ON CLASSICAL INEQUALITIES FOR AUTOCORRELATIONS AND AUTOCONVOLUTIONS

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Abstract. In this paper we study an autocorrelation inequality proposed by Barnard and Steinerberger [1]. The study of these problems is motivated by a classical problem in additive combinatorics. We establish the existence of extremizers to this inequality, for a general class of weights, including Gaussian functions (as studied by the second author and Ramos) and characteristic function (as originally studied by Barnard and Steinerberger). Moreover, via a discretization argument and numerical analysis, we find some almost optimal approximation for the best constant allowed in this inequality. We also discuss some other related problem about autoconvolutions.

1. Introduction

Motivated by an old problem in additive combinatorics about estimating the size of Sidon sets [7], many authors have studied the problem of finding the best constant \( c_{\text{max}} \) such that for any function \( f \in L^1(\mathbb{R}) \) supported in \([-1/4, 1/4]\) the following inequality holds

\[
\max_{-1/2 \leq t \leq 1/2} \int_{\mathbb{R}} f(t-x)f(x) \, dx \geq c_{\text{max}} \left( \int_{-1/4}^{1/4} f(x) \, dx \right)^2.
\]

The best known result so far was obtained by Cloninger and Steinerberger in [8], they proved that \( c \geq 1.28 \). Many other lower bounds were previously obtained in [6, 14, 17, 22, 18, 21]. Inspired by this question, two other related problems were proposed and studied by Barnard and Steinerberger in [1]. One of their results was the following: The inequality

\[
\int_{-1/2}^{1/2} \int_{\mathbb{R}} f(x)f(x+t) \, dx \, dt \leq 0.91 \|f\|_1 \|f\|_2,
\]

(1)

holds for any function \( f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \). It was also established in [1] via an example, that the best constant such that (1) holds is at least 0.8. The upper bound was recently improved by the second author and Ramos in [20], where they proved that this inequality still hold true when we write 0.865 instead of 0.91. A natural question is, what happen when we consider a different probability space, in particular, what happen when we consider Gaussian means. This question was also addressed in [20]:

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Weight Spectral Method Fixed Point

| Weight     | Lower bound | Upper bound | Difference | Lower bound |
|------------|-------------|-------------|------------|-------------|
| $\chi_{[-1/2,1/2]}$ | 0.8055809  | 0.8055896   | $< 9 \cdot 10^{-6}$ | 0.8055809   |
| $\exp(-\pi x^2)$ | 0.7152474  | 0.7152576   | $< 1.2 \cdot 10^{-5}$ | 0.7152475   |

Table 1. Upper and Lower bounds for the average problem. These bounds have been found with the algorithm described in Section 3.3, with a value of $\delta \approx 1.45 \cdot 10^{-3}$ and $\Delta \lambda \approx 0.001$. The implementation can be found at [10].

Proposition 1.1 ([20], Theorem 1.2). Let $a$ be a positive real number. For any $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, the following inequality holds

$$\left(\frac{a}{\pi}\right)^{1/2} \int_{\mathbb{R}} \int_{\mathbb{R}} f(x) f(x+t) e^{-at^2} \, dx \, dt \leq \left(\frac{8a}{27\pi}\right)^{1/4} \|f\|_1 \|f\|_2,$$

(2)

and $\left(\frac{8a}{27\pi}\right)^{1/4}$ can not be replaced by $\left(\frac{a}{\pi}\right)^{1/4}$.

In this paper, using Euler-Lagrange equations, in Section 2 we establish the existence of extremizers for (1), (2) and a more general class of weights (nonnegative functions $w : \mathbb{R} \to [0, +\infty)$). In Section 3, via a discretization argument, we find an almost optimal numerical approximation for the best constants allowed in these inequalities, see Table 1. Finally, in Section 4, we discuss a related problem about autoconvolution.

Related results about autoconvolution inequalities were recently obtained in [5]. We hope variations of the methods outlined in this paper can be applied to such, and other, related problems. As another example, a classical problem proposed by Erdos, the Minimum overlapping problem, can be reformulated in terms of autorrelations as observed by Haugland in [16], we believe that there is a strong relation between this problem and the one analyzed in this manuscript.

2. The extremizer exists and is compactly supported

Let $w(x) \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ be a symmetric decreasing weight with $\|w\|_{L^1(\mathbb{R})} = \|w\|_{L^\infty(\mathbb{R})} = 1$. Examples to keep in mind are $w(x) = \chi_{[-1/2,1/2]}$, or $w(x) = e^{-\pi x^2}$.

Let $C_{opt}$ be the smallest constant such that the inequality

$$\int_{\mathbb{R}^2} f(x) f(y) w(x-y) \, dx \, dy \leq C_{opt}(w) \|f\|_1 \|f\|_2$$

(3)

holds for all $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. After a change of variable, finding $C_{opt}$ for $a^{1/2} e^{-ax^2}$ is equivalent to find the optimal constant for which (2) holds. In particular, for $a = \pi$, (2) is equivalent to $0.707107 \approx \frac{1}{\sqrt{\pi}} \leq C_{opt}(e^{-\pi x^2}) \leq (\frac{2}{3})^{3/4} \approx 0.737788$ (compare to Table 1).

Our first theorem establishes the existence of extremizers with compact support for the inequality (3), as conjectured by the second author and Ramos (see [20], Conjecture 1.4).
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Figure 1. Numerical extremizers for the discretized problem. Extremizers are normalized so that \( \|f^*\|_1 \|f^*\|_2 = 1 \).

**Theorem 1.** Let \( w(x) \) be a symmetric decreasing weight in \( L^1(\mathbb{R}) \). Then there exists a bounded, symmetrically decreasing function \( f^* \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \) (with \( f^* \) depending on \( w \)), with compact support, such that

\[
\int_{\mathbb{R}^2} f(x)f(y)w(x-y)dxdy = C_{opt} \|f\|_1 \|f\|_2. \tag{4}
\]

We start observing that

\[
C_{opt} = \sup_{f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \atop \|f\|_1 \|f\|_2 \neq 0} \frac{\int_{\mathbb{R}^2} f(x)f(y)w(x-y)dxdy}{\|f\|_1 \|f\|_2} = \sup_{f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \atop \|f\|_1 \|f\|_2 \neq 0, f \geq 0} \frac{\int_{\mathbb{R}^2} f(x)f(y)w(x-y)dxdy}{\|f\|_1 \|f\|_2} = \sup_{f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \atop \|f\|_1 \|f\|_2 \neq 0, f \geq 0, f \text{ symmetric decreasing}} \frac{\int_{\mathbb{R}^2} f(x)f(y)w(x-y)dxdy}{\|f\|_1 \|f\|_2}.
\]

The last identity is a consequence of the well known Riesz rearrangement inequality:

**Lemma 2.1** (Riesz rearrangement inequality). Let \( f : \mathbb{R} \to \mathbb{R} \) be a nonnegative function, let \( f^* \) be the symmetric decreasing rearrangement of \( f \). That is:

\[
f^*(x) := \int_{a \geq 0} \chi_{[-\mu(\{f \geq a\})/2,\mu(\{f \geq a\})/2]}(x)da
\]
where $\mu$ is the usual Lebesgue measure. Then:

$$\int_{\mathbb{R}^2} f(x)f(y)w(x-y)dxdy \leq \int_{\mathbb{R}^2} f^*(x)f^*(y)w(x-y)dxdy$$  \hspace{0.5cm} (5)

with equality only if $f^*$ is a translation of $f$.

For $R > 0$, let

$$C_{\text{opt},R} := \sup_{f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \atop \|f\|_1 \neq 0} \int_{\mathbb{R}^2} f(x)f(y)w(x-y)dxdy \quad \text{subject to} \quad \|f\|_1 \leq R.$$  \hspace{0.5cm} (6)

A useful tool to establish Theorem 1 is the following local version of the theorem.

**Proposition 2.2.** There exists a function $f^* \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, supported in $[-R, R]$ such that

$$C_{\text{opt},R} := \frac{\int_{\mathbb{R}^2} f^*(x)f^*(y)w(x-y)dxdy}{\|f^*\|_1 \|f^*\|_2}.$$  \hspace{0.5cm} (7)

This follows directly from an adaptation of [20, Theorem 1.6].

**Proof of Theorem 1.** Let $R > 0$, sufficiently large. By the Proposition 2.2 we know that there exists a function $f^* \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ with support in $[-R, R]$ such that

$$C_{\text{opt},R} := \frac{\int_{\mathbb{R}^2} f^*(x)f^*(y)w(x-y)dxdy}{\|f^*\|_1 \|f^*\|_2}.$$  \hspace{0.5cm} (8)

We will show that the function $f^*$ do not depend on $R$ as soon as $R$ is large enough.

The function $f^*$ maximizes the functional

$$\mathcal{F}(g) := \log \left[ \int_{\mathbb{R}^2} g(x)g(y)w(x-y)dxdy \right] - \log \|g\|_1 - \frac{1}{2} \log \|g\|_2^2$$  \hspace{0.5cm} (9)

over the set of functions $g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ supported in $[-R, R]$. This leads to the following Euler-Lagrange equation inside the support of $f^*$:

$$0 = \nabla_f \mathcal{F}(f^*) = \frac{2f^* * w}{\int_{\mathbb{R}^2} f^*(x)f^*(y)w(x-y)dxdy} - \frac{1}{\|f^*\|_1} - \frac{f^*}{\|f^*\|_2^2}.$$  \hspace{0.5cm} (10)

Let $a > 0$ such that the support of $f^*$ is equal to $[-a, a]$. Integrating (8) from $-a$ to $a$ and rearranging, we have

$$\int_{[-a,a] \times \mathbb{R}} f^*(x)w(x-y)dxdy \frac{2}{C_{\text{opt},R} \|f^*\|_1 \|f^*\|_2} = \frac{1}{\|f^*\|_2} \int_{-a}^a f^*(x)dx + \frac{2a}{\|f^*\|_1} \frac{\|f^*\|_2}{\|f^*\|_1}.$$  \hspace{0.5cm} (11)

On the other hand

$$\int_{[-a,a] \times \mathbb{R}} f^*(x)w(y-x)dxdy = \|w\|_1 \|f^*\|_1.$$  \hspace{0.5cm} (12)

Combining both equalities we obtain

$$a = \frac{\|f^*\|_2^2}{\|f^*\|_2} \frac{\|w\|_1}{C_{\text{opt},R} \|f^*\|_1 \|f^*\|_2} \leq \frac{\|f^*\|_1 \|w\|_1}{\|f^*\|_2 C_{\text{opt},R}}.$$  \hspace{0.5cm} (13)

moreover, from Hölder’s inequality we know that $\|f^*\|_1 \leq \sqrt{2a} \|f^*\|_2$, and therefore
the discretized functions are supported in $(\delta_x - a, \delta_x + a)$.

The right hand side of (9) is decreasing in $C_{\text{opt},R}$. Therefore one can find bounds for $a$ by substituting $C_{\text{opt},R}$ even when its exact value is not known.

3. Approximation inequalities

\textbf{Notation:} Through this section $\langle f \rangle$ will denote a row vector, $|f|$ a column vector.

If $a$ is a matrix, $\langle f|a|g \rangle$ will be the vector-matrix-vector product. Moreover, $|1|$ will be the column vector with all ones. We will use the notation $\langle f|g \rangle$ as a dot product for $f$, $g$, with the dot product $\langle f|g \rangle = \delta_x \sum f_i g_i$, ($\delta_x$ is the discretization scale). If the discretized functions are supported in $(-a, a)$, then $|1| = 2a$. We denote by $K_w$ the convolution operator associated to $w$. 

\begin{equation}
\alpha \leq 2 \frac{\|1\|^2}{C_{\text{opt},R}}.
\end{equation}

\textbf{Finishing the proof.} The last remaining step is showing that the non-compactly-supported problem has a solution as well. That will follow if we show that

\begin{equation}
C_{\text{opt}} = \lim_{R \to \infty} C_{\text{opt},R}
\end{equation}

as in that case, an extremizer witnessing $C_{\text{opt},R}$ for $R$ large enough will witness $C_{\text{opt}}$ as well. Clearly $C_{\text{opt}} \geq \lim_{R \to \infty} C_{\text{opt},R}$ so we must show the opposite. Let $\epsilon > 0$. Let $g_0 \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ be an almost-extremizer, such that:

\begin{equation}
C_{\text{opt}} \leq \int_{\mathbb{R}^2} g_0(x)g_0(y)w(x-y)dxdy \frac{1}{\|g_0\|_1\|g_0\|_2} + \epsilon/2.
\end{equation}

Let $[-R_1, R_1]$ be the support of $g_1$. Then $C_{\text{opt}} \leq C_{\text{opt},R_1} - \epsilon$. \hfill $\Box$

\textbf{Remark 2.3.} Finer bounds for $a$ can be found by squaring and integrating (8), in the form of:

\begin{equation}
\frac{4}{C_{\text{opt},R}\|f^*\|^2\|f^*\|^2} \int_{-a}^a |w*f^*|^2dx = \int_{-a}^a \left| \frac{1}{\|f^*\|^1} + \frac{f^*}{\|f^*\|^2} \right|^2 dx,
\end{equation}

and using Young’s inequality in the left hand side

\begin{equation}
\|w\|^2\|f^*\|^2 \geq \int_{-a}^a |w*f^*|^2dx,
\end{equation}

which leads to

\begin{equation}
\frac{\|f^*\|^1}{\|f^*\|^2} \geq \sqrt{2a} \left( \frac{4\|w\|^2}{C_{\text{opt},R}} - 3 \right)^{-\frac{1}{2}}.
\end{equation}

This inequality gives a better bound for (1), giving a lower bound for the substracted term, namely the bound:

\begin{equation}
a \leq 2 \left( \frac{\|1\|^1}{C_{\text{opt},R}} - \sqrt{\frac{a}{2}} \left( \frac{4\|w\|^2}{C_{\text{opt},R}} - 3 \right)^{-\frac{1}{2}} \right)^2.
\end{equation}

The right hand side of (12) is decreasing in $C_{\text{opt},R}$. Therefore one can find bounds for $a$ by substituting $C_{\text{opt},R}$ even when its exact value is not known.
Let \( L^{1,2}([a,b]) := L^1([a,b]) \cap L^2([a,b]) \), and, for each function \( f \in L^{1,2}([a,b]) \) we define \( \|f\|_{L^{1,2}([a,b])} := \frac{1}{\lambda^{1/2}} L^{1,2}([a,b]) L^{2}([a,b]) \).

**Definition 3.1.** We define some spaces as the space of measurable functions (up to a.e. equivalence) with the following norms (with \( \lambda \) a parameter greater than zero):

- \( \|f\|_{H_\lambda ([a,b])}^2 = \lambda \|f\|_2^2 + \lambda^{-1} \left( \int_a^b f^2 \, dx \right) \).
- \( \|f\|_{B_\lambda ([a,b])}^2 = \lambda \|f\|_2^2 + \lambda^{-1} \left( \int_a^b |f| \, dx \right) \).

These spaces satisfy nice properties, including the following:

- \( \|f\|_{L^{1,2}([a,b])}^2 = \frac{1}{\lambda} \inf \|f\|_{B_\lambda ([a,b])}^2 \). This is a consequence of AM-GM inequality.
- \( \|f\|_{B_\lambda ([a,b])} \geq \|f\|_{H_\lambda ([a,b])} \) with equality if and only if \( f \) is nonnegative.
- \( H_\lambda \) is a Hilbert space. Since, by definition, \( \| \cdot \|_{H_\lambda ([a,b])} \) satisfies the parallelogram law.

**Remark 3.2.** Observe that by Hölder’s inequality and AM-GM inequality we have

\[
\|f + g\|_{H_\lambda ([a,b])}^2 - \|g\|_{H_\lambda ([a,b])}^2 = \lambda \|f\|_{H_\lambda ([a,b])}^2 + \lambda^{-1} \left( \lambda \int_a^b f^2 \, dx \right)
- \lambda \|f\|_{L^2([a,b])}^2 - \lambda^{-1} \left( \lambda \int_a^b |f| \, dx \right)
- \lambda \|g\|_{L^2([a,b])}^2 - \lambda^{-1} \left( \lambda \int_a^b |g| \, dx \right)

\leq 2\lambda \int_a^b f g \, dx + \frac{2}{\lambda} \left( \int_a^b f \, dx \right) \left( \int_a^b g \, dx \right)
- 2 \|f\|_{H_\lambda ([a,b])} \|g\|_{H_\lambda ([a,b])}

\leq 2\lambda \|f\|_{L^2([a,b])} \|g\|_{L^2([a,b])} + \frac{2}{\lambda} \left( \int_a^b f \, dx \right) \left( \int_a^b g \, dx \right)
- 2 \|f\|_{H_\lambda ([a,b])} \|g\|_{H_\lambda ([a,b])}

\leq 0, \text{ for any two functions } f, g \in H_\lambda ([a,b]).

This means that \( \| \cdot \|_{H_\lambda ([a,b])} \) satisfies the triangle inequality, so this is really a norm (since all the other properties trivially hold). Similarly we can see that \( \| \cdot \|_{B_\lambda} \) is in fact a norm.

The strategy to find the optimal value \( C_{\text{opt}} \) will be in two steps:

- First, by the addition of a parameter \( \lambda \) we will turn the optimization problem into a computationally tractable optimization problem.
- We will then show that a suitably discretized version of the computationally tractable problem is quantitatively close to the original continuous problem.

**A computationally tractable relaxation of the problem.** For any \( \lambda > 0 \), let

\[
c_\lambda := c_\lambda (w) := \max_{f \in L^1 \cap L^2} \frac{\langle f \mid w \rangle}{\lambda \langle f \mid f \rangle + \lambda^{-1} \langle f \mid 1 \rangle \langle 1 \mid f \rangle}.
\]

Observe that by Fubini’s theorem and Hölder’s inequality we have

\[
c_\lambda \leq \min \{2\lambda, 2/\lambda\}.
\]
Using the fact that \( \min_{\lambda > 0} \lambda a^2 + \lambda^{-1} b^2 = 2ab \) by AM-GM inequality. We can turn our original problem into:

\[
C_{\text{opt}} := c(w) := \max_{\lambda > 0} c_\lambda(w) = \max_{\lambda > 0} \max_{f \in L^1 \cap L^2} 2 \frac{\langle f | w | f \rangle}{\lambda(a + b) + \lambda^{-1} (1)(1)f}.
\]

**Lemma 3.3.** We have that

\[
\max_{\|f\|_{L^1(\{a,b\})} \leq 1} \langle f | K_w | f \rangle = 2 \max_{\lambda > 0} \max_{\|f\|_{H^1(\{a,b\})} \leq 1} \langle f | K_w | f \rangle
\]

moreover, the extremizers to \( \max_{\|f\|_{H^1(\{a,b\})} \leq 1} \langle f | K_w | f \rangle \) (which exist by the Hilbert theory) are symmetric decreasing nonnegative.

**Proof.** Observe that

\[
\max_{\|f\|_{L^1(\{a,b\})} \leq 1} \langle f | K_w | f \rangle = \max_{f \in L^1(\{a,b\})} \frac{\langle f | K_w | f \rangle}{\|f\|_{L^1(\{a,b\})}^2}
\]

\[
= 2 \max_{f \in L^1(\{a,b\})} \frac{\langle f | K_w | f \rangle}{\inf_{\lambda > 0} \|f\|_{H^1(\{a,b\})}^2}
\]

\[
= 2 \max_{f \in H^1(\{a,b\})} \frac{\langle f | K_w | f \rangle}{\inf_{\lambda > 0} \|f\|_{H^1(\{a,b\})}^2}
\]

\[
= 2 \max_{\lambda > 0} \max_{\|f\|_{H^1(\{a,b\})} \leq 1} \langle f | K_w | f \rangle.
\]

The last part of the statement follows from the Riesz rearrangement inequality. □

The problem of finding \( c_\lambda \) for a fixed \( \lambda \) becomes now essentially a problem about finding the spectrum of \( w \) (as a convolution operator) in a certain Hilbert space, with certain subtleties arising from the fact that we have a restriction to \( f \geq 0 \). These subtleties are addressed in Section 3.3.

**Relating a discretized version of the problem to the continuous problem.**

We start by defining our discretized spaces as the space of step functions on intervals of length \( \delta \):

**Definition 3.4.** Given \( \delta > 0 \) the set \( V_\delta \) will be the set of functions that are constant on intervals of the form \([n\delta, (n+1)\delta)\). Given a function \( f \) we define \([f]_\delta \in V_\delta\) by \([f]_\delta = \delta^{-1} \int_{\delta n}^{\delta(n+1)} f(s) ds\). We also define \( \{f\}_\delta := f - [f]_\delta \).

Observe that \( \|\{f\}_\delta\|_1 = \|f\|_1 \) and by Hölder’s inequality \( \|\{f\}_\delta\|_2 \leq \|f\|_2 \) for all \( f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \). Moreover, by the previous observation,

\[
\|\{f\}_\delta\|_{H^1} \leq \|f\|_{H^1}
\]

(15)

for all \( f \in H^1 \).

Our next lemma establishes smoothness properties for the extremizers on the interior of the support. Regularity (in the form of (3.5)) will be crucial in order to implement a numerical scheme to find the constants \( C_{\text{opt}} \).

**Lemma 3.5.** Let \( \hat{f} \) be an extremizer to the problem \( \max_{\|f\|_{H^1(\{a,b\})} \leq 1} 2\langle f | K_w | f \rangle \). Let \( c_\lambda = \max_{\|f\|_{H^1(\{a,b\})} \leq 1} 2\langle f | K_w | f \rangle \). Then \( \|\hat{f}\|_2 = \lambda^{-1/2} \) and \( \|\hat{f}'\|_2 \leq \frac{4}{c_\lambda \lambda^2} \).
In order to do this, we use two key properties: 

Thus, it suffices to bound Proposition 3.6. 

Then, by (15) we have 

This implies that on the support of \( f \), it holds that 

Therefore, since \( \hat{f} \geq 0 \) we have that 

This implies that on the support of \( f \), it holds that 

\[ |\hat{f}'| \leq \frac{2}{c_\lambda \lambda} |(w \ast \hat{f})'| \]

\[ \|\hat{f}'\|_2 \leq \frac{2}{c_\lambda \lambda} \|w'\|_1 \|\hat{f}\|_2 = \frac{2}{c_\lambda \lambda} \|w\|_\infty \|\hat{f}\|_2 = \frac{4}{c_\lambda \lambda} \|w\|_\infty \|\hat{f}\|_2 \leq \frac{4}{c_\lambda \lambda^{3/2}}. \]

\[ \Box \]

We are now ready to state the main result of the section:

**Proposition 3.6.** Let \( c_{\lambda, \delta} := \sup_{f \geq 0, f \in V_\delta, \|f\|_{H_\lambda} \leq 1} \langle f | K | f \rangle \) be the discretized version of \( c_\lambda \). Then:

\[ 0 \leq c_\lambda - c_{\lambda, \delta} \leq \frac{16 \delta^2}{\pi^2 c_\lambda \lambda^2} \] (18)

**Proof.** By construction, \( c_\lambda \geq c_{\lambda, \delta} \), so we can focus on the second inequality. Let \( f^* \) be an extremizer for \( c_\lambda \), we assume without loss of generality that \( \|f^*\|_{H_\lambda} = 1 \). Then, by (15) we have \( \|\langle f^* \rangle_\delta\|_{H_\lambda} \leq 1 \), moreover 

\[ c_\lambda - c_{\lambda, \delta} \leq K_w(f^*, f^*) - \frac{K_w([f^*]_\delta, [f^*]_\delta)}{\|f^*[f^*]_\delta\|_{H_\lambda}^2} \leq K_w(f^*, f^*) - K_w([f^*]_\delta, [f^*]_\delta). \]

Thus, it suffices to bound 

\[ K_w(f^*, f^*) - K_w([f^*]_\delta, [f^*]_\delta) = K_w([f^*]_\delta + f^*, \{f^*\}_\delta) = \langle ([f^*]_\delta + f^*) * w, \{f^*\}_\delta \rangle. \]

In order to do this, we use two key properties:

- (Orthogonality) \( \langle [f]_\delta, [g]_\delta \rangle = 0 \) for any two functions \( f, g \in L^1(\mathbb{R}) \).
- (Optimal Poincar\'e’s inequality) \( \|f\|_{L^2} \leq \frac{\delta}{\pi} \|f'\|_{L^2} \).
Using the orthogonality property and Young’s convolution inequality we obtain
\[ c_\lambda - c_{\lambda, \delta} \leq \| \{(f^* + [f^*]_\delta) * w\}_\delta \|_2 \| \{f^*\}_\delta \|_2. \]
Moreover, by the optimal Poincaré’s inequality and Young’s convolution inequality
\[ \| \{(f^* + [f^*]_\delta) * w\}_\delta \|_2 \leq \frac{\delta}{\pi} \| (f^* + [f^*]_\delta) * w' \|_2 \leq \frac{\delta}{\pi} \| (f^* + [f^*]_\delta) \|_2 \| w \|_{TV}. \]
Therefore, once again, by the optimal Poincaré inequality
\[ c_\lambda - c_{\lambda, \delta} \leq \frac{\delta^2}{\pi^2} \| f^* + [f^*]_\delta \|_2 \| w \|_{TV} \| f^* \|_2 \]
\[ \frac{4}{c_\lambda \lambda^{1/2}} \| w \|_{TV} = 2 \| w \|_\infty = 2, \]
and the triangle inequality, to obtain:
\[ c_\lambda - c_{\lambda, \delta} \leq \frac{16\delta^2}{\pi^2 c_\lambda \lambda^2}. \quad (19) \]

3.1. **Computational aspects I: Regularity with respect to \( \lambda \).** The computational strategy will be to find the value \( c_\lambda \) for a discrete subset of the possible \( \lambda \), and prove regularity properties of \( c_\lambda \) that guarantee that \( C_{\text{opt}} = c_\lambda^{*} \) is not far from the maximum of the values of \( c_\lambda \).

**Lemma 3.7.** The following properties hold:

(i) The Lipschitz constant of \( c_\lambda \) is at most 1.
(ii) If \( C_{\text{opt}} = c_\lambda^{*} \), then
\[ c_\lambda \geq \frac{2c_\lambda^{*}}{\lambda^{-1} \lambda^{*} + \lambda \lambda^{*^{-1}}} \]

**Proof.** Proof of (i): Let \( g \) be a function such that \( \| g \|_1 = 1 \), then \( \langle g | K_w | g \rangle \leq 1 \), and
\[ \left| \frac{d}{d\lambda} (\lambda \| g \|_2^2 + \lambda^{-1} \| g \|_1^2)^{-1} \right| = \lambda^{-1} (\lambda \| g \|_2^2 + \lambda^{-1} \| g \|_1^2)^{-2} \lambda \| g \|_2^2 - \lambda^{-1} \| g \|_1^2 | \leq 1. \]
The result follows from this.

**Proof of (ii):** We start observing that
\[ \lambda^{*} = \| f^* \|_1 \| f^* \|_2, \]
where \( f^* \) is an extremizer for our original problem (This is when the equality happen in AM-GM inequality). Then
\[ c_\lambda (\lambda^{-1} \| f^* \|_1^2 + \lambda^* \| f^* \|_2^2) \geq 2 \langle f^* | K_w | f^* \rangle = 2 \| f^* \|_1 \| f^* \|_2 c_\lambda^{*}. \]
Therefore
\[ c_\lambda \geq \frac{2\| f^* \|_1 \| f^* \|_2 c_\lambda^{*}}{\lambda^{-1} \lambda^{*} + \lambda \lambda^{*^{-1}}} = \frac{2c_\lambda^{*}}{\lambda^{-1} \lambda^{*} + \lambda \lambda^{*^{-1}}}. \]
3.2. Computational aspects II: Discretizing the convolution kernel. Let \( a, b \in \delta \mathbb{Z} \), and as previously, let \( V_\delta \) be the set of functions \( g : [a, b) \to \mathbb{R} \) that are constants on intervals of length \( \delta \). Let \( f : \delta \mathbb{Z} \cap [a, b) \to \mathbb{R} \), we define \( \|f\|_p^\delta := \left( \sum_{i \in \delta \mathbb{Z} \cap [a,b)} |f(i)|^p \right)^{1/p} \) (essentially a \( \delta \)-discretization of the \( L^p \) norm), and, we define \( E[f] := \sum_{i \in \delta \mathbb{Z}} f(i) \chi_{[i,i+\delta)} \in V_\delta \). Observe that we have the equality \( \|f\|_{\delta}^\delta = \|E[f]\|_{L^P} \). For any \( a, b \in \delta \mathbb{Z} \) this defines a natural isomorphism between \( V_\delta \) and functions with domain \( \delta \mathbb{Z} \cap [a, b) \). Let \( i : L^2(\delta \mathbb{Z} \cap [a,b)) \to L^2(\mathbb{R} \cap [a, b)) \cap V_\delta \) be the map given by this isomorphism i.e \( i f = E[f] \) for all \( f \in \delta \mathbb{Z} \cap [a, b) \) (Here we are considering \( \delta \mathbb{Z} \) with \( \delta \) times the counting measure as a measure). This isomorphism allows us to define the metrics induced in \( V_\delta \) by \( H_\lambda, B_\lambda \) as a metric for functions in \( \delta \mathbb{Z} \cap [a, b) \).

Given a function \( w \in L^1 \cap L^\infty([a, b]) \) there is a function in \( \tilde{w} \in L^1 \cap L^\infty(\delta \mathbb{Z} \cap [a, b]) \) such that for \( f, g \in L^2(\delta \mathbb{Z} \cap [a, b]) \) it holds that \[ \langle f, w \ast g \rangle_{L^2(\mathbb{R})} = \langle f, \tilde{w} \ast g \rangle_{L^2(\delta \mathbb{Z})}. \]

The function \( \tilde{w} \) is given explicitly by:

\[
\tilde{w}(s) = \langle \delta^{-1} 1_s, \tilde{w} \ast (\delta^{-1} 1_0) \rangle_{L^2(\delta \mathbb{Z})} = \langle \delta^{-1} \chi_{[s, s+\delta]}, w \ast (\delta^{-1} \chi_{[0, \delta)}) \rangle_{L^2(\mathbb{R})} = \int_s^{s+\delta} \int_{0}^{\delta} w(y-x) dx dy = \delta^{-2} \int_{s}^{s+\delta} \int_{0}^{\delta} w(t)(\delta - |t-s|) dt.
\]

In the two cases of special interest (when \( w \) is a Gaussian function or when \( w \) is the characteristic function of a set, we get more explicit values). In practice, however, the high stability of the integrals make it more accurate to perform the integrals numerically if \( \delta \) is small than to compute the difference numerically (of the order of \( 10^{-5} \)), for more details see the annotated code.

3.3. Computational aspects II: Solving the problem for a fixed \( \lambda \) at a discretization scale \( \delta \). What remains to do now is to give an algorithm that allows us to solve the extended problem for a fixed value of \( \lambda \). The key fact that we use is that, if we remove the constraint \( f \geq 0 \), then the solution could be readily found using the power method for self-adjoint finite dimensional linear operators (symmetric matrices). We focus first in the situation when the constraint \( f \geq 0 \) is removed, and then argue that we can assume we are in that situation.

Recalling that

\[
C_{\text{opt}}(w) = \max_{\lambda \leq 0} \max_{f \in L^1 \cap L^2([-a, a])} 2 \frac{\langle f|w|f \rangle}{\lambda \langle f|f \rangle + \lambda^{-1} \langle f|1|f \rangle}.
\]

Assume that the functions under consideration are supported in \([-a, a]\), so we have \( \langle 1|1 \rangle = 2a \). In an abuse of notation, we denote by \( w \) the convolution operator
associated to \( w \). The key observation is the following
\[
\lambda \langle f | f \rangle + \lambda^{-1} \langle f | 1 \rangle \langle 1 | f \rangle = \langle f | (\sqrt{\lambda} \text{Id} + b_{\lambda} | 1 \rangle | 1 \rangle \rangle^2 | f \rangle = \langle f | A_{\lambda}^2 | f \rangle,
\]
where \( b_{\lambda} \) is the unique positive solution to
\[
\lambda^{-1} = 2\sqrt{\lambda} b_{\lambda} + 2ab^2_{\lambda}
\]
and \( A_{\lambda} \) is a hermitian positive definite matrix. Let \( g := A_{\lambda} f \), then we have:
\[
2\langle f | w | f \rangle = 2\langle g | A_{\lambda}^{-1} w A_{\lambda}^{-1} | g \rangle = \lambda \langle f | f \rangle + \lambda^{-1} \langle f | 1 \rangle \langle 1 | f \rangle = \langle f | A_{\lambda}^2 | f \rangle,
\]
defining \( M_{\lambda} := 2A_{\lambda}^{-1} w A_{\lambda}^{-1} \) we have that
\[
C_{opt}(w) = \max_{\lambda > 0} \max \text{Spec}(M_{\lambda}).
\]
What remains to be shown is that the constraint \( f \geq 0 \) can indeed be dropped out. Let \( f^* \) be the solution to the discretized problem constrained to \( f \geq 0 \). We can assume (by the discrete version of Riesz rearrangement inequality, see [15, Chapter X]) that \( f^* \) is symmetric and non-increasing. Assume that, out of all potential extremizers to the discretized problem, \( f^* \) has minimal support.

Let \( \tilde{V} \) be the (finite dimensional) space of functions in \( \delta \mathbb{Z} \) with the same support as \( f^* \). Since \( f^* \) extremizes the rayleigh quotient \( \langle f | w | f \rangle \) it must be an eigenvector \( w \) restricted to the support of \( f \). It is the unique eigenvector: If there was another eigenvector \( g^* \), the function \( g^* - \alpha f^* \) would be an eigenvector as well, with a strictly smaller support if \( \alpha \) is chosen appropriately.

Let \( l \) be the number of \( \delta \)-intervals in the discretized shortest support. We can further constrain the optimization problem to functions supported in these \( l \) intervals without changing the value of the optimization. In this case, the extremizer function \( f^* \) will be an extremizer on the interior, and therefore, the largest eigenvalue of the bilinear form induced by \( w \). We can find such eigenvalue by the power method.

This shows the following algorithm will give the extreme value up to discretization errors:

(1) Choose \( \delta \) small enough for the desired error in Proposition 3.6 to hold. Let \( N \) be the number of \( \delta \)-intervals intersecting \([-a, a]\).

(2) Fix \( \lambda \) (and then repeat for a large enough set of \( \lambda \) for the desired tolerance in Proposition 3.6 to hold).

(3) For each natural \( 1 \leq k \leq N \) solve the unconstrained discretized problem (the eigenvalue problem) with the power method. If the solution satisfies the constraints, keep as a potential problem.

(4) The maximum for our original problem (up to the tolerance/error arising from the discretization) is the maximum over all the solutions given in the previous step, where the maximum is taken over all the \( \lambda \) and \( k \).

3.4. A conjectured iterative method. The Euler-Lagrange equations for \( f \) in the support of \( f \) can be written as:
\[
f^* \frac{\|f^*\|_2^2}{\|f^*\|_2^2} = \max \left( \frac{2f^* \ast w}{\int_{\mathbb{R}^2} f^*(x)f^*(y)w(x-y)dx dy} - \frac{1}{\|f^*\|_1}, 0 \right).
\]
This suggests a fixed point method (both in the discrete and continuous set-ups) to find an extremizer to the autoconvolution problem. This method converges in practice (see the last column in Table 1), and is hundreds of times faster than the method described in the previous section. We have however been unable to show convergence of such method.

This method bears resemblance to the numerical method in [4], where the authors were not able to show convergence of the method either.

4. A COMPARISON WITH THE MAXIMUM PROBLEM

At first glance this problem bears resemblance to the original maximum problem originally studied by Cilleruelo, Rusza and Vinuesa [7]. The problem of the mean (when \( w = \chi_{[-1/2,1/2]} \)) was indeed posed by Steinberger as a relaxation of the problem of the maximum, namely finding the largest constant such that:

\[
\max_{-1/2 \leq t \leq 1/2} \int_{\mathbb{R}} f(t - x)f(x) \, dx \geq c_{\max} \left( \int_{-1/4}^{1/4} f(x) \, dx \right)^2. \tag{21}
\]

this problem was studied using computational methods by Cloninger and Steinberger [8]. The complexity of numerical method proposed in [8] grows exponentially in the level of discretization of the function, as opposed to the polynomial cost in the method proposed in this work.

One may hope that the methods in this work could generalize to methods in the maximum problem, however we believe that the problems are genuinely different for two reasons:

- There is no reason to expect smooth (or Lipschitz) global extremizers to the maximum problem. In fact, the best known numerical extremizers [21, Figure 1] seem non-smooth, and are certainly not symmetrically decreasing. This prevents the gain of a \( \delta^2 \) in the discretization of the problem, which now has to be done much more carefully (8, Lemma 1).
- The (discretized) problem of the maximum seems to have multiple local extremizers of the functional in equation (21). This quite likely prevents fixed point methods to be efficient.

5. A REMARK ON ANOTHER AUTOCONVOLUTION PROBLEM

For any \( f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \) the following inequality holds:

\[
\|f * f\|_2 \leq \|f \ast f\|_1 \|f * f\|_\infty.
\]

Motivated by the regularizing effect of the autoconvolutions it was conjectured by Martin and O’Bryant [19, Conjecture 5.2.] that there is a universal constant \( 0 < c < 1 \) such that for all nonnegative \( f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \) the following improved inequality holds:

\[
\|f * f\|_2 \leq c \|f \ast f\|_1 \|f * f\|_\infty.
\]

In fact, they proposed \( c = \frac{\log 16}{4} \sim 1 - 0.1174 \). This was disproved by Matolcsi and Vinuesa [21], they observed that a necessary condition is \( c \geq 1 - 0.1107 \).

In this section we show that

**Theorem 5.1.** The best constant \( c \) such that for any \( f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \) the inequality

\[
\|f * f\|_2 \leq c \|f \ast f\|_1 \|f * f\|_\infty
\]

(22)
holds is 1.

We start with two results about cut-off functions in $\mathbb{R}$.

**Proposition 1.** There exists a smooth real function $\eta \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and some $n > 100$ such that the following hold:

- $\eta$ is symmetric nonnegative
- $\hat{\eta}$ is $C^{n-1}$, symmetric nonnegative, supported on $[-1,1]$ and positive on $(-1,1)$
- $\|\eta\|_1 = 1$
- $\eta(1-x) = cx^n(1+o(1))$, for $x \in [0, \frac{1}{10}]$, where the $o(1)$ term is smooth.

**Proof.** Let $f = (\frac{1}{4} - x^2)^m \chi_{[-1/2,1/2]}$ for $m > 100$. Let $\hat{\eta} = cf \ast f$, for $c$ that makes $\|\eta\|_1 = 1$ hold. $\square$

**Proposition 2.** With $\eta$ as given in the previous lemma, if $\phi : \mathbb{R} \to \mathbb{R}$ is a smooth function such that $\phi(x) = \phi(-x)$ for all $x \in \mathbb{R}$, $\phi(1) = 1 + o(1)$, and it does not vanish on $[-1,1]$, then $\phi \hat{\eta}$ admits a $C^{50}$ (complex) square root on $\mathbb{R}$.

**Proof.** By elementary complex analysis, we know there exists a unique $C^0$ complex square root $f$ of $\phi \hat{\eta}$. Moreover it is smooth in $(-1,1) \cup [-1,1]^c$. It suffices to show that this square root is $C^{50}$ near 1. By hypothesis, $\phi = (1 + o(1))$ near 1, where the $o(1)$ is a smooth term. Therefore $\phi \hat{\eta}(1-x) = cx^n(1 + o(1))$, where the $o(1)$ is a smooth term, and therefore admits a $C^{50}$ square root. $\square$

**Proof of Theorem 5.1.** Throughout the proof, let $\eta$ as given in the theorem above. Let $\eta_c(x) = c^{-1} \eta(cx)$. Let $g = \chi_{[-1/2,1/2]} \ast \eta_c$ for $c$ large enough.

We have that $\|g\|$ is a real-valued Schwartz function, and that

$$\|g\|_2 \geq (1 - \epsilon) \|g\|_1 \|g\|_\infty$$

for $\epsilon = \epsilon(c)$ as small as we want. We will show that there exists a test function $f$ such that the following inequality holds

$$\|f \ast f - g\|_* < \epsilon$$

for $*=1, 2, \infty$. This will finish the proof. Note that we can get $* = 2$ by interpolation. Let $z_1, \ldots, z_n$ be the positive zeros of $\hat{\eta}$. Note that since $\hat{\eta}(x) = \hat{\eta}(x)$, the negative zeros are $-z_1, \ldots, -z_n$. Let $\delta$ sufficiently small, and let

$$\hat{\tilde{g}}(w) := \hat{\eta_c}\left(\chi_{[-1/2,1/2]} + i\delta \sum_{j=1}^n \hat{\eta}(\delta^{-1}(w-z_j)) - \hat{\eta}(\delta^{-1}(w+z_j))\right).$$

Now we can take a square-root of $\hat{\tilde{g}}(w)$ by Proposition 2 and the result follows. $\square$

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