PROPERTIES OF BIHARMONIC SUBMANIFOLDS IN SPHERES

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ABSTRACT. In the present paper we survey the most recent classification results for proper biharmonic submanifolds in unit Euclidean spheres. We also obtain some new results concerning geometric properties of proper biharmonic constant mean curvature submanifolds in spheres.

1. Introduction

Biharmonic maps \( \phi : (M, g) \to (N, h) \) between Riemannian manifolds are critical points of the bienergy functional

\[
E_2(\phi) = \frac{1}{2} \int_M |\tau(\phi)|^2 \nu_g,
\]

where \( \tau(\phi) = \text{trace } \nabla d\phi \) is the tension field of \( \phi \) that vanishes for harmonic maps (see [17]). The Euler-Lagrange equation corresponding to \( E_2 \) is given by the vanishing of the bitension field

\[
\tau_2(\phi) = -J^\phi(\tau(\phi)) = -\Delta \tau(\phi) - \text{trace } R^N(d\phi, \tau(\phi))d\phi,
\]

where \( J^\phi \) is formally the Jacobi operator of \( \phi \) (see [24]). The operator \( J^\phi \) is linear, thus any harmonic map is biharmonic. We call proper biharmonic the non-harmonic biharmonic maps. Geometric and analytic properties of proper biharmonic maps were studied, for example, in [2, 25, 27].

The submanifolds with non-harmonic (non-minimal) biharmonic inclusion map are called proper biharmonic submanifolds. Initially encouraged by the non-existence results for proper biharmonic submanifolds in non-positively curved space forms (see, for example, [8, 13, 16, 21]), the study of proper biharmonic submanifolds in spheres constitutes an important research direction in the theory of proper biharmonic submanifolds.

The present paper is organized as follows.

Section 2 is devoted to the main examples of proper biharmonic submanifolds in spheres and to their geometric properties, mainly regarding the type and the order in the sense of Chen. Also, it gathers the most recent classification results for such submanifolds (for detailed proofs see [3]).

In Section 3 we present a series of new results concerning geometric properties of proper biharmonic constant mean curvature submanifolds in spheres. We begin with some identities which hold for proper biharmonic submanifolds with parallel mean curvature vector field (Propositions 3.1 and 3.2). We then obtain some necessary conditions that must be fulfilled by proper biharmonic constant mean curvature submanifolds (Corollary 3.5), and we end this section with a refinement, for hypersurfaces, of a result on the estimate of the mean curvature of proper biharmonic submanifolds in spheres (Theorem 3.7).

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The fourth section presents two open problems concerning the classification of proper biharmonic hypersurfaces and the mean curvature of proper biharmonic submanifolds in spheres.

In the last section we briefly present an interesting link between proper biharmonic hypersurfaces and $\mathcal{H}$-minimal hypersurfaces in spheres.

Other results on proper biharmonic submanifolds in spaces of non-constant sectional curvature can be found, for example, in [10, 15, 18, 19, 22, 30, 31].

2. Proper biharmonic submanifolds in spheres

The attempt to obtain classification results for proper biharmonic submanifolds in spheres was initiated with the following characterization theorem.

**Theorem 2.1** ([28]). (i) The canonical inclusion $\varphi : M^m \to S^n$ of a submanifold $M$ in an $n$-dimensional unit Euclidean sphere is biharmonic if and only if

\[
\begin{cases}
\Delta H + \text{trace } B(\cdot, A_H(\cdot)) - mH = 0 \\
4 \text{trace } A_{\nabla H}(\cdot) + m \text{grad}(|H|^2) = 0,
\end{cases}
\]

where $A$ denotes the Weingarten operator, $B$ the second fundamental form, $H$ the mean curvature vector field, $\nabla$ and $\Delta$ the connection and the Laplacian in the normal bundle of $M$ in $S^n$, and grad denotes the gradient on $M$.

If $M$ is a submanifold with parallel mean curvature vector field in $S^n$, then $M$ is biharmonic if and only if $\text{trace } B(\cdot, A_H(\cdot)) = mH$.

(ii) A hypersurface $M$ with nowhere zero mean curvature vector field in $S^{m+1}$ is biharmonic if and only if

\[
\begin{cases}
\Delta H - (m - |A|^2)H = 0 \\
2A(\text{grad}(|H|)) + m|H| \text{grad}(|H|) = 0.
\end{cases}
\]

If $M$ is a non-zero constant mean curvature hypersurface in $S^{m+1}$, then $M$ is proper biharmonic if and only if $|A|^2 = m$.

We note that the compact minimal, i.e. $H = 0$, hypersurfaces with $|A|^2 = m$ in $S^{m+1}$ are the Clifford tori $S^k(\sqrt{k/m}) \times S^{m-k}(\sqrt{(m-k)/m})$, $1 \leq k \leq m-1$ (see [14]).

Before presenting the basic examples of proper biharmonic hypersurfaces in spheres, together with some of their geometric properties, we recall the following definition (see, for example, [12]), which shall be used throughout the paper.

**Definition 2.2.** An isometric immersion of a compact manifold $M$ in $\mathbb{R}^n$, $\varphi : M \to \mathbb{R}^n$, is called of $k$-type if its spectral decomposition contains exactly $k$ non-zero terms, excepting the center of mass $\varphi_0 = \frac{1}{\text{vol}(M)} \int_M \varphi v_g$. More precisely,

$$
\varphi = \varphi_0 + \sum_{t=p}^q \varphi_t,
$$

where $\Delta \varphi_t = \lambda_t \varphi_t$ and $0 < \lambda_1 < \lambda_2 < \cdots \uparrow \infty$.

The pair $[p, q]$ is called the order of the immersion $\varphi : M \to \mathbb{R}^n$. 

2.1. The main examples of proper biharmonic submanifolds in spheres.

The hypersphere $S^m(1/\sqrt{2}) \subset S^{m+1}$.

Consider $S^m(a) = \left\{(x^1, \ldots, x^m, x^{m+1}, b) \in \mathbb{R}^{m+2} : |x| = a\right\} \subset S^{m+1}$, where $a^2 + b^2 = 1$. If $H$ is the mean curvature vector field of $S^m(a)$ in $S^{m+1}$, one gets $\nabla^2 H = 0$, $|H| = \frac{|b|}{a}$ and $|A|^2 = m \frac{b^2}{a^2}$.

Theorem 2.1 implies that $S^m(a)$ is proper biharmonic in $S^{m+1}$ if and only if $a = 1/\sqrt{2}$ (see [9]).

The generalized Clifford torus $S^{m_1}(1/\sqrt{2}) \times S^{m_2}(1/\sqrt{2}) \subset S^{m+1}$.

Consider $M = S^{m_1}(1/\sqrt{2}) \times S^{m_2}(1/\sqrt{2})$, $m_1 + m_2 = m$, $m_1 \neq m_2$, was the first example of proper biharmonic submanifold in $S^{m+1}$ (see [24]).

Inspired by these basic examples, two methods for constructing proper biharmonic submanifolds of codimension higher than one in $S^n$ were given.

**Theorem 2.3** ([8]). Let $M$ be a minimal submanifold of $S^{n-1}(a) \subset S^n$. Then $M$ is proper biharmonic in $S^n$ if and only if $a = 1/\sqrt{2}$.

**Remark 2.4.** (i) This result, called the composition property, proved to be quite useful for the construction of proper biharmonic submanifolds in spheres. For instance, it implies the existence of closed orientable embedded proper biharmonic surfaces of arbitrary genus in $S^4$ (see [8]).

(ii) All minimal submanifolds of $S^{n-1}(1/\sqrt{2}) \subset S^n$ are pseudo-umbilical, i.e. $A_H = |H|^2 \text{Id}$, with parallel mean curvature vector field in $S^n$ and $|H| = 1$.

(iii) Denote by $\phi : S^m(1/\sqrt{2}) \to S^{m+1}$ the inclusion of $S^m(1/\sqrt{2})$ in $S^{m+1}$ and by $i : S^{m+1} \to \mathbb{R}^{m+2}$ the canonical inclusion. Let $\varphi : S^m(1/\sqrt{2}) \to \mathbb{R}^{m+2}$, $\varphi = i \circ \phi$, be the inclusion of $S^m(1/\sqrt{2})$ in $\mathbb{R}^{m+2}$. Then

$$\varphi = \varphi_0 + \varphi_p,$$

where $\varphi_0, \varphi_p : S^m(1/\sqrt{2}) \to \mathbb{R}^{m+2}$, $\varphi_0(x, 1/\sqrt{2}) = (0, 1/\sqrt{2})$, $\varphi_p(x, 1/\sqrt{2}) = (x, 0)$ and $\Delta \varphi_p = 2m \varphi_p$.

Thus $S^m(1/\sqrt{2})$ is a 1-type submanifold of $\mathbb{R}^{m+2}$ with center of mass in $\varphi_0 = (0, 1/\sqrt{2})$ and eigenvalue $\lambda_p = 2m$, which is the first eigenvalue of the Laplacian on $S^m(1/\sqrt{2})$, i.e. $p = 1$.

Moreover, it is not difficult to verify that all minimal submanifolds in $S^m(1/\sqrt{2}) \subset S^{m+1}$, as submanifolds in $\mathbb{R}^{m+2}$, have the spectral decomposition given by (2.3).

Non pseudo-umbilical examples were also produced by proving the following product composition property.

**Theorem 2.5** ([8]). Let $M_1^{n_1}$ and $M_2^{n_2}$ be two minimal submanifolds of $S^{n_1}(a_1)$ and $S^{n_2}(a_2)$, respectively, where $n_1 + n_2 = n - 1$, $a_1^2 + a_2^2 = 1$. Then $M_1 \times M_2$ is proper biharmonic in $S^n$ if and only if $a_1 = a_2 = 1/\sqrt{2}$ and $m_1 \neq m_2$. 

Remark 2.6. (i) The proper biharmonic submanifolds of $S^n$ constructed as above are not pseudo-umbilical, but they still have parallel mean curvature vector field, thus constant mean curvature, and $|H| = \frac{|m_2 - m_1|}{m_1 + m_2} \in (0, 1)$.

(ii) Let $\varphi : S^{m_1}(1/\sqrt{2}) \times S^{m_2}(1/\sqrt{2}) \to \mathbb{R}^{m+2}$ be the inclusion of $S^{m_1}(1/\sqrt{2}) \times S^{m_2}(1/\sqrt{2})$ in $\mathbb{R}^{m+2}$, $m_1 < m_2$, $m_1 + m_2 = m$. Then

(2.4) $\varphi = \varphi_p + \varphi_q$,

where $\varphi_p, \varphi_q : S^{m_1}(1/\sqrt{2}) \times S^{m_2}(1/\sqrt{2}) \to \mathbb{R}^{m+2}, \varphi_p(x, y) = (x, 0), \varphi_q(x, y) = (0, y)$ and $\Delta \varphi_p = 2m_1 \varphi_p, \Delta \varphi_q = 2m_2 \varphi_q$.

Thus $S^{m_1}(1/\sqrt{2}) \times S^{m_2}(1/\sqrt{2})$ is a 2-type submanifold of $\mathbb{R}^{m+2}$ with eigenvalues $\lambda_p = 2m_1$ and $\lambda_q = 2m_2$, and it is mass-symmetric, i.e. it has center of mass in the origin.

Since the eigenvalues of the torus $S^{m_1}(1/\sqrt{2}) \times S^{m_2}(1/\sqrt{2})$ are obtained as the sum of eigenvalues of the spheres $S^{m_1}(1/\sqrt{2})$ and $S^{m_2}(1/\sqrt{2})$, we conclude that $p = 1$. Also, $q = 2$, i.e. $S^{m_1}(1/\sqrt{2}) \times S^{m_2}(1/\sqrt{2})$ has order $[1, 2]$ in $\mathbb{R}^{m+2}$, if and only if $m_2 \leq 2(m_1 + 1)$. Note that this holds, for example, for $S^1(1/\sqrt{2}) \times S^2(1/\sqrt{2}) \subset S^4$.

Moreover, it can be easily proved that all proper biharmonic submanifolds in $S^{m+1}$ obtained by means of the product composition property, as submanifolds in $\mathbb{R}^{m+2}$, have the spectral decomposition given by (2.4).

Other examples of proper biharmonic immersed submanifolds in spheres.

In [32] and [1] the authors studied the proper biharmonic Legendre immersed surfaces and the proper biharmonic 3-dimensional anti-invariant immersed submanifolds in Sasakian space forms. They obtained the explicit representations of such submanifolds in the unit Euclidean 5-dimensional sphere $S^5$.

Theorem 2.7 ([32]). Let $\phi : M^2 \to S^5$ be a proper biharmonic Legendre immersion. Then the position vector field $\varphi = i \circ \phi = \varphi(u, v)$ of $M$ in $\mathbb{R}^6$ is given by

$$\varphi(u, v) = \frac{1}{\sqrt{2}}(e^{iu}, ie^{-iu} \sin \sqrt{2}v, ie^{-iu} \cos \sqrt{2}v),$$

where $i : S^5 \to \mathbb{R}^6$ is the canonical inclusion.

Remark 2.8. The map $\phi$ is a full proper biharmonic Legendre embedding of a 2-dimensional flat torus $\mathbb{R}^2/\Lambda$ into $S^5$, where the lattice $\Lambda$ is generated by $(2\pi, 0)$ and $(0, \sqrt{2}\pi)$. It has constant mean curvature $|H| = 1/2$, it is not pseudo-umbilical and its mean curvature vector field is not parallel. Moreover, $\varphi = \varphi_p + \varphi_q$, where

$$\varphi_p(u, v) = \frac{1}{\sqrt{2}}(e^{iu}, 0, 0)$$

$$\varphi_q(u, v) = \frac{1}{\sqrt{2}}(0, ie^{-iu} \sin \sqrt{2}v, ie^{-iu} \cos \sqrt{2}v)$$

and $\Delta \varphi_p = \varphi_p, \Delta \varphi_q = 3\varphi_q$. Thus $\varphi$ is a 2-type immersion in $\mathbb{R}^6$ with eigenvalues 1 and 3. In this case, $p = 1$ and $q = 3$, i.e. $\varphi$ is a $[1, 3]$-order immersion in $\mathbb{R}^6$.

Theorem 2.9 ([1]). Let $\phi : M^3 \to S^5$ be a proper biharmonic anti-invariant immersion. Then the position vector field $\varphi = i \circ \phi = \varphi(u, v, w)$ of $M$ in $\mathbb{R}^6$ is given by

$$\varphi(u, v, w) = \frac{1}{\sqrt{2}}e^{iw}(e^{iu}, ie^{-iu} \sin \sqrt{2}v, ie^{-iu} \cos \sqrt{2}v).$$
Remark 2.10. The map $\varphi$ is a full proper biharmonic anti-invariant immersion from a 3-dimensional flat torus $\mathbb{R}^3/\Lambda$ into $\mathbb{S}^5$, where the lattice $\Lambda$ is generated by $(2\pi, 0, 0)$, $(0, \sqrt{2}\pi, 0)$ and $(0, 0, 2\pi)$. It has constant mean curvature $|H| = 1/3$, is not pseudo-umbilical, but its mean curvature vector field is parallel. Moreover, $\varphi = \varphi_p + \varphi_q$, where
\[
\varphi_p(u, v, w) = \frac{1}{\sqrt{2}} e^{iu}(e^{i\pi}(0, 0))
\]
\[
\varphi_q(u, v, w) = \frac{1}{\sqrt{2}} e^{iuv}(0, i e^{-iu} \sin \sqrt{2}v, i e^{-iu} \cos \sqrt{2}v)
\]
and $\Delta \varphi_p = 2\varphi_p$, $\Delta \varphi_q = 4\varphi_q$. Thus $\varphi$ is a 2-type submanifold of $\mathbb{R}^6$ with eigenvalues 2 and 4. It is easy to verify that $\varphi$ is a [2, 4]-order immersion in $\mathbb{R}^6$.

Since the immersion $\phi$ has parallel mean curvature vector field, one could ask weather its image arises by means of the product composition property. Indeed, it can be proved that, up to an orthogonal transformation of $\mathbb{R}^6$ which commutes with the usual complex structure, $\phi$ covers twice the proper biharmonic submanifold $\mathbb{S}^1(1/\sqrt{2}) \times \mathbb{S}^1(1/2) \times \mathbb{S}^1(1/2) \subset \mathbb{S}^5$.

2.2. Classification results. Some of the techniques used in order to obtain non-existence results in the case of non-positively curved space forms were adapted to the study of proper biharmonic submanifolds in spheres. Thus, in order to approach the classification problem for proper biharmonic hypersurfaces in spheres, the study was divided according to the number of principal curvatures. For submanifolds of higher codimension, supplementary conditions on the mean curvature vector field were imposed. This led to a series of rigidity results, which we enumerate below.

2.2.1. Proper biharmonic hypersurfaces. First, if $M$ is a proper biharmonic umbilical hypersurface in $\mathbb{S}^{m+1}$, i.e. all its principal curvatures are equal, then it is an open part of $\mathbb{S}^m(1/\sqrt{2})$.

Afterwards, proper biharmonic hypersurfaces with at most two distinct principal curvatures were considered.

Theorem 2.11 ([6]). Let $M$ be a hypersurface with at most two distinct principal curvatures in $\mathbb{S}^{m+1}$. If $M$ is proper biharmonic in $\mathbb{S}^{m+1}$, then it has constant mean curvature.

By using this result, the classification of such hypersurfaces was obtained.

Theorem 2.12 ([6]). Let $M^m$ be a hypersurface with at most two distinct principal curvatures in $\mathbb{S}^{m+1}$. Then $M$ is proper biharmonic if and only if it is an open part of $\mathbb{S}^m(1/\sqrt{2})$ or of $\mathbb{S}^{m_1}(1/\sqrt{2}) \times \mathbb{S}^{m_2}(1/\sqrt{2})$, $m_1 + m_2 = m$, $m_1 \neq m_2$.

Then followed the case of biharmonic hypersurfaces with at most three distinct principal curvatures. In order to solve this problem, the following property of proper biharmonic hypersurfaces in spheres was needed.

Proposition 2.13 ([6]). Let $M$ be a proper biharmonic hypersurface with constant mean curvature $|H|$ in $\mathbb{S}^{m+1}$, $m \geq 2$. Then $M$ has positive constant scalar curvature $s = m^2(1 + |H|^2) - 2m$.

First a non-existence result was obtained.

Theorem 2.14 ([5]). There exist no compact proper biharmonic hypersurfaces of constant mean curvature and with three distinct principal curvatures everywhere in the unit Euclidean sphere.
The proof relies on the fact that such hypersurfaces are isoparametric, i.e. have constant principal curvatures with constant multiplicities, and then, on the explicit expressions of the principal curvatures.

We note that, in [23], the authors classified the isoparametric proper biharmonic hypersurfaces in spheres.

**Theorem 2.15 ([23]).** Let $M^m$ be an isoparametric hypersurface in $S^{n+1}$. Then $M$ is proper biharmonic if and only if it is an open part of $S^m(1/\sqrt{2})$ or of $S^{m_1}(1/\sqrt{2}) \times S^{m_2}(1/\sqrt{2})$, $m_1 + m_2 = m$, $m_1 \neq m_2$.

Compact proper biharmonic hypersurfaces in $S^4$ were fully classified.

**Theorem 2.16 ([5]).** The only compact proper biharmonic hypersurfaces in $S^4$ are the hypersphere $S^3(1/\sqrt{2})$ and the torus $S^1(1/\sqrt{2}) \times S^2(1/\sqrt{2})$.

The proof uses the fact that a proper biharmonic hypersurface in $S^4$ has constant mean curvature, and thus constant scalar curvature, and a result in [11].

### 2.2.2. Proper biharmonic submanifolds of codimension higher than one.

In higher codimension, it was proved that the proper biharmonic pseudo-umbilical submanifolds, of dimension different from four, in spheres have constant mean curvature. This result leaded to the classification of proper biharmonic pseudo-umbilical submanifolds of codimension two.

**Theorem 2.17 ([6]).** Let $M^m$ be a pseudo-umbilical submanifold in $S^{m+2}$, $m \neq 4$. Then $M$ is proper biharmonic in $S^{m+2}$ if and only if it is minimal in $S^{m+1}(1/\sqrt{2})$.

Surfaces with parallel mean curvature vector field in $S^n$ were also investigated.

**Theorem 2.18 ([6]).** Let $M^2$ be a surface with parallel mean curvature vector field in $S^n$. Then $M$ is proper biharmonic in $S^n$ if and only if it is minimal in $S^{n-1}(1/\sqrt{2})$.

The above two results allowed the classification of proper biharmonic constant mean curvature surfaces in $S^4$.

**Theorem 2.19 ([4]).** The only proper biharmonic constant mean curvature surfaces in $S^4$ are the minimal surfaces in $S^3(1/\sqrt{2})$.

**Proof.** The key of the proof is to show that $\nabla^\perp H = 0$, in order to be able to apply Theorem 2.18.

We assume that $\nabla^\perp H \neq 0$ and consider $\{E_1, E_2\}$ tangent to $M$ and $\{E_3, E_4 = \frac{H}{|H|}\}$ normal to $M$, such that $\{E_1, E_2, E_3, E_4\}$ constitutes a local orthonormal frame field on $S^4$. Using the connection 1-forms w.r.t. $\{E_1, E_2, E_3, E_4\}$ and the tangent part of the biharmonic equation (2.1), we get $A_4 = 0$, where $A_4$ is the shape operator in direction of $E_4$. Then we identify two cases:

(i) If $A_3 = |H| \text{Id}$, then $M$ is pseudo-umbilical and, by Theorem 2.17, it is minimal in $S^3(1/\sqrt{2})$. This implies that $\nabla^\perp H = 0$, and we have a contradiction.

(ii) If $A_3 \neq |H| \text{Id}$, then the Gauss and Codazzi equations lead us to a contradiction and we conclude. \(\square\)

### 3. Properties of proper biharmonic submanifolds in spheres

We begin this section by presenting some general properties of proper biharmonic submanifolds with parallel mean curvature vector field in spheres, which are consequences of (2.1) and of the Codazzi and Gauss equations, respectively.
**Proposition 3.1.** Let $M$ be a proper biharmonic submanifold with parallel mean curvature vector field in $\mathbb{S}^n$. Then

(i) $|A_H|^2 = m|H|^2$, and it is constant,
(ii) $\text{trace} \nabla A_H = 0$,
(iii) $(\text{trace}(\nabla^1 B)(X, \cdot, A_H(\cdot)), H) = (\text{trace}(\nabla^1 B)(\cdot, X, A_H(\cdot)), H) = 0$, for all $X \in C(TM)$.

**Proposition 3.2.** Let $M$ be a proper biharmonic submanifold with parallel mean curvature vector field in $\mathbb{S}^n$. Let $p$ be an arbitrary point on $M$ and consider $\{e_i\}_{i=1}^m$ to be an orthonormal basis of eigenvectors for $A_H$ in $T_p M$. Denote by $\{a_i\}_{i=1}^m$ the eigenvalues of $A_H$ at $p$. Then, at $p$,

(i) $m|H|^2 = \sum_{i=1}^m a_i = \sum_{i=1}^m (a_i)^2$,
(ii) $(2m - 1)m|H|^2 = \frac{1}{2} \sum_{i,j=1}^m (a_i + a_j)(K_{ij} + |B(e_i, e_j)|^2)$,
(iii) $(m - 1 + m|H|^2)m|H|^2 = \sum_{i,j=1}^m a_i a_j (K_{ij} + |B(e_i, e_j)|^2)$,

where $K_{ij}$ denotes the sectional curvature of the 2-plane tangent to $M$ generated by $e_i$ and $e_j$.

For what concerns proper biharmonic constant mean curvature submanifolds in spheres, a partial classification result was obtained.

**Theorem 3.3** ([29]). Let $M$ be a proper biharmonic submanifold with constant mean curvature in $\mathbb{S}^n$. Then $|H| \in (0, 1]$. Moreover, if $|H| = 1$, then $M$ is a minimal submanifold of a hypersphere $\mathbb{S}^{n-1}(1/\sqrt{2}) \subset \mathbb{S}^n$.

Also, the properties regarding the type of the main examples previously presented are not casual. In fact, Theorem 3.3 was extended by establishing a general link between compact proper biharmonic constant mean curvature submanifolds in spheres and finite type submanifolds in the Euclidean space.

**Theorem 3.4** ([3]). Let $M^m$ be a compact constant mean curvature, $|H| \in (0, 1]$, submanifold in $\mathbb{S}^n$. Then $M$ is proper biharmonic if and only if

(i) $|H| = 1$ and $M$ is a 1-type submanifold of $\mathbb{R}^{n+1}$ with eigenvalue $\lambda = 2m$ and center of mass of norm equal to $1/\sqrt{2}$,

or

(ii) $|H| \in (0, 1)$ and $M$ is a mass-symmetric 2-type submanifold of $\mathbb{R}^{n+1}$ with eigenvalues $\lambda_p = m(1 - |H|)$ and $\lambda_q = m(1 + |H|)$.

This can be further used in order to obtain some necessary conditions that compact proper biharmonic submanifolds with constant mean curvature in spheres must fulfill.

**Corollary 3.5.** Let $M^m$ be a compact proper biharmonic constant mean curvature, $|H| \in (0, 1]$, submanifold in $\mathbb{S}^n$. Then

(i) $\lambda_1 \leq m(1 - |H|)$, where $\lambda_1$ is the first non-zero eigenvalue of the Laplacian on $M$,
(ii) $\text{Ricci}(X, X) \geq c\text{g}(X, X)$, for all $X \in C(TM)$, where $c > 0$, we have $c \leq (m - 1)(1 - |H|)$. 


2.1 implies that

\[ m \]

We shall denote, for convenience, \( t \) the squared norm of the Laplacian on \( M \), and thus \( \lambda_1 \leq m(1 - |H|) \).

(ii) The condition \( \text{Ricci}(X, X) \geq cg(X, X) \), for all \( X \in C(TM) \), implies, by a well-known result of Lichnerowicz (see [7]), that \( \lambda_1 \geq \frac{m}{m-1}c \). This, together with (i), leads to the conclusion. \( \square \)

We shall need the following result in order to obtain a refinement of Theorem 3.3.

**Theorem 3.6** ([26]). Let \( M \) be a compact hypersurface with constant normalized scalar curvature \( r = \frac{s}{m(m-1)} \) in \( S^{m+1} \). If

(i) \( r \geq 1 \),

(ii) the squared norm \( |B|^2 \) of the second fundamental form of \( M \) satisfies

\[
|B|^2 \leq (m - 1) \frac{m(r - 1) + 2}{m - 2} + \frac{m - 2}{m(r - 1) + 2},
\]

then either \( |B|^2 = m(r - 1) \) and \( M \) is a totally umbilical hypersurface; or

\[
|B|^2 = (m - 1) \frac{m(r - 1) + 2}{m - 2} + \frac{m - 2}{m(r - 1) + 2}
\]

and \( M = S^1(\sqrt{1 - c^2}) \times S^{m-1}(c) \), with \( c^2 = \frac{m-2}{m} \).

We get the following theorem.

**Theorem 3.7.** Let \( M^m, m \geq 4 \), be a compact proper biharmonic constant mean curvature hypersurface in \( S^{m+1} \). Then \( |H| \in (0, \frac{m-2}{m}) \cup \{1\} \). Moreover,

(i) \( |H| = 1 \) if and only if \( M = S^m(1/\sqrt{2}) \),

and

(ii) \( |H| = \frac{m-2}{m} \) if and only if \( M = S^1(1/\sqrt{2}) \times S^{m-1}(1/\sqrt{2}) \).

**Proof.** Since \( M \) is proper biharmonic with constant mean curvature \( |H| \), Theorem 2.1 implies that

\[
|B|^2 = |A|^2 = m.
\]

We shall denote, for convenience, \( t = m|H|^2 - 1 \).

Suppose that \( |H| \in (\frac{m-2}{m}, 1) \), which is equivalent to \( t \in \left( \frac{(m-4)(m-1)}{m}, m - 1 \right) \). By using Proposition 2.13, we obtain that

\[
r = 1 + \frac{t}{m - 1}.
\]

Condition (i) of Theorem 3.6 is equivalent to \( t \geq 0 \), which is satisfied. Also, using (3.2), since \( t < m - 1 \), the first inequality of (3.1) is satisfied. The second inequality of (3.1) becomes

\[
0 \leq mt^2 - (m^2 - 6m + 4)t - (m - 4)(m - 1)
\]

and it is satisfied since \( t > \frac{(m-4)(m-1)}{m} \). We are now in the hypotheses of Theorem 3.6 and we get \( r = 2 \), i.e. \( |H| = 1 \), or \( r = \frac{2(m-2)}{m} \), i.e. \( |H| = \frac{m-2}{m} \), thus we have a contradiction. Conclusively, we obtain \( |H| \in (0, \frac{m-2}{m}) \cup \{1\} \).

Case (i) is given by Theorem 3.3. It can also be proved by using Theorem 3.6.

For (ii), as we have already seen, if \( M = S^1(1/\sqrt{2}) \times S^{m-1}(1/\sqrt{2}) \), then \( |H| = \frac{m-2}{m} \). Conversely, if \( |H| = \frac{m-2}{m} \), then \( r = \frac{2(m-2)}{m} \), and we are in the hypotheses of Theorem 3.6 thus we conclude. \( \square \)
4. Open problems

In view of all the above results the following conjectures were proposed.

**Conjecture 4.1** (\[6\]). The only proper biharmonic hypersurfaces in $S^{m+1}$ are the open parts of hyperspheres $S^m(1/\sqrt{2})$ or of generalized Clifford tori $S^{m_1}(1/\sqrt{2}) \times S^{m_2}(1/\sqrt{2})$, $m_1 + m_2 = m$, $m_1 \neq m_2$.

**Conjecture 4.2** (\[6\]). Any proper biharmonic submanifold in $S^n$ has constant mean curvature.

5. Further remarks

There is an interesting link between the proper biharmonic hypersurfaces in $S^{m+1}$ and the $II$-minimal hypersurfaces. We briefly recall here the notion of $II$-minimal hypersurfaces (see [20]). We denote by $E$ the set of all hypersurfaces in a semi-Riemannian manifold $(N, h)$ for which the first, as well as the second, fundamental form is a semi-Riemannian metric. The critical points of the area functional of the second fundamental form

$$\text{Area}_{II} : E \to \mathbb{R}, \quad \text{Area}_{II}(M) = \int_M \sqrt{|\det A|} \, v_g$$

are called $II$-minimal. According to [20], we have

**Proposition 5.1.** Let $S^m(a)$ be the hypersphere of radius $a \in (0, 1)$ in $S^{m+1}$. The following are equivalent

(i) $S^m(a)$ is proper biharmonic,

(ii) $S^m(a)$ is $II$-minimal,

(iii) $a = 1/\sqrt{2}$.

**Proposition 5.2.** Let $M = S^{m_1}(a_1) \times S^{m_2}(a_2), \ a_1 \in (0, 1), \ a_1^2 + a_2^2 = 1$, be the generalized Clifford torus in $S^{m+1}, \ m_1 + m_2 = m$. The following are equivalent

(i) $M$ is proper biharmonic,

(ii) $M$ is $II$-minimal and non-minimal,

(iii) $a_1 = a_2 = 1/\sqrt{2}$ and $m_1 \neq m_2$.

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