Investigation of the maximum amplitude increase from the Benjamin–Feir instability

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Abstract

The Nonlinear Schrödinger (NLS) equation is used to model surface waves in wave tanks of hydrodynamic laboratories. Analysis of the linearized NLS equation shows that its harmonic solutions with a small amplitude modulation have a tendency to grow exponentially due to the so-called Benjamin–Feir instability. To investigate this growth in detail, we relate the linearized solution of the NLS equation to a fully nonlinear, exact solution, called soliton on finite background. As a result, we find that in the range of instability the maximum amplitude increase is finite and can be at most three times the initial amplitude.

Keywords: Nonlinear Schrödinger equation, Benjamin–Feir instability, soliton on finite background, maximum amplitude increase.

1 Introduction

This is an initial work on 'Extreme Waves in Hydrodynamics Laboratory'. Extreme waves here refer to very high amplitude, steep, waves that can appear suddenly from a relatively calm sea. Although extreme waves are very rare and unpredictable, they are still very dangerous to ships in case they meet.

We model the problem of extreme waves using dispersive wave modes. The specific property of dispersion is that waves with different wave length propagate with different phase velocities. In the following we assume that the wave field has a frequency spectrum that is localized around one frequency. Then the envelope of the wave field is described by the NLS equation [3]. The NLS equation is an amplitude equation for describing the change of envelope of a wave group. This equation is very instrumental in understanding various nonlinear wave phenomena: it arises in studies of unidirectional propagation of wave packets in a energy conserving dispersive medium at the lowest order of nonlinearity [4].
In this paper we analyze the behavior of the wave group envelope using both linear and nonlinear theories. Linear theory predicts exponential growth of the amplitude when certain conditions are satisfied—the Benjamin–Feir instability [4]. However, when the amplitude becomes large nonlinear effects must be taken into account, that, as it turns out, will prevent further exponential growth. The aim of this paper is to find the maximum amplitude of waves when the amplitude growth is triggered by the Benjamin–Feir instability. Also, we investigate how this maximum amplitude depends on the growth rate parameter from the Benjamin–Feir instability.

2 Modelling of Waves Envelope

2.1 Linear Theory

In linear theory of water waves, we can restrict the analysis of a surface elevation $\eta(x,t)$ to one-mode solution of the form $\eta(x,t) = ae^{i(kx-\omega t)} + c.c.$ (complex conjugate), where $a$ is a constant amplitude, $k$ is wavenumber, and $\omega$ is frequency. Then a general solution is a superposition of one-mode solutions. For surface waves on water of constant depth $h$, the parameters $k$ and $\omega$ are related by the following linear dispersion relation [3]:

$$\omega^2 = g k \tanh kh,$$

(1)

where $g$ is gravitational acceleration. We also write the linear dispersion relation as $\omega \equiv \Omega(k) = k \sqrt{\frac{g \tanh kh}{k}}$. This dispersion relation can be derived from the linearized equations of the full set of equations for water waves. The **phase velocity** is defined as $\Omega' = \frac{d\Omega}{dk}$ and the **group velocity** is defined as $\frac{d\Omega}{dk} = \Omega''$.

Since the linear wave system has elementary solutions of the form $e^{i(kx-\Omega(k)t)}$, it is often convenient to write the general solution of an initial value problem as an integral of its Fourier components [4]:

$$\eta(x,t) := \int_{-\infty}^{\infty} \alpha(k) e^{i(kx-\Omega(k)t)} dk,$$

(2)

where $\alpha(k)$ is the Fourier transform of $\eta(x,0)$. Writing dispersion relation $\Omega(k)$ as a power series (Taylor expansion) about a fixed wavenumber $k_0$ and neglecting $O(\kappa^3)$ terms, we find

$$\beta \kappa^2 = \Omega(k_0 + \kappa) - \Omega(k_0) - \Omega'(k_0) \kappa,$$

(3)

where $\beta = \frac{1}{2} \Omega''(k_0)$. Let us define $k = k_0 + \kappa$, $\tau = t$, and $\xi = x - \Omega(k_0) t$. Then equation (2) can be written like

$$\eta(x,t) = e^{i(k_0 x - \Omega(k_0) t)} \int_{-\infty}^{\infty} \alpha(k_0 + \kappa) e^{i\kappa \xi} e^{-i \beta \kappa^2 \tau} d\kappa.$$  

(4)

Denoting the integral in (4) with $\psi(\xi,\tau)$, we find that $\psi(\xi,\tau)$ satisfies

$$i \frac{\partial \psi}{\partial \tau} + \beta \frac{\partial^2 \psi}{\partial \xi^2} = 0.$$

(5)
This is the linear Schrödinger equation for narrow–banded spectra. Equation (5) is a partial differential equation that describes time evolution of the envelope of a linear wave packet [4]. Equation (5) has a monochromatic mode solution \( \psi(\xi, \tau) = e^{i(\kappa \xi - \nu \tau)} \) where \( \nu = \beta \kappa^2 \).

### 2.2 Nonlinear Theory

Assuming narrow-banded spectra, we consider the wave elevation in the following form

\[
\eta(x, t) = \psi(\xi, \tau) e^{i(k_0 x - \omega_0 t)} + c.c.,
\]

where \( \tau = t, \xi = x - \Omega(k_0) t, \psi(\xi, \tau) \) is a complex valued function (called the complex amplitude), \( k_0 \) and \( \omega_0 \) are central wavenumber and frequency, respectively. The evolution of the wave elevation is a weakly nonlinear deformation of a nearly harmonic wave with the fixed wavenumber \( k_0 \). If we substitute \( \eta \) to the equations which describes the physical motion of the water waves (see below), then one finds that the complex amplitude \( \psi(\xi, \tau) \) satisfies the nonlinear Schrödinger (NLS) equation.

As example, the NLS equation can be derived from the modified KdV equation

\[
\eta_t + i \Omega(-i \partial_x) \eta + \frac{3}{4} \partial_x \eta^2 = 0 \tag{2}
\]

With \( \tau = t \) and \( \xi = x - \Omega(k_0) t \), the corresponding NLS equation reads

\[
\frac{i}{\tau} \frac{\partial \psi}{\partial \tau} + \beta \frac{\partial^2 \psi}{\partial \xi^2} + \gamma |\psi|^2 \psi = 0, \beta, \gamma \in \mathbb{R},
\tag{6}
\]

where \( \beta \) and \( \gamma \) depend only on \( k_0 \):

\[
\beta = \frac{1}{2} \Omega'(k_0),
\tag{7}
\]

\[
\gamma = -\frac{9}{4} k_0 \left( \frac{1}{\Omega(k_0)} - \frac{1}{\Omega'(0)} + \frac{k_0}{2 \Omega(k_0) - \Omega(2k_0)} \right),
\tag{8}
\]

where \( \omega = \Omega(k) \) is the linear dispersion relation [6]. The coefficients \( \beta \) and \( \gamma \) in this paper have opposite signs compared to the corresponding coefficients in [6]. This equation arose as a model for packets of waves on deep water.

There are two types of NLS equations:

- If \( \beta \) and \( \gamma \) have the same sign, i.e. \( \beta \gamma > 0 \), then (6) is called the focusing NLS equation (an attractive nonlinearity, modulationally unstable [Benjamin–Feir instability]) [5].

- If \( \beta \) and \( \gamma \) have different signs, i.e. \( \beta \gamma < 0 \), then (6) is called the de-focusing NLS equation (a repulsive nonlinearity, stable solution) [5].

The wavenumber \( k_{\text{crit}} \), for which \( \beta \gamma = 0 \) holds, is called the critical wavenumber or the Davey–Stewartson value. Using the physical quantity \( g = 9.8 \text{ m/s}^2 \) and the water depth \( h = 5 \text{ m} \), then for wavenumbers \( k_0 > k_{\text{crit}} = 0.23 \), the product \( \beta \gamma > 0 \), and the NLS equation is of focusing type. The corresponding critical wavelength is 27.41 m.
In the following, we consider only focusing NLS equations. The NLS equation has a plane-wave solution
\[ A(\tau) = r_0 e^{i \gamma r_0^2 \tau}. \] (9)
In physical variables, the surface wave elevation is given as
\[ \eta(x, t) = 2 r_0 \cos(k_0 x - \omega_0 t + \gamma r_0^2 t). \]
Note that the corresponding phase velocity is \( \omega_0 - \gamma r_0^2 k_0 \). In the following section, we analyze the stability of this plane-wave solution.

3 Benjamin–Feir Instability

To investigate the instability of the NLS plane-wave solution, we perturb the \( \xi \)-independent function \( A(\tau) \) with a small perturbation of the form \( \epsilon(\xi, \tau) = A(\tau) B(\xi, \tau) \).

We look for the cases where under a small perturbation the amplitude of the plane wave solution grows in time [3]:
\[ \psi(\xi, \tau) = A(\tau)[1 + B(\xi, \tau)]. \] (10)
Substituting (10) into (6), and ignoring nonlinear terms we obtain the linearized NLS equation:
\[ i B_\tau + \beta B_{\xi \xi} + \gamma r_0^2 (B + B^*) = 0. \] (11)
We seek the solutions of (11) in the form
\[ B(\xi, \tau) = B_1 e^{(\sigma \tau + i \kappa \xi)} + B_2 e^{(\sigma^* \tau - i \kappa \xi)}, \] (12)
where \( B_1, B_2 \in \mathbb{C} \), \( k_0 + \kappa \) is local wavenumber, \( \kappa \) is modulation wavenumber, and \( \sigma \in \mathbb{C} \) is called the growth rate. If \( \text{Re} (\sigma) > 0 \), then the perturbed solution of the NLS equation grows exponentially. This is the criterion for the so-called Benjamin–Feir instability of a one-wave mode with modulation wavenumber \( \kappa \) [3].

Substituting the function \( B(\xi, \tau) \) in (12) into (11) yields a pair of coupled equations that can be written in matrix form as follows
\[
\begin{pmatrix}
i \sigma - \beta \kappa^2 + \gamma r_0^2 \\
\gamma r_0^2 \\
i \sigma - \beta \kappa^2 + \gamma r_0^2 \\
\end{pmatrix}
\begin{pmatrix}
B_1 \\

B_2 \\
B_2^* \\
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0 \\
0 \\
\end{pmatrix}
\] (13)
Nontrivial solution to (13) can exist only if the determinant of the left hand side matrix is zero. This condition reads as follows
\[ \sigma^2 = \beta \kappa^2 (2 \gamma r_0^2 - \beta \kappa^2). \] (14)
We have the following cases:

- The growth rate \( \sigma \) is real and positive if \( \kappa^2 < 2 \gamma r_0^2 \). This corresponds to Benjamin–Feir instability. For specified values of \( \kappa \), the perturbation amplitude is exponentially amplified in time [3].
The growth rate $\sigma$ is purely imaginary if $\kappa^2 > \frac{2\gamma}{\beta} \beta_0^2$. This corresponds to a plane-wave solution that has bounded amplitude for all time [3].

Thus, the range of instability is given by

$$0 < |\kappa| < |\kappa_{\text{crit}}| = \sqrt{\frac{2\gamma}{\beta} \beta_0}.$$  \hspace{1cm} (15)

It is easy to find that the ‘strongest’ instability occurs at $\kappa_{\text{max}} = \sqrt{\frac{2\gamma}{\beta} \beta_0}$, where the maximum growth rate is $\sigma_{\text{max}} = \gamma r_0^2$. Figure 1 shows the plot of growth rate $\sigma$ as a function of modulation wavenumber $\kappa$ for $r_0 = \beta = \gamma = 1$.

Figure 1: Growth rate $\sigma$ as a function of modulation wavenumber $\kappa$ for $r_0 = \beta = \gamma = 1$.

We can write the solution of the linearized NLS equation as

$$\psi(\xi, \tau) = r_0 e^{i \gamma r_0^2 \tau} [1 + e^{\sigma(\kappa) \tau} (B_1 e^{i \kappa \xi} + B_2 e^{-i \kappa \xi})],$$  \hspace{1cm} (16)

with $\sigma(\kappa) = \kappa \sqrt{\frac{2\beta \gamma}{\beta} \beta_0^2 - \beta^2 \kappa^2}$. Since this solution is obtained from the linearized NLS equation, it is only valid if amplitudes are small. When the time is increased, the amplitude increases exponentially and the linearized theory becomes invalid. Therefore, we cannot use the solution from the linearized equation to investigate the behavior of the maximum amplitude increase due to the Benjamin–Feir instability. Fortunately, there exists an exact solution to the NLS equation that describes the exact behavior of the wave profile and that corresponds to the Benjamin–Feir instability.

### 4 Modulational Instability

In this section we investigate the relation between the maximum amplitude of a certain solution of the NLS equation and the modulation wavenumber $\kappa$ in the
instability interval. For simplicity, we choose the amplitude \( r_0 = 1 \), and the coefficients \( \beta = \gamma = 1 \). An exact solution, the so-called soliton on finite background, in short SFB, (sometimes also called the second most important solution), of the NLS equation is given by [1]:

\[
\psi(\xi, \tau) := \left( \kappa^2 - 1 \right) \cosh(\sigma(\kappa) \tau) + \sqrt{\frac{2 - \kappa^2}{2}} \cos(\kappa \xi) + i \sigma(\kappa) \sinh(\sigma(\kappa) \tau) \cosh(\sigma(\kappa) \tau) - \sqrt{\frac{2 - \kappa^2}{2}} \cos(\kappa \xi) e^{i \tau},
\]

(17)

where \( 0 < \kappa < \sqrt{2} \) and \( \sigma(\kappa) = \kappa\sqrt{2 - \kappa^2} \).

For \( \kappa = 1 \), we have

\[
\psi(\xi, \tau) := \cos \xi + i \sqrt{2} \sinh \tau \sqrt{2 \cosh \tau - \cos \xi} e^{i \tau}.
\]

(18)

Figure 2 shows the plot of \(|\psi|\) from (18) as a function of \( \xi \) and \( \tau \). Note that \( \psi(\xi, \tau) \) is a \( 2\pi \)-periodic function with respect to \( \xi \) variable.

\[
\text{Figure 2: Plot of } |\psi(\xi, \tau)| \text{ for modulation wavenumber } \kappa = 1.
\]

In the following, we analyze the solution (17) in detail. The behavior for this SFB as \( \tau \to \pm \infty \) is given by

\[
\lim_{\tau \to \infty} |\psi(\xi, \tau)| = \sqrt{\left( \kappa^2 - 1 \right)^2 + \kappa^2 (2 - \kappa^2)} = 1.
\]

(19)

Because of this property the solution (17) is called as SFB. For a ‘normal’ soliton, the elevation vanishes at infinity: the ‘normal’ soliton is exponentially confined. For SFB, the solution is a similar elevation on top of the finite (nonzero) background level, here the normalized value 1.
Write the solution in the form $\psi(\xi, \tau) = u(\xi, \tau)e^{i\tau}$, where $u(\xi, \tau)$ describes the amplitude and the exponential part expresses oscillations in time. Let us investigate the behavior of $u(\xi, \tau)$ in time. For that consider the limiting behavior of $\partial_\tau u$ as $\tau \to \pm \infty$. We have

$$
\frac{\partial u}{\partial \tau}(\xi, \tau) = \frac{ia^2(\kappa) - \sigma(\kappa)\sqrt{2-\kappa^2}\cos(\kappa\xi)}{\left[\cosh(\sigma(\kappa)\tau) - \sqrt{2-\kappa^2}\cos(\kappa\xi)\right]^2}
$$

If $\xi \neq \frac{\pi}{2\kappa}$, then

$$
\frac{\partial u}{\partial \tau} \approx -2\kappa^2 e^{-\sigma(\kappa)\tau} \quad \text{if} \quad \tau \gg 0,
$$

and

$$
\frac{\partial u}{\partial \tau} \approx 2\kappa^2 e^{\sigma(\kappa)\tau} \quad \text{if} \quad \tau \ll 0.
$$

If $\xi = \frac{\pi}{2\kappa}$, then

$$
\frac{\partial u}{\partial \tau} \approx 4i\kappa^2 e^{-2\sigma(\kappa)\tau} \quad \text{for} \quad \tau \gg 0,
$$

and

$$
\frac{\partial u}{\partial \tau} \approx 4i\kappa^2 e^{2\sigma(\kappa)\tau} \quad \text{for} \quad \tau \ll 0.
$$

Figure 3: Plot amplitude maximum $\psi(\xi = 0, \tau = 0; \kappa)$ as a function of the modulation wavenumber $\kappa$.

Next, let us find the relation between the maximum amplitude of the exact solution \ref{eq:17} with modulation wavenumber $\kappa$, where $0 < \kappa < \sqrt{2}$. The maximum
value of the complex amplitude is at $\xi \equiv 0 \pmod{2\pi}$ and when $\tau = 0$. So, we have

$$|\psi|_{\text{max}} = |\psi(0, 0; \kappa)| = \frac{\kappa^2 - 1 + \sqrt{1 - \frac{1}{4} \kappa^2}}{1 - \sqrt{1 - \frac{1}{4} \kappa^2}}. \quad (25)$$

Using the approximation $\sqrt{1 + a} \approx 1 + \frac{1}{2} a$ for small $a$, and apply it to our formula ($a = -\frac{1}{2} \kappa^2$), we obtain

$$\lim_{\kappa \to 0} |\psi(0, 0; \kappa)| = 3. \quad (26)$$

As a result, the maximum factor of the amplitude amplification is

$$\lim_{\kappa \to 0} \frac{|\psi(0, 0; \kappa)|}{\lim_{\tau \to \pm \infty} |\psi(\xi, \tau; \kappa)|} = \frac{3}{1} = 3. \quad (27)$$

Figure 3 shows the plot of the maximum amplitude $|\psi|_{\text{max}}$ as a function of modulation wavenumber $\kappa$.

To summarize, Figure 4 compares the plot of the dispersion relation $\Omega$ and its quadratic approximation (leads to the NLS equation), the growth rate $\sigma$ (which is related to Benjamin–Feir instability), and the maximum amplitude of $\psi$ as functions of wavenumber $k$.

Figure 4: Dispersion relation $\Omega$ and its quadratic approximation, growth rate $\sigma$ (Benjamin–Feir instability), and maximum amplitude of $\psi$ as functions of wavenumber $k$. The case corresponds to $k_0 = 2\pi$. 
5 Conclusion

In this paper we modelled surface waves in wave tanks of hydrodynamics laboratories using the NLS equation. We analyzed the linearized NLS equation and obtained that its solutions have tendency to grow exponentially. We considered also the exact solution known as SFB of the NLS equation, that is the continuation of this linear instability. Using this, we found the maximum amplitude in space and time when the modulation wavenumber is in the interval of the Benjamin–Feir instability. The main result of this paper is that the maximum factor of the amplitude amplification due to the Benjamin–Feir instability is three. As we can see from Figure 4, the growth rate from the Benjamin–Feir instability does not determine the maximum amplitude amplification of the SFB. The results of this paper will be used in the further study of relating the Benjamin–Feir instability and the Phase–Amplitude equations.

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