Permanence, Extinction and Periodic Solution of a Non-autonomous Cooperative System with Stage Structure

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Abstract. This paper studies a class of non-autonomous two-species cooperative system with stage structure. Some sufficient conditions on the boundedness, permanence, extinction, periodic solution of the system are established by using the comparison method.

Keywords: Stage-structured cooperative system; Permanence; Extinction; Periodic solution.

1. Introduction

As we well known, in recent years the population dynamical systems are extensively studied[1-11]. Especially, the non-autonomous population cooperative systems has been extensively studied and excellent research results were obtained[1-10]. All of these studies considered the dynamical behavior of system by using different methods, for example, in [1,7], the authors studied global stability for the systems, in [1,2,4-6,8,10], the authors studied permanence for the systems, in [4,10], the authors studied extinction for the systems, in [2,3,9,10], the authors studied existence of positive periodic solutions for systems, in [3,5,11], the authors studied global attractivity for systems, in [4,7,9,10], the authors studied the population cooperative systems with stage structure and references cited therein. For example, Zhang et al. in [9], have studied the following non-autonomous stage-structured cooperative periodic system without delay

\[\begin{align*}
\dot{x}_1(t) &= \alpha(t)x_2(t) - r_1(t)x_1(t) - \beta(t)x_1(t) - \eta_1(t)x_1^2(t), \\
\dot{x}_2(t) &= \beta(t)x_1(t) - r_2(t)x_2(t) - \eta_2(t)x_2^2(t) + b(t)x_2(t)y(t), \\
\dot{y}(t) &= y(t)[R(t) - a(t)y(t) + c(t)x_2(t)].
\end{align*}\] (1)

By using the Mawhin's continuation theorem, the sufficient conditions on the existence of positive periodic solutions are established for system (1). But the authors in [9] only consider the existence of positive periodic solutions for system (1) and they did not consider the other dynamical behaviors for system (1). Therefore, in this paper we further study system (1) and we will establish some new sufficient conditions for the boundedness, permanence, extinction, periodic solution and global attractivity of system (1) by using the comparison method and the Lyapunov function method.

2. Preliminaries

In system (1), we have that \(x_1(t)\) represent the density of immaturity of species X at time t, \(x_2(t)\) represent the density of maturity of species X at time t, \(y(t)\) represent the density of species Y at time t. In this paper, we always assume that...
$H_1$: $r_i(t), \eta_i(t)(i = 1, 2), \alpha(t), \beta(t), b(t), R(t), a(t), c(t)$ are strictly positive $\omega$-periodic continuous functions with $\omega > 0$.

The following is a initial condition for system (1)

$$x_i(t) = \phi_i(t), y(t) = \phi(t), \forall t \in [0, +\infty), i = 1, 2$$

where $\phi_i(t)(i = 1, 2), y(t)$ are non-negative continuous functions defined on $[0, +\infty]$ satisfying $\phi_i(0) > 0(i = 1, 2), \phi(0) > 0$.

In this paper, for any $\omega$-periodic continuous function $z = x(t)$, $y(t)$ are strictly positive continuous functions with $\omega > 0$.

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Now, we present some useful lemmas.

**Definition 1.** System (1) is said to be permanent if there exist positive constants $m$, $M$ and $T^*$, such that each positive solution $(x_1(t), x_2(t), y(t))$ of system (1) with any positive initial value $\phi$, fulfill

$$m \leq x(t) \leq M$$

for all $t \geq T^*$, where $T^*$ may depend on $\phi$.

**Lemma 1.** (see [3]) If $a > 0$, $b > 0$ and $x(t) \geq (\leq) b - ax$, when $t \geq 0$, $x(0) \geq 0$ we have $x(t) \geq (\leq) b + (x(0) - b)e^{-at}$.

**Lemma 2.** (See [3]) Let $a > 0$, $b > 0$.

(I) If $x(t) \leq x(b - ax)$, then $\liminf_{t \to +\infty} x(t) \leq \frac{b}{a} t \geq 0 x(0) \geq 0$.

(II) If $x(t) \geq x(b - ax)$, then $\liminf_{t \to +\infty} x(t) \leq \frac{b}{a} t \geq 0 x(0) \geq 0$.

**Lemma 3.** (See [11]) If $a(t), b(t), c(t), d(t)$ and $f(t)$ are all $\omega$ periodic, then system

$$\dot{x}_1(t) = a(t)x_2(t) - b(t)x_1(t) - d(t)x_1^2(t),$$

$$\dot{x}_2(t) = c(t)x_1(t) - f(t)x_2^2(t),$$

has a positive $\omega$-periodic solution $(x_1^*(t), x_2^*(t))$ which is globally asymptotically stable with respect to $R^*_+ = \{ (x_1, x_2): x_1 > 0, x_2 > 0 \}$.

**Lemma 4.** ([11]) If there exist positive constants $m$ and $M$ for any $\Phi \in c_+^{\omega}[-\tau, 0]$ such that

$$m \leq \liminf_{t \to +\infty} x_i(t, 0, \Phi) \leq \limsup_{t \to +\infty} x_i(t, 0, \Phi) < M, \quad i = 1, 2 ... n$$

then the following periodic functional differential equation

$$\frac{dx}{dt} = F(t, x(t))$$

admits at least one positive $\omega$ periodic solution. Where $x(t) \in R^n$ and $F(t, x(t))$ is a n dimensional continuous functional, $x(t, 0, \Phi) = (x_1(t, 0, \Phi), x_2(t, 0, \Phi), ... x_n(t, 0, \Phi))$ is a solution of the functional differential equation with initial condition $x_0 = \Phi$.

3. Main Results

In this section, we will obtain some sufficient conditions for the ultimately boundedness, permanence, extinction and existence of periodic solution of system (1).

**Theorem 1.** Assume that $A_1^\omega > 0(i = 1, 2)$, where

$$A_1(t) = \eta_2(t) - \frac{b(t) + c(t)}{2}, \quad A_2(t) = a(t) - \frac{b(t) + c(t)}{2}$$

Then solutions of system (1) with initial condition (2) are ultimately bounded.

**Proof.** Suppose that $(x_1(t), x_2(t), y(t))$ is any solution of (1) with initial condition (2). Defining the function

$$V(t) = x_1(t) + x_2(t) + y(t),$$

and calculating the derivative of $V(t)$ along the positive solutions of system (1), we have
\[ \dot{V}(t) = \alpha(t)x_2(t) - r_1(t)x_1(t) - \beta(t)x_1(t) - \eta_1(t)x_1^2(t) + \beta(t)x_1(t) - r_2(t)x_2(t) \\
- \eta_2(t)x_2^2(t) + b(t)x_2(t)y(t) + y(t)[R(t) - a(t)y(t) + c(t)x_2(t)]. \]

By applying Lemma 2 to above differential inequality, for Theorem 2

Then form (8), we have

Since

Then form (6) and (7), we have

\[ - (r_1^I + \beta I)x_1(t) - r_2^I(t)x_2(t) - R Ly(t) - \eta_1^I(t)x_1^2(t) + \beta^M x_1(t) + \alpha^M x_2(t) \]

\[ = -k_1 V(t) - \eta_1^I x_1^2(t) + \beta^M x_1(t) + \alpha^M x_2(t) - A_1^I x_2^2(t) + 2R M y(t) - A_2^I y^2(t) \]  

(6)

Where \( k_1 = \min \{r_1^I + \beta I, r_2^I, R L\} \).

Then form (8), we have

\[ V(t) \leq k_2 - k_1 V(t) \]  

(9)

Then by Lemma 1, we have

This implies that any positive solutions of system (1) is ultimately bounded. Hence, there exist positive constant \( T_0 \) and \( M = \frac{k_2}{k_1} \) such that \( x_i(t) \leq M (i = 1, 2), y(t) \leq M, \quad t \geq T_0 \). This completes the proof.

**Theorem 2.** Assume that \( A_1^I > 0 (i = 1, 2) \) and \( b^L R L > \alpha^M r_2^M \) then system (1) is permanent.

**Proof.** Suppose \( z(t) = (x_1(t), x_2(t), y(t)) \) is any solution of system (1) with initial condition (2). Firstly, it follows from the third equation of system (1) that for \( t \geq 0 \), we have

\[ y(t) \geq y(t)(R L - \alpha^M y(t)). \]

By applying Lemma 2 to above differential inequality, for \( t \geq 0 \), we have

\[ \liminf_{t \to +\infty} y(t) \geq \frac{R L}{\alpha^M} =: m_3 \]

Then, there exists a constant \( T_1 > 0 \) such that \( y(t) \geq m_3 \) for \( t \geq T_1 \). Next from the first and second equation of system (1) that for \( t \geq T_1 \), we have

\[ \dot{x}_1(t) \geq \alpha(t)x_2(t) - (r_1(t) + \beta(t))x_1(t) - \eta_1(t)x_1^2(t). \]
\[ \dot{x}_2(t) \geq \beta(t)x_1(t) - \eta_2(t)x_2^2(t). \]

By lemma 3, the following auxiliary equation
\begin{align}
\dot{u}_1(t) &= \alpha(t)u_2(t) - (r_1(t) + \beta(t))u_1(t) - \eta_1(t)u_1^2(t) \\
\dot{u}_2(t) &= \beta(t)u_1(t) - \eta_2(t)u_2^2(t).
\end{align}

has a unique globally attractive positive \(\omega\)-periodic periodic solution \((\bar{x}_1(t), \bar{x}_2(t))\). Let \((u_1(t), u_2(t))\) be the solution of system (1) with \((u_1(T_1), u_2(T_1)) = (x_1(T_1), x_2(T_1))\) by comparison theorem, we have
\[ x_i(t) \geq u_i(t) (i = 1, 2), \quad t \geq T_1. \tag{11} \]

Also from the global attractivity of \((\bar{x}_1(t), \bar{x}_2(t))\), there exists a constant \(T_2 > 0\), such that
\[ |u_i(t) - \bar{x}_i(t)| < \frac{\bar{x}_i(t)}{2}, \quad t \geq T_2. \tag{12} \]

Eq.(12) combine with (11) leads to
\[ x_i(t) > \min_{0 \leq t \leq \omega} \left\{ \frac{\bar{x}_i(t)}{2} \right\} = m_i, \quad i = 1, 2, \quad t > T_2. \]

Therefore,
\[ \liminf_{t \to +\infty} x_i(t) \geq m_i, \quad i = 1, 2. \]

Thus, there exists a constant \(T^* > \max\{T_1, T_2\}\) such that \(x_i(t) \geq m_i\) for \(t \geq T^*\). This completes the proof of Theorem 2.

From, Theorem 2, we have the following result

**Corollary 1.** If \(H_1\) hold, then the species \(Y\) in system (1) is permanent.

On the existence of positive periodic solutions of system (1) we have the following result. As a direct result of Lemma 4, from Theorem 2, we have

**Corollary 2.** If the assumptions of Theorem 2 hold, then system (1) has at least one positive \(\omega\)-periodic solution.

**Theorem 3.** Immature species \(x_1(t)\) and mature species \(x_2(t)\) of system (1) are extinct if \(A^*_i > 0 (i = 1, 2)\) and \(r^*_1 - \alpha^* - b^* M > 0\).

**Proof.** Suppose \((x_1(t), x_2(t), y(t))\) be a positive solution of system (1) with initial conditions (2). Defining the function \(V(t) = x_1(t) + x_2(t)\) and calculating the derivative of \(V(t)\) along the positive solutions of system (1), we have
\[ \dot{V}(t) = \dot{x}_1(t) + \dot{x}_2(t) = \alpha(t)x_2(t) - r_1(t)x_1(t) + b(t)x_1(t) - \eta_1(t)x_1^2(t) + \beta(t)x_1(t) \]
\[ -r_2(t)x_2(t) - \eta_2(t)x_2^2(t) + b(t)x_2(t)y(t) \]
\[ \leq \alpha(t)x_2(t) - r_1(t)x_1(t) - r_2(t)x_2(t) + b(t)x_2(t)y(t) \]
\[ \leq \alpha(t)x_2(t) - r_1(t)x_1(t) - r_2(t)x_2(t) + b(t)x_2(t)y(t) \tag{13} \]

Then, it follows from (13) for \(t > T_0\)
\[ V(t) \leq -r^*_1 x_1(t) - (r^*_2 - \alpha^* - b^* M)x_2(t) \leq -kV(t). \]

Where \(k = \min\{r^*_1, r^*_2 - \alpha^* - b^* M\}\), which yields
\[ V(t) \leq V(0)e^{-kt}. \]

Then we have
\[ \lim_{t \to +\infty} V(t) = \lim_{t \to +\infty} (x_1(t) + x_2(t)) = 0. \tag{14} \]

From (14) there exists a constant \(T^{**} > T_0\) such that \(x_1(t) \to 0\) and \(x_2(t) \to 0\) for \(t > T^{**}\).This completes the proof.
4. Examples

Example 1. First, we consider the following system

\[
\begin{align*}
\dot{x}_1(t) &= (3.5 + 0.5 \cos(t))x_2(t) - (1.5 + 0.5 \cos(t))x_1(t) - (2.5 + 0.5 \cos(t))x_1(t) \\
& \quad - (0.5 + 0.4 \cos(t))x_1^2(t) \\
\dot{x}_2(t) &= (2.5 + 0.5 \cos(t))x_1(t) - (0.3 + 0.4 \cos(t))x_2(t) - (3 + 0.4 \cos(t))x_2(t) \\
& \quad - (0.8 + 0.1 \cos(t))x_2(t)y(t) \\
y(t) &= y(t)[3.5 + 0.5 \sin(t) - (2.5 + 0.4 \sin(t))y(t) + (0.15 + 0.1 \sin(t))x_2(t)]
\end{align*}
\]

directly calculation we get

\[
\begin{align*}
A_1^1 &\approx 2.0250, \quad A_2^1 \approx 1.5250, \quad b^L R^L \approx 2.1 > a^M r_2^M \approx 2.03.
\end{align*}
\]

It is clear that the conditions of Theorem 2 and Corollary 2 hold. Then system (15) is permanent and has a positive \(2\pi\)-periodic solution.

From Fig.1 and Fig.2 we can see system is permanent and has a positive \(2\pi\)-periodic solution.

Example 2. Next, we consider the following system

\[
\begin{align*}
\dot{x}_1(t) &= (1.5 + 0.5 \cos(t))x_2(t) - (1.5 + 0.5 \cos(t))x_1(t) - (2.5 + 0.5 \cos(t))x_1(t) \\
& \quad - (2.5 + 0.4 \cos(t))x_1^2(t) \\
\dot{x}_2(t) &= (2.5 + 0.5 \cos(t))x_1(t) - (3 + 0.5 \cos(t))x_2(t) - (3 + 0.4 \cos(t))x_2(t) \\
& \quad - (0.2 + 0.1 \cos(t))x_2(t)y(t) \\
y(t) &= y(t)[1.6 + 0.1 \sin(t) - (2.5 + 0.4 \sin(t))y(t) + (0.15 + 0.1 \sin(t))x_2(t)]
\end{align*}
\]

directly calculation we get

\[
\begin{align*}
A_1^1 &\approx 2.3250, \quad A_2^1 \approx 1.850, \quad k_1^L \approx 1.5, \quad k_2^L \approx 2.2933. \\
M &\approx 1.5289, \quad r_2^L - a^M - b^M M \approx 0.0413.
\end{align*}
\]

It is clear that the conditions of Theorem 3 and Corollary 1 hold. Then the immaturity and maturity species \(X\) of system (16) are extinct, the species \(Y\) of system (16) is permanent and has a positive \(2\pi\)-periodic solution.
Figure 3. Extinction and permanence.

From Fig.3, we can see the immaturity and maturity species $X$ of system (16) are extinct, species $Y$ of system (16) is permanent and has a positive $2\pi$-periodic solution.

5. Conclusion

In this paper, we have studied a class of non-autonomous two-species cooperative system with stage structure. By employing comparison method and useful inequality techniques, some sufficient conditions on the boundedness, permanence, extinction, periodic solution of the system are derived. Compared with previous research results in [9], we have obtained conditions on the boundedness, permanence, extinction of the system (1). In addition, in [9] the authors obtained some sufficient conditions on the existence of positive periodic solutions by means of the Mawhin's continuation theorem. However, in this paper we obtained by using of comparison method. Hence, the obtained results in this paper can be seen as the supplement and extension of the results obtained in [9] and other previously known related works.

Acknowledgement

This work was supported by the National Natural Science Foundation of China (Grant No. 11662020, 11861063).

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