[u].
\]
We begin by proving that
\[
\mathcal{E}_p^{(h)}[u] \leq E_p^{(h)}[u].
\]
(3.14)

We fix a positive orientation on $F_h$, the anti-clockwise orientation, and we choose as origin $A_1$. This induces a natural orientation on the vertices $P_j$, for $j = 1, \ldots, 3N$, where we set $N := 4^k$ and $P_1 = P_{3N+1} = A_1$. Let now $M_j$ be a segment of the $h$-th
extension (i.e. \(M_j \in F_h\)). From the definition of \(E_p^{(h)}\) given in 3.3, we get

\[
E_p^{(h)}[u] = \frac{4(p-1)h}{p} \sum_{j=1}^{3N} (u(P_{j+1}) - u(P_j))^p = \frac{4(p-1)h}{p} \sum_{j=1}^{3N} \left| \int_{M_j} |Du| \, d\ell \right|^p \\
\leq \frac{4(p-1)h}{p} \sum_{j=1}^{3N} |M_j|^{\frac{p}{2}} \int_{M_j} |Du|^p \, d\ell = \frac{1}{p} \left( \frac{4}{3} \right)^{(p-1)h} \sum_{j=1}^{3N} \int_{M_j} |Du|^p \, d\ell = E_p^{(h)}[u].
\]

From Proposition 3.8, \(v_h\) is in particular continuous on \(F_h\). Hence the function \(v_h\) is defined on the discrete set \(\mathcal{V}^h\), so we extend it to a continuous function \(Hv_h\) on \(F\). This extension is unique and it is obtained by constructing the discrete harmonic extension \(Hv_h|_{\mathcal{V}_*}\) of \(v_h|_{\mathcal{V}_*}\) to the set \(\mathcal{V}_*\) and then taking the unique continuous extension of \(Hv_h|_{\mathcal{V}_*}\) to \(F\). This iterative process is known as decimation in the physics literature (see [23, 24] and [9]). Then from 3.11 and 3.14 we have that

\[
E_p[Hv_h] = \sup_h E_p^{(h)}[v_h] \leq c. \quad (3.15)
\]

Moreover \(Hv_h \in D(E_p)\) and from 3.15 we have that \(\{Hv_h\}\) is a bounded sequence in \(D(E_p)\). Then there exists a subsequence, still denoted by \(Hv_h\), weakly converging to a function \(u^*\) in \(D(E_p)\) with

\[
E_p[u^*] \leq \lim_{h \to \infty} E_p[Hv_h] = \lim_{h \to \infty} E_p^{(h)}[v_h] \leq c \quad (3.16)
\]

(this follows from the lower semi-continuity of the norm, from 3.15 and 3.14). From Ascoli-Arzela Theorem it follows that

\[ Hv_h \to u^* \text{ uniformly in } C(F) \text{ and } u^* \in C(F). \quad (3.17) \]

We have to prove now that \(u^* = u|_F\) in \(L^p(F)\). Since \(\hat{v}_h\) weakly converges to \(u\) in \(W^{1,p}(\Omega)\), it strongly converges to \(u\) in \(W^{\sigma,p}(\Omega)\) for \(0 < \sigma < 1\). Hence \(\hat{v}_h|_F\) strongly converges to \(u\) in \(B^{\beta,p}_p(F)\) with \(\beta = \alpha - \frac{2-d_f}{p}\) and \(\hat{v}_h|_F \to u|_F\) in \(L^p(F)\) (in particular in \(L^2(F)\)).

Now let \(\varphi \in C(\Omega)\). Then

\[
\int_F (u^* - u|_F) \varphi \, d\mu = \int_F (u^* - Hv_h) \varphi \, d\mu + \delta_h \int_{F_h} v_h \varphi \, d\ell \\
- \int_F u|_F \varphi \, d\mu + \int_F Hv_h \varphi \, d\mu - \delta_h \int_{F_h} v_h \varphi \, d\ell.
\]

We note that the first integral tends to 0 as \(h\) goes to infinity from the weak convergence, and that the difference between the second and the third integral also tends to 0 since \(v_h \to u\) in \(\mathcal{H}\) by assumption. We have to estimate

\[
\int_F Hv_h \varphi \, d\mu - \delta_h \int_{F_h} v_h \varphi \, d\ell.
\]

We note that, for every \(P \in F_h\) and \(P^*_i \in \mathcal{V}^h\), from the uniform boundedness of \(v_h\) and the uniform Hölder continuity of \(v_h\) on \(F_h\) (see Proposition 3.8), for every \(\varepsilon > 0\) there exists \(\delta > 0\) such that for every \(h \geq \bar{h}\)

\[
A_1 := |\varphi(P)v_h(P) - \varphi(P^*_i)v_h(P^*_i)| \leq \varepsilon + c_{\varphi}c_H 3^{-\beta h}. \quad (3.18)
\]
Now we consider $P^*_i \in \mathcal{Y}^h \cap M^h$, where $M^h_i := \psi_{\mathcal{H}}(F_h)$. Then from 3.18 we have that
\[
\delta_h \int_{F_h} v_h \varphi \, d\ell \leq \delta_h \sum_{i=1}^{3N} \int_{M^h_i} |v_h(P^*)\varphi(P) - v_h(P^*_i)\varphi(P^*_i)| \, d\ell \\
+ \delta_h \sum_{i=1}^{3N} \int_{F_h} v_h(P^*_i)\varphi(P^*_i) \, d\ell \\
\leq \delta_h (\varepsilon c + c_{\mathcal{H}} 3^{-\beta h}) \delta_h^{-1} + \delta_h \sum_{i=1}^{3N} \int_{F_i} |v_h(P^*_i)\varphi(P^*_i)| \, d\ell.
\]

The first term vanishes as $h \to \infty$ for the arbitrariness of $\varepsilon$. Now we point out that
\[
\delta_h \sum_{i=1}^{3N} \int_{F_h} v_h(P^*_i)\varphi(P^*_i) \, d\ell = 4^{-h} \sum_{i=1}^{3N} v_h(P^*_i)\varphi(P^*_i) = 4^{-h} \sum_{i=1}^{3N} H v_h(P^*_i)\varphi(P^*_i) \\
= \mu(\psi_{\mathcal{H}}(F)) \sum_{i=1}^{3N} H v_h(P^*_i)\varphi(P^*_i) = \int_F H v_h(P^*_i)\varphi(P^*_i) \, d\mu.
\]

Hence we get, for $h \to \infty$,
\[
\int_F (H v_h(P) \varphi(P) - H v_h(P^*_i)\varphi(P^*_i)) \, d\mu \to 0
\]
since $H v_h$ is equi-Hölder continuous on $F$. We conclude the proof taking into account the liminf properties of the sum and Proposition 3.7.

**Proof of condition b).** We have to prove that for every $u \in \mathcal{H}$ there exists $\{v_h\}_{h \in \mathbb{N}}$ strongly converging to $u$ in $\mathcal{H}$ such that
\[
\Phi_p[u] \geq \lim_{h \to \infty} \Phi_p^{(h)}[v_h].
\]

We can suppose that $u \in D(\Phi_p)$. Indeed, if $u \notin D(\Phi_p)$ then $\Phi_p[u] = +\infty$ and from Lemma 2.16 it follows that there exists a sequence $\{v_h\}_{h \in \mathbb{N}}$ converging to $u$ in $\mathcal{H}$ and hence
\[
\lim_{h \to \infty} \Phi_p^{(h)}[v_h] \leq \Phi_p[u] = +\infty.
\]

Let then $u \in D(\Phi_p)$, i.e. $u \in W^{1,p}(\Omega)$ and $u|_F \in D(\mathcal{E}_p)$. For the case $p = 2$, we refer to [30]. Here we consider the case $p > 2$. Since $p > 2$, then $u$ belongs to $C(\overline{\Omega})$ (see [31]).

We extend by continuity $u$ to $\overline{T}$ and we denote its extension by $\hat{u}$. Following the same approach as in [30] with some suitable modifications, we introduce a quasi-uniform triangulation $\tau_h$ of $T$ made by equilateral triangles $T^j_h$ such that the vertices of the pre-fractal curve $F_h$ are nodes of the triangulation at the $h$-th level. Let $S_h$ be the space of all the functions being continuous on $\overline{T}$ and affine on the triangles of $\tau_h$. We denote by $\mathcal{M}_h$ the nodes of $\tau_h$, i.e. the set of the vertices of all $T^j_h$. For a given continuous function $u$, we denote by $I_h u$ the function which is affine on every $T^j_h \in \tau_h$ and which interpolates $u$ in the nodes $P_{j,i} \in \mathcal{M}_h \cap \overline{T}_h$. We set $w_h = I_h \hat{u}$ and we prove that $\{w_h\}$ strongly converges to $u$ in $\mathcal{H}$, which is equivalent to prove that (see Lemma 2.13) $(w_h, v_h)_{\mathcal{H}} \to (u, v)_H$ for every sequence $\{v_h\}$ weakly converging to a vector $v$ in $\mathcal{H}$.

We know that
\[
\|w_h - u\|_{W^{1,p}(T)} \to 0
\]
as $h$ goes to $\infty$ (see [12]) and hence $\|w_h - u\|_{W^{1,p}(\Omega)} \to 0$. 
From Theorem 2.3, there exists a constant $c$ independent of $h$ such that

\[ \|w_h - u\|_{L^2(F_h)} \leq c \delta_h^{\frac{1}{2}} \|w_h - u\|_{W^{1,p}(\Omega)}. \]

Then we have

\[
0 \leq \|(w_h, v_h)_{H_h} - (u, v)_{H}\|
\]

\[
= \int_{\Omega_h} w_h v_h \, d\mathcal{L}_2 + \delta_h \int_{F_h} w_h v_h \, d\ell - \int_{\Omega_h} uv \, d\mathcal{L}_2 - \int_{F} uv \, d\mu
\]

\[
= \|(w_h - u, v_h)_{L^2(\Omega_h)} + \delta_h \int_{F_h} (w_h - u) v_h \, d\ell + (u, v_h)_{H_h} - (u, v)_{H}\|
\]

\[
\leq \|(w_h - u, v_h)_{L^2(\Omega_h)}\| + \|\delta_h (w_h - u), \sqrt{\delta_h} v_h\|_{L^2(F_h)} + \|(u, v_h)_{H_h} - (u, v)_{H}\|
\]

\[
\leq \|w_h - u\|_{L^2(\Omega)} \|v_h\|_{L^2(\Omega)} + \sqrt{\delta_h} \|w_h - u\|_{L^2(F_h)} \sqrt{\delta_h} \|v_h\|_{L^2(F_h)}
\]

\[
+ \|(u, v_h)_{H_h} - (u, v)_{H}\|,
\]

The claim follows since $v_h \rightarrow v$ in $H$, therefore

\[
\sup_h \|v_h\|_{H_h} < \infty \text{ and } \sqrt{\delta_h} \|w_h - u\|_{L^2(F_h)} \leq c \|w_h - u\|_{H^1(\Omega)}.
\]

We now prove condition b) for the sequence $w_h$. We note that from Proposition 3.7

\[
\lim_{h \rightarrow \infty} \delta_h \int_{F_h} b|w_h|^p \, d\ell = \int_{F} b|u|^p \, d\mu.
\]

We have that

\[
\int_{\Omega_h} |Dw_h|^p \, d\mathcal{L}_2 \leq \int_{\Omega} |Dw_h|^p \, d\mathcal{L}_2,
\]

then, by taking the limit for $h \rightarrow \infty$, we have the thesis (since $\|D(w_h - u)\|_{L^p(\Omega)} \rightarrow 0$ for $h \rightarrow \infty$).

We have only to prove that

\[
\lim_{h \rightarrow \infty} \frac{\delta_h^{1-p}}{p} \int_{F_h} |Dw_h|^p \, d\ell \leq \mathcal{E}_p[u].
\]

We now show that

\[
\mathcal{E}_p^{(h)}[u] = \frac{\delta_h^{1-p}}{p} \int_{F_h} |Dw_h|^p \, d\ell,
\]

by using the parametrization of $F_h$ by means of the arc length with origin $P_1 = A$.

Since $w_h = I_h \hat{u}$, we have that

\[
w_h = m_j l + q_j \quad , \quad l \in [l_j, l_{j+1}],
\]

where $l_j = (j - 1) 3^{-h}$ for $j = 1, \ldots, 3N$. Hence we get

\[
\frac{\delta_h^{1-p}}{p} \int_{F_h} |Dw_h|^p \, d\ell = \frac{\delta_h^{1-p}}{p} \sum_{j=1}^{3N} m_j^p (l_{j+1} - l_j)
\]

\[
= \frac{4^{(p-1)h}}{p} \sum_{j=1}^{3N} (w_h(P_{j+1}) - w_h(P_j))^p = \mathcal{E}_p^{(h)}[w_h] = \mathcal{E}_p^{(h)}[u].
\]

The claim then follows from the monotonicity of $\mathcal{E}_p^{(h)}[u]$. 
Proposition 3.8. Let \( v_h \in D(\Phi^{(h)}_p) \) be weakly converging in \( \mathcal{H} \) to \( u \) with 
\[
\lim_{h \to \infty} \Phi^{(h)}_p[v_h] < \infty.
\]
Then \( v_h \) is uniformly bounded and Hölder continuous uniformly in \( h \) on \( F_h \).

Proof. We begin by proving the Hölder continuity. We start by assuming \( P, Q \in M_j \).
Then
\[
|v_h(P) - v_h(Q)| \leq \int_{PQ} |\nabla v_h| d\ell \leq \left( \int_{PQ} |\nabla v_h|^p d\ell \right)^{\frac{1}{p}} |P - Q|^{\frac{1}{p'}}.
\]
We point out that from 3.11 we have that
\[
\int_{F_h} |\nabla v_h|^p d\ell \leq c\delta_h^{-1},
\]
where \( c \) is independent of \( h \). Then, since \( |P - Q| \leq 3^{-h} \), by setting \( \beta := \frac{d_f}{p'} \) we get
from straightforward calculation
\[
|v_h(P) - v_h(Q)| \leq c |P - Q|^{\beta}.
\]
This proves the uniform Hölder continuity in \( h \) of \( v_h \). If \( P \) and \( Q \) do not belong to
the same segment, the proof can be carried out by a chain argument.

We now prove the uniform boundedness. From the uniform convergence of \( Hv_h \)
to \( u^* \) in \( C(F) \) (see 3.17), it follows that for every \( \varepsilon > 0 \) there exists \( \bar{h} > 0 \) such that,
for every \( h > \bar{h} \),
\[
|Hv_h(P) - u^*(P)| < \varepsilon \quad \forall P \in F.
\]
Then it holds that, since \( u^* \in C(F) \),
\[
\|Hv_h\|_{L^\infty(F_h)} \leq \|Hv_h - u^*\|_{L^\infty(F_h)} + \|u^*\|_{C(F)} \leq \varepsilon + \|u^*\|_{C(F)}.
\]
Now take \( \bar{P} \in \mathcal{V}_h \). Since \( Hv_h(\bar{P}) = v_h(\bar{P}) \), then
\[
|v_h(\bar{P})| = |Hv_h(\bar{P})| \leq \bar{c}.
\]
Let now \( P \in F_h \). Then
\[
|v_h(P)| \leq |v_h(P) - v_h(\bar{P})| + |v_h(\bar{P})|,
\]
hence the thesis follows. \( \square \)

In the following Theorem we deduce the G-convergence of the associated subdifferentials.

Theorem 3.9. \( \Phi^{(h)}_p \) M-converges to \( \Phi_p \) in \( \mathcal{H} \) if and only if \( \partial \Phi^{(h)}_p \) G-converges to \( \partial \Phi_p \).

For the proof see [36, Theorem 7.46]. This result will be crucial for the convergence of the solutions of the nonlinear abstract Cauchy problems.
4. Convergence of the solutions of the abstract Cauchy problems. We now consider the abstract homogeneous Cauchy problem

\[
(P) \left\{ \begin{array}{l}
\frac{du}{dt} + Au \ni 0, \quad t \in [0, T] \\
u(0) = u_0,
\end{array} \right.
\]

where \(A\) is the subdifferential of \(\Phi_p\), \(T\) is a fixed positive number, and \(u_0\) is a given function. We now recall some results on the properties of nonlinear semigroups generated by the (opposite of) subdifferential of a proper convex lower semicontinuous functional on a real Hilbert space (see Theorem 1 and Remark 2 in [6], see also [5]). According to [5, Section 2.1, chapter II], we say that a function \(u : [0, T] \to H\) is a strong solution of \((P)\) if \(u \in C([0, T]; H)\), \(u(t)\) is differentiable a.e. in \((0, T)\), \(u(t) \in D(A)\) a.e. and \(\frac{du}{dt} + Au \ni 0\) for a.e. \(t \in [0, T]\).

**Theorem 4.1.** Let \(\varphi : H \to (-\infty, +\infty]\) be a proper, convex, lower semicontinuous functional on a real Hilbert space \(H\), with effective domain \(D(\varphi)\). The subdifferential \(\partial \varphi\) is a maximal monotone \(m\)-accretive operator. Moreover, \(\overline{D(\varphi)} = \overline{D(\partial \varphi)}\). \(-\partial \varphi\) generates a (nonlinear) \(C_0\)-semigroup \(\{T(t)\}_{t \geq 0}\) on \(D(\varphi)\) in the following sense: for each \(u_0 \in \overline{D(\varphi)}\), the function \(u := T(\cdot)u_0\) is the unique strong solution of the problem

\[
\left\{ \begin{array}{l}
u \in C(\mathbb{R}_+; H) \cap W^{1,\infty}_{\text{loc}}((0, \infty); H) \text{ and } u(t) \in D(\varphi) \text{ a.e.}, \\
\frac{du}{dt} + \partial \varphi(u) \ni 0 \text{ a.e. on } \mathbb{R}_+,
\end{array} \right.
\]

\(u(0, x) = u_0(x)\).

In addition, \(-\partial \varphi\) generates a (nonlinear) semigroup \(\{\tilde{T}(t)\}_{t \geq 0}\) on \(H\), where for every \(t \geq 0\), \(\tilde{T}(t)\) is the composition of the semigroup \(T(t)\) on \(D(\varphi)\) with the projection on the convex set \(\overline{D(\varphi)}\).

In our case it turns out that, from Theorem 4.1, the subdifferentials \(\partial \Phi_p\) and \(\partial \Phi_p^{(h)}\) are maximal, monotone and \(m\)-accretive operators on \(H\) and \(H_h\) respectively. Then, if we denote with \(T_p(t)\) and \(T_p^{(h)}(t)\) the nonlinear semigroups generated by \(-\partial \Phi_p\) and \(-\partial \Phi_p^{(h)}\) respectively, these semigroups are strongly continuous and contractive on \(H\) and \(H_h\) (see Proposition 2.5 in [27] for the fractal case).

Now [27, Theorem 2.7] states the following result.

**Theorem 4.2.** If \(u_0 \in \overline{D(-A)}\), then \((P)\) has a unique strong solution \(u \in C([0, T]; H)\) defined as \(u = T_p(\cdot)u_0\) such that \(u \in W^{1,2}((\delta, T); H)\) for every \(\delta \in (0, T)\). Moreover \(u \in D(-A)\) a.e. for \(t \in (0, T)\), \(\sqrt{\frac{du}{dt}} \in L^2(0, T; H)\) and \(\Phi_p[u] \in L^1(0, T)\).

Moreover, from [27, Theorem 2.6] it can be proved that the solution \(u\) of problem \((P)\) solves the following problem \((\tilde{P})\) on \(\Omega\) for \(t \in (0, T)\) in the following weak sense:

\[
(\tilde{P}) \left\{ \begin{array}{l}
\frac{du}{dt} - \Delta_p u = 0, \quad \text{in } L^p(\Omega), \\
\left\langle \frac{du}{dt}, \psi \right\rangle_{L^2(\partial_\Omega; L^2(\partial_\Omega))} + \left\langle \frac{\partial u}{\partial n}, |Du|^{p-2}, \psi \right\rangle_{L^2(\partial_\Omega)} + \left\langle b |u|^{p-2} u, \psi \right\rangle_{L^p(\partial_\Omega; L^p(\partial_\Omega))} = 0, \quad \text{for every } \psi \in D(E_p), \\
u(0, x) = u_0(x) \quad \text{in } L^2(\Omega, m),
\end{array} \right.
\]
where $\alpha = 1 - \frac{2-d_f}{p}$. The second equation is the so-called Venttsel’ (or dynamical) boundary condition.

We now come to the pre-fractal case. For each $h \in \mathbb{N}$ fixed, we consider the abstract homogeneous Cauchy problem

$$
\begin{align*}
(P_h) \quad \left\{ \begin{array}{l}
\frac{du_h}{dt} + A_h u_h \ni 0, \\ u_h(0) = u_0^{(h)},
\end{array} \right. \quad t \in [0, T]
\end{align*}
$$

where $A_h$ is the subdifferential of $\Phi^{(h)}_p$, $T$ is a fixed positive number, and $u_0^{(h)}$ is a given function.

Before stating existence and uniqueness results we give a characterization of $A_h$.

**Theorem 4.3.** Let $u_h(t)$ belong to $D(\Phi^{(h)}_p)$ for a.e. $t \in (0, T)$, and $f$ be in $H_h$. Then $f \in \partial \Phi^{(h)}_p[u_h]$ if and only if

$$
\begin{align*}
\begin{cases}
-\Delta_p u_h = f &\text{in } L^p(\Omega_h), \\
\frac{\partial u_h}{\partial n} &\text{on } \partial \Omega_h, \\
-\partial_h^{1-p} (\Delta_p u_h, \psi)_{W^{-1,p}(\Omega_h), W^{1,p}(\Omega_h)} &\text{for every } \psi \in W^{1,p}(\Omega_h),
\end{cases}
\end{align*}
$$

where $\frac{\partial u_h}{\partial n}$ denotes the normal derivative across $F_h$.

**Proof.** Let $f \in \partial \Phi^{(h)}_p[u_h]$, i.e. $\Phi^{(h)}_p[v] - \Phi^{(h)}_p[u_h] \geq (f, v - u_h)_{H_h}$ for every $v \in D(\Phi^{(h)}_p)$:

$$
\begin{align*}
\int_{\Omega_h} f(v - u_h) \, d\mathcal{L}_2 + \delta_h \int_{F_h} f(v - u_h) \, d\ell \\
\leq \frac{1}{p} \int_{\Omega_h} \chi_{\Omega_h} (|Dv|^p - |Du_h|^p) \, d\mathcal{L}_2 + \frac{\delta_h}{p} \int_{F_h} b(|v|^p - |u_h|^p) \, d\ell \\
+ \frac{\delta_h^{1-p}}{p} \int_{F_h} (|Dv|^p - |Du_h|^p) \, d\ell.
\end{align*}

(4.1)

By choosing $v = u_h + t\psi$, with $\psi \in D(\Phi^{(h)}_p)$ and $0 < t \leq 1$ in 4.1, we obtain

$$
\begin{align*}
t \int_{\Omega_h} f \psi \, d\mathcal{L}_2 + t\delta_h \int_{F_h} f \psi \, d\ell \\
\leq \frac{1}{p} \int_{\Omega_h} \chi_{\Omega_h} (|D(u_h + t\psi)|^p - |Du_h|^p) \, d\mathcal{L}_2 + \frac{\delta_h}{p} \int_{F_h} b(|u_h + t\psi|^p - |u_h|^p) \, d\ell \\
+ \frac{\delta_h^{1-p}}{p} \int_{F_h} (|D(u_h + t\psi)|^p - |Du_h|^p) \, d\ell.
\end{align*}

(4.2)

Now, if $\psi \in D(\Omega_h)$, from 4.2 we have that

$$
\begin{align*}
\int_{\Omega_h} f \psi \, d\mathcal{L}_2 \leq \frac{1}{t} \int_{\Omega_h} \frac{(|D(u_h + t\psi)|^p - |Du_h|^p)}{t} \, d\mathcal{L}_2.
\end{align*}

$$

Then, by passing to the limit for $t \to 0^+$, we get

$$
\int_{\Omega_h} f \psi \, d\mathcal{L}_2 \leq \int_{\Omega_h} |Du_h|^{p-2}Du_h \, D\psi \, d\mathcal{L}_2.
$$
By taking $-\psi$ in 4.2 we obtain the opposite inequality, and hence we get

$$\int_{\Omega_h} f\psi \, d\mathcal{L}_2 = \int_{\Omega_h} |Du_h|^{p-2}Du_h \psi \, d\mathcal{L}_2.$$  

In order to apply Green formula for Lipschitz domains (see [8] and [4])

$$\int_{\Omega_h} |Du|^{p-2}Du \psi \, d\mathcal{L}_2 = \left\langle \frac{\partial u}{\partial \nu_h}, |Du|^{p-2} \psi \right\rangle_{w^{-\frac{1}{p'}}(F_h), w^{-\frac{1}{p'}}(F_h)} - \int_{\Omega_h} \Delta_p u \psi \, d\mathcal{L}_2$$

we ask that $w := |Du|^{p-2}Du \in (L^{p'}(\Omega_h))_2 := \{w \in (L^{p'}(\Omega_h))_2 : \text{div} \, w \in L^{p'}(\Omega_h)\}$. Since $p \geq 2$, then $p' \leq 2$, therefore if we choose $f \in L^2(\Omega_h)$ in particular $f \in L^{p'}(\Omega_h)$ (in particular $-\Delta_p u_h = f$ in $L^2(\Omega_h)$) then it holds a.e. in $\Omega_h$.

We go back to 4.2. Dividing by $t > 0$ and passing to the limit for $t \to 0^+$, we get

$$\int_{\Omega_h} f\psi \, d\mathcal{L}_2 + \delta_h \int_{F_h} f\psi \, d\mathcal{F} \leq \int_{\Omega_h} |Du_h|^{p-2}Du_h \psi \, d\mathcal{L}_2 + \delta_h \int_{F_h} b(|u_h|^{p-2}u_h \psi) \, d\mathcal{F} + \delta_h^{1-p} \int_{F_h} |Du_h|^{p-2}Du_h \psi \, d\mathcal{F}.$$  

As above, by taking $-\psi$ we obtain the opposite inequality, hence we get the equality. Then, by using Green formula for Lipschitz domains and since $-\Delta_p u_h = f$ in $L^{p'}(\Omega_h)$, we have

$$\delta_h \int_{F_h} f\psi \, d\mathcal{F} = \left\langle \frac{\partial u_h}{\partial \nu_h}, |Du_h|^{p-2} \psi \right\rangle_{w^{-\frac{1}{p'}}(F_h), w^{-\frac{1}{p'}}(F_h)} + \delta_h \int_{F_h} b(|u_h|^{p-2}u_h \psi) \, d\mathcal{F} + \delta_h^{1-p} \int_{F_h} |Du_h|^{p-2}Du_h \psi \, d\mathcal{F}. \quad (4.3)$$

We can define $\Delta_p$ as a variational operator $\Delta_p : W^{1,p}_0(F_h) \to W^{-1,p'}(F_h)$ in the following way:

$$\int_{F_h} |Dz|^{p-2}Dz \psi \, dw = -<\Delta_p z, w>_{W^{-1,p'}(F_h), W^{1,p}(F_h)} \quad (4.4)$$

for $z, w \in W^{1,p}_0(F_h)$. Then from 4.3 we have that

$$\delta_h f = \delta_h b|u_h|^{p-2}u_h - \delta_h^{1-p} \Delta_p u_h + \frac{\partial u_h}{\partial \nu_h} |Du_h|^{p-2} \quad (4.5)$$

holds in $W^{-\frac{1}{p'},p'}(F_h)$.

We want now to prove the converse. Let then $u_h \in D(\Phi_p^{(h)})$ be the weak solution of problem $(\tilde{P}_h)$. We have then to prove that $\Phi_p^{(h)}[v] - \Phi_p^{(h)}[u_h] \geq (f, v - u_h)_{H_n}$ for every $v \in D(\Phi_p^{(h)})$. By using the inequality

$$\frac{1}{p}(|a|^p - |b|^p) \geq |b|^{p-2}b(a - b) \quad (4.6)$$
one gets
\[ \Phi_p^{(h)}[v] - \Phi_p^{(h)}[u_h] \geq \int_{\Omega_h} |D u_h|^{p-2} D u_h D v \, d\mathcal{L}_2 - \int_{\Omega_h} |D u_h|^p \, d\mathcal{L}_2 \\
+ \delta h^{1-p} \int_{F_h} |D u_h|^{p-2} D u_h D v \, d\ell - \delta h^{1-p} \int_{F_h} |D u_h|^p \, d\ell \\
+ \delta h \int_{F_h} b u_h |p-2 u_h| v \, d\ell - \delta h \int_{F_h} b u_h |p| v \, d\ell. \quad (4.7) \]

Since \( u_h \) is the weak solution of \((\hat{P}_h)\), by using as test functions \( v \) and \( u_h \) we have
\[ \Phi_p^{(h)}[v] - \Phi_p^{(h)}[u_h] \geq (f,v)_{H_h} - (f,u_h)_{H_h}, \]
i.e. the thesis.

By proceeding as in [27, Theorem 2.6 and Theorem 2.7], one can prove the following result.

**Theorem 4.4.** If \( u_0^{(h)} \in \overline{D(-A_h)} \), then \((P_h)\) has a unique strong solution \( u_h \in C([0,T];H_h) \) defined as \( u_h = T^p_h(\cdot) u_0^{(h)} \) such that \( u_h \in W^{1,2}([\delta,T);H_h) \) for every \( \delta \in (0,T) \). Moreover \( u_h \in D(-A_h) \) a.e. for \( t \in (0,T) \), \( \sqrt{T u_h} \in L^2(0,T;H_h) \) and \( \Phi_p^{(h)}[u_h] \in L^1(0,T) \).

Moreover it follows that the solution \( u_h \) of problem \((P_h)\) solves for each \( h \in \mathbb{N} \) the following problem \((\hat{P}_h)\) on \( \Omega_h \) for \( t \in (0,T) \) in the following weak sense:

\[
(\hat{P}_h) \quad \begin{cases} 
\frac{d u_h}{dt} - \Delta_p u_h = 0, & \text{in } L^{p'}(\Omega_h) \\
\delta h \left< \frac{d u_h}{dt}, \psi_h \right>_{L^2(F_h),L^2(F_h)} + \left< \frac{\partial u_h}{\partial n_h}, |D u_h|^{p-2}, \psi_h \right>_{W^{\frac{2}{p'}}(F_h),W^{\frac{2}{p'}}(F_h)} \\
+ \delta h \left< b u_h |p-2 u_h|, \psi_h \right>_{L^{p'}(F_h),L^p(F_h)} - \delta h^{1-p} \left< \Delta_p u_h, \psi_h \right>_{W^{1,p}(F_h),W^{1,p}(F_h)} = 0 \\
\forall \psi_h \in W^{1,p}(F_h), \\
u_h(0,x) = u_0^{(h)}(x) & \text{in } L^2(\Omega) \cap L^2(\Omega, m_h). 
\end{cases}
\]

Theorem 3.5, Theorem 3.9 and Theorem 7.24 in [36] allow us to deduce that the pre-fractal solutions converge in a suitable sense to the limit fractal one.

**Theorem 4.5.** Let \( H_h, H, \Phi_p^{(h)}, \Phi_p \) and \( \delta_h \) be as in Theorem 3.5. Let \( T^p_h(t), u_0^{(h)} \) and \( u_0 \) be as in Theorems 4.2 and 4.4. If \( u_0^{(h)} \to u_0 \) strongly in \( H \), then \( T^p_h(t) u_0^{(h)} \to T^p_h(t) u_0 \) strongly in \( H \) for every \( t \geq 0 \).

**Remark 3.** We point out that the existence and uniqueness of the strong solution for problems \((P)\) and \((\hat{P}_h)\) can be proved also for the nonhomogeneous problems (see [27, Theorem 2.7] for the fractal case). But in this case the asymptotic behavior of the solutions is still an open problem.

**Acknowledgments.** The authors have been supported by the Gruppo Nazionale per l’Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).
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Received November 2016; revised July 2017.
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