Spectrum Structure and Behaviors of the Vlasov-Maxwell-Boltzmann Systems

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Abstract

The spectrum structures and behaviors of the Vlasov-Maxwell-Boltzmann (VMB) systems for both two species and one species are studied in this paper. The analysis shows the effect of the Lorentz force induced by the electro-magnetic field leads to some different structure of spectrum from the classical Boltzmann equation and the closely related Vlasov-Poisson-Boltzmann system. And the significant difference between the two-species VMB model and one-species VMB model are given. The structure in high frequency illustrates the hyperbolic structure of the Maxwell equation. Furthermore, the long time behaviors and the optimal convergence rates to the equilibrium of the Vlasov-Maxwell-Boltzmann systems for both two species and one species are established based on the spectrum analysis, and in particular the phenomena of the electric field dominating and magnetic field dominating are observed for the one-species Vlasov-Maxwell-Boltzmann system.

Key words. Vlasov-Maxwell-Boltzmann system, Lorentz force, spectrum structure, optimal convergence rates.

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1 Introduction

The Vlasov-Maxwell-Boltzmann system is a fundamental model in plasma physics for describing the time evolution of dilute charged particles, such as electrons and ions, under the influence of the self-induced Lorentz forces governed by Maxwell equations, cf. [2] for derivation and the physical background. In the literatures, there are two basic models, one is called two-species Vlasov-Maxwell-Boltzmann system that describes both the forces governed by Maxwell equations, and the other one called one-species Vlasov-Maxwell-Boltzmann system that takes account of the fact that the ion is much heavier than the electron and the electron moves faster than the ion so that the time evolution of electron can be considered under a fixed background of ion distribution.

\begin{align}
\begin{aligned}
\partial_t F_+ + v \cdot \nabla_x F_+ + (E + v \times B) \cdot \nabla_v F_+ &= Q(F_+, F_+) + Q(F_+, F_-), \\
\partial_t F_- + v \cdot \nabla_x F_- - (E + v \times B) \cdot \nabla_v F_- &= Q(F_-, F_+) + Q(F_-, F_-), \\
\partial_t E &= \nabla_x \times B - \int_{\mathbb{R}^3} (F_+ - F_-)dv, \\
\partial_t B &= -\nabla_x \times E, \\
\nabla_x \cdot E &= \int_{\mathbb{R}^3} (F_+ - F_-)dv, \\
\nabla_x \cdot B &= 0,
\end{aligned}
\end{align}

where $F_\pm = F_\pm(t,x,v)$ are number density distribution functions of charged particles, and $E(t,x), B(t,x)$ denote the electro and magnetic fields, respectively. Here, the operator $Q(\cdot, \cdot)$ describing the binary elastic collisions is given by

\begin{align}
Q(F,G) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} [(v - v_+) \cdot \omega](F(v')G(v') - F(v_+)G(v))dv_+d\omega,
\end{align}

where $v' = v - [(v - v_+) \cdot \omega]\omega$, $v_+ = v + [(v - v_+) \cdot \omega]\omega$, $\omega \in \mathbb{S}^2$.

The other one called one-species Vlasov-Maxwell-Boltzmann system that takes account of the fact that the ion is much heavier than the electron and the electron moves faster than the ion so that the time evolution of electron can be considered under a fixed background of ion distribution.

\begin{align}
\begin{aligned}
\partial_t F + v \cdot \nabla_x F + (E + v \times B) \cdot \nabla_v F &= Q(F, F), \\
\partial_t E &= \nabla_x \times B - \int_{\mathbb{R}^3} Fdv, \\
\partial_t B &= -\nabla_x \times E, \\
\nabla_x \cdot E &= \int_{\mathbb{R}^3} Fdv - n_0, \\
\nabla_x \cdot B &= 0,
\end{aligned}
\end{align}
where the time evolution of electrons is considered under the influence of a fixed ions background \( n_b(x) \), and \( F = F(x, v, t) \) is the number density function of electrons. And the operator \( Q(\cdot, \cdot) \) is defined by \( \| \cdot \|_1 \). The Vlasov-Maxwell-Boltzmann system has been intensively studied and many important progress has been made in \cite{6} \cite{3} \cite{10} \cite{11} \cite{19}. For instance, Guo \cite{11} has first established the global existence of classical solutions in three-dimensional torus when the initial data is a small perturbation of a global Maxwellian, and Strain \cite{19} proved the corresponding global existence result of classical solutions in \( \mathbb{R}^3 \). The diffusive limit for two-species VMB system was shown in \cite{11}. The recent important investigation of long time behavior of global solution near the global Maxwellian studied in \cite{2} \cite{5} shed light on the complicity of the Valslov-Maxwell-Boltzmann (VMB) system. Therein, it was shown by the method of compensated functions in \cite{5} that the total energy of the linearized one-species VMB system decays at the rate \( (1 + t)^{-\frac{1}{4}} \) (but the decay rate of the nonlinear system has not been obtained since the decay rate of the linear system obtained therein seems to be not enough to deal with the time evolution of the nonlinear terms) and in \cite{17} that the total energy of nonlinear two-species VMB system decays at the rate \( (1 + t)^{-\frac{1}{4}} \).

A natural and interesting question follows then, what is the main characters of the structures and time-asymptotical behaviors of the Valslov-Maxwell-Boltzmann system on the transport of charged particles under the influence of electromagnetic fields governed by the Maxwell equation and/or mutual interaction between charged particles. One of the methods to investigate this properties is the analysis of spectrum structures of the Valslov-Maxwell-Boltzmann system. However, in contrast to the works on Boltzmann equation \cite{8} \cite{15} \cite{16} \cite{20} \cite{21} \cite{22} and Vlasov-Poisson-Boltzmann system \cite{13} \cite{14}, the spectrum of the linearized Valslov-Maxwell-Boltzmann system around a global equilibrium has not been given despite of its importance. The main purpose of this paper is to fill in this gap.

The main purpose of this paper is to consider the structures of the linearized systems for the above VMB systems \cite{11} and \cite{13} around a global equilibrium so that some specific properties influenced by the electromagnetic fields and/or the mutual interaction between charged particles are revealed. These spectrum structures are important for understanding the behavior of the solutions to these systems both locally in space and globally in time. Indeed, the main purpose in the present paper is to continue the project to study the structures and behaviors on the transport of charged particles under the influence of electric fields, electromagnetic fields, or magnetic field. As it has already been studied in \cite{5} \cite{13} \cite{14} about the structure structures and behaviors of both one-species and two-species Vlasov-Poisson-Boltzmann systems, the influence of electric field gives rise to some complicated phenomena on the transport of charged particles. Indeed, it was shown in \cite{5} \cite{13} that the global distribution function to the one-species Vlasov-Poisson-Boltzmann system tends to the global Maxwellian at the optimal rate \( (1 + t)^{-\frac{1}{4}} \) in \( \mathbb{R}^3 \) which is slower than the Boltzmann equation and is caused mainly due to the slower but optimal decay \( (1 + t)^{-\frac{1}{4}} \) of the electric fields. On the other hand, with the influence of mutual interaction of charged particles included, the global distribution function and electric field of the two-species Vlasov-Poisson-Boltzmann system was proved in \cite{23} \cite{14} to converge to the equilibrium at the optimal rate \( (1 + t)^{-\frac{1}{4}} \) for the distribution functions and \( e^{-\mathcal{O}(1) t} \) for the electric field, where the key issue lies in the fact that the mutual interaction of charged particles leads to spectral gap.

In the present paper, we shall establish the structure of the spectrum in details for both two-species and one-species linearized VMB systems, analyze the corresponding semigroups to the linearized operators and show the optimal decay rates of global solutions to the linearized VMB systems, and finally obtain the (optimal) time-asymptotical behaviors of global solutions to the nonlinear VMB systems of both two-species and one-species types. To be more precisely, we first establish the structure of the spectrum in details for two linearized VMB systems and reveal the effect of electromagnetic fields on the distribution of spectrum of linearized operators. This effect gives rise to a completely different distribution of the spectrum of the linearized operators for both two-species and one-species VMB system. Indeed, the influence of electromagnetic fields on the transport of one-species charged particles causes the linearized VMB system admits higher order eigenvalues (spectrum) \( \lambda_6 = \lambda_7 = -\mathcal{O}(1)|\xi|^4 \) at lower frequency \( 0 < |\xi| \ll 1 \) besides those behaving like \( \lambda_j = ji - \mathcal{O}(1)|\xi|^2 \) for \( j = 0, \pm 1 \).
(refer to Theorem 2.3). This is carried out in terms of the relation between the rotational part of macroscopic velocity vector field and the electromagnetic fields and in turn causes the slower time-convergence rates of the global distribution to the global Maxwellian (refer to Theorems 2.10-2.12 for details). However, the mutual interaction of particles with different type of charges cancels this particular influence of electromagnetic fields and only the spectrum like $\lambda_j = -O(1)|\xi|^2$ is kept finally. In addition, the appearance of electromagnetic fields causes the additional spectrum (eigenvalues) around $\pm i|\xi|$ and in particular Re$\lambda = -O(1)|\xi|^{-1}$ at high frequency $|\xi| \gg 1$ for both one-species and two species VMB systems. This unfortunately leads to the loss of regularity of global solutions (refer to Theorem 2.2 and Theorem 2.3 for details).

Then, in terms of the analysis on spectrum structures and the semigroups of both two-species and one-species linearized VMB systems, we are further able to establish the optimal time convergence rates of the global solutions for both linearized systems and nonlinear systems in three-dimensional whole space. For two-species VMB system, we can observe some phenomena of wave propagation and magnetic field domination on long time behaviors of charge transport of charge transport due to the effect of magnetic filed and mutual interaction between the particles of two species. Indeed, for the global solution $(f_1,f_2,E,B)$ to the linearized two-species VMB system, we can show that the distribution function $f_1$, corresponding to the total summation of the distribution functions between the two species, is governed by the linearized Boltzmann equation and its optimal time decay rate $(1 + t)^{-\frac{5}{2}}$ in $L^2$-norm has been already established for instance in [20, 25]. Meanwhile, the magnetic field $B$ is also shown to tend to zero at the optimal time decay rates $(1 + t)^{-\frac{3}{2}}$ in $L^2$-norm, but the distribution function $f_2$, corresponding to the difference of the distribution functions between the two species, and the electric field $E$ decay at the faster optimal time rate $(1 + t)^{-\frac{3}{2}}$ in $L^2$-norm. In particular, the macroscopic part of the distribution function $f_2$ decay at exponentially and and microscopic part of the distribution function $f_2$ decays at the optimal rate $(1 + t)^{-\frac{3}{2}}$ (refer to Theorem 2.4 for details). Here we recall that the macroscopic part and microscopic part related to the distribution function $f_1$ decay at the different optimal rates $(1 + t)^{-\frac{3}{2}}$ and $(1 + t)^{-\frac{5}{2}}$ respectively as shown in [25]. These optimal algebraic time decay rates also established for the distribution function $(f_1,f_2,E,B)$ to the nonlinear two-species VMB system, and we are able to show that the distribution function $f_2$ and the electric field $E$ converge to zero state at the faster optimal time rate $(1 + t)^{-\frac{3}{2}}$ than the optimal rate $(1 + t)^{-\frac{3}{2}}$ for the function $f_1$ and the electric field $B$ (refer to Theorem 2.5 and Theorem 2.6 for details).

For one-species VMB system, some more subtle phenomena on long time behaviors of charge transport of charge transport are observed. Indeed, we can show that there are different long time behaviors of global solutions to one-species VMB system characterized and dominated by the effect of either the magnetic field or the electric field, which depends on whether the relation $\nabla_x \cdot E_0 = n_0$ holds or not with $n_0$ denoting the first moment of the initial distribution. We fist prove the phenomena of electric field dominating in the case that $\nabla_x \cdot E_0 \neq n_0$ for the linearized VMB system. Namely, we show that both the distribution function and the electric field tend to the equilibrium state at the optimal decay rate $(1 + t)^{-\frac{3}{2}}$ in $L^2$-norm which is slower than the faster optimal convergence rate $(1 + t)^{-\frac{5}{2}}$ in $L^2$-norm of the magnetic field to equilibrium state, and in particular the macroscopic density, momentum and energy, corresponding to the first three moments of the distribution function $f$, decay at the optimal rates $(1 + t)^{-\frac{3}{2}}$, $(1 + t)^{-\frac{1}{2}}$, and $(1 + t)^{-\frac{5}{2}}$ respectively (refer to Theorem 2.10 for details). This phenomena of electric field dominating has not been observed before. However, one can not extend these linear theory to the nonlinear VMB system although the global existence of strong solution can be established (refer to Theorem 2.12 for details), because these optimal time decay rates are too weak to be employed to control the expected long time rates of nonlinear terms (refer to Remark 2.14 for some verification).

In the case that $\nabla_x \cdot E_0 = n_0$ we prove the phenomena of magnetic field dominating for both linearized and nonlinear VMB system. Indeed, we are able to show that the magnetic field tend to zero at the optimal time rate $(1 + t)^{-\frac{3}{2}}$ in $L^2$-norm which is slower than the optimal time decay rate $(1 + t)^{-\frac{5}{2}}$ of the distribution function and the optimal time decay rate $(1 + t)^{-\frac{3}{2}}$ of the electric field in $L^2$-norm. In particular, the macroscopic
density, momentum and energy related to the distribution function $f$ are shown to decay at the different optimal rates $(1 + t)^{-\frac{1}{4}}$, $(1 + t)^{-\frac{1}{2}}$ and $(1 + t)^{-\frac{3}{4}}$ respectively (refer to Theorem 2.11 for details). Furthermore, we can also study rigorously the time-asymptotical behaviors of global solutions to the nonlinear VMB system and in particular obtain the optimal time decay rate $(1 + t)^{-\frac{1}{4}}$ of the distribution function $f$, the optimal time decay rate $(1 + t)^{-\frac{3}{4}}$ of the magnetic fields $B$, and the faster time decay rate $(1 + t)^{-\frac{1}{2}} \ln(1 + t)$ of electric field $E$ (refer to Theorem 2.12 for details). This gives more time information than those obtained by energy method in [7, 6] where only the upper bound of the decay rate of total energy was obtained. Here, we should mention that the time-convergence rate $(1 + t)^{-\frac{1}{4}}$ of the total energy of global solution for linearized one-species VMB system in [7] indeed corresponds to the case of the magnetic field dominating phenomena but without the above analysis in details made in the present paper, and the results corresponding to the electric field dominating phenomena is never observed before. In particular, we obtain the optimal decay rates of the solution to the nonlinear one-species VMB system (refer to Theorem 2.12 for details) which was not solved in [7].

The rest of this paper will be organized as follows. In Section 2 the main results on spectrum structures and time-asymptotic behaviors of global solutions for two-species VMB and one-species VMB are stated in section 2.1 and section 2.2 respectively. In Section 3 and Section 4 the spectrum structures of the two linearized systems for both two-species and one-species charge motion will be analyzed with detailed description in low and high frequency regions. Based on this analysis on the linearized operators, the decomposition of the corresponding semigroups generated by these operators will be given in Section 5 together with the optimal convergence rates to the equilibrium in time. The optimal convergence rates of the global solution to the original nonlinear system will be studied in the last section.

2 Main results

2.1 Two-species VMB system

We first consider the Cauchy problem for the two-species Valsov-Maxwell-Boltzmann system (1.1) as follows

\[
\begin{align*}
\partial_t F_+ + v \cdot \nabla_x F_+ + (E + v \times B) \cdot \nabla_v F_+ &= \mathcal{Q}(F_+, F_+) + \mathcal{Q}(F_+, F_-), \\
\partial_t F_- + v \cdot \nabla_x F_- - (E + v \times B) \cdot \nabla_v F_- &= \mathcal{Q}(F_-, F_+) + \mathcal{Q}(F_-, F_-), \\
\partial_t E &= \nabla_x \times B - \int_{\mathbb{R}^3} (F_+ - F_-) v dv, \\
\partial_t B &= -\nabla_x \times E, \\
\nabla_x \cdot E &= \int_{\mathbb{R}^3} (F_- - F_+) dv, \quad \nabla_x \cdot B = 0, \\
F_{\pm}(0, x, v) &= F_{\pm,0}(x, v), \quad E(0, x) = E_0(x), \quad B(0, x) = B_0(x).
\end{align*}
\]

(2.4)

In order to study the spectrum structure of the system (2.4), it is convenient to consider the following system for $F_1 = F_+ + F_-$ and $F_2 = F_+ - F_-$, that takes care of the cancellation in the original system:

\[
\begin{align*}
\partial_t F_1 + v \cdot \nabla_x F_1 + (E + v \times B) \cdot \nabla_v F_1 &= \mathcal{Q}(F_1, F_1), \\
\partial_t F_2 + v \cdot \nabla_x F_2 + (E + v \times B) \cdot \nabla_v F_1 &= \mathcal{Q}(F_2, F_1), \\
\partial_t E &= \nabla_x \times B - \int_{\mathbb{R}^3} F_2 dv, \\
\partial_t B &= -\nabla_x \times E, \\
\nabla_x \cdot E &= \int_{\mathbb{R}^3} F_2 dv, \quad \nabla_x \cdot B = 0, \\
F_1(0, x, v) &= F_{1,0} = F_{+,0} + F_{-,0}, \quad F_2(0, x, v) = F_{2,0} = F_{+,0} - F_{-,0}.
\end{align*}
\]

(2.5)
In the following, we will consider the spectrum of the operator by linearizing the system \((2.5)\) around an equilibrium state \((F_1^*, F_2^*, E^*, B^*) = (M(v), 0, 0, 0)\) with \(M(v)\) being the normalized Maxwellian given by
\[
M = M(v) = \frac{1}{(2\pi)^{3/2}} e^{-|v|^2/2}, \quad v \in \mathbb{R}^3.
\]
Set
\[
F_1 = M + \sqrt{M} f_1, \quad F_2 = \sqrt{M} f_2.
\]
Then the system \((2.5)\) for \((F_1, F_2, E, B)\) can be written as the following system for \((f_1, f_2, E, B)\):
\[
\begin{align*}
\partial_t f_1 + v \cdot \nabla_x f_1 - L f_1 &= \frac{1}{2} (v \cdot E) f_2 - (E + v \times B) \cdot \nabla_v f_2 + \Gamma(f_1, f_1), \\
\partial_t f_2 + v \cdot \nabla_x f_2 - L_1 f_2 - v \cdot \nabla M \cdot E &= \frac{1}{2} (v \cdot E) f_1 - (E + v \times B) \cdot \nabla_v f_1 + \Gamma(f_2, f_1), \\
\partial_t E &= \nabla_x \times B - \int_{\mathbb{R}^3} f_2 v \sqrt{M} dv, \\
\partial_t B &= -\nabla_x \times E, \\
f_1(0, x, v) &= \frac{F_{1,0} - M}{\sqrt{M}}, \quad f_2(0, x, v) = \frac{F_{2,0}}{\sqrt{M}}, \quad E(0, x) = E_0, \quad B(0, x) = B_0,
\end{align*}
\]
where
\[
L f = \frac{1}{\sqrt{M}} [Q(M, \sqrt{M} f) + Q(\sqrt{M} f, M)], \\
L_1 f = \frac{1}{\sqrt{M}} Q(\sqrt{M} f, M), \quad \Gamma(f, g) = \frac{1}{\sqrt{M}} Q(\sqrt{M} f, \sqrt{M} g).
\]
The linearized collision operators \(L\) and \(L_1\) can be written as, cf. [11 24],
\[
(L f)(v) = (K f)(v) - \nu(v) f(v), \quad (L_1 f)(v) = (K_1 f)(v) - \nu(v) f(v),
\]
\[
\nu(v) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} |(v - v_*) \cdot \omega| M_\omega d\omega dv_*,
\]
\[
(K f)(v) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} |(v - v_*) \cdot \omega| (\sqrt{M} f^* + \sqrt{M} f^*_*) \sqrt{M_\omega} d\omega dv_*
\]
\[
= \int_{\mathbb{R}^3} k(v, v_*) f(v_*) dv_*
\]
\[
(K_1 f)(v) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} |(v - v_*) \cdot \omega| \sqrt{M} \sqrt{M} f^* \sqrt{M_\omega} d\omega dv_*
\]
\[
= \int_{\mathbb{R}^3} k_1(v, v_*) f(v_*) dv_*
\]
where \(\nu(v)\) is called the collision frequency, \(K\) and \(K_1\) are self-adjoint compact operators on \(L^2(\mathbb{R}^3)\) with real symmetric integral kernels \(k(v, v_*)\) and \(k_1(v, v_*)\). The null space of the operator \(L\), denoted by \(N_0\), is a subspace spanned by the orthonormal basis \(\{\chi_j, j = 0, 1, \cdots, 4\}\) given by
\[
\chi_0 = \sqrt{M}, \quad \chi_j = v_j \sqrt{M} \quad (j = 1, 2, 3), \quad \chi_4 = \frac{(|v|^4 - 3) \sqrt{M}}{\sqrt{6}};
\]
and the null space of the operator \(L_1\), denoted by \(N_1\), is spanned only by \(\sqrt{M}\).

For later use, denote by \(L^2(\mathbb{R}^3)\) the Hilbert space of complex valued functions on \(\mathbb{R}^3\) with the inner product and the norm given by
\[
(f, g) = \int_{\mathbb{R}^3} f(v) g(v) dv, \quad \|f\| = \left( \int_{\mathbb{R}^3} |f(v)|^2 dv \right)^{1/2}.
\]
And let \(P_0, P_3\) be the projection operators from \(L^2(\mathbb{R}^3)\) to the subspace \(N_0, N_1\) with
\[
P_0 f = \sum_{i=0}^{4} (f, \chi_i) \chi_i, \quad P_1 = I - P_0,
\]
\[
(2.13)
\]
Without the loss of generality, we choose \( \nu \) where the operators \( B \) and \( D \) are given by
\[
(L f, f) = -\mu \| P_1 f \|^2, \quad f \in D(L),
\]
\[
(L_1 f, f) = -\mu \| P_r f \|^2, \quad f \in D(L_1),
\]
where \( D(L) \) and \( D(L_1) \) are the domains of \( L \) and \( L_1 \) given by
\[
D(L) = D(L_1) = \{ f \in L^2(\mathbb{R}^3) \mid \nu(v) f \in L^2(\mathbb{R}^3) \}. 
\]
In addition, for the hard sphere model, \( \nu \) satisfies
\[
\nu_0(1 + |v|) \leq \nu(v) \leq \nu_1(1 + |v|).
\]
Without the loss of generality, we choose \( \nu(0) \geq \nu_0 \geq \mu > 0 \) throughout this paper.

From the system (2.3)–(2.10) for \( (f_1, f_2, E, B) \), we have the following decoupled linearized system for \( f_1 \) and \( (f_2, E, B) \):
\[
\partial_t f_1 + v \cdot \nabla_x f_1 - L f_1 = 0, \quad \partial_t f_2 + v \cdot \nabla_x f_2 - L_1 f_2 - v \sqrt{M} \cdot E = 0, \quad \partial_t E = \nabla_x \times B - \int_{\mathbb{R}^3} f_2 v \sqrt{M} dv, \quad \partial_t B = -\nabla_x \times E, \quad \nabla_x \cdot E = \int_{\mathbb{R}^3} f_2 \sqrt{M} dv, \quad \nabla_x \cdot B = 0.
\]
The equation (2.18) is simply the linearized Boltzmann equation around a global Maxwellian so that its spectrum structure is well established since 1970s. Therefore, we only need to study the spectrum structure of the linear system (2.19)–(2.22) on \( (f_2, E, B) \).

For convenience of notations, rewrite the linearized system for \( f_1 \in L^2(\mathbb{R}^3 \times \mathbb{R}^3_0) \) and \( U = (f_2, E, B)^T \in L^2(\mathbb{R}^3 \times \mathbb{R}^3_0) \times L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3) \) as
\[
\partial_t f_1 = B_0 f_1, \quad f_1(0, x, v) = f_{1,0}(x, v),
\]
and
\[
\begin{aligned}
\partial_t U &= A_0 U, \quad t > 0, \\
\nabla_x \cdot E &= (f_2, \sqrt{M}), \quad \nabla_x \cdot B = 0, \\
U(0, x, v) &= U_0(x, v) = (f_{2,0}, E_0, B_0),
\end{aligned}
\]
where the operators \( B_0 \) and \( A_0 \) are operators on \( L^2(\mathbb{R}^3 \times \mathbb{R}^3_0) \) and \( L^2(\mathbb{R}^3 \times \mathbb{R}^3_0) \times L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3) \) defined by
\[
B_0 = L - (v \cdot \nabla_x),
\]
\[
A_0 = \begin{pmatrix} L_1 - (v \cdot \nabla_x) & v \sqrt{M} & 0 \\ -P_m & 0 & \nabla_x \times \\ 0 & -\nabla_x \times & 0 \end{pmatrix},
\]
with
\[
P_m f = (f, \nu \sqrt{M}).
\]
Take the Fourier transform in (2.24) with respect to $x$ to have
\[
\begin{cases}
\partial_t \hat{U} = \hat{A}_0(\xi)\hat{U}, & t > 0, \\
i(\xi \cdot \hat{E}) = (f_2, \sqrt{M}), & i(\xi \cdot \hat{B}) = 0, \\
\hat{U}(0, \xi, v) = \hat{U}_0(\xi, v) = (f_{2,0}, \hat{E}_0, \hat{B}_0),
\end{cases}
\] (2.28)
where $\hat{A}_0(\xi)$ is the Fourier transform of $A_0$.

Note that it is difficult to study the spectrum structure of the system (2.28) directly because of the constraints on $\hat{E}$ and $\hat{B}$ given in the second and third equations. One of the key observations in this paper is that by using the identity $F = (F \cdot y)y - y \times y \times F$ for any $F \in \mathbb{R}^3$ and $y \in \mathbb{S}^2$, we can firstly solve for $\hat{V} = (\hat{f}_2, \omega \times \hat{E}, \omega \times \hat{B})$ with $\omega = \xi/|\xi|$ so that by combining these two constraints, we have the full information on $\hat{U}$. For this reason, we consider the following system for $\hat{V}$:
\[
\begin{cases}
\partial_t \hat{V} = \hat{A}_1(\xi)\hat{V}, & t > 0, \\
\hat{V}(0, \xi, v) = \hat{V}_0(\xi, v) = (\hat{f}_{2,0}, \omega \times \hat{E}_0, \omega \times \hat{B}_0),
\end{cases}
\] (2.29)
with
\[
\hat{A}_1(\xi) = \left( \begin{array}{ccc}
\hat{B}_1(\xi) & -v\sqrt{M} \cdot \omega \times & 0 \\
-\omega \times P_m & 0 & i\xi \times \\
0 & -i\xi \times & 0
\end{array} \right).
\] (2.30)
Here, for $\xi \neq 0$,
\[
\hat{B}_1(\xi) = L_1 - i(v \cdot \xi) - \frac{i(v \cdot \xi)}{|\xi|^2} P_d.
\] (2.31)

Before further discussion, we will give a remark on the eigenvalues and eigenfunctions of the original system and the reduced system (2.29).

**Remark 2.1.** Set $F_+ = \frac{1}{2}M + \sqrt{M}f_+$, $F_- = \frac{1}{2}M + \sqrt{M}f_-$ to have
\[
\begin{cases}
\partial_t f_+ + v \cdot \nabla_x f_+ - \frac{1}{2}(L \pm L_1)f_+ - \frac{1}{2}f_-(L \mp L_1)f_+ + \frac{1}{2}v\sqrt{M} \cdot E = 0, \\
\partial_t E = \nabla_x \cdot B - \int_{\mathbb{R}^3} (f_+ - f_-)u\sqrt{M}dv, & \partial_t B = -\nabla_x \times E,
\end{cases}
\] (2.32)
\[
\nabla_x \cdot E = \int_{\mathbb{R}^3} (f_+ - f_-)\sqrt{M}dv, & \nabla_x \cdot B = 0.
\]

The eigenvalues of the system (2.32) are same as those of (2.23) and (2.29), and the eigenfunctions of the system (2.32) can be obtained as linear combinations of those for (2.23) and (2.29). In fact, let $\lambda$ be an eigenvalue with the corresponding eigenfunction denoted by $\phi$ of (2.23), and $\beta$ be an eigenvalue with the corresponding eigenvector denoted by $\Psi = (\psi, E, B)$ of (2.29). Then $U = (\phi, \psi, 0, 0)$ is the corresponding eigenvector with the eigenvalue $\lambda$, and $V = (\psi, -\psi, -\frac{i}{|\xi|^2}(\psi, \chi_0) - \frac{i}{|\xi|^2} \times E, -\frac{i}{|\xi|^2} \times B)$ is the corresponding eigenvector with the eigenvalue $\beta$.

By the above argument, from now on, we will focus on the system (2.29). It is interesting to find out that the spectrum structure depends on the decomposition of the asymptotics in low frequency and high frequency like the classical Boltzmann equation and the Vlasov-Poisson-Boltzmann system, but depends on the low-intermediate-high frequencies. This is mainly due to the hyperbolic structure of the Maxwell equations, in particular the effect of the magnetic field on the velocity field. More precisely, the spectrum of linearized operator contain an eigenvalue in low frequency located in a small neighborhood of the origin, and two eigenvalues in high frequency located in two small neighborhoods centered at the points $\lambda = \pm i|\xi|$ respectively. Except these eigenvalues, there is a spectral gap for the intermediate frequency. Precisely, we have the following result on the spectrum structure.
Theorem 2.2. There exist two constants \( r_0 > 0 \) and \( b_2 > 0 \) so that the spectrum \( \lambda \in \sigma(\hat{A}_1(\xi)) \subset \mathbb{C} \) for \( \xi = s\omega \) with \( s = |\xi| \leq r_0 \) and \( \omega \in S^2 \) consists of two points \( \{\lambda_j(s), j = 1, 2\} \) in the domain \( \text{Re}\lambda > -b_2 \). The spectrum \( \lambda_j(s) \) are \( C^\infty \) functions of \( s \) for \( |s| \leq r_0 \). In particular, the eigenvalues admit the following asymptotic expansion for \( |s| \leq r_0 \)

\[
\lambda_1(s) = \lambda_2(s) = -a_1s^2 + o(s^2),
\]

where \( a_1 > 0 \) is a constant defined in Theorem 3.1.

There exists a constant \( r_1 > 0 \) such that the spectrum \( \beta \in \sigma(\hat{A}_1(\xi)) \subset \mathbb{C} \) for \( s = |\xi| > r_1 \) consists of four eigenvalues \( \{\beta_j(s), j = 1, 2, 3, 4\} \) in the domain \( \text{Re}\lambda > -\mu/2 \). In particular, the eigenvalues satisfy

\[
\beta_1(s) = \beta_2(s) = -is + O(s^{-1/2}),
\]
\[
\beta_3(s) = \beta_4(s) = is + O(s^{-1/2}),
\]

\[
\frac{c_1}{s} \leq -\text{Re}\beta_j(s) \leq \frac{c_2}{s},
\]

for two positive constants \( c_1 \) and \( c_2 \).

For any \( r_1 > r_0 > 0 \), there exists \( \alpha = \alpha(r_0, r_1) > 0 \) such that for \( r_0 \leq |\xi| \leq r_1 \),

\[
\sigma(\hat{A}_1(\xi)) \subset \{ \lambda \in \mathbb{C} | \text{Re}\lambda(\xi) \leq -\alpha \}.
\]

Based on the spectrum structure given in the above theorem, the semigroup generated by the linearized operator of the Vlasov-Maxwell-Boltzmann system can be decomposed in three parts according to the frequency in low, high and intermediate regions so that the optimal time decay rates of the solution can be obtained for the linearized system. Note that a higher regularity on the initial data is needed because of the spectrum structure in the high frequency region. With the estimates on the semigroup, the optimal decay in time of the solution to the original nonlinear problem can also be obtained. Before stating results on nonlinear problem, let us first introduce the following notations.

Notations: The Fourier transform of \( f = f(x, v) \) is denoted by \( \hat{f}(\xi, v) = \mathcal{F} f(\xi, v) = \frac{1}{(2\pi)^{3/2}} \int f(x, v)e^{-ix\cdot\xi} \text{d}x \).

Set a weight function \( w(v) \) by

\[
w(v) = (1 + |v|^2)^{1/2},
\]

so that the Sobolev spaces \( H^N \) and \( H^N_w \) are given by

\[
H^N = \{ f \in L^2(\mathbb{R}^3_x \times \mathbb{R}^3_v) | \|f\|_{H^N} < \infty \}, \quad H^N_w = \{ f \in L^2(\mathbb{R}^3_x \times \mathbb{R}^3_v) | \|f\|_{H^N_w} < \infty \},
\]

equipped with the norms

\[
\|f\|_{H^N} = \sum_{|\alpha|+|\beta| \leq N} \|\partial_\alpha^\xi \partial_\beta^v f\|_{L^2(\mathbb{R}^3_x \times \mathbb{R}^3_v)}, \quad \|f\|_{H^N_w} = \sum_{|\alpha|+|\beta| \leq N} \|w\partial_\alpha^\xi \partial_\beta^v f\|_{L^2(\mathbb{R}^3_x \times \mathbb{R}^3_v)}.
\]

For \( q \geq 1 \), denote

\[
L^{2,q} = L^2(\mathbb{R}^3_v, L^q(\mathbb{R}^3_x)), \quad \|f\|_{L^{2,q}} = \left( \int_{\mathbb{R}^3} \left( \int_{\mathbb{R}^3} |f(x, v)|^q \text{d}x \right)^{2/q} \text{d}v \right)^{1/2}.
\]

In the following, denote by \( \| \cdot \|_{L^\infty} \) and \( \| \cdot \|_{L^2} \) the norms of the function spaces \( L^2(\mathbb{R}^3_x \times \mathbb{R}^3_v) \) and \( L^2(\mathbb{R}^3_x \times \mathbb{R}^3_v) \) respectively, and by \( \| \cdot \|_{L^2} \), \( \| \cdot \|_{L^2} \) and \( \| \cdot \|_{L^2} \) the norms of the function spaces \( L^2(\mathbb{R}^3_x), L^2(\mathbb{R}^3_v) \) and \( L^2(\mathbb{R}^3_x, H^m(\mathbb{R}^3_v)) \) respectively. For any integer \( m \geq 1 \), denote by \( \| \cdot \|_{H^m} \) and \( \| \cdot \|_{L^2(H^m)} \) the norms in the spaces \( H^m(\mathbb{R}^3_x) \) and \( L^2(\mathbb{R}^3_x, H^m(\mathbb{R}^3_v)) \) respectively. Moreover, a weighted Hilbert space \( L^2(\mathbb{R}^3_x, H^m(\mathbb{R}^3_v)) \) for \( \xi \neq 0 \) defined by

\[
L^2(\mathbb{R}^3_x, H^m(\mathbb{R}^3_v)) = \{ f \in L^2(\mathbb{R}^3_v) | \|f\|_\xi = \sqrt{\langle f, f \rangle_\xi} < \infty \},
\]
Theorem 2.4. Let $f, g \in L^2(\mathbb{R}^3) \times \mathbb{C}^3 \times \mathbb{C}^3$, define a weighted inner product and the corresponding norm by

$$(U, V)_\xi = (f, g)_\xi + (E_1, E_2) + (B_1, B_2), \quad \|U\|_\xi = \sqrt{(U, U)_\xi},$$

and another $L^2$ inner product and norm by

$$(U, V) = (f, g) + (E_1, E_2) + (B_1, B_2), \quad \|U\| = \sqrt{(U, U)}.$$ For simplicity, denote

$$(U, V) = (f, g) + (E_1, E_2) + (B_1, B_2), \quad \|U\| = \sqrt{(U, U)}.$$ For simplicity, denote

$$\|U\|_\xi^2 = \|f\|_{L^2_{\xi}}^2 + \|E\|_{L^2_{\xi}}^2 + \|B\|_{L^2_{\xi}}^2, \quad \|U\|_{L^2_\xi}^2 = \|f\|_{L^2_\xi}^2 + \|E\|_{L^2_\xi}^2 + \|B\|_{L^2_\xi}^2.$$ With the above preparation, we first state the estimates on the semigroup to linearized system.

**Theorem 2.3.** The semigroup $S(t, \xi) = e^{i\xi t}$ with $\xi = s\omega \in \mathbb{R}^3$ and $s = |\xi| \neq 0$ can be decomposed into

$$S(t, \xi)U = S_1(t, \xi)U + S_2(t, \xi)U + S_3(t, \xi)U, \quad U \in L^2(\mathbb{R}^3) \times \mathbb{C}^3 \times \mathbb{C}^3, \quad t > 0,$$

that has the following properties

$$S_1(t, \xi)U = \sum_{j=1}^2 e^{i\lambda_j(s)}(U, \Phi_j(s, \omega))\Phi_j(s, \omega)1(|\xi| \leq r_0),$$

$$S_2(t, \xi)U = \sum_{j=1}^4 e^{i\beta_j(s)}(U, \Phi_j(s, \omega))\Phi_j(s, \omega)1(|\xi| \geq r_1),$$

where $C^3_\xi = \{y \in \mathbb{C}^3 : y \cdot \xi = 0\}$. Here, $(\lambda_j(s), \Psi_j(s, \omega))$ and $(\beta_j(s), \Phi_j(s, \omega))$ are the eigenvalues and eigenvectors of the operator $\hat{A}_1(\xi)$ in low and high frequency regions with properties given in Theorem 2.19 and Theorem 2.21. And $S_3(t, \xi)U = S(t, \xi)U - S_1(t, \xi)U - S_2(t, \xi)U$ satisfies that there is a constant $\kappa_0 > 0$ independent of $\xi$ such that

$$\|S_3(t, \xi)U\|_\xi \leq C e^{-\kappa_0 t} \|U\|_\xi, \quad t \geq 0.$$ Then, we have the optimal time convergence rates of global solutions to linearized system [224].

**Theorem 2.4.** Let $(f_2(t), E(t), B(t))$ be a solution of the system [224]. If the initial data $U_0 = (f_0, E_0, B_0) \in L^2(\mathbb{R}^3) \times H^1(\mathbb{R}_x^3) \cap L^1(\mathbb{R}_x^3)$ for $l \geq 0$, then it holds for any $\alpha, \alpha' \in \mathbb{N}^3$ with $\alpha' \leq \alpha$ that

$$\|\partial^\alpha_x f_2(t)\|_{L^\infty_{\xi_{2,2}}} \leq C (1 + t)^{-\frac{\alpha}{2} - \frac{\alpha'}{2}} (\|\partial^\alpha_x U_0\|_{Z^2} + \|\partial^\alpha_x' U_0\|_{Z^1}) + C (1 + t)^{-m} \|\nabla_{\xi_{2,2}} \partial^\alpha_x U_0\|_{Z^2},$$

$$\|\partial^\alpha_x E(t)\|_{L^2_x} \leq C (1 + t)^{-\frac{\alpha}{2} - \frac{\alpha'}{2}} (\|\partial^\alpha_x U_0\|_{Z^2} + \|\partial^\alpha_x' U_0\|_{Z^1}) + C (1 + t)^{-m} \|\nabla_{\xi_{2,2}} \partial^\alpha_x U_0\|_{Z^2},$$

$$\|\partial^\alpha_x B(t)\|_{L^2_x} \leq C (1 + t)^{-\frac{\alpha}{2} - \frac{\alpha'}{2}} (\|\partial^\alpha_x U_0\|_{Z^2} + \|\partial^\alpha_x' U_0\|_{Z^1}) + C (1 + t)^{-m} \|\nabla_{\xi_{2,2}} \partial^\alpha_x U_0\|_{Z^2},$$

where $k = |\alpha - \alpha'|$ and $m \geq 0$. In particular, it holds for $f_2 = P_4 f_2 + P_4 f_2$ that

$$\|\partial^\alpha_x P_4 f_2(t)\|_{L^\infty_{\xi_{2,2}}} \leq C e^{-\kappa_0 t} \|\partial^\alpha_x U_0\|_{Z^2},$$

$$\|\partial^\alpha_x P_4 f_2(t)\|_{L^\infty_{\xi_{2,2}}} \leq C (1 + t)^{-\frac{\alpha}{2} - \frac{\alpha'}{2}} (\|\partial^\alpha_x U_0\|_{Z^2} + \|\partial^\alpha_x' U_0\|_{Z^1}) + C (1 + t)^{-m} \|\nabla_{\xi_{2,2}} \partial^\alpha_x U_0\|_{Z^2}.$$ Furthermore, if we assume that $l \geq 2$ and the Fourier transform $\hat{B}_0(\xi)$ of initial magnetic field $B_0(x)$ satisfies that

$$\inf_{|\xi| \leq r_0} |\hat{B}_0(\xi)| \geq d_0 > 0,$$ then

$$C_1 (1 + t)^{-\frac{\alpha}{2}} \leq \|f_2(t)\|_{L^\infty_{\xi_{2,2}}} \leq C_2 (1 + t)^{-\frac{\alpha}{2}},$$
for \( t > 0 \) large enough with \( C_2 \geq C_1 > 0 \) two constants.

Finally for the two-species system, we have the optimal time convergence rates of global solutions to the original nonlinear system \((2.6)\)–\((2.11)\) as follows.

**Theorem 2.5.** When \((f_{1,0}, f_{2,0}) \in H^N_w \cap L^{2,1} \) and \((E_0, B_0) \in H^{N+5}(\mathbb{R}^3_0) \cap L^1(\mathbb{R}^3_0)\) for \( N \geq 4\) satisfying \( \| (f_{1,0}, f_{2,0}) \|_{H^{N+\gamma}} + \| (E_0, B_0) \|_{H^{N+\gamma}(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)} \leq \delta_0 \) with \( \delta_0 > 0 \) being small, there exists a globally unique solution \((f_1, f_2, E, B)\) to the system \((2.6)\)–\((2.11)\) satisfying

\[
\| \partial_x^k f_1(t) \|_{L^2_x} + \| \partial_x f_1(t) \|_{L^2_x} \leq C \delta_0 (1 + t)^{-\frac{k}{2} - \frac{1}{2}},
\]

\[
\| \partial_x^k f_2(t) \|_{L^2_x} + \| \partial_x f_2(t) \|_{L^2_x} \leq C \delta_0 (1 + t)^{-\frac{k}{2} - \frac{1}{2}},
\]

for \( k = 0, 1 \). In particular, it holds for \( f_1 = P_0 f_1 + P_1 f_1 \) and \( f_2 = P_0 f_2 + P_1 f_2 \) that

\[
\| \partial_x^k (f_1(t), \chi_1) \|_{L^2_x} \leq C \delta_0 (1 + t)^{-\frac{k}{2} - \frac{1}{2}}, \quad j = 0, 1, 2, 3, 4,
\]

\[
\| \partial_x^k P_1 f_1(t) \|_{L^2_x} \leq C \delta_0 (1 + t)^{-\frac{k}{2} - \frac{1}{2}},
\]

\[
\| \partial_x^k P_1 f_2(t) \|_{L^2_x} \leq C \delta_0 (1 + t)^{-\frac{k}{2} - \frac{1}{2}},
\]

\[
\| \partial_x^k P_2 f_2(t) \|_{L^2_x} + \| \partial_x f_1(t) \|_{L^2_x} \leq C \delta_0 (1 + t)^{-\frac{k}{2} - \frac{1}{2}},
\]

\[
\| \partial_x^k (P_1 f_1, P_2 f_2)(t) \|_{H^N_w} + \| \nabla_x (P_0 f_1 P_0 f_2)(t) \|_{L^2_x(H^{N+\gamma-1})} + \| \nabla_x (E, B)(t) \|_{H^{N+\gamma-1}} \leq C \delta_0 (1 + t)^{-\frac{k}{2}},
\]

for \( k = 0, 1 \).

**Theorem 2.6.** Under the conditions given in Theorem 2.5, if we further assume that there exist positive constants \( d_0, d_1 > 0 \) and a small constant \( r_0 \) so that the Fourier transform \((f_{1,0}, f_{2,0}, E_0, B_0)\) of the initial data \((f_{1,0}, f_{2,0}, E_0, B_0)\) satisfies that \( \inf_{|\xi| \leq r_0} \| (f_{1,0}(\xi), \chi_0) \| \geq d_0, \inf_{|\xi| \leq r_0} \| (f_{1,0}(\xi), \chi_1) \| \geq d_1 \sup_{|\xi| \leq r_0} \| (f_{1,0}(\xi), \chi_0) \|, \sup_{|\xi| \leq r_0} \| (\hat{f}_{1,0}(\xi), \nu \sqrt{T}) \| = 0 \) and \( \inf_{|\xi| \leq r_0} \| \nu \times B_0(\xi) \| \geq d_0 \), then the global solution \((f_1, f_2, E, B)\) to the system \((2.6)\)–\((2.11)\) satisfies

\[
C_1 \delta_0 (1 + t)^{-\frac{k}{2}} \leq \| f_1(t) \|_{L^2_x} \leq C_2 \delta_0 (1 + t)^{-\frac{k}{2}},
\]

\[
C_1 \delta_0 (1 + t)^{-\frac{k}{2}} \leq \| f_2(t) \|_{L^2_x} \leq C_2 \delta_0 (1 + t)^{-\frac{k}{2}},
\]

\[
C_1 \delta_0 (1 + t)^{-\frac{k}{2}} \leq \| (f_1(t), \chi_1) \|_{L^2_x} \leq C_2 \delta_0 (1 + t)^{-\frac{k}{2}},
\]

\[
C_1 \delta_0 (1 + t)^{-\frac{k}{2}} \leq \| P_1 f_1(t) \|_{L^2_x} \leq C_2 \delta_0 (1 + t)^{-\frac{k}{2}},
\]

\[
C_1 \delta_0 (1 + t)^{-\frac{k}{2}} \leq \| P_2 f_2(t) \|_{L^2_x} \leq C_2 \delta_0 (1 + t)^{-\frac{k}{2}},
\]

for \( t > 0 \) large with two constants \( C_2 > C_1 > 0 \), and in particular

\[
C_1 \delta_0 (1 + t)^{-\frac{k}{2}} \leq \| f_1(t) \|_{L^2_x} \leq C_2 \delta_0 (1 + t)^{-\frac{k}{2}},
\]

\[
C_1 \delta_0 (1 + t)^{-\frac{k}{2}} \leq \| P_1 f_1(t) \|_{L^2_x} \leq C_2 \delta_0 (1 + t)^{-\frac{k}{2}},
\]

\[
C_1 \delta_0 (1 + t)^{-\frac{k}{2}} \leq \| P_2 f_2(t) \|_{L^2_x} \leq C_2 \delta_0 (1 + t)^{-\frac{k}{2}},
\]

for \( j = 0, 1, 2, 3, 4 \).

**Remark 2.7.** From the above theorems, Remark 2.1 and those estimates obtained in \((13)\) \((14)\), we justify the different time-asymptotic phenomena of charged particle transport at mesoscopic level that are determined by the irrotational electric field and electromagnetic field respectively. Indeed, it was shown in \((13)\) that the distribution
function converges to the global Maxwellian at the same optimal rate \((1+t)^{-\frac{1}{4}}\) as the irrotational electric field for the unipolar Vlasov-Poisson-Boltzmann in \(\mathbb{R}^3 \times \mathbb{R}^3\); and in \([14]\) it shows that the distribution function converges to the global Maxwellian at the optimal rate \((1+t)^{-\frac{1}{2}}\), which is slower than the exponential decay rate of the irrotational electric field for bipolar Vlasov-Poisson-Boltzmann equations in \(\mathbb{R}^3 \times \mathbb{R}^3\). While in the appearance of electro-magnetic force, the electric field decays at the optimal rate \((1+t)^{-\frac{1}{2}}\), which is faster than those of the distribution function and the magnetic field. In particular, it is the rotational vector field part of electric field for linear system that decays at algebraic rate, while the gradient vector field part of electric field decays exponentially in time.

### 2.2 One-species VMB system

We now turn to the one-species Vlasov-Maxwell-Boltzmann system where the time evolution of electrons is considered under the influence of a fixed ion background \(n_b(x)\). That is, consider

\[
\begin{align*}
\partial_t F + v \cdot \nabla_x F + (E + v \times B) \cdot \nabla_v F &= Q(F, F), \\
\partial_t E &= \nabla_x \times B - \int_{\mathbb{R}^3} F dv, \\
\partial_t B &= -\nabla_x \times E, \\
\nabla_x \cdot E &= \int_{\mathbb{R}^3} F dv - n_b, \quad \nabla_x \cdot B = 0, \\
F(x, v, 0) &= F_0(x, v), \quad E_0(x, 0) = E_0(x), \quad B_0(0, x) = B_0(x).
\end{align*}
\]

(2.42)

Here, \(F = F(x, v, t)\) is the number density function of electrons, and \(n_b(x) > 0\) is assumed to be a constant representing the spatially uniform density of the ionic background. Take \(n_b = 1\) without loss of generality.

The one-species VMB system \((2.42)\) has a stationary solution \((F^*, E^*, B^*) = (M(v), 0, 0)\) with \(M(v)\) being the normalized global Maxwellian defined above. Set

\[
F = M + \sqrt{M} f.
\]

Then the one-species VMB system \((2.42)\) for \((F, E, B)\) is reformulated in terms of \((f, E, B)\) into

\[
\begin{align*}
\partial_t f + v \cdot \nabla_x f - Lf - v\sqrt{M} \cdot E &= \frac{1}{2}(v \cdot E)f - (E + v \times B) \cdot \nabla_v f + \Gamma(f, f), \\
\partial_t E &= \nabla_x \times B - \int_{\mathbb{R}^3} f v \sqrt{M} dv, \\
\partial_t B &= -\nabla_x \times E, \\
\nabla_x \cdot E &= \int_{\mathbb{R}^3} f \sqrt{M} dv, \quad \nabla_x \cdot B = 0, \\
f(0, x, v) &= F_0 - \sqrt{M} f_0, \quad E_0(x, 0) = E_0(x), \quad B_0(0, x) = B_0(x).
\end{align*}
\]

(2.43) - (2.47)

Consider the linearized one-species Vlasov-Maxwell-Boltzmann system:

\[
\begin{align*}
\partial_t f + v \cdot \nabla_x f - Lf - v\sqrt{M} \cdot E &= 0, \\
\partial_t E &= \nabla_x \times B - \int_{\mathbb{R}^3} f v \sqrt{M} dv, \\
\partial_t B &= -\nabla_x \times E, \\
\nabla_x \cdot E &= \int_{\mathbb{R}^3} f \sqrt{M} dv, \quad \nabla_x \cdot B = 0.
\end{align*}
\]

(2.48) - (2.51)
Theorem 2.8. There exists a constant \( r_0 > 0 \) so that the spectrum \( \lambda \in \sigma(\hat{A}_3(\xi)) \subset \mathbb{C} \) for \( \xi = s\omega \) with \( |s| \leq r_0 \) and \( \omega \in S^2 \) consists of nine points \( \{ \lambda_j(s), -1 \leq j \leq 7 \} \) in the domain \( \text{Re} \lambda > -\mu/2 \). The spectrum \( \lambda_j(s) \) are \( C^\infty \) functions of \( s \) for \( |s| \leq r_0 \). In particular, the eigenvalues \( \lambda_j(s) \) have the following asymptotic expansions when \( |s| \leq r_0 \)

\[
\begin{align*}
\lambda_{\pm 1}(s) &= \pm i + (-a_1 \pm i b_1)s^2 + o(s^2), \quad \overline{\lambda_1} = \lambda_{-1}, \\
\lambda_0(s) &= -a_0s^2 + o(s^2), \\
\lambda_2(s) &= \lambda_3(s) = -i + (-a_2 - i b_2)s^2 + o(s^2), \quad \overline{\lambda_2} = \lambda_4, \\
\lambda_4(s) &= \lambda_5(s) = i + (-a_2 + i b_2)s^2 + o(s^2), \\
\lambda_6(s) &= \lambda_7(s) = -a_3s^4 + o(s^4),
\end{align*}
\]

where \( a_j > 0 \) (0 \( \leq j \leq 3 \)) and \( b_j > 0 \) (1 \( \leq j \leq 2 \)) are constants defined in Theorem 4.7.

There exists a constant \( r_1 > 0 \) such that the spectrum \( \beta \in \sigma(\hat{A}_3(\xi)) \subset \mathbb{C} \) for \( s = |\xi| > r_1 \) consists of four eigenvalues \( \{ \beta_j(s), j = 1, 2, 3, 4 \} \) in the domain \( \text{Re} \beta > -\mu/2 \). In particular, the eigenvalues satisfy

\[
\begin{align*}
\beta_1(s) &= \beta_2(s) = -is + O(s^{-1/2}), \\
\beta_3(s) &= \beta_4(s) = is + O(s^{-1/2}), \\
\frac{c_1}{s} \leq \text{Re} \beta_j(s) \leq \frac{c_2}{s},
\end{align*}
\]

for two positive constants \( c_1 \) and \( c_2 \).

For any \( r_1 > r_0 > 0 \), there exists \( \alpha = \alpha(r_0, r_1) > 0 \) such that for \( r_0 \leq |\xi| \leq r_1 \),

\[\sigma(\hat{A}_3(\xi)) \subset \{ \lambda \in \mathbb{C} \mid \text{Re} \lambda(\xi) \leq -\alpha \}.\]
Based on the spectrum of the linearized operator of one-species VMB system, we are able to analyze the corresponding semigroup for (2.54) below.

**Theorem 2.9.** The semigroup $S(t,\xi) = e^{t\hat{A}(t)}$ with $\xi = s\omega \in \mathbb{R}^3$ and $s = |\xi| \neq 0$ satisfies

$$S(t,\xi)U = S_1(t,\xi)U + S_2(t,\xi)U + S_3(t,\xi)U, \quad U \in L_2^2(\mathbb{R}_x^3) \times C_2^3 \times C_2^3, \quad t > 0,$$

where

$$S_1(t,\xi)U = \sum_{j=1}^{7} e^{\lambda_j(s)}(U, \Psi_j(s,\omega))_\xi \Psi_j(s,\omega)1_{|\xi| \leq r_0},$$

$$S_2(t,\xi)U = \sum_{j=1}^{4} e^{\beta_j(s)}(U, \Phi_j(s,\omega))_\xi \Phi_j(s,\omega)1_{|\xi| \geq r_1}.$$

Here, $(\lambda_j(s), \Psi_j(s,\omega))$ and $(\beta_j(s), \Phi_j(s,\omega))$ are the eigenvalues and eigenvectors of the operator $\hat{A}_3(\xi)$ given in Theorem 1.9 and Theorem 1.10 for $|\xi| \leq r_0$ and $|\xi| > r_1$ respectively, and $S_3(t,\xi)U = S(t,\xi)U - S_1(t,\xi)U - S_2(t,\xi)U$ satisfies that there exists a constant $\kappa_0 > 0$ independent of $\xi$ so that

$$\|S_3(t,\xi)U\|_\xi \leq C e^{-\kappa_0 t} \|U\|_\xi, \quad t \geq 0.$$

Then, we have the optimal time convergence rates of global solutions to linearized system.

**Theorem 2.10** (Electric Field Dominating). If the initial data $U_0 = (f_0, E_0, B_0) \in L^2(\mathbb{R}_x^3) \times H^1(\mathbb{R}_x^2) \times H^1(\mathbb{R}_x^2) \times H^1(\mathbb{R}_x^2)$ for $l \geq 0$ with $\nabla_x \cdot E_0 \neq (f_0, \chi_0)$ being held, then the unique solution $(f(t), E(t), B(t))$ to the system (2.52) satisfies that

$$\|\partial_x^m f(t)\|_{L^2_x,u} \leq C \delta(\alpha, \alpha')(1 + t)^{-\bar{\alpha} - \frac{1}{2} - \frac{1}{2}} + C(1 + t)^{-m - \frac{1}{2}} \|\nabla_x^m \partial_x^0 U_0\|_{L^2_x},$$

$$\|\partial_x^m E(t)\|_{L^2_x,u} \leq C \delta(\alpha, \alpha')(1 + t)^{-\bar{\alpha} - \frac{1}{2} - \frac{1}{2}} + C(1 + t)^{-m - \frac{1}{2}} \|\nabla_x^m \partial_x^0 U_0\|_{L^2_x},$$

$$\|\partial_x^m B(t)\|_{L^2_x,u} \leq C \delta(\alpha, \alpha')(1 + t)^{-\bar{\alpha} - \frac{1}{2} - \frac{1}{2}} + C(1 + t)^{-m - \frac{1}{2}} \|\nabla_x^m \partial_x^0 U_0\|_{L^2_x},$$

where and below $\delta(\alpha, \alpha') := \|\partial_x^0 U_0\|_{L^2_x} + \|\partial_x^0 U_0\|_{L^1_x}, k = |\alpha - \alpha'|$ and $m \geq 0$. In particular, it holds for $f = P_0 f + P_1 f$ that

$$\|\partial_x^m (f(t), \chi_0)\|_{L^2_x,u} \leq C \delta(\alpha, \alpha')(1 + t)^{-\bar{\alpha} - \frac{1}{2}},$$

$$\|\partial_x^m (f(t), v\sqrt{\gamma})\|_{L^2_x,u} \leq C \delta(\alpha, \alpha')(1 + t)^{-\bar{\alpha} - \frac{1}{2} + \frac{1}{2}},$$

$$\|\partial_x^m (f(t), \chi_4)\|_{L^2_x,u} \leq C \delta(\alpha, \alpha')(1 + t)^{-\bar{\alpha} - \frac{1}{2}},$$

$$\|\partial_x^m P_1 f(t)\|_{L^2_x,u} \leq C \delta(\alpha, \alpha')(1 + t)^{-\bar{\alpha} - \frac{1}{2} - \frac{1}{2}} + C(1 + t)^{-m - \frac{1}{2}} \|\nabla_x^m \partial_x^0 U_0\|_{L^2_x}.$$
**Theorem 2.11** (Magnetic Field Dominating). If the initial data $U_0 = (f_0, E_0, B_0) \in L^2(\mathbb{R}^3_x) \times H^1(\mathbb{R}^3_x) \cap L^1(\mathbb{R}^3_x) \times H^1(\mathbb{R}^3_x) \cap L^1(\mathbb{R}^3_x)$ for $l \geq 0$ with $\nabla_x \cdot E_0 = (f_0, \chi_0)$ being held, then there exists globally in time a unique solution $(f(t), E(t), B(t))$ to the system [2.62] which satisfies for any $\alpha, \alpha' \in \mathbb{N}^3$ with $\alpha' \leq \alpha$ that

$$
\|\partial_x^\alpha f(t)\|_{L^2_{x,v}} \leq C\delta(\alpha, \alpha')(1 + t)^{-\frac{\beta}{2} - \frac{1}{2}} + C(1 + t)^{\beta - \frac{1}{2}}\|\nabla_x^m \partial_x^\alpha U_0\|_{L^2_{x,v}},
$$

(2.71)

$$
\|\partial_x^\alpha E(t)\|_{L^2_x} \leq C\delta(\alpha, \alpha')(1 + t)^{-\frac{\beta}{2} - \frac{1}{2}} + C(1 + t)^{-\frac{\beta}{2} - \frac{1}{2}} + C(1 + t)^{-m}\|\nabla_x^m \partial_x^\alpha U_0\|_{L^2_{x,v}},
$$

(2.72)

$$
\|\partial_x^\alpha B(t)\|_{L^2_x} \leq C\delta(\alpha, \alpha')(1 + t)^{-\frac{\beta}{2} - \frac{1}{2}} + C(1 + t)^{-m}\|\nabla_x^m \partial_x^\alpha U_0\|_{L^2_{x,v}},
$$

(2.73)

where and below $\delta(\alpha, \alpha') =: \|\partial_x^\alpha U_0\|_{L^2} + \|\partial_x^\alpha U_0\|_{L^1}, k = |\alpha - \alpha'|$ and $m \geq 0$. In particular, it holds for $f = P_0 f + P_1 f$ that

$$
\|\partial_x^\alpha (f(t), \chi_0)\|_{L^2_x} \leq C\delta(\alpha, \alpha')(1 + t)^{-\frac{\beta}{2} - \frac{1}{2}},
$$

(2.74)

$$
\|\partial_x^\alpha f(t, v\sqrt{M})\|_{L^2_{x,v}} \leq C\delta(\alpha, \alpha')(1 + t)^{-\frac{\beta}{2} - \frac{1}{2}} + C(1 + t)^{-m}\|\nabla_x^m \partial_x^\alpha U_0\|_{L^2_{x,v}},
$$

(2.75)

$$
\|\partial_x^\alpha (f(t), \chi_0)\|_{L^2_x} \leq C\delta(\alpha, \alpha')(1 + t)^{-\frac{\beta}{2} - \frac{1}{2}},
$$

(2.76)

$$
\|\partial_x^\alpha P_1 f(t)\|_{L^2_{x,v}} \leq C\delta(\alpha, \alpha')(1 + t)^{-\frac{\beta}{2} - \frac{1}{2}} + C(1 + t)^{-m}\|\nabla_x^m \partial_x^\alpha U_0\|_{L^2_{x,v}}.
$$

(2.77)

Furthermore, if we also assume that $l \geq 2$ and there exist a constant $d_0 > 0$ such that the Fourier transform $\hat{U}_0 = (\hat{f}_0, \hat{E}_0, \hat{B}_0)$ of the initial data $U_0$ satisfies that $\inf_{|\xi| \leq r_0} |\hat{E}_0(\xi)| > d_0$, $\inf_{|\xi| \leq r_0} |\hat{B}_0(\xi)| > d_0$, $\inf_{|\xi| \leq r_0} |\hat{f}_0(\xi, \chi_0)| \geq d_0$ and $\sup_{|\xi| \leq r_0} |\hat{(f_0(\xi), V(\sqrt{M}))|} = 0$, then the global solution $(f(t), E(t), B(t))$ satisfies the following optimal time decay rates

$$
C_1(1 + t)^{-\frac{\beta}{2}} \leq \|f(t)\|_{L^2_{x,v}} \leq C_2(1 + t)^{-\frac{\beta}{2}},
$$

(2.78)

$$
C_1(1 + t)^{-\frac{\beta}{2}} \leq \|E(t)\|_{L^2_x} \leq C_2(1 + t)^{-\frac{\beta}{2}},
$$

(2.79)

$$
C_1(1 + t)^{-\frac{\beta}{2}} \leq \|B(t)\|_{L^2_x} \leq C_2(1 + t)^{-\frac{\beta}{2}},
$$

(2.80)

for $t > 0$ being large enough and $C_1 \geq C_2 > 0$ being two constants, and in particular

$$
C_1(1 + t)^{-\frac{\beta}{2}} \leq \|(f(t), \chi_0)\|_{L^2_x} \leq C_2(1 + t)^{-\frac{\beta}{2}},
$$

(2.81)

$$
C_1(1 + t)^{-\frac{\beta}{2}} \leq \|(f(t), \chi_0)\|_{L^2_x} \leq C_2(1 + t)^{-\frac{\beta}{2}}, \quad j = 1, 2, 3,
$$

(2.82)

$$
C_1(1 + t)^{-\frac{\beta}{2}} \leq \|(f(t), \chi_0)\|_{L^2_x} \leq C_2(1 + t)^{-\frac{\beta}{2}},
$$

(2.83)

$$
C_1(1 + t)^{-\frac{\beta}{2}} \leq \|P_1 f(t)\|_{L^2_{x,v}} \leq C_2(1 + t)^{-\frac{\beta}{2}}.
$$

(2.84)

Finally, we have the time convergence rates of global solutions to the nonlinear system [2.43]–[2.47] as follows.

**Theorem 2.12.** Assume that the initial data $f_0 \in H^{N+3}_w \cap L^{2,1}$ and $(E_0, B_0) \in H^{N+3}(\mathbb{R}^3_x) \cap L^1(\mathbb{R}^3_x)$ for $N \geq 4$ satisfy $\|f_0\|_{H^{N+3}_w \cap L^{2,1}} + \|(E_0, B_0)\|_{H^{N+3}(\mathbb{R}^3_x) \cap L^1(\mathbb{R}^3_x)} \leq \delta_0$ with $\delta_0 > 0$ being small enough. Then, the unique solution $(f, E, B)$ to the VMB system [2.43]–[2.47] exists globally in time and belongs to $H^{N+3}_w \times H^{N+3}_w \times H^{N+3}_w$.

Moreover, if it also holds $\nabla_x \cdot E_0 = (f_0, \chi_0)$, then the global solution satisfies for $k = 0, 1$ that

$$
\|\partial_x^k f(t)\|_{L^2_{x,v}} \leq C\delta_0(1 + t)^{-\frac{\beta}{2} - \frac{1}{2}},
$$

(2.85)

$$
\|\partial_x^k E(t)\|_{L^2_x} \leq C\delta_0(1 + t)^{-\frac{\beta}{2} - \frac{1}{2}} \ln(1 + t),
$$

(2.86)

$$
\|\partial_x^k B(t)\|_{L^2_x} \leq C\delta_0(1 + t)^{-\frac{\beta}{2} - \frac{1}{2}},
$$

(2.87)

and in particular

$$
\|\partial_x^k f(t, \chi_0)\|_{L^2_x} \leq C\delta_0(1 + t)^{-\frac{\beta}{2} - \frac{1}{2}},
$$

(2.88)

$$
\|\partial_x^k f(t, \chi_j)\|_{L^2_x} \leq C\delta_0(1 + t)^{-\frac{\beta}{2} - \frac{1}{2}}, \quad j = 1, 2, 3,
$$

(2.89)
the same rates
\( (1 + t)^{-\frac{3}{4} - \frac{5}{8}} \),
\( (2.90) \)
\[ \|P_1 f(t)\|_{H^\gamma} + \|\nabla_z P_0 f(t)\|_{L^2_z(H^\gamma - 1)} + \|\nabla_z (E, B)(t)\|_{H^\gamma - 1} \leq C\delta_0 (1 + t)^{-\frac{3}{8}}. \] 
\( (2.92) \)

Furthermore, if there also exist a constant \( d_0 > 0 \) and a small constant \( r_0 > 0 \) so that the Fourier transform
\( \hat{U}_0 = (f_0, \hat{E}_0, \hat{B}_0) \) of the initial data \( U_0 = (f_0, E_0, B_0) \) satisfies that
\( \inf_{|\xi| \leq r_0} |\hat{E}_0(\xi) \cdot \hat{E}_0(\xi)| \geq d_0, \inf_{|\xi| \leq r_0} |\hat{E}_0(\xi) \times \hat{E}_0(\xi)| \geq d_0, \inf_{|\xi| \leq r_0} |\hat{f}_0(\xi) \cdot (3, 0)| \geq d_0 \) and \( \sup_{|k| \leq r_0} \|\hat{(f_0(k), v)}\| \leq C \). Then, the global solution \((f, E, B)\) satisfies
\[ C_1 \delta_0 (1 + t)^{\frac{-3}{8}} \leq \|f(t)\|_{L^2_z} \leq C_2 \delta_0 (1 + t)^{-\frac{3}{8}}, \] 
\( (2.93) \)
\[ C_1 \delta_0 (1 + t)^{-\frac{3}{8}} \leq \|B(t)\|_{L^2_z} \leq C_2 \delta_0 (1 + t)^{-\frac{3}{8}}, \] 
\( (2.94) \)

for \( t > 0 \) large enough with two constants \( C_2 > C_1 \), and in particular
\[ C_1 \delta_0 (1 + t)^{-\frac{3}{8}} \leq \|f(t)\|_{L^2_z} \leq C_2 \delta_0 (1 + t)^{-\frac{3}{8}}, \quad j = 1, 2, 3, \] 
\( (2.95) \)
\[ C_1 \delta_0 (1 + t)^{-\frac{3}{8}} \leq \|f(t)\|_{L^2_z} \leq C_2 \delta_0 (1 + t)^{-\frac{3}{8}}, \] 
\( (2.96) \)
\[ C_1 \delta_0 (1 + t)^{-\frac{3}{8}} \leq \|P_1 f(t)\|_{L^2_z} \leq C_2 \delta_0 (1 + t)^{-\frac{3}{8}}. \] 
\( (2.97) \)

**Remark 2.13.** Let us give an example of the initial function \((f_0, E_0, B_0)\) which satisfies the assumptions of Theorem 2.12. For a positive constant \( d_0 \), we define \((f_0, E_0, B_0)\) as
\begin{align*}
    f_0(x, v) &= \frac{1}{(2\pi)^{3/2}} d_0 e^{\frac{v^2}{2d_0}} \int_{\mathbb{R}^3} |\xi| e^{-|\xi|^2/2} e^{ix \cdot \xi} d\xi \chi_0(v) + d_0 e^{\frac{v^2}{2d_0}} e^{-|v|^2/2} \chi_4(v), \\
    E_0(x) &= \frac{1}{(2\pi)^{3/2}} d_0 e^{\frac{v^2}{2d_0}} \int_{\mathbb{R}^3} \left( \frac{\xi}{|\xi|} + \frac{-(\xi_2, \xi_1, 0)}{(\xi_1^2 + \xi_2^2)^{1/2}} \right) e^{-|\xi|^2/2} e^{ix \cdot \xi} d\xi, \\
    B_0(x) &= \frac{1}{(2\pi)^{3/2}} d_0 e^{\frac{v^2}{2d_0}} \int_{\mathbb{R}^3} \frac{-(\xi_2, \xi_1, 0)}{(\xi_1^2 + \xi_2^2)^{1/2}} e^{-|\xi|^2/2} e^{ix \cdot \xi} d\xi.
\end{align*}

**Remark 2.14.** In the case of \( \nabla_z \cdot E_0 \neq (f_0, \sqrt{M}) \), the solutions of the nonlinear VMB system have no decay rates. The main reason is that the magnetic field decay too slow. For instance, we assume that \( f, E, B \) decay at the same rates \( (1 + t)^{-\frac{3}{8}}, (1 + t)^{-\frac{3}{8}} \) and \( (1 + t)^{-\frac{3}{8}} \) respectively as the linear solution, then we can obtain that the key quadratic nonlinear term decays at most at the rate \( (1 + t)^{-\frac{3}{8}} \). Thus, making use of the Duhamel’s principle to represent the solution via the semigroup, the corresponding nonlinear term to magnetic field \( B \) generated from the nonhomogeneous source decays as
\[ \int_0^t (1 + t - s)^{-\frac{3}{8}} (1 + t)^{-\frac{3}{8}} ds \leq C (1 + t)^{-\frac{3}{8}}. \]

Thus, the bootstrap argument breaks down and one can not expect the magnetic field to decay as \( (1 + t)^{-\frac{3}{8}} \) in the nonlinear case.

### 3 Spectral analysis for two species case

For the study on the spectrum structure of the two-species VMB, in the following subsection, we will first investigate some properties of the operator \( \tilde{A}_1(\xi) \) that lead to the description of its spectra and resolving. And then the asymptotics of its eigenvalues and eigenfunctions in low and high frequency regions will be given in the next two subsections.
3.1 Spectrum structure

First of all, note that \( P_4 \) is a self-adjoint operator satisfying \((P_4 f, P_4 g) = (P_4 f, g)\). Hence,

\[
(f, g)_\xi = (f, g + \frac{1}{|\xi|^2} P_4 g) = (f + \frac{1}{|\xi|^2} P_4 f, g).
\]

By \((3.7)\), we have for any \(f, g\)

\[
(\hat{B}_1(\xi)f, g)_\xi = (\hat{B}_1(\xi) f, g + \frac{1}{|\xi|^2} P_4 g) = (f, (L_1 + i (v \cdot \xi) + \frac{i(v \cdot \xi)}{|\xi|^2} P_4) g)_\xi = (f, \hat{B}_1(-\xi) g)_\xi.
\]

Also, note that \(\hat{B}_1(\xi)\) is a linear operator from the space \(L^2_\xi(\mathbb{R}^3)\) to itself, and for any \(y \in \mathbb{C}_\xi\),

\[
\left\| \frac{\xi}{|\xi|} \times \frac{\xi}{|\xi|} \times y \right\| = -y.
\]

Since \(L^2_\xi(\mathbb{R}^3) \times \mathbb{C}_\xi \times \mathbb{C}_\xi\) is an invariant subspace of the operator \(\hat{A}_1(\xi)\), \(\hat{A}_1(\xi)\) can be regarded as a linear operator on \(L^2_\xi(\mathbb{R}^3) \times \mathbb{C}_\xi \times \mathbb{C}_\xi\). Firstly, we have

**Lemma 3.1.** The operator \(\hat{A}_1(\xi)\) generates a strongly continuous contraction semigroup on \(L^2_\xi(\mathbb{R}^3) \times \mathbb{C}_\xi \times \mathbb{C}_\xi\) satisfying

\[
\|e^{it\hat{A}_1(\xi)} U\|_\xi \leq \|U\|_\xi, \quad \text{for } t > 0, \quad U \in L^2_\xi(\mathbb{R}^3) \times \mathbb{C}_\xi \times \mathbb{C}_\xi.
\]

**Proof.** We first show that both \(\hat{A}_1(\xi)\) and \(\hat{A}_1(\xi)^*\) are dissipative operators on \(L^2_\xi(\mathbb{R}^3)\). By \((3.2)\), we obtain for any \(U, V \in L^2_\xi(\mathbb{R}^3) \cap D(\hat{B}_1(\xi)) \times \mathbb{C}_\xi \times \mathbb{C}_\xi\) that \((\hat{A}_1(\xi) U, V)_\xi = (U, \hat{A}_1(\xi)^* V)_\xi\) with \(\hat{A}_1(\xi)^* = \hat{A}_1(-\xi)\).

Direct computation shows the dissipation of both \(\hat{A}_1(\xi)\) and \(\hat{A}_1(\xi)^*\), namely,

\[
\text{Re}(\hat{A}_1(\xi) U, V)_\xi = \text{Re}(\hat{A}_1(\xi)^* U, V)_\xi = (L_1 f, f) \leq 0.
\]

Since \(\hat{A}_1(\xi)\) is a densely defined closed operator, it follows from Corollary 4.4 on p.15 of [13] that the operator \(\hat{A}_1(\xi)\) generates a \(C_0\)-contraction semigroup on \(L^2_\xi(\mathbb{R}^3) \times \mathbb{C}_\xi \times \mathbb{C}_\xi\). \(\square\)

Define a \(6 \times 6\) matrix by

\[
B_3(\xi) = \begin{pmatrix} 0 & i\xi \times \xi \\ -i\xi \times 0 \\ \end{pmatrix}_{6 \times 6}.
\]

Since \(\mathbb{C}_\xi \times \mathbb{C}_\xi\) is an invariant subspace of the operator \(B_3(\xi)\), we can regard \(B_3(\xi)\) as an operator on \(\mathbb{C}_\xi \times \mathbb{C}_\xi\). Then we have

**Lemma 3.2.** For any \(\lambda \neq \pm i|\xi|\), the operator \(\lambda - B_3(\xi)\) is invertible on \(\mathbb{C}_\xi \times \mathbb{C}_\xi\) and satisfies

\[
\| (\lambda - B_3(\xi))^{-1} \| = \max_{j = \pm 1} |\lambda - ji| |\xi|^{-1}.
\]

**Proof.** First, we compute the eigenvalues of the operator \(B_3(\xi)\). For this, consider

\[
(\lambda - B_3(\xi)) X = 0, \quad X = (X_1, X_2) \in \mathbb{C}_\xi \times \mathbb{C}_\xi.
\]

It follows that

\[
\lambda X_1 - i\xi \times X_2 = 0, \quad (3.7)
\]

\[
\lambda X_2 + i\xi \times X_1 = 0. \quad (3.8)
\]

Multiplying \((3.7)\) by \(\lambda\) and using \((3.8)\) and \((3.3)\) give

\[
\lambda^2 X_1 + |\xi|^2 X_1 = 0, \quad (3.9)
\]
which implies that \( \lambda_j = ji|\xi| \) for \( j = \pm 1 \) are the eigenvalues of \( B_3(\xi) \). Thus \( \lambda - B_3(\xi) \) is invertible on \( \mathbb{C}_\xi^3 \times \mathbb{C}_\xi^3 \) for any \( \lambda \neq \pm i|\xi| \). Since

\[
(iB_3(\xi)X,Y) = (X,iB_3(\xi)Y), \quad \forall \ X,Y \in \mathbb{C}_\xi^3 \times \mathbb{C}_\xi^3,
\]

it follows that \( iB_3(\xi) \) is a self-adjoint operator on \( \mathbb{C}_\xi^3 \times \mathbb{C}_\xi^3 \) and hence

\[
\| (\lambda - B_3(\xi))^{-1} \| = \max_{j=\pm 1} |\lambda - ji|\xi||^{-1}.
\]

This completes the proof of the lemma. \( \square \)

Denote by \( \rho(\hat{A}_1(\xi)) \) the resolvent set and by \( \sigma(\hat{A}_1(\xi)) \) the spectrum set of \( \hat{A}_1(\xi) \). We have

**Lemma 3.3.** For each \( \xi \neq 0 \), the spectrum set \( \sigma(\hat{A}_1(\xi)) \) of the operator \( \hat{A}_1(\xi) \) in the domain \( \Re \lambda \geq -\nu_0 + \delta \) for any constant \( \delta > 0 \) consists of isolated eigenvalues \( \Sigma = \{ \lambda_j(\xi) \} \) with \( \Re \lambda_j(\xi) < 0 \).

**Proof.** Define

\[
G_1(\xi) = \begin{pmatrix} c(\xi) & 0 & 0 \\ 0 & 0 & i\xi \times \\ 0 & -i\xi \times & 0 \end{pmatrix}, \quad G_2(\xi) = \begin{pmatrix} K_1 - \frac{i(\nu(\xi))}{|\xi|} P_d & -v\sqrt{M} \cdot \omega \times 0 \\ -\omega \times P_m & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]

\[
c(\xi) = -\nu(v) - i(v \cdot \xi). \quad (3.11)
\]

It is obvious that \( \lambda - G_1(\xi) \) is invertible for \( \Re \lambda > -\nu_0 \) and \( \lambda \neq \pm i|\xi| \). Since \( G_2(\xi) \) is a compact operator on \( L_\xi^2(\mathbb{R}^3) \times \mathbb{C}_\xi^3 \times \mathbb{C}_\xi^3 \) for any fixed \( \xi \neq 0 \), \( \hat{A}_1(\xi) \) is a compact perturbation of \( G_1(\xi) \). Hence, by Theorem 5.35 on p.244 of [12], \( \hat{A}_1(\xi) \) and \( G_1(\xi) \) have the same essential spectrum where \( \sigma_e(G_1(\xi)) = \operatorname{Ran}(c(\xi)) \) and \( \sigma_d(G_1(\xi)) = \pm |\xi| \). Thus the spectrum of \( \hat{A}_1(\xi) \) in the domain \( \Re \lambda > -\nu_0 \) consists of discrete eigenvalues \( \lambda_j(\xi) \) with possible accumulation points only on the line \( \Re \lambda = -\nu_0 \).

We claim that for any discrete eigenvalue \( \lambda(\xi) \) of \( \hat{A}_1(\xi) \) in the region \( \Re \lambda \geq -\nu_0 + \delta \) for any constant \( \delta > 0 \), it holds that \( \Re \lambda(\xi) < 0 \) for \( \xi \neq 0 \). Indeed, set \( \xi = s\omega \) and let \( U = (f,E,B) \in L_\xi^2(\mathbb{R}^3) \times \mathbb{C}_\xi^3 \times \mathbb{C}_\xi^3 \) be the eigenvector corresponding to the eigenvalue \( \lambda \) so that

\[
\begin{cases}
\lambda f = L_1 f - is(v \cdot \omega)(f + \frac{1}{s} P_3 f) - v\sqrt{M} \cdot (\omega \times E), \\
\lambda E = -\omega \times (f,v\sqrt{M}) + i\xi \times B, \\
\lambda B = -i\xi \times E.
\end{cases} \quad (3.13)
\]

Taking the inner product \( (\cdot,\cdot)_\xi \) of \( \text{3.13} \) with \( U \), we have

\[
(L_1 f, f) = \Re \lambda \left( \| f \|^2 + \frac{1}{s^2} \| P_3 f \|^2 + |E|^2 + |B|^2 \right),
\]

which together with \( 2.16 \) implies \( \Re \lambda \leq 0 \).

Furthermore, if there exists an eigenvalue \( \lambda \) with \( \Re \lambda = 0 \), then it follows from the above that \( (L_1 f, f) = 0 \), namely, \( f = C_0 \sqrt{M} \in N_1 \). Substitute this into \( 3.13 \), we obtain

\[
\lambda C_0 \sqrt{M} = -i(v \cdot \omega) \left( s + \frac{1}{s} \right) C_0 \sqrt{M} - v\sqrt{M} \cdot (\omega \times E), \quad (3.14)
\]

which implies that \( C_0 = 0 \) and \( \omega \times E = 0 \). Therefore, \( f \equiv 0 \) and \( E \equiv 0 \). Substitute this into \( 3.13 \), we obtain \( B \equiv 0 \). This is a contradiction and thus it holds \( \Re \lambda < 0 \) for any discrete eigenvalue \( \lambda \in \sigma(\hat{A}_1(\xi)) \). \( \square \)

Now denote by \( T \) a linear operator on \( L_\xi^2(\mathbb{R}^3) \) or \( L_\xi^2(\mathbb{R}^3) \), and we define the corresponding norms of \( T \) by

\[
\| T \| = \sup_{\| f \|=1} \| T f \|, \quad \| T \|_\xi = \sup_{\| f \|_\xi=1} \| T f \|_\xi.
\]
Obviously,

\[(1 + |\xi|^{-2})^{-1}||T|| \leq ||T||_{\xi} \leq (1 + |\xi|^{-2})||T||. \tag{3.15}\]

Also, if \(T\) is a linear operator on \(L^2(\mathbb{R}^3_+) \times \mathbb{C}^3 \times \mathbb{C}_\xi^3 \) or \(L^2(\mathbb{R}^3_+) \times \mathbb{C}^3 \times \mathbb{C}_\xi^3\), then

\[||T|| = \sup_{||\xi||=1} ||TU||, \quad ||T||_{\xi} = \sup_{||\xi||=1} ||TU||_{\xi}. \]

We will make use of the following decomposition associated with the operator \(\hat{A}_1(\xi)\) for \(|\xi| > 0\)

\[\lambda - \hat{A}_1(\xi) = \lambda - G_1(\xi) - G_2(\xi) = (I - G_2(\xi)(\lambda - G_1(\xi))^{-1})(\lambda - G_1(\xi)), \tag{3.16}\]

where \(G_1(\xi), G_2(\xi)\) are defined by \(\ref{3.13}\). For \(\text{Re}\lambda > -\nu_0\) and \(\lambda \neq \pm i|\xi|\), we have

\[(\lambda - G_1(\xi))^{-1} = \begin{pmatrix} (\lambda - c(\xi))^{-1} & 0 \\ 0 & (\lambda - B_3(\xi))^{-1} \end{pmatrix}_{7 \times 7}, \tag{3.17}\]

\[G_2(\xi)(\lambda - G_1(\xi))^{-1} = \begin{pmatrix} X_1(\lambda, \xi) & X_2(\lambda, \xi) \\ X_3(\lambda, \xi) & 0 \end{pmatrix}_{7 \times 7}, \tag{3.18}\]

where \(B_3(\xi)\) is defined in \(\ref{3.3}\), and

\[X_1(\lambda, \xi) = (K_1 - \frac{i(v \cdot \xi)}{|\xi|^2} P_d)(\lambda - c(\xi))^{-1}, \tag{3.19}\]

\[X_2(\lambda, \xi) = (v \sqrt{M} \cdot \omega \times, 0 \times 3)_{1 \times 6}(\lambda - B_3(\xi))^{-1}, \tag{3.20}\]

\[X_3(\lambda, \xi) = \begin{pmatrix} -\omega \times P_m(\lambda - c(\xi))^{-1} \\ 0_{3 \times 1} \end{pmatrix}_{6 \times 1}. \tag{3.21}\]

Let \(K_1, K_4\) be the operators on the space \(X\) and \(Y\), and \(K_2, K_3\) be the operators \(Y \to X\) and \(X \to Y\) respectively. Let \(K\) be a matrix operator on \(X \times Y\) defined by

\[K = \begin{pmatrix} K_1 & K_2 \\ K_3 & K_4 \end{pmatrix}. \]

Then, we have

**Lemma 3.4.** If the norms of \(K_1, K_2, K_3\) and \(K_4\) satisfy

\[||K_1|| < 1, \quad ||K_4|| < 1, \quad ||K_2|| ||K_3|| < (1 - ||K_1||)(1 - ||K_4||),\]

then the operator \(I + K\) is invertible on \(X \times Y\).

**Proof.** Decompose \(I + K\) into

\[I + K = \begin{pmatrix} I + K_1 & K_2 \\ 0 & I + K_4 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ K_3 & 0 \end{pmatrix}. \]

Since

\[\begin{pmatrix} I + K_1 & K_2 \\ 0 & I + K_4 \end{pmatrix}^{-1} = \begin{pmatrix} (I + K_1)^{-1} & -(I + K_1)^{-1}K_2(I + K_4)^{-1} \\ 0 & (I + K_4)^{-1} \end{pmatrix},\]

it follows that

\[I + K = \begin{pmatrix} I + K_1 & K_2 \\ 0 & I + K_4 \end{pmatrix} \begin{pmatrix} I - (I + K_1)^{-1}K_2(I + K_4)^{-1}K_3 & 0 \\ 0 & (I + K_4)^{-1}K_3 \end{pmatrix} \begin{pmatrix} I + K_1 & K_2 \\ 0 & I + K_4 \end{pmatrix}\]

is invertible on \(X \times Y\) because \(||(I + K_1)^{-1}K_2(I + K_4)^{-1}K_3|| < 1\). \(\Box\)
Lemma 3.5. There exists a constant $C > 0$ such that

1. For any $\delta > 0$, we have
\[
\sup_{x \geq -r_0 + \delta} \| K_1(x + iy - c(\xi))^{-1} \| \leq C \delta^{-15/13}(1 + |\xi|)^{-2/13}.
\]  
\[
(3.22)
\]

2. For any $\delta > 0$, $r_0 > 0$, there is a constant $y_0 = (2r_0)^{5/3} \delta^{-2/3} > 0$ such that if $|y| \geq y_0$, we have
\[
\sup_{x \geq -r_0 + \delta, |\xi| \leq r_0} \| K_1(x + iy - c(\xi))^{-1} \| \leq C \delta^{-7/5}(1 + |y|)^{-2/5}.
\]  
\[
(3.23)
\]

3. For any $\delta > 0$, $r_0 > 0$, we have
\[
\sup_{x \geq -r_0 + \delta, \xi \in \mathbb{R}} \| P_m(x + iy - c(\xi))^{-1} \|_{L^2(\mathbb{R}) \to C^3} \leq C \delta^{-1/2}(1 + |\xi|)^{-1/2},
\]  
\[
(3.24)
\]
\[
\sup_{x \geq -r_0 + \delta, |\xi| \leq r_0} \| P_m(x + iy - c(\xi))^{-1} \|_{L^2(\mathbb{R}) \to C^3} \leq C(\delta^{-1} + 1)(r_0 + 1)|y|^{-1}.
\]  
\[
(3.25)
\]

4. For any $\delta > 0$, $r_0 > 0$, we have
\[
\sup_{x \geq -r_0 + \delta} \| (v \cdot \xi)|\xi|^2 P_0(x + iy - c(\xi))^{-1} \| \leq C \delta^{-1}|\xi|^{-1},
\]  
\[
(3.26)
\]
\[
\sup_{x \geq -r_0 + \delta} \| (v \cdot \xi)|\xi|^2 P_0(x + iy - c(\xi))^{-1} \| \leq C(\delta^{-1} + 1)(r_0 + 1)|y|^{-1}.
\]  
\[
(3.27)
\]

Proof. The proof of (3.22), (3.23), (3.26) and (3.27) can be found in Lemma 2.3 of [13]. We only need to prove (3.24) and (3.25). Since
\[
\| (x - iy + \nu(v) - i(v \cdot \xi))^{-1} v \sqrt{M} \|^2 \leq C \int_{\mathbb{R}^3} \frac{1}{(x + \nu_0)^2 + (y + (v \cdot \xi))^2} e^{-\frac{|v|^2}{4|x|^2}} dv \]
\[
= C \int_{\mathbb{R}^3} \frac{1}{(x + \nu_0)^2 + |v|^2} e^{-\frac{|v|^2}{4|x|^2}} dv \leq C(x + \nu_0)^{-1}|\xi|^{-1},
\]
we obtain
\[
|P_m(x + iy - c(\xi))^{-1} f| \leq \| (x - iy + \nu(v) - i(v \cdot \xi))^{-1} v \sqrt{M} \| |f| \leq C(x + \nu_0)^{-1/2}|\xi|^{-1/2} |f|.
\]
This proves (3.24). Since $P_m(\lambda - c(\xi))^{-1} = \frac{1}{\lambda} P_m + \frac{\lambda}{\lambda} P_m(\lambda - c(\xi))^{-1} c(\xi)$, it follows that $\| P_m(\lambda - c(\xi))^{-1} \| \leq |\lambda|^{-1} + C \delta^{-1}|\lambda|^{-1}(1 + |\xi|)$, which proves (3.25). \qed

With Lemma 3.5, we can investigate the spectrum set of the operator $\hat{A}_1(\xi)$ in the intermediate and high frequency regions.

Lemma 3.6. For the high and intermediate frequencies, the following statements hold.

1. For any $\delta_1, \delta_2 > 0$, there exists $R_1 = R_1(\delta_1, \delta_2) > 0$ such that for $|\xi| > R_1$,
\[
\sigma(\hat{A}_1(\xi)) \cap \{ \lambda \in \mathbb{C} | \text{Re}\lambda \geq -\nu_0 + \delta_1 \} \subset \bigcup_{j=\pm 1} \{ \lambda \in \mathbb{C} | |\lambda - ji\xi| \leq \delta_2 \}.
\]  
\[
(3.28)
\]

2. For any $r_1 > r_0 > 0$, there exists $\alpha = \alpha(r_0, r_1) > 0$ such that for $r_0 \leq |\xi| \leq r_1$,
\[
\sigma(\hat{A}_1(\xi)) \subset \{ \lambda \in \mathbb{C} | \text{Re}\lambda(\xi) \leq -\alpha \}.
\]  
\[
(3.29)
\]
Proof. We prove (3.28) first. By Lemma 5.6, 5.0 and (5.15), there is $R_1 = R_1(\delta_1, \delta_2) > 0$ such that for \( \text{Re}(\lambda) \geq -\nu_0 + \delta_1, \) \( \min_{j=\pm 1} |\lambda - j\xi| > \delta_2 \) and $|\xi| > R_1$,

$$\| (K_1 - \frac{iv(\xi)}{|\xi|} P_4) (\lambda - c(\xi))^{-1} \|_\xi \leq 1/2, \quad \| P_m (\lambda - c(\xi))^{-1} \|_{L^2(\mathbb{R}^3) \rightarrow C^0} \leq \delta_2/4, \quad \| (\lambda - B_3(\xi))^{-1} \| \leq \delta_2^{-1},$$

which leads to

$$\| X_1(\lambda, \xi) \|_\xi \leq 1/2, \quad \| X_2(\lambda, \xi) \|_{c^0 \rightarrow L^2(\mathbb{R}^3)} \| X_3(\lambda, \xi) \|_{L^2(\mathbb{R}^3) \rightarrow C^0} \leq 1/4.$$ 

This and Lemma 5.4 imply that the operator $I - G_2(\lambda)(\lambda - G_1(\xi))^{-1}$ is invertible on $L^2(\mathbb{R}^3) \times C_\xi \times C_\xi$ and thus $\lambda - \hat{A}_1(\xi)$ is invertible on $L^2(\mathbb{R}^3) \times C_\xi \times C_\xi$. Therefore, we have $\rho(\hat{A}_1(\xi)) \supset \{ \xi \in \mathbb{C} | \min_{j=\pm 1} |\lambda - j\xi| > \delta_2, \text{Re}(\lambda) \geq -\mu + \delta_1 \}$ for $|\xi| > R_1$ which gives (3.28).

Next, we turn to prove (3.29). For this, we first show that $\sup_{r_0 \leq |\xi| \leq r_1} |\text{Im}(\lambda(\xi))| < +\infty$ for any $\lambda(\xi) \in \sigma(\hat{A}_1(\xi))$ with $\text{Re}(\lambda) \geq -\nu_0 + \delta_1$. Indeed, by Lemma 5.6, 5.0 and (5.15), there exists $y_1 = y_1(r_0, r_1, \delta_1) > 0$ large enough such that for $\text{Re}(\lambda) \geq -\nu_0 + \delta_1, |\text{Im}\lambda| > y_1$ and $r_0 \leq |\xi| \leq r_1$,

$$\| (K_1 - \frac{iv(\xi)}{|\xi|} P_4) (\lambda - c(\xi))^{-1} \|_\xi \leq 1/6, \quad \| P_m (\lambda - c(\xi))^{-1} \|_{L^2(\mathbb{R}^3) \rightarrow C^0} \leq 1/6, \quad \| (\lambda - B_3(\xi))^{-1} \| \leq 1/6,$$

which leads to

$$\| X_1(\lambda, \xi) \|_\xi + \| X_2(\lambda, \xi) \|_{c^0 \rightarrow L^2(\mathbb{R}^3)} + \| X_3(\lambda, \xi) \|_{L^2(\mathbb{R}^3) \rightarrow C^0} \leq 1/2.$$ 

This implies that the operator $I - G_2(\lambda)(\lambda - G_1(\xi))^{-1}$ is invertible on $L^2(\mathbb{R}^3) \times C_\xi \times C_\xi$, which together with (3.10) yield that $\lambda - \hat{A}_1(\xi)$ is also invertible on $L^2(\mathbb{R}^3) \times C_\xi \times C_\xi$ when $\text{Re}(\lambda) \geq -\nu_0 + \delta_1, |\text{Im}\lambda| > y_1$ and $r_0 \leq |\xi| \leq r_1$. Note that it satisfies

$$\lambda - \hat{A}_1(\xi)^{-1} = (\lambda - G_1(\xi))^{-1}(I - G_2(\lambda)(\lambda - G_1(\xi))^{-1}).$$

Thus, we conclude that for $r_0 \leq |\xi| \leq r_1$,

$$\sigma(\hat{A}_1(\xi)) \cap \{ \lambda \in \mathbb{C} | \text{Re}(\lambda) \geq -\nu_0 + \delta_1 \} \subset \{ \lambda \in \mathbb{C} | \text{Re}(\lambda) \geq -\nu_0 + \delta_1, |\text{Im}\lambda| \leq y_1 \}. \quad (3.30)$$

Finally, we prove $\sup_{r_0 \leq |\xi| \leq r_1} \text{Re}(\lambda(\xi)) < 0$ by contradiction. If it does not hold, then there exists a sequence of $\{ \{\xi_n, \lambda_n, f_n, E_n, B_n\} \}$ satisfying $|\xi_n| \in [r_0, r_1], (f_n, E_n, B_n) \in L^2(\mathbb{R}^3) \times C_\xi \times C_\xi$ with $\|f_n\| + |E_n| + |B_n| = 1$, and $\lambda_n \in \sigma(\hat{A}_1(\xi_n))$ with $\text{Re}(\lambda_n) \rightarrow 0$ as $n \rightarrow \infty$. That is,

$$\begin{aligned}
\lambda_n f_n &= (L_1 - iv \cdot \xi_n - \frac{iv(\xi_n)}{|\xi_n|^2}) P_4 f_n - v \sqrt{M} \cdot (\omega_n \times E_n), \\
\lambda_n E_n &= -\omega_n \times (f_n, v \sqrt{M}) \pm i \xi_n \times B_n, \quad \lambda_n B_n = -i \xi_n \times E_n.
\end{aligned}$$

Rewrite the first equation as

$$(\lambda_n + v + iv(\xi_n)) f_n = K_1 f_n - i \frac{v \cdot \xi_n}{|\xi_n|^2} P_4 f_n - v \sqrt{M} \cdot (\omega_n \times E_n).$$

Since $K_1$ is a compact operator on $L^2(\mathbb{R}^3)$, there exists a subsequence $\{f_{n_j}\}$ of $\{f_n\}$ and $g_1 \in L^2(\mathbb{R}^3)$ such that

$$K_1 f_{n_j} \rightarrow g_1, \quad j \rightarrow \infty.$$ 

By using the fact that $|\xi_n| \in [r_0, r_1], P_4 f_n = C_0^1 \sqrt{M}$ with $|C_0^1| \leq 1$ and $|E_n| + |B_n| \leq 1$, there exists a subsequence of (still denoted by) $\{\{\xi_{n_j}, f_{n_j}, E_{n_j}, B_{n_j}\}\}$, and $(\xi_0, C_0, E_0, B_0)$ with $|\xi_0| \in [r_0, r_1]$ and $|C_0| \leq 1$ such that $(\xi_{n_j}, C_{n_j}^0, E_{n_j}, B_{n_j}) \rightarrow (\xi_0, C_0, E_0, B_0)$ as $j \rightarrow \infty$. In particular

$$i \frac{v \cdot \xi_0}{|\xi_0|^2} P_4 f_{n_j} \rightarrow g_2 = i \frac{v \cdot \xi_0}{|\xi_0|^2} C_0 \sqrt{M}, \quad \omega_{n_j} \times E_{n_j} \rightarrow \omega_0 \times E_0 =: Y_0, \quad j \rightarrow \infty.$$ 

Since $|\text{Im}\lambda_n| \leq y_1$ and $\text{Re}\lambda_n \rightarrow 0$, we can extract a subsequence of (still denoted by) $\{\lambda_{n_j}\}$ such that $\lambda_{n_j} \rightarrow \lambda_0$ with $\text{Re}\lambda_0 = 0$. Then

$$\lim_{j \rightarrow \infty} f_{n_j} = \lim_{j \rightarrow \infty} \frac{g_1 - g_2 - (v \cdot Y_0) \sqrt{M}}{\lambda_0 + v + iv(\xi_0)} = \frac{g_1 - g_2 - (v \cdot Y_0) \sqrt{M}}{\lambda_0 + v + iv(\xi_0)} := f_0 \quad \text{in} \quad L^2(\mathbb{R}^3).$$
Therefore, we can rewrite (3.32) as
\[
\lambda - \hat{A}_1(\xi) = \lambda - G_3(\xi) - G_4(\xi) = (I - G_4(\xi)(\lambda - G_3(\xi))^{-1})(\lambda - G_3(\xi)),
\]
where

\[
G_3(\xi) = \begin{pmatrix} Q(\xi) & 0 & 0 \\ 0 & 0 & i\xi \times \\ 0 & -i\xi \times & 0 \end{pmatrix}, \quad G_4(\xi) = \begin{pmatrix} Q_1(\xi) & -v\sqrt{M} \cdot \omega \times & 0 \\ -\omega \times P_m & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]
\[
Q(\xi) = L_1 - iP_{r}(v \cdot \xi)P_{r}, \quad Q_1(\xi) = iP_{d}(v \cdot \xi)P_{r} + iP_{r}(v \cdot \xi)(1 + \frac{1}{|\xi|^2})P_{d}.
\]

Lemma 3.7. Let $\xi \neq 0$ and $Q(\xi)$ defined by (3.34). We have

1. If $\lambda \neq 0$, then
\[
||\lambda^{-1}P_r(v \cdot \xi)(1 + \frac{1}{|\xi|^2})P_d||_L \leq C(|\xi| + 1)|\lambda|^{-1}.
\]

2. If $\text{Re}\lambda > -\mu$, then the operator $\lambda P_r - Q(\xi)$ is invertible on $N_1^+$ and satisfies
\[
||Q_r(\xi)^{-1}|| \leq (\text{Re}\lambda + \mu)^{-1},
\]
\[
||P_d(v \cdot \xi)P_r(\lambda P_r - Q(\xi))^{-1}P_r||_L \leq C(1 + |\xi|)^{-1}(1 + |\lambda|)^{-1},
\]
\[
||P_m(\lambda P_r - Q(\xi))^{-1}P_r||_{L_2^2(\mathbb{R}^3)} \leq C(1 + |\lambda|)^{-1}(1 + |\xi|).
\]

Proof. The proof of Lemma 3.7 can be found in [14]. Repeating a same argument as to the estimate (3.36), we can obtain (3.37) and (3.38). Hence, we omit the details for brevity.

By Lemmas 3.3, 3.4, and 3.7 we analyze the spectral and resolvent sets of the operator $\hat{A}_1(\xi)$.

Lemma 3.8. For any $\delta_1, \delta_2 > 0$, there are two constants $r_1 = r_1(\delta_1, \delta_2), y_1 = y_1(\delta_1, \delta_2) > 0$ such that for all $|\xi| \neq 0$ the resolvent set of $\hat{A}_1(\xi)$ satisfies
\[
\rho(\hat{A}_1(\xi)) \supset \{ \lambda \in \mathbb{C} \mid \text{Re}\lambda \geq -\mu + \delta_1, |\text{Im}\lambda| \geq y_1 \} \cup \{ \lambda \in \mathbb{C} \mid \text{Re}\lambda > 0 \}, \quad |\xi| \leq r_1,
\]
\[
\{ \lambda \in \mathbb{C} \mid \text{Re}\lambda \geq -\mu + \delta_1, \min_{j = \pm 1} |\lambda - ji| \geq \delta_2 \} \cup \{ \lambda \in \mathbb{C} \mid \text{Re}\lambda > 0 \}, \quad |\xi| \geq r_1.
\]

Proof. By Lemma 3.6 there exists $r_1 = r_1(\delta_1, \delta_2) > 0$ so that the second part of (3.30) holds. Thus we only need to prove the first part of (3.30). By Lemma 3.7 we have for $\text{Re}\lambda > -\mu$ and $\lambda \neq 0$ that the operator $\lambda - Q(\xi) = \lambda P_d + \lambda P_r - Q(\xi)$ is invertible on $L_2^2(\mathbb{R}^3)$ and it satisfies
\[
(\lambda P_d + \lambda P_r - Q(\xi))^{-1} = \lambda^{-1}P_d + (\lambda P_r - Q(\xi))^{-1}P_r,
\]
because the operator $\lambda P_d$ is orthogonal to $\lambda P_r - Q(\xi)$. Thus, for $\text{Re}\lambda > -\mu$ and $\lambda \neq 0, \pm i|\xi|$, the operator $\lambda - G_3(\xi)$ is invertible on $L_2^2(\mathbb{R}^3) \times C^2_\xi \times C^2_\xi$ and satisfies
\[
(\lambda - G_3(\xi))^{-1} = \begin{pmatrix} \lambda^{-1}P_d + (\lambda P_r - Q(\xi))^{-1}P_r & 0 \\ 0 & (\lambda - G_3(\xi))^{-1} \end{pmatrix}.
\]

Therefore, we can rewrite (3.32) as
\[
\lambda - \hat{A}_1(\xi) = (I - G_4(\xi)(\lambda - G_3(\xi))^{-1})(\lambda - G_3(\xi)),
\]
Substituting (3.51) into (3.49) and (3.48), we obtain the eigenvalue problem for 

For \( f \) By Lemma 3.7, (3.34) and (3.50), the microscopic part of the operator \( \hat{A}_1(\xi) \) is invertible on \( L^2_\xi(\mathbb{R}^3) \times C^1_\xi \times C^1_\xi \) and satisfies

\[
(\lambda - \hat{A}_1(\xi))^{-1} = (\lambda - G_3(\xi))^{-1}(I - G_4(\xi)(\lambda - G_3(\xi))^{-1})^{-1}.
\]

Therefore, \( \rho(\hat{A}_1(\xi)) \supset \{ \lambda \in \mathbb{C} | \Re \lambda \geq -\mu + \delta_1, |\Im \lambda| \geq y_1 \} \) for \( |\xi| \leq r_1 \). This completes the proof of the lemma.

\[\] 3.2 Asymptotics in low frequency

In this subsection, we study in the low frequency region, the asymptotics of the eigenvalues and eigenvectors of the operator \( \hat{A}_1(\xi) \). In terms of (3.34), the eigenvalue problem \( \hat{A}_1(\xi)U = \lambda U \) for \( U = (f, X, Y) \in L^2_\xi(\mathbb{R}^3) \times C^1_\xi \times C^1_\xi \) can be written as

\[
\lambda f = (L_1 - i(v \cdot \xi) - \frac{i(v \cdot \xi)}{\|\xi\|^2} P_d) f - v\sqrt{M} \cdot (\omega \times X),
\]

\[
\lambda X = -\omega \times (f, v\sqrt{M}) + i\xi \times Y, \quad \lambda Y = -i\xi \times X, \quad |\xi| \neq 0.
\]

Let \( f \) be the eigenfunction of (3.34), we rewrite \( f \) in the form \( f = f_0 + f_1 \), where \( f_0 = P_d f = C_0 \sqrt{M} \) and \( f_1 = (I - P_d) f = P_r f \). Then (3.47) gives

\[
\lambda f_0 = -P_d[i(v \cdot \xi)(f_0 + f_1)],
\]

\[
\lambda f_1 = L_1 f_1 - P_d[i(v \cdot \xi)(f_0 + f_1)] - \frac{i(v \cdot \xi)}{\|\xi\|^2} f_0 - v\sqrt{M} \cdot (\omega \times X).
\]

By Lemma (3.34) and (3.35), the microscopic part \( f_1 \) can be represented for any \( \Re \lambda > -\mu \) by

\[
f_1 = -i(\lambda P_r - Q(\xi))^{-1} P_r[i(v \cdot \xi)(1 + \frac{1}{\|\xi\|^2}) f_0 + v\sqrt{M} \cdot (\omega \times X)], \quad \Re \lambda > -\mu.
\]

Substituting (3.51) into (3.49) and (3.48), we obtain the eigenvalue problem for \( (C_0, X, Y) \) as

\[
\lambda C_0 = (1 + \frac{1}{\|\xi\|^2}) (R(\lambda, \xi)(v \cdot \xi) \sqrt{M}, v \cdot \xi) \sqrt{M}) C_0 + (R(\lambda, \xi)(v\sqrt{M} \cdot (\omega \times X)), v \cdot \xi) \sqrt{M}) C_0,
\]

\[
\lambda X = -\omega \times (R(\lambda, \xi)(v \cdot \xi) \sqrt{M}, v \cdot \xi) \sqrt{M}) C_0 - \omega \times (R(\lambda, \xi)(v\sqrt{M} \cdot (\omega \times X)), v\sqrt{M}) + i\xi \times Y,
\]

\[
\lambda Y = -i\xi \times X.
\]
where \( R(\lambda, \xi) = -(\lambda P_r - Q(\xi))^{-1} = [L_1 - \lambda P_r - iP_r(v \cdot \xi)P_r]^{-1} \).

By changing variable \( (v \cdot \xi) \to |\xi|v_1 \) and using the rotational invariance of the operator \( L_1 \), we have the following transformation.

**Lemma 3.9.** Let \( e_1 = (1, 0, 0) \), \( \xi = s\omega \) with \( s \in \mathbb{R}, \omega \in S^2 \). Then
\[
(R(\lambda, \xi)v_1\sqrt{M}, v_j\sqrt{M}) = \omega \omega_j(R(\lambda, se_1)\chi_1, \chi_1) + (\delta_{ij} - \omega_i\omega_j)(R(\lambda, se_1)\chi_2, \chi_2).
\] (3.55)

With (3.55) the equations (3.52)–(3.54) can be simplified as
\[
\lambda C = (1 + s^2)(R(\lambda, se_1)\chi_1, \chi_1)C_0, \quad (3.56)
\]
\[
\lambda X = (R(\lambda, se_1)\chi_2, \chi_2)X + i\xi \times Y, \quad (3.57)
\]
\[
\lambda Y = -i\xi \times X. \quad (3.58)
\]

Multiply (3.57) by \( \lambda \) and using (3.58) and (3.3), we obtain
\[
(\lambda^2 - (R(\lambda, se_1)\chi_2, \chi_2)\lambda + s^2)X = 0. \quad (3.59)
\]

Denote
\[
D_0(\lambda, s) =: \lambda - (1 + s^2)(R(\lambda, se_1)\chi_1, \chi_1),
\]
\[
D_1(\lambda, s) =: \lambda^2 - (R(\lambda, se_1)\chi_2, \chi_2)\lambda + s^2. \quad (3.60)
\]

The following result on \( D_0(\lambda, s) = 0 \) was proved in [14] in the study on the bipolar Vlasov-Poisson-Boltzmann system.

**Lemma 3.10 ([14]).** There are constants \( b_0 > 0 \) and \( r_0 > 0 \) with \( r_0 \) being small such that the equation \( D_0(\lambda, s) = 0 \) has no solution for \( \text{Re}\lambda \geq -b_0 \) and \( |s| \leq r_0 \).

**Lemma 3.11.** There are constants \( b_1, r_0, r_1 > 0 \) such that the equation \( D_1(\lambda, s) = 0 \) with \( \text{Re}\lambda \geq -b_1 \) has only one solution \( \lambda(s) \) for \( (s, \lambda) \in [-r_0, r_0] \times B_{r_1}(0) \) and it satisfies
\[
\lambda(0) = 0, \quad \lambda'(0) = 0, \quad \lambda''(0) = \frac{1}{(L_1^{-1}\chi_2, \chi_2)}.
\]

**Proof.** Since
\[
D_1(0, 0) = 0, \quad \partial_s D_1(0, 0) = 0, \quad \partial_\lambda D_1(0, 0) = -(L_1^{-1}\chi_2, \chi_2), \quad (3.62)
\]

the application of the implicit function theorem implies that there exist constants \( r_0, r_1 > 0 \) and a unique \( C^\infty \) function \( \lambda_0(s) \) such that \( D_1(\lambda_0(s), s) = 0 \) for \( (s, \lambda) \in [-r_0, r_0] \times B_{r_1}(0) \). In particular,
\[
\lambda_0(0) = 0 \quad \text{and} \quad \lambda_0'(0) = \frac{\partial_s D_1(0, 0)}{\partial_\lambda D_1(0, 0)} = 0. \quad (3.63)
\]

A direct computation gives \( \partial^2_s D(0, 0) = 2 \), which together with (3.62) yield
\[
\lambda_0''(0) = -\frac{\partial^2_s D(0, 0)}{\partial_\lambda D(0, 0)} = \frac{2}{(L_1^{-1}\chi_2, \chi_2)}. \quad (3.64)
\]

Let
\[
D_2(\lambda, s) = \frac{D_1(\lambda, s)}{\lambda - \lambda_0(s)}.
\]

Similarly to Lemma 3.10, we can prove that there is \( b_1 > 0 \) so that \( D_2(\lambda, 0) = \lambda - ((L_1 - \lambda P_r)^{-1}\chi_2, \chi_2) \neq 0 \) for \( \text{Re}\lambda \geq -b_1 \). By (3.60), we have \( |((L_1 - \lambda P_r)^{-1}\chi_1, \chi_1)| \leq C \) for \( \text{Re}\lambda \geq -b_1 \), which leads to \( \lim_{|\lambda| \to \infty} |D_2(\lambda, 0)| = \infty \). This together with the continuity of \( D_2(\lambda, 0) \) imply that there is a constant \( \delta_1 > 0 \) such that \( |D_2(\lambda, 0)| \geq \delta_1 \) for \( \text{Re}\lambda \geq -b_1 \).
Lemma 3.11, and choose $C$ satisfying $W$.

In particular, the eigenvalues admit the following asymptotic expansion for $\lambda$

Theorem 3.12. There exist two constants $r_0 > 0$ and $b_2 > 0$ so that the spectrum $\lambda \in \sigma(\hat{A}_1(\xi)) \subset \mathbb{C}$ for $\xi = s \omega$ with $|s| \leq r_0$ and $\omega \in S^2$ consists of two points $\{\lambda_j(s), j = 1, 2\}$ in the domain $\text{Re} \lambda > -b_2$. The spectrum $\lambda_j(s)$ and the corresponding eigenvector $\Psi_j(s, \omega) = (\psi_j, X_j, Y_j)(s, \omega)$ are $C^\infty$ functions of $s$ for $|s| \leq r_0$. In particular, the eigenvalues admit the following asymptotic expansion for $|s| \leq r_0$

$$\lambda_1(s) = -a_1 s^2 + o(s^2), \quad (3.65)$$

where

$$a_1 = \frac{1}{(L_1^{-1} \chi_2, \chi_2)} > 0. \quad (3.66)$$

The eigenvectors $\Psi_j = (\psi_j, X_j, Y_j)$ are orthogonal to each other and satisfy

$$\begin{cases} 
(\Psi_1(s, \omega), \Psi_j^*(s, \omega)) = (\psi, \overline{\psi}_j) - (X, \overline{X}_j) - (Y, \overline{Y}_j) = \delta_{ij}, & i \neq j = 1, 2, \\
(\psi_j, X_j, Y_j)(s, \omega) = (\psi_{j,0}, X_{j,0}, Y_{j,0})(\omega) + (\psi_{j,1}, X_{j,1}, Y_{j,1})(\omega)s + O(s^2), & |s| \leq r_0,
\end{cases} \quad (3.67)$$

where $\Psi_j^* = (\overline{\psi}_j, -\overline{X}_j, -\overline{Y}_j)$, and the coefficients $\psi_{j, n}, X_{j, n}, Y_{j, n}$ are given by

$$\begin{cases}
\psi_{j,0} = 0, & P_0 \psi_{j, n} = 0 (n \geq 0), \quad X_{j,0} = 0, \quad Y_{j,0} = iW_j, \\
\psi_{j,1} = -a_1 L_1^{-1} P_1 (v \cdot W_j) / \sqrt{M}, & X_{j,1} = a_1 \omega \times W_j, \quad Y_{j,1} = 0.
\end{cases} \quad (3.68)$$

Here, $W_j (j = 1, 2)$ are two orthonormal vectors satisfying $W_j \cdot \omega = 0$.

Proof. The eigenvalues $\lambda_j(s)$ and the eigenvectors $\Psi_j(s, \omega) = (\psi_j, X_j, Y_j)(s, \omega), j = 1, 2$, can be constructed as follows. Let $b_2 = \min\{b_0, b_1\}$ and take $\lambda_j = \lambda(s)$ to be the solution of the equation $D_1(\lambda, s) = 0$ defined in Lemma 3.11 and choose $C_0 = 0$, and $X_j = \omega \times W_j$ with $W_j, j = 1, 2$, being two linearly independent vectors satisfying $W_j \cdot \omega = 0$ and $W_1 \cdot W_2 = 0$. The corresponding eigenvectors $\Psi_j(s, \omega) = (\psi_j, X_j, Y_j)(s, \omega)$ are defined by

$$\begin{cases}
\psi_j(s, \omega) = -[L_1 - \lambda_j P_1 - is P_1 (v \cdot \omega) P_1]^{-1} P_1 (v \cdot W_j) / \sqrt{M}, \\
X_j(s, \omega) = \omega \times W_j, \quad Y_j(s, \omega) = \frac{is}{\lambda_j} W_j, \quad j = 1, 2,
\end{cases} \quad (3.69)$$

which satisfy $(\Psi_1(s, \omega), \Psi_2^*(s, \omega)) = 0$. We can normalize them by taking

$$(\Psi_j(s, \omega), \Psi_j^*(s, \omega)) = 1, \quad j = 1, 2.$$
The coefficients $W^j = b_j(s)T^j(\omega)$ for $j = 1, 2$ with $b_j \in \mathbb{R}$ and $T^j = (T^j_1, T^j_2, T^j_3) \in S^2$ are determined by the normalization condition as

\begin{equation}
\begin{aligned}
b_j(s)^2(D_j(s) - 1 + \frac{s^2}{\lambda_j(s)^2}) = 1, & \quad j = 1, 2, \\
|T^1| = |T^2| = 1, & \quad T^1 \cdot \omega = T^2 \cdot \omega = T^1, T^2 = 0,
\end{aligned}
\end{equation}

where $D_j(s) = (R(\lambda_j(s), s\epsilon_1)\chi_1, R(\lambda_j(s), -s\epsilon_1)\chi_1)$. To study the asymptotic expression of eigenvectors in the low frequency, we can take Taylor expansion of both eigenvalues and eigenvectors as

\[
\lambda_j(s) = \sum_{n=0}^{2} \lambda_{j,n}s^n + O(s^3), \quad (\psi_j, X_j, Y_j)(s, \omega) = \sum_{n=0}^{1}(\psi_{j,n}, X_{j,n}, Y_{j,n})(\omega)s^n + O(s^2).
\]

Substituting the above expansions into (3.70), we obtain $b_j(0) = 0$, $b'_j(0) = a_1$ and $b_j(-s) = -b_j(s)$. This and (3.69) give the expansion of $\Psi_j(s, \omega)$ for $j = 1, 2$, stated in (3.68). And then it completes the proof of the theorem.

### 3.3 Asymptotics in high frequency

We now turn to study the asymptotic expansions of the eigenvalues and eigenvectors in the high frequency region. Firstly, recalling the eigenvalue problem

\[
\begin{align*}
\lambda f &= B_1(\xi)f - v\sqrt{M} \cdot (\omega \times X), \\
\lambda X &= -\omega \times (f, v\sqrt{M}) + i\xi \times Y, \\
\lambda Y &= -i\xi \times X, \quad |\xi| \neq 0.
\end{align*}
\]

By Lemma 3.5 there is $R_0 > 0$ large enough such that the operator $\lambda - B_1(\xi)$ is invertible on $L^2_{\xi}(\mathbb{R}^3)$ for $\text{Re}\lambda \geq -\nu_0/2$ and $|\xi| > R_0$. Then it follows from (3.71) that

\[
f = (B_1(\xi) - \lambda)^{-1}v\sqrt{M} \cdot (\omega \times X), \quad |\xi| > R_0.
\]

Substituting (3.73) into (3.72) and using the transformation

\[
((B_1(\xi) - \lambda)^{-1}\chi_i, \chi_j) = \omega_i\omega_j((B_1(|\xi|e_1) - \lambda)^{-1}\chi_1, \chi_1) + (\delta_{ij} - \omega_i\omega_j)((B(|\xi|e_1) - \lambda)^{-1}\chi_2, \chi_2),
\]

we obtain

\[
\begin{align*}
\lambda X &= ((B_1(|\xi|e_1) - \lambda)^{-1}\chi_2, \chi_2)X + i\xi \times Y, \\
\lambda Y &= -i\xi \times X, \quad |\xi| > R_0.
\end{align*}
\]

Multiplying (3.74) by $\lambda$ and using (3.75) and (3.3), we obtain

\[
(\lambda^2 - ((B_1(|\xi|e_1) - \lambda)^{-1}\chi_2, \chi_2)\lambda + |\xi|^2)X = 0, \quad |\xi| > R_0.
\]

Denote

\[
D(\lambda, s) = \lambda^2 - ((B_1(s\epsilon_1) - \lambda)^{-1}\chi_2, \chi_2)\lambda + s^2, \quad s > R_0.
\]

Similar to the proof of Lemma 3.5 we can obtain

**Lemma 3.13.** For any $\delta > 0$, we have

\[
\sup_{x \geq -\nu_0 + \delta, y \in \mathbb{R}} ||(x + iy - c(\xi))^{-1}K_1|| \leq C\delta^{-15/13}(1 + |\xi|)^{-2/13},
\]

where $K_1$ is the operator defined in (3.31).
As consequence, it holds that for $|\xi| > R_0$

$$
\sup_{x \geq -\gamma_0 + \delta, y \in \mathbb{R}} \| (x + iy - B_1(\xi))^{-1} v \sqrt{M} \| \leq C\delta^{-1/2}(1 + |\xi|)^{-1/2}.
$$

(3.81)

We now study the equation (3.77) as follows.

**Lemma 3.14.** There is a constant $R_1 > 0$ such that the equation $D(\lambda, s) = 0$ has two solutions $\lambda_j(s)$, $j = \pm 1$, for $s > R_1$ satisfying

$$
|\lambda_j(s) - jis| \leq Cs^{-1/2} \to 0, \quad \text{as} \quad s \to \infty.
$$

In particular, there are two constants $c_1, c_2 > 0$ such that

$$
c_1 \frac{1}{s} \leq -\text{Re}\lambda_j(s) \leq c_2 \frac{1}{s}, \quad j = \pm 1.
$$

(3.82)

**Proof.** For any fixed $s > R_0$, we define a function of $\lambda$ as

$$
G_j^1(\lambda) = \frac{1}{2} \left( R_{22}(\lambda, s) + j \sqrt{R_{22}(\lambda, s^2 - 4s^2)} \right), \quad j = \pm 1, \quad s > R_0,
$$

(3.83)

where $R_{22}(\lambda, s) = ((B_j(s) - \lambda)^{-1} \chi_2, \chi_2)$. It is straightforward to verify that a solution of $D(\lambda, s) = 0$ for any fixed $s > R_0$ is a fixed point of $G_j^1(\lambda)$.

Consider an equivalent equation of (3.83) as

$$
G_j^2(\beta) = G_j^1(\lambda) - jis = \frac{1}{2} \left( B_j(\beta, s) + \frac{jB_j(\beta, s^2)}{\sqrt{B_j(\beta, s^2 - 4s^2 + is)}} \right), \quad j = \pm 1, \quad s > R_0,
$$

(3.84)

where $B_j(\beta, s) = ((B_j(s) - jis - \beta)^{-1} \chi_2, \chi_2)$. By (3.81), when $R_1 > 0$ is large enough and $\delta > 0$ is small enough, it holds for $s > R_1$ and $|\beta| \leq \delta$ that

$$
|G_j^2(\beta)| \leq \delta, \quad |G_j^2(\beta_1) - G_j^2(\beta_2)| \leq \frac{1}{2} |\beta_1 - \beta_2|.
$$

Thus $G_j^2(\beta)$ is a contraction mapping on $B_\delta(0)$ and there is a unique fixed point $\beta_j(s)$ of $G_j^2(\beta)$. Thus $\lambda_j(s) = jis + \beta_j(s)$ is the solution of $D(\lambda, s) = 0$ and $|\beta_j(s)| \leq Cs^{-1/2}$ because $|B_j(\beta, s)| \leq C \delta^{-1/2}$ due to (3.81).

We now turn to prove (3.84). For this, we decompose $\beta_j(s)$, $j = \pm 1$, into

$$
\beta_j(s) = \frac{1}{2} B_j(\beta_j, s) + \frac{jB_j(\beta_j, s^2)}{\sqrt{B_j(\beta_j, s^2 - 4s^2 + is)}} = I_1 + I_2.
$$

(3.85)

First, we estimate $I_1$. Since

$$
-\text{Re}I_1 = -\frac{1}{2} (L_1 - \text{Re}\beta_j) g_j = -\frac{1}{2} (L_1 g_j, g_j) + O \left( \frac{1}{\sqrt{s}} \right) (g_j, g_j),
$$

with $g_j = (L_1 - i(v_1 + j)s - \frac{i}{\sqrt{s}}P_d - \beta_j)^{-1} \chi_2$ for $j = \pm 1$, we obtain

$$
C_0(g_j, g_j) \leq -\text{Re}I_1 \leq C_1(\nu(v)g_j, g_j).
$$

(3.86)

Note that

$$
(L_1 - i(v_1 + 1)s - \frac{v_1}{s}P_d - \beta_j)^{-1} = (I - \nu + i(v_1 \pm 1)s)^{-1}(K_1 - \frac{v_1}{s}P_d - \beta_{\pm 1})^{-1}(-\nu - i(v_1 \pm 1)s)^{-1}

\begin{align*}
&= (-\nu - i(v_1 \pm 1)s)^{-1} + Z(\beta_{\pm 1}, s),
\end{align*}
$$
with

\[
Z(\beta_{\pm 1}, s) = Y(\beta_{\pm 1}, s)(I + Y(\beta_{\pm 1}, s))^{-1}(-\nu - i(v_1 \pm 1)s)^{-1},
\]

\[
Y(\beta_{\pm 1}, s) = (-\nu - i(v_1 \pm 1)s)^{-1}(K_1 - \frac{v_1}{s}P_d - \beta_{\pm 1}),
\]

we have

\[
(\nu(v)g_{\pm 1}, g_{\pm 1}) \leq 2(\nu(v + i(v_1 \pm 1)s)^{-1} - \chi_2, \nu + i(v_1 \pm 1)s)^{-1} - \chi_2) + 2(\nu Z(\beta_{\pm 1}, s)\chi_2, Z(\beta_{\pm 1}, s)\chi_2),
\]

\[
(g_{\pm 1}, g_{\pm 1}) \geq \frac{1}{2}((\nu + i(v_1 \pm 1)s)^{-1} - \chi_2, \nu + i(v_1 \pm 1)s)^{-1} - \chi_2) - (Z(\beta_{\pm 1}, s)\chi_2, Z(\beta_{\pm 1}, s)\chi_2).
\]

In the following, we will prove

\[
I_3 = ((\nu + i(v_1 \pm 1)s)^{-1} - \chi_2, \nu + i(v_1 \pm 1)s)^{-1} - \chi_2) \geq \frac{C_3}{s},
\] (3.87)

for some constant $C_3 > 0$. Indeed, by changing variable $(u_1, u_2, u_3) = ((v_1 \pm 1)s, v_2, v_3)$, we obtain for $s > 1$ that

\[
I_3 \geq \int_{R^3} \frac{1}{v_1^2(1 + |v|^2) + (v_1 \pm 1)^2s^2} v_1^2 M dv
\]

\[
\geq \frac{1}{s} \int_{R^3} \frac{1}{v_1^2(1 + (\frac{v_1}{s} \pm 1)^2)^2 + u_2^2 + u_3^2 + u_1^2} u_1^2 e^{-\frac{1}{2}(1 + \frac{v_1^2}{s})} e^{-\frac{u_2^2}{2}} e^{-\frac{u_3^2}{2}} du
\]

\[
\geq \frac{1}{s} \int_{R^3} \frac{1}{v_1^2(3 + 2u_1^2 + u_2^2 + u_3^2) + u_1^2} u_1^2 e^{-u_1^2} e^{-\frac{u_1^2}{2}} e^{-\frac{u_2^2}{2}} e^{-\frac{u_3^2}{2}} du \geq \frac{C_3}{s},
\]

As for

\[
I_4 = (\nu(v + i(v_1 \pm 1)s)^{-1} - \chi_2, \nu(v + i(v_1 \pm 1)s)^{-1} - \chi_2),
\] (3.88)

by the change variables $(u_1, u_2, u_3) = ((v_1 \pm 1)s, v_2, v_3)$, we obtain for $s > 1$ that

\[
I_4 \leq \int_{R^3} \frac{v_1^2(1 + |v|^2)}{v_1^2(1 + (\frac{v_1}{s} \pm 1)^2)^2} v_1^2 M dv \leq C \int_{R^3} \frac{1}{v_1^2 + (v_1 \pm 1)^2s^2} v_1^2 e^{-\frac{|v|^2}{2}} dv
\]

\[
\leq \frac{C}{s} \int_{R^3} \frac{1}{v_1^2 + u_1^2} u_1^2 e^{-\frac{u_1^2}{2}} e^{-\frac{u_2^2}{2}} e^{-\frac{u_3^2}{2}} du \leq \frac{C_4}{s},
\]

where $C_4 > 0$ is a constant.

By Lemma 3.13 for any $0 < \epsilon \ll 1$ there exists $s > R_1$ such that

\[
\|Y(\beta_{\pm 1}, s)\| \leq \|Y(\nu + i(v_1 \pm 1)s)^{-1}(K_1 - \frac{iv_1}{s}P_d)\| + O(\frac{1}{\sqrt{s}})\|Y(\nu + i(v_1 \pm 1)s)^{-1}\| \leq \epsilon,
\]

\[
\|\nu Y(\beta_{\pm 1}, s)\| \leq \|\nu(v + i(v_1 \pm 1)s)^{-1} (K_1 - \frac{iv_1}{s}P_d - \beta_{\pm 1}) \| \leq C.
\]

This and (3.88) lead to

\[
(\nu Z(\beta_{\pm 1}, s)\chi_2, Z(\beta_{\pm 1}, s)\chi_2) \leq \|\nu Y(\beta_{\pm 1}, s)\|\|Y(\beta_{\pm 1}, s)\|\|I + Y(\beta_{\pm 1}, s)^{-1}\|^2 \|\nu + i(v_1 \pm 1)s)^{-1} \chi_2\|^2
\]

\[
\leq C\epsilon I_4 \leq \frac{C\epsilon}{s}.
\]

Thus, by combining with (3.86), (3.87) and (3.88), there exist constants $C_5, C_6 > 0$ such that

\[
\frac{C_5}{s} \leq -Re I_1 \leq \frac{C_6}{s},
\] (3.89)

For $I_2$, we have

\[
|I_2| \leq \frac{C}{s} |B_{\pm 1}(\beta_{\pm 1}, s)|^2 \leq \frac{C}{s^2}.
\] (3.90)

Combining (3.85), (3.89) and (3.90), we obtain (3.82). The proof of the lemma is then completed. \qed
The following theorem gives the asymptotic expansions of eigenvalues and eigenvectors in the high frequency region.

**Theorem 3.15.** There exists a constant $r_1 > 0$ such that the spectrum $\sigma(\hat{A}_1(\xi)) \subset \mathbb{C}$ for $\xi = s\omega$ with $s = |\xi| > r_1$ and $\omega \in S^2$ consists of four eigenvalues $\{\beta_j(s), j = 1, 2, 3, 4\}$ in the domain $\text{Re}\lambda > -\mu/2$. In particular, the eigenvalues satisfy

$$
\beta_1(s) = \beta_2(s) = -is + O(s^{-1/2}),
$$
$$
\beta_3(s) = \beta_4(s) = is + O(s^{-1/2}),
$$
$$
\frac{c_1}{s} \leq -\text{Re}\beta_j(s) \leq \frac{c_2}{s},
$$

for two positive constants $c_1$ and $c_2$. The eigenvectors $\Phi_j(s, \omega) = (\phi_j, X_j, Y_j)(s, \omega)$ are orthogonal to each other and satisfy

$$
(\Phi_j(s, \omega), \Phi_j^*(s, \omega)) = (\phi_i, \phi_j) - (X_i, X_j) - (Y_i, Y_j) = \delta_{ij}, \quad 1 \leq i \neq j \leq 4,
$$

where $\Phi_j^* = (\overline{\phi}_j, -\overline{X}_j, -\overline{Y}_j)$. Moreover,

$$
\|\phi_j(s, \omega)\| = O\left(\frac{1}{\sqrt{s}}\right), \quad P_0\phi_j(s, \omega) = 0, \quad X_j(s, \omega) = O(1)i(\omega \times W^j), \quad Y_j(s, \omega) = O(1)iW^j.
$$

Here, $W^j$ $(j = 1, 2, 3, 4)$ are vectors satisfying $W^j \cdot \omega = 0$, $W^1 \cdot W^2 = 0$, $W^1 = W^3$, $W^2 = W^4$ and the normalization condition $(3.97)$.

**Proof.** The eigenvalue $\lambda_j(s)$ and the eigenvector $\Phi_j(s, \omega) = (\phi_j, X_j, Y_j)(s, \omega)$ can be constructed as follows. For $j = 1, 2, 3, 4$, we take $\lambda_1 = \lambda_2 = \lambda_3 = 1$ and $\lambda_4 = \lambda_1(s)$ to be the solution of the equation $D(\lambda, s) = 0$ defined in Lemma 3.14. Choose $X_j = \omega \times W^j$ with $W^1 = W^3$ and $W^2 = W^4$ as linearly independent vectors so that $W^1 \cdot \omega = 0$ and $W^1 \cdot W^2 = 0$. The corresponding eigenvectors $\Phi_j(s, \omega) = (\phi_j, X_j, Y_j)(s, \omega)$, $1 \leq j \leq 4$, are defined by

$$
\begin{align*}
\phi_j(s, \omega) &= -[L_4 - \beta_j - is(\nu \cdot \omega) - \frac{i(\nu \cdot \omega)}{s}P_0]^1(\nu \cdot W^j)\sqrt{M}, \\
X_j(s, \omega) &= \omega \times W^j, \quad Y_j(s, \omega) = \frac{is}{\lambda_j(s)}W^j,
\end{align*}
$$

which satisfy $(\Phi_1(s, \omega), \Phi_3^*(s, \omega)) = (\Phi_3(s, \omega), \Phi_3^*(s, \omega)) = 0$.

Rewrite the eigenvalue problem as

$$
\hat{A}_1(\xi)\Phi_j(s, \omega) = \beta_j(s)\Phi_j(s, \omega), \quad 1 \leq j \leq 4, \quad |s| \leq r_0.
$$

By taking the inner product $(\cdot, \cdot)_{\xi}$ of it with $\Phi_j^*(s, \omega)$, and using the fact that

$$
(\hat{A}_1(\xi)U, V)_{\xi} = (U, \hat{A}_1(\xi)^*V)_{\xi}, \quad U, V \in D(\hat{B}_1(\xi)) \times \mathbb{C}_\xi^2 \times \mathbb{C}_\xi^3,
$$

we have

$$
(\beta_j(s) - \beta_k(s))(\Phi_j(s, \omega), \Phi_k^*(s, \omega))_{\xi} = 0, \quad 1 \leq j, k \leq 4.
$$

Since $\beta_j(s) \neq \beta_k(s)$ for $j = 1, 2, k = 3, 4$ and $P_0\phi_j(s, \omega) = 0$, we have the orthogonal relation

$$
(\Phi_j(s, \omega), \Phi_k^*(s, \omega))_{\xi} = (\Phi_j(s, \omega), \Phi_j^*(s, \omega)) = 0, \quad 1 \leq j \neq k \leq 4.
$$

This can be normalized so that

$$
(\Phi_j(s, \omega), \Phi_j^*(s, \omega)) = 1, \quad j = 1, 2, 3, 4.
$$
Precisely, denote \( W_j = b_j(s)T^j(\omega) \) for \( j = 1, 2, 3, 4 \), with \( b_j \in \mathbb{R} \) and \( T^j = (T^j_1, T^j_2, T^j_3) \in \mathbb{S}^2 \), then the coefficients \( b_j, j = 1, \ldots, 4 \) are determined by the normalization condition

\[
\begin{cases}
\quad b_j(s)^2(D_j(s) - 1 + \frac{s^2}{\lambda_j(s)^2}) = 1, & j = 1, 2, 3, 4, \\
\quad |T^j| = 1, & T^j \cdot \omega = T^1 \cdot T^2 = 0, & T^1 = T^3, & T^2 = T^4,
\end{cases}
\]  

(3.97)

where \( D_j(s) = ((B_1(s e_1) - \beta_j(s))^{-1} \chi_1, (B_1(-s e_1) - \beta_j(s))^{-1} \chi_1) \). By substituting (3.81), (3.91) and (3.92) into (3.97) and (3.96), we obtain (3.95) so that the proof of the theorem is completed.

4 Spectral analysis for one-species case

In this section, we will study the spectrum structure of the linearized system \( 2.54 \) of one-species VMB. It is interesting to find out that its structure is very different in the low frequency region from the case of two-species. And this essential difference comes from the lack of the cancellation in the one-species system.

4.1 Spectral structure

Since \( L^2_\xi(\mathbb{R}^3_+)^2 \times C^1_\xi \times C^2_\xi \) is the invariant subspace of the operator \( \hat{A}_3(\xi) \), we can regard \( \hat{A}_3(\xi) \) as a linear operator on \( L^2_\xi(\mathbb{R}^3_+)^2 \times C^1_\xi \times C^2_\xi \). We have for any \( U, V \in L^2_\xi(\mathbb{R}^3_+) \cap D(B_2(\xi)) \times C^1_\xi \times C^2_\xi \),

\[
(\hat{A}_3(\xi)U, V)_\xi = (U, \hat{A}_3(-\xi)V)_\xi.
\]

Denote by \( \rho(\hat{A}_3(\xi)) \) the resolvent set and by \( \sigma(\hat{A}_3(\xi)) \) the spectrum set of \( \hat{A}_3(\xi) \). Similar to Lemmas 3.1 and 3.3 we have the following lemmas.

Lemma 4.1. The operator \( \hat{A}_3(\xi) \) generates a strongly continuous contraction semigroup on \( L^2_\xi(\mathbb{R}^3_+) \times C^1_\xi \times C^2_\xi \) satisfying

\[
\|e^{t\hat{A}_3(\xi)}U\|_\xi \leq \|U\|_\xi, \quad \text{for } t > 0, U \in L^2_\xi(\mathbb{R}^3_+) \times C^1_\xi \times C^2_\xi.
\]  

(4.1)

Lemma 4.2. For each \( \xi \neq 0 \), the spectrum set \( \sigma(\hat{A}_3(\xi)) \) of the operator \( \hat{A}_3(\xi) \) in the domain \( \text{Re}\lambda \geq -\nu_0 + \delta \) for any constant \( \delta > 0 \) consists of isolated eigenvalues \( \Sigma =: \{\lambda_j(\xi)\} \) with \( \text{Re}\lambda_j(\xi) < 0 \).

We will make use of the following decomposition associated with the operator \( \hat{A}_3(\xi) \) for \( |\xi| > 0 \)

\[
\lambda - \hat{A}_3(\xi) = \lambda - G_1(\xi) - G_5(\xi) = (I - G_5(\xi)(\lambda - G_1(\xi))^{-1})(\lambda - G_1(\xi)),
\]

(4.2)

where \( G_1(\xi) \) is defined by (3.11), and

\[
G_5(\xi) = \begin{pmatrix}
K - \frac{i(v \cdot \xi)}{|\xi|^2} P_d & -v \sqrt{M} \cdot \omega \times 0 \\
-\omega \times P_m & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]  

(4.3)

Here,

\[
G_5(\xi)(\lambda - G_1(\xi))^{-1} = \begin{pmatrix}
X_1^\dagger(\lambda, \xi) & X_2(\lambda, \xi) \\
X_3(\lambda, \xi) & 0
\end{pmatrix}_{7 \times 7},
\]

(4.4)

where \( X_2(\lambda, \xi) \) and \( X_3(\lambda, \xi) \) are defined by (3.20) and (3.21), and

\[
X_1^\dagger(\lambda, \xi) = (K - \frac{i(v \cdot \xi)}{|\xi|^2} P_d)(\lambda - c(\xi))^{-1}.
\]

(4.5)

By a similar argument as the one for Lemma 3.6 we can obtain the spectrum of the operator \( \hat{A}_3(\xi) \) in the intermediate and high frequency regions.
Lemma 4.3. In the high and intermediate regions of the frequency, we have

1. For any $\delta_1, \delta_2 > 0$, there exists $r_1 = r_1(\delta_1, \delta_2) > 0$ so that for $|\xi| > r_1$,

$$\sigma(\hat{\Lambda}_3(\xi)) \cap \{\lambda \in \mathbb{C} | \text{Re}\lambda \geq -\nu_0 + \delta_1\} \subset \bigcup_{j=\pm 1} \{\lambda \in \mathbb{C} | |\lambda - ji\xi| \leq \delta_2\}. \quad (4.6)$$

2. For any $r_1 > r_0 > 0$, there exists $\alpha = \alpha(r_0, r_1) > 0$ so that for all $r_0 \leq |\xi| \leq r_1$,

$$\sigma(\hat{\Lambda}_3(\xi)) \subset \{\lambda \in \mathbb{C} | \text{Re}\lambda(\xi) \leq -\alpha\}. \quad (4.7)$$

Then, we only need to study the spectrum and resolvent sets of $\hat{\Lambda}_3(\xi)$ in low frequency region. For this, we decompose $\lambda - \hat{\Lambda}_3(\xi)$ as

$$\lambda - \hat{\Lambda}_3(\xi) = \lambda P_A - G_0(\xi) + \lambda P_B - G_7(\xi) + G_8(\xi) + G_9(\xi), \quad (4.8)$$

where

$$G_0(\xi) = \begin{pmatrix} B_4(\xi) & -v\sqrt{M} \cdot \omega & 0 \\ -\omega \times P_m & 0 & i\xi \times \\ 0 & -i\xi \times & 0 \end{pmatrix}, \quad G_7(\xi) = \begin{pmatrix} B_3(\xi) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad G_8(\xi) = \begin{pmatrix} iP_1(v \cdot \xi)P_0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad G_9(\xi) = \begin{pmatrix} iP_9(v \cdot \xi)P_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (4.9)$$

$$B_4(\xi) = iP_0(v \cdot \xi)P_0 - \frac{i(v \cdot \xi)}{|\xi|^2} P_0, \quad B_3(\xi) = L - iP_1(v \cdot \xi)P_1, \quad (4.10)$$

and $P_A, P_B$ are the orthogonal projection operators defined by

$$P_A = \begin{pmatrix} P_0 & 0 & 0 \\ 0 & I_{3\times 3} & 0 \\ 0 & 0 & I_{3\times 3} \end{pmatrix}, \quad P_B = \begin{pmatrix} P_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (4.12)$$

It is straightforward to verify that $G_0(\xi)$ is a linear operator from the space $N_0 \times \mathbb{C}_\xi^3 \times \mathbb{C}_\xi^3$ to itself, which admits nine eigenvalues $\alpha_j(\xi)$ satisfying

$$\begin{cases} \alpha_j(\xi) = 0, & j = 0, 2, 3, \quad \alpha_{\pm 1}(\xi) = \pm i\sqrt{1 + \frac{3}{4}|\xi|^2}, \\
\alpha_4(\xi) = \alpha_5(\xi) = -i\sqrt{1 + |\xi|^2}, \quad \alpha_6(\xi) = \alpha_7(\xi) = i\sqrt{1 + |\xi|^2}. \end{cases} \quad (4.13)$$

Lemma 4.4. Let $\xi \neq 0$, we have the following properties for the linear operators $G_0(\xi)$ and $G_7(\xi)$ defined by (4.8).

1. If $\lambda \neq \alpha_j(\xi)$, then the operator $\lambda P_A - G_0(\xi)$ is invertible on $N_0 \times \mathbb{C}_\xi^3 \times \mathbb{C}_\xi^3$ and satisfies

$$\| (\lambda P_A - G_0(\xi))^{-1} \|_\xi = \max_{-1 \leq j \leq 7} \| (\lambda - \alpha_j(\xi))^{-1} \|, \quad (4.14)$$

$$\| G_0(\xi)(\lambda P_A - G_0(\xi))^{-1} P_A \|_\xi \leq C|\xi| \max_{-1 \leq j \leq 7} \| (\lambda - \alpha_j(\xi))^{-1} \|, \quad (4.15)$$

where $\alpha_j(\xi)$, $-1 \leq j \leq 7$, are the eigenvalues of $G_0(\xi)$ defined by (4.13).

2. If $\text{Re}\lambda > -\mu$, then the operator $\lambda P_B - G_7(\xi)$ is invertible on $N_0^+ \times \{0\} \times \{0\}$ and satisfies

$$\| (\lambda P_B - G_7(\xi))^{-1} \| \leq (\text{Re}\lambda + \mu)^{-1}, \quad (4.16)$$

$$\| G_7(\xi)(\lambda P_B - G_7(\xi))^{-1} P_B \|_\xi \leq C(1 + |\lambda|)^{-1}[ (\text{Re}\lambda + \mu)^{-1} + 1]|\xi| + |\xi|^2). \quad (4.17)$$
Proof. Since the operator $iG_0(\xi)$ is self-adjoint on $N_0 \times \mathbb{C}^3_\xi \times \mathbb{C}^3_\xi$, namely, 

$$(iG_0(\xi)U,V)_\xi = (U,iG_0(\xi)V)_\xi, \quad U,V \in N_0 \times \mathbb{C}^3_\xi \times \mathbb{C}^3_\xi.$$  

(4.18)

we can prove $4.13-4.15$. And by the dissipative of the operator $G_\gamma(\xi)$ on $N^\perp_\gamma \times \{0\} \times \{0\}$, we can prove $4.10-4.17$. 

With the help of Lemmas $4.2-4.4$, we obtain the spectral and resolvent sets of the operator $\hat{A}_3(\xi)$ by applying the similar argument as Lemma $4.8$.

**Lemma 4.5.** For any $\delta_1, \delta_2 > 0$, there are $r_1 = r_1(\delta_1, \delta_2), r_2 = r_2(\delta_1, \delta_2), y_1 = y_1(\delta_1, \delta_2) > 0$ such that

1. it holds for all $|\xi| \neq 0$ that the resolvent set of $\hat{A}_3(\xi)$ contains the following regions

$$\rho(\hat{A}_3(\xi)) \supset \left\{ \lambda \in \mathbb{C} | \text{Re}\lambda \geq -\mu + \delta_1, |\text{Im}\lambda| \geq y_1 \right\} \cup \left\{ \lambda \in \mathbb{C} | \text{Re}\lambda > 0, |\xi| \leq r_1 \right\}$$

$$\left\{ \lambda \in \mathbb{C} | \text{Re}\lambda \geq -\mu + \delta_1, \min_{j=1,2} |\lambda - j| |\xi| \geq \delta_2 \right\} \cup \left\{ \lambda \in \mathbb{C} | \text{Re}\lambda > 0, |\xi| \geq r_1 \right\}.$$  

(4.19)

2. it holds for $0 < |\xi| \leq r_2$ that the spectrum set of $\hat{A}_3(\xi)$ is located in the following domain

$$\sigma(\hat{A}_3(\xi)) \cap \{ \lambda \in \mathbb{C} | \text{Re}\lambda \geq -\mu + \delta_1 \} \subset \bigcup_{j=-1}^{7} \{ \lambda \in \mathbb{C} | |\lambda - \alpha_j(\xi)| \leq \delta_2 \},$$  

(4.20)

where $\alpha_j(\xi), -1 \leq j \leq 7$, are the eigenvalues of $G_0(\xi)$ defined in $4.13$.

### 4.2 Asymptotics in low frequency

We study the low frequency asymptotics of the eigenvalues and eigenvectors of the operator $\hat{A}_3(\xi)$ in this subsection. In terms of $(2.56)$, the eigenvalue problem $\hat{A}_3(\xi)U = \lambda U$ for $U = (f,X,Y) \in L^2(\mathbb{R}^3) \times \mathbb{C}^3_\xi \times \mathbb{C}^3_\xi$ can be written as

$$\begin{align*}
\lambda f &= (L - i(v \cdot \xi) - \frac{i(v \cdot \xi)}{|\xi|^2} P_0) f - v\sqrt{M} \cdot (\omega \times X), \\
\lambda X &= -\omega \times (f,v\sqrt{M}) + i\xi \times Y, \\
\lambda Y &= -i\xi \times X, \quad |\xi| \neq 0.
\end{align*}$$

(4.21)

By macro-micro decomposition, the eigenfunction $f$ of $(4.21)$ can be decomposed into $f = f_0 + f_1 =: P_0 f + P_1 f$. Then the first equation of $(4.21)$ gives rise to

$$\begin{align*}
\lambda f_0 &= -P_0[i(v \cdot \xi)(f_0 + f_1)] - \frac{i(v \cdot \xi)}{|\xi|^2} P_0 f_0 - v\sqrt{M} \cdot (\omega \times X), \\
\lambda f_1 &= L f_1 - P_1[i(v \cdot \xi)(f_0 + f_1)] \implies f_1 = -i(\lambda P_1 - B_2(\xi))^{-1} P_1(v \cdot \xi) f_0.
\end{align*}$$

(4.22)

(4.23)

Substituting $(4.22)$ into $(4.23)$, we obtain the eigenvalue problem for $(f_0, X, Y)$ as

$$\begin{align*}
\lambda f_0 &= -iP_0(v \cdot \xi) f_0 - \frac{i(v \cdot \xi)}{|\xi|^2} P_0 f_0 + P_0[i(v \cdot \xi) R^1(\lambda,\xi)\xi P_1(v \cdot \xi) f_0] - v\sqrt{M} \cdot (\omega \times X), \\
\lambda X &= -\omega \times (f_0,v\sqrt{M}) + i\xi \times Y, \\
\lambda Y &= -i\xi \times X,
\end{align*}$$

(4.24)

where $R^1(\lambda,\xi) = -(\lambda P_1 - B_2(\xi))^{-1} = [L - \lambda P_1 - iP_1(v \cdot \xi) P_1]^{-1}$.

To solve the eigenvalue problem $(4.24)$, we write $f_0 \in N_0$ as $f_0 = \sum_{j=0}^{4} W_j \chi_j$. Taking the inner product of the first equation of $(4.24)$ and $\chi_j$ for $j = 0, 1, 2, 3, 4$ respectively, we have the equations about $\lambda$ and $(W_0, W, W_4, X, Y)$ with $W = (W_1, W_2, W_3)$ and $X = (X_1, X_2, X_3), Y = (Y_1, Y_2, Y_3) \in \mathbb{C}^3_\xi$ for $\text{Re}\lambda > -\mu$:

$$\lambda W_0 = -i(W \cdot \xi),$$

(4.25)
\[ \lambda W_1 = -iW_0 \left( \xi + \frac{\xi}{|\xi|} \right) - i\sqrt{\frac{7}{3}}W_4 \xi + \sum_{j=1}^{3} W_j (R^1(\lambda, \xi)P_1(v \cdot \xi)x_j, (v \cdot \xi)x_i) \\
+ W_4(R^1(\lambda, \xi)P_1(v \cdot \xi)x_4, (v \cdot \xi)x_i) - (\omega \times X)_i, \]
(4.26)
\[
\lambda W_4 = -i\sqrt{\frac{7}{3}}(W \cdot \xi) + \sum_{j=1}^{3} W_j (R^1(\lambda, \xi)P_1(v \cdot \xi)x_j, (v \cdot \xi)x_4) \\
+ W_4(R^1(\lambda, \xi)P_1(v \cdot \xi)x_4, (v \cdot \xi)x_4), \]
(4.27)
\[
\lambda X = -\omega \times W + i\xi \times Y, \]
(4.28)
\[
\lambda Y = -i\xi \times X. \]
(4.29)

We apply the following transform to simplify the system (4.25)–(4.27).

**Lemma 4.6** ([13]). Let \( e_1 = (1, 0, 0), \xi = s\omega \) with \( s \in \mathbb{R}, \omega = (\omega_1, \omega_2, \omega_3) \in \mathbb{S}^2 \). Then, it holds for \( 1 \leq i, j \leq 3 \) and \( \Re \lambda > -\mu \) that

\[
(R^1(\lambda, \xi)P_1(v \cdot \xi)x_j, (v \cdot \xi)x_4) = s^2(\delta_{ij} - \omega_i\omega_4)(R^1(\lambda, se_1)P_1(v_1x_4), v_1x_2) \\
+ s^2\omega_j\omega_4(R^1(\lambda, se_1)P_1(v_1x_1), v_1x_1), \]
(4.30)
\[
(R^1(\lambda, \xi)P_1(v \cdot \xi)x_4, (v \cdot \xi)x_4) = s^2\omega_4(R^1(\lambda, se_1)P_1(v_1x_4), v_1x_1), \]
(4.31)
\[
(R^1(\lambda, \xi)P_1(v \cdot \xi)x_i, (v \cdot \xi)x_4) = s^2\omega_i(R^1(\lambda, se_1)P_1(v_1x_1), v_1x_4), \]
(4.32)
\[
(R^1(\lambda, \xi)P_1(v \cdot \xi)x_4, (v \cdot \xi)x_4) = s^2(R^1(\lambda, se_1)P_1(v_1x_4), v_1x_4). \]
(4.33)

With the help of (4.30)–(4.33), the equations (4.25)–(4.27) can be simplified as

\[
\lambda W_0 = -is(W \cdot \omega), \]
(4.34)
\[
\lambda W_i = -iW_0 \left( s + \frac{1}{s} \right) \omega_i - is\sqrt{\frac{7}{3}}W_4 \omega_i + s^2(W \cdot \omega)\omega_i R_{11} \\
+ s^2(W_i - (W \cdot \omega)\omega_i)R_{22} + s^2W_4 \omega_i R_{41} - (\omega \times X)_i, \]
(4.35)
\[
\lambda W_4 = -is\sqrt{\frac{7}{3}}(W \cdot \omega) + s^2(W \cdot \omega)R_{14} + s^2W_4 R_{44}, \]
(4.36)

where

\[
R_{ij} = R_{ij}(\lambda, s) = (R^1(\lambda, se_1)P_1(v_1x_i), v_1x_j). \]
(4.37)

Multiplying (4.35) by \( \omega_i \) and making the summation of resulted equations with respect to \( i = 1, 2, 3 \), we have

\[
\lambda (W \cdot \omega) = -iW_0 \left( s + \frac{1}{s} \right) - is\sqrt{\frac{2}{3}}W_4 + s^2(W \cdot \omega)R_{11} + s^2W_4 R_{41}. \]
(4.38)

Denote by \( U = (W_0, W \cdot \omega, W_4) \) a vector in \( \mathbb{R}^3 \). The system (4.34), (4.36) and (4.38) can be written as \( MU = 0 \) with the matrix \( M \) defined by

\[
M = \begin{pmatrix}
\lambda & is & 0 \\
i(s + \frac{1}{s}) & \lambda - s^2R_{11} & is\sqrt{\frac{2}{3}} - s^2R_{41} \\
0 & is\sqrt{\frac{2}{3}} - s^2R_{14} & \lambda - s^2R_{44}
\end{pmatrix}. \]
(4.39)

The equation \( MU = 0 \) admits a non-trivial solution \( U \neq 0 \) for \( \Re \lambda > -\mu \) if and only if it holds \( \det(M) = 0 \) for \( \Re \lambda > -\mu \).

Furthermore, by taking \( \omega \times \) to (4.35) and using (4.33), we have

\[
(\lambda - s^2R_{22})(\omega \times W) = X. \]
(4.40)
Moreover, \( \lambda \) denote \( C \) then we multiply above equation by \( \lambda \) and using (4.20) and (3.3) to have
\[
(\lambda^3 - s^2 R_{22} \lambda^2 + (1 + s^2) \lambda - s^2 R_{22})X = 0.
\] (4.41)

Denote
\[
D(\lambda, s) = \det(M), \quad D_0(\lambda, s) =: \lambda^3 - s^2 R_{22} \lambda^2 + (1 + s^2) \lambda - s^4 R_{22}.
\] (4.42)

The following two lemmas are about the solutions to the equations \( D(\lambda, s) = 0 \) and \( D_0(\lambda, s) = 0 \).

**Lemma 4.7.** There exist two small constants \( r_0 > 0 \) and \( r_1 > 0 \) so that the equation \( D(\lambda, s) = 0 \) admits \( C^\infty \) solution \( \lambda_j(s) \) \( (j = -1, 0, 1) \) for \( s, \lambda_j \in [-r_0, r_0] \times B_{r_1}(ji) \) that satisfy
\[
\lambda_j(0) = ji, \quad \lambda_j'(0) = 0, \quad \lambda_j''(0) = (L(L + iP_1)^{-1}P_1(v_1 \chi_1), (L + iP_1)^{-1}P_1(v_1 \chi_1))
\]
\[
\pm i\left\| (L + iP_1)^{-1}P_1(v_1 \chi_1) \right\|^2 + \frac{5}{3},
\] (4.44)
\[
\lambda_j'''(0) = 2(L^{-1}P_1(v_1 \chi_4), v_1 \chi_4).
\] (4.45)

Moreover, \( \lambda_j(s) \) is an even function and satisfies
\[
\lambda_j(s) = \lambda_{-j}(-s) = \lambda_{-j}(s) \quad \text{for} \quad j = 0, \pm 1.
\] (4.46)

In particular, \( \lambda_0(s) \) is a real function.

**Lemma 4.8.** There exist two small constants \( r_0 > 0 \) and \( r_1 > 0 \) so that the equation \( D_0(\lambda, s) = 0 \) admits \( C^\infty \) solution \( \lambda_j(s) \) \( (j = -1, 0, 1) \) for \( s, \lambda_j \in [-r_0, r_0] \times B_{r_1}(ji) \) that satisfy
\[
\lambda_j(0) = ji, \quad \lambda_j'(0) = 0, \quad \lambda_j''(0) = \lambda_j'''(0) = 0,
\] (4.47)
\[
\lambda_j'''(0) = (L(L + iP_1)^{-1}P_1(v_1 \chi_2), (L + iP_1)^{-1}P_1(v_1 \chi_2))
\]
\[
\pm i\left\| (L + iP_1)^{-1}P_1(v_1 \chi_2) \right\|^2 + 1,
\] (4.48)
\[
\lambda_j^{(4)}(0) = 24(L^{-1}P_1(v_1 \chi_2), v_1 \chi_2).
\] (4.49)

Moreover, \( \lambda_j(s) \) is an even function and satisfies
\[
\lambda_j(s) = \lambda_{-j}(-s) = \lambda_{-j}(s) \quad \text{for} \quad j = 0, \pm 1.
\] (4.50)

In particular, \( \lambda_0(s) \) is a real function.

**Proof.** Since \( D_0(\lambda, s) = 0 \) has three roots of the form \( (\lambda, s) = (ji, 0) \) for \( j = -1, 0, 1 \), with \( \lambda = ji \) being the solution to \( D_0(\lambda, 0) = \lambda(\lambda^2 + 1), \) and
\[
\partial_1 D_0(ji, 0) = 0, \quad \partial_1^2 D_0(ji, 0) = 1 - 3j^2 \neq 0,
\] (4.51)
the implicit function theorem implies that there exists small constants \( r_0, r_1 > 0 \) and a unique \( C^\infty \) function \( \lambda_j(s) : [-r_0, r_0] \to B_{r_1}(ji) \) such that \( D_0(\lambda_j(s), s) = 0 \) for \( s \in [-r_0, r_0] \), and in particular
\[
\lambda_j(0) = ji \quad \text{and} \quad \lambda_j'(0) = \frac{\partial_1 D_0(ji, 0)}{\partial_2 D_0(ji, 0)} = 0, \quad j = 0, \pm 1.
\]

Direct computation gives
\[
\partial_2^2 D_0(ji, 0) = 2j^2((L - jiP_1)^{-1}P_1(v_1 \chi_2), v_1 \chi_2) + 2ji.
\]
\[ \partial^2 D_0(0,0) = \partial^1 D_0(0,0) = 0, \quad \partial^4 D_0(0,0) = 24((L - jiP_1)^{-1}P_1(v_1\chi_2), v_1\chi_2), \]

which together with (4.51) yields

\[
\begin{cases}
\lambda_0''(0) = \lambda_0''(0) = 0, & \lambda_0^{(4)}(0) = -\frac{\partial^4 D_0(0,0)}{\partial^4 D_0(0,0)} = 24(1-P_1(v_1\chi_2), v_1\chi_2), \\
\lambda_{\pm 1}'(0) = \frac{\partial D_0(\pm \imath s)}{\partial \chi^\imath D_0(\pm \imath)} = ((L \mp \imath P_1)^{-1}P_1(v_1\chi_2), v_1\chi_2) \pm i.
\end{cases}
\] (4.52)

Finally, since \(D_0(\lambda, s) = D_0(\lambda, -s), D_0(\lambda, s) = D_0(\lambda, -s),\) we can obtain \(\lambda_0(s)\) by using the fact that \(\lambda_{\pm 1}(s) = \pm i + O(s^2)\) and \(\lambda_0(s) = O(s^4)\) as \(s \to 0.\)

With the help of Lemmas E.7, E.8 we are able to construct the eigenvector \(\Psi_j(s, \omega)\) corresponding to the eigenvalue \(\lambda_j\) at the low frequency. Indeed, we have

**Theorem 4.9.** There exists a constant \(r_0 > 0\) so that the spectrum \(\lambda \in \sigma(A_{s}(\xi)) \subset \mathbb{C}\) for \(\xi = sw\) with \(|s| \leq r_0\) and \(\omega \in \mathbb{S}^2\) consists of nine points \(\{\lambda_j(s), -1 \leq j \leq 7\}\) in the domain \(\text{Re}\lambda > -\mu/2.\) The spectrum \(\lambda_j(s)\) and the corresponding eigenvector \(\Psi_j(s, \omega) = (\psi_j, X_j, Y_j)(s, \omega)\) are \(C^\infty\) functions of \(s\) for \(|s| \leq r_0.\) In particular, the eigenvalues admit the following asymptotic expansion for \(|s| \leq r_0\)

\[
\begin{align*}
\lambda_{\pm 1}(s) &= \pm i + (-a_1 \pm ib_1)s^2 + o(s^2), & \lambda_1 = \lambda_{-1}, \\
\lambda_0(s) &= -a_0s^2 + o(s^2), \\
\lambda_{2}(s) &= \lambda_3(s) = -i + (-a_2 - ib_2)s^2 + o(s^2), & \lambda_2 = \lambda_4, \\
\lambda_4(s) &= \lambda_5(s) = i + (-a_2 + ib_2)s^2 + o(s^2), \\
\lambda_6(s) &= \lambda_7(s) = -a_3s^4 + o(s^4),
\end{align*}
\] (4.53)

where \(a_j > 0 \ (0 \leq j \leq 3)\) and \(b_j > 0 \ (1 \leq j \leq 2)\) are given by

\[
\begin{align*}
a_0 &= -(L^{-1}P_1(v_1\chi_4), v_1\chi_4), & a_3 &= -(L^{-1}P_1(v_1\chi_2), v_1\chi_2), \\
a_1 &= -\frac{1}{2}(L + iP_1)^{-1}P_1(v_1\chi_1), (L + iP_1)^{-1}P_1(v_1\chi_1), \\
a_2 &= -\frac{1}{2}(L + iP_1)^{-1}P_1(v_1\chi_2), (L + iP_1)^{-1}P_1(v_1\chi_2), \\
b_1 &= \frac{1}{4}([(L + iP_1)^{-1}P_1(v_1\chi_1)]]^2 + \frac{1}{8}), & b_2 &= \frac{1}{4}([(L + iP_1)^{-1}P_1(v_1\chi_2)]^2 + 1) + 1.
\end{align*}
\] (4.54)

The eigenvectors \(\Psi_j = (\psi_j, X_j, Y_j)\) are orthogonal to each other and satisfy

\[
\begin{align*}
(\Psi_j(s, \omega), \Psi_j^*(s, \omega))_\xi &= (\psi_j, \bar{\psi}_j)_\xi - (X_j, \bar{X}_j) - (Y_j, \bar{Y}_j) = \delta_{ij}, & -1 \leq i, j \leq 7, \\
(\psi_j, X_j, Y_j)(s, \omega) &= \sum_{n=0}^2(\psi_{j,n}, X_{j,n}, Y_{j,n})(\omega)s^n + o(s^n), & |s| \leq r_0.
\end{align*}
\] (4.55)

where \(\Psi_j^* = (\psi_j, -\bar{X}_j, -\bar{Y}_j)\), and the coefficients \((\psi_{j,n}, X_{j,n}, Y_{j,n})\) are given by

\[
\begin{align*}
\psi_{0,0} &= \chi_4, & \psi_{0,1} &= iL^{-1}P_1(v \cdot \omega)\chi_4, & \psi_{0,2} &= (P_1(v \cdot \omega))^\tau M, & X_0 = Y_0 = 0, \\
\psi_{\pm 1,0} &= \pm \sqrt{2}(v \cdot \omega) \sqrt{M}, & (\psi_{\pm 1,2}, \sqrt{M}) &= 0, & X_{\pm 1} = Y_{\pm 1} = 0, \\
\psi_{\pm 1,1} &= \pm \sqrt{2} \sqrt{M} \mp \sqrt{2} \chi_4 + \sqrt{2} (L \mp iP_1)^{-1}P_1(v \cdot \omega)^2 \sqrt{M}, \\
\psi_{j,0} &= \sqrt{2}(v \cdot W^j) \sqrt{M}, & (\psi_{j,n}, \sqrt{M}) &= 0 (n \geq 0), \\
P_1 \psi_{j,1} &= i\sqrt{2} L^{-1}P_1((v \cdot \omega)(v \cdot W^j)^\tau M), & j &= 2, 3, 4, 5, \\
X_{j,0} &= -i \sqrt{2} \omega \times W^j, & Y_{j,0} &= 0, & j &= 2, 3, & X_{j,0} = i\sqrt{2} \omega \times W^j, & Y_{j,0} &= 0, & j &= 4, 5, \\
\psi_{j,0} &= 0, & (\psi_{j,n}, \sqrt{M}) &= 0 (n \geq 0), \\
\psi_{j,1} &= (v \cdot W^j) \sqrt{M}, & X_{j,0} &= X_{j,1} = X_{j,2} = 0, & Y_{j,0} &= iW^j, & j &= 6, 7, \\
\end{align*}
\] (4.56)

Here, \(W^j (j = 2, 3, 4, 5, 6, 7)\) are normal vectors satisfying \(W^1 \cdot \omega = 0, W^1 \cdot W^2 = 0, W^2 = W^4 = W^6, W^3 = W^5 = W^7.\)
Proof. The eigenvalues $\lambda_j(s)$ and the eigenvectors $\Psi_j(s,\omega) = (\psi_j, X_j, Y_j)(s,\omega)$, $-1 \leq j \leq 7$, can be constructed as follows. For $j = 2, 3, 4, 5, 6, 7$, we take $\lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = \lambda_7 = \lambda_0(s)$ to be the solution of the equation $D_0(\lambda, s) = 0$ defined in Lemma 4.38 and choose $W_0 = W_4 = 0$, and $W = W^0$ to be the linearly independent vector so that $W_j \cdot \omega = 0$, $W^2 = W^3 = W^4 = W^5 = W^6 = W^7 = 0$ and $W^2 = W^4 = W^6$, $W^3 = W^5 = W^7$. And the corresponding eigenvectors $\Psi_j(s,\omega) = (\psi_j, X_j, Y_j)(s,\omega)$ are defined by
\[
\begin{align*}
\psi_j(s, \omega) &= (W^j \cdot v)\sqrt{M} + i[sL - \lambda_jP_1 - isP_1(v \cdot \omega)P_1]^{-1}P_1[(v \cdot \omega)(W^j \cdot v)\sqrt{M}], \\
X_j(s, \omega) &= (\lambda_j - s^2R_{22}(\lambda_j, s))\omega \times W^j, \\
Y_j(s, \omega) &= \frac{is}{\lambda_j}(\lambda_j - s^2R_{22}(\lambda_j, s))W^j,
\end{align*}
\]
which satisfy the orthonormal relation: $(\Psi_2, \Psi_3^\ast)_s = (\Psi_4, \Psi_5^\ast)_s = (\Psi_6, \Psi_7^\ast)_s = 0$.

For $j = -1, 0, 1$, we choose $\lambda_j = \lambda_j(s)$ to be a solution of $D(\lambda, s) = 0$ given by Lemma 4.4 and choose $X = Y = 0$, and denote by $\{a_j, b_j, d_j\} = \{W_0^j, (W \cdot \omega)^j, W_4^j\}$ a solution of system (4.34), (4.36), and (4.38) for $\lambda = \lambda_j(s)$. Then we define $\Psi_j(s, \omega) = (\psi_j(s, \omega), 0, 0)$ $(j = -1, 0, 1)$ where
\[
\begin{align*}
\psi_j(s, \omega) &= P_0\psi_j(s, \omega) + P_1\psi_j(s, \omega), \\
P_0\psi_j(s, \omega) &= a_j(s)\chi_0 + b_j(s)(v \cdot \omega)\sqrt{M} + d_j(s)\chi_4, \\
P_1\psi_j(s, \omega) &= isL - \lambda_jP_1 - isP_1(v \cdot \omega)P_1]^{-1}P_1[(v \cdot \omega)P_0\psi_j(s, \omega)].
\end{align*}
\]
Then following the similar argument as Theorem 3.4.2 we can obtain the expansion of $\Psi_j(s, \omega)$ in (4.55) and (4.56). Hence, we omit the detail for brevity.

### 4.3 Asymptotics in high frequency

The structure of the spectrum for the one-species VMB in high frequency region is similar to the two-species case so that we only sketch the key points here. Recalling the eigenvalue problem
\[
\begin{align*}
\lambda f &= B_2(\xi)f - v\sqrt{M} \cdot (\omega \times X), \\
\lambda X &= -\omega \times (f, v\sqrt{M}) + i\xi \times Y, \\
\lambda Y &= -i\xi \times X, \quad |\xi| \neq 0.
\end{align*}
\]
Similar to two species case, we obtain
\[
(\lambda^2 - ((B_2(|\xi|e_1) - \lambda)^{-1}\chi_2, \chi_2)\lambda + |\xi|^2)X = 0, \quad |\xi| > R_0. \tag{4.57}
\]
Denote
\[
D(\lambda, s) = \lambda^2 - ((B_2(se_1) - \lambda)^{-1}\chi_2, \chi_2)\lambda + s^2, \quad s > R_0. \tag{4.58}
\]

As Section 2.3, we can obtain

**Theorem 4.10.** There exists a constant $r_1 > 0$ so that the spectrum $\sigma(A_2(\xi)) \subset \mathbb{C}$ for $\xi = s\omega$ with $s = |\xi| > r_1$ and $\omega \in S^2$ consists of four points $\{\beta_j(s), j = 1, 2, 3, 4\}$ in the domain $\text{Re}\beta > -\mu/2$. In particular, the eigenvalues satisfy
\[
\begin{align*}
\beta_1(s) &= \beta_2(s) = -is + O(s^{-1/2}), \\
\beta_3(s) &= \beta_4(s) = is + O(s^{-1/2}), \\
c_1 \frac{1}{s} &\leq -\text{Re}\beta_j(s) \leq c_2 \frac{1}{s}. \tag{4.61}
\end{align*}
\]
The eigenvectors $\Phi_j = (\phi_j, X_j, Y_j)$ are orthogonal to each other and satisfy
\[
(\Phi_i(s, \omega), \Phi_j^\ast(s, \omega)) = (\psi_i, \psi_j^\ast) - (X_i, X_j^\ast) - (Y_i, Y_j^\ast) = \delta_{ij}, \quad 1 \leq i \neq j \leq 4. \tag{4.62}
\]
where $\Phi^*_j = (\phi^*_j, -X^*_j, -Y^*_j)$, and
\[
\begin{align*}
\|\phi_j(s, \omega)\| = O(1),
(\phi_j(s, \omega), \chi_0) = (\phi_j(s, \omega), \chi_4) = 0,
\end{align*}
\]
(4.63)

Here, $W^j$ $(j = 1, 2, 3, 4)$ are normal vectors satisfying $W^j \cdot \omega = 0$, $W^1 \cdot W^2 = 0$, $W^1 = W^3$, $W^2 = W^4$.

## 5 The linearized system

In this section, we consider the Cauchy problems (2.24) and (2.52) for the linearized Vlasov-Maxwell-Boltzmann equations in two and one-species and give the optimal time decay rates of the solution based on on spectrum structures obtained in the previous sections.

### 5.1 Semigroup for two-species

Before giving the theorem on the semigroup, we firstly prepare some lemmas on its properties.

**Lemma 5.1** ([13]). The operator $Q(\xi) = L_1 - iP(r, \omega)P_r$ generates a strongly continuous contraction semigroup on $N^1_T$, which satisfies for any $t > 0$ and $f \in N^1_T \cap L^2(\mathbb{R}^3)$ that

\[
\|e^{tQ(\xi)}f\| \leq e^{-\mu t}\|f\|.
\]
(5.1)

In addition, for any $x > -\mu$ and $f \in N^1_T \cap L^2(\mathbb{R}^3)$ it holds

\[
\int_{-\infty}^{+\infty} \|((x + i)y)P_r - Q(\xi)\| f^2 dy \leq \pi(x + \mu)^{-1}\|f\|^2.
\]
(5.2)

**Lemma 5.2.** The operator $B_3(\xi)$ generates a strongly continuous unitary semigroup on $C^6$, which satisfies for any $t > 0$ and $U \in C^6$ that

\[
|e^{tB_3(\xi)}U| = |U|.
\]
(5.3)

In addition, for any $x \neq 0$ and $U \in C^6$ it holds

\[
\int_{-\infty}^{+\infty} \|((x + iy) - B_3(\xi))^{-1}U\|^2 dy \leq \pi|x|^{-1}\|U\|^2.
\]
(5.4)

**Proof.** Since $iB_3(\xi)$ is a self-adjoint operator on $C^6$ satisfying (3.10), we can prove (5.3) and (5.4) by applying a similar argument as the one for Lemma 3.2 in [13].

**Lemma 5.3.** Let $r_0 > 0$ and $b_2 > 0$ be given in Lemma 5.1. Let $\alpha = (r_0, r_1) > 0$ with $r_1 > r_0$ and $\alpha(r_0, r_1)$ defined in Lemma 5.7. Then

\[
\sup_{0 < |\xi| < r_0, y \in \mathbb{R}} \|[(I - G_4(\xi))(-\frac{b_1}{2} + iy - G_3(\xi))^{-1}]^{-1}\| \leq C,
\]
(5.5)

\[
\sup_{r_0 < |\xi| < r_1, y \in \mathbb{R}} \|[(I - G_2(\xi))(-\frac{\alpha}{2} + iy - G_1(\xi))^{-1}]^{-1}\| \leq C.
\]
(5.6)

**Proof.** Let $\lambda = x + iy$. We prove (5.5) first. By (3.34) and (3.37), there exists $R > 0$ large enough such that if $\text{Re}\lambda \geq -\mu/2$, $|\text{Im}\lambda| \geq R$ and $|\xi| \leq r_0$, then (3.45) holds. This yields

\[
\|[(I - G_4(\xi))(-\frac{b_1}{2} + iy - G_3(\xi))^{-1}]^{-1}\| \leq 2.
\]

Thus, it remains to prove (5.5) for $|y| \leq R$. We will prove it by contradiction. Indeed, if (5.5) does not hold for $|y| \leq R$, namely, there are subsequences $\{\xi_n\}$, $\{\lambda_n = b_2/2 + iy_n\}$ with $|\xi_n| \leq r_0$, $|y_n| \leq R$, and $U_n = (f_n, e^{b_1}_n, B_1^n), V_n = (g_n, e^{b_1}_n, B_2^n)$ with $|U_n|_{\xi_n} \to 0$ ($n \to \infty$), $|V_n|_{\xi_n} = 1$ such that

\[
(I - G_4(\xi)(\lambda - G_3(\xi))^{-1})^{-1}U_n = V_n.
\]
Let $(a_n, b_n)^T = (\lambda_n - B_1(\xi_n))^{-1}(E_n^2, B_n^2)^T \iff (E_n^2, B_n^2)^T = \lambda_n(a_n, b_n)^T - (i\xi_n \times b_n, -i\xi_n \times a_n)^T$.

Then
\begin{align*}
P_d f_n &= P_d g_n + iP_d(v \cdot \xi_n)P_r(\lambda_n P_r - Q(\xi_n))^{-1}P_r g_n, \\
P_r f_n &= P_r g_n + i\lambda_n^{-1}P_r(v \cdot \xi_n)(1 + \frac{1}{|\xi_n|^2})P_d g_n - v\sqrt{M} \cdot (\omega_n \times a_n), \\
E_n^1 &= \lambda_n a_n - i\xi_n \times b_n - \omega_n \times ((\lambda_n P_r - Q(\xi_n))^{-1}P_r g_n, v\sqrt{M}), \\
B_n^1 &= \lambda_n b_n + i\xi_n \times a_n. 
\end{align*}

Substituting (5.8) into (5.7) and (5.9), we obtain
\begin{align*}
P_d f_n &= P_d g_n + iP_d(v \cdot \xi_n)P_r(\lambda_n P_r - Q(\xi_n))^{-1}P_r f_n \\
&\quad + \lambda_n^{-1}P_d(v \cdot \xi_n)P_r(\lambda_n P_r - Q(\xi_n))^{-1}P_r(v \cdot \xi_n)(1 + \frac{1}{|\xi_n|^2})P_d g_n, \\
E_n^1 &= \lambda_n a_n - i\xi_n \times b_n - \omega_n \times ((\lambda_n P_r - Q(\xi_n))^{-1}P_r f_n, v\sqrt{M}) \\
&\quad - \omega_n \times ((\lambda_n P_r - Q(\xi_n))^{-1}v\sqrt{M} \cdot (\omega_n \times a_n), v\sqrt{M}).
\end{align*}

Since $\|f_n\|_{\xi n} + |E_n^1| + |B_n^1| \to 0$ as $n \to \infty$, it follows from (5.11), (5.12) and (5.10) that
\begin{align*}
\lim_{n \to \infty} \sqrt{1 + \frac{1}{|\xi_n|^2} |C_n|} |\lambda_n + (|\xi_n|^2 + 1)((\lambda_n P_r - Q(|\xi_n|e_1))^{-1} \chi_1, \chi_1)| \to 0, \\
\lim_{n \to \infty} |\lambda_n a_n - i\xi_n \times b_n - ((\lambda_n P_r - Q(|\xi_n|e_1))^{-1} \chi_2, \chi_2)a_n| = 0, \\
\lim_{n \to \infty} |\lambda_n b_n + i\xi_n \times a_n| = 0,
\end{align*}
where $C_n = (g_n, \sqrt{M})$. Since $\sqrt{1 + \frac{1}{|\xi_n|^2} |C_n|} \leq 1, |\xi_n| \leq r_0, |g_n| \leq R$ and $|(a_n, b_n)| \leq 2b_2^{-1}(E_n^2, B_n^2) \leq 2b_2^{-1}$, there is a subsequence $\{(\xi_{n_j}, \lambda_{n_j}, C_{n_j})\}$ such that $\sqrt{1 + \frac{1}{|\xi_{n_j}|} |C_{n_j}|} \to A_0, a_{n_j} \to a_0, b_{n_j} \to b_0, \xi_{n_j} \to \xi_0, \lambda_{n_j} \to \lambda_0 = b_2/2 + iy \neq 0$. Thus
\begin{align*}
\frac{|A_0|}{|\lambda_0|} |\lambda_0 + (|\xi_0|^2 + 1)((\lambda_0 P_r - Q(|\xi_0|e_1))^{-1} \chi_1, \chi_1)| &= 0, \\
\lambda_0 a_0 - i\xi_0 \times b_0 - ((\lambda_0 P_r - Q(|\xi_0|e_1))^{-1} \chi_2, \chi_2)a_0 &= 0, \\
\lambda_0 b_0 + i\xi_0 \times a_0 &= 0.
\end{align*}

It is straightforward to verify that $(A_0, a_0, b_0) \neq 0$. Indeed, otherwise, we have $\lim_{j \to \infty}(\sqrt{1 + \frac{1}{|\xi_{n_j}|} |C_{n_j}|}, a_{n_j}, b_{n_j}) = 0$. This and (5.8) lead to $\lim_{j \to \infty} \|P_r g_{n_j}\| = 0$ and $\lim_{j \to \infty} (E_{n_j}^2, B_{n_j}^2) = 0$. Thus $\lim_{j \to \infty} \|V_{n_j}\|_{\xi_{n_j}} = 0$, which contradicts to $\|V_n\|_{\xi_n} = 1$. Therefore, (5.13), (5.14) and (5.15) imply that $\lambda_0$ is an eigenvalue of $\hat{B}(\xi_0)$ with $\text{Re}\lambda_0 = -b_2/2$ and $|\xi_0| \leq r_0$, which contradicts to Theorem 3.12 since we can assume $\text{Re}\lambda_j(s) \neq -b_2/2$ by taking a smaller $r_0$ if necessary.

By an argument similar to the one for Lemma 3.10, we can prove (5.6). And this completes the proof of the lemma.

With the help of Lemmas 3.3, 3.8 and Lemmas 5.1, 5.3 we have a decomposition of the semigroup $S(t, \xi) = e^{t\hat{B}(\xi)}$.

**Theorem 5.4.** The semigroup $S(t, \xi) = e^{t\hat{B}(\xi)}$ with $\xi = \omega \in \mathbb{R}^3$ and $s = |\xi| \neq 0$ has the following decomposition
\begin{equation}
S(t, \xi) U = S_1(t, \xi) U + S_2(t, \xi) U + S_3(t, \xi) U, \quad U \in L^2(\mathbb{R}^3_x) \times C^3 \times C^3, \quad t > 0,
\end{equation}
where
\[
S_1(t, \xi)U = \sum_{j=1}^{2} e^{i\lambda_j(t)}(U, \Psi_j(s, \omega))\Psi_j(s, \omega)1_{\{|\xi| \leq r_0\}},
\]
\[
S_2(t, \xi)U = \sum_{j=1}^{4} e^{i\beta_j(t)}(U, \Phi_j(s, \omega))\Phi_j(s, \omega)1_{\{|\xi| \geq r_1\}}.
\]

with \((\lambda_j(s), \Psi_j(s, \omega))\) and \((\beta_j(s), \Phi_j(s, \omega))\) being the eigenvalue and eigenvector of the operator \(\tilde{A}_1(\xi)\) given by Theorem 3.12 and Theorem 3.15 for \(|\xi| \leq r_0\) and \(|\xi| > r_1\) respectively, and \(S_3(t, \xi) =: S(t, \xi) - S_1(t, \xi) - S_2(t, \xi)\) satisfies that there exists a constant \(k_0 > 0\) independent of \(\xi\) such that
\[
\| S_3(t, \xi)U \|_{\xi} \leq C e^{-k_0 t} \|U\|_{\xi}, \quad t \geq 0.
\]

Proof. Since \(D(\tilde{B}_1(\xi)^2)\) is dense in \(L^2_2(\mathbb{R}^3)\), by Theorem 2.7 in [18], it is sufficient to prove the above decomposition for \(U = (f, E, B) \in D(\tilde{B}_1(\xi)^2) \times C^2_{\xi} \times C^2_{\xi}\). By Corollary 7.5 in [18], the semigroup \(e^{it\tilde{A}(\xi)}\) can be represented by
\[
e^{it\tilde{A}(\xi)}U = \frac{1}{2\pi i} \int_{\rho \to -\infty} e^{\lambda t} (\lambda - \tilde{A}(\xi)^{-1})^{-1} U d\lambda, \quad U \in D(\tilde{B}_1(\xi)^2) \times C^2_{\xi} \times C^2_{\xi}, \; \kappa > 0.
\]

It remains to analyze the resolvent \((\lambda - \tilde{A}(\xi))^{-1}\) for \(\xi \in \mathbb{R}^3\) in order to obtain the decomposition \(5.16\) for the semigroup \(e^{it\tilde{A}(\xi)}\).

By \(3.46\), we rewrite \((\lambda - \tilde{A}(\xi))^{-1}\) for \(|\xi| \leq r_0\) as
\[
(\lambda - \tilde{A}(\xi))^{-1} = (\lambda - G_3(\xi))^{-1} + Z_1(\lambda, \xi),
\]
with

\[
Z_1(\lambda, \xi) = (\lambda - G_3(\xi))^{-1}[I - Y_1(\lambda, \xi)]^{-1}Y_1(\lambda, \xi),
\]
\[
Y_1(\lambda, \xi) = G_4(\xi)(\lambda - G_3(\xi))^{-1}.
\]

Substituting \(5.21\) into \(5.20\), we have the following decomposition of the semigroup \(e^{it\tilde{A}(\xi)}\)
\[
e^{it\tilde{A}(\xi)}U = (e^{itQ(\xi)} P r f, 0, 0) - \frac{1}{2\pi i} \int_{\rho \to -\infty} e^{\lambda t} Z_2(\lambda, \xi)U d\lambda, \quad |\xi| \leq r_0,
\]
with
\[
Z_2(\lambda, \xi) = Z_1(\lambda, \xi) - Y_2(\lambda, \xi), \quad \text{with} \quad Y_2(\lambda, \xi) = \left(\begin{array}{cc}
\lambda^{-1}P_3 & 0 \\
0 & (\lambda - B_3(\xi))^{-1}
\end{array}\right).
\]

To estimate the last term on the right hand side of \(5.22\), let us denote
\[
X_{\kappa, N} = \frac{1}{2\pi i} \int_{\rho \to -\infty} e^{(\kappa + iy) t} Z_2(\kappa + iy, \xi)U1_{|\xi| \leq r_0} dy,
\]
where the constant \(N > 0\) is chosen large enough so that \(N > y_1\) with \(y_1\) defined in Lemma 4.8. Since \(Z_2(\lambda, \xi)\) is analytic in the domain \(\text{Re} \lambda > -b_2/2\) with only finite singularities at \(\lambda = \lambda_j(s) \in \sigma(\tilde{A}(\xi))\) for \(j = 1, 2\), we can shift the integration \(5.23\) from the line \(\text{Re} \lambda = \kappa > 0\) to \(\text{Re} \lambda = -b_2/2\). Then
\[
X_{\kappa, N} = X_{-\frac{b_2}{2} + \kappa, N} + H_N + 2\pi i \sum_{j=1}^{2} \text{Res} \left\{e^{\lambda t} Z_2(\lambda, \xi)U; \lambda_j(s)\right\} 1_{|\xi| \leq r_0},
\]
where \(\text{Res}\{f(\lambda); \lambda_j\}\) is the residue of \(f\) at \(\lambda = \lambda_j\) and
\[
H_N = \frac{1}{2\pi i} \left(\int_{-\frac{b_2}{2} + iN}^{\kappa + iN} - \int_{-\frac{b_2}{2} - iN}^{\kappa - iN}\right) e^{\lambda t} Z_2(\lambda, \xi)U1_{|\xi| \leq r_0} d\lambda.
\]
The right hand side of (5.24) is estimated as follows. By Lemma 3.7, we have

\[ \|H_N\|_{\xi} \to 0, \quad \text{as } N \to \infty. \tag{5.25} \]

By Cauchy Theorem, we obtain

\[ \lim_{N \to \infty} \left| \int_{\frac{b_2}{2} + \frac{iN}{2}}^{\frac{b_2}{2} + \frac{i(N + 1)}{2}} e^{\lambda t}\lambda^{-1} d\lambda \right| = \lim_{N \to \infty} \left| \int_{\frac{b_2}{2} - iN}^{\frac{b_2}{2} - i(N + 1)} e^{\lambda t}(\lambda - B_3(\xi))^{-1} d\lambda \right| = 0, \tag{5.26} \]

which leads to

\[ \int_{\frac{b_2}{2} + \frac{i\infty}{2}}^{\frac{b_2}{2} + \frac{i\infty}{2}} e^{\lambda t}Y_2(\lambda, \xi) d\lambda = \lim_{N \to \infty} \int_{\frac{b_2}{2} - iN}^{\frac{b_2}{2} + iN} e^{\lambda t}Y_2(\lambda, \xi) d\lambda = 0. \]

Thus

\[ \lim_{N \to \infty} X_{-\frac{b_2}{2}, N}(t) = X_{-\frac{b_2}{2}, \infty}(t) =: \int_{\frac{b_2}{2} + \frac{i\infty}{2}}^{\frac{b_2}{2} + \frac{i\infty}{2}} e^{\lambda t}Z_1(\lambda, \xi) U d\lambda. \tag{5.27} \]

By Lemma 5.3, it holds that

\[ \sup_{|\xi| \leq r_0, \eta \in \mathbb{R}} \| [I - Y_1(-\frac{b_2}{2} + iy, \xi)]^{-1} \|_{\xi} \leq 2. \]

Thus, we have for any \( U, V \in L^2_{\xi}(\mathbb{R}_t^+ \times C_{\xi}^{2} \times C_{\xi}^{2}) \),

\[ |(X_{-\frac{b_2}{2}, \infty}(t)U, V)_{\xi}| \leq Ce^{-\frac{b_2}{2}t} \int_{-\infty}^{+\infty} \|\lambda - G_3(\xi)\|^{-1}U\|_{\xi} \|\lambda - G_3(-\xi)\|^{-1}V\|_{\xi} \|d\lambda, \quad \lambda = -\frac{b_2}{2} + iy. \]

By (5.22) and (5.24), we have

\[ \int_{-\infty}^{+\infty} \|\lambda - G_3(\xi)\|^{-1}U\|_{\xi}^2 \|d\lambda \]

\[ = \int_{-\infty}^{+\infty} \|\lambda - Q(\xi)\|^{-1}P_r f\| + |\lambda|^{-1} \|P_d f\|_{\xi}^2 \|d\lambda + \int_{-\infty}^{+\infty} \|\lambda - B_3(\xi)\|^{-1}(E, B)^T \|_{\xi}^2 \|d\lambda \]

\[ \leq C(\|f\|_{\xi}^2 + \|(E, B)\|_{\xi}^2), \quad \lambda = -\frac{b_2}{2} + iy, \]

which yields

\[ |(X_{-\frac{b_2}{2}, \infty}(t)U, V)_{\xi}| \leq Ce^{-\frac{b_2}{2}t} \|U\|_{\xi} \|V\|_{\xi}, \]

\[ \tag{5.28} \]

\[ \text{Since } \lambda_j(s) \in \rho(Q(\xi)) \text{ and } Z_2(\lambda, \xi) = (\lambda - \hat{A}_1(\xi))^{-1} - \left( \begin{array}{cc} (\lambda P_r - Q(\xi))^{-1}P_r & 0 \\ 0 & 0 \end{array} \right), \]

\[ \text{by a similar argument as Theorem 3.4 in [13], we can prove} \]

\[ \text{Res}\{e^{At}Z_2(\lambda, \xi)U; \lambda_j(s)\} = \text{Res}\{e^{At}(\lambda - \hat{A}_1(\xi))^{-1}U; \lambda_j(s)\} = e^{\lambda_j(s)t}(U, \Psi_j^*(s, \omega))\Psi_j(s, \omega). \tag{5.29} \]

Therefore, we conclude from (5.22) - (5.28) that

\[ e^{\hat{A}_1(\xi)U} = (e^{tQ(\xi)}P_r f, 0, 0) + X_{-\frac{b_2}{2}, \infty}(t) + \sum_{j=1}^{2} e^{\lambda_j(s)}(U, \Psi_j^*(s, \omega))\Psi_j(s, \omega)1(|\xi| \leq r_0), \quad |\xi| \leq r_0. \tag{5.30} \]

By (5.30), we have for \( |\xi| > r_0 \) that

\[ (\lambda - \hat{A}_1(\xi))^{-1} = (\lambda - G_1(\xi))^{-1} + Z_3(\lambda, \xi), \tag{5.31} \]

with the operator \( Z_3(\lambda, \xi) \) defined by

\[ Z_3(\lambda, \xi) = (\lambda - G_1(\xi))^{-1}[I - Y_3(\lambda, \xi)]^{-1}Y_3(\lambda, \xi), \]

\[ Y_3(\lambda, \xi) =: G_2(\xi)(\lambda - G_1(\xi))^{-1}. \]
Similarly, in order to estimate the last term on the right hand side of (5.32), let us denote

\[ Y_\nu, N = \frac{1}{2\pi i} \int_{\gamma_N^c} e^{(\kappa + i\nu)y} Z_4(\kappa + i\nu, \nu) U (\kappa, \nu) d\lambda, \quad |\kappa| > r_1. \]  

(5.33)

Here,

\[ Z_4(\lambda, \xi) = Z_3(\lambda, \xi) - Y_3(\lambda, \xi), \quad \text{with} \quad Y_3(\lambda, \xi) = \begin{pmatrix} 0 & 0 \\ 0 & (\lambda - B_3(\xi))^{-1} \end{pmatrix}. \]

Similarly, in order to estimate the last term on the right hand side of (5.32), let us denote

\[ Y_{\kappa, N} = \frac{1}{2\pi i} \int_{\gamma_N^c} e^{(\kappa + i\nu)y} Z_4(\kappa + i\nu, \nu) U (\kappa, \nu) d\lambda, \quad \text{for} \quad |\kappa| > r_1. \]

(5.34)

where the constant \( N > y_1 \) for \( |\xi| \leq r_1 \) and \( N > 2|\xi| \) for \( |\xi| \geq r_1 \) with \( y_1, r_1 \) defined in Lemma 3.8. Since the operator \( Z_4(\lambda, \xi) \) is analytic in the domain \( \text{Re} \lambda \geq -\nu_0 := -\alpha(r_0, r_1)/2 > 0 \) and \( r_0 < |\xi| < r_1 \) with the constant \( \alpha(r_0, r_1) > 0 \) defined in Lemma 3.3 and is analytic except only finite singularities at \( \lambda = \lambda_j(s) \in \sigma(\hat{A}_1(\xi)) \) for \( j = 1, 2, 3, 4 \), in the domain \( \text{Re} \lambda \geq -\mu/2 \) and \( |\xi| \geq r_1 \), we can shift the integration of (5.33) for \( r_0 < |\xi| < r_1 \) from the line \( \text{Re} \lambda = \kappa > 0 \) to \( \text{Re} \lambda = -\nu_0 \), and shift the integration of (5.33) for \( |\xi| \geq r_1 \) from the line \( \text{Re} \lambda = \kappa > 0 \) to \( \text{Re} \lambda = -\mu/2 \) to obtain

\[ Y_{\kappa, N} = Y_{-\nu_0, N} + I_N, \]

(5.35)

with

\[ I_N = \frac{1}{2\pi i} \int_{\gamma_N^c} e^{(\kappa + i\nu)y} Z_4(\kappa + i\nu, \nu) U (\kappa, \nu) d\lambda, \]

\[ J_N = \frac{1}{2\pi i} \int_{\gamma_N^c} e^{(\kappa + i\nu)y} Z_3(\kappa + i\nu, \nu) U (\kappa, \nu) d\lambda. \]

By Lemma 3.5 and Lemma 5.3 it is straightforward to verify

\[ ||I_N|| \to 0, \quad ||J_N|| \to 0 \quad \text{as} \quad N \to \infty, \quad \text{(5.36)} \]

\[ \sup_{r_0 < |\xi| < r_1, y \in \mathbb{R}} \left \| I - Y_4(-\frac{by}{2} + iy, \xi) \right \|^{-1} \leq C, \quad \sup_{|\xi| > r_1, y \in \mathbb{R}} \left \| I - Y_4(-\frac{\mu}{2} + iy, \xi) \right \|^{-1} \leq 2. \quad \text{(5.37)} \]

By (5.36) and (5.37), it holds that

\[ Y_{-\nu_0, \infty}(t) = \lim_{N \to \infty} Y_{-\nu_0, N}(t) = \int_{-\infty}^{+\infty} e^{(-\nu_0 + iy)t} Z_4(-\nu_0 + iy, \xi) U dy, \]

\[ Y_{-\frac{\mu}{2}, \infty}(t) = \lim_{N \to \infty} Y_{-\frac{\mu}{2}, N}(t) = \int_{-\infty}^{+\infty} e^{(-\frac{\mu}{2} + iy)t} Z_3(-\frac{\mu}{2} + iy, \xi) U dy. \]

Then we have for any \( U, V \in L^2(\mathbb{R}^3) \times C^\infty_\xi \times C^\infty_\xi \),

\[ ||Y_{-\nu_0, \infty}(t) U (r_0 < |\xi| < r_1), V || \leq C e^{-\nu_0 t} \int_{-\infty}^{+\infty} ||(\lambda - G_1(\xi))^{-1} U || ||(\lambda - G_1(-\xi))^{-1} V || dy \]

\[ \leq C (\nu_0 - \nu_0) e^{-\nu_0 t} ||U|| ||V||, \quad \lambda = -\nu_0 + iy, \quad \text{(5.38)} \]

\[ ||Y_{-\frac{\mu}{2}, \infty}(t) U (|\xi| \geq r_1), V || \leq C e^{-\frac{\mu}{2} t} \int_{-\infty}^{+\infty} ||(\lambda - G_1(\xi))^{-1} U || ||(\lambda - G_1(-\xi))^{-1} V || dy \]

\[ \leq C (\nu_0 - \frac{\mu}{2}) e^{-\frac{\mu}{2} t} ||U|| ||V||, \quad \lambda = -\frac{\mu}{2} + iy, \quad \text{(5.39)} \]
where we have used (5.31) and the fact (cf. Lemma 2.13 of [22]) that
\[
\int_{-\infty}^{+\infty} \| (x + iy - c(\xi))^{-1} f \|_2^2 dy \leq \pi (x + \nu_0)^{-1} \| f \|_2^2, \quad x > -\nu_0.
\]
From (5.38), (5.39) and the fact that \( \| f \|_2^2 \leq \| f \|_2^2 \leq (1 + r_0^{-2}) \| f \|_2^2 \) for \( |\xi| > r_0 \), we have
\[
\| Y_{-\kappa_0,\infty}(t) 1_{\{ r_0 < |\xi| < r_1 \} } \|_\xi \leq C e^{-\kappa_0 t}, \quad \| Y_{-\kappa_0,\infty}(t) 1_{\{ |\xi| \geq r_1 \} } \|_\xi \leq C e^{-\frac{\kappa_0}{2} t}.
\] (5.40)

By \( \lambda_j(s) \in \rho(\epsilon(\xi)) \) and \( Z_4(\lambda, \xi) = (\lambda - \hat{A}_4(\xi))^{-1} - \left( \begin{array}{cc} (\lambda - c(\xi))^{-1} & 0 \\ 0 & 0 \end{array} \right) \), we can prove
\[
\text{Res}\{ e^{\hat{M}} Z_4(\lambda, \xi); \beta_j(s) \} = \text{Res}\{ e^{\hat{M}}(\lambda - \hat{A}_4(\xi))^{-1} U; \beta_j(s) \} = e^{\beta_j(s) t}(U, \Phi_j^s(s, \omega)) \Phi_j(s, \omega).
\] (5.41)

Therefore, we conclude from (5.32)–(5.41) that
\[
e^{\hat{A}_4(\xi) t} U = (e^{\text{c}(\xi) t} f, 0, 0) + Y_{-\kappa_0,\infty}(t) 1_{\{ r_0 < |\xi| < r_1 \} } + Y_{-\kappa_0,\infty}(t) 1_{\{ |\xi| \geq r_1 \} } + \sum_{j=1}^4 e^{\beta_j(s) t}(U, \Phi_j^s(s, \omega)) \Phi_j(s, \omega) 1_{\{ |\xi| \geq r_1 \} }, \quad |\xi| > r_0.
\] (5.42)

Combining (5.40) and (5.42) gives (5.16) with \( S_1(t, \xi) f, S_2(t, \xi) f \) and \( S_3(t, \xi) f \) defined by
\[
S_1(t, \xi) U = \sum_{j=1}^3 e^{\hat{A}_4(\xi) t} (U, \Psi_j^s(s, \omega)) \Phi_j(s, \omega) 1_{\{ |\xi| \leq r_0 \} },
\]
\[
S_2(t, \xi) U = \sum_{j=1}^4 e^{\beta_j(s) t}(U, \Phi_j^s(s, \omega)) \Phi_j(s, \omega) 1_{\{ |\xi| \geq r_1 \} },
\]
\[
S_3(t, \xi) U = (e^{Q(\xi) t} P_r f, 0, 0) 1_{\{ |\xi| \leq r_0 \} } + X_{-\kappa_0,\infty}(t) 1_{\{ |\xi| \leq r_0 \} } + Y_{-\kappa_0,\infty}(t) 1_{\{ r_0 < |\xi| < r_1 \} } + Y_{-\kappa_0,\infty}(t) 1_{\{ |\xi| \geq r_1 \} }.
\]

In particular, \( S_3(t, \xi) U \) satisfies (5.19) because of (5.31), (5.25), (5.40) and the estimate \( \| e^{\text{c}(\xi) t} 1_{\{ |\xi| > r_0 \} } \|_\xi \leq C e^{-\nu_0 t} \) coming from (5.12) and (2.17). This completes the proof of the theorem. \( \square \)

### 5.2 Optimal convergence rates for two-species

Based on the decomposition of the semigroup given in the previous subsection, we now study the optimal convergence rates of the solution of the linearized system to the equilibrium.

Set \( U = (f, E, B) \) with \( f = f(x, v) \), \( E = E(x) \) and \( B = B(x) \) and denote \( D_1 = \{ U \in L^2(\mathbb{R}_x^3 \times \mathbb{R}_v^3) \times L^2(\mathbb{R}_x^3) \times L^2(\mathbb{R}_x^3) \mid \| U \|_{D_1} < \infty \} \) \((D_1 = D_1)\) with the norm \( \| \cdot \|_{D_1} \) defined by
\[
\| U \|_{D_1} = \left( \int_{\mathbb{R}_x^3} (1 + |\xi|^2)^i \| \hat{U} \|_2^2 d\xi \right)^{1/2}
\]
\[
= \left( \int_{\mathbb{R}_x^3} (1 + |\xi|^2)^i \left( \int_{\mathbb{R}_x^3} |\hat{f}|^2 dv + |\hat{E}|^2 + |\hat{B}|^2 \right) d\xi \right)^{1/2},
\]
where \( \hat{f} = \hat{f}(\xi, v), \hat{E} = \hat{E}(\xi) \) and \( \hat{B} = \hat{B}(\xi) \).

For any \( U_0 = (f_0, E_0, B_0) \in L^2(\mathbb{R}_x^3 \times H^1(\mathbb{R}_v^3)) \times H^1(\mathbb{R}_x^3) \times H^1(\mathbb{R}_x^3) \), set
\[
e^{i\hat{A}_4(\xi) t} \hat{U}_0 = \left( \frac{i \hat{\xi}}{\| \hat{\xi} \|^2} \left( \frac{i \hat{\xi}}{\| \hat{\xi} \|^2} \hat{\xi} \right) \hat{V}_0_1, \sqrt{M} - \frac{\hat{\xi}}{\| \hat{\xi} \|} \cdot \left( \frac{i \hat{\xi}}{\| \hat{\xi} \|^2} \hat{V}_0_2, \frac{\hat{\xi}}{\| \hat{\xi} \|} \cdot \left( \frac{i \hat{\xi}}{\| \hat{\xi} \|^2} \hat{V}_0_3 \right) \right),
\] (5.43)

with
\[
e^{i\hat{A}_4(\xi) t} \hat{V}_0 = \left( \left( \frac{i \hat{\xi}}{\| \hat{\xi} \|^2} \hat{V}_0_1, \left( \frac{i \hat{\xi}}{\| \hat{\xi} \|^2} \hat{V}_0_2, \right) \right) \left( \frac{i \hat{\xi}}{\| \hat{\xi} \|^2} \hat{V}_0_3 \right) \right) \in L^2(\mathbb{R}_v^3) \times C_0^3 \times C_0^3,
\]
\[ V_0 = (\hat{f}_0, \frac{\xi}{|\xi|} \times \hat{E}_0, \frac{\xi}{|\xi|} \times \hat{B}_0). \]

Then \( e^{t\hat{a}_0}U_0 \) is the solution of the system (2.24). By Lemma 5.1, it holds that
\[ \|e^{t\hat{a}_0}U_0\|_{D^I} = \int_{\mathbb{R}^3} (1 + |\xi|^2)^{\frac{3}{2}} \|e^{\hat{a}_0(t)}\hat{V}_0\|^2 d\xi \leq \int_{\mathbb{R}^3} (1 + |\xi|^2)^{\frac{3}{2}} \|\hat{V}_0\|^2 d\xi = \|U_0\|_{D^I}. \]

This means that the linear operator \( \hat{a}_0 \) generates a strongly continuous contraction semigroup \( e^{t\hat{a}_0} \) on \( D^I \), and therefore, \( U(t) = e^{t\hat{a}_0}U_0 \) is a global solution to (2.24) for the linearized Vlasov-Maxwell-Boltzmann equations with initial data \( U_0 \in D^I \).

First of all, we have the upper bounds of the time decay rates given in

**Theorem 5.5.** Let \((f_2(t), E(t), B(t))\) be a solution of the system (2.24). If the initial data \( U_0 = (f_0, E_0, B_0) \in L^2(\mathbb{R}^3; H^1(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)) \times H^1(\mathbb{R}^3) \cap L^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3) \cap L^1(\mathbb{R}^3), \) then it holds for any \( \alpha, \alpha' \in \mathbb{N}^3 \) with \( \alpha' \leq \alpha \) that
\[
\begin{align*}
\|\partial_\alpha^p P_\alpha f(t)\|_{L^2_z} &\leq C e^{-\kappa t\xi} \|\partial_\alpha^p U_0\|_{Z^2}, \\
\|\partial_\alpha^p P_\alpha f(t)\|_{L^2_z} &\leq C(1 + t)^{-\frac{\alpha}{2} + \frac{\alpha'}{2}} \|\partial_\alpha^p U_0\|_{Z^2} + |\partial_\alpha^p B(t)|_{L^2_z} + |\partial_\alpha^p E(t)|_{L^2_z} + |\partial_\alpha^p H(t)|_{L^2_z} + C(1 + t)^{-\frac{\alpha}{2}} \|\nabla_x^m \partial_\alpha^p U_0\|_{Z^2},
\end{align*}
\]

where \( k = |\alpha - \alpha'| \) and \( m \geq 0 \).

**Proof.** By (5.43) and Theorem 5.1, we have for \( \omega = \xi/|\xi| \) that
\[
(\hat{f}_2(t, \omega \times \hat{E}(t), \omega \times \hat{B}(t))) = e^{t\hat{a}_1(\xi)}\hat{V}_0 = S_1(t, \xi)\hat{V}_0 + S_2(t, \xi)\hat{V}_0 + S_3(t, \xi)\hat{V}_0 = \sum_{k=1}^{3} (h_k(t), H_k(t), J_k(t)),
\]

where \( S_k(t, \xi)\hat{V}_0 = (h_k(t), H_k(t), J_k(t)) \in L^2_z(\mathbb{R}^3) \times C^k_z \times C^k_z, \ k = 1, 2, 3, \) and \( \hat{V}_0 = (\hat{f}_0, \omega \times \hat{E}_0, \omega \times \hat{B}_0) \). Then
\[
\begin{align*}
\|\partial_\alpha^p \hat{P}_\alpha f_2(t)\|_{L^2_z} &= \|\partial_\alpha^p \hat{P}_\alpha \hat{f}_2(t)\|_{L^2_z}, \\
&\leq \|\partial_\alpha^p P_\alpha h_1(t)\|_{L^2_z} + \|\partial_\alpha^p P_\alpha h_2(t)\|_{L^2_z} + \|\partial_\alpha^p P_\alpha h_3(t)\|_{L^2_z}, \quad (5.49)
\end{align*}
\]

By (5.19) and \( |\hat{E}_0| = \frac{1}{|\xi|} |(\hat{f}_0, \sqrt{M})|^2 + |\omega \times \hat{E}_0|^2, \) \( |\hat{B}_0| = |\omega \times \hat{B}_0|^2, \) we can estimate the terms on the right hand side of (5.49)–(5.51) as follows:
\[
\begin{align*}
\int_{\mathbb{R}^3} \frac{(\xi^2)^2}{|\xi|} |(h_3(t), \sqrt{M})|^2 + |H_3(t)|^2 + |J_3(t)|^2) d\xi &\leq C \int_{\mathbb{R}^3} e^{-2\kappa t\xi} \xi^2 \frac{1}{|\xi|^2} |(\hat{f}_0, \sqrt{M})|^2 + |\omega \times \hat{E}_0|^2 + |\omega \times \hat{B}_0|^2) d\xi \leq C e^{-2\kappa t\xi} \|\partial_\alpha^p f_0\|_{L^2_z} + \|\partial_\alpha^p E_0\|_{L^2_z} + \|\partial_\alpha^p B_0\|_{L^2_z}. 
\end{align*}
\]
In the low frequency region, by (5.17), we have

\[
S_1(t, \xi) \hat{V}_0 = \sum_{j=1}^{2} e^{\lambda_j(|\xi|)} \left\{ \left[ (\hat{f}_0, \psi_{j,0}) - (\omega \times \hat{E}_0, \chi_{j,0}) - (\omega \times \hat{B}_0, \gamma_{j,0}) \right] \left( \psi_{j,0}, X_{j,0}, Y_{j,0} \right) \right. \\
+ \left. |\xi| (\hat{T}_j(\xi) \hat{V}_0)_1, (\hat{T}_j(\xi) \hat{V}_0)_2, (\hat{T}_j(\xi) \hat{V}_0)_3 \right\} \mathbf{1}_{|\xi| \leq r_0},
\]

where \(T_j(\xi)\), \(j = 1, 2\), are the linear operators with the norm \(\|T_j(\xi)\|\) being uniformly bounded for \(|\xi| \leq r_0\).

By (3.67) and (3.68), we have

\[
(h_1(t), \sqrt{M}) = 0, \quad P_r h_1(t) = |\xi| \sum_{j=1}^{2} e^{\lambda_j(|\xi|)} (T_j(\xi) \hat{V}_0)_1, \quad \tag{5.53}
\]

\[
H_1(t) = |\xi| \sum_{j=1}^{2} e^{\lambda_j(|\xi|)} (T_j(\xi) \hat{V}_0)_2, \quad \tag{5.54}
\]

\[
J_1(t) = \sum_{j=1}^{2} e^{\lambda_j(|\xi|)} (\omega \times \hat{B}_0, W^j) W^j \quad \tag{5.55}
\]

where \(W^j, \ j = 1, 2\), is given by (3.68). Since

\[
\text{Re} \lambda_j(|\xi|) = a_j |\xi|^2 (1 + O(|\xi|)) \leq -\eta_1 |\xi|^2, \quad |\xi| \leq r_0,
\]

where \(\eta_1 > 0\) denotes a generic constant that will also be used later, we obtain by (5.53)–(5.55) that

\[
\|\xi^\alpha P_r h_1(t)\|_{L^2_{x}} \leq C \int_{|\xi| \leq r_0} (\xi^\alpha)^2 |\xi|^2 e^{-2\eta_0 |\xi|^2} \left( |\hat{f}_0|_{L^2_x}^2 + |\hat{E}_0|^2 + |\hat{B}_0|^2 \right) d\xi \\
\leq C (1 + t)^{-(\alpha + 2k)} \left( |\partial_x^e f_0|_{L^2_x}^2 + |\partial_x^e E_0|^2 + |\partial_x^e B_0|^2 \right), \quad \tag{5.57}
\]

\[
\|\xi^\alpha H_1(t)\|_{L^2_x} \leq C \int_{|\xi| \leq r_0} (\xi^\alpha)^2 |\xi|^2 e^{-2\eta_0 |\xi|^2} \left( |\hat{f}_0|_{L^2_x}^2 + |\hat{E}_0|^2 + |\hat{B}_0|^2 \right) d\xi \\
\leq C (1 + t)^{-(\alpha + 2k)} \left( |\partial_x^e f_0|_{L^2_x}^2 + |\partial_x^e E_0|^2 + |\partial_x^e B_0|^2 \right), \quad \tag{5.58}
\]

\[
\|\xi^\alpha J_1(t)\|_{L^2_x} \leq C \int_{|\xi| \leq r_0} (\xi^\alpha)^2 |\xi|^2 e^{-2\eta_0 |\xi|^2} \left( |\hat{f}_0|_{L^2_x}^2 + |\hat{E}_0|^2 + |\hat{B}_0|^2 \right) d\xi \\
\leq C (1 + t)^{-(\alpha + 2k)} \left( |\partial_x^e f_0|_{L^2_x}^2 + |\partial_x^e E_0|^2 + |\partial_x^e B_0|^2 \right). \quad \tag{5.59}
\]

with \(k = |\alpha - \alpha'|\).

In the high frequency region, by (5.18), we have

\[
S_2(t, \xi) \hat{V}_0 = \sum_{j=1}^{4} e^{i\beta_j(|\xi|)} \left[ (\hat{f}_0, \phi_j) - (\omega \times \hat{E}_0, \chi_j) - (\omega \times \hat{B}_0, \gamma_j) \right] \left( \phi_j, X_j, Y_j \right), \mathbf{1}_{|\xi| \geq r_1}, \tag{5.60}
\]

and in particular \((h_2(t), \sqrt{M}) = 0\). Since

\[
\text{Re} \beta_j(|\xi|) \leq -c_1 |\xi|^{-1}, \quad |\xi| \geq r_1, \quad \tag{5.61}
\]

we obtain by (7.60) and (5.39) that

\[
\|\xi^\alpha P_r h_2(t)\|_{L^2_x} \leq C \int_{|\xi| \geq r_1} (\xi^\alpha)^2 \frac{1}{|\xi|^2} e^{-2\eta_0 |\xi|^2} \left( |\hat{f}_0|_{L^2_x}^2 + |\hat{E}_0|^2 + |\hat{B}_0|^2 \right) d\xi \\
\leq C \sup_{|\xi| \geq r_1} \frac{1}{|\xi|^{2m+1}} e^{-2\eta_0 |\xi|^2} \int_{|\xi| \geq r_1} (\xi^\alpha)^2 |\xi|^2 |\hat{f}_0|_{L^2_x}^2 + |\hat{E}_0|^2 + |\hat{B}_0|^2 \right) d\xi, \quad \tag{5.62}
\]

\[
\|\xi^\alpha H_2(t)\|_{L^2_x} \leq C \int_{|\xi| \geq r_1} (\xi^\alpha)^2 \frac{1}{|\xi|^2} e^{-2\eta_0 |\xi|^2} \left( |\hat{f}_0|_{L^2_x}^2 + |\hat{E}_0|^2 + |\hat{B}_0|^2 \right) d\xi
\]
the proof of the theorem. The combination of (5.49)–(5.52), (5.57)–(5.59) and (5.65)–(5.67) leads to (5.44)–(5.47). And this completes H.-L. Li, T. Yang, M.-Y. Zhong
satisfies that

$$\inf_{|\xi| \geq r_1} \frac{1}{|\xi|^{2m} \prod_{j=1,2} (\xi_j^2 + 1 \parallel \nabla^2_\xi)_{\xi_j}^2} \leq C(1 + t)^{-2m},$$

it follows from (5.62)–(5.64) that

$$\|\xi^\alpha P_h(t)\|_{L^2(\xi)}^2 \leq C(1 + t)^{-(2m+1)} \left( \|\nabla_x^m \partial_x^\alpha f_0\|_{L^2(\xi)}^2 + \|\nabla_x^m \partial_x^\alpha E_0\|_{L^2(\xi)}^2 + \|\nabla_x^m \partial_x^\alpha B_0\|_{L^2(\xi)}^2 \right),$$

$$\|\xi^\alpha H(t)\|_{L^2(\xi)}^2 \leq C(1 + t)^{-(2m+1)} \left( \|\nabla_x^m \partial_x^\alpha f_0\|_{L^2(\xi)}^2 + \|\nabla_x^m \partial_x^\alpha E_0\|_{L^2(\xi)}^2 + \|\nabla_x^m \partial_x^\alpha B_0\|_{L^2(\xi)}^2 \right),$$

$$\|\xi^\alpha J(t)\|_{L^2(\xi)}^2 \leq C(1 + t)^{-(2m+1)} \left( \|\nabla_x^m \partial_x^\alpha f_0\|_{L^2(\xi)}^2 + \|\nabla_x^m \partial_x^\alpha E_0\|_{L^2(\xi)}^2 + \|\nabla_x^m \partial_x^\alpha B_0\|_{L^2(\xi)}^2 \right).$$

The combination of (5.69), (5.70), (5.71) and (5.72) leads to (5.73)–(5.75). And this completes the proof of the theorem.

In fact, the above time decay rates are optimal as shown in

**Theorem 5.6.** Let \((f_2(t), E(t), B(t))\) be a solution of the system (5.24). Assume that the initial data \(U_0 = (f_0, E_0, B_0) \in L^2(\mathbb{R}^3; H^1(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)) \times H^1(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)\) for \(l \geq 2\) and \(U_0 = (f_0, E_0, B_0)\) satisfies that \(\inf_{|\xi| \leq r_0} \frac{1}{|\xi|} \times B_0 \geq d_0 > 0\), then

$$C_1(1 + t)^{-\frac{3}{4}} \leq \|P_h(t)\|_{L^2(\xi)} \leq C_2(1 + t)^{-\frac{3}{4}},$$

$$C_1(1 + t)^{-\frac{3}{4}} \leq \|E(t)\|_{L^2(\xi)} \leq C_2(1 + t)^{-\frac{3}{4}},$$

$$C_1(1 + t)^{-\frac{3}{4}} \leq \|B(t)\|_{L^2(\xi)} \leq C_2(1 + t)^{-\frac{3}{4}},$$

for \(t > 0\) large enough with \(C_2 \geq C_1 > 0\) as two generic constants.

**Proof.** By Theorem 5.5, we only need to show the lower bounds of the time decay rates for the solution \((f(t), E(t), B(t))\) under the assumptions of Theorem 5.3. Indeed, in terms of Theorem 5.3, we have

$$\|P_h(t)\|_{L^2(\xi)} \geq \|P_h(t)\|_{L^2(\xi)} - \|P_h(t)\|_{L^2(\xi)} - \|P_h(t)\|_{L^2(\xi)}$$

$$\geq \|P_h(t)\|_{L^2(\xi)} - C(1 + t)^{-2} - Ce^{-Ct},$$

$$\|E(t)\|_{L^2(\xi)} \geq \frac{\sqrt{2}}{2} \left( \|h_1(t)\|_{L^2(\xi)} - \|h_2(t)\|_{L^2(\xi)} - \|h_3(t)\|_{L^2(\xi)} \right)$$

$$\geq \frac{\sqrt{2}}{2} \left( \|h_1(t)\|_{L^2(\xi)} - \|h_2(t)\|_{L^2(\xi)} - \|h_3(t)\|_{L^2(\xi)} \right)$$

$$\geq \frac{\sqrt{2}}{2} \left( \|h_1(t)\|_{L^2(\xi)} - \|h_2(t)\|_{L^2(\xi)} - \|h_3(t)\|_{L^2(\xi)} \right)$$

$$\geq \|J_1(t)\|_{L^2(\xi)} - C(1 + t)^{-2} - Ce^{-Ct},$$

$$\|B(t)\|_{L^2(\xi)} \geq \|J_1(t)\|_{L^2(\xi)} - \|J_2(t)\|_{L^2(\xi)} - \|J_3(t)\|_{L^2(\xi)}$$

$$\geq \|J_1(t)\|_{L^2(\xi)} - C(1 + t)^{-2} - Ce^{-Ct},$$

where we have used (5.39) and (5.65)–(5.67) for \(|\alpha| = 0\).

By (5.63) and \(\lambda_1(\xi) = \lambda_2(\xi)\), we have

$$P_h(t) = i\lambda_1(\xi) |\xi|^2 \hat{T}_1(t, \xi) \sum_{j=1,2} (\omega \times \hat{B}_j, W^j) L^{-1}_\xi \hat{V}_0(\xi, W^j) \chi_0 + \frac{1}{\xi^2} \hat{T}_3(t, \xi) \hat{U}_0,$$
where \( T_3(t, \xi) \tilde{U}_0 \) is a linear operator satisfying \( \| T_3(t, \xi) \tilde{U}_0 \|^2 \leq C e^{-2\eta_1 |\xi|^2 t} \| \tilde{U}_0 \|^2 \). Since the terms \( L_1^{-1}(v\cdot W^1) \sqrt{M} \) and \( L_1^{-1}(v \cdot W^2 \sqrt{M} \) are orthogonal, it follows from (5.73) that

\[
\| P, h_1(t) \|^2_{L^2} \geq \frac{1}{2} \| \xi \|^2 a_1^2 \| L_1^{-1} \chi \|^2_{L^2} e^{2 \text{Re} \lambda_j (|\xi|)^t} |\omega \times \hat{B}_0| - C |\xi|^4 e^{-2\eta_1 |\xi|^2 t} \| \tilde{U}_0 \|^2. \tag{5.75}
\]

Since

\[
\text{Re} \lambda_j (|\xi|) = a_j |\xi|^2 (1 + O(|\xi|)) \geq -\eta_2 |\xi|^2, \quad |\xi| \leq r_0,
\]

for some constant \( \eta_2 > 0 \), we obtain by (5.75) that

\[
\| P, h_1(t) \|^2_{L^2} \geq \frac{1}{2} d_0^2 a_1^2 \| L_1^{-1} \chi \|^2_{L^2} \int_{|\xi| \leq r_0} |\xi|^2 e^{-2\eta_1 |\xi|^2 t} d\xi = C \int_{|\xi| \leq r_0} e^{-2\eta_1 |\xi|^2 t} |\xi|^4 |\tilde{U}_0|^2 d\xi
\]

\[
= \frac{1}{2} d_0^2 a_1^2 \| L_1^{-1} \chi \|^2_{L^2} I_1 - C |\tilde{U}_0|^2 \|_{L^2}^2 (1 + t)^{-7/2}. \tag{5.76}
\]

Moreover, for time \( t \geq t_0 = \frac{1}{\eta_1} \), we have

\[
I_1 = \int_{|\xi| \leq r_0} |\xi|^2 e^{-2\eta_1 |\xi|^2 t} d\xi = 4\pi t^{-5/2} \int_{t^{-1} \leq r} e^{-2\eta_1 r^2} r^4 dr \geq C_0 (1 + t)^{-5/2}, \tag{5.77}
\]

where \( C_0 > 0 \) denotes a generic positive constant. We can substitute (5.76) and (5.77) into (5.71) to obtain (5.63).

By (5.54), we have

\[
H_1(t) = ia_1 |\xi|^2 \chi (|\xi|) t \sum_{j=1,2} (\omega \times \hat{B}_0, W^j)(\omega \times W^j) + |\xi|^2 T_4(t, \xi) \tilde{U}_0, \quad (h_1(t), \sqrt{M} = 0,
\]

where \( T_4(t, \xi) \) is a linear operator satisfying \( \| T_4(t, \xi) \tilde{U}_0 \|^2 \leq C e^{-2\eta_1 |\xi|^2 t} \| \tilde{U}_0 \|^2 \). Then

\[
|H_1(t)|^2 \geq \frac{1}{2} |\xi|^2 a_1^2 e^{2 \text{Re} \lambda_j (|\xi|)^t} |\omega \times \hat{B}_0|^2 - C e^{-2\eta_1 |\xi|^2 t} \| \tilde{U}_0 \|^2.
\]

Similar to (5.77), we get

\[
\| H_1(t) \|^2_{L^2} \geq \frac{1}{2} d_0^2 a_1^2 \int_{|\xi| \leq r_0} |\xi|^2 e^{-2\eta_1 |\xi|^2 t} d\xi - C \int_{|\xi| \leq r_0} |\xi|^4 e^{-2\eta_1 |\xi|^2 t} \| \tilde{U}_0 \|^2 d\xi
\]

\[
\geq C_0 (1 + t)^{-5/2} - C (1 + t)^{-7/2},
\]

which together with (5.72) lead to (5.63) for \( t > 0 \) being large enough.

By (5.55), we have

\[
J_1(t) = -e^{\lambda_j (|\xi|)^t} \sum_{j=1,2} (\omega \times \hat{B}_0, W^j) W^j + |\xi|^2 T_5(t, \xi) \tilde{U}_0,
\]

where \( T_5(t, \xi) \) is a linear operator satisfying \( \| T_5(t, \xi) \tilde{U}_0 \|^2 \leq C e^{-2\eta_1 |\xi|^2 t} \| \tilde{U}_0 \|^2 \). Then

\[
\| J_1(t) \|^2_{L^2} \geq \frac{1}{2} d_0^2 \int_{|\xi| \leq r_0} e^{-2\eta_1 |\xi|^2 t} d\xi - C \int_{|\xi| \leq r_0} |\xi|^2 e^{-2\eta_1 |\xi|^2 t} \| \tilde{U}_0 \|^2 d\xi
\]

\[
\geq C_0 (1 + t)^{-3/2} - C (1 + t)^{-5/2}.
\]

This and (5.73) give (5.70) for \( t > 0 \) being large enough. The proof is then completed.

For comparison, we include the known time decay rates of the global solution to the linearized Boltzmann equation (2.18) as follows, ref. [20, 25] and references therein.
Proof. By (4.18), we can prove (5.84) and (5.85) by applying a similar argument as Lemma 3.2 in [13].

In addition, for any \(C\) constants due to the lack of cancellation in the one-species model. However, in the following, we can see that the estimates are very different from the case of two-species mainly with the optimal convergence rates can be obtained for the one-species VMB based on its spectrum structure.

Corresponding to the linearized two-species VMB, the decomposition and estimation on the semigroup together with the optimal convergence rates can be obtained for the one-species VMB based on its spectrum structure.

5.3 Corresponding results for one-species

Corresponding to the linearized two-species VMB, the decomposition and estimation on the semigroup together with the optimal convergence rates can be obtained for the one-species VMB based on its spectrum structure. However, in the following, we can see that the estimates are very different from the case of two-species mainly due to the lack of cancellation in the one-species model.

Lemma 5.8 (13). The operator \(B_3(\xi) = L - i P_1(\cdot, \xi) P_1\) generates a strongly continuous contraction semigroup on \(N_0^+\) for any fixed \(|\xi| \neq 0\), which satisfies for any \(t > 0\) and \(f \in N_0^+ \cap L^2(\mathbb{R}_+^3)\) that

\[
\|e^{tB_3(\xi)} f\| \leq e^{-\mu t} \|f\|. 
\]

In addition, for any \(x > -\mu\) and \(f \in N_0^+ \cap L^2(\mathbb{R}_+^3)\), it holds

\[
\int_{-\infty}^{\infty} \|[(x + iy) P_1 B_3(\xi)]^{-1} f\|^2 dy \leq \pi (x + \mu)^{-1} \|f\|^2. 
\]

Lemma 5.9. The operator \(G_0(\xi)\) generates a strongly continuous unitary group on \(N_0 \times C^1_\xi \times C^1_\xi\) for any fixed \(|\xi| \neq 0\), which satisfies for \(t > 0\) and \(U \in N_0 \cap L^2(\mathbb{R}_+^3) \times C^1_\xi \times C^1_\xi\) that

\[
\|e^{tG_0(\xi)} U\|_\xi = \|U\|_\xi. 
\]

In addition, for any \(x \neq 0\) and \(f \in N_0 \cap L^2(\mathbb{R}_+^3)\), it holds

\[
\int_{-\infty}^{\infty} \|[(x + iy) P_1 - G_0(\xi)]^{-1} U\|^2 dy = \pi |x|^{-1} \|U\|^2_\xi. 
\]

Proof. Since the operator \(i G_0(\xi)\) is self-adjoint on \(N_0 \times C^1_\xi \times C^1_\xi\) with respect to the inner product \((\cdot, \cdot)_\xi\) defined by \([13, 18]\), we can prove (5.84) and (5.85) by applying a similar argument as Lemma 3.2 in [13].

By a similar argument as for Lemma 5.3, we have

Lemma 5.10. Given any constant \(r_0 > 0\). Let \(\alpha = \alpha(r_0, r_1) > 0\) with \(r_1 > r_0\) and \(\alpha(r_0, r_1)\) defined in Lemma 5.3. Then

\[
\sup_{r_0 < |\xi| < r_1, y \in \mathbb{R}} \|I - G_3(\xi)(-\frac{\alpha}{2} + iy - G_1(\xi))^{-1}\| \leq C. 
\]
With the help of Lemmas 4.3–4.5 and Lemmas 5.8–5.10, we have the decomposition of the semigroup $S(t, \xi) = e^{t\hat{k}_3(\xi)}$ as following, the detail of the proof is omitted for brevity.

**Theorem 5.11.** The semigroup $S(t, \xi) = e^{t\hat{k}_3(\xi)}$ with $\xi = s\omega \in \mathbb{R}^3$ and $s = |\xi| \neq 0$ has the following decomposition

$$S(t, \xi)U = S_1(t, \xi)U + S_2(t, \xi)U + S_3(t, \xi)U, \quad U \in L^2(\mathbb{R}^3_\xi) \times \mathbb{C}^3_\xi \times \mathbb{C}^3_\xi, \quad t > 0,$$

where

$$S_1(t, \xi)U = \sum_{j=1}^{7} e^{t\lambda_j(s)}(U, \Psi_j(s, \omega)) \xi \Psi_j(s, \omega)1_{(|\xi| \leq r_0)},$$

$$S_2(t, \xi)U = \sum_{j=1}^{4} e^{t\beta_j(s)}(U, \Phi_j(s, \omega)) \Phi_j(s, \omega)1_{(|\xi| > r_1)},$$

with $(\lambda_j(s), \Psi_j(s, \omega))$ and $(\beta_j(s), \Phi_j(s, \omega))$ being the eigenvalue and eigenvector of the operator $\hat{k}_3(\xi)$ given by Theorem 4.9 and Theorem 4.10, for $|\xi| \leq r_0$ and $|\xi| > r_1$ respectively, and $S_3(t, \xi) =: S(t, \xi) - S_1(t, \xi) - S_2(t, \xi)$ satisfies that there exists a constant $\kappa > 0$ independent of $\xi$ such that

$$\|S_3(t, \xi)U\|_{\xi} \leq Ce^{-\kappa t\|U\|_{\xi}}, \quad t \geq 0.$$

Based on the decomposition of the semigroup given in Theorem 5.11, we now study the optimal convergence rates of the solutions to the linearized system around an equilibrium.

For any $U_0 = (f_0, E_0, B_0) \in L^2(\mathbb{R}^3, H^1(\mathbb{R}^3)) \times H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$, define

$$e^{t\hat{k}_2(\xi)}\tilde{U}_0 = ((e^{t\hat{k}_3(\xi)})\tilde{V}_0)_1, -\frac{\xi}{|\xi|^2}(e^{t\hat{k}_3(\xi)})\tilde{V}_0)_2, \frac{\xi}{|\xi|^2} \times (e^{t\hat{k}_3(\xi)})\tilde{V}_0)_3, \quad \tilde{V}_0 = (f_0, \frac{\xi}{|\xi|} \times E_0, \frac{\xi}{|\xi|} \times B_0).$$

Then $e^{t\hat{k}_2}U_0$ is the solution of the system (2.52). By Lemma 4.1, it holds that

$$\|e^{t\hat{k}_2}U_0\|_{D^l} = \int_{\mathbb{R}^3} (1 + |\xi|^2 t)\|e^{t\hat{k}_3(\xi)}\tilde{V}_0\|_{D^l}^2 d\xi \leq \int_{\mathbb{R}^3} (1 + |\xi|^2 t)\|\tilde{V}_0\|_{D^l}^2 d\xi = \|U_0\|_{D^l}.$$ 

This implies that the linear operator $A_2$ generates a strongly continuous contraction semigroup $e^{t\hat{k}_2}$ in $D^l$, and therefore, $U(t) = e^{t\hat{k}_2}U_0$ is a global solution to the IVP (2.52) for the linearized one-species Vlasov-Maxwell-Boltzmann equation for $U_0 \in D^l$.

**Proof of Theorem 2.10.** By 4.91 and Theorem 5.11 we have for $\omega = \xi/|\xi|$ that

$$(\tilde{f}(t), \omega \times \tilde{E}(t), \omega \times \tilde{B}(t)) = e^{t\hat{k}_3(\xi)}\tilde{V}_0 = S_1(t, \xi)\tilde{V}_0 + S_2(t, \xi)\tilde{V}_0 + S_3(t, \xi)\tilde{V}_0 = \sum_{k=1}^{3} (h_k(t), H_k(t), J_k(t)), $$

with $\tilde{V}_0 = (f_0, \omega \times E_0, \omega \times B_0)$. Note that

$$\|\partial_\omega^s(f(t), \chi_j)\|_{L^2} \leq \frac{\|\xi^s(h_1(t), \chi_j)\|_{L^2} + \|\xi^s(h_2(t), \chi_j)\|_{L^2} + \|\xi^s(h_3(t), \chi_j)\|_{L^2}}{\|\xi^s(\chi_j)\|_{L^2}},$$

$$\|\partial_\omega^s E(t)\|_{L^2} \leq \frac{\|\xi^s(h_1(t), \chi_0)\|_{L^2} + \|\xi^s(h_2(t), \chi_0)\|_{L^2} + \|\xi^s(h_3(t), \chi_0)\|_{L^2}}{\|\xi^s(\chi_0)\|_{L^2}}.$$
and being uniformly bounded for $\|\xi^\alpha H_1(t)\|_{L^2} + \|\xi^\alpha H_2(t)\|_{L^2} + \|\xi^\alpha H_3(t)\|_{L^2}$.

(5.93)

\[ \|\partial_x^2 B(t)\|_{L^2} \leq \|\xi^\alpha J_1(t)\|_{L^2} + \|\xi^\alpha J_2(t)\|_{L^2} + \|\xi^\alpha J_3(t)\|_{L^2}. \]

(5.94)

By (5.93), we can estimate the last term on the right hand side of (5.92) as follows:

\[
\int_{\mathbb{R}} (\xi^2)\left(\|h_3(t)\|_{L^2} + \frac{1}{|\xi|^2} (h_3(t), \sqrt{M})\right)^2 + \|H_3(t)\|^2 + |J_3(t)|^2) \, d\xi \\
\leq C e^{-2\alpha t} (\|\partial_x^2 f_0\|^2_{L^2} + \|\partial_x^2 E_0\|_{L^2} + \|\partial_x^2 B_0\|_{L^2}).
\]

(5.95)

In the low frequency region, by (5.88), we have

\[ S_1(t, \xi \hat{V}_0) = \sum_{j=1}^{7} e^{\lambda_j(|\xi|)} [(f_0, \overline{\psi_j})_\xi - (\omega \times \hat{E}_0, \overline{X_j}) - (\omega \times \hat{B}_0, \overline{\chi_j})/(\psi_j, \chi_j)]_1_{|\xi| \leq r_0}, \]

From (5.88) and (5.90), the macroscopic part and microscopic part of $h_1(t)$ and $H_1(t), J_1(t)$ satisfy

\[ (h_1(t), \sqrt{M}) = \frac{1}{2} \sum_{j=1}^{5} e^{\lambda_j(|\xi|)} [\tilde{m}_0 - |\xi|_1 \tilde{n}_0] \omega + \frac{1}{2} \sum_{j=2, 3} e^{\lambda_j(|\xi|)} [(\tilde{m}_0 \cdot W^j) + i(\hat{E}_0, W^j)] W^j, \]

(5.96)

\[ (h_1(t), v\sqrt{M}) = \frac{1}{2} \sum_{j=1}^{5} e^{\lambda_j(|\xi|)} [(\tilde{m}_0 \cdot W^j) - i(\hat{E}_0, W^j)] W^j + \frac{1}{2} \sum_{j=6, 7} e^{\lambda_j(|\xi|)} [(\tilde{m}_0 \cdot W^j) + i(\hat{E}_0, W^j)] W^j,
\]

(5.97)

\[ (h_1(t), \chi_4) = \sqrt{\frac{\pi}{6}} \sum_{j=1}^{5} e^{\lambda_j(|\xi|)} [\tilde{m}_0 + e^{\lambda_j(|\xi|)}(\hat{q}_0 - T^{\frac{2}{3}} \hat{n}_0) + |\xi| \sum_{j=6, 7} e^{\lambda_j(|\xi|)} ((T_j(\xi) \hat{V}_0)_1, v\sqrt{M}), \]

(5.98)

\[ P_1 h_1(t) = -\frac{1}{2} \sum_{j=1}^{5} e^{\lambda_j(|\xi|)} [\tilde{m}_0 + e^{\lambda_j(|\xi|)}(\hat{q}_0 - T^{\frac{2}{3}} \hat{n}_0) + |\xi| \sum_{j=6, 7} e^{\lambda_j(|\xi|)} ((T_j(\xi) \hat{V}_0)_1, v\sqrt{M})], \]

(5.99)

\[ H_1(t) = -\frac{1}{2} e^{\lambda_j(|\xi|)} [i(\omega \times \tilde{m}_0) - (\omega \times \tilde{E}_0)] + \frac{1}{2} e^{\lambda_j(|\xi|)} [i(\omega \times \tilde{B}_0) - (\omega \times \tilde{E}_0)] \]

(5.100)

\[ J_1(t) = \frac{1}{2} \sum_{j=6, 7} e^{\lambda_j(|\xi|)} [(\omega \times \tilde{B}_0, W^j) W^j + |\xi| \sum_{j=6, 7} e^{\lambda_j(|\xi|)} (\tilde{m}_0 \cdot W^j) W^j], \]

(5.101)

where $(\tilde{m}_0, \tilde{n}_0, \tilde{q}_0)$ is the Fourier transform of the macroscopic density, momentum and energy $(n_0, m_0, q_0)$ of the initial data $f_0$ defined by $(n_0, m_0, q_0) = ((f_0, \chi_0), (f_0, v\sqrt{M}), (f_0, \chi_4))$, $W^j, 2 \leq j \leq 7$ is given by (4.55), and $T_j(\xi) = ((T_j(\xi) \hat{V}_0)_1, (T_j(\xi) \hat{V}_0)_2, (T_j(\xi) \hat{V}_0)_3), -1 \leq j \leq 7, are the linear operators with the norm $\|T_j(\xi)\|$ being uniformly bounded for $|\xi| \leq r_0$.

Since

\[ \text{Re} \lambda_j(|\xi|) = a_j|\xi|^2 (1 + O(|\xi|)) \leq -\eta_1|\xi|^2, \quad |\xi| \leq r_0, \quad j = -1, 0, 1, 2, 3, 4, 5, \]

\[ \text{Re} \lambda_k(|\xi|) = a_k|\xi|^2 (1 + O(|\xi|)) \leq -\eta_1|\xi|^4, \quad |\xi| \leq r_0, \quad k = 6, 7, \]
where $\eta_1 > 0$ denotes a generic constant, we obtain by (5.121), (5.101) that
\begin{align*}
||\xi^\alpha(h_1(t), \sqrt{M})||_{L^2_t}^2 & \leq C(1 + t)^{-3(3/4 + k)}||\partial_x^\alpha U_0||_{L^2_t}^2, \quad (5.102) \\
||\xi^\alpha(\beta_1(t), v \sqrt{M})||_{L^2_t}^2 & \leq C(1 + t)^{-1(3/4 + k)}||\partial_x^\alpha U_0||_{L^2_t}^2 + C(1 + t)^{-5(4/4 + k/2)}||\partial_x^\alpha U_0||_{L^2_t}^2, \quad (5.103) \\
||\xi^\alpha(h_1(t), \chi_4)||_{L^2_t}^2 & \leq C(1 + t)^{-3(3/4 + k)}||\partial_x^\alpha U_0||_{L^2_t}^2, \quad (5.104) \\
||\xi^\alpha P_{\epsilon} h_1(t)||_{L^2_t}^2 & \leq C(1 + t)^{-3(3/4 + k)}||\partial_x^\alpha U_0||_{L^2_t}^2 + C(1 + t)^{-7(4/4 + k/2)}||\partial_x^\alpha U_0||_{L^2_t}^2, \quad (5.105) \\
||\xi^\alpha H_1(t)||_{L^2_t}^2 & \leq C(1 + t)^{-3(3/4 + k)}||\partial_x^\alpha U_0||_{L^2_t}^2 + C(1 + t)^{-9(4/4 + k/2)}||\partial_x^\alpha U_0||_{L^2_t}^2, \quad (5.106) \\
||\xi^\alpha J_1(t)||_{L^2_t}^2 & \leq C(1 + t)^{-3(3/4 + k/2)}||\partial_x^\alpha U_0||_{L^2_t}^2 + C(1 + t)^{-5(2/4 + k)}||\partial_x^\alpha U_0||_{L^2_t}^2, \quad (5.107)
\end{align*}

with $k = |\alpha - \alpha'|$.

In the high frequency region, by (5.59), we have
\begin{equation}
S_2(t, \xi) \hat{V}_0 = \sum_{j=1}^{4} e^{it\lambda_j(|\xi|)}[(\hat{f}_0, \hat{\phi}_j) - (\omega \times \hat{E}_0, \hat{X}_j) - (\omega \times \hat{B}_0, \hat{Y}_j)](\hat{\phi}_j, \hat{X}_j, \hat{Y}_j)1(|\xi| \geq r_1), \quad (5.108)
\end{equation}

and in particular $(h_2(t), \sqrt{M}) = (h_2(t), \chi_4) = 0$. Since
$$
\text{Re} \lambda_j(|\xi|) \leq -c_1|\xi|^{-1}, \quad |\xi| \geq r_1,
$$
we obtain by (5.108) that
\begin{align*}
||\xi^\alpha(h_2(t), \sqrt{M})||_{L^2_t}^2 & \leq C(1 + t)^{-2(3m + 1)}(||\nabla^m \partial_x^\alpha f_0||_{L^2_t}^2 + ||\nabla^m \partial_x^\alpha E_0||_{L^2_t}^2 + ||\nabla^m \partial_x^\alpha B_0||_{L^2_t}^2), \quad (5.109) \\
||\xi^\alpha P_{\epsilon} h_2(t)||_{L^2_t}^2 & \leq C(1 + t)^{-2(3m + 1)}(||\nabla^m \partial_x^\alpha f_0||_{L^2_t}^2 + ||\nabla^m \partial_x^\alpha E_0||_{L^2_t}^2 + ||\nabla^m \partial_x^\alpha B_0||_{L^2_t}^2), \quad (5.110) \\
||\xi^\alpha H_2(t)||_{L^2_t}^2 & \leq C(1 + t)^{-2m}(||\nabla^m \partial_x^\alpha f_0||_{L^2_t}^2 + ||\nabla^m \partial_x^\alpha E_0||_{L^2_t}^2 + ||\nabla^m \partial_x^\alpha B_0||_{L^2_t}^2), \quad (5.111) \\
||\xi^\alpha J_2(t)||_{L^2_t}^2 & \leq C(1 + t)^{-2m}(||\nabla^m \partial_x^\alpha f_0||_{L^2_t}^2 + ||\nabla^m \partial_x^\alpha E_0||_{L^2_t}^2 + ||\nabla^m \partial_x^\alpha B_0||_{L^2_t}^2), \quad (5.112)
\end{align*}

Combining (5.92), (5.95), (5.102), (5.107) and (5.109)–(5.112) leads to $2.57 < 2.68$.

Now we turn to show the lower bound of time decay rates for the global solution under the assumptions of Theorem 2.10 Note that
\begin{align*}
||f(t, \chi_3)||_{L^2_t} \geq ||(h_1(t), \chi_3)||_{L^2_t} - C(1 + t)^{-1} - Ce^{-Ct}, \quad (5.113) \\
||E(t)||_{L^2_t} \geq \frac{\sqrt{2}}{2}||h_1(t), \sqrt{M}||_{L^2_t} + ||H_1(t)||_{L^2_t} - C(1 + t)^{-1} - Ce^{-Ct}, \quad (5.114) \\
||B(t)||_{L^2_t} \geq ||J_1(t)||_{L^2_t} - C(1 + t)^{-1} - Ce^{-Ct}, \quad (5.115)
\end{align*}

where we have used (5.305) and (5.109)–(5.112) for $|\alpha| = 0$.

By (5.90) and that fact that $\lambda_{-1}(|\xi|) = \lambda_{1}(|\xi|)$, we have
\begin{equation}
(h_1(t), \sqrt{M}) = e^{Re\lambda_1(|\xi|)t} \cos(Im\lambda_1(|\xi|)t)\hat{n}_0 + |\xi|T_{\delta}(t, \xi)\hat{V}_0,
\end{equation}

where $T_{\delta}(t, \xi)\hat{V}_0$ is the remainder term satisfying $||T_{\delta}(t, \xi)\hat{V}_0|| \leq Ce^{-\eta_1|\xi|^2}||\hat{V}_0||$. This leads to
\begin{equation}
||h_1(t, \sqrt{M})||^2 \geq \frac{1}{2} e^{2Re\lambda_1(|\xi|)t} \cos^2(Im\lambda_1(|\xi|)t)|\hat{n}_0|^2 - C|\xi|^2 e^{-2\eta_1|\xi|^2}||\hat{V}_0||^2. \quad (5.116)
\end{equation}

Since
\begin{equation}
\cos^2(Im\lambda_1(|\xi|)t) \geq \frac{1}{2} \cos^2[(1 + b_1|\xi|^2)t] - O(|\xi|^3t^2),
\end{equation}

and
\begin{equation}
\text{Re} \lambda_1(|\xi|) = a_j|\xi|^2(1 + O(|\xi|)) \geq -\eta_2|\xi|^2, \quad |\xi| \leq r_0, \quad j = -1, 0, 1, 2, 3, 4, 5,
\end{equation}
for some constant $\eta > 0$, we obtain by (5.113) that

$$\| (h_1(t), \sqrt{M}) \|_{L^2}^2 \geq \frac{d_0^2}{4} \int_{|\xi| \leq r_0} e^{-2\eta |\xi|^2} \cos(t + b_1 |\xi|^2) d\xi - C(1 + t)^{-5/2}$$

$$=: I_1 - C(1 + t)^{-5/2}. \quad (5.117)$$

Since it holds for $t \geq t_0 =: \frac{b_0^2}{r_0^2}$ with the constant $L \geq \sqrt{\frac{2r_0}{\eta}}$ that

$$I_1 = \frac{d_0^2}{4} t^{-3/2} \int_{|\xi| \leq r_0 \sqrt{L}} e^{-2\eta |\xi|^2} \cos(t + b_1 |\xi|^2) d\xi \geq \frac{\pi d_0^2}{4} (1 + t)^{-3/2} \int_0^L r^2 e^{-2\eta r^2} \cos(t + b_1 r^2) dr$$

$$\geq (1 + t)^{-3/2} \frac{\pi d_0^2 L}{4b_1} e^{-2\eta t^2} \int_0^L r^2 \cos^2(t + b_1 r^2) dr = (1 + t)^{-3/2} \frac{\pi d_0^2 L}{4b_1} e^{-2\eta t^2} \int_{t + b_1 L}^L \cos^2(y) dy$$

$$\geq (1 + t)^{-3/2} \frac{\pi d_0^2 L}{4b_1} e^{-2\eta L^2} \int_0^\pi \cos^2(y) dy \geq C_3 (1 + t)^{-3/2}, \quad (5.118)$$

where $C_3 > 0$ denotes a generic positive constant. We can substitute (5.117) and (5.118) into (5.113) with $j = 0$ to obtain (2.67).

By (5.97), we have

$$| (h_1(t), v\sqrt{M}) | \geq 1 |\frac{2\Re\lambda_1(\nu)|^t}{2} \sin^2(\text{Im}\lambda_1(\nu)|^t) |\bar{\eta}_0|^2 - C \xi^2 e^{-2|\xi|^4} \| \bar{V}_0 \|^2 - C |\xi|^2 e^{-2|\xi|^4} \| V_0 \|^2.$$

Then

$$\| (h_1(t), v\sqrt{M}) \|_{L^2}^2 \geq \frac{d_0^2}{4} \int_{|\xi| \leq r_0} \frac{1}{|\xi|^2} e^{-2\eta |\xi|^2} \sin^2(t + b_1 |\xi|^2) d\xi$$

$$- C \int_{|\xi| \leq r_0} |\xi|^2 e^{-2\eta |\xi|^2} \| \bar{V}_0 \|^2 d\xi - C \int_{|\xi| \leq r_0} |\xi|^2 e^{-2\eta |\xi|^2} \| V_0 \|^2 d\xi$$

$$\geq C_3 (1 + t)^{-1/2} - C(1 + t)^{-3/2} - C(1 + t)^{-5/4},$$

which together with (5.113) for $j = 1, 2, 3$ lead to (2.68) for $t > 0$ being large enough.

By (5.98) and the fact that $\lambda_0(\nu)|^t$ is real, we have

$$| (h_1(t), \chi_4) |^2 \geq \frac{1}{2} |\frac{2\Re\lambda_0(\nu)|^t}{2} |\eta_0|^2 - C \xi e^{-2|\xi|^2} (|\bar{\eta}_0|^2 + |\xi|^2 |\bar{V}_0|)^2),$$

which leads to

$$\| (h_1(t), \chi_4) \|_{L^2}^2 \geq \frac{1}{2} \int_{|\xi| \leq r_0} e^{-2\eta |\xi|^2} |\eta_0|^2 d\xi - C \int_{|\xi| \leq r_0} e^{-2\eta |\xi|^2} (|\bar{\eta}_0|^2 + |\xi|^2 |\bar{V}_0|)^2) d\xi$$

$$\geq C_3 \inf_{|\xi| \leq r_0} |\eta_0(\xi)| (1 + t)^{-3/2} - d_1 \sup_{|\xi| \leq r_0} |\bar{\eta}_0(\xi)| (1 + t)^{-3/2} - C(1 + t)^{-5/2}.$$

This and (5.113) with $j = 4$ lead to (2.69) for $t > 0$ being large enough.

By (5.99), we have

$$\| P_1 h_1(t) \|_{L^2}^2 \geq \frac{1}{2} |\frac{2\Re\lambda_1(\nu)|^t}{2} |\xi|^2 \sin(\text{Im}\lambda_1(\nu)|^t) t L \Psi + \cos(\text{Im}\lambda_1(\nu)|^t) t \Psi |^2 \|_{L^2}^2$$

$$- C |\xi|^2 e^{-2|\xi|^2} (|\bar{V}_0|)^2 - C |\xi|^4 e^{-2|\xi|^4} (|\bar{V}_0|)^2,$$

where $\Psi \in N_0^J$ is a non-zero real function given by

$$\Psi = (L - iP_1)^{-1} (L + iP_1)^{-1} P_1 (v \cdot \omega)^2 \sqrt{M} \neq 0.$$

Then

$$\| P_1 h_1(t) \|_{L^2}^2 \geq \frac{d_0^2}{4} \int_{|\xi| \leq r_0} e^{-2\eta |\xi|^2} \sin(t + b_1 |\xi|^2) t L \Psi + \cos(t + b_1 |\xi|^2) t \Psi |^2 \|_{L^2}^2 d\xi$$
\[ -C(1 + t)^{-5/2} - C(1 + t)^{-7/4} =: I_2 - C(1 + t)^{-5/2} - C(1 + t)^{-7/4}. \] (5.119)

Since it holds for time \( t \geq t_0 =: \frac{L^2}{r_0^2} \) with \( L \geq \sqrt{\frac{4\pi}{b_1}} \) that

\[
I_2 = \frac{d_1^2}{4}t^{-3/2} \int_{|\xi| \leq r_0 \sqrt{t}} e^{-2\eta_1|\xi|^2} \| \sin(t + b_1|\xi|^2)L\Psi + \cos(t + b_1|\xi|^2)\Psi \|^2_{L^2(\mathbb{R}^2)}d\xi
\geq \pi d_1^2(1 + t)^{-3/2} \int_0^L r^2 e^{-2\eta_2 r^2} \| \sin(t + b_1r^2)L\Psi + \cos(t + b_1r^2)\Psi \|^2_{L^2(\mathbb{R}^2)}dr
\geq \frac{L\pi d_1^2}{4b_1}(1 + t)^{-3/2} - C_3(1 + t)^{-3/2},
\] (5.120)

we can substitute (5.120) and (5.119) into (5.113) imply (2.70) for sufficiently large \( t > 0 \).

By (5.100), we obtain

\[
\frac{1}{\|\xi\|^2}|(h_1(t), \sqrt{M})|^2 + |H_1(t)|^2 \geq \frac{1}{2\|\xi\|^2}e^{2Re\lambda_1(|\xi|)t} \cos^2(\text{Im} \lambda_1(|\xi|)t)\|\tilde{n}_0\|^2 - C e^{-2\eta_1|\xi|^2t}\|\tilde{V}_0\|^2 - C\|\xi\|^6 e^{-2\eta_1|\xi|^4t}\|\tilde{V}_0\|^2,
\]
which leads to

\[
\|\frac{1}{\|\xi\|^2}(h_1(t), \sqrt{M})\|_{L^2_t}^2 + \|H_1(t)\|_{L^2_t}^2 \geq C_3(1 + t)^{-1/2} - C(1 + t)^{-3/2} - C(1 + t)^{-9/4}.
\]

This together with (5.114) lead to (2.65) for sufficiently large \( t > 0 \).

Finally, by (5.100) and the fact that \( \lambda_0(|\xi|) = \lambda_7(|\xi|) \) are real, we obtain

\[
|J_1(t)|^2 \geq \frac{1}{2} e^{2Re\lambda_j(|\xi|)t} |\omega \times \tilde{B}_0|^2 - C\|\xi\|^4 e^{-2\eta_j|\xi|^4t}\|\tilde{V}_0\|^2 - C\|\xi\|^2 e^{-2\eta_j|\xi|^2t}\|\tilde{V}_0\|^2.
\]

Since

\[
\text{Re} \lambda_j(|\xi|) = a_j|\xi|^4(1 + O(|\xi|)) \geq -\eta_2|\xi|^4, \quad |\xi| \leq r_0, \quad j = 6, 7,
\]
for some constant \( \eta_2 > 0 \), we have

\[
\|J_1(t)\|^2_{L^2_t} \geq \frac{d_1^2}{2} \int_{|\xi| \leq r_0} |\xi|^2 e^{-2\eta_2|\xi|^4t}d\xi - C \int_{|\xi| \leq r_0} |\xi|^4 e^{-2\eta_2|\xi|^4t}\|\tilde{V}_0\|^2d\xi
\geq C_3(1 + t)^{-3/4} - C(1 + t)^{-7/4} - C(1 + t)^{-5/2}.
\]

This together with (5.115) lead to (2.66). The proof is then completed.

**Proof of Theorem 2.41** In the case of \( \nabla_x \cdot E_0 = (f_0, \sqrt{M}) \), the high frequency term \( S_2(t, \xi)\tilde{V}_0 \) are same as those in the case of \( \nabla_x \cdot E_0 \neq (f_0, \sqrt{M}) \). The different part lie in the expansions of the low frequency term \( S_1(t, \xi)\tilde{V}_0 \). That is, in the low frequency region, by (5.88), we have

\[
S_1(t, \xi)\tilde{V}_0 = \sum_{j=-1}^7 e^{\text{Im} \lambda_j(|\xi|)t}(f_0, \sqrt{M}) + \frac{1}{\|\xi\|^2}i(\tilde{E}_0 \cdot \xi)(\psi_j, \sqrt{M}) - (\omega \times \tilde{E}_0, \mathbf{X}_j) - (\omega \times \tilde{B}_0, \mathbf{Y}_j)
\times (\psi_j, \mathbf{X}_j, \mathbf{Y}_j)1_{\{\xi \leq r_0\}},
\]
where we have used \((f_0, \sqrt{M}) = i(\tilde{E}_0 \cdot \xi) \) to replace \((f_0, \sqrt{M}) \). From (4.55) and (4.56), the macroscopic part and microscopic part of \( h_1(t) \) and \( h_2(t) \), \( h_3(t) \) satisfy

\[
(h_1(t), \sqrt{M}) = |\xi| \sum_{j=-1}^4 e^{\text{Im} \lambda_j(|\xi|)t}(T_j^1(\xi)\tilde{V}_0, \sqrt{M}),
\] (5.121)
Theorem 2.10. By (5.121), we have

\[ (h_1(t), v\sqrt{M}) = \frac{1}{2} \sum_{j=\pm 1} e^{\lambda_j(|\xi|)^2} [\tilde{\varphi}_0 - ji(\tilde{E}_0 \cdot \omega)] \cdot \omega + \frac{1}{2} \sum_{j=2,3} e^{\lambda_j(|\xi|)^2} [\tilde{\varphi}_0 \cdot W^j] + i(\tilde{E}_0 \cdot W^j)]W^j \]

\[ + \frac{1}{2} \sum_{j=4,5} e^{\lambda_j(|\xi|)^2} [i(\tilde{E}_0 \cdot W^j) - i(\tilde{E}_0 \cdot W^j)]W^j + i|\xi| \sum_{j=6,7} e^{\lambda_j(|\xi|)^2} (\omega \times \tilde{B}_0, W^j)W^j \]

\[ + |\xi| \sum_{j=1}^5 e^{\lambda_j(|\xi|)^2} (T_j(\xi)\tilde{V}_0), v\sqrt{M}) + |\xi|^2 \sum_{j=6,7} e^{\lambda_j(|\xi|)^2} (T_j(\xi)\tilde{V}_0), v\sqrt{M}), \]  
(5.122)

\[ (h_1(t), \chi_4) = e^{\lambda_0(|\xi|)^2}g_0 + |\xi| \sum_{j=-1}^4 e^{\lambda_j(|\xi|)^2} (T_j(\xi)\tilde{V}_0), \chi_4), \]  
(5.123)

\[ P_1h_1(t) = |\xi| \sum_{j=-1}^5 e^{\lambda_j(|\xi|)^2} P_1(T_j(\xi)\tilde{V}_0), + |\xi|^2 \sum_{j=6,7} e^{\lambda_j(|\xi|)^2} P_1(T_j(\xi)\tilde{V}_0), \]  
(5.124)

and \( H_1(t), J_1(t) \) are same as (5.100) and (5.101). Thus we obtain by (5.121)–(5.101) that

\[ \|\xi^\alpha (h_1(t), \sqrt{M})\|^2_{L^2_t} \leq C(1 + t)^{-5/2+k} \|\phi_0 U_0\|^2_{L^2_t}, \]

(5.125)

\[ \|\xi^\alpha (h_1(t), v\sqrt{M})\|^2_{L^2_t} \leq (1 + t)^{-5/2+k} \|\phi_0 U_0\|^2_{L^2_t} + C(1 + t)^{-1} \|\phi_0 U_0\|^2_{L^2_t}, \]

(5.126)

\[ \|\xi^\alpha (h_1(\overline{t}), \chi_4)\|^2_{L^2_t} \leq (1 + t)^{-5/2+k} \|\phi_0 U_0\|^2_{L^2_t}, \]

(5.127)

\[ \|\xi^\alpha P_1 h_1(t)\|^2_{L^2_t} \leq (1 + t)^{-5/2+k} \|\phi_0 U_0\|^2_{L^2_t} + C(1 + t)^{-7/4+k/2} \|\phi_0 U_0\|^2_{L^2_t}, \]

(5.128)

\[ \|\xi^\alpha H_1(t)\|^2_{L^2_t} \leq (1 + t)^{-3/2+k/2} \|\phi_0 U_0\|^2_{L^2_t} + C(1 + t)^{-9/4+k/2} \|\phi_0 U_0\|^2_{L^2_t}, \]

(5.129)

\[ \|\xi^\alpha J_1(t)\|^2_{L^2_t} \leq (1 + t)^{-3/2+k} \|\phi_0 U_0\|^2_{L^2_t} + C(1 + t)^{-5/2+k} \|\phi_0 U_0\|^2_{L^2_t}, \]

(5.130)

with \( k = |\alpha - \alpha'| \). Thus by (5.125)–(5.130) and (5.100)–(5.112), we can prove (2.74)–(2.78).

Now we turn to show the lower bound of time decay rates for the global solution under the assumptions of Theorem 2.10. By (5.112), we have

\[ (h_1(t), \sqrt{M}) = -|\xi| e^{Re\lambda_1(|\xi|)^t} \sin(\text{Im}\lambda_1(|\xi|)^t)(\tilde{E}_0 \cdot \omega) + |\xi|^2 T_\delta(t, \xi)\tilde{V}_0, \]

where \( T_\delta(t, \xi)\tilde{V}_0 \) is the remainder term satisfying \( ||T_\delta(t, \xi)\tilde{V}_0|| \leq C e^{-n_1|\xi|^2t}||\tilde{V}_0|| \). This leads to

\[ ||(h_1(t), \sqrt{M})||^2 \geq \frac{1}{2}|\xi|^2 e^{2Re\lambda_1(|\xi|)^t} \sin^2(\text{Im}\lambda_1(|\xi|)^t)(\tilde{E}_0 \cdot \omega)^2 - C|\xi|^4 e^{-2n_1|\xi|^2 t} ||\tilde{V}_0||^2. \]

It follows that

\[ ||(h_1(t), \sqrt{M})||^2_{L^2_t} \geq \frac{d^2}{4} \int_{|\xi| \leq r_0} |\xi|^2 e^{-2n_1|\xi|^2 t} \sin^2(t + b_1|\xi|^2 t) d\xi - (1 + t)^{-7/2} \]

\[ \geq C_3(1 + t)^{-5/2} - C(1 + t)^{-7/2}. \]

(5.131)

We can substitute (5.131) into (5.113) with \( j = 0 \) to obtain (2.81).

By (5.122) and (5.123), we have

\[ ||(h_1(t), v\sqrt{M})||^2 \geq \frac{1}{2}|\xi|^2 e^{2\lambda_0(|\xi|)^2} \omega \times \tilde{B}_0)^2 - C e^{-2n_1|\xi|^2 t} ||\tilde{E}_0||^2 + |\xi|^2 ||\tilde{V}_0||^2) - C|\xi|^4 e^{-2n_1|\xi|^2 t} ||\tilde{V}_0||^2, \]

\[ ||(h_1(t), \chi_4)||^2 \geq \frac{1}{2} e^{2\lambda_0(|\xi|)^2} g_0^2 - C|\xi|^2 e^{-2n_1|\xi|^2 t} ||\tilde{V}_0||^2, \]

which leads to

\[ ||(h_1(t), v\sqrt{M})||^2_{L^2_t} \geq C_3(1 + t)^{-5/4} - C(1 + t)^{-7/4} - C(1 + t)^{-3/2}, \]

\[ ||(h_1(t), \chi_4)||^2_{L^2_t} \geq C_3(1 + t)^{-3/2} - C(1 + t)^{-5/2}. \]
This together with (5.113) lead to (2.82) and (2.83) for \( t > 0 \) being large enough.

By (5.124), we have

\[
P_1 h_1(t) = |\xi|^2 \sum_{j=6,7} e^{\lambda_j(|\xi|) t} (\omega \times \hat{B}_0, W_j) L^{-1} P_1 (v \cdot \omega)(v \cdot W_j) \sqrt{M} \\
+ |\xi|^2 T_0(t, \xi) \hat{V}_0 + |\xi| T_{10}(t, \xi) \hat{V}_0,
\]

where \( T_0(t, \xi) \) and \( T_{10}(t, \xi) \) are the remainder terms satisfying \( \| T_0(t, \xi) \hat{V}_0 \| \leq C e^{-\eta_1|\xi|^2} \| \hat{V}_0 \| \) and \( \| T_{10}(t, \xi) \hat{V}_0 \| \leq C e^{-\eta_1|\xi|^2} \| \hat{V}_0 \| \). Then

\[
\| P_1 h_1(t) \|_{L^2_{t,x,v}}^2 \geq \frac{1}{2} \| L^{-1} P_1 (v \chi_2) \|_{L^2_{t}}^2 |\xi|^4 e^{2\lambda_0(|\xi|) t} |\omega \times \hat{B}_0| - C |\xi|^6 e^{-2\eta_1|\xi|^4} \| \hat{V}_0 \|^2 \\
- C |\xi|^2 e^{-2\eta_1|\xi|^2} \| \hat{V}_0 \|^2.
\]

This leads to

\[
\| P_1 h_1(t) \|_{L^2_{t,x,v}}^2 \geq C_3 (1 + t)^{-7/4} - C (1 + t)^{-9/4} - C (1 + t)^{-5/2},
\]

which together with (5.113) imply (2.84) for sufficiently large \( t > 0 \).

Finally, by (5.100) and (5.101) we obtain

\[
\frac{1}{|\xi|^2} \langle |h_1(t)|, \sqrt{M} \rangle^2 + \langle H_1(t) \rangle^2 \geq \frac{1}{2} e^{2 \text{Re} \lambda_1(|\xi|) t} \sin^2 (\text{Im} \lambda_1(|\xi|) t) |\hat{E}_0 \cdot \omega|^2 + \frac{1}{2} e^{2 \text{Re} \lambda_1(|\xi|) t} \cos^2 (\text{Im} \lambda_1(|\xi|) t) |\omega \times \hat{B}_0| - C |\xi|^6 e^{-2\eta_1|\xi|^4} \| \hat{V}_0 \|^2 \\
- C |\xi|^2 e^{-2\eta_1|\xi|^2} \| \hat{V}_0 \|^2,
\]

which lead to

\[
\frac{1}{|\xi|^2} \langle |h_1(t)|, \sqrt{M} \rangle^2 + \langle H_1(t) \rangle^2 \geq C (1 + t)^{-3/2} - C (1 + t)^{-5/2} - C (1 + t)^{-9/4}, \\
|J_1(t) \rangle^2 \geq C (1 + t)^{-3/4} - C (1 + t)^{-7/4} - C (1 + t)^{-5/2}.
\]

This together with (5.114) and (5.115) lead to (2.79) and (2.80). The proof is then completed. \( \square \)

### 6 The nonlinear system

In this section, we prove the large time decay rates of the solution to the Cauchy problem for Vlasov-Maxwell-Boltzmann systems with the estimates on the linearized problem obtained in Section 5.

#### 6.1 Energy estimates for two species

We first obtain some energy estimates. Let \( N \) be a positive integer and \( U = (f_1, f_2, E, B) \), and

\[
E_{N,k}(U) = \sum_{|\alpha| + |\beta| \leq N} \| w^k \partial_x^\alpha \partial_v^\beta (f_1, f_2) \|_{L^2_{t,x,v}}^2 + \sum_{|\alpha| \leq N} \| \partial_x^\alpha E(B) \|_{L^2_{t,x,v}}^2,
\]

\[
H_{N,k}(U) = \sum_{|\alpha| + |\beta| \leq N} \| w^k \partial_x^\alpha \partial_v^\beta (P_1 f_1, P_1 f_2) \|_{L^2_{t,x,v}}^2 + \sum_{1 \leq |\alpha| \leq N} \| \partial_x^\alpha E(B) \|_{L^2_{t,x,v}}^2 + \| E \|_{L^2_{t,x,v}}^2 \\
+ \sum_{|\alpha| \leq N-1} \| \partial_x^\alpha \nabla_x (P_0 f_1, P_0 f_2) \|_{L^2_{t,x,v}}^2 + \| P_0 f_2 \|_{L^2_{t,x,v}}^2, \\
D_{N,k}(U) = \sum_{|\alpha| + |\beta| \leq N} \| w^{3+k} \partial_x^\alpha \partial_v^\beta (P_1 f_1, P_1 f_2) \|_{L^2_{t,x,v}}^2 + \sum_{1 \leq |\alpha| \leq N-1} \| \partial_x^\alpha B \|_{L^2_{t,x,v}}^2 \\
+ \sum_{|\alpha| \leq N-1} \| \partial_x^\alpha \nabla_x (P_0 f_1, P_0 f_2) \|_{L^2_{t,x,v}}^2 + \| \partial_x^\alpha E \|_{L^2_{t,x,v}}^2 + \| P_0 f_2 \|_{L^2_{t,x,v}}^2.
\]
for \( k \geq 0 \). For brevity, we write \( E_N(U) = E_{N,0}(U) \), \( H_N(U) = H_{N,0}(U) \) and \( D_N(U) = D_{N,0}(U) \) for \( k = 0 \).

Firstly, by taking the inner product between \( \chi_j \) \((j = 0, 1, 2, 3, 4)\) and \( (2.6)\), we obtain a compressible Euler-Maxwell type system

\[
\partial_t n_1 + \text{div}_x m_1 = 0, \tag{6.4}
\]

\[
\partial_t m_1 + \nabla_x m_1 + \sqrt{\frac{2}{3}} \nabla_x q_1 = n_2 E + m_2 \times B - \int_{\mathbb{R}^3} v \cdot \nabla_x (P_1 f_1) v \sqrt{M} dv, \tag{6.5}
\]

\[
\partial_t q_1 + \sqrt{\frac{2}{3}} \text{div}_x m_1 = \frac{2}{3} E \cdot m_2 - \int_{\mathbb{R}^3} v \cdot \nabla_x (P_1 f_1) \chi_4 dv, \tag{6.6}
\]

where

\[(n_1, m_1, q_1) = ((f_1, \sqrt{M}), (f_1, v \sqrt{M}), (f_1, \chi_4)), (n_2, m_2) = ((f_2, \sqrt{M}), (f_2, v \sqrt{M})).\]

Taking the microscopic projection \( P_r \) on \( (2.6) \), we have

\[
\partial_t (P_r f_1) + P_r (v \cdot \nabla_x P_1 f_1) - L(P_r f_1) = -P_r (v \cdot \nabla_x P_0 f_1) + P_r G_1, \tag{6.7}
\]

where the nonlinear term \( G_1 \) is denoted by

\[
G_1 = \frac{1}{2} (v \cdot E) f_2 - (E + v \times B) \cdot \nabla_v f_2 + \Gamma(f_1, f_1). \tag{6.8}
\]

By \( (6.7) \), we can express the microscopic part \( P_r f_1 \) as

\[
P_r f_1 = L^{-1}[\partial_t (P_r f_1) + P_r (v \cdot \nabla_x P_1 f_1) - P_r G_1] + L^{-1} P_r (v \cdot \nabla_x P_0 f_1). \tag{6.9}
\]

Substituting \( (6.9) \) into \( (6.4) - (6.6) \), we obtain a compressible Navier-Stokes-Maxwell type system

\[
\partial_t n_1 + \text{div}_x m_1 = 0, \tag{6.10}
\]

\[
\partial_t m_1 + \partial_t R_1 + \nabla_x m_1 + \sqrt{\frac{2}{3}} \nabla_x q_1 = \kappa_1 (\Delta_x m_1 + \frac{1}{3} \nabla_x \text{div}_x m_1) + n_2 E + m_2 \times B + R_2, \tag{6.11}
\]

\[
\partial_t q_1 + \partial_t R_3 + \sqrt{\frac{2}{3}} \text{div}_x m_1 = \kappa_2 \Delta_x q_1 + \sqrt{\frac{2}{3}} E \cdot m_2 + R_4, \tag{6.12}
\]

where the viscosity and heat conductivity coefficients \( \kappa_1, \kappa_2 > 0 \) and the remainder terms \( R_1, R_2, R_3, R_4 \) are given by

\[
\kappa_1 = -(L^{-1} P_1 (v_1 \chi_2), v_1 \chi_2), \quad \kappa_2 = -(L^{-1} P_1 (v_1 \chi_4), v_1 \chi_4),
\]

\[
R_1 = (v \cdot \nabla_x L^{-1} P_1 f_1, v \sqrt{M}), \quad R_2 = -(v \cdot \nabla_x L^{-1} (P_1 (v \cdot \nabla_x P_1 f_1) - P_1 G_1), v \sqrt{M}),
\]

\[
R_3 = (v \cdot \nabla_x L^{-1} P_1 f_1, \chi_4), \quad R_4 = -(v \cdot \nabla_x L^{-1} (P_1 (v \cdot \nabla_x P_1 f_1) - P_1 G_1), \chi_4).
\]

By taking the inner product between \( \sqrt{M} \) and \( (2.7) \), we obtain

\[
\partial_t n_2 + \text{div}_x m_2 = 0. \tag{6.13}
\]

Taking the microscopic projection \( P_r \) on \( (2.7) \), we have

\[
\partial_t (P_r f_2) + P_r (v \cdot \nabla_x P_r f_2) - v \sqrt{M} \cdot E - L_1 (P_r f_2) = -P_r (v \cdot \nabla_x P_0 f_2) + P_r G_2, \tag{6.14}
\]

where the nonlinear term \( G_2 \) is denoted by

\[
G_2 = \frac{1}{2} (v \cdot E) f_1 - (E + v \times B) \cdot \nabla_v f_1 + \Gamma(f_2, f_1). \tag{6.15}
\]

By \( (6.14) \), we can express the microscopic part \( P_r f_2 \) as

\[
P_r f_2 = L_1^{-1}[\partial_t (P_r f_2) + P_r (v \cdot \nabla_x P_r f_2) - P_r G_2] + L_1^{-1} P_r (v \cdot \nabla_x P_0 f_2) - L_1^{-1} (v \sqrt{M} \cdot E). \tag{6.16}
\]
Substituting (6.16) into (6.13) and (2.8), we obtain
\[
\begin{align*}
\partial_t n_2 + \partial_t \text{div}_x R_5 &= -\kappa_3 n_2 + \kappa_3 \Delta_x n_2 - \text{div}_x R_6, \\
\partial_t E + \partial_t R_5 &= \nabla_x \times B + \kappa_3 \nabla_x n_2 - \kappa_3 E + R_6, \\
\partial_t B &= -\nabla_x \times E,
\end{align*}
\]
where the viscosity coefficient \(\kappa_3 > 0\) and the remainder terms \(R_5, R_6\) are defined by
\[
\kappa_3 = -(L^{-1}_1 \chi_1, \chi_1), \quad R_5 = (L^{-1}_1 P_f f_2, v \sqrt{M}), \\
R_6 = (L^{-1}_1 (P_f (v \cdot \nabla_x P_f f_2) - P_f G_2), v \sqrt{M}).
\]
The following lemma is from [3][9].

**Lemma 6.1** [3][9]. It holds that
\[
\|v^k \partial_x^\alpha \Gamma(f, g)\|_{L^2_v} \leq C \sum_{\beta_1 + \beta_2 \leq \beta} (\|\partial^\beta_v f\|_{L^2_v} \|v^{k+1} \partial^\beta_x g\|_{L^2_v} + \|v^{k+1} \partial^\beta_x f\|_{L^2_v} \|\partial^\beta_x g\|_{L^2_v}),
\]
for \(k \geq -1\), and
\[
\|\Gamma(f, g)\|_{L^{2,1}} \leq C(\|f\|_{L^2_{xx,v}} \|v g\|_{L^2_{xx,v}} + \|v f\|_{L^2_{xx,v}} \|g\|_{L^2_{xx,v}}).
\]

**Lemma 6.2** (Macroscopic dissipation). Let \((n_1, m_1, q_1)\) and \((n_2, E, B)\) be the strong solutions to (6.10)-(6.12) and (6.14)–(6.19) respectively. Then, there are two constants \(s_0, s_1 > 0\) such that
\[
\begin{align*}
\frac{d}{dt} \sum_{k \leq |\alpha| \leq N-1} s_0(\|\partial_x^\alpha (n_1, m_1, q_1)\|_{L^2_v}^2 + 2 \int_{\mathbb{R}^3} \partial_x^\alpha R_1 \partial_x^\alpha m_1 dx + 2 \int_{\mathbb{R}^3} \partial_x^\alpha R_3 \partial_x^\alpha q_1 dx) \\
+ \frac{d}{dt} \sum_{k \leq |\alpha| \leq N-1} 4 \int_{\mathbb{R}^3} \partial_x^\alpha m_1 \partial_x^\alpha \nabla_x n_1 dx + \sum_{k \leq |\alpha| \leq N-1} \|\partial_x^\alpha \nabla_x (n_1, m_1, q_1)\|_{L^2_v}^2 \\
\leq CE_N(U) D_N(U) + C \sum_{k \leq |\alpha| \leq N-1} \|\partial_x^\alpha \nabla_x P_1 f_1\|_{L^2_{xx,v}}^2,
\end{align*}
\]
\[
\begin{align*}
\frac{d}{dt} \sum_{k \leq |\alpha| \leq N-1} s_1(\|\partial_x^\alpha (n_2, E, B)\|_{L^2_v}^2 + 2 \int_{\mathbb{R}^3} \partial_x^\alpha \text{div}_x R_5 \partial_x^\alpha n_2 dx + 2 \int_{\mathbb{R}^3} \partial_x^\alpha R_5 \partial_x^\alpha E dx) \\
- \frac{d}{dt} \sum_{k \leq |\alpha| \leq N-2} 4 \int_{\mathbb{R}^3} \partial_x^\alpha E \partial_x^\alpha (\nabla_x \times B) dx \\
+ \sum_{k \leq |\alpha| \leq N-1} (\|\partial_x^\alpha n_2\|_{L^2_v}^2 + \|\partial_x^\alpha \nabla_x n_2\|_{L^2_v}^2 + \|\partial_x^\alpha E\|_{L^2_v}^2) + \sum_{k+1 \leq |\alpha| \leq N-1} \|\partial_x^\alpha B\|_{L^2_v}^2 \\
\leq CE_N(U) D_N(U) + C \sum_{k \leq |\alpha| \leq N} \|\partial_x^\alpha P_f f_2\|_{L^2_{xx,v}}^2,
\end{align*}
\]
with \(0 \leq k \leq N-2\).

**Proof.** First of all, we prove (6.20). Taking the inner product between \(\partial_x^\alpha m_1\) and \(\partial_x^\beta (6.11)\) with \(|\alpha| \leq N-1\), we have
\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \|\partial_x^\alpha m_1\|_{L^2_v}^2 &+ \int_{\mathbb{R}^3} \partial_x^\alpha \partial_t R_1 \partial_x^\alpha m_1 dx + \int_{\mathbb{R}^3} \partial_x^\alpha \nabla_x n_1 \partial_x^\alpha m_1 dx + \frac{2}{3} \int_{\mathbb{R}^3} \partial_x^\alpha \nabla_x q_1 \partial_x^\alpha m_1 dx \\
+ \kappa_1 (\|\partial_x^\alpha \nabla_x m_1\|_{L^2_v}^2 + \frac{1}{3} \|\partial_x^\alpha \text{div}_x m_1\|_{L^2_v}^2) \\
= \int_{\mathbb{R}^3} \partial_x^\alpha (n_2 E) \partial_x^\alpha m_1 dx + \int_{\mathbb{R}^3} \partial_x^\alpha (m_2 \times B) \partial_x^\alpha m_1 dx + \int_{\mathbb{R}^3} \partial_x^\alpha R_1 \partial_x^\alpha m_1 dx.
\end{align*}
\]

For the second and third terms on the left hand side of (6.22), we have
\[
\int_{\mathbb{R}^3} \partial_x^\alpha \partial_t R_1 \partial_x^\alpha m_1 dx = \frac{d}{dt} \int_{\mathbb{R}^3} \partial_x^\alpha R_1 \partial_x^\alpha m_1 dx
\]
Therefore, it follows from (6.22)–(6.25) that

\[ - \int_{\mathbb{R}^3} \partial_x^\alpha R_1 \partial_x^\alpha [\nabla_x n_1 - \sqrt{\frac{3}{\alpha}} \nabla_x q_1 + n_2 E + m_2 \times B - (v \cdot \nabla_x P_1 f_1, v \sqrt{M})] dx \]

\[ \geq \frac{d}{dt} \int_{\mathbb{R}^3} \partial_x^\alpha R_1 \partial_x^\alpha m_1 dx - \epsilon (\|\partial_x^\alpha \nabla_x n_1\|_{L^2}^2 + \|\partial_x^\alpha \nabla_x q_1\|_{L^2}^2) \]

\[ - C \sqrt{E_N(U)} D_N(U) - \frac{C}{\epsilon} \|\partial_x^\alpha \nabla_x P_1 P_1\|_{L^2}^2, \quad (6.23) \]

and

\[ \int_{\mathbb{R}^3} \partial_x^\alpha \nabla_x n_1 \partial_x^\alpha m_1 dx = - \int_{\mathbb{R}^3} \partial_x^\alpha n \partial_x^\alpha \text{div}_x m dx = \int_{\mathbb{R}^3} \partial_x^\alpha n_1 \partial_x^\alpha m_1 dx = \frac{d}{dt} \|\partial_x^\alpha n_1\|_{L^2}^2, \quad (6.24) \]

The first and second terms on the right hand side of (6.22) are bounded by \( C \sqrt{E_N(U)} D_N(U) \). The last term can be estimated by

\[ \int_{\mathbb{R}^3} \partial_x^\alpha R_2 \partial_x^\alpha m_1 dx \leq C \|\partial_x^\alpha \nabla_x P_1 f_1\|_{L^2}^2 + C(\|\partial_x^\alpha (E f_2)\|_{L^2}^2 + \|\partial_x^\alpha (B f_2)\|_{L^2}^2) 
\]

\[ + \|\partial_x^\alpha (B f_2)\|_{L^2}^2 + \|
\]

\[ \|\partial_x^\alpha \nabla_x m_1\|_{L^2}^2 + \frac{\kappa_1}{2} \|\partial_x^\alpha \nabla_x m_1\|_{L^2}^2 + C E_N(U) D_N(U), \quad (6.25) \]

where we have used Lemma 6.1 to obtain

\[ \|\partial_x^\alpha \nabla_x (f_1, f_2)\|_{L^2}^2 + \|\partial_x^\alpha (E f_2)\|_{L^2}^2 + \|\partial_x^\alpha (B f_2)\|_{L^2}^2 
\]

\[ \leq C \|f_1\|_{L^2}^2 \|\partial_x^\alpha \nabla_x f_1\|_{L^2}^2 + C \|E\|_{H^N}^2 + B \|\|B\|_{H^N}^2 \|\nabla_x f_2\|_{L^2}^2 
\]

\[ \leq C E_N(U) D_N(U). \quad (6.26) \]

Therefore, it follows from (6.22)–(6.25) that

\[ \frac{1}{2} \frac{d}{dt} \|\partial_x^\alpha m_1\|_{L^2}^2 + \|\partial_x^\alpha n_1\|_{L^2}^2 \] + \[ \frac{d}{dt} \int_{\mathbb{R}^3} \partial_x^\alpha R_1 \partial_x^\alpha m_1 dx + \frac{\sqrt{2}}{3} \int_{\mathbb{R}^3} \partial_x^\alpha \nabla_x q_1 \partial_x^\alpha m_1 dx 
\]

\[ + \frac{\kappa_2}{2} (\|\partial_x^\alpha \nabla_x m_1\|_{L^2}^2 + \frac{1}{3} \|\partial_x^\alpha \text{div}_x m_1\|_{L^2}^2) \]

\[ \leq C \sqrt{E_N(U)} D_N(U) + C \frac{1}{\epsilon} \|\partial_x^\alpha \nabla_x P_1 f_1\|_{L^2}^2 + \epsilon (\|\partial_x^\alpha \nabla_x n_1\|_{L^2}^2 + \|\partial_x^\alpha \nabla_x q_1\|_{L^2}^2). \quad (6.27) \]

Similarly, taking the inner product between \( \partial_x^\alpha q_1 \) and \( \partial_x^\alpha (6.12) \) with \( |\alpha| \leq N - 1 \), we have

\[ \frac{1}{2} \frac{d}{dt} \|\partial_x^\alpha q_1\|_{L^2}^2 \]

\[ + \frac{d}{dt} \int_{\mathbb{R}^3} \partial_x^\alpha R_3 \partial_x^\alpha q_1 dx + \frac{\sqrt{2}}{3} \int_{\mathbb{R}^3} \partial_x^\alpha \text{div}_x m_1 \partial_x^\alpha q_1 dx + \frac{\kappa_2}{2} \|\partial_x^\alpha \nabla_x q_1\|_{L^2}^2 \]

\[ \leq C \sqrt{E_N(U)} D_N(U) + \frac{C}{\epsilon} \|\partial_x^\alpha \nabla_x P_1 f_1\|_{L^2}^2 + \|\partial_x^\alpha \nabla_x m_1\|_{L^2}^2, \quad (6.28) \]

Again, taking the inner product between \( \partial_x^\alpha \nabla_x n_1 \) and \( \partial_x^\alpha (6.23) \) with \( |\alpha| \leq N - 1 \) gives

\[ \frac{d}{dt} \int_{\mathbb{R}^3} \partial_x^\alpha m_1 \partial_x^\alpha \nabla_x m_1 dx + \frac{\kappa_2}{2} \|\partial_x^\alpha \nabla_x m_1\|_{L^2}^2 \]

\[ \leq C \sqrt{E_N(U)} D_N(U) + \|\partial_x^\alpha \text{div}_x m_1\|_{L^2}^2 + \|\partial_x^\alpha \nabla_x q_1\|_{L^2}^2 + C \|\partial_x^\alpha \nabla_x P_1 f_1\|_{L^2}^2, \quad (6.29) \]

Taking the summation of 2s0 \( \sum_{k < |\alpha| < N - 1} (6.27) + (6.28) \) + 4 \( \sum_{k \leq |\alpha| < N - 1} (6.29) \) with \( s_0 > 0 \) large enough, \( \epsilon > 0 \) small enough and \( 0 \leq k \leq N - 1 \), we obtain (6.20).

Next, we turn to show (6.21). Taking the inner product between \( \partial_x^\alpha n_2 \) and \( \partial_x^\alpha (6.17) \) with \( |\alpha| \leq N - 1 \), we have

\[ \frac{d}{dt} \|\partial_x^\alpha n_2\|_{L^2}^2 + \frac{d}{dt} \int_{\mathbb{R}^3} \partial_x^\alpha \nabla_x R_5 \partial_x^\alpha n_2 dx + \kappa_3 \|\partial_x^\alpha n_2\|_{L^2}^2 + \kappa_3 \|\partial_x^\alpha \nabla_x n_2\|_{L^2}^2 \]
\[
\leq C \|\partial_x^\alpha \nabla_x P_f f_2\|_{L^2_{x,v}}^2 + C E_N(U) D_N(U). \tag{6.30}
\]

Similarly, taking the inner product between \(\partial_x^\alpha E\) and \(\partial_x^\alpha\) with \(|\alpha| \leq N - 1\) gives
\[
\frac{d}{dt} \|\partial_x^\alpha (E, B)\|_{L^2_{x,v}}^2 + 2 \frac{d}{dt} \int_{\mathbb{R}^3} \partial_x^\alpha R_0 \partial_x^\alpha E dx + \kappa_3 \|\partial_x^\alpha E\|_{L^2_{x,v}}^2 + \kappa_4 \|\partial_x^\alpha n_2\|_{L^2_{x,v}}^2 \\
\leq \frac{C}{\epsilon} (\|\partial_x^\alpha P_f f_2\|_{L^2_{x,v}}^2 + \|\partial_x^\alpha \nabla_x P_f f_2\|_{L^2_{x,v}}^2) + \epsilon \|\partial_x^\alpha (\nabla_x \times B)\|_{L^2_{x,v}}^2 + C E_N(U) D_N(U). \tag{6.31}
\]

And taking the inner product between \(\partial_x^\alpha \nabla_x \times B\) and \(\partial_x^\alpha\) with \(|\alpha| \leq N - 1\) gives
\[
- \frac{d}{dt} \int_{\mathbb{R}^3} \partial_x^\alpha E \partial_x^\alpha (\nabla_x \times B) dx - \|\partial_x^\alpha (\nabla_x \times E)\|_{L^2_{x,v}}^2 + \|\partial_x^\alpha (\nabla_x \times B)\|_{L^2_{x,v}}^2 = \int_{\mathbb{R}^3} \partial_x^\alpha m_2 \partial_x^\alpha (\nabla_x \times B) dx.
\]

This and the fact that \(\|\nabla_x \times E\|_{L^2_{x,v}}^{2} = \|\nabla_x B\|_{L^2_{x,v}}^{2}\) imply that
\[
- \frac{2}{\epsilon} \frac{d}{dt} \int_{\mathbb{R}^3} \partial_x^\alpha E \partial_x^\alpha (\nabla_x \times B) dx + \|\partial_x^\alpha \nabla_x B\|_{L^2_{x,v}}^2 \leq C \|\partial_x^\alpha P_f f_2\|_{L^2_{x,v}}^2 + 2 \|\partial_x^\alpha (\nabla_x \times E)\|_{L^2_{x,v}}^2. \tag{6.32}
\]

Taking the summation of \(s_1 \leq k \leq N - 1\) with \(s_1 > 0\) large enough, \(\epsilon > 0\) small enough and \(0 \leq k \leq N - 1\), we obtain (6.21). And this completes the proof of the lemma. \(\square\)

In the following, we shall estimate the microscopic terms to close the energy estimate.

**Lemma 6.3 (Microscopic dissipation).** Let \(N \geq 4\) and \((f_1, f_2, E, B)\) be a strong solution to Vlasov-Maxwell-Boltzmann system (2.6)–(2.11). Then, there are constants \(p_k > 0\), \(1 \leq k \leq N\) such that
\[
\frac{1}{2} \frac{d}{dt} \left( \sum_{1 \leq |\alpha| \leq N} \|\partial_x^\alpha (f_1, f_2)\|_{L^2_{x,v}}^2 + \|E, B\|_{L^2_{x,v}}^2 + \mu \|w^2 (P_1 f_1, P_2 f_2)\|_{L^2_{x,v}}^2 \right) \leq C \sqrt{E_N(U)} D_N(U), \tag{6.33}
\]

\[
\frac{1}{2} \frac{d}{dt} \left( \sum_{1 \leq |\alpha| \leq N} \|\partial_x^\alpha (f_1, f_2)\|_{L^2_{x,v}}^2 + \|\partial_x^\alpha (E, B)\|_{L^2_{x,v}}^2 + \mu \sum_{1 \leq |\alpha| \leq N} \|w^2 \partial_x^\alpha (P_1 f_1, P_2 f_2)\|_{L^2_{x,v}}^2 \right) \leq C \sqrt{E_N(U)} D_N(U), \tag{6.34}
\]

\[
\frac{d}{dt} \left( \sum_{1 \leq k \leq N} p_k \sum_{|\alpha| = k} \|\partial_x^\alpha \nabla_x (f_1, f_2)\|_{L^2_{x,v}}^2 + \|\partial_x^\alpha E\|_{L^2_{x,v}}^2 \right) + C \sqrt{E_N(U)} D_N(U). \tag{6.35}
\]

**Proof.** Taking the inner product between \(\partial_x^\alpha f_1\) and \(\partial_x^\alpha\) with \(|\alpha| \leq N\) \((N \geq 4)\), we have
\[
\frac{1}{2} \frac{d}{dt} \|\partial_x^\alpha f_1\|_{L^2_{x,v}}^2 = \int_{\mathbb{R}^3} (L \partial_x^\alpha f_1) \partial_x^\alpha f_1 dx dv \\
= \frac{1}{2} \int_{\mathbb{R}^6} \partial_x^\alpha (v \cdot E f_2) \partial_x^\alpha f_1 dx dv - \int_{\mathbb{R}^6} \partial_x^\alpha ((E + v \times B) \cdot \nabla_x f_2) \partial_x^\alpha f_1 dx dv + \int_{\mathbb{R}^6} \partial_x^\alpha \Sigma (f_1, f_1) \partial_x^\alpha f_1 dx dv \\
=: I_1 + I_2 + I_3. \tag{6.36}
\]

For \(I_1\), it holds that
\[
I_1 \leq C \sum_{1 \leq |\alpha| \leq |\alpha| - 1} \int_{\mathbb{R}^3} |v| \|\partial_x^\alpha E\|_{L^2_{x,v}} \|\partial_x^{\alpha - \alpha'} f_2\|_{L^2_{x,v}} \|\partial_x^\alpha f_1\|_{L^2_{x,v}} dx dv \\
+ C \int_{\mathbb{R}^3} |v| \|E\|_{L^2_{x,v}} \|\partial_x^\alpha f_2\|_{L^2_{x,v}} + \|\partial_x^\alpha E\|_{L^2_{x,v}} \|f_2\|_{L^2_{x,v}} \|\partial_x^\alpha f_1\|_{L^2_{x,v}} dx dv \\
\leq C \|E\|_{H^N_{x,v}} \|w^2 \nabla_x f_2\|_{L^2_{x,v}} \|w^2 \partial_x^\alpha f_1\|_{L^2_{x,v}} \leq C \sqrt{E_N(U)} D_N(U), \tag{6.37}
\]

for \(|\alpha| \geq 1\), and
\[
I_1 \leq \int_{\mathbb{R}^3} |v| \|E\|_{L^2_{x,v}} \|f_2\|_{L^2_{x,v}} \|f_1\|_{L^2_{x,v}} dx dv \leq C \sqrt{E_N(U)} D_N(U), \tag{6.38}
\]
for $|\alpha| = 0$. For $I_2$, it holds that

$$I_2 \leq C \sum_{1 \leq |\alpha| \leq |N|/2} \int_{\mathbb{R}^3} (||\partial_x^\alpha E||_{L^\infty} + |v|||\partial_x^\alpha B||_{L^\infty})||\partial_x^{-\alpha} \nabla_v f_2||_{L^2} ||\partial_x^\alpha f_1||_{L^2} dv$$

$$+ C \sum_{|\alpha| \geq |N|/2} \int_{\mathbb{R}^3} (||\partial_x^\alpha E||_{L^2} + |v|||\partial_x^\alpha B||_{L^2})||\partial_x^{-\alpha} \nabla_v f_2||_{L^2} ||\partial_x^\alpha f_1||_{L^2} dv$$

$$- \int_{\mathbb{R}^6} (E + v \times B) \partial_{x}^\alpha \nabla_v f_2 \partial_x^\alpha f_1 dx dv \leq C \sqrt{E_N(U)} D_N(U) - \int_{\mathbb{R}^6} (E + v \times B) \partial_{x}^\alpha \nabla_v f_2 \partial_x^\alpha f_1 dx dv. \quad (6.39)$$

For $I_3$, by Lemma 4.1, we obtain

$$I_3 \leq ||w^{-\frac{1}{2}} \partial_x^\alpha \Gamma(f_1, f_1)||_{L^2} ||w^{\frac{1}{2}} \partial_x^\alpha P_1 f_1||_{L^2} \leq C \sqrt{E_N(U)} D_N(U). \quad (6.40)$$

Therefore, it follows from (6.36)–(6.40) that

$$\frac{1}{2} \frac{d}{dt} ||\partial_x^\alpha f_1||_{L^2}^2 + \mu ||w^{\frac{1}{2}} \partial_x^\alpha P_1 f_1||_{L^2}^2 \leq C \sqrt{E_N(U)} D_N(U) - \int_{\mathbb{R}^6} (E + v \times B) \partial_{x}^\alpha \nabla_v f_2 \partial_x^\alpha f_1 dx dv. \quad (6.41)$$

Similarly, taking inner product between $\partial_x^\alpha f_2$ and $\partial_x^\alpha$ with $|\alpha| \leq N$ $(N \geq 4)$, we have

$$\frac{1}{2} \frac{d}{dt} ||\partial_x^\alpha f_2||_{L^2}^2 + ||\partial_x^\alpha (E, B)||_{L^2}^2 + \mu ||w^{\frac{1}{2}} \partial_x^\alpha P_1 f_2||_{L^2}^2 \leq C \sqrt{E_N(U)} D_N(U) - \int_{\mathbb{R}^6} (E + v \times B) \partial_{x}^\alpha \nabla_v f_1 \partial_x^\alpha f_2 dx dv. \quad (6.42)$$

Taking the summation of (6.41) and (6.42), for $|\alpha| = 0$ and $\sum_{1 \leq |\alpha| \leq N} [6.41] + [6.42]$, we obtain (6.33) respectively.

In order to close the energy estimate, we need to estimate the terms $\partial_x^\alpha \nabla_v f$ with $|\alpha| \leq N - 1$. For this, we rewrite (2.6) as

$$\partial_t (P_1 f_1) + v \cdot \nabla P_1 f_1 + (E + v \times B) \cdot \nabla_v P_1 f_2 - L(P_1 f_1)$$

$$= \Gamma(f_1, f_1) + \frac{1}{2} v \cdot E P_1 f_2 + P_0 v \cdot \nabla_v P_1 f_1 - \frac{1}{2} v \cdot E P_1 f_2 + (E + v \times B) \cdot \nabla_v P_1 f_2 - P_1 (v \cdot \nabla_v P_0 f_1), \quad (6.43)$$

and

$$\partial_t (P_2 f_2) + v \cdot \nabla P_2 f_2 - v \sqrt{M} \cdot E + (E + v \times B) \cdot \nabla_v P_1 f_1 + L_1 (P_2 f_2)$$

$$= \Gamma(f_2, f_1) + \frac{1}{2} v \cdot E P_1 f_1 + P_0 v \cdot \nabla_v P_2 f_2 - (v \cdot \nabla_v P_0 f_2 - \frac{1}{2} v \cdot E P_1 f_1 + (E + v \times B) \cdot \nabla_v P_0 f_1). \quad (6.44)$$

Let $1 \leq k \leq N$, and choose $\alpha, \beta$ with $|\beta| = k$ and $|\alpha| + |\beta| \leq N$. Taking inner product between $\partial_x^\alpha \partial_x^\beta P_1 f_1$ and (6.33), between $\partial_x^\alpha \partial_x^\beta P_2 f_2$ and (6.44) respectively, and then taking summation of the resulted equations, we have

$$\frac{d}{dt} \sum_{|\alpha| + |\beta| \leq |N|} ||\partial_x^\alpha \partial_x^\beta (P_1 f_1, P_2 f_2)||_{L^2}^2 + \mu \sum_{|\alpha| + |\beta| \leq |N|} ||w^{\frac{1}{2}} \partial_x^\alpha \partial_x^\beta (P_1 f_1, P_2 f_2)||_{L^2}^2$$

$$\leq C \sum_{|\alpha| \leq |N| - k} ||\partial_x^\alpha \nabla_v (P_0 f_0, P_4 f_2)||_{L^2}^2 + ||\partial_x^\alpha \nabla_v (P_1 f_1, P_2 f_2)||_{L^2}^2 + ||\partial_x^\alpha E||_{L^2}^2$$

$$+ C_k \sum_{|\alpha| + |\beta| \leq |N| - k} ||\partial_x^\alpha \partial_x^\beta (P_1 f_1, P_2 f_2)||_{L^2}^2 + C \sqrt{E_N(U)} D_N(U). \quad (6.45)$$

Then taking summation $\sum_{1 \leq k \leq N} P_k$ with constants $p_k$ chosen by

$$\mu p_k \geq 2 \sum_{1 \leq j \leq N - k} p_{k+j} C_{k+j}, \quad 1 \leq k \leq N - 1, \quad p_N = 1,$$

we obtain (6.35). The proof of the lemma is then completed.
With the help of Lemma 6.4, we have

**Lemma 6.4.** For $N \geq 4$, there are two equivalent energy functionals $E_N^f(\cdot) \sim E_N(\cdot)$, $H_N^f(\cdot) \sim H_N(\cdot)$ such that the following holds. If $E_N(U_0)$ is sufficiently small, then the Cauchy problem (2.6)–(2.11) of the two-species Vlasov-Maxwell-Boltzmann system admits a unique global solution $U = (f_1, f_2, E, B)$ satisfying

\[
\frac{d}{dt} E_N^f(U)(t) + \mu D_N(U)(t) \leq 0, \quad (6.46)
\]

\[
\frac{d}{dt} H_N^f(U)(t) + \mu D_N(U)(t) \leq C \|\nabla_x(n_1, m_1, q_1)\|^2_{L^2} + C \|\nabla_x B\|^2_{L^2}. \quad (6.47)
\]

**Proof.** Assume that

\[
E_N(U)(t) \leq \delta
\]

for $\delta > 0$ being small.

Taking the summation of $A_1[6.24] + [6.24]$ and $A_2[6.31] + [6.31]$ with $A_2 > C_0 A_1 > 0$ large enough and taking $k = 0$ in (6.20) and (6.21), we obtain (6.46).

Taking the inner product between $E$ and (6.18), between $f_2$ and (2.27) respectively, we have

\[
\frac{1}{2} \frac{d}{dt} \|E\|^2_{L^2} + \frac{d}{dt} \int_{\mathbb{R}^3} R_5 E dx + \kappa_3 \|E\|^2_{L^2} + \kappa_3 \|n_2\|^2_{L^2} \\
\leq \epsilon \|E\|^2_{L^2} + \frac{C}{\epsilon} \|\nabla_x B\|^2_{L^2} + C(\|P_2 f_2\|^2_{L^2} + \|\nabla_x P_2 f_2\|^2_{L^2}) + C \sqrt{E_N(U)} D_N(U),
\]

and

\[
\frac{1}{2} \frac{d}{dt} (\|f_2\|^2_{L^2} + \|E\|^2_{L^2}) - \int_{\mathbb{R}^3} (L_1 f_2) f_2 dx dv \leq \epsilon \|E\|^2_{L^2} + \frac{C}{\epsilon} \|\nabla_x B\|^2_{L^2} + C \sqrt{E_N(U)} D_N(U). \quad (6.49)
\]

Taking the summation of $s_2[6.48] + [6.49]$ with $s_2 > 0$ large enough and $\epsilon > 0$ small enough, we have

\[
\frac{d}{dt} s_2 (\|f_2\|^2_{L^2} + \|E\|^2_{L^2}) + 2 \frac{d}{dt} \int_{\mathbb{R}^3} R_5 E dx + \kappa_3 \|E\|^2_{L^2} + \kappa_3 \|n_2\|^2_{L^2} + \mu \|w^2 P_2 f_2\|^2_{L^2} \\
\leq C \|\nabla_x B\|^2_{L^2} + C \|\nabla_x P_2 f_2\|^2_{L^2} + C \sqrt{E_N(U)} D_N(U).
\]

Taking the inner product between [6.43] and $P_1 f_1$, we have

\[
\frac{d}{dt} \|P_1 f_1\|^2_{L^2} + \|w^2 P_1 f_1\|^2_{L^2} \leq C \|\nabla_x P_0 f_1\|^2_{L^2} + E_N(U) D_N(U).
\]

Taking the summation of $A_3[6.20] + [6.21] + A_4[6.30] + [6.31] + A_5[6.34] + [6.35]$ with $A_5 > C_0 A_4, A_4 > C_1 A_4$ large enough and taking $k = 1$ in (6.20) and (6.21), we obtain (6.47). \hfill \square

Repeating the proofs of Lemmas 6.2–6.3 we can show

**Lemma 6.5.** For $N \geq 4$, there are the equivalent energy functionals $E_{N,1}^f(\cdot) \sim E_{N,1}(\cdot)$, $H_{N,1}^f(\cdot) \sim H_{N,1}(\cdot)$ such that if $E_{N,1}(U_0)$ is sufficiently small, then the solution $U = (f_1, f_2, E, B)(t, x, v)$ to the two-species Vlasov-Maxwell-Boltzmann system (2.6)–(2.11) satisfies

\[
\frac{d}{dt} E_{N,1}^f(U)(t) + \mu D_{N,1}(U)(t) \leq 0, \quad (6.52)
\]

\[
\frac{d}{dt} H_{N,1}^f(U)(t) + \mu D_{N,1}(U)(t) \leq C \|\nabla_x(n_1, m_1, q_1)\|^2_{L^2} + C \|\nabla_x B\|^2_{L^2}. \quad (6.53)
\]

### 6.2 Convergence rates for two-species

Based on the above energy estimates and the convergence rates of the solution to the linearized system, the convergence rates of the solution to the two-species VMB can be summarized in the following theorem.
**Theorem 6.6.** Under the assumptions of Theorem 2.4 there exists a globally unique solution \((f_1, f_2, E, B)\) to the system (2.6)–(2.11) satisfying

\[
\|\partial_t^j(f_1(t), x_j)\|_{L^2} \leq C\delta_0(1 + t)^{-\frac{j}{2} - \frac{1}{4}}, \quad j = 0, 1, 2, 3, 4, \tag{6.54}
\]
\[
\|\partial_t^j P_0 f_1(t)\|_{L^2} \leq C\delta_0(1 + t)^{-\frac{j}{2} - \frac{1}{4}}, \tag{6.55}
\]
\[
\|\partial_t^j (f_2(t), \sqrt{\lambda})\|_{L^2} \leq C\delta_0(1 + t)^{-1}, \tag{6.56}
\]
\[
\|\partial_t^j P_E f_2(t)\|_{L^2} + \|\partial_t^j E(t)\|_{L^2} \leq C\delta_0(1 + t)^{-\frac{j}{2} - \frac{1}{4}}, \tag{6.57}
\]
\[
\|\partial_t^j B(t)\|_{L^2} \leq C\delta_0(1 + t)^{-\frac{j}{4}}. \tag{6.58}
\]
\[
\|(P_1 f_1, P_2 f_2)(t)\|_{H^{\frac{3}{2}}_x} + \|\nabla_x (P_0 f_1, P_2 f_2)(t)\|_{L^2} \leq C\delta_0(1 + t)^{-\frac{3}{4}}, \tag{6.59}
\]
for \(k = 0, 1\).

**Proof.** Let \((f_1, f_2, E, B)\) be a solution to the Cauchy problem (2.6)–(2.11) for \(t > 0\). We can represent the solution in terms of the semigroups \(e^{t\mathfrak{B}_0}\) and \(e^{t\mathfrak{A}_0}\) as

\[
f_1(t) = e^{t\mathfrak{B}_0} f_1(0) + \int_0^t e^{(t-s)\mathfrak{B}_0} G_1(s) ds, \tag{6.60}
\]
\[
(f_2(t), E(t), B(t)) = e^{t\mathfrak{A}_0} (f_2(0), E(0), B(0)) + \int_0^t e^{(t-s)\mathfrak{A}_0} (G_2(s), 0, 0) ds, \tag{6.61}
\]
where the nonlinear terms \(G_1\) and \(G_2\) are given by (6.8) and (6.14) respectively. Define a functional \(Q(t)\) for any \(t > 0\) by

\[
Q(t) = \sup_{0 \leq s \leq t} \left\{ \sum_{j=0}^4 \|\partial_t^j (f_1(s), x_j)\|_{L^2} (1 + s)^{\frac{j}{2} + \frac{1}{4}} + \|\partial_t^j P_1 f_1(s)\|_{L^2} (1 + s)^{\frac{j}{2} + \frac{1}{4}} \right. \\
\left. + \|\partial_t^j P_2 f_2(s)\|_{L^2} (1 + s)^{\frac{j}{2} + \frac{1}{4}} + \|\partial_t^j E(s)\|_{L^2} (1 + s)^{\frac{j}{2} + \frac{1}{4}} \right. \\
\left. + \|\partial_t^j B(s)\|_{L^2} (1 + s)^{\frac{j}{2} + \frac{1}{4}} + \|\nabla_x (P_0 f_1, P_2 f_2)(s)\|_{H^{\frac{3}{2}}_x} \right. \\
\right. + \|\nabla_x (E(s))(s)\|_{H^{\frac{3}{2}}_x}) (1 + s)^{\frac{j}{2}} \right\}. \tag{6.62}
\]

We claim that it holds under the assumptions of Theorem 6.6 that

\[
Q(t) \leq C\delta_0. \tag{6.63}
\]

It is straightforward to verify that the estimates (6.54)–(6.59) follow from (6.62). Hence, it remains to prove (6.62).

By Lemma 6.1 we can estimate the nonlinear term \(G_1(s), G_2(s)\) for \(0 \leq s \leq t\) in terms of \(Q(t)\) as

\[
\|G_1(s)\|_{L^2} \leq C \|w f_1\|_{L^{2,3}} \|f_1\|_{L^{2,6}} + ||E||_{L^{2}} \|w f_2\|_{L^{2,6}} + \|\nabla_v f_2\|_{L^{2,6}} = \leq C(1 + s)^{-\frac{1}{2}} Q(t)^2, \tag{6.64}
\]
\[
\|G_2(s)\|_{L^2} \leq C \|w f_2\|_{L^{2,3}} \|f_1\|_{L^{2,6}} + ||E||_{L^{2}} \|w f_2\|_{L^{2,6}} + \|\nabla_v f_1\|_{L^{2,6}} + \|B||_{L^{2}} \|w f_1\|_{L^{2,6}} \leq C(1 + s)^{-1} Q(t)^2, \tag{6.65}
\]
\[
\|G_2(s)\|_{L^2} \leq C \|w f_2\|_{L^{2,3}} \|f_1\|_{L^{2,6}} + ||E||_{L^{2}} \|w f_2\|_{L^{2,6}} + \|\nabla_v f_1\|_{L^{2,6}} + \|B||_{L^{2}} \|w f_1\|_{L^{2,6}} \leq C(1 + s)^{-1} Q(t)^2, \tag{6.66}
\]
and similarly
\[
\|G_2\|_{L^2_t(H^s_x)} \leq C(1+s)^{-\frac{d}{4}}Q(t)^2, \quad (6.67)
\]
for \(1 \leq k \leq N-1\). First, we consider the time decay rate of the macroscopic density, momentum and energy of \(f_1\). It follows from (5.78), (6.63) and (6.64) that
\[
\|\nabla_x f_1(t, \chi)\|_{L^2_t} \leq C(1+t)^{-\frac{d}{4}}(\|\nabla_x f_1(t, \chi)\|_{L^2_t} + \|f_1(t, \chi)\|_{L^2_t})
\]
\[
+ C \int_0^t (1+t-s)^{-\frac{d}{8}} \|\nabla_x f_1(s)\|_{L^2_x} + \|f_1(s)\|_{L^2_x})ds
\]
\[
\leq C\delta_0(1+t)^{-\frac{d}{4}} + C \int_0^t (1+t-s)^{-\frac{d}{8}} (1+s)^{-\frac{d}{8}} Q(t)^2 ds
\]
\[
\leq C\delta_0(1+t)^{-\frac{d}{4}} + C(1+t)^{-\frac{d}{4}} Q(t)^2, \quad k = 0, 1, \quad (6.68)
\]
where we have used
\[
\|\nabla_x f_1(t, \chi)\|_{L^2_t} + \|f_1(t, \chi)\|_{L^2_t} \leq C(1+s)^{-\frac{d}{4}} Q(t)^2, \quad k = 0, 1.
\]
Second, we estimate the microscopic part \(P_1 f_1(t)\) as follows. By (5.79), (6.63) and (6.64), we have
\[
\|P_1 f_1(t)\|_{L^2_t} \leq C(1+t)^{-\frac{d}{4}}(\|f_1(t, \chi)\|_{L^2_t} + \|f_1(t, \chi)\|_{L^2_t})
\]
\[
+ C \int_0^t (1+t-s)^{-\frac{d}{8}} \|f_1(s)\|_{L^2_x} + \|f_1(s)\|_{L^2_x})ds
\]
\[
\leq C\delta_0(1+t)^{-\frac{d}{4}} + C(1+t)^{-\frac{d}{4}} Q(t)^2, \quad (6.69)
\]
and
\[
\|\nabla_x P_1 f_1(t)\|_{L^2_t} \leq C(1+t)^{-\frac{d}{4}}(\|\nabla_x f_1(t, \chi)\|_{L^2_t} + \|f_1(t, \chi)\|_{L^2_t})
\]
\[
+ C \int_0^{t/2} (1+t-s)^{-\frac{d}{8}} \|\nabla_x G_1(s)\|_{L^2_x} + \|G_1(s)\|_{L^2_x})ds
\]
\[
+ C \int_{t/2}^t (1+t-s)^{-\frac{d}{8}} \|\nabla_x G_1(s)\|_{L^2_x} + \|\nabla_x G_1(s)\|_{L^2_x})ds
\]
\[
\leq C\delta_0(1+t)^{-\frac{d}{4}} + C(1+t)^{-\frac{d}{4}} Q(t)^2, \quad (6.70)
\]
where we have used
\[
\|\nabla_x G_1(s)\|_{L^2_x} + \|\nabla_x G_1(s)\|_{L^2_x} \leq C(1+s)^{-2} Q(t)^2.
\]
Next, we turn to consider the time decay rate of the \(f_2, E, B\). By (2.37), (2.34), (6.65), (6.66) and (6.67), we have
\[
\|P_1 f_2(t)\|_{L^2_t} + \|E(t)\|_{L^2_t} \leq C(1+t)^{-\frac{d}{4}}(\|U_0\|_{L^2} + \|U_0\|_{L^2} + \|\nabla_x U_0\|_{L^2})
\]
\[
+ C \int_0^t (1+t-s)^{-\frac{d}{8}} \|G_2(s)\|_{L^2_x} + \|G_2(s)\|_{L^2_x})ds
\]
\[
+ C \int_0^t (1+t-s)^{-2} \|\nabla_x G_2(s)\|_{L^2_x} ds
\]
\[
\leq C\delta_0(1+t)^{-\frac{d}{4}} + C(1+t)^{-\frac{d}{4}} Q(t)^2, \quad (6.71)
\]
and
\[
\|\nabla_x P_1 f_2(t)\|_{L^2_t} + \|\nabla_x E(t)\|_{L^2_t} \leq C(1+t)^{-\frac{d}{4}}(\|\nabla_x U_0\|_{L^2} + \|U_0\|_{L^2} + \|\nabla_x U_0\|_{L^2})
\]
\[
+ C \int_0^{t/2} (1+t-s)^{-\frac{d}{8}} \|\nabla_x G_2(s)\|_{L^2_x} + \|G_2(s)\|_{L^2_x})ds
\]
\[
+ C \int_{t/2}^t (1+t-s)^{-2} \|\nabla_x G_2(s)\|_{L^2_x} ds
\]
\[
\leq C\delta_0(1+t)^{-\frac{d}{4}} + C(1+t)^{-\frac{d}{4}} Q(t)^2,
\]
\[ + C \int_{\frac{t}{2}}^{t} (1 + t - s)^{-\frac{k}{2}} \langle \| \nabla_x G_2 \|_{L^2} + \| \nabla_x G_2 \|_{L^2} \rangle \, ds \]
\[ + C \int_{0}^{t} (1 + t - s)^{-2} \| \nabla_x^3 G_2 \|_{L^2} \, ds \]
\[ \leq C \delta_0 (1 + t)^{-\frac{3}{2}} + C (1 + t)^{-\frac{3}{2}} Q(t)^2. \]  

(6.72)

By (6.35), (6.65), (6.69) and (6.77), we have

\[ \| \nabla_x^k B(t) \|_{L^2} \leq C (1 + t)^{-\frac{3}{2}} \langle \| \nabla_x^k U_0 \|_{L^2} + \| U_0 \|_{L^2} + \| \nabla_x^2 U_0 \|_{L^2} \rangle \]
\[ + C \int_{0}^{t} (1 + t - s)^{-\frac{3}{2}} \langle \| \nabla_x^k G_2 \|_{L^2} + \| G_2 \|_{L^2} \rangle \, ds \]
\[ + C \int_{0}^{t} (1 + t - s)^{-2} \| \nabla_x^{k+2} G_2 \|_{L^2} \, ds \]
\[ \leq C \delta_0 (1 + t)^{-\frac{3}{2}} + C (1 + t)^{-\frac{3}{2}} Q(t)^2, \quad k = 0, 1. \]  

(6.73)

Finally, we estimate the higher order terms. Since

\[ d_1 H_{N,1}^f(U) \leq D_{N,1}(U) + C \sum_{|\alpha|=N} \| \partial_x^\alpha (E, B) \|_{L^2}^2, \]

(6.74)

for \( d_1 > 0 \), we still need to estimate the decay rate of \( \| \partial_x^\alpha (E, B) \|_{L^2}^2 \) for \( |\alpha| = N \). By Theorem 1.3 in \[7\], one has

\[ E_{k,1}(U)(t) \leq C (1 + t)^{-3/2} (E_{k+2,1}(U_0) + (\delta_0 + Q(t)^2)^2), \]

(6.75)

for any integer \( k \geq 4 \), where \( E_{k,1} \) is defined by (6.1). Then

\[ \| \nabla_x^N (E, B)(t) \|_{L^2} \leq C (1 + t)^{-\frac{3}{2}} \langle \| \nabla_x^N U_0 \|_{L^2} + \| U_0 \|_{L^2} + \| \nabla_x^{N+2} U_0 \|_{L^2} \rangle \]
\[ + C \int_{0}^{t} (1 + t - s)^{-\frac{3}{2}} \langle \| \nabla_x^N G_2(s) \|_{L^2} + \| G_2(s) \|_{L^2} \rangle \, ds \]
\[ + C \int_{0}^{t} (1 + t - s)^{-2} \| \nabla_x^{N+2} G_2(s) \|_{L^2} \, ds \]
\[ \leq C \delta_0 (1 + t)^{-\frac{3}{2}} + C (1 + t)^{-\frac{3}{2}} (\delta_0 + Q(t)^2)^2, \]

(6.76)

where we have used (6.79) to obtain

\[ \| G_2(s) \|_{L^2(H^{N+2})} \leq C E_{N+3,1}(U)(s) \leq C (E_{N+5,1}(U_0) + (\delta_0 + Q(t)^2)^2)(1 + s)^{-\frac{3}{2}}. \]

Then, by (6.53) and (6.74), we have

\[ \frac{d}{dt} H_{N,1}^f(U)(t) + d_1 \mu H_{N,1}^f(U)(t) \]
\[ \leq C \| \nabla_x(n_1, m_1, q_1)(t) \|_{L^2}^2 + C \| \nabla_x B(t) \|_{L^2}^2 + C \sum_{|\alpha|=N} \| \partial_x^\alpha (E, B)(t) \|_{L^2}^2. \]  

(6.77)

This and (6.76) give

\[ H_{N,1}^f(U)(t) \leq e^{-d_1 \mu t} H_{N,1}^f(U_0) + C \int_{0}^{t} e^{-d_1 \mu (t-s)} \| \nabla_x(n_1, m_1, q_1)(s) \|_{L^2}^2 + \| \nabla_x B(s) \|_{L^2}^2 \, ds \]
\[ + C \int_{0}^{t} e^{-d_1 \mu (t-s)} \sum_{|\alpha|=N} \| \partial_x^\alpha (E, B)(s) \|_{L^2}^2 \, ds \]
\[ \leq C (1 + t)^{-\frac{3}{2}} (\delta_0 + Q(t)^2) + (\delta_0 + Q(t)^2)^2. \]  

(6.78)
By summing (6.68)–(6.73) and (6.78), we have

\[ Q(t) \leq C(\delta_0 + Q(t)^2) + C(\delta_0 + Q(t)^2)^2, \]

which yields (6.74) when \( \delta_0 > 0 \) is chosen small enough. This completes the proof of the theorem.

Indeed, some of the above convergence rates can be shown to be optimal even for the nonlinear system.

**Theorem 6.7.** Under the assumption of Theorem 2.6, the global solution \((f_1, f_2, E, B)\) to the two-species Vlasov-Maxwell-Boltzmann system (2.6)–(2.11) satisfies

\[
C_1\delta_0(1 + t)^{-\frac{2}{\beta}} \leq \|\langle f_1(t), \chi_j \rangle\|_{L^2_x} \leq C_2\delta_0(1 + t)^{-\frac{2}{\beta}}, \quad j = 0, 1, 2, 3, 4, \tag{6.79}
\]

\[
C_1\delta_0(1 + t)^{-\frac{2}{\beta}} \leq \|P_1f_1(t)\|_{L^2_{x,v}} \leq C_2\delta_0(1 + t)^{-\frac{2}{\beta}}, \tag{6.80}
\]

\[
C_1\delta_0(1 + t)^{-\frac{2}{\beta}} \leq \|P_3f_2(t)\|_{L^2_{x,v}} \leq C_2\delta_0(1 + t)^{-\frac{2}{\beta}}, \tag{6.81}
\]

\[
C_1\delta_0(1 + t)^{-\frac{4}{\beta}} \leq \|E(t)\|_{L^2_x} \leq C_2\delta_0(1 + t)^{-\frac{4}{\beta}}, \tag{6.82}
\]

\[
C_1\delta_0(1 + t)^{-\frac{4}{\beta}} \leq \|B(t)\|_{L^2_x} \leq C_2\delta_0(1 + t)^{-\frac{4}{\beta}}, \tag{6.83}
\]

for \( t > 0 \) large with two constants \( C_2 > C_1 \).

**Proof.** By (6.60), (6.61), Theorem 5.7 and Theorem 6.6, we can establish the lower bounds of the time decay rates of macroscopic density, momentum and energy of the global solution \((f_1, f_2, E, B)\) to the system (2.6)–(2.11) and its microscopic part for \( t > 0 \) large enough. For example, it holds that

\[
\|\langle f_1(t), \chi_j \rangle\|_{L^2_x} \geq \|(e^{\delta_0}f_{1,0}, \chi_j)\|_{L^2_x} - \int_0^t \|(e^{(t-s)\delta_0}G_1(s), \chi_j)\|_{L^2_x} ds
\]

\[
\geq C_1\delta_0(1 + t)^{-\frac{2}{\beta}} - C_2\delta_0(1 + t)^{-\frac{2}{\beta}},
\]

\[
\|E(t)\|_{L^2_x} \geq \|(e^{\delta_0}(f_{2,0} + E_0, B_0))_2\|_{L^2_x} - \int_0^t \|(e^{(t-s)\delta_0}(G_2(s), 0, 0))_2\|_{L^2_x} ds
\]

\[
\geq C_1\delta_0(1 + t)^{-\frac{4}{\beta}} - C_2\delta_0(1 + t)^{-\frac{4}{\beta}}.
\]

Therefore, for \( \delta_0 > 0 \) small and \( t > 0 \) large, we can prove (6.79)–(6.83).

### 6.3 Corresponding results on one-species

Finally, we give the corresponding results on the one-species VMB. Let \( N \) be a positive integer and \( U = (f, E, B) \), and

\[
E^1_{N,k}(U) = \sum_{|\alpha| + |\beta| \leq N} \|u^k\partial_x^\alpha \partial_v^\beta f\|_{L^2_t L^2_x}^2 + \sum_{|\alpha| \leq N} \|\partial_x^\alpha (E, B)\|_{L^2_t L^2_x}^2,
\]

\[
H^1_{N,k}(U) = \sum_{|\alpha| + |\beta| \leq N} \|u^k\partial_x^\alpha \partial_v^\beta P_1f\|_{L^2_t L^2_x}^2 + \sum_{1 \leq |\alpha| \leq N} \|\partial_x^\alpha (E, B)\|_{L^2_t L^2_x}^2
\]

\[
+ \sum_{|\alpha| \leq N-1} \|\partial_x^\alpha \nabla_x P_0f\|_{L^2_t L^2_x}^2 + \|P_0f\|_{L^2_t L^2_x}^2,
\]

\[
D^1_{N,k}(U) = \sum_{|\alpha| + |\beta| \leq N} \|u^{k+\delta} \partial_x^\alpha \partial_v^\beta P_1f\|_{L^2_t L^2_x}^2 + \sum_{|\alpha| \leq N-1} \|\partial_x^\alpha \nabla_x P_0f\|_{L^2_t L^2_x}^2 + \|P_0f\|_{L^2_t L^2_x}^2
\]

\[
+ \sum_{1 \leq |\alpha| \leq N-1} \|\partial_x^\alpha E\|_{L^2_t L^2_x}^2 + \sum_{2 \leq |\alpha| \leq N-1} \|\partial_x^\alpha B\|_{L^2_t L^2_x}^2.
\]

for \( k \geq 0 \). For brevity, we write \( E^1_k(U) = E^1_{K,0}(U) \), \( H^1_k(U) = H^1_{K,0}(U) \) and \( D^1_k(U) = D^1_{K,0}(U) \) for \( k = 0 \).
Applying the similar argument as to derive the equation (6.10)–(6.12) and making use of the system (2.43)–(2.46), we can obtain a compressible Navier-Stokes-Maxwell type equations with inhomogeneous terms for the macroscopic density, momentum and energy \((n, m, q) = (f, \chi_0, (f, e\chi_0, (f, \chi_4))\) and \(E, B\) as follows

\[
\begin{align*}
\partial_t n + \text{div}_x m &= 0, \quad (6.84) \\
\partial_t n + \partial_t R_1 + \nabla_x n + \sqrt{\frac{2}{3}} \nabla_x q - E &= \kappa_1 \Delta_x m + \frac{1}{3} \nabla_x (\text{div}_x m) + nE + m \times B + R_2, \quad (6.85) \\
\partial_t q + \partial_t R_3 + \sqrt{\frac{2}{3}} \text{div}_x m &= \kappa_2 \Delta_x q + \sqrt{\frac{2}{3}} E \cdot m + R_4, \quad (6.86) \\
\partial_t E &= \nabla_x \times B - m, \quad (6.87) \\
\partial_t B &= -\nabla_x \times E, \quad (6.88)
\end{align*}
\]

where the viscosity and heat conductivity coefficients \(\kappa_1, \kappa_2 > 0\) and the remainder terms \(R_1, R_2, R_3, R_4\) are defined by

\[
\begin{align*}
\kappa_1 &= -(L^{-1}P_1(v_1\chi_2), v_1\chi_2), \quad \kappa_2 = -(L^{-1}P_1(v_1\chi_4), v_1\chi_4), \\
R_1 &= (v \cdot \nabla_x L^{-1}P_1 f, v\sqrt{M}), \quad R_2 = -(v \cdot \nabla_x L^{-1}(P_1(v \cdot \nabla_x P_1 f) - P_1 G), v\sqrt{M}), \\
R_3 &= (v \cdot \nabla_x L^{-1}P_1 f, \chi_4), \quad R_4 = -(v \cdot \nabla_x L^{-1}(P_1(v \cdot \nabla_x P_1 f) - P_1 G), \chi_4).
\end{align*}
\]

Here,

\[
G = \frac{1}{2} (v \cdot E) f - (E + v \times B) \cdot \nabla_x f + \Gamma(f, f).
\]

Similar to Section 5.1, we have the energy estimates of the one-species VMB system (2.43)–(2.47) as follows.

**Lemma 6.8** (Macroscopic dissipation). Let \((n, m, q, E, B)\) be the strong solutions to (6.84)–(6.88). Then, there are two constants \(s_0, s_1 > 0\) such that

\[
\begin{align*}
\frac{1}{2} \int \sum_{k \leq |\alpha| \leq N-1} s_0 (||\partial^\alpha_x (n, m, q)||^2_{L^2} + ||\partial^\alpha_x (E, B)||^2_{L^2} + \int \sum_{k \leq |\alpha| \leq N-1} \partial^\alpha_x R_1 \partial^\alpha_x m dx + 2 \int \partial^\alpha_x R_2 \partial^\alpha_x q dx) \\
+ \int \sum_{k \leq |\alpha| \leq N-1} s_1 \int \partial^\alpha_x m \partial^\alpha_x \nabla_x m dx - \int \sum_{k \leq |\alpha| \leq N-1} 8 \int \partial^\alpha_x E \partial^\alpha_x \nabla_x (E \times B) dx \\
- \int \sum_{k \leq |\alpha| \leq N-1} \frac{2}{3} \int \partial^\alpha_x E \partial^\alpha_x (\nabla_x \times B) dx - \int \sum_{k \leq |\alpha| \leq N-1} \frac{2}{3} \int \partial^\alpha_x E \partial^\alpha_x (\nabla_x \times B) dx \\
\leq C(\sqrt{E^1_N(U) + E^1_N(U)}D^1_N(U) + C \sum_{k \leq |\alpha| \leq N} ||\partial^\alpha_x P_1 f||^2_{L^2},
\end{align*}
\]

with \(0 \leq k \leq N - 3\).

**Lemma 6.9** (Microscopic dissipation). Let \(N \geq 4\) and \((f, E, B)\) be a strong solution to VMB system (2.43)–(2.47). Then, there are constants \(p_k > 0, 1 \leq k \leq N\) such that

\[
\begin{align*}
\frac{1}{2} \int \sum_{1 \leq |\alpha| \leq N} (||\partial^\alpha_x f||^2_{L^2} + ||\partial^\alpha_x (E, B)||^2_{L^2}) + \mu ||w^\alpha_x P_1 f||^2_{L^2, \omega} \leq C(\sqrt{E^1_N(U) + E^1_N(U)}D^1_N(U), \\
\frac{1}{2} \int \sum_{1 \leq |\alpha| \leq N} (||\partial^\alpha_x f||^2_{L^2, \omega} + ||\partial^\alpha_x (E, B)||^2_{L^2, \omega}) + \mu \sum_{1 \leq |\alpha| \leq N} ||w^\alpha_x P_1 f||^2_{L^2, \omega} \leq C(\sqrt{E^1_N(U)}D^1_N(U), \\
\frac{1}{2} \int ||P_1 f||^2_{L^2, \omega} + ||w^* P_1 f||^2_{L^2, \omega} \leq C ||\nabla_x P_0 f||^2_{L^2, \omega} + E^1_N(U)D^1_N(U),
\end{align*}
\]
\[
\frac{d}{dt} \sum_{1 \leq k \leq N} p_k \sum_{|\alpha|=k} \left| \frac{\partial^\alpha}{\partial x^\alpha} P f \right|^2_{L_x^2,v} + \mu \sum_{1 \leq k \leq N} p_k \sum_{|\alpha|=k} \left| \frac{\partial^\alpha}{\partial x^\alpha} P f \right|^2_{L_x^2,v} \\
\leq C \sum_{|\alpha| \leq N-1} \left| \frac{\partial^\alpha}{\partial x^\alpha} \nabla_x f \right|^2_{L_x^2,v} + C \sqrt{E_N^1(U) D_N^1(U)}.
\]

Lemma 6.10. Let \(N \geq 4\). Then, there are two equivalent energy functionals \(E_N^1(U) \sim E_N^1(U)\), \(H_N^1(U) \sim H_N^1(U)\) such that the following holds. If \(E_N^1(U_0)\) is sufficiently small, then the Cauchy problem \((2.43) - (2.47)\) of the one-species VMB system admits a unique global solution \(U = (f, E, B)\) satisfying

\[
\frac{d}{dt} E_N^1(U)(t) + \mu D_N^1(U)(t) \leq 0, \\
\frac{d}{dt} H_N^1(U)(t) + \mu D_N^1(U)(t) \leq C \left\| \nabla_x(n,m,q) \right\|^2_{L_x^2} + \left\| \nabla_x E \right\|^2_{L_x^2} + \left\| \nabla_x B \right\|^2_{L_x^2}.
\]

Lemma 6.11. Let \(N \geq 4\). There are the equivalent energy functionals \(E_N^{1,1}(U) \sim E_N^{1,1}(U)\), \(H_N^{1,1}(U) \sim H_N^{1,1}(U)\) such that if \(E_N^{1,1}(U_0)\) is sufficiently small, then the solution \(U = (f, E, B)(t, x, v)\) to the one-species VMB system \((2.43) - (2.47)\) satisfies

\[
\frac{d}{dt} E_N^{1,1}(U)(t) + \mu D_N^{1,1}(U)(t) \leq 0, \\
\frac{d}{dt} H_N^{1,1}(U)(t) + \mu D_N^{1,1}(U)(t) \leq C \left\| \nabla_x(n,m,q) \right\|^2_{L_x^2} + \left\| \nabla_x E \right\|^2_{L_x^2} + \left\| \nabla_x B \right\|^2_{L_x^2}.
\]

With these energy estimates, we can prove Theorems 2.1.12 for the nonlinear one-species VMB system \((2.43) - (2.47)\).

Proof of Theorem 2.1.2 Let \(f, E, B\) be a solution to the problem \((2.43) - (2.47)\) for \(t > 0\). We can represent its solution in terms of the semigroup \(e^{tA_2}\) by

\[
(f, E, B)(t) = e^{tA_2}(f_0, E_0, B_0) + \int_0^t e^{(s-t)A_2}(G, 0, 0)(s)ds,
\]

where the nonlinear term \(G\) is given by \((6.90)\). For this global solution \(f\), we define a functional \(Q(t)\) for any \(t > 0\) as

\[
Q(t) = \sup_{0 \leq s \leq t} \left\{ \left(1 + s\right)^{\frac{1}{4}} \left\| \frac{\partial^k}{\partial x^k} \left( f(s), \sqrt{M} \right) \right\|_{L_x^2} + \left(1 + s\right)^{\frac{1}{4}} \left\| \frac{\partial^k}{\partial x^k} \left( f(s), v \sqrt{M} \right) \right\|_{L_x^2} \\
+ \left(1 + s\right)^{\frac{1}{4}} \left\| \frac{\partial^k}{\partial x^k} \left( f(s), \chi \right) \right\|_{L_x^2} + \left(1 + s\right)^{\frac{1}{4}} \left\| \frac{\partial^k}{\partial x^k} P_1 f(s) \right\|_{L_x^2} \\
+ \left(1 + s\right)^{\frac{1}{4}} \left\| \nabla_x E(s) \right\|_{L_x^2} + \left(1 + s\right)^{\frac{1}{4}} \left\| \nabla_x B(s) \right\|_{L_x^2} \\
+ \left(1 + s\right)^{\frac{1}{4}} \left\{ \left\| P_1 f(s) \right\|_{H_x^\infty} + \left\| \nabla_x P_0 f(s) \right\|_{L_x^2(H_x^{\infty - i})} + \left\| \nabla_x E(s, B(s)) \right\|_{H_x^{\infty - 1}} \right\} \right\}
\]

In the case of \(\nabla_x \cdot E_0 = (f_0, \sqrt{M})\) and \(B_0 = 0\), we can obtain by \((5.124) - (5.123)\) and \((5.100) - (5.101)\) that

\[
\left\| \frac{\partial^k}{\partial x^k} (f(t), \chi) \right\|_{L_x^2} \leq C \left[ (1 + t)^{1 - \frac{k}{4}} + (1 + t)^{-\frac{k}{4}} \right] \left\{ \left\| \partial_x^k U_0 \right\|_{L_x^\infty} + \left\| \partial_x^k \alpha \right\|_{L_x^\infty} \right\}, \\
\left\| \partial_x^k (f(t), \chi_4) \right\|_{L_x^2} \leq C \left[ (1 + t)^{1 - \frac{k}{4}} + (1 + t)^{-\frac{k}{4}} \right] \left\{ \left\| \partial_x^k U_0 \right\|_{L_x^\infty} + \left\| \partial_x^k \nabla_x \chi \right\|_{L_x^\infty} \right\}.
\]

where \((f, E, B) = e^{tA_2}U_0\) with \(U_0 = (f_0, E_0, B_0)\), \(\alpha' \leq \alpha\) and \(k = |\alpha - \alpha'|\).

By Lemma 6.11 we can estimate the nonlinear term \(G(s)\) for \(0 \leq s \leq t\) in terms of \(Q(t)\) as

\[
\left\| G(s) \right\|_{L_x^2} \leq C \left\{ \left\| \nabla_x f \right\|_{L_x^2} + \left\| E \right\|_{L_x^2} \left\| \nabla_x f \right\|_{L_x^2} + \left\| B \right\|_{L_x^2} \right\}.
\]
In addition, the microscopic part $P$ and similarly for $E$, we can estimate the macroscopic energy 
\[
(1 + s)^{-\frac{3}{2} + \frac{3}{8}} (\|w f\|_{L_{x,v}^2} + \|\nabla_v f\|_{L_{x,v}^2} + \|B\|_{L_{x,v}^2} w \nabla_v f\|_{L_{x,v}^2}) \leq C(1 + s)^{-1} Q(t)^2,
\]
(6.100)
and similarly
\[
\|G(s)\|_{L_{x,v}^2(H_{x,v}^2)} \leq C(1 + s)^{-\frac{5}{4}} Q(t)^2,
\]
(6.101)
for $1 \leq k \leq N - 1$. Then, it follows from (2.76), (6.99) and (6.100) that
\[
\|\nabla^k_x(f(t), \sqrt{M})\|_{L_{x,v}^2} \leq C(1 + t)^{-\frac{3}{2} - \frac{k}{8}} \left( \|\nabla^k_x U_0\|_{L_{x,v}^2}^2 + \|U_0\|_{L_{x,v}^2} + \|\nabla^2 U_0\|_{L_{x,v}^2} \right)
\]
\[
+ C \int_0^t (1 + t - s)^{-\frac{3}{2} - \frac{k}{8}} \left( \|\nabla^k_x G(s)\|_{L_{x,v}^2} + \|\nabla^k_v G(s)\|_{L_{x,v}^2} \right) ds
\]
\[
\leq C \delta_0(1 + t)^{-\frac{3}{2} - \frac{k}{8}} + C(1 + t)^{-1 - \frac{3}{8}} Q(t)^2,
\]
(6.102)
for $k = 0, 1$.

Similarly, in terms of (2.76) and (5.95), we have
\[
\|\nabla^k_x(f(t), v \sqrt{M})\|_{L_{x,v}^2} \leq C(1 + t)^{-\frac{3}{2} - \frac{k}{8}} \left( \|\nabla^k_x U_0\|_{L_{x,v}^2}^2 + \|U_0\|_{L_{x,v}^2} + \|\nabla^2 U_0\|_{L_{x,v}^2} \right)
\]
\[
+ C \int_0^t (1 + t - s)^{-\frac{3}{2} - \frac{k}{8}} \left( \|\nabla^k_x G(s)\|_{L_{x,v}^2} + \|\nabla^k_v G(s)\|_{L_{x,v}^2} \right) ds
\]
\[
\leq C \delta_0(1 + t)^{-\frac{3}{2} - \frac{k}{8}} + C(1 + t)^{-1 - \frac{3}{8}} Q(t)^2,
\]
(6.103)
for $k = 0, 1$.

In terms of (2.76) and (5.96), we can estimate the macroscopic energy $(f(t), \chi_4)$ and its spatial derivative as
\[
\|\nabla^k_x(f(t), \chi_4)\|_{L_{x,v}^2} \leq C(1 + t)^{-\frac{3}{2} - \frac{k}{8}} \left( \|\nabla^k_x U_0\|_{L_{x,v}^2}^2 + \|U_0\|_{L_{x,v}^2} + \|\nabla^2 U_0\|_{L_{x,v}^2} \right)
\]
\[
+ C \int_0^t (1 + t - s)^{-\frac{3}{2} - \frac{k}{8}} \left( \|\nabla^k_x G(s)\|_{L_{x,v}^2} + \|G(s), \chi_4\|_{L_{x,v}^2} + \|\nabla_x G(s)\|_{L_{x,v}^2} \right) ds
\]
\[
\leq C \delta_0(1 + t)^{-\frac{3}{2} - \frac{k}{8}} + C(1 + t)^{-1 - \frac{3}{8}} Q(t)^2,
\]
(6.104)
for $k = 0, 1$, where we have used
\[
\|G(s)\|_{L_{x,v}^2} \leq \sqrt{\frac{3}{2}} \|E\|_{L_{x}^1} \leq C(1 + s)^{-\frac{3}{8}} Q(t)^2.
\]
In addition, the microscopic part $P_1 f(t)$ can be estimated by (2.77) and (5.97) as follows
\[
\|\nabla^k_x P_1 f(t)\|_{L_{x,v}^2} \leq C(1 + t)^{-\frac{3}{2} - \frac{k}{8}} \left( \|\nabla^k_x U_0\|_{L_{x,v}^2}^2 + \|U_0\|_{L_{x,v}^2} + \|\nabla^2 U_0\|_{L_{x,v}^2} \right)
\]
\[
+ C \int_0^t (1 + t - s)^{-\frac{3}{2} - \frac{k}{8}} \left( \|\nabla^k_x G(s)\|_{L_{x,v}^2} + \|\nabla^k_v G(s)\|_{L_{x,v}^2} \right) ds
\]
\[
\leq C \delta_0(1 + t)^{-\frac{3}{2} - \frac{k}{8}} + C(1 + t)^{-1 - \frac{3}{8}} Q(t)^2,
\]
(6.105)
for $k = 0, 1$.

Moreover, the electricity potential $E$ is bounded by
\[
\|\nabla^k_x E(t)\|_{L_{x,v}^2} \leq C(1 + t)^{-\frac{3}{2} - \frac{k}{8}} \left( \|\nabla^k_x U_0\|_{L_{x,v}^2}^2 + \|U_0\|_{L_{x,v}^2} + \|\nabla^2 U_0\|_{L_{x,v}^2} \right)
\]
By (6.53) and (6.74), we have
\[ 6 \delta_0 (1 + t)^{-\frac{3}{2}} + C(1 + t)^{-\frac{3}{2}} \frac{8}{5} \ln(1 + t) Q(t), \tag{6.106} \]
for \( k = 0, 1 \). And the magnetic potential \( B \) is bounded by
\[
\| \nabla_x^k B(t) \|_{L^2_z} \leq C (1 + t)^{-\frac{3}{2}} (\| \nabla_x^k U_0 \|_{L^2_z} + \| U_0 \|_{L^2_z} + \| \nabla_x^{k+1} U_0 \|_{L^2_z}) \\
+ C \int_0^t (1 + t - s)^{-\frac{3}{2}} (\| \nabla_x^k G(s) \|_{L^2_z} + \| G(s) \|_{L^2_z}) ds \\
+ C \int_0^t (1 + t - s)^{-1} \| \nabla_x^{k+1} G(s) \|_{L^2_z} ds \\
\leq C \delta_0 (1 + t)^{-\frac{3}{2}} + C (1 + t)^{-\frac{3}{2}} \frac{8}{5} Q(t)^2, \tag{6.107} \]
for \( k = 0, 1, 2 \).

Next, we estimate the higher order terms as below. By a similar argument as in Theorem 1.3 in [7], we have
\[
E_{1,k}^1 f(U)(t) \leq C (1 + t)^{-3/4} (E_{k+1,1}(U_0) + (\delta_0 + Q(t))^2), \tag{6.108} \]
for any integer \( k \geq 4 \). Then
\[
\| \nabla_x^N (E, B)(t) \|_{L^2_z} \leq C (1 + t)^{-\frac{3}{2}} (\| \nabla_x^N U_0 \|_{L^2_z} + \| U_0 \|_{L^2_z} + \| \nabla_x^{N+1} U_0 \|_{L^2_z}) \\
+ C \int_0^t (1 + t - s)^{-\frac{3}{2}} (\| \nabla_x^N G(s) \|_{L^2_z} + \| G(s) \|_{L^2_z}) ds \\
+ C \int_0^t (1 + t - s)^{-1} \| \nabla_x^{N+1} G(s) \|_{L^2_z} ds \\
\leq C \delta_0 (1 + t)^{-\frac{3}{2}} + C (1 + t)^{-\frac{3}{2}} (\delta_0 + Q(t))^2, \tag{6.109} \]
where we have used
\[
\| G(s) \|_{L^2_z(H^N)} \leq E_{N+2,1}(U)(s) \leq C (E_{N+3,1}(U_0) + (\delta_0 + Q(t))^2)(1 + s)^{-\frac{3}{2}}. \]

By (6.53) and (6.74), we have
\[
\frac{d}{dt} H_{1,1}^{1,f}(U)(t) + d_{1,\mu} H_{1,1}^{1,f}(U)(t) \\
\leq C (\| \nabla_x(n, m, q, E, B)(t) \|_{L^2_z}^2 + \| \nabla_x^2 B(t) \|_{L^2_z}^2) + C \sum_{|\alpha|=N} \| \partial_x^\alpha (E, B)(t) \|_{L^2_z}^2, \tag{6.110} \]
which and (6.108) lead to
\[
H_{1,1}^{1,f}(U)(t) \leq e^{-d_{1,\mu} t} H_{1,1}^{1,f}(U_0) + C \int_0^t e^{-d_{1,\mu} (t-s)} \sum_{|\alpha|=N} \| \partial_x^\alpha (E, B)(s) \|_{L^2_z}^2 ds \\
+ C \int_0^t e^{-d_{1,\mu} (t-s)} (\| \nabla_x(n, m, q, E, B)(s) \|_{L^2_z}^2 + \| \nabla_x^2 B(s) \|_{L^2_z}^2) ds \\
\leq C (1 + t)^{-\frac{3}{2}} (\delta_0 + Q(t))^2 + (\delta_0 + Q(t))^2. \tag{6.111} \]

By summing (6.102), (6.107) and (6.111), we have
\[
Q(t) \leq C (\delta_0 + Q(t))^2 + C (\delta_0 + Q(t))^2, \]
from which (2.56)–(2.57) can be verified provided that \( \delta_0 > 0 \) is small enough. Similarly, as Theorem 2.6 we can prove (2.93)–(2.97). \qed
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References

[1] Cercignani C., Illner R., Pulvirenti M.: The Mathematical Theory of Dilute Gases, AMS. Vol. 106. Springer-Verlag, New York (1994)
[2] Chapman S., Cowling T.G.: The mathematical theory of non-uniform gases, 3rd edition. Cambridge University Press, London (1970)
[3] R.J. Duan and R. M. Strain, Optimal time decay of the Vlasov-Poisson-Boltzmann system in $\mathbb{R}^3$, Arch. Ration. Mech. Anal., 199 (2011), no. 1, 291-328
[4] Duan R.J., Yang T.: Stability of the one-species Vlasov-Poisson-Boltzmann system. SIAM J. Math. Anal. 41, 2353-2387 (2010)
[5] Duan R.J., Ukai S.; Yang T., Zhao H.J.: Optimal decay estimates on the linearized Boltzmann equation with time-dependent forces and their applications. Comm. Math. Phys. 277, 189-236 (2008)
[6] Duan R.J.: Dissipative property of the Vlasov-Maxwell-Boltzmann System with a uniform ionic background. SIAM J. Math. Anal. 43(6), 2732-2757 (2011)
[7] Duan R.J., Strain R.M.: Optimal large-time behavior of the Vlasov-Maxwell-Boltzmann system in the whole space. Comm. Pure Appl. Math. 64, 1497-1546 (2011)
[8] Ellis R.S., Finsky M.A.: The first and second fluid approximations to the linearized Boltzmann equation. J. Math. pure et appl. 54, 125-156 (1975)
[9] Guo Y.: The Vlasov-Poisson-Boltzmann system near Maxwellians. Comm. Pure Appl. Math. 55, 1104-1135 (2002)
[10] Guo Y.: The Vlasov-Maxwell-Boltzmann system near Maxwellians. Invent. Math. 153(3), 593-630 (2003)
[11] Jang J.: Vlasov-Maxwell-Boltzmann diffusive limit. Arch. Ration. Mech. Anal. 194, 531-584 (2009).
[12] Kato T.: Perturbation Theory of Linear Operators. Springer, New York (1996)
[13] Li H.-L., Yang T., Zhong M.-Y.: Spectral analysis for the Vlasov-Poisson-Boltzmann system. preprint (2013).
[14] Li H.-L., Yang T., Zhong M.-Y.: Spectrum analysis and optimal decay rates of the bipolar Vlasov-Poisson-Boltzmann equations, preprint (2014).
[15] Liu T.-P., Yu S.-H.: The Greens function and large-time behavior of solutions for the one-dimensional Boltzmann equation. Comm. Pure Appl. Math. 57, 1543-1608 (2004)
[16] Liu T.-P., Yang T., Yu S.-H.: Energy method for the Boltzmann equation. Physica D, 188(3-4), 178-192 (2004)
[17] Markowich P.A., Ringhofer C.A., Schmeiser C.: Semiconductor Equations, Springer-Verlag, Vienna (1990)
[18] Pazy A.: Semigroups of Linear Operators and Applications to Partial Differential Equations. AMS Vol. 44. Springer-Verlag, New York, (1983)
[19] Strain R. M.: The Vlasov-Maxwell-Boltzmann system in the whole space. Comm. Math. Phys. 268(2), 543-567 (2006)
[20] Ukai S.: On the existence of global solutions of mixed problem for non-linear Boltzmann equation. Proceedings of the Japan Academy, 50, 179-184 (1974)
[21] Ukai S., Yang T.: The Boltzmann equation in the space $L^2 \cap L^\infty$: Global and time-periodic solutions. Analysis and Applications 4, 263-310 (2006)
[22] Ukai S., Yang T.: Mathematical Theory of Boltzmann Equation. Lecture Notes Series-No. 8, Hong Kong: Liu Bie Ju Center for Mathematical Sciences, City University of Hong Kong, March 2006.
[23] T. Yang, H.J. Yu, Optimal convergence rates of classical solutions for Vlasov-Poisson-Boltzmann system. Commun. Math. Phys. 301 (2011), 319-355.
[24] Alexander S., Yu S.-H.: On the Solution of a Boltzmann System for Gas Mixtures. Arch. Rational Mech. Anal. 195, 675-700 (2010)
[25] Zhong M.-Y.: Optimal time-decay rate of the Boltzmann equation. Sci. China Math. 57, 807-822 (2014)