Classification of certain types of maximal matrix subalgebras

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Abstract

Let $\mathcal{M}_n(K)$ denote the algebra of $n \times n$ matrices over a field $K$ of characteristic zero. A nonunital subalgebra $\mathcal{N} \subset \mathcal{M}_n(K)$ will be called a nonunital intersection if $\mathcal{N}$ is the intersection of two unital subalgebras of $\mathcal{M}_n(K)$. Appealing to recent work of Agore, we show that for $n \geq 3$, the dimension (over $K$) of a nonunital intersection is at most $(n-1)(n-2)$, and we completely classify the nonunital intersections of maximum dimension $(n-1)(n-2)$. We also classify the unital subalgebras of maximum dimension properly contained in a parabolic subalgebra of maximum dimension in $\mathcal{M}_n(K)$.

1 Introduction

Let $\mathcal{M}_n(F)$ denote the algebra of $n \times n$ matrices over a field $F$. For some interesting sets $\Lambda$ of subspaces $S \subset \mathcal{M}_n(F)$, those $S \in \Lambda$ of maximum dimension over $F$ have been completely classified. For example, a theorem of Gerstenhaber and Serezhkin [6, Theorem 1] states that when $\Lambda$ is the set of subspaces $S \subset \mathcal{M}_n(F)$ for which every matrix in $S$ is nilpotent, then each $S \in \Lambda$ of maximum dimension is conjugate to the algebra of all strictly upper triangular matrices in $\mathcal{M}_n(F)$. For another example, it is shown in [1, Prop. 2.5] that when $\Lambda$ is the set of proper unital subalgebras $S \subset \mathcal{M}_n(F)$ and $F$ is an algebraically closed field of characteristic zero, then each $S \in \Lambda$ of maximum dimension is a parabolic subalgebra of maximum dimension in $\mathcal{M}_n(F)$.

The goal of this paper is to classify the elements in $\Lambda$ of maximum dimension in the cases $\Lambda = \Gamma$ and $\Lambda = \Omega$, where the sets $\Gamma$ and $\Omega$ are defined below.

In Isaac’s text [3, p. 161], every ring is required to have a unity, but the unity in a subring need not be the same as the unity in its parent ring. Under this definition, a ring may have subrings whose intersection is not a subring. This motivated us to study examples of pairs of unital subrings in $\mathcal{M}_n(K)$ whose intersection $\mathcal{N}$ is nonunital, where $K$ is a field of characteristic zero. We call such $\mathcal{N}$ a nonunital intersection and we let $\Gamma$ denote the set of all nonunital intersections $\mathcal{N} \subset \mathcal{M}_n(K)$. Note that $\Gamma$ is closed under transposition and conjugation, i.e., if $\mathcal{N} \in \Gamma$, then $\mathcal{N}^T \in \Gamma$ and $S^{-1} \mathcal{N} S \in \Gamma$ for any invertible $S \in \mathcal{M}_n(K)$.

In order to define $\Omega$, we need to establish some notation. For brevity, write $\mathcal{M} = \mathcal{M}_n = \mathcal{M}_n(K)$. In the spirit of [2, p. viii], we define a subalgebra of
\( \mathcal{M} \) to be a vector subspace of \( \mathcal{M} \) over \( K \) closed under the multiplication of \( \mathcal{M} \) (cf. [2, p. 2]); thus a subalgebra need not have a unity, and the unity of a unital subalgebra need not be a unity of the parent algebra. Subalgebras \( \mathcal{A}, \mathcal{B} \subset \mathcal{M} \) are said to be similar if \( \mathcal{A} = \{ S^{-1}BS : B \in \mathcal{B} \} \) for some invertible \( S \in \mathcal{M} \). The notation \( \mathcal{M}[R_n] \) will be used for the subalgebra of \( \mathcal{M} \) consisting of those matrices whose \( n \)-th row is zero. Similarly, \( \mathcal{M}[R_n, C_n] \) indicates that the \( n \)-th row and \( n \)-th column are zero, etc. For \( 1 \leq i, j \leq n \), let \( E_{i,j} \) denote the elementary matrix in \( \mathcal{M} \) with a single entry 1 in row \( i \), column \( j \), and 0 in each of the other \( n^2 - 1 \) positions. The identity matrix in \( \mathcal{M} \) will be denoted by \( I \). For the maximal parabolic subalgebra \( \mathcal{P} := \mathcal{M}[R_n] + KE_{n,n} \) in \( \mathcal{M} \), define \( \Omega \) to be the set of proper subalgebras \( \mathcal{B} \) of \( \mathcal{P} \) with \( \mathcal{B} \neq \mathcal{M}[R_n] \).

We now describe Theorems 3.1–3.3, our main results. Theorem 3.1 shows that \( \dim \mathcal{N} \leq (n-1)(n-2) \) for each \( \mathcal{N} \in \Gamma \). Theorem 3.2 shows that up to similarity, \( \mathcal{W} := \mathcal{M}[R_n, R_{n-1}, C_n] \) and \( \mathcal{W}^T := \mathcal{M}[R_n, C_{n-1}, C_n] \) are the only subalgebras in \( \Gamma \) having maximum dimension \( (n-1)(n-2) \). In Theorem 3.3, we show that \( \dim \mathcal{B} \leq n^2 - 2n + 3 \) for each \( \mathcal{B} \in \Omega \), and we classify all \( \mathcal{B} \in \Omega \) of maximum dimension \( n^2 - 2n + 3 \).

The proofs of our theorems depend on four lemmas which are proved in Section 2. Lemma 2.1 shows that \( \mathcal{W} \) (and hence also \( \mathcal{W}^T \)) is a nonunital intersection of dimension \( (n-1)(n-2) \) when \( n \geq 3 \). Lemmas 2.2 and 2.3 show that \( \dim \mathcal{L} \leq n(n-1) \) for any nonunital subalgebra \( \mathcal{L} \subset \mathcal{M} \), and when equality holds, \( \mathcal{L} \) must be similar to \( \mathcal{M}[R_n] \) or \( \mathcal{M}[C_n] \). (Thus if \( \Lambda \) denotes the set of nonunital subalgebras \( \mathcal{L} \subset \mathcal{M} \), Lemmas 2.2 and 2.3 classify those \( \mathcal{L} \in \Lambda \) of maximum dimension.) Lemma 2.4 shows that if \( \mathcal{U} \subset \mathcal{M} \) is a subalgebra with unity different from \( I \), then some conjugate of \( \mathcal{U} \) is contained in \( \mathcal{M}[R_n, C_n] \).

## 2 Lemmas

Recall the definition \( \mathcal{W} := \mathcal{M}[R_n, R_{n-1}, C_n] \).

**Lemma 2.1.** For \( n \geq 3 \), \( \mathcal{W} \in \Gamma \) and \( \dim \mathcal{W} = (n-1)(n-2) \).

*Proof.* For \( n > 1 \), define \( A \in \mathcal{M} \) by \( A = I + E_{n,n-1} \). Note that \( A^{-1} = I - E_{n,n-1} \). A straightforward computation shows that for \( M \in \mathcal{M}[R_n, C_n] \), the conjugate \( AMA^{-1} \) is obtained from \( M \) by replacing the (zero) bottom row of \( M \) by the \( (n-1) \)-th row of \( M \). Since the bottom two rows of \( AMA^{-1} \)
are identical, it follows that

\[ \mathcal{A}M^{-1} \in \mathcal{M}[R_n, C_n] \cap \mathcal{A}\mathcal{M}[R_n, C_n]A^{-1} \] if and only if \( \mathcal{A}M^{-1} \in \mathcal{W} \).

Since \( \mathcal{W} = A^{-1}\mathcal{W}A \), this shows that \( \mathcal{W} \) is the intersection of the unital subalgebras \( A^{-1}\mathcal{M}[R_n, C_n]A \) and \( \mathcal{M}[R_n, C_n] \). To see that \( \mathcal{W} \) is nonunital, note that \( E_{1,n-1} \) is a nonzero matrix in \( \mathcal{W} \) for which \( E_{1,n-1}W \) is the zero matrix for each \( W \in \mathcal{W} \); thus \( \mathcal{W} \) cannot have a right identity, so \( \mathcal{W} \in \Gamma \). Finally, it follows from the definition of \( \mathcal{W} \) that \( \dim \mathcal{W} = (n-1)(n-2) \).

Remark: The same proof shows that \( \mathcal{W} \in \Gamma \) holds when the field \( K \) is replaced by an arbitrary ring \( R \) with \( 1 \neq 0 \). If moreover \( R \) happens to be commutative, then the dimension of the algebra \( \mathcal{W} \) over \( R \) is well-defined \([7, p. 483]\) and it equals \( (n-1)(n-2) \).

Lemma 2.2. For any nonunital subalgebra \( \mathcal{L} \subset \mathcal{M} \), \( \dim \mathcal{L} \leq n(n-1) \).

Proof. It cannot happen that \( \mathcal{L} + KI = \mathcal{M} \), otherwise \( \mathcal{L} \) would be a two-sided proper ideal of \( \mathcal{M} \), contradicting the fact that \( \mathcal{M} \) is a simple ring \([7, p. 280]\). Since \( \mathcal{L} + KI \) is a proper subalgebra of \( \mathcal{M} \) containing the unity \( I \), it follows from Agore \([1, Cor. 2.6]\) that \( \dim \mathcal{L} = -1 + \dim (\mathcal{L} + KI) \leq n(n-1) \).

Lemma 2.3. Any nonunital subalgebra \( \mathcal{L} \subset \mathcal{M} \) with \( \dim \mathcal{L} = n(n-1) \) must be similar to either \( \mathcal{M}[R_n] \) or \( \mathcal{M}[C_n] = \mathcal{M}[R_n]^T \).

Proof. Consider the two parabolic subalgebras \( \mathcal{P}, \mathcal{P}' \subset \mathcal{M} \) defined by

\[ \mathcal{P} = \mathcal{P}_K = \mathcal{M}[R_n] + KE_{n,n} \quad \mathcal{P}' = \mathcal{P}'_K = \mathcal{M}[C_1] + KE_{1,1} \, . \]

Note that \( \mathcal{P}' \) is similar to the transpose \( \mathcal{P}^T \). Since \( \mathcal{L} + KI \) is a proper subalgebra of \( \mathcal{M} \) of dimension \( n(n-1)+1 \), it follows from Agore \([1, Prop. 2.5]\) that \( \mathcal{L} + KI \) is similar to \( \mathcal{P} \) or \( \mathcal{P}' \), under the condition that \( K \) is algebraically closed. However, Nolan Wallach \([8]\) has proved that this condition can be dropped; see the Appendix. Thus, replacing \( \mathcal{L} \) by a conjugate if necessary, we may assume that \( \mathcal{L} + KI = \mathcal{P} \) or \( \mathcal{L} + KI = \mathcal{P}^T \). We will assume that \( \mathcal{L} + KI = \mathcal{P} \), since the proof for \( \mathcal{P}^T \) is essentially the same. It suffices to show that \( \mathcal{L} \) is similar to \( \mathcal{M}[R_n] \) or \( \mathcal{M}[C_1] \), since \( \mathcal{M}[C_1] \) is similar to \( \mathcal{M}[C_n] \).

Assume temporarily that each \( L \in \mathcal{L} \) has all entries 0 in its upper left \( (n-1) \times (n-1) \) corner. Then \( n = 2 \), because if \( n \geq 3 \), then every matrix in \( \mathcal{P} \) would have a zero entry in row 1, column 2, contradicting the definition of \( \mathcal{P} \).
Since $L \subset M[2]$ and both sides have dimension 2, we have $L = M[2]$, which proves the theorem under our temporary assumption.

When the temporary assumption is false, there exists $L \in L$ with the entry 1 in row $i$, column $j$ for some fixed pair $i, j$ with $1 \leq i, j \leq n - 1$. Since $E_{i,i}$ and $E_{j,j}$ are in $P = L + KI$ and $L$ is a two-sided ideal of $P$, we have $E_{i,j} = E_{i,i}LE_{j,j} \in L$. Consequently, $E_{a,b} = E_{a,i}E_{i,j}E_{j,b} \in L$ for all pairs $a, b$ with $1 \leq a \leq n - 1$ and $1 \leq b \leq n$. Therefore

$$M[2] = \sum_{a=1}^{n-1} \sum_{b=1}^{n} KE_{a,b} \subset L,$$

and since both $M[2]$ and $L$ have the same dimension $n(n-1)$, we conclude that $L = M[2]$. 

**Remark:** Any subalgebra $B \subset M$ properly containing $M[2]$ must also contain $I$. To see this, note that $B$ contains a nonzero matrix of the form

$$B := \sum_{i=1}^{n} c_i E_{n,i}, \quad c_i \in K.$$

If $c_j = 0$ for all $j < n$, then $E_{n,n} \in B$, so $I \in B$. On the other hand, if $c_j \neq 0$ for some $j < n$, then $E_{n,n} = c_j^{-1} BE_{j,n} \in B$, so again $I \in B$.

**Lemma 2.4.** Suppose that a subalgebra $U \subset M$ has a unity $e \neq I$. Then $S^{-1}US \subset M[2]$ for some invertible $S \in M$.

**Proof.** Let $r$ be the rank of the matrix $e$. Note that $e$ is idempotent, so by [5, p. 27], there exists an invertible $S \in M$ for which $S^{-1}eS = D_r$, where $D_r$ is a diagonal matrix with entries 1 in rows 1 through $r$, and entries 0 elsewhere. Replacing $U$ by $S^{-1}US$ if necessary, we may assume that $e = D_r$. Since $r \leq n - 1$, we have

$$U = eUe \subset eMe = D_rMD_r \subset D_{n-1}MD_{n-1} = M[2].$$

**3 Theorems**

Recall that $\Gamma$ is the set of all nonunital intersections in $M$.
Theorem 3.1. If \( N \in \Gamma \), then \( \dim N \leq (n - 1)(n - 2) \).

Proof. Let \( N \in \Gamma \), so that \( N = U \cap V \) for some pair of unital subalgebras \( U, V \subset \mathcal{M} \). Since \( N \) is nonunital, one of \( U, V \), say \( U \), does not contain \( I \). Thus \( U \) contains a unity \( e \neq I \). Define \( S \) as in Lemma 2.4. Replacing \( U, V, N \) by \( S^{-1}U S, S^{-1}V S, S^{-1}NS \), if necessary, we deduce from Lemma 2.4 that \( U \) is contained in \( \mathcal{M}[R_n, C_n] \). Since \( N \) is a nonunital subalgebra of \( U \subset \mathcal{M}[R_n, C_n] \), it follows from Lemma 2.2 with \((n - 1)\) in place of \( n \) that \( \dim N \leq (n - 1)(n - 2) \).

Theorem 3.2. Let \( n \geq 3 \). Then up to similarity, \( W \) and \( W^T \) are the only subalgebras of \( \mathcal{M} \) in \( \Gamma \) having dimension \( (n - 1)(n - 2) \).

Proof. By Lemma 2.1, every subalgebra of \( \mathcal{M} \) similar to \( W \) or \( W^T \) lies in \( \Gamma \) and has dimension \( (n - 1)(n - 2) \). Conversely, let \( N \in \Gamma \) with \( \dim N = (n - 1)(n - 2) \). We must show that \( N \) is similar to \( W \) or \( W^T \).

We may assume, as in the proof of Theorem 3.1, that \( N \) is a nonunital subalgebra of \( \mathcal{M}[R_n, C_n] \). Let \( \mathcal{L} \) be the subalgebra of \( \mathcal{M}_{n-1} \) consisting of those matrices in the upper left \( (n - 1) \times (n - 1) \) corners of the matrices in \( N \). Since \( \dim \mathcal{L} = \dim N = (n - 1)(n - 2) \), it follows from Lemma 2.3 that \( \mathcal{L} \) is similar to \( \mathcal{M}_{n-1}[R_{n-1}] \) or \( \mathcal{M}_{n-1}[C_{n-1}] \). Thus \( N \) is similar to \( W = \mathcal{M}[R_n, R_{n-1}, C_n] \) or \( W^T = \mathcal{M}[R_n, C_{n-1}, C_n] \).

Recall that \( \Omega \) denotes the set of proper subalgebras \( B \neq \mathcal{M}[R_n] \) in \( \mathcal{P} \).

Theorem 3.3. Let \( B \in \Omega \). Then \( \dim B \leq n^2 - 2n + 3 \). If \( B \) has maximum dimension \( n^2 - 2n + 3 \), then \( B \) is similar to one of

\[
\mathcal{M}E_{n,n} + \mathcal{M}[R_n, C_1] + KE_{1,1}, \quad \mathcal{M}E_{n,n} + \mathcal{M}[R_n, R_{n-1}] + KE_{n-1,n-1}.
\]

Proof. Let \( e \in \mathcal{M} \) denote the diagonal matrix of rank \( n - 1 \) with entry \( 0 \) in row \( n \) and entries \( 1 \) in the remaining rows. Because \( e \) is a left identity in \( \mathcal{M}[R_n] \) and \( Be \subset \mathcal{M}[R_n, C_n] \), it follows that \( Be \) is an algebra.

First suppose that \( Be = \mathcal{M}[R_n, C_n] \). Then \( \mathcal{P} = \mathcal{C} + \mathcal{D} \), where

\[
\mathcal{C} = B + KE_{n,n}, \quad \mathcal{D} = \mathcal{M}[R_n]E_{n,n}.
\]

We proceed to show that \( \mathcal{C} \cap \mathcal{D} \) is zero. Assume for the purpose of contradiction that there exists a nonzero matrix \( B \in \mathcal{C} \cap \mathcal{D} \). Then \( B \in \mathcal{B} \). We have \( BB = \mathcal{D} \), since the matrices in \( \mathcal{B} \) have all possible submatrices in their
upper left \((n - 1)\) by \((n - 1)\) corners. Thus \(D \subset B \subset C\), which implies that \(M[R_n] \subset B\) and \(P = C = B + KE_{n,n}\). If \(KE_{n,n} \subset B\), then \(B = P\), and if \(KE_{n,n}\) is not contained in \(B\), then \(B = M[R_n]\); either case contradicts the fact that \(B \in \Omega\).

Since \(C \cap D\) is zero,
\[
\dim B \leq \dim C = \dim P - \dim D = (n^2 - n + 1) - (n - 1) = n^2 - 2n + 2.
\]
Thus
\[
\dim B < n^2 - 2n + 3,
\]
so the desired upper bound for \(\dim B\) holds when \(B = M[R_n, C_n]\).

Next suppose that \(B\) is a proper subalgebra of \(M[R_n, C_n]\). We proceed to show that
\[
d := \dim B_e \leq (n - 1)(n - 2) + 1,
\]
by showing that
\[
\dim \mathcal{L} \leq (n - 1)(n - 2) + 1
\]
for every proper subalgebra \(\mathcal{L}\) of \(M_{n-1}\). If \(\mathcal{L}\) is nonunital, then
\[
\dim \mathcal{L} \leq (n - 1)(n - 2) < (n - 1)(n - 2) + 1
\]
by Lemma 2.2 (with \(n - 1\) in place of \(n\)). If \(\mathcal{L}\) contains a unit different from the identity of \(M_{n-1}\), then by Lemma 2.4 (with \(\mathcal{L}\) in place of \(U\)),
\[
\dim \mathcal{L} \leq \dim M[R_{n-1}, C_{n-1}] = (n - 2)^2 < (n - 1)(n - 2) + 1.
\]
If \(\mathcal{L}\) contains the identity of \(M_{n-1}\), then by Agore [1, Cor. 2.6],
\[
\dim \mathcal{L} \leq (n - 1)(n - 2) + 1.
\]
This completes the demonstration that \(d \leq (n - 1)(n - 2) + 1\).

Let \(B_1 e, B_2 e, \ldots, B_d e\) be a basis for \(B_e\), with \(B_i \in B\). Since \(B\) is a subspace of the vector space spanned by the \(d + n\) matrices
\[
B_1, \ldots, B_d, E_{1,n}, \ldots, E_{n,n},
\]
it follows that
\[
\dim B \leq d + n \leq (n - 1)(n - 2) + 1 + n = n^2 - 2n + 3.
\]
Thus the desired upper bound for \( \dim B \) holds in all cases.

The argument above shows that when we have the equality

\[
\dim B = d + n = (n - 1)(n - 2) + 1 + n = n^2 - 2n + 3,
\]

then

\[
B = Be + ME_{n,n}.
\]

Moreover, from the equality \( d = \dim Be = (n - 1)(n - 2) + 1 \), it follows from Agore [1, Prop. 2.5] (again appealing to the Appendix to dispense with the condition of algebraic closure) that there is an invertible matrix \( S \) in the set \( M[R_n, C_n] + E_{n,n} \) such that

\[
S^{-1}BeS \text{ is equal to one of } M[R_n, C_n, C_1] + KE_{1,1}, \quad M[R_n, C_n, R_{n-1}] + KE_{n-1,n-1}.
\]

Since \( S^{-1}ME_{n,n}S = ME_{n,n} \), we achieve the desired classification of \( \Omega \). \( \square \)

4 Appendix

Let \( F \) be a field of characteristic 0 with algebraic closure \( \overline{F} \). Given a proper subalgebra \( C \subset M_n(F) \) of maximum dimension, Agore [1, Prop. 2.5, Cor. 2.6] proved that the \( \overline{F} \)-span of \( C \) is similar over \( \overline{F} \) to the \( \overline{F} \)-span of \( D \) for some parabolic subalgebra \( D \) of maximum dimension in \( M_n(F) \). The purpose of this Appendix is to deduce that \( C \) is similar over \( F \) to \( D \).

**Lemma 4.1.** (Wallach) Let \( A \) be a subspace of \( M_n(F) \) of dimension \( n - 1 \) such that \( A \otimes_F \overline{F} \) has basis of one of the following two forms:

a) \( x_1 \otimes \lambda_1, x_2 \otimes \lambda_1, \ldots, x_{n-1} \otimes \lambda_1 \), with \( \lambda_1 \in (F^n)^*, x_j \in F^n \) and \( \lambda_1(x_j) = 0 \),

b) \( x_1 \otimes \lambda_1, x_1 \otimes \lambda_2, \ldots, x_1 \otimes \lambda_{n-1} \), with \( \lambda_j \in (F^n)^*, x_1 \in F^n \) and \( \lambda_j(x_1) = 0 \).

Then in case a) \( A \) is \( F \)-conjugate (i.e. under \( GL(n, F) \)) to the span of the matrices \( E_{i,n} \) with \( i = 1, \ldots, n - 1 \), and in case b) \( A \) is \( F \)-conjugate to the span of the matrices \( E_{n,i} \) with \( i = 1, \ldots, n - 1 \).

**Proof.** In either case, if \( X, Y \in A \) then \( XY = 0 \) and \( X \) has rank 1. For \( X \) of rank 1, we have \( XF^n = Fy \) for some \( y \neq 0 \). Thus there exists \( \mu \in (F^n)^* \) with \( Xz = \mu(z)y = (y \otimes \mu)(z) \) for all \( z \). We conclude that \( A \) has a basis over \( F \) of the form \( X_i = y_i \otimes \mu_i \) for \( i = 1, \ldots, n - 1 \).

We now assume that case a) is true (the argument for the other case is essentially the same). In case a), there exists \( z \in F^n \) such that

\[
\{X_1(z), \ldots, X_{n-1}(z)\}
\]
is linearly independent over $\bar{F}$. This implies that

$$\mu_1(z) \cdots \mu_{n-1}(z)y_1 \land \cdots \land y_{n-1} \neq 0.$$ 

Thus $y_1, \ldots, y_{n-1}$ are linearly independent. But $0 = X_iX_j = \mu_i(y_j)y_i \otimes \mu_j$. Thus $\mu_i(y_j) = 0$ for all $j = 1, \ldots, n - 1$. Let $\nu$ be a non-zero element of $(\bar{F}^n)^*$ such that $\nu(y_i) = 0$ for all $i = 1, \ldots, n - 1$. Then $\nu$ is unique up to non-zero scalar multiple. Thus $y_i \otimes \nu, i = 1, \ldots, n - 1$ is an $F$-basis of $\mathcal{A}$. Clearly there exists $g \in \text{GL}(n, F)$ such that $e_i, \ldots, e_n$ is the standard basis and $\xi_1, \ldots, \xi_n$ is the dual basis then $gy_i = e_i$ and $\nu \circ g = \xi_n$. This completes the proof in case a).

**Proposition 4.2. (Wallach)** Suppose that $\mathcal{L} \subset \mathcal{M}_n(F)$ is a subalgebra such that $\mathcal{L} \otimes_F \bar{F}$ is either:

a) conjugate to the parabolic subalgebra $\mathcal{P}_F$,

b) conjugate to the parabolic subalgebra $(\mathcal{P}_F)\dagger$.

In case a) $\mathcal{L}$ is $F$-conjugate to $\mathcal{P}_F$. In case b) $\mathcal{L}$ is $F$-conjugate to $\mathcal{P}_F\dagger$.

**Proof.** We just do case a) as case b) is proved in the same way. We look upon $\mathcal{L}$ as a Lie algebra over $F$. Then Levi’s theorem [4, p. 91] implies that $\mathcal{L} = S \oplus R$ with $S$ a semi-simple Lie algebra and $R$ the radical (the maximal solvable ideal). Thus $\mathcal{L} \otimes_F \bar{F} = S \otimes_F \bar{F} \oplus R \otimes_F \bar{F}$. Therefore $R \otimes_F \bar{F}$ is the radical of $\mathcal{L} \otimes_F \bar{F}$. If we conjugate $\mathcal{L} \otimes_F \bar{F}$ to $\mathcal{P}_F$ via $h \in \text{GL}(n, F)$, then we see that

$$h[R \otimes_F \bar{F}, R \otimes_F \bar{F}]h^{-1}$$

has basis $E_{i,n}, i = 1, \ldots, n - 1$. Thus hypothesis a) of Lemma 4.1 is satisfied for $\mathcal{A} = [R, R]$. There exists therefore $g \in \text{GL}(n, F)$ such that $g\mathcal{A}g^{-1}$ has basis $E_{i,n}, i = 1, \ldots, n - 1$. Assume that we have replaced $\mathcal{L}$ with $g\mathcal{L}g^{-1}$. Then $\mathcal{A}$ has basis $E_{i,n}, i = 1, \ldots, n - 1$. Since $[\mathcal{L}, \mathcal{A}] \subset \mathcal{A}$ and $\mathcal{P}_F$ is exactly the set of elements $X$ of $\mathcal{M}_n(F)$ such that $[X, \mathcal{A}] \subset \mathcal{A}$, we have $\mathcal{L} \subset \mathcal{P}_F$. Thus $\mathcal{L} = \mathcal{P}_F$, as both sides have the same dimension. □

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