STABLE COHOMOLOGY OF DISCRIMINANT COMPLEMENTS
FOR AN ALGEBRAIC CURVE

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Abstract. Let \( L \) be a degree \( n \geq 1 \) line bundle on a smooth projective complex algebraic curve \( X \). Let \( U(L) \) (resp. \( U(\mathcal{L}) \)) denote the set of algebraic (resp. \( C^\infty \)) sections of \( L \). We show that:

1. The inclusion \( U(L) \to U(\mathcal{L}) \) induces an isomorphism between \( H^*(U(L); \mathbb{Z}) \) and \( H^*(U(\mathcal{L}); \mathbb{Z}) \) for \(* < n\).
2. \( U(L) \) is aspherical.
3. \( \pi_1(U(L)) \) is a group closely related to the surface braid group \( Br_n(X) \).

1. Introduction

In this paper we are concerned with understanding the difference between spaces of algebraic and \( C^\infty \) sections of complex line bundles on a smooth projective algebraic curve \( X \) over \( \mathbb{C} \). We prove that these spaces have isomorphic cohomology in a range of degrees that grows with the degree of the line bundle.

Let \( X \) be a smooth projective algebraic curve over \( \mathbb{C} \) of genus \( g \). Let \( L \) be an algebraic line bundle on \( X \) of degree \( n \).

Let \( C^\infty(X, L) \) be the vector space of smooth sections of \( L \). Given \( s \in C^\infty(X, L) \), we say \( p \in X \) is a regular zero of \( s \) if \( s(p) = 0 \) and \( s'(p) \neq 0 \). If \( s \in C^\infty(X, L) \) is an algebraic section, then being a regular zero is equivalent to having index 1. Let \( U(L) = \{ s \in C^\infty(X, L) \mid \text{all zeroes of } s \text{ are isolated and of index 1} \} \).

Let \( U(L) := \{ s \in H^0(X, L) \mid \text{all zeroes of } s \text{ are regular} \} \).

There is a natural inclusion map \( i : U(\mathcal{L}) \to U(L) \). The aim of the present paper is to understand this inclusion at the level of cohomology. Our main theorem is as follows:

Theorem 1.1. Let \( X \) be a smooth projective complex algebraic curve of genus \( g \). Let \( n \geq 0 \). Let \( L \) be an algebraic line bundle of degree \( n \) on \( X \). Let \( i : U(\mathcal{L}) \to U(L) \) be the inclusion map. Then for all \( 0 \leq 2k \leq n - 2g \),

\[
i^* : H^k(U(L); \mathbb{Z}) \to H^k(U(\mathcal{L}); \mathbb{Z})
\]

is an isomorphism.

We also provide some qualitative understanding of the topology of the space \( U(\mathcal{L}) \) and relate it with more classical objects. Let \( n = \deg(L) \). Given a space \( M \), define

\[
P\text{Conf}_n M := \{ (x_1, \ldots, x_n) \in M^n \mid x_i \neq x_j \}.
\]

The permutation action of \( S_n \) on \( M^n \) restricts to an action on \( P\text{Conf}_n M \). Let

\[
U\text{Conf}_n M = P\text{conf}_n M/S_n
\]
be the unordered configuration space of \( n \) points on \( M \).

Define the \( n \) stranded surface braid group on a surface \( X \) as
\[
Br_n(X) := \pi_1(U\text{Conf}_n(X)).
\]
For our purposes we will need to define a group \( Br_n(X) \) that we call the extended surface braid group. This will be defined later on in Section 3 as \( \pi_1(U_n^{alg}) \) where \( U_n^{alg} \) is a space defined in Section 3 that is a \( \mathbb{C}^* \) bundle over \( U\text{Conf}_n X \).

Let \( \pi : \tilde{Br}_n(X) \to Br_n(X) \) be the projection. Now, \( U\text{Conf}_n(X) \subseteq \text{Sym}^n(X) \) has an Abel Jacobi map to \( \text{Pic}^n(X) \). This induces a map \( \alpha : Br_n(X) \to \mathbb{Z}^{2g} \). Let \( \mathcal{A} = \alpha \circ \pi \). Let \( K_n \subseteq Br_n(X) \) be the kernel of this map.

**Theorem 1.2.** Let \( n \geq 1 \). Let \( X \) be a smooth projective curve. Let \( \mathcal{L} \) be a line bundle of degree \( n \) on \( X \). The space \( U(\mathcal{L}) \) is a \( K(\pi,1) \). Furthermore,
\[
\pi_1(U(\mathcal{L})) \cong K_n.
\]

**Motivation** In [7] Vakil and Wood consider (among other things) the 'stable class' of the discriminant locus in the Grothendieck group of varieties \( K_0(\text{Var}) \). Let us recall the definition of the Grothendieck group of varieties. Let us fix a base field \( k \). Then we can consider the set
\[
\text{Var}_k = \{ X : X \text{is a variety over } k \}/\text{isomorphism}.
\]
We can form a monoid \( M \) out of \( \text{Var}_k \) as follows: let \( M \) be generated by elements of \( \text{Var}_k \), with the relation, if \( Y \subseteq X \), \([X] = [X - Y] + [Y] \) \( \in M \). The Grothendieck group \( K_0(\text{Var}_k) \) is the group completion of \( M \). It has a ring structure coming from the product of varieties. In the literature, the element \( \mathbb{A}^1 \) (often denoted \( \mathbb{L} \)) is sometimes inverted. Define \( \mathcal{M}_k = K_0(\text{Var}_k)[\frac{1}{\mathbb{L}}] \).

Consider a smooth variety \( X \) along with an ample line bundle \( \mathcal{L} \) on it.

**Theorem 1.3** (Vakil - Wood [7]). Let \( j \geq 1 \). Let \( U(\mathcal{L}^\otimes j) \) be the (open) variety of sections with smooth zero locus. Let \( \zeta_X \) be the Kapranov motivic zeta function, and let \( d \) be the dimension of \( X \). Then,
\[
\lim_{j \to \infty} \frac{[U(\mathcal{L}^j)]}{[H^0(X, \mathcal{L})]} = \frac{1}{\zeta_X(d+1)}.
\]
Here the limit is with respect to the dimension filtration in \( \mathcal{M}_k \).

While Theorem [13] seems to have nothing to do with the cohomology of the space \( U(\mathcal{L}^j) \), there is a specialisation map
\[
K_0(\text{Var}_k) \to \{ \text{Weighted Euler characteristics} \}.
\]
Thus, Theorem 1.3 implies that there is a stabilisation of Euler characteristics and one can hope for a stabilisation in cohomology as well.

In [6], Tommasi proves a cohomological result in the same vein as that of this paper, where she studies discriminant complements on \( \mathbb{P}^n \). Her set up is as follows. Let \( d, n \geq 1 \). Let \( X = \mathbb{P}^n \). Let \( \mathcal{L} = \mathcal{O}(d) \). Let
\[
U(\mathcal{L}) = \{ f \in H^0(X, \mathcal{L}) | f \text{ has only regular zeroes} \}.
\]

Then the main theorem of [6] stated in our notation is as follows:

**Theorem 1.4** (Tommasi [6]). Let \( d, n \geq 1 \). Let \( X = \mathbb{P}^n \), \( \mathcal{L} = \mathcal{O}(d) \). Let \( 0 \leq k \leq \frac{d+1}{2} \). Then
\[
H^k(U(\mathcal{L}); \mathbb{Q}) \cong H^k(GL_{n+1}(\mathbb{C}); \mathbb{Q}).
\]
Our motivation for the present paper was to understand if there are stability phenomena for discriminant complements over general varieties and whether cohomology in the stable range is dependent only on the topology of the variety. Theorem 1.1 shows that at least in the case of an algebraic curve there is some kind of stability phenomenon with cohomology in the stable range being purely topological in nature. We are currently working on extending these results to more general varieties.

**Relation to other work:** Orsola Tommasi has announced some results on homological stability for discriminant complements over arbitrary smooth projective varieties. We believe that the results in this paper are substantially different from hers. We focus on relating discriminant complements to spaces of $C^\infty$ sections, which is not the focus of her results.

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### 2. Smooth sections

The space $\mathcal{U}(\mathcal{L})$ is actually easy to understand topologically. There is a fibration $\pi: \mathcal{U}(\mathcal{L}) \to \mathcal{U}\text{Conf}_n X$ defined by: $\pi(f) = \{a|f(a) = 0\}$. We shall need to understand the fibres $\pi^{-1}\{a_1, \ldots, a_n\} \subseteq \mathcal{U}(\mathcal{L})$, but first we shall introduce some basic objects and prove some more technical lemmas. Let $Y, Z$ be based spaces. Let $C(Y, Z), C^*(Y, Z)$ denote the space of continuous maps from $Y$ to $Z$ and the space of based continuous maps from $Y$ to $Z$. Let $X$ be a $C^\infty$ manifold. Let $G = C^\infty(X, \mathbb{C}^*)$. For $p \in X$, let

$$G_p = \{f \in G|f(p) = 1\}.$$

Before we begin with stating and proving the propositions in this section, we note that they are mostly applications of the fact that $\mathbb{C}^*$ is a $K(\pi, 1)$ space and is covered by a contractible space. The space of continuous based maps into a $K(\pi, 1)$ has been classically studied first by Thom and then by many others. Lemma 2.4 is a bit more specific to our situation and is not an immediate application of the theory of $K(\pi, 1)$ spaces.

**Proposition 2.1.** with the above notation,

1. $G_p$ is weak homotopy equivalent to $H^1(X, Z)$.
2. $G$ is weak homotopy equivalent to $H^1(X, Z) \times \mathbb{C}^*$.

**Proof.** The space $\mathbb{C}^*$ is a $K(\pi, 1)$ and by the Proposition labelled Thom [4] in $C_* (X, \mathbb{C}^*)$ is homotopy equivalent to $H^1(X, Z)$ (i.e. each of its components is contractible and the set of components is in natural bijection with $H^1(X, Z)$). By Theorem 1.5 of [4], $C_* (X, \mathbb{C}^*)$ is weak homotopy equivalent to $C^\infty(X, \mathbb{C}^*)$. This establishes (1). To establish (2) we note that $G$ is homeomorphic to $G_p \times \mathbb{C}^*$, indeed an explicit homeomorphism is given $(f, \alpha) \in G_p \times \mathbb{C}^* \mapsto \alpha f \in G$.

**Proposition 2.2.** Let $D = \{z \in \mathbb{C}||z| \leq 1\}$. Let

$$S = \{f \in C^\infty(D - \{0\}, \mathbb{C}^*)|f(z) = 1 \text{ for } z \in S^1\}.$$

Then $S$ is contractible.
Furthermore, if $g.s$ Then $g.s$ By Lemma 2.3, there is a map $j$ (this is analogous to the proof of Proposition 2.2). There is an inclusion map between $\tilde{S}$ and $S'$ be defined by $\phi(f) = f$.

This implies that $S$ is contractible.

Lemma 2.3. Let $D$ be the closed unit disk in $\mathbb{C}$. Let
\[ F = \{ f \in C^\infty(D - \{0\}, \mathbb{C}^*) | f|_{\partial D} = 1 \} \]
Let
\[ \tilde{F} = \{ f \in C^\infty(D - \{0\}, \mathbb{C}^*) | f \text{ is nullhomotopic and } f(1) = 1 \} \]
Then $\tilde{F}$ deformation retracts to the point $f_0$, where $f_0(x) = 1$ for all $x$. Furthermore, the deformation retraction preserves the subset $F$.

Proof. Given $f \in \tilde{F}$ we can lift it to a unique map $\tilde{f} : D - \{0\} \to \mathbb{C}$, such that $\exp(0) = 1$. The straight line homotopy in $\mathbb{C}$ defines a homotopy between $\tilde{f}$ and the constant function. This in turn defines a homotopy $h$ between $f$ and $f_0$. This gives us our deformation retraction. It is easy to see that this preserves $F$. 

Lemma 2.4. Recall that $\mathfrak{G} = C^\infty(X, \mathbb{C}^*)$. Let $n \geq 1$. Let $\{a_1, \ldots, a_n\} \in \text{UConf}_n X$. Then,
1. There is a free action of $\mathfrak{G}$ (as a group under multiplication) on $\pi^{-1}(\{a_1, \ldots, a_n\})$.
2. The quotient $\pi^{-1}(\{a_1, \ldots, a_n\})/\mathfrak{G}$ is contractible.

Proof. We define our action as follows: if $s \in \pi^{-1}(\{a_1, \ldots, a_n\})$ and $g \in \mathfrak{G}$, define $g.s(x) = g(x)s(x)$. If $g.s(x) = 0$ then $s(x) = 0$ as $g(x) \neq 0$ for all $x \in X$. Furthermore, if $g.s = s$ then $g(x) = 1$ for $x \in X - \{a_1, \ldots, a_n\}$ and since $X - \{a_1, \ldots, a_n\}$ is dense, $g(x) = 1$ for all $x \in X$. This concludes the proof of (1).

Let $D_i$ be small disks surrounding the points $a_i$. Let $G_i = \{ f \in C^\infty(D_i, \mathbb{C}^*) | f|_{\partial D_i} = 1 \}$. We can identify each $G_i$ with the set of based maps from $S^2$ to $\mathbb{C}^*$. Since $\pi_1(S^2) = 0$, any based map $f : S^2 \to \mathbb{C}^*$ lifts to a unique map $\tilde{f} : S^2 \to \mathbb{C}$. Hence $G_i$ is homeomorphic to the space of based maps from $S^2$ to $\mathbb{C}$ and $\mathfrak{G}$ is contractible, $G_i$ is contractible.

Let $F_i = \{ f \in C^\infty(D_i - a_i, \mathbb{C}^*) : f|_{\partial D_i} = 1 \}$. By Proposition 2.2 $F_i$ is contractible. Let
\[ \tilde{F}_i = \{ f \in C(D_i - a_i, \mathbb{C}^*) | f|_{\partial D_i} \text{ is nullhomotopic} \} \]
Let
\[ \tilde{G}_i = \{ f \in C(D_i, \mathbb{C}^*) \} \]
Since $D_i$ is contractible, the space $\tilde{G}_i \simeq \mathbb{C}^*$ and the quotient $\tilde{F}_i/\tilde{G}_i$ is contractible (this is analogous to the proof of Proposition 2.2). There is an inclusion map $i : \tilde{F}_i/\tilde{G}_i \to \tilde{F}_i/\tilde{G}_i$ which is a homotopy equivalence as both spaces are contractible.

By Lemma 2.3 there is a map $j : \tilde{F}_i/\tilde{G}_i \to F_i/\tilde{G}_i$ satisfying the following properties.
1. There exists a homotopy $h : \tilde{F}_i \times [0, 1] \to \tilde{F}_i$ such that $h(f, 1) = f$, $h(f, 0) = j(f)$ and $h(f, t)|_{U_i} = f|_{U_i}$.
Proof. There is a fibration $p: \pi^{-1}((a_1, \ldots, a_n))/\mathcal{G} \to \text{Conf}_n \times X$. Let $A = \pi^{-1}((a_1, \ldots, a_n))/\mathcal{G}$. Fix an $s_0 \in \pi^{-1}((a_1, \ldots, a_n))$. Let $f : A \to \prod_{i=1}^n \tilde{F}_i/G_i$ be defined as follows:

$$\phi(s) = (s/s_0|_{D_1-a_1}, \ldots, s/s_0|_{D_n-a_n}).$$

We claim that $\phi$ is a homotopy equivalence. To prove this we first define $\psi : \prod_{i=1}^n F_i/G_i \to A$ as follows:

$$\psi(f_1, \ldots, f_n)(x) = \begin{cases} f_i(x)s_0(x) & \text{if } x \in D_i \\ s_0(x) & \text{otherwise.} \end{cases}$$

It is easy to see that $\psi \circ j \circ \phi \simeq \text{Id}$ (the homotopy $h$ mentioned above can be seen to define such a homotopy). Since $\prod_{i=1}^n F_i/G_i$ is contractible this implies (2). □

Remark: The action of $\mathcal{G}$ on $\pi^{-1}((a_1, \ldots, a_n))$ is in fact not transitive for any value of $n \geq 1$. The following example will illustrate this fact. Let $D$ be the closed unit disk in $\mathbb{C}$ which we identify with $\mathbb{R}^2$. Let $f : D \to \mathbb{C}$ be defined as $f(x, y) = (x, y)$. Let $g : g(x, y) = (2x, y)$. Let $E \to \mathbb{P}^1$ be the unique degree 1 line bundle on $\mathbb{P}^1$. Let $\phi : E|D \to D \times \mathbb{C}$ be a trivialisation. Let $f, \tilde{g}$ be $C^\infty$ sections on $\mathbb{P}^1$ of $E$ such that for $(x, y) \in D$, $\phi(f(x, y)) = ((x, y), f(x, y))$ and $\phi(\tilde{g}(x, y)) = ((x, y), g(x, y))$ (it follows from a standard obstruction theoretic argument that indeed exist such $f$ and $\tilde{g}$). If there exists $\tilde{h} \in C^\infty(X, \mathbb{C}^*)$ such that $\tilde{h}\tilde{f} = \tilde{g}$, then $\tilde{h}(0, 0) = \lim_{(x,y)\to(0,0)} \frac{a(x,y)}{f(x,y)}$. But this limit does not exist and hence $f$ and $\tilde{g}$ are not in the same $\mathcal{G}$ orbit.

Corollary 2.5. Let $n \geq 0$. Let $X$ be a smooth projective curve and $L$ a line bundle on it of degree $n$. Then $\text{U}(L)$ is a $K(\pi, 1)$.

Proof. There is a fibration

$$\pi^{-1}(\text{U}(L)) \to \text{U}(L) \to \text{UConf}_n X$$

Lemma 2.4 implies that $\pi^{-1}(\text{U}(L)) \simeq \mathcal{G}$. The space $\mathcal{G}$ is a $K(\pi, 1)$ by Proposition 2.4. Since $\text{U}(L)$ is the total space in a fibration with both base and fibre $K(\pi, 1)$ spaces is itself a $K(\pi, 1)$. □

3. Abel–Jacobi

In this section we will try to understand the space $U(L)$. Our method to understand the topology of $U(L)$ is by making it a subspace of a space $U_n$ which we shall construct.

We would like to remind the reader that to give a complex line bundle $L$, a holomorphic structure $h$ is equivalent to giving a Dolbeault operator $\partial_h : \Gamma(L) \to \Omega^{0,1} \otimes \Gamma(L)$. More details on Dolbeault operators and holomorphic structures may be found in Ch.3 of [H]. Let $\mathcal{H}_n$ be the space of holomorphic structures on $L$. The group $\mathcal{G}_p$ acts on $\mathcal{H}_n$ with trivial stabilizers. The quotient $\mathcal{H}_n/\mathcal{G}_p$ is naturally isomorphic to $\text{Pic}_n X$.
Let
\[ \mathcal{U}_n = \{(s, h) \in U(\mathcal{L}) \times \mathcal{H}_n | s \text{ is a algebraic section of } \mathcal{L} \text{ with respect to } h \}. \]
Note that the groups \( \mathfrak{G} \) and \( \mathfrak{G}_p \) act on this space \( \mathcal{U}_n \).

Let \( U^\text{alg}_n := \mathcal{U}_n / \mathfrak{G}_p \). There is a surjective map \( \pi : \mathcal{U}_n \to \mathcal{H}_n \) defined by \( \pi(s, h) = h \). Since \( \pi \) is equivariant with respect to the action of \( \mathfrak{G}_p \), it descends to a surjection
\[ \mathcal{A} : U^\text{alg}_n = \mathcal{U}_n / \mathfrak{G}_p \to \mathcal{H}_n / \mathfrak{G}_p = \text{Pic}_n X. \]
We observe that for \( \mathcal{L} \in \text{Pic}_n X \), we have the equality \( \mathcal{A}^{-1}(\mathcal{L}) = U(\mathcal{L}) \). This map \( \mathcal{A} \) can be seen as a section level version of the Abel-Jacobi map. We now wish to understand the topology of \( U^\text{alg}_n \).

**Proposition 3.1.** Let \( n \geq 1 \).

1. \( U^\text{alg}_n \) is a \( K(\pi, 1) \).
2. There is a short exact sequence
\[ 1 \to \mathbb{Z} \to \pi_1(U^\text{alg}_n) \to \text{Br}_n(X) \to 1. \]

**Proof.** There is a fibration \( \pi : U^\text{alg}_n \to \text{UConf}_n X \) defined by
\[ \pi(s, h) = \{a \in X | s(a) = 0\}. \]
If \( a = \{a_1 \ldots a_n\} \in \text{UConf}_n X \), then \( \pi^{-1}(a) \cong \mathbb{C}^* \), as algebraic sections of a line bundle are uniquely identified with their zeroes up to a scalar. Since \( \mathbb{C}^* \) and \( \text{UConf}_n X \) are \( K(\pi, 1) \) spaces, so is \( U^\text{alg}_n \). \( \square \)

### 3.1. An alternative definition of \( U^\text{alg}_n \)
In this subsection we will give an alternative definition of \( U^\text{alg}_n \). This will not be used in the rest of the paper.

Let \( X \) be a smooth projective curve of genus \( g \). Let \( n > g \). Let \( \text{Sym}^n X \) be the \( n \)th symmetric power of \( X \). Let \( \mathcal{P} \) denote the Poincare line bundle on \( X \times \text{Pic}_n X \), this is the unique line bundle on \( X \times \text{Pic}_n X \) such that \( \mathcal{P}|_{X \times \{p\}} = \mathcal{L} \) and \( \mathcal{P}|_{\{p\} \times \text{Pic}_n X} \). Let \( \pi : X \times \text{Pic}_n X \to \text{Pic}_n X \) denote the projection. The pushforward \( \pi_* \mathcal{P} \) defines a vector bundle \( E \) on \( \text{Pic}_n X \), sometimes called the Picard bundle. Let \( E_0 \subseteq E \) denote the zero section. We may identify \( \text{Sym}^n X \) with \( E - E_0 / \mathbb{C}^* \), i.e. \( \text{Sym}^n X \) is the projective space bundle associated to the vector bundle \( E \). Let \( \rho : E - E_0 \to \text{Sym}^n X \) denote the projection map. We then define \( U^\text{alg}_n \) to be \( \rho^{-1} (\text{UConf}_n X) \).

Let us emphasize that \( E - E_0 \to \text{Sym}^n X \) is not a trivial \( \mathbb{C}^* \) bundle. Indeed after restricting to a fibre of the projection \( \text{Sym}^n X \to \text{Pic}_n X \) the bundle \( \rho \) restricts to the bundle \( \mathbb{C}^{n-g+1} - \{0\} \to \mathbb{P}^{n-g} \) which is classically known to be non trivial. While it is possible that the bundle \( U^\text{alg}_n \to \text{UConf}_n X \) is a trivial \( \mathbb{C}^* \) bundle, we are unable to determine whether this is the case.

### 4. Comparing different fibres
To understand \( U(\mathcal{L}) \) we will analyze the map \( \mathcal{A} : U^\text{alg}_n \to \text{Pic}_n X \). We will prove that the map \( \mathcal{A} \) is similar to a homology fibration. More precisely, we have the following.

**Theorem 4.1.** Let \( n \geq 0 \). Let \( X \) be a smooth projective complex algebraic curve of genus \( g \), \( \mathcal{L} \) a line bundle of degree \( n \) on \( X \). Let \( 2k \leq n - g \). Let \( \mathcal{A} \) be the map
defined in Section 3. Let $W \subseteq \text{Pic}_n(X)$ be a small contractible neighbourhood of $\mathcal{L}$ homeomorphic to a ball. Let

$$i : \mathcal{A}^{-1}(\mathcal{L}) \to \mathcal{A}^{-1}(W)$$

be the inclusion map. Then

$$i^* : H^k(\mathcal{A}^{-1}(W); \mathbb{Z}) \to H^k(\mathcal{A}^{-1}(\mathcal{L}); \mathbb{Z})$$

is an isomorphism.

Before embarking on the proof of Theorem 3.2, we will need to set up some machinery.

There is a vector bundle $\pi : H^0(X, W) \to W$ defined as follows. Let

$$H^0(X, W) = \{(s, \mathcal{L}) | \mathcal{L} \in W, s \in H^0(X, \mathcal{L})\}.$$ 

Then $\mathcal{A}^{-1}(W)$ is an open subset of $H^0(X, W)$. The topology of the complement $\Sigma_W = H^0(X, W) - \mathcal{A}^{-1}(W)$ will be important for us to understand. It is immediate that $\Sigma_W = \{(f, \mathcal{L}) | \mathcal{L} \in W, f \in \Sigma(\mathcal{L})\}$, since for any $\mathcal{L} \in W$ $\mathcal{A}^{-1}(\mathcal{L})$ consists of all sections of $\mathcal{L}$ with regular zeroes.

We will create a relative stratification of $\Sigma^{-1}(W)$. This is similar to the stratification in Section 3. Let

$$\Sigma_{\mathcal{L}}^k = \{f \in \Sigma_{\mathcal{L}} | |\text{Sing}(f)| \geq k\}.$$ 

Let $N = \frac{d + n}{2}$. We stratify $\Sigma_{\mathcal{L}}$, the complement of $\mathcal{A}^{-1}(\mathcal{L})$ in $H^0(X, \mathcal{L})$ by

$$\Sigma_{\mathcal{L}}^k = \mathbb{P}\{f \in V | f \text{ has at least } k \text{ distinct singular zeroes}\},$$

for $k \leq N$. So $\Sigma_{\mathcal{L}}^1 \supset \Sigma_{\mathcal{L}}^2 \supset \ldots$.

Now we construct a cubical space $C$ that will be involved in understanding $\Sigma(\mathcal{L})$.

Let $N = \frac{d - 1}{2}$. Let $I$ be a subset of $\{1, \ldots, N - 1\}$. Let $I = \{i_1, \ldots, i_k\}$ let

$$C_I := \{(f, x_1, \ldots, x_k) | f \in \Sigma(\mathcal{L}), x_j \in U\text{Conf}_{i_j}(X) x_1 \subseteq x_2, \ldots, x_k \subseteq \text{Singular zeroes of } f\}.$$ 

We define

$$C_{I \cup (N)} := \{(f, x_1, \ldots, x_k) \in C_I | f \in \Sigma_{\mathcal{L}}^{\geq N}\}.$$ 

If $I \subseteq J$ then we have a natural map from $C_J \to C_I$ defined by restricting $p$. This gives $C$ the structure of a cubical space over the set $\{1, \ldots, N\}$. We can take the geometric realization of $C$ denoted by $|C|$. Then there is a map $\rho : |C| \to \Sigma(\mathcal{L})$, induced by the forgetful maps $C_I \to \Sigma(\mathcal{L})$.

$|C|$ is topologized in a non-standard way. The topology we give is analogous to the topology on $\mathcal{L}$ in Section 3. The primary reason we give $|C|$ this topology is to make $\rho$ proper. For $k < N$, there is an inclusion

$$i : U\text{Conf}_k(X) \to Gr(h^0(X, \mathcal{L}) - 2k, H^0(X, \mathcal{L})).$$

We define $L_k(\mathcal{L})$ to be the Zariski closure of the image. We will omit the $\mathcal{L}$ in our notation if there is only one line bundle that we are discussing. There is a relation, $<$ on the collection of all $L_k$, defined by $\lambda_1 < \lambda_2$ if as subspaces of $H^0(X, \mathcal{L})$, $\lambda_2 \subseteq \lambda_1$. Note that this extends the relation $\supseteq$ on the collection of all $U\text{Conf}_k(X)$.

Let $I = \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, N - 1\}$. Let $C_I = \{(f, \lambda_1, \ldots, \lambda_k) | \lambda_j \in L_{i_j}, \lambda_1 < \lambda_2 \cdots < \lambda_k \subseteq \text{Sing}(f)\}$.

Let $C_{I \cup N} = \{f, \lambda, \ldots, \lambda_k \in C_I | f \in \Sigma_{\mathcal{L}}^{\geq N}\}$. Then $C$ forms a cubical space in the same way that $C$ does.

Take the geometric resolution $|\tilde{C}|$. Now we will construct a map $|\tilde{C}| \to |C|$ that is the identity on $|C| \subseteq |\tilde{C}|$. This will exhibit $|C|$ as a quotient of $|\tilde{C}|$ and we will...
give it the quotient topology. Given \( \lambda \in L_k \), we can define \( \text{supp}(\lambda) \in \text{UConf}_n(\lambda)(X) \) by \( \text{supp}(\lambda) = \bigcap_{f \in \lambda} \text{Sing}(f) \).

This defines a map \( \text{supp} : |\bar{C}| \to |C| \) given by

\[
(f, \lambda_i, s_i) \in \bar{C}_I \times \Delta_I \mapsto (f, \text{supp}(\lambda_i), s'_i) \in C_I \times \Delta_J.
\]

Here \( J = \{ \text{supp}(\lambda_i) \} \) and \( s'_i = \sum_{n(\lambda_i)=j} s_i \).

The maps \( \bar{C}_I \to \Sigma_{\bar{I}} \) are proper and hence so is the induced map \( |\bar{C}| \to \Sigma_{\bar{I}} \).

**Proposition 4.2.** The map \( \rho : |C| \to \Sigma \) is a proper homotopy equivalence.

**Proof.** In our setting having proper contractible fibres implies that the map \( \rho \) is a proper homotopy equivalence, this follows by combining Theorem 1.1 and Theorem 1.2 of [3]. We note that if \( f \not\in \Sigma^{\geq N} \) the fibre \( \rho^{-1}(f) \) is the simplex with vertices labelled by the singular points. If \( f \in \Sigma^{\geq N} \) then the fibre is a cone. We have given \( |C| \) the quotient topology. The map \( \rho \) is a factor in the composite \( |\bar{C}| \to |C| \to \Sigma_{\bar{I}} \) which is proper, hence \( \rho \) itself is proper. \( \square \)

Now as in any geometric realization, \( |C| \) is filtered by

\[
F_n = \text{im}(\prod_{|I| \leq n} C_I \times \Delta_k).
\]

The \( F_n \) form an increasing filtration of \( |C| \), i.e. \( F_1 \subseteq F_2 \subseteq \ldots \subseteq F_{n+1} \subseteq \ldots \) and \( \bigcup_{n=1}^{\infty} F_n = |C| \).

We define

\[
B_n = \{ f \in \Sigma_{\bar{I}} | \text{f has at least } n \text{ singular zeroes} \}.
\]

**Proposition 4.3.** Let \( n < N \). Let \( \Delta_n \) be the interior of an \( n \) simplex. The space \( F_n - F_{n-1} \) is a \( \Delta_n \) bundle over the space \( B_n \). This is in turn a vector bundle over \( \text{UConf}_n(X) \).

**Proof.** The fact that \( B_n \) is the total space of a vector bundle over \( \text{UConf}_n(X) \) follows from Riemann-Roch, the fibres are all vector subspaces of \( H^0(X, \mathcal{L}) \) of codimension exactly \( 2(n+1) \). A point in \( F_n - F_{n-1} \) is a pair \( ((f, x_0, \ldots, x_n), s_0, \ldots, s_n) \) where the \( s_i \) are the simplicial coordinates. We have a map \( \pi : F_n - F_{n-1} \to B_n \) defined by

\[
(f, (x_1, \ldots, x_n), (s_0, \ldots, s_n)) \mapsto (f, (x_1, \ldots, x_n)).
\]

The map \( \pi \) expresses \( F_n - F_{n-1} \) as a \( \Delta_n \) bundle over \( B_n \). \( \square \)

Let \( e_d = \text{dim}_C(H^0(X, \mathcal{L})) \). We define a local coefficient system on \( \text{UConf}_n(X) \), denoted by \( \pm \mathbb{Z} \), in the following way. There is a homomorphism \( \pi_1(\text{UConf}_n(X)) \to S_n \) associated to the covering \( \text{PConf}_nX \to \text{UConf}_X \). We compose this with the sign homomorphism \( S_n \to \pm 1 \cong GL_1(\mathbb{Z}) \) to obtain our local system on \( \text{UConf}_n(X) \).

**Proposition 4.4.** Let \( d \geq 1 \). Let \( n < N \).

\[
\bar{H}_*(F_n + F_{n-1}, \mathbb{Z}) = \bar{H}_*(-(e_d - (2(n+1)))\text{UConf}_n+1(X); \pm \mathbb{Z}).
\]

**Proof.** By Proposition 4.3 the space \( F_k - F_{k-1} \) is a bundle over \( \text{UConf}_k(X) \). This fact implies that

\[
\bar{H}_*(F_k - F_{k-1}) \cong \bar{H}_*-(k+2e_d-2(n+1)(k+1))\text{UConf}_k(X), \mathbb{Z}(\sigma)).
\]

Here \( \mathbb{Z}(\sigma) \) is the local system obtained by the action of \( \pi_1(\text{UConf}_k(X)) \) on the fibres \( \bar{H}_*(\Delta^k, \mathbb{Z}) \) where in this case \( \Delta^k \) is the open \( k \) simplex corresponding to the fibres.
of the map \( F_k - F_{k-1} \to B_k \). But one observes that the action of \( \pi_1(\text{UCnf}_k(X)) \) on this open simplex is by permutation of the vertices which implies that \( Z(\sigma) = \pm \mathbb{Z} \).

As with any filtered space, there is a spectral sequence with \( E_1^{p,q} = \tilde{H}_{p+q}(F_p - F_{p-1}; \mathbb{Z}) \) converging to \( \tilde{H}_*(|C|; \mathbb{Z}) \). Now by Proposition 4.3 we know what \( E_1^{p,q} \) is for \( p < N \).

**Proposition 4.5.** \( \tilde{H}_*(|C| - F_N; \mathbb{Z}) \cong \tilde{H}_*([C]; \mathbb{Z}) \) for \( * \geq 2e_d - N \).

**Proof.** We first will try to bound \( \tilde{H}_*(F_N; \mathbb{Z}) \) and then use the long exact sequence of the pair \( (F_N, |C|) \). The space \( F_N \) is built out of locally closed subspaces

\[ \phi_k = \{(f, x_1, \ldots, x_k), p \in \Delta^k, x_i \text{ are singular zeroes of } f\}. \]

There exists a surjection \( \pi : \phi_k \to \text{UCnf}_k \). The map \( \pi \) is a fibre bundle with fibres \( \mathbb{Z}^{d-u-2k} \times \Delta^k \). The space \( \text{UCnf}_k \) has complex dimension \( k \). Therefore \( \tilde{H}_j(\phi_k; \mathbb{Z}) = 0 \) if \( j \geq 2e_d - N \geq 2(e_d - 2k) + 3k \). This implies \( \tilde{H}_j(F_N) = 0 \) if \( j \geq 2e_d - N \). The long exact sequence of the pair \( (F_N, |C|) \) implies that \( \tilde{H}_*(Y - F_N; \mathbb{Z}) \cong \tilde{H}_*([C]; \mathbb{Z}) \) for \( * \geq 2e_d - N \).

Now this simplicial resolution of \( \Sigma \) gives an associated spectral sequence for its Borel Moore homology with

\[ E_1^{p,q} = \tilde{H}_{p+q}(F_p - F_{p-1}) = \tilde{H}_{p-(e_d-(2)(q+1))}(\text{UCnf}_{p+1}(X), \mathbb{Z}) \]

for \( p < N \). Also, \( E_1^{p,q} = 0 \) if \( q \geq 2e_d - N \).

We will now construct a cubical space \( C \) which will be involved in understanding \( \Sigma(W) \). Our construction of \( C \) will be similar to that of \( C \). Let \( N = \frac{d-q}{2} \). Let \( I \) be a subset of \( \{1, \ldots, N - 1\} \). Say \( I = \{i_1, \ldots, i_k\} \) let

\[ C_I := \{(f, x_1, \ldots, x_k)|f \in \Sigma(W), x_j \in \text{UCnf}_{i_j}(X), x_1 \subseteq \ldots x_k \subseteq \text{ Singular zeroes of } f\}. \]

We define

\[ C_{I \cup \{N\}} := \{(f, x_1, \ldots, x_k) \in C_I, f \in \Sigma^\geq N(W)\}. \]

If \( I \subseteq J \) then we have a natural forgetful map from \( C_I \to C_J \). This gives \( C \) the structure of a cubical space over the set \( \{1, \ldots, N\} \). We can take the geometric realisation of \( C \) denoted by \( |C| \). Then there is a map \( \rho : |C| \to \Sigma(W) \), induced by the forgetful maps \( C_I \to \Sigma(W) \).

We again topologise \( |C| \) in a nonstandard way, this is entirely analogous to the way we topologise \( |C| \), so we will be brief in our description of it. We construct a bigger cubical space \( \tilde{C} \) such that for \( I = \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, N - 1\} \), \( \tilde{C}_I = \{(f, x_1, \ldots, x_k)|f \in \Sigma(L), L \in W, x_j \in L_{i_j}(L), x_1 < x_2 \ldots x_k < \text{Sing}(f)\} \). We define \( \tilde{C}_I \cup \{N\} \) analogously. We then form the geometric realisation, \( |\tilde{C}| \) and note that there is a surjective map \( |\tilde{C}| \to |C| \) and we give \( |C| \) the quotient topology with respect to this map.

**Proposition 4.6.** \( |C| \) is proper homotopy equivalent to \( \Sigma(W) \).

**Proof.** This is analogous to the proof of Proposition 4.2.

\( \mathcal{C} \) also has an ascending filtration, \( \mathcal{F}_n \) and this filtration gives us a spectral sequence for \( \tilde{H}_*(\mathcal{C}; \mathbb{Z}) \).

**Proposition 4.7.** \( \tilde{H}_*(|C| - \mathbb{F}_N; \mathbb{Z}) \cong \tilde{H}_*([C]; \mathbb{Z}) \) for \( * \geq 2e_d - N \).

**Proof.** This is analogous to the proof of Proposition 4.5.
So there is a spectral sequence with

\[ E_1^{p,q} = \check{H}_{p+q}(C_{0-\ldots-q}) = H_{p-(e_d-(2)(q+1))}^{p+1}(UConf_{p+1}(X), \mathbb{Z}) \]

for \( p < N \). Finally we have the main theorem of this section.

**Theorem 4.8.** The map \( \mathcal{A}^{-1}(\mathcal{L}) \to \mathcal{A}^{-1}(W) \) induces an isomorphism \( H_*(\mathcal{A}^{-1}(\mathcal{L}); \mathbb{Z}) \to H_*(\mathcal{A}^{-1}(W); \mathbb{Z}) \) for \( s < N = \frac{d}{2} \).

*Proof.* This proof involves studying Alexander duality of \( A^{-1}(\mathcal{L}) \) inside \( H^0(X, \mathcal{L}) \) and \( A^{-1}(W) \) inside \( H^0(X, W) \) (the space \( H^0(X, W) \) is a topological vector bundle over \( W \) and so is at least homeomorphic to an affine space).

Now we use the fact that under Alexander duality, intersection of Borel-Moore cycles turns into pullback in cohomology, namely the map

\[ H^*(A^{-1}(W)) \to H^*(A^{-1}(\mathcal{L})), \]

is Alexander dual to the map

\[ f : \check{H}_{s+2g}^{p+q}(\Sigma_{W}) \to \check{H}_{s}^{p}(\Sigma_{\mathcal{L}}) \]

given by intersecting cycles with \( \Sigma_{\mathcal{L}} \), i.e. \( f(\sigma) = \sigma \cap \Sigma_{\mathcal{L}} \).

To understand this map in Borel-Moore homology, we turn to our spectral sequences for \( \check{H}_*(\Sigma_{\mathcal{L}}; \mathbb{Z}) \) and \( H_*(\Sigma_W; \mathbb{Z}) \). Since our stratification of \( \Sigma_W \) is fiberwise we get a map of spectral sequences between the two spectral sequences. We have a map \( E_1^{p,q+2g} \to E_1^{p,q} \). It will suffice to show that this map is an isomorphism for \( p < N \). For \( p < N \), this map is given by the map

\[ \phi : \check{H}_{p+q+2g}^{p}(\mathbb{F}_p - \mathbb{F}_p, \mathbb{Z}) \to \check{H}_{p+q}^{p}(\mathbb{F}_p - \mathbb{F}_p, \mathbb{Z}) \]

induced by intersecting cycles. However, we have a diagram of fiber bundles as follows:

\[ \begin{array}{ccc}
K & \longrightarrow & K \times W \\
\downarrow & & \downarrow \\
F_p - F_{p-1} & \longrightarrow & \mathbb{F}_p - \mathbb{F}_{p-1} \\
\downarrow & & \downarrow \\
UConf_p(X) & \longrightarrow & UConf_pX
\end{array} \]

Using this diagram and the fact that the intersection map \( \check{H}_{s+2g}^{p+q}(K \times W) \to \check{H}_{s}(K) \) is an isomorphism (as \( W \) is homeomorphic to \( \mathbb{C}^g \)), \( \phi \) is an isomorphism. This implies the theorem. \( \square \)

**5. Homology fibration theorem**

Let us recall the usual homology fibration theorem:

**Theorem 5.1** ([2]). Let \( f : X \to Y \) be a map. Let \( Hf^{-1}(y) \) be the homotopy fibre of \( f \). Suppose \( f^{-1}(y) \to f^{-1}(U) \) is a homology equivalence for sufficiently small \( U \) open then \( f^{-1}(y) \to Hf^{-1}(y) \) is a homology equivalence.

For this paper we need the following analogous theorem:

**Theorem 5.2.** Let \( n \geq 0 \). Let \( f : X \to Y \) be a map such that for all \( y \in Y \) there exists an open neighbourhood \( U \) such that the inclusion \( j : f^{-1}(y) \to f^{-1}(U) \) induces an isomorphism

\[ j_* : H_k(f^{-1}(y); \mathbb{Z}) \to H_k(f^{-1}(U); \mathbb{Z}) \]

for all \( k \leq n \).
Then the natural map \( i : f^{-1}(y) \to Hf^{-1}(y) \) induces an isomorphism \( i_* : H_k(f^{-1}(y); \mathbb{Z}) \to H_k(Hf^{-1}(y); \mathbb{Z}) \) for \( k \leq n \).

**Proof.** This follows from the proof of Proposition 5 (which is the same as Theorem 5.1 of this paper) in [2]. \( \square \)

This implies the following theorem.

**Theorem 5.3.** For \( * \leq N \),

\[
H^*(A^{-1}(\mathcal{L}); \mathbb{Z}) \cong H^*(HA^{-1}(\mathcal{L}); \mathbb{Z}) \cong H^*(\pi; \mathbb{Z})
\]

where \( \pi \) is the previously described subgroup of the extended surface braid group.

**Proof.** First we note that \( HA^{-1}(\mathcal{L}) \) is a \( K(\pi, 1) \), where \( \pi \) is our previously described subgroup of the extended surface braid group. The map \( \mathcal{A} : U_{alg}^n \to Pic_n X \) has \( K(G, 1) \)s for both source and target and the induced map at the level of \( \pi_1 \) is given by \( \tilde{Br}_n X \to Br_n X \to \mathbb{Z}^{2g} \), since \( H\mathcal{A}^{-1}\mathcal{L} \) is the homotopy fibre it is also a \( K(G, 1) \) with fundamental group equal to

\[
K_n := \ker(\tilde{Br}_n X \to \mathbb{Z}^{2g}).
\]

By Theorem 5.2 and Theorem 4.8

\[
H^*(A^{-1}(\mathcal{L}); \mathbb{Z}) \cong H^*(HA^{-1}(\mathcal{L}); \mathbb{Z}) \text{ for } * \leq N.
\]

\( \square \)

6. Relating \( U(\mathcal{L}) \) and \( U(\mathcal{L}) \)

In this section we will relate the two spaces \( U(\mathcal{L}) \) and \( U(\mathcal{L}) \). We begin first with the following result.

**Proposition 6.1.** Let \( X \) be an algebraic curve. Let \( \mathcal{L} \) be a line bundle on \( X \) of degree \( n \). Let \( p \in X \) be a point. Then, \( U(\mathcal{L})/\mathcal{G}_p \simeq U_{alg}^n \).

**Proof.** Let

\[
S_{(x_1, \ldots, x_n)} = \{ f \in U(\mathcal{L}) : x_i \text{ are regular zeroes of } f \}/\mathcal{G}_p.
\]

We have a diagram of fiber bundles as follows.

\[
\begin{array}{cccc}
\mathbb{C}^* & \xrightarrow{f} & S_{(x_1, \ldots, x_n)} & \\
& & U_{alg}^n & \xrightarrow{U(\mathcal{L})/\mathcal{G}_p} \\
& \downarrow & & \downarrow \\
& \text{UConf}_n X & \xrightarrow{\sim} & \text{UConf}_n X
\end{array}
\]

By considering the long exact sequences of homotopy groups associated to these fiber bundles, it suffices to prove that the map \( f : \mathbb{C}^* \to S \) is a homotopy equivalence.

To prove this it suffices to prove that \( S/\mathbb{C}^* \) is contractible where \( \mathbb{C}^* \) is acting on \( S \) by \( z \cdot f(x) = z(f(x)) \). But

\[
S/\mathbb{C}^* = \{ f \in U(\mathcal{L}) | x_i \text{ are regular zeroes of } f \}/\mathcal{G}.
\]

This is contractible by (2) of Proposition 2.3. \( \square \)
We will need the following lemma to obtain our results.

**Lemma 6.2.** Let $X$ be an algebraic curve. Let $n \geq 1$. Let $a = \{a_1, \ldots, a_n\} \in UConf_n X$. Let $\alpha \in \pi_1(UConf_n X, a)$. Suppose $\mathcal{A}_\alpha(\alpha) \neq 0 \in H_1(X; \mathbb{Z})$. Let $P_\alpha$ denote the point-pushing map associated to $\alpha$. Let $c_i \in H_1(X - \{a_1 \ldots a_n\}; \mathbb{Z})$ be the puncture classes. Then there exists a class

$$
\gamma \in H_1(X - \{a_1 \ldots a_n\}; \mathbb{Z})
$$

such that $\gamma \cap \mathcal{A}_\alpha(\alpha) = 1$ and

$$
P_\alpha(\gamma) - \gamma = \sum m_i c_i,
$$

where $\sum m_i \neq 0$.

This can be deduced from a computation by Bena Tshishiku. For a reference see [3].

**Theorem 6.3.** The natural map $\rho : U(L) \to Pic_n X$ is nullhomotopic.

**Proof.** As $Pic_n X$ is a $K(\pi, 1)$ it suffices to prove that

$$
\rho_* : \pi_1(U(L)) \to \pi_1(Pic_n X) \cong H_1(X; \mathbb{Z})
$$

is trivial. Let $\pi : U(L) \to UConf_n X$ be defined by $\pi(s) = \{a \in X | s(a) = 0\}$. Let $\mathcal{A} : UConf_n X \to Pic_n X$ be the Abel-Jacobi map. Let $a = \{a_1, \ldots, a_n\} \in UConf_n X$. Let $\alpha \in \pi_1(UConf_n X, a)$. Suppose $\mathcal{A}(\alpha) \neq 0 \in H_1(X; \mathbb{Z})$. It suffices to show that $\alpha \not\sim \rho_* \pi_1(U(L))$. This is because the map $\rho$ factors through $\mathcal{A}$.

Let $Mod(X - \{a_1, \ldots, a_n\})$ be the mapping class group of the punctured surface $X - \{a_1, \ldots, a_n\}$. Associated to $\alpha$ there exists a point pushing map $P_\alpha \in Mod(X - \{a_1, \ldots, a_n\})$. Let $c_i \in H_1(X - \{a_1 \ldots a_n\}; \mathbb{Z})$ be the puncture classes.

Then by Lemma 6.2 there exists a class $[\gamma] \in H_1(X - \{a_1 \ldots a_n\})$ such that

$$
P_\alpha([\gamma]) - [\gamma] = \sum m_i c_i,
$$

where $m_i \in \mathbb{Z}$ satisfying $\sum m_i \neq 0$.

Let $f \in U(L)$ be such that $\pi(f) = a$. Suppose for the sake of contradiction that $\alpha \in \text{im}(\pi(U(L)), f)$ with $\alpha \neq 0$. Then there exists a loop in $U(L)$, which we will call $F_\alpha$ such that $\pi(F_\alpha) = \alpha$, i.e. $F_\alpha$ is a lift of $\alpha$.

Now for $s \in (0, 1)$, let $P_\alpha^s$ be the point-pushing homeomorphism along the path $\alpha|_{[0,s]}$. It is a well-defined element of

$$
\pi_0(\text{Homeo}((X, \alpha(0)), (X, \alpha(s)))).
$$

Now $P_\alpha^s(f)$ is a lift of $\alpha$ as a path (not a loop) to $U(L)$. Since the map $\pi$ is a fibration, any two paths that are lifts of $\alpha$ must have endpoints in the same component of $\pi^{-1}(\alpha)$. This would imply that $f$ (the endpoint of $F_\alpha$) and $P_\alpha^s(f)$ (the endpoint of $P_\alpha^s(f)$) would be in the same path component of $\pi^{-1}(\alpha)$. If that were so, then we would have

$$
\int_\gamma \frac{P_\alpha(f)}{|P_\alpha(f)|} - \int_\gamma \frac{f}{|f|} = 0.
$$

However we will now show that this is not the case.

We’d like to remind the reader that $\int_\gamma f/|f| = 1$. This is because the section $f$ has a zero of index 1 at each of the $a_i$s. Then we know that

$$
\int_\gamma \frac{P_\alpha(f)}{|P_\alpha(f)|} - \int_\gamma \frac{f}{|f|} = \int_{P_\alpha \gamma} \frac{f}{|f|} - \int_\gamma \frac{f}{|f|}.
$$
This proves that any $\alpha$ that lifts is forced to be trivial, which completes the proof. □

**Proposition 6.4.** Let $n \geq 1$, $p \in X$. There exists a homotopy equivalence $f : \text{Pic}_n X \to B\mathcal{G}_p$ that makes the following diagram commute up to homotopy:

\[
\begin{array}{ccc}
\mathbb{U}(\mathcal{L}) & \xrightarrow{\pi} & \mathbb{U}(\mathcal{L})/\mathcal{G}_p \\
U_n \mathcal{L} & \xrightarrow{i} & \mathbb{U}(\mathcal{L})/\mathcal{G}_p \\
\text{Pic}_n X & \xrightarrow{f} & B\mathcal{G}_p
\end{array}
\]

Here the map $g$ is the classifying map for the fibration $\mathbb{U}(\mathcal{L}) \to \mathbb{U}(\mathcal{L})/\mathcal{G}_p$.

**Proof.** The situation is as follows: $\pi : \mathbb{U}(\mathcal{L}) \to \mathbb{U}(\mathcal{L})/\mathcal{G}_p$ is a principal $\mathcal{G}_p$ bundle. Since $\mathcal{G}_p \simeq \mathbb{Z}^2 g$, we have an associated principal $\mathbb{Z}^2 g$ bundle $E \to \mathbb{U}(\mathcal{L})/\mathcal{G}_p$, where $E := \mathbb{U}(\mathcal{L})/(f_1 \sim f_2$ if $\pi(f_1) = \pi(f_2)$ and $f_1, f_2$ are in the same path component of $\pi^{-1}(\pi(f_1)))$.

Equivalently, if $(\mathcal{G}_p)_0$ is the identity component of $\mathcal{G}_p$, $E = \mathbb{U}(\mathcal{L})/(\mathcal{G}_p)_0$.

The quotient map $p : \mathbb{U}(\mathcal{L}) \to E$ is naturally a homotopy equivalence as the group $(\mathcal{G}_p)_0$ is contractible. We then have a diagram as follows:

\[
\begin{array}{ccc}
\mathbb{U}(\mathcal{L}) & \xrightarrow{p} & E \\
\mathbb{U}(\mathcal{L})/\mathcal{G}_p & \xrightarrow{=} & \mathbb{U}(\mathcal{L})/\mathcal{G}_p
\end{array}
\]

It suffices to prove that the natural map

$\alpha : \mathbb{U}(\mathcal{L})/\mathcal{G}_p \to \text{Pic}_n X$

satisfies the classifying space property for the fibration $E \to \mathbb{U}(\mathcal{L})/\mathcal{G}_p$. However by Proposition 6.3 the composite map $E \to \text{Pic}_n X$ is nullhomotopic and we can lift it to $\tilde{\text{Pic}}_n X$, the universal cover of $\text{Pic}_n X$. So we have a commutative diagram as follows:

\[
\begin{array}{ccc}
E & \xrightarrow{p} & \text{Pic}_n X \\
\mathbb{U}(\mathcal{L})/\mathcal{G}_p & \xrightarrow{\alpha} & \text{Pic}_n X
\end{array}
\]

Hence $\alpha$ is a classifying map and we are done. □

**Theorem 6.5.** $\mathbb{U}(\mathcal{L})$ is homotopy equivalent to $H\mathcal{A}^{-1}(\mathcal{L})$, the homotopy fibre of $\mathcal{A}$.

**Proof.** By Propositions 6.3 and 6.4 there is a diagram as follows:
Since the composite map $H\mathcal{A}^{-1}(\mathcal{L}) \to B\mathfrak{G}_p$ is null homotopic, by the properties of a fibre sequence we have a map $g : H\mathcal{A}^{-1}(\mathcal{L}) \to U(\mathcal{L})$ that commutes with the maps of the diagram. Since the maps $i$ and $f$ are homotopy equivalences, so is $g$. □

Now we can finally prove the theorems in the introduction of this paper.

\textbf{Proof of Theorem 1.2.} By Theorem 6.5 $U(\mathcal{L}) \simeq H\mathcal{A}^{-1}(\mathcal{L})$. So it suffices to prove that $H\mathcal{A}^{-1}(\mathcal{L})$ is a $K(\pi, 1)$ for $K_n$. However this follows from Theorem 6.3. □

\textbf{Proof of Theorem 1.1.} By Theorem 5.2 and Theorem 4.8 the map $f : U(\mathcal{L}) \to H\mathcal{A}^{-1}(\mathcal{L})$ induces an isomorphism $H^*(U(\mathcal{L})) \cong H^*(U(\mathcal{L}))$ for $* < n-(2g)$. But by Theorem 6.5 $H\mathcal{A}^{-1}(\mathcal{L}) \simeq \mathbb{U}(\mathcal{L})$ and it is easy to see that

$$i^* : H^*(\mathbb{U}(\mathcal{L}); \mathbb{Z}) \to H^*(U(\mathcal{L}); \mathbb{Z})$$

is the composition

$$H^*(\mathbb{U}(\mathcal{L}); \mathbb{Z}) \cong H^*(H\mathcal{A}^{-1}(\mathcal{L}); \mathbb{Z}) \to f^* H^*(U(\mathcal{L}); \mathbb{Z}).$$

□

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