A Statistical Mechanical Analysis on the Bound State Solution of an Energy-Dependent Deformed Hulthén Potential Energy

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Abstract In this article, we investigate the bound state solution of the Klein Gordon equation under mixed vector and scalar coupling of an energy-dependent deformed Hulthén potential in D dimensions. We obtain a transcendental equation after we impose the boundary conditions. We calculate energy spectra in four different limits and in arbitrary dimension via the Newton-Raphson method. Then, we use a statistical method, namely canonical partition function, and discuss the thermodynamic properties of the system in a comprehensive way. We find out that some of the thermodynamic properties overlap with each other, some of them do not.

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Key words: Klein-Gordon equation, energy-dependent deformed Hulthén potential energy, bound state solution, thermodynamic properties

1 Introduction

One of the major investigation areas in either relativistic or non-relativistic quantum mechanics is to obtain a solution of potential energies.\cite{1-4} This intense interest is based on the fact that the exact solution of the wave function has all the necessary information to define the physical system. Unfortunately, only a few numbers of potential energies have exact solutions. Unfortunately, only a few numbers of potential energies have exact solutions, for example, infinite well, finite well or barrier, Coulomb potential, and the harmonic oscillator. Beside these analytic solutions, semi exact solutions in case of $l = 0$, or approximate solutions in case of $l \neq 0$ are investigated comprehensively in many other potential energies such as Morse, Eckart, Rosen-Morse (RM), Manning-Rosen (MR), Pöschl-Teller (PT), Yukawa, Hylleraas, Hulthén, Woods-Saxon (WS), etc.

The Klein-Gordon (KG) equation is one of the fundamental relativistic wave equation that describes the motion of spin zero particles.\cite{14} Remarkable efforts have been executed to examine the solutions of the KG equation with a various number of potential energies. Yi et al. employed RM type vector and scalar potential energies to obtain the s-wave bound state energy spectra.\cite{15} Villalba et al. examined the bound state solution of a spatially one-dimensional cusp potential energy in the KG equation.\cite{16} Olgar et al. employed a supersymmetric technique to obtain a bound state solution of the s-wave KG equation with equal scalar and vector Eckart type potential energy.\cite{17} Only two years later, they applied the asymptotic interaction method (AIM), which is originally introduced by Ciftci et al.,\cite{18} to calculate an energy spectrum of the s-wave KG equation with the mixed scalar and vector generalized Hulthén potential in one dimension.\cite{19}

Then, he used AIM to investigate bound state solution of three different potential energies, namely linear, Morse, and Kratzer, in the KG equation.\cite{20} In 2010, Xu et al. studied the bound state solution of the KG equation with mixed vector and scalar PT potential energy with a non zero angular momentum parameter.\cite{21} Ikot et al. obtained an exact solution of the Hylleraas potential energy in the KG equation.\cite{22} Jia et al. examined the bound state solution of the KG equation with an improved version of the MR potential energy.\cite{23} Hou et al. studied the bound state solution of the s-wave KG equation with vector and scalar WS potential energy.\cite{24} Rojas et al. used the vector WS barrier in the KG equation and presented the continuum state solution.\cite{25} Later, Hassanabadi extended that study with an addition of scalar WS potential energy term.\cite{26} Arda et al. employed Nikiforov-Uvarov (NU) and studied the modified WS potential energy with position dependent mass in the KG equation in three dimensions.\cite{27} Badalov et al. used NU and Pekeris approximation to study any l state of the KG equation.\cite{28}
Bayrak et al. investigated the generalized WS potential energy in the KG equation for zero\cite{29} and non-zero\cite{30} values of the angular momentum parameter. One of the authors of this manuscript, Lütfioglu, with his collaborators examined the mixed vector and scalar generalized symmetric WS potential energies for the scattering case in the KG equation first under the equal magnitudes and signs (EMES), and then, in the equal magnitudes and opposite signs (EMOS).\cite{31} Later, he investigated the same problem in the bound state case.\cite{32} Beside these studies, multi-parameter exponential type potential energies\cite{33-35} and non central potentials\cite{36-37} are examined in the KG equation.

Recently, the investigation of different physical systems in one or three dimensions have been extended to higher dimensions to describe different phenomena not only in diverse fields of physics but in quantum chemistry, too.\cite{38} Chen et al. examined hydrogen type atoms by employing the Coulomb potential energy in KG equation in $D$ dimensions.\cite{39} Saad et al. applied AIM to study KG equation with unequal vector and scalar Kratzer potential energy in $D$ dimensions.\cite{40} In 2011, Hassanabadi et al. obtained an approximate solution by employing an equal scalar and vector generalized Kratzer potential to the $D$-dimensional KG equation for any angular momentum parameter.\cite{41} One year later, Hassanabadi et al. examined the Eckart potential in addition to modified Hylleraas potential energy in higher dimensional relativistic equations by supersymmetric quantum mechanic methods.\cite{42} Ibrahim et al. studied higher dimensional KG and Dirac equations with mixed equal scalar and vector RM potential energies by NU method.\cite{43} Ortakaya used pseudoharmonic oscillator potential energy in $D$-dimensional KG equation to obtain the bound state energy spectrum of CH, H$_2$ and HCl molecules.\cite{44} Antia et al. defined a combined potential energy function by addition of Mobius square potential to Yukawa potential energy. Then, they employed the NU method to solve the combined potential energy in high dimensional KG equation.\cite{45} Chen et al. obtained the relativistic bound state equation energy by employing the improved MR potential energy in $D$ spatial dimensions.\cite{46} Ikot et al. analyzed the improved MR potential energy for arbitrary angular momentum parameter in an approximate method in $D$ dimensions.\cite{47} Tan et al. and Jia et al. solved the $D$-dimensional KG equation with the improved and modified RM potential energy by employing supersymmetric WKB approximation.\cite{48-49} Xie et al. examined Morse potential energy in KG equation to derive the bound state energy equation in $D$ spatial dimensions.\cite{50} Ikot et al. employed NU method to analyze an exponential type molecule potential in the KG equation in $D$ dimensions.\cite{51}

In last decade, the prediction of the properties of a physical system by investigating their thermodynamic functions become popular. In this purpose, the scientist calculates the energy spectrum of the system in a relativistic or non-relativistic equation by proposing potential energy and then obtains the partition function. Ikhdair et al. solved the Schrödinger equation with the PT potential energy via AIM and discussed the thermodynamic functions.\cite{52} In 2014, Oyewumi et al. used the shifted Deng-Fan potential energy in the non-relativistic equation to analyze the statistical properties.\cite{53} One year later, Onate et al. defined the combination of hyperbolic and generalized PT potential energies and solved Dirac equation. They discussed the thermodynamic properties in non relativistic limit in addition to the spin symmetry (SS) and pseudospin symmetry (PSS) limits.\cite{54} In 2016, Arda et al. used the linear potential to investigate the thermodynamic quantities such as the Helmoltz free energy, and the mean energy with the specific heat function in both KG and Dirac equations.\cite{55} Onyeaju et al. studied the Dirac equation with the deformed Hylleraas in addition to WS potential energy and calculated the thermodynamic functions of some diatomic molecules.\cite{56} Then, Ikot et al. discussed the thermodynamic functions of diatomic molecules by using a general molecular potential.\cite{57} In another paper, Valencia-Ortega and Arias-Hernandez investigated the thermodynamic properties of diatomic molecules by adopting SO(2,1) anharmonic Eckart potential energy.\cite{58} Furthermore, Okorie with co-authors investigated thermodynamic functions by using modified Mobius square,\cite{59} modified Yukawa,\cite{60} quadratic exponential-type,\cite{61} shifted Tietz-Wei\cite{62-63} potential energies. In 2019, one of the authors of the present paper, Ikot, with his collaborators studied the thermodynamic properties of a q-deformed quantum oscillator in the scale of minimal length.\cite{64} The other author of the present paper, Lütfioglu, also contributed to the field by the studies via the investigation of the generalized symmetric WS potential energy in non relativistic\cite{65} and relativistic equations.\cite{66} With the non relativistic results, they obtained the thermodynamic properties of a nucleon in relatively small\cite{67} and big radius nuclei.\cite{68} Then, he compared the thermodynamic functions with excluding and including the surface effects in non-relativistic,\cite{69} and relativistic regimes.\cite{70} In a very recent article, they presented the variance of the thermodynamic functions in the existence of attractive or repulsive surface interaction terms.\cite{71} Besides these works, thermodynamic properties of molecules and dimers are examined in several articles by taking the vibrational and rotational partition functions into account.\cite{72-88} Our motivation is to determine the bound state solution of the energy-dependent deformed Hulthén potential in $D$-dimensional KG equation and discuss the
corresponding thermodynamic functions. Note that the energy-dependent potential energies have been investigated in both relativistic and non-relativistic wave equations since 1940.\cite{89-95} In recent times, Gupta et al. studied the Schrödinger equation with energy dependent harmonic oscillator potential energy function to describe quark systems.\cite{96} Ikot et al. examined energy dependent Yukawa potential energy with a Coloumb-like tensor interaction in the Dirac equation at the SS and PSS limits.\cite{97} Boumali et al. examined energy dependent harmonic oscillator in Schrödinger\cite{98} and KG equation\cite{99} to predict the Shannon entropy and Fisher information. The paper is organized as follows. In Sec. 2 we define the KG equation in an arbitrary dimension with the vector and the scalar potential energy coupling. Then, we describe q-deformed energy dependent Hulthén potential energy and obtain the radial wave function solution by employing a Greene-Aldrich approach to the centrifugal term. Furthermore, we derive the quantization condition. Before we end the section, we briefly give the normalization method in an energy dependent potential energy case. In Sec. 3 we state the thermodynamic functions such as Helmholtz free energy, entropy, internal energy, and specific heat. Then, in Sec. 4 we use the Newton-Raphson method to calculate energy spectra for various dimensions in the EMES, EMOS, pure vector and scalar limits. Moreover, we obtain the thermodynamic functions from the partition function. We demonstrate those functions within a comparison. In Sec. 5 we conclude the paper.

2 Solutions of the Klein-Gordon Equation in D Dimensions

We start by expressing the KG equation in D spatial dimensions with

\[ \left[ \hat{p}_\mu \hat{p}_\mu - (m_0 c^2)^2 \right] \phi(\vec{r}, t) = 0, \quad \mu = 0, 1, \ldots, D. \]  

(1)

Here, we use \( \hat{p}_\mu \), \( c \), and \( m_0 \) to denote the \((D + 1)\) momentum vector, the speed of light and the rest mass of the particle, respectively. Then, we employ a minimal coupling of the momentum vector to a \((D + 1)\) vector potential. Among the components of the vector potential, we only assume that the time component has non-zero value. This component, \( V_0 \), is called as “the vector potential” in the literature. In addition, we use a scalar potential, \( V_s \), coupling to the rest mass parameter term.

In this manuscript, we investigate the solution of the spherical symmetric potential energies that are time-independent. Therefore, we can separate the wave function into time and spatial components. Then, we decompose the spatial part of the wave function into radial and angular parts by employing the spherical symmetric nature of the potential energies. Finally, we obtain the radial equation as follows.

\[ \left[ \frac{d^2}{dr^2} - \frac{\gamma}{r^2} + \frac{1}{a^2 c^2} \left[ (E - V_0) - (m_0 c^2 + g V_s) \right] \right] \chi(r) = 0. \]  

(2)

Here \( \gamma \equiv [(D + 2l - 1)(D + 2l - 3)]/4 \), and \( l \) denotes the angular momentum quantum number. Furthermore, \( h \) represents the Planck constant, and \( g \) is the coupling constant that is nearly equal to one in the strong regime. Note that \( \chi(r) \equiv r F(r) \). In the rest of the article, we will use the natural units where \( h = c = 1 \).

2.1 Bound State Solutions

We examine q-deformed energy dependent vector and scalar Hulthén potential energy wells

\[ V_0(r) = -V_0 \frac{(1 + aE) e^{-\delta r}}{1 - q e^{-\delta r}}, \]  

(3)

\[ g V_s(r) = -S_0 \frac{(1 + aE) e^{-\delta r}}{1 - q e^{-\delta r}}, \]  

(4)

where \( V_0, S_0, a, \) and \( \delta \) are the vector potential depth, scalar potential depth, energy slope parameter, and the screening parameters, respectively.

In order to deal with the centrifugal term we adopt the Greene-Aldrich approximation scheme\cite{100}

\[ \frac{1}{r^2} \approx \delta^2 \frac{e^{-2\delta r}}{(1 - q e^{-\delta r})^2}. \]  

(5)

Here, \( \delta r < 1 \) and \( q \simeq 1 \). Note that for the validity, the deformation parameter value should not be higher than 1. Then, we substitute Eq. (3), Eq. (4), and Eq. (5) into Eq. (2) and we get

\[ \left\{ \frac{d^2}{dz^2} + E^2 - m^2 + 2(EV_0 + mS_0) \frac{(1 + aE) e^{-\delta r}}{1 - q e^{-\delta r}} + (V_0^2 - S_0^2) \frac{(1 + aE)^2 e^{-2\delta r}}{(1 - q e^{-\delta r})^2} - \gamma \delta^2 \frac{e^{-2\delta r}}{(1 - q e^{-\delta r})^2} \right\} \chi(z) = 0. \]  

(6)

We introduce a new coordinate transformation of the form \( z \equiv (1 - q e^{-\delta r})^{-1} \), and adopt the following abbreviations

\[ \varepsilon \equiv (1 + aE), \]  

(7)

\[ -\alpha^2 \equiv \frac{E^2 - m^2}{\delta^2}, \]  

(8)

\[ \sigma^2 \equiv \frac{2\varepsilon(EV_0 + mS_0)}{q\delta^2}, \]  

(9)

\[ \beta^2 \equiv \frac{1}{q^2} \left[ \frac{(V_0^2 - S_0^2)\varepsilon^2}{\delta^2} - \gamma \right]. \]  

(10)

We get

\[ \left\{ z(z - 1) \frac{d^2}{dz^2} + (2z - 1) \frac{d}{dz} + \beta^2 - \frac{\alpha^2}{z - 1} + \frac{\alpha^2 + \sigma^2 - \beta^2}{z} \right\} \phi(z) = 0. \]  

(11)

Then, we propose the following ansatz

\[ \phi(z) \equiv z^\mu (z - 1)^\nu u(z), \]  

(12)
where
\[ \mu^2 = \alpha^2 + \sigma^2 - \beta^2, \quad (13) \]
\[ \nu^2 = \alpha^2, \quad (14) \]
\[ \theta^2 = \frac{1}{4} - \beta^2. \quad (15) \]

We find that Eq. (11) turns into the following form
\[ z(1 - z) \frac{d^2 u(z)}{dz^2} + \left[ (1 + 2\mu) - z(2\mu + 2\nu + 2 + \theta) \right] \frac{du(z)}{dz} \]
\[ - \left[ (\mu + \nu + 1 + \theta)(\mu + \nu + 1/2) \right] u(z) = 0. \quad (16) \]
The solution can be expressed in terms of the hypergeometric functions \( _2F_1 \)
\[ \phi(z) = N_1 z^\mu (z - 1)^\nu _2F_1 (d, b, c, z) + N_2 z^{-\mu} (z - 1)^\nu _2F_1 \]
\[ \times (1 + d - c, 1 + b - c, 2 - c, z), \quad (17) \]

Then, we find
\[ \phi(z) = N_1 z^\mu (z - 1)^\nu \left[ \Gamma(c)\Gamma(c - d - b) \right] _2F_1 (d, b, d + b + c + 1, 1 - t) + (1 - t)^{c-d-1} \left[ \Gamma(c)\Gamma(d + b - c) \right] _2F_1 \]
\[ \times (c - d, c - b, c - d - b + 1, 1 - t). \quad (21) \]

After this identical transformation of the hypergeometric functions, the result values are equal to 1. Consequently, we get
\[ [N_1 \Gamma_1 + N_2 \Gamma_3] (z - 1)^\nu \left[ N_1 \Gamma_2 + N_2 \Gamma_4 \right] \times e^{-2\pi i\nu} (z - 1)^{-\nu}, \quad (23) \]

where
\[ \Gamma_1 \equiv \frac{\Gamma(1 + 2\mu)\Gamma(1 - 2\nu)}{\Gamma(1 + 2\mu + 2\nu)} \quad (24) \]
\[ \Gamma_2 \equiv \frac{\Gamma(\mu - \nu + 1/2 + \theta)\Gamma(\mu - \nu + 1/2 - \theta)}{\Gamma(\mu + 1/2 + \theta)\Gamma(\mu + 1/2 - \theta)} \quad (25) \]
\[ \Gamma_3 \equiv \frac{\Gamma(-\mu + 1/2 + \theta)\Gamma(-\mu + 1/2 - \theta)}{\Gamma(1 - 2\mu)\Gamma(-2\nu)} \quad (26) \]
\[ \Gamma_4 \equiv \frac{\Gamma(-\mu + 1/2 - \theta)\Gamma(-\mu - 1/2 - \theta)}{\Gamma(-\mu - 1/2 + \theta)\Gamma(1/2 + \nu)} \quad (27) \]

We assume that \( \nu = \alpha \). We are obliged to take \( N_2 = 0 \) and \( \Gamma_2 = 0 \) to avoid the singularity. We use the definition of the reciprocal of gamma function for negative integer as given in Ref. [101],
\[ \lim_{z \to -n} \frac{1}{\Gamma(-z)} = 0, \quad n = 0, 1, 2, \ldots, \quad (28) \]
to eradicate \( \Gamma_2 \). Although either \( \mu + \nu + 1/2 - \theta \) or \( \mu + \nu + 1/2 + \theta \) can be chosen to be equal to \( -n \), the symmetric structure of the wave functions under exchange of both parameters leads to obtain the same solution. We use the condition
\[ \mu + \nu + 1/2 - \theta = -n, \quad (29) \]
and we obtain
\[ \sqrt{m^2 - E^2} = \frac{\delta}{2} \left[ \Sigma - n + \beta^2 - \sigma^2 \right], \quad (30) \]
where
\[ \Sigma = \sqrt{1 - \left( \frac{V_0^2 - S_0^2}{q^2} \right)^2 \frac{\gamma}{q^2} - \frac{1}{2}}. \quad (31) \]

We find the unnormalized radial wave function as follows
\[ R_n(r) = \frac{N_n}{r} \left( \frac{1}{1 - q \cos \sigma} \right)^{\mu} \left( \frac{q}{e^{\theta r} - q} \right)^{\nu} F_1 \]
\[ \times \left( -n, n + 2\mu + 2\nu + 1, 1 + 2\mu, \frac{1}{1 - q \cos \sigma} \right). \quad (32) \]

### 2.2 Quantization

In this subsection, we take into account the boundary condition that dictates the radial wave function should go to zero at infinity. In that limit, the transformed coordinate \( z \) goes to 1. Therefore, we need to determine the behaviour of the hypergeometric function initially. We employ the following well-known property of the hypergeometric function\[^{[101]}\]
\[ \phi(z) = \frac{\Gamma(c)\Gamma(d + b - c)}{\Gamma(d)\Gamma(1 - b)} _2F_1 \]
\[ (1 - d, c - b, c - d - b + 1, 1 - t). \quad (22) \]

\[ |\chi_E(x)|^2 \left[ 1 - \frac{\partial^2}{\partial E^2} (E^2 - (E - V(x,E))^2 + (M + S(x,E))^2) \right] \bigg|_{E=E_n}. \]  

Consequently, in the problem one can calculate the normalization constant as follows

\[ \frac{1}{|N|^2} = \int_0^\infty r^2 dr \left( 1 + \frac{4aV_0 e^{1-5\beta r}}{1 - q e^{-5\beta r}} + \frac{2a^2(V_0^2 - S_0^2) e^{-2\beta r}}{(1 - q e^{-5\beta r})^2} \right) |R(r)|^2. \]  

Here, we skip calculating the normalization constant since it does not exist in our main motivation.

### 3 Thermodynamic Functions

One way to examine the thermodynamic properties of a physical system is to use the partition function. In the canonical ensemble, for a system that is in an equilibrium state, the partition function is defined with

\[ Z(\beta_T) = \sum_{n=0}^\infty e^{-\beta_T E_n}. \]  

Here, \( E_n \) represents the available microstate energy values. \( \beta_T \) is the reciprocal temperature function and it is inversely proportional to the multiplication of the Boltzmann constant with the absolute temperature. Thermodynamic functions such as Helmholtz free energy, \( F(\beta_T) \), entropy, \( S(\beta_T) \), internal energy, \( U(\beta_T) \), and specific heat, \( C_v(\beta_T) \), functions are obtained from the partition function as follows

\[ F(\beta_T) \equiv -\frac{1}{\beta_T} \ln Z(\beta_T), \]  

\[ S(\beta_T) = -k_B \frac{\partial F(\beta_T)}{\partial \beta_T}, \]  

\[ U(\beta_T) \equiv -\frac{\partial}{\partial \beta_T} \ln Z(\beta_T), \]  

\[ C_v(\beta_T) = k_B \frac{\partial U(\beta_T)}{\partial \beta_T}. \]

### 4 Results and Discussions

In this section, we construct the thermodynamic func-

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
\textbf{n} & \textbf{l} & \textbf{a = 1 (1/E)} & \textbf{a = 0 (1/E)} & \textbf{a = -1 (1/E)} \\
\hline
1 & 0 & \( -0.962 \ 938 \) & \( -0.999 \ 907 \) & \( -0.999 \ 967 \) \\
2 & 0 & \( -0.940 \ 966 \) & \( -0.999 \ 627 \) & \( -0.999 \ 869 \) \\
3 & 0 & \( -0.963 \ 843 \) & \( -0.999 \ 987 \) & \( -0.999 \ 997 \) \\
1 & 0 & \( -0.922 \ 427 \) & \( -0.999 \ 160 \) & \( -0.999 \ 705 \) \\
2 & 0 & \( -0.965 \ 764 \) & \( -0.999 \ 687 \) & \( -0.999 \ 907 \) \\
4 & 0 & \( -0.905 \ 797 \) & \( -0.999 \ 508 \) & \( -0.999 \ 476 \) \\
1 & 0 & \( -0.922 \ 848 \) & \( -0.999 \ 240 \) & \( -0.999 \ 742 \) \\
2 & 0 & \( -0.942 \ 664 \) & \( -0.999 \ 950 \) & \( -0.999 \ 987 \) \\
3 & 0 & \( -0.969 \ 016 \) & \( -0.998 \ 460 \) & \( -0.999 \ 967 \) \\
\hline
\end{tabular}
\caption{Energy spectrum for the EMES limit in three dimensions.}
\end{table}

Table 1 Energy spectrum for the EMES limit in three dimensions.

limits just after we calculate the energy spectra in different limits and dimensions. To calculate the energy spectra we solve the quantization condition numerically by the use of the Newton-Raphson method in the EMES limit, \( V_0 = S_0 = 0 \), in the pure vector limit, \( S_0 = 0 \), and in the pure scalar limit, \( V_0 = 0 \). Note that, since we study with the natural units, all units of the parameters of the system can be expressed in terms of energy or reciprocal energy. There are some parameters that are always kept as a constant in all limits, for instance, the mass and the deformation parameter. Both of them are equal to one. There are some other parameters, which we assign different values, i.e., a parameter, which is the measure of the energy dependence of the potential energy, is assumed to be equal to 1 \((1/E)\), 0, and \(-1 \((1/E)\). Note that, we calculate the spectra only in 3, 4, and 5 dimensions.

In the second part of this section, namely in Sub-

\section{4.1 EMES Limit}

We assume the energy depth parameters have equal values as given, \( V_0 = S_0 = 2 \((E)\). Moreover, the slope parameter is equal to 0.01 \((E)\). We tabulate the energy spectra in three dimensions in Table 1, in four dimensions in Table 2, and in five dimensions in Table 3, respectively.
Table 2 Energy spectrum for the EMES limit in four dimensions.

| n | l | \( a = 1 \) (1/E) | \( a = 0 \) (1/E) | \( a = -1 \) (1/E) |
|---|---|---|---|---|
| 1 | 0 | -0.977 245 | -0.999 987 | -0.999 997 | none | none |
| 2 | 0 | -0.951 600 | none | -0.999 820 | -0.999 940 | 0.999 940 |
| 1 | 0 | -0.979 644 | -0.999 382 | none | none | none |
| 3 | 0 | -0.931 576 | none | -0.999 447 | -0.999 809 | 0.999 809 |
| 1 | 0 | -0.952 636 | -0.999 987 | -0.999 972 | -0.999 993 | 0.999 993 |
| 2 | 0 | -0.985 401 | -0.995 733 | none | none | none |
| 4 | 0 | -0.914 064 | none | -0.998 887 | -0.999 613 | 0.999 613 |
| 1 | 0 | -0.932 297 | none | -0.999 572 | -0.999 867 | 0.999 867 |
| 2 | 0 | -0.954 443 | -0.999 765 | none | none | none |
| 3 | none | none | none | none | none | none |

Table 3 Energy spectrum for the EMES limit in five dimensions.

| n | l | \( a = 1 \) (1/E) | \( a = 0 \) (1/E) | \( a = -1 \) (1/E) |
|---|---|---|---|---|
| 1 | 0 | none | none | none | none |
| 2 | 0 | -0.963 843 | -0.999 987 | -0.999 988 | -0.999 997 | 0.999 997 |
| 1 | none | none | none | none | none |
| 3 | 0 | -0.941 524 | none | -0.999 709 | -0.999 907 | 0.999 907 |
| 1 | none | none | none | none | none |
| 2 | none | none | none | none | none |
| 4 | 0 | -0.922 848 | none | -0.999 240 | -0.999 742 | 0.999 742 |
| 1 | none | none | none | none | none |
| 2 | none | none | none | none | none |
| 3 | none | none | none | none | none |

4.2 EMOS Limit

In this limit, the energy depth parameters have negatively equal values. Here, we assume \( V_0 = 2 \) (E) and \( S_0 = -2 \) (E). Alike EMES limit, we choose the slope parameter to be equal to 0.01 (E). Then, we present the energy spectra in three dimensions in Table 4, in four dimensions in Table 5, and in five dimensions in Table 6, respectively.

Table 4 Energy spectrum for the EMOS limit in three dimensions.

| n | l | \( a = 1 \) (1/E) | \( a = 0 \) (1/E) | \( a = -1 \) (1/E) |
|---|---|---|---|---|
| 1 | 0 | none | none | none | none |
| 2 | 0 | none | none | none | none |
| 1 | none | none | none | none | none |
| 3 | 0 | none | none | none | none |
| 1 | none | none | none | none | none |
| 2 | none | none | none | none | none |
| 4 | 0 | none | none | none | none |
| 1 | none | none | none | none | none |
| 2 | none | none | none | none | none |
| 3 | none | none | none | none | none |

We see that when the energy dependence is fixed with \( \varepsilon = 1 - E \), (\( a = -1 \) (1/E)), most of the eigenvalues in the energy spectrum cannot be calculated. Therefore, we decide to calculate the spectrum for higher values of \( l \) parameter. Surprisingly, unlike \( \varepsilon = 1 + E \) case, the values of \( l \) parameter are not limited. In three, four and five dimensions we repeat the calculations and present them in Table 7, in Table 8, and in Table 9, respectively. We conclude that as the values of parameter \( l \) increase, energy eigenvalues converge.
Table 5  Energy spectrum for the EMOS limit in four dimensions.

| n | l  | \(a = 1 (1/E)\) | \(a = 0 (1/E)\) | \(a = -1 (1/E)\) |
|---|---|-----------------|-----------------|-----------------|
| 1 | 0  | -0.999 997     | 0.999 997       | 0.999 995       | 0.999 988 |
| 2 | 0  | none           | none            | none            | none       |
| 3 | 0  | none           | none            | none            | none       |
| 1 | 1  | -0.999 966     | 0.999 966       | 0.999 940       | 0.999 392  |
| 2 | 1  | -0.999 994     | 0.999 994       | 0.999 992       | 0.999 988  |
| 3 | 0  | none           | none            | none            | none       |
| 1 | 2  | -0.999 910     | 0.999 910       | 0.999 833       | 0.996 735  |
| 2 | 2  | -0.999 955     | 0.999 955       | 0.999 928       | 0.999 765  |
| 3 | 0  | none           | none            | none            | none       |
| 1 | 3  | none           | none            | none            | none       |
| 2 | 3  | none           | none            | none            | none       |
| 3 | 4  | none           | none            | none            | none       |

Table 6  Energy spectrum for the EMOS limit in five dimensions.

| n | l  | \(a = 1 (1/E)\) | \(a = 0 (1/E)\) | \(a = -1 (1/E)\) |
|---|---|-----------------|-----------------|-----------------|
| 1 | 0  | none           | none            | none            |
| 2 | 0  | -0.999 995     | 0.999 995       | 0.999 993       | 0.999 988  |
| 3 | 1  | none           | none            | none            | none       |
| 1 | 2  | -0.999 960     | 0.999 960       | 0.999 933       | 0.999 688  |
| 2 | 2  | none           | none            | none            | none       |
| 4 | 0  | none           | none            | none            | none       |
| 1 | 3  | -0.999 993     | 0.999 993       | 0.999 991       | 0.999 987  |
| 2 | 3  | -0.999 898     | 0.999 898       | 0.999 820       | 0.998 511  |
| 3 | 4  | none           | none            | none            | none       |

Table 7  Energy spectrum for the EMOS limit in three dimensions for the higher values of \(l\).

| \(l\) | \(E_{1l}\) | \(E_{2l}\) | \(E_{3l}\) | \(E_{4l}\) |
|-------|------------|------------|------------|------------|
| 0     | none       | none       | none       | none       |
| 1     | none       | 0.999 988  | none       | none       |
| 2     | 0.961 220  | none       | 0.999 688  | 0.999 988  |
| 3     | 0.939 873  | 0.958 831  | none       | 0.998 511  |
| 4     | 0.922 010  | 0.937 770  | 0.955 919  | none       |
| 5     | 0.906 141  | 0.919 970  | 0.935 276  | 0.952 608  |
| 10    | 0.841 848  | 0.851 190  | 0.860 976  | 0.871 257  |
| 50    | 0.542 193  | 0.544 537  | 0.546 944  | 0.549 416  |
| 100   | 0.308 370  | 0.308 232  | 0.308 136  | 0.308 080  |
| 500   | -0.488 246 | -0.492 937 | -0.497 594 | -0.502 218 |
| 1000  | -0.783 293 | -0.788 018 | -0.792 692 | -0.797 315 |
| 5000  | -0.989 668 | -0.991 032 | -0.992 300 | -0.993 473 |
| 10000 | -0.997 896 | -0.998 492 | -0.998 990 | -0.999 388 |
Table 8  Energy spectrum for the EMOS limit in four dimensions for the higher values of $l$.

| $l$ | $E_{1l}$ | $E_{2l}$ | $E_{3l}$ | $E_{4l}$ |
|-----|---------|---------|---------|---------|
| 0   | none    | none    | none    | none    |
| 1   | 0.974 544 | 0.999 392 | 0.999 988 | 0.999 765 |
| 2   | 0.949 943 | 0.971 654 | 0.996 735 | 0.999 765 |
| 3   | 0.930 632 | 0.947 749 | 0.968 168 | 0.991 779 |
| 4   | 0.913 876 | 0.928 572 | 0.945 107 | 0.964 299 |
| 5   | 0.898 742 | 0.911 841 | 0.926 166 | 0.942 099 |
| 10  | 0.836 270 | 0.845 350 | 0.854 842 | 0.864 790 |
| 50  | −0.488 753 | −0.493 446 | −0.498 104 | −0.502 729 |
| 1000| −0.783 454 | −0.788 178 | −0.792 852 | −0.797 475 |
| 5000| −0.989 670 | −0.991 034 | −0.992 302 | −0.993 475 |
| 10000| −0.997 896 | −0.998 492 | −0.998 990 | −0.999 388 |

Table 9  Energy spectrum for the EMOS limit in five dimensions for the higher values of $l$.

| $l$ | $E_{1l}$ | $E_{2l}$ | $E_{3l}$ | $E_{4l}$ |
|-----|---------|---------|---------|---------|
| 0   | none    | none    | none    | none    |
| 1   | 0.961 220 | 0.999 988 | 0.999 688 | 0.999 988 |
| 2   | 0.939 873 | 0.958 831 | none    | 0.998 511 |
| 3   | 0.922 010 | 0.937 770 | 0.955 919 | none    |
| 4   | 0.906 141 | 0.919 970 | 0.935 276 | 0.952 608 |
| 5   | 0.891 630 | 0.904 103 | 0.917 617 | 0.932 450 |
| 10  | 0.830 800 | 0.839 636 | 0.848 854 | 0.858 495 |
| 50  | 0.536 574 | 0.538 845 | 0.541 178 | 0.543 574 |
| 100 | 0.304 430 | 0.304 258 | 0.304 127 | 0.304 036 |
| 500 | −0.489 260 | −0.493 954 | −0.498 614 | −0.503 240 |
| 1000| −0.783 615 | −0.788 339 | −0.793 012 | −0.797 634 |
| 5000| −0.989 672 | −0.991 036 | −0.992 304 | −0.993 477 |
| 10000| −0.997 896 | −0.998 493 | −0.998 990 | −0.999 389 |

4.3 Pure Vector Limit

Table 10  Ground state energy spectra in the pure vector limit.

| $\delta$ (E) | $D = 3$(dim) | $D = 4$(dim) | $D = 5$(dim) |
|--------------|--------------|--------------|--------------|
| $a = 1$ (1/E)| none         | −0.999 738   | −0.978 390   |
| 0.05         | none         | 0.999 988    | 0.993 568    |
| 0.10         | −0.998 950   | −0.948 151   | none         |
| 0.982 143    | 0.979 277    | 0.973 296    | none         |
| $a = 0$ (1/E)| none         | 0.964 314    | 0.957 414    |
| 0.05         | 0.938 819    | 0.915 712    | 0.850 753    |
| 0.10         | none         | none         | none         |
| $a = -1$ (1/E)| none         | 0.973 259    | none         |
| 0.05         | none         | 0.775 983    | 0.635 179    |
| 0.10         | none         | none         | none         |
| 0.998 412    | 0.417 913    | none         |

In this limit, the scalar potential energy term is taken to be zero. Alike the previous limits, we assume that the $V_0 = 2 (E)$. Unlike, we examine two different values of the slope parameter and tabulate it in Table 10. We find that when the energy dependence is lost, only one value of energy appears in the spectrum.

4.4 Pure Scalar Limit

In this limit, the scalar potential energy term is equal to $2(E)$, while the vector potential energy term is zero. Alike the pure vector limit, we examine two different values of the slope parameter. We present the results in Table 11. We find that there is only one energy eigenvalue in pure scalar spectra unlike the vector limit. Moreover, when the potential energy does not depend on energy, ground state energy eigenvalues do not occur.
Table 11  Ground state energy spectra in the pure scalar limit.

| $E_{10}(E)$ | $\delta (E)$ | $D = 3$(dim) | $D = 4$(dim) | $D = 5$(dim) |
|-------------|-------------|-------------|-------------|-------------|
| $a = 1 (1/E)$ | 0.05 | -0.903 123 | -0.9765 612 | none |
| 0.10 | -0.731 329 | -0.916 126 | none |
| $a = 0 (1/E)$ | 0.05 | none | none | none |
| 0.10 | none | none | none |
| $a = -1 (1/E)$ | 0.05 | 0.903 123 | 0.976 512 | none |
| 0.10 | 0.731 329 | 0.916 126 | none |

4.5 Thermodynamic Properties

Therefore, we only employ Table 1, Table 2, and Table 3 to construct the partition function.

First, we use of the energy eigenvalues for $a = 1 (1/E)$, $a = 0 (1/E)$, and $a = -1 (1/E)$ in three dimensions from Table 1. We calculate the partition functions from Eq. (35) and plot them in the first column of Fig. 1. Then, we use the energy spectra in three, four and five dimensions for the $a = 1 (1/E)$ case from Table 1, Table 2, and Table 3. We present the plot of the partition functions in the second column of Fig. 1. We see that the partition functions in three and four dimensions overlap.

We obtain the Helmholtz free energy functions by employing Eq. (36). We demonstrate the three-dimensional results in the first column of Fig. 2. We see that Helmholtz free energy function for energy-dependent function cases has a very close appearance. We put forth the higher dimensional cases results in the second column of Fig. 2. We find out that the overlapping of the thermodynamic functions is still valid.

In this subsection, we use the EMES limit case results to examine the thermodynamic properties of the system.
We derive the entropy function from the Helmholtz free energy via Eq. (37). We show entropy functions versus lower temperature and relatively higher temperature in Fig. 3. The entropy function in three dimensions behaves like the entropy function of five dimensions at low temperatures, while it behaves like the entropy function obtained in four dimensions at relatively high temperatures. Another finding is, in three dimensions at a lower temperature the entropy functions for \( a = 0 \) and \( a = -1 \) (1/\( E \)) case act similar to each other while at a relatively high temperature not.

Then, we use Eq. (38) to compute the internal energy functions. We present internal energy functions in Fig. 4 versus temperature. We conclude that mean energy values are compatible with the results.

![Fig. 4](Color online) Comparison of the internal energy functions versus the temperature in the EMES limit.

![Fig. 5](Color online) Comparison of the specific heat functions versus the temperature in the EMES limit.
Finally, we achieve the specific heat function with the help of Eq. (39). We present them in Fig. 5 versus temperature. We conclude that at a relatively higher temperature in all dimensions the characteristic of the functions for \( a = 1 \) (1/E) case, remains the same. On the other hand, in three dimensions, the specific heat function of \( a = -1 \) (1/E) case, differs from others.

5 Conclusion

In this article, we investigated the bound state solutions of a mixed vector and scalar energy-dependent deformed Hulthén potential in the KG equation in arbitrary dimension. We obtained a transcendental equation, which yields to the quantization of the energy eigenvalues by the use of the necessary boundary conditions. Then, we employed the Newton-Raphson method to calculate energy spectra in the limits of the EMES, EMOS, pure vector and pure scalar. Finally, we used the canonical partition function definition and derived other thermodynamic functions, such as Helmholtz free energy, entropy, internal energy, and specific heat. Then, we discussed thermodynamic properties with energy dependency and dimensional effects.

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