Deep combinatorial optimisation for optimal stopping time problems and stochastic impulse control. Application to swing options pricing and fixed transaction costs options hedging.

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Abstract

A new method for stochastic control based on neural networks and using randomisation of discrete random variables is proposed and applied to optimal stopping time problems. Numerical tests are done on the pricing of American and swing options. An extension to impulse control problems is described and applied to options hedging under fixed transaction costs. The proposed algorithms seem to be competitive with the best existing algorithms both in terms of precision and in terms of computation time.

Keywords. Optimal stopping, American option, Swing option, Impulse control, Combinatorial optimisation, Neural network, Artificial intelligence, Fixed transaction costs.

1 Introduction

Motivation Optimal stopping problems are particularly important for risk management as they are involved in the pricing of American options. American-style options are used not only by traditional asset managers but also by energy companies to hedge “optimised assets” by finding optimal decisions to optimise their P&L and find their value. A common modelling of a power plant unit P&L is done using swing options which are American options allowing to exercise at most \( l \) times the option with possibly a constraint on the delay between two exercise dates (see Carmona and Touzi (2008) or Warin (2012) for gas storage modelling).

Formally, for \( T > 0 \), we are given a stochastic processes \((X_t)_{t \geq 0}\) defined on a probability space \((\Omega, \mathcal{F}, \mathbb{P}) = (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}\) and one wants to find an increasing sequence of \( \mathcal{F} \) stopping times \( \tau = (\tau_1, \tau_2, \ldots, \tau_l) \) that maximises the expectation of some objective function \( f \)

\[
\mathbb{E}_{\tau} \left( \sum_{i=1}^{l} f(\tau_i, X_{\tau_i})1_{\tau_i \leq T} \right).
\]

Numerical methods to solve the optimal stopping problem when \( l = 1, f(x, t) = e^{-rt}g(x) \) and \( X \) is Markovian include:

- Dynamic programming equation: the option price \( P_0 \) is computed using the following backward discrete scheme over a grid \( t_0 = 0 < t_1 < \ldots < t_N = T \):

\[
P_{N} = g(X_{T}),
\]

\[
P_{i} = \max(g(X_{t_i}), e^{-r(t_{i+1}-t_i)}\mathbb{E}(P_{i+1}|\mathcal{F}_{t_i})), \quad i = 0, \ldots, N - 1.
\]

One then needs to perform regression to compute the conditional expectations, see Longstaff and Schwartz (2001) or Bouchard and Warin (2012).
• Partial differential equation (PDE): a variational inequality derived from the Hamilton Jacobi Bellman equation is given by
\[
\min(-\partial_t + \mathcal{L})v + rv, v - g = 0, \quad v(x, T) = g(x)
\]
where \(\mathcal{L}\) is the infinitesimal generator of \(X\) (Shreve, 2004, Chapter 8, Section 3.3). A numerical scheme can be applied to solve this PDE and find the option value.

• Reflected Backward Stochastic Differential Equation (BSDE): the value of the American option is the solution of the reflected BSDE (see El Karoui et al. (1997)):
\[
Y_t = g(X_T) - r \int_t^T Y_s \, ds - \int_t^T Z_s \, dW_s + K_T - K_t, \\
Y_t \geq g(X_t), \quad 0 \leq t \leq T, \\
\int_0^T (Y_t - g(X_t)) \, dK_t = 0.
\]

Bouchard and Chassagneux (2008) provides a numerical scheme to solve these equations. These approaches generalise well for \(l \geq 1\), see Carmona and Touzi (2008) for dynamic programming principle or Bernhart et al. (2012) for the BSDE method. The non linear case where \(f\) is of the form \(\phi(\sum_{i=1}^l e^{-r\tau_i} g(X_{\tau_i}) 1_{\tau_i \leq T})\) is studied by Trabelsi (2013). The method proposed in the latter paper can also be related to the parametric valuation of American options: a decision rule or the exercise region is represented by a vector of parameters to be optimised (Glasserman, 2013, Chapter 8, Section 2). As in reinforcement learning, the estimation of optimal parameters is based on Monte Carlo simulations. This list of methods is not exhaustive and the reader can refer to (Glasserman, 2013, Chapter 8) for more details on numerical methods for American option pricing. All these algorithms suffer from the curse of dimensionality: the number of underlying is hardly above 5. However energy companies portfolio may trade derivatives involving more that 4 commodities at one time (e.g. swing options indexed on CO2, natural gas, electricity, volume, fuel) and traditional numerical methods hardly provide good solutions in a reasonable time.

Recently, neural network-based approaches have shown good results regarding stochastic control problems and PDE numerical resolution in high dimension (see Han et al. (2017b), Chan-Wai-Nam et al. (2019)). In Huré et al. (2018); Bachouch et al. (2018), the optimal policy is parameterised by a neural network which weights and bias minimise at each time step the right hand side of the dynamic programming equation, going backward. The value function to minimise at time \(t\) is either computed using all the optimal policies computed after \(t\) or using an approximation of the value function at time \(t+1\) from a neural network regression. This method can be used both for continuous and discrete actions. Numerical tests are performed on a gas storage in Bachouch et al. (2018). In Huré et al. (2019) and Han et al. (2017a), neural networks are used to solve BSDE. In Huré et al. (2019), the neural networks parameterising the solution and its gradient (or only the solution and the gradient is computed by numerical differentiation) minimise the \(L^2\) loss between the left hand-side and the right hand side of the Euler discretisation of the BSDE, going backward from the terminal value. Bachouch et al. (2018) and Huré et al. (2019) need to maximise one criteria by time step. The approach of Han et al. (2017a) is quite different: the neural network allows the parameterisation of the initial value of the BSDE and the gradient at each time step, and it minimises the distance between the terminal value obtained by the neural network and the terminal value of the BSDE, going forward. American put options prices are computed in Huré et al. (2019) up to dimension 40 with 160 time steps.

Fecamp et al. (2019) uses neural networks to parameterise the positions that need to be taken in order to hedge an option. One neural network is trained taking as entries the time and the value of the underlying(s) in order to minimise a risk criteria (\(L^2\) loss, value at risk,...). This risk criteria is estimated by Monte-Carlo simulations and the optimisation is done by gradient descent over the parameters of the neural network. This methodology shows very good performance, even in an incomplete market where there are (proportional) transaction costs and volume constraints. This last technique produces optimal control in a reasonable time even in high dimension but do not apply when there is a discrete control (e.g optimal stopping time). Neural network approaches have also been used in the context of swing options pricing in gas market in Barrera-Esteve et al. (2006). The definition of swing options slightly differs from ours as it considers a continuous control: the option owner buys a certain amount of gas between a minimum and a maximum quantity. It is however related to our problem as in continuous time, this option is bang-bang: it is optimal to exercise at the minimum or the maximum level at each date, that is choosing between two actions. Barrera-Esteve et al. (2006) directly models the policy by a neural network and optimise the objective function as in Fecamp et al. (2019).

Contrarily to Huré et al. (2018); Bachouch et al. (2018); Han et al. (2017a); Huré et al. (2019), the goal of this paper is to propose a reinforcement learning algorithm to solve optimal multi-exercise (rather than one single) stopping time problems with constraints on exercise times that does not need to derive a dynamic programming equation nor to find an equivalent BSDE of the problem. The only information needed is the dynamic of the state process \(X\) and the objective function. This kind of algorithm is called policy gradient and is well known in the area reinforcement learning, see Sutton et al. (2000) for instance. Although continuous control with reinforcement learning shows good results, the case of optimal stopping times is more difficult as it involves controls taking values in a discrete set of actions. The problem is similar
to a combinatorial optimisation one: at each time step, an action belonging to a finite set needs to be taken. One way to solve this problem is to perform a relaxation assuming that the control belongs to a continuous space. For instance, if one needs to price an American option, a decision represented by a value in \( \{0, 1\} \) and consisting in exercising or not must be taken. Relaxing the problem consists in searching for solutions in \([0, 1]\). Such method has been studied in Becker et al. (2019a) and Becker et al. (2019b) in the case of Bermudan options (with only one exercise) pricing with neural networks. They succeed in pricing Bermudan options in a high dimensional setting (up to 1000) with good accuracy.

On a very different combinatorial problem (namely traveller salesman) Bello et al. (2016) proposes an approach to solve with neural networks by using randomisation of discrete variables instead of relaxing the discrete setting. The probability for the action to take a discrete value is modelled by a neural network but the function to optimise is computed by Monte-Carlo sampling from the probability linked to the discrete actions. The difficulty then comes from the computation of the gradient: the trick which is common in reinforcement learning is to use the likelihood ratio method Sutton et al. (2000).

### Main results

Our approach follows the spirit of Fecamp et al. (2019): one directly parameterises the optimal policy by a neural network and maximises the objective function. We propose an algorithm using reinforcement learning as in Bello et al. (2016) in order to solve optimal stopping times problem as an combinatorial optimisation problem. While Bello et al. (2016) considers deterministic optimisation problem, the framework of this article is stochastic and involves a dynamic in time of the state process, with discrete decisions at each time step. Compared to the papers referenced above our approach presents the many advantages as it

- can solve multiple optimal stopping time problems;
- is independent from the dynamic of \( X \);
- allows to add in a flexible way any constraint on the stopping times;
- can then be associated with the one of Fecamp et al. (2019) considering continuous actions in order to solve stochastic impulse control problems, combining discrete and continuous controls;
- is able to choose any risk criteria to optimise even if it is not possible to derive a dynamic programming equation, see Fecamp et al. (2019) where hedging is done under an asymmetric risk criteria.

One of the proposed algorithm (Algorithm 1) allows to solve stopping time problems without any knowledge of the dynamic programming equation or of an equivalent BSDE. Numerical tests covering Bermudan and swing options are proposed and show good results in the pricing of 10 underlyings Bermudan option and also on 5 underlyings swing options having up to \( l = 6 \) exercise dates. An extension to stochastic impulse problems combining both continuous and discrete controls is proposed in Algorithm 2. Those problems are classical in finance but usually hard to solve. It is tested on a well known problem which is hedging with fixed transaction costs cases. Those fixed transaction costs can be seen as a fee to enter in the market or as the hedging operational cost. Algorithm 2 is applied to the hedging of a 3 underlying spread option with fixed transaction costs, a difficult problem to solve using stochastic control approaches. To our knowledge, this paper is the first to propose a neural network approach to solve multiple optimal stopping times and impulse control problems by modelling directly the policy, without the use of the dynamic programming equation. The theoretical convergence study of our algorithm is out of the scope of this paper.

### Organisation of the paper

The paper is organised as follow:

- Section 2 deals with optimal stopping times problems. Section 2.1 and Section 2.2 describe the problem we want to solve by the algorithm proposed in 2.3. The neural network architecture and hyper parameters are discussed in Section 2.4 and Section 2.5. Numerical tests on Bermudan and swing options are done in 2.6.
- In Section 3, an extension to impulse control is proposed. The optimal hedging with fixed transaction costs problem is described in Section 3.3. In Section 3.3.1, we compare our algorithm against stochastic control and against Whalley and Wilmott (1993) methodology on a call option with fixed transaction costs. Numerical tests are done on a spread option involving 3 risk factors in Section 3.3.2.

### 2 Optimal stopping

#### 2.1 Continuous time modelling

We are given a financial market operating in continuous time. Let \( (\Omega, \mathbb{F} = (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}) \) a filtered probability space and \( W \) a \( d \)-dimensional \( \mathbb{F} \)-Brownian motion. One assumes that \( \mathbb{F} \) satisfies the usual conditions of right continuity and completeness. Let \( T > 0 \) a finite horizon time and \( X = (X_1, X_2, \ldots, X_d) \) be the unique solution of the Stochastic Differential Equation (SDE):

\[
X_t = X_0 + \int_0^t \mu(s, X_s) ds + \int_0^t \sigma(s, X_s) dW_s, \quad t \in [0, T],
\]

(1)
with \( \mu : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d \) and \( \sigma : [0, T] \times \mathbb{R}^d \to \mathbb{R}^{d \times d} \) two measurable functions verifying \( |\mu(t, x) - \mu(t, y)| + \|\sigma(t, x) - \sigma(t, y)\| \leq K_1|x - y| \) and \( |\mu(t, x)| + \|\sigma(t, x)\| \leq K_2(1 + |x|) \) for \( x, y \in \mathbb{R}^d \) and \( t \in [0, T] \) \( (\cdot , \cdot ) \) denotes the Euclidian distance in \( \mathbb{R}^d \) and for a matrix \( A \in \mathbb{R}^{d \times d}, \|A\| = \sqrt{\text{tr}(AA^\top)} \) and \( K_1, K_2 \in \mathbb{R} \). Under these hypothesis, (1) has a unique strong solution which is Markovian. One could extend the modelling to more general Itô semi-martingales but for the sake of simplicity, we restrict ourselves to continuous Markovian diffusions. Using the notations of Carmona and Touzi (2008) and with \( X \) as defined in (1) for \( t \in [0, T] \) and \( X_t = X_T \) for \( t \geq T \), an optimal stopping time problem consists in solving the problem

\[
\sup_{\tau \in \mathcal{S}_T} \mathbb{E}_{\mathcal{F}_t}[\sum_{i=1}^t f(\tau_i, X_{\tau_i})1_{\tau_i \leq T}]
\]

(2)

where \( \mathcal{S}_T \) is the collection of all vectors of increasing stopping times \( \tau = (\tau_1, \ldots, \tau_t) \) such that for all \( i = 2, \ldots, t, \tau_i - \tau_{i-1} \geq \gamma \) a.s. on the set of events \( \{\tau_{i-1} \leq T\} \) and where \( f : [0, T] \times \mathbb{R}^d \to \mathbb{R} \) is a measurable function. \( l \in \mathbb{N}^* = \mathbb{N} \setminus \{0\} \) corresponds to the number of possible exercises and \( \gamma \geq 0 \) to the minimum delay between two exercise dates. The reader can refer to Ibáñez (2004); Carmona and Touzi (2008); Bernhart et al. (2012) for more information on swing options and methods to price them. The American option case corresponds to \( l = 1 \). One wants to find the optimal value (2) but also the optimal policy

\[
\tau^* \in \text{argmax}_{\tau \in \mathcal{S}_T} \mathbb{E}_{\mathcal{F}_t}[\sum_{i=1}^t f(\tau_i, X_{\tau_i})1_{\tau_i \leq T}]
\]

(3)

### 2.2 Discrete time modelling

In practice, one only considers optimal stopping on a discrete time grid (for instance, the valuation of a Bermudan option is used as a proxy of the American or swing option). Let us consider \( N + 1 \) exercise dates belonging to a discrete set \( \mathcal{D}_N = \{t_0 = 0 < t_1 < \ldots < t_N = T\} \), \( N \in \mathbb{N}^* \). The problem consists in finding

\[
\sup_{\tau \in \mathcal{S}_N} \mathbb{E}_{\mathcal{F}_t}\left[\sum_{i=1}^N f(\tau_i, X_{\tau_i})1_{\tau_i \leq T}\right]
\]

(4)

where \( \mathcal{S}_N \) is the set of stopping times belonging to \( \mathcal{S}_T \) such that \( \tau_i \in \mathcal{D}_N \) on \( \{\tau_i \leq T\} \), for \( i = 1, \ldots, l \). This discretisation is needed as it is needed in classical methods such as Longstaff and Schwartz (2001). The solution of (4) can then be approximated by the solution of

\[
\sup_{Y} \mathbb{E}_{\mathcal{F}_t}\left[\sum_{i=0}^N Y_if(t_i, X_{t_i})\right]
\]

(5)

where \( (Y_i)_{i=0,\ldots,N} \) is a sequence of \( (\mathcal{F}_{t_i})_{i=0,\ldots,N} \)-measurable random variables taking values in \( \{0, 1\} \) such that

\[
\sum_{i=0}^N Y_i \leq l
\]

(6)

and

\[
\sum_{i=(k-\gamma)+1}^{j} Y_i \leq 1 \text{ for } j \in \{0, \ldots, N\}
\]

(7)

where \( k_{l,\gamma} = \min\{i \leq j | t_j - t_i > \gamma\}1_{t_j - t_i > \gamma} - 1_{t_j \leq \gamma} \). Given a solution of Equation (5) \( (Y^*_i)_{i=0,\ldots,N} \), a proxy for the optimal control (3) is given by, on the event \( \{\sum_{i=0}^N Y_i = m\} \) with \( m \leq l \), for \( k \in \{1, \ldots, l\} \),

\[
\tau^*_k = t_k1_{k \leq m} + \infty 1_{k > m}
\]

with \( k^* = \min\{j \in \{0, \ldots, N\} | \sum_{i=0}^j Y^*_i \geq k\} \).

### 2.3 Algorithm description for optimal stopping times problems

As the \( Y_i^* \)s are discrete we cannot suppose that they are the output of a neural network which weights are optimised by applying a stochastic gradient descent (SGD). One idea to overcome this difficulty is to suppose that at each time step \( t_i, i \in \{0, \ldots, N\} \), the discrete variable \( Y_i \) follows a Bernoulli distribution conditionally on \( X_{t_i} \). In a non Markovian framework, one could consider that the probability for \( Y_i \) to be equal to 1 is a function of all the values of \( X_{t_j} \) for \( j \leq i \). In this case, one could use a Recurrent Neural Network to parameterise this function. In order to consider the different constraints on the sequence \( (Y_i)_{i=0,\ldots,N} \), the law of \( Y_i \) depends also on the realisations of \( Y_j \) for \( j \leq i - 1 \). The parameter of the Bernoulli distribution is parameterised by a neural network \( \mathbb{N}\mathbb{N} \) defined on \( [0, T] \times \mathbb{R}^d \times \Theta \) where \( \Theta \) represents the sets in which the bias and weights of the neural network lie.
Parameterisation without constraints on $Y$

Without constraints, the parameterisation is the following

$$P(Y_j = 1|X_{t_0}) = \expit \left( C \times \tanh \left( \text{NN}(t_j, X_{t_j}, \theta) \right) \right), \quad j = 0, \ldots, N,$$

with $\expit : \mathbb{R} \mapsto (0, 1)$ and $\expit(x) = \frac{1}{1 + \exp(-x)}$. $C \times \tanh(\text{NN}(x, \theta))$ outputs the logit (the inverse function of $\expit$) of $P(Y_j = 1|X_{t_0})$. The function $\tanh$ is not necessary and one could only consider $\text{NN}$ to parameterise the logit of the probability. To reduce the values taken by the logit, we bound the output of the neural using $\tanh$ and one chooses $C$ such that $\expit(-C) \approx 0$ and $\expit(C) \approx 1$. $C$ is given in Section 2.5.

Parameterisation with constraints (Eq. (6) and Eq. (7))

Now, let us consider the constraints (6) and (7). The parameterisation is the following

$$\begin{align*}
P(Y_j = 1|X_{t_0}) & = \expit \left( C \times \tanh(\text{NN}(t_j, X_{t_j}, \theta)) \right) \quad j = 0, \ldots, N, \\
P(Y_j = 1|X_{t_j}, X_{t_0}, \ldots, Y_{j-1}) & = \expit \left( C \times \tanh(\text{NN}(t_j, X_{t_j}, \theta)) - \lambda \sum_{i=1}^{j-1} Y_i \sum_{i=k_j,\gamma+1}^{j-1} Y_i \right), \quad j = 1, \ldots, N,
\end{align*}$$

where $\lambda > 0$ is a penalty term. We choose $\lambda$ such that $C - \lambda < -C$ in order to have a probability of 0 even in the case where the output of $\tanh \circ \text{NN}$ is 1. $\lambda$ is given in Section 2.5. The methodology can be extended to any constraint on the policy.

The neural network architecture is described in Section 2.4 and Figure 1. From now on, $P$ is replaced by $P_\theta$ to indicate the dependence of the law of $Y$ with $\theta$. To approximate a solution to (5) we search for a $\theta^*$ verifying:

$$\theta^* \in \arg \max_{\theta \in \Theta} \mathbb{E}_{P_\theta} \left( \sum_{i=0}^{N} Y_i f(t_i, X_{t_i}) \right).$$

The gradient of $\mathbb{E}_{P_\theta} \left( \sum_{j=0}^{N} Y_j f(t_j, X_{t_j}) \right)$ is computed using the likelihood ratio method:

$$\nabla_\theta \mathbb{E}_{P_\theta} \left( \sum_{j=0}^{N} Y_j f(t_j, X_{t_j}) \right) = \mathbb{E}_{P_\theta} \left( \sum_{j=0}^{N} Y_j f(t_j, X_{t_j}) \sum_{j=0}^{N} \nabla_\theta \log \left( \expit \left( C \times \tanh \left( \text{NN}(t_j, X_{t_j}, \theta) \right) - \lambda \sum_{i=0}^{j-1} Y_i \sum_{i=k_j,\gamma+1}^{j-1} Y_i \right) \right) \right).$$

(using the convention $\sum_{i=0}^{j-1} Y_i = 0$) which can be easily computed using backpropagation. Once we have this expression, the algorithm consists in applying a SGD by applying the derivative defined in Equation (10) (see Algorithm 1).

Finally, while on the training phase the actions are sampled from the outputted probability on the training set, they are chosen equal to 1 if the probability is greater than 0.5 and 0 otherwise on the test and validation sets.

Remark 2.1 If it is not possible to have access to exact simulations of $X$, it is possible to consider an Euler SDE discretisation of (1).

2.4 Neural network architecture

The neural network architecture is inspired by Chan-Wai-Nam et al. (2019) and consists in one single feed forward neural network which features are the time step $t_i$ and the current $X_{t_i}$ realisation. Let $t \in [0, T]$ and $x = (x_1, \ldots, x_d)^\top \in \mathbb{R}^d$. The neural network is defined as follow

$$\text{NN}(t, x, \theta) = A_L \circ \rho \circ A_{L-1} \circ \rho \ldots \circ A_1((t, x_1, \ldots, x_d)^\top)$$

where $A_i(g) = W_i g + \beta_i$ for $i = 1, \ldots, L$, $W_i \in \mathbb{R}^{(d+1) \times m}$, $W_i \in \mathbb{R}^{m \times m}$, $\beta_i \in \mathbb{R}^m$ for $i = 1, \ldots, L - 1$, $W_L \in \mathbb{R}^{m \times 1}$, $\beta_L \in \mathbb{R}$ for $L = 1, \ldots, L - 1$ and $\beta_L \in \mathbb{R}$. $L$ corresponds to the number of layers and $m$ to the number of neurons per layer (that we assume to be the same for every layer). The $W_i$ correspond to the weights and $\beta_i$ to the bias. The function $\rho$ is the activation function and is chosen as the ReLu function, that is $\rho(x) = \max(0, x)$. $\theta$ is then equal to $(W_1, \ldots, W_L, \beta_1, \ldots, \beta_L)$.

2.5 Hyper parameters

- The likelihood ratio estimator of the gradient has high variance and choosing a higher number of batch size $N_{\text{batch}} (>1000)$ allows for a better estimation. The drawback is that it tends to slow down the algorithm. To reduce the variance, one could also use a baseline function as in Zhao et al. (2011).
Algorithm 1 Algorithm for optimal stopping. The lines in blue are the main difference with classical Adam algorithm (Kingma and Ba (2014)). The green part implements the exponential decay of the learning rates.

1: $\alpha_0$ : Initial stepsize
2: $r_\alpha \in [0,1]$ : Stepsize reduction
3: $\beta_1, \beta_2 \in [0,1]$ : Exponential decay rates for the moment estimates,
4: $N_{\text{iter}}$ : number of iterations
5: $N_{\text{batch}}$ : number of simulations at each gradient descent iteration (batch size)
6: $\theta_0$ randomly chosen
7: $m_0 \leftarrow 0$
8: $v_0 \leftarrow 0$
9: $i_{\text{iter}} \leftarrow 0$
10: for $i_{\text{iter}} = 0 \ldots N_{\text{iter}}$ do
11:    for $u = 0 \ldots N$ do
12:        $X_u \leftarrow N_{\text{batch}}$ samples simulations of $X_{t_u}$
13:        $Y_u \leftarrow N_{\text{batch}}$ of a Bernouilli r.v. with parameter
14:        $$expit \left( C \times \tanh \left( \text{NN}(t_u, X_u, \theta_{i_{\text{iter}}}) \right) \right) - \lambda I_{\sum_{i=1}^{u-1} y_i \geq 1} \sum_{i=1}^{u-1} y_i \geq 1$$
15:      end for
16:      $m_{i_{\text{iter}}} \leftarrow m_{i_{\text{iter}}-1} + (1 - \beta_1)g_{i_{\text{iter}}}$ (update biased first moment estimate)
17:      $v_{i_{\text{iter}}} \leftarrow \beta_2 v_{i_{\text{iter}}-1} + (1 - \beta_2)g_{i_{\text{iter}}}^2$ (update biased second raw moment estimate)
18:      $\hat{m}_{i_{\text{iter}}} \leftarrow \frac{m_{i_{\text{iter}}}}{1 - \beta_1^{i_{\text{iter}}}}$ (computes bias-corrected first moment estimate ($\beta_1^{i_{\text{iter}}}$ stands for $\beta_1$ to the power of $i_{\text{iter}}$))
19:      $\hat{v}_{i_{\text{iter}}} \leftarrow \frac{v_{i_{\text{iter}}}}{1 - \beta_2^{i_{\text{iter}}}}$ (computes bias-corrected second raw moment estimate ($\beta_2^{i_{\text{iter}}}$ stands for $\beta_2$ to the power of $i_{\text{iter}}$))
20:      $\theta_{i_{\text{iter}}} \leftarrow \theta_{i_{\text{iter}}-1} - \alpha_{i_{\text{iter}}} \hat{m}_{i_{\text{iter}}} / (\sqrt{\hat{v}_{i_{\text{iter}}}} + \epsilon)$ (update parameters)
21:      $\alpha_{i_{\text{iter}}} \leftarrow r_{\alpha} \alpha_{i_{\text{iter}}-1}$ (update stepsize)

- The learning rate ($\alpha$ in Algorithm 1) is modified at each time step. We initially choose a learning rate of $\alpha_0 (0.001$ or $0.005$), which we find to be a reasonable starting value. At each time step, we decrease it by multiplying it by $r_\alpha \in [0,1]$ (0.98$^{1/100}$ when $\alpha_0 = 0.001$ and 0.96$^{1/100}$ when $\alpha_0 = 0.005$) (technique inspired from Bello et al. (2016)).
- As in Chan-Wai-Nam et al. (2019), in our case, regularisation which is classically used to avoid overfitting is not relevant and we won’t use it as our data is not redundant and thus the network does not experience overfitting.
- Since we use the same network at each time step, we use a mean-variance normalisation over all the time steps to center all the inputs ($t_i, X_{t_i}$) for all $t_i$’s with the same coefficients. The scaling and recentering coefficients are estimated on pre-simulated data that is just used to this end.
- We use Xavier initialisation Glorot and Bengio (2010) for the weights and a normal initialisation for the bias.
- The number of layers is chosen equal to 3. The number of neurons per layer is constant (but can vary from a case to another).
- Every 100 steps, the objective value is computed over a testing set. The parameters kept at the end are the ones minimising those evaluations. The objective function is finally evaluated on a validation set.
- $C = 10$ and the penalisation parameter $\lambda$ is set to 25.
- The library used is Abadi et al. (2015) and the algorithm runs on a laptop with 8 cores of 2.50 GHz, a RAM memory of 15,6 Go and without GPU acceleration.
2.6 Numerical results

In this section Algorithm 1 is applied to the valuation of Bermudan and swing options. The function \( f(s, x) \) is of the form \( e^{-rs}g(x) \) where \( g \) is the payoff of the option and \( r \geq 0 \) is the risk free rate. We place ourselves in the Black-Scholes framework: \( \mu(s, x) = (r - \delta)x \), with \( \delta \geq 0 \) corresponding to the dividend rate and \( \sigma(s, x) = \text{diag}(x)\sqrt{\Sigma^T} \) with \( \Sigma \) a positive definite matrix. We choose to work with a regular time grid \( t_i = \frac{i}{N} \) for \( i = 0, \ldots, N \). The probability measure corresponds to the risk neutral probability and finding the value of the option consists in solving Problem (5).

2.6.1 Bermudan options

In this section, we assume that \( l = 1 \) (only one exercise) and we consider different options to price. For all the cases, we choose an initial learning rate \( \alpha_0 = 0.001 \), decaying with a rate of 0.98 every 100 steps (that is \( \alpha = 0.98^{1/100} \)), a test set of size 500,000 and a validation set of size 4,096,000 (500,000 is chosen high to have very accurate optimisation and 4,096,000 is chosen as in Becker et al. (2019b)).

Put option with \( d = 1 \), payoff \( g(x) = (K - x)^+ \), \( K = 90 \), \( S_0 = 95 \), \( r = 0.02 \), \( \delta = 0 \), \( \sqrt{\Sigma} = 0.3 \), \( N = 10 \), \( T = 1 \). We consider a batch size equal to \( N_{\text{batch}} = 5,000 \), a neural network with a depth of 3 layers having 10 neurons each and \( N_{\text{iter}} = 5,000 \) iterations. The reference value is given in Bouchard and Chassagneux (2008).

Max-call option with \( d \in \{2, 10\} \), payoff \( g(x) = \max((x_i)_{i=1,\ldots,d}) - K)^+ \), \( K = 100 \), \( S_0 = 100 \), \( i = 1, \ldots, d \), \( r = 0.05 \), \( \delta = 0.1 \), \( \sqrt{\Sigma} = 0.2I_d \) (\( I_d \) is the identity matrix with size \( d \times d \)), \( N = 9 \), \( T = 3 \). We consider a batch size equal \( N_{\text{batch}} = 5,000 \) for \( d = 2 \) and \( N_{\text{batch}} = 12,000 \) for \( d = 10 \), a neural network with 3 layers of size 30 for \( d = 2 \) and 70 for \( d = 10 \) and \( N_{\text{iter}} = 10,000 \) iterations. The reference values are given in Becker et al. (2019b).

Strangle spread option with \( d = 5 \), payoff \( g(x) = -\left(K_1 - \frac{1}{5} \sum_{i=1}^{5} x_i\right)^+ + \left(K_2 - \frac{1}{5} \sum_{i=1}^{5} x_i\right)^+ + \left(K_4 - \frac{1}{5} \sum_{i=1}^{5} x_i\right)^+ - \left(K_3 - \frac{1}{5} \sum_{i=1}^{5} x_i\right)^+ \), \( K_1 = 75 \), \( K_2 = 90 \), \( K_3 = 110 \), \( K_4 = 125 \), \( S_0 = 100 \), \( i = 1, \ldots, 5 \).
In the following cases, we use an initial learning rate $\alpha_0 = 0.005$, decaying with a rate of 0.96 every 100 steps (that is $r_n = 0.96^{1/100}$), a test set of size 500,000 and a validation set of size 4,096,000.

We compare in Table 2 the results obtained by Algorithm 1 with the results of Ibáñez (2004) in the case of a put option with $d = 1$, $g(x) = (K - x)^+$, $K = 40$, $S_0 \in \{35, 40, 45\}$, $r = 0.0488$, $\delta = 0$, $\Sigma = 0.25$, $N = 12$, $T = 0.25$, $\gamma = \frac{K}{S_0}$ (only one time step for delay) and $\ell \in \{1, 2, 3, 4, 5, 6\}$. We consider a batch size equal to $N_{\text{batch}} = 2,000$, a neural networks with 3 layers size 10 and $N_{\text{iter}} = 5,000$ iterations. Every case takes around 2 minutes to converge, see Table 3. The algorithm gives very accurate results in a short period of time (less than two minutes) for the valuation of the swing options.

| Use case / Method | Algorithm 1 | Reference | Difference | Time (s) |
|-------------------|-------------|-----------|------------|----------|
| Bermudan put      | 0.0603      | 0.0603    | 0.0%       | 155.2    |
| Max-call, $d = 2$ | 13.8934     | 13.8990   | 0.04%      | 452.7    |
| Max-call, $d = 10$| 38.2115     | 38.2780   | 0.17%      | 2948.4   |
| Strangle spread   | 11.7830     | 11.7940   | 0.09%      | 5211.7   |

Table 1: Results obtained on different Bermudan options pricing with Algorithm 1. A reference value is given (Bouchard and Chassagneux (2008) for the put option and Becker et al. (2019b) for the other options). The algorithm succeeds in pricing Bermudan options in dimension relatively high (up to 10) and also with a high number of time steps (up to 50). The computing time is more sensitive to the number of time steps than to the dimension: the number of neural network estimation is equal to the number of time steps. The increase of computing time when dimension increases is mostly caused by a need to increase the batch size and a more important simulation time. Algorithm 1 succeeds in pricing Bermudan options and solves problems that are usually hard to solve and very expensive in term of computation time as they suffer from the curse of dimensionality.

### 2.6.2 Swing options

In the following cases, we use an initial learning rate $\alpha_0 = 0.005$, decaying with a rate of 0.96 every 100 steps (that is $r_n = 0.96^{1/100}$), a test set of size 500,000 and a validation set of size 4,096,000.

We compare in Table 2 the results obtained by Algorithm 1 with the results of Ibáñez (2004) in the case of a put option with $d = 1$, $g(x) = (K - x)^+$, $K = 40$, $S_0 \in \{35, 40, 45\}$, $r = 0.0488$, $\delta = 0$, $\Sigma = 0.25$, $N = 12$, $T = 0.25$, $\gamma = \frac{K}{S_0}$ (only one time step for delay) and $\ell \in \{1, 2, 3, 4, 5, 6\}$. We consider a batch size equal to $N_{\text{batch}} = 2,000$, a neural networks with 3 layers size 10 and $N_{\text{iter}} = 5,000$ iterations. Every case takes around 2 minutes to converge, see Table 3. The algorithm gives very accurate results in a short period of time (less than two minutes) for the valuation of the swing options.

| $l / S_0$ | 35 | 40 | 45 |
|----------|----|----|----|
| 1        | (5.1, 5.114, 0.282%) | (1.775, 1.774, 0.045%) | (0.409, 0.411, 0.408%) |
| 2        | (10.165, 10.195, 0.291%) | (3.478, 3.48, 0.047%) | (0.769, 0.772, 0.388%) |
| 3        | (15.181, 15.23, 0.323%) | (5.09, 5.111, 0.414%) | (1.084, 1.089, 0.492%) |
| 4        | (20.151, 20.23, 0.39%) | (6.622, 6.661, 0.593%) | (1.361, 1.358, 0.187%) |
| 5        | (25.16, 25.2, 0.159%) | (8.075, 8.124, 0.602%) | (1.573, 1.582, 0.557%) |
| 6        | (30.085, 30.121, 0.12%) | (9.448, 9.502, 0.567%) | (1.756, 1.756, 0.019%) |

Table 2: Comparison of results obtained by Algorithm 1 with the ones of Ibáñez (2004) for different initial values $S_0$ and different number of executions $l$. The first value corresponds to the swing option value obtained with our method, the second value to the one in Ibáñez (2004) and the third value is the relative difference in %.

To assess the performance of Algorithm 1 in high dimension, let us consider the pricing of the geometrical put option with payoff $g(x) = \left(K - \prod_{i=1}^{t} x_i\right)^+$. Let $d = 5$, $K = 40$, $S_0^{t} = 40^{1/5}$, $r = 0.0488$, $\delta = \frac{\Sigma}{T}$, $\Sigma = \frac{0.25}{\sqrt{2}} I_5$, $N = 12$, $T = 0.25$, $\gamma = \frac{T}{\Sigma}$ and $\ell \in \{1, 2, 3, 4, 5, 6\}$. Parameters are chosen in order to have an option value equal to the the one dimensional case put option value: the product of the components of $X$ follows a Black-Scholes dynamic with drift parameter equal to 0.0488 and volatility equal to 0.25. It allows to have a reference value (from Ibáñez (2004)) while considering a high dimensional case. We consider a batch size equal to $N_{\text{batch}} = 8,000$, a neural networks with 3 layers of size 30 and $N_{\text{iter}} = 5,000$ iterations.
Table 3: Time in seconds for training and predicting with Algorithm 1 to price swing options for different initial values $S_0$ and different number of executions $l$.

| $l / S_0$ | 35     | 40     | 45     |
|-----------|--------|--------|--------|
| 1         | 121.4  | 121.9  | 126.1  |
| 2         | 124.0  | 124.5  | 126.0  |
| 3         | 126.0  | 125.3  | 122.4  |
| 4         | 127.2  | 127.7  | 122.9  |
| 5         | 117.0  | 122.7  | 122.8  |
| 6         | 124.4  | 122.7  | 116.1  |

Results are given in Table 4. The algorithm succeeds in pricing this option with 5 underlyings very quickly (around 10 minutes). The hyperparameters have been chosen in order to have very accurate results. To reduce the time, it is possible to change some of the hyperparameters. In order to show that Algorithm 1 can give good results in a very short period of time, let us consider a neural network with 3 layers of 20 neurons each (instead of 30), $N_{\text{iter}} = 2,000$ (instead of 5,000), $N_{\text{batch}} = 3,000$ (instead of 8,000), a validation set size of 100,000 (instead of 4,096,000) and let us omit the evaluation on the test set. Results are given in Table 5. In less than 2 minutes, it is possible to obtain results with a 2% accuracy.

Table 4: Comparison of results obtained by Algorithm 1 for the pricing of a 5 dimensional put option with the reference values reported in Ibáñez (2004). The time in seconds corresponds to the time of training and predicting.

| Use case / Method | Algorithm 1 | Reference | Difference | Time |
|-------------------|-------------|-----------|------------|------|
| $l = 1$           | 1.7748      | 1.774     | 0.04%      | 555.9|
| $l = 2$           | 3.4770      | 3.480     | 0.09%      | 694.7|
| $l = 3$           | 5.0897      | 5.111     | 0.42%      | 758.5|
| $l = 4$           | 6.6229      | 6.661     | 0.57%      | 725.8|
| $l = 5$           | 8.0588      | 8.124     | 0.8%       | 685.7|
| $l = 6$           | 9.4311      | 9.502     | 0.75%      | 812.8|

Table 5: Comparison of results obtained by Algorithm 1 with suboptimal hyperparameters for the pricing of a 5 dimensional put option with the reference values reported in Ibáñez (2004). The time in seconds corresponds to the time of training and predicting.

| Use case / Method | Algorithm 1 | Reference | Difference | Time |
|-------------------|-------------|-----------|------------|------|
| $l = 1$           | 1.7744      | 1.774     | 0.05%      | 91.9 |
| $l = 2$           | 3.4441      | 3.480     | 1.03%      | 89.6 |
| $l = 3$           | 5.0310      | 5.111     | 1.57%      | 91.3 |
| $l = 4$           | 6.5903      | 6.661     | 1.06%      | 92.8 |
| $l = 5$           | 8.0362      | 8.124     | 1.08%      | 89.9 |
| $l = 6$           | 9.2934      | 9.502     | 2.2%       | 90.3 |

Let us now consider the case of put option with $d = 1$, $g(x) = (K - x)^+$, $K = 100$, $S_0 = 100$, $r = 0.05$, $\delta = 0$, $\Sigma = 0.3$, $N = 50$, $T = 1$, $\gamma = 0.25$ and $l \in \{1, 2, 3, 4, 5\}$. Delay constraint is now present and a higher number of dates is considered. We consider a batch size equal to $N_{\text{batch}} = 5,000$, a neural networks with 3 layers size 10 and $N_{\text{iter}} = 10,000$ iterations. We compare in Table 6 the results obtained with Algorithm 1 to the ones obtained with Carmona and Touzi (2008). The algorithm gives satisfying results but in this very situation we can see that the relative error increases with the number of exercises dates.

The different use cases show that Algorithm 1 is able to solve optimal stopping time problems in a reasonable time, even when the dimension is high and also for multi-exercise. The algorithm is simple and allows us to find an optimal policy without any knowledge on the dynamic programming equation. While the time increases a little when dimension increases, it increases a lot more with the number of time steps and the algorithm can have troubles to converge. To confirm all those results, one should study the convergence of the algorithm which is out of the scope of this paper.
3 Extension to stochastic impulse control

In this section we present how the algorithm of Section 2.3 can be combined to the method described in Fecamp et al. (2019) for impulse control problems. Impulse control has a lot of applications in economics and finance (real options valuation in energy markets, and optimal order execution in illiquid markets...). In Section 3.3, we apply the algorithm to an option hedging under fixed transaction costs.

An impulse control is a sequence \((\tau, \zeta) = (\tau_i, \zeta_i)_{i \geq 1}\) with \((\tau_i)\) a sequence of increasing \(\mathcal{F}\) stopping times and \((\zeta_i)_{i \geq 1}\) a sequence of \((\mathcal{F}_{\tau_i})_{i \geq 1}\)-measurable random variables taking values in \(E \subset \mathbb{R}^d\). Let \(A\) the set of impulse control. A stochastic impulse control problem is the optimisation problem

\[
\sup_{(\tau, \zeta) \in A} \mathbb{E}_\tau(f(\tau, \zeta, X))
\]

with \(X\) the solution of (1). As for the optimal stopping time problems, the impulse control problems can be associated to a BSDE or to a quasi-variationnal inequality, see Kharroubi et al. (2010) and can only be solved numerically most of the time.

We consider the discretisation of problem (11) and we search for

\[
\sup_{(\tau, \zeta) \in A^N} \mathbb{E}_\tau(f(\tau, \zeta, X))
\]

where \(A^N \subset A\) contains the control with \(\tau\) taking values in a discrete set \(\mathcal{D}_N\). This problem is equivalent to

\[
\sup_{(Y, Z)} \mathbb{E}_\tau(f_N(Y, Z, X))
\]

where \((Y_i, Z_i)_{i=0,\ldots,N}\) is a sequence of \((\mathcal{F}_{\tau_i})_{i=0,\ldots,N}\)-measurable random variables with \(Y_i\) taking values in \(\{0, 1\}\) and \(Z_i\) taking values in \(E\).

**Remark 3.1** More generally, the process \(X\) can depend on the control if jumps are present in its dynamic but this case is not considered here.

### 3.1 Algorithm extension to impulse control

Z is a continuous control and can be optimised as done in Fecamp et al. (2019). A neural network \(\text{NN}_Z\) with vector of parameters \(\theta_Z\) outputs the optimal continuous control:

\[
Z_{t_i} = \text{NN}_Z(t_i, X_{t_i}, \theta_Z)
\]

with \(\theta_Z \in \Theta_Z\) the set in which the bias and weights of \(\text{NN}_Z\) lie. \(\text{NN}_Z\) is chosen as a Recurrent Neural Network, see Section 3.2, but for ease of notation, the dependence with the memory of the neural network is omitted.

\(Y\) is again supposed to be Bernoulli distributed with a probability parameterised by a neural network \(\text{NN}_Y\) with vector of parameters \(\theta_Y\). Omitting the constraints (which can be easily added as in Equation (9)), the probability is parameterised as in Equation (8):

\[
P(Y_j = 1|X_{t_j}) = \exp\left(C \times \tanh(\text{NN}_Y(t_j, X_{t_j}, \theta_Y))\right), \quad j = 0, \ldots, N,
\]

with \(\theta_Y \in \Theta_Y\) the set in which the bias and weights of \(\text{NN}_Y\) lie.

Algorithm 2 finds the optimal parameters of the neural networks. The algorithm is quite similar to Algorithm 1 except there is a gradient descent over the parameters \(\theta_Z\). The computation of this gradient is easily done using backpropagation. In Algorithm 2, the two gradients are concatenated in order to have one gradient updated with one learning rate: it is possible (and it is used in some of the results hereafter) to consider two different learning rates and to update each gradient separately.
Algorithm 2 Algorithm for impulse control problems. Main differences with Algorithm 1 are in blue.

1: \( \alpha_0 \): Initial stepsize
2: \( r_\alpha \in [0,1] \): Stepsize reduction
3: \( \beta_1, \beta_2 \in [0,1] \): Exponential decay rates for the moment estimates,
4: \( N_{\text{iter}} \): number of iterations
5: \( N_{\text{batch}} \): number of simulations at each gradient descent iteration (batch size)
6: \( \theta_0 = (\theta_{Z,0}, \theta_{Y,0}) \) randomly chosen
7: \( m_0 \leftarrow 0 \)
8: \( v_0 \leftarrow 0 \)
9: \( i_{\text{iter}} \leftarrow 0 \)
10: for \( i_{\text{iter}} = 0 \ldots N_{\text{iter}} \) do
11: for \( u = 0 \ldots N \) do
12: \( X_u \leftarrow N_{\text{batch}} \) samples simulations of \( X_{tu} \)
13: \( \Delta_u \leftarrow N_{\text{NN}}(t_u, X_u, \theta_{Z,i_{\text{iter}}}) \)
14: \( Y_u \leftarrow N_{\text{batch}} \) of a Bernoulli r.v. with parameter \( \expit(C \times \tanh(\text{NN}_Y(t_u, X_u, \theta_{Y,i_{\text{iter}}})) \)
15: \( i_{\text{iter}} \leftarrow i_{\text{iter}} + 1 \)
16: \( g_{i_{\text{iter}},z} \leftarrow \nabla_{\theta_z} \frac{1}{N_{\text{batch}}} \sum_{n=1}^{N_{\text{batch}}} f_N(Y^n, Z^n, X^n) \) (get objective function gradient w.r.t \( \theta_Z \))
17: \( g_{i_{\text{iter}},y} \leftarrow \frac{1}{N_{\text{batch}}} \sum_{n=1}^{N_{\text{batch}}} f_N(Y^n, Z^n, X^n) \left( \sum_{u=0}^{N} \nabla_{\theta_y} \log(\expit(C \times \tanh(\text{NN}_Y(t_u, X_u^n, \theta_{Y,i_{\text{iter}}}-1)))) \right) \)
18: \( g_{i_{\text{iter}}} \leftarrow (g_{i_{\text{iter}},z}, g_{i_{\text{iter}},y}) \) (concatenate gradients)
19: \( m_{i_{\text{iter}}} \leftarrow m_{i_{\text{iter}}-1} + (1 - \beta_1)g_{i_{\text{iter}}} \) (update biased first moment estimate)
20: \( v_{i_{\text{iter}}} \leftarrow \beta_2 v_{i_{\text{iter}}-1} + (1 - \beta_2)g^2_{i_{\text{iter}}} \) (update biased second raw moment estimate)
21: \( \hat{m}_{i_{\text{iter}}} \leftarrow m_{i_{\text{iter}}} / (1 - \beta_1) \) (computes bias-corrected first moment estimate (\( \beta_1 \) stands for \( \beta_1 \) to the power of \( i_{\text{iter}} \))
22: \( \hat{v}_{i_{\text{iter}}} \leftarrow v_{i_{\text{iter}}} / (1 - \beta_2) \) (computes bias-corrected second raw moment estimate (\( \beta_2 \) stands for \( \beta_1 \) to the power of \( i_{\text{iter}} \))
23: \( \theta_{i_{\text{iter}}} \leftarrow \theta_{i_{\text{iter}}-1} - \alpha_{i_{\text{iter}}} \hat{m}_{i_{\text{iter}}} / (\sqrt{\hat{v}_{i_{\text{iter}}} + \epsilon}) \) (update parameters)
24: \( \alpha_{i_{\text{iter}}} \leftarrow r_\alpha \alpha_{i_{\text{iter}}-1} \) (update stepsize)

3.2 Architecture of the neural networks

We propose to follow Fecamp et al. (2019) but we use two recurrent neural networks, one for \( \text{NN}_Z \) and another one for \( \text{NN}_Y \). The recurrent cells are LSTM cells Hochreiter and Schmidhuber (1997) among which a series of feedforward layers are placed (see Figure 2). One could give as inputs of \( \text{NN}_Y \) the outputs of \( \text{NN}_Z \) but from our tests it appears that it does not improve the results.

3.3 Numerical results: application to fixed transaction costs

In this section we consider the problem of hedging an European style option paying \( g(X_T) \) under fixed transaction costs: at each date \( t_j, j \in \{0, \ldots, N\} \) and for each asset \( k \in \{1, \ldots, d\} \), the portfolio manager pays a constant fee \( c_k^i \) that does not depend on the volume of underlying \( k \) that is purchased or sold (given that the volume is not null). We search for an impulse control \((Y, Z)\) with \( Y_1 = 1 \) if there is an hedge at time \( t_i \), \( i \in \{0, \ldots, N\} \) and \( Z_k^i \) corresponds to the quantity of asset \( k, k \in \{1, \ldots, d\} \) to buy or sell at time \( t_i \). Let us consider the self-financing portfolio with value at time \( t_j, j \in \{0, \ldots, N\} \),

\[
\Pi_{t_J} = P_0 + \sum_{k=1}^{d} \sum_{i=1}^{j} \Delta_{i-1}^k (X_{t_i}^k - X_{t_{i-1}}^k) \tag{14}
\]

where \( \Delta_i \) corresponds to the amount of asset held at time \( t_i \), \( P_0 \) corresponds to the option premium and where

\[
\begin{align*}
\Delta_0 & = Z_0 \\
\Delta_i & = \Delta_{i-1} + Y_i Z_i, \quad i = 1, \ldots, N - 1.
\end{align*}
\]

The last control \((Y_N, Z_N)\) does not appear in the replication portfolio and nor in the optimisation problem. The transaction costs paid until time \( t_j, j \in \{0, \ldots, N\} \), generated by the strategy is equal to

\[
C_{t_J} = \sum_{i=0}^{j-1} \sum_{k=1}^{d} Y_i c_k^i. \tag{15}
\]
One searches for an impulse control \((Y, Z)\) and an initial premium \(P_0\) that minimises the moment-based criteria:

\[
L_f(X, Y, Z) = \beta \mathbb{E}_\mathbb{P}(C_T) + (1 - \beta)\mathbb{E}_\mathbb{P}(E_T^2)
\]

where

\[
E_T = \Pi_T - g(X_T)
\]

is the replication error and \(\beta \in [0, 1]\) quantifies the trade-off between the average costs \(\mathbb{E}_\mathbb{P}(C_T)\) ad the \(L^2\) loss \(\mathbb{E}_\mathbb{P}(E_T^2)\). One could consider different risk criteria as in Fecamp et al. (2019) but the mean-variance choice is convenient for comparison to usual stochastic control methods as a dynamic stochastic programming equation can be derived. Practitioners prefer to use the homogeneous criterion \(\beta \mathbb{E}_\mathbb{P}(C_T) + (1 - \beta)\sqrt{\mathbb{E}_\mathbb{P}(E_T^2)}\) which can be addressed by Algorithm 2 but not by dynamic programming approaches.

### 3.3.1 European call option

In this section, we hedge an European call option (dimension 1, payoff \((x - K)^+\)) in the Black-Scholes framework \((X) = \text{the solution of } (1) \text{ with } \mu(t, x) = 0 \text{ and } \sigma(t, x) = \sigma x\) with

- **Case 1:** \(X_0 = 1, K = 1, T = 1, \sigma = 0.25\), \(c_j = c = 0.001\) for every \(j \in \{0, \ldots, N\}\), \(\beta = 0.25\), and \(N = 10\),
- **Case 2:** \(X_0 = 1, K = 1, T = 2, \sigma = 0.25\), \(c_j = c = 0.0005\) for every \(j \in \{0, \ldots, N\}\), \(\beta = 0.25\), \(N = 20\).

The hedging dates belong to the regular grid \(D_N = \{\frac{i}{N}, i = 0, \ldots, N\}\). To hedge those options, let us consider Algorithm 2 with a batch size of \(N_{\text{batch}} = 5,000\), a test set size of 50,000, a validation set size of 100,000, an initial learning rate of \(\alpha_0 = 0.001\), a decay of \(r_\alpha = 0.98^{1/100}\) (only for the neural networks associated with \(Y\)), \(N_{\text{iter}} = 20,000\) iterations, a LSTM cell with 60 units followed by 3 layers of 20 neurons for \(NN_Y\) and a LSTM cell with 50 units followed by 3 layers of 10 neurons for \(NN_Z\). \(P_0\) is a trainable variable that is optimised. We compare Algorithm 2 to three strategies:

- The Black-Scholes strategy which consists in hedge at each time step with the Black-Scholes delta. This strategy is naive as it does not take into account the transaction costs. \(P_0\) is equal to \(\mathbb{E}(g(X_t))\).
- The strategy described in Whalley and Wilmott (1993). In this latter paper, the cost function is the exponential utility function defined (with the notations defined in (14), (17) and (15)) by:

\[
U_\gamma(\delta, q) = \mathbb{E}_\mathbb{P}\left(e^{-\gamma(C_T + E_T)}\right).
\]

The strategy results in a traditional Black-Scholes strategy with boundaries. Assuming zero interest rate, boundaries are given by \(\Delta_{BS,t} \pm H_t(\gamma)\) where

\[
H_t(\gamma) = \left(\frac{12c(G_{BS,t}^2)}{\gamma}\right)^{\frac{1}{4}}
\]
Figure 3: Comparison of losses between Whalley-Wilmott, traditional Black-Scholes and Algorithm 2 with various risk aversion in (18). Left: Case 1 (10 dates); Right: Case 2 (20 dates) (Neural network stands for Algorithm 2).

| Case   | Black-Scholes       | Whalley-Wilmott | Dynamic programming | Algorithm 2       |
|--------|---------------------|-----------------|---------------------|-------------------|
| 10 dates | 0.00302 (0.01/0.00069) | 0.00169 (0.0036/0.0011) | 0.00169 (0.0030/0.0.0013) | 0.00170 (0.0032/0.0012) |
| 20 dates | 0.00277 (0.01/0.00036) | 0.00114 (0.0027/0.0006) | 0.00119 (0.0021/0.0009) | 0.00134 (0.0024/0.001) |

Table 7: Loss (Cost/L^2 loss) comparison between Whalley-Wilmott strategy, dynamic programming and Algorithm 2 on a call option with fixed transaction costs.

and where $\Delta_{BS,t}$ and $\Gamma_{BS,t}$ are the traditional Black-Scholes $\Delta$ and $\Gamma$ at time $t$. The strategy is then the following:

$$\Delta_0 = \begin{cases} 
\Delta_{BS,t_0} & \text{if } |\Delta_{BS,t_0}| \geq H_{t_0}(\gamma), \\
0 & \text{otherwise}
\end{cases}$$

$$\Delta_j = \begin{cases} 
\Delta_{BS,t_j} & \text{if } |\Delta_{BS,t_j} - \Delta_{j-1}| \geq H_{t_j}(\gamma), \\
\Delta_{j-1} & \text{otherwise}
\end{cases}, \quad j = 1, \ldots, N - 1$$

and the optimal premium is given by $P_0 = \mathbb{E}_T(g(X_T))$. Despite its desirable properties, exponential utility cost function is difficult to deal with when it is used in a numerical resolution algorithm and one prefers the criteria (16). As we do not know the equivalence between the risk aversion $\gamma$ for the criteria (18) and the one for the criteria (16) $\beta$, we consider all the strategies given by Whalley and Wilmott (1993) varying $\gamma$ and choose the one minimising (16). In practice, we make $\gamma$ vary on a regular grid with 1000 points between 0 and 30.

• A dynamic programming approach, solved with the library Gevret (2016).

In Figure 3, the losses between a Black-Scholes strategy that does not take into account costs, the neural network strategy computed with Algorithm 2 and the Whalley-Wilmott strategy for multiple risk aversion are compared. The optimal risk aversion with respect to criteria (16) is equal to 22.82 for 10 dates and 20.29 for 20 dates. Results are given in Table 7. The Whalley-Wilmott, Dynamic programming and Algorithm 2 improve a lot the loss compared to the Black-Scholes strategy in both cases. Hedging positions from 4 simulations and the different strategies are given in Figure 4 for case 1 and in Figure 5 for case 2. We can see the intuitive behaviour consisting in waiting for the transaction-cost-free delta to change sufficiently before doing anything. When 10 hedging dates are considered, the three strategies give similar losses. However, the hedging error and costs are not the same: Dynamic Programming and Algorithm 2 hedge more than Whalley-Wilmott. We can see that Algorithm 2 performs well. Considering 20 dates, the neural network is slightly sub optimal: on Figure 5, one can see that the continuous control is not perfect. However we notice that the loss is still decreasing after 20,000 iterations but very slowly. Computation times are comparable between Dynamic programming and Algorithm 2, see 8: a high precision is needed in the regression for the Dynamic programming algorithm due to the lack of smoothness of the problem around 0 while the convergence requires a batch size sufficiently high and 20,000 iterations for Algorithm 2. However, one expects for Algorithm 2 to be more robust when dimension increases.

| Case   | Stochastic control | Algorithm 2 |
|--------|--------------------|-------------|
| 10 dates | 2928               | 4004        |
| 20 dates | 14310              | 9423        |

Table 8: Time (in seconds) comparison between dynamic programming and Algorithm 2 on a one-dimensional call option with fixed transaction costs.
Figure 4: Comparison of bought quantities between Black-Scholes, Whalley-Wilmott, Dynamic programming and Algorithm 2 for case 1 (Neural network stands for Algorithm 2).

Figure 5: Comparison of bought quantities between Black-Scholes, Whalley-Wilmott, Dynamic programming and Algorithm 2 for case 2 (Neural network stands for Algorithm 2).
3.3.2 3-dimensional spread option

We consider the hedging of the spread option on $N = 10$ dates with payoff

$$(X_{1,T} - \frac{1}{2}(X_{2,T} + X_{3,T}) - K)^+$$

where

$$dX_{i,t} = \sigma_i X_{i,t} dW_{i,t}, t \in [0,T], \ i = 1,\ldots,3,$$

$W$ is a multivariate Brownian motion with correlation matrix

$$
\begin{pmatrix}
1 & 0.3 & 0.6 \\
0.3 & 1 & 0.2 \\
0.6 & 0.2 & 1
\end{pmatrix},
$$

$(\sigma_1, \sigma_2, \sigma_3) = (0.25, 0.1, 0.1), (X_{1,0},X_{2,0},X_{3,0}) = (1,0.6,0.6), K = 0.4, T = 1$. The time mesh is $D_N = \{ \frac{i}{N}, i = 0,\ldots,N \}$, the costs are given by $(c_1^j,c_2^j,c_3^j) = (0.002,0.001,0.0)$ for every $j \in \{0,\ldots,N\}$ and $\beta = 0.1$.

In Table 9 the loss induced by Algorithm 2 is compared to a naive strategy ignoring fixed transaction costs and computed using algorithm in Fecamp et al. (2019). Hyperparameters for Algorithm 2 are: a LSTM cell of 50 units followed by 3 layers of 10 neurons for $\mathbb{NN}_Z$, a LSTM cell of 60 units followed by 3 layers of 30 neurons for $\mathbb{NN}_Y$, batch size of 8,000, test size of 50,000, validation size of 100,000, $N_{iter} = 10,000$ iterations and an initial learning rate $\alpha$ equals to 0.001 and step size reduction $r_{\alpha} = 0.98^{1/100}$ (only for $\mathbb{NN}_Y$). The use of Algorithm 2 decreases global loss by 25% as it hedges itself 2.06 times in average instead of the 10 times in the naive strategy.

In this case, and due to the curse of dimensionality, stochastic control tools need hours on a cluster whereas our methodology runs on a personal laptop in one hour. Moreover, speed would be increased if a GPU would have been used.

Figure 6 shows the delta for 4 different simulations and compare the two strategies.

|                  | Loss   |
|------------------|--------|
| Algorithm 2      | 0.00268 (0.00637/0.00227) |
| Transaction costs free strategy | 0.00356 (0.03/0.000629) |

Table 9: Loss (Cost/$L^2$ loss) for the hedging of a 3 markets spread.
Algorithm 2 extends Algorithm 1 to deal with impulse control problems and shows very good performance on hedging with fixed transaction costs. This can be used to minimise a criteria which is a tradeoff between costs and a moment-based criteria. As we can also add constraints on the stopping times, instead of considering a penalty on the costs, one could also limit the total cost for each trajectory (by limiting the number of exercises).

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