A FINITE \( \mathbb{Q} \)-BAD SPACE

SERGEI O. IVANOV AND ROMAN MIKHAILOV

Abstract. We prove that, for a free noncyclic group \( F \), the second homology group \( H_2(\hat{F}_\mathbb{Q}, \mathbb{Q}) \) is an uncountable \( \mathbb{Q} \)-vector space, where \( \hat{F}_\mathbb{Q} \) denotes the \( \mathbb{Q} \)-completion of \( F \). This solves a problem of A. K. Bousfield for the case of rational coefficients. As a direct consequence of this result, it follows that a wedge of two or more circles is \( \mathbb{Q} \)-bad in the sense of Bousfield–Kan. The same methods as used in the proof of the above result serve to show that \( H_2(\hat{F}_\mathbb{Z}, \mathbb{Z}) \) is not a divisible group, where \( \hat{F}_\mathbb{Z} \) is the integral pronilpotent completion of \( F \).

1. Introduction

In the foundational work [1], A. K. Bousfield and D. M. Kan introduced the concept of \( R \)-completion of a space for a commutative ring \( R \). For a space \( X \), there is an \( R \)-completion functor \( X \mapsto R_\infty X \) such that a map between two spaces \( f : X \to Y \) induces an isomorphism of reduced homology \( \tilde{H}_*(X, R) \cong \tilde{H}_*(Y, R) \) if and only if it induces a homotopy equivalence \( R_\infty X \cong R_\infty Y \). Thus, \( R \)-completion can be viewed as an approximation of the \( R \)-homology localization of a space, defined in [2]. For certain classes of spaces, such as nilpotent spaces, \( R \)-completion and \( R \)-homology localization coincide.

The \( R \)-completion functor for spaces is closely related to the \( R \)-completion functor for groups. For a group \( G \), denote by \( \{ \gamma_i(G) \}_{i \geq 1} \) the lower central series of \( G \). We will consider the pronilpotent completion \( \hat{G}_\mathbb{Z} \) of \( G \) as well as the \( \mathbb{Q} \)-completion \( \hat{G}_\mathbb{Q} \) defined as

\[
\hat{G}_\mathbb{Z} = \limleftarrow G/\gamma_i(G), \quad \hat{G}_\mathbb{Q} = \limleftarrow G/\gamma_i(G) \otimes \mathbb{Q}.
\]

Here \( G/\gamma_i(G) \otimes \mathbb{Q} \) is the Malcev \( \mathbb{Q} \)-localization of the nilpotent group \( G/\gamma_i(G) \). One can find the definition of \( \mathbb{Z}/p \)-completion \( \hat{G}_{\mathbb{Z}/p} \) in [1], [3]. In this paper we do not use \( \mathbb{Z}/p \)-completion and work only over \( \mathbb{Z} \) or \( \mathbb{Q} \). It is shown in [1, Ch.4] that \( R \)-completion of a connected space \( X \) can be constructed explicitly as \( \hat{W}(GX)_R \), where \( G \) is the Kan loop simplicial group, \( (GX)_R \) is the \( R \)-completion of \( GX \) and \( \hat{W} \) is the classifying space functor.

A space \( X \) is called \( R \)-good if the map \( X \to R_\infty X \) induces an isomorphism of reduced homology \( \tilde{H}_*(X, R) \cong \tilde{H}_*(R_\infty X, R) \), and called \( R \)-bad otherwise. In other words, for \( R \)-good spaces \( R \)-homology localization and \( R \)-completion coincide.

There are a lot of examples of \( R \)-good and \( R \)-bad spaces. The key example of [1] is the projective plane \( \mathbb{R}P^2 \), which is \( \mathbb{Z} \)-bad. This fact implies that some finite wedge of circles is also \( \mathbb{Z} \)-bad. It is shown in [3] that a wedge of two circles is \( \mathbb{Z} \)-bad. In [1], Bousfield proved that, for any prime \( p \), a wedge of circles is \( \mathbb{Z}/p \)-bad, thus providing first example of a finite \( \mathbb{Z}/p \)-bad space. For \( R \) a subring of the rationals or \( \mathbb{Z}/n \), \( n \geq 2 \), and a free group \( F \), there is a weak equivalence ([1, 5.3])

\[
R_\infty K(F, 1) \cong K(\hat{F}_R, 1).
\]
Therefore, the question of \( R \)-goodness of a wedge of circles is reduced to the question of nontriviality of the higher \( R \)-homology of the \( R \)-completion of a free group. The same question naturally appears in the theory of \( HR \)-localizations of groups. In [3, Problem 4.11], Bousfield posed the following problem:

**Problem. (Bousfield)** Does \( H_2(\hat{F}_R, R) \) vanish when \( F \) is a finitely generated free group and \( R = \mathbb{Q} \) or \( R = \mathbb{Z}/n \)?

In the recent paper [7], the authors show that, for \( R = \mathbb{Z}/n \), \( H_2(\hat{F}_R, R) \) is an uncountable group, solving the above problem for the case \( R = \mathbb{Z}/n \). The key step in [7] substantially uses the theory of profinite groups. Hence the method given in [7] cannot be directly transferred to the case \( R = \mathbb{Q} \).

In this paper we answer Bousfield’s problem over \( \mathbb{Q} \). Our main results are the following theorems.

**Theorem 1.** For a finitely generated noncyclic free group \( F \), \( H_2(\hat{F}_\mathbb{Q}, \mathbb{Q}) \) is uncountable.

Moreover, we prove that the image of the map \( H_2(\hat{F}_\mathbb{Z}, \mathbb{Z}) \to H_2(\hat{F}_\mathbb{Q}, \mathbb{Q}) \) is uncountable.

**Theorem 2.** For a finitely generated noncyclic free group \( F \) and a prime \( p \), \( H_2(\hat{F}_\mathbb{Z}, \mathbb{Z}/p) \) is uncountable. In particular, \( H_2(\hat{F}_\mathbb{Z}, \mathbb{Z}) \) is not divisible.

Theorem 2 answers a problem posted in [3]. As mentioned above, \( \mathbb{Q}_\infty K(F, 1) = K(\hat{F}_\mathbb{Q}, 1) \). Therefore, Theorem 1 implies the following:

**Corollary.** A wedge of \( \geq 2 \) circles is \( \mathbb{Q} \)-bad.

As far as known to the authors, this is the first known example of a finite \( \mathbb{Q} \)-bad space.

The proof is organized as follows. In Section 2 we discuss technical results about power series. The main result of Section 2, Proposition 2.1, states that the kernel of the natural map between a rational power series ring and the coinvariants of the diagonal action of the rationals on the exterior square \( \mathbb{Q}[[x]] \to \Lambda^2(\mathbb{Q}[[x]])_{\mathbb{Q}} \), given by \( f \mapsto f \wedge 1 \), is countable. (In the proof of the proposition we use the fact that the group algebra \( \mathbb{Q}[[\mathbb{Q}]] \) is countable. In the similar statement for the \( \mathbb{Z}/p \)-completion we should consider the mod-\( p \) group algebra of the group of \( p \)-adic integers \( \mathbb{Z}/p[\mathbb{Z}_p] \), which is uncountable. So this method fails for \( \mathbb{Z}/p \)-completions.) Here In Section 3, we consider the integral lamplighter group:

\[
\mathcal{L}\mathcal{G} = \langle a, b \mid [a, a^b] = 1, i \in \mathbb{Z} \rangle,
\]

which is isomorphic to the wreath product of two infinite cyclic groups, as well as its \( p \)-analog \( \mathbb{Z}/p \wr C \), where \( C \) denotes an infinite cyclic group. The group \( \mathcal{L}\mathcal{G} \) is metabelian; therefore, its completions \( \hat{\mathcal{L}\mathcal{G}}_{\mathbb{Z}} \) and \( \hat{\mathcal{L}\mathcal{G}}_{\mathbb{Q}} \) can be easily described (see (3.1), (3.2)), and the homology group \( H_2(\hat{\mathcal{L}\mathcal{G}}_{\mathbb{Q}}, \mathbb{Q}) \) is isomorphic to the natural coinvariant quotient of the exterior square \( \Lambda^2(\mathbb{Q}[[x]]) \). The key step in the proof of the main results occurs in Section 4, in Proposition 4.1. Let \( F = F(a, b) \) be a free group of rank two with generators \( a, b \). We construct (see Proposition 4.1) an uncountable collection of elements \( r_a, s_q \in \hat{F}_\mathbb{Z} \) such that \( [r_a, a][s_q, b] = 1 \) in \( \hat{F}_\mathbb{Z} \). One can consider the group homology \( H_2(\hat{F}_\mathbb{Z}, \mathbb{Z}) \) as a kernel of the
commutator map \( \hat{F}_Z \wedge \hat{F}_Z \rightarrow \hat{F}_Z, a \wedge b \mapsto [a, b] \) where \( \hat{F}_Z \wedge \hat{F}_Z \) is the non-abelian exterior square of \( \hat{F}_Z \). Therefore, the pairs of elements \( (r_q \wedge a)(s_q \wedge b) \in \hat{F}_Z \wedge \hat{F}_Z \) define certain elements of \( H_2(\hat{F}_Z, \mathbb{Z}) \). Next we consider the following natural maps between homology groups of different completions, which are induced by the standard projection \( F \rightarrow \mathcal{L}G \):

\[
\begin{array}{ccc}
H_2(\hat{F}_Q, \mathbb{Q}) & \longrightarrow & H_2(\hat{F}_Z, \mathbb{Z}) & \longrightarrow & H_2(\hat{F}_Z, \mathbb{Z}/p) \\
\downarrow & & \downarrow & & \downarrow \\
H_2(\mathcal{L}G_Q, \mathbb{Q}) & \longrightarrow & H_2(\mathcal{L}G_Z, \mathbb{Z}/p),
\end{array}
\]

and show, in the final Section 5, that the sets of images of the elements \( (r_q \wedge a)(s_q \wedge b) \) in \( H_2(\mathcal{L}G_Q, \mathbb{Q}) \) and \( H_2(\mathcal{L}G_Z, \mathbb{Z}/p) \) are uncountable. Theorems 1 and 2 follow.

2. Technical results about power series

We denote by \( C \) an infinite cyclic group written multiplicatively as \( C = \langle t \rangle \). For a commutative ring \( R \) we denote by \( R[[x]] \) the ring of formal power series over \( R \) and by \( R[C] \) the group algebra of \( C \). Consider the multiplicative homomorphism

\[
\tau : C \rightarrow R[[x]], \quad \tau(t) = 1 + x.
\]

The induced ring homomorphism is denoted by the same letter

\[
\tau : R[C] \rightarrow R[[x]].
\]

**Lemma 2.1.** If we denote by \( I \) the augmentation ideal of \( R[C] \) and set \( R[C]^\wedge = \varprojlim R[C]/I^i \), then \( \tau(I^n) \subseteq x^n \cdot R[[x]] \) and \( \tau \) induces isomorphisms

\[
R[C]/I^n \cong R[x]/x^n
\]

\[
R[C]^\wedge \cong R[[x]].
\]

**Proof.** If we set \( x = t - 1 \), we obtain \( R[C] = R[x, (1 + x)^{-1}] \) and \( I = x \cdot R[C] \). Observe that the image of the element \( 1 + x \) in \( R[x]/x^n \) is invertible. Since localization at the element \( 1 + x \) is an exact functor, the short exact sequence \( x^n \cdot R[x] \rightarrow R[x] \rightarrow R[x]/x^n \) gives the short exact sequence \( (x^n \cdot R[x])_{1+x} \rightarrow R[C] \rightarrow R[x]/x^n \). It follows that \( R[C]/x^n \cong R[x]/x^n \). The assertion follows. \( \square \)

Denote by \( \sigma \) the antipode of the group ring \( R[C] \):

\[
\sigma : R[C] \rightarrow R[C], \quad \sigma(\sum a_i t^i) = \sum a_i t^{-i}.
\]

Obviously \( \sigma(I^n) = I^n \), and hence it induces a continuous involution

\[
\hat{\sigma} : R[C]^\wedge \rightarrow R[C]^\wedge.
\]

Composing this involution with the isomorphism \( R[C]^\wedge \cong R[[x]] \) we obtain a continuous involution

\[
\tilde{\sigma} : R[[x]] \rightarrow R[[x]]
\]

such that

\[
\tilde{\sigma}(x) = -x + x^2 - x^3 + x^4 - \ldots.
\]
Consider the case $R = \mathbb{Q}$. Note that the set $1 + x \cdot \mathbb{Q}[[x]]$ is a group and there is a unique way to define $r$-power map $f \mapsto f^r$ for $r \in \mathbb{Q}$ that extends the usual power map $f \mapsto f^n$ such that $f^{r_1 r_2} = (f^{r_1})^{r_2}$ (see Lemma 4.4 of [4]). This map is defined by the formula

\[ f^r = \sum_{n=0}^{\infty} \binom{r}{n} (f - 1)^n, \]

where $\binom{r}{n} = (r-1) \ldots (r-n+1)/n!$. Denote by $C \otimes \mathbb{Q}$ the group $\mathbb{Q}$ written multiplicatively as powers of $t$: $C \otimes \mathbb{Q} = \{ t^r \mid r \in \mathbb{Q} \}$. Consider the multiplicative homomorphism

\[ \tau : C \rightarrow \mathbb{Q}[[x]] : \]

\[ \tau(t^r) = (1 + x)^r. \]

The induced ring homomorphism is denoted by the same letter

\[ \tau : \mathbb{Q}[C \otimes \mathbb{Q}] \rightarrow \mathbb{Q}[[x]]. \]

This homomorphism allows us to consider $\mathbb{Q}[[x]]$ as a $\mathbb{Q}[C \otimes \mathbb{Q}]$-module. We claim that the homomorphism $\tau : \mathbb{Q}[C \otimes \mathbb{Q}] \rightarrow \mathbb{Q}[[x]]$ respects the involutions:

\[ \tau \circ \sigma_{C \otimes \mathbb{Q}} = \tilde{\sigma} \circ \tau, \]

where $\sigma_{C \otimes \mathbb{Q}}$ is the antipode on $\mathbb{Q}[C \otimes \mathbb{Q}]$. Indeed, we have that $(1 + x)^{-1} = \tilde{\sigma}(1 + x) = \tilde{\sigma}((1 + x)^{1/n})^n$ and then $\tilde{\sigma}((1 + x)^{1/n}) = (1 + x)^{-1/n}$, which implies $\tilde{\sigma}((1 + x)^r) = (1 + x)^{-r}$ for any $r \in \mathbb{Q}$, and hence $\tau(\sigma_{C \otimes \mathbb{Q}}(t^r)) = \tilde{\sigma}(\tau(t^r))$ for any $r \in \mathbb{Q}$.

**Proposition 2.1.**

1. Denote by $\Lambda^2(\mathbb{Q}[[x]])$ the exterior square of $\mathbb{Q}[[x]]$ considered as a $C \otimes \mathbb{Q}$-module with the diagonal action. Consider the space of $C \otimes \mathbb{Q}$-coinvariants $(\Lambda^2(\mathbb{Q}[[x]]))_{C \otimes \mathbb{Q}}$. Then the kernel of the homomorphism

\[ \theta : \mathbb{Q}[[x]] \rightarrow (\Lambda^2(\mathbb{Q}[[x]]))_{C \otimes \mathbb{Q}}, \]

\[ \theta(f) = f \wedge 1 \]

is countable.

2. Let $p$ be a prime. Denote by $\Lambda^2(\mathbb{Z}/p[[x]])$ the exterior square of $\mathbb{Z}/p[[x]]$ considered as a $C$-module with the diagonal action. Consider the space of $C$-coinvariants $(\Lambda^2(\mathbb{Z}/p[[x]]))_C$. Then the kernel of the homomorphism

\[ \theta : \mathbb{Z}/p[[x]] \rightarrow (\Lambda^2(\mathbb{Z}/p[[x]]))_C, \]

\[ \theta(f) = f \wedge 1 \]

is countable.

**Proof.**

1. Consider the linear map

\[ \alpha : \Lambda^2(\mathbb{Q}[[x]]) \rightarrow \mathbb{Q}[[x]]^{\otimes 2}, \]

\[ \alpha(f \wedge g) = f \otimes g - g \otimes f. \]

Note that this is a homomorphism of $\mathbb{Q}[C \otimes \mathbb{Q}]$-modules, where the action of $C \otimes \mathbb{Q}$ is defined diagonally in both cases. Hence, it induces a linear map:

\[ \alpha_{C \otimes \mathbb{Q}} : (\Lambda^2(\mathbb{Q}[[x]]))_{C \otimes \mathbb{Q}} \rightarrow (\mathbb{Q}[[x]]^{\otimes 2})_{C \otimes \mathbb{Q}}. \]

Next, we consider the homomorphism

\[ (\mathbb{Q}[[x]]^{\otimes 2})_{C \otimes \mathbb{Q}} \rightarrow \mathbb{Q}[[x]] \otimes_{\mathbb{Q}[C \otimes \mathbb{Q}]} \mathbb{Q}[[x]], \]

\[ (f \otimes g) = f \otimes \tilde{\sigma}(g), \]

\[ \beta : (\mathbb{Q}[[x]]^{\otimes 2})_{C \otimes \mathbb{Q}} \rightarrow \mathbb{Q}[[x]] \otimes_{\mathbb{Q}[C \otimes \mathbb{Q}]} \mathbb{Q}[[x]], \]

\[ (f \otimes g) = f \otimes \tilde{\sigma}(g), \]
which is well defined because $\tau_Q$ respects the involutions (2.2): $f^{r'} \otimes \tilde{\sigma}(g^{r'}) = f^{r'} \otimes \tilde{\sigma}(g)^{r'} = f \otimes \tilde{\sigma}(g)$. Denote by $K$ the subfield of the field of Laurent power series $Q((x))$ generated by the image of $\tau_Q$. Then there is a map

$$\gamma : Q[[x]] \otimes_{Q[C \otimes Q]} Q[[x]] \longrightarrow Q((x)) \otimes_K Q((x)).$$

The composition

$$\gamma \circ \beta \circ \alpha_{C \otimes Q} \circ \theta_Q : Q[[x]] \rightarrow Q((x)) \otimes_K Q((x))$$

sends $f$ to $f \otimes 1 - 1 \otimes \tilde{\sigma}(f)$. Note that for any vector spaces $V, U$ over any field and any elements $v_1, v_2 \in V$ and $u_1, u_2 \in U$, if $v_1$ and $v_2$ are linearly independent and $u_1 \neq 0, u_2 \neq 0$, then $v_1 \otimes u_1$ and $v_2 \otimes u_2$ are linearly independent in $V \otimes U$. It follows that for any $f \in Q[[x]] \setminus K$ we have that $f \otimes 1 - 1 \otimes \tilde{\sigma}(f) \neq 0$ in $Q((x)) \otimes_K Q((x))$. Therefore $\text{Ker}(\theta_Q) \subseteq K$. Since the fraction field of the countable algebra $Q$ is countable, $K$ is countable. The assertion follows.

(2) The proof is the same. \hfill \Box

3. Completions of lamplighter groups $\mathcal{LG}$ and $\mathcal{LG}(p)$

Recall the definition of the tensor square for a non-abelian group $[3]$. For a group $G$, the tensor square $G \otimes G$ is the group generated by the symbols $g \otimes h$, $g, h \in G$, satisfying the following defining relations:

$$fg \otimes h = (gf^{-1} \otimes h^{f^{-1}})(f \otimes h),$$
$$f \otimes gh = (f \otimes g)(f^{g^{-1}} \otimes h^{g^{-1}}),$$

for all $f, g, h \in G$. The exterior square $G \wedge G$ is defined as

$$G \wedge G := G \otimes G/(g \otimes g, \ g \in G).$$

The images of the elements $g \otimes h$ in $G \wedge G$ will be denoted by $g \wedge h$. If $G = E/R$ for a free group $E$, there is a natural isomorphism $G \wedge G \cong \frac{[E,E]}{[R,E]}$.

For any group $G$, there is a natural short exact sequence

$$0 \longrightarrow H_2(G, \mathbb{Z}) \longrightarrow G \wedge G \stackrel{[-,-]}{\longrightarrow} [G,G] \longrightarrow 1$$

(see $[3]$ (2.8) and $[6]$). Let $g_1, \ldots, g_n, h_1, \ldots, h_n \in G$ be elements such that $[g_1, h_1] \ldots [g_n, h_n] = 1$. Then the element $(g_1 \wedge h_1) \ldots (g_n \wedge h_n)$ defines an element in $H_2(G, \mathbb{Z})$:

$$(g_1 \wedge h_1) \ldots (g_n \wedge h_n) \in H_2(G, \mathbb{Z}).$$

If $R$ is a commutative ring, then the image of $(g_1 \wedge h_1) \ldots (g_n \wedge h_n)$ in $H_2(G, R)$ is denoted by

$$((g_1 \wedge h_1) \ldots (g_n \wedge h_n)) \otimes R \in H_2(G, R).$$

We will consider two versions of the lamplighter group. The integral lamplighter group

$$\mathcal{LG} = \mathbb{Z} : C = \langle a, b \mid [a, a^b] = 1, \ i \in \mathbb{Z} \rangle$$

and the $p$-lamplighter group for a prime $p$

$$\mathcal{LG}(p) = \mathbb{Z}/p : C = \langle a, b \mid [a, a^b] = a^p = 1, \ i \in \mathbb{Z} \rangle.$$

Observe that $\mathcal{LG} = \mathbb{Z}[C] \rtimes C$ and $\mathcal{LG}(p) = \mathbb{Z}/p[C] \rtimes C$. Using Lemma 2.1 and $[6]$ Prop. 4.7, we obtain

$$\widehat{\mathcal{LG}}_\mathbb{Z} = \mathbb{Z}[[x]] \rtimes C$$

(3.1)
and

\[ \mathcal{L}G_Q = \mathbb{Q}[x] \times (\mathbb{C} \otimes \mathbb{Q}), \quad \mathcal{L}G(p)_Z = \mathbb{Z}/p[[]]\times \mathbb{C}, \]

where \( C \) acts on \( \mathbb{Z}[x] \) and \( \mathbb{Z}/p[[]] \) via \( \tau \) and \( C \otimes \mathbb{Q} \) acts on \( \mathbb{Q}[x] \) via \( \tau_Q \).

**Proposition 3.1.** There are isomorphisms

\[
\left( \Lambda^2(\mathbb{Q}[[]]) \right)_{C \otimes \mathbb{Q}} \cong H_2(\mathcal{L}G_Q, \mathbb{Q}),
\]

\[
\left( \Lambda^2(\mathbb{Z}/p[[]]) \right)_{C} \cong H_2(\mathcal{L}G(p)_Z, \mathbb{Z}/p)
\]

in both cases given by

\[
(f \wedge f') \mapsto ((f, 1) \wedge (f', 1)) \otimes R,
\]

where \( R = \mathbb{Q} \) and \( R = \mathbb{Z}/p \) respectively.

**Proof.** Consider the short exact sequence \( \mathbb{Q}[[]] \to \mathcal{L}G_Q \to (\mathbb{C} \otimes \mathbb{Q}) \) and the associated spectral sequence \( E \). Since \( \mathbb{Q} = \lim_{\to} \mathbb{Z} \) and homology commutes with direct limits, we have \( H_n(C \otimes \mathbb{Q}, -) = 0 \) for \( n \geq 2 \). It follows that \( E_{i,2}^2 = 0 \) for \( i \geq 2 \) and hence there is a short exact sequence

\[
0 \to E_{0,2}^2 \to H_2(\mathcal{L}G_Q, \mathbb{Q}) \to E_{1,1}^2 \to 0.
\]

Observe that the action of \( C \) on \( \mathbb{Q}[[]] \) has no invariants. Then

\[
E_{1,1}^2 = H_1(C \otimes \mathbb{Q}, \mathbb{Q}[[]]) = \lim_{\to} H_1(C \otimes \frac{1}{n!} \mathbb{Z}, \mathbb{Q}[[]]) = \lim_{\to} \mathbb{Q}[[]]^{C \otimes \frac{1}{n!} \mathbb{Z}} = 0.
\]

It follows that the map

\[
H_2(\mathbb{Q}[[]], \mathbb{Q})_{C \otimes \mathbb{Q}} = E_{0,2}^2 \to H_2(\mathcal{L}G_Q, \mathbb{Q})
\]

is an isomorphism. The map is induced by the map \( \mathbb{Q}[[]] \to \mathcal{L}G_Q \) that sends \( f \in \mathbb{Q}[[]] \) to \( f, 1 \in \mathcal{L}G_Q \). Then the isomorphism \((3.3)\) sends \( f \wedge f' \) to \( ((f, 1) \wedge (f', 1)) \otimes \mathbb{Q} \). Using the isomorphism \( \Lambda^2(\mathbb{Q}[[]]) \cong H_2(\mathbb{Q}[[]], \mathbb{Q}) \) we obtain the assertion.

The second isomorphism can be proved similarly. \( \square \)

4. **Completion of a free group**

For elements of groups or Lie rings, we will use the left-normalized notation \([a_1, \ldots, a_n] := [[a_1, \ldots, a_{n-1}], a_n]\) and the following notation for Engel commutators

\[
[a_0, b] := a, \quad [a_{i+1}, b] = [[a_i, b], b]
\]

for \( i \geq 0 \).

For all elements \( a, b \) of a Lie ring, the Jacobi identity implies that

\[
[a, b, a, b] + [b, [a, b], a] + [[a, b], [a, b]] = 0.
\]

It follows that

\[
[\mathbf{a}, \mathbf{b}, \mathbf{a}] = [\mathbf{a}, \mathbf{b}, \mathbf{a}].
\]

The following lemma is a generalization of this identity.

**Lemma 4.1.** Let \( L \) be a Lie ring, \( a, b \in L \) and \( n \geq 1 \). Then

\[
[[a, 2n, b], a] = \left[ \sum_{i=0}^{n-1} (-1)^i [[a, 2n-1-i, b], [a_i, b]], b \right].
\]
\[ \text{The Jacobi identity implies that} \]

\[ \left[[a_{2n-1}, b], [a_i b]\right] + \left[[a_{2n-1-i}, b], [a_{i+1} b]\right] = \left[[a_{2n-1-i}, b], [a_i b], b]\right] \]

for \(0 \leq i \leq n - 1\). Taking the alternating sum of these identities and using the fact that \([[a_n, b], [a_n b]] = 0\), we obtain the assertion. \(\square\)

**Corollary 4.1.** Let \(F = F(a, b)\) be a free group with generators \(a, b\). For any \(n \geq 1\),

\[ \left[[a_{2n}, b], a\right] \equiv \left[\prod_{i=0}^{n-1} \left[[a_{2n-1-i}, b], [a_i b]\right]^{(-1)^i}, b\right] \mod \gamma_{2n+3}(F). \]

We denote by \(F\) the free group on two variables \(F = F(a, b)\) and denote by \(\varphi : F \to LG\) the obvious epimorphism to the integral lamplighter group. It induces a homomorphism between pronilpotent completions

\[ \hat{\varphi} : \hat{F}_Z \to \hat{LG}_Z. \]

Note that

\[ \varphi([u, v]) = 1 \text{ for } u, v \in \langle a \rangle^F, \]

where \(\langle a \rangle^F\) is the normal subgroup of \(F\) generated by \(a\).

**Proposition 4.1.** For any sequence of integers \(q = (q_1, q_2, \ldots)\), there exists a pair of elements \(r_q, s_q \in \gamma_3(\hat{F}_Z)\) such that

1. \([r_q, a][s_q, b] = 1;\]
2. \(\hat{\varphi}(s_q) = 1;\]
3. \(\hat{\varphi}(r_q) = \prod_{i=3}^{\infty} [a_{i-1} b]^{q_i} \), where \(n_{2i+1} = q_i\) for \(i \geq 1\) and \(n_{2i}\) are some integers

(we control only odd terms of the product).

**Proof.** We claim that there exist sequences of elements \(r_q^{(3)}, r_q^{(4)}, \ldots \in F\) and \(s_q^{(3)}, s_q^{(4)}, \ldots \in F\) such that

1. \([\prod_{i=3}^{k} r_q^{(i)}, a][\prod_{i=3}^{k} s_q^{(i)}, b] \in \gamma_{k+2}(F);\]
2. \(\varphi(s_q^{(k)}) = 1;\]
3. \(\varphi(\prod_{i=3}^{k} r_q^{(i)}) = \prod_{i=3}^{k} [a_{i-1} b]^{n_i} \mod \gamma_{k+1}(LG), \) where \(n_{2i+1} = q_i\) for \(2i + 1 \leq k.\)

Then we take \(r_q = \prod_{i=3}^{\infty} r_q^{(i)}\) and \(s_q = \prod_{i=3}^{\infty} s_q^{(i)}\) and the assertion follows. So it is sufficient to construct such elements \(r_q^{(k)}, s_q^{(k)}\) inductively.

In order to prove the base case we set

\[ r_q^{(3)} := [a, b, b]^{q_1}, \quad s_q^{(3)} := [a, b, a]^{-q_1}. \]

**Corollary 4.1** with \(n = 1\), implies that

\[ [r_q^{(3)}, a][s_q^{(3)}, b] \in \gamma_5(F). \]

Clearly \(s_q^{(3)}, r_q^{(3)} \in \gamma_3(F), \) \(\varphi(s_q^{(3)}) = 1\) and \(\varphi(r_q^{(3)}) = [a_{2} b]^{q_1} \).

In order to prove the inductive step, assume that we already constructed

\[ r_q^{(3)}, \ldots, r_q^{(k)}, s_q^{(3)}, \ldots, s_q^{(k)}, \]

such that
with the properties (0)-(3). Construct \( r_q^{(k+1)} \) and \( s_q^{(k+1)} \). Note that any element of \( \gamma_{k+2}(F)/\gamma_{k+3}(F) \) can be presented as \([A,a][B,b] \cdot \gamma_{k+3}(F)\), where \( A,B \in \gamma_{k+1}(F) \). Then

\[
(4.4) \quad \prod_{i=3}^{k} r_q^{(i)}(a) \prod_{i=3}^{k} s_q^{(i)}(b) \equiv [A,a][B,b] \mod \gamma_{k+3}(F).
\]

Using that the images of \([A^{-1},a],[B^{-1},b]\) are in the center of \( F/\gamma_{k+3}(F) \), that \( \prod_{i=3}^{k} r_q^{(i)}, \prod_{i=3}^{k} s_q^{(i)} \in \gamma_3(F) \) and the identity \([xy,z] = [x,z]y \cdot [y,z]\) we obtain

\[
(4.5) \quad \prod_{i=3}^{k} r_q^{(i)} A^{-1}, a] \cdot \prod_{i=3}^{k} s_q^{(i)} B^{-1}, b] \in \gamma_{k+3}(F).
\]

Next we prove that

\[ \varphi(B) = 1. \]

Since \( B \in \gamma_{k+1}(F) \) we have

\[ B \equiv [a,k\ b] \equiv c \mod \gamma_{k+2}(F), \]

where \( e \in \mathbb{Z} \) and \( c \) is a product of powers of other basic commutators of weight \( k + 1 \). All these other basic commutators contain at least twice \( a \). It follows that \( \varphi(c) = 1 \). Since \( A \in \gamma_3(F) \subseteq \langle a \rangle F \), we have \( \varphi([A,a]) = 1 \). Moreover, \( \varphi([\prod_{i=3}^{k} r_q^{(i)},A][\prod_{i=3}^{k} s_q^{(i)},B]) = 1 \). Then

\[ [a_{k+1},b]^e \in \gamma_{k+3}(\mathcal{L}G). \]

This implies that \( e = 0 \) and hence \( \varphi(B) = 1 \).

If \( k \) is odd, we do not care about (3) and we just take

\[ r_q^{(k+1)} = A^{-1}, \quad s_q^{(k+1)} = B^{-1}. \]

Indeed, it is easy to check that the properties (0)-(2) are satisfied and the property (3) automatically follows.

Suppose now that \( k \) is even, say \( k = 2k' \). Consider the image of the element \( \prod_{i=3}^{k} r_q^{(i)} \cdot A^{-1} \) in the quotient \( \mathcal{L}G/\gamma_{k+2}(\mathcal{L}G) \). By the induction hypothesis,

\[ \varphi(\prod_{i=3}^{k} r_q^{(i)}) \equiv \prod_{i=3}^{k} [a_{i-1}b]^{n_i} \cdot c' \mod \gamma_{k+2}(\mathcal{L}G), \]

where \( c' \in \gamma_{k+1}(\mathcal{L}G) \). Since the quotient \( \gamma_{k+1}(\mathcal{L}G)/\gamma_{k+2}(\mathcal{L}G) \) is cyclic with generator \([a_{k'}b] \cdot \gamma_{k+2}(\mathcal{L}G)\),

\[ c' \equiv [a_{k'}b]^y \mod \gamma_{k+2}(\mathcal{L}G) \]

for some \( y \in \mathbb{Z} \). For \( n \geq 1 \), denote

\[ z_n := \prod_{i=0}^{n-1} [(a_{2n-1-i},b],[a,\ b)]^{(-1)^i}. \]

Corollary 4.1 implies that

\[ [(a_{k}b),a][z_{-1},b] \in \gamma_{k+3}(F). \]

We set

\[ r_q^{(k+1)} := A^{-1}[a_{k}b][q_{k'}]^{-e}, \quad s_q^{(k+1)} := B^{-1}z_{-1}^{(q_{k'})^{-e}}. \]

Now

\[ \prod_{i=3}^{k+1} r_q^{(i)}[a][\prod_{i=3}^{k+1} s_q^{(i)},b] \in \gamma_{k+3}(F) \]
and
\[ \varphi(\prod_{i=3}^{k+1} r_q^{(i)}) = \prod_{i=3}^{k+1} [a_{i-1} b]^{n_i}. \]

The properties (0) and (2) are obvious. \( \square \)

5. Proof of Theorems 1 and 2

Let \( F \) be a free group of rank \( \geq 2 \) and \( p \) be a prime. We will show that the image of the homomorphism \( H_2(\hat{F}_Z, \mathbb{Z}) \to H_2(\hat{F}_Q, \mathbb{Q}) \) is uncountable. The proof that the image of the map \( H_2(\hat{F}_Z, \mathbb{Z}) \to H_2(\hat{F}_Z, \mathbb{Z}/p) \) is uncountable is similar.

Since the free group with two generators is a retract of a free group of higher rank, it is enough to prove this only for \( F = F(a, b) \). The map
\[ H_2(\hat{F}_Z, \mathbb{Z}) \to H_2(\hat{L}\hat{G}_Q, \mathbb{Q}) \]
factors through \( H_2(\hat{F}_Q, \mathbb{Q}) \). Then it is enough to prove that the image the map (5.1) is uncountable.

For \( q \in \{0, 1\}^\mathbb{N} \) we denote by \( r_q, s_q \) some fixed elements of \( \hat{F}_Z \) satisfying properties (1), (2), (3) of Proposition 4.1. Then
\[ \hat{\varphi}(r_q) = \prod_{i=3}^\infty [a_{i-1} b]^{n_i(q)}, \]
where \( n(q)_{2i+1} = q_i \) and
\[ [r_q, a][s_q, b] = 1, \quad \hat{\varphi}(s_q) = 1. \]

Set
\[ f_q = \sum_{i=3}^\infty n_i(q)x^{i-1} \in \mathbb{Z}[x]. \]

If we consider \( \hat{L}\hat{G}_Z \) as the semidirect product \( \mathbb{Z}[x] \rtimes C \), we obtain that \( [a_{i-1} b] = (x^{i-1}, 1) \) and hence
\[ \hat{\varphi}(r_q) = (f_q, 1). \]

If we denote by \( \hat{\varphi}_Q \) the composition of \( \hat{\varphi} \) with the map \( \hat{L}\hat{G}_Z \to \hat{L}\hat{G}_Q \), we obtain
\[ \hat{\varphi}_Q(r_q) = (f_q^Q, 1), \]
where \( f_q^Q \) is the image of \( f_q \) in \( \mathbb{Q}[x] \). Consider the map
\[ \Theta_Q : \mathbb{Q}[x] \to H_2(\hat{L}\hat{G}_Q, \mathbb{Q}) \]
given by
\[ f \mapsto ((f, 1) \cup 1) \otimes \mathbb{Q}. \]

Observe that this map is the composition of the map from Proposition 2.11 and the isomorphism from Proposition 3.11. Therefore the kernel of \( \Theta_Q \) is countable. Set
\[ A := \{ f_q^Q \mid q \in \{0, 1\}^\mathbb{N} \} \subseteq \mathbb{Q}[x]. \]

Using that \( f_q^Q = \sum_{i=3}^\infty n_i(q)x^{i-1} \), where \( n_{2i+1}(q) = q_i \), we obtain that \( A \) is uncountable. Using that the kernel of \( \Theta_Q \) is countable, we obtain that its image
\[ \Theta_Q(A) = \{ ((f_q^Q, 1) \cup 1) \otimes \mathbb{Q} \mid q \in \{0, 1\}^\mathbb{N} \} \subseteq H_2(\hat{L}\hat{G}_Q, \mathbb{Q}) \]
is uncountable. Finally, observe that any element \( ((f_q^Q, 1) \cup 1) \otimes \mathbb{Q} \) of \( \Theta_Q(A) \) has a preimage in \( H_2(\hat{F}_Z, \mathbb{Z}) \) given by \( (r_q \cup a)(s_q \cup b) \), and then \( \Theta_Q(A) \) lies in the image of \( H_2(\hat{F}_Z, \mathbb{Z}) \to \).
This implies that the groups \( H_2(\hat{F}_Q, \mathbb{Q}) \) and \( H_2(\hat{F}_Z, \mathbb{Z}/p) \cong H_2(\hat{F}_Z, \mathbb{Z}) \otimes \mathbb{Z}/p \) are uncountable and Theorems 1 and 2 follow.

**References**

[1] A. K. Bousfield, D. M. Kan: *Homotopy limits, completions and localizations*, Lecture Notes in Mathematics, vol. 304, 1972.

[2] A. K. Bousfield: The localization of spaces with respect to homology, *Topology* 14 (1975), 311–335.

[3] A. K. Bousfield: Homological localization towers for groups and \( \pi \)-modules, Mem. Amer. Math. Soc., vol. 10, no. 186, 1977.

[4] A. K. Bousfield: On the \( p \)-adic completions of nonnilpotent spaces, *Trans. Amer. Math. Soc.* 331 (1992), 335–359.

[5] R. Brown, J.-L. Loday: Van Kampen theorems for diagrams of spaces, *Topology* 26 (1987), 311–335.

[6] S. O. Ivanov, R. Mikhailov: On a problem of Bousfield for metabelian groups, Adv. Math. 290 (2016), 552–589.

[7] S. O. Ivanov, R. Mikhailov: On discrete homology of a free pro-\( p \)-group, preprint [arXiv:1705.09131](https://arxiv.org/abs/1705.09131).

[8] C. Miller: The second homology of a group, *Proc. Amer. Math. Soc.* 3 (1952), 588–595.