On the Hypermultiplet Moduli Space of Heterotic Compactifications with Small Instantons

Eugene Perevalov

Department of Mathematics
Harvard University
Cambridge, MA 02138, USA

ABSTRACT

We explore a relation between four-dimensional $N = 2$ heterotic vacua induced by Mirror Symmetry via Heterotic/Type II duality. It allows us to compute the $\alpha'$ corrections to the hypermultiplet moduli space of heterotic compactifications on $K3 \times T^2$ in the limit of large base of the elliptic $K3$. We concentrate on the case of point-like instantons on orbifold singularities leading to low-dimensional hypermultiplet moduli spaces.

1 pereval@math.harvard.edu
1. **Introduction**

When a heterotic string is compactified on a $K3 \times T^2$ manifold the result is an $N = 2$ supergravity coupled to Yang-Mills with matter. The moduli space of the above theory is parametrized by the VEVs of scalar components of vector and hypermultiplets and locally has the following form.

$$\mathcal{M} \cong \mathcal{M}_H \times \mathcal{M}_V,$$

where $\mathcal{M}_H$ is a quaternionic manifold and $\mathcal{M}_V$ is a special Kähler manifold. The geometry of the moduli space encodes the information about the effective low-energy theory and is of primary interest. In general, it receives corrections of two types: due to finite size of the compactification manifold and to nonzero string coupling ($\alpha'$ and $g_s$ corrections, respectively). However, since the string coupling is determined by the VEV of the dilaton which is a scalar component of a vector multiplet in heterotic compactifications, $\mathcal{M}_H$ is unaffected by $g_s$ corrections. $\mathcal{M}_V$ on the other hand is expected to be corrected by the quantum effects. What allows one to compute these corrections and obtain exact expressions for the geometry of $\mathcal{M}_V$ are two kinds of dualities believed to hold in string theory: Mirror Symmetry (see e.g.,[1] and Heterotic/Type II duality[2,3].

Heterotic/Type II duality relates heterotic strings compactified on $K3 \times T^2$ to Type IIA strings on a Calabi–Yau threefold. The key fact which allows us to sum up $g_s$ corrections to $\mathcal{M}_V$ is that in type IIA compactifications the dilaton resides in a hypermultiplet and (in the low energy limit) cannot affect the geometry of vector multiplets. Thus if one can find a type IIA compactification dual to a heterotic model in question the corresponding $\mathcal{M}_V$ will be $g_s$-exact on the type IIA side. At this stage however one still has to compute the $\alpha'$ corrections since for type IIA vector multiplets correspond to Kähler class parameters of the Calabi–Yau threefold and those are corrected by the world-sheet instantons. The predicament is resolved by turning to mirror symmetry conjecture which in its "full" form states that the type IIA string compactified on a Calabi–Yau manifold $M$ is physically equivalent to the type IIB string compactified on the mirror manifold $W$. Thus the Kähler class parameters of $M$ get replaced by the complex structure parameters of $W$ which do not suffer from the corrections due to finite size of the manifold. In this way one can manage to obtain the exact low energy form of the vector multiplet moduli space of a $N = 2$ heterotic model in four dimensions.
The question we address in this paper is given such a heterotic compactification how to compute \( \alpha' \) corrections to the classical moduli space of hypermultiplets. For this purpose we are going to use the concept of a \( c \)-map introduced in[4]. The \( c \)-map relates the vector multiplet moduli space resulting from a compactification of type IIA theory on a Calabi–Yau threefold \( M \) to the hypermultiplet moduli space obtained by the compactification of type IIB theory on the same manifold (or, equivalently, by type IIA on the mirror \( W \)). Thus it converts special Kähler manifolds into quaternionic ones. For our purpose we need a heterotic version of the \( c \)-map. Namely given a heterotic model in four dimensions we would like to know what the "mirror" model is (the original one being equivalent to the type IIA on a Calabi–Yau threefold \( M \) the "mirror" one is defined as the dual to the type IIA string theory compactified on the mirror manifold \( W \)). The answer to this question is provided in[5] (see also[6]) and is roughly speaking that in order to obtain the "mirror" model we have to replace all finite size instantons with structure group \( G \) by point-like instantons[7,8] sitting on an orbifold singularity of the \( K3 \) corresponding to \( G \) and vice versa.\(^1\) Thus we can using the \( c \)-map compute \( \alpha' \) corrections to the metric on the moduli space of hypermultiplets of our original model from that on the moduli space of vector multiplets of the "mirror" model. An important point to note here is that by doing this we do not obtain the exact metric on the moduli space since the \( c \)-map does not capture the quantum effects on the type II side which, under the type II/heterotic duality, become part of \( \alpha' \) corrections in the heterotic model.\(^2\) Recalling that the type IIA dilaton is mapped to the size of the base of the elliptic fibration \( S_H \to B_H \) where \( S_H \) is the heterotic \( K3 \)[9], we conclude that this method allows us to compute the \( \alpha' \) corrections when the base \( B_H \) is large with additional corrections arising away from this limit.

This paper is organized as follows. In §2 we briefly review how the exact expressions on the moduli space of vector multiplets can be obtained with the help of Heterotic/TypeII duality and Mirror Symmetry. In §3 we describe the \( c \)-map. In §4 we use the \( c \)-map and the results of [5] to compute the \( \alpha' \) corrections to the metric (or rather the holomorphic function which encodes it) on the moduli space of hypermultiplets of heterotic compactifications

\(^1\) There as well as in this paper we restrict ourselves to the heterotic compactifications which can be lifted to six dimensions, \( i.e. \), with trivial vector bundles over the \( T^2 \) component of the compactification manifold. More precisely, we consider models where the bundle over \( K3 \times T^2 \) has the form of a direct product of bundles over the corresponding factors of the compactification space, with the factor over the \( T^2 \) being trivial (no Wilson lines).

\(^2\) I am grateful to E. Witten for pointing this out to me.
in the regime of large $B_H$. In particular we concentrate on the simpler case of point-like instantons where the classical moduli space is just that of sigma models on $K3$ orbifolds and we do not have to deal with the bundle moduli. §5 summarizes our results.

2. Exact results on vector multiplet moduli space

In this section we briefly summarize the use of string dualities for the purpose of obtaining exact expressions for the moduli space of vector multiplets in four-dimensional heterotic compactifications with $N = 2$. As we have already mentioned $\mathcal{M}_V$ has the structure of a special Kähler manifold meaning that both the Kähler potential and the Wilsonian gauge couplings of the vector multiplets in the $N = 2$ effective lagrangian are determined by a single holomorphic function of the moduli — the prepotential $F(X)$ which is (in supergravity) homogeneous of second degree in complex fields $X^I$ where $I = 0, 1, \ldots n_V$ ($n_V$ being the number of vector multiplets). The Kähler potential is given in terms of the prepotential by

$$K = -\log \left(iX^I F_I(X) - iX^I \overline{F}_I(X) \right),$$

where $F_I(X) = \frac{\partial F(X)}{\partial X^I}$. In fact the physical moduli parametrize an $n_V$-dimensional manifold. A convenient choice of coordinates is $X^0(z) = 1$, $X^A(z) = z^A$, $A = 1, \ldots n_V$. In these (special) coordinates the Kähler potential becomes

$$K = -\log \left(2(F + \overline{F}) - (z^A - \overline{z}^A)(F_A - \overline{F}_A) \right),$$

where now $F(z)$ is an arbitrary holomorphic function of $z^A$ related to $F(X)$ via $F(X) = i(X^0)^2 F(z)$.

For $N = 2$ heterotic vacua the prepotential receives contributions at the string tree, one loop and the non-pertubative level:

$$F^{(0)}(S, M) + F^{(1)}(M) + F^{(np)}(e^{-8\pi^2 S}, M),$$

where $S$ stands for the dilaton and $M$ for the rest of the vector moduli. The fact that the dilaton arises in the universal sector together with the Peccei-Quinn symmetry and the form of the Kähler potential (2.2) constrain the tree level contribution to the prepotential to be\[10\]

$$F = -S(T U - \phi^i \phi^i),$$

\[3\]
where $T$ and $U$ are the usual torus moduli and $i = 1, \ldots, n_V - 3$. It follows that the tree level Kähler potential has the form

$$K^{(0)} = - \log(S + \overline{S}) - \log(\text{Re}T \text{Re}U - \text{Re}\phi^i \text{Re}\phi^i)$$

(2.5),

and the metric derived from it is the natural metric on the coset space

$$\mathcal{M}^{(0)}_V = \frac{SU(1,1)}{U(1)} \times \frac{SO(2, n_V - 1)}{SO(2) \times SO(n_V - 1)}.$$  

(2.6)

where the first factor corresponds to the dilaton. The one loop contribution to the prepotential can be computed using the perturbative quantum symmetries and the singularity structure corresponding to the gauge symmetry enhancement loci (see e.g., [11]). But we will use the dual type IIA model to obtain the fully corrected form of the prepotential. In order to do that we need to find the type II compactification dual to the heterotic one we are considering. Having done that we can write the corresponding exact form of the prepotential as $(t_A = -iz^A$ are the Kähler class parameters and $q_A = e^{-2\pi t_A}$).

$$\mathcal{F} = -\frac{1}{6} C_{ABC} t_A t_B t_C + \frac{1}{(2\pi)^3} \sum_{d_1, \ldots, d_n} n_{d_1, \ldots, d_n} \text{Li}_3\left(\prod_{i=1}^n q_A^{d_A}\right).$$

(2.7)

Here the $C_{ABC}$ are the classical intersection numbers of the Calabi–Yau threefold and $n_{d_1, \ldots, d_n}$ are the rational instanton numbers of multidegree $d_1, \ldots, d_n$. Note that the cubic part of the prepotential generated at the tree level on the type II side corresponds to the sum of the tree and one loop level contributions on the heterotic side. The rational instanton numbers can be found by replacing, by mirror symmetry, the type IIA compactified on $M$ by type IIB on the mirror Calabi–Yau threefold $W$ where the vector multiplets correspond to the complex structure parameters and the moduli space is not corrected by the world-sheet instantons and is exact at the string tree level. They have been calculated explicitly in a number of low-dimensional examples[12-14].

Consider the heterotic vacuum obtained by taking a vector bundle with structure group $E_8 \times E_8$ with the instanton number 14 in one factor and 10 in the other. Then the primordial gauge symmetry is broken completely and we get a model with 3 vector multiplets (including the dilaton) in four dimensions (the so-called STU model). The number of neutral hypermultiplets can be computed from the formula

$$n_H = h_1 k_1 - \text{dim} G_1 + h_2 k_2 - \text{dim} G_2 + 20$$

(2.8)

with $h_i$, $k_i$, $G_i$ being the dual Coxeter number, the instanton number and the structure group for the two components of the vector bundle, and the universal 20 arising from
the number of the (smooth) K3 moduli. In our case this gives 244. The dual type II vacuum results from the compactification of type IIA string theory on the Calabi–Yau threefold which can be constructed as a hypersurface in the toric variety defined by the data displayed in Table 2.1\(^3\) for \(n = 2\). The resulting Calabi–Yau manifold is an elliptic fibration over the Hirzebruch surface \(\mathbb{F}_2\).

|     | \(s\) | \(t\) | \(u\) | \(v\) | \(x\) | \(y\) | \(w\) | Degrees |
|-----|------|------|------|------|------|------|------|---------|
| \(\lambda\) | 1    | 1    | \(n\) | 0    | \(2n + 4\) | \(3n + 6\) | 0    | \(6n + 12\) |
| \(\mu\)    | 0    | 0    | 1    | 1    | 4    | 6    | 0    | 12      |
| \(\nu\)    | 0    | 0    | 0    | 0    | 2    | 3    | 1    | 6       |

**Table 2.1:** The scaling weights for the toric variety.

The Hodge numbers of the Calabi–Yau threefold are \(h_{11} = 3\), \(h_{21} = 243\), in agreement with the heterotic spectrum. The classical intersection numbers for this Calabi–Yau manifold are easily calculated and the resulting cubic prepotential is

\[
-\mathcal{F}_{\text{cubic}} = \frac{4}{3} t_1^3 + t_1^2 t_2 + 2 t_1^2 t_3 + t_1 t_2 t_3 + t_1 t_3^2.
\] (2.9)

The rational instanton numbers for this model have been calculated in [14] and in[15].

3. **Review of the c-map**

Originally the c-map was defined in [4] as a one which transforms the low-energy effective lagrangian for a type IIA superstring into the corresponding lagrangian for a type IIB superstring compactified on the same (2,2) superconformal system. If we restrict ourselves to cases where the abstract superconformal system can be represented by a Calabi–Yau manifold with Hodge numbers \(h_{11}\) and \(h_{21}\) then the spectrum of a type IIA compactification

---

\(^3\) One starts with homogeneous coordinates \(s, t, u, v, x, y, w\) removes the loci \(\{s = t = 0\}\), \(\{u = v = 0\}\), \(\{x = y = w = 0\}\), and takes the quotient by three scalings \((\lambda, \mu, \nu)\) with the exponents from Table 2.1.
contains \( h_{11} \) vector multiplets and \( h_{21} + 1 \) hypermultiplets (1 counting the dilaton) and the corresponding numbers for the type IIB compactifications are \( h_{21} \) and \( h_{11} + 1 \), respectively. Since in \( N = 2 \) supergravity the scalars of the vector multiplets parametrize a special Kähler manifold (a Kähler manifold with a holomorphic prepotential) and the scalars in the hypermultiplets a quaternionic manifold, the \( c \)-map can be seen as a transformation sending a special Kähler manifold of complex dimension \( h_{11} \) into a quaternionic manifold of real dimension \( 4(h_{11} + 1) \) and a quaternionic manifold of dimension \( 4(h_{21} + 1) \) into a special Kähler manifold of dimension \( h_{21} \), \( i.e., \) the \( c \)-map has the general form

\[
c : K_{h_{11}} \otimes Q_{h_{21} + 1} \rightarrow K_{h_{21}} \otimes Q_{h_{11} + 1},
\]

where \( K_n \) is the set of all special Kähler \( n \)-folds and \( Q_m \) the set of quaternionic spaces of quaternionic dimension \( m \) (real dimension \( 4m \)). But the fundamental operation as was shown in [4] is not \( c \) but \( s_n \)

\[
s_n : K_n \rightarrow Q_{n+1}
\]

mapping a special Kähler \( n \)-fold into a quaternionic space of real dimension \( 4(n + 1) \).

The image of the map \( s_n \) is some proper subset of \( Q_{n+1} \) denoted by \( \tilde{Q}_{n+1} \) and called dual-quaternionic:

\[
\tilde{Q}_{n+1} \equiv \text{Im}(s_n) \subset Q_{n+1}.
\]

Thus \( s_n \) is an isomorphism between \( K_n \) and \( \tilde{Q}_{n+1} \). It was proven in [4] that in type II theories the hypermultiplet moduli spaces belong to \( \tilde{Q}_n \) for appropriate \( n \). The low-energy effective lagrangian then is specified by a point in the following spaces:

\[
K_{h_{11}} \otimes \tilde{Q}_{h_{21} + 1} \quad \text{(type IIA)}
\]

\[
\tilde{Q}_{h_{11} + 1} \otimes K_{h_{21}} \quad \text{(type IIB)}
\]

For any special Kähler space the \( c \)-map can be constructed in the following way. Given a four-dimensional \( N = 2 \) supergravity reduce it dimensionally to three dimensions. The result is an \( N = 4 \) supergravity. After this dimensional reduction, the hypermultiplet moduli in three dimensions parametrize the same quaternionic manifold they did in four dimensions. As for vector moduli, we get one extra scalar from the fourth component of the vector. And moreover, in three dimensions, a vector is dual to a scalar so we get one more. Thus after the duality transformation the bosonic part of of the vector multiplet sector reduces to a theory of \( 4n + 4 \) scalars if we have started with \( n \) vector multiplets in four dimensions. The four additional scalars come from the gravitational sector. These \( 4n + 4 \) scalars parametrize a quaternionic manifold as required by \( N = 4 \) supersymmetry.
Thus the resulting moduli space in three dimensions has the form of a product of two quaternionic manifolds:

\[ \mathcal{M}_3 \cong \mathcal{M}_3^{(1)} \times \mathcal{M}_3^{(2)}, \tag{3.5} \]

where \( \mathcal{M}_3^{(1)} \) is the 4D hypermultiplet manifold, \( \mathcal{M}_3^{(2)} \) arises from the 4D vector multiplets. Then the c-map just corresponds to the interchange of \( \mathcal{M}_3^{(1)} \) and \( \mathcal{M}_3^{(2)} \) or, more precisely, the result of the action of \( s_n \) on the special Kähler space in question is \( \mathcal{M}_3^{(2)} \), the quaternionic space resulting from the dimensional reduction of the vector multiplet sector to 3D.

Before we give the explicit result of this transformation in a general case let us consider the simpler one of symmetric special Kähler spaces. All such spaces allowed in \( N = 2 \) supergravity[16] are listed in the first column of Table 3.1. There are two general families and five exceptional models related to the Jordan algebras. In these cases the special Kähler and quaternionic spaces related by the c-map are simply those associated to the same Jordan algebra. This fact was noticed by physicists in[17]. For example the Kähler space \( \frac{U(1,n)}{U(n) \times U(1)} \) and the quaternionic one \( \frac{E_{8(-24)}}{E_6 \times SO(2)} \) are associated to the Jordan algebra \( J_3^C \), the Jordan algebra of the hermitian \( 3 \times 3 \) complex matrices.

| Kähler space, \( \mathcal{M} \) | \( \dim \mathcal{C} \mathcal{M} \) | Quaternionic space |
|---------------------------------|-----------------|-------------------|
| \( \frac{U(1,n)}{U(n) \times U(1)} \) | \( n \) | \( \frac{U(2,n+1)}{U(2) \times U(n+1)} \) |
| \( \frac{SU(1,1)}{U(1)} \times \frac{SO(n-1,2)}{SO(n-1) \times SO(2)} \) | \( n \geq 2 \) | \( \frac{SO(n+1,4)}{SO(n+1) \times SO(4)} \) |
| \( \frac{SU(1,1)}{U(1)} \) | 1 | \( \frac{G_2(+2)}{SO(4)} \) |
| \( \frac{Sp(6,\mathbb{R})}{U(3)} \) | 6 | \( \frac{F_4(+4)}{USp(6) \times SU(2)} \) |
| \( \frac{U(3,3)}{U(3) \times U(3)} \) | 9 | \( \frac{E_{6(+2)}}{SU(6) \times SU(2)} \) |
| \( \frac{SO^*(12)}{U(6)} \) | 15 | \( \frac{E_{7(-5)}}{SO(12) \times SU(2)} \) |
| \( \frac{E_{7(-26)}}{E_6 \times SO(2)} \) | 27 | \( \frac{E_{8(-24)}}{E_7 \times SU(2)} \) |

**Table 3.1:** The c-map for symmetric Kähler spaces.
A full classification of the quaternionic spaces with a transitive solvable group of motions (normal quaternionic spaces) was given in mathematics literature\cite{18}. More precisely, the normal dual-quaternionic spaces were characterized and their images under the inverse s-map constructed. This approach is more general than that based on Jordan algebras since it provides the c-map for homogeneous and not just symmetric spaces.

The c-map in the general case was worked out in\cite{19}. Let us recall the results.

The bosonic part of of the $N = 2$ lagrangian for vector multiplets is\cite{20,21}
\[
\mathcal{L} = \frac{1}{2} R - K_{A\bar{B}} \partial_{\mu} z^A \partial^\mu \bar{z}^B + \frac{1}{4} \text{Re} \mathcal{N}_{IJ} F_{\mu\nu}^I F^{\mu\nu, J} + \frac{1}{4} \text{Im} \mathcal{N}_{IJ} F_{\mu\nu}^I \tilde{F}^{\mu\nu, J}, \tag{3.6}
\]
with
\[
\mathcal{N}_{IJ} = \frac{1}{2} F_{IJ} - \frac{(Nz)_I (Nz)_J}{(zNz)}, \quad \mathcal{N}_{IJ} = \frac{1}{2} (F_{IJ} + \bar{F}_{IJ})
\]
and $K$ as in (2.2).

The dimensional reduction to three dimensions yields the lagrangian for the scalar manifold of the following form
\[
\mathcal{L} = - K_{A\bar{B}} \partial_{\mu} z^A \partial^\mu \bar{z}^B - \left[ S + \bar{S} + \frac{1}{2} (C + \bar{C}) R^{-1} (C + \bar{C}) \right]^{-2} \times \left| \partial_\mu S + (C + \bar{C}) R^{-1} \partial_\mu C - \frac{1}{4} (C + \bar{C}) R^{-1} \partial_\mu N R^{-1} (C + \bar{C}) \right|^2
\]
\[
+ \left[ S + \bar{S} + \frac{1}{2} (C + \bar{C}) R^{-1} (C + \bar{C}) \right]^{-1} \left( \partial_\mu C - \frac{1}{2} \partial_\mu N R^{-1} (C + \bar{C}) \right)
\]
\[
\times R^{-1} \left( \partial_\mu \bar{C} - \frac{1}{2} \partial_\mu \bar{N} R^{-1} (C + \bar{C}) \right), \tag{3.7}
\]
where $R_{IJ} = \text{Re} \mathcal{N}_{IJ}$.

Eq. (3.7) defines a manifold for the $2(n + 1)$ complex scalar fields $S$, $z^A$ and $C_I$ which is a dual-quaternionic manifold $\tilde{M}_V$ of (real) dimension $4n_V + 4$ which is the image under the $s_n$ map of the special Kähler manifold $M_V$ of complex dimension $n_V$. (3.7) can be rewritten as
\[
\mathcal{L} = - K_{A\bar{B}} \partial_{\mu} z^A \partial^\mu \bar{z}^B - \tilde{K}_{S\bar{S}} D_\mu S D^\mu \bar{S} - \tilde{K}_{S\bar{C}_I} D_\mu S D^\mu \bar{C}_I - \tilde{K}_{C_I S} D_\mu C_I D^\mu \bar{S} - \tilde{K}_{C_I \bar{C}_J} D_\mu C_I D^\mu \bar{C}_J, \tag{3.8}
\]
where
\[
D_\mu C = \partial_\mu C - \frac{1}{2} \partial_\mu N R^{-1} (C + \bar{C}),
\]
\[
D_\mu S = \partial_\mu S + \frac{1}{4} (C + \bar{C}) R^{-1} \partial_\mu N R^{-1} (C + \bar{C}),
\]
\[ \tilde{K} = -\log(S + \overline{S} + \frac{1}{2}(C + \overline{C})R^{-1}(C + \overline{C})). \]

The resulting space enjoys the following properties:

- At each fixed \( z^A \) the Ramond-Ramond scalars \( C_I \) parametrize a \( \frac{SU(1,n_V+2)}{SU(1) \times U(n_V+2)} \) manifold.
- At each point \( (\text{Re} C_I = 0, \text{Im} C_I = \text{const}) \) the \( z^A, S \) fields parametrize a \( \frac{SU(1,1)}{U(1)} \times \mathcal{M}_V \) manifold where \( \mathcal{M}_V \) is the original Kähler manifold of the vector multiplets.
- The quaternionic manifold \( \tilde{M}_V \) has at least \( 2n_V + 4 \) isometries acting on all but \( z^A \) coordinates.

4. **Moduli space of hypermultiplets in heterotic compactifications**

Our aim in this section is twofold: to give some evidence in favor of the conjecture of [5] and to use it in conjunction with the exact results on the moduli space of vector multiplets and the \( c \)-map transformation to compute the \( \alpha' \) corrections to the moduli space of hypermultiplets in the regime of large \( B_H \). For the sake of simplicity, we will discuss only the models where all instantons are point-like although our method should work as well for any heterotic \( K3 \times T^2 \) compactification.\(^4\)

We will illustrate our approach by one particular example. Consider a heterotic obtained by taking a \( K3 \) with two \( E_8 \) singularities with 14 point-like instantons located at one singularity and 10 at the other. The resulting spectrum in four dimensions consists of 243 vector multiplets and 4 hypermultiplets.\(^5\)

4.1. **The classical moduli space**

\(^4\) Provided, as we have already mentioned in §1, that the bundle over \( K3 \times T^2 \) factorizes into a product of bundles over \( K3 \) and \( T^2 \), the latter being trivial.

\(^5\) The corresponding vacuum in six dimensions obtained by decompactifying the \( T^2 \) has 97 tensor multiplets, \( E_8^8 \times F_4^8 \times G_2^{16} \times SU(2)^{16} \) gauge group (for a total of 2672 vector multiplets) with the hypermultiplet content including [22]\( \{\frac{1}{2}(7,2) + \frac{1}{2}(1,2)\} \) for each of the 16 factors of \( G_2 \times SU(2) \) as well as 4 neutral hypermultiplets. Thus for this vacuum, \( H = H_c + H_0 = 128 + 4 = 132 \) and the anomaly cancellation condition is satisfied as \( H - V = 132 - 2672 = -2540 = 273 - 29 \cdot 97 = 273 - 29T \).
The classical hypermultiplet moduli space in this example is simply that of sigma models on a $K3$ orbifold with two $E_8$ singularities which can be determined as follows. As is well known (see e.g., [23]) $H_2(S, \mathbb{Z})$ for a $K3$ surface $S$ is a 22-dimensional self-dual even lattice of signature $(3,19)$ and one can choose a basis of 2-cycles so that the inner product on the basis elements of $H_2(S, \mathbb{Z})$ forms the matrix

$$
\begin{pmatrix}
-E_8 \\
-E_8 \\
& U \\
& U \\
& U
\end{pmatrix}
$$

(4.1)

where $-E_8$ denotes the $8 \times 8$ matrix given by minus the Cartan matrix of the Lie algebra $E_8$ and $U$ is the "hyperbolic plane"

$$
U \cong \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
$$

There is the natural embedding

$$
\Gamma_{3,19} \cong H_2(S, \mathbb{Z}) \subset H_2(S, \mathbb{R}) \cong \mathbb{R}^{3,19}.
$$

To "measure" the complex structure on our $K3$ surface we can use the periods of the holomorphic 2-form $\Omega$

$$
\omega_i = \int_{e_i} \Omega.
$$

Dividing $\Omega \in H^2(S, \mathbb{C})$ as $\Omega = x + iy$ where $x, y \in H^2(S, \mathbb{R})$ and noticing that

$$
\int_S \Omega \wedge \Omega = (x + iy). (x + iy) = (x.x - y.y) + 2ix.y = 0
$$

and

$$
\int_S \Omega \wedge \overline{\Omega} = (x + iy). (x - iy) = (x.x + y.y) = \int_S \|\Omega\|^2 > O,
$$

10
we conclude that the vectors $x$ and $y$ span a space-like 2-plane in $H^2(S, \mathbb{R})$. If in addition we want to specify a Ricci-flat metric on $S$ we need to choose a Kähler form, $J$, which represents another direction in $H^2(S, \mathbb{R})$. This third direction is also space-like since

$$\int_S J \wedge J = \text{Vol}(S) > 0,$$

and it is perpendicular to $\Omega$ because the Kähler form is of type (1,1). Thus $\Omega$ and $J$ together span a space-like 3-plane $\Sigma$. As we rotate $\Sigma$ in $H^2(S, \mathbb{R})$ with respect to the fixed lattice $\Gamma_{3,19}$ we obtain inequivalent Ricci-flat metrics on the $K3$ surface. Thus the moduli space of Ricci-flat metrics has the form of a Grassmanian of oriented space-like 3-planes in $\mathbb{R}^{3,19}$ (modulo the effect of diffeomorphisms acting on the lattice $H^2(S, \mathbb{Z})$) which locally isomorphic to

$$\frac{SO(3,19)}{SO(3) \times SO(19)} \times \mathbb{R}_+,$$

where the $\mathbb{R}_+$ factor describes the overall volume. We still need to incorporate the $B$-field. Since $b_2(S) = 22$ it will add 22 moduli to our 58 for a total of 80. Taking into account the holonomy of the moduli space and the facts about conformal field theory we obtain a moduli space which is locally of the form

$$\frac{SO(4,20)}{SO(4) \times SO(20)}$$

(see [23] for more detail).

Now to determine the form of the moduli space of sigma models on a $K3$ orbifold with two $E_8$ singularities we need to notice that in this case the 3-plane $\Sigma$ has to be perpendicular to all 16 roots of the two $E_8$’s in (4.1). Thus the moduli of the Ricci-flat metrics on the orbifold will be given by a Grassmanian of a space-like 3-plane within the space $\mathbb{R}^{3,3}$, i.e., locally isomorphic to

$$\frac{SO(3,3)}{SO(3) \times SO(3)} \times \mathbb{R}_+.$$  

As to the $B$-field moduli, since we have shrunk 16 2-cycles to zero volume we are left with only 6 of them for a total of $9 + 1 + 6 = 16$ moduli. Thus instead of (4.3) we get

$$\frac{SO(4,4)}{SO(4) \times SO(4)},$$

(4.5)
4.2. The heterotic "mirror" model

To apply the c-map we need to find the heterotic compactification "mirror" to the one we have just described in the sense explained in §1. The results of [5] tell us that the model we are looking for can be constructed by taking a vector bundle with the structure group $E_8 \times E_8$ (corresponding to the singularities of the orbifold) with 14 instantons in one $E_8$ factor and 10 in the other. The resulting spectrum in four dimensions consists of 3 vector and 244 hypermultiplets.\(^6\) Now it’s the vector multiplet moduli space which is of interest to us. Using (2.4) and (2.6) we find that the tree level prepotential reads

$$F^{(0)} = -STU,$$  \hspace{1cm} (4.6)

and the corresponding moduli space

$$M^{(0)}_V = \frac{SU(1,1)}{U(1)} \times \frac{SO(2,2)}{SO(2) \times SO(2)}.$$  \hspace{1cm} (4.7)

Note that (4.7) and (4.5) appear in the same row (second, for $n = 2$) of Table 3.1 which means that the c-map acting on the tree level vector multiplet moduli space of our "mirror" model gives precisely the classical moduli space for the hypermultiplets for the original one thus providing some additional evidence for the conjecture of [5]. To go beyond the classical moduli space for the hypermultiplets we have to use the quantum corrected prepotential for vectors on the "mirror" side. To do that we take the type IIA dual of the heterotic "mirror" model which in this case is furnished by the Calabi–Yau threefold described at the end of §2, namely the elliptic fibration over $\mathbb{F}_2$ with degenerate fibers of type $I_1$ and $II$ only (see [23]). The exact prepotential for this Calabi–Yau manifold reads

$$F = -\left(\frac{4}{3}t_1^3 + t_1^2t_2 + 2t_1^2t_3 + t_1t_2t_3 + t_1t_3^2\right)$$

$$+ \frac{1}{(2\pi)^3} \sum_{d_1,d_2,d_3} n_{d_1,d_2,d_3} \text{Li}_3(e^{-2\pi \sum_{A=1}^3 d_A t_A}),$$  \hspace{1cm} (4.8),

where some of the instanton numbers are given in Table 4.1 [15]. Using the map between type II and heterotic moduli [11]:

$$t_1 = U, \quad t_2 = S - T, \quad t_3 = T - U,$$  \hspace{1cm} (4.9)

\(^6\) In six dimensions, the corresponding model has 1 tensor, 244 hypermultiplets and no vectors so that the anomaly cancellation condition holds as $H - V = 244 - 0 = 244 = 273 - 29 = 273 - 29T$. 

12
Table 4.1: Some of the instanton numbers for rational curves of
multidegree $[d_1, d_2, d_3]$ for the Calabi–Yau manifold elliptically fibered
over $\mathbb{F}_2$ with $(h_{11}, h_{21}) = (3, 243)$.

| $[0, 0, 1]$ | $-2$ | $[0, 1, 1]$ | $-2$ | $[0, 1, 2]$ | $-4$ | $[0, 1, 3]$ | $-6$ |
| $[0, 1, 4]$ | $-8$ | $[0, 1, 5]$ | $-10$ | $[0, 2, 3]$ | $-6$ | $[0, 2, 4]$ | $-32$ |
| $[1, 0, 0]$ | $480$ | $[1, 0, 1]$ | $480$ | $[1, 1, 1]$ | $480$ | $[1, 1, 2]$ | $1440$ |
| $[1, 1, 3]$ | $2400$ | $[1, 1, 4]$ | $3360$ | $[1, 2, 3]$ | $2400$ | $[2, 0, 0]$ | $480$ |
| $[2, 0, 2]$ | $480$ | $[2, 2, 2]$ | $480$ | $[3, 0, 0]$ | $480$ | $[3, 0, 3]$ | $480$ |
| $[4, 0, 4]$ | $480$ | $[5, 0, 0]$ | $480$ | $[6, 0, 0]$ | $480$ | $[0, 1, 0]$ | $0$ |

we can write the exact prepotential for the heterotic "mirror" model:

$$
\mathcal{F} = -STU - \frac{1}{3} U^3 + \frac{1}{(2\pi)^3} \sum_{l,m,k} n_{l+m+k,l,m+l} \text{Li}_3(e^{-2\pi i(lS+mT+kU)}).
$$

(4.10)

4.3. Corrections to the metric on the hypermultiplet moduli space

Now that we have the expression (4.10) we can in principle write down the $\alpha'$ corrections to
the metric on the moduli space of hypermultiplets of our original heterotic compactification
in the regime of large $B_H$. Let us rewrite (4.10) in a form that can be directly substituted
in (3.8). We use the special coordinates associated to the heterotic moduli.

$$
\mathcal{F} = -iz^1z^2z^3
- \frac{1}{3}i(z^3)^3 + \frac{1}{(2\pi)^3} \sum_{m,k} n_{m+k,0,m} \text{Li}_3(e^{-2\pi i(mz^2+kz^3)})
+ \frac{1}{(2\pi)^3} \sum_{l>0,m,k} n_{l+m+k,l,m+l} \text{Li}_3(e^{-2\pi i(lz^3+ mz^2+kz^3)}).
$$

(4.11)

The expression on the first line of (4.11) is the tree level heterotic prepotential for the
"mirror" model and as we know gives rise, via (3.8), to the classical moduli space of sigma
models on the $K3$ orbifold with two $E_8$ singularities. Note that this is an indication
of the fact that under the $c$-map the heterotic dilaton of the "mirror" model maps to
the hypermultiplet governing the overall area $A$ of the $K3$. The piece of $\mathcal{F}$ on the next line of (4.11) corresponds to one-loop correction to the ”mirror” prepotential and, after the $c$-map (or, more precisely, the $s_3$ transformation), will give us the $\alpha'$ corrections to the classical metric of the order of $\frac{1}{A^p}$. And the third line of that equation gives us the contribution to the ”mirror” heterotic prepotential arising non-perturbatively in $g_s$. On the hypermultiplet side, it presumably will give us the $\alpha'$ corrections to the metric of the order $e^{-A}$.

5. Discussion

If we represent an $N = 2$ string vacuum in four dimensions as a compactification of heterotic strings on a $K3 \times T^2$ space the hypermultiplet moduli space is free of corrections due to string coupling. What remains to compute is the corrections due to finite size of the compactification manifold. For that purpose one can use the fact that, combining Heterotic/Type II duality with Mirror Symmetry it is possible to find another heterotic compactification which is in a sense ”mirror” to the original one. Namely, it produces a vacuum which is not equivalent to the one we have started with but related to it in the following way. Suppose the type IIA dual to the original heterotic model is determined by a Calabi–Yau manifold $M$. Then the type IIA dual to the ”mirror” heterotic model is given by the mirror Calabi–Yau manifold $W$. This operation produces different but not unrelated physics. The relation between the two string vacua is that there is a well-defined operation, due to [4], transforming a special Kähler space into a quaternionic one that maps the moduli space of vector multiplets of the ”mirror” model to that of hypermultiplets of the original one. This operation however is valid only at string tree level in type II theories and hence cannot capture the quantum corrections which are present in the geometry of hypermultiplets on type II side. As a result when we apply it to heterotic models, although we start from the exact prepotential for the ”mirror” model the $c$-map fails to produce the exact form of the hypermultiplet moduli space metric, missing the part of the $\alpha'$ corrections which would be mapped to the quantum corrections on the type II side under the heterotic/type II duality. Thus all we can hope for is to obtain the $\alpha'$ corrections to the moduli space of our heterotic compactification in the regime when the base of the elliptic fibration $S_H \to B_H$ is large and we can ignore the quantum corrections on the type II side not captured by the $c$-map. Note that since all $\alpha'$ corrections disappear in the limit of large overall volume of the $K3$, in order for our results to make sense we need to make the area of the elliptic fiber sufficiently small so that the corrections resulting from the last two
lines in (4.11) dominate the corrections related to the type II dilaton and not captured by the $c$-map.

Specifically, heterotic compactifications involving small instantons provide us with models with a few hypermultiplet moduli whose "mirrors" have correspondingly low-dimensional vector multiplet moduli spaces for which exact results have in fact been obtained previously. We have found, in one particular example, that the tree-level vector multiplet moduli space of the "mirror" produces, under the $c$-map, precisely the classical moduli space of the hypermultiplets. Thus the corrections which on the "mirror" side are due to finite string coupling are responsible to the corrections to the classical hypermultiplet moduli space which are attributed to finite size of the $K3$. We find that in a sense the $g_s$ expansion on the "mirror" maps into the $\alpha'$ expansion on the original heterotic model.\footnote{In the limit of large $B_H$}

We have not written down the resulting metric explicitly mainly because the expressions seem to be quite complicated. Let us note however that as it is the case with the vector multiplet moduli spaces the dual-quaternionic manifolds which obtain are fully specified by a single holomorphic function we have written down explicitly. As we have said already the biggest omission is our failure to compute the $\alpha'$ corrections away from the large $B_H$ limit. In order to do that one would have to find the "quantum" $c$-map which in effect amounts to computing the quantum corrections to the moduli space of hypermultiplets on the type II side. Some interesting results in this direction were obtained in\cite{24} and subsequent progress made in\cite{25} and\cite{26}. Another point worth mentioning is that in the present paper we were dealing with the local form of the moduli space only leaving completely out the questions of discrete identifications due to dualities. We leave these issues for future work.

Acknowledgements

It's a pleasure to thank P. Candelas, X. de la Ossa and S. T. Yau for helpful discussions. I am grateful to E. Witten and P. Mayr for pointing out an error in the original version of the paper. This work was supported by the Department of Energy grant.
References

1. S.-T. Yau, editor, Essays on Mirror Manifolds, International Press, 1992
2. S. Kachru and C. Vafa, Nucl. Phys. B450 (1995) (69), hep-th/9505103.
3. S. Ferrara, J. A. Harvey, A. Strominger and C. Vafa, Phys. Lett. 361B (1995) 59, hep-th/9505162.
4. S. Cecotti, S. Ferrara and L. Girardello, Intern. Journ. Mod. Phys. A4 10 (1989) 2475.
5. E. Perevalov and G. Rajesh, Phys. Rev. Lett., 79 (1997) 2931, hep-th/9706005.
6. P. Berglund and P. Mayr, hep-th/9811217.
7. E. Witten, Nucl. Phys. B460 (1996) 541, hep-th/9511030.
8. P. S. Aspinwall and D. R. Morrison, Nucl. Phys. B503 (1997) 533, hep-th/9705104.
9. P. S. Aspinwall, hep-th/9802194.
10. S. Ferrara and A. van Proeyen, Class. Quant. Grav. 6 (1989) 243.
11. D. Lüst, hep-th/9803072 and references therein.
12. P. Candelas, X. de la Ossa, P. Green and L. Parkes, Nucl. Phys. B359 (1991) 21.
13. P. Candelas, X. de la Ossa, A. Font, S. Katz and D. Morrison, Nucl. Phys. B416 (1994) 481.
14. S. Hosono, A. Klemm, S. Theisen and S.-T. Yau, Commun. Math. Phys. 167 (1995) 301.
15. P. Berglund, S.Katz, A. Klemm and P. Mayr, Nucl. Phys. B483 (1997) 209, hep-th/9605154.
16. E. Cremmer and A. van Proeyen, Class. Quant. Grav. 2 (1985) 445.
17. M. Gunyadin, G. Sierra and P. K. Townsend, Phys. Lett. 133B (1983) 72.
18. D. V. Alekseevskii, Math USSR Izvestija 9 (1975) 297.
19. S. Ferrara and S. Sabharwal, Nucl. Phys. B332 (1990) 317.
20. B. de Wit and A. van Proeyen, Nucl. Phys. B245 (1984) 89.
21. E. Cremmer et. al. Nucl. Phys. B250 (1985) 385.
22. P. Candelas, E. Perevalov and G. Rajesh, Nucl. Phys. B519 (1998) 225.
23. P. S. Aspinwall, [hep-th/9611137].
24. K. Becker, M. Becker and A. Strominger, Nucl. Phys. B456 (1995) 130, [hep-th/9507158].
25. H. Ooguri, C. Vafa, Phys. Rev. Lett. 77 (1996) 3296, [hep-th/9608079].
26. A. Strominger, Phys. Lett. 421B (1998) 139, [hep-th/9706195].