ANOTHER PROOF THAT $\text{MM}^{++}$ IMPLIES WOODIN’S AXIOM (*)

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Abstract. Let $\text{MM}^{++}(\kappa)$ state that the forcing axiom $\text{MM}^{++}$ can be instantiated only for stationary set preserving posets of size at most $\kappa$. We give a detailed account of Asperò and Schindler’s proof that $\text{MM}^{++}(\kappa)+$ there are class many Woodin cardinals implies Woodin’s axiom (*) if $\diamondsuit_\kappa$ holds and $\kappa > \aleph_2$. Our presentation takes advantage of the notion of consistency property: specifically we rephrase Asperò and Schindler’s forcing as a specific instantiation of the notion of “consistency property” used by Makkai, Keisler, Mansfield and others in the study of infinitary logics. We also reorganize the order of presentation of the various parts of the proof. Taken aside these variations, our account is quite close to [1].

I assume throughout this note that the reader is familiar with the key properties of stationary set preserving forcings, knows the definition of $\text{MM}^{++}$ and what are its most important applications, and is familiar with the theory of $\mathbb{P}_{\text{max}}$ (for example as exposed in [3]).

I will try to make this paper as self-contained as possible.

What I will do is follow very closely Asperò and Schindler’s proof, while reorganizing their presentation relating explicitly their forcing machinery to the notion of consistency property introduced by Makkai and others to produce models of theories in infinitary logics.

I give right away a fast account of the proof strategy of Asperò and Schindler’s result, more details and precise definitions will follow later on.

Following the proof pattern of Asperò and Schindler’s result I set up their machinery in a slightly different terminology to design for each $A \in \mathcal{P}(\omega_1) \setminus L(\mathbb{R})$ and $D$ dense subset of $\mathbb{P}_{\text{max}}$ universally Baire in the codes a stationary set preserving forcing $P_{D,A}$ which instantiates the $\Sigma_1$-sentence:

There is a $\text{NS}$-correct iteration of some $(N,I,a) \in D$ which maps $a$ to $A$.

Asperò and Schindler show that everything can be done assuming $V$ models $\text{NS}_{\omega_1}$ is saturated + there are class many Woodin cardinals with $P_{D,A}$ of size $\kappa$ for any $\kappa$ such that:

- $\kappa \geq 2^{(2^{\omega_1})};$
- $\diamondsuit_\kappa$ holds.

Now if $\text{MM}^{++}(\kappa)$ holds, $\text{NS}_{\omega_1}$ is saturated, and $2^{\omega_1} = \aleph_2$. Moreover $\diamondsuit_\kappa$ can be established for $\kappa = \aleph_3$ by a stationary set preserving forcing of size $2^{\aleph_2}$ which does not add new subsets of $\aleph_2$. Putting everything together, we get that $\text{MM}^{++}(2^{\aleph_2})+\text{there are class many Woodin cardinals}$ implies the following

For any $A \in \mathcal{P}(\omega_1) \setminus L(\mathbb{R})$ such that $L[A]$ computes correctly $\omega_1$, and any $D$ dense subset of $\mathbb{P}_{\text{max}}$ universally Baire in the codes there is a $\text{NS}$-correct iteration of some $(N,I,a) \in D$ which maps $a$ to $A$.

This is an even stronger version of (*).

I thank Boban Veličković for suggesting the idea to relate Asperò and Schindler’s forcing to consistency properties and for giving me key hints for the proof of Lemma 2.10 and Fact 2.4. I also thank Ilijas Farah and Ben de Bondt for the many useful comments. I take full responsibility for the errors or gaps remaining.

$\Sigma_1^1$ in a signature admitting predicate symbols for all the universally Baire sets, for the predicate $\text{NS}_{\omega_1}$ and for all properties defined by a $\Delta_0$-formula. We will be more precise later on.
The following is the key consequence of the existence of class many Woodin cardinals we will need:

\[ (H_{\omega_1}^V, \in, B) < (H_{\omega_1}^V[G], \in, B^V[G]) \]

for all \( B \) universally Baire set in \( V \) and all generic extensions \( V[G] \) of \( V \).

This property holds assuming class many Woodin cardinals.

The presentation is structured as follows:

- **Section 1** gives the main definitions and a review of the results in the literature on which one leverages to establish that \( \text{MM}^{++} \) implies (\( * \)). This is just to make this paper completely self-contained. The reader familiar with \( P_{\text{max}} \), \( \text{MM}^{++} \) etc. can skip this part; proper references to the results presented in this section are in any case given at the appropriate stages of this paper.

- **Section 2** states the main result (which is just rephrasing in our set up Schindler’s and Aspero’s main theorem) while introducing the key concept of semantic certificate\(^2\). Roughly a semantic certificate for \( A \in \mathcal{P}(\omega_1) \setminus L(\mathbb{R}) \) and \( D \in L(\mathbb{R}) \) dense subset of \( P_{\text{max}} \) existing in some generic extension \( V[G] \) of \( V \) (possibly collapsing \( \omega_1^V \) to countable) is a tuple \( (J, r, f, T) \) such that \( J \) is an iteration of length \( \omega_1^V \) of some \( P_{\text{max}} \) condition \( (N, I, a) \in D^V[G] \) mapping \( a \) to \( A \) but with no evident obstruction to the fact that \( J \) is \( \text{NS} \)-correct and belongs to a stationary set preserving extension of \( V \) (the other elements \( r, f, T \) play a crucial role which will be outlined at the proper stage).

The main result states that there is a stationary set preserving forcing \( P_{D,A} \) of size \( \kappa \) generically adding a semantic certificate \( (J_G, r_G, f_G, T_G) \) for \( D, A \) such that in any generic extension \( V[G] \) of \( V \) by \( P_{D,A} \), \( J_G \) is really a \( \text{NS} \)-correct iteration in \( V[G] \).

In 2.1 we prove the first approximations to this result, i.e. the existence of semantic certificates\(^3\) in generic extensions collapsing \( \omega_2^V \) to countable (cfr. Lemma 2.10 — corresponding to [1, Lemma 3.2]).

- **Section 3** proves the main result:

  - In Section 3.1 we introduce the notion of forcing given by consistency properties and establish some general facts about these types of forcings. Specifically a consistency property given by a class of structures \( C \) for a signature \( \mathcal{L} \) is the family of finite sets of atomic (or negated atomic) \( \mathcal{L} \)-sentences which are realized by some structure in \( C \). Now this is naturally a forcing notion \( P_C \) ordered by reverse inclusion. When things are properly organized a generic filter for \( P_C \) produces a term model in \( C \).

  - In Section 3.2 we introduce and analyze a specific instance of these forcings: the forcing \( P_\kappa^* \) (which corresponds to the forcing \( P_0 \) in the sequence of forcings \( \{ P_\alpha : \alpha \leq \kappa \} \) of [1]) and show that it gives an elegant way to produce generic term-models whose transitive collapse are semantic certificates (cfr. Fact 3.21 — corresponding to [1, Lemma 3.3]). Unfortunately this forcing cannot be provably stationary set preserving since it has size continuum and Woodin has shown that \( \text{MM}^{++}(\kappa) \) is consistent with the negation of (\( * \)) [7, Thm. 10.90].

  - In Section 3.3 we introduce and analyze the forcings \( \{ P_\alpha : \alpha \in C \cup \{ \kappa \} \} \) indexed by a club subset \( C \) of \( \kappa \) (corresponding to the sequence of forcings \( \{ P_\alpha : \alpha \leq \kappa \} \) of [1]), and we prove the main Theorem 3.33 (corresponding

\(^2\)Our notion of semantic certificate is slightly different from the one appearing in [1], but in spirit it has the same effects.

\(^3\)This corresponds to [1, Lemma 3.2]. Here we also pay attention to explain how to get the tree \( T \) appearing in [1, equ. (8) pag. 16] and its role in their proof.
to [1, Lemma 3.8, Corollary 3.9]) stating that $P_\kappa$ is stationary set preserving and generically adds a $\text{NS}$-correct iteration witnessing that $G_A \cap D$ is non empty\(^4\). The forcings $P_\alpha$ are also consistency properties of the form $P_{\mathcal{W}_\alpha}$, but now $\mathcal{W}_\alpha$ is designed so that $P_{\mathcal{W}_\alpha}$ adds generically countable substructures of $H^V_\kappa$ which can be used to seal $P_\alpha$-names for clubs\(^5\). $\diamondsuit_\kappa$ is then used to show that this sealing process is completed at stage $\kappa$ as all $P_\alpha$-names for clubs are guessed at some stage $\alpha < \kappa$ and sealed by the trace on $P_\alpha$ of the generic filter for $P_\kappa$. This roughly outlines why $P_\kappa$ should be stationary set preserving.

I decided to be overly cautious and to outline where the arguments require the existence of class many Woodin cardinals, and why they cannot go through with weaker assumptions. This might be annoying for some readers, I apologize for that.

Let us remark once more that:

(A) This work is just a re-elaboration of Asperó and Schindler’s result, the key ideas are all appearing in [1]. Here we reformulate them in a slightly different terminology and give an alternative presentation of this proof.

(B) This work wouldn’t exist without Boban Velickovic’s suggestion to relate Asperó and Schindler’s work to the notion of consistency property used in infinitary logic (consistency properties are particularly useful if one wants to produce models of a certain first order theory omitting certain prescribed types). This is somewhat implicit in Asperó and Schindler’s work and we make it explicit here.

It is also convenient before proceeding to introduce the following short-hand notation to denote $\mathcal{L}$-structures for a first order signature $\mathcal{L}$:

**Notation 1.** Given a signature $\tau$, we denote an $\mathcal{L}$-structure $\mathcal{M} = (M, R^M : R \in \tau)$ by $(M, \tau^M)$.

Later on we will need a similar notation for multi-sorted structures (see Section 3.2).

1. **Main result and background material on $\mathbb{P}$-max, universally baire sets and $\text{MM}^{++}$**

Let us first set up the proper language and terminology in order to deal with the $\mathbb{P}$-max technology and to state our main result.

$\text{ZFC}^-$ denotes ZFC minus the powerset axiom.

**Definition 1.1.** A countable transitive set $M$ is an iterable structure if it models $\text{ZFC}^- + \text{there exists an uncountable cardinal}$ and for some transitive $N \supseteq M$:

- $N$ models $\text{ZFC} + \text{NS}_{\omega_1}$ is precipitous,
- $\omega_2^N$ is a countable ordinal in $V$,
- $\text{otp}(N \cap \text{Ord}) \geq \omega_1^V$,
- $H^N_{\omega_2^N} = H^M_{\omega_2}$,
- if $A \in N$ is a maximal antichain of $(\mathcal{P}(\omega_1)/\text{NS})^M$, then $A \in M$.

For example if $N$ is transitive, models $\text{ZFC}$, $\omega_2^N$ is a countable ordinal, and $\text{otp}(N \cap \text{Ord}) \geq \omega_1$, then $M = H^N_{\omega_2^N}$ is an iterable structure if $N$ models $\text{NS}_{\omega_1}$ is saturated; while so is $M = H^N_{\omega_2}$ for any countable (in $V$) cardinal (of $N$) $\kappa > (2^{\aleph_2})^N$ in $N$ if $N$ just models $\text{NS}_{\omega_1}$ is precipitous (see [3, Lemma 1.5] for the ratio of this definition).

**Definition 1.2.** [3, Def. 1.2] Let $M$ be an iterable structure. Let $\gamma$ be an ordinal less than or equal to $\omega_1$. An iteration $J$ of $M$ of length $\gamma$ consists of a family of transitive models as

\[ G_A \text{ is the } P_{\text{max}} \text{-filter given by the conditions } (N, I, a) \text{ which correctly iterate } a \text{ to } A, \text{ more details later.} \]

\[ \text{Roughly these countable structures can be used to witness that a given stationary set of } V \text{ on } \omega_1^V \text{ is met by a given } P_{\text{max}} \text{-name for a club subset of } \omega_1^V. \]
\((M_\alpha : \alpha \leq \gamma)\), sets \(\langle G_\alpha : \alpha < \gamma \rangle\) and a commuting family of elementary embeddings
\[\langle j_{\alpha\beta} : M_\alpha \rightarrow M_\beta : \alpha \leq \beta \leq \gamma \rangle\]

such that:
1. \(M_0 = M\),
2. each \(G_\alpha\) is an \(M_\alpha\)-generic filter for \(\mathcal{P}(\omega_1)/\text{NS}_{\omega_1}M_\alpha\),
3. each \(j_{\alpha\alpha}\) is the identity mapping,
4. each \(j_{\alpha\alpha+1}\) is the ultrapower embedding induced by \(G_\alpha\),
5. for each limit ordinal \(\beta \leq \gamma\), \(M_\beta\) is the direct limit of the system \(\{M_\alpha, j_{\alpha\beta} : \alpha \leq \delta < \beta\}\),

and for each \(\alpha < \beta\), \(j_{\alpha\beta}\) is the induced embedding.

An iteration
\[\mathcal{J} = \langle j_{\alpha\beta} : M_\alpha \rightarrow M_\beta : \alpha \leq \beta \leq \omega_1 \rangle\]

existing in \(V\) is \(\text{NS}\)-correct if
\[M_{\omega_1} \cap \text{NS}^V_{\omega_1} = \text{NS}^{M_{\omega_1}}_{\omega_1}\].

From now we will just write correct in place of \(\text{NS}\)-correct.

We adopt the convention to denote an iteration \(\mathcal{J}\) just by \(\langle j_{\alpha\beta} : \alpha \leq \beta \leq \gamma\rangle\), we also stipulate that if \(X\) denotes the domain of \(j_{0\alpha}\), \(X_\alpha\) or \(j_{0\alpha}(X)\) denotes the domain of \(j_{\alpha\beta}\) for any \(\alpha \leq \beta \leq \gamma\).

**Fact 1.3.** [3, Lemma 1.5, Lemma 1.6] Assume \((M,\varepsilon)\) is iterable. Then for any ordinal \(\gamma \leq \omega_1^V\) any iteration
\[\mathcal{J} = \langle j_{\alpha\beta} : M_\alpha \rightarrow M_\beta : \alpha \leq \beta < \gamma \rangle\]
induced by filters \(\langle G_\alpha : \alpha < \gamma \rangle\) is such that the structure \(M_\gamma\) obtained in accordance with

Def. 1.2 is well-founded hence isomorphic to its transitive collapse.

A trivial fact is that for an iteration
\[\mathcal{J} = \langle j_{\alpha\beta} : N_\alpha \rightarrow N_\beta : \alpha \leq \beta \leq \gamma \rangle\]
and \(\alpha < \beta\), \(N_\alpha\) and \(N_\beta\) are both transitive but in general it holds that \(N_\alpha \subseteq N_\beta\).

Notice that \(M\) is iterable if and only if it is \(0\)-iterable.

**Definition 1.4.** [3, Def. 2.1] \(\mathbb{P}_{\text{max}}\) is the subset of \(H_{\omega_1}\) given by the pairs \((M, a)\) such that

- \(M\) is iterable, countable, and models Martin’s axiom,
- \(a \in \mathcal{P}(\omega_1)^M \setminus L(\mathbb{R})^M\), and there exists \(r \in \mathcal{P}(\omega) \cap M\) such that \(\omega_1^M = \omega_1^{L[a,r]}\).

\((M, a) \leq (N, b)\) if there exists \(\mathcal{J} = \langle j_{\alpha\beta} : \alpha \leq \beta \leq \omega_1^M \rangle\) in \(M\) iteration of \(N\) of length \(\omega_1^M\) such that \(j_{0\omega_1^M}(b) = a\) and \((M, \varepsilon)\) models that \(\mathcal{J}\) is correct.

Note that \(\mathbb{P}_{\text{max}}\) is a definable class in \((H_{\omega_1}, \varepsilon)\); in particular it belongs to any transitive model of \(\text{ZFC}^-\) containing \(\mathcal{P}(\omega)\).

Our definition of \(\mathbb{P}_{\text{max}}\) is slightly different than the one given in [3], but it defines an equivalent of the poset defined in [3, Def. 2.1] in view of the following\(^6\):

**Fact 1.5.** Let \(\mathbb{P}_{\text{max}}^0\) be the forcing \(\mathbb{P}_{\text{max}}\) as defined in [3, Def. 2.1]. Assume there are class many Woodin cardinals.

Then for every condition \((M, I, a)\) in \(\mathbb{P}_{\text{max}}^0\) there is a condition \((N, b)\) in \(\mathbb{P}_{\text{max}}\) and an iteration \(\langle j_{\alpha\beta} : \alpha \leq \beta \leq \omega_1^N \rangle\) in \(N\) of \((M, I)\) according to [3, Def. 1.2] such that
\[j_{0\omega_1^N}(I) = \text{NS}^{N_{\omega_1}}_{\omega_1} \cap j_{0\omega_1^N}[M] \text{ and } j_0(a) = b.\]

\(^6\)I.e. \((M_\gamma, E_\gamma)\) is the direct limit of \(\mathcal{J}\) if \(\gamma\) is limit, or the ultrapower of \((M_\alpha, \varepsilon)\) by \(G_\alpha\) if \(\gamma = \alpha + 1\), see [3, Section 1] for details.

\(^7\)Much weaker large cardinals assumptions are needed, we don’t spell the optimal hypothesis.
Conversely assume $N$ is iterable and $\delta$ is Woodin in $N$. Let $G$ be $N$-generic for some $P \in N$ forcing $\text{NS}_{\omega_1}$ is precipitous and Martin’s axiom. Then $(H^N, \text{NS}_{\omega_1}, a) \in \mathbb{P}_{\text{max}}^0$ for all $a \in \mathcal{P}(\omega_1)^N \setminus L(\mathbb{R})^N$ such that there exists $r \in \mathcal{P}^{\omega_1} \cap N[G]$ with $\omega_1^M = \omega_1^{L[a,r]}$ with $\kappa = (2^{\aleph_1})^+$ in $N[G]$.

**Proof.** Only the first part is non-trivial. Let $\gamma > \delta$ be two Woodin cardinals. Let $X \prec V_\gamma$ be countable with $\delta, (M, I, a) \in X$. Let $N_0$ be the transitive collapse of $X$, and $N$ be a generic extension of $N_0$ by a forcing collapsing $\delta$ to become $\omega_2$ and forcing $\text{NS}_{\omega_1}$ is precipitous and Martin’s axiom. Since $\gamma$ is Woodin, there are class many measurables in $V_\gamma$, hence $N_0$ is iterable and so is $N$ (by [3, Thm. 4.10]).

By [3, Lemma 2.8] there is in $N$ the required iteration $(j_{\alpha\beta} : \alpha \leq \beta \leq \omega_1^N)$ of $(M, I)$ and we can set $b = j_{\omega_1^N}(a)$. \qed

In particular (at the prize of assuming the right large cardinal assumptions) the forcings $\mathbb{P}_{\text{max}}$ and $\mathbb{P}_{\text{max}}^0$ are equivalent.

Let from now on $\text{UB}$ denote the class of universally Baire sets.

**Definition 1.6.** $(\ast)$-UB holds in $V$ if:

- There are class many Woodin cardinals.
- For any $A \in \mathcal{P}(\omega_1) \setminus L(\text{UB})$ such that $\omega_1^A = \omega_1^V$:
  - the set
    
    \[ G_A = \{(M_0, a) \in \mathbb{P}_{\text{max}} : \exists J \text{ correct iteration of } M_0 \text{ such that } j_{\omega_1}(a) = A \text{ and } \text{NS}_{\omega_1}^L(\text{UB}) \cap M_{\omega_1} = \text{NS}_{\omega_1}^L(a) \} \]

  is a filter on $\mathbb{P}_{\text{max}}$;
  - $G_A$ meets all dense subsets of $\mathbb{P}_{\text{max}}$ which are universally Baire in the codes\footnote{D is universally Baire in the codes if there is a universally Baire set $B$ such that $D = \text{Cod}[B]$ (See Def. 1.12). Note that assuming the existence of class many Woodin cardinals any set of reals definable in $L(\mathbb{R})$ is universally Baire.};
  - $\mathcal{P}(\omega_1) \subseteq L(\mathbb{R})[A]$.
- $\text{NS}_{\omega_1}$ is precipitous in $V$.

Note that the above is expressible as a schema of $\Pi_2$-sentences in the signature $\tau_{\text{NS}_{\omega_1}, \text{UB}}$ which admits predicate symbols for all $\Delta_0$-properties, all universally Baire sets, and the non-stationary ideal $\text{NS}_{\omega_1}$, and constant symbols for $\omega_1, \omega, 0$ (see Proposition 1.24).

The rest of this paper will give a proof of the following remarkable result:

**Theorem 1.7** (Asperó, Schindler [1]). Assume there are class many Woodin cardinals and $\text{MM}^{++}(\kappa)$ holds for $\kappa = 2^{\omega_2}$. Then so does $(\ast)$-UB.

Our proof is a variation of Asperó and Schindler’s argument which reorganizes slightly their presentation while following its streamline.

**Strategy of the proof.** We will use the following result:

**Theorem 1.8.** [3, Thm. 5.1, Thm. 6.3, Thm 7.7] Assume in $V$ there are class many Woodin cardinals. Let $G$ be $L(\mathbb{R})$-generic for $\mathbb{P}_{\text{max}}$. Then in $L(\mathbb{R})[G]$ it holds that:

- $\psi_{\text{AC}}$ and Martin’s axiom hold and therefore, $2^\omega = 2^{\omega_1} = \omega_2$.
- $\text{NS}_{\omega_1}$ is saturated.
- For any $A \in \mathcal{P}(\omega_1)^{L(\mathbb{R})[G]} \setminus L(\text{UB})$
  
  \[ G_A = \{(M_0, a) \in \mathbb{P}_{\text{max}} : \exists J \text{ correct iteration of } M_0 \text{ such that } j_{\omega_1}(a) = A \text{ and } \text{NS}_{\omega_1}^{L(\text{UB})} \cap M_{\omega_1} = \text{NS}_{\omega_1}^{M_{\omega_1}} \} \]

  is an $L(\mathbb{R})$-generic filter for $\mathbb{P}_{\text{max}}$ and such that $L(\mathbb{R})[G_A] = L(\mathbb{R})[G]$.

\footnote{See Section 1.3 below for details.}
In particular to establish that $(\ast)$-UB holds in $V$ it suffices to prove that:

1. There are class many Woodin cardinals;
2. $\text{NS}_{\omega_1}^V$ is saturated;
3. For some fixed $A$ in $\mathcal{P}(\omega_1)^V \setminus L(\text{UB})$, $G_A$ is a filter for $\mathbb{P}_{\text{max}}$ meeting all dense subsets of $\mathbb{P}_{\text{max}}$ universally Baire in the codes, and $\mathcal{P}(\omega_1)^V \subseteq \mathcal{P}(\omega_1)^{L(\mathbb{R})[G_A]}$ ($G_A$ will also be $L(\mathbb{R})$-generic since all sets of reals definable in $L(\mathbb{R})$ are universally Baire in the codes assuming class many Woodin cardinals).

Assuming $\text{MM}^{++}(c)$ and the existence of class many Woodin cardinals, the first two conditions are automatically met. The proof of [3, Lemma 7.6] (but not its statement) actually shows that if $V$ models that $\text{NS}_{\omega_1}$ is saturated, then $G_A$ (as computed in $V$) is a filter on $\mathbb{P}_{\text{max}}$ for any $A \in \mathcal{P}(\omega_1)^V \setminus L(\mathbb{R})$ such that $L[A]$ computes correctly $\omega_1$.

Clearly $L(\mathbb{R})[A] = L(\mathbb{R})[G_A]$. Also $\text{MM}^{++}(c)$ entails that $\mathcal{P}(\omega_1)^V \subseteq L(\mathbb{R})[A]$ (since for example it entails $\psi_{\text{AC}}$ [3, Section 6] or Martin’s axiom). So the unique missing point is to check that $G_A$ is a filter meeting all dense subsets of $\mathbb{P}_{\text{max}}$ universally Baire in the codes assuming $\text{MM}^{++}$.

There is a natural strategy:

**Strategy 1.9.** Given $D \subseteq \mathbb{P}_{\text{max}}$ dense and universally Baire in the codes and $A \in \mathcal{P}(\omega_1)^V \setminus L(\text{UB})$, find a forcing $\mathbb{P}_{D,A}$ such that if $G$ is $V$-generic for $\mathbb{P}_{D,A}$ in $V[G]$ there is an iteration

$$
\mathcal{K} = \{k_{\alpha,\beta} : M_\alpha \rightarrow M_\beta : \alpha \leq \beta \leq \omega_1^V\}
$$

such that in $V[G]$:

1. $(M_0, A \cap \omega_1^{M_0}) \in D$,
2. $k_{0,\omega_1^V}(A \cap \omega_1^{M_0}) = A$,
3. $H_{\omega_2}^V \subseteq M_{\omega_1^V}$ and $\text{NS}_{\omega_1^V}^{M_{\omega_1^V}} \cap V = \text{NS}_{\omega_1^V}^V$,
4. $\text{NS}_{\omega_1^V}^{M_{\omega_1^V}} = \text{NS}_{\omega_1^V}^{V[G]} \cap M_{\omega_1^V}$.

Assume the above is possible. Then $\mathbb{P}_{D,A}$ is stationary set preserving since:

$$
\text{NS}_{\omega_1^V}^V = \text{NS}_{\omega_1^V}^{M_{\omega_1^V}} \cap V = \text{NS}_{\omega_1^V}^{V[G]} \cap M_{\omega_1^V} \cap H_{\omega_2}^V = \text{NS}_{\omega_1^V}^{V[G]} \cap H_{\omega_2}^V = \text{NS}_{\omega_1^V}^{V[G]} \cap V.
$$

Now the statement

$$
\exists \mathcal{K} [\mathcal{K} \text{ is a correct iteration}] \wedge ((M_0, a) \in D) \wedge k_{0,\omega_1^V}(a) = A
$$

is a $\Sigma_1$-formula in parameter $A$ in the language $\tau_\text{ST} \cup \{B, \omega_1, \text{NS}_{\omega_1}\}$ where $\tau_\text{ST}$ is a signature admitting predicate symbols for all $\Delta_0$-functions and relations and $B$ is a universally Baire set coding $D$ in some absolute manner.

If $\mathbb{P}_{D,A}$ has size $\kappa \geq c$, and $\text{MM}^{++}(\kappa)$ holds in $V$ this $\Sigma_1$-statement is reflected to $V$.

Let us first address the issues of defining properly:

- What is the meaning of “$\tau_\text{ST}$ is a signature admitting predicate symbols for all $\Delta_0$-functions and relations and $B$ is a universally Baire set coding $D$ in some absolute manner”?
- What is the meaning of “The formula

$$
\exists \mathcal{K} [\mathcal{K} \text{ is a correct iteration}] \wedge ((M_0, a) \in D) \wedge k_{0,\omega_1^V}(a) = A
$$

is a $\Sigma_1$-formula in parameter $A$ in the language $\tau_\text{ST} \cup \{B, \omega_1, \text{NS}_{\omega_1}\}$”.

- Why $\text{MM}^{++}$ holding in $V$ entails that $\Sigma_1$-formulae in parameter $A \in H_{\omega_2}$ for the language $\tau_\text{ST} \cup \{B, \omega_1, \text{NS}_{\omega_1}\}$ holding in a generic extension by a stationary set preserving forcing are reflected to $V$.
1.1. The signature $\tau_{ST}$.

**Notation 1.10.**

- $\tau_{ST}$ is the extension of the first order signature \{\in\} for set theory which is obtained by adjoining predicate symbols $R_\phi$ of arity $n$ for any $\Delta_0$-formula $\phi(x_1, \ldots, x_n)$, and constant symbols for $\omega$ and $\emptyset$.
- $\text{ZFC}^-$ is the $\in$-theory given by the axioms of ZFC minus the power-set axiom.
- $T_{ST}$ is the $\tau_{ST}$-theory by the axioms

$$\forall \vec{x} (R_{\forall x \in \phi}(y, \vec{x}) \leftrightarrow \forall x (x \in y \to R_\phi(y, x, \vec{x}))$$
$$\forall \vec{x} [R_\phi \land \psi(\vec{x}) \leftrightarrow (R_\phi(\vec{x}) \land R_\psi(\vec{x}))]$$
$$\forall \vec{x} [R_\neg \phi(\vec{x}) \leftrightarrow \neg R_\phi(\vec{x})]$$

for all $\Delta_0$-formulae $\phi(\vec{x})$, together with the $\Delta_0$-sentences

$$\forall x \in \emptyset \neg(x = x),$$

$\omega$ is the first infinite ordinal

(the former is an atomic $\tau_{ST}$-sentence, the latter is expressible as the atomic sentence for $\tau_{ST}$ stating that $\omega$ is a non-empty limit ordinal and all its elements are successor ordinals or 0).

- $\text{ZFC}_{ST}$ is the $\tau_{ST}$-theory $\text{ZFC}^- \cup T_{ST}$, accordingly we define $\text{ZFC}_{ST}$.

In $\text{ZFC}_{ST}$ many absolute concepts (such as that of being a function) are now expressed by an atomic formula, while certain more complicated ones (for example those defined by means of transfinite recursion over an absolute property, such as $x$ is the transitive closure of $y$) can still be expressed by means of $\text{ZFC}_{ST}$-provably $\Delta_1$-properties of $\tau_{ST}$ (i.e. properties which are $\text{ZFC}_{ST}$-provably equivalent at the same time to a $\Pi_1$-formula and to a $\Sigma_1$-formula), hence are still absolute between any two models (even non-transitive) $M, N$ of $\text{ZFC}_{ST}$ of which one is a substructure of the other. On the other hand many definable properties have truth values which may vary depending on which model of $\text{ZFC}_{ST}$ we work in (for example $\kappa$ is an uncountable cardinal is a $\Pi_1 \setminus \Sigma_1$-property in $\text{ZFC}_{ST}$ whose truth value may depend on the choice of the model of $\text{ZFC}_{ST}$ to which $\kappa$ belongs).

1.2. Absolute codings of subsets of $H_{\omega_1}$ and Shoenfield’s absoluteness. We define second order number theory as the first order theory of the structure $(\mathcal{P}(\mathbb{N}) \cup \mathbb{N}, \in, \subseteq, =, \mathbb{N})$.

$\Pi_1^1$-sets (respectively $\Sigma_1^1$, $\Delta_1^1$) are the subsets of $\mathcal{P}(\mathbb{N}) \equiv \mathbb{2}^\mathbb{N}$ defined by a $\Pi_1^1$-formula (respectively by a $\Sigma_1^1$-formula, at the same time by a $\Delta_1^1$-formula in the appropriate language), if the formula defining a set $A \subseteq (\mathbb{2}^\mathbb{N})^n$ has some parameter $\vec{r} \in (\mathbb{2}^\mathbb{N})^{<\omega}$ we accordingly write that $A = \Pi_1^1(\vec{r})$ (respectively $\Sigma_1^1(\vec{r})$, $\Delta_1^1(\vec{r})$). If the formula defining a set $A \subseteq (\mathbb{2}^\mathbb{N})^n$ uses an extra-predicate symbol $B \subseteq (\mathbb{2}^\mathbb{N})^k$ we write that $A = \Pi_1^1(B)$ (respectively $\Sigma_1^1(B)$, $\Delta_1^1(B)$).

$A \subseteq (\mathbb{2}^\mathbb{N})^N$ is projective if it is defined by some $\Pi_1^1$-property for some $n$. Similarly we define the notion of being projective in $\vec{r} \in (\mathbb{2}^\mathbb{N})^{<\omega}$ or $B \subseteq (\mathbb{2}^\mathbb{N})^k$.

**Remark 1.11.** $A \subseteq (\mathbb{2}^\mathbb{N})^k$ is projective in some $\vec{r} \in (\mathbb{2}^\mathbb{N})^{<\omega}$ if and only if it is obtained by a Borel subset of $(\mathbb{2}^\mathbb{N})^m$ by successive applications of the operations of projection on one coordinate and complementation.

**Definition 1.12.** Given $a \in H_{\omega_1}$, $\vec{r} \in \mathbb{2}^\mathbb{N}$ codes $a$, if (modulo a recursive bijection of $\mathbb{N}$ with $\mathbb{N}^2$), $\vec{r}$ codes a well-founded extensional relation on $\mathbb{N}$ whose transitive collapse is the transitive closure of $\{a\}$.
• Cod : $2^\mathbb{N} \to H_{\omega_1}$ is the map assigning $a$ to $r$ if and only if $r$ codes $a$ and assigning the emptyset to $r$ otherwise.

• WFE is the set of $r \in 2^\mathbb{N}$ which (modulo a recursive bijection of $\mathbb{N}$ with $\mathbb{N}^2$) are a well founded extensional relation on $\mathbb{N}$ whose transitive collapse is the transitive closure of $\{a\}$.

The following are well known facts\(^{10}\).

**Remark 1.13.** The map Cod is defined by a $\mathsf{ZFC^-}$-provably $\Delta_1$-property (with no parameters) over $H_{\omega_1}$ and is surjective. Moreover WFE is a $\Pi_1$-subset of $2^\mathbb{N}$. Therefore if $N$ is a transitive model of $\mathsf{ZFC^-}$ existing in some transitive model $W$ of $\mathsf{ZFC}$, $N$ computes correctly Cod and WFE, i.e., $\operatorname{Cod}^N = \operatorname{Cod}^W \cap N$ and $\operatorname{WFE}^N = \operatorname{WFE}^W \cap N$.

**Lemma 1.14.** Assume $B \subseteq 2^\omega$. Let $A \subseteq 2^\mathbb{N}$ be a $\Sigma^1_{n+1}(B)$ set. Then $A$ is $\Sigma_n$-definable in the structure $(H_{\omega_1}, \tau^V_{ST}, B)$ in the language $\tau_{ST} \cup \{B\}$. Conversely if $A$ is $\Sigma_n$-definable in the structure $(H_{\omega_1}, \tau^V_{ST}, B)$ in the language $\tau_{ST} \cup \{B\}$, then $\operatorname{Cod}^{-1}[A]$ is a $\Sigma^1_{n+1}(B)$ set.

**Definition 1.15.** Let $M,W$ be transitive models of $\mathsf{ZFC^-}$. The pair $(M,W)$ is a canonical pair if $H^M_{\omega_1} \subseteq H^W_{\omega_1}$ and $\operatorname{otp}(M \cap \mathcal{O}) \geq \omega^W_{\omega_1}$.

Note that $M$ may not be a subset of $W$ (and this case will occur in our proofs).

**Lemma 1.16.** Assume $(M,W)$ is a canonical pair. Then

$$(H^M_{\omega_1}, \tau^M_{ST}) \prec_1 (H^W_{\omega_1}, \tau^W_{ST})$$

**Proof.** Use Shoenfield’s absoluteness Lemma for $\Sigma^1_n$-properties [2, Thm. 25.20] between the transitive models of $\mathsf{ZFC^-}$ $M \subseteq W$ and the observation that $\Sigma_1$-definable subsets of $H_{\omega_1}$ corresponds to $\Sigma^1_n$-properties (see [2, Lemma 25.25]).\(\square\)

1.3. Universally Baire sets.

**Definition 1.17.** Let $V$ be a transitive model of $\mathsf{ZFC}$. $B \subseteq 2^\omega$ is universally Baire in $V$ if there are class sized trees $S_B,T_B$ on $(2 \times \mathcal{O})^{<\omega}$ such that:

• $S_B \cap (2 \times \alpha)^{<\omega}$ and $T_B \cap (2 \times \alpha)^{<\omega}$ are elements of $V$ for any ordinal $\alpha$;

• $B = p[T_B]$ (i.e. $r \in B$ if and only if there is $f \in \mathcal{O}^{\omega}$ such that $(r \upharpoonright n, f \upharpoonright n) \in T_B$ for all $n \in \omega$);

• $p[T_B] \cap p[S_B] = \emptyset$;

• for all $G$ $V$-generic for some forcing notion $P \in V$, $p[T_B]^V[G] \cup p[S_B]^V[G] = (2^\omega)^V[G]$.

Trees $T$ and $S$ satisfying the two latter conditions for all $G$ $V$-generic for a specific forcing notion $P$ are said to be projecting to complements for $P$.

**Notation 1.18.** Given a universally Baire set $B$ and $G$ $V$-generic for some forcing notion $P$ of $V$, we let $B^V[G] = p[T_B]^V[G]$.

**Theorem 1.19.** Assume in $V$ there are class many Woodin cardinals. Let $\mathsf{UB}^V$ be the family of universally Baire sets of $V$ and $\tau_{\mathsf{UB}^V} = \tau_{ST} \cup \mathsf{UB}^V$. Let $G$ be $V$-generic for some forcing notion $P \in V$ and $H$ be $V[G]$-generic for some forcing $Q$ in $V[G]$.

Then

$$(H^V_{\omega_1}, \tau^V_{ST}, A^V[G] : A \in \mathsf{UB}^V) \prec (H^V_{\omega_1}[H], \tau^V_{ST}[G], A^V[G][H] : A \in \mathsf{UB}^V).$$

For a proof of this theorem see for example [5, Thm. 4.7].

In particular assume $B$ is universally Baire and such that

$$(H_{\omega_1}, \tau^V_{ST}, B) \models D = \operatorname{Cod}[B] \text{ is a dense subset of } \mathcal{P}_{\operatorname{max}}.$$
Then 
\[(H_{\omega_1}^{V[G]}, \tau_{ST}^{V[G]}, B^{V[G]}) \models \text{Cod}[B^{V[G]}] \] 
is a dense subset of \(\mathbb{P}_{\max}\)
and \(\text{Cod}[B^{V[G]}] \cap H_{\omega_1}^{V} = D\).

**Definition 1.20.** \(A \subseteq H_{\omega_1}\) is **universally Baire in the codes** if \(A = \text{Cod}[B]\) for some universally Baire set \(B\).

**Notation 1.21.** Given \(A\) a family of universally Baire sets of \(V\), \(\tau_A = \tau_{ST} \cup A\) and \(T_A\) is the theory of the \(\tau_A\)-structure 
\[(H_{\omega_1}^{V}, \tau_{ST}, A : A \in A).\]

1.4. **MM\(^{++}\) and the reflection of \(\Sigma_1\)-sentences in \(\tau_{ST} \cup UB \cup \{\omega_1, NS_{\omega_1}\}\).** We now show that \(MM^{++}\) entails a strong form of BMM which reflects not only \(\Sigma_1\)-properties, but also \(\Sigma_1\)-properties expressed in the first order language expanding \(\tau_{ST}\) with predicate symbols for \(NS_{\omega_1}\) and universally Baire sets. Let

\[\tau_{NS_{\omega_1}, UB} = \tau_{ST} \cup UB \cup \{\omega_1, NS_{\omega_1}\}\].

Given a Woodin cardinal \(\delta\), \(T_\delta\) denotes the full stationary tower of height \(\delta\) (denoted as \(\mathbb{P}_\delta\) in [4]) and for \(K\) V-generic for \(T_\delta\) \(\text{Ult}(V,K)\) denotes the generic ultrapower and \(j_K : V \rightarrow \text{Ult}(V,K)\) the generic ultrapower embedding (see [4, Chapter 2] for the key definitions and results).

By [6, Thm. 2.12] assuming the existence of class many Woodin cardinals \(MM^{++}(\kappa)\) can be formulated as follows:

**Definition 1.22.** \(MM^{++}(\kappa)\) holds if for any (some9 Woodin cardinal \(\delta > \kappa\) and for any stationary set preserving forcing \(P\) of size \(\kappa\), there is a stationary set \(T_P \in V_\delta\) and a complete embedding \(i : P \rightarrow T_\delta \upharpoonright T_P\) such that whenever \(G\) is V-generic for \(P:\)

- The quotient forcing \(T_\delta \upharpoonright T_P / i[G]\) is stationary set preserving in \(V[G]\),
- Letting \(H\) be \(V[G]\)-generic for \(T_\delta \upharpoonright T_P / i[G]\) and \(K\) be the V-generic filter for \(T_\delta \upharpoonright T_P\) induced by the forcing isomorphism of \(T_\delta \upharpoonright T_P\) with \(P * (T_\delta \upharpoonright T_P / i[G])\), we have that the critical point of \(j_K : V \rightarrow \text{Ult}(V,K)\) is \(\omega_2\), \(j_K(\omega_2) < \delta = j_K(\delta)\) and \(\text{Ult}(V,K)^{< \delta} \subseteq \text{Ult}(V,K)\) holds in \(V[K]\).

**Theorem 1.23.** Assume \(MM^{++}(\kappa)\) holds in \(V\) and there are class many Woodin cardinals.

Then for any stationary set preserving forcing \(P \in V\) of size \(\kappa\) and any G V-generic for \(P\)
\[(H_{\omega_2}^{V}, \tau_{ST}^{V}, NS_{\omega_1}^{V}, UB^{V}) \prec (H_{\omega_2}^{V[G]}, \tau_{ST}^{V[G]}, NS_{\omega_1}^{V[G]}, A^{V[G]} : A \in UB^{V}).\]

We sketch a proof:

**Proof.** Assume \(\phi(x, y)\) is a quantifier free formula in the signature \(\tau_{NS_{\omega_1}, UB}\). Let \(B_1, \ldots, B_k \in UB^{V}\) be the universally Baire sets appearing in the formula \(\phi(x, y)\). Fix \(A \in H_{\omega_2}^{V}\), and assume that \(P\) is stationary set preserving of size \(\kappa\) and forces \(\exists y \phi(A, y)\).

Let \(K\) be V-generic for \(T_\delta \upharpoonright T_P\) and \(G, H \in V[K]\) be such that \(G\) is V-generic for \(P\) and \(H\) is \(V[G]\)-generic for \(T_\delta \upharpoonright T_P / i[G]\) and \(V[K] = V[G][H]\).

Now:

- \(H_{\omega_2}^{V[G][H]} = H_{\omega_2}^{\text{Ult}(V,K)}\) and \(j_K(\text{NS}_{\omega_1}^{V[G]} = \text{NS}_{\omega_1}^{\text{Ult}(V,K)} = \text{NS}_{\omega_1}^{V[G][H]}\): since \(V[G][H] = V[K]\), \(j(\omega_2) < \delta\) and \(V[K]\) models that \(\text{Ult}(V,K)^{< \delta} \subseteq \text{Ult}(V,K)\).
- \(j_K(B) = B^{V[G][H]}\) for all universally Baire sets \(B \in V\): assume \(S_B, T_B \in V\) are trees on \((2 \times \text{Ord})^{< \omega}\) projecting to complements in any forcing extension with \(B = p[T_B]\). Then \(j_K(S_B), j_K(T_B)\) in \(\text{Ult}(V,K)\) are trees projecting to complements containing \(j_K[S_B], j_K[T_B]\): therefore if \(p\) is the projection map of \((2 \times \text{Ord})^{< \omega}\) on \(2^{\omega}\),
\[B^{V[K]} = p[T_B] = p[j_K[T_B]] = p[j_K(T)] \subseteq j_K(B),\]
This gives that the identity map defines an elementary embedding of the \(\tau_{NS_{\omega_1}, UB^V}\)-structure

\[
(H^V_{\omega_2}, \tau^V_{\omega_1}, NS_{\omega_1}, UB^V)
\]

into the \(\tau_{NS_{\omega_1}, UB^V}\)-structure

\[
(H_{\omega_2}^{|G[H]|}, \tau_{\omega_1}^{|G[H]|}, NS_{\omega_1}^{|G[H]|}, B^{|G[H]|} : B \in UB^V).
\]

Since \(H\) is \(V[G]\)-generic for a stationary set preserving forcing in \(V[G]\) and \(B^{|G[H]|} \cap V[G] = B^{|G|}\) for all universally Baire sets in \(V\), we get that

\[
(H_{\omega_2}^{|G[H]|}, \tau_{\omega_1}^{|G[H]|}, NS_{\omega_1}^{|G[H]|}, B^{|G[H]|} : B \in UB^V)
\]

is a substructure of

\[
(H_{\omega_2}^{|G[H]|}, \tau_{\omega_1}^{|G[H]|}, NS_{\omega_1}^{|G[H]|}, B^{|G[H]|} : B \in UB^V).
\]

By choice of \(P\)

\[
(H_{\omega_2}^{|G[H]|}, \tau_{\omega_1}^{|G[H]|}, NS_{\omega_1}^{|G[H]|}, B^{|G[H]|} : B \in UB^V) \models \exists y \phi(A, y).
\]

Therefore since \(\Sigma_1\)-properties are preserved by superstructures

\[
(H_{\omega_2}^{|G[H]|}, \tau_{\omega_1}^{|G[H]|}, NS_{\omega_1}^{|G[H]|}, B^{|G[H]|} : B \in UB^V) \models \exists y \phi(A, y).
\]

We conclude that

\[
(H^V_{\omega_2}, \tau^V_{\omega_1}, NS_{\omega_1}, UB^V) \models \exists y \phi(A, y).
\]

The proof is completed. \(\square\)

1.5. \(G_A \cap D\) is a \(\Sigma_1\)-property in the signature \(\tau_{ST} \cup UB \cup \{\omega_1, NS_{\omega_1}\}\). The statement

\[
x = \{j_{\alpha\beta} : M_\alpha \to M_\beta : \alpha \leq \beta \leq \omega_1\}
\]

is an atomic formula in free variable \(x\) for the signature \(\tau_{ST} \cup \{\omega_1\}\) given by the conjunctions of:

1. \(x\) is a function,
2. \(\text{dom } x = \omega_1\),
3. \(\forall \alpha \in \omega_1 \ x(\alpha)\) is an ordered pair \((M_\alpha, G_\alpha)\) such that:
   - \(M_\alpha\) is a transitive set,
   - \((M_\alpha, \in)\) models ZFC + \(NS_{\omega_1}\) is precipitous,
   - \(G_\alpha\) is \(M_\alpha\)-generic for \(P(\omega_1)^{M_\alpha}/NS_{\omega_1}^{M_\alpha}\),
   - \(M_{\alpha+1} = \text{Ult}(M_\alpha, G_\alpha)\) where \(\text{Ult}(M_\alpha, G_\alpha)\) is the generic ultrapower induced by \(G_\alpha\) on \(M_\alpha\),
   - if \(\alpha\) is a limit ordinal, \(M_\alpha\) is the direct limit of the iteration \(x \restriction \alpha\).

We leave to the reader to check that the above properties are formalizable by \(\Delta_0\)-formulae. Therefore

\[
x = \{j_{\alpha\beta} : M_\alpha \to M_\beta : \alpha \leq \beta \leq \omega_1\}
\]

is the \(\tau_{ST} \cup \{\omega_1, NS_{\omega_1}\}\)-atomic formula given by the conjunctions of:

- \(x\) is an iteration of length \(\omega_1\)
- \(\forall z [z \subseteq \omega_1 \land z \in M_\omega \to (M_\omega \models z \text{ is stationary} \leftrightarrow \neg NS_{\omega_1}(z))]\).

Given \(A \in P(\omega_1) \setminus L(UB)\) and \(D\) dense subset of \(P_{\text{max}}\) universally Baire in the codes with \(D = \text{Cod}[B]\) and \(B \in UB\), \(G_A \cap D\) is non-empty is given by the \(\tau_{ST} \cup \{A, \omega_1, NS_{\omega_1}, B\}\) \(\Sigma_1\)-formula:

\[
\exists x \exists r [(x\text{ is a correct iteration of length } \omega_1) \land B(r) \land \text{Cod}(r) = (M_0, A \cap \omega_1^{M_0}) \cup j_{0\omega_1}(A \cap \omega_1^{M_0}) = A].
\]
**Proposition 1.24.** In any model of ZFC+there are class many Woodin cardinals+NSω1 is precipitous (⋆)-UB is expressible by an axiom schema of Π2-sentences for the signature τNSω1,UB.

This axiom system is given by the Π2-axioms of ZFC̸ST enriched with the Π1-sentence for τ̸ST U {ω1}

\[ ∀f \ [(f \text{ is a function}) \land (\text{dom}(f) = ω)] \rightarrow (\text{ran}(f) \neq ω_1), \]

the Π2-sentence for τ̸ST U {ω1, NSω1}

\[ ∀S \ [\text{NS}_ω(1) \implies \exists C \ [(\text{C is a club subset of } ω_1) \land (C \cap S = \emptyset)] \]

and (for any universally Baire set B such that Cod[B] = D is a dense subset of Pmax) the Π2-sentences for τ̸ST U {ω1, NSω1, B}

\[ ∀A \]

\[ [(ω_1 \text{ is the first uncountable cardinal}]^{L[A]} \land ∀r∀β [(r \subseteq ω \land β \text{ is an ordinal}) \rightarrow A \notin L_β[r])] \]

→

\[ \exists(N_0, a_0) \exists J \exists r(B(r) \land \text{Cod}(r) = (N_0, a_0) \land J \text{ is a correct iteration of } N_0 \text{ with } j_{ω_1}(a_0) = A) \]

\]

Proof. The unique thing to check is that NSω1 is saturated follows from the above axioms, but this holds by [3, Thm. 5.1].

\]

2. MAIN RESULT AND SEMANTIC CERTIFICATES

The key concept introduced by Asperò and Schindler is that of a semantic certificate.

Given a dense set D of Pmax universally Baire in the codes and \( A \in \mathcal{P}(ω^V_1) \setminus L(\mathbb{R}^V) \) it is not at all transparent how one can produce (even in an arbitrary forcing extension \( V[G] \) of \( V \) possibly not stationary set preserving) an iteration

\[ J = \{j_{αβ} : N_α \rightarrow N_β : α ≤ β \leq ω^V_1\} \]

respecting the constraints \( (N_0, A \cap ω^N_1) \in D^V[G] \) and \( j_{0ω_1}(A \cap ω^N_1) = A \). Semantic certificates (to be defined below) impose even stronger constraints on \( J \).

**Convention 2.1.** Throughout the rest of this paper we assume NSω1 is saturated, Martin’s axiom, and \( 2^{κ_0} = 2^{κ_1} = κ_2 \) holds in \( V \).

Note that all these assumptions hold assuming MM++ and are also a consequence of (⋆)-UB.

To introduce the notion of semantic certificate we must first select the class of transitive models of ZFC and the type of iterations of countable structures we will consider in the remainder of this paper, these are the canonical pairs introduced in Def. 1.15.

**Remark 2.2.** We will be interested in the following three combinations of canonical pairs (cfr. Def. 1.15) \( M, W \):

- \( W = M[G] \) is a generic extension of \( M \).
- \( M \) is the standard universe of sets \( V \) and \( W = M \), where \( j : V \rightarrow M \) is an elementary embedding definable in \( V[G] \) with critical point \( ω^V_1 \) for \( V[G] \) a generic extension of \( V \).
- \( W = V[G][H] \) is a generic extension of \( V \) with \( G \) \( V \)-generic for Coll(ω, ω^V_1), \( H \) \( V[G] \)-generic for Coll(ω, η) with η large enough, \( j : V \rightarrow M \) an elementary embedding definable in \( V[G] \) with critical point \( ω^V_1 \) and \( ω^M_1 = ω^V_1 \), and \( M = M[H] \).

Note that in the second case we can just assume that \( H_ω^V \subseteq M \), in general the transitive classes \( M, V \) definable in \( V[G] \) do not have much else in common.
Definition 2.3. Let:

- $V$ be a transitive model of $\text{ZFC + NS}_{\omega_1}$ is saturated,
- $B$ be a universally Baire set in $V$ and $D$ in $V$ be a dense subset of $\mathbb{P}_{\text{max}}$ such that $\text{Cod}(B) = D$,
- $A \in \mathcal{P}(\omega_1)^V \setminus L(\mathbb{R})^V$ be such that $\omega_1^{L[A]} = \omega_1^V$.

Let $T$ be a witnessing tree for $A, D$ if:

- $T$ is a tree on $L_\omega(2, \omega_2^V)$;
- $p[T] \subseteq B^{V[G]}$ holds in all generic extensions $V[G]$ of $V$;
- whenever $G$ is $V$-generic for $\text{Coll}(\omega, \omega_1^V)$ for some $\eta \geq \lambda$, there is $r \in p[T]^{V[G]}$ such that $\text{Cod}(r) = (N_0, a_0)$ refines $(H_{\omega_2^V}, A)$ in the $\mathbb{P}_{\text{max}}$ order of $V[G]$.

The following Fact encapsulates the unique place in the proof of Thm. 1.7 where the assumption that there are class many Woodin cardinals is essentially used and cannot be significantly reduced.

Fact 2.4. Assume $V$ models $\text{NS}_{\omega_1}$ is saturated + there are class many Woodin cardinals.

Then there is in $V$ a witnessing tree $T$ on $(2 \times \omega_2^V)^{<\omega}$ for $A, D$ for any $D$ dense subset of $\mathbb{P}_{\text{max}}$ with $D = \text{Cod}(B)$ for some $B \in \text{UB}^V$, and $A \in \mathcal{P}(\omega_1)^V \setminus L(\mathbb{R})$ such that $\omega_1^{L[A]} = \omega_1^V$.

Proof. Let $G$ be $V$-generic for $\text{Coll}(\omega, \omega_1^V)$. Then $(H_{\omega_2^V}, A)$ is a $\mathbb{P}_{\text{max}}$-condition in $V[G]$ (any iteration of $X$ in $V[G]$ of length at most $\omega_1^{V[G]}$ can be extended to an iteration of $V$, by [3, Lemma 1.5, Lemma 1.6]).

Find $(N, b) \leq (H_{\omega_2^V}, A)$ in $\text{Cod}(B^{V[G]})$ (which is dense in $\mathbb{P}_{\text{max}}^{V[G]}$ by Thm. 1.19), $r \in B^{V[G]}$ such that $\text{Cod}(r) = (N, b)$, $h \in \lambda^\omega$ such that $(r, h)$ is a branch through $T_B$ in $V[G]$ (where $T_B, S_B$ are class sized trees on $(2 \times \text{Ord})^{<\omega}$ witnessing the universal Baireness of $B$ in $V$ according to Def. 1.17).

Let $\dot{r}$ and $\dot{h}$ be $\text{Coll}(\omega, \omega_1^V)$-names$^{11}$ for $r$ and $h$ such that every $p$ in $\text{Coll}(\omega, \omega_1^V)$ forces that $(\dot{r}_G, \dot{h}_G)$ is a branch through $T_B$ such that $\text{Cod}(\dot{r}_G) \leq_{\mathbb{P}_{\text{max}}} (H_{\omega_2^V}, A)$ holds in $V[G]$ whenever $G$ is $V$-generic with $p \in G$.

Let for each $n$

$$Y_n = \left\{ \beta \in \text{Ord} : \exists p \in \text{Coll}(\omega, \omega_1^V) p \vdash \dot{h}(<\beta) = \beta \right\}$$

and $Z = \bigcup_{n \in \omega} Y_n$. Then $Z \in V$ has size at most $\omega_1^V$ and $p$ belongs to the projection of $T^* = T_B \cap (2 \times Z)^{<\omega}$ in $V[G]$. We copy in $V$ $T^*$ to a tree $T$ on $(2 \times \omega_2^V)^{<\omega}$ via an injection $g : Z \to \lambda$ in $V$, and we let $\dot{f}$ be the $\text{Coll}(\omega, \omega_1^V)$-name on $T$ obtained as an isomorphic image of $\dot{h}$ via $g$.

We get that:

- $\dot{r}, \dot{f}$ are $\text{Coll}(\omega, \omega_1^V)$-names such that whenever $H$ is $V$-generic for $\text{Coll}(\omega, \omega_1^V)$,
- $(\dot{r}_H, \dot{f}_H)$ is a branch through $T$ such that $\text{Cod}((\dot{r}_H) = \text{Cod}(r) \leq_{\mathbb{P}_{\text{max}}} (H_{\omega_2^V}, A)$ is a $\mathbb{P}_{\text{max}}$-condition refining $(H_{\omega_2^V}, A)$,

The map $g$ grants that $p[T]^{V[K]} \subseteq p[T_B]^{V[K]} = B^{V[K]}$ holds in any forcing extension $V[K]$ of $V$ for any notion of forcing in $V$.

Hence $T$ works. \qed

Remark 2.5. We use here in an essential way the existence of class many Woodin cardinals (specifically Thm. 1.19). We could also replace this assumption with the existence of a

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$^{11}$To prove the existence of such names independently of the choice of the $V$-generic filter $G$ one uses the homogeneity of $\text{Coll}(\omega, \omega_1^V)$. 
Woodin cardinal \( \delta \) which is a limit of Woodin cardinals (and then use Thm. [4, Thm 3.1.6]). What is crucially needed to prove the Fact is that the \( \Pi_2 \)-sentence for \( \tau_{ST} \)

\[
\forall x \ [x \notin P_{\text{max}} \lor \exists y \ (B(y) \land \text{Cod}(y) \leq P_{\text{max}} x)]
\]

holding in the structure

\((H^V_{\omega_1}, \tau_{ST}, B)\)

remains true in the structure

\((H^V_{\omega_1}[G], \tau_{ST}, B^V[G])\).

To see that the above is a \( \Pi_2 \)-sentence for \( \tau_{ST} \) use [3, Remark 1.4] and [2, Lemma 25.25].

**Definition 2.6.** Let \( M \) be a transitive model of \( \text{ZFC} + \text{NS}_{\omega_1} \) is saturated:

- \( T \in M \) be a tree on \( (2 \times \omega_2^M)^{\omega} \);
- \( A \in \mathcal{P}(\omega_1)^M \setminus L(\mathbb{R})^M \) be such that \( \omega_1^L[A] = \omega_1^M \).

A tuple \((J, r, f, T)\) existing in some transitive outer model \( W \) of \( M \) is a weak semantic certificate for \( A \) if:

1. \((r, f)\) is a branch through \( T \).
2. \( J = \{ j_{\alpha \beta} : N_{\alpha} \to N_{\beta} : \alpha \leq \beta \leq \omega_1^M \} \) is an iteration with \( N_0 \) iterable in \( W \).
3. \( \text{Cod}(r) = (N_0, A \cap \omega_1^{N_0}) \) holds.
4. \( j_{\omega_1 M}(A \cap \omega_1^{N_0}) = A \).
5. \( H_{\omega_1}^M \subseteq N_{\omega_1} \).
6. \( \text{NS}^M_{\omega_1} = \text{NS}^{N_{\omega_1}}_{\omega_1} \cap \mathcal{P}(\omega_1)^M \).

Let \( D \) in \( M \) be a dense subset of \( P_{\text{max}}^M \) such that \( D = \text{Cod}[B] \) for some fixed \( B \in UB^M \).

A weak semantic certificate \((J, r, f, T)\) for \( A \) existing in \( W \) is a semantic certificate for \( A, D \) (relative to \( M \)) if:

7. \( M \) models that \( T \) is a witnessing tree for \( A, D \).
8. \( W \) is some generic extension of \( M \).

Condition 5 and 6 are essential in many stages of the proof, for example we already use them in the proof of Corollary 2.8 right below. Condition 7 hides all the large cardinal strength needed in our proofs. Condition 8 makes the property:

*There exists a semantic certificate for \( A, D \)*

first order expressible without any reference to \( D \) or to the universally Baire set \( B \) which codes \( D \): the relevant information on \( B \) and \( D \) is now encoded in the witnessing tree \( T \) (whose projection may not be defining a universally Baire set of \( M \), but it is still defining a subset of \( B^{M[G]} \) in any generic extension of \( M \)). This is going to be very handy, see Remark 2.9 below.

There is no reason to expect that the statement *there exists a weak semantic certificate for \( A \) in some transitive outer model of \( \text{any} \) reasonable way in \( V \), since the class of transitive outer models of \( V \) is not first order definable.*

The main result of the paper is the following:

**Theorem 2.7.** Assume \( V \) models \( \text{NS}_{\omega_1} \) is saturated and there are class many Woodin cardinals. Let \( \kappa > \omega_2 \) be a regular cardinal such that \( \diamondsuit_{\kappa} \) holds.

Let \( A \in \mathcal{P}(\omega_1) \setminus L(\mathbb{R}) \) and \( D \) dense subset of \( P_{\text{max}} \) be such that:

- \( D = \text{Cod}[B] \) for some universally Baire set \( B \),
- \( \omega_1^L[A] = \omega_1^V \).

Then there is a stationary set preserving forcing \( P_{D,A} \) in \( V \) of size \( \kappa \) producing in its generic extensions \( V[H] \) a semantic certificate \((J_H, r_H, f_H, T)\) for \( A, D \) such that

\[ \text{NS}^V[H] \cap N_{\omega_1^V} = \text{NS}^{N_{\omega_1}}_{\omega_1}, \]
where
\[ \mathcal{J}_H = \{ j_{\alpha\beta} : N_\alpha \to N_\beta : \alpha \leq \beta \leq \omega_1^V \}. \]

**Corollary 2.8.** Assume \( \text{MM}^{++} \) there are class many Woodin cardinals. Then \((\ast)\)-UB holds.

The corollary is immediate:

**Proof.** By \( \text{MM}^{++} \), \( \text{NS}_{\omega_1} \) is saturated. Find \( \kappa > \omega_2 \) such that \( \mathcal{M}_\kappa \) holds in \( V \). Fix \( A \in \mathcal{P}(\omega_1)^V \) as prescribed by the theorem, and let \( D \) vary among the universally Baire in the codes dense subsets of \( \mathcal{P}_{\text{max}} \). If \( H \) is \( V \)-generic for \( P_{D,A} \), then in \( V[H] \) there is
\[ \mathcal{J}_H = \{ j_{\alpha\beta} : N_\alpha \to N_\beta : \alpha \leq \beta \leq \omega_1 \} \]
iteration such that \( (N_0, A \cap \omega_1^{N_0}) \in D \) and \( j_{0\omega_1}(A \cap \omega_1^{N_0}) = A \) (cfr. Condition 4 of Def. 2.6). Now
\[ H_{\kappa_4}^V \subseteq N_{\omega_1^V}, \]
and \( \text{NS}_{\omega_1}^V = \text{NS}_{\omega_1}^{N_{\omega_1}} \cap H_{\omega_2} \) (by Conditions 5 and 6 of a semantic certificate). Therefore in \( V[H] \) every stationary subset of \( \omega_1 \) in \( V \) is stationary in \( N_{\omega_1^V} \), hence also in \( V[H] \) (by Thm. 3.33). We conclude that \( P_\kappa \) is stationary set preserving.

Since \( \text{NS}_{\omega_1}^{N_{\omega_1}} = \text{NS}_{\omega_1}^{V[H]} \cap N_{\omega_1^V} \), \( \mathcal{J}_H \) is a correct iteration in \( V[H] \) with
\[ (N_0, A \cap \omega_1^{N_0}) = \text{Cod}(r_H), \quad r_H \in p[T] \subseteq p[T_B] = B^{V[H]}. \]

By the results of Section 1.4, the existence of a correct iteration \( \mathcal{J} \) such that \( j_{0\omega_1}(A \cap \omega_1^{N_0}) = A \) and \( (N_0, A \cap \omega_1^{N_0}) = \text{Cod}(r) \) with \( r \in B^{V[H]} \) can be reflected to \( V \), by \( \text{MM}^{++} \). This concludes the proof. \( \square \)

The rest of the paper is dedicated to the proof of Thm. 2.7.

First of all let us outline some key observations on semantic certificates.

**Remark 2.9.** Assume \((M, W)\) is a canonical pair and \( H_{\omega_2}^M \) is countable in \( W \). The statement
\( (\mathcal{J}, r, f, T) \) is a weak semantic certificate for \( A \).

is first order expressible in the structure \((H_{\omega_1}^W, \tau_2^W)\) by the sentence \( \phi(A, T, H_{\omega_2}^M, \omega_1^M) \) in parameters (the countable in \( W \) sets) \( A, T, H_{\omega_2}^M, \omega_1^M \):

\( (1) \)
\[ ((r, f) \text{ is a branch of } T) \land \]
\[ \land \mathcal{J} = \{ j_{\alpha\beta} : N_\alpha \to N_\beta : \alpha \leq \beta \leq \omega_1^M \} \text{ is an iteration} \land \]
\[ \land j_{0\omega_1}(A \cap \omega_1^{N_0}) = A \land \]
\[ \land H_{\omega_2}^M \subseteq N_{\omega_1^M} \land \]
\[ \land \text{NS}_{\omega_1}^M = \text{NS}_{\omega_1}^{N_{\omega_1}^M} \cap H_{\omega_2}^M \land \]
\[ \land \text{Cod}(r) = (N_0, A \cap \omega_1^{N_0}) \]

The first five properties listed above of \( \mathcal{J}, r, f, T \) are expressed by \( \Delta_0 \)-formulae in the relevant parameters (for example we already showed that this is the case for \( \mathcal{J} \) is an iteration of length \( \omega_1^M \)). The last assertion is a provably \( \Delta_1 \)-property in the signature \( \tau_2^W \).

The unique delicate part (hiding significant large cardinal strength) for a semantic certificate is the request \( p[T] \subseteq B^{V[G]} \) holds in all generic extensions of \( V \). This is expressible by the \( \Pi_1^1(B, T) \)-property
\[ \forall r \in 2^\omega \forall f \in \kappa^\omega \left[ (\forall n \in \omega(r \upharpoonright n, f \upharpoonright n) \in T) \implies B(r) \right]. \]
In principle $\Pi^1_1(B,T)$-properties are not universally Baire, even if $B$ is universally Baire and $T$ is countable.

Now if $W$ is any transitive outer model of $M$, it is not at all clear whether we can make sense of $B^W$ for a universally Baire set $B$ of $M$. The notion of witnessing tree $T$ show that we do not have to worry about this issue, and we can just check whether in this outer model weak semantic certificates $(J, r, f, T)$ for $A$ with certain properties (depending only on the parameters $A, H^{M, T}$) exist in $W$.

Most of our arguments will run as follows:

- We fix in advance in $V$ a witnessing tree $T$ for $A, D$.
- We define in some generic extension $V[G]$ of $V$ a generic $\tilde{k} : V \to \bar{M}$ with $\omega_1^{\bar{M}} = \omega^{V[G]}_1$.
- We consider the canonical pair $(\bar{M}, V[G])$ and we also produce in $V[G]$ a weak semantic certificate $(J, r, f, \tilde{k}(T))$ for $\tilde{k}(A)$ with certain extra properties (independent of the universally Baire set $B$) which can be expressed as $\Sigma^1_2(\tilde{k}(T), \tilde{k}(A), \tilde{k}(H^{V[G]})$-facts in $\bar{M}[H]$ whenever $H$ is a $\bar{M}$-generic filter with $\tilde{k}(A), \tilde{k}(H^{\omega_2}), \tilde{k}(T)$ countable in $\bar{M}[H]$.
- Since these properties of $(J, r, f, \tilde{k}(T))$ (holding in $V[G]$) still hold in $H^{V[H]}_\omega$ for any $V$-generic extension $V[H]$ in which $\tilde{k}(A), \tilde{k}(H^{\omega_2}), \tilde{k}(T)$ are countable, we reflect them to $(H^{ST}_\omega, T^{ST}_H)$ using Shoenfield’s absoluteness between $(H^{ST}_\omega, T^{ST}_H)$ and $(H^{V[H]}_\omega, T^{V[H]}_H)$.
- Finally (since $\tilde{k}(T)$ is in $\bar{M}$ a witnessing tree for $\tilde{k}(A), \tilde{k}(D)$), we use elementarity of $\tilde{k}$ to reflect these properties to $V$ in the form:

  Semantic certificates for $A, D$ with the required properties exist in any generic extension of $V$ by $\text{Coll}(\omega, \omega_2^V)$.

The first example of this modular argument is given by Lemma 2.10.

2.1. Existence of semantic certificates. The first key result of Asperó and Schindler is that semantic certificates for $A, D$ exist in some generic extension of $V$ (the proof below produces them in a generic extension collapsing $\omega_2$ to countable).

Lemma 2.10. Assume in $V$ $\text{NS}_{\omega_1}$ is saturated and Martin’s axiom, $X$ is a transitive model of $\text{ZFC}^-$ and $T$ is a witnessing tree on $(2 \times \omega_2)^{<\omega}$ for $A, D$ as given by Fact 2.4, with $A \in \mathcal{P}(\omega_1) \setminus L(\mathbb{R})$ such that $\omega_{1}^{L[A]} = \omega_1^{V}$ and $D = \text{Cod}[B]$ dense subset of $\mathcal{P}_{\text{max}}$ with $B$ universally Baire.

Then for any $G$ $V$-generic for $\text{Coll}(\omega, \omega_2^V)$, there is in $V[G]$ a semantic certificate $(J, r, f, T)$ for $A, D$.

Proof. Let $G$ be $V$-generic for $\text{Coll}(\omega, \omega_2^V)$. In $V[G]$ $H^{V}_{\omega_2}$ is countable and iterable. This gives that $(H^{V}_{\omega_2}, A)$ is a $\mathbb{P}^{V[G]}_{\text{max}}$ condition.

Since $T$ is a witnessing tree for $A, D$, there is $(N, b) = \text{Cod}(\dot{r}_G) \leq (H^{V}_{\omega_2}, A)$ with $r = \dot{r}_G \in p[T]$.

Since $N$ is countable and iterable in $V[G]$ and $\gamma = (\omega_3)^V = \omega_1^{V[G]}$, we can find in $V[G]$ an iteration

$\mathcal{J} = \left\{ j_{\alpha\beta} : N_\alpha \to N_\beta : \alpha \leq \beta \leq \gamma = \omega_1^{V[G]} \right\}$

of $N = N_0$ (just apply [3, Lemma 2.8] in $V[G]$ to $N$).

Let $\mathcal{K} = \left\{ k_{\alpha\beta} : \alpha \leq \beta \leq \omega_1^V \right\}$ be the iteration witnessing that $(N, b) \leq (H^{V}_{\omega_2}, A)$ and

$\bar{\mathcal{K}} = \left\{ \bar{k}_{\alpha\beta} : \alpha \leq \beta \leq \gamma \right\} = j_{0^\gamma}(\mathcal{K})$.

Extend in $V[G]$ $\bar{\mathcal{K}}$ to a full iteration of $V$ applying [3, Lemma 1.5, Lemma 1.6], and let $M = \bar{k}(V)$ with $\bar{k}$ the unique extension of $\bar{k}_{0^\gamma}$ to $V$. 

Let $H$ be $V$-generic for $\text{Coll}(\omega, \eta)$ with $\eta = \bar{k}(\omega^V_2)$ and $G \in V[H]$. Now observe that:

- $H$ is $M$-generic for $\text{Coll}(\omega, \eta)$, since $M \subseteq V[G]$ and $H$ is also $V[G]$-generic for $\text{Coll}(\omega, \eta)$.
- $\bar{k}(T) = \text{tree in } (2 \times \bar{k}(\omega^V_2))^{<\omega}$ in $\bar{M}$ which is countable in $M[H]$ and in $V[H]$, and $r \in p[\bar{k}(T)]_{V[G]}$.
- $\mathcal{J}, \bar{k}(H^V_{\omega_1})$ are in $H^V_{\omega_1}$, and $H^\bar{M}_V$ models that $\mathcal{J}$ is an iteration such that $(N_0, b) = \text{Cod}(r)$, $r \in p[\bar{k}(T)]$, and $j_{0T}(b) = j_{0T}(\bar{k}(\omega^N_1)(A)) = \bar{k}(A)$.

Consider the $\Sigma_1$-sentences in signature $\tau_{ST}$ and parameters $\bar{k}(A), \bar{k}(T), \bar{k}(H^V_{\omega_2}), \omega^M_1$ (which are in $\bar{M}$ and are countable in $M[H]$)

$$\text{There is a branch } (r, \bar{k}[f]) \text{ of } \bar{k}(T) \text{ and an iteration } \mathcal{J} = \{ j_{\alpha\beta} : N_\alpha \rightarrow N_\beta : \alpha \leq \beta \leq \omega^M_1 \}$$

such that:

- $\text{Cod}(r) = (N_0, a_0)$,
- $j_{0\omega^2}(A \cap \omega^N_{\omega_0}) = \bar{k}(A)$,
- $\bar{k}(H^V_{\omega_2}) \subseteq N_{\omega_1}^{\omega^M}$,
- $\text{NS}_{\omega_1}(H^V_{\omega_2}) = \text{NS}_{\omega_1}^{\omega^M} \cap \bar{k}(H^V_{\omega_2})$.

This sentence is true in $V[H] \supseteq M[H]$ and all its parameters are in $H^\bar{M}_V$. Therefore it holds in $H^\bar{M}_V$ by Shoenfield’s absoluteness (cfr. Lemma 1.16).

By homogeneity of $\text{Coll}(\omega, \eta)$ we conclude that $M$ models that: “in any $M$-generic extension for $\text{Coll}(\omega, \bar{k}(\omega^V_2))$ there is a weak semantic certificate for $\bar{k}(A)$ as witnessed by $\bar{k}(T)$”. By elementarity of $\bar{k}$ we get that the same holds in $V$ replacing $\bar{k}(A), \bar{k}(T)$ by $A, T$.

**Remark 2.11.** Provided one is given in advance the witnessing tree $T$, this Lemma does not need the existence of class many Woodin cardinals to go through. One just needs to be able to define in $V[G]$ the iterations $\mathcal{J}$ and $\mathcal{K}$. This property of $\mathbb{P}_{\text{max}}$-conditions holds if $V[G]$ satisfies very mild large cardinal properties (see [3, Section 2]).

### 3. THE PROOF

We will show that STRATEGY 1.9 works. $P_{D,A}$ is construed in what follows as the top element of an increasing chain of posets

$$\{ P_\alpha : \alpha \in C \}$$

each of size $\kappa$, where $\kappa > \omega_2$ is regular, $C$ is a club subset of $\kappa$, and $\check{\omega}_\kappa$ holds.

**Convention 3.1.** Fix from now on:

(A) $A \in \mathbb{P}(\omega_1) \setminus L(\mathbb{R})$ such that $\omega_1^{L[A]} = \omega^V_1$, and $B$ universally Baire set such that $\text{Cod}[B] = D$ is a dense subset of $\mathbb{P}_{\text{max}}$ with $N$ a model of ZFC+Martin’s axiom for any of its conditions $(N, b)$.

(B) $T_B, S_B$ trees on $(2 \times \text{Ord})^{<\omega}$ projecting to complements in every set sized forcing extension of $V$ and such that $B = p[T_B]$.

(C) $\kappa > \omega_2$ a regular cardinal such that $\check{\omega}_\kappa$ holds.

(D) $T$ a witnessing tree on $(2 \times \omega^V_2)^{<\omega}$ for $A, D$ according to Def. 2.3.

We first focus on designing a forcing generically adding a weak semantic certificate for $A$ as witnessed by $T$ and prove that this forcing notion accomplishes all requests set forth in STRATEGY 1.9, except possibly that of being stationary set preserving. To overcome this problem one is then led to define a transfinite sequence of posets

$$\{ P_\alpha : \alpha \in C \cup \{ \kappa \} \}$$
with $P_{\min(C)}$ generically adding a weak semantic certificate for $A$ as witnessed by $T$, and such that for $\alpha < \beta$ $P_\beta$ extends $P_\alpha$ adding many “sufficiently $P_\alpha$-generic” conditions. One then ensures using $\varnothing_\kappa$ that at many stages $\lambda \in C$ the $P_\kappa$-names for clubs are guessed by $P_\kappa$-names for clubs. Now the sequence $\{P_\alpha : \alpha \in C\}$ is designed in order that at all stages $\lambda < \beta$ there are $P_\beta$-conditions which “seal” the $P_\lambda$-name $A_\lambda$ for a club given by the Diamond sequence $\{A_\alpha : \alpha < \kappa\}$ at $\lambda$. Here sealing means that for any fixed stationary set $S$ existing in $V$ one can admit as $P_\beta$-conditions many which will guarantee that any $P_\beta$-generic filter $G$ including them will be such that $G \cap P_\alpha$ is a (possibly not $P_\alpha$-generic) filter ensuring that $(A_\alpha)_{G \cap P_\alpha \cap S}$ is non-empty. In particular the $P_\lambda$ are more and more likely to be stationary set preserving as $\lambda$ increases; $\varnothing_\kappa$ will be used to catch one’s tail and be able to infer that $P_\kappa$ is stationary set preserving because all potential $P_\kappa$-names for clubs have been sealed at some previous stage.

The forcings $P_\alpha$ we borrow from Asper`o and Schindler are here defined as “consistency properties”, a central concept in the analysis of infinitary logics.

3.1. Consistency properties as forcing notions. It will be crucial for our purposes to focus on consistency properties for multi-sorted signatures for relational languages.

**Multi-sorted logics.** Multi-sorted first order logic extends first order logic by introducing sorts for elements of its structures, by specifying for each variable and constant symbol in the language the sort of this symbol, and for any function and relation symbol a declaration of the sort of its entries (and of its output for function symbols).

A typical relational signature for multi-sorted logics reads as $\langle c_k^i : k \in K_i, i \in I; R_j : j \in J \rangle$ where the upper index of each constant symbol denotes the sort of that symbol, and each relation symbol $R_j$ comes with a arity $n_j$ and a map $s^j : n_j \to I$ which specifies for each entry of the relation its sort.

Formulae of multi-sorted logic for relational language are built as in usual first order logic with the further extra-clauses that:

- $(t = s)$ is allowed only if $t, s$ are terms of the same sort,
- $R_j(t_1, \ldots, t_n)$ is allowed if and only if $t_k$ is of sort $s^j(k)$ for all $k$,
- variable symbols are sorted and quantifiers are bounded to range over the given sort.

A structure $\mathcal{M} = (M_i : i \in I, R_j^M : j \in J, (c_k^i)^M : k \in K_i, i \in I)$ is a multi sorted structure for the multi-sorted signature $\langle c_k^i : k \in K_i, i \in I; R_j : j \in J \rangle$ in which each $M_i$ denotes the domain of the sort $i \in I$, each $(c_k^i)^M$ is an element of $M_i$, and for all $j \in J, a_1, \ldots, a_n, R_j^M(a_1, \ldots, a_n)$ holds if and only if $a_k \in M_{s^j(k)}$ for $k = 1, \ldots, n_j$.

Multi-sorted logic can be fragmentarily added with a fragment of first order logic by identifying the multi-sorted structure $\mathcal{M} = (M_i : i \in I, R_j^M : j \in J, (c_k^i)^M : k \in K_i, i \in I)$ for the multi-sorted signature $\langle c_k^i : k \in K_i, i \in I; R_j : j \in J \rangle$ with the first order structure:

$\mathcal{M}^* = \bigcup_{i \in I} M_i, M_i : i \in I, R_j^M : j \in J, (c_k^i)^M : k \in K_i, i \in I$ for the signature $\langle c_k^i : k \in K_i, i \in I; R_j : j \in J, X_i : i \in I \rangle$ with each $X_i$ a new unary predicate symbol interpreted by $M_i$ in the structure $\mathcal{M}$. But there is an inherent advantage in multi-sorted logic: for example by forbidding the formulae $(t = s)$ for $t, s$ of different sorts one may render certain concepts which are first order definable in

$\mathcal{M}^* = \bigcup_{i \in I} M_i, M_i : i \in I, R_j^M : j \in J, (c_k^i)^M : k \in K_i, i \in I$
undefinable in

\[ \mathcal{M} = \langle M_i : i \in I, R^M_j : j \in J, (c^M_k : k \in K_i, i \in I) \rangle. \]

For example, this makes the notion of morphism for multi-sorted structure much easier to handle than for first order structures (a morphism \( k : \mathcal{M} \to \mathcal{N} \) of multi-sorted structures is a multi-map mapping the domain of the sort \( i \) in \( \mathcal{M} \) into the domain of the sort \( i \) in \( \mathcal{N} \) and respecting the interpretation of atomic formulae).

We will crucially use this fact in the key part of the proof of Lemma 3.40 (cfr. Claim 2). For the moment to appreciate the extra-flexibility given by multi-sorted logic we present the following trivial example for the multi-sorted signature \( \{0, 1; A_0, A_1\} \) with 0, 1 sorts and the \( A_i \)'s unary predicates with sort \( i \). Consider the two \( \{0, 1; A_0, A_1\} \)-structures

\[ \mathcal{M}_0 = (\mathbb{N}, \mathbb{N}; \mathbb{N}, \mathbb{N}) \]
\[ \mathcal{M}_1 = (X, \mathbb{N} \setminus X; X, \mathbb{N} \setminus X) \]

where \( X \) is an infinite cofinite subset of \( \mathbb{N} \). It is clear that these structures are isomorphic as multi-sorted structures \( \{0, 1; A_0, A_1\} \) On the other hand (following the identification suggested above of a multi-sorted structure \( \mathcal{M} \) with a first order structure \( \mathcal{M}^* \)) we get that

\[ \mathcal{M}_0^* = (\mathbb{N}, \mathbb{N}, \mathbb{N}) \]
\[ \mathcal{M}_1^* = (\mathbb{N}, X, \mathbb{N} \setminus X) \]

It is immediate to check that these two structures are not even elementarily equivalent.

The point being that in multi-sorted logic we do not have to worry about the intersection of domains of different sorts \( i, j \) in a multi-sorted structure \( \mathcal{M} \), these two domains may be (partly) overlapping, but multi-sorted logic is not able to detect this overlap.

We will use the extra-flexibility to produce a certain embedding \( k : \mathcal{M} \to \mathcal{N} \) of multi-sorted structures both having interpretation of the sorts given by partially overlapping domains. The collation of the maps of \( i \) defined on the domains of the various sorts for \( \mathcal{M} \) may not even be a function: there could be overlap of the domain \( M_i \) of sort \( i \) with \( M_j \) of sort \( j \) and for \( a \in M_i \cap M_j \) the multimap \( k \), may assign a value using its \( i \)-th component different from the value assigned by its \( k \)-th component.

**Notation 3.2.** Let \( \mathcal{L} = S \cup R \cup K \) be a multi-sorted relational signature with \( S \) its set of sorts, \( R \) its set of relation symbols and \( K \) its set of constant symbols.

We denote an \( \mathcal{L} \)-structure

\[ \mathcal{M} = (M_s : s \in S, R^M : R \in R, c^M : c \in K) \]

by \((S^M, R^M, K^M)\).

**Definition 3.3.** Given a signature \( \mathcal{L} = S \cup R \cup K \) the \( \mathcal{L}_{\infty, \omega} \)-logic for \( \mathcal{L} \) is the smallest family of formulae containing the \( \mathcal{L} \)-atomic formulae and closed under quantifications, negation, infinitary disjunctions and conjunctions, i.e.:

- If \( \phi \) is an atomic \( \mathcal{L} \)-formula, it is also an \( \mathcal{L}_{\infty, \omega} \)-formula for \( \mathcal{L} \).
- If \( \{ \phi_i(\bar{x}_{ij}) : i < \gamma, j < \eta \} \) is a set of \( \mathcal{L}_{\infty, \omega} \)-formulae for \( \mathcal{L} \) with each \( \phi_i(\bar{x}_{ij}) \) in displayed free variables \( \{x_{ij} : j < \omega\} \),

\[ \bigwedge_{i < \gamma} \phi_i(\bar{x}_{ij}), \]
\[ \bigvee_{i < \gamma} \phi_i(x_0, \ldots, x_n) \]

are \( \mathcal{L}_{\infty, \omega} \)-formulae for \( \mathcal{L} \) in displayed free variables \( \{x_{ij} : i < \gamma, j < \eta\} \).
- If \( \phi(y, \bar{x}) \) is an \( \mathcal{L}_{\infty, \omega} \)-formula for \( \mathcal{L} \), so are \( \exists y \phi(y, \bar{x}) \) and \( \forall y \phi(y, \bar{x}) \).
- If \( \phi \) is an \( \mathcal{L}_{\infty, \omega} \)-formula for \( \mathcal{L} \), so is \( \neg \phi \).
The semantics is defined on $\mathcal{L}$-structures $\mathcal{M} = (M_s : s \in \mathcal{S}, R^\mathcal{M} : R \in \mathcal{R}, c^\mathcal{M} : c \in \mathcal{K})$ and assignments of the free variables to $M = \bigcup_{s \in \mathcal{S}}$ by the obvious rules:

- $\mathcal{M} \models R_j(\vec{x})[\vec{x}/\vec{a}]$ if and only if $R^\mathcal{M}_j(\vec{a})$ holds.

- $\mathcal{M} \models \neg \phi(\vec{x})[\vec{x}/\vec{a}]$ if and only if $\mathcal{M} \not\models \phi(\vec{x})[\vec{x}/\vec{a}]$.

- $\mathcal{M} \models \bigvee_{i<\gamma} \phi_i(\vec{x}_i)[\vec{x}_i/\vec{a}_i : i < \gamma]$ if and only if for some $i < \gamma$ $\mathcal{M} \models \phi_i(\vec{x}_i)[\vec{x}_i/\vec{a}_i]$.

- $\mathcal{M} \models \bigwedge_{i<\gamma} \phi_i(\vec{x}_i)[\vec{x}_i/\vec{a}_i : i < \gamma]$ if and only if for all $i < \gamma$ $\mathcal{M} \models \phi_i(\vec{x}_i)[\vec{x}_i/\vec{a}_i]$.

- $\mathcal{M} \models \forall x \phi(x, \vec{y})[\vec{y}/\vec{b}]$ with $x$ of sort $s$ if and only if for all $a \in M_s$ $\mathcal{M} \models \phi(x, \vec{y})[x/a, \vec{y}/\vec{b}]$.

- $\mathcal{M} \models \exists x \phi(x, \vec{y})[\vec{y}/\vec{b}]$ with $x$ of sort $s$ if and only if for some $a \in M_s$ $\mathcal{M} \models \phi(x, \vec{y})[x/a, \vec{y}/\vec{b}]$.

**Notation 3.4.** A $\mathcal{L}_{\infty, \omega}$-formula $\psi(\vec{x})$ is of $\bigwedge \bigvee$-type if it is logically equivalent to a formula

$$\bigwedge_{i \in I} \bigvee_{j \in J} \phi_{ij}(\vec{x})$$

with each $\phi_{ij}$ an atomic or negated atomic formula.

**Definition 3.5.** A $\mathcal{L}_{\infty, \omega}$-theory $\Sigma$ for the signature $\mathcal{L} = (\mathcal{S}, \mathcal{R}, \mathcal{K})$ (with $\mathcal{K} = \{c^s_j : j \in \mathcal{K}_s, s \in \mathcal{S}\}$) is **standard** if it contains:

**$\mathcal{L}$-equality axioms:**

1. $(c = c)$,
2. $(c = d) \rightarrow (d = c)$,
3. $[(c = d) \land (d = e)] \rightarrow (c = e)$,
4. $[R(c_1, \ldots, c_n) \land \bigwedge_{j=1}^n (d_j = c_j)] \rightarrow R(d_1, \ldots, d_n)$.

for all constant symbols $c, d, e, c_1, \ldots, c_n, d_1, \ldots, d_n$ of $\mathcal{L}$ and all permitted formulae of type $(c = d), R(c_1, \ldots, c_n)$ of the multi-sorted signature $\mathcal{L}$. 
\(\mathcal{L}\)-quantifier elimination axioms:
\[
( \bigwedge_{j \in K_s} \psi(c_j^s)) \leftrightarrow \exists x \psi(x),
\]
for all \(\mathcal{L}_{\infty,\omega}\)-formulae \(\psi(x)\) in displayed free variable \(x\) of sort \(s \in \mathcal{S}\):
\[
\forall x \bigwedge_{j \in K_s} (x = c_j^s)
\]
for all free variables \(x\) of sort \(s\) and for all sorts \(s \in \mathcal{S}\).

\(\Sigma\) is of \(\bigwedge \bigvee\)-type if it is standard and all other axioms of \(\Sigma\) are sentences of \(\bigwedge \bigvee\)-type.

**Remark 3.6.** For a standard \(\mathcal{L}_{\infty,\omega}\)-theory, the quantifier elimination axioms give that any \(\Pi_2\)-sentence \(\forall x \exists y \psi(x, y)\) with \(\psi(x, y)\) quantifier free is equivalent to a \(\bigwedge \bigvee\)-type sentence.

By passing to infinitary logic we will be able to express the \(\Pi_2\)-fragment of an ordinary multi-sorted \(\mathcal{L}\)-theory \(T\) with the Henkin property (i.e. for any \(\mathcal{L}\)-formula \(\psi(x)\) there is some constant symbol \(c \in \mathcal{L}\) with \((\exists x \psi(x)) \rightarrow \psi(c) \in T\)) using quantifier free sentences of \(\bigwedge \bigvee\)-type on the \(\mathcal{L}_{\infty,\omega}\)-theory extending \(T\) with the equality and quantifier elimination axioms.

**Consistency properties.** We will be interested in studying certain forcing notions whose properties are best described by means of standard theories.

**Definition 3.7.** Let \(\mathcal{L} = \mathcal{S} \cup \mathcal{R} \cup \mathcal{K}\) be a multi-sorted signature.

- A \(\nu : \mathcal{L} \rightarrow M \cup \bigcup_{n<\omega} \mathcal{P}(M^n)\) existing in some generic extension of \(V\) is a generic \(\mathcal{L}\)-assignment if \(M = (\mathcal{S}^M, \mathcal{R}^M, \mathcal{K}^M)\) is an \(\mathcal{L}\)-structure with \(\mathcal{S}^M = \{M_s : s \in \mathcal{S}\}\), \(M = \bigcup \mathcal{S}^M\), and:
  - \(\nu \upharpoonright \mathcal{K}\) maps correctly and surjectively the constants of sort \(s\) to \(M_s\),
  - \(\nu \upharpoonright \mathcal{R}\) maps correctly the relation symbols \(R_j\) of \(\mathcal{R}\) of arity \(n_j\) and type \(s_j\) to subsets of \(M_{s_j(0)} \times \cdots \times M_{s_j(n_j)}\).
- Given a generic assignment \(\nu\), \(\Sigma_\nu\) is the family of \(\mathcal{L}_{\infty,\omega}\)-sentences for \(\mathcal{L}\) realized in the structure \(M\).

**Warning 3.8.** In certain circumstances occurring below, when the interpretation of a certain relation symbol \(R\) in \(\mathcal{R}\) is clear, we consider as an \(\mathcal{L}\)-structure also one where the interpretation of \(R\) is not explicitly given (because the information conveyed on the interpretations of the other symbols suffices to determine the natural interpretation of \(R\)).

The same occurs for generic assignments \(\nu\), we will consider just the assignment restricted to the constants (and eventually to some predicates), the assignment \(\nu\) on other predicates will be clear from the context.

This is a standard practice (consider for example the omission of the interpretation for the relation symbol \(=\) in the usual notation for first order structures).

The following definition encapsulates the key provisions of a *consistency property* generated by a class of structures according to [?], and gives a powerful tool to produce “generic” models omitting a prescribed family of types.

**Definition 3.9.** Given a family of generic assignments \(\mathcal{A}\) for a signature \(\mathcal{L} = \mathcal{S} \cup \mathcal{R} \cup \mathcal{K}\):

- The forcing \(P_\mathcal{A}\) consists of all those finite sets of atomic or negated atomic \(\mathcal{L}\)-sentences which are contained in some \(\Sigma_\nu\) with \(\nu \in \mathcal{A}\).
- The order on \(P_\mathcal{A}\) is given by reverse inclusion.
- A set of \(\mathcal{L}\)-sentences \(\Sigma\) is \(\mathcal{A}\)-consistent if \(p \in P_\mathcal{A}\) for every finite subset \(p\) of \(\Sigma\).
- Given a maximally \(\mathcal{A}\)-consistent set of sentences \(\Sigma\), the term model \(M_\Sigma\) for \(\mathcal{L}\) is defined as follows:
  - \([c]_\Sigma = \{d \in \mathcal{K} : (c = d) \in \Sigma\}\);
  - \(s^\Sigma = \{[c]_\Sigma : c\) of sort \(s\}\);
\[ R^\Sigma([c_1]^\Sigma, \ldots, [c_n]^\Sigma) \text{ if and only if } R(c_1, \ldots, c_n) \in \Sigma. \]

The canonical assignment \( \nu^\Sigma \) maps \( c \) to \([c]^\Sigma\), \( R \) to \( R^\Sigma \).

Note that a filter \( H \) on \( P_A \) is maximal if and only if \( \Sigma_H = \cup H \) is maximally \( A \)-consistent. It is also clear that \( A \)-consistent sets are finitely satisfiable, hence consistent in the ordinary sense.

**Lemma 3.10.** Given a family of generic assignments \( A \) for a signature \( L = S \cup R \cup K \), the following holds for the forcing \( P_A \):

1. \( \mathcal{M}_\Sigma \) is a well defined \( L \)-structure for all maximally \( A \)-consistent \( \Sigma \).
2. Assume \( H \) is \( V \)-generic for \( P_A \), \( \Sigma_H = \cup H \).

Then for any \( L \)-sentence
\[ \bigwedge_{i \in I} \bigvee_{j \in J_i} \psi_{ij} \]
of type \( \bigwedge, \bigvee \), TFAE:

\( A \) \( \mathcal{M}_{\Sigma_H} \models \bigwedge_{i \in I} \bigvee_{j \in J_i} \psi_{ij}. \)

\( B \) For all \( i \in I \) and some \( p \in H \), \( D_i \) is dense below \( p \), where
\[ D_i = \{ q \in P_A : \psi_{ij} \in q \text{ for some } j \in J_i \}. \]

**Notation 3.11.** Given a maximal filter \( H \) on \( P_A \) denote \( \mathcal{M}_{\Sigma_H} \) by \( \mathcal{M}_H \), \([c]^\Sigma_H\) by \([c]^H \), \( R^\Sigma_H \) by \( R^H \), \( s^\Sigma_H \) by \( s^H \), \( \nu^\Sigma_H \) by \( \nu^H \).

\( \mathcal{M}_H \), \( \nu^H \), \( R^H \), \( s^H \), \([c]^H \) denote the canonical \( P_A \)-names for the corresponding objects determined by the \( V \)-generic filter \( H \) for \( P_A \).

**Proof.**

1. It is clear that \([c]^\Sigma\) is an equivalence relation, and also that \( R^\Sigma \) is independent of the representative chosen in \([c]^\Sigma\). We spell out some details:
   \( \neg \Sigma \) **is an equivalence relation:** Assume \((c = d), (d = e) \in \Sigma_H \). The sets
   \( \{(c = d), (d = e), \neg (c = e)\}, \{(c = d), \neg (d = e)\}, \{\neg (c = c)\} \) are not even consistent, therefore any maximally \( A \)-consistent set (henceforth consistent) containing \((c = d), (d = e) \) must also contain \((c = e), (d = c), (c = c)\).

2. \( R^\Sigma \) **is well defined:** Assume \([c_i]^\Sigma = [d_i]^\Sigma \) for \( i = 1, \ldots, n \) then:
   \[ R^\Sigma([c_i]^\Sigma, \ldots, [c_n]^\Sigma) \]
   if and only if
   \[ R(c_1, \ldots, c_n) \in \Sigma \]
   even consistent
   \[ R(d_1, \ldots, d_n) \in \Sigma. \]

2. Assume \( H \) is \( V \)-generic for \( P_A \).

One direction is clear: if for some \( p \in H \) \( D_i \) is predense below \( p \) for all \( i \in I \), then \( H \cap D_i \neq \emptyset \) for all \( i \in I \), hence for all \( i \) there is \( j \) such that \( \psi_{ij} \in \Sigma_H \). This immediately yields that
\[ \mathcal{M}_H \models \bigwedge_{i \in I} \bigvee_{j \in J_i} \psi_{ij}. \]

For the converse assume
\[ \mathcal{M}_H \models \bigwedge_{i \in I} \bigvee_{j \in J_i} \psi_{ij}. \]
Then some \( p \in H \) is such that
\[
p \models_{\mathcal{A}} \mathcal{M}_H \models \bigwedge_{i \in I} \bigvee_{j \in J_i} \psi_{ij}^\gamma.
\]
By definition (since \( Z = \{(i, j) : i \in I, j \in J\} \) is in \( V \) and so is \( \bigwedge_{i \in I} \bigvee_{j \in J_i} \psi_{ij} \))
\[
p \models_{\mathcal{A}} \forall \tau \in \tilde{I} \exists \sigma \in \tilde{J} \mathcal{M}_H \models \psi_{\tau \sigma}^\gamma.
\]
By definition of the forcing relation (using the fact that \( \tilde{Z}_H = Z \) for all \( V \)-generic filters \( H \), and the forcing clauses for existential formulae), this holds if and only if
\[
D_i^* = \left\{ q \in P : \exists j (q \models \mathcal{M}_H \models \psi_{ij}^\gamma) \right\}
\]
is open dense below \( p \) for all \( i \in I \).

Now if \( q \models \mathcal{M}_H \models \psi_{ij}^\gamma \) and \( H \) is \( V \)-generic for \( P_A \) with \( q \in H \), we get that \( \psi_{ij} \in \Sigma_H \), therefore \( q \cup \{ \psi_{ij} \} \in H \).

Hence we conclude that
\[
D_i = \{ q \in P : \exists j (\psi_{ij} \in q) \}
\]
is dense below \( p \) for all \( i \in I \).

\[\Box\]

**Remark 3.12.** Note that the Lemma holds in this generality just for sentences of \( \bigwedge \bigvee \)-type. For example consider a sentence
\[
\bigvee_{j \in J} \bigwedge_{i \in I_j} \psi_{ij}
\]
Running the proof as before for this sentence we get to the conclusion that
\[
\mathcal{M}_H \models \bigvee_{j \in J} \bigwedge_{i \in I_j} \psi_{ij}
\]
if and only if for some \( j \in J \) and some \( q \in H \)
\[
q \models \mathcal{M}_H \models \bigwedge_{i \in I_j} \psi_{ij}^\gamma.
\]
Now this is in general false because typically a condition \( q \in P_A \) can decide only finitely many atomic \( L \)-sentences (the unique effective method to check whether \( q \) forces the truth in \( \mathcal{M}_H \) of an atomic \( L \)-sentence is to check that it belongs to \( q \)). On the other hand the above conditions require \( q \) to decide the infinitely many atomic \( L \)-sentences \( \psi_{ij} \) for all \( i \in I_j \).

**Definition 3.13.** Given a family of generic assignments \( \mathcal{A} \), we let \( \Sigma_\mathcal{A} \) denote the family of \( L \)-sentences \( \psi \) in \( V \) of type \( \bigwedge \bigvee \) such that
\[
\emptyset \models_{\mathcal{A}} \mathcal{M}_H \models \psi^\gamma.
\]
The following observation is very useful:

**Fact 3.14.** Assume \( p \in P_A \) and \( \psi := \bigwedge_{i \in I} \bigvee_{j \in J_i} \psi_{ij} \) is a sentence of \( \bigwedge \bigvee \)-type such that any \( \mathcal{A} \)-assignment \( \nu \) with \( p \subseteq \Sigma_\nu \) has \( \psi \) in \( \Sigma_\nu \). Then \( \mathcal{M}_H \models \psi \) for any \( H \) \( V \)-generic for \( p \).

In particular if a family of \( L \)-assignments \( \mathcal{A} \) is axiomatized by a standard \( L \)-theory \( \Sigma \), \( \mathcal{M}_H \models \Sigma \) for any \( H \) \( V \)-generic for \( \Sigma \).

**Proof.** The sets
\[
D_i = \{ q \in P : \exists j (\psi_{ij} \in q) \}
\]
required to check \( \mathcal{M}_H \models \psi \) are dense below \( p \).

\[\Box\]
3.2. The forcing $P_0^*$. First of all it is convenient to isolate the smallest transitive fragment $X_{(j,r,f,T)}$ of $V[G]$ having as elements all the objects required to define a semantic certificate $(J,r,f,T)$ existing in $V[G]$ and computing correctly all the properties of this semantic certificate we are interested in. Now $P_0^*$ will be obtained by gluing together finite pieces of information describing in an appropriate multi-sorted signature the properties of the structures $(X_{(j,r,f,T)},\in)$. In fact $P_0^*$ will be $P_A$ for a given family $A$ of generic assignment $\nu : L^0 \rightarrow X_{(j,r,f,T)}$ satisfying certain natural prescriptions. $L^0$ will contain the sorts, predicates and constants needed to encode what could be true or false in a generic structure $(X_{(j,r,f,T)},\in)$. A generic filter for $P_0^*$ will add a term model for $L^0$. By the result of the previous section it is to be expected that if things are organized properly the term model defined by the generic filter for $P_0^*$ will be isomorphic to the multi-sorted presentation in signature $L^0$ of a semantic certificate (to get this we just have to ensure that the $L^0$-theory associated to semantic certificates is of $\bigwedge \bigvee$-type and then apply Fact 3.14, with some extra care to grant well-foundedness of the generic term model). If $P_0^*$ were stationary set preserving, one would be (almost) done. However this basic set-up is not yet sufficient to argue that $P_0^*$ is stationary set preserving, just to infer that generic filters for $P_0^*$ add semantic certificates $(J,r,f,T)$ for $A,D$. For the moment however let us analyze in detail the forcing $P_0^*$ to get accustomed with the more intricate arguments of the next section where the chain of forcings $\{P_\alpha : \alpha \in C \cup \{\kappa\}\}$ will be introduced, with each one adding a semantic certificate, and with $P_\kappa$ being also stationary set preserving.

Definition 3.15. Given a semantic certificate $(J,r,f,T)$ for $A,D$ existing in some $V[K]$, let

$$X_{(j,r,f,T)} = \{j_{\alpha\beta} : N_\alpha \rightarrow N_\beta : \alpha \leq \beta \leq \omega_1^V\}. $$

$$X_{(j,r,f,T)}$$ is the set

$$\left( \bigcup_{i \leq \omega_1^V} N_a \right) \cup \{N_\alpha : \alpha \leq \omega_1^V\} \cup \{j_{\alpha\beta} : \alpha \leq \beta \leq \omega_1^V\} \cup \{G_\alpha : \alpha < \omega_1\} \cup \{r,f\}.$$

$X_{(j,r,f,T)}$ is the multi sorted structure of sorts $N_\alpha : \alpha \leq \omega_1^V, \{r,f\}$ and binary relations $\in | N_\alpha$ of type $(N_\alpha,N_\alpha)$, unary relations $G_\alpha$ of type $N_\alpha$, binary relations $j_{\alpha\beta}$ of type $(N_\alpha,N_\beta)$, unary relation $T$ of type $12 \ N_{\omega_1}$, binary relation $\sqsubseteq^2 | T$ of type $13 \ (N_{\omega_1},N_{\omega_1})$, ternary relation br$_T$ of type $14 \ (N_{\omega_1},\{r,f\},\{r,f\})$, ternary relation Cod$^*$ of type $15 \ (N_0,N_0,\{r,f\})$, satisfaction predicates $16 \ Sat_{N_\alpha}$ of type $(N_\alpha,N_\alpha)$ for $\alpha \leq \omega_1$,

$$(N_\alpha : \alpha \leq \omega_1^V, \{r,f\} ; \in | N_\alpha, Sat_{N_\alpha} : \alpha \leq \omega_1^V, G_\alpha : \alpha < \omega_1^V, j_{\alpha\beta} : \alpha \leq \beta \leq \omega_1^V, \sqsubseteq^2 | T, br_T, Cod^*). \)

Remark 3.16.

- The list

$$\{N_\alpha : \alpha \leq \omega_1\} \cup \{j_{\alpha\beta} : \alpha \leq \beta \leq \omega_1^V\} \cup \{G_\alpha : \alpha < \omega_1\} \cup \{r,f\}$$

gives the $\in$-maximal element of $(X_{(j,r,f,T)},\in)$ and the list is not redundant, i.e any element in the above list does not belong to any of the others, and any $\in$-maximal element of the graph of the $\in$-relation over $(X_{(j,r,f,T)},\in)$ is in the above list.

---

12Denoting the extension of $T$, which is a subset of $N_{\omega_1}$.
13$(s,t) \sqsubseteq^2 (u,v)$ if and only if $(s,t),(u,v) \in T$, $s \sqsubseteq u, t \sqsubseteq v$.
14With $br_T(x,y,z)$ holding just if $x = (s,t) \in T, y = r, z = f$ and $s \sqsubseteq u, t \sqsubseteq f$.
15Recall Cod$(r) = (N_0,A \cap \omega_1^{N_0}); Cod^*(x,y,z)$ holds if and only if $x \in N_0, y = A \cap \omega_1^{N_0}, z = r$.
16To be represented by the pairs $\langle \psi, (a_1, \ldots, a_m) \rangle$ with $\psi(x_1, \ldots, x_m)$ an $\epsilon$-formula in displayed free variables and $(a_1, \ldots, a_m) \in N^{\in}_m$ such that

$$(N_\alpha, \in) \models \psi(x_1, \ldots, x_m)[x_1/a_1, \ldots, x_m/a_m].$$
• In essence $X(J, r, f, T)$ is a set containing all the elements and predicates of the multi-sorted structure $\mathcal{X}(J, r, f, T)$ except a few objects which can be easily reconstructed from the remaining ones (e.g. the sort $\{r, f\}$, the predicates $\text{Sat}_{N_\alpha}$, $\text{br}_T$, $\text{Cod}^\dagger$).
• $H_{\omega_2}$, $T$ are both subsets of $N_\omega^V$ (by Condition 5 for the weak semantic certificate for $A$ as witnessed by $T$).
• Note that $X(J, r, f, T)$ is not transitive: if $\langle a, b \rangle = \{\{a\}, \{a, b\}\} \in j_{a\beta}$, $\{a\} \in N_\alpha$, but $\{a, b\} \in N_\alpha \cup N_\beta$ may not be neither in $N_\alpha$, nor in $N_\beta$. However with the exception of the elements in the graph of $j_{a\beta}$, all other elements of $X(J, r, f, T)$ are contained in some $N_\alpha$. 
• $X(J, r, f, T)$ has size bounded by $|\omega_2|^{|V|}$.

The key point is that (as we will see) the $\mathcal{L}_{\infty, \omega}$-theory of the structures $\mathcal{X}(J, r, f, T)$ completely describes the notion of semantic certificate up to type isomorphism.

Given a semantic certificate $(J, r, f, T)$ for $A, D$ as witnessed by $T$ with

$$J = \{j_{a\beta} : N_\alpha \to N_\beta : \alpha \leq \beta \leq \omega_1^V\},$$

$\mathcal{L}^0$ is in $V$ the multi-sorted signature with sorts, constants, and predicate symbols to be interpreted by the relevant objects (or subsets — see Warning 3.8) of $X(J, r, f, T)$ which define the associated multi-sorted structure $\mathcal{X}(J, r, f, T)$.

**Definition 3.17.** $\mathcal{L}^0$ consists of the following sorts, constants, and predicate symbols:

**Sorts:**

(a) $\dot{N}_i$ to denote the extension of $N_i$ for each $i \leq \omega_1^V$,

(b) $\{r, f\}$ to denote the set given by the elements of the selected branch $(r, f)$ of $T$.

**Constants:**

(c) $\dot{x}$ of sort $\dot{N}_{\omega_1^V}$ for $x \in H_{\omega_2}$ to denote the elements of $H_{\omega_2}$.

(d) symbols $\{\dot{e}_{n,i} : n \in \omega\}$ of sort $\dot{N}_i$ for $i < \omega_1^V$ needed to denote the elements of $N_i$ (but only for $i < \omega_1^V$);

(e) symbols $\{\dot{e}_{j, \omega_1^V} : j < \omega_1^V\}$ of sort $\dot{N}_{\omega_1^V}$;

(f) symbols $\dot{r}, \dot{f}$ of sort $\{r, f\}$ to be interpreted in some $V[G]$ by the selected branch $(r, f)$ through $T$ such that $\text{Cod}(r) = (N_0, a_0)$.

**Predicates:**

(g) A unary predicate symbol $\dot{T}$ of type $\dot{N}_{\omega_1}$ to denote the extension of the (2-dimensional) witnessing tree $T \subseteq \dot{N}_{\omega_1}$;

(h) Unary predicate symbols $\{\dot{G}_j : j < \omega_1\}$ (each of type $\dot{N}_j$ for $j < \omega_1^V$) to denote the $N_\alpha$-generic filters such that $\dot{N}_{\alpha+1}$ is the ultrapower of $N_\alpha$ by $G_\alpha$;

(i) Binary predicate symbols $\dot{j}_{a\beta}$ of type $(\dot{N}_\alpha, \dot{N}_\beta)$ to denote $j_{a\beta}$ for $\alpha \leq \beta \leq \omega_1^V$;

(j) A ternary predicate symbol $\dot{br}_T$ of type $(\dot{N}_{\omega_1}, \{r, f\}, \{r, f\})$ to denote the initial segment relation between elements of $T$ and the branch $(r, f)$;

(k) For each $\alpha \leq \omega_1^V$ a binary predicate $\varepsilon_{N_\alpha}$ of type $(\dot{N}_\alpha, \dot{N}_\alpha)$ to denote the $\varepsilon$-relation restricted to $N_\alpha$;

(l) For each $\alpha \leq \omega_1^V$ a satisfaction predicate $\text{Sat}_{N_\alpha}(\langle \psi(x_1, \ldots, x_m) \rangle)$ of type$^{17}$ $(\dot{N}_\alpha, \dot{N}_\alpha)$ to be represented by the pairs $\langle \upharpoonright \psi \upharpoonright, (a_1, \ldots, a_m) \rangle$ with $\psi(x_1, \ldots, x_m)$ an $\varepsilon$-formula in displayed free variables and $(a_1, \ldots, a_m) \in N_\alpha^m$.

---

$^{17}$It is clear that the interpretation of this predicate symbol will subsume the interpretation of $\varepsilon_{N_\alpha}$ by considering the formula $\text{Sat}_{N_\alpha}(\langle x \in y \rangle)$. However it is convenient (just for notational simplicity) to have a special symbol to denote $\varepsilon|_{N_\alpha}$. 
such that
\[(N_\alpha, \in) \models \psi(x_1, \ldots, x_m)[x_1/a_1, \ldots, x_m/a_m];\]

\[(m)\] A ternary predicate symbol \(\text{Cod}^*\) of type \((\hat{N}_0, N_0, \{r, f\})\) to encode the fact that \(\text{Cod}(r) = (N_0, a)\) by holding just of the triples \(x, y, z\) with \(y = A \cap \omega_1^N_0\) and \(z = r\).

Note that \(L^0 \in V\) has size \(\omega_2\) in \(V\).

**Definition 3.18.** Let \((J, r, f, T)\) be a weak semantic certificate for \(A, D\) with
\[J = \{j_{\alpha\beta} : N_{\alpha} \to N_{\beta} : \alpha \leq \beta \leq \omega_1^V\}.\]

A map \(\nu\) with domain \(L^0\) is an admissible interpretation for \((J, r, f, T)\) if it respects the following constraints on the interpretations of constants and predicates:

**Sorts:**
- (a) \(\nu(\hat{N}_i) = N_i\) for each \(i \leq \omega_1^V\).
- (b) \(\nu(\{r, f\}) = \{r, f\}\).

**Constant symbols:**
- (c) \(\nu(\hat{x}) = x\) for every \(x \in H_{\omega_2}\).
- (d) \(\nu(\{c_{\alpha n} : n \in \omega\}) = N_\alpha\) and \(\nu(c_{0n}) = A \cap \omega_1^N_0\) for all \(\alpha < \omega_1^V\).
- (e) \(\nu(\{\hat{c}_{j,\omega_1^V} : j \in \omega_1^V\}) = N_{\omega_1^V}\) and \(\nu(c_{0,\omega_1^V}) = A\).
- (f) \(\nu(\hat{r}) = r \in p[T], \nu(f) = f\) with \((r, f)\) a branch of \(T\) and \(\text{Cod}(r) = (N_0, A \cap \omega_1^N_0)\).

**Predicates:**
- (g) \(\nu(\hat{T}) = T\).
- (h) \(\nu(G_\alpha) = G_\alpha\) for \(\alpha < \omega_1^V\).
- (i) \(\nu(j_{\alpha\beta}) = j_{\alpha\beta}\) for all \(\alpha \leq \beta \leq \omega_1^V\).
- (j) \(\nu(\text{br}_T) = \{(s, t, r, f) : s \subseteq r, t \subseteq f\}\).
- (k) \(\nu(\text{Sat}_{N_\alpha})\) as prescribed by (l) of Def. 3.17.
- (l) \(\nu(\text{Cod}^*) = \{(x, A \cap \omega_1^N_0, r) : x \in N_0\}\).

**Warning 3.19.** A key point is that once we fixed the structure \(X(\hat{J}, r, f, T)\) which is the target of \(\nu\), we have a certain freedom in changing the value of the assignments on constant symbols (for example for all those of the form \(\hat{c}_{n,\alpha}\) for \(n > 0\) and \(\alpha < \omega_1^V\)), while maintaining that the (so modified assignment) is still admissible; but we have no freedom in changing the interpretation of the predicate symbols. In particular many of the constant symbols could also be considered sorted variables.

- Given an admissible assignment \(\nu, \Sigma_\nu\) is the set of \(L_\infty^\omega\)-sentences for \(L^0\) realized in the structure \(X(\hat{J}, r, f, T)\) by the assignment \(\nu\).
- A set \(\Sigma\) of \(L_\infty^\omega\)-sentences for \(L^0\) is generically consistent if in some generic extension \(V[G]\) of \(V\) there is a semantic certificate \((J, r, f, T)\) for \(A, D\), and an admissible assignment \(\nu : L^0 \to X(\hat{J}, r, f, T)\) such that \(\Sigma \subseteq \Sigma_\nu\).

The first key observation is that the \(L^0\)-isomorphism types of a semantic certificate define an elementary class for \(L_\infty^\omega\) which is axiomatized by a standard theory (cfr. Def. 3.5).

To accomodate our \(L_\infty^\omega\)-axiomatization of the notion of semantic certificate it is convenient to detail more the nature of the map \(\text{Cod} : 2^\omega \to H_{\omega_1}\).

**Convention 3.20.** We assume the following properties of \(\text{Cod}\) in case \(\text{Cod}(r) = (N, a) \in \mathbb{P}_{\text{max}}^\omega\):
• $r$ is identified (modulo a recursive bijection $n \mapsto \langle k_n, j_n \rangle$ of $\omega$ with $\omega^2$) with a well-founded extensional relation $E_r$ with domain $\omega$;

• the Mostowski collapsing map $\pi$ of the structure $(\omega, E_r)$ has as range $\text{trcl} \{ \{ N, a \} \}$ and maps 0 to $N_1$, 1 to $\{ N \}$, 2 to $\{ N, a \}$, 3 to $\{ N, a \}$, 4 to $\{ \{ N, a \} \}$, 5 to $a$.

**Fact 3.21.** Let $\Sigma^0$ be the $L^0$ theory for $L_{\omega, \omega}$ given by the $L^0$-equality axioms and the $L^0$-quantifier elimination axioms (cfr. Def. 3.5), and the following list of axioms:

(I) Axioms to characterize the extensions of $\check{T}$ and of $H^V_{\omega_1}$ inside $N_{\omega_1}$:

(a) For all $x, y \in H^V_{\omega_2}$

\[
(\check{x} \in \check{y}) \text{ if } x \in y,
\]

\[
(\check{x} \notin \check{y}) \text{ if } x \notin y.
\]

(b) The infinitary conjunction of infinitary disjunctions

\[
\bigwedge_{x \in H^V_{\omega_2}} \bigvee_{j < \omega^V} (\check{x} = \check{c}_{j, \omega_1}).
\]

(c) For all constant symbols $c$ of sort $\check{N}_{\omega_1}^V$

\[
\check{T}(c) \leftrightarrow \bigvee_{(s,t) \in T} (c = \langle \check{s}, \check{t} \rangle)
\]

(II) The following axioms to describe the properties of an iteration $J$:

(a) For all $\alpha \leq \beta \leq \omega^V_1$

\[
\bigwedge_{n < \omega} \bigvee_{j < \omega^V_1} (\check{j}_n^\omega \check{c}_n^\omega) = \check{c}_j^\omega,
\]

\[
\bigwedge_{n < \omega} \bigvee_{m < \omega} (\check{j}_n^\omega) = \check{c}_m^\omega.
\]

(b) For all $\alpha < \beta \leq \omega^V_1$ and constant symbols $\check{c}_n^\omega, \check{c}_m^\omega$ for $n, m < \omega$

\[
\left[ \check{G}_\alpha(\check{c}_n^\omega) \wedge \check{j}_n^\omega(\check{c}_n^\omega) = \check{c}_m^\omega \right] \leftrightarrow \left[ \text{Sat}_{N^\omega}(\check{c}_n^\omega) \wedge \text{Sat}_{N^\omega}(\check{c}_m^\omega) \right].
\]

(c) For all $\alpha < \beta \leq \omega^V_1$ and constant symbols $\check{c}_m^\omega$ for $m < \omega$

\[
\bigvee_{n, k < \omega} \left[ (\check{c}_k^\omega = \check{j}_n^\omega(\check{c}_n^\omega)) \wedge \text{Sat}_{N^\omega}(\check{c}_k^\omega(\check{c}_n^\omega) = \check{c}_m^\omega(\check{c}_n^\omega)) \right].
\]

(d) For all $\alpha < \omega^V_1$

\[
\bigwedge_{n < \omega} \text{Sat}_{N^\omega}(\check{c}_n^\omega) \text{ is a dense subset of } \mathcal{P}(\check{\omega}_1) \rightarrow \bigvee_{m < \omega} \left( \check{G}_\alpha(\check{c}_m^\omega) \wedge \check{c}_m^\omega \in \check{c}_n^\omega \right).
\]

(III) The following sentences describing the properties an iteration $J$, a real $r$ and an $f : \omega \rightarrow \kappa$ must satisfy to grant that $(J, r, f, T)$ is a semantic certificate:

(a) The sentence

\[
\bigwedge_{n < \omega} \bigvee_{(s,t) \in T \cap (2 \times \omega^V_1)^n} \text{br}_T(\langle s, t \rangle, \check{r}, \check{f})
\]

\[
\bigwedge_{(s,t), (u,v) \in T} \left[ \text{br}_T(\langle s, t \rangle, \check{r}, \check{f}) \wedge \text{br}_T(\langle u, v \rangle, \check{r}, \check{f}) \right] \rightarrow \left( \langle s, t \rangle \sqsubset^2 \langle u, v \rangle \lor \langle u, v \rangle \sqsubset^2 \langle s, t \rangle \right)
\]

\[\text{This sentence states that } j_{\alpha \beta}[N_a] = \{ j_{\alpha \beta}(f)(\omega^V_1) : f \in N_a, \text{dom } f = \omega^V_1 \}.\]
(b) The axioms

\[ \bigwedge_{n < \omega} \text{Cod}^\omega(c_{n,0}, c_{0,0}, \tilde{r}), \]

\[ \bigwedge_{n \in \omega, k_n, j_n \geq 5, (s,t) \in T} (br_{T}(\langle s,t \rangle, \tilde{r}, \tilde{f}) \land s(n) = \bar{1}) \rightarrow \text{Sat}_{N_0}(\langle c_{k_n-5,0} \in c_{j_n-5,0} \rangle), \]

\[ \bigwedge_{n \in \omega, k_n, j_n \geq 5, (s,t) \in T} (br_{T}(\langle s,t \rangle, \tilde{r}, \tilde{f}) \land s(n) = \bar{0}) \rightarrow \text{Sat}_{N_0}(\langle c_{k_n-5,0} \notin c_{j_n-5,0} \rangle). \]

(c) The sentence

\[ \tilde{j}_{\omega_1}(\epsilon_{0,0}) = A. \]

(d) For all constant symbols \( c \) of sort \( N_{\omega_1} \) and \( y \in H_{\omega_2} \)

\[ c \in \tilde{y} \leftrightarrow \bigvee_{x \in y} \tilde{x} = c. \]

(e) The infinitary conjunction

\[ \bigwedge_{S \in (P(\omega_1) \cup \text{NS}_{\omega_1})^V} \text{Sat}_{N_{\omega_1}^V}(\langle \tilde{S} \text{ is stationary} \rangle). \]

Then:

(1) \( \Sigma^0 \) is standard (cfr. Def. 3.5).

(2) \( \Sigma^0 \) is realized by any admissible assignment \( \nu \).

(3) Whenever \( \mathcal{N} = ((\omega^0)^N, (\mathbb{R}^0)^N, (\mathbb{K}^0)^N) \) is a \( \mathcal{L}^0 \)-model of \( \Sigma^0 \) existing in some transitive model \( W \), we have that:

- The domain of each sort of \( (\omega^0)^N \) has as extension the interpretations of the constant symbols of \( \mathbb{K}^0 \) of that sort.
- The sort \( N_{\alpha} \) of \( \mathcal{N} \) is well founded and extensional for all \( \alpha \leq \omega_1^V \).
- The transitive collapse of the sorts \( N_{\alpha} \) of \( \mathcal{N} \) define a semantic certificate \( (J, r, f, T) \) for \( A, D \) existing in \( W \) with \( r(n) = i \) and \( f(n) = j \) if and only if for some \( (s,t) \in T \) with \( s(n) = i \) and \( f(n) = j \), \( br_{T}(\langle s,t \rangle, r, f) \).

Proof. The proof that \( \Sigma^0 \) is standard and that \( \Sigma^0 \subseteq \Sigma_{\nu} \) for any admissible assignment \( \nu \) is left to the reader. It consists in the tyrosome checking that all the above sentences are of \( \bigwedge \bigvee \)-type and are in \( \Sigma_{\nu} \) for any admissible assignment \( \nu \).

Let \( W \) be a transitive model of \( ZFC \) and \( \mathcal{N} \) be in \( W \) a structure satisfying \( \Sigma^0 \). The axiom for quantifier elimination grants that \( \mathcal{N} \) has as domain of its sorts exactly the interpretation of the constant symbols of \( \mathcal{L}^0 \) of that sort. The axioms of Type (I) grant that

\[ \{ \tilde{x}^N : x \in H_{\omega_2} \}, \in N_{\omega_1}^V \]

is isomorphic to \( (H_{\omega_2}, \in) \) and also that

\[ \{ \tilde{c}^N : \mathcal{N} \models T(c) \}, (\subseteq^2)^V \]

is isomorphic to

\[ (T, \subseteq^2). \]

These axioms combined with axioms (III)a grant that:

---

19. These axioms formalize that \( \text{Cod}(r) = (N_0, a_0) \) by imposing that \( (N_0, \in) \) is isomorphic to \( E_r \upharpoonright [5; \infty) \) in accordance with notation 3.20, and that \( c_{0,0} \) is assigned to \( A \cap N_0^N \).

20. This axioms grants that \( (H_{\omega_2}, \in) \) sits inside \( (N_{\omega_1}^V, \in) \) as a transitive subclass.
• The unique infinite sequence \((r, f) \in V[G]\) given by \(r(n) = i\) and \(f(n) = \alpha\) if 
\[ \mathcal{N} \models \bar{r} (\bar{n}) = i \land \bar{f} (\bar{n}) = \bar{\alpha} \]
is a branch through \(T\). Therefore in \(V[G]\) \(\text{Cod}(r) = (N_0, a_0)\) with \(a_0 = A \cap \omega_1^{N_0}\).

• The structure 
\[ \{c_{a,0}^N : n \in \omega\}, \in_N^N \]
is well-founded and its transitive collapse is exactly \(N_0\) by Axioms (III)b.

Now the axioms of type (II) grant that an iteration \(\mathcal{J}\) of \(N_0\) can be defined so to extend the partial \(\in\)-isomorphism \(\Theta\) so far defined between the multi-sorted structures \((H_{\omega_2}^V, N_0; \in | H_{\omega_2}, \in | N_0)\) and 
\[ \{\{\check{x} : x \in H_{\omega_2}^V\}, \{\check{c}_{a,0} : n \in \omega\}, \in_{N_1}^V, \in_{N_0}\} \]
to a full isomorphism \(\Theta : \mathcal{X}(\mathcal{J}, r, f, T) \rightarrow N\) with 
\[ \mathcal{J} = \{j_{\alpha\beta} : N_\alpha \rightarrow N_\beta : \alpha \leq \beta < \omega_1^V\} \]
the unique iteration of \(N_0\) of length \(\omega_1^V\) such that \(j_{\omega_1}(A \cap \omega_1^{N_0}) = A\).

Specifically these axioms impose inductively the following for all \(\alpha < \omega_1^V\):

- \(\{c_{j,\alpha}^N : j \in \omega\}, \in_{M_\alpha}^N\) is isomorphic to \((N_\alpha, \in)\),
- The Mostowski collapsing map of \(\{c_{\cdot,\alpha}^N : j \in \omega\} \) onto \(N_\alpha\), maps \(\{c^N : \mathcal{N} \models G_\alpha(c)\}\) onto some \(G_\alpha\) which is \(N_\alpha\)-generic for \(\beta(\omega_1, \mathcal{N})^{N_\alpha}\).

This occurs because of Axioms (II)b, (II)c holding in \(\mathcal{M}\). Using the fact that there is at most one iteration in \(V[G]\) satisfying the constraints given by these axioms and using [3, Lemma 2.7] we conclude that \(\Theta\) extends to a full isomorphism of the \(L^0\)-structure 
\[ (N_i : i \leq \omega_1^V, \{r, f\} : \in_{N_i}, i \leq \omega_1^V, G_i : i < \omega_1^V, j_{\alpha\beta} : \alpha \leq \beta \leq \omega_1^V, \text{Sat}_{N_\alpha} : \alpha \leq \omega_1^V) \]
with \((\langle S \rangle^V, \langle R \rangle \setminus \{\check{T}, \text{Cod}^*, \text{br}_T\})^{N_\alpha}\). Now the structure given by (2) uniquely determines the iteration 
\[ \mathcal{J} = \{j_{\alpha\beta} : N_\alpha \rightarrow N_\beta : \alpha \leq \beta \leq \omega_1^V\} \].

Finally use the type (III) axioms to get that 
- \(A \cap \omega_1^{N_0} \in N_0\) and \(j_{\omega_1}(A \cap \omega_1^{N_0}) = A\) (by Axioms (III)b and (III)c),
- \(H_{\omega_2}^V \subseteq N_{\omega_1}^V\) (by Axiom (III)d),
- \(\text{NS}_{\omega_1}^V = \text{NS}_{\omega_1}^{N_{\omega_1}^V} \cap V\) (by Axiom (III)e).

This shows that \((\mathcal{J}, r, f, T)\) is a semantic certificate for \(A, D\) as witnessed by \(T\) in \(V[G]\). \(\Box\)

**Remark 3.22.** In this proof we use crucially the fact that a semantic certificate existing in \(V[G]\) is witnessed by a branch through \(T\). Notice that in principle \(\check{N}_{\alpha}\) could be such that \(\check{B}^N\) (whatever definition one may try to give to \(\check{B}^N\)) and \(B^V[G]\) may not have much in common. However this is not the case for the properties which can be described by \(L^0\)-sentences in parameter \(T\). This is one of the reasons to define the tree \(T\).

Note also that we forgot to add the extensionality axiom for the interpretation of the \(\in\)-relation in the above list; so either one adds it, or one oberves that for the models of \(\Sigma^0_1\) the domain is exactly given by the interpretation of the constants, hence extensionality comes for free since the axioms grants that extensionality holds for the image under the transitive collapse of the \(\in_{\mathcal{N}}\)-relation and this map is injective.

**Definition 3.23.** Let \(\mathcal{A}\) be the family of admissible assignments according to Def. 3.17. \(P_0^*\) is \(P_{\mathcal{A}}\).

The key consequence of Fact 3.21, Lemma 3.10, and Fact 3.14 is the following:
Lemma 3.24. Given \( H \) \( V \)-generic for \( P_0^* \), \( \mathcal{M}_H \) models \( \Sigma^0 \).

Therefore \( \mathcal{M}_H \) is well founded and its unique isomorphism with \( X(J_H,r_H,F_H,T) \) gives a uniquely defined semantic certificate \( (J_H,r_H,f_H,T) \) for \( A,D \).

Moreover the map \( \nu_H \colon c \mapsto \pi_{\alpha,H}(\langle c \rangle_H) \) (where \( \pi_{\alpha,H} : N^\mathcal{M}_H \rightarrow X(J_H,r_H,F_H,T) \) is the Mostowski collapse of \( e_{\mathcal{N}_\alpha} \)) is an admissible interpretation on the constants of \( L^0 \) and can be naturally extended to a full interpretation of \( L^0 \) by mapping \( \nu_H(X) = \nu_H[X^\mathcal{M}_H] \) for \( X \) predicate symbol of \( L^0 \).

Proof. By Lemma 3.10 and Fact 3.14, \( \mathcal{M}_H \) realizes all sentences of type \( \bigwedge \bigvee \) holding in \( \Sigma_\nu \) for all admissible \( \nu \). \( \Sigma^0 \) is a subset of this family of sentences by Fact 3.21. Hence \( \mathcal{M}_H \) models \( \Sigma^0 \).

The remaining items of the Lemma are almost self-evident. \( \square \)

By Lemma 2.10 \( P_0^* \) is non-empty; the problem is to check whether it is also stationary set preserving. This natural question has a negative answer since \( P_0^* \) has size \( \kappa_2 \) and Woodin has shown that \( \text{MM}^+(2^{\aleph_0}) \) is consistent with the negation of \((*)\) (cfr. [7, Thm. 10.90]). To sidestep this difficulty, Asperò and Schindler are led to the definition of the sequence of forcings \( \{ P_\alpha : \alpha \leq \kappa \} \), and we also follow their proof-pattern.

3.3. The forcing \( P_\kappa \). We fix (or already have fixed) in \( V \):

Notation 3.25.

\begin{itemize}
  \item \( \kappa > \omega_2 \) regular and such that \( \dot{\omega}_\kappa \) holds.
  \item \( T \) tree on \( (2 \times \omega^\kappa)^{<\omega} \), such that there are semantic certificates \( (J,r,f,T) \) for \( A,D \) in any generic extension \( V[G] \) of \( V \) by \( \text{Coll}(\omega,\omega_2) \).
  \item \( C \subseteq \kappa \) club and \( \{ Q_\alpha,A_\alpha : \alpha \in \kappa \} \) a \( \dot{\omega}_\kappa \) sequence such that:
    \begin{enumerate}
      \item For all \( \eta \in C \)
        \[ (Q_\eta,\in,T,R \cap Q_\eta) \prec (H_\kappa,\in,T,R), \]
        where \( R \) is a fixed well ordering of \( H_\kappa \) in type \( \kappa \).
      \item For all \( \eta \) limit point of \( C \)
        \[ Q_\eta = \bigcup_{\alpha \in C \cap \eta} Q_\alpha. \]
    \end{enumerate}
  \item For any \( Y \subseteq H_\kappa \) there are stationarily many \( \eta < \kappa \) such that \( Y \cap Q_\eta = A_\eta \).
\end{itemize}

The forcings \( \{ P_\alpha : \alpha \in C \cup \{ \kappa \} \} \) are obtained from \( P_0^* \) adding new constant and predicate symbols so to have a stronger control on the type of structures generated by their \( V \)-generic filters. The key idea is to incorporate inside the extended language the names of the countable elementary substructures of \( V \) which can be used to seal a \( P_\kappa \)-name for a club. Specifically we will add to \( L^0 \) a sort \( \dot{Q} \) to denote (a subset of) \( H_\kappa \) and unary predicates \( \{ \dot{X}_i : i < \omega^\kappa \} \) in this new sort which will be used to ensure the following:

\begin{enumerate}
  \item Whenever \( H \) is \( V \)-generic for the forcing \( P_\kappa \), for “almost” all\(^\text{21}\) \( \eta \in C \) and \( \alpha < \omega^\kappa \):
    \begin{itemize}
      \item \( \dot{X}_\alpha = (\dot{X}_\alpha)_H \prec (Q_\eta,\in,P_\eta,A_\eta); \)
      \item \( H \cap P_\eta \) is \( X_\alpha \)-generic\(^\text{22}\) for \( P_\eta \);
      \item \( X_\alpha \cap \omega^\kappa = \alpha \).
    \end{itemize}
\end{enumerate}

\(^\text{21}\)More precisely for any \( S^* \subseteq \kappa \) stationary subset of \( \kappa \) in \( V \) and any \( S \subseteq \omega^\kappa \) stationary subset of \( \omega^\kappa \) in \( V \), one can find \( \eta \in S^* \) and \( \alpha < \omega^\kappa \) as prescribed.

\(^\text{22}\)It meets inside \( X_\alpha \) all dense sets of \( P_\eta \) definable in \( X_\alpha \).
We will also ensure that the layering of the posets \( P_\eta \) is continuous at limit stages, and that the \( P_\eta \) are contained in \( Q_\eta \) for all \( \eta \in C \), so to get that
\[
(4) \quad (Q_\eta, \in, P_\eta, A_\eta) \prec (H_\kappa, \in, P_\kappa, X),
\]
holds in \( V \) for stationarily many \( \eta \) and any \( X \subseteq H_\kappa \).

Now assume \( S \) is a stationary subset of \( \omega_1 \) in \( V \) and \( \dot{C} \) is a \( P_\kappa \)-name for a club. Let \( H \) be \( V \)-generic for \( P_\kappa \). By 3 and 4, we can find \( \eta \in C \) and \( \alpha < \omega_1^V \) so that \( \dot{C} \cap Q_\eta = A_\eta \) is a \( P_\eta \)-name for a club and
\[
X_\alpha = (\dot{X}_\alpha)_H \prec (Q_\eta, \in, P_\eta, A_\eta) \prec (H_\kappa, \in, P_\kappa, \dot{C}),
\]
\( \alpha = X_\alpha \cap \omega_1^V \in S \). \( H \cap P_\eta \) is \( X_\alpha \)-generic for \( P_\eta \). Then we should expect that \( \dot{C}_H \) will have \( \alpha \in S \) as a limit point (note that \( (\dot{X}_\alpha)_H \) will be countable in \( V[H] \) and will not exist in \( V \)). This will grant that \( S \cap \dot{C}_H \) is non-empty. By repeating this argument for all stationary subsets of \( \omega_1^V \) in \( V \) and all \( P_\kappa \)-names for a club subset of \( \omega_1^V \), we will get that \( P_\kappa \) is stationary set preserving. On the other hand the forcing \( P_\kappa \) will also add a semantic certificate for \( A, D \) as witnessed by \( T \). This allows to infer, combining the information so far given, that \( \text{MM}^{++} \) instantiated for \( P_\kappa \) can be used to assert that \( G_A \cap D \) is non-empty.

We are now going to describe close approximations to the generic objects added by the various \( P_\eta \); then we will define the signature which characterizes these generic objects as models of a standard theory of \( \bigwedge \setminus \Delta \)-type for \( L_1^\kappa \). The forcings \( P_\eta \) will be of the form \( P_{W_\eta} \) for suitably chosen families of generic assignments \( W_\eta \).

We assume and recall the following facts, notations and conventions:

**Notation 3.26.** Given a weak semantic certificate \((\mathcal{J}, r, f, T)\) for \( A, D \), we let \( X_{(J,r,f,T)} \) be the set:
\[
\left( \bigcup_{i \leq \omega_1^V} N_\alpha \right) \cup \{ N_\alpha : \alpha \leq \omega_1^V \} \cup \{ j_{\alpha \beta} : \alpha \leq \beta \leq \omega_1^V \} \cup \{ G_\alpha : \alpha < \omega_1 \} \cup \{ r, f \},
\]
\( X_{(J,r,f,T)} \) be the structure:
\[
\langle N_\alpha : \alpha \leq \omega_1^V, \{ r, f \} ; \in \rangle N_\alpha, \text{Sat}_N_\alpha : \alpha \leq \omega_1^V, G_\alpha : \alpha < \omega_1^V, j_{\alpha \beta} : \alpha \leq \beta \leq \omega_1^V, \in^2 \rangle T, \text{br}_T, \text{Cod}^*.\]

Recall that we have fixed in \( V \) the diamond sequence \( \{ Q_\lambda : \lambda \in C \} \) and the witnessing tree \( T \).

**Definition 3.27.** For \( \lambda \in C \cup \{ \kappa \} \) a tuple
\[
(\mathcal{J}, r, f, F, K)
\]
existing in some transitive ZFC-model \( W \supseteq Q_\lambda \) is a weak \( \lambda \)-precertificate (relative to \( V \)) if

- \((\mathcal{J}, r, f, T)\) is a weak semantic certificate for \( A \) (relative to \( V \)) existing in \( W \) as witnessed by \( T \) as in Def. 2.6.
- \( K \subseteq \omega_1^V \).
- \( F \) a function with domain \( \omega_1^V \) defined by
\[
F : \omega_1^V \to C \cap \lambda \times \mathcal{P}(Q_\lambda)^W \times Q_\lambda
\]
\[
\alpha \mapsto \langle \lambda_\alpha, X_\alpha, Z_\alpha \rangle
\]
with \( \langle \lambda_\alpha, X_\alpha, Z_\alpha \rangle = (F_0(\alpha), F_1(\alpha), F_2(\alpha)) \) and:
- \( F \upharpoonright (\omega_1^V \setminus K) \) constantly with value \( \langle 0, 1, 0 \rangle \).
- \( F_0 \upharpoonright K \) and \( F_1 \upharpoonright K \) being \( \subseteq \)-increasing,
- for all \( \alpha \in K \), \( F_2(\alpha) = Z_\alpha \subseteq Q_{F_0(\alpha)} \) is in \( Q_\lambda \) and \( F_1(\alpha) = X_\alpha \) is the domain of an elementary substructure of
\[
(Q_{F_0(\alpha)}, \in, F_2(\alpha), A_{F_0(\alpha)})
\]
\( \alpha = X_\alpha \cap \omega_1^V \).

- \((J, r, f, F, K)\) is a \(\lambda\)-precertificate for \(A, Q_\lambda\) (relative to \(V\)) if it exists in a generic extension of \(V\).

We denote by \(X^\lambda_{(J, r, f, F, K)}\) the set
\[
X_{(J, r, f, F, K)} \cup \{K, F_0, Q_\lambda\} \cup \{F_1(\alpha), F_2(\alpha) : \alpha \in K\}.
\]
and by \(X^\lambda_{(J, r, f, F, K)}\) the multi-sorted structure obtained extending \(X_{(J, r, f, F, K)}\) with the sort \(Q_\lambda\), the constants \(\{Z_\alpha : \alpha < \omega_1^V\}\) of sort \(Q_\lambda\), the binary predicates \(F_0, \in \| Q_\lambda\) both of type \((Q_\lambda, Q_\lambda)\), the unary predicates \(K, \{X_\alpha : \alpha < \omega_1^V\}\) each of type \(Q_\lambda\).

\(X^\lambda_{(J, r, f, F, K)}\) is the smallest set existing in \(W\) and containing all the information needed to reconstruct the weak semantic certificate \((J, r, f, T_\lambda)\) for \(A, D\), the function \(F\), the selected set \(K\). \(X^\lambda_{(J, r, f, F, K)}\) is the multi-sorted structure which will encode this information using multi-sorted logic.

**Remark 3.28.** Note that letting
\[
J = \{j_{\alpha \beta} : N_\alpha \rightarrow N_\beta : \alpha \leq \beta \leq \omega_1^V\},
\]
and \((G_i : i < \omega_1^V)\) list the \(N_i\)-generic ultrafilters associated to \(J\),

- The top elements in the \(\varepsilon\)-graph of \(X^\lambda_{(J, r, f, F, K)}\) are exactly the elements of the following set:
\[
\{r, f, K, F_0\} \cup \{X_\alpha : \alpha < \omega_1^V\} \cup \{N_i : i \leq \omega_1^V\} \cup J \cup \{G_\alpha : \alpha < \omega_1^V\}.
\]

- \(X^\lambda_{(J, r, f, F, K)}\) is not transitive for the same reason for which transitivity fails for \(X_{(J, r, f, T_\lambda)}\), but the new sorts, constants, and predicates denote elements or subsets of \(Q_\lambda\), hence they will not have an \(\varepsilon\)-ill-founded extension in models of \(\Sigma^0\) extended with axioms granting that the sort \(Q_\lambda\) gets interpreted by an isomorphic copy itself.

Note also that the extra requirement imposed on a weak \(\lambda\)-precertificate are expressible by \(\Delta_0\)-properties in parameters \(C \cap \lambda, \omega_1^V, 0, \in \cap Q_\lambda, F_0, K, \{X_\alpha, Z_\alpha : \alpha < \omega_1^V\}\), i.e.:

- (i) \(K \subseteq \omega_1^V\),

- (ii)
\[
(F_0 \text{ is a function}) \land (\text{dom}(F_0) = \omega_1^V) \land (\text{ran} F_0 \subseteq C \cap \lambda \cup \{0\}) \land
\]
\[
\land \forall \alpha \in \beta \in \omega_1^V \ [(\alpha \in K \land \beta \in K) \rightarrow (X_\alpha \subseteq X_\beta \land F_0(\alpha) \in F_0(\beta))]
\]

- (iii) \(\forall \alpha \in \omega_1^V \setminus K \ [(X_\alpha = Z_\alpha = 1) \land (F_0(\alpha) = 0)]\),

- (iv) \(\forall \alpha \in K \ [(X_\alpha \subseteq Q_{F_0(\alpha)}) \land (X_\alpha \cap \omega_1^V = \alpha) \land (Z_\alpha \subseteq Q_{F_0(\alpha)}) \land (Z_\alpha \in Q_\lambda)]\),

- (v)
\[
\forall \alpha \in K \forall n, m \in \omega \forall \bar{z} \in X_\alpha^{\omega} \ [
\gamma (X_\alpha, \in, Z_\alpha \cap X_\alpha, A_{F_0(\alpha)\cap X_\alpha}) \models \phi_n(\bar{z}) \leftrightarrow \gamma (Q_{F_0(\alpha)}, \in, Z_\alpha, A_{F_0(\alpha)}) \models \phi_n(\bar{z})].
\]

Combining these observations with those in Remark 2.9, we get that all the conditions needed to check whether \((J, r, f, F, K)\) is a weak \(\lambda\)-precertificate are expressible by provably \(\Delta_1\)-properties in \((H^W, \tau^W)\) for every \(W\) transitive model of \(ZFC\) in which all the relevant parameters are countable.

---

23 \(X_\alpha\) will not be in \(V\), and will be countable in \(V[G]\). In the right set up (i.e. when the posets \(\{P_\beta : \beta \in C \cap \lambda_\alpha\}\) have been defined), \(Z_\alpha\) will be assigned to the poset \(P_\lambda\). The very definition of \(P_\lambda\) is done by a recursion based on the notion of \(\lambda\)-precertificate and using the sequence \(\{P_\beta : \beta \in C \cap \lambda_\alpha\}\), in particular we cannot for the moment assign \(Z_\alpha\) to its canonical intended meaning. Hence we let \(Z_\alpha\) range over all possible elements of \(Q_\lambda\).
Definition 3.29. Let $\mathcal{L}^1$ be the extension of $\mathcal{L}^0$ (given in Def. 3.17) adding the following sort, constants and predicates:

Sort:

(n) A sort $\dot{Q}$ to denote the element of $Q_\lambda$ for some $\lambda \in C \cup \{\kappa\}$.

Constants:

(o) symbols $\dot{x}$ of sort $\dot{Q}$ for each $x \in H_\kappa$ to be interpreted by $x$.

(p) symbols $\dot{c}_{n,\alpha}$ of sort $\dot{Q}$ for each $n < \omega, \alpha < \omega^V_1$ to be interpreted as the elements of the various $X_\alpha$.

(q) symbols $\dot{Z}_\alpha$ of sort $\dot{Q}$ for each $\alpha < \omega_1^V$ to be interpreted as the sets $F_2(\alpha)$.

Predicates:

(r) a binary predicate symbol $\dot{\varepsilon}_\dot{Q}$ of type $(\dot{Q}, \dot{Q})$ to be interpreted by the $\varepsilon$-relation restricted to the interpretation of $Q$,

(s) a unary predicate symbol $\dot{K}$ of type $\dot{Q}$ to be interpreted by the selected subset of $\omega_1^V$,

(t) a binary predicate symbol $\dot{F}_0$ of type $(\dot{Q}, \dot{Q})$ to be interpreted by the selected map $\alpha \mapsto \lambda_\alpha$ from $\omega_1^V$ to $C$,

(u) unary predicate symbols $\dot{X}_\alpha$ of type $\dot{Q}$ for each $\alpha < \omega_1$ to be interpreted by the sets $F_1(\alpha)$,

(v) Let $\{\in, U_0, U_1\}$ be the signature with $U_0, U_1$ unary predicate symbols; we introduce the satisfaction predicates $\text{Sat}_{i,\alpha}(\psi(x_1, \ldots, x_m)^\gamma)$ of type $(\dot{Q}, \dot{Q})$ with $i = 0, 1$ and $\alpha < \omega_1^V$ to be represented by the pairs $(\psi^\gamma, (a_1, \ldots, a_m))$ with $\psi(x_1, \ldots, x_m)$ an $\{\in, U_0, U_1\}$-formula in displayed free variables and $(a_1, \ldots, a_m) \in H^\kappa_1$ such that for all $\alpha < \omega_1^V$:

(i) $\text{Sat}_{0,\alpha}(\psi(a_1, \ldots, a_m)^\gamma)$ holds if and only if $\alpha \in K, a_1, \ldots, a_m \in X_\alpha$ and

\[ (X_\alpha, \in, Z_\alpha \cap X_\alpha, A_{F_0(\alpha)} \cap X_\alpha) \models \psi(x_1, \ldots, x_m)[x_1/a_1, \ldots, x_m/a_m]. \]

(ii) $\text{Sat}_{1,\alpha}(\psi(a_1, \ldots, a_m)^\gamma)$ holds if and only if $\alpha \in K, F_0(\alpha) = \eta, a_1, \ldots, a_m \in Q_\eta$, and

\[ (Q_\eta, \in, Z_\alpha, A_\eta) \models \psi(x_1, \ldots, x_m)[x_1/a_1, \ldots, x_m/a_m]. \]

For $\lambda \in C \cup \{\kappa\}$ $\mathcal{L}^1_\lambda$ restricts $\mathcal{L}^1$ omitting the constant symbols $\dot{x}$ for any $x \notin Q_\lambda$, so that $\mathcal{L}^1_\kappa$ is $\mathcal{L}^1$.

$\mathcal{R}^1_\lambda$ denotes the relations symbols of $\mathcal{L}^1_\lambda$ and $\mathcal{K}^1_\lambda$ its constant symbols.

$\Delta_\lambda$ the family of atomic sentences of $\mathcal{L}^1_\lambda$.

Note that (with a slight abuse of notation, see Warning 3.19) $\mathcal{L}^1_\lambda(\mathcal{L}, r, f, F, K)$ can be considered an $\mathcal{R}^1_\lambda$-structure whenever $(\mathcal{J}, r, f, F, K)$ is a $\lambda$-precertificate.

Notation 3.30. From now on we assume $\mathcal{L}^1_\lambda$ being a definable class in $(H_\kappa, \in, R)$. We may (and will) also assume that $\mathcal{L}^1_\lambda \subseteq Q_\lambda$ is a definable class in $(Q_\lambda, \in, R \cap Q_\lambda)$ for all $\lambda \in C$.

Definition 3.31.

- Given a $\lambda$-precertificate $(\mathcal{J}, r, f, F, K)$ existing in $V[G]$, a function

\[ \nu : \mathcal{L}^1_\lambda \rightarrow X^1_{(\mathcal{J}, r, f, F, K)} \]

is a $\lambda$-admissible interpretation for $(\mathcal{J}, r, f, F, K)$ if:

- $\nu \upharpoonright \mathcal{L}^0$ respects the interpretation of the symbols of $\mathcal{L}^0$ according to the prescription set forth in Def. 3.18;

- $\nu$ respects the following assignments to the symbols of $\mathcal{L}^1 \setminus \mathcal{L}^0$:

  (n) $\nu(\dot{Q}) = Q_\lambda$,

  (o) $\nu(\dot{x}) = x$ for all $x \in Q_\lambda$,
3.9 and 3.31 Def. 3.31 (v)(ii)

\[ \text{Assume Theorem 3.33.} \]

\[ \text{We define by transfinite recursion on } \alpha \text{ occurring in a } \lambda\text{-precertificate to admit generic filters for larger and larger fragments of } P_\lambda \text{ by letting } K \text{ and } F_0 \text{ grow in size.} \]

**Definition 3.32.** We define by transfinite recursion on \( \alpha \in C \cup \{ \kappa \} \) the forcings \( P_\alpha \) and the notions of (weak) \( P_\alpha \)-certificate \((J,r,f,F,K)\) and (weak) \( P_\alpha \)-witness \( \nu \) for the certificate \((J,r,f,F,K)\) as follows:

(A) \( P_{\min(C)} = P_{\max(C)} \) according to\(^2^4\) Def. 3.39 and 3.31.

(B) A weak \( P_{\min(C)} \)-certificate is a weak \( \min(C) \)-precertificate, and a weak \( P_{\min(C)} \)-witness for the \( P_{\min(C)} \)-certificate is a \( \min(C) \)-admissible interpretation according to Def. 3.31.

(C) For \( \eta \in C \cup \{ \kappa \} \) a weak \( \eta \)-precertificate \((J,r,f,F,K)\) existing in \( W \) is a weak \( P_\eta \)-certificate, and a weak \( \eta \)-admissible \( \nu : L^1_\eta \to X^\eta_0(J,r,f,F,K) \) existing in \( W \) is a weak \( P_\eta \)-witness for \((J,r,f,F,K)\) if the following holds for all \( \alpha \in K \):

(i) \[
F : \omega^Y_1 \to \eta \times \mathcal{P}(Q_\eta)^W \times Q_\eta
\]
\[
\alpha \mapsto \langle \lambda_\alpha, X_\alpha, Z_\alpha \rangle = \langle F_0(\alpha), F_1(\alpha), F_2(\alpha) \rangle
\]
with \( P_{\lambda_\alpha} = Z_\alpha \).

(ii) \[
E \cap X_\alpha \cap [\Sigma_\nu]^{<\omega} \neq \emptyset
\]
for any dense subset \( E \) of \( P_{\lambda_\alpha} \) definable in
\[
(Q_{\lambda_\alpha}, \in, P_{\lambda_\alpha}, A_{\lambda_\alpha})
\]
using parameters in \( X_\alpha \).

(D) A \( P_\eta \)-certificate (respectively a \( P_\eta \)-witness) is a weak \( P_\eta \)-certificate (respectively a weak \( P_\eta \)-witness) existing in a generic extension of \( V \).

(E) For \( \eta \in C \cup \{ \kappa \} \), \( W_\eta \) is the family of \( P_\eta \)-witnesses and \( P_\eta \) is \( P_{W_\eta} \) according to Def. 3.39.

We can now state in precise terms the main technical result of this paper:

**Theorem 3.33.** Assume \( H \) is \( V \)-generic for \( P_r \). Let
\[
J_H = \{ j_{\alpha\beta} : \mathcal{N}_\alpha^H \to \mathcal{N}_\beta^H : \alpha \leq \beta \leq \omega^Y_1 \}
\]
be the iteration induced by \( H \) and accordingly define \( r_H, f_H, T \). Then \((J_H, r_H, f_H, T)\) is a semantic certificate for \( A, D \) and for all \( S \in \mathcal{P}(\omega_1)^{\mathcal{N}_1} \)
\[
(N_{\omega_1}, \in) \models S \text{ is stationary} \iff (V[H], \in) \models S \text{ is stationary.}
\]

\(^2^4\)By the requirements set forth for \( K \) in Def. 3.27, the \( \min(C) \)-precertificate witnessing \( p \in P_{\min(C)} \) always assigns \( \emptyset \) to \( K \).
In particular (since $H_{\omega_2}^V \subseteq N_{\omega_1}$ and $\text{NS}^V = \text{NS}^{N_{\omega_1}} \cap H_{\omega_2}^V$) $P_\kappa$ is stationary set preserving.

The key to the proof of Thm. 3.33 will be a combination of straightforward generalizations to the various $P_\lambda$ of the results on forcings induced by consistency properties which we already obtained for $P_\alpha^\ast$, and of a sophisticated variation of Lemma 2.10 which will be used to run a master condition argument in the style of those one uses to establish the properness of a given poset.

Let us start outlining basic properties of the various forcings $P_\lambda$.

**Remark 3.34.**

(i) The key condition for the various $P_\eta$ is (C).

(ii) The definition of the sequence $\{P_\eta : \eta \in C \cup \{\kappa\}\}$ is first order because “existence in some generic extension” is first order definable in $(V, \in)$ (while “existence in a transitive outer model of ZFC” is apparently not). On the other hand once we know what $P_\eta$ is in $V$, it makes perfect sense to check whether in some outer transitive model $W$ a certain tuple is a weak $P_\eta$-certificate or $P_\eta$-witness.

(iii) $\{P_\eta : \eta \in C \cup \{\kappa\}\}$ can be defined in any sufficiently correct $\in$-structure containing $\langle H_\kappa, R, T, C, \dot{\kappa}\rangle$.

More precisely this recursive definition can be carried in the structure $\langle H_\kappa, \in, R, T, C, \dot{\kappa}\rangle$.

It will be convenient to relativize this definition to $\bar{M}$ where $\bar{k} : V \rightarrow \bar{M}$ is a generic embedding existing in a generic extension of $V$. In particular it makes sense to check in outer models of $\bar{M}$ whether there are weak $\bar{k}(P_\eta)$-witnesses $\nu$, and weak $\bar{k}(P_\eta)$-certificates $(\mathcal{J}, r, f, F, K)$ with $(\mathcal{J}, r, f, \bar{k}(T))$ a weak semantic certificate for $\bar{k}(A)$. This will occur in the statement of Claim 2 to follow.

(iv) Each $P_\eta$ is a subset of $[\Delta_\eta]^{<\omega} \subseteq Q_\eta$.

(v) $\nu : L_\lambda^\eta \rightarrow X(\mathcal{J}, r, f, F, K)$ is a weak $P_\lambda$-witness is expressible by a provably $\Delta_1$-property $\psi(\nu, A, \{Q_\eta : \eta \in C \cap \lambda + 1\}, T, \{Q_\eta, P_\eta : \eta \in C \cap \lambda\}, \nu)$ in $\tau_{ST}$: it adds to the request that the target of $\nu$ is a weak $\lambda$-precertificate, and that $\nu$ is a $\lambda$-assignment, formulae for Condition (C) given by

$$\forall \alpha \in K (F_2(\alpha) = P_{\bar{F}_0(\alpha)})$$

and

$$\forall \alpha \in K \forall \eta \in (C \cap \lambda) \forall \exists \in X_\alpha^{<\omega} \exists \bar{z} \in X_\alpha^{<\omega}$$

$$[(F_0(\alpha) = \eta \wedge [(Q_\eta, \in, P_\eta, A_\eta) \models \forall x \in P_\eta \exists y \in P_\eta (x \leq y \wedge \psi(y, \bar{z}))]])$$

$$\rightarrow$$

$$\exists \bar{z} \in ([\Sigma_\nu]^{<\omega} \cap X_\alpha) ((Q_\eta, \in, P_\eta, A_\eta) \models \psi(\bar{z}, \bar{z}))$$

(vi) For all $\lambda < \eta \in C \cup \{\kappa\}$,

(a) $P_\lambda = P_\eta \cap Q_\lambda$ holds:

- If $\nu$ is a weak $P_\lambda$-witness for the weak $P_\lambda$-certificate $(\mathcal{J}, r, f, F, K)$, then it is easily checked that $\nu^*$ is a weak $P_\eta$-witness (where $\nu^*(\bar{Q}) = Q_\eta$, $\nu^*(\in_\bar{Q}) = \in_\bar{Q}$, and the other assignments can be left unchanged).

- If $\nu$ is a weak $P_\eta$-witness and $(\mathcal{J}, r, f, F, K)$ its associated weak $P_\eta$-certificate, letting $\gamma = \sup \{\alpha \in K : F_0(\alpha) < \lambda\}$, $K^* = K \cap \gamma$, the map $F^*$ such that $F^* \upharpoonright K^* = F \upharpoonright K^*$ and defined by $\alpha \mapsto \langle 0, 1, 1 \rangle$ for all other $\alpha < \omega^V_1$ is such that $(\mathcal{J}, r, f, F^*, K^*)$ is a weak $P_\lambda$-certificate; moreover letting $\nu^*(c) = \nu(c)$ for all $c$ not among $\{\bar{F}_0, \bar{K}, \bar{X}_\alpha, \bar{Z}_\alpha : \alpha < \omega^V_1\}$, $\nu^*(\bar{Q}) = Q_\lambda \nu^*(\bar{K}) = K \cap \gamma$, $\nu^*(\bar{F}_0) = F_0^\ast$, $\nu^*(\bar{X}_\alpha) = 0$ and $\nu^*(\bar{Z}_\alpha) = 0$
if $\alpha \notin K^*$, $\nu^*(\tilde{X}_\alpha) = \nu(\tilde{X}_\alpha)$ and $\nu^*(\tilde{Z}_\alpha) = \nu(\tilde{Z}_\alpha)$ for $\alpha \in K \cap \gamma$, we get that $\nu^*$ is a weak $P_\lambda$-witness.

(vii) We should not expect that $P_\lambda$ is a complete suborder of $P_\kappa$ for $\lambda \in C$. Condition (C) is inserted exactly to infer that in some cases the inclusion of $P_\lambda$ into $P_\kappa$ is sufficiently well behaved; for example we will see that whenever $r_{\alpha,\lambda} = \{(\tilde{a} \in \tilde{K}), (\tilde{F}_0(\tilde{a}) = \tilde{\lambda})\}$ is in $P_\kappa$, the map $r \mapsto r \cup r_{\alpha,\lambda}$ is well defined on a sufficiently large fragment of $P_\lambda$ and can be used to seal a $P_\kappa$-name $\tilde{C}$ for a club at stage $\lambda$ if $A_\lambda = \tilde{C} \cap Q_\lambda$.

All properties for being a $P_\lambda$-certificate (with the exception of Condition (C)(ii) of Def. 3.32) are expressible by $L^1_\lambda$-sentences of $\bigwedge \bigvee$-type, henceforth will be true in the term model given by a generic filter for $P_\lambda$ (which is a maximally $\omega_1$-consistent set of atomic $L^1_\lambda$-sentences). In order to assert that also (C)(ii) is satisfied in the generic term model we use a density argument.

On the other hand (as for $P^*_0$), the $L^1_\lambda$-structures whose domain is a $P_\lambda$-certificate can still be characterized by a $\Sigma^\omega_1$-theory; to do so however we must expand the language adding a predicate symbol for the filter $[\Sigma_\nu \cap \Delta_\lambda]^<\omega$ given by the $P_\lambda$-witness in order to formalize Condition (C)(ii) of Def. 3.32 in the extended signature.

Definition 3.35. $L^1^*_\lambda$ is obtained from $L^1$ adding:

(w) A unary predicate symbol $\tilde{H}$ of type $\tilde{Q}$ to be interpreted by $[\Sigma_\nu \cap \Delta_\lambda]^<\omega$ by any $\lambda$-admissible assignment $\nu$.

$L^1^*_\lambda = L^1_\lambda \cup \{\tilde{H}\}$

Notation 3.36. From now on we identify a $\lambda$-admissible $\nu$ by its restriction to the fragment given by

$L^* = \{\tilde{X}_\alpha, \tilde{Z}_\alpha : n < \omega, \alpha < \omega^*_1\} \cup \{\tilde{K}, \tilde{F}_0, \tilde{r}, \tilde{f}\}$.

We also consider that the $\lambda$-admissible $\nu$ are defined on the whole $L^1^*_\lambda$.

Remark 3.37. It is clear that any $\nu_0$ defined on $L^*$ admits at most one extension to a $\lambda$-admissible $\nu : L^1^*_\lambda \rightarrow X(\mathcal{J}, r, f, F, K)$: if $\nu(\tilde{r}) = r$, $\nu(\tilde{f}) = f$, $(r, f) \in T$, and $\text{Cod}(r) = (N_0, a_0)$, there is at most one iteration $\mathcal{J}$ of $N_0$ such that $j_{\alpha \omega^*_1}(a_0) = A$ such that $(r, f, T)$ is weak semantic certificate for $A$; now there is at most one $\lambda$-precertificate $(\mathcal{J}, r, f, F, K)$ such that $F_0 = \nu(\tilde{F}_0)$, $K = \nu(\tilde{K})$, $F_1(\alpha) = X_\alpha = \nu(\tilde{X}_\alpha)$, $F_2(\alpha) = Z_\alpha$ for all $\alpha < \omega^*_1$.

We now introduce a $\Sigma^\omega_1$-theory whose models are (modulo isomorphism) exactly the $P_\lambda$-witnesses. We will use axioms expressible in the signature $L^1_\lambda$ to capture all the properties characterizing a $\lambda$-precertificate satisfying also Condition (C)(i) of Def. 3.23. We will need the extra predicate symbol $\tilde{H}$ to axiomatize Condition (C)(ii) of Def. 3.23.

Lemma 3.38. Let $\Sigma^\lambda_\lambda$ be the $L^1_\lambda$-theory for $\Sigma^{\omega_1}_\omega$ containing:

- The $L^1_\lambda$-equality axioms and the $L^1_\lambda$-quantifier elimination axioms.
- All axioms of $\Sigma^0$.
- The following list of axioms:
  
  (IV) For all constant symbols $c$ of sort $\tilde{Q}$

  $\bigvee_{x \in Q_\lambda} \tilde{x} = c$.

25But not on all of $P_\lambda$: for example if $\gamma > \alpha$ and if $(\tilde{\gamma} \in \tilde{K})$ is in $r$ then $r \cup r_{\alpha,\lambda}$ is not in $P_\kappa$. 

(b) For all \( x, y \in \mathbb{Q}_\lambda \)
\[
(\bar{x} \in \bar{y}) \text{ if } x \in y,
\]
\[
(\bar{x} \notin \bar{y}) \text{ if } x \notin y.
\]

(c) For all constant symbols \( c \) of sort \( \dot{Q} \)
\[ c \in \dot{K} \rightarrow c \in \dot{\omega}_1^V. \]

(d) For all \( \alpha \in \omega_1^V \)
\[
(\bar{\alpha} \notin \dot{K}) \rightarrow \left[ (\dot{F}_0(\bar{\alpha}) = \bar{0}) \land \left( \bigwedge_{n<\omega} \bar{e}_{n,\alpha} = \bar{0} \right) \land \dot{Z}_\alpha = \bar{0} \right].
\]

(e) For all constant symbols \( c, d, e \) of sort \( \dot{Q} \)
\[
(\dot{F}_0(c) = d \land c \in \dot{K}) \rightarrow \bigvee_{\eta \in C \cap \lambda} (d = \bar{\eta}),
\]
\[
[(\dot{F}_0(c) = d) \land (\dot{F}_0(c) = e)] \rightarrow (d = e),
\]

(f) For all constant symbols \( c \) of sort \( \dot{Q} \) and \( \alpha < \omega_1^V \)
\[ c \in \dot{X}_\alpha \leftrightarrow \bigvee_{n<\omega} \bar{e}_{n,\alpha} = c. \]

(g) For all constant symbols \( c \) of sort \( \dot{Q} \) and \( \alpha < \omega_1^V \)
\[ \bar{\alpha} \in \dot{K} \rightarrow \left[ (c \in \dot{X}_\alpha \land c < \dot{\omega}_1^V) \leftrightarrow c \in \bar{\alpha} \right]. \]

(h) For all constant symbols \( c \) of sort \( \dot{Q} \) and \( \alpha < \beta < \omega_1^V \)
\[ (\bar{\alpha} \in \dot{K} \land \bar{\beta} \in \dot{K}) \rightarrow (c \in \dot{X}_\alpha \rightarrow c \in \dot{X}_\beta). \]

(i) For all constant symbols \( c, d \) of sort \( \dot{Q} \) and \( \alpha < \beta < \omega_1^V \)
\[ \left[ \bar{\alpha} \in \dot{K} \land \bar{\beta} \in \dot{K} \land \dot{F}_0(\bar{\alpha}) = c \land \dot{F}_0(\bar{\beta}) = d \right] \rightarrow c \in d. \]

(j) For \( \alpha < \omega_1^V, \eta < \kappa, \mathcal{R} = \{\in, U_0, U_1\} \) the signature extending \( \{\in\} \) with unary predicates \( U_0, U_1 \), let \( \Psi_{\alpha\eta} \) be the sentence
\[
\bigwedge_{\psi \in \text{Form}_R, n<\omega, i_1, \ldots, i_n<\omega} \left[ \text{Sat}_{\alpha,\eta}(\bar{r}(e_{i_1,\alpha}, \ldots, e_{i_n,\alpha})) \leftrightarrow \text{Sat}_{1,\alpha}(\bar{r}(e_{i_1,\alpha}, \ldots, e_{i_n,\alpha})) \right]
\]
where \( \text{Form}_R \) denotes the set of \( \mathcal{R} \)-formulae.
We add the axioms
\[
\left[ \left( \bar{\alpha} \in \dot{K} \right) \land (\dot{F}_0(\bar{\alpha}) = \bar{\eta}) \right] \rightarrow \Psi_{\alpha\eta}
\]
for all \( \alpha < \omega_1^V, \eta \in C \cap \lambda. \)

(k) For all \( \alpha < \omega_1^V, \eta \in C \cap \lambda \)
\[ (\bar{\alpha} \in \dot{K} \land \dot{F}_0(\bar{\alpha}) = \bar{\eta}) \rightarrow \left[ \left( \bigwedge_{n<\omega} \bigvee_{x \in Q_\eta} \bar{e}_{n,\alpha} = \bar{x} \right) \land \dot{Z}_\alpha = \bar{Z}_\eta \right]. \]

For any atomic sentence \( \psi \in \Delta_\alpha \), let \( x_\psi = \{\psi\} \). \( \Sigma^1_\lambda \) adds to \( \Sigma^1_\lambda \) the axioms:

(V)

(a) for all atomic sentences \( \psi \in \Delta_\lambda \), the axiom
\[ \psi \leftrightarrow (\bar{x}_\psi \in \dot{H}) \]
Lemma 3.39. Let \( n \in \{\in, U_0, U_1\} \) the axiom
\[
\left[ (\hat{\alpha} \in \hat{K}) \land (F_0(\hat{\alpha}) = \hat{\eta}) \land \text{Sat}_{0,\alpha}(\forall x [U_0(x) \rightarrow \exists y (U_0(y) \land x \subseteq y \land \psi(y, c_1, \ldots, c_n))]^\eta) \right] 
\]
\[
\bigvee_{n<\omega} \left[ \text{Sat}_{0,\alpha}(\forall x [\in, c_1, \ldots, c_n]^\eta) \land \in, c_1, \ldots, c_n \in \hat{H} \right]
\]

Then for any \( \lambda \in C \cup \{\kappa\} \):
1. \( \Sigma^1_\lambda \) and \( \Sigma^1_* \) are standard theories of \( \bigwedge \bigvee \) -type. Moreover any model of \( \Sigma^1_\lambda \) is such that \( \in, Q \) and \( \in, \eta \) for \( \alpha < \omega^V_1 \) are interpreted by well-founded extensional relations whose transitive collapses determines a unique \( \lambda \)-precertificate \((J, r, f, F, K)\).

2. Given a \( \lambda \)-admissible \( \nu : \mathcal{L}^\lambda_\lambda \rightarrow X_{(J, r, f, F, K)} \), let \( \nu^* \) extend \( \nu \) to \( \mathcal{L}^\lambda_\lambda \) by letting \( \nu(H) = [\Sigma_\nu \cap \Delta_\lambda]^{<\omega} \). Then \( \Sigma^1_* \subseteq \Sigma^\delta_\nu \), if and only if \( \nu \) is a weak \( P_\lambda \)-witness for some weak \( P_\lambda \)-certificate \((J, r, f, F, K)\).

3. Assume \( N \) is a \( \Sigma_{\omega, \omega}^1 \) model of \( \Sigma^1_* \) existing in some transitive model \( W \), and let \( \pi^*_{N} \) denote the Mostowski collapsing map of the structure \((s^N, \in, s)\) for each sort \( s \) of \( \mathcal{L}^\lambda_\lambda \).

Then the multi-map defined on each \( c \) of sort \( s \) by
\[
\nu_N : c \mapsto \pi^*_N(c^N)
\]

and naturally defined on the predicates\(^{26}\) of \( N \) can be uniquely extended to a weak \( P_\lambda \)-witness for the weak \( P_\lambda \)-certificate \((J, r, f, F, K)\).

Proof. Along the lines of the proof of Fact 3.21, using also Remark 3.37.

1: The key point is that the new sorts, constants, and relation symbols define elements or subsets of the structure \((Q_\lambda, \in)\), hence they cannot produce witnesses to the ill-foundedness of \( \in, \eta \).

2: Left to the reader: the new axioms are exactly added to check that any \( \nu^* \) witnessing them is such that \( \nu^* \upharpoonright \mathcal{L}^\lambda_\lambda \) is a weak \( P_\lambda \)-witness.

3: Left to the reader: same argument as for the previous item.

This is the first non-trivial result for the forcings \( P_\lambda \):

Lemma 3.39. Let \( \mathcal{M}_H \) be the canonical term model for \( \mathcal{L}^\lambda_\lambda \) induced by \( \Sigma_H \) whenever \( H \) is a \( V \)-generic filter \( H \) for \( P_\lambda \) (according to Def. 3.9).

Then \( \mathcal{M}_H \) models \( \Sigma^1_\lambda \) and \( \in, \eta, \pi^*_{\mathcal{M}_H} \) is well-founded and extensional on each sort \( s \) for which \( \in, s \) is a predicate symbol of \( \mathcal{L}^\lambda_\lambda \).

Moreover, let:

- \((J_H, r_H, f_H, F_H, K_H)\) be the precertificate induced by \( H \) via the Mostowski collapse \( \pi^*_{\mathcal{M}_H} \) of each sort \( s \) of \( \mathcal{M}_H \) for which \( \in, s \) is in \( \mathcal{L}^\lambda_\lambda \) with \( F_H(\alpha) = \langle \eta^H, X^H, Z^H \rangle \) for all \( \alpha < \omega^V_1 \),
- \( \nu_H : c \mapsto \pi^*_{\mathcal{M}_H}(\in, \eta, \pi^*_{\mathcal{M}_H} \in, \eta, \pi^*_{\mathcal{M}_H} \eta) \) be the associated admissible assignment naturally extended\(^{27}\) to the relation symbols of \( \mathcal{L}^\lambda_\lambda \).

Then \((J_H, r_H, f_H, F_H, K_H)\) is a \( P_\lambda \)-certificate and \( \nu_H \) is a \( P_\lambda \)-witness such that \( H = [\Sigma_{\nu_H}]^{<\omega} \cap \Delta_\lambda \). Hence the natural extension of \( \mathcal{M}_H \) to an \( \mathcal{L}^\lambda_\lambda \)-structure makes it a model of \( \Sigma^1_* \).

\(^{26}\)I.e. in accordance with Warning 3.8.

\(^{27}\)I.e. in accordance with Warning 3.8.
Proof. By Lemma 3.10 and Fact 3.38 \( \mathcal{M}_H \) is a \( L^1_\lambda \)-structure which models \( \Sigma_1^1 \) whenever \( H \) is \( V \)-generic for \( P_\lambda \), since \( \Sigma_1^1 \) is standard of \( \bigwedge \bigvee \) type, and is contained in \( \Sigma_\nu \) for all \( P_\nu \)-witnesses \( \nu \). By Fact 3.38 the map \( \nu_H : L^1_\lambda \to X_{(j_H,r_H,f_H,F_H,K_H)}^\lambda \) is a \( \lambda \)-admissible interpretation.

We must now check that the natural extension of \( \nu_H \) to a \( L^1_\lambda \)-assignment \( \nu_H^* \) obtained by letting \( \nu_H^*(\hat{H}) = H \) satisfies also the missing axioms of \( \Sigma_1^1 \). For this we cannot use a syntactic argument, since the axioms we must check are not part of the theory \( \Sigma_H \) one uses to define \( \mathcal{M}_H \) and \( \nu_H \) (the symbol \( \hat{H} \) is not part of the signature for the theory \( \Sigma_H \)).

The Axioms (V)a are in \( \Sigma_\nu_H \) almost by definition.

We must also check that Axioms (V)b are also in \( \Sigma_\nu_H \). For this we use a density argument on \( P_\lambda \):

Assume \( \alpha \in K_H \) and \( F^H_0(\alpha) = \eta \) for some \( \eta \in C \cap \lambda \). Let \( D^* \) be a dense subset of \( P_\eta \) definable in \( (Q_\eta , \in , P_\eta , A_\eta) \) with parameters \( a_1 , \ldots , a_n \in X^H_\omega \). Find \( i_1 , \ldots , i_n \) such that \( \hat{a}_j = \hat{e}_{j,\alpha} \in \Sigma_\nu_H \) for all \( j = 1 , \ldots , n \). Then

\[
p_0 = \{ \hat{a}_j = \hat{e}_{j,\alpha} : j = 1 , \ldots , n \} \cup \{ (\hat{F}_0(\hat{\alpha}) = \hat{\eta}) : \hat{\alpha} \in \hat{K} \} \in H
\]

Now consider the set

\[
E = \{ p \leq p_0 : \text{for some } q_0 \subseteq p, q_0 \in D^* \text{ and there is } n_p \text{ such that } (q_0 = e_{n_p,\alpha}) \in p \}
\]

We claim that \( E \in V \) is dense in \( P_\lambda \) below \( p_0 \): Assume \( p \in P_\lambda \) refines \( p_0 \). Then any \( P_\lambda \)-witness \( \nu \) with \( p \subseteq \Sigma_\nu \) and associated \( P_\lambda \)-certificate \( (\mathcal{J} , r) , f , F , K \) with \( F_1(\alpha) = X_\alpha \) is such that there is some \( q_0 \in \Sigma_\nu \) such that \( X_\alpha \in X^\omega \), since \( D^* \) is a dense subset of \( P_\eta \cap Q_\eta \) definable in parameters \( a_1 , \ldots , a_n \in X_\alpha \) in the structure \( (Q_\eta , \in , P_\eta , A_\eta) \), by Condition (C)(ii) applied to \( \nu \).

Since \( \Sigma_1^1 \subseteq \Sigma_\nu \), \( \hat{q}_0 = \hat{e}_{m,\alpha} \in \Sigma_\nu \) for some \( m < \omega \). Now we let \( q = p \cup q_0 \cup \{ (\hat{q}_0 = \hat{e}_{m,\alpha}) \} \).

Then \( q \in E \) refines \( p \).

Therefore \( E \) is dense below \( p_0 \).

Since \( p_0 \in H \) and \( E \) is dense below \( p_0 \), we can find \( p^* \leq p_0 \in E \cap H \). Since \( p^* \subseteq \Sigma_\nu_H \), we conclude that Condition (C)(ii) is satisfied by \( \nu_H \) and \( X_{(j_H,r_H,f_H,F_H,K_H)} \) for \( E \) and \( X^H_\omega \) by \( q_{p^*} \in \Sigma_\nu_H \cap D \cap X^H_\omega \). \( q_{p^*} \) witnesses that \( \mathcal{M}_H \) satisfies the conclusion of Axiom (V)b instantiated for \( D \) and \( \alpha \).

The following is a key observation explaining how \( \diamond_\lambda \) and Condition (C)(ii) can be used for proving that \( P_\kappa \) is stationary set preserving.

**Lemma 3.40.** Assume \( \hat{X} \) is a \( P_\kappa \)-name for an unbounded subset of \( \omega_1^V \). Then there are stationarily many \( \eta < \kappa \) such that \( A_\eta = \hat{X} \cap Q_\lambda \) and

\[
(Q_\eta , \in , P_\eta , A_\eta) \prec (H_\kappa , \in , P_\kappa , \hat{X})
\]

Assume further that for some such \( \eta \), \( r_{a\eta} = \{ \hat{\alpha} \in \hat{K} : \hat{F}_0(\hat{\alpha}) = \hat{\eta} \} \) is a condition in \( P_\kappa \) for some \( \alpha < \omega_1^V \).

Then

\[
r_{a\eta} \Vdash_{P_\kappa} \hat{\alpha} \text{ is a limit point of } \hat{X}.
\]

**Proof.** \( \hat{X} \) is naturally represented by a subset of \( P_\kappa \times \omega_1 \) identifying the pairs \( (p,\alpha) \) such that \( p \Vdash_\kappa \alpha \in \hat{X} \). Hence \( \hat{X} \subseteq H_\kappa \). By \( \diamond_\kappa \) there are stationarily many \( \eta < \kappa \) such that

\[
(Q_\eta , \in , P_\eta , A_\eta) \prec (H_\kappa , \in , P_\kappa , \hat{X}).
\]

Let \( H \) be \( P_\kappa \)-generic with \( r_{a\eta} \in H \). By the previous Lemma we obtain that \( H \) induces a unique \( P_\kappa \)-certificate \( (J_H , r_H , f_H , F_H , K_H) \) and a \( P_\kappa \)-witness \( \nu_H \) associated to \( (J_H , r_H , f_H , F_H , K_H) \) such that \( H = [\Sigma_\nu_H]^{<\omega} \cap \Delta_\kappa \).
Now the sets

\[ D_\xi = \{ p \in P_\kappa : \text{there is } \omega_1^Y > \gamma > \xi (p \Vdash_{P_\kappa} \gamma \in \check{X}) \} \]

are dense in \( P_\kappa \) and definable in parameter \( \xi \) in the structure \((H_\kappa, \in, P_\kappa, \check{X})\) for any \( \xi < \omega_1^Y \).

By assumption

\[(5) \quad (Q_\eta, \in, P_\eta, A_\eta) \prec (H_\kappa, \in, P_\kappa, \check{X}),\]

while (letting \( F_H(\alpha) = \langle \eta, X_\alpha, Z_\alpha \rangle \)) Condition (C) of Def. 3.32 ensures that:

(i) \( Z_\alpha = P_\eta \),
(ii) \( (X_\alpha, \in, Z_\alpha \cap X_\alpha, A_\eta \cap X_\alpha) \prec (Q_\eta, \in, P_\eta, A_\eta) \),
(iii) \( X_\alpha \cap \omega_1^\alpha = \alpha \),
(iv) \( A \cap X_\alpha \cap H \) is non-empty for all \( A \) dense subset of of \( P_\eta \) definable by parameters in \( X_\alpha \).

By 5, we get that \( D_\xi \cap P_\eta \) is a dense subset of \( P_\eta \) definable in \( (Q_\eta, \in, P_\eta, A_\eta) \) by parameter \( \xi < \omega_1^Y \). By (i), (ii), (iii), we get that \( D_\xi \cap P_\eta \) is definable in

\[(X_\alpha, \in, Z_\alpha \cap X_\alpha, A_\eta \cap X_\alpha)\]

for all \( \xi < \alpha \). By (iv) we get that there is some \( q \in H \cap D_\xi \cap X_\alpha \). Now observe that for any \( q \) in this set, the least \( \gamma > \xi \) such that \( q \) forces \( \gamma \in D_\xi \) is such that \( \gamma < \alpha \) (by (ii) and (iii)). This gives that \( \alpha \) is a limit point of \( X_H \) in \( V[H] \). We are done. \( \square \)

We are now ready to complete the proof of Theorem 2.7.

**Proof.** Fix \( \check{C} P_\kappa \)-name for a club subset of \( \omega_1^Y \).

Towards a contradiction assume that for some \( H \) \( V \)-generic for \( P_\kappa \) there is \( S \in N_{\omega_1}^H \) such that

\[ \check{C}_H \cap S = \emptyset, \]

but

\[ (N_{\omega_1}^H, \in) \models S \subseteq \omega_1^Y \text{ is stationary.} \]

Since \( S \in N_{\omega_1}^H \), \( S = \nu_H(\hat{\epsilon}_{j,\omega_1^Y}) \), for some \( j < \omega_1^Y \).

This gives that

\[ \text{Sat}_{N_{\omega_1}^H}(\hat{\epsilon}_{j,\omega_1^Y} \subseteq \omega_1^Y \text{ is stationary}) \in \Sigma_H. \]

Note that the latter is an atomic sentence of \( \mathcal{L}^1 \). Therefore

\[ p = \{ \text{Sat}_{N_{\omega_1}^H}(\hat{\epsilon}_{j,\omega_1^Y} \subseteq \omega_1^Y \text{ is stationary}) \} \in H \]

and some \( p_0 \in H \) refining \( p \) forces

\[ \check{C} \cap \nu_H(\hat{\epsilon}_{j,\omega_1^Y}) = \emptyset. \]

Then for any \( V \)-generic filter \( H \) for \( P_\kappa \) with \( p_0 \in H \), \( V[H] \) models that

\[ \check{C}_H \cap \nu_H(\hat{\epsilon}_{j,\omega_1^Y}) = \emptyset \]

and

\[ (N_{\omega_1}^H, \in) \models \nu_H(\hat{\epsilon}_{j,\omega_1^Y}) \subseteq \omega_1 \text{ is stationary.} \]

Our aim is to find \( r \leq p_0 \) forcing that \( \check{C} \cap \nu_H(\epsilon_{j,\omega_1^Y}) \) is non-empty, thus reaching a contradiction.

By \( \check{D}_\kappa \),

\[ S_{\check{C}} = \{ \lambda < \kappa : A_\lambda = \check{C} \cap Q_\lambda \} \]

is stationary.

Pick \( \lambda \in S_{\check{C}} \) with \( p_0 \in P_\lambda = P_\kappa \cap Q_\lambda \) and

\[ (Q_\lambda, \in, P_\lambda, A_\lambda) \prec (H_\kappa, \in, P_\kappa, \check{C}). \]
Note that for $\alpha < \omega_1^V$ the sets
\[ D_\alpha = \{ p \in P_\kappa : \exists \xi > \alpha (p \Vdash_P \xi \in \mathcal{C}) \} \]
are dense open in $P_\kappa$, hence such that $D_\alpha \cap Q_\lambda$ is dense in $P_\lambda$, and definable in parameter $\alpha$ in the structure $(Q_\lambda, \in, P_\lambda, A_\lambda)$.

Now to find this $r$ we proceed along the lines of the proof of Lemma 2.10 and use Lemma 3.40 in the generic ultrapower $\bar{M}$ to be defined below.

Let $G$ be $V$-generic for $\text{Coll}(\omega, \kappa)$. Now $2^\lambda \leq \kappa$ (by $\diamondsuit_\kappa$), and $p_0 \in P_\lambda = P_\kappa \cap Q_\lambda$ with $P_\lambda$ in $V$ of size $|\lambda| < \kappa$. Hence we can fix in $V[G]$ a $V$-generic filter $H$ for $P_\lambda$ with $p_0 \in H$. By Lemma 3.39, the structure $\mathcal{M}_H$ for $\mathcal{L}_\lambda^1$ induced by $H$ is such that all the relations $\in_s$ are well founded, and the transitive collapse of their domains via the Mostowski map $\pi_H^1$ extend to a multi-map which is an isomorphism of $\mathcal{M}_H$ with the $\mathcal{L}_\lambda^1$-structure
\[ \mathcal{X}(\mathcal{J}_H, r_H, f_H, F_H, K_H) \]
with $(\mathcal{J}_H, r_H, f_H, F_H, K_H)$ a $P_\lambda$-certificate and $\nu_H : c \mapsto \pi_H^1([c]_H)$ for $c$ a constant of sort $s$ a $P_\lambda$-witness. We take note that $(\mathcal{J}_H, r_H, f_H, T)$ is a semantic certificate for $A, D$. To simplify notation we let $\nu_H = \nu$ in what follows.

Let us denote
\[ \mathcal{J}_H = \{ j_{\alpha \beta} : N_\alpha \to N_\beta : \alpha \leq \beta \leq \omega_1^V \} \]

Note that all these objects (i.e. $A, \bar{C}, S, T, \diamondsuit_\kappa, P_\lambda, \nu, \mathcal{J}_H, r_H, F_H, K_H, \ldots$) belong to $\mathcal{H}_{\omega_1^V[G]}$.

Let now $G_0$ be in $V[G]$ an $\mathcal{N}_{\omega_1^V}$-generic filter for $(\mathcal{P}(\omega_1) / \mathcal{N}_{\omega_1^V})^{N_{\omega_1^V}}$ with $S = \nu_H(\dot{c}_{j_{\omega_1^V}}) \in G_0$ ($G_0$ exists because $S$ is stationary in $\mathcal{N}_{\omega_1^V}$). Let $k : N_{\omega_1^V} \to N_{\omega_1}^*$ be the ultrapower embedding induced by $G_0$. This gives that $\omega_1^V \in k(S)$. Extend $k$ in $V[G]$ to an iteration
\[ K = \{ k_{\alpha \beta} : N_\alpha^* \to N_\beta^* : \alpha \leq \beta \leq \omega_1^V[G] = \rho \} \]
of $N_{\omega_1^V} = N_0^*$ with $k = k_{01}$ using the fact that $N_0$ (and therefore also $N_{\omega_1} = N_0^*$) is iterable in $V[G]$.

Note that
\[ \omega_1^V \in k_{0\rho}(S) = k_{0\rho}(\nu_H(\dot{c}_{j_{\omega_1^V}})). \]

Since $\mathcal{H}_{\omega_2} \subseteq N_{\omega_1^V}, \mathcal{N}_{\omega_1^V} = \mathcal{N}_{\omega_1^V}^{N_{\omega_1^V}} \cap V$, and $V$ models $\mathcal{N}_{\omega_1^V}$ is saturated, we can extend uniquely $k_{0\rho} \upharpoonright \mathcal{H}_{\omega_2} \to \text{an elementary} \bar{k} : V \to \bar{M}$ applying [3, Lemma 1.5, Lemma 1.6] to\(^{28}\) $k_{0\rho} \upharpoonright \mathcal{H}_{\omega_2}^V$.

Let us denote by
\[ \bar{\mathcal{J}} = \{ j_{\alpha \beta} : N_\alpha \to N_\beta : \alpha \leq \beta \leq \omega_1^V[G] = \rho \} \]

\(^{28}\)More precisely, these assumptions on $V$ and $\mathcal{N}_{\omega_1^V}$ give that all maximal antichains of $(\mathcal{P}(\omega_1) / \mathcal{N}_{\omega_1^V})^\mathcal{N}_{\omega_1^V}$ are in $\mathcal{N}_{\omega_1^V}$ (hence $G_0 \cap V$ is $V$-generic for $(\mathcal{P}(\omega_1) / \mathcal{N}_{\omega_1^V})^V$). Inductively, letting $\{ G_\alpha : \alpha < \rho \}$ be the family of filters on $(\mathcal{P}(\omega_1) / \mathcal{N}_{\omega_1^V})^{k_0(\mathcal{N}_{\omega_1^V})}$ defining $K$, and assuming the existence of $k_0 : V \to \bar{M}_\alpha$ unique extension to $V$ of $k_0 \cap \mathcal{H}_{\omega_2}^V$, one can check (by elementarity of $k_0$ and of $k_0^\rho$) that
\[ \mathcal{N}_{\omega_1^V}^{\bar{M}_\alpha} = \mathcal{N}_{\omega_1^V}^{k_0(\mathcal{N}_{\omega_1^V})} \cap k_0(\mathcal{H}_{\omega_2}^V), \]
and that $\bar{M}_\alpha$ models $\mathcal{N}_{\omega_1^V}$ is saturated; this gives that all maximal antichains of $(\mathcal{P}(\omega_1) / \mathcal{N}_{\omega_1^V})^{\bar{M}_\alpha}$ are maximal antichains of $(\mathcal{P}(\omega_1) / \mathcal{N}_{\omega_1^V})^{\mathcal{N}_{\omega_1^V}}$; therefore one can also inductively infer that $G_\alpha \cap \bar{M}_\alpha$ is $\bar{M}_\alpha$-generic for $(\mathcal{P}(\omega_1) / \mathcal{N}_{\omega_1^V})^{\bar{M}_\alpha}$. We conclude that $\{ G_\alpha \cap \bar{M}_\alpha : \alpha < \rho \}$ gives rise to a unique iteration
\[ \{ k_{\alpha \beta} : \alpha \leq \beta \leq \rho \} \]
of $V$ such that $k_{0\beta} \cap \mathcal{H}_{\omega_2} = k_{0\beta} \cap \mathcal{H}_{\omega_2}^V$ for all $\beta \leq \rho$. We can set $\bar{k} = k_{0\rho}$. 
the iteration obtained letting \( \bar{j}_{\alpha\beta} = j_{\alpha\beta} \) if \( \beta \leq \omega^V_1, \bar{j}_{\omega^V_1 + \alpha} = k_{\alpha\beta}, \bar{j}_{\omega^V_1} = k_{\alpha\beta} \circ j_{\omega^V_1} \) if \( \beta \geq \omega^V_1 \geq \alpha \).

We can easily check this first property of \( \tilde{J} \) (recall Def. 2.6):

**Claim 1.** \((\tilde{J}, r_H, \tilde{k}[f_H], \tilde{k}(T))\) is in \( V[G] \) a weak semantic certificate for \( \tilde{k}(A) \).

**Proof.** The \( \Delta_1 \)-properties outlined in Remark 2.9 are easily checked for \((\tilde{J}, r_H, \tilde{k}[f_H], \tilde{k}(T))\) using the elementarity of \( \bar{j}_{0\theta} \) and of \( \tilde{k} \).

\( \square \)

Our aim will be to reinforce this conclusion to the following:

**Claim 2.** Let:

- \( f^* = \tilde{k}[f_H] \);
- \( K^* = \tilde{K}_H \cup \{ \omega^V \} \);
- for \( \alpha \in \omega^V_1 \):
  \[
  F_0^*(\alpha) = \tilde{k}((F_H)_0(\alpha)),
  F_1^*(\alpha) = \tilde{k}((F_H)_1(\alpha)),
  F_2^*(\alpha) = \tilde{k}((F_H)_2(\alpha)),
  \]
- \( F_0^*(\omega^V) = \tilde{k}(\lambda),
  F_1^*(\omega^V) = \tilde{k}(Q),
  F_2^*(\omega^V) = \tilde{k}(P) \);
- \( F^*(\beta) = (0, 1, 0) \) for all \( \rho > \beta > \omega_1^V \).

Then in \( V[G] \) there is a \( \tilde{k}(\kappa) \)-assignment

\[
\bar{v} : \tilde{k}(L^1_{\kappa}) \to X(\tilde{J}, r_H, f^*, F^*, K^*)
\]

such that:

- \( \Sigma_\varphi \supseteq \tilde{k}[\cup H] \supseteq \tilde{k}(p_0) \),
- \( (\omega^V_1 \in \tilde{K}, (\tilde{F}_0(\omega^V_1) = \tilde{k}(\lambda)) \in \Sigma_\varphi, \)
- \( \bar{v} \) is a weak \( \tilde{k}(P) \)-witness making \((\tilde{J}, r_H, f^*, F^*, K^*)\) a weak \( \tilde{k}(P) \)-certificate relative to\(^{29} \tilde{M} \).

Suppose we succeed to prove the Claim. Let \( \tilde{G} \) be \( V[G] \)-generic for \( \text{Coll}(\omega, \rho) \). Then all the parameters needed to check the above are in \( H^V_{\omega_1^G} \).

Now

\[
(H^V_{\omega_1^G} / \tilde{M})
\]

is a \( \Sigma_1 \)-substructure of

\[
(H^V_{\omega_1} / \tilde{M})
\]

Therefore it models

- There are a weak \( \tilde{k}(P) \)-certificate \((\tilde{J}^*, r^*, f^*, F^*, K^*)\) and a weak \( \tilde{k}(P) \)-witness \( \bar{v} \) for \((\tilde{J}, r^*, f^*, F^*, K^*)\) such that
  \[
  p^* = \{ (\omega^V_1 \in K^*), (F_0^*(\omega^V_1) = \tilde{k}(\lambda)) \cup \tilde{k}(p_0) \} \subseteq \Sigma_\varphi.
  \]
- This gives that \( \tilde{M} \) models that \( \text{Coll}(\omega, \rho) \) forces that there are a weak \( \tilde{k}(P) \)-certificate \((\tilde{J}^*, r^*, f^*, F^*, K^*)\) and \( \tilde{k}(P) \)-witness \( \bar{v} \) with \( p^* \subseteq \Sigma_\varphi. \)

We conclude that \( \tilde{M} \) models that \( p^* = \tilde{k}(p_0) \cup \{ \omega^V_1 \in \tilde{K}, \tilde{F}_0(\omega^V_1) = \tilde{k}(\lambda) \} \) is in \( \tilde{k}(P) \) as witnessed by \( \bar{v} \) and \((\tilde{J}^*, r^*, f^*, F^*, K^*)\).

We also observe that

\[
\tilde{M} \models p^* \models \omega^V_1 \in \tilde{k}(\tilde{C})
\]

by applying in \( \tilde{M} \) Lemma 3.40 to \( \tilde{k}(\tilde{C}) \), \( \tilde{k}(\lambda) \), and observing that \((\tilde{F}_0(\omega^V_1) = \tilde{k}(\lambda)), (\omega^V_1 \in \tilde{K}) \) is in \( p^* \).

---

\(^{29}\)See Remark 3.34(iii).
Once $8$ is proved, by elementarity of $\bar{k}$, we get that in $V$ there is a condition in $P_{\kappa}$ refining $p_0$ and forcing that $\nu_H(\dot{c}_{j,\omega^j_1}) \cap \dot{C}$ is non-empty (since we already know by $6$, that $\omega^j_1 \in k_0(S)$). This is the desired contradiction.

To complete the proof of the Theorem we need only to prove Claim 2.

3.3.1. Proof of Claim 2. First of all we easily check that $(\bar{J}, r_H, f^*, F^*, K^*)$ is a $\bar{k}(\kappa)$-precertificate relative to $\bar{M}$ which satisfies also Condition (C) (i) of Def. 3.32.

This is not hard: $(\bar{J}, r_H, f^*, k(T))$ is a weak semantic certificate for $\bar{k}(A)$ by Claim 1. Now all the other requirements needed to establish that $(\bar{J}, r_H, f^*, F^*, K^*)$ is a $\bar{k}(\kappa)$-precertificate relative to $\bar{M}$ are easily checked using the elementarity of $\bar{k}$, for example:

- if $\alpha \in K_H$, and $F_H(\alpha) = (\eta, X_\alpha, P_\eta)$ (it is the case that $Z_\alpha = P_\eta$ since $(J_H, r_H, f_H, K_H, F_H)$ is a $P_\lambda$-certificate with $\nu_H$ a $P_\lambda$-witness)
  
  $$(X_\alpha, \in, P_\eta \cap X_\alpha, A_\eta \cap X_\alpha) \prec (Q_\eta, \in, P_\eta, A_\eta);$$

  by elementarity of $\bar{k}$

  $$(\bar{k}[X_\alpha], \in, \bar{k}[P_\eta \cap X_\alpha], \bar{k}[A_\eta \cap X_\alpha]) \prec (\bar{k}[Q_\eta], \in, \bar{k}[P_\eta], \bar{k}(A_\eta)).$$

- Similarly if $\alpha = \omega^j_\nu$, $X_\alpha = \bar{k}[Q_\lambda]$ and
  
  $$(\bar{k}[Q_\lambda], \in, \bar{k}[P_\lambda], \bar{k}[A_\lambda]) = (\bar{k}[Q_\lambda], \in, \bar{k}(P_\lambda), \bar{k}(A_\lambda) \cap \bar{k}(Q_\lambda)) \prec (\bar{k}(Q_\lambda), \in, \bar{k}(P_\lambda), \bar{k}(A_\lambda)).$$

Now Condition (C) (ii) would also be easy to check if we can ensure the existence of a $\bar{\nu}$ satisfying $7$ as in the Claim.

Assume this is the case. Then for $\alpha \in K_H$ with $F_0(\alpha) = \eta$ and $E$ dense subset of $\bar{k}(P_\eta) \cap \bar{k}[X_\alpha]$ definable in

$$\bar{k}[X_\alpha], \in, \bar{k}[P_\eta \cap X_\alpha], \bar{k}[A_\eta \cap X_\alpha],$$

we have that $E = \bar{k}(U) \cap k[X_\alpha]$ for some $U$ dense subset of $P_\eta$ definable in the structure

$$(X_\alpha, \in, P_\eta \cap X_\alpha, A_\eta \cap X_\alpha).$$

Then $H \cap U \cap X_\alpha \neq \emptyset$, giving that $k[H] \cap k(U) \cap k[X_\alpha] \neq \emptyset$.

Similarly since $H$ is $V$-generic for $P_\lambda$, we get that $k[H] \cap k(U) \cap k[Q_\lambda] \neq \emptyset$ for any $U \in V$ dense subset of $P_\lambda$.

This would show that $(\bar{J}, r_H, f^*, F^*, K^*)$ is a weak $\bar{k}(P_\kappa)$-certificate relative to $\bar{M}$ as witnessed by $\bar{\nu}$.

So we are left with the definition of $\bar{\nu}$.

We modify in $V[G]$, $\nu_H$ (from now on denoted as $\nu$) to a weak $\bar{k}(P_\kappa)$-certificate

$$\bar{\nu} : \bar{k}(L^1_{\kappa}) \to X_{(J, r_H, f^*, F^*, K^*)}$$

witnessing that $^{30} (\bar{J}, r_H, f^*, F^*, K^*) \in V[G]$ is a weak $\bar{k}(P_\kappa)$-certificate.

We first define $\bar{\nu}$ on

$$\bar{k}[L^1_{\kappa}] \cup \{X_{\omega^j_\nu}, \dot{Z}_{\omega^j_\nu}\}$$

according to the following five types of sort, constant, or predicate symbols:

(a) $\bar{\nu}($c$)$ is defined ad hoc if $c \in L^1$ is a sort, constant, or predicate symbol of $L^1$ among $\{r, f\}, F_0, K, X_i$ for each $i < \omega^j_\nu$, and the constant symbols $\dot{r}, \dot{f}$, or if $c$ is either $X_{\omega^j_\nu}$ or if $c$ is br$T$.

(b) $\bar{\nu}(\bar{k}(c)) = \bar{\nu}(\nu(c))$ if $c$ is any symbol of $L^1 \setminus L^0$ such that $c$ has not been already considered in the list (a).

^{30}To avoid a heavy notation we will free to identify certain symbols $c$ of $L^1$ whose intended meaning is transparent with the corresponding symbol $k(c)$ of $k(L^1)$. For example we write $K$ instead of $k(K)$. Similarly we write $c_{\omega^j_\nu}$ with the intended meaning of denoting the constants which represents the elements of the $\omega^j_\nu$-th point of the sequence $k(\{X_\alpha : \alpha < \omega^j_\nu\})$, etc.
(c) \( \bar{\nu}(\bar{k}(c)) = \nu(c) \) if \( c \) is a sort or predicate symbol of \( L^0 \) among \( \bar{N}_\alpha, \bar{G}_\alpha, \text{Sat}_{\bar{N}_\alpha}, \in_{\bar{N}_\alpha}, \text{Cod}^*, J_{\alpha\beta}, \)

or \( c \) is a constant of sort \( \bar{N}_\alpha \) for \( \alpha \leq \beta < \omega^1 \).

(d) \( \bar{\nu}(c) = j_{\omega^1, \rho}(c) \) if \( c \) is a constant symbol of \( L^0 \) and \( \nu(c) \in N_{\omega_1} \) (e.g. \( c \) is \( \dot{c}_{i, \omega^1} \) for some \( i < \omega^1 \), or \( c = \dot{x} \) for some \( x \in H_{\omega_1} \).

(e) \( \bar{\nu}(\dot{j}_{\alpha \omega^1}) = j_{\alpha \rho} \) for \( \alpha \leq \omega^1 \), \( \bar{\nu}(\dot{N}_{\alpha \omega^1}^\nu) = N_{\alpha \rho}^\nu \), and

\[
\bar{\nu}(\text{Sat}_{\bar{N}_{\alpha \omega^1}^\nu}) = \{ (\bar{\nu}\psi, (a_1, \ldots, a_n)) : (N_{\alpha \rho}^\nu, \in) \models \psi(a_1, \ldots, a_n) \}.
\]

\( \bar{\nu} \) is defined as follows on the symbols of type (a):

(i) \( \bar{\nu}(\bar{K}) = K^* = K_H \cup \{ \omega^1 \}, \)

(ii) \( \bar{\nu}(X_i) = \bar{k}[F_1(i)] \) for all \( i \in K_H, \)

(iii) \( \bar{\nu}(X_{\omega^1}) = \bar{k}(Q_\lambda), \)

(iv) \( \bar{\nu}(\bar{Z}_{\omega^1}) = \bar{k}(P_\lambda), \)

(v) \( \bar{\nu}(\dot{F}_0)(\alpha) = \bar{k}(F_H)(\alpha) \) for \( \alpha \in K_H, \)

(vi) \( \bar{\nu}(\dot{F}_0)(\omega^1) = \bar{k}(\lambda), \)

(vii) \( \bar{\nu}(\dot{F}_0)(\beta) = 0 = \bar{\nu}(\bar{Z}_\beta), \bar{\nu}(\dot{X}_\beta) = 1 \) if \( \beta \notin K^*, \)

(viii) \( \bar{\nu}(\bar{r}) = r_H, \)

(ix) \( \bar{\nu}(\bar{f}) = \bar{k}[f_H] = f^*, \)

(x) \( \bar{\nu}(\bar{T}) = \bar{k}(T), \)

(xi) \( \bar{\nu}(\bar{b}_{\bar{r}T}) = \{ ((r_H \upharpoonright n, f^* \upharpoonright n), r_H, f^*) : n \in \omega \}. \)

We must therefore check that:

(1) \( \bar{\nu} \) is well defined (i.e. \( \bar{\nu} \) respects the equivalence class \([\cdot]_H \) on the constant symbols of \( L^1_\lambda \)) and is consistent with the constraints set forth in Def. 3.18 for the symbols in its domain in order to be \( \bar{k} \)-admissible for \((\bar{J}, r_H, f^*, F^*, K^*) \) relative to \( M \) (the clauses of Def. 3.31 must be satisfied).

(2) Any extension of \( \bar{\nu} \) in \( V[G] \) to a total assignment \( \bar{\nu} : \bar{k}(L^1) \rightarrow X_{(J, r_H, f^*, F^*, K^*)} \)

respecting the constraints of Def. 3.31 witnesses that \((\bar{J}, r_H, f^*, F^*, K^*) \) is a weak \( \bar{k}(P_\kappa) \)-certificate relative to \( M \), hence \( \bar{\nu} \) is also a weak \( \bar{k}(P_\kappa) \)-witness.

The key to establish items 1 and 2 is the following

**Subclaim 1.** For \( \psi(c_1, \ldots, c_n) \) an atomic \( L^1_\lambda \)-sentence

\[
\psi(c_1, \ldots, c_n) \in \Sigma_H \text{ if and only if } \lambda^k�_{(J, r_H, f^*, F^*, K^*)} \models \psi(\bar{\nu}(c_1), \ldots, \bar{\nu}(c_n)).
\]

Suppose for the moment the Subclaim is proved and let us complete the proof of the Theorem. By the Subclaim \( \bar{\nu} \) can be extended to the whole of \( \bar{k}(L^1_\lambda) \) so to maintain the constraints set forth in Def. 3.31 to be a \( \bar{k}(\kappa) \)-assignment because these constraints are easily checked for \( \bar{\nu} \) on its partial domain. Then all conditions required to make \( \bar{\nu} \) a weak \( \bar{k}(P_\kappa) \)-witness are met, and we are done.

So we are left with the proof of Subclaim 1.

**Proof.** It is essentially a tiresome matter of checking all possible cases for all atomic sentences of \( L^1_\lambda \), appealing to the fact that \( \bar{k}, \bar{j}_{\omega^1, \rho} \) are elementary maps, and the observation that the cases for predicates with a multi-sorted type are easy to handle. Let \( S^1 \) denote the sorts of \( L^1 \) and define the multimap \( (j^*_s : s \in S^1) \) by

\[
 j^*_s : \nu(c) \mapsto \bar{\nu}(c)
\]

for \( c \) a constant symbol of \( L^1 \) of sort \( s \). Now:

(i) Sorts are mapped to sorts:

\[
 \bullet \bar{\nu}(\bar{Q}) = \bar{k}(Q_\lambda) = \bar{k}(\nu(\bar{Q})),
\]
(i) For atomic formulae given by a predicate $R_j$ whose type $s_j$ has entries all of the same sort, one easily checks the preservation of the formula appealing to the properties of the multi-map $(\tilde{j}_s^*: s \in S^1)$.

- $(F_0)^\nu(c) = \nu(d)$ holds if and only if $F_0^\nu(k(\nu(c))) = \tilde{k}(\nu(d))$ if and only if $F_0^\nu(\nu(c)) = \tilde{k}(d)$ for $\nu(c) \in Q_\lambda$; similarly one handles the case of the formula $K(c)$.
- $\text{Sat}_{\alpha,\omega}^\nu(\forall \psi(\nu(c_1), \ldots, \nu(c_m)))$ holds if and only if
  $$(Q_\lambda, \epsilon, P_\lambda, A_\lambda) \models \psi(\nu(c_1), \ldots, \nu(c_m))$$
if and only if
$$(\tilde{k}(Q_\lambda), \epsilon, \tilde{k}(P_\lambda), \tilde{k}(A_\lambda)) \models \psi(\tilde{k}(\nu(c_1)), \ldots, \tilde{k}(\nu(c_m)))$$
for all $c$ of sort $\tilde{Q}$ and $\tilde{k}$ is elementary.
- $\text{Sat}_{\alpha,\omega}^\nu(\forall \psi(\nu(c_1), \ldots, \nu(c_m)))$ holds if and only if
  $$(N_\alpha, \epsilon) \models \psi(\nu(c_1), \ldots, \nu(c_m))$$
if and only if
$$(N_\alpha, \epsilon) \models \psi(\tilde{j}_\omega^\nu(\nu(c_1)), \ldots, \tilde{j}_\omega^\nu(\nu(c_m)))$$
for all $c$ of sort $\tilde{N}_\omega$ and $\tilde{j}_\omega^\nu$ is elementary.

- For $\alpha < \omega$ $\text{Sat}_{\alpha,\omega}^\nu(\forall \psi(\nu(c_1), \ldots, \nu(c_m)))$ holds if and only if
  $$(N_\alpha, \epsilon) \models \psi(\nu(c_1), \ldots, \nu(c_m))$$
if and only if $\text{Sat}_{\omega,\omega}^\nu(\forall \psi(\nu(c_1), \ldots, \nu(c_m)))$ holds, given that $\tilde{\nu}(c) = \tilde{\nu}(c)$ for all $c$ of sort $\tilde{N}_\omega$.

(ii) We leave to the reader to handle the other atomic formulae whose type has entries of just one fixed sort.

(iii) It remains to handle the case of formulae whose predicate has a multi-sorted type. There are just three clauses in Def. 3.17 and Def. 3.29 defining them:

- $(a)$ $j_{\alpha,\beta}(c) = d$ (Clause (i)),
- $(b)$ $br_T(c, d, e)$ (Clause (j)),
- $(c)$ Cod$^*(c, d, e)$ (Clause (m)),

We handle them as follows:

(a) We observe that:
- if $\beta < \omega_1^\nu$, $\nu(c) = \tilde{\nu}(c)$ and $\nu(d) = \tilde{\nu}(d)$,
- if $\beta = \omega_1^\nu$, $\nu(c) = \tilde{\nu}(c)$ and $\tilde{j}_\omega^\nu(\nu(d)) = \tilde{\nu}(d)$.

It is immediate to check that this type of formulae is preserved in both cases by the multi-map $(\tilde{j}_N^\nu, j_N^\nu)$.

(b) It is immediate to check their preservation observing that $\tilde{\nu}(\tilde{j}(c)) = \tilde{j}_\omega^\nu(\nu(c))$.

(c) It is immediate to check their preservation observing that $\tilde{\nu}(\tilde{j}(c)) = \tilde{\nu}(c)$ for all constants $c$ of sort $\tilde{N}_0$.

The remaining details are left to the reader. □

We now closed all open ends of the proof and Theorem 3.33 is proved. □
Remark 3.41. Subclaim 1 is the very reason why one has to resort to multi-sorted logic to formalize the notion of $P_\lambda$-certificate in infinitary logic. Consider the first order structures $(\mathcal{X}(J, r, H, F, K, H))^+ \cdot (\mathcal{X}(\mathcal{J}, r, H, F, K, H))^+$ associated respectively to $\mathcal{X}(J, r, H, F, K, H)$ and $\mathcal{X}(\mathcal{J}, r, H, F, K, H)$. In this case there is overlap between the domains of the various sorts $s$ (for example $\nu(Q) \cap \nu(N_{\omega_1}) \supseteq H_{\omega_2}$), and these domains could be mapped in possibly different way by the corresponding maps $j_s^\ast$. For example if $x \in N_{\omega_1} \setminus H_{\omega_2}$ it is possible that $x = \nu(\hat{x})$ and $x = \nu(c_{i, \omega_1})$, while it is a priori conceivable that $j_Q^\ast(\hat{x}) = k(x) \neq j_{\omega_1}^\ast(x) = j_{\omega_1}^\ast(c_{i, \omega_1})$.

But this potential conflict is avoided exactly because we resorted to multi-sorted logic: the critical first order formula $\hat{c} = c_{i, \omega_1}$ is not among the ones whose preservation should be checked in Subclaim 1 exactly because it is instantiated for constants of different sort.

4. Comparing the proofs

The reader familiar with Asperó and Schindler’s proof will notice that our proof is just rephrasing their argument taking advantage of the relation between density argument and the use of consistency properties to produce models of sentences for infinitary multi-sorted logic. A small twist with respect to their proof is also given by our use of the apparatus set forth in 3.1 to establish many properties of the generic extension by $P_\lambda$, by means of a syntactic analysis of the $\mathcal{L}_{\omega_1, \omega}$-theory of the $P_\lambda$-certificates. This makes a large swath of density arguments easy consequences of Fact 3.14.

Taken aside these considerations, the proof presented here rephrases with a different terminology the arguments given in [1].

5. On the consistency strength of (*)&

This part is joint work with Ralf Schindler.

For $P$ a poset of regular uncountable size $\kappa$ $\text{FA}^+ (P)$ states that for any $P$-name $\bar{T}$ for a stationary subset of $\kappa$, there is $G$ filter such that

\[
\{ \alpha < \kappa : \exists p \in G \mathrel{p} \models \bar{\alpha} \in \bar{T} \}
\]

is stationary in its supremum.

Theorem 5.1 (Schindler, V.). Assume $\kappa = 2^\theta = 2^\theta > \omega_1$.

Let $\mathcal{L}_2 = \mathcal{L}_1 \cup \{ \bar{R} \}$ with $\bar{R}$ a binary predicate symbol of type $(\bar{Q}, \bar{Q})$.

Let $\Sigma_2^\lambda$ enlarge $\Sigma_1^\lambda$ with the axioms

(VI)

(a) For all $\delta < \omega_1$ and $\eta < \lambda$,

\[
\left[ \delta \in K \wedge \bar{R}(\bar{\delta}, \bar{\eta}) \rightarrow \bigwedge_{\gamma < \lambda} \left( \bar{X}_\delta(\bar{\gamma}) \rightarrow \bar{\gamma} < \bar{\eta} \wedge (\bar{\gamma} < \bar{\eta} \rightarrow \bigvee_{n < \omega} \bar{\gamma} \in \bar{e}_{n, \delta} ) \right) \right]
\]

Let $\bar{P}_\lambda$ be exactly the forcing defined by $P_\lambda$ with the additon that a $\bar{P}_\eta$-witness is obtained by an $\eta$-precertificate $\nu$ as the unique extension of $\nu$ to a $\bar{\nu} : \mathcal{L}_2^\lambda \rightarrow \mathcal{X}(\mathcal{J}, r, f, F, K)$ such that:

- $\Sigma_2^\lambda \subseteq \Sigma_\nu$,
- Condition (C)/(ii) is satisfied, now with with $\bar{P}_\eta$ replacing $P_\eta$.

Assume $\text{FA}^+_{\eta} (\bar{P}_\eta)$. Then $\square_{\kappa}$ fails.

These axioms states that for $\delta \in K \mathrel{R}(\delta, \eta)$ holds if and only if $\eta$ is the supremum of $X_{\delta} \cap \text{Ord}$. 

\[\text{FA}^+_{\eta} (\bar{P}_\eta)\]
Note that the unique change from $P_\lambda$ to $\bar{P}_\lambda$ is that in the new forcing it is possible to express by an atomic formula the statement $\sup(X_\delta) = \eta$ for any $\delta < \omega_1^Y$, $\eta < \lambda$. On the other hand, it is not clear whether the inclusion map of $P_\lambda$ into $\bar{P}_\lambda$ is a complete embedding: if $r \in P_\lambda$, $D$ is dense in $P_\lambda$, and $s \in D$ is an extension of $r \upharpoonright P_\eta = r \cap \Delta_\eta$, the unique extension of a $P_\eta$-witness $\nu$ for $s$ to an assignment $\bar{\nu}$ with $\Sigma_{\bar{\nu}} \supseteq \Sigma_\eta^P$ may not satisfy all the atomic sentences for $s \cup r$ or (even if it satisfied all these atomic sentences) it might not be a $P_\eta$-certificate.

**Proof.** Under our assumptions on $\kappa$, Shelah has proved that $\dot{Q}_\kappa$ is witnessed by the set of points of countable cofinality below $\kappa$ (see [?]).

Now the proof is just a slight refinement of the proof of Thm. 3.33. Assume towards a contradiction that $(C_\alpha : \alpha < \kappa)$ is a square sequence, i.e., is such that $\otp(C_\alpha) \leq \theta$ for all $\alpha$, and $c_\alpha = c_\beta \cap \alpha$ whenever $\alpha$ is a limit point of $c_\beta$. By a standard diagonalization argument we can find $\xi < \theta$ such that $\dot{Q}_\kappa$ is witnessed by $S_\xi$, where

$$S_\xi = \{ \eta < \kappa : \text{cof}(\eta) = \omega \text{ and } \otp(C_\eta) = \xi \}.$$ 

If one parses through that proof one realizes that the following holds:

**Claim 3.** Given $p \in \bar{P}_\kappa$ and a $\bar{P}_\kappa$-name $\dot{C}$ for a club subset of $\kappa$, there are $\delta < \omega_1$ and $\lambda \in S_\xi$ such that

$$p \cup \left\{ \bar{K}(\delta), \bar{F}_0(\delta) = \bar{\lambda}, \bar{R}(\delta, \bar{\lambda}) \right\}$$

is in $\bar{P}_\kappa$ and forces $\lambda \in \dot{C}$. The proof of this claim is (modulo the variations spelled out below) exactly the one given in the proof of Thm. 3.33 to infer that on top of any condition $p \in P_\kappa$, and for any $\dot{e}_{i, \omega_1}$ which $p$ forces to be stationary for $\mathcal{N}_{\omega_1}$ and for any $P_\kappa$-name $\dot{C}$ for a club subset of $\omega_1$ one can always find $\delta < \omega_1$ and $\lambda$ such that

$$p \cup \left\{ \bar{K}(\delta), \bar{F}_0(\delta) = \bar{\lambda} \right\}$$

is in $P_\kappa$ and $p$ forces $\delta \in \dot{e}_{i, \omega_1} \cap \dot{C}$. The key changes are the following:

- A $\bar{P}_\kappa$-name for a subset of $\kappa$ is still given by a subset of $\mathcal{H}_\kappa$ (now we need $\kappa$-many antichains of $P_\kappa$ to decide $\dot{C}$), hence we can find $\lambda \in S_\xi$ such that $\dot{C} \cap Q_\lambda = A_\lambda$.
- Claim 2 is now reinforced adding the further request that
  $$\bar{\nu} : \bar{k}(\Sigma_1^A) \to \mathcal{X}(\bar{j}_{i, \bar{r}_n, j^*, F^*, K^*})$$

is such that its unique extension to a model of $\bar{k}(\Sigma_1^A)$ is such that $\bar{R}((\omega_1^Y, \bar{k}(\lambda))) \in \Sigma_{\bar{\nu}}$. The proof that this stronger form of the claim holds is exactly the same, with the following further addition: since $\lambda$ has countable cofinality in $\nu$, $\bar{k}(\lambda)$ has countable cofinality in $\bar{M}$, hence $\mathcal{V}[G]$ models that $\bar{k}[\lambda]$ is cofinal in $\bar{k}(\lambda)$. This gives that $\bar{R}((\omega_1^Y, \bar{k}(\lambda))) \in \Sigma_{\bar{\nu}}$ assuming the $\bar{\nu}$ given in Claim 2 has been extended to $\bar{k}(\Sigma_1^A)$ in the unique possible way which makes it a model of $\bar{k}(\Sigma_1^A)$.

- The same definability argument given in the proof of the main theorem yields that
  $$\bar{k}(p) \cup \left\{ \bar{K}(\bar{\omega}_1), \bar{F}_0(\bar{\omega}_1) = \bar{k}(\lambda), \bar{R}(\bar{\omega}_1, \bar{k}(\lambda)) \right\}$$

forces that $k(\lambda) \in k(\dot{C})$ holds in $\bar{M}$ for $\bar{k}(P_\kappa)$.

This gives that

**Claim 4.** $\dot{S}$ is a $P_\kappa$-name for a stationary set, where

$$\dot{S} = \left\{ (\bar{\lambda}, p) : \lambda \in S_\xi, \text{ and } \bar{K}(\delta), (\bar{F}_0(\delta) = \bar{\lambda}), \bar{R}(\delta, \bar{\lambda}) \in p \text{ for some } \delta \in \omega_1^Y \right\}.$$
Now by $\text{FA}^{+1}(\bar{P}_\kappa)$ we can find a $H$ filter on $\bar{P}_\kappa$ such that

$$S = \dot{S}_H = \{ \lambda \in S_\xi : \text{exists } p \in H \text{ forcing } \dot{\lambda} \in \dot{S} \}$$

is stationary in its supremum $\eta$. Then $S \cap C_\eta$ is stationary. If $\alpha < \gamma \in S \cap C_\eta$ are limit points of $C_\eta$, we get that:

$$C_\alpha = C_\eta \cap \alpha = C_\gamma \cap \gamma \cap \alpha = C_\gamma \cap \alpha$$

hence

$$\xi = \text{otp}(C_\alpha) = \text{otp}(C_\gamma \cap \alpha) < \text{otp}(C_\gamma) = \xi,$$

which is a contradiction. □

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