Formation of antihydrogen in antiproton - positron collision

S.Roy, S. Ghosh Deb and C. Sinha
Theoretical Physics Department,
Indian Association for the Cultivation of Science, Kolkata-700032, India.

1 Abstract:

A quantum mechanical approach is proposed for the formation of antihydrogen ($\bar{H}$) in the ground and excited states (2s, 2p) via the mechanism of three body recombination (TBR) inside a trapped plasma of antiproton ($\bar{p}$) and positrons ($e^+$) or in the collision between the two beams of them. Variations of the differential (DCS) as well as the total (TCS) formation cross sections are studied as a function of the incident energies of both the active and the spectator $e^+$s. Significantly large cross sections are found at very low incident energies in the TBR process as compared to other processes leading to antihydrogen. The present ($\bar{H}$) formation cross section decreases with increasing positron energy (temperature) but no simple power law could be predicted for it covering the entire energy range, corroborating the experimental findings qualitatively. The formation cross sections are found to be much higher for unequal energies of the two $e^+$s than for equal energies, as expected physically.

2 Introduction:

Production of antihydrogen, the simplest and the most stable bound state of antimatter is one of the current topics nowadays both from the experimental and the theoretical perspectives mainly because its study provides various fundamental differences between matter and antimatter. Particularly, cold antihydrogen ($\bar{H}$) atom is an ideal system for studying the fundamental symmetries in physics e.g., the CPT invariance theorem in the standard quantum field theory and the gravitational weak equivalence principle for antimatter. The major challenge facing the $\bar{H}$ research is the production of cold and trapped ground state $\bar{H}$ that
is needed for the precise laser spectroscopy. Apart from these, there are many important practical applications of the $\bar{H}$ out of which the followings are worthy to be mentioned. First, the antihydrogen may also be used for igniting inertial confinement fusion pellets, the feasibility of which was already investigated [1]. Second, the antihydrogen finds important applications in the propulsion system [2].

In view of the recent technological advances in the cooling and trapping mechanism of antiprotons ($\bar{p}$) and positrons ($e^+$), the long term goal for the production of cold and trapped $\bar{H}$, necessary for the high precision spectroscopic studies has now become possible. This has motivated theoretical workers to venture different processes producing antihydrogen. The most important of these processes is the following three body recombination (TBR) $\bar{p} + e^+ + e^+ \rightarrow \bar{H} + e^+$ (I) in which a spectator particle carries away the excess energy and the momentum released in the recombination. The above reaction poses to be more efficient by orders of magnitude [3, 4] compared to other $\bar{H}$ production processes, e.g., the radiative recombination (RR) [5 - 7], the three body charge transfer between the Ps and the $\bar{p}$ [8 - 19]. The main reason for this is due to the following. The spectator positron in the TBR process efficiently carries off the extra energy, unlike the other $\bar{H}$ production reactions. Another important advantage of the process (I) is that the reactants are stable charged particles which can be held in a trap for cooling and then subsequently for the recombination to occur. In fact, it is found experimentally [3, 4] that the TBR in the trapped plasma of antiproton and positrons happens to be the most efficient $\bar{H}$ production reaction at low and intermediate energies. However, the main disadvantage of the TBR is that the $\bar{H}$ is favourably formed in the excited states [3], although for the high precision spectroscopic studies, the ground state $\bar{H}$ is highly needed. In contrast, in the RR process, although the ground state is favoured, the cross-section itself is much lower [3, 4].
Regarding the experimental situation for the TBR process, the three main International Groups are working on it with much endeavour at Cern, e.g., ATHENA [20 - 25], ATRAP [26 - 30] and Harvard [31 - 33] while another group from Riken [34, 35] is also concentrating on the experiments of cold and trapped $\bar{H}$ production. In all the experiments attention is being paid mainly to the temperature dependence of the $\bar{H}$ production at extreme low energies in the range of meV.

As for the theoretical situation, the first detailed study for the TBR is due to Robicheaux [36] in the framework of classical trajectory Monte Carlo (CTMC) method. However, in this calculation the Author introduced some fraction of electrons along with the $e^+$ as well as a strong magnetic field in order to make the process feasible. It was noted [36] that the $\bar{H}$ formation reduces substantially in presence of the $e^-$s. Prior to and also following this work [36], there exist some calculations by the same Author [37, 38] that mainly study the temperature dependence of the TBR process based on some statistical models.

The present work addresses the study of the $\bar{H}$ formation cross sections (both differential and total) through the TBR mechanism in the collision between the positron and the $\bar{p}$ plasma. To our knowledge, this is the first quantum mechanical attempt along this direction in the TBR process. Although experimentally the TBR process favours the $\bar{H}$ formation in highly excited states, for the theoreticians it is much easier to calculate the cross sections in the ground and low lying states. Thus in the absence of any experimental cross section data, the present theoretical estimates of the ground and excited (2s, 2p) states $\bar{H}$ formation cross-sections might give some stimulus and guidelines to the future detailed experiments.

The present model corresponds to the following situations. We consider an ensemble of weakly correlated positrons and the $e^+$ plasma density is assumed to be low enough so that the
\( e^+ - e^+ \) interaction can be treated as a perturbation. The antiproton is treated as a stationary ionic target located at the origin of coordinates which corresponds to the experimental situation of a cold and trapped \( \bar{p} \). The latter assumption should be quite legitimate when the positron velocity is much faster than that of the antiproton which happens to be the case due to large mass difference between the two. Since the recombination reaction requires a third body for the energy and momentum conservation of the process, another \( e^+ \) (electron) of the plasma serves this purpose and the process becomes a TBR one. Further, there is a probability of exchange between the active and the spectator positrons which is also incorporated in the present model.

### 3 Theory:

The present study deals with the following three body recombination process:

\[
\bar{p} + e^+ + e^+ \rightarrow \bar{H} + e^+. \tag{1}
\]

In the present formulation the \( \bar{p} \) is assumed to be stationary (target). The prior form of the transition amplitude \( T_{if} \) for this process is given by

\[
T_{if} = \langle \Psi_{f} (\vec{r}_1, \vec{r}_2)(1 + P) \mid V_i \mid \psi_i (\vec{r}_1, \vec{r}_2) \rangle, \tag{2}
\]

where \( P \) denotes the exchange operator corresponding to the interchange of the positrons in the final channel. \( V_i \) in equation (2) is the initial channel perturbation which is the part of the total interaction not diagonalised in the initial state and \( \psi_i \) is the corresponding asymptotic wave function. The final channel wave function \( \Psi_f \) satisfies the three body Schrödinger equation obeying the incoming wave boundary condition:

\[
(H - E)\Psi_f = 0. \tag{3}
\]

The total Hamiltonian (\( H \)) of the system can be written as

\[
H = -\frac{1}{2} \nabla_1^2 - \frac{1}{2} \nabla_2^2 - \frac{1}{r_1} - \frac{1}{r_2} + \frac{1}{r_{12}}, \tag{4}
\]
where \( \vec{r}_1 \) and \( \vec{r}_2 \) represent the position vectors of the active \( e^+ \) (to be transferred) and the spectator \( e^+ \)s respectively. The atomic unit (a.u.) is used throughout the work.

The initial channel asymptotic wave function \( \psi_i \) in equation (2) satisfies the following Schrödinger equation

\[
(-\frac{1}{2}\nabla_1^2 - \frac{1}{2}\nabla_2^2 - \frac{1}{r_1} - \frac{1}{r_2} - E) \psi_i = 0
\]  

and is given by

\[
\psi_i = N_j e^{i\vec{k}_j \cdot \vec{r}_j} _1F_1 [ i\alpha_j, 1, -i (k_j r_j - \vec{k}_j \cdot \vec{r}_j)]
\]  

with \( N_j = \exp(-\frac{\pi \alpha_j}{2}) \Gamma(1-i\alpha_j) \); \( j = 1, 2 \); \( \alpha_j = -\frac{1}{k_j} \); \( \vec{k}_j \) denotes the incident momentum of the active or the spectator \( e^+ \) respectively. The approximated final state wave function \( \Psi_f^- \) is chosen in the framework of eikonal approximation as follows:

\[
\Psi_f^- = \phi_f(r_1) e^{i\vec{k}_f \cdot \vec{r}_2} \exp [i\eta_f \int_{z_1}^\infty (\frac{1}{r_{12}} - \frac{1}{r_2})dz']
\]  

with \( \eta_f = \frac{1}{k_f} \), \( k_f \) being the final momentum of the spectator \( e^+ \); \( \phi_f(r_1) \) represents the bound state wave function of the \( \bar{H} \) atom.

Finally the differential cross section for the process (1) is given by

\[
\frac{d\sigma}{d\Omega} = \frac{k_f}{k_1 k_2} \left[ \frac{1}{4} (|f + g|^2) + \frac{3}{4} (|f - g|^2) \right],
\]  

where \( f \) and \( g \) corresponds to the direct and the exchange amplitudes respectively.

Using the following contour integral representations of the eikonal phase factors [39] as well as the coulomb functions [40] and after much analytical reductions [41, 42], the transition matrix element (2) is finally reduced [43] to a three dimensional integral which is evaluated numerically by using different quadrature methods. The eikonal phase factor is of the form

\[
y^{\pm(i\eta - n)} = \frac{(-1)^{n+1}}{2i \sin(\pm i\eta)} \int_{c}^{\infty} (-\lambda)^{\pm i\eta \pm 1} \exp(-\lambda y)dy \quad (9a)
\]  

where the contour \( c \) has a branch cut from 0 to \( \infty \) [39]; the confluent hypergeometric function:
\[ _1F_1 (i\alpha, 1, z) = \frac{1}{2\pi i} \int_{\Gamma_1}^{(0^+, 1^+)} p(\alpha, t) \exp(zt) dt \]  

(9b)

with \( p(\alpha, t) = t^{-1+i\alpha}(t-1)^{-i\alpha} \), \( \Gamma_1 \) is a closed contour encircling the two points 0 and 1 once anti-clockwise [40]. At the point where the contour crosses the real axis to the right side of 1, arg \( t \) and arg \( (t-1) \) are both zero.

4 Results and Discussions:

We have computed the \( \bar{H} \) formation cross sections both differential and total for the TBR process (1) in the framework of the coulomb distorted eikonal approximation (CDEA), where distortions have been included in both the channels. The exchange between the active and the spectator \( e^+ \)s is also incorporated. Since the present process (1) is an exothermic reaction it can occur even at zero incident energy. However, our results are not converged below 5 eV due to computational problems and are therefore not reported here. Furthermore, it may be mentioned that the present model might not yield very reliable results at extreme low energies.

Figs (1 - 5) exhibit the present differential cross sections (DCS) in the ground and excited states (2s and 2p) for different incident energies of both the positrons e.g., \( E_1 = E_2 = 10, 20, 25, 30 \) and \( 50 \) eV respectively. The figures reveal that the \( \bar{H} \) formation (in all the states) is strongly favoured in the forward directions and as such the DCS are presented upto \( 60^0 \) only, beyond which the cross sections become negligible. At very low incident energy, the magnitude of the formation cross section is found to be largest for the 2p state and smallest for the 1s state while the 2s lies in between, i.e., \( 2p > 2s > 1s \) (vide fig. 1). This trend of the DCS is noted upto 15 eV (not shown in figure), although with increasing incident energy, the maximum of the 2p DCS decreases and tends towards the 2s maximum so that at \( E_1 = E_2 = 20 \) eV, the 2s overtakes the 2p (vide fig. 2). The DCS peak in this case is in the order \( 2s > 2p > 1s \). In contrast, at intermediate and high incident energies (\( \sim 25 \) eV onwards), the
DCS is maximum for the 1s state and minimum for the 2p state while the 2s lies in between (i.e. $1s > 2s > 2p$, figs. 3 - 5).

As for the position of the DCS maximum, at low incident energies ($\sim$ upto 15 eV), the 1s and 2p maxima lie at some lower scattering angles ($\sim 20^0$) while the 2s maximum occurs at the extreme forward ($\sim 0^0$). With increasing energy, the DCS maxima for these two states (1s and 2p) move towards the extreme forward ($\sim 0^0$), while the 2s maximum moves in the reverse direction (vide figs. 2 - 4). However, at high incident energies (e.g., $\sim$ 40 eV onwards), all the partial DCS maxima are finally peaked at extreme forward $0^0$ (vide fig. 5), as expected.

Figs. 6(a) - 6(c) again exhibit the partial DCS but for some unequal energies of the two incident $e^+$ s (i.e. $E_1 \neq E_2$) along with a case for $E_1 = E_2$ (15 eV) for the sake of comparison. The following interesting features are noted from the figures. All the partial DCS are found to be much higher (by a factor of $\sim$ 2 to 2.5) when the energy of the active $e^+$ ($E_1$) is greater than that of the spectator one ($E_2$), i.e., when $E_1 > E_2$. The DCS for unequal ($E_1 \neq E_2$) energies lie much above than those for equal energies ($E_1 = E_2$). This is quite expected physically due to strong repulsion between the two $e^+$ s at equal energies.

Next we come over to the total cross sections (TCS) for the $\bar{H}$ formation displayed in figs. 7 and 8 for different sets of incident energies. Figure 7 displays the partial TCS when the two incident $e^+$s share equal energy ($E_1 = E_2$). As in the case of DCS, at low and intermediate incident energies ($\sim$ upto 20 eV), the partial TCS follows the order $2p > 2s > 1s$ (inset of fig.7), while beyond 25 eV it is in the decreasing order with excitation of the $\bar{H}$ state, i.e., $1s > 2s > 2p$. The dominance of the 2p TCS at low incident energies could probably be attributed to the long range polarization effects which is much stronger for the 2p state than for any other states. In fact, a major contribution to the polarization effect that mainly dominates at lower incident energies, comes from the lowest lying p state (i.e., 2p state).
Fig. 7 also indicates that although all the partial TCS decrease monotonically with increasing incident energy, they do not follow any simple power law (e.g. $\sim T^{-9/2}$), corroborating the experimental findings [21, 23].

Figures 8(a) - 8(c) display the partial TCS against the active $e^+$ energy ($E_1$) for some fixed values of $E_2$ (spectator) while the insets exhibit the reverse one, i.e. TCS vs $E_2$ for fixed $E_1$. As in the case of DCS, for a fixed sum of $E_1$ and $E_2$, the partial TCS is found to be larger when $E_1 > E_2$ than for $E_1 < E_2$. Further, the TCS against $E_2$ falls off much more rapidly than the TCS versus $E_1$ (cf. figs. 8 with their insets). Regarding the relative magnitude of the partial TCS, for lower energy of the spectator $e^+$, e.g., $E_2 = 10$ eV (fig. 8(a)), the general trend of the TCS follows the order $2p > 2s > 1s$ at low and intermediate $E_1$ while at higher $E_1$, the above order changes to $2s > 2p > 1s$. Similar behaviour is noted in fig. 8(b) (for $E_2 = 15$ eV) as in fig. 8(a) with some exceptions at higher $E_1$. At intermediate $E_2$ (25 eV, fig. 8(c)), the partial TCS follow different orders for different ranges of $E_1$, e.g., at lower $E_1$ the $2p$ dominates while at higher $E_1$ the $1s$ dominates. However, at higher $E_2$ ($\sim E_2 \geq 50$ eV), the $1s$ cross section dominates throughout the range of $E_1$ except at very low energies ($E_1 \sim 5 - 10$ eV) where the $2s$ is most prominent (not shown in figure).

Figure 9 demonstrates a comparison between the present TCS and the corresponding experimental results of Merrison et al [44] for the $Ps(1s)+\bar{p} \rightarrow \bar{H}(1s)+e^+$ process. The experimental Ps energies are converted to the antiproton energies following the relation: $E_\bar{p}(Kev) = \frac{K_{Ps} \times 6.8 \times 918}{1000}$. As was anticipated [3, 4], the present TBR cross sections are found to lie much above the experimental data [44] for the above-said process.

For the sake of some numerical measures, we have displayed in Table I the present partial ($1s$, $2s$, $2p$) TCS along with some other existing theoretical results due to Mitroy et al [14, 17] for the process $\bar{p} + Ps \rightarrow \bar{H}(n, l, m) + e^+$ using unitarized Born approximation [17] and close coupling approximation.
Results due to Sinha et al [19] for the above process in the eikonal approximation both with and without laser field are also included in the Table I. The incident energies are chosen in accordance with their [14, 17] calculations. The field assisted (FA) results [19] are presented for the field strength 0.01 a.u. and the frequency 0.043 a.u.

Table I

| Energy (eV) | Mitroy et al [14] 1s (2s+2p) | Mitroy et al [17] 1s 2s 2p | Sinha et al [19] FF (1s) FA (1s) | Present results 1s 2s 2p |
|------------|--------------------------------|---------------------------|-------------------------------|------------------------|
| 13.60      | 1.923 8.44                    | 1.46 0.66 3.81            | 0.92 8.32                     | 373.07 705.4 530.06    |
| 20.40      | -                              | 0.94 0.254 1.76           | 0.47 3.99                     | 68.25 59.62 36.13      |
| 25.84      | 0.735 1.729                    | -                         | 0.29 2.05                     | 23.16 15.43 11.048    |
| 34.00      | -                              | 0.394 0.08 0.396          | 0.16 0.65                     | 5.14 3.29 1.21        |
| 43.52      | 0.24 0.28                      | -                         | 0.08 0.19                     | 1.38 0.73 0.21        |
| 54.40      | -                              | 0.13 0.03 0.06            | 0.04 0.09                     | 0.507 0.212 0.651    |
| 63.92      | 0.078 0.053                    | -                         | 0.02 0.05                     | 0.165 0.064 0.034    |

Table I again confirms (as in fig.9) that the present TBR cross sections are much larger than all the other processes leading to antihydrogen throughout the energy range considered.
Table II displays the probable power laws obeyed by the partial as well as the sum TCS for different incident energy ranges of the $e^+$ corresponding to figures 7 ($E_1 = E_2$) and 8 ($E_1 \neq E_2$).

| Energy Range (in eV) | Power law obeyed |
|----------------------|------------------|
| $E_1 = E_2$          | $1s$ | $2s$ | $2p$ | $1s + 2s + 2p$ |
| 5 - 10               | $\sim E^{-2.9}$ | $\sim E^{-3.7}$ | $\sim E^{-4.8}$ | $\sim E^{-4.5}$ |
| 10 - 25              | $\sim E^{-3.9}$ | $\sim E^{-5.3}$ | $\sim E^{-6.0}$ | $\sim E^{-5.4}$ |
| 25 - 50              | $\sim E^{-5}$   | $\sim E^{-6.1}$ | $\sim E^{-6.7}$ | $\sim E^{-5.9}$ |
| $E_1 \leq E_2$ (10 eV) |     |     |     |               |
| 5 - 10               | $\sim E_1^{-1.6}$ | $\sim E_1^{-1.3}$ | $\sim E_1^{-1.8}$ | $\sim E_1^{-1.6}$ |
| $E_1 \geq E_2$ (10 eV) |     |     |     |               |
| 10 - 25              | $\sim E_1^{-1.5}$ | $\sim E_1^{-1.7}$ | $\sim E_1^{-2.1}$ | $\sim E_1^{-1.9}$ |
| 25 - 50              | $\sim E_1^{-1.5}$ | $\sim E_1^{-1.8}$ | $\sim E_1^{-2.3}$ | $\sim E_1^{-1.9}$ |

As is revealed from the table, the low energy partial TCS (e.g., $E_1 = E_2 \sim 5 - 10$ eV) falls off much slowly as compared to the intermediate and high energies and the slope of the $1s$ TCS (see also fig. 7) is much less than that of the others (2s, 2p and $1s + 2s + 2p$). Another important feature should be noted from this Table that for $E_1 \neq E_2$, the power of the exponent decreases as compared to the $E_1 = E_2$ case throughout the energy range. This again indicates the better efficiency of the $\bar{H}$ production for unequal energies ($E_1 \neq E_2$) of the active and the passive $e^+$ s over a wider energy ranges ($E_1$).
5 Conclusions:

The salient features of the present study are as follows:

At very low incident energies the present TBR cross section for the $\bar{H}$ formation in the $2p$ state is found to be the dominant process among the three states $1s$, $2s$, $2p$ while at intermediate and high incident energies, the ground state ($1s$) cross section dominates for both equal and unequal energies of the two positrons with some exceptions for the latter case ($E_1 > E_2$).

Substantially high cross sections are noted in the TBR model than in the other RR / charge transfer processes leading to antihydrogen.

The partial TCS is found to be significantly higher when the active $e^+$ energy is greater than that of the spectator $e^+$ (i.e., $E_1 > E_2$) than for $E_1 = E_2$ or for $E_1 < E_2$.

For a more efficient production of $\bar{H}$ for a wider energy range, the unequal ($E_1 > E_2$) distribution of energy between the active and the spectator positrons could be suggested rather than the equal one ($E_1 = E_2$).

The present $\bar{H}$ formation cross section decreases with increasing $e^+$ energy (i.e., temperature) but does not follow any simple scaling law (e.g., $\sim T^{-9/2}$), corroborating the experimental findings. However, both the partial and the sum TCS obey different power laws for different incident energy ranges.

Finally, the present results might be important for the future detailed $\bar{H}$ experiments.

6 References:

[1] Andre Gsponer and Jean - Pierr Hurni, arXiv: physics/0507125v2 [physics.plasm-ph] 17 July, (2005).
[2] Special issue of the Journal of the British Interplanetary Society on antimatter propulsion. JBIS 35 (1982) 387. - R. L. Forward: Making and storing antihydrogen for propulsion, Workshop on the Design of a low Energy Antimatter Facility in the USA, University of Wisconsin, October 3 - 5 (1985).

[3] G. Gabrielse, S. L. Rolston, L. Haarsma and W. Kells, Phys.Letters A, 129, 38 (1988).

[4] G. Gabrielse, Advances in Atomic, Molecular and Optical Physics, 50, 155 (2005).

[5] S. M. Li, Z.J. Chen, Q.Q. Wang, Z.F. Zhou, Eur. Phys. J. D., 7, 39 (1999).

[6] Shu-Min Li, Yan-Gang Miao, Zi-Fang Zhou, Ji Chen, Yao-Yang Liu, Phys. Rev. A, 58, 2615 (1998).

[7] Saverio Bivona, Riccardo Burlon, Gaetano Ferrante and Claudio Leone, Optics Express, 14, 3715 (2006).

[8] J. W. Humberston, M. Charlton, F.M. Jacobsen, B.I. Deutch, J. Phys. B, 20, L25 (1987).

[9] J. W. Darewych, J. Phys. B, 20, 5917 (1987).

[10] S. N. Nahar and J. M. Wadehra, Phys. Rev. A, 37, 4118, (1988).

[11] M. Charlton, Phys. Lett. A, 143, 143 (1990).

[12] S. Tripathi, C.Sinha and N. C. Sil, Phys. Rev. A, 42, 1785 (1990).

[13] J. Mitroy and A. T. Stelbovics, Phys. Rev. Lett., 72, 3495 (1994).

[14] J. Mitroy and G. Ryzhikh, J. Phys. B., 30, L371 (1997).

[15] S. Tripathi, R. Biswas and C. Sinha, Phys. Rev. A, 51 , 3584 (1995).

[16] J. Mitroy, Phys. Rev. A, 52 , 2859 (1995).

[17] J. Mitroy and A. T. Stelbovics, J. Phys. B, 27, L79 (1994).

[18] D. B. Cassidy, J. P. Merrison, M. Charlton, J. Mitroy and G. Ryzhikh, J. Phys. B, 32, 1923 (1999).
[19] A. Chattopadhyay, C. Sinha, Phys. Rev. A, 74, 022501 (2006).
[20] M. Amoretti et al, Nature, 419, 456 (2002).
[21] M. Amoretti et al, Phys. Rev. Lett., 91, 055001 (2003).
[22] M. Amoretti et al, Phys. lett. B., 578, 23 (2004).
[23] M. Amoretti et al, Phys. lett. B., 583, 59 (2004).
[24] N. Madsen et al, Phys. Rev. Lett., 94, 033403 (2005).
[25] M. Amoretti et al, Phys. Rev. Lett., 97, 213401 (2006).
[26] G. Gabrielse et al, Phys. Rev. Lett., 89, 213401 (2002).
[27] G. Gabrielse et al, Phys. Rev. Lett., 89, 233401 (2002).
[28] G. Gabrielse et al, Phys. Lett. B., 548, 140 (2002).
[29] G. Gabrielse et al, Phys. Rev. Lett., 93, 073401 (2004).
[30] G. Gabrielse et al, Phys. Rev. Lett., 98, 113002 (2007).
[31] G. Gabrielse et al, Phys. Lett. B., 507, 1 (2001).
[32] A. Speck et al, Phys. Lett. B, 597, 257 (2004).
[33] T. Pohl et al, Phys. Rev. Lett., 97, 143401 (2006).
[34] G. Andresen et al, Phys. Rev. Lett., 98, 023402 (2007).
[35] A Torii Hiroyuki et al. Private communication.
[36] F. Robicheaux, Phys. Rev. A, 70, 022510 (2004).
[37] F. Robicheaux, Phys. Rev. A, 73, 033401 (2006).
[38] F. Robicheaux, J. Phys. B, 40, 271 (2007).
[39] I. S. Gradshteyn and I. M. Ryzhik ; Tables and Integrals, Series and Products ( Academic New York, 1980 )p. 933.
[40] A. Messiah ; Quantum Mechanics ( North - Holland, Amsterdam, 1966), Vol. 1, p.481.
[41] R. Biswas and C. Sinha, Phys. Rev. A, 50, 354 (1994).

[42] B. Nath and C. Sinha, Eur. Phys. J. D., 6, 295 (1999).

[43] B. Nath and C. Sinha, J. Phys. B., 33, 5525 (2000).

[44] J. P. Merrison, H. Bluhme, J. Chevallier, B. I. Deutch, P. Hvelplund, L. V. Jorgensen, H. Knudsen, M. R. Poulsen and M. Charlton, Phys. Rev. Lett., 78, 2728 (1997).