Multi-headed symmetrical superpositions of coherent states

Bo Lan and Xue-xiang Xu
College of Physics and Communication Electronics, Jiangxi Normal University, Nanchang 330022, China
xuxuxiang@jxnu.edu.cn

Based on $N$ different coherent states with equal weights and phase-space rotation symmetry, we introduce $N$-headed incoherent superposition states (NIHSSs) and $N$-headed coherent superposition states (NHCSSs). These $N$ coherent states are associated with $N$-order roots of the same complex number. We study and compare properties of NIHSSs and NHCSSs, including average photon number, Mandel Q parameter, quadrature squeezing, Fock matrix elements and Wigner function. Among all these states, only 2HCSS (i.e., Schrodinger cat state) presents quadrature-squeezing effect. Our theoretical results can be used as a reference for researchers in this field.

**Keywords:** Schrodinger cat state; coherent state; superposition; phase-space rotation symmetry; Wigner function

I. INTRODUCTION

The superposition principle is one of the pillars upon which the entire structure of quantum mechanics is built[1]. Generally, superpositions include operator superposition and state superposition. For example, two-operator superposition $c_1O_1 + c_2O_2$ is superposing from operators $O_1$ and $O_2$ with respective weights $c_1$ and $c_2$, such as the delated photon addition $a_A + e^{i\phi}a_B[2]$, ancilla-assisted photon subtraction $a_A^2 - a_B^2[3]$ and photon addition-subtraction sequences $aa^† - e^{i\theta}a^1a[4]$. Two-state superposition in terms of states $|\psi_1\rangle$ and $|\psi_2\rangle$ deviates coherent superposition $c_1|\psi_1\rangle + c_2|\psi_2\rangle$ and incoherent superposition $c_1|\psi_1\rangle \langle \psi_1 | + c_2|\psi_2\rangle \langle \psi_2 |$ with respective weights $c_1$ and $c_2$. Similarly, multi-operator superposition and multi-state superposition can be generalized.

In quantum mechanics, the superposition principle is the origin of nonclassical properties of quantum states[5]. So superposition becomes an important way to generating new quantum states in quantum state engineering. Among them, coherent-state superpositions are of great importance for many quantum subjects[6]. For instance, the Schrodinger cat state is a superposition of two coherent states with coherent amplitudes of the same magnitude but different phases[7]. Moreover, the Schrodinger cat state can exhibit different nonclassical properties having foundational applications in quantum information processing[8, 9].

In recent several decades, the quantum superpositions involving more than two component coherent states have been attracting increasing interests of researchers. Yukawa et al. considered the superposition of three coherent states with different phases[10]. Raimond et al. proposed the superposition of four coherent states with different phases[11]. This state was also called as four-headed cat state and applied to quantum phase estimation[12 13] or state preparation[14]. Vlastakis et al. implemented the superpositions of up to four coherent states[15].

In this paper, we shall unify all these multi-component coherent-state superpositions appearing in a large number of literatures. Unlike previous literatures, we consider the link between $N$ coherent states in quantum physics and $N$-order roots of complex number $\alpha$ in mathematics. Using these coherent states, we construct superposition states with equal weights in coherent and incoherent way. These states will present distinctive properties. The paper is organized as follows: In Sec.II, we introduce the background for this paper. In Sec.III, we introduce the coherent-state superpositions with equal weights, i.e., the focus of the paper. In Sec. IV, we study and compare their statistical properties, Fock matrix elements and Wigner functions, respectively. Conclusions are summarized in the last section.

II. BACKGROUND

A complex number $\alpha$ may be written in the general form $\alpha = x + iy = re^{i\theta}$, where $x = \text{Re} \alpha$, $y = \text{Im} \alpha$, $r = |\alpha|$, and $\theta = \text{arg} \alpha$ are called as real part, imaginary part, modulus and argument of $\alpha$, respectively. For a fixed complex number $\alpha$, one can take argument $\theta = \theta_p + 2k\pi$ for arbitrary integer $k$, where $\theta_p$ is the principal argument of $\alpha$ and varies from $0$ to $2\pi$[16]. According to de Moivre theorem, the $N$-order roots of $\alpha$ may be written as

$$N^{\alpha} = r^{\frac{1}{N}} e^{\frac{2k\pi i \theta_p}{N}}, \quad (k = 0, 1, \cdots, N - 1),$$

which are just $N$ complex numbers having same modulus $r^{1/N}$ but different phases $(2k\pi + \theta_p)/N$. Noting that the quantity $r^{1/N}$ represents the positive $N$-order root of modulus $r$. In particular, $\sqrt[N]{\alpha}$ is just $\alpha$ for $N = 1$.

Geometrically, complex numbers can be shown on a so-called complex plane. In complex plane, complex number $z = x + iy$ can be drawn as a vector from $(0, 0)$ to $(x, y)$, where one can go to $x$ point on the real axis and to $y$ point on the imaginary axis. Similarly, $N$-order roots of complex number $\alpha$ can be distributed symmetrically around the origin in complex plane. Moreover, the
Therefore, we can obtain the operator \( \alpha \) can be verified for \( N \geq 2 \). Indeed, they have \( N \)-fold rotation symmetry and possesses \( 2\pi/N \) rotational symmetry, as shown in Fig.1. Recently, discrete rotational symmetry was also used to define bosonic rotation codes\(^{[17]}\). As we all know, coherent state \(|\alpha\rangle\) with amplitude \( \alpha \) can be generated by operating the displacement operator \( D(\alpha) = e^{\alpha a^\dagger - \alpha^* a} \) on the vacuum \(|0\rangle\)\(^{[18]} \).\(^{[19]} \). That is to say, one to one correspondence can be established between a coherent state and a complex number. Therefore, we can obtain \( N \) different coherent states \(|\mathbb{P} e^{i\frac{2k\pi + \theta_p}{N}}\rangle\) in terms of above \( N \)-order roots of \( \alpha \).

### III. COHERENT-STATE SUPERPOSITIONS WITH EQUAL WEIGHTS

Employing equal-weight superpositions of above \( N \) coherent states, we introduce multi-headed quantum states in this section.

**Superposition way I:** A \( N \)-headed incoherent superposition state (NHICSS)

\[
\rho_{\text{c}} = \frac{1}{N} \sum_{k=0}^{N-1} \left| \mathbb{P} e^{i\frac{2k\pi + \theta_p}{N}} \right\rangle \left\langle \mathbb{P} e^{i\frac{2k\pi + \theta_p}{N}} \right|.
\]

is introduced by superposing above \( N \) coherent states with equal weights in incoherent way. Obviously, this state is the equal-weight incoherent mixture of those \( N \) coherent states. Recently, this state has been used as the resource of quantum key distribution\(^{[20]} \)\(^{[21]} \).

**Superposition way II:** A \( N \)-headed coherent superposition state (NHCSS)

\[
|\psi_c\rangle = \frac{1}{\sqrt{N_c}} \sum_{k=0}^{N-1} \left| \mathbb{P} e^{i\frac{2k\pi + \theta_p}{N}} \right\rangle,
\]

is introduced by superposing above \( N \) coherent states with equal weights in coherent way. Its density operator, i.e. \( \rho_c = |\psi_c\rangle \langle \psi_c| \), can be further written as

\[
\rho_c = \frac{1}{N_c} \sum_{k_1=0}^{N-1} \sum_{k_2=0}^{N-1} \left| \mathbb{P} e^{i\frac{2k_1\pi + \theta_p}{N}} \right\rangle \left\langle \mathbb{P} e^{i\frac{2k_2\pi + \theta_p}{N}} \right|.
\]

Here

\[
N_c = \sum_{k_1=0}^{N-1} \sum_{k_2=0}^{N-1} e^{i\frac{2\pi k_1 k_2}{N}} (\mathbb{P} e^{i\frac{2\pi (k_1 + k_2)\theta_p}{N}} - 1).
\]

is the normalization factor. In particular, \( N_c = 1 \) for case \( N = 1 \) and \( N_c = 2 + 2e^{-2\theta_p} \) for case \( N = 2 \). Moreover, for case \( N = 1 \), \( \rho_c \) and \( \rho_{\text{c}} \) will reduce to the coherent state \(|\alpha\rangle\). By the way, the NHCSS \(|\psi_c\rangle\) is the eigenstate of the operator \( a^N \).

In contrast to coherent state with \( N = 1 \), we call states with \( N \geq 2 \) as multi-head cases. In the field of quantum optics, the 2HCSS is often called as the Schrodinger cat state\(^{[22]} \). Following this naming rule, NHCSSs (i.e. high-order cat states) are called the generalized cat states\(^{[23]} \). Different from other literatures, we introduce NHICSSs and NHCSSs associated with \( N \)-order roots of the same \( \alpha \). For example, the Schrodinger cat state is not defined as \(|\alpha\rangle + |\alpha\rangle\) but as \(|\sqrt{\alpha\rangle} + |\sqrt{-\alpha\rangle}\) (2HCSS) in our work.

### IV. STATISTICAL PROPERTIES

In order to calculate statistical properties, we firstly derive \( \langle a^h a^l \rangle \) (\( h, l \) are integers) for NHICSSs and NHCSSs. For NHICSS, we have the general expression

\[
\langle a^h a^l \rangle_{\rho_{\text{c}}} = \frac{r^{2N}}{N} \sum_{k=0}^{N-1} e^{i\frac{2k\pi + \theta_p}{N} - ih \frac{2k\pi + \theta_p}{N}}.
\]

Using Eq. (7), we list \( \langle a^1 \rangle \), \( \langle a \rangle \), \( \langle a^1 a \rangle \), \( \langle a^2 \rangle \), and \( \langle a^2 a^2 \rangle \) for NHICSS in table I. Moreover, we find \( \langle a^4 \rangle = |\alpha|^{2/N} \) and \( \langle a^2 a^2 \rangle = |\alpha|^{4/N} \) for NHICSS in table I. For NHCSS, we have the general expression

\[
\langle a^h a^l \rangle_{\rho_c} = \frac{r^{2N}}{N} \sum_{k_1=0}^{N-1} \sum_{k_2=0}^{N-1} e^{i\frac{2k_1\pi + \theta_p}{N} - ih \frac{2k_1\pi + \theta_p}{N}} (e^{i \frac{2k_2\pi + \theta_p}{N}} - 1)
\]

\[
\times e^{i\frac{2k_1 k_2 + \theta_p}{N} - il \frac{2k_1 k_2 + \theta_p}{N} - \frac{i\theta_p}{2} (k_1 + k_2) \theta_p},
\]

is the resource of quantum key distribution\(^{[20]} \)\(^{[21]} \).
Using Eq(8), we list $\langle a^\dagger \rangle$, $\langle a \rangle$, $\langle a^\dagger a \rangle$, $\langle a^\dagger 2a \rangle$, $\langle a^2 \rangle$, and $\langle a^\dagger a^\dagger 2a \rangle$ for NHCSS in table II. Some expressions of $\langle a^\dagger a \rangle$ and $\langle a^\dagger 2a \rangle$ in Table II, which are related to $|\alpha|$ but not $\theta_p$, haven’t been given yet because of their complexity. But we can easily provide their numerical results through the computing software of Mathematica. Interestingly, we we have $\langle a^\dagger \rangle = \langle a \rangle = \langle a^\dagger 2 \rangle = \langle a^2 \rangle = 0$ for cases $N \geq 3$ in both table I and table II.

**TABLE I: Expectation values $\langle a^\dagger a \rangle$ for some NHICSSs with same $\alpha$.**

| Case | $\langle a^\dagger \rangle$ | $\langle a \rangle$ | $\langle a^\dagger a \rangle$ | $\langle a^\dagger 2a \rangle$ | $\langle a^2 \rangle$ | $\langle a^\dagger a^\dagger 2a \rangle$ |
|------|-----------------|----------------|-----------------|-----------------|----------------|----------------|
| $N=1$ | $|\alpha|^2\alpha^2$ | $|\alpha|^4\alpha^4$ | $|\alpha|^4\alpha^4$ | $|\alpha|^4\alpha^4$ | $|\alpha|^4\alpha^4$ | $|\alpha|^4\alpha^4$ |
| $N=2$ | $|\alpha|^2\alpha^2$ | $|\alpha|^4\alpha^4$ | $|\alpha|^4\alpha^4$ | $|\alpha|^4\alpha^4$ | $|\alpha|^4\alpha^4$ | $|\alpha|^4\alpha^4$ |
| $N=3$ | $|\alpha|^2\alpha^2$ | $|\alpha|^4\alpha^4$ | $|\alpha|^4\alpha^4$ | $|\alpha|^4\alpha^4$ | $|\alpha|^4\alpha^4$ | $|\alpha|^4\alpha^4$ |
| $N=4$ | $|\alpha|^2\alpha^2$ | $|\alpha|^4\alpha^4$ | $|\alpha|^4\alpha^4$ | $|\alpha|^4\alpha^4$ | $|\alpha|^4\alpha^4$ | $|\alpha|^4\alpha^4$ |
| $N=5$ | $|\alpha|^2\alpha^2$ | $|\alpha|^4\alpha^4$ | $|\alpha|^4\alpha^4$ | $|\alpha|^4\alpha^4$ | $|\alpha|^4\alpha^4$ | $|\alpha|^4\alpha^4$ |
| $N=6$ | $|\alpha|^2\alpha^2$ | $|\alpha|^4\alpha^4$ | $|\alpha|^4\alpha^4$ | $|\alpha|^4\alpha^4$ | $|\alpha|^4\alpha^4$ | $|\alpha|^4\alpha^4$ |
| ... | ... | ... | ... | ... | ... | ... |

**TABLE II: Expectation values $\langle a^\dagger a^\dagger 2a \rangle$ for some NHCSSs with same $\alpha$.**

| Case | $\langle a^\dagger \rangle$ | $\langle a \rangle$ | $\langle a^\dagger a \rangle$ | $\langle a^\dagger 2a \rangle$ | $\langle a^2 \rangle$ | $\langle a^\dagger a^\dagger 2a \rangle$ |
|------|-----------------|----------------|-----------------|-----------------|----------------|----------------|
| $N=1$ | $|\alpha|^2\alpha^2$ | $|\alpha|^4\alpha^4$ | $|\alpha|^4\alpha^4$ | $|\alpha|^4\alpha^4$ | $|\alpha|^4\alpha^4$ | $|\alpha|^4\alpha^4$ |
| $N=2$ | $|\alpha|^2\alpha^2$ | $|\alpha|^4\alpha^4$ | $|\alpha|^4\alpha^4$ | $|\alpha|^4\alpha^4$ | $|\alpha|^4\alpha^4$ | $|\alpha|^4\alpha^4$ |
| $N=3$ | $|\alpha|^2\alpha^2$ | $|\alpha|^4\alpha^4$ | $|\alpha|^4\alpha^4$ | $|\alpha|^4\alpha^4$ | $|\alpha|^4\alpha^4$ | $|\alpha|^4\alpha^4$ |
| $N=4$ | $|\alpha|^2\alpha^2$ | $|\alpha|^4\alpha^4$ | $|\alpha|^4\alpha^4$ | $|\alpha|^4\alpha^4$ | $|\alpha|^4\alpha^4$ | $|\alpha|^4\alpha^4$ |
| $N=5$ | $|\alpha|^2\alpha^2$ | $|\alpha|^4\alpha^4$ | $|\alpha|^4\alpha^4$ | $|\alpha|^4\alpha^4$ | $|\alpha|^4\alpha^4$ | $|\alpha|^4\alpha^4$ |
| $N=6$ | $|\alpha|^2\alpha^2$ | $|\alpha|^4\alpha^4$ | $|\alpha|^4\alpha^4$ | $|\alpha|^4\alpha^4$ | $|\alpha|^4\alpha^4$ | $|\alpha|^4\alpha^4$ |
| ... | ... | ... | ... | ... | ... | ... |

**Average photon number:** Light intensity can be described by average photon number $\bar{n} = \langle n \rangle = \langle a^\dagger a \rangle$. In Fig.2, we plot $\bar{n}$ as a function of $|\alpha|$ for NHICSSs and NHCSSs. As shown in Fig.2(a) for NHICSS, in the regime of $0 < |\alpha| < 1$, the larger $N$ is, the larger $\bar{n}$ is; while in the regime of $|\alpha| > 1$, the larger $N$ is, the smaller $\bar{n}$ is. This can also be checked by $\bar{n} = |\alpha|^2/N$. While from Fig.2 (b) for NHCSS, in the whole range of $|\alpha|$, the larger $N$ is, the smaller $\bar{n}$ is.

**Mandel Q parameter:** We examine the Mandel Q parameter $M_Q = \langle a^\dagger 2a^\dagger a \rangle / \langle a^\dagger a \rangle - \langle a^\dagger a \rangle$, which indicate that the distribution is Poissonian, super-Poissonian and sub-Poissonian, if $M_Q = 0$, $M_Q > 0$ or $M_Q < 0$, respectively. Obviously, coherent state and NHICSSs are Poissonian, which can be verified by $M_Q = |\alpha|^4/|\alpha|^2 - |\alpha|^2 = 0$ and $M_Q = |\alpha|^{4/N}/|\alpha|^{2/N} - |\alpha|^{2/N} = 0$. While for NHCSSs, we plot $M_Q$ as a function of $|\alpha|$ in Fig.3. For 2HCSS, the distribution is super-Poissonian when $|\alpha| < 11.7069$ and Poissonian when $|\alpha| > 11.7069$. For 3HCSS, the distribution is super-Poissonian when $|\alpha| < 5.23972$, sub-Poissonian when $5.23972 < |\alpha| < 17.1512$, and Poissonian when $|\alpha| > 17.1512$. For 4HCSS, the distribution is super-Poissonian when $|\alpha| < 9.8696$ and $39.4784 < |\alpha| < 48.8264$, sub-Poissonian when $9.8696 < |\alpha| < 48.8264$, and Poissonian when $|\alpha| > 48.8264$. Moreover, all these states will tend to Poisson distribution if $|\alpha|$ is large enough.

**Quadrature squeezing effect:** We explore squeezing of quadrature amplitude defining from quadratures $X_1 = (a + a^\dagger)/\sqrt{2}$ and $X_2 = (a - a^\dagger)/(\sqrt{2}i)$. Their variances can be expressed as $\langle \Delta X_j \rangle^2 = \langle a^\dagger a \rangle - |\langle a^\dagger \rangle|^2 + |\langle a \rangle|^2 - |\langle a^\dagger a \rangle|^2 - |\langle a \rangle|^2$.
Re\(|\langle a^2 \rangle - \langle a^4 \rangle \rangle + 0.5\) with \(+ \rightarrow j = 1\) and \(- \rightarrow j = 2\), respectively. A quantum state exhibits quadrature squeezing if \(\Delta^2 X_1 < 0.5\) or \(\Delta^2 X_2 < 0.5\). In fact, we know \(\Delta^2 X_1 = 0.5\) and \(\Delta^2 X_2 = 0.5\) for coherent (vacuum) state. Obviously, NHCSs and NHICSSs with \(N = 2\) have no squeezing due to \(\langle a^2 \rangle = \langle a^4 \rangle + 0.5 \geq 0\). 2HICSS has no squeezing due to \(\langle a^2 \rangle = \langle a^4 \rangle + 0.5 \geq 0\). But 2HCSS has the possibility of squeezing in certain range of \(|\alpha|\), which can be seen from Fig.4 and \(\langle X^2 \rangle = |\alpha| \tan h |\alpha| \pm \text{Re}(\alpha^*) + 0.5\).

V. FOCK MATRIX ELEMENTS

A density operator \(\rho\) can be written as \(\rho = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} p_{mn} |m\rangle \langle n|\) with \(p_{mn} = \langle m| \rho |n\rangle\). Here, the coefficients \(p_{mn}\) are called Fock matrix elements (FME) corresponding to component \(|m\rangle \langle n|\) [24, 25]. Of course, the coefficients \(p_{mn}\) show the photon number distribution (PND).

FMEs of NHICSS can be expressed as

\[ p_{mn}^{\text{IC}} = \frac{\alpha^m - \alpha^n}{N!}, \quad m, n = 0, 1, 2, \ldots \]

PNDs for NHICSS can be written as \(p_{nm}^{\text{IC}} = e^{2m/N} e^{-r^2/N} / m!\) (a Poissonian distribution).

FMEs of NHCSS can be expressed as

\[ p_{mn}^{\text{IC}} = \frac{\alpha^m + \alpha^n - \delta_{m,n}}{N!} \sum_{k=0}^{N} e^{i m k \pi / N} e^{-i n k \pi / N}, \quad m, n = 0, 1, 2, \ldots \]

From Eq.(4), NHCSS can expanded in the photon-number basis

\[ \psi_c = \frac{N e^{-r^2/N / 2}}{\sqrt{N!}} \sum_{s=0}^{\infty} \frac{\alpha^s}{\sqrt{(N-s)!}} |N-s\rangle. \]

so PNDs for NHCSS can be written as \(p_{mn}^{\text{IC}} = |\langle m| \psi_c \rangle|^2 = N^2 r^2 e^{-r^2/N} \delta_{m,N-s} / (N|N-s|!\].

VI. WIGNER FUNCTION

Wigner function is extremely useful in quantum optics because it contains complete information about quantum state[26]. Wigner function of a quantum state \(\rho\) can be written as \(W(\beta) = \frac{1}{2} \text{Tr} [\rho D(\beta) \Pi D(\beta)^\dagger]\) with complex-number coordinate \(\beta = (x + iy) / \sqrt{2}\) in phase space, where \(\Pi = (-1)^a \alpha a^\dagger \alpha \) denotes the photon number parity operator and \(D(\beta) = e^{\beta a^\dagger - \beta^* a}\) is the displacement operator[13]. In normal-order form, \(W(\beta)\) can

FIG. 4: \(\langle X^2 \rangle\) as a function of \(|\alpha|\) for 2HCSS. The squeezing effect is shown in the grey region.

FIG. 5: FMEs for (a) 3HICSS and (b) 3HCSS with \(\alpha = 1 + i\).

FIG. 6: PNDs for (a) 3HICSS and (b) 3HCSS with \(\alpha = 1 + i\).
be written as [27]

\[ W_{\rho}(\beta) = \sqrt{\frac{2}{\pi}} e^{-2\left(\alpha^* - \beta^*\right)(\alpha - \beta)} \rho. \]  

(12)

For NHICSS, we have

\[ W_{\rho_{\alpha\alpha}}(\beta) = \frac{2}{\pi N} \sum_{k_1=0}^{N-1} \sum_{k_2=0}^{N-1} e^{\frac{k_1 - k_2}{\sqrt{N}}} e^{e^{2\pi k_1/\sqrt{N}} - \beta} \langle \alpha | \beta \rangle. \]  

(13)

For NHCSS, we have

\[ W_{\rho_{\alpha\alpha}}(\beta) = \frac{2}{\pi N} \sum_{k_1=0}^{N-1} \sum_{k_2=0}^{N-1} e^{\frac{k_1 - k_2}{\sqrt{N}}} e^{e^{2\pi k_1/\sqrt{N}} - \beta} \langle \alpha | \beta \rangle. \]  

(14)

In particularly, for case \( N = 1 \), Eqs. (13) and (14) will reduce to Wigner function of coherent state \( |\alpha\rangle \)

\[ W_{|\alpha\rangle}(\beta) = \frac{2}{\pi} e^{-2|\alpha - \beta|^2}, \]  

(15)

which have a Gaussian form with the center \( \alpha \) in phase space. For case \( N = 2 \), Eq. (13) will reduce to

\[ W_{\rho_{\alpha\alpha}}(\beta) = \frac{e^{-2|\sqrt{\pi} e^{\hat{\theta}/2} - \beta|^2}}{\pi} + \frac{e^{-2|\sqrt{\pi} e^{-\hat{\theta}/2} - \beta|^2}}{\pi}. \]  

(16)

Similarly, Eq. (14) will reduce to

\[ W_{\rho_{\alpha\alpha}}(\beta) = \frac{e^{-2|\sqrt{\pi} e^{\hat{\theta}/2} - \beta|^2}}{\pi (1 + e^{-2\hat{\theta}})} + \frac{e^{-2|\sqrt{\pi} e^{-\hat{\theta}/2} - \beta|^2}}{\pi (1 + e^{2\hat{\theta}})} + \frac{e^{2|\sqrt{\pi} e^{\hat{\theta}/2} - \beta|^2}}{\pi (1 + e^{-2\hat{\theta}})} + \frac{e^{2|\sqrt{\pi} e^{-\hat{\theta}/2} - \beta|^2}}{\pi (1 + e^{2\hat{\theta}})}. \]  

(17)

where the last two terms owe to the interference between \( \sqrt{|\alpha|} \) and \( -\sqrt{|\alpha|} \).

Using Eqs. (15), (16) and (17), we depict Wigner functions for coherent state, 2HICSS, 2HCSS, respectively in Fig.7. As represented pictorially in Fig.7(a), the Wigner function of coherent state \( |\alpha\rangle \) with \( \alpha = |\alpha| e^{i\hat{\theta}} \) have a positive Gaussian peak, whose center of shaded circle (representing “area of uncertainty”) locates at distance \( |\alpha| \) from the origin and at angle \( \hat{\theta} \) above the position axis. From Fig.7(b), Wigner function of 2HICSS has a characteristic shape that consists of two positive Gaussian peaks. From the Fig.7(c), we can see that Wigner function of 2HCSS has more hills and dips [28].

According to Eq. (13), we plot Wigner functions for 3HICSS, 4HICSS and 5HICSS in Fig.8. Clearly, the distribution of peaks is same as the distribution of \( N \) roots of \( \alpha \) in complex plane. According to Eq. (14), we plot Wigner functions for 3HCSS, 4HCSS and 5HCSS in Fig.9.

Compared with Fig.8, surfaces in Fig.9 have remarkably exhibit multiple areas of negativity in phase space, which indicate the nonclassicality of the NHCSSs. Indeed, multiple areas of negativity result from higher interference effects of coherent states. Of course, Wigner functions in both Fig.8 and Fig.9 possess distinctive rotational \( 2\pi/N \)

FIG. 7: Wigner functions for (a) coherent state \( |\alpha\rangle \); (b) 2HICSS; (c) 2HCSS with \( \alpha = 1 + i \).

FIG. 8: Wigner functions for (a) 3HICSS; (b) 4HICSS; (c) 5HICSS with \( \alpha = 1 + i \).

FIG. 9: Wigner functions for (a) 3HCSS; (b) 4HCSS; (c) 5HCSS with \( \alpha = 1 + i \).
To conclude, we have introduced NHICSSs and NHCSSs by superposing $N$ coherent states associated with $N$-order roots of complex number $\alpha$. Therefore, our research belongs to mathematical physics. In fact, these quantum states have also been widely studied in previous literatures. But different from those literatures, we have unified all the states in the standard form based on the same complex number. Some properties, including average photon number, Mandel Q parameter, quadrature squeezing effect, Fock matrix elements (photon number distributions) and Wigner function, have been studied for these quantum states in detail. Analytical expressions have been given and numerical results have been analyzed.

Our main results show that: (1) Light intensity of NHCSS will decrease as $N$ increases for any $|\alpha|$; but that of NHICSS will increase ($|\alpha| < 1$) and decrease ($|\alpha| > 1$) as $N$ increases. (2) NHICSS remains the Poissonian character of the original coherent state, but NHCSS may present Poissonian, sub-Poissonian and super-Poissonian in different $|\alpha|$. (3) Only 2HCSS may present the quadrature squeezing effect. (4) NHICSSs include all photon components, but NHCSSs only include photon components with $|N \cdot s|$; (5) Wigner functions of NHICSSs have no negative region, but Wigner functions of NHCSSs have negative regions due to the interference effect. In addition, the parity has been discussed incidentally.

By the way, we have only considered superpositions of coherent states with the equal weights. In principle, many other coherent-state superpositions will generate by setting arbitrary superposition weights. Moreover, quantum states related to our considered NHICSSs and NHCSSs are useful for quantum metrology [30–32], quantum error correction [33, 34] and quantum key distribution [35, 37]. We believe that these theoretical results will provide further references for relevant researchers.

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[1] P. A. Dirac, *The principle of quantum mechanics* (Cambridge University Press, Cambridge, 1930).
[2] N. Biagi, L. S. Costanzo, M. Bellini, and A. Zavatta, Phys. Rev. Lett. 124, 033604 (2020).
[3] H. Takahashi, K. Wakui, S. Suzuki, M. Takeoka, K. Hayasaka, A. Furusawa, and M. Sasaki, Phys. Rev. Lett. 101, 233605 (2008).
[4] A. Zavatta, V. Parigi, M. S. Kim, H. Jeong, and M. Bellini, Phys. Rev. Lett. 103, 140406 (2009).
[5] F. R. Cardoso, D. Z. Rossatto, G. P. L. M. Fernandes, G. Higgins, and C. J. Villas-Boas, Phys. Rev. A 103, 062405 (2021).
[6] N. Akhtar, B. C. Sanders, and C. Navarrete-Benlloch, Phys. Rev. A 103, 053711 (2021).
[7] E. Schrodinger, Naturwissenschaften 23, 807 (1935).
[8] G. Tatsi, L. Mazzarella, and J. Jeffers, Phys. Rev. A 103, 023709 (2021).
[9] K. K. Mishra, D. Yadav, G. Shukla, and D. K. Mishra, Phys.
[10] M. Yukawa, K. Miyata, T. Mizuta, H. Yonezawa, P. Marek, R. Filip, and A. Furusawa, Opt. Express 21, 5529 (2013).
[11] J. M. Raimond, C. Sayrin, S. Gleyzes, I. Dotsenko, M. Brune, S. Haroche, P. Facchi, and S. Pascazio, Phys. Rev. Lett. 105, 213601 (2010).
[12] S. Y. Lee, C. W. Lee, H. Nha, and D. Kaszlikowski, J. Opt. Soc. Am. 32, 061186 (2015).
[13] D. A. R. Dalvit, R. L. de Matos Filho, and F. Toscano, New J. Phys. 8, 010276 (2006).
[14] L. Y. Jiang, Q. Guo, X. X. Xu, M. Cai, W. Yuan, and Z. L. Duan, Opt. Commun. 369, 179 (2016).
[15] B. Vlastakis, G. Kirchmair, Z. Leghtas, S. E. Nigg, L. Frunzio, S. M. Girvin, M. Mirrahimi, M. H. Devoret, and R. J. Schoelkopf, Science 342, 607 (2013).
[16] C. Harper, Analytic Methods in Physics (Wiley-VCH, Berlin, 1999).
[17] A. L. Grimsmo, J. Combes, and B. Q. Baragiola, Phys. Rev. X 10, 011058 (2020).
[18] R. J. Glauber, Phys. Rev. 131, 2766 (1963).
[19] M. O. Scully and M. S. Zubairy, Quantum Optics (Cambridge University Press, 1997).
[20] A. Denys, P. Brown, and A. Leverrier, Quantum 5, 540 (2021).
[21] P. Papanastasiou, C. Lupo, C. Weedbrook, and S. Pirandola, Phys. Rev. A 98, 012340 (2018).
[22] C. C. Gerry and P. L. Knight, Introductory Quantum Optics (Cambridge University Press, 2005).
[23] A. Z. Goldberg and K. Heshami, arXiv: 2106.03862 (2021).
[24] H. L. Zhang, H. C. Yuan, and X. X. Xu, Phys. Scr. 95, 045101 (2020).
[25] G. Pierobon, G. Cariolaro, and G. Dattoli, J. Math. Phys. 62, 082101 (2021).
[26] W. P. Schleich, Quantum Optics in Phase Space (WILEY-VCH Verlag GmbH, Berlin, 2001).
[27] Z. M. Liu and L. Zhou, Optik 142, 1 (2017).
[28] D. V. Sychev, A. E. Ulano, A. A. Pushkina, M. W. Richards, I. A. Fedorov, and A. I. Lvovsky, Nature Photon. 57, 379 (2017).
[29] R. J. Birrittella, P. M. Alsing, and C. C. Gerry, AVS Quantum Sci. 3, 014701 (2021).
[30] A. Z. Goldberg, A. B. Klimov, M. Grassl, G. Leuchs, and L. L. Sanchez-Soso, AVS Quantum Sci. 2, 044701 (2020).
[31] D. A. R. Dalvit, R. L. de Matos Filho, and F. Toscano, New J. Phys. 8, 276 (2006).
[32] N. Akhtar, B. C. Sanders, and C. Navarrete-Benlloch, Phys. Rev. A 103, 053711 (2021).
[33] M. Bergmann, and P. van Loock, Phys. Rev. A 94, 042332 (2016).
[34] A. L. Grimsmo, J. Combes, and B. Q. Baragiola, Phys. Rev. X 10, 011058 (2020).
[35] D. Sych and G. Leuchs, New J. Phys. 12, 053019 (2010).
[36] P. Papanastasiou and S. Pirandola, Phys. Rev. Res. 3, 013047 (2021).
[37] J. Lin, T. Upadhyaya, and N. Lutkenhaus, Phys. Rev. X 9, 041064 (2019).