Classical Propagation of Light in Spatio-Temporal Periodic Media

B. S. Alexandrov,¹ K.Ø. Rasmussen,¹ A.T. Findikoglu,¹ A.R. Bishop,¹ and I. Z. Kostadinov²

¹Los Alamos National Laboratory, Los Alamos, New Mexico 87544
²Ohio State University, Columbus, Ohio 43210

(Dated: July 22, 2018)

We analyze the propagation of electromagnetic waves in media where the dielectric constants undergo rapid temporal periodic modulation. Both spatially homogeneous and periodic media are studied. Fast periodic temporal modulation of the dielectric constant of a homogeneous medium leads to existence of photonic band-gap like phenomena. In the presence of both spatial and temporal periodicity the electromagnetic spectrum is described in a four-dimensional cube, defining an effective Brillouin zone. In the case of incommensurability between space and time periodicities, completely dispersed point spectra exist.

The advent of materials whose electric permittivity ε is periodically modulated at the nanometer length scale has introduced the important concept of photonic band-gap structures. This has in turn made traditional solid-state concepts like reciprocal space, Brillouin zones, dispersion relations, Bloch wave functions, etc. directly applicable to the field of classical electromagnetic wave propagation. During the last decades the attractive possible applications of such photonic band gap materials have driven intense theoretical and experimental studies of the propagation of electromagnetic waves in spatially periodic and disordered dielectric structures. This has provided the field of photonics with a wide range of new applications, mostly related to guided light modes. Recently a magnetic photonic crystal was made by periodically modulating the magnetic permeability μ of specially constructed material.

Although there have also been tentative studies of effects arising from low frequency modulation of the dielectric constant, this possibility has been largely ignored. However, as we show here, essentially all fruitful concepts from the now mature field of photonic band-gap materials can be applied in the case of fast temporal modulation of a material’s dielectric response. Further, the combination of spatial and temporal modulation of the dielectric response introduces intriguing new concepts to the field of classic electromagnetic wave propagation. The results are also valid for periodic modulation of the magnetic permeability μ of spatially homogeneous or periodic magnetic media.

First, it is straightforward to realize that temporal modulation leads to photonic band structures similar to spatial modulation: In a material with a time dependent dielectric response ε(t), the electromagnetic waves, V(x, t), are described by the wave equation:

\[ \frac{\partial^2 V(x,t)}{\partial x^2} = \frac{\varepsilon_0}{c^2} \frac{\partial^2 \varepsilon(t)}{\partial t^2} V(x,t), \]  

(1)

where ε₀ is the vacuum dielectric constant and c the speed of light in vacuum. By simple separation of variables V(x, t) = \( \sum_k u_k(x)u_k(t) \), the solution of this equation may be expressed in the form:

\[ V(x,t) = \sum_k \varepsilon(t) u_k(t) e^{ikx} = \sum_k U_k(t)e^{ikx}. \]  

(2)

After introducing the dimensionless variables, x → x/L and t → t/τ, the equation for U_k(t) becomes:

\[ \frac{d^2 U_k(t)}{dt^2} + k^2 s^2 f(t) U_k(t) = 0, \]  

(3)

where s = \( \frac{c}{\varepsilon_0 \tau} \) and f(t) = \( \varepsilon^{-1}(t) \). With \( \varepsilon(t) \) periodic so that f(t) is periodic, the solutions will be of the Floquet-Bloch type

\[ U_k(t) = \exp(-i\omega t) U_k^\omega(t), \]  

(4)

which is the central property for photonic band-gap theory.

By casting the problem in this form, it is reduced to determining the dispersion relation k = k(ω). As is the case for traditional photonic crystals, the dispersion relation will contain the frequency regime where electromagnetic wave propagation is possible, as well as the "photonic gaps" where the propagation of electromagnetic waves is damped (or amplified). A possible realization of temporal dielectric modulation is to apply standing waves with time dependent intensity, in the form of one, two or three perpendicular laser beams, on a dielectric slab, as illustrated in Fig. 1. This concept is widely applied to create optical lattices within Bose-Einstein condensates. Applying the beams on a spatially homogeneous or multi-dimensionally modulated dielectric slabs, a variety of space-time dielectric structures can be generated.

First, we study the simple case of periodic dielectric modulation of a linear and homogeneous medium by modeling it (see Fig. 2) as a series of pulses characterized by the two times τ and \( \frac{\gamma}{\tau} \). The parameter 1/τ is the duration of a single pulse of the dielectric function of the material ε(t), and τ is the period of repetition of the pulses. If the standing wave laser beam consists
of ultrashort pulses, a single pulse duration \( \varepsilon(t) \) in the range \( \gamma = 10^{-12} \)–\( 10^{-15} \) sec can be realized. Similarly, a pulse separation \( c\tau \) comparable to the wavelength of the light in the medium is realistic. Specifically, we study the propagating waves in a 1 + 1 space-time scalar wave equation (11) by assuming the dielectric constant to be of the form:

\[
\varepsilon(t) = 1 + \sum_{m=1}^{N} g(t - m\tau) \\
= 1 + \sum_{m=1}^{N} \frac{\varepsilon_{\text{temp}} \gamma \tau}{2} \exp(-\gamma|t - m\tau|). 
\]

The specific choice (see Fig. 2) of the function \( g(t) \) is motivated by its convenient form for analytical treatment: For narrow pulse widths (\( \gamma \to \infty \)), \( g(t) \) reduces to a \( \delta \)-function and the model is then similar to the Kronig-Penney model. Applying this expression in Eq. (6) for the Fourier amplitude \( \tilde{U}_k(\omega) \):

\[
(\omega^2 - \kappa^2 s^2) \tilde{U}_k(\omega) = \sum_{m=1}^{N} \frac{\omega^2 \varepsilon \gamma \tau}{(\gamma^2 + \omega^2)} U_k(m),
\]

where we have used the approximation

\[
\int dt e^{-\gamma|t-n|} U_k(t) f(t) \approx U_k(n) \int dt e^{-\gamma|t-n|} f(t),
\]

which amounts to assuming the wave function \( U_k(t) \) to be constant during the pulse duration \( 1/\gamma \) (see Fig. 2). For the discrete set of amplitudes \( \tilde{U}_k(\omega) \), when we integrate both sides of Eq. (4) and employ the relationship

\[
U_k(m) \equiv U_k(t = m) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega m} \tilde{U}_k(\omega),
\]

we obtain the matrix equation:

\[
U_k(m) = \sum_{n=1}^{N} \Lambda_{n,m} U_{k_n}(m), 
\]

\[
\Lambda_{n,m} = \frac{\varepsilon_{\text{temp}} \gamma^2}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\omega(n-m)}}{(\omega^2 - \kappa^2 s^2)(\omega^2 + \gamma^2)} d\omega.
\]

Without further approximations we find after some manipulation (see supplement for details) that the dispersion relation of this system can be expressed as:

\[
\cos \omega = h_{\text{temp}}^2 - \frac{1}{2} \left[ \cosh \gamma + \cos \kappa s + \eta_{\text{temp}}(\cosh \gamma + \frac{k}{\gamma} \sin \kappa s) \right],
\]

\[
h_{\text{temp}}^0 = \cosh \gamma \cos \kappa s + \eta_{\text{temp}}(\sinh \gamma \cos \kappa s + \frac{k}{\gamma} \cosh \gamma \sin \kappa s),
\]

and

\[
\eta_{\text{temp}} = \frac{\gamma^3}{2(\gamma^2 + \kappa^2 s^2)} \varepsilon_{\text{temp}}.
\]

In Fig. 3 we illustrate this dispersion relation, Eq. (9). The width of the gap is very small because of the realistic smallness of the modulation \( \varepsilon_{\text{temp}} / \varepsilon_0 = 0.01 \) of the dielectric constant. One can easily see that the widths of the gaps are proportional to \( \varepsilon_{\text{temp}} \). Figure 4 shows the width of the band-gap as a function of \( \tau \) and \( \gamma \). This width is much larger for high frequencies but is relatively insensitive to \( \gamma \). At this point it is instructive to note that for a traditional one dimensional photonic crystal, similarly defined by the dielectric constant

\[
\varepsilon(x) = 1 + \sum_{m=1}^{N} \frac{\varepsilon_{\text{spat}} \rho L_x}{2} \exp(-\rho|x - mL_x|),
\]

we obtain the dispersion relation by simply interchanging \( k \) with \( \omega \) and \( ks \) with \( \omega / s \).
FIG. 3: Spectrum of a "time photonic crystal" with $\tau = 4 \times 10^{-12}$ sec; $\gamma = 4$; $\varepsilon_0 = 11$; $\varepsilon_t = 0.01$. The "band-gap" is displayed in the upper inset.

Turning now to the more general question of multidimensional periodicity, we assume the dielectric constant $\varepsilon(r, t)$ to be a periodic function of both space and time, and solve the Maxwell equations for the electric field $\vec{E}(r, t)$. Again we have the Bloch type solutions with respect to space and time variables and so we introduce Bloch-Floquet parameters. The final dispersion relation will then relate $k$ and $\omega$. This means that in the case of combined spatial and temporal periodicity the electromagnetic excitations spectrum is described by separated points represented by $(k, \omega)$, all belonging to an equivalent four-dimensional cube defining the four-dimensional Brillouin zone. For simplicity we consider the dielectric constant $\varepsilon(r, t)$ to be of the type $\varepsilon(r, t) = \varepsilon(r)\varepsilon(t)$, i.e. we assume independent spatial and temporal modulations of the dielectric media. The general solution of the wave equation is then

$$\vec{E}(r, t) = \sum_n \vec{V}_n(r) U_n(t),$$

(11)

where for $\vec{V}_n(r)$ and $U_n(t)$ we have

$$\frac{d^2}{dt^2} \varepsilon(t) U_n(t) = -\nu^2 s^2 U_n(t)$$

(12)

and

$$\nabla \times \nabla \times \vec{V}_n(r) = -\nu^2 \varepsilon(r) \vec{V}_n(r)$$

(13)

with

$$\nabla \cdot \varepsilon(r) \vec{V}_n(r) = 0$$

and $s = \frac{cT}{L_x\sqrt{\varepsilon_0}} = \frac{L_x}{L_T}$ and $\nu$ being constants. In order to simplify the problem, we consider the wave equation for the scalar wave amplitude $V(x, t)$. In this case of periodicity in one space dimension and in time, we have to take into account that the separation constant $-\nu^2$ is the same in both equations (12) and (13).

We can now establish the dispersion relation between the normalized frequency $\omega$ and the normalized wave-number $k$. For simplicity we first examine the case of $s = 1$, where the temporal and spatial periodicities are equal, $L_T = L_x$. We find that the dispersion of such a media is dominated by a vacuum -like dispersion relation i.e. $\omega \approx sk$. The spectrum in this case is given mostly by the diagonal of the unit square, resembling in this sense a homogeneous medium. The actual dispersion depicted in Fig. 4 shows this overall behavior but, as can be clearly seen in the inset, the underlying structure is very complex. In fact the dispersion relation is a dense set of discrete points clustering around two separate branches. From this case we can easily construct the overall structure of the dispersion relation resulting from rational values of $s \neq 1$. For an integer values of $s > 1$ the relation $\omega \approx sk$ results in values of $\omega > 1$ and so this part of the dispersive branch is folded back into the standard zone. Similarly, for integer values of $s^{-1} > 1$ the roles of $\omega$ and $k$ are reversed and the $k$ values have

FIG. 4: Width of the band-gaps as a percent of the mid-gap frequency for three frequencies $\tau = 10^{-13}$ sec - the lower curve, $\tau = 10^{-14}$ sec - the middle position curve, and $\tau = 5 \times 10^{-15}$ sec - the upper curve. Triangles correspond to $\gamma = 5$, and squares to $\gamma = 3$. The inset presents band-gaps for microwave region. An accessible [18] experimental setup: $\tau = 10^{-8}$ sec $\gamma = 10$, $\varepsilon_{\text{temp}} = 0.2\varepsilon_0$, $\varepsilon_0 = 300$ was simulated.
FIG. 5: Dispersion relation of the temporally and spatially modulated medium with \( \tau = 3 \times 10^{-13} \) sec; \( \gamma = 3; \varepsilon_0 = 11; L_x = 3 \times 10^{-5} \) m; \( \rho = 3 ; \varepsilon_x = 0.01; \varepsilon_t = 0.01 \) s=1. The inset shows a blow-up illustrating the underlying complex structure of the dispersion relation.

FIG. 6: Spectrum of the "space-time photonic crystal" with \( \tau = 3 \times 10^{-13} \) sec; \( \gamma = 3; \varepsilon_0 = 11; \varepsilon_x = 0.01; \varepsilon_t = 0.01 \) s = \( \frac{\sqrt{2}}{3} \). In the inset we represent a point spectrum - the incommensurability case of two basic lengths, where \( s = \sqrt{2} \) and \( \tau = 3 \times 10^{-13} \) sec.

to be folded back into the standard zone. Therefore, for rational values of \( s = M/N \), where \( M \) and \( N \) are integers leads to the repetition of \( M \) branches along the \( k \) axis and \( N \) branches along the \( \omega \) axis. This is illustrated in Fig. 5, where the \( s = 2/3 \) case is displayed. When \( s \) is not a rational number, every single line determining the dispersion relations has a definite width. This means that in these cases any \( \omega \) corresponds to an infinite number of \( k \) values, and vice-versa. This is completely new situation, which has no analog in photonic or electronic band structures. The reason for this broadening is the existence of small but finite band-gaps. The folding of the dispersion curves leads to a slight displacement of the equivalent curves. The largest displacements appear for the intervals of \( \nu \) where gaps are large. Then, at the larger values of \( \nu \), the curves become closer and when \( \nu \) tends to infinity they approach the corresponding straight line branches. This means that, when \( s \) is not a rational number, incommensurability between the spatial and temporal periodicities exists. In this case the dispersion relation consists of separated points and it is intriguingly complex, as is illustrated for \( s = \sqrt{2} \) in the inset of Fig. 6. Finally, we emphasize that all these effects exist when the time variation of \( \varepsilon(t) \) is rapid. When \( \varepsilon(t) \) is a slowly varying function of time compared to the amplitude \( U_k(t) \), we can simply ignore its time dependence \( 21 \).

In summary, we have studied the effects of time variation of the dielectric constant of different electromagnetic media, which leads to the existence of band-gap like phenomena. The physical reason for these effects is the necessity of synchronization of the phases of the propagating waves with the external time periodicity of the medium. In cases of simultaneous space and time periodicity, we found an electromagnetic wave spectrum essentially described in an equivalent two-dimensional, three-dimensional or even four-dimensional cube defining the Brillouin zone. The folding of the dispersion relation when \( s = M/N \), with \( M \) and \( N \) integer, and broadening of dispersion curves have been demonstrated. When \( s \neq M/N \) the effects of incommensurability of the two internal lengths ( space \( L_x \) and time \( L_t = \frac{c \tau}{\varepsilon_0} \) exist, and point like spectra are exhibited. Modulation of the photonic band structure is suggested.

One of us, B.S.A. is grateful to Prof. E.Yablonovitch and Prof. M. Balkanski for the useful discussions. I.Z.K. and B.S.A. would like to acknowledge the assistance of Dr.V.Popov and Prof.M.Mateev. Work at Los Alamos National Laboratory is performed under the auspices of the US DoE.

[1] S. John, Phys. Rev. Lett. 53, 2169 (1984)
[2] P. W. Anderson, Phil. Mag. B 52, 5050 (1985)
[3] E. Yablonovitch , Phys. Rev. Letters , 58 , 2059 (1987)
[4] S. John, Phys.Rev.Lett. 58, 2486 (1987)
[5] K. M. Ho, C. T. Chan, and C. M. Soucoulis, Phys. Rev. Lett. 65,3152 (1990)
[6] K. M. Leung and Y. F. Lin, Phys. Rev. Lett. 65, 2636 (1990)
[7] Photonic Band Structure, Special Issue, J. of Mod. Optics.V.41, No.2 (1994)
[8] Yurii A. Vlasov, Martin OBoyle, Hendrik F. Hamann and
The physical reason for the band-gap like phenomena is the necessity of synchronization of the phases of the propagating waves with the external time periodicity of the medium. The additional energy for amplification will be proportional to the $\frac{\partial P}{\partial t}$, where $P$ is the induced polarization.

An experimental test of these ideas could be made at microwave frequencies. The response of most nonlinear dielectric materials is much stronger at microwave frequencies than at optical frequencies. For example, in $Sr_{1-x}Ba_xTiO_3$ compounds, it is possible to achieve $\varepsilon_f = 0.2\varepsilon_0$ with $\varepsilon_0 = 300$ for an applied electric field of about $10^6 V/m$ at frequencies up to several tens of GHz. In addition, at microwave frequencies, network analysis equipment and techniques could be employed to achieve high spectral resolution, good signal sensitivity, and large dynamic range.