The bulk-edge correspondence for the split-step quantum walk on the one-dimensional integer lattice

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Abstract

Suzuki’s split-step quantum walk on the one-dimensional integer lattice can be naturally viewed as a chirally symmetric quantum walk. Given the unitary time-evolution of such a chirally symmetric quantum walk, we can separately introduce well-defined indices for the eigenvalues $\pm 1$. The bulk-edge correspondence for Suzuki’s split-step quantum walk is twofold. Firstly, we show that the multiplicities of the eigenvalues $\pm 1$ coincide with the absolute values of the associated indices. Note that this can be viewed as the symmetry protection of bound states, and that the indices we consider are robust in the sense that these depend only on the asymptotic behaviour of the parameters of the given model. Secondly, we show that such bound states exhibit exponential decay at spatial infinity.

Keywords: Chiral symmetry, Bulk-edge correspondence, Split-step quantum walk, Symmetry protection of bound states

1. Introduction

Quantum walk theory is widely recognised as a natural quantum-mechanical counterpart of the classical random walk theory [Gud88, ADZ93, Mey96, ABN+01]. In this paper, we shall focus on the well-known discrete-time quantum walk on the one-dimensional integer lattice $\mathbb{Z}$. This is a two-state quantum walk model, and so $\ell^2(\mathbb{Z}, \mathbb{C}^2) = \ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z})$ is the underlying state Hilbert space. The primitive form of the split-step quantum walk discussed in [BBD10, KBF+12, Kit12] can be characterised by the following unitary time-evolution:

\begin{equation}
U_{\text{kit}} := \begin{pmatrix} 1 & 0 \\ 0 & L \end{pmatrix} \begin{pmatrix} \cos \theta_2 & -\sin \theta_2 \\ \sin \theta_2 & \cos \theta_2 \end{pmatrix} \begin{pmatrix} L^* & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \theta_1 & -\sin \theta_1 \\ \sin \theta_1 & \cos \theta_1 \end{pmatrix},
\end{equation}

where $L$ is the bilateral left-shift operator on $\ell^2(\mathbb{Z})$, and where $\theta_j = (\theta_j(x))_{x \in \mathbb{Z}}$ is a $\mathbb{R}$-valued sequence for each $j = 1, 2$. We refer to [1] as the time-evolution operator of Kitagawa’s split-step quantum walk throughout this paper, where each trigonometric sequence is viewed as a bounded multiplication operator on $\ell^2(\mathbb{Z})$. It turns out that the evolution operator of Kitagawa’s split-step quantum walk can be naturally generalised to that of Suzuki’s split-step quantum walk [FFS17, FFS18, FFS19, Tan21, NOW21], given explicitly by the following formula:

\begin{equation}
U_{\text{suz}} := \begin{pmatrix} p & qL \\ L^*q^* & -p(-1) \end{pmatrix} \begin{pmatrix} a & b^* \\ b & -a \end{pmatrix}, \quad \Gamma := \begin{pmatrix} p & qL \\ L^*q^* & -p(-1) \end{pmatrix},
\end{equation}

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where we always assume that \( R \)-valued sequences \( p = (p(x))_{x \in \mathbb{Z}}, a = (a(x))_{x \in \mathbb{Z}} \) and \( C \)-valued sequences \( q = (q(x))_{x \in \mathbb{Z}}, b = (b(x))_{x \in \mathbb{Z}} \) satisfy \( p(x)^2 + |q(x)|^2 = 1 \) and \( a(x)^2 + |b(x)|^2 = 1 \) for each \( x \in \mathbb{Z} \). Indeed, it is not difficult to show that \( \mathbf{1} \) can be made unitarily equivalent to \( \mathbf{2} \), provided that we appropriately define \( p, q, a, b \) in terms of \( \theta_1, \theta_2 \) (see Lemma \( \mathbf{1} \) for details).

A notable advantage of using Suzuki’s split-step quantum walk lies in the fact that the associated time-evolution operator \( U = U_{\text{suz}} \) exhibits chiral symmetry with respect to the unitary self-adjoint operator \( \Gamma \) defined by the second equality in \( \mathbf{2} \):

\[
U^{*} = \Gamma U. \tag{3}
\]

In fact, given any pair \((\Gamma, U)\) of an abstract unitary operator \( U \) and an abstract unitary self-adjoint operator \( \Gamma \) satisfying \( \mathbf{4} \), we can introduce the following two indices according to \( \mathbf{CGG}^{+18}, \mathbf{CGS}^{+18}, \mathbf{CGWW21} \):

\[
\text{ind}_{+}(\Gamma, U) := \text{Trace}(\Gamma|_{\ker(U-1)}), \quad \text{ind}_{-}(\Gamma, U) := \text{Trace}(\Gamma|_{\ker(U+1)}), \tag{4}
\]

where \( \ker(U-1) \) and \( \ker(U+1) \) are \( \Gamma \)-invariant subspaces by \( \mathbf{5} \). Note that \( \text{ind}_{+}(\Gamma, U) \) is well-defined, if the essential spectrum of \( U \), defined by \( \sigma_{\text{ess}}(U) := \{ z \in \mathbb{C} \mid U - z \text{ is not Fredholm} \} \), does not contain \( 1 \). Indeed, the restriction \( \Gamma|_{\ker(U-1)} \) becomes a unitary self-adjoint operator on the finite-dimensional vector space \( \ker(U-1) \) in this case. Similarly, \( \text{ind}_{-}(\Gamma, U) \) is well-defined, if \( -1 \notin \sigma_{\text{ess}}(U) \). The following bulk-edge correspondence is one of the main results of the present article:

**Theorem 1.1.** Let \( U = U_{\text{suz}} \) be the evolution operator of Suzuki’s split-step quantum walk given by \( \mathbf{2} \), and let us assume the existence of the following two-sided limits:

\[
p(\pm \infty) := \lim_{x \to \pm \infty} p(x) \in (-1, 1), \quad a(\pm \infty) := \lim_{x \to \pm \infty} a(x) \in (-1, 1). \tag{5}
\]

Let \( |p(x)| < 1 \) and \( |a(x)| < 1 \) for each \( x \in \mathbb{Z} \). Then \( p(\ast) \neq \pm a(\ast) \) for each \( \ast = \pm \infty \) if and only if \( \pm 1 \notin \sigma_{\text{ess}}(U) \).

In this case, the following two assertions hold true:

(i) We have \( \dim \ker(U \mp 1) = |\text{ind}_{\pm}(\Gamma, U)| \), where

\[
\text{ind}_{\pm}(\Gamma, U) = \begin{cases} +1, & p(-\infty) \mp a(-\infty) < 0 \Leftrightarrow p(+\infty) \mp a(+\infty), \\ -1, & p(+\infty) \mp a(+\infty) < 0 \Leftrightarrow p(-\infty) \mp a(-\infty), \\ 0, & \text{otherwise}. \end{cases} \tag{6}
\]

(ii) If \( (-1)^{j}(p(-\infty) \mp a(-\infty)) < 0 < (-1)^{j}(p(+\infty) \mp a(+\infty)) \) for some \( j = 1, 2 \), then any non-zero vector \( \Psi \in \ker(U \mp 1) \) admits the following unique representation:

\[
\Psi = \left( \frac{a^{\mp 1}(\ast)}{b} \psi \right), \quad \psi \in \ker \left( L + \frac{p(\ast)}{q} a \mp (-1)^{j} \frac{a^{\mp 1}(\ast)}{b} \right). \tag{7}
\]

Moreover, the eigenstate \( \Psi \) characterised by \( \mathbf{7} \) exhibits exponential decay. More precisely, there exist positive constants \( c_{\pm}^{+}, c_{\pm}^{-}, \kappa_{\pm}^{+}, \kappa_{\pm}^{-}, x_{\pm} \), such that

\[
\kappa_{\pm}^{+} e^{-c_{\pm}^{+}|x|} \leq \| \Psi(x) \| \leq \kappa_{\pm}^{-} e^{-c_{\pm}^{-}|x|}, \quad |x| \geq x_{\pm}. \tag{8}
\]

Note first the equality \( \dim \ker(U \mp 1) = |\text{ind}_{\pm}(\Gamma, U)| \) in Theorem \( \mathbf{1.1} \) can be understood as the existence of eigenstates corresponding to \( \pm 1 \) by chiral symmetry. Here, the robustness of \( \text{ind}_{\pm}(\Gamma, U) \) is ensured by the formula \( \mathbf{8} \) which depends only on the asymptotic values \( \mathbf{6} \). It is also shown in Theorem \( \mathbf{1.1} \) \( \mathbf{2} \) that such symmetry protected eigenstates can be uniquely characterised by the explicit formula \( \mathbf{7} \), and that they exhibit exponential decay in the sense of \( \mathbf{8} \).

Theorem \( \mathbf{1.1} \) can be classified as an index theorem for 2-phase chirally symmetric quantum walks on the one-dimensional integer lattice \( \mathbb{Z} \), since we assume the existence of the two-sided limits as in \( \mathbf{5} \). Index theory of such 2-phase quantum walks can be found in the extensive literature \( \mathbf{CGS}^{+18}, \mathbf{CGG}^{+18}, \mathbf{CGS}^{+18}, \mathbf{CGWW21}, \mathbf{Suz19}, \mathbf{ST19}, \mathbf{Mat20}, \mathbf{AFST20}, \mathbf{Tan21}, \mathbf{CGWW21} \). As such, Theorem \( \mathbf{1.1} \) may not seem novel at first glance.
Note, however, that the ultimate purpose of the present article is to generalise both Theorem 1.1(i),(ii) by replacing the 2-phase assumption \([5]\) with the following significantly weakened assumption:

\[
\sup_{x \in \mathbb{Z}} |p(x)| < 1, \quad \sup_{x \in \mathbb{Z}} |a(x)| < 1. \tag{9}
\]

For example, the new assumption \([9]\) will allow us to consider the case where \(p\) and \(a\) are periodic. Unlike the 2-phase case, it seems unrealistic to extract any useful information about \(\sigma_{\text{ess}}(U_{\text{sus}})\) from the new abstract assumption \([9]\). As such, it is desirable to also generalise \([3]\).

The present article is organised as follows. In \([3]\) we develop new index theory for abstract unitary operators \(U\) exhibiting chiral symmetry in the sense of \([3]\) in full generality. Note that this somewhat elementary construction is beyond the scope of the existing literature \([CGG+18, CGS+18, Suz19, Tan21, CGWW21]\), since it makes use of neither the notion of a Fredholm operator, nor any local structure of the underlying Hilbert space. We show that our indices coincide with \([4]\), if the essential spectrum of \(U\) has a spectral gap at \(\pm 1\). The purpose of \([3]\) is to prove Theorem 1.1 with \([3]\) replaced by \([9]\) (see Theorem 3.1 for more details). As we shall see, Theorem 1.1(ii) is a natural extension of \([FFS18, Theorem 5.1]\) which states that non-trivial vectors in the so-called birth eigenspaces exhibit exponential decay. However, we do not make use of the spectral mapping theorem for chirally symmetric unitary operators discussed in \([SS16, SS19]\); see \([SS19, Lemma 2.2]\) for details. For example, it is shown in this section that the decay rates \(c_1^k, c_2^k\) in \([3]\) depend on the gaps of the essential spectrum under the 2-phase assumption \([5]\), and that the index formulas in \([ST19, Mat20, AFST20, Tan21]\) can be easily derived from Theorem 1.1(ii).

2. Indices for chirally symmetric unitary operators

By operators we always mean everywhere-defined bounded linear operators between Banach spaces throughout this paper. Recall that the (Fredholm) essential spectrum of an operator \(X\) on a Hilbert space \(\mathcal{H}\) is defined by \(\sigma_{\text{ess}}(X) := \{z \in \mathbb{C} \mid X - zI\text{ is not Fredholm}\}\). If \(X\) is normal, then \(\sigma_{\text{ess}}(X) = \sigma(X) \setminus \sigma_{\text{dis}}(X)\), where \(\sigma_{\text{dis}}(X)\) is the discrete spectrum of \(X\). Note that the equality \(\ker X = \ker X^* X\) shall be repeatedly used without any further comment.

A chiral pair on \(\mathcal{H}\) is any pair \((\Gamma, U)\) of a unitary self-adjoint operator \(\Gamma : \mathcal{H} \to \mathcal{H}\) and an operator \(U : \mathcal{H} \to \mathcal{H}\), satisfying the chiral symmetry condition \([3]\). Note that the underlying Hilbert space \(\mathcal{H}\) admits a \(\mathbb{Z}_2\)-grading of the form \(\mathcal{H} = \ker(\Gamma - 1) \oplus \ker(\Gamma + 1)\), and that \(\Gamma = 1 \oplus (-1)\) with respect to this orthogonal decomposition, where 1 denotes the identity operator on a Hilbert space throughout this paper. The operator \(U\) can then be written as \(U = R + iQ\), where \(R, Q\) are the real and imaginary parts of \(U\) respectively. We have:

\[
R = \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix}_{\ker(\Gamma - 1) \oplus \ker(\Gamma + 1)}, \quad Q = \begin{pmatrix} 0 & Q_2 \\ Q_1 & 0 \end{pmatrix}_{\ker(\Gamma - 1) \oplus \ker(\Gamma + 1)}. \tag{10}
\]

Here, the first equality follows from the commutation relation \([\Gamma, R] := \Gamma R - R \Gamma = 0\), whereas the second equality follows from the anti-commutation relation \([\Gamma, Q] := \Gamma Q + Q \Gamma = 0\) (see \([Suz19, Lemma 2.2]\) for details). Since \(R, Q\) are self-adjoint, we have \(R_j^* = R_j\) for each \(j = 1, 2\), and \(Q_j = Q_j^*\). The following formula shall be referred to as the standard representation of \(U\) with respect to \(\Gamma\) throughout this paper;

\[
U = \begin{pmatrix} R_1 & iQ_2 \\ iQ_1 & R_2 \end{pmatrix}_{\ker(\Gamma - 1) \oplus \ker(\Gamma + 1)}. \tag{11}
\]

With \([11]\) in mind, we introduce the following formal indices:

\[
\ind_{\pm}(\Gamma, U) := \dim \ker(R_1 \mp 1) - \dim \ker(R_2 \mp 1), \tag{12}
\]

\[
\ind(\Gamma, U) := \dim \ker Q_1 - \dim \ker Q_2. \tag{13}
\]
Lemma 2.1. Given a chiral pair \((\Gamma, U)\) with \(\Gamma\) being the standard representation of \(U\), we have

\[
\ker(U \mp 1) = \ker(R_1 \mp 1) \oplus \ker(R_2 \mp 1),
\]

(14)

\[
\ker Q_j = \ker(R_j - 1) \oplus \ker(R_j + 1), \quad j = 1, 2.
\]

(15)

Moreover, the following assertions hold true:

(i) The index \(\text{ind}_\pm(\Gamma, U)\) is a well-defined integer, if \(\dim \ker(U \mp 1) < \infty\). In this case,

\[
|\text{ind}_\pm(\Gamma, U)| \leq \dim \ker(U \mp 1).
\]

(16)

(ii) The index \(\text{ind}(\Gamma, U)\) is a well-defined integer, if \(\dim \ker(U - 1) \oplus \ker(U + 1) < \infty\). In this case,

\[
\text{ind}(\Gamma, U) = \text{ind}_+(\Gamma, U) + \text{ind}_-(\Gamma, U).
\]

(17)

Note that \(\dim \ker(U \mp 1) < \infty\) is a weaker assumption than \(\pm 1 \not\in \sigma_{\text{ess}}(U)\).

Proof. Since \(U = R + iQ\) is unitary and since \([R, Q] = 0\), we have \(R^2 + Q^2 = 1\). Firstly, this matrix equality implies \(R_j^2 + Q_j^2 = 1\) for each \(j = 1, 2\), and so (15) follows. Secondly, the same equality implies \((U \mp 1)^*(U \mp 1) = 2(1 \mp R)\). We obtain (14) from

\[
\ker(U \mp 1) = \ker(U \mp 1)^*(U \mp 1) = \ker(1 \mp R) = \ker(R \mp 1),
\]

(18)

where \(\ker(R \mp 1) = \ker(R_1 \mp 1) \oplus \ker(R_2 \mp 1)\), since \(R = R_1 \oplus R_2\).

(i) It follows from (14) that if \(\dim \ker(U \mp 1) < \infty\), then \(\dim \ker(R_j \mp 1) < \infty\) for each \(j = 1, 2\), and so \(\text{ind}_\pm(\Gamma, U)\) is well-defined. We have

\[
|\text{ind}_\pm(\Gamma, U)| \leq \dim \ker(R_1 \mp 1) + \dim \ker(R_2 \mp 1) = \dim \ker(U \mp 1).
\]

(ii) It follows from (15) that

\[
\dim \ker Q_j = \dim \ker(R_j - 1) + \dim \ker(R_j + 1), \quad j = 1, 2.
\]

(19)

If \(\dim \ker(U - 1) \oplus \ker(U + 1) < \infty\), then \(\dim \ker(R_j - 1) \oplus \ker(R_j + 1) < \infty\) for each \(j = 1, 2\) by (18). It follows from (19) that

\[
\text{ind}(\Gamma, U) = \dim \ker Q_1 - \dim \ker Q_2
\]

\[
= \dim \ker(R_1 - 1) + \dim \ker(R_1 + 1) - (\dim \ker(R_2 - 1) + \dim \ker(R_2 + 1))
\]

\[
= \dim \ker(R_1 - 1) - \dim \ker(R_2 - 1) + \dim \ker(R_1 + 1) - \dim \ker(R_2 + 1)
\]

\[
= \text{ind}_+(\Gamma, U) + \text{ind}_-(\Gamma, U).
\]

\[
\square
\]

Lemma 2.2. Let \((\Gamma_0, U_0), (\Gamma, U)\) be two chiral pairs on Hilbert spaces \(\mathcal{H}_0, \mathcal{H}\) respectively. If \((\Gamma_0, U_0), (\Gamma, U)\) are unitarily equivalent in the sense that \((\Gamma_0, U_0, (\epsilon^*\Gamma', \epsilon^*U')\) for some unitary operator \(\epsilon: \mathcal{H}_0 \rightarrow \mathcal{H}\), then the following assertions hold true:

(i) If \(\dim \ker(U_0 \mp 1) = \dim \ker(U \mp 1)\) is finite, then \(\text{ind}_\pm(\Gamma_0, U_0) = \text{ind}_\pm(\Gamma, U)\).

(ii) If \(\dim \ker(U_0 - 1) \oplus \ker(U_0 + 1) = \dim \ker(U - 1) \oplus \ker(U + 1)\) is finite, then \(\text{ind}(\Gamma_0, U_0) = \text{ind}(\Gamma, U)\).

Proof. The details of what follows can be found in the proof of [AFST20, Lemma 2]. Firstly, there exists a unitary operator \(\epsilon_j: \ker(\Gamma_0 + (-1)^j) \rightarrow \ker(\Gamma + (-1)^j)\) for each \(j = 1, 2\), such that the given unitary operator \(\epsilon\) can be identified with the direct sum \(\epsilon_1 \oplus \epsilon_2: \ker(\Gamma_0 - 1) \oplus \ker(\Gamma_0 + 1) \rightarrow \ker(\Gamma - 1) \oplus \ker(\Gamma + 1)\). Secondly, if \(U\) admits the standard representation of the form (11), then the standard representation of \(U_0\) is given by the following formula:

\[
U_0 = \begin{pmatrix}
\epsilon_1^* R_1 \epsilon_1 & i \epsilon_2^* Q_2 \epsilon_2 \\
if \epsilon_2^* Q_1 \epsilon_1 & \epsilon_2^* R_2 \epsilon_2
\end{pmatrix}_{|\ker(\Gamma_0 - 1) \oplus \ker(\Gamma_0 + 1)|}.
\]
Proposition 2.3. Let \( \Gamma, U \) be a chiral pair, and let \( \Gamma' := \Gamma U \). Then the following assertions hold true:

(i) If \( U \) admits the standard representation of the form \([11]\), then

\[
\ker(R_j \mp 1) = \ker(U \mp 1) \cap \ker(\Gamma + (-1)^j) = \ker(\Gamma + (-1)^j) \cap \ker(\Gamma' \mp (-1)^{j+1}), \quad j = 1, 2. \tag{20}
\]

(ii) The pair \( (\Gamma', U) \) is also a chiral pair. Moreover, if \( \ker(U \mp 1) \) is finite-dimensional, then

\[
\ind_{\pm}(\Gamma, U) = \pm \ind_{\pm}(\Gamma', U). \tag{21}
\]

(iii) If \( \ker(U - 1) \oplus \ker(U + 1) \) is finite-dimensional, then

\[
\ind(\Gamma', U) = \ind_{+}(\Gamma', U) - \ind_{-}(\Gamma', U). \tag{22}
\]

Proof. (i) We have

\[
U \mp 1 = \begin{pmatrix} R_1 \mp 1 & iQ_2 \\ iQ_1 & R_2 \mp 1 \end{pmatrix}.
\]

It follows from this equality that

\[
\ker(U \mp 1) \cap \ker(\Gamma + (-1)^j) = \ker(R_j \mp 1) \cap \ker Q_j = \ker(R_j \mp 1),
\]

where the last equality follows from \([15]\). Similarly, we have

\[
\Gamma' \mp 1 = \Gamma U \mp 1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} R_1 & iQ_2 \\ iQ_1 & R_2 \end{pmatrix} \mp 1 = \begin{pmatrix} R_1 \mp 1 & iQ_2 \\ -iQ_1 & -(R_2 \pm 1) \end{pmatrix}.
\]

We obtain

\[
\ker(\Gamma - 1) \cap \ker(\Gamma' \mp 1) = \ker(R_1 \mp 1) \cap \ker Q_1 = \ker(R_1 \mp 1),
\]

\[
\ker(\Gamma + 1) \cap \ker(\Gamma' \mp 1) = \ker(R_2 \pm 1) \cap \ker Q_2 = \ker(R_2 \pm 1).
\]

The above identities can be written as the single formula \([20]\).

(ii) Note that \( (\Gamma', U) \) is a chiral pair, since \( \Gamma' U \Gamma' = \Gamma'(\Gamma U)\Gamma' = \Gamma' \Gamma = U^* \). Let \( \ker(U \mp 1) \) be finite-dimensional, and let

\[
m_{j,\pm} := \dim(\ker(\Gamma + (-1)^j) \cap \ker(\Gamma' \mp (-1)^{j+1})),
\]

\[
m'_{j,\pm} := \dim(\ker(\Gamma' + (-1)^j) \cap \ker(\Gamma \mp (-1)^{j+1})).
\]

It follows from (i) that \( \ind_{\pm}(\Gamma, U) = m_{1,\pm} - m_{2,\pm} \) and \( \ind_{\pm}(\Gamma', U) = m'_{1,\pm} - m'_{2,\pm} \). The formula \([21]\) is an immediate consequence of the following equalities:

\[
m'_{1,++} = m_{1,++}, \quad m'_{2,++} = m_{2,++},
\]

\[
m'_{1,--} = m_{1,--}, \quad m'_{2,--} = m_{2,--}.
\]

(iii) This follows from (i) and (ii). \(\square\)
Remark 2.4. If \( \dim \ker(U \mp 1) < \infty \), then it follows from the first equality in \eqref{gamma} that
\[
\text{ind}_\pm (\Gamma, U) = \dim \ker(U \mp 1) \cap \ker(\Gamma - 1) - \dim \ker(U \mp 1) \cap \ker(\Gamma + 1).
\]
Note that the right hand side can be written as \( \text{Trace}(\Gamma|_{\ker(U \mp 1)}) \), where \( \ker(U - 1) \) and \( \ker(U + 1) \) are \( \Gamma \)-invariant subspaces by the chiral symmetry condition \eqref{chiral}. That is, the previously mentioned formula \eqref{formula} is consistent with \eqref{trace}.

Corollary 2.5. Let \((\Gamma, U)\) be a chiral pair, and let \(\ker(U - 1) \oplus \ker(U + 1)\) be finite-dimensional. Then we have the following formulas:
\[
\text{ind}_\pm (\Gamma, U) = \text{ind}_\mp (\Gamma, U), \quad \text{ind} (\Gamma, -U) = \text{ind} (\Gamma, U), \quad \text{ind} (-\Gamma, U) = -\text{ind} (\Gamma, U).
\]

Proof. If \( U \) admits the standard representation of the form \eqref{standard} with respect to \( \Gamma \), then the standard representation of \(-U\) is
\[
-U = \begin{pmatrix} -R_1 & -iQ_2 \\ -iQ_1 & -R_2 \end{pmatrix} \ker(\Gamma - 1) \oplus \ker(\Gamma + 1).
\]
It follows that \( \text{ind}_\pm (\Gamma, -U) = \text{ind}_\mp (\Gamma, U) \), and so \( \text{ind} (\Gamma, -U) = \text{ind} (\Gamma, U) \) by \eqref{ind}. Similarly, the standard representation of \( U \) with respect to \(-\Gamma\) is
\[
U = \begin{pmatrix} R_2 & iQ_1 \\ iQ_2 & R_1 \end{pmatrix} \ker(\Gamma + 1) \oplus \ker(\Gamma - 1).
\]
We have \( \text{ind}_\pm (-\Gamma, U) = \dim \ker(R_2 \mp 1) - \dim \ker(R_1 \mp 1) = -\text{ind}_\mp (\Gamma, U) \), and so \( \text{ind} (-\Gamma, U) = -\text{ind} (\Gamma, U) \).\qed

3. The Bulk-edge Correspondence

We are now in a position to state the following generalisation of Theorem 1.1.

Theorem 3.1. Let \( U = U_{\text{suz}} \) be the evolution operator of Suzuki’s split-step quantum walk given by \eqref{suz}, and let \eqref{_trace} hold true. Let us introduce the following notation:
\[
\Lambda(\kappa) := \frac{1 + \kappa}{1 - \kappa}, \quad \kappa \in (-1, 1),
\]
\[
\delta_{j, \pm} (y) := \frac{\sqrt{\Lambda((-1)^{j} \gamma(y))\Lambda((1)^{j}\gamma(y))}}{\pm e^{i \text{Arg} \gamma(y) + \text{Arg} \delta(y)}}, \quad y \in \mathbb{Z},
\]
\[
\Delta_{j, \pm} := \sum_{x=1}^{\infty} \left( \prod_{y=1}^{x} |\delta_{j, \pm} (-y)|^{-2} \right) + \sum_{x=1}^{\infty} \left( \prod_{y=0}^{x-1} |\delta_{j, \pm} (y)|^{2} \right),
\]
where \( j = 1, 2 \), and where \( \text{Arg} \) denotes the principal argument of a non-zero complex number \( w \). Then \( \Delta_{1, \pm} \) and \( \Delta_{2, \pm} \) cannot be simultaneously finite, and the following assertions hold true:

(i) The dimension of \( \ker(U \mp 1) \) is at most 1. More explicitly, we have
\[
|\text{ind}_\pm (\Gamma, U)| = \dim \ker(U \mp 1),
\]
\[
\text{ind}_\pm (\Gamma, U) = \begin{cases} +1, & \Delta_{1, \pm} < \infty, \\
-1, & \Delta_{2, \pm} < \infty, \\
0, & \Delta_{1, \pm} = \Delta_{2, \pm} = \infty. \end{cases}
\]
(ii) If \( \Delta_{j, \pm} < \infty \) for some \( j = 1, 2 \), then we have the linear isomorphism \( \tau_{j, \pm} : \ker(L - \delta_{j, \pm}) \to \ker(U \mp 1) \) defined by
\[
\tau_{j, \pm}(\psi) := \left( \frac{\sqrt{\Lambda(\mp(-1)^j a(x))}}{\psi} \right)^{1/x}, \quad \psi \in \ker(L - \delta_{j, \pm}).
\]
That is, \( \dim \ker(L - \delta_{j, \pm}) = \dim \ker(U \mp 1) = 1 \) according to (28) to (29).

(iii) For each \( j = 1, 2 \), let
\[
\delta^{\downarrow}_{j, \pm} := \min \left\{ \liminf_{x \to \infty} \left( \prod_{y=1}^{x} |\delta_{j, \pm}(-y)|^{-2} \right)^{1/x}, \liminf_{x \to \infty} \left( \prod_{y=0}^{x-1} |\delta_{j, \pm}(y)|^{2} \right)^{1/x} \right\},
\]
\[
\delta^{\uparrow}_{j, \pm} := \max \left\{ \limsup_{x \to \infty} \left( \prod_{y=1}^{x} |\delta_{j, \pm}(-y)|^{-2} \right)^{1/x}, \limsup_{x \to \infty} \left( \prod_{y=0}^{x-1} |\delta_{j, \pm}(y)|^{2} \right)^{1/x} \right\},
\]
\[
\Lambda^{\downarrow}_{j, \pm} := \inf_{x \in \mathbb{Z}} \Lambda(\mp(-1)^j a(x)) + 1,
\]
\[
\Lambda^{\uparrow}_{j, \pm} := \sup_{x \in \mathbb{Z}} \Lambda(\mp(-1)^j a(x)) + 1.
\]

If \( 0 < \delta^{\downarrow}_{j, \pm} \leq \delta^{\uparrow}_{j, \pm} < 1 \) for some \( j = 1, 2 \), then \( \Delta_{j, \pm} < \infty \). Moreover, in this case, for any \( \epsilon > 0 \) satisfying \( 0 < \delta^{\downarrow}_{j, \pm} - \epsilon < \delta^{\uparrow}_{j, \pm} + \epsilon < 1 \), there exists \( x_{\pm} \in \mathbb{N} \) with the property that if \( \psi \in \ker(L - \delta_{j, \pm}) \) is a non-zero vector, then \( \Psi := \tau_{j, \pm}(\psi) \) given by (30) exhibits the following exponential decay:
\[
\Lambda^{\downarrow}_{j, \pm} \left( \delta^{\downarrow}_{j, \pm} - \epsilon \right)^{|x|} \leq \frac{\|\Psi(x)\|^2}{|\psi(0)|^2} \leq \Lambda^{\uparrow}_{j, \pm} \left( \delta^{\uparrow}_{j, \pm} + \epsilon \right)^{|x|}, \quad |x| \geq x_{\pm}.
\]

**Remark 3.2.** We have the following remarks:

(i) Note that the function \( \Lambda \) defined by (25) is a bijection from \((-1, 1)\) onto \((0, \infty)\), and that the graph of \( \Lambda \) is given by the following figure;

![Figure 1: This figure represents the graph of \( t = \Lambda(s) \).](image)

That is, \( \Lambda_{j, \pm} \) defined by (27) is either a finite positive number or \( +\infty \). The formula (29) is a complete classification of \( \operatorname{ind}_{\mp}(\Gamma, U) \), since \( \Lambda_{1, \pm} \) and \( \Lambda_{2, \pm} \) cannot be finite at the same time according to Theorem 3.1.
(ii) It is in general difficult to compute $\delta_{j,\pm}^\pm$, $\delta_{j,\pm}^\pm$, but the following estimates may be useful:\footnote{The estimation \ref{31} to \ref{32} can be easily proved by the well-known fact (see, for example, \cite[Theorem 3.37]{Rud76}) that given a sequence $(\alpha(x))_{x \in \mathbb{N}}$ of positive numbers, we have\footnote{The estimation \ref{40} to \ref{42} is valid.}:

\[
\lim_{x \to \infty} \frac{\alpha(x+1)}{\alpha(x)} \leq \liminf_{x \to \infty} \alpha(x)^{1/s} \leq \limsup_{x \to \infty} \alpha(x)^{1/s} \leq \limsup_{x \to \infty} \frac{\alpha(x+1)}{\alpha(x)}.
\]

\[
\delta_{j,\pm}^\pm \geq \min \left\{ \liminf_{x \to \infty} |\delta_{j,\pm}(-x)|^{-2}, \liminf_{x \to \infty} |\delta_{j,\pm}(x)|^2 \right\}, \quad (36)
\]

\[
\delta_{j,\pm}^\pm \leq \max \left\{ \limsup_{x \to \infty} |\delta_{j,\pm}(-x)|^{-2}, \limsup_{x \to \infty} |\delta_{j,\pm}(x)|^2 \right\}. \quad (37)
\]

3.1. Proof of the bulk-edge correspondence

The purpose of the current section is to prove Theorem 3.1. We can then obtain Theorem 1.1 as an immediate corollary. In what follows we shall make use of the following obvious properties of $\Lambda$ without any further comment. For each $s, s' \in (-1, 1)$, we have

\[
\Lambda(-s) = \Lambda(s)^{-1}, \quad (38)
\]

\[
\Lambda(s)\Lambda(s') = \Lambda \left( \frac{s + s'}{1 + ss'} \right), \quad (39)
\]

\[
\Lambda(s)\Lambda(s') \lesssim 1 \text{ if and only if } s + s' \lesssim 0, \quad (40)
\]

where $1 + ss' > 0$ in \ref{40}, and where the notation $\lesssim$ in \ref{40} simultaneously denotes the three binary relations $>$, $=$, $\leq$.

Lemma 3.3. If \ref{40} holds true, then we have the following well-defined linear isomorphisms:

\[
\ker(L - \delta_{j,\pm}) \ni \psi \mapsto \left( \frac{a \overline{\psi}}{\psi} \right) \in \ker(I' + (-1)^j) \cap \ker(I' + (-1)^{j+1}), \quad j = 1, 2, \quad (41)
\]

where the bounded sequence $\delta_{j,\pm}$ is defined by \ref{26}.

That is,

\[
\ker(I' + (-1)^j) \cap \ker(I' + (-1)^{j+1}) = \left\{ \left( \frac{a \overline{\psi}}{\psi} \right) \mid \psi \in \ker(L - \delta_{j,\pm}) \right\}, \quad j = 1, 2.
\]

Proof. It follows from \ref{40} that the following sequences are bounded:

\[
\frac{-b^*}{a + 1} = \frac{a + 1}{b}, \quad \frac{-q}{p + 1} = \frac{p + 1}{q^*}. \quad (42)
\]

For each $\psi_1, \psi_2 \in \ell^2(\mathbb{Z})$, let us first consider the following $\mathbb{C}^2$-vector:

\[
\left( I' + 1 \right) \left( \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right) = \left( \begin{array}{cc} a + 1 & b^* \\ b & -(a + 1) \end{array} \right) \left( \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right) = \left( \frac{(a + 1)\psi_1 + b^*\psi_2}{b\psi_1 - (a + 1)\psi_2} \right),
\]

where $(a \mp 1)\psi_1 + b^*\psi_2 = 0$ if and only if $b\psi_1 - (a - 1)\psi_2 = 0$ by the first equality in \ref{12}. It follows that the following equality holds true:

\[
\ker(I' + 1) = \ker \left( \left( \begin{array}{cc} a + 1 & b^* \\ b & -(a + 1) \end{array} \right) \mp 1 \right) = \left\{ \left( \frac{a + 1}{b} \psi \right) \mid \psi \in \ell^2(\mathbb{Z}) \right\}.
\]
It follows that
\[ \ker(\Gamma \mp 1) = \ker \begin{pmatrix} 1 & 0 \\ 0 & L^* \end{pmatrix} \left\{ \begin{pmatrix} p & q \\ q^* & -p \end{pmatrix} \mp 1 \right\} \begin{pmatrix} 1 & 0 \\ 0 & L^* \end{pmatrix} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & L^* \end{pmatrix} \begin{pmatrix} p & q \\ q^* & -p \end{pmatrix} \right\} \psi \in \ell^2(\mathbb{Z}) \),
\]
where the last equality follows from the second equality in (42). We obtain the following equalities for each \( j = 1, 2 \):
\[
\ker(\Gamma' \mp (-1)^{j+1} \gamma) = \left\{ \begin{pmatrix} a \mp (-1)^j \gamma \\ \psi b \end{pmatrix} \bigg| \psi \in \ell^2(\mathbb{Z}) \right\},
\]
\[
\ker(\Gamma' \mp (-1)^{j}) = \left\{ \begin{pmatrix} \gamma \psi \pm \frac{q}{p} \psi \\ \psi \end{pmatrix} \bigg| \psi \in \ell^2(\mathbb{Z}) \right\}.
\]

Next, we show that (11) is a well-defined linear transform. Note that the bounded sequence \( \delta_{j, \pm} \) consists of:
\[
\Lambda((-1)^j p) = \frac{1 + (-1)^j p}{1 - (-1)^j p} \times \frac{1 + (-1)^j q}{1 + (-1)^j p} = \frac{(1 + (-1)^j p)^2}{|q|^2},
\]
\[
\Lambda(\mp (-1)^j a) = \frac{1 \mp (-1)^j a}{1 \pm (-1)^j a} \times \frac{1 \mp (-1)^j a}{1 \pm (-1)^j a} = \frac{(1 \mp (-1)^j a)^2}{|b|^2},
\]
where \( 1 + (-1)^j p \) and \( 1 \mp (-1)^j a \) are sequences of positive numbers. We obtain
\[
\delta_{j, \pm} = \pm e^{-i(\theta + \phi)} \sqrt{\Lambda((-1)^j p)\Lambda(\mp (-1)^j a)} = \pm \frac{1 + (-1)^j p}{q} \frac{1 \mp (-1)^j a}{b} = \frac{p + (-1)^j a}{b} \frac{-q}{b}. \quad (45)
\]
It follows from (45) that given \( \psi \in \ell^2(\mathbb{Z}) \), we have that \( \psi \in \ker(L - \delta_{j, \pm}) \) if and only if the following equality holds true:
\[
\frac{-q}{p + (-1)^j a} L \psi = \frac{-a \mp (-1)^j \gamma}{b} \psi.
\]
It follows from (43) to (44) that (11) is a well-defined bijective linear transform.

**Lemma 3.4.** Let \( \delta = (\delta(x))_{x \in \mathbb{Z}} \) be a bounded sequence of non-zero complex numbers, and let
\[
\Delta := \sum_{x=1}^{\infty} \left( \prod_{y=1}^{x} |\delta(-y)|^{-2} \right) + \sum_{x=1}^{\infty} \left( \prod_{y=0}^{x-1} |\delta(y)|^2 \right).
\]
Then the following assertions hold true:

(i) We have
\[
\dim \ker (L - \delta) = \begin{cases} 1, & \Delta < \infty, \\ 0, & \Delta = \infty. \end{cases} \quad (46)
\]

(ii) Let
\[
\delta^\dagger := \min \left\{ \liminf_{x \to \infty} \left( \prod_{y=1}^{x} |\delta(-y)|^{-2} \right)^{1/x}, \liminf_{x \to \infty} \left( \prod_{y=0}^{x-1} |\delta(y)|^2 \right)^{1/x} \right\},
\]
\[
\delta^\dagger := \max \left\{ \limsup_{x \to \infty} \left( \prod_{y=1}^{x} |\delta(-y)|^{-2} \right)^{1/x}, \limsup_{x \to \infty} \left( \prod_{y=0}^{x-1} |\delta(y)|^2 \right)^{1/x} \right\}. \quad (47)
\]
If \( 0 < \delta^\dagger \leq \delta^\dagger < 1 \), then \( \dim \ker (L - \delta) = 1 \). Moreover, in this case, for any \( \epsilon > 0 \) satisfying \( 0 < \delta^\dagger - \epsilon \leq \delta^\dagger + \epsilon < 1 \), there exists \( x_\epsilon \in \mathbb{N} \), such that for any \( \psi \in \ker (L - \delta) \) we have
\[
|\psi(0)|^2 (\delta^\dagger - \epsilon) |x| \leq |\psi(x)|^2 \leq |\psi(0)|^2 (\delta^\dagger + \epsilon) |x|, \quad |x| \geq x_\epsilon. \quad (48)
\]
Note that (i) shows that \( \dim \ker (L - \delta) \) depends only on \( |\delta| \). As for (ii), if \( 0 < \delta^\uparrow \leq \delta^\downarrow < 1 \), then (50) can be rewritten as

\[
|\psi(0)|^2 e^{\log(\delta^\downarrow - \epsilon)|x|} \leq |\psi(x)|^2 \leq |\psi(0)|^2 e^{\log(\delta^\uparrow + \epsilon)|x|}, \quad |x| \geq x_\epsilon,
\]

where \( \epsilon > 0 \) is any number satisfying \( 0 < \delta^\downarrow - \epsilon \leq \delta^\uparrow + \epsilon < 1 \).

**Proof.** (i) We need to solve a difference equation of the form:

\[
\psi(x + 1) = \delta(x) \psi(x), \quad \forall x \in \mathbb{Z}.
\]

(50) Since each \( \delta(x) \) is non-zero, such a solution is uniquely determined by the initial value \( \psi(0) \). In particular, if \( \psi, \psi' \in \ker (L - \delta) \) are non-zero vectors, then \( \psi(0), \psi'(0) \) are non-zero, and so the linear combination \( \psi'(0) \psi - \psi(0) \psi' \) is the zero vector. It follows that \( \psi, \psi' \) are linearly independent, and so \( \dim \ker (L - \delta) \leq 1 \).

Suppose that we have a bounded sequence \( \psi = (\psi(x))_{x \in \mathbb{Z}} \) satisfying (50). We have

\[
\psi(x) = \prod_{y=0}^{x-1} \delta(y) \psi(0), \quad \psi(-x) = \prod_{y=1}^{x} \delta(-y)^{-1} \psi(0), \quad x \geq 1.
\]

(51) Since \( \sum_{x \in \mathbb{Z}} |\psi(x)|^2 = |\psi(0)|^2 + \sum_{x \in \mathbb{N}} |\psi(-x)|^2 + \sum_{x \in \mathbb{N}} |\psi(x)|^2 \), we get

\[
\sum_{x \in \mathbb{Z}} |\psi(x)|^2 = |\psi(0)|^2 + |\psi(0)|^2 \sum_{x \in \mathbb{N}} \prod_{y=1}^{x} |\delta(-y)|^{-2} + |\psi(0)|^2 \sum_{x \in \mathbb{N}} \prod_{y=0}^{x-1} |\delta(y)|^2.
\]

That is, \( \dim \ker (L - \delta) = 1 \) if and only if \( \Delta := \sum_{x=1}^{\infty} \left( \prod_{y=0}^{x} |\delta(-y)|^{-2} \right) + \sum_{x=1}^{\infty} \left( \prod_{y=0}^{x-1} |\delta(y)|^2 \right) < \infty. \)

(ii) If \( 0 < \delta^\downarrow \leq \delta^\uparrow < 1 \), then \( \dim \ker (L - \delta) = 1 \) by the root test. Let \( \epsilon > 0 \) be any number satisfying \( 0 < \delta^\downarrow - \epsilon \leq \delta^\uparrow + \epsilon < 1 \). It follows that there exists \( x_\epsilon \in \mathbb{N} \), such that

\[
\delta^\downarrow - \epsilon < \min \left\{ \inf_{x \geq x_\epsilon} \left( \prod_{y=1}^{x} |\delta(-y)|^{-2} \right)^{1/x}, \inf_{x \geq x_\epsilon} \left( \prod_{y=0}^{x-1} |\delta(y)|^2 \right)^{1/x} \right\},
\]

(52)

\[
\delta^\uparrow + \epsilon > \max \left\{ \sup_{x \geq x_\epsilon} \left( \prod_{y=1}^{x} |\delta(-y)|^{-2} \right)^{1/x}, \sup_{x \geq x_\epsilon} \left( \prod_{y=0}^{x-1} |\delta(y)|^2 \right)^{1/x} \right\}.
\]

(53)

Let \( \psi \in \ker (L - \delta) \), and let \( |x| \geq x_\epsilon \). On one hand, if \( x \geq x_\epsilon \), then \( |\psi(x)|^2 = \prod_{y=0}^{x-1} |\delta(y)|^2 |\psi(0)|^2 \), and so

\[
(\delta^\downarrow - \epsilon)^2 |\psi(0)|^2 < |\psi(x)|^2 < (\delta^\uparrow + \epsilon)^2 |\psi(0)|^2.
\]

On the other hand, if \( -x \geq x_\epsilon \), then \( |\psi(x)|^2 = \prod_{y=1}^{x} |\delta(-y)|^{-2} |\psi(0)|^2 \), and so

\[
(\delta^\downarrow - \epsilon)^{-2} |\psi(0)|^2 < |\psi(x)|^2 < (\delta^\uparrow + \epsilon)^{-2} |\psi(0)|^2.
\]

The claim follows. \( \square \)

**Proof of Theorem 3.7** It follows from Proposition 2.3(i) that

\[
\ker(U \equiv 1) = \bigoplus_{j=1,2} \ker(\Gamma + (-1)^j) \cap \ker(\Gamma' \equiv (-1)^{j+1}).
\]

The linear isomorphisms of the form (40) allow us to let

\[
m_{j,+} := \dim (\ker(\Gamma + (-1)^j) \cap \ker(\Gamma' \equiv (-1)^{j+1})) = \dim \ker (L - \delta_{j,+}), \quad j = 1, 2.
\]
Let δ := δ_{j,±} and let Δ := Δ_{j,±}. It then follows from Lemma [3,4] that

\[ m_{j,±} = \begin{cases} 
1, & \text{Δ}_{j,±} < \infty, \\
0, & \text{otherwise}.
\end{cases} \]  

(54)

As in Proposition [2,3(i)], we obtain the following formulas:

\[ \text{ind}_±(I, U) = m_{1,±} - m_{2,±}, \]  

(55)

\[ \dim \ker(U \mp 1) = m_{1,±} + m_{2,±}, \]  

(56)

where each \( m_{j,±} \) is either 0 or 1. Assume the contrary that \( \Delta_{j,±} < \infty \) for each \( j = 1, 2 \). In this case, for each \( j = 1, 2 \), we have \( \prod_{y=0}^{\infty} |\delta_{1,±}(y)|^2 \to 0 \) as \( x \to \infty \). Therefore, \( \prod_{y=0}^{\infty-x} |\delta_{1,±}(y)|^2 |\delta_{2,±}(y)|^2 \to 0 \) as \( x \to \infty \).

Note, however, that this is impossible, since for each \( y = 0, \ldots, x-1 \) we have

\[ |\delta_{1,±}(y)|^2 |\delta_{2,±}(y)|^2 = \Lambda(p(y))^{-1} \Lambda(\mp a(y))^{-1} \Lambda(p(y)) \Lambda(\mp a(y)) = 1. \]

This contradiction shows \( \Delta_{1,±} + \Delta_{2,±} = \infty \).

(i) If \( \Delta_{1,±} = \Delta_{2,±} = \infty \), then we get the trivial equalities \( \text{ind}_±(I, U) = 0 = \dim \ker(U \mp 1) \). On one hand, if \( \Delta_{1,±} < \infty \), then \( \text{ind}_±(I, U) = 1 - 0 = 1 \) and \( \dim \ker(U \mp 1) = 1 + 0 \). On the other hand, if \( \Delta_{2,±} < \infty \), then \( \text{ind}_±(I, U) = 0 - 1 = 0 \) and \( \dim \ker(U \mp 1) = 0 + 1 \). Thus, the formulas (28) to (29) have been verified.

(ii) It is obvious that (30) defines a linear isomorphism.

(iii) If \( 0 < \delta_{j,±}^1 < \delta_{j,±}^2 < \infty \) for some \( j = 1, 2 \), then \( \Delta_{j,±} < \infty \) by the root test. Let \( \epsilon > 0 \) be any number satisfying \( 0 < \delta_{j,±}^1 - \epsilon < \delta_{j,±}^2 + \epsilon < 1 \). It follows from Lemma [3,4(ii)] that there exists \( x_± \in \mathbb{N} \), such that for any non-zero \( \psi \in \ker(L - \delta_{j,±}) \), we have

\[ (\delta_{j,±}^1 - \epsilon) |x| \leq \frac{|\psi(x)|^2}{|\psi(0)|^2} \leq (\delta_{j,±}^2 + \epsilon) |x|, \quad |x| \geq x_±. \]

Let \( \Psi := \tau_{j,±}(\psi) \) be defined by (31). With (30) in mind, we have \( \|\Psi(x)\|^2 = (\Lambda(\mp(-1)^j a(x)) + 1) |\psi(x)|^2 \) for each \( x \in \mathbb{Z} \). The claim follows.

3.2. The anisotropic case

Proof of Theorem [7,4] Let \( U = U_{\text{aux}} \) be the evolution operator of Suzuki’s split-step quantum walk given by (2), and let us assume the existence of the two-sided limits of the form (5). Let \( |p(x)| < 1 \) and \( |a(x)| < 1 \) for each \( x \in \mathbb{Z} \). It is shown in [Tan21, Theorem B(ii)] that

\[ \text{σ}_{\text{ess}}(U) = \bigcup_{\pm = \infty} \{ z \in \mathbb{T} \mid \text{Re} z \in I(*) \}, \]  

(57)

\[ I(*) := [p(*)a(*) - \sqrt{1 - p(*)^2} \sqrt{1 - a(*)^2}, p(*)a(*) + \sqrt{1 - p(*)^2} \sqrt{1 - a(*)^2}], \quad * = \pm \infty, \]  

(58)

where \( \mathbb{T} \) is the unit-circle in the complex plane. Since \( p(*), a(*) \in (-1, 1) \), we can uniquely write \( p(*) = \sin \theta(*) \) and \( a(*) = \sin \phi(*) \) for some \( \theta(*) , \phi(*) \in (-\pi/2, \pi/2) \). We get

\[ p(*)a(*) \pm \sqrt{1 - p(*)^2} \sqrt{1 - a(*)^2} = \sin \theta(*) \sin \phi(*) \pm \cos \theta(*) \cos \phi(*) = \pm \cos (\theta(*) \mp \phi(*)), \]

where \( \mp \in I(*) \) if and only if \( \pm = \cos(\theta(*) \mp \phi(*)) \) if and only if \( \theta(*) \mp \phi(*) = 0 \). Since \( (-\pi/2, \pi/2) \ni x \mapsto \sin x \in (-1, 1) \) is a bijective odd function, the last equality is equivalent to \( p(*) \mp a(*) \). That is, \( p(*) \neq \pm a(*) \) for each \( * = \pm \infty \) if and only if \( \pm \not\in \sigma_{\text{ess}}(U) \). From here on, we assume \( \pm \not\in \sigma_{\text{ess}}(U) \) and prove Theorem 3.1(ii).

(i) Since \( p(*) , a(*) \in (-1, 1) \), it follows from the continuity of \( \Lambda \) that \( \lim_{x \to *} \Lambda(p(x)) \Lambda(\mp a(x)) = \Lambda(p(*)) \Lambda(\mp a(*)) \). That is,

\[ \lim_{x \to *} \Lambda(p(x)) \Lambda(\mp a(x)) \leq 1 \text{ if and only if } \iff p(*) \mp a(*) \leq 0, \]
where $\preceq$ simultaneously denotes the two binary relations $>$ and $<$. For each $j = 1, 2$, we get

$$
\Delta_{j, \pm} = \sum_{x=1}^{\infty} \left( \prod_{y=0}^{x-1} (\Lambda(p(-y))\Lambda(p^y(-y))^{|-1|^{j+1}}) + \sum_{x=1}^{\infty} \left( \prod_{y=0}^{x-1} (\Lambda(p(-y))\Lambda(p^y(-y))^{|-1|^{j+1}}) \right) \right).
$$

It follows from the ratio test that

$$
\Delta_{j, \pm} < \infty \text{ if and only if } (-1)^j (p(+\infty) \mp a(+\infty)) < 0 < (-1)^j (p(-\infty) \mp a(-\infty)), \quad j = 1, 2.
$$

It is now easy to see that (29) becomes (6).

(ii) We have

$$
\delta_{j, \pm}^i = \min \left\{ (\Lambda(p(-\infty))\Lambda(p^y(-\infty)))^{(-1)^{j+1}}, (\Lambda(p(+\infty))\Lambda(p^y(+\infty)))^{(-1)^{j+1}} \right\}, \quad (59)
$$

$$
\delta_{j, \pm}^f = \max \left\{ (\Lambda(p(-\infty))\Lambda(p^y(-\infty)))^{(-1)^{j+1}}, (\Lambda(p(+\infty))\Lambda(p^y(+\infty)))^{(-1)^{j+1}} \right\}. \quad (60)
$$

It follows from Theorem 3.1(iii) that for each $j = 1, 2$, we have $0 < \delta_{j, \pm}^i \leq \delta_{j, \pm}^f < 1$ if and only if $\Delta_{j, \pm} < \infty$. Moreover, in this case, for any $\epsilon > 0$ satisfying $0 < \delta_{j, \pm}^i - \epsilon < \delta_{j, \pm}^f + \epsilon < 1$, there exists $x_{\pm} \in \mathbb{N}$ with the property that if $\psi \in \ker(L - \delta_{j, \pm})$ is a non-zero vector, then $\Psi := \tau_{j, \pm}(\psi)$ given by (30) exhibits the following exponential decay;

$$
A_{j, \pm}^\pm \left( \delta_{j, \pm}^i - \epsilon \right) |x| \leq \frac{\|\Psi(x)\|^2}{\psi(0)^2} \leq A_{j, \pm}^\pm \left( \delta_{j, \pm}^f + \epsilon \right) |x|, \quad |x| \geq x_{\pm}. \quad (61)
$$

We obtain (8), if we let

$$
k_{j, \pm}^i := |\psi(0)|^2 \Lambda_{j, \pm}^i, \quad (62)
$$

$$
k_{j, \pm}^f := |\psi(0)|^2 \Lambda_{j, \pm}^f, \quad (63)
$$

$$
c_{j, \pm}^i := -\log \left( \delta_{j, \pm}^i - \epsilon \right), \quad (64)
$$

$$
c_{j, \pm}^f := -\log \left( \delta_{j, \pm}^f + \epsilon \right). \quad (65)
$$

4. Discussion and Concluding Remarks

4.1. Kitagawa’s split-step quantum walk

We show that (1) can be made unitarily equivalent to (2), provided that we appropriately define $p, q, a, b$ in terms of $\theta_1, \theta_2$.

Lemma 4.1. Let

$$
p := \sin \theta_2(\cdot + 1), \quad q := \cos \theta_2(\cdot + 1), \quad a := -\sin \theta_1, \quad b := \cos \theta_1.
$$

Then

$$
\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} U_{\text{kit}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} p & qL \\ -qL^* & -p(\cdot - 1) \end{pmatrix} \begin{pmatrix} a & b^* \\ b & -a \end{pmatrix}.
$$

Proof. Let $\sigma_1$ be the first Pauli matrix. Given any $\mathbb{R}$-valued sequence $\theta = (\theta(x))_{x \in \mathbb{Z}}$, we consider the following rotation matrix;

$$
R(\theta) := \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}.
$$
It is obvious that \( R(\theta)R(\phi) = R(\theta + \phi) \) for any \( \mathbb{R} \)-valued sequences \( \theta, \phi \). If we let \( \Gamma := \sigma_1(1 \oplus L)R(\theta_2)(L^* \oplus 1) \) and \( \Gamma' := R(\theta_1)\sigma_1 \), then

\[
\sigma_1 U_{\text{sin}} \sigma_1 = \Gamma \Gamma'.
\]

We have

\[
R(\theta_j)\sigma_1 = \begin{pmatrix}
\cos \theta_j \\
\sin \theta_j \\
\sin \theta_j \\
\cos \theta_j
\end{pmatrix} = \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
-\sin \theta_j \\
\cos \theta_j \\
\cos \theta_j \\
\sin \theta_j
\end{pmatrix}.
\]

Now

\[
\Gamma = \sigma_1(1 \oplus L)R(\theta_2)(L^* \oplus 1) = \begin{pmatrix}
p & qL \\
L^*q^* & -p(-1)
\end{pmatrix}.
\]

4.2. Spectral gaps and decay rates

We shall impose the assumption of Theorem 1.1 throughout this subsection. Recall that any non-zero vector \( \Psi := \tau_j \), \( \pm \) given by (30) exhibits the following exponential decay;

\[
\Lambda \downarrow j, \pm (\delta \downarrow j, \pm - \epsilon) |x| \leq ||\Psi(x)|| \leq \Lambda \uparrow j, \pm (\delta \uparrow j, \pm + \epsilon) |x|, |x| \geq x_\pm,
\]

where

\[
\delta \downarrow j, \pm = \min \left\{ (\Lambda(p(-\infty))\Lambda(\mp a(-\infty)))^{(-1)^{j+1}}, (\Lambda(p(+\infty))\Lambda(\mp a(+\infty)))^{(-1)^{j-1}} \right\},
\]

\[
\delta \uparrow j, \pm = \max \left\{ (\Lambda(p(-\infty))\Lambda(\mp a(-\infty)))^{(-1)^{j+1}}, (\Lambda(p(+\infty))\Lambda(\mp a(+\infty)))^{(-1)^{j-1}} \right\}.
\]

4.2.1. Half-gapped examples

**Example 4.2** (half-gapped case). Let \( 0 < p_0 < 1 \), and let

\[
p(-\infty) := -p_0, \quad a(-\infty) := \pm p_0,
p(+\infty) := p_0, \quad a(+\infty) := \mp p_0.
\]

Since \( \pm a(-\infty) = p_0 \neq p(-\infty) \) and \( \pm a(+\infty) = -p_0 \neq p(+\infty) \), the essential spectrum of the operator \( U \) has a spectral gap at \( \pm 1 \). Moreover,

\[
p(-\infty) \mp a(-\infty) = -2p_0 < 0 < 2p_0 = p(+\infty) \mp a(+\infty).
\]

We get \( \text{ind} \pm (\Gamma, U) = \dim \ker(U \mp 1) = 1 \). It follows from (57) to (58) that the essential spectrum of \( U \) is given explicitly by

\[
\sigma_{\text{ess}}(U) = \{ z \in \mathbb{T} \mid \Re z \in I_\pm \}, \quad I_\pm := [\mp p_0^2 - (1 - p_0^2), \mp p_0^2 + (1 - p_0^2)].
\]

More precisely, the essential spectrum \( \sigma_{\text{ess}}(U) \) can be classified into the following two distinct cases:
\( (a(-\infty), a(+\infty)) = (p_0, -p_0) \) \( (a(-\infty), a(+\infty)) = (-p_0, p_0) \)

\[ \sigma_{\text{ess}}(U) = \{ z \in \mathbb{T} \mid \Re z \in I_+ \} \]

\[ \sigma_{\text{ess}}(U) = \{ z \in \mathbb{T} \mid \Re z \in I_- \} \]

Figure 2: The black connected regions depict \( \sigma_{\text{ess}}(U) \). We have \(-1 \in \sigma_{\text{ess}}(U)\) in Case (i), whereas \(+1 \in \sigma_{\text{ess}}(U)\) in Case (ii).

We have \( I_+ = [-1, 1 - 2p_0^2] \) in Case (i), and \( I_- = [-1 + 2p_0^2, 1] \) in Case (ii). This motivates us to introduce the gap width \( \Omega(p_0) := 2p_0^2 \).

Moreover, we have

\[ \delta_{1,\pm} = \Lambda(-p_0)^2. \]

Note that \( \Omega(p_0) = 2p_0^2 \) increases as \( p_0 \to 1 \). In this case, the convergence rates \( \delta_{1,\pm} = \Lambda(-p_0)^2 \) decrease, since \( \Lambda(-p_0)^2 \to 0 \).

4.2.2. Double-gapped examples

We let \( p(x) = 0 \) for each \( x \in \mathbb{Z} \). We have the following index formula;

\[ \text{ind}_\pm(I, U) = \begin{cases} +1, & \mp a(-\infty) < 0 < \mp a(+\infty), \\ -1, & \mp a(+\infty) < 0 < \mp a(-\infty), \\ 0, & \text{otherwise}. \end{cases} \quad (67) \]

Note that we have

\[ \delta_{j,\pm}^\dagger = \min \left\{ \Lambda(\mp(-1)^j a(+\infty)), \frac{1}{\Lambda(\mp(-1)^j a(-\infty))} \right\}, \quad (68) \]

\[ \delta_{j,\pm}^\ddagger = \max \left\{ \Lambda(\mp(-1)^j a(+\infty)), \frac{1}{\Lambda(\mp(-1)^j a(-\infty))} \right\}, \quad (69) \]

It follows from (57) to (58) that

\[ \sigma_{\text{ess}}(U) = \bigcup_{* = \pm\infty} \{ z \in \mathbb{T} \mid \Re z \in I(*) \}, \quad (70) \]

\[ I(*) := [-\sqrt{1 - a(*)^2}, \sqrt{1 - a(*)^2}], \quad * = \pm\infty. \quad (71) \]
Fig. 3 motivates us to introduce the following gap width;

\[
\Omega := \min\{1 - \sqrt{1 - a(-\infty)^2}, 1 - \sqrt{1 - a(+\infty)^2}\}.
\]

Suppose that \(\mp a(-\infty) < 0 < \mp a(+\infty)\) holds true. Then \(\Lambda(\mp a(-\infty)) < 1 \text{ and } \frac{1}{\Lambda(\mp a(+\infty))} < 1\). Note that \(\Omega\) increases as \(|a(\ast)| \to 1\) for each \(\ast = \pm\infty\). This corresponds to \(\mp a(-\infty) \to -1\) and \(\mp a(+\infty) \to +1\). Thus, \(\Lambda(\mp a(-\infty)) \to 0\) and \(\frac{1}{\Lambda(\mp a(+\infty))} \to 0\). It follows that \(\delta_{2,\pm} \to 0\) and \(\delta_{1,\pm} \to 0\).

4.3. The square of the evolution operator

Theorem 3.1 gives a concrete quantum walk example with the property that the estimate (16) becomes an equality. The purpose of the current subsection is to show that this is not always the case. Our counter example is based on the following simple proposition.

**Proposition 4.3.** If \((\Gamma, U)\) is an abstract chiral pair on a Hilbert space \(H\), then \((\Gamma, U^2)\) and \((\Gamma' \Gamma' \Gamma', U^2)\) are unitarily equivalent chiral pairs. Moreover, the following assertions hold true:

(i) If \(\ker(U^2 - 1) = \ker(U - 1) \oplus \ker(U + 1)\) is finite-dimensional, then

\[
\text{ind}_+(\Gamma, U^2) = \text{ind}(\Gamma, U).
\]

(ii) If \(\ker(U^2 + 1) = \ker(U - i) \oplus \ker(U + i)\) is finite-dimensional, then

\[
\text{ind}_-(\Gamma, U^2) = 0.
\]

(iii) If \(\ker(U^2 - 1) \oplus \ker(U^2 + 1)\) is finite-dimensional, then \(\text{ind}(\Gamma, U^2) = \text{ind}(\Gamma, U)\).

**Proof.** Note that \((\Gamma, U^2)\) and \((\Gamma' \Gamma' \Gamma', U^2)\) are chiral pairs, since \(U^2 = \Gamma'(\Gamma' \Gamma')\). We have

\[
(\Gamma, U^2) = (\Gamma, \Gamma' \Gamma' \Gamma') \cong (\Gamma' \Gamma' \Gamma', \Gamma' (\Gamma' \Gamma' \Gamma') \Gamma') = (\Gamma' \Gamma' \Gamma', (U^2)^*) \cong (\Gamma' \Gamma' \Gamma', U^2),
\]

where \(\cong\) represents unitary equivalence. If \(U\) admits the standard representation of the form (11), then \(U^2\) admits the following standard representation;

\[
U^2 = \begin{pmatrix}
2R_1^2 - 1 & 2iQ_1R_2 \\
2iQ_1R_1 & 2R_2^2 - 1
\end{pmatrix}.
\]

It follows that

\[
\text{ind}_\pm(\Gamma, U^2) := \dim \ker((2R_1^2 - 1) \mp 1) - \dim \ker((2R_2^2 - 1) \mp 1).
\]
(i) If $\ker(U^2 - 1) = \ker(U - 1) \oplus \ker(U + 1)$ is finite-dimensional, then $\ker(U - 1) = \ker(R - 1)$ and $\ker(U + 1) = \ker(R + 1)$ are finite-dimensional. In this case,

$$\text{ind}_+ (\Gamma, U^2) = \dim \ker((2R^2 - 1) - 1) - \dim \ker((2R^2 - 1) - 1)$$

$$= \dim \ker(R^2 - 1) - \dim \ker(R^2 - 1)$$

$$= \dim \ker(R_1 - 1) + \dim \ker(R_1 + 1) - (\dim \ker(R_2 - 1) + \dim \ker(R_2 - 1))$$

$$= \dim \ker(R_1 - 1) - \dim \ker(R_2 - 1) + \dim \ker(R_1 + 1) - \dim \ker(R_2 - 1)$$

$$= \text{ind}_+ (\Gamma, U) + \text{ind}_- (\Gamma, U)$$

where the last equality follows from (78). We get $\text{ind}_- (\Gamma, U^2) = 0$. (ii) If $\ker(U^2 + 1) = \ker(U - i) \oplus \ker(U + i)$ is finite-dimensional, then it follows from (74) that

$$\text{ind}_- (\Gamma, U^2) = \dim \ker((2R^2 - 1) + 1) = \ker((2R^2 - 1) + 1) = \ker((2R^2 - 1) + 1) = \ker((2R^2 - 1) + 1)$$

and $\ker \Gamma, U$ be the evolution operator of Suzuki’s split-step quantum walk, and let $\Gamma, U$ be defined by the right hand side of (77) (resp. of (78)). We are required to show $\text{ind}_+ (\Gamma, U) = \text{ind}_- (\Gamma, U) = 1$. Thus $\text{ind}_+ (\Gamma, U^2) = 0$. On the other hand, we get

$$\text{dim} \ker(U^2 - 1) \oplus \ker(U + 1)) = |\text{ind}_+ (\Gamma, U)| + |\text{ind}_- (\Gamma, U)| = 2.$$

Then $\dim \ker(U^2 - 1) = 2$ and $\text{ind}_+ (\Gamma, U^2) = 0$. That is, $|\text{ind}_+ (\Gamma, U^2)| \neq \dim \ker(U^2 - 1)$.

### 4.4 A New Derivation of the Existing Index Formulas

The purpose of the current subsection is to give an alternative derivation of the following existing index formulas by making use of (9);

**Theorem 4.6** ([Tan21], Theorem B). Under the assumption of Theorem 1.1, let $-1, +1 \notin \sigma_{\text{ess}}(U)$. Then

$$\text{ind}_+ (\Gamma, U) =
\begin{cases}
0, & \text{if } |p(-\infty)| < |a(-\infty)| \text{ and } |p(+\infty)| < |a(+\infty)|, \\
+\text{sign } p(+\infty), & \text{if } |p(-\infty)| < |a(-\infty)| \text{ and } |p(+\infty)| > |a(+\infty)|, \\
-\text{sign } p(-\infty), & \text{if } |p(-\infty)| > |a(-\infty)| \text{ and } |p(+\infty)| < |a(+\infty)|, \\
+\text{sign } p(+\infty) - \text{sign } p(-\infty), & \text{if } |p(-\infty)| > |a(-\infty)| \text{ and } |p(+\infty)| > |a(+\infty)|,
\end{cases}
$$

(77)

$$\text{ind}_- (\Gamma', U') =
\begin{cases}
-\text{sign } a(+\infty), & \text{if } |p(-\infty)| < |a(-\infty)| \text{ and } |p(+\infty)| < |a(+\infty)|, \\
+\text{sign } a(-\infty), & \text{if } |p(-\infty)| < |a(-\infty)| \text{ and } |p(+\infty)| > |a(+\infty)|, \\
-\text{sign } a(+\infty), & \text{if } |p(-\infty)| > |a(-\infty)| \text{ and } |p(+\infty)| < |a(+\infty)|, \\
0, & \text{if } |p(-\infty)| > |a(-\infty)| \text{ and } |p(+\infty)| > |a(+\infty)|,
\end{cases}
$$

(78)

where $\text{sign}$ denotes the sign function $\text{sign} : \mathbb{R} \to \{-1, 1\}$. We let $\text{sign} 0 := 1$ by convention.

**Proof.** Let $i_+$ (resp. $i_-$) be defined by the right hand side of (77) (resp. of (78)). We are required to show $i_+ = \text{ind}_+ (\Gamma, U)$ and $i_- = \text{ind}_- (\Gamma', U')$. Note first that (9) can be rewritten as

$$\text{ind}_\pm (\Gamma, U) =
\begin{cases}
0, & \text{ if } p(-\infty) < \pm a(-\infty) \text{ and } p(+\infty) < \pm a(+\infty), \\
+1, & \text{ if } p(-\infty) < \pm a(-\infty) \text{ and } p(+\infty) > \pm a(+\infty), \\
-1, & \text{ if } p(-\infty) > \pm a(-\infty) \text{ and } p(+\infty) < \pm a(+\infty), \\
0, & \text{ if } p(-\infty) > \pm a(-\infty) \text{ and } p(+\infty) > \pm a(+\infty).
\end{cases}
$$

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Note also that we have $i_{\pm} = \text{ind}_+(I, U) \pm \text{ind}_-(I, U)$ in each of the 16 cases defined by Table 1. The claim follows from Proposition 2.3(iii).

| Cases | $-\infty$ | $+\infty$ | $\text{ind}_+(I, U)$ | $\text{ind}_-(I, U)$ | $i_+$ | $i_-$ |
|-------|-----------|-----------|----------------------|----------------------|-------|-------|
| Case 1 | $p(-\infty) < +a(-\infty)$ | $p(\infty) < +a(\infty)$ | 0 | 0 | 0 |
| Case 2 | $p(-\infty) < +a(-\infty)$ | $p(\infty) < -a(\infty)$ | +1 | −1 | 0 | +2 |
| Case 3 | $p(-\infty) < +a(-\infty)$ | $p(\infty) > a(\infty)$ | +1 | 0 | +1 | +1 |
| Case 4 | $p(-\infty) < +a(-\infty)$ | $-p(\infty) > a(\infty)$ | 0 | −1 | −1 | +1 |
| Case 5 | $p(-\infty) < +a(-\infty)$ | $p(\infty) < a(\infty)$ | −1 | +1 | 0 | −2 |
| Case 6 | $p(-\infty) < -a(-\infty)$ | $p(\infty) < -a(\infty)$ | 0 | 0 | 0 | 0 |
| Case 7 | $p(-\infty) < -a(-\infty)$ | $p(\infty) > a(\infty)$ | 0 | +1 | 0 | −2 |
| Case 8 | $p(-\infty) < -a(-\infty)$ | $-p(\infty) > a(\infty)$ | −1 | 0 | −1 | −1 |
| Case 9 | $p(-\infty) > a(-\infty)$ | $p(\infty) < a(\infty)$ | −1 | 0 | +1 | −1 |
| Case 10 | $p(-\infty) > a(-\infty)$ | $p(\infty) < a(\infty)$ | 0 | −1 | −1 | −1 |
| Case 11 | $p(-\infty) > a(-\infty)$ | $p(\infty) > a(\infty)$ | 0 | 0 | 0 | 0 |
| Case 12 | $p(-\infty) > a(-\infty)$ | $-p(\infty) > a(\infty)$ | −1 | 0 | +1 | +1 |
| Case 13 | $p(-\infty) < a(-\infty)$ | $p(\infty) < a(\infty)$ | 0 | +1 | +1 | −1 |
| Case 14 | $p(-\infty) > a(-\infty)$ | $p(\infty) > a(\infty)$ | +1 | 0 | +1 | +1 |
| Case 15 | $p(-\infty) > a(-\infty)$ | $+p(\infty) > a(\infty)$ | +1 | 0 | +1 | +2 |
| Case 16 | $p(-\infty) > a(-\infty)$ | $-p(\infty) > a(\infty)$ | 0 | 0 | 0 | 0 |

Table 1: Classification of the indices

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