WEIGHTED ROOTED TREES AND DEFORMATIONS OF OPERADS

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Abstract. We will define an operad $B^0$ on planar rooted trees. $B^0$ is analogous to the NAP-operad in the non-planar tree setting. We will define a family of “current-preserving” operads $B^\lambda$ depending on a scalar parameter $\lambda$, which can be seen as a deformation of the operad $B^0$. Forgetting the extra “current-preserving” notion above gives back the Brace operad for $\lambda = 1$ and the $B^0$ operad for $\lambda = 0$. A natural map from non-planar rooted trees to planar ones gives back the current-preserving interpolation between NAP and pre-Lie investigated in a previous article [11].

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Contents

1. Background on operads 1
2. A description of Pre Lie and Brace operads 2
2.1. Rooted trees and planar rooted trees 2
2.2. The pre-Lie operad 3
2.3. The Brace operad 3
3. A description of the NAP-operad and its counterpart in the planar rooted trees setting 4
3.1. The NAP operad 4
3.2. An operad of planar rooted trees analogous to NAP 4
4. The notion of current-preserving operads 5
4.1. Structure of current-preserving operads 5
4.2. Current-preserving operads associated to ordinary operads 6
5. A family of current-preserving operads 6
5.1. Interpolation between NAP and Pre-Lie 6
5.2. Interpolation between Brace and $B^0$ 8
References 13

1. Background on operads

In this section, we review the material needed for this article. We refer to Ginzburg and Kapranov [6] or J. L. Loday [8]. Let $K$ be a field of characteristic zero. An operad (in the symmetric monoidal category of $k$-vector spaces) is given by a collection of vector spaces $(\mathcal{O}(n))_{n \geq 0}$, a right action of the symmetric group $S_n$ on $\mathcal{O}(n)$, and a collection of compositions:
Example 1. Any vector space yields an operad \( \text{End}_V \) with \( \text{End}_V(n) = \text{Hom}(V^\otimes n, V) \) where:

\[
(f \circ_i g)(a_1, \ldots, a_{n+m-1}) = f(a_1, \ldots, a_{i-1}, g(a_i, \ldots, a_{i+m-1}), a_{i+m}, \ldots, a_{n+m-1})
\]

An algebra over an operad \( \mathcal{O} \), or \( \mathcal{O} \)-algebra, is a vector space \( V \) together with an operad morphism from \( \mathcal{O} \) to \( \text{End}_V \). This is equivalent to giving linear maps

\[
\mathcal{O}(n) \otimes_{S_n} V^\otimes n \rightarrow V;
\]

satisfying associativity conditions with respect to the compositions.

An important point in the theory of operad is the following theorem:

**Theorem 1.** [11] Chapter 5, sect 5.7.1. The free \( \mathcal{O} \)-algebra generated by \( V \) is the space \( \mathcal{O}(V) = \bigoplus_{n \geq 0} \mathcal{O}(n) \otimes_{S_n} V^\otimes n \).

In the remainder of this article, we describe operads by species formalism, i.e: we replace the set \( \{1, 2, ..., n\} \) by any finite set \( A \) of cardinal \( n \). For more details see sections 2 and 3 of [11].

2. A DESCRIPTION OF PRE LIE AND BRACE OPERADS

2.1. Rooted trees and planar rooted trees.

- A rooted tree \( T \) is a finite graph, without loops, with a special vertex called the root of \( T \). The set of rooted trees will be denoted by \( \mathcal{T} \). Let \( D \) be a nonempty set. A rooted tree decorated by \( D \) is a rooted tree with an application from the set of its vertices into \( D \). The set of rooted trees decorated by \( D \) will be denoted by \( \mathcal{T}^D \). Following the notation of Connes and Kreimer [4], any tree \( T \) writes \( T = B_+(r, T_1 \cdots T_k) \) where \( r \) is the root (or the decoration of the root) and \( T_1, \ldots, T_k \) are trees. So we have

\[
B_+(r, T_1 \cdots T_k) = B_+(r, T_{\sigma(1)} \cdots T_{\sigma(k)}) \quad \forall \sigma \in S_k.
\]
The vector space spanned by $T^D$ will be denoted by $RT^D$. We denote by $RT$ the species of rooted trees: for any finite set $A$ the vector space $RT(A)$ is spanned by the rooted trees with $|A|$ vertices, together with a bijection from the set of vertices onto $A$.

- A planar rooted tree $T$ is a rooted tree with an embedding into the plane. The set of planar rooted trees will be denoted by $T_P$. Let $D$ be a nonempty set. A planar rooted tree decorated by $D$ is a planar tree with an application from the set of its vertices into $D$. The set of planar rooted trees decorated by $D$ will be denoted by $T^D_P$.

\[ B_+(r, T_1 T_2 \cdots T_k) \neq B_+(r, T_2 T_1 \cdots T_k). \]

We draw the planar tree in the disk:

\[ D_+ = \{ (x, y) \in \mathbb{R}^2; y > 0 \text{ and } x^2 + y^2 < 1 \}, \]

but the root is drawn in $x = y = 0$.

The vector space spanned by $T^D_P$ will be denoted by $PRT^D$. We denote by $PRT$ the species of planar rooted trees: for any finite set $A$ the vector space $PRT(A)$ is spanned by the rooted trees with $|A|$ vertices, together with a bijection from the set of vertices onto $A$.

2.2. The pre-Lie operad. We describe the pre-Lie operad in terms of non-planar labelled rooted trees, following [3]. Let $A$ and $B$ be two finite sets. Let $v \in A$. We define the partial composition $\circ_v : RT(A) \otimes RT(B) \to RT((A - \{v\}) \amalg B)$, as follows:

\[
(1) \quad T \circ_v S = \sum_{f : E(T, v) \to B} T \circ^f_v S,
\]

where $T \circ^f_v S$ is the tree of $RT((A - \{v\}) \amalg B)$ obtained by replacing the vertex $v$ of $T$ by the tree $S$ and connecting each edge $a$ in $E(T, v)$ at the vertex $f(a)$ of $S$. If $v$ is not the root of $T$, the edge going down from $v$ is now going down from the root of $S$. The root of the new tree is the root of $T$ if it is different from vertex $v$, and of $S$ else (see details in [3]). The unit is the tree with a single vertex. These partial compositions define an operad which is the pre-Lie operad.

2.3. The Brace operad. We describe the brace operad by the planar labelled rooted trees (for more details see [1]). Let $T$ be a labelled planar rooted tree. Let $s$ be a vertex of $T$. Let $B_c(s)$ be a little disk of center $s$. The pair $(s, \alpha)$ is called an angle of $T$ if $\alpha$ is a connected component of $B_c(s) \bigcap (D_+ \setminus T)$. We denote by $Ang(T)$ the set of angles of $T$. Naturally, from left to right we set a total order on $Ang(T)$ as follows: considering an angle as a direction from a vertex, one can draw a path from every angle to a point of the upper part of the unit circle. We order then these points clockwise.
Example 2.

Let $T, S$ be labelled planar rooted trees. Let $v$ be a vertex of $T$. We denote by $E(T, v)$ the totally ordered set (from left to right) of the incoming edges on $v$. We can consider the set of increasing functions from $E(T, v)$ to $\text{Ang}(S)$. We define

$$T \odot_v S = \sum_{f : E(T, v) \to \text{Ang}(S)} T \odot_v^f S,$$

where $f$ is an increasing function and $T \odot_v^f S$ is the planar tree obtained by substitution of $S$ on vertex $v$ of $T$, plugging the incoming edges on $S$ according to map $f$.

These partial compositions defined above, define a structure of an operad which is the Brace operad $[1]$.

3. A description of the NAP-operad and its counterpart in the planar rooted trees setting

3.1. The NAP operad. we describe the NAP operad by the non-planar labelled rooted trees $[7]$. We define the partial compositions $\odot_v : \mathcal{RT}(A) \otimes \mathcal{RT}(B) \to \mathcal{RT}((A - \{v\}) \amalg B)$, as follows:

$$T \odot_v S = T \odot_v^{f_0} S,$$

where $T \odot_v^{f_0} S$ is the labelled rooted tree of $\mathcal{RT}((A - \{v\}) \amalg B)$ obtained by replacing the vertex $v$ of $T$ by the tree $S$ and connecting each edge $a$ in $E(T, v)$ at the root of $S$. The unit is the tree with a single vertex.

3.2. An operad of planar rooted trees analogous to NAP. In this section, we describe an operad of planar labelled rooted trees. This operad is the planar analogue of NAP.

We define the partial compositions $\odot_v : \mathcal{PRT}(A) \otimes \mathcal{PRT}(B) \to \mathcal{PRT}((A - \{v\}) \amalg B)$, as follows:

$$T \odot_v S = \sum_{f_0 : E(T, v) \to \text{Ang}^0(S)} T \odot_v^{f_0} S,$$

where $\text{Ang}^0(S)$ is the set of angles starting from the root of $S$ and $f_0$ is an increasing function from $E(T, v)$ to $\text{Ang}^0(S)$.

Proposition 2. The partial compositions introduced above define a structure of an operad. We will denote this operad by $\mathcal{B}^0$. The unit is the tree with a single vertex.

Proof. We easily verify the unity, associativity and equivariance axioms. We omit the proof, as we will give a proof of a more general result later on. \[\square\]
Definition 1. We will denote the symmetrization operator of trees by \( \varphi \) from the space of non-planar labelled rooted trees to the space of planar labelled rooted trees, by induction we define \( \varphi \):

\[
\varphi(T) = B_+(r, \varphi(T_1) \sqcup \varphi(T_2) \sqcup \cdots \sqcup \varphi(T_k)),
\]

where \( \sqcup \) is the shuffle product.

i.e: \( \varphi(T) \) is the sum of all planar representations of \( T \).

Theorem 3. \( \varphi \) is a morphism of operads from Pre-Lie to Brace \([1]\). Similarly \( \varphi \) is a morphism of operads from NAP to \( B^0 \).

Proof. The first assertion is proved by F. Chapoton \([1\text{ Prop 4}]\), the second assertion follows immediately by considering only the terms of minimal potential energy (see Definition 4 below). \( \square \)

Definition 2. We define

\[
(T \star S) = (T \odot v) \cdot w S.
\]

Equivalently, if \( T = B_+(r, T_1 \ldots T_n) \) then

\[
(T \star S) = \sum_{i=0}^{n} B_+(r, T_1 \ldots T_i ST_{i+1} \ldots T_n).
\]

Proposition 4. The space \( (T, \star) \) is a right non-associative permutative algebra \([5]\). i.e: for any planar rooted trees \( T, S, U \), we have:

\[
(T \star S) \star U = (T \star U) \star S
\]

4. The notion of current-preserving operads

4.1. Structure of current-preserving operads. Let \( \mathcal{O} \) be an operad and \( G \) be a commutative semigroup, with additively denoted binary law. We say that \( \mathcal{O} \) has a structure of \( G \)-current-preserving operad, if moreover \( \mathcal{O}_A = \prod \mathcal{O}_{A,W} \), \( W : A \to G \) where :

- The right action of the symmetric group \( AutA \) verifies:

\[
\mathcal{O}_{A,W} \cdot \sigma = \mathcal{O}_{A,W \cdot \sigma}, \quad \forall \sigma \in AutA.
\]

- For any finite sets \( A, B \) and \( v \in A \), we have:

\[
\circ_v : \mathcal{O}_{A,W} \otimes \mathcal{O}_{B,X} \to \mathcal{O}_{A-\{v\} \sqcup B \sqcup X},
\]

with image zero if \( \sum_{b \in B} X(b) \neq W(v) \). Here, \( W \sqcup X \) is defined by: \( W \sqcup X(a) = W(a), \forall a \in A \), and \( W \sqcup X(b) = X(b), \forall b \in B \).

Example 3. The model of current-preserving operads is the operad \( \text{Endop}(V) \) where \( V \) is a \( G \)-graded vector space \([11]\). Current-preserving operads are colored operads, with an extra structure given by semigroup law on the set of colors.
4.2. Current-preserving operads associated to ordinary operads. Given an operad \( \mathcal{O} \) and any commutative semigroup \( G \), we define a \( G \)-current-preserving operad \( \mathcal{O}^G \) as follows: for any finite set \( A \), we have:

\[
\mathcal{O}^G_A := \prod_{W:A \to G} \mathcal{O}_{A,W},
\]

where \( \mathcal{O}_{A,W} \) is nothing but a copy of \( \mathcal{O}_A \). The partial compositions of \( (\alpha, W) \in \mathcal{O}_{A,W} \) and \( (\beta, X) \in \mathcal{O}_{B,X} \) are defined for any \( a \in A \) by:

\[
(\alpha, W) \circ_a^G (\beta, X) = (\alpha \circ_a \beta, W \sqcup X),
\]

if \( \sum_{b \in B} X(b) = W(a) \), and \( (\alpha, W) \circ_a^G (\beta, X) = 0 \) if \( \sum_{b \in B} X(b) \neq W(a) \). There is a natural morphism of operads (in the ordinary sense) \( \phi^G : \mathcal{O} \to \mathcal{O}^G \) given for any finite set \( A \) and for any \( \alpha \in \mathcal{O}_A \) by:

\[
\phi^G(\alpha) := \sum_{W:A \to G} (\alpha, W).
\]

The algebras on \( \mathcal{O}^G \) are nothing but \( G \)-graded algebras on \( \mathcal{O} \). The morphism of operads \( \phi \) simply reflects the forgetful functor from \( G \)-graded \( \mathcal{O} \)-algebras to \( \mathcal{O} \)-algebras. Hence, any ordinary operad gives rise to a \( G \)-current-preserving operad naturally associated with it.

5. A FAMILY OF CURRENT-PRESERVING OPERADS

5.1. Interpolation between NAP and Pre-Lie. We give here a family of \( \mathbb{N}^* \)-current-preserving operads \( (\mathcal{O}^\lambda)_{\lambda \in K} \), where \( \mathbb{N}^* \) is the additive semi-group \( \{1, 2, 3, \ldots\} \) of positive integers. We have shown that this family interpolates between the \( \mathbb{N}^* \)-current-preserving version of the NAP operad and the \( \mathbb{N}^* \)-current-preserving version of the pre-Lie operad (see details in [11]).

Definition 3. We introduce non-planar rooted trees with weights on their vertices:

\[
\begin{align*}
\begin{array}{cccc}
\text{\textcircled{1}} & \text{\textcircled{2}} & \text{\textcircled{3}} & \ldots \\
\end{array}
\end{align*}
\]

For a non-planar rooted tree \( T \) and a weight function \( W : v \mapsto W(v) \in \mathbb{N}^* \), we define the weight of \( (T, W) \) by:

\[
|T| = \sum_{v \in v(T)} W(v),
\]

where \( v(T) \) denotes the set of vertices of \( T \). Sometimes we will also use the notation \( |v| \) instead of \( W(v) \).

Example 4. \( \begin{array}{cccc}
\begin{array}{cccc}
\text{\textcircled{1}} & \text{\textcircled{2}} & \text{\textcircled{3}} & \ldots \\
\end{array}
\end{array} \) are the non-planar rooted trees with weight less or equal to 3.

We draw non-planar rooted trees with labels and numbers on their vertices, each number refers to the weight of the vertex.
Definition 4. We define the potential energy of a weighted non-planar rooted tree \((T, W)\) by:

\[
d(T) = \sum_{v \in v(T)} W(v)h(v),
\]

where \(h(v)\) is the height of \(v\) in \(T\), i.e. the distance from \(v\) to the root of \(T\) counting the number of edges.

This notion of potential energy matches the physical intuition: if a branch of a tree is moved down, the potential energy decreases by a multiple of its weight.

For any finite set \(A\), let \(O_A\) be the completed vector space spanned by the non-planar rooted trees with \(|A|\) vertices of any weight, labelled by \(A\). Namely:

\[
O_A := \prod_{W: A \to \mathbb{N}^*} O_{A,W},
\]

where \(O_{A,W}\) is the vector space spanned by the non-planar rooted trees with \(|A|\) vertices, labelled by \(A\) and with weight function \(W\). For any weighted non-planar rooted tree \(S \in O_A\) and any vertex \(v\) of \(S\), \(E(S,v)\) denotes the set of edges of \(S\) arriving at the vertex \(v\) of \(S\). Let \(B\) be another finite set and \(T \in O_B\) another weighted rooted tree with \(|B|\) vertices. Let \(\lambda\) be an element of the field \(K\). We define the partial compositions by:

\[
S \circ_{v,\lambda} T = \begin{cases} 
\sum_{f:E(S,v) \to v(T)} \lambda^{d(S \circ f T) - d(S \circ f^0 T)} S \circ f T & \text{if } |T| = |v| \\
0 & \text{otherwise},
\end{cases}
\]

where \(S \circ f T\) is the element of \(O_{(A-\{v\}) \coprod B}\) obtained by replacing the vertex \(v\) by \(S\) and connecting each edge of \(E(S,v)\) to its image by \(f\) in \(v(T)\). Here \(f_0\) is the map from \(E(S,v)\) to \(v(T)\) which sends each edge \(a\) of \(E(S,v)\) to the root of \(T\). The tree \(S \circ f^0 T\) has therefore the smallest potential energy in the above sum. Unit is given by:

\[
e = \sum_{n \geq 1} \otimes^n,
\]

where \(\otimes\) is the tree with one single vertex of weight \(n\) (this infinite sum makes sense as \(O_1\) is a direct product). The right action of the symmetric groups is given by permutation of the labels.
Example 5. Let us consider
\[ S = (a_1, 1) (b_3, 1) (c_2, 1) (d_1, 2) \] and \[ T = (e_1, 1) \]. Here letters \( a, b, c, \ldots \) are labels of vertices, which are of weight 1, 2 or 3. We have:

\[ S \circ_{b, \lambda} T = \]

\[ f_0 : \begin{cases} c \to e \\ d \to e \end{cases} \]

\[ f : \begin{cases} c \to e \\ d \to h \end{cases} \]

\[ f_0 : \begin{cases} c \to h \\ d \to e \end{cases} \]

\[ f : \begin{cases} c \to h \\ d \to h \end{cases} \]

Theorem 5. The partial compositions defined above \[ \Pi \] yield a structure of \( \mathbb{N}^{+} \)-current-preserving operad on the species \( A \mapsto \Theta_A \), denoted by \( \Theta^\lambda \).

Remark 6. With the notations of Sec 4.2, the current-preserving operad \( \Theta^\lambda \) is naturally associated with NAP for \( \lambda = 0 \) and with Pre-Lie for \( \lambda = 1 \) \[ \Pi \].

5.2. Interpolation between Brace and \( B^0 \). We keep the notations of the sec. 5.1 but we replace non-planar rooted trees by planar rooted trees and \( \Theta \) by \( B \).

Let \( A \) be a finite set. Let \( S \) be a weighted planar rooted tree with \( |A| \) vertices. Let \( B \) another finite set and \( T \) another weighted planar rooted tree with \( |B| \) vertices. Let \( \lambda \) be an element of the field \( K \). We define the partial compositions:

\[ S \circ_{v, \lambda} T = \sum_{f : E(S,v) \to \text{Ang}(T)} \lambda^{d(S \circ_{v}^{f_0} T) - d(S \circ_{v}^{f} T)} S \circ_{v}^{f} T \text{ if } |T| = |v| \]

where \( S \circ_{v}^{f} T \) is the weighted planar rooted tree of \( B_{A-(v)_{\Pi B}} \) obtained by replacing the vertex \( v \) by \( T \) and connecting each edge of \( E(S,v) \) to its image by \( f \) in \( \text{Ang}(T) \). Here, \( f_0 \) is any increasing map from \( E(S,v) \) to \( \text{Ang}^0(T) \). The trees \( S \circ_{v}^{f_0} T \) have, therefore, the smallest potential energy in the above sum.

Remark 7. \( f_0 \) is not unique but the energy \( d(S \circ_{v}^{f_0} T) \) is the same for any \( f_0 : E(S,v) \to \text{Ang}^0(T) \).
Example 6. Let us consider $S = (a,1)$ and $T = (e,1)$. Here letters $a,b,c...$ are labels of vertices, which are of weight 1, 2 or 3. We have:

$$S \circ_{b,\lambda} T =$$

\[
\begin{align*}
&\sum \lambda^{d(S \circ f_{v} T) - d(S \circ g_{w} T)} (S \circ f_{v} T) \circ_{w,\lambda} U \\
&= \sum \lambda^{d(S \circ f_{v} T) - d(S \circ g_{w} T)} (S \circ f_{v} T) \circ_{w,\lambda} U \\
&= \sum \lambda^{d(T \circ_{v} S) - d(T \circ_{w} S)} S \circ_{v,\lambda} (T \circ g_{w} U) \\
&= \sum \lambda^{d(T \circ_{v} S) - d(T \circ_{w} S)} S \circ_{v,\lambda} (T \circ g_{w} U)
\end{align*}
\]

where:

$$A(f,g) = d(S \circ f_{v} T) - d(S \circ g_{w} U) + d((S \circ f_{v} T) \circ g_{w} U) - d((S \circ f_{v} T) \circ g_{w} U).$$

Similarly we have:

$$S \circ_{v,\lambda} (T \circ_{w,\lambda} U) =$$

\[
\begin{align*}
&\sum \lambda^{d(T \circ_{v} S) - d(T \circ_{w} S)} S \circ_{v,\lambda} (T \circ g_{w} U) \\
&= \sum \lambda^{d(T \circ_{v} S) - d(T \circ_{w} S)} S \circ_{v,\lambda} (T \circ g_{w} U) \\
&= \sum \lambda^{d(T \circ_{v} S) - d(T \circ_{w} S)} S \circ_{v,\lambda} (T \circ g_{w} U)
\end{align*}
\]

Theorem 8. The partial compositions defined above yield a structure of $\mathbb{N}^*$-current-preserving operad on the species $A \mapsto \mathcal{B}_{A}$, denoted by $\mathcal{B}^{\lambda}$.

Proof. We prove nested associativity first, and then disjoint associativity.

- Nested associativity:

  Let $S, T, U$ be three weighted planar trees, let $v$ be a vertex of $S$ and $w$ be a vertex of $T$ such that $|T| = |v|$ and $|U| = |w|$.

  Show $(S \circ_{v,\lambda} T) \circ_{w,\lambda} U = S \circ_{v,\lambda} (T \circ_{w,\lambda} U)$ where $v$ is a vertex of $S$ and $w$ a vertex of $T$.

  We have:

  $$(S \circ_{v,\lambda} T) \circ_{w,\lambda} U = \sum \lambda^{d(S \circ f_{v} T) - d(S \circ g_{w} T)} (S \circ f_{v} T) \circ_{w,\lambda} U$$

  $$= \sum \lambda^{d(S \circ f_{v} T) - d(S \circ g_{w} T)} (S \circ f_{v} T) \circ_{w,\lambda} U$$

  $$= \sum \lambda^{d(T \circ_{v} S) - d(T \circ_{w} S)} S \circ_{v,\lambda} (T \circ g_{w} U)$$

  $$= \sum \lambda^{d(T \circ_{v} S) - d(T \circ_{w} S)} S \circ_{v,\lambda} (T \circ g_{w} U)$$

  where:

  $$A(f,g) = d(S \circ f_{v} T) - d(S \circ g_{w} U) + d((S \circ f_{v} T) \circ g_{w} U) - d((S \circ f_{v} T) \circ g_{w} U).$$
where we have set:

\[ B(\tilde{f}, \tilde{g}) = d(T \circ_{\tilde{w}} U) - d(T \circ_{\tilde{w}_0} U) + d(S \circ_{\tilde{v}} (T \circ_{\tilde{w}} U)) - d(S \circ_{\tilde{v}_0} (T \circ_{\tilde{w}} U)). \]

In order to show \((S \circ_{v,\lambda} T) \circ_{w,\lambda} U = S \circ_{v,\lambda} (T \circ_{w,\lambda} U)\), we have to prove the following lemma:

**Lemma 9.** There is a natural bijection \((f, g) \mapsto (\tilde{f}, \tilde{g})\) such that

\[ (S \circ_{\tilde{v}} T) \circ_{\tilde{w}} U = S \circ_{\tilde{v}} (T \circ_{\tilde{w}} U). \]

**Proof.** Let \(v\) be a vertex of \(S\) and \(w\) be a vertex of \(T\) such that \(|T| = |v|\) and \(|U| = |w|\). We denoted by \(\text{Ang}_w(T)\) the set of angles of \(T\) issued from the vertex \(w\). Let \(f : E(S, v) \rightarrow \text{Ang}(T)\) and \(g : E(S \circ_{\tilde{v}} T, w) \rightarrow \text{Ang}(U)\) be a two increasing functions.

We look for \(\tilde{g} : E(T, w) \rightarrow \text{Ang}(U)\) and \(\tilde{f} : E(S, v) \rightarrow \text{Ang}(T \circ_{\tilde{w}} U) = \text{Ang}(U) \cup \text{Ang}(T) \setminus \{\text{Ang}_w(T)\}\) such that the equation (11) is checked.

Let \(e\) be an edge of \(T\) arriving at \(w\), thus \(e\) is an edge of \(S \circ_v T\) arriving at \(w\). We set \(\tilde{g}(e) = g(e)\). Similarly we define \(\tilde{f}\) in a unique way:

\[
\tilde{f} : E(S, v) \rightarrow \text{Ang}(T \circ_{\tilde{w}} U) = \text{Ang}(U) \cup \text{Ang}(T) \setminus \{\text{Ang}_w(T)\}
\]

\[
e \mapsto \tilde{f}(e) = \begin{cases} 
  f(e) & \text{if } f(e) \notin \text{Ang}_w(T) \\
  g(e) & \text{if } f(e) \in \text{Ang}_w(T)
\end{cases}
\]

Conversely, we assume that we have the pair \((\tilde{f}, \tilde{g})\) and look for the pair \((f, g)\) such that equation (11) is verified. We have \(\tilde{f} : E(S, v) \rightarrow \text{Ang}(U) \cup \text{Ang}(T) \setminus \{\text{Ang}_w(T)\}\) and \(\tilde{g} : E(T, w) \rightarrow \text{Ang}(U)\). We then define:

\[
f : E(S, v) \rightarrow \text{Ang}(T)
\]

\[
e \mapsto f(e) = \begin{cases} 
  \tilde{f}(e) & \text{if } \tilde{f}(e) \notin \text{Ang}(U) \\
  w & \text{if } \tilde{f}(e) \in \text{Ang}(U)
\end{cases}
\]

and

\[
g : E(S \circ_{\tilde{v}} T, w) \rightarrow \text{Ang}(U)
\]

\[
e \mapsto g(e) = \begin{cases} 
  \tilde{g}(e) & \text{if } e \in T \\
  \tilde{f}(e) & \text{if } e \in S \text{ and } f(e) \in \text{Ang}_w(T)
\end{cases}
\]

Proof of Theorem 5 (continued): To show \((S \circ_{v,\lambda} T) \circ_{w,\lambda} U = S \circ_{v,\lambda} (T \circ_{w,\lambda} U)\), it remains to show the equality \(A(f, g) = B(\tilde{f}, \tilde{g})\). We set:

\[
A'(f, g) = A(f, g) + d((S \circ_{\tilde{v}} T) \circ_{\tilde{w}} U),
\]

\[
B'(\tilde{f}, \tilde{g}) = B(\tilde{f}, \tilde{g}) + d(S \circ_{\tilde{v}} (T \circ_{\tilde{w}} U)),
\]

\[
e(f) = d(S \circ_{\tilde{v}} T) - d(S \circ_{\tilde{v}_0} T),
\]

\[
e(\tilde{g}) = d(T \circ_{\tilde{w}} U) - d(T \circ_{\tilde{w}_0} U).
\]

We have:

\[\epsilon(f) = \sum_{e \in E(S, v)} h(f(e)) \cdot |B_e|,\]
where \( h(f(e)) \) is the distance between \( f(e) \) and the root of \( T \) in the new tree \( S \circ_v^f T \) and \(|B_e|\) is the weight of the branch above \( e \). Similarly:

\[
d((S \circ_v^f T) \circ_w^{g_0} U) - d((S \circ_v^f T) \circ_w^{g_0} U) = \sum_{e \in E(S,v)} h(f(e)) |B_e| = \epsilon(f).
\]

Here \( g_0 \) is not involved because it was connected to the root of \( U \), so \( A'(f, g) = d((S \circ_v^f T) \circ_w^{g_0} U) \). By the same computation with \( \tilde{g} \) instead of \( f \) we show \( B'(\tilde{f}, \tilde{g}) = d(S \circ_v^{\tilde{f}} (T \circ_w^{\tilde{g}} U)) \). So by Lemma [9] we have \( A'(f, g) = B'(\tilde{f}, \tilde{g}) \) that is to say \( A(f, g) + d((S \circ_v^{\tilde{f}} T) \circ_w^{g_0} U) = B(\tilde{f}, \tilde{g}) + d(S \circ_v^{\tilde{f}} (T \circ_w^{g_0} U)) \), which proves that \( A(f, g) = B(\tilde{f}, \tilde{g}) \) because by Lemma [9] we have \( d((S \circ_v^{\tilde{f}} T) \circ_w^{g_0} U) = d(S \circ_v^{\tilde{f}} (T \circ_w^{g_0} U)) \).

- **Disjoint associativity:** let \( v, w \) be two disjoint vertices of \( S \) such that \(|v| = |T|\) and \(|w| = |U|\), show that:

\[
(S \circ_v^{\lambda} T) \circ_w^{\lambda} U = (S \circ_w^{\nu} U) \circ_v^{\lambda} T.
\]

We have

\[
(S \circ_v^{\lambda} T) \circ_w^{\lambda} U = \sum_{f: E(S,v) \rightarrow \text{Ang}(T)} \chi^{d(S \circ_v^{\nu} T) - d(S \circ_v^{\nu} T)} (S \circ_v^{\nu} T) \circ_w^{\lambda} U
\]

\[
= \sum_{f: E(S,v) \rightarrow \text{Ang}(T)} \left( \sum_{g: E(S,\tilde{v}) \rightarrow \text{Ang}(T)} \chi^{d(S \circ_v^{\nu} T) - d(S \circ_v^{\nu} T)} (S \circ_v^{\nu} T) \circ_w^{g_0} U \right)
\]

\[
= \sum_{f: E(S,v) \rightarrow \text{Ang}(T)} \left( \sum_{g: E(S,\tilde{v}) \rightarrow \text{Ang}(T)} \chi^{C(f, g)} (S \circ_v^{\nu} T) \circ_w^{g_0} U \right),
\]

where:

\[
k(f) = d(S \circ_v^{\nu} T) - d(S \circ_v^{\nu} T),
\]

\[
C(f, g) = k(f) + d((S \circ_v^{\nu} T) \circ_w^{g_0} U) - d((S \circ_v^{\nu} T) \circ_w^{g_0} U).
\]

Similarly we find:

\[
(S \circ_w^{\nu} U) \circ_v^{\lambda} T = \sum_{\tilde{g}: E(S,w) \rightarrow \text{Ang}(U)} \sum_{f: E(S,\tilde{v}, U) \rightarrow \text{Ang}(T)} \chi^{D(\tilde{f}, \tilde{g})} (S \circ_w^{\tilde{g}} U) \circ_v^{\tilde{f}} T,
\]

where:

\[
D(\tilde{f}, \tilde{g}) = k(\tilde{g}) + d((S \circ_w^{\tilde{g}} U) \circ_v^{\tilde{f}} T) - d((S \circ_w^{\tilde{g}} U) \circ_v^{\tilde{f}} T),
\]

\[
k(\tilde{g}) = d(S \circ_w^{\tilde{g}} U) - d(S \circ_w^{\tilde{g}} U).
\]

In order to prove (13), we need the following lemma:

**Lemma 10.** We have a natural bijection \((f, g) \mapsto (\tilde{f}, \tilde{g})\) such that

\[
(S \circ_v^{f} T) \circ_w^{g_0} U = (S \circ_v^{\tilde{f}} U) \circ_v^{\tilde{g}} T.
\]

**Proof.** Let \( f: E(S,v) \rightarrow \text{Ang}(T) \) and \( g: E(S,\tilde{v}, T, w) \rightarrow \text{Ang}(U) \). We look for \( \tilde{g}: E(S,\tilde{v}, w) \rightarrow \text{Ang}(U) \) and \( \tilde{f}: E(S,\tilde{v}, U, v) \rightarrow \text{Ang}(T) \) such that the Equation (14) is verified. Let \( \tilde{g} \) be the restriction of \( g \) on the edges \( e \) from \( S \) and \( \tilde{f} = f \). Here \( E(S,\tilde{v}, U, v) = E(S, v) \) because the vertices \( v \) and \( w \) are disjoint. \( \square \)
Proof of Theorem 5 (end): Thus to show disjoint associativity, it remains to show that for any pair \((f, g)\) and \((\tilde{f}, \tilde{g})\) we have \(C(f, g) = D(\tilde{f}, \tilde{g})\). We set \(C'(f, g) = C(f, g) + d((S \circ_{v_0} T) \circ_{w_0} U)\) and \(D'(\tilde{f}, \tilde{g}) = D(\tilde{f}, \tilde{g}) + d((S \circ_{w_0} U) \circ_{v_0} T)\). We have:

\[
\begin{align*}
k(\tilde{g}) &= d(S \circ_{w_0} U) - d(S \circ_{w_0} U) - d((S \circ_{w_0} U) \circ_{v_0} T) = \sum_{e \in E(S, w)} h(\tilde{g}(e)).|B_e|,
\end{align*}
\]

where \(h(\tilde{g}(e))\) is the distance between the root of \(U\) and \(\tilde{g}(e)\) in the new tree \(S \circ_{w_0} U\). We also have:

\[
d((S \circ_{w_0} U) \circ_{v_0} T) - d((S \circ_{w_0} U) \circ_{v_0} T) = k(\tilde{g})
\]

because we changed the vertex \(v\) by a tree of the same weight, and because \(\tilde{f}_0\) is defined by grafting onto the root. This proves:

\[
D'(\tilde{f}, \tilde{g}) = d((s \circ_{w_0} U) \circ_{v} T).
\]

By Lemma 10 again, \(C'(f, g) = D'(\tilde{f}, \tilde{g})\) and \(C(f, g) = D(\tilde{f}, \tilde{g})\), which proves disjoint associativity.

The partial compositions defined on \(O^\lambda\) hence verify the axioms of a current-preserving operad.

Remark 11. With the notations of Sec 4.2, the current-preserving operad \(B^\lambda\) is naturally associated with \((B^0)^{\mathbb{N}^*}\) for \(\lambda = 0\) and with \(\text{Brace}^{\mathbb{N}^*}\) for \(\lambda = 1\).

Theorem 12. The map \(\varphi\) introduced in Definition 1 is a morphism of current-preserving operad from \(O^\lambda\) to \(B^\lambda\) i.e:

\[
\varphi(S \circ_{v, \lambda} T) = \varphi(S) \circ_{v, \lambda} \varphi(T).
\]

Proof. By definition of partial compositions on planar trees and non-planar trees, and as the shuffle product permute the branches in all possible ways, then we verify that \(\varphi\) is a current-preserving operad morphism. The key point is the following: for any \(f : E(S, v) \to v(T)\), choosing a planar representative of \(S \circ_{v_0} T\) amounts to choosing planar representatives \(\tilde{S}\) and \(\tilde{T}\) of \(S, T\) respectively, together with an increasing map \(\tilde{f} : E(\tilde{S}, v) \to \text{Ang}(\tilde{T})\) above \(f\), i.e such that the following diagram commutes:

\[
\begin{array}{ccc}
E(\tilde{S}, v) & \xrightarrow{\tilde{f}} & \text{Ang}(\tilde{T}) \\
\downarrow{f} & & \downarrow{v(T)} \\
\end{array}
\]

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