Zeros of tree-level amplitudes at multi-boson thresholds

M.B. Voloshin
Theoretical Physics Institute, University of Minnesota
Minneapolis, MN 55455
and
Institute of Theoretical and Experimental Physics
Moscow, 117259

Abstract

Propagation of particles with emission of arbitrary number of identical bosons all being at rest is considered. It is shown that in certain models the tree-level amplitudes for production of \( n \) scalar bosons by two incoming particles are all equal to zero at the threshold starting from some small number \( n \). In particular this nullification occurs for production of massive scalars by two Goldstone bosons in the linear sigma model for \( n > 1 \) and also for production of Higgs bosons in the Standard Model by gauge bosons and/or by fermions, provided that the ratio of their masses to that of the Higgs boson take special discrete values.
Based on the technique recently suggested by Lowell Brown\textsuperscript{1} it has been found\textsuperscript{2, 3} that the on-mass-shell scattering amplitudes in the \(\lambda \phi^4\) theory for production of \(n\) particles of the field \(\phi\) by two incoming ones display a peculiar behavior at the \(n\) particle threshold. Namely, the tree-level result for those amplitudes is strictly zero for \(n > 4\) in the theory without spontaneous symmetry breaking (positive mass term)\textsuperscript{2} and for \(n > 2\) in the theory with spontaneous symmetry breaking (negative mass term)\textsuperscript{3}.

In this paper few more examples of the same behavior are given, where the initial particles are different from the bosons being produced. The processes considered can be generically written as \(\chi \chi \rightarrow n \phi\) and \(f \bar{f} \rightarrow n \phi\) where \(\chi\) stands for either a different than \(\phi\) scalar boson or a massive gauge boson and \(f\) stands for a fermion. In particular it is found that in a linear sigma model the amplitude of production of \(n\) massive \(\sigma\) particles by two incoming Goldstone bosons vanishes at the threshold for any \(n\) larger than 1. As a consequence of this nullification the amplitude of production of more than one Higgs boson by longitudinal massive vectors in the Standard Model is vanishing at the threshold to the leading order in the ratio of the vector boson and the Higgs boson masses. This behavior for the particular case \(n = 2\) was observed in some previous direct calculations\textsuperscript{4}. Furthermore for the transversal vector bosons it will be demonstrated that if the ratio of the masses satisfies the relation

\begin{equation}
4m_V^2/m_H^2 = N(N + 1)
\end{equation}

with \(N\) being an integer, the tree-level amplitudes of the processes \(V_T V_T \rightarrow n H\) are all zero at the threshold for \(n > N\). For the fermions in the Standard Model the “magic” mass ratio is of the form

\begin{equation}
m_f/m_H = N/2
\end{equation}

with an integer \(N\), in which case the nullification at the threshold occurs for the amplitudes of the processes \(f \bar{f} \rightarrow n H\) for all \(n\) greater than \(N\) and also for \(n = N\). Notice that both in the case of transversal vector bosons and in the case of fermions this implies that the nullification of the production amplitudes occurs in fact at all kinematically possible thresholds for on-mass-shell processes due to the obvious constraint \(2m_V \leq n m_H\) or \(2m_f \leq n m_H\), provided of course that the relation (1) or (2) holds.

To arrive at these conclusions we calculate at the tree level the propagator \(D_n(p)\) of a quantum particle with emission of \(n\) on-mass-shell bosons of the field \(\phi\) all being at rest and \(p\) being the final momentum in the propagator after the emission (see Fig. 1). As a most direct approach to this calculation we use a combination of the Brown’s technique and the
one based on the generating function method of solving recursion relations for the propagators $D(p)$. To formulate the recursion relations we introduce the notation $-iV(\phi)$ for the vertex of interaction of the propagating particle ($\chi$ or $f$) with arbitrary external field $\phi$ and for the beginning we consider the propagator of a bosonic field $\chi$ whose mass in the absence of interaction with $\phi$ is $m_{0\chi}$. The recursion equations for the propagators $D_n(p)$ arise from graph shown in Fig. 2 and can be written as

$$D_n(p) = -i d(p + nq) \sum_{n_1} \frac{n!}{n_1!(n-n_1)!} V_{n_1} D_{n-n_1}(p) ,$$

where $q$ is the momentum of each of the produced $\phi$-bosons, in their rest frame $q = (M, 0)$ with $M$ being the mass of the $\phi$ boson, $d(p)$ is the propagator in the absence of interaction with the field $\phi$: $d(p) = i/(p^2 - m_{0\chi}^2)$, and $V_n$ are the matrix elements for production of $n$ bosons at rest by the operator $V(\phi(x))$:

$$V_n = \langle n|V(\phi(0))|0 \rangle .$$

Notice, that in a theory where the field $\phi$ develops a vacuum expectation value $v = \langle 0|\phi|0 \rangle$ the “bare” propagator $d(p)$ does not coincide with the propagator

$$D_0(p) = \frac{i}{p^2 - m_{0\chi}^2 - V_0} ,$$

since the physical mass of the $\chi$ boson receives also a contribution from $V_0 = V(v)$: $m_{\chi}^2 = m_{0\chi}^2 + V_0$.

The recursion equations (3) are converted into a differential equation by introducing generating functions $V(z)$ and $D(p; z)$ related to the $V_n$ and $D_n(p)$ as

$$V(z) = \sum_{n=0} z^n V_n \quad \text{and} \quad D(p; z) = -i \sum_{n=0} z^n D_n(p) .$$

In terms of the generating functions the equation (3) is equivalent to the $n$-th term of the Taylor series expansion of the differential equation

$$\left( M^2 z^2 \frac{d^2}{dz^2} + (2(p \cdot q) + M^2) z \frac{d}{dz} + p^2 - m_{0\chi}^2 - V(z) \right) D(p; z) = 1$$

with the initial condition at $z = 0$ that $D(p; 0) = D_0(p)$ (the consistency of this condition is guaranteed by the inhomogeneous term in the right hand side) and that the solution has Taylor expansion in positive powers of $z$. Alternatively one can specify the latter condition
by calculating explicitly the propagator $D_1(p)$, i.e. with emission of one particle, which determines the first derivative $dD(p; z)/dz$ at $z = 0$.

At the tree level the generating function $V(z)$ is determined by the generating function $\Phi(z)$ for the matrix elements of the field $\phi$ itself,

$$\Phi(z) = \sum \langle n | \phi(0) | 0 \rangle z^n / n! , \tag{8}$$

$V(z) = V(\Phi(z))$. The function $\Phi(z)$ is known for a class of theories\[6,5,1]. In this paper the analysis is restricted to a sub-class of probably the most physical interest i.e. where the Lagrangian density for the field $\phi$ is that of the $\lambda \phi^4$ theory

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2 - \frac{1}{4} \lambda \phi^4 \tag{9}$$

with or without spontaneous symmetry breaking, i.e. with either sign of the $m^2$. For the case of no symmetry breaking one has\[6,5,1]

$$\Phi(z) = \frac{z}{1 - (\lambda/8M^2) z^2} \tag{10}$$

with $M = m$ and for the case of negative $m^2$ the generating function is given by\[6,5,1]

$$\Phi(z) = v \frac{1 + z/2v}{1 - z/2v} , \tag{11}$$

where $v = |m|/\sqrt{\lambda}$ and also the physical mass of the bosons of the field $\phi$ is $M = \sqrt{2} |m|$.

We shall further restrict the generality of the present discussion by requiring that the vertex function $V(\phi)$ has the simple form

$$V(\phi) = \xi \phi^2 \tag{12}$$

which still covers quite a few interesting cases.

Let us first consider the unbroken symmetry case. Upon substitution into equation (12) of the generating function $V(z)$ as determined by eqs.(12) and (11) the equation takes the form

$$\left( M^2 z^2 \frac{d^2}{dz^2} + (2(p \cdot q) + M^2) z \frac{d}{dz} + p^2 - m^2 + \frac{\xi z^2}{[1 - (\lambda/8M^2) z^2]^2} \right) D(p; z) = 1 . \tag{13}$$

It is convenient to introduce the notation $\epsilon = (p \cdot q)/M^2$ and $\omega = \sqrt{\epsilon^2 - (p^2 - m^2)} / M$, where the square root of a positive number is taken to be positive. In the rest frame of the
produced bosons $\epsilon$ is the energy $p_0$ of the final particle in the propagator in units of $M$ and $\omega^2$ is related to its spatial momentum $p$, namely it is $p^2 + m_0^2 \chi$ in units of $M^2$. Thus the difference $\epsilon^2 - \omega^2$ is the measure of by how much the momentum $p$ is off shell.

The solution of the equation (13) can be sought in the form

$$D(p; z) = y^{-\epsilon/2} f(p; y),$$

where $y = -\lambda z^2/(8M^2)$, in terms of which eq.(13) is rewritten as

$$\left(4y^2 \frac{d^2}{dy^2} + 4y \frac{d}{dy} - \omega^2 + 8 \frac{\xi}{\lambda} \frac{y}{(1+y)^2} \right) f(p; y) = \frac{y^{\epsilon/2}}{M^2}. \quad (15)$$

For the broken symmetry case, using in the same manner the generating function (11) instead of (10) and seeking the solution in the form (14) in terms of the variable $y = -z/(2v)$, one arrives at the equation essentially identical to eq.(15) up to rescaling of some terms:

$$\left(4y^2 \frac{d^2}{dy^2} + 4y \frac{d}{dy} - 4\bar{\omega}^2 + 8 \frac{\xi}{\lambda} \frac{y}{(1+y)^2} \right) f(p; y) = 4 \frac{y^{\epsilon/2}}{M^2}. \quad (16)$$

where $\bar{\omega}^2 = \epsilon^2 - (p^2 - m_0^2 - \xi v^2)/M^2$, and it can be also reminded that the expression for $M$ in terms of $|m|$ differs from that in the case of unbroken symmetry by factor $\sqrt{2}$. Therefore the solution of the equation (16) is obtained from that of eq.(15) by replacing $\omega \rightarrow 2\bar{\omega}$ and by overall rescaling of the solution by the factor 4.

The operator in the homogeneous left hand side of eq.(15) is related to the well known exactly solvable Schrödinger operator with the Pöschl-Teller potential. Namely, the substitution $y = e^{2r}$ converts the operator into

$$\frac{d^2}{d\tau^2} - \omega^2 + \frac{s(s+1)}{(\cosh \tau)^2}, \quad (17)$$

with $s(s+1) = 2\xi/\lambda$.

The properties of the operator (17) are well known from Quantum Mechanics (see e.g. in the textbook [8]). In particular these properties are quite special when the parameter $s$ is integer, $s = N$, and there is no reflection of waves in the one-dimensional Quantum Mechanical problem. It is in this special case of integer $s$ that the infinite series of zeros of the threshold amplitudes arises and which will be considered here. The solution of the equation (13) for $D(p; z)$ can be written explicitly in terms of the hypergeometric function $F(-s, \omega - s, \omega + 1, -y)$, which in the case $s = N$ is reduced to a Jacobi polynomial of the $N$-th power. We therefore define the function
\[ F_N(\omega, y) = \frac{\Gamma(1 + \omega)}{\Gamma(N + 1 + \omega)} y^{-\omega/2} (1 + y)^{N+1} \frac{d^N}{dy^N} \left( \frac{y^{\omega+N}}{(1+y)^{N+1}} \right) \] (18)

and write the explicit formula for \( D(p; z) \) in the unbroken symmetry case in terms of the variable \( y \) in the form

\[
D(p, z(y)) = \frac{y^{-\epsilon/2}}{4\omega M^2} \left[ F_N(-\omega, y) \int_0^y u^{\tilde{z}-1} F_N(\omega, u) \, du + F_N(\omega, y) \int_y^\infty u^{\tilde{z}-1} F_N(-\omega, u) \, du \right],
\] (19)

where \( y \) is assumed to be positive. Notice, that \( F_N(\omega, y) \) is regular at small \( y \): \( F_N(\omega, y) \approx y^{\omega/2} \), while \( F_N(-\omega, y) \) is singular at \( y \to 0 \). At \( y \to \infty \) the behavior of these functions is switched. Also in its dependence on \( \omega \) the function \( F_N(-\omega, y) \) develops simple poles at \( \omega = 1, 2, \ldots, N \) due to the factor \( \Gamma(1-\omega)/\Gamma(N+1-\omega) \) in its definition.

We can now proceed to considering the amplitude of production \( n \phi \)-bosons by two incoming particles. In terms of the propagator \( D_n(p) \) this amplitude corresponds to negative \( p_0 \) (the final line in the propagator is that of an incoming particle), so that \( \epsilon = -n/2 \). The on-mass-shell amplitude is given by the double pole of this propagator when \( \omega = -\epsilon = n/2 \).

The only possibility for the expression in the right hand side of the equation (19) to develop a double pole for positive \( \omega \) is when one pole term comes from the function \( F_N(-\omega, y) \) and the other one comes from the divergence of the integral. At negative \( \epsilon \) the first of the integrals in eq.(19) indeed develops single poles, so the double poles are possible only for the values \( \omega = 1, 2, \ldots, N \), i.e. only as long as \( n \leq 2N \) and thus the on-mass-shell threshold amplitudes are all zero for \( n > 2N \).

For the case of broken symmetry as we have seen the generating function \( D \) for the propagators is determined by the same operator as for the unbroken symmetry case with \( \omega \) rescaled by factor 2 (eq.(16)). Therefore the same consideration leads one to the conclusion that in this case the on-mass-shell threshold amplitudes are zero for all \( n \) larger than \( N \).

This nullification of the amplitudes at the thresholds is the one observed for production of the \( \phi \) bosons by particles of the same field, whose propagation is described by the interaction (12) with \( \xi = 3\lambda \), so that \( 2\xi/\lambda = 6 \) and thus \( N = 2 \) both in the case of no spontaneous symmetry breaking\(^2\)\(^,\) and in the case of broken symmetry\(^3\). Here we consider few more applications corresponding to different integer values \( s = N \).
Linear sigma model.

In the Lagrangian density of the linear sigma model

\[ \mathcal{L} = \frac{1}{2} (\partial_\mu \sigma)^2 + \frac{1}{2} (\partial_\mu \vec{\pi})^2 - \frac{\lambda}{4} (\sigma^2 + \vec{\pi}^2 - v^2)^2 \]  

the field \( \sigma \) plays the role of \( \phi \) and the \( \pi \) components play the role of \( \chi \) in the previous discussion. The interaction between the “pions” and the \( \sigma \) in terms of our previous definition (12) corresponds to \( \xi = \lambda \), thus \( 2\xi/\lambda = 2 \), so that \( N = 1 \). Therefore one immediately concludes that all the tree-level amplitudes of the processes \( 2\pi \rightarrow n\sigma \) vanish at the threshold except for \( n = 1 \).

Higgs boson production by massive vector bosons.

For a heavy Higgs boson in the standard model to the leading order in the ratio of the masses \( m_V/m_H \) the amplitudes of production of Higgs bosons by longitudinally polarized massive vector bosons are given by those of the linear sigma model, where the \( \pi \) components describe the longitudinally polarized gauge bosons. Therefore from the previous paragraph one concludes that in this order the threshold amplitudes for \( V_L V_L \rightarrow nH \) vanish at the threshold for \( n > 1 \). For the particular case \( n = 2 \) this behavior was observed in direct calculations\[4\] of Feynman graphs.

For the transversal components of the gauge bosons the Lagrangian in the Proca gauge is equivalent to that of a scalar interacting with the properly normalized Higgs field \( \phi \) with the constant such that \( 2\xi/\lambda = 4 m_V^2/m_H^2 \). Therefore we conclude that if this mass ratio has a value corresponding to the equation (12) the tree-level amplitudes for the processes \( V_T V_T \rightarrow nH \) vanish at the threshold for \( n > N \). Notice however that for the on-mass-shell processes of this type only the values of \( n \) larger than \( N \) are possible, due to the kinematical constraint \( 2m_V \leq n m_H \). Therefore for all physically possible on-mass-shell processes of this kind the amplitudes vanish at the threshold.

Production of scalar bosons by fermions.

Let us turn now to the case when the \( \phi \) bosons are produced by fermions due to Yukawa interaction corresponding to the vertex function

\[ V(\phi) = h\phi \]  

(21)
where $h$ is the coupling constant. The analog of the bosonic equation (7) for the fermion propagator generating function reads as

$$\left( (\gamma \cdot q)z \frac{d}{dz} + (\gamma \cdot p) - m_{0f} - h\Phi(z) \right) D(p; z) = 1 ,$$

(22)

where $m_{0f}$ is the fermion mass term in the absence of interaction with $\phi$, and naturally the generating function $D$ in this case is a matrix acting on bispinors.

The solution of the equation (22) can be sought in the form

$$D(p; z) = \left( (\gamma \cdot q)z \frac{d}{dz} + (\gamma \cdot p) + m_{0f} + h\Phi(z) \right) F(p; z) ,$$

(23)

and the equation for the function $F(p; z)$ thus reads as

$$\left( M^2 z^2 \frac{d^2}{dz^2} + (2(p \cdot q) + M^2)z \frac{d}{dz} + p^2 - (m_{0f} + h\Phi(z))^2 + (\gamma \cdot q)hz \frac{d\Phi(z)}{dz} \right) F(p; z) = 1 .$$

(24)

This equation takes a familiar from the previous discussion form in the case relevant to the Standard Model, when $m_{0f} = 0$, i.e. when the fermion gets all of its mass due to the spontaneous symmetry breaking. Substituting then the explicit expression (11) for $\Phi(z)$ one finds that the potential term in the equation (24) has the form

$$- h^2\Phi(z)^2 + (\gamma \cdot q)hz(d\Phi(z)/dz) = -m_f^2 - M^2 \left( 4 \frac{m_f^2}{M^2} - 2 \frac{(\gamma \cdot q) m_f}{M} \right) \frac{z/2v}{(1 - z/2v)^2} ,$$

(25)

where $m_f = hv$ is the physical mass of the fermion. The spin operator $(\gamma \cdot q)/M$ in the rest frame of produced bosons is simply $\gamma_0$. Therefore its eigenvalues in this frame are either $+1$ or $-1$ and a free fermion wave function at a non-zero spatial momentum contains both eigenstates (upper and lower components of the Dirac bispinor in the standard representation). Therefore comparing with the previously discussed bosonic case with spontaneously broken symmetry, we conclude that the nullification of the threshold amplitudes for the scattering $f \bar{f} \rightarrow n H$ for $n > N$ occurs when the coefficient takes exceptional values for both eigenstates, i.e. the following two relations hold simultaneously:

$$4 \frac{m_f^2}{M^2} + 2 \frac{m_f}{M} = N(N + 1) \quad \text{and} \quad 4 \frac{m_f^2}{M^2} - 2 \frac{m_f}{M} = N_1(N_1 + 1)$$

(26)

It is assumed for simplicity that the coupling is scalar, though the following discussion can be readily generalized to the case of arbitrary mixture of scalar and pseudoscalar couplings.
with both $N$ and $N_1$ being integer. This obviously is equivalent to the “magic” relation (2).

It is interesting to notice that unlike the case of Higgs boson production by transversal vector bosons the result for fermions that the threshold amplitudes are zero for $n > N$ still leaves potentially non-zero one kinematically possible amplitude, i.e. for $n = N$, which corresponds to the degenerate situation when the fermion and antifermion being exactly at rest produce $N$ Higgs bosons all being also at rest. This amplitude however is ruled out by the gamma-matrix structure, provided that the Yukawa coupling is scalar as in eq.(21). Indeed in this case in the absence of any spatial momenta the gamma-matrix structure of the amplitude of the scattering $2 \to N$ in the rest frame can only be given by $A \gamma_0 + B$ with $A$ and $B$ being ordinary numbers. On the other hand the bispinors for the static fermion and antifermion are orthogonal different eigenvectors of the matrix $\gamma_0$ (the fermion bispinor has only upper components non-zero, while in that for the antifermion non-zero are the lower components). Therefore the amplitude is vanishing for the on-mass-shell fermion-antifermion pair.

It is not known at present to what extent these unexpected infinite series of zeros of the production amplitudes at the thresholds are affected by the loop corrections. Neither it is clear whether this unexpected behavior is a manifestation of a hidden symmetry.

I am thankful to L.B. Okun and V.A. Novikov for bringing to my attention the papers [4] where the threshold nullification to the leading order in $m_V/m_H$ of the amplitudes of the scattering $V_L V_L \to H H$ was found by a conventional calculation of Feynman graphs. This work is supported in part by the DOE grant DE-AC02-83ER40105.

References

[1] L.S. Brown, Univ. Washington preprint UW/PT-92-16, September 1992, to be published in Phys. Rev. D.

[2] M.B. Voloshin, Minnesota preprint TPI-MINN-92/45-T, September 1992, submitted to Phys. Rev. D.

[3] B.H. Smith, Minnesota preprint TPI-MINN-92/50-T, September 1992, submitted to Phys. Rev. D.
[4] D.A. Dicus et. al., Phys. Lett. B200, 187 (1988);  
A. Abbassabi et. al., Phys. Lett. B213, 386 (1988);  
A. Dobrovolskaya and V. Novikov, Z. Phys. C 52, 427 (1991), Orsay preprint LPTHE Orsay 92/37, July 1992.

[5] E.N. Argyres, R.H.P. Kleiss and C.G. Papadopoulos, CERN preprint CERN-TH-6496 (1992)

[6] M.B. Voloshin, Minnesota preprint TPI-MINN-92/1-T, January 1992, Nucl. Phys. B, to be published.

[7] M.B. Voloshin, Minnesota preprint TPI-MINN-92/46-T, September 1992, submitted to Phys. Rev. D.

[8] L.D. Landau and E.M. Lifshits, Quantum Mechanics, Non-Relativistic Theory, Third edition. Pergamon Press. Problems to Sects. 23 and 25 and mathematical appendices.

Figure captions

Fig. 1. The propagator $D_n(p)$ with emission of $n$ on-mass-shell particles all being at rest. The circle represents the sum of all tree graphs.

Fig. 2. The recursion equation (3) for the propagators $D_n(p)$. The filled circle corresponds to the sum of all tree graphs originating from one-fold interaction, described by the vertex function $V(\phi)$, of the propagating particle with the field $\phi$. 
Figure 1

\[ n = p + nq \]

Figure 2

\[ n = \sum_{n_1}^{n_1} V (n - n_1) \]