Lyapunov optimizing measures and periodic measures for $C^2$ expanding maps

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Abstract

We prove that there exists an open and dense subset $U$ in the space of $C^2$ expanding self-maps of the circle $T$ such that the Lyapunov minimizing measures of any $T \in U$ are uniquely supported on a periodic orbit.

This answers a conjecture of Jenkinson-Morris in the $C^2$ topology.

1 Introduction

1.1 Main theorems

The ergodic optimization problem has connections with Lagrangian Mechanics, Thermodynamical Formalism, Multifractal Ananlysis, and Control Theory (see [14]). In the generic chaotic setting, it has been conjectured by Yuan and Hunt [22] that for an Axiom A or uniformly expanding system $T$ and a (topologically) generic smooth function $f$, there exists an optimal periodic orbit. Contreras [8] has made substantial contributions to Yuan-Hunt’s conjecture. Later on, the papers [11,12,16] progressed a lot in this direction.

In Yuan-Hunt’s conjecture, the function $f$ is not strongly related to the system $T$. The aim of this paper is to consider the optimal measures of some quantities very related to the dynamical system. One of the most interesting quantities may be the Lyapunov exponent. The measures optimize Lyapunov exponents are said to be Lyapunov optimal measures. This notion was given by Contreras-Lopes-Thieullen [9]. We will show that the Lyapunov minimizing/maximizing measures of generic 1-dimensional expanding self-maps are supported on periodic orbits for the $C^2$ topology.

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Let $T = \mathbb{R}/\mathbb{Z}$ be the circle and $T : T \to T$ be a $C^1$ self-map. Let $\mathcal{M}_{\text{inv}}(T)$ (resp. $\mathcal{M}_{\text{erg}}(T)$) be the set of $T$-invariant (resp. $T$-ergodic) Borel probability measures. For any $\mu \in \mathcal{M}_{\text{inv}}(T)$, define its Lyapunov exponent as

$$\lambda_T(\mu) := \int \log |DT|d\mu.$$ 

Lyapunov exponents are very important dynamical quantities. We are interested in seeking which measures minimize or maximize the Lyapunov exponents. Define

$$\alpha_T := \inf_{\nu \in \mathcal{M}_{\text{inv}}(T)} \lambda_T(\nu), \quad \beta_T := \sup_{\nu \in \mathcal{M}_{\text{inv}}(T)} \lambda_T(\nu).$$

An invariant measure $\mu$ is said to be a Lyapunov minimizing measure if $\alpha_T = \lambda_T(\mu)$; it is said to be a Lyapunov maximizing measure if $\beta_T = \lambda_T(\mu)$.

Lyapunov minimizing/maximizing measures may be very difficult to describe for any self-map. However, it was imagined for these measures are periodic for generic expanding self-maps. A self-map $T : T \to T$ is expanding if there are $C > 0$ and $\lambda > 1$ such that $\|DT^n(x)\| \geq C\lambda^n$ for any $x \in T$.

For any two self-maps $S$ and $T$, the $C^k$-distance between $S$ and $T$ is defined to be

$$d_{C^k}(S, T) = \sum_{i=0}^{k} d_{C^0}(D^iS, D^iT).$$

Given $\chi \in (0, 1]$, the $C^{k,\chi}$-distance between $S$ and $T$ is defined to be

$$d_{C^{k,\chi}}(S, T) = d_{C^k}(S, T) + \sup_{x \neq y} \frac{d_{C^0}(D^kS, D^kT)}{|x - y|^\chi}.$$ 

Let $\mathcal{E}^k(T)$ ($\mathcal{E}^{k,\chi}(T)$) be the space of $C^k$ ($C^{k,\chi}$) expanding self-maps endowed with the $C^k$-distance ($C^{k,\chi}$-distance).

**Theorem A.** There is a dense open set $U \subset \mathcal{E}^2(T)$ such that the Lyapunov minimizing measure of $T \in U$ is unique and supported on a periodic orbit.

The proof of Theorem A is mainly based on a Lipschitz-$C^1$ version.

**Theorem B.** There is a dense open set $U \subset \mathcal{E}^{1,1}(T)$ such that the Lyapunov minimizing measure of $T \in U$ is unique and supported on a periodic orbit.

One can also get a Hölder-$C^1$ version of Theorem B.

**Theorem C.** Assume that $\chi \in (0, 1]$. There is a dense open set $U \subset \mathcal{E}^{1,\chi}(T)$ such that the Lyapunov minimizing measure of $T \in U$ is unique and supported on a periodic orbit.

Since the proof of Theorem C follows almost the same line of the proof of Theorem B, it is omitted.

Theorem A answers a conjecture of Jenkinson-Morris [15, Conjecture 1] positively in the $C^2$ topology.
Conjecture 1.1. [15] For integer $k \geq 2$, a generic $T \in \mathcal{E}^k$ has a unique Lyapunov minimizing measure, and this measure is supported on a periodic orbit of $T$.

Note that the conjecture of Jenkinson-Morris when $k > 2$ is still open.

Contreras-Lopez-Thieullen [9] has proved that for any $T$ in some dense open subset of $\bigcup_{1 > \beta > \alpha} \mathcal{E}^{1+\beta}$ endowed with the $C^{1,\alpha}$-distance, the Lyapunov minimizing/maximizing measures of $T$ are unique and periodic.

In $C^1$ topology, the situation is completely different. It has been proved by Jenkinson-Morris [15] that for generic $C^1$ expanding self-maps on $\mathbb{T}$, the Lyapunov minimizing is unique but has full support.

1.2 Discussions in the higher-dimensional case

In the higher-dimensional case, one interesting problem is to consider the ergodic optimization problem of the upper Lyapunov exponents. Let $M$ be a $d$-dimensional Riemannian manifold without boundary. Let $T: M \to M$ be a $C^1$ self-map. Given an ergodic measure $\mu$ of $T$, as in [4, Section C.1], there is a measurable filtration for $\mu$-almost every point $x \in M$,

$$T_x M = E_1(x) \supset E_2(x) \supset \cdots \supset E_k(x) \supset E_{k+1} = \{0\}$$

and constants $\lambda_1 > \lambda_2 > \cdots > \lambda_k$ such that for any $1 \leq i \leq k$ and for any $v \in E_i \setminus E_{i-1}$, one has that

$$\lim_{n \to \infty} \frac{1}{n} \log \| DT^n (v) \| = \lambda_i.$$  

$\lambda_1$ is said to be the upper Lyapunov exponent of $\mu$. An invariant measure $\mu$ is said to be the maximizing measure of $\lambda_1$ if $\lambda_1(\mu) = \sup_{\nu \in \mathcal{M}_{inv}(T)} \lambda_1(\nu)$. An invariant measure $\mu$ is said to be the minimizing measure of $\lambda_1$ if $\lambda_1(\mu) = \inf_{\nu \in \mathcal{M}_{inv}(T)} \lambda_1(\nu)$. By Cao [7], the maximizing measure of $\lambda_1$ does exist. Symmetrically, one knows the existence of minimizing measure of $\lambda_k$ (the lower Lyapunov exponent). We have the following conjectures:

**Conjecture 1.2.** Let $M$ be a $d$-dimensional compact Riemannian manifold. For an integer $k \geq 2$, for a $C^k$ generic expanding self-map $T$, the upper Lyapunov exponent $\lambda_1$ has a unique Lyapunov maximizing measure, and this measure is supported on a periodic orbit of $T$.

For $k = 1$, for a $C^1$ generic expanding self-map $T$, the upper Lyapunov exponent admits a unique maximizing measure, which has zero entropy and full support.

Although we do not know the existence of minimizing measure of the upper Lyapunov exponent $\lambda_1$, one can still formulate the following conjecture.

**Conjecture 1.3.** For generic expanding self-map $T$ on a manifold $M$, the minimizing measure of the upper Lyapunov exponent $\lambda_1$ exists, is unique and has zero entropy.
One can still have results in the higher-dimensional case on the sum of Lyapunov exponents. Given an ergodic measure \( \mu \), denote by 
\[
\lambda_{\text{sum}} = \sum_{i=1}^{k} \lambda_i (\dim E_i - \dim E_{i+1}).
\]

By [19, Proposition 1.3, Theorem 1.6], to find the optimal measures of \( \lambda_{\text{sum}} \) is equivalent to find the optimal measures of the continuous function \( \log |\text{Det}(T)| \). Thus, it is essentially the same as the one-dimensional case. One has the following theorems.

Denote by \( \mathcal{E}^k(M) \) and \( \mathcal{E}^{k,\chi}(M) \) the spaces of \( C^k \) expanding maps and of \( C^{k,\chi} \) expanding maps, respectively.

**Theorem D.** There is a dense open set \( \mathcal{U} \subset \mathcal{E}^2(M) \) such that the minimizing measure with respect to \( \lambda_{\text{sum}} \) of \( T \in \mathcal{U} \) is unique and supported on a periodic orbit.

**Theorem E.** Assume that \( \chi \in (0,1] \). There is a dense open set \( \mathcal{U} \subset \mathcal{E}^{1,\chi}(M) \) such that the minimizing measure with respect to \( \lambda_{\text{sum}} \) of \( T \in \mathcal{U} \) is unique and supported on a periodic orbit.

The proof of these two theorems follow almost the same lines of the above ones, hence are omitted.

In the \( C^1 \) topology, the optimization problem of \( \lambda_{\text{sum}} \) has been considered in [17].

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## 2 Proof of Theorem A

In this section, we are going to prove Theorem A. Denote by \( \mathcal{L}^-(T)/\mathcal{L}^+(T) \) the set of Lyapunov minimizing/maximizing measures of \( T \), respectively.

We need the following version of Mañe’s Lemma.

**Lemma 2.1.** Let \( T \) be a \( C^{1,1} \) expanding self-map of \( \mathbb{T} \). Then there exists a Lipschitz map \( f \) from \( \mathbb{T} \) to \( \mathbb{R} \) such that

\[
\bigcup_{\nu \in \mathcal{L}^-(T)} \text{supp}(\nu) \subset \{ y \in \mathbb{T} : F(y) = \inf_{x \in \mathbb{T}} F(x) = \alpha(T) \},
\]

where \( F(x) = f(T(x)) - f(x) + \log \| DT(x) \| \).

**Proof.** Notice that \( \log \| DT \| \) is Lipschitz since \( T \) is \( C^{1,1} \). This Lemma follows immediately from the classical Mañe’s Lemma for the expanding self-map \( T \) and the Lipschitz function \( \log \| DT \| \) (see [5,6,9,10,20] for various versions and approaches). \( \square \)

Expanding self-maps on the circle are structurally stable: this was proved by Shub [21]. We can also have the information on the conjugacy maps, see [15, Lemma 2] and [18, Proposition 5.1.6] for a precise proof.
Theorem 2.2. Let $S_0$ be a $C^1$ expanding self-map of $\mathbb{T}$. For any $\bar{\varepsilon}_0 > 0$, there is $\bar{\varepsilon} > 0$ such that for any $S$, if $d_{C^1}(S, S_0) < \bar{\varepsilon}$, then there is a homeomorphism $\pi_S : \mathbb{T} \to \mathbb{T}$ such that

- $d_{C^0}(\pi_S, \text{Id}) < \bar{\varepsilon}_0$,
- $\pi_S \circ S_0 = S \circ \pi_S$.

Let $T$ be a $C^1$ expanding self-map of $\mathbb{T}$ with $\|DT(x)\| > 1$ for all $x \in \mathbb{T}$. Let $\Gamma$ be a periodic orbit of $T$. Define the gap of $\Gamma$ by

$$G(\Gamma) = \begin{cases} \frac{1}{20 \max_{x \in \mathbb{T}} |DT(x)|}, & \text{if } \#\Gamma = 1, \\ \min_{x, y \in \Gamma, x \neq y} d(x, y), & \text{others}. \end{cases}$$

One has the following expansive-like Lemma.

**Lemma 2.3.** Let $T$ be a $C^1$ expanding self-map of $\mathbb{T}$ with $\|DT(x)\| > 1$ for each $x \in \mathbb{T}$ and $\Gamma$ be a periodic orbit of $T$. If $z \in \mathbb{T}$ satisfies

$$d(T^i z, \Gamma) < \frac{G(\Gamma)}{2 \max_{x \in \mathbb{T}} \|DT(x)\|} \text{ for all } i \in \mathbb{N} \cup \{0\},$$

then $z \in \Gamma$.

**Proof.** We will prove by contradiction and assume that $z \notin \Gamma$. One can find $p \in \Gamma$ such that $d(z, p) = d(z, \Gamma)$. If $\#\Gamma = 1$, by the expansion property, one can find $m \in \mathbb{N}$ such that

$$d(T^m(z), T^m(p)) \geq \frac{1}{2 \max_{x \in \mathbb{T}} \|DT(x)\|} > \frac{G(\Gamma)}{2 \max_{x \in \mathbb{T}} \|DT(x)\|}.$$ 

Now one can assume that $\#\Gamma > 1$. By the expansion property, there is $m \in \mathbb{N}$ such that

$$\frac{G(\Gamma)}{2 \max_{x \in \mathbb{T}} \|DT(x)\|} \leq d(T^m(z), T^m(p)) < \frac{G(\Gamma)}{2}.$$

For any $q \in \Gamma \setminus \{p\}$, one has that

$$d(T^m(z), T^m(q)) \geq d(T^m(p), T^m(q)) - d(T^m(z), T^m(p)) \geq G(\Gamma) - G(\Gamma)/2 \geq G(\Gamma)/2.$$ 

Thus

$$d(T^m(z), \Gamma) = d(T^m(z), T^m(p)) \geq \frac{G(\Gamma)}{2 \max_{x \in \mathbb{T}} \|DT(x)\|}.$$

This contradicts to the assumption. \hfill \square

### 2.1 Periodic orbits with large inner distance

To prove Theorem A and Theorem B we need the following proposition which is a particular case of [11, Proposition 3.1].

**Proposition 2.4.** Let $T$ be a $C^1$ expanding self-map of $\mathbb{T}$ with $\|DT(x)\| > 1$ for all $x \in \mathbb{T}$ and $E \subset \mathbb{T}$ be a nonempty compact invariant subset of $T$. Then for any $C > 0$, there exists a periodic orbit $\Gamma$ of $T$ depending on $C$ such that

$$G(\Gamma) > C \cdot \sum_{x \in \Gamma} d(x, E).$$
2.2 The perturbation result

Given a point $x$, let $\delta_x$ be the Dirac $\delta$-measure supported on the point $x$. For a self map $S$ and a periodic orbit $\Gamma$ of $S$, denote by

$$\delta_\Gamma = \frac{1}{\#\Gamma} \sum_{x \in \Gamma} \delta_x.$$ 

**Theorem 2.5.** Let $T$ be a $C^{1,1}$ expanding self-map of $\mathbb{T}$. For any $\varepsilon > 0$, there are

- a periodic orbit $\Gamma_T$ of $T$;
- an open set $U_{\varepsilon, T}$ in the $\varepsilon$-neighborhood of $T$ for the $C^{1,1}$-topology;

such that for any $S \in U_{\varepsilon, T}$, $\delta_{\Gamma_S}$ is the unique Lyapunov minimizing measure of $S$.

Theorem B can be deduced from Theorem 2.5 directly. \hfill \square

One has the following more precise version of Theorem 2.5.

**Theorem 2.6.** Let $T$ be a $C^{1,1}$ expanding self-map of $\mathbb{T}$. For any $\varepsilon > 0$, there are

- a periodic orbit $\Gamma_T$ of $T$;
- a map $h : \mathbb{T} \to \mathbb{R}$ satisfying $h$ is supported in a small neighborhood of $\Gamma_T$, $\|h\|_{C^{1,1}} < \varepsilon/2$, and $S_0 = T + h$ can be regarded as an $\varepsilon/2$-perturbation of $T$ for the $C^{1,1}$-topology;
- a neighborhood $U_{\varepsilon, T}$ of $S_0$ such that $U_{\varepsilon, T}$ is contained in the $\varepsilon$-neighborhood of $T$;

such that for any $S \in U_{\varepsilon, T}$, $\delta_{\Gamma_S}$ is the unique Lyapunov minimizing measure of $S$.

Based on the above Lipschitz-$C^1$ version, one has the following differentiable $C^2$ version.

**Theorem 2.7.** Let $T$ be a $C^2$ expanding self-map of $\mathbb{T}$. For any $\varepsilon > 0$, there are

- a periodic orbit $\Gamma_T$ of $T$;
- an open set $U_{\varepsilon, T}$ in the $\varepsilon$-neighborhood of $T$ for the $C^2$-topology;

such that for any $S \in U_{\varepsilon, T}$, $\delta_{\Gamma_S}$ is the unique Lyapunov minimizing measure of $S$.

Theorem A can be deduced from Theorem 2.7 directly. \hfill \square

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1One can do this perturbation for the lift of $T$ from $\mathbb{R}$ to $\mathbb{R}$ and then pull the perturbation back to $\mathbb{T}^1$. 
2.3 The proof of Theorem 2.6

This subsection is devoted to the proof of Theorem 2.6. So now we are under the assumptions of Theorem 2.6.

Recall that when $X, Y$ are two metric spaces, $f : X \to Y$ is a map, the Lipschitz constant of $f$ is defined to be

$$\text{Lip}(f) = \sup_{x_1, x_2 \in X, x_1 \neq x_2} \frac{d_Y(f(x_1), f(x_2))}{d_X(x_1, x_2)}.$$

**Proof.** The proof can be divided into several steps. Up to changing the metric on $T$, without loss of generality, one can assume that $\|DT(x)\| > 1$ for any $x \in T$.

**The cohomological equation.** By Lemma 2.1 there exists $f \in \text{Lip}(T, \mathbb{R})$ such that (2.1) holds for

$$F_T(x) = f(T(x)) - f(x) + \log \|DT(x)\|.$$

By Lemma 2.1, one has that

$$F_T(x) \geq \alpha(T) \quad \forall x \in T, \text{ and } F_T|_{\text{supp}(\mu)} = \alpha(T), \quad \forall \mu \in \mathcal{L}^-(T). \quad (2.2)$$

For any other self-map $S$, denote by

$$F_S(x) = f(S(x)) - f(x) + \log \|DS(x)\|.$$

**Fix constants.** We fix a constant $K$ independent of the perturbation such that

- $K > \max\{2 \max_{x \in T} \|DT(x)\|, 10\}$.
- $K > \text{Lip}(f) \cdot (\max_{x \in T} \|DT(x)\| + 1) + \text{Lip}(DT) > 2\text{Lip}(f)$.
- $K > \lambda/(\lambda - 1)$, where $\lambda = \inf_{x \in T} \|DT(x)\|$.

**Reduce $\epsilon$.** By reducing $\epsilon$ if necessary, one has that for any $S$ satisfying $d_{C^{1,1}}(S, T) < \epsilon$, one has that $S$ is still an expanding self-map with $\|DS(x)\| > 1$ for any $x \in T$, and we have

1. $$\|DS(x)\| > \frac{\|DT(x)\| + 1}{2} > 1 > 2/K, \quad \forall x \in T. \quad (2.3)$$

2. $$K > 2 \max_{x \in T} \|DS(x)\|, \quad K > \min_{x \in T} \frac{\|DS(x)\|}{\|DS(x)\| - 1}. \quad (2.4)$$

3. $$K > \text{Lip}(f) \cdot (\max_{x \in T} \|DS(x)\| + 1) + \text{Lip}(DS). \quad (2.5)$$
A small constant $\rho_\varepsilon$ and big constants $L_\varepsilon, C_\varepsilon$. Fix $L_\varepsilon \in \mathbb{N}$ such that

$$L_\varepsilon \cdot \varepsilon > 4K^6. \quad (2.6)$$

Fix $\rho_\varepsilon > 0$ sufficiently small such that

$$\rho_\varepsilon \cdot K^{L_\varepsilon} < \frac{1}{2K}. \quad (2.7)$$

Fix $C_\varepsilon$ sufficiently large such that

$$\varepsilon \cdot \rho_\varepsilon \cdot C_\varepsilon > 6K^5. \quad (2.8)$$

Now one has the following estimate.

**Claim.**

$$\frac{L_\varepsilon \cdot \varepsilon \cdot \rho_\varepsilon \cdot C_\varepsilon}{K^4} > 3K + \rho_\varepsilon \cdot C_\varepsilon \cdot K^2. \quad (2.9)$$

**Proof.** By (2.6), one has that

$$\frac{L_\varepsilon \cdot \varepsilon \cdot \rho_\varepsilon \cdot C_\varepsilon}{2K^4} > \rho_\varepsilon \cdot C_\varepsilon \cdot K^2.$$

By (2.8), one has that

$$\frac{L_\varepsilon \cdot \varepsilon \cdot \rho_\varepsilon \cdot C_\varepsilon}{2K^4} > 3K.$$

Combining the above inequalities together, one can conclude. \hfill \square

**Periodic orbits with large inner distance; constants $G^*$ and $d_*$.** Fix $\mu_T \in \mathcal{L}^-(T)$. Consider $E = \text{supp}(\mu_T)$. By Proposition 2.4 there is a periodic orbit $\Gamma_T$ of $T$ such that

$$G(\Gamma_T) > C_\varepsilon \cdot \sum_{x \in \Gamma_T} d(x, \text{supp}(\mu_T)). \quad (2.10)$$

Note that by the definition, this is still valid when $E$ is a fixed point: in this case $d(x, \text{supp}(\mu_T)) = 0$, but $G(\Gamma) > 0$. By the Lipschitz property of $F_T$ and the choice of $K$, one has that for any $x \in \Gamma_T$,

$$|F_T(x) - \alpha(T)| \leq K \cdot d(x, \text{supp}(\mu_T)). \quad (2.11)$$

Denote by $G^* = G(\Gamma_T)$ and $d_* = \sum_{x \in \Gamma_T} d(x, \text{supp}(\mu_T))$. The inequality (2.10) can be read as $G^* > C_\varepsilon d_*$.  

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The perturbation map $h$. Assume that $\Gamma_T = \{p_1, p_2, \ldots, p_{\tau(p)}\}$. In a local chart, one defines the perturbations in the following way.

Since $K > \max_{x \in T} \|DT(x)\|$, one has that for any $x \in T$, $1/\|DT(x)\| > 1/K$. Thus, for any $\gamma \in [0, 1]$, one has that

$$\frac{1}{\|DT(x)\| - \gamma \cdot \varepsilon \cdot \rho_\varepsilon \cdot G^*/(2K)} > 1/K.$$ 

Consequently, we choose $\gamma_i \in [0, 1]$ such that if we are in the interval $[\|DT(p_i)\| - \gamma_i \cdot \varepsilon \cdot \rho_\varepsilon \cdot G^*/K^3, \|DT(p_i)\|]$, one has that

$$\frac{1}{\|DT(p_i)\| - \gamma_i \cdot \varepsilon \cdot \rho_\varepsilon \cdot G^*/(2K)} \, dz = \varepsilon \cdot \rho_\varepsilon \cdot G^*/K^4. \quad (2.12)$$

Define a real valued function $h(x)$ on $T$, such that in local charts, one has the following expression:

$$h(x) = \begin{cases} 
-\frac{\varepsilon}{2K \cdot \rho_\varepsilon \cdot G^*} (x - p_i)(p_i + \rho_\varepsilon \cdot G^* - x)^2 \cdot \gamma_i, & \text{if } x \in (p_i, p_i + \rho_\varepsilon \cdot G^*), \\
0, & \text{if } x = p_i, \\
-\frac{\varepsilon}{2K \cdot \rho_\varepsilon \cdot G^*} (x - p_i)(x - p_i + \rho_\varepsilon \cdot G^* - x)^2 \cdot \gamma_i, & \text{if } x \in (p_i - \rho_\varepsilon \cdot G^*, p_i), \\
0, & \text{otherwise}.
\end{cases}$$

**Lemma 2.8.** $h$ has the following properties:

1. $h(p_i) = 0$, $h(p_i \pm \rho_\varepsilon \cdot G^*) = 0$ and $Dh(p_i \pm \rho_\varepsilon \cdot G^*) = 0$.

2. $Dh(p_i) = -\gamma_i \cdot \varepsilon \cdot \rho_\varepsilon \cdot G^*/(2K)$.

3. $\|h\|_{C^1} < \varepsilon/2$, $\text{Lip}(Dh) < \varepsilon/2$.

**Proof.** Without loss of generality, one can assume that $p_i = 0$ in a local chart. One has the following calculation:

$$Dh(x) = \begin{cases} 
-\frac{\varepsilon}{2K \cdot \rho_\varepsilon \cdot G^*} \left[ (\rho_\varepsilon \cdot G^* - x)^2 - 2x(\rho_\varepsilon \cdot G^* - x) \right] \cdot \gamma_i, & \text{if } x \in (0, \rho_\varepsilon \cdot G^*), \\
-\frac{\varepsilon}{2K \cdot \rho_\varepsilon \cdot G^*} \cdot \gamma_i, & \text{if } x = 0, \\
-\frac{\varepsilon}{2K \cdot \rho_\varepsilon \cdot G^*} \left[ (x + \rho_\varepsilon \cdot G^*)^2 + 2x(\rho_\varepsilon \cdot G^* - x) \right] \cdot \gamma_i, & \text{if } x \in (-\rho_\varepsilon \cdot G^*, 0), \\
0, & \text{otherwise}.
\end{cases}$$

From the expression, one knows that $h(0) = 0$, $h(\pm \rho_\varepsilon \cdot G^*) = 0$, $Dh(\pm \rho_\varepsilon \cdot G^*) = 0$ and $Dh(0) = -\gamma_i \cdot \varepsilon \cdot \rho_\varepsilon \cdot G^*/(2K)$. From the expression of $h$, one knows that $\|h\|_{C^1} < \varepsilon/2$. By a simple calculation, one has that the whole interval,

$$\|Dh\| \leq \frac{\varepsilon}{2K \cdot \rho_\varepsilon \cdot G^*} 2(\rho_\varepsilon \cdot G^*)^2 < \varepsilon/2.$$ 

One calculates the second derivative of $h$, which are not well-defined on $\pm \rho_\varepsilon \cdot G^*$:

$$D^2h(x) = \begin{cases} 
-\frac{\varepsilon}{2K \cdot \rho_\varepsilon \cdot G^*} \left[ 6x - 4\rho_\varepsilon \cdot G^* \right] \cdot \gamma_i, & \text{if } x \in (0, \rho_\varepsilon \cdot G^*), \\
\text{not well defined}, & \text{if } x = 0, \\
-\frac{\varepsilon}{2K \cdot \rho_\varepsilon \cdot G^*} \left[ 6x + 4\rho_\varepsilon \cdot G^* \right] \cdot \gamma_i, & \text{if } x \in (-\rho_\varepsilon \cdot G^*, 0), \\
0, & \text{otherwise except } \pm \rho_\varepsilon \cdot G^*.
\end{cases}$$
On each interval, one has that $\|D^2 h\| < \varepsilon/2$ from the expression. Thus, one knows that $\text{Lip}(Dh) < \varepsilon/2$. 

**The perturbation $S_0$.** Note that when we work in local charts, we can write

$$S_0(x) = T(x) + h(x).$$

It is clear that when $\varepsilon$ is small enough, $S_0$ is an expanding self-map.

**Lemma 2.9.** $S_0$ has the following properties:

- $d_{C^1} (S_0, T) < \varepsilon/2$.
- $DS_0(p_i) = DT(p_i) - \gamma_i \cdot \varepsilon \cdot \rho_\varepsilon \cdot G^*/(2K)$ for each $p_i \in \Gamma_T$.
- $\Gamma_T$ is still the periodic orbit of $S_0$, and $T|\Gamma_T = S_0|\Gamma_T$.

**Proof.** These properties follows from the properties of $h$ directly.

For $S_0$, one has that for any $p \in \Gamma_T$, which is also a periodic point of $S_0$,

$$F_{S_0}(p) = f(T(p)) - f(p) + \log \|DS_0(p)\|$$

$$= f(T(p)) - f(p) + \log \|DT(p)\| - \int_{[DS_0(p)]} \frac{1}{z} dz$$

$$\leq f(T(p)) - f(p) + \log \|DT(p)\| - \int_{[DT(p)]} \frac{1}{z} dz$$

$$= F_T(x) - \varepsilon \cdot \rho_\varepsilon \cdot G^*/K^4.$$  \hfill (2.13)

Thus, for any $p, q \in \Gamma_{S_0}$, one has that

$$|F_{S_0}(p) - F_{S_0}(q)| \leq |F_T(p) - F_T(q)|\leq K(d(p, \text{supp}(\mu_T)) + d(q, \text{supp}(\mu_T))).$$ \hfill (2.14)

**Choose the constants $\tilde{\varepsilon}_0 > \tilde{\varepsilon} > 0$ and find the neighborhood $\mathcal{U}$.** Take $\tilde{\varepsilon}_0 \in (0, \varepsilon/2)$ such that

$$(\rho_\varepsilon \cdot G^* + \tilde{\varepsilon}_0) \cdot K^L \varepsilon < \frac{G^* - 2\tilde{\varepsilon}_0}{2K},$$ \hfill (2.15)

and

$$L_\varepsilon (\varepsilon \cdot \rho_\varepsilon \cdot G^*/K^4 - K \cdot d_* - 2\tilde{\varepsilon}_0 \cdot K) \leq \left(\rho_\varepsilon G^* + \tilde{\varepsilon}_0\right) K^2$$

$$+ \tau(\Gamma_S) \left(4\tilde{\varepsilon}_0 \cdot K + 2Kd_* / \tau(\Gamma_S) \right).$$ \hfill (2.16)

$$\varepsilon \cdot \rho_\varepsilon \cdot G^*/K^4 - K \cdot d_* - 2\tilde{\varepsilon}_0 \cdot K > 0.$$ \hfill (2.17)
Claim. One can choose $\tilde{z}_0$ such that Inequalities (2.15), (2.16) and (2.17) hold.

Proof of the Claim. These come from (2.7), (2.9) and (2.8) by noticing $G* > C_\varepsilon d_*$.  

By Theorem 2.2 there is $\tilde{\varepsilon} > 0$ such that for any $S$, if $d_{C^1}(S,S_0) < \tilde{\varepsilon}$, then there is a homeomorphism $\pi_S : \mathbb{T} \to \mathbb{T}$ such that

$$
d_{C^0}(\pi_S,Id) < \tilde{z}_0 \text{ and } \pi_S \circ S_0 = S \circ \pi_S. \quad (2.18)
$$

Consequently, $\Gamma_S = \pi_S(\Gamma_{S_0})$ is a periodic orbit of $S$.

Without loss of generality, one can assume that $\tilde{\varepsilon} < \tilde{z}_0$.

One has the following estimate on $\Gamma_S$:

**Lemma 2.10.** For any two distinct $x, y \in \Gamma_S$, one has that $d(x, y) > G* - 2\tilde{z}_0$.

*Proof.* By the definition of $G*$, one has that $d(\pi_S^{-1}(x), \pi_S^{-1}(y)) > G*$. One can conclude by noticing that $d(x, \pi_S^{-1}(x)) < \tilde{z}_0$ and $d(y, \pi_S^{-1}(y)) < \tilde{z}_0$.  

We take $U$ to be the $\tilde{\varepsilon}$-neighborhood of $S_0$ in the $C^{1,1}$-topology.

**Claim.** $U$ is contained in the $\varepsilon$-neighborhood of $T$ in the $C^{1,1}$-topology.

*Proof of the Claim.* This follows from the fact that $0 < \tilde{\varepsilon} < \tilde{z}_0 < \varepsilon/2$.  

For any $S, R \in U$, for any $x \in \mathbb{T}$, by (2.3), one has that

$$
|\log \|DS(x)\| - \log \|DR(x)\| |
\leq \max\left\{\frac{1}{\inf_{w \in \mathbb{T}} \|DS(w)\|}, \frac{1}{\inf_{w \in \mathbb{T}} \|DR(w)\|}\right\} \cdot d_{C^0}(DS(x), DR(x))
\leq K/2 \cdot d_{C^0}(DS(x), DR(x)).
$$

Hence, together with the fact that $K > 2\text{Lip}(f)$,

$$
|F_S(x) - F_R(x)|
\leq |f(S(x)) - f(R(x))| + |\log \|DS(x)\| - \log \|DR(x)\| |
\leq \text{Lip}(f) \cdot d_{C^0}(S,R) + K/2 \cdot d_{C^0}(DS(x), DR(x))
\leq \tilde{z}_0 \cdot K. \quad (2.19)
$$
The average on \( \Gamma_S \). For any \( S \in \mathcal{U} \), denote by
\[
A_{\Gamma_S} = \int F_S d\delta_S = \frac{\sum_{z \in \Gamma_S} F_S(z)}{\# \Gamma_S}.
\]

Clearly, for \( S_0 \), by (2.13), one has that
\[
A_{\Gamma_{S_0}} = \frac{\sum_{z \in \Gamma_T} F_T(z)}{\# \Gamma_T} = \alpha(T) + K \cdot \frac{d_*}{\# \Gamma_T} - \varepsilon \cdot \rho \cdot G^*/K^4.
\]

Thus, for any \( S \in \mathcal{U} \), one has that
\[
A_{\Gamma_S} \leq A_{\Gamma_{S_0}} + K \cdot \tilde{\varepsilon}_0 \leq \alpha(T) + K \cdot \tilde{\varepsilon}_0 + K \cdot \frac{d_*}{\# \Gamma_T} - \varepsilon \cdot \rho \cdot G^*/K^4.
\]

Estimates for \( \Gamma_S \). For any \( x, y \in \Gamma_S \), one has that
\[
|F_S(x) - F_S(y)| \leq |F_S(x) - F_{S_0}(x)| + |F_S(y) - F_{S_0}(y)| + |F_{S_0}(x) - F_{S_0}(y)|
\]
\[
\leq 2\tilde{\varepsilon}_0 \cdot K + |F_{S_0}(x) - F_{S_0}(\pi_S^{-1}(x))| + |F_{S_0}(y) - F_{S_0}(\pi_S^{-1}(y))|
\]
\[
+ |F_{S_0}(\pi_S^{-1}(x)) - F_{S_0}(\pi_S^{-1}(y))|
\]
\[
\leq 2\tilde{\varepsilon}_0 \cdot K + 2\tilde{\varepsilon}_0 \cdot K + K(d(\pi_S^{-1}(x), \text{supp}(\mu_T)) + d(\pi_S^{-1}(y), \text{supp}(\mu_T)))
\]
\[
= 4\tilde{\varepsilon}_0 \cdot K + K(d(\pi_S^{-1}(x), \text{supp}(\mu_T)) + d(\pi_S^{-1}(y), \text{supp}(\mu_T))).
\]

Thus, for each \( x \in \Gamma_S \), one has that
\[
|F_S(x) - A_{\Gamma_S}| \leq \frac{1}{\tau(\Gamma_S)} \sum_{y \in \Gamma_S} |F_S(x) - F_S(y)|
\]
\[
\leq \frac{1}{\tau(\Gamma_S)} \sum_{y \in \Gamma_S} \left( 4\tilde{\varepsilon}_0 \cdot K + K(d(\pi_S^{-1}(x), \text{supp}(\mu_T)) + d(\pi_S^{-1}(y), \text{supp}(\mu_T))) \right)
\]
\[
= 4\tilde{\varepsilon}_0 \cdot K + K(d(\pi_S^{-1}(x), \text{supp}(\mu_T)) + \frac{1}{\tau(\Gamma_S)} \sum_{y \in \Gamma_S} d(\pi_S^{-1}(y), \text{supp}(\mu_T)))
\]
\[
= 4\tilde{\varepsilon}_0 \cdot K + K(d(\pi_S^{-1}(x), \text{supp}(\mu_T)) + d_*/\tau(\Gamma_S)).
\]

Define \( \tilde{F}_S(x) = F_S(x) - A_{\Gamma_S} \) for all \( x \in \mathbb{T} \). Thus,
\[
\frac{\sum_{z \in \Gamma_S} \tilde{F}_S(z)}{\# \Gamma_S} = \int \tilde{F}_S d\delta_S = 0.
\]
Moreover, recall that \( F_S = f \circ S - f + \log \| DS \| \), one has
\[
\text{Lip}(\tilde{F}_S) = \text{Lip}(F_S) \leq \text{Lip}(f) \max_{x \in \mathbb{T}} \| DS(x) \| + \text{Lip}(f) + \frac{1}{\min_{x \in \mathbb{T}} \| DS(x) \|} \text{Lip}(DS) \\
\leq \text{Lip}(f)(\max_{x \in \mathbb{T}} \| DS(x) \| + 1) + \text{Lip}(DS) < K
\]
(2.25)

**Domains away from the periodic orbit.** Put
\[
\mathcal{F}_T = \{ x \in \mathbb{T} : d(x, \Gamma_T) > \rho \cdot G^* \}.
\]
Then \( \mathcal{F}_T \) is an open subset of \( \mathbb{T} \). By the definition of \( h(x) \), one can see that \( h(x) = 0 \) for any \( x \in \mathcal{F}_T \) and hence
\[
F_{S_0} \mid_{\mathcal{F}_T} = F_T \mid_{\mathcal{F}_T}.
\]
(2.26)

**Estimates in \( \mathcal{F}_T \).** We give a lower bound of \( \tilde{F}_S \) in \( \mathcal{F}_T \).

**Claim.**
\[
\tilde{F}_S(x) \geq \varepsilon \cdot \rho \cdot G^*/K^4 - K \cdot d_* - 2\varepsilon_0 \cdot K > 0, \quad \forall x \in \mathcal{F}_T.
\]
(2.27)

**Proof of the Claim.** Since \( F_{S_0}(x) \geq F_T(x) \geq \alpha(T) \) for any \( x \in \mathcal{F}_T \), one has that
\[
F_S(x) \geq F_{S_0}(x) - \varepsilon_0 \cdot K \geq \alpha(T) - \varepsilon_0 \cdot K, \quad \forall x \in \mathcal{F}_T.
\]
(2.28)

This implies that
\[
\tilde{F}_S(x) = F_S(x) - A_{\Gamma_S} \geq \alpha(T) - \varepsilon_0 \cdot K - 2\varepsilon_0 \cdot K > 0.
\]

**Estimates outside of \( \mathcal{F}_T \)**

**Lemma 2.11.** If \( x \notin \mathcal{F}_T \cup \Gamma_S \), then there exists \( N(x) \in \mathbb{N} \) such that
\[
x, S(x), \ldots, S^{N(x)-1} x \in \{ z \in \mathbb{T} : d(z, \Gamma_S) \leq (\rho \cdot G^* + \varepsilon_0) K^L \}
\]
and \( \sum_{i=0}^{N(x)-1} \tilde{F}_S(S^i(x)) > 0 \).
Proof. Now we assume that \( x \notin \mathcal{F}_T \cup \Gamma_S \). This implies that \( d(x, \Gamma_T) \leq \rho_\varepsilon \cdot G^* \). By the property of \( \pi_S \) (2.18) and the fact \( \Gamma_T = \Gamma_{S_0} \), one has that
\[
d(x, \Gamma_S) \leq d(x, \Gamma_T) + \tilde{\varepsilon}_0 \leq \rho_\varepsilon \cdot G^* + \tilde{\varepsilon}_0.
\]
Notice that
\[
\rho_\varepsilon \cdot G^* + \tilde{\varepsilon}_0 < (\rho_\varepsilon \cdot G^* + \tilde{\varepsilon}_0)K^{L_\varepsilon} \leq \frac{G^* - 2\tilde{\varepsilon}_0}{2K}. \tag{2.29}
\]
By Lemma 2.3, Lemma 2.10 and the assumption \( x \notin \Gamma_S \), there exists \( n \in \mathbb{N} \cup \{0\} \) such that
\[
d(S^n(x), \Gamma_S) \geq \frac{G(\Gamma_S)}{2 \max_{x \in T} \|DS(x)\|} \geq \frac{G^* - 2\tilde{\varepsilon}_0}{2K} > \rho_\varepsilon \cdot G^* + \tilde{\varepsilon}_0.
\]
Notice that \( S \) is uniformly expanding and the expanding rate of \( S \) is not larger than \( K \). Thus, there exists \( m(x) \in \mathbb{N} \), the minimal natural number such that
\[
\rho_\varepsilon \cdot G^* + \tilde{\varepsilon}_0 < d(S^m(x), \Gamma_S) \leq (\rho_\varepsilon \cdot G^* + \tilde{\varepsilon}_0) \cdot K,
\]
i.e.,
\[
m(x) = \min \{ m \in \mathbb{N} : d(S^m(x), \Gamma_S) > \rho_\varepsilon \cdot G^* + \tilde{\varepsilon}_0 \}.
\]
By the minimality of \( m(x) \), one has that
\[
d(S^l(x), \Gamma_S) \leq \rho_\varepsilon \cdot G^* + \tilde{\varepsilon}_0, \quad \forall \ 0 \leq l \leq m(x) - 1. \tag{2.30}
\]
Since the expanding rate of \( S \) is bounded by \( K \), one has
\[
S^{m(x)}(x), S^{m(x)+1}(x), \ldots, S^{m(x)+L_\varepsilon - 1}(x)
\]
\[\in \{ z \in T : \rho_\varepsilon \cdot G^* + \tilde{\varepsilon}_0 < d(z, \Gamma_S) \leq (\rho_\varepsilon \cdot G^* + \tilde{\varepsilon}_0)K^{L_\varepsilon} \}.
\]
Choose \( L(x) \in \mathbb{N} \) to be the minimal integer such that \( d(S^{m(x)+L(x)}(x), \Gamma_S) > (\rho_\varepsilon \cdot G^* + \tilde{\varepsilon}_0)K^{L_\varepsilon} \). Clearly by the definition, one has that \( L(x) \geq L_\varepsilon \). Now we set \( N(x) = m(x) + L(x) \).

Now we are going to prove \( \sum_{i=0}^{m(x)+L(x)-1} \tilde{F}_S(S^i(x)) > 0 \). Let \( m \in \{0, 1, \ldots, m(x) - 1\} \) be the maximal number such that
\[
S^m(x) \notin \mathcal{F}_T.
\]
Then by (2.27), one has that
\[
\tilde{F}_S(S^i(x)) > \varepsilon \cdot \rho_\varepsilon \cdot G^* \quad \frac{K^4}{K^4} - 2\tilde{\varepsilon}_0 \cdot K - K \cdot d_* > 0, \quad \forall \ m \leq i \leq m(x) + L(x) - 1. \tag{2.31}
\]
Choose \( p_S \in \Gamma_S \) such that
\[
d(x, \Gamma_S) = d(x, p_S).
\]
Claim. For $0 \leq i \leq m - 1$, one has that
\[ d(S^i(x), \Gamma_S) = d(S^i(x), S^i(p_S)). \]

Proof of the Claim. Given $0 \leq i \leq m$, one knows that by Lemma 2.10,
\[ d(S^i(p_S), \Gamma_S \setminus \{S^i(p_S)\}) > G^* - 2\tilde{\varepsilon}_0. \]
Thus, it suffices to prove that $d(S^i(x), S^i(p_S)) < G^*/2 - \tilde{\varepsilon}_0$. By the fact that $i \leq m - 1 < m(x)$, one has that
\[ d(S^i(x), S^i(p_S)) \leq \rho_\varepsilon \cdot G^* + \tilde{\varepsilon}_0 \leq (\rho_\varepsilon \cdot G^* + \tilde{\varepsilon}_0)K^\varepsilon \overset{\text{2.13}}{\leq} \frac{G^* - 2\tilde{\varepsilon}_0}{2K} < G^*/2 - \tilde{\varepsilon}_0. \]

By the above claim, we have
\[
\sum_{i=0}^{m-1} d(S^i(x), S^i(p_S)) \leq \sum_{i=0}^{m-1} \frac{d(S^{m-1}(x), S^{m-1}(p_S))}{(\min_{w \in T} \|DS(w)\|)^i} \\
\leq \frac{(\rho_\varepsilon \cdot G^* + \tilde{\varepsilon}_0) \cdot \min_{w \in T} \|DS(w)\|}{\min_{w \in T} \|DS(w)\| - 1} \overset{\text{2.15}}{\leq} (\rho_\varepsilon \cdot G^* + \tilde{\varepsilon}_0)K.
\]
Therefore,
\[
\sum_{i=0}^{m-1} (\tilde{F}_S(S^i(x)) - \tilde{F}_S(S^i(p_S))) \overset{\text{2.25}}{\geq} -K \sum_{i=0}^{m-1} d(S^i(x), S^i(p_S)) \geq -K^2(\rho_\varepsilon \cdot G^* + \tilde{\varepsilon}_0). \tag{2.32}
\]

Claim. One has that
\[
\sum_{i=0}^{m-1} \tilde{F}_S(S^i(p_S)) \geq -\tau(\Gamma_S) \left(4\tilde{\varepsilon}_0 \cdot K + 2Kd_\varepsilon/\tau(\Gamma_S)\right). \tag{2.33}
\]

Proof of the claim. Assume that $m = Q\tau(\Gamma_S) + r$ for some nonnegative integer $Q$ and $0 \leq r \leq \tau(\Gamma_S) - 1$. When $r = 0$, one has that
\[
\sum_{i=0}^{m-1} \tilde{F}_S(S^i(p_S)) = Q \cdot \sum_{i=0}^{\tau(\Gamma_S)-1} \tilde{F}_S(S^i(p_S)) = 0 \\
\geq -\tau(\Gamma_S) \left(4\tilde{\varepsilon}_0 \cdot K + 2Kd_\varepsilon/\tau(\Gamma_S)\right).
\]
When $r \geq 1$,
\[
\sum_{i=0}^{m-1} \tilde{F}_S(S^i(p_S)) = Q \cdot \sum_{i=0}^{\tau(\Gamma_S)-1} \tilde{F}_S(S^i(p_S)) + \sum_{i=Q\tau(\Gamma_S)}^{r-1} \tilde{F}_S(S^i(p_S)) \tag{2.24}
\]
\[
= \sum_{i=0}^{r-1} \tilde{F}_S(S^i(p_S))
\]
\[
= \sum_{i=0}^{r-1} F_S(S^i(p_S)) - rA_{\Gamma_S}
\]
\[
\geq - \sum_{z \in \Gamma_S} |F_S(z) - A_{\Gamma_S}|
\]
\[
\geq - \sum_{z \in \Gamma_S} (4\varepsilon_0 \cdot K + K(d(\pi_S^{-1}(z), \text{supp}(\mu_T)) + d_*/\tau(\Gamma_S))
\]
\[
\geq -\tau(\Gamma_S) (4\varepsilon_0 \cdot K + 2Kd_*/\tau(\Gamma_S)).
\]
\[\square\]

Therefore, by applying (2.31), (2.32) and (2.33), we have that
\[
\sum_{i=0}^{N(x)-1} \tilde{F}_S(S^i(x)) = \sum_{i=m}^{m(x)+L(x)-1} \tilde{F}_S(S^i(x)) + \sum_{i=0}^{m-1} (\tilde{F}_S(S^i(x)) - \tilde{F}_S(S^i(p_S))) + \sum_{i=0}^{m-1} \tilde{F}_S(S^i(p_S))
\]
\[
\geq (L(x) + m(x) - m)(\varepsilon \cdot \rho \cdot G^*/K^4 - K \cdot d_* - 2\varepsilon_0 \cdot K)
\]
\[
- \rho \cdot G^* + \varepsilon_0 \cdot K^2 - \tau(\Gamma_S) (4\varepsilon_0 \cdot K + 2Kd_*/\tau(\Gamma_S))
\]
\[
\geq L_e (\varepsilon \cdot \rho \cdot G^*/K^4 - K \cdot d_* - 2\varepsilon_0 \cdot K)
\]
\[
- \rho \cdot G^* + \varepsilon_0 \cdot K^2 - \tau(\Gamma_S) (4\varepsilon_0 \cdot K + 2Kd_*/\tau(\Gamma_S))
\]
\[
\geq 0.
\]
\[\square\]

**Ergodic measures.** Now we check for measures. Take an $S$-ergodic probability measure $\mu \neq \delta_{\Gamma_S}$. To conclude, it suffices to prove that
\[
\int F_Sd\mu > \int F_Sd\delta_{\Gamma_S},
\]
which is equivalent to to show that
\[
\int \tilde{F}_Sd\mu > \int \tilde{F}_Sd\delta_{\Gamma_S} \tag{2.24}
\]
\[
= 0.
\]
Let $x$ be a generic point of $\mu$. Then $S^i(x) \notin \Gamma_S$ for all $i \in \mathbb{N} \cup \{0\}$ since $\mu \neq \delta_{\Gamma_S}$. By Lemma 2.11 for $x \notin F_T$, one can define $N(x)$. So for any $y \notin \Gamma_S$, define

$$I(y) = \begin{cases} 1 & \text{if } y \in F_T, \\ N(y) & \text{if } y \notin F_T. \end{cases}$$

Claim.

$$\sum_{i=0}^{I(y)-1} \widetilde{F}_S(S^i(y)) > 0, \quad \forall y \notin \Gamma_S. \quad (2.34)$$

Proof of the Claim. By Lemma 2.11 one has that $\sum_{i=0}^{I(y)-1} \widetilde{F}_S(S^i(y)) > 0$ for any $y \notin F_T \cup \Gamma_S$. If $y \in F_T$, $I(y) = 1$. One only has to show that $\widetilde{F}_S(y) > 0$ for $y \in F_T$. This is given by Inequality (2.27). □

Now we define an index sequence $\{j_n\}_{n \in \mathbb{N}}$ by induction on $n$. Put

$$j_1 = 0, \quad \text{and } j_n = j_{n-1} + I(S^{j_{n-1}}(x)) \text{ for } n \geq 2.$$ 

The index sequence $\{j_n\}_{n \in \mathbb{N}}$ is well defined since $S^i(x) \notin \Gamma_S$ for all $i \in \mathbb{N} \cup \{0\}$.

Set $F_1 = \{ z \in T : d(z, \Gamma_S) > (\rho \cdot G^* + \xi_0)K^{L_\epsilon} \}$. By the definition, one can check that $F_1 \subset F_T$.

Claim.

$$\mu(F_1) > 0. \quad (2.35)$$

Proof. By Lemma 2.3, there is $n \in \mathbb{N}$ such that

$$d(S^n(x), \Gamma_S) \geq \frac{G(\Gamma_S)}{2\max_{z \in T} \|DS(z)\|} > \frac{G(\Gamma_S)}{2K} \overset{\text{Lemma 2.10}}{>} \frac{G^* - 2\xi_0}{2K} \overset{\text{2.15}}{>} (\rho \cdot G^* + \xi_0)K^{L_\epsilon}.$$ 

In other words, $S^n(x) \in F_1$. Since $S^n(x)$ is also a generic point of $\mu$, one has that $\mu(F_1) > 0$. □

Put

$$\mathcal{N} = \{ i \in \mathbb{N} \cup \{0\} : S^i(x) \in F_1 \}.$$ 

By the ergodicity of $\mu$, we have that

$$\liminf_{N \to +\infty} \frac{\#(\mathcal{N} \cap [0, N-1])}{N} \geq \mu(F_1). \quad (2.36)$$

By the fact that $F_1 \subset F_T$ and the definition of $I$, one has that

$$\mathcal{N} \subset \{ j_n : j_{n+1} - j_n = 1, n \in \mathbb{N} \}. \quad (2.37)$$
Therefore, we have
\[
\int \tilde{F}_s d\mu = \lim_{m \to +\infty} \frac{1}{j_{m+1}} \sum_{i=0}^{j_{m+1}-1} \tilde{F}_s(S^i(x)) = \lim_{m \to +\infty} \frac{1}{j_{m+1}} \sum_{n=1}^{m} \sum_{i=j_n}^{j_{n+1}-1} \tilde{F}_s(S^i(x))
\]
\[
= \lim_{m \to +\infty} \frac{1}{j_{m+1}} \left( \sum_{n=1}^{m} \sum_{i=j_n}^{j_{n+1}-1} \tilde{F}_s(S^i(x)) + \sum_{n=1}^{m} \sum_{i=j_n}^{j_{n+1}-1} \tilde{F}_s(S^i(x)) \right)
\]
\[
\geq \lim_{m \to +\infty} \frac{1}{j_{m+1}} \left( \sum_{n=1}^{m} \sum_{i=j_n}^{j_{n+1}-1} \tilde{F}_s(S^i(x)) \right) \geq \lim_{m \to +\infty} \frac{1}{j_{m+1}} \left( \sum_{n=1}^{m} \sum_{i=j_n}^{j_{n+1}-1} \tilde{F}_s(S^i(x)) \right) \geq \lim_{m \to +\infty} \frac{1}{j_{m+1}} \left( \sum_{n=1}^{m} \sum_{i=j_n}^{j_{n+1}-1} \tilde{F}_s(S^i(x)) \right) \geq \mu(\mathcal{F}_1) \left( \frac{\varepsilon \cdot \rho \cdot G^*}{K^4} - 2\tilde{\varepsilon}_0 \cdot K - K \cdot d_\star \right) > 0.
\]

Hence one can conclude. \(\square\)

### 2.4 Proof of Theorem 2.7

Let \(T\) be a \(C^2\) expanding self-map of \(\mathbb{T}\). Consider the \(C^{1,1}\) map \(h\) defined in the proof of Theorem 2.6. For \(\delta > 0\) we let
\[
h_\delta(x) = \frac{1}{2\delta} \int_{-\delta}^{\delta} h(x + s) ds.
\]
One has that
\[
Dh_\delta(x) = \frac{1}{2\delta} \int_{-\delta}^{\delta} Dh(x + s) ds.
\]
By a simple calculation, one has that

**Claim.** For any \(x \in \mathbb{T}\), \(D^2h_\delta(x) = 1/(2\delta)(Dx(x + \delta) - Dx(x - \delta))\).

By Theorem 2.6, \(\|D^2h_\delta\|_{C^0} \leq \text{Lip}(Dh) < \varepsilon/2\). Clearly, \(\|h_\delta\|_{C^1} < \varepsilon/2\). Thus \(S_\delta = T + h_\delta\) is an \(\varepsilon/2\)-perturbation of \(T\) for \(\delta\) small enough in the \(C^2\) topology.

Any \(C^2\)-small perturbation of \(S_\delta\) is contained in a neighborhood of \(S_0\) in the \(C^{1,1}\)-neighborhood. Thus by Theorem 2.6, for any \(S\) sufficiently close to \(S_\delta\), the Lyapunov minimizing measure of \(S\) is supported on \(\Gamma_S\). Hence the proof of Theorem 2.7 is complete. \(\square\)
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