A GAMMA FUNCTION IN TWO VARIABLES

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Abstract. We introduce a gamma function $\Gamma(x, z)$ in two complex variables which extends the classical gamma function $\Gamma(z)$ in the sense that $\lim_{x \to 1} \Gamma(x, z) = \Gamma(z)$. We will show that many properties which $\Gamma(z)$ enjoys extend in a natural way to the function $\Gamma(x, z)$. Among other things we shall provide functional equations, a multiplication formula, and analogues of the Stirling formula with asymptotic estimates as consequences.

1. Introduction

Throughout, let $\mathbb{N}, \mathbb{Z}, \mathbb{R}$, and $\mathbb{C}$ be the sets of positive integers, integers, real numbers, and complex numbers respectively. Further, let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mathbb{Z}_0 = \mathbb{Z} \setminus \mathbb{N}, \mathbb{R}^+ = \mathbb{R} \setminus \{r \in \mathbb{R} : r \leq 0\}$, and $\mathcal{D} = \mathbb{C} \setminus \{x \in \mathbb{R} : x \leq 0\}$. The gamma function $\Gamma(z)$ is one of the most important special functions in mathematics with applications in many disciplines like Physics and Statistics. It was first introduced by Euler in the integral form

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, dt.\quad (1)$$

Well-known equivalent definitions for the gamma function include the following three forms:

$$\Gamma(z) = \left(z e^\gamma \prod_{n=1}^\infty (1 + \frac{z}{n})e^{-\frac{z}{n}}\right)^{-1},\quad (2)$$

$$\Gamma(z) = \lim_{n \to \infty} \frac{n^zn!}{(z)n+1},\quad (3)$$

$$\Gamma(z) = \frac{1}{z} \prod_{n=1}^\infty (1 + \frac{1}{n})^z(1 + \frac{z}{n})^{-1},\quad (4)$$

where $\gamma$ is the Euler-Mascheroni constant

$$\gamma = \lim_{n \to \infty} \left(1 + \frac{1}{2} + \ldots + \frac{1}{n} - \log n\right)$$

and $(z)_n$ is the Pochhammer symbol

$$(z)_n = \begin{cases} 1 & \text{if } n = 0, \\ z(z+1)\ldots(z+n-1) & \text{if } n \in \mathbb{N}. \end{cases}$$
The gamma function satisfies the basic functional equation \( \Gamma(z + 1) = z \Gamma(z) \). Barnes [2] and Post [11] investigated the theory of difference equations of the more general form \( \phi(z+1) = f(z)\phi(z) \) under conditions on the function \( f(z) \) and obtained generalized gamma functions as solutions. See also Barnes [2] where multiple gamma functions have been introduced. Many mathematicians considered concrete cases of generalized gamma functions. Dilcher [7] introduced for any nonnegative integer \( k \) the function

\[
\Gamma_k(z) := \lim_{n \to \infty} \exp\left\{ \frac{\log^{k+1} n}{k+1} z \right\} \prod_{j=1}^{n} \exp\left\{ \frac{1}{k+1} \log^{k+1} (j + z) \right\}
\]

which for \( k = 0 \) becomes \( \Gamma(z) \), see formula (3). Díaz and Pariguan [4] extended the integral representation (1) to the function

\[
\Gamma_k(z) = \int_{0}^{\infty} t^{z-1} e^{-tk} dt \quad (k \in \mathbb{R}^+)
\]

which for \( f(t) = t \) clearly gives \( \Gamma(z) \).

In this paper we present a gamma function \( \Gamma(x,z) \) in two complex variables which is meromorphic in both variables and which satisfies \( \lim_{x \to 1} \Gamma(x,z) = \Gamma(z) \).

Our motivation is to extend the Weierstrass form (2) in much the same way the Hurwitz zeta function

\[
\zeta(x,s) = \sum_{n=0}^{\infty} \frac{1}{(n + x)^s}
\]

extends the Riemann zeta function

\[
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}
\]

So our definition involves the infinite product

\[
\prod_{n=0}^{\infty} (1 + \frac{z}{n + x})^{-1} e^{\frac{n}{n+x}} \quad \text{rather than} \quad \prod_{n=1}^{\infty} (1 + \frac{z}{n})^{-1} e^{\frac{n}{n}}
\]

and in order to maintain valid the analogues of properties of \( \Gamma(z) \) the factor \( e^{-z \gamma} \) will be replaced by \( e^{-z \gamma(x)} \), where \( \gamma(x) \) is defined as follows.

**Definition 1.** For \( x \in \mathbb{D} \setminus \mathbb{Z}^+ \) let the function \( \gamma(x) \) be

\[
\gamma(x) = \lim_{n \to \infty} \left( \frac{1}{x} + \frac{1}{x + 1} + \ldots + \frac{1}{x + n - 1} - \log n \right) = \frac{1}{x} + \sum_{n=1}^{\infty} \left( \frac{1}{x + n} - \log \frac{n + 1}{n} \right).
\]

Note that \( \gamma(1) = \gamma \) and that \( \gamma(x) = \gamma_0(x) = -\psi(x) \) where

\[
\gamma_0(x) = \lim_{n \to \infty} \left( \frac{1}{x} + \frac{1}{x + 1} + \ldots + \frac{1}{x + n} - \log(n + x) \right)
\]

is the zeroth Stieltjes constant and

\[
\psi(x) = \log' \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}
\]
A GAMMA FUNCTION IN TWO VARIABLES 3

is the digamma function. For an account of these functions we refer to Coffey [3] and Dilcher [6]. It is easily seen that the function \( \gamma(x) \) represents an analytic function on \( \mathbb{C} \setminus \mathbb{Z} \) and that

\[
\gamma(x + 1) = \frac{-1}{x} + \gamma(x).
\]

In section 2 we study the function \( G(x, z) \) represented as an infinite product. This prepares the ground for section 3 where we introduce the gamma function \( \Gamma(x, z) \) along with some of its basic properties including functional equations and a formula for the modulus \( |\Gamma(n + i, n + i)| \) for \( n \in \mathbb{N}_0 \). Section 4 is devoted to the analogues of the forms (3) and (4) together with their consequences such as values at half-integers and residues at poles. In section 5 we give the analogue of the Gauss’ duplicate formula. Further in section 6 we present the analogue of the Stirling’s formula leading to asymptotic estimates for our function. Finally in section 7 we give series expansions in both variables and as a result we provide recursive formulas for the coefficients of the series in terms of the Riemann-Hurwitz zeta functions.

2. The function \( G(x, z) \)

Definition 2. For \( x \in \mathbb{C} \setminus \mathbb{Z} \) and \( z \in \mathbb{C} \) let the function \( G(x, z) \) be defined as follows

\[
G(x, z) = \prod_{n=0}^{\infty} \left( 1 + \frac{z}{n + x} \right) e^{-\frac{z}{n + x}}.
\]

Note that \( G(x, z) \) is entire in \( z \) for fixed \( x \in \mathbb{C} \setminus \mathbb{Z} \) and that \( \lim_{z \to 0} G(x, z) = G(x, 0) = 1 \).

Proposition 1. We have:

(a) \( G(x, z - 1) = (z + x - 1) e^{\gamma(x)} G(x, z) \).

(b) \( G(x - 1, z) = \frac{z + x - 1}{x - 1} e^{-\frac{z}{x - 1}} G(x, z) \).

Proof. (a) Clearly the zeros of \( G(x, z) \) are \(-x, -(x + 1), -(x + 2), \ldots\) and the zeros of \( G(x, z - 1) \) are \(-(x - 1), -(x), -(x + 1), -(x + 2), \ldots\). Then by the theory of Weierstrass products, we can write

\[
G(x, z - 1) = e^{g(x, z)}(z + x - 1) \prod_{n=0}^{\infty} \left( 1 + \frac{z}{n + x} \right) e^{-\frac{z}{n + x}}
\]

for an entire function \( g(x, z) \). Taking logarithms and differentiating with respect to \( z \) we find

\[
\frac{d}{dz} \log G(x, z - 1) = \frac{d}{dz} g(x, z) + \frac{1}{z + x - 1} + \sum_{n=0}^{\infty} \left( \frac{1}{z + x + n} - \frac{1}{x + n} \right).
\]

On the other hand, from the definition of \( G(x, z) \) we have

\[
\frac{d}{dz} \log G(x, z - 1) = \sum_{n=0}^{\infty} \left( \frac{1}{z + x + n - 1} - \frac{1}{x + n} \right)
\]

\[
= \frac{1}{z + x - 1} - \frac{1}{x} + \sum_{n=0}^{\infty} \left( \frac{1}{z + x + n} - \frac{1}{x + n} \right) + \sum_{n=0}^{\infty} \left( \frac{1}{x + n} - \frac{1}{x + n + 1} \right).
\]
which gives

$$\frac{d}{dz} \log G(x, z - 1) = \frac{1}{z + x - 1} + \sum_{n=0}^{\infty} \left( \frac{1}{z + x + n} - \frac{1}{x + n} \right).$$

Then the relations (6) and (7) imply that \( \frac{d}{dz} g(x, z) = 0 \) and so \( g(x, z) \) is independent of \( z \), say \( g(x, z) = g(x) \). It remains to prove that \( g(x) = \gamma(x) \). From \( G(x, z - 1) = (z + x - 1)e^{\theta(x)}G(x, z) \) and \( G(x, 0) = 1 \) we get

$$e^{-g(x)} = xG(x, 1) = x \prod_{n=0}^{\infty} \left( \frac{x + n + 1}{x + n} \right) e^{-\frac{x}{\pi n}}.$$ 

Furthermore,

$$x \prod_{m=0}^{n-1} \left( \frac{x + m + 1}{x + m} \right) e^{-\frac{x}{\pi m}} = (x + n)e^{-\left( \frac{1}{x} + \frac{1}{x+1} + \ldots + \frac{1}{x+n-1} \right)} + ne^{-\left( \frac{1}{x} + \frac{1}{x+1} + \ldots + \frac{1}{x+n-1} \right)},$$

which yields

$$e^{-g(x)} = \lim_{n \to \infty} x \prod_{m=0}^{n-1} \left( \frac{x + m + 1}{x + m} \right) e^{-\frac{x}{\pi m}} = \lim_{n \to \infty} ne^{-\left( \frac{1}{x} + \frac{1}{x+1} + \ldots + \frac{1}{x+n-1} \right)},$$

or equivalently,

$$g(x) = \lim_{n \to \infty} \left( \frac{1}{x} + \frac{1}{x+1} + \ldots + \frac{1}{x+n-1} - \log n \right) = \gamma(x),$$

as desired.

Part (b) follows directly by the definition of \( G(x, z) \). This completes the proof. \( \square \)

**Proposition 2.** If \( x \in \mathbb{C} \setminus \mathbb{Z} \), then

$$G(x, -z)G(-x, z) = \frac{(z - x) \sin \pi(z - x)}{x \sin \pi x} e^{z \cot(\pi x) + \frac{\pi}{2}}.$$ 

**Proof.** As the zeros of \( \sin(z - x) \) are \( x, \pi + x, -\pi + x, 2\pi + x, -2\pi + x, \ldots \), by the theory of Weierstrass products we have

$$\sin(z - x) = (z - x)e^{g(x,z)} \prod_{n=1}^{\infty} \left( 1 - \frac{z}{n\pi + x} \right) e^{\frac{z}{n\pi + x}} \prod_{n=1}^{\infty} \left( 1 + \frac{z}{n\pi - x} \right) e^{-\frac{z}{n\pi - x}},$$

for an entire function \( g(x, z) \). We find \( e^{g(x,z)} \) as follows. Setting

$$f_n(x, z) = e^{g(x,z)}(z - x) \prod_{k=1}^{n} \left( 1 - \frac{z}{k\pi + x} \right) e^{\frac{z}{k\pi + x}} \left( 1 + \frac{z}{k\pi - x} \right) e^{-\frac{z}{k\pi - x}},$$

we have \( \sin(z - x) = \lim_{n \to \infty} f_n(x, z) \). Taking logarithms and differentiating we obtain

$$\frac{f_n'(x, z)}{f_n(x, z)} = \frac{d}{dz} g(x, z) + \frac{1}{z - x} + \sum_{k=1}^{n} \left( \frac{1}{k\pi + x - z} + \frac{1}{k\pi - x + z} + \frac{1}{k\pi + x} - \frac{1}{k\pi - x} \right)$$

$$= \frac{d}{dz} g(x, z) + \frac{1}{z - x} + \sum_{k=1}^{n} \frac{2(x - z)}{(k\pi)^2 - (x - z)^2} - \sum_{k=1}^{n} \frac{2x}{(k\pi)^2 - x^2}.$$
But as is well-known,
\[
\lim_{n \to \infty} f'_n(x, z) = \cot(z - x) = \frac{1}{z - x} + \sum_{n=1}^{\infty} \frac{2(z - x)}{(z - x)^2 - (n\pi)^2}.
\]
Thus
\[
\frac{d}{dz} g(x, z) = \sum_{n=1}^{\infty} \frac{2x}{(n\pi)^2 - x^2} = \frac{1}{x} - \cot x,
\]
and hence \(g(x, z) = z(\frac{1}{x} - \cot x) + h(x)\). Now equation (8) gives
\[
\frac{\sin(z - x)}{z - x} = e^{h(x)} e^{\frac{z}{x} - z \cot x} \prod_{n=1}^{\infty} \left(1 - \frac{z}{n\pi + x}\right) e^{\frac{z}{n\pi + x}} \left(1 + \frac{z}{n\pi - x}\right) e^{-\frac{z}{n\pi - x}},
\]
which by letting \(z \to 0\) implies
\[
e^{h(x)} = \frac{\sin x}{x}.
\]
Therefore we have
\[
\frac{\sin(z - x)}{z - x} = \frac{\sin x}{x} e^{\frac{z}{x} - z \cot x} \prod_{n=1}^{\infty} \left(1 - \frac{z}{n\pi + x}\right) e^{\frac{z}{n\pi + x}} \left(1 + \frac{z}{n\pi - x}\right) e^{-\frac{z}{n\pi - x}}.
\]
In particular,
\[
\frac{\sin \pi(z - x)}{\pi(z - x)} = \frac{\sin \pi x}{\pi x} e^{\frac{x}{z} - \pi x \cot \pi x} \prod_{n=1}^{\infty} \left(1 - \frac{z}{n\pi + x}\right) e^{\frac{z}{n\pi + x}} \left(1 + \frac{z}{n\pi - x}\right) e^{-\frac{z}{n\pi - x}}
\]
(9)
\[
= \frac{\sin \pi x}{\pi x} e^{\frac{x}{z} - \pi x \cot \pi x} G(x, -z) G(-x, z) \left(1 - \frac{z}{x}\right)^{-2} e^{-\frac{z}{x}}
\]
or equivalently
\[
G(x, -z) G(-x, z) = \frac{(z - x) \sin \pi(z - x)}{x \sin \pi x} e^{z \cot(\pi x) + \frac{x}{z}}
\]
This completes the proof. \(\square\)

**Corollary 1.** If \(x \in \mathbb{C} \setminus \mathbb{Z}\), then
\[
\prod_{n=1}^{\infty} \left(1 - \frac{z^2}{(n + x)^2}\right) \left(1 - \frac{z^2}{(n - x)^2}\right) = \left(\frac{x}{\sin \pi x}\right)^2 \frac{\sin^2 \pi z - \sin^2 \pi x}{z^2 - x^2}.
\]

**Proof.** By the first identity in (8) we have
\[
\frac{\sin \pi(z - x)}{\pi(z - x)} \frac{\sin \pi(z + x)}{\pi(z + x)} = \left(\frac{\sin \pi x}{\pi x}\right)^2 \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{(n + x)^2}\right) \left(1 - \frac{z^2}{(n - x)^2}\right),
\]
which means that
\[
\prod_{n=1}^{\infty} \left(1 - \frac{z^2}{(n + x)^2}\right) \left(1 - \frac{z^2}{(n - x)^2}\right) = \frac{x^2}{z^2 - x^2} \frac{\sin^2 \pi z - \sin^2 \pi x}{\sin^2 \pi x},
\]
which completes the proof. \(\square\)
3. The function $\Gamma(x, z)$

Throughout for any $x \in \mathbb{C}$ let

$$S_x = \mathbb{C} \setminus \{-x + n : n \in \mathbb{N}_0 \cup \{-1\}\}.$$

**Definition 3.** For $x \in \mathbb{C} \setminus \mathbb{Z}^-_0$ and $z \in S_x$ let the function $\Gamma(x, z)$ be defined as follows.

$$\Gamma(x, z) = \left( (z + x - 1)e^{z \gamma(x)}G(x, z) \right)^{-1}.$$

Note that for fixed $x \in \mathbb{C} \setminus \mathbb{Z}^-_0$ the function $\Gamma(x, z)$ is meromorphic with simple poles at $z \in S_x$ and that $\lim_{x \to 1} \Gamma(x, z) = \Gamma(1, z) = \Gamma(z)$.

**Proposition 3.** We have

(a) $\Gamma(x, z + 1) = (z + x - 1) \Gamma(x, z)$, \hspace{1cm} ($x \in \mathbb{C} \setminus \mathbb{Z}^-_0, z + 1 \in S_x$)

(b) $\Gamma(x + 1, z) = \frac{z + x - 1}{x} \Gamma(x, z)$, \hspace{1cm} ($x + 1 \in \mathbb{C} \setminus \mathbb{Z}^-_0, z \in S_x$)

(c) $\Gamma(x + 1, z + 1) = \frac{(z + x - 1)(z + x)}{x} \Gamma(x, z)$, \hspace{1cm} ($x + 1 \in \mathbb{C} \setminus \mathbb{Z}^-_0, z + 1 \in S_{x+1}$).

**Proof.**

(a) We have

$$\Gamma(x, z + 1) = \left( (z + x)e^{(z+1) \gamma(x)}G(x, z + 1) \right)^{-1}$$

$$= \left( (z + x)e^{\gamma(x)}G(x, z + 1)e^{z \gamma(x)} \right)^{-1}$$

$$= \left( G(x, z)e^{z \gamma(x)} \right)^{-1}$$

$$= (z + x - 1) \Gamma(x, z),$$

where the fourth identity follows by Proposition I(a).

(b) We have

$$\Gamma(x + 1, z) = \left( (z + x)e^{z(x+1)}G(x + 1, z) \right)^{-1}$$

$$= \left( (z + x)e^{\frac{z}{x} + \gamma(x)}G(x + 1, z) \right)^{-1}$$

$$= \left( (z + x)e^{-\frac{z}{x} + \gamma(x)}G(x + 1, z)e^{z \gamma(x)} \right)^{-1}$$

$$= \frac{1}{x} \left( e^{z \gamma(x)}G(x, z) \right)^{-1}$$

$$= \frac{z + x - 1}{x} \Gamma(x, z),$$

where the second identity follows from the relation (I) and the fourth identity from Proposition I(b).

(c) This part follows by a combination of part (a) and part (b).
Corollary 2. Let $x \in \mathbb{C}\backslash\mathbb{Z}_0^-$ and let $n \in \mathbb{N}$. Then we have

(a) $\Gamma(x, 1) = 1$,
(b) $\Gamma(x, 0) = \frac{1}{x-1}, \quad (x \neq 1)$
(c) $\Gamma(x, n) = (x)_{n-1}, \quad (n \geq 2)$
(d) $\Gamma(x, -n) = \frac{1}{(x-n-1)_{n+1}}$
(e) $\Gamma(n, z) = \frac{(z)_{n-1}}{(n-1)!} \Gamma(z), \quad (n \geq 2)$.

Proof. (a) As $G(x, 0) = 1$, we have by Proposition 3(a)

$$1 = xe^{\gamma(x)}G(x, 1),$$

and thus by definition

$$\Gamma(x, 1) = \left(xe^{\gamma(x)}G(x, 1)\right)^{-1} = 1.$$
Then by virtue of part (a) we get
\[
\frac{z-x-1}{-x} \Gamma(-x, z)(-z + x - 1) \Gamma(x, -z) = \frac{-\sin \pi x}{(z-x) \sin \pi(z-x)}.
\]
or equivalently
\[
\Gamma(-x, z) \Gamma(x, -z) = \frac{-x \sin \pi x}{((z-x)^3 - (z-x)) \sin \pi(z-x)}.
\]
This completes the proof. \(\square\)

**Corollary 3.** If \(n \in \mathbb{N}_0\) and \(z \notin \mathbb{N}_0\), then
\[
\lim_{x \to -n} \Gamma(x, z) = 0.
\]

**Proof.** By Proposition 4(a) we have
\[
\Gamma(x, z) = \Gamma(x, 1 - (1 - z)) = \frac{1}{(1 - z - x) \sin \pi(1 - z - x) \Gamma(1 - x, 1 - z)}.
\]
Then
\[
\lim_{x \to -n} \Gamma(x, z) = \frac{-\sin \pi n}{(1 - z + n) \sin \pi(1 - z + n) \Gamma(1 + n, 1 - z)} = 0.
\]

**Corollary 4.** If \(n \in \mathbb{N}_0\), then
\[
|\Gamma(n + i, n + i)|^2 = |\Gamma(n - i, n - i)|^2 = \frac{5 \prod_{k=0}^{2n-2} 4 + k^2}{\prod_{k=0}^{n-1} 1 + k^2} \frac{e^\pi}{10(e^{2\pi} + 1)}.
\]

**Proof.** First note that
\[
(10) \quad \Gamma(\bar{x}, \bar{z}) = \Gamma(x, z),
\]
from which the first identity immediately follows. As to the second formula, using identity (10) and Proposition 4(b) we obtain
\[
|\Gamma(i, i)|^2 = |\Gamma(i, i)| = \Gamma(i, i) \Gamma(\bar{i}, \bar{i}) = \frac{-i \sin \pi i}{((2i)^3 - (2i)) \sin 2\pi i} = \frac{e^\pi}{10(e^{2\pi} + 1)},
\]
which gives the result for \(n = 0\). If \(n > 1\) we have by Proposition 4(c)
\[
|\Gamma(n + i, n + i)|^2 = \Gamma(n + i, n + i) \Gamma(n - i, n - i)
\]
\[
= \frac{(2i - 1)(2i + 1)(2i + 2n - 2) (-2i - 1)(-2i)(-2i + 2n - 2)}{i(i + 1) \ldots (i + n - 1)(-i + 1) \ldots (-i + n - 1)} |\Gamma(i, i)|^2
\]
\[
= \frac{(1)^n(2i - 1)(2i + 1)(2i + 2n - 2)(2i - 2n - 2)}{(-1)^n(i)(i + 1)(i + n - 1)(-i + n - 1)(i - (n - 1))} |\Gamma(i, i)|^2
\]
\[
= \frac{(4 + 1)(4 + 1) \ldots (4 + (2n - 2)^2)}{1(1 + 1) \ldots (1 + (n - 1)^2)} \frac{e^\pi}{10(e^{2\pi} + 1)}.
\]
This completes the proof. \(\square\)
4. Analogues of Euler’s formulas, residues, and values at half-integers

**Proposition 5.** We have

(a) \[ \Gamma(x, z) = \lim_{n \to \infty} \frac{n^x(x + 1) \ldots (x + n - 1)}{(z + x - 1)(z + x) \ldots (z + x + n - 1)} = \lim_{n \to \infty} \frac{n^x(x)_n}{(z + x - 1)_{n+1}}. \]

(b) \[ \Gamma(x, z) = \frac{x}{(z + x - 1)(z + x)} \prod_{n=1}^{\infty} \left( 1 + \frac{1}{n} \right)^z \left( 1 + \frac{z}{x + n} \right)^{-1}. \]

**Proof.** (a) We have

\[ \Gamma(x, z) = \lim_{n \to \infty} \left( z + x - 1 \right) e^{x(1 + \frac{1}{1!} + \ldots + \frac{1}{n!} - \log n)} \prod_{k=0}^{n-1} \left( 1 + \frac{z}{x + k} \right) e^{-x} \]

\[ = \lim_{n \to \infty} \left( z + x - 1 \right) e^{-x \log n} \prod_{k=0}^{n-1} \left( \frac{z + x + k}{x + k} \right)^{-1} \]

\[ = \lim_{n \to \infty} \left( n^x(z + x - 1)(z + x) \ldots (z + x + n - 1) \right)^{-1} \]

\[ = \lim_{n \to \infty} \frac{x}{(z + x - 1)(z + x)} \prod_{n=1}^{\infty} \left( 1 + \frac{1}{n} \right)^z \left( 1 + \frac{z}{x + n} \right)^{-1}. \]

(b) By the previous proof we have

\[ \Gamma(x, z) = \frac{1}{z + x - 1} \lim_{n \to \infty} n^x \prod_{k=0}^{n-1} \left( 1 + \frac{z}{x + k} \right) \]

\[ = \frac{1}{z + x - 1} \lim_{n \to \infty} \prod_{k=1}^{n} \left( 1 + \frac{1}{k} \right)^z \prod_{k=0}^{n-1} \left( 1 + \frac{z}{x + k} \right) \]

\[ = \frac{x}{(z + x - 1)(z + x)} \prod_{n=1}^{\infty} \left( 1 + \frac{1}{n} \right)^z \left( 1 + \frac{z}{x + n} \right)^{-1}. \]

**Corollary 5.** If \( x, x + z \in \mathbb{C} \setminus \mathbb{Z}_0^- \), then

\[ \Gamma(x, z) \Gamma(x + z, -z) = \frac{1}{(x - 1)(z + x - 1)}. \]

**Proof.** By Proposition 5(a) we have

\[ \Gamma(x, z) = \lim_{n \to \infty} \frac{n^x(x)_n}{(z + x - 1)_{n+1}} = \frac{1}{x - 1} \lim_{n \to \infty} \frac{n^x(x - 1)_{n+1}}{(z + x - 1)_{n+1}} = \frac{1}{z + x - 1} \lim_{n \to \infty} \frac{n^x(x)_{n+1}}{(z + x)_{n+1}}, \]

where the last identity follows since \( \lim_{n \to \infty} \frac{x^+}{z + x + n} = 1 \). Then

\[ \Gamma(x + z, -z) = \frac{1}{x - 1} \lim_{n \to \infty} \frac{n^{-x}(x + z)_{n+1}}{(x)_{n+1}} = \frac{1}{(x - 1)(z + x - 1) \Gamma(x, z)}. \]

This completes the proof.
Corollary 6. If \( k, l \in \mathbb{N}_0 \) such that \( k + l \neq 0 \), then
\[
\Gamma \left( \frac{2k+1}{2}, \frac{2l+1}{2} \right) = \frac{2}{\sqrt{\pi} (2k-1) (-4)^{k+1} (l+1)! (-l-\frac{1}{2})_{k+l}}
\]

Proof. On the one hand we have by Corollary 5
\[
\Gamma \left( \frac{2k+1}{2}, \frac{2l+1}{2} \right) \Gamma \left( k + l + 1, \frac{2l+1}{2} \right) = \frac{2}{(2k-1)(k+l)}.
\]

On the other hand by Corollary 2(e) and the well-known fact that
\[
\Gamma(1/2 - k) = \frac{\sqrt{\pi}(-4)^k k!}{(2k)!}
\]
we have
\[
\Gamma \left( k + l + 1, -\frac{2l+1}{2} \right) = \frac{(-l-1/2)_{k+l}}{(k+l)!} \Gamma(-l-1/2) = \frac{(-l-1/2)_{k+l}}{(k+l)!} \frac{\sqrt{\pi}(-4)^{l+1} (l+1)!}{(2l+2)!}.
\]

Now combine these identities to deduce the required formula. \(\square\)

Corollary 7. If \( x \in \mathbb{C} \setminus \mathbb{Z} \) and \( m \in \mathbb{N}_0 \cup \{-1\} \), then the residue of \( \Gamma(x, z) \) at \( z = -(x+m) \) is
\[
\frac{1}{(x-1)(z+x-1) \Gamma(z+x,-z)}
\]
if \( m = -1 \)
\[
\frac{1}{(x-1)^m(z+x-1)(m+1)! \Gamma(x+2m+1)}
\]
otherwise.

Proof. Suppose first that \( m = -1 \). By Corollary 5 and Proposition 3(c) we obtain
\[
\Gamma(x, z) = \frac{1}{(x-1)(z+x-1) \Gamma(z+x,-z)}.
\]

Then
\[
\lim_{z \to -(x-1)} (z + (x - 1)) \Gamma(x, z) = \lim_{z \to -(x-1)} \frac{1}{(x-1)(x-1) \Gamma(1,x-1)} = \frac{1}{(x-1) \Gamma(x-1)}.
\]

Suppose now that \( m \neq -1 \). Then repeatedly application of Proposition 3(c) yields
\[
\Gamma(x, z) = \frac{1}{(x-1)(z+x-1) \Gamma(z+x,-z)}
\]
\[
= \frac{1}{(z+x-1)(z+x) \Gamma(z+x+1,-z+1)}
\]
or equivalently,
\[
(z + x + m) \Gamma(x, z) = \frac{1}{(z+x-1)(m+1) \Gamma(z+x+1,-z+1)}
\]
Thus
\[
\lim_{z \to -(x+m)} (z + x + m) \Gamma(x, z) = \frac{1}{(x)^{2m+1} \Gamma(1,x+2m+1)}
\]
which implies the desired result since \( \Gamma(1,x+2m+1) = \Gamma(x+2m+1) \). \(\square\)
Note that if \( x = 1 \) and \( m = -1, 0, 1, 2, \ldots \), then Corollary 7 agrees with the well-known fact that the residue of \( \Gamma(z) \) at \( z = -(m + 1) \) is
\[
\frac{(-1)^{m+1}}{(m+1)!}.
\]

5. An analogues of the Gauss’ multiplication formula

**Proposition 6.** If \( x \in \mathbb{C} \setminus \mathbb{Z}_0 \), then the function

\[
f(x, z) = \frac{n^n \Gamma(x, z) \Gamma(x, z + \frac{1}{n}) \Gamma(x, z + \frac{2}{n}) \cdots \Gamma(x, z + \frac{n-1}{n})}{n \Gamma(n(x - 1) + 1, nz)}
\]

is independent of \( z \).

**Proof.** By Proposition 5 we have

\[
f(x, z) = \frac{n^n \prod_{k=0}^{n-1} \lim_{m \to \infty} \frac{(z + \frac{k}{n} + x + m - 1)(z + \frac{k}{n} + x + m - 2)}{(nz + (x - 1) + kn)(nz + (x - 1) + (k + 1)n)}}
\]

\[
= \lim_{m \to \infty} \frac{n^{mn-1} (x)_{m-1} (n(z + 1))^{n-1}}{(n(x - 1))_{mn-1}}
\]

where the last identity follows as

\[
\frac{(n(z + 1))^{n-1}}{(n(x - 1))^{mn-1}} = 1.
\]

This completes the proof. \(\square\)

**Corollary 8.** We have

\[
\Gamma(x, z) \Gamma(x, z + \frac{1}{2}) \Gamma(1 - x, z) \Gamma(1 - x, z + \frac{1}{2}) = 2^{2-4z} \Gamma(2x - 1, 2z) \Gamma(1 - 2x, 2z) \tan \frac{\pi x}{x - \frac{1}{2}}.
\]

**Proof.** Taking \( z = \frac{1}{n} \) in Proposition 6 we obtain

\[
f(x, z)f(1 - x, z) = f(x, \frac{1}{n})f(1 - x, \frac{1}{n})
\]

\[
= \frac{\Gamma(x, \frac{1}{n}) \Gamma(x, \frac{2}{n}) \cdots \Gamma(x, \frac{n-1}{n}) \Gamma(1 - x, \frac{1}{n}) \Gamma(1 - x, \frac{2}{n}) \cdots \Gamma(1 - x, \frac{n-1}{n})}{\Gamma(n(x - 1) + 1, 1)}
\]

\[
= \frac{\Gamma(x, \frac{1}{n}) \Gamma(1 - x, \frac{n-1}{n}) \Gamma(x, \frac{2}{n}) \Gamma(1 - x, \frac{n-1}{n}) \cdots \Gamma(x, \frac{n-1}{n}) \Gamma(1 - x, \frac{1}{n})}{\prod_{k=1}^{n-1} ((\frac{k}{n} - x) \sin \pi(\frac{k}{n} - x))}
\]

\[= \frac{(-1)^{n-1} \sin \pi x}{\prod_{k=1}^{n-1} ((\frac{k}{n} - x) \sin \pi(\frac{k}{n} - x))},\]
where the last identity follows by Proposition 3(a). Now if \( n = 2 \), then Proposition 6 combined with the previous formula gives

\[
2^{4z-2} \frac{\Gamma(x, z + \frac{1}{2}) \Gamma(1-x, z) \Gamma(1-x, z + \frac{1}{2})}{\Gamma(2x-1, 2z) \Gamma(1-2x, 2z)} = -\sin \pi x (\frac{\pi}{2} - x) \sin(\frac{\pi}{2} - \pi x),
\]

or equivalently,

\[
\Gamma(x, z) \Gamma(x, z + \frac{1}{2}) \Gamma(1-x, z) \Gamma(1-x, z + \frac{1}{2}) = 2^{2-4z} \Gamma(2x-1, 2z) \Gamma(1-2x, 2z) \tan \pi x.
\]

This completes the proof. \( \square \)

6. AN ANALOGE OF THE STIRLING’S FORMULA

In this section we essentially use the same ideas as in Lang [8, p. 422-427] to derive a formula for \( \log \Gamma(x, z) \) leading to asymptotic formulas for \( \Gamma(x, z) \) which are analogues to the Stirling’s formula. For \( t \in \mathbb{R} \), let \( P(t) = t - [t] - \frac{1}{2} \) and for convenience for \( z \in \mathcal{D} \) let

\[
I_n(z) = \int_0^n P(t) z^t dt, \quad \text{and} \quad I(z) = \lim_{n \to \infty} I_n(z) = \int_0^\infty P(t) z^t dt.
\]

Proposition 7. If \( x \in \mathbb{R}^+ \) and \( z \in \mathbb{R}^+ \cap S_x \), then

\[
\log \Gamma(x, z) = (z + x - \frac{3}{2}) \log(z + x - 1) - z + 1 - (x - \frac{1}{2}) \log x + I(x) - I(z + x - 1).
\]

Proof. We have with the help of Euler’s summation formula

\[
\log \frac{(z + x - 1)(z + x) \cdots (z + x + n - 1)}{x(x+1) \cdots (x+n)} = \sum_{k=0}^n \log(z + x - 1 + k) - \sum_{k=0}^n \log(x + k)
\]

\[
= \int_0^n \log(z + x - 1 + t) dt + \frac{1}{2} [(z + x - 1 + n) + \log(z + x - 1)] + I_n(z + x - 1)
\]

\[- \int_0^n \log(x + t) dt - \frac{1}{2} [(x + n) + \log x] - I_n(x)
\]

\[
= [(z + x - 1 + t) \log(z + x - 1 + t) - (z + x - 1 + t)]_0^n - [(x + t) \log(x + n + t) - (x + t)]_0^n
\]

\[+
\frac{1}{2} [(z + x - 1 + n) + \log(z + x - 1)] - \frac{1}{2} [(x + n) + \log(x)] + I_n(z + x - 1) - I_n(x),
\]

which after routine calculations becomes

\[
\log \frac{(z + x - 1)(z + x) \cdots (z + x + n - 1)}{x(x+1) \cdots (x+n)} = \log n^x \log \left(1 + \frac{z + x - 1}{n}\right)
\]

\[+ (x + n - \frac{3}{2}) \log \left(1 + \frac{z + x - 1}{n}\right) - (z + x - \frac{3}{2}) \log(z + x - 1) - (x + n - \frac{1}{2}) \log \left(1 + \frac{x}{n}\right)
\]

\[+ \left(x - \frac{1}{2}\right) \log x - \log n + I_n(z + x - 1) - I_n(x).
\]

Equivalently,

\[
\log \frac{(z + x - 1)(z + x) \cdots (z + x + n - 1)}{n^x(x+1) \cdots (x+n-1)} = \log(x + n) + z \log \left(1 + \frac{z + x - 1}{n}\right)
\]

\[+ (x + n - \frac{3}{2}) \log \left(1 + \frac{z + x - 1}{n}\right) - (z + x - \frac{3}{2}) \log(z + x - 1) - (x + n + \frac{1}{2}) \log \left(1 + \frac{x}{n}\right)
\]

\[+ \left(x - \frac{1}{2}\right) \log x - \log n + I_n(z + x - 1) - I_n(x).
\]
\[ + \left( x - \frac{1}{2} \right) \log x - \log n + I_n(z + x - 1) - I_n(x) \]

\[ = z \log \left( 1 + \frac{z + x - 1}{n} \right) + (x + n - \frac{3}{2}) \log \left( 1 + \frac{z + x - 1}{n} \right) - (z + x - \frac{3}{2}) \log(z + x - 1) \]

\[ - (x + n - \frac{1}{2}) \log \left( 1 + \frac{x}{n} \right) + \left( x - \frac{1}{2} \right) \log x + I_n(z + x - 1) - I_n(x). \]

Now use the fact that

\[ \log \left( 1 + \frac{z}{n} \right) = \frac{z}{n} + O \left( \frac{1}{n^2} \right) \]

and Proposition 5(a) and take \( \lim_{n \to \infty} \) on both sides to get

\[ \log \frac{1}{\Gamma(x, z)} = (z + x - 1 - \frac{3}{2}) \log(z + x - 1 - x + (x - \frac{1}{2}) \log x + I(z + x - 1) - I(x), \]

implying the required identity. \( \square \)

**Corollary 9.** Let \( x \in \mathbb{R}^+ \) and \( z \in \mathbb{R}^+ \cap S_x \). Then

(a) for \( x \to \infty \) we have

\[ \Gamma(x, z) \sim (z + x - 1)^{z + x - \frac{3}{2}} e^{1-z} x^{1/2-x}, \]

(b) for \( z \to \infty \) we have

\[ \Gamma(x, z) \sim (z + x - 1)^{z + x - \frac{3}{2}} e^{1-z} x^{1/2-x} + I(x). \]

**Proof.** Combine Proposition 7 with the fact that \( \lim_{z \to \infty} I(z) = 0. \) \( \square \)

7. **Series expansions and recursive formulas for the coefficients**

To use the property \( \log(z_1 z_2) = \log z_1 + \log z_2 \), we suppose in this section that \( x \in \mathbb{R}^+ \) and \( z \in \mathbb{R}^+ \cap S_x \).

**Proposition 8.** If \( |z| < \inf(1, |x|) \), then

\[ \log \Gamma(x, z + 1) = -z \gamma(x) - \sum_{m=2}^{\infty} \frac{(-1)^{m-1}}{m} \zeta(m, x) z^m. \]

**Proof.** On the one hand, we have by definition

\[ \log \Gamma(x, z) = -\log(z + x - 1) - z \gamma(x) - \sum_{n=0}^{\infty} \left( \log(1 + \frac{z}{x + n}) - \frac{z}{x + n} \right). \]

On the other hand, by Proposition 5(a) we have

\[ \log \Gamma(x, z + 1) = -\log(z + x - 1) + \log \Gamma(x, z). \]
Combining these two relations we obtain

\[
\log \Gamma(x, z + 1) = -z \gamma(x) - \sum_{n=0}^{\infty} \left( \log(1 + \frac{z}{x + n}) - \frac{z}{x + n} \right)
= -z \gamma(x) - \sum_{n=0}^{\infty} \left( \frac{\sum_{m=1}^{\infty} \frac{(-1)^{m-1} z^m}{m (x + n)^m}}{x + n} \right)
= -z \gamma(x) - \sum_{m=2}^{\infty} \frac{(-1)^{m-1}}{m} z^m \sum_{n=0}^{\infty} \frac{1}{(x + n)^m}
= -z \gamma(x) - \sum_{m=2}^{\infty} \frac{(-1)^{m-1}}{m} \zeta(m, x) z^m.
\]

\[\square\]

**Corollary 10.** If \(|z| < \inf(1, |x|)\) and

\[
\Gamma(x, z + 1) = \sum_{m=0}^{\infty} a_m(x) z^m,
\]

then \(a_0(x) = 1\) and for \(m > 0\) we have

\[
a_m(x) = \frac{1}{m} \left( -a_{m-1}(x) \gamma(x) + \sum_{k=0}^{m-2} (-1)^m a_k(x) \zeta(m-k, x) \right).
\]

**Proof.** Clearly if \(z = 0\), then \(\Gamma(x, 1) = a_0(x) = 1\) by Corollary 2(a). Differentiating the power series with respect to \(z\) gives

(11) \[
\frac{d}{dz} \Gamma(x, z + 1) = \sum_{m=1}^{\infty} ma_m(x) z^{m-1}.
\]

Further in Proposition 8 differentiating with respect to \(z\) yields

(12) \[
\frac{d}{dz} \log \Gamma(x, z + 1) = \frac{d}{dz} \frac{\Gamma(x, z + 1)}{\Gamma(x, z + 1)} = -\gamma(x) - \sum_{m=2}^{\infty} (-1)^{m-1} \zeta(m, x) z^{m-1}.
\]

Next combining (11) with (12) gives

\[
\sum_{m=1}^{\infty} ma_m(x) z^{m-1} = \left( \sum_{m=0}^{\infty} a_m(x) z^m \right) \left( -\gamma(x) + \sum_{m=2}^{\infty} (-1)^m \zeta(m, x) z^{m-1} \right).
\]

Now the desired identity follows by equating the coefficients. \(\square\)

**Proposition 9.** If \(|x - 1| < \inf(1, |z + 1|)\), then

\[
\log \Gamma(x + 1, z) = \sum_{n=1}^{\infty} \left( z \log \frac{n + 1}{n} - \log \frac{z + n}{n} \right)
+ \sum_{n=2}^{\infty} \frac{z(x - 1)}{n(n + z)} + \sum_{m=2}^{\infty} \frac{(-1)^m \zeta(m, z) - \zeta(m) - \frac{1}{z^m + 1}}{m} (x - 1)^m.
\]
Proof. Note that it is easily checked that

\begin{align}
- z \gamma(x) + \sum_{n=2}^{\infty} \left( \frac{z}{x+n-1} - \frac{n+z}{n} \right) \\
= - \frac{z}{x} + \log(1+z) + \sum_{n=1}^{\infty} \left( z \log \frac{n+1}{n} - \log \frac{n+z}{n} \right).
\end{align}

Now combining the definition of $\Gamma(x,z)$ with Proposition 3(b) yields

$$\log \Gamma(x+1,z) = -\log x - z \gamma(x)$$

$$- \sum_{n=0}^{\infty} \left( \log(x+n+z) - \log(x+n) - \frac{z}{x+n} \right)$$

$$= -z \gamma(x) - \log(x+z) + \frac{z}{x}$$

$$- \sum_{n=2}^{\infty} \left( \log(x-1+n+z) - \log(x-1+n) - \frac{z}{x-1+n} \right)$$

$$= -z \gamma(x) - \log(x+z) + \frac{z}{x}$$

$$- \sum_{n=2}^{\infty} \left( \log(n+z) + \log(1 + \frac{x-1}{n+z}) - \log n - \log(1 + \frac{x-1}{n}) - \frac{z}{x-1+n} \right)$$

$$= -z \gamma(x) + \sum_{n=2}^{\infty} \left( \frac{z}{x-1+n} - \log \frac{n+z}{n} \right) - \log(x+z) + \frac{z}{x}$$

$$- \sum_{n=2}^{\infty} \left( \log(1 + \frac{x-1}{n+z}) - \log(1 + \frac{x-1}{n}) \right)$$

$$= \log(1+z) - \log(x-1+z+1) + \sum_{n=1}^{\infty} \left( z \log \frac{n+1}{n} - \log \frac{n+z}{n} \right)$$

$$+ \sum_{n=2}^{\infty} \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} \left( \frac{1}{n^m} - \frac{1}{(n+z)^m} \right) (x-1)^m$$

$$= -\log(1 + \frac{x-1}{z+1}) + \sum_{n=1}^{\infty} \left( z \log \frac{n+1}{n} - \log \frac{n+z}{n} \right) + \sum_{n=2}^{\infty} \frac{z(x-1)}{n(n+z)}$$

$$+ \sum_{m=2}^{\infty} \frac{(-1)^{m-1}}{m} \left( -1 + \zeta(m) + \frac{1}{z^m} + \frac{1}{(z+1)^m} - \zeta(m,z) \right) (x-1)^m$$

$$= \sum_{n=1}^{\infty} \left( z \log \frac{n+1}{n} - \log \frac{n+z}{n} \right)$$

$$+ \sum_{n=2}^{\infty} \frac{z(x-1)}{n(n+z)} + \sum_{m=2}^{\infty} \frac{(-1)^{m-1}}{m} \left( -1 + \zeta(m) + \frac{1}{z^m} - \zeta(m,z) \right) (x-1)^m,$$

where the fifth identity follows with the help of (13). This completes the proof. □
Corollary 11. If $|x-1| < \inf(1, |z+1|)$ and 

$$\Gamma(x+1, z) = \sum_{m=0}^{\infty} b_m(z)(x-1)^m,$$

then $b_0(z) = z \Gamma(z)$ and for $m > 0$

$$b_m(z) = \frac{1}{m} b_{m-1}(z) + \sum_{n=2}^{\infty} \frac{z}{n(n+z)} + \frac{1}{m} \sum_{k=0}^{m-2} (-1)^{m-k} b_k(z) (\zeta(m-k, z) - \zeta(m-k) - z^{-(m-k)+1}).$$

Proof. Taking $x = 1$, we have $b_0(z) = \Gamma(2, z) = z \Gamma(z)$ by Corollary 2(e). Further, by Proposition 9 we have

$$\frac{d}{dx} \log \Gamma(x+1, z) = \frac{d}{dx} \frac{\Gamma(x+1, z)}{\Gamma(x+1, z)},$$

$$= \sum_{n=2}^{\infty} \frac{z}{n(n+z)} + \sum_{m=2}^{\infty} (-1)^{m-2} (\zeta(m, z) - \zeta(m) - z^{-m} + 1)(x-1)^{m-1}.$$ 

On the other hand, it follows from the assumption that

$$\frac{d}{dx} \Gamma(x+1, z) = \sum_{m=1}^{\infty} mb_m(z)(x-1)^{m-1}.$$ 

Then from (14) and (15) we get

$$\sum_{m=1}^{\infty} mb_m(z)(x-1)^{m-1} = \left( \sum_{m=0}^{\infty} b_m(x-1)^m \right) \times$$

$$\left( \sum_{n=2}^{\infty} \frac{z}{n(n+z)} + \sum_{m=2}^{\infty} (-1)^{m-2} (\zeta(m, z) - \zeta(m) - z^{-m} + 1)(x-1)^{m-1} \right),$$

and the result follows by equating the coefficients. \qed

References

[1] E.W. Barnes, On the theory of the multiple gamma functions, Trans. Cambridge. Phil. Soc. 19 (1904), 374-425.
[2] E.W. Barnes, The Linear Difference Equation of the First Order, Proc. London Math. Soc. 2 (2) (1905), 438-469.
[3] Mark W. Coffey, Integral and series representations of the digamma and polygamma functions, Analysis 32 (4) (2012), 317-337.
[4] R. Díaz and E. Pariguan, On hypergeometric functions and Pochhammer k-symbol Divulga-ciones Matemáticas 15 (2) (2007), 179-192.
[5] R. Díaz and C. Teruel, q,k-Generalized Gamma and Beta Functions, Journal of Nonlinear Mathematical Physics 12 (1) (2005), 118-134.
[6] K. Dilcher, Generalized Euler constants for arithmetical progressions, Math. Comp. 59 (1992), 259-282.
[7] K. Dilcher, On generalized gamma functions related to the Laurent coefficients of the Riemann zeta function, Aequationes Mathematicae 48 (1994), 55-85
[8] S. Lang, Complex Analysis, Fourth Edition, Springer, 1998.
[9] T. G. Loc and T. D. Tai, The generalized gamma functions, (2011), arXiv:1105.6002
[10] J. E. Marsden and M. J. Hoffman, Basic complex analysis, Third edition, Freeman, 1998.
[11] E.L. Post, The Generalized gamma functions, Ann. Math. 20 (2) (1919), 202-217.
[12] E. T. Whittaker and G. N. Watson, *Course of modern analysis*, Fourth edition, Cambridge University Press, 1996.

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