Tesi di Laurea Magistrale

Approximate Hermitian-Yang-Mills structures on semistable Higgs bundles

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Strutture Hermitian-Yang-Mills approssimate su fibrati di Higgs semistabili

Sunto

Si introducono le nozioni di struttura (debole) Hermitian-Yang-Mills e di struttura Hermitian-Yang-Mills approssimata su fibrati di Higgs, quindi si costruisce il funzionale di Donaldson per fibrati di Higgs su varietà di Kähler compatte e si presentano alcune proprietà di base di questo funzionale. In particolare, si prova che il suo gradiente può essere scritto in termini della curvatura media della connessione di Hitchin-Simpson e si studiano alcune proprietà dell’equazione di evoluzione associata al funzionale di Donaldson. Successivamente si affronta il problema dell’esistenza di strutture Hermitian-Yang-Mills approssimate su fibrati di Higgs e si studia la relazione tra l’esistenza di tali strutture e la nozione algebro-geometrica di semistabilità di Mumford-Takemoto. In particolare, si prova che per un fibrato di Higgs su una superficie di Riemann compatta le nozioni di esistenza di strutture Hermitian-Yang-Mills approssimate e di semistabilità sono equivalenti.

Infine, usando il flusso del calore associato al funzionale di Donaldson, si prova che la semistabilità di un fibrato di Higgs su una varietà di Kähler compatta e l’esistenza di strutture Hermitian-Yang-Mills approssimate sono equivalenti in qualunque dimensione. Come conseguenza di questo fatto, si deduce che diversi risultati riguardanti l’esistenza di strutture Hermitian-Yang-Mills approssimate su fibrati di Higgs possono essere espressi in termini di semistabilità.

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Abstract

We review the notions of (weak) Hermitian-Yang-Mills structure and approximate Hermitian-Yang-Mills structure for Higgs bundles. Then, we construct the Donaldson functional for Higgs bundles over compact Kähler manifolds and we present some basic properties of it. In particular, we show that its gradient flow can be written in terms of the mean curvature of the Hitchin-Simpson connection. We also study some properties of the evolution equation associated to that functional. Next, we study the problem of the existence of approximate Hermitian-Yang-Mills structure and its relation with the algebro-geometric notion of Mumford-Takemoto semistability and we show that for a Higgs bundle over a compact Riemann surface, the notion of approximate Hermitian-Yang-Mills structure is in fact the differential-geometric counterpart of the notion of semistability.
Finally, using Donaldson heat flow, we show that the semistability of a Higgs bundle over a compact Kähler manifold (of every dimension) implies the existence of an approximate Hermitian-Yang-Mills structure. As a consequence of this we deduce that many results about Higgs bundles written in terms of approximate Hermitian-Yang-Mills structures can be translated in terms of semistability.
"Adesso è più normale
adesso è meglio,
adesso è giusto, giusto, è giusto
che io vada"

E quando poi sparí del tutto
a chi diceva "è stato un male"
a chi diceva "è stato un bene"
raccomandò "non vi conviene
venire con me dovunque vada,
ma c'è amore un po' per tutti
e tutti quanti hanno un amore
sulla cattiva strada
sulla cattiva strada"

Fabrizio de André
La cattiva strada
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Introduction

Some historical background

The notion of holomorphic vector bundle is common to some branches of mathematics and theoretical physics. In particular, this notion plays a fundamental role in complex differential geometry, algebraic geometry, conformal string and Yang-Mills theories. Moreover, the study of holomorphic vector bundles involves techniques from geometric analysis, partial differential equations and topology. In this thesis we study Higgs bundles and some of their main properties. We restrict our study to the case when the complex manifold is compact Kähler. On the one hand, complex manifolds provide a rich class of geometric objects, which behave rather differently than real smooth manifolds.

In complex geometry, the Hitchin-Kobayashi correspondence asserts that the notion of (Mumford-Takemoto) stability, originally introduced in algebraic geometry, has a differential-geometric equivalent in terms of special metrics. In its classical version, this correspondence is established for holomorphic vector bundles over compact Kähler manifolds and says that such bundles are polystable if and only if they admit a Hermitian-Einstein\(^1\) structure. This correspondence also holds for Higgs bundles.

The history of this correspondence probably starts in 1965, when Narasimhan and Seshadri \[30\] proved that a holomorphic bundle over a Riemann surface is stable if and only if it corresponds to a projective irreducible representation of the fundamental group of the surface. Then, in the 80’s Kobayashi \[23\] introduced for the first time the notion of Hermitian-Einstein structure in a holomorphic vector bundle, as a generalization of Kähler-Einstein metric in a tangent bundle. Shortly after, Kobayashi \[24\] and Lübke \[29\] proved that a bundle with a irreducible Hermitian-Einstein structure must be necessarily stable. Donaldson \[12\] showed that the result of Narasimhan and Seshadri \[30\] can be formulated in term of metrics and proved that the concepts of stability and existence of Hermitian-Einstein metrics are equivalent for holomorphic vector bundles over Riemann surfaces. Then, Kobayashi and Hitchin conjectured that the equivalence should be true in general for holomorphic vector bundles over Kähler manifolds.

The existence of a Hermitian-Einstein structure in a stable holomorphic vec-

\(^1\)In the literature Hermitian-Einstein, Einstein-Hermite and Hermitian-Yang-Mills are all synonymous. Sometimes also the terminology Hermitian-Yang-Mills-Higgs is used.
tor bundle was proved by Donaldson for projective algebraic surfaces in [13] and for projective algebraic manifolds in [14]. Finally, Uhlenbeck and Yau showed in [34] proved for general compact Kähler manifolds using some techniques from analysis and Yang-Mills theory. Hitchin [21], while studying the self-duality equations over a compact Riemann surface, introduced the notion of Higgs field and showed that the result of Donaldson for Riemann surfaces could be modified to include the presence of a Higgs field. Following the result of Hitchin, Simpson [31] defined a Higgs bundle to be a holomorphic vector bundle together with a Higgs field and proved the Hitchin-Kobayashi correspondence for such objects. Actually, using some sophisticated techniques in partial differential equations and Yang-Mills theory, he proved the correspondence even for non-compact Kähler manifolds, if they satisfy some analytic conditions. As an application of this, Simpson [32] later studied in detail a one-to-one correspondence between stable Higgs bundles over a compact Kähler manifold with vanishing Chern classes and irreducible representations of the fundamental group of that Kähler manifold.

The Hitchin-Kobayashi correspondence has been extended in several directions. Lübke and Teleman [28] studied the correspondence for compact complex manifolds. Bando and Siu [1] extended the correspondence to torsion-free sheaves over compact Kähler manifolds and introduced the notion of admissible Hermitian metric for such objects. Following the ideas of Bando and Siu, Biswas and Schumacher [3] introduced the notion of admissible Hermitian-Yang-Mills metric in the Higgs case and generalized this extension to torsion-free Higgs sheaves.

In [13] and [14] Donaldson introduced a functional, which is indeed known as the Donaldson functional, and later Simpson [31] extended this functional in his study of the Hitchin-Kobayashi correspondence for Higgs bundles. Kobayashi in [25] constructed the same functional in a different form and showed that it plays a fundamental role in a possible extension of the Hitchin-Simpson correspondence. In fact, he proved in [25] that for holomorphic vector bundles over projective algebraic manifolds, the counterpart of semistability is the notion of approximate Hermitian-Yang-Mills structure.

The correspondence between semistability and the existence of approximate Hermitian-Yang-Mills structure in the ordinary case has been originally proposed by Kobayashi. In [25] he proved that for a holomorphic vector bundle over a compact Kähler manifold a boundedness property of the Donaldson functional implies the existence of a Hermitian-Einstein structure and that implies the semistability of the bundle. Then, using some properties of the Donaldson functional and the Mehta-Ramanathan Theorem, he established the boundedness property of the Donaldson functional for semistable holomorphic bundles over compact algebraic manifolds. As a consequence of this, he obtained the correspondence between semistability and the existence of approximate Hermitian-Einstein structures when the base manifold was projective. Then he conjectured that all three conditions (the boundedness property, the existence of an approximate Hermitian-Einstein structure and the semistability) should be equivalent in general, that is, independently from whether the manifold was algebraic or not.
In the Higgs case and when the manifold is one-dimensional (a compact Riemann surface), the boundedness property of the Donaldson functional follows from the semistability in a similar way to the classical case, since we need to consider only Higgs subbundles and their quotients and we have a decomposition of the Donaldson functional in terms of these objects. The existence of approximate Hermitian-Einstein metrics for semistable holomorphic vector bundles has been recently studied in [22] using some techniques developed by Buchdahl [7], [8] for the desingularization of sheaves in the case of compact complex surfaces. One of the main difficulties in the study of this correspondence in higher dimensions arises from the notion of stability, since for compact Kähler manifolds with dimension greater or equal than two, it is necessary to consider subsheaves and not only subbundles. On the other hand, properties of the Donaldson functional commonly involve holomorphic bundles. Although all these difficulties appears also in the Higgs case, using Donaldson heat flow, Li and Zhanh [17] recently showed that the semistability of a Higgs bundle over a compact Kähler manifold implies the existence of approximate Hermitian-Einstein structure. It remains to be proved that, in dimension greater or equal than two, the semistability of the Higgs bundle implies the boundedness property of the Donaldson functional. In order to prove this, it seems natural to introduce first the notion of admissible Hermitian metrics on Higgs sheaves. Then, to define the Donaldson functional for such objects using these metrics and finally, following [22] to study how this functional defined for a semistable Higgs bundle can be decomposed in terms of Higgs subsheaves and their quotients.

About this thesis

This thesis is organized as follows. In Chapter 1 we start with some basic definitions and results, in particular we introduce complex manifolds and almost complex structures on differentiable manifolds. Then we summarize the main properties of complex differential forms over complex manifolds. Finally, we introduce Kähler manifolds. After studying some properties of these objects, we give an example of complex manifolds which cannot be given a Kählerian structure.

In Chapter 2 we present some definitions and results on principal fibre bundles over a differentiable manifold M. In particular we study connections and curvatures in principal fibre bundles and present some important results such as the structure equation and the Bianchi identity.

Although our primary interest lies in holomorphic vector bundles, we begin Chapter 3 with the study of connections in differentiable complex bundles. Then, we introduce connections in complex vector bundles over complex manifolds and we characterize those complex vector bundles which admit holomorphic structures. Moreover, we define Hermitian structures in complex vector bundles over (real or complex) manifolds and we study those connections which are compatible with the holomorphic structure and the Hermitian metric. Finally, we summarize the main properties of subbundles and quotient bundles. We study how Hermitian metrics on holomorphic vector bundles (over complex
manifolds) induce metrics on subbundles and quotient bundles. Moreover, we
give the definition and study the main properties of the second fundamental
form of subbundles and quotient bundles.

In Chapter 4 we introduce Chern classes. First of all we define the first
Chern class of a line bundle. Then, using the splitting principle, we define
higher Chern classes for complex vector bundles of any rank. Moreover, we
present the axiomatic approach to Chern classes: this enables us to separate
differential geometry aspects of Chern classes from their topological aspects.
Finally, via de Rahm theory, we give a representation of Chern classes in terms
of the curvature form of a connection in a complex vector bundle.

In Chapter 5 we summarize some basic properties of coherent sheaves over
compact Kähler manifolds. Then, we review the definition of singularity sets for
coherent sheaves and briefly comment some of their main properties concerning
to the codimension of these singularity sets. We review the construction of the
determinant bundle of a coherent sheaf and we write some facts on determinant
bundles that are used through this work. Moreover, we present some useful
analytic results such as the Fredholm alternative Theorem and the Maximum
principle for parabolic equations.

In Chapter 6 we study the basics results of Higgs sheaves. These results are
important mainly because the notion of stability in higher dimension (greater
than one) makes reference to Higgs subsheaves and not only Higgs subbundles.
Moreover, we summarized some properties of metrics and connections
on Higgs bundles and introduce the space of Hermitian structures, which is the
space where the Donaldson functional is defined. Finally, we construct the Don-
aldson functional for Higgs bundles over compact Kähler manifolds following a
construction similar to that of Kobayashi and we present some basic properties
of it. In particular, we prove that the critical points of this functional are
precisely the Hermitian-Yang-Mills structures, and we show also that its gradi-
ent flow can be written in terms of the mean curvature of the Hitchin-Simpson
connection. We also establish some properties of the solution of the evolution
equation associated with that functional. Next, we study the problem of the ex-
istence of approximate Hermitian-Yang-Mills structure and its relation with the
algebraic-geometric notion of Mumford-Takemoto semistability. We prove that
if the Donaldson functional of a Higgs bundle over a compact Kähler manifold
is bounded from below, then there exists an approximate Hermitian-Yang-Mills
structure on it. This fact, together with a result of Bruzzo and Graña Otero [6],
implies the semistability of the Higgs bundle. Then we show that a semistable
but non-stable Higgs bundle can be included into a short exact sequence with a
stable Higgs subsheaf and a semistable Higgs quotient. We use this result in the
final part of this chapter when we show that for a Higgs bundle over a compact
Riemann surface, the notion of approximate Hermitian-Yang-Mills structure is
in fact the differential-geometric counterpart of the notion of semistability.

In Chapter 7, using Donaldson heat flow, we show that the semistability of a
Higgs bundle over a compact Kähler manifold (of every dimension) implies the
existence of an approximate Hermitian-Yang-Mills structure; this proof is due
to Jiayu and Zhang [17] and essentially follows from Simpson [31].
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Chapter 1

Basics on Complex Manifolds

1.1 Complex manifolds

In this first chapter we start with some basic definitions and results, in particular we introduce complex manifolds and almost complex structures on differentiable manifolds.

**Definition 1.1.1.** Let $M$ a $T^2, N^2$ topological space. A complex $n$-atlas $\mathcal{A}$ is a collection of local charts $\{U_\alpha, \phi_\alpha\}$ such that:

1. $\{U_\alpha\}$ is a countable open cover of $M$,
2. $\phi_\alpha : U_\alpha \to \mathbb{C}^n$ are homeomorphisms of $U_\alpha$ with an open subset of $\mathbb{C}^n$,
3. If $U_\alpha \cap U_\beta \neq \emptyset$, the transition functions $\phi_\alpha \circ \phi^{-1}_\beta|_{\phi_\beta(U_\alpha \cap U_\beta)} : \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta)$ are biholomorphisms.

**Definition 1.1.2.** Let $M$ a $T^2, N^2$ topological space. Two complex $n$-atlases $\mathcal{A} = \{U_\alpha, \phi_\alpha\}$ and $\mathcal{B} = \{V_\beta, \psi_\beta\}$ are equivalent if every transition function $\phi_\alpha \circ \phi^{-1}_\beta|_{\phi_\beta(U_\alpha \cap V_\beta)} : \phi_\beta(U_\alpha \cap V_\beta) \to \phi_\alpha(U_\alpha \cap V_\beta)$ is a biholomorphism. A complex structure on $M$ is an equivalence class $\mathcal{A}$ of complex $n$-atlases.

**Definition 1.1.3.** Let $M$ be a $T^2, N^2$ topological space and let $\mathcal{A}$ a complex structure on $M$. We say that the pair $(M, \mathcal{A})$ is a complex manifold. We also define the complex dimension of $M$ as $\dim_{\mathbb{C}} = n$.

**Definition 1.1.4.** Let $M$ and $N$ be complex manifolds with $\dim_{\mathbb{C}} M = m$ and $\dim_{\mathbb{C}} N = n$. Let $f : M \to N$ be an application. $f$ is holomorphic if for every point $P \in M$ and local charts $(U_\alpha, \phi_\alpha)$ of $P$ in $M$ and $(V_\beta, \psi_\beta)$ of $f(P)$ in $N$ such that $f(U_\alpha) \subseteq V_\beta$ the representative $\psi_\beta \circ f \circ \phi^{-1}_\alpha$ is holomorphic.

Holographic functions have some interesting properties, see [10] and [16] for more details.
Theorem 1.1.1. (Open mapping Theorem) Let $M$ and $N$ be complex manifolds and let $f : M \rightarrow N$ be a holomorphic function. If $f$ is not constant, then $f$ is an open mapping.

Definition 1.1.5. Let $(M, g)$ be a Riemannian manifold and let $f$ be a $C^\infty$ function on $M$. The gradient of $f$ is the unique vector field $\nabla f \in \chi(M)$ such that for every vector field $X \in \chi(M)$

$$g(\nabla f, X) = df(X).$$

A point $P \in M$ is said to be a critical point of $f$ if $\nabla f (P) = 0$.

Note 1.1.2. In local coordinates we have

$$(\nabla f) = g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j}.$$  

Definition 1.1.6. Let $(M, g)$ be a Riemannian manifold of (real) dimension $n$ and let $X \in \chi(M)$ be a vector field on $M$. Let $\nabla$ be the Levi-Civita connection associated with the Riemannian metric $g$. The divergence of $X$ the function on $M$ obtained by contraction of the $(1, 1)$-tensor field $\nabla X$, i.e., the function

$$\text{div}(X) = C_1^1(\nabla X).$$

Here $C_1^1$ denote the contraction of the $(1, 1)$-tensor field $\nabla X$.

Note 1.1.3. In local coordinates we have

$$\text{div}(X) = \sum_{k=1}^n \left( \frac{\partial X^k}{\partial x^k} + \sum_{h=1}^n \Gamma^h_{kh} X^h \right),$$

where $\Gamma^h_{ij}$ are the Christoffel’s symbols of the Levi-Civita connection $\nabla$ associated with $g$.

Definition 1.1.7. Let $(M, g)$ be a Riemannian manifold of (real) dimension $n$ and let $f$ be a $C^\infty$ complex valued function on $M$. We define the Laplace-Beltrami operator as the divergence of the gradient:

$$\Delta f = \text{div}(\nabla f).$$

Definition 1.1.8. Let $M$ be a complex manifold of complex dimension $n$ and let $f : M \rightarrow \mathbb{C}$ be a $C^\infty$ function. We say that $f$ is harmonic if, regarded as a function on the real Riemannian manifold $M$ of real dimension $2n$, $\Delta f = 0$.

Proposition 1.1.4. Let $M$ be a compact complex manifold and let $f : M \rightarrow \mathbb{C}$ be a harmonic function. Then $f$ must be a constant.

We now define and study almost complex manifolds.

Definition 1.1.9. Let $M$ be a differentiable manifold. A pair $(M, J)$ is called an almost complex manifold if $J$ is a tensor field of type $(1, 1)$ such that at each point $P \in M$, we have: $J^2_P = -1_P$. The tensor field $J$ is called the almost complex structure of $M$. 

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Let us look at the tangent space of a complex manifold $M$ with $\dim \mathbb{C} M = n$. The tangent space $T_p M$, regarded as a $2n$-real vector space, is spanned by $2n$-real vectors
\[
\{\partial/\partial x^1, \ldots, \partial/\partial x^n, \ldots, \partial/\partial y^1, \ldots, \partial/\partial y^n\},
\]
where $z^k = x^k + iy^k$ are the complex coordinates in a complex chart $(U, \phi)$. With the same coordinates the dual space $T_p^* M$ as $2n$-dimensional (real) vector space is spanned by
\[
\{dx^1, \ldots, dx^n, dy^1, \ldots, dy^n\}.
\]
Let us define $2n$-vectors
\[
\partial/\partial z^k = \frac{1}{2} \{\partial/\partial x^k - i\partial/\partial y^k\},
\]
(1.1)
\[
\partial/\partial \bar{z}^k = \frac{1}{2} \{\partial/\partial x^k + i\partial/\partial y^k\},
\]
(1.2)
where $1 \leq k \leq n$. Clearly they form a basis of the $2n$-dimensional (complex) vector space $T_p M^\mathbb{C} = T_p M \otimes \mathbb{C}$. Note that $\partial/\partial z^k$ and $\partial/\partial \bar{z}^k$.

Proposition 1.1.5. Let $M$ be a complex manifold of (complex) dimension $n$. Regarded as a $2n$-differentiable manifold, $M$ admits an almost complex structure $J$.

Proof. Let $M$ be a complex manifold and define a linear map $J_P : T_p M \rightarrow T_p M$ by:
\[
J_P \left( \frac{\partial}{\partial x^k} \right)_p = \frac{\partial}{\partial y^k} \bigg|_p \quad \text{and} \quad J_P \left( \frac{\partial}{\partial y^k} \right)_p = -\frac{\partial}{\partial x^k} \bigg|_p.
\]

$J_P$ is a real tensor of type $(1, 1)$ and $J_P^2 = -\mathbb{I}_p$ where $\mathbb{I}_p$ is the identity map on $T_p M$. With respect to the basis $\{\partial/\partial x^1, \ldots, \partial/\partial x^n, \ldots, \partial/\partial y^1, \ldots, \partial/\partial y^n\}$, $J_P$ takes the form:
\[
\begin{pmatrix}
0 & -\mathbb{I}_p \\
\mathbb{I}_p & 0
\end{pmatrix}.
\]
We only have to show that the action of $J_P$ is independent of the chart. In fact, let $(U, \phi)$ and $(V, \psi)$ be overlapping charts with local coordinates $z^k = x^k + iy^k$ and $w^k = u^k + iv^k$. On $U \cap V$, the functions $z^k = z^k(w)$ satisfy the Cauchy-Riemann relations. Then we find:
\[
J \left( \frac{\partial}{\partial u^k} \right) = J \left( \frac{\partial x^h}{\partial u^k} \frac{\partial}{\partial x^h} + \frac{\partial y^h}{\partial u^k} \frac{\partial}{\partial y^h} \right) = \frac{\partial y^h}{\partial u^k} \frac{\partial}{\partial y^h} + \frac{\partial x^h}{\partial u^k} \frac{\partial}{\partial x^h} = \frac{\partial}{\partial v^k}.
\]
We also find $J(\partial/\partial v^k) = -\partial/\partial u^k$ and this completes the proof. \qed
Let \((M, J)\) be an almost complex manifold. \(J_P\) may be defined on \(T_P M^\mathbb{C}\) by setting
\[
J_P(X + iY) = J_PX + iJ_PY.
\]
Then we have the identities
\[
J_P(\partial/\partial z^k) = i\partial/\partial z^k \quad \text{and} \quad J_P(\partial/\partial \bar{z}^k) = -i\partial/\partial \bar{z}^k.
\]
Thus we have an expression for \(J_P\) in (anti-) holomorphic bases,
\[
J_P = idz^k \otimes \partial/\partial z^k - id\bar{z}^k \otimes \partial/\partial \bar{z}^k
\]
whose components are given by
\[
\begin{pmatrix}
i1_P & 0 \\
0 & -i1_P
\end{pmatrix}.
\]
Let \(Z \in T_P M^\mathbb{C}\) be a vector of the form \(Z = Z^k \partial/\partial z^k\). Then \(Z\) is an eigenvector of \(J_P\), in fact we have \(J_PZ = iZ\). In the same way if \(W = W^k \partial/\partial \bar{z}^k\), it satisfies \(J_PW = -iW\). In this way \(T_P M^\mathbb{C}\) is decomposed into a direct sum
\[
T_P M^\mathbb{C} = T_P M^{C^+} \oplus T_P M^{C^-},
\]
(1.3)
where
\[
T_P M^{C^\pm} = \{ Z \in T_P M^\mathbb{C} | J_PZ = \pm iZ \}.
\]
Now \(Z \in T_P M^\mathbb{C}\) is uniquely decomposed as \(Z = Z^+ + Z^-\) and \(T_P M^{C^+}\) is spanned by \(\{ \partial/\partial z^k \}\), while \(T_P M^{C^-}\) by \(\{ \partial/\partial \bar{z}^k \}\). In the same way \(TM^\mathbb{C} = TM \otimes \mathbb{C}\) is decomposed into a direct sum
\[
TM^\mathbb{C} = TM^{C^+} \oplus TM^{C^-}
\]
where
\[
TM^{C^\pm} = \{ Z \in TM^\mathbb{C} | JZ = \pm iZ \}.
\]
**Definition 1.1.10.** \(Z \in T_P M^{C^+}\) is called a vector of type \((1, 0)\), while \(W \in T_P M^{C^-}\) is called a vector of type \((0, 1)\).

We can easy verify that
\[
T_P M^{C^-} = \overline{T_P M^{C^+}}.
\]
**Note 1.1.6.** We have the following identity for the dimensions of these complex vector spaces:
\[
dim_{\mathbb{C}} T_P M^{C^+} = \dim_{\mathbb{C}} T_P M^{C^-} = \frac{1}{2} \dim_{\mathbb{C}} T_P M^\mathbb{C} = \dim_{\mathbb{C}} M.
\]
**Definition 1.1.11.** Let \(M\) be a complex manifold of (complex) dimension \(n\). We define \(\mathcal{X}(M)^\mathbb{C} = \mathcal{X}(M) \otimes \mathbb{C}\).

Given a complex vector field \(Z \in \mathcal{X}(M)^\mathbb{C}\), \(Z\) is naturally decomposed as \(Z = Z^+ + Z^-\). \(Z^+\) is called the component of type \((1, 0)\), while \(Z^-\) of type \((0, 1)\). Accordingly once \(J\) is given, \(\mathcal{X}(M)^\mathbb{C}\) is decomposed uniquely as
\[
\mathcal{X}(M)^\mathbb{C} = \mathcal{X}(M)^{C^+} \oplus \mathcal{X}(M)^{C^-}
\]
(1.4)
Definition 1.1.12. Let \((M, J)\) be an almost complex manifold. If the Lie bracket of any vector fields of type \((1, 0)\) \(X, Y \in \mathcal{X}(M)^{C_+}\) is again of type \((1, 0)\), i.e. \([X, Y] \in \mathcal{X}(M)^{C_+}\), the almost complex structure \(J\) is said to be integrable.

Theorem 1.1.7. Let \((M, J)\) be an almost complex manifold. If the almost complex structure \(J\) is integrable then \(M\) admits a complex structure and such complex structure is unique up to biholomorphism.

1.2 Complex differential forms

In this section we define the main properties of complex differential forms over a complex manifold.

Definition 1.2.1. Let \(M\) be an \(n\)-dimensional real differentiable manifold. We define \(A^p = \Gamma(M, T^*M \otimes \mathbb{C})\) the space of \(\mathcal{C}^\infty\) complex \(p\)-forms over \(M\). In local coordinates \(\omega \in A^p\) has the form:

\[
\omega = \omega_{j_1...j_p} dx^{j_1} \wedge \ldots \wedge dx^{j_p},
\]

here \(\omega_{j_1...j_p}\) are \(\mathcal{C}^\infty(U, \mathbb{C})\) functions on the local chart \((U; x^1, \ldots, x^n)\).

Now we define complex differential forms on a complex manifold \(M\) and then we introduce the Dolbeault complex and the Dolbeault cohomology groups.

Definition 1.2.2. Let \(M\) be a complex manifold of (complex) dimension \(n\) and let \(p, q \geq 0\). We define \(A^{p,q} = \Gamma(M, \bigwedge_p T^*M^{C_+} \otimes \bigwedge_q T^*M^{C_-})\) the space of \((p, q)\)-forms over \(M\). As in the previous definition, in local coordinates \(\omega \in A^{p,q}\) has the form:

\[
\omega = \omega_{i_1...i_p,j_1...j_q} dz^{i_1} \wedge \ldots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \ldots \wedge d\bar{z}^{j_q},
\]

where \(\omega_{i_1...i_p,j_1...j_q}\) are \(\mathcal{C}^\infty(U, \mathbb{C})\) functions on \((U; z^1, \ldots, z^n, \bar{z}^1, \ldots, \bar{z}^n)\).

So we have \(A^r = \sum_{p+q=r} A^{p,q}\) and the exterior differential \(d\) has the form \(d = d' + d''\), where \(d' : A^{p,q} \rightarrow A^{p+1,q}\) and \(d'' : A^{p,q} \rightarrow A^{p,q+1}\). Now we can define the Dolbeault cohomology groups.

Definition 1.2.3. Let \(M\) be a complex manifold of (complex) dimension \(n\) and let \(r \geq 0\). The sequence of \(\mathbb{C}\)-linear maps

\[
A^{r,0} \xrightarrow{d''} A^{r,1} \xrightarrow{d''} \ldots \xrightarrow{d''} A^{r,n-1} \xrightarrow{d''} A^{r,n}
\]

is the Dolbeault complex.

The set of \(d''\)-closed \((p, q)\)-forms is called the \(Z^{p,q}(M)\) and its element are called \((p, q)\)-cocycles, while the set of \(d''\)-exact \((p, q)\)-forms is denoted by \(B^{p,q}(M)\) and its element are called \((p, q)\)-coboundaries. Since \(d'' \circ d'' = 0\), we have \(B^{p,q}(M) \subseteq Z^{p,q}(M)\). The complex vector space

\[
H^{p,q}(M) = Z^{p,q}(M) / B^{p,q}(M)
\]

is called the \((p, q)\)th Dolbeault cohomology group of the complex manifold \(M\).
In the real case every closed differential form is locally exact (Poincaré’s Lemma). As seen in [16] in the complex case we have a similar result due to Grothendieck. Every \(\mathbb{d}''\)-closed complex \((p,q)\)-form, with \(q \geq 1\), is locally \(\mathbb{d}''\)-exact.

**Definition 1.2.4.** Let \(r_1, \ldots, r_n \in (0, +\infty]\). A polydisc in \(\mathbb{C}^n\) is a set \(\Delta = \{(z^1, \ldots, z^n) \in \mathbb{C}^n | |z^j| < r_j\}\).

**Theorem 1.2.1.** (Dolbeault’s Lemma) Let \(\Delta \subseteq \mathbb{C}^n\) a polydisc with compact closure \(\overline{\Delta}\), i.e. \(\Delta\) is bounded. Let \(\zeta\) a complex \((p,q)\)-form on an open neighborhood \(U\) of \(\overline{\Delta}\) and assume \(q \geq 1\). Then \(\zeta\) is locally \(\mathbb{d}''\)-exact, i.e. there exists a complex \((p,q-1)\)-form \(\eta\) on \(\Delta\) such that \(\mathbb{d}''\eta = \zeta\).

**Note 1.2.2.** \(Z^p,0(M)\) is the set of holomorphic \(p\)-forms.

### 1.3 Kähler manifolds

Let \(M\) be a complex manifold with \(\text{dim}_\mathbb{C} M = n\) and let \(g\) be a Riemann metric on \(M\) as a differentiable manifold. Take \(Z = X + iY\) and \(W = U + iV\) and extend \(g\) to \(T_PM^\mathbb{C}\) so that

\[
g_P(JPZ, JPY) = g_P(X, Y)
\]

at each point \(P \in M\) and for any \(X, Y \in T_PM\), it is said to be a Hermitian metric. The pair \((M, g)\) is called a Hermitian manifold.
Proof. Let \( g \) be any Riemannian metric of a complex manifold \( M \). Define a new metric \( \tilde{g} \) by
\[
\tilde{g}_P(X, Y) = \frac{1}{2} [g_P(X, Y) + g_P(JX, JY)].
\]
Clearly \( \tilde{g}_P(JX, JY) = \tilde{g}_P(X, Y) \). Moreover, \( \tilde{g} \) is positive definite provided that \( g \) is. Hence \( \tilde{g} \) is a Hermitian metric on \( M \).

Let \( g \) be a Hermitian metric on a complex manifold \( M \). From (1.6), we find that \( g_{ij} = g_{\overline{j}i} \). Thus the Hermitian metric \( g \) takes the form
\[
g = g_{ij} dz^i \otimes d\overline{z}^j + g_{\overline{j}i} dz^i \otimes dz^j.
\]

Note 1.3.2. Take \( X, Y \in T_P M^{\mathbb{C}+} \). Define an inner product \( h_P \) in \( T_P M^{\mathbb{C}+} \) by
\[
h_P(X, Y) = g_P(X, Y).
\]
It is easy to see that \( h_P \) is a positive-definite Hermitian form in \( T_P M^{\mathbb{C}+} \). In fact:
\[
h(X, Y) = g(X, Y) = g(JX, JY) = -g(JY, X) = -\omega(Y, X),
\]
hence \( \omega \) is a 2-form.

Definition 1.3.2. Let \((M, J, g)\) a Hermitian manifold. The 2-form \( \omega \) is called the Kähler form of the Hermitian metric \( g \).

Definition 1.3.3. A Kähler manifold \((M, \omega)\) is a Hermitian manifold \((M, g)\) whose Kähler form \( \omega \) is closed: \( d\omega = 0 \). The metric \( g \) is called the Kähler metric of \( M \).

Proposition 1.3.3. Let \((M, g)\) a Hermitian manifold. The Kähler form \( \omega \) is a complex 2-form of type \((1, 1)\).

Proof. Let \( J \) be the almost complex structure and let \( X, Y \in TM^{\mathbb{C}+} \). From the definition of \( TM^{\mathbb{C}+} \) we have \( JX = iX \) and \( JY = iY \). Then we have
\[
\omega(X, Y) = \omega(-iJX, -iJY) = g(J(-iJX), -iJY) =
\]
\[
= (-i)^2 g(J^2 X, JY) = -g(-X, JY) =
\]
\[
= g(X, JY) = g(JY, X) = \omega(Y, X) =
\]
\[
= -\omega(X, Y).
\]
Hence, \( \omega(X, Y) = 0 \). In the same way, if \( X, Y \in TM^{\mathbb{C}−} \), \( \omega(X, Y) = 0 \), proving that the Kähler form \( \omega \) is of type \((1, 1)\). 

\( \square \)
Note 1.3.4. Let \((M, g)\) a Hermitian manifold and let \(\omega\) its Kähler form. In local coordinates we have
\[
\omega = ig_{ij}dz^i \wedge d\bar{z}^j. \tag{1.8}
\]

**Proposition 1.3.5.** Let \((M, g)\) a Hermitian manifold. The Kähler form \(\omega\) is real, i.e., \(\overline{\omega} = \omega\).

**Proof.** In local coordinates we have
\[
\overline{\omega} = \overline{ig_{ij}dz^i \wedge d\bar{z}^j} = -i\overline{g}_{ij}d\bar{z}^i \wedge dz^j = -ig_{ji}dz^j \wedge d\bar{z}^i = \omega.
\]

**Proposition 1.3.6.** A Hermitian manifold \((M, g)\) of complex dimension 1 is Kähler.

**Proof.** In local coordinates the Kähler form is \(\omega = ig_{ij}dz^i \wedge d\bar{z}^j\). Then we have
\[
d\omega = i\frac{\partial g_{ij}}{\partial z^k}dz^k \wedge dz^i \wedge d\bar{z}^j + i\frac{\partial g_{ij}}{\partial \bar{z}^k}dz^i \wedge d\bar{z}^j \wedge d\bar{z}^j = 0.
\]

**Proposition 1.3.7.** A Hermitian manifold \((M, g)\) of (complex) dimension \(n\) regarded as a \(2n\)-dimensional differentiable manifold is always orientable.

**Proof.** Let \(\omega\) the Kähler form. Then \(\omega^n\) is a \(2n\) non-vanishing real form, in fact
\[
\omega^n = (i)^n n! \det(g_{ij}) dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^n
\]
and \(\det(g_{ij})\) is nowhere vanishing.

**Proposition 1.3.8.** Let \((M, g)\) be a Hermitian manifold. The following conditions are equivalent:

1. \(g\) is a Kähler metric, i.e., \(d\omega = 0\),
2. \(\frac{\partial g_{ij}}{\partial z^k} = \frac{\partial g_{ij}}{\partial z^l}\),
3. \(\frac{\partial g_{ij}}{\partial \bar{z}^k} = \frac{\partial g_{ij}}{\partial \bar{z}^l}\),
4. There exists a locally defined real function \(f\) such that \(\omega = i\theta d^\alpha f\), i.e.,
\[
g_{ij} = \frac{\partial^2 f}{\partial z^i \partial \bar{z}^j}.
\]

**Proof.** See [25] for a detailed proof.
1.4 The Hopf manifold

Let $\Delta$ be the discrete group generated by $z^k \mapsto 2z^k$, $1 \leq k \leq n$. The quotient manifold $(\mathbb{C}^n \setminus \{0\})/\Delta$ is a complex manifold called the Hopf manifold. It is homeomorphic to $S^1 \times S^{2n-1}$.

Now, we want to prove that for $n \geq 2$ the Hopf manifold cannot be given a Kählerian structure. In fact compact Kähler manifolds have strong topological restrictions. Perhaps the simplest among them is the following:

**Proposition 1.4.1.** The second Betti number of a compact Kähler manifold $(M,\omega)$ of (complex) dimension $n$ is strictly positive.

**Proof.** Let $(M,\omega)$ be a compact Kähler manifold of (complex) dimension $n$. Since $\omega$ is closed, it determines an element $u = [\omega] \in H^2_{de}(M)$. Consider now the $2n$-form $\omega^n$, this determines an element $u^n = [\omega]^n = [\omega^n]$ in $H_{de}^{2n}(M)$. Integrating the volume form $\frac{\omega^n}{n!}$ we find

$$\int_M \frac{\omega^n}{n!} = \text{Vol}(M) > 0,$$

hence $u^n \neq 0$. Therefore $u \neq 0$, and then $H_{de}^2(M) \neq 0$.

**Corollary 1.4.2.** Let $n \geq 2$. The Hopf manifold $S^1 \times S^{2n-1}$ cannot be given a Kählerian structure.

**Proof.** Since $n \geq 2$, $H^1_{de}(S^{2n-1}) = H^2_{de}(S^{2n-1}) = 0$. From Künneth formula we have

$$H^2_{de}(S^1 \times S^{2n-1}) = \bigoplus_{p+q=2} [H^p_{de}(S^1) \otimes H^q_{de}(S^{2n-1})] =$$

$$\oplus [H^0_{de}(S^1) \times H^2_{de}(S^{2n-1})] \oplus$$

$$\oplus [H^1_{de}(S^1) \times H^1_{de}(S^{2n-1})] \oplus$$

$$\oplus [H^0_{de}(S^1) \times H^0_{de}(S^{2n-1})] =$$

$$= 0 \otimes \mathbb{R} \oplus [\mathbb{R} \otimes 0] \oplus [\mathbb{R} \otimes 0] = 0.$$

Then, for $n \geq 2$ the second Betti number of the Hopf manifold $S^1 \times S^{2n-1}$ is 0, and this shows that for $n \geq 2$ the Hopf manifold cannot be given a Kählerian structure.
Chapter 2
Principal Fibre Bundles

In this chapter we present some definitions and results on principal fibre bundles over a differentiable manifold $M$. See [27] for more details.

2.1 Basics on principal $G$-bundles

Definition 2.1.1. Let $M$ be a differentiable manifold and $G$ a Lie group. A (differentiable) principal fibre bundle over $M$ with group $G$ consists of a manifold $P$ and an action $G$ on $P$ satisfying the following conditions:

1. $G$ acts freely on $P$ on the right,
2. $M$ is the quotient space of $P$ by the equivalence relation induced by $G$, i.e., $M = P/G$ and the canonical projection $\pi : P \to M$ is differentiable,
3. $P$ is locally trivial, that is, every point $x$ of $M$ has a neighborhood $U$ such that $\pi^{-1}(U)$ is isomorphic with $U \times G$ in the sense that there is a diffeomorphism $\psi : \pi^{-1}(U) \to U \times G$ such that $\psi(u) = (\pi(u), \varphi(u))$ where $\varphi$ is a mapping of $\pi^{-1}(U)$ into $G$ satisfying $\varphi(ua) = (\varphi(u))a$ for all $u \in \pi^{-1}(U)$ and $a \in G$.

A principal fibre bundle will be denoted by $P(M,G)$, or simply $P$. We call $P$ the total space, $M$ the base space, $G$ the structure group and $\pi$ the projection. For each $x \in M$, $\pi^{-1}(x)$ is a closed submanifold of $P$ called the fibre over $x$. If $u \in \pi^{-1}(x)$ is a point in the fibre over $x$, then $\pi^{-1}(x)$ is the set of point $ua$ with $a \in G$. Every fibre is diffeomorphic to $G$.

Definition 2.1.2. Let $P(M,G)$ and $Q(N,H)$ be principal fibre bundles over the (differentiable) manifolds $M$ and $N$ respectively. A morphism of principal fibre bundles consist of a $C^\infty$ mapping $f : P \to Q$ and a homomorphism $\psi : G \to H$ such that $f(ua) = f(u)\psi(a)$ for all $u \in P$ and $a \in G$. For the sake of simplicity, we shall denote $f$ and $\psi$ by the same letter $f$. Every morphism $f : P \to Q$ maps each fibre of $P$ into a fibre of $Q$ and hence it induces a mapping of $M$ into $N$, which will be also denoted by $f$.

Definition 2.1.3. A morphism $f : P(M,G) \to Q(N,H)$ is called an imbedding if $(P,f)$ is a regular submanifold of $Q$ and if $f : G \to H$ is a homomorphism of groups. If $f : P \to Q$ is an imbedding, then $(M,f)$ is a regular
submanifold of \( N \). By identifying \( P \) with \( f(P) \), \( G \) with \( f(G) \) and \( M \) with \( f(M) \), we say that \( P(M, G) \) is a subbundle of \( Q(N, H) \).

**Definition 2.1.4.** Given a principal fibre bundle \( P(M, G) \), the action of \( G \) on \( P \) induces a homomorphism \( \sigma \) of the Lie algebra \( g \) of \( G \) into the Lie algebra \( X(P) \) of vector fields on \( P \) (see Proposition 4.1 in [27]). For each \( A \in g \), \( A^* = \sigma(A) \) is called the fundamental vector field corresponding to \( A \).

In order to relate our intrinsic definition of a principal fibre bundle to the definition and the construction by means of an open cover, we need the concept of transition functions. By (3) for a principal fibre bundle \( P(M, G) \) it is possible to choose an open covering \( \{ U_\alpha \} \) of \( M \) where each \( \pi^{-1}(U_\alpha) \) provided with a diffeomorphism \( u \mapsto (\pi(u), \varphi_\alpha(u)) \) of \( \pi^{-1}(U_\alpha) \) onto \( U_\alpha \times G \) such that \( \varphi_\alpha(ua) = (\varphi_\alpha(u))a \) for all \( u \in \pi^{-1}(U_\alpha) \) and \( a \in G \).

If \( u \in \pi^{-1}(U_\alpha \cap U_\beta) \), then \( \varphi_\beta(ua)(\varphi_\alpha(ua))^{-1} = \varphi_\beta(u)(\varphi_\alpha(u))^{-1} \), which shows that \( \varphi_\beta(u)(\varphi_\alpha(u))^{-1} \) depends only on \( \pi(u) \) not on \( u \). Hence, if \( U_\alpha \cap U_\beta \neq \emptyset \), we can define a mapping \( \psi_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow G \) by \( \psi_{\alpha\beta}(\pi(u)) = \varphi_\beta(u)(\varphi_\alpha(u))^{-1} \). The family of mappings \( \psi_{\alpha\beta} \) are called transition functions of the bundle \( P(M, G) \) corresponding to the open cover \( \{ U_\alpha \} \) of \( M \). It is easy to verify that

\[
\psi_{\alpha\beta}(x) \circ \psi_{\beta\gamma}(x) \circ \psi_{\gamma\alpha}(x) = I_G \text{ for } x \in U_\alpha \cap U_\beta \cap U_\gamma.
\]

Conversely, we have the following:

**Theorem 2.1.1.** Let \( M \) be a differentiable manifold, \( \{ U_\alpha \} \) an open cover of \( M \) and \( G \) a Lie group. Given a mapping \( \psi_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow G \) for every nonempty \( U_\alpha \cap U_\beta \) such that

\[
\psi_{\alpha\beta}(x) \circ \psi_{\beta\gamma}(x) \circ \psi_{\gamma\alpha}(x) = I_G \text{ for } x \in U_\alpha \cap U_\beta \cap U_\gamma,
\]

we can construct a (differentiable) principal fibre bundle \( P(M, G) \) with transition functions \( \psi_{\alpha\beta} \).

**Definition 2.1.5.** Let \( P(M, G) \) a principal fibre bundle over the differentiable manifold \( M \) and let \( f : N \rightarrow M \) be a \( C^\infty \) mapping. The pull-back of \( P \) (via \( f \)) is the principal fibre bundle over \( N \) defined as the fibre product \( f^*P = N \times_M P = \{(n, u) \in N \times P | f(n) = \pi(u) \} \).

### 2.2 Connections in a principal fibre bundle

**Definition 2.2.1.** Let \( P(M, G) \) be a principal fibre bundle over a manifold \( M \) with structure group \( G \). For each \( u \in P \), let \( T_uP \) be the tangent space of \( P \) at the point \( u \) and \( G_u \subseteq T_uP \) the subspace of \( T_uP \) consisting of vectors tangent to the fibre through \( u \). A connection \( \Gamma \) in \( P \) is a \( C^\infty \) assignment of a subspace \( Q_u \) of \( T_uP \) for each \( u \in P \) such that

1. \( T_uP = G_u \oplus Q_u \),
2. \( Q_u = (R_u)_*Q_u \) for every \( u \in P \) and \( a \in G \).
The second condition means that the distribution \( u \rightarrow Q_u \) is invariant under the action of \( G \) over the total space \( P \). We call \( G_u \) the vertical subspace and \( Q_u \) the horizontal subspace of \( T_u P \). In this way a vector \( X_u \in T_u P \) can be uniquely written as direct sum of his vertical and horizontal component:

\[
X_u = Y_u \oplus Z_u \quad \text{where} \quad Y_u \in G_u, \quad Z_u \in Q_u.
\]

**Definition 2.2.2.** Given a connection \( \Gamma \) in a principal fibre bundle \( P \), we define a 1-form \( \omega \) on \( P \) with values in the Lie algebra \( g \) of \( G \) as follows. For each \( X_u \in T_u P \), we define \( \omega(X) \) to be the unique \( A \in g \) such that \( (A^*)_u \) is equal to the vertical component of \( X_u \). The form \( \omega \) is called the connection form of the given connection \( \Gamma \).

**Proposition 2.2.1.** The connection form \( \omega \) of a connection \( \Gamma \) in \( P \) satisfies the following conditions:

1. \( \omega(A^*) = A \) for every \( A \in g \),
2. \( (R_u)^* \omega = \text{ad}_{a^{-1}} \omega \), that is \( \omega((R_u)_*, X) = \text{ad}_{a^{-1}}(\omega(X)) \) for every \( a \in G \) and every vector field \( X \) on \( P \), where \( \text{ad} \) denotes the adjoint representation of \( G \) on \( g \).

The projection \( \pi : P \rightarrow M \) induces a linear mapping \( \pi : T_u P \rightarrow T_x M \) for each \( u \in P \), where \( x = \pi(u) \). When a connection is given, \( \pi \) maps the horizontal subspace \( Q_u \) isomorphically onto \( T_x M \).

**Definition 2.2.3.** Given a connection \( \Gamma \) in \( P \), the horizontal lift of a vector field \( X \) on \( M \) is the unique vector field \( X^* \) on \( P \) which is horizontal and which projects onto \( X \), i.e. \( \pi(X^*_u) = X_{\pi(u)} \) for every \( u \in P \).

**Proposition 2.2.2.** Let \( \Gamma \) be a connection in \( P \) and let \( X \) be a vector field on \( M \), there is a unique horizontal lift \( X^* \) of \( X \). The lift \( X^* \) is invariant by \( R_u \) for every \( a \in G \). Conversely, every horizontal vector field \( X^* \) on \( P \) invariant by \( G \) is the lift of a vector field \( X \) on \( M \).

**Proposition 2.2.3.** Let \( X^* \) and \( Y^* \) be the horizontal lifts of \( X \) and \( Y \) respectively. Then

1. \( X^* + Y^* \) is the horizontal lift of \( X + Y \).
2. For every function \( f \in C^\infty(M, \mathbb{R}) \), \( f^* X^* \) is the horizontal lift of \( fX \) where \( f^* \) is the pull-back function \( f^* = f \circ \pi \).
3. The horizontal component of \([X^*, Y^*]\) is the horizontal lift of \([X, Y]\).

## 2.3 Curvature form and structure equation

**Definition 2.3.1.** Let \( P(M, G) \) be a principal fibre bundle and \( \rho \) a representation of \( G \) on a finite dimensional vector space \( V \), i.e., a group homomorphism \( \rho : G \rightarrow GL(V) \). A pseudotensorial form of degree \( r \) on \( P \) of type \((\rho, V)\) is a \( V \)-valued \( r \)-form \( \varphi \) on \( P \) such that

\[
R^*_a \varphi = \rho(a^{-1}) \cdot \varphi \quad \text{for} \quad a \in G.
\]

Such a form is called a tensorial form if it is horizontal in the sense that \( \varphi(X_1, \ldots, X_r) = 0 \) whenever at least one of the tangent vectors \( X_i \) is vertical, i.e., tangent to a fibre.
By the above definition we see that if $\Gamma$ is a connection in a principal fibre bundle $P$ with structure group $G$, then its connection form $\omega$ is a pseudotensorial 1-form of type $(\text{ad}, g)$. Now we present the main results of this section (see [27], p.76-78 for details). Let $\Gamma$ be a connection in $P$ and let $G_u$ and $Q_u$ be the vertical and the horizontal subspaces of $T_uP$, respectively. Let $h : T_uP \rightarrow Q_u$ be the projection.

**Proposition 2.3.1.** If $\varphi$ is a pseudotensorial $r$-form on $P$ of type $(\rho, V)$, then

1. The form $\varphi h$ defined by $(\varphi h)(X_1, \ldots, X_r) = \varphi(hX_1, \ldots, hX_r)$ is a tensorial form of type $(\rho, V)$,

2. $d\varphi$ is a pseudotensorial $(r+1)$-form of type $(\rho, V)$.

**Definition 2.3.2.** The $(r+1)$-tensorial form $D\varphi = (d\varphi) h$ is called the exterior covariant derivative of $\varphi$ and $D$ is called exterior covariant differentiation.

**Definition 2.3.3.** Let $\Gamma$ be a connection in $P$ and let $\omega$ be its connection form. By the above Proposition, $\Omega = D\omega$ is a tensorial 2-form of type $(\text{ad}, g)$. It is called the curvature form of $\Gamma$.

**Theorem 2.3.2.** (Structure equation) Let $\Gamma$ be a connection in a principal fibre bundle $P$ and let $\omega$ and $\Omega$ be its connection form and curvature form respectively. Then

$$\Omega(X,Y) = d\omega(X,Y) + \frac{1}{2}[\omega(X), \omega(Y)].$$

**Theorem 2.3.3.** (Bianchi’s identity) Let $\Gamma$ be a connection in a principal fibre bundle $P$ and let $\omega$ and $\Omega$ its connection form and its curvature form respectively. Then the exterior covariant derivative of the curvature is 0, i.e.,

$$D\Omega = 0.$$
Chapter 3

Connections in Vector Bundles

Although our primary interest lies in holomorphic vector bundles, we begin this chapter with the study of connections in differentiable complex vector bundles.

**Definition 3.0.4.** Let $M$ be a differentiable manifold of (real) dimension $n$. A $C^\infty$ complex vector bundle over $M$ of rank $r$ is a differentiable manifold $E$ together with a surjective $C^\infty$ map $\pi : E \to M$ such that there exists a countable open covering $\{U_\alpha\}$ satisfying:

1. There are diffeomorphisms $\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}^r$ such that the diagrams commute

   $\begin{tikzcd}
   \pi^{-1}(U_\alpha) \ar[r, \phi_\alpha] \ar[d, \pi] & U_\alpha \times \mathbb{C}^r \ar[d, pr_1] \\
   U_\alpha
   \end{tikzcd}$

   Here $pr_1$ is the projection on the first factor.

2. When $U_\alpha \cap U_\beta \neq \emptyset$, $\phi_\beta \circ \phi_\alpha^{-1}$ is an automorphism of $(U_\alpha \cap U_\beta) \times \mathbb{C}^r$ with the condition that $\forall P \in U_\alpha \cap U_\beta$

   $\phi_\beta \circ \phi_\alpha^{-1}(P, *) : \mathbb{C}^r \to \mathbb{C}^r$

   is $\mathbb{C}$-linear.

**Note 3.0.4.** From (1) it immediately follows that if $U_\alpha \cap U_\beta \neq \emptyset$, then $\phi_\beta \circ \phi_\alpha^{-1}(P, *) : \mathbb{C}^r \to \mathbb{C}^r$ is invertible. In fact it is a bijective $\mathbb{C}$-linear endomorphism of $\mathbb{C}^r$.

**Note 3.0.5.** In the same manner, if $M$ is a complex manifold, we can define a holomorphic vector bundle $E$ over $M$ in the obvious way.
As in the case of principal fibre bundles, we have the following result:

**Theorem 3.0.6.** Let $M$ be a differentiable manifold, $\{U_\alpha\}$ an open covering of $M$. Given a $C^\infty$ mapping $\psi_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(r, \mathbb{C})$ for every nonempty $U_\alpha \cap U_\beta \cap U_\gamma$ such that for every $x \in U_\alpha \cap U_\beta \cap U_\gamma$

$$\psi_{\alpha\beta}(x) \circ \psi_{\beta\gamma}(x) \circ \psi_{\gamma\alpha}(x) = 1_{C^r} \quad (3.1)$$

we can construct a complex vector bundle of rank $r$ over the manifold $M$ with transition functions $\psi_{\alpha\beta}$.

**Definition 3.0.5.** Let $(E, M, \pi_E)$ and $(F, N, \pi_F)$ be complex vector bundles of rank $r$ and $p$ over the differentiable manifolds $M$ and $N$, respectively. A morphism of complex vector bundles is a pair of $C^\infty$ mapping $f : E \to F$ and $\tilde{f} : M \to N$ such that

1. The following diagram commutes

$$\begin{array}{ccc}
E & \xrightarrow{f} & F \\
\pi_E & \downarrow & \pi_F \\
M & \xrightarrow{\tilde{f}} & N \\
\end{array}$$

2. For every $P \in M$ the mapping $f|_{E_P} : E_P \to F_{\tilde{f}(P)}$ is $\mathbb{C}$-linear on the fibres.

Finally, if $f$ is a diffeomorphism, we say that the complex vector bundles $(E, M, \pi_E)$ and $(F, N, \pi_F)$ are isomorphic.

**Definition 3.0.6.** Let $(E, M, \pi_E)$ be a complex vector bundle of rank $r$ over the differentiable manifold $M$. A subbundle of $E$ is a complex vector bundle $(S, M, \pi_S)$ over $M$ with a morphism $i : S \to E$ such that the following diagram commutes

$$\begin{array}{ccc}
S & \xrightarrow{i} & E \\
\pi_S & \downarrow & \pi_E \\
M & \xrightarrow{\text{Id}_M} & M \\
\end{array}$$

**Definition 3.0.7.** Let $(E, M, \pi)$ be a complex vector bundle of rank $r$ over the differentiable manifold $M$ and let $f : N \to M$ be a $C^\infty$ mapping. The pull-back of $E$ (via $f$) is the complex vector bundle over $N$ defined as the fibre product $f^*E = N \times_M E = \{(n, e) \in M \times E | f(n) = \pi(e)\}$.

**Note 3.0.7.** From the above definition we can deduce that the following diagram commutes

$$\begin{array}{ccc}
f^*E & \xrightarrow{pr_2} & E \\
pr_1 & \downarrow & \pi \\
M & \xrightarrow{f} & N \\
\end{array}$$
is commutative. In particular \((f^*E, M, pr_1)\) is a complex vector bundle over \(M\) of rank \(r\) and \(pr_2 : f^*E \rightarrow E\) is a morphism of complex vector bundles.

**Note 3.0.8.** In the same way we can define morphisms, subbundles and pull-back bundles of holomorphic vector bundles over complex manifolds.

### 3.1 Connections in complex vector bundles (over real manifolds)

Let \(M\) be a differentiable manifold of (real) dimension \(n\) and let \(E\) a \(C^\infty\) complex vector bundle of rank \(r\) over \(M\). We make use of the following notation:

1. \(A^p\) is the space of \(C^\infty\) complex \(p\)-forms over \(M\),
2. \(A^p(E) = \Gamma(M, \bigwedge^p T^*M \otimes E)\) is the space of \(C^\infty\) complex \(p\)-forms over \(M\) with values in \(E\).

**Definition 3.1.1.** A connection \(D\) in \(E\) is a \(\mathbb{C}\)-linear homomorphism \(D : A^0(E) \rightarrow A^1(E)\) such that

\[
D(f\sigma) = \sigma \otimes df + fD\sigma \quad \text{for } f \in A^0 \text{ and } \sigma \in A^0(E).
\]  

By abuse of notation we omit \(\otimes\) and write \(D(f\sigma) = \sigma df + fD\sigma\).

**Definition 3.1.2.** Let \(s = (s_1, \ldots, s_r)\) be a local frame field of \(E\) over an open set \(U \subseteq M\), then given a connection \(D\), we can write

\[
Ds_i = \sum_{j=1}^{r} s_j \omega^j_i.
\]  

We call the matrix 1-form \(\omega = (\omega^j_i)\) the connection form of \(D\) with respect to the frame field \(s\).

Considering \(s = (s_1, \ldots, s_r)\) as a row vector, we can rewrite (3.3) in matrix notation as follows:

\[
Ds = s\omega.
\]

**Definition 3.1.3.** If \(\zeta = \zeta^i s_i\) is an arbitrary section of \(E\) over \(U\), then (3.2) and (3.3) imply

\[
D\zeta = \sum_{i=1}^{r} s_i d\zeta^i + \sum_{i,j=1}^{r} s_i \zeta^j \omega^j_i.
\]

We call \(D\zeta\) the covariant derivative of \(\zeta\).

Evaluating \(D\) on a tangent vector \(X\) of \(M\) at the point \(x\), we obtain an element of the fibre \(E_x\) denoted by

\[
DX\zeta = (D\zeta)(X) \in E_x
\]

**Definition 3.1.4.** A section \(\zeta\) of \(E\) is said to be parallel if \(D\zeta = 0\). If \(c = c(t), 0 < t < a\), is a curve in \(M\), a section \(\zeta\) defined along \(c\) if

\[
D_{c(t)}\zeta = 0 \quad \text{for } 0 \leq t \leq a,
\]  

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In terms of the local frame field \( s \), (3.4) can be written as a system of ordinary differential equations
\[
\frac{d\zeta^i}{dt} + \sum_{j=1}^{r} \omega^j_i (c'(t)) \zeta^j = 0.
\]

We shall now study how the connection form \( \omega \) changes when we change the local frame field \( s \). Let \( s' = (s'_1, \ldots, s'_r) \) be another local frame field over \( U \). It is related to \( s \) by
\[
s = s'a,
\]
where \( a : U \rightarrow GL(r, \mathbb{C}) \) is a matrix-valued function on \( U \). Let \( \omega' \) be the connection form of \( D \) with respect to \( s' \). Then
\[
\omega = a^{-1} \omega' a + a^{-1} da. \tag{3.5}
\]
In fact we have,
\[
s\omega = Ds = D(s'a) = (Ds')a + s'da = s'\omega'a + s'da = s(a^{-1} \omega'a + a^{-1} da).
\]
We extend a connection \( D : A^0(E) \rightarrow A^1(E) \) to a \( \mathbb{C} \)-linear map
\[
D : A^p(E) \rightarrow A^{p+1}(E), \quad \text{for } p \geq 0.
\]
by setting
\[
D(\eta \otimes s) = d\eta \otimes s + (-1)^p \eta \otimes Ds \quad \text{for } \eta \in A^p \quad \text{and } s \in A^0(E).
\]

**Definition 3.1.5.** Using this extended \( D \), we define the curvature \( R \) of the connection \( D \) to be
\[
R = D \circ D : A^0(E) \rightarrow A^2(E).
\]
Then \( R \) is \( A^0 \)-linear. In fact, if \( f \in A^0 \) and \( \sigma \in A^0(E) \), then
\[
D^2(fs) = D(df \otimes s + fDs) = D(df \otimes s) + D(fDs) = -df \otimes Ds + df \otimes Ds + fD^2s = D^2s.
\]
Hence \( R \) is a 2-form on \( M \) with values in \( \text{End}(E) \), i.e., \( R \in A^2(\text{End}(E)) \).

**Definition 3.1.6.** Using the matrix notations of (3.3) the curvature form \( \Omega \) of \( D \) with respect to the frame field \( s \) is defined by
\[
s\Omega = D^2s.
\]
Then
\[
\Omega = d\omega + \omega \wedge \omega. \tag{3.6}
\]
In fact,
\[
s\Omega = D(s\omega) = Ds \wedge \omega + sd\omega = s(\omega \wedge \omega + d\omega).
\]
Exterior differentiation of (3.6) gives Bianchi identity:
\[
d\Omega = \Omega \wedge \omega - \omega \wedge \Omega = [\Omega, \omega]. \tag{3.7}
\]
If \( \omega' \) is the connection form of \( D \) relative to another frame field \( s' = sa^{-1} \), the corresponding curvature form \( \Omega' \) is related to \( \Omega \) by
\[
\Omega = a^{-1} \Omega a. \tag{3.8}
\]
In fact,
\[ s\Omega = D^2 s = D^2(s'a) = D(Ds'a + s'da) = \\
D^2 s'a - Ds' \wedge da + Ds' \wedge da = \\
= s'\Omega a = s(a^{-1}\Omega a). \]

Let \( \{ U_\alpha \} \) be an open cover of \( M \) with local frame field \( s_\alpha \) on each \( U_\alpha \). If \( U_\alpha \cap U_\beta \neq \emptyset \), then
\[ s_\alpha = s_\beta g_{\beta\alpha} \quad \text{on } U_\alpha \cap U_\beta. \]
where \( g_{\beta\alpha} : U_\alpha \cap U_\beta \to GL(r, \mathbb{C}) \) is the \( \mathcal{C}^\infty \) transition function. Then we have:
\[ \omega_\alpha = g_{\beta\alpha}^{-1} \omega_\beta g_{\beta\alpha} + g_{\beta\alpha}^{-1} d g_{\beta\alpha} \quad \text{on } U_\alpha \cap U_\beta. \]
(3.9)

Conversely, given a system of \( gl(r, \mathbb{C}) \)-valued \( 1 \)-forms \( \omega_\alpha \) on \( U_\alpha \) satisfying (3.9), we obtain a connection \( D \) in \( E \) having \( \{ \omega_\alpha \} \) as connection forms. If \( \Omega_U \) is the curvature form of \( D \) relative to \( s_\alpha \), then (3.8) means
\[ \Omega_\alpha = g_{\beta\alpha}^{-1} \Omega_\beta g_{\beta\alpha} \quad \text{on } U_\alpha \cap U_\beta. \]

**Definition 3.1.7.** Let \( E \) be a \( \mathcal{C}^\infty \) complex vector bundle over a real manifold \( M \). Let \( E^* \) be the dual vector bundle of \( E \). The duality pairing
\[ \langle \cdot, \cdot \rangle : E^*_x \times E_x \to \mathbb{C} \]
induces a duality pairing
\[ \langle \cdot, \cdot \rangle : A^0(E^*) \times A^0(E) \to A^0. \]

Given a connection \( D \) in \( E \), we define a connection, also denoted by \( D \), in \( E^* \) by the following formula:
\[ d\langle \zeta, \eta \rangle = \langle D\zeta, \eta \rangle + \langle \zeta, D\eta \rangle \quad \text{for } \zeta \in A^0(E) \quad \text{and } \eta \in A^0(E^*). \]

**Proposition 3.1.1.** Let \( E \) and \( F \) two complex vector bundles over the same manifold \( M \). Let \( D_E \) and \( D_F \) be connections in \( E \) and \( F \), respectively. Then we can define connections
1. \( D_E \oplus D_F \) in the direct sum \( E \oplus F \) in the obvious way,
2. \( D_E \otimes D_F \) in the tensor product \( E \otimes F \), by \( D_E \otimes D_F = D_E \otimes I_F + I_E \otimes D_F \)
If we denote the curvatures of \( D_E \) and \( D_F \) by \( R_E \) and \( R_F \), then we have
1. \( D_E \oplus D_F \) has curvature \( R_E \oplus R_F \),
2. \( D_E \otimes D_F \) has curvature \( R_E \otimes I_F + I_E \otimes R_F \).

If \( s = (s_1, \ldots, s_r) \) is a local frame field of \( E \) and \( t = (t_1, \ldots, t_p) \) is a local frame field of \( F \) and \( \omega_E, \omega_F, \Omega_E, \Omega_F \), are the connections and the curvature forms with respect to these frame fields, then in a natural manner the connection and curvature forms of \( D_E \oplus D_F \) are given by
\[
\begin{pmatrix}
    \omega_E & 0 \\
    0 & \omega_F
\end{pmatrix}
\quad \text{and} \quad 
\begin{pmatrix}
    \Omega_E & 0 \\
    0 & \Omega_F
\end{pmatrix},
\]

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while those of $D_{E \otimes F}$ are given by
\[
\omega_E \otimes I_p + I_r \otimes \omega_F \quad \text{and} \quad \Omega_E \otimes I_p + I_r \otimes \Omega_F.
\]

Here $I_r$ and $I_p$ denote the identity matrices of rank $r$ and $p$. All these formulas extend in an obvious way to the direct sum and the tensor product of any number of vector bundles and so they give formulas for the connection and curvature in
\[
E^{\otimes p} \otimes E^{\otimes q} = E \otimes \cdots \otimes E^* \otimes \cdots \otimes E^*
\]

Let $E$ be a complex vector bundle over $M$ and let $N$ be another manifold. Given a $\mathcal{C}^\infty$ mapping $f : N \rightarrow M$, we obtain an induced vector bundle $f^* E$ over $N$.

Since in the category of $\mathcal{C}^\infty$ bundles there is an isomorphism $f^* E \cong f^{-1} E \otimes_f A_M A_N$, if $D$ is a connection on $E$, there is a pull-back connection $f^* D$ on $f^* E$ defined by
\[
(f^* D)(\sum \varphi_j s_j) = \sum \varphi_j D(s_j) + \sum d\varphi_j \otimes s_j,
\]
where $\varphi_j \in A^0_N$ are $\mathcal{C}^\infty$ functions on $N$ and $s_j \in \Gamma(M, E)$ are $\mathcal{C}^\infty$ sections of $E$.

If $\omega$ and $\Omega$ are respectively the connection form and the curvature form of $D$ over a local frame field $(U_\alpha, s_\alpha)$, then $f^* \omega$ and $f^* \Omega$ are the connection form and the curvature form with respect to the pull-back local frame field $(f^{-1} U, f^* s)$.

### 3.2 Connections in complex vector bundles (over complex manifolds)

Let $M$ be a complex manifold with (complex) dimension $n$ and $E$ a $\mathcal{C}^\infty$ complex vector bundle of rank $r$ over $M$. In addition to the notations $A^p$ and $A^p(E)$ introduced in the previous section, we use the following:

- $A^{p,q} = \Gamma(M, \bigwedge_p T^* M^{C^+} \otimes \bigwedge_q T^* M^{C^-})$ the space of complex $(p,q)$-forms over $M$,
- $A^{p,q}(E) = \Gamma(M, \bigwedge_p T^* M^{C^+} \otimes \bigwedge_q T^* M^{C^-} \otimes E)$ the space of complex $(p,q)$-forms over $M$ with values in $E$.

Let $D$ be a connection in $E$. We can write $D = D' + D''$, where
\[
D' : A^{p,q}(E) \rightarrow A^{p+1,q}(E) \quad \text{and} \quad D'' : A^{p,q}(E) \rightarrow A^{p,q+1}(E).
\]

Decomposing $D$ according to the bidegree, we have, for $\sigma \in A^0(E)$ and $\phi \in A^{p,q}$,
\[
D'(\sigma \phi) = D'\sigma \wedge \phi + \sigma d'\phi,
\]
\[
D''(\sigma \phi) = D''\sigma \wedge \phi + \sigma d''\phi.
\]

Let $R$ be the curvature of $D$, i.e., $R = D \circ D \in A^2(\text{End}(E))$. Then
\[
R = D' \circ D' + (D' \circ D'' + D'' \circ D') + D'' \circ D'',
\]
where $D' \circ D' \in A^{2,0}(\text{End}(E))$, and $D'' \circ D'' \in A^{0,2}(\text{End}(E))$, while $D' \circ D'' + D'' \circ D' \in A^{1,1}(\text{End}(E))$.

Let $s$ be a local frame field of $E$ and let $\omega$ and $\Omega$ be the connection and the curvature forms of $D$ with respect to $s$. We can write

$$\omega = \omega^{1,0} + \omega^{0,1},$$

$$\Omega = \Omega^{2,0} + \Omega^{1,1} + \Omega^{0,2}.$$

Let $M$ be a complex manifold of (complex) dimension $n$, and let $E$ be a holomorphic vector bundle of rank $r$ over $M$. Let $\{U_\alpha\}$ be an open cover which trivializes $E$ and let $s_\alpha = (s_\alpha^1, \ldots, s_\alpha^r)$ be a holomorphic frame field on $U_\alpha$. Let $\zeta \in A^0(E)$ be a $C^\infty$ section. On $U_\alpha$ we write

$$\zeta = \sum \zeta^j_\alpha s_j^\alpha,$$

where $\zeta^j_\alpha$ are $C^\infty(U_\alpha, \mathbb{C})$ functions. Then we set

$$d'' E(\zeta) = \sum_j d''(\zeta^j_\alpha)s_j^\alpha \quad \text{on } U_\alpha.$$

**Proposition 3.2.1.** $d'' E : \Gamma(M, E) \rightarrow \Gamma(M, E \otimes TM^C)$ is well defined and

$$d'' E(f \zeta) = d''(f) \otimes \zeta + f d'' E(\zeta) \quad \text{for } f \in A^0, \zeta \in A^0(E).$$

Moreover, $d'' E \circ d'' E = 0$.

**Proof.** First of all we prove that $d'' E$ is well defined. Let $s_\beta = (s_\beta^1, \ldots, s_\beta^r)$ be a holomorphic frame field on $U_\beta$. On the overlapping open set $U_\alpha \cap U_\beta$, $s_\beta$ is related to $s_\alpha$ by

$$s_\alpha = s_\beta \cdot a_{\alpha \beta},$$

where $a = a_{\alpha \beta} : U_\alpha \cap U_\beta \rightarrow GL(r, \mathbb{C})$ is a holomorphic matrix-valued function on $U_\alpha \cap U_\beta$. $a$ is a holomorphic matrix-valued function, so that $d''(a_{\alpha \beta}^j) = 0$. Then we have

$$d'' E(\zeta) = \sum d''(\zeta^j_\alpha)s_j^\alpha = \sum d''(\zeta^j_\alpha)a_{\alpha \beta}^j s_j^\beta =$$

$$= \sum d''(\zeta^j_\alpha a_{\alpha \beta}^j)s_j^\beta = \sum d''(\zeta^j_\beta)s_j^\beta = d'' E(\zeta),$$

and this proves that $d'' E$ is well defined.

In order to prove that $d'' E(f \zeta) = d''(f) \otimes \zeta + f d'' E(\zeta)$, let $f \in A^0$ and $\zeta \in A^0(E)$. After a straightforward computation we find

$$d'' E(f \zeta) = \sum d''(f \zeta^j_\alpha)s_j^\alpha =$$

$$= \sum d''(f) \zeta^j_\alpha s_j^\alpha + \sum f d''(\zeta^j_\alpha)s_j^\alpha =$$

$$= d''(f) \otimes \sum \zeta^j_\alpha s_j^\alpha + f \sum d''(\zeta^j_\alpha)s_j^\alpha =$$

$$= d''(f) \otimes \zeta + f d'' E(\zeta).$$

Finally, from $d'' \circ d'' = 0$ we have $d'' E \circ d'' E = 0$, and this completes the proof. \qed
We shall now describe those complex vector bundles which admit holomor-
phic structures. Let $A^{p,q}$ be the sheaf of complex $(p,q)$-forms over $M$ and let $A^{p,q}(E) = E \otimes A^{p,q}$ be the sheaf of complex $(p,q)$-forms over $M$ with values in $E$. Then we have the following

**Theorem 3.2.2.** Let $M$ be a complex manifold of (complex) dimension $n$ and let $E$ be a complex vector bundle of rank $r$ over $M$. The following conditions are equivalent:

1. There exists a holomorphic vector bundle structure on $E$,
2. There exists an operator $d''_E : A^{0,0}(E) \to A^{0,1}(E)$ such that
   
   \[(a) \quad d''_E(f\zeta) = d''(f) \otimes \zeta + f d''_E(\zeta) \quad \text{for} \quad f \in A^{0,0}, \zeta \in A^{0,0}(E),\]
   
   \[(b) \quad d''_E \circ d''_E = 0.\]

Moreover, we have the following results:

**Proposition 3.2.3.** Let $E$ be a holomorphic vector bundle over a complex manifold $M$. Then there exists a connection $D$ such that

$$D'' = d''_E$$

For such a connection, the $(0,2)$-component $D'' \circ D''$ of the curvature $R$ vanishes.

**Proof.** Let $\{U\}$ be a locally finite open cover of $M$ and let $\{\rho_U\}$ a partition of unity subordinate to $\{U\}$. Let $s_U$ be a holomorphic frame field of $E$ on $U$ and let $D_U$ be the flat connection in $E|_U$ defined by $D_U(s_U) = 0$. Then $D = \sum \rho_U D_U$ is a connection in $E$ with the property that $D'' = d''_E$. The second assertion is obvious, since $d''_E \circ d''_E = 0$.

**Proposition 3.2.4.** Let $E$ be a $C^\infty$ complex vector bundle over a complex manifold $M$. If $D$ is a connection in $E$ such that $D'' \circ D'' = 0$, then there is a unique holomorphic vector bundle structure in $E$ such that $D'' = d''_E$.

**Proof.** See [25] for a detailed proof.

**Proposition 3.2.5.** For a connection $D$ in a holomorphic vector bundle $E$, the following conditions are equivalent:

1. $D'' = d''_E$,
2. For every local holomorphic section $s$, $Ds$ is of degree $(1,0)$,
3. With respect to a local holomorphic frame field, the connection form $\omega$ is of degree $(1,0)$.

### 3.3 Connections in Hermitian vector bundles

**Definition 3.3.1.** Let $E$ be a $C^\infty$ complex vector bundle over a (real or complex) manifold $M$. A Hermitian structure or Hermitian metric $h$ in $E$ is a $C^\infty$ field of Hermitian inner products in the fibre of $E$. Thus,

1. $h(\zeta, \eta)$ is $C$-linear in $\zeta$, where $\zeta, \eta \in E_x$,
2. \( h(\zeta, \eta) = h(\eta, \zeta) \),
3. \( h(\zeta, \zeta) > 0 \) for \( \zeta \neq 0 \),
4. \( h(\zeta, \eta) \) is a \( C^\infty \) function if \( \zeta \) and \( \eta \) are \( C^\infty \) sections.

We call \((E, h)\) a Hermitian vector bundle.

Given a local frame field \( s_\alpha = (s_1, \ldots, s_r) \) of \( E \) over \( U_\alpha \), we set
\[
h_{ij} = h(s_i, s_j) \quad \text{for } i, j = 1, \ldots, r
\]
and
\[
H_\alpha = (h_{ij})
\]
Then \( H_\alpha \) is a positive definite Hermitian matrix at every point of \( U_\alpha \). When we are working with a single frame field, we often drop the subscript \( \alpha \).

**Definition 3.3.2.** A connection \( \mathcal{D} \) in \((E, h)\) is called an \( h \)-connection if it preserves \( h \), i.e., if it makes \( h \) parallel in the following sense:
\[
d(h(\zeta, \eta)) = h(D\zeta, \eta) + h(\zeta, D\eta) \quad \text{for } \zeta, \eta \in \Omega^0(E).
\]
Let \( \omega = (\omega^i_j) \) be the connection form relative to the local frame field \( s_U \). Then setting \( \zeta = s_i \) and \( \eta = s_j \) in (3.11), we obtain
\[
dh_{ij} = h(Ds_i, s_j) + h(s_i, Ds_j) = \omega^a_i h_{aj} + h_{ij} \omega^b_j.
\]
So in matrix notation we have
\[
dH = \omega^T H + H \overline{\omega}.
\]
Applying \( d \) to (3.3) we obtain
\[
\Omega^T H + H \Omega = 0.
\]
If \( E \) is a holomorphic vector bundle over a complex manifold \( M \), then a Hermitian structure \( h \) determines a natural \( h \)-connection satisfying \( D'' = d''_E \).
Namely, we have

**Proposition 3.3.1.** Given a Hermitian structure \( h \) in a holomorphic vector bundle \( E \) over a complex manifold \( M \), there is a unique \( h \)-connection \( D'' \) such that \( D'' = d''_E \).

**Proof.** 1. First all we prove the uniqueness. Let \( D \) be such a connection and let \( s_\alpha = (s_1, \ldots, s_r) \) be a local frame field on \( U_\alpha \). Since \( Ds_i = D's_i \), the connection form \( \omega_\alpha = (\omega^i_j) \) is of degree \( (1, 0) \). From (3.12) we obtain
\[
d'h_{ij} = \omega^a_i h_{aj},
\]
or, in matrix notation,
\[ d'H_\alpha = \omega^\alpha_\alpha H_\alpha. \]

This determines the connection form \( \omega_\alpha \), i.e.,
\[ \omega^\alpha_\alpha = d'H_\alpha H_\alpha^{-1}. \]

Hence, we have proved uniqueness.

2. Now, we want to prove existence. Let \( \omega^\alpha_\alpha = d'H_\alpha H_\alpha^{-1} \). By a straightforward calculation we can see that the collection \( \{ \omega_\alpha \} \) satisfies (3.9)
\[ \omega_\alpha = g^{-1}_{\beta\alpha} \omega_\beta g_{\beta\alpha} + g^{-1}_{\beta\alpha} dg_{\beta\alpha} \quad \text{on } U_\alpha \cap U_\beta, \]
and this completes the proof. \( \square \)

**Definition 3.3.3.** The connection given by the previous Proposition is called the Hermitian connection of the holomorphic Hermitian vector bundle \( (E,h) \).

**Proposition 3.3.2.** The curvature of the Hermitian connection in a holomorphic vector bundle is of degree \((1,1)\). If \( (E,h) \) is a \( \mathcal{C}^\infty \) complex vector bundle over a complex manifold \( M \) with an Hermitian structure \( h \) and \( D \) is an \( h \)-connection whose curvature is of degree \((1,1)\), then there is a unique holomorphic structure in \( E \) which makes \( D \) the Hermitian connection of the vector bundle \( (E,h) \).

**Proof.** Let \( D \) be a Hermitian connection in the holomorphic Hermitian vector bundle \( (E,h) \) over the complex manifold \( M \). Its connection form is given locally by \( \omega^\alpha_\alpha = d'H_\alpha H_\alpha^{-1} \), and its curvature \( R \) has no \((0,2)\)-components since \( D'' \circ D'' = d'^E \circ d''_E = 0 \). By (3.13) it has no \((2,0)\)-components either. So the curvature is a \((1,1)\)-form with values in \( \text{End}(E) \)
\[ R = D' \circ D'' + D'' \circ D' \in A^{1,1}(E). \]

The second part of this Proposition follows 3.2.4 (See [25], p. 9 and p. 12 for more details). \( \square \)

With respect to a local holomorphic frame field the connection form \( \omega = \omega^i_j \) is of degree \((1,0)\). Since the curvature form \( \Omega \) is equal to the \((1,1)\)-component of \( d\omega + \omega \wedge \omega \), we obtain
\[ \Omega = d''\omega. \]

From \( \omega^\alpha_\alpha = d'H_\alpha H_\alpha^{-1} \) we obtain
\[ \Omega^i = d''d'HH^{-1} + d'H H^{-1} \wedge d''HH^{-1}. \]

In local coordinates we write
\[ \Omega^i_j = R^i_{j\alpha\beta}dz^\alpha \wedge dz^\beta. \]
3.4 Subbundles and quotient bundles

Let $E$ be a holomorphic vector bundle of rank $r$ over a complex manifold $M$ of (complex) dimension $n$ and let $S$ be a holomorphic subbundle of rank $p$ of $E$. Then the quotient bundle $Q = E/S$ is a holomorphic vector bundle of rank $r - p$. We can express this situation as an exact sequence

$$0 \rightarrow S \rightarrow E \rightarrow Q \rightarrow 0.$$ 

Let $h$ be a Hermitian structure in $E$. Restricting $h$ to $S$, we obtain a Hermitian structure $h_S$ in $S$. Taking the orthogonal complement of $S$ in $E$ with respect to $h$, we obtain a complex subbundle $S^\perp$ of $E$.

**Note 3.4.1.** The complex subbundle $S^\perp$ of $E$ may not be a holomorphic subbundle of $E$ in general. Thus

$$E = S \oplus S^\perp$$

is merely a $C^\infty$ orthogonal decomposition of $E$.

As a $C^\infty$ complex vector bundle, $Q$ is naturally isomorphic to $S^\perp$. Hence, we obtain also a Hermitian structure $h_Q$ in a natural way.

**Definition 3.4.1.** Let $D$ denote the Hermitian connection in $(E, h)$. We define $D_S$ and $A$ by

$$D\zeta = D_S \zeta + A\zeta \quad \text{for } \zeta \in A^0(S), \quad (3.14)$$

where $D_S \zeta \in A^1(S)$ and $A\zeta \in A^1(S^\perp)$.

**Proposition 3.4.2.** Under the hypothesis of the previous definition, we have the following results:

1. $D_S$ is the Hermitian connection of $(S, h_S)$,

2. $A$ is a $(1,0)$-form with values in $\operatorname{Hom}(S, S^\perp)$, i.e., $A \in A^{1,0}(\operatorname{Hom}(S, S^\perp))$.

**Proof.** Let $f$ be a function on $M$. Replacing $\zeta$ by $f\zeta$ in (3.14), we obtain

$$D(f\zeta) = D_S(f\zeta) + A(f\zeta).$$

On the other hand,

$$D(f\zeta) = df\zeta + fD\zeta = df\zeta + fD_S\zeta + fA\zeta.$$ 

Comparing the components of the two decompositions of $D(f\zeta)$, we conclude

$$D_S(f\zeta) = df\zeta + fD_S\zeta \quad \text{and} \quad A(f\zeta) = fA\zeta. \quad (3.15)$$

The first equality says that $D_S$ is a connection and the second says that $A$ is a $1$-form with values in $\operatorname{Hom}(S, S^\perp)$. If $\zeta$ in $D\zeta = D_S\zeta + A\zeta$ is holomorphic, then $D\zeta$ is a $(1,0)$-form with values in $E$. Hence, $D_S\zeta$ is a $(1,0)$-form with values in $S$ while $A$ is a $(1,0)$-form with values in $\operatorname{Hom}(S, S^\perp)$. Finally, if $\zeta, \zeta' \in A^0(S)$, then...
\[ d(h(\zeta, \zeta')) = h(D\zeta, \zeta') + h(\zeta, D\zeta') = \\
= h(DS\zeta + A\zeta, \zeta') + h(\zeta, DS\zeta' + A\zeta') = \\
= h(DS\zeta, \zeta') + h(\zeta, DS\zeta'), \]

and this proves that \( DS \) preserves \( h_S \).

**Definition 3.4.2.** We call \( A \in A^{1,0}(\text{Hom}(S, S^\perp)) \) the second fundamental form of \( S \) in \((E, h)\). With the identification \( Q = S^\perp \), we can consider \( A \) as an element of \( A^{1,0}(\text{Hom}(S, Q)) \).

**Definition 3.4.3.** Similarly, we define \( DS^\perp \) and \( B \) by setting

\[ D\eta = B\eta + D_{S^\perp}\eta \quad \text{for} \quad \eta \in A^0(S^\perp), \]

where \( B\eta \in A^1(S) \) and \( D_{S^\perp}\eta \in A^1(S^\perp) \). Under the identification \( Q = S^\perp \), we may consider \( D_{S^\perp} \) as a mapping \( A^0(Q) \rightarrow A^1(Q) \). The we write \( D_Q \) in place of \( D_{S^\perp} \).

**Proposition 3.4.3.** In the hypothesis of the previous definitions and constructions we have the following results:

1. \( D_Q \) is the Hermitian connection of \((Q, h_Q)\),

2. \( B \) is a \((0,1)\)-form with values in \( \text{Hom}(S^\perp, S) \), i.e., \( B \in A^{0,1}(\text{Hom}(S^\perp, S)) \),

3. \( B \) is the adjoint of \(-A \), i.e.,

\[ h(A\zeta, \eta) + h(\zeta, B\eta) = 0 \quad \text{for} \quad \zeta \in A^0(S) \quad \text{and} \quad \eta \in A^0(S^\perp). \]

**Proof.** The proof is very similar to the previous one. See [25] for more details.
Chapter 4

Chern Classes

4.1 Chern classes of a line bundle

In this section we define the first Chern class of a line bundle. We start with some useful definitions.

**Definition 4.1.1.** Consider $\mathbb{P}^n(\mathbb{C})$ with the open covering $\{U_i\}$, where $U_i = \{z^i \neq 0\}$ and consider the holomorphic line bundle over $\mathbb{P}^n(\mathbb{C})$ with transition functions $g_{ij} = \frac{z^i}{z^j}$. We will call it the universal line bundle over $\mathbb{P}^n(\mathbb{C})$.

**Definition 4.1.2.** Let $L$ be the universal line bundle over $\mathbb{P}^n(\mathbb{C})$, and let $H = \{a_0 z^0 + \cdots + a_n z^n = 0\}$ be an hyperplane in $\mathbb{P}^n(\mathbb{C})$. If we take non-homogeneous coordinates we have

$$
\sum a_k z^k = z^i \left( a_0 \frac{z^0}{z^i} + \cdots + a_i + \cdots + a_n \frac{z^n}{z^i} \right) \quad \text{on } U_i.
$$

So on $U_i \cap U_j$ we have

$$
\left( a_0 \frac{z^0}{z^i} + \cdots + a_i + \cdots + a_n \frac{z^n}{z^i} \right) = \left( a_0 \frac{z^0}{z^j} + \cdots + a_i + \cdots + a_n \frac{z^n}{z^j} \right) \frac{z^i}{z^j} \frac{z^j}{z^i} = \sum a_k z^k \frac{z^j}{z^i} = \frac{z^j}{z^i}.
$$

Because of this origin the latter bundle, to be denoted by $H$, is called the hyperplane section bundle of $\mathbb{P}^n(\mathbb{C})$. It is the dual bundle of the universal line bundle over $\mathbb{P}^n(\mathbb{C})$.

**Definition 4.1.3.** Let $M$ be a differentiable manifold. We define the Picard group of $M$ as

$$
\text{Pic}(M) = \{\text{isomorphism classes of complex line bundles over } M\}
$$

This is a group with the operation $[L_1] \otimes [L_2] = [L_1 \otimes L_2]$.

**Definition 4.1.4.** Let $M$ be a differentiable manifold and let $\mathcal{A}$ (resp. $\mathcal{A}^*$) be the sheaf of germs of $C^\infty$ complex functions (resp. nowhere vanishing complex functions) over $M$. Consider the exact sequence of sheaves:

$$
0 \to \mathcal{Z} \to \mathcal{A} \to \mathcal{A}^* \to 1,
$$

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where \( j : \mathbb{Z} \to \mathcal{A} \) is simply the natural injection and \( e : \mathcal{A} \to \mathcal{A}^* \) is the exponential map
\[
e(f) = \exp(2\pi i f) \quad \text{for} \quad f \in \mathcal{A}.
\]
This induces an exact sequence of cohomology groups
\[
\cdots \xrightarrow{j^*} H^1(M, \mathcal{A}) \xrightarrow{e^*} H^1(M, \mathcal{A}^*) \xrightarrow{\delta} H^2(M, \mathbb{Z}) \xrightarrow{j^*} H^2(M, \mathcal{A}) \xrightarrow{e^*} \cdots
\]
where \( \delta \) is the connecting homomorphism. Since \( \mathcal{A} \) is a fine sheaf over the differentiable manifold \( M \), we have
\[
H^p(M, \mathcal{A}) = 0 \quad \text{for} \quad p \geq 1.
\]
Then \( \delta : H^1(M, \mathcal{A}^*) \to H^2(M, \mathbb{Z}) \) is a group isomorphism. Identifying \( H^1(M, \mathcal{A}^*) \) with the Picard group \( \text{Pic}(M) \) of \( M \), (see Theorem 5.1.4 for details), we can define the first Chern class of a complex line bundle \( L \) over \( M \) by
\[
c_1(L) = \delta(L), \quad L \in H^1(M, \mathcal{A}^*).
\]
We also set \( c_0(L) = 1 \) and \( c(L) = 1 + c_1(L) \).

**Remark 4.1.1.** From the above definition we immediately see that

1. Two line bundles over \( M \) are isomorphic, if and only if their first Chern classes coincide,
2. For two line bundles \( L_1 \) and \( L_2 \) over \( M \) we have
\[
c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2).
\]

**Note 4.1.2.** Let \( \{U_\alpha\} \) be an open cover which trivializes the complex line bundle \( L \), and let be \( g_{\alpha\beta} \) its transition functions. From the definition of the connecting homomorphism we can deduce an explicit formula for a Čech cocycle representing \( c_1(L) \) with respect to the open cover \( \{U_\alpha\} \) :
\[
\{c_1(L)\}_{\alpha\beta\gamma} = \frac{1}{2\pi i} (\ln g_{\alpha\beta} + \ln g_{\beta\gamma} + \ln g_{\gamma\alpha}). \quad (4.1)
\]

**Note 4.1.3.** If \( M \) is a complex manifold, we can define Chern classes of holomorphic line bundles in a similar way. (See Chern [11], Chapter VI for more details).

### 4.2 Chern classes of a complex vector bundle

In this section we define higher Chern classes for complex vector bundles of any rank. We proceed in two steps:

1. We first define Chern classes of vector bundles that are direct sums of line bundles,
2. We show that we can always reduce the computation of Chern classes to the previous case.
Definition 4.2.1. For $i = 1, \ldots, k$ let $\sigma_i$ denote the symmetric function of order $i$ in $k$ arguments, defined as

$$\sigma_i(x_1, \ldots, x_k) = \sum_{1 \leq j_1 < \cdots < j_i \leq k} x_{j_1} \cdots x_{j_i}.$$

Definition 4.2.2. Let $M$ be a differentiable manifold and let $E$ be a complex vector bundle of rank $r$ over $M$. Let us assume $E = L_1 \oplus \cdots \oplus L_r$ is the direct sum of complex line bundles over $M$. For $i = 1, \ldots, k$ we define the $i$-th Chern class of $E$ as

$$c_i(E) = \sigma_i(c_1(L_1), \ldots, c_1(L_k)) \in H^{2i}(M, \mathbb{Z}),$$

where

$$\sigma_i(c_1(L_1), \ldots, c_1(L_k)) = \sum_{1 \leq j_1 < \cdots < j_i \leq k} c_1(L_{j_1}) \cup \cdots \cup c_1(L_{j_i})$$

and $\cup$ denotes the cup product.

Step 2 relies on the following result, sometimes called the splitting principle. (See Bott-Tu [4], p. 273-278, for more details). Our proof uses the de Rham cohomology of $M$, but with some little changes one can prove the splitting principle also for the cohomology with coefficient in $\mathbb{Z}$.

Theorem 4.2.1. (Splitting principle) Let $E$ be a complex vector bundle of rank $r$ over a differentiable manifold $M$. There exists a differentiable manifold $N$ (also called splitting manifold) and a $C^\infty$ mapping $f : N \rightarrow M$ such that

1. the pull-back bundle $f^* E$ is a direct sum of line bundles,
2. the cohomology morphism $f^*: H^*(M) \rightarrow H^*(N)$ is injective,
3. the Chern classes $c_i(f^* E)$ lie in the image of the cohomology morphism $f^*$.

Proof. Let $\tau : E \rightarrow M$ be a $C^\infty$ complex vector bundle of rank $r$ over a differentiable manifold $M$. Our goal is to construct the splitting manifold $N = F(E)$ and the splitting map $f : F(E) \rightarrow E$. We prove this Theorem by induction on the rank of $E$.

1. If $E$ has rank 1, there is nothing to prove.
2. If $E$ has rank 2, we can take as a splitting manifold $F(E)$ the projective bundle $\mathbb{P}(E)$, which is by definition the fibre bundle over $M$ whose fibre at a point $P \in M$ is the projective space $\mathbb{P}(E_P)$ and whose transition functions are $\tilde{g}_{\alpha \beta} : U_{\alpha} \cap U_{\beta} \rightarrow \text{PGL}(r, \mathbb{C})$, induced from the transition functions $g_{\alpha \beta}$ of $E$. As on the projectivization of a vector space, on $\mathbb{P}(E)$ there are several tautological bundles: the pull-back bundle $\pi^* E$, the universal subbundle $S_E$ and the universal quotient bundle $Q_E$. 

\[
\begin{array}{ccccccc}
0 & \longrightarrow & S_E & \longrightarrow & \pi^* E & \longrightarrow & Q_E & \longrightarrow & 0 \\
\tau^* & & & & \downarrow \tau & & & & \\
\mathbb{P}(E) & & & & E & & & & M \\
\pi & & & & \downarrow \tau & & & & \\
& & & & M & & & & \\
\end{array}
\]
Here the universal subbundle $S_E$ over $\mathbb{P}(E)$ is defined by

$$S_E = \{(l_P, v) \in \pi^*E | v \in l_P\},$$

while the quotient bundle $Q_E$ is determined by the tautological exact sequence

$$0 \to S_E \to \pi^*E \to Q_E \to 0.$$  

If $E$ has rank 2 we have $\pi^*E = S_E \oplus Q_E$, which is a direct sum of line bundles.

3. Now suppose $E$ has rank 3. Over $\mathbb{P}(E)$ the line bundle $S_E$ splits off as before. The quotient bundle $Q_E$ over $\mathbb{P}(E)$ has rank 2 and so can be split into a direct sum of line bundles when pulled back to $\mathbb{P}(Q_E)$.

$$\beta^*S_E \oplus S_E \oplus Q_{Q_E}$$

$$\begin{array}{c}
\mathbb{P}(Q_E) \\
\downarrow \\
\mathbb{P}(E) \\
\downarrow \\
\alpha \\
\tau \\
M
\end{array}$$

4. The pattern is now clear, we split off one subbundle at a time by pulling back to the projectivization of a quotient bundle (see Figure 4.1 for details).

Setting $F(E) = \mathbb{P}(Q_{n-2})$, this is the splitting manifold and this completes the proof.

\[\square\]

**Definition 4.2.3.** Let $E$ be a complex vector bundle of rank $r$ over a differentiable manifold $M$. The $i$-th Chern class $c_i(E)$ of $E$ is the unique class in $H^{2i}(M, \mathbb{Z})$ such that $f^*(c_i(E)) = c_i(f^*E)$. We also set $c_0(E) = 1$ and we define the total Chern class of $E$ as $c(E) = \sum_{i=0}^r c_i(E)$.

**Proposition 4.2.2.** The Chern classes of a complex vector bundle $E$ of rank $r$ over a differentiable manifold $M$ satisfy the following properties:

1. if two vector bundles $E$ and $F$ over $M$ are isomorphic, their Chern classes coincide,

2. (Naturality): if $f : N \to M$ is a differentiable map and $E$ is a complex vector bundle over $M$, then

$$f^*(c_i(E)) = c_i(f^*E),$$

3. (Whitney product formula): if $E$ and $F$ are complex vector bundles over $M$, then

$$c(E \oplus F) = c(E) \cup c(F)$$
4. (Normalization): if $L$ is the universal line bundle over $\mathbb{P}^1(\mathbb{C})$, then $-c_1(L)$ is the positive generator of $H^2(\mathbb{P}^1(\mathbb{C}), \mathbb{Z})$; in other words, $c_1(L)$ integrated on the fundamental 2-cycle $\mathbb{P}^1(\mathbb{C})$ is equal to $-1$.

Proof. In view of the splitting principle, it is enough to prove the properties (1) and (2) when $E$ and $F$ are line bundles.

1. Follows from the definition of the first Chern class of a line bundle.

2. Let $E$ be a line bundle over $M$ and let $f : N \rightarrow M$ be a differentiable map. Let $\{U_\alpha\}$ be an open cover which trivializes the line bundle $E$ with transition functions $g_{\alpha\beta}$. Then $\{f^{-1}(U_\alpha)\}$ is an open cover which trivializes the pull-back bundle $f^*E$, with transition functions $\tilde{g}_{\alpha\beta} = g_{\alpha\beta} \circ f$. From (4.1) we deduce that:

$$\{c_1(f^*E)\}_{\alpha\beta\gamma} = \frac{1}{2\pi i} (\ln \tilde{g}_{\alpha\beta} + \ln \tilde{g}_{\beta\gamma} + \ln \tilde{g}_{\gamma\alpha}) = \frac{1}{2\pi i} (\ln g_{\alpha\beta} \circ f + \ln g_{\beta\gamma} \circ f + \ln g_{\gamma\alpha} \circ f) = \frac{1}{2\pi i} f^* (\ln g_{\alpha\beta} + \ln g_{\beta\gamma} + \ln g_{\gamma\alpha}) = \{f^* c_1(E)\}_{\alpha\beta\gamma}.$$  

and this completes the proof of (2).

3. By the splitting principle we can assume $E = L_1 \oplus \cdots \oplus L_r$ and $F = L_{r+1} \oplus \cdots \oplus L_{r+q}$ are direct sum of line bundles. Then

$$c(E \oplus F) = c(L_1 \oplus \cdots \oplus L_r \oplus L_{r+1} \oplus \cdots \oplus L_{r+q}) = \sum_{i=0}^{r+q} \sigma_i(L_1, \ldots, L_r, L_{r+1}, \ldots, L_{r+q}) = \prod_{i=1}^{r+q} (1 + c_1(L_i)) = \left[ \prod_{k=1}^{r} (1 + c_1(L_k)) \right] \left[ \prod_{k=1}^{q} (1 + c_1(L_{r+k})) \right] = c(E) \cup c(F).$$

4. It follows from a general results: for any divisors $D \in \text{Div}(X)$ in a compact Riemann surface

$$\int_X c_1(D) = \deg(D),$$

where $c_1(D)$ is the Chern class of the line bundle associated with the divisor $D$. Then, if $H$ is the hyperplane line bundle over $\mathbb{P}^1(\mathbb{C})$, we find

$$\int_{\mathbb{P}^1(\mathbb{C})} c_1(H) = \deg H = 1.$$
Proposition 4.2.3. If \( E = M \times \mathbb{C} \) is the trivial line bundle over the differentiable manifold \( M \), then \( c_1(E) = 0 \).

Proof. It follows immediately from (4.1).

Definition 4.2.4. Let \( E \) be a complex vector bundle of rank \( r \) over the differentiable manifold \( M \) with transition functions \( g_{\alpha \beta} \). The determinant bundle of \( E \) is the line bundle \( \det(E) \) with transition functions \( \tilde{g}_{\alpha \beta} \det(g_{\alpha \beta}) \).

Proposition 4.2.4. Let \( E \) be a complex vector bundle of rank \( r \) over the differentiable manifold \( M \), and let \( \det(E) \) its determinant bundle. Then we have

\[
\c_1(E) = \c_1(\det(E)).
\]

Proof. 1. If \( E \) is a line bundle the result is trivial, since \( E = \det(E) \).

2. From the splitting principle we may assume \( E = L_1 \oplus \cdots \oplus L_r \) is direct sum of line bundles. From (1) we have \( \det(L_1 \otimes \cdots \otimes L_r) = c_1(L_1 \otimes \cdots \otimes L_r) \) and then we have

\[
c_1(\det(E)) = c_1(\det(L_1 \otimes \cdots \otimes L_r)) =
\]
\[
= c_1(L_1 \otimes \cdots \otimes L_r) =
\]
\[
= c_1(L_1) + \cdots + c_1(L_r) = c_1(E).
\]
**Corollary 4.2.5.** Let $E$ be a complex line bundle over $M$ and let $E^*$ its dual bundle. Then $c_1(E^*) = -c_1(E)$.

**Proof.** Obviously $E \otimes E^*$ is isomorphic to the trivial line bundle $M \times \mathbb{C}$ over $M$. Then $0 = c_1(E \otimes E^*) = c_1(E) + c_1(E^*)$ and this completes the proof. □

**Corollary 4.2.6.** Let $E$ be a complex vector bundle of rank $r$ over the differentiable manifold $M$, and let $E^*$ be its dual bundle. Then for $i = 1, \ldots, r$ we have $c_i(E^*) = (-1)^i c_i(E)$.

**Proof.** From the splitting principle we may assume $E = L_1 \oplus \cdots \oplus L_r$ is direct sum of line bundles over $M$. Then $E^* = L_1^* \oplus \cdots \oplus L_r^*$. From the previous corollary and elementary properites of symmetric functions we obtain

$$c_i(E^*) = \sigma_i(c_1(L_1^*), \ldots, c_1(L_r^*))$$

$$= (-1)^i \sigma_i(c_1(L_1), \ldots, c_1(L_r)) = (-1)^i c_i(E).$$

□

### 4.3 Axiomatic approach to Chern classes

In order to minimize topological prerequisites, in this section we take the axiomatic approach to Chern classes. This enables us to separate differential geometry aspects of Chern classes from their topological aspects. We consider the category of complex vector bundles over real manifolds. For more references see Kobayashi [25] and Kobayashi-Nomizu vol. 1 [26].

**Axiom 4.3.1.** For each complex vector bundle $E$ over $M$ and for each integer $0 \leq i \leq \text{rk}(E)$, the $i$-th Chern class $c_i(E) \in H^{2i}(M, \mathbb{R})$ is given and $c_0(E) = 1$. We set

$$c(E) = \sum_{i=0}^{\text{rk}(E)} c_i(E),$$

and call $c(E)$ the total Chern class of $E$.

**Axiom 4.3.2.** (Naturality) Let $E$ be a complex vector bundle over $M$ and let $f : N \rightarrow M$ be a $C^\infty$ mapping. Then

$$c(f^*E) = f^*(c(E)) \in H^*(M, \mathbb{R}).$$

**Axiom 4.3.3.** (Whitney sum formula) Let $E_1, \ldots, E_q$ be complex line bundles over $M$ and let $E_1 \oplus \cdots \oplus E_q$ be their Whitney sum. Then

$$c(E_1 \oplus \cdots \oplus E_q) = c(E_1) \cup \cdots \cup c(E_q).$$

**Axiom 4.3.4.** (Normalization) If $L$ is the universal line bundle over $\mathbb{P}^1(\mathbb{C})$, then $-c_1(L)$ is the positive generator of $H^2(\mathbb{P}^1(\mathbb{C}), \mathbb{Z})$, that is, the one compatible with the orientation of $\mathbb{P}^1(\mathbb{C})$. In other words, $c_1(L)$ integrated on the foundamental 2-cycle $\mathbb{P}^1(\mathbb{C})$ is equal to $-1$. 44
4.4 Chern classes in terms of curvature

In the previous section we introduced the $i$-th Chern class $c_i(E)$ of a complex vector bundle as an element of $H^{2i}(M, \mathbb{R})$. Via the de-Rham theory we should be able to represent $c_i(E)$ by a closed 2i-form $\gamma_i$. In this section we shall construct such a $\gamma_i$ using the curvature form of a connection in $E$. For convenience, we imbed $H^{2i}(M, \mathbb{R})$ into $H^{2i}(M, \mathbb{C})$ and represent $c_i(E)$ by a closed complex 2i-form $\gamma_i$.

Definition 4.4.1. Let $V$ be the Lie algebra $\mathfrak{gl}(r, \mathbb{C})$ of the linear group $GL(r, \mathbb{C})$, i.e., the Lie algebra of all $r \times r$ complex matrices. Let $G$ be $GL(r, \mathbb{C})$ acting on $\mathfrak{gl}(r, \mathbb{C})$ by the adjoint action, i.e.,

$$X \in \mathfrak{gl}(r, \mathbb{C}) \rightarrow aXa^{-1} \in \mathfrak{gl}(r, \mathbb{C}), \quad a \in GL(r, \mathbb{C}).$$

Now we define homogeneous polynomials $f_k$ on $\mathfrak{gl}(r, \mathbb{C})$ of degree $k = 1, \ldots, r$ by

$$\det \left( I_r - \frac{1}{2\pi i} X \right) = 1 + f_1(X) + f_2(X) + \cdots + f_r(X), \quad X \in \mathfrak{gl}(r, \mathbb{C}). \quad (4.2)$$

Since

$$\det \left( I_r - \frac{1}{2\pi i} aXA^{-1} \right) = \det \left( a \left( I_r - \frac{1}{2\pi i} X \right) a^{-1} \right) = \det \left( I_r - \frac{1}{2\pi i} X \right)$$

the polynomials $f_1, \ldots, f_r$ are $GL(r, \mathbb{C})$-invariants. It is known that these polynomials generate the algebra of $GL(r, \mathbb{C})$-invariant polynomials on $\mathfrak{gl}(r, \mathbb{C})$.

Since $GL(r, \mathbb{C})$ is a connected Lie group, the $GL(r, \mathbb{C})$-invariance can be expressed infinitesimally. In fact, we have the following result:

Proposition 4.4.1. A symmetric $\mathbb{C}$-multilinear k-form $f$ on $\mathfrak{gl}(r, \mathbb{C})$ is $GL(r, \mathbb{C})$-invariant if and only if

$$\sum_{j=1}^k f(X_1, \ldots, [Y, X_j], \ldots, X_k) = 0 \quad \text{for} \quad X_j, Y \in \mathfrak{gl}(r, \mathbb{C}). \quad (4.3)$$

Definition 4.4.2. Let $E$ be a complex vector bundle of rank $r$ over a differentiable manifold $M$ of (real) dimension $n$. Let $D$ be a connection in $E$ and let $R$ be its curvature. Choosing a local frame field $s = (s_1, \ldots, s_r)$, we denote the connection form and the curvature form of $D$ by $\omega$ and $\Omega$ respectively. Given a $GL(r, \mathbb{C})$-invariant symmetric multilinear form $f$ of degree $k$ on $\mathfrak{gl}(r, \mathbb{C})$, we set

$$\gamma = f(\Omega) = f(\Omega, \ldots, \Omega).$$

Proposition 4.4.2. $\gamma$ is independent of the choice of the local frame field $s$ and hence is a globally defined differential form of degree $2k$.

Proof. If $s' = sa^{-1}$ is another frame field, then the corresponding curvature form is given by $a\Omega a^{-1}$. Since $f$ is $GL(r, \mathbb{C})$-invariant, it immediately follows that $\gamma$ is a globally defined differential form of degree $2k$. \qed
**Proposition 4.4.3.** $\gamma$ is closed, i.e., $d\gamma = 0$. Then $\gamma$ represents a cohomology class in $H^{2k}(M, \mathbb{C})$.

**Proof.** Using the Bianchi identity $d\Omega = [\Omega, \omega]$ and (4.3) we have

\[
d\gamma = df(\Omega, \ldots, \Omega) = f(d\Omega, \ldots, \Omega) + \cdots + f(\Omega, \ldots, d\Omega) = f([\Omega, \omega], \ldots, \Omega) + \cdots + f(\Omega, \ldots, [\Omega, \omega]) = 0.
\]

Now we show that the cohomology class is well defined.

**Proposition 4.4.4.** The cohomology class of $\gamma$ does not depend on the choice of the connection $D$.

**Proof.** We consider two connections $D_0$ and $D_1$ in $E$ and connect them by a segment of connections

\[D_t = (1 - t)D_0 + tD_1, \quad 0 \leq t \leq 1.\]

Let $\omega_t$ and $\Omega_t$ be the connection form and the curvature form of $D_t$ with respect to a local frame field $s$. Then we write

\[\omega_t = \omega_0 + t\alpha, \quad \text{where} \quad \alpha = \omega_1 - \omega_0,
\]

and

\[\Omega_t = d\omega_t + \omega_t \wedge \omega_t.
\]

Then

\[
\frac{d\Omega_t}{dt} = d\alpha + \alpha \wedge \omega_t + \omega_t \wedge \alpha = D_t \alpha.
\]

Finally we set

\[\phi = k \int_0^1 f(\alpha, \Omega_t, \ldots, \Omega_t)dt.
\]

From (3.5) we see that the difference $\alpha$ of two connection forms transforms in the same way as the curvature form under a transformation of the local frame field $s$. It follows that $f(\alpha, \Omega_t, \ldots, \Omega_t)$ is independent of $s$ and hence a globally defined $(2k-1)$-form on $M$. Therefore, $\phi$ is a $(2k-1)$-form on $M$. From Bianchi identity $D_t \Omega_t = 0$, we obtain

\[kd\alpha(\Omega_t, \ldots, \Omega_t) = kD_t f(\alpha, \Omega_t, \ldots, \Omega_t) = kf(D_t \alpha, \Omega_t, \ldots, \Omega_t) = kf\left(\frac{d\Omega_t}{dt}, \Omega_t, \ldots, \Omega_t\right) = \frac{d}{dt} f(\Omega_t, \Omega_t, \ldots, \Omega_t).
\]

Hence,

\[d\phi = \int_0^1 \frac{d}{dt} f(\Omega_t, \Omega_t, \ldots, \Omega_t) = f(\Omega_1, \ldots, \Omega_1) - f(\Omega_0, \ldots, \Omega_0),
\]

which proves that the cohomology class of $\gamma$ does not depend on the connection $D$. \qed
Definition 4.4.3. Using the $GL(r, \mathbb{C})$-invariant polynomials $f_k$ defined by (4.2), we may define
\[ \gamma_k = f_k(\Omega), \quad k = 1, \cdots, r. \] (4.4)
In other words,
\[ \det \left( I_r - \frac{1}{2\pi i} \Omega \right) = 1 + \gamma_1 + \gamma_2 + \cdots + \gamma_r. \] (4.5)

Note 4.4.5. After some linear algebra calculations we find
\[ \gamma_k = \frac{(-1)^k}{(2\pi i)^k} \sum \delta^{i_1 \cdots i_k}_{j_1 \cdots j_k} \Omega^{i_1} \wedge \cdots \wedge \Omega^{i_k}. \]
In particular
\[ \gamma_i = -\frac{1}{2\pi i} \text{tr}(\Omega), \]
and
\[ \gamma_2 = -\frac{1}{8\pi^2} (\text{tr}(\Omega) \wedge \text{tr}(\Omega) - \text{tr}(\Omega \wedge \Omega)). \]

Theorem 4.4.6. Let $E$ be a complex vector bundle of rank $r$ over a differentiable manifold $M$ of (real) dimension $n$. The $k$-th Chern class $c_k(E)$ of a complex vector bundle $E$, as a cohomology class in $H^{2k}(M, \mathbb{C})$, is represented by the closed 2k-form $\gamma_k$ defined by (4.4) or (4.5).

Proof. We have to show that the cohomology classes represented by the closed 2k-forms $\gamma_k$ satisfy the four axioms given in the previous section.

1. Axiom 1 is trivially satisfied. We simply need to set $\gamma_0 = 1$.

2. For Axiom 2, in the vector bundle $f^* E$ induced from $E$ by the $\mathcal{C}^\infty$ mapping $f : N \rightarrow M$, we use the connection $f^* D$ induced from a connection $D$ in $E$. Then its curvature form is given by $f^* \Omega$. Since $f_k(f^* \Omega) = f^*(f_k(\Omega)) = f^* \gamma_k$, Axiom 2 is satisfied.

3. To verify Axiom 3, let $D_1, \ldots, D_q$ be connections in line bundles $E_1, \ldots, E_q$ respectively and let $\Omega_1, \ldots, \Omega_q$ be their curvature forms. We use the connection $D = D_1 \otimes \cdots \otimes D_q$ in $E = E_1 \otimes \cdots \otimes E_q$, then its curvature form is diagonal with diagonal entries $\Omega_1, \ldots, \Omega_q$. Hence
\[ \det \left( I_r - \frac{1}{2\pi i} \Omega \right) = \left( 1 - \frac{1}{2\pi i} \Omega_1 \right) \wedge \cdots \wedge \left( 1 - \frac{1}{2\pi i} \Omega_q \right), \]
which establishes Axiom 3.

4. We take a natural Hermitian structure in the tautological line bundle $L$ over $\mathbb{P}^1(\mathbb{C})$, i.e., the one arising from the natural inner product in $\mathbb{C}^2$. Since a fibre of $L$ is a complex line through the origin of $\mathbb{C}^2$, each element $\zeta \in L$ is represented by a vector $(\zeta^0, \zeta^1)$ in $\mathbb{C}^2$. Then the Hermitian structure $h$ is defined by:
\[ h(\zeta, \zeta) = |\zeta^0|^2 + |\zeta^1|^2 \]
Considering $[\zeta^0, \zeta^1]$ as a homogeneous coordinate in $\mathbb{P}^1(\mathbb{C})$, let $z = \zeta^1 / \zeta^0$ be the inhomogeneous coordinate in the local chart $U_0 = \mathbb{P}^1(\mathbb{C}) - \{0, 1\}$. Let $s$ be the frame field of $L$ over $U_0$ defined by
\[ s(z) = (1, z) \in L_z \subseteq \mathbb{C}^2. \]
With respect to $s$, $h$ is given by the function
\[ H(z) = h(s(z), s(z)) = 1 + |z|^2. \]
Hence the connection form and the curvature form of the Hermitian connection $D$ are given by
\[ \omega = \frac{\pi dz}{1 + |z|^2}, \quad \Omega = -\frac{dz \wedge d\bar{z}}{(1 + |z|^2)^2}. \]
So
\[ \gamma_1 = \frac{dz \wedge d\bar{z}}{2\pi i (1 + |z|^2)^2}. \]
Using polar coordinates $(r, t)$ defined by $z = re^{2\pi it}$, $0 \leq t \leq 1$ we write
\[ \gamma_1 = -\frac{2rdr \wedge dt}{(1 + r^2)^2} \quad \text{on } U_0. \]
Then integrating $\gamma_1$ over $\mathbb{P}^1(\mathbb{C})$, we obtain
\[ \int_{\mathbb{P}^1(\mathbb{C})} \gamma_1 = \int_{U_0} \gamma_1 = \int_{0}^{1} \int_{0}^{\infty} -\frac{2rdr \wedge dt}{(1 + r^2)^2} = \int_{0}^{1} dt \left( \int_{0}^{\infty} -\frac{2rdr}{(1 + r^2)^2} \right) = \int_{0}^{1} \frac{1}{1 + r^2} \bigg|_{r=0}^{r=\infty} = \int_{0}^{1} -dt = -1. \]
and this verifies Axiom 4.
Chapter 5

Algebraic and Analytic Tools

In this chapter we present some algebraic notions such as coherent sheaves, torsion-free sheaves and locally-free sheaves over a compact Kähler manifold $(X, \omega)$. Moreover, we introduce the notion of $\omega$-stable and $\omega$-semistable torsion-free sheaf of $\mathcal{O}_X$-modules over a compact Kähler manifold $(X, \omega)$. For more detailed definitions and proofs see [19], [25] and [35].

On the other hand, in the last section of this chapter we present some useful analytic tools which are involved in the proofs of the main results of this work: the Fredholm alternative Theorem and the Maximum Principle. See [15] and [25] for more details.

5.1 Torsion-free and locally-free analytic coherent sheaves

Definition 5.1.1. Let $X$ be complex manifold of (complex) dimension $n$ and let $\mathcal{O} = \mathcal{O}_X$ be the structure sheaf of $X$, i.e., the sheaf of germs of holomorphic functions on $X$. We write

\[ \mathcal{O}^p = \mathcal{O} \otimes \cdots \otimes \mathcal{O} \]

An analytic sheaf over $X$ is a sheaf of $\mathcal{O}_X$-modules over $X$.

Definition 5.1.2. Let $\mathcal{S}$ be an analytic sheaf over a complex manifold $X$ of (complex) dimension $n$. $\mathcal{S}$ is coherent if, for every point $P \in X$, there exists a neighborhood $U$ of $P$ in $X$ and an exact sequence of sheaves

\[ \mathcal{O}^q|_U \longrightarrow \mathcal{O}^p|_U \longrightarrow \mathcal{S}|_U \longrightarrow 0. \]

Definition 5.1.3. Let $\mathcal{S}$ be an analytic sheaf over a complex manifold $X$ of (complex) dimension $n$. We have the following definitions:

1. $\mathcal{S}$ is free if it is isomorphic to a direct sum of copies of the structure sheaf $\mathcal{O}_X$,

2. $\mathcal{S}$ is locally-free if $X$ can be covered by open sets $\{U\}$ for which $\mathcal{S}|_U$ is a free $\mathcal{O}|_U$-module.
If $S$ is locally-free, the rank of $S$ on such an open set is the number of copies of the structure sheaf needed (finite or infinite). If $X$ is connected, the rank of a locally-free sheaf is the same everywhere. A locally-free sheaf of rank 1 is also called an invertible sheaf.

**Proposition 5.1.1.** Let $S$ be a coherent sheaf over a complex manifold $X$ of (complex) dimension $n$. Then $S$ has finite rank.

*Proof.* It follows essentially from Oka Lemma and the Syzygy Theorem. (See Gunning-Rossi [16] for details). □

**Definition 5.1.4.** Let $S$ be an analytic sheaf over a complex manifold $X$ of (complex) dimension $n$. $S$ is torsion-free if for every point $P \in X$, the corresponding stalk $S_P$ is a torsion-free $\mathcal{O}_P$-module.

**Proposition 5.1.2.** Every subsheaf of an analytic torsion-free sheaf is also torsion-free.

*Proof.* Let $F \subseteq S$ be a subsheaf of the analytic sheaf $S$. For every point $P \in X$ the stalk $F_P$ is an $\mathcal{O}_P$-submodule of $S_P$. Since $S_P$ is a torsion-free $\mathcal{O}_P$-module, we conclude that $F_P$ is a torsion-free $\mathcal{O}_P$-module, and this completes the proof. □

**Proposition 5.1.3.** Let $S$ be a coherent sheaf over a complex manifold $X$ of (complex) dimension $n$. If $S$ is locally-free then it is torsion-free.

*Proof.* Let $P \in X$ a point of $X$. We only have to show that the stalk $S_P$ is a torsion-free $\mathcal{O}_P$-module. Since $S$ is locally-free, we can find a neighborhood $U \subseteq X$ of $P$ in $X$ such that $S|_U = \mathcal{O}^n|_U$. Since $S$ is coherent, according to Proposition 5.1.1, the rank on $U$ is finite. If we consider the stalk at the point $P$ we have $S_P = \mathcal{O}_P^n$ and then, since $\mathcal{O}_P$ is UFD, we have that the stalk $S_P$ is a torsion-free $\mathcal{O}_P$-module. □

The following result demonstrates a deep relationship between vector bundles and locally-free sheaves. For the proof see Wells [35].

**Theorem 5.1.4.** Let $X$ be a complex connected manifold of (complex) dimensions $n$. There is a one-to-one correspondence between isomorphism classes of holomorphic vector bundles over $X$ and isomorphism classes of locally-free coherent sheaves over $X$.

**Definition 5.1.5.** Let $S$ be a coherent sheaf over a complex manifold $X$ of complex dimension $n$. The singularity set of the sheaf $S$ is

$$S = S_{n-1}(S) = \{ P \in X | S_P \text{ is not free} \}$$

**Theorem 5.1.5.** Let $S$ be coherent torsion-free sheaf over a complex manifold $X$ of (complex) dimension $n$. The singularity set $S$ is a closed analytic subset of $X$ of dimension $\dim_{\mathbb{C}} S \leq n - 2$.

*Proof.* See Kobayashi [25] p. 154-159 for a detailed proof. □

**Corollary 5.1.6.** Let $S$ be a coherent torsion-free sheaf over a complex manifold $X$ of (complex) dimension 1, i.e., a Riemann surface. Then $S$ is locally-free.

*Proof.* Clearly $S$ is locally-free outside the singularity set $S$. From the previous Theorem, since $X$ is a Riemann surface, we conclude that $S$ is empty and this completes the proof. □
5.2 Stable and semistable sheaves

According to Theorem 5.1.4, every exact sequence
\[ 0 \to E_m \to \cdots \to E_0 \to 0 \]  
(5.1)
of holomorphic vector bundles over a complex manifold \( X \) of (complex) dimension \( n \), induces an exact sequence of locally-free coherent sheaves over \( X \)
\[ 0 \to \mathcal{E}_m \to \cdots \to \mathcal{E}_0 \to 0, \]  
(5.2)
where \( \mathcal{E}_i \) denotes the sheaf \( \mathcal{O}(E_i) \) of germs of holomorphic sections of \( E_i \). Conversely, every exact sequence (5.2) of locally-free sheaves over a complex manifold \( X \) comes from an exact sequence (5.1) of the corresponding holomorphic vector bundles.

**Definition 5.2.1.** Given a coherent sheaf \( S \) over a complex manifold \( X \), we shall define its determinant bundle \( \det S \).

Let
\[ 0 \to \mathcal{E}_n \to \cdots \to \mathcal{E}_0 \to S|_U \to 0 \]  
(5.3)
be a locally-free resolution of \( S|_U \), where \( U \) is a small open set in the base manifold \( X \). Thanks to Syzygy Theorem and Oka Lemma such a resolution always exists. Let \( E_i \) denote the holomorphic vector bundle corresponding to the sheaf \( \mathcal{E}_i \). We set
\[ \det S|_U = \bigotimes_{i=0}^n (\det E_i)^{(1)} \]  
(5.4)
One should check that \( \det S|_U \) is independent of the choice of the resolution (5.3), for details see Kobayashi [25], p. 163-165.

**Proposition 5.2.1.** Let \( S \) be a coherent sheaf over a complex manifold \( X \) of complex dimension \( n \). Then \( \det S \) is a line bundle over \( X \).

**Proof.** From (5.4) we know that \( \det S \) is locally a holomorphic line bundle over \( X \) and then, thanks to Theorem 5.1.4, it is a locally-free sheaf over \( X \) of rank 1. Hence, also from Theorem 5.1.4 we deduce that \( \det S \) is in one-to-one correspondence with an isomorphism class of holomorphic line bundle over \( X \).

**Definition 5.2.2.** Let \( \det S \) be the determinant bundle of a coherent sheaf \( S \) over a complex manifold \( X \) of complex dimension \( n \). The degree of \( S \) is defined by
\[ \deg(S) = \int_M c_1(S) \wedge \omega^{n-1}, \]  
(5.5)
while the slope of \( S \) is defined by
\[ \mu(S) = \frac{\deg(S)}{\text{rk}(S)} = \frac{\int_M c_1(S) \wedge \omega^{n-1}}{\text{rk}(S)}. \]  
(5.6)
**Definition 5.2.4.** Let $S$ be a torsion-free coherent sheaf over a compact Kähler manifold $(X,\omega)$ of (complex) dimension $n$. $S$ is $\omega$-semistable if for every coherent subsheaf $S' \subseteq S$ with $0 < \text{rk}(S') < \text{rk}S$, the inequality

$$\mu(S') \leq \mu(S)$$

holds. If moreover the strict inequality

$$\mu(S') < \mu(S)$$

holds for all coherent subsheaf $S'$ with $0 < \text{rk}(S') < \text{rk}(S)$, we say that $S$ is $\omega$-stable.

### 5.3 Analytic tools

**Theorem 5.3.1.** (Fredholm alternative) Let $H$ be a Hilbert space and let $K : H \rightarrow H$ be a compact bounded linear operator. Then

1. $\ker(I - K)$ is finite dimensional (and hence it is closed),
2. $\text{Im}(I - K)$ is closed,
3. $\text{Im}(I - K) = \ker(I - K)^\perp$,
4. $\ker(I - K) = 0$ if and only if $\text{Im}(I - K) = H$,
5. $\dim \ker(I - K) = \dim \ker(I - K^*)$.

**Theorem 5.3.2.** (Maximum principle for parabolic equations) Let $M$ be a compact Riemannian manifold and let $f : M \times [0,a) \rightarrow \mathbb{R}$ a function of class $C^1$ with continuous laplacian $\Delta f$ satisfying the inequality

$$\partial_t f + c\Delta f \leq 0, \quad (c > 0).$$

Set $F(t) = \max_M f(x,t)$. Then $F(t)$ is a monotone decreasing function of $t$.

**Lemma 5.3.3.** Let $(X,\|\cdot\|_X)$ be a Banach space and let $\{x_m\} \subseteq X$ be a sequence such that $x_m \rightharpoonup x$ in $X$. Then

$$\|x\|_X \leq \liminf_m \|x_m\|_X.$$ 

**Lemma 5.3.4.** Let $(X,\|\cdot\|_X)$ and $(Y,\|\cdot\|_Y)$ be Banach spaces and let $\{x_m\} \subseteq X$, $\{L_m\} \subseteq \mathcal{L}(X,Y)$ be sequences such that

1. $x_m \rightarrow x$ in $X$,
2. $L_m \rightarrow L$ in $\mathcal{L}(X,Y)$.

Then

$$L_m x_m \rightarrow L x \quad \text{in} \ Y.$$
Proof. First, from \( x_m \longrightarrow x \) in \( X \) we deduce that the sequence \( \{x_m\} \) is bounded in \( X \), then there exists \( M > 0 \) such that
\[
\|x_m\|_X \leq M
\]
and \( M \) does not depend on \( m \). Let \( \varphi \in Y' \), then \( \varphi L_m \longrightarrow \varphi L \) in \( X' \), in fact
\[
\|\varphi L_m - \varphi L\|_{X'} \leq \|\varphi\|_{Y'} \|L_m - L\|_{\mathcal{L}(X,Y)} \longrightarrow 0.
\]
So that, since \( x_m \rightharpoonup x \) in \( X \) and since \( \varphi L, \varphi L_m \in X' \),
\[
|\langle \varphi, L_m x_m \rangle - \langle \varphi, L x \rangle| \leq |\langle \varphi, L_m x_m \rangle - \langle \varphi, L x_m \rangle| + |\langle \varphi, L x_m \rangle - \langle \varphi, L x \rangle| \leq
\leq \|\varphi L_m - \varphi L\|_{X'} \|x_m\|_X + |\langle \varphi L, x_m \rangle - \langle \varphi L, x \rangle| \leq
\leq M\|\varphi L_m - \varphi L\|_{X'} + |\langle \varphi L, x_m \rangle - \langle \varphi L, x \rangle| \longrightarrow 0.
\]
\( \square \)
Chapter 6

Approximate Hermitian-Yang-Mills Metrics and Semistability

In this chapter we review the notions of (weak) Hermitian-Yang-Mills structure and approximate Hermitian-Yang-Mills structure for Higgs bundles. Then, we construct the Donaldson functional for Higgs bundles over compact Kähler manifolds and present some basic properties of it. In particular, we study the properites of the Donaldson heat flow and we establish a relation between this gradient flow and the mean curvature of the Hitchin-Simpson connection. We also study some properties of the solutions of the evolution equation associated with that functional. Finally, we study the problem of the existence of approximate Hermitian-Yang-Mills structutres and its relation with the notion of semistability.

In particular in this chapter we show that for a Higgs bundle \( \mathcal{E} = (E, \phi) \) over a compact Riemann surface \( X \) with Kähler form \( \omega \), the following conditions are equivalent:

1. There exists an approximate Hermitian-Yang-Mills metric structure,
2. The Higgs bundle \( \mathcal{E} = (E, \phi) \) is \( \omega \)-semistable.

6.1 Higgs sheaves and Higgs bundles

We start this section with some basic definitions.

**Definition 6.1.1.** Let \( (X, \omega) \) be a compact Kähler manifold of (complex) dimension \( n \). We define \( \Omega^1_X \) the holomorphic cotangent bundle to \( X \). It is the dual of the holomorphic tangent bundle to \( X \).

**Note 6.1.1.** From the decomposition \( TX^\mathbb{C} = TX^\mathbb{C}^+ \oplus TX^\mathbb{C}^- \) we identify the holomorphic tangent bundle to \( X \) with the sheaf of holomorphic sections of \( TX^\mathbb{C}^+ \).

**Definition 6.1.2.** Let \( (X, \omega) \) be a compact Kähler manifold of (complex) dimension \( n \). A Higgs sheaf \( \mathcal{E} \) over \( X \) is a coherent sheaf \( E \) over \( X \), together
with a morphism of $\mathcal{O}_X$-modules $\phi : E \to E \otimes \Omega^1_X$, such that the morphism $\phi \wedge \phi : E \to E \otimes \Omega^2_X$ vanishes, i.e., $\phi \wedge \phi = 0$. The morphism $\phi$ is called the Higgs field of $\mathcal{E}$.

**Definition 6.1.3.** A Higgs sheaf $\mathcal{E}$ is said to be torsion-free if the sheaf $E$ is torsion free. A Higgs bundle $\mathcal{E}$ is just a Higgs sheaf in which the sheaf $E$ is locally-free.

**Definition 6.1.4.** A Higgs subsheaf $\mathfrak{F}$ is a subsheaf $\mathfrak{F}$ of $E$ such that $\phi(\mathfrak{F}) \subseteq \mathfrak{F} \otimes \Omega^1_X$.

**Definition 6.1.5.** Let $(X, \omega)$ be a compact Kähler manifold and let $E = (E, \phi)$ be a Higgs sheaf over $X$. A section $s \in \Gamma(X, E)$ is $\phi$-invariant if there exists a section $\lambda$ of $\Omega^1_X$ such that $\phi(s) = s \otimes \lambda$.

**Definition 6.1.6.** Let $(X, \omega)$ be a Kähler manifold of (complex) dimension $n$ and let $E_1$ and $E_2$ be two Higgs sheaves over $X$. A morphism between $E_1$ and $E_2$ is a map $f : E_1 \to E_2$ such that the diagram

$$
\begin{array}{ccc}
E_1 & \xrightarrow{\phi_1} & E_1 \otimes \Omega^1_X \\
\downarrow{f} & & \downarrow{f \otimes \text{Id}} \\
E_2 & \xrightarrow{\phi_2} & E_2 \otimes \Omega^1_X 
\end{array}
$$

is commutative. We will denote such a morphism by $f : \mathcal{E}_1 \to \mathcal{E}_2$. A sequence of Higgs sheaves is a sequence of their corresponding coherent sheaves where each map is a morphism of Higgs sheaves. A short exact sequence of Higgs sheaves is defined in the obvious way.

**Note 6.1.2.** Using local coordinates on $X$ we can write $\phi = \phi_\alpha dz^\alpha$, where the index takes values $\alpha = 1, \ldots, n$ and each $\phi_\alpha$ is an endomorphism of $E$. The condition $\phi \wedge \phi = 0$ is then equivalent to the commutativity of the endomorphisms $\phi_\alpha$.

**Note 6.1.3.** Let $(X, \omega)$ be a compact Kähler manifold of (complex) dimension $n$ and let $\mathcal{E} = (E, \phi)$ be a Higgs bundle of rank $r$ over $X$. If $(s_1, \ldots, s_r)$ is a local frame field on $X$ and $\eta^1, \ldots, \eta^r$ is its dual, using local coordinates on $X$ one can write $\phi = \phi^\gamma_{\alpha\beta} s_\gamma \otimes \eta^\beta \otimes dz^\alpha$, where the indexes take values $\alpha = 1, \ldots, n$ and $\beta, \gamma = 1, \ldots, r$ while $\phi^\gamma_{\alpha\beta}$ are functions locally defined on $X$.

We give the definitions of dual Higgs bundle and Higgs pull-back bundle.

**Definition 6.1.7.** Let $(X, \omega)$ be a compact Kähler manifold of (complex) dimension $n$ and let $\mathcal{E} = (E, \phi)$ be a Higgs bundle of rank $r$ over $X$. Let us consider the Higgs field $\phi$ as a section of $\text{End}(E) \otimes \Omega^1_X$. Since $\text{End}(E^*) \cong \text{End}(E)$ there is a natural dual morphism $\phi^* : E^* \to E^* \otimes \Omega^1_X$. From this it follows that $\mathcal{E}^* = (E^*, \phi^*)$ is a Higgs bundle, called the dual Higgs bundle of $\mathcal{E} = (E, \phi)$.

**Definition 6.1.8.** Let $(X, \omega)$ be a compact Kähler manifold of (complex) dimension $n$ and let $\mathcal{E} = (E, \phi)$ be a Higgs bundle of rank $r$ over $X$. Let $Y$ be
another compact Kähler manifold and let \( f : Y \rightarrow X \) be a holomorphic map. Since in the category of holomorphic bundles there is an isomorphism

\[ f^* E \cong f^{-1} E \otimes f^{-1} \mathcal{O}_X \otimes \mathcal{O}_Y, \]

if \( \phi \) is a Higgs field on \( E \), there is a pull-back Higgs field \( f^* \phi \) on \( f^* E \) defined by

\[ (f^* \phi)(\sum \phi_j s_j) = \sum \phi_j \phi(s_j), \]

where \( \phi_j \in A^0_Y \) are \( C^\infty \) functions on \( Y \) and \( s_j \in \Gamma(X, E) \) are \( C^\infty \) sections of \( E \).

**Definition 6.1.9.** Let \( (X, \omega) \) be a compact Kähler manifold of (complex) dimension \( n \) and let \( E \) be a Higgs sheaf over \( X \) of rank \( r \). The degree of \( E \) is defined by

\[ \deg(E) = \int_X c_1(E) \cdot \omega^{n-1}, \]

and the slope of \( E \) is defined by

\[ \mu(E) = \frac{\deg(E)}{\text{rk}(E)} = \frac{\int_X c_1(E) \cdot \omega^{n-1}}{\text{rk}(E)}. \]

As in the ordinary case (see Kobayashi [25] for more details) there is a notion of stability for Higgs sheaves, which depends on the Kähler form \( \omega \) and makes reference only to Higgs subsheaves. Namely we have:

**Definition 6.1.10.** Let \( (X, \omega) \) be a compact Kähler manifold of (complex) dimension \( n \) and let \( E \) be a Higgs sheaf over \( X \) of rank \( r \). \( E \) is \( \omega \)-stable (resp. \( \omega \)-semistable) if it is torsion-free and for any Higgs subsheaf \( \mathcal{F} \) with \( 0 < \text{rk}(\mathcal{F}) < \text{rk}(E) \) one has the inequality \( \mu(\mathcal{F}) < \mu(E) \) (resp. \( \mu(\mathcal{F}) \leq \mu(E) \)).

Let \( (X, \omega) \) be a compact Kähler manifold of (complex) dimension \( n \) and let \( E = (E, \phi) \) be a Higgs bundle of rank \( r \) over \( X \). Let \( h \) be an Hermitian metric on this bundle. Let \( D_h = D'_h + D'' \) be the Hermitian connection on \( E \). From section 3.3 we already know that this connection exists and is the unique connection compatible with the metric \( h \) and the holomorphic structure of the bundle \( E \). (For more details see [26].) Here \( D'_h \) and \( D'' = d''_E \) are the components of type \((1, 0)\) and \((0, 1)\). Using this decomposition of \( D_h \) and the Higgs field \( \phi \), Simpson [31] introduced a connection on \( \mathcal{E} \) in the following way:

\[ D'' = D'' + \phi, \quad D'_h = D'_h + \overline{\phi}_h, \]

where \( \overline{\phi}_h \) is the usual adjoint of the Higgs field with respect to the hermitian structure \( h \), and it is defined by the formula

\[ h(\overline{\phi}_h s, s') = h(s, \phi s'), \]

where \( s \) and \( s' \) are sections of the Higgs bundle.

**Note 6.1.4.** \( D'_h \) and \( D'' \) are not of type \((1, 0)\) and \((0, 1)\).
Definition 6.1.11. The resulting connection $D_h = D'_h + D''$ is called the Hitchin-Simpson connection. Clearly
\[ D_h = D_h + \phi + \overline{\phi}_h \]
depends on the Higgs field $\phi$ and there is an extra dependence on $h$ via $\overline{\phi}_h$.

Definition 6.1.12. The curvature of the Hitchin-Simpson connection is defined by $R_h = D_h \circ D_h$ and we say that the pair $(E, h)$ is Hermitian flat if this curvature vanishes.

From the previous definition we immediately have
\[ R_h = (D_h + \phi + \overline{\phi}_h) \wedge (D_h + \phi + \overline{\phi}_h), \]
then using the decomposition $D_h = D'_h + D''$ and defining
\[ [\phi, \overline{\phi}_h] = \phi \wedge \overline{\phi}_h + \overline{\phi}_h \wedge \phi \]
we obtain the following formula of the Hitchin-Simpson curvature in terms of the curvature of the Hermitian connection $D_h$
\[ R_h = R_h + D'_h(\phi) + D''(\overline{\phi}_h) + [\phi, \overline{\phi}_h]. \]

In the previous formula, $D'_h(\phi)$ and $D''(\overline{\phi}_h)$ are the components of type $(2, 0)$ and $(0, 2)$, respectively, while the $(1, 1)$ component is given by
\[ R^{1,1}_h = R_h + [\phi, \overline{\phi}_h]. \]

We denote by $\text{Herm}(E)$ the space of Hermitian forms in $E$ and by $\text{Herm}^+(E)$ the space of Hermitian structures (i.e., the positive definite Hermitian forms) in $E$. For any Hermitian structure $h$ it is possible to identify $\text{Herm}(E)$ with the tangent space of $\text{Herm}^+(E)$ at the point $h$ (see Kobayashi [25] for more details). Hence
\[ \text{Herm}(E) \cong T_h \text{Herm}^+(E). \] (6.1)

If $v$ denotes an element in $\text{Herm}(E)$, one defines the endomorphism $h^{-1}v$ by setting $s' \mapsto h^{-1}vs'$, where $h^{-1}vs'$ is the unique section of $E$ such that
\[ v(s, s') = h(s, h^{-1}vs') \quad \text{for all } s \in \Gamma(X, E). \]

We can also define a Riemann structure in $\text{Herm}^+(E)$. From (6.1), for any $v, v' \in \text{Herm}(E)$ we define the inner product
\[ (v, v')_h = \int_X \text{tr}(h^{-1}v \cdot h^{-1}v') \frac{\omega^n}{n!} \]

We have some natural properties associated with tensor products and direct sums. In particular we have

Proposition 6.1.5. Let $E_1$ and $E_2$ two Higgs bundles with Higgs fields $\phi_1$ and $\phi_2$ respectively. Then
1. The pair $E_1 \otimes E_2 = (E_1 \otimes E_2, \phi)$ is a Higgs bundle with Higgs field $\phi = \phi_1 \otimes I_2 + I_1 \otimes \phi_2$.
2. If \( pr_i : E_1 \oplus E_2 \to E_i \) with \( i = 1, 2 \) denote the natural projections, then \( E_1 \oplus E_2 = (E_1 \oplus E_2, \phi) \) is a Higgs bundle with \( \phi = pr_1^* \phi_1 + pr_2^* \phi_2 \).

Let \((X, \omega)\) be a compact Kähler manifold of (complex) dimension \( n \) and let \( \mathcal{E} = (E, \phi) \) be a Higgs bundle of rank \( r \) over \( X \). In a similar way as in the ordinary case (see Kobayashi, [25]) we have a notion of Hermitian-Yang-Mills metric structure for the Higgs bundle \( \mathcal{E} \). Let us consider the usual operator \( * : A^{p,q} \to A^{p+1,q+1} \) and the operator \( L : A^{p,q} \to A^{p+1,q+1} \) defined by \( L \varphi = \omega \wedge \varphi \), where \( \varphi \in A^{p,q} \) is a form on \( X \) of type \((p,q)\). Then we define as usual the operator \( A = * \circ L^{-1} : A^{p,q} \to A^{p+1,q-1} \).

**Definition 6.1.13.** Consider now a Hermitian metric \( h \in \text{Herm}^+(\mathcal{E}) \) and let \( \mathcal{R}_h \) be its Hitchin-Simpson curvature. We can define the mean curvature of the Hitchin-Simpson connection \( D_h \) just by contraction of this curvature with the operator \( iA \). In other words,

\[
K_h = iAR_h, \tag{6.2}
\]
or equivalently

\[
i nR_h \wedge \omega^{n-1} = K_h. \tag{6.3}
\]

**Note 6.1.6.** \( K_h \) is selfadjoint with respect to the Hermitian metric \( h \), i.e., \( h(K_h s, s') = h(s, K_h s') \) for any section \( s, s' \in \Gamma(X, E) \).

**Definition 6.1.14.** We define the operators

\[
\Box_0 = iAD'd' \quad \text{and} \quad \Box_h = iAD''D'_h.
\]

Note that operator \( \Box_h \) depends on the Kähler form \( \omega \) via the action of the operator \( A \) and also on the metric \( h \), while \( \Box_0 \) depends on the Kähler form \( \omega \) via the application of \( A \).

**Proposition 6.1.7.** \( K_h \in A^0(\text{End}(E)) \). In particular the \((2,0)\) and \((0,2)\) components of \( \mathcal{R}_h \) do not contribute to \( K_h \).

**Proof.** We already know that \( \mathcal{R}_h \in A^2(\text{End}(E)) \) and that the operator \( iA \) kills the components of \((2,0)\) and \((0,2)\) type in \( \mathcal{R}_h \). Then, from (6.2) we have

\[
K_h = iAR_h = iA(R_h + [\phi, \bar{\phi}_h] + D'_h(\phi) + D''(\bar{\phi}_h)) = \\
iA(R_h + [\phi, \bar{\phi}_h]) + iAD'_h(\phi) + iAD''(\bar{\phi}_h) = \\
iA(R_h + [\phi, \bar{\phi}_h]) = iAR_h^{1,1}. \]

**Definition 6.1.15.** We say that a Hermitian metric \( h \in \text{Herm}^+(\mathcal{E}) \) is a weak Hermitian-Yang-Mills structure with factor \( \gamma \) for \( \mathcal{E} \) if

\[
K_h = \gamma I_E,
\]

where \( \gamma \) is a real valued function on \( X \) and \( I_E \) is the identity endomorphism on \( E \). If \( \gamma = c \) is a real positive constant, we say that \( h \) is a Hermitian-Yang-Mills structure.

**Note 6.1.8.** The mean curvature can be considered also a Hermitian form, by defining

\[
K_h(s, s') = h(s, K hs') \tag{6.4}
\]

where \( s, s' \) are section of the Higgs bundle \( \mathcal{E} = (E, \phi) \).

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6.2 Weak Hermitian-Yang-Mills structures: elementary results

Let \((X, \omega)\) be a compact Kähler manifold of (complex) dimension \(n\). From the expression of the curvature of tensor product and direct sum of Higgs bundles over \(X\), we immediately have the following

**Proposition 6.2.1.** 1. If \(h_1\) and \(h_2\) are two weak Hermitian-Yang-Mills structures with factors \(\gamma_1\) and \(\gamma_2\) for Higgs bundles \(E_1\) and \(E_2\) over \(X\), then \(h_1 \otimes h_2\) is a weak Hermitian-Yang-Mills structure with factor \(\gamma_1 + \gamma_2\) for the tensor product bundle \(E_1 \otimes E_2\).

2. The metric \(h_1 \oplus h_2\) is a weak Hermitian-Yang-Mills structure with factor \(\gamma\) for the Whitney sum \(E_1 \oplus E_2\) if and only if both metrics \(h_1\) and \(h_2\) are weak Hermitian-Yang-Mills structures with the same factor \(\gamma\) for \(E_1\) and \(E_2\).

**Proof.** 1. As in the classical case consider the formula for the curvature of the Hitchin-Simpson connection in a tensor product

\[
R_{1\otimes 2} = R_1 \otimes I_2 + I_1 \otimes R_2.
\]

Hence, taking the trace with respect to \(\omega\) (that is, applying the operator \(i\Lambda\)) we have the following expression involving the mean curvature of a tensor product

\[
K_{1\otimes 2} = K_1 \otimes I_2 + I_1 \otimes K_2.
\]

2. To prove (2) we use the following identity

\[
K_{1\oplus 2} = K_1 \oplus K_2.
\]

\(\square\)

From the previous result and the definition of the dual Higgs bundle we have the following

**Corollary 6.2.2.** Let \(h \in \text{Herm}^+ (E)\) be a (weak) Hermitian-Yang-Mills structure with factor \(\gamma\) for the Higgs bundle \(E\) over \(X\). Then

1. The induced metric on the tensor product \(E \otimes \mathcal{E}^* \otimes \mathcal{E}^* \otimes \mathcal{E}\) is a (weak) Hermitian-Yang-Mills structure with factor \((p - q)\gamma\),

2. The induced Hermitian metric on \(\bigwedge^p \mathcal{E}\) is a (weak) Hermitian-Yang-Mills structure with factor \(p\gamma\) for every \(0 \leq p \leq r = \text{rk}(\mathcal{E})\).

In general, if \(h\) is a weak Hermitian-Yang-Mills structure with factor \(\gamma\), the slope \(\mu(\mathcal{E})\) can be written in terms of \(\gamma\). In fact, we have

**Proposition 6.2.3.** Let \((X, \omega)\) be a compact Kähler manifold of (complex) dimension \(n\) and let \(E = (E, \phi)\) be a Higgs bundle of rank \(r\) over \(X\). If \(h \in \text{Herm}^+ (E)\) is a weak Hermitian-Yang-Mills structure with factor \(\gamma\), then

\[
\mu(\mathcal{E}) = \frac{1}{2n\pi} \int_X \gamma \omega^n.
\]
Proof. Let \( \mathcal{R}_h \) be the Hitchin-Simpson curvature of the Hitchin-Simpson connection \( D_h \) associated with the Hermitian connection \( D_h \). From (6.3) we have the identity
\[
in \mathcal{R}_h \wedge \omega^{n-1} = K_h \omega^n.
\]
Taking the trace we obtain
\[
intr \mathcal{R}_h \wedge \omega^{n-1} = \text{tr} K_h \omega^n.
\]
By hypothesis \( h \) is a weak Hermitian-Yang-Mills structure with factor \( \gamma \). Integrating over \( X \) we obtain
\[
\mu(\mathcal{E}) = \frac{1}{r} \int_X c_1(\mathcal{E}) \wedge \omega^{n-1} = \frac{1}{r} \int_X \frac{1}{2\pi i} \text{tr}(\mathcal{R}_h) \wedge \omega^{n-1} = -\frac{1}{2\pi i} \int_X \text{tr}(\mathcal{R}_h) \wedge \omega^{n-1} = \frac{1}{2\pi i} \int_X \text{tr}(K_h) \omega^n = \frac{1}{2\pi} \int_X r \gamma \omega^n = \frac{1}{2\pi} \int_X r \gamma \omega^n.
\]
\( \square \)

Lemma 6.2.4. Let \( h \) be a weak Hermitian-Yang-Mills structure with factor \( \gamma \) for \( \mathcal{E} \) and let \( a = a(x) \) be a real positive definite function on \( X \), then \( h' = ah \) is a weak Hermitian-Yang-Mills structure with factor \( \gamma' = \gamma + \Box_0(\ln a) \).

Proof. Clearly \( h' \) defines another Hermitian metric on \( \mathcal{E} \). Since \( h' \) is a conformal change of \( h \), we have in particular \( \overline{\phi}_{h'} = \overline{\phi}_h \), in fact, for every sections \( s, s' \) of \( \mathcal{E} \) we have
\[
h(\phi s, s') = h(s, \overline{\phi}_h s')
\]
\[
h'(\phi s, s') = h'(s, \overline{\phi}_{h'} s').
\]
Since \( h' = ah \), we have
\[
h(\phi s, s') = h(s, \overline{\phi}_{h'} s'),
\]
and this proves that \( \overline{\phi}_{h'} = \overline{\phi}_h \). Then from (6.3), \( K' = K + \Box_0(\ln a) \) (see Kobayashi [25] for the classical case). Since taking the wedge product with \( \omega^{n-1} \) kills the \((2,0)\) and \((0,2)\) components, we obtain
\[
K' \omega^n = in \mathcal{R}' \wedge \omega^{n-1} = in (R' + [\phi, \overline{\phi}_{h'}]) \wedge \omega^{n-1} = in (R' + [\phi, \overline{\phi}_h]) \wedge \omega^{n-1} = K' \omega^n = in [\phi, \overline{\phi}_h] \wedge \omega^{n-1} = (K + \Box_0(\ln a) I_E) \omega^n = (K + \Box_0(\ln a) I_E) \omega^n = (\gamma + \Box_0(\ln a) I_E) \omega^n.
\]
\( \square \)

Lemma 6.2.5. If \( h \in \text{Herm}^+(\mathcal{E}) \) is a weak Hermitian-Yang-Mills structure with factor \( \gamma \), then there exists a conformal change \( h' = ah \) such that \( h' \) is a Hermitian-Yang-Mills structure with constant factor \( c \), given by
\[
c \int_X \omega^n = \int_X \gamma \omega^n.
\]
(6.6)
Such a conformal change is unique up to homotety.
Proof. Since $X$ is compact the integrals $\int_X \gamma \omega^n$ and $\int_X \omega^n$ are finite real numbers. Then there exists a finite real number $c$ such that

$$c \int_X \omega^n = \int_X \gamma \omega^n.$$  

Hence,

$$\int_X (c - \gamma) \omega^n = 0.$$  

It is sufficient to prove that there exists a function $u$ satisfying the equation

$$\Box_0 u = c - \gamma. \tag{6.7}$$

In fact, setting $h' = e^u h$, from the previous Lemma it follows that $h'$ is a weak Hermitian-Yang-Mills structure with factor $\gamma' = \gamma + (c - \gamma) = c$. Now from Hodge theory and Fredholm alternative Theorem we know that (6.7) has a solution if and only if $c - \gamma$ is orthogonal to all $\Box_0$-harmonic functions. Since $X$ is a compact complex manifold, a function on $X$ is $\Box_0$-harmonic if and only if it is constant. So (6.7) has solution if and only if

$$\int_X (c - \gamma) \omega^n = 0.$$  

But this equality always holds from the choice of the constant $c$, and this completes the proof.

From the previous Lemma we immediately see that if a Higgs bundle admits a weak Hermitian-Yang-Mills structure, then it also admits, by an appropriate conformal change of the metric, a Hermitian-Yang-Mills structure. In particular, if the Higgs bundle has rank 1, we have the following

**Corollary 6.2.6.** Let $(X, \omega)$ be a compact Kähler manifold of (complex) dimension $n$ and let $\mathcal{E} = (E, \phi)$ be a Higgs bundle of rank 1 over $X$. Then $\mathcal{E}$ admits a Hermitian-Yang-Mills structure.

**Proof.** Let $h \in \operatorname{Herm}^+(\mathcal{E})$ be a Hermitian metric on $E$, which always exists from a partition of unity argument. Since $\mathcal{E}$ has rank 1 and since $\mathcal{K}_h$ is selfadjoint, we have

$$\mathcal{K}_h = \gamma I_E,$$

where $\gamma$ is a positive real function on the manifold $X$. Then the thesis comes from Lemma 6.2.5.

6.3 Approximate Hermitian-Yang-Mills metrics

In this section we define approximate Hermitian-Yang-Mills metric structures on Higgs bundles, and study some of their properties.

**Definition 6.3.1.** Let $(X, \omega)$ be a compact Kähler manifold of (complex) dimension $n$ and let $\mathcal{E} = (E, \phi)$ be a Higgs bundle of rank $r$ over $X$. We define a positive real constant $c$ as

$$c = \frac{2\pi \mu(\mathcal{E})}{(n - 1)! \operatorname{Vol}(X)}. \tag{6.8}$$
Definition 6.3.2. Let $(X, \omega)$ be a compact Kähler manifold of (complex) dimension $n$ and let $\mathcal{E} = (E, \phi)$ be a Higgs bundle of rank $r$ over $X$. Let $h$ be a Hermitian metric on $\mathcal{E}$ and let $K$ be its Hitchin-Simpson mean curvature. We define the length of the endomorphism $K - cI_E$ by the formula

$$|K - cI_E|^2 = \text{tr} [(K - cI_E) \cdot (K - cI_E)].$$  

(6.9)

Since $K$ is selfadjoint with respect to the metric $h$ and $c$ is real, $|K - cI_E|^2$ is a real function on $X$ and $|K - cI_E|^2 \geq 0$.

Definition 6.3.3. In the hypotheses of the previous definition, we introduce the following norms:

$$\|K - cI_E\|_{L^1} = \int_X |K - cI_E|^n \omega^n / n!$$

$$\|K - cI_E\|_{L^2}^2 = \int_X |K - cI_E|^2 \omega^n / n!$$

(6.10)

$$\|K - cI_E\|_{L^\infty} = \max_X |K - cI_E|.$$

Definition 6.3.4. Let $(X, \omega)$ be a compact Kähler manifold of (complex) dimension $n$ and let $\mathcal{E} = (E, \phi)$ be a Higgs bundle of rank $r$ over $X$. We say that $\mathcal{E}$ admits an approximate Hermitian-Yang-Mills structure if for any $\epsilon > 0$ there exists a metric $h_\epsilon$ such that

$$\max_X |K_{h_\epsilon} - cI_E| < \epsilon.$$

Here $K_{h_\epsilon}$ is the mean curvature of the Hitchin-Simpson connection associated with $h_\epsilon$.

This notion satisfies some simple properties with respect to tensor product and direct sums.

Proposition 6.3.1. Let $(X, \omega)$ be a compact Kähler manifold of (complex) dimension $n$. If the Higgs bundles $\mathcal{E}_1$ and $\mathcal{E}_2$ over $X$ admit approximate Hermitian-Yang-Mills structures, so does their tensor product $\mathcal{E}_1 \otimes \mathcal{E}_2$. Furthermore, if $\mu(\mathcal{E}_1) = \mu(\mathcal{E}_2)$, so does their Whitney sum.

Proof. 1. Assume that $\mathcal{E}_1$ and $\mathcal{E}_2$ admit approximate Hermitian-Yang-Mills structures with factors $c_1$ and $c_2$, respectively. Let $\epsilon > 0$. Then there exist $h_1$ and $h_2$ such that

$$\max_X |K_1 - c_1 I_{E_1}| < \epsilon / 2, \quad \max_X |K_2 - c_2 I_{E_2}| < \epsilon / 2,$$

where $K_1$ and $K_2$ are the mean curvature endomorphisms of the Hitchin-Simpson connection associated with $h_1$ and to $h_2$, respectively. Defining $h = h_1 \otimes h_2$ and setting $c = c_1 + c_2$, $I = I_{E_1} \otimes I_{E_2}$, the mean curvature

$$K = K_1 \otimes I_{E_2} + I_{E_1} \otimes K_2$$

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satisfies the inequality
\[
\max_X |K - cI| = \max_X |K_1 \otimes I_{E_2} + I_{E_1} \otimes K_2 - c_1 I_{E_1} \otimes I_{E_2} - c_2 I_{E_1} \otimes I_{E_2}| \leq \\
\leq \max_X |K_1 \otimes I_{E_2} - c_1 I_{E_1} \otimes I_{E_2}| + \\
+ \max_X |I_{E_1} \otimes K_2 - c_2 I_{E_1} \otimes I_{E_2}| \leq \\
\leq \max_X |K_1 - c_1 I_{E_1}| + \max_X |K_2 - c_2 I_{E_2}| < \\
< \epsilon/2 + \epsilon/2 = \epsilon,
\]
and this completes the proof.

2. If \( \mu(\mathcal{E}_1) = \mu(\mathcal{E}_2) \), necessarily \( c_1 = c_2 = c \). Defining this time \( h = h_1 \oplus h_2 \), from \( K = K_1 \oplus K_2 \) we have
\[
\max_X |K - cI_{E_1 \oplus E_2}| = \max_X |K_1 \oplus K_2 - cI_{E_1 \oplus E_2}| \leq \\
\leq \max_X |K_1 - cI_{E_1}| + \max_X |K_2 - cI_{E_2}| = \\
= \max_X |K_1 - cI_{E_1}| + \max_X |K_2 - cI_{E_2}| < \\
< \epsilon/2 + \epsilon/2 = \epsilon.
\]
Herm\(^+\)(\(\mathcal{E}\)) is a connected Riemannian manifold (see Chapter VI in [25] for more details), we connect \(h\) and \(k\) by a curve \(h_t, 0 \leq t \leq 1\), in Herm\(^+\)(\(\mathcal{E}\)) so that \(h_0 = k\) and \(h_1 = h\). We set

\[
Q_1 = \ln(\det(k^{-1}h)), \quad Q_2 = i \int_0^1 \text{tr}(v_t \cdot \mathcal{R}_t) \, dt.
\]

where \(v_t = h_t^{-1} \partial_t h_t\) and \(\mathcal{R}_t\) denotes the curvature of the Hitchin-Simpson connection associated with \(h_t\). Since \(h\) and \(k\) are Hermitian structures, \(\det(k^{-1}h)\) is a strictly positive real function.

Notice that \(Q_1(h,k)\) does not involve the curve \(h_t\). On the other hand to define \(Q_2(h,k)\) we use explicitly the curve \(h_t\).

**Definition 6.4.1.** Let \((X,\omega)\) be a compact Kähler manifold of (complex) dimension \(n\) and let \(\mathcal{E} = (E,\phi)\) be a Higgs bundle of rank \(r\) over \(X\). Let \(h,k \in \text{Herm}^+(\mathcal{E})\). We define the Donaldson functional by

\[
\mathcal{L}(h,k) = \int_X \left[ Q_2(h,k) - \frac{c}{n} Q_1(h,k) \omega \right] \wedge \frac{\omega^{n-1}}{(n-1)!},
\]

where \(c\) is, as usual, the constant given by

\[
c = \frac{2\pi\mu(\mathcal{E})}{(n-1)!\text{Vol}(X)}.
\]

**Note 6.4.1.** Notice that the \((2,0)\) and \((0,2)\) components of \(\mathcal{R}_t\) do not contribute to \(\mathcal{L}(h,k)\). In fact, if \(\mathcal{R}_h\) is the curvature of the Hitchin-Simpson connection associated with the Hermitian metric \(h\), from \(D_h'(\phi) \wedge \omega^{n-1} = D''(\alpha_h) \wedge \omega^{n-1} = 0\) we have

\[
\mathcal{R}_h \wedge \omega^{n-1} = (R_h + D_h'(\phi) + D''(\alpha_h) + [\phi, \overline{\phi}_h]) \wedge \omega^{n-1} = (R_h + [\phi, \overline{\phi}_h]) \wedge \omega^{n-1} = \mathcal{R}_h^{1,1} \wedge \omega^{n-1}.
\]

We want to prove that the Donaldson functional does not depend on the curve joining the metrics \(h\) an \(k\). First of all we prove the following result

**Lemma 6.4.2.** Let \((X,\omega)\) be a compact Kähler manifold of (complex) dimension \(n\). Let \(\eta \in \text{d}'A^{0,1} + \text{d}''A^{1,0}\), one has

\[
\int_X \eta \wedge \omega^{n-1} = 0.
\]

**Proof.** We write \(\eta = \text{d}(\alpha + \beta) = \text{d}'\alpha + \text{d}''\beta\) where \(\alpha \in A^{1,0}\) and \(\beta \in A^{0,1}\). Since \(X\) is Kähler, i.e., \(\omega = 0\), we have

\[
\text{d}[(\alpha + \beta) \wedge \omega^{n-1}] = [\text{d}(\alpha + \beta)] \wedge \omega^{n-1} = \eta \wedge \omega^{n-1}.
\]

From Stokes’ Theorem, since \(\partial X = \emptyset\) we conclude

\[
\int_X \eta \wedge \omega^{n-1} = \int_X \text{d}[(\alpha + \beta) \wedge \omega^{n-1}] = \int_{\partial X} (\alpha + \beta) \wedge \omega^{n-1} = 0.
\]

\(\square\)
The following Lemma and the subsequent Propositions are straightforward generalizations of a result of Kobayashi \( \text{(see [25], Chapter VI)} \) to the Higgs case. Part of the proof is similar to the proof presented in [25]. However, some differences arise because of the term involving the commutator of the Higgs field in the Hitchin-Simpson curvature.

**Lemma 6.4.3.** Let \( h_t \) (for \( a \leq t \leq b \)) be any differentiable curve in \( \text{Herm}^+(\mathcal{E}) \) and \( k \) any fixed Hermitian structure of \( \mathcal{E} \). Then the \((1, 1)\) component of

\[
\int_0^1 \text{tr}(v_t R_t) dt + Q_2(h_a, k) - Q_2(h_b, k)
\]

is an element in \( d'A^{0,1} + d''A^{1,0} \).

**Proof.** Following [25], we consider the domain \( \Delta \) in \( \mathbb{R}^2 \) defined by

\[
\Delta = \{(t, s) | a \leq t \leq b, 0 \leq s \leq 1\},
\]

and let \( h : \Delta \to \text{Herm}^+(\mathcal{E}) \) be a smooth mapping such that \( h(t, 0) = k \) and \( h(t, 1) = h_t \) for \( a \leq t \leq b \). Let \( h(a, s) \) and \( h(b, s) \) line segments from \( k \) to \( h_a \) and from \( h_b \) to \( k \), respectively. There is a simple expression for \( h \) given by

\[
h(t, s) = sh_t + (1 - s)k.
\]

Define the endomorphisms \( u = h^{-1} \partial_s h, \ v = h^{-1} \partial_t h \) and we put

\[
\mathcal{R} = d''(h^{-1}d'h) + [\phi, \phi h]
\]

and

\[
\Psi = i \text{tr}[\mathcal{R}],
\]

where \( \partial_t \mathcal{R} = h^{-1} \partial_t (d'h \mathcal{R}) \) is the exterior differential of the smooth mapping \( h \) in the domain \( \Delta \). Hence \( \Psi \) can be written in the form

\[
\Psi = i \text{tr}[(uds + vdt)\mathcal{R}], \quad (6.12)
\]

Applying Stokes’ Theorem to \( \Psi \) (which is considered here as a 1-form in the domain \( \Delta \)) we get

\[
\int_\Delta \partial \Psi = \int_{\partial \Delta} \Psi.
\]

The right hand side of the above expression can be computed straightforwardly from definition. In fact, after a short computation we obtain

\[
\int_{\partial \Delta} \Psi = - \int_{t=a}^{t=b} \Psi|_{s=0} + \int_{s=0}^{s=1} \Psi|_{t=a} + \int_{t=a}^{t=b} \Psi|_{s=1} - \int_{s=0}^{s=1} \Psi|_{t=b} =
\]

\[
i \int_a^b \text{tr}(v_t R_t) dt + Q_2(h_a, k) - Q_2(h_b, k).
\]

Therefore, we need to show that the left hand side of \( (6.12) \) is an element of \( d'A^{0,1} + d''A^{1,0} \), and hence, it suffices to show that \( \partial \Psi \in d'A^{0,1} + d''A^{1,0} \). Now, from the definition of \( \Psi \) we have

\[
\partial \Psi = i \text{tr}[(uds + vdt)\mathcal{R} - (uds + vdt)d'h] =
\]

\[
= i \text{tr}[(\partial_s v - \partial_t u)\mathcal{R} - u\partial_t \mathcal{R} + v\partial_s \mathcal{R}]ds \wedge dt.
\]

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On the other hand, a simple computation shows that

\[ \partial_t u = -vu + h^{-1}\partial_t \partial_h h, \quad \partial_v v = -uv + h^{-1}\partial_v \partial_h h, \]

\[ \partial_t R = d''D'v + [\phi, \partial_h \bar{\phi}_h], \quad \partial_v R = d''D'u + [\phi, \partial_h \bar{\phi}_h]. \]

Replacing these expressions in the formula for \( \tilde{d}\Psi \) and writing \( R = R + [\phi, \bar{\phi}_h] \) (since the (2, 0) and (0, 2) components do not contribute), we have

\[ \tilde{d}\Psi = i\text{tr}[(vu - uv)R - ud''D'v + vd''D'u]ds \wedge dt + \]

\[ + i\text{tr}[v[\phi, \partial_h \bar{\phi}_h] - u[\phi, \partial_v \bar{\phi}_h]] + (vu - uv)[\phi, \bar{\phi}_h]ds \wedge dt. \]

The first trace in the expression above does not depend on the Higgs field \( \phi \) (in fact, it is the same expression that it is found in [25] for the classical case). Then we only have to show that the second trace is identically zero.

First of all, we need explicit expressions for \( \partial_t \bar{\phi}_h \) and \( \partial_v \bar{\phi}_h \). Omitting the parameter \( t \) for simplicity, from (31) we know that

\[ \bar{\phi}_{h_s + \delta s} = u_0^{-1}\bar{\phi}_{h_s}, \quad \bar{\phi}_{h_s} + u_0^{-1}[\bar{\phi}_{h_s}, u_0] \]

where \( u_0 \) is a selfadjoint endomorphism such that \( h_{s+\delta s} = h_s + u_0 \). Now

\[ h_{s+\delta s} = h_s + \partial_t h_s + O(\delta s^2) \]

and hence, at the first order in \( \delta s \), we obtain \( u_0 = 1 + \delta s \) and consequently \( \partial_v \bar{\phi}_h = [\bar{\phi}_h, v] \). In a similar way we obtain the formula \( \partial_t \bar{\phi}_h = [\bar{\phi}_h, t] \). Therefore, using these relations, the Jacobi identity and the cyclic property of the trace, we see that the second trace is identically zero. On the other hand, the term involving the curvature \( R \) can be written in terms of \( u, v \) and their covariant derivatives. So, finally, we get

\[ \tilde{d}\Psi = -i\text{tr}[vD''u + ud''D']ds \wedge dt. \]

As it is shown in [25] in the classical case, defining the (0, 1)-form \( \alpha = i\text{tr}[vD''u] \) we finally obtain

\[ \tilde{d}\Psi = -[d''\alpha + d''\pi + id''d'tr(vu)]ds \wedge dt \]

and hence \( \tilde{d}\Psi \) is an element of \( d'A^{0, 1} + d''A^{1, 0} \).

As a consequence of the above Lemma we have an important result for piecewise differentiable closed curves. Namely, we have

**Proposition 6.4.4.** Let \( h_t, \alpha \leq t \leq \beta \), be a piecewise differentiable closed curve in \( \text{Herm}^+ (\mathfrak{g}) \). Then

\[ i \int_\alpha^\beta \text{tr}(v_t \cdot R^{1, 1}_t)dt = 0 \mod d'A^{0, 1} + d''A^{1, 0}. \]

**Proof.** Let \( \alpha = a_0 < a_1 < \cdots < a_p = \beta \) be the values of \( t \) where the curve \( h_t \) is not differentiable. Now take a fixed metric \( k \) in \( \text{Herm}^+ (\mathfrak{g}) \). We have

\[ i \int_\alpha^\beta \text{tr}(v_t \cdot R^{1, 1}_t)dt = \sum_{j=1}^p i \left( \int_{a_{j-1}}^{a_j} \text{tr}(v_t \cdot R^{1, 1}_t)dt \right). \]
From the previous Lemma, for each \( j = 1, \ldots, p \), we have
\[
\int_{a_{j-1}}^{a_j} \text{tr}(v_t \cdot R_{1,1}) dt = 0 \mod d'A^{0,1} + d''A^{1,0},
\]
and this completes the proof. \( \square \)

**Corollary 6.4.5.** The Donaldson functional \( L(h, k) \) does not depend on the curve joining the Hermitian metrics \( h \) and \( k \) in \( \text{Herm}^+(E) \).

**Proof.** Let \( \gamma_1 \) and \( \gamma_2 \) be two differentiable curves from \( h \) to \( k \), and let \( L_{\gamma_1}(h, k) \) and \( L_{\gamma_2}(h, k) \) be the Donaldson functionals computed along the curves \( \gamma_1 \) and \( \gamma_2 \), respectively. We have to show that
\[
L_{\gamma_1}(h, k) = L_{\gamma_2}(h, k).
\]
From the definition of Donaldson functional we have
\[
L_{\gamma_1}(h, k) - L_{\gamma_2}(h, k) = \int_X \left[ Q_{\gamma_1}^2(h, k) - \frac{c_n}{n} Q_{\gamma_1}^1(h, k) \omega \right] \wedge \omega^{n-1} / (n-1)! +
\]
\[
- \int_X \left[ Q_{\gamma_2}^2(h, k) - \frac{c_n}{n} Q_{\gamma_2}^1(h, k) \omega \right] \wedge \omega^{n-1} / (n-1)! =
\]
\[
= \int_X \left[ (Q_{\gamma_1}^2 - Q_{\gamma_2}^2) - \frac{c_n}{n} (Q_{\gamma_1}^1 - Q_{\gamma_2}^1) \omega \right] \wedge \omega^{n-1} / (n-1)!.\]
Since
\[
Q_{\gamma_1}^1(h, k) = Q_{\gamma_2}^1(h, k) = \ln(\det(k^{-1}h)),
\]
we have
\[
L_{\gamma_1}(h, k) - L_{\gamma_2}(h, k) = \int_X \left[ (Q_{\gamma_1}^2(h, k) - Q_{\gamma_2}^2(h, k)) - \frac{c_n}{n} (Q_{\gamma_1}^1 - Q_{\gamma_2}^1) \omega \right] \wedge \omega^{n-1} / (n-1)! =
\]
\[
= \int_X \left[ i \int_{\gamma_1} \text{tr}(v_t \cdot R_t) dt - i \int_{\gamma_2} \text{tr}(v_t \cdot R_t) dt \right] \wedge \omega^{n-1} / (n-1)! =
\]
\[
= \int_X \left[ i \int_{\gamma_1 - \gamma_2} \text{tr}(v_t \cdot R_t) dt \right] \wedge \omega^{n-1} / (n-1)!.\]
But \( \gamma_1 - \gamma_2 \) is a piecewise differentiable closed curve in \( \text{Herm}^+(E) \), then by the previous Proposition
\[
i \int_{\gamma_1 - \gamma_2} \text{tr}(v_t \cdot R_t) = 0 \mod d'A^{0,1} + d''A^{1,0},
\]
so from Lemma 6.4.2 we finally have
\[
L_{\gamma_1}(h, k) - L_{\gamma_2}(h, k) = \int_X \left[ i \int_{\gamma_1 - \gamma_2} \text{tr}(v_t \cdot R_t) dt \right] \wedge \omega^{n-1} / (n-1)! = 0,
\]
and this completes the proof. \( \square \)

**Proposition 6.4.6.** For any Hermitian metric \( h \in \text{Herm}^+(E) \) and any real constant \( a > 0 \), the Donaldson functional satisfies \( L(h, ah) = 0 \).
Proof. Let \( r = \text{rk}(\mathcal{E}) \) be the rank of the Higgs bundle \( \mathcal{E} = (E, \phi) \). Clearly
\[
Q_1(h, ah) = \ln \det[(ah)^{-1} h] = -r \ln a.
\]
Now, let us consider the curve \( h_t = e^{t \ln a(1-t)} h \) from \( ah \) to \( h \). For this curve \( v_t = h_t^{-1} \partial_t h_t = -\ln a I_E \) and
\[
\mathcal{R}_t^{1,1} = d''(h_t^{-1} d'h_t) + [\phi, \overline{\phi}_t] = d''(h^{-1} d'h) + [\phi, \overline{\phi}],
\]
where \( \overline{\phi}_t = \overline{\phi}_{h_t} \) is just an abbreviation. Therefore the \((1,1)\) component of \( Q_2(h, ah) \) becomes
\[
Q_2^{1,1}(h, ah) = \int_0^1 \text{tr}(v_t \cdot \mathcal{R}_t^{1,1}) dt = i \int_0^1 \text{tr}[-\ln a(R + [\phi, \overline{\phi}])] dt = -i(\ln a)\text{tr} R.
\]
and hence, from the above formula we obtain
\[
\mathcal{L}(h, ah) = \int_X \left[ Q_2(h, ah) - \frac{c}{n} Q_1(h, ah) \right] \wedge \frac{\omega^{n-1}}{(n-1)!} = \\
= \int_X \left[ Q_2^{1,1}(h, ah) - \frac{c}{n} Q_1(h, ah) \right] \wedge \frac{\omega^{n-1}}{(n-1)!} = \\
= \int_X \left[ -i(\ln a)\text{tr} R + \frac{c}{n} r \ln a \right] \wedge \frac{\omega^{n-1}}{(n-1)!} = \\
= \frac{i \ln \ln a}{n!} \int_X \text{tr} R \wedge \omega^{n-1} + cr(\ln a)\text{Vol}(X) = \\
= \frac{2\pi r}{(n-1)!} \int_X \text{tr} R \wedge \omega^{n-1} + cr(\ln a)\text{Vol}(X) = \\
= \frac{2\pi r \ln a}{(n-1)!} \int_X c_1(\mathcal{E}) \wedge \omega^{n-1} + cr(\ln a)\text{Vol}(X) = \\
= \frac{2\pi r \ln a}{(n-1)!} \mu(\mathcal{E}) + \frac{2\pi \mu(\mathcal{E})}{(n-1)!\text{Vol}(X)} r \ln a\text{Vol}(X) = 0.
\]

\[\square\]

Lemma 6.4.7. For any differentiable curve \( h_t \) in the Riemannian manifold \( \text{Herm}^n(\mathcal{E}) \) and any fixed point \( k \in \text{Herm}^n(\mathcal{E}) \) we have
\[
\partial_t Q_1(h_t, k) = \text{tr}(v_t), \\
\partial_t Q_2^{1,1}(h_t, k) = i\text{tr}(v_t \cdot \mathcal{R}_t^{1,1}) \mod d'A^{0,1} + d''A^{1,0}.
\]

Proof. 1. Since \( k \) does not depend on \( t \), we get
\[
\partial_t Q_1(h_t, k) = \partial_t \ln(\det k^{-1}) + \partial_t \ln(\det h_t) = \partial_t \ln(\det h_t) = \text{tr}(v_t). \quad (6.13)
\]
2. It suffices to consider \( b \) as a variable in the equation
\[
i \int_0^1 \text{tr}(v_t \cdot \mathcal{R}_t) dt + Q_2(h_b, k) - Q_2(h_b, k) = 0 \mod d'A^{0,1} + d''A^{1,0},
\]
and differentiate this expression with respect to \( b \).
Let $h$ be a metric in $\text{Herm}^+(\mathcal{E})$ and let $\mathcal{K}$ be the mean curvature of the Hitchin-Simpson connection associated with $h$. Notice that the endomorphism $\mathcal{K} \in A^0(\text{End}(E))$ can be written as $\mathcal{K} = h^{-1}\mathcal{K}(\cdot,\cdot)$, where $\mathcal{K}(\cdot,\cdot)$ denotes the mean curvature as a Hermitian form.

**Theorem 6.4.8.** Let $(X,\omega)$ be a compact Kähler manifold of (complex) dimension $n$ and let $\mathcal{E} = (E,\phi)$ be a Higgs bundle of rank $r$ over $X$. Let $\text{Herm}^+(\mathcal{E})$ be the Riemannian manifold of the Hermitian metric on $E$, and let $h$ be a fixed element in $\text{Herm}^+(\mathcal{E})$. Then, $h$ is a critical point of $\mathcal{L}$, i.e., a critical point of the function $\mathcal{L}(s,k) \to \mathbb{R}$, if and only if $\mathcal{K} - \text{ch} = 0$, if and only if $h$ is an Hermitian-Yang-Mills structure for $\mathcal{E}$.

**Proof.** By using the above Lemma, from $\int_{\mathcal{R}}^{1,1} \omega^{n-1} = \mathcal{K}_t \omega^n$ we have the following formula for the derivative with respect to $t$ of the Donaldson functional

$$
\frac{d}{dt}\mathcal{L}(h_t,k) = \frac{d}{dt} \int_X \left[ Q_2(h_t,k) - \frac{c}{n} Q_1(h_t,k) \omega \right] \wedge \frac{\omega^{n-1}}{(n-1)!} =
$$

$$
= \int_X \left[ \partial_t Q_2(h_t,k) - \frac{c}{n} \partial_t Q_1(h_t,k) \omega \right] \wedge \frac{\omega^{n-1}}{(n-1)!} =
$$

$$
= \int_X \left[ i\text{tr}(v_t \cdot \mathcal{R}^{1,1}_t) - \frac{c}{n} \text{tr}(v_t) \omega \right] \wedge \frac{\omega^{n-1}}{(n-1)!} =
$$

$$
= \int_X \text{tr}(v_t \cdot \mathcal{K}_t) - c\text{tr}(v_t) \wedge \frac{\omega^n}{n!} =
$$

$$
= \int_X \text{tr}[(\mathcal{K}_t - cI_E)v_t] \wedge \frac{\omega^n}{n!}.
$$

We consider the endomorphism $\mathcal{K}_t$ as a Hermitian form by defining $\mathcal{K}_t(s,s') = h_t(s,\mathcal{K}_t s')$, where $s$ and $s'$ are section of the Higgs bundle $\mathcal{E} = (E,\phi)$. Since $v_t = h_t^{-1} \partial_t h_t$, for any fixed Hermitian metric $k \in \text{Herm}^+(\mathcal{E})$ and any differentiable curve $h_t$ in $\text{Herm}^+(\mathcal{E})$ we obtain

$$
\frac{d}{dt}\mathcal{L}(h_t,k) = (\mathcal{K}_t - \text{ch}_t, \partial_t h_t),
$$

where $\mathcal{K}_t$ is considered here as a form and $(\cdot,\cdot)$ is the inner product in the Riemannian manifold $\text{Herm}^+(\mathcal{E})$. For each $t$, we can consider $\partial_t h_t \in \text{Herm}(\mathcal{E})$ as a tangent vector of $\text{Herm}^+(\mathcal{E})$ at $h_t$. (See [25], Chapter VI for more details). Therefore, the differential $d\mathcal{L}$ of the functional evaluated at $\partial_t h_t$ is given by

$$
d\mathcal{L}(\partial_t h_t) = \frac{d}{dt}\mathcal{L}(h_t,k).
$$

Then, the gradient of $\mathcal{L}(s,k)$ is given by $\nabla \mathcal{L} = \mathcal{K} - \text{ch}$, and this completes the proof. Notice that here $\mathcal{K}$ still denotes the mean curvature as an Hermitian form.

Now, using the decomposition $\mathcal{D} = \mathcal{D}' + \mathcal{D}''$, we show that all critical points of $\mathcal{L}$ correspond to an absolute minimum.

**Theorem 6.4.9.** Let $k$ be a fixed Hermitian structure of the Higgs bundle $\mathcal{E}$ and let $h_0$ be a critical point of $\mathcal{L}(h,k)$. The Donaldson functional attains an absolute minimum at $h_0$. 69
Proof. The second derivative of $\mathcal{L}$ is

$$\partial_t^2 \mathcal{L}(h_t, k) = \partial_t \int_X \text{tr}[(\mathcal{K}_t - c I_E) v_t] \frac{\omega^n}{n!} =$$

$$= \int_X \text{tr}[(\partial \mathcal{K}_t \cdot v_t + (\mathcal{K}_t - c I_E) \partial_t v_t)] \frac{\omega^n}{n!}.$$ 

Here $\mathcal{K}_t$ is an element of $A^0(\text{End}(E))$. Since $h_0$ is a critical point of the Donaldson functional, $\mathcal{K}_t - c I_E = 0$ at $t = 0$, hence

$$\partial_t^2 \mathcal{L}(h_t, k)|_{t=0} = \int_X \text{tr}(\partial_t \mathcal{K}_t \cdot v_t) \frac{\omega^n}{n!} \bigg|_{t=0}.$$ 

On the other hand, $\partial_t \mathcal{K}_t$ can be written in terms of $v_t$. In fact, we have

$$D'' D'_{h_t} v_t = D''(D'_{h_t} v_t + [\phi_{h_t}, v_t]) =$$

$$= D'' D'_{h_t} v_t + [\phi, D'_{h_t} v_t] + D''[\phi_{h_t}, v_t] + [\phi, [\phi_{h_t}, v_t]].$$ 

Since $\partial_t \phi_{h_t} = [\phi_{h_t}, v_t]$ we get

$$\partial_t R_t^{1,1} = \partial_t R_t + [\phi, \partial_t \phi_{h_t}] = D'' D'_{h_t} v_t + [\phi, [\phi_{h_t}, v_t]].$$ 

Therefore, since the operator $iA$ kills the $(2,0)$ and $(0,2)$ components, we have

$$iA[\phi, D'_{h_t} v_t] = 0 \quad \text{and} \quad iAD''[\phi_{h_t}, v_t] = 0.$$ 

From the previous equations and since the linear operators $iA$ and $\partial_t$ commute, we obtain

$$\partial_t \mathcal{K}_t = \partial_t iA R_t^{1,1} = iA \partial_t R_t^{1,1} =$$

$$= iA(D'' D'_{h_t} v_t + [\phi, [\phi_{h_t}, v_t]]) =$$

$$= iA(D'' D'_{h_t} v_t + [\phi, [\phi_{h_t}, v_t]]) + iA[\phi, D'_{h_t} v_t] + iAD''[\phi_{h_t}, v_t] =$$

$$= iA(D'' D'_{h_t} v_t + [\phi, D'_{h_t} v_t] + D''[\phi_{h_t}, v_t] + [\phi, [\phi_{h_t}, v_t]]) =$$

$$= iAD'' D'_{h_t} v_t.$$ 

Hence, replacing this in the expression for the second derivative of $\mathcal{L}$ we find

$$\partial_t^2 \mathcal{L}(h_t, k)|_{t=0} = \int_X \text{tr}(iA D'' D'_{h_t} v_t \cdot v_t) \frac{\omega^n}{n!} \bigg|_{t=0} = \|D'_{h_t} v_t\|_{L^2}^2.$$ 

That is, $h_0$ must be a local minimum of $\mathcal{L}$. Now, suppose in addition that $h_1$ is an arbitrary element of $\text{Herm}^+(\mathcal{E})$ and assume $h_1$ is a geodesic which joins the point $h_0$ and $h_1$. Hence, $\partial_t v_t = 0$ (see [25] for more details). Therefore, for such a geodesic we have

$$\partial_t^2 \mathcal{L}(h_t, k) = \int_X \text{tr}(\partial_t \mathcal{K}_t \cdot v_t) \frac{\omega^n}{n!}.$$ 

Following the same procedure we have done before, but this time for $t$ arbitrary, we get for $0 \leq t \leq 1$

$$\partial_t^2 \mathcal{L}(h_t, k) = \|D'_{h_t} v_t\|_{L^2}^2 \geq 0.$$ 

Note that the right hand side implicitly depends on $t$ via $D'_{h_t}$. It follows that $\mathcal{L}(h_0, k) \leq \mathcal{L}(h_1, k)$. If we assume $h_1$ is also a critical point of $\mathcal{L}$, we necessarily obtain the equality $\mathcal{L}(h_0, k) = \mathcal{L}(h_1, k)$, so the local minimum defined for any critical point of $\mathcal{L}$ is an absolute minimum.
6.5 The evolution equation

For the construction of approximate Hermitian-Yang-Mills structures, the standard procedure is to start with a fixed Hermitian metric $h_0$ and try to find from it an approximate Hermitian-Yang-Mills metric structure using a curve $h_t, 0 \leq t < +\infty$. In other words, we try to find the approximate structure by deforming $h_0$ through the 1-parameter family of Hermitian metrics $h_t$.

Theorem 6.5.1. Given a Hermitian metric $h_0$ on the Higgs bundle $\mathcal{E}$, the non-linear evolution problem

$$
\begin{cases}
\partial_t h_t = -(\mathcal{K}_t - c h_t) \\
h(0) = h_0
\end{cases}
$$

has a unique smooth solution defined for every positive time $0 \leq t < +\infty$. Notice that here $\mathcal{K}_t$ is the Hermitian form associated with the mean curvature of the Hitchin-Simpson connection of $h_t$.

Proof. See Kobayashi [25], p. 205-223 for details.

Note 6.5.2. Let $k$ be a fixed Hermitian metric on the Higgs bundle $\mathcal{E}$. We have $\nabla \mathcal{L} = \mathcal{K} - c h$, where $\mathcal{K}$ is the Hermitian form associated with the mean Hitchin-Simpson curvature of $h$. Hence, we can rewrite the evolution problem in the gradient form

$$
\begin{cases}
\partial_t h_t = -\nabla \mathcal{L} \\
h(0) = h_0
\end{cases}
$$

Here $\nabla \mathcal{L}$ is a vector field on the Riemannian manifold $\text{Herm}^+(\mathcal{E})$. This non-linear evolution problem is called the Donaldson heat flow problem.

In this section we study some properties of the solutions of the Donaldson heat flow problem. In particular, we are interested in the study of the mean curvature when the parameter $t$ goes to infinity.

Proposition 6.5.3. Let $h_t, 0 \leq t < +\infty$ be the solution of the Donaldson heat flow with initial condition $h_0$. Then:

1. For any fixed Hermitian metric $k \in \text{Herm}^+(\mathcal{E})$, the functional $\mathcal{L}(h_t, k)$ is a monotone decreasing function of $t$; that is,

$$
\frac{d}{dt} \mathcal{L}(h_t, k) = -\|\mathcal{K}_t - c I_E\|^2_{L^2} \leq 0, \quad (6.14)
$$

2. $\max_X |\mathcal{K}_t - c I_E|^2 = \|\mathcal{K}_t - c I_E\|^2_{L^\infty}$ is a monotone decreasing function,

3. If $\mathcal{L}(h_t, k)$ is bounded from below, i.e., there exists a real constant $A$ such that $\mathcal{L}(h_t, k) \geq A > -\infty$ for $0 \leq t < +\infty$, then

$$
\max_X |\mathcal{K}_t - c I_E|^2 \rightarrow 0 \quad \text{as} \quad t \rightarrow +\infty.
$$

Here $\mathcal{K}_t \in A^0(\text{End}(E))$ is the Hitchin-Simpson mean curvature endomorphism.
Proof. The proofs of (2) and (3) are similar to the proofs in the classical case [25], but this time we need to work with the operator \( \Box_h = iAD''D_h' \) instead of the operator \( \Box_h = iAD''D_h' \).

1. From a previous calculation we already know that
\[
\frac{d}{dt} \mathcal{L}(h_t, k) = (K_t - c h_t, \partial_t h_t),
\]
where \( K_t \) is now thought as a Hermitian form. Since \( h_t \) is the solution of the Donaldson heat flow, from the definition of the Riemannian structure in Hermitian (E) we get
\[
\frac{d}{dt} \mathcal{L}(h_t, k) = -(K_t - c h_t, K_t - c h_t) = -\|K_t - c I_E\|^2_{L^2}.
\]

2. Let \( K_t \in A^0(\text{End}(E)) \) be the mean curvature of the Hitchin-Simpson connection associated with the metric \( h_t \) and let \( v_t = h_t^{-1} \partial_t h_t \). Consider the operator \( \Box_h = iAD''D_h' \) which depends on the Kähler form \( \omega \) via the adjoint multiplication \( A \) and also on the metric \( h \). Using \( \Box_h \), we can rewrite
\[
iAD''D_h' v_t = iA\partial_t R_{t, 1}^{1, 1} = \partial_t K_t
\]
as
\[
\partial_t K_t = \Box_h v_t,
\]
where \( \Box_h = \Box_h \) and the subscript \( t \) reminds us the dependence on the metric \( h_t \). From the evolution equation we have \( v_t = -(K_t - c I_E) \), and hence we get \( \Box_h v_t = -\Box_h K_t \). Therefore we obtain
\[
\partial_t K_t = -\Box_h K_t
\]
or
\[
(\partial_t + \Box_h) K_t = 0.
\]
On the other hand, since \( \text{tr}(AB) = \text{tr}(BA) \),
\[
D''D_h' |K_t - c I_E|^2 = D''D_h' \text{tr}[(K_t - c I_E) \cdot (K_t - c I_E)] =
= 2\text{tr}[(K_t - c I_E) \cdot D''D_h' K_t] + 2\text{tr}[(D''K_t \cdot D_h' h_t K_t].
\]

Applying the \( iA \) operator (this will kill the \( (2, 0) \) and \( (0, 2) \) components), from \( \text{tr}(AB) = \text{tr}(BA) \) and since the operator \( iA \) and the trace commute we get
\[
\Box_h |K_t - c I_E|^2 = iAD''D_h' |K_t - c I_E|^2 =
= iAD''D_h' \text{tr}[(K_t - c I_E) \cdot (K_t - c I_E)] =
= 2\text{tr}[(K_t - c I_E) \cdot \Box_h K_t] +
+ iA\text{tr}[D''h_t(K_t - c I_E) \cdot D''(K_t - c I_E)] +
+ iA\text{tr}[D''K_t(K_t - c I_E) \cdot D_h' h_t(K_t - c I_E)] =
= 2\text{tr}[(K_t - c I_E) \cdot \Box_h K_t] + iA\text{tr}[D''h_tK_t \cdot D''h_tK_t] +
+ iA\text{tr}[D''K_t \cdot D''h_tK_t] =
= 2\text{tr}[(K_t - c I_E) \cdot \Box_h K_t] + 2iA\text{tr}[D''h_tK_t \cdot D''h_tK_t] =
= -2\text{tr}[(K_t - c I_E) \cdot \partial_t K_t] - 2|D''K_t|^2 =
= -\partial_t |K_t - c I_E|^2 - 2|D''K_t|^2,
\]
where $|D''K_t|^2 = -i\text{At}r(D''h_t \cdot D''K_t)$ is a positive real valued function on $X$. (See p. 225 in [25] for details in the classical case). So, finally we obtain

$$
(\partial_t + \Box_t)|K_t - cI_E|^2 = \partial_t|K_t - cI_E|^2 + \Box_t|K_t - cI_E|^2 = \\
= \partial_t|K_t - cI_E|^2 - \partial_t|K_t - cI_E|^2 - 2|D''K_t|^2 = \\
= -2|D''K_t|^2 \leq 0,
$$

(6.15)

and (2) follows from the Maximum Principle 5.3.2.

3. Finally (3) follows from (1) and (2) in a similar way to the classical case [25]. Integrating the equality (6.14) from 0 to $s$, we obtain

$$
\mathcal{L}(h_s, k) - \mathcal{L}(h_0, k) = -\int_0^s ||K_t - cI_E||_{L^2}^2 dt.
$$

Since $\mathcal{L}(h_s, k)$ is bounded below by a constant $A$ independent of $s$ and since is a monotone decreasing of $s$, there exists the limit

$$
\lim_{s \to +\infty} \mathcal{L}(h_s, k) = L,
$$

where $L$ is a finite real number. Hence

$$
\int_0^{+\infty} ||K_t - cI_E||_{L^2}^2 dt = \lim_{s \to +\infty} \mathcal{L}(h_s, k) - \mathcal{L}(h_0, k) = L - \mathcal{L}(h_0, k) < +\infty.
$$

In particular we deduce

$$
||K_t - cI_E||_{L^2}^2 \longrightarrow 0 \quad \text{as} \quad s \longrightarrow +\infty. \quad (6.16)
$$

Let $\chi = \chi(x, y, t)$ be the heat kernel for the differential operator $\partial_t + \Box_t$. Set

$$
\int X \chi(x, y, t - t_0)||K_t - cI_E||^2(y) dy,
$$

where $dy$ is the volume form $dy = \omega^n$. Then $u(x, y)$ is of class $C^\infty$ on $X \times (t_0, +\infty)$ and extends to a continuous function on $X \times [t_0, +\infty)$. From the definition of the heat kernel we immediately have

$$
(\partial_t + \Box_t)u(x, t) = 0 \quad \text{for} \quad (x, t) \in X \times (t_0, +\infty),
$$

and

$$
u(x, t_0) = f(x, t_0) = (||K_{t_0} - cI_E||^2)(x).
$$

Combined with the inequality (6.15) this yields

$$
(\partial_t + \Box_t)(||K_t - cI_E||^2 - u(x, t)) \leq 0 \quad \text{for} \quad (x, t) \in X \times (t_0, +\infty).
$$
By the Maximum Principle 5.3.2 and the properties of \( u(x,t) \) we find
\[
\max_X (|K_{t_0} - cI_E|^2 - u(x,t)) \leq \max_X (|K_{t_0} - cI_E|^2 - u(x,t_0)) = 0, \quad t \geq t_0.
\]
Hence
\[
\max_X |K_{t_0+a} - cI_E|^2 \leq \max_X u(x,a,t_0+a) = \max_X \int_X \chi(x,y,a)|K_{t_0} - cI_E|^2(y)dy \leq C_a \int_X |K_{t_0} - cI_E|^2(y)dy = C_a \|K_{t_0} - cI_E\|_{L^2},
\]
where
\[
C_a = \max_{X \times X} \chi(x,y,a).
\]
Fix \( a \), say \( a = 1 \), and let \( t_0 \to +\infty \). Since \( \mathcal{L}(h_t,k) \) is bounded below, using (6.16) we conclude
\[
\max_X |K_{t_0+1} - cI_E|^2 \leq C_1 \|K_{t_0} - cI_E\|_{L^2} \to 0,
\]
and this completes the proof.

\[\square\]

**Corollary 6.5.4.** The Donaldson functional is a real valued function.

**Proof.** Let \( h_t, 0 \leq t \leq 1 \), be a curve in \( \text{Herm}^+(E) \) such that \( h_0 = k \) and \( h_1 = h \). From a previous calculation we already know that
\[
\frac{d}{dt} \mathcal{L}(h_t,k) = (K_t - c_h, \partial_t h_t)
\]
where \( K_t \) is now thought as a Hermitian form and \((\cdot,\cdot)\) is the inner product of \( \text{Herm}^+(E) \). So that \( (K_t - c_h, \partial_t h_t) \in \mathbb{R} \) for any \( t \in [0,1] \) (see p.196-197 in [25] for details), and then
\[
\mathcal{L}(h_k) = \int_0^1 \frac{d}{dt} \mathcal{L}(h_t,k) dt = \mathbb{R}.
\]

\[\square\]

At this point we introduce the main result of this section. This establishes a relation among the boundedness property of Donaldson functional, semistability and the existence of approximate Hermitian-Yang-Mills metric structures on the Higgs bundle \( \mathcal{E} \).

**Theorem 6.5.5.** Let \((X,\omega)\) be a compact Kähler manifold of (complex) dimension \( n \) and let \( \mathcal{E} = (E,\phi) \) be a Higgs bundle of rank \( r \) over \( X \). We have implications (1) \( \to \) (2) \( \to \) (3) for the following statements:

1. for any fixed Hermitian metric structure \( k \in \text{Herm}^+(\mathcal{E}) \), there exists a constant \( B \) such that \( \mathcal{L}(h,k) \geq B \) for all Hermitian metrics \( h \in \text{Herm}^+(\mathcal{E}) \),
2. $\mathcal{E}$ admits an approximate Hermitian-Yang-Mills structure, i.e., for each $\epsilon > 0$ there exists an Hermitian metric $h$ in $\mathcal{E}$ such that

$$\max_X |K - cI_E| < \epsilon,$$

where $h$ depends on $\epsilon$ and $K$ is the mean curvature endomorphism of the Hitchin-Simpson connection associated with the metric $h$.

3. $\mathcal{E}$ is $\omega$-semistable.

Proof. 1. Assume (1). The Donaldson functional is bounded below by a constant $B$. Let $h_0$ be a fixed Hermitian metric in $\mathcal{E}$ and let $h_t$ be the solution of the Donaldson heat flow with initial condition $h_0$. Then from the previous Proposition there exists a real constant $B$ such that

$$L(h_t, h_0) \geq B > -\infty$$

for every positive time $0 \leq t < +\infty$. From Theorem 6.5.5,

$$\max_X |K_t - cI_E|^2 \to 0 \quad \text{as} \quad t \to +\infty,$$

where $K_t$ is the mean curvature endomorphism $K_t \in A^0(\text{End}(\mathcal{E}))$. Hence, there exists an approximate Hermitian-Yang-Mills structure.

2. On the other hand, that (2) implies (3) has been proved in [6] by Bruzzo and Graña Otero. Here we reproduce their proof. Assume (2) and let $\mathcal{F}$ be a proper nontrivial Higgs subsheaf of $\mathcal{E}$. Then $\text{rk}(\mathcal{F}) = p$ for some $0 < p < r = \text{rk}(\mathcal{E})$ and the inclusion $\mathcal{F} \to \mathcal{E}$ induces a morphism $\text{det}(\mathcal{F}) \to \Lambda^p \mathcal{E}$. Tensoring by $(\text{det}(\mathcal{F}))^{-1}$ we have a nonzero section $s$ of the Higgs bundle

$$\mathcal{G} = \Lambda^p \mathcal{E} \otimes (\text{det}(\mathcal{F}))^{-1}.$$ 

If $\psi$ represents the Higgs field naturally defined on $\mathcal{G}$ by the Higgs field of $\mathcal{E}$ and $\mathcal{F}$, then $s$ is $\psi$-invariant, i.e., $\psi(s) = s$. Now, since by hypothesis $\mathcal{E}$ admits an approximate Hermitian-Yang-Mills structure, from Proposition 6.3.1 we know that so does $\mathcal{G}$ and in particular

$$c_0 = \frac{2\pi \mu(\mathcal{E}) - \mu(\mathcal{F})}{(n - 1)!\text{Vol}(X)}.$$ 

From 6.3.3, since $s$ is a nonzero $\psi$-invariant section, $\deg(\mathcal{G}) \geq 0$ and so $c_0 \geq 0$. Hence $\mu(\mathcal{E}) \geq \mu(\mathcal{F})$, showing that $\mathcal{E}$ is $\omega$-semistable.

\[\square\]

6.6 The one-dimensional case

In this section we establish a boundedness property for the Donaldson functional for semistable Higgs bundles over compact Riemann surfaces. As a consequence we get that in the one-dimensional case all three conditions in Theorem 6.5.5 are equivalent.

We introduce some properites that will be useful in proving some statements using induction on the rank of Higgs bundles. This section is essentially an
extension to Higgs bundles of the classical case. (See p. 226-233 in [25] for more details).

Now, let \((X, \omega)\) be a compact Kähler manifold of (complex) dimension \(n\) and let

\[
0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}'' \longrightarrow 0
\]

(6.17)

be an exact sequence of Higgs bundles over \(X\). As in the ordinary case (see Chapter I, §6 in [25]), a Hermitian metric \(h\) in \(\mathcal{E}\) induces Hermitian structures \(h'\) and \(h''\) in \(\mathcal{E}'\) and \(\mathcal{E}''\), respectively. We have also a second fundamental form \(A_h \in A^{1,0}(\text{Hom}(E', E''))\) and its adjoint \(B_h \in A^{0,1}(\text{Hom}(E'', E'))\), where \(B_h^* = -A_h\). As usual, some properties which hold in the ordinary case, also hold in the Higgs case.

**Proposition 6.6.1.** Given an exact sequence (6.17) and a pair of Hermitian structures \(h\) and \(k\) in \(\mathcal{E}\), the function \(Q_1(h, k)\) and the form \(Q_2(h, k)\) satisfy the following relations:

\[
\begin{align*}
Q_1(h, k) &= Q_1(h', k') + Q_1(h'', k''), \quad (6.18) \\
Q_2(h, k) &= Q_2(h', k') + Q_2(h'', k'') - \text{itr}[B_h \wedge B_h^* - B_k \wedge B_k^*] \\
&\quad \mod d'A^{0,1} + d''A^{1,0}. \quad (6.19)
\end{align*}
\]

**Proof.**

1. (6.18) is straightforward from the definition of \(Q_1\). Since \(h = h' \oplus h''\) and \(k = k' \oplus k''\), we immediately have

\[
\begin{align*}
Q_1(h, k) &= \ln(\det(k^{-1}h)) = \\
&= \ln(\det(k'^{-1}h' \oplus k''^{-1}h'')) = \\
&= \ln(\det(k'^{-1}h')) + \ln(\det(k''^{-1}h'')) = Q_1(h', k') + Q_1(h'', k'').
\end{align*}
\]

2. On the other hand, (6.19) follows from an analysis similar to the ordinary case. Since the sequence (6.17) in particular is an exact sequence of holomorphic vector bundles over the complex manifold \(X\), for any Hermitian metric \(h\) we have a splitting of the exact sequence by \(C^\infty\) homomorphisms \(\mu_h : E \longrightarrow E'\) and \(\lambda_h : E'' \longrightarrow E\). In particular

\[
B_h = \mu_h \circ d'' \circ \lambda_h.
\]

We consider now a curve of Hermitian structures \(h_t\) for \(0 \leq t \leq 1\) such that \(h_0 = k\) and \(h_1 = h\). Corresponding to \(h_t\) we have a family of homomorphisms \(\mu_t\) and \(\lambda_t\). We define the homomorphism \(S_t : E'' \longrightarrow E'\) given by

\[
S_t = \lambda_t - \lambda_0.
\]

A short computation shows that \(\partial_t B_t = d''(\partial_t S_t)\). Choosing convenient orthonormal local frame fields for \(\mathcal{E}'\) and \(\mathcal{E}''\) (see [25] for more details), the endomorphism \(v_t = h_t^{-1} \partial_t h_t\) can be presented by the matrix

\[
v_t = \begin{pmatrix}
\nu_t' & -\partial_t S_t \\
-(\partial_t S_t)^* & \nu_t''
\end{pmatrix}.
\]

Here \(\nu_t'\) and \(\nu_t''\) are the natural endomorphisms associated with \(h_t'\) and \(h_t''\), respectively. Now, from the ordinary case we have

\[
R_t = \begin{pmatrix}
R_t' - B_t \wedge B_t^* & D'B_t \\
-DB_t^* & R_t'' - B_t' \wedge B_t
\end{pmatrix}
\]

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where $R'_t$ and $R''_t$ are the Hermitian curvatures of $E'$ and $E''$, respectively.

Now, $R_{t}^{1,1} = R_t + [\phi, \bar{\phi}_h_t]$. Since $E'$ and $E''$ are Higgs subbundles of $E$, we obtain an expression for the $(1, 1)$-component of the Hitchin-Simpson curvature

$$R_{t}^{1,1} = \left( \begin{array}{c} R_t^{1,1} - B_t \wedge B_t^* \\ D' B_t \\ - D'' B_t^* \\ R''_t^{1,1} - B_t^* \wedge B_t \end{array} \right),$$

where $R_{t}^{1,1} = R_t + [\phi, \bar{\phi}_h_t]_E'$ and $R''_{t}^{1,1} = R''_t + [\phi, \bar{\phi}_h_t]_E''$. Hence we can compute the trace

$$\text{tr}(v_t \cdot R_{t}^{1,1}) = \text{tr}(v'_t \cdot R'_t^{1,1}) + \text{tr}(v''_t \cdot R''_t^{1,1}) + \text{tr}(\partial_t S_t \cdot D' B_t) - \text{tr}((\partial_t S_t)^* \cdot D' B_t) + \text{tr}(v'_t \cdot B_t \wedge B_t^*) - \text{tr}(v''_t \cdot B_t^* \wedge B_t).$$

The last four terms are the same as in the ordinary case [25]. After a short computation we finally get that

$$\text{tr}(v_t \cdot R_{t}^{1,1}) = \text{tr}(v'_t \cdot R'_t^{1,1}) + \text{tr}(v''_t \cdot R''_t^{1,1}) - \partial_t \text{tr}(B_t \wedge B_t^*) \mod d'A^{0,1} + d''A^{1,0}.$$

Then, multiplying the last expression by $i$ and integrating from $t = 0$ to $t = 1$ we finally obtain (6.19).

\[ \square \]

**Definition 6.6.1.** In the hypothesis of the previous Proposition, we define $|B|$ as the nonnegative real function on the compact Kähler manifold $(X, \omega)$ that satisfies

$$|B|^2 \omega^n = -\text{intr}(B \wedge B^*) \wedge \omega^{n-1}.$$

**Lemma 6.6.2.** Let $(X, \omega)$ be a compact Kähler manifold of (complex) dimension $n$. Given an exact sequence of Higgs bundles

$$0 \rightarrow E' \rightarrow E \rightarrow E'' \rightarrow 0$$

over $X$, with $\mu(E) = \mu(E')$ and a pair of Hermitian structures $h$ and $k$ in $E$, we have the identity

$$\mathcal{L}(h, k) = \mathcal{L}(h', k') + \mathcal{L}(h'', k'') + \|B_h\|^2_{L^2} - \|B_k\|^2_{L^2}. \quad (6.20)$$

**Proof.** First of all, from $\mu(E) = \mu(E')$ we have $c = c' = c''$. From Lemma 6.4.2
and from (6.18) and (6.19) we obtain

\[
\mathcal{L}(h, k) = \int_X \left[ Q_2(h, k) - \frac{c}{n} Q_1(h, k) \omega \right] \wedge \frac{\omega^{n-1}}{(n-1)!} = \\
= \int_X \left[ Q_2(h', k') - \frac{c'}{n} Q_1(h', k') \omega \right] \wedge \frac{\omega^{n-1}}{(n-1)!} + \\
+ \mathcal{L}(h, k) = \int_X \left[ Q_2(h'', k'') - \frac{c''}{n} Q_1(h'', k'') \omega \right] \wedge \frac{\omega^{n-1}}{(n-1)!} + \\
+ \int_X -i \text{tr} [B_h \wedge B_{k}^* - B_k \wedge B_{h}^*] \wedge \frac{\omega^{n-1}}{(n-1)!} = \\
= \mathcal{L}(h', k') + \mathcal{L}(h'', k'') + \\
+ \int_X |B_h|^2 \omega^n/n! - \int_X |B_k|^2 \omega^n/n! = \\
= \mathcal{L}(h', k') + \mathcal{L}(h'', k'') + \|B_h\|^2_{L^2} - \|B_k\|^2_{L^2}.
\]

\[\square\]

In dimension greater or equal than two, the notion of stability (resp. semistability) depends on the Kähler form, as the degree depends on it. Now, in dimension one, the degree does not depend on the Kähler form and hence the notion of stability (resp. semistability) does not depend on it and we can establish all our results without any explicit reference to \(\omega\). Since the degree and the rank of any Higgs sheaf is the same degree and rank of the corresponding coherent sheaf, we have the following (see [25], Ch. V, Lemma 7.3).

**Lemma 6.6.3.** Let consider the exact sequence of Higgs sheaves

\[0 \rightarrow \mathcal{F} \rightarrow \mathcal{E} \rightarrow \mathcal{G} \rightarrow 0,\]

then

\[\text{rk}(\mathcal{F})(\mu(\mathcal{E}) - \mu(\mathcal{F})) + \text{rk}(\mathcal{G})(\mu(\mathcal{E}) - \mu(\mathcal{G})) = 0.\]

From Lemma 6.6.3 it follows that the condition of stability (resp. semistability) can be written in terms of quotient Higgs sheaves instead of Higgs subsheaves. To be precise we have

**Corollary 6.6.4.** Let \((X, \omega)\) be a compact Kähler manifold of (complex) dimension \(n\) and let \(\mathcal{E}\) be a torsion-free Higgs sheaf over \(X\). Then \(\mathcal{E}\) is \(\omega\)-stable (resp. semistable) if for every quotient Higgs sheaf \(\mathcal{G}\) with \(0 < \text{rk}(\mathcal{G}) < \text{rk}(\mathcal{E})\) it follows \(\mu(\mathcal{G}) < \mu(\mathcal{E})\) (resp. \(\mu(\mathcal{G}) \leq \mu(\mathcal{E})\)).

From the definition of degree, one has that any torsion Higgs sheaf \(\mathcal{T}\) has \(\deg(\mathcal{T}) \geq 0\). Therefore, in a similar way to the classical case, this implies that in the definition of stability (resp. semistability) we do not have to consider all quotient Higgs sheaves. To be precise we have

**Proposition 6.6.5.** Let \((X, \omega)\) be a compact Kähler manifold of (complex) dimension \(n\) and let \(\mathcal{E}\) be a torsion-free Higgs sheaf over \(X\). Then

1. \(\mathcal{E}\) is \(\omega\)-stable (resp. semistable) if and only if \(\mu(\mathcal{F}) < \mu(\mathcal{E})\) (resp. \(\leq\)) for any Higgs subsheaf \(\mathcal{F}\) with \(0 < \text{rk}(\mathcal{F}) < \text{rk}(\mathcal{E})\) and such that the quotient \(\mathcal{E}/\mathcal{F}\) is torsion-free.
2. $\mathcal{E}$ is $\omega$-stable (resp. semistable) if and only if $\mu(\mathcal{E}) < \mu(\mathcal{G})$ (resp. $\leq$) for any torsion-free quotient Higgs sheaf $\mathcal{G}$ with $0 < \text{rk}(\mathcal{G}) < \mu(\mathcal{E})$.

Proof. Is true in one direction. For the converse, suppose the inequality between slopes in (1) (resp. in (2)) holds for proper Higgs subsheaves with torsion-free quotient (resp. for torsion-free quotient Higgs sheaves) and let consider an exact sequence of Higgs sheaves

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{E} \rightarrow \mathcal{G} \rightarrow 0.$$ 

Let $\mathcal{E} = (E, \phi)$ and denote by $\phi_E$ and $\phi_G$ the Higgs fields of $\mathcal{F}$ and $\mathcal{G}$ respectively. That is $\mathcal{F} = (F, \phi_F)$ and $\mathcal{G} = (G, \phi_G)$. Now, let $T$ be the torsion subsheaf of $G$. Since the Higgs field satisfies $\phi(T) \subseteq T \otimes \Omega^1_X$, the pair $\mathcal{T} = (T, \phi|_T)$ is a Higgs subsheaf of $\mathcal{E}$ with Higgs quotient, say $\mathcal{G}_1$. Then if we define $\mathcal{F}_1$ by the kernel of the Higgs morphism $\mathcal{E} \rightarrow \mathcal{G}_1$, we have the following commutative diagram of Higgs sheaves

in which all rows and columns are exact. From this diagram we have that $\mathcal{F}$ is a Higgs subsheaf of $\mathcal{G}_1$ with $\mathcal{T} = \mathcal{F}_1/\mathcal{F}$. Since $\mathcal{T}$ is a torsion Higgs sheaf $\deg(\mathcal{T}) \geq 0$ and we also obtain

$$\deg(\mathcal{G}) = \deg(\mathcal{F}) + \deg(\mathcal{G}_1) \geq \deg(\mathcal{G}_1)$$

and

$$\deg(\mathcal{F}_1) = \deg(\mathcal{F}) + \deg(\mathcal{T}) \geq \deg(\mathcal{T}).$$

Now, since $\mathcal{T}$ is torsion we have $\text{rk}(\mathcal{G}) = \text{rk}(\mathcal{G}_1)$ and $\text{rk}(\mathcal{F}_1) = \text{rk}(\mathcal{F})$ and hence finally we obtain

$$\mu(\mathcal{F}) \leq \mu(\mathcal{F}_1) \quad \text{and} \quad \mu(\mathcal{G}_1) \leq \mu(\mathcal{G}).$$

At this point, the converse direction in (1) and (2) follows from the hypothesis and the last two inequalities. \qed
Now we can establish a boundedness property for the Donaldson functional for semistable Higgs bundles in the one-dimensional case, i.e., for compact Riemann surfaces. Namely we have

**Theorem 6.6.6.** Let \((X, \omega)\) be a compact Riemann surface and let \(\mathcal{E} = (E, \phi)\) be a Higgs bundle of rank \(r\) over \(X\). If \(\mathcal{E}\) is \(\omega\)-semistable, then for any fixed Hermitian metric \(k \in \text{Herm}^+(\mathcal{E})\) the set \(\{L(h, k) | h \in \text{Herm}^+(\mathcal{E})\}\) is bounded below.

**Proof.** Fix \(k\) and assume \(\mathcal{E}\) is \(\omega\)-semistable. The proof runs by induction on the rank of \(\mathcal{E}\).

1. If \(\text{rk}(\mathcal{E}) = 1\), from Corollary 6.2.6 there exists a Hermitian-Yang-Mills structure \(h_0\). The Donaldson functional must attain an absolute minimum at \(h_0\), i.e., for any other Hermitian metric \(h\)

\[
L(h, k) \geq L(h_0, k).
\]

Then the set \(\{L(h, k) | h \in \text{Herm}^+(\mathcal{E})\}\) is bounded below.

2. Let us assume \(\text{rk}(\mathcal{E}) \geq 2\) and let us assume the thesis true for every Higgs bundle \(\mathcal{F}\) over \(X\) such that \(0 < \text{rk}(\mathcal{F}) < \text{rk}(\mathcal{E})\).

We have to distinguish between two cases.

(a) If \(\mathcal{E}\) is \(\omega\)-stable, there exists a Hermitian-Yang-Mills structure \(h_0\) on it (see [31] for a detailed proof) and the Donaldson functional must attain an absolute minimum at \(h_0\), i.e., for any other Hermitian metric \(h\)

\[
L(h, k) \geq L(h_0, k).
\]

Hence the set \(\{L(h, k) | h \in \text{Herm}^+(\mathcal{E})\}\) is bounded below.

(b) Suppose \(\mathcal{E}\) is \(\omega\)-semistable, but not \(\omega\)-stable, with \(\text{rk}(\mathcal{E}) \geq 2\). Among all proper nontrivial Higgs subsheaves with \(0 < \text{rk}(\mathcal{F}) < \text{rk}(\mathcal{E})\), torsion-free quotient and the same slope of \(\mathcal{E}\) we choose one, say \(\mathcal{E}'\), with minimal rank. Since \(\mu(\mathcal{E}') = \mu(\mathcal{E})\), the sheaf \(\mathcal{E}'\) is necessarily \(\omega\)-stable. If not, there exists a proper Higgs subsheaf \(\mathcal{F}'\) of \(\mathcal{E}'\) with \(\mu(\mathcal{F}') \geq \mu(\mathcal{E}')\). Since \(\mathcal{F}'\) is clearly a subsheaf of \(\mathcal{E}\) and \(\mathcal{E}\) is \(\omega\)-semistable, we obtain \(\mu(\mathcal{E}') \leq \mu(\mathcal{F}') \leq \mu(\mathcal{E}) = \mu(\mathcal{E}')\), which is a contradiction, because \(\mathcal{E}'\) was chosen of minimal rank. Now let \(\mathcal{E}'' = \mathcal{E}' / \mathcal{F}'\). Using Lemma 7.3 in [23] it follows that \(\mu(\mathcal{E}) = \mu(\mathcal{E}') = \mu(\mathcal{E}'')\) and \(\mathcal{E}''\) is \(\omega\)-semistable. Hence we have the following exact sequence of Higgs sheaves

\[
0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}'' \longrightarrow 0,
\]

where both \(\mathcal{E}'\) and \(\mathcal{E}''\) are torsion-free. Since \(\dim_c X = 1\), from Corollary 5.1.6 we deduce that they are also locally-free. Hence the above sequence is in fact an exact sequence of Higgs bundles.

Assume now \(h\) is an arbitrary metric on \(\mathcal{E}\), by applying Lemma 6.6.2 to the metrics \(h\) and \(k\) we obtain

\[
L(h, k) = L(h', k') + L(h'', k'') + \|B_h\|^2_{L^2} - \|B_k\|^2_{L^2} \geq L(h', k') + L(h'', k'') - \|B_k\|^2_{L^2},
\]

(6.21)
where $h', k'$ and $h'', k''$ are the Hermitian structures induced by $h$ and $k$ in $\mathcal{E}'$ and $\mathcal{E}''$, respectively. By induction, $\mathcal{L}(h', k')$ and $\mathcal{L}(h'', k'')$ are bounded below by constants depending only on $k'$ and $k''$. Then, from (6.21) it follows that $\mathcal{L}(h, k)$ is bounded below by a constant depending only on $k$.

As a consequence of this Theorem we obtain the following

**Theorem 6.6.7.** Let $(X, \omega)$ be a compact Riemann surface and let $\mathcal{E} = (E, \phi)$ be a Higgs sheaf of rank $r$ over $X$. The following are equivalent:

1. $\mathcal{E}$ is $\omega$-semistable,
2. $\mathcal{E}$ admits an approximate Hermitian-Yang-Mills structure.

**Proof.** Since $X$ is a Riemann surface, from 5.1.6 we deduce that $(E, \phi)$ is a Higgs bundle over $X$. Hence, the thesis comes from the previous result and Theorem 6.5.5.

As a consequence of this Theorem we immediately deduce that in the one-dimensional case, many results about Higgs bundles written in terms of approximate Hermitian-Yang-Mills structures can be translated in terms of semistability. In particular we have the following

**Corollary 6.6.8.** If $(X, \omega)$ is a compact Riemann surface and $\mathcal{E}_1$, $\mathcal{E}_2$ are $\omega$-semistable Higgs bundles over $X$, then so is their tensor product $\mathcal{E}_1 \otimes \mathcal{E}_2$. Furthermore, if $\mu(\mathcal{E}_1) = \mu(\mathcal{E}_2)$, so is the Whitney sum $\mathcal{E}_1 \oplus \mathcal{E}_2$.

**Proof.** 1. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be $\omega$-semistable Higgs bundles over $X$. From the previous Theorem $\mathcal{E}_1$ and $\mathcal{E}_2$ admit approximate Hermitian-Yang-Mills metric structures, so does their tensor product $\mathcal{E}_1 \otimes \mathcal{E}_2$ (Proposition 6.3.1). Hence, using the above Theorem we conclude that $\mathcal{E}_1 \otimes \mathcal{E}_2$ is $\omega$-semistable.

2. It is similar to (1), using the second part of Proposition 6.3.1.

**Corollary 6.6.9.** If $\mathcal{E}$ is $\omega$-semistable, then so is the tensor product bundle $\mathcal{E}^{\otimes p} \otimes \mathcal{E}^{* \otimes q}$ and the exterior product bundle $\bigwedge^p \mathcal{E}$ whenever $0 \leq p \leq r = \text{rk}(\mathcal{E})$. 

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Chapter 7

General Case

In the previous chapter we have proved the equivalence between semistability and the existence of approximate Hermitian-Yang-Mills metric structures for Higgs bundles over compact Riemann surfaces. In this chapter we extend this result to higher dimensions. Let \((X, \omega)\) be a compact Kähler manifold of (complex) dimension \(n\) and let \(\mathcal{E} = (E, \phi)\) be a Higgs bundle of rank \(r\) over \(X\). Let \(h_t\) be the solution of the Donaldson heat flow with initial metric \(h_0\), i.e., the solution of the non-linear evolution problem

\[
\begin{cases}
\partial_t h_t = -\nabla \mathcal{L} \\
h(0) = h_0
\end{cases}
\]

From Theorem 6.5.1 we know that this problem has a unique solution \(h_t\) defined for every positive time \(0 \leq t < +\infty\), and \(\mathcal{L}(h_t, h_0)\) is a real monotone decreasing function on \(t\) for \(0 \leq t < +\infty\). If we assume that \((E, \phi)\) is \(\omega\)-semistable, we can distinguish between two cases:

1. \(\mathcal{L}(h_t, h_0)\) is bounded below. From Theorem 6.5.5 there exists an approximate Hermitian-Yang-Mills structure.

2. \(\mathcal{L}(h_t, h_0)\) is not bounded below, i.e., \(\mathcal{L}(h_t, h_0) \to -\infty\) for \(t \to +\infty\). Under the assumption that \((E, \phi)\) is \(\omega\)-semistable, we can still prove the existence of approximate Hermitian-Yang-Mills metrics.

7.1 Some useful results

**Lemma 7.1.1.** Let \((X, \omega)\) be a compact Kähler manifold of (complex) dimension \(n\) and let \(\mathcal{E} = (E, \phi)\) be a Higgs bundle of rank \(r\) over \(X\). Let \(h \in \text{Herm}^+(E)\) be a Hermitian metric and let \(\mathcal{R}\) and \(\mathcal{K}\) be its Hitchin-Simpson curvature and mean curvature, respectively. Then

\[
\int_X \text{tr}(\mathcal{K} - cI_E) \frac{\omega^n}{n!} = 0 \tag{7.1}
\]

**Proof.** First we note that \(iA \text{tr}(\mathcal{R}) = \text{tr}(iA\mathcal{R})\), since \(iA\) and the trace are linear differential operator. From Stokes’ formula and the representation of Chern
classes in terms of curvature we have:

\[
\frac{1}{2\pi} \int_X \text{tr}(\mathcal{K} - cI_E) \frac{\omega^n}{n!} = \frac{1}{2\pi} \int_X \text{tr}(\mathcal{K}) \frac{\omega^n}{n!} - \frac{cr \text{Vol}(X)}{2\pi} =
\]

\[
= \frac{1}{2\pi} \int_X \text{tr}(i\mathcal{R}) \frac{\omega^n}{n!} - \frac{\mu(E)r \text{Vol}(X)}{(n-1)!\text{Vol}(X)} =
\]

\[
= \frac{1}{2\pi} \int_X i\text{At}(\mathcal{R}) \frac{\omega^n}{n!} - \frac{r}{(n-1)!} \frac{1}{r} \int_X c_1(E) \wedge \omega^{n-1} =
\]

\[
= \int_X \frac{i}{2\pi} \text{tr}(\mathcal{R}) \wedge A \omega^n - \int_X c_1(E) \wedge \frac{\omega^{n-1}}{(n-1)!} =
\]

\[
= \int_X -\frac{1}{2\pi i} \text{tr}(\mathcal{R}) \wedge \frac{\omega^{n-1}}{(n-1)!} - \int_X c_1(E) \wedge \frac{\omega^{n-1}}{(n-1)!} =
\]

\[
= \int_X c_1(E) \wedge \frac{\omega^{n-1}}{(n-1)!} - \int X c_1(E) \wedge \frac{\omega^{n-1}}{(n-1)!} = 0.
\]

Given a metric \( h \in \text{Herm}^+(E) \), let \( \mathcal{K} \) be the mean curvature of the Hitchin-Simpson connection associated with \( h \). There always exists a real positive function \( a = a(x) \) on \( X \) such that, setting \( h' = ah \), \( \text{tr}(\mathcal{K}' - cI_E) = 0 \). To be more precise, we have the following:

**Lemma 7.1.2.** Let \((X, \omega)\) be a compact Kähler manifold of (complex) dimension \( n \) and let \( \mathcal{E} = (E, \phi) \) be a Higgs bundle of rank \( r \) over \( X \). Let \( h \in \text{Herm}^+(E) \) be a Hermitian metric on \( E \), and let \( \mathcal{K} \) be the mean curvature of the Hitchin-Simpson connection associated with \( h \). By an appropriate conformal change of the metric \( h \), we can assume \( \text{tr}(\mathcal{K} - cI_E) = 0 \).

**Proof.** Consider a real positive function \( a = a(x) \) on \( X \) such that, setting \( h' = ah \), \( \text{tr}(\mathcal{K}' - cI_E) = 0 \). To be more precise, we have the following:

1. From the identity \( \mathcal{K}' \omega^n = \mathcal{K} \omega^n + \Box_0 \ln(a) I_E \omega^n \) (7.2) and since the Kähler form is nowhere vanishing, taking the trace in the previous identity we obtain

\[
\text{tr}(\mathcal{K}' - cI_E) = \text{tr}(\mathcal{K} - cI_E) + r \Box_0 \ln(a).
\]

(7.3)

Hence, it suffices to prove that there is a function \( u \) satisfying the equation

\[
\Box_0 u = -\frac{1}{r} \text{tr}(\mathcal{K} - cI_E).
\]

(7.4)

In fact, setting \( h' = e^u h \), from (7.3) it follows that \( \text{tr}(\mathcal{K}' - cI_E) = 0 \). Now from Hodge theory and Fredholm alternative Theorem we know that (7.4) has a solution if and only if \( \text{tr}(\mathcal{K} - cI_E) \) is orthogonal to all \( \Box_0 \)-harmonic functions. Since \( X \) is a compact complex manifold, a function on \( X \) is \( \Box_0 \)-harmonic if and only if it is constant. So (7.4) has solution if and only if

\[
\int_X \text{tr}(\mathcal{K} - cI_E) \frac{\omega^n}{n!} = 0.
\]

But this equality always holds from the previous Lemma and this completes the proof. □
Lemma 7.1.3. Let $h_t$ be a solution of the Donaldson heat flow. Then $\|K_t - cI_E\|_{L^2}^2$ is a monotone decreasing function of $t$ for $0 \leq t < +\infty$.

Proof. From a previous calculation we have
\[ \frac{d}{dt} \mathcal{L}(h_t, h_0) = -\|K_t - cI_E\|_{L^2}^2 \]
and
\[ \partial_t^2 \mathcal{L}(h_t, h_0) = \|D' v_t\|_{h_t}^2 \geq 0. \]
Hence,
\[ \frac{d}{dt} \|K_t - cI_E\|_{L^2}^2 = -\|D' v_t\|_{h_t}^2 \leq 0 \]
and this completes the proof.

Lemma 7.1.4. Let $h_t, 0 \leq t < +\infty$, be the solution of the Donaldson heat flow. We have
\[ (\partial_t + \square_t) \tr(K_t - cI_E) = 0, \]
where $K_t$ is the mean curvature of the Hitchin-Simpson connection associated with $h_t$ and $\square_t = \square_{h_t}$. Here the subscript $t$ remember us the dependence on the metric $h_t$.

Proof. We already know that
\[ (\partial_t + \square_t)K_t = 0. \]
Since the trace and the linear operators $\square_t$ and $\partial_t$ commute, we obtain
\[ (\partial_t + \square_t) \tr(K_t - cI_E) = \tr([\partial_t + \square_t](K_t - cI_E)) = \tr((\partial_t + \square_t)K_t) = 0. \]

Let $\text{Herm}^+_\text{int}(E)$ denote the set of all Hermitian metrics $h$ satisfying
\[ \|K_h\|_{L^1} = \int_X |K_h|^{\omega^n} w < +\infty \]
where $K_h$ is the mean curvature of the Hitchin-Simpson connection of $h$. If $X$ is compact this space coincides with $\text{Herm}^+(E)$. The space $\text{Herm}^+_\text{int}(E)$ was studied by Simpson in [31]. It is an analytic manifold, which in general is not connected, and has the following properties. If $k \in \text{Herm}^+_\text{int}(E)$ is a fixed element, then any other metric in the same connected component is given by $h = k \exp(a)$ with $a$ a smooth endomorphism of $E$ which is selfadjoint with respect to $k$. Moreover Simpson showed in [31] that for any Higgs bundle $E = (E, \phi)$ over a compact Kähler manifold $X$ the solution of the Donaldson heat flow remains in the same connected component of the initial metric. To be precise, if $k$ is a fixed metric in $\text{Herm}^+_\text{int}(E)$, then the unique solution of the Donaldson heat flow $h_t$ with $h_0 = k$ is contained in the same connected component as $k$.

Now, let $(X, \omega)$ be a compact Kähler manifold of (complex) dimension $n$ and let $E = (E, \phi)$ be a Higgs bundle of rank $r$ over $X$. Let $h_0 \in \text{Herm}^+(E)$ be an initial metric with the condition $\tr(K_0 - cI_E) = 0$, and let $h_t, 0 \leq t < +\infty$, be the corresponding solution of the Donaldson heat flow. Then $h_t$ will be of the form $h_t = h_0 \exp(S(t))$ for some section $S(t)$ of $\text{End}(E)$ over $X$, and $S(t)$ will be selfadjoint with respect to $h_0$. 

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Definition 7.1.1. Let \((X, \omega)\) be a compact Kähler manifold of (complex) dimension \(n\) and let \(\mathcal{E} = (E, \phi)\) be a Higgs bundle of rank \(r\) over \(X\). Let \(h_0 \in \text{Herm}^+(E)\) be an initial metric with the condition \(\text{tr}(\mathcal{K}_0 - cI_E) = 0\), where \(\mathcal{K}_0\) is the mean curvature of the Hitchin-Simpson connection associated with \(h_0\). Let \(h_t, 0 \leq t < +\infty\), be the solution of the Donaldson heat flow with initial condition \(h_0\), and let \(S(t)\) be a section of \(\text{End}(E)\) such that \(h_t = h_0 \exp(S(t))\) and \(S(t)\) is selfadjoint with respect to \(h_0\). Then \(|S(t)|^2 = \text{tr}(S(t) \cdot S(t))\) is a nonnegative real valued function on \(X\) and we have the following norms:

1. \(\|S(t)\|_{L^1} = \int_X |S(t)|^n \omega^n\)
2. \(\|S(t)\|_{L^2} = \left(\int_X |S(t)|^2 \omega^n\right)^{\frac{1}{2}}\)
3. \(\|S(t)\|_{L^\infty} = \max_X |S(t)|\)

Moreover, since \(h_t\) is differentiable for \(0 < t < +\infty\), so does \(S(t)\).

From the initial condition \(\text{tr}(\mathcal{K}_0 - cI_E) = 0\) and Lemma 7.1.4, using the theory of heat equations on Riemannian manifolds, we deduce that \(\text{tr}(\mathcal{K}_t - cI_E) = 0\). From \(v_t = -(\mathcal{K}_t - cI_E)\) and

\[
\frac{\partial}{\partial t} \text{tr} e^{S(t)} = \frac{\partial}{\partial t} \det e^{S(t)} = \frac{\partial}{\partial t} \det(h_0^{-1} h_t) = \frac{\partial}{\partial t} \det h_t = \frac{1}{\det h_0} \frac{\partial}{\partial t} \det h_t = \frac{1}{\det h_0} \text{tr}(h_t^{-1} \partial_t h_t) = \frac{1}{\det h_0} \text{tr} e^{S(t)} = \frac{1}{\det h_0} \text{tr}(\mathcal{K}_t - cI_E) = 0
\]

we deduce that \(\text{tr} S(t)\) is constant. Since \(e^{\text{tr} S(0)} = \det e^{S(0)} = \det(h_0^{-1} h_0) = 1\), one has \(\text{tr} S(0) = 0\), and hence

\[
\text{tr} S(t) = 0 \quad \text{for} \quad 0 \leq t < +\infty. \quad (7.5)
\]

Now, after introducing two useful results we prove one of the most important inequalities of this section.

Lemma 7.1.5. Let \(A, B \in M_{n,n}(\mathbb{C})\) such that \(A\) and \(B\) are selfadjoint and assume that \(B\) is positive semi-definite. Then

\[
|\text{tr}(AB)| \leq \sqrt{\text{tr}(AA) \cdot \text{tr}(BB)}.
\]

Proof. Since \(B\) is selfadjoint and positive semi-definite we obtain

\[
\text{tr}(B^2) \leq \text{tr}^2(B).
\]

In fact, let \(\lambda_1, \ldots, \lambda_n \in [0, +\infty]\) be the eigenvalues of \(B\). Hence,

\[
\text{tr}(B^2) = \sum_{1 \leq i \leq n} \lambda_i^2 \leq \left( \sum_{1 \leq i \leq n} \lambda_i \right)^2 = \text{tr}^2(B).
\]
Finally, from the Cauchy-Schwarz inequality and since $A$ and $B$ are selfadjoint we conclude
\[
|\text{tr}(AB)| = |\text{tr}(AB^*)| \leq \sqrt{\text{tr}(AA^*)} \sqrt{\text{tr}(BB^*)} = \\
= \sqrt{\text{tr}(AA)} \sqrt{\text{tr}(BB)} \leq \sqrt{\text{tr}(AA)} |\text{tr}(B)|.
\]
\hfill \Box

\textbf{Lemma 7.1.6.} Let $a_1, \ldots, a_n \in \mathbb{R}$. Then
\[
\sqrt{a_1^2 + \cdots + a_n^2} \leq n^{1/2} \ln(e^{a_1} + \cdots + e^{a_n} + e^{-a_1} + \cdots + e^{-a_n}).
\]
\textbf{Proof.} Since $f(x) = \ln(x)$ is a monotone increasing function of $x$ on $(0, +\infty)$, for $1 \leq i \leq n$ we have
\[
|a_i| = \ln(e^{a_i}) \leq \ln(e^{a_1} + \cdots + e^{a_n} + e^{-a_1} + \cdots + e^{-a_n}).
\]
Hence,
\[
\sqrt{a_1^2 + \cdots + a_n^2} \leq n^{1/2} \max_{1 \leq i \leq n} |a_i| \leq n^{1/2} \ln(e^{a_1} + \cdots + e^{a_n} + e^{-a_1} + \cdots + e^{-a_n}).
\]
\hfill \Box

\textbf{Lemma 7.1.7.} Let $(X, \omega)$ be a compact Kähler manifold of (complex) dimension $n$ and let $\mathfrak{E} = (E, \phi)$ be a Higgs bundle of rank $r$ over $X$. Let $h_0 \in \text{Herm}^+(E)$ be a metric with the condition $\text{tr}(\mathcal{K}_0 - cIE) = 0$, where $\mathcal{K}_0$ is the mean curvature of the Hitchin-Simpson connection of $h_0$. Let $h_t$ be the solution of the Donaldson heat flow with initial condition $h_0$ and let $S(t)$ be a section of $\text{End}(E)$ such that $h_t = h_0 \exp(S(t))$ and $S(t)$ is selfadjoint with respect to $h_0$. The following inequality holds:
\[
\left( \frac{1}{\sqrt{\text{rk}(E)}} \|S(t)\|_{L^1} - \text{Vol}(X) \ln(2\text{rk}(E)) \right) \|\mathcal{K}_t - cIE\|_{L^2} \leq -\sqrt{\text{Vol}(X)} \mathcal{L}(h_t, h_0).
\]
\textbf{(7.6)}

\textbf{Proof.} Let $V = \text{Vol}(X)$ and $r = \text{rk}(E)$. First of all, from the Hölder inequality we have
\[
\|\mathcal{K}_s - cIE\|_{L^1} = \int_X |\mathcal{K}_s - cIE| \frac{\omega^n}{n!} = \int_X 1 \cdot |\mathcal{K}_s - cIE| \frac{\omega^n}{n!} \leq \\
\leq (V)^{1/2} \left( \int_X |\mathcal{K}_s - cIE|^2 \frac{\omega^n}{n!} \right)^{1/2} = (V)^{1/2} \|\mathcal{K}_s - cIE\|_{L^2}
\]
\textbf{(7.7)}

Set $H_t = e^{S(t)} = h_0^{-1} h_t$ and let $\lambda_1(t), \ldots, \lambda_r(t)$ be the eigenvalues of $S(t)$. Since $S(t)$ is selfadjoint with respect to $h_0$ we deduce that, for each $t \geq 0$, they are real valued functions on $X$ and then, using Lemma \textbf{7.1.6} we obtain
\[
\|S(t)\| = \sqrt{\lambda_1^2(t) + \cdots + \lambda_r^2(t)} \leq \\
\leq r^{1/2} \ln(e^{\lambda_1(t)} + \cdots + e^{\lambda_r(t)} + e^{-\lambda_1(t)} + \cdots + e^{-\lambda_r(t)}) = \\
= r^{1/2} \ln(\text{tr}H_t + \text{tr}H_t^{-1}),
\]
\textbf{(7.8)}
so that

\[ r^{-1/2} |S(t)| \leq \ln(\text{tr}H_t + \text{tr}H_t^{-1}). \]  

(7.9)

Since \( S(t) \) is selfadjoint with respect to \( h_0 \), it follows that \( H_t = e^{S(t)} \) and \( H_t^{-1} = e^{-S(t)} \) are also selfadjoint with respect to \( h_0 \). Moreover, from the definition of \( H_t \) and the properties of the exponential of matrices we deduce that \( H_t \) and \( H_t^{-1} \) are positive definite and then \( \text{tr}H_t > 0 \) and \( \text{tr}H_t^{-1} > 0 \). In fact

\[ \text{tr}H_t = e^{\lambda_1(t)} + \ldots + e^{\lambda_r(t)} > 0 \]

and

\[ \text{tr}H_t^{-1} = e^{-\lambda_1(t)} + \ldots + e^{-\lambda_r(t)} > 0. \]

We also notice that \( H_t^{-1} \partial_t H_t = h_t^{-1} h_0 \partial_t (h_0^{-1} h_t) = h_t^{-1} \partial_t h_t \) and hence \( H_t^{-1} \partial_t H_t \) is selfadjoint with respect to \( h_0 \).

By direct calculation, from Lemma \([7.1.5]\) since \( \partial_t \) and the trace commute and since the trace is \( GL(r, \mathbb{C}) \)-invariant we find

\[
\begin{align*}
\frac{\partial}{\partial t} \ln(\text{tr}H_t + \text{tr}H_t^{-1}) &= \frac{\partial_t (\text{tr}H_t + \text{tr}H_t^{-1})}{\text{tr}(H_t + H_t^{-1})} = \frac{\partial_t \text{tr}H_t + \partial_t \text{tr}H_t^{-1}}{\text{tr}(H_t + H_t^{-1})} = \\
&= \frac{\text{tr}(\partial_t H_t) + \text{tr}(\partial_t H_t^{-1})}{\text{tr}(H_t + H_t^{-1})} = \frac{\text{tr}(H_t^{-1} \cdot \partial_t H_t) + \text{tr}(H_t^{-1} \cdot \partial_t H_t^{-1})}{\text{tr}(H_t + H_t^{-1})} \\
&\leq \frac{|\text{tr}(H_t^{-1} \cdot \partial_t H_t H_t) - \text{tr}(H_t^{-1} \cdot \partial_t H_t H_t^{-1})|}{\text{tr}(H_t + H_t^{-1})} = \frac{|\text{tr}(H_t^{-1} \cdot \partial_t h_t H_t) + |\text{tr}(H_t^{-1} \cdot \partial_t h_t H_t^{-1})|}{\text{tr}(H_t + H_t^{-1})} \\
&\leq \frac{|\sqrt{\text{tr}(h_t^{-1} \cdot \partial_t h_t \cdot h_t^{-1} \cdot \partial_t h_t) + |\text{tr}(H_t^{-1} \cdot \partial_t h_t H_t^{-1})|}{\text{tr}(H_t + H_t^{-1})} = \frac{|\sqrt{\text{tr}(h_t^{-1} \cdot \partial_t h_t \cdot h_t^{-1} \cdot \partial_t h_t) + |\text{tr}(H_t^{-1} \cdot \partial_t h_t H_t^{-1})|}{\text{tr}(H_t + H_t^{-1})} = \frac{|\sqrt{\text{tr}(h_t^{-1} \cdot \partial_t h_t \cdot h_t^{-1} \cdot \partial_t h_t) + |\text{tr}(H_t^{-1} \cdot \partial_t h_t H_t^{-1})|}{\text{tr}(H_t + H_t^{-1})} = |K_t - cI_E|. \quad (7.10)
\end{align*}
\]

On one hand, since \( \|K_t - cI_E\|_{L^2} \) is a nonnegative real valued monotone decreasing function on \( t, 0 \leq t < +\infty \), we have

\[ t\|K_t - cI_E\|_{L^2}^2 \leq \int_0^t \|K_s - cI_E\|_{L^2}^2 ds = -\mathcal{L}(h_t, h_0), \]  

(7.11)

and then, for \( t > 0 \)

\[ \|K_t - cI_E\|_{L^2} \leq t^{-1/2}(-\mathcal{L}(h_t, h_0))^{1/2}. \]  

(7.12)
On the other hand, since $X$ is compact we can use the Fubini-Tonelli Theorem. Integrating over $X$, from (7.10) and (7.9) and from the Hölder inequality we have

$$r^{-1/2}||S(t)||_{L^1} - V \ln(2r) = \int_X r^{-1/2}|S(t)| - \ln(2r)\frac{\omega^n}{n!} \leq$$

$$\leq \int_X |\ln(\text{tr}H_t + \text{tr}H_t^{-1}) - \ln(2r)|\frac{\omega^n}{n!} =$$

$$= \int_X \frac{\omega^n}{n!} \int_0^t \frac{\partial}{\partial s} \ln(\text{tr}H_s + \text{tr}H_s^{-1}) ds =$$

$$= \int_0^t ds \int_X \frac{\partial}{\partial s} \ln(\text{tr}H_s + \text{tr}H_s^{-1})\frac{\omega^n}{n!} \leq$$

$$\leq \int_0^t ds \int_X |\mathcal{K}_s - cI_E|\frac{\omega^n}{n!} =$$

$$= \int_0^t \|\mathcal{K}_s - cI_E\|_{L^1} ds =$$

$$= \int_0^t 1 \cdot \|\mathcal{K}_s - cI_E\|_{L^1} ds \leq$$

$$(\int_0^t ds)^{1/2} \left( \int_0^t \|\mathcal{K}_s - cI_E\|_{L^2}^2 ds \right)^{1/2} =$$

$$= t^{1/2} \left( \int_0^t \|\mathcal{K}_s - cI_E\|_{L^2}^2 ds \right)^{1/2} \leq$$

$$\leq t^{1/2} \left( \int_0^t V||\mathcal{K}_s - cI_E||_{L^2}^2 ds \right)^{1/2} =$$

$$= V^{1/2}t^{1/2} \left( \int_0^t \|\mathcal{K}_s - cI_E\|_{L^2}^2 ds \right)^{1/2} =$$

$$= V^{1/2}t^{1/2}(−\mathcal{L}(h_t, h_0))^{1/2}.$$

Combining (7.13) and (7.12), since $||\mathcal{K}_t - cI_E||_{L^2} \geq 0$ finally we obtain

$$(r^{-1/2}||S(t)||_{L^1} - V \ln(2r))||\mathcal{K}_t - cI_E||_{L^2} \leq (V^{1/2}t^{1/2}(−\mathcal{L}(h_t, h_0))^{1/2}) \cdot t^{-1/2}(−\mathcal{L}(h_t, h_0))^{1/2} =$$

$$= -V^{1/2}\mathcal{L}(h_t, h_0)$$

and this proves (7.10). □

Now we review some constructions involving Hermitian matrices. Let $E$ be a Higgs bundle with a fixed Hermitian metric $k$, and let $S = S(E)$ be the real vector bundle of selfadjoint endomorphisms of $E$. Suppose $\varphi : \mathbb{R} \to \mathbb{R}$ is a smooth function. Then we define a map of fibre bundles over $X$

$$\varphi : S \to S$$

as follows: suppose $s \in S$, then, at each point in $X$, choose an orthonormal frame $\{e_i\}$ for $E$ with $s(e_i) = \lambda_i e_i$, and set

$$\varphi(s)(e_i) = \varphi(\lambda_i)e_i.$$
Suppose $\Psi : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is a smooth function of two variables. Then we define a map of fibre bundles

$$\Psi : S \rightarrow S(\text{End}(E)),$$

where $S(\text{End}(E))$ consists of elements of $\text{End}(\text{End}(E))$ which are selfadjoint with respect to the usual metric $\text{tr}(A \cdot B^*)$. The function $\Psi$ is described as follows. Suppose $s \in S$ and $A \in \text{End}(E)$. Choose an orthonormal frame field $\{e_i\}$ of eigenvectors of $s$ with eigenvalues $\lambda_i$. Let $\{\hat{e}_i\}$ be the dual frame field in $E^*$, and write $A = \sum_{i,j} A_{ij} \hat{e}_i \otimes e_j$. Then set

$$\Psi(s)(A) = \sum_{i,j} \Psi(\lambda_i, \lambda_j) A_{ij} \hat{e}_i \otimes e_j.$$

Again this is well defined, smooth and linear in $A$.

If the functions $\varphi$ and $\Psi$ are analytic, then we can express the constructions above as a power series. If

$$\varphi(\lambda) = \sum a_n \lambda^n$$

then

$$\varphi(s) = \sum a_n s^n.$$ 

If

$$\Psi(\lambda_1, \lambda_2) = \sum b_{mn} \lambda_1^m \lambda_2^n$$

then

$$\Psi(s)(A) = \sum b_{mn} s^m A s^n.$$ 

Now, suppose $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ is a smooth function. Define $d\varphi : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ by

$$d\varphi(\lambda_1, \lambda_2) = \frac{\varphi(\lambda_1) - \varphi(\lambda_2)}{\lambda_1 - \lambda_2},$$

which is taken as $(d\varphi/d\lambda)(\lambda_1)$ if $\lambda_1 = \lambda_2$. If $s \in S$, then $D'\varphi(s) = d\varphi(s)(D's)$ where the right side uses the obvious extension to form-coefficient in the second variable. To see this for example when $\varphi$ is analytic, note that if $\varphi(\lambda) = \lambda^n$ then

$$d\varphi(\lambda_1, \lambda_2) = \sum_{i+j=n-1} \lambda_1^i \lambda_2^j$$

whereas

$$D'(s^n) = \sum_{i+j=n-1} s^i D'(s^j).$$ 

The construction $\varphi(s)$ and $\Psi(S)$ retain the same positivity properties as $\varphi$ and $\Psi$. For example if $\varphi(\lambda) > 0$ for all $\lambda$, then $\varphi(s)$ is positive definite for all $s$. And if $\Psi(\lambda_1, \lambda_2) > 0$ for all $\lambda_1, \lambda_2$ then $\text{tr}(\Psi(A) \cdot A^*) > 0$ for all $s$ and all $A \in \text{End}(E)$.

We will describe how these constructions behave with respect to Sobolev norms. Fix a Hermitian metric $k$ on a Higgs bundle $E$. Using the metric we can define the space

$$L^p_0(S) = \{s \in S | \int_X |s|^p \frac{\omega^n}{n!} < +\infty\},$$

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where $s$ is selfadjoint with respect to the metric $k$ and $|s|^2 = \text{tr}(s \cdot s)$. In particular $L^2_0(S)$ is a Hilbert space with the inner product
\[
\langle s, u \rangle = \int_X \text{tr}(s \cdot u) \frac{\omega^n}{n!} \quad (7.14)
\]
and then
\[
\|s\|_{L^2}^2 = \int_X \text{tr}(s \cdot s) \frac{\omega^n}{n!}. \quad (7.15)
\]

Note 7.1.8. In the above formulas $s$ and $u$ are selfadjoint with respect to the metric $k$ since they are element of $S$. Then $\int_X \text{tr}(s \cdot u) \frac{\omega^n}{n!} \in \mathbb{R}$, in fact, since the Kähler form is real, from $\text{tr}(A) = \text{tr}(A^*)$ and $\text{tr}(AB) = BA$, we have
\[
\int_X \text{tr}(s \cdot u) \frac{\omega^n}{n!} = \int_X \text{tr}(s \cdot u) \frac{\bar{\omega}^n}{n!} = \int_X \text{tr}(s \cdot \bar{u}) \frac{\omega^n}{n!} = \int_X \text{tr}(u^* \cdot s^*) \frac{\omega^n}{n!} = \int_X \text{tr}(u \cdot s) \frac{\omega^n}{n!} = \int_X \text{tr}(s \cdot u) \frac{\omega^n}{n!}.
\]

From (7.14) and (7.15) we can deduce the following

**Lemma 7.1.9.** Let $k$ be a fixed Hermitian metric and let $K_k$ be the mean curvature of the Hitchin-Simpson connection associated with the metric $k$. If $u_m \rightharpoonup u_\infty$ in $L^2_0(S)$, then
\[
\int_X \text{tr}(u_m \cdot K_k) \frac{\omega^n}{n!} \longrightarrow \int_X \text{tr}(u_\infty \cdot K_k) \frac{\omega^n}{n!}, \quad (7.16)
\]
Moreover, if $\text{tr}(u_m) = 0$ for each $m$, then $\text{tr}(u_\infty) = 0$.

**Proof.** From (7.14) and (7.15) we immediately deduce that
\[
s \longmapsto \int_X \text{tr}(s \cdot K_k) \frac{\omega^n}{n!}
\]
is a continuous linear functional on $L^2_0(S)$ and then (7.16) follows from the definition of weak convergence in Hilbert spaces. In order to prove that $\text{tr}(u_\infty) = 0$ if $\text{tr}(u_m) = 0$ for each $m$ let us consider the set
\[
\Omega = \{ x \in X | \text{tr}(u_\infty(x)) > 0 \}.
\]
Since $u_\infty$ is a continuous section of $\text{End}(E)$ and it is selfadjoint with respect to the metric $k$ we have that $\text{tr}(u_\infty)$ is a continuous real valued function on $X$, so that $\Omega$ is an open subset of $X$. Then, since $X$ is compact and the volume form $\frac{\omega^n}{n!}$ defines a finite measure on $X$, if we apply (7.16) to the open set $\Omega$ we obtain
\[
\int_{\Omega} \text{tr}(u_m) \frac{\omega^n}{n!} = \int_{\Omega} \text{tr}(u_m \cdot I_E) \frac{\omega^n}{n!} \longrightarrow \int_{\Omega} \text{tr}(u_\infty \cdot I_E) \frac{\omega^n}{n!} = \int_{\Omega} \text{tr}(u_\infty) \frac{\omega^n}{n!}.
\]
Since $\text{tr}(u_m) = 0$ we deduce that
\[
\int_{\Omega} \text{tr}(u_\infty) \frac{\omega^n}{n!} = 0.
\]
But the continuous function $\text{tr}(u_\infty)$ is strictly positive on the open set $\Omega$ and then we conclude that $\Omega = \emptyset$. In a similar way the open set $\Theta = \{ x \in X | \text{tr}(u_\infty(x)) < 0 \}$ is empty, so $\text{tr}(u_\infty) = 0$ on $X$. \qed
Let $L^p_1(S)$ denote the space of sections $s \in S$ such that $s \in L^p(S)$ and $D''s \in L^p(E)$. Note that this is a condition on $D''E s = D''s$ and also a growth condition involving the Higgs field $\phi$ if $X$ is noncompact. For a given number $b$, denote the closed subspaces of sections $s \in S$ with $|s| \leq b$ by $L^p_{0,b}$ and $L^p_{1,b}$.

Finally let $P(S)$ be the normed space of smooth sections $s \in S$ with norm

$$\|s\|_p = \max_X |s| + \|D''s\|_{L^2} + \|D'_{L^1} s\|_{L^1}.$$ 

The constructions $\varphi$ and $\Psi$ behave in a rather delicate fashion on $L^p(S)$ and $L^p_1(S)$ as it is shown in the following Proposition. They behave better on $P(S)$ since their $\|\cdot\|_{L^\infty}$-norm is controlled.

**Proposition 7.1.10.** Let $\varphi$ and $\Psi$ be functions as above.

1. The map $\varphi$ extends to a continuous nonlinear map

$$\varphi : L^p_{0,b}(S) \longrightarrow L^p_{0,b'}(S)$$

for some $b'$.

2. The map $\Psi$ extends to a map

$$\Psi : L^p_{0,b}(S) \longrightarrow \text{Hom}(L^p(\text{End}(E)), L^q(\text{End}(E)))$$

for $q \leq p$, and for $q < p$ it is continuous in the norm operator topology.

3. The map $\varphi$ extends to a map

$$\varphi : L^p_{1,b}(S) \longrightarrow L^p_{0,b'}(S)$$

for $q \leq p$, and it is continuous for $q < p$. The formula $D'' \varphi(s) = d\varphi(s)(D''s)$ holds in this context.

4. If $\varphi$ and $\Psi$ are analytic with infinite radius of convergence, the maps

$$\varphi : P(S) \longrightarrow P(S),$$

$$\Psi : P(S) \longrightarrow P(\text{End}(\text{End}(E)))$$

are analytic.

For the proof see Proposition 4.1 in [31].

Now, let $k \in \text{Herm}^+(E)$ be a fixed Hermitian structure. We already know that any Hermitian metric $h$ will be of the form $k \exp(v)$ for some section $v$ of $\text{End}(E)$ over $X$. Moreover, $v$ is selfadjoint with respect to $k$. We can join $k$ to $h$ by the geodesic $h_\tau = k \exp(\tau v)$ where $0 \leq \tau \leq 1$. Note that here $v_\tau = h^{-1}_\tau \partial_\tau h_\tau = v$ is constant, i.e., it does not depend on $\tau$. (See Chapter VI, §1 and §2 in [25] for more details). Now, in the proof of Theorem [6.4.9] we got an expression for the second derivative $\partial^2_{\tau} \mathcal{L}(h_\tau, k)$ for any curve $h_\tau = k \exp(\tau v)$, namely:

$$\partial^2_{\tau} \mathcal{L}(h_\tau, k) = \int_X \text{tr}[\partial_\tau \mathcal{K}_\tau \cdot v_\tau + (\mathcal{K}_\tau - cI_E) \cdot \partial_\tau v_\tau ] \frac{\omega^n}{n!},$$
where $\mathcal{K}_\tau$ is the mean curvature endomorphism of the Hitchin-Simpson connection associated with the metric $h_\tau = k \exp(\tau v)$. Notice that in our case, the chosen curve is such that $h_0 = k$, since it is also a geodesic $\partial_\tau v_\tau = 0$ we have

$$
\partial^2_\tau \mathcal{L}(h_\tau, k) = \int_X \text{tr}(\partial_\tau \mathcal{K}_\tau \cdot v) = \frac{\omega^n}{n!} = ||\mathcal{D}' h_\tau, v||^2_{h_\tau}. \tag{7.17}
$$

Therefore, following [33], the idea is to find a simple expression for $||\mathcal{D}' h_\tau, v||^2_{h_\tau}$ or equivalently for $||\mathcal{D}' h_\tau, v||^2_{h_\tau}$ and to integrate it twice with respect to $\tau$. We can do this using local coordinates, indeed, at any point of $X$ we can choose a local frame field so that $h_0 = I$ and $v = \text{diag}(\beta_1, \ldots, \beta_r)$. In particular, using such a local frame field we have, for $h_\tau = k \exp(\tau v), h_\tau^{ij} = e^{-\beta_1^\tau} \delta_{ij}$, and hence, (after a short computation) we obtain

$$
||\mathcal{D}' h_\tau, v||^2_{h_\tau} = \int_X \sum_{i,j=1}^r e^{(\beta_i - \beta_j)\tau} |\mathcal{D}' h_\tau|_{ij}^2 \frac{\omega^{n-1}}{(n-1)!}. \tag{7.18}
$$

Now, at $\tau = 0$ the functional $\mathcal{L}(h_\tau, k)$ vanishes and since $h_0 = k$ is not a Hermitian-Yang-Mills structure, we have

$$
\partial_\tau \mathcal{L}(h_\tau, k)|_{\tau=0} = \int_X \text{tr}[\mathcal{K}_0 - c I_E] \cdot v \frac{\omega^n}{n!}. \tag{7.17}
$$

Then, integrating (7.17) twice we obtain

$$
\mathcal{L}(h_\tau, k) = \tau \int_X \text{tr}[\mathcal{K}_0 - c I_E] \cdot v \frac{\omega^n}{n!} + \int_X \sum_{i,j=1}^r \Psi_\tau(\beta_i, \beta_j) |\mathcal{D}' h_\tau|_{ij}^2 \frac{\omega^{n-1}}{(n-1)!}, \tag{7.18}
$$

where $\Psi_\tau$ is the analytic function given by

$$
\Psi_\tau(\beta_i, \beta_j) = \frac{e^{(\beta_j - \beta_i)\tau} - (\beta_j - \beta_i)\tau - 1}{(\beta_j - \beta_i)^2}.
$$

In particular, at $\tau = 1$ the expression (7.18) corresponds (up to a constant term) to the definition of the Donaldson functional given by Simpson [31]. In fact, setting $\Psi(x_1, x_2) = \Psi_1(x_1, x_2)$ for $(x_1, x_2) \in \mathbb{R} \times \mathbb{R}$, for two metrics in the same component $k$ and $h = e^s$, Simpson defines in [31] the Donaldson functional as

$$
\mathcal{L}(h, k) = \int_X \text{tr}(s \cdot \mathcal{K}_k) \frac{\omega^n}{n!} + \int_X \langle \Psi(s)(\mathcal{D}' h, s, \mathcal{D}' h) |_{ij} \frac{\omega^{n-1}}{(n-1)!}. \tag{7.18}
$$

Notice also that if the initial metric $k = h_0$ is Hermitian-Yang-Mills, the first term of the right hand side of (7.18) vanishes and the functional coincides with the Donaldson functional in [33].

Now, let $h_0 \in \text{Herm}^+(E)$ be a metric with the condition $\text{tr}(\mathcal{K}_0 - c I_E) = 0$ and let $h_t$ be the solution of the Donaldson heat flow with initial metric $h_0$. Since $\text{tr}(S(t)) = 0$ and since $\text{tr}(AB) = \text{tr}(BA)$, from (7.18) we have

$$
\mathcal{L}(h_t, h_0) = \int_X \text{tr}(S(t) \cdot \mathcal{K}_0) \frac{\omega^n}{n!} + \int_X \sum_{i,j=1}^r \Psi_1(\beta_i(t), \beta_j(t)) |\mathcal{D}' h_t|_{ij}^2 \frac{\omega^{n-1}}{(n-1)!}, \tag{7.19}
$$

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where $\beta_1(t), \ldots, \beta_r(t)$ are the eigenvalues of $S(t)$. Following Simpson in [31] we can rewrite (7.19) in the equivalent form

$$L(h_t, h_0) = \int_X \text{tr}(S(t) \cdot K_0) \frac{\omega^n}{n!} + \int_X \langle \Psi(S(t))(\mathcal{D}''S(t)), \mathcal{D}''S(t) \rangle h_0 \frac{\omega^{n-1}}{(n-1)!}. \quad (7.20)$$

From the previous constructions it follows that if the Donaldson functional $L(h_t, h_0)$ is not bounded below, then $\|S(t)\|_{L^\infty} \to +\infty$ as $t \to +\infty$. Namely we have

**Lemma 7.1.11.** Let $(X, \omega)$ be a compact Kähler manifold of (complex) dimension $n$ and let $\mathfrak{E} = (E, \phi)$ be a Higgs bundle of rank $r \geq 2$ over $X$. Let $h_0 \in \text{Herm}^+(E)$ be a Hermitian metric such that $\text{tr}(K_0 - cI_E) = 0$, where $K_0$ is the mean curvature of the Hitchin-Simpson connection associated with $h_0$. Let $h_t$ be the solution of the Donaldson heat flow with initial metric $h_0$ and assume $h_t = h_0 e^{S(t)}$, where $S(t)$ is a section of $\text{End}(E)$ and it is selfadjoint with respect to $h_0$. If

$$L(h_t, h_0) \to -\infty \quad \text{as} \quad t \to +\infty,$$

then

$$\|S(t)\|_{L^\infty} \to +\infty \quad \text{as} \quad t \to +\infty.$$

**Proof.** We will show that if the limit does not hold, there is a sequence $t_m \to +\infty$ such that $L(h_{t_m}, h_0)$ is bounded below. Suppose the required limit does not hold. Hence, we can find a positive constant $B > 0$ and a sequence $t_m \to +\infty$ such that

$$\|S(t_m)\|_{L^\infty} \leq B$$

From the theory of the Gerschgorin circles (see [2] for details), there exists a positive radius $R > 0$, which does not depend on $m$, such that

$$|\beta_i^{(m)}| \leq R \quad \text{for all} \quad m \geq 0, \quad 1 \leq i \leq r,$$

where $\beta_i^{(m)}$ are the eigenvalues of $S(t_m)$. Since

$$\Psi(x_1, x_2) = \frac{e^{(x_2-x_1)} - (x_2 - x_1)}{(x_2 - x_1)^2} - 1$$

is analytic with infinite radius of convergence, there exists a constant $C$ such that

$$\sum_{i,j=1}^r |\Psi(\beta_i^{(m)}, \beta_j^{(m)})| \leq C$$

and $C$ does not depend on $m$. Therefore, from (7.19) and since $X$ is compact and we can locally consider $\mathcal{D}'' = D'' + \phi$ as a matrix of forms of degree 1 which
does not depend on $m$, we have
\[
\mathcal{L}(h_{t_m}, h_0) \geq -|\mathcal{L}(h_{t_m}, h_0)| = \\
= -\left| \int_X \text{tr}(S(t_m) \cdot K_0) \frac{\omega^n}{n!} + \right. \\
+ \int_X \sum_{i,j=1}^r \Psi_1(\beta_i^{(m)}, \beta_j^{(m)}) |D'' S(t_m)^j|^2 \frac{\omega^{n-1}}{(n-1)!} \bigg| \\
\geq -\left| \int_X \text{tr}(S(t_m) \cdot K_0) \frac{\omega^n}{n!} \\
+ \int_X \sum_{i,j=1}^r \Psi_1(\beta_i^{(m)}, \beta_j^{(m)}) |D'' S(t_m)^j|^2 \frac{\omega^{n-1}}{(n-1)!} \bigg| \\
\geq -C_1\|S(t_m)\|_{L^2} - C \int_X |D'' S(t_m)^j|^2 \frac{\omega^{n-1}}{(n-1)!} \\
\geq -C_1\|S(t_m)\|_{L^2} - CC_2C_1\|S(t_m)\|_{L^\infty} \geq \\
\geq -C_1 + CC_2B
\]
and this estimate does not depend on $m$. Therefore $\mathcal{L}(h_{t_m}, h_0)$ is bounded below as $t_m \to +\infty$, which is a contradiction. \hfill \square

Now, let $\{s_m\}$ be a sequence of sections in $S$ with $\text{tr}(s_m) = 0$ such that
\[
\|s_m\|_{L^1} \to +\infty
\]
and let us assume
\[
\max_X |s_m| \leq C_1\|s_m\|_{L^1} + C_2
\]
where $C_1$ and $C_2$ do not depend on $m$. Set $t_m = \|s_m\|_{L^1}$ and $u_m = t_m^{-1}s_m$, so $\|u_m\|_{L^1} = 1$. Since $t_m \to +\infty$, from
\[
\|u_j\|_{L^\infty} = \frac{\|S(t_j)\|_{L^\infty}}{\|S(t_j)\|_{L^1}} \leq C_1 + \frac{C_2}{\|S(t_j)\|_{L^1}}
\]
we conclude that $\max_X |u_m| \leq C$, where $C$ does not depend on $m$. Moreover, since $\text{tr}$ is linear, from $u_m = t_m^{-1}s_m$ and $\text{tr}(s_m) = 0$ we deduce that $\text{tr}(u_m) = 0$. 94
Lemma 7.1.12. Up to extracting a subsequence, $u_m \rightharpoonup u_\infty$ weakly in $L^2_1(S)$. The limit $u_\infty$ is not 0. If $\Phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a nonnegative smooth function with compact support such that $\Phi(x_1, x_2) \leq (x_1 - x_2)^{-1}$ whenever $x_1 > x_2$, then

$$
\int_X \text{tr}(u_\infty \cdot K_k) \frac{\omega^n}{n!} + \int_X \langle \Phi(u_\infty)(D''u_\infty), D''u_\infty \rangle_k \frac{\omega^{n-1}}{n-1!} \leq
$$

$$
\liminf_m \left[ \int_X \text{tr}(u_m \cdot K_k) \frac{\omega^n}{n!} + \int_X \langle l_m \Psi(l_m u_m)(D''u_m), D''u_m \rangle_k \frac{\omega^{n-1}}{n-1!} \right],
$$

(7.21)

where $k$ is a Hermitian metric on the Higgs bundle $(E, \phi)$.

Proof. In Proposition 5.3 and Lemma 5.4 in [31] Simpson proved that, up to considering a subsequence, $u_m \rightharpoonup u_\infty$ weakly in $L^2_1(S)$ and $u_\infty \neq 0$. Hence, we have to prove the estimate (7.21). First, assume $\Phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a nonnegative smooth function with compact support $K$ such that $\Phi(x_1, x_2) < (x_1 - x_2)^{-1}$ whenever $x_1 > x_2$. Then, there exists $l > 0$ such that

$$
\Psi(x_1, x_2) < l\Psi(lx_1, lx_2) \quad \text{for all} \quad (x_1, x_2) \in \mathbb{R} \times \mathbb{R}.
$$

(7.22)

To see this fix $(a, b) \in \mathbb{R} \times \mathbb{R}$. Hence, since as $l \to +\infty$ the quantity $l\Psi(la, lb)$ increases monotonically to $(a - b)^{-1}$ if $a > b$ and to $+\infty$ if $a \leq b$, there exists $l_{(a, b)} > 0$ such that

$$
\Phi(a, b) < l_{(a, b)}\Psi(l_{(a, b)}a, l_{(a, b)}b)
$$

so that, in order to prove (7.22), it suffices to take

$$
l = \max_{(a, b) \in K} l_{(a, b)}.
$$

From (7.22) and since $l_m \to +\infty$, for $m \gg 0$

$$
\|\Phi^{1/2}(u_m)D''u_m\|_{L^2}^2 = \int_X \langle \Phi(u_\infty)(D''u_\infty), D''u_\infty \rangle_k \frac{\omega^{n-1}}{n-1!} \leq
$$

$$
\int_X \langle l_m \Psi(l_m u_m)(D''u_m), D''u_m \rangle_k \frac{\omega^{n-1}}{n-1!}
$$

and then

$$
\liminf_m \|\Phi^{1/2}(u_m)D''u_m\|_{L^2}^2 \leq \liminf_m \int_X \langle l_m \Psi(l_m u_m)(D''u_m), D''u_m \rangle_k \frac{\omega^{n-1}}{n-1!},
$$

(7.23)

We already know that $\max |u_m| \leq C$, where $C$ does not depend on $m$. Then, since $X$ is compact, $\{u_m\} \subseteq L^2_{0, b}(S)$.

Now, consider the compact immersion of Sobolev spaces

$$
L^2_1(S) \hookrightarrow L^2_0(S).
$$

(7.24)

From $u_m \rightharpoonup u_\infty$ weakly in $L^2_1(S)$ and from (7.24) we can deduce that, up to consider a subsequence, $u_m \rightharpoonup u_\infty$ strongly in $L^2_{0, b}(S)$.

So, we can apply (7.1.10) (b) to conclude that for any $q < 2$

$$
\Phi^{1/2}(u_m) \to \Phi^{1/2}(u_\infty) \quad \text{strongly in} \ \text{Hom}(L^2, L^q).
$$

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Moreover, from \( u_m \to u_\infty \) weakly in \( L^2(S) \), we have \( D^n u_m \to D^n u_\infty \) weakly in \( L^2_0(S) \) and hence, from Lemma 5.3.4 we find that

\[
\Phi^{1/2}(u_m)D^n u_m \to \Phi^{1/2}(u_\infty)D^n u_\infty \quad \text{weakly in } L^2_0(S)
\]

for any \( q < 2 \). So that, from Lemma 5.3.3

\[
\|\Phi^{1/2}(u_\infty)D^n u_\infty\|_{L^q}^2 \leq \liminf_m \|\Phi^{1/2}(u_m)D^n u_m\|_{L^q}^2 \quad (7.25)
\]

for any \( q < 2 \).

On the other hand, from Lemma 7.1.9 we know that

\[
\int_X \text{tr}(u_m \cdot \mathcal{K}_k) \frac{\omega^n}{n!} \to \int_X \text{tr}(u_\infty \cdot \mathcal{K}_k) \frac{\omega^n}{n!} \quad (7.26)
\]

Set \( V = \text{Vol}(X) \). From (7.23), (7.25) and (7.26) we have

\[
\int_X \text{tr}(u_\infty \cdot \mathcal{K}_k) \frac{\omega^n}{n!} + \|\Phi^{1/2}(u_\infty)D^n u_\infty\|_{L^q}^2 \leq \int_X \text{tr}(u_\infty \cdot \mathcal{K}_k) \frac{\omega^n}{n!} + \liminf_m \|\Phi^{1/2}(u_m)D^n u_m\|_{L^q}^2 \leq \int_X \text{tr}(u_\infty \cdot \mathcal{K}_k) \frac{\omega^n}{n!} + \liminf_m V^{\frac{2-q}{q}} \|\Phi^{1/2}(u_m)D^n u_m\|_{L^q}^2 \leq \int_X \text{tr}(u_\infty \cdot \mathcal{K}_k) \frac{\omega^n}{n!} + V^{\frac{2-q}{q}} \liminf_m \int_X \langle l_m \Psi(l_m u_m)(D^n u_m), D^n u_m \rangle \frac{\omega^{n-1}}{(n-1)!} = (1 - V^{\frac{2-q}{q}}) \int_X \text{tr}(u_\infty \cdot \mathcal{K}_k) \frac{\omega^n}{n!} + V^{\frac{2-q}{q}} \liminf_m \int_X \langle l_m \Psi(l_m u_m)(D^n u_m), D^n u_m \rangle \frac{\omega^{n-1}}{(n-1)!} = (1 - V^{\frac{2-q}{q}}) \int_X \text{tr}(u_\infty \cdot \mathcal{K}_k) \frac{\omega^n}{n!} + V^{\frac{2-q}{q}} \liminf_m \int_X \text{tr}(u_m \cdot \mathcal{K}_k) \frac{\omega^n}{n!} + \int_X \langle l_m \Psi(l_m u_m)(D^n u_m), D^n u_m \rangle \frac{\omega^{n-1}}{(n-1)!}.
\]

This works for any \( q < 2 \) and then, in the limit, using some continuity and boundedness arguments, this implies inequality (7.21).

Since (7.21) holds for all nonnegative smooth functions \( \Phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) with compact support such that \( \Phi(x_1, x_2) < (x_1 - x_2)^{-1} \) whenever \( x_1 > x_2 \), again using some continuity and boundedness arguments, we can conclude that the inequality in the Lemma also holds if we assume \( \Phi(x_1, x_2) \leq (x_1 - x_2)^{-1} \) whenever \( x_1 > x_2 \).

**Lemma 7.1.13.** In the hypothesis of the previous Lemma, let \( \tilde{\Phi} : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) be a nonnegative smooth function such that

\[
\tilde{\Phi}(\lambda_i, \lambda_j) = \frac{1}{\lambda_i - \lambda_j} \quad \text{if} \quad \lambda_i > \lambda_j,
\]

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where \( \lambda_1 < \cdots < \lambda_l \) are the distinct eigenvalues of \( u_\infty \). Then
\[
\int_X \text{tr}(u_\infty \cdot K_k) \frac{\omega^n}{n!} + \int_X \langle \Phi(u_\infty)(D''u_\infty), D''u_\infty \rangle k \frac{\omega^{n-1}}{(n-1)!} \leq \\
\leq \liminf_m \left[ \int_X \text{tr}(u_m \cdot K_k) \frac{\omega^n}{n!} + \int_X \langle l_m \Psi(l_m u_m)(D''u_m), D''u_m \rangle k \frac{\omega^{n-1}}{(n-1)!} \right].
\]

**Proof.** We can construct a nonnegative smooth function \( \Phi : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) with compact support such that
1. \( \Phi(\lambda, \lambda) = \tilde{\Phi}(\lambda, \lambda) \), where \( \lambda_1 < \cdots < \lambda_l \) are the distinct eigenvalues of \( u_\infty \),
2. \( \Phi(x_1, x_2) \leq (x_1 - x_2)^{-1} \) whenever \( x_1 > x_2 \).

Hence, from (1) and the above Lemma we conclude that
\[
\int_X \text{tr}(u_\infty \cdot K_k) \frac{\omega^n}{n!} + \int_X \langle \Phi(u_\infty)(D''u_\infty), D''u_\infty \rangle k \frac{\omega^{n-1}}{(n-1)!} = \\
= \int_X \text{tr}(u_\infty \cdot K_k) \frac{\omega^n}{n!} + \int_X \langle \Phi(u_\infty)(D''u_\infty), D''u_\infty \rangle k \frac{\omega^{n-1}}{(n-1)!} \leq \\
\leq \liminf_m \left[ \int_X \text{tr}(u_m \cdot K_k) \frac{\omega^n}{n!} + \int_X \langle l_m \Psi(l_m u_m)(D''u_m), D''u_m \rangle k \frac{\omega^{n-1}}{(n-1)!} \right].
\]

The notion of weak subbundle of a holomorphic vector bundle was introduced in [34], and we can make a similar definition.

**Definition 7.1.2.** Let \((X, \omega)\) be a compact Kähler manifold of (complex) dimension \( n \) and let \( \mathcal{E} = (E, \phi) \) be a Higgs sheaf of rank \( r \) over \( X \). Let \( k \) be a Hermitian metric on \( E \) and let \( D_k = D'_k + D''_k \) be the Hitchin-Simpson connection associated with \( k \). A \( L^2_1(S) \)-subbundle of \( E \) is a section of \( \pi \in L^2_1(S) \) such that

1. \( \pi^2 = \pi = \pi^* k \),
2. \( (I_E - \pi) D''(\pi) = 0 \).

Following Uhlenbeck and Yau in [34] one can prove that \( \pi \) is smooth outside a subvariety of complex codimension greater than or equal to 2 and that it defines a Higgs subsheaf of \( E \). In fact, since \( D'' = D'' + \phi \), we can separate the components of type \((0,1)\) from the components of type \((1,0)\). Hence we have

1. \( (I_E - \pi) D''(\pi) = 0 \),
2. \( (I_E - \pi) \phi(\pi) = 0 \).

From [34], (1) can be identified as the holomorphic condition, while (2) can be identified as the Higgs subsheaf condition.
Lemma 7.1.14. (Key Lemma) Let $(X, \omega)$ be a compact Kähler manifold of (complex) dimension $n$ and let $E = (E, \phi)$ be a Higgs bundle of rank $r \geq 2$ over $X$. Let $h_t$ be the solution of the Donaldson heat flow with initial condition $h_0$, and suppose $\text{tr}(\mathcal{K}_0 - cI_E) = 0$, where $\mathcal{K}_0$ is the mean curvature endomorphism of the Hitchin-Simpson connection associated with $h_0$. Let us assume $(E, \phi)$ is $\omega$-semistable and $\mathcal{L}(h_t, h_0)$ is not bounded below, i.e. $\mathcal{L}(h_t, h_0) \to -\infty$ as $t \to +\infty$. Then
\[ -\frac{\mathcal{L}(h_t, h_0)}{\|S(t)\|_{L^1}} \to 0 \quad \text{as} \quad t \to +\infty, \]
where $S(t)$ is a section of $\text{End}(E)$ such that $h_t = h_0 e^{S(t)}$ and $S(t)$ is selfadjoint with respect to $h_0$.

Proof. Let $\mathcal{K}_t$ be the mean curvature of the Hitchin-Simpson connection associated with the metric $h_t$. We will show that if the estimate does not hold, there is a Higgs subsheaf which contradicts semistability. Suppose the required estimate does not hold. Hence, we can find a positive constant $\epsilon > 0$ and a sequence $t_m \to +\infty$ such that
\[ -\frac{\mathcal{L}(h_{t_m}, h_0)}{\|S(t_m)\|_{L^1}} \geq \epsilon > 0. \] (7.27)
Since $\max_X |\mathcal{K}_t - cI_E|$ is a monotone decreasing function (see Proposition 6.5.3 for details), from
\[ \max_X |\mathcal{K}_t| \leq \max_X |\mathcal{K}_t - cI_E| + c \]
we deduce that $\max_X |\mathcal{K}_t|$ is uniformly bounded with respect to $t$. So from Lemma (3.1)(d) in [31] and the hypothesis that $\max_X |\mathcal{K}_t|$ are uniformly bounded, we have the following Simpson’s estimate (p. 885 in [31])
\[ \|S(t)\|_{L^\infty} \leq C_1 \|S(t)\|_{L^1} + C_2, \] (7.28)
where $C_1$ and $C_2$ depend only on the curvature of the initial metric $h_0$ and the Kähler form $\omega$. Since $\mathcal{L}(h_t, h_0)$ is not bounded below, $\mathcal{L}(h_{t_m}, h_0) \to -\infty$ as $t_m \to +\infty$ and then $\|S(t_m)\|_{L^1} \to +\infty$. In fact, from Lemma 7.1.11 and (7.28) we have
\[ \frac{\|S(t_m)\|_{L^\infty} - C_2}{C_1} \leq \|S(t_m)\|_{L^1} \to +\infty \quad \text{as} \quad t_m \to +\infty. \]
Set $u_m = t_m^{-1}S(t_m)$, where $t_m = \|S(t_m)\|_{L^1}$, then $\|u_m\|_{L^1} = 1$. Since $S(t)$ is selfadjoint with respect to $h_0$, the $u_m$ are also selfadjoint with respect to $h_0$. From the hypothesis $\text{tr}(\mathcal{K}_0 - cI_E) = 0$ and from (7.5) we deduce that
\[ \text{tr}u_m = |t_m|^{-1}|S(t_m)| = |t_m|^{-1}\text{tr}S(t_m) = 0. \] (7.29)
Since $\|S(t_m)\|_{L^1} \to +\infty$, from (7.28) we find
\[ \|u_m\|_{L^\infty} = \frac{\|S(t_m)\|_{L^\infty}}{\|S(t_m)\|_{L^1}} \leq C_1 + \frac{C_2}{\|S(t_m)\|_{L^1}}, \]
and then for $m$ largely enough, $\|u_m\|_{L^\infty} \leq C_1 + 1$. 98
Hence, up to extracting a subsequence, from Lemma 5.4 in [31] \( u_m \to u_\infty \)
weakly in \( L^2 \), and the limit is not 0.

Moreover, from (7.29) and from Lemma 7.1.9 we deduce that \( u_\infty = 0 \). From
Lemma 5.5 in [31] we know that the eigenvalues of \( u_\infty \) are real and constant
almost everywhere, in other words, there are \( \lambda_1 < \cdots < \lambda_l \) which are the distinct
eigenvalues of \( u_\infty(x) \) for almost all \( x \in X \).

Since \( u_\infty = 0 \) and since \( u_\infty \) is not 0 we must have \( l \geq 2 \), otherwise if
\( u_\infty = 0 \) and \( l = 1 \) it follows that \( u_\infty = 0 \) contradicting the fact that \( u_\infty \neq 0 \).

The weak limit \( u_\infty \) gives rise to a flag of \( L^2_\ast(S) \)-subbundles. For any integer
\( 1 \leq \alpha < l \), define \( C^\infty \) functions \( P_\alpha : \mathbb{R} \to \mathbb{R} \) such that
\[
P_\alpha(x) = \begin{cases} 1 & \text{if } x \leq \lambda_\alpha \\ 0 & \text{if } x \geq \lambda_{\alpha+1}
\end{cases}
\]
and set
\[
\alpha(x) = P_\alpha(u_\infty).
\]
From the definition of \( P_\alpha \) it follows that, if \( \lambda_1 < \cdots < \lambda_l \) are the distinct
eigenvalues of \( u_\infty \),
\[
P_\alpha(\lambda_i) = \begin{cases} 1 & \text{if } i \leq \alpha \\ 0 & \text{if } i > \alpha
\end{cases}
\]
(7.30)

We contend that the \( \alpha \), \( 1 \leq \alpha \leq l \), are \( L^2_\ast(S) \)-subbundles of \( E \). In fact we have

1. The \( \alpha \) are in \( L^2_\ast(S) \) by Proposition 4.1(c) in [31],
2. From (7.30) \( P_\alpha^2 = \alpha \) vanishes at \( \lambda_1, \ldots, \lambda_l \) and then \( \alpha^2 = \alpha \),
3. From §4 in [31] \( D_\ast(\alpha) = dP_\alpha(u_\infty)(D_\ast u_\infty) \). Set \( \Phi_\alpha(y_1, y_2) = (1-P_\alpha)(y_2) \cdot 
\]
d\( \Phi_\alpha(y_1, y_2) \). It is easy to see that \( (I_E - \alpha)D_\ast = \Phi_\alpha(u_\infty)(D_\ast u_\infty) \). On
the other hand, \( \Phi_\alpha(\lambda_i, \lambda_j) = 0 \) if \( \lambda_i > \lambda_j \), in fact
(a) If \( \lambda_i \leq \lambda_\alpha \), then \( 1 - P_\alpha(\lambda_i) = 0 \),
(b) If \( \lambda_i > \lambda_j \), then \( P_\alpha(\lambda_i) = P_\alpha(\lambda_j) = 0 \)
and then
\[
d_\alpha(\lambda_i, \lambda_j) = \frac{P_\alpha(\lambda_i - P_\alpha(\lambda_j))}{\lambda_i - \lambda_j} = 0.
\]
By Lemma 5.6 in [31], \( \Phi_\alpha(u_\infty)(D_\ast u_\infty) = 0 \), so we conclude that
\( (I_E - \alpha)D_\ast = \Phi_\alpha(u_\infty)(D_\ast u_\infty) = 0 \),
and then the \( \alpha \) are \( L^2_\ast(S) \)-subbundles.

By Uhlenbeck and Yau’s regularity result of \( L^2_\ast \)-subbundles (see [34] for a
detailed proof), \( \alpha \) represents a coherent torsion-free Higgs subsheaf \( E_\alpha \) of \( (E, \phi) \).
Set
\[
\nu = \lambda_1 \deg(E) - \sum_{\alpha=1}^{l-1} (\lambda_{\alpha+1} - \lambda_\alpha) \deg(E_\alpha).
\]
Now, choose an orthonormal frame field \( \{ e_j \} \) of eigenvectors of \( u_\infty \) with eigenvalues \( \lambda_j \). Here \( \lambda_1 \leq \cdots \leq \lambda_r \) are the all eigenvalues of \( u_\infty \). From (7.30) we have

\[
\left[ \lambda_I E - \sum_{\alpha=1}^{l-1} (\lambda_{\alpha+1} - \lambda_\alpha) \pi_\alpha \right] (e_j) = \lambda_I e_j - \sum_{\alpha=1}^{l-1} (\lambda_{\alpha+1} - \lambda_\alpha) P_\alpha (\lambda_j) e_j = \\
= \lambda_I e_j - \sum_{\alpha=j}^{l-1} (\lambda_{\alpha+1} - \lambda_\alpha) e_j = \\
= \lambda_j e_j = u_\infty (e_j).
\]

Then we can write

\[
u = \nu - \lambda_I \deg(E) + \lambda_I \deg(E) = \\
= [\nu - \lambda_I \deg(E)] + \lambda_I \deg(E) \mu(E) = \\
= [\nu - \lambda_I \deg(E)] + \sum_{\alpha=1}^{l-1} (\lambda_{\alpha+1} - \lambda_\alpha) \deg(E_\alpha) \mu(E) = \\
= - \sum_{\alpha=1}^{l-1} (\lambda_{\alpha+1} - \lambda_\alpha) \deg(E_\alpha) + \sum_{\alpha=1}^{l-1} (\lambda_{\alpha+1} - \lambda_\alpha) \deg(E_\alpha) \mu(E) = \\
= \sum_{\alpha=1}^{l-1} (\lambda_{\alpha+1} - \lambda_\alpha) \deg(E_\alpha) \mu(E) - \sum_{\alpha=1}^{l-1} (\lambda_{\alpha+1} - \lambda_\alpha) \deg(E_\alpha) \mu(E_\alpha) = \\
= \sum_{\alpha=1}^{l-1} (\lambda_{\alpha+1} - \lambda_\alpha) \deg(E_\alpha) \mu(E) - \mu(E_\alpha). \tag{7.32}
\]

On the other hand, from the Chern-Weil formula (Lemma 3.2 in [31]) we have

\[
\deg(E_\alpha) = \int_X \tr(\pi_\alpha K_0) \frac{\omega^n}{n!} - \int_X |D'\pi_\alpha|^2 \frac{\omega^{n-1}}{(n-1)!},
\]

while from the definition of Chern classes in terms of curvature and from (6.3) we can write

\[
\deg(E) = \int_X \tr(K_0) \frac{\omega^n}{n!}.
\]

Therefore, from (7.31) and since \( D'(\pi_\alpha) = dP_\alpha(u_\infty)(D'\omega_\infty) \), we find
\[ \nu = \int_X \text{tr}(u_\infty K_0) \frac{\omega^n}{n!} + \int_X \left( \sum_{\alpha=1}^{i-1} (\lambda_{\alpha+1} - \lambda_\alpha) (dP_\alpha)^2 (u_\infty, D'' u_\infty, D'' u_\infty) h_0 \frac{\omega^{n-1}}{(n-1)!} \right) \cdot \frac{\omega^n}{n!}. \]

In fact, after a straightforward computation, we have

\[ \nu = \lambda_i \deg(E) - \sum_{\alpha=1}^{i-1} (\lambda_{\alpha+1} - \lambda_\alpha) \deg(E_\alpha) = \]

\[ = \lambda_i \int_X \text{tr}(K_0) \frac{\omega^n}{n!} - \sum_{\alpha=1}^{i-1} (\lambda_{\alpha+1} - \lambda_\alpha) \left[ \text{tr}(\pi_\alpha K_0) \frac{\omega^n}{n!} - \int_X |D'' \pi_\alpha h_0 \frac{\omega^{n-1}}{(n-1)!}| \right] = \]

\[ = \int_X \text{tr}(\lambda_i I_E K_0) \frac{\omega^n}{n!} - \int_X \sum_{\alpha=1}^{i-1} (\lambda_{\alpha+1} - \lambda_\alpha) \text{tr}(\pi_\alpha K_0) \frac{\omega^n}{n!} + \]

\[ + \int_X \sum_{\alpha=1}^{i-1} (\lambda_{\alpha+1} - \lambda_\alpha) |D'' \pi_\alpha| h_0 \frac{\omega^{n-1}}{(n-1)!} = \]

\[ = \int_X \text{tr}(\lambda_i I_E K_0) \frac{\omega^n}{n!} - \int_X \sum_{\alpha=1}^{i-1} \text{tr}((\lambda_{\alpha+1} - \lambda_\alpha) \pi_\alpha K_0) \frac{\omega^n}{n!} + \]

\[ + \int_X \sum_{\alpha=1}^{i-1} (\lambda_{\alpha+1} - \lambda_\alpha) |D'' \pi_\alpha| h_0 \frac{\omega^{n-1}}{(n-1)!} = \]

\[ = \int_X \text{tr}((\lambda_i I_E - \sum_{\alpha=1}^{i-1} (\lambda_{\alpha+1} - \lambda_\alpha) \pi_\alpha) \cdot K_0) \frac{\omega^n}{n!} + \]

\[ + \int_X \sum_{\alpha=1}^{i-1} (\lambda_{\alpha+1} - \lambda_\alpha) (dP_\alpha)^2 (u_\infty, D'' u_\infty, D'' u_\infty) h_0 \frac{\omega^{n-1}}{(n-1)!} = \]

\[ = \int_X \text{tr}(u_\infty \cdot K_0) \frac{\omega^n}{n!} + \]

\[ + \int_X \sum_{\alpha=1}^{i-1} (\lambda_{\alpha+1} - \lambda_\alpha) (dP_\alpha)^2 (u_\infty, D'' u_\infty, D'' u_\infty) h_0 \frac{\omega^{n-1}}{(n-1)!} \]

From (7.30), if \( \lambda_i > \lambda_j \), we have

\[ \sum_{\alpha=1}^{i-1} (\alpha_{\alpha+1} - \lambda_\alpha) (dP_\alpha)^2 (\lambda_i, \lambda_j) = \sum_{\alpha=j}^{i} (\lambda_{\alpha+1} - \lambda_\alpha) \left( \frac{P_\alpha(\lambda_i) - P_\alpha(\lambda_j)}{\lambda_i - \lambda_j} \right)^2 = \]

\[ = \frac{1}{(\lambda_i - \lambda_j)^2} \sum_{\alpha=j}^{i} (\lambda_{\alpha+1} - \lambda_\alpha) = \]

\[ = \frac{1}{(\lambda_i - \lambda_j)^2} \cdot (\lambda_i - \lambda_j) = \frac{1}{(\lambda_i - \lambda_j)}. \]

Finally, if we apply Lemma 7.1.13 to the function

\[ \Phi(x_1, x_2) = \sum_{\alpha=1}^{i-1} (\lambda_{\alpha+1} - \lambda_\alpha) (dP_\alpha)^2 (x_1, x_2), \]

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from [7.20] we obtain

\[ 0 > -\epsilon \geq \liminf_m \frac{\mathcal{L}(h_{t_m}, h_0)}{\|S(t_m)\|_{L^1}} = \]

\[ = \liminf_m \left[ \frac{1}{\|S(t_m)\|_{L^1}} \left( \int_X \text{tr}(S(t_m) \cdot \mathcal{K}_0) \frac{\omega^n}{n!} + \int_X \langle \Psi(S(t_m)) (\mathcal{D}' S(t_m)), \mathcal{D}' S(t_m) \rangle_{h_0} \frac{\omega^n}{(n-1)!} \right) \right] = \]

\[ = \liminf_m \left[ \frac{1}{t_m} \left( \int_X \text{tr}(S(t_m) \cdot \mathcal{K}_0) \frac{\omega^n}{n!} + \int_X \langle \Psi(l_m u_m) (\mathcal{D}' S(t_m)), \mathcal{D}' S(t_m) \rangle_{h_0} \frac{\omega^n}{(n-1)!} \right) \right] = \]

\[ = \liminf_m \left( \int_X \text{tr}(u_m \cdot \mathcal{K}_0) \frac{\omega^n}{n!} + \int_X \langle l_m \Psi(l_m u_m) (\mathcal{D}' u_m), (\mathcal{D}' u_m)_{h_0} \rangle_{h_0} \frac{\omega^n}{(n-1)!} \right) \]

\[ \geq \int_X \text{tr}(u_{\infty} \mathcal{K}_0) \frac{\omega^n}{n!} + \int_X \langle \sum_{\alpha=1}^{l-1} (\lambda_{\alpha+1} - \lambda_{\alpha}) (dP_{\alpha})^2 (u_{\infty}) (\mathcal{D}' u_{\infty}), \mathcal{D}' u_{\infty} \rangle_{h_0} \frac{\omega^n}{(n-1)!} = \nu. \]

On the other hand, [7.32] and the \( \omega \)-semistability imply \( \nu \geq 0 \), so we get a contradiction.

**Lemma 7.1.15.** Let \((X, \omega)\) be a compact Kähler manifold of (complex) dimension \( n \) and let \( \mathcal{E} = (E, \phi) \) be a Higgs bundle of rank \( r \) over \( X \). Let \( h_t \) be the solution of the Donaldson heat flow with initial condition \( h_0 \), and suppose \( \text{tr}(\mathcal{K}_0 - cI_E) = 0 \). Let us assume \((E, \phi)\) is \( \omega \)-semistable and \( \mathcal{L}(h_t, h_0) \) is not bounded below, i.e., \( \mathcal{L}(h_t, h_0) \rightarrow -\infty \) as \( t \rightarrow +\infty \). Then

\[ \|\mathcal{K}_t - cI_E\|_{L^2} \rightarrow 0 \quad \text{as} \quad t \rightarrow +\infty. \]

**Proof.** From the hypothesis we know that \( \lim_{t \rightarrow +\infty} \mathcal{L}(h_t, h_0) = -\infty \), then from Lemma [7.1.14] \( \|S(t)\|_{L^1} \rightarrow +\infty \) as \( t \rightarrow +\infty \). Otherwise there exists \( \epsilon > 0 \) and a sequence \( t_j \rightarrow +\infty \) such that \( \|S(t_j)\|_{L^1} \leq \epsilon \) and then

\[ -\mathcal{L}(h_{t_j}, h_0) \geq -\mathcal{L}(h_{t_j}, h_0) \rightarrow +\infty \quad \text{as} \quad j \rightarrow +\infty, \]

contradicting [7.1.14]. For \( t \) large enough,

\[ \left( \frac{1}{\sqrt{\text{tr}(E)}} \|S(t)\|_{L^1} - \text{Vol}(X) \ln(2\text{rk}(E)) \right) \]

(7.34)
is positive, so we can divide both terms of (7.6) by (7.34) obtaining
\[ \|K_t - cI_E\|_{L^2} \leq -\sqrt{\text{Vol}(X)} \mathcal{L}(h_t, h_0) \left( \frac{1}{\sqrt{\text{rk}(E)}} \|S(t)\|_{L^1} - \text{Vol}(X) \ln(2\text{rk}(E)) \right). \]

Now, applying again Lemma 7.1.14
\[ 0 \leq \|K_t - cI_E\|_{L^2} \leq -\sqrt{\text{Vol}(X)} \mathcal{L}(h_t, h_0) \left( \frac{1}{\sqrt{\text{rk}(E)}} \|S(t)\|_{L^1} - \text{Vol}(X) \ln(2\text{rk}(E)) \right) \to 0 \]
as \( t \to +\infty \) and this completes the proof.

### 7.2 Proof of the main Theorem

In this section we prove the equivalence between semistability and the existence of approximate Hermitian-Yang-Mills metric structures for Higgs bundles in every dimension. First of all we need to prove a preliminary result under the assumption \( \text{tr}(K_0 - cI_E) = 0 \).

**Proposition 7.2.1.** Let \((X, \omega)\) be a compact Kähler manifold of (complex) dimension \( n \) and let \( \mathcal{E} = (E, \phi) \) be a Higgs bundle of rank \( r \) over \( X \). Let \( h_t \) be the solution of the Donaldson heat flow with initial condition \( h_0 \) and let \( K_0 \) be the mean curvature of the Hitchin-Simpson connection associated with \( h_0 \). Let us assume \( h_0 \) satisfies the condition \( \text{tr}(K_0 - cI_E) = 0 \) and the Donaldson functional \( \mathcal{L}(h_t, h_0) \) is not bounded below, i.e., \( \mathcal{L}(h_t, h_0) \to -\infty \) as \( t \to +\infty \).

If \( (E, \phi) \) is \( \omega \)-semistable, then
\[ \max_X |K_t - cI_E| \to 0. \]

So there exists an approximate Hermitian-Yang-Mills metric structure on the semistable Higgs bundle \((E, \phi)\).

**Proof.** We follow Kobayashi’s argument (see [25], p.224-226 for details). Let \( \chi = \chi(x, y, t) \) be the heat kernel for the differential operator \( \partial_t + \square_t \), where \( \square_t = \square_{h_t} \) and the subscript \( t \) remember us the dependence on the metric \( h_t \). Set
\[ f(x, t) = (|K_t - cI_E|^2)(x) \quad \text{for} \quad (x, t) \in X \times [0, +\infty). \]
Now fix \( t_0 \in [0, +\infty) \) and set
\[ u(x, t) = \int_X \chi(x, y, t - t_0)(|K_t - cI_E|^2)(y)dy \]
where \( dy \) is the volume form \( dy = \frac{\omega^n}{n!} \). Then \( u(x, y) \) is of class \( C^\infty \) on \( X \times (t_0, +\infty) \) and extends to a continuous function on \( X \times [t_0, +\infty) \). From the definition of the heat kernel we immediately have
\[ (\partial_t + \square_t)u(x, t) = 0 \quad \text{for} \quad (x, t) \in X \times (t_0, +\infty), \]
and
\[ u(x, t_0) = f(x, t_0) = (|K_{t_0} - cI_E|^2)(x). \]

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Combined with the inequality (6.15) this yields
\[(\partial_t + \Box_t)(|K_t - cI_E|^2 - u(x,t)) \leq 0 \quad \text{for} \quad (x,t) \in X \times (t_0, +\infty).
\]

By the Maximum Principle 5.3.2 and the properties of $u(x,t)$ we find
\[
\max_X (|K_t - cI_E|^2 - u(x,t)) \leq \max_X (|K_{t_0} - cI_E|^2 - u(x,t_0)) = 0, \quad t \geq t_0.
\]

Hence,
\[
\max_X |K_{t_0+a} - cI_E|^2 \leq \max_X u(x,a,t_0 + a) = \max_X \int_X \chi(x,y,a)|K_{t_0} - cI_E|^2(y)dy \leq C_a \int_X |K_{t_0} - cI_E|^2(y)dy = C_a \||K_{t_0} - cI_E||_L^2,
\]
where
\[
C_a = \max_X \chi(x,y,a).
\]
Fix $a$, say $a = 1$, and let $t_0 \to +\infty$. Using Lemma 7.1.15 we conclude
\[
\max_X |K_{t_0+1} - cI_E|^2 \leq C_1 \||K_{t_0} - cI_E||_L^2 \to 0,
\]
and this completes the proof. □

Now, using the previous Proposition and Lemma 7.1.2 we can give the proof of the main Theorem of this section:

**Theorem 7.2.2.** Let $(X,\omega)$ be a compact Kähler manifold of (complex) dimension $n$ and let $E = (E,\phi)$ be a Higgs bundle of rank $r$ over $X$. If $(E,\phi)$ is $\omega$-semistable, then it admits an approximate Hermitian-Yang-Mills structure.

**Proof.** Let $h_0 \in \text{Herm}^+(E)$ be a fixed Hermitian metric structure on $E$, and let $K_0$ be the mean curvature of the Hitchin-Simpson connection associated with $h_0$. From Lemma 7.1.2 we may assume $\text{tr}(K_0 - cI_E) = 0$. Let $h_t$ be a solution of the Donaldson heat flow with initial condition $h_0$. We already know that $h_t$ is defined for every positive time $0 \leq t < +\infty$ and $\mathcal{L}(h_t, h_0)$ is a real monotone decreasing function of $t$ for $0 \leq t < +\infty$. Now we can distinguish between three cases:

1. If $\text{rk}(E) = 1$, from $\text{tr}(K_0 - cI_E) = 0$ we deduce that $K_0 = cI_E$. So that $h_0$ is a Hermitian-Yang-Mills metric.

2. If $\text{rk}(E) \geq 2$ and $\mathcal{L}(h_t, h_0)$ is bounded below, the thesis comes from Theorem 6.5.5.

3. If $\text{rk}(E) \geq 2$ and $\mathcal{L}(h_t, h_0)$ is not bounded below, since $\text{tr}(K_0 - cI_E) = 0$, the thesis comes from Proposition 7.2.1. □

Hence, from the previous result and Theorem 6.5.5 we have the following
Corollary 7.2.3. Let $(X, \omega)$ be a compact Kähler manifold of (complex) dimension $n$ and let $\mathcal{E} = (E, \phi)$ be a Higgs bundle of rank $r$ over $X$. The following conditions are equivalent:

1. $\mathcal{E}$ is $\omega$-semistable,
2. $\mathcal{E}$ admits an approximate Hermitian-Yang-Mills structure.

As a consequence of this we deduce that many results about Higgs bundles written in terms of approximate Hermitian-Yang-Mills structures can be translated in terms of semistability. In particular we have the following:

Corollary 7.2.4. If $(X, \omega)$ is a compact Kähler manifold of (complex) dimension $n$ and $\mathcal{E}_1, \mathcal{E}_2$ are $\omega$-semistable Higgs bundles over $X$, then so is their tensor product $\mathcal{E}_1 \otimes \mathcal{E}_2$. Furthermore, if $\mu(\mathcal{E}_1) = \mu(\mathcal{E}_2)$, so is the Whitney sum $\mathcal{E}_1 \oplus \mathcal{E}_2$.

Proof. 1. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be $\omega$-semistable Higgs bundles over $X$. From Theorem 7.2.2, $\mathcal{E}_1$ and $\mathcal{E}_2$ admit approximate Hermitian-Yang-Mills metric structures, so do their tensor product $\mathcal{E}_1 \otimes \mathcal{E}_2$ (Proposition 6.3.1). Hence, using the above corollary we conclude that $\mathcal{E}_1 \oplus \mathcal{E}_2$ is $\omega$-semistable.

2. It is similar to (1), using the second part of Proposition 6.3.1.

Corollary 7.2.5. If $\mathcal{E}$ is $\omega$-semistable, then so is the tensor product bundle $\mathcal{E}^\otimes p \otimes \mathcal{E}^\ast \otimes q$ and the exterior product bundle $\wedge^p \mathcal{E}$ whenever $0 \leq p \leq r = \text{rk}(\mathcal{E})$. 

\[\square\]
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