TIME ANALYTICITY WITH HIGHER NORM ESTIMATES FOR THE 2D NAVIER-STOKES EQUATIONS

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Abstract. This paper establishes bounds on norms of all orders for solutions on the global attractor of the 2D Navier-Stokes equations, complexified in time. Specifically, for periodic boundary conditions on $[0, L]^2$, and a force $g \in D(A^{-\frac{1}{2}})$, we show there is a fixed strip about the real time axis on which a uniform bound $\|A^\alpha u\| < m_\alpha \nu \kappa_0^\alpha$ holds for each $\alpha \in \mathbb{N}$. Here $\nu$ is viscosity, $\kappa_0 = \frac{2\pi}{L}$, and $m_\alpha$ is explicitly given in terms of $g$ and $\alpha$. We show that if any element in $A$ is in $D(A^\alpha)$, then all of $A$ is in $D(A^\alpha)$, and likewise with $D(A^\alpha)$ replaced by $C^\infty(\Omega)$. We demonstrate the universality of this "all for one, one for all" law on the union of a hierarchal set of function classes. Finally, we treat the question of whether the zero solution can be in the global attractor for a nonzero force by showing that if this is so, the force must be in a particular function class.

1. Introduction

It has been known for nearly half a century that all solutions in the global attractor $A$ of the incompressible 2D Navier-Stokes equations (NSE) can be extended to analytic functions in a uniform strip $S$ straddling the real axis in the complexified time plane (e.g. [3,13,17,18,21]). Moreover, these solutions are uniformly bounded in the norm $|A^\alpha \cdot|$, where $A$ is the Stokes operator, $|\cdot| = |\cdot|_{L_2(\Omega)}$, and $\Omega = [0, L]^2$ is the spatial domain, with periodic boundary conditions. This is so, at least if the external force $g$ in the equation is complex-analytic in such a strip. On the other hand, as remarked in [4], if $0 \in A$, then by inserting 0 into the NSE, one sees that $g$ must be in the domain $D(A)$, so that in fact the solution is in $D(A^2)$. This in turn implies that $g$ is in $D(A^2)$, and so on by induction. Thus, $g$ must be in $D(A^\alpha)$ for any $\alpha \in \mathbb{N}$.

Analyticity in time has a number of applications. It allows for the use of the Cauchy integral formula in deriving bounds. It offers a route to proving backward uniqueness (see [10,23] for compressible flow, and [6] for an application in 3D). The width of the strip of analyticity affects the approximation of the global attractor by various purely algebraic methods [8,10,14].

In this paper we carry out rather intensive estimates during the inductive process described in the opening paragraph to establish uniform bounds in $|A^\alpha \cdot|$ on a strip $S$ of a specific width $\delta$ for all $\alpha \in \mathbb{N}$. This implies that if $0 \in A$, then all elements in $A$, as well as $g$, are in $C^\infty(\Omega)$. We show that the bounds in $|A^\alpha \cdot|$ can be sharpened to

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some extent, by reducing the width of the strip according to $\delta_{n+1} = \delta_n/2$. Moreover, we prove the following “all for one, one for all” law \[19\]: regardless of whether $0 \in \mathcal{A}$: if any element in $\mathcal{A}$ is in $\mathcal{D}(A^0)$, then all of $\mathcal{A}$ is in $\mathcal{D}(A^0)$, and likewise with $\mathcal{D}(A^0)$ replaced by $C^\infty(\Omega)$.

It is expected that this law is somewhat universal in that it would hold for a variety of subsets of $H$, the natural phase space for the NSE (see \[2.1\]). We explore a particular family of function classes $\mathcal{C}(\sigma) \subset C^\infty(\Omega)$ for which all functions $u$ satisfy $\sup_{\alpha \in \mathbb{N}} |A^{\alpha/2} u| \exp(-\sigma \alpha^2/2) < \infty$. Indeed, we show in Section \[11\] that the “all for one, one for all” law holds for $\cup_{\sigma \in \mathbb{N}} \mathcal{C}(\sigma)$. These classes are shown to be truly hierarchal: $\mathcal{C}(\sigma_1) \not\subset \mathcal{C}(\sigma_2)$ and moreover $\cup_{\sigma \in \mathbb{N}} \mathcal{C}(\sigma) \not\subset C^\infty(\Omega)$.

The distinction of the zero element in $H$ for a non-zero force $g$ is intriguing. It is clear from the discussion in the opening paragraph that if $g \not\in \mathcal{D}(A)$, then $0 \not\in \mathcal{A}$. In addition, we have a lower bound on $|u|$ for $u \in \mathcal{A}$ in the case $g \in \mathcal{D}(A^0)$ which is valid for forces heavily weighted in the higher Fourier modes (see Theorem 12.2 in \[4\]). The higher modes in the force must be more heavily weighted as the Grashof number $G$ (see \[3.3\]) is increased. Due to its connection to the dissipation length scale, $G$ must be large for a 2D flow to be turbulent \[12\]. It is unknown whether there exists any nonzero force for which $0 \in \mathcal{A}$. In this paper we find a particular value $\sigma^*$, such that if $0 \in \mathcal{A}$, then $g \in \mathcal{C}(\sigma^*)$. This narrows somewhat the search for such a special force.

2. Main results

We consider the Navier-Stokes equations (NSEs) on $\Omega = [0, L]^2$

$\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla)u + \nabla p = F,$

$\nabla \cdot u = 0,$

$u(x, 0) = u_0(x),$

$\int_{\Omega} u \, dx = 0,$

$\int_{\Omega} F \, dx = 0,$

where $u : \mathbb{R}^2 \to \mathbb{R}^2$ and $p : \mathbb{R}^2 \to \mathbb{R}$ are unknown, $\Omega$-periodic functions, and $\nu > 0$ is the kinematic viscosity of the fluid, $L > 0$ is the period, $p$ is the pressure, and $F$ is the “body” force as in \[21\] \[3\] \[22\]. We introduce the phase space $H$ as the subspace of $L^2(\Omega)^2$ consisting of the closure of the set of all $\mathbb{R}^2$-valued trigonometric polynomials $v$ such that

$$\nabla \cdot v = 0 \quad \text{and} \quad \int_{\Omega} v(x) \, dx = 0.$$  

The scalar product in $H$ is taken to be

$$\langle u, v \rangle = \int_{\Omega} u(x) \cdot v(x) \, dx$$

with associated norm $|u| = (u, u)^{1/2}$.

Let $\mathcal{P} : L^2(\Omega)^2 \to L^2(\Omega)^2$ be the orthogonal projection (called the Helmholtz-Leray projection) with range $H$, and define the Stokes operator as $A = -\mathcal{P} \Delta$ (= $-\Delta$, under periodic boundary conditions), which is positive, self-adjoint with a compact inverse. As a consequence, the space $H$ has an orthonormal basis $(w_j)_{j=1}^\infty$ of eigenfunctions of $A$, namely, $Aw_j = \lambda_j w_j$, with $0 < \lambda_1 = (2\pi/L)^2 \leq \lambda_2 \leq \lambda_3 \leq \cdots$. 

(cf. [3, 22]). The powers \( A^\sigma \) are defined by
\[
A^\sigma v = \sum_{j=1}^{\infty} \lambda_j^\sigma (v, w_j)w_j, \quad \sigma \in \mathbb{R},
\]
where \((\cdot, \cdot)\) is the \(L^2\)-scalar product. The domain of \( A^\sigma \) is denoted \( \mathcal{D}(A^\sigma) \).

The NSEs can be written as a differential equation (which will be referred to as the NSE) in the real Hilbert space \( H \) in the following form
\[
\frac{du}{dt} + \nu Au + B(u, u) = g, \quad u \in H,
\]
where the bilinear operator \( B \) and the driving force \( g \) are defined as
\[
B(u, v) = \mathcal{P}((u \cdot \nabla)v) \quad \text{and} \quad g = \mathcal{P}F.
\]

We recall that the global attractor \( \mathcal{A} \) of the NSE is the collection of all elements \( u_0 \) in \( H \) for which there exists a solution \( u(t) \) of NSE, for all \( t \in \mathbb{R} \), such that \( u(0) = u_0 \) and \( \sup_{t \in \mathbb{R}} |u(t)| < \infty \).

To give another definition of \( \mathcal{A} \), we need to recall several concepts. First, as is well-known, for any \( u_0, f \in H \), there exists a unique continuous function \( u \) from \([0, \infty)\) to \( H \) such that \( u(0) = u_0, u(t) \in \mathcal{D}(A), t \in (0, \infty) \), and \( u \) satisfies the NSE for all \( t \in (0, \infty) \). Therefore, one can define the map \( S(t) : H \to H \) by
\[
S(t)u_0 = u(t)
\]
where \( u(\cdot) \) is as above. Since \( S(t_1)S(t_2) = S(t_1 + t_2) \), the family \( \{S(t)\}_{t \geq 0} \) is called the “solution” semigroup. Furthermore, a compact set \( B \) is called absorbing if for any bounded set \( \bar{B} \subset H \) there is a time \( \tilde{t} \geq 0 \) such that \( S(t)B \subset \bar{B} \) for all \( t \geq \tilde{t} \). The attractor can be now defined by the formula
\[
\mathcal{A} = \bigcap_{t \geq 0} S(t)\bar{B},
\]
where \( \bar{B} \) is any absorbing compact subset of \( H \).

Let \( H_\mathbb{C} \) be the complex Hilbert space \( H \otimes \mathbb{C} = H + iH \) (see discussion following (3.17) for more details). Similarly, for any linear subspace \( D \) of \( H \) we denote \( D \otimes \mathbb{C} \) by \( D_\mathbb{C} \). For \( \delta > 0 \) we define the strip
\[
\mathcal{S}(\delta) := \{\zeta \in \mathbb{C} : |\Re(\zeta)| < \delta \}.
\]

**Theorem 2.1.** If \( 0 \notin \mathcal{A} \), then there exists \( \delta > 0 \) and \( \tilde{R}_\alpha \in [0, \infty), \alpha \in \mathbb{N}, \) such that for any solution \( u(\cdot) \) in \( \mathcal{A} \), the function \( A_\mathbb{C}^\alpha u(\zeta) \) is \( H_\mathbb{C} \)-valued analytic in the strip \( \mathcal{S}(\delta) \), and \( |A_\mathbb{C}^\alpha u(\zeta)| \leq \tilde{R}_\alpha \nu \kappa_0^\alpha \), where \( u(\zeta) \) satisfies the NSE with complexified time \( (3.15) \).

The proof of Theorem 2.1 will provide specific estimates for \( \delta \) and \( \tilde{R}_\alpha \), (see Remark 6.6).

**Corollary 2.2.** If \( 0 \notin \mathcal{A} \), then \( \mathcal{A} \cup \{g\} \subset C^\infty([0, L]^2) \).

The next result does not assume \( 0 \notin \mathcal{A} \), but rather, that \( g \) is smooth to a certain extent.

**Proposition 2.3.** Assume that \( g \in \mathcal{D}(A_\mathbb{C}^\alpha) \) for some \( \alpha \in \mathbb{N} \). Then \( \mathcal{A} \subset \mathcal{D}(A_\mathbb{C}^\alpha) \) and any solution \( u(\cdot) \) in \( \mathcal{A} \) can be extended in the strip \( \mathcal{S}(\delta_\alpha) \), where \( \delta_\alpha > 0 \) depends on \( g \) and \( \alpha \), to a \( \mathcal{D}(A_\mathbb{C}^\alpha) \)-valued analytic function such that
\[
\sup\{|A_\mathbb{C}^\alpha u(\zeta)| : \zeta \in \mathcal{S}(\delta_\alpha)\} \leq m_\alpha \nu \kappa_0^\alpha
\]
where \( m_\alpha \) is a non-dimensional parameter which, along with \( \delta_\alpha \), depends only on \( g \) and \( \alpha \).

The utility of the above proposition lies in the explicit estimates for the coefficients \( \delta_\alpha \) and \( m_\alpha \). Its proof is given in Section 8.

In fact, by using the techniques in the proof of Theorem 2.1 and Proposition 2.3 we will obtain the following generalization of Theorem 2.1

**Theorem 2.4.** If \( \mathcal{A} \cap C^\infty([0,L]^2) \neq \emptyset \), then \( \mathcal{A} \cup \{g\} \subset C^\infty([0,L]^2) \).

### 3. Preliminary material

Under periodic boundary conditions, we may express an element \( u \in H \) as a Fourier series expansion

\[
  u(x) = \sum_{k \in \mathbb{Z}^2 \setminus \{0\}} \hat{u}(k) e^{i\kappa_0 k \cdot x},
\]

where \( \kappa_0 = 2\pi/L \), \( \hat{u}(0) = 0 \), \( (\hat{u}(k))^* = \hat{u}(-k) \) and \( k \cdot \hat{u}(k) = 0 \). Parseval’s identity reads as

\[
  |u|^2 = \sum_{k \in \mathbb{Z}^2 \setminus \{0\}} |\hat{u}(k)|^2 = \sum_{k \in \mathbb{Z}^2 \setminus \{0\}} |\hat{u}(k)\hat{v}(k)|^2,
\]

as well as

\[
  (u, v) = \sum_{k \in \mathbb{Z}^2 \setminus \{0\}} \hat{u}(k) \cdot \hat{v}(-k).
\]

The following inequalities will be repeatedly used in this paper

\[
  \begin{align*}
  \kappa_0 |u| &\leq |A^\frac{1}{2} u|, \quad \text{for } u \in \mathcal{D}(A^\frac{1}{2}), \\
  |u|_{L^4(\Omega)} &\leq c_L |u|^\frac{1}{2} |A^\frac{1}{2} u|^\frac{3}{2}, \quad \text{for } u \in \mathcal{D}(A^\frac{1}{2}), \\
  |u|_\infty &\leq c_A |u|^\frac{1}{2} |Au|^\frac{3}{2}, \quad \text{for } u \in \mathcal{D}(A).
  \end{align*}
\]

known respectively as the Poincaré, Ladyzhenskaya and Agmon inequalities. Both \( c_L \) and \( c_A \) are absolute constants. By Theorem 9.2 and 9.3 in [9], we have that

\[
  c_L \leq \left( \left( \frac{1}{2\pi} \right)^2 + \frac{1}{\sqrt{2\pi}} + 2 \right)^\frac{1}{2},
\]

\[
  c_A \leq \left( \left( \frac{1}{2\pi} \right)^2 + \frac{1}{\sqrt{2\pi}} + 2 + 4\sqrt{2} \right)^\frac{3}{2}.
\]

In fact, (3.3) can be given in the following stronger form

We stress that our estimates will depend on the generalized Grashof number

\[
  G = \frac{|g|}{\nu^2 \kappa_0^2}.
\]

We also recall that if \( G < c_L^2 \) then \( \mathcal{A} \) contains only one point \( u_0 \in \mathcal{D}(A) \) which satisfies

\[
  \nu Au_0 + B(u_0, u_0) = g
\]

(see Proposition 2.1 in [4]). Note that in this case both Theorem 2.2 and 2.1 are trivially valid. Therefore, throughout this paper we will assume that \( G \) satisfies

\[
  G \geq \frac{1}{c_L^2}.
\]
We recall the following algebraic properties of the bilinear operator $B(u,v)$ where $u,v \in \mathcal{D}(A)$ from \cite{[2]}
\begin{equation}
(B(u,v),w) = -(B(u,w),v),
\end{equation}
\begin{equation}
(B(u,u),Au) = 0,
\end{equation}
\begin{equation}
(B(Av,v),u) = (B(u,v),Av),
\end{equation}
\begin{equation}
(B(u,v),Av) + (B(v,u),Av) + (B(v,v),Au) = 0.
\end{equation}

From \eqref{3.8} and \eqref{3.9}, it easily follows that if $u \in \mathcal{D}(A^{3/2})$ then $B(u,u) \in \mathcal{D}(A)$ and
\begin{equation}
AB(u,u) = B(u,Au) - B(Au,u).
\end{equation}

From \eqref{2.3} by $u$ and $Au$, respectively, integrate over $\Omega$, and apply the relations \eqref{3.6} and \eqref{3.7}, then we have the following inequalities
\begin{equation}
\frac{1}{2} \frac{d}{dt}|u|^2 + \nu \kappa_0^2 |u|^2 \leq \frac{1}{2} \frac{d}{dt}|u|^2 + \nu |A \dot{u}|^2 = (g, u) \leq \frac{|g|^2}{2 \nu \kappa_0^2} + \nu \kappa_0^2 |u|^2,
\end{equation}
\begin{equation}
\frac{1}{2} \frac{d}{dt}|A \dot{u}|^2 + \nu |Au|^2 = (g, Au) \leq \frac{|g|^2}{2 \nu} + \frac{\nu |Au|^2}{2}.
\end{equation}

\eqref{3.11} and \eqref{3.12} are called the balance equations for the energy and enstrophy, respectively. Applying Gronwall’s lemma to \eqref{3.11} and \eqref{3.12} we obtain, for all $t \geq t_0$, that
\begin{equation}
|u(t)|^2 \leq e^{-\nu \kappa_0^2 (t-t_0)} |u(t_0)|^2 + (1 - e^{-\nu \kappa_0^2 (t-t_0)}) \nu G^2,
\end{equation}
\begin{equation}
|A \dot{u}(t)|^2 \leq e^{-\nu \kappa_0^2 (t-t_0)} |A \dot{u}(t_0)|^2 + (1 - e^{-\nu \kappa_0^2 (t-t_0)}) \nu \kappa_0^2 G^2.
\end{equation}

From \eqref{3.13} we see that the closed ball $B$ of radius $2 \nu \kappa_0 G$ in $\mathcal{D}(A^{3/2})$, by the Sobolev embedding theorem, is an absorbing set in $H$. Therefore, we can define the global attractor $A$ as in \cite{[2]}.

In the next lemma, we list several necessary estimates involving $B(\cdot,\cdot)$.

**Lemma 3.1.** The following hold in the appropriate space,
\begin{equation}
[(B(u,u), A^2 u)] \leq 2 \nu \kappa_0^2 |Au||A \dot{u}|^2, u \in \mathcal{D}(A^2),
\end{equation}
\begin{equation}
[(B(u,u), A^3 u)] \leq \sqrt{2}(\sqrt{2c_A} + c_A)|u|^2 |Au| |A \dot{u}|^2, u \in \mathcal{D}(A^3).
\end{equation}

Relations \eqref{3.15} and \eqref{3.16} are straightforward applications of the Ladyzhenskaya and Amgnon’s inequalities.

Now we consider the NSE with complexified time and the corresponding solutions in $H \subset \mathcal{D}(A)$ as in \cite{[7]} and \cite{[10]}. We recall that
\begin{equation}
H \subset \{ u + iv : u, v \in H \},
\end{equation}
and that $H$ is a Hilbert space with respect to the following inner product
\begin{equation}
(u + iv, u' + iv')_H = (u, u')_H + (v, v')_H + i[(v, u')_H - (u, v')_H],
\end{equation}
where $u, u', v, v' \in H$. The extension $A_H$ of $A$ is given by
\begin{equation}
A_H(u + iv) = Au + iAv,
\end{equation}
for $u, v \in \mathcal{D}(A)$; thus $\mathcal{D}(A^2_H) = \mathcal{D}(A)_H$. Similarly, $B(\cdot,\cdot)$ can be extended to a bounded bilinear operator from $\mathcal{D}(A^2_H) \times \mathcal{D}(A_H)$ to $H$ by the formula
\begin{equation}
B_H(u + iv, u' + iv') = B(u, u') - B(v, v') + i[B(u, v') + B(v, u')].
\end{equation}
for \( u, v \in D(A^{1/2}) \), \( u', v' \in D(A) \). We should note that (3.9)-(3.10) do not hold in the complex case.

The Navier-Stokes equations with complex time are defined as

\[
(3.18) \quad \frac{du(\zeta)}{d\zeta} + \nu A_C u(\zeta) + B_C(u(\zeta), u(\zeta)) = g,
\]

where \( \zeta \in \mathcal{S}(\delta) \) (see (2.3)), \( u(\zeta) \in H_C \), and \( \frac{du(\zeta)}{d\zeta} \) denotes the derivative of \( H_C \)-valued analytic functions \( u(\zeta) \).

For notational simplicity, in the following considerations we drop the subscript \( \mathbb{C} \) for the inner products, norms, and the operators just defined.

\section{4. Supplementary estimates}

We now obtain estimates for the nonlinear terms with complexified time, observing that neither relation (3.7) nor Lemma 3.1 hold in this case. We will use the Ladyzhenskaya and Agmon inequalities as described before, noting the additional factor 2.

\[
(4.1) \quad |u|_{L^4} \leq 2c_L |u|^{1/2} |A^{1/2} u|^{3/2}, \quad \text{where} \quad u \in D(A^{1/2})_C
\]

and

\[
(4.2) \quad |u|_{\infty} \leq 2c_A |u|^{1/2} |Au|^{1/2}, \quad \text{where} \quad u \in D(A)_C.
\]

In the present complex case, the analogue of Lemma 3.1 is

\textbf{Lemma 4.1.} For \( u \in D(A^{\alpha})_C \), where \( \alpha = 1, 2 \) or 3,

\[
(4.3) \quad |(B(u, u), Au)| \leq 4c_L^2 |u|^{1/2} |A^{1/2} u| |Au|^{3/2},
\]

\[
(4.4) \quad |(B(u, u), A^2 u)| \leq 2(2c_L^2 + c_A) |u|^{1/2} |Au|^{3/2} |A^{1/2} u|,
\]

\[
(4.5) \quad |(B(u, u), A^3 u)| \leq 2(2c_L^2 + c_A) |u|^{1/2} |Au|^{1/2} |A^{3/2} u|.
\]

\textbf{Proof.} To obtain the first inequality, use (4.1) for the first two terms and the \( L^2 \) norm for the third term. For the second inequality, we use integration by parts to get

\[
(4.6) \quad |B(u, u), A^2 u| \leq \sum_{j=1,2} \left[ |(B(D_j u, u), D_j Au)| + |(B(u, D_j u), D_j Au)| \right].
\]

Using (4.1) and (4.2), we have

\[
(4.7) \quad \sum_j |(B(D_j u, u), D_j Au)| \leq 4c_L^2 |A^{1/2} u| |Au| |A^{1/2} u|
\]

and

\[
(4.8) \quad \sum_j |(B(u, D_j u), D_j Au)| \leq 2c_A |u|^{1/2} |Au| |A^{1/2} u|,
\]

for \( u \in D(A^{1/2}) \).

Now apply the interpolating relation \( |A^{1/2} u| \leq |Au|^{1/2} |u|^{1/2} \) in (4.7) to arrive at (4.4).
For the last inequality, we use the same method as above. Integrating by parts, using (4.1) and (4.2), and then interpolating, we have

$$|B(u, A^\frac{1}{2}u)| \leq \sum_{j=1,2} \left[ |(B(D_j u, D_j A^2 u)| + |(B(u, D_j u), D_j A^2 u)| \right]$$

$$\leq 4c_2^2 |A^\frac{1}{2}u||Au||A^\frac{1}{2}u| + 2c_A |u|^\frac{1}{2}|Au|^\frac{3}{2}|A^\frac{5}{2}u|$$

$$\leq 2(2c_2^2 + c_A) |u|^\frac{1}{2}|Au|^\frac{3}{2}|A^\frac{5}{2}u|,$$

thus obtaining (4.5).

Concerning the existence of $D(A^\frac{1}{2})_\mathbb{C}$-valued analytic extensions of the solutions of the NSE, one can consult Section 7 in Part I of [21] and Chapter 12 in [3]. However, for our work, we need the following observations.

**Remark 4.2.** Many of the differential relations to follow are of the form

$$\frac{d}{d\rho} \Phi(u(t_0 + \rho e^{i\theta})) = \Psi(u(t_0 + \rho e^{i\theta})),$$

where $\Phi(u)$ and $\Psi(u)$ are explicit functions of $u$ in a specified subspace of $H$. Often the definition of $\Psi(u)$ involves many terms. Therefore in the sequel we will make the following abuse of notation

$$\frac{d}{d\rho} \Phi(u(t_0 + \rho e^{i\theta})) = \Psi(u).$$

**Remark 4.3.** Let $\alpha \in \mathbb{N}$. To solve the equation (3.18) in a strip $S(\delta_\alpha)$ and to insure that $u(\zeta)$ is a $D(A^{\alpha/2})_\mathbb{C}$-valued analytic function (equivalently, $A^{\alpha/2}u(\zeta)$ is $H_\mathbb{C}$-valued analytic), the proof for the case $\alpha = 1$ presented in [21] and [3] shows that it suffices to establish the following fact:

For any $t_0 \in \mathbb{R}$, $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and solution $u(\zeta)$ of the equation (3.18) in $S(\delta_\alpha)$, the solution of the equation

$$\frac{d}{d\rho} u(t_0 + \rho e^{i\theta}) + \nu(\cos \theta) Au + B(u, u) = g, \quad g \in D(A^{\frac{\alpha+1}{2}})$$

satisfies, for

$$0 \leq \rho \leq \frac{\delta_\alpha}{\sin \pi/4} = \sqrt{2}\delta_\alpha,$$

the following conditions

$$u(t_0 + \rho e^{i\theta}) \in D(A^{\frac{\alpha+1}{2}})_\mathbb{C},$$

and $\sup |A^{\frac{\alpha+1}{2}}u(t_0 + \rho e^{i\theta})|$ is finite and independent of $t_0$, $\rho$ and $\theta$.

This can be rigorously established with “the Galerkin approximation, for which analyticity in time is trivial because it is a finite-dimensional system with a polynomial nonlinearity. The crucial part, then, is to obtain suitable a priori estimates for the solution in a complex time region that is independent of the Galerkin approximation.” (see [11] Chapter II, Section 8, Page 63). The justification for this procedure is given in the Appendix.

Using the procedure described in Remark 4.3, we will prove Theorem 2.1 by induction on $\alpha$. In order to start a uniform recurrent process we need $\alpha \geq 3$. In the following Lemmas 4.4, 4.8 and 4.9 we obtain the necessary estimates for $\alpha = 1, 2, 3$. We stress that the case $\alpha = 1$ was treated in Theorem 12.1 in [3], while the cases
\[ \alpha = 1,2 \text{ were already established in } [4], \text{ Theorem 11.1, although with different estimates.} \]

**Lemma 4.4.** If \( u(\cdot) \) is a solution of the NSE in the attractor \( A \), then

1. \( u(\cdot) \) can be extended to a \( D(A^\frac{1}{2})_C \)-valued analytic function in the strip \( S(\delta_1) \), where

\[
\delta_1 := \frac{1}{16 \cdot 24^1 \nu \kappa_0^2 G^2}
\]

and

\[
|A^{\frac{1}{2}} u(\zeta)| \leq \hat{R}_1 \nu \kappa_0, \quad \forall \ \zeta \in S(\delta_1),
\]

where

\[
\hat{R}_1 = \sqrt{2G}.
\]

2. Moreover, defining

\[
R_2 = 2137G^3 \nu^4
\]

we have

\[
|Au(t)| \leq R_2 \nu \kappa_0^2, \quad \forall \ t \in \mathbb{R}.
\]

**Proof.** First, according to Remark 4.3 to prove the statement (i) it is sufficient to establish the estimates (4.11) and (4.12) for \( \delta_1 \) chosen as in (4.10).

Taking the inner product of both sides in (3.9) with \( Au(t_0 + \rho e^{i\theta}) \), we obtain

\[
\frac{1}{2} \frac{d}{d\rho} |A^{\frac{1}{2}} u(t_0 + \rho e^{i\theta})|^2 = \Re(e^{i\theta}(g, Au)) - \nu \cos \theta |Au|^2 - \Re(e^{i\theta}(B(u, u), Au))
\]

\[
\leq |g||Au| - \sqrt{\nu^2 |Au|^2 + |(B(u, u), Au)|}.
\]

Using the Cauchy-Schwarz inequality and then the Young’s inequality \(|ab| \leq |a|^p/p + |b|^q/q \) with \( p = 2, q = 4 \), for the term \(|g||Au|\), we obtain

\[
\frac{d}{d\rho} |A^{\frac{1}{2}} u(t_0 + \rho e^{i\theta})|^2 \leq \frac{\nu}{\sqrt{2}} |Au|^2 \leq \frac{\sqrt{2} |g|^2}{\nu} + 2(B(u, u), Au)|.
\]

We use Lemma 4.4 for the bilinear term in the above relation to get the following inequality

\[
\frac{d}{d\rho} |A^{\frac{1}{2}} u(t_0 + \rho e^{i\theta})|^2 \leq \frac{\nu}{\sqrt{2}} |Au|^2 \leq \frac{\sqrt{2} |g|^2}{\nu} + 8c_0^2 |u|^2 |A^{\frac{1}{2}} u||Au|^2.
\]

Using Young’s inequality again, with \( p = 4 \) and \( q = 4/3 \) for the last term, we have

\[
8c_0^2 |u|^2 |A^{\frac{1}{2}} u||Au|^2 \leq \frac{3}{4} \left( \frac{\nu^2}{\sqrt{3}} \left| Au \right|^2 \right)^{\frac{2}{3}} + \frac{1}{4} \left( \frac{3}{\nu^2 \sqrt{2}} \right)^{\frac{2}{3}} 8c_0^2 |u|^2 |A^{\frac{1}{2}} u|^4,
\]

and hence,

\[
\frac{d}{d\rho} |A^{\frac{1}{2}} u(t_0 + \rho e^{i\theta})|^2 \leq \frac{\nu}{2 \sqrt{2}} |Au|^2 \leq \frac{\sqrt{2} |g|^2}{\nu} + \frac{8^3 \sqrt{2}}{\nu^{3/2}} |u|^2 |A^{\frac{1}{2}} u|^4.
\]

From (3.1) and (4.15), we obtain

\[
\frac{d}{d\rho} |A^{\frac{1}{2}} u(t_0 + \rho e^{i\theta})|^2 \leq \frac{\sqrt{2} |g|^2}{\nu} + \frac{8^3 \sqrt{2}}{\nu^{3/2} \kappa_0^2} |A^{\frac{1}{2}} u|^6.
\]
The above inequality has the form

\[
\frac{d\phi}{d\rho} \leq \gamma + \beta \phi^3,
\]

where

\[
\phi(\rho) := |A^{\frac{1}{2}} u(t_0 + \rho e^{i\theta})|^2, \quad \gamma = \sqrt{2} |g|^2 / \nu, \quad \beta = \frac{24^3 c_0^8}{\nu^3 K_0^2 \sqrt{2}}.
\]

Integrating (4.16), we obtain

\[
\int_{\phi(0)}^{\phi(\rho)} \frac{d\phi}{(\gamma + \beta \phi(0))^3} \leq \int_{\phi(0)}^{\phi(\rho)} \frac{d\phi}{(\gamma + \beta \phi(\rho))^3} \leq \rho,
\]

and hence

\[
\frac{1}{2 \beta^3 (\gamma + \beta \phi(0))^2} \leq \frac{1}{2 \beta^3 (\gamma + \beta \phi(\rho))^2} \leq \rho.
\]

Thus, if

\[
\rho \leq \frac{1}{4 \beta^3 (\gamma + \beta \phi(0))^2}
\]

then \(\phi(\rho)\) satisfies

\[
\gamma + \beta \phi(\rho) \leq \sqrt{2} (\gamma + \beta G^2 (\nu K_0))^2,
\]

that is

\[
|A^{\frac{1}{2}} u(t_0 + \rho e^{i\theta})|^2 \leq (\sqrt{2} - 1) \left(\frac{\gamma}{\beta}\right)^{\frac{1}{3}} + \sqrt{2} G^2 (\nu K_0)^2
\]

\[
\leq \frac{2^{\frac{1}{3}} |g| (\nu K_0)^{2/3}}{24 c_0^{\frac{4}{3}}} + \sqrt{2} G^2 (\nu K_0)^2
\]

\[
\leq \left(\sqrt{2} + \frac{2^{\frac{1}{3}}}{24}\right) G^2 (\nu K_0)^2 \leq 2 G^2 (\nu K_0)^2.
\]

Note that in the third inequality above we used (3.5).

If \(\delta_1\) is defined as in (4.11) and if \(\rho \leq \sqrt{2} \delta_1\), then (4.18) holds. Consequently, (4.19) also holds. That is

\[
|A^{\frac{1}{2}} u(t_0 + \rho e^{i\theta})| \leq \hat{R}_1 \nu K_0,
\]

where \(\hat{R}_1\) is defined in (4.12).

Since \(\theta \in [-\frac{\pi}{4}, \frac{\pi}{4}]\) and \(t_0 \in \mathbb{R}\) are arbitrary, we infer

\[
|A^{\frac{1}{2}} u(\zeta)| \leq \hat{R}_1 \nu K_0, \quad \text{for} \ \zeta \in S(\delta_1).
\]

With this estimate, the proof of statement (i) is concluded.
It remains to prove statement (ii). Integrating (4.15) and applying (3.5), we obtain
\[
\frac{\nu}{2\sqrt{2}} \int_0^{\sqrt{2} \delta_1} |Au(t_0 + e^{i\pi/4})|^2 \, dp \leq |A^u u(t_0)|^2 + \int_0^{\sqrt{2} \delta_1} (\gamma + \beta |A^u u(t_0 + \rho e^{i\pi/4})|^2) \, dp,
\]
\[
\leq 2G^2 \nu^2 \kappa_0^2 + \sqrt{2} \gamma \delta_1 + 8\sqrt{2} G^6 \nu^6 \kappa_0^6 \beta_1
\]
\[
= \left[ 2 + \frac{1}{8 \cdot 24^3} G^2 \kappa_0^2 \right] G^2 \nu^2 \kappa_0^2
\]
\[
\leq \left[ 2 + \frac{1}{8 \cdot 24^3} + \frac{1}{2} \right] G^2 \nu^2 \kappa_0^2 \leq 2\sqrt{2} G^2 \nu^2 \kappa_0^2,
\]
i.e.,
\[
(4.20) \quad \int_0^{\sqrt{2} \delta_1} |Au(t_0 + e^{i\pi/4})|^2 \, dp \leq 8G^2 \nu \kappa_0^2.
\]
Since $Au(\zeta)$ is an analytic function in $D(t_0, \delta) := \{s_1 + is_2 : |s_1 - t_0|^2 + s_2^2 \leq \delta_1^2\}$, it satisfies the mean value property
\[
Au(t_0) = \frac{1}{\pi \delta_1^2} \iint_{D(t_0, \delta_1)} Au(s_1 + is_2) \, ds_1 ds_2,
\]
from which we deduce
\[
|Au(t_0)| \leq \frac{1}{\pi \delta_1^2} \iint_{D(t_0, \delta_1)} |Au(s_1 + is_2)| \, ds_1 ds_2.
\]
In order to exploit the estimate (4.20), we replace the disk $D(t_0, \delta_1)$ by the polygon $abcdef$ as shown in the figure below.

![Diagram](image)

Then, by using Schwarz reflection principle, we obtain
\[
|Au(t_0)| \leq \frac{1}{\pi \delta_1^2} \int_{abcdef} |Au(s_1 + is_2)| ds_1 ds_2
\]
\[
= \frac{2}{\sqrt{2\pi \delta_1^2}} \int_{t_0 - \delta_1}^{t_0 + \delta_1} \int_0^{\sqrt{2} \delta_1} |Au(t + \rho e^{i\pi/4})| \, dp \, dt
\]
\[
\leq \frac{2}{\sqrt{2\pi \delta_1^2}} \int_{t_0 - \delta_1}^{t_0 + \delta_1} \left( \int_0^{\sqrt{2} \delta_1} |Au(t + \rho e^{i\pi/4})|^2 \, dp \right)^{\frac{1}{2}} \left( \int_{\sqrt{2} \delta_1}^{\sqrt{2} \delta_1} \right)^{\frac{1}{2}} \, dt
\]
\[
\leq \frac{12 \cdot 2^8 \cdot G^2 \nu \kappa_0^2}{\pi},
\]
that is
\[
|Au(t_0)| \leq \frac{6 \cdot 2^8 \cdot 3 \sqrt{3}}{2^4 \pi} G^5 c_{\Delta}^4 \nu \kappa_0^2 \leq R_2 \nu \kappa_0^2.
\]
This completes the proof of the statement (ii) and Lemma 4.3. \qed
Corollary 4.5. For all \( u^0 \in \mathcal{A} \), we have

\[
|A^\perp u^0| \leq R_1 \nu \kappa_0, \quad R_1 := G,
\]

and

\[
|Au^0| \leq R_2 \nu \kappa_0^2.
\]

Proof. Let \( u^0 \in \mathcal{A} \) and denote by \( u(t) \), \( t \in \mathbb{R} \), the solution of the NSE satisfying \( u(0) = u^0 \). Then, according to Lemma 4.4, 4.14 holds; in particular, for \( t = 0 \). This yields (4.22). The estimate (4.21) follows from (3.14) with \( t = 0 \) and \( t_0 \to -\infty \).

Remark 4.6. It is worth comparing (4.14) with (4.24)

\[
1 - \left| \frac{u}{\kappa_0} \right| \leq \frac{\kappa_0}{\kappa_0} (4.22).
\]

Proof. Let \( u^0 \in \mathcal{A} \) and denote by \( u(t) \), \( t \in \mathbb{R} \), the solution of the NSE satisfying \( u(0) = u^0 \). Then, according to Lemma 4.4, 4.14 holds; in particular, for \( t = 0 \). This yields (4.22). The estimate (4.21) follows from (3.14) with \( t = 0 \) and \( t_0 \to -\infty \).

Remark 4.6. It is worth comparing (4.14) with

\[
|Au| \leq \frac{\kappa_0}{\kappa_0} \left( \frac{\nu}{\kappa_0} \right)^2 \left( \frac{2\Lambda_1^2 + c_2^2 G^2}{\nu} \right).
\]

from Theorem 3.1 of [4], where \( \Lambda_1 := \frac{\Lambda_1^2}{\kappa_0(|\theta|)} \). When \( \Lambda_1^2 > \frac{21372}{2} c_2^2 G^2 \) the estimate (4.22) is better than (4.23). Moreover, there are cases when \( \Lambda_1 \) is large, \( \mathcal{A} = \{ u^0 \} \), and thus \( |Au| \leq \nu \kappa_0^2 \).

Corollary 4.7. If \( 0 \in \mathcal{A} \) then \( g \in \mathcal{D}(A^\perp) \), and

\[
|A^\perp g| \leq \frac{R_1 \nu \kappa_0}{\delta_1},
\]

where \( \delta_1 \) and \( R_1 \) are defined as in (4.10) and (4.12) respectively.

Proof. Let \( u(t) \), \( t \in \mathbb{R} \), be a solution of NSE such that \( u(0) = 0 \). According to Theorem 11.1 in [4], if \( 0 \in \mathcal{A} \), then \( g \in \mathcal{D}(A) \). We evaluate the NSE at \( t_0 = 0 \) and \( \theta = 0 \) to obtain

\[
\frac{du(\zeta)}{d\zeta} \bigg|_{t_0=0} = g.
\]

Since \( u(\zeta) \) is a \( \mathcal{D}(A^\perp) \)-valued analytic function, its derivative \( \frac{du(\zeta)}{d\zeta} \in \mathcal{D}(A^\perp)_C \) for all \( \zeta \in \mathcal{S}(\delta_1) \). Thus, \( g \in \mathcal{D}(A^\perp) \), and then from

\[
A^\perp g = A^\perp \frac{du(\zeta)}{d\zeta} \bigg|_{\zeta=0} = \frac{dA^\perp u(\zeta)}{d\zeta} \bigg|_{\zeta=0} = \frac{1}{2\pi i} \int_{\partial D(0, \delta)} \frac{A^\perp u(z)}{z^2} \, dz,
\]

where \( \delta \in (0, \delta_1) \), we obtain

\[
|A^\perp g| \leq \frac{R_1 \nu \kappa_0}{\delta}.
\]

Letting \( \delta \to \delta_1 \) in (4.25), we deduce (4.24) \( \square \).

We will now establish estimates for the case \( \alpha = 2 \).

Lemma 4.8. If \( 0 \in \mathcal{A} \) and if \( u(t) \), \( t \in \mathbb{R} \), is any solution of the NSE in \( \mathcal{A} \), then \( u(t) \) can be extended to a \( \mathcal{D}(A)_C \)-valued analytic function \( u(\zeta) \), for \( \zeta \in \mathcal{S}(\delta_2) \), where

\[
\delta_2 := \min \left\{ \delta_1, 16^{-1} \left[ \left( 2c_2^2 + c_A \right)^2 + c_2^2 \frac{\nu^2}{\delta_0^2} + \left( 2c_2^2 + c_A \right)^4 R_1^2 R_2^2 \nu^2 \kappa_0^2 \right] \right\},
\]

and \( \delta_1 \), \( R_1 \) and \( R_2 \) are defined in (4.10), (4.12) and (4.13), respectively. Furthermore,

\[
|Au(\zeta)| \leq R_2 \nu \kappa_0^2, \quad \text{for} \quad \zeta \in \mathcal{S}(\delta_2),
\]
where

\begin{equation}
\tilde{R}_2 := \left( \frac{3(\sqrt{2} \cdot 16^2 \cdot 24^6 e^{16})^{2/3}}{4(2c_L^2 + c_A)^{4/3}} G^6 + 4\tilde{R}_2^2 \right)^{\frac{1}{2}}.
\end{equation}

**Proof.** Applying again the short procedure in Remark 4.3, we take the inner product of the NSE with the function \( A^2 u(t_0 + \rho e^{i\theta}) \) and obtain

\[
\frac{1}{2} \frac{d}{d\rho} |Au(t_0 + \rho e^{i\theta})|^2 = \Re(e^{i\theta} (g, A^2 u)) - \nu (\cos \theta) |A\dot{\bar{u}}|^2 - \Re(e^{i\theta} (B(u, u), A^2 u)).
\]

Proceding as in the proof of Lemma 4.4, we obtain

\[
\frac{1}{2} \frac{d}{d\rho} |Au(t_0 + \rho e^{i\theta})|^2 \leq |(A\bar{g}, A\dot{\bar{u}})| - \nu |A\dot{\bar{u}}|^2 + |(B(u, u), A^2 u)|.
\]

Using Young’s inequality \(|ab| \leq \frac{1}{p} |a|^p + \frac{1}{q} |b|^q\), with \( p = q = 2 \) for the term \(|(A\bar{g}, A\dot{\bar{u}})|\), we obtain

\[
\frac{d}{d\rho} |Au(t_0 + \rho e^{i\theta})|^2 \leq \sqrt{2} \frac{|A\bar{g}|^2}{\nu} - \nu \sqrt{2} |A\dot{\bar{u}}|^2 + 2|B(u, u), A^2 u)|.
\]

We use Lemma 4.4 to obtain

\[
\frac{d}{d\rho} |Au(t_0 + \rho e^{i\theta})|^2 \leq \sqrt{2} \frac{|A\bar{g}|^2}{\nu} - \nu \sqrt{2} |A\dot{\bar{u}}|^2 + 4(2c_L^2 + c_A)^2 |u|^2 |Au||A\dot{\bar{u}}| |\dot{\bar{u}}|^2.
\]

Using Young’s inequality again, we obtain

\[
\frac{d}{d\rho} |Au(t_0 + \rho e^{i\theta})|^2 + \frac{1}{2\sqrt{2}} |A\dot{\bar{u}}|^2 \leq \sqrt{2} \frac{|A\bar{g}|^2}{\nu} + \frac{8\sqrt{2} (2c_L^2 + c_A)^2 |u||Au|^3}{\nu}.
\]

Using Poincaré’s inequality and the bound on \(|A\dot{\bar{u}}|^2\) obtained in Lemma 4.4 we obtain

\begin{equation}
\frac{d}{d\rho} |Au(t_0 + \rho e^{i\theta})|^2 + \frac{1}{2\sqrt{2}} |A\dot{\bar{u}}|^2 \leq \sqrt{2} \frac{|A\bar{g}|^2}{\nu} + 8(2c_L^2 + c_A)^2 R_1 |Au|^3.
\end{equation}

As before, we ignore the term containing \(|A\dot{\bar{u}}|^2\) to get the inequality

\begin{equation}
\frac{d\phi_2(\rho)}{d\rho} \leq \gamma_2 + \beta_2 (\phi_2(\rho))^2,
\end{equation}

where

\[
\phi_2(\rho) = |Au(t_0 + \rho e^{i\theta})|^2, \quad \gamma_2 = \sqrt{2} \frac{\tilde{R}_2^2 \nu \kappa_0^2}{\delta_1}, \quad \beta_2 = 8 \sqrt{2} (2c_L^2 + c_A)^2 \tilde{R}_1.
\]

From (4.30), we obtain the analogue of the relation (4.17), namely

\[
\frac{2}{\beta_2 ((\gamma_2/\beta_2)^{2/3} + \phi_2(0)))^{\frac{3}{2}}} < \frac{2}{\beta_2 ((\gamma_2/\beta_2)^{2/3} + \phi_2(\rho)))^{\frac{3}{2}}} \leq \rho.
\]

We observe that if

\[
\rho < \frac{1}{\beta_2 ((\gamma_2/\beta_2)^{2/3} + \phi_2(0)))^{\frac{3}{2}}},
\]

then

\[
((\gamma_2/\beta_2)^{2/3} + \phi_2(\rho)))^{\frac{3}{2}} \leq 2 ((\gamma_2/\beta_2)^{2/3} + \phi_2(0)))^{\frac{3}{2}}
\]

and hence,

\[
|Au(t_0 + \rho e^{i\theta})|^2 \leq 3 (\gamma_2/\beta_2)^{2/3} + 4 |Au(t_0)|^2.
\]
By \((4.14)\), we obtain
\[
|A u(t_0 + \rho e^{i\theta})|^2 \leq \left( \frac{3(\sqrt{2} \cdot 16^2 \cdot 24^3 c_L^4) / 4(2c_L^2 + c_A)^{4/3}}{G^6 + 4R^2} \right) \nu^2 \kappa_0^4,
\]
Thus, if we define \(\delta_2\) by \((4.26)\) and \(\tilde{R}_2\) by \((4.28)\), then we obtain \((4.27)\). □

We now consider the case \(\alpha = 3\) after which we can proceed by induction for all \(\alpha > 3\). Let \(\delta_3 = \delta_2/2\), where \(\delta_2\) is defined as in Lemma \((4.8)\) and let \(r = (2\sqrt{2} - \sqrt{3})\delta_3\). Then, given any \(\zeta\) in \(S(\delta_4)\), there is a real \(t_0\) such that \(D(\zeta, r)\) is in the sector of \(D(t_0, 2\sqrt{3}\delta_3)\) where \(\theta\) varies from \(-\pi/4\) to \(\pi/4\), as shown in the next figure.

Using \((4.29)\) and the notation from \((4.30)\) we obtain, for \(\theta \in [-\pi/4, \pi/4]\),
\[
\left(4.31\right) \quad \frac{\nu}{2\sqrt{2}} \int_0^{2\sqrt{3}\delta_3} |A^\frac{3}{2}(t_0 + \rho e^{i\theta})|^2 d\rho \leq |A u(t_0)|^2 + \int_0^{2\sqrt{3}\delta_3} (\gamma_2 + \beta_2 |A u|^3) d\rho \\
\leq N_2 (\nu \kappa_0^2)^2,
\]
where
\[
\left(4.32\right) \quad N_2 := R_1^2 + \frac{2\delta_2 \tilde{R}_1^2}{\delta_1 \nu \kappa_0^2} + 16(2c_L^2 + c_A)^2 \tilde{R}_1 \delta_2 \nu \kappa_0^2.
\]

Using the mean value theorem for the analytic function \(A^\frac{3}{2} u(\zeta)\) (as we did before for \(A u(\zeta)\) in \(D(\zeta, r)\)) we obtain
\[
|A^\frac{3}{2} u(\zeta)| \leq \frac{1}{\pi r^2} \int_{\{s_1 + i s_2 \in D(\zeta, r)\}} |A^\frac{3}{2} u(s_1 + i s_2)| d s_1 d s_2 \\
\leq \frac{1}{\pi r^2} \int_{-\pi/4}^{\pi/4} \int_0^{2\sqrt{3}\delta_3} |A^\frac{3}{2} u(t_0 + \rho e^{i\theta})| d\rho d\theta \\
\leq \frac{1}{\pi r^2} \int_{-\pi/4}^{\pi/4} d\theta \left( \int_0^{2\sqrt{3}\delta_3} |A^\frac{3}{2} u(t_0 + \rho e^{i\theta})|^2 d\rho \right)^{\frac{1}{2}} \left( \int_0^{2\sqrt{3}\delta_3} \rho^2 d\rho \right)^{\frac{1}{2}} \\
\leq \frac{1}{\pi r^2} \frac{\pi}{2} (2\sqrt{2}N_2 \nu \kappa_0^2) \frac{1}{\sqrt{3}} \left( \frac{2\sqrt{2}\delta_3}{\delta_3} \right)^{\frac{3}{4}} \\
= \frac{4}{\sqrt{3}(2\sqrt{2} - \sqrt{3})^2} N_2 \frac{\nu \kappa_0^2}{\delta_3^2} < 4N_2 \frac{\nu \kappa_0^2}{\delta_3^2}.
\]

Now to obtain for \(A^\frac{3}{2} u(t), t \in \mathbb{R}\), an estimate analogous to \((4.14)\), we use an argument similar to that involving the polygon abcdef in the proof of Lemma \((4.3)\) we note that now the roles of \(\delta_1\) and \((4.20)\) are played by \(\delta_3\) and \((4.31)\), respectively.
In this manner, we obtain

\begin{equation}
|A^{\frac{\alpha}{2}} u(t)| \leq R_3 \nu \kappa_0^3,
\end{equation}

where

\begin{equation}
R_3 := \frac{12\sqrt{2}}{\pi} \left( \frac{N_3}{\delta_3 \nu \kappa_0^2} \right)^{\frac{1}{2}},
\end{equation}

and

\begin{equation}
N_3 := R_2^2 + \frac{2\delta_1 \tilde{R}_1^2}{\delta_1^2 \nu \kappa_0^2} + 16(2c_A^2 + c_A)^2 \tilde{R}_1 \delta_3 \nu \kappa_0^2.
\end{equation}

We sum up the results obtained above in the following

**Lemma 4.9.** If \(0 \in A\) and if \(u(t), t \in \mathbb{R}\) is any solution of the NSE in \(A\), then \(u(t)\) can be extended to a \(D(A^{\frac{\alpha}{2}})\)-valued analytic function \(u(\zeta)\), for \(\zeta \in S(\delta_3)\), where \(\delta_3 := \frac{\delta_2}{2}\), and \(\delta_2\) is defined as in \((4.26)\), for which the following estimates holds

\begin{equation}
|A^{\frac{\alpha}{2}} u(\zeta)| \leq \tilde{R}_3 \nu \kappa_0^3, \quad \text{for } \zeta \in S(\delta_3),
\end{equation}

where

\begin{equation}
\tilde{R}_3 := 4 \frac{N_2^{\frac{1}{2}}}{\delta_3^{\frac{1}{2}} \nu^{\frac{1}{2}} \kappa_0^{\frac{3}{2}}},
\end{equation}

and \(N_2\) is defined in \((4.32)\).

Moreover, \(u(t)\) satisfies the relation \((4.35)\).

**Remark 4.10.** Lemmas [4.4] [4.8] and [4.9] establish the validity of Theorem [2.1] for the case \(\alpha \in \{1, 2, 3\}\).

5. Two more Estimates

In this section we present an extension of the estimates given in Lemma [3.1] and Lemma [4.1] to the powers \(A^\alpha (\alpha \in \mathbb{Z}, \alpha > 3)\) of \(A\) using the method of Constantin (see [2]), but with \(\ln(\kappa_0^{-1} A^{\frac{1}{2}})\) replacing \(A^\theta\) with \(\theta \in (0, 1)\).

**Lemma 5.1.** Let \(u \in D(A^{\frac{\alpha}{2}}), v \in D(A^{\frac{\alpha+1}{2}}), w \in D(A^\alpha)\), and \(\alpha > 3\), then

\begin{equation}
\|(H(u, v), A^\alpha w)\| \leq 2^\alpha c_A \left( |u|_{A^{\frac{\alpha}{2}}} |A^{\frac{\alpha}{2}} v| + |A^{\frac{\alpha}{2}} u| |A^{\frac{\alpha}{2}} v| |v|_{A^{\frac{\alpha}{2}} w} + |A^{\frac{\alpha}{2}} v| |v|_{A^{\frac{\alpha}{2}} w} \right) |A^{\frac{\alpha}{2}} w|.
\end{equation}

**Proof.** Fix \(\alpha > 3\). To simplify the exposition, we denote \(\tilde{u} := A^{\frac{\alpha}{2}} u\) and \(u \in D(A^{\frac{\alpha}{2}})\).
Then for any \( u \in \mathcal{D}(A^{\frac{1}{2}}) \), \( v \in \mathcal{D}(A_{\mathbb{R}}^{\frac{1}{2}}) \), \( w \in \mathcal{D}(A^\alpha) \), we have
\[
|(B(u, v), A^\alpha w)| \leq L^2 \kappa_0^{1+2\alpha} \sum_{h,j,k \in \mathbb{Z}^2 \setminus \{0\}} |\hat{u}(h)||\hat{v}(j)||\hat{w}(k)||k|^{2\alpha}
\]
\[
= L^2 \kappa_0^{1+\alpha} \sum_{h,j,k \in \mathbb{Z}^2 \setminus \{0\}} |\hat{u}(h)||\hat{v}(j)||\hat{w}(k)||k|^\alpha
\]
\[
\leq L^2 \kappa_0^{1-\alpha} \sum_{h,j,k \in \mathbb{Z}^2 \setminus \{0\}} |\hat{u}(h)||\hat{v}(j)||\hat{w}(k)|(|h| + |j|)^\alpha|h|^{-\alpha}|j|^{-\alpha}
\]
\[
= L^2 \kappa_0^{1-\alpha} \sum_{h,j,k \in \mathbb{Z}^2 \setminus \{0\}} |\hat{u}(h)||\hat{v}(j)||\hat{w}(k)|e^{\alpha[\ln(|h|+|j|)-\ln|h|\ln|j|]}
\]
\[
= L^2 \kappa_0^{1-\alpha} \sum_{h,j,k \in \mathbb{Z}^2 \setminus \{0\}} \ldots + L^2 \kappa_0^{1-\alpha} \sum_{h,j,k \in \mathbb{Z}^2 \setminus \{0\}} \ldots
\]
\[
=: I_1 + I_2,
\]
where the notation is self-explanatory.

For \( I_1 \), since \( \ln(|h|+|j|)-\ln|h|\ln|j| \) is decreasing with respect to \(|j|\), we have
\[
I_1 \leq L^2 \kappa_0^{1-\alpha} \sum_{h,j,k \in \mathbb{Z}^2 \setminus \{0\}} |\hat{u}(h)||\hat{v}(j)||\hat{w}(k)|e^{\alpha[\ln(2-\ln|h|)]}
\]
\[
= L^2 \kappa_0 \sum_{h,j,k \in \mathbb{Z}^2 \setminus \{0\}} |\hat{u}(h)||\hat{v}(j)||\hat{w}(k)|e^{\alpha\ln 2}
\]
\[
\leq 2^\alpha L^2 \kappa_0 \sum_{h,j,k \in \mathbb{Z}^2 \setminus \{0\}} |\hat{u}(h)||\hat{v}(j)||\hat{w}(k)|
\]

By estimating \( I_2 \) in a same way, we obtain
\[
|(B(u, v), A^\alpha w)| \leq 2^\alpha L^2 \kappa_0 \sum_{h,j,k \in \mathbb{Z}^2 \setminus \{0\}} |\hat{u}(h)||\hat{v}(j)||\hat{w}(k)|
\]

We define the auxiliary functions \( U \) and \( \tilde{U} \) by \( U := \sum_{k \in \mathbb{Z}^2 \setminus \{0\}} |\hat{u}(k)|e^{i\omega_k x} \) and \( \tilde{U} := \sum_{k \in \mathbb{Z}^2 \setminus \{0\}} |\hat{u}(k)|e^{i\omega_k x} \) and in a similar way, the functions \( \hat{V}, \hat{\tilde{V}}, \hat{W}, \hat{\tilde{W}} \). Then we have that
\[
|(B(u, v), A^\alpha w)| \leq 2^\alpha \int_{[0,L]^2} [(U \cdot (-\Delta)^{\frac{1}{2}} \hat{V}) + (\tilde{U} \cdot (-\Delta)^{\frac{1}{2}} \hat{V})]\hat{W} d^2 x
\]
\[
\leq 2^\alpha [\|U\|_{L^\infty} (\Delta)^{\frac{1}{2}} \hat{V}|_{L^2} + |\tilde{U}|_{L^2} (\Delta)^{\frac{1}{2}} \hat{V}|_{L^\infty}][\hat{W}|_{L^2}]
\]
Using Agmon’s inequality, we obtain
\[
|(B(u, v), A^\alpha w)| \leq 2^\alpha c_{\mathcal{A}} [\|U\|_{L^2}^2 (\Delta)^{\frac{1}{2}} |\Delta U|_{L^2}^2 (\Delta)^{\frac{1}{2}} \hat{V}|_{L^2}^2
\]
\[
+ |\tilde{U}|_{L^2}^2 (\Delta)^{\frac{1}{2}} |\Delta \tilde{U}|_{L^2}^2 (\Delta)^{\frac{1}{2}} \hat{V}|_{L^2}^2][\hat{W}|_{L^2}
\]
\[
= 2^\alpha c_{\mathcal{A}} \left( \|u\|_{L^2}^2 |A^\frac{1}{2} u|_{L^2}^2 |A^\frac{1}{2} v|_{L^2}^2 + |A^\frac{1}{2} u||A^\frac{1}{2} v|_{L^2}^2 (|A^\frac{1}{2} v|_{L^2}^2) |A^\frac{1}{2} w|_{L^2}^2 \right).
\]
Lemma 5.2. Let \( u \in \mathcal{D}(A^\frac{\alpha}{2})_C, \) \( v \in \mathcal{D}(A^{\frac{\alpha+1}{2}})_C, \) \( w \in \mathcal{D}(A^\alpha)_C \) and \( \alpha > 3, \) then
\[
|B(u, v), A^\alpha w) \leq 2^{\alpha+2} \bar{T}_C \left( |u|^{\frac{3}{2}} |Au|^{\frac{1}{2}} |A^{\frac{\alpha+2}{2}} v| + |A^\frac{\alpha}{2} u| |A^\frac{\alpha}{2} v|^{\frac{1}{2}} |A^\frac{\alpha}{2} w| \right). 
\]

The proof of Lemma 5.2 is omitted since it is similar to the one above, and the only difference is in the constant.

6. Induction

The standing assumption in this section is that 0 \( \in \mathcal{A}. \)

Under this assumption we will obtain, for all \( \alpha > 3 \) and for any solution \( u(t), \) \( t \in \mathbb{R}, \) in \( \mathcal{A}, \) estimates of the form
\[
|A^\frac{\alpha}{2} u(t)| \leq R_\alpha \nu \kappa_0^\alpha, \quad \text{for } t \in \mathbb{R},
\]
and for its analytic extension
\[
|A^\frac{\alpha}{2} u(\zeta)| \leq \tilde{R}_\alpha \nu \kappa_0^\alpha, \quad \text{for } \zeta \in \mathcal{S}(\delta_\alpha),
\]
where \( \delta = \delta_3 \) (i.e., \( \delta_\alpha = \delta_3 \) for all \( \alpha \geq 3 \)).

Note that (6.1), (6.2) were already established for \( \alpha = 1, 2, 3 \) in Section 5. For the values of \( R_\alpha, \tilde{R}_\alpha \) see (4.12), (4.13), (4.28), (4.34), (4.35). Therefore we will assume that (6.1) and (6.2) are valid for some \( \alpha \in \mathbb{N}, \alpha \geq 3 \) and prove by induction (starting with \( \alpha = 3 \)) that they are also valid for \( \alpha + 1. \) For this we need the following

Lemma 6.1. Let \( u(t) \) be the solution of the NSE satisfying \( u(0) = 0 \) and let \( u(\zeta) \) be its \( \mathcal{D}(A^\frac{\alpha}{2})_C \)-analytic extension in \( \mathcal{S}(\delta_\alpha), \) then
\[
g \in \mathcal{D}(A^\frac{\alpha}{2}),
\]
and
\[
G_{\alpha+1} := \frac{|A^\frac{\alpha}{2} g|}{\nu^2 \kappa_0^2} \leq \frac{\tilde{R}_\alpha}{\nu \kappa_0^2 \delta}.
\]

Proof. Since 0 \( \in \mathcal{A}, \) we have \( g = \frac{du}{dt} u(t) \in \mathcal{D}(A^\frac{\alpha}{2}), \) and since \( u(\zeta) \) satisfies (6.2) for \( t \in \mathbb{R}, \) also that
\[
|A^\frac{\alpha}{2} \frac{du(\zeta)}{d\zeta} | \leq \frac{1}{2\pi} \int_{|\zeta-t|=\delta} \frac{A^\frac{\alpha}{2} u(\zeta)}{(\zeta-t)^2} d\zeta \leq \frac{\tilde{R}_\alpha \nu \kappa_0^\alpha}{\delta},
\]
for \( t \in \mathbb{R}, \) in particular \( t = 0. \) Therefore, \( G_{\alpha+1} \leq \frac{\tilde{R}_\alpha}{\nu \kappa_0^2 \delta}. \)

Lemma 6.2. Let \( u(t), t \in \mathbb{R} \) be any solution of the NSE in \( \mathcal{A}. \) If \( u(t) \) satisfies (6.1), and its analytic extension satisfies (6.2) for some \( \alpha \geq 3, \) then (6.1) also holds for \( \alpha + 1, \) i.e.,
\[
|A^\frac{\alpha+1}{2} u(t)| \leq R_{\alpha+1} \nu \kappa_0^{\alpha+1}, \quad \forall \ t \in \mathbb{R},
\]
where
\[
R_{\alpha+1}^2 = \frac{36}{\pi^2} \left( \frac{1}{\delta \nu \kappa_0^2} + \frac{4}{\nu^2 \kappa_0^4 \delta^2} + 2 \sqrt{2} \Gamma_\alpha \right) \tilde{R}_\alpha^2.
\]
and \( \Gamma_\alpha \) is defined in (6.7) and (6.9).

Moreover, we have \( R_{\alpha+1}^2 > \tilde{R}_\alpha^2. \)
Proof. Let $t_0 \in \mathbb{R}$ be arbitrary and $\rho \in [0, \sqrt{2}\delta)$. Once again, we follow the procedure outlined in Remark 4.3. So, taking inner product in both sides of (4.9) with $A^\alpha u$, we get

$$\frac{1}{2} \frac{d}{d\rho}|A^{\frac{\alpha}{2}}u(t_0 + \rho e^{i\theta})|^2 \leq |(g, A^\alpha u)| - \nu \cos \theta |A^{\frac{\alpha+1}{2}}u|^2 + |(B(u, u), A^\alpha u)|.$$

For $\alpha = 3$, since

$$(B(u, u), A^3 u) = \sum_{j=1,2} [(B(D_j u, u), D_j A^2 u) + (B(u, D_j u), D_j A^2 u)]$$

we have

$$|(B(u, u), A^3 u)| \leq 4c_2^2 |A^{\frac{3}{2}} u|^{\frac{3}{2}} |Au| |A^{\frac{1}{2}} u|^{\frac{1}{2}} |A^2 u| + 8c_2^2 |A^{\frac{3}{2}} u|^{\frac{3}{2}} |A^{\frac{1}{2}} u|^{\frac{1}{2}} |Au| |A^2 u|$$

and consequently

$$(6.6) \quad \frac{1}{2} \frac{d}{d\rho}|A^{\frac{3}{2}}u(t_0 + \rho e^{i\theta})|^2 + \frac{3\nu \cos \theta}{4} |A^2 u|^2 \leq \frac{1}{\nu \cos \theta} |Ag|^2 + |(B(u, u), A^3 u)|$$

It follows that

$$(6.7) \quad \Gamma_3 := 3^3 \cdot 2^5 c_L \tilde{R}_1^2.$$

For $\alpha > 3$, by Young’s inequality and Lemma 5.2, we obtain

$$(6.8) \quad \frac{1}{2} \frac{d}{d\rho}|A^{\frac{\alpha}{2}}u(t_0 + \rho e^{i\theta})|^2 + \frac{3}{4} \nu \cos \theta |A^{\frac{\alpha+1}{2}}u|^2 \leq \frac{1}{\nu \cos \theta} |A^{\frac{\alpha-1}{2}}g|^2 + 2^{\alpha+\frac{3}{2}} c_A |u|^\frac{3}{2} |Au| |A^{\frac{\alpha}{2}} u| |A^{\frac{\alpha}{2}} u|$$

$$+ 2^{\alpha+3} c_A |A^{\frac{3}{2}} u|^{\frac{3}{2}} |A^{\frac{1}{2}} u|^{\frac{1}{2}} |A^{\frac{3}{2}} u|^2,$$
and

\[
\frac{1}{2} \frac{d}{dp} |A^\frac{\alpha+1}{2} u(t_0 + pe^{i\theta})|^2 + \frac{1}{2} \nu \cos \theta |A^\frac{\alpha+1}{2} u|^2 \\
\leq \frac{1}{\nu \cos \theta} |A^\frac{\alpha+1}{2} g|^2 + \frac{1}{\nu \cos \theta} \left( 2^{\alpha+1} c_A |u| \frac{1}{2} |Au| \frac{1}{2} |A^\frac{\alpha+1}{2} u| \right)^2 \\
+ 2^{\alpha+1} c_A |A^\frac{\alpha+1}{2} u| \frac{1}{2} |A^\frac{\alpha+1}{2} u|^2 \\
\leq \frac{1}{\nu \sqrt{2}} |A^\frac{\alpha+1}{2} g|^2 + \left( \frac{2^{\alpha+1} c_A}{\nu \cos \theta} \right)^2 \left( R_1 \nu \kappa_0 \right)^2 + 2^{\alpha+2} c_A \sqrt{R_1 \nu \kappa_0} \sqrt{R_2 \nu \kappa_0} |A^\frac{\alpha+1}{2} u|^2 \\
\leq \frac{2}{\nu \sqrt{2}} |A^\frac{\alpha+1}{2} g|^2 + \Gamma_\alpha \nu \kappa_0^2 |A^\frac{\alpha+1}{2} u|^2,
\]

where

\[
(6.9) \quad \Gamma_\alpha := 2^{\alpha+1} c_A \left[ 2^{\alpha+2} c_A \left( R_1 \right) + \sqrt{R_1 \nu \kappa_0} \right].
\]

It is not hard to prove that \( \Gamma_\alpha \geq 1 \) for all \( \alpha \geq 3 \).

Since \( \cos \theta \geq \frac{\sqrt{2}}{2} \), for \( \theta \in [-\pi/4, \pi/4] \), we obtain

\[
(6.10) \quad \frac{1}{\nu \sqrt{2}} \int_0^{2\theta} |A^\frac{\alpha+1}{2} u(t_0 + pe^{i\theta})|^2 d\rho \leq |A^\frac{\alpha+1}{2} u(t_0)|^2 + \frac{4\delta}{\nu} \left( \frac{1}{\nu \cos \theta} \right)^2 |A^\frac{\alpha+1}{2} g|^2 \\
+ \frac{4\delta}{\nu} \left( \frac{2^{\alpha+1} c_A}{\nu \cos \theta} \right)^2 |A^\frac{\alpha+1}{2} u|^2 \\
\leq \frac{2\sqrt{2}}{\nu \sqrt{2}} \kappa_0^2 R_1 \nu^2 \kappa_0^2 + 4\delta G^2 \nu^3 \kappa_0^2 (\alpha + 1) + 2\nu \kappa_0^2 \Gamma_\alpha \sqrt{2\delta} \nu \kappa_0^2 \kappa_0^2
\]

It follows that

\[
\nu \sqrt{2} \int_0^{2\theta} |A^\frac{\alpha+1}{2} u(t_0 + pe^{i\theta})|^2 d\rho \leq |A^\frac{\alpha+1}{2} u(t_0)|^2 + \frac{4\delta}{\nu} \left( \frac{1}{\nu \cos \theta} \right)^2 |A^\frac{\alpha+1}{2} g|^2 \\
+ 2\nu \kappa_0^2 \Gamma_\alpha \int_0^{2\theta} \left| A^\frac{\alpha+1}{2} u(\zeta) \right|^2 d\rho \\
\leq \frac{2\sqrt{2}}{\nu \sqrt{2}} \kappa_0^2 R_1 \nu^2 \kappa_0^2 + 4\delta G^2 \nu^3 \kappa_0^2 (\alpha + 1) + 2\nu \kappa_0^2 \Gamma_\alpha \sqrt{2\delta} \nu \kappa_0^2 \kappa_0^2
\]

and

\[
\int_0^{2\theta} |A^\frac{\alpha+1}{2} u(t_0 + pe^{i\theta})|^2 d\rho \leq \sqrt{2} \nu \kappa_0^2 R_1 \nu^2 \kappa_0^2 + 4\delta G^2 \nu^3 \kappa_0^2 (\alpha + 1) + 4\Gamma_\alpha \delta \nu \kappa_0^2 \nu^2 \kappa_0^2 (\alpha + 1)
\]

\[= N_\alpha.\]
Since \( u(\zeta) = D(A^{\frac{\alpha}{2}}) \) \( \mathbb{C} \)-valued analytic in \( S(\delta_{\alpha}) \), we obtain (as we have done in the proof of Lemma 4.4) the following successive relations

\[
|A^{\frac{\alpha}{2}} u(t_0)| = \left| \frac{1}{\pi \delta^2} \int_{D(t_0, \delta)} A^{\frac{\alpha}{2}} u(\zeta) d\zeta d\mathfrak{R}(\zeta) \right|
\leq \frac{1}{\pi \delta^2} \int_{D(t_0, \delta)} |A^{\frac{\alpha}{2}} u(\zeta)| d\zeta d\mathfrak{R}(\zeta)
\leq \frac{1}{\pi \delta^2} \int_{\delta} |A^{\frac{\alpha}{2}} u(\zeta)| d\zeta d\mathfrak{R}(\zeta)
= \frac{2}{\sqrt{2\pi \delta^2}} \int_{t_0+\delta} dt \int_{t_0-\delta} \left( \int_{0}^{\sqrt{\delta}} |A^{\frac{\alpha}{2}} u(\zeta)|^2 d\zeta \right)^{\frac{1}{2}} (\sqrt{2\delta})^{\frac{1}{2}} \leq \frac{2}{\sqrt{2\pi \delta^2}} 3\alpha(\delta_\alpha \sqrt{2\delta})^{\frac{1}{2}} = \frac{6 \cdot 2^{\frac{1}{2}}}{\sqrt{2\pi}} (\frac{N_{\alpha}}{\delta})^{\frac{1}{2}},
\]

that is, using (4.9),

\[
|A^{\frac{\alpha}{2}} u(t_0)|^2 \leq \frac{18\sqrt{2}}{\pi^2} N_{\alpha}^2 \delta
= \frac{18\sqrt{2}}{\pi^2} \left( \frac{\sqrt{2} R_{a}^{2} \nu_{\kappa_{0}}^{2\alpha}}{\delta} + 4 \sqrt{2} G^{2} \nu_{\kappa_{0}}^{2(\alpha+1)} + 4 \Gamma_{\alpha} \tilde{R}_{a} \nu_{\kappa_{0}}^{2(\alpha+1)} \right)
\leq \frac{36}{\pi^2} \nu_{\kappa_{0}}^{2(\alpha+1)} \left( \frac{R_{a}^{2}}{\delta \nu_{\kappa_{0}}^{2}} + 4 \tilde{R}_{a}^{2} \nu_{\kappa_{0}}^{2(\alpha+1)} + 2 \sqrt{2} \Gamma_{\alpha} \tilde{R}_{a} \nu_{\kappa_{0}}^{2(\alpha+1)} \right)
\leq \frac{36}{\pi^2} \nu_{\kappa_{0}}^{2(\alpha+1)} \left( \frac{1}{\delta \nu_{\kappa_{0}}^{2}} + 4 \nu_{\kappa_{0}}^{2(\alpha+1)} + 2 \sqrt{2} \Gamma_{\alpha} \tilde{R}_{a} \nu_{\kappa_{0}}^{2(\alpha+1)} \right) \tilde{R}_{a}^{2}
= R_{\alpha+1} \nu_{\kappa_{0}}^{2(\alpha+1)} \tilde{R}_{a}^{2},
\]

where

\[
R_{\alpha+1} = \frac{36}{\pi^2} \left( \frac{1}{\delta \nu_{\kappa_{0}}^{2}} + 4 \nu_{\kappa_{0}}^{2(\alpha+1)} \right) \tilde{R}_{a}^{2},
\]

It is easy to check that

\[
R_{\alpha+1} > \tilde{R}_{a}.
\]

We conclude the proof by observing that \( t_0 \in \mathbb{R} \) is arbitrary.

We next extend the result in Lemma 6.2 to the strip \( S(\delta_{\alpha}) \).

**Lemma 6.3.** If \( 0 \in A \) and if the solution \( u(t), t \in \mathbb{R} \) of the NSE in \( A \) satisfies (6.7), (6.3) for \( \alpha \), then \( u(\zeta) \) is a \( \mathcal{D}(A^{\frac{\alpha}{2}}) \) \( \mathbb{C} \)-valued analytic function, and (6.2) also holds for \( \alpha + 1 \). In particular, we have

\[
|A^{\frac{\alpha}{2}} u(\zeta)| \leq \tilde{R}_{\alpha+1} \nu_{\kappa_{0}}^{\alpha+1}, \quad \forall \zeta \in S(\delta_{\alpha}).
\]

where

\[
\tilde{R}_{\alpha+1} = \beta^{\Gamma_{\alpha+1}} \frac{36 \sqrt{2}}{\pi^2} \Gamma_{\alpha}(1 + \epsilon_{\alpha}) \tilde{R}_{a}^{2},
\]

\[
(6.11)
\]
\[ \beta := e^{2\sqrt{\delta \nu} \kappa_0^2}, \]

\[ \epsilon_\alpha = \frac{1}{2\sqrt{2\Gamma_\alpha \delta \nu \kappa_0^2}} + \frac{\sqrt{2}}{\Gamma_\alpha \nu^2 \kappa_0^4 \delta^2} + \frac{\pi^2}{72\nu^2 \kappa_0^4 \delta^4 \Gamma_\alpha \Gamma_{\alpha + 1}}, \]

and \( \Gamma_\alpha \) is defined in (6.9).

Moreover, the following inequality holds

\[ \tilde{R}_{\alpha + 1} > R_{\alpha + 1}. \]

**Proof.** Let \( t_0 \in \mathbb{R} \) be arbitrary and \( \rho \in [0, \sqrt{2}\delta) \). By virtue of Remarks 4.3 and A.4 we can assume that \( u(\zeta) \) is \( \mathcal{D}(A^{\frac{\alpha + 1}{2}}) \) -valued analytic. Taking the inner product of (3.18) with \( A^{\alpha + 1}u \), as in the proof of Lemma 6.2, we get

\[ \frac{1}{2} \frac{d}{d\rho} |A^{\frac{\alpha + 1}{2}} u(t_0 + \rho e^{i\theta})|^2 + \frac{\sqrt{2}}{4} |A^{\frac{\alpha + 1}{2}} u(\zeta)|^2 \]

\[ \leq \frac{\sqrt{2}}{\nu} |A^\frac{\alpha}{2} g|^2 + \nu \kappa_0 \Gamma_{\alpha + 1} |A^{\frac{\alpha + 1}{2}} u|^2, \]

where the Lemma 6.1 is used.

It follows that

\[ \frac{d}{d\rho} |A^{\frac{\alpha + 1}{2}} u(t_0 + \rho e^{i\theta})|^2 \leq \frac{4}{\nu \sqrt{2}} |A^\frac{\alpha}{2} g|^2 + 2\nu \kappa_0 \Gamma_{\alpha + 1} |A^{\frac{\alpha + 1}{2}} u|^2. \]

Since \( \rho \in [0, \sqrt{2}\delta) \), we have, by (6.3),

\[ |A^{\frac{\alpha + 1}{2}} u(\zeta)|^2 \leq e^{2\nu \kappa_0^2 \Gamma_{\alpha + 1} \rho} |A^{\frac{\alpha + 1}{2}} u(t_0)|^2 + \frac{\sqrt{2}}{\nu \sqrt{2}} \frac{|A^{\frac{\alpha}{2}} g|^2}{2\nu \kappa_0^2 \Gamma_{\alpha + 1}} \left( e^{2\nu \kappa_0^2 \Gamma_{\alpha + 1} \rho} - 1 \right) \]

\[ \leq e^{2\nu \kappa_0^2 \Gamma_{\alpha + 1} \rho} \left[ |A^{\frac{\alpha + 1}{2}} u(t_0)|^2 + \frac{\sqrt{2}}{\nu \kappa_0^2 \Gamma_{\alpha + 1}} \right] \]

\[ \leq e^{2\sqrt{\delta \nu} \kappa_0^2 \Gamma_{\alpha + 1}} \left[ R_{\alpha + 1}^2 \nu^2 \kappa_0^2 \Gamma_{\alpha + 1}^2 + \frac{\sqrt{2}}{\nu \kappa_0^2 \Gamma_{\alpha + 1}} \right] \]

\[ \leq \beta^{\Gamma_{\alpha + 1}} \left[ \frac{36}{\pi^2} \left( \frac{1}{\delta \nu \kappa_0^2} + \frac{4}{\nu^2 \kappa_0^4 \delta^2} + 2\nu \Gamma_{\alpha + 1} \right) + \frac{\sqrt{2}}{\nu \kappa_0^2 \delta^2 \Gamma_{\alpha + 1}} \right] \]

\[ \beta^{\Gamma_{\alpha + 1}} 72\sqrt{2} \frac{\Gamma_{\alpha + 1}}{\pi^2} (1 + \epsilon_\alpha) \bar{R}_0^2 \]

\[ =: \tilde{R}_{\alpha + 1}^{2 \nu^2 \kappa_0^2 \Gamma_{\alpha + 1}}. \]

While (6.13) shows that \( \tilde{R}_\alpha \) increases with \( \alpha \), the next result provides an explicit upper bound.
Proposition 6.4. For \( \alpha > 3 \),

\[
\tilde{R}_{\alpha+1}^2 \leq C(g)\beta_1^{4^\alpha+1} \beta_2^{(\alpha+1)^2+\frac{\beta}{2}(\alpha+1)},
\]

where

\[
C_1 := \prod_{\alpha=3}^\infty (1 + \epsilon_\alpha), \quad C_3 := 4\left[2^\frac{\alpha}{2}c_A^2 \tilde{R}_1 \tilde{R}_2 + 2^\frac{\alpha}{2}c_A \sqrt{\tilde{R}_1 \tilde{R}_3}\right],
\]

\[
C_2 := 3^32^{-7}c_L^8 \tilde{R}_1^2 \prod_{\gamma=3}^\infty (1 + \eta_\gamma), \quad \eta_\gamma = \sqrt{\tilde{R}_1 \tilde{R}_3} \frac{2^\gamma c_A \tilde{R}_1 \tilde{R}_2}{2^\gamma+2 \alpha},
\]

\[
\beta_1 := \beta C_3, \quad \beta_2 := \max\left\{\frac{72\sqrt{7}}{\pi^2}, c_A^2 \tilde{R}_1 \tilde{R}_2\right\},
\]

and

\[
C(g) := C_1 C_2 \tilde{R}_2^{2\beta - 19/2}.
\]

Proof. Since \( \sum_{\alpha=3}^\infty \epsilon_\alpha \) is convergent, we have \( C_1 := \prod_{\alpha=3}^\infty (1 + \epsilon_\alpha) < \infty \). Due to the definition of \( \Gamma_\alpha \) in (6.9), we have

\[
\prod_{\gamma=3}^\alpha \Gamma_\gamma = 3^32^\frac{3\alpha}{2}c_L^8 \tilde{R}_1^2 \prod_{\gamma=4}^\alpha 2^{2\gamma+7/2} (c_A^2 \tilde{R}_1 \tilde{R}_2) \left(1 + \frac{\sqrt{\tilde{R}_1 \tilde{R}_3}}{2^\gamma+2 c_A \tilde{R}_1 \tilde{R}_2}\right)
\[
= 3^32^\frac{3\alpha}{2}c_L^8 \tilde{R}_1^2 \prod_{\gamma=4}^\alpha 2^{2\gamma+7/2} (c_A^2 \tilde{R}_1 \tilde{R}_2) [1 + \eta_\gamma]
\[
= 3^32^\frac{3\alpha}{2}c_L^8 \tilde{R}_1^2 2^{2\gamma+7/2} (c_A^2 \tilde{R}_1 \tilde{R}_2) \prod_{\gamma=4}^\alpha [1 + \eta_\gamma]
\[
< 3^32^{-7}c_L^8 \tilde{R}_1^2 2^{2\gamma+7/2} \eta_\alpha (c_A^2 \tilde{R}_1 \tilde{R}_2) \prod_{\gamma=4}^\alpha [1 + \eta_\gamma]
\[
= 2^{\alpha^2+\frac{4\alpha}{3}} (c_A^2 \tilde{R}_1 \tilde{R}_2) \prod_{\gamma=4}^\alpha [1 + \eta_\gamma]
\]

and

\[
\sum_{\gamma=4}^{\alpha+1} \Gamma_\gamma = \sum_{\gamma=4}^{\alpha+1} \left[2^{2\gamma+7/2} c_A^2 \tilde{R}_1 \tilde{R}_2 + 2^{\gamma+1} c_A \sqrt{\tilde{R}_1 \tilde{R}_3}\right]
\[
= 2^{7/2}c_A^2 \tilde{R}_1 \tilde{R}_2 \sum_{\gamma=4}^{\alpha+1} 2^{2\gamma} + 2^\frac{\alpha}{2} c_A \sqrt{R_1 R_3} \sum_{\gamma=4}^{\alpha+1} 2^\gamma
\[
= 2^{7/2}c_A^2 \tilde{R}_1 \tilde{R}_2 \frac{4^{\alpha+2} - 4^3}{3} + 2^\frac{\alpha}{2} c_A (2^{\alpha+2} - 2^4) \sqrt{R_1 R_3}
\[
\leq \left(2^{\frac{5}{2}} c_A^2 \tilde{R}_1 \tilde{R}_2 + 2^\frac{\alpha}{2} c_A \sqrt{R_1 R_3}\right) 4^{\alpha+2}
\[
= C_3 4^{\alpha+1}.
\]
It follows from the recursion relation (6.13) that
\[
\tilde{R}_{\alpha+1}^2 := \beta^2(1 + \epsilon_\alpha)\tilde{R}_\alpha^2
\]
\[
= \beta^{\sum_{\gamma=1}^{\alpha+1} \Gamma_\gamma}(\frac{72\sqrt{2}}{\pi^2})\alpha(1 + \epsilon_\gamma)\tilde{R}_3^2
\]
\[
\leq \beta^{C_3 4^{\alpha+1}} C_2 C_1 \tilde{R}_3^2 \max(\frac{72\sqrt{2}}{\pi^2}, 2, c^2 A \tilde{R}_1 \tilde{R}_2)^{\alpha^2 + \frac{2}{3}} \alpha^{-5}
\]
\[
= C(g) \beta_1^{4^{\alpha+1}} \beta_2^{(\alpha+1)^2 + \frac{2}{3}(\alpha+1)}.
\]

\[\square\]

**Remark 6.5.** Theorem 2.1 is now a direct consequence of Lemmas 4.4, 4.8, 4.9, and Proposition 6.4.

**Remark 6.6.** The explicit upper bound of \( \tilde{R}_\alpha \) given in Proposition 6.4 provides information about the possible dependence of \( \tilde{R}_\alpha \) upon \( \alpha \). On the other hand, \( \delta_\alpha = \delta_3 \), for all \( \alpha > 3 \).

7. The Class \( \mathcal{C}(\sigma) \)

The estimates for \( \{A^\frac{A}{\partial t} u(t), t \in \mathbb{R}\} \) can be slightly improved by shrinking the width \( \delta_\alpha \) of the strip \( \mathcal{S}(\delta_\alpha) \) in the induction argument for \( \alpha > 3 \).

**Theorem 7.1.** Let \( 0 \in \mathcal{A} \) and let
\[
\delta_{\alpha+1} := \frac{\delta_\alpha}{2}
\]
for \( \alpha \in \mathbb{N}, \alpha \geq 3 \). Then for any solution in \( u(t) \in \mathcal{A}, t \in \mathbb{R} \), one has
\[
|A^\frac{A}{\partial t} u(\zeta)| \lesssim \tilde{R}_{\alpha+1}^{\alpha+1} \nu^{\alpha+1}, \quad \forall \zeta \in \mathcal{S}(\delta_{\alpha+1}),
\]
for all \( \alpha \geq 3 \), where the constants \( \tilde{R}_{\alpha+1}, \alpha \geq 3 \), are redefined in the following way
\[
\tilde{R}_{\alpha+1}^2 := \frac{1024\sqrt{2}}{\pi^2} \Gamma_\alpha(1 + \xi_\alpha)\tilde{R}_\alpha^2,
\]
with
\[
\xi_\gamma = \frac{1}{4\sqrt{2}\nu \delta_\alpha \Gamma_\alpha} + \frac{1}{\sqrt{2}\nu \delta_\alpha \Gamma_\alpha\delta_{\alpha+1}}.
\]
Furthermore, we have the following estimate
\[
\tilde{R}_{\alpha+1}^2 \lesssim \tilde{C}(g) \beta_3^{(\alpha+1)^2},
\]
where
\[
\beta_3 := \max\left(\frac{1024\sqrt{2}}{\pi^2}, c^2 A \tilde{R}_1 \tilde{R}_2\right), \quad \tilde{C}(g) := C_2 C_3 \tilde{R}_3^2 \beta_3^{-3/8},
\]
and
\[
C_4 := \prod_{\gamma=1}^{\infty}(1 + \xi_\gamma).
\]
Proof. As done in the proof of Lemma 4.9, we can easily prove that under the new definition (7.2), the relation (7.1) is true.

Then, we obtain (as in the proof of Proposition 6.4)

\[
\tilde{R}_{\alpha+1}^2 := \frac{1024\sqrt{2}}{\pi^2} \Gamma_{\alpha}(1 + \xi_{\alpha}) \tilde{R}_{\alpha}^2 \\
< \left(\frac{1024\sqrt{2}}{\pi^2}\right)^{\alpha-2} \tilde{R}_{3}^2 \prod_{\gamma=3}^{\alpha} \Gamma_{\gamma} \sum_{\gamma=3}^{\infty} (1 + \xi_{\gamma}) \\
\leq \left(\frac{1024\sqrt{2}}{\pi^2}\right)^{\alpha-3} 2^{\alpha^2 + \frac{\alpha}{2}} \left(c_4 \tilde{R}_{1} \tilde{R}_{2}\right)^{\alpha-2} C_2 C_4 \tilde{R}_{3}^2 \\
\leq \tilde{C}(g) \gamma^5 (\alpha+1)^2.
\]

□

The estimates in Theorem 7.1 identify the role of the subset of \(C^\infty([0,L]^2) \cap H\) defined below.

Definition 7.2. \[C(\sigma) := \{ u \in C^\infty([0,L]^2) \cap H : \exists c_0 = c_0(u) \in \mathbb{R} \text{ such that } \frac{|A^\frac{\nu}{2} u|^2}{v^2 \kappa_0^2} \leq c_0 e^{\sigma \alpha^2}, \alpha \in \mathbb{N}\} \]

Remark 7.3. The main conclusion of Theorem 7.1 can be given in the following succinct formulation

\[0 \in A \Rightarrow A \subset C(\frac{3}{2} \ln \beta_3).\]

Remark 7.4. An equivalent definition of the class \(C(\sigma)\) is

\[(7.4) \quad C(\sigma) = \{ u \in C^\infty([0,L]^2) \cap H : |u|_{\sigma} := \sup \{|A^\frac{\nu}{2} u| e^{\sigma \alpha^2}, \alpha \in \mathbb{N}\} < \infty\}.\]

It is easy to check that \(u \mapsto |u|_{\sigma}\) is a norm on \(C(\sigma)\). Obviously, \(C(\sigma)\) equipped with this norm is a Banach space.

Moreover, Theorem 7.1 has the following corollary

Corollary 7.5. If \(0 \in A\), then \(g \in C(\frac{5}{2} \ln \beta_3)\).

Proof. Since \(\delta_{\alpha} = \frac{1}{2\pi} \delta_3 > \frac{1}{2\pi} \delta_3\), by (6.3) and (7.3) we get

\[|A^\frac{\nu}{2} g| \leq \tilde{R}_3 \nu_0^2 \frac{2}{\delta_3} \leq \tilde{R}_3 \nu_0^2 \frac{2}{\delta_3} \leq \tilde{R}_3 \nu_0^2 \frac{2}{\delta_3} \beta_3^{\frac{1}{2}} \alpha^2.\]

and then

\[
\frac{|A^\frac{\nu}{2} g|^2}{v^2 \kappa_0^2} \leq \tilde{C}(g) \frac{5}{2} \beta_3^{\frac{5}{2}} \alpha^2,
\]

where both sides are dimensionless.

Consequently, it follows that

\[g \in C(\frac{5}{2} \ln \beta_3).\]

□
Remark 7.6. We can prove that if \( v \in H \) satisfies
\[
|e^{b \ln (\kappa_0^{-1} A^{\frac{1}{2}} + 0)^{2}}| < \infty, \quad a > e, \quad b > 0
\]
then
\[
(7.5) \quad v \in \mathcal{C}(1/b).
\]
Indeed, noting the following relation
\[
|A^\frac{1}{2} v| = |A^\frac{1}{2} e^{-b \ln (\kappa_0^{-1} A^{\frac{1}{2}} + 0)^{2}} e^{b \ln (\kappa_0^{-1} A^{\frac{1}{2}} + 0)^{2}} v|
\]
\[
\leq |A^\frac{1}{2} e^{-b \ln (\kappa_0^{-1} A^{\frac{1}{2}} + 0)^{2}}|_{o_p} |e^{b \ln (\kappa_0^{-1} A^{\frac{1}{2}} + 0)^{2}} v|,
\]
we obtain that
\[
|A^\frac{1}{2} e^{-b \ln (\kappa_0^{-1} A^{\frac{1}{2}} + 0)^{2}} u|^2 = \sum_{k \in \mathbb{Z} \setminus \{0\}} |k|^{2a} e^{-2b \ln (|k| + a)} \sum_{k \in \mathbb{Z} \setminus \{0\}} |\hat{u}(k)|^2
\]
\[
\leq \sup_{k \in \mathbb{Z} \setminus \{0\}} |k|^{2a} e^{-2b \ln (|k| + a)} \sum_{k \in \mathbb{Z} \setminus \{0\}} |\hat{u}(k)|^2
\]
\[
= \sup_{k \in \mathbb{Z} \setminus \{0\}} |k|^{2a} e^{-2b \ln (|k| + a)} |u|^2,
\]
and (7.5) follows from Definition 7.2.

8. Proof of Proposition 2.3

In this section we will use the short procedure given in Remark 4.3 (see also Remark A.4). Namely, we assume that the solution \( u(t) \) and its analytic extension \( u(\zeta) \) exists and then establish the necessary a priori estimates. In addition, for simplicity, we use the following notation
\[
\rho_\beta := \rho_{\max}(G, \frac{|A^\frac{1}{2} u_0|}{\nu K_0^\beta}),
\]
(8.1)
\[
M_{\alpha, \beta} := M_\alpha(G, G_{\alpha-1}, \frac{|A^\frac{1}{2} u_0|}{\nu K_0^\beta}),
\]
where
\[
G_\alpha := \frac{|A^\frac{1}{2} g|}{\nu K_0^{\alpha^2}}.
\]
\( \rho_{\max}(\cdot, \cdot) \) is defined in (A.7) and \( M_\alpha(\cdot, \cdot) \) is defined in (A.10), (8.3), (8.10) and (8.12).

Using Corollary A.2 and Lemma A.3 we easily obtain
Proposition 8.1. \( u(\zeta) \) is \( \mathcal{D}(A^{\frac{1}{2}})_{\mathbb{C}} \)-valued analytic function in \( \Pi(t_0, \rho_1) \) and satisfies
\[
|A^{\frac{1}{2}}u(\zeta)| \leq [M_{1,1}^2 + \sqrt{2}G^2]^{\frac{1}{2}} \nu \kappa_0,
\]
for any \( \zeta \in \Pi(t_0, \rho_1) \), where \( \Pi(\cdot, \cdot), M_{1,1} \) are defined in (8.1) and (8.1).

To prove Proposition 8.1, we need to prove first the following

Proposition 8.2. Assume \( \beta \in \mathbb{N}, u_0 \in \mathcal{D}(A^{\frac{1}{2}}) \), and \( g \in \mathcal{D}(A^{\frac{1}{2}}) \) then \( u \) is \( \mathcal{D}(A^{\frac{3+1}{2}})_{\mathbb{C}} \)-valued analytic function in the interior of \( S(t_0, \rho_3) \) and there exists \( M_{\beta, \beta} \) such that
\[
|A^{\frac{3}{2}}u(\zeta)| \leq M_{\beta, \beta} \nu \kappa_0 \beta, \quad \zeta \in S(t_0, \rho_3),
\]
holds, where \( S(\cdot, \cdot) \) is defined in (8.8) and \( M_{\beta, \beta} \) is defined in (8.3).

Proof. For \( \beta = 1 \), (8.2) is a direct corollary of Lemmas A.1 A.3 and 4.4.

For \( \beta = 2 \), we will need a supplementary estimate for the term \( (B(u, u), A^2u) \).

Integrating by parts we have
\[
(B(u, u), A^2u) = \sum_{j=1,2} \left[ (B(D_j u, u), D_j A u) + (B(u, D_j u), D_j A u) \right].
\]

Using interpolation \( |Au| \leq |A^{\frac{1}{2}}u| |A^{\frac{3}{2}}u| \) and \( |A^{\frac{3}{2}}u| \leq |u| |A^{\frac{3}{2}}u| \), we infer that
\[
|B(u, u), A^2u| \leq 4c_L^2 |(A^{\frac{1}{2}}u)Au| |A^{\frac{3}{2}}u| + |u| |A^{\frac{1}{2}}u| |Au| |A^{\frac{3}{2}}u| ^2
\]
\[
\leq 8c_L^2 |u| |A^{\frac{1}{2}}u| |Au| |A^{\frac{3}{2}}u| ^2
\]
\[
\leq \frac{8c_L^2}{\nu} |A^{\frac{3}{2}}u| |Au| |A^{\frac{3}{2}}u| ^2.
\]

Using (8.8), we obtain
\[
\frac{1}{2} \frac{d}{dt} |Au(t_0 + \rho \sin \theta)|^2 + \frac{3\nu \cos \theta}{4} |A^{\frac{1}{2}}u|^2 \leq \frac{1}{\nu \cos \theta} |A^{\frac{3}{2}}g|^2 + |(B(u, u), A^2u)|
\]
\[
\leq \frac{|A^{\frac{1}{2}}g|^2}{\nu \cos \theta} + \frac{1}{|A^{\frac{1}{2}}g|^2} \left( \frac{2}{\nu \cos \theta} \right)^{3/4} \frac{8c_L^2}{\nu \kappa_0} |A^{\frac{3}{2}}u| |Au| |A^{\frac{3}{2}}u| ^2
\]
\[
\leq \frac{|A^{\frac{1}{2}}g|^2}{\nu \cos \theta} + \frac{2^{23/4} c_L^4}{\nu^3 \kappa_0^2} |A^{\frac{3}{2}}u| |Au| |A^{\frac{3}{2}}u| ^2 + \frac{\nu \cos \theta}{4} |A^{\frac{3}{2}}u|^2,
\]
whence
\[
\frac{d}{dp} |Au(t_0 + \rho \sin \theta)|^2 + \frac{\sqrt{2}c_L}{2} |A^{\frac{1}{2}}u|^2 \leq \frac{\sqrt{2}c_L |A^{\frac{3}{2}}g|^2}{\nu} + \frac{3}{4} \frac{2^{23/4} c_L^4}{\nu^3 \kappa_0^2} |A^{\frac{3}{2}}u| |Au|^2,
\]
since \( |\theta| \leq \frac{\pi}{2} \).

By Lemmas A.1 and A.3 for all \( \zeta \in S(t_0, \rho_2) \), we have \( |A^{\frac{1}{2}}u(\zeta)| \leq M_{1,2} \nu \kappa_0 \). It follows that
\[
\frac{d}{dp} |Au(t_0 + \rho \sin \theta)|^2 + \frac{\sqrt{2}c_L}{2} |A^{\frac{1}{2}}u|^2 \leq a + b |Au|^2,
\]
where \( a = 2\sqrt{2}c_L |A^{\frac{1}{2}}g|^2 / \nu \) and \( b = 3 \cdot 2^{25/4} c_L^4 \nu \kappa_0^2 M_{1,2}^4 \). Then, we obtain
\[
|Au(t_0 + \rho \sin \theta)|^2 \leq e^{b_0} |Au(t_0)|^2 + \frac{a}{b} (e^{b_0} - 1) \leq e^{b_0} [ |Au|^2 + \frac{a}{b} ] \leq e^{b_0} [ |Au|^2 + \frac{a}{b} ],
\]
that is, 
\[ |Au(\zeta)| \leq M_{2,2}\nu\kappa^2_0, \quad \forall \zeta \in S(t_0, \rho_2), \]
where \( M_{2,2} = M_2(G, G_1, \frac{|Au_0|}{\nu\kappa_0}) \) and
\begin{equation}
M_2(G, G_1, \frac{|Au_0|}{\nu\kappa_0}) := e^{3\gamma_0^2\nu^2\kappa^2_0M_1^2\rho_2^2} \left[ \frac{|Au_0|^2}{\nu^2\kappa_0^2} + \frac{G^2_1}{3^3 \cdot 2^1 \kappa_0^6 \cdot M_1^4} \right]^{\frac{1}{2}}.
\end{equation}

Furthermore, from (8.4) we obtain
\begin{equation}
\int_0^{\rho_2} |A^{\frac{3}{2}}u(t_0 + \rho e^{i\theta})|^2 d\rho \leq \gamma,
\end{equation}
where
\[ \gamma := \frac{4|A^{\frac{3}{2}}g|^2}{\nu^2} \rho_2^2 + \frac{\sqrt{\pi}}{\nu}|Au_0|^2 + 3\cdot 2^{13} \kappa_0^6 \cdot M_1^2 \cdot M_2^2 \cdot \rho_2^2. \]

For any \( \zeta = t_0 + \rho e^{i\theta} \), \( |\theta| < \frac{\pi}{4} \) and \( 0 < \rho < \rho_2 \), we can choose \( r > 0 \) such that
\begin{equation}
D(\zeta, r) \subset S(t_0, \rho_2).
\end{equation}
From (8.6), by the same method used in the proof of Lemma 4.4 we obtain that
\begin{equation}
|A^{\frac{3}{2}}u(\zeta)| = \frac{1}{2\pi r^2} \int_{D(\zeta, r)} A^{\frac{3}{2}} u(\xi) d\mathcal{R}(\xi) d\mathcal{I}(\xi) \leq \frac{\rho^2_{\gamma}^2 \gamma^2}{2\sqrt{8\gamma^2}}.
\end{equation}

Therefore, \( u(\zeta) \) is a \( \mathcal{D}(A)^{\frac{3}{2}} \)\( \nu \)-valued analytic function in the interior of the sector \( S(t_0, \rho_2) \).

For \( \beta = 3 \), from (6.0) we proceed as in the case \( \beta = 2 \) to obtain
\begin{equation}
|A^{\frac{3}{2}}u(\zeta)| \leq M_{3,3}\nu\kappa^3_0, \quad \forall \zeta \in S(t_0, \rho_3),
\end{equation}
where \( M_{3,3} = M_3(G, G_2, \frac{|Au_0|}{\nu\kappa_0}) \) and
\begin{equation}
M_3(G, G_2, \frac{|Au_0|}{\nu\kappa_0}) := e^{3\gamma_0^2\nu^2\kappa^3_0M_4^2\rho_3^2} \left[ \frac{|Au_0|^2}{\nu^2\kappa_0^6} + \frac{G^2_2}{3^3 \cdot 2^{15} \kappa_0^8 \cdot M_1^8} \right]^{\frac{1}{2}}.
\end{equation}

Furthermore, the case \( \beta = 3 \) can be treated as the previous case \( \beta = 2 \) by deducing from (8.9) and (8.10) the sequence of relations analogous to the sequence of (8.6), (8.7) and (8.8). We obtain that \( u(\zeta) \) is a \( \mathcal{D}(A)_\nu \)\( \nu \)-valued analytic function in the interior of the sector \( S(t_0, \rho_3) \).

For the cases \( \beta > 3 \), by Lemma 5.2 we have
\begin{equation}
\frac{1}{2} d\rho \left| A^{\frac{\beta}{2}}u(t_0 + \rho e^{i\theta}) \right|^2 + \frac{1}{2\nu \cos \theta} |A^{\frac{\beta+1}{2}}u| \leq \frac{|A^{\frac{\beta+1}{2}}g|^2}{\nu \cos \theta} + \frac{(2^{3+2}c_A|u|\sigma_3|Au|\sigma_3|A^{\frac{\beta}{2}}u|)^2}{\nu \cos \theta} + 2^{3+2}c_A|A^{\frac{\beta}{2}}u|^2 |A^{\frac{\beta}{2}}u|^2.
\end{equation}

It follows that for any \( \zeta \in S(t_0, \rho_3) \), we have
\[ |A^{\frac{\beta}{2}}u(\zeta)| \leq M_{1,3}\nu\kappa_0, |Au(\zeta)| \leq M_{2,3}\nu\kappa_0^2, |A^{\frac{\beta}{2}}u(\zeta)| \leq M_{3,3}\nu\kappa_0^3. \]
Therefore, we obtain
\[
\frac{1}{2} \frac{d}{dt} |A^{\hat{\beta}} u(t_0 + \rho e^{i\theta})|^2 + \frac{1}{2} \nu \cos \theta |A^{\hat{\beta} + 1} u|^2 \\
\leq \frac{|A^{\hat{\beta} + 1} g|^2}{\nu \cos \theta} + \frac{1}{\nu \cos \theta} \left\{ (2^{\beta + \frac{3}{2}} c_A (M_{1,\beta} M_{2,\beta})^{\frac{1}{2}} \nu_k)^2 |A^{\hat{\beta}} u|^2 + 2^{\beta + \frac{3}{2}} c_A (M_{1,\beta} M_{2,\beta})^{\frac{1}{2}} \nu_k^2 |A^{\hat{\beta}} u|^2 \right\} \\
\leq \sqrt{\frac{\gamma}{\nu}} |A^{\hat{\beta}} g|^2 + \gamma_\beta \nu_k^2 |A^{\hat{\beta}} u|^2,
\]
where
\[
\gamma_\beta = 2^{2\beta + 7/2} c_A (M_{1,\beta} M_{2,\beta}) + 2^{\beta + 1} c_A \sqrt{M_{1,\beta} M_{2,\beta}}.
\]
and consequently
\[
|A^{\hat{\beta}} u(\zeta)| \leq M_{\beta,\beta} \nu_k^\beta, \quad \forall \ \zeta \in S(t_0, \rho_\beta),
\]
where \(M_{\beta,\beta} = M_\beta(G, G_{\alpha-1}, |A^{\hat{\beta}} u|_{\nu_k^\beta})\) and
\[
(8.12) \quad M_\alpha(G, G_{\alpha-1}, |A^{\hat{\beta}} u|_{\nu_k^\beta}) := e^{\gamma_\beta \nu_k^\beta} \nu_\rho \left[ \frac{|A^{\hat{\beta}} u|_{\nu_k^\beta}}{\nu^2 \nu_k^2} + \sqrt{2G_\alpha^2} \right]^{\frac{1}{\gamma_\beta}}, \quad \forall \ \alpha > 3.
\]

Furthermore, using the same method as in the case \(\beta = 2\) or 3, we can deduce that \(u(\zeta)\) is a \(D(A^{\hat{\alpha}+1})\)-valued analytic function in the interior of this sector \(S(t_0, \rho_\beta)\). \(\square\)

We are now ready to give the proof of Proposition 2.3. First, the case \(\alpha = 1\) is a direct consequence of Lemma 4.4. Suppose that Proposition 2.3 is valid for some \(\alpha \geq 1\). This means that if \(g \in D(A^{\hat{\beta} + 1})\), then \(A \subset D(A^{\hat{\beta} + 1})\) and moreover any solution \(u(\zeta)\) in \(A\) can be extended to an \(D(A^{\hat{\alpha}+1})\)-valued analytic function in the strip \(S(\delta_\alpha)\) such that
\[
(8.13) \quad \sup \{|A^{\hat{\alpha}+1} u(\zeta)| : \zeta \in S(\delta_\alpha)\} \leq m_{\alpha+1} \nu_k^\alpha < \infty.
\]

We will prove, under the assumption \(g \in D(A^{\hat{\beta}})\), that \(A \subset D(A^{\hat{\alpha}+1})\) and that any solution \(u(\zeta)\) in \(A\) can be extended to a \(D(A^{\hat{\alpha}+1})\)-valued analytic function in some strip \(S(\delta_{\alpha+1})\) satisfying
\[
(8.14) \quad \sup \{|A^{\hat{\alpha}+1} u(\zeta)| : \zeta \in S(\delta_{\alpha+1})\} \leq m_{\alpha+1} \nu_k^{\alpha+1} < \infty.
\]

The estimate in (8.13) is in particular valid for \(t \in \mathbb{R}\). Therefore, applying Proposition 8.2 for \(\beta = \alpha + 1\) and each \(t_0 \in \mathbb{R}\), we obtain by defining
\[
m_{\alpha+1} := M_{\alpha+1}(G, G_\alpha, m_\alpha),
\]
and
\[
\delta_{\alpha+1} := \frac{1}{\sqrt{2}} \rho_{\max}(G, m_\alpha)
\]
that the solution \(u(\zeta)\) in \(A\) can be extended to a \(D(A^{\hat{\alpha}+1})\)-valued analytic function in the strip \(S(\delta_{\alpha+1})\) and relation (8.14) holds. Thus the proof is completed by induction.
9. The “All for one, one for all” law

Proposition 9.1. Let $\alpha \in \mathbb{N}$ be fixed. Then

$$\mathcal{A} \cap \mathcal{D}(A^\frac{\alpha}{2}) \neq \emptyset \Rightarrow \mathcal{A} \subset \mathcal{D}(A^\frac{\alpha}{2}).$$

Furthermore, $g \in \mathcal{D}(A^\frac{\alpha}{2\alpha - 2})$, $\alpha > 2$.

Proof. Note that the case $\alpha = 1, 2$ are trivially true. We now consider the case $\alpha = 3$.

Let $u^0 \in \mathcal{A} \cap \mathcal{D}(A^\frac{3}{2})$ and let $u(t)$ denote the solution of the NSE such that $u(0) = u^0$. We already know that this solution extends to a $\mathcal{D}(A)^{\mathcal{C}}$ valued analytic solution $u(\zeta)$ in a strip of $\mathcal{S}(\delta_0) := \{\zeta \in \mathbb{C} : |\Im(\zeta)| < \delta, \delta > 0\}$. Therefore from

$$\frac{d}{d\zeta}u(\zeta) + \nu Au(\zeta) + B(u(\zeta), u(\zeta)) = g, \quad \zeta \in \mathcal{S}(\delta_0),$$

we obtain

$$(9.1) \quad g = \frac{d}{d\zeta}u(\zeta)_{|\zeta=0} + \nu Au^0 + B(u^0, u^0),$$

where $\frac{d}{d\zeta}u(\zeta)_{|\zeta=0} \in \mathcal{D}(A)$, $\nu Au^0 \in \mathcal{D}(A^\frac{3}{2})$.

Since for $\forall \ w \in \mathcal{D}(A)$ we have

$$|B(u^0, u^0), Aw| = | \sum_{j=1,2} (B(D_ju^0, u^0) + B(u^0, D_ju^0), D_jw)|$$

$$\leq (c_1^2 |A^\frac{3}{2} Au^0| |Au^0| + c_A |u^0|^\frac{3}{2} |Au^0|^\frac{3}{2})(|D_1w| + |D_2w|)$$

$$\leq \sqrt{2}(c_L^2 + c_A) |u^0|^\frac{3}{2} |Au^0|^\frac{3}{2} |A^\frac{3}{2} w|,$$

we get that

$$B(u^0, u^0) \in \mathcal{D}(A^\frac{3}{2}),$$

and

$$|A^\frac{3}{2} B(u^0, u^0)| \leq \sqrt{2}(c_L^2 + c_A) |u^0|^\frac{3}{2} |Au^0|^\frac{3}{2}.$$

Thus from (9.1) and (9.2) we infer that

$$g \in \mathcal{D}(A^\frac{3}{2}).$$

Proposition 2.3 now yields

$$\mathcal{A} \subset \mathcal{D}(A^\frac{3}{2}),$$

and that there exists a $\delta_3 \in (0, \delta_2)$ such that any solution $u(t), t \in \mathbb{R}$ in $\mathcal{A}$ extends to a $\mathcal{D}(A^\frac{3}{2})^{\mathcal{C}}$ valued analytic function in the strip $\mathcal{S}(\delta_3) := \{\zeta \in \mathbb{C} : |\Im(\zeta)| < \delta_3\}$.

We will proceed now by induction on $\alpha$; the induction assumption will be that for some $\alpha \geq 3$

$$(9.3) \quad \mathcal{A} \cap \mathcal{D}(A^\frac{\alpha}{2}) \neq \emptyset \Rightarrow \mathcal{A} \subset \mathcal{D}(A^\frac{\alpha}{2}).$$

Assume that there exists a $u^0 \in \mathcal{A} \cap \mathcal{D}(A^\frac{\alpha-1}{2})$ and that $u(t)$ is the solution of the NSE in $\mathcal{A}$ satisfying $u(0) = u^0$. The induction assumption implies that $Au \in \mathcal{D}(A^\frac{\alpha}{2})$ and $\frac{dAu}{dt}|_{t=0} = \frac{dAu}{dt}|_{t=0} \in \mathcal{D}(A^\frac{\alpha}{2})$.

By Lemma 5.1 we have that for $\forall \ w \in \mathcal{D}(A^{\alpha-1})$

$$|B(u^0, u^0), A^{\alpha-1}w| \leq 2^{\alpha-1}c_A \left(|u^0|^\frac{\alpha}{2} |Au^0|^\frac{\alpha}{2} |A^\frac{\alpha}{2} u^0| + |A^{\frac{\alpha-1}{2}} u^0| |A^\frac{\alpha}{2} u^0|^\frac{\alpha}{2} |A^\frac{\alpha}{2} w| \right).$$
It follows that
\[ B(u^0, u^0) \in \mathcal{D}(A_{\alpha+1}^{\frac{1}{2}}). \]
Thus we have that \( g \in \mathcal{D}(A_{\alpha+1}^{\frac{1}{2}}) \). Using Proposition 2.3 we finally obtain that (9.3) also holds for \( \alpha + 1 \). This completes the proof.

\[ \Box \]

Observe that since
\[ H \cap C^\infty([0, L]^2; \mathbb{R}^2) = \bigcap_{\alpha \in \mathbb{N}} \mathcal{D}(A_{\alpha}^{\frac{1}{2}}), \]
Theorem 2.4 is a direct consequence of Proposition 9.1.

**Remark 9.2.** Proposition 9.1 and Theorem 2.4 assert that if there is one point in the attractor \( \mathcal{A} \) belonging to a certain class (namely \( H \cap H^\alpha([0, L]^2)(\alpha \in \mathbb{N}) \) or \( H \cap C^\infty([0, L]^2) \)) then all points of \( \mathcal{A} \) belong to this class. We will show that for the class defined in Definition 7.2 the “All for one, one for all” law is also valid. Moreover, we expect that this law is “almost” universal, in that it holds for a variety of subsets of \( H \).

10. \( \cup_{\sigma > 0} \mathcal{C}(\sigma) \nsubseteq C^\infty([0, L]^2; \mathbb{R}^2) \cap H \)

A natural question one may ask regarding the newly defined classes \( \mathcal{C}(\sigma), \sigma > 0 \), is: will the union of all the classes \( \mathcal{C}(\sigma) \) for \( \sigma \) ranging in \((0, \infty)\) actually be the same as the family of \( C^\infty \) functions in \( H \)? The answer is no.

**Theorem 10.1.** \( \cup_{\sigma > 0} \mathcal{C}(\sigma) \nsubseteq C^\infty([0, L]^2; \mathbb{R}^2) \cap H \)

To prove this theorem, we assume that \( \cup_{\sigma > 0} \mathcal{C}(\sigma) = C^\infty([0, L]^2; \mathbb{R}^2) \cap H \) and we will arrive at a contradiction. To this end, we need the following observation:

**Proposition 10.2.** There exists a 1-1 correspondence between \( \dot{\mathcal{C}}_{per}([0, L]^2; \mathbb{R}) \) and \( C^\infty([0, L]^2; \mathbb{R}^2) \cap H \).

In the above proposition, \( \dot{\mathcal{C}}_{per}([0, L]^2; \mathbb{R}) \) consists of elements in \( C^\infty_{per}([0, L]^2; \mathbb{R}) \) with zero-average.

**Proof.** Let \( \psi \) be any given element in \( \dot{\mathcal{C}}_{per}([0, L]^2; \mathbb{R}) \). Define \( u = (u_1, u_2) \) by setting \( u_1 = \frac{\partial \psi}{\partial x_2}, u_2 = -\frac{\partial \psi}{\partial x_1} \), then one immediately sees that \( u_1, u_2 \) both belong to \( C^\infty_{per}([0, L]^2; \mathbb{R}) \), and \( \nabla \cdot u = 0 \), thus we get an element \( u \) in \( C^\infty([0, L]^2; \mathbb{R}^2) \cap H \). (We recall that \( \psi \) is called the stream function corresponding to \( u = (u_1, u_2) \)).

Conversely, given \( u = (u_1, u_2) \), with \( u_1, u_2 \in C^\infty_{per}([0, L]^2; \mathbb{R}) \) and \( \nabla \cdot u = 0 \), we will get the stream function \( \psi \) by uniquely solving
\[ u_1 = \frac{\partial \psi}{\partial x_2}, u_2 = -\frac{\partial \psi}{\partial x_1}. \]

From the above equation, we have \( \frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1} = \Delta \psi \). When we express this equality in the form of Fourier series expansion, we uniquely determine the coefficient \( \hat{\psi}(k) \) for \( \psi \) expressed in terms of the coefficients of \( u_1, u_2 \) as follows:

\[ \hat{\psi}(k) = -\frac{k_2 \hat{u}_1(k) - k_1 \hat{u}_2(k)}{|k|^2}, \]

where \( k = (k_1, k_2) \) is the wave number.
and hence the stream function corresponding to the function \( u = (u_1, u_2) \) is
\[
\psi(x) = \sum_{k \in \mathbb{Z}^2 \setminus \{0\}} \hat{\psi}(k)e^{i\sigma_0 k \cdot x} = \sum_{k \in \mathbb{Z}^2 \setminus \{0\}} \frac{k_2 \hat{u}_1(k) - k_1 \hat{u}_2(k)}{|k|^2}e^{i\sigma_0 k \cdot x}.
\]
Therefore, the one-to-one correspondence between these two families is established. 

**Proof of Theorem 10.1.** Given an arbitrary \( u \in C^\infty([0, L]^2; \mathbb{R}^2) \cap H \), suppose that \( u \in \mathcal{C}(\sigma) \), for some \( \sigma > 0 \). Let \( \psi \) be the corresponding stream function, invoking the relation (10.1), one can show that
\[
|(-\Delta)^{\frac{\alpha}{2}} \psi|_{L^2} \leq \left| A \frac{\alpha}{2} u \right|_{L^2} \leq \sqrt{c_0}e^{\sigma(\alpha-1)^2/2}|\nu|^{\alpha-1},
\]
where the constant \( c_0 \) is from the definition of the class. Equivalently,
\[
|(-\Delta)^{\frac{\alpha}{2}} \psi|_{L^2} \leq \sqrt{c_0}e^{\sigma\alpha^2/2}\nu^{\alpha}, \quad \forall \, \alpha \geq 0.
\]
It follows from the above inequality and the Sobolev embedding theorem that,
\[
|(-\Delta)^{\frac{\alpha}{2}} \psi|_{L^\infty} \leq c_1|(-\Delta)^{\frac{\alpha}{2}} \psi|_{H^s} \leq c_2|(-\Delta)^{\frac{\alpha}{2}} \psi|_{L^2} \leq c_3e^{\sigma\alpha^2/2}\nu^{\alpha}, \quad \forall \, \alpha \geq 1.
\]
where \( c_1, c_2, c_3 \) are non-dimensional constants.

The one-to-one correspondence in Proposition 10.2 and the inequality given above imply that, for any \( \psi \in \dot{C}_{per}^\infty([0, L]^2; \mathbb{R}) \), we have
\[
(-\Delta)^{\frac{\alpha}{2}} \psi|_{L^\infty} \leq ce^{\sigma\alpha^2/2}\nu^{\alpha}, \quad \forall \, \alpha \geq 1.
\]
for some constant \( c \). Notice that \( |(-\Delta)^{\frac{\alpha}{2}} \psi|_{L^\infty} \) controls the magnitude of all the derivatives of the function \( \psi \) up to order \( \alpha - 1 \), so this inequality says that the coefficients in the Taylor series expansion of every function \( \psi \in \dot{C}_{per}^\infty([0, L]^2; \mathbb{R}) \) must have an estimate of the form \( e^{\sigma\alpha^2/2} \).

However, we can easily construct a function \( \eta \in \dot{C}_{per}^\infty([0, L]^2; \mathbb{R}) \) which does not have this kind of estimate for the coefficients of its Taylor series expansion. To find such a \( C^\infty \) function, we choose \( x_0 \) to be the center of the square \( [0, L]^2 \), and for any given \( C^\infty \) function \( \psi \) consider first the following power series
\[
\sum_{\beta \in \mathbb{N}^2} \frac{1}{\beta!} e^{\sigma|\beta|^2/2}(x - x_0)^\beta
\]

It follows from Borel’s Theorem (see page 381 in [20]) that this series defines a \( C^\infty \) function, denoted by \( \psi \) on, say the disk \( D(x_0, r_0) \), for some \( r_0 < \frac{L}{2\sqrt{2}} \). Moreover, we extend the definition \( \psi \) to \( \mathbb{R}^2 \) by setting \( \psi = 0 \) outside this disk.

Then choose another function \( \psi \in C^\infty([0, L]^2) \) such that \( \psi = 1 \) on \( D(x_0, 2r_0/3) \), with \( \text{supp}(\psi) \subset D(x_0, 2r_0/3) \) and \( 0 \leq \psi \leq 1 \). Finally, let \( \eta_1 = \psi \psi \), and define \( \eta = \eta_1 - \int_{[0,L]^2} \eta_1(x)dx \), then extend the definition of \( \eta \) to \( \mathbb{R}^2 \) by periodicity, to get \( \eta \in \dot{C}_{per}^\infty([0, L]^2; \mathbb{R}) \). Now the function \( \eta \) has coefficients in its Taylor expansion which contradict the estimates in (10.2). Hence, the assumption \( \cup_{\sigma > 0} \mathcal{C}(\sigma) = C^\infty([0, L]^2) \cap H \) is not true, and the theorem is proved. 

A second question regarding the classes defined in this paper concerns the possibility that \( \mathcal{C}(\sigma_1) = \mathcal{C}(\sigma_2) \), for some \( \sigma_1 < \sigma_2 \). Equivalently, does the family of the classes \( \{\mathcal{C}(\sigma)\}_{\sigma > 0} \) form an increasing (or decreasing) family? We answer this question by proving the following proposition.
Proposition 10.3. For the family of classes \( \{C(\sigma)\}_{\sigma > 0} \), we have,
\[
C(\sigma_1) \subsetneq C(\sigma_2), \quad \forall \ \sigma_1 < \sigma_2.
\]

Proof. We assume by contradiction that
\[
C(\sigma_1) = C(\sigma_2),
\]
for some \( \sigma_1 < \sigma_2 \). Since both \( C(\sigma_1) \) and \( C(\sigma_2) \) are Banach spaces, if it happens that these two are the same, then their norms are comparable; i.e., there exist \( m > 0, M > 0 \) such that
\[
m \leq \frac{|u|_{C(\sigma_1)}}{|u|_{C(\sigma_2)}} \leq M,
\]
holds for all \( u \), where \( | \cdot |_{C(\sigma)} \) is defined in (7.4).

Initially, in the definition of the class \( C(\sigma) \), we allow only \( \alpha \in \mathbb{N} \); however, we are going to show that we could actually extend this definition by taking \( \alpha \in \mathbb{R} \). Indeed, for any \( 0 < \beta < \alpha + 1 \), by interpolation we have
\[
|A^\frac{2}{\alpha} u| \leq |A^\frac{2}{\alpha} u|^{\alpha+1-\beta} |A^{\frac{2}{1-\beta}} u|^\beta.
\]

One could easily obtain, using the estimate \( |A^\frac{2}{\alpha} u|^2 \leq c_0 e^{\sigma^2 \nu^2 k_0^{2\alpha}} \), the following
\[
|A^\frac{2}{\alpha} u|^2 \leq c_0' e^{\sigma^2 \nu^2 k_0^{2\beta}}
\]
where \( c_0' = c_0 e^{\sigma^4 / \beta} \). Notice that the change of constant \( c_0 \) does not depend on the values of \( \alpha \) and \( \beta \).

Assuming that such a modification in our definition of the class \( C(\sigma) \) has been made, we take in particular the function \( u(x) = \sum_{\{k: |k|^2 = \Lambda \}} \hat{u}(k) e^{i\kappa x} \), for some fixed \( \Lambda \). After a direct calculation, one gets
\[
\frac{|u|_{C(\sigma_1)}}{|u|_{C(\sigma_2)}} = e^{\frac{\left(\ln \Lambda_1\right)^2}{2\sigma_1^2} - \frac{\left(\ln \Lambda_2\right)^2}{2\sigma_2^2}} = e^{\frac{\left(\ln \Lambda_1\right)^2}{2\sigma_1^2\sigma_2^2} (\ln \Lambda_2)^2}.
\]

As \( \Lambda \to \infty \), we have that
\[
\frac{|u|_{C(\sigma_1)}}{|u|_{C(\sigma_2)}} \to \infty.
\]
Thus, we see that the constant \( M > 0 \) in (10.3) cannot exist. \( \square \)

11. “One for all and all for one” Law for \( \bigcup_{\sigma > 0} C(\sigma) \)

The main result in this section is simple to state.

Theorem 11.1. If \( \mathcal{A} \cap \bigcup_{\sigma > 0} C(\sigma) \neq \emptyset \), then \( \mathcal{A} \subset \bigcup_{\sigma > 0} C(\sigma) \).

The proof is a consequence of the following three lemmas.

Lemma 11.2. Let \( u(t), t \in \mathbb{R} \) be any solution of the NSE in \( \mathcal{A} \). If \( g \in C^\infty([0,L]^2) \),
then for any \( \alpha \in \mathbb{N} \), \( u \) is a \( D(A^\frac{2}{\alpha})_{|C'} \)-valued analytic function in some strip \( S(\delta_\alpha) \) and satisfies
\[
|A^\frac{2}{\alpha} u(\zeta)| \leq M_\alpha \nu k_0^{\alpha}, \quad \forall \ \zeta \in S(\delta_\alpha).
\]

In particular, for \( \alpha \geq 3 \), we can choose
\[
\delta_{\alpha+1} := \frac{\delta_\alpha}{2},
\]
(11.1)
Proof. Since \( \alpha \) where \( \Gamma \), to obtain that
\[
M_{\alpha + 1} := \frac{8\sqrt{2}}{\pi} \left( \frac{M_{\alpha}^2}{\delta_{\alpha} \nu K_0} + 4G_{\alpha-1}^2 + \sqrt{2} \Gamma_\alpha M_{\alpha}^2 \right)^{1/2},
\]
(11.2)

\[
G_{\alpha} := \frac{|A^g|}{\nu^2 K_0^{\alpha+2}},
\]
(11.3)
and
\[
\Gamma_\alpha := \begin{cases} \frac{3^2 \cdot 2^{31/2} c_A M_1^2}{2^{4/3} c_A [2^{\alpha/2} c_A M_1 + \sqrt{M_1 M_3}]} & \text{if } \alpha = 3, \\ \frac{3^2 \cdot 2^{31/2} c_A M_1^2}{2^{4/3} c_A [2^{\alpha/2} c_A M_1 + \sqrt{M_1 M_3}]} & \text{otherwise.} \end{cases}
\]
(11.4)

The only difference is that \( \delta_2, M_2, \) and \( M_3 \) depend on \( G_1 \).

For \( \alpha \geq 3 \), we have the following inequality (6.10)
\[
\frac{1}{2} \frac{d}{dp} |A^g u(t_0 + \rho e^{it})|^2 + \frac{\sqrt{2}}{4} |A^g u(\zeta)|^2 \leq \frac{2}{\nu \sqrt{2}} |A^{\mathbb{R}^1} g|^2 + \frac{\nu K_0^2 \Gamma_\alpha}{\nu K_0^2} |A^g u|^2,
\]
where \( \Gamma_\alpha \) is defined as in the Lemma \([6.2]\) and the only difference is that it depends on \( M_\alpha \), not \( \tilde{R}_\alpha \), see (11.4). It follows that
\[
\int_0^{\sqrt{2} \delta_\alpha} |A^{\mathbb{R}^1} u(\zeta)|_{\xi = \zeta + \rho e^{it}} d\rho \leq \sqrt{2} \left(M_\alpha + 4G_{\alpha-1}^2 + \sqrt{2} \nu K_0^2 \Gamma_\alpha M_{\alpha}^2 \right) \nu K_0^{\alpha-1}.
\]

Proceeding as in the proof of Theorem \([7.1]\) i.e. choosing \( \delta_{\alpha+1} \) as in (11.1), we obtain that for \( \forall \zeta \in S(\delta_{\alpha+1}) \), there is a disk \( D(\zeta, \delta_{\alpha+1}) \) contained in \( S(\delta_\alpha) \). It follows that
\[
|A^{\mathbb{R}^1} u(\zeta)| \leq \frac{2 \delta_\alpha}{\delta_{\alpha+1}^2} \int_0^{\sqrt{2} \delta_\alpha} |A^{\mathbb{R}^1} u(\zeta)|_{\xi = \zeta + \rho e^{it}} d\rho
\]
\[
\leq \frac{2 \delta_\alpha}{\delta_{\alpha+1}^2} \left( \int_0^{\sqrt{2} \delta_\alpha} |A^{\mathbb{R}^1} u(\zeta)|^2_{\xi = \zeta + \rho e^{it}} d\rho \right)^{1/2} \left( \sqrt{2} \delta_\alpha \right)^{1/2}
\]
\[
\leq \frac{8 \sqrt{2}}{\pi} \left( \frac{M_{\alpha}^2}{\delta_{\alpha} \nu K_0} + 4G_{\alpha-1}^2 + \sqrt{2} \Gamma_\alpha M_{\alpha}^2 \right)^{1/2} \nu K_0^{\alpha-1}
\]
\[
= M_{\alpha+1} \nu K_0^{\alpha-1}.
\]
\[
\square
\]

Remark 11.3. Notice that \( M_\alpha \) depends only on \( g \).

Lemma 11.4. If \( A\cap C(\sigma) \neq \emptyset \) for some \( \sigma > 0 \), then \( g \in C^\infty([0, L]^2) \) and \( G_\alpha \) defined in (11.2) satisfies the following estimate
\[
G_{\alpha}^2 \leq \gamma_2 e^{\sigma_2 \alpha^2}, \forall \alpha \in \mathbb{N}, \alpha \geq \alpha_1,
\]
where
\[
\alpha_1 := \max\{\left\lfloor \log_4 \frac{2}{c_4} \right\rfloor + 1, 4\},
\]
and \( \gamma_2, \sigma_2, c_4 \) are defined in (11.11), (11.12), (11.9).
Proof. Let \( u_0 \in A \cap C(\sigma) \) and \( u(t) \) be the solution of the NSE with \( u(0) = u_0 \). By Proposition 9.1 we infer that \( g \in C^\infty([0,L]^2) \). Then applying Lemma 11.2, we know that \( u \) is a \( D(\mathcal{A}^{1/2}) \)-valued analytic function in the strip \( S(\delta_0) \) and satisfies

\[
(11.5) \quad \left| \frac{d}{dp} \mathcal{A}^{1/2} u(0) \right| = \left| \frac{1}{2\pi i} \int_{\partial D(0,\delta_0)} \frac{\mathcal{A}^{1/2} u(\xi)}{\xi^2} \, d\xi \right| \leq \frac{M_0 \nu \kappa_0^\alpha}{\delta_\alpha}.
\]

Since \( u_0 \in C(\sigma) \), we obtain that there exists \( c_0 \) such that

\[
\frac{|\mathcal{A}^{1/2} Au_0|^2}{(\nu \kappa_0^\alpha)^2} \leq c_0 e^{\sigma (\alpha + 2)^2} \leq c_0 e^{8\sigma \alpha^2}.
\]

That is

\[
(11.6) \quad |\mathcal{A}^{1/2} Au_0| \leq c_1 e^{\sigma \alpha^2 \nu_k \kappa_0^{\alpha + 2}},
\]

where

\[
c_1 := \sqrt{c_0} e^{4\sigma}.
\]

Using Lemma 5.2 we have

\[
(11.7) \quad |\mathcal{A}^{1/2} B(u_0, u_0)|^2 \leq 2^{2\sigma \alpha^2} \left( |u_0||Au_0||A^{1/2} u_0|^2 + |\mathcal{A}^{1/2} u_0||A^{1/2} u_0|^2 \right)
\]

\[
\leq 2^{2\sigma \alpha^2} c_A \left( e^{\sigma((\alpha+1)^2+1)} + e^{\sigma(\alpha^2+5)} \right) \nu \kappa_0^{2\alpha+4}
\]

\[
\leq 2^{2\sigma \alpha^2} c_A \nu \kappa_0^{2\alpha+4}
\]

\[
= 2^{2\sigma \alpha^2} c_A \nu \kappa_0^{2\alpha+4},
\]

where

\[
c_2 := \sqrt{2c_A} c_0 e^{\sigma (1+\frac{1}{8}(2+\ln 4)^2)}.
\]

From (11.5), (11.6), (11.7), we obtain that

\[
|\mathcal{A}^{1/2} g| = \left| \frac{d}{dp} \mathcal{A}^{1/2} u(0) + \nu \mathcal{A}^{1/2} Au_0 + \mathcal{A}^{1/2} B(u_0, u_0) \right|
\]

\[
\leq \left| \frac{M_0}{\delta_\alpha \nu \kappa_0^\alpha} + c_3 e^{\sigma \alpha^2} \right| \nu_0 \kappa_0^{\alpha + 2},
\]

i.e.

\[
(11.8) \quad G_\alpha \leq \frac{M_0}{\delta_\alpha \nu \kappa_0^\alpha} + c_3 e^{\sigma \alpha^2},
\]

where

\[
c_3 := c_1 + c_2.
\]
Applying Lemma 11.2 and (11.8), we infer that for $\alpha > 3$, $M_\alpha$ defined in (11.2) satisfies

\[
M^{\alpha+1}_0 \leq \frac{128}{\pi^2} \left( \frac{M^2_\alpha}{\delta_1 \nu K_0^2} + \frac{8[ M^{\alpha-1}_0/c_3 \sigma(\alpha-1)^2 + \sqrt{2} \Gamma_\alpha M^2_\alpha]}{\delta_2 (\nu^2 K_0^4)} \right)
\]

\[
\leq \frac{128}{\pi^2} \left( \frac{M^2_\alpha}{\delta_1 \nu K_0^2} + \frac{8[ M^{\alpha-1}_0/c_3 \sigma(\alpha-1)^2 + \sqrt{2} \Gamma_\alpha M^2_\alpha]}{\delta_2 (\nu^2 K_0^4)} \right)
\]

\[
= c_4^{\alpha+1} M^2_\alpha + c_5 e^{2\sigma(\alpha-1)^2},
\]

where

\[
c_4 = \frac{128}{\pi^2} \left( \frac{1}{\delta_3 \nu K_0^2} + \frac{1}{\delta_3 \nu^2 K_0^4} + 2^2 c_4 M^2_1 M_2 + c_4 \sqrt{M_1 M_3} \right), \quad c_5 = \frac{2^8 \pi^2 c_3^2}{3}.
\]

It follows that

\[
M^2_\alpha \leq c_4^{\alpha} M^{\alpha-1}_\alpha + c_5 e^{2\sigma(\alpha-2)^2}
\]

\[
\leq c_4^{\alpha} [c_4^{\alpha}(c_4^{\alpha-1}) M^{\alpha-2}_\alpha + c_5 e^{2\sigma(\alpha-3)^2}] + c_5 e^{2\sigma(\alpha-2)^2}
\]

\[
\leq (c_4^{\alpha})^2 M^{\alpha-2}_\alpha + c_5 (1 + c_4^{\alpha}) e^{2\sigma(\alpha-2)^2}
\]

\[
\ldots
\]

\[
\leq (c_4^{\alpha})^{\alpha-4} M^2_\alpha + c_5 [1 + \ldots + (c_4^{\alpha})^{\alpha-4}] e^{2\sigma(\alpha-2)^2}
\]

\[
= (c_4^{\alpha})^{\alpha-4} M^2_\alpha + c_5 \left( c_4^{\alpha-3} - 1 \right) e^{2\sigma(\alpha-2)^2}.
\]

For $\alpha \geq \alpha_1$ (i.e. $c_4^{\alpha} \geq 2$) we have

\[
M^2_\alpha \leq (c_4^{\alpha})^{\alpha-4} M^2_\alpha + 2 c_5 (c_4^{\alpha})^{\alpha-4} e^{2\sigma(\alpha-2)^2}
\]

\[
= (M^2_\alpha + 2 c_5) c_4^{\alpha-4} (c_4^{\alpha})^{\alpha-4} e^{2\sigma(\alpha-2)^2}
\]

\[
\leq \gamma_1 e^{2\sigma\alpha^2},
\]

where

\[
\sigma_1 := \ln 4 + 2\sigma, \quad \gamma_1 := (M^2_\alpha + 2 c_5) c_4^{\alpha-4} e^{\frac{\ln 4 + 2\sigma}{\alpha_1}}.
\]

Therefore, by (11.8), we infer that for $\alpha \geq \alpha_1$, $G_\alpha$ satisfies

\[
G^2_\alpha \leq \left[ \frac{M^2_\alpha}{\delta_1 \nu K_0^2} + c_3 e^{\sigma^2 \alpha^2} \right] \leq \frac{2 M^2_\alpha}{\delta_1 \nu^2 K_0^4} + 2 c_3^{\alpha} e^{2\sigma\alpha^2}
\]

\[
\leq \frac{\gamma_1 e^{2\sigma_1 \alpha^2 - 2\gamma_1 (\ln 4)^2}}{\delta_3 \nu^2 K_0^4} + 2 c_3^{\alpha} e^{2\sigma\alpha^2} \leq \gamma_2 e^{2\sigma\alpha^2},
\]

where

\[
\gamma_2 := \frac{\gamma_1 e^{2\gamma_1 (\ln 4)^2}}{\delta_3 \nu^2 K_0^4} + 2 c_3^{\alpha},
\]

(11.11)

\[
\sigma_2 := \max\{3\sigma_1, 2\sigma\}.\]
Remark 11.5. By defining
\[ c_0 := \max \{ \gamma_2, \frac{|A_{\alpha}^{g}|^2}{\nu^4 \kappa_0^2}, \ldots, \frac{|A_{\alpha}^{g}|^2}{\nu^4 \kappa_{2\alpha+1}^2} \}, \]
we see that \( g \in C(\sigma_2) \).

Combining the Lemma 11.2 and Lemma 11.4, we obtain

**Lemma 11.6.** If \( A \cap C(\sigma) \neq \emptyset \) for some \( \sigma > 0 \), then for \( u(t), t \in \mathbb{R} \), any solution of the NSE in \( A \), we obtain that
\[ |A_{\alpha}^{g} u(t)| \leq M_\alpha \nu^\alpha_0, \ \forall \ t \in \mathbb{R}, \ \forall \ \alpha \in \mathbb{N}, \ \alpha > \alpha_1 \]
and
\[ M_\alpha^2 \leq \gamma_3 e^{\sigma_\alpha_2}, \]
where \( \sigma_3, \gamma_3 \) are defined in (11.13), (11.14).

**Proof.** From Lemma 11.4, we know that \( g \in C^\infty([0, L]^2) \) and has the following estimates
\[ G_\alpha^2 \leq \gamma_2 e^{\sigma_\alpha_2}, \ \forall \ \alpha \geq \alpha_1, \]
and for \( \alpha > \alpha_1 \), we have
\[ M_{\alpha+1} := \frac{8\sqrt{2}}{\pi} \left( \frac{M_\alpha^2}{\delta_\alpha \nu^\alpha_0} + 4 G_{\alpha-1}^2 + \sqrt{2} \Gamma_\alpha M^2_\alpha \right)^\frac{1}{2}. \]
It follows that
\[ M_{\alpha+1}^2 \leq \frac{128}{\pi^2} \left( \frac{M_\alpha^2}{\delta_\alpha \nu^\alpha_0} + 4 \gamma_2 e^{\sigma_\alpha_2 (\alpha-1)^2} + \sqrt{2} \Gamma_\alpha M^2_\alpha \right) \]
\[ \leq \frac{128}{\pi^2} \left( \frac{M_\alpha^2}{\delta_\alpha \nu^\alpha_0} + 4 \sqrt{2} \Gamma_\alpha M^2_\alpha \right) + \frac{2^7}{\pi^2} \gamma_2 e^{\sigma_\alpha_2 (\alpha-1)^2} \]
\[ = c_6 4^{\alpha+1} M^2_\alpha + c_7 e^{\sigma_\alpha_2 (\alpha-1)^2}, \]
where
\[ c_6 := \frac{128}{\pi^2} \left( \frac{1}{\delta_3 \nu^\alpha_0} + 2^2 c_\Lambda^2 M_1 M_2 + c_A \sqrt{M_1 M_3} \right), \quad c_7 := \frac{2^7}{\pi^2} \gamma_2. \]
Then we obtain the analogue of the relation (11.10)
\[ M^2_\alpha \leq \gamma_3 e^{\sigma_\alpha_2}, \]
where
\[ \sigma_3 := 2 \ln 4 + 2 \sigma_2, \]
(11.13) \[ \gamma_3 := (M_4^2 + 2 c_7) c_6^{-4} e^{4 \sigma_2} e^{-\frac{\ln c_6^{-4} \ln 4 - 4 \sigma_2}{e^{2 \sigma_2}}}. \]

Now, we can pass to the proof of Theorem 11.1.
proof of Theorem 11.7. Let \( u_0 \in \mathcal{A} \), and \( u(t), t \in \mathbb{R} \), be the solution of the NSE with \( u(0) = u_0 \). From Lemma 11.6, we obtain that

\[
\frac{|A^\perp u(t)|^2}{\nu^2 \kappa_0^{2\alpha}} \leq M_0^2 \leq \gamma_3 e^{\sigma_3 \alpha^2}, \quad \forall \ t \in \mathbb{R}, \ \forall \ \alpha \in \mathbb{N}, \alpha > \alpha_1.
\]

By choosing

\[
\gamma_4 := \max\{\gamma_3, \frac{|A^\perp u_0|^2}{\nu^2 \kappa_0^{2\alpha}}, \ldots, \frac{|A^\perp u(t)|^2}{\nu^2 \kappa_0^{2\alpha}}\},
\]

we infer that

\[
\frac{|A^\perp u(t)|^2}{\nu^2 \kappa_0^{2\alpha}} \leq \gamma_4 e^{\sigma_3 \alpha^2}, \quad \forall \ \alpha \in \mathbb{N},
\]

i.e.

\[
u_0 \in \mathcal{C}(\sigma_3).
\]

Since \( u_0 \notin \mathcal{A} \) is arbitrary, the proof is complete. \( \square \)

APPENDIX A.

Let \( P_\kappa \) denote the orthogonal projection of \( H \) onto

\[
\text{span}\{w_j : Aw_j = \lambda_j w_j, j \leq \kappa\}.
\]

Let \( g \in H, u_0 \in D(A^\perp), t_0 \in \mathbb{R} \) and let \( u(t) \) for \( t \geq t_0 \) be the solution of the NSE with “initial data” \( u(t_0) = u_0 \); moreover, for \( \kappa \in \mathbb{N} \) we denote by \( u_\kappa(t) \in P_\kappa H, t \geq t_0 \) the solution of the following ODE

(A.1) \[
\frac{du_\kappa(t)}{dt} + \nu A\kappa u(t) + P_k B(u_k(t), u_\kappa(t)) = P_k g, \quad t \geq t_0,
\]

(A.2) \[
u_\kappa(t_0) = P_k u_0,
\]

usually called a Galerkin approximation of the solution \( u(t) \) (e.g. see [21], Chapter 2). Standard ODE theory guarantees the existence of a unique solution \( u_\kappa(t) \) for \( \{t \in \mathbb{R} : |t - t_0| < \epsilon_0\} \) (where \( \epsilon_0 = \epsilon_0(u_0) > 0 \)) and then for all \( t \geq t_0 \) provided that for every \( t_1 \in [t_0, \infty) \), \( u_\kappa(t) \) exists on \( [t_0, t_1) \) and

(A.3) \[
sup_{t_1 \in [t_0, t_1)} |u_\kappa(t)| < \infty.
\]

The validity of the property (A.3) is obtained in the following way. From (A.1) we infer

\[
\frac{1}{2} \frac{d}{dt} |u_\kappa(t)|^2 + \nu |A^\perp u_\kappa(t)|^2 (g, u_\kappa(t)) \leq |g| |u_\kappa(t)|, \quad t \in [t_0, t_1),
\]

\[
\frac{1}{2} \frac{d}{dt} |A^\perp u_\kappa(t)|^2 + \nu |A u_\kappa(t)|^2 (g, A u_\kappa) \leq \frac{|g|^2}{2\nu} + \frac{\nu |A u_\kappa|^2}{2},
\]

and then

\[
\frac{d}{dt} |u_\kappa(t)|^2 + \nu \kappa_0^2 |u_\kappa(t)|^2 \leq \frac{|g|^2}{\nu \kappa_0^2}, \quad t \in [t_0, t_1),
\]

\[
\frac{d}{dt} |A^\perp u_\kappa(t)|^2 + \nu |A u_\kappa(t)|^2 \leq \frac{|g|^2}{\nu},
\]

from which it follows that

\[
|u_\kappa(t)|^2 \leq e^{-\nu \kappa_0^2(t-t_0)} u_0^2 + (1 - e^{-\nu \kappa_0^2(t-t_0)}) \nu^2 G^2,
\]

(A.4) \[
|A^\perp u_\kappa(t)|^2 \leq e^{-\nu \kappa_0^2(t-t_0)} |A^\perp u_0|^2 + (1 - e^{-\nu \kappa_0^2(t-t_0)}) \nu^2 \kappa_0^2 G^2,
\]
for $t \geq t_0$. In addition, it is not hard to prove that $u_\kappa(t) \to u(t)$ in $H$ uniformly for $t \in [t_0, t_1]$, \forall $t_1 \in \mathbb{R}$.

The complexified version of $\text{(A.1)}$ and $\text{(A.2)}$ has the following form

$$dV(\zeta) + \nu AV(\zeta) + P_\kappa B(V(\zeta), V(\zeta)) = P_\kappa g,$$

where $t_0 \in \mathbb{R}$, $V(\zeta) \in P_\kappa H_C$ and $V^0 \in \mathcal{D}(\hat{A}_\kappa)$. We will study the initial value problem for this equation. The classical form of Cauchy’s existence theorem (e.g. see [1] Chapter 11, Section 5) ensures that the complex differential system $\text{(A.5)}$ has a unique analytic solution $V(\zeta)$ defined in some neighborhood $\{ \zeta \in \mathbb{C} : |\zeta - t^0| < \epsilon_0 \}$ of $t^0$ satisfying the condition $\text{(A.6)}$.

To extend the domain of existence for $V(\zeta)$ we proceed in the following manner (see also [2], [3]): from $\text{(A.5)}$ we obtain

$$\frac{d}{d\rho}V(t^0 + \rho e^{i\theta}) + e^{i\theta} [\nu AV(\zeta) + P_\kappa B(V(\zeta), V(\zeta))]|_{\zeta = t^0 + \rho e^{i\theta}} = e^{i\theta} P_\kappa g,$$

where it is convenient to take $|\theta| \leq \frac{\pi}{4}$. Then proceeding as in the proof of Lemma 4.4 we obtain that if

$$\rho < \min\{\epsilon_0, \rho_1\},$$

where

$$\rho_1 = \rho_{\text{max}}(G, \frac{|A^0 V^0|}{\nu_0}) := \sqrt{2} \left\{ 4 \cdot 24^3 \cdot c_L \left[ \frac{2^4}{24} G^2 + \frac{|A^0 V^0|^2}{\nu_0^2} \right]^2 \right\}^{-1},$$

then

$$|A^0 V(t)|^2 \leq \frac{2^4}{24} G^2 (\nu_{\kappa_0})^2 + \sqrt{2} |A^0 V^0|^2.$$
does not depend on \( \kappa \) and depends only on \( G \) and \( \frac{1}{\nu\kappa_0^2} \).

Clearly, in the case \( t^0 \geq t_0, V(t^0) = u_\kappa(t^0) \), the restriction of \( V(\zeta) \) to some interval \( (t^0, t^1) \) coincides with the Galerkin approximation \( u_\kappa(t) \) defined in (A.1) and (A.2). Therefore, letting the initial time \( t^0 \) vary over the whole \( [t_0, \infty) \), we obtain the following.

**Corollary A.2.** For all \( \kappa \in \mathbb{N} \), \( u_\kappa(\zeta) \) is \( \mathcal{D}(A^\pm_\kappa) \)-valued analytic analytic in

\[
(A.11) \quad \Pi(t_0, \rho_1) := \{ \zeta \in \mathbb{C} : \Re(\zeta) \geq t_0, \Im(\zeta) \leq \min\{\Re(\zeta), \frac{\rho_1}{\sqrt{2}}\} \}
\]

and the relation

\[
|A^\pm_\kappa u_\kappa(\zeta)| \leq [M_{1,1}^2 + \sqrt{2}G^2]^\pm \nu\kappa_0
\]

holds for \( \forall \, \zeta \in \Pi(t_0, \rho_1) \).

We conclude our consideration by presenting the justification of the Remark 4.3.

For this purpose we need the following

**Lemma A.3.** Let \( \alpha \in \mathbb{N} \) and \( N_\alpha \) be a domain containing an interval \( (t_0, t_\alpha) \subset \mathbb{R} \). Furthermore, let \( u_\kappa(\zeta) \) be the (complex) Galerkin approximation of solution \( u(t) \) on \( [t_0, +\infty) \) of the NSE satisfying \( u(t_0) = u_0 \). If each \( u_\kappa \) is \( P_\kappa H_C \)-valued analytic in \( N_\alpha \) such that

\[
(A.12) \quad \sup\{|A^\pm_\kappa u_\kappa(\zeta)|, \zeta \in N_\alpha\} \leq M_\alpha \nu\kappa_0^\alpha < \infty,
\]

then \( u(\zeta) \) is a \( \mathcal{D}(A^\pm_\kappa) \)-valued analytic function in \( N_\alpha \) and

\[
\sup\{|A^\pm_\kappa u(\zeta)|, \zeta \in N_\alpha\} \leq M_\alpha \nu\kappa_0^\alpha .
\]

**Proof.** Note that \( \lim_{\kappa \to \infty} |u_\kappa(t) - u(t)| = 0 \) for all \( t \geq t_0 \). Therefore, for any \( h \in H_C \), \( (u_\kappa(t), h) \to (u(t), h) \) for every \( t \geq t_0 \). Applying Vitali’s theorem and using (A.12) we obtain that \( (u_\kappa(\zeta), h) \) is converging to a \( \mathbb{C} \)-valued analytic function \( u_h(\zeta) \). It is easy to show that \( u(\zeta, h) \) is antilinear in \( h \) and that

\[
|u(\zeta, h)| \leq M_\alpha \nu|h|, \quad \forall \, h \in H_C.
\]

Therefore the Riesz–Fréchet theorem yields \( V(\zeta) \in H_C \) such that

\[
u(\zeta, h) = (V(\zeta), h), \quad \forall \, h \in H_C, \forall \, \zeta \in N_\alpha.
\]

This shows that \( V(\zeta) \) is weakly analytic in \( N_\alpha \) and therefore is also strongly analytic (i.e. \( V(\zeta) \) is a \( H_C \)-valued analytic function) (e.g. see page 377, 399 [1]; page 93 [12]). Moreover for \( h \in \mathcal{D}(A^\pm_\kappa) \) and \( \zeta \in N_\alpha \), we have that

\[
|(V(\zeta), A^\pm_\kappa h)| = \lim_{\kappa \to \infty} |(u_\kappa(\zeta), A^\pm_\kappa h)| = \lim_{\kappa \to \infty} |(A^\pm_\kappa u_\kappa(\zeta), h)| \leq M_\alpha \nu\kappa_0^\alpha|h| .
\]

Thus \( V(\zeta) \in \mathcal{D}(A^\pm_\kappa) \) and \( |A^\pm_\kappa V(\zeta)| \leq M_\alpha \nu\kappa_0^\alpha \), since \( A^\pm_\kappa \) is self-adjoint. Now

\[
(A^\pm_\kappa V(\zeta), h) = (V(\zeta), A^\pm_\kappa h),
\]

for every \( h \in \mathcal{D}(A^\pm_\kappa) \). Since \( \mathcal{D}(A^\pm_\kappa) \) is dense in \( H_C \), for any \( h^0 \in H_C \) there exists a sequence \( \{h^m\} \subset \mathcal{D}(A^\pm_\kappa) \) such that \( h^m \to h^0 \), and hence

\[
(A.13) \quad |(A^\pm_\kappa V(\zeta), h^0) - (A^\pm_\kappa V(\zeta), h^m)| \leq M_\alpha \nu\kappa_0^\alpha|h^0 - h^m|
\]

which implies that \( (A^\pm_\kappa V(\zeta), h^m) \to (A^\pm_\kappa V(\zeta), h^0) \) uniformly in \( \zeta \in N_\alpha \) and therefore \( A^\pm_\kappa V(\zeta) \) is also weakly analytic. Hence \( A^\pm_\kappa V(\zeta) \) is an \( H_C \)-valued analytic function.
For any $h \in D(A^{1/2})$, by using Ladyzhenskaya’s inequality and the analyticity of $A^{1/2} V(\zeta)$, we have that as $\xi \to 0$,
\[
\frac{1}{\xi} ((B(V(\zeta + \xi), V(\zeta + \xi), h), (B(V(\zeta), V(\zeta), h))
\]
\[
= \frac{1}{\xi} [(B(V(\zeta + \xi) - V(\zeta), V(\zeta)), h) + (B(V(\zeta), V(\zeta + \xi) - V(\zeta)), h)
\]
\[
+ (B(V(\zeta + \xi) - V(\zeta), V(\zeta + \xi) - V(\zeta)), h)]
\]
\[
= (B\left(\frac{V(\zeta + \xi) - V(\zeta)}{\xi}, V(\zeta), h\right) + (B(V(\zeta), \frac{V(\zeta + \xi) - V(\zeta)}{\xi}, h)
\]
\[
+ \xi (B(V(\zeta + \xi) - V(\zeta), V(\zeta + \xi) - V(\zeta), h))
\]
\[
\to (B\left(\frac{dV(\zeta)}{d\zeta}, V(\zeta), h\right) + (B(V(\zeta), \frac{dV(\zeta)}{d\zeta}, h),)
\]
and hence $(B(V(\zeta), V(\zeta)), h)$ is analytic.

But for $\zeta = t \in (t_0, t_\alpha)$, $V(t) = u(t)$, so the following equation
\[
\left(\frac{dV(\zeta)}{d\zeta}, h\right) + \nu(A^{1/2} V(\zeta), A^{1/2} h) + (B(V(\zeta), V(\zeta), h) = (g, h),
\]
holds for $t \in (t_0, t_\alpha)$ and hence it holds for $\zeta \in N_\alpha$ by analyticity. Now it follows that $V(\zeta)$ satisfies the complexified NSE (3.18) in $N_\alpha$. We obtain, in particular, $V(\zeta) = u(\zeta)$ for all $\zeta \in N_\alpha$.

\[\text{Remark A.4. It should be clear that, for the solution } u(\zeta) \text{ of the NSE (3.18) considered in Remark 4.3 the differential inequalities involving } \frac{d}{dt}|A^{1/2} u(t_0 + \rho e^{i\theta})|^2 \text{ used to establish estimates independent of } t_0, \rho \text{ and } \theta \text{ for } |A^{1/2} u(\zeta)|, \text{ are “strikingly similar” to those involving } \frac{d}{dt}|A^{1/2} u_\kappa(t_0 + \rho e^{i\theta})|^2, \text{ where } u_\kappa(\zeta) \text{ is the solution of (A.5) satisfying (A.2). These estimates and the classical form of Cauchy’s existence theorem imply the existence of } u_\kappa(\zeta) \text{ as well as the estimates given in the Proposition 8.1 and 8.2. Thus, due to the considerations already made in this Appendix, the latter estimates imply both the existence of } u(\zeta) \text{ and the estimates of } |A^{1/2} u(\zeta)| \text{ obtained by using the short procedure given in Remark 4.3. Because of this fact, the latter estimates are usually referred as “a priori estimates” (e.g. see [21], [3]).}

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