AN INTRODUCTION TO RELATIVE CALABI–YAU STRUCTURES

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ABSTRACT. These are notes taken by the second author for a series of three lectures by the first author on absolute and relative Calabi–Yau completions and Calabi–Yau structures given at the workshop of the International Conference on Representations of Algebras which was held online in November 2020. Such structures are relevant for (higher) representation theory as well as for the categorification of cluster algebras with coefficients. After a quick reminder on dg categories and their Hochschild and cyclic homologies, we present examples of absolute and relative Calabi–Yau completions (in the sense of Yeung). In many examples, these are related to higher preprojective algebras in the sense of Iyama–Oppermann. We conclude with the definition of relative (left and right) Calabi–Yau structures after Brav–Dyckerhoff.

1. Introduction

In 1957, the Italo-American geometer Eugenio Calabi conjectured [5] that each Kähler manifold whose first Chern class vanishes admits a Ricci flat metric. His conjecture was proved twenty years later by Shing-Tung Yau [34]. Algebraic varieties with trivial canonical bundle are now called Calabi–Yau varieties. The proper ones among these are characterized by the fact that their bounded derived category admits a Serre functor isomorphic to a power (the dimension) of the suspension functor. Following Kontsevich, a Hom-finite (algebraic) triangulated category with this property is now called a Calabi–Yau triangulated category. In these notes, we are concerned with a relative version of this notion that was first sketched by Toën [27] in 2014 and fully developed by Brav and Dyckerhoff in [3, 4]. One of the key features of their notion of (left) relative Calabi–Yau structure is a gluing construction analogous to that in cobordism of manifolds. Wai-Kit Yeung showed how to construct large classes of examples using relative Calabi–Yau completions in [19, 35] and advocated the idea that these should be viewed as noncommutative conormal bundles. This was justified using Kontsevich–Rosenberg’s criterion by Bozec–Calaque–Scherotzke in [2].

In the representation theory of quivers and finite-dimensional algebras, the motivations for studying relative Calabi–Yau structures come from at least three sources:

- applications in the study of Fukaya categories (cf. for example Brav–Dyckerhoff’s [3, 4]),
- the categorification of cluster algebras with coefficients (as in the work of Geiss–Leclerc–Schröer [7], Leclerc [21], Jensen–King–Su [13], Pressland [24, 23, 22, 25] ... as well as [32, 33, 31]),
- its close links with higher Auslander–Reiten theory (to be illustrated below, cf. also [32, 33]).

In these notes, after an informal illustration of the main notions and a quick reminder on dg (=differential graded) algebras and their derived categories, our first aim will be to...
present many examples of dg algebras and morphisms endowed with Calabi–Yau structures, respectively relative Calabi–Yau structures. These will be obtained using (relative) Calabi–Yau completions and should illustrate the relevance of these for (higher) Auslander–Reiten theory. Our second aim will be to sketch the foundations of the subject for which we will need to recall the necessary material on Hochschild and cyclic homology.

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2. Intuition and first examples

In this purely introductory section, we informally discuss the key notions and constructions to be developed in the sequel.

There is a close analogy (which actually goes deeper) between the notion of orientation of a (real, smooth) $n$-dimensional manifold $M$ and the notion of (absolute) $n$-Calabi–Yau structure on a dg algebra $A$. This extends to a relative setting\(^1\) where we obtain a close analogy between an $n$-dimensional manifold with boundary $\partial M \subset M$ both endowed with compatible orientations and a **relative** $n$-Calabi–Yau structure on a morphism $B \to A$ of dg algebras.

\[^1\]Notice that we do not write the relative setting because the setting considered here is not the relative one in the usual sense of algebraic geometry.
to this morphism, we find the algebra morphism from the polynomial algebra $k[t]$ to the Auslander algebra of the truncated polynomial algebra $k[x]/(x^3)$ which takes the unique object to the indecomposable projective $P = k[x]/(x^3)$ and the indeterminate $t$ to the multiplication by $x$.

In Figure 2, we consider an example of a relative 3-Calabi–Yau completion. The given morphism is the inclusion of the path algebra $A$ of the linearly oriented $A_3$-quiver into its Auslander algebra. The 3-Calabi–Yau completion is a morphism from the 2-dimensional Ginzburg algebra of type $A_3$ to a relative Jacobian algebra, namely the one associated with the ice quiver in the lower half of the figure endowed with the potential given by the difference between the sum of the 3-cycles rotating clockwise and those rotating counterclockwise.

3. Complements on dg algebra resolutions

Let $k$ be a field. Let $Q$ be a graded quiver, i.e. a quiver $(Q_0, Q_1, s, t)$ endowed with a function $|\cdot| : Q_1 \to \mathbb{Z}$. For $n \in \mathbb{Z}$ we write $Q_1^n$ for the set of arrows $\alpha$ of $Q$ of degree $|\alpha| = n$. The associated path algebra $kQ$ then becomes a graded algebra if we define the degree of a path $a_1 \ldots a_l$ to be the sum of the degrees of the composable arrows $a_i$. As a typical example, let $A$ be the path algebra of the quiver

$$
1 \xrightarrow{b} 2 \xrightarrow{a} 3
$$

subject to the relation $ab = 0$ (all arrows in degree 0). Let $\tilde{A}$ be the graded path algebra of the graded quiver

$$
1 \xrightarrow{b} 2 \xrightarrow{a} 3
$$

where $a$ and $b$ have degree 0 and $c$ has degree $-1$. We endow the graded path algebra $\tilde{A}$ with the unique algebra differential such that $d(c) = ab$. Then the algebra morphism $\tilde{A} \to A$ taking $a$ to $a$, $b$ to $b$ and $c$ to zero is compatible with the differential (that of $A$ being zero) and is in fact a quasi-isomorphism of dg algebras. This is an example of a cofibrant dg algebra resolution.

**Proposition 3.1.** For each quotient $A = kQ/I$ of a path algebra with finite $Q_0$, there is a dg algebra quasi-isomorphism

$$
\tilde{A} \xrightarrow{\varepsilon} A
$$
where \( \tilde{A} = (k\tilde{Q}, d) \) for a non positively graded quiver \( \tilde{Q} \) with \( \tilde{Q}_0 = Q_0, \tilde{Q}_1^0 = Q_1 \) such that \( \varepsilon \) induces the identity \( \tilde{Q}_0 = Q_0 \) and the canonical projection \( k\tilde{Q}_1 \rightarrow kQ_1 \). The morphism \( \varepsilon \) is a cofibrant dg algebra resolution.

In general, there is no ‘minimal choice’ for \( \tilde{A} \). However, there is if \( kQ/I \) is a monomial algebra, cf. [26], or if we work in the setup of pseudocompact dg algebras, cf. [29].

4. Derived categories and the inverse dualizing bimodule

Let \( k \) be a perfect field and \( A \) a dg algebra. Recall that \( \mathcal{C}A \) denotes the category of right dg \( A \)-modules, \( \mathcal{H}A \) the associated category up to homotopy and \( \mathcal{D}A \) the derived category, i.e. the localization of \( \mathcal{C}A \) or \( \mathcal{H}A \) at the class of all quasi-isomorphisms. It is triangulated with suspension functor \( \Sigma : \mathcal{D}A \rightarrow \mathcal{D}A \) given by the shift of dg modules and each short exact sequence of dg modules yields a canonical triangle. The derived category \( \mathcal{D}A \) admits arbitrary (set-indexed) coproducts. The perfect derived category \( \text{per} (A) \) is its subcategory of compact objects. It equals the thick subcategory of \( \mathcal{D}A \) generated by the free module \( A_A \) of rank 1.

The perfectly valued derived category \( \text{pvd} (A) \) is the full subcategory of \( \mathcal{D}A \) whose objects are the dg modules \( M \) whose underlying dg \( k \)-module is perfect. Equivalently, we may require that we have

\[
\sum_{p \in \mathbb{Z}} \dim \text{Hom}(P, \Sigma^p M) < \infty
\]
for each perfect dg module $P$ or that we have

$$\sum_{p \in \mathbb{Z}} \dim H^p(M) < \infty.$$ 

The dg algebra $A$ is proper if it belongs to $\text{pvd}(A)$ or, in other words, if the sum of its homologies is finite-dimensional. For example, if $A = A^0$ is concentrated in degree 0, then $A$ is proper if and only if $A^0$ is finite-dimensional and in this case, we have

$$\text{pvd}(A) = \mathcal{D}^b(\text{mod} A^0),$$

where mod $A^0$ denotes the category of $k$-finite-dimensional right $A^0$-modules.

The dg algebra $A$ is connective if $H^pA = 0$ for all $p > 0$. In this case, the canonical morphism $\tau_{\leq 0}A \to A$ of dg algebras is a quasi-isomorphism so that for most purposes, we may assume that $A_p = 0$ for $p > 0$. We say that $A$ is a stalk dg algebra or simply stalk algebra if $H^p(A) = 0$ for all $p \not= 0$. In this case, $A$ is linked to the dg algebra $H^0(A)$ concentrated in degree 0 via the two dg algebra quasi-isomorphisms

$$H^0(A) \leftarrow \tau_{\leq 0}A \to A.$$

Let $A$ be a dg algebra. Its enveloping algebra is

$$A^e = A \otimes A^{\text{op}}$$

so that right $A^e$-modules identify with $A$-$A$-bimodules via the rule

$$m.(a \otimes b) = (-1)^{|b||a|}b ma.$$

For a dg bimodule $M$, its bimodule dual $M^\vee$ is defined by

$$M^\vee = \text{RHom}_{A^e}(M, A^e).$$

We still view it as an object of $\mathcal{D}(A^e)$ using the canonical isomorphism $(A^e)^{op} = A^e$. More explicitly, the derived Hom-space is computed using the outer bimodule structure on $A^e$ and the bimodule structure on $\text{RHom}$ comes from the inner bimodule structure on $A^e$.

Notice that if $P$ is perfect in $\mathcal{D}(A^e)$, we have a canonical isomorphism

$$P \xrightarrow{\sim} (P^\vee)^\vee.$$

Indeed, it suffices to check this for $P = A_A$ and then it is clear.

The dg algebra $A$ is (homologically) smooth if the identity bimodule $A_A A$ is perfect in $\mathcal{D}(A^e)$. For example, if $A = A^0$ is concentrated in degree 0, then $A$ is smooth if and only if $A_A A$ has a bounded resolution by finitely generated projective bimodules. If $A = A^0$ is finite-dimensional, then $A$ is smooth if and only if it is of finite global dimension (here we use the assumption that $k$ is perfect!).

**Lemma 4.1.**

a) If $A$ is smooth, the category $\text{pvd}(A)$ is Hom-finite.

b) If $Q$ is a finite non positively graded quiver, then for any choice of algebra differential $d$ on the graded path algebra $kQ$, the dg algebra $(kQ, d)$ is smooth.

Suppose that $A$ is a smooth dg algebra. Its inverse dualizing bimodule is

$$\Omega_A = A^\vee = \text{RHom}_{A^e}(A, A^e).$$

Denote by $D$ the duality $\text{Hom}_k(?, k)$ over the ground field. The following is Lemma 4.1 of [18].
Lemma 4.2. For $L \in \text{pvd}(A)$ and $M \in D_A$, we have a canonical isomorphism

$$D\text{Hom}_{D_A}(L, M) \cong \text{Hom}_{D_A}(M \otimes_A \Omega_A, L).$$

Corollary 4.3. The functor $S^{-1}L = L \otimes_A \Omega_A$ induces an inverse Serre functor on $\text{pvd}(A)$, i.e. an autoequivalence such that we have isomorphisms

$$D\text{Hom}_{D_A}(L, M) \cong \text{Hom}_{D_A}(S^{-1}M, L)$$

which are bifunctorial in $L, M \in \text{pvd}(A)$.

5. Calabi–Yau completions

Fix a perfect field $k$. Let $A$ be a dg $k$-algebra. Fix an integer $n \in \mathbb{Z}$. By definition, a bimodule $n$-Calabi–Yau structure on $A$ is an isomorphism $\Sigma^n \Omega_A \Rightarrow A$ in $D(A^e)$. Notice that by Lemma 4.2, the category $\text{pvd}(A)$ is then an $n$-Calabi–Yau category, i.e. it is Hom-finite and $\Sigma^n$ is a Serre functor. We say that $A$ is a bimodule $n$-Calabi–Yau dg algebra if it is endowed with a bimodule $n$-CY structure.

Exercise 5.1. Suppose that $A$ is bimodule $n$-Calabi–Yau and $H^0(A)$ is not semi-simple.

a) Show that if $A$ is connective and $H^0A$ is finite-dimensional, then $H^pA \neq 0$ for infinitely many $p < 0$.

b) Show that if $A$ is a stalk dg algebra, then $H^0A$ is infinite-dimensional.

5.2. Absolute Calabi–Yau completions. Suppose that $B$ is a smooth dg algebra. Let $n \in \mathbb{Z}$ be an integer and put

$$\omega = \Sigma^{n-1} \Omega_B.$$

We may and will assume that $\omega$ is cofibrant as a dg $B$-bimodule. Following [19], we define the $n$-Calabi–Yau completion of $B$ to be the tensor dg algebra

$$\Pi_n B = T_B(\omega) = B \oplus \omega \oplus (\omega \otimes_B \omega) \oplus \cdots \oplus \omega^{\otimes_B p} \oplus \cdots.$$ 

It is not hard to check that up to quasi-isomorphism, $\Pi_n B$ does not depend on the choice of $\omega$ in its homotopy class.

Theorem 5.3 ([19, 35, 20]). The dg algebra $\Pi_n B$ is smooth and carries a canonical bimodule $n$-Calabi–Yau structure. In particular, the category $\text{pvd}(\Pi_n B)$ is $n$-Calabi–Yau.

For example, consider $B = k$. Then $\Omega_B = k$ and $\Pi_n(B) = k[t]$, where $t$ is of degree $1 - n$ and $\Pi_nB$ carries the zero differential. In particular, $\Pi_n k$ is concentrated in degree 0 iff $n = 1$. As another example, let $Q$ be a connected non Dynkin quiver and $B = kQ$. Then the 2-Calabi–Yau completion $\Pi_2 B$ has its homology concentrated in degree 0 and is quasi-isomorphic to the preprojective algebra of $Q$.

Connective dg algebras form a particularly important class. It is natural to ask to which extent this class is stable under forming CY-completions. This question is not hard to answer: Suppose that $B$ is a smooth and connective dg algebra. Define the bimodule dimension $d$ of $B$ to be the supremum of the integers $p$ such that $H^p \Omega_B \neq 0$. Then $\Pi_n B$ is connective if and only if $n \geq d + 1$.

Let us investigate 2-CY-completions. Let $B$ be a smooth dg algebra. For $n = 2$, the bimodule $\omega$ is $\omega = \Sigma \Omega_B$ and the functor

$$\otimes_B \omega = \otimes_B(\Sigma \Omega_B) = \Sigma \circ (\otimes_B \Omega_B)$$

where
induces the composed functor

\[ \tau^{-1} = S^{-1}\Sigma \]

in the perfect derived category \( \text{pvd}(B) \), where \( S^{-1} \) is the inverse Serre functor. We denote this functor by \( \tau^{-1} \) because if \( B \) is a finite-dimensional algebra of finite global dimension, then it is the inverse Auslander–Reiten translation of the category \( \text{pvd}(B) = \mathcal{D}^b(\text{mod} B) \). For arbitrary smooth and proper \( B \), the restriction of \( \Pi_2(B) \) to a dg \( B \)-module is

\[ (\Pi_2(B))|_B = T_B(\omega)|_B = B \oplus (B \otimes_B \omega)^L \oplus (B \otimes_B \omega \otimes_B \omega)^L + \cdots = \bigoplus_{p \geq 0} \tau^{-p}B. \]

Notice that in general, it will not be perfectly valued. Now suppose that \( B = kQ \) for a connected non Dynkin quiver \( Q \). Then we know that \( \tau^{-p}B \) lies in \( \text{mod} B \) for all \( p \geq 0 \) so that \( \Pi_2(B) \) is a stalk dg algebra:

\[ \Pi_2(B) \cong H^0(\Pi_2 B) = T_B(H^0 \omega). \]

It follows that \( \Pi_2(B) \) is quasi-isomorphic to the classical preprojective algebra of \( Q \) by the description of this algebra as a tensor algebra due to Baer–Geigle–Lenzing [1]. If \( Q \) is an arbitrary finite acyclic quiver and \( B = kQ \), then \( \Pi_2(B) \) can be described as the 2-dimensional Ginzburg algebra of \( B \). For example, for the quiver

1 \( \xrightarrow{b} \) 2 \( \xrightarrow{a} \) 3

the dg algebra \( \Pi_2(B) \) is given by the graded quiver

\[ Q : \begin{array}{ccc} 1 & \xleftarrow{\bar{b}} & 2 & \xrightarrow{a} & 3 \\ & t_1 & t_2 & t_3 & \\ \bar{b} & \downarrow \bar{b} & \downarrow \bar{b} & \downarrow \bar{b} & \end{array} \]

with the arrows \( a, \bar{b}, \bar{a} \) and \( \bar{b} \) in degree 0 and the three loops \( t_i \) in degree \(-1\). The differentials of the loops yield the preprojective relations:

\[ dt_1 = -\bar{b}b, \quad dt_2 = \bar{b}\bar{a} - \bar{a}a, \quad dt_3 = a\bar{a}. \]

Thus, we always have an isomorphism between \( H^0(\Pi_2(B)) \) and the classical preprojective algebra but the homologies \( H^p(\Pi_2(B)) \) are non zero in infinitely many degrees \( p < 0 \) unless all connected components of \( Q \) are non Dynkin.

Let us now consider an example of a 3-CY-completion. Consider the Auslander algebra \( B \) of the equioriented \( A_3 \)-quiver given by

Using a cofibrant dg algebra resolution of \( B \) (cf. 3.1) it is not hard to check that \( \Pi_3(B) \) is quasi-isomorphic to the 3-dimensional Ginzburg algebra of the quiver with potential
$(R, W)$, where $R$ is the quiver of the ‘relation completion’ of $B$

and $W$ the difference of the sum of the 3-cycles of $R$ turning clockwise minus the unique 3-cycle turning counterclockwise. For completeness, let us recall how to construct the 3-dimensional Ginzburg algebra $\Gamma(R, W)$ associated with a quiver with potential $(R, W)$: starting from $R$, construct a quiver $\tilde{R}$ as follows:

a) for each arrow $a : i \to j$ of $R$, add an arrow $\overline{a} : j \to i$ of degree $-1$ and

b) for each vertex $i$ of $R$, add a loop $t_i : i \to i$ of degree $-2$.

Now define the differential on the graded path algebra $k\tilde{R}$ by

$$d(t_i) = e_i \sum_{a \in R_1} (a\overline{a} - \overline{a}a) e_i$$

for each vertex $i$ of $\tilde{R}$ and

$$d\overline{a} = \partial a W$$

for each arrow $a$ of $R$. Here $\partial a : HH_0(kR) \to kR$ is the cyclic derivative defined on a path $p$ by

$$\partial a p = \sum_{p = uav} vu$$

where the sum ranges over all decompositions $p = uav$ with paths $u, v$ of length $\geq 0$. For example, starting from

with the potential $W = abc$ we obtain the quiver $\tilde{R}$

with the differential determined by

$$d(t_1) = c\overline{a} - \overline{b}b, \ d(t_2) = a\overline{a} - \overline{c}c, \ d(t_3) = \overline{b}b - \overline{a}a$$

and

$$d(\overline{a}) = bc, \ d(\overline{b}) = ca, \ d(\overline{c}) = ab.$$
6. Relative Calabi–Yau completions

Let \( k \) be a perfect field and \( A \) and \( B \) smooth dg \( k \)-algebras. Let \( f : B \to A \) be a dg algebra morphism. Recall that we do not require \( f \) to preserve the unit. A typical example would be the inclusion of a finite-dimensional representation-finite algebra of finite global dimension into its Auslander algebra. Let \( n \) be an integer.

Following Yeung [35], we make the following definitions:

- a) the **relative inverse dualizing bimodule** is the bimodule dual of the cone over the natural morphism \( A \otimes_B A \to A \) considered as an object in \( \mathcal{D}(A^e) \).

- b) the **\( n \)-dimensional relative derived preprojective algebra** of \( B \to A \) is 
  \[ \Pi_n(A, B) = T_A(\omega), \]
  where \( \omega \) is a cofibrant resolution of \( \Sigma^{n-1}\Omega_{A,B} \).

- c) the **relative \( n \)-Calabi–Yau completion** of \( f : B \to A \) is the canonical morphism of dg algebras 
  \[ \Pi_{n-1}(B) \to \Pi_n(A, B). \]

Let us explain how the canonical morphism in c) is obtained: By construction, we have a triangle in \( \mathcal{D}(A^e) \)
\[ \Omega_{A,B} \to A^\vee \to (A \otimes_B A)^\vee \to \Sigma \Omega_{A,B}. \]

This yields morphisms
\[ \Omega_B = B^\vee \to A \otimes_B L_B \otimes_B A \cong (A \otimes_B A)^\vee \to \Sigma \Omega_{A,B}. \]

The canonical morphism between the tensor algebras is induced by their composition. We will define the notion of a relative (left) \( n \)-CY structure below. Assuming it we can state the following theorem.

**Theorem 6.1** (Yeung [35], Bozec–Calaque–Scherotzke [2]). The dg algebra \( \Pi_n(A, B) \) is smooth and the morphism 
\[ \Pi_{n-1}(B) \to \Pi_n(A, B) \]
carries a canonical relative (left) \( n \)-CY structure.

Let us emphasize that all constructions and theorems generalize easily from the setting of dg algebras and morphisms to that of dg categories and dg functors and are proved in this generality in the references.

7. Examples of relative 2-Calabi–Yau completions

Let us recall the following example from section 2: Let us take \( B \) to be \( k \), the algebra \( A \) to be the path algebra of the quiver \( 1 \to 2 \to 3 \) and \( i \) the morphism given by the inclusion of the vertex 3 into this quiver as in Figure 3. If we apply the relative 2-Calabi–Yau completion to this morphism, we find a morphism of dg algebras \( \Pi_1(B) \to \Pi_2(A, B) \) which, remarkably, are both stalk algebras: the dg algebra \( \Pi_1(B) \) is quasi-isomorphic to the polynomial algebra \( k[t] \) and the dg algebra \( \Pi_2(A, B) \) to the truncated polynomial algebra \( k[x]/(x^3) \). The dg functor \( \Pi_1(B) \to \Pi_2(A, B) \) takes the unique object to the indecomposable projective \( P = k[x]/(x^3) \) and the indeterminate \( t \) to the multiplication by \( x \).

More generally, let us consider a finite quiver \( Q \) and a subset \( F \subset Q_0 \) of frozen vertices. We consider \( F \) as a subquiver with empty set of arrows. Then we have a natural algebra...
morphism $kF \to kQ$. We then find that $\Pi_2(kQ, kF)$ is the 2-dimensional relative Ginzburg algebra of $(Q, F)$. For example, for the quiver

$$
\bullet \xrightarrow{b} \bullet \xrightarrow{a} \bullet
$$

we find that $\Pi_2(kQ, kF)$ is given by

$$
\bullet \xrightarrow{t_1} \bullet \xrightarrow{t_2} \bullet
$$

where the arrows $a, b, \pi$ and $\beta$ are in degree 0, the loops $t_i$ in degree $-1$ and the differential is determined by

$$
d(t_1) = -\beta b, \quad d(t_2) = b \beta - \pi a.
$$

Thus, the algebra $H^0 \Pi_2(kQ, kF)$ is the relative preprojective algebra: it has the same quiver as the classical preprojective algebra but no relations at the frozen vertices. In our example here, we see that remarkably, $\Pi_2(kQ, kF)$ is a stalk algebra. Here is another example:

Here we start from a ‘framed’ quiver of type $D_4$ and obtain again a relative preprojective algebra (in particular a stalk algebra) that could be called the ‘Nakajima algebra’ because its (stable) representations (up to isomorphism) with a given dimension vector identify with the points of the corresponding (regular) Nakajima quiver variety.

As a final example, let us consider

$$
\bullet \xrightarrow{a} \bullet \xrightarrow{b} \bullet \xrightarrow{c} \bullet
$$

Again, the dg algebra $\Pi_2(A, B)$ is a stalk algebra. Let us consider its variant $\tilde{\Pi}_2(A, B)$ where we replace the path algebra with the completed path algebra. This is easily seen to be isomorphic to the Auslander algebra of the Bass order

$$
B_3 = \begin{bmatrix}
R & Rx^3 \\
R & R
\end{bmatrix},
$$
where $R = k[[x]]$ whose indecomposables are the $[R, Rx^i], 0 \leq i \leq 4$. For more information on Bass orders, we refer to page 72, of [9].

8. Examples of relative 3-Calabi–Yau completions

Consider the following example of a 3-CY-completion:

![Diagram showing a 3-dimensional Ginzburg algebra]

Here, the morphism $B \to A$ on the left hand side is the embedding of the path algebra of the quiver $A_2$ into its Auslander algebra. On the right hand side, we have the (absolute) 2-dimensional Ginzburg algebra $\Pi_2(B)$ at the top and the 3-dimensional relative preprojective algebra $\Pi_3(A, B)$ at the bottom. It is quasi-isomorphic to the relative 3-dimensional Ginzburg algebra of the ice quiver with potential $(R, W)$ with $R$ given by

![Diagram showing an ice quiver]

and the potential $W = abc$. The construction of the relative 3-dimensional Ginzburg algebra is similar to that of the absolute one but

- we do not add reversed arrows $\overline{b}$ for frozen arrows $b$ and
- we do not add loops $t_i$ for frozen vertices $i$.

In the example here, the relative 3-dimensional algebra turns out to be a stalk algebra, i.e. quasi-isomorphic to the corresponding relative Jacobian algebra, which is given by

![Diagram showing a stalk algebra]

Note that the second example in section 2 is the analogous example for equioriented $A_4$ instead of $A_2$. In the next section, we will exhibit a general framework into which these examples fit.
9. Higher Auslander Algebras and the Stalk Property in Dimension $\geq 3$

Let $n \geq 1$ be an integer and $B$ a finite-dimensional algebra of finite global dimension over a perfect field $k$. Denote by $S$ the Serre functor of the bounded derived category $D^b(B) = D^b(\text{mod } B)$. Explicitly, we have

$$S = \tau \otimes_B DB.$$ 

The composition of functors

$$\text{mod } B \xrightarrow{\text{can}} D^b(B) \xrightarrow{\Sigma^n S^{-1}} D^b(B) \xrightarrow{H^0} \text{mod } B$$

equals by definition $\tau_n^{-1}$, the higher inverse Auslander-Reiten translation introduced by Iyama [10]. Define

$$\mathcal{M} = \text{add } \{ \tau_n^{-p} B \mid p \geq 0 \}.$$ 

The motivating example is the case where $n = 1$ and $B$ is the path algebra $kQ$ of a Dynkin quiver $Q$. Then $B$ is 1-dimensional and $\text{mod } (B) = \mathcal{M}$ is 2-dimensional and in the previous section, we have seen examples where $\Pi_3(\text{mod } (B), \text{proj } (B))$ is concentrated in degree 0.

Put $B = \text{proj } B$ and $A = \mathcal{M}$. As shown by Iyama in [10], if $\text{mod } B$ admits an $n$-cluster-tilting object $M$, then the category $\mathcal{M}$ equals $\text{add } (M)$. Let us assume that the category $\mathcal{A}$ (considered as a dg category concentrated in degree 0) is smooth (this assumption can be weakened to ‘local smoothness’). We consider the question of when the relative $(n+2)$-dimensional derived preprojective algebra $\Pi_{n+2}(\mathcal{A}, B)$ is a stalk category, i.e. has its homology concentrated in degree 0. Recall that we have

$$\Pi_{n+2}(\mathcal{A}, B) = T_B(\omega),$$

where $\omega = \Sigma^{n+1} \Omega_{A,B}$ and $\Omega_{A,B}$ is a cofibrant replacement of the bimodule dual of the cone over

$$A \otimes_B A \rightarrow A.$$

For $M \in \mathcal{M}$, let us abbreviate

$$M^\wedge = \text{Hom}_B(?, M) |_{\mathcal{A}}$$

considered as a finitely generated projective module over $\mathcal{A} = \mathcal{M}$. We have the following key lemma.

**Key lemma 9.1.** Suppose that $\mathcal{M}$ is $n$-rigid, i.e. we have

$$\text{Ext}^i_B(L, M) = 0$$

for all $L, M$ in $\mathcal{M}$ and all $1 \leq i \leq n - 1$. Then for $M \in \mathcal{M}$, we have a canonical isomorphism

$$M^\wedge \otimes_A \omega \xrightarrow{\sim} (\tau_n^{-1} M)^\wedge.$$ 

A proof can be found in Proposition 8.6 of [33].

**Corollary 9.2.** If $\mathcal{M}$ is $n$-rigid, then the dg algebra $\Pi_{n+2}(\mathcal{M}, \text{proj } B)$ is concentrated in degree 0.

Notice that the $n$-rigidity assumption holds if

- $B$ is $n$-representation finite in the sense of Iyama–Oppermann [12] or
- $B^{op}$ is $n$-complete in the sense of [11] or
- $B$ is $n$-representation infinite in the sense of Herschend–Iyama–Oppermann [8]
10. Absolute and relative Calabi–Yau structures

We have announced in the preceding sections that the $n$-CY-completion $\Pi_n(B)$ of a smooth dg algebra carries a canonical (left) $n$-Calabi–Yau structure and that the relative $n$-CY-completion of a morphism $B \to A$ between smooth dg algebras carries a canonical (left) relative $n$-CY-structure. Our aim in this section is to define these structures and their right counterparts.

10.1. A reminder on cyclic homology. Let $k$ be a field and $A$ a $k$-algebra (associative, with 1 but in general noncommutative). We consider $A$ as a bimodule over itself, i.e. as a module over the enveloping algebra $A_e = A \otimes A^{op}$. Recall that its bar resolution $R$ is the complex whose component in homological degree $p \geq 0$ is $A \otimes A \otimes A^{p} \otimes A$ and whose differential $b'$ is defined for $p \geq 1$ by

$$b'(a_0 \otimes a_1 \cdots \otimes a_p \otimes a_{p+1}) = \sum_{i=0}^{p} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{p+1}.$$ 

The Hochschild chain complex is $CA = R \otimes_{A_e} A$. Its component in (homological) degree $p$ is isomorphic to $A \otimes A^{p}$ and we write $b$ for its differential. It represents the derived tensor product $A \otimes_{A_e} A$ in the derived category of vector spaces and its homology in (homological) degree $n$ is the $n$th Hochschild homology $HH_n(A) = H_n(R \otimes_{A_e} A) = H_n(A \otimes_{A_e} A) = \text{Tor}_n^A(A, A)$.

For example, we find that we have $HH_0(A) = A/[A, A]$, where $[A, A]$ denotes the subspace of $A$ generated by the commutators $[a, b] = ab - ba, a, b \in A$.

It is a remarkable fact due to Connes [6] that the Hochschild chain complex admits a canonical ‘circle action up to homotopy’. Algebraically, following Kassel [14], this is encoded by the structure of a mixed complex, i.e. an action of the dg algebra $\Lambda = k[\varepsilon]/(\varepsilon^2)$, where $\varepsilon$ is of cohomological degree $-1$, endowed with the differential $d = 0$. Notice that $\Lambda$ is isomorphic to the singular homology $H_*(S^1, k)$ of the circle $S^1$ and quasi-isomorphic to the dg algebra of singular chains on $S^1$. It carries the structure of graded commutative Hopf algebra with coproduct given by $\Delta(\varepsilon) = 1 \otimes \varepsilon + \varepsilon \otimes 1$. Connes has given explicit formulas for the $\Lambda$-action on $CA$ which involve the unit $1 \in A$. Since we need functoriality with respect to algebra morphisms which do not necessarily preserve the unit, we replace $CA$ with a quasi-isomorphic complex $MA$, which we call the mixed complex of $A$, and which is obtained as the total complex of the first two columns of the Connes–Quillen bicomplex.

Here, as above, the symbol $b'$ denotes the differential of the bar resolution and $b$ the differential of the Hochschild chain complex while $t$ is defined by

$$t(a_1 \otimes \cdots \otimes a_n) = (-1)^{n-1} a_2 \otimes \cdots \otimes a_n \otimes a_1 \quad \text{and} \quad N = 1 + t + \cdots + t^{n-1}.$$
We write $d$ for the differential of $MA$ and $d' : MA \to \Sigma^{-1}MA$ for its endomorphism of degree $-1$ induced by $N$. Then we have

$$d^2 = 0, \quad d'^2 = 0 \quad \text{and} \quad dd' + d'd = 0$$

so that $MA$ becomes a dg module over $\Lambda$, where $\varepsilon$ acts via $d'$. In the sequel, we consider $MA$ as an object in the derived category of mixed complexes, i.e. the derived category $D\Lambda$ of dg $A$-modules. It is then clear from the construction that $MA$ is functorial in $A$ even with respect to algebra morphisms which do not necessarily respect the unit, for example the inclusion $A \hookrightarrow M_2(A)$ sending $a \in A$ to the matrix

$$\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}.$$  

We have a canonical inclusion $CA \to MA$, which is a quasi-isomorphism because the second column is the augmented bar resolution, which is acyclic. Thus, we have an isomorphism

$$A \overset{L}{\otimes}_{A^e} A \cong MA$$

in $Dk$. In particular, we have isomorphisms

$$HH_n(A) = H^{-n}MA = H^{-n}(MA \overset{L}{\otimes}_{A} \Lambda) = H^{-n}(\text{RHom}_{A}(\Lambda, MA)).$$

One defines cyclic homology as the ‘coinvariants’ of the homotopy circle action

$$HC_n(A) = H^{-n}(MA \overset{L}{\otimes}_{A} k)$$

and negative cyclic homology as the ‘invariants’

$$HN_n(A) = H^{-n}(\text{RHom}_{A}(k, MA)).$$

Many authors write $HC_n^{-}(A)$ for $HN_n(A)$ but we prefer not putting the essential information into a barely visible exponent. The augmentation $\Lambda \to k$ yields canonical maps

$$HN_n(A) \longrightarrow HH_n(A) \longrightarrow HC_n(A).$$

For a morphism of algebras $B \to A$ (not necessarily preserving the unit), we define the relative mixed complex $M(A, B)$ as the cone over the morphism $MB \to MA$. This yields the definitions of relative cyclic, Hochschild and negative homologies denoted respectively by $HC_n(A, B)$, $HH_n(A, B)$ and $HN_n(A, B)$.

It is easy to extend these constructions from algebras to dg algebras and further to dg categories. For example, if $\mathcal{A}$ is a dg category, the bar resolution of the identity bimodule $\mathcal{A} (?, -)$ is given by the sum total complex of the double complex whose $p$th term is given by

$$\prod \mathcal{A}(A_p, -) \otimes \mathcal{A}(A_{p-1}, A_p) \otimes \cdots \otimes \mathcal{A}(?, A_0),$$

where the sum is over all sequences of objects $A_0, \ldots, A_p$ of $\mathcal{A}$ and the differential is given by the same formula as in the case of an algebra.

We recall the following facts from [16]:

**Theorem 10.2.** a) Let $\mathcal{A}$ and $\mathcal{B}$ be dg categories and $F : \mathcal{A} \to \mathcal{B}$ a Morita functor, i.e. a dg functor such that the restriction $F_* : D\mathcal{B} \to D\mathcal{A}$ is an equivalence. Then $MF : MA \to MB$ is an isomorphism in $Dk$. 


b) Let
\[ 0 \longrightarrow A \xrightarrow{F} B \xrightarrow{G} C \longrightarrow 0 \]
be an exact sequence of dg categories, i.e. the induced sequence
\[ 0 \longrightarrow \mathcal{D}A \xrightarrow{F^*} \mathcal{D}B \xrightarrow{G^*} \mathcal{D}C \longrightarrow 0 \]
is an exact sequence of triangulated categories, where \( F^* \) is left adjoint to the restriction \( F_* \). Then there is a canonical triangle in \( \mathcal{D}A \)
\[ MA \xrightarrow{ME} MB \xrightarrow{MG} MC \longrightarrow \Sigma MA. \]

Notice that a) implies derived Morita invariance for the three invariants and that b) yields long exact sequences.

10.3. Absolute left and right Calabi–Yau-structures. Let \( \mathcal{A} \) be a smooth dg category. We have canonical isomorphisms in \( Dk \)
\[ \mathcal{A} \otimes_{\mathcal{A}^e} \mathcal{A} \cong \mathcal{A} \otimes_{\mathcal{A}^e} (\mathcal{A}^\vee)^\vee \cong \text{RHom}_{\mathcal{A}^e}(\mathcal{A}^\vee, \mathcal{A}). \]
We deduce a canonical map
\[ HN_n(\mathcal{A}) \rightarrow HH_n(\mathcal{A}) \rightarrow H^{-n}(\mathcal{A} \otimes_{\mathcal{A}^e} \mathcal{A}) \rightarrow \text{Hom}_{\mathcal{D}A^e}(\mathcal{A}^\vee, \Sigma^{-n}\mathcal{A}). \]

**Definition 10.4** (Kontsevich). A left \( n \)-CY-structure on \( \mathcal{A} \) is a class in \( HN_n(\mathcal{A}) \) whose image in \( \text{Hom}_{\mathcal{D}A^e}(\mathcal{A}^\vee, \Sigma^{-n}\mathcal{A}) \) under the above map is an isomorphism.

We see that if \( \mathcal{A} \) carries a left \( n \)-CY-structure, it is in particular a bimodule \( n \)-Calabi–Yau. The negative cyclic homology group appears naturally when one studies the deformations of bimodule \( n \)-CY-algebras, cf. [30]. One can show that each \( n \)-CY-completion of a smooth dg category carries a canonical left \( n \)-CY-structure, cf. [20, 2].

Now let \( \mathcal{A} \) be a proper dg category (i.e. we have \( \mathcal{A}(X,Y) \in \text{per}(k) \) for all \( X,Y \in \mathcal{A} \)). Let \( \mathcal{D}A^{op} \) denote the dg bimodule
\[ (X,Y) \mapsto \mathcal{D}A(Y,X). \]
We have canonical isomorphisms
\[ D(\mathcal{A} \otimes_{\mathcal{A}^e} \mathcal{A}) \cong \text{Hom}_k(\mathcal{A} \otimes_{\mathcal{A}^e} \mathcal{A}, k) \cong \text{RHom}_{\mathcal{A}^e}(\mathcal{A}, \mathcal{D}A^{op}). \]
We deduce morphisms
\[ DHC_{-n}(\mathcal{A}) \rightarrow DHH_{-n}(\mathcal{A}) \cong H^{-n}\text{RHom}_{\mathcal{A}^e}(\mathcal{A}, \mathcal{D}A^{op}) = \text{Hom}_{\mathcal{D}A^e}(\mathcal{A}, \Sigma^{-n}\mathcal{D}A^{op}). \]

**Definition 10.5** (Kontsevich). A right \( n \)-CY-structure on \( \mathcal{A} \) is a class in \( DHC_{-n}(\mathcal{A}) \) which yields an isomorphism \( \mathcal{A} \cong \Sigma^{-n}\mathcal{D}A^{op} \).

If \( \mathcal{A} \) carries a right \( n \)-CY-structure, we have the Serre duality formula
\[ \mathcal{D}A(X,Y) \cong \Sigma^n\mathcal{A}(Y,X) \]
bifunctorially in \( X,Y \in \mathcal{A} \). In particular we see that \( \mathcal{A} \) is perfectly \( n \)-Calabi–Yau, i.e. the perfect derived category \( \text{per}(\mathcal{A}) \) is \( n \)-Calabi–Yau as a triangulated category. The definition above extends to ‘componentwise proper’ dg categories, i.e. dg categories \( \mathcal{A} \) such that \( HP^\mathcal{A}(X,Y) \) is finite-dimensional for all \( p \in \mathbb{Z} \) and all \( X,Y \in \mathcal{A} \). These are important for our applications. For example, if \( Q \) is a finite acyclic quiver, then the canonical dg enhancement
\[ (\mathcal{C}_Q)_{dg} = \text{per}(\Gamma_Q)/\text{pvd}(\Gamma_Q) \]
of the classical cluster category $\mathcal{C}_Q$ is componentwise proper but not proper. Here the subscripts dg on the right hand side denote the corresponding subcategories of the dg derived category of $\Gamma_{Q,0} = \Pi_3(kQ)$, i.e. the canonical dg enhancement of the derived category. For ‘componentwise proper’ dg categories endowed with a right $n$-CY-structure, the perfect derived category per $(\mathcal{A})$ is still a Hom-finite $n$-CY triangulated category.

Suppose that $\mathcal{A}$ is a dg category which is both smooth and proper. Then, as shown by Toën–Vaquié in [28], the Yoneda functor yields a quasi-equivalence

$$\mathcal{A} \xrightarrow{\sim} \operatorname{pvd}_{dg}(\mathcal{A}).$$

Now suppose that moreover $\mathcal{A}$ is augmented (in the sense of [15]) and $\operatorname{pvd}_{dg}(\mathcal{A})$ is generated, as a triangulated category, by the image of the restriction functor $\operatorname{pvd}(\mathcal{A}) \to \operatorname{pvd}(\mathcal{A})$, where $\mathcal{A} \to \overline{\mathcal{A}}$ is the augmentation. Then we have a Morita equivalence [17]

$$\mathcal{A}^! \xrightarrow{\sim} \operatorname{pvd}_{dg}(\mathcal{A}).$$

We deduce an isomorphism

$$\operatorname{HN}_n(\mathcal{A}) \xrightarrow{\sim} \operatorname{HN}_n(\mathcal{A}^!) \xrightarrow{\sim} \operatorname{DHC}_{-n}(\mathcal{A}).$$

**Proposition 10.6.** This isomorphism establishes a bijection between the left and the right $n$-Calabi–Yau structures on $\mathcal{A}$.

Left CY-structures are inherited by dg localizations whereas right CY-structures are inherited by full dg subcategories. More precisely, we have the following proposition.

**Proposition 10.7 ([19]).** Let $F : \mathcal{A} \to \mathcal{B}$ be a dg functor. Suppose that $F$ is a localization, i.e. the functor $F^* : \mathcal{D} \mathcal{A} \to \mathcal{D} \mathcal{B}$ is a Verdier localization. If $\mathcal{A}$ is smooth, then so is $\mathcal{B}$ and the image under the induced morphism $\operatorname{HN}_n(\mathcal{A}) \to \operatorname{HN}_n(\mathcal{B})$ of a left $n$-CY-structure on $\mathcal{A}$ is a left $n$-CY-structure on $\mathcal{B}$.

**Remark 10.8.** As a consequence, if $\Gamma = \Gamma_{Q,0} = \Pi_3(kQ)$ for an acyclic quiver $Q$, then per$_{dg}(\Gamma)$ carries a left $3$-CY-structure and so does its localization $(\mathcal{C}_Q)_{dg}$, the dg cluster category. This seems to be contradictory with the well-known fact that the cluster category is $2$-Calabi–Yau as a triangulated category. The explanation is that the $2$-Calabi–Yau property of the triangulated category $\mathcal{C}_Q$ comes from a right $2$-Calabi–Yau structure on $(\mathcal{C}_Q)_{dg}$ (cf. below). The fact that $2 \neq 3$ is not a contradiction because $(\mathcal{C}_Q)_{dg}$ is smooth (as a localization of per$_{dg}(\Gamma)$) but not proper (only componentwise proper).

**Theorem 10.9** (Brav–Dyckerhoff [3]). Let $\mathcal{A}$ be a smooth dg category. Each left $n$-CY structure on $\mathcal{A}$ yields a canonical right $n$-CY structure on $\operatorname{pvd}_{dg}(\mathcal{A})$.

For example, the canonical left $3$-CY structure on $\Gamma$ as above yields a right $3$-CY structure on $\operatorname{pvd}_{dg}(\Gamma)$ which is responsible for the $3$-CY property of the triangulated category $\operatorname{pvd}(\Gamma)$. The right $3$-CY structure on $\operatorname{pvd}_{dg}(\Gamma)$ yields the right $2$-CY structure on $(\mathcal{C}_Q)_{dg}$ via the connecting morphism in cyclic homology associated with the exact sequence

$$0 \to \operatorname{pvd}_{dg}(\Gamma) \to \text{per}_{dg}(\Gamma) \to (\mathcal{C}_Q)_{dg} \to 0,$$

cf. [16].

10.10. **The derived category of morphisms.** Let $k$ be a perfect field and $A$ a dg $k$-algebra. Let $I = kA_2$ (the letter $I$ stands for ‘interval’). Then the objects of $\mathcal{D}(I^{op} \otimes A)$ identify with morphisms $f : M_1 \to M_2$ of dg $A$-modules. Each object gives rise to a triangle in $\mathcal{D}(A)$

$$M_1 \xrightarrow{f} M_2 \longrightarrow C(f) \longrightarrow \Sigma M_1,$$
functorial in $(M_1 \to M_2)$ considered as an object of $\mathcal{D}(I^{op} \otimes A)$. For two objects $f : M_1 \to M_2$ and $f' : M'_1 \to M'_2$, consider a diagram whose rows are triangles of $\mathcal{D}(A)$

$$
\begin{array}{ccc}
M_1 & \xrightarrow{f} & M_2 \\
\downarrow{\alpha} & & \downarrow{\beta} \\
M'_1 & \xrightarrow{f'} & M'_2
\end{array}
\quad
\begin{array}{ccc}
C(f) & \xrightarrow{\Sigma M_1} & \Sigma M_1 \\
\downarrow{\epsilon} & & \downarrow{\sigma} \\
C(f') & \xrightarrow{\Sigma M'_1} & \Sigma M'_1
\end{array}
$$

It is well-known (and easy to check) that for a given morphism $b$, there are morphisms $a$ and $c$ making the diagram commutative if and only if we have $g'ba = 0$ and that in this case, the pair $(a, b)$ lifts to a morphism of $\mathcal{D}(I^{op} \otimes A)$. This statement is refined by the following lemma

**Lemma 10.11.** We have a canonical isomorphism bifunctorial in $f, f' \in \mathcal{D}(I^{op} \otimes A)$

$$
\text{RHom}_{I^{op} \otimes A}(f, f') \cong \text{fib} \left( \text{RHom}_A(M_2, M'_2) \to \text{RHom}_A(M, C(f')) \right),
$$

where we write $\text{fib}(g)$ for $\Sigma^{-1}C(g)$.

A proof may be found in Lemma 3.1 of [33].

10.12. **Definition of relative Calabi–Yau structures.** Let $n \in \mathbb{Z}$ be an integer and $f : B \to A$ a morphism between smooth dg algebras. With each class in relative Hochschild homology $HH_n(A, B)$, we will associate a morphism of triangles of $\mathcal{D}(A^c)$

$$
\begin{array}{ccc}
\text{fib}(\mu) & \xrightarrow{\nu} & A \otimes_B A \\
\downarrow & & \downarrow \\
\Sigma^{1-n}A^\vee & \xrightarrow{\Sigma^{1-n}\mu^\vee} & \Sigma^{1-n}(A \otimes_B A)^\vee \\
\downarrow & & \downarrow \\
\Sigma^{1-n}\text{cof}(\mu^\vee) & \xrightarrow{} & \ldots
\end{array}
$$

(10.12.1)

For this, we observe that by Lemma 10.11, we have an isomorphism

$$
\text{RHom}(\mu^\vee, \nu) \cong \text{fib} \left( \text{RHom}_{A^c}(A \otimes_B A^\vee, A \otimes_B A) \to \text{RHom}_{A^c}(A^\vee, A) \right).
$$

Moreover, we have canonical isomorphisms

$$
A \otimes_{A^c} A \cong \text{RHom}_{A^c}(A^\vee, A)
$$

and

$$
A \otimes_{A^c} A \cong (A \otimes_B A) \otimes_{A^c} (A \otimes_B A) \cong \text{RHom}_{A^c}((A \otimes_B A)^\vee, A \otimes_B A).
$$

Thus, we get the following chain of morphisms

$$
\text{RHom}_{I^{op} \otimes A^c}(\mu^\vee, \nu) \cong \text{fib} \left( A \otimes_{A^c} A \to A \otimes_{A^c} A \right)
$$

$$
\cong \left( \Sigma^{-1}HH(A, B) \right)
$$

where $HH(A, B)$ denotes the relative Hochschild complex of $B \to A$. By taking homology in homological degree $n$, we see that with each class in $HN_n(A, B)$, there is associated a natural morphism of triangles 10.12.1.

**Definition 10.13** (Brav–Dyckerhoff [3]). A relative left $n$-CY-structure on $B \to A$ is a class in $HH_n(A, B)$ whose associated morphism of triangles 10.12.1 is an isomorphism.
Notice that for $B = 0$, we recover the absolute notion. If $A$ and $B$ are concentrated in degree 0, one easily deduces that $A$ is bimodule internally $n$-Calabi–Yau (in the sense of Pressland [24]) with respect to the image $e$ of $1_B$ in $A$.

Suppose that $B \rightarrow A$ carries a left relative $n$-CY structure. Put $A = \text{pvd}_{dg}(A)$ and $B = \text{pvd}_{dg}(B)$ and let $r : A \rightarrow B$ be the restriction functor.

**Theorem 10.14** (Brav–Dyckerhoff [3]). The functor $r : A \rightarrow B$ inherits a canonical relative right $n$-CY structure, i.e. there is a canonical class in $\text{DHC}_{1-n}(B, A)$ which yields an isomorphism of triangles

$$
\begin{array}{cccccccc}
\Sigma^{n-1}A & \overset{\Sigma^{n-1}r}{\longrightarrow} & \Sigma^{n-1}B|_A & \longrightarrow & \Sigma^{n-1}\cof (r) & \longrightarrow & \ldots \\
\downarrow & & \downarrow & & \downarrow & & \\
\text{fib}(Dr) & \longrightarrow & DB^{op}|_A & \overset{Dr}{\longrightarrow} & DA & \longrightarrow & \ldots
\end{array}
$$

where we write $\cof (r)$ instead of $C(r)$ to emphasize the duality with $\text{fib}(Dr)$.

We refer to [3] for the construction of the class in $\text{DHC}_{1-n}(B, A)$ and that of the corresponding morphism of triangles in the theorem. Concretely, for objects $L$ and $M$ of $\text{pvd}(A)$, the above diagram becomes

$$
\begin{array}{cccccccc}
\text{RHom}_A(L, \Sigma^{n-1}M) & \overset{\text{res}}{\longrightarrow} & \text{RHom}_B(L, \Sigma^{n-1}M) & \longrightarrow & C(L, \Sigma^{n-1}M) & \longrightarrow & \ldots \\
\downarrow & & \downarrow & & \downarrow & & \\
\text{DC}(M, L) & \longrightarrow & \text{DRHom}_B(M, L) & \overset{\text{Dres}}{\longrightarrow} & \text{DRHom}_A(M, L) & \longrightarrow & \ldots
\end{array}
$$

Here, we write $C(M, L)$ for the cone over the morphism

$$\text{RHom}_A(M, L) \rightarrow \text{RHom}_B(M, L)$$

and $C(L, \Sigma^{n-1}M)$ for the cone over the morphism

$$\text{RHom}_A(L, \Sigma^{n-1}M) \rightarrow \text{RHom}_B(L, \Sigma^{n-1}M).$$

If we have $B = 0$, we recover an isomorphism

$$\text{DRHom}_A(M, L) \cong \text{RHom}_A(L, \Sigma^nM)$$

so the theorem is a generalization of Theorem 10.9.

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