Symmetry of stochastic non-variational differential equations

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Chapter 1

Getting started

1.1 Introduction and Background

Symmetry based methods to attack (in particular, nonlinear) deterministic differential equations were introduced long ago, and have witnessed a flourishing development – both in the theory and in concrete applications – in the last two decades.

Analogous methods for stochastic differential equations are much less widespread and actually at a less advanced stage of development. Lately there seem to be a growing interest in applying the symmetry approach to the study of stochastic differential equations.

The purpose of this text is to provide a short review – and a guide to the literature – of the topic directed to people familiar with stochastic differential equations (SDEs) but not with symmetry methods.\textsuperscript{1}

In order to keep the text short (and also in view of the fact symmetry considerations, in particular those related to the Noether theorem, are already routinely present in this specific field), I decided not to treat at all (symmetric) stochastic variational problems; some bibliographic notes are however provided on this topic too.

Symmetry attracted mathematicians since the very beginning of Mathematics; it just suffices to think of the study or regular polygons and polyhedra by the Greeks (and, of course, their development of spherical geometry, which they needed to sail across the Mediterranean \cite{185}).

However, for a long time Symmetry was confined to the study of geometrical objects and concepts. The way to apply symmetry to the study of analytical objects, and in particular equations, was paved by René Descartes (1596-1650) with his merging of geometry and analysis in “Géométrie Analytique”. In fact, Evariste Galois (1811-1832) introduced the concept of a group in the study of solutions (or impossibility of solution in algebraic terms) of algebraic equations.\footnote{A shorter review has been provided by E. Bibbona in a talk \cite{21}, but this has never been published as an article.}
The modern theory of Symmetry was laid down by Sophus Lie (1842-1899) [150, 151, 201]. Even in this case, the motivation behind the work of Lie was not in pure algebra, but instead in the effort to solve differential equations.

I will assume the reader knows some (very) basic notions about Lie groups, and illustrate how the theory of symmetry helps in determining solutions of (deterministic) differential equations, both ODEs and PDEs, staying within the classical theory (i.e. disregarding the many – and rather relevant – extension of it developed in the last two decades). I will then present some brief discussion of more and less recent attempts to extend this theory to the study of stochastic differential equations.²

As mentioned above this review is directed primarily to people familiar with stochastic equations, but not with symmetry methods for studying differential equations; thus while introducing the latter – and actually keeping their discussion to a rather basic level – I will assume the reader is familiar with Stochastic Differential Equations (in this case too we will need only the very basics of the theory). Here and there I will also point out some topics which are not central to the development of the theory but would deserve investigation and promise results.

I will adopt a rather “applied” point of view. Many of us are rather familiar with solving equations with spherical symmetry (e.g. the central force problem in Mechanics when thinking of ODEs, or the isotropic wave equation for PDEs), and we all know that the key step there is to pass to spherical coordinates. These should be seen as the most familiar example of symmetry-adapted coordinates, and I will advocate the idea (which is admittedly a XIX century one) that symmetry considerations should allow to determine symmetry-adapted coordinates even in cases where these are not immediately obvious.

A more modern and intrinsic approach to symmetry of differential equations is provided in many books, see e.g. [8, 60, 88, 131, 174, 175, 198], and the reader is referred to them for a complete exposition of the theory and of its applications.³

I should also mention that I will mainly speak of continuous symmetries. Discrete ones can also exist but, apart from cases where they are obvious (typically, reflection symmetries), they are difficult to detect. The reason is the one well understood by Lie: for continuous actions one can base considerations on the tangent space and thus reduce to linear problems, while for fully discrete actions linearization is not possible.⁴

²It is maybe worth mentioning that symmetry considerations for Markov processes made their way into the literature earlier on; see e.g. [105, 106, 149].
³I should add that the bibliography on symmetry methods for differential equations is by now rather extended; thus the selection of these books is just according to my personal taste and the need to provide a reasonably short list. Many other good books also exist (some of these will be quoted below for specific issues), and the reader can find them more suited to his/her taste.
⁴It should also be mentioned – and stressed – that we will consider symmetries for continuous equations. Discrete, or differential-difference, deterministic equations can and have been also studied in terms of symmetry [145, 147, 148], but we will not touch upon these. As far as I know, no such study has been performed for discrete, or differential-difference, stochastic
Symmetry of stochastic equations is by now not only a mathematical subject, but also an applied one. I will not speak of concrete applications, but I would like here to mention that current research topic include such diverse application as Fluid Mechanics [11, 68, 115, 116] and Financial Mathematics [9, 79, 104, 141, 142, 143, 144, 202].

Last but not least, another very important topic will be absent from my discussion (as specified also in the title). This is \textit{symmetry of variational problems}, i.e. the beautiful theory laid down by Emmy Noether (1882-1935) in 1915 (and published in 1918) [170], see also [127], and which played such a great role in the fundamental Physics of the second half of XX century\footnote{One often forgets that Noether was not motivated by Classical Mechanics, but by General Relativity; correspondingly, her original Theorem was of much wider scope than it is usually taught in the Mechanics courses. A nice discussion of Noether theorem is provided by Olver [174], while the work of Noether is discussed at length (and her original work provided in a reliable translation) together with its influence in the book by Kosmann-Schwarzbach [127].} – and earlier on in Mechanics. This is also developed in the stochastic framework, as we will briefly recall (mainly to point out some relevant literature) below; see in particular Remark 3.6.

1.2 Practicalities, notation, plan of the work

Equations and Propositions are numbered consecutively through the full body of the work; on the other hand, footnotes, Examples and Remarks are numbered by Chapter. The end of Remarks and Examples is marked respectively by the symbols \(\odot\) and \(\odot\). I have tried to avoid too many cross-references among Chapters, which in some case led to repeated equations.

Some (rather standard) notations are routinely used; thus \(\partial_i = \left(\frac{\partial}{\partial x^i}\right)\) throughout the work, and summation over repeated indices is always implied (if not explicitly stated otherwise).

As stated above, the work is primarily directed to people dealing with stochastic equations but not familiar with symmetry techniques. Thus Chapter 2 provides a quick but self-contained introduction to the latter (limiting of course to the basic aspects; references are provided for several further development). On the other hand, we assume reader are familiar with stochastic differential equations, and will not discuss these in their general features, going straight to their symmetry properties. Albeit investigation of symmetry aspects of SDEs started by considering equations in Stratonovich form, we will first consider equations in Ito form (Chapter 3) and only afterwards consider equations in Stratonovich form (Chapter 4); I think there is no need to justify giving a prominent role to Ito, i.e. properly defined, equations. We will deviate from the historical development of the subject also by putting in the same Chapter old and recent results. On the other hand we will devote a separate discussion
(Chapter 5) to a very recent development, i.e. random symmetries (of both types of equations).

Work on symmetry of SDEs concentrated so far to a large extent on what would be the proper definition of symmetry in this case, and how symmetries can be actually determined; this is also true of Chapters 3, 4 and 5 here. But one, in particular if not familiar with the symmetry approach, should also wonder what is the use of all this. Our final Chapter 6 provides a partial answer to this, illustrating several applications of symmetry considerations.

I would like to stress that the answer here is only partial not due to laziness by the author, but due to the fact the theory is under development. Actually, this is just the main motivation which led me to write these notes, in the hope they can attract new practitioners to this field and thus contribute somehow to the development of the field.

1.3 Acknowledgements

This work was stimulated by the workshop “Stochastics and Symmetry” (Milano, October 2015); I would like to thank the organizers, and in particular Sergio Albeverio, for their kind invitation.

My older works on symmetry of stochastic equations were performed in collaboration with prof. N. Rodriguez Quintero (now at Universidad de Sevilla), while some of the recent results reported here were obtained with F. Spadaro (now at EPFL Lausanne); I warmly thank both of them. I would also like to thank L. Peliti and A. Vulpiani for encouragement; C. Lunini for remarks and discussions about Section 4.4 and Chapter 5; and J.C. Zambrini for a critical reading of, and interesting remarks on, the whole manuscript.
Chapter 2

Deterministic equations

2.1 The geometry of differential equations

The key idea in describing the geometrical meaning of a differential equation (or better is associating a geometrical object to a differential equation) is to introduce jet spaces (or more precisely jet bundles) [8, 60, 88, 120, 131, 174, 175, 178, 188].

We will denote the space (the bundle) of independent and dependent variables as the phase space (phase bundle) $M$; note that in the case of ODEs we are actually referring to the complete phase space, rather than the reduced one (only dependent variables) which is often used in the case of autonomous dynamical systems. Here I will consider the case where the DEs (differential equations) at hand are defined in $\mathbb{R}^q$, with dependent variables taking also value in $\mathbb{R}^p$, and do not have boundary conditions, so that the geometry is the simplest possible one. (Hence I will speak of “spaces” rather than “manifolds” or “bundles”.) I will also assume, for ease of language and discussion, that the DEs under consideration do not involve non-algebraic functions of the derivatives.

The jet space can be thought of as the space of dependent ($u^1, \ldots, u^p$) and independent ($x^1, \ldots, x^q$) variables, together with the partial derivatives of the $u$ with respect to the $x$. In principle one can – and for certain questions (e.g. in the case of PDEs for some side conditions) should – consider the infinite order jet space [131, 174, 175, i.e. partial derivatives of all orders. But if we are dealing with a DE of order $n$, then for most questions it will suffice to consider the jet space of order $n$, $J^n M$ i.e. to consider partial derivatives of order $k \leq n$ only. These will be denoted as $u^a_J$, where $J = (J_1, \ldots, J_q)$ (here $j_i \geq 0$) is a multi-index of order $|J| = j_1 + \ldots + j_q$, and $u^a_J := \partial u^a / (\partial x_1^{j_1} \ldots \partial x_q^{j_q})$.

A differential equation (or system thereof) $\Delta$ is then a standard equation in $J^n M$, and hence it describes as usual a manifold in it; this is also called the solution manifold for $\Delta$, and denoted as $S_\Delta \subset J^n M$. This is a geometrical object, and we can now apply geometrical tools to study it. E.g., we can consider
maps or vector fields which leave them invariant.¹

There is of course a problem: the variables \( u^a_j \) represents derivatives of the \( u^a \) w.r.t. the \( x^i \), hence they are not really independent variables – albeit our description considered them as such. In order to take this fact into account, the jet space should be equipped with an additional structure, the contact structure. This is associated to the name of Elie Cartan (1869-1951) [45, 46, 47]; jet bundles are associated with the name of his pupil Charles Ehresmann (1905-1979) [80].

The information carried by the contact structure is that e.g. \( u^a_i \) is the derivative of \( u^a \) w.r.t. \( x^i \). This can be expressed by introducing the one-forms

\[
\omega^a := du^a - \sum_{i=1}^q u^a_i dx^i
\]

and looking at their kernel (or annihilator), i.e. the set of vector fields \( X \) such that \( \iota_X \omega = 0 \).

We will thus consider general (smooth) vector fields in \( M \), but as for vector fields in \( J^n M \) only those compatible with the contact structure will respect the special nature of the \( u^a_j \) variables.

Note that if we think of a vector field (VF) as an infinitesimal transformation of the \( x \) and \( u \) variables, once this is defined the transformations of the derivatives are also implicitly defined. This rather trivial observation can be made precise; the procedure of extending a VF in \( M \) to aVF in \( J^n M \) by requiring the preservation of the contact structure is also called prolongation. Correspondingly, there is a prolongation formula providing the components of the prolonged VF in terms of the components of the original one.

This is better expressed in recursive form, but we need first to introduce some notation. We will write

\[
\partial_i := \frac{\partial}{\partial x^i}, \quad \partial_a := \frac{\partial}{\partial u^a}, \quad \partial^a_j := \frac{\partial}{\partial u^a_j}.
\]

Thus a VF in \( M \) will be written in components as

\[
X = \xi^i(x, u) \partial_i + \varphi^a(x, u) \partial_a.
\]

Here and in the following, the Einstein summation convention will be routinely employed.

We will also write \( D_i \) for the substantial derivative w.r.t. \( x^i \), i.e.

\[
D_i = \partial_i + u^a_i \partial_a + u^a_{ij} \partial^a_j + \ldots,
\]

and denote by \( u^{[k]} \) the set of all the derivatives of the \( u \) of order \( k \).

A vector field in \( J^n M \) will then be written (with \( \psi^a_0 \equiv \varphi^a \)) as

\[
Y = \xi^i(x, u, \ldots, u^{[n]}) \partial_i + \sum_{|J|=0}^n \psi^a_J(x, u, \ldots, u^{[n]}) \partial^a_J.
\]

¹For obvious reasons, one is primarily interested in maps defined in the “physical” phase manifold and then lifted to the Jet manifold; as for the way of performing this lift, see below.
This is the prolongation of a vector field in $M$ if and only if
\[ \xi^i(x,u,...,u^{[n]}) = \xi^i(x,u) , \quad \psi^a_j(x,u,...,u^{[n]}) = \psi^a_j(x,u,...,u^{[J]}) \]
and moreover the components satisfy the prolongation formula
\[ \psi^a_{J,i} = D_i \psi^a_j - u^a_{J,k} D_i \xi^k(x,u) . \]
We will refer again to standard books [8, 60, 88, 131, 174, 175, 198] for its derivation.

A VF $X$ defined in $M$ is then an infinitesimal Lie-point symmetry (more precisely, the generator of a one-parameter local group of symmetries) if its prolongation, also written $X^{(n)}$, satisfies
\[ X^{(n)} : S_\Delta \rightarrow TS_\Delta . \]

This definition is consistent and precise; its drawbacks is that it only concerns the manifold $S_\Delta$, while when considering a DE we would usually be interested in its solutions. It turns out an equivalent characterization of symmetries of DE is to map solutions into (generally, different) solutions.

In order to understand this, we should characterize geometrically solutions to a given DE $\Delta$. A function $u = u(x)$ can also be seen as a section of the bundle $(M,\pi_0,X)$, i.e. of the phase manifold (or just space) seen as a bundle over the manifold (or just space) of independent variables. This is just the set of points
\[ \gamma_f = \{(x,u) : u = f(x)\} \in M . \]
This section is naturally lifted to sections $\gamma_f^{(k)}$ in Jet spaces, simply by computing derivatives. E.g., at first order we have
\[ \gamma_f^{(1)} = \{(x,u,u_x) : u = f(x) , \ u_x = f'(x)\} . \]
With this construction, a function $u = f(x)$ is a solution to $\Delta$ if and only if
\[ \gamma_f^{(n)} \subset S_\Delta \subset J^n M . \]
The details of the prolongation operation guarantee that $X^{(n)}$ will transform (locally) sections into sections, hence (if it is tangent to $S_\Delta$) solutions into solutions.

We conclude that indeed a symmetry maps solutions into (generally, different) solutions. In the case a solution is mapped into itself, we say it is an invariant solution.

It is immediately apparent from this discussion that a first use of symmetry can be that of generating new solutions from known ones.

This is surely of interest, but it is not the only way in which knowing the symmetry of a differential equation can help in determining (all or some of) its solutions, as we will discuss below; see sects.2.3 and 2.5.

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\(^2\)E.g., the fundamental solution of the heat equation can be generated in this way starting from the trivial (constant) solution; see [174] and Example 2.14 below.
Finally, we should mention that in some cases one considers *generalized symmetries*, corresponding to the action of generalized vector fields. These are vector fields acting in \( M \) but with coefficients which depend also on derivatives of the \( u \) with respect to the \( x \); thus they are properly defined only in the space of sections of the bundle, or in infinite order jet spaces. We will not enter in the details of how one makes sense of these, referring the reader to the literature [8, 60, 88, 131, 174, 175, 198], but some results in Sect. 2.3.2 and Sect. 4.2 will be stated in terms of these. In the rest of the paper we will just consider standard vector fields.

### 2.2 Determining the symmetry of a differential equation

Before discussing how symmetry are used, we should briefly discuss how one can determine the (continuous) symmetries of a given differential equation.\(^3\)

The first step is to consider a general VF of the form (2.3), with \( p \) and \( q \) (i.e. the number of dependent and independent variables) as suitable for the equation (or system thereof) under consideration; one should then apply the prolongation formula (2.7) and thus obtain the prolonged vector field \( Y = X^{(n)} \) corresponding to \( X \), see (2.5); here of course contact is made with (2.3) by choosing \( \psi^a = \varphi^a \). In this way we obtain a vector field which depends on the unknown functions \( \xi^i(x,u) \), \( \varphi^a(x,u) \).

We should then determine what are the conditions on these function which guarantee that (2.8) is satisfied. As \( S_\Delta \) is identified as the zero-level set of the function \( \Delta : J^s M \to \mathbb{R}^s \) (here \( s \) is the number of equations to be satisfied, i.e. the dimension of the system \( \Delta \)), we just have to apply \( Y \) (considered as a differential operator) on \( \Delta \) (considered as a set of functions) and require that \( Y(\Delta) = 0 \) whenever \( \Delta = 0 \).

Note that we could as well require that \( X(\Delta) = 0 \) in general; this will however be too strong a requirement (it corresponds to requiring that all the level set – not just the zero one – of \( \Delta \) are invariant), and in this case one also speaks of strong symmetries.

**Remark 2.1.** Actually a rather strict relation exists between general symmetries of a given equation and strong symmetries of some equivalent equation, as shown by Carinena, Del Olmo and Winternitz [37]; the reader is referred to their work for details.

When writing down the condition \( Y(\Delta) = 0 \) on \( \Delta = 0 \), we should recall that the \( (x,u,u_J) \) should be considered as independent variables; moreover all the dependencies on the \( u_J \) with nonzero \( J \) are completely explicit. Thus the vanishing of \( Y(\Delta^J) \) amounts to the vanishing *separately* of the coefficients of

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\(^3\)As mentioned above, determination of discrete symmetries – when not trivial – is in general a harder problem, and a non-algorithmic one (but see [99, 117, 118, 119] for constructive albeit non-general methods).
different monomials in the $u_J$. In this way we have a (usually, rather large) set of equations, known as the (symmetry) determining equations, to be satisfied by the unknown functions $\xi, \varphi$.

These are coupled PDEs, but the relevant point is that they are linear; this of course descends from the fact we are considering the infinitesimal action of vector fields, i.e. linearized transformations, and thus touches to the core of Lie theory.

Despite being in large number, these equations do have a hierarchical structure: those corresponding to monomials with high order derivatives of the $\xi, \varphi$ functions will be rather simple and are easily solved. One then has an ansatz for the $\xi, \varphi$, and further equations get simpler. Proceeding systematically one is often able to solve apparently very complex systems. When systems are too large – or the scientist too lazy – one can also resort to computer programs written in symbolic manipulation languages (e.g. the package symmgrp [51]; see also [112] and [113] for other software).

In fact, the key point here is that the solution of the determining equations, and hence the determination of the continuous Lie-point symmetries, is completely algorithmic.\(^4\)

Needless to say, it may also be rather complex computationally, so that before the introduction of symbolic manipulation computer languages their solution could be just too difficult in practice, albeit possible in principle.

Detailed examples of actual computations with solution of the determining equations for relevant differential equations are given in any book on symmetries of DEs, e.g. [8, 60, 88, 131, 174, 175, 198], and the reader is referred to them.

It is also worth mentioning that the concept of symmetry of a differential equation has been extended (from the Lie-point framework considered here) in several directions; this is not the place to discuss such extensions, for which the reader is – as usual – referred to the literature.\(^5\)

We will now suppose to have determined the symmetries of our differential equation $\Delta$, and turn to the problem of how to use it to obtain information on the solutions of $\Delta$.

As we will see in a moment, the key idea is the same for ODEs and PDEs, and amounts to the use of symmetry-adapted coordinates; but the scope of the application of symmetry methods is rather different in the two cases.

From now on, for the sake of discussion, I will restrict to the scalar case (a single equation for a single dependent variable); the general case is considered

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\(^4\)The exception to this is for dynamical systems, i.e. systems of first order ODEs (the reason is obvious: in this case we lack the above-mentioned hierarchical structure in the determining equations), discussed in Sect. 2.4 below. As suggested by Stephani [198], for these it is useful to use the theory of characteristics in reverse, i.e. transform the dynamical system into an equivalent first order quasilinear PDE. A different approach goes through combining symmetry analysis with perturbation theory [56, 57, 60], and in particular (Poincaré-Birkhoff) normal forms [14, 81, 124]; this should be compared with the approach to normal forms for stochastic differential equations, as also briefly mentioned later on (Sect. 5.1).

\(^5\)Readers interested in certain real-world applications can also be alerted about the extension of the symmetry approach to difference or differential-difference equations, see e.g. [145, 147, 148, 219].
Example 2.1. Let us consider the (linear, homogeneous) equation
\[ x^2 \frac{d^2 u}{dx^2} + x \frac{du}{dx} + u = 0 \] (2.12)
for \( u = u(x) \). We consider vector fields of the form
\[ X = \xi(x,u) \partial_x + \varphi(x,u) \partial_u ; \]
their second prolongation will be written in the form
\[ X^{(2)} = Y = \xi(x,u) \frac{\partial}{\partial x} + \varphi(x,u) \frac{\partial}{\partial u} + \psi(x,u,u_x) \frac{\partial}{\partial u_x} + \chi(x,u,u_x,u_{xx}) \frac{\partial}{\partial u_{xx}} ; \]
here the functions \( \psi \) and \( \chi \) are explicitly computed (in terms of the unknown functions \( \xi \) and \( \varphi \)) via the prolongation formula (2.7). We obtain
\[ \psi(x,u,u_x) = \varphi_x + u_x \varphi_u - u_x \xi - u_x^2 \xi_u , \]
and a more involved formula (which we do not report here) for \( \chi \).

Let us first look for strong symmetries of the equation (2.12). By applying \( Y \) on the equation \( \Delta \) we obtain an expression of the form
\[ Y(\Delta) = \alpha_0(x,u) + \alpha_1(x,u) u_x + \alpha_2(x,u) u_x^2 + \alpha_3(x,u) u_x^3 + \beta_1(x,u) u_{xx} + \beta_2(x,u) u_x u_{xx} , \] (2.13)
where \( \alpha_i \) and \( \beta_j \) are explicit functions of the unknown \( \xi \), \( \varphi \). As the dependencies on \( u_x \) and \( u_{xx} \) are explicit, we require the vanishing of the \( \alpha_i \) and \( \beta_j \). With straightforward algebra, we obtain that necessarily
\[ \xi(x,u) = c_1 x , \quad \varphi(x,u) = \varphi(x) \] (2.14)
where \( \varphi(x) \) is an arbitrary solution to (2.12) (as discussed in Sect. 2.6 below, this corresponds to the fact (2.12) is linear, and we will consider these symmetries as trivial\(^7\)); actually as (2.12) can be solved exactly, we have
\[ \varphi(x) = k_1 \sin(\log |x|) + k_2 \cos(\log |x|) . \]
Thus the only nontrivial symmetry is in this case generated by
\[ X = x \partial_x , \]

---

\( ^6 \)Also, several of our examples will concern linear equations; this is just for ease of discussion and get simple computations, but the theory is primarily concerned with nonlinear equations.

\( ^7 \)They are associated to the (linear) superposition principle for solutions to linear equations.
which corresponds to a scale transformation
\[ x \to \lambda x ; \]

needless to say, this symmetry was immediately apparent from the form of our equation (2.12).

Let us now look for standard (as opposed to strong) symmetries of the same equation (2.12). In this case we have to restrict \( Y(\Delta) \) to the solution manifold \( S(\Delta) \), which we do by substituting for \( u_{xx} \) according to (2.12); we obtain now
\[
[Y(\Delta)]_{S\Delta} = \gamma_0(x, u) + \gamma_1(x, u) u_x + \gamma_2(x, u) u_x^2 + \gamma_3(x, u) u_x^3 , \tag{2.15}
\]

where \( \gamma_i \) are explicit functions of the unknown \( \xi, \phi \). As the dependencies on \( u_x \) are explicit, we require the vanishing of the \( \gamma_i \). The computation is made easier by starting from \( \gamma_3 = 0 \) and \( \gamma_2 = 0 \), which allows to make also the dependencies on \( u \) fully explicit. Proceeding by standard computations, we obtain in the end that there are six symmetry generators, i.e.
\[
X_1 = x \partial_x , \\
X_2 = u \partial_u , \\
X_3 = x u \sin(\log |x|) \partial_x + u^2 \cos(\log |x|) \partial_u , \\
X_4 = x u \cos(\log |x|) \partial_x - u^2 \sin(\log |x|) \partial_u , \\
X_5 = 2 x \sin(\log |x|) \cos(\log |x|) \partial_x + u \cos(2 \log |x|) \partial_u , \\
X_6 = x \cos(2 \log |x|) \partial_x - 2 u \sin(\log |x|) \cos(\log |x|) \partial_u ,
\]

apart from symmetries of the form
\[ X_\phi = \phi(x) \partial_u \]

with \( \phi \) a solution to (2.12) (see above for these).

Example 2.2. Let us consider the ODE
\[
\left( \frac{d^2 u}{dx^2} \right) = - \sin(u) \tag{2.16}
\]

for \( u = u(x) \); this describes uniform (in space: here \( x \) actually represents time) solutions to the sine-Gordon equation (2.63) to be met later on. In this case there is only one symmetry generator, which is the obvious one, \( X = \partial_x \); this is actually a strong symmetry.\(^8\)

Example 2.3. Consider [198] the (second order, nonlinear) ODE
\[
\frac{d^2 u}{dx^2} = (x - u) \left( \frac{du}{dx} \right)^3 . \tag{2.17}
\]

\(^8\)In this and the following examples we do not provide details of the computations (which would be space-consuming) but the reader is invited to perform them in order to get accustomed with the approach.
This has the maximal number of symmetry generators for a second order ODE, i.e. eight, and hence an eight-parameter Lie symmetry group.\(^9\)

The generators are

\[
\begin{align*}
X_1 &= \frac{1}{2} \left[ (1 - 2u^2 + 4xu - 2x^2) \sin(u) + 2(x - u) \cos(u) \right] \partial_x + \left[ (x - u) \cos(u) \right] \partial_u , \\
X_2 &= \frac{1}{2} \left[ (2u^2 - 4xu + 2x^2 - 1) \cos(u) + 2(x - u) \sin(u) \right] \partial_x + \left[ (x - u) \sin(u) \right] \partial_u , \\
X_3 &= (x - u) \partial_x , \quad X_4 = \partial_x + \partial_u , \\
X_5 &= -\left[ \cos(u) \cos(u) + (u - x) \sin(u) \right] \partial_x - \cos^2(u) \partial_u , \\
X_6 &= \left[ \frac{1}{2} \left( (x - u) \cos(2u) + \sin(2u) \right) \right] \partial_x + \sin(u) \cos(u) \partial_u , \\
X_7 &= \cos(u) \partial_x , \quad X_8 = \sin(u) \partial_x .
\end{align*}
\]

\[\diamond\]

**Example 2.4.** The standard example for determination of symmetries of a PDE is the heat equation; this is discussed in virtually any book on symmetries of differential equations (see e.g. those mentioned above). We will refrain from considering it, referring the reader e.g. to [174]. \[\diamond\]

**Example 2.5.** We will instead consider the KdV equation \([35, 78]\)

\[
\frac{d}{dt}u + u_{xxx} + 6uu_x = 0 .
\]

(2.18)

We write vector fields in the form

\[
X = \xi(x,t,u) \partial_x + \tau(x,t,u) \partial_t + \varphi(x,t,u) \partial_u ,
\]

and proceed according to the general method. With some standard computations we get that there are four symmetry generators, given by

\[
\begin{align*}
X_1 &= \partial_x , \quad X_2 = \partial_t ; \\
X_3 &= x \partial_x + 3t \partial_t - 2u \partial_u , \quad X_4 = 6t \partial_x + \partial_u .
\end{align*}
\]

\[\diamond\]

### 2.3 Symmetry and ODEs

#### 2.3.1 Reduction

If we have an ODE \(\Delta\) of order \(n\) and this admits a Lie-point symmetry, the equation can be reduced to an equation of order \(n - 1\). The solutions to the\footnote{This is also the case for \(d^2u/dx^2 = 0\), in which case one gets the (eight parameters) group of projective transformations.}
original and to the reduced equations are in correspondence through a \textit{quadrature}, i.e., an integration; this of course introduces an integration constant, hence the correspondence is certainly not one-to-one.

It should be noted that in the case of multiple symmetries we do not always have as many reductions as symmetries: this depends on the Lie algebraic structure of the symmetry algebra – i.e. of the Lie algebra of the symmetry vector fields \cite{8, 60, 88, 131, 174, 175, 198}.

In this case one can describe the reduction procedure and the reason why it works in rather simple terms (the notation and discussion are simplified by dealing with scalar equations, as we do here).

We will start by considering an ODE of order \(n\), which we write quite generally as

\[
\Delta := F(x, u; u_x, u_{xx}, \ldots, u^{[n]}) = 0;
\] (2.19)

here \(F\) is a smooth function of its arguments.

Suppose we have determined a vector field (2.3) which is a symmetry of \(\Delta\), and assume moreover (for ease of discussion) it is actually a strong symmetry, i.e. \(X(\Delta) = 0\). We will then change variables via an invertible map

\[
(x, u) \rightarrow (y, z)
\] (2.20)

(here \(y\) should be thought of as the independent variable, \(z\) as the dependent one), so that in the new variables \(X\) is written as

\[
\hat{X} = \frac{\partial}{\partial z}.
\] (2.21)

(The notation \(\hat{X}\) should not cause confusion: \(X\) and \(\hat{X}\) are the same geometrical object, but expressed in different systems of coordinates; we introduce a different notation since the prolongation depends on the coordinates and on which variable is considered as the independent one, and we will need to prolong the vector field.)

By doing this we also have to write the equation in the new variables, which yields

\[
\Delta := G(y, z; z_y, z_{yy}, \ldots, z^{[n]}) = 0.
\] (2.22)

The detailed form of \(G\) will depend on the change of variables, but as both \(X\) and \(S_\Delta\) are geometrical object, the tangency condition (2.8) does not depend on the coordinates we are using. That is, we know that necessarily \(\hat{X}^{\mathrm{(n)}}(G) = 0\); on the other hand, the prolongation formula (2.7) guarantees that with \(\hat{X}\) as in (2.21), we just have \(\hat{X}^{(2)} = \hat{X} = \partial_z\).

But if this the case, it just means \(G\) does not depend on \(z\),

\[
G(y, z; z_y, z_{yy}, \ldots, z^{[n]}) = H(y; z_y, z_{yy}, \ldots, z^{[n]}).
\] (2.23)

\[10\] This is also related to a recent development, i.e. so called \textit{twisted symmetries} of differential equations \cite{166, 167, 168}; for these see also \cite{91, 92}. In this context one should also mention \textit{solvable structures} \cite{18, 111, 190}.
We can then perform a new change of coordinates (involving only the dependent coordinate)

\[ w := z_y. \]  

(2.24)

In these coordinates, we write the equation as

\[ H(y, w; w_y, ..., w_{[n-1]}) = 0. \]  

(2.25)

That is, we have reduced the equation to one of lower order.

Suppose now we are able to determine a solution \( w = h(y) \) to the reduced equation. This identifies solutions \( z = g(y) \) to the original equation (in “intermediate” coordinates) simply by integrating (2.24),

\[ z(y) = \int w(y) \, dy; \]  

(2.26)

note a constant of integration will appear here.

We can finally go back to the original coordinates; this is done simply by inverting the map (2.20).

It should be noted that the reduced equation could still be too hard to solve; the method can only guarantee that we are reduced to a problem of lower order, i.e. hopefully simpler than the original one.\(^{11}\)

We should also mention that other applications of the symmetry approach to ODEs are also possible; among many, we would like to recall here the study and determination of nonlinear superposition principle [16, 36, 38, 39, 40, 43], similar to the familiar one for the Riccati equation [44, 133, 182]. This is related to the well known Wei-Norman method [42, 53, 54, 216, 217]; the stochastic counterpart of this, including nonlinear superposition principle, has been studied in [139].

As mentioned above, standard symmetry techniques often fail in the (relevant!) case of dynamical systems. In this framework, one is often interested in the situation near a known solution, and this is investigated by perturbation techniques; most of these are based in a way or another on the approach pioneered by Poincaré and also known as the method of normal forms. The interplay between symmetry and perturbations (which is, as well known, of paramount importance in Quantum Mechanics [75, 136, 157, 205, 218]) has also been investigated in the literature, both in general and for the specific case of the normal forms approach [14, 60, 81, 124], see e.g. [94, 102, 103].\(^{12}\)

Example 2.6. Let us consider the equation

\[ x^2 u'' + xu' + u = 0. \]

\(^{11}\)It should be stressed that this is not the only possible strategy; in several situations, it is actually convenient to increase the dimension of the system, embedding a nonlinear problem into a linear one. This is done e.g. in solving the Calogero system [33, 35, 173], or more generally in the Kazhdan-Kostant-Sternberg approach [41, 125]. Also, one can try not to reduce (or increase) the order of the system, but take it into a more convenient form, e.g. to deal with an autonomous system [71, 76, 172].

\(^{12}\)See also the proceedings of the series of conferences on “Symmetry and Perturbation Theory” (SPT) [193]; and related volumes [194].
seen in Example 2.1 above; in this case the (linear) equation is easily solved, and the solutions are

\[ u(x) = c_1 \sin(\log |x|) + c_2 \cos(\log |x|) \, . \]

We will use symmetry to obtain these solutions, and we deal with the symmetry generated by

\[ X = x \partial_x \, . \]

The associated change of variables \((x, u) \to (y, z)\), putting the VF in the form \(X = \partial_z\), is

\[ x = e^y, \quad u = y; \quad \text{[with inverse \quad y = u, \quad z = \log(x)]} \, . \]

With these, and recalling \(y\) is the new independent variable, \(z\) the new dependent one, we have

\[ \begin{align*}
  du &= dy, \quad dx = e^y \, dz; \\
  \frac{du}{dx} &= 1/(e^y \, dz/dy); \\
  d^2u/dx^2 &= -(e^{-2z}/z_y)(1 + z_{yy}/z_y^2) ,
\end{align*} \]

and hence the equation is rewritten as

\[ - \left( \frac{1}{z_y} + \frac{z_{yy}}{z_y^3} \right) + \frac{1}{z_y} + y = 0 ; \]

passing now to the variable \(w = z_y\), we get the first order separable equation

\[ \frac{w_y}{w^3} = y . \]

This is immediately solved, yielding

\[ w(y) = \pm \frac{1}{\sqrt{k_1 - y^2}} \, . \]

Going back to the variable \(z\), we have to solve

\[ \frac{dz}{dy} = \pm \frac{1}{\sqrt{k_1 - y^2}} \, , \]

which provides

\[ z(y) = \pm \arctan \left( \frac{y \sqrt{k_1 - y^2}}{y^2 - k_1} \right) + k_2 \, . \]

Inverting the original change of coordinates, i.e. going back to the \((x, u)\) coordinates, this reads

\[ \log|\!|x|\!| = \pm \arctan \left[ \frac{u \sqrt{k_1 - u^2}}{u^2 - k_1} \right] + k_2 \, , \]
which is inverted to give

\[
\begin{align*}
\frac{u}{\sqrt{1 + \tan^2(k_2 - \log x)}} &= \pm \sqrt{k_1} \sin(k_2 - \log x) \\
\frac{u}{\sqrt{1 + \tan^2(k_2 - \log x)}} &= \pm \sqrt{k_1} [\sin(k_2) \cos(\log x) - \cos(k_2) \sin(\log x)] \\
&= c_1 \sin(\log x) + c_2 \cos(\log x).
\end{align*}
\]

Example 2.7. Let us consider the equation

\[ u'' = (x - u)(u')^3 \]

of Example 2.3 above. Among its eight symmetries, we consider

\[ X_3 = (x - u) \partial_x \]

and the associated change of variables

\[ t = u, \quad s = \log(x - u); \quad [u = t, \quad x = t + e^s] . \]

This entails

\[
\begin{align*}
\frac{du}{dx} &= \frac{1}{1 + e^s s_t} \\
\frac{d^2u}{dx^2} &= -\frac{e^s}{(1 + e^s s_t)^3} (s_{tt} + s_t^2).
\end{align*}
\]

In this way the equation reads

\[
\frac{d^2s}{dt^2} + \left(\frac{ds}{dt}\right)^2 + 1 = 0;
\]

this can be solved directly, yielding

\[ s(t) = \hat{k}_2 + \log[\cos(t - k_1)] ; \]

or we can complete our general procedure by passing to the dependent variable

\[ w := (ds/dt), \quad \text{in terms of which we get a first order (separable) equation,} \]

\[ \frac{dw}{dt} + w^2 + 1 = 0 . \]

This gives

\[ w(t) = -\tan(t - k_1) = \tan(k_1 - t) \]

and therefore we get \( s(t) \) as above.
Going back to the original variables, and writing $k_2 = \log(k_2)$, this reads
\[
\log(x - u) = \log(k_2) + \log[\cos(k_1 - u)] ,
\]
namely
\[
(x - u) = k_2 \cos(k_1 - u) ,
\]
which provides the solution (in implicit form) to our original equation, as can be checked by explicit computation.

\textbf{Example 2.8.} In this and the following example, taken from Olver [174], we will actually consider classes of (second order) ODEs. We start by considering a general autonomous second order equation,
\[
F(u, u_x, u_{xx}) = 0 .
\]  
(2.27)
This is invariant under $X = \partial_x$, i.e. translations of the independent variable. To let this fit into our scheme, we must therefore invert the role of dependent and independent variables; our change of coordinates will be
\[
x = z , \ u = y ;
\]
with this we get
\[
\frac{du}{dx} = \frac{1}{z_y} , \ \frac{d^2 u}{dx^2} = - \frac{z_{yy}}{z_y} \frac{dz}{dy} = - \frac{z_{yy}}{z_y^2} u_x = - \frac{z_{yy}}{z_y^2} .
\]
The original equation reads therefore
\[
F(y, \frac{1}{z_y}, - \frac{z_{yy}}{z_y}) := H(y, z_y, z_{yy}) = 0 .
\]  
(2.28)
with the new change of dependent variable $z_y = w$ this reads
\[
H(y, w, w_y) = 0 .
\]  
(2.29)

\textbf{Example 2.9.} Consider a general linear homogeneous second order equation,
\[
\frac{d^2 u}{dx^2} + p(x) \frac{du}{dx} + q(x) u = 0 .
\]  
(2.30)
Being linear, this is invariant under scale transformations in $u$, and these are generated by
\[
X = u \partial_u .
\]
The associated change of variables is
\[
x = y , \ u = e^z ; \ \text{[with inverse $y = x , \ z = \log(u)$]} .
\]
In these variables we have \( X = \partial_z \); moreover,
\[
u_x = e^z z_y, \quad u_{xx} = e^w (z_{yy} + z_y^2) .
\]
Therefore the equation reads now
\[
e^w [z_{yy} + z_y^2 + p(y) z_y + q(y)] = 0 .
\]
We can eliminate the factor \( e^w \) (which is never zero); by the usual change of dependent variable \( w = z_y \) we further rewrite the equation as
\[
w_y + w^2 + p(y) w + q(y) = 0 ;
\] (2.31)

\[\Delta\]

\[\Box\]

2.3.2 Conserved quantities

As well known, Noether theory \([127, 170, 174]\) provides the connection between symmetry and conservation laws (for PDEs) or directly conserved quantities (for ODEs), in the case of variational systems. However, even for non-variational systems of ODEs there are conserved quantities associated to (some) symmetries, as first noticed by Hojman \([114]\).

This fact played an important role in early investigation of symmetries of stochastic differential equations (see in particular the works by Misawa and by Albeverio and Fei, to be discussed in Sect. 4.2 below), and it is thus worth discussing it in some detail in the present context.

Following Hojman (and reverting to consider \( t \) as the independent variable, \( x^i \) as the dependent ones), we consider a system of second order ODEs for \( x^i(t) \in V \), with \( V \) an \( n \)-dimensional manifold; as the result is local, we can just consider \( V = \mathbb{R}^n \). Our system will be written, at least locally, in the form
\[
\ddot{x}^i = F_i(x, \dot{x}, t),
\] (2.32)

with \( F \) a smooth function (which will be required to satisfy some additional condition). We will consider symmetry (generalized) vector fields \([174]\) of the form
\[
X = \varphi^i(x, \dot{x}, t) \frac{\partial}{\partial x^i};
\] (2.33)

this might be the evolutionary representative \([174]\) of a vector field
\[
X_0 = \tau(x, t) \partial_t + \xi^i(q, x) \partial_{x^i},
\]
but this is not necessarily the case. 13

Note that the statement that \( X \) is a symmetry of (2.32) means in this case that its components satisfy
\[
D_t^2 (\varphi^i) = \varphi^j \frac{\partial F^i}{\partial x^j} + (D_t \varphi^j) \frac{\partial F^i}{\partial x^j} .
\] (2.34)

\[\footnote{Note that the evolutionary representative of \( X_0 \) is \( X_0 = (\xi^i - \tau \dot{x}^i) \partial_t \), which sets a severe limitation on the form of the \( Q^i \) in \( X \) for the latter to be in fact an evolutionary representative.}\]
This can be checked by direct explicit computation, and the same holds for the following proposition \[114\].

**Proposition 1.** Let \( X \) as in (2.33) be a symmetry of eq. (2.32), and let \( F \) in (2.32) satisfy

\[
\frac{\partial F^i}{\partial \dot{x}^i} = - D_t[\log(\lambda)] \tag{2.35}
\]

for some smooth function \( \lambda = \lambda(x) \). Then the quantity

\[
J_\lambda := \frac{1}{\lambda} \left[ \frac{\partial (\lambda \varphi^i)}{\partial x^i} + \frac{\partial [\lambda (D_t \varphi^i)\lambda]}{\partial \dot{x}^i} \right] \tag{2.36}
\]

is conserved under the flow of (2.32).

**Remark 2.2.** It should be noted that for \( \lambda \) constant, so that (2.35) reduces to

\[
\frac{\partial F^i}{\partial \dot{x}^i} = 0 \tag{2.37}
\]

the conserved quantity identified by Proposition 1 is just

\[
J_0 := \frac{\partial \varphi^i}{\partial x^i} + \frac{\partial (D_t \varphi^i)}{\partial \dot{x}^i}. \tag{2.38}
\]

\(\Diamond\)

**Example 2.10.** Consider a harmonic oscillator (which of course would admit a variational description) in polar coordinates \((r, \theta)\) \[114\], so that (2.32) is now

\[
\ddot{r} = - \omega^2 r + r \dot{\theta}^2,
\]

\[
\ddot{\theta} = - \frac{2}{r} \dot{r} \dot{\theta}.
\]

The (generalized) vector field

\[
X = r^3 \dot{\theta} \partial_r
\]

is a (generalized) symmetry for these equations. On the other hand, eq. (2.35) is satisfied by choosing \( \lambda = r^2 \). In this case one obtains

\[
J_\lambda = 6r^2 \dot{\theta},
\]

which is proportional to the angular momentum (which is of course itself conserved). \(\Diamond\)
2.4 Dynamical systems

I declared above that I would consider scalar equations; an exception is however in order, i.e. to consider dynamical systems. By this we mean systems of first order ODEs of the form

\[ \frac{dx^i}{dt} = f^i(x, t) . \] (2.39)

These may represent equations in \( M_0 = \mathbb{R}^n \), or the \( x^i (i = 1, ..., n) \) can be more generally local coordinates on a \( n \)-dimensional manifold \( M_0 \), the restricted phase manifold; we will also consider \( M = M_0 \times \mathbb{R} \) (the second factor representing time), the phase manifold.\(^{15}\)

Note that (unless we consider problem focusing on the zeroes of \( f \), as e.g. in normal forms theory \([14, 81, 124]\)) we can always reduce to considering autonomous dynamical systems, i.e. to the case of

\[ \frac{dx^i}{dt} = f^i(x) \] (2.40)

simply by adding a new variable \( x^0 \) with evolution equation \( dx^0/dt = 1 \).

Moreover, in applications one is quite often concerned from the beginning with autonomous systems.

Here we will just give some basic result, also in order to ease comparison with the results for stochastic dynamical systems to be considered later on. A specific discussion of symmetries for dynamical systems is provided e.g. in \([60]\) (see in particular Chapter III there), to which the reader is referred for further detail.

2.4.1 Symmetry of dynamical systems

To the dynamical system (2.40) is naturally associated a vector field in \( M_0 \), i.e.

\[ X_f = f^i(x) \partial_i ; \] (2.41)

note that for a general dynamical system (2.39) we need to consider vector fields defined in \( M \), i.e.

\[ X_f = f^i(x, t) \partial_i . \]

We can also associate to the same dynamical system (2.39) or (2.40) a full dynamical vector field (always defined in \( M \), taking into account also the flow of time; this reads

\[ Z_f = \partial_t + f^i(x, t) \partial_i . \] (2.42)

When looking for symmetries of a dynamical system, one should consider general vector fields of the form

\[ X = \tau(x, t) \partial_t + \varphi^i(x, t) \partial_i . \] (2.43)

\(^{14}\)Note that, at difference with other sections but conforming to the general use in the literature, here we denote the independent variable as \( t \) and the dependent ones as \( x^i \).

\(^{15}\)In the dynamical systems literature these are sometimes referred to as, respectively, the phase manifold and the augmented (or extended) phase manifold.
The general procedure, described in Sect. 2.2, would produce an under-determined system of \( n \) equations for the \( n+1 \) functions \((\tau; \varphi^1, ..., \varphi^n)\); for (2.40) these read

\[
\frac{\partial \varphi^i}{\partial t} + f^j \frac{\partial \varphi^i}{\partial x^j} - \varphi^j \frac{\partial f^i}{\partial x^j} = \left( \frac{\partial \tau}{\partial t} + f^j \frac{\partial \tau}{\partial x^j} \right) f^i .
\] (2.44)

In this context, it is quite natural to look for more restricted classes of symmetries: that is,

(i) those which act on \( t \) just by a reparametrization (that is, with \( \tau = \tau(t) \) only), also designed as fiber-preserving symmetries;

(ii) those which do not act on \( t \) (automorphisms, possibly time-dependent, of \( M_0 \): that is, with \( \tau = 0 \)), also designed as time-preserving symmetries;

(iii) or even which neither act nor depend on \( t \) (automorphisms of \( M_0 \): that is, with \( \tau = 0, \varphi^i_t = 0 \)); these are also designed as Lie point time-independent (LPTI) symmetries.

In discussing these, and more generally symmetries of dynamical systems, it is useful to introduce the Lie-Poisson bracket of two \((C^\infty)\) functions defined on \( M_0 \) (which is again a \( C^\infty \) function on \( M \)). This reads

\[
\{f,g\} := (f \cdot \nabla)g - (g \cdot \nabla)f ; \tag{2.45}
\]

in components, we have

\[
\{f,g\}^i = f^j \frac{\partial g^i}{\partial x^j} - g^j \frac{\partial f^i}{\partial x^j} . \tag{2.46}
\]

The bracket is obviously antisymmetric, \( \{g,f\} = -\{f,g\} \), and satisfies the Jacobi identity.

This bracket has an immediate relation with the commutator of the vector fields \( X_f \) and \( X_g \) associated to the functions \( f \) and \( g \). In fact,

\[
[X_f,X_g] = [f^j \partial_j, g^m \partial_m] = [f^j (\partial_j g^i) - g^i (\partial_j f^j)] \partial_i = \{f,g\}^i \partial_i = X_{\{f,g\}} .
\]

In other words,

\[
\{f,g\} = h \iff [X_f,X_g] = X_h .
\]

With this notation, we have

**Proposition 2.** The general determining equations (2.44) for symmetries of an autonomous dynamical system (2.40) read

\[
\varphi^i_t + \{f,\varphi\}^i = (Z_f \tau) f^i . \tag{2.47}
\]

For symmetries of class (i) above these reduce to

\[
\varphi^i_t + \{f,\varphi\}^i = \tau_t f^i . \tag{2.48}
\]
for class (ii) we just have
\[ \phi^i_t + \{f, \phi\}^i = 0 ; \tag{2.49} \]
and for LPTI, i.e. class (iii), symmetries we get
\[ \{f, \phi\}^i = 0 . \tag{2.50} \]

Note that (2.48) can be reduced to the form (2.49) by defining
\[ \psi = \phi - \tau f ; \]
in fact with this (2.48) just reads
\[ \psi^i_t + \{f, \psi\}^i = 0 . \tag{2.51} \]

In the following, we will denote by \( G_f \) the set of time-preserving symmetries for \( f \), that is of vector fields \( X = \varphi'(x,t)\partial_t \) satisfying (2.49).

### 2.4.2 Constants of motion, and the module structure

Assume now that the dynamical system (2.40) admits a conserved quantity (or constant of motion, or first integral) \( \alpha \), i.e. a smooth function \( \alpha : M \rightarrow \mathbb{R} \) such that \( \mathcal{Z}_f(\alpha) = 0 \); we will denote by \( I_f \) the set of these functions (clearly sums and products of such functions still give functions in \( I_f \)). In the case of \( \alpha \) not depending on time, this also reads \( X_f(\alpha) = 0 \). In this case, if \( X \) is a (time-preserving) symmetry for (2.40), then \( \tilde{X} = \alpha X \) is also a symmetry for the same dynamical system. In fact,
\[
\tilde{\varphi}^i_t + \{f, \tilde{\varphi}\}^i = \partial_t (\alpha \varphi^i) + \{f, \alpha \varphi\}^i = \alpha \varphi^i_t + \alpha \varphi^i_t + \alpha \{f, \varphi\}^i + f^j \partial_j (\alpha) = \alpha (\varphi^i_t + \{f, g\}^i) + \mathcal{Z}_f(\alpha) = 0 .
\]

We conclude that the set \( G_f \) of (time-preserving) symmetry generators for a given dynamical system has, beside the structure of algebra, also the structure of a module over \( I_f \). More precisely, we have the following [212]

**Proposition 3.** The set \( G_f \) is a finitely generated module over \( I_f \).

**Remark 2.3.** In the presence of nontrivial \( I_f \), the set \( G_f \) will be infinite dimensional as a Lie algebra, and finite dimensional as a Lie module.

**Remark 2.4.** Our discussion, and Proposition 3, also hold for the set \( \Gamma_f \) of general symmetries for (2.39), i.e. of vector fields of the form (2.43) satisfying
(2.44). In fact, consider the vector field $\tilde{X} = \alpha X$ as above, and assume $X$ is a symmetry for (2.39). Then (2.44) yields
\[
\frac{\partial \tilde{\varphi}^i}{\partial t} + f^j \frac{\partial \tilde{\varphi}^i}{\partial x^j} - \tilde{\varphi}^j \frac{\partial f^i}{\partial x^j} - \left( \frac{\partial \tau}{\partial t} + f^j \frac{\partial \tau}{\partial x^j} \right) f^i
\]
\[
= \alpha \left[ \frac{\partial \varphi^i}{\partial t} + f^j \frac{\partial \varphi^i}{\partial x^j} - \varphi^j \frac{\partial f^i}{\partial x^j} - \left( \frac{\partial \tau}{\partial t} + f^j \frac{\partial \tau}{\partial x^j} \right) f^i \right]
\]
\[
+ \alpha \varphi^i + \varphi^j f^j \partial_j (\alpha) - \tau (\alpha + f^j \partial_j (\alpha)) f^i
\]
\[
= (\varphi^i - \tau f^i) Z_f (\alpha) = 0 ;
\]
the term in square brackets was cancelled since it is zero by the assumption that $X$ is a symmetry.

Example 2.11. Consider the simple system
\[
\frac{dx}{dt} = [1 + \exp[-(x^2 + y^2)]] y
\]
\[
\frac{dy}{dt} = - [1 + \exp[-(x^2 + y^2)]] x .
\]
This obviously admits $X_0 = y \partial_x - x \partial_y$ (i.e. rotations) as a symmetry, and $\rho = (x^2 + y^2)$ as a conserved quantity. In fact, it is easy to check that any vector field of the form
\[
X_k = \rho^k X_0
\]
is also a symmetry.

2.4.3 Orbital symmetries

It is well known that studying dynamical systems it is often fruitful to focus on trajectories (also called solution orbits) rather than on full solutions (which include the law of displacement along trajectories, i.e. the time parametrization of solution curves).

Thus it appears one could have some advantage in considering, beside full symmetries (which map solutions into solutions) and invariant full solutions, also so called orbital symmetries and invariant trajectories. This approach was pursued in particular by Walcher [214, 215] (see also [62]), and here we will give some basic notions about it.

First of all we should characterize dynamical systems of the form (2.40) which have the same solution orbits, i.e. introduce an equivalence relation in the set of systems of this form. It is clear that (2.40) will have the same trajectories as any system of the form (where $\mu (x) \neq 0$ whenever $f(x) \neq 0$)
\[
\frac{dx^i}{dt} = \mu (x) f^i (x) ;
\]
(note that one could also consider a multiplier $\mu$ depending on time as well, but this would lead us into the realm of time-dependent dynamical systems (2.39).
Now we recall that we defined symmetries of an equation $\Delta$ as vector fields $X$ such that their prolongation (in this case, $X^{(1)}$) leaves the solution manifold $S_\Delta$ invariant. Dealing with equivalence classes of vector fields, we should require that $X^{(1)}$ maps $S_\Delta$ into a possibly different manifold $S_{\tilde{\Delta}}$ which is the solution manifold for a possibly different equation $\tilde{\Delta}$, in the same equivalence class as $\Delta$.

By standard computations, we obtain that (the first prolongation of) a generic vector field

$$X = \tau(x,t) \partial_t + \varphi^i(x,t) \partial_i$$

maps the dynamical system (2.40) into a new dynamical system

$$dx^i/dt = g^i(x,t)$$

with

$$g^i = f^i - \varepsilon \left[ \varphi^i_t + \{ \varphi, f \}^i - (D_t \tau) f^i \right].$$  (2.54)

For what we have seen above, the new system is in the same equivalence class as the old one if and only if $g$ is proportional to $f$ through a scalar function, i.e. if and only if there is a function $\mu$ such that

$$\varphi^i_t + \{ \varphi, f \}^i - (D_t \tau) f^i = \mu(x,t) f^i.$$  (2.55)

As the last term in the l.h.s. is surely of the required form (that is, proportional to $f$ and thus can be absorbed into the definition of $\mu$), we are reduced to study the equation

$$\varphi^i_t + \{ \varphi, f \}^i = \mu(x,t) f^i.$$  (2.56)

Note that this is the same as the equation determining standard (time-preserving) symmetries, except that in that case we required $\mu \equiv 0$.

We would like to recall an equivalent (local) characterization of the equivalence relation we are considering here; this is taken from [62] (see also [58, 59] for somewhat related results).

**Proposition 4.** Two dynamical systems are locally orbit-equivalent if and only if they admit the same first integrals, and hence the same invariant sets, near any non-stationary point.

### 2.5 Symmetry and PDEs

In the case of PDEs we will, for the sake of definiteness, consider an equation with two independent variables $(x,t)$ and a dependent one, $u$. Again to keep things simple we will focus on the case of a second order equation, say

$$\Delta := F(x,t,u; u_x, u_t, u_{xx}, u_{xt}, u_{tt}) = 0.$$  (2.57)

The vector fields can in this framework be written as

$$X = \xi(x,t,u) \partial_x + \tau(x,t,u) \partial_t + \varphi(x,t,u) \partial_u.$$  (2.58)
we refrain from writing down explicitly its second order prolongation $Y = X^{(2)}$ (see e.g. [88, 174] for the explicit expression).

The approach in the case of PDEs is in a way at the opposite as the one for ODEs. That is, if we have determined a vector field of the form (2.58) which is a symmetry for $\Delta$, we will perform a change of coordinates

$$ (x, t; u) \rightarrow (y, s; v) $$

(2.59)

(here $v$ should be thought as the new dependent variable) such that in the new coordinate the VF $X$ reads as\(^{16}\)

$$ \tilde{X} = \frac{\partial}{\partial y} ; $$

(2.60)

that is, it should be all along one of the independent coordinates (as opposed to the ODE case, where it was set to be along the dependent one).

In the new coordinates, the equation will be written as

$$ \tilde{\Delta} := G(y, s, v; v_y, v_s, v_{yy}, v_{ys}, v_{ss}) = 0 . $$

(2.61)

Again the tangency condition (2.8) holds independently of the used coordinates, and again prolongation formula guarantees that $\tilde{X}^{(2)} = \tilde{X}$.

Now our goal will not be to obtain a general reduction of the equation, but instead to obtain a (reduced) equation which determines the invariant solutions to the original equation (by this we mean $X$-invariant, of course).

In the new coordinates, this is just obtained by imposing $v_y = 0$, i.e. $v = v(s)$. Note that the reduced equation will have (one) less independent variables than the original one. Needless to say, this is specially good when we started from a PDE with two independent variables, but it is useful in general. On the other hand, this reduced equation will not have solutions in correspondence with general solution to the original equation: only the invariant solutions will be common to the two equations (note also that in this case, contrary to the ODE case, we do not need to solve any “reconstruction problem”).

In this sense, symmetry is just providing a way to make educated guesses (or educated ansatzes) about the functional dependence of some classes of special solutions.\(^{17}\)

**Remark 2.5.** It should be mentioned that one can also determine invariant solutions under symmetry groups which are not symmetries of the equation, see e.g. the approach by Levi and Winternitz in terms of “conditional symmetries” [146, 177] (these are also related to the “non-classical method” [66, 176]; see [181] for a discussion of such relation) or more generally the so called “partial symmetries” [61]. The latter also admit an asymptotic formulation [93, 96, 97].

---

\(^{16}\)Again the notation $\tilde{X}$ should not cause confusion, this is the same geometrical object as $X$ expressed in different coordinates.

\(^{17}\)It will of course provide much more, but we do not want to enter in the details of the theory [8, 60, 88, 131, 174, 175, 198].
Remark 2.6. Here we only consider standard Lie-point symmetries, i.e. those corresponding to standard vector fields. These have been generalized in many ways (which cannot be discussed here), including generalized symmetries [8, 60, 88, 131, 174, 175, 198], the non-classical method [65, 66, 67, 176] potential symmetries [22, 28, 181, 186, 226], and nonlocal ones [1, 10, 132] (also in relation [48, 49, 50] to so called solvable structures [18, 111, 190]); in these cases the transformations considered are not generated by standard vector fields (the names correspond to the kind of transformations considered).

More recently so called “twisted symmetries” have been introduced by Muriel and Romero [166, 167, 168], see also [48, 91, 92]; in this case one considers standard vector fields but a deformation of the prolongation operation. The interested reader is referred to the literature for more details. As far as I know, no corresponding extensions exist for stochastic equations.

Example 2.12. As mentioned above, symmetries can be used to look for invariant solutions to a given PDE in terms of a simpler (reduced) equation; in particular this might be an ODE.

We will consider the KdV equation (2.18); its symmetries were computed in Example 2.4 above. In particular we will look at solutions invariant under $X = X_2 - vX_1$; these are obviously functions of the form $u(x,t) = \eta(x - vt) : = \eta(z)$, i.e. travelling waves (with speed $v$). Inserting this ansatz into the KdV equation, we get a reduced (ordinary) equations, which reads

$$-v\eta_z + \eta_{zzz} + 6\eta \eta_z = 0 .$$

(2.62)

In this way we achieved our task (reduction to an equation with less independent variables). We also note this is immediately integrated once, yielding (the $k_i$ will be integration constants)

$$-v\eta + \eta_{zz} + 3\eta^2 + k_1 = 0 .$$

Multiplying this by $\eta_z$ and integrating again, we get

$$-\frac{v}{2} \eta^2 + \frac{1}{2} \eta_z^2 + \eta^3 + k_1 \eta + k_2 = 0 .$$

This equation is solved in terms of special (elliptic) functions; see e.g. [174] (example 3.4) for details. For $k_1 = k_2 = 0$ we get

$$\eta(z) = \frac{v}{2} \frac{1}{\cosh^2(\beta)} ,$$

where $\beta = -(\sqrt{v}/2)(k_3 \pm x)$. This is of course the well known one-soliton solution; the simplest writing is obtained by choosing $v = 2$ and $k_3 = 0$ (the sign in $\beta$ is then inessential, due to parity of $\cosh^2(x)$), in which case we get

$$\eta = \frac{1}{\cosh^2(x)} .$$
Example 2.13. As a second example of this procedure, consider the sine-Gordon equation
\[ u_{tt} - u_{xx} = -\sin(u) \]  
(2.63)
This is autonomous, hence surely invariant under both of
\[ X_1 = \partial_t, \quad X_2 = \partial_x \]
and any linear combination thereof (reduction under \( X_2 \) gives – upon a slight change of notation – eq. (2.16), considered in Example 2.2 above). Looking for solutions invariant under \( X = X_2 - vX_1 \) amounts to looking for travelling waves with speed \( v \). Writing
\[ u(x, t) = \eta(x - vt) := \eta(z) \]
the sine-Gordon equation is reduced to
\[ \frac{d^2 \eta}{dz^2} = \frac{1}{1 - v^2} \sin(\eta), \]  
(2.64)
i.e. to the motion of a particle of unit mass in an effective potential
\[ W(\eta) := \frac{1}{1 - v^2} \cos(\eta). \]  
(2.65)
Note that this is qualitatively different depending on \( v^2 < 1 \) or \( v^2 > 1 \); in particular it turns out that nontrivial solutions will not be able to satisfy the natural boundary conditions (inherited from a finite energy condition) in the case \( v^2 > 1 \) [31, 32].

Example 2.14. The one-dimensional heat (or diffusion) equation
\[ u_t = u_{xx} \]  
(2.66)
has a symmetry algebra spanned (beside the infinite factor related to its linear character, see Sect. 2.6) by six vector fields:
\[
\begin{aligned}
X_1 & = \partial_x, \quad X_2 = \partial_t, \quad X_3 = u \partial_u, \quad X_4 = x \partial_x + 2t \partial_t; \\
X_5 & = 2t \partial_x - x u \partial_u, \quad X_6 = 4t \partial_t + 4t x \partial_x - (x^2 + 2t) u \partial_u;
\end{aligned}
\]
Note the first four generators correspond to rather obvious (translation or scaling) symmetries, \( X_5 \) is related to Galilean boosts to moving coordinate frames (see [174], example 2.41), while \( X_6 \) is nontrivial.

There is no doubt that \( u = c \) is a (highly trivial!) solution to the heat equation; on the other hand if we act on this by \( X_6 \), it gets transformed into the one-parameter family of nontrivial solutions
\[ u(x, t; s) = \frac{c}{\sqrt{1 + 4st}} \exp\left[-s \frac{x^2}{1 + 4st}\right], \]
where \( s \) is the group parameter.

Choosing \( s = \pi c^2 \) we get

\[
    u(x, t) = \frac{c}{\sqrt{1 + 4\pi c^2 t}} \exp \left[ -\pi c^2 \frac{x^2}{1 + 4\pi c^2 t} \right];
\]

by a time translation (generator \( X_2 \)) of an amount

\[
    \delta t = \left( \frac{1 - c}{c} \right) t - \frac{1}{4c^2 \pi}
\]

this is transformed into the fundamental solution

\[
    u(x, t) = \frac{1}{\sqrt{4\pi t}} \exp \left[ -\frac{x^2}{4t} \right].
\]

Thus by (two) symmetry transformations we have mapped the trivial solution \( u = c \) into the fundamental solution of the heat equation.

\[\Diamond\]

**Example 2.15.** The sine-Gordon equation (2.63) is invariant under the Lorentz group (with the units used in (2.63), the limit speed is \( c = 1 \)). Thus we can determine static solutions \( u(x, t) = \eta(x) \) (as suggested by our notation, these are obtained as in Example by setting \( v = 0 \)) and then transform them into moving solution by a Lorentz boost. More generally, this applies to any Lorentz-invariant equation.

\[\Diamond\]

### 2.6 Symmetry and linearization

It is interesting to note that, as shown by Bluman and Kumei [25, 134] (see also [23, 26, 27]), the (algorithmic) symmetry analysis is also able to detect if a nonlinear equation can be linearized by a change of coordinates.\(^\text{18}\)

The reason can be made quite clear intuitively: a linear equation has a rather large symmetry algebra, corresponding to the (linear) superposition principle. That is, we know *a priori* that any transformation adding to the independent variable an arbitrary solution of the equation itself, or actually a linear combination of solutions, will map solutions into solutions.

On the other hand, such a property cannot be destroyed by a change of coordinates. Thus if an equation is linear in some coordinates but is expressed in a different system of coordinates (in which it is nonlinear), there will be trace

\(^{18}\)In the Calogero classification of integrable systems [34, 35], these would be \( C \)-integrable; it is remarkable that \( S \)-integrable equations are also characterized by (generalized) symmetry. We cannot touch upon this topic here, and the reader is just referred e.g. to (chapter 5 in) the book by Olver [174], or to [35, 41, 78].
of it being linearizable. In fact, the trace will show up in the symmetry computation in that the determining equations will admit, beside other solutions, vector fields of the form

$$X_\alpha = \alpha(x,t) \partial_u,$$  \hspace{1cm} (2.67)

with $\alpha(x,t)$ an arbitrary solution of some linear equation. Then the nonlinear equation under study can be transformed exactly to this linear equation (the symmetry also gives a hint about how to obtain this, i.e. about the required transformation).\(^{19}\)

**Example 2.16.** The Burgers equation

$$v_t = v_{xx} + 2vv_x = D_x(v_x + v^2)$$

can be rewritten in “potential form” setting (note this will introduce an integration to go back to the original coordinates) $v := u_x$; it reads then

$$u_t = u_{xx} + u^2_x.$$  

This equation admits six symmetries (which correspond to those of the heat equations) plus a family of the form mentioned above, i.e. symmetry VF s of the form

$$X = \alpha(x,t) e^{-u} \partial_u.$$  

To eliminate the factor $e^{-u}$, it suffices to set $u = \log w$, i.e. $w = e^u$. With this, we get

$$X_\alpha = \alpha(x,t) \partial_w.$$  

Actually, writing the equation in terms of $w$ we get, in view of

$$u_t = \frac{1}{w} w_t, \quad u_x = \frac{1}{w} w_x, \quad u_{xx} = \frac{w_{xx} w - w^2_x}{w^2},$$  

just the heat equation. Thus the symmetry analysis led us to “discover” the well known Hopf-Cole transformation [24, 174].  \(\diamondsuit\)

---

\(^{19}\)This question has also been studied in the context of perturbation theory for dynamical systems [17, 98].
Chapter 3

Ito stochastic equations

3.1 Symmetry, SDEs, diffusion equations

It is now time to come to the central issue of interest in this paper, i.e. consider stochastic differential equations (SDEs) \([12, 19, 83, 85, 109, 122, 123, 135, 153, 171, 200, 208]\) and their symmetries. Albeit the early works on this matter \([3, 161, 162, 163, 164, 165]\) considered Stratonovich equations\(^1\), I will first (and mainly) focus on SDEs in Ito form,

\[
\begin{align*}
\text{dx}^i &= f^i(x,t) \, dt + \sigma^i_j(x,t) \, dw^j; \\
\end{align*}
\]

(3.1)

note that I will only consider ordinary (as opposed to partial\(^2\)) SDEs.

In (3.1), as customary, \(f^i\) and \(\sigma^i_j\) are smooth functions, \(\sigma\) a nonzero matrix, and the \(w^k\) are independent homogeneous standard Wiener processes, satisfying

\[
\langle |w^i(t) - w^j(t)|^2 \rangle = \delta^{ij} \delta(t-s). 
\]

(3.2)

It is well known that to the Ito equation (3.1) is associated a diffusion (Fokker-Planck or Chapman-Kolmogorov\(^3\)) equation, which reads

\[
\begin{align*}
\frac{\partial u}{\partial t} + A^{ij} u_{ij} + B^i u_i + C u &= 0, \\
\end{align*}
\]

(3.3)

where \(u_i = (\partial u/\partial x^i)\) (and similarly for \(u_{ij}\)), while the coefficients \(A, B, C\) are functions of the independent variables \((x, t)\), given by

\[
\begin{align*}
A^{ij} &= -\frac{1}{2} (\sigma \sigma^T)^{ij} , \\
B^i &= f^i - \partial_j (\sigma \sigma^T)^{ij} , \\
C &= (\partial_i \cdot f^i) - \frac{1}{2} \partial_{ij}^2 (\sigma \sigma^T)^{ij} . \\
\end{align*}
\]

\(^1\)These will be considered later on, in Chapter 4.

\(^2\)For an attempt at considering symmetry of partial SDEs, see [156].

\(^3\)In the following we will always refer to this as the Fokker-Planck (FP) equation, conforming to the Physics community notation.
Remark 3.1. Here I will consider only (infinitesimal generators of) continuous symmetries. Discrete symmetries of stochastic equations are considered e.g. in the Appendix to [89].

Remark 3.2. As the Fokker-Planck equation (3.3) is linear, it will have the symmetries of the form $X_\alpha$ with $\alpha$ an arbitrary solution of (3.3) itself, see Sect. 2.6. We will consider these as trivial symmetries in our context.

Remark 3.3. A more careful analysis would uncover a delicate question: when dealing with the Fokker-Planck equation in a probabilistic context, we are interested only in solutions $u(x,t)$ which satisfy the normalization condition
\[ \int_{-\infty}^{+\infty} u(x,t) \, dx = 1 \quad \forall t . \]
This condition does not correspond to a linear space, and hence the linear superposition principle does not apply here. Correspondingly, the symmetries $X_\alpha$ mentioned in Remark 3.2 should not be considered as acceptable in the present context (see [100], appendix B, for details). Thus, excluding them from consideration is correct, and not only for their “trivial” nature.

Remark 3.4. In the following we will consider transformations acting also on the independent variable $t$; it should be recalled that this will also have an effect on the Wiener processes $w^i(t)$. This is discussed in very simple terms in [100] (see Appendix A there). In particular, if $t \to s = t + \varepsilon \tau(t)$, then
\[ w^i(t) \to \bar{w}^i(s) = \sqrt{1 + \varepsilon \tau'(s)} \, w^i(s) ; \]
this implies in particular
\[ d\bar{w}^i = (1 + \varepsilon'(2/d\tau)) \, dw^i . \]
More generally, it is known (see e.g. [171], Theorem 8.20; or [210] chap.4) that under a general time change $t \to s = t + \varepsilon \tau(x,t)$ we have
\[ w^i(t) \to \bar{w}^i(s) = \sqrt{1 + \varepsilon(d\tau/dt)} \, w^i(s) , \quad (3.4) \]
which of course implies (for $(d\tau/dt)$ limited, as we always assume)
\[ d\bar{w}^i = [1 + \varepsilon(1/2)(d\tau/dt)] \, dw^i . \quad (3.5) \]
This will be of use in the following.

Remark 3.5. As well known, the Ito equation (3.1) is in a (rather nontrivial, see e.g. [122, 153, 200]) sense equivalent to the Stratonovich equation
\[ dx^i = b^i(x,t) \, dt + \sigma^i_k(x,t) \circ dw^k \quad (3.6) \]

\[ \text{Note, in passing, that a similar problem arises when one considers solutions } \psi(x,t) \text{ to the Schroedinger equation, as these should satisfy } ||\psi|| = 1, \text{ with of course } ||.|| \text{ the } L_2 \text{ norm.} \]
with
\[ b^i(x,t) := f^i(x,t) - \frac{1}{2} \left[ \frac{\partial (\sigma^T)^i_j(x,t)}{\partial x^k} \sigma^{kj} \right]. \quad (3.7) \]

In the following it will be convenient to set
\[ s^i(x,t) := \frac{1}{2} \left[ \frac{\partial (\sigma^T)^i_j(x,t)}{\partial x^k} \sigma^{kj} \right], \quad (3.8) \]
so that (3.7) reads simply
\[ b^i(x,t) := f^i(x,t) - s^i(x,t). \quad (3.9) \]

However, one should be careful about this equivalence, which – as mentioned above – goes through some subtle points (see e.g. [200], chapter 8).

We will discuss the relation between symmetries of an Ito equation and of the equivalent Stratonovich one in Sect. 4.4 below, after discussing also symmetries of equations in Stratonovich form.

Remark 3.6. Here again I will not consider variational problems [69, 110, 179, 220, 221, 222, 223, 224, 225] and their symmetries\(^5\). It should however be stressed that stochastic versions of Noether theory exist [4, 15, 161, 203, 204]. The relation between symmetries and conserved quantities for stochastic non variational systems has been considered by several authors (see in particular [2, 3, 5, 6, 162, 163, 164, 165]) and will be discussed in Sect. 6.1 below.

\(^5\)One should also mention that in recent times there has been a surge of interest on variational principles for stochastic fluid dynamics, also in connection with stochastic soliton equations [115, 116].

3.2 Transformation of an Ito equation, and symmetries

We will start by providing the (skeleton of the) fundamental computation concerning transformation of an Ito equation under the action of a vector field; see also [101] for details.

We consider a fully general vector field in the \((x,t)\) space, i.e.
\[ X = \tau(x,t) \partial_t + \xi^i(x,t) \partial_i. \quad (3.10) \]

Under the action of this we have, recalling also Remark 3.4 above and in particular (3.5), and working always at first order in \(\varepsilon\),
\[ x \rightarrow x^i + \varepsilon \xi^i(x,t) \]
\[ t \rightarrow t + \varepsilon \tau(x,t) \]
\[ dw^k(t) \rightarrow dw^k(t) + \varepsilon \frac{1}{2} (\partial_t \tau) dw^k(t). \]
Thus the Ito equation

\[ dx^i - f^i(x, t) \, dt - \sigma^i_k \, dw^k(t) = 0 \quad (3.11) \]

is mapped into \(^6\)

\[
\begin{align*}
&dx^i + \varepsilon \, d\xi^i = [f^i + \varepsilon (\xi^i \partial \xi^i + \tau \partial \xi^i)] \, (dt + \varepsilon \, d\tau) \quad (3.12) \\
&\quad + [\sigma^i_k + \varepsilon (\xi^i \partial \sigma^i_k + \tau \partial \sigma^i_k) \, [1 + \varepsilon (1/2)(\partial \tau)]] \, dw^k.
\end{align*}
\]

We now just have to use Ito formula to evaluate the differentials \(d\xi^i\) and \(d\tau\), which gives

\[
\begin{align*}
&d\xi^i = \left( \frac{\partial \xi^i}{\partial t} \right) dt + \left( \frac{\partial \xi^i}{\partial x^j} \right) dx^j + \frac{1}{2} \left( \frac{\partial^2 \xi^i}{\partial x^j \partial x^m} \right) \sigma^j_k \sigma^m_k \, dt, \\
&d\tau = \left( \frac{\partial \tau}{\partial t} \right) dt + \left( \frac{\partial \tau}{\partial x^j} \right) dx^j + \frac{1}{2} \left( \frac{\partial^2 \tau}{\partial x^j \partial x^m} \right) \sigma^j_k \sigma^m_k \, dt.
\end{align*}
\]

Plugging these into (3.12), and restricting to the flow of (3.1) – that is, substituting for \(dx^i\) according to it – we get that of course terms of order zero in \(\varepsilon\) cancel out, while first order terms yield a contribution

\[
\begin{align*}
&\left[ \frac{\partial \xi^i}{\partial t} + f^j \frac{\partial \xi^i}{\partial x^j} - \xi^j \frac{\partial f^i}{\partial x^j} - \frac{\partial (\tau f^i)}{\partial t} - f^j \frac{\partial \tau}{\partial x^j} f^i \\
&\quad + \frac{1}{2} \left( \frac{\partial^2 \xi^i}{\partial x^j \partial x^m} + f^i \frac{\partial^2 \tau}{\partial x^j \partial x^m} \right) \sigma^j_k \sigma^m_k \right] \, dt \\
&\quad + \left[ \sigma^j_k \frac{\partial \xi^i}{\partial x^j} - \xi^j \frac{\partial \sigma^i_k}{\partial x^j} - \tau \frac{\partial \sigma^i_k}{\partial t} - f^i \sigma^j_k \frac{\partial \tau}{\partial x^j} \\
&\quad - \frac{1}{2} \sigma^j_k \frac{\partial \tau}{\partial t} \right] \, dw^k \\
\quad := \alpha^i(x, t) \, dt + \beta^i_k(x, t) \, dw^k. \quad (3.13)
\end{align*}
\]

In other words, the action of the vector field (3.10) maps the original equation (3.11) into the (generally, different) equation

\[
\begin{align*}
dx^i - [f^i(x, t) + \varepsilon \alpha^i(x, t)] \, dt - [\sigma^i_k(x, t) + \varepsilon \beta^i_k(x, t)] \, dw^k, \quad (3.15)
\end{align*}
\]

with \(\alpha^i\) and \(\beta^i_k\) defined above.

Obviously, the transformed equation (3.15) is the same as the original one (3.11) if and only if \(\alpha^i(x, t) = 0, \beta^i_k(x, t) = 0\) for all \(i\) and \(k\).

Thus, in view of (3.13) and (3.14), we have that:

**Proposition 5.** The determining equations for general symmetries (with generator of the form (3.10)) of the Ito equation (3.1) read

\[
\begin{align*}
&\frac{\partial \xi^i}{\partial t} + f^j \frac{\partial \xi^i}{\partial x^j} - \xi^j \frac{\partial f^i}{\partial x^j} - \frac{\partial (\tau f^i)}{\partial t} - f^j \frac{\partial \tau}{\partial x^j} f^i \\
&\quad + \frac{1}{2} \left( \frac{\partial^2 \xi^i}{\partial x^j \partial x^m} + f^i \frac{\partial^2 \tau}{\partial x^j \partial x^m} \right) \sigma^j_k \sigma^m_k \, dt \\
&\quad + \left[ \sigma^j_k \frac{\partial \xi^i}{\partial x^j} - \xi^j \frac{\partial \sigma^i_k}{\partial x^j} - \tau \frac{\partial \sigma^i_k}{\partial t} - f^i \sigma^j_k \frac{\partial \tau}{\partial x^j} \\
&\quad - \frac{1}{2} \sigma^j_k \frac{\partial \tau}{\partial t} \right] \, dw^k.
\end{align*}
\]

\(^6\)Here all functions are to be thought as depending on \(x\) and \(t\); that is, \(f^i = f^i(x, t)\) and the like.
\[
+ \frac{1}{2} \left( \frac{\partial^2 \xi^i}{\partial x^j \partial x^m} + f^i \frac{\partial^2 \tau}{\partial x^j \partial x^m} \right) \sigma^j_k \sigma^m_k = 0 ;
\]

\[
\sigma^j_k \frac{\partial \xi^i}{\partial x^j} - \xi^3 \frac{\partial \sigma^i_k}{\partial x^j} - \tau \frac{\partial \sigma^j_k}{\partial t} - f^i \sigma^j_k \frac{\partial \tau}{\partial x^j} - \frac{1}{2} \sigma^i_k \frac{\partial \tau}{\partial t} = 0. \tag{3.16}
\]

These equations are general but also rather involved. It turns out to be more convenient to consider special classes of transformations, as we will do below. We stress that limitation to such simpler types of transformations is not only convenient, but also justified physically (and mathematically, as we argue in a moment, see Remark 3.7), as will be discussed below.

### 3.3 Lie-point (fiber-preserving) symmetries

#### 3.3.1 Symmetries of the Ito equation

As we have seen above, it is possible to write down explicitly the determining equations for general symmetries of the Ito equation, but these are rather involved. Moreover, on physical grounds one wants to consider general transformations in space (i.e. in the \(x\) variables), but the transformation in \(t\) should not depend on the point of space we are at \([90, 100]\); that is we would like to consider fiber-preserving symmetries (see Sect. 2.4.1).

**Remark 3.7.** It should also be mentioned that for general stochastic processes (in particular, those of non-bounded variation) a time change depending on the spatial coordinates would cause the measure of the transformed process to be not absolutely continuous w.r.t. the original one. This is a feature one would certainly not like to allow\(^7\). See also the discussion in Sect. 5.1.3 in this respect.

We will thus consider vector fields of the form

\[
X_0 = \tau(t) \partial_t + \xi^i(x, t) \partial_i , \tag{3.17}
\]

where of course \(\partial_i = \partial/\partial x^i\). We have then the following result, which can be proved by direct computation \([100]\) or specializing the computations and result of Sect. 3.2.\(^8\)

---

\(^7\)More generally, dealing with space-dependent time maps opens a number of quite delicate problems \([70, 85, 123, 153, 171]\) – which we are not willing to discuss in this context. Thus the reader might think of the discussion to be given in Chapter 5 and considering time maps which depend on \(x\) (which suffices to make this a random time change) and/or on the realization \(w(t)\) of the Wiener process, apart from \(t\) itself, as purely formal.

\(^8\)These are the first example of a large list of determining equations for stochastic equations; these will always be in the form of a couple of (systems of) equations. To avoid any confusion or awkward phrasing in referring to them, we will use a notation with (a) and (b) for the two (sets of) equations; thus e.g. eq.(3.18.a) and (3.18.b).
Proposition 6. The determining equation for fiber-preserving symmetries of the Ito equation (3.1) are

\[
\begin{align*}
(\partial_t \xi^i) &+ [(f^j \cdot \partial_j) \xi^i - (\xi^j \cdot \partial_j) f^i] - \partial_t (\tau f^i) + \frac{1}{2} (\sigma \sigma^T)^{jk} \partial^2_{jk} \xi^i = 0 \\
(\sigma^j_k \cdot \partial_j) \xi^i &- (\xi^j \cdot \partial_j) \sigma^j_k - \tau \partial_t \sigma^j_k - \frac{1}{2} (\partial_t \tau) \sigma^j_k = 0.
\end{align*}
\]

(3.18)

Remark 3.8. One is sometimes willing to further restrict the set of allowed transformation, and consider only simple symmetries; these have generator

\[ X_0 = \xi^i(x,t) \partial_t . \]

(3.19)

The corresponding determining equations for simple symmetries of the Ito equation (3.1) read

\[
\begin{align*}
(\partial_t \xi^i) &+ [(f^j \cdot \partial_j) \xi^i - (\xi^j \cdot \partial_j) f^i] + \frac{1}{2} (\sigma \sigma^T)^{jk} \partial^2_{jk} \xi^i = 0 , \\
(\sigma^j_k \cdot \partial_j) \xi^i &- (\xi^j \cdot \partial_j) \sigma^j_k = 0 ;
\end{align*}
\]

(3.20)

they are obtained from (3.18) just by setting \( \tau = 0 \).

3.3.2 Symmetries of the associated diffusion equation

A step forward in considering symmetry for SDEs (independently from a variational origin) was done when symmetries of an Ito equation were associated and compared to symmetries of the corresponding diffusion equation.

The idea behind this – in terms of solutions – is that a sample path should be mapped into an equivalent one, where equivalence is meant in statistical sense. We thus have two types of symmetries for the one-particle process described by a SDE: the equation can be invariant under the map, or it may be mapped into a different equation which has the same associated diffusion equation. In this way one is to a large extent considering the symmetries of the associated Fokker-Planck (FP) equation, and this had been studied in detail in the literature [63, 64, 84, 126, 128, 129, 130, 184, 187, 191, 192].

In the same way as for symmetries of the Ito equation, it is possible to write down explicitly the determining equations for symmetries of the associated FP equation. Here again we consider fiber-preserving symmetries (see Sect. 2.4.1), i.e. vector fields of the form (3.17). When dealing with the FP equation we will consider vector fields of the form

\[ X = \tau(t) \partial_t + \xi^i(x,t) \partial_i + \varphi(x,t,u) \partial_u . \]

(3.21)

Note that having \( \tau \) and \( \xi^i \) independent of \( u \) is needed in order to be able to project vector fields defined in the \((x,t,u)\) space down to the \((x,t)\) space; this will be required to compare symmetries of the Ito and of the associated FP
equation (and more generally vector fields of the form $X$ and those of the form $X_0$). We have the:

**Proposition 7.** Consider the Ito equation (3.1); the determining equations for nontrivial symmetries of the associated Fokker-Planck equation are

$$
\begin{align*}
\partial_t(\tau A^{ik}) + (\xi^m \partial_m A^{ik} - A^{im} \partial_m \xi^k - A^{mk} \partial_m \xi^i) &= 0 \\
\partial_t(\tau B^i) - \left[ \partial_i \xi^i + (B^m \partial_m \xi^i - \xi^m \partial_m B^i) \right] + (A^{ik} \partial_k \beta + A^{mi} \partial_m \beta) - A^{mk} \partial_m \xi^i &= 0 \\
\partial_t(\tau C) + \partial_i \beta + A^{ik} \partial_k \beta + B^i \partial_i \beta + \xi^m \partial_m C &= 0.
\end{align*}
$$

(3.22)

Note that here “nontrivial symmetries” should be understood in the sense of Remark 3.2 above. Proposition 7 was proved in [100] by direct computation.

### 3.3.3 Symmetry of Ito versus Fokker-Planck equations

It follows immediately from comparison of (3.18) and (3.22) that:

**Proposition 8.** Symmetries of an Ito equation, i.e. solutions to (3.18), can be extended to (projectable) symmetries of the associated Fokker-Planck equation, i.e. solutions to (3.22), while the converse is not necessarily true.

The determining equations allow, at least in principle, to find the symmetries of a given Ito equation; moreover, in view of Proposition 8 these can be obtained refining the list of symmetries of the associated FP equation. As the latter is a standard (deterministic) PDE, its symmetries are determined by standard techniques.

Let us look at the statement of Proposition 8 in more detail. First of all, we should not consider all symmetries of the FP equation, but only those which:

(i) can be projected to the space in which the Ito equation is defined, and are projectable\(^9\) in this; and

(ii) preserve the normalization condition for the probability measure.

The first requirement leads to consider vector fields

$$
X = \tau(t) \partial_t + \xi(x,t) \partial_x + \varphi(x,t,u) \partial_u := X_0 + \varphi(x,t,u) \partial_u.
$$

(3.23)

It turns out that the second requirement is satisfied if and only if

$$
\varphi(x,t,u) = \alpha(x,t) + \beta(x,t) u
$$

(3.24)

\(^9\)That is, the change in time will be the same at all points of space. More general settings are also possible, see e.g. [13] and Chapter 5 below.
with moreover
\[ \int \alpha(x, t) \, dx = 0 \quad (\forall t) \; ; \quad \beta(x, t) = -\text{div} [\xi(x, t)] \, . \tag{3.25} \]

This specification gives a constructive meaning to the statement that the symmetry of (3.1) can be extended to a symmetry of the associated FP equation.

On the other hand [100], Proposition 8 has another simple consequence (actually a corollary):

**Proposition 9.** Consider the Ito equation (3.1) and the associated FP equation. A vector field in the form (3.23) and which is a symmetry of the associated FP equation is also a symmetry of the Ito equation under consideration if and only if \( \Gamma = 0 \), where
\[ \Gamma = \left[ \sigma_j^m \partial_m \xi^k - \xi^m \partial_m \sigma_j^k - \tau \partial_t \sigma_j^k - \frac{1}{2} (\partial_t \tau \sigma_j^k) \right] \, . \tag{3.26} \]

**Remark 3.9** Most recently the “diffusive” approach to symmetries of SDEs has been reconsidered by F. De Vecchi in his (M.Sc.) thesis [72], making contact with so called “second order geometry” developed by Meyer and Schwartz [82, 158, 159, 160, 189]. This introduces some interesting Geometry of second order differential operators with no constant part. Note that his approach is entirely through the associated diffusion (hence deterministic) equation. The interested reader is referred to his paper [73] (announced in [74]).

**Example 3.1.** Let us consider the Ito equation
\[ dx^i = \sigma_0 \, dw \, , \tag{3.27} \]
i.e. a free particle moving in one dimension under the action of a constant noise (here \( \sigma_0 \) is a real constant). In this case \( f = 0 \) and \( s = \sigma_0 \). The corresponding FP equation is just the heat equation
\[ u_t = (\sigma_0/2) \, u_{xx} \, . \]

As well known, the infinitesimal symmetries of the latter are generated by
\[
\begin{align*}
V_1 &= \partial_t \, , \quad V_2 = \partial_x \, , \quad V_3 = 2t \partial_t + x \partial_x \, ; \\
V_4 &= u \partial_u \, ; \quad V_5 = \sigma_0^2 t \partial_x - \sigma_0 xu \partial_u \, , \quad V_6 = t^2 \partial_t + xt \partial_x - \frac{1}{2} \left( t + \frac{x^2}{\sigma_0^2} \right) u \partial_u \, ; \\
V_\alpha &= \alpha(x, t) \partial_u \, .
\end{align*}
\]

Here \( \alpha(x, t) \) is any solution to the heat equation itself, and the infinite dimensional algebra spanned by the \( V_\alpha \) corresponds to the fact the equation is linear, see Sect. 2.6.

\[ ^{10}\text{For application of the Meyer-Schwartz approach, see also [138].} \]
It is easy to check that $V_1, V_2, V_3$ are also symmetries of the Ito equation (3.27); these vector field do not act on $u$. Note also that for these $\beta = -\partial_x \xi$, which in this one dimensional case means $\beta = -\text{div}(\xi)$, see (3.25); the condition is not satisfied by $V_4, V_5, V_6$. One can actually check that $V_1, V_2, V_3$ span the symmetry algebra of (3.27): in this case the (3.18) read

$$\xi_t + \frac{1}{2} \sigma_0^2 \xi_{xx} = 0,$$

$$\xi_x - \frac{1}{2} \tau_t = 0.$$  

(3.28)

The latter implies (as $\tau_x = 0$) that $\xi_{xx} = 0$, hence $\xi = a(t)x + b(t)$. Now the first of the above equations (3.28) require that $a$ and $b$ are actually constant, that is $\xi = c_3 x + c_2$. With this, (3.28.b) enforces in turn $\tau = 2c_3 t + c_1$ (the $c_k$ are real constants). We get exactly the $V_1$ (associated to $c_1$), $V_2$ (associated to $c_2$) and $V_3$ (associated to $c_3$).

Example 3.2. Consider next a two-dimensional example,

$$dx = y \, dt$$
$$dy = -k^2 y \, dt + \sqrt{2k^2} \, dw(t);$$  

(3.29)

the associated FP equation is the Kramers equation

$$u_t = k^2 u_{yy} - y u_x + k^2 y u_x + k^2 u.$$  

(3.30)

This is linear, hence we will have symmetries $V_\alpha$ with $\alpha(x, t)$ an arbitrary solution. Apart from these, the infinitesimal symmetry generators of the Kramers equation are, as determined in [191],

$$V_1 = \partial_t, \quad V_2 = \partial_x, \quad V_3 = e^{-k^2 t} \left( k^{-2} \partial_x - \partial_y \right);$$

$$V_4 = u \partial_u, \quad V_5 = \frac{t}{2} \partial_x + \partial_y - \frac{1}{2} \left( y + k^2 x \right) u \partial_u;$$

$$V_6 = e^{k^2 t} \left( k^{-2} \partial_x + \partial_y - y u \partial_u \right).$$

Note that $V_1, V_2, V_3$ (which do not act on $u$, i.e. are of the form $X_0$) satisfy $\beta = -\text{div}(\xi)$, while $V_4, V_5, V_6$ do violate this condition. One can check [100] that $V_1, V_2, V_3$ span the full symmetry algebra for (3.29).

Example 3.3. In the two examples above there was full correspondence between admissible symmetries of the FP equation and symmetries of the Ito equation. We now consider an example (again two-dimensional, and again taken from [100]) in which this is not the case. This is the equation

$$dx^1 = \cos(t) \, dw^1(t) - \sin(t) \, dw^2(t)$$
$$dx^2 = \sin(t) \, dw^1(t) + \cos(t) \, dw^2(t);$$  

(3.31)
this is also written in vector notation as
\[ dx = R(t) \, dw(t) , \] (3.32)
where \( R(t) = \sigma(t) \) is the matrix of rotation by an angle \( t \).

The associated FP equation is just the two-dimensional heat equation,
\[ u_t = \frac{1}{2} u_{xx} . \] (3.33)

It is immediate to check that \( \partial_t \) is (of course) a symmetry of (3.33) but it does not satisfy (the second of) the determining equations (3.18) and hence is not a symmetry of (3.31).

Note that we would have exactly the same situation for any orthogonal \( \sigma \) matrix not constant in time.

\[ \diamond \]

**Example 3.4.** Let us consider the case of Ito equations with linear drift \([7]\), i.e.
\[ dx^i = M^i_j \, x^j \, dt + \sigma^i_j \, dw^j ; \] (3.34)
we will further simplify the situation by considering the case of a constant and invertible \( \sigma \) matrix. This system represents an Ornstein-Uhlenbeck process \([169, 183]\) and, writing \( A = -(1/2)\sigma\sigma^T \), the associated FP equation is just
\[ u_t + M^i_k + M^i_j x^k u_i - A^i_j u_{ij} = 0 . \] (3.35)

In this case it results
\[ \xi^i(x,t) = L^i_j(t) \, x^j + P^i(t) ; \]
the matrix \( L \) satisfies \( L = (1/2)\tau I \).

For \( M \neq I \) one gets \( \tau = 0 \), hence \( L = 0 \), and \( P(t) = \exp[Mt]P_0 \). In the special case \( M = I \), one gets \( \tau = c_1 e^t + c_2 \), hence \( L = (c_1/2)e^t I \), and \( P(t) = e^t P_0 \).

As for \( \beta \) in the symmetry of the associated FP equation, we note that \( \text{div}(\xi) = \text{Tr}(L) \), thus \( \beta = 0 \) for \( M \neq I \), and \( \beta = (n/2)\tau_t \) for \( M = I \).  

\[ \diamond \]

### 3.4 W-symmetries

The theory can be extended to consider also transformations acting on the Wiener processes, also called \( W \)-symmetries. More specifically, we will consider (infinitesimal) maps
\[
\begin{align*}
t &\to s = t + \varepsilon \tau(t) \\
x^i &\to y^i = x^i + \varepsilon \xi^i(x,t) \\
w^k &\to z^k = w^k + \varepsilon \mu^k(w,t) .
\end{align*}
\] (3.36)
Note that here as well, as in Sect. 3.3, we are just considering fiber-preserving transformations.\footnote{More general ones – with the cautionary notes already seen in Remark 3.7– will be considered in Chapter 5.}

The restrictions on $\tau$ and $\xi$ in (3.36) are the same as before (see Sect. 3.3) and do not need further comments; as for the transformation undergone by the Wiener processes, i.e. the function $\mu = \mu(w,t)$, we have allowed this to depend on time and on the processes themselves, but not on the space variables $x^i$. The rationale for this is that we think of the (time-dependent) stochastic processes $w^i(t)$ as independent of the position $x(t)$ reached by the test particle, and we would like this property to be still valid for the transformed processes $z(t)$ (see also Remark 3.7 above).

It should be stressed that $\mu(w,t)$ in (3.36) cannot be an arbitrary function. In fact, if we want the transformed processes $z^i(t)$ to be still Wiener ones\footnote{Some works in the literature (about symmetry of SDEs) do only require that the transformed processes have the same mean and variance of the original ones, with no check on the fate of higher moments. I am not understanding the idea behind this choice (which was indeed questioned by other authors [206]), and will not comment on these.}, it turns out [89] that the only possibility is to have

$$z^i(t) = M^i_j w^j(t)$$

with $M$ a constant orthogonal matrix, $MM^+ = I$. In terms of infinitesimal generators, we get

$$\mu^i(w,t) = B^i_k(w,t) w^k \quad (B^T = -B). \quad (3.38)$$

Having established the class of allowed transformations, one can compute their effect on the Ito equation (3.1). This is summarized in the following statements [89], the second (Proposition 11) being an immediate corollary of the first (Proposition 10).

**Proposition 10.** Under the map (3.36), with $\mu$ of the form (3.38), the Ito equation (3.1) is mapped into a new Ito equation

$$dy^i = F^i(y,s) ds + S^i_k(y,s) dz^k$$

with $F^i = f^i + \varepsilon(\delta f)^i$, $S^i_k = \sigma^i_k + \varepsilon(\delta \sigma)^i_k$. The first variations are given explicitly by

$$(\delta f)^i = \partial_t \xi^i + [(f^j \partial_j) \xi^i - (\xi^j \partial_j) f^i] - \partial_t (\tau f^i) + A^{jk} \partial^{\alpha}_{jk} \xi^i, \quad (3.39)$$

$$(\delta \sigma)^i_k = \left[(\sigma^j_k \partial_j) \xi^i - (\xi^j_k \partial_j) \sigma^i_k \right] - \tau \partial_t \sigma^i_k - \frac{1}{2} \left(\partial_t \tau \right) \sigma^i_k - \sigma^i \tau \sigma^p B^p_k. \quad (3.39)$$

**Proposition 11.** The map (3.36), with $\mu$ as in (3.38), is a symmetry of the Ito equation (3.1) if and only if the quantities defined in (3.39) satisfy $(\delta f)^i = 0$, 

\((\delta \sigma)_k^i = 0\) for all \(i\) and \(k\). In other words, the determining equations for W-symmetries of (3.1) are

\[
\begin{align*}
\partial_t \xi^i + \left[ (f^j \partial_j) \xi^i - (\xi^j \partial_j) f^i \right] - \partial_k (\tau f^i) + A^{jk} \delta^2_{jk} \xi^i &= 0 \quad (3.40) \\
\left[ (\sigma^j \partial_j) \xi^i - (\xi^j \partial_j) \sigma^i \right] - \tau \partial_t \sigma^i_k - \frac{1}{2} \left( \partial_t \tau \right) \sigma^i_k - \sigma^i_p B^p_k &= 0.
\end{align*}
\]

**Remark 3.10.** Note that the possibility of action on \(W\) can *not* be used to balance the change in \(W\) induced by transformation of \(t\), as the latter is *not* an orthogonal action.

**Remark 3.11.** Finally we note that this is definitely *not* the most general allowable transformation. For example, in their work on normal forms\(^{13}\) for stochastic dynamical systems [13], L. Arnold and P. Imkeller considered a more general class. As it was remarked already in [89, 100], albeit the restriction to (3.36) is well justified physically, from a mathematical point of view (i.e. to obey just the internal coherence requirements, disregarding the applications physicists are primarily interested in) one should extend the theory by including the class of transformation considered there. We will come back to this point later on in Chapter 5.

We will now consider some examples. In order to better compare W-symmetries with standard ones, we will consider some situations already discussed above in the Examples of Sect. 3.3.

**Example 3.5.** We start by considering again Example 3.2. In this case the determining equations (3.40) require \(B = 0\), i.e. we have no new vector fields allowing W-symmetries beside standard ones.

**Example 3.6.** Let us consider the equations

\[
\begin{align*}
dx &= (a_1/x) \, dt + dw_1(t) \\
dy &= a_2 \, dt + dw_2(t).
\end{align*}
\]

This has four standard symmetry generators, i.e.

\[
X_1 = \partial_t, \quad X_2 = \partial_x, \quad X_3 = \partial_y; \quad X_4 = 2t \partial_t + x \partial_x + (y + a_2t) \partial_y,
\]

and a proper W-symmetry, generated by

\[
X_5 = (a_2 t - y) \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} + w_2 \frac{\partial}{\partial w_1} - w_1 \frac{\partial}{\partial w_2}.
\]

\(^{13}\)As well known, normal forms - and transformation to normal forms – is intimately connected with symmetries [14, 60, 213].
Example 3.7. We consider again the equation (3.31), i.e. Example 3.3. In this case, beside the standard symmetries already obtained in Sect. 3.3, we also have a W-symmetry,

$$X = \left( y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} \right) + \left( w_2 \frac{\partial}{\partial w_1} - w_1 \frac{\partial}{\partial w_2} \right).$$

This represents a simultaneous identical rotation in the $(x, y)$ and the $(w_1, w_2)$ planes.

Example 3.8. Let us consider the setting of Example 3.4. With the notation introduced above in this Section, the symmetries satisfy

$$ (\tau_1) \tilde{M} + [B, \tilde{M}] = (1/2) \tau_{tt} I, \quad (3.41) $$

where we have written $\tilde{M} := \sigma^{-1} M \sigma$. In particular, setting $\tau = 0$ we get new W-symmetries corresponding to matrices $B$ which commute with $\tilde{M}$ (e.g. $B = c \tilde{M}$).

Example 3.9. Finally let us consider

$$dx^i = -(1 - \lambda |x|^2) x^i dt + dw^i, \quad (3.42)$$

with $\lambda \neq 0$. It is easily seen that the only standard symmetry generator is $V_0 = \partial_t$; allowing W-symmetries we also have simultaneous rotations in the $x$ and the $w$ spaces.

3.5 Symmetries of random dynamical systems

Up to now we have considered symmetries of an Ito equation (3.1) describing a one-particle process. However in many situations, in particular in Physics, we are interested in many-particles processes; we also refer to these as random dynamical systems. One would also like to study symmetries of these, and in particular of those described by the same Ito equation, i.e. by an ensemble of non-interacting identical particles (with different initial positions) undergoing the stochastic process described by (3.1).\(^{14}\)

It should be stressed that the resulting equations will be covariant under any permutation of the particles, as follows from considering identical ones.

Not surprisingly, it turns out that – for a given Ito equation – any symmetry of the one particle process is also a symmetry of the associated random dynamical system, while the converse is not true\(^{15}\). This can be seen as a corollary to

\(^{14}\)It should be mentioned that for processes with independent increments, the two-particle process embodies the full information needed to determine the $N$ particle process, see [12] (sect.2.3.9).

\(^{15}\)Here we are referring to “symmetries” in the sense of infinitesimal generators of continuous symmetries; the statement is (trivially) true also for the permutation symmetry mentioned above.
the following Proposition 12; in order to state this, it is convenient to set some ad hoc notation.

We will consider \( N \) copies (\( i = 1, \ldots, N \)) of the Ito equation in \( \mathbb{R}^n \) (here \( i = 1, \ldots, n \)),

\[
dy^i_a = \varphi^i(y_a, t) \, dt + \rho^i_k(y_a, t) \, dw^k(t) ; \tag{3.43}
\]

these can be written as a single Ito equation in the form (3.1) by setting (in block notation)

\[
f = \begin{pmatrix} \varphi(1) \\ \vdots \\ \varphi(N) \end{pmatrix}, \quad \sigma = \begin{pmatrix} \rho(1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ \rho(N) & \cdots & 0 \end{pmatrix}, \tag{3.44}
\]

where we have set, for ease of notation,

\[
\varphi(k) = \varphi(y_k, t), \quad \rho(k) = \rho(y_k, t). \tag{3.45}
\]

We stress that the resulting \( N \) particle equation, which is a Ito equation in dimension \( d = N \cdot n \), depends only on \( n \) Wiener processes; this distinguishes the situation from the general one, i.e. (in the presently used language) that of a \( d \)-dimensional one-particle process.

Note that with the notation (3.44) we have

\[
\frac{1}{2} \sigma \sigma^T = \frac{1}{2} \begin{pmatrix} \rho(1)\rho^T(1) & \cdots & \rho(1)\rho^T(N) \\ \vdots & \ddots & \vdots \\ \rho(N)\rho^T(1) & \cdots & \rho(N)\rho^T(N) \end{pmatrix} := \begin{pmatrix} A(y_1, y_1) & \cdots & A(y_1, y_N) \\ \vdots & \ddots & \vdots \\ A(y_N, y_1) & \cdots & A(y_N, y_N) \end{pmatrix}.
\]

We will set a similar notation for the (symmetry) vector fields. That is, we will set in general, and again in block notation,

\[
\xi(t; y_1, \ldots, y_N) = \begin{pmatrix} \xi(1)(t; y_1, \ldots, y_N) \\ \vdots \\ \xi(N)(t; y_1, \ldots, y_N) \end{pmatrix}.
\]

As recalled above, the equations should be covariant under \( S_N \), the group of permutation of the \( N \) identical particles. This means that \( \xi(1)(t; y_1, \ldots, y_N) \) should be invariant under the permutations which do not affect \( y_1 \) (thus under a subgroup \( S_{N-1} \)), and that one has (considering for simplicity just the cyclic permutations group \( Z_N \subset S_N \))

\[
\xi(k; t; y_1, \ldots, y_N) = \xi(1)(t; y_k, \ldots, y_{k-1}).
\]

Finally, we will write for short

\[
\frac{\partial}{\partial y_a^i} \Delta_{(a,b)} = \frac{\partial^2}{\partial y_a^i \partial y_b^j}.
\]

With these notations, we have the
Proposition 12. The symmetry generators for the $N$ particle process defined by the Ito equation (3.43) satisfy the determining equations (no sum on $a$)

$$
\begin{align*}
\partial_t \xi_i^a & - \partial_t (\tau \varphi_i^a) + \left[ (\varphi_j^a) \partial_j^a \right] \xi_i^a - (\xi_j^a) \partial_j^a \varphi_i^a \right] \\
+ & A(y(a), y(a)) \Delta_{(a,a)} \xi_i^a + \sum_{b \neq a} (\varphi_j^b) \partial_j^b \xi_i^a \\
+ & \sum_{b,c \neq (a,a)} A(y(b), y(c)) \Delta_{(b,c)} \xi_i^a = 0,
\end{align*}
$$

(3.46)

$$
\begin{align*}
\left[ (\rho_i^a) \partial_j^a \xi_j^a - (\xi_i^a) \partial_j (\rho_i^a) \right] & - \tau (\partial_i \tau) (\rho_i^a) - \frac{1}{2} \left( \partial_i \tau \right) (\rho_i^a) \right] \\
- & (\rho_i^a) \Delta_{M} + \sum_{b \neq a} \left[ (\rho_i^b) \partial_j^b \xi_j^a \right] = 0.
\end{align*}
$$

(3.47)

Remark 3.12. The approach – and most of the notation – developed for $N$ non-interacting particles can to a large extent be applied to the case of interacting particle as well. Albeit these can be treated (as recalled above) within the frame of a general stochastic process – and Ito equation – in a suitably large space, one would expect that specializing to the case of $N$ identical (interacting) particles would give some results which are definitely non-generic\textsuperscript{16}. Similar extensions would be possible considering sets of different families of particles; in this case the arguments based on the covariance under the full $S_N$ permutation group should be accordingly modified.

Example 3.10. We start by considering, as in Example 3.1 above, the equation

$$
dx = dw;$$

(3.48)

that is, we have $\varphi = 0, \rho = 1$. As we are in one dimension, $B = 0$. We will consider the two-particle process associated to this, and write $y_{(1)} = x, y_{(2)} = y$ and $\xi_{(1)}(x, y, t) = \xi_{(2)}(y, x, t) = \xi(x, y, t)$. The determining equations (3.46), (3.47) are now simply

$$
\begin{align*}
2 \eta_t + (\xi_{xx} + 2 \xi_{xy} + \xi_{yy}) & = 0 \\
\tau_t & = 2 (\xi_x + \xi_y).
\end{align*}
$$

We observe that

$$
\xi(x, y, t) = f(x - y), \quad \tau = 0
$$

is a solution to these, for any smooth function $f$. These are obviously not symmetries (nor meaningful) for the one-particle process defined by the same Ito equation.

\textsuperscript{16}Such an extension of this formalism, to the best of my knowledge, has not been developed; here again one would have an interesting project.
Example 3.11. Let us now consider again the Example 3.2, which we now write as
\[
\begin{align*}
  dx &= y \, dt \\
  dy &= -k^2 \, dt + \sqrt{2k^2} \, dw 
\end{align*}
\]
we will consider the two-particle process defined by this system, and write \( y_{(k)} = (x_k, y_k) \). In this case we have
\[
\varphi = \begin{pmatrix} x_2 \\ -k^2 x_2 \end{pmatrix}, \quad \rho = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{2k^2} \end{pmatrix},
\]
and will set
\[
\eta = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}.
\]
Writing
\[
\zeta := k^2 (x_1 - y_1) + (x_2 - y_2),
\]
we get solutions for any arbitrary smooth function of \( \zeta \), in the form
\[
\eta = h(\zeta) \begin{pmatrix} x_1 - y_1 \\ 1 \end{pmatrix}.
\]
\[\diamondsuit\]

Example 3.12. Finally we consider (similarly to Examples 3.4 and 3.8 above) the Ito equation with constant linear drift
\[
\begin{align*}
  dx^i &= M^i_j x^j \, dt + \sqrt{2s^2} \, dw^i. 
\end{align*}
\]
As in previous Examples, we will consider the two-particle process – and use the simplified notation set above – and just look for solutions with \( \tau = 0, \xi_1 = \xi_2 = \xi \) and \( \partial_t \xi = 0 \).

It turns out that a special (and simple) class of solutions is provided by
\[
\xi^i = L^i_k (x^k - y^k) + B^i_k x^k
\]
for \( L \) and \( B \) matrices commuting with \( M \), i.e. for \( [L, M] = 0 = [B, M] \). \[\diamondsuit\]
Chapter 4

Stratonovich stochastic equations

It is well known that, albeit the proper formulation of Stochastic Differential Equations corresponds to the Ito formalism, in many applications it is also convenient to consider the so called Stratonovich formalism [12, 109, 171, 180, 199, 200].

The main advantage of this is that a Stratonovich SDE behaves nicely, i.e. in the “usual” way, under coordinate transformations. As symmetry refers to invariance under transformations, it is not surprising at all that the first attempts to consider symmetry in the framework of SDEs [3, 162, 163] focused on Stratonovich equations, and that the theory of symmetry of Stratonovich SDEs is neatly formulated – as we will see below.

We will now analyze symmetries of Stratonovich SDEs. Before doing this, an important remark is necessary: one would be tempted to guess that the symmetries of a Stratonovich and of the “equivalent” Ito equations should coincide. We will see that in general\footnote{There are classes of SDEs, e.g. linear ones, for which there is indeed equivalence. Similarly this holds for certain classes of symmetries.} this is not the case, even in quite simple examples; this was already remarked by Unal [206], and will be discussed in some detail below, see Sect. 4.4.

The reason for this non correspondence appears to be rooted in the non-trivial character of the “equivalence” between Ito and associated Stratonovich equations; this is discussed e.g. in [200] (see Chapter 8 in there).

4.1 Transformation of a Stratonovich equation

Similarly to what we did in Sect. 3.2 for Ito equations, we will first derive the explicit expression for the transformation of a SDE in Stratonovich form, which
we write again (for ease of reference) as
\[ dx^i = b^i(x,t) dt + \sigma^i_k(x,t) \circ dw^k(t), \quad (4.1) \]
under the action of a general vector field
\[ X = \tau(x,t) \partial_t + \varphi^i(x,t) \partial_i; \quad (4.2) \]
this induces the map
\[ x^i \rightarrow x^i + \varepsilon \varphi^i(x,t); \quad t \rightarrow t + \varepsilon \tau(x,t). \quad (4.3) \]

We should first of all recall that, as already mentioned in Remark 3.4, the action on \( t \) induces an action on the increments \( dw^k \) of the Wiener processes \( w^k(t) \); more precisely, we have \([171]\)
\[ dw^k \rightarrow dw^k + \varepsilon \left( \frac{1}{2} \frac{d\tau}{dt} dw^k. \quad (4.4) \right) \]
Thus under (4.3) the equation (4.1) is mapped into
\[ d(x^i + \varepsilon \varphi^i) = b^i(x + \varepsilon \varphi, t + \varepsilon \tau) d(t + \varepsilon \tau) + \sigma^i_k(x + \varepsilon \varphi, t + \varepsilon \tau) \circ dw^k(t + \varepsilon \tau). \quad (4.5) \]
(Here it is understood that \( \varphi = \varphi(x,t), \tau = \tau(x,t). \))

Performing all computations at order \( \varepsilon \), and noticing that on the flow of (4.1) we have
\[ d\varphi^i = \frac{\partial \varphi^i}{\partial x^j} dx^j + \frac{\partial \varphi^i}{\partial t} dt \]
\[ = \left( \frac{\partial \varphi^i}{\partial x^j} \right) \left( b^j(x,t) dt + \sigma^j_k(x,t) \circ dw^k(t) \right) + \left( \frac{\partial \varphi^i}{\partial t} \right) dt; \quad (4.6) \]
\[ d\tau = \frac{\partial \tau}{\partial x^j} dx^j + \frac{\partial \tau}{\partial t} dt \]
\[ = \left( \frac{\partial \tau}{\partial x^j} \right) \left( b^j(x,t) dt + \sigma^j_k(x,t) \circ dw^k(t) \right) + \left( \frac{\partial \tau}{\partial t} \right) dt, \quad (4.7) \]
eq (4.5) reads
\[ dx^i = b^i(x,t) dt + \sigma^i_k(x,t) \circ dw^k \]
\[ - \varepsilon \left[ \left( (\partial_t \varphi^i) - \partial_t (\tau b^i) \right) + \left( b^j (\partial_j \varphi^i) - \varphi^i (\partial_j b^i) \right) - b^j (\partial_j \tau) b^i \right] dt \]
\[ + \varepsilon \left[ \tau (\partial_t \sigma^i_k) + (1/2)(\partial_t \tau) \sigma^i_k - \left( \sigma^j_k (\partial_j \varphi^i) - \varphi^j (\partial_j \sigma^i_k) \right) \right] \]
\[ + b^j (\partial_j \tau) \sigma^i_k) \circ dw^k(t). \quad (4.8) \]
This coincides with (4.1) if and only if both terms in square brackets vanish; that is, we have \([101]\):
Proposition 13. The determining equations for deterministic symmetries of the Stratonovich equation (4.1) are

\[
\begin{align*}
((\partial_t \varphi^i) - \partial_i (\tau b^i)) & + (b^j (\partial_j \varphi^i) - \varphi^j (\partial_j b^i)) - b^i (\partial_i \tau) b^i = 0, \\
(\tau (\partial_t \sigma^i_k) + (1/2)(\partial_i \tau) \sigma^j_k) & - \left(\sigma^j_k (\partial_j \varphi^i) - \varphi^j (\partial_j \sigma^i_k)\right) \\
+ b^i (\partial_i \tau) \sigma^j_k = 0. 
\end{align*}
\]

(4.9)

Remark 4.1. The equations reported in [101] (and obtained in the same way) are apparently different; but one should note that in there, based on physical motivation\(^2\), only the case \(\tau = \tau(t)\) was considered. The equations do of course coincide in the case \(\tau = \tau(t)\), to be discussed in detail below.

4.2 Strong symmetries

As mentioned above, the first attempts to use symmetry in the context of SDEs [3, 162, 163] involved Stratonovich equations, and had quite strong requirements for a map to be considered a symmetry of the SDE. They were based on the idea of a symmetry as a map taking solutions into solutions; the problem is that while in the deterministic case it is quite clear what is meant by “solution” (one just has to distinguish between general and special solutions), and hence by “mapping a solution to a solution” or by “invariant solution”, in the stochastic case this can be interpreted in several ways.

Thus, the first approach by Misawa [162, 163, 164], then extended and generalized by Albeverio and Fei [3] (see also [154, 165]), required that for any given realization of the Wiener process any sample path satisfying the equation would be mapped to another such sample path. It is not surprising that the presence of symmetries was then basically related to situations where, in suitable coordinates, the evolution of some of the coordinates is deterministic and not stochastic. However, uncovering this fact in other (i.e. non-adapted) coordinates may be not simple; thus this works gave nontrivial results, and in particular showed that one can have symmetries – and, under certain additional conditions, related conserved quantities – also in the case of SDEs, pretty much as in Hojman’s work [114] mentioned above, see Sect. 2.3.2.

It should be stressed that in this case one has quantities which are always conserved under the stochastic flow. This means that the level manifolds of the conserved quantities are always invariant, i.e. that both the drift and the stochastic term are zero in directions transversal to these manifolds. This is by all means a rather strong requirement.

On the other hand, the requirement can be met in practice, and when this is the case it is surely relevant to be able to detect the associated conservation laws (note that once again this can be better seen by passing to adapted coordinates).

\(^2\text{Recall also Remark 3.7 in this respect.}\)
4.2.1 Time-preserving strong symmetries

A simple computation – in practice, a specialization of the general one presented in Sect. 4.1 to the case \( \tau = 0 \) – taking advantage of the favorable properties of stochastic differential equations in Stratonovich form (i.e. the fact we can just use the chain rule), shows that under the map

\[ x^i \mapsto x^i + \varepsilon \varphi^i(x, t) \quad (4.10) \]

the equation (4.1) is mapped into

\[
\begin{align*}
\frac{dx^i}{dt} &= b^i(x,t) dt + \sigma^i_k(x,t) \circ dw^k(t) \\
&+ \varepsilon \left[ \left( \frac{\partial \varphi^i}{\partial t} + b^j \frac{\partial \varphi^i}{\partial x^j} - \varphi^j \frac{\partial b^i}{\partial x^j} \right) dt + \\
&+ \left( \sigma^j_k \frac{\partial \varphi^i}{\partial x^j} - \varphi^j \frac{\partial \sigma^i_k}{\partial x^j} \right) \circ dw^k(t) \right];
\end{align*}
\]

(4.11)

where arguments are not indicated, it is understood that \( b, \sigma, \varphi \) should be thought of as functions of \( x \) and \( t \). Note that (4.10) can be thought as the action of a vector field

\[ X = \varphi^i(x,t) \partial_i. \quad (4.12) \]

We say that the vector field \( X \) is a strong symmetry\(^3\) for (4.1) if it leaves the equation invariant. By looking at the discussion above, we immediately have

**Proposition 14.** The vector field \( X \) defined in (4.12) is a (simple) strong symmetry for (4.1) if it satisfies

\[
\begin{align*}
\partial_t \varphi^i + (b^i \partial_j \varphi^i - \varphi^j \partial_i b^j) &= 0, \\
\sigma^j_k \partial_j \varphi^i - \varphi^j \partial_j \sigma^i_k &= 0.
\end{align*}
\]

(4.13)

**Remark 4.2.** The equations (4.13) have been first determined by Misawa [162, 163], and hence are also known as Misawa equations.\(^\circ\)

**Remark 4.3.** Given a Stratonovich equation (4.1), one associates to this \( n+1 \) vector fields (also called Misawa vector fields):

\[ X_0 = \partial_t + b^i(x, t) \partial_i; \quad X_k = \sigma^i_k(x, t) \partial_i. \quad (4.14) \]

Note these are associated with the deterministic \( (X_0) \) and the random part (in the Stratonovich decomposition) of (4.1); passing to the Ito formalism – as mentioned in Remark 3.5 – the two would mix.\(^4\)

\(^3\)Needless to say, one should not make confusion with the notion of strong symmetry in the deterministic context considered in Sect. 2.2.

\(^4\)It should be stressed that the Ito formalism is well adapted to the stochastic framework in that it separates the drift and the martingale in the stochastic process \( x(t) \); the Stratonovich formalism is convenient from other points of view, but one should remember that \( b(x, t) \) is not the drift.
Remark 4.4. With the notation introduced above (see Sect. 2.4.1) the condition stated in Proposition 14 can also be written as

\[ \partial_t \varphi^i + \{b, \varphi\}^i_k = 0, \]
\[ \{\sigma, \varphi\}^i_k = 0. \]  
(4.15)

By looking at our discussion in the deterministic case – and at (4.14) – it is easily seen that, equivalently, \( X \) is a strong symmetry if and only if

\[ [X, X_0] = 0 = [X, X_k], \]  
(4.16)

i.e. if and only if it commutes with the Misawa vector fields associated to equation (4.1).

Example 4.1. Consider the three-dimensional case\(^5\) (depending on a single Wiener process)

\[
\begin{align*}
    dx &= - (y - z) \, dt + (z - y) \circ dw \\
    dy &= - (z - x) \, dt + (x - z) \circ dw \\
    dz &= - (x - y) \, dt + (y - x) \circ dw
\end{align*}
\]

The vector field

\[
X = (|x|^2 / 2) \left( \partial_x + \partial_y + \partial_z \right)
\]

is a strong symmetry for this system.

Example 4.2. The system considered in Example 4.1 above actually admits many symmetries (which now we denote by \( Y \), to avoid confusion with the \( X_k \)); e.g. for those of the form

\[
Y_k = \eta_k(x, y, z) \left( \partial_x + \partial_y + \partial_z \right),
\]

we can choose

\[
\begin{align*}
    \eta_0 &= (x + y + z), & \eta_1 &= (x^2 + y^2 + z^2), & \eta_2 &= (xy + yz + zx), \\
    \eta_3 &= [x^2 (y + z) + y^2 (z + x) + z^2 (x + y) + 3 xyz], \\
    \eta_4 &= [(x^3 + y^3 + z^3) - 3 xyz].
\end{align*}
\]

\(\diamondsuit\)

\(^5\)This example was suggested by Misawa in his original paper \([163]\), and it thus became a standard example for checking subsequent results. As the reader will note, we are not infringing this tradition, and will be repeatedly considering it.
4.2.2 Extended symmetries

Following Misawa [162, 163] we started by considering symmetries of the form (4.12); the reason to consider only such transformations is that acting on $t$ means acting on the Wiener processes as well. One could consider this case as well, proceeding as seen above in the general case (see Sect. 4.1), but a clever way to circumvent the problem was devised by Albeverio and Fei [3].

Consider a Stratonovich equation (4.1) and a strong symmetry of it in the form (4.12). Then we consider symmetries of the symmetry vector field, i.e.

$$S = \alpha^0(x,t) \partial_t + \alpha^i(x,t) \partial_i \quad (4.17)$$

such that

$$S(x^i) = \varphi^i(x,t) = \alpha^i(x,t) ; \quad (4.18)$$

note that we are not imposing any condition on $\alpha^0$, thus in practice we are just considering extensions of the symmetry vector field $X$, which acts only on dependent variables, to a vector field which also acts on the independent variable.

It is immediate from (4.16) to see that

$$[X_0, S] = [X_0, S_0] + [X_0, X] = [X_0, S_0] ;$$
$$[X_k, S] = [X_k, S_0] + [X_k, X] = [X_k, S_0] .$$

Let us then consider the set $\mathcal{L}$ of vector fields $L$ satisfying the relations

$$[X_0, L] = T^0 X_0 + T^m X_m ;$$
$$[X_k, L] = R^0_k X_0 + R^m_k X_m . \quad (4.19)$$

**Proposition 15.** The set $\mathcal{L}$ is a Lie algebra; any symmetry vector field $X$ for the equation (4.1) satisfies $X \in \mathcal{L}$.

Thus $\mathcal{L}$ represents an extension of the set of strong symmetries (in the sense of Misawa) for the Stratonovich equation under consideration; we will refer to the vector fields $L$ in $\mathcal{L}$ as extended symmetries for (4.1). Proposition 15 is then rephrased saying that the set of extended symmetries of a (Stratonovich) stochastic equation has the structure of a Lie algebra.

**Remark 4.5.** Proposition 15 follows from a simple computation. If we have vector fields $L_i$ satisfying

$$[X_0, L_i] = T^0_i X_0 + T^m_i X_m ,$$
$$[X_k, L_i] = R^0_{ik} X_0 + R^m_{ik} X_m ,$$
and consider $[X_0, [L_i, L_j]]$, it follows from Jacobi identity and some explicit computation that

$$[X_0, [L_i, L_j]] = ([X_0, L_i], L_j) - ([X_0, L_j], L_i) = \Theta^0_{ij} X_0 + \Theta^m_{ij} X_m ,$$
where
\[
\Theta^0_{ij} = (T^n_i R^n_j - T^n_j R^n_i) + (L_i(T^n_j) - L_j(T^n_i)),
\]
\[
\Theta^m_{ij} = (T^n_i T^n_j - T^n_j T^n_i) + (T^n_i R^n_j - T^n_j R^n_i) + (L_i(T^n_j) - L_j(T^n_i)).
\]

Similarly, and again using the Jacobi identity,
\[
[X_k [L_i, L_j]] = [[X_k, L_i], L_j] - [[X_k, L_j], L_i] = \Gamma^0_{ij} X_0 + \Gamma^m_{ij} X_m,
\]
where
\[
\Gamma^0_{ij} = \left( (R^0_i T^0_j - R^0_j T^0_i) + (R^0_k R^0_j - R^0_k R^0_i) + (L_i(R^0_j) - L_j(R^0_i)) \right),
\]
\[
\Gamma^m_{ij} = \left( (R^m_i T^m_j - R^m_j T^m_i) + (R^m_k R^m_j - R^m_k R^m_i) + (L_i(R^m_j) - L_j(R^m_i)) \right).
\]
These relations express the algebraic structure of the set $\mathcal{L}$.

**Example 4.3.** Consider again Example 4.1 (see also Example 4.2). This admits several vector fields $L$ satisfying the relations $[X_0, L] = 0 = [X_k, L]$; e.g. we can consider
\[
L_0 = (x + y + z)(\partial_x + \partial_y + \partial_z),
\]
\[
L_1 = (x^2 + y^2 + z^2)(\partial_x + \partial_y + \partial_z)
\]
\[
L_2 = (xy + yz + zx)(\partial_x + \partial_y + \partial_z)
\]
\[
L_3 = [x^2(y + z) + y^2(z + x) + z^2(x + y) + 3xyz](\partial_x + \partial_y + \partial_z),
\]
\[
L_4 = [(x^3 + y^3 + z^3) - 3xyz](\partial_x + \partial_y + \partial_z),
\]
...

These form a Lie algebra; e.g. considering only the first vector fields we have
\[
[L_0, L_1] = -L_1 + 4L_2,
[L_0, L_2] = 2L_1 + L_2,
[L_1, L_2] = 2L_4,
[L_0, L_3] = 6L_3 + 2L_4,
[L_0, L_4] = 0,
\]

\[\diamondsuit\]

### 4.3 W-symmetries of Stratonovich equations

One could consider, similarly to what was done for Ito equations (see Sect. 3.4), W-symmetries for Stratonovich equations. One considers again maps of the form (3.36), see the discussion in Sect. 3.4 for the reasons of this limitation, and with standard computations it turns out the Stratonovich equation (4.1) is mapped into an equation
\[
dx^i = \left[ b^i(x, t) + \varepsilon(\delta b)^i(x, t) \right] dt + \left[ \sigma^i_k(x, t) + \varepsilon(\delta \sigma)^i_k(x, t) \right] \circ dw^k, (4.20)
\]
where the first order variations are given explicitly by

\[
\delta b_i = \partial_t \phi_i + b_j \partial_j b_i - \partial_t (\tau b_i) - b_j b_i (\partial_t \tau) - \sigma_i (\partial_i h^k), \tag{4.21}
\]

\[
\delta \sigma_{ik} = \sigma_j \partial_j \phi_i - \phi_j \partial_j \sigma_{ik} - b_i \sigma_j \partial_j \tau - \sigma_m (\partial h^k / \partial w^m) - \tau (\partial_i \sigma_k) - (1/2) \sigma_{lk} (\partial_k \tau). \tag{4.22}
\]

The determining equations for W-symmetries of the Stratonovich equation (4.1) are therefore given by the vanishing of \(\delta b_i\) and \(\delta \sigma_{ik}\) as given by (4.21).

The remarks presented in Sect. 3.4 do also apply in this case.

### 4.4 Symmetries of Ito versus Stratonovich equations

As well known, and recalled above (Remark 3.5), there is a correspondence between stochastic differential equations in Stratonovich and in Ito form. In particular, the Stratonovich equation (4.1) and the Ito equation (3.1) are equivalent if and only if the coefficients \(b\) and \(f\) satisfy the relation

\[
f_i(x,t) = b_i(x,t) + \frac{1}{2} \left[ \frac{\partial}{\partial x^k} (\sigma^T)^i_k (x,t) \right] \sigma^{kj} := b_i(x,t) + \rho^i(x,t). \tag{4.23}
\]

Note this involves implicitly the metric (to raise the index in \(\sigma\)); as we work in \(\mathbb{R}^n\) we do not need to worry about this. Moreover, for \(\sigma\) (and hence \(\sigma^T\)) a constant matrix, we get \(\rho = 0\) i.e. \(b_i = f_i\).

Note also that \(\sigma\) is the same in the Ito and the corresponding Stratonovich equations, i.e. in (4.1) and in (3.1); thus (4.22) can be used in both directions. In particular, we can immediately use it to rewrite the determining equations for symmetries (of different types) of the Stratonovich equation (4.1) in terms of the coefficients in the equivalent Ito equation.

One would be tempted to expect that symmetries of an Ito equation and those of the corresponding Stratonovich one are just the same, and thus study the former via the latter. This would be particularly attractive in view of the fact that the determining equations (4.9) for symmetries of Stratonovich equations are substantially simpler than the determining equations (3.16) for symmetries of Ito equations; and similarly for simple symmetries, see (4.13) and (3.20).

Unfortunately, this way of proceeding would in general give incorrect results; in order to understand this fact is convenient to first discuss the case of simple symmetries.

The determining equations (4.13) for simple symmetries of (4.1) are immediately rewritten in terms of the coefficients \(f_i\) of the equivalent Ito equation as \((i, k = 1, \ldots, n)\)

\[
\partial_t \phi_i + [f_j (\partial_j \phi_i) - \varphi^j (\partial_j f^i)] - [\rho^j (\partial_j \varphi^i) - \varphi^j (\partial_j \rho^i)] = 0,
\]

\[
\sigma_i^j (\partial_j \phi_i) - \varphi^j (\partial_j \sigma_i^k) = 0; \tag{4.23}
\]

\(\text{The same holds at the level of determining equations for random symmetries, as it will be seen in Chapter 5 (see in particular Sect. 5.5).}\)
where \( \rho^i(x,t) \) is defined in (4.22).

It is immediate to check that the equations (4.23) do not coincide with the equations (3.20), which we rewrite here for ease of reference:

\[
(\partial_t \xi^i) + \left[ (f^j \cdot \partial_j) \xi^i - (\xi^j \cdot \partial_j) f^i \right] + \frac{1}{2} (\sigma \sigma^T)^{jk} \partial^2_{jk} \xi^i = 0 ,
\]

\[
(\sigma^j_k \cdot \partial_j) \xi^i - (\xi^j \cdot \partial_j) \sigma^i_k = 0 .
\] (4.24)

More precisely, the equations (4.23.b) are just the same as the equations (4.24.b), while equations (4.23.a) and (4.24.a) are different. The difference corresponds to

\[
\delta^i := \phi^i (\partial_j \rho^i) - \rho^i (\partial_j \phi^i) - \frac{1}{2} \Delta \phi^i \neq 0 .
\] (4.25)

Note that this inequality generally holds even in one dimension. In fact, in the one-dimensional case we get, using the definition (4.22), \( \rho = (1/2)(\sigma \sigma_x) \), and recalling that now \( \Delta \phi = \sigma^2 \phi_{xx} \) (as \( \phi \) is just a function of \( x \) and \( t \)),

\[
\delta = \frac{1}{2} \left[ \phi \sigma^2_x - \sigma^2 \phi_{xx} + \sigma (\sigma_{xx} \phi - \sigma_x \phi_x) \right] .
\] (4.26)

Thus a given vector field \( X = \phi^i \partial_i \) is a symmetry for both the Ito and the corresponding Stratonovich equation if and only if \( \phi \) satisfies the system made of both (4.23) and (4.24); this is actually a system of three (sets of) equations, as follows from the identity of (4.23.b) and (4.24.b):

\[
\phi^i_t + f^j \partial_j \phi^i - \phi^j \partial_j f^i = -\frac{1}{2} \Delta \phi^i ,
\] (4.27)

\[
\phi^i_t + f^j \partial_j \phi^i - \phi^j \partial_j f^i = - (\phi^i \partial_j \rho^i - \rho^i \partial_j \phi^i) ,
\] (4.28)

\[
\sigma^j_k \partial_j \phi^i - \phi^j \partial_j \sigma^i_k = 0 .
\] (4.29)

We can of course rearrange the first two equations and use one of them (say (4.28) to deal with a first order equation) and their difference, which in the present notation is just \( \delta^i = 0 \).

Despite the fact (4.27) and (4.28) are different, they could still admit the same solutions when restricted to the set of solutions to (4.29). The latter is a linear equation, and can in principle (and sometimes also in practice, see the Examples below) be solved by the method of characteristics; we will denote by \( \mathcal{S} \) the space of solutions to (4.29). In terms of \( \delta^i \), this observation means that albeit in general \( \delta^i \neq 0 \), it may vanish when restricted to \( \mathcal{S} \).

It may also happen that \( \delta^i \) is not zero when restricted to \( \mathcal{S} \), but it is zero when restricted to \( \mathcal{S} \) and to the solution set of (4.28) (or of (4.27), equivalently); this will happen in particular if there are no nontrivial symmetries.

In fact it happens that the identity of symmetries of an Ito and of the equivalent Stratonovich equations always holds for simple deterministic symmetries, while for general deterministic symmetries this is the case only if the function \( \tau \) in (4.2) satisfies a certain condition. (The situation for random symmetries is quite different, as we will see in Sect.5.5.)
The question was studied by Unal [206], who gave a complete answer in the case of deterministic symmetries.\footnote{The proof of his theorems are based on quite involved computations which are not reported in his paper; they have recently been checked and confirmed [152].}

**Proposition 16.** The simple deterministic symmetries of an Ito equation and of the associated Stratonovich equation are always the same. A general deterministic symmetry (4.2) of an Ito equation is also a symmetry of the associated Stratonovich equation (or vice versa) if and only if the function $\tau$ in (4.2) satisfies the condition

$$
\sigma^k_m \sigma^{im} \partial_k \left[ (\partial_t \tau) + f^i \partial_i \tau + \frac{1}{2} \sigma^p_q \sigma^{qj} (\partial^2_{pq} \tau) \right].
$$

**Example 4.4.** Consider the Ito (and the corresponding Stratonovich) equation, with $\sigma = x$ and hence $\rho = x/2$,

$$
\begin{cases}
 dx^i &= x \, dt + x \, dw \quad \text{(Ito)} \\
 dx^i &= (x/2) \, dt + x \circ dw \quad \text{(Stratonovich)}.
\end{cases}
$$

In this case the system (4.27)-(4.29) reads

$$
\begin{align*}
\varphi_t + x \varphi_x - \varphi &= -(1/2) x^2 \varphi_{xx} \\
\varphi_t + x \varphi_x - \varphi &= -(1/2) (\varphi - x \varphi_x) \\
x \varphi_x - \varphi &= 0;
\end{align*}
$$

obviously the first two equations do not coincide.

However, the last equation yields

$$
\varphi(x, t) = \alpha(t) x,
$$

and now, on the space of these functions (i.e. of solutions to the last equation) the first two equations do both read

$$
\alpha' x + x \alpha - \alpha x = 0,
$$

which by the way yields $\alpha' = 0$ and hence

$$
\varphi(x, t) = ax
$$

with $a$ a constant. \hfill \Box

**Example 4.5.** Consider now the Ito (and the corresponding Stratonovich) equation

$$
\begin{cases}
 dx^i &= x^2 \, dt + x \, dw \quad \text{(Ito)} \\
 dx^i &= (x^2 - x/2) \, dt + x \circ dw \quad \text{(Stratonovich)}.
\end{cases}
$$
In this case the system \((??)\) reads

\[
\begin{align*}
\varphi_t + x^2 \varphi_x - 2x \varphi &= -\frac{1}{2} x^2 \varphi_{xx} \\
\varphi_t + x^2 \varphi_x - 2x \varphi &= -\frac{1}{2} (\varphi - x \varphi_x) \\
x \varphi_x - \varphi &= 0;
\end{align*}
\]

again the last equation yields \(\varphi = \alpha(t)x\), and with this \textit{ansatz} the first two equations both read

\[
\alpha' x + x^2 \alpha - 2x^2 \alpha = 0.
\]

However, now the equation enforces \(\alpha(t) = 0\), and hence there are no nontrivial symmetries.

The (possible) lack of correspondence between the symmetries of an Ito equation and of the corresponding Stratonovich equation might seem rather surprising at first; however, the notion of correspondence between an Ito and the associated Stratonovich equation is not so trivial, as discussed e.g. in the last chapter of the book by Stroock \([200]\) (see in particular Sect. 8.1.2 there), and thus the difference between the symmetries of the two is not so strange, after all.

\textbf{Remark 4.6.} It should also be stressed that the above discussion only hints at having possibly different symmetries for a Stratonovich and the corresponding Ito equations; but it does not rule out the possibility that symmetries to the two are in correspondence, albeit not identical\(^8\). In the case of random symmetries, to be discussed in Chapter 5, we will show explicit examples where symmetries of an Ito and of the corresponding SDE are not the same, but are in one-to-one correspondence.

On the other hand, an Ito equation and the associated Stratonovich equation do carry the same \textit{statistical} information. In view of the discussion and results in Sect. 3.3 (and in \([100]\)), we would expect there is a correspondence between symmetries of the Fokker-Planck equation which are also symmetries of the Ito equation and symmetries of the equivalent Stratonovich equation. This is indeed the case, as shown by Spadaro; see \([101]\) for details of the proof.

\textbf{Proposition 17.} \textit{Given an Ito equation and the associated Fokker-Planck equation, the symmetries of the latter which are also symmetries of the Ito equation, are also symmetries of the associated Stratonovich equation.}

\textbf{Remark 4.7.} As stressed by Unal \([206]\) (and confirmed by Lunini \([152]\)), in order to really obtain different \textit{deterministic} symmetries in the Ito and Stratonovich cases, one should consider rather complex situations. Note that in

\(^8\)Once again, we have here an interesting project to be pursued.
the case of random symmetries, one can have different symmetries even in the simple case, as shown in Sect. 5.5.

Remark 4.8. This review focuses on Ito and Stratonovich equations. However it is also possible to interpolate between these two classes of equations; in fact, one can have intermediate type equations depending on a continuous parameter \( \alpha \in [0, 1] \) so that for \( \alpha = 0 \) we have Stratonovich equations and for \( \alpha = 1 \) Ito equations (see e.g. [137], where applications are also considered). It would be interesting to study the symmetries of these intermediate cases, and ascertain if the whole family admits – at least in the case of simple deterministic symmetries – the same ones.
Chapter 5

Random symmetries

5.1 Random diffeomorphisms

In the case of deterministic differential equations it is entirely natural to consider transformations generated by smooth, deterministic vector fields. But, in the case of stochastic differential equations the restriction to deterministic generators (as we have considered so far) is not that obvious. In fact, one could argue that the transformations to be considered should be stochastic as well.

This point of view was adopted by L. Arnold and P. Imkeller in their seminal work on normal forms for stochastic differential equations [13] (see also [12]), and they considered random diffeomorphisms as generators of the normalizing transformations. We will follow the same approach for symmetries of SDEs; i.e. consider, beside the usual (deterministic) diffeomorphisms, random diffeomorphisms as well.\(^1\)

5.1.1 Random maps

Arnold and Imkeller [13] define a near-identity random map \(h : \Omega \times M \to M\), with \(M\) a smooth manifold and \(\Omega\) a probability space, as a measurable map such that: (i) \(h(\omega, \cdot) \in C^\infty(M)\); (ii) \(h(\omega, 0) = 0\); (iii) \((Dh)(\omega, 0) = \text{id}\).

Property (i) means that we can consider this as a family of diffeomorphisms (i.e., passing to generators, of vector fields) on \(M\), depending on elements \(\omega\) of the probability space \(\Omega\). Note that this dependence is rather arbitrary, in particular no request of smoothness is present.

We will also refer to the generator of such a map, with a slight abuse of notation, as a random diffeomorphism. Note that random diffeomorphisms (as well as random maps) only act in the underlying smooth manifold \(M\), i.e. they do not act (but see next Section) on the elements of the probability space \(\Omega\).\(^2\)

\(^1\)This Chapter will follow a recent paper by the author and F. Spadaro [101]. I am indebted to C. Lunini for a number of questions and remarks on these matters.

\(^2\)In our case, \(M = \mathbb{R} \times M_0\), with \(\mathbb{R}\) corresponding to the time coordinate, is the phase manifold for the system, while \(\Omega\) will be the path space for the \(n\)-dimensional Wiener process.
With local coordinates \(x^i\) on \(M_0\), we want to consider general random diffeomorphisms generated by vector fields of the form

\[
X = \tau(x, t; w) \partial_t + \varphi^i(x, t; w) \partial_i .
\]  

(5.1)

A \textit{time-preserving} random diffeomorphism will be characterized by having \(\tau = 0\), while the \textit{fibration-preserving} ones (with reference to the fibration \(M \to \mathbb{R}\)) will be characterized by \(\partial_i \tau = 0\) for all \(i\), i.e. \(\tau = \tau(t; w)\).

We will start by considering “simple” (i.e. time-preserving) random symmetries in order to tackle the key problem in the simplest setting \([52, 101]\); later on (Sect. 5.2.2) we will consider the general case. Simple random symmetries will be time-preserving, i.e. have as generator

\[
Y = \varphi^i(x, t; w) \partial_i .
\]

(5.2)

In the following it will be convenient to use the notation

\[
\triangle f := \sum_{k=1}^{n} \frac{\partial^2 f}{\partial w^k \partial w^k} + \sum_{j,k=1}^{n} (\sigma \sigma^T)^{jk} \frac{\partial^2 f}{\partial x^j \partial x^k} .
\]

(5.3)

Note that for a function depending only on the \((x, t)\) variables – as in the case of deterministic symmetries – the first term on the r.h.s. vanishes identically.

We recall once again that, as mentioned above at several points (or see e.g. Theorem 8.20 in the book by Oksendal \([171]\)), a transformation of time will induce a transformation of the Wiener processes. In particular a map \(t \to t + \varepsilon \tau(x, t)\) will induce the map

\[
dw^k \to dw^k + \varepsilon \frac{1}{2} \left(\frac{d\tau}{dt}\right) dw^k := dw^k + \varepsilon \delta w^k .
\]

(5.4)

\section*{5.1.2 Random \(W\)-maps}

One can also consider general vector fields in the \((x, t; w)\) space \([101]\), i.e.

\[
Y = \tau(x, t; w) \partial_t + \varphi^i(x, t; w) \partial_i + h^k(x, t; w) \hat{\partial}_k .
\]

(5.5)

Here we started to use the shorthand notation

\[
\hat{\partial}_k := \partial / \partial w^k ,
\]

(5.6)

which will be routinely used also in the following. We also write \(X = \tau \partial_t + \varphi^i \partial_i\) for the restriction of \(Y\) in (5.5) to the \((x, t)\) space.

Note that in (5.5) we are considering also the possibility of direct action on the \(w^k\) variables (apart from the action induced by a change in time), as in the approach to \(W\)-symmetries \([89]\).
As already pointed out in Sect. 3.4, the requirement that the transformed processes \( \tilde{w}^k(t) = w^k(t) + \varepsilon h^k(x,t,w) \) are still Wiener processes, implies that \( \tilde{w}^k = M_k^T w^k \) with \( M \) an orthogonal matrix, and hence that necessarily

\[
h^k = B^k(x,t;w) w^k \tag{5.7}
\]

with \( B \) a (real) antisymmetric matrix. This will be assumed from now on. Note moreover that if \( B \) does not depend on \( w \), then \( \Delta(h^k) \) reduces to its “deterministic” part; and that, as discussed in Sect. 3.4, \( B \) should not actually depend on the \( x \) nor on the \( w \) variables.

### 5.1.3 Random time changes

If we allow a time map which depend on the state of the Wiener processes \( w^k(t) \), or even just on the \( x^i(t) \), we are actually allowing a random time change. This point seems to have originated some confusion in part of the literature devoted to symmetry of SDEs, so we will briefly discuss it in the present subsection.

On physical grounds one would be specially interested in the case where the change of time does not depend neither on the realization of the stochastic processes \( w^k(t) \) nor on the spatial coordinates \( x^i \); i.e. in the case of fiber-preserving maps. These will be obtained from the general case by simply setting \( \tau = \tau(t) \).

It should also be recalled that, as already mentioned in Remark 3.7, considering a time change which depends on \( x \) and/or \( w \) (and is therefore a random time change) will in general destroy the absolute continuity of the measure of the transformed process w.r.t. that of the original one, and will give raise to a number of quite delicate questions [55, 70, 85, 122, 123, 135, 153, 171, 210]; thus the discussion concerning such transformations should to a large extent be considered, from the mathematical point of view, as a formal one.

It is maybe worth providing some further detail on this point (also to explain to which extent our discussion will not be just formal, and what are the underlying problems). When considering a time change depending on \( x \) one should bear in mind that \( x(t) \) follows a SDE and is therefore a random process; this also holds, of course, for \( w(t) \). Such time changes are thus in general not acceptable. To make a long story short and non-rigorous (see e.g. [85, 122, 123, 153, 171, 200] for a precise discussion), one should consider time changes described by integrals of functions of \( x \) and/or \( w \); the integration has of course a regularizing effect. Thus one should consider time changes of the form

\[
t \to \tilde{t} = \beta(x,t,w) = \int_0^t \gamma(x,s,w) ds \; ;
\]

\[3\]The case where time changes depends on the dependent coordinates has been analyzed by several authors; see e.g. [86, 87, 155, 196, 197]. A further complication comes from the fact that in some of these papers the authors considered transformations which could map the Wiener processes driving the SDE into a process of different nature. As far as I know this point was first raised by Unal [206] and his remarks originated a debate which I will not report here: we want transformations leaving the SDE, and a fortiori the nature of the processes driving it, unchanged.
in our case the function $\gamma$ should be of the form
\[
\gamma(x, s, w) = 1 + \varepsilon \theta(x, s, w),
\]
and the relation between $\theta$ and $\tau$ is
\[
\theta = d\tau/dt.
\]
Under the map (5.8), the standard Wiener process $w(t)$ with increments $dw$
is mapped into a Wiener process $z(t)$ with increments
\[
 dz = \int_0^t \sqrt{\gamma(x, s, w)} \, dw(s).
\]
A brief self-contained discussion is provided e.g. in Chapter 8 of [171].

5.2 Ito equations

5.2.1 Simple random symmetries of Ito equations

We will now consider the case of simple random symmetries, i.e. vector fields
of the form (5.2); under this the Ito equation
\[
 dx^i = f^i(x, t) \, dt + \sigma^i_k(x, t) \, dw^k
\]
is in general mapped into a new, generally different, Ito equation. The equation
remains invariant if and only if the component of the vector field (5.2) satisfy
appropriate relations, i.e. the determining equations. More precisely, as shown
in [101],

**Proposition 18.** The determining equations for simple random symmetries for
the Ito equation (5.11) are
\[
\begin{align*}
(\partial_i \varphi^i) + f^i(\partial_j \varphi^i) - \varphi^j(\partial_j f^i) &= -\frac{1}{2} (\triangle \varphi^i), \\
(\tilde{\partial}_k \varphi^i) + \sigma^j_k(\partial_j \varphi^i) - \varphi^j(\partial_j \sigma^i_k) &= 0.
\end{align*}
\]

**Remark 5.1.** Apparently the only difference w.r.t. the determining equations
for (simple) deterministic symmetries (3.20) is the presence of the $\tilde{\partial}_k \varphi^i$ term in
the second equation. But one should however recall that – despite the formal
analogy – the term $\triangle \varphi^i$ does now also include derivatives w.r.t. the $w^k$ variables,
which are of course absent in (3.20).

**Remark 5.2.** The solutions to the determining equations should then be evaluated
on the flow of the evolution equation (the Ito SDE); this can lead some
function to get less general, or even trivial; see in this respect Examples 5.1 and
5.3 below.
**Example 5.1.** We start by considering a rather trivial example, i.e. the equation (3.27) of Example 3.1. In this case ($f = 0, \sigma = \sigma_0$) we just have a system of two equations for the single function $\varphi = \varphi(x,t;w)$, and (5.12) read

$$
\partial_t \varphi = -\frac{1}{2} \triangle \varphi, \quad \partial_w \varphi = -\sigma_0 \left( \partial_x \varphi \right).
$$

The solution to the second of these is $\varphi(x,t,w) = \psi(z,t)$, where $F$ is an arbitrary (smooth) function of $z := x - \sigma_0 w$ and $t$. Now the first equation reads

$$
\psi_t + \sigma_0^2 \psi_{zz} = 0,
$$

and Fourier transforming we write $\psi = \rho_k(t) \exp[i k z]$ and obtain $\rho_k(t) = r_k \exp[\sigma_0^2 k^2 t]$, with $r_k$ a constant. It should also be noted that $dz = 0$ on solutions to our equation (3.27) (see Remark 5.2).

**Example 5.2.** We consider another one-dimensional example, i.e.

$$
dx = dt + x \, dw; \quad (5.13)
$$

this was considered in [100], where it was shown it admits no deterministic symmetries. The determining equations (5.12) read now

$$
\varphi_t + \varphi_x = -\frac{1}{2} \left( \varphi_{ww} + x^2 \varphi_{xx} \right), \quad \varphi_w + x \varphi_x = \varphi;
$$

the second equation implies

$$
\varphi(x,t,w) = x \psi(z,t), \quad z := x e^{-w}.
$$

Plugging this into the first equation we get

$$
\psi + z \psi_z + x \left( \psi_t + \frac{3}{2} z \psi_z + z^2 \psi_{zz} \right),
$$

which yields

$$
\psi(z,t) = c \frac{e^{-t/2}}{z},
$$

with $c$ a constant; we hence have a simple random symmetry identified by

$$
\varphi = e^{w-t/2}.
$$

**Example 5.3.** We pass to consider examples in dimension two. The first case we consider is a system related to work by Finkel [84], i.e.

$$
\begin{align*}
dx_1 &= (a_1/x_1) \, dt + dw_1, \\
dx_2 &= a_2 \, dt + dw_2;
\end{align*} \quad (5.14)
$$

\footnote{We will write, here and below, the explicit vector indices (in $x$, $w$, $\varphi$) as lower ones in order to avoid any possible misunderstanding.}
here \( a_1, a_2 \) are two non-zero real constants.

In this case the second set of determining equations (5.12) imply that, setting 
\[ z_k := w_k - x_k, \]
\[
\varphi_1(x_1, x_2, t; w_1, w_2) = \eta_1(t, z_1, z_2), \quad \varphi_2(x_1, x_2, t; w_1, w_2) = \eta_2(t, z_1, z_2). 
\]

Plugging these into the first set of equations in (5.12), and recalling that \( \eta_i = \eta_i(t, z_1, z_2) \), and hence the coefficient of different powers of \( x_1 \) must vanish separately, we have 
\[
\eta_1(t, z_1, z_2) = 0, \quad \eta_2(t, z_1, z_2) = \eta(t, z). 
\]

finally plugging this into the equation for \( \eta_2 \) we obtain that \( \xi \) is an arbitrary function of \( \zeta = a_2 t + z_2 \). In conclusion, we got
\[
\varphi_1 = 0, \quad \varphi_2 = \eta(t, z_2). 
\]

Note that, again, \( \zeta \) (and hence \( \eta(\zeta) \)) is trivially constant, i.e. \( d\zeta = 0 \), on solutions to (5.14) (and again see Remark 5.2 in this respect). ♦

**Example 5.4.** We will consider another two-dimensional example, which is an Ornstein-Uhlenbeck type process related to the Kramers equation:
\[
\begin{align*}
 dx &= y \, dt \\
 dy &= -k^2 y \, dt + \sqrt{2} k \, dw(t).
\end{align*}
\]

this is the system considered in Example 3.2. Note we have a single Wiener process \( w(t) \), and correspondingly we will look for solutions \( \varphi^j = \varphi^j(x_1, x_2, t; w) \).

As in the previous example, we will start from the second set of equations in (5.12); for our system these read
\[
\begin{align*}
 \frac{\partial \varphi_1}{\partial w} &= 0, \quad \frac{\partial \varphi_1}{\partial x_2} = 0, \quad \frac{\partial \varphi_2}{\partial w} = 0, \quad \frac{\partial \varphi_2}{\partial x_2} = 0.
\end{align*}
\]

These of course rule out any possible dependence on \( w \), i.e. show that there is no simple random symmetry. ♦

**Example 5.5.** In applications one is often faced with \( n \)-dimensional system, depending on \( n \) Wiener processes,
\[
\begin{align*}
 dx^i &= (M^i_j \, x^j) \, dt + \sigma^i_j \, dw^j(t),
\end{align*}
\]

with \( M \) and \( \sigma \) constant matrices. It is also frequent that \( \sigma \) is diagonal.

In this case (for general, i.e. non necessarily diagonal, \( \sigma \)) the determining equations for simple random symmetries read
\[
\begin{align*}
 (\partial_i \varphi^i) + M^i_q \, x^q \, (\partial_j \varphi^j) - M^i_j \, \varphi^j + \frac{1}{2} \varphi_{ij} &= 0, \\
 (\hat{\partial}_k \varphi^i) + \sigma^j_k (\partial_j \varphi^i) &= 0.
\end{align*}
\]  

(5.15)
We start by considering the second set of equations; these yield
\[ \varphi^i(x^1, ..., x^n, t, w^1, ..., w^n) = \psi^i(z^1, ..., z^n; t), \]
where we have defined \( z^k := x^k - \sigma^k \xi \). For functions of this form we get immediately that
\[ \Delta \varphi^i = 2 \left( \sigma^l_k \sigma^m_k \frac{\partial^2 \psi^i}{\partial z^l \partial z^m} \right), \]
and hence the first set of determining equations read simply
\[
\frac{\partial \psi^i}{\partial t} + \left( M^j_k x^k \right) \left( \frac{\partial \psi^i}{\partial z^j} \right) - M^i_j \psi^j + \sigma^l_k \sigma^m_k \frac{\partial^2 \psi^i}{\partial z^l \partial z^m}. \tag{5.16}
\]
This requires in particular
\[ (M^T)^j_p \left( \frac{\partial \psi^i}{\partial z^j} \right) = 0, \]
which means \( \nabla \psi^i \) (for all \( i = 1, ..., n \)) is in the kernel of \( M^T \). If \( M \) is non singular, necessarily \( \nabla \psi^i = 0 \), hence \( \psi^i(z^1, ..., z^n; t) = \eta^i(t) \); moreover, substituting this into the equation again, we get the \( \eta^i(t) \) satisfy
\[ \frac{\partial \eta^i}{\partial t} = M^i_j \eta^j, \]

\[ \eta^i(t) = (\exp[Mt])^i_j \eta^j(0). \]

5.2.2 General random symmetries of Ito equations

We can now pass to consider general symmetries (note these could possibly also be W-symmetries) of Ito equations. The vector field (5.5) induces – taking into account also the discussion of the previous Section 5.1.2 and in particular eq.(5.4) – the infinitesimal map
\[
x^i \rightarrow x^i + \varepsilon \varphi^i(x, t; w), \quad \tau^k \rightarrow t + \delta \tau^k, \quad w^k \rightarrow w^k + \varepsilon \delta w^k, \tag{5.17}
\]

With this, the Ito equation (5.11) will again be mapped into a new, generally different, equation. It is convenient to introduce some further notation; we define the Misawa vector fields\(^5\) \( Y_\mu \) and the second order operator \( L \) by
\[
Y_0 := \partial_t + f^j \partial_j, \quad Y_k := \hat{\partial}_k + \sigma^l_k \partial_l ; \quad L := Y_0 + \frac{1}{2} \Delta. \tag{5.18}
\]

With these, the condition for the Ito equation (5.11) to be invariant are readily determined [101].

\(^5\)Note we are extending the definition of Misawa vector fields given previously, to include the \( \hat{\partial}_k \) terms.
Proposition 19. The determining equation for general random $W$-symmetries of (5.11) are

$$
X(f^i) - L(\varphi^i) + f^i L(\tau) + \sigma^i_k L(h^k) = 0 ,
X(\sigma^i_k) - Y_k(\varphi^i) + f^i Y_k(\tau) + \sigma^i_m Y_k(h^m) = 0 .
$$

(5.19)

The equations (5.19) can also be rewritten, using the explicit form of $L$ and $\psi$, as

$$
X(f^i) - Y_0(\varphi^i) + f^i Y_0(\tau) + \sigma^i_k Y_0(h^k) = \frac{1}{2} \left[ \Delta(\varphi^i) + f^i \Delta(\tau) + \sigma^i_k \Delta(h^k) \right] ,
X(\sigma^i_k) - Y_k(\varphi^i) + f^i Y_k(\tau) + \sigma^i_m Y_k(h^m) = -\frac{1}{2} (\partial_t \tau) \sigma^i_k .
$$

(5.20)

In the case of general symmetries not acting directly on the $w$ variables, i.e. on the Wiener processes (and thus excluding the case of $W$-symmetries) the equations (5.19) reduce to

$$
X(f^i) - L(\varphi^i) + f^i L(\tau) = 0 ,
X(\sigma^i_k) - Y_k(\varphi^i) + f^i Y_k(\tau) = 0 ,
$$

(5.21)
i.e. using the explicit form of $L$ and $\psi$, to

$$
X(f^i) - Y_0(\varphi^i) + f^i Y_0(\tau) = \frac{1}{2} \left[ \Delta(\varphi^i) + f^i \Delta(\tau) \right] ,
X(\sigma^i_k) - Y_k(\varphi^i) + f^i Y_k(\tau) = -\frac{1}{2} (\partial_t \tau) \sigma^i_k .
$$

(5.22)

Remark 5.3. This system is over-determined for all $n > 1$, and in general we will have no symmetries; even in the case there are symmetries, the equations are not always easy to deal with, despite being linear, due to the high dimension. For $n = 1$ the counting of equations and unknown functions would suggest we always have symmetries, but the solutions could be only local in some of the variables.

Remark 5.4. It is easily checked that in the case of deterministic vector fields, i.e. $\varphi = \varphi(x,t)$, $\tau = \tau(x,t)$, $h = 0$, the equations (5.19) reduce to the determining equations for deterministic symmetries seen in previous Chapters. Similarly, in the case of simple random symmetries, i.e. $\varphi = \varphi(x,t;w)$, $\tau = h = 0$, we get the equations (5.12) derived above, and for $\varphi$ not depending on $h$ these further reduce to the determining equations for simple (deterministic) symmetries.

\*\*\*It appears there is no simple way to write these in terms of commutation with Misawa vector fields. This should not be surprising, as that possibility – even in the deterministic case – is peculiar to Stratonovitch equations.\*\*\*
Remark 5.5. It is also worth considering random fiber-preserving symmetries, i.e. require $\tau = \tau(t)$ in (5.22) above. In this case we write the determining equations as

$$\partial_t \varphi^i + f^j \partial_j \varphi^i - \varphi^j \partial_j f^i - \tau \partial_t f^i - f^i \partial_t \tau + \frac{1}{2} \triangle \varphi^i = 0 ,$$

$$\hat{\partial} \varphi^i + \sigma^j_k \partial_j \varphi^i - \varphi^j \partial_j \sigma^i_k - \tau \partial_t \sigma^i_k - \frac{1}{2} (\partial_t \tau) \sigma^i_k = 0 ,$$

(5.23)

having reverted to a more explicit notation.

Example 5.6. We will consider again the equations of Example 5.2, i.e.

$$dx = dt + x dw ;$$

(5.24)

we have seen this does not admit any deterministic symmetry, and it admits a simple random symmetry, identified by $\varphi = \exp[w - t/2]$. We will now check this admits some more general random symmetry; in order to keep computations simple, we will restrict to the time-independent case $\tau = 0$ and $\varphi_t = h_t = 0$. (Note $\varphi_t = 0$ rules out the simple random symmetry obtained above.)

In this case the equations (5.19) read

$$x h_x + x^2 h_{xx} - \varphi_x - \frac{1}{2} (\varphi_{ww} + x^2 \varphi_{xx} + x h_{ww}) = 0$$

$$\varphi - \varphi_w - x \varphi_x + x h_w + x^2 h_x = 0 .$$

The second equation requires

$$\varphi(x, w) = x \ (h(x, w) + \eta(z)) , \quad z := w - \log(|x|) ;$$

plugging this into the first one we get

$$-\eta(z) + \eta'(z) + \frac{1}{2} \eta''(z) - x , \eta''(z) = h(x, w) - x^2 h_x(x, w) .$$

Solutions to these are provided by

$$h(x, w) = e^{1/x} \beta(w) + k , \quad \eta(z) = -k ,$$

with $k$ an arbitrary constant and $\beta$ an arbitrary smooth function.

The random symmetries we obtained in this way are

$$Y = \left[ x e^{1/x} \beta(w) \right] \partial_x + \left[ e^{1/x} \beta(w) + k \right] \partial_w .$$

(5.25)

Example 5.7. We consider the system

$$dx_1 = [1 - (x_1^2 + x_2^2)] x_1 \ dt + dw_1$$

$$dx_2 = [1 - (x_1^2 + x_2^2)] x_2 \ dt + dw_2 ;$$
this is manifestly covariant under simultaneous rotations in the \((x_1, x_2)\) and the 
\((w_1, w_2)\) planes [89].

In order to simplify (slightly) the computations, we will look for symmetries which are time-preserving and time-independent; that is, we assume again \(\tau = 0\), 
\((\partial_t \varphi^i) = 0 = (\partial_t h^k)\). Setting \(z_k := x_k - w_k\), the first set of \((5.19)\) provides

\[
\begin{align*}
h_1(x_1, x_2, w_1, w_2) &= \varphi_1(x_1, x_2, w_1, w_2) + \rho_1(z_1, z_2) \\
h_2(x_1, x_2, w_1, w_2) &= \varphi_2(x_1, x_2, w_1, w_2) + \rho_2(z_1, z_2),
\end{align*}
\]

where the \(\rho_i\) are arbitrary smooth functions of \((z_1, z_2)\).

Plugging these into the first set of \((5.19)\) we obtain two equations involving 
\(\varphi^i\) and derivatives of the \(\rho^i\). These equations can then be solved for the 
\(\varphi^i\) in terms of the \(\rho^i\), yielding some complicate expression we do not report.

This shows we have random symmetries in correspondence with arbitrary 
functions \(\rho_i(z_1, z_2)\). When these are linear,

\[
\begin{align*}
\rho_1 &= r_{10} + r_{11} z_1 + r_{12} z_2 ; \\
\rho_2 &= r_{20} + r_{21} z_1 + r_{22} z_2 ,
\end{align*}
\]

and writing \(\chi := [-1 + 3(x_1^2 + x_2^2)]\), the resulting random symmetries can be explicitly identified via standard simple computations. With the choice

\[
r_{10} = 0, r_{20} = 0 ; \\
r_{11} = 0, r_{12} = 1, r_{21} = -1, r_{22} = 0
\]

we get just simultaneous rotations (by the same angle) in the \((x_1, x_2)\) and 
\((w_1, w_2)\) planes [89].

\[\Box\]

5.3 Stratonovich equations

As in the case of Ito equations, we have first (in Chapter 4) considered deter-
mistic transformations for Stratonovich SDEs. But again, as in the Ito case, one would like to consider random diffeomorphisms as well. This will be done in the present Section.

5.3.1 Simple random symmetries of Stratonovich equa-
tions

We will start by considering simple random symmetries, i.e. generators of the form \((5.2)\), for the Stratonovich equation

\[
dx^i = b^i(x, t) dt + \sigma^i_k(x, t) \circ dw^k .
\]

Under the action of \(Y\), the equation \((5.26)\) is mapped into another equation of the same type. More precisely, the latter is

\[
dx^i + \varepsilon d\varphi^i = (b^i + \varepsilon \varphi^j \partial_j b^i) dt + (\sigma^i_k + \varepsilon \varphi^j \partial_j \sigma^i_k) \circ dw^k ;
\]

\[\Box\]
taking into account (5.26) and expanding the term $d\phi$, we have that terms of first order in $\varepsilon$ cancel out if and only if

$$
(\partial_t \varphi^i) dt + (\partial_j \varphi^i) dx^j + (\hat{\partial}_k \varphi^i) \circ dw^k = (\varphi^j \partial_j b^i) dt + (\varphi^j \partial_j \sigma^i_k) \circ dw^k .
$$

(5.28)

Note that the last term in the l.h.s. is the only difference with respect to the computation in the deterministic case.

Considering now $x$ on the solutions to (5.26), we get with a simple rearrangement [101]

$$
[\partial_t \varphi^i + b^j (\partial_j \varphi^i) - \varphi^j (\partial_j b^i)] dt + \left[ (\hat{\partial}_k \varphi^i) + \sigma^j_k (\partial_j \varphi^i) - \varphi^j (\partial_j \sigma^i_k) \right] \circ dw^k = 0 ;
$$

it follows immediately that

**Proposition 20.** The determining equations for simple random symmetries (of the form (5.2)) of the Stratonovich SDE (5.26) are

$$
\partial_t \varphi^i + b^j (\partial_j \varphi^i) - \varphi^j (\partial_j b^i) = 0 \quad (i = 1, \ldots, n)
$$

$$
\hat{\partial}_k \varphi^i + \sigma^j_k (\partial_j \varphi^i) - \varphi^j (\partial_j \sigma^i_k) = 0 \quad (i, k = 1, \ldots, n) .
$$

(5.29)

**Remark 5.6.** In order to express these determining equations in compact terms, it is convenient to consider the Misawa vector fields (5.18) associated with the SDE; in terms of these, the determining equations (5.29) read simply

$$
[Y, Y_\mu] = 0 \quad (\mu = 0, 1, \ldots, n) .
$$

(5.30)

These can be compared with (4.16) for deterministic symmetries.

**Remark 5.7.** The equations (5.29) should be compared with the corresponding determining equations for simple random symmetries of Ito equations, (5.12). Here too – as in the deterministic case – the second set of equations is just the same in the two cases.

**Remark 5.8.** More precisely, the determining equations (5.29) for simple random symmetries of (5.26) are immediately rewritten in terms of the coefficients $f^i$ of the equivalent Ito equation as $(i, k = 1, \ldots, n)$

$$
\partial_t \varphi^i + [f^i (\partial_j \varphi^i) - \varphi^j (\partial_j f^i)] - \rho^j (\partial_j \varphi^i) - \varphi^j (\partial_j \rho^i) = 0 ,
$$

$$
\hat{\partial}_k \varphi^i + [\sigma^j_k (\partial_j \varphi^i) - \varphi^j (\partial_j \sigma^i_k)] = 0 ;
$$

(5.31)

where $\rho^j (x, t)$ is defined in (4.22).

Note that the equations (5.31) can be expressed in the compact form (5.30) of commutation with the vector fields $Y_\mu$ defined in (5.18), except that now the same vector field $Y_0$ should now better (but equivalently) be defined as

$$
Y_0 = \partial_t + [f^i (x, t) + \rho^i (x, t)] \partial_i .
$$

(5.32)
It is immediate to check that the equations (5.31) do not coincide with the equations (5.12) determined above. The difference is due to
\[\delta^i := \varphi^j (\partial_j \rho^i) - \rho^j (\partial_j \varphi^i) - \frac{1}{2} \triangle \varphi^i \neq 0. \tag{5.33}\]
This inequality generally holds (for \(\sigma_x \neq 0\)) even in one dimension. Actually, as we have seen above (Sect. 4.4), this is true even for deterministic vector fields, i.e. for \(\tilde{\partial}_k \varphi^i \equiv 0\). But, as in the deterministic case, one should evaluate \(\delta^i\) on the space \(S\) of solutions to the (common) second set of determining equations.

5.3.2 Time-changing random symmetries of Stratonovich equations

The computations presented in Section 5.3.1 above can be extended to cover the case where the considered transformations act on time as well; in this case one should, as usual, take into account the map induced on the Wiener processes. Here we will consider only fiber-preserving maps, i.e. \(\tau = \tau(t, w)\). Proceeding in the standard way \cite{101}, we have that

**Proposition 21.** The determining equations for random fiber-preserving symmetries of the Stratonovich equation (5.26) are
\[
(\partial_t \varphi^i) + b^i (\partial_j \varphi^i) - \varphi^j (\partial_j b^i) - (\partial_t \tau b^i) = 0,
\]
\[
\tilde{\partial}_k \varphi^i + \sigma^j_k (\partial_j \varphi^i) - \varphi^j (\partial_j \sigma^i_k) - \tau (\partial_t \sigma^i_k) - (1/2)(\partial_t \tau) \sigma^i_k = 0. \tag{5.34}\]

**Remark 5.9.** If we want to express these in terms of commutation properties, introducing the vector fields
\[Z_0 = \partial_t + b^i (x, t; w) \partial_i, \quad Z_k = \tilde{\partial}_k + \sigma^i_k (x, t; w) \partial_i,\]
the determining equations (5.34) are rewritten – recalling we assume \(\tau\) does not depend on the \(x\) variables – as
\[\left[ Z_0, Z \right] = (\partial_t \tau) Z_0; \quad \left[ Z_k, Z \right] = (\tilde{\partial}_k \tau) \partial_i + \frac{1}{2} \sigma^i_k (\partial_t \tau) \partial_i.\]
Thus, in this case we do not obtain that the determining equations amount to a simple condition in terms of commutators (contrary to the case of simple symmetries, see Remark 5.6). 

---

\(^7\)This expression is formally equal to the one seen in Sect. 4.4, but one should recall that now \(\triangle\) also contains derivatives w.r.t. the \(w^k\) variables.

\(^8\)Actually, for random symmetries one could argue that the random transformations should also be set in Stratonovich form, and thus be inherently different from those considered for Ito equations.

\(^9\)We stress that here we are not considering maps acting directly on the Wiener processes; that is, here we are not considering \(W\)-symmetries; these will be considered later on in Sect. 5.4 (where we also give general formulas for possibly non fiber-preserving maps).
Remark 5.10. The determining equations (5.34) for random fiber-preserving symmetries of (5.26) can be rewritten in terms of the coefficients $f^i$ of the equivalent Ito equation as

\[ \partial_t \varphi^i + [f^j(\partial_j \varphi^i) - \varphi^j(\partial_j f^i)] = [\rho^i(\partial_j \varphi^i) - \varphi^j(\partial_j \rho^i)] + (f^i + \rho^i)(\partial_t \tau) , \]

\[ \hat{\partial}_k \varphi^i + [\sigma^j_k(\partial_j \varphi^i) - \varphi^j(\partial_j \sigma^i_k)] = \tau \partial_t \sigma^i_k + \frac{1}{2} (\partial_t \tau) \sigma^i_k ; \tag{5.35} \]

here $i,k = 1, ..., n$ and $\rho^i(x,t)$ is defined in (4.22).

These equations should be compared with the corresponding determining equations for simple random symmetries of the equivalent Ito equation, see (5.23). Once again the second set of equations coincide in the two cases.

It is immediate to check that the first set of equations (5.35) do not coincide with the corresponding equations in (5.23) determined above. However, as already remarked for deterministic and simple random symmetries, one should consider these equations restricted to the space $S$ of solutions to the (common) second set of equations.

Example 5.8. Let us consider the equation

\[ dx = -x dt + x \circ dw ; \]

in this case the Misawa vector fields are

\[ \begin{align*}
Y_0 &= \partial_t - x \partial_x ; \\
Y_1 &= \partial_w + x \partial_x .
\end{align*} \]

The requirement that $X := \varphi(x,t,w)\partial_x$ commutes with both $Y_0$ and $Y_1$ yields

\[ \varphi(x,t,w) = e^{-t} \eta(z) , \quad z := (e^w/x) . \]

Example 5.9. Let us consider the system

\[ \begin{align*}
\begin{align*}
\frac{dx_1}{dt} &= -x_2 dt + \alpha x_1 \circ dw_1 \\
\frac{dx_2}{dt} &= -x_1 dt + \alpha x_2 \circ dw_2 .
\end{align*}
\]

The Misawa vector fields are now

\[ \begin{align*}
Y_0 &= \partial_t - x_2 \partial_1 + x_1 \partial_2 ; \\
Y_1 &= \hat{\partial}_1 + \alpha \partial_1 , \\
Y_2 &= \hat{\partial}_2 + \alpha \partial_2 .
\end{align*} \]

Requiring the vector field

\[ X = \varphi^1(x_1,x_2,t,w_1,w_2) \partial_1 + \varphi^2(x_1,x_2,t,w_1,w_2) \partial_2 \]

to commute with $Y_1$ and $Y_2$ enforces

\[ \begin{align*}
\varphi^1 &= x_1 \eta^1(z_1,z_2,t) , \\
\varphi^2 &= x_2 \eta^2(z_1,z_2,t) ,
\end{align*} \]
where we have defined $z_k := [(aw_k - \log|x_k|)/a]$. Requiring now that $X$ also commutes with $Y_0$, we get that actually it must be $\eta^1 = \eta^2 = c$; thus in the end the only simple random symmetry of the system under consideration is

$$X = \partial_1 + \partial_2;$$

this is actually, obviously, a simple deterministic symmetry.

**Example 5.10.** We consider again the equation

$$dx = dt + x\, dw,$$

as in Example 5.2 above. The corresponding Stratonovich equation is

$$dx = \left[1 - \frac{x}{2}\right] dt + x \circ dw;$$

the determining equations (5.29) for simple random symmetries of this Stratonovich equation read

$$\partial_t \varphi + [1 - (x/2)] (\partial_x \varphi) + (1/2) \varphi = 0$$

$$\partial_w \varphi + x (\partial_x \varphi) - \varphi = 0.$$

It is immediate to check these, or more precisely the first of these, do not correspond to the equations obtained in Example 5.2. However this set of equations does admit a solution, which is just the same as that obtained in Example 5.2:

$$\varphi(x, t, w) = c_0 \exp[w - t/2].$$

This fact will be discussed below, see Sect.5.5.

**Example 5.11.** When dealing with symmetries of Stratonovich equations, it is customary to consider the Misawa system [161]

$$\begin{align*}
  dx_1 &= (x_3 - x_2) dt + (x_3 - x_2) \circ dw \\
  dx_2 &= (x_1 - x_3) dt + (x_1 - x_3) \circ dw \\
  dx_3 &= (x_2 - x_1) dt + (x_2 - x_1) \circ dw;
\end{align*}$$

it is well known – and immediately apparent – that this admits the simple symmetry generated by

$$X = (1/2)(x_1^2 + x_2^2 + x_3^2) \left(\partial_1 + \partial_2 + \partial_3\right)$$

(and many others, as discussed by Albeverio and Fei [3]; see Sect. 4.2.2). Note that this involves only one Wiener process, which will induce a non-symmetric expression for the equivalent Ito system.

Using (4.22), the equivalent system of Ito equations turns out to be

$$\begin{align*}
  dx_1 &= (1/2)(3x_3 - x_2 - 2x_1) dt + (x_3 - x_2) dw \\
  dx_2 &= (x_1 - x_3) dt + (x_1 - x_3) dw \\
  dx_3 &= (x_2 - x_1) dt + (x_2 - x_1) dw.
\end{align*}$$
It is immediate to check that the determining equations (5.12) are not satisfied by \( X \); more precisely, the second set of (5.12) are (of course) satisfied, while the first set is not: in fact, we get (for all \( i = 1, 2, 3 \))

\[
\partial_t \varphi_i + f^j (\partial_j \varphi_i) - \varphi^j (\partial_j f^i) + \frac{1}{2} (\Delta \varphi_i) = F(x),
\]

where

\[
F(x) = 2 x_1^2 + 3 x_2^2 + 3 x_3^2 - \left( \frac{5}{2} x_1 x_2 + 3 x_2 x_3 + \frac{5}{2} x_1 x_3 \right).
\]

5.4 Random W-symmetries for Stratonovich equations

In the previous Sect. 5.3.2 we have considered transformations acting on time, and through this on the Wiener processes, but not directly on the \( w \) variables. In order to complete our discussion, we should allow also for this possibility, i.e. consider (random) W-symmetries as well; this is precisely the subject of the present Section.

We will thus consider again a map of the general form (5.17). Under this, the Stratonovich equation (5.26) is mapped into (all functions should be thought as functions of \( (x, t) \) or \( (x, t; w) \) as appropriate)

\[
dx^i + \varepsilon d\varphi^i = \left[ b^j + \varepsilon \left( \varphi^j \partial_j b^i + \tau \partial_t b^i + h^k \partial_k b^i \right) \right] (dt + \varepsilon dr) \\
+ \left[ \sigma^i_k + \varepsilon \left( \varphi^j \partial_j \sigma^i_k + \tau \partial_t \sigma^i_k \right) \right] \left[ dw^k + \varepsilon \left( (1/2)(\partial_t \sigma^i_k) dw^k + dh^k \right) \right].
\]

The terms of order \( \varepsilon \) provide the equation

\[
d\varphi^i = \left( \varphi^j \partial_j b^i + \tau \partial_t b^i \right) dt + b^i d\tau + (1/2)(\partial_t \tau) \sigma^i_k dw^k \\
+ \sigma^i_k dh^k + \left( \varphi^j \partial_j \sigma^i_k + \tau \partial_t \sigma^i_k \right) dw^k.
\]

We should now expand the differentials \( d\tau, d\varphi^i, dh^k \), which gives

\[
d\tau = (\partial_t \tau) dt + (\partial_j \tau) dx^j + (\partial_k \tau) dw^k,
\]

and the like for \( d\varphi^i \) and \( dh^k \) (note we are not yet considering the restrictions of \( h^k \) which result from the discussion of Sect. 3.4; these will be introduced in a moment). Doing this, the equation results in the vanishing of the expression

\[
\left[ \partial_t \varphi^i - \tau \partial_t b^i - \varphi^j \partial_j b^i - b^i \partial_t \tau - \sigma^i_k \partial_k h^k \right] dt \\
+ \left[ \partial_k \varphi^i - b^i \partial_k \tau - (1/2)(\partial_t \tau) \sigma^i_k - \sigma^i_m \partial_m h^m - \varphi^j \partial_j \sigma^i_k - \tau \partial_t \sigma^i_k \right] dw^k \\
+ \left[ (\partial_j \varphi^i) - b^i (\partial_j \tau) - \sigma^i_k (\partial_j h^k) \right] dx^j.
\]
Now the determining equations (5.38) read simply

\[
\left[ \partial_t \varphi^i + (b^i \partial_j \varphi^i - \varphi^j \partial_j b^i) - \partial_t (\tau b^i) - b^i b^j (\partial_j \tau) - \sigma_k^i (\partial_i h^k) - \sigma^i_k (\partial_j h^k b^i) \right] dt \\
+ \left[ \hat{\partial}_k \varphi^i + \left( \sigma^j_k \partial_j \varphi^i - \varphi^j \partial_j \sigma^i_k \right) - b^i \left( \hat{\partial}_k \tau + \sigma^i_k (\partial_j \tau) \right) - \sigma^i_m (\hat{\partial}_k h^m) - \sigma^i_m (\partial_j h^m) \sigma^j_k \right] dw^k 
\]

(5.36)

It is now time to recall the discussion of Sect. 3.4 and its conclusions, i.e. that the \( h^k \) should not depend on the \( x \) (and actually be linear in the \( w \)). Taking this into account, the equations (5.36) get slightly simplified and we end up with

\[
\left[ \partial_t \varphi^i + (b^i \partial_j \varphi^i - \varphi^j \partial_j b^i) - \partial_t (\tau b^i) - b^i b^j (\partial_j \tau) - \sigma_k^i (\partial_i h^k) \right] dt \\
+ \left[ \hat{\partial}_k \varphi^i + \left( \sigma^j_k \partial_j \varphi^i - \varphi^j \partial_j \sigma^i_k \right) - b^i \left( \hat{\partial}_k \tau + \sigma^i_k (\partial_j \tau) \right) - \sigma^i_m (\hat{\partial}_k h^m) - (\tau \partial_i \sigma^i_k + (1/2) \sigma^i_k \partial_i \tau) \right] dw^k 
\]

(5.37)

hence we have the

**Proposition 22.** The determining equations for general random symmetries (including possibly \( W \)-symmetries) of the Stratonovich equation (5.26) are

\[
\partial_t \varphi^i + (b^i \partial_j \varphi^i - \varphi^j \partial_j b^i) - \partial_t (\tau b^i) - b^i b^j (\partial_j \tau) - \sigma_k^i (\partial_i h^k) = 0 , \\
\hat{\partial}_k \varphi^i + \left( \sigma^j_k \partial_j \varphi^i - \varphi^j \partial_j \sigma^i_k \right) - b^i \left( \hat{\partial}_k \tau + \sigma^i_k (\partial_j \tau) \right) - \sigma^i_m (\hat{\partial}_k h^m) - (\tau \partial_i \sigma^i_k + (1/2) \sigma^i_k \partial_i \tau) = 0 . 
\]

(5.38)

**Remark 5.11.** Not surprisingly, it appears that in this case too (see Remark 5.9) there is no simple way to express the determining equations (5.38) in terms of commutation properties with the Misawa vector fields.

**Example 5.12.** Let us consider again, as in Example 5.8, the equation

\[
dx = -x dt + x \circ dw .
\]

Now the determining equations (5.38) read simply

\[
\varphi_t - x \varphi_x + \varphi + x \tau_t - x^2 \tau_x - x h_t = 0 \\
\varphi_w + x \varphi_x - \varphi + x (\tau_w + x \tau_x) - (1/2) x \tau_t - x h_w = 0 .
\]

This is an under-determined system of two equations for the three unknown functions \( \varphi, \tau, h \); we will look for solutions with \( \tau \equiv 0 \), which yields the simplified equations

\[
\varphi_t - x \varphi_x + \varphi = x h_t \\
\varphi_w + x \varphi_x - \varphi = x h_w .
\]
Setting
\[ \varphi(x, t, w) = \psi(t, w) x , \quad h(x, t, w) = \psi(t, w) x , \quad h(x, t, w) = \psi(t, w) \]
we get a family of solutions. Note that, as discussed above, \( h \) – and therefore \( \psi \) – should actually be linear in \( w \); thus we will in fact have \( \psi(t, w) = \beta(t)w \), with \( \beta \) an arbitrary function. \( \diamond \)

**Example 5.13.** Let us consider again the equation studied in Example 5.10, i.e.
\[ dx = \left(1 - \frac{x}{2}\right) dt + x \circ dw ; \]
thus we have \( b = \left(1 - \frac{x}{2}\right), \sigma = x \). The determining equations will again be under-determined, and again we will look for solutions with \( \tau \equiv 0 \); moreover, we set \( h = \eta(t) w \). In this frame, (5.38) read
\[
\begin{align*}
\varphi_t + \left(1 - \frac{x}{2}\right) \varphi_x + \frac{1}{2} \varphi &= x w \eta' , \\
\varphi_w + x \varphi_x - \varphi &= x \eta .
\end{align*}
\]
The second equation yields immediately
\[ \varphi(x, t, w) = x \left[ \eta(t) \log(|x|) + \psi(t, z) \right] , \quad z = xe^{-w} . \]
Plugging this into the first equation we obtain that necessarily \( \eta(t) = 0 \) (as seen by considering the coefficients of terms with \( \log(|x|) \)), thus in this case we get no new symmetries with respect to those found in Example 5.10. \( \diamond \)

### 5.5 Random symmetries of Ito versus Stratonovich equations

In Sect.4.4 we have compared the determining equations for deterministic symmetries of an Ito and of the corresponding Stratonovich equation. Here we aim at doing the same for random symmetries\(^{10}\). We will actually focus on simple ones, and just consider scalar equations\(^{11}\); we will see by concrete examples that one can indeed have different symmetries (as well as just coinciding ones) in the two cases.

We first of all rewrite, for ease of reference, the determining equations for simple random symmetries of scalar Ito and Stratonovich equations; for the Ito case we have
\[
\begin{align*}
\varphi_t + f \varphi_x - \varphi f_x &= -\frac{1}{2} \left[ \varphi_{ww} + \sigma^2 \varphi_{xx} \right] \\
\varphi_w + \sigma \varphi_x - \varphi \sigma_x &= 0
\end{align*}
\]
\(^{10}\)This section is based on ongoing work with C. Lunini [95].

\(^{11}\)A more exhaustive analysis, considering more general random symmetries, is also possible along the same lines; but the simple case will already present the different situations which can occur in the general ones, see below.
while in the Stratonovich case (using the expression of \( \rho \) in terms of \( \sigma \)) the equations are

\[
\begin{align*}
\varphi_t + f \varphi_x - \varphi f_x &= -\frac{1}{2} \left[ \varphi \sigma_x^2 + \varphi \sigma \sigma_{xx} - \sigma \sigma_x \varphi_x \right] \\
\varphi_w + \sigma \varphi_x - \varphi \sigma_x &= 0.
\end{align*}
\] (5.40)

The first equations in the two sets are obviously different in general, as stressed by Unal [206]. The second equation in both sets is just the same; note also it is a linear equation, which makes it solvable, and that its solutions are a linear space. We will denote by \( S \) the space of solutions to this equation; needless to say, this depends on the function \( \sigma = \sigma(x,t) \).

We can then compare the first equations in the systems (5.39) and (5.40) when restricted to \( S \). This allows to identify situations in which the equations (and hence their solutions) coincide, situations in which the equations are different but they both admit only the trivial solution \( \varphi(x,t,w) = 0 \), and situations in which the two equations are different – and indeed they do not admit any common solution.

The condition for the two equations to coincide is immediately apparent from (5.39) and (5.40), and is just

\[
\left[ \varphi_{ww} + \sigma^2 \varphi_{xx} \right]_S = \left[ \varphi \sigma_x^2 + \varphi \sigma \sigma_{xx} - \sigma \sigma_x \varphi_x \right]_S.
\] (5.41)

This is a kind of “compatibility equation”.

If we consider a given \( \sigma(x,t) \), the second equation in (5.39) and (5.40) can be solved, providing a concrete expression for functions in \( S \). One can then proceed to solve the compatibility condition (5.41). Note that this provides a set of functions \( \varphi(x,t,w) \). By considering either one of the (first equations in) (5.39) or (5.40) considered as equations for \( f(x,t) \) one can determine the cases in which the Ito and the equivalent Stratonovich equation have the same symmetry.

**Remark 5.12.** We note that we are guaranteed to have only common solutions if (5.41) is satisfied. In fact, the condition to have a common simple random symmetry to an Ito and the associated Stratonovich equation is that both (5.39) and (5.40) are satisfied, i.e. that \( \varphi \) satisfies the system

\[
\begin{align*}
\varphi_t + f \varphi_x - \varphi f_x &= \frac{1}{2} \left[ \varphi_{ww} + \sigma^2 \varphi_{xx} \right] \\
\varphi_t + f \varphi_x - \varphi f_x &= -\frac{1}{2} \left[ \varphi \sigma_x^2 + \varphi \sigma \sigma_{xx} - \sigma \sigma_x \varphi_x \right] \\
\varphi_w + \sigma \varphi_x - \varphi \sigma_x &= 0.
\end{align*}
\] (5.42) (5.43) (5.44)

By considering the difference of (5.42) and (5.43), we reduce to a system which is just made of (5.39) (or equivalently (5.40)) and the compatibility equation (5.41). We conclude that common solutions are possible only if (5.41) is satisfied on solutions to the other equations in the system.
Note that this contains \( \varphi \) itself; thus it just provides a check (to know if a symmetry of, say, the Ito equation is also a symmetry for the associated Stratonovich one) once solutions have been determined. On the other hand, as we show in the Examples below, it allows to provide a general discussion and in particular to ascertain when (that is, for which \( f(x, t) \), given that a certain \( \sigma(x, t) \) has been assigned) there can be nontrivial common symmetries. ⊓⊔

**Remark 5.13.** As already mentioned, the scalar one-dimensional situation is rich enough to present the different possibilities, as shown by the following examples; one should nevertheless remark that (at least in principle) in higher dimensions one could have a situation where some of the symmetries are common, and some are different. Once again, we have a problem worth exploring. ⊓⊔

**Example 5.14.** Let us consider the case \( \sigma = 1 \). Now the equation (5.39.b), and the identical equation (5.40.b), reads

\[
\varphi_w + \varphi_x = 0 ,
\]

and its general solution is

\[
\varphi(x, t, w) = \psi(z, t) , \quad z := x - w .
\]

The compatibility equation (5.41) is therefore

\[
\psi_{zz} = 0 ,
\]

with general solution

\[
\psi(z, t) = \alpha(t) + \beta(t) z .
\]

By considering the determining equations as equations for \( f(x, t) \), with \( \sigma = 1 \) and \( \varphi \) as determined above, it turns out that the equations admitting such a symmetry are characterized by

\[
f(x, t) = F(t) + G(t) x ;
\]

the functions \( F \) and \( G \) are related to the \( \alpha \) and \( \beta \) characterizing \( \psi \) and hence \( \varphi \) by several equations, which we do not write explicitly. Note that this discussion shows that when \( f \) is not of the above (linear) form, there will be no nontrivial common symmetries for the Ito and the corresponding Stratonovich equations.\footnote{12}{The solution sets may coincide in that they reduce to the trivial one.}

\[\Diamond\]

**Example 5.15.** A similar discussion can be conducted for the case \( \sigma = x \). Now the second equation in (5.39) and (5.40) reads

\[
\varphi_w + x \varphi_x - \varphi = 0 ,
\]
and its general solution is
\[ \varphi(x, t, w) = x \psi(z, t), \quad z := x e^{-w}. \tag{5.46} \]
The compatibility equation (5.41) is therefore
\[ [x \varphi_x + x^2 \varphi_{xx} + \varphi_{ww} - \varphi]_S = 0 \]
which using our functional form for \( \varphi \) reads explicitly
\[ z \psi_z + z^2 \psi_{zz} = 0, \]
with general solution \( \psi(z, t) = \alpha(t) + (1/z) \beta(t) \) and hence
\[ \varphi(x, t, w) = x \alpha(t) + e^{-w} \beta(t). \tag{5.47} \]
Correspondingly, the \( f(x, t) \) admitting a symmetry identified by (5.47) are identified by plugging the latter into (5.39) or (5.40), which yields \( f_{xx} = 0 \), i.e.
\[ f(x, t) = g(t) + h(t) x. \]
Again the functions \( g, h \) are linked to the \( \alpha, \beta \) by certain simple relations which we do not write explicitly.
This discussion shows immediately what are the \( f(x, t) \) which – for the given \( \sigma(x, t) \) – can give common solutions to the determining equations (5.39) and (5.40); and again we just get linear functions. ♦

**Example 5.16.** In Example 5.2 and Example 5.10 we have considered the Ito equation

\[ dx = dt + x dw \]

and the associated Stratonovich one; we obtained that albeit the determining equations are not the same, their solutions do coincide and are given by

\[ \varphi = c_0 \exp[w - t/2]. \]

The discussion of the previous Example 5.15 explains why and how this happens. ♦

**Example 5.17.** Motivated by Example 5.14, let us consider the linear Ito equations

\[ dx = (\alpha(t) + \beta(t) x) dt + dw. \tag{5.48} \]
The second of (5.39) will give (5.45). With this, the other determining equation reads simply

\[ \psi_t + (\alpha + \beta x) \psi_z - \beta \psi + \psi_{zz} = 0. \tag{5.49} \]
This implies \( \psi_t = 0 \), i.e. \( \psi(x, t) = \theta(t) \), and the equation is thus reduced to

\[ \theta'(t) = \beta(t) \theta(t), \tag{5.50} \]
with solution
\[ \theta(t) = \theta(0) \exp \left[ \int_0^t \beta(s) \, ds \right]. \]

We thus have as solution to the determining equations
\[ \varphi(x, t, w) = \varphi(t) = c_0 \exp \left[ \int_0^t \beta(s) \, ds \right]. \]

Let us now consider the corresponding Stratonovich equations and the determining equations (5.40) for their simple symmetries. The first equation reads
\[ \psi_t + (\alpha + \beta x) \psi_z - \beta \psi = 0 \]  \hspace{1cm} (5.51)

This is of course still different from (5.49), but it still requires \( \psi_z = 0 \) and hence \( \psi(z, t) = \theta(t) \); the equation for \( \theta \) is again (5.50), and we thus have that simple symmetries are common to the Ito and the corresponding Stratonovich equations.

Example 5.18. Consider now the Ito equation
\[ dx = x^2 \, dt + dw; \]
according to the discussion in Example 5.14, this should not admit common symmetries for this and the corresponding Stratonovich equation.

In facts, the first of (5.39) reads now, assuming again (5.45),
\[ \psi_t + x^2 \psi_z + \psi_{zz} - 2x \psi = 0; \]
this enforces \( \psi(z, t) = 0 \). If we consider the first of (5.40) we get
\[ \psi_t + x^2 \psi_z - 2x \psi = 0, \]
which also enforces \( \psi(z, t) = 0 \); again the equations are different and again they admit the same solutions – but now they are just the trivial one, i.e. provide no symmetry. Note that this discussion is easily extended to the case where \( x^2 \) is replaced by a general polynomial of order \( n > 1 \).

Example 5.19. Finally, let us consider
\[ dx = e^{-kx} \, dt + dw. \]

Now the first of (5.39) reads
\[ \psi_t + e^{-kx} (\psi_z + k \psi) + \psi_{zz} = 0. \]
This requires
\[ \psi(z, t) = \gamma(t) e^{-kz}, \]
which should moreover satisfy
\[ \left[ \gamma'(t) + k^2 \gamma(t) \right] e^{-kz} = 0 ; \]
this in turn enforces \( \gamma(t) = c_0 \exp[-k^2 t] \). We thus have
\[ \varphi_{\text{ito}}(x,t,w) = c_0 \exp[-k(x-w) - k^2 t] . \]

Let us now consider the corresponding Stratonovich equation; the first of (5.40) reads
\[ \psi_z + e^{-kx} \psi_z + k e^{-kx} \psi = 0 . \]
This again requires \( \psi_z = -k \psi \) and hence \( \psi(z,t) = \gamma(t) \exp(-kz) \), but this should now moreover satisfy
\[ \gamma'(t) e^{-kz} = 0 , \]
which of course provides \( \gamma(t) = c_0 \). Thus we get
\[ \varphi_{\text{strat}}(x,t,w) = c_0 \exp[-k(x-w)] . \]

That is, in this case the Ito and the Stratonovich equations do admit non-trivial symmetries, but they are not the same. On the other hand, there is a (simple) one to one correspondence between the two sets.

\[ \Diamond \]

**Remark 5.14.** As mentioned in Remark 4.8, one could consider intermediate cases between Ito and Stratonovich equations depending on a continuous parameter \( \alpha \) [137]. This opens the possibility to study how the symmetries of these intermediate equations depend on such a parameter, and possibly if there is a correspondence between different symmetries in the extreme – i.e. the Ito and the Stratonovich – cases we are here considering, as was the case in Example 5.19 above.

\[ \bigcirc \]
Chapter 6

Use of symmetries for studying SDEs

So far, our discussion – like most of the literature devoted to symmetries of stochastic differential equations – focused on determining what is the “right definition” for symmetries of a SDE rather than on the use of these symmetries.

It may look surprising that applications of symmetries – except for extension of the Noether theorem to the stochastic framework [4, 161, 203, 204], which is of course an extremely important application! – were not more actively pursued neither at the time of the early works of Misawa and Albeverio & Fei [3, 162, 163, 164], nor more recently.

In my opinion one of the reason for this is quite simply that the way to proceed for “applications” was (at least in principle) clear: if we have a symmetry, we should use (as in the deterministic case, and as done in stochastic Noether theory) symmetry-adapted coordinates.

In this chapter we will consider – rather briefly, which also keeps us in line with the general literature – several applications of this general idea. There is no need to stress that a lot of work remains to be done in this direction, both in terms of general theory and in terms of concrete applications to specific problems, but as we will see below (Sect. 6.4) there is a close analogue of the symmetry reduction holding for deterministic ODEs, see Propositions 33 and 34. These appear to be the gateway for much of this future work.

We will actually follow, for once, the time development of the subject. We start from considering Stratonovich type equation, as in the early works of Misawa and Albeverio and Fei [3, 162, 163, 164] mentioned above (Sect. 6.1); we will then also consider Ito equations (Sect. 6.2) and the problem of linearization of SDEs (Sect. 6.3), which is also a special case of the more general problem of reducing a SDE to a simpler (i.e. more convenient) form, see Sect. 6.4. The reduction of SDEs will also be considered in this context.

Remark 6.1. In the previous chapters, see in particular Chap. 3, we have also considered the relations between symmetries of a SDE and those of the
associated Fokker-Planck equation; symmetries of the Fokker-Planck equation can of course be used to determine explicit solutions to it (as discussed in Chap. 2). However this falls within the realm of deterministic equations and we will thus not discuss this aspect, referring the reader to the literature [63, 64, 84, 126, 128, 129, 130, 184, 187, 191, 192].

6.1 Stratonovich equations. Strong symmetries and strongly conserved quantities

As already mentioned, the early work by Misawa, and then by Albeverio and Fei, on symmetries of non-variational stochastic equations were prompted by the attempt to extend Hojman’s results (see Sect. 2.3.2) to the realm of SDE.

In fact, under an additional condition, a strong symmetry\(^1\) of a SDE is related to a strongly conserved quantity for the flow of the SDE under study. This is a stochastic counterpart to the Hojman theorem (Proposition 1 above, see Sect. 2.3.2).

A strongly conserved quantity \(J\) for the Stratonovich equation (4.1), which we rewrite here once again as

\[
\begin{align*}
\mathrm{d}x^i &= b^i(x,t) \, \mathrm{d}t + \sigma^i_k \circ \mathrm{d}w^k \\
&= b^i(x,t) \, \mathrm{d}t + \sigma^i_k \, \mathrm{d}w^k 
\end{align*}
\] (6.1)

for ease of reference, is a smooth function \(J(x,t)\) which is invariant under both \(X_0\) and all of the \(X_k\) defined in (4.14), thus under all the Misawa vector field associated to (6.1), i.e. such that

\[
X_\mu(J) = 0 \quad \forall \mu = 0, \ldots, n .
\] (6.2)

Remark 6.2. The name is justified by the fact that if \(J\) is a strongly conserved quantity for (6.1) then \(\mathrm{d}J \equiv 0\) on solutions to (6.1). Indeed, under (6.1) we have in general

\[
\mathrm{d}J = [X_0(J)] \, \mathrm{d}t + [X_k(J)] \circ \mathrm{d}w^k(t) ,
\] (6.3)

hence the above condition (6.2) guarantees that \(\mathrm{d}J \equiv 0\).

6.1.1 Time-preserving strong symmetries

We will first consider time-preserving strong symmetries; the following result was shown by Misawa [163].

Proposition 23. Let the vector field

\[
X = \varphi^i(x,t) \partial_i
\] (6.4)

\(^1\)These were defined in Sect. 4.2.
be a strong symmetry for the equation (6.1); assume moreover there exists a function $\lambda(x,t)$ such that

$$\frac{\partial b^i}{\partial x^i} = \text{div}(b) = -X_0(\lambda);$$
$$\frac{\partial \sigma^i_k}{\partial x^i} = \text{div}(\sigma_k) = -X_k(\lambda) \quad (k = 1, \ldots, n).$$

(6.5)

Then the quantity

$$J_\lambda := \frac{\partial \varphi^i}{\partial x^i} + X(\lambda) = \text{div}(\varphi) + X(\lambda)$$

(6.6)

is conserved under the flow of (6.1).

**Remark 6.3.** Note that for $\lambda$ a constant (or more generally a strongly conserved quantity), and hence $\text{div}(b) = 0 = \text{div}(\sigma)$, the conserved quantity is just $J_0 = \text{div}(\varphi)$. ⊙

**Proposition 24.** If $J$ is a conserved quantity for (6.1) and $X = \varphi^i(x,t)\partial_i$ a symmetry for it, then $X(J)$ is also conserved [3].

**Remark 6.4.** In fact, if $J$ satisfies (6.2) we immediately have, using (4.16),

$$X_k[X(J)] = X[X_k(J)] = 0,$$

for all $k = 0, 1, \ldots, n$. ⊙

**Remark 6.5.** On the other hand, as already seen above (see Proposition 15) it is apparent from (4.16) (see Remark 4.1) that strong symmetries of a given equation (6.1) form a Lie algebra [3]. This, together with the previous Proposition 24, means that we have a Lie algebraic structure for the set of conserved quantities under (6.1), see again [3] for a discussion. ⊙

**Remark 6.6.** In the same way, we observe that multiplying a symmetry vector field by a (nontrivial) constant of motion we obtain a (new) symmetry vector field. In fact, let $X$ satisfy (4.16), $J$ be a conserved quantity, and consider $Y = JX$. Then

$$[X_k, Y] = (X_k(J)) X + J [X_k, X] = 0.$$  

(6.7)

Thus, exactly as in the case of deterministic dynamical systems [60], the set of symmetries of a given stochastic differential equation (6.1) has also the structure of a Lie module over the algebra of constants of motion, or conserved quantities (it seems this fact went unnoticed so far). ⊙

**Example 6.1.** Let us consider again, as in Example 4.1, the Misawa example; i.e. the system with

$$b = -\begin{pmatrix} y-z \\ z-x \\ x-y \end{pmatrix}, \quad \sigma = -\begin{pmatrix} y-z & 0 & 0 \\ z-x & 0 & 0 \\ x-y & 0 & 0 \end{pmatrix}.$$  

(6.8)
Equation (6.5) is satisfied with $\lambda$ any constant. As already observed, the vector field $X = (|x|^2/2)(\partial_x + \partial_y + \partial_z)$ is a strong symmetry; the associated conserved quantity according to Proposition 23 is just

$$J_0 = |x|^2.$$ 

That is, the stochastic process lives on a sphere of constant radius, set by initial conditions. ♦

**Example 6.2.** Let us consider again Example 6.1 above, but from the point of view of Proposition 24 [3]. As observed earlier on, this admits many symmetries, e.g. those of the form

$$Y_\eta = [\eta(x,y,z)] (\partial_x + \partial_y + \partial_z),$$

where we can e.g. choose

$$\eta_0 = (x + y + z),$$
$$\eta_1 = (x^2 + y^2 + z^2),$$
$$\eta_2 = (xy + yz + zx),$$
$$\eta_3 = [x^2(y + z) + y^2(z + x) + z^2(x + y) + 3xyz].$$

The associated conserved quantities $J_k$ are $J_0 = 1$ (that is, a trivial one), $J_1 = J_2 = |x|^2 = (x^2 + y^2 + z^2) = \eta_1$ (as in Example 6.1), and

$$J_3 = 2(x^2 + y^2 + z^2) + 7(xy + yz + zx) = 2\eta_1 + 7\eta_2;$$

note that in view of $J_1$, this means that $\tilde{J}_3 = (xy + yz + zx) = \eta_2$ is conserved.

Actually the $\eta_k$ are also themselves conserved quantities on the flow of the system, as easily checked by direct computation.

We want now to check that symmetry maps conserved quantities into conserved quantities. In fact, writing $Y_k = Y_{\eta_k}$ for short, we have (beside, of course, $Y_k(J_0) = 0$) at first steps

$$Y_0(J_1) = 2(\eta_0 + 2\eta_2), \quad Y_1(J_1) = 2\eta_0 \eta_1,$$
$$Y_2(J_1) = 2\eta_0 \eta_2, \quad Y_3(J_1) = 2\eta_0 \eta_3;$$
$$Y_0(J_2) = 2(\eta_0 + 2\eta_2), \quad Y_1(J_2) = 2\eta_0 \eta_1,$$
$$Y_2(J_2) = 2\eta_0 \eta_2, \quad Y_3(J_2) = 2\eta_0 \eta_3;$$
$$Y_0(J_3) = 2(\eta_0 + 2\eta_2), \quad Y_1(J_3) = 2\eta_0 \eta_1,$$
$$Y_2(J_3) = 2\eta_0 \eta_2, \quad Y_3(J_3) = 2\eta_0 \eta_3.$$

As the $\eta_k$ are conserved, this verifies indeed Proposition 24. ♦

### 6.1.2 Extended strong symmetries

In Proposition 23, the symmetry vector field was bound to be of the form (6.4). When considering vector fields of the form

$$X = \tau(x,t) \partial_t + \varphi^i(x,t) \partial_i,$$ (6.9)
this result extends [3] to

**Proposition 25.** Let \( X \) be a vector field of the form (6.9) and let it be a strong symmetry for the equation (6.1); assume moreover there exists a function \( \lambda(x, t) \) such that

\[
\frac{\partial b^i}{\partial x^i} = -X_0(\lambda) ; \quad \frac{\partial \sigma^i_k}{\partial x^i} = -X_k(\lambda) \quad (k = 1, \ldots, n) .
\]  

(6.10)

Then the quantity

\[
K_\lambda := \frac{\partial \tau}{\partial t} + \frac{\partial \phi^i}{\partial x^i} + X(\lambda) - X_0(\tau)
\]

(6.11)

is conserved under the flow of (6.1).

**Remark 6.7.** The constructions of this Section are based on conserved quantities. We have seen in sect.2.4.3 (see in particular Proposition 3 there) that in the case of deterministic dynamical systems there is an essential relation between conserved quantities and orbital symmetries [214, 215]. I am not aware of any study of this question in the context of stochastic dynamical systems; quite clearly this would have some interest.

**Example 6.3.** Let us consider, as suggested in [3], the system (defined for \( t > 0 \), say \( t \geq 1 \)) given by

\[
\begin{align*}
    dx &= \left(\frac{x}{t}\right) dt + \left(\frac{x}{t}\right) (z - y) \circ dw \\
    dy &= \left(\frac{y}{t}\right) dt + \left(\frac{y}{t}\right) (x - z) \circ dw \\
    dz &= \left(\frac{z}{t}\right) dt + \left(\frac{z}{t}\right) (y - x) \circ dw 
\end{align*}
\]

this is again involving a single Wiener process. We consider the vector field

\[
X = \tau \partial_t := (x + y + z) \partial_t ;
\]

its relation with the Misawa vector fields is given by

\[
[X_0, X] = \frac{1}{t} (x + y + z) \ X_0 := \rho_0 X_0 ; \quad [X_1, X] = \rho_0 X_1 .
\]

In this case, eq. (6.10) is satisfied with \( \lambda = -3 \log(t) \). The associated conserved quantity is

\[
J = \frac{-4}{t} (x + y + z) = \frac{-4}{t} \tau .
\]

In fact, we have

\[
dJ = -4\frac{d\tau}{t} + 4\frac{\tau^2}{t^2} dt = \frac{4}{t} (dx + dy + dz) + 4\frac{\tau^2}{t^2} dt
\]

\[
= \frac{-4}{t} \left[ \frac{\tau}{t} dt + \frac{1}{t} [(xz - xy) + (yx - yz) + (zy - zx)] \circ dw \right] + \frac{4\tau^2}{t^2} dt
\]

\[
= 0 .
\]
6.2 Ito equations. Adapted coordinates

The experience built with deterministic differential equation – as well as common sense – suggests that in the presence of a symmetry it is convenient to reformulate the equation under study in terms of symmetry-adapted coordinates. It is quite natural to expect this also holds when dealing with stochastic differential equations, albeit in this case one does not (yet?) have as detailed results and theorems as in the deterministic case.

A substantial advance was provided by R. Kozlov [130] who noted that the same kind of results relating symmetry and reduction for ODEs also hold for SDEs, albeit with one (not so weak) extra condition: that is, only symmetries not acting on the time can actually be used. The precise results in this direction will be discussed in Sect. 6.4, where we review Kozlov’s work.

In this section we will instead briefly reconsider some of the examples discussed above from the point of view of symmetry adapted coordinates, obtaining other kind of results; as mentioned above, in Sect. 6.3 and even more in Sect. 6.4 we will see other applications – again based on the idea of symmetry-adapted coordinates – of the symmetry analysis of Ito equations.

Example 6.4. Let us consider again, as in Example 6.1, the system (6.8)

\[ \begin{align*}
   dr & = 0 \\
   d\vartheta & = \alpha(\vartheta, \varphi) \, dt + \alpha(\vartheta, \varphi) \circ dw \\
   d\varphi & = \beta(\vartheta, \varphi) \, dt + \beta(\vartheta, \varphi) \circ dw ;
\end{align*} \]

where of course \( \vartheta \in [0, 2\pi] \), \( \varphi \in [-\pi/2, \pi/2] \), and writing
\[\begin{align*}
   \alpha(\vartheta, \varphi) & := 1 - (\sin \vartheta + \cos \vartheta) \tan \varphi , \\
   \beta(\vartheta, \varphi) & := \sin \vartheta - \cos \vartheta ,
\end{align*}\]

the system considered in this example reads simply

\[\begin{align*}
   dr & = 0 \\
   d\vartheta & = \alpha(\vartheta, \varphi) \, dt + \alpha(\vartheta, \varphi) \circ dw \\
   d\varphi & = \beta(\vartheta, \varphi) \, dt + \beta(\vartheta, \varphi) \circ dw ;
\end{align*}\]

thus the conservation of \( r \) is completely explicit. ◊

Example 6.5. Let us consider again the system (3.29) seen in Example 3.2 and related to the Kramers equation, i.e.

\[\begin{align*}
   dx & = y \, dt \\
   dy & = -k^2 y \, dt + \sqrt{2k^2} \, dw(t) .
\end{align*}\]

By focusing on the vector field
\[ V_3 = e^{-k^2t} \left( k^{-2} \partial_x - \partial_y \right) \]

in its symmetry algebra (see Example 3.2), we pass to coordinates
\[ p = x , \quad q = y + k^2 x . \]
With these, we also have
\[ dx = dp, \quad dy = dq - k^2 dp; \]
thus the stochastic systems reads
\[
\begin{align*}
    dp &= (q - k^2 p) \, dt \\
    dq &= \sqrt{2k^2} \, dw(t).
\end{align*}
\]
In other words, using symmetry-adapted coordinates led us to decompose the system into a stochastic and a deterministic part.

\[\Box\]

**Example 6.6.** We consider again the system
\[
\begin{align*}
    dx_1 &= [1 - (x_1^2 + x_2^2)] x_1 \, dt + dw_1 \\
    dx_2 &= [1 - (x_1^2 + x_2^2)] x_2 \, dt + dw_2;
\end{align*}
\]
seen in Example 5.6.

We change coordinates as suggested by the symmetries determined in there; we will thus set
\[
    x_1 = \rho \cos(\vartheta), \quad x_2 = \rho \sin(\vartheta); \quad w_1 = \chi \cos(\lambda), \quad w_2 = \chi \sin(\lambda).
\]
With these coordinates, the system reads simply
\[
\begin{align*}
    d\rho &= (1 - \rho^2) \rho \, dt + \cos(\lambda - \vartheta) \, d\chi - \chi \sin(\lambda - \vartheta) \, d\lambda, \\
    d\vartheta &= (1/\rho) \left[ \sin(\lambda - \vartheta) \, d\chi + \chi \cos(\lambda - \vartheta) \, d\lambda \right].
\end{align*}
\]
The invariance under simultaneous rotations in the \((x_1, x_2)\) and \((w_1, w_2)\) planes (i.e. simultaneous shifts in the angles \(\vartheta\) and \(\lambda\)) is now completely explicit.

\[\Box\]

### 6.3 Ito equations. Linearization

Transforming an equation to linear form is a convenient way to solve it, and thus the problem of linearizing a given SDE was considered by several authors. I am not aware of a direct analogue of the Bluman-Kumei theorem mentioned in Sect. 2.6 above; but I will be mentioning here some results on this topic. I will actually only consider the case of a one-dimensional SDE.

In particular, Grigoriev, Ibragimov, Meleshko and Kovalev [107] considered the problem of linearizing a SDE via a smooth deterministic change of the dependent variable – they called this **strong linearization** – i.e. via a map
\[ y = \varphi(x, t); \tag{6.12} \]

\[\footnote{For other approaches to (symmetry-related) linearization of SDEs, see [129, 130, 211].}\]
we assume $\varphi_x \neq 0$, so the map is invertible, and we denote the inverse as $x = \psi(y, t)$. Under this map, the Ito equation

$$dx = f(x, t) \, dt + g(x, t) \, dw$$

(6.13)

is mapped into a new Ito equation for $y$,

$$dy = F(y, t) \, dt + G(y, t) \, dw$$

(6.14)

with

$$F(y, t) = \varphi_t + f \varphi_x + \frac{1}{2} g^2 \varphi_{xx}, \quad G(y, t) = g \varphi_x,$$

(6.15)

and the functions on the r.h.s. of these formulas should be understood to be expressed in terms of $y$ via $x = \psi(y, t)$.

One would like to get a linear equation for $y$, i.e. to have

$$F(y, t) = \alpha_0(t) + \alpha_1(t) y; \quad G(y, t) = \beta_0(t) + \beta_1(t) y.$$  

(6.16)

This is, of course, not always possible, so the problem lies in identifying the equations (that is, the functions $f$ and $g$) which are linearizable, and the linearizing map $\varphi$.

Comparing (6.15) and (6.16) it is clear that this amounts to studying conditions for the existence of solutions – and if these are satisfied, one would also like to find such solutions – to the system

$$\varphi_t + f \varphi_x + \frac{1}{2} g^2 \varphi_{xx} = \alpha_0 + \alpha_1 \varphi,$$

$$g \varphi_x = \beta_0 + \beta_1 \varphi.$$  

(6.17)

These can be considered as identifying the partial derivatives $\varphi_t$ and $\varphi_x$; the request that the mixed partial derivatives determination through the two definitions should coincide (that is, $\partial_x \varphi_t = \partial_t \varphi_x$) yields the solvability condition

$$\alpha_0 \beta_1 + (\beta_0)_t - \beta_0 (\alpha_1 + \mu) + \varphi [(\beta_1)_t - \beta_1 \mu] = 0,$$

(6.18)

where we have defined $\mu = \mu(x, t)$ by

$$\mu := \frac{1}{g} \left( g_t + f g_x + \frac{1}{2} g^2 g_{xx} - g f_x \right).$$

(6.19)

Note that $\mu$ is defined entirely in terms of the coefficients of the original SDE.

The situation is simpler when $\mu_x = 0$; in this case one has [107]

**Proposition 26.** Let the function $f$ and $g$ in equation (6.13) be such that $\mu$ defined in (6.19) satisfies $\mu_x = 0$. Then the Fokker-Planck equation associated to (6.13) is equivalent to the heat equation, and (6.13) is reduced to the linear SDE

$$dy = h(t) \, dw,$$

$$h(t) = \exp \left[ \int \mu(t) \, dt \right];$$

(6.20)
the linearizing map \( y = \varphi(x, t) \) is the solution to the compatible system of PDEs

\[
\varphi_t = h \left( \frac{g_x}{2} - \frac{f}{g} \right), \quad \varphi_x = \frac{h}{g}.
\]

(6.21)

The general case, i.e. \( \mu_x \neq 0 \), corresponds to a more involved situation. One can show [107] that a different condition also leads to similar results; the condition is now expressed by two equations:

\[
\begin{align*}
\mu_{xxx} - \frac{1}{g \mu_x} \left( g \mu_x \mu_{xx} - g_{xx} \mu_x^2 - g_x \mu_x \mu_{xx} \right) &= 0, \quad (6.22) \\
\mu_{xxt} - \frac{1}{g \mu_x} \left[ (g \mu_{xt} + g \mu_{x} - g_t \mu_x) \mu_{xx} \right. \\
&\quad \left. - (g_{xt} - g_x \mu + g \mu_x) \mu_x^2 \right] = 0. \quad (6.23)
\end{align*}
\]

With this, one has [107]:

**Proposition 27.** Let the function \( f \) and \( g \) in equation (6.13) be such that \( \mu \) defined in (6.19) satisfies (6.22), (6.23). Then (6.13) is reduced to the linear SDE

\[
dy = \alpha(x, t) \, dt + \beta(x, t) \, y \, dw,
\]

(6.24)

where

\[
\begin{align*}
\alpha(x, t) &= \exp \left[ \int \left( \beta \frac{g_x - \beta}{2} + \frac{g_x - f \beta}{g} + \frac{\mu_{xt}}{\mu_x} - 2 \frac{g \mu_x}{\beta} - \mu \right) \, dt \right], \\
\beta(x, t) &= - \frac{g_x \mu_x + g \mu_{xx}}{\mu_x};
\end{align*}
\]

the linearizing map \( y = \varphi(x, t) \) is given by

\[
\varphi = \frac{\alpha \beta}{\beta_t - \beta \mu}.
\]

(6.25)

**Remark 6.8.** Second order SDE have been analyzed along the same lines in a number of publications; see e.g. [107, 128, 130, 155, 195, 196, 197]. We will just refer the interested reader to these.

**Remark 6.9.** Linearization of SDE, and more generally reduction of SDE to an analogue of Poincaré-Dulac normal form [14, 81, 124, 212], has been considered by Arnold and Imkeller [12, 13]; they provided a complete solution to the problem, and hence I will just refer the reader to their work.

In this context, it may be noted that they mainly focused on the formal aspects of the theory and the determination of a (formal) Taylor series for the normalizing transformation, while as for the existence of an actual random diffeomorphism (see below) whose Taylor series coincides with the one they determine, they refer to a theorem by Borel as used and proved by Vanderbauwhede.
(see [209], page 142); it is possible that further progress can be obtained by looking at the problem of normal form convergence in a more specific way.

**Remark 6.10.** It should be stressed that Arnold and Imkeller did consider maps more general than those considered by Grigoriev *et al.*, i.e. *random diffeomorphisms* (see Chapter 5); this puts on equal footing the dynamical SDE and the change of variables required by the normalizing procedure, which can thus be seen as the flow of an equation of the same type, i.e. a SDE [101].

**Remark 6.11.** Perturbative linearization is also related to *approximate symmetries* (normalization corresponds to having the linear part of the system as a symmetry for the full system); in this respect one should consider [121] (see also [207]).

### 6.4 Transformation of an Ito equation to simpler form. Reduction

The approach considered in the previous Section 6.3 aimed at linearization of a SDE. More generally, one can aim at transforming a SDE into a convenient form. This is in general a simpler (than the original) form, but it may also be an equation which has already been studied and which is thus just more convenient, albeit possibly not simpler than the original one.

As symmetries are present – or absent – independently of the coordinates used, they are a natural tool to investigate if two equations might be transformed one into the other. This point of view has been advocated by several authors, and here we are specially interested in the results obtained by R. Kozlov in a series of papers [128, 129, 130], which we follow quite closely.

The idea was to classify possible symmetry groups of SDEs of different orders (in particular, low order ones); apart from the interest of the classification in itself, as a byproduct one would like to detect the possibility of transforming the equation into a simpler form.

**Remark 6.12.** It should be stressed that in this section – and actually in the whole Chapter – the symmetries are always meant to be *deterministic* ones; as

---

3This parallels the approach to Poincaré normalization through Lie transforms, i.e. through the time-one flow of a dynamical system [20, 29, 30].

4This remark goes back to Moser, who used it in the framework of normalization theory to advocate that a (Poincaré-Birkhoff) normal form could be actually (and not just formally) conjugated to the original form of the system under study only if the latter has a symmetry (this since the normal form admits its linear – or quadratic in the Hamiltonian case – part as a symmetry).

5One should mention that the reduction (and reconstruction) problem was also considered by Lazaro-Camí and Ortega [140] through a more geometric approach (they were also giving substantial attention to the Hamiltonian – thus variational –setting); we will not discuss their work here.

6It is maybe worth recalling that a similar approach was pursued by Lie in dealing with ordinary differential equations [151] (see also [108]); in most cases a maximal dimensional (for the order of the equation) symmetry group is a guarantee that the equation can be linearized.
far as I know there is no study along these lines considering random symmetries as well. This is a topic which is surely worth investigating.

6.4.1 First order scalar SDEs

In this Section, at difference with the other parts of this work, we will also consider higher order SDEs.

The case of a single first order equation is of course specially relevant, and we will hence start by considering it; the reader should be warned it is also in a way a degenerate one, as will be clear from comparing the results obtained here with those in Sect. 6.4.2 and Sect. 6.4.3 below.

For an equation of first order, i.e. a standard Ito equation

$$dx = f(x,t) \, dx + g(x,t) \, dw(t) \, , \quad (6.26)$$

we have the following results\(^7\) [128].

**Proposition 28.** If a scalar SDE (6.26) admits a fiber-preserving symmetry, with generator $X = \tau(t) \partial_t + \xi(x,t) \partial_x$, then there exists a fiber-preserving change of coordinates $s = s(t), \ y = y(x,t)$ which maps $w(t)$ into the Wiener process $\omega(s)$ and the equation into

$$dy = \varphi(y) \, ds + \gamma(y) \, d\omega(s) \, . \quad (6.27)$$

**Proposition 29.** A scalar SDE (6.26) can have a symmetry algebra of dimension $r = 0, 1, 2, 3$. In the case $r = 3$, the equation can be transformed into a Brownian motion equation, $dy = d\omega(s)$, via a change of variables with non-random time transformation.

**Proposition 30.** A scalar SDE (6.26) can have a symmetry algebra of dimension $r = 3$ if and only if it admits a symmetry generator of the form

$$X_* = \xi(x,t) \, \partial_x \, . \quad (6.28)$$

This in turn is the case if and only if the functions appearing in (6.26) satisfy the relation

$$\left[ \frac{g_t}{g} - g \left( \frac{f}{g} \right)_x + \frac{1}{2} \, g \, g_{xx} \right]_x = 0 \, . \quad (6.29)$$

It is interesting to note that the presence of a symmetry of the form (6.28) leads to integrability of the equation (see also Sect. 6.4.3 in this respect). In fact, while in the case of a nonzero $\tau$ we have to take into account the effect

\(^7\)Unfortunately, the proofs of these are not always given in full detail. After the completion of this paper, C. Lunini has provided detailed proofs of Kozlov’s theorem; she has also shown that the condition in Proposition 28 is not only sufficient but also necessary for the possibility to map an equation of the type (6.26) into one of the type (6.27) [152].
on the Wiener process, in the case \( \tau = 0 \) we do not have to worry about this. Passing to symmetry-adapted coordinates \((y, t)\) means the symmetry vector field will be expressed in these as

\[
X_* = \partial_y ,
\]

while the equation will be written as

\[
dy = \phi(y, t) \, dt + \gamma(y, t) \, dw(t) .
\] (6.31)

But in this case the determining equations for symmetries of the equation (which is by hypothesis satisfied by \( X_* = \eta(y, t) \partial_y \)) read simply

\[
\begin{align*}
\frac{\partial \eta}{\partial t} + \phi \frac{\partial \eta}{\partial y} - \eta \frac{\partial \phi}{\partial y} &= 0 \\
\gamma \frac{\partial \eta}{\partial y} - \eta \frac{\partial \gamma}{\partial y} &= 0 ;
\end{align*}
\] (6.32)

as in the symmetry-adapted coordinates we have \( \eta = 1 \), these actually read

\[
\frac{\partial \phi}{\partial y} = 0 , \quad \frac{\partial \gamma}{\partial y} = 0 .
\]

In other words, the equation (6.31) is actually

\[
dy = \phi(t) \, dt + \gamma(t) \, dw(t) ,
\] (6.33)

and is therefore promptly integrated to give

\[
y(t) = y(t_0) + \int_{t_0}^t \phi(s) \, ds + [w(t) - w(t_0)] .
\] (6.34)

**Example 6.7.** The equation [128]

\[
dx = f(t) \, dt + g(t) \, dw(t)
\] (6.35)

admits a three-dimensional symmetry algebra, with generators

\[
\begin{align*}
X_1 &= (1/g^2) [\partial_t + f \partial_x] , \\
X_2 &= \partial_x , \\
X_3 &= (2G/g^2) [\partial_t + f \partial_x] + (x - F) \partial_x ,
\end{align*}
\]

where we have written

\[
F(t) := \int f(t) \, dt , \quad G(t) := \int g(t) \, dt .
\]

These vector fields satisfy the commutation relations

\[
[X_1, X_2] = 0 , \quad [X_1, X_3] = 2X_1 , \quad [X_2, X_3] = X_2 .
\]
With the change of variables
\[ s = \int g^2(t) \, dt \quad y = x - \int f(t) \, dt, \quad (6.36) \]
the equation is mapped into the Brownian motion equation
\[ dy = d\omega(s); \quad (6.37) \]
here \( \omega(s) \) is the Wiener process obtained from \( w(t) \) via the above change of variables. This equation admits the symmetries
\[ X_1 = \partial_s, \quad X_2 = \partial_y, \quad X_3 = 2s\partial_s + y\partial_y. \]
Solution to the original equation (6.35) are obtained by solutions to (6.37), which read
\[ y(s) = y(s_0) + [\omega(s) - \omega(s_0)], \quad (6.38) \]
by inverting the change of variables (6.36); with this the function (6.38) is mapped into
\[ x(t) = x(t_0) + \int_{t_0}^t f(s) \, ds + \int_{t_0}^t g(s) \, dw(s), \quad (6.39) \]
which provides a solution to the original equation (6.37). Needless to say, this solution could be derived immediately from (6.35), but the example shows how to use the Kozlov result and procedure.

\[ \diamond \]

Example 6.8. Let us consider the vector field \( X = \partial_t + x\partial_x \), which of course is not of the form required by Proposition 30.\(^8\)

In view of (3.18), the more general Ito equation which admits \( X \) as a symmetry is
\[ dx = x\varphi(z) \, dt + x\gamma(z) \, dw(t), \quad (6.40) \]
where we have written as \( z \) the characteristic function for \( X \), i.e.
\[ z := x \, e^{-t}. \]
Passing to symmetry adapted coordinates means passing from \((x,t)\) to \((y,z)\), where \( y = y(x,t) \) is the solution to
\[ X(y) = \partial_t y + x\partial_x y = 1. \]
The latter is given by
\[ y = \log(x) + \beta(z), \]
\[^8\]We note that this vector field generates the one-parameter group (here \( \lambda \) is the group parameter) \( t \to t + \lambda, \ x \to e^\lambda x \), so that we have a special behavior in \( x = 0 \). Thus we should consider separately the domains \( x > 0 \) and \( x < 0 \); we will consider just the former.
with $\beta$ an arbitrary smooth function; we set $\beta(z) \equiv 0$. Thus our change of coordinates and the inverse one are given by

$$y = \log(x), \quad z = xe^{-t}; \quad x = e^y, \quad t = y - \log(z).$$

In these coordinates, the equation (6.40) reads

$$dy = -\frac{\varphi(z)}{[1 - \varphi(z)]} \, dz + \frac{\gamma(z)}{[1 - \varphi(z)]} \, dw(y - \log(z)).$$

Actually, it is more convenient to use coordinates $(y,t)$, so that the Wiener process depends on these variables in a simple way. In such coordinates, $z = \exp(y-t)$ and our equation (6.40) reads

$$dy = \varphi(z) \, dt + \gamma(z) \, dw(t) = \varphi[\exp(y-t)] \, dt + \gamma[\exp(y-t)] \, dw(t).$$

In general, this equation is not integrable.

Thus this example shows concretely that a symmetry which is not of the form (6.28) (as stipulated by Kozlov) does not, in general, imply the integrability of the equation.

\[\diamondsuit\]

### 6.4.2 Higher order scalar SDEs

The results discussed above for a first order scalar SDE can be generalized to the case of scalar higher order SDEs\(^9\), i.e. equations of the form

$$dx^{n-1} = f(x, x^{(1)}, \ldots, x^{(n-1)}, t) \, dt + g(x, x^{(1)}, \ldots, x^{(n-1)}, t) \, dw(t). \quad (6.41)$$

Albeit we have not considered this kind of equations in our discussion\(^10\), it is worth mentioning the results obtained by Kozlov [130].

**Proposition 31.** The scalar SDE (6.41) of order $n$ admits a symmetry algebra of dimension at most $(n+2)$. If (6.41) admits a symmetry algebra of dimension $r = n+2$, then it can be mapped into the higher order Brownian motion equation

$$dy^{(n-1)} = \gamma_0 \, d\omega(t).$$

The symmetry subalgebra for the scalar SDE (6.41) generated by vector fields of the form $X = \xi(x,t) \partial_x$ has dimension $r_0 \leq n$.

**Remark 6.13.** A complete classification of symmetry algebras for equations (6.41) of order $n = 2$ and $n = 3$ is provided in [130].

---

\(^9\)A single equation of order $n$ could of course also be mapped into a system of $n$ first order equations, and analyzed as a system; see Sect. 6.4.3.

\(^10\)Symmetries of higher order SDEs have been considered by several authors; see e.g. [155, 195, 211].
6.4.3 Systems of SDEs

The same kind of approach can be pursued for systems of SDEs. The case of a degenerate diffusion matrix $\sigma^i_j$ would introduce several degenerations (and subcases to be considered) in the problem, so one likes to consider the case where the diffusion matrix has full rank [129].

More precisely, one considers the system

$$dx^i = f^i(x^1, ..., x^n; t) dt + \sigma^i_k(x^1, ..., x^n; t) dw^k(t)$$  \hspace{1cm} (6.42)

with $i = 1, ... n, k = 1, ..., m$ and assumes $n \leq m$ with $\sigma$ of rank $n$. In this case symmetries are fiber-preserving [129], i.e. of the form $X = \tau(t) \partial_t + \xi^i(x,t) \partial_i$ (where as usual $\partial_i = \partial/\partial x^i$); this guarantees we will have a non-random change of time.

**Proposition 32.** The $n$-dimensional system of SDEs (6.42) admits a symmetry algebra of dimension $r \leq (n + 2)$. The symmetry subalgebra generated by vector fields of the form $X = \xi^i(x,t) \partial_i$ has dimension $r_0 \leq n$.

The full classification of symmetry properties for two dimensional systems of SDEs ($n = 2$, hence $r \leq 4$) is provided in [129].

In the case of systems one can also use symmetry properties to reduce (as for ordinary differential equations) the dimension of the system; in the case of a sufficiently large algebra with a suitable structure (again as for ODEs) one can infer integrability of the equation. It should be stressed that in this case (at difference with ODEs) only symmetries acting in the space of dependent variables can actually be used.\footnote{Looking back at the examples considered above, we note that in Example 6.7 we had symmetries of this form (and the equation could be integrated), while in Example 6.8 we had a symmetry but it was not of the required form, and correspondingly the equation could not, in general, be integrated.}

**Proposition 33.** If the $n$-dimensional system (6.42) admits a symmetry generated by a vector field of the form

$$X_r = \xi^i(x,t) \partial_i,$$

then there exists a regular change of variables $y = y(x,t)$ which maps the system into a system

$$dy^i = \varphi^i(y^1, ..., y^{n-1}; t) dt + \rho^i_k(y^1, ..., y^{n-1}; t) dw^k(t)$$  \hspace{1cm} (6.43)

of dimension $(n - 1)$ plus a “reconstruction equation”

$$y^n(t) = y^n(t_0) + \int_{t_0}^{t} \varphi^n(y^1, ..., y^{n-1}; s) ds + \int_{t_0}^{t} \rho^i_k(y^1, ..., y^{n-1}; s) dw^k(s).$$  \hspace{1cm} (6.44)
Proposition 34. Let the \( n \)-dimensional system (6.42) admits a solvable symmetry group of dimension \( r \), acting regularly\(^\text{12}\). Then the system can be reduced to a system of dimension \( q = (n - r) \). The solutions of the original system are in correspondence with solutions to the reduced system and are obtained by the latter via quadratures. If \( r = n \), the original system is integrable and its general solution is obtained by quadratures.

Example 6.9. Consider the system (6.42) for \( m = n = 2 \) and \( \sigma \) constant, \( \sigma = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \), with of course \( \det(\sigma) \neq 0 \), and \( f \) linear\(^\text{13}\),

\[
 f_i = a_i + b_i x_1 + c_i x_2 .
\]

This turns out [129] to admit two symmetries of the required form, which are actually of the type

\[
 X_i = A_{ij}(t) \partial_j \quad [\det(A) \neq 0] .
\]

With the change of coordinates

\[
 y_1 = \frac{A_{22} x_1 - A_{21} x_2}{A_{11} A_{22} - A_{12} A_{21}} , \quad y_2 = \frac{A_{11} x_2 - A_{12} x_1}{A_{11} A_{22} - A_{12} A_{21}}
\]

and writing

\[
 \alpha_1 = \frac{A_{22} a_1 - A_{21} a_2}{A_{11} A_{22} - A_{12} A_{21}} , \quad \alpha_2 = \frac{A_{11} a_2 - A_{21} a_1}{A_{11} A_{22} - A_{12} A_{21}} ;
\]

\[
 \beta_{11} = \frac{A_{22} S_{11} - A_{21} S_{21}}{A_{11} A_{22} - A_{12} A_{21}} , \quad \beta_{12} = \frac{A_{22} S_{12} - A_{21} S_{22}}{A_{11} A_{22} - A_{12} A_{21}} ;
\]

\[
 \beta_{21} = \frac{A_{11} S_{21} - A_{21} S_{11}}{A_{11} A_{22} - A_{12} A_{21}} , \quad \beta_{22} = \frac{A_{11} S_{22} - A_{21} S_{12}}{A_{11} A_{22} - A_{12} A_{21}} ;
\]

the system is mapped into

\[
 dy_1 = \alpha_1 \, dt + \beta_{11} \, dw_1(t) + \beta_{12} \, dw_2(t) \\
 dy_2 = \alpha_2 \, dt + \beta_{21} \, dw_1(t) + \beta_{22} \, dw_2(t)
\]

which is readily integrated. \(\diamondsuit\)

---

\(^\text{12}\)This requires the group orbits to be \( r \)-dimensional manifolds with regular embedding in the ambient space; in other words, points which are on the same orbit and nearby in the space should also be nearby along the orbit. See e.g. [77, 174] for regular group action.

\(^\text{13}\)We write all indices as lower ones in order to avoid any confusion.
Chapter 7

Conclusions

The symmetry approach is a general way to attack deterministic differential equations (and actually one can set in terms of it all the different solution methods for differential equations, provided suitable generalizations of the concept of symmetry are considered); in the deterministic framework it proved invaluable both for the theoretical study of differential equations and for obtaining concrete solutions.

The theory is comparatively much less advanced in the case of stochastic differential equations. There is now some general agreement on what the “right” – that is, useful – definition of symmetry for stochastic differential equations is, but only few applications have been considered, most of these concerning “integrable” equations. There is ample space for considering new applications, first and foremost considering “non integrable” equations.

Correspondingly, there is ample space for concrete applications, i.e. applying the approaches already existing or to be developed to new concrete stochastic systems.

Albeit here we have mentioned it only in passing, symmetry theory flourished and expanded its role by considering generalization of the “standard” (i.e. Lie-point) symmetries in several directions, some of them classical and some developed only in recent years. As far as I know, there is no attempt in this direction for stochastic systems yet; any work in this direction is very likely to collect success and relevant results.

It should also be stressed that actually in dealing with SDEs one could legitimately consider more general classes of transformations, as already done in normal forms theory for stochastic dynamical systems [12, 13]. A first attempt in this direction was proposed only very recently [101], and here again there is ample space for new work and interesting results.

Summarizing in a single sentence: the first attempts to use symmetry in the analysis of stochastic equations were promising; time is now ripe for extending fully fledged symmetry theory to stochastic systems.

I hope that the present text, aiming at providing an overview of the state of the art, can be of help in promoting work on this fascinating subject.
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Erratum for
“Symmetry of stochastic non-variational
differential equations”
(Physics Report 686 (2017), 1-62)

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In my recent paper [1], due to a regrettable and rather trivial mistake, a mixed derivatives term is missing in the expression (5.3) for the Ito Laplacian – which is essentially a Taylor expansion. The correct formula is, of course

$$\Delta f := \sum_{k=1}^{n} \frac{\partial^2 f}{\partial w^k \partial w^k} + \sum_{j,k=1}^{n} (\sigma \sigma^T) \frac{\partial^2 f}{\partial x^j \partial x^k} + 2 \sum_{j,k=1}^{n} \sigma^{ik} \frac{\partial^2 f}{\partial x^j \partial w^k}. \quad (5.3)$$

(The reader is alerted that the same mistake found its origin in a previous paper of mine [2], on which some of this review was based.)

This error has no consequence on our general discussion – conducted in terms of the $\Delta$ operator – except for what is said below; but it does affect the specific computations occurring in most of the concrete examples of Section 5.

The error in (5.3) has some more substantial consequence in Remark 5.8 and Section 5.5.

The part of Remark 5.8 following eq.(5.32) is simply wrong: once the correct formula for $\Delta(\varphi)$ is used, the quantity $\delta^i$ defined in eq.(5.33) is exactly zero, in any dimension, as proved in [3].

All the discussion in Sect.5.5 should be revised in the light of this fact; in particular, $\delta^i = 0$ means that for simple (deterministic or random) symmetries, there is a full equivalence between an Ito and the corresponding Stratonovich equation. (In the deterministic case, this was proved by Unal, see ref.200 in [1].)

Note this holds in any dimension (while in Section 5.5 we only considered the scalar case); the correct statement concerning the matter considered in this Section is therefore as follows:

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“The simple (deterministic or random) symmetries of an Ito equation and those of the corresponding Stratonovich one do coincide”.

I apologize to the readers, and thank the anonymous Referee of [3] for pointing out the mistake.

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