Graph-Based Minimum Dwell Time and Average Dwell Time Computations for Discrete-Time Switched Linear Systems

Ferruh İlhan and Özkan Karabacak

Istanbul Technical University, Electronics and Communication Engineering Department, 34469, Maslak, Istanbul, Turkey.

Abstract

Discrete-time switched linear systems where switchings are governed by a digraph are considered. The minimum (or average) dwell time that guarantees the asymptotic stability can be computed by calculating the maximum cycle ratio (or maximum cycle mean) of a doubly weighted digraph where weights depend on the eigenvalues and eigenvectors of subsystem matrices. The graph-based method is applied to systems with defective subsystem matrices using Jordan decomposition. In the case of bimodal switched systems scaling algorithms that minimizes the condition number can be used to give a better minimum (or average) dwell time estimates.

Keywords: Switched systems, minimum dwell time, average dwell time, asymptotic stability.

1. Introduction

Problems on the stability of switched linear systems have been attracted many researchers in the last two decades [1]. A well-known approach to the stability of switched linear systems is to impose constraints on the set of switching signals so as to guarantee the stability. Mainly, two kinds of constraints are considered: minimum dwell time constraint and average dwell time constraint. In
the former approach, intervals between two consecutive switchings are assumed to be larger than or equal to a number called minimum dwell time, whereas in the latter approach these intervals are assumed to be larger than or equal to a number on average, the number being called average dwell time.

In literature, there are many methods to find a minimum dwell time and/or average dwell time for a given switched system that guarantees stability. However, in general, there is no method that estimates the smallest possible minimum (or average) dwell time for a given switched systems. The most efficient methods can find relatively good values for minimum (or average) dwell time, but they depend on the linear matrix inequalities (LMI) and therefore do not give any insight between the subsystem properties and the minimum (or average) dwell time.

Finding a minimum (or average) dwell time explicitly depending on the subsystem properties has been achieved in [2] for continuous-time switched systems with non-defective subsystem matrices, where a minimum dwell time is found as a function of subsystem eigenvalues and eigenvectors along a specific switching cycle (namely, the cycle with maximum cycle ratio, to be explained soon). This method is based on considering a switched system as a doubly-weighted digraph (called switching graph) with specific weights that provide upper bounds on the norm of any solution. A switching signal is then viewed as a walk in the switching graph. Since any walk can be decomposed into cycles and a path of bounded length, conditions can be imposed on the cycles only. The so-called maximum cycle ratio of the switching graph then provides a minimum dwell time estimate, whereas the so-called maximum cycle mean of the switching graph provides an averaged dwell time estimate. This method can be found preferable when switched system where switchings are governed by a digraph are considered. For, in this case, already existing constraints on the switchings have been naturally considered by the switching graph, giving rise to much smaller dwell-time estimates.

Switched systems where switchings are governed by a digraph can be seen in control engineering applications [3, 4, 5]. Theoretically, such systems are
first considered in the switched system literature, up to our knowledge, in [6], where stability conditions are reduced to the conditions on strongly connected components of the digraph. Then, one of the authors of this paper presents sufficient conditions on the stability of constrained switched systems using the properties of the digraph that governs the switchings [2]. This method has been improved in [7] so as to cover systems with defective subsystem matrices. Recently, in [8], stabilization of switched systems where switchings are governed by a digraph has been considered.

In this paper, we apply the method in [2] to discrete-time switched linear systems and waive the non-defectiveness condition by considering the Jordan form of subsystem matrices. In addition, we improve the minimum dwell time estimate for bimodal systems by applying a scaling algorithm that minimizes the condition number of matrices. We consider discrete-time switched linear systems of the form

$$x(t + 1) = A_{\sigma(t)}x(t), \sigma : [0, \infty) \to \{1, \ldots, M\}, \sigma \in \mathcal{S}, t \in \mathbb{N},$$

where \(\{A_i \in \mathbb{R}^{N \times N}\}_{i \in \{1, \ldots, M\}}\) is a finite set of Schur stable subsystem matrices, \(\{1, \ldots, M\}\) is the index set for subsystems and \(\mathcal{S}\) denotes the set of admissible switching signals. An admissible switching signal \(\sigma(t) \in \mathcal{S}\) is a piecewise right-continuous function of the form \(\sigma(t) : [0, \infty) \to \{1, 2, \ldots, M\}\) with finite number discontinuities in finite time intervals. Here, \(M\) is the number of subsystems. Consider switching instants \(t_0, t_1, \ldots\), where \(t_k \in \mathbb{Z}^+, t_0 = 0\) and denote the active subsystem in the interval \(\{t_k, \ldots, t_{k+1} - 1\}\) by \(\sigma_k\). Let \(N_{\sigma}(t)\) denote the number of switchings before time \(t\). Two different sets of switching signals are considered:

$$\mathcal{S}_{\min}[\tau] = \{\sigma | t_k - t_{k-1} \geq \tau \}$$

$$\mathcal{S}_{\text{ave}}[\tau, N_0] = \{\sigma | N_{\sigma}(t) \leq N_0 + \frac{t}{\tau} \}$$

\(\mathcal{S}_{\min}[\tau]\) consists of the switching signals where the intervals between two consecutive switchings are always larger than \(\tau\), whereas \(\mathcal{S}_{\text{ave}}[\tau, N_0]\) is the set of switching signals with the property that at each time \(t\) the number of past switchings \(N_{\sigma}(t)\) satisfies the average dwell time condition \(N_{\sigma}(t) \leq N_0 + \frac{t}{\tau}\).
In the sequel, we first explain how the switching graph arises naturally considering a bound on the norm of the solution of the linear switched system (1). In Section 2, we define the switching graph which will be used to estimate a minimum (or average) dwell time. Based on the switching graph, methods for minimum dwell time and average dwell time computation are given in Section 3 and Section 4 respectively. Finally, we will discuss on possible future research in this area in Section 6.

2. Switching Graph

For a given initial condition \(x(0)\), the solution of the discrete-time switched linear system (1) can be written as

\[
x(t) = A^{(t-t_n)} \left( \prod_{k=1}^{n} A^{(t_k-t_{k-1})}_{\sigma_k} \right) x(0),
\]

for \(t \in \{t_n, \ldots, t_{n+1} - 1\}\). Let \(\| \cdot \|\) denote the 2-norm for vectors and spectral norm for matrices. Using norm inequalities and eigen-decomposition \(A_i = V_i D_i V_i^{-1}\), we have

\[
\|x(t)\| = \|A^{(t-t_n)}_{\sigma_{n+1}} \left( \prod_{k=1}^{n} A^{(t_k-t_{k-1})}_{\sigma_k} \right) x(0)\|
\]

\[
= \| V_{\sigma_{n+1}}^{t-t_n} \left( \prod_{k=1}^{\sigma_{n+1}} V_{\sigma_k}^{-1} D_{\sigma_k}^{(t_k-t_{k-1})} \right) V_{\sigma_1}^{-1} x(0)\|
\]

\[
\leq \| V_{\sigma_{n+1}} \| \| V_{\sigma_1}^{-1} \| \| D_{\sigma_{n+1}} \| \| V_{\sigma_{n+1}} \|^{(t-t_n)} \left( \prod_{k=1}^{n} \| V_{\sigma_{k+1}}^{-1} V_{\sigma_k} \| \| D_{\sigma_k} \| \| V_{\sigma_k} \|^{(t_k-t_{k-1})} \right) \| x(0) \|.
\]

Let \(\rho_i\) be the spectral radius of the subsystem \(i\). Using \(\|D_i\| = \rho_i\) and writing the term in parenthesis in exponential form, we have

\[
\|x(t)\| \leq \| V_{\sigma_{n+1}} \| \| V_{\sigma_1}^{-1} \| \| \rho_i^{(t-t_n)} \| \sum_{k=1}^{n} \left( \ln \| V_{\sigma_{k+1}}^{-1} V_{\sigma_k} \| + (t_k-t_{k-1}) \ln \rho_{\sigma_k} \right) \| x(0) \|
\]

\[
\leq \gamma \rho_i^{(t-t_n)} \sum_{k=1}^{n} \left( \ln \| V_{\sigma_{k+1}}^{-1} V_{\sigma_k} \| + (t_k-t_{k-1}) \ln \rho_{\sigma_k} \right) \| x(0) \|,
\]

where \(\gamma = \max_{i,j \in \{1, \ldots, M\}} \| V_i \| \| V_j^{-1} \|\). Note that \(\ln \rho_i\) is negative for all \(i\), since each subsystem is Schur stable. Using \(\rho_i^{(t-t_n)} < 1\) and \(\tau \leq t_k - t_{k-1}\) for \(\sigma \in \mathcal{S}_{\min} [\tau]\), we can write

\[
\|x(t)\| \leq \gamma e^{\sum_{k=1}^{n} \left( \ln \| V_{\sigma_{k+1}}^{-1} V_{\sigma_k} \| + \tau \ln \rho_{\sigma_k} \right)} \| x(0) \|
\]

\[
= \gamma e^{\alpha(n)} \| x(0) \|,
\]
where
\[
\alpha(n) = \sum_{k=1}^{n} \ln \|V_{\sigma_{k+1}}^{-1} V_{\sigma_k}\| + \tau \ln \rho_{\sigma_k}.
\] (11)

In the following, we show that the function \(\alpha(n)\) can be seen as the weight of a walk (of length \(n\)) on a doubly weighted digraph called the switching graph.

2.1. Definition of a switching graph

For some switched systems, transitions between subsystems are restricted by a digraph, called switching graph, of which nodes represent subsystems and directed edges represent admissible transitions between subsystems. As a consequence of this idea, a switching signal \(\sigma\) can be viewed as a walk on the switching graph. It can be easily seen that each transition from subsystem \(i\) to subsystem \(j\) adds a value to \(\alpha\) function which is a function of the ordered pair \((i,j)\), namely \(\ln \|V_{\sigma_j}^{-1} V_{\sigma_i}\| + \tau \ln \rho_{\sigma_i}\). These values can be assigned as the weights of directed edges for all admissible transitions between subsystems. Consider \(\omega_{ij}^+ = \ln \|V_{\sigma_j}^{-1} V_{\sigma_i}\|\) as the gain of the transition from subsystem \(i\) to subsystem \(j\) and \(\omega_{ij}^- = -\ln \rho_{\sigma_i}\) as the loss of the transition from subsystem \(i\) to subsystem \(j\). Then, we have a doubly weighted switching graph as follows.

A switching graph of a switched linear system (1) is a doubly weighted digraph
\[
\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \omega^+, \omega^-, \}.
\] (12)

Here, \(\mathcal{V}\) is the set of nodes which is isomorphic to the index set of subsystems. \(\mathcal{E}\) is the set of the directed edges which represent admissible transitions between subsystems. This set is given by \(\mathcal{E} = \{(i,j) | i \neq j, i, j \in \mathcal{V}\}\) in the case of no restriction imposed on transitions between subsystems, namely we have a fully connected switching graph. For a switched system with four subsystems, the switching graph is shown in Figure 2.1. \(\omega^+\) and \(\omega^-\) are the weight functions defined on the set \(\mathcal{E}\) as
\[
\omega_{ij}^+ = \ln \|V_{\sigma_j}^{-1} V_{\sigma_i}\| \quad \text{(13)}
\]
\[
\omega_{ij}^- = -\ln \rho_{\sigma_i} \quad \text{(14)}
\]
\( \omega^+ \) is well-defined when the eigenvector matrix is composed of the eigenvectors with unit Euclidean norm and in the case of \( k \)-multiple eigenvalues, eigenvectors are chosen as any \( k \) orthogonal vectors that span the corresponding eigenspace. In this case, the eigenvector matrix is well-defined up to a multiplication from right by a unitary matrix. Consider the eigenvector matrices \( V_i \) and \( V_j \). Choose different eigenvector matrices as \( V_i' = V_i \cdot U_i \) and \( V_j' = V_j \cdot U_j \).

\[
\omega^+_{ij} = \ln \| V_j^{-1} V_i' \| = \ln \| U_j^{-1} V_i U_j \| = \ln \| V_j^{-1} V_i \|. 
\]

where the final equation comes from the fact that the spectral norm is unitarily invariant. Note that choosing eigenvectors with a different Euclidean norm can change the value of \( \omega^+ \). This case is of concern in Section 3.3 and Section 4.3.

Assume that the switchings in the switched system under consideration should respect a directed graph. In this case, the set of admissible switching signals \( S \) is restricted by the directed edges of the switching graph. We denote \( S_{\text{min}}[\tau] \) with a switching graph as

\[
S_{\text{\#min}}[\tau] = \{ \sigma \in S_{\text{min}}[\tau] | (\sigma_k, \sigma_{k+1}) \in \mathcal{E} \}. 
\]
This set of switching signals contains the switching signals that respect the given digraph $G$ satisfying the minimum dwell time property. Similarly we have

$$S_{G, \text{ave}}[\tau, N_0] = \{ \sigma \in S_{\text{ave}}[\tau, N_0] | (\sigma_k, \sigma_{k+1}) \in \mathcal{E} \}.$$  

(17)

### 2.2. Maximum cycle ratio

Consider a weighted digraph $G = \{V, E, \omega\}$. A walk $W$ can be defined as

$$W = (p_1, p_2), (p_2, p_3), \ldots, (p_s, p_{s+1})$$

where $(p_i, p_{i+1}) \in \mathcal{E}$ for all $i = 1, \ldots, s$. A path is a walk $\{1, \ldots, M\} = (p_1, p_2), (p_2, p_3), \ldots, (p_s, p_{s+1})$ where $p_1, \ldots, p_s$ are distinct and a cycle is a path with $p_{s+1} = p_1$. The weight of a walk $W$ is defined as

$$\omega(W) = \sum_{k=1}^{s} \omega_{p_k p_{k+1}}.$$ 

Now consider a doubly weighted graph $G = \{V, \mathcal{E}, \omega^+, \omega^-\}$. The ratio of a cycle $C$ on $G$ is defined as $\nu(C) = \frac{\omega^+(C)}{\omega^-(C)}$. The maximum cycle ratio $\nu$ is defined as

$$\nu(G) = \max_{C \in C} \nu(C) = \max_{C \in C} \frac{\omega^+(C)}{\omega^-(C)}$$

(18)

where $C$ denotes the set of all cycles on $G$.

Similarly, the mean of a cycle is defined as $\mu(C) = \frac{\omega^+(C)}{|C|}$, where $|C|$ is the length of the cycle $C$. The maximum cycle mean $\mu$ is defined as

$$\mu(G) = \max_{C \in C} \mu(C) = \max_{C \in C} \frac{\omega^+(C)}{|C|}.$$ 

(19)

Optimum (minimum or maximum) cycle ratio, also known as profit-to-time ratio, and optimum cycle mean have been considered in graph theory literature\cite{9, 10, 11, 12}, and has many applications in different areas such as scheduling problems\cite{13, 14, 15} and performance analysis of digital systems\cite{16}. There are many algorithms that can be used to find the optimum cycle ratio and optimum cycle mean for a given doubly weighted digraph (See\cite{12}). In terms of practical complexity, one of the fastest algorithms is given in\cite{11}. In the sequel, we use this algorithm for which a C code is available in Ali Dasdan’s personal web page\cite{17}.
3. Minimum Dwell Time Computation

In this section, we show that an estimate for a minimum dwell time can be given as the maximum cycle ratio of the switching graph. Three different cases where subsystems matrices are non-defective, defective and where the switched system consists of two subsystems (bimodal case) are considered.

3.1. Non-defective case

Let us consider switched linear system

$$x(t + 1) = A_{\sigma(t)}x(t), \sigma : [0, \infty) \to \{1, \ldots, M\}, \sigma \in S_{\mathcal{G},\min}[\tau], t \in N$$  \hspace{1cm} (20)

where each $A_i$ is a Schur stable and non-defective subsystem matrix.

**Theorem 1.** The switched linear system (20) is asymptotically stable if

$$\tau > \nu(\mathcal{G}).$$  \hspace{1cm} (21)

**Proof.** Note that a switching signal $\sigma$ can be represented by a walk $W$ on the switching graph. If the length of the walk is finite, the last subsystem stays active forever which guarantees the asymptotic stability of the switched linear system. Hence, walks with infinite length are considered, which represent switching signals having infinitely many switchings. Using Eq. 11, it is seen that $\alpha(n)$ is the weight of the walk

$$W_n := (\sigma_1, \sigma_2), (\sigma_2, \sigma_3), \ldots, (\sigma_n, \sigma_{n+1})$$  \hspace{1cm} (22)

for the weight function $\omega_{ij} = \omega^+_{ij} + \tau \omega^-_{ij}$. Using the fact that any walk on a digraph with $M$ nodes can be decomposed into cycles and a path of the length at most $M - 1$, $\alpha(n)$ is decomposed as $\alpha(n) = \alpha_1(n) + \alpha_2(n) + \cdots + \alpha_M(n)$. Here $\alpha_k(n)$ is the weight of the path and $\alpha_k(n)$ is the sum of the weights of all $k$-cycles. Note that the assumption $\tau > \nu(\mathcal{G})$ implies $\omega(C) = \omega^+(C) - \tau \omega^-(C) < 0$ for any cycle $C$. Namely, weights of all cycles are negative. Since $V$ is finite, $\alpha(n)$ is bounded, and therefore, $\alpha(n) \to -\infty$ as $n \to \infty$. This is valid for all $\sigma \in S_{\mathcal{G},\min}[\tau]$. Hence, switched linear system (20) is asymptotically stable. \blacksquare
3.2. Defective case

In this subsection, we generalize Theorem 1 to switched systems with defective subsystem matrices. For this purpose, Jordan matrix decomposition of the form $A_i = P_i J_i P_i^{-1}$ where $P_i$ is the generalized eigenvector matrix of $A_i$ and $J_i$ is the Jordan matrix is used. Eq. (7) is true when $V$ and $D$ are rewritten as $P$ and $J$, respectively. Then, we have

$$\|x(t)\| \leq \|P_{\sigma_{n+1}}\| \|P_{\sigma_1}^{-1}\| \|J_{\sigma_{n+1}}\|^{(t-t_n)} \left( \prod_{k=1}^n \|P_{\sigma_{k+1}}^{-1} P_{\sigma_k}\| \|J_{\sigma_k}\|^{(t_k-t_{k-1})} \right) \cdot \|x(0)\|. \quad (23)$$

Assume that $\|J_i\| < 1$ for each $i = 1, \ldots, M$. Then, we have

$$\|x(t)\| \leq \|P_{\sigma_{n+1}}\| \|P_{\sigma_1}^{-1}\| e^{\left( \sum_{k=1}^n \ln \|P_{\sigma_{k+1}}^{-1} P_{\sigma_k}\| + \tau \ln \|J_{\sigma_k}\| \right)} \|x(0)\| \quad (24)$$

$$= \gamma e^{\alpha(n)} \|x(0)\|, \quad (25)$$

where $\gamma = \max_{i,j \in \{1, \ldots, M\}} \|P_i\| \|P_j^{-1}\|$, $\alpha(n) = \sum_{k=1}^n \ln \|P_{\sigma_{k+1}}^{-1} P_{\sigma_k}\| + \tau \ln \|J_{\sigma_k}\|$ is a function of $n$. Let us define new weights as,

$$\omega_{ij}^+ = \ln \|P_j^{-1} P_i\|, \quad (26)$$

$$\omega_{ij}^- = -\ln \|J_i\|. \quad (27)$$

As a consequence, we have the counterpart of Theorem 1:

**Theorem 2.** The switched linear system (20) is asymptotically stable if $\|J_i\| < 1$ for all $i = 1, \ldots, M$ and

$$\tau > \nu(\mathcal{G}), \quad (28)$$

where the cycle ratio is found using the weights given in (26) and (27).

The constraint $\|J_i\| < 1$ seems very restrictive. Hence it may not be satisfied for a set of Schur stable subsystem matrices. In this case, one can use the fact that 1’s above diagonal in Jordan matrix are conventional and can be replaced by any sufficiently small $\epsilon$. Assume that the Jordan form $J$ consists of one Jordan block, matrices with a Jordan form that consists of more than one Jordan block
can be treated similarly. Then, by a change of the generalized eigenvector matrix from \( P = [p_0|p_1|p_2|\ldots] \) to \( P_\epsilon = [p_0|\epsilon p_1|\epsilon^2 p_2|\ldots] \), the matrix \( A \) can be written as

\[
A = P_\epsilon \cdot J_\epsilon \cdot P_\epsilon^{-1},
\]

where \( J_\epsilon \) is in Jordan form with \( \epsilon \)'s in place of 1's. It is known that \( \| J_\epsilon \| \leq \| D_\epsilon \| + \| N_\epsilon \| \) where \( D_\epsilon \) denotes diagonal part of \( J_\epsilon \) and \( N_\epsilon \) denotes nilpotent part of \( J_\epsilon \). Then, \( \| N_\epsilon \| = \epsilon \). Choose \( \epsilon < 1 - \| D_\epsilon \| \). Therefore, it is always possible to find \( J_\epsilon \) such that \( \| J_\epsilon \| < 1 \).

### 3.3. Bimodal case

Theorem 1 can be enhanced for bimodal switched systems, namely switched systems with two subsystems. Since there is only one cycle in the bimodal case, the maximum cycle ratio is found as

\[
\nu(G) = \frac{\ln(\| V_2^{-1}V_1 \| \cdot \| V_1^{-1}V_2 \|)}{-\ln(\rho_1 \rho_2)} = \frac{\ln(\kappa(V_2^{-1}V_1))}{-\ln(\rho_1 \rho_2)},
\]

where \( \kappa \) denotes the condition number for the spectral norm, namely \( \kappa(A) = \| A \| \cdot \| A^{-1} \| \). Therefore, using Theorem 1, the bimodal switched system is stable if

\[
\tau > \frac{\ln(\kappa(V_2^{-1}V_1))}{-\ln(\rho_1 \rho_2)}.
\]

It is known that eigenvectors can be scaled by any nonzero scalar. Then, an eigenvector matrix multiplied from right by a nonsingular diagonal matrix is also an eigenvector matrix. Let \( D \) denote the set of nonsingular diagonal matrices. Consider the eigenvector matrices \( V_1, V_2 \). Let \( \tilde{V}_1, \tilde{V}_2 \) be the new eigenvector matrices obtained from scaling columns of \( V_1, V_2 \) using \( D_1, D_2 \in D \), respectively. Then, we have

\[
\tilde{V}_2^{-1}\tilde{V}_1 = D_2^{-1}V_2^{-1}V_1D_1.
\]

Note that \( \tilde{V}_2^{-1}V_1 \) is obtained by scaling rows and columns of \( V_2^{-1}V_1 \). Hence, the condition (31) in Theorem 1 can be replaced by a stronger condition

\[
\tau > \frac{\ln(\min_{D_L, D_R \in D} \kappa(D_LV_2^{-1}V_1D_R))}{-\ln(\rho_1 \rho_2)}.
\]
There is no analytical method for minimizing the condition number for the spectral norm by scaling rows and columns, but algorithmic methods are available [18, 19].

Corollary 1. The switched linear system (20) with $M = 2$ is asymptotically stable if

$$
\tau > \frac{\ln(\min_{D_L, D_R \in \mathcal{D}} \kappa(D_L V_2^{-1} V_1 D_R))}{-\ln(\rho_1 \rho_2)}.
$$

(34)

One can use any other $p$-norm in Inequality (7). Since $p$-norm is submultiplicative and $\|D\|_p = \rho(D)$ for a diagonal matrix $D$, where $\rho$ denotes the spectral radius; one can similarly obtain the condition

$$
\tau > \frac{\ln \kappa_p(V_2^{-1} V_1)}{-\ln(\rho_1 \rho_2)},
$$

(35)

where, $\kappa_p(A) = \|A\|_p \|A^{-1}\|_p$. There is an analytical method [20] for minimizing the condition number for norms $\|\cdot\|_1$ and $\|\cdot\|_\infty$ by scaling rows and columns. According to this method, for $p = 1, \infty$,

$$
\min_{D_L, D_R \in \mathcal{D}} \kappa_p(D_L A D_R) = \rho(|A| A^{-1})
$$

(36)

where $|A|$ denotes the matrix whose elements are absolute value of the corresponding elements of $A$, and $\rho$ denotes spectral radius. Hence, we have the following result:

Corollary 2. The switched linear system (20) with $M = 2$ is asymptotically stable if

$$
\tau > \frac{\ln(\rho(|V_2^{-1} V_1| V_1^{-1} V_2))}{-\ln(\rho_1 \rho_2)}.
$$

(37)

Remark 1. It is well-known that switched linear systems with simultaneously triangularizable subsystem matrices are stable under arbitrary switching [21]. Corollary 2 can be used to give a simple proof of this result for bimodal systems. Let $A_1$ and $A_2$ be simultaneously triangularizable subsystem matrices of a bimodal switched system. Since simultaneous similarity transformation applied to $A_1$ and $A_2$ does not affect the condition [37], we can assume that $V_1$ and $V_2$ are triangular. Therefore, $S := V_2^{-1} V_1$ is also triangular. Let
$s_i$ denote the $i$th diagonal element of $S$. Then, the $i$th diagonal element of the matrix $|S||S^{-1}|$ is equal to $|s_i| \cdot \frac{1}{1/s_i} = 1$. This implies that all eigenvalues of $|S||S^{-1}|$ are equal to one, since $|S||S^{-1}|$ is triangular. Hence, we have $\rho(|S||S^{-1}|) = \rho(|V_2^{-1}V_1||V_1^{-1}V_2|)) = 1$. Substituting this in (37), we get $\tau > 0$.

4. Average Dwell Time Computation

In this section, the average dwell time problem is considered, namely finding the smallest possible value $\tau$ for which the switched system (1) is asymptotically stable for the average dwell time set (3). We first consider the case of non-defective subsystems. Next, we show that for the case of defective subsystems, the approach in Section 3.2 can be applied similarly. Finally, for bimodal switched systems one can improve the following computation by using condition number minimizing methods as in Section 3.3.

4.1. Non-defective case

For the case of non-defective subsystem matrices, a bound on the average dwell time can be found using the maximum cycle mean of the switching graph and the largest spectral radius $\rho_{\text{max}} = \max_i \rho_i$ as follows:

**Theorem 3.** Let $\{A_i\}_{i=1,\ldots,M}$ be a family of non-defective Schur stable matrices and $G$ be a switching graph. Then the switched linear system given by

$$x(t+1) = A_{\sigma(t)}x(t), \sigma : [0, \infty) \to \{1, \ldots, M\}, \sigma \in \mathcal{S}_{G, \text{ave}}[\tau, N_0]$$

is asymptotically stable for all $N_0$ if

$$\tau > \frac{\mu(G)}{-\ln \rho_{\text{max}}}. \tag{39}$$

**Proof.** By the assumption that the switching signal has infinitely many switching and subsystems are non-defective, for $t \in \{t_n, \ldots, t_{n+1} - 1\}$ we have the inequality (8), which can be written as

$$\|x(t)\| \leq \gamma e^{\alpha t} + \frac{1}{\ln \rho_{\text{max}}} \|x(0)\|, \tag{40}$$

12
where \( \alpha(n) = \sum_{k=1}^{n} \ln \| V_{\sigma_k+1} V_{\sigma_k} \| \). Consider the walk associated to the switching signal \( \sigma \)

\[
W_n := (\sigma_1, \sigma_2, \sigma_3, \ldots, (\sigma_n, \sigma_{n+1})
\]

(41)

it is seen that the \( \omega^+ \)-weight of the walk \( W_n \) is equal to \( \alpha(n) \). Since a walk can be decomposed into cycles and a path of length less than \( M \) where \( M \) is the number of nodes,

\[
\alpha(n) = \alpha_*(n) + \alpha_2(n) + \alpha_3(n) + \cdots + \alpha_M(n)
\]

(42)

where \( \alpha_*(n) \) is the \( \omega^+ \)-weight of the path of the length less than \( M \), say \( M_* \) and \( \alpha_k(n) \) is the sum of the \( \omega^+ \)-weights of all cycles of length \( k \). Defining

\[
\bar{\gamma} = \gamma e^{\max_{\mathcal{W}} \omega^+(\mathcal{W})}
\]

where \( \mathcal{W} \) changes over all possible paths, we get

\[
\| x(t) \| \leq \bar{\gamma} e^{\alpha_2(n)+\alpha_3(n)+\cdots+\alpha_M(n)+t \ln \rho_{\max}} \| x(0) \|.
\]

(43)

Consider the maximum cycle mean of the switching graph \( \mathcal{G} \), namely \( \mu(\mathcal{G}) = \max_{C \in \mathcal{C}} \frac{\omega^+(C)}{|C|} \), where \( |C| \) denotes the length of the cycle and \( \mathcal{C} \) is the set of all cycles in \( \mathcal{G} \). Then, it is obtained that \( \omega^+(C) \leq |C| \mu(\mathcal{G}) \). Since \( \alpha_2(n), \ldots, \alpha_M(n) \)
are cycle weights, we get

\[
\alpha_2(n) + \cdots + \alpha_M(n) \leq (N_*(t) - M_*) \mu(\mathcal{G})
\]

\[
\leq (N_0 - M_*) \mu(\mathcal{G}) + \frac{t \mu(\mathcal{G})}{\tau}.
\]

(44) (45)

Substituting this into (43) and defining \( \tilde{\gamma} = \bar{\gamma} e^{N_0 \mu(\mathcal{G})} \), we obtain

\[
\| x(t) \| \leq \tilde{\gamma} e^{(\frac{\omega^+}{\tau} + \ln \rho_{\max})t} \| x(0) \|.
\]

(46)

Since \( \frac{\mu(\mathcal{G})}{\tau} + \ln \rho_{\max} < 0 \) by assumption, we conclude that \( \| x(t) \| \to 0 \) □

4.2. Defective case

Let us generalize Theorem 3 to any linear switched systems. For this purpose, Jordan matrix decomposition of the form \( A_i = P_i J_i P_i^{-1} \) has been used where \( P_i \) is the generalized eigenvector matrix and \( J_i \) is the Jordan form
A_i. Define $\omega_{ij}^+ = \ln \| P_j^{-1} P_i \|$. Manipulating Eq. 23 with the assumption that $\| J_i \| < 1$, we have

$$\| x(t) \| \leq e^{\alpha(n) + \tau \ln J_{\max} \| x(0) \|},$$

where $\alpha(n) = \sum_{k=1}^{n} \ln \| P_{\sigma_{k+1}}^{-1} P_{\sigma_k} \|$ and $\| J_{\max} \| = \max_i \| J_i \|$. Hence, it is obtained that $\tau > \frac{-\mu(G)}{-\ln \rho_{\max}}$. This shows that in the case of defective subsystem matrices average dwell time can be computed by replacing $V$ and $\rho_{\max}$ in Theorem 3 with $P$ and $\| J_{\max} \|$, respectively.

### 4.3. Bimodal case

Similar to the minimum dwell time case, it is noted that the average dwell time for bimodal switched systems can be improved. As there is only one cycle in a bimodal system, $\mu(G) = \frac{\omega^+(C)}{2}$. Then, the average dwell time can be found as $\tau > \frac{-\mu(G)}{-\ln \rho_{\max}} = \frac{-\omega^+(C)}{-2 \ln \rho_{\max}} = \frac{-\ln \kappa_p (V_0^{-1} V_1)}{-2 \ln \rho_{\max}}$. Hence, the method in Subsection 3.3 can be applied to the computation of average dwell time.

### 5. Illustrative Examples

We apply the obtained minimum dwell time computation method to two illustrative examples and compare the results with two different methods in literature: The method of Morse given in [22], which finds a minimum dwell time guaranteeing each subsystem to be contractive, and the method of Geromel & Colaneri given in [23], which uses an LMI approach based on a multiple Lyapunov function technique. We skip the comparison of the obtained average dwell time computation to other methods in literature since the methods for the average dwell time computation in literature either requires a specified convergence rate as in [24] or exist in mode-dependent form as in [25], namely for each subsystem a certain average dwell time condition is imposed.

**Example 1.** Consider a switched system consisting of four linear subsystems whose matrices are given by

$$A_k = (U^{-1})^k \cdot A \cdot U^k, \quad k = 0, \ldots, 3$$

(48)
Here,

\[ A = \begin{pmatrix} -0.2 & 1 & 0 \\ -1 & 1.4 & 0 \\ 0 & 0 & -0.4 \end{pmatrix} \]  

and

\[ U = \begin{pmatrix} 1.2 & 0 & 0 \\ 0 & \cos\left(\frac{\pi}{3}\right) & \sin\left(\frac{\pi}{3}\right) \\ 0 & -\sin\left(\frac{\pi}{3}\right) & \cos\left(\frac{\pi}{3}\right) \end{pmatrix} . \]

Assume that the switchings respect one of the switching graphs: fully connected \( G_1 \) (Fig. 2.1), one-sided ring \( G_2 \) (Fig. 2) and two-sided ring \( G_3 \) (Fig. 3).

For different switching graphs, minimum dwell time values are computed using Theorem 1 in Table 5. It can be seen that the results are better than the results obtained by the method of Morse [22]. Comparing the results with the method of Geromel & Colaneri [23], it can be seen that only for the switching graph \( G_3 \), Theorem 1 gives a worse result.

Let us consider a bimodal system for which we can use Corollary 1 and the two-sided equilibration method in [19] to compute a better value for the minimum dwell time than the one obtained by Theorem 1.
Example 2. Let $A_1$ and $A_2$ given below be the subsystem matrices of a switched system:

$$A_1 = \begin{pmatrix}
-0.38 & 0.2 & 0.1 \\
-0.16 & 0.72 & 0.16 \\
-0.24 & 0.24 & 0.8
\end{pmatrix}$$

$$A_2 = \begin{pmatrix}
-0.8 & -0.07 & 0.04 \\
0.1 & -1 & 0.05 \\
-0.1 & -0.06 & -0.34
\end{pmatrix}$$

$\tau$ is calculated as 7 using the condition given in Theorem 1. However, applying Corollary 1 $\tau$ is calculated as 1 which is equal to the value found by linear matrix inequalities method [23]. Here, we use the two sided equilibration method in [19] which is based on the idea that the condition number can be reduced by making norms of rows as well as norms of columns equal. The row scaling matrix $D_R$ and the column scaling matrix $D_L$ are calculated as below:

$$D_R = \begin{pmatrix}
0.8528 & 0 & 0 \\
0 & 0.7178 & 0 \\
0 & 0 & 1.9789
\end{pmatrix}$$

$$D_L = \begin{pmatrix}
1.8805 & 0 & 0 \\
0 & 0.4709 & 0 \\
0 & 0 & 1.8803
\end{pmatrix}$$
Table 1: The minimum dwell time values computed for Example

| Switching Graph | Minimum dwell time | Geromel | Morse | Colaneri |
|-----------------|-------------------|---------|-------|----------|
| $G_1$           | $\tau$           | 7       | 8     | 7        |
| $G_2$           | $\tau$           | 7       | 8     | 7        |
| $G_3$           | $\tau$           | 5       | 8     | 2        |

6. Conclusion

A method for the computation of the minimum dwell time that guarantees the asymptotic stability of a switched system has been presented. The method is applicable to systems where switchings are governed by a digraph. The graph-theoretical nature of the method allows fast computation of an estimate of the minimum dwell time using the maximum cycle ratio algorithms in graph theory. We note that there are many problems that can be considered for the switched systems where switchings are governed by digraphs. The role that the nature of the switching digraph plays on the dynamics of the switched system should be considered further.

We have shown that the average dwell time can be computed using the minimum cycle mean of the switching graph. This approach can be improved in two different ways: Firstly, one can introduce the mode-dependent average dwell time as in [25], and try to find sufficient conditions on the mode-dependent average dwell times for a given switching graph. Secondly, one can consider a preassumed convergence rate as in [24] to calculate the average dwell time of a given switching graph in a less conservative method.
References

[1] J. P. Hespanha. Uniform stability of switched linear systems: extensions of LaSalle’s invariance principle. *IEEE Trans. on Auto. Cont.*, 49(4):470–482, 2004.

[2] Ö. Karabacak. Dwell time and average dwell time methods based on the cycle ratio of the switching graph. *Syst. Control Lett.*, 62(11):1032–1037, 2013.

[3] Y. Hou, C. Dong, and Q. Wang. Stability analysis of switched linear systems with locally overlapped switching law. *J. Guid. Control Dyn.*, 33(2):396–403, 2010.

[4] L. Xu, Q. Wang, W. Li, and Y. Hou. Stability analysis and stabilisation of full-envelope networked flight control systems: switched system approach. *IET Control Theory A.*, 6(2):286–296, 2012.

[5] W. Zhang, Y. Hou, X. Liu, and Y. Zhou. Switched control of three-phase voltage source PWM rectifier under a wide-range rapidly varying active load. *IEEE Trans. Power Electron.*, 27(2):881–890, 2012.

[6] J. L. Mancilla-Aguilar, R. Garcia, E. Sontag, and Y. Wang. Uniform stability properties of switched systems with switchings governed by digraphs. *Nonlinear Anal.-Theor.*, 63(3):472–490, 2005.

[7] Ö. Karabacak, F. İlhan, and I. Öner. Explicit sufficient stability conditions on dwell time of linear switched systems. In *Proc. of the 53rd Conf. on Decision and Control*.

[8] A. Kundu, N. Balachandran, and D. Chatterjee. Algorithmic synthesis of stabilizing switching signals for discrete-time switched linear systems. arXiv:1405.1857 [cs.SY], 2014.
[9] R. W. Karp. Characterization of minimum cycle mean in a digraph. *Discrete Math.*, 23(3):309–311, 1978.

[10] M. V. Golitschek. Optimal cycles in doubly weighted graphs and approximation of bivariate functions by univariate ones. *Numer. Math.*, 39(1):65–84, 1982.

[11] A. Dasdan and R. K. Gupta. Faster maximum and minimum mean cycle algorithms for system-performance analysis. *IEEE T. Comput. Aid. D.*, 17(10):889–899, 1998.

[12] A. Dasdan. Experimental analysis of the fastest optimum cycle ratio and mean algorithms. *ACM Trans. Des. Autom. Electron. Syst.*, 9(4):385–418, 2004.

[13] P. Bouyer, E. Brinksma, and K. G. Larsen. Staying alive as cheaply as possible. In Alur, R and Pappas, GJ, editor, *Hybrid Systems: Computation and Control, Proceedings*, volume 2993 of *Lect. Notes Comput. Sc.*, pages 203–218, 2004.

[14] R. Leus and W. Herroelen. Scheduling for stability in single-machine production systems. *J. Sched.*, 10(3):223–235, 2007.

[15] V. Kats and E. Levner. Cyclic routing algorithms in graphs: Performance analysis and applications to robot scheduling. *Comput. Ind. Eng.*, 61(2, SI):279–288, 2011.

[16] R. Lu and C. K. Koh. Performance analysis of latency-insensitive systems. *IEEE T. Comput. Aid. D.*, 25(3):469–483, 2006.

[17] Ali Dasdan’s web page. [http://www.dasdan.net/ali/publications.php#journals_submitted](http://www.dasdan.net/ali/publications.php#journals_submitted)

[18] R. D. Braatz ve M. Morari. Minimizing the euclidean condition number. *SIAM J. on Contr. Optim.*, 32(6):1763–1768, 1994.
[19] C. S. Liu. A two-side equilibration method to reduce the condition number of an ill-posed linear system. *CMES-Comp. Model. Eng.*, 91(1):17–42, 2013.

[20] F. L. Bauer. Optimally scaled matrices. *Numer. Math.*, 5:73–87, 1963.

[21] Y. Mori, T. Mori, and Y. Kuroe. A solution to the common Lyapunov function theorem for continuous-time systems. In *Proc. of the 36. Conf. on Decision and Control*, pages 3530–3531, San Diego, California, 1997.

[22] S. Morse. Supervisory control of families of linear set-point controllers-part 1: exact matching. *IEEE Trans. on Auto. Cont.*, 41(10):1413–1431, 1996.

[23] J. C. Geromel and P. Colaneri. Stability and stabilization of discrete time switched systems. *Int. J. Control*, 79(7):719–728, 2006.

[24] W. A. Zhang and L. Yu. Stability analysis for discrete-time switched time-delay systems. *Automatica*, 45(10):2265 – 2271, 2009.

[25] J. Zhang, Z. Han, F. Zhu, and J. Huang. Stability and stabilization of positive switched systems with mode-dependent average dwell time. *Nonlinear Analysis: Hybrid Systems*, 9(0):42 – 55, 2013.