Gluing and lifting exact model structures for the recollement of exact categories

Jiangsheng Hu\textsuperscript{a}, Haiyan Zhu\textsuperscript{b}\textsuperscript{*} and Rongmin Zhu\textsuperscript{c}

\textsuperscript{a}School of Mathematics, Hangzhou Normal University, Hangzhou 311121, China
\textsuperscript{b}College of Science, Zhejiang University of Technology, Hangzhou 310023, China
\textsuperscript{c}School of Mathematical Sciences, Huaqiao University, Quanzhou 362021, China
E-mails: jiangshenghu@hotmail.com, hyzhu@zjut.edu.cn and rongminzhu@hotmail.com

Abstract

In this paper, we first provide an explicit procedure to glue together hereditary exact model structures for the recollement of exact categories. To that end, we use the notion of cotorsion pairs and we investigate the gluing of complete hereditary cotorsion pairs along the recollement of exact categories. Moreover, we study liftings of recollements of hereditary exact model structures to recollements of their associated homotopy categories. This leads to a new method to produce recollements of triangulated categories. Applications are given to contraderived categories, projective stable derived categories and stable categories of Gorenstein injective modules over an upper triangular matrix ring.

Keywords: recollement; exact model structure; cotorsion pair; homotopy category.

2020 Mathematics Subject Classification: 18G10, 18G25, 16D90.

1. Introduction

The notion of a cotorsion pair goes back to [33], which has been defined originally in the category of abelian groups, and then in an abelian category or an exact category. It got an enormous impulse thanks to the discovery by Hovey [22] of the one-to-one correspondence between abelian model structures and certain cotorsion pairs in abelian categories. Later on, Gillespie demonstrated in [13] that the above Hovey’s one-to-one correspondence naturally carries over to a correspondence between hereditary exact model structures and cotorsion pairs in a weakly idempotent complete exact category. For short, we will call this a WIC exact category in this paper. Therefore, the theory of exact model structures concerns the case of when \( \mathcal{C} \) is a WIC exact category and there is a model structure on \( \mathcal{C} \) that is compatible with the exact structure. The upshot of working with a hereditary exact model structure is that its homotopy category is canonically triangulated, in fact, it coincides with the stable category of a Frobenius category (see [13, 15, 21] for examples).

Recollements were first introduced in the setting of triangulated categories by Beilinson, Bernstein and Deligne [4] and then generalized to the level of abelian categories (see for instance [10, 11, 27, 31]). Recently, Wang, Wei and Zhang [35] give a generalization of recollements of abelian categories, which they called recollements of exact categories. Roughly speaking, a recollement is a short exact sequence of triangulated or exact categories where the functors involving

\textsuperscript{*}Corresponding author.

Jiangsheng Hu was supported by the NSF of China (12171206) and the Natural Science Foundation of Jiangsu Province (BK20211358). Haiyan Zhu was supported by Zhejiang Provincial Natural Science Foundation of China (LY18A010032) and the NSF of China (12271481). Rongmin Zhu was supported by the NSF of China (12201223).
admit both left and right adjoints. Such a recollement situation of exact categories is denoted throughout the paper by the following diagram

\[
\begin{array}{ccc}
A & \xrightarrow{i^*} & C \xleftarrow{j_*} \\
\downarrow{i!} & & \downarrow{j^!} \\
\downarrow{i^!} & & \downarrow{j_*}
\end{array}
\]

of WIC exact categories and additive functors satisfying the compatibility conditions in [35, Definition 3.1]. In this case one says that \((A, C, B)\) is a recollement of exact categories.

It should be noted that recollements of exact categories (in particular abelian categories) appear quite naturally in various settings and are omnipresent in representation theory (see [11, 31]). For instance, any idempotent element of a ring \(R\) induces a recollement situation between the module categories over the rings \(R, R/ReR\) and \(eRe\) (see [31, Example 2.7]). Moreover, it has been shown in Theorem 3.5 and Example 3.6 that a recollement of abelian categories under some mild conditions can induce a nontrivial recollement of exact categories (it is no longer a recollement of abelian categories).

However, the existence of recollements of triangulated categories often is difficult to establish and then plays an important role in geometry of singular spaces [4], representation theory [1, 9, 25] and homological conjectures [6, 7, 20, 37]. In the recent ten years, there are a lot of interesting work on the constructions of recollements of triangulated categories such as derived categories of ordinary rings or differential graded rings, stable categories of Frobenius categories, or more generally, homotopy category of exact model structures (see for instance [2, 6, 7, 8, 12, 15, 16]). Motivated by the discussion so far, we study the following general questions.

**Question 1.1.** Let \((A, C, B)\) be a recollement of exact categories.

1. How can we glue together hereditary exact model structures in \(A\) and \(B\) to obtain a hereditary exact model structure in \(C\)?

2. When these model structures can be lifted to a recollement of their associated triangulated homotopy categories?

In order to answer these questions, we first need to take up the following question because of the one-to-one correspondence between exact model structures and certain complete cotorsion pairs in a WIC exact category (see [13, Corollary 3.4]).

**Question 1.2.** Giving a recollement \((A, C, B)\) of exact categories, how can we glue together complete hereditary cotorsion pairs in \(A\) and \(B\) to obtain a complete hereditary cotorsion pair in \(C\)?

Recall that for a right exact functor \(T : B \to A\) between abelian categories, there exists an abelian category, denoted by \((T \downarrow A)\), consisting of all triples \((Y, X, \varphi)\) where \(\varphi : T(Y) \to X\) is a morphism in \(A\). We note that this new abelian category is called a *comma category* in [29] and forming a comma category along a given functor is a standard way to glue two categories. We refer to [23, 31] for a detailed discussion on this matter. Recently, Hu and Zhu characterized when complete hereditary cotorsion pairs in abelian categories \(A\) and \(B\) can induce complete hereditary cotorsion pairs in \((T \downarrow A)\) (see [23, Proposition 3.4]). If \((A, C, B)\) is a recollement of abelian categories and \(i^!\) is an exact functor in Question 1.2, then the abelian category \(C\) is equivalent to the comma category \((i^!j_! \downarrow A)\) (see [10, Proposition 8.9] or [11, Proposition 3.1]). So [23, Proposition 3.4] gives an
answer to Question 1.2 provided that \((\mathcal{A}, \mathcal{C}, \mathcal{B})\) is a recollement of abelian categories such that \(i^d\) is an exact functor.

The main aim of this paper is to provide more answers to the above questions. To state our results precisely, we first introduce some notation and definitions.

Let \(T : \mathcal{D}_1 \to \mathcal{D}_2\) be a functor between WIC exact categories, and let \(\mathcal{Y}\) be a subcategory of \(\mathcal{D}_1\). The functor \(T\) is called \(\mathcal{Y}\)-exact if \(T\) preserves the exactness of the admissible exact sequence \(B \to B' \to Y\) in \(\mathcal{D}_1\) with \(Y \in \mathcal{Y}\). Here we denote admissible monomorphisms by \(\hookrightarrow\) and denote admissible epimorphisms by \(\to\).

Let \(\mathcal{X}\) be a subcategory of \(\mathcal{A}\) and \(\mathcal{Y}\) a subcategory of \(\mathcal{B}\) in the recollement (1.1). We set
\[
\mathcal{M}_{\mathcal{X}}^\mathcal{Y} := \{C \in \mathcal{C} \mid i^*(C) \in \mathcal{X}, j^*(C) \in \mathcal{Y}\},
\]
\[
\mathcal{M}_\mathcal{Y} := \{C \in \mathcal{C} \mid i^*(C) \in \mathcal{X}, j^*(C) \in \mathcal{Y}, \varepsilon_C : j,j^*(C) \to C\text{ is an admissible monomorphism}\},
\]
where \(\varepsilon : j,j^* \to 1_{\mathcal{C}}\) is the counit of the adjoint pair \((j_!, j^*)\) in the recollement (1.1).

The next result conveys that one can glue together complete hereditary cotorsion pairs from \(\mathcal{A}\) and \(\mathcal{B}\) to \(\mathcal{C}\) in the recollement (1.1), which provides a partial answer to Question 1.2 properly.

**Theorem 1.1.** Let \((\mathcal{A}, \mathcal{C}, \mathcal{B})\) be a recollement of exact categories with \(i^d\) an exact functor. Assume that \((\mathcal{U}', \mathcal{V}')\) and \((\mathcal{U}'', \mathcal{V}'')\) are complete hereditary cotorsion pairs in \(\mathcal{A}\) and \(\mathcal{B}\), respectively. Set \(\mathcal{U} = \mathcal{M}_{\mathcal{U}'}^\mathcal{V}'\) and \(\mathcal{V} = \mathcal{M}_{\mathcal{V}'}^{\mathcal{V}''}\). If \(\mathcal{C}\) has enough projective and injective objects and \(j_!\) is \(\mathcal{U}'\)-exact, then \((\mathcal{U}, \mathcal{V})\) is a complete hereditary cotorsion pair in \(\mathcal{C}\).

A few comments on Theorem 1.1 are in order. First, it generalizes Lemma 3.3 and Proposition 3.4 in [23]. More precisely, in [23], one of key arguments in the proof is that all objects in the comma category \((T \downarrow \mathcal{A})\) can be represented clearly by the objects in \(\mathcal{A}\) and \(\mathcal{B}\), while in our general context we do not have this fact and therefore must avoid this kind of arguments. So, the idea of proving Theorem 1.1 will be different from the one in [23] (see Remark 4.7).

Second, it should be pointed out that the condition “\((\mathcal{A}, \mathcal{C}, \mathcal{B})\) is a recollement of exact categories with \(i^d\) an exact functor” is natural and very often met. More specifically, one can construct the desired recollement with \(i^d\) an exact functor from any recollement of abelian categories (see Theorem 3.5 and Example 3.6). During the course of the proof of Theorem 3.5, for any recollement (1.1) of exact categories, we will show that \(i^d\) is an exact functor if and only if \(i^*j_!, j^* = 0\), which refines a result obtained by Franjou and Pirashvili in [10] (see Proposition 3.3 andRemark 3.4).

Last, we employ an example in [38] to illustrate Theorem 1.1 as follows: (1) The exactness of the functor \(i^d\) cannot be omitted in general; (2) The condition “\(\mathcal{C}\) has enough projective and injective objects and \(j_!\) is \(\mathcal{U}'\)-exact” really occurs (see Example 4.8).

Our next result, providing a partial answer to Question 1.1, can be stated as follows. Its proof is based on Theorem 1.1 and a deep result of Gillespie about constructing a hereditary exact model structure from two cotorsion pairs in a WIC exact category (see [14, Theorem 1.1]).

**Theorem 1.2.** Let \((\mathcal{A}, \mathcal{C}, \mathcal{B})\) be a recollement of exact categories with \(i^d\) an exact functor, and let \(\mathcal{M}_\mathcal{A} = (\mathcal{U}_1', \mathcal{V}_1', \mathcal{V}_2')\) and \(\mathcal{M}_\mathcal{B} = (\mathcal{U}_1'', \mathcal{W}_1'', \mathcal{V}_2'')\) be hereditary exact model structures on \(\mathcal{A}\) and \(\mathcal{B}\), respectively. Set \(\mathcal{U}_1 = \mathcal{M}_{\mathcal{U}_1'}^{\mathcal{U}_1''}, \mathcal{V}_1 = \mathcal{M}_{\mathcal{W}_1'}^{\mathcal{V}_2'}, \mathcal{U}_2 = \mathcal{M}_{\mathcal{U}_2'}^{\mathcal{V}_2''}\) and \(\mathcal{V}_2 = \mathcal{M}_{\mathcal{W}_2'}^{\mathcal{V}_2''}\). Assume that \(\mathcal{C}\) has enough projective and injective objects, \(j_!\) is \(\mathcal{U}'_1\)-exact and \(\mathcal{U}_1 \cap \mathcal{V}_1 = \mathcal{U}_2 \cap \mathcal{V}_2\).
(1) There is a hereditary exact model structure \( M_C = (U_1, W, V_2) \) on \( C \), where the class \( W \) is given by

\[
W = \{ X \in C \mid \exists \text{ an admissible exact sequence } X \rightarrowtail R \twoheadrightarrow Q \text{ with } R \in V_1, \ Q \in U_2 \} = \{ X \in C \mid \exists \text{ an admissible exact sequence } R' \rightarrowtail Q' \twoheadrightarrow X \text{ with } R' \in V_1, \ Q' \in U_2 \}.
\]

(2) We have the following recollement of triangulated categories

\[
\begin{array}{ccc}
\text{Ho}(\mathcal{M}_A) & \xrightarrow{L(i^*)} & \text{Ho}(\mathcal{M}_C) \\
\downarrow{L(i_*)} & & \downarrow{L(j_*)} \\
\text{Ho}(\mathcal{M}_B) & \xleftarrow{R(j^*)} & \text{Ho}(\mathcal{M}_G), \\
\end{array}
\]

where \( L(i^*), L(i_*), L(j), L(j_*), R(i_*), R(j_*), R(i^!), R(j^!) \) are the total derived functors of those in (1.1).

Recently, Gao, Koenig and Psaroudakis [11] showed that ladders of certain height of recollements of abelian categories allow to construct recollements of triangulated categories; Georgios and Psaroudakis [12] used Quillen model structures to show a systematic method to lift recollements of projective (resp., injective) hereditary abelian model structures (see [12, Setup 6.5]) to recollements of their associated homotopy categories. It should be noted that our work offers a different perspective. More precisely, we first glue together complete hereditary cotorsion pairs for the recollement of exact categories, then provide an explicit procedure to glue together hereditary exact model structures (not necessarily projective or injective abelian model structures) for this recollement. As a result, we can build recollements of triangulated categories by lifting recollement of hereditary exact model categories to recollements of their associated homotopy categories.

As an application of Theorem 1.2, for any upper triangular matrix ring, we obtain recollements of contraderived categories, projective stable derived categories and stable categories of Gorenstein injective modules. (see Corollaries 5.1, 5.2 and 5.6).

The structure of this paper is organized as follows. In Section 2, we give some terminologies and some preliminary results which are needed for our proof. In Section 3, we provide a method to construct recollements of exact categories from recollements of abelian categories. In Section 4, we prove Theorems 1.1 and 1.2 mentioned in the introduction. Finally, some applications of Theorem 1.2 on upper triangular matrix rings are given in Section 5.

2. Preliminaries

The assumptions, the notation, and the definitions from this section will be used throughout the paper.

2.1. Exact categories. The concept of an exact category is originally due to Quillen [32], but the common reference for a simple axiomatic description is [24, Appendix A] and an extensive treatment of the concept is also given in [5]. Roughly speaking, an exact category is a pair \( (A, \mathcal{E}) \) where \( A \) is an additive category and \( \mathcal{E} \) is a class of “short exact sequences”: That is, an actual kernel-cokernel pair \( A \rightarrowtail B \xrightarrow{\alpha} C \). In what follows, we call such a sequence an admissible exact sequence, and call \( A \rightarrow B \) (resp., \( B \rightarrow C \)) an admissible monomorphism (resp., admissible epimorphism). Many
authors use the alternate terms conflation, inflation and deflation. The class $\mathcal{E}$ of admissible exact sequences must satisfy exact axioms, for details, we refer the reader to [5, Definition 2.1], which are inspired by the properties of short exact sequences in any abelian category.

We often write $\mathcal{A}$ instead of $(\mathcal{A}, \mathcal{E})$ when we consider only one exact structure on $\mathcal{A}$. An exact category $\mathcal{A}$ is called weakly idempotent complete if every split monomorphism has a cokernel and every split epimorphism has a kernel (see for instance [13, Definition 2.2]). For convenience, we will call this a WIC exact category.

**Lemma 2.1.** [13, Proposition 2.3] The following are true for any WIC exact category:

(1) If $gf$ is an admissible monomorphism, then $f$ is an admissible monomorphism.

(2) If $gf$ is an admissible epimorphism, then $g$ is an admissible epimorphism.

Next, we recall the following definition, which is a particular case of Definitions 2.9 and 2.12 in [35].

**Definition 2.2.** Let $\mathcal{A}$ be a WIC exact category. A sequence $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathcal{A}$ is said to be right exact if there exist an admissible exact sequence $K \xrightarrow{h_2} B \xrightarrow{g} C$ and an admissible epimorphism $h_1: A \to K$ such that $f = h_2 h_1$. Moreover, an additive covariant functor $F: \mathcal{A} \to \mathcal{B}$ between exact categories is called a right exact functor if it takes those right exact sequences in $\mathcal{A}$ to sequences of the same ilk in $\mathcal{B}$. Dually, one can also define the left exact sequences and left exact functors.

Let $n$ be a natural number. Following [5], an $n + 1$-term sequence

$$X_n \xrightarrow{f_n} X_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0$$

in $\mathcal{A}$ is called exact if it satisfies the following conditions:

(1) Both $X_n \hookrightarrow X_{n-1} \twoheadrightarrow \ker(f_{n-2})$ and $\ker(f_1) \twoheadrightarrow X_1 \to X_0$ are admissible exact sequences.

(2) $\ker(f_i) \hookrightarrow X_i \twoheadrightarrow \ker(f_{i-1})$ is also an admissible exact sequence for any $2 \leq i \leq n - 2$.

Similarly, one can also define the exact sequence $\cdots \to X_2 \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0$ with infinite length.

### 2.2. Cotorsion pairs in exact categories

In this section, we always assume that $\mathcal{A}$ is an exact category. Recall from [13] that a pair of subcategories $(\mathcal{X}, \mathcal{Y})$ of $\mathcal{A}$ is said to be a cotorsion pair if

$$\mathcal{X} = \perp \mathcal{Y} := \{ X \in \mathcal{A} | \text{Ext}^1_\mathcal{A}(X, Y) = 0 \text{ for each } Y \in \mathcal{Y} \},$$

$$\mathcal{Y} = \mathcal{X}^\perp := \{ Y \in \mathcal{A} | \text{Ext}^1_\mathcal{A}(X, Y) = 0 \text{ for each } X \in \mathcal{X} \}.$$ 

A cotorsion pair $(\mathcal{X}, \mathcal{Y})$ is called complete [13] if for each $M \in \mathcal{A}$ there exist admissible exact sequences in $\mathcal{A}$

$$Y_M \hookrightarrow X_M \xrightarrow{f_M} M \text{ and } M \xrightarrow{g_M} Y_M \twoheadrightarrow X_M$$

such that $X_M, Y_M \in \mathcal{X}$ and $Y_M, Y^M \in \mathcal{Y}$. In this case, $f_M$ is called a special $\mathcal{X}$-precover, while $g_M$ is called a special $\mathcal{Y}$-preenvelope.

Recall that an object $P$ in $\mathcal{A}$ is called projective provided that any admissible epimorphism ending at $P$ splits. The exact category $\mathcal{A}$ is said to have enough projective objects provided that each object $X$ fits into an admissible epimorphism $d: P \to X$ with $P$ projective. Dually one has the notions of injective objects and exact categories with enough injective objects.

The following lemma is essentially taken from [18, Lemma 5.20], where a variation of it appears. The proof given there carries over to the present situation.
Lemma 2.3. (Salce’s Lemma) Assume that \( A \) has enough projective objects and injective objects, and \((\mathcal{X}, \mathcal{Y})\) is a cotorsion pair in \( A \). Then the following are equivalent.

1. \((\mathcal{X}, \mathcal{Y})\) is complete.
2. \(\mathcal{X}\) is special precovering in \( A \).
3. \(\mathcal{Y}\) is special preenveloping in \( A \).

A cotorsion pair \((\mathcal{X}, \mathcal{Y})\) in \( A \) is called hereditary [13] if \(\mathcal{X}\) is closed under taking kernels of admissible epimorphisms between objects of \(\mathcal{X}\) and if \(\mathcal{Y}\) is closed under taking cokernels of admissible monomorphisms between objects of \(\mathcal{Y}\). In this case, we say \(\mathcal{X}\) is resolving, and \(\mathcal{Y}\) is coresolving.

The following lemma is essentially taken from [15, Lemma 2.3], where a variation of it appears.

Lemma 2.4. Let \((\mathcal{X}, \mathcal{Y})\) be a cotorsion pair in \( A \). If \( A \) has enough projective objects or enough injective objects, then \((\mathcal{X}, \mathcal{Y})\) is hereditary if and only if \(\text{Ext}_A^2(X,Y) = 0\) for every \(X \in \mathcal{X}\), \(Y \in \mathcal{Y}\).

2.3. Exact model structures. Recall from [13] that an exact model structure on a WIC exact category \( A \) is a model structure in the sense of [21, Definition 1.1.3] in which each of the following holds.

1. A map is a (trivial) cofibration if and only if it is an admissible monomorphism with a (trivially) cofibrant cokernel.
2. A map is a (trivial) fibration if and only if it is an admissible epimorphism with a (trivially) fibrant kernel.

Gillespie showed in [13] that the correspondence between model structures and cotorsion pairs from [22] carries over to the case of WIC exact categories as below.

Theorem 2.5. [13, Corollary 3.4] Let \( A \) be a WIC exact category. There is a one-to-one correspondence between exact model structures on \( A \) and complete cotorsion pairs \((Q, R \cap W)\) and \((Q \cap W, R)\) where \(W\) is a thick subcategory of \( A \). Given a model structure, \(Q\) is the class of cofibrant objects, \(R\) the class of fibrant objects and \(W\) the class of trivial objects. Conversely, given the cotorsion pairs with \(W\) thick, a cofibration (resp., trivial cofibration) is an admissible monomorphism with a cokernel in \(Q\) (resp., \(Q \cap W\)), and a fibration (resp., trivial fibration) is an admissible epimorphism with a kernel in \(R\) (resp., \(R \cap W\)).

Due to the above Hovey’s one-to-one correspondence, we will often not distinguish between the Hovey triple and the actual model structure on a WIC exact category \( A \). For example, we may say that \(\mathcal{M} = (Q, W, R)\) is an exact model structure and understand this to mean the model structure associated to the Hovey triple \((Q \cap W, R)\) on \( A \). On the other hand, we may say that an exact model structure is hereditary if its associated Hovey triple is hereditary.

Let \( W \) be the class of weak equivalences. The homotopy category of the model category is the localization \(\mathcal{C}[W^{-1}]\) and is denoted by \(\text{Ho}(\mathcal{M})\). By [17, Section 4.2], we know that if \(\mathcal{M} = (Q, W, R)\) is a hereditary Hovey triple, then \(\text{Ho}(\mathcal{M})\) is a triangulated category and it is triangle equivalent to the stable category \((Q \cap R)/\omega\), where \(\omega = Q \cap W \cap R\) is the class of projective-injective objects.

2.4. Recollements of triangulated categories. Loosely, a recollement is an “attachment” of two triangulated categories. The standard reference is [4]. Let \( T', T, T'' \) be triangulated categories. We give the definition that appeared in [26] based on localization and colocalization sequences.
Definition 2.6. Let \( T' \xrightarrow{F} T \xrightarrow{G} T'' \) be a sequence of triangulated functors between triangulated categories. We say it is a localization sequence when there exist right adjoints \( F_\rho \) and \( G_\rho \) giving a diagram of functors as below with the listed properties.

\[
\begin{array}{cccccc}
T' & \xrightarrow{F} & T & \xrightarrow{G} & T'' \\
\end{array}
\]

\( F_\rho \) and \( G_\rho \) with the analogous properties.

A colocalization sequence is the dual. That is, there must exist left adjoints \( F_\lambda \) and \( G_\lambda \) with the analogous properties.

2.5. Recollements of exact categories. In this subsection, we recall the definition of a recollement situation in the context of exact categories (see [35]). For an additive functor \( F : A \to C \) between additive categories, we denote by \( \text{im}F = \{ C \in C \mid C \cong F(A) \text{ for some } A \in A \} \) the essential image of \( F \) and by \( \ker F = \{ A \in A \mid F(A) = 0 \} \) the kernel of \( F \).

Definition 2.8. [35, Definition 3.1] Let \( A, B, C \) be three WIC exact categories. A recollement of \( C \) relative to \( A \) and \( B \), denoted by \( (A, C, B) \), is a diagram

\[
\begin{array}{cccccc}
A & \xleftarrow{i_*} & i^* & \xrightarrow{i^!} & C & \xrightarrow{j_*} \xrightarrow{j^*} B \\
\end{array}
\]

given by two exact functors \( i_* \) and \( j^* \), two right exact functors \( i^* \), \( j_! \) and two left exact functors \( i^! \), \( j_* \), which satisfies the following conditions:

- (1) \( (i^*, i_*), (i^!, i^!) \), \( (j_!, j_*) \) and \( (j^*, j^!) \) are adjoint pairs;
- (2) \( i_* \), \( j_! \) and \( j_* \) are fully faithful;
- (3) \( \text{im}i_* = \text{ker}j^* \);
- (4) For any \( C \in C \), there exists an exact sequence in \( C \)

\[
i_* i^!(C) \xrightarrow{\sigma_C} C \xrightarrow{\nu_C} j_* j^*(C) \to i_*(A)
\]
with $A \in \mathcal{A}$, where $\sigma_C$ and $\eta_C$ are given by the adjunction morphisms;

(5) For any $C \in \mathcal{C}$, there exists an exact sequence in $\mathcal{C}$

$$i_*(A') \rightarrow j_! j^*(C) \xrightarrow{\varepsilon_C} C \xrightarrow{\delta_C} i^* C$$

with $A' \in \mathcal{A}$, where $\varepsilon_C$ and $\delta_C$ are given by the adjunction morphisms.

In this case one says that $(\mathcal{A}, \mathcal{C}, \mathcal{B})$ is a recollement of exact categories.

If the categories $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ are abelian, then Definition 2.8 coincides with the definition of recollement of abelian categories. We refer to [31, Section 2.1] for examples of recollements of abelian categories. For examples of recollement of exact categories, we refer to Section 3.

**Notation for units and counits.** Throughout, we denote by $\delta : 1_C \rightarrow i_! i^*$ (resp., $\eta : 1_C \rightarrow j_! j^*$), the unit of the adjoint pair $(i^*, i_*)$ (resp., $(j^*, j_*)$), and by $\sigma : i_* i^! \rightarrow 1_C$ (resp., $\varepsilon : j_! j^* \rightarrow 1_C$), the counit of the adjoint pair $(i_*, i^!)$ (resp., $(j_!, j^*)$).

We list some properties of recollements (see [35, Lemma 3.3]), which will be used in the sequel.

**Lemma 2.9.** The following are true for any recollement $(\mathcal{A}, \mathcal{C}, \mathcal{B})$ of exact categories.

(1) $i^* j_! = 0 = i_* j_!$.

(2) All the natural transformations

$$i^! i_* \rightarrow 1_A, \ 1_A \rightarrow i^! i_*, \ 1_B \rightarrow j^* j_!, \ j^* j_* \rightarrow 1_B$$

are natural isomorphisms.

(3) $i^*$ preserves projective objects and $i^!$ preserves injective objects.

(4) $j_!$ preserves projective objects and $j^*$ preserves injective objects.

(5) If $i_!$ is exact, then $j_*$ is exact.

In the following sections, we always assume that $(\mathcal{A}, \mathcal{C}, \mathcal{B})$ is a recollement of exact categories defined in Definition 2.8, where $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are WIC exact categories.

### 3. Constructing recollements of exact categories from recollements of abelian categories

We begin this section with the following easy observation.

**Lemma 3.1.** Let $C$ be an object in $\mathcal{C}$.

(1) If $i^* j_* j^*(C) = 0$, then $\eta_C : C \rightarrow j_* j^*(C)$ is an admissible epimorphism. Thus, there exists an admissible exact sequence $i_* i^! (X) \xrightarrow{\sigma_C} X \xrightarrow{\eta_C} j_* j^*(X)$ in $\mathcal{C}$.

(2) If $i^! j_! j^*(C) = 0$, then $\varepsilon_C : j_! j^*(C) \rightarrow C$ is an admissible monomorphism. Thus, there exists an admissible exact sequence $j_! j^*(X) \xrightarrow{\varepsilon_C} X \xrightarrow{\delta_C} i_! i^* (X)$ in $\mathcal{C}$.

**Proof.** We only prove (1), and the proof of (2) is similar. Note that there exists an exact sequence $i_* i^! (C) \xrightarrow{\sigma_C} C \xrightarrow{\eta_C} j_* j^*(C) \rightarrow i_*(A)$ in $\mathcal{C}$ with $A \in \mathcal{A}$. Hence $i^* (f) : i^* j_* j^*(C) \rightarrow i^* i_*(A)$ is an admissible epimorphism in $\mathcal{A}$. Since $i^* j_* j^*(C) = 0$ by hypothesis, it follows that $A \cong i^* i_*(A) = 0$. So $\eta_C : C \rightarrow j_* j^*(C)$ is an admissible epimorphism, as desired.

**Lemma 3.2.** Let $M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3$ be an admissible exact sequence in $\mathcal{C}$.

(1) If $\eta_{M_i} : M_i \rightarrow j_* j^*(M_i)$ is admissible epic for $i = 1, 2, 3$, then $i_!(M_1) \xrightarrow{i_!(f)} i_!(M_2) \xrightarrow{i_!(g)} i_!(M_3)$ is an admissible exact sequence in $\mathcal{A}$. 
(2) If $\varepsilon_{M_i} : j_i j^*(M_i) \to M_i$ is admissible monic for $i = 1, 2, 3$, then $i^*(M_1) \xrightarrow{i^*(f)} i^*(M_2) \xrightarrow{i^*(g)} i^*(M_3)$ is an admissible exact sequence in $\mathcal{A}$.

Proof. We only prove (1), and the proof of (2) is similar. Note that we have a left exact sequence $i^!(M_1) \xrightarrow{i^!(f)} i^!(M_2) \xrightarrow{i^!(g)} i^!(M_3)$ in $\mathcal{A}$. If we set $N := \text{coker}(i^!(f))$, to prove the exactness of the left sequence $i^!(M_1) \xrightarrow{i^!(f)} i^!(M_2) \xrightarrow{i^!(g)} i^!(M_3)$, it suffices to show that $i^!(M_3) \cong N$. By hypothesis, we have the following commutative diagram

$$
\begin{array}{c}
i_\ast i^!(M_1) \xrightarrow{\eta_{M_1}} M_1 \xrightarrow{\eta_{M_1}} j_\ast j^* (M_1) \\
i_\ast i^!(M_2) \xrightarrow{\eta_{M_2}} M_2 \xrightarrow{\eta_{M_2}} j_\ast j^* (M_2) \\
i_\ast i^!(M_3) \xrightarrow{\eta_{M_3}} M_3 \xrightarrow{\eta_{M_3}} j_\ast j^* (M_3),
\end{array}
$$

where all rows are admissible exact sequences. Since $j_\ast j^* (g) \eta_{M_2} = \eta_{M_2} g$ is admissible epic, so is $j_\ast j^* (g)$. Hence the sequence $i_\ast i^!(M_1) \xrightarrow{i_\ast i^!(f)} i_\ast i^!(M_2) \xrightarrow{i_\ast i^!(g)} i_\ast i^!(M_3)$ in the above diagram is an admissible exact sequence by [5, Corollary 8.13]. Since $i_\ast$ is an exact functor, $i_\ast i^!(M_1) \xrightarrow{i_\ast i^!(f)} i_\ast i^!(M_2) \xrightarrow{i_\ast i^!(g)} i_\ast (N)$ is an admissible exact sequence in $\mathcal{C}$. Thus $i_\ast (N) \cong i_\ast i^!(M_3)$, and so $N \cong i^* i_\ast (N) \cong i^* i_\ast i^!(M_3) \cong i^! (M_3)$, as desired. \hfill $\square$

**Proposition 3.3.** Let $\mathcal{A}, \mathcal{C}, \mathcal{B}$ be a recollement of exact categories.

(1) The following conditions are equivalent.
- (a) $i^!$ is an exact functor.
- (b) $i^* j_\ast = 0$.
- (c) $i^* j_\ast j^* = 0$.

(2) The following conditions are equivalent.
- (a) $i^*$ is an exact functor.
- (b) $i^* j^* = 0$.
- (c) $i^* j^* j_\ast = 0$.

Proof. We only prove (1), and the proof of (2) is similar.

(a) $\Rightarrow$ (b). The proof is model on that of Proposition 8.8 in [10]. Let $X$ be an object in $\mathcal{B}$. It follows that $\delta_{j_\ast (X)} : j_\ast (X) \to i_\ast i^* j_\ast (X)$ is an admissible epimorphism. Since $i^!$ is an exact functor, $i^!(\delta_{j_\ast (X)}) : i^!(j_\ast (X)) \to i^!(i_\ast i^* j_\ast (X))$ is also admissible epic. Note that $i^!(j_\ast) = 0$ by Lemma 2.9(1). Thus $i^* j_\ast (X) \cong i^! i_\ast i^* j_\ast (X) = 0$, as desired.

(b) $\Rightarrow$ (c) is trivial.

(c) $\Rightarrow$ (a) holds by Lemmas 3.1 and 3.2. \hfill $\square$

**Remark 3.4.** We note that Proposition 3.3 not only generalizes [10, Proposition 8.8] from recollements of abelian categories to the setting of exact categories, but also refines it by deleting two superfluous assumptions “with enough projectives” and “with enough injectives”. Also, our proof here is different from that in [10].

We are now in a position to state and prove the main result of this section, which provides a method to construct recollements of exact categories from recollements of abelian categories.
**Theorem 3.5.** Given the following recollement of abelian categories

\[
\begin{array}{ccc}
A & \xrightarrow{i^*} & C \\
\downarrow{j_1} & & \downarrow{j_1} \\
\downarrow{j} & & \downarrow{j} \\
B, & \xleftarrow{j^*} & C \\
\end{array}
\]

we set \( C_1 := \{ C \in C \mid i^* j_*(C) = 0 \} \) and \( B_1 := \{ B \in B \mid B \cong j^*(C) \text{ for some } C \in C_1 \} \).

1. The following are equivalent:
   a. \( i^* : C \to A \) is an exact functor;
   b. \( C_1 = C \);
   c. \( B_1 = B \).

2. If \( i^* : C \to A \) is not an exact functor, then the recollement (3.1) can induce the following recollement of exact categories:

\[
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{i^*} & C_1 \\
\downarrow{j_1} & & \downarrow{j_1} \\
\downarrow{j} & & \downarrow{j} \\
\mathcal{B}_1 & \xleftarrow{j^*} & C \\
\end{array}
\]

such that both \( i^* : C_1 \to \mathcal{A} \) and \( j_* : \mathcal{B}_1 \to C_1 \) are exact functors.

**Proof.** (1) \( a \Rightarrow b \) follows from Proposition 3.3(1) and \( b \Rightarrow c \) is trivial. To prove \( c \Rightarrow a \), it suffices to show \( i^* j_*(B) = 0 \) for any \( B \in B \) by Proposition 3.3(1). Since \( B_1 = B \) by (c), there exists \( C \in C_1 \) such that \( B \cong j^*(C) \). So \( i^* j_*(B) \cong i^* j_*(C) = 0 \), as desired.

(2) We first claim that \( B_1 \) and \( C_1 \) are WIC exact categories with exact structures induced from \( B \) and \( C \), respectively. By the additive of the functors \( i^*, j_* \) and \( j^* \), it suffices to check that \( B_1 \) and \( C_1 \) are closed under extensions in \( B \) and \( C \), respectively. Let \( 0 \to N_1 \xrightarrow{j_1} N_2 \xrightarrow{j_2} N_3 \to 0 \) be an exact sequence in \( B \) with \( N_1, N_3 \in B_1 \). Then there exist \( M_1, M_3 \in C_1 \) such that \( N_1 \cong j_*(M_1) \) and \( N_3 \cong j_*(M_3) \). Thus there exists \( M_2 \in C \) such that \( N_2 \cong j_*(M_2) \) by Lemma 2.9(3). Thanks to [10, Proposition 4.3], we have the following commutative diagram with exact rows and columns

\[
\begin{array}{ccccccc}
0 & \to & j_1(N_1) & \to & j_*(N_1) & & \\
& & \downarrow & & \downarrow & & \\
0 & \xrightarrow{i_* j_1} & j_1(N_2) & \to & j_*(N_2) & \to & i_* j_*(N_2) \\
& & \downarrow{j(g)} & & \downarrow{j_*(g)} & & \\
0 & \xrightarrow{i_* j_1} & j_1(N_3) & \to & j_*(N_3) & \to & 0 \\
& & \downarrow & & \downarrow & & \\
& & 0 & & 0 & &
\end{array}
\]

Since \( j_1(g) \) is epic, so is \( j_*(g) \). This implies that \( 0 \to j_*(M_1) \to j_*(M_2) \to j_*(M_3) \to 0 \) is exact in \( C \). Applying the functor \( i^* \), we have an exact sequence in \( \mathcal{A} \)

\[
i^* j_*(M_1) \to i^* j_*(M_2) \to i^* j_*(M_3) \to 0.
\]
Example 3.6.⊗

By Lemma 2.6(ii), it suffices to show the condition (5) of Definition 2.8. Let C be an object in C1. Note that there exists an exact sequence in C

$$0 \to i_*(A) \to j_!j^*(C) \xrightarrow{\varepsilon} C \to i^*i^!(C) \to 0$$

(3.3)

with $A \in \mathcal{A}$. It is easy to check that each term in the above exact sequence belongs to C1. If we set $K := \text{im}(\varepsilon_C)$, we need to show that $K$ is in C1. Applying $j^*$ to the sequence (3.3), it follows that $j^*(K) \cong j^*(C)$. Thus $i^*j_*j^*(C) \cong i^*j_*j^*(C) = 0$, so $K \in C_1$.

Finally, it follows from Proposition 3.3(1) that $i^! : C_1 \to \mathcal{A}$ is exact. So $j_* : B_1 \to C$ is also exact by Lemma 2.9(5). This completes the proof. \qed

The following example, due to Zhang-Cui-Rong [38], shows the recollement of abelian categories can induce a nontrivial recollement of exact categories provided that $i^!$ is not an exact functor.

Example 3.6. Let $A = B$ be the path algebra $k(1 \to 2)$, where $\text{char} k \neq 2$. Write the conjunction of paths from right to left. Thus $e_1e_2 = 0$ and $e_2e_1 \cong k$. Take $M = N = A e_2 \otimes_k e_1 A$. Then $M \otimes_A N = 0 = N \otimes_A M$. Let $A$ be the Morita ring $(\begin{smallmatrix} A & N \\ N & A \end{smallmatrix})$. By [38, Section 2.4], we obtain the recollement

$$\xymatrix{ & \text{Mod-A} \ar[rr]^{i^*} \ar[dr]_{i_*} & & \text{Mod-A} \ar[rr]^{j^*} \ar[dr]_{j_*} & & \text{Mod-A} \\ \text{Mod-A} & & \text{Mod-A} & & \text{Mod-A} }$$

(3.4)

where $i^*$ is given by $(\begin{smallmatrix} Y \\ X \end{smallmatrix}) f,g \mapsto \text{coker} g$; $i_*$ is given by $X \mapsto (\begin{smallmatrix} Y \\ X \end{smallmatrix})_{0,0}$; $j^*$ is given by $(\begin{smallmatrix} Y \\ X \end{smallmatrix}) f,g \mapsto \text{Ker} \tilde{f}$, where $\tilde{f} := \eta_XY(f)$ and $\eta_XY$ is the adjunction isomorphism $\text{Hom}_A(A \otimes_A X, Y) \cong \text{Hom}_A(X, \text{Hom}_A(N, Y))$; $j_*$ is given by $Y \mapsto (N \otimes_A Y)_{0,1}$; $j^*$ is given by $(\begin{smallmatrix} Y \\ X \end{smallmatrix}) f,g \mapsto Y$; $j_*$ is given by $Y \mapsto \left(\begin{smallmatrix} \text{Hom}_A(N,Y) \\ Y \end{smallmatrix}\right)_{Y,0}$, where $\epsilon$ is the counit $N \otimes_A \text{Hom}_A(N,-) \to 1_{\text{Mod-A}}$. In this case, we have $B_1 = \{ Y \in \text{Mod-A} | \text{Hom}_A(N,Y) = 0 \}$.

The Auslander-Reiten quiver $\Gamma(\text{mod-A})$ of the module category mod-A has the form

$$\xymatrix{ & A e_1 \ar[rd]_{\pi} & \\ S_2 & S_1. }$$

Since $A N$ is isomorphic to the simple left $A$-module $A e_2 = S_2$, it follows that

$$B_1 = \{ Y \in \text{Mod-A} | \text{Hom}_A(N,Y) = 0 \} = \text{Add}(S_1).$$

Thus we have

$$C_1 = \{(\begin{smallmatrix} Y \\ X \end{smallmatrix}) f,g \in \text{Mod-A} | i^*j_*j^*(\begin{smallmatrix} Y \\ X \end{smallmatrix}) f,g = 0\}$$

$$= \{(\begin{smallmatrix} Y \\ X \end{smallmatrix}) f,g \in \text{Mod-A} | \text{Hom}_A(N,Y) = 0\} = \{(\begin{smallmatrix} Y \\ X \end{smallmatrix}) f,g \in \text{Mod-A} | Y \in B_1\}.$$

This informs us that $A = \text{Mod-A} \subsetneq C_1 \subsetneq \text{Mod-A} = C$ and $0 \neq B_1 \subsetneq \text{Mod-A} = B$, as desired.
4. PROOFS OF THE MAIN RESULTS

In this section, we prove the main results mentioned in the introduction. We keep the notation introduced in the previous sections.

4.1. Proof of Theorem 1.1. In what follows, we always assume that \( \mathcal{U}', \mathcal{V}' \) are subcategories of \( \mathcal{A} \) and \( \mathcal{U}'', \mathcal{V}'' \) are subcategories of \( \mathcal{B} \). Denote by \( \mathcal{U} = \{ C \in \mathcal{C} \mid i^*(C) \in \mathcal{U}' \} \), \( j^*(C) \in \mathcal{U}'' \), \( \varepsilon_M \) : \( j_i^*(M) \to M \) is an admissible monomorphism \( \) and by \( \mathcal{V} = \{ C \in \mathcal{C} \mid i^!(C) \in \mathcal{V}' \} \), \( j^*(C) \in \mathcal{V}'' \).

The following result is crucial to the proof of Theorem 1.1.

**Proposition 4.1.** Assume that the WIC exact category category \( \mathcal{C} \) in the recollement \((1.1)\) has enough projective and injective objects. If \( i^! \) is an exact functor and \( j^*_i \) is \( (\mathcal{V}'') \)-exact, then \( (\mathcal{U}', \mathcal{V}') \) and \( (\mathcal{U}'', \mathcal{V}'') \) are hereditary cotorsion pairs in \( \mathcal{A} \) and \( \mathcal{B} \), respectively if and only if \( (\mathcal{U}, \mathcal{V}) \) is a hereditary cotorsion pair in \( \mathcal{C} \).

To prove Proposition 4.1, we need some preparations.

**Lemma 4.2.** Let \( i_* (A) \to M \to N \) be an admissible exact sequence in \( \mathcal{C} \). If \( \varepsilon_N : j_i^*(N) \to N \) is admissible monic, then \( i^* i_* (A) \to i^*(M) \to i^*(N) \) is an admissible exact sequence in \( \mathcal{A} \).

**Proof.** Since \( i^* \) is right exact, we have a right exact sequence \( i^* i_* (A) \to i^*(M) \to i^*(N) \) in \( \mathcal{A} \). If we set \( K := \ker (i^*(M) \to i^*(N)) \), we only need to show that \( i^* i_* (A) \cong K \). Note that we have the following commutative diagram

\[
\begin{array}{ccc}
    j_i j_i^*(A) & \to & j_i j_i^*(M) \\
    \downarrow & & \downarrow \\
    i_* (A) & \to & M \\
    \downarrow & & \downarrow \\
    i_* i_* (A) & \to & i_* i^*(M) & \to & i_* i^*(N)
\end{array}
\]

such that all columns, and both the first and third rows are right exact. Since \( \varepsilon_N : j_i^*(N) \to N \) is an admissible monomorphism, it follows from the snake lemma (see [5, Exercise 8.15]) that the right exact sequence \( i_* i_* (A) \to i_* i^*(M) \to i_* i^*(N) \) in the above commutative diagram is admissible exact. Since \( i_* (K) \to i_* i^*(M) \to i_* i^*(N) \) is also an admissible exact sequence in \( \mathcal{C} \), we have \( i_* (K) \cong i_* i^* (A) \). So \( K \cong i^* i_* (A) \) by noting that \( i_* \) is fully faithful. This completes the proof. \( \square \)

**Lemma 4.3.** If \( i^! \) is an exact functor, then any object \( M \in \mathcal{C} \) gives the following admissible exact sequence

\[
\begin{array}{ccc}
    i_* i^! j_i j_i^* (M) & \to & j_i j_i^* (M) \\
    \downarrow & & \downarrow \varepsilon_M \\
    j_i j_i^* (M) \oplus i_* i^! (M) & \to & M
\end{array}
\]

**Proof.** Note that \( i^* j_i j_i^* (M) = 0 \) by Proposition 3.3(1). It follows from Lemma 3.1 that \( \eta_M : M \to j_i j_i^* (M) \) is an admissible epimorphism. Thus we have the following commutative diagram

\[
\begin{array}{ccc}
    i_* i^! j_i j_i^* (M) & \to & j_i j_i^* (M) \\
    \downarrow & & \downarrow \varepsilon_M \\
    j_i j_i^* (M) & \to & M \\
    \downarrow & & \downarrow \eta_M \\
    i_* i^! (M) & \to & j_i j_i^* (M)
\end{array}
\]

\[
\varepsilon_M
\]

\[
\eta_M
\]

\[
\eta_M
\]

\[
\eta_M
\]

\[
\eta_M
\]
Thanks to [5, Proposition 2.12], we have the desired admissible exact sequence. □

**Proposition 4.4.** Assume that $\mathcal{C}$ has enough projective and injective objects and $M$ is an object in $\mathcal{C}$. If $i^!$ is an exact functor, then $\text{Ext}_A^1(M, i_*(I)) = 0$ for any injective object $I$ in $A$ if and only if $\varepsilon_M : j_*j^*(M) \to M$ is an admissible monomorphism.

**Proof.** Since $\mathcal{C}$ has enough projective and injective objects by hypothesis, it follows from [35, Lemma 3.3(5)] that $A$ has enough projective and injective objects. By Lemma 4.3, we have the following admissible exact sequence

$$i_*i^!j_*j^*(M) \xrightarrow{(\sigma_{ijj^*(M)}, i_*i^!(\varepsilon_M))} j_*j^*(M) \oplus i_*i^!(M) \xrightarrow{(-\varepsilon_M, \sigma_M)} M.$$

"⇒". Let $f : i^!j_*j^*(M) \to I$ be an admissible monomorphism in $A$ with $I$ injective. Since $\text{Ext}_A^1(M, i_*(I)) = 0$ by hypothesis, there exists a morphism $(g, h) : j_*j^*(M) \oplus i_*i^!(M) \to i_*(I)$ such that

$$(g, h)(i_*i^!(\varepsilon_M)) = f.$$

Thus $g\sigma + hi_*i^!(\varepsilon_M) = i_*(f)$. Since $g \in \text{Hom}_C(j_*j^*(M), i_*(I)) \cong \text{Hom}_C(i^!j_*j^*(M), I) = 0$, we have

$$i_*(f) = hi_*i^!(\varepsilon_M).$$

Note that $i_*(f)$ is admissible monic. Thus $i_*i^!(\varepsilon_M)$ is also admissible monic. So $\varepsilon_M : j_*j^*(M) \to M$ is an admissible monomorphism by the commutative diagram in Lemma 4.3.

"⇐". Let $I$ be an injective object in $A$. Then we have the following exact sequence

$$\text{Hom}_C(j_*j^*(M) \oplus i_*i^!(M), i_*(I)) \xrightarrow{\text{Hom}_C(i_*i^!j_*j^*(M), i_*(I))} \text{Ext}_A^1(M, i_*(I)) \xrightarrow{\text{Ext}_A^1(i_*i^!j_*j^*(M), i_*(I)).}$$

Note that $\text{Ext}_A^1((i_*i^!j_*j^*(M), i_*(I)) \cong \text{Ext}_A^1(i_*i^!j_*j^*(M), i_*(I)) \cong \text{Ext}_A^1(i_*i^!j_*j^*(M), i_*(I) = 0$ by [35, Lemma 3.3(7)]. To prove $\text{Ext}_A^1(M, i_*(I)) = 0$, it suffices to show that

$$\text{Hom}_C((\sigma_{ijj^*(M)}, i_*(I)) : \text{Hom}_C(j_*j^*(M) \oplus i_*i^!(M), i_*(I)) \xrightarrow{\text{Hom}_C(i_*i^!j_*j^*(M), i_*(I))} \text{Hom}_C(i_*i^!j_*j^*(M), i_*(I))$$

is an epimorphism. Since $i_*i^!j_*j^*(M) = 0$, $i_*i^!i_*i^!(M) \cong i_*i^!(M)$ and $i_*i^!j_*j^*(M) \cong i_*i^!j_*j^*(M)$, we have the following commutative diagram

$$\text{Hom}_C(j_*j^*(M) \oplus i_*i^!(M), i_*(I)) \xrightarrow{\cong} \text{Hom}_C(i_*i^!j_*j^*(M), i_*(I)) \xrightarrow{\cong} \text{Hom}_A(i_*i^!j_*j^*(M), I) \xrightarrow{\cong} \text{Hom}_A(i_*i^!j_*j^*(M), I).$$

Note that $\varepsilon_M : j_*j^*(M) \to M$ is an admissible monomorphism by hypothesis. Then we have an admissible exact sequence $j_*j^*(M) \to M \to i_*i^!(M) \to in C$. Since $i_*i^!(\varepsilon_M) : i_*i^!j_*j^*(M) \to i_*i^!(M)$ is an admissible monomorphism, it follows that $\text{Hom}_A(i^!(\varepsilon_M), I) : \text{Hom}_A(i_*i^!j_*j^*(M), I) \to \text{Hom}_A(i_*i^!j_*j^*(M), I)$ is an epimorphism. So $\text{Hom}_C((\sigma_{ijj^*(M)}, i_*(I))$ is an epimorphism, as desired. □

**Lemma 4.5.** The following are true for any recollement $(\mathcal{A}, \mathcal{C}, \mathcal{B})$ of exact categories.
Proof. By [35, Lemma 3.3(5)], we only prove (1). Assume that \( M_1 \to M_2 \to M_3 \) is an admissible exact sequence in \( \mathcal{C} \). Let \( P \) be a projective object in \( \mathcal{B} \). Thus we have the following commutative diagram

\[
\begin{array}{ccc}
\text{Hom}_{\mathcal{C}}(j_!(P), M_2) & \longrightarrow & \text{Hom}_{\mathcal{C}}(j_!(P), M_3) \\
\downarrow \cong & & \downarrow \cong \\
\text{Hom}_{\mathcal{B}}(P, j^*(M_2)) & \longrightarrow & \text{Hom}_{\mathcal{A}}(P, j^*(M_3)).
\end{array}
\]

Since \( j^*: \mathcal{C} \to \mathcal{B} \) is an exact functor, it follows that \( \text{Hom}_{\mathcal{B}}(P, j^*(M_2)) \to \text{Hom}_{\mathcal{B}}(P, j^*(M_3)) \) is an epimorphism. So \( \text{Hom}_{\mathcal{C}}(j_!(P), M_2) \to \text{Hom}_{\mathcal{C}}(j_!(P), M_3) \) is an epimorphism by the commutative diagram above, as desired. \( \square \)

**Lemma 4.6.** Assume that \( i^l \) is an exact functor and \( N \) is an object in \( \mathcal{C} \).

1. If \( \mathcal{A} \) has enough projective objects and \( M \) is an object in \( \mathcal{A} \), then \( \text{Ext}_{\mathcal{C}}^i(i_*(M), N) \cong \text{Ext}_{\mathcal{A}}^i(M, i^l(N)) \) for any \( i \geq 1 \).
2. If \( \mathcal{C} \) has enough projective objects and \( U \) is an object in \( \mathcal{C} \), then \( \text{Ext}_{\mathcal{C}}^i(U, i_*(N)) \cong \text{Ext}_{\mathcal{A}}^i(i^l(U), N) \) and \( \text{Ext}_{\mathcal{C}}^i(U, j_*(N)) \cong \text{Ext}_{\mathcal{B}}^i(j^*(U), N) \) for any \( i \geq 1 \).
3. Assume that \( \mathcal{B} \) has enough projective objects and \( \mathcal{L} \) is a resolving subcategory of \( \mathcal{B} \) which contains projective objects. If \( j_! \) is \( \mathcal{L} \)-exact, then \( \text{Ext}_{\mathcal{C}}^i(j_!(L), N) \cong \text{Ext}_{\mathcal{B}}^i(L, j^*(N)) \) for any object \( L \) in \( \mathcal{L} \) and all \( i \geq 1 \).

**Proof.**

1. Let \( \cdots \to P_2 \to P_1 \to P_0 \to M \) be an exact sequence in \( \mathcal{A} \) with each \( P_i \) projective. It follows from Lemma 4.5(2) that \( \cdots \to i_*(P_2) \to i_*(P_1) \to i_*(P_0) \to i_*(M) \) is an exact sequence in \( \mathcal{C} \) with each \( i_*(P_i) \) projective. For any integer \( i \geq 1 \), we have the following commutative diagram

\[
\begin{array}{ccc}
\text{Hom}_{\mathcal{C}}(i_*(P_{i-1}), N) & \longrightarrow & \text{Hom}_{\mathcal{C}}(i_*(P_i), N) \\
\downarrow \cong & & \downarrow \cong \\
\text{Hom}_{\mathcal{A}}(P_{i-1}, i^l(N)) & \longrightarrow & \text{Hom}_{\mathcal{A}}(P_i, i^l(N))
\end{array}
\]

which implies \( \text{Ext}_{\mathcal{C}}^i(i_*(M), N) \cong \text{Ext}_{\mathcal{A}}^i(M, i^l(N)) \).

2. The proof is similar to that of (1).

3. Let \( L \) be an object in \( \mathcal{L} \). Then there exists an exact sequence \( \cdots \to Q_2 \to Q_1 \to Q_0 \to L \) in \( \mathcal{B} \) with each \( Q_i \) projective. Note that \( j_! \) is \( \mathcal{L} \)-exact by hypothesis. Since \( \mathcal{L} \) is a resolving subcategory of \( \mathcal{B} \) which contains projective objects, it follows that \( \cdots \to j_!(Q_2) \to j_!(Q_1) \to j_!(Q_0) \to j_!(L) \) is an exact sequence in \( \mathcal{C} \) with each \( j_!(Q_i) \) projective by Lemma 4.5(1). For any integer \( i \geq 1 \), we
have the following commutative diagram

\[
\begin{array}{ccc}
\text{Hom}_C(j_i(Q_{i-1}), N) & \rightarrow & \text{Hom}_C(j_i(Q_i), N) \\
\downarrow \cong & & \downarrow \cong \\
\text{Hom}_G(Q_{i-1}, j^*(N)) & \rightarrow & \text{Hom}_G(Q_i, j^*(N)) \\
\end{array}
\]

which implies that \( \text{Ext}^1_C(j_i(L), N) \cong \text{Ext}^1_B(L, j^*(N)) \).

We are now in a position to prove Proposition 4.1.

**Proof of Proposition 4.1.** Note that \( \mathcal{C} \) has enough projective objects by hypothesis. Since \( j_* : \mathcal{B} \rightarrow \mathcal{C} \) is an exact functor by Lemma 2.9(5), it follows from [35, Lemma 3.3(6)] that \( \mathcal{B} \) has enough projective objects.

"⇒". Assume that \((\mathcal{U}', \mathcal{V}')\) and \((\mathcal{U}'', \mathcal{V}'')\) are hereditary cotorsion pairs in \( \mathcal{A} \) and \( \mathcal{B} \), respectively. For lucidity, we divide the proof into 3-steps.

**Step 1:** \( \text{Ext}^2_B(U, V) = 0 \) for every \( U \in \mathcal{U} \) and \( V \in \mathcal{V} \). Since \( i^*(U) \in \mathcal{U}' \) and \( i^!(V) \in \mathcal{V}' \), it follows from Lemma 4.6(1) that \( \text{Ext}^2_C(i_*i^*(U), V) \cong \text{Ext}^2_A(i^*(U), i^!(V)) = 0 \). Note that \( j^*(U) \in \mathcal{U}'' \) and \( j^!(V) \in \mathcal{V}'' \). Applying Lemma 4.6(3), it follows that \( \text{Ext}^2_C(j_*j^*(U), V) = \text{Ext}^2_B(j^*(U), j^!(V)) = 0 \).

Since \( j_*j^*(U) \rightarrow U \rightarrow i_*i^*(U) \) is admissible exact in \( \mathcal{C} \), this means \( \text{Ext}^2_C(U, V) = 0 \), as desired.

**Step 2:** \( \mathcal{U}'' \subseteq \mathcal{V} \). Let \( N \) be an object in \( \mathcal{U}'' \). To prove \( N \in \mathcal{V} \), it suffices to show \( i^!(N) \in \mathcal{V}' \) and \( j^*(N) \in \mathcal{V}' \). Let \( M \) be an object in \( \mathcal{U}' \). Then \( i_*(M) \in \mathcal{U} \). It follows from Lemma 4.6(1) that \( \text{Ext}^1_A(M, i^!(N)) \cong \text{Ext}^1_C(i_*i^*(M), N) = 0 \). So \( i^!(N) \) belongs to \( \mathcal{V}' \). On the other hand, we assume that \( L \) is an object in \( \mathcal{U}'' \). Then \( j_i(L) \in \mathcal{U} \). Applying Lemma 4.6(3), we obtain \( \text{Ext}^1_C(L, j^*(N)) \cong \text{Ext}^1_C(j_i(L), N) = 0 \). So \( j^*(N) \) belongs to \( \mathcal{V}' \).

**Step 3:** \( \mathcal{V}' \subseteq \mathcal{U} \). Let \( M \) be an object in \( \mathcal{V}' \). To prove \( M \in \mathcal{U} \), it suffices to show \( i^*(M) \in \mathcal{U}' \), \( j^*(M) \in \mathcal{U}'' \) and \( \varepsilon_M : j_*j^*(M) \rightarrow M \) is an admissible monomorphism. Let \( N \) be an object in \( \mathcal{V}' \). Then \( i_*(N) \) is in \( \mathcal{V} \). By Proposition 4.4, one can check that \( \varepsilon_M : j_*j^*(M) \rightarrow M \) is an admissible monomorphism. Thus we have an admissible exact sequence \( j_*j^*(M) \rightarrow M \rightarrow i_*i^*(M) \) in \( \mathcal{C} \), whence we obtain the following exact sequence

\[
\text{Hom}_C(j_*j^*(M), i_*(N)) \rightarrow \text{Ext}^1_C(i_*i^*(M), i_*(N)) \rightarrow \text{Ext}^1_C(M, i_*(N)).
\]

Note that \( \text{Hom}_C(j_*j^*(M), i_*(N)) \cong \text{Hom}_G(j^*(M), j^*(i_*(N))) = 0 \) and \( \text{Ext}^1_C(M, i_*(N)) = 0 \). It follows that \( \text{Ext}^1_C(i_*i^*(M), i_*(N)) = 0 \), and therefore \( \text{Ext}^1_A(i^*(M), N) \cong \text{Ext}^1_A(i^*(M), i^!i_*(N)) \cong \text{Ext}^1_C(M, i_*(N)) = 0 \) by Lemma 4.6(1). This implies \( i^*(M) \in \mathcal{U}' \).

Next we assume that \( L \) is an object in \( \mathcal{V}' \). Then \( j_i(L) \) is in \( \mathcal{V} \). It follows from Proposition 3.3(1) and Lemma 3.1(1) that \( i_*i^!(M) \rightarrow M \rightarrow j_*j^!(M) \) is an exact sequence in \( \mathcal{C} \). Thus we have the following exact sequence

\[
\text{Hom}_C(i_*i^!(M), j_*(L)) \rightarrow \text{Ext}^1_C(j_*j^!(M), j_*(L)) \rightarrow \text{Ext}^1_C(M, j_*(L)).
\]

Since \( \text{Hom}_C(i_*i^!(M), j_*(L)) \cong \text{Hom}_A(i^!(M), j^!(j_*(L))) = 0 \) and \( \text{Ext}^1_C(M, j_*(L)) = 0 \), it follows that \( \text{Ext}^1_C(j_*j^!(M), j_*(L)) = 0 \). Let \( \xi : L \rightarrow D \rightarrow j^*(M) \) be an admissible exact sequence in \( \mathcal{B} \). Since \( i^! : \mathcal{C} \rightarrow \mathcal{A} \) is exact by hypothesis, it follows from Lemma 2.9(5) that \( j_* : \mathcal{B} \rightarrow \mathcal{C} \) is also exact, and therefore \( j_*\xi : j_!(L) \rightarrow j_*D \rightarrow j_*j^!(M) \) is an admissible exact sequence in \( \mathcal{C} \). This implies that \( j_*(\xi) \) is split. Since \( j_* \) is a fully faithful functor, the sequence \( \xi \) is split. It follows that \( \text{Ext}^1_C(j^*(M), L) = 0 \). Thus \( j^*(M) \in \mathcal{U}'' \), as desired.
“⇐”. Assume that \((\mathcal{U}, \mathcal{V})\) is a hereditary cotorsion pair in \(\mathcal{C}\). It is easy to check that \(\mathcal{U}' = i^*(\mathcal{U})\), \(\mathcal{U}'' = j^*(\mathcal{U})\), \(\mathcal{V}' = i^*(\mathcal{V})\) and \(\mathcal{V}'' = j^*(\mathcal{V})\). By Lemma 4.6(1), we obtain \(\operatorname{Ext}^1_{\mathcal{A}}(i^*(U), i^*(V)) \cong \operatorname{Ext}^1_{\mathcal{C}}(i_*i^*(U), V)\) for every \(U \in \mathcal{U}, \mathcal{V} \in \mathcal{V}\) and \(i = 1, 2\). Since \(U \in \mathcal{U}\), it follows that \(i_*i^*(U) \in \mathcal{U}\). This means \(\operatorname{Ext}^1_{\mathcal{A}}(i^*(U), i^*(V)) = 0\) for \(i = 1, 2\), and therefore \(\operatorname{Ext}^1_{\mathcal{A}}(U', V') = 0\) for every \(U' \in \mathcal{U}', \mathcal{V}' \in \mathcal{V}'\) and \(i = 1, 2\). Similarly, one can prove \(\operatorname{Ext}^1_{\mathcal{B}}(U'', V'') = 0\) for every \(U'' \in \mathcal{U}'', \mathcal{V}'' \in \mathcal{V}''\) and \(i = 1, 2\)

Next we will prove that \((\mathcal{U}', \mathcal{V}')\) is a cotorsion pair in \(\mathcal{A}\). It suffices to show \((\mathcal{U}')^\perp \subseteq \mathcal{V}'\) and \((\mathcal{V}')^\perp \subseteq \mathcal{U}'\). Let \(N\) be an object in \((\mathcal{U}')^\perp\). For any \(U \in \mathcal{U}\), one has \(\operatorname{Ext}^1_{\mathcal{A}}(U, i_*(N)) \cong \operatorname{Ext}^1_{\mathcal{C}}(i^*(U), N)\) by Lemma 4.6(2). Since \(i^*(U) \in \mathcal{U}'\), we obtain \(\operatorname{Ext}^1_{\mathcal{C}}(i^*(U), N) = 0\). Thus \(\operatorname{Ext}^1_{\mathcal{C}}(U, i_*(N)) = 0\), whence \(i_*(N) \in \mathcal{V}'\). So \(N \cong i^*i_*(N)\) belongs to \(\mathcal{V}'\) and \((\mathcal{U}')^\perp \subseteq \mathcal{V}'\). On the other hand, for the containment \((\mathcal{V}')^\perp \subseteq \mathcal{U}'\), we assume that \(M\) is an object in \((\mathcal{V}')^\perp\). For any \(V \in \mathcal{V}\), one has \(\operatorname{Ext}^1_{\mathcal{A}}(i_*(M), V) \cong \operatorname{Ext}^1_{\mathcal{C}}(M, i^*(V))\) by Lemma 4.6(1). Since \(i^*(V) \in \mathcal{V}'\), we obtain \(\operatorname{Ext}^1_{\mathcal{C}}(M, i^*(V)) = 0\). It follows that \(\operatorname{Ext}^1_{\mathcal{C}}(i_*(M), V) = 0\). Thus \(i_*(M) \in \mathcal{U}\). This implies that \(M \cong i^*i_*(M)\) belongs to \(\mathcal{U}'\), as desired.

Finally, to prove that \((\mathcal{U}'', \mathcal{V}'')\) is a cotorsion pair in \(\mathcal{B}\), it suffices to show \((\mathcal{U}'')^\perp \subseteq \mathcal{V}''\) and \((\mathcal{V}'')^\perp \subseteq \mathcal{U}''\). Let \(Y\) be an object in \((\mathcal{U}'')^\perp\). For any \(U \in \mathcal{U}\), one has \(\operatorname{Ext}^1_{\mathcal{B}}(U, j_*(Y)) \cong \operatorname{Ext}^1_{\mathcal{C}}(j^*(U), Y)\) by Lemma 4.6. Since \(j^*(U) \in \mathcal{U}''\), we obtain \(\operatorname{Ext}^1_{\mathcal{C}}(j^*(U), Y) = 0\). Thus \(\operatorname{Ext}^1_{\mathcal{C}}(U, j_*(Y)) = 0\), whence \(j_*(Y) \in \mathcal{V}'\). So \(Y = j^*j_*(Y)\) belongs to \(\mathcal{V}''\) and \((\mathcal{U}'')^\perp \subseteq \mathcal{V}''\). On the other hand, for the containment \((\mathcal{V}'')^\perp \subseteq \mathcal{U}''\), we assume that \(X\) is an object in \((\mathcal{V}'')^\perp\). For any \(V \in \mathcal{V}\), one has \(\operatorname{Ext}^1_{\mathcal{B}}(j_*(X), V) \cong \operatorname{Ext}^1_{\mathcal{C}}(j^*(X), V)\) by Lemma 4.6(3). Since \(j^*(V) \in \mathcal{V}'\), we have \(\operatorname{Ext}^1_{\mathcal{C}}(j^*(X), V) = 0\). It follows that \(\operatorname{Ext}^1_{\mathcal{B}}(j_*(X), V) = 0\), and therefore \(j_*(X) \in \mathcal{U}\). So \(X \cong j^*j_*(X)\) belongs to \(\mathcal{U}''\), as desired. \(\square\)

Now we can prove Theorem 1.1 in the introduction.

**Proof of Theorem 1.1.** Let \(C\) be an object in \(\mathcal{C}\). Then there exists an admissible exact sequence \(j^*(C) \xrightarrow{f_1} V'' \xrightarrow{g_1} U''\) in \(\mathcal{B}\) with \(V'' \in \mathcal{V}''\) and \(U'' \in \mathcal{U}''\). Since \(j_1\) is \(\mathcal{U}''\)-exact, we have the following pushout diagram in \(\mathcal{C}\)

\[
\begin{array}{ccc}
C & \xrightarrow{i^*(1)} & D \\
\downarrow{\alpha_1} & & \downarrow{\alpha_2} \\
\mathcal{U}' & \xrightarrow{j_1(U'')} & \mathcal{U}''
\end{array}
\]

Since \(i^1\) is an exact functor by hypothesis, we have an admissible exact sequence \(i^1(C) \xrightarrow{i^1(\alpha_1)} i^1(D) \xrightarrow{i^1(\alpha_2)} i^1j_1(U'')\) in \(\mathcal{A}\). Note that there exists an admissible exact sequence \(i^1(D) \xrightarrow{f_2} V' \xrightarrow{g_2} U'\) in \(\mathcal{A}\) with \(V' \in \mathcal{V}'\) and \(U' \in \mathcal{U}'\). Thus we have the following pushout diagram in \(\mathcal{A}\)

\[
\begin{array}{ccc}
i^1(C) & \xrightarrow{i^1(\alpha_1)} & i^1(D) \\
\downarrow{h_1} & & \downarrow{h_2} \\
V' & \xrightarrow{g_2} & X \\
\downarrow{g_2} & & \downarrow{g_2} \\
\mathcal{U}' & \xrightarrow{j_1(U'')} & \mathcal{U}''
\end{array}
\]
By Lemma 4.3, there is an admissible exact sequence in $C$

$$i_*i^!j^!j^*(C) \xrightarrow{\varphi_1} j^*j^!(C) \oplus i_*i^!(C) \xrightarrow{\psi_1} C,$$

where $\varphi_1 = \left( \frac{\sigma_{j^*j^!(C)}}{i_*i^!(\varepsilon_C)} \right)$ and $\psi_1 = (\varepsilon_C, \sigma_C)$.

Note that $\sigma_{j^!(V')} : i_*i^!j^!(V') \to j^!(V')$ and $\sigma_{j^!(U')} : i_*i^!j^!(U') \to j^!(U')$ are admissible monomorphisms. Then we have the following pushout diagrams

$$\begin{array}{ccc}
i_*i^!j^!(V') & \xrightarrow{\sigma_{j^!(V')}} & j^!(V') \\
\downarrow i_*(f_2i^!(\beta_1)) & & \downarrow a_1 \\
i_*(V') & \xrightarrow{b_1} & V, \\
\end{array} (4.1)$$

$$\begin{array}{ccc}
i_*i^!j^!(U') & \xrightarrow{\sigma_{j^!(U')}} & j^!(U') \\
\downarrow i_*(\gamma_1) & & \downarrow a_2 \\
i_*(X) & \xrightarrow{b_2} & U, \\
\downarrow i_*(\gamma_2) & & \downarrow \\
i_*(U') & \xrightarrow{\psi_2} & i_*(U'), \\
\end{array} (4.2)$$

Thus there exist admissible exact sequences in $C$

$$i_*i^!j^!(V') \xrightarrow{\varphi_2} j^!(V') \oplus i_*(V') \xrightarrow{\psi_2} V,$$

$$i_*i^!j^!(U') \xrightarrow{\varphi_3} j^!(U') \oplus i_*(X) \xrightarrow{\psi_3} U,$$

where $\varphi_2 = \left( \frac{\sigma_{j^!(V')}}{i_*i^!(f_2i^!(\beta_1))} \right)$, $\varphi_3 = \left( \frac{\sigma_{j^!(U')}}{i_*i^!(\gamma_1)} \right)$, $\psi_2 = (-a_1, b_1)$ and $\psi_3 = (-a_2, b_2)$. Applying $j^*$ to the commutative diagrams (4.1) and (4.2), we conclude that $j^*(a_1) : j^*j^!(V') \to j^*(V)$ and $j^*(a_2) : j^*j^!(U') \to j^*(U)$ are isomorphisms. Since $i_*(\gamma_1)$ is admissible monic, so is $a_2$. Hence we have the following diagram

$$\begin{array}{ccc}
i^!j^j^*(U') & \xrightarrow{i^!j^j^*(a_2)} & i^!j^j^*(U) \\
\downarrow i^!(\varepsilon_{j^!(U')}) & & \downarrow i^!(\varepsilon_U) \\
i^!j^j^!(U') & \xrightarrow{i^!(a_2)} & i^!(U), \\
\end{array}$$

such that $i^!(\varepsilon_{j^!(U')})$ and $i^!j^j^*(a_2)$ are isomorphisms. Since $i^!(a_2)$ is an admissible monomorphism, so is $i^!(\varepsilon_U)$. Note that there exists an object $A \in \mathcal{A}$ such that $i^*(A) \to j^!j^*(U)$ is a left exact sequence in $C$. Thus $i^!i^*(A) \to i^!j^j^*(U) \xrightarrow{i^!(\varepsilon_U)} i^!(U)$ is also a left exact sequence in $\mathcal{A}$, whence $A \cong i^!i^*(A) = 0$. So $\varepsilon_U : j^j^!(U) \to U$ is an admissible monomorphism. Applying the snake lemma (see [5, Corollary 8.13]), we have the following commutative diagram such that all rows and columns
are admissible exact sequences in $\mathcal{C}$

\[
\begin{array}{c}
i_*^l j_!^*(C) \xrightarrow{\varphi_1} j_!^*(C) \oplus i_*^l(C) \xrightarrow{\psi_1} C \\
i_*^l j_!(f_1) \downarrow \downarrow \quad \downarrow \downarrow \\
i_*^l j_!(V'') \xrightarrow{\varphi_2} j_!(V'') \oplus i_*^l(V') \xrightarrow{\psi_2} V \\
i_*^l j_!(g_1) \downarrow \downarrow \quad \downarrow \downarrow \\
i_*^l j_!(U'') \xrightarrow{\varphi_3} j_!(U'') \oplus i_*^l(X) \xrightarrow{\psi_3} U
\end{array}
\quad (4.3)
\]

Applying $j^*$ to the diagram (4.3) above, we obtain $j^*(V) \cong j^*_i j_!(V'') \cong V''$ and $j^*(U) \cong j^*_i j_!(U'') \cong U''$. If we set $\zeta : 1_A \to i_*^l$ be the unit of the adjoint pair $(i_*^l, i^l)$, then we obtain $i^l(j_!(V'')) \cong i^l(i_*^l(V))$. This means that $i^l(j_!(V''))$ is an admissible epimorphism. Since $i^l(j_!(V''))$ is an admissible monomorphism, it follows that $i^l(j_!(V''))$ is an isomorphism. Applying $i^l$ to the commutative diagram (4.1) yields that $i^l(V) \cong i^l(i_*^l(V')) \cong V' \in \mathcal{V}$, and therefore we obtain $V \in \mathcal{V}$. Next, applying $i^*$ to the second column in the diagram (4.2) leads to a right exact sequence $i^* j_!(U'') \xrightarrow{i^*(g_2)} i^*(U) \rightarrow i^* i_*(U')$. Since $i^* j_! = 0$, it follows that $i^*(U) \cong i^* i_*(U') \cong U' \in \mathcal{U}$. This implies $U \in \mathcal{U}$. So $(\mathcal{U}, \mathcal{V})$ is a complete cotorsion pair by Lemma 2.3, as desired.

By [31, Example 2.12], the comma category $\mathcal{C} = (\downarrow T \downarrow \mathcal{A})$ defined in introduction can induce the following recollement of abelian categories:

\[
\begin{array}{c}
A \xrightarrow{i_*^l} C \xleftarrow{j^*} B,
\end{array}
\quad (4.4)
\]

where $i^*(\begin{pmatrix} A \\ B \end{pmatrix})_f = \text{coker} f$, $i^*(\begin{pmatrix} A \\ B \end{pmatrix})_f = A$ and $j^*(\begin{pmatrix} A \\ B \end{pmatrix})_f = B$ for any $\begin{pmatrix} A \\ B \end{pmatrix} 
\text{id}$ for any $A \in \mathcal{A}$, and $j^!(B) = \begin{pmatrix} T(B) \\ B \end{pmatrix}$ and $j_*(B) = \begin{pmatrix} 0 \\ B \end{pmatrix}$ for any $B \in \mathcal{B}$.

**Remark 4.7.** Assume that $T : \mathcal{B} \to \mathcal{A}$ is a right exact functor between abelian categories with enough projective and injective objects. Note that $i^!$ is an exact functor in the above recollement (4.4). It follows from Proposition 3.3(1) that $\mathcal{C}_1 = \{C \in \mathcal{C} \mid i^* j_! j^*(C) = 0\} = \mathcal{C}$ and $\mathcal{B}_1 = \{B \in \mathcal{B} \mid B \cong j^*(C) \text{ for some } C \in \mathcal{C}_1\} = \mathcal{B}$. So Theorem 1.1 here is just Lemma 3.3 and Proposition 3.4 in [23]. Note that, in [23], one of the key arguments in the proof is that all objects in $(\downarrow T \downarrow \mathcal{A})$ can be represented clearly by the objects in $\mathcal{A}$ and $\mathcal{B}$, while in our general context we do not have this fact and therefore must avoid this kind of arguments. So, the idea of proving Theorem 1.1 will be different from the one in [23].

We end this section with the following example which illustrates Theorem 1.1.

**Example 4.8.** Let $A = B$ be the path algebra $k(1 \to 2)$, where $\text{char} k \neq 2$. Take $M = N = Ae_2 \oplus_k e_1 A$. The Auslander-Reiten quiver $\Gamma(\text{mod-}A)$ of the module category $\text{mod-}A$ has the form

\[
\begin{array}{c}
\sigma \\
Ae_1 \xrightarrow{\pi} S_2 \xrightarrow{\pi} S_1.
\end{array}
\]
Keep the notation of Example 3.6. Thus we have $B_1 = \{ Y \in \text{Mod-}A \mid \text{Hom}_A(N,Y) = 0 \} = \text{Add}(S_1)$, which implies

$$C_1 = \{ (\frac{X}{Y})_{f,g} \in \text{Mod-}A \mid Y \in B_1 \} = \text{Add}\{ \left( \frac{S_2}{S_1} \right)_{0,1}, \left( \frac{Ae_1}{S_1} \right)_{0,\sigma}, \left( \frac{S_1}{S_1} \right)_{0,0}, \left( \frac{Ae_1}{0} \right)_{0,0}, \left( \frac{S_2}{0} \right)_{0,0}, \left( \frac{S_1}{0} \right)_{0,0} \}.$$ 

Thus $\text{Add}\{ \left( \frac{S_2}{S_1} \right)_{0,1}, \left( \frac{Ae_1}{0} \right)_{0,0}, \left( \frac{S_2}{0} \right)_{0,0} \}$ is the class of projective objects of $C_1$. It is an easy exercise to show that $C_1$ has enough projective objects. Denote by $\text{Proj}(C_1)$ the class of projective objects of $C_1$. Similarly, one can show that $C_1$ has enough injective objects.

Take $(U', V') = (\text{Proj}(C_1), C)$ and $(U'', V'') = (B_1, B_1)$. Clearly, $j_i$ is $U''$-exact since $U''$ is the class of projective objective of $B_1$. If we set $V = C_1$ and

$$U = \left\{ \left( \frac{X}{Y} \right)_{f,g} \in \text{Mod-}A \mid \text{cok} \text{er} g \in \text{Proj } A, \ Y \in B_1, \ g \text{ is monomorphic} \right\},$$

then $(U, V) = (\mathcal{M}_{U''}, \mathcal{N}_{V''})$ is a projective cotorsion pair in $C_1$ by Theorem 1.1.

Finally, we consider the recollelement (3.4) of module categories in Example 3.6. It is clear that $i^! : \text{Mod-}A \to \text{Mod-}A$ is not an exact functor. Moreover, if we set $(U', V') = (U'', V'') = (\text{Add}\{S_2, Ae_1\}, \text{Mod-}A)$ in , then

$$\mathcal{M}_{U''} = \left\{ \left( \frac{X}{Y} \right)_{f,g} \in \text{Mod-}A \mid \text{cok} \text{er} g \in \text{Proj } A, \ Y \in \text{Proj } A, \ g \text{ is monomorphic} \right\}$$

and $\mathcal{N}_{V''} = \text{Mod-}A$. If follows from [38, Theorem 4.4] that $i^!$ is no longer a cotorsion pair in $\text{Mod-}A$. So the exactness of the functor $i^!$ in Theorem 1.1 cannot be omitted in general.

4.2. **Proof of Theorem 1.2.** We begin this subsection with the following lemma, which provides us a method to construct a Hovey triple from two cotorsion pairs in a WIC exact category.

**Lemma 4.9.** [14, Theorem 1.1] Let $C$ be a WIC exact category and suppose $(Q, \tilde{R})$ and $(\tilde{Q}, R)$ are complete hereditary cotorsion pairs over $C$ with (1) $\tilde{Q} \subseteq Q$, (2) $Q \cap \tilde{R} = \tilde{Q} \cap R$. Then there exists a unique exact model structure $(Q, W, R)$, and its class $W$ of trivial objects is given by

$$W = \{ X \in C \mid \exists \text{ an admissible exact sequence } X \to R \to Q \text{ with } R \in \tilde{R}, \ Q \in \tilde{Q} \} = \{ X \in C \mid \exists \text{ an admissible exact sequence } R' \to Q' \to X \text{ with } R' \in \tilde{R}, \ Q' \in \tilde{Q} \}.$$ 

Let $\mathcal{M}_A = (U_1', W_1', V_2')$ be a hereditary exact model structure on $A$. Then we have two complete hereditary cotorsion pairs $(U_1', V_1')$, $(U_2', V_2')$ in $A$, where $V_1' = W' \cap V_2'$, $U_2' = U_1' \cap W'$. Let $\mathcal{M}_B = (U_1'', W'', V_2'')$ be an exact model structure on $B$. Similarly, we obtain two complete hereditary cotorsion pairs $(U_1'', V_1'')$, $(U_2'', V_2'')$ in $B$, where $V_1'' = W'' \cap V_2''$, $U_2'' = U_1'' \cap W''$. Therefore, if $j_i$ is $U_i''$-exact, from Theorem 1.1 we will obtain two complete hereditary cotorsion pairs $(U_1, V_1)$ and $(U_2, V_2)$ in $C$, where $U_1 = \mathcal{M}_{U_1''}, V_1 = \mathcal{N}_{V_1''}, U_2 = \mathcal{M}_{U_2''}$ and $V_2 = \mathcal{N}_{V_2''}$. If $U_1 \cap V_1 = U_2 \cap V_2$, then by Lemma 4.9, there exists a unique class $W$, such that $\mathcal{M}_C = (U_1, W, V_2)$ is a Hovey triple in $C$, and $W \cap V_2 = V_1, U_1 \cap W = U_2$.

A **Quillen map** of model categories $M \to N$ consists of a pair of adjoint functors $(L, R) : M \rightleftarrows N$ such that $L$ preserves cofibrations and trivial cofibrations (it is equivalent to require that $R$ preserves fibrations and trivial fibrations). In this case the pair $(L, R)$ is also called a **Quillen adjunction.** A Quillen map induces adjoint total derived functors between the homotopy categories [30]. The class of weak equivalences is the most important class of morphisms in a model category. The following important characterization is proved in [13, Corollary 3.4].
**Lemma 4.10.** Let $\mathcal{M} = (\mathcal{Q}, \mathcal{W}, R)$ be a Hovey triple. Then a morphism is a weak equivalence if and only if $f = p_i$, where $i$ is an admissible monomorphism with cokernels $Q \cap W$ and $p$ is an admissible epimorphism with ker $p \in R \cap W$.

Before giving our main result, we need the following crucial result.

**Proposition 4.11.** Let $(\mathcal{A}, \mathcal{C}, \mathcal{B})$ be a recollement of exact categories with $i^!$ exact, and let $\mathcal{M}_A = (\mathcal{U}_1', \mathcal{W}', \mathcal{V}_2')$ and $\mathcal{M}_B = (\mathcal{U}_1'', \mathcal{W}'', \mathcal{V}_2'')$ be hereditary exact model structures on $\mathcal{A}$ and $\mathcal{B}$, respectively. We set $\mathcal{U}_1 = \mathcal{M}_A^{U_1}$, $\mathcal{V}_1' = \mathcal{M}_1^{W'' \cap W_2}$, $\mathcal{U}_2 = \mathcal{M}_B^{U'' \cap W''}$ and $\mathcal{V}_2' = \mathcal{M}_B^{V''}$. Assume that $j_i$ is $\mathcal{U}_1'$-exact and $\mathcal{U}_1 \cap \mathcal{V}_1 = \mathcal{U}_2 \cap \mathcal{V}_2$. Then the following hold.

1. There is a hereditary exact model structure $\mathcal{M}_C = (\mathcal{U}_1, \mathcal{W}, \mathcal{V}_2)$ on $\mathcal{C}$, where the class $\mathcal{W}$ is given by

$$\mathcal{W} = \{X \in \mathcal{C} \mid \exists\text{ an admissible exact sequence } X \rightarrow R \rightarrow Q \text{ with } R \in \mathcal{V}_1, \ Q \in \mathcal{U}_2\}$$

2. We have the following localization sequence of triangulated categories

$$\text{Ho}(\mathcal{M}_A) \xrightarrow{L(i_*)} \text{Ho}(\mathcal{M}_C) \xrightarrow{L(j^*)} \text{Ho}(\mathcal{M}_B)$$

where $L(i_*)$, $L(j^*)$, $R(i^!)$ and $R(j_*)$ are the total derived functors of those in (1.1).

3. Then we have the following colocalization sequence of triangulated categories

$$\text{Ho}(\mathcal{M}_A) \xrightarrow{L(j^!)} \text{Ho}(\mathcal{M}_C) \xrightarrow{L(j_*)} \text{Ho}(\mathcal{M}_B)$$

where $L(i^!), L(j_!)$, $R(i_*)$ and $R(j^*)$ are the total derived functors of those in (1.1).

**Proof.** (1) Let $\mathcal{V}_1' = \mathcal{W}' \cap \mathcal{V}_2'$, $\mathcal{U}_2' = \mathcal{U}_1' \cap \mathcal{W}'$ and $\mathcal{V}_1'' = \mathcal{W}'' \cap \mathcal{V}_2''$, $\mathcal{U}_2'' = \mathcal{U}_1'' \cap \mathcal{W}''$. If $j_i$ is $\mathcal{U}_1'$-exact, then by Theorem 1.1, $(\mathcal{U}_1, \mathcal{V}_1')$ and $(\mathcal{U}_2, \mathcal{V}_2')$ are two complete cotorsion pairs in $\mathcal{C}$. Because $\mathcal{U}_2 = \mathcal{M}_A^{U'' \cap W''}$, $\mathcal{U}_1' = \mathcal{M}_A^{U'' \cap W''}$ and $\mathcal{U}_2 = \mathcal{M}_B^{U'' \cap W''}$, by Lemma 4.9, there exists a unique class $\mathcal{W}$, such that $\mathcal{M}_C = (\mathcal{U}_1, \mathcal{W}, \mathcal{V}_2)$ is a hereditary exact model structure on $\mathcal{C}$.

(2) We first claim that $(i_*, i^!)$ and $(j^!, j_*)$ are Quillen adjunctions. Since (trivial) cofibrations equal admissible monomorphisms with (trivially) cofibrant cokernels and (trivial) fibrations equal admissible epimorphisms with (trivially) fibrant kernels, the inclusions $i_*U_1' \subseteq U_1$ and $i_*U_2' \subseteq U_1 \cap W'$ imply that $i_*$ preserves cofibrations and trivial cofibrations. Thus $(i_*, i^!)$ is a Quillen adjunction. Similarly, $j^!$ is a left adjoint and preserves cofibrations and trivial cofibrations. Hence $(j^!, j_*)$ is a Quillen adjunction by the definition. By [30, Proposition 16.2.2], the total derived functors $L(i_*)$ and $R(i^!)$ exist and form an adjoint between $\text{Ho}(\mathcal{M}_A)$ and $\text{Ho}(\mathcal{M}_C), L(j^!)$ and $R(j_*)$ exist and form an adjoint between $\text{Ho}(\mathcal{M}_C)$ and $\text{Ho}(\mathcal{M}_B)$. That is, we have the following diagram

$$\text{Ho}(\mathcal{M}_A) \xrightarrow{L(i_*)} \text{Ho}(\mathcal{M}_C) \xrightarrow{L(j^!)} \text{Ho}(\mathcal{M}_B).$$
In general, the right derived functor is defined on objects by first taking a fibrant replacement and then applying the functor. Similarly, the left derived functor is defined by first taking a cofibrant replacement and then applying the functor. So we have computed \((L(i_*), R(i_!)) = (i_*Q_A, i_!R_C)\) and \((L(j^*), R(j_*)) = (j^*Q_C, j_*R_B)\). Here, the notation such as \(Q_A\) means to take a special \(\mathcal{U}_1\)-precover. Similarly, the notation \(R_C\) means to take a special \(\mathcal{V}_2\)-preenvelope. Recall from [13, Proposition 4.4 and Section 5] that the distinguished triangles in \(\text{Ho}(\mathcal{M})\) are, up to isomorphism, the images in \(\text{Ho}(\mathcal{M})\) of distinguished triangles in \((\mathcal{Q} \cap \mathcal{R})/\omega\) under the equivalence \((\mathcal{Q} \cap \mathcal{R})/\omega \to \text{Ho}(\mathcal{M})\). By an argument similar to that in [12, Corollary 2.10], we see that these four functors are triangulated functors.

In order to show that the diagram (4.5) is a localization sequence, it remains to show

(i) \(R(i^!) \circ L(i_*) \cong 1_{\text{Ho}(\mathcal{M}_A)}\).

(ii) \(L(j^!) \circ R(j_* \cong 1_{\text{Ho}(\mathcal{M}_B)}\).

(iii) The essential image of \(L(i_*)\) equals the kernel of \(L(j^!)\).

To prove (i), let \(f : X \to Y\) be a homomorphism in \(\mathcal{A}\). Using the completeness of the cotorsion pair \((\mathcal{U}_1 \cap \mathcal{W}', \mathcal{V}_2')\), we get the following commutative diagram

\[
\begin{array}{c}
X \xrightarrow{q} X' \xrightarrow{} Z_1 \\
\downarrow f \downarrow \quad \downarrow f' \downarrow \\
Y \xrightarrow{q'} Y' \xrightarrow{} Z_1',
\end{array}
\]

where \(X', Y' \in \mathcal{V}_2', Z_1, Z_1' \in \mathcal{U}_1 \cap \mathcal{W}'\). Note that \(X'\) and \(Y'\) are fibrant replacements of \(X\) and \(Y\) in \(\mathcal{M}_A\), respectively. So both \(q\) and \(q'\) are natural isomorphisms in \(\text{Ho}(\mathcal{M}_A)\). The functor \(i_*Q_A\) acts by \(\bar{f} \mapsto \hat{f}\), where \(\hat{f}\) is any map making the diagram below commute

\[
\begin{array}{c}
i_*K \xrightarrow{i_*H_1} i_*X' \\
\downarrow f \quad \downarrow i_*f' \downarrow \\
i_*K' \xrightarrow{i_*H_1'} i_*Y',
\end{array}
\]

where the rows are admissible exact sequences, \(H_1, H_1' \in \mathcal{U}_1\), and \(K, K' \in \mathcal{W}' \cap \mathcal{V}_2'\). Moreover, we obtain \(H_1, H_1' \in \mathcal{U}_1 \cap \mathcal{V}_2\) since \(\mathcal{V}_2\) is closed under extensions. Now applying \(i!R_C\) to \(\hat{f}\) gives us \(\overline{f}\) in the next commutative diagram

\[
\begin{array}{c}
i!i_*H_1 \xrightarrow{p} i!L_1 \xrightarrow{} i!C_1 \\
\downarrow \overline{f} \downarrow \overline{f} \\
i!i_*H_1' \xrightarrow{p'} i!L_1' \xrightarrow{} i!C_1',
\end{array}
\]

where \(L_1, L_1' \in \mathcal{V}_2, C_1, C_1' \in \mathcal{U}_1 \cap \mathcal{W} = \mathcal{U}_2\). Since \(i_*H_1, i_*H_1' \in \mathcal{V}_2\) and \(\mathcal{V}_2\) is coresolving, we get that \(C_1, C_1' \in \mathcal{U}_2 \cap \mathcal{V}_2 = \mathcal{U}_1 \cap \mathcal{V}_1\) by hypotheses.

Furthermore, it is easy to check the inclusions \(i_*((\mathcal{W}' \cap \mathcal{V}_2')) = i_*((\mathcal{V}_2')) \subseteq \mathcal{V}_1 = \mathcal{W} \cap \mathcal{V}_2\) and \(i!(\mathcal{U}_1 \cap \mathcal{V}_1) \subseteq i!(\mathcal{V}_1) \subseteq \mathcal{V}_1' = \mathcal{W}' \cap \mathcal{V}_2\). Thus \(i!j, i!j', p\) and \(p'\) are all weak equivalences in \(\mathcal{M}_A\) by
Lemma 4.10. So, in $\text{Ho}(\mathcal{M}_A)$, we have a commutative diagram

$$
\begin{array}{c}
X \xrightarrow{q} X' \xrightarrow{\nu' \gamma} i_x' X' \xrightarrow{\nu' j} i_x H_1 \xrightarrow{p} i_x L_1 \\
Y \xrightarrow{q} Y' \xrightarrow{\nu' \gamma} i_x' Y' \xrightarrow{\nu' j} i_x H_1' \xrightarrow{p'} i_x L_2,
\end{array}
$$

where $\nu : 1_A \rightarrow i_x'$ is the unit of the adjoint pair $(i_x, i_x')$. This diagram gives rise to a natural isomorphism: $R(i_x') \circ L(i_x) \cong 1_{\text{Ho}(\mathcal{M}_A)}$.

Next we prove (ii). Let $X \in \text{Ho}(\mathcal{M}_B)$ be any object. Using the completeness of the cotorsion pair $(\mathcal{U}_B', \mathcal{V}_B)$, we obtain an admissible exact sequence $X \rightarrow E \rightarrow L$ with $E \in \mathcal{V}_B'$ and $L \in \mathcal{U}_B'$. Note that $E$ is a fibrant replacement of $X$ in $\mathcal{M}_B$, so we have a natural isomorphism $L \cong E$ in $\text{Ho}(\mathcal{M}_B)$. By Lemma 2.9(5), $j_*$ is exact. Then the functor $R(j_*) = j_* R_B$ acts by $X \rightarrow j_* E$, and $j_* E$ is in the admissible exact sequence $j_* X \rightarrow j_* E \rightarrow j_* L$. Now applying $j_* Q_C$ to $j_* E$ gives us $j_* N$ in the next admissible exact sequence

$$
j_* K \rightarrow j_* N \xrightarrow{\mu} j_* j_* E,
$$

where $N$ is a cofibrant replacement of $j_* E$, $N \in \mathcal{U}_1$, $K \in \mathcal{W} \cap \mathcal{V}_2$. By inclusion $j_* (\mathcal{W} \cap \mathcal{V}_2) = j_* (\mathcal{V}_1) \subseteq \mathcal{V}_B'' = \mathcal{W}'' \cap \mathcal{V}_2'$, we see that $\mu$ is a weak equivalence in $\mathcal{M}_B$. Hence, we have isomorphisms $L(j_*) \circ R(j_*)(X) \cong j_* N \cong j_* j_* E \cong E \cong X$ in $\text{Ho}(\mathcal{M}_B)$. By an argument similar to that in (i), we see that these isomorphisms are natural.

For (iii), let $X$ belongs to the essential image of $L(i_*)$. Then there exist an object $Y \in \mathcal{U}_1$, such that $X \cong i_* Y$ in $\text{Ho}(\mathcal{M}_C)$. Since $i_* Y \in \mathcal{U}_1$, $L(j_*)(Y) = j_* i_* Y = 0$. Hence $X$ is contained in kernel of $L(j_*)$.

Conversely, let $X$ belongs to the kernel of $L(j_*)$. Then $L(j_*)(X)$ is a zero object in $\text{Ho}(\mathcal{M}_B)$, that is, $L(j_*)(X) \in \mathcal{U}_B'' \cap \mathcal{V}_B'' \cap \mathcal{W}''$. We claim that there exists $Y \in \mathcal{A}$ such that $L(i_*)(Y) \cong X$ in $\text{Ho}(\mathcal{M}_C)$.

Notice that the functor $L(j_*)$ acts by $X \rightarrow j_* P$, where $j_* P$ is in an admissible exact sequence $j_* K \rightarrow j_* P \rightarrow j_* X$ in $\mathcal{C}$. Here $P \in \mathcal{U}_1$, $K \in \mathcal{V}_1 = \mathcal{W} \cap \mathcal{V}_2$. So $j_* P \cong L(j_*)(X) \in \mathcal{U}_B'' \cap \mathcal{V}_B'' \cap \mathcal{W}'' \subseteq \mathcal{U}_B''$. Now consider the admissible exact sequence

$$
j_* j_* P \rightarrow P \xrightarrow{\rho} i_* i_* P.
$$

One has $j_* j_* P \in j_*(\mathcal{U}_B'') \subseteq \mathcal{U}_2 = \mathcal{U}_1 \cap \mathcal{W}$. It follows that $\rho$ is a weak equivalence in $\mathcal{M}_C$. Define $Y := i_* (P)$. Since $i_* (P) \in i_* (\mathcal{U}_1) \subseteq \mathcal{U}_1$, we have $L(i_*)(Y) = L(i_*)(i_* (P)) = i_* i_* P \cong P \cong X$ in $\text{Ho}(\mathcal{M}_C)$. Hence the desired result follows immediately.

(3) The proof is similar to that of (2), and so we omit it here.

Now, we are ready to prove the main result of this paper.

**Proof of Theorem 1.2.** By Proposition 4.11, we only need to show that there are natural isomorphisms $L(i_*) \cong R(i_*)$ and $L(j_*) \cong R(j_*)$. Let $\mathcal{V}_1' := \mathcal{W'} \cap \mathcal{V}_2'$, $\mathcal{U}_2' := \mathcal{U}_1' \cap \mathcal{W'}$ and $\mathcal{V}_1'' := \mathcal{W}'' \cap \mathcal{V}_2'$, $\mathcal{U}_2'' := \mathcal{U}_1'' \cap \mathcal{W}''$. Let $f : X \rightarrow Y$ be a morphism in $\text{Ho}(\mathcal{M}_C)$. The functor $L(i_*)$ acts by $f \mapsto f$, where $\overline{f}$ is any morphism making the diagram below commute

$$
\begin{array}{c}
i_* K_1 \xrightarrow{\overline{i_* P_1}} i_* X \\
\overline{j_1} \downarrow \quad \downarrow i_* f \\
i_* K_2 \xrightarrow{\overline{i_* P_2}} i_* Y.
\end{array}
$$

\hfill \square
Here all rows are admissible exact sequences, $P_1, P_2 \in \mathcal{U}'_1$, and $K_1, K_2 \in \mathcal{W}' \cap \mathcal{V}'_2 = \mathcal{V}'_1$. The functor $R(i_*)$ acts by $f \mapsto \hat{f}$, where $\hat{f}$ is any morphism making the next diagram commute

\[
\begin{array}{ccc}
  i_*X & \xrightarrow{q_1} & i_*D_1 \\
  \downarrow i_*f & & \downarrow \hat{f} \\
  i_*Y & \xrightarrow{q_2} & i_*D_2
\end{array}
\]

where $D_1, D_2 \in \mathcal{V}'_2$, $C_1, C_2 \in \mathcal{U}'_1 \cap \mathcal{W}' = \mathcal{U}'_2$. Note that $i_*(\mathcal{V}'_1) \subseteq \mathcal{V}_1 = \mathcal{W} \cap \mathcal{V}_2$ and $i_*(\mathcal{U}'_2) \subseteq \mathcal{U}_2 = \mathcal{U}_1 \cap \mathcal{W}$. Then $j_1, j_2, q_1$ and $q_2$ are weak equivalences in $\mathcal{M}_C$. Hence in $\text{Ho}(\mathcal{M}_C)$, we have a commutative diagram

\[
\begin{array}{ccc}
i_*P_1 & \xrightarrow{j_1} & i_*X \xrightarrow{q_1} i_*D_1 \\
\downarrow T & & \downarrow \hat{f} \\
i_*P_2 & \xrightarrow{j_2} & i_*Y \xrightarrow{q_2} i_*D_2
\end{array}
\]

giving rise to a natural isomorphism $L(i_*) \cong R(i_*)$. The proof of the natural isomorphism $L(j^*) \cong R(j^*)$ is similar. This completes the proof. $\square$

5. APPLICATIONS TO UPPER TRIANGULAR MATRIX RINGS

Throughout this section, for any ring $R$, all $R$-modules are understood to be left $R$-modules and $C(R)$ is the category of chain complexes of $R$-modules. We denote by $\text{Mod-}R$ the class of $R$-modules.

Let $\Lambda = \left( \begin{array}{cc} R & M \\ 0 & S \end{array} \right)$ be an upper triangular matrix ring, where $R$ and $S$ are rings and $R M S$ is an $R$-$S$-bimodule. If we set $T := M \otimes_S - : \text{Mod-}S \to \text{Mod-}R$, then $T$ induces a functor $T : C(S) \to C(R)$ by $X^\bullet \mapsto M \otimes_S X^\bullet$. Note that in this case, $\mathcal{C}(\Lambda) = (T \downarrow C(S))$, where $(T \downarrow C(S)) = \{ (X^\bullet)_0 | X^\bullet \in C(R), Y^\bullet \in C(S), \phi : T Y^\bullet \to X^\bullet \text{ in } C(R) \}$. Therefore, by [31, Example 2.12], we obtain the recollement

\[
\begin{array}{ccc}
C(R) & \xrightarrow{i^*} & C(\Lambda) \xrightarrow{j_!} C(S),
\end{array}
\]

where $i^*$ is given by $(\begin{smallmatrix} X^\bullet \\ Y^\bullet \end{smallmatrix})_0 \mapsto \text{coker} \phi$; $i_*$ is given by $X^\bullet \mapsto (\begin{smallmatrix} X^\bullet \\ 0 \end{smallmatrix})$; $j^!$ is given by $(\begin{smallmatrix} X^\bullet \\ Y^\bullet \end{smallmatrix})_0 \mapsto X^\bullet$; $j_!$ is given by $Y^\bullet \mapsto (M \otimes_S Y^\bullet)_{id}$; $j^*$ is given by $(\begin{smallmatrix} X^\bullet \\ Y^\bullet \end{smallmatrix})_0 \mapsto Y^\bullet$; $j_*$ is given by $Y^\bullet \mapsto (\begin{smallmatrix} 0 \\ Y^\bullet \end{smallmatrix})$. Note that the functor $i_*, j^!, j_*$ defined above are exact.

5.1. The category of chain complexes. For a given class $\mathcal{X}$ of $R$-modules, we have the following classes of chain complexes in $C(R)$.

(1) $\tilde{\mathcal{A}}_R$ denotes the class of all exact chain complexes $X$ with cycles $Z_n X \in \mathcal{X}$.

(2) $\text{dw} \tilde{\mathcal{A}}_R$ denotes the class of all chain complexes $X$ with components $X_n \in \mathcal{X}$.

(3) $\text{ex} \tilde{\mathcal{A}}_R$ denotes the class of all exact chain complexes $X$ with components $X_n \in \mathcal{X}$.

Denote by $\text{Proj-}R$ the class of projective modules. It follows that the projective cotorsion pair $(\text{Proj-}R, \text{Mod-}R)$ in $\text{Mod-}R$ can be lifted to a complete hereditary cotorsion pair $(\text{dw} \tilde{\mathcal{P}}_R, \mathcal{W}_{\text{ctr},R})$ in $C(R)$. The complexes in $\mathcal{W}_{\text{ctr},R}$ have been called contraacyclic. From [17, Proposition 6.5], we know that the triple $\mathcal{M}_{\text{ctr},R}^{\text{proj}} = (\text{dw} \tilde{\mathcal{P}}_R, \mathcal{W}_{\text{ctr},R}, C(R))$ is a hereditary abelian model structure on $C(R)$ and its homotopy category, $\text{Ho}(\mathcal{M}_{\text{ctr},R}^{\text{proj}})$, called as the contraderived category over $R$, is
equivalent to $\mathbf{K}(\text{Proj-}R)$, where $\mathbf{K}(\text{Proj-}R)$ is the chain homotopy category of all complexes of projective modules.

Now we have the following result.

**Corollary 5.1.** Let $\Lambda = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ be an upper triangular matrix ring. If $\text{pd}_R M < \infty$ and $\text{pd}_S M < \infty$, then we have the following recollements of triangulated categories:

$$
\begin{align*}
\text{Ho}(\mathcal{M}^{\text{proj}}_{\text{ctr}, R}) & \xrightarrow{i_+} \text{Ho}(\mathcal{M}^{\text{proj}}_{\text{ctr}, A}) & \xrightarrow{j_+} \text{Ho}(\mathcal{M}^{\text{proj}}_{\text{ctr}, S}) \\
\text{K}(\text{Proj-}R) & \xrightarrow{F_R} \text{K}(\text{Proj-}A) & \xrightarrow{F_A} \text{K}(\text{Proj-}S) \\
\text{K}^b(\text{Proj-}R) & \xrightarrow{D^b(i_+)} \text{K}^b(\text{Proj-}A) & \xrightarrow{D^b(j_+)} \text{K}^b(\text{Proj-}S).
\end{align*}
$$

(5.2)

Here, the notation such as $Q_{\Lambda}$ means to take a special $dw\mathcal{P}_{\Lambda}$-precover. The functors such as $F_R$ is the triangulate equivalence $\text{Ho}(\mathcal{M}^{\text{proj}}_{\text{ctr}, R}) \to \text{K}(\text{Proj-}R)$, and $F_R^{-1}$ is the inverse of $F_R$. The notation such as $L^b(i^*), D^b(i_+), D^b(i^\pi), L^b(j_+), D^b(j_+), D^b(j^*)$ are the derived functors of those in (5.1).

**Proof.** If $\text{pd}_R M < \infty$ and $\text{pd}_S M < \infty$, then by [28], we have the recollement in the last row. By [17, Proposition 6.5], it suffices to construct the recollement (5.2). From [17, Proposition 6.5], we know that the triple $\mathcal{M}^{\text{proj}}_{\text{ctr}, R} = (dw\mathcal{P}_R, \mathcal{W}_{\text{ctr}, R}, \mathcal{C}(R))$ and $\mathcal{M}^{\text{proj}}_{\text{ctr}, S} = (dw\mathcal{P}_S, \mathcal{W}_{\text{ctr}, S}, \mathcal{C}(S))$ are hereditary abelian model structures on $\mathcal{C}(R)$ and $\mathcal{C}(S)$, respectively. Note that the Hovey triple $(dw\mathcal{P}_R, \mathcal{W}_{\text{ctr}, R}, \mathcal{C}(R))$ induces two complete cotorsion pairs $(dw\mathcal{P}_R, \mathcal{W}_{\text{ctr}, R}, \mathcal{W}_{\text{ctr}, R}, \mathcal{C}(R))$. Similarly, we have two complete cotorsion pairs $(dw\mathcal{P}_S, \mathcal{W}_{\text{ctr}, S}, \mathcal{C}(S))$ and $(\mathcal{P}_S, \mathcal{C}(S))$ in $\mathcal{C}(S)$. Applying Theorem 1.1, we obtain two cotorsion pairs $(\mathcal{U}_1, \mathcal{V}_1)$ and $(\mathcal{U}_2, \mathcal{V}_2)$ in $\mathcal{C}(A)$, where

$$
\begin{align*}
\mathcal{U}_1 &= \{ (\chi^\phi_{\omega})_{\phi} \in \mathcal{C}(\Lambda) \mid \text{coker}\phi \in dw\mathcal{P}_R, Y^\bullet \in dw\mathcal{P}_S, M \otimes_S Y^\bullet \to X^\bullet \text{ is a monomorphism} \} \\
\mathcal{V}_1 &= \{ (\chi^\phi_{\omega})_{\phi} \in \mathcal{C}(\Lambda) \mid X^\bullet \in \mathcal{W}_{\text{ctr}, R}, Y^\bullet \in \mathcal{W}_{\text{ctr}, S} \} \\
\mathcal{U}_2 &= \{ (\chi^\phi_{\omega})_{\phi} \in \mathcal{C}(\Lambda) \mid \text{coker}\phi \in \mathcal{P}_R, Y^\bullet \in \mathcal{P}_S, M \otimes_S Y^\bullet \to X^\bullet \text{ is a monomorphism} \} \\
\mathcal{V}_2 &= \mathcal{C}(\Lambda).
\end{align*}
$$

From [19, Theorem 3.1] we know that a $\Lambda$-module $X = (\chi^\phi_{\omega})_{\phi}$ is projective if and only if $Y$ is projective in $S\text{-Mod}$, $\text{coker}\phi$ is projective in $R\text{-Mod}$ and $\phi : M \otimes_S Y \to X$ is monic. Therefore, one can show $\mathcal{U}_1 = dw\mathcal{P}_\Lambda$. Moreover, since $(\mathcal{U}_1, \mathcal{V}_1)$ and $(\mathcal{U}_2, \mathcal{V}_2)$ are cotorsion pairs, we have $\mathcal{V}_1 = \mathcal{W}_{\text{ctr}, A}$ and $\mathcal{U}_2 = \mathcal{P}_\Lambda$. Hence $\mathcal{U}_1 \cap \mathcal{V}_1 = \mathcal{U}_2 \cap \mathcal{V}_2 = \mathcal{P}_\Lambda$. Thus Theorem 1.2 yields the desired recollement. Note that in each model structures above, all objects are fibrant. Therefore, we have $L(i^*) = i^*Q_{\Lambda}$,
Gluing and lifting exact model structures for the recollement of exact categories

\[ R(i_*) = i_*, \quad R(i^!) = i^!, \quad L(j_0) = j_0Q_S, \quad R(j_0^*) = j_0^* \text{ and } R(j_0) = j_0. \] Here, the notation such as \( Q_\Lambda \) means to take a special \( dw\tilde{P}_\Lambda \)-precover. One can check that all diagrams are commutative. \( \square \)

It is shown in [3, Proposition 2.2.1(1)] that the projective cotorsion pair \((\text{Proj}-R, \text{Mod}-R)\) in \( \text{Mod}-R \) can be lifted to a complete hereditary cotorsion pair \((\text{ex}\tilde{P}_R, \text{V}_{\text{prj},R})\) in \( C(R) \). By [17, Proposition 7.3], there is a hereditary abelian model structure \( M_{\text{stb},R}^{\text{proj}} = (\text{ex}\tilde{P}_R, \text{V}_{\text{prj},R}, C(R)) \) on \( C(R) \), which is called as the \textit{exact Proj model structure} on \( C(R) \). Moreover, its homotopy category \( \text{Ho}(M_{\text{stb},R}^{\text{proj}}) \), called as the \textit{projective stable derived category} over \( R \), is equivalent to \( K_{\text{ex}}(\text{Proj}-R) \), where \( K_{\text{ex}}(\text{Proj}-R) \) is the chain homotopy category of all exact complexes of projective modules. Thus we have the following result.

**Corollary 5.2.** Let \( \Lambda = \left( \begin{array}{c|c} R & M \\ \hline 0 & S \end{array} \right) \) be an upper triangular matrix ring. If \( M \) has finite flat dimension as a right \( S \)-module, then we have the following recollements of triangulated categories:

\[
\begin{array}{ccc}
\text{Ho}(M_{\text{stb},R}^{\text{proj}}) & \xrightarrow{i_*} & \text{Ho}(M_{\text{stb},R}^{\text{proj}}) \\
\text{K}_{\text{ex}}(\text{Proj}-R) & \xleftarrow{F_R} & \text{K}_{\text{ex}}(\text{Proj}-R)
\end{array}
\]

\[
\begin{array}{ccc}
\text{Ho}(M_{\text{stb},R}^{\text{proj}}) & \xrightarrow{j_*} & \text{Ho}(M_{\text{stb},S}^{\text{proj}}) \\
\text{K}_{\text{ex}}(\text{Proj}-\Lambda) & \xleftarrow{F_S} & \text{K}_{\text{ex}}(\text{Proj}-\Lambda)
\end{array}
\]

\[
\begin{array}{ccc}
\text{K}_{\text{ex}}(\text{Proj}-R) & \xleftarrow{F_{R^{-1}}} & \text{K}_{\text{ex}}(\text{Proj}-R) \\
\text{K}_{\text{ex}}(\text{Proj}-\Lambda) & \xleftarrow{F_{S^{-1}}} & \text{K}_{\text{ex}}(\text{Proj}-\Lambda)
\end{array}
\]

Here, the notation such as \( Q_\Lambda \) means to take a special \( \text{ex}\tilde{P}_\Lambda \)-precover. The functors such as \( F_R \) is the triangulate equivalence \( \text{Ho}(M_{\text{stb},R}^{\text{proj}}) \to \text{K}_{\text{ex}}(\text{Proj}-R) \), and \( F_R^{-1} \) is the inverse of \( F_R \).

**Proof.** It suffices to construct the recollement (5.3) by [17, Proposition 6.5]. From [17, Proposition 6.5] we know that the triple \( M_{\text{stb},R}^{\text{proj}} = (\text{ex}\tilde{P}_R, \text{V}_{\text{prj},R}, C(R)) \) and \( M_{\text{stb},S}^{\text{proj}} = (\text{ex}\tilde{P}_S, \text{V}_{\text{prj},S}, C(S)) \) are hereditary abelian model structures on \( C(R) \) and \( C(S) \), respectively. Note that the Hovey triple \( (\text{ex}\bar{P}_R, \text{V}_{\text{prj},R}, C(R)) \) induces two complete cotorsion pairs \( (\text{ex}\tilde{P}_R, \text{V}_{\text{prj},R}) \) and \( (\bar{P}_R, C(R)) \). Similarly, we have two complete cotorsion pairs \( (\text{ex}\tilde{P}_S, \text{V}_{\text{prj},S}) \) and \( (\bar{P}_S, C(S)) \) in \( C(S) \). Applying Theorem 1.1, we obtain two cotorsion pairs \( (U_1, V_1) \) and \( (U_2, V_2) \) in \( C(\Lambda) \), where

\[
U_1 = \{ (\bar{X}^*)_{\phi} \in C(\Lambda) \mid \text{coker}\phi \in \text{ex}\tilde{P}_R, Y^* \in \text{ex}\tilde{P}_S, M \otimes_S Y^* \to X^* \text{ is a monomorphism} \};
\]

\[
V_1 = \{ (\bar{X}^*)_{\phi} \in C(\Lambda) \mid X^* \in \text{V}_{\text{prj},R}, Y^* \in \text{V}_{\text{prj},S} \};
\]

\[
U_2 = \{ (\bar{X}^*)_{\phi} \in C(\Lambda) \mid \text{coker}\phi \in \bar{P}_R, Y^* \in \bar{P}_S, M \otimes_S Y^* \to X^* \text{ is a monomorphism} \};
\]

\[
V_2 = C(\Lambda).
\]

Consider the exact sequence of complexes \( 0 \to M \otimes_S Y^* \to X^* \to \text{coker}\phi \to 0 \). Since \( M \) has finite flat dimension as a right \( S \)-module by hypothesis, \( X^* \) is an exact complex. This means that \( U_1 = \text{ex}\tilde{P}_\Lambda \). Moreover, since \( (U_1, V_1) \) and \( (U_2, V_2) \) are cotorsion pairs, we have \( V_1 = \text{V}_{\text{prj},\Lambda} \) and \( U_2 = \bar{P}_\Lambda \). Hence \( U_1 \cap V_1 = U_2 \cap V_2 = \bar{P}_\Lambda \). Thus Theorem 1.2 yields the desired recollement. Note that in each model structures above, all objects are fibrant. Therefore, we have \( L(i^*) = i^*Q_\Lambda, \)
\(R(i_s) = i_s, R(j_t) = j_t, L(j_i) = j_iQ_S, R(j^*) = j^*\) and \(R(j_i) = j_i\). Here, the notation such as \(Q\) means to take a special \(\text{ex} \mathcal{P}_\Lambda\)-precover. One can check that all diagrams are commutate. \(\square\)

5.2. The category of Gorenstein injective modules. Let \(\Lambda = \left(\begin{array}{cc} R & M \\ 0 & S \end{array}\right)\) be an upper triangular matrix ring, where \(R\) and \(S\) are rings and \(RM_S\) is an \(R\)-\(S\)-bimodule. Recall that the monomorphism category \(\text{Mon}(\Lambda)\) induced by the bimodule \(RM_S\) is the subcategory of \(\text{Mod}-\Lambda\) consisting of \((\begin{array}{c} X \\ Y \end{array})\) such that \(\phi: M \otimes_SY \to X\) is a monomorphism in \(\text{Mod}-R\). For any class \(C\) of \(R\)-modules and any class \(D\) of \(S\)-modules, let \(\text{Rep}(C, D)\) denote the following class of \(\Lambda\)-modules:

\[
\text{Rep}(C, D) = \{N = (\begin{array}{c} X \\ Y \end{array}) \in \text{Mod}-\Lambda \mid X \in C, Y \in D\}.
\]

For any ring \(R\), an \(R\)-module \(X\) is Gorenstein injective if there exists an exact complex of injective \(R\)-modules \(E^\bullet = \cdots \to E^{-1} \to E^0 \to E^1 \to \cdots\) such that for any injective module \(I\), the complex \(\text{Hom}_R(I, E^\bullet)\) is still exact, and such that \(X \cong \ker(E^0 \to E^1)\). Such an exact complex of injective modules \(E^\bullet\) is called a totally acyclic complex of injective modules. For any ring \(R\), we denote by \(\mathcal{GI}(R)\) (resp., \(\mathcal{I}(R)\)) the class of Gorenstein injective (resp., injective) \(R\)-modules and by \(\mathcal{GI}(\text{Mon}(\Lambda))\) the class of Gorenstein injective objects in \(\text{Mon}(\Lambda)\). Recently, Šaroch and Šťovíček [34] have proved that \((\mathcal{GI}(R), \mathcal{GI}(\mathcal{I}(R)))\) is a complete hereditary cotorsion pair for any ring \(R\).

The following result give a characterization of injective objects in \(\text{Mon}(\Lambda)\).

**Lemma 5.3.** Let \(\Lambda = \left(\begin{array}{cc} R & M \\ 0 & S \end{array}\right)\) be an upper triangular matrix ring. If \(MS\) is flat, then \(X = \left(\begin{array}{c} X_1 \\ X_2 \end{array}\right)_\phi\) is an injective object in \(\text{Mon}(\Lambda)\) if and only if \(X \in \text{Rep}(\mathcal{I}(R), \mathcal{I}(S))\) and \(\phi\) is monic.

**Proof.** The “if” statement follows from the proof of [36, Proposition 2.2(2)]. For the “only if” statement, suppose that \(X = \left(\begin{array}{c} X_1 \\ X_2 \end{array}\right)_\phi\) is an injective object in \(\text{Mon}(\Lambda)\). We have \(\text{Ext}_R^1(U, X_1) \cong \text{Ext}_\Lambda^1((\begin{array}{c} U \\ 0 \end{array}), \left(\begin{array}{c} X_1 \\ X_2 \end{array}\right)_\phi) = 0\) for every \(R\)-module \(U\). Therefore, \(X_1\) is an injective \(R\)-module. Since \(MS\) is flat, by [39, Lemma 3.2(5)], we have \(\text{Ext}_\Lambda^1(V, X_2) \cong \text{Ext}_\Lambda^1(MS(V), \left(\begin{array}{c} X_1 \\ X_2 \end{array}\right)_\phi) = 0\) for any \(S\)-module \(V\). It follows that \(X_2\) is an injective \(S\)-module. \(\square\)

A bimodule \(RM_S\) is called cocompatible if the following three conditions hold:

\((C1)\) \(MS\) is flat:
\((C2)\) \(M \otimes_SY \in \mathcal{I}(R)\) for each injective right \(S\)-module \(Y\);
\((C3)\) the functor \(M \otimes_S-\) preserves totally acyclic complex of injective modules.

**Example 5.4.** (1) If \(R = S\) and \(M = \bigoplus_{\text{finite}} R\), then \(RM_R\) is a cocompatible bimodule.

(2) For an ring \(R\), let \(S := \bigoplus_{\text{finite}} R\), and \(RM_S := RS_S\). Then \(RM_S\) is a cocompatible bimodule.

(3) For an ring \(S\), let \(R := \bigoplus_{\text{finite}} S\), and \(RM_S := RS_S\). Then \(RM_S\) is a cocompatible bimodule.

(4) Suppose that \(R\) and \(S\) are quasi-Frobenius rings. If \(RM_S\) is a bimodule with \(RM\) injective and \(MS\) finitely generated projective, then \(RM_S\) is a cocompatible bimodule. In fact, any injective left \(S\)-injective module is an direct summand of \(\text{Hom}_Z(S, Q/Z)^I\) for some set \(I\). To show \((C2)\), it is only need to show that \(M \otimes_S \text{Hom}_Z(S, Q/Z)^I\) is injective. Since \(MS\) is finitely presented, we have an isomorphism \(M \otimes_S \text{Hom}_Z(S, Q/Z)^I \cong \text{Hom}_Z(\text{Hom}_S(M, S), Q/Z)^I\). By the fact that \(RM\) is injective and \(S_S\) is injective, we obtain that \(\text{Hom}_S(M, S)\) is a flat right \(R\)-module. Therefore, \(\text{Hom}_Z(\text{Hom}_S(M, S), Q/Z)^I\) is injective. \((C3)\) holds since each injective left \(R\)-module is projective.
Lemma 5.5. Let $\Lambda = \left( \begin{array}{l} R \\ 0 \\ S \end{array} \right)$ be an upper triangular matrix ring and $X = \left( \begin{array}{l} X_1 \\ X_2 \end{array} \right)_{\phi^X}$ a $\Lambda$-module. If $RM_S$ is cocompatible, then $X \in \mathcal{GI}(\text{Mon}(\Lambda))$ if and only if $X \in \text{Rep}(\mathcal{GI}(R), \mathcal{GI}(S))$ and $\phi^X$ is monic.

Proof. “⇒”. If $X = \left( \begin{array}{l} X_1 \\ X_2 \end{array} \right)_{\phi^X} \in \mathcal{GI}(\text{Mon}(\Lambda))$, then there exists an exact sequence of injective objects in $\text{Mon}(\Lambda)$: $E^* = \left( E^*_1 \rightarrow E^*_2 \rightarrow \cdots \right)$ such that $\ker \partial^i_1 \cong X_1$. By Lemma 5.3, $E^*_1$ and $E^*_2$ are injective modules for each $i \in \mathbb{Z}$. So we have an exact sequence of injective $R$-modules $E^*_1$ such that $\ker \partial^0_1 \cong X_1$. We know from Lemma 5.3 that $(\mathcal{I})$ is injective in $\text{Mon}(\Lambda)$ for each injective $R$-module $I$, so the complex $\text{Hom}_R(I, E^*_1) \cong \text{Hom}_{\text{Mon}(\Lambda)}((\mathcal{I}), (E^*_1))$ is exact. It follows that $X_1 \in \mathcal{GI}(\mathcal{R})$. Similarly, we have an exact sequence of injective $S$-modules $E^*_2$ such that $\ker \partial^0_1 \cong X_2$. Let $E$ be an injective $S$-module. Since $RM_S$ is cocompatible, $M \otimes_S E$ is an injective $R$-module. Applying Lemma 5.3 again, we get that $(M \otimes_S E)^{\bullet}$ is injective in $(\mathcal{I})$, so the complex $\text{Hom}_R(E, E^*_1) \cong \text{Hom}_{\text{Mon}(\Lambda)}((M \otimes_S E)^{\bullet}, (E_i^*))$ is exact. It follows that $X_2 \in \mathcal{GI}(\mathcal{S})$.

“⇐”. Consider the exact sequence $0 \rightarrow (M \otimes_S X_2) \rightarrow (X_1 \rightarrow X_2) \rightarrow (\ker \phi^X) \rightarrow 0$. Since the class of Gorenstein injective objects is closed under extensions, it only need to show that $(M \otimes_S X_2)$ and $(\ker \phi^X)$ are belong to $\mathcal{GI}(\text{Mon}(\Lambda))$. Let $\left( \begin{array}{l} I_1 \\ I_2 \end{array} \right)_{\phi^I}$ be an injective object in $\text{Mon}(\Lambda)$. Since $RM_S$ satisfies $(C2)$, $M \otimes_S I_2$ is an injective $R$-module. So the exact sequence $0 \rightarrow M \otimes_S I_2 \rightarrow I_1 \rightarrow \ker \phi^I \rightarrow 0$ splits. It follows that $\ker \phi^I$ is an injective $R$-module and we have $\left( \begin{array}{l} I_1 \\ I_2 \end{array} \right)_{\phi^I} \cong (M \otimes_S I_2) \oplus (\ker \phi^I)$.

Since $X_2$ is Gorenstein injective, there exists a totally acyclic complex of injective modules $E^*_2$. Therefore, by condition $(C3)$, the complex $\text{Hom}_{\text{Mon}(\Lambda)}(\left( \begin{array}{l} \ker \phi^I \\ 0 \end{array} \right), (M \otimes_S E^*_2)^{\bullet}) \cong \text{Hom}_R(\ker \phi^I, M \otimes_S E^*_2)^{\bullet}$ is exact. On the other hand, by the fact that the functor $V \rightarrow (M \otimes_S Y)^{\bullet}$ is fully faithful, the complex $\text{Hom}_{\text{Mon}(\Lambda)}((M \otimes_S I_2)^{\bullet}, (M \otimes_S E^*_2)^{\bullet}) \cong \text{Hom}_S(I_2, E^*_2)^{\bullet}$ is exact. Together, we showed that for each injective object $\left( \begin{array}{l} I_1 \\ I_2 \end{array} \right)_{\phi^I}$ in $\text{Mon}(\Lambda)$, $\text{Hom}_{\text{Mon}(\Lambda)}((\mathcal{I}), (E_i^*))$ is exact. Now, we have $(M \otimes_S X_2) \cong \ker (\partial^0_1 \otimes S_{\phi^X})$, which means that $(M \otimes_S X_2) \in \mathcal{GI}(\text{Mon}(\Lambda))$.

It suffices to show $(\ker \phi^X) \in \mathcal{GI}(\text{Mon}(\Lambda))$. Consider the exact sequence $M \otimes_S E^*_2$, we have $M \otimes_S X_2 \cong \ker \partial^0_1(M \otimes_S E^*_2)$. So $M \otimes_S X_2 \in \mathcal{GI}(\mathcal{R})$. Since $\mathcal{GI}(\mathcal{R})$ is coresolving, $\ker \phi^X \cong \ker \partial^0_1$. It follows that $(\ker \phi^X) \cong \ker (\partial^0_1)$. Note that for each injective object $\left( \begin{array}{l} I_1 \\ I_2 \end{array} \right)_{\phi^I}$ in $\text{Mon}(\Lambda)$, we have isomorphisms $\text{Hom}_{\text{Mon}(\Lambda)}((M \otimes_S I_2)^{\bullet}, (C^*) \cong \text{Hom}_{\text{Mon}(\Lambda)}(j(I_2), (C^*) \cong \text{Hom}_S(I_2, j^*((C^*)) = 0$ and $\text{Hom}_{\text{Mon}(\Lambda)}(\left( \begin{array}{l} \ker \phi^I \\ 0 \end{array} \right), (C^*) \cong \text{Hom}_R(\ker \phi^I, C^*)$. Therefore, the complex $\text{Hom}_{\text{Mon}(\Lambda)}(\left( \begin{array}{l} I_1 \\ I_2 \end{array} \right)_{\phi^I}, (C^*)) \cong \text{Hom}_{\text{Mon}(\Lambda)}((M \otimes_S I_2)^{\bullet} \oplus (\ker \phi^I), (C^*))$ is exact. It follows that $(\ker \phi^X) \in \mathcal{GI}(\text{Mon}(\Lambda))$, as desired.

Corollary 5.6. Let $\Lambda = \left( \begin{array}{l} R \\ 0 \\ S \end{array} \right)$ be an upper triangular matrix ring. Denote by $\mathcal{I}(\text{Mon}(\Lambda))$ the class of injective objects in $\text{Mon}(\Lambda)$. If $RM_S$ is cocompatible, then we have the following recollement
of triangulated categories:

\[
\begin{array}{ccc}
\mathcal{GI}(R) & \xrightarrow{i_*} & \mathcal{GI}(\text{Mon}(\Lambda)) \\
\mathcal{I}(R) & \xrightarrow{i^*} & \mathcal{I}(\text{Mon}(\Lambda)) \\
\mathcal{I}(S) & \xrightarrow{j_*} & \mathcal{GI}(S) \\
\end{array}
\]

Here, the notation such as \( R_S \) means to take a special \( \mathcal{GI}(S) \)-preenvelope.

Proof. Note that \( \mathcal{GI}(R) \cap \mathcal{GI}(S) = \mathcal{I}(R) \) by [15, Theorem 5.2]. Therefore, by Lemma 4.9, there exists a category \( \mathcal{W}(R) \) of modules, such that \((\text{Mod-}R, \mathcal{W}(R), \mathcal{GI}(R))\) forms a hereditary abelian model structure. Let \( \mathcal{M}_R = (\text{Mod-}R, \mathcal{W}(R), \mathcal{GI}(R)) \) and \( \mathcal{M}_S = (\text{Mod-}S, \mathcal{W}(S), \mathcal{GI}(S)) \) be abelian model structures on \( \text{Mod-}R \) and \( \text{Mod-}S \), respectively. Since \( \mathcal{M}_S \) is flat, applying Theorem 1.1, we obtain two cotorsion pairs \((U_1, V_1)\) and \((U_2, V_2)\) in \( \text{Mod-}\Lambda \), where

\[
\begin{align*}
U_1 &= \{ (\chi, \phi) \in \text{Mod-}\Lambda \mid \phi \text{ is a monomorphism} \} = \text{Mon}(\Lambda); \\
V_1 &= \{ (\chi, \phi) \in \text{Mod-}\Lambda \mid X \in \mathcal{I}(R), Y \in \mathcal{I}(S) \} = \text{Rep}(\mathcal{I}(R), \mathcal{I}(S)); \\
U_2 &= \{ (\chi, \phi) \in \text{Mod-}\Lambda \mid \text{coker} \phi \in \mathcal{W}(R), Y \in \mathcal{W}(S), \phi \text{ is a monomorphism} \}; \\
V_2 &= \{ (\chi, \phi) \in \text{Mod-}\Lambda \mid X \in \mathcal{GI}(R), Y \in \mathcal{GI}(S) \} = \text{Rep}(\mathcal{GI}(R), \mathcal{GI}(S)).
\end{align*}
\]

In the following, we claim that \( U_1 \cap V_1 = U_2 \cap V_2 = \mathcal{I}(\text{Mon}(\Lambda)) \). Let \((\chi, \phi) \in U_2 \cap V_2\). We show that \((\chi, \phi) \in U_1 \cap V_1\). Note that \( \mathcal{W}(R) \cap \mathcal{GI}(R) = \mathcal{I}(R) \) and \( \mathcal{W}(S) \cap \mathcal{GI}(S) = \mathcal{I}(S) \). So it suffices to prove that \( X \in \mathcal{W}(R) \). Consider the exact sequence \( 0 \to M \otimes_S Y \to X \to \text{coker} \phi \to 0 \). Since \( R \mathcal{M}_S \) is cocomplete, \( M \otimes_S Y \in \mathcal{I}(R) \in \mathcal{W}(R) \). Hence \( X \in \mathcal{W}(R) \) since \( \mathcal{W}(R) \) is closed under extensions. Conversely, let \((\chi, \phi) \in U_1 \cap V_1\). It remains to prove that \( \text{coker} \phi \in \mathcal{W}(R) \). By the argument above, we see that in this case, \( M \otimes_S Y \in \mathcal{I}(R) \in \mathcal{W}(R) \). Therefore, the claim follows by the fact that \( \mathcal{W}(R) \) is a thick subcategory. Note that the triangular matrix ring \( \Lambda \) can induce a recollement situation between the module categories over the rings \( R, \Lambda \) and \( S \) (see [31, Example 2.7] for instance). Thus Lemma 5.3, Lemma 5.5 and Theorem 1.2 yield the desired recollement. Note that in each model structures above, all objects are cofibrant. Therefore, we have \( L(i^*) = i^*, L(i_*) = i_*, R(i^*) = i^* \mathcal{I}(\Lambda), L(j_*) = j_*, L(j^*) = j^* \) and \( R(j_*) = j_* \mathcal{R}_S \). Here, the notation such as \( \mathcal{R}_S \) means to take a special \( \mathcal{GI}(S) \)-preenvelope.

\[\square\]

References

[1] L. Angeleri Hügel, S. Koenig, Q.H. Liu, Recollements and tilting objects, J. Pure Appl. Algebra, 215 (2011) 420-438.
[2] S. Bazzoni, M. Tarantino, Recollements from cotorsion pairs, J. Pure Appl. Algebra, 223 (2019) 1833-1855.
[3] H. Becker, Models for singularity categories, Adv. Math. 254 (2014) 187-232.
[4] A.A. Beilinson, J. Bernstein, P. Deligne, Faisceaux pervers, Astérisque 100, Soc. Math. France, Paris, 1982.
[5] Theo Bühler, Exact categories, Expo. Math. 28 (2010) 1-69.
[6] H.X. Chen, C.C. Xi, Good tilting modules and recollements of derived module categories, Proc. London Math. Soc. 104 (2012) 959-996.
[7] H.X. Chen, C.C. Xi, Recollements of derived categories III: finitistic dimensions, J. London Math. Soc. 95 (2017) 633-658.
[8] H.X. Chen, C.C. Xi, Recollements of derived categories I: Construction from exact contexts, J. Pure Appl. Algebra 225 (2021) 106648.
[9] E. Cline, B. Parshall, L. Scott, Algebraic stratification in representation categories, J. Algebra 117 (1988) 504-521.
Gluing and lifting exact model structures for the recollement of exact categories

[10] V. Franjou, T. Pirashvili, *Comparison of abelian categories recollements*, Doc. Math. 9 (2004) 41-56.
[11] N. Gao, S. Koenig, C. Psaroudakis, *Ladders of recollements of abelian categories*, J. Algebra 579 (2021) 256-302.
[12] D. Georgios, C. Psaroudakis, *Lifting recollements of abelian categories and model structures*, J. Algebra 623 (2023) 395-446.
[13] J. Gillespie, *Model structures on exact categories*, J. Pure Appl. Algebra 215 (2011) 2892-2902.
[14] J. Gillespie, *How to construct a Hovey triple from two cotorsion pairs*, Fund. Math. 230 (2015) 281-289.
[15] J. Gillespie, *Gorenstein complexes and recollements from cotorsion pairs*, Adv. Math. 291 (2016) 859-911.
[16] J. Gillespie, *Exact model structures and recollements*, J. Algebra 458 (2016) 265-306.
[17] J. Gillespie, *Hereditary abelian model categories*, Bull. Lond. Math. Soc. 48 (2016) 895-922.
[18] R. Göbel, J. Trlifaj, *Approximations and Endomorphism Algebras of Modules*, 2nd revised and extended ed., GEM 41, W. de Gruyter, Berlin 2012.
[19] A. Haghany, K. Varadarajan, *Study of modules over formal triangular matrix rings*, J. Pure Appl. Algebra 147 (1) (2000) 41-58.
[20] D. Happel, *Reduction techniques for homological conjectures*, Tsukuba J. Math. 17 (1993) 115-130.
[21] M. Hovey, *Model categories*, Mathematical Surveys and Monographs, vol. 63 (American Mathematical Society, Providence, RI, 1999).
[22] M. Hovey, *Cotorsion pairs, model category structures, and representation theory*, Math. Z. 241 (2002) 553-592.
[23] J.S. Hu, H.Y. Zhu, *Special precovering classes in comma categories*, Sci. China Math. 65 (2022) 933-950.
[24] B. Keller, *Chain complexes and stable categories*, Manuscripta Math. 67(4) (1990) 379-417.
[25] S. Koenig, *Tilting complexes, perpendicular categories and recollements of derived module categories of rings*, J. Pure Appl. Algebra 73 (1991) 211-232.
[26] H. Krause, *The stable derived category of a Noetherian scheme*, Compos. Math. 141 (5) (2005) 1128-1162.
[27] N.J. Kuhn, *The generic representation theory of finite fields: a survey of basic structure*, in: *Infinite Length Modules*, Bielefeld, 1998, in: Trends Math., Birkhäuser, Basel, 2000, pp. 193-212.
[28] P. Liu, M. Lu, *Recollements of singularity categories and monomorphism categories*, Commun. Algebra 43(6)(2015) 2443-2456.
[29] N. Marmaridis, *Comma categories in representation theory*, Comm Algebra, 11 (1983) 1919-1943.
[30] J.P. May, K. Ponto, *More concise algebraic topology, Localization, completion, and model categories*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago IL, 2002.
[31] C. Psaroudakis, *Homological theory of recollements of abelian categories*, J. Algebra 398(2014) 63-110.
[32] D. Quillen, *Higher algebraic K-theory. I*, Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), Lecture Notes in Math., vol. 341, Springer verlag, 1973, pp. 85-147.
[33] L. Salce, *Cotorsion theories for abelian groups*, Symposia Math. 23 (1979) 11-32.
[34] J. Šaroch, J. Štovíček, *Singular compactness and definability for Σ-cotorsion and Gorenstein modules*, Selecta Math. (N.S.) 26 (2) (2020) No. 23, 40 pp.
[35] L. Wang, J.Q. Wei, H.C. Zhang, *Recollements of extriangulated categories*, Colloq. Math. 167 (2022) 239-259.
[36] B.L. Xiong, P. Zhang, Y.H. Zhang, *Bimodule monomorphism categories and RSS equivalences via cotilting modules*, J. Algebra 503 (2018) 21-55.
[37] P. Zhang, *Gorenstein-projective modules and symmetric recollements*, J. Algebra 388 (2013) 65-80.
[38] P. Zhang, J. Cui, S. Rong, Cotorsion pairs and model structures on Morita rings, arXiv:2208.05684v2, 2022.
[39] R.M. Zhu, Y.Y. Peng, N.Q. Ding, *Recollements associated to cotorsion pairs over upper triangular matrix rings*, Publ. Math. Debrecen 98 (1-2) (2021) 83-113.