DETERMINANTS, THEIR APPLICATIONS TO MARKOV PROCESSES, AND A RANDOM WALK PROOF OF KIRCHHOFF’S MATRIX TREE THEOREM

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ABSTRACT. Kirchhoff’s matrix tree theorem is a well-known result that gives a formula for the number of spanning trees in a finite, connected graph in terms of the graph Laplacian matrix. A closely related result is Wilson’s algorithm for putting the uniform distribution on the set of spanning trees. We will show that when one follows Greg Lawler’s strategy for proving Wilson’s algorithm, Kirchhoff’s theorem follows almost immediately after one applies some elementary linear algebra. We also show that the same ideas can be applied to other computations related to general Markov chains and processes on a finite state space.

1. INTRODUCTION

Markov chains and Markov processes on a finite state space are completely determined by a matrix. In the case of chains, it is a transition matrix $P$ whose $(i,j)$ entry specifies the probability that the chain will go in one step from state $i$ to state $j$. In the case of Markov processes, it is a $Q$-matrix of the form $R(P-I)$, where $R$ is a diagonal matrix with non-negative entries and $P$ is a transition matrix. The $i$th diagonal entry of $R$ gives the rate at the process leaves the state $i$ and $P$ gives the distribution of where it will go when it leaves.

At least in theory, every question that one can ask about the chain or process can be answered in terms of $P$ or $Q$. However, in practice, it is often difficult to write down a transparent expression that gives the answer. For example, a quantity of probabilistic interest is the stationary distribution, that is, a distribution that is left invariant by the chain or process. Every Markov chain or process on a finite state space admits at least one stationary distribution, and sometimes it has many. A necessary and sufficient condition for it to have precisely one is that there exist a point that is accessible from every point. When one has such a situation, one would like to have a simple expression for this unique distribution in terms of $P$ or $Q$, and, as an application of the ideas here, we will give one.

Remark. The usual procedure for finding stationary distributions is to look for solutions to $\pi P \equiv P^*\pi = \pi$ in the case of Markov chains and $\pi Q = Q^*\pi \equiv 0$ in the case of Markov processes. Among other places, this procedure is discussed in the books [5] and [9].

A quite different application of our considerations is to Wilson’s algorithm and Kirchhoff’s matrix tree theorem. Given a finite, connected graph, there are lots of spanning trees (i.e., subgraphs that contain no cycles and include all vertices). In 1847, Gustav Kirchhoff [6] gave a formula for the number of spanning trees, and in 1996 David Wilson [11] gave an algorithm for generating a spanning tree uniformly at random (without knowing the actual number of spanning trees). Kirchhoff’s formula expresses the number in terms of the graph Laplacian matrix $L \equiv D - A$, where $D$ is the diagonal matrix whose $i$th diagonal entry is the degree (i.e., the number of vertices to which it is connected by an edge) of the vertex $i$ and $A$ is the adjacency matrix, the matrix whose...
(i, j) entry is 1 if there is an edge between i and j and is 0 otherwise. Obviously, \(-L = D(P - I)\) where \(P = D^{-1}A\) is a transition matrix, and so \(-L\) is a \(Q\)-matrix. Wilson’s algorithm uses the chain determined by the \(P\) to explore the graph. Since, with probability 1, his algorithm produces a spanning tree, and since the probability of its producing any particular one is the same for all spanning trees, the number of spanning trees must be equal to the reciprocal of the probability that Wilson’s algorithm produces a particular one. Thus, one can recover Kirchhoff’\'s result if one can show that the probability that Wilson’s algorithm produces a particular spanning tree is the reciprocal of Kirchhoff’\’s expression in terms of \(L\). Following a strategy developed by Greg Lawler, we will show how this can be done.

**Remark.** Even today, Kirchhoff’\’s result has to be considered a sophisticated application of matrix algebra, but in 1847, when matrix algebra was in its infancy, it was a remarkable achievement. Most modern proofs are based on the Cauchy-Binet formula and have no obvious connection to probability theory; for an easily accessible account, see [1]. Lawler’\’s proof [7] that Wilson’\’s algorithm works is very different from Wilson’\’s own proof [11]; see also [4] for a detailed exposition of Wilson’\’s technique. Our proof is based on the same idea as Lawler’\’s.

**Example.** In order to illustrate the concepts, notation, and proof of the matrix tree theorem via Wilson’\’s algorithm, we will work with the following example. Consider the graph \(\Gamma\) shown below having vertex set \(\{x_1, x_2, x_3, x_4, x_5, x_6\}\) and graph Laplacian matrix \(L = D - A\) as given.

\[
L = \begin{bmatrix}
3 & -1 & -1 & -1 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 & 0 \\
-1 & -1 & 4 & -1 & 0 & -1 \\
-1 & 0 & -1 & 3 & -1 & 0 \\
0 & 0 & 0 & -1 & 2 & -1 \\
0 & 0 & -1 & 0 & -1 & 2
\end{bmatrix}
\]

It can be easily determined that this graph has 29 spanning trees if one considers the degree of \(x_3\) in the spanning tree: \(\deg(x_3) = 4\) for 2 spanning trees, \(\deg(x_3) = 3\) for 10 spanning trees, 13 spanning trees have \(\deg(x_3) = 2\), and 4 spanning trees have \(\deg(x_3) = 1\).

2. **Two theorems about determinants**

In this section, we will be dealing with the vector space \(\mathbb{C}^N\) and will be using the Hermitian inner product \(\langle x, y \rangle = x_1\overline{y_1} + \cdots + x_N\overline{y_N}\) and norm \(|x|^2 = \langle x, x \rangle\). Given an \(N \times N\) matrix \(M \in \mathbb{C}^N \otimes \mathbb{C}^N\) and a subset \(\Delta \subseteq \{1, \ldots, N\}\) with \(n\) elements, denote by \(M^\Delta\) the \((N - n) \times (N - n)\) matrix obtained from \(M\) by removing the rows and columns corresponding to the indices in \(\Delta\). Assuming that \(\det[M] \neq 0\), Cramer’s rule states that the \((i, i)\) entry of \(M^{-1}\) is

\[
(M^{-1})_{ii} = \frac{\det[M^{(i)}]}{\det[M]}
\]

for all \(i = 1, \ldots, N\). The following theorem now follows immediately from (1) by induction.

**Theorem 2.1.** Let \(M\) be a non-degenerate \(N \times N\) matrix. Suppose that \((\sigma(1), \ldots, \sigma(N))\) is a permutation of \((1, \ldots, N)\). Set \(\Delta_1 = \emptyset\) and, for \(j = 2, \ldots, N\), let \(\Delta_j = \Delta_{j-1} \cup \{\sigma(j - 1)\} = \{\sigma(1), \ldots, \sigma(j - 1)\}\). If \(M^{\Delta_j}\) is non-degenerate for all \(j = 1, \ldots, N\), then

\[
\prod_{j=1}^n (M^{\Delta_j})_{\sigma(j)\sigma(j)}^{-1} = \det[M]^{-1}.
\]

Obviously, the interest in (2) is not that it gives an efficient way of computing \(\det[M]^{-1}\), but that it shows that the product on the left is independent of the permutation \(\sigma\).
Example. We illustrate how to use the notation of Theorem 2.1 to do a computation. Suppose that $M$ is the non-degenerate $3 \times 3$ matrix

$$
M = \begin{bmatrix}
1 & 2 & 3 \\
9/10 & 2/5 & -1/10 \\
1/10 & -2/5 & -9/10 \\
-1/5 & 4/5 & -1/5 \\
\end{bmatrix}
$$

so that $M^{-1} = \begin{bmatrix}
1 & 0 & -1/2 \\
1/4 & -1/4 & 1 \\
0 & -1 & -1/2 \\
\end{bmatrix}$. Observe that $\det[M] = 1/4$. We will now calculate this determinant using (2). Let $\sigma$ be any permutation of $\{1, 2, 3\}$, say $\{2, 3, 1\}$, so that $\Delta_1 = \emptyset$, $\Delta_2 = \{2\}$, $\Delta_3 = \{2, 3\}$. We now find $(M^{\Delta_i})^{-1} = M^{-1}$,

$$
(M^{\Delta_2})^{-1} = \begin{bmatrix}
1 & 2 & 3 \\
9/10 & 1/10 & -1/10 \\
-1/5 & -1/5 & -1/5 \\
\end{bmatrix}^{-1} = \begin{bmatrix}
1 & 3 \\
9/10 & -1 \\
-1/5 & -9/2 \\
\end{bmatrix}
$$

and so

$$
\prod_{j=1}^{3} (M^{\Delta_j})_{\sigma(j)}^{-1} = (M^{\Delta_1})_{22}^{-1} (M^{\Delta_2})_{33}^{-1} (M^{\Delta_3})_{11}^{-1} = \frac{1}{4} \cdot \frac{9}{2} \cdot \frac{10}{9} = \frac{5}{4}.
$$

The next theorem relies on the Jordan-Chevalley decomposition [3] of a matrix into its semi-simple and nilpotent parts. Namely, what we need to know is that if $\lambda_1, \ldots, \lambda_N$ are the eigenvalues (equivalently, the roots, counting multiplicity, of the characteristic polynomial) of $M$, then there exist matrices $B$ and $C$ and a basis $(v_1, \ldots, v_N)$ for $\mathbb{C}^N$ such that $M = B + C$ where $C$ is nilpotent (i.e., $C^N = 0$), $Bv_i = \lambda_i v_i$ for $1 \leq i \leq N$, and $B$ commutes with $C$.

**Theorem 2.2.** Suppose that $M$ is an $N \times N$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_N$, assume that $\lambda_j = 0$ if and only if $j = 1$, and set $\Pi_M = \lambda_2 \cdots \lambda_N$, the product of the non-zero eigenvalues of $M$. Then there is a unique $v_1^*$ such that $M^* v_1^* = 0$ and $\langle v_1, v_1^* \rangle = 1$. Moreover

$$
\det[M^{(i)}] = (v_1^*)_i (v_1^*)_i.
$$

**Proof.** Let $B$, $C$, and $(v_1, \ldots, v_N)$ be the quantities described above. Because the kernel of $M$ has the same dimension as that of $M^*$, we know that $v_1^*$ is uniquely determined by the conditions $M^* v_1^* = 0$ and $\langle v_1, v_1^* \rangle = 1$. Now choose $(v_2, \ldots, v_N)$ so that $(v_1^*, v_2^*, \ldots, v_N^*)$ is the dual basis for $(v_1, \ldots, v_N)$ (i.e., $(v_i, v_j^*) = \delta_{ij}$). For $\alpha \notin \{-\lambda_1, -\lambda_2, \ldots, -\lambda_N\}$,

$$
(\alpha I + M)^{-1} = (\alpha I + B)^{-1} + \sum_{n=1}^{N-1} (-1)^n (\alpha I + B)^{-(n+1)} C^n.
$$

To verify this, multiply the right side of (4) by $(\alpha I + B) + C$, expand, and note that necessarily $C^N = 0$ since $C$ is nilpotent. Hence, because $Cv_1 = Mv_1 - Bv_1 = 0$,

$$
(\alpha I + M)_{\bar{i}\bar{i}}^{-1} = \sum_{j=1}^{N} \sum_{j=1}^{N} \langle \alpha I + M \rangle_{\bar{j}\bar{j}}^{-1} v_j^* v_j (v_j^*)_i (v_j^*)_i
$$

$$
= \sum_{j=1}^{N} \frac{v_j^*_i (v_j^*_i)}{\alpha + \lambda_j} + \sum_{n=1}^{N-1} (-1)^n \sum_{j=2}^{N} \frac{(v_j^*_i)}{\alpha + \lambda_j^{n+1}} \sum_{j=1}^{N} \langle C^n v_j^*, v_j^* \rangle (v_j)_i.
$$

If we multiply both sides of the preceding equality by $\alpha$ and let $\alpha \searrow 0$, then, since $\lambda_j = 0$ iff $j = 1$, we see that only the first term on the right hand side survives; that is,

$$
\lim_{\alpha \searrow 0} \alpha (\alpha I + M)_{\bar{i}\bar{i}}^{-1} = (v_1^*_i) (v_1^*_i).
$$
By combining this with (1), and observing that
\[ \det[\alpha I + M] = \alpha \prod_{j=2}^{N} (\alpha + \lambda_j) \]
we get (3) as required.

By summing over \( i \) in (3), we see that
\[ \Pi_M = \sum_{i=1}^{N} \det[M^{(i)}]. \]
However, this an inefficient way to prove (5), which holds without any assumptions on \( M \) other than that \( \det[M] = 0 \). Indeed, when \( \det[M] = 0 \), the right side of (5) is then the constant term in the polynomial \(-\lambda^{-1} \det[M - \lambda I]\).

**Remark.** Let \( \text{cof}[M] \) be the cofactor matrix for \( M \), namely the matrix with \((i, j)\) entry equal to \((-1)^{i+j} \) times the determinant of the matrix obtained from \( M \) by removing the \( i \)th column and \( j \)th row. The same argument that led to (3) shows that if \( \lambda \) is a simple (i.e., it is different from all the others) eigenvalue of \( M \) with eigenvector \( v \), then
\[ v^* = \frac{\text{cof}[M - \lambda I]v}{|v|^2 \Pi_M - \lambda I}. \]
In particular, when \( M \) is normal and therefore \( v^* = |v|^{-2}v \), then \( v \) is an eigenvector for \( M - \lambda I \) with eigenvalue \( \Pi_{M - \lambda I} \).

### 3. A BRIEF ACCOUNT OF MARKOV CHAINS AND PROCESSES

As we said, a transition matrix \( P \) on a finite state space \( V \) is the key ingredient in the construction of Markov chains. Namely, a Markov chain with transition matrix \( P \) is a sequence, best thought of as a path \( \{X_n : n \geq 0\} \), of \( V \)-valued random variables with the property that, for any \( n \geq 1 \) and \( x_0, \ldots, x_n \in V \),
\[ \mathbb{P}(X_m = x_m \text{ for } 0 \leq m \leq n) = \mathbb{P}(X_m = x_m \text{ for } 0 \leq m < n) P_{x_{n-1}x_n}. \]
Equivalently, in probabilistic terminology, the conditional probability given \( X_0, \ldots, X_{n-1} \) that \( X_n = x_n \) is \( P_{X_{n-1}x_n} \). It is easy to verify that if one knows the distribution of \( X_0 \), then the distribution of \( \{X_n : n \geq 0\} \) is uniquely determined by (6). We will use the notation \( \mathbb{P}_x \) to denote the distribution of the chain when \( X_0 \equiv x \) and \( \mathbb{E}_x \) to denote expectations with respect to \( \mathbb{P}_x \). Starting from (6), it is not hard to show that, for any \( n_1 \) and \( n_2 \),
\[ \mathbb{P}(X_m = x_m \text{ for } 0 \leq m \leq n_1 + n_2) = \mathbb{P}(X_m = x_m \text{ for } 0 \leq m \leq n_1) \mathbb{P}_{x_{n_1}}(X_{n_1+m} = x_m \text{ for } 1 \leq m \leq n_2) \]
Given a \( \Delta \subseteq V \), let \( \xi^\Delta \equiv \inf\{n \geq 0 : X_n \in \Delta\} \) be the first time that the chain visits \( \Delta \). Thus \( \xi^\Delta = \infty \) for paths that never visit \( \Delta \). Say that \( \Delta \) is accessible from \( x \notin \Delta \) if there exist an \( n \geq 1 \) and points \( x_0, \ldots, x_n \in V \) such that \( x_0 = x, x_n \in \Delta \), and \( P_{x_{m-1}x_m} > 0 \) for \( 1 \leq m \leq n \).

**Lemma 3.1.** If \( \Delta \) is accessible from each \( x \in V \), then \( \max_{x \in V} \mathbb{E}_x[\xi^\Delta] < \infty \).

**Proof.** By the accessibility assumption, we know that, for each \( x \in V \), there is an \( n \) and an \( \theta \in (0,1) \) such that \( \mathbb{P}_x(\xi^\Delta > n) \leq \theta \), and because \( V \) is finite, we can choose one \( n \) and \( \theta \) that works simultaneously for all \( x \in V \). Hence, by (7),
\[ \mathbb{P}_x(\xi^\Delta > (k+1)n) = \sum_{y \in V} \mathbb{P}_x(\xi^\Delta > kn \text{ and } X_{kn} = y) \mathbb{P}_y(\xi^\Delta > n) \leq \theta \mathbb{P}_x(\xi^\Delta > kn). \]
By induction, this means that \( \mathbb{P}_x(\xi^\Delta > kn) \leq \theta^k \), and so the asserted result follows. \( \square \)
Continuing under the conditions in Lemma 3.1, our next goal is to show that \((I - P)^\Delta\) is invertible and that
\[
\mathbb{E}_x \left[ \sum_{n=0}^{\xi^\Delta} 1_{\{y\}}(X_n) \right] = ((I - P)^\Delta)_{xy}^{-1} \quad \text{for } x, y \in V \setminus \Delta.
\]

Perhaps the most elementary way to check this is to first observe that
\[
\mathbb{E}_x \left[ \sum_{n=0}^{\xi^\Delta} 1_{\{y\}}(X_n) \right] = \sum_{n=0}^{\infty} \mathbb{P}_x(X_n = y \& \xi^\Delta > n).
\]

Second, note that, because \(y \notin \Delta\), we have \(\mathbb{P}_x(X_n = y \& \xi^\Delta > n) = \mathbb{P}_x(X_n^{\Delta\Delta} = y)\). Finally, \(\{X_n^{\Delta\Delta} : n \geq 0\}\) is the chain starting at \(x\) determined that the transition matrix \(P'\) whose \((x', y')\) entry equals \(P_{x'y'}\) if \(x' \in V \setminus D\) and equals \(\delta_{x', y'}\) if \(x' \in \Delta\). Hence, by \([6]\), \(\mathbb{P}_x(X_n^{\Delta\Delta} = y) = (P')_x^y\), and it is an easy matter to see that \((P')^n_x^y = (P^\Delta)^n_{xy}\). Thus, we now know that
\[
\mathbb{E}_x \left[ \sum_{n=0}^{\xi^\Delta} 1_{\{y\}}(X_n) \right] = \sum_{n=0}^{\infty} (P^\Delta)^n_{xy}.
\]

In particular, since the left hand side is dominated by \(\mathbb{E}_x[\xi^\Delta]\), the series on the right converges. Finally, knowing that this series converges, an elementary argument shows that
\[
(I - P^\Delta) \sum_{n=0}^{\infty} (P^\Delta)^n = I
\]
and therefore that \(I + P^\Delta + (P^\Delta)^2 + \cdots\) is the inverse of \((I - P)^\Delta\).

A closely related consideration is the following. Given \(x \in V\), define \(\{\tau_x^{(m)} : m \geq 0\}\) inductively so that \(\tau_x^{(0)} = 0\) and \(\tau_x^{(m)} = \inf\{n > \tau_x^{(m-1)} : X_n = x\}\). If \(x \in V \setminus D\), then it is clear that \(\tau_x^{(m)} < \xi^\Delta\) if and only if
\[
\sum_{n=0}^{\xi^\Delta} 1_{\{x\}}(X_n) > m.
\]

At the same time, using \([7]\), one can check that \(\mathbb{P}_x(\tau_x^{(m)} < \xi^\Delta) = \mathbb{P}_x(\tau_x^{(m-1)} < \xi^\Delta) \mathbb{P}_x(\tau_x^{(1)} < \xi^\Delta)\), and so \(\mathbb{P}_x(\tau_x^{(m)} < \xi^\Delta) = \mathbb{P}_x(\tau_x^{(1)} < \xi^\Delta)^m\). Combining these, we arrive at
\[
((I - P)^\Delta)^{-1}_{xx} = \mathbb{E}_x \left[ \sum_{n=0}^{\xi^\Delta} 1_{\{y\}}(X_n) \right] = \frac{1}{1 - \tau_\Delta(x)} \quad \text{where } \tau_\Delta(x) = \mathbb{P}_x(\tau_x^{(1)} < \xi^\Delta).
\]

Analogous results hold for the Markov processes associated with a \(Q\)-matrix. Indeed, if \(Q = R(P - I)\) and \(\{X_n : n \geq 0\}\) is a Markov chain with transition matrix \(P\) and initial distribution \(\mu\), then a Markov process \(\{X(t) : t \geq 0\}\) determined by \(Q\) with initial distribution is obtained from \(\{X_n : n \geq 0\}\) by randomizing the time that it stays at a point. More precisely, one can always choose \(R\) and \(P\) so that \(F_x = 0\) for all \(x \in V\). With this choice, the unparameterized paths of \(\{X(t) : t \geq 0\}\) are exactly the same as those of \(\{X_n : n \geq 0\}\). The difference is that, instead of remaining in a state for time 1, if \(X(s) = x\), then it stays at \(x\) for an exponential holding time with rate constant \(R_x\). That is, the probability has not left \(x\) before some time \(t > s\) is \(e^{-(t-s)R_x}\). On the other hand, when it leaves a point, the distribution of where it goes is exactly the same as that for the chain.
Fortunately, we do not need to know much about these processes. Based on the preceding description, one can show that the probability that the process started from \( x \) will be at \( y \) at time \( t \) is \( (e^tQ)_{xy} \), where

\[
e^tQ = \sum_{k=0}^{\infty} \frac{t^k Q^n}{n!}.
\]

Starting from this, it is easy to see that \( \pi \) is a stationary distribution for the process if and only if \( \pi Q \equiv Q^* \pi = 0 \).

### 4. Computing stationary distributions for Markov chains and processes

Let \( P \) be a transition matrix on the finite state space \( V \). A point \( y \) is said to be accessible from \( x \) if \( \{ y \} \) is accessible from \( x \), and \( x \) is said to communicate with \( y \) if each is accessible from the other. It should be clear that communication is an equivalence relation, and, for a given \( x \in V \), we use \([x]\) to denote the communication class of \( x \) (i.e., the set of \( y \) that communicate with \( x \)). Further, one says that \( x \) is recurrent or transient depending on whether \( P_x(\tau_x < \infty) = 1 \) or \( P(\tau_x < \infty) < 1 \).

The following facts are not hard and their proofs can be found in §3.1 of [10]. If \( x \) is recurrent and \( y \) is accessible from \( x \), then \( y \) is also recurrent and \( x \) communicates with \( y \). Thus, no transient state is accessible from any recurrent one. From (6) with \( \Delta = 0 \), one sees that if \( x \) is transient, then the expected length of time that the chain starting at \( x \) spends at \( x \) is finite. Thus, at least one state in \( V \) must be recurrent, and every transient state must have a recurrent state to which it is accessible.

We next need to know that there always exists at least one stationary distribution.

**Lemma 4.1.** If \( P \) is a transition matrix and \( x \) is a recurrent state, then \( E_x[\tau_x] < \infty \). In addition, if, for \( y \in V \),

\[
\mu_y = \frac{1}{E_x[\tau_x]} E_x \left[ \sum_{m=0}^{\tau_x-1} 1_{\{y\}}(X_m) \right],
\]

then \( \mu \) is a stationary distribution for \( P \), and \( \mu_y = 0 \) for \( y \notin [x] \).

**Proof.** Since \( P_{x'y} = 0 \) for \( x' \in [x] \) and \( y \notin [x] \), without loss in generality we may and will assume that \( V = [x] \). By (9), \( E_x[\tau_x] < \infty \), and so it is clear that \( \mu \) is a probability distribution. To prove that \( \mu P = \mu \), observe that

\[
E_x[\tau_x] \mu_y = E_x \left[ \sum_{m=0}^{\tau_x-1} 1_{\{y\}}(X_m) \right] = E_x \left[ \sum_{m=1}^{\tau_x} 1_{\{y\}}(X_m) \right]
\]

\[
= \sum_{m=1}^{\infty} \mathbb{P}_x(X_m = y \& \tau_x \geq m)
\]

\[
= \sum_{m=1}^{\infty} \sum_{z \in V} \mathbb{P}_x(X_{m-1} = z \& \tau_x > m - 1 \& X_m = y)
\]

\[
= \sum_{z \in V} \sum_{m=0}^{\infty} \mathbb{P}_x(X_m = z \& \tau_x > m) P_{zy}
\]

\[
= \sum_{z \in V} E_x \left[ \sum_{m=0}^{\tau_x-1} 1_{\{y\}}(X_m) \right] P_{zy}
\]

\[
= E_x[\tau_x] (\mu P)_y,
\]

where, in the passage to the last line, we have used (6). \( \square \)

We now have everything that we need to prove the following application of Theorem 2.2.
Theorem 4.2. If \( Q = R(P - I) \) is a Q-matrix, where \( R \) is a diagonal matrix with positive diagonal entries and \( P \) is a transition matrix, then \( \det(-Q^{(x)}) \geq 0 \) for all \( x \in V \). Moreover, if the null space of \( Q \) is one dimensional, then \( \det(-Q^{(x)}) > 0 \) if and only if \( x \) is recurrent for \( P \), and therefore \( \text{dim}(\text{Null}(Q)) = 1 \) if and only if \( \Pi_Q > 0 \). Finally, if \( \text{dim}(\text{Null}(Q)) = 1 \) and is positive when \( x \) is recurrent for \( P \), it suffices to handle the case when \( R \) is the identity. To that end, set \( M = I - P \) and note that for any \( \Delta \subseteq V \) and \( \alpha > 0 \),

\[
(\alpha I + M)^{\Delta} = \sum_{n=0}^{\infty} \alpha^{-n-1}(P^{\Delta})^n.
\]

Hence \( (\alpha I + M)^{\Delta} x > 0 \) for all \( x \in V \setminus \Delta \), and so by Theorem 2.1, \( \det((\alpha I + M)^{\{x\}}) > 0 \). After letting \( \alpha \searrow 0 \), it follows that \( \det(M^{\{x\}}) \geq 0 \).

Next assume that \( \text{Null}(P - I) \) is one dimensional. Then \( \text{Null}((P - I)^{\ast}) \) is also one dimensional, and so there is only one stationary distribution for the Markov chain determined by \( P \). Hence, by Lemma 1.1 and the preceding discussion, every recurrent state must communicate with every other one and be accessible from every transient state. Moreover, if \( x \) is a recurrent state and \( x \in \Delta \subseteq V \), then Lemma 3.1 says that \( (M^{\Delta})^{-1} < \infty \) for all \( y \in V \setminus \Delta \), and therefore, again by Theorem 2.1, \( \det(M^{\{x\}}) > 0 \). In addition, since \( \det(M^{\{x\}}) \geq 0 \) for all \( x \) and there must exist a recurrent \( x \), it follows from (5) that \( \Pi_Q > 0 \). Finally, because \( QI = 0 \), Theorem 2.2 says that the right hand side of (10) is the unique \( \mu \) satisfying \( \mu Q = 0 \) and \( (1, \mu) = 1 \).

What remains is to prove that \( \det(-Q^{(x)}) = 0 \) if \( \Pi_Q > 0 \) and \( x \) is transient, and again it suffices to handle the case when \( R = I \). But if \( \Pi_Q > 0 \), then we know that the \( \pi \) in (10) satisfies \( \pi P = \pi \). Thus, for any \( x \) and \( n \geq 1 \),

\[
n\pi x = \sum_{m=0}^{n-1} (\pi P^m) x = E_{\pi} \left[ \sum_{m=0}^{n-1} 1_{\{x\}}(X_m) \right],
\]

where \( E_{\pi} \) denotes expectation with respect to the distribution \( P_{\pi} \) of the Markov chain determined by \( P \) with initial distribution \( \pi \). Hence,

\[
n\pi x \leq E_{\pi} \left[ \sum_{m=0}^{\infty} 1_{\{x\}}(X_m) \right]
\]

for all \( n \geq 1 \). Further, by (7),

\[
E_{\pi} \left[ \sum_{m=0}^{\infty} 1_{\{x\}}(X_m) \right] = E_{\pi} \left[ \sum_{m=\xi(x)}^{\infty} 1_{\{x\}}(X_m) \right] = \sum_{m=\xi(x)}^{\infty} \sum_{k=0}^{m} P_{\pi}(X_m = x \& \xi^{(x)} = k)
\]

\[
= \sum_{k=0}^{\infty} \sum_{m=k}^{\infty} P_{\pi}(\xi^{(x)} = k) P_x(X_{m-k} = x)
\]

\[
= \left( \sum_{k=0}^{\infty} P_{\pi}(\xi^{(x)} = k) \right) E_x \left[ \sum_{m=0}^{\infty} 1_{\{x\}}(X_m) \right]
\]

\[
= P_{\pi}(\xi^{\Delta} < \infty) E_x \left[ \sum_{m=0}^{\infty} 1_{\{x\}}(X_m) \right].
\]
Thus, if \( x \) is transient and therefore
\[
\mathbb{E}_x \left[ \sum_{m=0}^{\infty} 1_{\{x\}}(X_m) \right] < \infty,
\]
then \( \pi_x \) and therefore \( \det(-Q^{\{x\}}) \) must be 0.

**Remark.** Once one knows that there is some state that is accessible from every other state, another proof that \( \Pi - Q > 0 \) can be based on the following argument. Under these circumstances, Doeblin’s theorem (Theorem 2.2.1 of [10]) implies that
\[
\lim_{\alpha \downarrow 0} \alpha(aI - Q)^{-1}v = \langle v, \pi \rangle
\]
for all \( v \). Now suppose that 0 were not a simple eigenvalue of \( Q \). Then there would exist a \( v \neq 0 \) such that \( \langle v, \pi \rangle = 0 \) and \( Q^n v = 0 \) for some \( n \geq 1 \). But this would mean that
\[
(aI - Q)^{-1}v = \sum_{m=0}^{n} \alpha^{-m-1}Q^m v
\]
and therefore that \( Q^m v = 0 \) for all \( m \geq 1 \), from which is follows that
\[
v = \lim_{\alpha \downarrow 0} \alpha(aI - Q)^{-1}v = \langle v, \pi \rangle = 0.
\]
Hence, 0 must be a simple eigenvalue of \( Q \). Furthermore, because it is real, its non-real eigenvalues come in conjugate pairs. Finally, if \( -Qv = \lambda v \), then \( e^{tQ}v = e^{-\lambda t}v \) and so, since \( \|e^{tQ}v\| \leq \|v\| \), the real part of \( \lambda \) must be non-negative. (Here \( \| \cdot \| \) denotes the uniform norm.) In particular, if \( \lambda \) is real and different from 0, it must be strictly positive.

**Corollary 4.3.** Assume that \( Q = P - I \) where \( P \) is a transition matrix with the property that the only solutions \( v \) to \( Pv = v \) are constant multiples of 1. Then, for any recurrent \( x \),
\[
\mathbb{P}_x(\tau_x \leq \tau_y) = \frac{\det((I - P)\{y\})}{\det((I - P)\{x,y\})}.
\]

**Proof.** By Lemma 4.1, one expression for the unique stationary distribution \( \pi \) is
\[
\pi_y = \frac{1}{\mathbb{E}_x[\tau_x]} \mathbb{E}_x \left[ \sum_{m=0}^{\tau_x-1} 1_{\{y\}}(X_m) \right].
\]
Hence, since
\[
\mathbb{E}_x \left[ \sum_{m=0}^{\tau_x-1} 1_{\{x\}}(X_m) \right] = 1,
\]
we find
\[
\frac{\pi_y}{\pi_x} = \mathbb{E}_x \left[ \sum_{m=0}^{\tau_x-1} 1_{\{y\}}(X_m) \right].
\]
By Theorem 4.2, \( \pi_y/\pi_x = \det((I - P)\{y\})/\det((I - P)\{x\}) \). At the same time, proceeding as in the proof of Lemma 4.1, one sees that
\[
\mathbb{E}_x \left[ \sum_{m=0}^{\tau_x-1} 1_{\{y\}}(X_m) \right] = \mathbb{P}_x(\tau_y \leq \tau_x) \mathbb{E}_y \left[ \sum_{m=0}^{\tau_y-1} 1_{\{y\}}(X_m) \right].
\]
Finally, by (9) and (1),
\[
\mathbb{E}_y \left[ \sum_{m=0}^{\tau_y-1} 1_{\{y\}}(X_m) \right] = ((I - P)^{\{x\}})^{-1}yy = \frac{\det((I - P)^{\{x,y\}})}{\det((I - P)^{\{x\}})}.
\]
After combining these, we arrive at the asserted equation.

Remark. As a consequence of (11), we see that \( \det((I - P)^{(y)}) \leq \det((I - P)^{(x,y)}) \) when \( x \) is recurrent and \( y \in V \). This observation has the following generalization. Namely, given any transition matrix \( P \) and \( \Delta \subseteq V \) and \( y \in V \setminus \Delta \), one has that \( \det((I - P)^{\Delta}) \leq \det((I - P)^{\Delta \cup \{y\}}) \). To see this, first observe that the argument with which we proved in Theorem 4.2 that \( \det((I - P)^{(x)}) \geq 0 \) can be used to show that \( \det((I - P)^{\Delta}) \geq 0 \) for any \( \Delta \subseteq V \). Thus there is nothing to do when \( \Delta \subseteq V \) and \( \det((I - P)^{\Delta}) = 0 \). On the other hand, if \( \det((I - P)^{\Delta}) > 0 \), then one can use (9) together with (1) to identify the ratio of \( \det((I - P)^{\Delta \cup \{y\}}) \) to \( \det((I - P)^{\Delta}) \) as

\[
\mathbb{E}_y \left[ \sum_{m=0}^{\infty} 1_{\{y\}}(X_m) \right] \geq 1.
\]

By induction, this means that \( \det((I - P)^{\Delta_1}) \leq \det((I - P)^{\Delta_2}) \) when \( \Delta_1 \subseteq \Delta_2 \).

In the case of a three-state Markov chain, it is easy to perform the calculations in Theorem 4.2 and Corollary 4.3.

Example. Let \( P = (p_{ij})_{1 \leq i,j \leq 3} \) be a transition matrix. If we set

\[
D(i, j, k) = p_{ji}(1 - p_{kk}) + p_{jk}p_{ki},
\]

then

\[
\det((I - P)^{(1)}) = D(1, 2, 3), \quad \det((I - P)^{(2)}) = D(2, 3, 1), \quad \text{and} \quad \det((I - P)^{(3)}) = D(3, 1, 2). \]

Hence, by Theorem 2.1, \( P \) has a unique stationary distribution \( \pi \) if and only if

\[
\Pi \equiv D(1, 2, 3) + D(2, 3, 1) + D(3, 1, 2) > 0,
\]

in which case,

\[
\pi_1 = \frac{D(1, 2, 3)}{\Pi}, \quad \pi_2 = \frac{D(2, 3, 1)}{\Pi}, \quad \text{and} \quad \pi_3 = \frac{D(3, 1, 2)}{\Pi}.
\]

Furthermore, if \( D(1, 2, 3) > 0 \), then

\[
\mathbb{P}_1(\tau_2 < \tau_1) = \frac{D(2, 3, 1)}{1 - p_{33}} \quad \text{and} \quad \mathbb{P}_1(\tau_3 < \tau_1) = \frac{D(3, 1, 2)}{1 - p_{22}},
\]

and similarly for \( \mathbb{P}_2 \) and \( \mathbb{P}_3 \).

5. Wilson’s Algorithm and Kirchhoff’s Formula

Let \( \Gamma = (V, E) \) be a connected \( N \)-vertex graph in which no vertex has an edge to itself and any two vertices are connected by at most one edge. Recall that the graph Laplacian matrix \( L = D - A \), where \( A \) is the adjacency matrix for \( \Gamma \) and \( D \) is the diagonal matrix of degrees. Think of \(-L\) as a \( Q \)-matrix. Clearly, the connectedness of \( \Gamma \) implies that all the vertices communicate with one another and are therefore all recurrent. Hence, by Theorem 4.2, \( \det(L^\Delta) > 0 \) for all non-empty \( \Delta \subseteq V \). In addition, if \( (x_1, \ldots, x_N) \) is any ordering of the vertices, then, by Theorem 2.1

\[
(12) \quad \frac{1}{\det(L^{(x_1)})} = \prod_{m=1}^{N-1} (L^{(x_1, \ldots, x_m)})^{-1} x_{m+1} x_{m+1}.
\]

At the same time, since \( \pi L = 0 \) when \( \pi_x = 1/N \) for all \( x \in V \), Theorem 4.2 says that

\[
\det(L^{(x)}) = \frac{\Pi L}{N} \quad \text{for all} \ x \in V.
\]
By combining these, we see that, for any ordering \((x_1, \ldots, x_N)\) of the elements of \(V\),

\begin{equation}
\prod_{m=1}^{N-1} \left( \mathcal{L}^{(x_1, \ldots, x_m)} \right)_{x_{m+1}x_{m+1}}^{-1} = \frac{1}{\det(\mathcal{L}^{(x_1)})} = \frac{N}{\Pi_L}.
\end{equation}

In order to explain the relevance of the preceding to Wilson’s algorithm, we have to explain what his algorithm is. Set \(P = \mathcal{D}^{-1}A\). Given an ordering \((x_1, \ldots, x_N)\) of the vertices, take \(\Delta_1 = \{x_N\}\) and run a Markov chain \(\{X_n : n \geq 1\}\) with transition matrix \(P\) starting from \(x_1\). Consider the segment \(\{X_n : 0 \leq n \leq \Delta_1\}\), and let \((Y_1,1, \ldots, Y_1,K_1)\) be the successive points visited by the path obtained from this segment by erasing all of its loops (i.e., cycles). Note that both \(K_1\) and \(Y_1,k\), \(1 \leq k \leq K_1\), will be random. If \(Y_1,1, \ldots, Y_1,K_1\) is a spanning tree with a single branch running from \(x_1\) to \(x_N\), in which case the algorithm terminates. If \(K_1 < N\), set \(\Delta_2 = \{Y_1,1, \ldots, Y_1,K_1\}\), and take \(x_{2,1}\) to be the first vertex from \((x_1, \ldots, x_N)\) that is not in \(\Delta_2\). Run the Markov chain starting from \(x_{2,1}\), consider the segment \(\{X_n : 0 \leq n \leq \Delta_2\}\), and let \((Y_2,1, \ldots, Y_2,K_2)\) be the successive vertices visited by its loop erasure. If \(\{Y_1,1, \ldots, Y_1,K_1\} \cup \{Y_2,1, \ldots, Y_2,K_2\} = V\), then the algorithm stops and \(\{Y_1,1, \ldots, Y_1,K_1\} \cup \{Y_2,1, \ldots, Y_2,K_2\}\) are the vertices of a spanning tree that has two branches if \(Y_2,K_2 = Y_1,K_1\) and three branches otherwise. One continues running the algorithm in this way until it produces a spanning tree. Since \(\mathbb{P}_x(\Delta < \infty) = 1\) for all \(x \in V\) and \(\Delta \subseteq V\), with probability 1 it will produce a tree after no more than \(N\) runs.

Wilson’s theorem says that the probability of his algorithm producing any particular spanning tree is the same for all spanning trees. In view of the preceding description, proving his theorem comes down to the following computation. Let \(\emptyset \neq \Delta \subseteq V\) be given, and assume that \((y_1, \ldots, y_K)\) is a given ordering of \(K\) distinct vertices from \(V \setminus \Delta\).

**Question.** What is the probability \(\mathcal{P}^\Delta(y_1, \ldots, y_K)\) that \((y_1, \ldots, y_K)\) will be the successive points visited by the loop erasure of the segment \(\{X_n : 0 \leq n \leq \Delta\}\) of the Markov chain started at \(y_i\)?

If one knows the answer, then one knows how to compute the probability \(\mathcal{P}(T)\) that Wilson’s algorithm produces the spanning tree \(T\). Indeed, suppose that the branch structure of \(T\) is

\((y_1,1, \ldots, y_1,K_1) \cdots (y_L,1, \ldots, y_L,K_L),\)

where

1. for each \(1 \leq \ell \leq L\), the vertices \(y_{\ell,1}, \ldots, y_{\ell,K_\ell}\) are distinct, and
   \[V = \{y_{\ell,k} : 1 \leq \ell \leq L \text{ and } 1 \leq k \leq K_L\},\]
2. and for each \(2 \leq \ell \leq L\),
   \[\{y_{\ell,1}, \ldots, y_{\ell,K_\ell}\} \cap \bigcup_{j=1}^{\ell-1} \{y_{j,1}, \ldots, y_{j,K_j}\} = \{y_{\ell,K_\ell}\}.\]

Then

\[\mathcal{P}(T) = \prod_{\ell=1}^{L} \mathcal{P}^{\Delta_\ell}(y_{\ell,1}, \ldots, y_{\ell,K_\ell}),\]

where \(\Delta_1 = \{y_{1,K_1}\}\) and \(\Delta_\ell = \Delta_{\ell-1} \cup \{y_{\ell,1}, \ldots, y_{\ell,K_\ell}\}\) for \(2 \leq \ell \leq L\).

Hence, if we can show that

\[\mathcal{P}^\Delta(y_1, \ldots, y_K) = \prod_{k=1}^{K-1} \left( \mathcal{L}^{\Delta \cup \{y_j : 1 \leq j < k\}} \right)_{y_k y_k}^{-1},\]

then we will know that

\begin{equation}
\mathcal{P}(T) = \prod_{\ell=1}^{L} \prod_{k=1}^{K_\ell-1} \left( \mathcal{L}^{\Delta_\ell \cup \{y_{\ell,j} : 1 \leq j < k\}} \right)_{y_{\ell,k} y_{\ell,k}}^{-1},
\end{equation}
which in conjunction with (13) would mean that

\[(15)\]
\[\mathcal{P}(T) = \frac{1}{\det(\mathcal{L}(y_1, y_K))} = \frac{N}{\prod_k \mathcal{L}_k}.
\]

The equality in (14) was proved by Lawler. To derive it, for \(m = (m_1, \ldots, m_{K-1}) \in \mathbb{N}^K\), let \(\sigma_1^{(m)}\) be the time of the \(m_1\)th visit of \(\{X_n : n \geq 0\}\) to \(y_1\), and, for \(2 \leq k < K\), let \(\sigma_k^{(m)}\) be the time of the \(m_k\)th visit of \(\{X_n : n \geq \sigma_k^{(m)}\}\) to \(y_k\). Also, define \(\sigma_1^{(m)}\) to be time of the \((m_1 + 1)\)th visit of \(\{X_n : n \geq 0\}\) to \(y_1\) and \(\sigma_k^{(m)}\) to be the time of the \((m_k + 1)\)th visit of \(\{X_n : n \geq \sigma_k^{(m)}\}\) to \(y_k\). Then

\[\mathcal{P}^\Delta(y_1, \ldots, y_K) = \sum_{m \in \mathbb{N}^K} \mathbb{P}_{y_1}(B_{K-1}^{(m)}),\]

where, for any \(1 \leq k < K\),

\[B_k^{(m)} = \{\sigma_j^{(m)} < y_k^{(m)} & X_{\sigma_j^{(m)}+1} = y_{j+1} \text{ for } 1 \leq j \leq k\}.\]

To compute \(\mathbb{P}_{y_1}(B_{k}^{(m)})\), define \(\zeta_1^{(m)} = \zeta^\Delta\) and, for \(1 \leq k < K\), let \(\zeta_k^{(m)}\) be the time of the first visit of \(\{X_n : n \geq \sigma_k^{(m)}\}\) to \(\Delta_k \equiv \Delta \cup \{y_1, \ldots, y_{k-1}\}\). Then

\[B_k^{(m)} = \{\sigma_j^{(m)} < \zeta_k^{(m)} & X_{\sigma_j^{(m)}+1} = y_{j+1} \text{ for } 1 \leq j \leq k\}.
\]

Using (7), one can show that \(\mathbb{P}_{y_1}(B_1) = \frac{1}{\mathcal{L}(y_1, y_2)} \mathbb{P}_{y_1}(\tau_{y_1}^{(m)} < \zeta^\Delta)\), where \(\tau_{y_1}^{(m)}\) is the time of the \(m\)th visit to \(x\) by \(\{X_n : n \geq 0\}\). For \(2 \leq k \leq K\), note that

\[B_k^{(m)} = B_{k-1}^{(m)} \cap \{\sigma_k^{(m)} < \zeta_k^{(m)} & X_{\sigma_k^{(m)}+1} = y_{k+1}\},\]

and again use (7) to see that

\[\mathbb{P}_{y_1}(B_k^{(m)}) = \frac{1}{\mathcal{L}(y_1, y_{k+1})} \mathbb{P}_{y_1}(\tau_{y_k}^{(m_k)} < \zeta^\Delta) \mathbb{P}_{y_1}(B_{k-1}^{(m)}).
\]

By induction, this shows that

\[\mathbb{P}_{y_1}(B_K^{(m)}) = \prod_{k=1}^{K-1} \frac{\mathbb{P}_{y_k}(\tau_{y_k}^{(m_k)} < \zeta^\Delta)}{\mathcal{L}(y_{k+1}, y_{k+1})},\]

and therefore, after summing over \(m \in \mathbb{N}^K\) and applying (9), we have that

\[\mathcal{P}^\Delta(y_1, \ldots, y_K) = \prod_{k=1}^{K-1} \frac{\mathcal{L}(y_{k+1}, y_{k+1})^{-1}}{\mathcal{L}(y_{k+1}, y_{k+1})} = \prod_{k=1}^{K-1} (\mathcal{L}(y_{k+1}, y_{k+1}))^{-1}\]

\[= \frac{N}{\prod_k \mathcal{L}_k}.
\]

The equation (15) contains a proof that Wilson’s algorithm yields the uniform distribution on the spanning trees. In addition, since \(\mathcal{P}(T)\) is the reciprocal of the number of spanning trees, Kirchhoff’s formula follows at once as well, namely,

\[(16)\]
\[\text{the number of spanning trees in } \Gamma = \det(\mathcal{L}(x)) = \frac{\prod_k \mathcal{L}_k}{N}.
\]

**Example (continued).** Suppose that we implement Wilson’s algorithm to generate a spanning tree of \(\Gamma\). We start with our original graph as shown in the picture on the left having vertex set \(\{x_1, x_2, x_3, x_4, x_5, x_6\}\). Start a simple random walk at \(x_1\) and stop it when it first reaches \(x_6\). Assume the loop-erasure of this path is \([x_1, x_3, x_6]\), and add this branch to the spanning tree. For the second branch, since \(x_2\) is the first vertex in \(V\) not visited in the first branch, start a random walk at \(x_2\) and stop it when it reaches \(\{x_1, x_3, x_6\}\). Assume that the loop-erasure of this path is \([x_2, x_3]\) and add this branch to the spanning tree. Finally, start a simple random walk at \(x_4\) and stop it when it reaches \(\{x_1, x_3, x_6\} \cup \{x_2, x_3\}\). Assume that the loop-erasure of this path is \([x_4, x_5, x_6]\), and add this third
Moreover, this completes the construction of the spanning tree shown below on the right. In the general notation from the proof above, we have \([y_{1,1}, y_{1,2}, y_{1,3}] = [x_1, x_3, x_6], \) \([y_{2,1}, y_{2,2}] = [x_2, x_3], \) \([y_{3,1}, y_{3,2}, y_{3,3}] = [x_4, x_5, x_6], \) and \(\Delta_1 = \{y_{1,1}\} = \{y_{1,3}\} = \{x_6\}. \) We will now check that the probability that Wilson’s algorithm produces this spanning tree is \(1/29,\) which is the reciprocal of the number of spanning trees of \(\Gamma.\)

![Graph of spanning tree]

We know from (15) that the probability of this particular spanning tree being generated is

\[
P(T) = \frac{1}{\det[\mathcal{C}^{x_6}]} = \frac{\det\left((I - P)^{(x_6)}\right)^{-1}}{\det[\mathcal{D}^{x_6}]} = \det\left((I - P)^{(x_6)}\right)^{-1} \prod_{j=1}^{5} \frac{1}{d_{x_j}}.
\]

Since the order in which the remaining vertices were added to the spanning is \([x_1, x_3, x_2, x_4, x_5],\) we know from Theorem 2.1 that

\[
\det\left((I - P)^{(x_6)}\right)^{-1} = \left((I - P)^{(x_6)}\right)^{-1}_{x_1} \cdot \left((I - P)^{(x_6,x_1)}\right)^{-1}_{x_3} \cdot \left((I - P)^{(x_6,x_1,x_3)}\right)^{-1}_{x_2} \cdot \left((I - P)^{(x_6,x_1,x_3,x_2)}\right)^{-1}_{x_4} \cdot \left((I - P)^{(x_6,x_1,x_3,x_2,x_4)}\right)^{-1}_{x_5}.
\]

The transition matrix for simple random walk on the graph \(\Gamma\) is

\[
P = \begin{bmatrix}
1 & 0 & 2/3 & 1/3 & 1/3 & 1
x_2 & 1/2 & 0 & 0 & 0 & 1/3 \\
x_3 & 1/4 & 1/4 & 0 & 0 & 1/3 \\
x_4 & 1/3 & 0 & 0 & 0 & 1/2 \\
x_5 & 0 & 0 & 0 & 1/2 & 0 \\
x_6 & 0 & 0 & 0 & 1/2 & 0
\end{bmatrix}
\]

and so we find \((I - P)^{(x_6)}\)^{-1} equals

\[
\begin{bmatrix}
x_1 & 1 & -1/3 & -1/3 & -1/3 & 0 \\
x_2 & 1/2 & 0 & 0 & 0 & 1/3 \\
x_3 & 1/4 & 1/4 & 0 & 0 & 1/3 \\
x_4 & 1/3 & 0 & 0 & 0 & 1/2 \\
x_5 & 0 & 0 & 0 & 1/2 & 1
\end{bmatrix}^{-1} = \begin{bmatrix}
x_1 & 93/29 & 50/29 & 76/29 & 60/29 & 20/29 \\
x_2 & 75/29 & 74/29 & 80/29 & 54/29 & 18/29 \\
x_3 & 57/29 & 40/29 & 84/29 & 48/29 & 16/29 \\
x_4 & 60/29 & 36/29 & 64/29 & 78/29 & 26/29 \\
x_5 & 30/29 & 18/29 & 32/29 & 39/29 & 42/29
\end{bmatrix}
\]

Moreover, \((I - P)^{(x_6,x_1)}\)^{-1} equals

\[
\begin{bmatrix}
x_2 & 1 & -1/2 & 0 & 0 \\
x_3 & -1/4 & 1 & -1/4 & 0 \\
x_4 & 0 & -1/3 & 1 & -1/3 \\
x_5 & 0 & 0 & -1/2 & 1
\end{bmatrix}^{-1} = \begin{bmatrix}
x_2 & 36/31 & 20/31 & 6/31 & 2/31 \\
x_3 & 10/31 & 40/31 & 12/31 & 4/31 \\
x_4 & 4/31 & 16/31 & 42/31 & 14/31 \\
x_5 & 2/31 & 8/31 & 21/31 & 38/31
\end{bmatrix}
\]
\[(I - P)^{\{x_6,x_1,x_3\}})^{-1} = \begin{pmatrix} x_2 & x_4 & x_5 \\ x_4 & 1 & 0 \\ x_5 & 0 & -1/2 \end{pmatrix}^{-1} = \begin{pmatrix} x_2 & x_4 & x_5 \\ x_4 & 0 & 6/5 \\ x_5 & 0 & 3/5 \end{pmatrix},
\]

\[(I - P)^{\{x_6,x_1,x_2,x_3\}})^{-1} = x_5 \begin{pmatrix} x_5 \\ 6/5 \\ 3/5 \end{pmatrix}, \quad \text{and} \quad ((I - P)^{\{x_6,x_1,x_2,x_4\}})^{-1} = x_5 \begin{pmatrix} x_5 \\ 1 \end{pmatrix},
\]

so that
\[
\text{det} \left[ ((I - P)^{\{x_6\}})^{-1} \right] = \frac{93}{29} \cdot \frac{40}{31} \cdot \frac{6}{5} \cdot 1 = \frac{144}{29}.
\]

Hence,
\[
\mathcal{P}(\mathcal{T}) = \frac{144}{29} \cdot \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{4} \cdot \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{29}.
\]

But we already knew that there are 29 spanning trees of \(\Gamma\), so we have verified in this case that Wilson’s algorithm does, in fact, produce a spanning tree uniformly at random.

### 6. Cayley’s theorem

If \(\Gamma = (V, E)\) is the complete graph on \(N + 1\) vertices so that there is an edge connecting each vertex to every other, then Cayley’s theorem states that the number of spanning trees of \(\Gamma\) is \((N + 1)^{N-1}\). This formula was first discovered in 1860 by Carl Wilhelm Borchardt, although it is now universally named after Arthur Cayley \([2]\) who extended Borchardt’s original results. Of course, Cayley’s theorem is easily derived from Kirchhoff’s matrix tree theorem by computing \(\text{det}[\mathcal{L}]\) using elementary column operations to bring the matrix into lower triangular form.

The easiest way to prove Cayley’s theorem via Wilson’s algorithm is to use \([9]\). Start a simple random walk at \(x\), and suppose that \(\Delta \subseteq V \setminus \{x\}\) is any nonempty collection of vertices with \(|\Delta| = m\). Recall that \(r_\Delta(x)\) is the probability that simple random walk starting at \(x\) returns to \(x\) before entering \(\Delta\). Let \(r_\Delta(x; k)\) be the probability that simple random walk starting at \(x\) returns to \(x\) in exactly \(k\) steps without entering \(\Delta\) so that
\[
r_\Delta(x) = \sum_{k=2}^{\infty} r_\Delta(x; k)
\]

because it takes the simple random walk at least 2 steps to return to its starting point. Since each vertex has an edge to every other vertex, we have partitioned the vertex set into three pieces, namely \(V_1 = \{x\}\), \(V_2 = \Delta\) which has cardinality \(m\), and \(V_3\) which has cardinality \(N - m\). Thus, the probability that simple random walk starting at \(x\) returns to \(x\) in exactly \(k\) steps without entering \(\Delta\) is
\[
r_\Delta(x; k) = \mathbb{P}_x \{S_1 \in V_3, S_2 \in V_3, \ldots, S_{k-1} \in V_3, S_k = x\} = \frac{N - m}{N} \left( \frac{N - 1 - m}{N} \right)^{k-2} \frac{1}{N}
\]

and so
\[
r_\Delta(x) = \frac{(N - m)}{N^2} \sum_{k=2}^{\infty} \left( \frac{N - 1 - m}{N} \right)^{k-2} = \frac{N - m}{N(m + 1)}.
\]

By \([9]\),
\[
((I - P)^\Delta)_{xx}^{-1} = \frac{1}{1 - r_\Delta(x)} = \frac{N(m + 1)}{m(N + 1)}.
\]
Suppose that we now label the vertices of $\Gamma$ as $V = \{y_1, \ldots, y_{N+1}\}$, and set $\Delta_j = \{y_1, \ldots, y_j\}$ for $j = 1, \ldots, N$. Since $|\Delta_j| = j$, we have from Theorem 2.1 combined with (17),

$$\det \left( \left( (I - P)^\{y_1\} \right)^{-1} \right) = \prod_{j=1}^{N} \frac{N^j (N+1)!}{j(N+1)^N N!} = \frac{N^N}{(N+1)^{N-1}}.$$ 

Since each of the $(N+1)$ vertices has degree $N$, we conclude that the number of spanning trees of the complete graph on $N+1$ vertices is

$$\det(L^{y_1}) = \frac{\det(D^{y_1})}{\det \left( \left( (I - P)^\{y_1\} \right)^{-1} \right)} = \frac{N^N}{(N+1)^{N-1}} = (N+1)^{N-1}.$$ 

**Remark.** Lyons and Peres [8] use Wilson’s algorithm to prove Cayley’s theorem. However, their proof of Wilson’s algorithm is via cycle-popping and so their derivation is quite different than ours. Moreover, they do not extend their proof of Cayley’s theorem to a proof of Kirchhoff’s matrix tree theorem.

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