Causal Spin Foams

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Abstract

I discuss how to impose causality in spin-foam models, separating forward and backward propagation, turning a given triangulation to a ‘causal set’. I show that the criteria proposed to identify the forward, causal sector of the theory are equivalent. Essential to the argument is the closure condition for each 4-simplex.

1 Introduction

The spin-foam models of quantum gravity\[1\] are an attempt to realize the idea of quantizing Regge Calculus \[2\], discretizing space-time in a simplicial complex, and calculating a ‘partition function’ integrating over all possible configurations. However, a partition function is not really an approximation to a path integral: it does not distinguish the past from the future. This is just like the Wheeler De Witt equation, which unlike the Schrödinger equation knows no time, and is expected to give a ‘forward’ and a ‘backward’ propagation\[3\][4]. But I would argue that the two should be separated, and that the forward is the causal one.

A symptom that something is missing to the model comes from the asymptotic analysis of the model of Engle-Livine-Pereira-Rovelli\[7\] and Freidel-Krasnov\[8\] performed by J.Barrett et al.\[9\] and Mu Xin Han et al.\[10\]. A saddle-point expansion gives in the limit for the contribution of each triangle $\Delta_{ab}$, shared by tetrahedra $e_a$ and $e_b$ in a 4-simplex $v$, a Regge-like expression:

$$N_{ab} e^{iA_{ab} \Theta_{ab}} + N_{ab} e^{-iA_{ab} \Theta_{ab}}$$ (1.1)

$A_{ab}$ the area of the triangle, $\Theta_{ab}$ the (hyperbolic) dihedral angle between $e_a$ and $e_b$. The second term in (1.1) is there because in the expansion one finds two saddle points, related by a parity reflection; very much like what one finds in the simple model studied in [3]; this is the ‘cosine problem’ [11].

Some time ago D.Oriti and E.Livine[5], in an analysis of the original model[1], emphasized the importance of introducing an element of causality in the model; and indeed the approach to quantum gravity based on dynamical triangulations became ‘causal dynamical triangulation’[6] when it was decided that causality was the missing ingredient.

A possible way to impose causality on a spinfoam model has been suggested by M.Cortes and L.Smolin[12], based on the work of W.Wieland[13]. Simplifying, for each tetrahedron $e$ one has a closure constraint for the area tensors $S_{IJ}^I$ of its triangles, and for each 4-simplex $v$ a closure constraint for the volume vectors $V_{ev}^I$ of its tetrahedra[14]

$$\sum_{f \in e} S_{IJ}^I = 0; \quad \sum_{e \in v} V_{ev}^I = 0$$ (1.2)

i.e. the sum of the oriented areas and of the oriented volumes must be zero; I shall define these quantities more precisely below. It is also commonly assumed that all tetrahedra
are space-like\(^1\) i.e. all the \(V_e^I\) are time-like, the actual volume being \(V_v = \frac{1}{6}\sqrt{-V_e^0 V_{el}}\). But if the \(V_e^I\) are assumed time-like, they can only sum to zero for each \(v\) if some \(V_e^0\) are positive, some negative; some tetrahedra must be oriented forward, some backward, the closure condition becoming a sort of Kirchhoff law for each 4-simplex. A given triangulation, if this orientation is dictated a priori, becomes a sort of ‘causal set’ in the sense of R. Sorkin\(^15\), or an ‘energetic causal set’ \(^12\); a simple detailed example will be shown in the last section of this paper.

In this way the model becomes ‘causal’; but to give a single term in (1.1) the configurations over which one integrates have to be limited to the forward oriented ones; proposals on how to identify them have been made and investigated in \(^10\) \(^17\) \(^19\). I will show that the different formulations are equivalent. In \(^10\) it is also shown that (under appropriate non-degeneracy hypotheses) the configurations that satisfy the saddle point conditions also satisfy the 4-simplex closure. So it is not just 4-simplex closure, but the preordained orientation of all tetrahedra what limits the configurations over which one integrates.

I assume that I am given a triangulation of 4-space to a simplicial complex \(\mathcal{K}\), with dual skeleton the 2-complex \(\mathcal{C}\), made of 4-simplices/vertices \(v\), tetrahedra/edges \(e\), triangles/faces \(f\). The only boundaries of \(\mathcal{K}\) are an initial and a final triangulation of \(S^3\); the simplest case is the pentachoron, which has 5 vertices, for which I give details in the last section.

A combinatorial notion of orientation for a 4-simplex is given by an ordering of its vertices \((abced)\), or \((P_1, ..., P_5)\), which induces an orientation of its 5 tetrahedra \(
\{(abed), (abe)c, (abde), (aced), (bcde)\}\), that in turn determines the orientation of the triangles e.g. \((abed) : (bced), (cad), (abed), (bac)\). With these rules, each triangle within a 4-simplex is in two tetrahedra with opposite orientation. Even permutations of vertices do not change orientation, odd ones reverse it. \(\mathcal{K}\) must be orientable, meaning that an order of the vertices can be chosen for each 4-simplex such that each tetrahedron belongs to two 4-simplices with opposite orientation. For example:

\[
(abced') : \quad abcd \quad ab'c e \quad abde' \quad ace'd \quad bcde' \\
(abd'c'e') : \quad ab'dc \quad ab'd e \quad ab'e' c \quad ade' c \quad bdc'e'
\]

Regge’s original idea\(^2\) was that each 4-simplex \(v\) is a chunk of 4-space with a flat inside, curvature residing in the bones (triangles/faces) \(f\); \(e^I_\mu\) is a tetrad 1-form in a coordinate patch covering \(v\); Lorentz transformations connect the frames in the tetrahedron \(e = v \cap v'\), overlap of two 4-simplices. The spacetime curvature shows up when going round a bone with successive transformations one does not come back to the original frame.

The 4-volume form \(e^0 \wedge e^1 \wedge e^2 \wedge e^3\) characterizes the ‘geometric’ orientation of the 4-simplex; integrated, it gives for each 4-simplex a positive 4-volume \(V_v\), and each tetrahedron \(e \in v\) a 3-volume 4-vector \(V_e^I = \int e^I_{JKL} e^J \wedge e^K \wedge e^L\). The 4 triangles that bound each tetrahedron have area tensors \(S^{IJ}_{f}\), with the crucial property that \(\eta_{IJ} V_e^I S^{f}_{f} V_e^J = 0\); these area tensors are chosen as independent variables instead of the tetrads. Within a 4-simplex \((1, ..., 5)\) tetrahedra can be labeled by the vertex they do not include, triangles

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\(^1\)in a previous paper\(^16\) I worried that time-like tetrahedra might be needed to model specific spacetimes. Ignoring the background one may well decide a-priori their absence, crucial to the argument that follows.

\(^2\)another point of view\(^19\) is that the second term in (1.1) accounts for the contribution of ‘anti-spacetimes’ fluctuations, regions of negative lapse function; from the point of view of this paper and of \(^10\), 4-simplices with negative \(V_v\) separated from the rest by degenerate 4-simplices. I find difficult to reconcile this with the overall causal structure of the model.
by the tetrahedra they border. One can easily derive that classically, for each 4-simplex, area tensors, 3-volume vectors and 4 volume are related as:

\[
\epsilon^{IJKL} V_{eK} V_{eL} = 2 V_4^I S_e^{IJ}, \quad \epsilon^{IJKL} V_2^I V_3^J V_4^K V_5^L = V_4^3
\]  

(1.3)

Since we assume all tetrahedra to be space-like, all the 4-vectors \( V_I \) are time-like, pointing in opposite time directions for a tetrahedron shared by \( v \) and \( v' \); this links the ‘combinatorial’ and the ‘geometric’ notion of orientation.

2

For a given triangulation the spinfoam amplitude \( A(K) \) is the integral over all holonomies \( g_{ev'} \in SL(2, \mathbb{C}) \) of the product of the face amplitudes \( A_f \) associated to each face/triangle. Each holonomy is factorized as \( g_{ev'} = g_{ve} g_{ev'} \), \( g_{ve} = g_{ev}^{-1} \) if \( e = v \cap v' \), and a ‘simplicity projector’ is inserted for each \( e \) in the chain. For example, for a face of 3 steps:

\[
A_f = \text{Tr} P_j g_{e_1 v_1 e_2} g_{e_2 v_2 e_3} g_{e_3 v_3 e_1}
\]

the trace taken in a rep. \( (j_f, \gamma_{j_f}) \) of \( SL(2, \mathbb{C}) \), and

\[
P_j = \sum_m |(j, \gamma j)m \rangle \langle (j, \gamma j)m| := \sum_m |jm\rangle(jm|.
\]  

(2.1)

This form reflects the key points of the EPRL model: one only sums over the ‘\( \gamma \)-simple’ representations of the \( SL(2, \mathbb{C}) \), i.e. those with indices \( (j, \gamma j) \), \( j \) a (half)-integer, and over the lowest \( SU(2) \) within their decomposition; the areas of the triangles \( A_f = \sqrt{\frac{1}{2} S_f^{IJ} S_f^{IJ}} \) are quantized and given by \( \gamma j \).

Explicitly, the general expression for the spinfoam amplitude is:

\[
A(K) = \int \prod dg_{ev} \prod_f A_f = \int \prod dg_{ev} \prod_{jj} d_{jj} \sum_{mm'} |(jjm|g_{ve}^{-1} g_{ve'}|jjm')\ldots(jjm''|g_{ve}^{-1} g_{ve'}|jjm)\rangle
\]  

(2.2)

where link/tetrahedron \( e \) enters \( v \), \( e' \) leaves it. In the appendix I show how to express the projectors in terms of coherent states, and to rewrite this expression as:

\[
A(K) = \sum_{jj} \int \prod dg_{ev} \prod_{ve} dn_{ef} <n_{ef}|g_{ve}^{-1} g_{ve'}|n'_{ef}>^{2jj} \angle <n'_{ef}|g_{ve}^{-1} g_{ve'}|n'_{ef}>^{jj(\gamma + 1) + 1} <n_{ef}|g_{ve}^{-1} g_{ve'}|n_{ef}>^{-jj(\gamma - 1) + 1} = \sum_{jj} \int \prod dg_{ev} \prod_{ve} dn_{ef} e^{iS}
\]  

(2.3)

The last line above prepares the ground for the saddle point analysis of the large \( j \) behaviour, which we expect to be dominated by the ‘critical configurations’, where \( ReS = \frac{\partial S}{\partial g_{ev}} = \frac{\partial S}{\partial n_{ef}} = 0 \). We now have two sets of integrals, over the link group element \( g_{ev} \) and
over the \( \mathbf{n}_{ef} \), which can be interpreted as normal to the triangle. The two sets give independent descriptions of the geometry, which are linked for critical configurations which extremise \( S \).

For a 4-simplex \( v \) the volume vectors of the five tetrahedra can be taken, up to a proportionality constant, as

\[
V^I_v = \epsilon_v g^I_{veJ} T^J
\]

where \( T^I = (1, 0, 0, 0) \), and \( \epsilon_v = \pm 1 \) determines the orientation of the tetrahedron in \( v \), which must be preassigned; if \( e = v \cap v' \), \( \epsilon_{ve} = -\epsilon_{v'e} \). By eq. \((1.3)\) these vectors determine the 4-volume of the 4-simplex \( V_4 \), and the area tensors of the triangles \( S_{IJ}^{(ee')} \). On the other hand from \((2.3)\) we see that the quantum theory gives the triangles quantized areas, with unit normals \( \mathbf{n}_{ef} \), so that

\[
\ast S_{IJ}^{(ee')} = A^{(f)} g_{eeK} g_{eeL} (T \wedge (0, \mathbf{n}_{ef}) \theta_{KL}^{(ff)})
\]

Do these two descriptions agree? Eq. \((1.3)\) written for \( \ast S_{ee'}^{[IJ]} \) reads

\[
V^I_v V^J_{e'} - V^J_v V^I_{e'} = -2 V_4 \ast S_{ee'}^{[IJ]} \tag{2.6}
\]

If we multiply this equation by \( \ast S_{ee'IJ}^{(ef)} \) and replace in it \((2.5)\) we obtain:

\[
V_4 A_{ee'}^2 = A_{ee'} \epsilon_{ee'} (g_{ee} g_{ee'} - g_{ee} g_{ee'}) T^K T^L \eta_{IM} \eta_{JN} g^{LK} g^{MN} (T \wedge (0, \mathbf{n}_{ee'}))^{PQ} = \\
= A_{ee'} \left( -\epsilon_{ee'} (g_{ee} g_{ee'} - g_{ee} g_{ee'}) \right)^{PQ} \tag{2.7}
\]

which is my key equation. According to J.Engle, to get the ‘proper vertex amplitude’ \([17]\), with the desired asymptotic behaviour \([18]\), one should limit the integrations to the configurations for which the RHS of this equation is positive. This agrees nicely with what MuXin Han et al. find \([10]\): in the forward time-oriented sector of the theory the 4-volumes \( V_4 \) of all 4-simplices are positive. We see therefore that the criteria proposed to identify the forward, causal sector of the theory are equivalent.

In conclusion, to give a causal structure to a spin-foam theory we must use orientable triangulations, with a-priori given orientations for the tetrahedra, and limit the integrations to give positive 4-volumes to all 4-simplices, or equivalently to proper vertex amplitudes.

### 3 An example: the evolution of the pentachoron.

The ‘pentachoron’ is a simple model for \( S^3 \), with an exotic name: five points all connected to each other. One could think of more ambitious models along the same lines, with 19 or 124 vertices \([20][21]\). Following the rules explained in \([20]\), a triangulation evolving a pentachoron \((abcde)\) at \( t=0 \) to a later pentachoron \((a'b'c'd'e')\) at \( t=1 \) can be realized connecting with edges all the vertices of the first to the vertices of the second, but omitting the edges \((aa'), (bb'), (cc'), (dd'), (ee')\); in this way we realize a division of the spacetime \( S^3 \otimes R \) between \( t=0 \) and \( t=1 \) in 30 4-simplices. This triangulation can be proved to be orientable, i.e. it can be organized so that each of the 70 tetrahedra is in two 4-simplices with opposite orientation, for ex.

\[
\begin{align*}
(abcde') : & \quad abcd \quad abe'c \quad abde' \quad ace'd \quad bcde' \\
(abd'e') : & \quad abdc \quad abe'd \quad ab'c'e \quad ade'c \quad bd'e'
\end{align*}
\]

\((3.1)\)
and can therefore be described by a graph like the one below; this graph is meant to explain the sense in which this evolution can be regarded as a mini-causal-set: the pentagons represent 4-simplices, the oriented lines linking them the tetrahedra they share or, in the dual interpretation, the discrete spin connection $g_{vv'}$:

However, one should not take it too literally; the initial pentachoron (abcde) comes before the final (a'b'c'd'e'), but that notion does not apply to the intermediate stages; there is no sense in which (abcde') comes 'before' (abdc'e'). This may well be the more interesting lesson to be drawn from the example.

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Appendix: The spinfoam amplitude

In this appendix I shall give a quick derivation of the expression for the spinfoam amplitude, following [22][10]. One uses the unitary irreducible representations $(k,\nu)$ of $\text{SL}(2,\mathbb{C})$, which act as $g \triangleright f(z) = f(zg)$ on functions of the spinor $z = \begin{pmatrix} z_0 \\ z_1 \end{pmatrix}$ such that $f(\lambda z) = \lambda^{-1+i\nu+k} \bar{\lambda}^{-1+i\nu-k} f(z)$. The Hilbert space $\mathcal{H}_{k,\nu}$ of such functions can be realized as the space of the functions $f \in L_2(\text{SU}(2))$ such that $f(e^{i\sigma_3 u}) = e^{2ik\varphi} f(u)$. A complete orthonormal set is: $\{|(k,\nu);jm\rangle\}$, $j \geq k$, $m \in (-j,j)$.

Writing $g = \begin{pmatrix} a & c \\ b & d \end{pmatrix} = kh = \begin{pmatrix} \lambda^{-1} & \mu \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} \beta & -\bar{\alpha} \\ \bar{\alpha} & \beta \end{pmatrix}$, $h \in \text{SU}(2)$, and defining: $ug := k(g,u)h(g,u)$, the action of $g \in \text{SL}(2,\mathbb{C})$ on the basis for this space will be:

$$<u|g|(k,\nu);jm> = \sqrt{d_j} \lambda(g,u)^{-k+i\nu-1} \bar{\lambda}(g,u)^{k+i\nu-1} D_{km}^j(h(g,u)) \quad (3.2)$$

with $d_j = 2j+1$, having taken $(\alpha,\beta)$ as the spinor, and $\mathcal{H}_{k,\nu} = \bigoplus_{j \geq k} \mathcal{H}_j$, $\mathcal{H}_j$ the space on which the $j$-th representation of $\text{SU}(2)$ acts. In the expressions that follow the $\lambda$-s appear in pairs and may be taken real. The 'simplicity constraints' limit the representations that appear in the amplitude to be of the 'gamma-simple' type $(k,\nu) = (j,\gamma_j)$, and the $\text{SU}(2)$ representation to be the lowest. With these preliminaries, in the expression (2.2) for the ELPR/FK spinfoam amplitude, using the (3.2) and inserting a complete set, each matrix
element can be written as:

\[ (jm|g^{-1}g'|jm') = d_j \int_{SU(2)} du \frac{D_j^{jm}(h(g, u))D_j^{jm}(h(g', u))}{\lambda(g, u)^{2\gamma_j + 2}\lambda(g', u)^{-2\gamma_j + 2}} = d_j \int_{SU(2)} du \frac{<jj|h(g, u)^{\dagger}|jj><jj|h(g', u)|jm'>}{\lambda(g, u)^{2\gamma_j + 2}\lambda(g', u)^{-2\gamma_j + 2}} \]  

(3.3)

therefore in the expression of \(A_f\) the trace of the product can be rearranged as a product of matrix elements of the form:

\[ <jj|h(g', u')h(g, u)^{\dagger}|jj> = <\frac{1}{2}\lambda(g, u)^{-1}\frac{1}{2} > > 2j = \]

\[ <\frac{1}{2}\lambda(g', u')^{-1}\frac{1}{2} > > 2j = \]

\[ \left( \frac{<\frac{1}{2}\lambda(g', u')^{-1}\frac{1}{2} >}{\lambda(g', u')\lambda(g, u)} \right)^{2j} = \left( \frac{<\sigma|g^{\dagger-1}g^{-1}|\sigma>}{\lambda(g', u')\lambda(g, u)} \right)^{2j}. \]

Here I have used: \(k(g, u)|\frac{1}{2}\lambda > = \lambda(g, u)^{-1}\frac{1}{2} > , \lambda(g, u) = [\frac{1}{2}\lambda|g^{\dagger-1}g^{-1}|\frac{1}{2} >]^{1/2},\) and defined the coherent states \(|n> = |u\frac{1}{2}\lambda >\) with \(\sigma \cdot n = u\sigma_3 u^\dagger.\) Replacing these expressions in \((2.2),\) we obtain the equation \((2.3)\) given in the text:

\[ A(K) = \sum_{j_f} d_{j_f} \int \prod_{ve} dg_{ve} \prod_{v \in f} d_{ne_f} \frac{<n_f|g^{\dagger-1}g^{-1}|n_f>}{\lambda(g_{ve}, u')^{2j_f(i\gamma+1)+2}\lambda(g_{ve}, u)^{-2j_f(i\gamma-1)+2}}. \]

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