A dilemma of nonequivalent definitions of differential operators in noncommutative geometry

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Abstract. In contrast with differential operators on modules over commutative and graded commutative rings, there is no satisfactory notion of a differential operator in noncommutative geometry.

1. Introduction

Let $K$ be a commutative ring and $A$ a $K$-ring (a unital algebra with $1 \neq 0$). Let $P$ and $Q$ be $A$-modules. We address the notion of a $K$-linear $Q$-valued differential operator on $P$. If a ring $A$ is commutative, there is the conventional definition (Definition 1, Section 2) of a differential operator on $A$-modules (Grothendieck [9], Krasil’schik et al [14] and Giachetta et al [10]) (see Koszul [13], Akman [1] and Akman et al [2] for the equivalent ones). This definition is straightforwardly extended to modules over graded commutative rings (Definition 5, Section 3), i.e., supergeometry (Giachetta et al [10]). However, there are several nonequivalent definitions of differential operators in noncommutative geometry (Dubois-Violette et al [7], Borowiec [4], Lunts et al [17] and Giachetta et al [10]). We show that none of them is satisfactory at all (Section 5).

The main problem is that derivations of a noncommutative ring $A$ fail to form an $A$-module. Let us recall that, given a $K$-algebra $A$ and an $A$-bimodule $Q$, a $Q$-valued derivation of $A$ is defined as a $K$-module morphism $u : A \to Q$ which obeys the Leibniz rule

$$u(ab) = u(a)b + au(b), \quad a, b \in A. \quad (1)$$

However, there is another convention such that the Leibniz rule takes the form $u(ab) = bu(a) + au(b)$ (Lang [16]). A graded commutative ring is a particular noncommutative ring. However, the definition of a derivation of a graded commutative algebra (Bartocci et al [3]) also differs from the expression (1). Therefore, supergeometry is not a particular case of noncommutative geometry.

The Chevalley–Eilenberg differential calculus over a $K$-ring $A$ provides an important example of differential operators on modules over $A$. If $A$ is the $\mathbb{R}$-ring of smooth real functions on a smooth manifold $X$, this differential calculus is the de Rham complex of exterior forms on $X$. 
Throughout the paper, a two-sided \( \mathcal{A} \)-module \( P \) is called the \((\mathcal{A} - \mathcal{A})\)-bimodule. A \((\mathcal{A} - \mathcal{A})\)-bimodule is said to be a \( \mathcal{A} \)-bimodule if it is central over the center \( \mathcal{Z}_\mathcal{A} \) of \( \mathcal{A} \), i.e., \( ap = pa, \ a \in \mathcal{Z}_\mathcal{A}, \ p \in P \). If \( \mathcal{A} \) is commutative, a \( \mathcal{A} \)-bimodule is simply called an \( \mathcal{A} \)-module.

2. Differential operators on modules over a commutative ring

Let \( \mathcal{A} \) be a commutative \( \mathcal{K} \)-ring. Given \( \mathcal{A} \)-modules \( P \) and \( Q \), the \( \mathcal{K} \)-module \( \text{Hom}_\mathcal{K}(P, Q) \) of \( \mathcal{K} \)-module homomorphisms \( \Phi : P \to Q \) is endowed with the two different \( \mathcal{A} \)-module structures

\[
(a\Phi)(p) := a\Phi(p), \quad (\Phi \cdot a)(p) := \Phi(ap), \quad a \in \mathcal{A}, \quad p \in P. \tag{2}
\]

We further refer to the second one as the \( \mathcal{A} \cdot \)-module structure. Let us put

\[
\delta_a \Phi := a\Phi - \Phi \cdot a, \quad a \in \mathcal{A}. \tag{3}
\]

**Definition 1.** An element \( \Delta \in \text{Hom}_\mathcal{K}(P, Q) \) is called a \( k \)-order \( Q \)-valued differential operator on \( P \) if

\[
\delta_{a_0} \circ \cdots \circ \delta_{a_k} \Delta = 0
\]

for any tuple of \( k + 1 \) elements \( a_0, \ldots, a_k \) of \( \mathcal{A} \).

The set \( \text{Diff}_k(P, Q) \) of these operators inherits the \( \mathcal{A} \)- and \( \mathcal{A}^* \)-module structures (2). In particular, zero order differential operators obey the condition

\[
\delta_a \Delta(p) = a\Delta(p) - \Delta(ap) = 0, \quad a \in \mathcal{A}, \quad p \in P,
\]

and, consequently, they are \( \mathcal{A} \)-module morphisms \( P \to Q \). First order differential operators \( \Delta \) satisfy the condition

\[
\delta_b \circ \delta_a \Delta(p) = ba\Delta(p) - b\Delta(ap) - a\Delta(bp) + \Delta(abp) = 0, \quad a, b \in \mathcal{A}. \tag{4}
\]

Let \( P = \mathcal{A} \). Any zero order \( Q \)-valued differential operator \( \Delta \) on \( \mathcal{A} \) is defined by its value \( \Delta(1) \). There is an isomorphism \( \text{Diff}_0(\mathcal{A}, Q) = Q \) via the association

\[
Q \ni q \mapsto \Delta_q \in \text{Diff}_0(\mathcal{A}, Q),
\]

where the zero order differential operator \( \Delta_q \) is given by the equality \( \Delta_q(1) = q \). A first order \( Q \)-valued differential operator \( \Delta \) on \( \mathcal{A} \) fulfils the condition

\[
\Delta(ab) = b\Delta(a) + a\Delta(b) - ba\Delta(1), \quad a, b \in \mathcal{A}.
\]

It is called a \( Q \)-valued derivation of \( \mathcal{A} \) if \( \Delta(1) = 0 \), i.e., the Leibniz rule (1) holds. If \( \Delta \) is a derivation of \( \mathcal{A} \), then \( a\Delta \) is well for any \( a \in \mathcal{A} \). Hence, derivations of \( \mathcal{A} \) form a left \( \mathcal{A} \)-module \( \mathfrak{d}(\mathcal{A}, Q) \). Any first order differential operator on \( \mathcal{A} \) falls into the sum

\[
\Delta(a) = a\Delta(1) + [\Delta(a) - a\Delta(1)]
\]
of the zero order differential operator \( a \Delta(1) \) and the derivation \( \Delta(a) - a \Delta(1) \). Thus, there is the \( \mathcal{A} \)-module decomposition

\[
\text{Diff}_1(\mathcal{A}, Q) = Q \oplus \mathfrak{d}(\mathcal{A}, Q).
\]  

(5)

Let \( P = Q = \mathcal{A} \). The module \( \mathfrak{d}\mathcal{A} \) of derivations of \( \mathcal{A} \) is also a Lie algebra over the ring \( \mathcal{K} \) with respect to the Lie bracket

\[
u \circ u' - u' \circ u, \quad u, u' \in \mathfrak{d}\mathcal{A}.
\]

Accordingly, the decomposition (5) takes the form

\[
\text{Diff}_1(\mathcal{A}) = \mathcal{A} \oplus \mathfrak{d}\mathcal{A}.
\]

For instance, let \( \mathcal{A} = C^\infty(X) \) be an \( \mathbb{R} \)-ring of smooth real functions on smooth manifold \( X \). Let \( P \) be a projective \( C^\infty(X) \)-modules of finite rank. In accordance with the well-known Serre–Swan theorem, \( P \) is isomorphic to the module \( \Gamma(E) \) of global sections of some vector bundle \( E \to X \). In this case, Definition 1 restarts familiar theory of linear differential operators on vector bundles over a manifold \( X \). Let \( Y \to X \) be an arbitrary bundle over \( X \). The theory of (nonlinear) differential operators on \( Y \) is conventionally formulated in terms of jet manifolds \( J^kY \) of sections of \( Y \to X \) (Palais [18], Krasil’shchik et al [14], Bryant et al [5] and Giachetta et al [11]). Accordingly, there is the following equivalent reformulation of Definition 1 in terms of the jet modules \( J^k(P) \) of a module \( P \) (Krasil’shchik et al [14] and Giachetta et al [10]). If \( P = \Gamma(E) \), these jet modules are isomorphic to modules of sections of the jet bundles \( J^kE \to X \).

Given an \( \mathcal{A} \)-module \( P \), let us consider the tensor product \( \mathcal{A} \otimes_\mathcal{K} P \) of \( \mathcal{K} \)-modules \( \mathcal{A} \) and \( P \). We put

\[
\delta^b(a \otimes p) := (ba) \otimes p - a \otimes (bp), \quad p \in P, \quad a, b \in \mathcal{A}.
\]

Let \( \mu^{k+1} \) be the submodule of \( \mathcal{A} \otimes_\mathcal{K} P \) generated by elements \( \delta^b \circ \cdots \circ \delta^1(a \otimes p) \).

**Definition 2.** The \( k \)-order jet module \( J^k(P) \) of a module \( P \) is the quotient of the \( \mathcal{K} \)-module \( \mathcal{A} \otimes_\mathcal{K} P \) by \( \mu^{k+1} \). The symbol \( c \otimes_k p \) stands for its elements.

In particular, the first order jet module \( J^1(P) \) consists of elements \( c \otimes_1 p \) modulo the relations

\[
\delta^a \circ \delta^b(1 \otimes_1 p) = ab \otimes_1 p - b \otimes_1 (ap) - a \otimes_1 (bp) + 1 \otimes_1 (abp) = 0.
\]

The \( \mathcal{K} \)-module \( J^k(P) \) is endowed with the \( \mathcal{A} \) - and \( \mathcal{A}^\star \)-module structures

\[
b(a \otimes_k p) := ba \otimes_k p, \quad b \cdot (a \otimes_k p) := a \otimes_k (bp).
\]

There exists the module morphism

\[
J^k : P \ni p \mapsto 1 \otimes_k p \in J^k(P)
\]  

(6)
of the $A$-module $P$ to the $A^\bullet$-module $J^k(P)$ such that, seen as an $A$-module, $J^k(P)$ is generated by elements $J^k p$, $p \in P$. The following holds (Krasil’shchik et al [14]).

**Theorem 3.** Any $k$-order $Q$-valued differential operator $\Delta$ of on an $A$-module $P$ uniquely factorizes

$$\Delta : P \xrightarrow{J^k} J^k(P) \longrightarrow Q$$

through the morphism $J^k$ (6) and some $A$-module homomorphism $f^\Delta : J^k(P) \to Q$.

Theorem 3 shows that $J^k(P)$ is the representative object of the functor $Q \to \text{Diff}_k(P,Q)$. Its proof is based on the fact that the morphism $J^k$ (6) is a $k$-order $J^k(P)$-valued differential operator on $P$. Let us denote $J : P \ni p \mapsto 1 \otimes p \in A \otimes P$. Then, for any $f \in \text{Hom}_A(A \otimes P, Q)$, we obtain

$$\delta_{b_0} \circ \cdots \circ \delta_{b_0} (f \circ J)(p) = f(\delta_{b_0} \circ \cdots \delta_{b_0} (1 \otimes p)). \quad (7)$$

The correspondence $\Delta \mapsto f^\Delta$ defines an $A$-module isomorphism

$$\text{Diff}_k(P, Q) = \text{Hom}_A(J^k(P), Q). \quad (8)$$

This isomorphism leads to the above mentioned equivalent reformulation of Definition 1.

**Definition 4.** A $k$-order $Q$-valued differential operator on a module $P$ is an $A$-module morphism of the $k$-order jet module $J^k(P)$ of $P$ to $Q$.

As was mentioned above, the Chevalley–Eilenberg differential calculus over a commutative ring $A$ provides an important example of differential operators over $A$ (Giachetta et al [10]). It is a subcomplex

$$0 \to \mathcal{K} \xrightarrow{\text{in}} A \xrightarrow{d} \mathcal{O}^1[\mathfrak{d}A] \xrightarrow{d} \mathcal{O}^2[\mathfrak{d}A] \xrightarrow{d} \cdots \quad (9)$$

of the Chevalley–Eilenberg complex $C^*[\mathfrak{d}A, A]$ of the Lie $K$-algebra $\mathfrak{d}A$ (Fuks [8]) where $\mathcal{O}^k[\mathfrak{d}A]$ are modules of $A$-multilinear (but not all $K$-multilinear) skew-symmetric maps

$$\phi^k : \times \mathfrak{d}A \to A \quad (10)$$

and $d$ is the Chevalley–Eilenberg coboundary operator

$$d\phi^k(u_0, \ldots, u_k) = \sum_{i=0}^k (-1)^i u_i(\phi^k(u_0, \ldots, \widehat{u}_i, \ldots, u_k)) + \sum_{i<j} (-1)^{i+j} \phi^k([u_i, u_j], u_0, \ldots, \widehat{u}_i, \ldots, \widehat{u}_j, \ldots, u_k), \quad (11)$$

where the caret $\hat{}$ denotes omission. A direct verification shows that if $\phi^k$ is an $A$-multilinear map, so is $d\phi^k$. The graded module $\mathcal{O}^*[\mathfrak{d}A]$ is provided with the structure of a graded $A$-algebra with respect to the product

$$\phi \wedge \phi' = \sum_{i_1 < \cdots < i_r, j_1 < \cdots < j_s} \text{sgn}^{i_1 \cdots i_r, j_1 \cdots j_s} \phi(u_{i_1}, \ldots, u_{i_r}) \phi'(u_{j_1}, \ldots, u_{j_s}), \quad (12)$$

$$\phi, \phi' \in \mathcal{O}^*[\mathfrak{d}A], \quad u_k \in \mathfrak{d}A,$$
where \( \text{sgn} \) is the sign of a permutation. This product obeys the relations

\[
\begin{align*}
  d(\phi \wedge \phi') &= d(\phi) \wedge \phi' + (-1)^{|\phi|} \phi \wedge d(\phi'), \quad \phi, \phi' \in O^*[\mathcal{A}], \\
  \phi \wedge \phi' &= (-1)^{|\phi||\phi'|} \phi' \wedge \phi.
\end{align*}
\]  

Using the first one, one can easily justify that the Chevalley–Eilenberg differential \( d \) obeys the relations (4) and, thus, it is a first order \( O^{k+1}[\mathcal{A}] \)-valued differential operator on \( O^k[\mathcal{A}] \), \( k \in \mathbb{N} \). In particular, we have

\[
(da)(u) = u(a), \quad a \in \mathcal{A}, \quad u \in \mathcal{A}.
\]  

It follows that \( d(1) = 0 \) and \( d \) is a \( O^1[\mathcal{A}] \)-valued derivation of \( O^0[\mathcal{A}] = \mathcal{A} \).

For instance, the Chevalley–Eilenberg differential calculus over the \( \mathbb{R} \)-ring \( C^\infty(X) \) is the de Rham complex of exterior forms on a smooth manifold \( X \) where \( d \) is the exterior differential.

### 3. Differential operators in supergeometry

As was mentioned above, Definition 1 is straightforwardly extended to linear differential operators on modules over a graded commutative ring.

Unless otherwise stated, by a graded structure throughout this Section is meant a \( \mathbb{Z}_2 \)-graded structure, and the symbol \([\ ]\) stands for the \( \mathbb{Z}_2 \)-graded parity. Recall that an algebra \( \mathcal{A} \) is called graded if it is endowed with a grading automorphism \( \gamma \) such that \( \gamma^2 = \text{Id} \). A graded algebra falls into the direct sum \( \mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1 \) of two \( \mathbb{Z} \)-modules \( \mathcal{A}_0 \) and \( \mathcal{A}_1 \) of even and odd elements such that \( \gamma(a) = (-1)^{|a|}a, \quad a \in \mathcal{A} \), and

\[
[aa'] = ([a] + [a']) \text{mod } 2, \quad a, a' \in \mathcal{A}.
\]

If \( \mathcal{A} \) is a graded ring, then \([1] = 0\). A graded algebra \( \mathcal{A} \) is called graded commutative if

\[
aa' = (-1)^{|a||a'|}a'a.
\]

Of course, a commutative ring is a graded commutative ring where \( \mathcal{A} = \mathcal{A}_0 \). Given a graded algebra \( \mathcal{A} \), a left graded \( \mathcal{A} \)-module \( Q \) is a left \( \mathcal{A} \)-module such that

\[
[aq] = ([a] + [q]) \text{mod } 2.
\]

A graded module \( Q \) is split into the direct sum \( Q = Q_0 \oplus Q_1 \) of two \( \mathcal{A}_0 \)-modules \( Q_0 \) and \( Q_1 \) of even and odd elements. Similarly, right graded modules are defined. If \( \mathcal{A} \) is a graded commutative ring, a graded \( \mathcal{A} \)-module can be provided with a graded \( \mathcal{A} \)-bimodule structure by letting

\[
qa := (-1)^{|a||q|}aq, \quad a \in \mathcal{A}, \quad q \in Q.
\]
Let $K$ be a commutative ring and $A$ a graded commutative $K$-ring. Let $P$ and $Q$ be graded $A$-modules. The graded $K$-module $\text{Hom}_K(P, Q)$ of graded $K$-module homomorphisms $\Phi : P \rightarrow Q$ can be endowed with the two graded $A$-module structures
\[
(a\Phi)(p) := a\Phi(p), \quad (\Phi \bullet a)(p) := \Phi(ap), \quad a \in A, \quad p \in P,
\]
called $A$- and $A^\bullet$-module structures, respectively. Let us put
\[
\delta_a \Phi := a\Phi - (-1)^{|a||b|}\Phi \bullet a, \quad a \in A.
\]
(17)
The following generalizes Definition 1 (Giachetta et al [10]).

**Definition 5.** An element $\Delta \in \text{Hom}_K(P, Q)$ is said to be a $k$-order $Q$-valued graded differential operator on $P$ if
\[
\delta_{a_0} \circ \cdots \circ \delta_{a_k} \Delta = 0
\]
for any tuple of $k + 1$ elements $a_0, \ldots, a_k$ of $A$.

The set $\text{Diff}_k(P, Q)$ of these operators inherits the graded module structures (16). In particular, zero order graded differential operators obey the condition
\[
\delta_a \Delta(p) = a\Delta(p) - (-1)^{|a|\Delta} \Delta(ap) = 0, \quad a \in A, \quad p \in P
\]
i.e., they are graded $A$-module morphisms $P \rightarrow Q$. First order graded differential operator $\Delta$ satisfy the condition
\[
\delta_a \circ \delta_b \Delta(p) = ab\Delta(p) - (-1)^{|b|\Delta} b\Delta(ap) - (-1)^{|a|\Delta} a\Delta(bp) + (-1)^{|b|\Delta+|a|\Delta} ab\Delta(1) = 0, \quad a, b \in A, \quad p \in P.
\]

Let $P = A$. Any zero order $Q$-valued graded differential operator $\Delta$ on $A$ is defined by its value $\Delta(1)$. Then there is a graded $A$-module isomorphism $\text{Diff}_0(A, Q) = Q$ via the association
\[
Q \ni q \mapsto \Delta_q \in \text{Diff}_0(A, Q),
\]
where $\Delta_q$ is given by the equality $\Delta_q(1) = q$. A first order $Q$-valued graded differential operator $\Delta$ on $A$ fulfills the condition
\[
\Delta(ab) = \Delta(a)b + (-1)^{|a|\Delta} a\Delta(b) - (-1)^{|b|\Delta+|a|\Delta} ab\Delta(1), \quad a, b \in A.
\]
It is called a $Q$-valued graded derivation of $A$ if $\Delta(1) = 0$, i.e., the graded Leibniz rule
\[
\Delta(ab) = \Delta(a)b + (-1)^{|a|\Delta} a\Delta(b), \quad a, b \in A,
\]
holds (cf. the Leibniz rule (1)). One obtains at once that any first order graded differential operator on $A$ falls into the sum
\[
\Delta(a) = \Delta(1)a + [\Delta(a) - \Delta(1)a]
\]
of a zero order graded differential operator \( \Delta(1)a \) and a graded derivation \( \Delta(a) - \Delta(1)a \).

If \( \Delta \) is a graded derivation of \( A \), then \( a\Delta \) is so for any \( a \in A \). Hence, graded derivations of \( A \) constitute a graded \( A \)-module \( \mathfrak{d}(A, Q) \).

Let \( P = Q = A \). The module \( \mathfrak{d}A \) of graded derivation of \( A \) is also a Lie \( \mathcal{K} \)-superalgebra with respect to the superbracket

\[
[u, u'] = u \circ u' - (-1)^{|u||u'|} u' \circ u,
\]

\( u, u' \in A \).

We have the graded \( A \)-module decomposition

\[
\text{Diff}_1(A) = A \oplus \mathfrak{d}A.
\]

Since \( \mathfrak{d}A \) is a Lie \( \mathcal{K} \)-superalgebra, let us consider the Chevalley–Eilenberg complex \( C^*[\mathfrak{d}A; A] \) of \( \mathfrak{d}A \) where the graded commutative ring \( A \) is a regarded as a \( \mathfrak{d}A \)-module (Fuks [8]). It reads

\[
0 \to A \xrightarrow{\delta^0} C^1[\mathfrak{d}A; A] \xrightarrow{\delta^1} \cdots C^k[\mathfrak{d}A; A] \xrightarrow{\delta^k} \cdots
\]

where \( C^k[\mathfrak{d}A; A] \) are \( \mathfrak{d}A \)-modules of \( \mathcal{K} \)-linear graded morphisms of the graded exterior products \( \Lambda^k \mathfrak{d}A \) of the \( \mathcal{K} \)-module \( \mathfrak{d}A \) to \( A \). Let us bring homogeneous elements of \( \Lambda^k \mathfrak{d}A \) into the form

\[
\varepsilon_1 \wedge \cdots \wedge \varepsilon_r \wedge \varepsilon_{r+1} \wedge \cdots \wedge \varepsilon_k, \quad \varepsilon_i, \varepsilon_j \in \mathfrak{d}A_0, \quad \varepsilon_j \in \mathfrak{d}A_1.
\]

Then the coboundary operators of the complex (19) are given by the expression

\[
\delta^{r+s-1}c(\varepsilon_1 \wedge \cdots \wedge \varepsilon_r \wedge \varepsilon_{r+1} \wedge \cdots \wedge \varepsilon_s) = \sum_{i=1}^{r} (-1)^{i-1} \varepsilon_i c(\varepsilon_1 \wedge \cdots \varepsilon_{i-1} \wedge \varepsilon_i \wedge \varepsilon_{i+1} \wedge \cdots \wedge \varepsilon_s) +
\]

\[
\sum_{j=1}^{s} (-1)^{r} \varepsilon_i c(\varepsilon_1 \wedge \cdots \wedge \varepsilon_r \wedge \varepsilon_{r+1} \wedge \cdots \varepsilon_{j} \wedge \cdots \wedge \varepsilon_s) +
\]

\[
\sum_{1 \leq i < j \leq r} (-1)^{i+j} c([\varepsilon_i, \varepsilon_j] \wedge \varepsilon_1 \wedge \varepsilon_r \wedge \varepsilon_{r+1} \wedge \cdots \varepsilon_{i} \wedge \varepsilon_{j} \wedge \cdots \wedge \varepsilon_s) +
\]

\[
\sum_{1 \leq i < j \leq s} (-1)^{i+j} c([\varepsilon_i, \varepsilon_j] \wedge \varepsilon_1 \wedge \varepsilon_r \wedge \varepsilon_{r+1} \wedge \cdots \varepsilon_{i} \wedge \varepsilon_{j} \wedge \cdots \wedge \varepsilon_s).
\]

The subcomplex \( O^*[\mathfrak{d}A] \) of the complex (19) of \( A \)-linear morphisms is the above mentioned graded Chevalley–Eilenberg differential calculus over a graded commutative \( \mathcal{K} \)-ring \( A \). Its coboundary operators \( \delta^k \) are first order \( O^*k + 1[\mathfrak{d}A] \)-valued graded differential operators on \( O^*k[\mathfrak{d}A] \) (Giachetta et al [10]).

One may hope that Definition 4 is also extended to modules over graded commutative rings. One has introduced jets of these modules (Giachetta et al [10]), but the corresponding generalization of Theorem 3 has not been studied.
4. Noncommutative differential calculus

Turning to differential operators over a noncommutative ring, let us start with a few examples of noncommutative differential calculus.

In a general setting, a \(\mathbb{Z}\)-graded algebra \(\Omega^*\) over a commutative ring \(K\) is defined as a direct sum

\[
\Omega^* = \bigoplus_k \Omega^k, \quad k = 0, 1, \ldots,
\]

of \(K\)-modules \(\Omega^k\), provided with an associative multiplication law \(\alpha \cdot \beta\), \(\alpha, \beta \in \Omega^*\), such that \(\alpha \cdot \beta \in \Omega^{\|\alpha\| + \|\beta\|}\), where \(\|\alpha\|\) denotes the degree of an element \(\alpha \in \Omega^{\|\alpha\|}\). It follows that \(\Omega^0\) is a (noncommutative) \(K\)-algebra \(A\), and \(\Omega^*\) is an \((A - A)\)-algebra. A graded algebra \(\Omega^*\) is called a differential graded algebra or a differential calculus over \(A = \Omega^0\) if it is a cochain complex of \(K\)-modules

\[
0 \to K \to A \xrightarrow{\delta} \Omega^1 \xrightarrow{\delta} \cdots \Omega^k \xrightarrow{\delta} \cdots
\]

with respect to a coboundary operator \(\delta\) which obeys the graded Leibniz rule

\[
\delta(\alpha \cdot \beta) = \delta\alpha \cdot \beta + (-1)^{\|\alpha\|}\alpha \cdot \delta\beta.
\]

In particular, \(\delta : A \to \Omega^1\) is a \(\Omega^1\)-valued derivation of a \(K\)-algebra \(A\). One considers the minimal differential graded subalgebra \(\Omega^* A\) of a differential calculus \(\Omega^*\) which contains \(A\). It is called the minimal differential calculus over \(A\). Seen as an \((A - A)\)-algebra, \(\Omega^* A\) is generated by the elements \(\delta a, a \in A\), and consists of monomials \(\alpha = a_0 \delta a_1 \cdots \delta a_k, a_i \in A\), whose product obeys the juxtaposition rule

\[
(a_0 \delta a_1) \cdot (b_0 \delta b_1) = a_0 \delta(a_1 b_0) \cdot \delta b_1 - a_0 a_1 \delta b_0 \cdot \delta b_1.
\]

Let us generalize the Chevalley–Eilenberg differential calculus over a commutative ring in Section 2 to a noncommutative \(K\)-ring \(A\). For this purpose, let us consider derivations \(u \in dA\) of \(A\). They obey the Leibniz rule (1). The set of these derivations \(dA\) is both a \(\mathbb{Z}A\)-bimodule and a Lie \(K\)-algebra with respect to the Lie bracket \([u, u']\), \(u, u' \in dA\). It is readily observed that derivations preserve the center \(Z_A\) of \(A\).

Let us consider the Chevalley–Eilenberg complex \(C^* [dA, A]\) of the Lie algebra \(dA\) with coefficients in the ring \(A\), regarded as a \(dA\)-module. This complex contains a subcomplex \(\mathcal{O}^*[dA]\) of \(\mathbb{Z}A\)-multilinear skew-symmetric maps with respect to the Chevalley–Eilenberg coboundary operator \(d\) (11). Its terms \(\mathcal{O}^k[dA]\) are \(A\)-bimodules. In particular,

\[
\mathcal{O}^1[dA] = \text{Hom}_{\mathbb{Z}A}(dA, A).
\]

The graded module \(\mathcal{O}^*[dA]\) is provided with the product (12) which obeys the relation (13) and makes \(\mathcal{O}^*[dA]\) into a differential graded algebra. One can think of its elements as being noncommutative generalization of exterior forms on a manifold. However, it should
be noted that, if $A$ is not commutative, there is nothing like the graded commutativity (14) in general. The minimal Chevalley–Eilenberg differential calculus $O^\ast A$ over $A$ consists of the monomials $a_0 da_1 \wedge \cdots \wedge da_k$, $a_i \in A$, whose product $\wedge$ obeys the juxtaposition rule

$$(a_0 da_1) \wedge (b_0 db_1) = a_0 d(a_1 b_0) \wedge db_1 - a_0 a_1 db_0 \wedge db_1, \quad a_i, b_i \in A.$$ 

For instance, it follows from the product (12) that, if $a, a' \in Z_A$, then

$$da \wedge da' = -da' \wedge da, \quad ada = (da')a. \quad (22)$$

**Proposition 6.** There is the duality relation

$$\mathfrak{d}A = \text{Hom}_{A-\mathcal{A}}(O^1 A, A). \quad (23)$$

**Proof.** It follows from the definition (11) of the Chevalley–Eilenberg coboundary operator that

$$(da)(u) = u(a), \quad a \in A, \quad u \in \mathfrak{d}A. \quad (24)$$

This equality yields the morphism

$$\mathfrak{d}A \ni u \mapsto \phi_u \in \text{Hom}_{A-\mathcal{A}}(O^1 A, A), \quad \phi_u(da) := u(a), \quad a \in A.$$ 

This morphism is a monomorphism because the module $O^1 A$ is generated by elements $da$, $a \in A$. At the same time, any element $\phi \in \text{Hom}_{A-\mathcal{A}}(O^1 A, A)$ induces the derivation $u_\phi(a) := \phi(da)$ of $A$. Thus, there is a morphism

$$\text{Hom}_{A-\mathcal{A}}(O^1 A, A) \rightarrow \mathfrak{d}A,$$

which is a monomorphism since $O^1 A$ is generated by elements $da$, $a \in A$.

A different differential calculus over a noncommutative ring is often used in noncommutative geometry (Connes [6] and Landi [15]). Let us consider the tensor product $A \otimes K$ of $K$-modules. It is brought into an $A$-bimodule with respect to the multiplication

$$b(a \otimes a')c := (ba) \otimes (a'c), \quad a, a', b, c \in A.$$ 

Let us consider its submodule $\Omega^1(A)$ generated by the elements $1 \otimes a - a \otimes 1$, $a \in A$. It is readily observed that

$$d : A \ni a \mapsto 1 \otimes a - a \otimes 1 \in \Omega^1(A) \quad (25)$$

is a $\Omega^1(A)$-valued derivation of $A$. Thus, $\Omega^1(A)$ is an $A$-bimodule generated by the elements $da$, $a \in A$, such that the relation

$$(da)b = d(ab) - adb, \quad a, b \in A, \quad (26)$$
holds. Let us consider the tensor algebra $\Omega^*(A)$ of the $A$-bimodule $\Omega^1(A)$. It consists of the monomials $a_0da_1\cdots da_k$, $a_i \in A$, whose product obeys the juxtaposition rule
\[(a_0da_1)(b_0db_1) = a_0d(a_1b_0)db_1 - a_0a_1db_0b_1, \quad a_i, b_i \in A,\]
because of the relation (26). The operator $d$ (25) is extended to $\Omega^*(A)$ by the law
\[d(a_0da_1\cdots da_k) := da_0da_1\cdots da_k,\]
that makes $\Omega^*(A)$ into a differential graded algebra.

Of course, $\Omega^*(A)$ is a minimal differential calculus. One calls it the universal differential calculus over $A$ because of the following property (Landi [15]). Let $P$ be an $A$-bimodule. Any $P$-valued derivation $\Delta$ of $A$ factorizes as $\Delta = f^A \circ d$ through some $(A - A)$-module homomorphism $f^A : \Omega^1(A) \to P$. Moreover, let $A'$ be another $K$-algebra and $(\Omega^*, \delta')$ its differential calculus over a $K$-ring $A'$. Any homomorphism $A \to A'$ is uniquely extended to a morphism of differential graded algebras $\rho^* : \Omega^*(A) \to \Omega^*$ such that $\rho^{k+1} \circ d = \delta' \circ \rho^k$. Indeed, this morphism factorizes through the morphism of $\Omega^*(A)$ to the minimal differential calculus in $\Omega^*$ which sends $da \to \delta'\rho(a)$. Elements of the universal differential calculus $\Omega^*(A)$ are called universal forms. However, they can not be regarded as the noncommutative generalization of exterior forms because, in contrast with the Chevalley–Eilenberg differential calculus, the monomials $da$, $a \in Z_A$, of the universal differential calculus do not satisfy the relations (22).

It seems natural to regard derivations of a noncommutative $K$-ring $A$ and the Chevalley–Eilenberg coboundary operator $d$ (11) as particular differential operators in noncommutative geometry.

5. Differential operators in noncommutative geometry

As was mentioned above, Definition 1 provides a standard notion of differential operators on modules over a commutative ring. However, there exist its different generalizations to modules over a noncommutative ring.

Let $P$ and $Q$ be $A$-bimodules over a noncommutative $K$-ring $A$. The $K$-module $\text{Hom}_K(P, Q)$ of $K$-linear homomorphisms $\Phi : P \to Q$ can be provided with the left $A$- and $A^*$-module structures (2) and the similar right module structures
\[(\Phi a)(p) := \Phi(p)a, \quad (a \bullet \Phi)(p) := \Phi(pa), \quad a \in A, \quad p \in P.\]
(27)
For the sake of convenience, we will refer to the module structures (2) and (27) as the left and right $A - A^*$ structures, respectively. Let us put
\[\overline{\delta}_a\Phi := \Phi a - a \bullet \Phi, \quad a \in A, \quad \Phi \in \text{Hom}_K(P, Q).\]
(28)
It is readily observed that
\[\delta_a \circ \overline{\delta}_b = \overline{\delta}_b \circ \delta_a, \quad a, b \in A.\]
The left $\mathcal{A}$-module homomorphisms $\Delta : P \to Q$ obey the conditions $\delta_a \Delta = 0$, for all $a \in \mathcal{A}$ and, consequently, they can be regarded as left zero order $Q$-valued differential operators on $P$. Similarly, right zero order differential operators are defined.

Utilizing the condition (4) as a definition of a first order differential operator in noncommutative geometry, one however meets difficulties. If $P = \mathcal{A}$ and $\Delta(1) = 0$, the condition (4) does not lead to the Leibniz rule (1), i.e., derivations of the $\mathcal{K}$-ring $\mathcal{A}$ are not first order differential operators. In order to overcome these difficulties, one can replace the condition (4) with the following one (Dubois-Violette et al [7]).

**Definition 7.** An element $\Delta \in \text{Hom}_\mathcal{K}(P, Q)$ is called a first order differential operator on a bimodule $P$ over a noncommutative ring $\mathcal{A}$ if it obeys the condition

\[
\begin{align*}
\delta_a \circ \delta_b \Delta &= \delta_b \circ \delta_a \Delta = 0, \quad a, b \in \mathcal{A}, \\
da \Delta(p)b - a \Delta(pb) - \Delta(ap)b + \Delta(apb) &= 0, \quad p \in P.
\end{align*}
\] (29)

First order $Q$-valued differential operators on $P$ make up a $\mathcal{Z}_\mathcal{A}$-module $\text{Diff}_1(P, Q)$. If $P$ is a commutative bimodule over a commutative ring $\mathcal{A}$, then $\delta_a = \delta_a$ and Definition 7 comes to Definition 1 for first order differential operators.

In particular, let $P = \mathcal{A}$. Any left or right zero order $Q$-valued differential operator $\Delta$ is uniquely defined by its value $\Delta(1)$. As a consequence, there are left and right $\mathcal{A}$-module isomorphisms

\[
Q \ni q \mapsto \Delta^R_{\eta}(a) \in \text{Diff}^R_0(\mathcal{A}, Q), \quad \Delta^R_{\eta}(a) = qa, \quad a \in \mathcal{A},
\]
\[
Q \ni q \mapsto \Delta^L_{\eta}(a) \in \text{Diff}^L_0(\mathcal{A}, Q), \quad \Delta^L_{\eta}(a) = aq.
\]

A first order $Q$-valued differential operator $\Delta$ on $\mathcal{A}$ fulfills the condition

\[
\Delta(ab) = \Delta(a)b + a \Delta(b) - a \Delta(1)b.
\] (30)

It is a derivation of $\mathcal{A}$ if $\Delta(1) = 0$. One obtains at once that any first order differential operator on $\mathcal{A}$ is split into the sums

\[
\Delta(a) = a \Delta(1) + |\Delta(a) - a \Delta(1)|,
\]
\[
\Delta(a) = \Delta(1)a + |\Delta(a) - \Delta(1)a|
\]

of the derivations $\Delta(a) - a \Delta(1)$ or $\Delta(a) - \Delta(1)a$ and the left or right zero order differential operators $a \Delta(1)$ and $\Delta(1)a$, respectively. If $u$ is a $Q$-valued derivation of $\mathcal{A}$, then $au$ (2) and $ua$ (27) are so for any $a \in \mathcal{Z}_\mathcal{A}$. Hence, $Q$-valued derivations of $\mathcal{A}$ constitute a $\mathcal{Z}_\mathcal{A}$-module $\mathfrak{d}(\mathcal{A}, Q)$. There are two $\mathcal{Z}_\mathcal{A}$-module decompositions

\[
\text{Diff}_1(\mathcal{A}, Q) = \text{Diff}^L_0(\mathcal{A}, Q) \oplus \mathfrak{d}(\mathcal{A}, Q),
\]
\[
\text{Diff}_1(\mathcal{A}, Q) = \text{Diff}^R_0(\mathcal{A}, Q) \oplus \mathfrak{d}(\mathcal{A}, Q).
\]
They differ from each other in the inner derivations \( a \mapsto aq - qa \).

Let \( \text{Hom}^R_A(P,Q) \) and \( \text{Hom}^L_A(P,Q) \) be the modules of right and left \( A \)-module homomorphisms of \( P \) to \( Q \), respectively. They are provided with the left and right \( A - A^\ast \)-module structures (2) and (27), respectively.

**Proposition 8.** An element \( \Delta \in \text{Hom}_K(P,Q) \) is a first order \( Q \)-valued differential operator on \( P \) in accordance with Definition 7 iff it obeys the condition

\[
\Delta(apb) = (\hat{\partial} a)(p)b + a\Delta(p)b + a(\hat{\partial} b)(p), \quad p \in P, \ a, b \in A,
\]

where \( \hat{\partial} \) and \( \hat{\partial} \) are \( \text{Hom}^R_A(P,Q) \)- and \( \text{Hom}^L_A(P,Q) \)-valued derivations of \( A \), respectively. Namely,

\[
(\hat{\partial} a)(pb) = (\hat{\partial} a)(p)b, \quad (\hat{\partial} b)(ap) = a(\hat{\partial} b)(p).
\]

**Proof.** It is easily verified that, if \( \Delta \) obeys the equalities (31), it also satisfies the equalities (29). Conversely, let \( \Delta \) be a first order \( Q \)-valued differential operator on \( P \) in accordance with Definition 7. One can bring the condition (29) into the form

\[
\Delta(apb) = [\Delta(ap) - a\Delta(p)]b + a\Delta(p)b + a[\Delta(pb) - \Delta(p)b],
\]

and introduce the derivations

\[
(\hat{\partial} a)(p) := \Delta(ap) - a\Delta(p), \quad (\hat{\partial} b)(p) := \Delta(pb) - \Delta(p)b.
\]

For instance, let \( P \) be a differential calculus over a \( K \)-ring \( A \) provided with an associative multiplication \( \circ \) and a coboundary operator \( d \). Then \( d \) exemplifies a \( P \)-valued first order differential operator on \( P \) by Definition 7. It obeys the condition (31) which reads

\[
d(apb) = (da \circ p)b + a(dp)b + a((-1)^{|p|}p \circ db).
\]

For instance, let \( P = \mathcal{O}^* A \) be the Chevalley–Eilenberg differential calculus over \( A \). In view of the relations (21) and (23), one can think of derivations \( u \in \partial A \) as being vector fields in noncommutative geometry. A problem is that \( \partial A \) is not an \( A \)-module. One can overcome this difficulty as follows (Borowiec [4]).

Given a noncommutative \( K \)-ring \( A \) and an \( A \)-bimodule \( Q \), let \( d \) be a \( Q \)-valued derivation of \( A \). One can think of \( Q \) as being a first degree term of a differential calculus over \( A \). Let \( Q^*_R \) be the right \( A \)-dual of \( Q \). It is an \( A \)-bimodule:

\[
(bu)(q) := bu(q), \quad (ub)(q) := u(bq), \quad b \in A, \quad q \in Q.
\]

One can associate to each element \( u \in Q^*_R \) the \( K \)-module morphism

\[
\hat{u} : A \in a \mapsto u(da) \in A.
\]
This morphism obeys the relations
\[ (\hat{bu})(a) = bu(da), \quad \hat{a}(ba) = \hat{a}(b)a + (\hat{ub})(a). \] (33)

One calls \((Q^*_R, u \mapsto \hat{u})\) the \(A\)-right Cartan pair, and regards \(\hat{u}\) (32) as an \(A\)-valued first order differential operator on \(A\) (Borowiec [4]). Let us note that \(\hat{u}\) (32) need not be a derivation of \(A\) and fails to satisfy Definition 7, unless \(u\) belongs to the two-sided \(A\)-dual \(Q^* \subset Q^*_R\) of \(Q\). Morphisms \(\hat{u}\) (32) are called into play in order to describe (left) vector fields in noncommutative geometry (Borowiec [4] and Jara et al [12]).

In particular, if \(Q = O^1A\), then \(au\) for any \(u \in \mathfrak{d}A\) and \(a \in A\) is a left noncommutative vector field in accordance with the relation (15).

Similarly, the \(A\)-left Cartan pair is defined. For instance, \(ua\) for any \(u \in \mathfrak{d}A\) and \(a \in A\) is a right noncommutative vector field.

If \(A\)-valued derivations \(u_1, \ldots, u_r\) of a noncommutative \(K\)-ring \(A\) or the above mentioned noncommutative vector fields \(\hat{u}_1, \ldots, \hat{u}_r\) on \(A\) are regarded as first order differential operators on \(A\), it seems natural to think of their compositions \(u_1 \circ \cdots \circ u_r\) or \(\hat{u}_1 \circ \cdots \circ \hat{u}_r\) as being particular higher order differential operators on \(A\). Let us turn to the general notion of a differential operator on \(A\)-bimodules.

By analogy with Definition 1, one may try to generalize Definition 7 by means of the maps \(\delta_a\) (3) and \(\overline{\delta}_a\) (28). A problem lies in the fact that, if \(P = Q = A\), the compositions \(\delta_a \circ \delta_b\) and \(\overline{\delta}_a \circ \overline{\delta}_b\) do not imply the Leibniz rule and, as a consequence, compositions of derivations of \(A\) fail to be differential operators.

This problem can be solved if \(P\) and \(Q\) are regarded as left \(A\)-modules (Lunts et al [17]). Let us consider the \(\mathcal{K}\)-module \(\text{Hom}_K(P, Q)\) provided with the left \(A - \mathcal{A}^*\) module structure (2). We denote by \(Z_0\) its center, i.e., \(\delta_a \Phi = 0\) for all \(\Phi \in Z_0\) and \(a \in A\). Let \(I_0 = \overline{Z}_0\) be the \(A - \mathcal{A}^*\) submodule of \(\text{Hom}_K(P, Q)\) generated by \(Z_0\). Let us consider:

(i) the quotient \(\text{Hom}_K(P, Q)/I_0\),
(ii) its center \(Z_1\),
(iii) the \(A - \mathcal{A}^*\) submodule \(\overline{Z}_1\) of \(\text{Hom}_K(P, Q)/I_0\) generated by \(Z_1\),
(iv) the \(A - \mathcal{A}^*\) submodule \(I_1\) of \(\text{Hom}_K(P, Q)\) given by the relation \(I_1/I_0 = \overline{Z}_1\).

Then we define the \(A - \mathcal{A}^*\) submodules \(I_r, r = 2, \ldots, \) of \(\text{Hom}_K(P, Q)\) by induction as \(I_r/I_{r-1} = \overline{Z}_r\), where \(\overline{Z}_r\) is the \(A - \mathcal{A}^*\) module generated by the center \(Z_r\) of the quotient \(\text{Hom}_K(P, Q)/I_{r-1}\).

**Definition 9.** Elements of the submodule \(I_r\) of \(\text{Hom}_K(P, Q)\) are said to be left \(r\)-order \(Q\)-valued differential operators on an \(A\)-bimodule \(P\) (Lunts et al [17]).

**Proposition 10.** An element \(\Delta \in \text{Hom}_K(P, Q)\) is a differential operator of order \(r\) in accordance with Definition 9 if it is a finite sum
\[ \Delta(p) = b_i \Phi^i(p) + \Delta_{r-1}(p), \quad b_i \in A, \] (34)

where \(\Delta_{r-1}\) and \(\delta_a \Phi^i\) for all \(a \in A\) are \((r - 1)\)-order differential operators if \(r > 0\), and they vanish if \(r = 0\).
Proof. If \( r = 0 \), the statement is a straightforward corollary of Definition 9. Let \( r > 0 \).

The representatives \( \Phi_r \) of elements of \( Z_r \) obey the relation

\[
\delta_c \Phi_r = \Delta'_{r-1}, \quad c \in A,
\]

where \( \Delta'_{r-1} \) is an \((r - 1)\)-order differential operator. Then representatives \( \Phi_r \) of elements of \( Z_r \) take the form

\[
\Phi_r(p) = \sum_i c'_i \Phi^i(c_i p) + \Delta''_{r-1}(p), \quad c_i, c'_i \in A,
\]

where \( \Phi^i \) satisfy the relation (35) and \( \Delta''_{r-1} \) is an \((r - 1)\)-order differential operator. Due to the relation (35), we obtain

\[
\Phi_r(p) = b_i \Phi^i(p) + \Delta''_{r-1}(p), \quad b_i = c_i c'_i, \quad \Delta''_{r-1} = - \sum_i c'_i \delta_{c_i} \Phi^i + \Delta''_{r-1}.
\]

Hence, elements of \( I_r \) modulo elements of \( I_{r-1} \) take the form (36), i.e., they are given by the expression (34). The converse is obvious.

If \( A \) is a commutative ring, Definition 9 comes to Definition 1. Indeed, the expression (34) shows that \( \Delta \in \text{Hom}_K(P, Q) \) is an \( r \)-order differential operator iff \( \delta_a \Delta \) for all \( a \in A \) is a differential operator of order \( r - 1 \).

Proposition 11. If \( P \) and \( Q \) are \( A \)-bimodules, the set \( I_r \) of \( r \)-order \( Q \)-valued differential operators on \( P \) is provided with the left and right \( A - A^* \) module structures.

Proof. This statement is obviously true for zero order differential operators. Using the expression (34), one can prove it for higher order differential operators by induction.

Let \( P = Q = A \). Any zero order differential operator on \( A \) in accordance with Definition 9 takes the form \( a \mapsto cac' \) for some \( c, c' \in A \).

Proposition 12. Let \( \Delta_1 \) and \( \Delta_2 \) be \( n \)- and \( m \)-order \( A \)-valued differential operators on \( A \), respectively. Then their composition \( \Delta_1 \circ \Delta_2 \) is an \((n + m)\)-order differential operator.

Proof. The statement is proved by induction as follows. If \( n = 0 \) or \( m = 0 \), the statement issues from the fact that the set of differential operators possesses both left and right \( A - A^* \) structures. Let us assume that \( \Delta \circ \Delta' \) is a differential operator for any \( k \)-order differential operators \( \Delta \) and \( s \)-order differential operators \( \Delta' \) such that \( k + s < n + m \). Let us show that \( \Delta_1 \circ \Delta_2 \) is a differential operator of order \( n + m \). Due to the expression (34), it suffices to prove this fact when \( \delta_a \Delta_1 \) and \( \delta_a \Delta_2 \) for any \( a \in A \) are differential operators of order \( n - 1 \) and \( m - 1 \), respectively. We have the equality

\[
\delta_a(\Delta_1 \circ \Delta_2)(b) = a(\Delta_1 \circ \Delta_2)(b) - (\Delta_1 \circ \Delta_2)(ab) = \\
\Delta_1(a \Delta_2(b)) + (\delta_a \Delta_1 \circ \Delta_2)(b) - (\Delta_1 \circ \Delta_2)(ab) = \\
(\Delta_1 \circ \delta_a \Delta_2)(b) + (\delta_a \Delta_1 \circ \Delta_2)(b),
\]
whose right-hand side, by assumption, is a differential operator of order \( n + m - 1 \).

Any derivation \( u \in \mathfrak{d} \mathcal{A} \) of a \( K \)-ring \( \mathcal{A} \) is a first order differential operator in accordance with Definition 9. Indeed, it is readily observed that

\[
(\delta_a u)(b) = au(b) - u(ab) = -u(a)b, \quad b \in \mathcal{A},
\]

is a zero order differential operator for all \( a \in \mathcal{A} \). The compositions \( au, u \cdot a \) (2), \( ua, a \cdot u \) (27) for any \( u \in \mathfrak{d} \mathcal{A}, a \in \mathcal{A} \) and the compositions of derivations \( u_1 \circ \cdots \circ u_r \) are also differential operators on \( \mathcal{A} \) in accordance with Definition 9.

At the same time, noncommutative vector fields do not satisfy Definition 9 in general. First order differential operators by Definition 7 also need not obey Definition 9, unless \( P = Q = \mathcal{A} \).

By analogy with Definition 9 and Proposition 10, one can define differential operators on right \( \mathcal{A} \)-modules as follows.

**Definition 13.** Let \( P \) and \( Q \) be seen as right \( \mathcal{A} \)-modules over a noncommutative \( K \)-ring \( \mathcal{A} \). An element \( \Delta \in \text{Hom}_K(P, Q) \) is said to be a right zero order \( Q \)-valued differential operator on \( P \) if it is a finite sum \( \Delta = \Phi^i b_i, b_i \in \mathcal{A} \), where \( \delta_a \Phi^i = 0 \) for all \( a \in \mathcal{A} \). An element \( \Delta \in \text{Hom}_K(P, Q) \) is called a right differential operator of order \( r > 0 \) on \( P \) if it is a finite sum

\[
\Delta(p) = \Phi^i(p) b_i + \Delta_{r-1}(p), \quad b_i \in \mathcal{A},
\]

where \( \Delta_{r-1} \) and \( \delta_a \Phi^i \) for all \( a \in \mathcal{A} \) are right \( (r - 1) \)-order differential operators.

Definition 9 and Definition 13 of left and right differential operators on \( \mathcal{A} \)-bimodules are not equivalent, but one can combine them as follows.

**Definition 14.** Let \( P \) and \( Q \) be bimodules over a noncommutative \( K \)-ring \( \mathcal{A} \). An element \( \Delta \in \text{Hom}_K(P, Q) \) is a two-sided zero order \( Q \)-valued differential operator on \( P \) if it is either a left or right zero order differential operator. An element \( \Delta \in \text{Hom}_K(P, Q) \) is said to be a two-sided differential operator of order \( r > 0 \) on \( P \) if it is brought both into the form

\[
\Delta = b_i \Phi^i + \Delta_{r-1}, \quad b_i \in \mathcal{A},
\]

and

\[
\Delta = \Phi \delta_i + \Delta_{r-1}, \quad \delta_i \in \mathcal{A},
\]

where \( \Delta_{r-1}, \Delta_{r-1} \) and \( \delta_a \Phi^i, \delta_a \Phi^i \) for all \( a \in \mathcal{A} \) are two-sided \( (r - 1) \)-order differential operators.

One can think of this definition as a generalization of Definition 7 to higher order differential operators.

It is readily observed that two-sided differential operators described by Definition 14 need not be left or right differential operators, and *vice versa*. At the same time, \( \mathcal{A} \)-valued derivations of a \( K \)-ring \( \mathcal{A} \) and their compositions obey Definition 14.
In conclusion, note that Definition 4 is also generalized to a certain class of first order differential operators on modules over a noncommutative ring.

The notion of a jet is extended to a module $P$ over a noncommutative ring $A$. One can follow Definition 2 in order to define the left jets of a two-sided module $P$ over a noncommutative ring $A$. However, the relation (7) fails to hold if $k > 0$. Therefore, no module of left jets is the representative object of left differential operators. The notion of a right jet of $P$ meets the similar problem. Given an $A$-module $P$, let us consider the tensor product $A \otimes K P \otimes K A$ of $K$-modules $A$ and $P$. We put

$$\delta^b(a \otimes p \otimes c) := (ba) \otimes p \otimes c - a \otimes (bp) \otimes c,$$

$$\delta^c(a \otimes p \otimes c) := a \otimes p \otimes (cb) - a \otimes (pb) \otimes c, \quad p \in P, \quad a, b, c \in A.$$

Let us denote by $\mu^1$ the two-sided $A$-submodule of $A \otimes K P \otimes K A$ generated by elements of the type $\delta^c \circ \delta^b(1 \otimes p \otimes 1)$. One can define the first order two-sided jet module $J^1(P) = A \otimes K P \otimes K A/\mu^1$ of $P$. Let us denote

$$J : P \ni p \mapsto 1 \otimes p \otimes 1 \in A \otimes P \otimes A.$$

Then the equality

$$\delta_{b_0} \circ \cdots \circ \delta_{b_k}(f \circ J)(p) = f(\delta_{b_0} \circ \cdots \circ \delta_{b_k}(1 \otimes p \otimes 1))$$

holds for any $f \in \text{Hom}_{A-A}(A \otimes P, Q)$. It follows that $J^1(P)$ is the representative object of the functor $Q \rightarrow \text{Diff}_1(P, Q)$ where the $K$-module $\text{Diff}_1(P, Q)$ consists of two-sided first order differential operators $\Delta$ by Definition 14 which obey the condition $\delta^c \circ \delta^b \Delta = 0$ for all $c, b \in A$. They are the first order differential operators by (Dubois-Violette et al [7]).

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