Quantization of $gl(1, \mathbb{R})$ Generalized Chern-Simons Theory in 1+1 Dimensions

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Abstract

We present a quantization of previously proposed generalized Chern-Simons theory with $gl(1, \mathbb{R})$ algebra in 1+1 dimensions. This simplest model shares the common features of generalized CS theories: on-shell reducibility and violations of regularity. On-shell reducibility of the theory requires us to use the Lagrangian Batalin-Vilkovisky and/or Hamiltonian Batalin-Fradkin-Vilkovisky formulation. Since the regularity condition is violated, their quantization is not straightforward. In the present case we can show that both formulations give an equivalent result. It leads to an interpretation that a physical degree of freedom which does not exist at the classical level appears at the quantum level.
1 Introduction

Although three forces in nature are described as gauge theories and have been quantized, the quantization of gravity is not yet achieved. Among the difficulties, nonrenormalizability of Einstein’s gravity is the most difficult one. There are still some belief that superstring theory describes real nature containing gravity. There is, however, another approach to describe the gravity theory.

Witten has shown that three dimensional gravity can be described by a Chern-Simons theory with an appropriate algebra which possesses a non-degenerate invariant bilinear form (ref.[1]). At the same time he has shown that Chern-Simons theory in three dimensions is exactly solvable (ref.[2]). This fact ensures the consistency and finiteness of the three dimensional gravity. Furthermore, it has also been shown that the three dimensional conformal gravity can be also treated by the Chern-Simons theory (ref.[3]).

The extension of the Chern-Simons theory into arbitrary dimensions, which was given by one of the authors (N.K.) and Watabiki, was aimed to make a similar approach to gravity as in the three dimensional case (refs.[4]-[6]). Although their action keeps a formal correspondence with the ordinary Chern-Simons theory, it can be used in arbitrary dimensions. The gauge symmetries are also formally similar to those of ordinary Chern-Simons theory, or in other words, the extension is done by keeping these similarities, but their explicit expressions are very complicated. This theory might be useful in formulating four dimensional gravity. It was shown that topological conformal gravity can be formulated by four-dimensional generalized Chern-Simons theory (ref.[6]). However this theory is generally on-shell reducible and its equations of motion violate the regularity in the sense we will explain later. Thus the quantization of this system is not straightforward.

In this paper we investigate the quantization of the simplest model defined in two dimensions with \( gl(1,\mathbb{R}) \) algebra. Though this model is quite simple, it shares the common characteristics, reducibility and violation of the regularity. The quantization of the simplest model may give a clue for the quantization of general cases. We apply methods developed for quantizing reducible systems with cares for the violation of the regularity.

After the introduction of the model in Sec. 2, we present a quantization based on a Lagrangian formulation, \( à la \) Batalin and Vilkovisky (refs.[7]) in Sec. 3. Their
treatment gives a systematic way to deal with reducible gauge symmetries and obtain a BRST invariant gauge fixed action by solving so-called master equation. Even with the violation of the regularity we can utilize this formulation for the present model. However, the meaning of the physical space defined by the BRST cohomology is not obvious in the Lagrangian formulation. The meaning of the BRST cohomology and thus the structure of the quantization are clearer in the Hamiltonian formalism. Therefore we investigate the model in the Hamiltonian formulation in Sect. 4. The systematic treatment in the Hamiltonian formalism is given by Batalin, Fradkin and Vilkovisky (refs.[8]-[11]), as an extension of the Dirac’s treatment of the constrained systems (ref.[12]) and a comprehensive explanation can be found in ref. [13]. We use this formulation with a suitable generalization. We conclude in Sect. 5 with summaries and discussions.

2 \( gl(1, \mathbb{R}) \) model in two dimensions

The generalized Chern-Simons theory which was introduced by N.K. and Watabiki (refs.[4]-[6]) is the generalization of the ordinary three dimensional Chern-Simons theory into arbitrary dimensions. The generalized theory highly respects the structure of the three dimensional ordinary Chern-Simons theory. Indeed, the action of the generalized Chern-Simons theory possesses the following form,

\[
S = \frac{1}{2} \int_M Tr(AQA + \frac{2}{3}A^3),
\]

where \( Q \) corresponds to the ordinary exterior derivative and \( A \) to a Lie algebra valued gauge field one form. However \( A \) is defined to contain arbitrary forms to extend the system to arbitrary dimensions. (See refs.[4]-[6] for details.)

The simplest case defined in two dimension is given in ref.[5] based on the \( gl(1, \mathbb{R}) \) algebra. The action expanded into components is given by

\[
S = \int d^2 x \left( \epsilon^{\mu\nu} \partial_\mu \omega_\nu \phi - \frac{1}{2} \epsilon^{\mu\nu} B_{\mu\nu} \phi^2 \right),
\]

where \( \phi, \omega_\mu \) and \( B_{\mu\nu} \) are scalar, vector and antisymmetric tensor fields, respectively, and \( \epsilon^{01} = -\epsilon_{01} = 1 \) in 1 + 1 dimension. This lagrangian possesses gauge symmetries

\[
\delta_g \phi = 0,
\]
\[
\delta_g \omega_\mu = \partial_\mu v + 2\phi u_\mu,
\]
\[
\delta_g B = \epsilon^{\mu\nu} \partial_\mu u_\nu,
\]
where $B$ is defined by $B \equiv \frac{1}{2} \epsilon^{\mu\nu} B_{\mu\nu}$. Here gauge fields and gauge parameters are real variables and thus the corresponding algebra is $gl(1, \mathbb{R})$. Although the action has a simple form, it has some unusual properties which are common to generalized CS theories. The equations of motion of this theory are given by

$$
\begin{align*}
\phi & : \quad \epsilon^{\mu\nu} \partial_\mu \omega_\nu - 2\phi B = 0, \\
\omega_\mu & : \quad \epsilon^{\mu\nu} \partial_\nu \phi = 0, \\
B & : \quad -\phi^2 = 0.
\end{align*}
$$

Generally, equations of motion are called regular if any function of field variables vanishing at a stationary point of the Lagrangian can be written as their “linear” combination, where the coefficients of the combination could be field dependent. From this definition, it is clear that the regularity is violated in the present case. $\phi$ itself vanishes due to eq.(2.8) while $\phi$ can not be written as a “linear” combination of the above equations of motion unless we accept a singular coefficient like $\frac{1}{\phi}$.

In addition this system is on-shell reducible since the gauge transformations (2-3)-(2-5) are invariant under the transformation

$$
\begin{align*}
\delta_\lambda v & = -2\phi \lambda, \\
\delta_\lambda u_\mu & = \partial_\mu \lambda,
\end{align*}
$$

with the on-shell condition $\partial_\mu \phi = 0$. Since the transformations (2.9) and (2.10) are not reducible any more, the Lagrangian (2.2) is called a first-stage reducible system.

These two aspects, the violation of the regularity and the on-shell reducibility, are common features in the generalized Chern-Simons theory.

3 \quad gl(1, \mathbb{R}) model in the Lagrangian formalism

In this section we present a quantization of the model based on the Lagrangian formalism. We can rewrite the action (2.2) into the following form by using $B \equiv \frac{1}{2} \epsilon^{\mu\nu} B_{\mu\nu}$,

$$
S_0 = \epsilon^{\mu\nu} \partial_\mu \omega_\nu \phi - B\phi^2,
$$

where the integration symbol is omitted. The gauge symmetries are given in the previous section, eqs.(2.3)-(2.5), and this is a first-stage reducible system.

According to the procedure given by Batalin and Vilkovisky (refs.[7]), we introduce ghost fields $C$ and $C_\mu$ corresponding to the gauge freedom $v$ and $u_\mu$ and a ghost for
ghost $C_1$ to the reducible degree $\lambda$, which are fermionic and bosonic, respectively. In addition, a minimal set of anti-fields $\Phi^*_{\text{min}} = (\phi^*, \omega^* \nu, B^*, C^*, C^*_\mu, C^*_1)$ are introduced corresponding to the field variables $\Phi_{\text{min}} = (\phi, \omega_\mu, B, C, C_\mu, C_1)$. The Grassmann parities of the anti-fields are opposite to that of corresponding fields, as usual.

Now the minimal action is obtained by solving the master equation

$$
(S_{\text{min}}(\Phi_{\text{min}}, \Phi^*_{\text{min}}), S_{\text{min}}(\Phi_{\text{min}}, \Phi^*_{\text{min}})) = 0,
$$

$(X, Y) = \frac{\partial_r X}{\partial \Phi^+_A} \frac{\partial_l Y}{\partial \Phi^-_A} - \frac{\partial_r X}{\partial \Phi^-_A} \frac{\partial_l Y}{\partial \Phi^+_A}$,

(3.2)

with the boundary conditions

$$
S_{\text{min}}(\Phi_{\text{min}}, \Phi^*_{\text{min}})|_{\Phi^*_{\text{min}} = 0} = S_0,
$$

(3.4)

$$
\frac{\partial_l S_{\text{min}}(\Phi_{\text{min}}, \Phi^*_{\text{min}})}{\partial \phi^*}|_{\Phi^*_{\text{min}} = 0} = 0,
$$

(3.5)

$$
\frac{\partial_l S_{\text{min}}(\Phi_{\text{min}}, \Phi^*_{\text{min}})}{\partial \omega^{\mu*}}|_{\Phi^*_{\text{min}} = 0} = \partial_\mu C + 2\phi C_\mu,
$$

(3.6)

$$
\frac{\partial_l S_{\text{min}}(\Phi_{\text{min}}, \Phi^*_{\text{min}})}{\partial B^*}|_{\Phi^*_{\text{min}} = 0} = \epsilon^{\mu\nu} \partial_\mu C_\nu,
$$

(3.7)

$$
\frac{\partial_l S_{\text{min}}(\Phi_{\text{min}}, \Phi^*_{\text{min}})}{\partial C^*_\mu}|_{\Phi^*_{\text{min}} = 0, \text{on-shell}} = 0,
$$

(3.8)

$$
\frac{\partial_l S_{\text{min}}(\Phi_{\text{min}}, \Phi^*_{\text{min}})}{\partial C^*_1}|_{\Phi^*_{\text{min}} = 0, \text{on-shell}} = \partial_\mu C_1,
$$

(3.9)

where $r$ and $l$ denote right and left derivatives, respectively. It should be noted that the boundary conditions (3.8) and (3.9) are required only on-shell. It is straightforward to solve the master equation perturbatively in the order of antighost fields by taking into account the fact that $\phi$ equals zero on the shell. Though it is not assured that the master equation possesses a solution when the regularity of equations of motion is violated, there is a solution in the present case. We find

$$
S_{\text{min}} = \epsilon^{\mu\nu} \partial_\mu \omega_\nu \phi - B \phi^2 + \omega^{\mu*} (\partial_\mu C + 2\phi C_\mu - \epsilon_{\mu\nu} \omega^{*\nu} C_1),
$$

$$
+ B^* \epsilon^{\mu\nu} \partial_\mu C_\nu + C^{*\mu} \partial_\mu C_1 - 2C^{*\mu} \phi C_1.
$$

(3.10)

We next introduce a nonminimal action, which must be added to the minimal one to make a gauge fixing, as

$$
S_{\text{nonmin}} = \tilde{C}^* r + \tilde{C}^{*,*}_\mu b^\mu + \tilde{C}^{*,*}_1 b_1 + \eta^{*,*} \pi,
$$

(3.11)

* Though it is not manifestly stated in refs.[7] that these conditions should be required only on-shell, they are enough from the definition of on-shell reducibility.
where auxiliary fields $b, b^\mu, \bar{C}_1$ and $\eta$ are bosonic and $b_1, \pi, \bar{C}$ and $\bar{C}_\mu$ are fermionic, respectively. Their corresponding antifields possess the opposite Grassmann parities. A gauge fixing is done by choosing a suitable gauge fermion $[7]$. We can adopt, for example, the following gauge fermion $\Psi$, which leads to a Landau-type gauge fixing:

$$\Psi = \bar{C}^\mu \partial^\nu B_{\mu \nu} + \bar{C}^\mu \partial_\mu \eta + \bar{C}_1 \partial^\mu C_\mu + \bar{C} \partial^\mu \omega_\mu, \quad (3.12)$$

where we assume that the base manifold is flat with the metric $\eta_{\mu \nu} = \text{diag}(-1, +1)$ for simplicity. The anti-fields can be eliminated by substituting the derivative of the $\Psi$ into them. Then the action turns out to be

$$S = S(\Phi, \Phi^*)|_{\Phi^* = \frac{\partial \Psi}{\partial \Phi}},$$

$$= \epsilon^{\mu \nu} \partial_\mu \omega_\nu \phi - B \phi^2 - \partial^\mu \bar{C} (\partial_\mu C + 2 \phi C_\mu) - \epsilon_{\mu \nu} \partial^\mu \bar{C} \partial^\nu \bar{C} C_1 + \frac{1}{2} (\partial^\mu \bar{C}^\nu - \partial^\nu \bar{C}^\mu) (\partial_\mu C_\nu - \partial_\nu C_\mu) - \partial^\mu \bar{C}_1 \partial_\mu C_1 - b \partial^\mu \omega_\mu + b^\mu (\partial_\mu \eta - \epsilon_{\mu \nu} \partial^\nu B) - b_1 \partial^\mu C_\mu - \partial_\mu \bar{C}_1 \pi. \quad (3.13)$$

It should be noted that the fields $\bar{C}, \bar{C}^\mu, C_1, \bar{C}_1$ and $b_1$ are treated as purely imaginary fields to make the action hermitian, following to the convention of ghost fields $C, C_\mu$ being real variables. This action is invariant under the following on-shell nilpotent BRST transformation when applied as a right variation:

$$s\Phi = (\Phi, S(\Phi, \Phi^*))|_{\Phi^* = \frac{\partial \Psi}{\partial \Phi}} \quad (3.14)$$

$$s\phi = 0, \quad (3.15)$$

$$s\omega_\mu = \partial_\mu C + 2 \phi C_\mu + 2 \epsilon_{\mu \nu} \partial^\nu \bar{C} C_1, \quad (3.16)$$

$$sB = \epsilon^{\mu \nu} \partial_\mu C_\nu, \quad (3.17)$$

$$sC = -2 \phi C_1, \quad (3.18)$$

$$sC_\mu = \partial_\mu C_1, \quad (3.19)$$

$$sC_1 = 0, \quad (3.20)$$

$$s\bar{C} = b, \quad sb = 0, \quad (3.21)$$

$$s\bar{C}^\mu = b^\mu, \quad sb^\mu = 0, \quad (3.22)$$

$$s\bar{C}_1 = b_1, \quad sb_1 = 0, \quad (3.23)$$

$$s\eta = \pi, \quad s\pi = 0. \quad (3.24)$$

It is easy to see that the action does not receive a quantum correction and is also a solution of the quantum master equation. This fact can be naturally understood.
from a perturbative calculation. Nonvanishing connected $n$ point functions for $n \geq 3$ include only three different three-point functions of unphysical fields, which exist only at the tree level.

4 $gl(1, \mathbb{R})$ model in the Hamiltonian formalism

The same model as in the previous section is investigated by the Hamiltonian formalism in this section. From the action (3.1) we obtain canonical momenta as

$$\pi_{\phi} = 0,$$  \hspace{1cm} (4.1)

$$\pi_{\omega_0} = 0,$$  \hspace{1cm} (4.2)

$$\pi_{\omega_1} = \phi,$$  \hspace{1cm} (4.3)

$$\pi_B = 0.$$  \hspace{1cm} (4.4)

All of these equations give primary constraints. The canonical Hamiltonian following from the Lagrangian becomes

$$H_C = \phi \partial_1 \omega_0 + B \phi^2.$$  \hspace{1cm} (4.5)

This Hamiltonian and the above constraints with Lagrange multipliers define the following total Hamiltonian,

$$H_T = \phi \partial_1 \omega_0 + B \phi^2 + \lambda_{\phi} \pi_{\phi} + \lambda_{\omega_0} \pi_{\omega_0} + \lambda_{\omega_1} (\pi_{\omega_1} - \phi) + \lambda_B \pi_B.$$  \hspace{1cm} (4.6)

According to the ordinary Dirac’s procedure, we have to check the consistency of the constraints. The results are

$$\partial_0 \pi_{\phi} = -\partial_1 \omega_0 - 2\phi B + \lambda_{\omega_1} = 0,$$  \hspace{1cm} (4.7)

$$\Rightarrow \lambda_{\omega_1} = \partial_1 \omega_0 + 2\phi B;$$  \hspace{1cm} (4.8)

$$\partial_0 \pi_{\omega_0} = \partial_1 \phi = 0,$$  \hspace{1cm} (4.9)

$$\partial_0 (\pi_{\omega_1} - \phi) = -\lambda_{\phi} = 0,$$  \hspace{1cm} (4.10)

$$\Rightarrow \lambda_{\phi} = 0;$$  \hspace{1cm} (4.11)

$$\partial_0 \pi_B = -\phi^2 = 0.$$  \hspace{1cm} (4.12)

Two lagrange multipliers are determined. After the substitution of these expressions into the total Hamiltonian, we obtain

$$H_T = \pi_{\omega_1} \partial_1 \omega_0 + 2\phi \pi_{\omega_1} B - B \phi^2 + \lambda_{\omega_0} \pi_{\omega_0} + \lambda_B \pi_B.$$  \hspace{1cm} (4.13)
At the same time, we have found secondary constraints

\[- \phi^2 = 0, \quad (4.14)\]
\[\partial_1 \phi = 0, \quad (4.15)\]

The consistency check of these constraints gives no other relations. After all, we have obtained the set of constraints eqs. (4.1)-(4.4) and eqs. (4.15) and (4.14).

It should be noted that the constraints (4.15) and (4.14) violates Dirac’s regularity condition. The regularity in Hamiltonian formulation can be stated similarly to the Lagrangian case by replacing equations of motion by constraint equations. Constraints are called regular if any function of canonical variables vanishing on the constraint surface can be written as their “linear” combination, where the coefficients of the combination could be dependent on canonical variables. In the present case eq.(4.14) implies that \( \phi \) vanishes on the constraint surface. However \( \phi \) itself can not be written as a ”linear” combination of (4.14) and (4.15) unless we admit expressions like \( -\frac{1}{\phi} G_1 \) or \( \int dx^1 G_2 \). The former expression includes a singular coefficient and the latter requires to specify a boundary condition. Thus they are not acceptable.

We can replace these two constraints by a single one

\[\phi = 0, \quad (4.16)\]

which is equivalent to eq.(4.14) in the present case. Then we can separate the constraints into the first class and second class. With some redefinitions we can obtain a set of constraints as

\[
\text{second class} \quad \pi_\phi = 0, \quad \phi = 0, \quad (4.17) \\
\text{first class} \quad \pi_{\omega_0} = 0, \quad (4.18) \\
\quad \pi_{\omega_1} = 0, \quad (4.19) \\
\quad \pi_B = 0. \quad (4.20)
\]

These constraints imply that no dynamical variables exist, as is expected from the topological nature of the generalized Chern-Simons action.

Now the quantization of this system is trivial since there is no dynamical degree of freedom. (We treat a flat base manifold here and do not take into account possible finite degrees of freedom depending on the topology of the base manifold.) By adopting gauge fixing conditions \( \omega_0 = \omega_1 = B = 0 \), all variables and thus Hamiltonian \( H_T \) itself vanish identically. Thus we can conclude this theory is completely empty.
This is, however, not the end of the story. In more general cases with non-abelian gauge groups, constraints of the form \( \phi_1^2 - \phi_2^2 - \phi_3^2 = 0 \) can appear. In these cases, we can not adopt constraints which satisfy the regularity condition and at the same time give a constraint surface without singularities. In the above example, we can linearize the constraint equation with respect to \( \phi_1 \) by choosing one of the branches of \( \phi_1 = \pm \sqrt{\phi_2^2 + \phi_3^2} \). It, however, gives a singular constraint surface with a conical singularity at \( \phi_i = 0 \). Therefore it seems rather natural in the generalized Chern-Simons theory to adopt a quantization method different from the usual one, \textit{i.e.}, a quantization based on regularity violating constraints that follow directly from the Lagrangian. In the following we perform the Hamiltonian BRST quantization à la Batalin, Fradkin and Vilkovisky by using the regularity violating constraints. It is interesting that in this treatment of Hamiltonian formulation with a suitable choice of gauge condition, we can show that the gauge fixed action thus obtained is just the same as the result of Lagrangian formulation in Sect. 3. Though the Lagrangian and Hamiltonian constructions are formally equivalent in usual cases as shown in refs.\[14\] and \[15\], we can show the equivalence of both formulations when applied to the present model only if we adopt the regularity violating constraints in the Hamiltonian formalism. The physical meaning of the quantization in the Lagrangian formulation was not obvious in the previous section while it becomes clear in the Hamiltonian formulation from the above mentioned equivalence.

We first rearrange the constraints (4.1)-(4.4), (4.14) and (4.15) into (4.1)-(4.4) and

\[
G_1 \equiv -\pi_\omega^2 = 0, \tag{4.21}
\]

\[
G_2 \equiv \partial_1 \pi_\omega = 0, \tag{4.22}
\]

so that the first and second class constraints are separated. Now we can carry on without the variables \( \phi \) and \( \pi_\phi \) thanks to the second class constraints (4.1) and (4.3) if we replace all \( \phi \) by \( \pi_\omega \), and set \( \pi_\phi = 0 \). We further adopt gauge conditions for the first class constraints (4.2) and (4.4) as \( \omega_0 = B = 0 \), for simplicity. Though this is not inevitable, it turns out that this is the simplest way to lead to the gauge fixed Lagrangian of the form (3.13). Then we can also eliminate \( \omega_0, \pi_\omega, B \) and \( \pi_B \) from the system. After the above manipulation, we have two phase space variables \( \omega_1 \) and \( \pi_\omega \) with the first class constraints (4.21) and (4.22). Then the total Hamiltonian (4.13) vanishes completely. It should be noted that these constraints violate the regularity condition but only in the constant mode of \( \pi_\omega \).
Now following to the procedure in the Hamiltonian formalism, we define a Koszul-Tate differential $\delta$ and Grassmann odd ghost momenta $P_1$ and $P_2$ as

$$
\begin{align*}
\delta \omega_1 &= \delta \pi_{\omega_1} = 0, \\
\delta P_1 &= -\pi_{\omega_1}^2, \\
\delta P_2 &= \partial_1 \pi_{\omega_1}.
\end{align*}
$$

In the case the regularity condition is not violated, Koszul-Tate differential is defined so that its homology is a set of functions on the constraint surface, i.e. those not vanishing on the surface. We follow the usual procedure and give a comment afterwards on what the violation of regularity causes. Constraints (4.21) and (4.22) are not independent due to the following relation

$$
\partial_1 G_1 + 2\pi_{\omega_1} G_2 = 0,
$$

which shows that they are reducible. In the present case there is only one independent relation and thus the system is called first stage reducible. Then it is necessary to introduce one more ghost momentum $P$, which is Grassmann even, as

$$
\delta P = -(\partial_1 P_1 + 2\pi_{\omega_1} P_2).
$$

This differential is nilpotent by definition. Note that due to the violation of the regularity condition the $\delta$-closed constant mode of $\pi_{\omega_1}$ is not $\delta$ exact while “classical” constraints (4.21) requires $\pi_{\omega_1} = 0$ on the surface. This is an interesting phenomenon that a degree of freedom which does not exist at the “classical” level appears at the “quantum” level.

We next define an extended longitudinal differential $D$ in the following way. First the longitudinal differential $d$ is defined from the form of the gauge transformations as

$$
\begin{align*}
d\omega_1 &= \partial_1 \eta_2 + 2\pi_{\omega_1} \eta_1, \\
d(\text{others}) &= 0.
\end{align*}
$$

where we introduce two fermionic ghost fields of ghost number 1 corresponding to the first class constraints which are generators of the gauge transformation. Due to the reducibility (4.26), ghost fields appear only in the combination of the right hand side of eq.(4.28). Thus there exists a cohomology of $d$ in the ghost number 1 sector of the
ghost combination orthogonal to the above one. In order to kill this cohomology, we have to introduce an auxiliary differential $\Delta$ and one bosonic ghost $\eta$

\begin{align}
\Delta \eta_1 &= \partial_1 \eta, \\
\Delta \eta_2 &= -2\pi_{\omega_1} \eta, \\
\Delta (\text{others}) &= 0.
\end{align}

(4.30) \quad (4.31) \quad (4.32)

Now three ghosts $\eta_1$, $\eta_2$ and $\eta$ are introduced. This makes possible to extend the phase space to include $P$’s and $\eta$’s. Then $D$ is obtained by the sum of these two differentials,

\[ D = \Delta + d. \]

(4.33)

This differential $D$ is nilpotent in the space of ghosts and constraint surface, in other words, nilpotent modulo $\delta$-exact term.

Finally the BRST differential $s$ is defined by $\delta$ and $D$,

\[ s = \delta + D + \left( \begin{array}{c} \delta \eta_1 = \delta \eta_2 = \delta \eta = 0, \\
DP_1 = DP_2 = DP = 0. \end{array} \right) \]

(4.34) \quad (4.35) \quad (4.36)

The $s$ is determined by the requirement of the nilpotency of $s$. Since $D$ is nilpotent modulo $\delta$-exact term, we can realize the nilpotency of $s$ by choosing

\begin{align}
\left( \begin{array}{c} s \omega_1 = 2\eta P_2, \\
\left( \begin{array}{c} s \text{ (others)} = 0. \end{array} \right) \end{array} \right)
\end{align}

(4.37) \quad (4.38)

The action of the differential $s$ is listed as follows:

\begin{align}
s P_1 &= -\pi_{\omega_1}^2, \\
s P_2 &= \partial_1 \pi_{\omega_1}, \\
s P &= -(\partial_1 P_1 + 2\pi_{\omega_1} P_2), \\
s \omega_1 &= \partial_1 \eta_2 + 2\pi_{\omega_1} \eta_1 + 2\eta P_2, \\
s \pi_{\omega_1} &= 0, \\
s \eta_1 &= \partial_1 \eta, \\
s \eta_2 &= -2\pi_{\omega_1} \eta, \\
s \eta &= 0.
\end{align}

(4.39) \quad (4.40) \quad (4.41) \quad (4.42) \quad (4.43) \quad (4.44) \quad (4.45) \quad (4.46)
The extended phase space is defined to include the above ghosts and ghost momenta with a canonical structure

\[
[P_i, \eta_j] = -\delta_{ij}, \quad [P, \eta] = -1,
\]  

where \([,]\) represents the graded Poisson bracket which will be replaced by the graded commutation relation multiplied by \(-i\) as in usual cases. By using this canonical relation, the nilpotent BRST charge \(\Omega^{Min}\) is defined by

\[
\Omega^{Min} = \eta_1 \pi_{\omega_1}^2 - \eta_2 \partial_1 \pi_{\omega_1} + \eta(\partial_1 P_1 + 2\pi_{\omega_1} P_2),
\]

which realizes \(sX = [X, \Omega^{Min}]\).

In order to fix the gauge, we have to extend the phase space further and introduce the following set of canonical variables and their momenta,

\[
\lambda_1, \lambda_2, \quad b_1, b_2, \quad \bar{C}_1, \bar{C}_2, \quad \rho_1, \rho_2,
\]

\[
\lambda_0, \quad b_0, \quad \bar{C}_0, \quad \rho_0,
\]

\[
\lambda, \quad b, \quad \bar{C}, \quad \rho.
\]

Two sets with indices 1 and 2 correspond to two first class constraints and the other two sets to one reducible condition (4.26). Their statistics are bosonic for \(b_i, \lambda_i, \bar{C}\) and \(\rho\), and fermionic for \(\bar{C}_i, \rho_i, b\) and \(\lambda\). The canonical structure is defined by

\[
[\rho_i, \bar{C}_j] = [b_i, \lambda_i] = -\delta_{ij},
\]

\[
[\rho, \bar{C}] = [b, \lambda] = -1.
\]

The action of BRST differential is also extended to these variables as

\[
s\lambda_a = -\rho_a, \quad s\rho_a = 0
\]

\[
s\bar{C}_a = b_a, \quad sb_a = 0,
\]

where \(a\) denotes \(i = 0, 1, 2\) or nothing. The corresponding extended BRST charge is given by

\[
\Omega = \Omega^{Min} + \Omega^{Nonmin},
\]

\[
\Omega^{Nonmin} = -\sum_{i=0}^{2} \rho_i b_i + \rho b.
\]
Now the gauge fixed action $S$ is obtained by a Legendre transformation from the Hamiltonian in the extended phase space:

$$
S = \dot{\omega}_1 \pi_{\omega_1} + \sum_{i=1,2} \dot{\eta}_i P_i + \dot{\eta} P + \sum_{i=0}^{2} (\dot{\lambda}_i b_i + \dot{\tilde{C}}_i \rho_i) + \dot{\lambda} b + \dot{\tilde{C}} \rho - H_K, \quad (4.55)
$$

$$
H_K = [K, \Omega], \quad (4.56)
$$

where $K$ is called a gauge-fixing fermion. The gauge fixed Hamiltonian $H_K$ consists of gauge-fixing and ghost parts only since the total Hamiltonian of the system have vanished. The gauge-fixing fermion generally takes the form $K = \tilde{C}_a \chi_a - P_i \lambda_i$ with $\chi_a$ being functions of phase space variables corresponding to a gauge choice. (In the present case, there are four terms in the first sum and three in the second.) There is no systematic way to find $K$ so as to yield a covariant expression. Here, however, we can use the result in the Lagrangian formulation as a clue. Actually we want to show that the two formulations give an equivalent result. Therefore we plug the above form into (4.55) and compare the form with the expression (3.13). We have found that the following gauge-fixing fermion $K$ works as desired:

$$
K = -\tilde{C}_1 \partial_1 \lambda_0 + \tilde{C}_2 \partial_1 \omega_1 - \tilde{C}_0 \partial_1 \lambda_1 - \tilde{C} \partial_1 \eta_1 - \dot{P}_1 \lambda_1 - \dot{P}_2 \lambda_2 - \dot{P} \lambda. \quad (4.57)
$$

Indeed after integrating out the momentum variables $P_1, P_2, P, \rho_1, \rho_2, \rho$ with this gauge-fixing fermion, the action becomes

$$
S = \dot{\omega}_1 \pi_{\omega_1} + \sum_{i=0}^{3} \dot{\lambda}_i b_i + \dot{\tilde{C}} \lambda_0 + \dot{\tilde{C}}_1 (\dot{\eta}_1 + \partial_1 \lambda_1) - \dot{\tilde{C}}_2 (\dot{\eta}_2 + 2 \eta \partial_1 \dot{C}_2 - 2 \pi_{\omega_1} \lambda) + \dot{\tilde{C}}_0 \rho_0
$$

$$
- \dot{\tilde{C}} \dot{\eta} - \dot{\tilde{C}}_1 \partial_1 \rho_0 - \dot{\tilde{C}}_2 \partial_1 (\dot{\eta}_2 + 2 \pi_{\omega_1} \eta_1) + \dot{\tilde{C}}_0 \partial_1 (\dot{\eta}_1 - \partial_1 \lambda_1) - \dot{\tilde{C}} \partial_1 \partial_1 \eta
$$

$$
+ b_1 \partial_1 \lambda_0 - b_2 \partial_1 \omega_1 + b_0 \partial_1 \lambda_1 + b \partial_1 \eta_1 - \lambda_1 \pi_{\omega_1}^2 + \lambda_2 \partial_1 \pi_{\omega_1}. \quad (4.58)
$$

If we rename the variables as,

$$
\pi_{\omega_1} \rightarrow \phi, \quad (4.59)
$$

$$
(\lambda_1, \lambda_2, \lambda_0, \lambda) \rightarrow (B, \omega_0, \eta, C_{\mu=0}), \quad (4.60)
$$

$$
(b_1, b_2, b_0, b) \rightarrow (b_{\mu=1}^\mu, -b, b_{\mu=0}^\mu, -b_1), \quad (4.61)
$$

$$
(\eta_1, \eta_2, \eta) \rightarrow (C_{\mu=1}, C, C_1), \quad (4.62)
$$

$$
(\tilde{C}_1, \tilde{C}_2, \tilde{C}_0, \tilde{C}) \rightarrow (C_{\mu=1}, -\tilde{C}, \tilde{C}_{\mu=0}, -\tilde{C}_1), \quad (4.63)
$$

$$
\rho_0 \rightarrow -\pi, \quad (4.64)
$$

this action completely coincides with the gauge fixed action (3.13) in the Lagrangian formalism. Note that we have not taken care of the fact that $\tilde{C}, \tilde{C}_{\mu}, C_1, \tilde{C}_1$ and $b_1$
should be treated as purely imaginary fields. The equivalence, however, continues to hold even if we insert some \( i \)'s to take it into account.

The physical space is defined by a cohomology of BRST differential. As we have mentioned above, there is a constant mode of \( \phi(=\pi_{\omega_1}) \) which belongs to the homology of the Koszul-Tate differential \( \delta \). At the same time this constant mode belongs to the cohomology of the differential \( D \). Therefore this constant mode linear in \( \phi \) is a physical degree of freedom while a constant mode of \( \phi^2 \) is not. This implies that a physical degree of freedom which does not exist at the classical level appears at the quantum level.

\section{Conclusions}

We have investigated the quantization of the generalized Chern-Simons theory in the simplest model with \( gl(1,\mathbb{R}) \) algebra in 1+1 dimensions both in the Lagrangian and Hamiltonian formulations. Since this type of theory always violates the regularity condition, its quantization would be rather naturally defined by adopting regularity violating equations. Thus in this simplest model the square of the zero form field \( \phi \), which appears as the multiplier of the highest form field and, at first sight, looks singular, is treated as one of reducible functions. We have found that the BRST invariant gauge-fixed action obtained from the Lagrangian BV formulation coincides with that from the BFV Hamiltonian formulation when we adopt these regularity violating constraints. It is surprising that a physical degree of freedom which does not exist at the classical level appears at the quantum level due to the violation of the regularity. We know that a zero form field plays an important role in more realistic generalized CS theories as emphasized in the classical discussions (refs.[4]-[6]). In particular the constant part of the zero form field became a physical order parameter of the generalized CS theory. There might be some connections between these two facts. It is thus important to investigate more realistic cases with non-abelian gauge algebra and/or in higher space-time dimensions to see a physical implication of this type of quantization.

We have taken into account neither the metric dependence, which appear in the gauge fixing part, nor the dependence on the global topology. These points will also be discussed in the future publication.
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