Vainshtein screening in Horndeski theories nonminimally and kinetically coupled to ordinary matter

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We study the Vainshtein screening mechanism in Horndeski theories in the presence of a scalar field \( \phi \) nonminimally and kinetically coupled to ordinary matter field. A general interacting Lagrangian describing this coupling is characterized by energy transfer \( f_1 \) and momentum exchange \( f_2 \). For a spherically symmetric configurations on top of the cosmological background, we investigate the static perturbations in linear and nonlinear regimes with respect to the scalar field perturbation. In the former regime, the parametrized post-Newtonian parameter generally deviates from unity as long as the matter coupling or \( G_{4, \phi} \) exists. On the other hand, in the latter regime, we show that the nonlinear self-interaction term of scalar field successfully activates the Vainshtein mechanism even in the presence of the couplings \( f_1 \) and \( f_2 \). The gravitational potentials recover the Newtonian behavior deep inside the Vainshtein radius. The bounds on coupling terms not to substantially change the Vainshtein radius are also given.

I. INTRODUCTION

The late-time cosmic acceleration is discovered from the observation of Type Ia supernovae in 1998 [1, 2]. The unknown source of this accelerated expansion is named dark energy (DE). The simplest candidate for DE is the cosmological constant \( \Lambda \). The Λ-Cold-Dark-Matter (ΛCDM) model [3, 4] describes the standard cosmic evolution. However, there are some unsettled problems in the concordance model. One is the cosmological constant problem [5, 6]. The measured energy density of \( \Lambda \) is too small compared to the theoretical prediction. Another is the coincidence problem, which is why the present values of energy densities of \( \Lambda \) and matter field have the same order of magnitude. Moreover, there exists non-negligible disagreement for the present value of matter fluctuation amplitude \( \sigma_8 \) between low- and high-redshift measurements [7–11]. There is also the similar tension for the today’s Hubble constant \( H_0 \) [12–19]. For the high-redshift measurement, the values of these quantities are obtained from the observation of Cosmic-Microwave-Background by assuming the ΛCDM scenario. These problems may indicate the necessity of a new cosmological scenario other than the ΛCDM model.

If the cosmological constant is not the origin of DE, then we need to find an alternative for explaining the late-time cosmic acceleration. In this context, the dynamical dark energy models based on scalar-tensor theories are well studied (see Refs. [20–23] for reviews). For scalar-tensor theories with a single scalar field, there is a framework which can treat the several theories in a unified manner, the so-called Horndeski theory [24–27] which is the most general scalar-tensor theory with second-order equations of motion. After the gravitational wave event GW170817 [28] with its electromagnetic counterpart GRB170817A [29], the tensor propagation speed \( c_t \) is tightly constrained to be quite close to that of light \( c \). If we demand that \( c_t = c \) in Horndeski theories, then the resultant theory uncoupled to CDM predicts the enhancement of the growth of matter perturbations relevant to the scale of galaxy clusterings. As a consequence, the aforementioned \( \sigma_8 \) tension tends to be even worse with respect to the ΛCDM model.

In recent years, it is suggested that an interaction between DE and CDM can alleviate the \( \sigma_8 \) tensions [30–60]. Such a coupling has been introduced at first in order to resolve the coincidence problem [61, 62]. In the phenomenological approaches [63–74], the interaction terms between CDM and scalar field \( \phi \) associated with DE are added to their background equations of motion by hand. In this case, however, there is ambiguity in the way of defining perturbative quantities [35, 75, 76]. In order to overcome the difficulties, the Lagrangian formulation of interactions is proposed in Refs. [77–85] in which CDM is described as a perfect fluid by the so-called Schutz-Sorkin action [86–88]. The interaction between DE and CDM consists of energy transfer and momentum exchange. The former is characterized by a CDM number density \( n_c \) coupled to scalar field \( \phi \) and its kinetic term \( X = -\partial_\mu \phi \partial^\mu \phi /2 \), while the latter is implemented with a scalar product of the CDM four-velocity \( u^\mu_c \) and field derivative \( \partial_\mu \phi \), i.e., \( Z = u^\mu_c \partial_\mu \phi \). In Ref. [82], the cosmological perturbations are investigated with the general coupling Lagrangian \( f(n_c, \phi, X, Z) \). It was shown that the coupling must have the form \( f(n_c, \phi, X, Z) = -f_1(\phi, X, Z) \rho_c(n_c) + f_2(\phi, X, Z) \) under a condition for CDM velocity to vanish, where \( \rho_c \) is the CDM energy density. Between these two types of interactions, it is known that momentum exchange can suppress the growth of CDM density contrast [65–70, 72, 77–81, 89] and may alleviate the \( \sigma_8 \) tension. However, since actual observations are executed targeting baryon in galaxies, it is difficult to conclude that the \( \sigma_8 \) tension can be completely relaxed as long as the scalar field couples only to CDM. Furthermore, in the context of compactified higher-dimensional theories, for instance, a scalar field generally couples to ordinary matter [90–92]. It is therefore more natural to take into account the scalar field coupled not only to CDM but also to baryon.

On the other hand, the local gravity experiments in the solar system agree with general relativity in high precision [93]. Thus, in scalar-tensor theories, the propagation of scalar field coupled to baryon (the so-called fifth force) must be screened on solar-system scales. One of several such screening scenarios is the Vainshtein mechanism [94–101] in which derivative self-interaction of a scalar degree of freedom suppresses the propagation of fifth force at short distances. The Vainshtein screening
effect in Horndeski theories with the uncoupled matter is studied in Refs. [102–104]. The analysis includes the case for subclass of Horndeski theories satisfying \( c_t = c \), and shows that the Newton gravity can be recovered on sufficient small scales by virtue of the self-interaction term of scalar field.

In this paper, we study the local gravity behavior of matter-coupled Horndeski theories with the general interaction Lagrangian \( f(n, \phi, X, Z) \). The subclass of Horndeski theories satisfying \( c_t = c \) is adopted in the DE sector. The matter sector is treated as a pressureless perfect fluid, which is described by a Schutz-Sorkin action. In the static and spherically symmetric spacetime perfect fluid, which is described by a Schutz-Sorkin action. The matter sector is treated as a pressureless matter-coupled Horndeski theories with the general interaction Lagrangian satisfying \( f(n, \phi, X, Z) \). The bounds on coupling terms are derived for subclass of Horndeski theories satisfying \( f(n, \phi, X, Z) \).

In Sec. II, the background equations of motion are derived on the flat Friedmann-Lemaître-Robertson-Walker space-time. In Refs. [103, 105, 106], by solving these equations, we derive general formula of the Vainshtein radius and show that the screening effect successfully works even in the presence of a scalar field \( \phi \) nonminimally and kinetically coupled to ordinary matter field. We also discuss the constraints on the coupling not to change the Vainshtein radius significantly.

This paper is organized as follows. In Sec. II, the background equations of motion are derived on the flat Friedmann-Lemaître-Robertson-Walker space-time. In Sec. III, we obtain the scalar perturbation equations under the quasi-static approximation inside the sub-Hubble scales. We then explore the solutions to linearized perturbation equations. In Sec. IV, we will find small scale solutions where the nonlinear terms for scalar field have dominant contributions in perturbation equations. We determine the Vainshtein radius and show the deviation of gravitational potentials from GR is screened deep inside the Vainshtein radius even in the presence of the coupling. The bounds on coupling terms are derived for the concrete models. Sec. V is devoted to conclusions.

We use the natural unit where the speed of light \( c \), the reduced Planck constant \( \hbar \), and the Boltzmann constant \( k_B \) are equivalent to 1.

II. HORNDESKI THEORIES WITH THE MATTER COUPLING

A. Total action describing the Horndeski theories coupled to nonrelativistic matter

We consider the subclass of Horndeski theories coupled to the matter field described by the following action:

\[
S = \int d^4 x \sqrt{-g} \mathcal{L}_H - \int d^4 x \left[ \sqrt{-g} \rho(n) + J^\mu \partial_\mu \ell \right] + \int d^4 x \sqrt{-g} f(n, \phi, X, Z) ,
\]

where \( g \) is the determinant of metric tensor \( g_{\mu\nu} \), and

\[
\mathcal{L}_H = G_4(\phi) R + G_2(\phi, X) + G_3(\phi, X) \Box \phi ,
\]

is the Lagrangian density describing the subclass of Horndeski theories in which the propagation speed of gravitational waves is strictly equal to that of light [23, 107–115]. Here, \( G_2 \) and \( G_3 \) are functions of scalar field \( \phi \) and its kinetic term,

\[
X = -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi ,
\]

with the covariant operator \( \nabla_\mu \) and the d’Alembertian \( \Box = g^{\mu\nu} \nabla_\mu \nabla_\nu \), while \( G_4 \) depends only on \( \phi \) and represents the coupling between the scalar field and the Ricci scalar \( R \). For \( G_4(\phi) \neq \) constant, this nonminimal coupling gives rise to the modification of gravity. In the presence of the nonlinear self-interaction term \( G_3(\phi, X) \), the Vainshtein screening suppresses such modification at short distances [102–104].

The second integral of the right-hand-side of Eq. (2.1) corresponds to the Schutz-Sorkin action [86–88] describing the perfect fluid. We treat this perfect fluid as general nonrelativistic matter including not only CDM but also baryon. In Eq. (2.1), the energy density \( \rho \) is a function of fluid number density \( n \), and the vector field \( J^\mu \) is related to the fluid four velocity \( u^\mu \), as

\[
u^\mu = \frac{J^\mu}{n\sqrt{g}}.
\]

The four velocity obeys the relation \( u^\mu u_\mu = -1 \), which leads to the expression of \( n \) in terms of \( J^\mu \) as

\[
n = \sqrt{\frac{g_{\mu\nu} J^\mu J^\nu}{g}} .
\]

The scalar quantity \( \ell \) is a Lagrange multiplier with the notation \( \partial_\mu \ell = \partial f/\partial x^\mu \). This quantity is related to the conservation of particle number on the cosmological background as we will see later.

The third integral of the right-hand-side of Eq. (2.1) represents the interaction between the scalar field and matter. The function \( f \) depends on \( n, \phi, X \), and the quantity

\[
Z = u^\mu \nabla_\mu \phi ,
\]

characterizing momentum exchange. In Ref. [82], it has been shown that the coupling \( f \) generally affects the fluid sound speed. Such an effect causes an increase of pressure for the nonrelativistic matter which can prevent the successful structure formation. This situation can be avoided if the coupling \( f \) satisfies the following condition:

\[
f_{nn} = 0 ,
\]

where the comma in subscripts represents a partial derivative with respect to the scalar quantity represented in the index, e.g., \( f_n = \partial f/\partial n \) and \( f_{nn} = \partial^2 f/\partial n^2 \). Under this condition, the functional form of the coupling \( f \) is restricted to

\[
f(n, \phi, X, Z) = -f_1(\phi, X, Z) n + f_2(\phi, X, Z) .
\]
Since the energy density of nonrelativistic matter linearly depends on its number density, the above coupling can also be expressed as

\[ f(n, \phi, X, Z) = -f_1(\phi, X, Z) \rho(n) + f_2(\phi, X, Z). \]  

(2.9)

In this section, we keep to use the general functional form \( f(n, \phi, X, Z) \) for the sake of generality. We will adopt Eqs. (2.7)-(2.9) when needed in the following section.

**B. Cosmological perturbations and background equations of motion**

We study spherically symmetric perturbations on the cosmological background in the Newtonian gauge,

\[ ds^2 = -(1 + 2\alpha)dt^2 + a(t)^2(1 + 2\beta) \left[ dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \right], \]

(2.10)

where \( a(t) \) is the time-dependent scale factor, \( r \) is comoving distance, \( \theta \) is polar angle, and \( \varphi \) is azimuthal angle. The scalar perturbations \( \alpha, \beta \) depend on only \( t \) and \( r \) since we consider spherically symmetric overdensity. The scalar field is decomposed into the time-dependent background part \( \phi(t) \) and the perturbed part \( \delta \phi \), as

\[ \phi(t, r) = \bar{\phi}(t) + \delta \phi(t, r). \]

(2.11)

Similarly, we decompose the energy density of matter,

\[ \rho(t, r) = \bar{\rho}(t) + \delta \rho(t, r), \]

(2.12)

as well as the temporal and spatial components of \( J^\mu \),

\[ J^t = \sqrt{-g} \left[ \bar{n}(t) + \delta n(t, r) \right], \]
\[ J^r = \sqrt{-g} a(t)^2 \delta \phi J(t, r), \]
\[ J^\theta = J^\varphi = 0, \]

(2.13)

where \( \delta J, \delta \phi \) are scalar perturbations. We hereafter omit the bar. The matter velocity potential \( v \) is defined by

\[ u_r = -\partial_r v. \]

(2.14)

We can replace \( \delta j \) to \( v \) for linear perturbations since a substitution of Eqs. (2.13) and (2.14) into Eq. (2.4) results in the following relation,

\[ \partial_t \delta J = -n \partial_v v. \]

(2.15)

We henceforth discuss velocity potential using \( v \) instead of \( \delta j \). Expanding Eq. (2.5) for the line element (2.10) up to linear order in perturbations, the perturbed number density of matter fluid \( \delta n \) is given by

\[ \delta n = \delta J - 3n \beta. \]

(2.16)

Hence, \( \delta J \) can be eliminated by using relation (2.16) with \( \delta \rho = \rho_n \delta n \), as

\[ \delta J = \frac{\delta \rho}{\rho_n} + 3n \beta. \]

(2.17)

**Variation of the action (2.1) in terms of \( J^\mu \), it follows that**

\[ \partial_\mu \ell = (\rho_n - f_n)u_\mu + \frac{f}{n} (\nabla_\mu \phi + Z u_\mu). \]

(2.18)

On using Eqs. (2.11) and (2.14), the leading order terms of Eq. (2.18) are given by

\[ \ell = -(\rho_n - f_n), \]

(2.19)

for the temporal component on the background, where a dot represents the derivative with respect to \( t \), and

\[ \partial_\mu \ell = -(\rho_n - f_n) \partial_\mu v + \frac{f}{n} \left( \partial_\mu \delta \phi - \delta \phi \partial_\mu v \right), \]

(2.20)

for the radial component at the linear perturbation level. Integrating Eq. (2.19) for \( t \) and Eq. (2.20) with respect to \( r \), we obtain

\[ \ell = - \int_r^0 \left[ \rho_n (\bar{t}) - f_n (\bar{t}) \right] d\bar{t} - \left( \rho_n - f_n + \frac{\dot{\phi} f}{n} \right) v \]

\[ + \frac{f}{n} \delta \phi. \]

(2.21)

Hence, the quantity \( \ell \) can be expressed in terms of background quantities and other perturbation variables. Additionally, varying the action (2.1) with respect to \( \ell \) results in the following constraint,

\[ \partial_\mu J^\mu = 0. \]

(2.22)

On using Eq. (2.13), the constraint (2.22) leads to \( na^3 = \) constant, which corresponds to the conservation of total particle number of matter fluid. This relation is equivalent to the continuity equation,

\[ \dot{\rho} + 3H (\rho + P) = 0, \]

(2.23)

where \( H = \dot{a}/a \) is the Hubble-Lemaître expansion rate, and

\[ P = n \rho_n - \rho, \]

(2.24)

is the pressure of matter fluid. We can derive the background equations of motion by expanding the action (2.1) up to linear order in time-dependent perturbations \( \alpha, \beta, \delta \phi \) and eliminating \( \delta j, \delta J, \ell \) with using Eqs. (2.15), (2.17), and (2.21). Then, it follows that

\[ 3M_{pl}^2 H^2 = \rho_{DE} + \rho, \]

(2.25)

\[ M_{pl}^2 \left( 2\dot{H} + 3H^2 \right) = - P_{DE} - P, \]

(2.26)

\[ \rho_{DE} + 3H (\rho_{DE} + P_{DE}) = 0, \]

(2.27)

where \( M_{pl} \) is the reduced Planck mass related to the Newton gravity constant \( G \) as \( M_{pl} = (8\pi G)^{-1/2} \). The quantities \( \rho_{DE} \) and \( P_{DE} \) correspond to the energy density and the pressure of scalar field associated with dark energy defined by

\[ \rho_{DE} = -G_2 + \dot{\phi}^2 G_{2,XX} + \dot{\phi}^2 (G_{3,\phi} - 3H \dot{\phi} G_{3,X}) \]

\[ + 3H^2 (M_{pl}^2 - 2G_4) - 6H \dot{\phi} G_{4,\phi} - f + \dot{\phi}^2 f_{XX} + \dot{\phi} f_{XZ}, \]

(2.28)

\[ P_{DE} = G_2 + \dot{\phi}^2 (G_{3,\phi} + \dot{\phi} G_{3,X}) \]

\[ + (2\dot{H} + 3H^2) (2G_4 - M_{pl}^2) + 2 \left( \dot{\phi}^2 + 2H \dot{\phi} \right) G_{4,\phi} \]

\[ + 2\dot{\phi}^2 G_{4,\phi \phi} + f - nf_n, \]

(2.29)

respectively.
III. PERTURBATION EQUATIONS AND THE SOLUTIONS IN THE LINEAR REGIME

Let us derive the scalar perturbation equations and clarify how the matter coupling affects the gravitational potentials deep inside the Hubble radius.

A. Perturbation equations under the quasi-static approximation

We expand the action (2.1) up to quadratic order in scalar perturbations. Variations of the expanded action with respect to $\alpha$, $\beta$, $\delta \phi$, $\nu$, $\delta \rho$, $\delta \ell$ lead to perturbation equations. In what follows, we adopt the same procedure taken in Refs. [103, 105, 106]. First, we adopt the following approximation to perturbation equations deep inside the Hubble radius, i.e., $a r \ll 1 / H$, where $r$ is the typical distance. For the quantity $\xi$ representing $\alpha$, $\beta$, or $\delta \phi$, we ignore time derivatives $\dot{\xi}$ and non-derivative terms $\xi$ compared to spatial derivatives $\xi' / a$ since $|\xi' / a| \gg |\xi / (a r)|$, whereas the time derivative $|\xi|$ takes the order of $|H\xi|$. We apply $|\xi' / a|^2 \ll |\xi'/a| (a r)$ due to $|\xi'/a| \ll 1 / (a r)$ while we keep the higher-order derivatives of $\delta \phi$ since the terms can grow up at short distances. Second, in order to focus on a static overdensity, we eliminate $t$-dependence in matter perturbations $\delta \rho$ and $\nu$. From the Poisson equation, we deal with $\delta \rho, \nu$ as comparable to spatial derivatives $\xi' / a$. The equations for $\delta \rho$ and $\nu$ correspond to the $t$ and $r$ components of Eq. (2.18). The combination of these equations is equivalent to Eq. (3.38) in Ref. [82], which is needed to discuss the time evolution of $\delta \rho$. In the following, we do not use them, since we consider static overdensity, but focus on the equation for $\delta \ell$. After an integration over $r$, this equation is written as,

$$v' = \frac{6 a^2 H (1 + \dot{\rho} / \rho) A r}{\rho + P} = \frac{6 a^2 H A r}{\rho},$$

with

$$A = \frac{M(r)}{8 \pi \rho^3}, \quad M(r) = 4 \pi \int \bar{r}^2 \bar{\rho}(\bar{r}) d\bar{r},$$

where $\bar{r}$ is the physical distance defined by $\bar{r} = a r$, and we used the conditions for dust-like matter: $\dot{\rho} / \rho = 0$ and $P = 0$. We apply the relation (3.1) in order to eliminate the velocity potential $\nu$ from the other perturbation equations. In the discussion below, we use $\bar{r}$ instead of $r$ itself but omit the bar for the sake of simplicity.

After the process, the reduced equations for $\alpha$, $\beta$, $\delta \phi$ can be integrated over $r$ once. We rewrite the perturbed quantities with the notation, $x = \delta \phi' / r$, $y = \alpha' / r$, and $z = \beta' / r$, following the same notation adopted in Refs. [103, 105, 106]. Consequently, we obtain

$$\frac{c_1 x}{2} + q_t z + \left(1 + \frac{c_2 n}{\rho} \right) A = 0,$$

$$\frac{\dot{q}_t x}{\phi} + q_t (y + z) + \frac{c_3 n^2 A}{2 \rho} = 0,$$

$$\frac{(c_1 \phi - \dot{q}_t) x^2}{\phi} + c_4 x - \frac{c_1 y}{2} - \frac{\dot{q}_t z}{\phi} + \frac{[c_5 n^2 + c_7] H - c_9 n}{2 \rho} A = 0,$$

where

$$q_t = 2 G \rho,$$

$$c_1 = G_3, X \phi^2 + 2 G_4, \phi,$$

$$c_2 = f, X_n \phi^2 + f, Z_n \phi - f, n,$$

$$c_3 = 6 f, n n,$$

$$c_4 = G_2, X + 2 G_4, \phi - 4 G_3, X \phi H - 2 G_3, X \phi,$$

$$c_5 = 6 (f, X_{nn} \phi + f, Z_{nn}),$$

$$c_6 = 2 \left[ \phi f, X_{nn} + f, X_{nn} \phi + (f, X_n + f, Z_{nn}) \phi \right] + 3 f, Z_{nn} H - f, n \phi,$$

$$c_7 = 6 f, z.$$}

Due to the conditions (2.7), we find

$$c_3 = c_5 = 0.$$}

We eliminate them in the following discussion.

B. The linear regime solutions in the presence of matter coupling

At large distances from the origin of overdensity, the quantity $A$ is much smaller than unity so that the linear terms in Eq. (3.5) should dominate over the quadratic term of $x$. As a result, all the Eqs. (3.3)-3.5 settle down to be linear, then we can easily obtain the solution on large scales as

$$\alpha' = \left[ G_N \left(1 + \frac{c_2 n}{\rho} \right) + \frac{C_1 (\phi c_1 - 2 \dot{q}_t)}{16 \pi H q_t^2} \right] M r^2,$$

$$\beta' = - \left[ G_N \left(1 + \frac{c_2 n}{\rho} \right) + \frac{C_2 \phi c_1}{16 \pi H q_t^2} \right] M r^2,$$

$$\delta \phi' = \frac{C_3 \phi}{8 \pi H q_t} M r^2,$$

with

$$G_N = \frac{1}{8 \pi q_t},$$

$$C = \frac{2 q_t H (H c r q_t - (c_1 c_2 + c_9 q_t) n - c_1 \phi q_t)}{c_1^2 - 2 c_4 q_t \phi - 4 \dot{q}_t c_1}.$$
where $G_N$ corresponds to the gravitational coupling. Since the coefficients $c_2$, $c_6$, $c_7$ and the last term in the definition of $c_4$ arise from the interaction between the scalar field and matter affects the gravitational potentials via the term $c_M/r$ and the quantity $C$ in Eqs. (3.15)-(3.16). Without them, Eqs. (3.15)-(3.17) reduce to ones given by Eqs. (33)-(35) in Ref. [103]. In this linear regime, the parametrized post-Newtonian (PPN) parameter, defined by $\gamma = -\beta/\alpha$, is given as

$$
\gamma = \left[1 - \frac{2C\dot{\phi}}{16\pi G_N(1 + 4\rho c_7^2 \rho H^2q_t^2 + C c_1)}\right]^{-1}. \tag{3.20}
$$

The gravity in the linear regime is obviously modified when $G_{4,\phi} \neq 0$, while there seems no modification for the GR-like case characterized by $G_{4,\phi} = 0$ since $\gamma = 1$. However if we may consider the case with $f = -f_1(\phi)\rho(n)$ as a theory in Einstein frame, we will see that the modification of gravity exists even though $G_{4,\phi} = 0$. Let us consider the simple action in Einstein frame described by

$$
S_E = \int d^4x \sqrt{-g} \left( \frac{M_{pl}^2}{2} R - \frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi \right) + \hat{S}_m(\Psi_m, \hat{g}_{\mu\nu}, f_1(\phi)), \tag{3.21}
$$

where a hat represents the quantity in Einstein frame, and $\Psi_m$ is a matter field. If we conformally transform the metric as $g_{\mu\nu} = \Omega^2(\phi) \hat{g}_{\mu\nu}$ such that the matter coupling vanishes after the transformation, i.e., the Jordan frame, the resultant action can be written as

$$
S_J = \int d^4x \sqrt{-g} \left( \frac{M_{pl}^2}{2} \Omega^4 R - \frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi \right) + S_m(\Psi_m, g_{\mu\nu}). \tag{3.22}
$$

The energy-momentum $T^\mu_\nu$ derived from the action (3.21) and $T^\mu_\nu$ from (3.22) are connected to each other as

$$
T^\mu_\nu = \Omega^{-4}T^\mu_\nu. \tag{3.23}
$$

Meanwhile, for the case $f = -f_1(\phi)\rho(n)$, the coupled energy-momentum tensor $T^\mu_\nu$ is expressed by (see, for instance, Eq. (2.27) in Ref. [82])

$$
\hat{T}^\mu_\nu = f_1 T^\mu_\nu. \tag{3.24}
$$

Comparing Eqs. (3.23)-(3.24), we obtain

$$
\Omega^2 = f_1^{-1/2}. \tag{3.25}
$$

The resultant Jordan frame action (3.22) with (3.25) is included in Eq. (2.1) with $G_4 = (M_{pl}^2/2)f_1^{-1/2}$ and $f = 0$. Hence, $\gamma \neq 1$ holds in Jordan frame. This can also be interpreted as follows. When we decompose the conformal factor as $\Omega(\phi + \delta \phi) = \Omega(\phi)(1 + \delta \Omega(\phi, \delta \phi))$, the gravitational potential in Jordan frame are given by $\alpha = \tilde{\alpha} - \delta \Omega$. We also obtain $\beta$ in the same way, and they lead to $\gamma \neq 1$. Hence, $f_1(\phi)$ affects the gravitational law even for $G_{4,\phi} = 0$. In a similar manner, $f_2$ can also change the linear regime gravity. Thus, we need the screening mechanism at short distances in the presence of the coupling.

IV. SCREENING MECHANISM IN THE NONLINEAR REGIME

A. The nonlinear regime solutions in the presence of matter coupling

Let us proceed to the analysis at short distances where the nonlinear term of scalar field in Eq. (3.5) becomes non-negligible. On using Eqs. (3.3)-(3.4) to eliminate $y$ and $z$ from Eq. (3.5), the resultant equation includes only $x$ and $A$. Rewriting them to $\delta \phi$ and $M$, it reduces to

$$
\frac{B}{2H\dot{\phi}^2} \delta \phi'^2 + \delta \phi' - \frac{\dot{\phi}C}{8\pi G M} = 0, \tag{4.1}
$$

where the quantity $C$ is given by Eq. (3.19), and

$$
B = \frac{8Hq_t}{\phi} \left( \dot{\phi} - \phi c_1 \right) - 2\phi c_1 q_t - 4\phi c_1. \tag{4.2}
$$

Solving the above equation for $\delta \phi'$, we obtain

$$
\delta \phi' = \frac{H\dot{\phi} r}{B} \left( \sqrt{1 + \frac{BCM}{4\pi H^2 q_t r^3}} - 1 \right). \tag{4.3}
$$

Here, we took the branch in which the solution is consistent with the linear regime solution (3.17) in the large scale limit $r \to \infty$. The above solution also reduces to Eq. (3.17) in the limit $B \to 0$ which corresponds to $G_{3,X} \to 0$ since $\dot{q_t} - \phi c_1 = -\dot{\phi} G_{3,X}$. The vanishing of self-interaction term $G_3$ leads to the absence of nonlinear term in Eq. (4.1), so that Eqs. (3.17) and (4.3) coincide with each other. We consider the case with $G_{3,X} \neq 0$ in what follows. As is clear from Eq. (4.3), the second term in the square root of Eq. (4.3) gets larger on smaller scales and exceeds unity inside the radius $r_V$ given by

$$
r_V = \left( \frac{BCM}{4\pi H^2 q_t} \right)^{1/3}. \tag{4.4}
$$

This is the so-called Vainshtein radius. Under the condition $r \ll r_V$, Eq. (4.3) reduces to

$$
\delta \phi' \simeq \frac{H\dot{\phi} r_V}{B} \left( \frac{r_V}{r} \right)^{1/2}, \tag{4.5}
$$

which shows that the solution deep inside the Vainshtein radius, $\delta \phi' \propto r^{-1/2}$, varies more slowly than the linear regime solution, $\delta \phi' \propto r^{-2}$. The qualitatively different behavior of $\delta \phi$ in the region $r \ll r_V$ compared to the linear regime enables the Vainshtein mechanism at work. Indeed, substituting Eq. (4.5) into Eqs. (3.3)-(3.4) and solving them in terms of the derivatives of gravitational potentials, $\alpha'$ and $\beta'$, we obtain

$$
\alpha' = \frac{G_N M}{r^2} \left[ 1 + \frac{c_2 \rho}{\rho} + \frac{C(\dot{\phi}c_1 - 2\dot{q}_t)}{Hq_t} \left( \frac{r}{r_V} \right)^{3/2} \right], \tag{4.6}
$$

$$
\beta' = -\frac{G_N M}{r^2} \left[ 1 + \frac{c_2 \rho}{\rho} + \frac{C\dot{\phi}c_1}{Hq_t} \left( \frac{r}{r_V} \right)^{3/2} \right]. \tag{4.7}
$$
The solutions (4.5)-(4.7) settle down to Eqs. (42), (44), and (45) in Ref. [103], when the coupling vanishes. Eqs. (4.6)-(4.7) are integrated to give,

\[ \alpha = -G_N M \frac{r}{r^2} \left[ 1 + \frac{c_2 n}{\rho} - \frac{2 C (\dot{\varphi}_1 - 2 \ddot{q}_1)}{H q_1} \left( \frac{r}{r_V} \right)^{3/2} \right], \]

\[ \beta = \frac{G_N M}{r^2} \left[ 1 + \frac{c_2 n}{\rho} - \frac{2 C \dot{\varphi}_1}{H q_1} \left( \frac{r}{r_V} \right)^{3/2} \right]. \]

(4.8), (4.9)

Obviously, the terms proportional to \((r/r_V)^{3/2}\) in the solutions of gravitational potentials, as well as the derivative of scalar field perturbation given by Eq. (4.5), become negligible deep inside the Vainshtein radius, \(r \ll r_V\). While the term \(c_2 n/\rho\) sourced by the matter coupling remains even in this regime, it can be absorbed into the definition of \(M\) both in Eqs. (4.8)-(4.9) such as \(\tilde{M} = (1 + c_2 n/\rho) M\). Indeed, the PPN parameter reduces to \(\gamma \simeq 1\) in the regime \(r \ll r_V\). Consequently, we have found that the Vainshtein screening efficiently works even in the presence of the coupling \(f(n, \varphi, X, Z)\).

**B. Bounds on the coupling terms in concrete theories**

We now investigate a bound on the coupling \(f_1\) and \(f_2\) in the context of gravity at short distances. In order to put detailed bounds on these functions, we need to specify the functional forms of them together with the Horndeski functions \(G_2, G_3, G_4\), and study the background evolution in the model so as to determine the present value of background quantities, e.g., \(\varphi, \dot{\varphi}, H\). However, this is beyond the scope of our paper. Alternatively, we put rough bounds on the functions \(f_1\) and \(f_2\) by requiring that the coupling terms do not drastically change the Vainshtein radius. In doing so, we focus on the theories described by \(G_2 = X\) and \(G_3 = \kappa_3 X\), where \(\kappa_3\) is a constant, for the sake of simplicity. First we consider the pure momentum exchange, i.e., \(f_1 = 0\) and \(f_2 \neq 0\). In this case, the Vainshtein radius defined by Eq. (4.4) reduces to

\[ r_V = (8\kappa_3 G_4 M/\pi \rho) \left[ \left( \kappa_3 \dot{\varphi}^2 - 2 G_{4,\phi} \right) \rho - 12 H G_4 f_2, Z \right] / \left\{ \kappa_3 \dot{\varphi}^4 + 4 \left[ 2(2\dot{\varphi}H + \ddot{\varphi})G_4 - G_{4,\phi} \dot{\varphi}^2 \right] \kappa_3 - 4(1 + f_{2,X}) G_4 - 12 G^2_{4,\phi} \right\}^{1/2}. \]

(4.10)

In order for the momentum exchange \(f_2\) not to significantly modify the Vainshtein radius, the coupling should satisfy the following two conditions:

\[ O(f_{2,X}) \lesssim O(1), \]

\[ O(f_{2,Z} M^2/\phi^2) \lesssim O \left( \frac{(\kappa_3 \dot{\varphi}^2 - 2 G_{4,\phi}) H}{G_4} \times 10^{-1} \right), \]

(4.11), (4.12)

where we used the observational bounds on the matter density parameter at present epoch, \(\Omega_{m,0} \simeq 0.3\), such that

\[ O \left( \frac{\rho}{12 M^2_\text{Pl} H^2} \right) = O \left( \frac{\Omega_{m,0}}{4} \right) \simeq O(10^{-1}). \]

(4.13)

Second, we consider the pure energy transfer, i.e., \(f_1 = f_1(\phi)\) and \(f_2 = 0\). In this case, the Vainshtein radius reduces to

\[ r_V = (8\kappa_3 G_4 M/\pi \rho) \left[ \left( \kappa_3 \dot{\varphi}^2 - 2 G_{4,\phi} \right) (1 + f_1) + 4 G_4 f_1, \phi \right] / \left\{ \kappa_3 \dot{\varphi}^4 + 4 \left[ 2(2\dot{\varphi}H + \ddot{\varphi})G_4 - G_{4,\phi} \dot{\varphi}^2 \right] \kappa_3 - 4 G_4 - 12 G^2_{4,\phi} \right\}^{1/2}. \]

(4.14)

In order for \(f_1\) not to modify the Vainshtein radius drastically, the bound on \(f_1\) is given as

\[ O(f_1) \lesssim O(1), \]

(4.15)

and \(f_1\) must not be a steep function with respect to \(\phi\).

**V. CONCLUSIONS**

We studied the Vainshtein screening of coupled DE and matter theories. In Sec. II, we introduced the total action (2.1) describing such theories. For the DE sector, we adopted the subclass of Horndeski theories (2.2) in which the speed of gravitational waves is equal to that of light. The matter was dealt with as a pressureless perfect fluid which is described by the Schutz-Sorkin action. These two components are coupled to each other through the general interacting Lagrangian \(f(n, \varphi, X, Z) = -f_1(\varphi, X, Z) \rho(n) + f_2(\varphi, X, Z)\) satisfying the condition (2.7). We adopted the perturbed metric with the Newtonian gauge (2.10) to derive the equations of motion (2.25)-(2.29) on the cosmological background.

In Sec. III, we obtained the perturbation equations (3.3)-(3.5) for the spherically symmetric and static configurations on top of the cosmological background. We adopted the quasi-static approximation deep inside the Hubble scale following the methods suggested in Refs. [103, 105, 106]. In this approximation scheme, time- and non-derivative terms of \(\alpha, \beta, \delta \varphi\) were neglected compared to their spatial derivatives. The perturbations \(\delta \rho, \nu\) of the perfect fluid were considered as static, to focus on the local gravity on the solar system scales. By solving the perturbation equations (3.3)-(3.5) in the linear regime where the quadratic term of \(x\) becomes negligible, we found the solutions (3.15)-(3.17) which leads the PPN parameter \(\gamma\) given by Eq. (3.20). The PPN parameter reduces to \(\gamma = 1\) when \(G_{4,\phi} = 0\) holds. However, as long as the coupling exists, \(\gamma\) can deviate from unity through a conformal transformation (see also Ref. [116]). Hence, the screening mechanism needs to be at work on small scales.

In Sec. IV, we derived the closed form equation (4.1) for the perturbation of scalar field, and determined the Vainshtein radius \(r_V\) in Eq. (4.4) inside which the non-linear term gives the dominant contribution. Deep inside the Vainshtein radius, i.e., \(r \ll r_V\), the solutions of a perturbed scalar field and two gravitational potentials reduce to those given in Eqs. (4.5)-(4.7). We showed that the modification of gravity depends on \(r\) in the form of \((r/r_V)^{3/2}\) being negligible in the region \(r \ll r_V\), and confirmed that the PPN parameter \(\gamma\) reduced to unity in this regime. Thus, we have found that the Vainshtein mechanism sufficiently works even in the presence of the
coupling \( f(n, \phi, X, Z) = -f_1(\phi, X, Z) \rho(n) + f_2(\phi, X, Z) \).

Furthermore, by requiring that the coupling does not drastically change the Vainshtein radius, we put rough bounds on the functions \( f_1 \) and \( f_2 \) in concrete theories. For the pure momentum exchange characterized by \( f_1 = 0 \) and \( f_2 \neq 0 \), we specified the two bounds on \( f_2 \), Eqs. (4.11)-(4.12). We also derived the similar bound (4.15) on \( f_1 \) in the pure energy transfer case satisfying \( f_1 = f_2(\phi) \) and \( f_2 = 0 \). In order to make an accurate estimate of the Vainshtein radius and put more precise constraints on \( f_1 \) and \( f_2 \) from the local gravity experiments, we need to specify a model and solve the background equations (2.25)-(2.29) numerically. It would be interesting to clarify how the coupling \( f \) decrease/increase the \( rv \) in the viable interacting models, but the discussion is beyond the scope of this paper.

We note that, in the viable coupled dark energy models such as those studied in Refs. [80, 81, 117], a field potential plays key role for the late-time cosmic acceleration. In the case where the field potential gives the dominant contributions even at short distances, one can consider another type of screening scenario, the chameleon mechanism [118, 119]. One can also consider the case in which both the chameleon potential and the non-linear kinetic term associated with Vainshtein mechanism exist [120]. Then, it would be of interest to study whether the chameleon mechanism can also work sufficiently even in the presence of the general coupling \( f(n, \phi, X, Z) \).

In this paper, we focused on quasi-static perturbations in small scales paying attention to the gravitational experiments within the solar system. In doing so, we neglected the time-dependence in the matter density contrast. However, the DE-matter interaction affects the evolution of density fluctuation in galaxy cluster scales. It would be of interest to put constraint on the coupling from Large Scale Structure observations.

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