A Unified Model for Two Localisation Problems: Electron States in Spin-Degenerate Landau Levels, and in a Random Magnetic Field

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Abstract

A single model is presented which represents both of the two apparently unrelated localisation problems of the title. The phase diagram of this model is examined using scaling ideas and numerical simulations. It is argued that the localisation length in a spin-degenerate Landau level diverges at two distinct energies, with the same critical behaviour as in a spin-split Landau level, and that all states of a charged particle moving in two dimensions, in a random magnetic field with zero average, are localised.
Two apparently disparate localisation problems of great current interest arise in the context of the quantum Hall effect. The first concerns the delocalisation transition in Landau levels that are spin-degenerate, while the second involves localisation in the presence of a random magnetic field. In this paper we show that both physical situations can be represented by a single model, and that conclusions about one carry implications for the other. We present the results of extensive numerical simulations, which elucidate the behaviour of our unifying model.

The mobility edge in spin-split Landau levels represents the best characterised example to date of critical behaviour at a metal-insulator transition. Interpretation of the experiments starts from a picture for the dependence on energy, $E$, of eigenstate properties, which is derived from scaling ideas \[1\] and supported by numerical simulations \[2–5\]. In this picture, almost all eigenstates within a disorder-broadened Landau level are Anderson localised, but the localisation length, $\xi(E)$, diverges at a critical energy, $E_c$, which, in the simplest case, lies at the band centre: $\xi(E) \sim |E - E_c|^{-\nu}$. Scaling has been observed as this zero-temperature critical point is approached, as a function of temperature \[6\], of sample size \[7\] and of frequency \[8\]. Early measurements of the ratio $\kappa = z/\nu$, where $z$ and $\nu$ are the dynamic scaling exponent and the localisation length exponent, respectively, obtained the value \[6\], $\kappa \simeq 0.42$, while subsequent experiments on mesoscopic Hall bars have allowed $z$ and $\nu$ to be determined independently, yielding the value \[7\] $\nu = 2.3 \pm 0.1$, in excellent accord with the results of numerical simulations using a variety of models and techniques \[3–5\].

By contrast, in sufficiently disordered samples at low magnetic field, it is possible for the disorder-broadening of Landau levels to exceed the Zeeman energy, so that each orbital level becomes spin-degenerate. Under these conditions different critical behaviour is reported \[6–8\]. Assuming, as for spin-split Landau levels, that the localisation length diverges only at one energy, a value of the ratio $\kappa$ smaller by a factor of 2, $\kappa \simeq 0.21$, is obtained from scaling both with temperature \[6\] and frequency \[8\]. Measurements on mesoscopic samples with spin-degenerate Landau levels are in qualitative accord, giving a larger value \[7\] for $\nu$, $\nu \simeq 6.5$.

We present below results of numerical simulations that are consistent with these observations. Supposing, as in the experimental analysis, that the localisation length diverges only at the centre of a spin-degenerate Landau level, we obtain, for localisation lengths in our model similar in magnitude to those probed experimentally, a much faster divergence than in spin-split Landau levels, with $\nu \simeq 5.8$ as the best fit under these assumptions. We argue, however, that such an analysis is, in fact, incorrect. We believe, instead, that there are two energies at which the localisation length diverges in a spin-degenerate Landau level, displaced symmetrically by small amounts to either side of the Landau level centre, and that the localisation length between these mobility edges is large but not infinite. Indeed, re-analysing out numerical data under this new assumption, we obtain a substantially better fit to a power-law divergence of the localisation length, with the same exponent value, $\nu \simeq 2.3$, as found in spin-split Landau levels.

Interest in localisation in two dimensions, in the presence of a random magnetic field with an average value of zero, arises both from the Chern-Simons theory of electrons in a half-filled Landau level \[9,10\] and from gauge theories of doped Mott insulators \[11\]. Recent simulations of localisation in a random magnetic field have resulted in conflicting interpretations, some authors \[3,12\] arguing that there exists a range of energies for which
states are extended, while others [13] suggest that all states are localised.

The data we present below indicate that in our model for localisation in a random magnetic field, all states are localised. Moreover, we argue that should extended states exist in this problem, it would have implications for the quantum Hall effect that would require revision of the usually accepted scaling flow diagram [1].

Our calculations use an extension of the ‘network model’, introduced previously by one of us [3]. As a representation of a spin-split Landau level, the original network model is based on a semiclassical picture of electron motion in a smooth, two-dimensional random potential, under a strong, perpendicular magnetic field. Taking the magnetic length to be much shorter than the correlation length of the potential, electron motion can be separated into two components: a rapid cyclotron orbit and a slow drift of the guiding centre along equipotential lines. At this level, the spatial extent of eigenstates depends only on classical percolation properties of the equipotential lines. Quantum effects are introduced by allowing tunnelling between disjoint portions of the same equipotential, near saddle points of the potential. Electrons propagate coherently through the system, so that interference between different tunnelling paths is automatically included.

Formally, the model for a spin-split Landau level consists of a set of links, representing portions of an equipotential, which meet at nodes, representing saddle points. For simplicity, the links and nodes are arranged on a square lattice. Each link is characterised by a direction for flow of probability flux, which is the direction of the corresponding guiding centre drift, and by the phase shift that an electron acquires on traversing the link. Randomness is introduced by choosing these link phases from a uniform distribution. Each node is described by a scattering matrix which, after allowing for the constraints of unitarity, contains only one important quantity, the ‘node parameter’, \( \theta \). Current amplitudes on the four links meeting at a given node are related by (referring to Fig. 1)

\[
\begin{pmatrix}
\psi_{\text{in},R} \\
\psi_{\text{out},R}
\end{pmatrix}
= \begin{pmatrix}
\cosh \theta & \sinh \theta \\
\sinh \theta & \cosh \theta
\end{pmatrix}
\begin{pmatrix}
\psi_{\text{out},L} \\
\psi_{\text{in},L}
\end{pmatrix}
\]

(1)

where, for brevity, we have chosen a gauge in which all amplitudes \( \{ \psi \} \) have the same phase. Tunnelling is turned off in two limits: \( \theta = 0 \), when \( \psi_{\text{out},R} = \psi_{\text{in},L} \) and \( \psi_{\text{in},R} = \psi_{\text{out},L} \); and \( \theta \to \infty \), when \( \psi_{\text{in},R} = \psi_{\text{out},R} \) and \( \psi_{\text{out},L} = -\psi_{\text{in},L} \). More generally, it is useful to note the following duality transformation: if the amplitudes \( \psi_{\text{out},R} \) and \( \psi_{\text{out},L} \) are permuted in (1), \( \theta \) should be replaced by \( \theta' \), where \( \sinh \theta' = 1/\sinh \theta \). Thus tunnelling is maximal at \( \sinh \theta = \sinh \theta' = 1 \).

In the generalisation of this network model which we study below, two quantum-mechanical fluxes are carried by each link. Scattering between these two channels is included by replacing the link phases of the original model with \( U(2) \) matrices that transform the two incident current amplitudes on a link into outgoing ones. We choose these matrices randomly and independently on each link, with the Haar measure. At the nodes, we suppose (without loss of generality [14]) that tunnelling conserves the channel index, with parameters \( \theta_1, \theta_2 \) for each channel respectively. For simplicity, we do not consider randomness in \( \theta_1 \) or \( \theta_2 \).

As a representation of a spin-degenerate Landau level, the two channels correspond to the two possible spin orientations and the non-zero \( U(2) \) mixing arises from spin-orbit scattering. In the absence of Zeeman splitting, \( \theta_1 = \theta_2 \equiv \theta \), and on sweeping the Fermi energy through the Landau level, the system follows a line in parameter space from \( \theta = 0 \) to \( \theta = \infty \).
With Zeeman splitting, this line is displaced: schematically, one can take \( \theta_1 = (1 + g)\theta \) and \( \theta_2 = (1 - g)\theta \), where \( g \) represents the electron \( g \)-factor, with \( |g| < 1 \).

The same model also represents electron motion in a random magnetic field that has a correlation length large compared to the typical cyclotron radius. In this semiclassical limit it is again useful to consider electron guiding centres, which under these conditions drift along contours of the magnetic field. Extended states, if any exist, must be associated with the percolation of the zero field contour. Drift along the zero field contour has been discussed in detail in Refs [15,16]. A mapping to our network model is most easily established by considering a special case, in which the field switches abruptly between two values, \( \pm B_0 \), as the contour is crossed, and by restricting attention to Fermi energies lying between the energies of the lowest two Landau levels in a uniform field of strength \( B_0 \). Generalisations will be treated elsewhere [14]. In these circumstances [10], two modes propagate along the contour in the same direction, arising ultimately from symmetric and antisymmetric linear combinations of the lowest Landau levels on either side. In our model, a portion of this contour corresponds to a link, which hence must support two channels. Where two different portions of contour approach within a magnetic length of each other, tunnelling can occur between them, as described by the nodes of the model. Moreover, meandering of the contour will result in scattering between the modes, represented by the \( U(2) \) matrices. We identify the line in the parameter space \( (\theta_1, \theta_2) \) which has the symmetry of the random field problem from the condition that the average Hall conductance vanish: in order that the average circulation around plaquettes of the network model be zero, we require \( \sinh \theta_1 = 1/\sinh \theta_2 \) [17].

We are now in a position to discuss the phase diagram for both problems in terms of our unifying model. In the absence of scattering between channels, the model would consist of two uncoupled networks, each as studied in Ref [3]. States are localised, except (in the respective networks) on the lines \( \sinh \theta_1 = 1 \) and \( \sinh \theta_2 = 1 \). In this paper, we are concerned with the result of coupling the two networks. First, consider, as in Ref [18], traversing the spin-degenerate Landau level on the line: \( \theta_1 = \theta_2 \equiv \theta \). Along this line, the Hall conductance, \( \sigma_{xy} \), measured at short distances in units of \( e^2/h \), must vary smoothly between \( \sigma_{xy} = 0 \) at \( \theta = 0 \) and \( \sigma_{xy} = 2 \) as \( \theta \to \infty \). In particular, \( \sigma_{xy} = 1 \) at \( \sinh \theta_1 = \sinh \theta_2 = 1 \), which is the centre of the spin-degenerate Landau level, and the Khmelnitskii flow diagram [1] suggests that scaling takes the system to a localisation fixed point. Additionally, one expects [18] to find two isolated points either side of the Landau level centre, with \( \sigma_{xy} = 1/2 \) and \( 3/2 \) respectively, from which the system scales towards a delocalisation fixed point. Since any trajectory from \( \theta_1 = \theta_2 = 0 \) to \( \theta_1 = \theta_2 = \infty \) must, on this analysis, share these features, one is led to the phase diagram of Fig. 2(a). In this phase diagram, there are two distinct mobility edges in the spin-degenerate Landau level and all states on the random field line are localised. An alternative scenario requires that two delocalisation fixed points in the scaling flow diagram coalesce if the Zeeman energy is small enough, and yields the phase diagram of Fig. 2(b). In this event, states are delocalised only at the centre of the spin-degenerate Landau level (with, potentially, critical properties in a new universality class) and there exists an entire region of extended states on the random field line. Clearly, one can also imagine more exotic possibilities, which we do not discuss.

To identify which phase diagram actually pertains, we have carried out numerical simulations, employing standard techniques (see [3] and references therein). We measure the
localisation length $\xi_M$ in networks which are up to $M = 128$ links wide and $1.2 \times 10^6$ links long, using periodic boundary conditions to avoid extended edge states. Localisation lengths in the thermodynamic limit, $\xi_\infty$, are extracted via a conventional one-parameter scaling analysis.

We discuss first our results on the random field line, $\sinh \theta_1 = 1/\sinh \theta_2$. The localisation length remains finite everywhere on this line (Fig. 3), including, notably, the point, $\sinh \theta_1 = \sinh \theta_2 = 1$, that coincides with the spin-degenerate Landau level centre. Such behaviour is expected from the phase diagram of Fig. 2(a), but is incompatible with that of Fig. 2(b).

Consider next a Landau level without Zeeman splitting: $\theta_1 = \theta_2 \equiv \theta$. On this line, over a narrow range on either side of the level centre, the bulk localisation length is so large that we are unable to determine it reliably. The divergence in the localisation length, as the level centre ($\sinh \theta = 1$) is approached from the low energy tail ($\theta \ll 1$) is examined in Fig. 4, comparing the two alternative hypotheses represented by Fig. 2. According to the former, one must simultaneously determine the position, $\theta_c$, of the lower mobility edge and the value, $\nu$, of the critical exponent. Since large uncertainties are associated with such a two-parameter fit, we simply demonstrate that there exists a choice for $\theta_c$ for which the data are consistent with the exponent value, $\nu = 2.3$, obtained in simulations of a spin-split Landau level. Supposing, alternatively (as in the experimental analysis), that there is a mobility edge at the level centre, our data fit less well to a power law and require a much larger exponent, $\nu \approx 5.8$, in at least qualitative accord with experiment. We suggest that it would be of considerable interest to analyse experiments on spin-degenerate Landau levels with the assumption that there are two distinct mobility edges, as in Fig. 2(a).

Finally, we attempt a precise determination of the critical exponent, $\nu$, in our model. To do so, we study the line $\sinh \theta_2 = 2$, which is equivalent to introducing Zeeman splitting. In addition, in order to reduce the localisation length to measurable values in the range $\sinh \theta_1 \approx 1$, it is necessary to decrease the coupling between channels on each link. We achieve this by abandoning an isotropic distribution for the $U(2)$ matrices, and restricting their off-diagonal elements to have modulus 0.3. The localisation length diverges as the region $\sinh \theta_1 \approx 1$ is approached from either side. We fit power laws to each divergence, assuming a mobility edge at $\theta_1 = \theta_c$, obtaining exponents $\nu_-(\theta_c)$ and $\nu_+ (\theta_c)$ on each side. Solving the equation $\nu = \nu_-(\theta_c) = \nu_+ (\theta_c)$, we obtain $\nu = 2.45$ and $\sinh \theta_c = 1.00$. This exponent value is remarkably close to the most precise determination for spin-split Landau levels, $\nu = 2.34 \pm 0.04$.

In summary, our numerical study has shown that, in two dimensions, all states are localised in a random magnetic field and that the spin-degenerate Landau levels have a pair of delocalisation transitions in the same universality class as the spin-split system. We have further demonstrated that these conclusions are mutually consistent when cast in a wider parameter space containing both quantum Hall systems of arbitrary Zeeman splitting and the random field problem, as summarised in the phase diagram of Fig. 2(a).

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[17] At the point $\sinh \theta_1 = \sinh \theta_2 = 1$, the line representing the random field problem crosses that representing a spin degenerate Landau level at the mid-point in energy of the Landau level. Here, we have the apparently paradoxical requirement that the average Hall conductivity is zero for the former system, but $e^2/h$ for the latter. The paradox is resolved by noting that different boundary conditions apply in the two cases. Edge states carry no Hall current in the first case, but contribute unit Hall conductance in the second case. See [14].

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FIGURES

FIG. 1. Node parameters. Diagrams indicate tendency for transmission and reflection for different node parameters $\theta$.

FIG. 2. Schematic phase diagrams: (a) supported by our results (b) alternative with unconventional critical line (bold line). (sd: spin-degenerate network, rf: random field network)

FIG. 3. Random field network. Bulk localisation length $\xi_\infty$ as a function of sinh $\theta$. Insets: one-parameter scaling function $\xi_M/M = f(\xi_\infty/M)$ and schematic diagram of network.

FIG. 4. Spin-degenerate Landau level. A better power-law fit to the form $\xi_\infty \sim |\sinh \theta - \sinh \theta_c|^{-\nu}$ is obtained for sinh $\theta_c = 1.35$. Inset: schematic diagram of network.
\[
\sinh \theta < 1 \quad \text{sinh } \theta > 1
\]

\[\text{FIG. 1}\]

\[\text{FIG. 2}\]
