POLARIZATION OF THE MASSLESS FERMIONIC VACUUM IN THE BACKGROUND OF A SINGULAR MAGNETIC VORTEX IN 2+1-DIMENSIONAL SPACE-TIME

(Ukrainian Journal of Physics, 43, no.12, 1513-1525 (1998))

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Abstract

Effects of the configuration of an external static magnetic field in the form of a singular vortex on the vacuum of a quantized massless spinor field are determined. The most general boundary conditions at the punctured singular point which make the twodimensional Dirac Hamiltonian to be self-adjoint are employed.

1 Introduction

A study of effects of singular external fields (zero-range potentials) in quantum mechanics has a long history and has been comprehensively conducted (see [1] and references therein). Contrary to this, effects of singular external fields in quantum field theory are at the initial stage of consideration, and much has to be elucidated. Singular background can act on the vacuum of a second-quantized spinor field in a rather unusual manner: the leakage of quantum numbers from the singularity point occurs. This is due, apparently, to the fact that a solution to the Dirac equation, unlike that to the Klein-Gordon one, does not obey a condition of regularity at the singularity point. It is necessary then to specify a boundary condition at this point, and the least restrictive, but physically acceptable, condition is such that guarantees self-adjointness of the Dirac Hamiltonian. Thus, effects of polarization of the vacuum by a singular background appear to depend on the choice of the boundary condition at the singularity point, and the set of permissible boundary conditions is labelled, most generally, by a self-adjoint extension parameter.

As examples of singular background configurations, one can consider a pointlike magnetic monopole in threedimensional space and a pointlike magnetic vortex in twodimensional space. While in the first case there is a leakage of charge to the vacuum, which results in the monopole becoming the dyon violating the Dirac quantization condition and CP symmetry [2, 3, 4], in the second case the situation is much more complicated, since there is a leakage of both charge and other quantum numbers to the vacuum. For particular choices of the boundary condition at the singularity point it has been shown that charge [5, 6], current [7] and angular momentum [8] are induced in the vacuum. The vacuum quantum numbers under general boundary conditions which are compatible with self-adjointness have been considered in Refs. [9, 10, 11].

Thus far the effects of polarization of the massive fermionic vacuum have been studied. In an irreducible representation of the massive fermion vacuum have been studied. In an irreducible representation of the Clifford algebra in 2+1-dimensional space-time, the mass term violates the symmetry under both space and time parity transformations. Our interest will be in the parity of the third type, which is similar to the axial symmetry in even-dimensional space-times (see Refs. [12, 13, 14]). This parity is also violated by the mass term in the irreducible representation.
In the absence of the mass term all above symmetries are formally present. Thus our concern is, in particular, whether the polarization of the massless vacuum in a singular background respect these formal symmetries? It will be shown that, although the parity anomaly is absent, the parity breaking condensate emerges in the vacuum. In a parity-invariant model with two species of fermions composing a reducible representation of the Clifford algebra, this condensate is transformed into the chiral symmetry breaking condensate. Also all other characteristics of the massless fermionic vacuum in a singular background are determined.

2 Quantization of Spinor Field and Boundary Condition at the Location of the Vortex

The operator of the second-quantized spinor field is presented in the form

\[ \Psi(x, t) = \sum_{E\lambda > 0} e^{-iE_\lambda t} \langle x | \lambda \rangle a_\lambda + \sum_{E\lambda < 0} e^{-iE_\lambda t} \langle x | \lambda \rangle b_\lambda^+, \]

where \( a_\lambda \) and \( b_\lambda \) are the spinor particle (antiparticle) creation and annihilation operators satisfying the anticommutation relations

\[ [a_\lambda, a_\lambda^+] = [b_\lambda, b_\lambda^+] = \langle \lambda | \lambda' \rangle, \]

and \( \langle x | \lambda \rangle \) is the solution to the stationary Dirac equation

\[ H \langle x | \lambda \rangle = E_\lambda \langle x | \lambda \rangle, \]

\( H \) is the Dirac Hamiltonian, \( \lambda \) is the set of parameters (quantum numbers) specifying a state, \( E_\lambda \) is the energy of a state; symbol \( \sum \) means the summation over discrete and the integration (with a certain measure) over continuous values of \( \lambda \). The ground state \( \langle \text{vac} \rangle \) is defined conventionally by the equality

\[ a_\lambda \langle \text{vac} \rangle = b_\lambda \langle \text{vac} \rangle = 0. \]

In the case of quantization of a massless spinor field in the background of a static vector field \( V(x) \), the Dirac Hamiltonian takes the form

\[ H = -i\alpha[\partial - iV(x)], \]

where

\[ \alpha = \gamma^0, \quad \beta = \gamma^0, \]

\( \gamma \) and \( \gamma^0 \) are the Dirac \( \gamma \) matrices. In the 2+1-dimensional space-time \( (x, t) = (x^1, x^2, t) \), the Clifford algebra has two inequivalent irreducible representations which can be differed in the following way:

\[ i\gamma^0 \gamma^1 \gamma^2 = s, \quad s = \pm 1. \]

Choosing the \( \gamma^0 \) matrix in the diagonal form

\[ \gamma^0 = \sigma_3, \]

one gets

\[ \gamma^1 = e^{i\sigma_3 x^1}i\sigma_1 e^{-i\sigma_3 x^1}, \quad \gamma^2 = e^{i\sigma_3 x^1}i\sigma_2 e^{-i\sigma_3 x^1}, \]
where $\sigma_1, \sigma_2$ and $\sigma_3$ are the Pauli matrices, and $\chi_1$ and $\chi_{-1}$ are the parameters that are varied in the interval $0 \leq \chi_s < 2\pi$ to go over to the equivalent representations.

The configuration of the external field $V(x) = (V_1(x), V_2(x))$ is chosen to be

$$V_1(x) = -\Phi^{(0)} \frac{x^2}{(x^1)^2 + (x^2)^2}, \quad V_2(x) = \Phi^{(0)} \frac{x^1}{(x^1)^2 + (x^2)^2}, \quad (2.10)$$

which corresponds the magnetic field strength in the form of a singular vortex

$$\partial \times V(x) = 2\pi \Phi^{(0)} \delta(x), \quad (2.11)$$

where $\Phi^{(0)}$ is the total flux (in $2\pi$ units) of the vortex – i.e., of the thread that pierces the plane $(x^1, x^2)$ at the origin. The wave function on the plane with the punctured singular point $x = 0$ obeys the most general condition (see [9] for more details)

$$< r, \varphi + 2\pi | = e^{2\pi \Upsilon} < r, \varphi | , \quad (2.12)$$

where $r = \sqrt{(x^1)^2 + (x^2)^2}$ and $\varphi = \arctan(x^2/x^1)$ are the polar coordinates, and $\Upsilon$ is a continuous real parameter which varies in the range $0 \leq \Upsilon < 1$. It can be shown (see, for example, [8, 9]) that $\Upsilon$ as well as $\Phi^{(0)}$ is changed under singular gauge transformations, whereas the difference $\Phi^{(0)} - \Upsilon$ remains invariant. Thus, physically sensible quantities depend on the gauge invariant combination $\Phi^{(0)} - \Upsilon$ which will be for brevity denoted as the reduced vortex flux in the following.

A solution to the Dirac equation (2.3) with Hamiltonian (2.5) in background (2.10), that obeys the condition (2.12), can be presented as

$$< x|E,n >= \begin{pmatrix} f_n(r,E)e^{i(n+\Upsilon)\varphi} \\ g_n(r,E)e^{i(n+\Upsilon+s)\varphi} \end{pmatrix}, \quad n \in \mathbb{Z}, \quad (2.13)$$

where $\mathbb{Z}$ is the set of integer numbers, and the radial functions $f_n$ and $g_n$ satisfy the system of equations

$$e^{-i\chi_s}[\partial_r + s(n - \Phi^{(0)} + \Upsilon)r^{-1}]f_n(r,E) = Eg_n(r,E),$$
$$e^{i\chi_s}[\partial_r + s(n - \Phi^{(0)} + \Upsilon + s)r^{-1}]g_n(r,E) = Ef_n(r,E). \quad (2.14)$$

When the reduced vortex flux $\Phi^{(0)} - \Upsilon$ is integer, the requirement of square integrability for the wave function (2.13) provides its regularity everywhere on the plane $(x^1, x^2)$, rendering partial Dirac Hamiltonians for every value of $n$ to be essentially self-adjoint. When $\Phi^{(0)} - \Upsilon$ is fractional, the same is valid only for $n \neq n_0$, where

$$n_0 = \|\Phi^{(0)} - \Upsilon\| + \frac{1}{2} - \frac{1}{2}s, \quad (2.15)$$

$\|u\|$ is the integer part of a quantity $u$ (i.e., the greatest integer that is less than or equal to $u$). For $n = n_0$, each of the two linearly independent solutions to system (2.14) meets the requirement of square integrability. Any particular solution in this case is characterized by at least one (at most both) of the radial functions being divergent as $r^{-p}$ ($p < 1$) for $r \to 0$. If one of the two linearly independent solutions is chosen to have a regular upper and an irregular lower component, then the other one has a regular lower and an irregular upper component. Therefore, contrary to the case of $n \neq n_0$, the partial Dirac Hamiltonian in the case of $n = n_0$
is not essentially self-adjoint \textsuperscript{[1]}. The Weyl - von Neumann theory of self-adjoint operators (see, e.g., Refs.\[16, 17\]) has to be employed in order to consider the possibility of a self-adjoint extension in the case of \(n = n_0\). It can be shown that the self-adjoint extension exists indeed and is parametrized by one continuous real variable denoted in the following by \(\Theta\). Thus, the partial Dirac Hamiltonian in the case of \(n = n_0\) is defined on the domain of functions obeying the condition

\[
\cos(s \frac{\Theta}{2} + \frac{\pi}{4}) \lim_{r \to 0} (\mu r)^F f_{n_0} = -e^{i\chi s} \sin(s \frac{\Theta}{2} + \frac{\pi}{4}) \lim_{r \to 0} (\mu r)^{1-F} g_{n_0},
\]  

(2.16)

where \(\mu > 0\) is the parameter of the dimension of inverse length and

\[
F = s\|\Phi^{(0)} - \Upsilon\| + \frac{1}{2} - F s,
\]  

(2.17)

\(\|u\| = u - [u]\) is the fractional part of a quantity \(u\), \(0 \leq \|u\| < 1\); note here that Eq.(2.16) implies that \(0 < F < 1\), since, for \(F = \frac{1}{2} - \frac{1}{2}s\), both \(f_{n_0}\) and \(g_{n_0}\) obey the condition of regularity at \(r = 0\). Note also that Eq.(2.16) is periodic in \(\Theta\) with the period of \(2\pi\); therefore, without a loss of generality, all permissible values of \(\Theta\) will be restricted in the following to the range \(-\pi \leq \Theta \leq \pi\).

All solutions to the Dirac equation in the background of a singular magnetic vortex correspond to the continuous spectrum and, therefore, obey the orthonormality condition

\[
\int d^2x < E, n|\mathbf{x} > < \mathbf{x}'|E', n' > = \frac{\delta(E - E')}{\sqrt{|EE'|}} \delta_{nn'}.
\]  

(2.18)

In the case of \(0 < F < 1\) one can get the following expressions corresponding to the regular solutions with \(sn > sn_0\):

\[
\begin{pmatrix}
  f_n \\
  g_n
\end{pmatrix}
= \frac{1}{2\sqrt{\pi}}
\begin{pmatrix}
  J_{l-F}(kr) e^{i\chi s} \\
  \text{sgn}(E) J_{l+1-F}(kr)
\end{pmatrix},
\]  

(2.19)

\(l = \text{s}(n - n_0)\),

the regular solutions with \(sn < sn_0\):

\[
\begin{pmatrix}
  f_n \\
  g_n
\end{pmatrix}
= \frac{1}{2\sqrt{\pi}}
\begin{pmatrix}
  J_{l+F}(kr) e^{i\chi s} \\
  -\text{sgn}(E) J_{l-1+F}(kr)
\end{pmatrix},
\]  

(2.20)

\(l' = \text{s}(n_0 - n)\),

and the irregular solution:

\[
\begin{pmatrix}
  f_{n_0} \\
  g_{n_0}
\end{pmatrix}
= \frac{1}{2\sqrt{\pi}[1 + \sin(2\nu_E) \cos(m\pi)]}
\times
\left(
\begin{pmatrix}
  [\sin(\nu_E)J_{-F}(kr) + \cos(\nu_E)J_F(kr)] e^{i\chi s} \\
  \text{sgn}(E)[\sin(\nu_E)J_{-1-F}(kr) - \cos(\nu_E)J_{1+F}(kr)]
\end{pmatrix}
\right);  
\]  

(2.21)

here \(k = |E|\), \(J_\rho(u)\) is the Bessel function of order \(\rho\) and

\[
\text{sgn}(u) = \begin{cases} 
  1, & u > 0 \\
  -1, & u < 0
\end{cases}.
\]  

\textsuperscript{1}A corollary of the theorem proven in Ref.\[15\] states that, for the partial Dirac Hamiltonian to be essentially self-adjoint, it is necessary and sufficient that a non-square-integrable solution exist.
Substituting the asymptotic form of Eq.(2.21) at \( r \to 0 \) into Eq.(2.16), one arrives at the relation between the parameters \( \nu_E \) and \( \Theta \):

\[
\tan(\nu_E) = \text{sgn}(E)\left(\frac{k}{2\mu}\right)^{2F-1} \frac{\Gamma(1-F)}{\Gamma(F)} \tan\left(s \frac{\Theta}{2} + \frac{\pi}{4}\right),
\]

(2.22)

where \( \Gamma(u) \) is the Euler gamma function.

Using an explicit form of solutions (2.19) – (2.21), all vacuum polarization effects can be determined.

3 Fermion Number

In the second-quantized theory in 2+1-dimensional space-time the operator of the fermion number is given by the expression

\[
\hat{N} = \int d^2 x \frac{1}{2} [\Psi^+(x,t), \Psi(x,t)] = \sum [a^{+}_\lambda a_\lambda - b^{+}_\lambda b_\lambda - \frac{1}{2}\text{sgn}(E_\lambda)],
\]

(3.1)

and, consequently, its vacuum expectation value takes the form

\[
\mathcal{N} \equiv \langle \text{vac}|\hat{N}|\text{vac}\rangle = -\frac{1}{2} \int d^2 x \text{tr} < x|\text{sgn}(H)|x > .
\]

(3.2)

From general arguments, one could expect that the last quantity vanishes due to cancellation between the contributions of positive and negative energy solutions to the Dirac equation (2.3). Namely this happens in a lot of cases. That is why every case of a nonvanishing value of \( \mathcal{N} \) deserves a special attention.

Considering the case of the background in the form of a singular magnetic vortex (2.10) – (2.11), one can notice that the contribution of the regular solutions (2.19) and (2.20) is cancelled upon summation over the sign of energy, whereas the irregular solution (2.21) yields a nonvanishing contribution to \( \mathcal{N} \) (3.2). Defining the vacuum fermion number density

\[
\mathcal{N}_x = -\frac{1}{2} \text{tr} < x|\text{sgn}(H)|x > ,
\]

(3.3)

we get

\[
\mathcal{N}_x = -\frac{1}{8\pi} \int dkk \left\{ A\left(\frac{k}{\mu}\right)^{2F-1} \left[ L_{(+)} + L_{(-)} \right] \left[ J_{2-F}(kr) + J_{2-F}(kr) \right] + J_{1-F}(kr) J_{F}(kr) - J_{1-F}(kr) J_{1-F}(kr) \right\} +
\]

\[
\left[ L_{(+)} - L_{(-)} \right] \left[ J_{F}(kr) J_{1-F}(kr) - J_{1-F}(kr) J_{F}(kr) \right] +
\]

\[
+ A^{-1}\left(\frac{k}{\mu}\right)^{1-2F} \left[ L_{(+)} + L_{(-)} \right] \left[ J_{F}(kr) + J_{2F}(kr) \right] \right\},
\]

(3.4)

where

\[
A = 2^{1-2F} \frac{\Gamma(1-F)}{\Gamma(F)} \tan \left( s \frac{\Theta}{2} + \frac{\pi}{4} \right),
\]

(3.5)

\[
L_{(\pm)} = 2^{-1}\{\cos(F\pi) \pm \cosh[(2F - 1) \ln(\frac{k}{\mu}) + \ln A] \}^{-1}.
\]

(3.6)
Transforming the integral in Eq.(3.4), we get the final expression

$$\mathcal{N}_x = \frac{-\sin(F\pi)}{2\pi^3 r^2} \int_0^\infty dw \frac{K_F^2(w) - K_{1-F}^2(w)}{\cosh((2F-1) \ln(\frac{w}{\mu r}) + \ln A)},$$

(3.7)

where $K_\rho(w)$ is the Macdonald function of order $\rho$. The vacuum fermion number density (3.7) vanishes at half integer values of the reduced vortex flux ($F = \frac{1}{2}$) as well as at $\cos \Theta = 0$. Otherwise, at large distances from the vortex we get

$$\mathcal{N}_{x_{r \to \infty}} = -(F - \frac{1}{2}) \frac{\sin(F\pi)}{2\pi^2 r^2} \left\{ \begin{array}{ll}
(\mu r)^{2F-1} A^{-1} \frac{\Gamma(\frac{3}{2} - F) \Gamma(\frac{3}{2} - 2F)}{\Gamma(2 - F)\Gamma(1 - F)}, & 0 < F < \frac{1}{2} \\
(\mu r)^{1-2F} A \frac{\Gamma(F + \frac{1}{2}) \Gamma(2F - \frac{1}{2})}{\Gamma(1 + F)}, & \frac{1}{2} < F < 1
\end{array} \right..$$

(3.8)

Integrating Eq. (3.7) over the plane ($x^1, x^2$), we obtain the total vacuum fermion number

$$\mathcal{N} = -\frac{1}{2} \text{sgn}_0 \left[ (F - \frac{1}{2}) \cos \Theta \right],$$

(3.9)

where

$$\text{sgn}_0(u) = \left\{ \begin{array}{ll}
\text{sgn}(u), & u \neq 0 \\
0, & u = 0
\end{array} \right..$$

4 Current

Let us regard the 2-dimensional space ($x^1, x^2$) as a 1+1-dimensional space-time with the Wick-rotated time axis. The Clifford algebra in this space-time has the exact irreducible representation with the above $\alpha$ matrices playing now the role of the $\gamma$ matrices and the $\beta$ matrix playing the role similar to that of the $\gamma^5$ matrix in a 3+1-dimensional space-time. Introducing the mass parameter $M$ to tame the infrared divergence, one can define the trace of two-point causal Green’s function with the $\alpha$ matrix inserted between the field operators:

$$J(x, x'|M) = <\text{vac}|T\Psi^+(x', 0)\alpha \Psi(x, 0)|\text{vac}>=
= \text{tr} <x|\alpha(H - iM)^{-1}|x'>,$$

(4.1)

where $T$ is the symbol of time ordering in 1+1-dimensional space-time. Inserting the damping factor $(E^2 + M^2)^{-z}$ (where Re $z > 0$) into the integral corresponding to Eq.(4.1), we define the matrix element

$$J(x, x'; z|M) = \text{tr} <x|\alpha(H + iM)(H^2 + M^2)^{-1-z}|x'>.$$

(4.2)

Returning to the massless theory in a 2+1-dimensional space-time ($x^1, x^2, t$), let us define the vacuum current in the conventional way (compare with Eqs.(3.1) – (3.3))

$$j(x) = <\text{vac}|\frac{1}{2} [\Psi^+(x, t), \alpha \Psi(x, t)]^-|\text{vac}>= -\frac{1}{2} \text{tr} <x|\alpha \text{sgn}(H)|x>.$$

(4.3)

One can notice the relation

$$j(x) = -\frac{1}{2} J(x, x; -\frac{1}{2}|0),$$. 

(4.4)
so that in the following the matrix element (4.2) in the coincidence limit \( x' = x \) will be regarded as a generalized current.

In the background of a singular magnetic vortex (2.10) – (2.11) the radial component

\[
J_r(x, x; z|M) = r^{-1}[x^1J_1(x, x; z|M) + x^2J_2(x, x; z|M)]
\]  

(4.5)

vanishes, whereas the angular component

\[
J_\varphi(x, x; z|M) = r^{-1}[x^1J_2(x, x; z|M) - x^2J_1(x, x; z|M)]
\]  

(4.6)

is nonvanishing. The contribution of the regular solutions (2.19) and (2.20) to Eq.(4.6) is given by the expression

\[
[J_\varphi(x, x; z|M)]_{\text{REG}} = \frac{s}{\pi} \int_0^\infty dk \frac{k^2}{(k^2 + M^2)^{1+z}} \left[ \sum_{l=1}^\infty J_{l-F}(kr)J_{l+F}(kr) - \sum_{l'=1}^\infty J_{l'+F}(kr)J_{l'-F}(kr) \right].
\]  

(4.7)

Performing the summation over \( l \) and \( l' \), we get

\[
[J_\varphi(x, x; z|M)]_{\text{REG}} = \frac{s}{\pi} \int_0^\infty dk \frac{k^2}{(k^2 + M^2)^{1+z}} \left\{ FJ_F(kr)J_{-F}(kr) - (1 - F)J_{1-F}(kr)J_{-F}(kr) \right\}
\]  

\[ - \frac{1}{2} kr[J_F^2(kr) + J_{-1+F}^2(kr) - J_{-F}^2(kr) - J_{1-F}^2(kr)] \right\}.
\]  

(4.8)

Transforming the integral in the last expression, we get

\[
J_\varphi(x, x; z|M)]_{\text{REG}} = \frac{s \sin(z\pi)}{\pi^2} \int_{|M|}^\infty dw \frac{w^2}{(w^2 - M^2)^{1+z}} \times
\]  

\[ \left\{ I_F(w)K_{1-F}(w) - I_{1-F}(w)K_F(w) + \frac{2 \sin(F\pi)}{\pi} [wK_F^2(w) - wK_{1-F}^2(w) - (2F - 1)K_F(w)K_{1-F}(w)] \right\},
\]  

(4.9)

where \( I_\rho(w) \) is the modified Bessel function of order \( \rho \). The contribution of the irregular solution (2.21) to Eq.(4.6) is given by the expression

\[
[J_\varphi(x, x; z|M)]_{\text{IRREG}} = \frac{s}{2\pi} \int_0^\infty dk \frac{k}{(k^2 + M^2)^{1+z}} \times
\]  

\[ \times \left\{ A \left( \frac{k}{\mu} \right)^{2F-1} \left[ (k + iM)L_{(+)} - (k - iM)L_{(-)} \right]J_{-F}(kr)J_{1-F}(kr) + [(k + iM)L_{(+)} + (k - iM)L_{(-)}] [J_F(kr)J_{1-F}(kr) - J_{-F}(kr)J_{1+F}(kr)] - \right\}
\]  

\[ - A^{-1} \left( \frac{k}{\mu} \right)^{1-2F} \left[ (k + iM)L_{(+)} - (k - iM)L_{(-)} \right] J_F(kr)J_{-1+F}(kr) \right\},
\]  

(4.10)
where $A$ and $L(\pm)$ are given by Eqs. (3.5) and (3.6), respectively. Transforming the integral in Eq. (4.10), we get
\[
[J_\varphi(x, x; z)|M]\text{IRREG} = \frac{s \sin(z\pi)}{\pi^2} \frac{1}{r^{1+2z}} \int_{|M|r}^\infty dw \frac{w^2}{(w^2 - M^2 r^2)^{1+z}} \times
\]
\[
\times \left( I_{1-F}(w)K_F(w) - I_F(w)K_{1-F}(w) - \frac{2\sin(F\pi)}{\pi}K_F(w)K_{1-F}(w) \times
\]
\[
\times \{\tanh[(2F-1)\ln(\frac{w}{\mu r}) + \ln A] - \frac{iMr}{w \cosh[(2F-1)\ln(\frac{w}{\mu r}) + \ln A]}\} \right). \tag{4.11}
\]
Summing Eqs. (4.9) and (4.11), we get
\[
J_\varphi(x, x; z)|M = \frac{2s \sin(F\pi)}{\pi^3} \frac{\sin(z\pi)}{\pi} \frac{1}{r^{1+2z}} \int_{|M|r}^\infty dw \frac{w^2}{(w^2 - M^2 r^2)^{1+z}} \times
\]
\[
\times \left( w[K_F^2(w) - K_{1-F}^2(w)] - K_F(w)K_{1-F}(w)\{2F-1 + \tanh[(2F-1)\ln(\frac{w}{\mu r}) + \ln A] - \frac{iMr}{w \cosh[(2F-1)\ln(\frac{w}{\mu r}) + \ln A]}\} \right). \tag{4.12}
\]
Note that Eqs.(4.9) and (4.11) are valid at $-\frac{1}{2} < \text{Re} z < 0$, and their sum, Eq.(4.12), can be continued analytically to the half plane $\text{Re} z < 0$. This may look somewhat embarrassing, since the initial motivation, as presented above, was to consider the case of $\text{Re} z > 0$. However, the situation can be cured by means of analytic continuation using partial integration. Namely, integrating Eq. (4.12) by parts, we get the expression
\[
J_\varphi(x, x; z)|M = \frac{s \sin(F\pi)}{\pi^2} \frac{\sin(z\pi)}{\pi} \frac{1}{r^{1+2z}} \int_{|M|r}^\infty dw (w^2 - M^2 r^2)^{-z} \times
\]
\[
\times \left[ w[K_F^2(w) - K_{1-F}^2(w)] + [K_F^2(w) + K_{1-F}^2(w)]\{w \tanh[(2F-1)\ln(\frac{w}{\mu r}) + \ln A]- \frac{iMr}{\cosh[(2F-1)\ln(\frac{w}{\mu r}) + \ln A]}\}ight.
\]
\[
- \frac{K_F(w)K_{1-F}(w)}{\cosh[(2F-1)\ln(\frac{w}{\mu r}) + \ln A]} \times
\]
\[
\times \left( \frac{2F-1}{\cosh[(2F-1)\ln(\frac{w}{\mu r}) + \ln A]} + \frac{iMr}{w} \{2F-1)\tanh[(2F-1)\ln(\frac{w}{\mu r}) + \ln A] + 1\} \right), \tag{4.13}
\]
which is continued analytically to the domain $\text{Re} z < 1$. Integrating Eq.(4.12) by parts $N$ times, one can get the expression for the generalized current which is continued analytically to the domain $\text{Re} z < N$.

Taking into account Eq.(4.4), we get the following expression for the vacuum current:
\[
J_\varphi(x) = \frac{s \sin(F\pi)}{\pi r^2} \left\{ \frac{(F - \frac{1}{2})^2}{4 \cos(F\pi)} - \frac{1}{\pi^2} \int_0^\infty dw w K_F(w)K_{1-F}(w) \times
\]
\[
\times \tanh[(2F-1)\ln(\frac{w}{\mu r}) + \ln A]\right\}; \tag{4.14}
\]
recall that the radial component $j_r(x)$ is vanishing, see Eq. (4.5). At $\cos \Theta = 0$ we get
\[
j_{\varphi}(x) = \frac{s \tan(F \pi)}{4 \pi r^2} (F - \frac{1}{2})(F - \frac{1}{2} \pm 1), \quad \Theta = \pm s \frac{\pi}{2}.
\] (4.15)

At half integer values of the reduced vortex flux ($F = \frac{1}{2}$), taking into account the relation
\[
A|_{F=\frac{1}{2}} = \tan(s \frac{\Theta}{2} + \frac{\pi}{4}),
\] (4.16)
we get
\[
j_{\varphi}(x)|_{F=\frac{1}{2}} = -\frac{\sin \Theta}{4 \pi r^2}.
\] (4.17)

If $\cos \Theta \neq 0$ and $F \neq \frac{1}{2}$, then at large distances from the vortex we get
\[
j_{\varphi}(x)_{r \to \infty} = \frac{s \tan(F \pi)}{4 \pi r^2} |F - \frac{1}{2}|(|F - \frac{1}{2}| - 1).
\] (4.18)

5 Parity Breaking Condensate

Since the twodimensional massless Dirac Hamiltonian (2.5) anticommutes with the $\beta$ matrix
\[
[H, \beta]_+ = 0,
\] (5.1)
the Dirac equation (2.3) is invariant under the parity transformation
\[
E_\lambda \to -E_\lambda, \quad <x|\lambda \to \beta <x|\lambda >.
\] (5.2)

However, this invariance is violated by the boundary condition (2.16), unless $\cos \Theta = 0$. Consequently, the parity breaking condensate emerges in the vacuum:
\[
C_x = <\text{vac}|\frac{1}{2}[^{\uparrow} \Psi(x, t), \beta \Psi(x, t)]_+|\text{vac}>=
\]
\[
= -\frac{1}{2} \text{tr} <x|\beta \text{sgn}(H)|x >.
\] (5.3)

Let us start with the regularized condensate
\[
C_x(z|M) = -\frac{1}{2} \text{tr} <x|\beta H(H^2 + M^2)^{-\frac{1}{2}-\frac{1}{2}}|x >.
\] (5.4)

The contribution of the regular solutions (2.19) and (2.20) to Eq. (5.4) is cancelled upon summation over the sign of energy. Thus, only the contribution of the irregular solution (2.21) to Eq. (5.4) survives:
\[
C_x(z|M) = -\frac{1}{8\pi} \int_0^\infty \frac{dk k^2}{(k^2 + M^2)^{\frac{1}{2}+\frac{1}{2}} - \frac{1}{2}} \left\{ A\left(\frac{k}{\mu}\right)^{2F-1} [L_+ + L_-][J_{2-F}(kr) - J_{1-F}(kr)] - 
\right.
\]
\[
- J_{1-F}(kr) + 2[L_+ - L_-][J_{-F}(kr)J_{F}(kr) + J_{1-F}(kr)J_{-1+F}(kr)] + 
\]
\[
+ A^{-1}\left(\frac{k}{\mu}\right)^{1-2F} [L_+ + L_-][J_{F}(kr) - J_{1+F}(kr)] \right\},
\] (5.5)
where $A$ and $L_{(\pm)}$ are given by Eqs. (3.5) and (3.6), respectively. Transforming the integral in Eq. (5.5), we get

$$C_\mathbf{x}(z|M) = -\frac{\sin(F\pi)}{2\pi^3} \cos(z\pi)r^{2(z-1)} \int_{|M|r}^\infty dw w^2 (w^2 - M^2 r^2)^{-\frac{3}{2}} \times$$

$$\times \frac{K_F^2(w) + K_{1-F}^2(w)}{\cosh((2F-1)\ln(\frac{w}{\mu r}) + \ln A)},$$

(5.6)

where $\text{Re} z < \frac{1}{2}$. Then the vacuum condensate (5.3) is given by the following expression:

$$C_\mathbf{x} \equiv C_\mathbf{x}(0|0) = -\frac{\sin(F\pi)}{2\pi^3 r^2} \int_{0}^\infty dw w \frac{K_F^2(w) + K_{1-F}^2(w)}{\cosh((2F-1)\ln(\frac{w}{\mu r}) + \ln A)}.$$  

(5.7)

Evidently, Eq. (5.7) vanishes if $\cos \Theta = 0$. At half integer values of the reduced vortex flux ($F = \frac{1}{2}$), we get

$$C_\mathbf{x}|_{F=\frac{1}{2}} = -\frac{\cos \Theta}{4\pi^2 r^2}.$$  

(5.8)

At large distances from the vortex we get

$$C_\mathbf{x}_{x^\to\infty} = -\frac{\sin(F\pi)}{2\pi^3 r^2} \left\{ \begin{array}{ll}
(\mu r)^{2F-1} A^{-1} & \Gamma(\frac{3}{2} - F) \Gamma(\frac{3}{2} - 2F) \\
(\mu r)^{1-2F} A & \Gamma(F + \frac{1}{2}) \Gamma(2F - \frac{1}{2})
\end{array} \right\}, \quad 0 < F < \frac{1}{2}$$

(5.9)

$$1 \leq F < 1$$

Integrating Eq. (5.7) over the plane $(x^1, x^2)$, we obtain the total vacuum condensate

$$\mathcal{C} \equiv \int d^2 x C_\mathbf{x} = -\frac{\text{sgn}(\cos \Theta)}{4|F - \frac{1}{2}|}.$$  

(5.10)

Thus, the total vacuum condensate is infinite at $F = \frac{1}{2}$ if $\cos \Theta \neq 0$.

### 6 Absence of Parity Anomaly

Let us define the generalized twodimensional axial current

$$J^3(x, x; z|M) = \text{itr} < x|\alpha(\beta H + iM)(H^2 + M^2)^{-1-z}|x >.$$  

(6.1)

Owing to the twodimensionality, the components of the current (6.1) are related to the components of the generalized current considered in Section 4 (Eq. (4.2) at $x' = x$)

$$J^3_\varphi = sJ_\varphi, \quad J^3_\varphi = -sJ_r.$$  

(6.2)

In the background of a singular magnetic vortex (2.10) – (2.11), the divergence of the current $\mathbf{J}$ is vanishing (see Eq. (4.12))

$$\nabla \cdot J(x, x; z|M) = 0,$$  

(6.3)
whereas the divergence of the current $J^3$ is nonvanishing, owing to the relation
\[ \partial \cdot J^3(x, x; z|M) = s \partial \times J(x, x; z|M). \] (6.4)

Let us consider the effective action in the Euclidean 1+1-dimensional space-time
\[ S_{\text{eff}}^{(1+1)}[V(x)] = -\int d^2x \text{tr} <x| \ln(H\tilde{M}^{-1})|x>, \] (6.5)
where $\tilde{M}$ is the parameter of the dimension of mass. The invariance of action (6.5) under the gauge transformation,
\[ V(x) \to V(x) + \partial \Lambda(x), \quad <x| \to e^{i\Lambda(x)} <x|, \quad |x| \to e^{-i\Lambda(x)} |x|, \] (6.6)
is stipulated by the conservation law
\[ \lim_{M \to 0} \lim_{z \to 0} \partial \cdot J^3(x, x; z|M) = 0. \] (6.7)

If the conservation law
\[ \lim_{M \to 0} \lim_{z \to 0} \partial \cdot J^3(x, x; z|M) = 0 \] (6.8)
holds, then action (6.5) is invariant under the localized version of the parity transformation (compare with Eq. (5.2))
\[ V(x) \to V(x) + \partial \beta \Lambda(x), \quad <x| \to e^{i\beta \Lambda(x)} <x|, \quad |x| \to |x| e^{i\beta \Lambda(x)}. \] (6.9)
The breakdown of the latter symmetry,
\[ \lim_{M \to 0} \lim_{z \to 0} \partial \cdot J^3(x, x; z|M) \neq 0, \] (6.10)
is denoted as a parity anomaly, i.e., the axial anomaly in 1+1-dimensional space-time.

Note that in classical theory both gauge and localized parity symmetries are conserved. This is reflected by the formal invariance of action (6.5) under both transformations (6.6) and (6.9). However, in quantum theory, in order to calculate the effective action in a certain background, one has to use regularization which in fact breaks the localized parity symmetry. Namely, one has to substitute $\ln[(H - iM)\tilde{M}^{-1}]$ for $\ln(H\tilde{M}^{-1})$ into Eq. (6.5), where the regulator mass $M$ is the symmetry breaking parameter. That is why the generalized currents (4.2) at $x' = x$ and (6.1) come into play. The divergence of the latter current can be presented in the form
\[ \partial \cdot J^3(x, x; z|M) = 2\tilde{\zeta}_x(z|M) - 2M^2 \tilde{\zeta}_x(z + 1|M) - \\
-4iMC_x(z + \frac{1}{2}|M), \] (6.11)
where $C_x(z|M)$ is the regularized condensate (5.4) and
\[ \tilde{\zeta}_x(z|M) = \text{tr} <x|\beta(H^2 + M^2)^{-z}|x> \] (6.12)
is the modified (by insertion of the $\beta$ matrix) zeta function density.
In the background of a singular magnetic vortex the conservation law (6.7) holds, as a consequence of Eq. (6.3). Incidentally, the regularized condensate is given by Eq. (5.6). As to the modified zeta function density, the following expression can be obtained similarly to the above:

\[ \tilde{\zeta}_x(z|M) = \frac{\sin(F\pi)}{\pi^3} \sin(z\pi)r^{2(z-1)} \int_{|M|r}^{\infty} dw \, (w^2 - M^2r^2)^{-z} \times \]

\[ \times \left\{ K_F^2(w) - K_{1-F}^2(w) + [K_F^2(w) + K_{1-F}^2(w)] \tanh[(2F - 1) \ln\left(\frac{w}{\mu r}\right) + \ln A] \right\}, \quad (6.13) \]

where \( \text{Re} \, z < 1 \). Extending the domain of definition in \( z \) in Eqs. (5.6) and (6.13) by means of integration by parts, we get

\[ \tilde{C}_x(z|M) = -\frac{\sin(F\pi)}{\pi^3} \frac{\cos(z\pi)}{1 - 2z} r^{2(z-1)} \int_{|M|r}^{\infty} \frac{dw}{w} \frac{w^2 - M^2r^2)^{\frac{1}{2}-z}}{\cosh[(2F - 1) \ln\left(\frac{w}{\mu r}\right) + \ln A]} \times \]

\[ \times \left\{ (F - \frac{1}{2})[K_F^2(w) - K_{1-F}^2(w)] + 2wK_F(w)K_{1-F}(w) + (F - \frac{1}{2})[K_F^2(w) + K_{1-F}^2(w)] \tanh[(2F - 1) \ln\left(\frac{w}{\mu r}\right) + \ln A] \right\}, \quad (6.14) \]

where \( \text{Re} \, z < \frac{3}{2} \) and

\[ \tilde{\zeta}_x(z|M) = \frac{\sin(F\pi)}{\pi^3} \frac{\sin(z\pi)}{1 - z} r^{2(z-1)} \int_{|M|r}^{\infty} \frac{dw}{w} \frac{w^2 - M^2r^2}{}^{1-z} \times \]

\[ \times \left\{ \frac{1}{2}[K_F^2(w) - K_{1-F}^2(w)] + [FK_F^2(w) + (1 - F)K_{1-F}^2 + 2wK_F(w)K_{1-F}(w)] \right\} \times \]

\[ \times \tanh[(2F - 1) \ln\left(\frac{w}{\mu r}\right) + \ln A](F - \frac{1}{2})[K_F^2(w) + K_{1-F}^2(w)] \tanh[(2F - 1) \ln\left(\frac{w}{\mu r}\right) + \ln A] \right\}, \quad (6.15) \]

where \( \text{Re} \, z < 2 \). Using the last representations, we get

\[ \lim_{M \to 0} M\tilde{C}_x(z + \frac{1}{2}|M) = \lim_{M \to 0} M^2\tilde{\zeta}_x(z + 1|M) = 0, \quad \text{Re} \, z < 1, \quad (6.16) \]

and, consequently,

\[ \lim_{M \to 0} \partial \cdot J^3(x, x; z|M) = 2\tilde{\zeta}_x(z|0), \quad \text{Re} \, z < 1. \quad (6.17) \]

One can easily get

\[ \tilde{\zeta}_x(z|0) = \frac{\sin(F\pi)}{\pi^3} \sin(z\pi)r^{2(z-1)} \left\{ \sqrt{\frac{2}{3}} \frac{\Gamma(1 - z)}{\Gamma((\frac{3}{2}) - z)} (F - \frac{1}{2})\Gamma(F - z)\Gamma(1 - F - z) + \right\}

\[ + \int_0^{\infty} dw \, w^{1-2z} [K_F^2(w) + K_{1-F}^2(w)] \tanh[(2F - 1) \ln\left(\frac{w}{\mu r}\right) + \ln A] \right\}; \quad (6.18) \]

in particular, at half integer values of the reduced vortex flux:

\[ \tilde{\zeta}_x(z|0)|_{F = \frac{1}{2}} = \frac{s \sin \Theta}{2\pi^\frac{3}{2}} \frac{\Gamma((\frac{3}{2}) - z)}{\Gamma(z)} r^{2(z-1)}, \quad (6.19) \]
and for \( \cos \Theta = 0 \):
\[
\tilde{\zeta}_x(z|0) = \pm \frac{\sin(F\pi)}{2\pi^{\frac{3}{2}}} \frac{\Gamma(\frac{3}{2} - z \pm F \mp \frac{1}{2})\Gamma(\frac{1}{2} - z \mp F \pm \frac{1}{2})}{\Gamma(z)\Gamma(\frac{3}{2} - z)} r^{2(z-1)}, \quad \Theta = \pm \frac{\pi}{2}.
\] (6.20)

Consequently, we obtain
\[
\tilde{\zeta}_x(0|0) = 0, \quad x \neq 0.
\] (6.21)

Thus, the anomaly is absent everywhere on the plane with the puncture at \( x = 0 \). This looks rather natural, since the twodimensional anomaly density \( 2\tilde{\zeta}_x(0|0) \) is usually identified with the quantity \( \frac{1}{\pi} \partial \times V(x) \) \[18, 19, 20\], and the last quantity in the present case vanishes everywhere on the punctured plane, see Eq.(2.11). We see that the natural expectations are confirmed, provided that the boundary conditions at the puncture are chosen to be physically acceptable, i.e., compatible with the self-adjointness of the Hamiltonian \[\] \[\] \[\] \[\]; we conclude that the leakage of the anomaly, unlike that of the vacuum condensate or of the vacuum fermion number, does not happen.

We might finish here the discussion of the anomaly problem in the background of a singular magnetic vortex. However, there remains a purely academic question: what is the anomaly density in background (2.10) – (2.11) on the whole plane (without puncturing \( x = 0 \))? Just due to a confusion persisting in the literature \[21, 22\], we shall waste now some time to clarify this, otherwise inessential, point.

The background field strength (2.11), when considered on the whole plane, is interpreted in the sense of a distribution (generalized function), i.e., a functional on a set of suitable test functions \( f(x) \):
\[
\int d^2x \frac{1}{\pi} \partial \times V(x) f(x) = 2\Phi^{(0)} f(0); \quad (6.22)
\]
here \( f(x) \) is a continuous function. In particular, choosing \( f(x) = 1 \), one gets
\[
\int d^2x \frac{1}{\pi} \partial \times V(x) = 2\Phi^{(0)}. \quad (6.23)
\]
Considering the anomaly density on the whole plane, one is led to study different limiting procedures as \( r \to 0 \) and \( z \to 0 \) in Eq.(6.18). So, the notorious question is, whether the anomaly density \( 2\tilde{\zeta}_x \) can be interpreted in the sense of a distribution which coincides with the distribution \( \frac{1}{\pi} \partial \times V(x) \)? The answer is resolutely negative, and this will be immediately demonstrated below.

First, using the explicit form (6.18), we get
\[
\int d^2x 2\tilde{\zeta}_x(z|0) = \begin{cases} \infty, & z \neq 0 \\ 0, & z = 0 \end{cases} \quad (6.24)
\]
therefore, the anomaly functional cannot be defined on the same set of test functions as that used in Eq.(6.22) (for example, the test functions have to decrease rapidly enough at large (small) distances in the case of \( z > 0 \) (\( z < 0 \))). Moreover, if one neglects the requirement of self-consistency, permitting a different set of test functions for the anomaly functional, then even this will not save the situation. Let us use the test functions which depend on \( z \) and are adjusted in such a way that the quantity
\[
\mathcal{A} = \lim_{z \to 0} \int d^2x 2\tilde{\zeta}_x(z|0) f(x; z) \quad (6.25)
\]
is finite. Certainly, this quantity can take values in a rather wide range, but it cannot be made equal to the right-hand side of Eq. (6.23). Really, the only source of the dependence on $\Phi^{(0)}$ in the integral in Eq. (6.25) is the factor $\hat{\zeta}(z|\Phi^{(0)})$, and the latter, as is evident from Eq. (6.18), depends rather on $\{\Phi^{(0)}\}$, than on $\Phi^{(0)}$ itself, thus forbidding the linear dependence of $\mathcal{A}$ on $\Phi^{(0)}$. In particular, let us choose the test function $f(x; z)$ with the asymptotics at small distances:

$$f(x; z) = \left[1 + (\bar{\mu}r)^{-2}\right]^{z-1}e^{-2\bar{\mu}r},$$

where $\bar{\mu}$ is the parameter of dimension of mass, and the asymptotics at large distances providing the vanishing of the integral in Eq. (6.25) at the upper limit. Choosing the case of $\cos \Theta = 0$ for simplicity and taking into account Eq. (6.20), we get

$$A = -2(F - \frac{1}{2} \pm \frac{1}{2}), \quad \Theta = \pm \frac{s\pi}{2},$$

which differs clearly from $2\Phi^{(0)}$.

Thus, in a singular background the conventional relation between the anomaly density and the background field strength is valid only in the space with punctured singularities. If the singularities are not punctured, then the anomaly density and the background field strength can be interpreted in the sense of distributions, but, contrary to the assertion of the authors of Refs. [21, 22], the conventional relation is not valid.

### 7 Angular Momentum

Let $\hat{M}$ be an operator in the first-quantized theory, which commutes with the Dirac Hamiltonian

$$[\hat{M}, H] = 0.$$  

(7.1)

Then, in the second-quantized theory, the vacuum expectation value of the dynamical variable corresponding to $\hat{M}$ is presented in the form

$$\mathcal{M} = \int d^2x \mathcal{M}_x,$$

(7.2)

where

$$\mathcal{M}_x = \langle \text{vac} | \frac{1}{2} [\Psi^\dagger(x, t), \hat{M} \Psi(x, t)]_\text{vac} \rangle = -\frac{1}{2} \text{tr} x |\hat{M}\text{ sgn}(H)|x\rangle.$$  

(7.3)

The commutation relation (7.1) is the evidence of invariance of the theory with $\hat{M}$ being the generator of the symmetry transformations. Since, in the background of a singular magnetic vortex (2.10) – (2.11), there is invariance with respect to rotations around the location of the vortex, one can take $\hat{M}$ as the generator of rotations – the operator of angular momentum in the first-quantized theory (see [3] for more details):

$$\hat{M} = -ix \times \partial - \gamma + \frac{1}{2}s\beta.$$  

(7.4)

Note that the eigenvalues of the operator $\hat{M}$ (7.4) on spinor functions satisfying condition (2.12) are half integer.
Decomposing Eq.(7.4) into the orbital and spin parts, we get in the second-quantized theory
\[ \mathcal{M}_x = \mathcal{L}_x + \mathcal{S}_x, \] (7.5)
where
\[ \mathcal{L}_x = \frac{1}{2} \text{tr} < \mathbf{x} | (i \mathbf{x} \times \partial + \Upsilon) \text{sgn}(H) | \mathbf{x} > \] (7.6)
and
\[ \mathcal{S}_x = -\frac{1}{4} s \text{tr} < \mathbf{x} | \beta \text{sgn}(H) | \mathbf{x} > . \] (7.7)
Since the vacuum spin density (7.7) is related to the vacuum condensate (5.3),
\[ \mathcal{S}_x = \frac{1}{2} s C_x, \] (7.8)
there remains only the vacuum orbital angular momentum density (7.6) to be considered.

Let us start, as in Section 5, with the regularized quantity
\[ \mathcal{L}_x(z|M) = \frac{1}{2} \text{tr} < \mathbf{x} | (i \mathbf{x} \times \partial + \Upsilon) H (H^2 + M^2)^{-\frac{1}{2}-z} | \mathbf{x} > . \] (7.9)
The contribution of the regular solutions (2.19) and (2.20) to Eq.(7.9) is cancelled upon summation over the sign of energy, whereas the contribution of the irregular solution (2.21) to Eq. (7.9) survives
\[ \mathcal{L}_x(z|M) = \frac{1}{8\pi} \int_0^\infty \frac{dk k^2}{(k^2 + M^2)^{1+z}} \left\{ A \left( \frac{k}{\mu} \right)^{2F-1} [L_+(+) + L_-(\text{-})] [n_0 J^2_{-F} (kr) + 
+ (n_0 + s) J^2_{1-F} (kr)] + 2 [L_+(+) - L_-(\text{-})] [n_0 J_{-F} (kr) J_F (kr) - 
- (n_0 + s) J_{1-F} (kr) J_{1+F} (kr)] + A^{-1} \left( \frac{k}{\mu} \right)^{1-2F} [L_+(+) + L_-(\text{-})] [n_0 J^2_F (kr) + 
+ (n_0 + s) J^2_{-1+F} (kr)] \} , \] (7.10)
where \( A \) and \( L_\pm \) are given by Eqs.(3.5) and (3.6). Transforming the integral in Eq.(7.10), we get
\[ \mathcal{L}_x(z|M) = -\frac{\sin(F\pi)}{2\pi^3} \cos(z\pi)r^{2(z-1)} \int_{|M|r}^\infty dw \frac{w^2 (w^2 - M^2 r^2)^{-\frac{1}{2}-z} \times 
\times n_0 K^2_F (w) - (n_0 + s) K^2_{1-F} (w)}{\cosh[(2F - 1) \ln \left( \frac{w}{\mu r} \right) + \ln A]}, \] (7.11)
where Re \( z < \frac{1}{2} \). Then we get
\[ \mathcal{L}_x \equiv \mathcal{L}_x(0|0) = -\frac{\sin(F\pi)}{2\pi^3 r^2} \int_0^\infty dw \frac{n_0 K^2_F (w) - (n_0 + s) K^2_{1-F} (w)}{\cosh[(2F - 1) \ln \left( \frac{w}{\mu r} \right) + \ln A]}. \] (7.12)
Summing Eqs.(7.12) and (7.8), taking into account Eqs.(5.7) and (2.15), we obtain the following expression for the vacuum angular momentum density in the background of a singular magnetic vortex (2.10) – (2.11):

\[ \mathcal{M}_x = (\Phi^{(0)} - \Upsilon) + \frac{1}{2} \mathcal{N}_x, \]  

(7.13)

where the vacuum fermion number density \( \mathcal{N}_x \) is given by Eq.(3.7). Thus, the total vacuum angular momentum takes the form (see Eq.(3.9))

\[ \mathcal{M} = -\frac{1}{2} (\Phi^{(0)} - \Upsilon) + \frac{1}{2} s \beta. \]  

(7.14)

Concluding this section, let us note that relation (7.1) remains to be valid if a constant is added to the operator \( \hat{M} \). Thus, a definition which is alternative to Eq.(7.4) has been proposed for the angular momentum in the first-quantized theory [24]:

\[ \hat{M}' = -i x \times \partial - \Phi^{(0)} + \frac{1}{2} s \beta. \]  

(7.15)

Then, in the second-quantized theory, we get

\[ \mathcal{M}'_x = -s (F - \frac{1}{2}). \mathcal{N}_x \]  

(7.16)

and

\[ \mathcal{M}' = \frac{1}{2} s |F - \frac{1}{2}| \text{sgn}_0(\cos \Theta). \]  

(7.17)

Various arguments pro and contra the physical meaningfulness of the operator \( \hat{M}' \) (7.15) are known in the literature (see Refs. [23, 24, 25]). However, the crucial point is that the operator \( \hat{M} \) is the generator of rotations, while the operator \( \hat{M}' \) is not (the eigenvalues of operator \( \hat{M}' \) on spinor functions are not half integer).

8 Conclusion

We have determined the effects of polarization of the massless fermionic vacuum by a singular magnetic vortex in 2+1-dimensional space-time. If the quantized massless fermion field belongs to an irreducible representation of the Clifford algebra, then fermion number, current, parity breaking condensate, spin and angular momentum are induced in the vacuum. We have demonstrated that the parity anomaly is not induced in the vacuum.

All boundary conditions at the location of the vortex provide the Dirac Hamiltonian to be self-adjoint. The condition \( \cos \Theta = 0 \) is distinguished, since it corresponds to one of the two components of a solution to the Dirac equation being regular for all \( n \); if \( \Theta = \frac{s \pi}{2} \), then the lower components are regular, and, if \( \Theta = -\frac{s \pi}{2} \), then the upper components are regular. For this boundary condition the vacuum current only is nonvanishing, while all other vacuum polarization effects are vanishing. At half integer values of the reduced vortex flux the vacuum current and condensate are nonvanishing, the total condensate being infinite unless \( \cos \Theta = 0 \), while other vacuum polarization effects are vanishing. Note also that the vacuum fermion number, spin and angular momentum change their sign, while the vacuum current and condensate remain unchanged if one goes over to the inequivalent representation of the Clifford algebra.
Finally, let us consider the case of a quantized massless fermion field belonging to the reducible representation composed as a direct sum of two inequivalent irreducible representations. It follows immediately from the above that, in the last case, the vacuum current and condensate only are induced in the vacuum. However, now the vacuum condensate breaks chiral symmetry rather than parity. It should be noted that chiral symmetry breaking in the background of regular configurations of an external magnetic field has been extensively discussed in the literature \[23,27\]. One concludes that chiral symmetry breaking occurs also in the background of a singular configuration of an external magnetic field, as a result of leakage from the point of singularity.

The research was supported by the State Foundation for Fundamental Research of Ukraine (project 2.4/320) and the Swiss National Science Foundation (grant CEEC/NIS/96-98/7 IP 051219).

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