Dirac’s method for the two-dimensional damped harmonic oscillator in the extended phase space

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The system of two-dimensional damped harmonic oscillator is revisited in the extended phase space. We show that the system is singular. The classical Hamiltonian is proportional to the first-class constraint. We pursue with the Dirac’s canonical quantization procedure by fixing the gauge and provide a reduced phase space description of the system.

I. INTRODUCTION

The Hamiltonians of most real physical systems are explicitly time dependent and do not provide directly a conserved quantity. Succeeding to isolate an invariant helps to know about a fundamental system property. For instance in the case of an autonomous Hamiltonian system the Hamiltonian itself represent an invariant. Many approaches have been developed to identify conserved quantities for explicitly time-dependent systems. The first one being developed by Noether in the context of the Lagrangian formalism [1]. The invariant for the one-dimensional time-dependent harmonic oscillator has been derived by Lewis [2]. It has been demonstrated later that the Lewis procedure follows from Noether Theorem [3] and that has been extended by Chattopadhyay to derive invariants for certain one-dimensional non-linear systems [4]. An other approach of finding conserved quantities for explicitly time-dependent has been developed by Leach by performing a finite time dependent canonical transformation [5]. A third way of finding exact invariants for time-dependent classical Hamiltonians has been derived by Lewis and Leach by using direct Ansätze with different powers in the canonical momentum [6].

The invariants for time-dependent Hamiltonian systems are still investigated and of interest in the literature. We have been first interested in finding the class of invariants for the two-dimensional time-dependent Landau problem and harmonic oscillator in a magnetic field [3] where we considered an isotropic two-dimensional harmonic oscillator with arbitrarily time-dependent mass $M(t)$ and frequency $\Omega(t)$ in an arbitrarily time-dependent magnetic field $B(t)$. Two commuting invariant observables (in the sense of Lewis and Riesenfeld) $L, I$ have been derived in terms of some solutions of an auxiliary ordinary differential equation and an orthonormal basis of the Hilbert space consisting of joint eigenvectors $\varphi_\lambda$ of $L, I$.

Recently we studied a system of two non-interacting damped oscillators with equal time-dependent coefficients of friction and equal time-dependent frequencies [6]. The system is described by the Lagrangian function

$$L(x_1, x_2, \dot{x}_1, \dot{x}_2, t) = f^{-1}(t) \left(\frac{m}{2} (\dot{x}_1^2 + \dot{x}_2^2) - \frac{m\omega^2(t)}{2} (x_1^2 + x_2^2)\right), \quad (1)$$

where $f$ is an arbitrary function such that $f(t) = e^{-\int_0^t \eta(t') dt'}$, we assume that the function $f$ is twice differentiable and the canonical coordinates are $x_1, x_2$. The canonical momenta are respectively given by
\[ p_1 = \frac{\partial L}{\partial \dot{x}_1} = mf^{-1}(t)\dot{x}_1; \quad (2) \]
\[ p_2 = \frac{\partial L}{\partial \dot{x}_2} = mf^{-1}(t)\dot{x}_2. \quad (3) \]

In the canonical formalism the dynamics of the system is governed by the classical Hamiltonian

\[ H(x_1, x_2, p_1, p_2, t) = p_1 \dot{x}_1 + p_2 \dot{x}_2 - L(x_1, x_2, \dot{x}_1, \dot{x}_2, t) \quad (4) \]

that is equivalent to

\[ H(x_1, x_2, p_1, p_2, t) = \frac{f(t)}{2m}(p_1^2 + p_2^2) + f^{-1}(t) \frac{ma^2(t)}{2} (x_1^2 + x_2^2), \quad (5) \]

The dynamical state of the system is defined by the values of the canonical coordinates and momenta at any given time \( t \). The coordinates and momenta satisfy a set of Poisson Brackets relations

\[ \{x_1, p_1\}_{PB} = 1; \quad \{x_2, p_2\}_{PB} = 1; \quad \{x_1, p_2\}_{PB} = \{x_2, p_1\}_{PB} = 0, \quad (6) \]

and

\[ \{f, g\}_{PB} = \frac{\partial f}{\partial x_1} \frac{\partial g}{\partial p_1} + \frac{\partial f}{\partial x_2} \frac{\partial g}{\partial p_2} - \frac{\partial f}{\partial p_1} \frac{\partial g}{\partial x_1} - \frac{\partial f}{\partial p_2} \frac{\partial g}{\partial x_2} \quad (7) \]

where \( f \) and \( g \) are any functions of the \( x_i \)'s and \( p_i \)'s, \( i = 1, 2 \). At the quantum level the dynamic invariant method formulated by Lewis-Riesenfeld \[10\] has been used to construct an exact invariant operator. The exact solutions for the corresponding time-dependent Schrödinger equations are provided. The solutions have been used to derive the generators of the \( su(1,1) \) Lie algebra that enable to construct and study the properties of the coherent states à la Barut-Girardello and à la Perelomov.

In this paper we revisite the model in \[9\] in the extended phase space. The idea is not new in the literature. For instance we refer to the following works that we meet : the one by Struckmeier on Hamiltonian dynamics on the symplectic extended phase space for autonomous and non-autonomous systems \[11\], the work done by Baldiotti and al. on the quantization of the damped harmonic oscillator \[12\], the work by Menouar and al. on the quantization of the time-dependent singular potential systems; non-central potential in three dimension \[14\] and the recent paper by Garcia-Chung and al. on the Dirac’s method for time-dependent Hamiltonian systems in the extended phase space \[15\]. In the extended phase space that is to consider the time \( t \) as a dynamical variable with a corresponding conjugate momentum, the Lagrangian of the system is singular characterized then by the presence of constraints. We identify the constraint and apply the Dirac method of quantization. Our procedure is close to the discussions in the paper \[15\] with the difference that we propose to illuminate the Dirac’s method of quantization of the system with no intention of finding the invariants due to the results in \[9\]. An advantage of extending the phase space is that the
symplectic group of the system is also enlarged giving place to study the canonical transformation in the extended phase space such that the final dynamical description of the reduced phase space is no longer time-dependent. An invariant of the system can be obtained by applying in the extended phase space a finite canonical transformation to the initial Hamiltonian of the system.

The quantum Hamiltonian given in [9] is straightforward through canonical quantization as the Lagrangian (1) is regular. Let’s briefly recall here the procedure of canonical quantization of the system described by (1,5,6). The Hessian matrix \( M \) of the Lagrangian function is given by

\[
M = \begin{bmatrix}
\frac{\partial^2 L}{\partial \dot{x}_1^2} & \frac{\partial}{\partial x_1} \left( \frac{\partial L}{\partial \dot{x}_2} \right) \\
\frac{\partial}{\partial \dot{x}_2} \left( \frac{\partial L}{\partial x_1} \right) & \frac{\partial^2 L}{\partial \dot{x}_2^2}
\end{bmatrix} = \begin{bmatrix}
f^{-1}m & 0 \\
0 & f^{-1}(m)
\end{bmatrix}
\] (8)

and the determinant

\[
det \, M = \det \begin{vmatrix}
f^{-1}m & 0 \\
0 & f^{-1}(m)
\end{vmatrix} = m^2 f^{-2}(t) \neq 0 \tag{9}
\]

The Lagrangian (1) is called regular or standard since its Hessian matrix satisfies the equation (9). The system described by the Lagrangian (1) does not involve constraints and we assume that the phase space is flat and admits the procedure of canonical quantization which consists in demanding that to the classical canonical pairs \( (x_1, p_1), (x_2, p_2) \) that satisfy the Poisson brackets in equation (6) we associate the operators \( \hat{x}_1, \hat{x}_2, \hat{p}_1, \hat{p}_2 \) acting both on the Hilbert space of the states \( \mathcal{H} \) and obey the canonical commutation relations

\[
[\hat{x}_1, \hat{p}_1] = i\hbar; \quad [\hat{x}_2, \hat{p}_2] = i\hbar; \quad [\hat{x}_1, \hat{p}_2] = 0; \quad [\hat{x}_2, \hat{p}_1] = 0. \tag{10}
\]

where the commutator of two operators is given by \([\hat{f}, \hat{g}] = \hat{f}\hat{g} - \hat{g}\hat{f}\). We assign to the classical Hamiltonian \( H(x_1, x_2, p_1, p_2, t) \) in equation (5) which is a function of the dynamical variables \( x_1, x_2, p_1, p_2 \) an operator \( \hat{H}(\hat{x}_1, \hat{x}_2, \hat{p}_1, \hat{p}_2, t) \) which is obtained by replacing the dynamical variables with the corresponding operators. Other classical dynamical quantities in Quantum Mechanics are similarly associated with quantum operators that act on the Hilbert space of states.

The organization of the paper is the following: in section II we apply the Dirac’s method for constrained system to the model in the extended phase space. Concluding remarks are given in section III.

II. THE MODEL IN THE EXTENDED PHASE SPACE

In this section we would like to perform the Dirac method’s of quantization to constrained systems. The reader interested in knowing more about the method may check in the literature, for instance one may read in [16–23]. Here, a constrained system is one in which there exists a relationship between the system’s degrees of freedom that holds for all times.

We consider the time integral of the Lagrangian in equation (11) as the action

\[
S \left[ x_1(t), x_2(t), \frac{dx_1(t)}{dt}, \frac{dx_2(t)}{dt} \right] = \int_{t_1}^{t_2} L(x_1, x_2, \dot{x}_1, \dot{x}_2) dt, \tag{11}
\]
In order to extend the phase space we consider the time parameter $t$ as an additional degree of freedom for the system described by $S$ in (11). We consider the arbitrary time scaling transformation $t = t_\tau(\tau)$, where the parameter $\tau$ plays the role of the new time parameter. The function $t_\tau(\tau)$ is so chosen that it gives a smooth one-to-one correspondence of the domain $\tau$ and $t$. This transformation changes also the dependency of the coordinates and require the following redefinitions

$$x_1(t_\tau(\tau)) = x_{1,\tau}(\tau); \quad x_2(t_\tau(\tau)) = x_{2,\tau}(\tau), \quad (12)$$

consequently we have a new functional expression for the action $S$ denoted by

$$S_\tau \left[ x_{1,\tau}, x_{2,\tau}, t_\tau, \frac{dx_{1,\tau}}{d\tau}, \frac{dx_{2,\tau}}{d\tau}, \frac{dt_\tau}{d\tau} \right] = \int_{\tau_1}^{\tau_2} \frac{m}{2} f^{-1}(t_\tau) \left( \left( \frac{\dot{x}_{1,\tau}}{t_\tau} \right)^2 + \left( \frac{\dot{x}_{2,\tau}}{t_\tau} \right)^2 \right)\frac{m\omega^2(t_\tau)}{2} \left( x_{1,\tau}^2 + x_{2,\tau}^2 \right) t_\tau d\tau, \quad (13)$$

where the notations $\dot{x}_{i,\tau}$, $i = 1,2$ and $\dot{t}_\tau$ are respectively $\frac{dx_i}{dt}, i = 1,2$ and $\frac{dt_\tau}{d\tau} t_\tau$. The generalized configuration variables on the extended phase space are given by $x_{1,\tau}, x_{2,\tau}, t_\tau$ and their velocities are respectively given by $\dot{x}_{1,\tau}, \dot{x}_{2,\tau}, \dot{t}_\tau$. We consider the boundary conditions $x_{1,\tau}(t_1) = x_1$, $x_{2,\tau}(t_2) = x_2$, $t_\tau(t_1) = t_1$, $t_\tau(t_2) = t_2$

The integrand of the equation (13) thus defines the extended Lagrangian $L_\tau$

$$L_\tau(x_{1,\tau}, x_{2,\tau}, t_\tau, \dot{x}_{1,\tau}, \dot{x}_{2,\tau}, \dot{t}_\tau) = f^{-1}(t_\tau) m \left( \dot{x}_{1,\tau}^2 + \dot{x}_{2,\tau}^2 \right) - \frac{m\omega^2(t_\tau)}{2} f^{-1}(t_\tau) \left( x_{1,\tau}^2 + x_{2,\tau}^2 \right) \quad (14)$$

Let’s determine first the Hessian matrix $M_\tau$ of the Lagrangian function $L_\tau$

$$M_\tau = \begin{bmatrix}
\frac{\partial^2 L}{\partial^{2} x_{1,\tau}} & \frac{\partial^2 L}{\partial x_{1,\tau} \partial x_{2,\tau}} & \frac{\partial^2 L}{\partial^{2} x_{2,\tau}} \\
\frac{\partial^2 L}{\partial x_{2,\tau} \partial x_{1,\tau}} & \frac{\partial^2 L}{\partial x_{2,\tau} \partial x_{2,\tau}} & \frac{\partial^2 L}{\partial^{2} x_{2,\tau}} \\
\frac{\partial^2 L}{\partial t_\tau \partial x_{1,\tau}} & \frac{\partial^2 L}{\partial t_\tau \partial x_{2,\tau}} & \frac{\partial^2 L}{\partial^{2} t_\tau}
\end{bmatrix}, \quad (15)$$

that is equivalent to

$$M_\tau = \begin{bmatrix}
\frac{f^{-1}(t_\tau)m}{t_\tau} & 0 & -\frac{f^{-1}(t_\tau)m x_{1,\tau}}{t_\tau^2} \\
0 & \frac{f^{-1}(t_\tau)m}{t_\tau} & -\frac{f^{-1}(t_\tau)m x_{2,\tau}}{t_\tau^2} \\
-\frac{f^{-1}(t_\tau)m x_{1,\tau}}{t_\tau^2} & -\frac{f^{-1}(t_\tau)m x_{2,\tau}}{t_\tau^2} & \frac{f^{-1}(t_\tau)m (x_{1,\tau}^2 + x_{2,\tau}^2)}{t_\tau^2}
\end{bmatrix}. \quad (16)$$
It is easy to show that the determinant (Hessian) of the matrix $M_\tau$ is zero, that means that the Lagrangian $L_\tau$ is singular and a singular Lagrangian theory necessarily involves constraints. Let’s determine now the corresponding conjugate momenta of the configuration variables $x_{1,\tau}, x_{2,\tau}, t_\tau$

$$p_{1,\tau} = \frac{\partial L_\tau}{\partial \dot{x}_{1,\tau}} = \frac{f^{-1}(t_\tau) m \dot{x}_{1,\tau}}{t_\tau};$$  \hfill (17)

$$p_{2,\tau} = \frac{\partial L_\tau}{\partial \dot{x}_{2,\tau}} = \frac{f^{-1}(t_\tau) m \dot{x}_{2,\tau}}{t_\tau};$$  \hfill (18)

$$p_\tau = \frac{\partial L_\tau}{\partial \dot{t}_\tau} = -\frac{f(t_\tau)}{2m} (p_{1,\tau}^2 + p_{2,\tau}^2) - \frac{m \omega^2(t_\tau) f^{-1}(t_\tau)}{2} (x_{1,\tau}^2 + x_{2,\tau}^2).$$  \hfill (19)

The momentum $p_\tau$ is expressed in term of the fundamental variables and a constraint arise as

$$\phi = p_\tau + \frac{f(t_\tau)}{2m} (p_{1,\tau}^2 + p_{2,\tau}^2) + \frac{m \omega^2(t_\tau) f^{-1}(t_\tau)}{2} (x_{1,\tau}^2 + x_{2,\tau}^2) \sim 0.$$  \hfill (20)

We derive the extended Hamiltonian $H_\tau$ as the Legendre transform of the extended Lagrangian $L_\tau$

$$H_\tau(x_{1,\tau}, x_{2,\tau}, t_\tau, p_{1,\tau}, p_{2,\tau}, p_\tau) = p_{1,\tau} \dot{x}_{1,\tau} + p_{2,\tau} \dot{x}_{2,\tau} + p_\tau \dot{t}_\tau - L$$ \hfill (21)

that is explicitly

$$H_\tau = \left( \frac{f(t_\tau)}{2m} (p_{1,\tau}^2 + p_{2,\tau}^2) + p_\tau + \frac{m \omega^2(t_\tau) f^{-1}(t_\tau)}{2} (x_{1,\tau}^2 + x_{2,\tau}^2) \right) t_\tau.$$  \hfill (22)

A first remark is that

$$H_\tau = t_\tau \phi \sim 0$$  \hfill (23)

The use of $\sim$ sign instead of the $=$ sign is due to Dirac and has a special meaning: two quantities related by a $\sim$ sign are only equal after all constraints have been imposed. Two such quantities are weakly equal to one another. It is important to note that the Poisson brackets in any expression must be worked out before any constraint are set to zero.

We have now an extended phase space determined by $x_{1,\tau}, x_{2,\tau}, p_{1,\tau}, p_{2,\tau}, p_\tau$. The simplectic structure is determined by the non vanishing Poisson Brackets

$$\{x_{1,\tau}, p_{1,\tau}\}_{PB} = 1; \quad \{x_{2,\tau}, p_{2,\tau}\}_{PB} = 1; \quad \{t_\tau, p_\tau\}_{PB} = 1$$  \hfill (24)
and the Poisson Brackets for two arbitrary smooth functions \( f \) and \( g \) in this extended phase space takes the following form

\[
\{ f, g \}_{PB} = \frac{\partial f}{\partial x_1, \tau} \frac{\partial g}{\partial p_{1, \tau}} + \frac{\partial f}{\partial x_2, \tau} \frac{\partial g}{\partial p_{2, \tau}} + \frac{\partial f}{\partial t, \tau} \frac{\partial g}{\partial p_{1, \tau}} - \frac{\partial f}{\partial p_{1, \tau}} \frac{\partial g}{\partial x_1, \tau} - \frac{\partial f}{\partial p_{2, \tau}} \frac{\partial g}{\partial x_2, \tau} - \frac{\partial f}{\partial x_1, \tau} \frac{\partial g}{\partial t, \tau} \tag{25}
\]

The constraint \( \phi \) is a primary constraint and the only one as there are no secondary constraints generated. We have in presence a first-class constraint. Let’s recall that a dynamical variable \( R \) is said to be first-class if it has weakly vanishing Poisson brackets with all constraints. The Hamiltonian \( H_\tau \) is a first-class Hamiltonian. We set the total Hamiltonian to be

\[
H_{\tau T} = \lambda \phi , \tag{26}
\]

where \( \lambda \) is a Lagrange multiplier, note that \( \lambda \) depends only on time. The Hamiltonian equations of motion derived with this Poisson bracket and the Hamiltonian \( H_{\tau T} \) are given by

\[
\dot{x}_{1, \tau} = \{ x_{1, \tau}, H_{\tau T} \}_{PB} = \lambda \frac{f(t_\tau)}{m} p_{1, \tau}; \tag{27}
\]

\[
\dot{x}_{2, \tau} = \{ x_{2, \tau}, H_{\tau T} \}_{PB} = \lambda \frac{f(t_\tau)}{m} p_{2, \tau}; \tag{28}
\]

\[
\dot{p}_{1, \tau} = \{ p_{1, \tau}, H_{\tau T} \}_{PB} = -\lambda m \omega^2(t_\tau) f^{-1}(t_\tau) x_{1, \tau}; \tag{29}
\]

\[
\dot{p}_{2, \tau} = \{ p_{2, \tau}, H_{\tau T} \}_{PB} = -\lambda m \omega^2(t_\tau) f^{-1}(t_\tau) x_{2, \tau}; \tag{30}
\]

\[
\dot{t}_\tau = \{ t_\tau, H_{\tau T} \}_{PB} = \lambda; \tag{31}
\]

\[
\dot{p}_\tau = \{ p_\tau, H_{\tau T} \}_{PB};
\]

\[
= \lambda \left( \frac{\dot{f}(t_\tau)}{2m} (p_{1, \tau}^2 + p_{2, \tau}^2) + \frac{m f^{-1}(t_\tau) \omega(t_\tau)}{2} [2 \omega(t_\tau) - \omega(t_\tau) f(t_\tau) f^{-1}(t_\tau)] (x_{1, \tau}^2 + x_{2, \tau}^2) \right) \tag{32}
\]

The total Hamiltonian is proportional to the constraint \( \phi \) and the coefficient of proportionality is a Lagrange multiplier denoted by \( \lambda \). The Lagrange multiplier is independent of the phase space points. This kind of Lagrange multiplier is referred to as non-canonical gauge. This particular case of constrained system in which the total Hamiltonian is null when the constraint is strongly set to zero are usually called reparametrization invariant system \[24\]. The fact that we have only first class constraint implies that all phase space functions will evolve by gauge transformation and the
system at a given time will gauge equivalent to the system at any other time. To quantize such a theory we need to choose between the Dirac and the canonical quantization procedure. If we choose the canonical quantization we face the fact that we have no Schrödinger equations because the total Hamiltonian must necessarily annihilate physical states. The solution is to impose a supplementary constraint $\gamma$ that depends on the time variable. The process in which a value for the Lagrange multiplier $\lambda$ is fixed is usually called *fixing the gauge*. For instance, the most common gauge fixing is the case in which $\lambda = \frac{t_2 - t_1}{\tau_2 - \tau_1}$. This gauge solves the equation in (31) $t_\tau = \lambda$, that means

$$t_\tau = \frac{(t_2 - t_1)(\tau - \tau_1)}{\tau_2 - \tau_1} + t_1$$

(33)

with $t_\tau(\tau_1) = t_i$, $i = 1,2$ hold.

The gauge fixing condition leads to an additional constraint surface

$$\eta = t_\tau - \frac{(t_2 - t_1)(\tau - \tau_1)}{\tau_2 - \tau_1} + t_1 \sim 0.$$  

(34)

and

$$\{\phi, \eta\} \sim 0$$

(35)

The constraints $\phi$ and $\eta$ are second-class constraints. Let’s recall that a dynamical variable $R$ is said to be second-class if it has weakly non vanishing Poisson brackets with all the constraints. Let’s define now the Dirac brackets. The matrix of the constraints is given by

$$\Delta = \begin{pmatrix} \{\phi, \phi\} & \{\phi, \eta\} \\ \{\eta, \phi\} & \{\eta, \eta\} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

(36)

The matrix $\Delta$ is obviously invertible and its inverse is given by

$$C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

(37)

The Dirac brackets of two extended phase space quantities $f$ and $g$ is given by

$$\{f, g\}_{DB} = \{f, g\}_{PB} - \{(f, \phi)_{PB}(g, \eta)_{PB} - \{f, \eta\}_{PB}(\phi, g)_{PB}\}$$

(38)

The Poisson bracket (25) is then replaced by the Dirac bracket (38). On this base, let’s calculate the Dirac brackets of the fundamental variables in the extended phase space. The non vanishing Dirac brackets are the following

$$\{x_1, \tau, p_1, \tau\}_{DB} = 1;$$

(39)
\[ \{x_{2,\tau}, p_{2,\tau}\}_{DB} = 1; \quad (40) \]

\[ \{x_{1,\tau}, p_{\tau}\}_{DB} = -\frac{f(t)}{m} p_{1,\tau}; \quad (41) \]

\[ \{x_{2,\tau}, p_{\tau}\}_{DB} = -\frac{f(t)}{m} p_{2,\tau}; \quad (42) \]

\[ \{p_{1,\tau}, p_{\tau}\}_{DB} = m \omega^2(t\tau) f^{-1}(t\tau) x_{1,\tau}; \quad (43) \]

\[ \{p_{2,\tau}, p_{\tau}\}_{DB} = m \omega^2(t\tau) f^{-1}(t\tau) x_{2,\tau}. \quad (44) \]

Comparing the Poisson brackets in equation (24) and the Dirac brackets in equations (39 - 44), we can note the differences that are essentially \( \{t, p_{\tau}\}_P = 1 \) while \( \{t, p_{\tau}\}_D = 0 \) and \( \{p_{1,\tau}, p_{\tau}\}_P = 0 \) while \( \{p_{1,\tau}, p_{\tau}\}_D = m \omega^2(t\tau) f^{-1}(t\tau) x_{1,\tau} \) and \( \{p_{2,\tau}, p_{\tau}\}_D = m \omega^2(t\tau) f^{-1}(t\tau) x_{2,\tau} \). When the constraints are fulfilled that means \( \phi = 0 \) and \( \eta = 0 \), we have the coordinates \( x_{1,\tau}, x_{2,\tau}, p_{1,\tau}, p_{2,\tau} \) selected as the physical degree of freedom using \( \tau \) as the time parameter or instead we can use \( x_1, x_2, p_1, p_2 \) with \( t \) as time parameter in accordance with the initial description. In that situation, \( t_{\tau} = \frac{(t_2 - t_1)(\tau - \tau_1)}{\tau_2 - \tau_1} + t_1 \) and \( p_{\tau} = -\hat{H}(x_1, x_2, p_1, p_2, \tau) \), where \( \hat{H} \) is the Hamiltonian in (3). The dynamic of the system is then generated by the Hamiltonian \( \hat{H} \) and the nonvanishing Dirac brackets \( \{x_{1,\tau}, p_{1,\tau}\}_{DB} = 1; \{x_{2,\tau}, p_{2,\tau}\}_{DB} = 1; \)

The canonical quantization procedure as described in section (I) for an unconstrained system is to promote the phase space variable \( x_1, x_2, p_1, p_2 \) to operators \( \hat{x}_1, \hat{x}_2, \hat{p}_1, \hat{p}_2 \) that act on elements of a Hilbert space, which we denote \( |\psi\rangle \). The commutator between phase space variables

\[ [\hat{f}, \hat{g}] = i\hbar \{f, g\}_{DB} \quad (45) \]

and the quantum level Hamiltonian is given by

\[ \hat{H} = \frac{f(t)}{2m} (p_1^2 + p_2^2) + f^{-1}(t) \frac{m \omega^2(t)}{2} (\hat{x}_1^2 + \hat{x}_2^2), \quad (46) \]

with

\[ [\hat{x}_1, \hat{p}_1] = i\hbar; \quad [\hat{x}_2, \hat{p}_2] = i\hbar; \quad [\hat{x}_1, \hat{p}_2] = 0; \quad [\hat{x}_2, \hat{p}_1] = 0. \quad (47) \]
III. CONCLUDING REMARKS

As we have already pointed out in the introduction our aim in this paper is to illuminate the Dirac’s quantization procedure for the model in the extended phase space. The result is that at the quantum level the model is reduced to the initial quantum Hamiltonian. The system can be studied as in [9] by means of the Levis-Riesenfeld procedure of finding invariants hermitian operators. The invariant operator in [9] is given by

$$\hat{I}(t) = \frac{1}{2} \left[ (mf^{-1}\dot{\rho}x_1 - \rho\dot{p}_1)^2 + \frac{\nu^2}{\rho^2} \dot{x}_1^2 + (mf^{-1}\dot{\rho}x_2 - \rho\dot{p}_2)^2 + \frac{\nu^2}{\rho^2} \dot{x}_2^2 \right],$$

(48)

where the function $\rho$ is the solution of the so-called Ermakov-Pinney equation [11]

$$\ddot{\rho} + \eta \dot{\rho} + \omega^2 \rho = \frac{\nu^2 f^2}{m^2 \rho^4}$$

(49)

An alternative of finding invariants of the system described by (1,5) is to study the canonical transformation in the extended phase space such that the final dynamical description of the reduced phase space is no longer time dependent. This method is discussed in [15]. The canonical transformation is a generalization of the Struckmeier transformation [25]. For the present case we consider a coordinate transformation of the form

$$\begin{bmatrix} x_1, \tau \\ x_2, \tau \\ t, \tau \\ p_1, \tau \\ p_2, \tau \end{bmatrix} = \begin{bmatrix} A_1(Q_1, T) \\ A_2(Q_2, T) \\ B(T) \\ C_1(Q_1, T)P_1 + D_1(Q_1, T) \\ C_2(Q_2, T)P_2 + D_2(Q_2, T) \\ F(Q_1, Q_2, T, P_1, P_2, P_T) \end{bmatrix}$$

(50)

The canonical transformation matrix resulting from (50) is given by

$$\mathcal{M} = \begin{bmatrix} \frac{\partial x_1, \tau}{\partial Q_1} & \frac{\partial x_1, \tau}{\partial Q_2} & \frac{\partial x_1, \tau}{\partial T} & \frac{\partial x_1, \tau}{\partial P_1} & \frac{\partial x_1, \tau}{\partial P_2} & \frac{\partial x_1, \tau}{\partial P_T} \\ \frac{\partial x_2, \tau}{\partial Q_1} & \frac{\partial x_2, \tau}{\partial Q_2} & \frac{\partial x_2, \tau}{\partial T} & \frac{\partial x_2, \tau}{\partial P_1} & \frac{\partial x_2, \tau}{\partial P_2} & \frac{\partial x_2, \tau}{\partial P_T} \\ \frac{\partial t, \tau}{\partial Q_1} & \frac{\partial t, \tau}{\partial Q_2} & \frac{\partial t, \tau}{\partial T} & \frac{\partial t, \tau}{\partial P_1} & \frac{\partial t, \tau}{\partial P_2} & \frac{\partial t, \tau}{\partial P_T} \\ \frac{\partial p_1, \tau}{\partial Q_1} & \frac{\partial p_1, \tau}{\partial Q_2} & \frac{\partial p_1, \tau}{\partial T} & \frac{\partial p_1, \tau}{\partial P_1} & \frac{\partial p_1, \tau}{\partial P_2} & \frac{\partial p_1, \tau}{\partial P_T} \\ \frac{\partial p_2, \tau}{\partial Q_1} & \frac{\partial p_2, \tau}{\partial Q_2} & \frac{\partial p_2, \tau}{\partial T} & \frac{\partial p_2, \tau}{\partial P_1} & \frac{\partial p_2, \tau}{\partial P_2} & \frac{\partial p_2, \tau}{\partial P_T} \\ \frac{\partial F}{\partial Q_1} & \frac{\partial F}{\partial Q_2} & \frac{\partial F}{\partial T} & \frac{\partial F}{\partial P_1} & \frac{\partial F}{\partial P_2} & \frac{\partial F}{\partial P_T} \end{bmatrix},$$

(51)

that is equivalent to
\[
M = \begin{bmatrix}
A_1' & 0 & \dot{A}_1 & 0 & 0 & 0 \\
0 & A_2' & \dot{A}_2 & 0 & 0 & 0 \\
0 & 0 & \dot{B} & 0 & 0 & 0 \\
C_1' P_1 + D_1' & 0 & \dot{C}_1 P_1 + \dot{D}_1 & C_1 & 0 & 0 \\
0 & C_2' P_2 + D_2' & \dot{C}_2 P_2 + \dot{D}_2 & 0 & C_2 & 0 \\
\frac{\partial F}{\partial Q_1} & \frac{\partial F}{\partial Q_2} & \frac{\partial F}{\partial T} & \frac{\partial F}{\partial P_1} & \frac{\partial F}{\partial P_2} & \frac{\partial F}{\partial P_T}
\end{bmatrix}.
\] (52)

where \( A_i' = \frac{\partial A_i}{\partial Q_i}; \quad C_i' = \frac{\partial C_i}{\partial Q_i}; \quad D_i' = \frac{\partial D_i}{\partial Q_i}, \quad i = 1, 2 \) and the dot notation is used for the derivative with respect to \( T \). We would like to solve \( M^T J M = J \) where \( J \) is the matrix

\[
J = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
-1 & 0 & 0 & 0 & 0 & 1 \\
0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0
\end{bmatrix}.
\] (53)

Solving \( M^T J M = J \) leads to a system of differential equations

\[
\dot{A}_1 (C_1' P_1 + D_1') + \dot{B} \frac{\partial F}{\partial Q_1} = (\dot{C}_1 P_1 + \dot{D}_1) A_1'; \quad (54)
\]

\[
\dot{A}_2 (C_2' P_2 + D_2') + \dot{B} \frac{\partial F}{\partial Q_2} = (\dot{C}_2 P_2 + \dot{D}_2) A_2'; \quad (55)
\]

\[
C_1 \dot{A}_1 + \dot{B} \frac{\partial F}{\partial P_1} = 0; \quad (56)
\]

\[
C_2 \dot{A}_2 + \dot{B} \frac{\partial F}{\partial P_2} = 0; \quad (57)
\]

\[
\dot{B} \frac{\partial F}{\partial P_T} = 1; \quad (58)
\]

\[
C_1 A'_1 = 1; \quad (59)
\]
whose general solution is given by

\[ C_1 = \frac{1}{A_1'}, \quad C_2 = \frac{1}{A_2'}, \quad t_\tau = B(T); \]  

\[ F = \frac{P_\tau}{B} - \frac{\dot{A}_1 A_1'}{A_1' B} P_1 - \frac{\dot{A}_2 A_2'}{A_2' B} P_2 + \frac{1}{B} \left[ \int \left( \dot{D}_1 A_1' - \dot{A}_1 D_1' \right) dQ_1 + \int \left( \dot{D}_1 A_2' - \dot{A}_2 D_2' \right) dQ_2 \right], \]  

where the functions \( A_i(Q, T), B(T), D_i(Q, T), i = 1, 2 \) are arbitrary. We can now write \( x_{1, \tau}, x_{2, \tau}, t_\tau, p_{1, \tau}, p_{2, \tau}, p_\tau \) in terms of the new coordinates as

\[
\begin{bmatrix}
  x_{1, \tau} \\
  x_{2, \tau} \\
  t_\tau \\
  p_{1, \tau} \\
  p_{2, \tau} \\
  p_\tau
\end{bmatrix} =
\begin{bmatrix}
  A_1(Q, T) \\
  A_2(Q, T) \\
  B(T) \\
  \frac{1}{A_1'} P_1 + D_1 \\
  \frac{1}{A_2'} P_2 + D_2 \\
  \frac{P_\tau B}{-A_1' A_2'} P_1 - \frac{A_1 A_2'}{A_2'} P_2 + \frac{1}{B} \left[ \int \left( \dot{D}_1 A_1' - \dot{A}_1 D_1' \right) dQ_1 + \int \left( \dot{D}_1 A_2' - \dot{A}_2 D_2' \right) dQ_2 \right]
\end{bmatrix}.
\]

The new variables are time independent since the time variable is \( \tau \). A new Hamiltonian of the system can be derived in terms of these new variables that is also an invariant of the system since autonomous.

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