Phase shift analysis of PT-symmetric nonhermitian extension of $A_{N-1}$ Calogero model without confining interaction

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Abstract

We discuss a many-particle quantum system, which is obtained by adding some nonhermitian but PT (i.e. combined parity and time reversal) invariant interaction to the $A_{N-1}$ rational Calogero model without confining potential. This model gives rise to scattering states with continuous real spectrum. The scattering phase shift is determined through the exchange statistics parameter. We find that, unlike the case of usual Calogero model, the exclusion and exchange statistics parameter differ from each other in the presence of PT invariant interaction.

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1 Introduction

The subject of exactly solvable many particle quantum mechanical systems with long-range interactions is closely connected to diverse subjects like fractional statistics, random matrix theory, level statistics for disordered systems, Yangian algebra etc. and generated a lot of interest in past years. The $A_{N-1}$ Calogero model (related to $A_{N-1}$ Lie algebra) is the simplest example of such a dynamical model, describing $N$ particles on a line and with Hamiltonian given by \[ H = -\frac{1}{2} \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + \frac{g}{2} \sum_{j \neq k} \frac{1}{(x_j - x_k)^2} + \frac{\omega^2}{2} \sum_{j=1}^{N} x_j^2, \] (1.1)

where $g$ is the coupling of long-range interaction and $\omega$ is the coupling of harmonic confining interaction. This $A_{N-1}$ Calogero model can be solved exactly to obtain the complete set of discrete energy eigenvalues and corresponding bound state eigenfunctions. The complete set of energy eigenvalues can be written as

\[ E_{n_1, n_2, \cdots, n_N} = \frac{N\omega}{2} \left[ 1 + (N - 1)\nu \right] + \omega \sum_{j=1}^{N} n_j, \] (1.2)

where $n_j$s are non-negative integer valued quantum numbers with $n_j \leq n_{j+1}$ and $\nu$ is a real positive parameter related to the coupling constant of long range interaction as

\[ g = \nu^2 - \nu. \] (1.3)

This spectrum in (1.2) is same as that for a $N$ number of free bosonic oscillators apart from a constant shift for all energy levels. This spectrum can be expressed exactly in the form of the energy eigenvalues of free bosonic oscillators: $E_{\tilde{n}_1, \tilde{n}_2, \cdots, \tilde{n}_N} = \frac{N\omega}{2} + \omega \sum_{j=1}^{N} \tilde{n}_j$, where $\tilde{n}_j \equiv n_j + \nu(j-1)$ are quasi-excitation numbers. These quasi-excitation numbers are no longer integers and they satisfy a modified selection rule given by $\tilde{n}_{j+1} - \tilde{n}_j \geq \nu$. This selection rule restricts the difference between the quasi-excitation numbers to be at least $\nu$ apart. As a consequence, the Calogero model (1.1) provides a microscopic realization for generalized exclusion statistics (GES) [2] with $\nu$ representing the corresponding GES parameter [3, 4, 5].

The Calogero model without confining potential, which is obtained by setting $\omega = 0$ in eqn.(1.1), is also studied in Ref.[1]. Unlike the earlier case, the spectrum of this model is continuous and only scattering states occur. Due to such scattering, particle momenta in an outgoing $N$-particle plane wave get rearranged (reversely ordered) in terms of the momenta in the incoming plane wave. The corresponding momentum independent scattering phase shift is given by $\theta_{sc} = \pi \nu \frac{N(N-1)}{2}$, which is simply $\nu \pi$ times the total number of two-body exchanges that is needed for rearranging $N$ particles in the reverse order. Thus it is natural to identify $\nu$ as the exchange statistics parameter in this case [4]. It may be noted that this exchange statistics parameter coincides with the exclusion statistics parameter as obtained earlier in the presence of confining potential.
In the last few years, theoretical investigations on different nonhermitian Hamiltonians have received a major boost because many such systems, whenever they are invariant under combined parity and time reversal (PT) transformation, lead to either real (when PT symmetry is unbroken) or pairs of complex conjugate energy eigenvalues (when PT symmetry is spontaneously broken) \[6, 7, 8, 9, 10\]. Such property of energy eigenvalues in nonhermitian PT invariant systems can be related to the pseudo-hermiticity \[9\] or anti-unitary symmetry \[10\] of the corresponding Hamiltonians. As concrete examples of PT symmetric quantum mechanics, the Hamiltonians of only one particle in one space dimension have mostly been considered in the literature so far. However, nonhermitian but PT invariant extension of some exactly solvable many particle quantum mechanical system in one space dimension have also been considered recently \[11, 12, 13, 14\]. The PT transformation for such \(N\)-particle system can be written as

\[
i \to -i, \quad x_j \to -x_j, \quad p_j \to p_j
\] (1.4)

where \(j \in [1, 2 \cdots N]\) and \(x_j (p_j \equiv -i \frac{\partial}{\partial x_j})\) denotes coordinate (momentum) operator of the \(j\)-th particle. In particular, an extension of \(A_{N-1}\) Calogero model with confining term is proposed by adding a momentum dependent long-range interaction \((\delta \sum_{j\neq k} \frac{1}{(x_j-x_k)} \frac{\partial}{\partial x_j})\) to the Hamiltonian (1.1):

\[
H_{ext} = -\frac{1}{2} \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + g \sum_{j \neq k} \frac{1}{(x_j - x_k)^2} + \delta \sum_{j \neq k} \frac{1}{(x_j - x_k)} \frac{\partial}{\partial x_j} + \frac{\omega^2}{2} \sum_{j=1}^{N} x_j^2,
\] (1.5)

where \(\delta\) is a real parameter \[5, 12\]. It is shown that this nonhermitian, PT invariant model can be solved exactly and within certain range of the related parameters it yields a real spectrum

\[
E_{n_1n_2 \cdots n_N} = \frac{N \omega}{2} [1 + (N - 1) \tilde{\nu} + \omega \sum_{j=1}^{N} n_j].
\] (1.6)

Here \(\tilde{\nu} = \nu' - \delta\) and \(\nu'\) is a real positive parameter which is related to the coupling constants \(g\) and \(\delta\) as

\[
g = \nu'^2 - \nu' (1 + 2\delta).
\] (1.7)

In analogy with the case of original Calogero model, the energy eigenvalues (1.6) can be rewritten exactly in the form of energy spectrum for \(N\) free oscillators: \(E_{n_1n_2 \cdots n_N} = \frac{N \omega}{2} + \omega \sum_{j=1}^{N} \tilde{n}_j\), where \(\tilde{n}_j \equiv n_j + \tilde{\nu}(j-1)\) are quasi excitation numbers satisfying a modified selection rule given by \(\tilde{n}_{j+1} - \tilde{n}_j \geq \tilde{\nu}\). Consequently, the extended Calogero model (1.5) also provides a microscopic realization for GES, where \(\tilde{\nu}\) represents the exclusion statistics parameter.

In this context, one can naturally ask whether the exclusion and exchange statistics parameters are same or not in the case of above described nonhermitian, PT invariant extension of the Calogero model. This motivates us to find out the exchange statistics parameter associated with such extension of Calogero model in the absence of confining interaction, whose Hamiltonian is obtained by putting \(\omega = 0\) in eqn.(1.5). In Sec.2 of this
article, we briefly review the construction of scattering eigenstates and phase shift for the $A_{N-1}$ Calogero model in absence of confining potential. By following the same procedure, in Sec.3 we find out the scattering eigenstates for the extended Calogero model without confining potential and also calculate the related phase shift. Using this scattering phase shift, we determine the exchange statistics parameter for the extended Calogero model. Section 4 is kept for conclusions.

2 Scattering states of Calogero model without confining interaction

In this section we briefly discuss the scattering state solutions and corresponding phase shift of $A_{N-1}$ Calogero model in the absence of confining interaction. Such a Calogero model is described by the Hamiltonian given in (1.1) with $\omega = 0$ as

$$H_0 = -\frac{1}{2} \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + \frac{g}{2} \sum_{j \neq k} \frac{1}{(x_j - x_k)^2}. \tag{2.1}$$

Following [1], the eigenvalue problem for the above Hamiltonian can be solved to obtain scattering states within a sector of configuration space corresponding to a definite ordering of particles like $x_1 \geq x_2 \geq \cdots \geq x_N$. The zero energy ground state wavefunction of this model is given by

$$\psi_{gr} = \prod_{j<k} (x_j - x_k)^\nu, \tag{2.2}$$

where $\nu$ is a real positive parameter satisfying the relation (1.3).

Next, we consider the general eigenvalue equation associated with the Hamiltonian (2.1):

$$H_0 \psi = p^2 \psi, \tag{2.3}$$

where $p$ is real and positive. Solutions of this eigenvalue equation can be written in the form

$$\psi = \psi_{gr} \tau(x_1, x_2, \cdots x_N), \tag{2.4}$$

where $\tau(x_1, x_2, \cdots x_N)$ satisfies the following differential equation,

$$-\frac{1}{2} \sum_{j=1}^{N} \frac{\partial^2 \tau}{\partial x_j^2} - \nu \sum_{j \neq k} \frac{1}{(x_j - x_k)} \frac{\partial \tau}{\partial x_j} = p^2 \tau. \tag{2.5}$$

Further, to separate the ‘radial’ and ‘angular’ part of the eigenfunction, one assumes that $\tau(x_1, x_2, \cdots x_N) = P_{k,q}(x) \chi(r)$, where the radial variable $r$ is defined as

$$r^2 = \frac{1}{N} \sum_{i \neq j} (x_i - x_j)^2 \tag{2.6}$$
and $P_{k,q}(x)$s are translationally invariant, symmetric, k-th order homogeneous polynomials satisfying the differential equations

$$\sum_{j=1}^{N} \frac{\partial^2 P_{k,q}(x)}{\partial x_j^2} + \nu \sum_{j \neq k} \frac{1}{(x_j - x_k)} \left( \frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_k} \right) P_{k,q}(x) = 0. \quad (2.7)$$

Note that the index $q$ in $P_{k,q}(x)$ can take any integral value ranging from 1 to $g(N,k)$, where $g(N,k)$ is the number of independent polynomials which satisfy eqn.(2.7) for a given $N$ and $k$ [1]. One should further note at this point that the translational invariance and homogeneity property of the polynomial $P_{k,q}(x)$ enforce it to satisfy the following relations,

$$\sum_{j=1}^{N} \frac{\partial P_{k,q}(x)}{\partial x_j} = 0, \quad \sum_{j=1}^{N} x_j \frac{\partial P_{k,q}(x)}{\partial x_j} = k P_{k,q}(x). \quad (2.8)$$

By substituting $P_{k,q}(x) \chi(r)$ in the place of $\tau(x_1, x_2, \cdots x_N)$ in eqn.(2.5), and using the relations (2.7), (2.8), one obtains

$$-\frac{d^2 \chi(r)}{dr^2} - \frac{(1+2b)}{r} \frac{d\chi(r)}{dr} = p^2 \chi(r), \quad (2.9)$$

where $b = \frac{N-3}{2} + k + \frac{N(N-1)\nu}{2}$. The above equation admits a solution of the form

$$\chi(r) = r^{-b} J_b(pr),$$

where $J_b(pr)$ denotes the Bessel function. Hence the scattering state eigenfunctions of $H_0$ with eigenvalue $p^2$ are finally obtained as

$$\psi = \prod_{j<k} (x_j - x_k)^\nu r^{-b} J_b(pr) P_{k,q}(x). \quad (2.10)$$

Next we want to discuss the scattering phase shift for the above model. For this purpose, one has to construct a more general eigenfunction which in the asymptotic limit (i.e., $r \to \infty$ limit) can be expressed in terms of an incoming free particle wavefunction ($\psi_+$) and an outgoing free particle wavefunction ($\psi_-$), where the incoming wavefunction will be of the form

$$\psi_+ = \exp \left[ i \sum_{j=1}^{N} p_j x_j \right], \quad (2.11)$$

with $p_j \leq p_{j+1}$, $p^2 = \sum_{j=1}^{N} p_j^2$ and $\sum_{j=1}^{N} p_j = 0$. This can be achieved by taking appropriate linear superposition of all degenerate eigenfunctions (with eigenvalue $p^2$) of the form (2.10):

$$\psi_{gen} = \prod_{j<k} (x_j - x_k)^\nu \sum_{k=0}^{g(N,k)} \sum_{q=1}^{\nu} C_{kq} r^{-b} J_b(pr) P_{k,q}(x), \quad (2.12)$$

where $C_{kq}$s are expansion coefficients depending on particle momenta. We assume that each of the momenta $p_i$ is a product of a radial part $p$ and an angular part $\alpha_i$, i.e., $p_i = p \alpha_i$. Now by matching the dimensions of the right hand sides of equations (2.11) and (2.12), it
can be shown that \( C_{kq} = p^{\frac{n-N}{2}} \tilde{C}_{kq}(\alpha_i) \), where \( \tilde{C}_{kq}(\alpha_i) \) depends only on the angular parts of the momenta. By using this relation for \( C_{kq} \) and also the asymptotic properties of Bessel function at \( r \to \infty \) limit,

\[
J_b(pr) \to \frac{1}{\sqrt{2\pi pr}} \left\{ e^{-i(n+\frac{1}{2})\frac{\pi}{2}+i pr} + e^{i(n+\frac{1}{2})\frac{\pi}{2}-i pr} \right\},
\]

one can write down the asymptotic form from of \( \psi_{gen} \) (2.12) as

\[
\psi_{gen} \sim \psi_+ + \psi_- ,
\]

(2.13)

where

\[
\psi_{\pm} = (2\pi r)^{-\frac{1}{2}} p^{(n-\frac{1}{2})} \prod_{j<k} (x_j - x_k) \nu^{-A} \sum_{k=0}^{\infty} \sum_{q=1}^{\tilde{C}_{kq}(\alpha_i)} r^{-k} P_{k,q}(x)e^{\pm i(b+\frac{1}{2})\frac{\pi}{2} \pm ipr} ,
\]

(2.14)

with \( A = b - k = \frac{N-3}{2} + \frac{N(N-1)\nu}{2} \) and \( n = \frac{3-N}{2} \). Now, to get the expression for the phase shift due to scattering, the outgoing wavefunction \( \psi_- \) has to be expressed in terms of the incoming wavefunction \( \psi_+ \). For this purpose let us consider a special permutation of the particle coordinates \( T \), defined as

\[
Tx_i = x_{N-i+1} , \quad i = 1, 2, \ldots, N
\]

such that the set \( \{-Tx\} \) belongs to the same sector as that of the set \( \{x\} \), with the particle ordering \( x_1 \geq x_2 \geq \cdots \geq x_N \). It is also to be noted here that the symmetry and the homogeneity of \( P_{k,q}(x) \) enables us to write the relation,

\[
P_{k,q}(-Tx) = e^{-ik\pi} P_{k,q}(x).
\]

Hence, by taking advantage of the above facts and finally using eqn. (2.11), it can be shown that the outgoing wavefunction \( \psi_- \) can be written as

\[
\psi_- = e^{-i\pi(A+n)} \psi_+ \left( x \to -Tx, \ p \to -p \right)
\] \[= e^{-i\pi \nu \frac{N(N-1)}{2}} \exp \left[i \sum_{j=1}^{N} x_j p_{N+1-j} \right].
\]

Comparing (2.11) with (2.15), one finds that the momenta of the incoming plane wave gets rearranged (reversely ordered) in the scattering process. Moreover, the outgoing plane wave acquires a momentum independent phase shift given by \( \pi \nu \frac{N(N-1)}{2} \). Thus \( \nu \) can be identified with the exchange statistics parameter associated with this Calogero model.

3 Scattering phase shift of extended Calogero Model without confining interaction

Here we aim to study the \( PT \) symmetric nonhermitian extension of the Calogero model without confining interaction. We extend the Hamiltonian (2.1) by adding an extra term
\[ \frac{1}{2} \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + \frac{g}{2} \sum_{j \neq k} \frac{1}{(x_j - x_k)^2} + \delta \sum_{j \neq k} \frac{1}{x_j - x_k} \frac{\partial}{\partial x_j} \]

(2.16)

Since \( H_{ext} \) (1.5) reduces to the above Hamiltonian at \( \omega = 0 \) limit, the exchange statistics parameter of extended Calogero model can be determined from the phase shift analysis of the scattering states associated with the Hamiltonian (2.16). Now we follow the same procedure as in the previous section to study the scattering states of this extended model.

We start by observing that the zero energy ground state wave function of the Hamiltonian (2.16) is quite similar in form with the ground state wave function (2.2) of the original Calogero model:

\[ \psi_{gr} = \prod_{j<k}(x_j - x_k)^{\nu'}, \]

(2.17)

where the modified exponent \( \nu' \) is related to the coupling constants \( g \) and \( \delta \) through the relation (1.7). For the purpose of obtaining nonsingular ground state eigenfunction at the limit \( x_i \rightarrow x_j \), \( \nu' \) should be a non-negative exponent. Due to eqn.(1.7), this condition restricts the ranges of coupling constants \( g \) and \( \delta \) as (i) \( \delta \geq -\frac{1}{2}, 0 > g \geq -(\delta + \frac{1}{2})^2 \), and (ii) \( g \geq 0 \) with arbitrary value of \( \delta \). Next, we consider the general eigenvalue equation associated the Hamiltonian (2.16) given by

\[ H_{ext} \psi = p^2 \psi, \]

(2.18)

where \( p \) is a real positive parameter. It is easy to see that the solutions of this eigenvalue equation can be written in the form \( \psi = \psi_{gr} \tau'(x_1, x_2 \cdots x_N) \), where \( \psi_{gr} \) represents the modified ground state eigenfunction (2.17) and \( \tau'(x_1, x_2 \cdots x_N) \) satisfies a differential equation like

\[ -\frac{1}{2} \sum_{j=1}^{N} \frac{\partial^2 \tau'}{\partial x_j^2} - (\nu' - \delta) \sum_{j \neq k} \frac{1}{x_j - x_k} \frac{\partial \tau'}{\partial x_j} = p^2 \tau'. \]

(2.19)

Next we assume that \( \tau'(x_1, x_2 \cdots x_N) \) can be factorised as

\[ \tau'(x_1, x_2 \cdots x_N) = P'_{k,q}(x)\chi'(r), \]

(2.20)

where \( r \) is the radial variable defined in (2.6) and \( P'_{k,q}(x) \)s are translationally invariant, symmetric, k-th order homogeneous polynomials satisfying the differential equations

\[ \sum_{j=1}^{N} \frac{\partial^2 P'_{k,q}(x)}{\partial x_j^2} + (\nu' - \delta) \sum_{j \neq k} \frac{1}{x_j - x_k} \left( \frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_k} \right) P'_{k,q}(x) = 0. \]

(2.21)

Note that the form of eqn.(2.21) is same as eqn.(2.7) apart from the fact that here \( \nu \) is replaced by \( \nu' - \delta \). Hence it is clear that \( P'_{k,q}(x) \) can be obtained from any given expression
of $P_{k,q}(x)$ by simply substituting the parameter $\nu$ with $\nu' - \delta$. So the index $q$ in $P'_{k,q}(x)$ can also take values ranging from 1 to $g(N,k)$. Substituting the factorised form (2.20) of $\tau(x_1, x_2 \cdots x_N)$ in the differential eqn.(2.19) and making use of the properties of $P'_{k,q}(x)$ we obtain the equation satisfied by the 'radial' part of the wavefunction as

$$\frac{\partial^2 \chi'(r)}{\partial r^2} - \frac{1 + 2\nu'}{r} \frac{\partial \chi'(r)}{\partial r} = p^2 \chi'(r) \quad (2.22)$$

with $\nu' = \frac{N-3}{2} + k + (\nu' - \delta)\frac{N(N-1)}{2}$. The solution of eqn.(2.22) can be expressed through the Bessel function: $\chi'(r) = r^{-\nu'} J_{\nu'}(pr)$. Hence the scattering state eigenfunctions of $H_{ext}$ (2.16) with real positive eigenvalue $p^2$ are obtained as

$$\psi = \prod_{j<k} (x_j - x_k)^{\nu'} r^{-\nu'} J_{\nu'}(pr) P'_{k,q}(x). \quad (2.23)$$

Next, we carry out the calculation of phase shift in the same manner as in the previous section. First, we aim to construct a more general eigenfunction of $H_{ext}$ such that in the asymptotic limit it can be expressed in terms of an incoming wave ($\psi_+$) of the form (2.11) and an outgoing wave ($\psi_-$). For this purpose one has to take appropriate linear superposition of all degenerate eigenfunctions (with eigenvalue $p^2$) of the form (2.23):

$$\psi_{gen} = \prod_{j<k} (x_j - x_k)^{\nu'} \sum_{k=0}^{\infty} \sum_{q=1}^{g(N,k)} C'_{kq} r^{-\nu'} J_{\nu'}(pr) P'_{k,q}(x), \quad (2.24)$$

where $C'_{kq}$'s are expansion coefficients which are functions of particle momenta. Once again by doing dimensional analysis, we obtain

$$C'_{kq} = p^{\frac{(3-N)}{2} + \frac{N(N-1)\delta}{2}} \tilde{C}'_{kq}(\alpha_i),$$

where $\tilde{C}'_{kq}(\alpha_i)$ depends only on the angular parts of the momenta. By using this explicit expression for $C'_{kq}$ and the asymptotic properties of Bessel function at $r \to \infty$, we obtain the asymptotic form of $\psi_{gen}$ (2.24) as $\psi_{gen} \sim \psi_+ + \psi_-$, where

$$\psi_\pm = (2\pi r)^{-\frac{3}{2}} p^{n'-\frac{1}{2}} \prod_{j<k} (x_j - x_k)^{\nu'} r^{-\nu'} \sum_{k=0}^{\infty} \sum_{q=1}^{g(N,k)} \tilde{C}'_{kq}(\alpha_i) r^{-k} P'_{k,q}(x) e^{\pm i(\nu'+\frac{1}{2})\frac{\pi}{2} + ipr}.$$  

In the above expression $A' = b' - k = \frac{N-3}{2} + (\nu' - \delta)\frac{N(N-1)}{2}$ and $n' = \frac{3-N}{2} + \frac{N(N-1)\delta}{2}$. Following the same procedure as in the case of $A_{N-1}$ Calogero model without confining term, we find that the outgoing wavefunction ($\psi_-$) can be written as

$$\psi_- = e^{-i\pi(A'+n')} \psi_+ \left( x \rightarrow -T x, \ p \rightarrow -p \right) = e^{-i\pi\nu'\frac{N(N-1)}{2}} \exp \left[ i \sum_{j=1}^{N} x_j p_{N+1-j} \right]. \quad (2.25)$$

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Comparing eqns.(2.25) and (2.11), we observe that the momenta of the incoming plane wave gets rearranged (reversely ordered) in the scattering process. Furthermore, the outgoing plane wave now acquires a momentum independent phase shift given by \( \pi \nu' \frac{N(N-1)}{2} \), which is \( \pi \nu' \) times the number of two body exchanges needed for rearranging \( N \)-particles in the reverse order. Thus \( \nu' \) can be identified with the exchange statistics parameter associated with this extended Calogero model.

4 Conclusion

Here we study a nonhermitian but \( PT \) invariant extension of the \( A_{N-1} \) Calogero model without confining interaction. Unlike the case of \( A_{N-1} \) Calogero model with confining term, here we find only scattering states with continuous, real eigenvalues. We also calculate the scattering phase shift explicitly for this model. The exchange statistics parameter for this model is determined through the scattering phase shift of plane waves. Such exchange statistics parameter is given by \( \nu' \), which is related to the coupling constants through eqn.(1.7). However, it is found earlier that the exclusion statistics parameter associated with \( PT \) invariant extension of Calogero model in the presence of confining interaction is given by \( \tilde{\nu} \equiv \nu' - \delta \). Thus we surprisingly find that, in contrary to the case of original Calogero model, the exclusion and exchange statistics parameters differ from each other in the presence of \( PT \) invariant interaction. In particular, within a range of coupling constant given by \( \delta > 0, \ 0 > g > -\delta(1 + \delta) \), the value of exclusion statistics parameter \( \tilde{\nu} \) becomes negative. As a result, the ground state energy does not have any lower bound in the thermodynamic limit (i.e., \( N \to \infty \) limit). On the other hand, the exchange statistics parameter \( \nu' \) is positive and well defined even within the above mentioned range of the coupling constants.

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