Algebraic integrable dynamical systems, 2+1-dimensional models in wholly discrete space-time, and inhomogeneous models in 2-dimensional statistical physics

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Abstract

This paper is devoted to constructing and studying exactly solvable dynamical systems in discrete time obtained from some algebraic operations on matrices, to reductions of such systems leading to classical field theory models in 2+1-dimensional wholly discrete space-time, and to connection between those field theories and inhomogeneous models in 2-dimensional statistical physics.
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To the reader

This is an English version of my recent dissertation, or at least of its substantial part. The Russian version contained also a long Introduction. Here I decided to drop it: it seemed not very easy for me to write an exciting enough Introduction in English. I hope that it will be clear from the Contents and the text itself what this all is about. I think also that it is enough to read a very short Section 1 in order to decide whether this paper is any interesting. However, the Bibliography retains all references from the Russian text.

Acknowledgements. I owe to A.B. Shabat the idea of “local reductions”, i.e. reductions to multidimensional field theories, for dynamical systems of algebraic nature. L.D. Faddeev drew my attention to the work [30] and urged me to do more research in its direction. V.V. Sokolov gave me a valuable consultation on dynamical systems associated with orthogonal or symplectic matrices. I express my deepest gratitude to them all.
Chapter 1

Dynamical system connected with the dimer model

1. Definition of the dynamical system. Gauge invariance

In this chapter we consider the following dynamical system in discrete time. Let

\[ L = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \]

be a block matrix, \( A, \ldots, D \) being \( n \times n \) matrices consisting of complex numbers. Consider the following two operations: construction of the inverse matrix

\[ L \rightarrow L^{-1} \]

and the block transposing

\[ L = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \rightarrow L^t = \begin{pmatrix} A & C \\ B & D \end{pmatrix}. \]

Now let a (birational) mapping \( f \) be a composition of these two operations:

\[ f(L) = (L^{-1})^t. \] (1.1)

Let us introduce the discrete integer-valued time \( \tau \), and let the matrix \( L \) depend on \( \tau \) so that

\[ L(\tau + 1) = f(L(\tau)). \] (1.2)

This dynamical system has been already mentioned in literature \[30\]. In the present chapter, the integrability of this system is demonstrated, assuming that the “motion” is considered up to a “gauge transformation” (see below). We also demonstrate what conditions (reduction) must be imposed on the matrix \( L \) in order to obtain a meaningful evolution equation in 2+1-dimensional space-time. This equation is a 2+1-dimensional version of Toda lattice in discrete time. The evolution is of hyperbolic character: perturbations propagate not faster than fixed “light speed”. The solution to a Cauchy problem can be constructed in theta functions according to a rather simple, in principle, scheme. The remarkable property of the 2+1-dimensional model is that its “integrals of motion” are nothing else than a statistical sum of the well-known flat dimer model (in our case, the statistical sum of dimer model depends on two “spectral parameters”).

Returning then back to the general dynamical system (1.2), we consider its connections with a discrete analogue of the Lax \( L, A \) pair and possible generalizations coming therefrom.

Definition 1.1. We will call the gauge transformation of the matrix \( L \) the following transformation of its blocks:

\[ A \rightarrow GAH, \ldots, D \rightarrow GDH, \] (1.3)

with \( G \) and \( H \) non-degenerate \( n \times n \) matrices.
It is clear that the transformation (1.3) commutes with the shift by two time units. Two matrices \( L \) and \( L' \) connected by the transformation (1.3) will be called gauge equivalent. Thus, dynamics (1.2) induces a dynamics on the set of gauge invariance classes of matrices \( L \).

2. Vacuum curves and vacuum vectors

It turns out that the dynamics (1.2) preserves the so-called vacuum curve \( \Gamma \) of the operator \( L \) (the bases being fixed, we make no difference between a linear operator and its matrix). To be exact, \( \Gamma \) remains unchanged under the transformation \( f \circ f \), and undergoes a simple transformation under \( f \). The curve \( \Gamma \) together with the class of linear equivalence of the pole divisor of the vacuum vectors (see below) determines the matrix \( L \) up to a gauge transformation. The set of those classes of linear equivalence is isomorphic to a complex torus—the Jacobian of the curve \( \Gamma \). The dynamics (1.2) linearizes on the Jacobian, i.e. the transformation \( f \) corresponds to a constant shift on the torus.

Now, let us discuss these facts in detail.

The vacuum curve of the operator \( L \) is an algebraic curve in the space \( \mathbb{C}^2 \) of two variables \( u, v \). Here are two equivalent definitions of it [42].

**Definition 2.1.** Consider the relation

\[
L(U \otimes X) = V \otimes Y, \tag{2.1}
\]

wherein

\[
U = \begin{pmatrix} u \\ 1 \end{pmatrix}, \quad V = \begin{pmatrix} v \\ 1 \end{pmatrix}
\]

are two-dimensional vectors, \( X \) and \( Y \) are \( n \)-dimensional vectors. For a generic matrix \( L \), the non-zero solutions \((U, V, X, Y)\) of the relation (2.1) are parametrized, up to a scalar factor in \( X \) and \( Y \), by points of an algebraic curve \( \Gamma \) of genus \( g = (n - 1)^2 \) given by an equation of the form

\[
P(u, v) = 0, \tag{2.2}
\]

\( P(u, v) \) being a polynomial of degree \( n \) in each variable, i.e.

\[
P(u, v) = \sum_{j,k=1}^{n} a_{jk} u^j v^k. \tag{2.3}
\]

\( \Gamma \) is called the vacuum curve of the operator \( L \).

**Definition 2.2.** The vacuum curve of the operator \( L \) is the curve \( \Gamma \) in \( \mathbb{C}^2 \) given by the equation

\[
P(u, v) = \det(V^\perp LU) = \det(uA + B - uvC - vD) = 0, \tag{2.4}
\]

where

\[
V^\perp = (1, -v).
\]

Let us denote the points of the vacuum curve by the letter \( z = (u, v) \in \Gamma \). Then \( U = U(z) \) and \( V = V(z) \) are meromorphic vectors on \( \Gamma \) with the pole divisors \( D_U \) and \( D_V \) of degree \( n \), while \( X = X(z) \) and \( Y = Y(z) \), if normalized by, e.g., the condition that their \( n \)th coordinates equal unity, become meromorphic vectors with pole divisors \( D_X \) and \( D_Y \) of degree \( n^2 - n \) [42]. Under this normalization, a meromorphic scalar factor \( h(z) \) must be added into (2.4):

\[
L(U(z) \otimes X(z)) = h(z)V(z) \otimes Y(z). \tag{2.5}
\]

The linear equivalence of divisors

\[
D_U + D_X \sim D_V + D_Y
\]
holds and is provided by the function \( h(z) \) in the sense that \( h(z) \) has its poles in the points of \( D_U + D_X \) and zeros in the points of \( D_V + D_Y \).

As is shown in the paper [12], the vacuum curve equation \( P(u, v) = 0 \) and the class of linear equivalence of divisor \( D_X \) or \( D_Y \) determine a generic matrix \( L \) to within a gauge transformation, and vice versa, the gauge transformations do not change the vacuum curve and the classes of linear equivalence of divisors. In other words, the correspondence

\[
(\text{class of gauge equivalence of } L) \leftrightarrow (\Gamma, \text{ the class of } D_X)
\]

is a birational isomorphism.

We will call \( X(z) \) the vacuum vector and \( Y(z) \) the covacuum vector in the point \( z \) of the curve \( \Gamma \). \( X(z) = X(u, v) \) generates the (one-dimensional) kernel of the matrix

\[
uA + B - uvC - vD.
\]

(2.6)

The Definition 2.1 allows one to trace what happens with the vacuum curve and vacuum vectors under the transformation \( L \to L^{-1} \), while the Definition 2.2 allows one to trace what happens under the transformation \( L \to L^t \). Namely, it is seen from the relation

\[
L^{-1}(V(z) \otimes Y(z)) = h(z)^{-1}U(z) \otimes X(z)
\]

that the vacuum curve equation for the matrix \( L^{-1} \) is

\[
P(v, u) = 0,
\]

while its vacuum vector in the point \((v, u)\) coincides with the covacuum vector of the initial matrix \( L \). As for the block transposing, the vacuum curve equation for the matrix \( L^t \)

\[
det(uA + C - uvB - vD) = 0
\]

may be rewritten as

\[
u^n v^n \det(v^{-1}A - B + u^{-1}v^{-1}C - u^{-1}D) = 0,
\]

i.e.

\[
u^n v^n P(-v^{-1}, -u^{-1}) = 0.
\]

The vacuum vector of the matrix \( L^t \) in the point \((-v^{-1}, -u^{-1})\) of its vacuum curve coincides with the vacuum vector \( X(u, v) \) of the matrix \( L \).

Combining these considerations, one finds out that the vacuum curve \( \tilde{\Gamma} \) of the matrix \((L^{-1})^t\) is given by equation

\[
u^n v^n P(-u^{-1}, -v^{-1}) = 0,
\]

while the vacuum vector \( \tilde{X}(-u^{-1}, -v^{-1}) \) coincides with the vector \( Y(u, v) \) of the matrix \( L \).

Identifying the curves \( \Gamma \) and \( \tilde{\Gamma} \) by means of the isomorphism

\[
(u, v) \leftrightarrow (-u^{-1}, -v^{-1}),
\]

one sees that

\[
D_X \sim D_Y \sim D_X + D_U - D_V,
\]

which means that, in essence, the transformation \((L^{-1})^t\) results in adding a fixed element of the Picard group, namely the equivalence class of the divisor \( D_U - D_V \), to the pole divisor \( D_X \) of the vacuum vectors. It is clear also that after two transformations one returns to the initial curve:

\[
\tilde{\Gamma} = \Gamma.
\]
3. Reduction to evolution equation in the 2+1-dimensional space-time

The dynamical system of the previous section admits an interesting reduction, i.e., some special choice of the matrices $A, \ldots, D$ that is in agreement with the evolution. In this section, it will be convenient to treat the matrices $A, \ldots, D$ as linear operators acting from the linear space $\mathcal{H}_1$ into the linear space $\mathcal{H}_2$ (of the same finite dimension). This being the situation at the moment $\tau$, the operators act, of course, from $\mathcal{H}_2$ into $\mathcal{H}_1$ at the moment $\tau + 1$, and so on.

Let each of the spaces $\mathcal{H}_1$, $\mathcal{H}_2$ be a direct sum of $lm/2$ identical subspaces of dimension $d$, where $l, m$ are even numbers. Let us imagine these subspaces as situated at the vertices of the square lattice on the torus of the sizes $l \times m$ (which will mean the periodic boundary conditions in both discrete space variables). Let the subspaces be arranged in checkerboard fashion, as in Fig. 1.1, where the empty circles correspond to subspaces of the space $\mathcal{H}_1$, while the filled circles correspond to those of the space $\mathcal{H}_2$.

Let then the operators $A, \ldots, D$ be such that the image of each of the mentioned $d$-dimensional subspaces with respect to, say, operator $A$ lies in the $d$-dimensional subspace of $\mathcal{H}_2$ at which points the arrow marked “$A$” that links these two subspaces (Fig. 1.1). Analogously, the restrictions on $B, C, D$ are depicted in Fig. 1.1 (see also formula (3.11) for non-degenerate $A, \ldots, D$). Thus, to each link of the lattice a $d \times d$ matrix is attached that is a block of one of the “large” matrices $A, \ldots, D$. Let us shade half of the squares of the lattice in a checkerboard way, as in Fig. 1.1. One can verify that the evolution of the system may be described as follows.

At the first step, each of the four $d \times d$ matrices that correspond to the arrows surrounding each shaded square is transformed into a matrix expressed through just these four matrices. This goes according to the following formulae, in which the $d \times d$ blocks are somewhat freely denoted by the same letters $A, \ldots, D$ as the “large” matrices:

$$ A \rightarrow (A - BD^{-1}C)^{-1}, \quad (3.1) $$
$$ B \rightarrow (B - AC^{-1}D)^{-1}, \quad (3.2) $$
$$ C \rightarrow (C - DB^{-1}A)^{-1}, \quad (3.3) $$
$$ D \rightarrow (D - CA^{-1}B)^{-1}. \quad (3.4) $$
However, the formulae (3.1–3.4) apply equally to the “large” matrices.
After the transformation (3.1–3.4), all the arrows reverse, and at the second step the non-shaded squares are engaged in the same way according to the same formulae (3.1–3.4). Then everything is repeated. Thus, the evolution is of hyperbolic nature: each local perturbation spreads not faster than one unit of length per unit of time.

Let us clarify the symmetries of vacuum curves and divisors $D_X$ in this “reduced” model. Let us introduce two integer-valued coordinates $\xi, \eta$ for the vertices of the lattice, so that $\xi$ increases by 1 in passing from a vertex one step to the right, and $\eta$ increases by 1 in passing one step upwards. $\xi$ and $\eta$ are defined modulo $l$ and $m$ respectively. A $d$-dimentional subspace of $H_1$ or $H_2$ will be denoted $H_{\xi\eta}$ if it corresponds to a vertex with coordinates $\xi, \eta$. Consider a linear transformation in spaces $H_1$ and $H_2$ consisting in multiplying the vectors of each subspace $H_{\xi\eta}$ by $\omega_1^\xi$, $\omega_1$ being a fixed primitive root of the $l$-th degree of unity:

$$\omega_1^l = 1.$$  

This corresponds to the following transformation of the operators $A,\ldots, D$ (from now on we speak of each of these operators “as a whole”, not of their blocks):

$$A \rightarrow \omega_1 A, \quad B \rightarrow B, \quad C \rightarrow C, \quad D \rightarrow \omega_1^{-1} D.$$  

(3.5)

Consider also another linear transformation in $H_1$ and $H_2$, consisting in multiplying the vectors of each subspace $H_{\xi\eta}$ by $\omega_2^\eta$, $\omega_2$ being a fixed primitive root of the $m$-th degree of unity:

$$\omega_2^m = 1.$$  

This corresponds to the following transformation:

$$A \rightarrow A, \quad B \rightarrow \omega_2 B, \quad C \rightarrow \omega_2^{-1} C, \quad D \rightarrow D.$$  

(3.6)

The vacuum curve of the operator $L$, which is given by equation (2.4)

$$P(u,v) = \det(uA + B - uvC - vD) = 0,$$

must be invariant under the transformations (3.5), (3.6). This leads to the invariance of the polynomial $P(u,v)$ with respect to the following transformations $g_1$ and $g_2$:

$$g_1(u,v) = (\omega_1 u, \omega_1^{-1} v),$$  

(3.7)

$$g_2(u,v) = (\omega_2^{-1} u, \omega_2^{-1} v).$$  

(3.8)

This invariance, then, leads to the following statement: only those coefficients $a_{jk}$ are non-zero in the vacuum curve equation (see (2.2), (2.3)) for the “reduced” model, for which

$$j - k \equiv 0 (\text{mod } l),$$  

$$j + k \equiv 0 (\text{mod } m).$$  

(3.9)

As for the divisor $D_X$, let us recall that it consists of such points in the curve $\Gamma$ in which vanishes the last coordinate of the vector $X$ (see (42), the latter being an eigenvector of the matrix (2.6) with zero eigenvalue:

$$(uA + B - uvC - vD)X(u,v) = 0.$$  

(3.10)

This immediately leads to the conclusion: the divisor $D_X$ is invariant with respect to the transforma- 
m$

Under some additional condition, the inverse statement also holds: if the curve $\Gamma$ and divisor $D_X$ are invariant under the transformations (3.7), (3.8), then the corresponding $L$-operator comes from a “reduced” model described in this section. For instance, this is true if $l/2$ and $m/2$ are relatively prime numbers. If these numbers are not relatively prime, some conditions are to be
imposed on the divisor $D_X$. To avoid going into details of this latter case, let us not consider it here.

Thus, let an operator $L = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be given, $A, \ldots, D$ being $n \times n$ matrices, $n = (lm/2)d$, $l$ and $m$ even, and $l/2$ and $m/2$ being relatively prime. Let the vacuum curve $\Gamma$ of the operator $L$ and the divisor $D_X$ be invariant under the action of the group $G$ generated by its elements $g_1, g_2$ (3.7, 3.8), $\omega_1$ and $\omega_2$ being primitive roots of degrees $l$ and $m$ of unity. Then the linear space in which operators $A, \ldots, D$ act decomposes into a direct sum of $lm/2$ $d$-dimensional subspaces $H_{\xi \eta}$, $\xi$ and $\eta$ being integers modulo $l$ and $m$ respectively and such that $\xi + \eta$ is an even number, and the following equalities between the images of these subspaces hold (in a “generic” case of non-degenerate $A, \ldots, D$):

$$AH_{\xi-1, \eta+1} = B H_{\xi \eta} = CH_{\xi, \eta+2} = DH_{\xi+1, \eta+1}. \quad (3.11)$$

The equalities (3.11) mean exactly that one is in the situation of Fig. 1.1.

Let us prove the above statements. First, the natural projection from the curve $\Gamma$ to its factor $\Gamma/G$ has no branch points (here the fact that $l/2$ and $m/2$ are relatively prime is used to demonstrate that ramification does not occur when $u$ or $v$ equals zero or infinity). Thus, the $n$-dimensional linear space of meromorphic functions $x(z) = x(u, v)$ whose pole divisor is $D_X$ decomposes into a direct sum of subspaces of equal dimensions corresponding to the characters of (commutative) group $G$. Each of these subspaces consists of functions $x(z)$ satisfying relations

$$x(gz) = \chi_{\xi \eta}(g)x(z),$$

the character $\chi_{\xi \eta}$ being a scalar factor

$$\chi_{\xi \eta}(g) = \omega_1^{g_1 \xi} \omega_2^{g_2 \eta},$$

where

$$g = g_1^a g_2^b.$$

The equality $g_1^2 g_2^2 = 1$ means that $\xi + \eta$ must be an even number.

The components of the vector $X(z)$ are exactly the functions $x(z)$. In an appropriate basis, $d$ components correspond to each character $\chi_{\xi \eta}$. Let us denote $H_{\xi \eta}$ the set of vectors with other components equal to zero. Now, the equalities (3.11) are to be proved to end this section.

Consider the decomposition of vector $X(u, v)$ into a sum

$$X(u, v) = \sum_{\xi, \eta} X_{\xi, \eta}(u, v),$$

where $X_{\xi, \eta} \in H_{\xi \eta}$. Then

$$X_{\xi, \eta}(g(u, v)) = \chi_{\xi, \eta}(g)X_{\xi, \eta}(u, v).$$

Consider the sum

$$\sum_{g \in G} \chi_{\xi, \eta}(g^{-1})g\{ (uA + B - uvC - vD)X(u, v) \} = 0 \quad (3.12)$$

(which is equal to zero because of (3.11)). The action of $g$ upon the braces in (3.12) means that each $u$ and $v$ in the braces is transformed according to (3.7), (3.8), i.e. $u$ changes into $\chi_{1, -1}(g)u$, and $v$ changes into $\chi_{-1, -1}(g)v$. The equality (3.12) gives thus, after cancelling a factor equal to the order of group $G$,

$$uAX_{\xi-1, \eta+1}(u, v) + BX_{\xi, \eta}(u, v) - uvCX_{\xi, \eta+2}(u, v) - vX_{\xi+1, \eta+1}(u, v)D = 0. \quad (3.13)$$

Let us set $u = 0$ in (3.13). Then $v$ can take $n$ different values $v_j$ satisfying relation $P(0, v_j) = 0$. To these values $v_j$ correspond $d$ linearly independent vectors $X_{\xi \eta}(0, v_j)$, and also $d$ vectors $X_{\xi+1, \eta+1}(0, v_j)$. Thus, the equalities

$$BX_{\xi \eta}(0, v_j) = v_j DX_{\xi+1, \eta+1}(0, v_j)$$

that result from (3.13) give

$$BH_{\xi \eta} = DH_{\xi+1, \eta+1}.$$

Analogously, one can as well obtain the rest of equalities (3.11).
4. Connection to dimer model

As has been demonstrated, the integrals of motion of the dynamical system of Section 1 and its reductions (if the *even* degrees of the transformation \([1.1]\) are considered) are the coefficients \(a_{jk}\) of the vacuum curve \((2.3)\). These coefficients are determined up to a common factor, so they may be divided by \(a_{00}\). As one can see, the resulting coefficients are those of the polynomial

\[
\det(1 + uAB^{-1} - vDB^{-1} - uvCB^{-1}).
\]  

(4.1)

In other words, the determinant \((4.1)\) is an integral of motion for any \(u,v\).

Let us turn now to the model from section 3, that is to the model in 2+1-dimensional discrete space-time with periodic boundary conditions, and let the dimension \(d\) of the linear space corresponding to each vertex be equal to 1. Each of the “small” matrices \(A,B,C,D\) corresponding to the links will then be a single (depending on the link) number \(a,b,c\) or \(d\). It is well known that the determinant of any \(N \times N\) matrix is a sum of its matrix elements products corresponding in a certain way to the permutations of \(N\) objects, while each permutation decomposes into a product of the cyclic ones. In our situation, the cyclic permutations correspond to the non-selfintersecting closed paths (contours) going along the arrows of the following diagram (Fig. 1.2) (thus, general permutations correspond to the sets of non-intersecting paths). To each closed path corresponds the product of the weights \(ua, -vd, -uvc, vb^{-1}\) on its links, and, to get right signs for the terms of which the determinant \((4.1)\) is made up, one should add a minus sign to each such product containing an *even* number of the factors \(b^{-1}\).

**Remark 4.1.** Another way to obtain right signs is: to multiply each \(b\) by \(-1\) and then multiply each product corresponding to a closed path (and containing any number of \(b\)'s) by \(-1\).

It turns out that the determinant \((4.1)\) is connected with the statistical sum of the well known dimer model \([69]\). Let us define the correspondence between the sets of paths and the dimer configurations as follows. Let the empty set of paths correspond to the “standard” dimer configuration, the dimers being placed on the “\(B\)-links” (Fig. 1.3). For a non-empty set of paths, let us change the standard configuration along all the paths, replacing each dimer by a free link and vice versa. One can verify that this is a bijective correspondence.

The statistical sum being considered, let the weights \(-b\) (not \(b^{-1}\)) correspond to the “\(B\)-links”, while to the other links correspond the unchanged weights \(ua, -vd, -uvc\). Then one can see that the statistical sum, if multiplied by \(\prod_{\text{all links}} (-b^{-1})\) (let us call the result the normalized statistical sum), consists of the same terms as the determinant \((4.1)\), up to different signs of some of them. Let us emphasize that the dimer model is, of course, *inhomogeneous*: the weights \(a,b,c,d\) are different for different links.

Let us study these signs in detail. Note that the conditions of non-intersecting and non-selfintersecting impose strong restrictions on the possible path configurations. Every closed path on the torus is homologically equivalent to a linear combination with integer coefficients of two
basis cycles a and b whose intersection number is 1 (I use the boldface font for cycles, because the letters a, b... are already in use). If the torus is cut along a closed non-selfintersecting path c not equivalent to zero, the result will be homeomorphic to the lateral surface of a cylinder (this follows, e.g., from [68], chapter 1, section 3). Then the contour d going along a generatrix of the cylinder in a properly chosen direction has the intersection number 1 with the contour c. The intersection number being bilinear and integer-valued, we find that if the contour c is homologically equivalent to a sum \( la + mb \), then \( l \) and \( m \) cannot have common divisors (not equal to \( \pm 1 \)). Thus, the following lemma is valid.

**Lemma 4.1.** Every closed non-selfintersecting path on the torus is homologically equivalent to a linear combination of the basis paths a and b with relatively prime integer coefficients.

Now let us pass to the case of several contours on the torus. If they do not intersect, their intersection numbers equal 0 (of course) and thus their homological classes must be proportional to one another. This together with Lemma 4.1 leads to the following lemma.

**Lemma 4.2.** Several closed non-intersecting and non-selfintersecting paths going along the arrows on the torus, as in Fig. 1.2, are necessarily all homologically equivalent to one another.

If two paths are homologically equivalent, then the terms of the same degrees in \( u \) and \( v \) correspond to them (one can see in Fig. 1.2 that the different ways round an “elementary square” yield the same degrees of \( u \) and \( v \)). Let the basis paths a and b yield the terms proportional to \( x = u^{\alpha_1}v^{\beta_1}, y = u^{\alpha_2}v^{\beta_2} \) correspondingly (with the factors of proportionality not depending on \( u, v \)). According to Lemma 4.1, the determinant (4.1) and the statistical sum of the dimer model are polynomials in \( x, y \). The following lemma sums up this section.

**Lemma 4.3.** Let \( f(x, y) \) and \( s(x, y) \) be the determinant (4.1) and the normalized statistical sum of the dimer model considered as functions of \( x \) and \( y \). Then

\[
\begin{align*}
    s(x, y) &= \frac{1}{2}(-f(x, y) + f(-x, y) + f(x, -y) + f(-x, -y)) \tag{4.2} \\
    f(x, y) &= \frac{1}{2}(-s(x, y) + s(-x, y) + s(x, -y) + s(-x, -y)) \tag{4.3}
\end{align*}
\]

**Proof.** If the normalized statistical sum consists of the terms

\[ c_{jk} x^j y^k = c_{jk} (u^{\alpha_1}v^{\beta_1})(u^{\alpha_2}v^{\beta_2})^k, \]

then the determinant consists of the same terms multiplied by

\[ (-1)^{\text{number of contours}} = (-1)^{\text{g.c.d.}(j,k)} = (-1)^{j+k+j+k} \]

(here Remark 4.1 and Lemmas 4.1 and 4.2 are used). This means that the signs of all the terms must be changed except where both numbers \( j \) and \( k \) are even. This is exactly what the formulae (4.2, 4.3) do. The lemma is proved.
5. Equation of motion for “physical” variables

For the dynamical system connected with the dimer model, to each link of the square lattice corresponds a complex number denoted \( a, b, c \) or \( d \), as was explained in the beginning of Section 4. Recall that the dependence of those numbers on a link of the lattice is implied, but usually not indicated explicitly, not to overload the formulae and figures such as Fig. 1.2. We study the evolution of the dynamical system up to the gauge transformation (1.3). It is easily seen that, for the dimer model, those transformations can be described as follows. Let us call the elementary gauge transformation corresponding to a given vertex of the lattice and a non-zero complex number \( \lambda \) the following: take the numbers \( a, b, c \) and \( d \) corresponding to the four incoming and outgoing links for the given vertex, and multiply them all by \( \lambda \). The general gauge transformation for the dimer model will be a composition of any number of elementary transformations (which, of course, commute with each other).

Remark 5.1. There exists also a discrete group of transformations of the form (1.3) compatible with the reduction of the system to the dimer model which consists of translations of the lattice: \( G = H^{-1} \) in (1.3) is in this case the operator of translation by an integer vector \((\xi, \eta)\), with \(\xi + \eta\) even. Here we, however, do not consider such transformations.

In this section, we will introduce, instead of \( a, b, c, d \), new variables \( \Omega \) which can be called “physical” in the sense that \( \Omega \) does not change under (general) gauge transformations. We will write out the “equation of motion” for \( \Omega \), which turns out to be a 2+1-dimensional variant of the Toda lattice in discrete time. The remarkable and not a priori evident property of this equation is the completely equal status of the three coordinates: spatial \( \xi, \eta \) and the time \( \tau \). We will consider also the questions of how to come back from \( \Omega \)’s to the variables \( a, b, c, d \), and whether it is possible to express the statistical sum in terms of \( \Omega \).

So, let a number \( \Omega \) correspond to each two-dimensional cell (square) of the lattice, which is a “multiplicative circulation” composed from the numbers \( a, b, c, d \) on the links bordering the cell as follows:

\[
\Omega = \varepsilon_a \varepsilon_b \varepsilon_c \varepsilon_d,
\]

where

\[
\varepsilon_a = -\varepsilon_b = -\varepsilon_c = \varepsilon_d = \pm 1,
\]

the sign “+” corresponding to non-shaded squares on Fig. 1.1 and the sign “−” corresponding to shaded squares. In other words, let us move around the square anticlockwise, and take the number on each link in the degree one or minus one if we pass that link in or against its direction correspondingly.

Remark 5.2. This construction is described naturally in terms of cohomology theory. Namely, the torus \( \mathbb{T}^2 \) with the square lattice on it and the given orientation (direction) for all links of the lattice is a CW-complex ([6], chapter 1, §4). The “field” \( a, b, c, d \) is a 1-cochain with coefficients in the multiplicative group \( \mathbb{C}^* \) of non-zero numbers (we assume that in the “generic” situation, which we prefer to consider, no one of the numbers \( a, b, c, d \) equals zero), while its “rotor” \( \Omega \) is the coboundary of that cochain. This cohomological interpretation will be of use for us later, when we will consider the reconstruction of the “field” \( a, b, c, d \) from \( \Omega \).

Let us examine the change of circulations \( \Omega \) in one step of evolution. Here one must consider separately the squares of the form \( \begin{array}{c} a \\ d \end{array} \begin{array}{c} c \\ b \end{array} \) (i.e. shaded ones in Fig. 1.1) and the squares \( \begin{array}{c} d \\ a \end{array} \begin{array}{c} c \\ b \end{array} \) (non-shaded in Fig. 1.1). The former we will call the active squares, the latter — the non-active ones.

The circulation around an active square is

\[
\Omega = a^{-1}cd^{-1}b,
\]

(5.1)
The transformation \([5.1-5.4]\), which in our case can be written as the transformation

\[
\begin{align*}
a & \rightarrow d (ad - bc)^{-1}, \\
b & \rightarrow -c (ad - bc)^{-1}, \\
c & \rightarrow -b (ad - bc)^{-1}, \\
d & \rightarrow a (ad - bc)^{-1}
\end{align*}
\]  

(5.2)

with the simultaneous reversing of all the arrows, changes the circulation in an obvious way as

\[\Omega \rightarrow \Omega^{-1}.\]  

(5.3)

Convert now the formulae \([5.2]\) for active squares to the following form which we need for examining what happens in the neighboring non-active squares:

\[
\begin{align*}
a & \rightarrow a^{-1}(1 - \Omega)^{-1}, \\
b & \rightarrow b^{-1}(1 - \Omega^{-1})^{-1}, \\
c & \rightarrow c^{-1}(1 - \Omega^{-1})^{-1}, \\
d & \rightarrow d^{-1}(1 - \Omega)^{-1}.
\end{align*}
\]  

(5.4)

This done, consider the formulae \([5.4]\) not for one active square, but for four active squares adjacent to the chosen non-active one. To be exact, let us consider the formula

\[
\Omega = \Omega(\xi, \eta) = ac^{-1}db^{-1},
\]

\[\xi, \eta \text{ being the integer-valued coordinates of, say, south-western vertex of this square, are transformed by one step of evolution as}
\]

\[
\begin{align*}
\Omega(\xi, \eta) & \rightarrow \Omega(\xi,\eta) \cdot (1 - \Omega(\xi,\eta - 1)) \cdot (1 - \Omega(\xi,\eta + 1)) \times \\
& \times (1 - \Omega^{-1}(\xi - 1,\eta))^{-1} \cdot (1 - \Omega^{-1}(\xi + 1,\eta))^{-1}.
\end{align*}
\]  

(5.5)

After the step of evolution, the active squares become non-active and vice versa. Thus, with the proper choice of the origin for the integer-valued time coordinate \(\tau\), the active squares have the \(\text{odd} \ \sum \xi + \eta + \tau\) of their three coordinates. As for the non-active squares, they can be totally eliminated from consideration when deriving the “equation of motion” for \(\Omega\) because, according to \([5.3]\), the circulation around a non-active square is the same as that around the same square in the previous moment \(\tau\), when it was active. Writing out explicitly the dependence of \(\Omega\) on the time, we get from \([5.3]\) and \([5.3]\) the following “equation of motion” containing only the circulations around active squares:

\[
\begin{align*}
\Omega(\xi, \eta, \tau + 1) & = \\
& = \frac{(1 - \Omega(\xi, \eta, \tau - 1)) (1 - \Omega(\xi, \eta + 1, \tau))(1 - \Omega^{-1}(\xi, \eta, \tau))}{(1 - \Omega^{-1}(\xi - 1, \eta, \tau)) (1 - \Omega^{-1}(\xi + 1, \eta, \tau))},
\end{align*}
\]  

(5.6)

\[\xi + \eta + \tau \ \text{even}.
\]

Note the following compact form of equation \([5.6]\). Introduce the “discrete pseudo-Laplacians” \(\Delta_{\tau,\xi}\) and \(\Delta_{\eta,\xi}\) acting on functions \(F\) on the cubic lattice, by the formulae

\[
\begin{align*}
(\Delta_{\tau,\xi}F)(\xi,\eta,\tau) & = F(\xi,\eta,\tau - 1) + F(\xi,\eta,\tau + 1) - \\
& - F(\xi - 1,\eta,\tau) - F(\xi + 1,\eta,\tau); \\
(\Delta_{\eta,\xi}F)(\xi,\eta,\tau) & = F(\xi,\eta - 1,\tau) + F(\xi,\eta + 1,\tau) - \\
& - F(\xi - 1,\eta,\tau) - F(\xi + 1,\eta,\tau).
\end{align*}
\]  

(5.7)

(5.8)

By means of these operators, \([5.6]\) is rewritten simply as

\[\Delta_{\tau,\xi} \ln \Omega = \Delta_{\eta,\xi} \ln (\Omega - 1).\]  

(5.9)
It is seen from (5.9) that the spatial coordinate $\eta$ and the time $\tau$ in our model are present at an equal status. Namely, the equation (5.9) obviously does not change under simultaneous changes

$$\tau \leftrightarrow \eta, \quad \Omega \leftrightarrow 1 - \Omega.$$  \hspace{1cm} (5.10)

The equal rights between the two spatial axes must exist, of course, too. It is described, as one can easily verify, by the substitutions

$$\xi \leftrightarrow \eta, \quad \Omega \leftrightarrow \Omega - 1.$$  \hspace{1cm} (5.11)

It is implied, of course, that the interchanges of coordinate axes in (5.10, 5.11) act only on the subscripts of pseudo-Laplacians of the type (5.7, 5.8). Those interchanges generate the symmetric group $S_3$ acting on the set $(\xi, \eta, \tau)$, and the corresponding transformations of $\Omega$ generate a group of linear-fractional transformations isomorphic to $S_3$.

**Remark 5.3.** The equation (5.6) or (5.9) can be obtained by a simple transformation from the “discrete Toda field” equation from [62] (see also [59, 60, 61]), at least if we do not take into account the boundary conditions. That discrete field in [62] depends on an integer-valued coordinate $k$ and two more coordinates $u$ and $v$, each taking values with steps $h$, and is written as $f_k(u,v)$. The field equation has the form (formula (8) from [62])

$$\exp(f_k(u + h, v + h) - f_k(u, v + h) - f_k(u, v + h) + f_k(u, v)) =$$

$$= \left(1 + h^2 \exp(f_{k+1}(u + h, v) - f_k(u, v + h))\right) \times$$

$$\times \left(1 + h^2 \exp(f_k(u + h, v) - f_{k-1}(u, v + h))\right).$$  \hspace{1cm} (5.12)

Divide (5.12) by the equation obtained from (5.12) by the change

$$k \to k - 1, \quad u \to u - h, \quad v \to v + h,$$

and introduce a new variable

$$g_k(u, v) = -h^2 \exp(f_k(u + h, v) - f_{k-1}(u, v + h)).$$

We then get the equation

$$g_k(u + h, v) g_k(u - h, v) = \frac{(1 - g_{k+1}(u, v))(1 - g_{k-1}(u - h, v + h))}{(1 - g_k(u, v))(1 - g_k(u - h, v + h))}.$$

Finally, setting

$$\eta = k, \quad \xi = -h^{-1}u + h^{-1}v, \quad \tau = h^{-1}u + h^{-1}v,$$

$$\Omega(\xi, \eta, \tau) = g_k(u, v),$$

we get for $\Omega$ equation (5.6).

Thus, the dynamical system connected with the inhomogeneous dimer model is a variant of the Toda lattice.

Let us now pay attention to the question to what degree can the “field” $a, b, c, d$ be reconstructed from its “rotor” $\Omega$. First of all, this can be done only if $\Omega$ is indeed a coboundary (see Remark 5.2), i.e. if the corresponding to $\Omega$ cohomology class in the group $H^2(T^2, C^*)$ vanishes. The mentioned cohomology group, according to the book [18], is isomorphic to $C^*$. Constructively speaking, all this looks as follows: two “fields” $\Omega$ belong to the same cohomology class if and only if the products of their values over all two-dimensional cells are equal (recall that the group $C^*$ is multiplicative), and the field $\Omega$ is a coboundary if this product equals 1.

Assuming that this last condition is satisfied, we can find a cochain $a, b, c, d$ whose coboundary is $\Omega$. It is natural to want to determine that cochain to within the (general) gauge transformations
introduced in the beginning of this section. It is easy to see that, in cohomological terms, the
gauge transformations are changes of the 1-cochain \(a, b, c, d\) by coboundaries of 0-cochains. From
its coboundary, however, \(\Omega\) can be restored only to within a cocycle. To find \(a, b, c, d\) up to a
cocycle, one must fix two more parameters because, according to [68], the group \(H^1(T^2, C^*)\) is
isomorphic to \(C^* \oplus C^*\). Those two parameters are nothing else than the values of cochain \(a, b, c, d\) on two (arbitrarily chosen) basis cycles on the torus, i.e. the products of numbers corresponding
to the links, taken in degrees \(\pm 1\), over all links entering a basis cycle. Here, as well as earlier, the
sign \(+\) is taken if the direction of our way around the cycle coincides with the direction of a link,
and the sign \(-\) is taken if it does not.

Comparing this situation to the representation of the motion integrals (4.1) as a sum over the
sets of closed paths on the torus given in Section 4, we see that the determinant (4.1) and, conse-
quently, the statistical sum are not determined uniquely by a field \(\Omega\). They are determined only
up to a two-parameter transformation which, as one can see from the considerations of Section 4,
can be interpreted as a renormalization of the variables \(u, v\):

\[ u \rightarrow \text{const}_1 \cdot u, \quad v \rightarrow \text{const}_2 \cdot v. \]

6. The discrete analog of Lax pair and a generalization of the dynamical system

Now let us return from the reduction of Section 3 to general matrices \(L = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\). Let us
consider the evolution described in Section 1 from another viewpoint. Denote

\[ (L^{-1})^t = \begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix}. \]

This means that

\[ \begin{pmatrix} \tilde{A} & \tilde{C} \\ \tilde{B} & \tilde{D} \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = 1. \]  

(6.1)

It follows from the equality (6.1) that

\[ \tilde{A}A + \tilde{C}C = \tilde{B}B + \tilde{D}D, \]

\[ \tilde{A}B + \tilde{C}D = 0, \]

\[ \tilde{B}A + \tilde{D}C = 0. \]

These three equations are equivalent to the fact that the following equality holds for any complex
\(u\):

\[ -(\tilde{A} - u\tilde{B})^{-1}(\tilde{C} - u\tilde{D}) = (uA + B)(uC + D)^{-1}. \]

(6.3)

Vice versa, from (6.3) follows

\[ \tilde{L}^tL = \begin{pmatrix} F & 0 \\ 0 & F \end{pmatrix}, \]

\( F \) being equal to both sides of (6.3), i.e.

\[ \tilde{L} = \begin{pmatrix} F & 0 \\ 0 & F \end{pmatrix}(L^{-1})^t. \]

It is clear that with any choice of \(F\) the matrix \(\tilde{L}\) belongs to the same equivalence class. The
formula (6.3) defines the same evolution in the space of these classes as it was in Section 4 with
the agreement that the operators without a tilde correspond to the moment of time \(\tau\), while those
with a tilde correspond to the moment \(\tau + 1\).

The formula (6.3) suggests the following generalization. Let, from now on, \(A(u)\) and \(B(u)\) be
matrices depending polynomially on \(u\):

\[ A(u) = A_0 + A_1 u + \cdots + A_m u^m, \]

(6.4)
\[ B(u) = B_0 + B_1 u + \cdots + B_m u^m. \]  

(6.5)

We will look for matrices \( \tilde{A}(u), \tilde{B}(u) \)—the matrix polynomials of the same degrees \( m_A \) and \( m_B \) in \( u \)—that satisfy, for any \( u \), the equation

\[ \tilde{B}(u)^{-1} \tilde{A}(u) = A(u) B(u)^{-1}. \]  

(6.6)

The relation \[(6.6)\] provides what is called a discrete analog of the Lax \( L, A \)-pair. Let us forget for a moment that we are already using the letters \( L \) and \( A \) for other purposes, and remind that the Lax \( L,A \)-pair is a pair of operators depending on the time and other parameters and having usually some special form, the evolution of operator \( L \) in the case of discrete time being described by the formula

\[ L(\tau + 1) = A(\tau) L(\tau) A(\tau)^{-1}. \]  

(6.7)

On the other hand, let us rewrite (returning to our notations) \[(6.6)\] in the following form:

\[ \tilde{B}(u)^{-1} \tilde{A}(u) = A(u) B(u)^{-1} A(u) A(u)^{-1}. \]  

(6.8)

Comparing \[(6.7)\] and \[(6.8)\], we see that, in our case, \( B(u)^{-1} A(u) \) plays the role of Lax \( L \)-operator, while \( A(u) \) — the role of Lax \( A \)-operator.

Let \( v \) be an eigenvalue of both sides of \[(6.6)\]. Let \( Y(u,v) \) be the corresponding eigenvector normalized, as in Section 2, so that its last coordinate equals unity, and let \( X(u,v) \) be the vector proportional to \( B(u)^{-1} Y(u,v) \) and normalized in the same way. One can verify that this may be described by the following formula (\( h(u,v) \) being a scalar factor):

\[ \left( \begin{array}{c} A(u) \\ B(u) \end{array} \right) X(u,v) = h(u,v) \left( \begin{array}{c} v \\ 1 \end{array} \right) \otimes Y(u,v), \]  

(6.9)

which is in obvious analogy to \[ (2.5) \]. The divisor equivalence is

\[ mD_u + D_X \sim D_v + D_Y, \]  

(6.10)

\( D_u \) and \( D_v \) being pole divisors of the functions \( u \) and \( v \), \( m = \max(m_A, m_B) \).

For a given \( u \), the eigenvalues \( v \) come from the equation

\[ P(u,v) = \det \left( A(u) - v B(u) \right) = 0. \]  

It defines an algebraic curve \( \Gamma \)—“generalized vacuum curve”. Let us calculate the genus \( g \) of the curve \( \Gamma \). First, we need to know the number of branch points of the projection

\[ (u,v) \longrightarrow u \]  

(6.11)

of the curve \( \Gamma \) onto the complex plane.

Consider \( P(u,v) \) as a polynomial in \( v \):

\[ P(u,v) = a_0(u) + a_1(u) + \cdots + a_n(u)v^n. \]  

(6.12)

One can verify that \( a_j(u) \) has a degree

\[ \deg a_j(u) = (n - j)m_A + jB. \]  

(6.13)

From this one can deduce that the discriminant of \( P(u,v) \) considered as a polynomial in \( v \) is a polynomial of degree

\[ b = (m_A + m_B) n (n - 1) \]  

in \( u \). The mapping \[(6.11)\] being \( n \)-sheeted and the number of branch points equaling \( b \), one obtains from the Riemann—Hurwitz formula that

\[ g = (n - 1) \left( \frac{m_A + m_B}{2} - n - 1 \right). \]  

(6.14)
So, the following construction has been described. Given two polynomial matrix functions \( A(u) \) and \( B(u) \), one considers the meromorphic matrix function \( A(u)B(u)^{-1} \) (or else \( B(u)^{-1}A(u) \)), and from this function the algebro-geometrical objects arise: the generalized vacuum curve \( \Gamma \) and the linear equivalence class of the pole divisor \( D_Y \) (or, respectively, \( D_X \)) of the eigenvectors of the mentioned meromorphic matrix function. Instead of the pair \((A(u), B(u))\), it is sufficient to indicate its equivalence class with respect to gauge transformations

\[
A(u) \rightarrow GA(u)H, \quad B(u) \rightarrow GB(u)H; \quad (6.15)
\]

instead of the function \( A(u)B(u)^{-1} \), its equivalence class with respect to transformations

\[
A(u)B(u)^{-1} \rightarrow GA(u)B(u)^{-1}G^{-1}
\]

will suffice. Then it turns out that the correspondence between such equivalence classes (either of the pairs \((A(u), B(u))\) or the functions \( A(u)B(u)^{-1} \)) and the abovementioned algebro-geometrical objects is a birational isomorphism, the divisors \( D_X \) and \( D_Y \) being of degree \( g + n - 1 \), as in Section 2.

The easiest way to show this is to start from a given curve \( \Gamma \) defined by the equation

\[
P(u, v) = \sum_{j=0}^{n} \sum_{k=0}^{(n-j)m_{A}+jm_{B}} a_{jk}v^{j}u^{k} = 0
\]

(compare with (6.12, 6.13)) and a divisor \( D_X \) in it of degree \( g + n - 1 \).

The number of coefficients \( a_{jk} \) minus one common factor equals

\[
(n + 1) \left( \frac{m_{A} + m_{B}}{2}n + 1 \right) - 1. \quad (6.16)
\]

The linear equivalence class of divisor \( D_X \) is defined, as is known, by \( g \) parameters. Adding up the expressions (6.16) and (6.14), one gets the total of

\[
(m_{A} + m_{B})n^{2} + 1 \quad (6.17)
\]

parameters.

Then, the gauge equivalence class of the pair \((A(u), B(u))\) is constructed out of relation (6.10). To give more details, one must at first choose a divisor \( D_Y \) satisfying the equivalence (6.10). Then the poles and zeros of the function \( h(u,v) \) are determined. For \( X(u,v) \) and \( Y(u,v) \) one must take columns consisting each of \( n \) linearly independent meromorphic functions with corresponding pole divisors. The arbitrariness in these constructions leads exactly to the fact that \( A(u) \) and \( B(u) \) are determined up to a transformation (6.13).

The pair \((A(u), B(u))\), up to a scalar common factor, is determined by \((m_{A} + m_{B} + 2)n^{2} - 1 \) parameters (see (6.4, 6.5)). In taking the gauge equivalence class, the number of parameters is reduced by \( 2(n^{2} - 1) \). The result is again (6.17). This means that, indeed, to a generic pair \((A(u), B(u))\) corresponds a divisor \( D_X \) of degree \( g + n - 1 \) and the correspondence

\[
(\text{gauge equivalence class of the pair } (A(u), B(u))) \leftrightarrow (\Gamma, \text{ class of } D_X)
\]

is a birational isomorphism.

Now let us recall that \( Y(u,v) \) was defined as an eigenvector of the operator \( A(u)B(u)^{-1} \), while \( X(u,v) \), as is easily seen, is an eigenvector of \( B(u)^{-1}A(u) \). The relation (6.6) means that for the pair \((A(u), B(u))\) its vector \( X(u,v) \) is nothing else than \( Y(u,v) \), i.e. the equivalence holds

\[
D_X \sim D_X + (mD_{u} - D_{v}). \quad (6.18)
\]

Now, assuming that if a quantity without a tilde corresponds to the moment of time \( \tau \) then that with a tilde corresponds to \( \tau + 1 \), one comes to a conclusion that to the adding of unity to the time corresponds a constant shift (6.18) in the Jacobian of the curve \( \Gamma \). Thus, the dynamics of the system in this section, as well as in Section 2, linearizes.
Chapter 2

Dynamical system connected with the transposition of three matrices

7. Definition of dynamical system

Let

\[
\mathcal{A} = \begin{pmatrix} A & B & C \\ D & F & G \\ H & J & K \end{pmatrix}
\]

be a block matrix acting in the linear space of \(m + n + r\)-dimensional complex column vectors, so that, for example, \(A, F\) and \(K\) are square matrices of sizes \(m \times m, n \times n, r \times r\) respectively. Consider the problem of factorization of matrix \(\mathcal{A}\) into a product of the form

\[
\mathcal{A} = \mathcal{A}_1 \mathcal{A}_2 \mathcal{A}_3,
\]

where

\[
\mathcal{A}_1 = \begin{pmatrix} A_1 & B_1 & 0 \\ C_1 & D_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathcal{A}_2 = \begin{pmatrix} A_2 & 0 & B_2 \\ 0 & 1 & 0 \\ C_2 & 0 & D_2 \end{pmatrix},
\]

\[
\mathcal{A}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & A_3 & B_3 \\ 0 & C_3 & D_3 \end{pmatrix}.
\]

The factorization (7.2) may be seen as a generalization of the factorization of an orthogonal rotation in the 3-dimensional space into rotations through the “Euler angles”. However, at the moment we consider no orthogonality conditions.

One can obtain from the factorization (7.2) other factorizations of the same kind by using the following transformation of the triple \(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3:\)

\[
\mathcal{A}_1 \rightarrow \mathcal{A}_1 \begin{pmatrix} M_1^{-1} & 0 & 0 \\ 0 & M_2^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathcal{A}_2 \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & M_3 \end{pmatrix}, \quad \mathcal{A}_3 \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & M_3^{-1} \end{pmatrix},
\]

\[
M_1, M_2, M_3 \text{ being arbitrary non-degenerate matrices of proper sizes.}
\]

Lemma 7.1. For a generic matrix \(\mathcal{A}\) factorization (7.2) is (if exists) unique to within the transformations (7.4).
Let \( \mathcal{A} = \mathcal{A}_1 \mathcal{A}_2 \mathcal{A}_3 \) be another factorization of the same kind. From (7.2) and (7.5) one finds
\[
\mathcal{A}_2' = \mathcal{A}_1'' \mathcal{A}_2 \mathcal{A}_3'',
\]
where
\[
\mathcal{A}_1'' = (\mathcal{A}_1')^{-1} \mathcal{A}_1, \quad \mathcal{A}_3'' = \mathcal{A}_3 (\mathcal{A}_3')^{-1}.
\]
Let us denote the blocks in the dashed matrices by the same letters \( A_1, \ldots, D_3 \) as in equalities (7.3) with proper number of dashes added to them. The relation (7.6) is rewritten as
\[
\begin{pmatrix}
A' \\
B' \\
C'
\end{pmatrix} = \begin{pmatrix}
A'' A_2 & A'' B_2 C_3'' + B_1'' A_3'' \\
C'' A_2 & C'' B_2 C_3'' + D_2'' A_3'' \\
D_2' & D_2 D_3'
\end{pmatrix}.
\]
From here, one obtains at once the equalities
\[
C_2' = C_2, \quad C_1' = 0, \quad C_3'' = 0.
\]
Taking this into account, one finds from the block in 2nd row and 2nd column that
\[
D_1'' A_3'' = 1.
\]
Thus, \( D_1'' \) and \( A_3'' \) are non-degenerate. Now the blocks just above the main diagonal yield
\[
B_1'' = 0, \quad B_3'' = 0.
\]
So, the matrices \( \mathcal{A}_1'' \) and \( \mathcal{A}_3'' \) are block–diagonal. In is easy to see that this means exactly that \( \mathcal{A}_1', \mathcal{A}_2', \mathcal{A}_3' \) are obtained from \( \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3 \) by the transformation (7.4). The lemma is proved.

Now let us construct, starting from the block matrix \( \mathcal{A} \), new matrix \( \mathcal{B} \) by following means: factorize \( \mathcal{A} \) into the product (7.2) and set
\[
\mathcal{B} = \mathcal{A}_3 \mathcal{A}_2 \mathcal{A}_1.
\]
From the above considerations it is seen that the matrix \( \mathcal{B} \) is determined to within the transformations
\[
\mathcal{B} \rightarrow \begin{pmatrix}
M_1 & 0 & 0 \\
0 & M_2 & 0 \\
0 & 0 & M_3
\end{pmatrix} \mathcal{B} \begin{pmatrix}
M_1^{-1} & 0 & 0 \\
0 & M_2^{-1} & 0 \\
0 & 0 & M_3^{-1}
\end{pmatrix}.
\]
Let us call such transformations, as applied to the block matrices here, the gauge transformations. The following simple but important observation is valid: if matrix \( \mathcal{A} \) itself undergoes a gauge transformation, this in no way affects the set of matrices \( \mathcal{B} \) obtained from formula (7.9).

It will be shown in Section 10 that factorization (7.2) does exist for a generic matrix \( \mathcal{A} \). This factorization will be constructed by means of algebraic geometry. Taking this into account, we are ready now to define the dynamic system that we are going to examine. Let \( \mathcal{M} \) be the set of block matrices (7.1) taken to within gauge transformations (7.10), or, using stricter language, the set of equivalence classes of such matrices with respect to transformations (7.10). The set \( \mathcal{M} \) will be our “phase space”. Then, the birational mapping \( f \) is defined on the set \( \mathcal{M} \) that brings into correspondence with a matrix \( \mathcal{A} \), factorized into the product (7.2), the matrix \( \mathcal{B} \) factorized into the product (7.9). Let us now bring into consideration the “discrete time” \( \tau \) taking integer values and say that to the transition from time \( \tau \) to time \( \tau + k \) corresponds the mapping
\[
f \circ \cdots \circ f_k.
\]
8. Connection with Clifford algebras and “twisted” Yang–Baxter equation

The dynamical system described in Section 7 deals with linear operators acting in the direct sum of three vector spaces. Now we will show that one can pass on from the direct sum to a tensor product of (other) spaces, and connect our dynamical system with the “twisted”, or generalized, Yang–Baxter equation (about which see [50, 72, 82]). The arising solutions of that equation belong to the “free fermion” type. This will be also seen in Section 13, where we, basically, will also arrive at a “twisted” Yang–Baxter equation. In the present section, we consider the most general situation, and use algebraic means, while in Section 13 we will consider some reduction of our model, and will use some topological considerations as well.

Let us begin with the constructing, out of a given square matrix $R$ of size $(n_1 + n_2) \times (n_1 + n_2)$, of the operator $R$ acting in the tensor product of spaces $E_1$ and $E_2$ having dimensions $2^{n_1}$ and $2^{n_2}$ correspondingly.

Consider the Clifford algebra generated by fermionic creation and annihilation operators $a_j^\pm$, where $1 \leq j \leq N$, i.e. the algebra with relations

$$
\begin{align*}
  a_j^+ a_k^+ + a_k^+ a_j^- &= a_j^- a_k^- + a_k^- a_j^+ = 0, \\
  a_j^+ a_k^- + a_k^- a_j^+ &= \delta_{jk},
\end{align*}
$$

(8.1)

$\delta_{jk}$ being the Kronecker delta symbol. Such algebra can be realized by operators in a $2^N$-dimensional space, and than there exists in that space a non-zero vector (“vacuum”) $\Omega$ such that

$$a_j^- \Omega = 0 \quad \text{for all } j.
$$

Let us have constructed two such algebras, with $N = n_1$ and $n_2$, acting in spaces $E_1$ and $E_2$. We are going to explain how to obtain out of them, in the spirit of [83], one “large” algebra acting in $E_1 \otimes E_2$.

Consider the particle number operator

$$
N_1 = \sum_{j=1}^{n_1} a_j^+ a_j^- \quad \text{(8.2)}
$$

acting in $E_1$. The eigenvalues of this operator are integers, so one can introduce the operator

$$
Z_1 = (-1)^{N_1} \quad \text{(8.3)}
$$

Then let us define the operators $b_j^\pm$ in $E_1 \otimes E_2$ by formulae

$$
\begin{align*}
  b_j^+ &= a_j^+ \otimes 1 \quad \text{if } 1 \leq j \leq n_1, \\
  b_j^- &= Z_1 \otimes a_j^- \quad \text{if } n_1 + 1 \leq j \leq n_1 + n_2.
\end{align*}
$$

(8.4)

(8.5)

(it is understood that the left factor in the tensor products acts in $E_1$, while the right one—in $E_2$).

A simple check shows that the operators (8.4, 8.3) satisfy the same commutation relations (8.1), and the vacuum is now the tensor product of vacua in $E_1$ and $E_2$.

Let us now associate to a matrix $R$ the automorphism $\varphi$ of the Clifford algebra acting in $E_1 \otimes E_2$ which acts on the creation and annihilation operators in the following way:

$$
\varphi(b_j^-) = \sum_{k=1}^{n_1+n_2} R_{jk} b_k^-, \\
\varphi(b_j^+) = \sum_{k=1}^{n_1+n_2} ((R^{-1})^T)_{jk} b_k^+.
$$

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According to the theory of Clifford algebras (as it can be found, e.g., in [50]), \( \varphi \) is an inner automorphism, i.e. it can be represented in the form

\[
\varphi(A) = \tilde{R}A\tilde{R}^{-1}, \tag{8.6}
\]

\( A \) being an arbitrary element of the algebra, \( \tilde{R} \in \text{Aut} E_1 \otimes E_2 \). To conclude the construction of operator \( R \), it remains to define, similarly to (8.2), the particle number operator

\[
N_2 = \sum_{j=1}^{n_2} a^+_j a^-_j
\]

in the space \( E_2 \), and to set

\[
R = \tilde{R}. (-1)^{N_1 \otimes N_2}. \tag{8.7}
\]

**Theorem 8.1.** Let the matrices \( A_1, A_2, A_3 \) be as in Section 4 (formulae (7.3)), and let there be given also the “primed” matrices \( A'_1, A'_2, A'_3 \) of similar form, e.g.

\[
A'_1 = \begin{pmatrix} A'_1 & B'_1 & 0 \\ C'_1 & D'_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]

etc. Let the equality

\[
A_1 A_2 A_3 = A'_3 A'_2 A'_1. \tag{8.8}
\]

hold. We will associate to the matrix \( A_1 \) or, to be exact, to its “nontrivial” part \( R = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \), the operator \( \tilde{R} = R_{12} \) acting in \( E_1 \otimes E_2 \) according to the construction described by formulae (8.1–8.7), setting \( n_1 = m, n_2 = n \). In the same way, let us associate to the matrix \( A'_1 \) the operator \( R' = R'_{12} \).

Consider also a space \( E_3 \) of dimension \( 2^r \) and, replacing \( E_2 \) by \( E_3 \) and \( n_2 \) by \( r \) in the construction (8.4–8.7), associate to the matrices \( A_3, A'_3 \) and \( A''_3 \) the operators \( L = L_{13} \) and \( L' = L'_{13} \), correspondingly (the letter \( L \) plays the role of \( R \) from (8.7), the subscripts of \( L \) and \( L' \) denote the numbers of spaces \( E_j \) in whose tensor product an operator acts). Finally, to the matrices \( A_3, A'_3 \) and \( A''_3 \) we will associate similarly the operators \( M = M_{13} \) and \( M' = M'_{13} \).

Under these conditions, the “twisted” Yahg–Baxter equation holds:

\[
RLM = M'L'R', \tag{8.9}
\]

where we imply, as usual, that each operator is multiplied tensorly by the unity operator in the lacking space, e.g. \( L \) is multiplied by 1 \( \in \text{Aut} E_2 \).

The equality (8.8) must be interpreted as the decomposition (7.2) of the operator \( A \) obtained at the previous step of evolution from Section 4 as the product

\[
A = A'_3 A'_2 A'_1
\]

of type (7.9). Thus, (8.8) can be regarded as a description of a step of that evolution, while (8.9) is a reformulation of (8.8) in which the direct sums of vector spaces are replaced by tensor products.

**Proof of Theorem 8.1.** To begin with, introduce operators \( \tilde{R}, \tilde{L}, \tilde{M}, \tilde{M}', \tilde{L}', \tilde{R}' \) as follows. Generalize the construction (8.4–8.5) for the tensor product \( E_1 \otimes E_2 \) of three spaces, joining the creation and annihilation operators acting in them in a single algebra by multiplying them, if necessary, tensorly by operators of type (8.3), so that the upper relations (8.1) hold. We get

\[
c^+_j = a^+_j \otimes 1 \otimes 1 \quad \text{if} \quad 1 \leq j \leq m, \tag{8.10}
\]

\[
c^+_j = Z_1 \otimes a^+_j \otimes 1 \quad \text{if} \quad m + 1 \leq j \leq m + n, \tag{8.11}
\]

\[
c^+_j = Z_1 \otimes Z_2 \otimes a^+_j \quad \text{if} \quad m + n + 1 \leq j \leq m + n + r. \tag{8.12}
\]
The operators \( \hat{R}, \ldots, \hat{R}' \) in space \( \mathcal{E}_1 \otimes \mathcal{E}_2 \otimes \mathcal{E}_3 \) are determined by the following conditions: 1) each of them doesn’t change the vacuum \( \Omega_1 \otimes \Omega_2 \otimes \Omega_3 \) and 2) consider a column of height \( m + n + r \) made of the annihilation operators \( c_j^- \) \( \text{[8.10] [8.12]} \). The action of each of the operators \( \hat{R}, \ldots, \hat{R}' \) on that column by conjugation of every element, e.g.

\[
c_j^- \rightarrow \hat{R}c_j^- \hat{R}^{-1},
\]
is equivalent to the action of the corresponding matrix \( \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_j', \mathcal{A}_j'' \) or \( \mathcal{A}_j' \) on the whole column, e.g.

\[
\begin{pmatrix}
c_1 \\
\vdots \\
c_{m+n+r}
\end{pmatrix} \rightarrow \mathcal{A}_1 \begin{pmatrix}
c_1 \\
\vdots \\
c_{m+n+r}
\end{pmatrix}.
\]

Such operators \( \hat{R}, \ldots, \hat{R}' \), according to the theory of Clifford algebras \([56]\), exist and are unique. Besides, the equality

\[
\hat{R} \hat{L} \mathcal{M} = \mathcal{M}' \hat{L}' \hat{R}',
\]
holds, because both sides of \( \text{(8.13)} \) preserve the vacuum and act identically on the annihilation operator column, namely:

\[
\begin{aligned}
&\begin{pmatrix}
c_1 \\
\vdots \\
c_{m+n+r}
\end{pmatrix} \rightarrow \mathcal{A}_1 \mathcal{A}_2 \mathcal{A}_3 \begin{pmatrix}
c_1 \\
\vdots \\
c_{m+n+r}
\end{pmatrix} = \mathcal{A}_2' \mathcal{A}_3' \mathcal{A}_1' \begin{pmatrix}
c_1 \\
\vdots \\
c_{m+n+r}
\end{pmatrix}.
\end{aligned}
\]

The following lemma shows how the operators in \( \text{(8.13)} \) (they don’t have, generally, the form required in Yang–Baxter equation) are connected with the operators in relation \( (8.9) \) that we are proving. More exactly, we will connect the operators “with hats” with the operators “with tildes” which, we remind, were defined each in its own tensor product of two spaces.

**Lemma 8.1.**

\[
\hat{R} = \tilde{R}, \quad \mathcal{M} = \tilde{\mathcal{M}}, \quad \mathcal{M}' = \tilde{\mathcal{M}}', \quad \hat{R}' = \tilde{R}',
\]
while \( \hat{L} \) and \( \hat{L}' \) are connected with \( \tilde{L} \) and \( \tilde{L}' \) as follows:

\[
\tilde{L}' = (-1)^{N_1 N_2} \tilde{L} (1) (-1)^{N_1 N_2},
\]
where \( (1) \) denotes the prime or its absence, and \( N_1 N_2 \) is, strictly speaking, \( N_1 \otimes N_2 \otimes 1 \).

**Proof of Lemma 8.1** is based on the fact that LHS’s and RHS’s of \( \text{(8.14)} \) and \( \text{(8.13)} \) define (by conjugations) the same automorphisms of the Clifford algebra acting in \( \mathcal{E}_1 \otimes \mathcal{E}_2 \otimes \mathcal{E}_3 \). This is checked by direct calculations for each of the equalities \( \text{(8.14)} \) \( \text{(8.15)} \) and for each set of operators \( \text{(8.10)} \) \( \text{(8.11)} \) \( \text{(8.12)} \) separately. We will remark only that the role of factors \(-1)^{N_1 N_2} \) in \( \text{(8.15)} \) is seen from equalities of type

\[
(-1)^{N_1 N_2} (Z_1 \otimes a_k^+ \otimes 1) = (1 \otimes a_k^+ \otimes 1) (-1)^{N_1 N_2},
\]
which demonstrate how an operator, in this case, \( Z_1 \), disappears or appears after a conjugation with \(-1)^{N_1 N_2} \).

To prove the proof of Theorem 8.1, let us write out the equality \( \text{(8.13)} \) in terms of operators \( \hat{R}, \ldots, \hat{R}' \) defined by equalities of type \( \text{(8.7)} \), using \( \text{(8.14)} \) and \( \text{(8.13)} \):

\[
\hat{R} \hat{L} (1)^{N_1 (N_2 + N_3)} \mathcal{M} (1)^{N_2 N_3} = \mathcal{M} (1)^{N_2 (N_1 + N_3)} \hat{L}' (1)^{N_1 (N_2 + N_3)} \hat{R}' (1)^{N_1 N_2}.
\]

All the degrees of minus one in LHS and RHS of \( \text{(8.17)} \) can be conveyed through other factors to the right using the fact that \( \mathcal{M} \) commutes with \( N_2 + N_3 \) (i.e. “preserves the total number of particles of second and third kinds”) and with \( N_1 \), similarly \( \hat{L}' \) commutes with \( N_1 + N_3 \) and with \( N_2 \), and \( \hat{R}' \) commutes with \( N_1 + N_2 \) and with \( N_3 \). This done, the factor \((1)^{N_1 N_2 N_3 + N_1 N_3 + N_2 N_3} \) arising in both LHS and RHS, is canceled, and we come to \( \text{(8.9)} \). The theorem is proved.
9. Invariant algebraic curve of matrix $A$ and some divisors in it

The dynamical system of Section 7 turns out to be completely integrable. To be exact, an invariant curve $\Gamma$ can be constructed out of matrix $A$, together with a divisor $D$ on it. In terms of these algebro-geometrical objects, the evolution is as follows: $\Gamma$ does not change, while $D$—more precisely, its linear equivalence class—depends linearly on the discrete time $\tau$.

Let us start from the definition of the curve $\Gamma$. The word “invariant” in this definition will be justified in Section 10.

**Definition 9.1.** The invariant curve $\Gamma$ of the operator $A$ of the form (7.1) is an algebraic curve in $\mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1$ (i.e. in the space of three complex variables $u, v, w$, each allowed also to take value $\infty$) given by equations

$$\det(A - \begin{pmatrix} u1_m & 0 & 0 \\ 0 & v1_n & 0 \\ 0 & 0 & w1_r \end{pmatrix}) = 0,$$

$$v = uw. \tag{9.2}$$

Here a subscript of each $1$ means the size of corresponding unity matrix, while $0$ denotes rectangular zero matrices of different sizes. Strictly speaking, equations (9.1), (9.2) define the “finite part” of the curve $\Gamma$, the whole curve $\Gamma$ being its closure in Zariski topology.

The equality (9.1) means that a column vector $X' = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$ exists, with $X, Y, Z$ column vectors of dimensions $m, n$ and $r$, correspondingly, such that

$$A \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} uX \\ vY \\ wZ \end{pmatrix}. \tag{9.3}$$

Such vectors $X'$ form a one-dimensional holomorphic bundle over $\Gamma$.

The next lemma shows the structure of the zero and pole divisors of functions $u, v, w$. For these divisors, the notations $(u)_0, (u)_\infty$ etc. are used.

**Lemma 9.1.** There exist such effective divisors (i.e. finite sets of points) $D_1, \ldots, D_6$ in the curve $\Gamma$ that

$$(u)_\infty = D_1 + D_2, \quad (v)_\infty = D_3 + D_4, \quad (w)_\infty = D_3 + D_4,$$

$$(u)_0 = D_4 + D_6, \quad (v)_0 = D_5 + D_6, \quad (w)_0 = D_5 + D_6. \tag{9.4}$$

$D_3$ and $D_5$ are of degree $m$, $D_2$ and $D_4$ are of degree $n$, $D_1$ and $D_6$ are of degree $r$. Generally, all points included in divisors $D_1, \ldots, D_6$ are different from each other.

**Proof.** Consider, e.g., the case $u = 0, w \neq 0, w \neq \infty$. Then, according to (9.2), $v = 0$. The equality (9.3) turns into the following system:

$$\begin{cases} AX + BY + CZ = 0, \\ DX + FY + GZ = 0, \\ HX + JY + KZ = wZ. \end{cases}$$

One can express $X$ and $Y$ through $Z$ (e.g., $Y = -(F - DA^{-1}B)^{-1}(G - DA^{-1}C)Z$) and then substitute these expressions into the third one. One will come to an equation of the form

$$KZ = wZ \tag{9.5}$$

which has $r$ characteristic roots $w_1, \ldots, w_r$, different from each other in general case. This is how $r$ points $(0, 0, w_1), \ldots, (0, 0, w_r)$ of divisor $D_6$ are obtained. The other divisors in (9.4) arise in a similar way. The lemma is proved.

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The vector \( \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \) in (9.3) is determined up to a scalar factor which may depend on the point in the curve \( \Gamma \). So, this vector can be normalized by setting its first coordinate identically equal to unity (cf. [42]).

However, \( X, Y \) and \( Z \) taken separately satisfy stronger restrictions, as the following lemma shows.

In the lemma, \((f)\) denotes the divisor of a function \( f \) (zeros enter with the + sign, poles with the − sign, as usual).

**Lemma 9.2.** The column vector \( X \) consists of functions \( f \) such that
\[
(f) + D - (u)_{\infty} \geq 0; \quad (9.6)
\]
the column vector \( Y \) consists of functions \( f \) such that
\[
(f) + D - (v)_{\infty} \geq 0; \quad (9.7)
\]
the column vector \( Z \) consists of functions \( f \) such that
\[
(f) + D - (w)_{\infty} \geq 0. \quad (9.8)
\]

**Proof.** One can see immediately from the formula (9.3) that the vector \( uX \) entering into R.H.S. cannot grow faster than the vector \( \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \) in L.H.S. in such points where \( u = \infty \). This is exactly what the inequality (9.6) states. The inequalities (9.7) and (9.8) are proved similarly.

Now the time has come to make it sure that the curve \( \Gamma \), for a generic matrix \( A \), is a smooth irreducible curve. One may wish also to calculate its genus in some simple way. To do that, we are now going to examine a relatively simple particular case of the matrix \( A \), although at the time “generic” enough to make sure that such its features as genus and the degree of divisors are the same for matrices in some Zariski neighborhood.

Thus, let all the matrix elements of \( A \) equal zero except the ones lying, first, in the main diagonal and, second, in the “broken” diagonal parallel to the main one (for these latter matrix elements, the difference between the numbers of a column and a row must be some constant modulo \( m + n + r \)). The elements in the main diagonal will be denoted as \( a_1, \ldots, a_m, f_1, \ldots, f_n, k_1, \ldots, k_r; \) and let the elements in the broken diagonal be all equal to the same complex number \( s \):

\[
A = \begin{pmatrix}
  a_1 & s & & & \\
  & \ddots & \ddots & \ddots & \\
  & & \ddots & f_1 & s \\
  & & & \ddots & f_n \\
  & & & & \ddots \\
  & & & & & \ddots \\
  s & & & & k_1 \\
  & s & & & k_2 \\
  & & s & & k_3 \\
  & & & s & k_4 \\
  & & & & \ddots \\
  \end{pmatrix}. \quad (9.9)
\]

It does not matter through which blocks exactly the “broken” diagonal passes.

For the finite \( u, w \), the curve \( \Gamma \) now examined is given by equation (resulting from the substitution of (9.3) and (9.2) into (1.1))
\[
F(uw) \equiv \prod_{\alpha=1}^{m} (a_\alpha - u) \cdot \prod_{\beta=1}^{n} (f_\beta - uw) \cdot \prod_{\gamma=1}^{r} (k_\gamma - w) \pm s^{m+n+r} = 0. \quad (9.10)
\]
As is known, in singular points
\[
\begin{align*}
\frac{\partial F}{\partial u} &= 0, \\
\frac{\partial F}{\partial w} &= 0.
\end{align*}
\tag{9.11}
\]
The system (9.11) has a finite number of solutions, and, changing \(s\) in (9.10), one can make these solutions not to lie in the curve \(\Gamma\), which thus will be free of singularities for finite \(u, w\). It is an easy exercise to show that there are no singularities when \(u\) or \(w\) is infinite as well.

Returning now to general matrices \(A\) and curves \(\Gamma\), let us note that it cannot be that the system (9.11) or its substitute in the neighborhood of infinite \(u\) or \(w\) possesses solutions in the curve \(\Gamma\) in general case and does not possess them in a particular case. Thus, the smoothness of \(\Gamma\) for a generic \(A\) is clear. As for irreducibility, to prove it let us examine the natural projection of \(\Gamma\) onto the Rimann sphere \(\mathbb{C} P^1\) of the variable \(u\). This projection is an \((n + r)\)-sheet cover, and if \(\Gamma\) consisted of two or more components, the sheets of the cover would split into groups belonging to each component. To prove that it is not so in the general case, it is enough to present an example where it is not so. To do this, take \(A\) of the form (9.9) and, moreover, put \(f_1 = \cdots = f_n = k_1 = \cdots = k_r = 0\). Equation (9.10) then becomes
\[
w^{n+r}u^n \prod_{\alpha=1}^{m} (a_\alpha - u) \pm s^{n+r} = 0.
\]
Let \(a_1 \neq 0\) and not coincide with other \(a_\alpha\). Then in a neighborhood of the point \((u, w) = (a_1, \infty)\) the variable \(w\) behaves, up to a nonzero factor, like
\[
w^{-1} \sim (a_1 - u)^{1/(n+r)}.
\]
From here one sees that all the mentioned \(n + r\) sheets belong to a single component, i.e. the irreducibility of \(\Gamma\) is proved.

Now let us denote the number of branch points of the cover \(\Gamma \to \mathbb{C} \ni u\) as \(b\). Then the genus of the curve \(\Gamma\), according to the Riemann—Hurwitz formula, is
\[
g = 1 - n - r + \frac{b}{2}.
\tag{9.12}
\]
Our next aim is to express \(b\) and \(g\) through \(m, n\) and \(r\).

**Lemma 9.3.** The degree of the vector \(
\begin{pmatrix}
X \\
Y \\
Z
\end{pmatrix}
\) pole divisor \(D\) is \(m + b/2\).

**Proof.** Write out the equation (9.3) “explicitly”:
\[
\begin{align*}
AX + BY + CZ &= uX, \\
DX + FY + GZ &= vY, \\
HX + JY + KZ &= wZ.
\end{align*}
\tag{9.13}
\]
Expressing \(X\) through \(Y\) and \(Z\) by means of the first of these equations and substituting into the rest, one finds:
\[
\begin{pmatrix}
w^{-1} (D(u - A)^{-1}B + F) \\
w^{-1} (D(u - A)^{-1}C + G)
\end{pmatrix}
\begin{pmatrix}
Y \\
Z
\end{pmatrix}
= w
\begin{pmatrix}
Y \\
Z
\end{pmatrix}.
\tag{9.14}
\]
To a generic \(u\) correspond \(n + r\) different \(w = w_1, \ldots, w_{n+r}\), and the corresponding \(n + r\) vectors \(
\begin{pmatrix}
Y \\
Z
\end{pmatrix}
\) are linearly independent as eigenvectors of the matrix in L.H.S. of (9.14). An easy check shows that in the “suspicious”, from the standpoint of equation (9.14), points \(u = 0, \infty\), and also in points where \(\det(u - A) = 0\), there exist as well \(n + r\) linearly independent vectors \(
\begin{pmatrix}
Y \\
Z
\end{pmatrix}
\) —solutions of limit cases of the system (9.14).
Consider a determinant

\[
d = \begin{vmatrix} Y(w_1) & \ldots & Y(w_{n+r}) \\ Z(w_1) & \ldots & Z(w_{n+r}) \end{vmatrix}.
\] (9.15)

Given \( u \), it changes its sign under odd permutations of \( w \)'s.

However, \( d^2 \) is a function of \( u \) only. From the above one sees that \( d^2(u) \) vanish in branch points where to a given \( u \) correspond less than \( n + r \) values of \( v \) or \( w \). This yields \( b \) zeros of the function \( d^2(u) \). There is, however, one more cause for this function to vanish. According to Lemmas 9.2 and 9.3, \( Y \) and \( Z \) must vanish as a whole in the points of divisor \( D_3 \) — the common part of divisors \( (v)_\infty \) and \( (w)_\infty \). The degree of divisor \( D_3 \) is \( m \), so we get \( 2m \) more zeros of \( d^2(u) \).

The number of function \( d^2(u) \) poles equals the number of its zeros, i.e. \( b + 2m \). Thus, the meromorphic vector \( \begin{pmatrix} Y \\ Z \end{pmatrix} \) has \( b/2 + m \) poles, as desired. It remains just to note that vector \( X \), in accord with formula (9.6), has no poles that \( \begin{pmatrix} Y \\ Z \end{pmatrix} \) doesn’t have, and on the other hand doesn’t vanish in points of divisor \( D_3 \). The lemma is proved.

Let us turn again to matrices \( A \) of the form (9.9). For such matrix, it is easy to find the vector

\[
\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \mathcal{X}
\]

in a given point \((u, v, w) \in \Gamma \). Let us assume that the “broken” diagonal is the one adjacent to the main diagonal, so that there is only one letter \( s \) in the lower left corner. Then the following holds for the vector \( \mathcal{X} \) coordinates:

\[
(a_1 - u)X_1 + sX_2 = 0, \\
(a_2 - u)X_2 + sX_3 = 0, \\
\ldots \ldots \ldots \\
(a_m - u)X_m + sY_1 = 0, \\
(f_1 - v)Y_1 + sY_2 = 0, \\
\ldots \ldots \ldots \\
(f_n - v)Y_n + sZ_1 = 0, \\
(k_1 - w)Z_1 + sZ_2 = 0, \\
\ldots \ldots \ldots \\
(k_n - w)Z_n + sX_1 = 0.
\]

From here the ratios between vector \( \mathcal{X} \) coordinates are readily seen. Assuming the normalization condition \( X_1 = 1 \), one finds out that \( \mathcal{X} \) has the poles a) of the order \( m \) in \( n \) points \((u, v, w) = (\infty, f_\beta, 0) \) and b) of the order \( m + n \) in \( r \) points \((u, v, w) = (\infty, \infty, k_\gamma) \). In all, \( \mathcal{X} \) possesses thus \( mn + mr + nr \) poles, taking their multiplicities into account. Recalling Lemma 9.3 and formula (9.13), one can now find the genus \( g \) of the curve as well. As a matrix \( A \) of the form (9.9) is “generic enough”, the results on the degree of divisor \( \mathcal{D} \) of the vector \( \mathcal{X} \) and genus \( g \) of the curve apply also to curves corresponding to generic matrices \( A \). Let us formulate them as the following lemma.

**Lemma 9.4.** For a generic matrix \( A \), the genus of the curve \( \Gamma \) is

\[
g = mn + mr + nr - m - n - r + 1,
\] (9.16)

while the degree of divisor \( \mathcal{D} \) of the meromorphic vector \( \mathcal{X} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \) is

\[
mn + mr + nr = g + m + n + r - 1.
\] (9.17)

Thus, in this section we have constructed, for a given matrix \( A \), an algebraic curve \( \Gamma \) and a bundle of vectors \( \mathcal{X} \) over it, and calculated the genus \( g \) of the curve and the degree of the bundle (i.e. the divisor \( \mathcal{D} \) degree). As a helpful tool, a matrix \( A \) of special simple form (9.9) was used which, from many viewpoints, was “generic enough”. In Section 10 we will study how these objects behave under evolution introduced in Section 7.
10. Evolution in terms of divisors

In this section it is shown, at first, that there exists a one-to-one correspondence (more precisely, a birational isomorphism) between the set of block matrices $A$ (7.1) taken up to gauge transformations (7.4), and the set of pairs (an algebraic curve, a linear equivalence class of divisors on it) of a certain kind. This correspondence has, in essence, been constructed in Section 9, and here are some missing details. Then, it is explained which divisors and why correspond to the factors $A_1$, $A_2$ and $A_3$ in (7.2) taken separately. Finally, it is demonstrated that to the matrix $B$ (7.9) obtained from $A$ by reversing the order of its factors, the same curve $\Gamma$ corresponds, but the divisor undergoes some constant shift. Thus, the motion linearizes in the Jacobian of curve $\Gamma$. Let us proceed to a detailed consideration.

Equations (9.1, 9.2) define, for a block matrix $A$, an algebraic curve $\Gamma$. Those equations can obviously be written as

\[\sum_{i=0}^{m} \sum_{j=0}^{n} \sum_{k=0}^{r} a_{ijk} u^i v^j w^k = 0, \tag{10.1}\]

\[v = uw.\]

Besides, a linear bundle over $\Gamma$ has been constructed in Section 9—the bundle of vectors \(\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}\). That means that the divisor $D$ of the bundle is determined, up to linear equivalence, whose degree is $g + m + n + r - 1$, $g$ being the curve’s genus (9.17). Gauge transformations (7.4) do not change a pair $(\Gamma, \text{class of divisor } D)$.

Now let us show how to construct the matrix $A$ starting from coefficients $a_{ijk}$ of the curve (10.1) (arbitrary complex numbers in “general position”) and a divisor $D$ of degree $g + m + n + r - 1$. Note that genus $g$ of the curve $\Gamma$ defined by formulae (10.1) without any (a priori) connection with block matrices is given by the same formula (9.16). This can be seen, e.g., by starting again from the “simple” curve (9.10) of Section 9 whose genus is known. Define now the meromorphic column vectors $X, Y$ and $Z$, guided by Lemma 9.2: for components of vector $X$, take $m$ linearly independent meromorphic functions on $\Gamma$ satisfying relation (9.6), and for $Y$ and $Z$ take, similarly, $n$ functions satisfying (9.7) and $r$ functions satisfying (9.8).

Note also that Lemma 9.1 about divisors $(u)_\infty, (v)_\infty, (w)_\infty$ entering in formulae (9.6–9.8) remains valid for curves defined by an “abstract” system (10.1), which is immediately seen on substituting zero or infinity for $u, v, \text{or } w$ in (10.1). It is clear now that relation (7.4) determines unambiguously the matrix $A$ (cf. a similar construction in paper [42]). Another choice of linearly independent functions for components of $X, Y$ and $Z$ leads, of course, to a gauge transformation (7.11). The vectors $X, Y, Z$ change under it to $M_1X, M_2Y, M_3Z$. On the other hand, if divisor $D$ is changed to another divisor belonging to the same linear equivalence class, the vectors $X, Y$ and $Z$ are just multiplied by a scalar meromorphic function $h$ having zeros in the points of the first divisor and poles—in the points of the second one.

Hence, the following theorem has been proved.

**Theorem 10.1.** If block matrices $A$ and $\hat{A}$ of the form (7.4) are connected by a gauge transformation

\[
\hat{A} = \begin{pmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & M_3 \end{pmatrix} A \begin{pmatrix} M_1^{-1} & 0 & 0 \\ 0 & M_2^{-1} & 0 \\ 0 & 0 & M_3^{-1} \end{pmatrix} \tag{10.2}
\]

then they have a common invariant curve $\Gamma$ given by equations of the form (10.1), and the holomorphic bundles of the vectors $X$ corresponding to them by formula (9.3) are isomorphic.

Conversely, if two matrices $A$ and $\hat{A}$ of the form (7.4) have the same invariant curve $\Gamma$ and the corresponding vector $X$ and $\hat{X}$ bundles are isomorphic, then (10.2) holds. Being properly
Figure 2.4. Factorization of matrix \( \mathcal{A} \) and the divisors

normalized (say, by a condition that the first coordinate identically equals unity), \( \mathcal{X} \) and \( \hat{\mathcal{X}} \) become meromorphic vectors with linear equivalent pole divisors \( D \) and \( \hat{D} \) of degree \( g + m + n + r - 1 \), and

\[
\hat{\mathcal{X}} = \begin{pmatrix} M_1 X \\ M_2 Y \\ M_3 Z \end{pmatrix} \cdot h(u, v, w), \tag{10.3}
\]

where \( h \) is a scalar meromorphic function whose divisor \( (h) = (h)_0 - (h)_\infty \) satisfies equality

\[
(h) = D - \hat{D}.
\]

Examine now each multiplier in factorization (7.2) separately. Lemma 7.1 shows that factorization (7.2), if exists, is unique to within the transformations (7.4). Let us demonstrate how to construct this factorization by algebro-geometrical means.

Consider the following figure (Fig. 2.4).

The meaning of the numbers standing near the edges in this figure is as follows: if those numbers are \( jk \), then the meromorphic vector corresponding to the edge consists of such functions \( f \) whose zero and pole divisor \( (f) \) satisfies inequality

\[
(f) + D - D_j - D_k \geq 0.
\]

Those inequalities must be in agreement with Lemmas 9.1 and 9.2. In particular, the matrix \( \mathcal{A}_3 \) will be defined by equality (notations of formulae (7.3) are used),

\[
\begin{pmatrix} A_3 & B_3 \\ C_3 & D_3 \end{pmatrix} \begin{pmatrix} Y \\ Z \end{pmatrix} = \begin{pmatrix} Y' \\ Z' \end{pmatrix}, \tag{10.4}
\]

where the meromorphic vector \( Y \) consists of functions \( f \) such that

\[
(f) + D - D_1 - D_3 \geq 0
\]

(formulae (9.7) and (9.4)); \( Z \) of functions such that

\[
(f) + D - D_3 - D_4 \geq 0
\]
(formulae (9.8) and (9.4)); \( Y' \) and \( Z' \) consist by definition of such linearly independent functions that
\[(f) + D - D_3 - D_6 \geq 0\]
for \( Y' \), and
\[(f) + D - D_2 - D_3 \geq 0\]
for \( Z' \). It is easy to see that (10.4) is a correct definition for matrix \( A_3 \), because the components of each of the vectors \( \begin{pmatrix} Y \\ Z \end{pmatrix} \) and \( \begin{pmatrix} Y' \\ Z' \end{pmatrix} \) form a basis in the space of meromorphic functions \( f \) such that
\[(f) + D - D_3 \geq 0.\]

Next, let
\[
\begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix} \begin{pmatrix} X \\ Z' \end{pmatrix} = \begin{pmatrix} X' \\ wZ \end{pmatrix},
\]
(10.5)
\[
\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \begin{pmatrix} X' \\ Y' \end{pmatrix} = \begin{pmatrix} uX \\ vY \end{pmatrix},
\]
(10.6)
where \( X' \) consists of functions \( f \) such that
\[(f) + D - D_2 - D_3 \geq 0.\]

It is shown in much the same way as above that equalities (10.5) and (10.6) do correctly define the matrices \( A_2 \) and \( A_1 \). What remains is to check the validity of equality (7.2) for \( A_1, A_2 \) and \( A_3 \) given by these definitions. To do this, observe that (10.4–10.6) together yield
\[
A_1A_2A_3 \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} uX \\ vY \\ wZ \end{pmatrix}.
\]
(10.7)
The equality (7.3) follows from comparing (10.7) with (10.3).

Note that the arbitrariness in choosing \( X', Y', \) and \( Z' \) corresponds, of course, to transformations (7.4).

Now let us pass to matrix \( B \), a product of the same three factors in the inverse order. The formulae (7.9) and (10.4–10.6) together yield (if one multiplies both sides of (10.5) by \( u \), and both sides of (10.4) by \( v \ )):
\[
B \begin{pmatrix} X' \\ Y' \\ uZ' \end{pmatrix} = \begin{pmatrix} uX' \\ vY' \\ uZ' \end{pmatrix}.
\]
(10.8)
Compare the divisors of meromorphic vectors in L.H.S.’s of (10.8) and (10.3). An easy calculation shows that
\[D_{X'} - D_X = D_{Y'} - D_Y = D_{(uZ')} - D_Z = D_1 - D_6.\]
One sees hence that the same curve \( \Gamma \) corresponds to the operator \( B \) as to the operator \( A \), while the divisor \( D \) changes to \( D + D_1 - D_6 \).

Thus, in this section the name “invariant” has been justified for the curve \( \Gamma \); it has been shown not to change under the evolution of Section 7. At the same time, it was demonstrated how to construct the factorization (7.2). Finally, it was shown that the evolution is described in algebro-geometrical terms as a linear, with respect to discrete time, change of (the linear equivalence class of) divisor \( D \); it changes by \( D_1 - D_6 \) per each unit of time.
11. Algebro-geometrical objects in the orthogonal and symplectic cases

In this section, the algebro-geometrical devices developed earlier are supplemented with necessary means for studying the evolution of orthogonal and symplectic matrices $A$. It turns out (Section 13, that the condition of orthogonality or symplecticity is an admissible, i.e. compatible with the evolution, reduction of the dynamical system defined in Section 7. We will start, however, with some consideration for generic matrices $A$, namely, looking at what happens with the invariant curve $\Gamma$ and the bundle of vectors $\mathcal{X}$ (see formula (9.3)) under two operations: inversion $A \to A^{-1}$ and transposing $A \to A^T$.

As for the inversion, here virtually everything is seen from formula (9.3), especially if one rewrites it as

$$A^{-1} \begin{pmatrix} uX \\ vX \\ wZ \end{pmatrix} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}. \quad (11.1)$$

It is clear that the curve $\Gamma$, basically, does not change; the roles of column vectors $X,Y,Z$ are played by $uX,vY$ and $wZ$, while the roles of meromorphic functions $u,v,w$ on $\Gamma$ are played by $u^{-1},v^{-1}$, and $w^{-1}$. The exact formulation is given in the following lemma.

**Lemma 11.1.** If to a matrix $A$ corresponds a curve $\Gamma$ given by equations $P(u,v,w) = 0$ and $v = uw$ (see (9.1) or (10.4)), and a divisor $D$ in it (see Lemma 10.2 and the paragraph before it), then to the matrix $A^{-1}$ corresponds the curve $\Gamma^{-1}$ given by equations $P(u^{-1},v^{-1},w^{-1}) = 0$ and $v = uw$, and the divisor $D^{-1}$ in it that is the image of $D$ under the natural isomorphism

$$\Gamma \to \Gamma^{-1}: \quad (u,v,w) \mapsto (u^{-1},v^{-1},w^{-1}). \quad (11.2)$$

**Proof.** It remains to remind that $D$ is the pole divisor common for the meromorphic column vectors in LHS and RHS of (9.3) or (11.1). Hence the validity of the statement in lemma about the divisor $D^{-1}$ is clear, and thus the lemma is proved. Let us add, however, that, for the curve $\Gamma^{-1}$, the roles of divisors $D_1, D_2, D_3$ and $D_5$ (let them be listed in this order) are played by the images of divisors $D_0, D_1, D_4, D_2, D_3, D_5$ and $D_3$. This is due to the fact that the role of divisor $(u)_\infty$ is played by the image of $(u)_0$, the role of $(u)_0$ is played by $(u)_\infty$, and similarly for the functions $v$ and $w$. The connection between those divisors is given by formulae (9.4).

Consider now the transposing $A \to A^T$. The determinant is not changed under transposing, thus it follows from (9.1) that the invariant curve of matrix $A^T$ is the same $\Gamma$. Somewhat more complicated is to find column vectors $\tilde{X}, \tilde{Y}, \tilde{Z}$, satisfying the equation

$$A^T \begin{pmatrix} \tilde{X} \\ \tilde{Y} \\ \tilde{Z} \end{pmatrix} = \begin{pmatrix} \tilde{u}X \\ \tilde{v}Y \\ \tilde{w}Z \end{pmatrix}. \quad (11.3)$$

We will obtain them here using the direct constructive method from the author’s work [8]. Note however that we might act also in another way, adapting to our case the idea of divisor duality and of the scalar product of meromorphic functions on an algebraic curve associated to this duality, ascending to the work [6] and described also in more recent works [65, 66, 67]. We will pay tribute to this elegant idea in Section 13 in the framework of the “local” approach to the evolution of orthogonal matrices, while returning now to the methods of [8].

Transpose (11.3):

$$\begin{pmatrix} \tilde{X}^T & \tilde{Y}^T & \tilde{Z}^T \end{pmatrix} A = \begin{pmatrix} \tilde{u}X^T & \tilde{v}Y^T & \tilde{w}Z^T \end{pmatrix}. \quad (11.4)$$

Multiply both sides of (11.4) from the right (scalarly) by $\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$, i.e. by the column vector appearing in (9.3), and use the formula (9.3) to exclude $A$ from the LHS. One gets, after collecting similar terms,

$$(u - \tilde{u})\tilde{X}^TX + (v - \tilde{v})\tilde{Y}^TY + (w - \tilde{w})\tilde{Z}^TZ = 0. \quad (11.5)$$
In formula (11.5), \((u, v, w)\) and \((\tilde{u}, \tilde{v}, \tilde{w})\) are so far arbitrary points in the curve \(\Gamma\) (one and the same, as was explained above). Set now \(\tilde{u} = u\). To a given (generic) \(u\) correspond \(n + r\) different \(w\)'s (recall the formulae (10.1)). Choose now \(w\) and \(\tilde{w}\) different: \(w \neq \tilde{w}\). Then (11.5), on being multiplied by \(\frac{\tilde{w}}{w - \tilde{w}}\), yields
\[
\tilde{v}Y^TY + \tilde{w}Z^TZ = 0, \quad \text{where} \quad \tilde{v} = uw.
\]

Let \(\tilde{w}\) be fixed, and \(w\) take other \(n + r - 1\) possible values which we will denote \(w_2, \ldots, w_{n+r}\). Then the components of vector \(\left(\frac{vY}{wZ}\right)\) are proportional to the cofactors of the determinant first column entries
\[
\Delta(u, \tilde{w}) = \begin{vmatrix}
Y(u, uw, \tilde{w}) & Y(u, uw_2, w_2) & \cdots & Y(u, uw_{n+r}, w_{n+r}) \\
Z(u, uw, \tilde{w}) & Z(u, uw_2, w_2) & \cdots & Z(u, uw_{n+r}, w_{n+r})
\end{vmatrix}, \tag{11.6}
\]

Choose the normalization as follows:
\[
\left(\frac{\tilde{v}Y}{\tilde{w}Z}\right) = \frac{1}{\Delta(u, \tilde{w})} \cdot \left(\text{vector of the first column entries} \right) \cdot \left(\text{cofactors of } \Delta(u, \tilde{w})\right). \tag{11.7}
\]

We have already met the determinant \(\Delta(u, \tilde{w})\) under the name of \(d\) (formula (9.13)). Here, however, we consider it not as a two-valued function of a point \((u, uw, \tilde{w}) \in \Gamma\) rather than a function of \(u\). This leads to a large multiplication of this determinant zeroes, some of them being also zeroes of all the minors—components of the vector from (11.7), while others being not. The zeroes of \(\Delta(u, \tilde{w})\) are of interest for us, of course, as candidates for being poles of the vector \(\left(\frac{\tilde{v}Y}{\tilde{w}Z}\right)\). Note that, although the enumerator and denominator in the RHS of (11.7) are two-valued functions, the resulting fraction is obviously single-valued.

Consider first the “branch points”, i.e. the points where either a) \(w_j = \tilde{w}\) for some \(j\), or b) \(w_j = w_k\) for some unequal \(j\) and \(k\). There are \(b\) points of the first type (see formula (9.13)), and those points form the ramification divisor \(D_{\text{ram}}\), while the points of the second type are of no interest for us, because all the minors in such points have, too, zeroes of the same character (namely as the square root of a local parameter in a curve), so that zeroes in the RHS of (11.7) cancel one another. Next, we know that in some points the columns of determinant \(\Delta(u, w)\) must vanish as a whole. Such points, too, are divided into a) those where the first column vanishes and b) others. The first are again \(m\), and they form the divisor \(D_3\), while the second are again of no interest, because the minors also vanish in them.

Other candidates for being the poles of the vector \(\left(\frac{\tilde{v}Y}{\tilde{w}Z}\right)\) might be the poles of the enumerator in the RHS of (11.7). However, in a point where a column of minors has a pole, the corresponding column of \(\Delta(u, \tilde{w})\) has a pole, too. Recalling also the linear independence of columns of \(\Delta(u, \tilde{w})\) (see the proof of Lemma 9.3), we get a pole of \(\Delta(u, \tilde{w})\) that cancels the pole of minors.

Thus, the poles of the vector \(\left(\frac{\tilde{v}Y}{\tilde{w}Z}\right)\) can only be the points of the divisor
\[
D_{\text{ram}} + D_3.
\]

We must consider also points where all coordinates of this vector vanish. Of all candidates the only survivors here are the first order poles of \(\Delta(u, \tilde{w})\). There are \(mn + mr + nr\) such poles, and they form the divisor denoted in this paper as \(D\). Other candidates in zeroes of all coordinates of the vector \(\left(\frac{\tilde{v}Y}{\tilde{w}Z}\right)\) are sifted away by arguments like those used above when searching for the poles.

Summarize these considerations in the following lemma.
Lemma 11.2. The vector \((\tilde{v}\hat{Y} \ w\hat{Z})\) from (11.7) consists of functions \(f\) whose divisors obey the relation
\[
(f) + \mathcal{D}_{\text{ram}} + \mathcal{D}_3 - \mathcal{D} \geq 0.
\]

It remains to clarify the following. Those functions, in principle, may obey some stronger constraints, such as
\[
(f) + \mathcal{D}_{\text{unknown}} \geq 0, \quad \text{where } \mathcal{D}_{\text{unknown}} < \mathcal{D}_{\text{ram}} + \mathcal{D}_3 - \mathcal{D}.
\]

It turns out that they do not. This is seen from the degrees of divisors:
\[
\deg (\mathcal{D}_{\text{ram}} + \mathcal{D}_3 - \mathcal{D}) = b + m - (mn + mr + nr) = g + n + r - 1
\]
(recall (9.12) and (9.16)), and this is exactly what we need.

Denote now as \(\hat{\mathcal{D}}\) the divisor of singularities common for the vectors
\[
\begin{pmatrix}
\tilde{X} \\
\hat{Y} \\
\tilde{Z}
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
u \tilde{X} \\
\hat{v}\hat{Y} \\
\hat{w}\hat{Z}
\end{pmatrix}
\]
(we mean here that poles enter with the plus sign!). As, for the divisor \(\hat{\mathcal{D}}\) and vectors \(\tilde{X}, \hat{Y}, \tilde{Z}\), the same relations as (9.6–9.8) hold, one can derive that \(\hat{\mathcal{D}}\) is obtained from the divisor of singularities of the vector \((\tilde{v}\hat{Y} \ w\hat{Z})\), by adding \(\mathcal{D}_5\) (for illustration, Fig. 2.4 again can be used). Thus, we got the following: under the transposing of a matrix \(A\), the divisor \(\mathcal{D}\) changes to
\[
\hat{\mathcal{D}} = \mathcal{D}_{\text{ram}} + \mathcal{D}_3 + \mathcal{D}_5 - \mathcal{D}. \quad (11.8)
\]

Here \(\mathcal{D}_{\text{ram}}\) is the ramification divisor with regard to the variable \(u\). In the following theorem, this result is formulated in more symmetric and elegant form.

Theorem 11.1. Let to a matrix \(A\) correspond an algebraic curve \(\Gamma\) and a divisor \(\mathcal{D}\) in it (as was described in Sections 9 and 10), while to the transposed matrix \(A^T\) — the same curve and a divisor \(\hat{\mathcal{D}}\). Then
\[
\hat{\mathcal{D}} + \mathcal{D} \sim \mathcal{D}_{\text{can}} + \mathcal{D}_1 + \mathcal{D}_2 + \mathcal{D}_3 + \mathcal{D}_4 + \mathcal{D}_5 + \mathcal{D}_6, \quad (11.9)
\]
where \(\mathcal{D}_{\text{can}}\) is a canonical divisor in \(\Gamma\), and divisors \(\mathcal{D}_1, \ldots, \mathcal{D}_6\) are defined in Lemma 9.4.

Proof. As is known [78], canonical divisor is the divisor of any differential in \(\Gamma\). Take the differential \(du\) and find its divisor \((du) = (du)_0 - (du)_\infty\). It is not difficult to understand that the zeroes of the form \(du\) are situated exactly in the branch points of \(\Gamma\) considered as a covering over the Riemann sphere of variable \(u\):
\[
(du)_0 = \mathcal{D}_{\text{ram}}, \quad (11.10)
\]

while the poles of \(du\) coincide with the poles of function \(u\), but are of order 2:
\[
(du)_\infty = 2(u)_\infty = 2(\mathcal{D}_1 + \mathcal{D}_2), \quad (11.11)
\]
according to Lemma 9.1. It follows from (11.10) and (11.11) that, for any canonical divisor \(\mathcal{D}_{\text{can}}\),
\[
\mathcal{D}_{\text{ram}} \sim \mathcal{D}_{\text{can}} + 2(\mathcal{D}_1 + \mathcal{D}_2).
\]

For more symmetry, recall that \(\mathcal{D}_1 + \mathcal{D}_2 \sim (u)_\infty \sim (u)_0 \sim \mathcal{D}_4 + \mathcal{D}_6\), after which (11.8) turns into (11.9). The theorem is proved.

Theorem 11.2. If a block matrix \(A\) of the form (7.1) is such that \(A^T = MA^{-1}M^{-1}\), with \(M\) having the form
\[
M = \begin{pmatrix}
M_1 & 0 & 0 \\
0 & M_2 & 0 \\
0 & 0 & M_3
\end{pmatrix}
\]
(11.12)
with equalities (11.16) and (11.17) taken into account, in the following form:

\[ R : (u, v, w) \longleftrightarrow (u^{-1}, v^{-1}, w^{-1}), \quad (11.13) \]

while the divisor \( D \) and its image \( D^t \) under involution \( I \) obey the equivalence

\[ D + D^I \sim D_{can} + D_1 + \ldots + D_6. \quad (11.14) \]

Conversely, if a matrix \( A \) is such that its curve \( \Gamma \) possesses involution (11.13), and the divisor equivalence (11.14) holds, then \( A^T \) and \( A^{-1} \) are gauge equivalent.

**Proof** follows at once from Lemma [11.1] and Theorems [11.1] and [10.1].

Further properties of the block-diagonal matrix \( M \) (11.12) providing the gauge equivalence between \( A^T \) and \( A^{-1} \) come out during some algebro-geometrical examination, which we will now perform. These properties determine, roughly speaking, whether matrix \( A \) is "in essence" symplectic or orthogonal. Calculate \( M \), using meromorphic vectors

\[
\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}
\]

of matrices \( A^T \) and \( A^{-1} \) respectively, connected, according to Theorem [10.1], by the relation

\[ \tilde{X}(u, v, w) = MA\tilde{X}(u, v, w)h(u, v, w) \quad (11.15) \]

(cf. formulae [10.2], [10.3]). To be exact, we will now deal with blocks \( M_2 \) and \( M_3 \), so that the already obtain formulae (11.6) and (11.7) will be of use.

For a given generic complex number \( u \), there exist \( n + r \) points in the curve \( \Gamma \) with coordinates \((u, uw_1, w_1), \ldots, (u, uw_{n+r}, w_{n+r})\). To avoid bulky formulas, we will omit the middle coordinate, and write those points as \((u, w_1), \ldots, (u, w_{n+r})\). One sees from (11.13) that \( \tilde{Y} \) and \( \tilde{Z} \) are given in all those points at once by the formula (where, of course, \( v_1 = uw_1 \) and so on)

\[
\begin{pmatrix} v_1\tilde{Y}(u, w_1) & \ldots & v_{n+r}\tilde{Y}(u, w_{n+r}) \\ w_1\tilde{Z}(u, w_1) & \ldots & w_{n+r}\tilde{Z}(u, w_{n+r}) \end{pmatrix}^T =
\begin{pmatrix} Y(u, w_1) & \ldots & Y(u, w_{n+r}) \\ Z(u, w_1) & \ldots & Z(u, w_{n+r}) \end{pmatrix}^{-1}. \quad (11.16)
\]

For the vectors \( \tilde{Y} \) and \( \tilde{Z} \) corresponding to the inverse matrix, the relations

\[ v\tilde{Y}(u, w) = Y(u^{-1}, w^{-1}), \quad w\tilde{Z}(u, w) = Z(u^{-1}, w^{-1}). \quad (11.17) \]

hold. This follows from (11.13), if we replace there \((u, v, w)\) by \((u^{-1}, v^{-1}, w^{-1})\) (the replacement applies, of course, to the arguments of vector functions \( X, Y, Z \) implied in (11.13) as well).

From the formula (11.15) follows (as before, we omit \( v \) in the triples \((u, v, w)\))

\[
\begin{pmatrix} v\tilde{Y}(u, w) \\ w\tilde{Z}(u, w) \end{pmatrix} = \begin{pmatrix} M_2 & 0 \\ 0 & M_3 \end{pmatrix} \begin{pmatrix} v\tilde{Y}(u, w) \\ w\tilde{Z}(u, w) \end{pmatrix} \cdot h(u, w). \quad (11.18)
\]

Giving \( w \) all \( n + r \) possible values, we obtain \( n + r \) linearly independent columns in the LHS and RHS, which allows us to express \( M_2 \) and \( M_3 \) from (11.13). It is convenient to write the result, with equalities (11.16) and (11.17) taken into account, in the following form:

\[
\begin{pmatrix} M_2^{-1} & 0 \\ 0 & M_3^{-1} \end{pmatrix} = \begin{pmatrix} Y(u^{-1}, w_1^{-1}) & \ldots & Y(u^{-1}, w_{n+r}^{-1}) \\ Z(u^{-1}, w_1^{-1}) & \ldots & Z(u^{-1}, w_{n+r}^{-1}) \end{pmatrix},
\]

32
Recall that $h$ performs the divisor equivalence between $\tilde{D}$, given by the formula (11.8), and $D^I$ —the image of $D$ under involution $I$, in the sense that

$$(h) = D^I - \tilde{D}.$$  

Substituting here (11.8), we find

$$(h) = D + D^I - D_{\text{ram}} - D_3 - D_5.$$  

The divisor in the RHS of (11.20) is obviously invariant with respect to involution $I$, thus the divisor $(h)$ of function $h$ possesses the same property. If the curve $\Gamma$ is irreducible, and this is exactly the fact in the general position, then $h$ is determined by its divisor up to a constant factor. This means that under the involution $I$ the function $h$ is multiplied by a constant which, evidently, must equal $\pm 1$:

$$h(u, w) = \pm h(u^{-1}, w^{-1}).$$  

Return to the equality (11.19). It has a constant matrix, not depending on a point of the curve, in its LHS. Take the matrix transpose of that equality, and change $(u, w) \leftrightarrow (u^{-1}, w^{-1})$. We get:

$$\begin{pmatrix} M_2 & 0 \\ 0 & M_3 \end{pmatrix} = \pm \begin{pmatrix} M_2 & 0 \\ 0 & M_3 \end{pmatrix}^T,$$

where the sign coincides with that in (11.21). It is clear that, for the similar reasons, the equality $M_1 = \pm M_1^T$ also holds. As a result, the following lemma is proved.

**Lemma 11.3.** If a block matrix $A$ has the property

$$A^T = MAM^{-1},$$

where $M$ is a block diagonal matrix, then $M$ is symmetric or antisymmetric if the function $h$ in the curve $\Gamma$ with the zero and pole divisor (11.20) is even or odd with respect to the involution $I$, correspondingly.

It follows from (11.22) that to a gauge transformation $A \rightarrow \mathcal{N}AN^{-1}, \quad \mathcal{N} = \text{diag}(N_1, N_2, N_3)$, corresponds the transformation

$$\mathcal{M} \rightarrow (\mathcal{N}^{-1})^T \mathcal{M} \mathcal{N}^{-1}$$

of matrix $\mathcal{M}$. A symmetrical $\mathcal{M}$ can be reduced by such a transformation to an identity matrix, and for an antisymmetrical one each diagonal block $M_i$ can be reduced to the standard form

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

consisting of half-size blocks. The relation (11.23) means, after this transformation, the orthogonality of $A$ in the first case and its symplecticity—in the second case.

Let us summarize these considerations in the following theorem.

**Theorem 11.3.** In the notations and under the assumptions of Theorem 11.2, a matrix $A$ with an irreducible curve $\Gamma$ is gauge equivalent to an orthogonal or symplectic matrix, if the function $h$ with the zero and pole divisor (11.20) is even or odd, correspondingly, with respect to the involution $I$ (11.13).

Thus, in this section we studied the algebro-geometrical objects in a specific case of orthogonal or symplectic matrices $A$. Now it would be not very hard work to show that those objects retain their specific form under the evolution, thus proving that a matrix $A$ retains its property to be gauge equivalent to an orthogonal or symplectic matrix. We will prefer, however, to go another way, considering the decomposition (7.2) in orthogonal and symplectic cases.
12. **Matrix factorization in case of orthogonality or symplecticity, and conservation of those properties under evolution**

It is natural to expect that an orthogonal or symplectic matrix \( A \) can be factorized in a product of matrices of the same kind (orthogonal or symplectic) \( A_1, A_2, A_3 \) (formulae (7.2, 7.3)). Indeed, the following lemma holds.

**Lemma 12.1.** Under conditions of Lemma 11.3, for the matrix \( A \) such a factorization (7.2, 7.3) exists that for each matrix \( A_i \) separately, \( i = 1, 2, 3 \), a relation like (11.22) is valid, i.e.

\[
A_i^T = MA_i^{-1}M^{-1}.
\]  

(12.1)

**Proof.** Consider, to begin, an arbitrary factorization \( A = A_1A_2A_3 \) of matrix \( A \), not requiring that (12.1) hold. Then there are obvious factorizations

\[
A^T = A_1^T A_2^T A_3^T
\]  

(12.2)

and

\[
A^{-1} = A_3^{-1} A_2^{-1} A_1^{-1}.
\]  

(12.3)

One more factorization of matrix \( A^T \), besides (12.2), into a product of three matrices with zero and unity blocks in the same places, can be obtained from (11.22) and (12.3) in the following way:

\[
A^T = MA_3^{-1}M^{-1} = M A_3^{-1} A_2^{-1} A_1^{-1} M^{-1} = \]

\[
(M A_3^{-1} M^{-1}) (M A_2^{-1} M^{-1}) (M A_1^{-1} M^{-1}).
\]  

(12.4)

Applying Lemma 7.1 to the two factorizations (12.2) and (12.4), we deduce that the following relations, for some nondegenerate matrices \( F_1, F_2, F_3 \) of proper sizes, must hold:

\[
A_3^T = (M A_3^{-1} M^{-1}) \begin{pmatrix} 1 & 0 & 0 \\ 0 & F_2 & 0 \\ 0 & 0 & F_3 \end{pmatrix},
\]  

(12.5)

\[
A_2^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & F_3^{-1} \end{pmatrix} (M A_2^{-1} M^{-1}) \begin{pmatrix} F_1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},
\]  

(12.6)

\[
A_1^T = \begin{pmatrix} F_1^{-1} & 0 & 0 \\ 0 & F_2^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} (M A_1^{-1} M^{-1}).
\]  

(12.7)

Our aim now is to find such a transformation of type (7.4) for matrices \( A_1, A_2, A_3 \) that the relations (12.1) hold. Consider at first a matrix

\[
\tilde{A}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & K_2 & 0 \\ 0 & 0 & K_3 \end{pmatrix} A_3
\]  

(12.8)

with some nondegenerate \( K_2 \) and \( K_3 \). Relations (12.8) and (12.5) together yield the following connection between \( \tilde{A}_3^T \) and \( \tilde{A}_3^{-1} \):

\[
\tilde{A}_3^T = M \tilde{A}_3^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & K_2 & 0 \\ 0 & 0 & K_3 \end{pmatrix} M^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & K_2^T & 0 \\ 0 & 0 & K_3^T \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & F_2 & 0 \\ 0 & 0 & F_3 \end{pmatrix}.
\]  

(12.9)

This will turn into the required relation

\[
\tilde{A}_3^T = M \tilde{A}_3^{-1} M^{-1}
\]  

(12.10)
if we manage to obey, by a proper choice of \( K_2 \) and \( K_3 \), the following equalities:

\[
K_2 M_2^{-1} K_2^T F_2 = M_2^{-1}, \tag{12.11}
\]

\[
K_3 M_3^{-1} K_3^T F_3 = M_3^{-1}. \tag{12.12}
\]

Rewrite (12.11) in the form

\[
K_2 M_2^{-1} K_2^T = M_2^{-1} F_2^{-1}. \tag{12.13}
\]

In the following Lemma 12.2 we will demonstrate that the RHS of (12.13) is symmetric or antisymmetric in case \( M_2 \) is symmetric or antisymmetric correspondingly. Assuming, for a while, this fact without proof, we find that the equation (12.13) is always solvable with respect to \( K_2 \)—this is a simple consequence from the properties of quadratic forms and antisymmetric bilinear forms and their matrices, see, e.g., §§90 and 91 of the manual \( [79] \) (one must take into account here also the nondegeneracy of matrices \( M_2^{-1} \) and \( F_2^{-1} \)).

Similarly, the equation (12.12) is solvable with respect to \( K_3 \). Thus, we managed to obey the relation (12.10). Then, solving the equation for \( K_1 \), similar to (12.11) and (12.12), we see that the relations

\[
\tilde{A}_2^T = MA_2^{-1} \mathcal{M}^{-1}
\]

and

\[
\tilde{A}_1^T = MA_1^{-1} \mathcal{M}^{-1}
\]

also hold, for

\[
\tilde{A}_2^T = \begin{pmatrix} K_1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} A_2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K_3^{-1} \end{pmatrix}
\]

and

\[
\tilde{A}_1 = A_1 \begin{pmatrix} K_1^{-1} & 0 & 0 \\ 0 & K_2^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]

We don’t need any more the initial matrices \( A_i \) without tildes in the rest of this Proof. Rename the new matrices \( \tilde{A}_i \) into \( A_i \), and with this Lemma 12.1 is proved. Recall, however, that its proof was based on the following lemma (and its analogs arising from changing the subscript 2 to 1 or 3).

**Lemma 12.2.** If, in the notations of Lemma 12.1

\[
M_2 = \pm M_2^T, \tag{12.14}
\]

then

\[
(M_2^{-1} F_2^{-1}) = \pm (M_2^{-1} F_2^{-1})^T, \tag{12.15}
\]

with the same sign as in (12.14).

**Proof.** Take the equality (12.13) and apply to its both sides, first, the transposing, and second, the matrix inversion. We will get, in the LHS, \( A_3^{-1} \), while in the RHS—some expression containing \( A_3^T \), which we have no need to write down here. This done, express again \( A_3^T \) through \( A_2^{-1} \) by multiplying the LHS and RHS of the obtained equality by suitable matrices and the interchanging LHS and RHS. The result will be the following:

\[
A_3^T = MA_3^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & F_2^T & 0 \\ 0 & 0 & F_3^T \end{pmatrix} \mathcal{M}^{-1}. \tag{12.16}
\]

Comparing (12.16) with (12.13), we find, in particular, that

\[
M_2^{-1} F_2 = F_2^T M_2^{-1}. \tag{12.17}
\]
It is easy to see that the equality (12.17) is equivalent to (12.15), and with this the proof of Lemma 12.2 is over, and the proof of Lemma 12.1 is thus complete, too.

The following theorem, which summarizes the results of this section, immediately follows from Lemma 12.1.

**Theorem 12.1.** The property of a matrix \( A \) of the form (7.1) to be gauge equivalent to an orthogonal or symplectic matrix and, consequently, to be factorable in a product (7.2, 7.3) of matrices of the same kind, is conserved under the evolution described in Section 7.

*Proof* follows at once from the definition of this evolution given in the end of Section 7.
Chapter 3

Inhomogeneous 6-vertex model

13. Dynamical system connected with the 6-vertex model on the kagome lattice

Consider now a reduction of the system defined in Section 7 which leads to a dynamical system in $2 + 1$-dimensional fully discrete space-time. Let the linear space in which matrix $A$ (7.1) acts have a basis enumerated by edges of a triangular lattice on the torus (Fig. 3.5), so that those components of the vector $\mathbf{X} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$ corresponding to horizontal edges form the vector $X$, while those corresponding to oblique edges form the vector $Y$, and the ones corresponding to vertical edges form the vector $Z$. As the number of each of those three types of edges is the same, all the blocks in the matrix $A$ have the same sizes:

$$m = n = r \quad (= \text{the number of lattice edges}). \quad (13.1)$$

Impose the following “locality” condition on matrix $A$: let the vector corresponding to any given edge of the lattice be transformed under the action of $A$ into a linear combination of just three vectors, corresponding to the edges coming upwards, to the right and northeastwards from the vertex that is the upper, right, or northeastern end of the considered “incoming” edge (Fig. 3.6). Thus, only those elements of matrix $A$ are not zeros that correspond to “local” transitions of Fig. 3.6.

The factorization of a “local” matrix $A$ into the product (7.2) corresponds to each vertex represented by a small circle in Fig. 3.5 being converted into a triangle of the type shown in Fig. 2.4, so that the lattice transforms into a kagome lattice (Fig. 3.7). The triangles arising from the vertices-circles are shaded in Fig. 3.7.

One can easily see that the “locality” property of matrix $A$ is preserved by a step of evolution, if, of course, the proper gauge is taken. The detailed description of the step of evolution from the “local” viewpoint and the description of gauge transformations preserving the “locality” of $A$ are given in the beginning of Section 14. Here we will concentrate on the values conserved under the evolution.

Let us return to the triangle lattice of Figure 3.5. Express the “integral of motion”

$$I(u, w) = \det(1 - \mathcal{A} \begin{pmatrix} u^{-1} & 0 & 0 \\ 0 & u^{-1}w^{-1} & 0 \\ 0 & 0 & w^{-1} \end{pmatrix}) \quad (13.2)$$

in terms of paths going along the edges of this lattice ($I(u, w)$ is indeed an integral of motion with any $u, w$, because the equality $I(u, w) = 0$ determines the invariant curve, and a possible multiplicative constant is fixed by the fact that the constant term in (13.2) equals unity).

As it known, the determinant of a matrix is an alternating sum of its elements’ products, each summand corresponding to some permutation of the matrix columns, while each permutation
Figure 3.5. The triangular lattice

Figure 3.6. The vectors corresponding to “incoming” edges are transformed by $A$ into linear combinations of those corresponding to “outgoing” edges

Figure 3.7. The kagome lattice
factorizes into a product of cyclic ones. As applied to our matrix $A$, it means that the determinant \((13.2)\) is a sum each term of which corresponds to a set of closed trajectories going along the arrows according to Fig. 3.6 (recall that the lattice is situated on the torus!). The trajectories of each given set can have intersections and self-intersections, but none of the edges may be passed through twice or more by one or several trajectories.

To be exact, to each trajectory corresponds a product of entries of the matrix

$$A = \begin{pmatrix}
    u^{-1} & 0 & 0 \\
    0 & u^{-1}w^{-1} & 0 \\
    0 & 0 & w^{-1}
\end{pmatrix}$$

corresponding to transitions through a vertex to a neighboring edge according to Fig. 3.6, multiplied (the product as a whole) by \((-1)\). To each set of trajectories (including, of course, the empty set) corresponds the product of the mentioned values corresponding to its trajectories. A direct check shows that all the minus signs, including that in formula \((13.2)\), have been taken into account correctly.

It is easy also to describe the determinant $I(u, w)$ in terms of the kagome lattice obtained on factorizing the matrix $A$ into the product \((7.2)\). This description almost repeats two preceding paragraphs. Let us formulate it as the following lemma.

**Lemma 13.1.** $I(u, w)$ is a sum over sets of trajectories on the kagome lattice; the direction of motion is upwards, to the right, or northeastwards; none of the edges is passed through twice by trajectories of a given set; to the vertices of types $\bullet\bullet$, $\bullet\uparrow$, $\uparrow\uparrow$, if a trajectory passes through them, correspond the factors equalling matrix elements of matrices $A_1$, $A_2$, $A_3$ respectively; besides, to each move to the right through a lattice period corresponds a factor $u^{-1}$, and to each move upwards—a factor $w^{-1}$ (and both of them to a diagonal move); finally, to each set corresponds one more factor, $(-1) (\text{number of trajectories})$.

Now let us link $I(u, w)$ with the statistical sum of inhomogeneous 6-vertex model on the kagome lattice. Let each edge of the kagome lattice be able to take one of two states, which will be depicted below as either presence or absence of an arrow on the edge (the arrow will always be directed upwards, to the right, or northeastwards). A “Boltzmann weight” will correspond to each vertex as follows: if there are no arrows on the edges meeting at the vertex, the weight will be 1; if there is exactly one arrow coming into the vertex and exactly one going out of it, the weight is the difference between the products of weights corresponding to the intersecting and non-intersecting paths through the vertex:

$$\text{Weight} \left( \begin{array}{c}
\bullet\bullet \\
\uparrow\uparrow
\end{array} \right) = \text{Weight} \left( \begin{array}{c}
\bullet\bullet \\
\uparrow\uparrow
\end{array} \right) \cdot \text{Weight} \left( \begin{array}{c}
\bullet\bullet \\
\uparrow\uparrow
\end{array} \right) - \text{Weight} \left( \begin{array}{c}
\bullet\bullet \\
\uparrow\uparrow
\end{array} \right) \cdot \text{Weight} \left( \begin{array}{c}
\bullet\bullet \\
\uparrow\uparrow
\end{array} \right);$$

\[ (13.3) \]

in the rest of cases the weight is zero.

A weight will also correspond to each edge of the kagome lattice: weight 1 to an edge without an arrow, and weights $u^{-1/2}$, $w^{-1/2}$ or $u^{-1/2}w^{-1/2}$ to a horizontal, vertical or oblique edge having an arrow. If needed, the edge weights can be included in the vertex weights, but we will not do that here.

The statistical sum $S(u, w)$ of our 6-vertex model is, of course, a sum of products of vertex and edge weights over all arrow configurations. The next lemma is the key statement.
Lemma 13.2. The statistical sum $S(u,w)$ is a sum over the same sets of trajectories as the determinant $I(u,w)$, and to each set corresponds the same summand up to, maybe, a minus sign. To be exact, the number of trajectories in the exponent of $(-1)$ in Lemma 13.1 changes to the number of intersections (self-intersections included) of a given set of trajectories.

Proof. is evident from the statistical sum definition.

Each closed path on the torus is homologically equivalent to a linear combination of two basis cycles $a$ and $b$. The same is true for a set of paths (trajectories), regarded as a formal sum of them. Different sets may be homologically equivalent to a given cycle $la + mb$ but, as the following lemma shows, they have something in common.

Lemma 13.3. For any set of trajectories on the torus homologically equivalent to a cycle $la + mb$ ($a, b$ being basis cycles, $l, m$—integers),

$$(\text{number of intersections}) - (\text{number of trajectories}) \equiv lm - l - m \pmod{2}.$$  \hfill (13.4)

Proof. may consist in the following simple consideration: 1) if the set consists of $l$ trajectories going along $a$, and $m$ ones going along $b$, (13.4) is obviously true, 2) under deformations of trajectories, the number of intersections changes only by even numbers, 3) with elimination of an intersection \(\begin{array}{c}\hline \hline \end{array}\) or inverse operation, the LHS of (13.4) may change also only by an even number. Starting from an arbitrary set and applying the transformations 2) and 3), one can arrive at a set of type 1), so the lemma is proved.

Let the following products of edge weights correspond to the basis cycles: $x = u^{\alpha_1}w^{\beta_1}$ for $a$ and $y = u^{\alpha_2}w^{\beta_2}$ for $b$. Denote

$$s(x, y) = S(u, w), \quad f(x, y) = I(u, w).$$  \hfill (13.5)

Theorem 13.1. The statistical sum of the inhomogeneous 6-vertex model on the kagome lattice defined in this section is invariant with respect to the evolution of the reduced $2 + 1$-dimensional model (for all $u, w$) and is connected with the determinant $I(u, w)$ \hfill (13.2), whose vanishing defines the invariant curve of the model, by relations (in the notations of (13.5))

$$s(x, y) = \frac{1}{2} (-f(x, y) + f(-x, y) + f(x, -y) + f(-x, -y)), \quad (13.6)$$

$$f(x, y) = \frac{1}{2} (-s(x, y) + s(-x, y) + s(x, -y) + s(-x, -y)). \quad (13.7)$$

Proof. It follows from Lemmas \hfill (13.1) \hfill (13.2) \hfill (13.3) that in the expansions of $s(x, y)$ and $f(x, y)$ in powers of $x$ and $y$ the coefficients near $x^my^n$ coincide if $lm - l - m$ is even, and differ in their signs in the opposite case. This is exactly what the formulae (13.6,13.7) are about.

Remark 13.1. From the physical viewpoint, the determinant $I(u, w)$ can be regarded as a statistical sum of the same model, but with other boundary conditions than for $S(u, w)$. This can be explained as follows. Let us regard our torus as obtained by identification of “opposite sides” of a plane domain that can be obtained by cutting the torus along the basis cycles $a$ and $b$. We will consider those cycles as components of the plane domain boundary. Of course, we can determine the numbers $l$ and $m$, for each set of trajectories from Lemmas \hfill (13.1) \hfill (13.2) \hfill (13.3), if we just know how the trajectories intersect with the boundary (and we may know nothing about the behavior of trajectories inside the domain). And it is exactly the numbers $l$ and $m$ that determine, according to the proof of Theorem \hfill (13.1), whether a sign must be changed of a given summand in a “usual” statistical sum $S(u, w)$ in order to get $I(u, w)$.
We considered in Section 13 an inhomogeneous 6-vertex model satisfying the “free fermion” condition, on the kagome lattice. The lattice itself was situated on a torus, thus, in particular, it was finite. The evolution was in fact “local”, i.e. a given weight in a moment $\tau$ was influenced only by weights in a few neighboring points in the moment $\tau - 1$. However, the evolution was described “globally”, just as a particular case of the evolution from Section 6.

In this section, we will describe the evolution in local terms, the description being valid, because of its localness, for an infinite in both spatial directions lattice as well. The evident solitonic character of the model stimulates one to study, in particular, the “finite-gap” quasiperiodic solutions on this infinite lattice. Recall however that the algebraic curve appearing in Section 13 was associated with a “global” object—the statistical sum of the model regarded as a function of two parameters, $u$ and $w$. In this section we develop another approach, namely, in the spirit of the usual theory of finite-gap solutions, we start from a given algebraic curve with some marked points in it and construct a solution out of those objects.

It is natural to begin with a definition of evolution of matrices corresponding to vertices of a finite or infinite kagome lattice independent of any global objects. More precisely, we will assume, to be in accord with the evolution definition given in Section 7, that before a step of evolution there are given some products of such matrices (of size $2 \times 2$), while the matrices themselves appear and disappear within the step. Hence, at the beginning of a step, to each triangle of the form $\begin{pmatrix} a_1 & b_1 & 0 \\ c_1 & d_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ contained in the kagome lattice (and shaded in Fig. 3.7 situated on p. 38) a $3 \times 3$-matrix of complex numbers must correspond. The step begins with this matrix being factorized into a product of three matrices of the following form:

$$
\begin{pmatrix} a_1 & b_1 & 0 \\ c_1 & d_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_2 & 0 & b_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & a_3 & b_3 \\ 0 & c_3 & d_3 \end{pmatrix}.
$$

(14.1)

This done, we will assume that in our triangle the matrix $\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$ corresponds to the vertex $\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$ corresponds to the vertex $\begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix}$ corresponds to the vertex $\begin{pmatrix} 1 & 0 & 0 \\ 0 & a_3 & b_3 \\ 0 & c_3 & d_3 \end{pmatrix}$ (14.2) corresponds to the triangle as a whole. Finally, turn inside out all the triangles of the form $\begin{pmatrix} \hat{a}_1 & \hat{b}_1 \\ \hat{c}_1 & \hat{d}_1 \end{pmatrix}$, to the vertex $\begin{pmatrix} \hat{a}_2 & \hat{b}_2 \\ \hat{c}_2 & \hat{d}_2 \end{pmatrix}$, and to the vertex $\begin{pmatrix} \hat{a}_3 & \hat{b}_3 \\ \hat{c}_3 & \hat{d}_3 \end{pmatrix}$. Then let the matrix

$$
\begin{pmatrix} 1 & 0 & 0 \\ 0 & \hat{a}_3 & \hat{b}_3 \\ 0 & \hat{c}_3 & \hat{d}_3 \end{pmatrix} \begin{pmatrix} \hat{a}_2 & 0 & \hat{b}_2 \\ 0 & 1 & 0 \\ \hat{c}_2 & 0 & \hat{d}_2 \end{pmatrix} \begin{pmatrix} \hat{a}_1 & \hat{b}_1 \\ \hat{c}_1 & \hat{d}_1 \end{pmatrix}.
$$

(14.2)

in our lattice, converting them into triangles of the form (e.g. moving all oblique lines one lattice period right), putting the same matrix (14.2) in correspondence to each “converted” triangle as to the initial one. The description of a step of evolution is over.

The same can be presented in a somewhat other way, starting from the triangular lattice of Fig. 3.5 (p. 38). By the beginning of an evolution step, to each $\textit{circle}$—a vertex of that lattice—a
3 × 3-matrix must correspond that is to be factorized in a product (14.1), which corresponds to a “decomposition” of the circle in a triangle \(\square\). In these terms, the step of evolution ends with the triangles \(\triangle\) being “packed” in new circles.

Just as in Section 7, the evolution is defined up to “gauge transformations”. To explain what the gauge transformations look like now, note that each entry of the abovementioned 2 × 2-matrices naturally corresponds to a certain pair “incoming edge, outgoing edge” for the given vertex (Figure 3.6 on p. 38 will remind which edges are called incoming and outgoing). If we mark those pairs of edges with arrows, we will obtain the following pictures for entries of the matrix \(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}\):

\[
\begin{align*}
  a_1 : & \quad \includegraphics[width=1cm]{arrow} \\
  b_1 : & \quad \includegraphics[width=1cm]{arrow} \\
  c_1 : & \quad \includegraphics[width=1cm]{arrow} \\
  d_1 : & \quad \includegraphics[width=1cm]{arrow}
\end{align*}
\]

Now it is seen that to each edge a set of elementary gauge transformations naturally corresponds parameterized by the set of nonzero complex numbers. Namely, we can multiply by a nonzero constant the matrix entries in the vertex corresponding to the given edge as an outgoing one, and divide by the same constant the entries in the neighboring vertex corresponding to the given edge as an incoming one. Nonzero numbers can be put in correspondence to all edges, and the elementary gauge transformations can be performed for all simultaneously. The resulting transformation will be called simply a gauge transformation on the whole lattice.

**Lemma 14.1.** 1) The result of the step of evolution described earlier in this section is determined to within a gauge transformation.

2) The result of the evolution step is not changed if a gauge transformation is applied to the initial state.

3) For a finite-dimensional system, the evolution coincides with that defined in Section 7, and the gauge transformations considered here are such in the sense of Section 7, as well (but there are more gauge transformations in Section 7 because they include also “non-local”, from the standpoint of model on the kagome lattice, transformations).

**Proof** follows immediately from the definitions given here and in Section 7.

Passing on to the “local” algebro-geometrical description of evolution, let us consider first the following abstract divisor evolution on an infinite in both spatial directions lattice. At the moment, we don’t need to know the exact structure of those divisors or an algebraic manifold where they belong. Thus, let us temporarily understand by “divisors” just elements of some abelian group \(\mathcal{G}\). Let six elements \(D_1, \ldots, D_6\) be fixed in that group.

Let a divisor correspond to each edge of the triangular lattice in Fig. 3.5 by the beginning of a step of evolution, with the following condition fulfilled: for each lattice vertex (a circle in Fig. 3.5) an element \(D \in \mathcal{G}\) can be indicated such that the divisors corresponding to edges abutting on that vertex are as shown in Fig. 3.8.

Thus, \(D\) depends linearly on the coordinates of a vertex, increasing by \(D_1 + D_2 - D_4 - D_6\) when moving one lattice period to the right, and by \(D_3 + D_4 - D_5 - D_2\) when moving one lattice period up.

A step of evolution begins with every circle being decomposed into a triangle such as depicted in Fig. 3.9, where the numbers near an edge serve as a brief notation for a divisor corresponding to it, e.g. “12” denotes \(D - D_1 - D_2\), and so on, compare Fig. 2.4. Thus, the kagome lattice arises from the triangular one, and then we take the triangles of the form \(\triangle\) in it and “pack” them in circles—the vertices of the new triangular lattice. With this, the divisor evolution step is over. As the following lemma states, everything is ready for a next step.
Figure 3.8. Divisors on edges abutting on a triangular lattice vertex

Figure 3.9. Divisors on edges of the kagome lattice
Lemma 14.2. The divisors corresponding to edges abutting on each vertex of the triangular lattice obtained in the end of the previous paragraph are again of the form of Fig. 3.3, with $D$ depending on the vertex.

Proof follows from an easy direct calculation.

Now let an algebraic curve $\Gamma_0$ of genus $g_0$ be given, and the abelian group $G$ be the group of all divisors on $\Gamma_0$. Let $D_1, \ldots, D_6$ be divisors on $\Gamma_0$, consisting each of one point (note in parentheses that, of course, one can consider a “vector” model as well, $D_1, \ldots, D_6$ consisting in that case each of several points). Let divisors $D$ corresponding to vertices of triangular lattice (see Fig. 3.3) be of the degree $g_0 + 2$. If all those algebro-geometrical objects are generic, the Riemann–Roch theorem shows that to each edge of triangular (Fig. 3.8) or kagome (Fig. 3.9) lattice corresponds a one-dimensional space of meromorphic functions $\mathbf{f}$ such that

\[(f) + D - D_j - D_k \leq 0,\]

if a divisor $D - D_j - D_k$ corresponds to the given edge.

It remains to explain how the matrices are constructed corresponding to vertices of our two lattices. Fix a nonzero function $\mathbf{f}$ satisfying (14.3) for every $j$ and $k$, and denote it as $f_{jk}$. Then, say, the matrix $A_{\text{local}}$ corresponding to a vertex in Fig. 3.8 is found from the relation

\[
A_{\text{local}} \begin{pmatrix} f_{12} \\ f_{13} \\ f_{34} \end{pmatrix} = \begin{pmatrix} f_{46} \\ f_{56} \\ f_{52} \end{pmatrix}
\]

that must hold in each point of the curve $\Gamma_0$. Similarly, the matrix $\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$ (see (14.1) and the graphical analog of that relation, Figure 3.9) is found from the relation

\[
\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} f_{26} \\ f_{36} \end{pmatrix} = \begin{pmatrix} f_{46} \\ f_{56} \end{pmatrix}
\]

etc. Relations of type (14.4–14.5) determine the matrices correctly due to the fact that all meromorphic functions in every such relation lie in a linear space of needed dimension. For example, in relation (14.3), divisors of all functions satisfy the condition

\[(f_{jk}) + D - D_6 \geq 0,
\]

hence the Riemann–Roch theorem, together with equalities

\[
\deg D = g_0 + 2, \quad \deg D_6 = 1,
\]

shows that the space of such functions is two-dimensional.

Note that other choice of any function $f_{jk}$, i.e. its multiplication by a nonzero constant, corresponds to an elementary gauge transformation of matrices.

Lemma 14.3. The described above divisor evolution generates, by means of formulae of type (14.4) and (14.5), the matrix evolution described in the beginning of this section.

Proof. We must show that the operation of decomposition of a triangular lattice vertex in a triangle of the kagome lattice, and the inverse operation of “packing”, when applied to divisors, generate the similar operations on matrices. E.g., if a matrix $A_{\text{local}}$ is determined by the relation (14.4), and we have to factorize that matrix in a product (14.1), then the values $a_1, \ldots, d_3$ obtained from (14.5) and two similar relations must give the solution to the problem. It is worth while to write out explicitly those two relations similar to (14.3):

\[
\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \begin{pmatrix} f_{12} \\ f_{23} \end{pmatrix} = \begin{pmatrix} f_{26} \\ f_{52} \end{pmatrix}, \quad \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix} \begin{pmatrix} f_{13} \\ f_{34} \end{pmatrix} = \begin{pmatrix} f_{36} \\ f_{23} \end{pmatrix}.
\]
If we now let the product (14.1) act on the column from the LHS of (14.4), and use (14.5) and (14.6), we will get exactly the column in the RHS of (14.4), which means that the factorization of a matrix is done with the help of divisors in our local case with equal success as in the global case of Section 10, cf. formulae (10.4–10.6). Certainly, the similar considerations are valid also for the “packing” operation. The lemma is proved.

Thus, we have shown how to construct solutions to the problem of matrix evolution on the infinite lattice in the spirit of standard finite-gap integration theory. In Section 15 we will connect this approach with the “global” approach of Sections 9 and 10. Explicit expression for the solution in multidimensional theta functions is presented in Section 16.

15. Connection between “local” and “global” curves in the periodic case

We continue to consider divisors from Section 14 corresponding to edges of the infinite triangular lattice. Let there exist integers \( \xi_1 \) and \( \eta_1 \) such that the divisor equivalence holds in the curve \( \Gamma_0 \)

\[
\xi_1(D_1 + D_2 - D_4 - D_6) + \eta_1(D_3 + D_4 - D_5 - D_2) \sim 0. \tag{15.1}
\]

(compare with the text between lemmas 14.1 and 14.2). As each of the divisors \( D_1, \ldots, D_6 \) consists now of just one point, it is convenient to introduce for those points notations \( P_1, \ldots, P_6 \). Note also that \( D_j \)'s in (15.1) are now other divisors than the “global” ones denoted by the same letters in Sections 9–11. Equivalence (15.1) means that there exists a function \( g_1 \) on \( \Gamma_0 \) that has a zero of multiplicity \( \xi_1 \) in the point \( P_1 \), a zero of multiplicity \( \xi_1 - \eta_1 \) in the point \( P_2 \), \ldots, a pole of multiplicity \( \xi_1 \) in the point \( P_6 \). Thus, meromorphic functions \( f \), out of which we will construct matrices according to relations of type (14.4), can be put in correspondence to lattice edges in such a way that the multiplying of a function \( f \) by \( g_1 \) will correspond to a lattice translation by the vector \( (\xi_1, \eta_1) \). As for the matrices like \( A_{\text{local}} \) from (14.4), they will obviously be periodic with period \( (\xi_1, \eta_1) \).

**Lemma 15.1.** Let there exist two linearly independent vectors with integer entries \( (\xi_1, \eta_1) \) and \( (\xi_2, \eta_2) \) such that the following divisor equivalences in the curve \( \Gamma_0 \) hold: (15.1) and a similar one

\[
\xi_2(D_1 + D_2 - D_4 - D_6) + \eta_2(D_3 + D_4 - D_5 - D_2) \sim 0. \tag{15.2}
\]

Then matrices corresponding to lattice edges are, in a proper gauge, doubly periodic in coordinates, with periods \((\xi_1, \eta_1)\) and \((\xi_2, \eta_2)\).

**Proof** is obtained by adding to the above consideration for \((\xi_1, \eta_1)\) a similar consideration for \((\xi_2, \eta_2)\).

Thus, conditions (15.1) and (15.2) are sufficient for the model on infinite lattice to become, in essence, a model on a torus introduced in Section 13. Our next task is to learn how to pass on from the curve \( \Gamma_0 \) to the “global” curve \( \Gamma \), and by that calculate the statistical sum of the model on the torus starting from a “small” curve \( \Gamma_0 \). To be concrete, consider the problem of constructing vectors \[
\begin{pmatrix}
X \\
Y \\
Z
\end{pmatrix}
\]
such that

\[
A \begin{pmatrix}
X \\
Y \\
Z
\end{pmatrix} = \begin{pmatrix}
uX \\
uwY \\
wZ
\end{pmatrix}, \tag{15.3}
\]

the components of vectors \( X, Y, \) and \( Z \) corresponding to horizontal, oblique, and vertical edges respectively; \( A \) being a global operator, as in Section 13, and \( u \) and \( w \) being some so far unknown values.
It turns out that the solution of (15.3) is obtained if \( u \) and \( w \) are introduced as multivalued functions on \( \Gamma_0 \) by the formulae
\[
\begin{align*}
u^\xi w^\eta &= g_1, \\
u^\xi w^\eta_2 &= g_2,
\end{align*}
\]
where the function \( g_2 \) is constructed out of (15.2) in the same way as \( g_1 \) out of (15.3). The vectors \( X, Y \) and \( Z \) are built now as follows. Choose some vertex to be the origin of integer-valued coordinates \((\xi, \eta)\). For each of the vectors \( X, Y \) and \( Z \), one component—the one corresponding to the incoming edge abutting on the origin of coordinates (recall Fig. 3.6 on p. 38)—is defined as the value of meromorphic function \( f \) corresponding to this edge as in Section 14, in some point of \( \Gamma_0 \). Next, we take some values \( u \) and \( w \) for this point according to (15.4), and for an incoming edge at a point with coordinates \((\xi, \eta)\) we take as a component of the vector \( X, Y \) or \( Z \) the value of the corresponding function \( f \) multiplied by \( u^{-\xi} w^{-\eta} \). As a result, as was required, the vector components are not changed under a translation by the periods \((\xi_1, \eta_1)\) and \((\xi_2, \eta_2)\) (we assume of course that a “quasiperiodic” gauge was chosen for functions \( f \), as in Lemma 15.1).

**Lemma 15.2.** The vectors \( X, Y \) and \( Z \) constructed in the previous paragraph satisfy indeed the equation (15.3) for any point of the curve \( \Gamma_0 \) and any pair \((u, w)\) lying above it (i.e. satisfying (15.4)).

**Proof.** Indeed, it is easy to see that (15.3) is equivalent to the totality of relations of type (14.4) for all lattice vertices, because the factors \( u, uw \) and \( w \) in (15.3) are canceled by factors arising because of multiplying the functions \( f \) by \( u^{-\xi} w^{-\eta} \).

Having constructed the solutions of equation (15.3), we arrive at some algebraic dependence between \( u \) and \( w \) arising from (15.4) and an algebraic dependence between the meromorphic functions \( g_1 \) and \( g_2 \) on the curve \( \Gamma_0 \). We expect that it will be, maybe under some additional conditions, the same dependence as given by the “global” equations (9.1, 9.2), because solutions of (15.4) do satisfy, of course, the system (9.1, 9.2). However, we don’t know at the moment whether our local construction does not lead to some singular operators \( A \), e.g. those having an invariant curve that contains several components. In such case, one can conceive a situation where (15.4) together with the dependence between \( g_1 \) and \( g_2 \) yields only a part of the curve \( \Gamma \), while all \( \Gamma \) must be known, of course, for calculating the statistical sum. Besides, we must attach the exact meaning to the words “algebraic dependence between \( g_1 \) and \( g_2 \)”.

It turns out that there exist simple conditions that guarantee the irreducibility of \( \Gamma \). They are as follows:

(i) irreducibility of \( \Gamma_0 \),

(ii) the parallelogram built on vectors \((\xi_1, \eta_1)\) and \((\xi_2, \eta_2)\) must be the minimal parallelogram of periods (i.e. the parallelogram of minimal area among the parallelograms built on two linearly independent vectors of periods), and

(iii) at least one of the pairs \((\xi_1, \xi_2)\), \((\eta_1, \eta_2)\), \((\xi_1 - \eta_1, \xi_2 - \eta_2)\) consists of relatively prime numbers—the condition without which the proof of the following lemma fails.

**Lemma 15.3.** Consider the mapping \( \phi : \Gamma_0 \to CP_1 \times CP_1 \) given by the formula
\[
z \mapsto (g_1(z), g_2(z)).
\]
Under the above conditions (i) and (iii), \( \phi \) is a birational isomorphism of the curve \( \Gamma_0 \) on its image \( \phi(\Gamma_0) \).

**Proof.** Compose the following table of zero and pole multiplicities for functions \( g_1 \) and \( g_2 \) in points \( P_1, \ldots, P_6 \).

|     | \( P_1 \) | \( P_2 \) | \( P_3 \) | \( P_4 \) | \( P_5 \) | \( P_6 \) |
|-----|-----------|-----------|-----------|-----------|-----------|-----------|
| \( g_1 \) | \( \xi_1 \) | \( \xi_1 - \eta_1 \) | \( \eta_1 \) | \( -\xi_1 + \eta_1 \) | \( -\eta_1 \) | \( -\xi_1 \) |
| \( g_2 \) | \( \xi_2 \) | \( \xi_2 - \eta_2 \) | \( \eta_2 \) | \( -\xi_2 + \eta_2 \) | \( -\eta_2 \) | \( -\xi_2 \) |
This table is made up according to formulae (15.1) and (15.2). An integer in a table cell is the zero multiplicity of a function in a point if it is positive, or minus pole multiplicity if it is negative. Functions \( g_1 \) and \( g_2 \) have no other zeros or poles.

Suppose that the mapping \( \phi \) is a \( q \)-sheeted covering. Let us replace \( \phi(\Gamma_0) \) by its nonspecial model \( \Gamma_{ns} \) which exists according to \( \S 7A \) of the manual [80]. Somewhat freely, we will denote the covering \( \Gamma_0 \to \Gamma_{ns} \) by the same letter \( \phi \).

Let us prove that a point \( P_j \) in which zero or pole multiplicities of functions \( g_1 \) and \( g_2 \) are relatively prime (such a point exists according to the condition (iii)) cannot be a branch point of covering \( \phi \). Indeed, due to the indicated relative primality, a local parameter in the point \( P_j \) can be chosen as a product of some degrees of \( g_1 \) and \( g_2 \), i.e. is uniquely determined by an underlying point from a neighborhood of \( \phi(P_j) \), which cannot happen in a branch point.

The point \( P_j \) from the previous paragraph also cannot have the same image as another point \( P \in \Gamma_0 \), i.e. \( \phi(P_j) \neq \phi(P) \) when \( P \neq P_j \). To show this, we note that \( P \) can only be one of the points \( P_1, \ldots, P_6 \), because only there zeros and poles of \( g_1 \) and \( g_2 \) are situated. However, by virtue of the linear independence of periods \( (\xi_1, \eta_1) \) and \( (\xi_2, \eta_2) \), and the above table, the multiplicities of zeros or poles of \( g_1 \) and \( g_2 \) in points \( \phi(P_j) \) and \( \phi(P) \) cannot coincide (even if \( P \) is a branch point).

Hence \( \phi(P_j) \neq \phi(P) \).

Combining the results of two preceding paragraphs, we see that \( P_j \) is the only point lying above \( \phi(P_j) \), and not a branch point. Hence, the number of sheets \( q = 1 \), and the lemma is proved.

We continue to consider the algebraic dependence between functions \( g_1 \) and \( g_2 \). In view of the irreducibility of \( \Gamma_0 \), this dependence can be expressed as

\[
P_0(g_1, g_2) = 0,
\]

(15.5)

\( P_0 \) being an irreducible polynomial in two variables over the field of complex numbers. By Lemma 15.3, the curve given by equation (15.5) is birationally isomorphic to \( \Gamma_0 \).

Theorem 15.1. Assume conditions (i), (iii) on p. 46 Connection between “local” and “global” curves in the periodic case lemma 15.2. Then, equation (15.5), which describes the algebraic dependence between functions \( g_1 \) and \( g_2 \) realizing the equivalences (15.4), (15.5) of divisors on \( \Gamma_0 \), after substituting in it the expressions (15.4) of \( g_1 \) and \( g_2 \) through \( u \) and \( w \) gives exactly the invariant curve of global operator \( A \) on a torus with periods \( (\xi_1, \eta_1) \) and \( (\xi_2, \eta_2) \).

Proof. Note that the equation of curve \( \Gamma \) has as well the form

\[
P(g_1, g_2) = P(u^{\xi_1}w^{\eta_1}, u^{\xi_2}w^{\eta_2}) = 0,
\]

(15.6)

where \( g_1 \) and \( g_2 \) correspond to going around two basis cycles on the torus, see Section 13. We will prove that equations (15.5) and (15.6) define the same curve.

According to the table on p. 46, functions \( g_1 \) and \( g_2 \) have \( l_1 \) and \( l_2 \) poles respectively in \( \Gamma_0 \), where

\[
l_j = 2 \cdot \max\{|\xi_j|, |\eta_j|, |\xi_j - \eta_j|\}, \quad j = 1, 2.
\]

(15.7)

Hence any generic value is taken by a variable \( g_j \) in \( l_j \) points of the curve, so that \( l_j \) values of the other variable correspond to it. This situation can be described by equation (15.5) only if \( P_0 \) has degree \( l_2 \) in \( g_1 \) and degree \( l_1 \) in \( g_2 \).

Passing on to equation (15.6), note that it is natural to suppose that the degrees of \( g_1 \) and \( g_2 \) (but not \( u \) and \( w \)) in it can be negative, if the expansion of the cycle associated with a given trajectory (see Section 13, three paragraphs starting from the one containing formula (13.3)), in basis cycles has one or two negative coefficients. It is not hard to check that the difference between the maximal and minimal degrees of, say, \( g_2 \) does not exceed the number of lines in the triangular lattice of Fig. 3.3 that intersect with a vector with coordinates \( (\xi_1, \eta_1) \) (it is convenient that we imagine it as situated “generically”, e.g. having a point with irrational coordinates as its origin).

Indeed, the difference between the greatest and least intersection numbers of any trajectory going along the edges on the torus always to the right, upwards, or to the north-east, and the basis cycle corresponding to the vector \( (\xi_1, \eta_1) \), cannot exceed, of course, the mentioned number of lines. This
number of lines is, however, nothing else but $t_1$ from formula (15.4), because the vector $(\xi_1, \eta_1)$, evidently, intersects with $[\xi_1, \text{vertical}, [\eta_1, \text{horizontal}, and [\xi_1 - \eta_1]$ oblique lines.

Thus, we see that, having multiplied $P(g_1, g_2)$ (if needed) by its common denominator, we get a polynomial of the same degrees $t_{2,1}$ in variables $g_{1,2}$ as $P_0(g_1, g_2)$. Hence it is clear that the equation (15.5) together with (15.4) yields indeed the whole “global” curve $\Gamma$. The theorem is proved.

The following lemma describing more precisely the structure of $\Gamma$ will be the last in this section. It is here that we will use the condition (ii), not needed for us before.

**Lemma 15.4.** Consider a covering $\Gamma'$ of the curve $\Gamma_0$ defined by multivalued functions $u$ and $w$ according to (15.4). Let the conditions (i) and (ii) on p. 46Connection between “local” and “global” curves in the periodic caselemma.15..2 hold (while we will not use condition (iii) that guarantees that $\Gamma'$ and $\Gamma$ are isomorphic). Then the curve $\Gamma'$ is irreducible.

**Proof.** The system (15.4) defines an extension of the field of meromorphic functions on $\Gamma_0$. This extension is normal, because there exist automorphisms of $\Gamma'$ mapping its any sheet into any other, defined by an evident formula

$$(u, w) \rightarrow (\omega_1 u, \omega_2 w), \quad (15.8)$$

$\omega_1$ and $\omega_2$ being some root of unity. There are $|\xi_1 \eta_2 - \xi_2 \eta_1|$ such automorphisms, and if they all enter in the Galois group, then this group acts transitively on solutions of the system (15.4), and both this system and, hence, the curve $\Gamma'$ are irreducible (see a chapter on Galois theory in the book [72] or [84]). In this case, besides, the extension defined by system (15.4) is $|\xi_1 \eta_2 - \xi_2 \eta_1|$-dimensional over the function field on $\Gamma_0$.

If, otherwise, the curve $\Gamma'$ is reducible, then this extension is less then $|\xi_1 \eta_2 - \xi_2 \eta_1|$-dimensional over the function field on $\Gamma_0$. Consider all possible products $u^\xi w^\eta$ of integer degrees of $u$ and $w$ for $\xi$ and $\eta$ lying within one parallelogram of periods (it is implied that every such parallelogram contains $|\xi_1 \eta_2 - \xi_2 \eta_1|$ integer points, and that one can cover all the plane with such parallelograms, with periods $(\xi_1, \eta_1)$ and $(\xi_2, \eta_2)$, while other details of their disposition are not important). The products $u^\xi w^\eta$ are multiplied under the action of Galois group elements by some characters of that group, and as there are $|\xi_1 \eta_2 - \xi_2 \eta_1|$ products, and fewer characters, then within one parallelogram of periods there are products to which correspond the same character. This makes clear that there exists a product $y_0 = u^\xi w^\eta$ invariant under the Galois group for which $(\xi_0, \eta_0)$ is not an integer linear combination of periods $(\xi_1, \eta_1)$ and $(\xi_2, \eta_2)$. The existence of such function $y_0$ on $\Gamma_0$ means, however, that $(\xi_0, \eta_0)$ is a period, which contradicts to the minimality of parallelogram built on $(\xi_1, \eta_1)$ and $(\xi_2, \eta_2)$. Thus, an assumption of reducibility of $\Gamma'$ has led to a contradiction. The lemma is proved.

Let us sum up the results. In this section, a situation was described when matrices in vertices of the triangular lattice constructed by method of Section 14 depend doubly periodically on the coordinates, so that a model on an infinite lattice becomes a model on a torus. For such a model, a statistical sum exists that is a polynomial in $u$ and $w$ closely connected with the equation of invariant curve of a “global” matrix $A$ composed of matrices situated in lattice vertices. To deal with this situation, we presented, under some restrictions, a principle of constructing a “global” invariant curve (usually, of a very high genus) out of a curve $\Gamma_0$ (that may be, e.g., elliptic). The mentioned restrictions guarantee the irreducibility of algebraic curves whose components otherwise might be lost or confused. A constructive realization of the mentioned principle will be presented in Section 17.

Note that the statement about obtaining the whole $\Gamma$ was deduced without using condition (ii) and Lemma 15.4. We used only the irreducibility of $P$ as a polynomial in $g_1$ and $g_2$. Condition (ii), guaranteeing the irreducibility of $P$ as a polynomial in $u$ and $w$, will not be needed for the explicit calculation of statistical sum in Section 15. It appears however that a not complicated Lemma 15.4 gives a useful information on the curve $\Gamma$. Later we will use this lemma to calculate the number of components of some reducible curves $\Gamma$ (Remark 17.2).
16. Expression of a finite-gap solution in theta functions and a multilinear form of the equation

In this section, we will express the meromorphic functions $f_{jk}$ entering in formulae (14.4–14.6) and defined on the curve $\Gamma_0$, through a theta function $\theta(z)$ defined on the Jacobian $\text{Jac}(\Gamma_0)$ of that curve. Recall (see §1 in Chapter II of the manual [85]) that

$$\theta(z) = \theta(z, \Omega) = \sum_{n \in \mathbb{Z}^{g_0}} \exp(\pi i n^T \Omega n + 2\pi i n^T z), \quad (16.1)$$

where $z$ is a complex column vector of height $g_0$ equal to the genus of curve $\Gamma_0$; the components of vector $n$ of height $g_0$ run through all integer values; and $\Omega$ is the matrix of periods of $\Gamma_0$. We will obtain, from the expressions for $f_{jk}$, very simple expressions for matrix elements of matrices entering in LHS’s of (14.5, 14.6). This, in its turn, will lead us to constructing some analogs of Hirota’s $\tau$-function satisfying a six-linear homogeneous equation. As is known, recently there appear more and more integrable equations for which one cannot construct a bilinear Hirota representation. It is replaced, according to papers [86, 87], by its multilinear analog.

Pass on to a detailed account. As was explained in Section 14, meromorphic functions correspond to edges of the kagome lattice, and out of those functions matrices can be found corresponding to vertices of type $\bigcup$, $\bigcap$ or $\bigcup$, according to the formulae (14.5, 14.6). We will write out a general expression for such functions through the theta function (16.1). Choose one of the lattice edges as “initial” and divide the meromorphic functions on all edges by the function corresponding to the initial edge. Hence, we change all divisors satisfying relations of type

$$(f_{on \ a \ given \ edge}) + D_{on \ a \ given \ edge} \geq 0 \quad (16.2)$$

by a constant divisor.

Now to the initial edge a function satisfies identically equal to unity, and also an effective divisor $D_{ini}$, i.e. a formal sum of $g_0$ points in $\Gamma_0$. According to §3 in Chapter II of the book [85], we can choose a vector $z_0 \in \mathbb{C}^{g_0}$ and a point $P_0 \in \Gamma_0$ in such a way that the multivalued function

$$\varphi(P) = \theta(z_0 + \int_{P_0}^P \omega) \quad (16.3)$$

of a point $P \in \Gamma_0$ will have exactly $D_{ini}$ as its zero divisor. Recall that $\omega$ is a vector composed of holomorphic differentials on $\Gamma_0$.

As for the rest of the edges, to each of them a divisor of type

$$D_{on \ a \ given \ edge} = D_{ini} + \sum_{j=1}^k R_j - \sum_{j=1}^k Q_j \quad (16.4)$$

corresponds, where $k$ is some number (increasing as we are moving off the initial edge), and each of the points $R_j$ and $Q_j$ coincides with one of the points $P_1, \ldots, P_6$. Correctness of (16.4) follows immediately from comparing divisors on neighboring edges, see e.g. Fig. 3.3 (from where it is seen also what the points $R_j$ and $Q_j$ exactly are; we will be concerned with this somewhat later).

The explicit expression for a function satisfying (16.2) with a divisor (16.4) is as follows (necessary explanations are given just below the formula):

$$f_{on \ a \ given \ edge}(P) = \frac{\theta(z_0 + \int_{P_0}^P \omega - \sum_{j=1}^k \int_{Q_j}^P \omega) \prod_{j=1}^k \theta(e + \int_{R_j}^P \omega)}{\prod_{j=1}^k \theta(e + \int_{R_j}^P \omega)} \cdot \frac{\prod_{j=1}^k \theta(e + \int_{Q_j}^P \omega)}{\prod_{j=1}^k \theta(e + \int_{Q_j}^P \omega)}. \quad (16.5)$$
Here \( e \in \mathbb{C}^{\partial_0} \) is any such vector that
\[
\theta(e) = 0,
\]
while the function
\[
E_e(x,y) = \theta(e + \int_x^y \omega)
\]
of two points \( x, y \in \Gamma \), called \textit{principal form}, does not identically vanish. It remains to indicate the integration paths in (16.5), more precisely, their homological classes. Connect all points \( R_j \) and \( Q_j \), \( 1 \leq j \leq k \), and also \( P \), with \( P_0 \) by arbitrary paths. This done, we will assume that each path \( Q_jR_j \) consists of two parts \( Q_jP_0 \) and \( P_0R_j \), and, similarly, that paths \( R_jP \) and \( Q_jP \) also “pass through the point \( P_0 \).”

One can verify by standard means that (16.5) defines a single-valued function on the curve \( \Gamma_0 \), so that the choice of path \( P_0P \) in fact plays no rôle. As for the equality (16.2), it follows from the fact that all “superfluous” zeros of principal forms in the numerator and denominator of the right-hand fraction in (16.5) cancel each other \([85]\).

Now pass on to calculating matrix elements corresponding to transitions between neighboring edges of kagome lattice, by formulas like (14.5, 14.6), indicating \textit{en passant} concretely points \( R_j \) and \( Q_j \) in formulas (16.4, 16.5). To begin, consider a situation depicted in Fig. 3.10. This figure is, in essence, a modification of Fig. 3.9. The thick edge is the initial one. Near other edges, the divisors \( \sum_{j=1}^k R_j - \sum_{j=1}^k Q_j \) corresponding to them are written out (recall that the poles of functions are situated in points \( R_j \)) while the zeros—in points \( Q_j \). The system (14.3), after substituting the functions according to (16.5) and Fig. 3.10 and multiplying by the common denominator, acquires the following form:

\[
\begin{pmatrix}
a_1 & b_1 \\
c_1 & d_1
\end{pmatrix}
\begin{pmatrix}
\theta(z_0 + \int_{P_0}^P \omega) \theta(e + \int_{P_2}^P \omega) \\
\theta(z_0 + \int_{P_0}^P \omega - \int_{P_3}^P \omega) \theta(e + \int_{P_3}^P \omega)
\end{pmatrix}
\]
$$\theta(z_0 + \int_{p_0}^{p_1} \omega - \int_{p_3}^{p_4} \omega) \theta(e + \int_{p_4}^{p_5} \omega)$$

$$\theta(z_0 + \int_{p_0}^{p_1} \omega - \int_{p_3}^{p_4} \omega) \theta(e + \int_{p_4}^{p_5} \omega)$$

(16.7)

The equality (16.7) must hold for any point $P \in \Gamma_0$. Recalling (16.6), we see that $a_1$ and $c_1$ are easily found if we set $P = P_3$, while $b_1$ and $d_1$ — if we set $P = P_2$. It is convenient to introduce notations

$$z(P) = \int_{p_0}^{p_1} \omega, \quad z_j = \int_{p_0}^{p_j} \omega, \quad 1 \leq j \leq 6.$$ 

We get

$$a_1 = \frac{\theta(z_0 + z_3 - z_2 + z_4) \theta(e + z_3 - z_4)}{\theta(z_0 + z_3) \theta(e + z_3 - z_2)},$$

(16.8)

$$b_1 = \frac{\theta(z_0 + z_4) \theta(e + z_2 - z_4)}{\theta(z_0 + z_3) \theta(e + z_2 - z_3)},$$

(16.9)

$$c_1 = \frac{\theta(z_0 + z_3 - z_2 + z_5) \theta(e + z_3 - z_5)}{\theta(z_0 + z_3) \theta(e + z_3 - z_2)},$$

(16.10)

$$d_1 = \frac{\theta(z_0 + z_5) \theta(e + z_2 - z_5)}{\theta(z_0 + z_3) \theta(e + z_2 - z_3)}.$$ 

(16.11)

Matrix elements $a_2, \ldots, d_3$ are found similarly. We will write out for them only systems of type (16.7), because explicit expressions like (16.8) (16.11) are obtained from them in an obvious way by using (16.6). So,

$$\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \begin{pmatrix} \theta(z_0 + z(P) - z_6 + z_1) \theta(e + z(P) - z_1) \\ \theta(z_0 + z(P) - z_6 + z_3) \theta(e + z(P) - z_3) \end{pmatrix} =$$

$$= \begin{pmatrix} \theta(z_0 + z(P)) \theta(e + z(P) - z_6) \\ \theta(z_0 + z(P) - z_6 + z_5) \theta(e + z(P) - z_5) \end{pmatrix},$$

(16.12)

$$\begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix} \begin{pmatrix} \theta(z_0 + z(P) - z_2 - z_6 + z_3 + z_1) \theta(e + z(P) - z_1) \\ \theta(z_0 + z(P) - z_2 - z_6 + z_3 + z_4) \theta(e + z(P) - z_4) \end{pmatrix} =$$

$$= \begin{pmatrix} \theta(z_0 + z(P) - z_2 + z_3) \theta(e + z(P) - z_6) \\ \theta(z_0 + z(P) - z_6 + z_3) \theta(e + z(P) - z_2) \end{pmatrix}.$$ 

(16.13)

No we can extend Figure 3.10 in any direction and calculate matrix elements in some more vertices. Fortunately, not complicated calculations like those done above show that the formulae (16.7) (16.13) remain valid under translations by lattice periods, if a change

$$z_0 \rightarrow z_0 + \xi(P_1 + P_5 - P_1 - P_2) + \eta(P_2 + P_5 - P_3 - P_4),$$

(16.14)

is done in them, where the integers $\xi$ and $\eta$ show how many lattice periods horizontally and vertically we have moved off the “initial” triangle in Fig. 3.10.

Perfectly similar considerations, with using the divisor evolution from Section 14, are valid for the dependence of matrix elements on the time. The result consists, of course, in adding in (16.14) a linear dependence of $z_0$ on the time $\tau$. We will formulate this as follows: if divisors on edges around two vertices of the same type (for example, of type \(\rightarrow\)) differ by $\sum_{j=1}^{l} R_j - \sum_{j=1}^{l} Q_j$, then

the corresponding values $z$ differ by $\sum_{j=1}^{l} \int_{R_j}^{Q_j} \omega$.  

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Let us end this section by some simple remarks about an analog of Hirota’s bilinear representation for our dynamical system. As was indicated in the beginning of this section, only a six-linear representation could be found, which is obtained if we write the entries of matrices like

\[
\begin{pmatrix}
  a_1 & b_1 \\
  c_1 & d_1
\end{pmatrix}
\]

in the form

\[
a_1 = \frac{\alpha_1}{t_1}, \quad b_1 = \frac{\beta_1}{t_1}, \quad c_1 = \frac{\gamma_1}{t_1}, \quad d_1 = \frac{\delta_1}{t_1}
\]

etc. and then multiply every equality of the form

\[
\begin{pmatrix}
  a_1' & b_1' \\
  c_1' & d_1'
\end{pmatrix}
\]

by the following. Recall the formulae (16.8–16.11), where only \(z\) and \(t\) are free variables. The reasonableness of the obtained equation is substantiated by the following. Recall the formulae (16.8–16.11), where only \(z_0\) depends on coordinates and time, and set \(t_1 = \theta(z_0 + z_3)\). Similarly, choose the rest of \(t_j\) equal to theta functions of \(z_0 + \text{const}\) situated in the denominators of formulae of type (16.5–16.11). We get for each of the values \(\alpha_j, \beta_j, \gamma_j, \delta_j, t_j\) that it is in the “finite-gap” case a theta function multiplied by a harmless constant, which is a distinguishing feature of a Hirota \(\tau\)-function.

17. Invariant curve equation in the thermodynamical limit

As soon as we began in Section 14 to consider an inhomogeneous 6-vertex model on the infinite lattice whose Boltzmann weights are constructed out of an algebraic curve \(\Gamma_0\) (of not very high genus) and some divisors in it and depend on coordinates quasiperiodically, we naturally come to the problem of existence and calculation of the thermodynamic limit of specific free energy, i.e. the limit of the logarithm of statistical sum divided by lattice area. In this work, some steps in this direction are proposed. Just now we will continue to consider the double periodic case of Section 14 assuming conditions (15.1, 15.2) of linear dependence with integer coefficients for divisors \(P_i - P_6\), \(P_2 - P_3\) and \(P_3 - P_5\). The vectors of periods \((\xi_1, \eta_1)\) and \((\xi_2, \eta_2)\) can be, however, as big as desired.

We will see that there exists an elegant integral representation for the real part of the limit of the ratio of the logarithm of determinant (16.3) to the area of the lattice “covering” the lattice on the torus with periods \((\xi_1, \eta_1)\) and \((\xi_2, \eta_2)\), with both dimensions of the covering lattice tending to infinity. It follows from Theorem (13.1) on the “physical level of rigor” that this limit coincides with the specific free energy (see also Remark 13.1). We will present some confirmation for this in a simple particular case in the concluding part of this section, while now returning to the mathematical level of rigor.

Let meromorphic functions \(g(z)\) and \(h(z)\) be given on a smooth irreducible algebraic curve \(\Gamma_0\). Then those functions obey an algebraic dependence

\[
P(g, h) = 0,
\]

\(P\) being an irreducible polynomial. Standard methods of the theory of functions of a complex variable provide in a not complicated way an integral representation for the absolute value of \(P(g, h)\).

Introduce the following notations: \(k\) and \(l\) will be the numbers of poles, with regard to their multiplicities, of functions \(g(z)\) and \(h(z)\) respectively. By \(g_j(h_0)\), where \(1 \leq j \leq l\), we will denote the values of function \(g\) in those points where \(h(z) = h_0\) (we take those points in any order and also with regard to their multiplicities). Similarly, \(h_j(g_0)\), with \(1 \leq j \leq k\), will be the values of \(h\)
in points where \( g(z) = g_0 \). The following integral representation (a version of the Cauchy integral formula) takes place:

\[
2 \sum_{j=1}^l \ln \left| \frac{g_0 - g_j(h_0)}{g_0 - g_j(\infty)} \right| = 2 \sum_{j=1}^k \ln \left| \frac{h_0 - h_j(g_0)}{h_0 - h_j(\infty)} \right| = \frac{1}{2\pi i} \int_{\Gamma_0} \frac{dg(z) \wedge dh(z)}{(g(z) - g_0)(h(z) - h_0)}. \tag{17.2}
\]

Let us prove, for the sake of completeness, the formula \( (17.2) \). Consider a real-valued function \( \varphi(z) = 2 \ln |g(z) - g_0| \) defined on \( \Gamma_0 \), except, of course, the points where \( g(z) = g_0 \) or \( \infty \). As the holomorphic part of the function \( \varphi \) differential coincides with \( dg(z)/(g(z) - g_0) \), the integral in \( (17.2) \) can be rewritten as

\[
\frac{1}{2\pi i} \int_{\Gamma_0} \frac{d\varphi(z) \wedge dh(z)}{h(z) - h_0}. \tag{17.3}
\]

By Stokes theorem, the integral \( (17.3) \) equals

\[
- \frac{1}{2\pi i} \sum_{\text{singular points}} \oint_{\partial \Omega} \varphi(z) \frac{dh(z)}{h(z) - h_0},
\]

where the sum is taken over all singular points of the integrand in \( (17.3) \), while the integrals under summation sign—along infinitesimal contours around those points in the anticlockwise direction. As the singularities of function \( \varphi \) have the character of modulus of local parameter logarithm, the integrals along infinitesimal contours around them vanish unless the singularities of \( \varphi \) coincide with the points where \( h(z) = h_0 \) or \( \infty \). The integrals around those points, divided by \(-2\pi i\), equal \( 2 \ln |g_j(h_0) - g_0| \) and \( 2 \ln |g_j(\infty) - g_0| \) respectively. Hence

\[
2 \sum_{j=1}^l \ln \left| \frac{g_0 - g_j(h_0)}{g_0 - g_j(\infty)} \right| = \frac{1}{2\pi i} \int_{\Gamma_0} \frac{dg(z) \wedge dh(z)}{(g(z) - g_0)(h(z) - h_0)}. \tag{17.4}
\]

The remaining equality in \( (17.2) \) follows from the obvious equalness in status of functions \( g \) and \( h \).

To link the equalities \( (17.2) \) to the polynomial \( P(g, h) \) from formula \( (17.1) \), note that

\[
\lim_{g' \to \infty, h' \to \infty} \frac{P(g, h)P(g', h')}{P(g', h)P(g, h')} = \prod_{j=1}^l \frac{g - g_j(h)}{g - g_j(\infty)} = \prod_{j=1}^k \frac{h - h_j(g)}{h - h_j(\infty)}. \tag{17.4}
\]

Equalities \( (17.4) \) are proved by directly comparing the zeros and poles of LHS and RHS’s in variable \( g \) or \( h \). Let \( P(g, h) \) be normalized so that the coefficient at the term of highest degrees in both variables, i.e. \( g'h^k \), equals unity. We have

\[
\frac{P(g', h)}{(g')^l} \xrightarrow{g' \to \infty} \prod_{j=1}^k (h - h_j(\infty)), \tag{17.5}
\]
\[
\frac{\mathcal{P}(g,h')}{(h')^k} \xrightarrow{h' \to \infty} \prod_{j=1}^l \frac{1}{g_j(\infty)}. \tag{17.6}
\]

Formulae (17.2) and (17.4–17.6) together yield
\[
|\mathcal{P}(g_0, h_0)|^2 = \exp \left( \frac{1}{2\pi i} \int_\Gamma \frac{dg(z) \wedge d\overline{h(z)}}{(g(z) - g_0)(h(z) - h_0)} \right) \times \prod_{j=1}^k (h_0 - h_j(\infty)) \prod_{p=1}^l (g_0 - g_p(\infty)) \right|^2. \tag{17.7}
\]

We will see later that, in order to pass to a “thermodynamical limit” in formula (17.7), we must consider the following situation. Let functions \(g(z)\) and \(h(z)\) on \(\Gamma_0\) be represented in the form
\[
g(z) = u^\xi, \quad h(z) = w^\eta, \tag{17.8}
\]
where \(\xi\) and \(\eta\) are integers, while \(u\) and \(w\) are functions on some covering of \(\Gamma_0\) (this can always be done, of course). Let \(a\) and \(b\) be positive integers divisible by \(\xi\) and \(\eta\) respectively. Introduce functions
\[
G(z) = u^a = g(z)^{\xi/a}, \quad H(z) = w^b = h(z)^{\eta/b}. \tag{17.9}
\]

Consider a limit
\[
\lim_{a \to \infty} \int_\Gamma \frac{dG(z) \wedge d\overline{H(z)}}{(G(z) - G_0)(H(z) - H_0)}, \tag{17.10}
\]
where
\[
G_0 = u_0^a, \quad H_0 = u_0^b \tag{17.11}
\]
with some \(u_0\) and \(w_0\) not depending on \(a\) and \(b\). Changing the order of passage to limit and integration (the legitimacy of which operation is verified by standard means), we find that the limit (17.10) equals
\[
\int_\Delta d\ln u(z) \wedge d\ln w(z), \tag{17.12}
\]
\(\Delta\) being that part of \(\Gamma_0\) where
\[
|u| \geq |u_0|, \quad |w| \geq |w_0|. \tag{17.13}
\]

Let us now link the above considerations to Section 15 in the following way. Let us take functions \(g_1(z)\) and \(g_2(z)\) from Section 15 as \(g(z)\) and \(h(z)\) respectively. Assume for simplicity that in formulae (15.1, 15.2)
\[
\eta_1 = \xi_2 = 0, \tag{17.14}
\]
and denote
\[
\xi = \xi_1, \quad \eta = \eta_2.
\]
Thus, the torus becomes “rectangular”, with periods \((\xi, 0)\) and \((0, \eta)\), while formulae (15.4) turn into (17.8). Assume also conditions (i–iii) on p. 46Connection between “local” and “global” curves in the periodic case lemma.15.2, observing that the last of them will mean just that \(\xi\) and \(\eta\) are relatively prime.

It follows from the table on p. 46 and formula (17.14) that \(g_1(z)\) has zeros of multiplicity \(\xi\) in points \(P_1\) and \(P_2\), and poles of multiplicity \(\xi\) in \(P_3\) and \(P_5\), while \(g_2(z)\) has zeros of multiplicity \(\eta\) in points \(P_3\) and \(P_5\), and poles of multiplicity \(\eta\) in \(P_2\) and \(P_5\). Hence, it is clear that functions \(u\) and \(w\) on a \(\xi\eta\)-sheeted nonramified covering of \(\Gamma_0\) possess the following zeros and poles of multiplicity one:
Remark 17.1. Functions \( u \) and \( w \), thus, can be expressed in a simple way through principal forms on \( \Gamma_0 \), but we will not need those expressions here.

As will be clear soon, the limit
\[
\lim_{a \to \infty} \lim_{b \to \infty} \frac{1}{ab} \ln |\tilde{P}(G_0, H_0)|,
\]
is important for us. Here the polynomial \( \tilde{P}(G,H) \) defines the algebraic dependence between functions (17.9) which can be written as
\[
\tilde{P}(G,H) = \frac{a}{\xi} \prod_{p=1}^{a/\xi} \frac{b}{\eta} \prod_{q=1}^{b/\eta} P(g \exp \frac{2\pi ip\xi}{a}, h \exp \frac{2\pi i q\eta}{b}) = 0.
\]
(17.15)

Substituting \( G \) and \( H \) in place of \( g \) and \( h \) in (17.7), taking into account the equality of expressions (17.10) and (17.12), and knowing the poles of functions \( G(z) \) and \( H(z) \) (see (17.9) and the table before Remark 17.1), we find:
\[
\lim_{a \to \infty} \lim_{b \to \infty} \frac{1}{ab} \ln |\tilde{P}(G_0, H_0)| =
\]
\[
= \frac{1}{4\pi i} \int_{\Delta} d \ln u(z) \wedge d \ln w(z) + \ln |u_0| + \ln |w_0| +
\]
\[
+ \ln (\max\{|u_0|, |u(P_5)|\}) + \ln (\max\{|w_0|, |w(P_4)|\}),
\]
(17.16)
where domain \( \Delta \subset \Gamma_0 \) is, as before, defined by inequalities (17.13), while \( u_0 \) and \( w_0 \), of course, obey (17.11).

Pass now on to explaining why the polynomial (17.15) arises when we take a lattice on the torus \( a \times b \) that covers torus \( \xi \times \eta \).

Theorem 17.1. Let the invariant curve corresponding to the model on “rectangular” torus of sizes \( \xi \times \eta \) be given by equality
\[
\mathcal{P}(g, h) = \mathcal{P}(u^\xi, w^\eta) = 0.
\]
(17.17)
Let numbers \( a \) and \( b \) be divisible by \( \xi \) and \( \eta \) respectively. Then the invariant curve corresponding to model on the lattice on torus \( a \times b \), with matrix elements in the vertices “raised” from the lattice \( \xi \times \eta \) according to the natural covering, is given by equation (17.15), where (17.8) must be substituted.

Remark 17.2. Thus, the invariant curve in this case consists of \( \frac{ab}{\xi \eta} \) copies of the covering of \( \Gamma_0 \) defined by multifunctions \( u \) and \( w \) (17.8). Recalling conditions (i–ii) on p. 46 Connection between “local” and “global” curves in the periodic case, we see that under these conditions the invariant curve contains exactly \( \frac{ab}{\xi \eta} \) components.

Proof of Theorem 17.1. Recall that an invariant curve is birationally isomorphic to a set of such pairs \( (u, w) \) for which there exists a vector \( X = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \) such that
\[
\mathcal{A} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} uX \\ uwY \\ wZ \end{pmatrix}
\]
(17.18)
(see Definition 9.1 and formula (9.3) after it). If we deal with the torus $a \times b$, then each of column vectors $X$, $Y$, and $Z$ has $ab$ components. Recall that when considering equalities of type \((17.18)\), we pass from the kagome lattice to the triangular one, “packing” triangles in circles, as was explained in Section 13. Let \((x, y)\) be integer-valued coordinates of a given vertex of triangular lattice. The equality \((17.18)\) remains valid under the transformation

$$X \to \exp(\frac{2\pi i c_1}{a}x)X,$$

$$Y \to \exp(\frac{2\pi i c_1}{a}x + \frac{2\pi i c_2}{b}y)Y,$$

$$Z \to \exp(\frac{2\pi i c_2}{b}y)Z; \quad (17.19)$$

where $c_1$ and $c_2$ are any integers. If matrix elements of $A$ were raised from the lattice $\xi \times \eta$, then among the vectors $X$ there are periodic ones with periods $(\xi, 0)$ and $(0, \eta)$, and pairs $(u, w)$ corresponding to such vectors form the curve \((17.17)\). Clearly, all vectors $X$ must be periodic with periods $(a, 0)$ and $(0, b)$, and they all can be obtained from vectors with periods $(\xi, 0)$ and $(0, \eta)$ by transformation \((17.19)\) with proper $c_1$ and $c_2$. It is not difficult to see that the corresponding transformations \((17.20)\) of pairs $(u, w)$ provide exactly $\frac{\xi\eta}{\xi_0^2}$ isomorphic components of the invariant curve according to \((17.15)\). The theorem is proved.

Note that “integral of motion” $I(u, w)$ \((13.2)\) from Section 13 differs from polynomial $\tilde{P}(G, H)$ by a factor whose absolute value is $|uw|^{-2ab}$ (see \((17.15)\) and a remark about normalization of $P(g, h)$ before formula \((17.5)\)). Hence we will get

$$\lim_{a \to \infty} \lim_{b \to \infty} \frac{1}{ab} \ln |I(u_0, w_0)|$$

if we subtract $(2 \ln |u_0| + 2 \ln |w_0|)$ from the RHS of \((17.16)\). Thus, the desired thermodynamic limit has been calculated.

**Remark 17.3.** Logarithmic differentials of multifunctions $u(z)$ and $w(z)$ on the curve $\Gamma_0$, that occupy a key position in formula \((17.16)\), astonishingly belong to the same type as differentials playing a key role in the algebro-geometrical string theory \([88, 89, 90]\). Namely, they are differentials of the third kind (i.e. possessing only simple poles) on $\Gamma_0$, with all their periods pure imaginary (because absolute values of $u(z)$ and $w(z)$ are single-valued). The algebraic techniques used in this work forced us, however, to impose the integer linear dependence conditions \((15.1, 15.2)\) on the divisors.

Besides, formula \((17.16)\) reminds of string theory because the double integral in it can be interpreted as the area of domain $\Delta$ in Kähler metric corresponding to imaginary part of the 2-form $d \ln u(z) \wedge d \ln w(z)$ (changing the form to its imaginary part does not change the integral in \((17.14)\), because both sides in that formula are real). As is known, the “Nambu–Hara–Goto” action in string theory is proportional to the “world sheet” area \([91]\).

In the remaining part of this section we will show how our method works in case of a homogeneous model, choosing a curve of genus 0 as $\Gamma_0$:

$$\Gamma_0 = CP^1. \quad (17.21)$$

Clearly, conditions \((15.1, 15.2)\) of divisor equivalence in this case always satisfied, and any integer vector is a period (in the sense of Lemma \((15.3)\)).

To simplify the task still more, we will consider a model on square lattice. We will obtain it by “removing oblique lines” from the triangular lattice. For this purpose, we will set

$$P_2 = P_4 \quad (17.22)$$
and see how this will affect the form of matrix $A_{\text{local}}$ corresponding to a vertex of Fig. 3.8 type and determined from the relation (14.4). The latter is worth being written out here once more:

$$A_{\text{local}} \begin{pmatrix} f_{12} \\ f_{13} \\ f_{34} \end{pmatrix} = \begin{pmatrix} f_{46} \\ f_{56} \\ f_{52} \end{pmatrix}$$

(17.23)

(recall that divisors $D_1, \ldots, D_6$ on Fig. 3.8 consist each of one point $P_1, \ldots, P_6$, and each meromorphic function $f_{jk}$ equals zero in points $P_j$ and $P_k$).

**Lemma 17.1.** Assume condition (17.22) and normalize functions $f_{jk}$ on all oblique edges by a condition

$$f_{jk}(P_2) = 1.$$

Then, matrices $A_{\text{local}}$ in all vertices of triangular lattice have the form

$$A_{\text{local}} = \begin{pmatrix} a & 0 & c \\ d & 1 & g \\ h & 0 & k \end{pmatrix}.$$  

(17.24)

**Proof** follows immediately from considering relation (17.23) in the point $P_2$, taking into account the fact that function $f_{jk}$ equals zero in that point if 2 or 4 is contained among the numbers $j$ and $k$, while otherwise equals unity. Note that we do not use conditions (17.21) here.

It follows from the form (17.24) of matrix $A_{\text{local}}$ that every closed path on triangular lattice to which a zero weight ( = product of all matrix elements, associated with that path, of matrices $A_{\text{local}}$) corresponds, goes either entirely through vertical and horizontal edges or is just an oblique line (wound around the torus, of course). The determinant $I(u, w)$ (13.2) factorizes in a product of a factor corresponding to paths through the square lattice and depending on $a, c, h, k$ and, certainly, on $u$ and $w$, and a factor corresponding to oblique paths and depending only on $u$ and $w$.

It follows from the table on p. 55 together with (17.22) that functions $u$ and $w$ on $\Gamma_0 = CP_1$ have each one zero and one pole. The matrix $\begin{pmatrix} a & c \\ h & k \end{pmatrix}$ is obtained from a relation like the first of relations (14.6), which in our case can be written as

$$\begin{pmatrix} a & c \\ h & k \end{pmatrix} \begin{pmatrix} f_{12} \\ f_{23} \end{pmatrix} = \begin{pmatrix} uf_{12} \\ wf_{23} \end{pmatrix},$$

whence

$$\left| \begin{array}{cc} a - u & c \\ h & k - w \end{array} \right| = 0.$$  

(17.25)

We will simplify our task still more by considering the “usual” 6-vertex model, assuming conditions

$$a = k, \quad c = h, \quad c^2 - a^2 = 1$$

(see Section 13 the text between Lemmas 13.1 and 13.2 on the connection between Boltzmann weights and matrix elements). Then (17.23) turns into

$$u = \frac{aw + 1}{w - a}.$$  

(17.26)

Let us assume that $u_0$ and $w_0$ in (17.16) are real and positive. For concreteness, let $u_0$ and $w_0$ be not very different from unity, and

$$-1 < a < 1.$$

The integration domain $\Delta$ in (17.16) for this case (a lune between two circumferences) is depicted, in the complex plane of variable $w$, in Fig. 3.11.
Figure 3.11. Integration domain $\Delta$ in the double integral in formula (17.16) in case of a homogeneous model on square lattice, depicted in the complex plane of variable $w$.

Transform the double integral in (17.16) into a single integral. We have

$$
\int \int \Delta \, d \ln u \land d \ln w = \int \int \Delta \, d \ln \left| \frac{u}{u_0} \right|^2 \land d \ln w = \\
\oint_{\partial \Delta} \ln \left| \frac{u}{u_0} \right|^2 \, d \ln w = \int_{-\phi_0}^{\phi_0} \ln \left| \frac{u}{u_0} \right|^2 \, d \ln w. \tag{17.27}
$$

Here we used the fact that $\ln \left| \frac{u}{u_0} \right|^2$ is a single-valued function, moreover equalling zero in the arc $CDA$; $\partial \Delta$ is the boundary of $\Delta$, i.e. $DCABC$. Divide the integral (17.27) by $4\pi i$, as required in formula (17.16), and transform it further, using, in particular, (17.26):

$$
\frac{1}{4\pi i} \int_{-\phi_0}^{\phi_0} \ln \left| \frac{u}{u_0} \right|^2 \, d \ln w = \int_{-\phi_0}^{\phi_0} \ln \left| \frac{aw_0e^{i\phi} + 1}{u_0(w_0e^{i\phi} - a)} \right| \, d\phi, \tag{17.28}
$$

where $\phi_0$ found from conditions

$$
0 < \phi_0 < \pi, \quad \left| \frac{aw_0e^{i\phi} + 1}{u_0(w_0e^{i\phi} - a)} \right| = 1.
$$

The integral (17.28) is the essential, i.e. depending on $a$, part of the limit in the LHS of (17.16). Of course, it coincides, to within an additive constant (to be exact, a function depending only on $u_0$ and $w_0$), with the specific free energy of the homogeneous “free fermionic” 6-vertex model on the square lattice, see e.g. references [76, 43].
18. A “local” approach to orthogonal matrices

In this section we will show what symmetry conditions must be imposed on the “local” algebro-geometrical objects from Section 14 in order to obtain the description of evolution of orthogonal matrices. In contrast to direct calculations of Section 11, here we simply present constructions made in the spirit of works [64, 65, 66, 67], and prove their necessary properties.

Let, as in Section 14, there be given an algebraic curve $\Gamma_0$ and six divisors in it consisting each of one point, namely $P_1, \ldots, P_6$. Let, besides, the curve $\Gamma_0$, like the “global” curve $\Gamma$ from Section 11, possess an involution under which $P_1, \ldots, P_6$ are transformed as follows: $P_1 \leftrightarrow P_6$, $P_2 \leftrightarrow P_5$, $P_3 \leftrightarrow P_4$. Consider meromorphic differentials $\psi$ on $\Gamma_0$ possessing the following properties: a) $\psi$ has poles (of not higher order than 1) only in points $P_1, \ldots, P_6$ and b) $\psi$ transforms into $-\psi$ under the action of involution $I$. It follows from those properties and from the fact that the dimension of the linear space of holomorphic differentials on $\Gamma_0$ equals $g_0$ (the genus of the curve) that the dimension of linear space of differentials $\psi$ equals $g_0 + 3$.

Fix now a concrete differential $\psi$ with non-zero residues in all points $P_1, \ldots, P_6$, and consider a divisor $D$ on $\Gamma_0$ such that $D + D^I$ is exactly the zero divisor of $\psi$ (here $D^I$ is the image of $D$ under involution $I$). Thus,

$$D + D^I \sim D_{\text{can}} + P_1 + \ldots + P_6,$$

(18.1)

where $D_{\text{can}}$ is any canonical divisor on $\Gamma_0$. The relation (18.1) is a “local” analog of relation (11.14). Divisor $D$ consists of $g_0 + 2$ points, while the dimension of linear space of meromorphic functions $f$ satisfying condition

$$(f) + D \geq 0,$$

(18.2)

generically, equals 3.

Define the scalar product $\langle f, g \rangle$ of two functions satisfying (18.2) as the sum of residues of differential $fg^I\psi$ in points $P_1, P_2, P_5$ ($g^I$ is, of course, the image of $g$ under evolution $I$):

$$\langle f, g \rangle = \sum_{P_1, P_2, P_5} \text{Res} fg^I\psi.

(18.3)

We will study the properties of this scalar product step by step. First let us verify that it is symmetric, i.e. that

$$\langle f, g \rangle = \langle g, f \rangle.

(18.4)

Apply involution $I$ to all objects entering in RHS of (18.3), which will cause no change to that RHS as a whole. We get:

$$\langle f, g \rangle = \sum_{P_1, P_2, P_3} \text{Res} (-f^Ig^I) = \sum_{P_1, P_2, P_3} \text{Res} f^Ig^I\psi.

(18.5)

The right-hand equality in (18.5) follows from the fact that the sum of differential $f^Ig^I\psi$ residues, taken over all its poles $P_1, \ldots, P_6$, equals zero. Further, the rightmost side of (18.5) obviously equals $\langle g, f \rangle$, so (18.4) is proved.

Consider now scalar products of functions $f_{jk}$ satisfying conditions

$$f_{jk} + D - P_j - P_k \geq 0.

(18.6)

In other words, function $f_{jk}$ has a pole divisor $D$ and must have zeros in points $P_j$ and $P_k$. We will prove the equalities

$$\langle f_{26}, f_{36} \rangle = \langle f_{56}, f_{46} \rangle = 0,

(18.7)

$$\langle f_{12}, f_{23} \rangle = \langle f_{26}, f_{52} \rangle = 0,

(18.8)

$$\langle f_{13}, f_{34} \rangle = \langle f_{36}, f_{23} \rangle = 0.

(18.9)

Let us prove, for example, that $\langle f_{26}, f_{36} \rangle = 0$. We have $f_{36} = f_{51}$, so meromorphic differential $f_{26}f_{36}\psi$ has no poles at all in points $P_1, P_2, P_5$, whence the scalar product (18.3) indeed equals 0. The remaining equalities (18.7–18.9) are proved similarly.
If now we manage to normalize all functions $f_{jk}$ entering in equalities (18.7–18.9) so that
\[ \langle f_{jk}, f_{jk} \rangle = 1, \]
and turn to relations (14.5, 14.6) defining “local” matrices \( \begin{pmatrix} a_l & b_l \\ c_l & d_l \end{pmatrix} \) for \( l = 1, 2, 3 \), we will see that those matrices are orthogonal ((18.7), (18.8) and (18.9) are responsible, together with (18.10), for orthogonality of local matrices with \( l = 1, 2 \) and 3 respectively). At the same time, (18.10) will show the non-degeneracy of scalar product (18.3). Thus consider, e.g., a scalar square of function $f_{12}$. We have $f_{12}^I = f_{64}$, so there remains only a residue in point $P_5$ in formula (18.3). Generically, $f_{12}(P_3)f_{64}(P_5) \neq 0$, so this residue does not vanish, and we can divide $f_{12}$ by its square root.

Similarly, the scalar squares of other functions entering in (18.7–18.8) are non-zero, hence those functions can be normalized according to (18.10).

As a result, we have proved so far the orthogonality of three matrices \( \begin{pmatrix} a_l & b_l \\ c_l & d_l \end{pmatrix} \) defined by relations (14.5, 14.6) and corresponding to vertices of some chosen triangle in kagome lattice, depicted in Fig. 3.9 (p. 13). To all (nine) edges in Fig. 3.9 meromorphic functions corresponded whose divisors satisfied equality (18.2). We have introduced a scalar product in the three-dimensional space of such functions by formula (18.3), and normalized the functions according to (18.10).

We can deal similarly with any other triangle of the form of Fig. 3.9. If such a triangle is \( \xi \) horizontal lattice periods and \( \eta \) vertical periods far from the one that we have considered, then divisor $D$ must be replaced by
\[ D_{(\xi,\eta)} = D + \xi (P_1 + P_2 - P_4 - P_6) + \eta (P_3 + P_4 - P_5 - P_2). \]
Remarkably, the scalar product in the space of meromorphic functions $f$ such that
\[ (f) + D_{(\xi,\eta)} \geq 0 \]
may (and must) be introduced again by formula (18.3) with the same differential $\psi$ as for functions satisfying (18.2). The point is that “superfluous” poles of function $f$ are exactly compensated by zeros of function $g^I$ and vice versa, so that differential $fg^I \psi$ can, as before, have only first order poles in points $P_1, \ldots, P_6$.

The analogs of orthogonality relations (18.7–18.9) for all lattice vertices are proved by as before.

We will formulate the results of this section as a following theorem.

**Theorem 18.1.** If a curve $\Gamma_0$ in the situation of Section 14 possesses an involution $I$ mapping the points $P_1, P_2, P_3$ in $P_5, P_4, P_5$ respectively, while a divisor $D$ satisfies condition (18.1), then under a proper gauge all square matrices in LHS’s of equalities (14.4–14.6) are orthogonal.

**Proof.** It remains to give two simple clarifications. First, the gauge mentioned in the theorem is fixed by normalization conditions of type (18.10). Second, we considered in this section only a situation at one moment of discrete time. However, a simple calculation based on divisor evolution described between Lemmas 14.1 and 14.2 shows that our considerations are valid for any time (“superfluous” poles of each function in (18.3) are compensated by zeros of another function as well as before).

### 19. Reduction to Ising model

It is known that Onsager’s “star–triangle” transformation ([63], see also manual [76]), which can be graphically represented as
\[ \bigcirc \rightarrow \bigtriangleup. \]

(19.1)
converts the statistical mechanical Ising model on a plane hexagonal lattice to the Ising model on a triangular lattice. Imagine now the obtained triangular lattice as made up of triangles of the form \( \nabla \). If we apply the triangle–star
\[ \nabla \rightsquigarrow \Upsilon, \quad (19.2) \]
transformation to those triangles, we will come to hexagonal lattice again. If Ising model is inhomogeneous, i.e. a coupling along a given edge depends on the edge, then transformations \((19.1)\) and \((19.2)\) applied alternately lead to some evolution of those coefficients on alternating hexagonal and triangular lattices. A hypothesis looks natural that this evolution is completely integrable.

In this, not very large, section we will show that the mentioned evolution of Ising model coupling coefficients is isomorphic to the evolution of orthogonal matrices in vertices of triangular lattice. The latter evolution was considered from a “local” algebro-geometrical viewpoint in Section 18.

The direct way to Ising model turns out to go via considering the orthogonal matrices \( v^+ \) and \( v^− \) of the form
\[ \bigg( \begin{array}{c} a_1 \ b_1 \\ b_1 \ -a_1 \\ 0 \ 0 \ 1 \end{array} \bigg) \quad \bigg( \begin{array}{c} a_2 \ 0 \ b_2 \\ 0 \ 1 \ 0 \\ b_2 \ 0 \ -a_2 \end{array} \bigg) \quad \bigg( \begin{array}{c} 1 \ 0 \ 0 \\ 0 \ a_3 \ b_3 \\ 0 \ b_3 \ -a_3 \end{array} \bigg) =
\]
\[ \bigg( \begin{array}{c} a_1' \ 0 \ b_1' \\ 0 \ a_3' \ b_3' \ b_3' \ -a_3' \end{array} \bigg) \quad \bigg( \begin{array}{c} a_2' \ 0 \ b_2' \\ 0 \ 1 \ 0 \\ b_2' \ 0 \ -a_2' \end{array} \bigg) \quad \bigg( \begin{array}{c} a_1'' \ b_1'' \ 0 \\ b_1'' \ -a_1'' \ 0 \\ 0 \ 0 \ 1 \end{array} \bigg), \quad (19.3) \]
where for all \( j \)
\[ a_j^2 + b_j^2 = 1, \quad (a_j')^2 + (b_j')^2 = 1. \]
The relation \((19.3)\) corresponds to description of matrix evolution given in Section 14: at every step a \( 3 \times 3 \) matrix obtained as a product of the form as in RHS of \((19.3)\) is factorized in a product of the form as in the LHS of that formula.

Introduce Ising model coupling coefficients \( K_1, K_2, K_3, L_1, L_2, L_3 \) by formulae
\[ \exp(\pm 2K_1) = b_1 \mp ia_1, \quad \exp(\pm 2K_2) = \frac{i(b_2 \mp 1)}{a_2}, \]
\[ \exp(\pm 2K_3) = b_3 \mp ia_3, \quad (19.4) \]
\[ \exp(\pm 2L_1) = \frac{i(b_1' \mp 1)}{a_1'}, \quad \exp(\pm 2L_2) = b_2' \mp ia_2', \]
\[ \exp(\pm 2L_3) = \frac{i(b_3' \mp 1)}{a_3'}. \quad (19.5) \]
The following lemma is principal in this section.

**Lemma 19.1.** If \((19.3)\) holds, the coefficients \( K_1, \ldots, L_3 \) satisfy the following star–triangle relations (we reproduce formulae \((6.4.8a–d)\) from the book \([76]\); coefficients \( K_j \) belong to a triangle, \( L_j \) —to a star, \( R \) is some numeric factor):
\[ 2 \cosh(L_1 + L_2 + L_3) = R \exp(K_1 + K_2 + K_3), \quad (19.6) \]
\[ 2 \cosh(-L_1 + L_2 + L_3) = R \exp(K_1 - K_2 - K_3), \quad (19.7) \]
\[ 2 \cosh(L_1 - L_2 + L_3) = R \exp(-K_1 + K_2 - K_3), \quad (19.8) \]
\[ 2 \cosh(L_1 + L_2 - L_3) = R \exp(-K_1 - K_2 + K_3). \quad (19.9) \]
Figure 3.12. Star–triangle transformation is shown by thick lines, while the corresponding matrix transformation—by thin lines

Lemma 19.1 is illustrated by Fig. 3.12.

Proof of Lemma 19.1 will be done using the following parameterization of relations (19.3) with variables $k, \lambda, \mu$, where $k$ is the modulus of all elliptic functions:

\begin{align*}
a_1 &= \text{sn} \lambda, \quad a_2 = k \text{sn} (\lambda + \mu), \quad a_3 = \text{sn} \mu, \quad \text{(19.10)} \\
b_1 &= \text{cn} \lambda, \quad b_2 = \text{dn} (\lambda + \mu), \quad b_3 = \text{cn} \mu, \quad \text{(19.11)} \\
a_1' &= k \text{sn} \lambda, \quad a_2' = \text{sn} (\lambda + \mu), \quad a_3' = k \text{sn} \mu, \quad \text{(19.12)} \\
b_1' &= \text{dn} \lambda, \quad b_2' = \text{cn} (\lambda + \mu), \quad b_3' = \text{dn} \mu. \quad \text{(19.13)}
\end{align*}

The fact that (19.10–19.13) is really a parameterization of (19.3) is verified directly using elementary properties of elliptic functions. Then, from the same properties it follows that formulae for $K_2$ and $L_2$ from (19.4) and (19.7) may be rewritten as follows:

\begin{align*}
\exp(\pm 2K_2) &= \text{cn} \kappa \mp i \text{sn} \kappa, \\
\exp(\pm 2L_2) &= \frac{i(\text{dn} \kappa \mp 1)}{k \text{sn} \kappa}, \quad \text{(19.14)}
\end{align*}

where

\begin{equation*}
\lambda + \kappa + \mu = iI'. \quad \text{(19.15)}
\end{equation*}

Here we have denoted the half-period of elliptic functions by $I'$, following R. Baxter’s book [76], although it is usually denoted $K'$. Our formulas (19.4) and (19.7) coincide now, with regard to $I'$, with parameterization (7.8.5) from [76] for relations (19.6–19.9), while (19.15) coincides with formula (7.13.4) from [76] if our $\lambda, \kappa, \mu$ equal Baxter’s $iu_1, iu_2, iu_3$. The lemma is proved.

We have described the transformation (19.1) by means of orthogonal matrices, and illustrated this with Figure 3.12. Similarly, transformation (19.2) together with the transformation of orthogonal matrices is illustrated by Figure 3.13. The latter differs from Fig. 3.12 in the fact that now

Figure 3.13. Triangle–star transformation is shown by thick lines, while the corresponding matrix transformation—by thin lines
to a kagome lattice triangle $\quad$ corresponds an Ising triangle, while to a triangle $\quad$ corresponds a star. One can reduce this situation to the previous one by transposing the products of orthogonal matrices (recall that matrices themselves are symmetric), corresponding to LHS and RHS of Fig. 3.13 and then changing their places. Then it is clear that relations of the form (19.4, 19.5) between matrix elements and coefficients $K_j$ and $L_j$ for triangle and star respectively work in the case of Fig. 3.13 as well.
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