WDVV Equations in Seiberg-Witten theory 
and associative algebras

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This is a short review of the results on the associativity algebras and WDVV equations found recently 
for the Seiberg-Witten solutions of \( N = 2 \) 4d SUSY gauge theories. The presentation is mostly based on 
the integrable treatment of these solutions.

1. What is WDVV. More than two years ago N.Seiberg and E.Witten \([1]\) proposed a new 
way to deal with the low-energy effective actions 
of \( N = 2 \) four-dimensional supersymmetric gauge 
theories, both pure gauge theories (i.e. containing 
only vector supermultiplet) and those with matter 
hypermultiplets. Among other things, they have 
shown that the low-energy effective actions (the 
end-points of the renormalization group flows) fit 
into universality classes depending on the vacuum 
of the theory. If the moduli space of these 
vacua is a finite-dimensional variety, the effective 
can be essentially described in terms of system with \( \text{finite-dimensional phase space} \) \((\# \text{ of} \) 
degrees of freedom is equal to the rank of the 
gauge group), although the original theory lives 
in a many-dimensional space-time. These effective 
theories turn out to be integrable. Integrable 
structure behind the Seiberg-Witten (SW) approach 
has been found in \([2]\) and later examined in 
detail in \([3]\).

The second important property of the SW framework which merits the adjective "topological" 
has been recently revealed in the series of papers 
\([4,5]\) and has much to do with the associative 
algebras. Namely, it turns out that the prepotential 
of SW theory satisfies a set of Witten-Dijkgraaf-
Verlinde-Verlinde (WDVV) equations. These equations 
have been originally presented in \([6]\) (in a different form, see below)

\[
F_i F_j^{-1} F_k = F_k F_j^{-1} F_i
\]

where \( F_i \)'s are matrices with the matrix elements 
that are the third derivatives of the unique function \( F \) of many variables \( a_i \)'s (prepotential in the 
SW theory) that given on a moduli space:

\[
(F_i)_{jk} = \frac{\partial^3 F}{\partial a_i \partial a_j \partial a_k}, \quad i, j, k = 1, ..., n
\]

Although generally there is a lot of solutions to the 
matrix equations \([6]\), it is extremely non-trivial 
task to express all the matrix elements through 
the only function \( F \). In fact, there have been only 
known the two different classes of the non-trivial 
solutions to the WDVV equations, both being intimately 
related to the two-dimensional topological 
theories of type A (quantum cohomologies \([7]\)) and 
of type B (\( N = 2 \) SUSY Landau-Ginzburg (LG) 
theories that were investigated in a variety of papers, see, for example, \([8]\) and references therein). 
Thus, the existence of a new class of solutions connected 
with the four-dimensional theories looks quite striking. It is worth noting that both the 
two-dimensional topological theories and the SW 
theories reveal the integrability structures related 
the WDVV equations. Namely, the function \( F \) plays the role of the (quasiclassical) \( \tau \)-function of 
some Whitham type hierarchy \([8,2,3]\).

In this brief review, we will describe the results 
of papers \([4,5]\) that deal with the structure and 
origin of the WDVV equations in the SW theories. 
To give some insight of the general structure of 
the WDVV equations, let us consider the simplest 
non-trivial examples of \( n = 3 \) WDVV equations 
in topological theories. The first example is the 
\( N = 2 \) SUSY LG theory with the superpotential
W'(\lambda) = \lambda^3 - q [8]. In this case, the prepotential reads as
\begin{equation}
F = \frac{1}{2}a_1a_2^2 + \frac{1}{2}a_1^2a_3 + \frac{q}{2}a_2a_3^2
\end{equation}
and the matrices $F_i$ (the third derivatives of the prepotential) are
\begin{align*}
F_1 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \\
F_2 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & q \end{pmatrix}, \\
F_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & q \\ 0 & q & 0 \end{pmatrix}.
\end{align*}
One can easily check that these matrices do really satisfy the WDVV equations (4).

The second example is the quantum cohomologies of $\mathbb{CP}^2$. In this case, the prepotential is given by the formula [7]
\begin{equation}
F = \frac{1}{2}a_1a_2^2 + \frac{1}{2}a_1^2a_3 + \sum_{k=1}^{\infty} N_k a_3^{3k-1} (3k - 1)! e^{ka_2}
\end{equation}
where the coefficients $N_k$ (describing the rational Gromov-Witten classes) counts the number of the rational curves in $\mathbb{CP}^2$ and are to be calculated. Since the matrices $F$ have the form
\begin{align*}
F_1 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \\
F_2 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & F_{222} & F_{223} \\ 0 & F_{232} & F_{233} \end{pmatrix}, \\
F_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & F_{223} & F_{233} \\ 0 & F_{233} & F_{333} \end{pmatrix}
\end{align*}
the WDVV equations are equivalent to the identity
\begin{equation}
F_{333} = F_{223}^2 - F_{222}F_{233}
\end{equation}
which, in turn, results into the recurrent relation defining the coefficients $N_k:
\begin{equation}
\frac{N_k}{(3k - 4)!} = \sum_{a+b=k} \frac{a^2b(3b-1)b(2a-b)}{(3a-1)!(3b-1)!} N_aN_b.
\end{equation}
The crucial feature of the presented examples is that, in both cases, there exists a constant matrix $F_1$. Following [8], one can consider it as a flat metric on the moduli space. In fact, in its original version, the WDVV equations have been written in a slightly different form, that is, as the associativity condition of some algebra. We will discuss this later, and now just remark that, having distinguished (constant) metric $\eta \equiv F_1$, one can naturally rewrite (5) as the equations
\begin{equation}
C_iC_j = C_jC_i
\end{equation}
for the matrices $(C_i)_{jk} = \eta^{-1}F_{ij}$, i.e. $C_{ik} = \eta^{ij}F_{jk}$. Formula (3) is equivalent to (4) with $j = 1$. Moreover, this particular relation is already sufficient [4,5] to reproduce the whole set of the WDVV equations (4).

Let us also note that, although the WDVV equations can be fulfilled only for some specific choices of the coordinates $a_i$ on the moduli space, they still admit any linear transformation. This defines the flat structures on the moduli space, and we often call $a_i$ flat coordinates.

In fact, the existence of the flat metric is not necessary for (4) to be true, how we explain below. Moreover, the SW theories give exactly an example of such a case, where there is no distinguished constant matrix. This matrix can be found in topological theories because of existence of their field theory interpretation where the unity operator is always presented.

2. Perturbative SW prepotentials. Before going into the discussion of the WDVV equations for the complete SW prepotentials, let us note that the leading perturbative part of them should satisfy the equations (4) by itself (since the classical quadratic piece does not contribute into the third derivatives). In each case it can be checked by the straightforward calculation. On the other hand, if the WDVV equations are fulfilled for perturbative prepotential, it is a necessary condition for them to hold for complete prepotential.

To determine the one-loop perturbative prepotential from the field theory calculation, let us note that there are two origins of masses in $\mathcal{N} = 2$ SUSY YM models: first, they can be generated by vacuum values of the scalar $\phi$ from the gauge supermultiplet. For a supermultiplet in representation $R$ of the gauge group $G$ this contribution to the prepotential is given by the analog of the Coleman-Weinberg formula (from now on, we omit the classical part of the prepotential from all expressions):
\begin{equation}
F_R = \pm \frac{1}{4} \text{Tr}_R \phi^2 \log \phi,
\end{equation}
and the sign is “+” for vector supermultiplets (normally they are in the adjoint representation) and “−” for matter hypermultiplets. Second, there are bare masses $m_R$ which should be added to $\phi$ in \( F \). As a result, the general expression for the perturbative prepotential is

$$F = \frac{1}{4} \sum_{\text{vector multiplets}} \text{Tr}_A (\phi + M_n I_A)^2 \log (\phi + M_n I_A) - \frac{1}{4} \sum_{\text{hyper multiplets}} \text{Tr}_R (\phi + m_R I_R)^2 \log (\phi + m_R I_R) + f(m)$$

where the term $f(m)$ depending only on masses is not fixed by the (perturbative) field theory but can be read off from the non-perturbative description, and $I_R$ denotes the unit matrix in the representation $R$.

As a concrete example, let us consider the $SU(n)$ gauge group. Then, say, perturbative prepotential for the pure gauge theory acquires the form

$$F_{\text{pert}}^V = \frac{1}{4} \sum_{ij} (a_i - a_j)^2 \log (a_i - a_j)$$

This formula establishes that when v.e.v.’s of the scalar fields in the gauge supermultiplet are non-vanishing (perturbatively $a_i$ are eigenvalues of the vacuum expectation matrix $\langle \phi \rangle$), the fields in the gauge multiplet acquire masses $m_{rr'} = a_r - a_{r'}$ (the pair of indices $(r, r')$ labelling a field in the adjoint representation of $G$). In the $SU(n)$ case, the eigenvalues are subject to the condition $\sum_i a_i = 0$. Analogous formulas for the adjoint matter contribution to the prepotential is

$$F_{\text{pert}}^A = \frac{1}{4} \sum_{ij} (a_i - a_j + M)^2 \log (a_i - a_j + M)$$

while the contribution of the fundamental matter reads as

$$F_{\text{pert}}^F = -\frac{1}{4} \sum_i (a_i + m)^2 \log (a_i + m)$$

The perturbative prepotentials are discussed in detail in \[5\]. In that paper is also contained the check of the WDVV equations for these prepotentials. Here we just list the results.

i) The WDVV equations always hold for the pure gauge theories $F_{\text{pert}} = F_{\text{pert}}^V$.

ii) If one considers the gauge supermultiplets interacting with the matter hypermultiplets in the first fundamental representation with masses $m_{\alpha}$ $F_{\text{pert}} = F_{\text{pert}}^V + r F_{\text{pert}}^F + K F_{\text{pert}}(m)$ (where $r$ and $K$ are some undetermined coefficients), the WDVV equations do not hold unless $K = r^2/4$, $F_{\text{pert}}(m) = \frac{1}{4} \sum_{\alpha, \beta} (m_{\alpha} - m_{\beta})^2 \log (m_{\alpha} - m_{\beta})$, the masses being regarded as moduli (i.e. the equations \[6\] contain the derivatives with respect to masses).

iii) If in the theory the adjoint matter hypermultiplets are presented, i.e. $F_{\text{pert}} = F_{\text{pert}}^V + F_{\text{pert}}^A + f_A(m)$, the WDVV equations never hold.

From the investigation of the WDVV equations for the perturbative prepotentials, one can learn the following lessons:

- masses are to be regarded as moduli
- as an empiric rule, one may say that the WDVV equations are satisfied by perturbative prepotentials which depend only on the pairwise sums of the type $(a_i \pm b_j)$, where moduli $a_i$ and $b_j$ are either periods or masses.

This is the case for the models that contain either massive matter hypermultiplets in the first fundamental representation (or its dual), or massless matter in the square product of those. Troubles arise in all other situations because of the terms with $a_i \pm b_j \pm c_k \pm \ldots$ (The inverse statement is wrong – there are some exceptions when the WDVV equations hold despite the presence of such terms – e.g., for the exceptional groups.)

- at value $r = 2$, like $a_i$’s lying in irrep of $G$, masses $m_{\alpha}$’s can be regarded as lying in irrep of some $\tilde{G}$ so that if $G = A_n, C_n, D_n$, $\tilde{G} = A_n, D_n, C_n$ accordingly. This correspondence “explains” the form of the mass term in the prepotential $f(m)$.

Our last example of the perturbative prepotential is related to the 5d SUSY YM theory discussed by N.Nekrasov (see the last reference of \[3\]). The theory is considered compactified onto the circle of radius $R$, so that in four-dimensions it can be

\[3\]This general rule can be easily interpreted in D-brane terms, since the interaction of branes is caused by strings between them. The pairwise structure $(a_i \pm b_j)$ exactly reflects this fact, $a_i$ and $b_j$ should be identified with the ends of string.
seen as a gauge theory of the infinitely many vector supermultiplets with masses $M_k = k/R$. For the sake of simplicity, we put $R = 1$. Then, the perturbative prepotentials in this theory reads as

\[ F_{\text{pert}} = \frac{1}{4} \sum_{i,j} \left( \frac{1}{3} \delta_{ij} - \frac{1}{2} Li_3 \left( e^{-2a_{ij}} \right) \right) - \frac{n}{4} \sum_{i>j>k} a_i a_j a_k \]

\[ (8) \]

where $a_{ij} \equiv a_i - a_j$ and $Li_3(x)$ is the standard three-logarithm. The first sum in this expression tends to the usual logarithmic prepotential $F_{\text{pert}}$ as $R \to 0$, while the second one vanishes. It deserves mentioning that the second (cubic) term does not come from any field theory calculation, but has a non-perturbative nature. It is similar to the $U^3$-terms of the perturbative prepotential $F_{\text{pert}}$ of the heterotic string [9]. The presence of these terms turns to be absolutely crucial for the WDVV equations to hold in this case.

3. Associativity conditions. In the context of the two-dimensional LG topological theories, the WDVV equations arose as associativity condition of some polynomial algebra. We will prove below that the equations in the SW theories have the same origin. Now we briefly remind the main ingredients of this approach in the standard case of the LG theories.

In this case, one deals with the chiral ring formed by a set of polynomials $\{ \Phi_i(\lambda) \}$ and two co-prime (i.e. without common zeroes) fixed polynomials $Q(\lambda)$ and $P(\lambda)$. The polynomials $\Phi$ form the associative algebra with the structure constants $C_{ij}^k$ given with respect to the product defined by modulo $P'$:

\[ \Phi_i \Phi_j = C_{ij}^k \Phi_k Q' + (\ast) P' \to C_{ij}^k \Phi_k Q' \]

\[ (9) \]

the associativity condition being

\[ (\Phi_i \Phi_j) \Phi_k = \Phi_i (\Phi_j \Phi_k), \]

\[ (10) \]

i.e.

\[ C_i C_j = C_j C_i, \quad (C_i)^j_k = C^j_i \]

\[ (11) \]

Now, in order to get from these conditions the WDVV equations, one needs to choose properly the flat moduli [8]:

\[ a_i = -\frac{n}{i(n-i)} \text{res} \left( P^{i/n} dQ \right), \quad n = \text{ord}(P) \]

Then, there exists the prepotential whose third derivatives are given by the residue formula

\[ F_{ijk} = \text{res}_{P^r=0} \frac{\Phi_i \Phi_j \Phi_k}{P^r} \]

\[ (12) \]

On the other hand, from the associativity condition \[1\] and residue formula \[12\], one obtains that

\[ F_{ijk} = (C_i)^j_k F_{Q'1k}, \quad \text{i.e.} \quad C_i = F_i F_{Q}^{-1} \]

\[ (13) \]

Substituting this formula for $C_i$ into \[1\], one finally reaches the equations of the WDVV type. The choice $Q' = \Phi_1$ gives the standard equations \[4\]. In two-dimensional topological theories, there is always the unity operator that corresponds to $Q' = 1$ and leads to the constant metric $F_{Q'}$.

Thus, from this short study of the WDVV equations in the LG theories, we can get three main ingredients necessary for these equations to hold. These are:

- associative algebra
- flat moduli (coordinates)
- residue formula

We will show that in the SW theory only the first ingredient requires a non-trivial check.

4. SW theories and integrable systems. Now we turn to the WDVV equations in the SW construction \[1\] and show how they are related to integrable system underlying the corresponding SW theory. The most important result of \[1\], from this point of view, is that the moduli space of vacua and low energy effective action in SYM theories are completely given by the following input data:

- Riemann surface $\mathcal{C}$
- moduli space $\mathcal{M}$ (of the curves $\mathcal{C}$)
- meromorphic 1-form $dS$ on $\mathcal{C}$

How it was pointed out in [2,3], this input can be naturally described in the framework of some underlying integrable system. Let us consider a concrete example – the $SU(n)$ pure gauge SYM theory that can be described by the periodic Toda
chain with \( n \) sites. This integrable system is entirely given by the Lax operator
\[
L(w) = \begin{pmatrix}
    p_1 & e^{q_1 - q_2} & w \\
    e^{q_1 - q_2} & p_2 & \vdots \\
    \vdots & \ddots & \ddots \\
    \frac{1}{w} & \ldots & p_n
\end{pmatrix}
\] (14)

The Riemann surface \( C \) of the SW data is nothing but the spectral curve of the integrable system, which is given by the equation
\[
\det (L(w) - \lambda) = 0
\]
Taking into account (14), one can get from this formula the equation
\[
w + \frac{1}{w} = P(\lambda) = \prod_{i=1}^{n} (\lambda - \lambda_i), \quad \sum_i \lambda_i = 0
\] (15)
where the ramification points \( \lambda_i \) are Hamiltonians (integrals of motion) parametrizing the moduli space \( \mathcal{M} \) of the spectral curves. The replace \( Y \equiv w - 1/w \) transforms the curve (13) to the standard hyperelliptic form \( Y^2 = P^2 - 4 \), the genus of the curve being \( n - 1 \).

As to the meromorphic 1-form \( dS = \lambda \frac{dw}{w} = \lambda \frac{dp}{p} \), it is just the shorten action "\( p dq \)" along the non-contractible contours on the Hamiltonian tori. Its defining property is that the derivatives of \( dS \) with respect to the moduli (ramification points) are holomorphic differentials on the spectral curve.

Now let us describe the general integrable framework for the SW construction and start with the theories without matter hypermultiplets. First, matter hypermultiplets. First, we introduce bare spectral curve \( E \) that is torus \( y^2 = x^3 + g_2 x^2 + g_3 \) for the UV finite SYM theories with the associated holomorphic 1-form \( d\omega = dx/y \). This bare spectral curve degenerates into the double-punctured sphere (annulus) for the asymptotically free theories: \( x \to w + 1/w \), \( y \to w - 1/w \), \( d\omega = dw/w \). On this bare curve, there is given a matrix-valued Lax operator \( L(x, y) \). The dressed spectral curve is defined from the formula
\[
\det (L - \lambda) = 0
\]
This spectral curve is a ramified covering of \( E \) given by the equation
\[
\mathcal{P}(\lambda; x, y) = 0
\] (16)
In the case of the gauge group \( G = SU(n) \), the function \( \mathcal{P} \) is a polynomial of degree \( n \).

Thus, the moduli space \( \mathcal{M} \) of the spectral curve is given just by coefficients of \( \mathcal{P} \). The generating 1-form \( dS \mathrel{\approx} \lambda d\omega \) is meromorphic on \( C \) (hereafter the equality modulo total derivatives is denoted by "\( \mathrel{\approx} \)").

The prepotential and other "physical" quantities are defined in terms of the cohomology class of \( dS \):
\[
a_i = \oint_{A_i} dS, \quad a_i^P = \frac{\partial F}{\partial a_i} = \oint_{B_i} dS,
\] (17)
\[
A_I \circ B_J = \delta_{IJ}
\]
The first identity defines here the appropriate flat moduli, while the second one – the prepotential. The derivatives of the generating differential \( dS \) give holomorphic 1-differentials:
\[
\frac{\partial dS}{\partial a_i} = d\omega_i
\] (18)
and, therefore, the second derivative of the prepotential is the period matrix of the curve \( C \):
\[
\frac{\partial^2 F}{\partial a_i \partial a_j} = T_{ij}
\] (19)
The latter formula allows one to identify prepotential with logarithm of the \( \tau \)-function of Whitham hierarchy [8]: \( F = \log \tau \).

So far we reckoned without massive hypermultiplets. In order to include them, one just needs to consider the surface \( C \) with punctures. Then, the masses are proportional to residues of \( dS \) at the punctures, and the moduli space has to be extended to include these mass moduli. All other formulas remain in essence the same (see [4,5] for more details).

By the present moment, the correspondence between SYM theories and integrable systems is built through the SW construction in most of known cases that are collected in the table.

Note that the only theory, when the SW approach is applied but the corresponding integrable system still remains unknown is the UV finite SYM theory with the matter hypermultiplets in the first fundamental representation.

\footnote{In the table we considered only the classical groups (see below).}
Table. SUSY gauge theories $\leftrightarrow$ integrable systems correspondence

| SUSY physical theory | 4d pure gauge SYM theory, gauge group $G$ | 4d SYM with fundamental matter | 4d SYM with adjoint matter | 5d pure gauge SYM theory |
|----------------------|------------------------------------------|---------------------------------|---------------------------|--------------------------|
| Underlying integrable system | Toda chain for the dual affine $G^\vee$ | Rational spin chain of XXX type | Calogero-Moser system | Relativistic Toda chain |
| Bare spectral curve | sphere | sphere | torus | sphere |
| Dressed spectral curve | hyperelliptic | hyperelliptic | non-hyperelliptic | hyperelliptic |
| Generating meromorphic 1-form $dS$ | $\lambda \frac{dw}{w}$ | $\lambda \frac{dw}{w}$ | $\lambda \frac{dy}{y}$ | $\log \lambda \frac{dw}{w}$ |

To complete this table, we describe the dressed spectral curves in each case in more explicit terms. Let us note that in all but adjoint matter cases the curves are hyperelliptic and can be described by the general formula

$$P(\lambda, w) = 2P(\lambda) - w - \frac{Q(\lambda)}{w}$$  \hspace{1cm} (20)

Here $P(\lambda)$ is characteristic polynomial of the algebra $G$ itself, i.e.

$$P(\lambda) = \det(G - \lambda I) = \prod_i (\lambda - \lambda_i)$$  \hspace{1cm} (21)

where determinant is taken in the first fundamental representation and $\lambda_i$’s are the eigenvalues of the algebraic element $G$. For the pure gauge theories with the classical groups, $Q(\lambda) = \lambda^{2s}$ and

$$A_{n-1} : P(\lambda) = \prod_{i=1}^n (\lambda - \lambda_i), \quad s = 0;$$

$$B_n : P(\lambda) = \lambda^n \prod_{i=1}^n (\lambda^2 - \lambda_i^2), \quad s = 2;$$

$$C_n : P(\lambda) = \prod_{i=1}^n (\lambda^2 - \lambda_i^2), \quad s = -2;$$

$$D_n : P(\lambda) = \prod_{i=1}^n (\lambda^2 - \lambda_i^2), \quad s = 2$$  \hspace{1cm} (22)

For exceptional groups, the curves arising from the characteristic polynomials of the dual affine algebras do not acquire the hyperelliptic form. Therefore, in this case, the line ”dressed spectral curve” in the table has to be corrected.

In order to include $n_F$ massive hypermultiplets in the first fundamental representation one can just change $\lambda^{2s}$ for $Q(\lambda) = \lambda^{2s} \prod_{i=1}^{n_F} (\lambda - m_i)$ if $G = A_n$ and for $Q(\lambda) = \lambda^{2s} \prod_{i=1}^{n_F} (\lambda^2 - m_i^2)$ if $G = B_n, C_n, D_n$ [10].

At last, the 5d theory is just described by $Q(\lambda) = \lambda^{n/2}$.

In the Calogero-Moser case, the spectral curve is non-hyperelliptic, since the bare curve is elliptic. Therefore, it can be described as some covering of the hyperelliptic curve. We do not go into further details here, just referring to [5].

5. WDVV equations in SW theories. As we already discussed, in order to derive the WDVV equations along the line used in the context of the LG theories, we need three crucial ingredients: flat moduli, residue formula and associative algebra. However, the first two of these are always contained in the SW construction provided the underlying integrable system is known. Indeed, one can derive (see [4,5]) the following residue formula

$$F_{ijk} = \text{res}_{dw=0} \frac{d\omega_i d\omega_j d\omega_k}{d\omega d\lambda}$$  \hspace{1cm} (23)

where the proper flat moduli $a_i$’s are given by formula (17). Thus, the only point is to be checked
is the existence of the associative algebra. The residue formula (23) hints that this algebra is to be the algebra \( \Omega^3 \) of the holomorphic differentials \( d\omega_i \). In the forthcoming discussion we restrict ourselves to the case of pure gauge theory, the general case being treated in complete analogy.

Let us consider the algebra \( \Omega^1 \) and fix three differentials \( dQ, d\omega, d\lambda \in \Omega^1 \). The product in this algebra is given by the expansion

\[
d\omega_i d\omega_j = C^k_{ij} d\omega_k dQ + (+)d\omega + (+)d\lambda \tag{24}
\]

that should be factorized over the ideal spanned by the differentials \( d\omega \) and \( d\lambda \). This product belongs to the space of quadratic holomorphic differentials:

\[
\Omega^1 \cdot \Omega^1 \oplus \Omega^1 \cdot (dQ \oplus d\omega \oplus d\lambda) \tag{25}
\]

Since the dimension of the space of quadratic holomorphic differentials is equal to \( 3g - 3 \), the l.h.s. of (24) with arbitrary \( d\omega_i \)'s is the vector space of dimension \( 3g - 3 \). At the same time, at the r.h.s. of (24) there are \( g \) arbitrary coefficients \( C^k_{ij} \) in the first term (since there are exactly so many holomorphic 1-differentials that span the arbitrary holomorphic 1-differential \( C^k_{ij} d\omega_k \), \( g - 1 \) arbitrary holomorphic differentials in the second term (one differential should be subtracted to avoid the double counting) and \( g - 2 \) holomorphic 1-differentials in the third one. Thus, totally we get that the r.h.s. of (24) is spanned also by the basis of dimension \( g + (g - 1) + (g - 2) = 3g - 3 \).

This means that the algebra exists in the general case of the SW construction. However, this algebra generally is not associative. This is because, unlike the LG case, when it was the algebra of polynomials and, therefore, the product of the two belonged to the same space (of polynomials), product in the algebra of holomorphic 1-differentials no longer belongs to the same space but to the space of quadratic holomorphic differentials. Indeed, to check associativity, one needs to consider the triple product of \( \Omega^1 \):

\[
\Omega^1 \cdot \Omega^1 \cdot \Omega^1 \in \Omega^3 = \Omega^1 \cdot (dQ) \oplus \Omega^2 \cdot d\omega \oplus \Omega^1 \cdot d\lambda \tag{26}
\]

Now let us repeat our calculation: the dimension of the l.h.s. of this expression is \( 5g - 5 \) that is the dimension of the space of holomorphic 3-differentials. The dimension of the first space in expansion of the r.h.s. is \( g \), the second one is \( 3g - 4 \) and the third one is \( 2g - 4 \). Since \( g + (3g - 4) + (2g - 4) = 6g - 8 \) is greater than \( 5g - 5 \) (unless \( g \leq 3 \)), formula (20) does not define the unique expansion of the triple product of \( \Omega^1 \) and, therefore, the associativity spoils.

The situation can be improved if one considers the curves with additional involutions. As an example, let us consider the family of hyperelliptic curves: \( y^2 = Pol_{2g+2}(\lambda) \). In this case, there is the involution, \( \sigma: y \to -y \) and \( \Omega^1 \) is spanned by the \( \sigma \)-odd holomorphic 1-differentials \( \frac{x^{i-1} dx}{y^i} \), \( i = 1, ..., g \). Let us also note that both \( dQ \) and \( d\omega \) are \( \sigma \)-odd, while \( d\lambda \) is \( \sigma \)-even. This latter fact means that \( d\lambda \) can be only meromorphic unless there are punctures on the surface (which is, indeed, the case in the presence of the mass hypermultiplets). Thus, formula (24) can be replaced by that without \( d\lambda \)

\[
\Omega^1_+ = \Omega^1_\cdot (dQ) \oplus \Omega^1_+ \cdot d\omega \tag{27}
\]

where we expanded the space of holomorphic 2-differentials into the parts with definite \( \sigma \)-parity: \( \Omega^2 = \Omega^2_+ \oplus \Omega^2_- \), which are manifestly given by the differentials \( \frac{x^{i-1}(dx)^2}{y^i} \), \( i = 1, ..., 2g - 1 \) and \( \frac{x^{i-1}(dx)^2}{y^i} \), \( i = 1, ..., g - 2 \) respectively. Now it is easy to understand that the dimensions of the l.h.s. and r.h.s. of (27) coincide and are equal to \( 2g - 1 \).

Analogously, in this case, one can check the associativity. It is given by the expansion

\[
\Omega^3_+ = \Omega^1_+ \cdot (dQ)^2 \oplus \Omega^2_+ \cdot d\omega \tag{28}
\]

where both the l.h.s. and r.h.s. have the same dimension: \( 3g - 2 = g + (2g - 2) \). Thus, the algebra of holomorphic 1-differentials on hyperelliptic curve is really associative. This completes the proof of the WDVV equations in this case.

Now let us briefly list when there exist the associative algebras basing on the spectral curves discussed in the previous section. First of all, it exists in the theories with the gauge group \( A_n \), both in the pure gauge 4d and 5d theories and in the theories with fundamental matter, since, in accordance with s.4, the corresponding spectral curves are hyperelliptic ones of genus \( n \).

Equally the theories with the gauge groups \( SO(n) \) or \( Sp(n) \) are described by the hyperelliptic curves. The curves, however, are of higher genus
2n − 1. This would naively destroy all the reasoning of this section. The arguments, however, can be restored by noting that the corresponding curves (see (22)) have yet another involution, \( \rho : \lambda \to -\lambda \). This allows one to expand further the space of holomorphic differentials into the pieces with definite \( \rho \)-parity: \( \Omega^1_- = \Omega^1_- \oplus \Omega^1_+ \) etc. so that the proper algebra is generated by the differentials from \( \Omega^1_- \). One can easily check that it leads again to the associative algebra.

Consideration is even more tricky for the exceptional groups, when the corresponding curves are non-hyperelliptic. However, additional involutions allow one to get associative algebras in these cases too.

The situation is completely different in the adjoint matter case that is described by the Calogero-Moser integrable system. Since, in this case, the curve is non-hyperelliptic and has no evident involutions, one needs to include into consideration both the differentials \( d\omega \) and \( d\lambda \) for algebra to exist. However, under these circumstances, the algebra is no longer associative how it was demonstrated above. This can be done also by direct calculation for several first values of \( n \) (see [5]).

6. Concluding remarks. To conclude these short notes, let us emphasize that, along with already mentioned problem of lack of the WDVV equations for the Calogero-Moser system, no counterpart of the WDVV equations is also known yet for the Calabi-Yau manifolds (the naive one is just empty [7]). The way to resolve these problems might be to construct higher associativity conditions like it has been done by E.Getzler in the elliptic case [11], that is to say, for the elliptic Gromov-Witten classes. It may deserve mentioning that the WDVV equations in the type A topological theories themselves do still wait for the explanation in terms of associative algebras. All these problems are to be resolved in order to establish to what extent there is a really deep reason for the WDVV equations to emerge.

I am grateful to E.Akhmedov, A.Gorsky, S.Gukov, I.Krichever, A.Losev, A.Marshakov, A.Morozov, N.Nekrasov and I.Polyubin for useful discussions. This work was supported by grant INTAS-RFBR-95-0690.

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