On the effective theory of vortices in two-dimensional spinless chiral p-wave superfluids

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We propose a $U(1) \times Z_2$ effective gauge theory for vortices in a $p_x + ip_y$ superfluid in two dimensions. The combined gauge transformation binds $U(1)$ and $Z_2$ defects so that the total transformation remains single-valued and manifestly preserves the particle-hole symmetry of the action. The $Z_2$ gauge field introduces a complete Chern-Simons term in addition to a partial one associated with the $U(1)$ gauge field. The theory reproduces the known physics of vortex dynamics such as a Magnus force proportional to the superfluid density. It also predicts a universal Abelian phase, exp(i$\pi/8$), upon the exchange of two vortices, modified by non-universal corrections due to the partial Chern-Simons term that are screened in a charged superfluid.

The two-dimensional spinless chiral p-wave superfluid is the minimal model for describing properties of many realizations of topological superfluids and superconductors, including topological insulator-superconductor interfaces [1,3], the layered material Sr$_2$RuO$_4$ [4,6] and some cold atom systems [7,8], as well as certain spin models admitting anyon excitations [9]. Vortex defects of the phase of the pairing order parameter thus endow them with non-Abelian exchange statistics [10,13], thus making them potential candidates for topological quantum information processing [14,16]. In addition, they are expected to admit a quantized Abelian exchange phase that plays an important role in proposals for universal topological quantum computation with vortices [17]. It is therefore quite important to formulate a cogent theory that accounts for the dynamics of vortices.

In previous work on this system a low-energy effective action has been derived by the standard gradient expansion method [18-24], shedding light on the collective response of the superfluid to external electromagnetic fields. However, in this derivation vortices have been generally left out. It appears then that the Abelian exchange phase of vortices, while surmised from the conformal properties of its edge states or the properties of candidate bulk wavefunctions [10,11], has never been derived from a microscopic model [24,25]. Consequently, it remains unclear whether bulk vortices in a chiral p-wave superfluid or superconductor exhibit this exchange phase and, if so, to what degree it is universal or how it may affect their physics.

To answer these questions, in this paper we derive a $U(1) \times Z_2$ effective gauge theory that handles vortex defects properly. The $U(1)$ gauge field is governed by an action that is identical to the one previously derived by gradient expansion, including a partial Chern-Simons (CS) term. Interestingly, a $Z_2$ gauge field emerges in the effective theory governed by a new full Abelian CS term. We show that the coefficient of the partial CS term is not a universal quantity and depends on the details of dispersion and higher-energy behavior of the system. The full CS term of the $Z_2$ gauge field is, on the other hand, a truly topological term with a quantized coefficient. We calculate the exchange angle of two vortices due to each $Z_2$ field and show that the new CS term dictates a universal Abelian statistics phase of the vortices equal to $e^{i\pi/8}$. In contrast, for neutral superfluids, the partial CS term spoils the quantization of the exchange phase by adding a long distance non-universal correction. For charged superfluids, screening effects exponentially diminish this contribution over the effective penetration depth. This sets a low bound for the distance between vortices during exchange processes required for topological quantum computation.

We start with the action for a spinless chiral p-wave superconductor [26], $Z = \int \mathcal{D}(\bar{\eta}, \eta) e^{iS}$, where $\eta = (\phi, \bar{\phi})^T$ and $\bar{\eta} = (\phi, \bar{\phi})$ are the Nambu spinors with Grassmann variables $\phi(r)$ and $\bar{\phi}(r)$ in the coordinate space $r = (r, t)$. In the following we will interchangeably use $z \equiv t$ as the third coordinate. The action is $S = \frac{1}{2} \int d^3r \bar{\eta} \tilde{G}^{-1} \eta$, with $\tilde{G}^{-1} = i\partial_t - \mathcal{H}$ the inverse Green’s function matrix and the Bogoliubov-deGennes Hamiltonian density [27],

$$\mathcal{H} = \begin{pmatrix} \xi_p - A_t & e^{i\theta/2} - \xi_p A_t \\ -e^{-i\theta/2} \Delta(p) & -\Delta(p) e^{i\theta/2} \end{pmatrix} . \quad (1)$$

Here, $\xi_p$ is the dispersion of excitations above the ground state, $p = -i\nabla$ is the momentum operator, $\Delta(p)$ is the amplitude and $e^{i\theta(r,t)}$ is the phase of the superconducting order parameter (including vortices), and $A = (A_x, A_y)$ is the electromagnetic gauge field. (In a neutral superfluid, $A = 0$.) We assume $e = c = \hbar = 1$. In the continuum, $\xi_p = p^2/2m - \epsilon_F$ with $\epsilon_F$ the Fermi energy and $\Delta(p) = v(p_x + ip_y)$ with $v$ the slope of the pairing order parameter in momentum space.

In order to keep track of the winding number around each vortex we define $\theta(r,t) = \sum_{j=1}^{n} \theta_j(r,t)$, where $2\pi\ell_j < \theta_j = \arg(r-x_j) \leq 2\pi(\ell_j+1)$ is the phase around the vortex located at $x_j(t)$ and $\ell_j$ is its winding number. We take the branch cut of $\arg(r)$ to be the positive real axis and index the Riemann sheets with the branch...
number $\ell$ \cite{28}.

The partition function is invariant under a unitary transformation, $U$, of the inverse Green’s function with a Jacobian of unit modulus; that is, $U = e^{i\tau_0}e^{i\theta}e^{i\tau_0}$, where $\tau_x$, $\tau_y$, and $\tau_z$ are Pauli matrices in the Nambu space. We demand that $U$ respect the particle-hole symmetric structure of the spinor fields. This means that $U$ must transform $(\eta, \bar{\eta})$ in such a way that ensures one spinor remains the conjugate transpose of the other and each component of the spinor is the conjugate of the other. The requirement is equivalent to the condition $U^\dagger = \tau_x U \tau_x$. In the operator language, this is the condition to maintain the fermionic commutations relations under the Bogoliubov transformation. It can be shown that any such $U$ is composed of a finite product of the following building blocks: $\tau_x, \tau_y, e^{i\theta \tau_x}$ and $e^{i\theta \tau_y}$ where $\mu \in \mathbb{R}$ and $m \in \mathbb{Z}$. The actual number of distinct sequences can be reduced through use of the commutations relations between the generators and is ultimately finite.

To proceed further, it is convenient to gauge away the phase of the superconducting order parameter. This will add space-time gradients of $\theta(x, t)$ to the electromagnetic potential in the kinetic term. A naive transformation, $e^{i\tau_0}e^{i\theta/2}$, which involves only the phase of the order parameter, leads to multi-valuedness in the presence of vortices.

To avoid this problem Anderson \cite{29} suggested using the transformations $e^{-i(\tau_x+1)/2}e^{i\theta}$, resulting in the superfluid velocity appearing as an effective gauge field in either the electron or the hole component of the Hamiltonian. This gauge choice becomes possible when opposite spins are associated with the two components of the Nambu spinor.

Franz and Tešanović \cite{30, 31} developed the transformation $e^{i(\tau_x+1)\theta}e^{i\tau_y}e^{i\tau_z}$ for a periodic bipartite vortex lattice, where $A$ and $B$ are two sublattices. The vortices should be assigned to the subsets in such a way that the effective magnetic field vanishes on average. Physically, a vortex assigned to subset $A$ will be seen by electrons and be invisible to holes, while vortex assigned to subset $B$ will be seen holes and be invisible to electrons. Inevitably, in both transformation, particle-hole symmetric structure of the spinors cannot be maintained without additional constraints on the ensemble of allowed partitions of $\theta$.

Instead, we suggest the following transformation

$$U = e^{i\tau_0 e^{i\theta}(x,t)/2}e^{i\gamma(x,t)}$$

where $\theta$ is the phase function and $\gamma = \pi \sum_j \xi_j$ keeps the transformation properly single-valued by supplying the required sign each time the winding number in $\theta$ changes as it evolves in space and time. Our transformation is similar in spirit to the Franz-Tešanović transformation, especially as formulated in Ref. \cite{32} but it manifestly preserves the particle-hole symmetry of the action. Upon applying this gauge transformation two gauge fields appear in the action: $a_\mu = A_\mu - \partial_\mu / 2$ couples only to the kinetic energy terms, with opposite signs for particles and holes, and $b_\nu = \partial_\nu / 2$ couples minimally to momentum, both in the kinetic energy and in the pairing term. The $b$ gauge field is associated with the vortex branch cuts and its associated current is proportional to the vortex current. Thus,

$$G^{-1} = i\partial_t - b_0 + \tau_x a_1 - \tau_\mu h^{\mu}(p - b, a),$$

where the 3-vector $h(p, a) = (\Re \Delta(p), \Im \Delta(p), \xi_p - \tau_\mu a)$.

We can now integrate out the fermion fields to find the effective action, $S_{\text{eff}} = \frac{4}{3} \text{Tr} \log G$, where $\text{Tr}(\cdot)$ stands for $\int d\theta dt (\tau | \text{tr}(\cdot) | \tau, t)$ and $\tau$ is the trace over the Nambu space. A tedious but straightforward calculation yields \cite{28}, to second order in the gauge fields,

$$S_{\text{eff}} = \int d\theta dt \left[ \kappa_a \rho A_\mu - \rho_0 a_\mu - \rho_0 a_\mu a_\mu + \frac{\kappa_b}{8\pi} \xi^{\mu\nu}\partial_\mu b_\nu b_\nu \right],$$

where $\xi^{\mu\nu}$ is the antisymmetric tensor and latin indices $i, j$ run over the spatial components. The coefficients appearing in Eq. (4) are found in terms of $g(k) \equiv h(k, 0)$ as follows: $n = \frac{1}{8\pi} \int d\mu (1 - \frac{\nu}{|g|})$ is the superfluid density; $\rho_0 = \frac{1}{16\pi} \int d\mu (g^2 + g^2_0)$; $\rho_1 = \frac{1}{16\pi} \int d\mu (1 - \frac{\nu}{|g|}) \partial_\mu b_\mu b_\mu$. The coefficient of the partial CS term for $a$,

$$\kappa_a = \frac{1}{4\pi} \int \frac{\xi^{\nu\lambda\mu\nu\lambda}}{|g|^3} dk,$$

is non-universal and depends on the details of the system. The coefficient of the full CS term for $b$, on the other hand,

$$\kappa_b = \frac{1}{4\pi} \int \frac{\xi^{\nu\lambda\nu\lambda}}{|g|^3} dk,$$

is the Pontryagin charge of the field $g^\mu(k)$ and is therefore always an integer.

In the continuum, $\xi_k = \kappa^2/2m - \epsilon_F$ and $\Delta(k) = \nu(k_x + ik_y)$, we find the values $n \sim (mv)^2 \log \left( \frac{\Lambda}{mv^2} \right)$ with $\Lambda$ an energy cut-off, $\rho_1 = m\kappa_\nu/4\pi$, and $\rho_\nu = (n/2m)\delta_{ij}$ reflects the Galilean invariance in the continuum \cite{34}.

The coefficients $\kappa_a^\infty = \left[ 1 - 2\frac{\epsilon_F}{mv} \Theta(-\epsilon_F) \right]^{-1}$ and $\kappa_b^\infty = \Theta(\epsilon_F)$, where $\Theta$ is the step function. Note that this extends the results obtained in Refs. \cite{20, 21} to the strong pairing regime, $\epsilon_F < 0$. For comparison, we have also calculated these coefficient for a system on the square lattice. In this case, $\xi_k = \frac{1}{m\pi^2} (2 - \cos k_x d - \cos k_y d - \epsilon_F$ and $\Delta(k) = \frac{1}{2} (\sin k_x d + i \sin k_y d)$, where $d$ is the lattice spacing. The coefficients $\kappa_{a,b}^{\infty}$ are plotted in Fig. 3 as a function of $md^2\epsilon_F/4$. We observe that $\kappa_b^{\infty}$ acquires the values $\pm 1$ in the topological regime $0 < \epsilon_F < 4/(md^2)$ \cite{35} and zero otherwise. In contrast, $\kappa_a^{\infty}$ and $\kappa_a^{\infty}$ are clearly non-universal and vary with $\epsilon_F$, showing derivative discontinuities when crossing into the topological regime. The
sign change of $\kappa^a_{ \mathrm{sq}}$ on the lattice signals a sign reversal in the Hall response of the superconductor. The action in Eq. (1) is our central result. It now exemplifies by the physical significance of each term appearing in the action. The first term gives rise to the Magnus force on a moving vortex. To see this, note that for a moving vortex $\partial_t \theta = -\dot{x} \cdot \nabla \theta$, where $x(t)$ is the position of the vortex. So, the first term yields $\int dt \, \dot{x} \cdot A_M$ with $A_M = -\int dr \, n \nabla \theta / 2$. Therefore, the vortex is subject to a Lorenz-like force $\dot{x} \times B_M$ where the Magnus flux $B_M = \nabla x \times A_M = \pi n \hat{z}$ is proportional to the superfluid density. The contribution from the electromagnetic gauge field $A_t$ in a superconductor vanishes due to the overall charge neutrality of the system. The second and third terms, in conjunction with the Maxwell Lagrangian, give rise to the usual screening of vortices through the Meissner effect. The second term also contributes to the mass of the vortex by generating a term $\int dt \frac{1}{2} m_v x^2$ in the action, where $m_v = \int dr \rho_s (\nabla \theta / 2 - A)^2$.

The fourth and fifth terms, as we now show, carry significant information about the dynamics of vortices. Previous work on the effective low-energy theory of the p-wave superconductor, using only the U(1) part of our transformation, yielded an action similar to that of an s-wave superconductor but with an additional partial CS term. Stone and Roy attributed this partial CS term to the existence of a Hall-like response to external fields. They recognized that the Hall current depends on the external field primarily through its effect in modifying the density. Note that the partial CS term we derive here is different from the one appearing in the literature, since in our case $\nabla \times \nabla \theta$ is explicitly nonzero due to the presence of vorticity in $\theta$. Moreover, the full CS term derived here is entirely absent in previous work. As we show now, both of these terms have significant contributions to the exchange statistics of vortices.

Abelian exchange statistics characterizes the manner in which the wavefunction transforms under the interchange of indistinguishable excitations. Specifically, the statistics is reflected in the wavefunction as a boundary condition, $|\psi(x_1, x_2)\rangle = e^{i\epsilon x} |\psi(x_2, x_1)\rangle$, with $x_1$ and $x_2$ the positions of the excitations and $e^{i\epsilon x}$ the phase accumulated during their exchange. The exchange angle $\chi$ can be obtained by evaluating the Berry’s phase accumulated upon the exchange of two particles. A full CS term in the Lagrangian produces a quantized Berry’s phase that depends solely on the topology of the ground state and has long been used to describe fractional exchange statistics of excitations.

At first sight, the $Z_2$ nature of the $b$ gauge field in our effective theory seems to make the calculation of the exchange phases due to the full CS term tricky. However, this is similar to the situation encountered in the singular string gauge, in which the gauge field is zero everywhere except on a string emanating from the vortex. One may show that the string gauge is continuously connected to a smooth gauge without changing the winding numbers along the process. Therefore, we can calculate the Berry’s phase contribution of the $b$ gauge field in the usual way by writing $b = b_1 + b_2$, where $b_1$ and $b_2$ are associated with the two vortices, and considering the cross terms between them. Both cross terms contribute equally since, by partial integration, $\int e^{i\lambda x} b_{1\lambda} \partial_\mu b_{2\nu} = \int e^{i\lambda x} b_{2\lambda} \partial_\mu b_{1\nu}$. Assuming for simplicity that only vortex 2 is moving, we have $e^{i\lambda x} \partial_\mu b_{1\nu} = \pi \delta(r) \delta^\lambda_x$, and

$$\chi_b = \frac{\kappa_b}{4} \int d\mathbf{r} d\mathbf{t} \, \delta(r) b_{2t} = \pi \kappa_b$$

which is quantized to $\pi/8$ in the weak pairing (topological) regime, as advertised.

The partial CS term in Eq. (4) also contributes to the Berry’s phase, albeit not in a quantized fashion due to the non-universal behavior of $\kappa_a$. We write again $a = a_1 + a_2$ for two vortices and consider the cross terms in the CS term between $a_1$ and $a_2$. In a superfluid the external electromagnetic gauge field is absent and we have $a_1 = -\frac{1}{2} \partial_\mu \arg(r - x_1(t))$, where $x_1(t)$ is the position of vortex 1, and similarly for $a_2$. The calculation is simplified by assuming that only vortex 2 moves, so that $a_1 = 0$. Then, only one of the cross terms contributes and

$$\chi^a_{sf} = \frac{\kappa_a}{8\pi} \int d\mathbf{r} d\mathbf{t} a_{2t} (\nabla \times A)_1 = \frac{\pi \kappa_a}{16}.$$
a vortex at the origin where $\lambda$ is the (effective) penetration depth. This will modify the result by a geometric phase,

$$\frac{\kappa_a}{8\pi} \int dt d\mathbf{r} a_2(\nabla \times \mathbf{A}_1)_z = -\frac{\pi \kappa_a}{16} \left[ 1 - (1 + R/\lambda)e^{-R/\lambda} \right],$$

for a circular exchange at distance $R$. So, in a superconductor the total exchange angle due to the partial CS term is

$$\chi_a = \frac{\pi \kappa_a}{16} (1 + R/\lambda)e^{-R/\lambda}. \quad (9)$$

When the distance between the vortices is much longer than $\lambda$ this exchange angle vanishes exactly. However, at distances smaller or comparable to $\lambda$, non-universal contributions to the exchange phase will occur.

Therefore, the total exchange angle $\chi = \chi_a + \chi_b$ depends on the details of the dispersion and, in particular, is different in a chiral p-wave superfluid from that in a superconductor due to screening effects.

In conclusion, we have derived an effective action of vortices in a spinless chiral p-wave superfluid by properly treating the vortex branch cuts and revealed an Abelian $Z_2$ gauge structure for the chiral p-wave superfluid. In principle, our transformation is applicable to any pairing symmetry and arbitrary distribution of vortices. In the s-wave case we have checked that this does not produce additional terms in the action. In the d-wave case a similar approach has been used to formulate an effective theory of cuprate superconductors [39][40], but no CS term was found.

In this work, we restricted our attention to the Abelian gauge transformations [4]. This is enough to infer the Abelian exchange phase of vortices. It can also be used to deduce the existence of zero energy Majorana modes: the number density of zero modes is given, mod 2, by

$$2 \langle \eta(r)\eta(r) \rangle = 4 \langle \delta S_{\text{eff}}/\delta b_0 \rangle = \kappa_b \sum \delta(\mathbf{r} - \mathbf{x}_j(t)),$$

which is quantized and equal to the vortex density in the weak pairing regime. A natural question for future work is whether the other parts of the full group of gauge transformations harbor additional physics. Indeed, as is well known, the zero energy Majorana modes endow the vortices with the non-Abelian statistics of Ising anyons [11][12]. It would be interesting to see if such a non-Abelian representation emerges in the gauge structure of the effective vortex action by using the entire group of gauge transformations.

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SUPPLEMENTAL MATERIAL

In this Supplemental Material we provide details regarding the derivation of the effective action appearing in the main text, Eq. 4. We write the Green’s function as $G_0(k) = (k^2 - g(k) \cdot \tau)^{-1}$ and perform a perturbative expansion to second order in the gauge fields, writing $iS_{\text{eff}} = \log \text{Pf}(G^{-1}) = \frac{1}{2} \log \text{Det}(G^{-1}) = \frac{1}{2} \text{Tr} \log(G^{-1})$,

$$S_{\text{eff}} = -\frac{i}{2} \text{Tr} \log (G^{-1}_0) - \frac{i}{2} \text{Tr} \log(1 + G_0V)$$

$$\simeq -\frac{i}{2} \text{Tr} \log (G^{-1}_0) - \frac{i}{2} \text{Tr}(G_0V) + \frac{i}{4} \text{Tr}(G_0VG_0V),$$

(S1)

where $V$ is the part of $G^{-1}$ that depends explicitly on $a$ and $b$, i.e. $V = -b + \tau_z a_t - \tau_x h^x (p - b, a) + \tau_y h^y (p, 0)$. The fields $a$ and $b$ couple via their associated currents

$$f^a = \delta^a \tau_z + \partial_k g_x (1 - \delta^a),$$

$$f^b = \partial_k g_y G^{-1}.$$  

(S2)

To calculate traces, we use the following formulas

$$\text{tr}\{\tau_{\mu} \tau_{\nu}\} = 2\delta_{\mu\nu},$$

$$\text{tr}\{\tau_{\lambda} \tau_{\mu} \tau_{\nu}\} = 2i\epsilon_{\lambda\mu\nu},$$

$$\text{tr}\{\tau_{\lambda} \tau_{\mu} \tau_{\nu} \tau_{\sigma}\} = 2(\delta_{\lambda\mu} \delta_{\nu\sigma} - \delta_{\lambda\nu} \delta_{\mu\sigma} + \delta_{\lambda\sigma} \delta_{\mu\nu}).$$

The non-vanishing terms

We proceed to derive the coefficients of the five terms appearing in the action, Eq. 4.

The coefficient $n$. The coefficient multiplying $a_0$ is $n = -\frac{i}{2(2\pi^2)} \int d^3k \text{tr}(G_0 \partial_z)$. Since it contains an integration over a single Green’s function, care should be taken in its calculation. The correct analytical structure requires that the Green’s function is multiplied by an exponent $e^{i\tau_{\mu} k_{\mu}}$, where $\eta \rightarrow 0$, leading to the expression

$$n = -\frac{i}{2(2\pi^2)} \sum_{a=1}^{\infty} \int d^3k \frac{sk_t + g_z}{k_t^2 - |g|^2 - i\eta} e^{is\eta k_t}.\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad (S4)$$

Using contour integration over $k_t$ (see Fig. S1) where $\lambda \equiv \sqrt{|g|^2 - i\eta}$ one obtains the expression

$$n = \frac{1}{8\pi^2} \int dk \left(1 - \frac{g_z}{|g|}\right). \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad (S5)$$

The coefficient $\rho_t$. Writing the appropriate second order correlator,

$$\rho_t = \frac{i}{32\pi^3} \int d^3k \text{tr} (G_0 j^t \partial_t G_0 j^t)$$

$$= \frac{i}{16\pi^3} \int d^3k \frac{k_t^2 - g_x^2 - g_y^2 + \beta^2}{(k_t^2 - |g|^2 + i\eta)^2}$$

$$= \frac{1}{16\pi^2} \int dk \frac{g_x^2 + g_y^2}{|g|^3},$$

(S7)

where we used the integral

$$\int_{-\infty}^{\infty} dk_t \frac{\alpha k_t^2 + \beta}{(k_t^2 - |g|^2 + i\eta)^2} = \frac{i\pi(-\alpha |g|^2 + \beta)}{2|g|^3}, \quad\quad\quad (S8)$$

with $\alpha = 1$ and $\beta = g_x^2 - g_y^2$. For p-wave superfluids in the infinite system limit,

$$\rho_t = \frac{1}{16\pi^2} \int dk \frac{v^2 k^2}{(\xi^2 + v^2 k^2)^{3/2}} = \frac{m\kappa^\infty_a}{4\pi}, \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad (S9)$$

where $\kappa^\infty_a = \left(1 - \frac{\epsilon_{x-\epsilon_y}}{m^2\pi^2}\right)^{-1}$ coincides with the coefficient of the partial CS term, to be derived below.
The coefficient $\rho_{ij}$. Formally, this coefficient has contributions both from first order and second order in the gradient expansion. The first order contribution is

$$-\frac{i}{2(2\pi)^3} \int d^3k \text{tr} \left[ G_0 \left( -\frac{\tau_3}{2m} \delta_{ij} \right) \right] = -n \frac{\kappa}{2m} \delta_{ij}, \quad (S10)$$

For $g_z = \xi$, we can write $\delta_{ij}/m = \partial_k \partial_k g_z$, to obtain the form in the main text. The second order contribution exactly vanishes following the integration over $k_t$,

$$\frac{i}{32\pi^2} \int d^3k \text{tr} \left( G_0 j^i_\alpha G_0 j^i_\alpha \right) = \frac{i}{16\pi^2} \int d^3k \frac{\partial g_z}{\partial k_i} \frac{\partial g_z}{\partial k_j} \frac{k_t^2}{k_t^2 + |g|^2} = 0. \quad (S11)$$

The coefficient $\kappa_\alpha$. To calculate $\kappa_\alpha$ we consider the correlator of $j^i_\alpha$ and $j^i_\alpha$ to first order in $q_i$ (no summation convention)

$$\frac{i q_i}{64\pi^2} \int d^3k \text{tr} \left[ \partial_k G_0 \tau_3 G_0 \partial_k g_z \right] = \frac{i q_i}{64\pi^2} \int d^3k \text{tr} \left\{ \left[ \partial_k G_0, \partial_k G_0 \right] \frac{\partial g_m}{\partial k_i} \frac{\partial g_m}{\partial k_j} \right\}$$

$$= -\frac{q_i}{16\pi^2} \sum_{\ell m} \int d^3k \frac{1}{(k_t^2 - |g|^2 + i\eta)^2} \epsilon_{\ell m g} \frac{\partial g_m}{\partial k_i} \frac{\partial g_m}{\partial k_j}$$

$$= \frac{-i q_i}{32\pi^2} \sum_{\ell m} \int d^3k \frac{1}{|g|^3} \epsilon_{\ell m g} \frac{\partial g_m}{\partial k_i} \frac{\partial g_m}{\partial k_j}. \quad (S12)$$

For the infinite system p-wave superfluid this results in (no summation convention)

$$\frac{i q_i \epsilon_{ij}}{32m\pi^2} \int d^3k \frac{v^2 k_t^2}{(k_t^2 + v^2 k^2)^{3/2}} = \frac{i q_i \epsilon_{ij}}{16\pi} \kappa_\alpha. \quad (S13)$$

The coefficient $\kappa_b$. For convenience we consider one of the correlators giving rise to the CS coefficient

$$\frac{i q_i}{64\pi^2} \int d^3k \text{tr} \left[ \partial_k G_0 \partial_k g \cdot \tau G_0 \partial_k g \cdot \tau \right] = \frac{i q_i}{32\pi^2} \int d^3k \frac{\epsilon_{\mu \nu \lambda} g^\mu \partial_k g^\nu \partial_k g^\lambda}{|g|^3}. \quad (S14)$$

For the infinite system p-wave superfluid, we get

$$\kappa_b^\infty = \frac{1}{4\pi} \int d^3k \frac{\left( k_t^2 + \epsilon_F \right) v^2}{\left( v^2 k^2 + \left( k_t^2 - \epsilon_F \right)^2 \right)^{3/2}} = \Theta(\epsilon_F). \quad (S15)$$

The vanishing terms

We provide an argument for the decoupling of the $a$ and $b$ fields, as well as for the vanishing of all mass terms for the field $b$.

The decoupling of the fields $a$ and $b$. It can be shown that the integrand of the correlator describing the coupling between $a$ and $b$

$$\int d^3k \text{tr} \left[ G_0(k + \frac{q}{2}) j^i_\alpha(k) G_0(k - \frac{q}{2}) j^i_\alpha(k) \right], \quad (S16)$$

is always odd under $k \rightarrow -k$. Therefore, it vanishes to all orders in $q$ following an integration over $k$.

The absence of mass terms for the field $b$. In first order in the gradient expansion we find the following contribution to the mass of $b$

$$-\frac{i}{2(2\pi)^3} \int d^3k \text{tr} \left[ G_0 \left( -\frac{\tau_3}{2m} \right) \right] = -\frac{n}{2m}. \quad (S17)$$

Another contribution appears in second order (no summation convention),

$$\frac{i}{32\pi^2} \int d^3k \text{tr} \left( G_0 j^i_\alpha G_0 j^i_\alpha \right) = \frac{1}{16\pi^2} \int d^3k \frac{|g|^2 (\partial_k g \cdot \partial_k g) - (\partial_k g)^2}{|g|^3}. \quad (S18)$$

where in the infinite system p-wave superfluid we get, after integration over the angle of $k$,

$$\frac{1}{16\pi} \int_0^{\Lambda_b} d|k| |k|^2 \left( \frac{k^4}{2m^2} + \epsilon_F^2 + v^2 k^2 \right). \quad (S19)$$

While each is formally divergent, the sum of the two contributions, Eq. (S17) and Eq. (S19), now converges to zero,

$$\lim_{\Lambda_b \rightarrow \infty} \left[ \frac{n}{2m} - \frac{1}{16\pi} \int d|k| |k|^2 \left( \frac{k^4}{2m^2} + \epsilon_F^2 + v^2 k^2 \right) \right] = \lim_{\Lambda_b \rightarrow \infty} \left( \frac{n}{2m} - \frac{\partial_m n}{4} - \frac{|\epsilon_F| + mv^2}{8\pi} \right) \kappa_b^\infty = 0. \quad (S20)$$