Mismatch of conductivity anisotropy in the mixed and normal states of type–II superconductors.

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We have calculated the Bardeen-Stephen contribution to the vortex viscosity for uniaxial anisotropic superconductors within the time-dependent Ginzburg-Landau (TDGL) theory. We focus our attention on superconductors with a mismatch of anisotropy of normal and superconducting characteristics. Exact asymptotics for the Bardeen-Stephen contribution have been derived in two limits: (i) \( l_{\text{Eab}} \ll \xi_{\text{ab}}, l_{\text{Ec}} \ll \xi_{\text{c}}, l_{\text{Eab}} \ll \xi_{\text{ab}} \) where \( l_{\text{Eab}}, l_{\text{Ec}} \) and \( \xi_{\text{ab}}, \xi_{\text{c}} \) are the electric field penetration lengths and the coherence lengths in the \( ab \) plane and in the direction of the \( c \) axis. Also we suggest a variational procedure which allows us to calculate the vortex viscosity for superconductors with arbitrary parameters \( \xi \) and \( l_{\text{E}} \). The approximate analytical result is compared with numerical calculations. Finally, using a generalized TDGL theory, we prove that the viscosity anisotropy and, thus, the flux-flow conductivity anisotropy may depend on temperature.

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I. INTRODUCTION

The existence of a non-zero electrical resistivity in type–II superconductors in the mixed state is connected with the motion of magnetic flux vortices. It can be observed in the presence of a sufficiently large transport current so that pinning is suppressed. In the stationary flux-flow regime the Lorentz force acting on an isolated vortex is balanced by the intrinsic viscous drag force:

\[
\frac{\phi_0}{c} (\mathbf{j}_{\text{tr}} \times \mathbf{n}) = \eta \mathbf{V}_L
\]  

(1)

Here \( \phi_0 \) is the flux quantum, \( \mathbf{j}_{\text{tr}} \) is the transport current density, \( \mathbf{n} \) is the unit vector along the magnetic field, \( \mathbf{V}_L \) is the vortex velocity and \( \eta \) is a viscosity coefficient. As vortices move, the magnetic field in the sample becomes nonstationary and a macroscopic electrical field \( \mathbf{E} \) is induced, which is connected with the transport current via Ohm’s law: \( \mathbf{E} = \mathbf{j}_{\text{tr}} / \sigma \). For weak average magnetic fields, \( B \ll H_c2 \), where \( H_c2 \) is the upper critical field, the flux-flow conductivity \( \sigma \) is

\[
\sigma = \frac{c^2 \eta}{B \phi_0}.
\]  

(2)

The presence of a finite conductivity implies that current flow is accompanied by dissipation. It has been shown that there are two main mechanisms of dissipation: losses due to relaxation of the order parameter\textsuperscript{8} and ohmic losses associated with normal currents flowing through the vortex core\textsuperscript{9}.

For an anisotropic superconductor Eq. (2) is generalized as follows:

\[
\dot{\sigma} = \frac{c^2}{B \phi_0} \begin{pmatrix} \eta_{yy} & -\eta_{yx} \\ -\eta_{xy} & \eta_{xx} \end{pmatrix}
\]  

(3)

with the \( z \)-axis along the magnetic field. The peculiar structure of the conductivity tensor is explained by the fact that the \( x \)-component of the electric field depends on the \( y \)-component of the vortex velocity, and vice versa.

It can be seen from Eq. (3) that the flux-flow conductivity is determined by the magnetic field and the viscosity tensor \( \eta \). A rigorous approach to the problem of viscosity evaluation has been first suggested by Schmid\textsuperscript{3} and was later developed by Gor’kov and Kopnin\textsuperscript{10} (see also Ref. \textsuperscript{5} for review). Their method is based on the time-dependent Ginzburg-Landau (TDGL) theory. Within this model the flux-flow conductivity has been evaluated for isotropic superconductors in several papers\textsuperscript{6–8}. Both viscosity components due to order parameter relaxation and ohmic losses (frequently called the Bardeen-Stephen contribution) have been derived.

Theoretical studies of free flux flow in anisotropic materials have been stimulated by the discovery of high-temperature superconductors which appeared to possess rather strong anisotropy. A number of papers have addressed this problem using different models and approximations\textsuperscript{9,11}. The procedure of viscosity calculation can be essentially simplified in the limit of dirty uniaxial superconductors with the ratio \( s_0 = m_c/\sigma_c \eta_{ab} \sigma_{ab} \) equal to unity. Here \( \sigma_c, \sigma_{ab} \) and \( m_c, m_{ab} \) are the normal conductivities and Cooper pair masses in the direction of the anisotropy axis \( c \) and in the perpendicular \( ab \)-plane, respectively. The condition \( s_0 = 1 \) allows to reduce the problem of anisotropic vortex dynamics to an isotropic one by means of a scaling transformation.\textsuperscript{10} Yet this is not true in the case \( s_0 \neq 1 \), i.e, for a mismatch of anisotropies of Cooper pair masses and normal conductivities. Such a mismatch is theoretically possible in the relatively clean limit\textsuperscript{12} and it may have been experimentally observed in a new class of Fe-based pnictide superconductors. According to Ref. \textsuperscript{13} the ration \( \sigma_{ab}/\sigma_c \) in PrFeAsO\textsubscript{1–x}K\textsubscript{x}Fe\textsubscript{2}As\textsubscript{2} is close to 120, whereas \( m_c/m_{ab} \) in the same compound is about 25, as determined in Ref. \textsuperscript{14} from upper critical field measurements. In Refs. \textsuperscript{15} and \textsuperscript{16} anisotropies of the same order in Ba\textsubscript{1–x}K\textsubscript{x}Fe\textsubscript{2}As\textsubscript{2} are reported. However, existing experimental data for the
pnictides are contradictory. In Ref. [17] a relatively low resistivity anisotropy in BaFe$_2$As$_2$ is given: $\sigma_{ab}/\sigma_c \sim 2 - 3$. In some work, [18] an anisotropy mismatch has not been clearly detected. Previous calculations of the vortex viscosity tensor accounted for the anisotropy mismatch only on the basis of a simplified model of a step-like order parameter profile within the vortex core. [9] Of course, a detailed comparison with experimental data demands these calculations to be generalized for a more realistic order parameter profile.

In this paper we evaluate analytically the viscosity tensor for a realistic gap profile within the core, focusing our attention on the case $s_0 \neq 1$ and considering both standard TDGL model and its generalization for superconductors with a finite gap. [12] In section II we derive the basic equations following the approach of Gor’kov and Kopnin. [5] In Section III we develop approximate methods based on different assumptions about the ratio of the electric field penetration depth to the coherence length. The results of preceding works [5, 9] are revised and improved.

In the end of this section we consider a variational principle which provides us with a simple general relation for the Bardeen–Stephen contribution. In section IV the problem is considered in the framework of a generalized TDGL theory. Here we derive our main result: we predict that the flux-flow conductivity anisotropy may depend on temperature in superconductors with the parameter $s_0 \neq 1$.

II. BASIC EQUATIONS

Following Gor’kov and Kopnin, [5] we start the analysis of vortex motion with the TDGL equation for the superconducting order parameter $\psi$:

$$
\gamma \left( \frac{\hbar \partial \psi}{\partial t} + 2i e \Phi \psi \right) = - \frac{\delta F}{\delta \psi^*}, \tag{4}
$$

$$
F = \int \left[ \left( i \hbar \nabla - \frac{2e}{c} A \right) \psi \frac{\hat{m} - 1}{2} - i \hbar \nabla - \frac{2e}{c} A \right) \psi^* + a |\psi|^2 + \frac{1}{2} b |\psi|^4 \right] d^3r.
$$

Here $F$ is the usual GL free energy, $\gamma$ is a relaxation constant and $A$ and $\Phi$ are the vector and scalar potentials, respectively. We consider uniaxial anisotropic superconductors, so the Cooper pair mass $\hat{m}$ is a tensor with components $m_{ij} = m_{ab}(\delta_{ij} + \mu i \nu j)$ where $\nu$ is the unit vector along the $c$-axis, $\mu = m_{cc}/m_{ab} - 1$. Eq. (4) is supplemented by the equation for the current density

$$
\text{div} \mathbf{j} = 0, \tag{5}
$$

where

$$
\mathbf{j} = 2e |\psi|^2 \hat{m}^{-1} \left( \hbar \nabla \theta - \frac{2e}{c} A \right) - \hat{\sigma}_n \left( \nabla \Phi + \frac{1}{c} \frac{\partial A}{\partial t} \right). \tag{6}
$$

Here $\theta = \arg(\psi)$ and $\hat{\sigma}_n$ is the normal-state conductivity tensor with components $\sigma_{n ij} = \sigma_{ab} \delta_{ij} + (\sigma_c - \sigma_{ab}) \nu_i \nu_j$. For simplicity we will consider only superconductors with a large Ginzburg-Landau parameter $\kappa = \lambda / \xi \gg 1$, where $\lambda$ is the London penetration length and $\xi$ is the superconducting coherence length. One can prove that $|2eA/c|/|\hbar \nabla \theta| \ll 1$ at distances much smaller than $\lambda$ from the vortex axis in the gauge where $A = 0$ on the vortex axis and $\text{div} A = 0$. Imposing the additional condition $l_\xi^2 / \lambda \xi \ll 1$, where $l_\xi$ is the electric field penetration depth (see Eq. (12)), one can neglect the term

$$
\frac{1}{c} \frac{\partial A}{\partial t}
$$

in Eq. (6).

Let us consider the orientation of the internal magnetic field at an angle $\varphi$ to the crystallographic $c$-axis. We choose the coordinate frame $(x, y, z)$ with the $z$-axis coinciding with the vortex axis and with the $c$-axis lying in the $xz$-plane (see Fig. 1). In this frame the functions $\psi$ and $\Phi$ do not depend on $z$, and the tensor $\hat{\eta}$ is diagonal.

The derivation of the force balance equation [1] and the explicit expression for the viscosity may be found in Refs. [5] and [9]. However, in Appendix A we outline the calculations for the reader’s convenience.

The components of the Bardeen-Stephen contribution, $\hat{\eta}_{\text{bs}}$, are given by

$$
\eta_x = -2 \frac{|a|}{b} \gamma \hbar \int f^2(x_1 \rho_1 \frac{y_1}{\rho_1} \left( u^2 \Phi_x - \frac{y_1}{\rho_1} \right) \right) dx_1 dy_1, \tag{7}
$$

$$
\eta_y = -2 \frac{|a|}{b} \gamma \hbar \int f^2(x_1 \rho_1 \frac{x_1}{\rho_1} \left( u^2 \Phi_y - \frac{x_1}{\rho_1} \right) \right) dx_1 dy_1. \tag{8}
$$

Here

$$
\eta_x = \left[m(\varphi)/m_{ab} \right]^{1/2} (\eta_{\text{bs}})_{xx}, \tag{9}
$$

$$
\eta_y = \left[m_{ab}/m(\varphi) \right]^{1/2} (\eta_{\text{bs}})_{yy}. \tag{10}
$$

FIG. 1. The coordinate frame

\[ \text{\copyright 2023 American Physical Society. } \]
\[ m(\varphi) = \frac{m_{ab}(1 + \mu)}{1 + \mu \cos^2 \varphi}, \quad u = \frac{\xi_{ab}}{l_{Eab}}, \]

\[ (x_1, y_1) = \xi_{ab}^{-1} \left( \sqrt{\frac{m(\varphi)}{m_{ab}}} \rho_1 x, \rho_1 y \right), \quad \rho_1 = \sqrt{x_1^2 + y_1^2}, \quad (11) \]

\[ \xi_{ab} = \sqrt{\frac{\hbar}{2m_{ab} |a|}}, \quad l_{Eab} = \left[ \frac{\hbar \sigma_{ab}}{\left( 8e^2 \gamma |a| / b \right)^{1/2}} \right]. \quad (12) \]

The function \( f(\rho) \) describes the profile of the dimensionless order parameter modulus in a static isotropic vortex. This function satisfies the relation

\[ \frac{1}{\rho} \frac{d}{d\rho} \left( \rho f' \right) - \frac{f}{\rho^2} + f - f^3 = 0, \quad (13) \]

which follows from Eqs. (A8) and (A17). The boundary conditions are \( f(0) = 0, \ f(\infty) = 1 \). The functions \( \Phi_x \) and \( \Phi_y \) in Eqs. (7) and (8) should be determined from the linear equations

\[ s \frac{\partial^2 \Phi_x}{\partial x_1^2} + \frac{\partial^2 \Phi_x}{\partial y_1^2} = \left( u^2 \Phi_x - \frac{y_1}{\rho_1^2} \right) f^2(\rho_1), \quad (14) \]

\[ s \frac{\partial^2 \Phi_y}{\partial x_1^2} + \frac{\partial^2 \Phi_y}{\partial y_1^2} = \left( u^2 \Phi_y - \frac{x_1}{\rho_1^2} \right) f^2(\rho_1), \quad (15) \]

where

\[ s(\varphi) = 1 + \left( \frac{m_c \sigma_c}{m_{ab} \sigma_{ab}} - 1 \right) \frac{\sin^2 \varphi}{1 + \mu \cos^2 \varphi} > 0. \quad (16) \]

The electric potential can be expressed in terms of \( \Phi_x \) and \( \Phi_y \) via

\[ \Phi = \left( \Phi_x \sqrt{\frac{m(\varphi)}{m_{ab}}} V_{Lx} - \Phi_y V_{Ly} \right) \frac{4\pi e \hbar}{b \sigma_{ab}} \sqrt{\frac{|a|}{2m_{ab}}}. \quad (17) \]

Note that there is a relation connecting the components \( \eta_x \) and \( \eta_y \):

\[ \eta_y(s, u) = \eta_x \left( \frac{1}{s}, \frac{u}{\sqrt{s}} \right). \quad (18) \]

In the next section we consider some limiting cases.

### III. APPROXIMATE SOLUTIONS

#### A. The \( l_E \ll \xi \) limit

Consider such materials that the electric field penetration length is much smaller than the coherence length:

\[ l_{Eab} \ll \xi_{ab}, \quad l_{Ec} \ll \xi_c. \quad (19) \]

This limiting case is more close to gapless superconductors with a high concentration of magnetic impurities which are characterized by the ration \( \xi / l_E = \sqrt{12} \). The conditions (19) impose the following restrictions on the parameters \( s \) and \( u \): \( u \gg 1, \ s \ll u^2 \). In this subsection we will analyse the case \( s \ll 1 \). The case \( 1 \ll s \ll u^2 \) can be considered in a similar way by dividing Eqs. (14) and (15) by \( s \). We shall search the asymptotics of the viscosity when \( u \to \infty \) neglecting small terms of order of higher than \( u^{-2} \) (however, it will be shown that one should keep terms of the order of \( u^{-2} \)).

Our approximation is based on the fact that the characteristic length scale for the functions \( \Phi_x \) and \( \Phi_y \) is \( u^{-1} \). Hence, the unknown functions reach their asymptotic behavior at distances \( \rho \ll 1 \) from the vortex axis, where the order parameter profile \( f(\rho) \) is well approximated by the first several terms of its Taylor series:

\[ f^2(\rho) \approx k_2 \rho^2 + k_4 \rho^4 + k_6 \rho^6. \]

We substitute this expansion into Eq. (14) and introduce the new variables \( \tilde{\rho} = \rho \sqrt{u} \), \( \tilde{\Phi}_x = \Phi_x u^{1/2} \):

\[ \frac{\partial^2 \tilde{\Phi}_x}{\partial y^2} + s \frac{\partial^2 \tilde{\Phi}_x}{\partial x^2} = \left( k_2 \tilde{\rho}^2 + k_4 \tilde{\rho}^4 + \ldots \right) \left( \tilde{\Phi}_x - \frac{\tilde{y}}{\tilde{\rho}} \right). \quad (20) \]

Further the tilde over \( \tilde{x} \) and \( \tilde{y} \) will be omitted. The solution of Eq. (20) can be expanded in the powers of \( u^{-1} \):

\[ \tilde{\Phi}_x = \Phi_x^{(0)} + u^{-1} \Phi_x^{(1)} + R_x, \quad (21) \]

where \( \Phi_x^{(0)} \) and \( \Phi_x^{(1)} \) satisfy the following relations:

\[ \frac{\partial^2 \Phi_x^{(0)}}{\partial y^2} + s \frac{\partial^2 \Phi_x^{(0)}}{\partial x^2} = k_2 \rho^2 \Phi_x^{(0)} - k_2 y, \quad (22) \]

\[ \frac{\partial^2 \Phi_x^{(1)}}{\partial y^2} + s \frac{\partial^2 \Phi_x^{(1)}}{\partial x^2} = k_4 \rho^4 \Phi_x^{(1)} + k_4 \rho^4 \left( \Phi_x^{(0)} - \frac{y}{\rho^2} \right), \quad (23) \]

and \( R_x \) is a remainder term. It is proved in Appendix B that an analogous expansion can be made in the integral in the rhs of Eq. (7):

\[ \eta_x = -2 \frac{|a|}{b} \gamma \hbar \left[ \frac{I_{1x}(s)}{u} + \frac{I_{2x}(s)}{u^2} + o(u^{-2}) \right], \quad (24) \]

where

\[ I_{1x}(s) = \int k_2 y \left( \Phi_x^{(0)} - \frac{y}{\rho^2} \right) dx \, dy, \quad (25) \]

\[ I_{2x}(s) = \int \frac{y}{\rho^2} \left[ k_4 \rho^4 \left( \Phi_x^{(0)} - \frac{y}{\rho^2} \right) + k_2 \rho^2 \Phi_x^{(1)} \right] dx \, dy. \quad (26) \]
The viscosity component $\eta_y$ can be calculated similarly:

$$\eta_y = -2\frac{|a|}{b} \gamma \hbar \left[ \frac{I_{1y}(s)}{u} + \frac{I_{2y}(s)}{u^2} + o(u^{-2}) \right].$$

(27)

Using Eq. (18) we obtain

$$I_{1y}(s) = I_{1x}(s^{-1}) \sqrt{s}, \quad I_{2y}(s) = I_{2x}(s^{-1}) s.$$  

(28)

In principle, the functions $I_{1x}(s)$ and $I_{2x}(s)$ can be determined by numerical calculations, however, in section III C we present some analytical expressions for these functions.

In Ref. 9 the $u \gg 1$ limit was considered using the Bardeen-Stephen model [3]. This approach is essentially based on the assumption about a step-like order parameter profile within the core and does not allow to obtain a leading term of the order of $u^{-3}$ in the expansion [24].

The particular case $s = 1$ has been considered in a number of works mentioned above. It corresponds to isotropic superconductors, or anisotropic superconductors with no anisotropy mismatch: $(m_c \sigma_c)/(m_{ab} \sigma_{ab}) = 1$. If $s = 1$, Eqs. (22) and (23) can be solved exactly:

$$\Phi_x^{(0)} = \frac{1}{\rho^2} \exp(-\sqrt{2} \rho^2/2) y,$$

$$\Phi_x^{(1)} = \frac{k_4 y}{u k_2} \left( \frac{1}{4} + \frac{\sqrt{k_2} \rho^2}{8} \right) \exp(-\sqrt{k_2} \rho^2/2).$$

After some integration we obtain a simple relation for the viscosity:

$$\eta_x = \eta_y = 2\pi \frac{|a|}{b} \hbar \alpha_2(u),$$

(29)

$$\alpha_2(u) \approx \sqrt{k_2} \frac{k_2}{u} + \frac{k_4}{2k_2 u^2} = \frac{0.583}{u} - \frac{1}{8u^2}.$$  

(30)

Here the value $\sqrt{k_2} = 0.583$ was taken from Ref. 7 and the relation $k_4 = -k_2/4$ follows from Eq. (13).

It is appropriate to recall here the result obtained by Hu [3] and derived from the exact solution of Eq. (A12) with an approximate order parameter profile:

$$f(\rho) = \frac{\rho}{\sqrt{\rho^2 + \rho^2}}.$$  

(32)

According to Schmid and Hu, the optimal value of $\delta$ is $\sqrt{2}$ which follows from a variational principle. We can compare different values of $\alpha_2(u)$. When $u = \sqrt{T_2}$ Eq. (31) yields $\alpha_2 = 0.186$, Eq. (30) yields $\alpha_2 = 0.158$, while the numerical result is $\alpha_2 = 0.159$ [3]. Our formula gives an error less than 1%. If we keep only the term of order $u^{-1}$ in Eq. (30), we will get a 6% error which increases with decreasing $u$.

### B. The $l_E \gg \xi$ limit.

Consider the range of parameters $s \gg u^2$ and $u \gg 1$. In terms of $lE$, $\xi$ and $\varphi$ these conditions read

$$l_E \gg \xi, \quad l_E \gg \xi, \quad \cos^2 \varphi \ll \frac{\sigma_{I_2}^2}{\sigma_{ab} \sigma_{ab}}.$$  

(31)

Thus, the magnetic field must make a small angle with the $ab$-plane.

When $s \gg u^2$, the term $u^2 \Phi_x$ in Eq. (14) is negligible compared to $g_1/\rho^2$ in the region $\rho \ll \sqrt{s}/u$, so we immediately obtain from Eq. (7)

$$\eta_x \sim 2 \pi \hbar y \left[ \ln \frac{u}{\sqrt{s}} - 1.475 \right].$$  

(33)

$$\eta_y = 2\pi \hbar y \left[ \ln \frac{u}{\sqrt{s}} - 0.475 \right].$$  

(34)

Note that in Ref. 9 in the $u \ll 1$ limit similar expressions containing $\ln u^{-1}$ have been derived. This similarity is not accidental: the presence of the logarithm $\ln(lE/\xi)$ is a characteristic feature of the $l_E \gg \xi$ limit.

### C. A variational principle

In this subsection we suggest a simple variational procedure for the calculation of the viscosity tensor. According to Ref. 5 a general expression for the dissipation function $W[\Phi]$ reads:

$$W[\Phi] = \nabla \Phi \cdot \nabla \Phi + 2\gamma \left[ \hbar \frac{\partial \Phi}{\partial t} + 2ie \Phi \right]^2.$$  

(35)

The electric potential should be found from Eq. (A12) which can be viewed as a condition of zero variational derivative of the functional

$$\int_{z=0} W[\Phi]|d^2\rho|.$$  

Thus, the minimum of the functional above equals the loss power per unit length of a moving vortex:

$$\nu_L \bar{\nu}_L = \min_{\Phi} \int_{z=0} W[\Phi]|d^2\rho|.$$  

(36)

This relation allows us to apply the direct variational method to our problem.

For the sake of convenience we rewrite Eq. (36) in our rescaled coordinate frame separately for both components of the Bardeen-Stephen contribution:

$$\eta_x = \tilde{\eta}(s, 1, u), \quad \eta_y = \tilde{\eta}(1, s, u),$$  

(37)
\[ \dot{\eta}(s_x, s_y, u) = 2 \left[ \frac{a}{b} \gamma h u^2 \cdot \min \int \frac{\partial \phi}{\partial x} \right]^2 \]

\[ + s_y \left( \frac{\partial \phi}{\partial y} \right)^2 + f^2(\rho) \left( u^2 \phi - \frac{y}{\rho^2} \right)^2 \right] dx dy. \tag{38} \]

Eqs. (37) and (38) have two important consequences. First, the viscosity tensor is positively defined when \( \sigma_{ab} > 0 \) and \( \sigma_x > 0 \). Second, the components of \( \partial \phi \) increase as the conductivity increases:

\[ \frac{\partial \eta_i}{\partial \sigma_{ab}} > 0, \quad \frac{\partial \eta_i}{\partial \sigma_x} > 0, \quad i = x, y. \]

We can obtain an upper estimate for the viscosity components if we substitute a trial function into Eq. (38). In order to find an appropriate trial function consider the exact equation for \( \dot{\phi} \):

\[ s_x \frac{\partial^2 \phi}{\partial x^2} + s_y \frac{\partial^2 \phi}{\partial y^2} = \left( u^2 \phi - \frac{y}{\rho^2} \right) f^2(\rho). \tag{39} \]

The solution of this equation is an even function of \( x \) and an odd function of \( y \), so its Fourier series has the form

\[ \phi = \sum_{n=0}^{\infty} \phi_{2n+1}(\rho) \sin(2n+1)\chi, \tag{40} \]

where \( \chi \) is the polar angle in the \( xy \) plane. When \( \rho \) is sufficiently large, \( \phi \approx y/(u^2\rho^2) \), that means, that the series in Eq. (40) contains only the first term. Thus, the trial function

\[ \phi = \frac{4\hat{\phi}(\rho)}{s_x + 3s_y} \sin \chi \tag{41} \]

has the correct parity and the correct asymptotics. Let us substitute this function into Eq. (38):

\[ \dot{\eta} \approx 2 \frac{|a|}{b} \gamma h \pi \hat{\phi}^2 \min \phi \int_0^\infty \left[ \frac{\partial \hat{\phi}}{\partial \rho} \right]^2 + \frac{\hat{\phi}^2}{\rho^2} \]

\[ + \frac{f^2(\rho)}{u^2} \left( \hat{u}^2 \hat{\phi} - \frac{1}{\rho} \right)^2 \right] d\rho. \tag{42} \]

where

\[ \hat{u} = u \left( \frac{s_x}{4} + \frac{3s_y}{4} \right)^{-1/2}. \]

The differential equation for \( \hat{\phi} \) is

\[ -\frac{1}{\rho} \frac{d}{d\rho} \left( \frac{\partial \hat{\phi}}{\partial \rho} \right) + \frac{\hat{\phi}}{\rho^2} + f^2(\rho) \left( \hat{u}^2 \hat{\phi} - \frac{1}{\rho} \right) = 0. \tag{43} \]

Note that we obtain exactly the same equation if we substitute \( \Phi_x = \hat{\phi}(\rho) \sin \chi \) into Eq. (14) when \( s = 1 \) and \( u = \hat{u} \). This means that the trial function (11) reduces our problem to an isotropic one. Unfortunately, an exact solution of Eq. (43) is unknown. However, Schmidt found a solution with an approximate order parameter profile [see Eq. (32)]:

\[ \bar{\phi} = K_1(\hat{u} \delta) \frac{\sqrt{\delta^2 + \rho^2} \sqrt{K_1(\hat{u} \sqrt{\delta^2 + \rho^2})}}{\delta K_1(\hat{u} \delta) \hat{u}^2 \rho}. \]

Using this function and the expression (32) for \( f \) we can calculate the rhs of Eq. (42):

\[ \dot{\eta} \approx 2 \frac{|a|}{b} \gamma h \pi \frac{K_0(\delta \hat{u})}{\delta \hat{u} K_1(\delta \hat{u})}. \tag{44} \]

We take \( \delta = f'(0) - 1 = k_2^{-1/2} \) in order to obtain the correct asymptotics when \( u \to \infty, s = 1 \) (this asymptotics is determined by \( f'(0) \), see subsection III A). Finally, combining (37) and (44) we get approximate relations for the components of \( \dot{\eta} \):

\[ \eta_x \approx 2 \pi \frac{|a|}{b} \gamma h \pi f'(0) \frac{2u}{u^2} \sqrt{s+3} \frac{K_0(\frac{2u}{f'(0)\sqrt{s+3}})}{K_1(\frac{2u}{f'(0)\sqrt{s+3}})}; \tag{45} \]

\[ \eta_y \approx 2 \pi \frac{|a|}{b} \gamma h \pi f'(0) \frac{2u}{u^2} \sqrt{3s+1} \frac{K_0(\frac{2u}{f'(0)\sqrt{3s+1}})}{K_1(\frac{2u}{f'(0)\sqrt{3s+1}})}. \tag{46} \]

No restrictions on the parameters \( s \) and \( u \) are implied here.

Let us check if these relations are in accordance with the results from subsections III A and III B. Expanding \( \eta_x \) in the form (45) in the powers of \( u^{-1} \) when \( u \gg 1 \) and \( s \ll 1 \) we obtain the following expressions for the coefficients \( I_{1x} \) and \( I_{2x} \), which were introduced in subsection III A (see Eq. (24)):

\[ I_{1x}(s) = -\frac{\pi k_2^{1/2} s + 3}{2}, \quad I_{2x}(s) = \frac{\pi k_2(s + 3)}{8}. \tag{47} \]

When \( s = 1 \)

\[ \eta_x = \eta_y = 2 \pi \frac{|a|}{b} \gamma h \pi \frac{f'(0)}{u} + O(u^{-2}), \]

which should be compared with Eq. (30). The perfect agreement between the exact and approximate result is not surprising, because the trial function (11) is the exact solution of our variational problem in the isotropic case.

In order to check whether Eqs. (45) and (46) are applicable for \( s \neq 1 \) we used numerical calculations. We solved Eq. (39) in the region \( x > 0, y > 0 \) with the boundary conditions

\[ \frac{\partial \phi}{\partial x} \bigg|_{x=0} = 0, \quad \phi \bigg|_{y=0} = 0. \]

A sufficiently large \( 450 \times 450 \) mesh with a \( 0.03 \times 0.03 \) unit cell has been used. The numerical algorithm applied was the method of steepest descent. After the determination
of the function \( \phi(\rho) \) numerical integration has been performed.

When \( s = 0 \) Eqs. (47) and (28) give
\[
\eta_x = 2\frac{|a|}{b} \gamma h \frac{1.59}{u} + O(u^{-2}),
\]
\[
\eta_y = 2\frac{|a|}{b} \gamma h \frac{0.92}{u} + O(u^{-2}).
\]

These analytical expressions are in a good agreement with the asymptotics derived by numerical calculations:
\[
\eta_x = 2\frac{|a|}{b} \gamma h \frac{1.58}{u} + O(u^{-2}),
\]
\[
\eta_y = 2\frac{|a|}{b} \gamma h \frac{0.86}{u} + O(u^{-2}).
\]

When \( s \gg u^2 \) Eqs. (45) and (46) give
\[
\eta_x \approx \eta_y \approx 2\pi h \gamma |a| \frac{\ln \sqrt{s}}{u},
\]
which coincides with the main logarithmic term in Eqs. (33) and (34).

One can see that the agreement between the exact and approximate asymptotics is quite well. This is a strong argument in favor of the applicability of Eqs. (45) and (46) for intermediate values of \( s \) and \( u \).

In Fig. 2 we plot the analytical and numerical \( \varphi \) dependencies of the diagonal components of the full viscosity (\( \hat{\eta} = \hat{\eta}_{\rho\rho} + \hat{\eta}_{\phi\phi} \)).

### IV. TEMPERATURE DEPENDENCE OF THE VISCOILITY ANISOTROPY.

Within the framework of the TDGL equation (1) the viscosity anisotropy \( \eta_{xx}/\eta_{yy} \) does not depend on temperature. However, the region of applicability of Eq. (4) is limited by gapless superconductivity. In this section we consider a more general approach based on the generalized TDGL equation (19) (see also Ref. 12 for review):
\[
2\pi h \gamma \sqrt{1 + q |\psi|^2/|\psi_\infty|^2} \frac{\partial |\psi|}{\partial t} = -\frac{\delta F}{\delta |\psi|}. \tag{48}
\]

\[
\frac{\gamma |\psi|^2}{\sqrt{1 + q |\psi|^2/|\psi_\infty|^2}} \left( \hbar \frac{\partial \theta}{\partial t} + 2e \Phi \right) = \frac{\hbar^2}{2} \nabla \left( |\psi|^2 \hat{m}^{-1} \nabla \theta \right), \tag{49}
\]

\[
q = \frac{32\pi^2 r_p^2 T_c (T_c - T)}{2 \zeta(3) \hbar^2}, \quad |\psi_\infty|^2 = \frac{|a|}{b}.
\]

Here \( T_c \) is the critical temperature and \( \tau_{ph} \) is the electron-phonon mean free time. In the isotropic case Eqs. (48) and (49) are valid for dirty superconductors, when the temperature is close to \( T_c \) and variations of the order parameter in space and in time are sufficiently slow.

The main relations for the viscosity can be derived in the same way as described in section II. As a result, we find that the viscosity still comprises two terms representing two mechanisms of dissipation, but the viscosity components undergo some changes. For example, Eq. (A19) is modified as follows:
\[
(\hat{\eta}'_{\rho\rho})_{ij} = 2\pi h \gamma \frac{|a|}{b} \delta_{ij} \int_0^\infty \left( \frac{df}{d\rho} \right)^2 \rho \sqrt{1 + q f^2(\rho)} d\rho. \tag{50}
\]

In order to obtain the counterparts of Eqs. (7), (8), (14) and (15) one should make the following substitutions in...
now reads but they are slightly modified. For example, Eq. (29)
be derived again within the generalized TDGL theory,
below). It is obvious that all main relations from sec. III can
be derived within the generalized TDGL theory, when
we leave Eqs. (37) unchanged. Hence,
\[ \frac{\partial}{\partial T} \left( \frac{\eta_{oh}}{\eta_0} \right)_{xx} > 0; \quad \frac{\partial}{\partial T} \left( \frac{\eta_{oh}}{\eta_0} \right)_{yy} > 0. \]
At sufficiently low temperatures, when \( q \gg 1, s < 1 \) and
\( u \sim 1 \), it may happen that
\[ (\eta_{io})_{xx} \gg (\eta_{oh})_{xx}, \quad (\eta_{io})_{yy} \gg (\eta_{oh})_{yy}. \]
Then the viscosity anisotropy is determined by the relaxational term:
\[ \frac{\eta_{xx}}{\eta_{yy}} \approx \frac{(\eta_{io})_{xx}}{(\eta_{io})_{yy}} = \frac{1 + \mu}{1 + \mu \cos^2 \varphi}. \]
Note that when \( s \neq 1 \), \( \eta_{xx} \neq \eta_{yy} \), so
\[ \frac{\eta_{xx}}{\eta_{yy}} \neq \frac{1 + \mu}{1 + \mu \cos^2 \varphi}, \]
when \( q < 1 \). We have proved that within the generalized
tDGL theory the viscosity anisotropy and the flux-flow
conductivity anisotropy do depend on temperature. The
schematic \( T \) dependence of the ratio \( \eta_{xx}/\eta_{yy} \) is plotted in Fig. 3.

V. CONCLUSION

By solving the time-dependent Ginzburg-Landau equation
we analyzed the viscous flux-flow in anisotropic superconductors. The Bardeen-Stephen contribution to the
viscosity tensor \( \tilde{\eta} \) has been calculated in the \( l_E \gg \xi \) and
\( l_{Ec} \gg \xi \) limits. We emphasize that in these calculations
we did not use any simplifying assumptions concerning
the shape of the order parameter in a static vortex. We
suggested a variational procedure, which allowed us to
derive the relations (45) and (46) suitable for arbitrary
electric field penetration lengths (\( l_{Eab} \) and \( l_{Ec} \)), coherence
lengths (\( \xi_{ab} \) and \( \xi_{c} \)) and orientation of the magnetic
field. Our results may be useful for interpretation of experimental data on flux flow conductivity in isotropic and anisotropic superconductors in weak magnetic fields
(\( B \ll H_c2 \)).

Viscous flux-flow has also been examined within a general-
ized TDGL theory. We found that the viscosity anisotropy may depend on temperature and, thus, the
flux-flow conductivity anisotropy may be altered by heat-
ing or cooling the sample. We hope that this effect will
be observed experimentally in the near future.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{Fig3.png}
\caption{Schematic temperature dependence of the viscosity
anisotropy for \( \varphi = \pi/2 \). The parameter \( \epsilon \) is of the order of
unity.}
\end{figure}
VI. ACKNOWLEDGEMENTS

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APPENDIX A

In this appendix we derive Eqs. (7), (8), (14) and (15). In Eq. (4) it is convenient to make a scaling of the variables: $x' = x/m(\varphi/m_{ab})^{1/2}$, $y' = y$, $z' = z$. We rewrite Eq. (4) in the form

$$\gamma \hbar \frac{\partial |\psi|}{\partial t} = \frac{\hbar^2}{2m_{ab}} \left[ \nabla^2 |\psi| - |\psi| (\nabla^2 \theta)^2 \right] - a|\psi| - b|\psi|^3,$$

(A1)

$$\gamma |\psi|^2 \left( \frac{\partial \theta}{\partial t} + 2e\Phi \right) = \frac{\hbar^2}{2m_{ab}} \nabla'(|\psi|^2 \nabla' \theta).$$

(A2)

According to Eq. (5), the two-dimensional current $j' = (j_x|m(\varphi)/m_{ab})^{1/2}, j_y)$ satisfies the relation

$$\text{div}'j' = 0.$$  

(A3)

It follows from Eqs. (A2) and (A3) that

$$\gamma |\psi|^2 \left( \frac{\partial \theta}{\partial t} + 2e\Phi \right) = \frac{\hbar}{4e} \nabla' (\tilde{\sigma}_n \nabla' \Phi),$$

(A4)

where we introduced the tensor $\tilde{\sigma}_n$ with components

$$\tilde{\sigma}_{nx'n'} = s(\varphi)\sigma_{ab}, \quad \tilde{\sigma}_{ny'n'} = \sigma_{ab}, \quad \tilde{\sigma}_{nx'y'} = \tilde{\sigma}_{ny'x'} = 0,$$

(A5)

with $s(\varphi)$ given by Eq. (16). For a moving vortex one should search the solution of Eqs. (A1), (A2) and (A3) in the form $\psi = \psi(\rho' - \tilde{\mathbf{V}}_L t), \Phi = \Phi(\rho' - \tilde{\mathbf{V}}_L t)$, where $\tilde{\mathbf{V}}_L = (V_{Lx}|m(\varphi)/m_{ab})^{1/2}, V_{Ly})$ and $\rho' = (x', y')$. We expand $|\psi|$ and $\theta$ in powers of $V_L$ up to the first order term, assuming the vortex velocity to be sufficiently small:

$$|\psi| \approx \psi_0(\rho' - \tilde{\mathbf{V}}_L t) + \psi_3(\rho' - \tilde{\mathbf{V}}_L t),$$

(A6)

$$\theta \approx \theta_0(\rho' - \tilde{\mathbf{V}}_L t) + \theta_3(\rho' - \tilde{\mathbf{V}}_L t).$$

(A7)

Here $\psi_0(\rho')$ and $\theta_0(\rho')$ correspond to a static vortex. The functions $\psi_3, \theta_3$ and $\Phi$ are of the order of $V_L$. We substitute (A6) and (A7) into Eqs. (A1), (A3) and (A4):

$$-a\psi_0 - b\psi_3^0 + \frac{\hbar^2}{2m_{ab}} \left[ \nabla'^2 \psi_0 - \psi_0(\nabla'\theta)^2 \right] = 0,$$

(A8)

$$\frac{\hbar^2}{2m_{ab}} \left[ \nabla'^2 \psi_0 - \psi_0(\nabla'\theta)^2 - 2\psi_0(\nabla'\theta_0) \cdot \nabla'\theta_0 \right] - a\psi_0 - 3b\psi_3^0 \psi_0 = -\gamma \hbar (\tilde{\mathbf{V}}_L \nabla') \psi_0,$$

(A9)

$$\text{div}'j'_0 = 0, \quad j'_0 = \frac{2e\hbar\psi_3^0}{m_{ab}} \nabla'\theta_0,$$

(A10)

$$\text{div}'j'_1 = 0, \quad j'_1 = \frac{2e\hbar}{m_{ab}} \left( 2\psi_0\psi_1 \nabla'\theta_0 + \psi_0^2 \nabla'\theta_1 \right) - \tilde{\sigma}_n \nabla'\Phi,$$

(A11)

Now we introduce some new notations: $\psi_d = (\mathbf{d}\nabla')\psi_0$, $\theta_d = (\mathbf{d}\nabla')\theta_0$, $j'_d = (\mathbf{d}\nabla')j'_0$, where $\mathbf{d}$ is an arbitrary vector. A simple equation connecting $\psi_d$ and $\theta_d$ can be obtained by applying the operator $\mathbf{d}\nabla'$ to Eq. (A10):

$$\frac{\hbar}{2m_{ab}} \left[ \nabla'^2 \psi_d - \psi_d(\nabla'\theta_0)^2 - 2\psi_0(\nabla'\theta_0) \cdot \nabla'\theta_d \right] - a\psi_d - 3b\psi_3^0 \psi_d = 0$$

(A13)

The vector $j'_d$ satisfies the obvious relation $\text{div}'j'_d = 0$. Let us multiply Eq. (A9) by $\psi_d$, subtract Eq. (A13) multiplied by $\psi_1$ and integrate the resulting equation over a large volume containing the whole vortex. After some simple algebra and integration by parts we obtain

$$-\gamma \hbar \int (\tilde{\mathbf{V}}_L \nabla') \psi_0 \psi_d d^3r' = \frac{\hbar}{4e} \int [(j'_1 + \tilde{\sigma}_n \nabla' \Phi) \nabla' \theta_d - j'_d \nabla'\theta_1] d^3r'$$

$$= \frac{\hbar}{4e} \int (\tilde{\sigma}_n \nabla' \Phi) \nabla' \theta_d d^3r' + \frac{\hbar}{2e} \int \left( j'_1 - j'_d \right) dS.$$  

(A14)

Here $S$ is a surface far from the vortex axis. At large distances $r' \gg \xi_{ab}$ we have

$$j'_1 \approx \frac{2e\hbar |a|}{2mb_{ab}} \nabla'\theta_1 = j'_{tr}, \quad \theta_1 = \frac{b_{mb}}{2mb_{ab}} (j'_{tr} \cdot \rho') + \text{const},$$

(A15)

where $j'_{tr}$ is the transport current which is constant. If we calculate the surface integral in Eq. (A14) and make some simple transformations, we obtain the force balance equation:

$$\frac{\pi \hbar}{e} (\mathbf{d} \cdot (j'_{tr} \times \mathbf{n})) = -2\pi \gamma \hbar (\mathbf{d} \cdot \tilde{\mathbf{V}}_L) \int_0^\infty \left( \frac{df}{d\rho} \right)^2 \rho d\rho$$

$$+ \frac{\hbar \sigma_{ab}}{2e} \int \left[ s(\varphi) \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} \right] (\mathbf{d} \cdot \nabla'\theta_0) d^2\rho',$$

(A16)

where $\mathbf{n}$ is a unit vector along the magnetic field. If we compare Eqs. (11) and (A15), we can see that the viscosity tensor in the frame $(x', y', z')$ should be defined as follows:

$$\mathbf{d} \cdot \tilde{n} \tilde{\mathbf{V}}_L = 2\pi \gamma \hbar (\mathbf{d} \cdot \tilde{\mathbf{V}}_L) \left( \frac{a}{b} \right) \int_0^\infty \left( \frac{df}{d\rho} \right)^2 \rho d\rho$$

$$+ \frac{\hbar \sigma_{ab}}{2e} \int \left[ s(\varphi) \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} \right] (\mathbf{d} \cdot \nabla'\theta_0) d^2\rho',$$

(A16)

where we introduced the function

$$f(\rho) = \psi_0(\rho\xi_{ab}) \sqrt{\frac{b}{a}}.$$
The components of the viscosity $\tilde{\eta}$ in the frame $(x, y, z)$ are given by

$$
\eta_{xx} = [m(\varphi)/m_h]^{1/2} \eta'_{x'x'}, \quad \eta_{yy} = [m_h/m(\varphi)]^{1/2} \eta'_{y'y'}.
$$

(A16)

The rhs of Eq. (A16) contains two terms, representing two mechanisms of dissipation. The viscosity due to relaxation of the order parameter is (229).

$$(\eta_{f,0})_{ij} = 2\pi \hbar \frac{|a|}{b} \alpha_1 \delta_{ij}, \quad (A19)$$

The second term in the rhs of Eq. (A16) defines the ohmic viscosity tensor $\eta'_h$, which is to be evaluated:

$$
\mathbf{d} \cdot \eta'_h \nabla L = -\frac{\hbar \sigma_{ab}}{2e} \int \left[ \frac{\partial^2 \Theta}{\partial x'^2} - \frac{\partial^2 \Phi}{\partial y'^2} \right] (dx') (d\rho) (d\rho').
$$

(A20)

Now, if we substitute $\Phi$ in the form (17) into Eqs. (A12) and (A20) and switch to the coordinates $(x_1, y_1)$ (see Eq. (11)) we obtain Eqs. (7), (8), (14) and (15).

**APPENDIX B**

In this Appendix we will derive Eq. (24). First, we divide the integral in Eq. (7) into two parts:

$$
\eta_x = \eta_{x1} + \eta_{x2}, \quad (B1)
$$

$$
\eta_{x1} = -2 \frac{|a|}{b} \gamma \hbar \int_{\rho_1 < \rho_0 / \sqrt{u}} \frac{f^2(\rho_1)}{\rho_1^3} \left( u^2 \Phi_x - \frac{y_1}{\rho_1^2} \right) dx_1 dy_1, \quad (B2)
$$

$$
\eta_{x2} = -2 \frac{|a|}{b} \gamma \hbar \int_{\rho_1 > \rho_0 / \sqrt{u}} \frac{f^2(\rho_1)}{\rho_1^3} \left( u^2 \Phi_x - \frac{y_1}{\rho_1^2} \right) dx_1 dy_1, \quad (B3)
$$

where $\rho_0 = u^{1/6+\delta}$, $\delta \ll 1/6$. Note that the rhs of Eq. (14) is small when $\rho_1 > \rho_0 / \sqrt{u} \gg u^{-1/2}$, so it can be accounted for by perturbation theory:

$$
\Phi_x = \frac{y_1}{u^2 \rho_1^2} + \Phi_{x1} + \Phi_{x2} + \ldots, \quad (B4)
$$

$$
\Phi_{x1} = \frac{1}{u^4 f^2(\rho_1)} \left( \frac{\partial^2}{\partial y_1^2} + s \frac{\partial^2}{\partial x_1^2} \right) \frac{y_1}{\rho_1^2},
$$

$$
\Phi_{x2} = \frac{1}{u^6} \left[ \frac{1}{f^2(\rho_1)} \left( \frac{\partial^2}{\partial y_1^2} + s \frac{\partial^2}{\partial x_1^2} \right) \right]^2 \frac{y_1}{\rho_1^2}.
$$

The main contribution to the integral in Eq. (B3) is determined by small $\rho_1$. The order of $(\rho_0^6)^{-1} \ll u^{-2}$, the integrals of higher-order terms are also negligibly small, hence

$$
\eta_{x2} \approx -2 \frac{|a|}{b} \gamma \hbar \frac{I_{0x}}{u}, \quad (B5)
$$

$$
I_{0x} = \int_{\rho > \rho_0} \frac{y}{\rho^2} \left( \frac{\partial^2}{\partial y^2} + s \frac{\partial^2}{\partial x^2} \right) y \frac{u}{\rho^2} \ dx \ dy.
$$

Let us consider the component $\eta_{x1}$. In the new variables introduced in subsection IIIA Eq. (B2) reads

$$
\eta_{x1} = -2 \frac{|a|}{b} \gamma \hbar \int_{\rho < \rho_0} f^2 \left( \frac{\rho}{\sqrt{u}} \right) \frac{y}{\rho^2} \left( \Phi_x - \frac{y}{\rho^2} \right) dx \ dy. \quad (B6)
$$

Now we estimate the term $R_x$ introduced in Eq. (21). It satisfies the following relation:

$$
\frac{\partial^2 R_x}{\partial \rho^2} + s \frac{\partial^2 R_x}{\partial x^2} - u f^2 \left( \frac{\rho}{\sqrt{u}} \right) R_x
$$

$$
= \left[ u f^2 \left( \frac{\rho}{\sqrt{u}} \right) - k_2 \rho^2 \right] \left( \Phi_x - \frac{y}{\rho^2} \right) + \left( u f^2 \left( \frac{\rho}{\sqrt{u}} \right) - k_2 \rho^2 \Phi_x^{(1)} \right), \quad (B7)
$$

Note that when $\rho \ll \sqrt{u}$ the source in the rhs of (B7) can be presented as $S(x, y) u^{-2}$, where $S(x, y)$ is some function independent of $u$. Since (B7) is a screening equation, the function $R_x(x, y, u)$ for small $\rho$ does not depend on the behavior of the source in the area of big $\rho$ and can be presented as $R_x = R_x(x, y) u^{-2}$. On the other hand, when $\rho \gg 1$ the derivatives in the lins of Eq. (B7) are small, hence in the area $1 \ll \rho \ll \sqrt{u}$

$$
R_x \approx \frac{1}{u^2} \left[ -k_2 \rho^2 \left( \Phi_x^{(0)} - \frac{y}{\rho^2} \right) - \frac{k_4 \rho^4}{k_2} \right], \quad (B8)
$$

$$
|R_x| \leq \text{const} \frac{\rho^2}{nu^2}. \quad (B8)
$$

Now we substitute $\Phi_x$ in the form (21) into (B6):

$$
\eta_{x1} = -2 \frac{|a|}{b} \gamma \hbar \left( I_{1x} + I_{2x} + I_{3x} \right), \quad (B9)
$$

where

$$
I_{1x} = \int_{\rho \ll \rho_0} k_2 y \left( \Phi_x^{(0)} - \frac{y}{\rho^2} \right) dx \ dy, \quad (B10)
$$

$$
I_{2x} = \int_{\rho \ll \rho_0} \frac{y}{\rho^2} \left[ k_4 \rho^4 \left( \Phi_x^{(0)} - \frac{y}{\rho^2} \right) + 2 \rho^2 \Phi_x^{(1)} \right] dx \ dy, \quad (B11)
$$

$$
I_{3x} = \int_{\rho \ll \rho_0} f^2 \left( \frac{\rho}{\sqrt{u}} \right) \left( \Phi_x^{(0)} - \frac{y}{\rho^2} \right) \frac{y}{\rho^2} dx \ dy
$$

$$
+ \int_{\rho \ll \rho_0} f^2 \left( \frac{\rho}{\sqrt{u}} \right) \frac{\Phi_x^{(1)} y}{\rho^2} dx \ dy
$$

$$
+ \int_{\rho \ll \rho_0} f^2 \left( \frac{\rho}{\sqrt{u}} \right) R_x \frac{y}{\rho^2} dx \ dy. \quad (B12)\]
One can easily prove that
\[
\Phi^{(0)}_y = \frac{y}{\rho^2} + \frac{1}{k_2\rho^2} \left( \frac{\partial^2}{\partial y^2} + s \frac{\partial^2}{\partial x^2} \right) \frac{y}{\rho^2} + O(\rho^{-9}),
\]

\[
k_{40}\Phi_x^{(0)} - \frac{y}{\rho^2} + k_2\rho^2\Phi_y^{(1)} = O(\rho^{-5}).
\]

From Eqs. (B13), (B14) and (B18) we can see that all integrals in Eq. (B12) are of the order \(\rho_0^2/\rho^3\). Thus \(|I_{3z}| \ll u^{-2}\), so it can be neglected. Also we can integrate in Eqs. (B10) and (B11) over the whole \(xy\) plane, since
\[
|I_{0x} + I_{1x} - I_{1x}(s)| \ll u^{-1},
\]

\[
|I'_{2x} - I_{2x}(s)| \ll 1,
\]

Finally, taking into account Eqs. (B1), (B5) and (B9) we obtain Eq. (24).

**APPENDIX C**

In this Appendix we consider in detail the derivation of Eqs. (B3) and (B4). We will present here the calculations for the \(\eta_y\) component, since the calculations for the \(\eta_x\) component are less complicated. First, we rewrite Eq. (15) in the form
\[
\int s^2 \frac{\partial^2 \Phi_x}{\partial x^2} - u^2 \Phi_y + \frac{x}{\rho} f^2(\rho) = -\frac{\partial^2 \Phi_x}{\partial y^2} - u^2 \Phi_y (1 - f^2(\rho)). \tag{C1}
\]

The index “1” is omitted. It will be proved below that the terms in the rhs of Eq. (C1) give a small contribution to the viscosity, so they can be neglected. Then the solution of Eq. (C1) has the form
\[
\Phi_y \approx \Phi_{y0} = \int_{-\infty}^{+\infty} \frac{x' f^2(\rho') \exp(-u|x-x'|/\sqrt{s})}{2u\sqrt{s}} dx',
\]

where \(\rho' = \sqrt{x'^2 + y'^2}\). Consider a quantity \(y_0\) in the range \(1 \ll y_0 \ll \sqrt{s}/u\) (for example, \(y_0 = s^{1/4}/u^{1/2}\)). We divide the integral in Eq. (C1) into three parts:
\[
\eta_y = -2 \frac{|a|}{b} \gamma h \left[ \int |y| < y_0 f^2(\rho) \frac{x}{\rho} u^2 \Phi_y \, dx \, dy - \int |y| < y_0 f^2(\rho) \frac{x}{\rho} \, dx \, dy \right] - 4 \frac{|a|}{b} \gamma h \int_{y > y_0} f^2(\rho) \frac{x}{\rho} \left( u^2 \Phi_y - \frac{x}{\rho} \right) \, dx \, dy. \tag{C3}
\]

Using the inequality
\[
f^2(\rho) \leq \frac{\rho^2}{A_1 + \rho^2}, \tag{C4}
\]

where \(A_1\) is some constant, we can estimate the first integral:
\[
\left| \int_{|y| < y_0} f^2(\rho) \frac{X}{\rho^2} u^2 \Phi_y \, dx \, dy \right| \leq \text{const} \frac{y_0}{\sqrt{s}} \left( \ln \frac{\sqrt{s}}{u} \right)^2 \ll 1
\]

\[
\tag{C5}
\]

Here and further “const” denotes a constant independent of any parameters. The second term in Eq. (C3) has the following asymptotics when \(y_0 \gg 1\):
\[
\int_{|y| < y_0} f^2(\rho) \frac{X}{\rho^2} \, dx \, dy \approx \frac{x}{\rho} \ln y_0 + C_y. \tag{C6}
\]

The constant \(C_y\) will be evaluated below. The third integral in Eq. (C3) can be simplified if we take into account that \(\rho > y_0, \rho' > y_0\) and \(y_0 \gg 1\), so we can substitute unity instead of \(f^2\):
\[
\int_{y > y_0} f^2(\rho) \frac{x}{\rho} \left( u^2 \Phi_y - \frac{x}{\rho} \right) \, dx \, dy
\]

\[
\approx \int_{y_0}^{\infty} \frac{\rho}{\sqrt{s}} \int_{-\infty}^{+\infty} \frac{y \exp(-u|x|/\sqrt{s})}{x^2 + 4u^2} \, dx \, dy
\]

\[
= \frac{x}{2} \int_{y_0}^{\infty} \frac{\rho}{\sqrt{s}} \int_{0}^{2\pi} \frac{\rho y \exp(-u|x|/\sqrt{s})}{\sqrt{s}x^2 + 4u^2} \, dx \, dy
\]

\[
= -\frac{\pi}{\sqrt{s}u} \int_{0}^{\infty} \frac{\rho y}{\sqrt{s}u^2 + 4u^2} \, dx
\]

\[
\approx -\frac{\pi}{\sqrt{s}u} \int_{0}^{\infty} \ln \frac{s^{3/4}u}{x^2 + 4u^2} \, dx = \frac{\pi}{\sqrt{s}} \ln y_0 - \frac{\pi}{\sqrt{s}} \ln \frac{s^{3/4}u}{x^2 + 4u^2} + \frac{\pi}{4} C_y, \tag{C7}
\]

where \(C\) is the Euler constant:
\[
C = -\int_{0}^{\infty} \ln x \cdot e^{-x} \, dx \approx 0.577.
\]

Using (C3) - (C7) we obtain
\[
\eta_y = 2\pi h \gamma \frac{|a|}{b} \left( \frac{1}{2} \ln \frac{s^{3/4}u}{4u^2} + \frac{C_y}{\pi} - C \right). \tag{C8}
\]

The component \(\eta_x\) can be calculated in a similar way:
\[
\eta_x = 2\pi h \gamma \frac{|a|}{b} \left( \frac{1}{2} \ln \frac{s^{3/4}u}{4u^2} + \frac{C_x}{\pi} - C \right), \tag{C9}
\]

\[
C_x = \lim_{y_0 \to \infty} \left( \int_{|y| < y_0} f^2(\rho) \frac{y^2}{\rho^3} \, dx \, dy - \pi \ln y_0 \right). \tag{C10}
\]

Now we evaluate \(C_x\) and \(C_y\). Here the constant \(g_4'\) from Ref. 7 will be useful:
\[
g_4' = \int_{0}^{\infty} \left[ f^2(\rho) - \frac{\rho^2}{1 + \rho^2} \right] \rho^{-1} \, d\rho = -0.3982. \tag{C11}
\]

It is easy to check that
\[
C_x = \pi g^4 + \lim_{y_0 \to \infty} \int_{|y| < y_0, \rho > y_0} \frac{y^2 \rho^4}{4} \, dx \, dy = \pi \left( g_4' + \ln 2 - \frac{1}{2} \right). \tag{C12}
\]

Similarly,
\[
C_y = \pi \left( g_4' + \ln 2 + \frac{1}{2} \right). \tag{C12}
\]

Finally, the components of the viscosity take the form
\[
\eta_x = 2\pi h \gamma \frac{|a|}{b} \left( \ln \frac{s^{3/4}u}{4u^2} + g_4' - C - \frac{1}{2} \right), \tag{C12}
\]

\[
\eta_y = 2\pi h \gamma \frac{|a|}{b} \left( \frac{1}{2} \ln \frac{s^{3/4}u}{4u^2} + g_4' - C - \frac{1}{2} \right). \tag{C12}
\]
\[
\eta_y = 2\pi \hbar \gamma \frac{|a|}{b} \left( \ln \frac{\sqrt{s}}{u} + g'_4 - C + \frac{1}{2} \right). \quad (C13)
\]

If we substitute \(C\) and \(g'_4\) with their numerical values, we obtain Eqs. (33) and (34).

Now it is necessary to prove our assumption concerning the rhs of Eq. (C1). Consider it as a perturbation. The first order correction to the approximate solution \(\Phi_{y0}\) has the form

\[
\Phi_{y1} = R^{(1)}_{y} + R^{(2)}_{y},
\]

\[
R^{(1)}_{y} = \int_{-\infty}^{+\infty} u \Phi_{y0}(x', y) [1 - f^2(\rho')] \exp\left( -\frac{u |x - x'|}{\sqrt{s}} \right) dx',
\]

\[
R^{(2)}_{y} = \frac{1}{2u\sqrt{s}} \int_{-\infty}^{+\infty} \partial^2 \Phi_{y0}(x', y) \exp\left( -\frac{u |x - x'|}{\sqrt{s}} \right) dx'.
\]

The contribution of \(\Phi_{y1}\) to \(\eta_y\) is equal to

\[
\Delta \eta_y = -2\frac{|a|}{b} \hbar (I^{(1)}_{y} + I^{(2)}_{y}),
\]

where

\[
I^{(1)}_{y} = \int f^2(\rho) \frac{x}{\rho^2} u^2 R^{(1)}_{y} dx dy,
\]

\[
I^{(2)}_{y} = \int f^2(\rho) \frac{x}{\rho^2} u^2 R^{(2)}_{y} dx dy.
\]

We will show that \(|I^{(1)}_{y}| \ll 1\) and \(|I^{(2)}_{y}| \ll 1\) when \(s \gg u^2\).

A simple estimate for \(|\Phi_{y0}|\) can be obtained with the help of (C4):

\[
|\Phi_{y0}| \leq \text{const} \ln \frac{\sqrt{s}}{u}.
\]

Using the inequality

\[
\frac{\rho^2}{A_2 + \rho^2} < f^2(\rho)
\]

and Eq. (C18) we can estimate \(I^{(1)}_{y}\):

\[
|I^{(1)}_{y}| \leq \text{const} \frac{u}{\sqrt{s}} \ln \frac{\sqrt{s}}{u} \ll 1.
\]

For all \(x'\) and \(y\) we can write

\[
\left| \frac{\partial^2}{\partial y^2} \left( \frac{1}{x'^2 + y^2} f^2(\rho') \right) \right| \leq \frac{\text{const}}{(A_3 + x'^2 + y^2)^2},
\]

whence

\[
|I^{(2)}_{y}| \leq \text{const} \int \frac{|x| \exp\left( -\frac{u |x - x'|}{\sqrt{s}} \right)}{(A_3 + x'^2 + y^2)(A_4 + x'^2)} dx' dx'' dx.
\]

\[
\leq \text{const} \frac{u}{\sqrt{s}} \int \frac{\exp\left( -\frac{|x'|^2 - |x''|^2}{\sqrt{s}} \right)}{(x'^2 + A_3)} dx' dx'' \leq \frac{\text{const}}{\sqrt{s}u} \ln \frac{\sqrt{s}}{u} \ll 1
\]

when \(u \gtrsim 1\).

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