Photon addition and subtraction are well-known to render Gaussian states non-Gaussian. We evaluate how their optical nonclassicality is affected by the photon addition/subtraction process. It is known that photon addition always transforms a Gaussian state into a nonclassical state. We show that photon subtraction transforms classical states into classical ones and nonclassical ones into nonclassical ones. For a quantitative analysis of the resulting nonclassicality, we use a recently introduced nonclassicality witness, the quadrature coherence scale, and compute the relative nonclassicality gain of the single photon added/subtracted Gaussian states with respect to the original Gaussian states. For arbitrary single-mode Gaussian states, we show that this relative gain can be substantial in both cases. Whereas the gain is always higher for photon-added states, it is only slightly so for a large parameter range and in particular in the limit of high squeezing, where both gains are identical. We also analyze the Wigner negative volume of the photon-added/subtracted states. Whereas photon-added Gaussian states are never Wigner positive, we show that single mode photon-subtracted states are Wigner positive if and only if they are weakly nonclassical or classical. Our analysis relies on explicit and general expressions for the characteristic and Wigner functions of photon added/subtracted multi-mode Gaussian states that we obtain simply.

1 Introduction

Gaussian states are prominent in continuous-variable quantum information as they are relatively easy to produce experimentally and simple to study theoretically. Nevertheless, non-Gaussian states or operations are essential for performing certain quantum information tasks and in particular, are needed to achieve universal photonic quantum computation[1, 2]. One possible method for state de-Gaussification is photon addition or subtraction. This technique is of interest because it allows the engineering of various quantum states. For example, cat states with small amplitude can be prepared by subtracting a photon from a vacuum squeezed state with a fidelity close to one [3, 4, 5]. In addition, such states can be generated experimentally [3, 6, 7, 8]. For a review on photon addition and subtraction, see [9].

In this paper, we will concentrate on one-photon addition/subtraction on multi-mode Gaussian states. The resulting non-Gaussian states generally exhibit many nonclassical properties. Some of those are inherited from the Gaussian mother state to which a photon is added or from which it is subtracted. Others are due to the addition/subtraction process itself. It is our goal in this paper to study to which extent the photon addition/subtraction enhances the optical nonclassicality of the Gaussian mother state.

In quantum optics, a state is said to be optically classical if it can be written as a mixture of coherent states [10]. Equivalently, a state is optically classical if its Glauber-Sudarshan $P$-function is non-negative everywhere in phase space. We recall the precise definition below, see Eq. (9). This definition is difficult to use directly in order to detect the optical nonclassicality of general states: both theoretically and experimentally, the $P$-function is hard to determine for many states. As a result, a variety of measures and witnesses of optical nonclassicality have been proposed over the years [11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35]. In this paper, we will use as a witness of optical nonclassicality the quadrature coherence scale $C(\rho)$ (QCS) of the state $\rho$, whose square is defined as [33, 36]

$$C^2(\rho) = \frac{1}{2n\mathcal{P}^2} \sum_{j=1}^{2n} \text{Tr}[\rho, \hat{r}_j][\hat{r}_j, \rho]$$

where $\hat{r} = (\hat{x}_1, \hat{p}_1, \cdots, \hat{x}_n, \hat{p}_n)$ is the vector of the position and momentum quadratures, $\mathcal{P} = \text{Tr}\rho^2$ is the purity of the state $\rho$ and $n$ the number of modes. It was shown in [33] that the QCS is a witness of optical nonclassicality. Indeed, if $C^2 > 1$ then the state is non-
classical. Since the reverse is not necessarily true, this witness is not faithful. It nevertheless provides both an upper and lower bound for a suitably defined distance from the set of optically classical states $C_{cl}$ [33] and as such it defines an optical nonclassicality measure. In particular, the larger the QCS, the farther the state is from $C_{cl}$. The evaluation of the QCS on large families of benchmark states in [33, 36, 37, 38] has confirmed the efficiency of the QCS as an optical nonclassicality measure.

In [36] it was further shown that the quadrature coherence scale measures the spread of the coherences of the quadratures of the state and that it provides a time scale for decoherence: states with a large QCS decohere faster, corroborating the idea that they are more nonclassical. Finally, in [38], it is shown that strongly entangled Gaussian states necessarily have a large QCS. This last result links entanglement to optical nonclassicality.

We will follow [33, 36] in referring to optically nonclassical states $\rho$ for which the QCS is less than 1 as weakly nonclassical states, the others being strongly nonclassical. In other words, we have that

$$C^2(\rho) \leq 1$$

if and only if $\rho$ is classical or weakly nonclassical. Here and in what follows, we drop the adjective “optical” from the expression “optical nonclassicality”.

In this paper we are interested in the question how much nonclassicality is acquired or lost by the photon-added/subtracted state $\rho_\pm$ as compared to the initial state $\rho$? We will measure this gain or loss with the relative nonclassicality gain $R_\pm$ defined as

$$R_\pm = \frac{C^2(\rho_\pm) - C^2(\rho^G)}{C^2(\rho^G)}.$$  (2)

It provides the percentage gain in nonclassicality as a result of the photon-addition/subtraction process. To compute $R_\pm$ we first establish a simple, explicit and straightforward formula for the characteristic function of an arbitrary photon-added/subtracted state. We show that, if the Wigner and/or characteristic functions of the mother state $\rho$ are known, then it is straightforward to obtain them also for the corresponding photon-added/subtracted states [see Eq. (15)-(16)]. For photon-added/subtracted Gaussian states, this immediately provides a simple explicit closed form expression for those functions [see Eq. (17) and (20)].

Using these general results, we show that photon-subtracted Gaussian states are classical if and only if the Gaussian mother state is. And we provide a simple criterion for their Wigner positivity. This is in contrast with the case of photon-added Gaussian states, which are known to always have some negativity in their Wigner function, and hence to always be nonclassical.

We illustrate our approach with several examples. We first consider photon addition/subtraction to general single mode Gaussian states. Those are characterized by a temperature parameter $0 \leq q < 1$ and a squeezing parameter $r \geq 0$. The photon addition/subtraction process de-Gaussifies any such squeezed thermal (SqTh) single mode Gaussian state. Our detailed analysis of $R_\pm(q, r)$ reveals the following features.

First of all, while we show that the relative nonclassicality gain is larger for photon addition than for photon subtraction for all values of $q$ and $r$, it turns out that, for large values of $r$, $R_+(q, r)$ and $R_-(q, r)$ converge to a common limiting value that decreases with $q$. Whereas the relative nonclassicality gain decreases to this limiting value with increasing $r$ for photon addition, it increases with increasing $r$ for photon subtraction. It diminishes with growing $q$ in both cases but this noise-sensitivity is larger in the case of photon subtraction. This shows that adding or removing a photon has rather similar effects on the nonclassicality provided the initial Gaussian state is sufficiently strongly squeezed. Second, there are nevertheless some notable differences between the two processes. It turns out that the nonclassicality gain can in fact be negative, so that nonclassicality can be lost in the photon addition/subtraction process. This is not surprising for photon subtraction and occurs in that case even at small $q$, whenever the squeezing is small. But nonclassicality loss also occurs more surprisingly for photon addition with very noisy states (large $q$), even for arbitrarily large squeezing. Finally, we show that photon subtraction leads to a classical state if the initial Gaussian state is classical, to a weakly nonclassical state if the initial Gaussian state is weakly nonclassical and to a strongly nonclassical one if the initial Gaussian state is strongly nonclassical.

Quantitatively, with a squeezing parameter in the range $1 \leq r \leq 2$, and with $q \leq 0.2$, the squeeze factor, expressed in dB, reaches values in the range 7 to 15 for photon addition. In this same parameter range, the relative nonclassicality gain for photon addition is at least 20% and can be as high as 175-200% for low $q$. Similar values are attained for photon subtraction, in the same parameter range for $r$ but provided $q < 0.15$.

The Wigner negative volume of a state, defined in [22], is a frequently used nonclassicality witness. From the explicit expression we have for the Wigner function, it is readily computed for all photon-added/subtracted single mode Gaussian states. We observe it is not a good indicator of the nonclassicality of those states, in particular at high values of squeezing. This is a reflection of the fact that all Gaussian states have a positive Wigner function so that the Wigner neg-
The Fourier transform of the characteristic function gives the Wigner function
\[ W(\alpha) = \frac{1}{(\pi)^2} \int \chi(z) e^{(\xi^\dagger \alpha - z \cdot \alpha)} d^2nz \]
where \( d^2nz = d^nRe(z)d^nIm(z), \alpha = (\alpha_1 \cdots \alpha_n)^T \), and \( \alpha_j = \alpha_{j1} + i \alpha_{j2} = \frac{1}{\sqrt{2}}(x_j + ip_j) \in \mathbb{C} \). It is normalized so that \( \int W(\alpha)d^2n\alpha = 1 \).

For later reference, we recall that a state \( \rho \) is said to be (optically) classical \([10]\) if and only if there exist a positive function \( P(z) \) so that
\[ \rho = \int P(z)|z\rangle\langle z|dz. \]
Here \( |z\rangle = D(z)|0\rangle \) are the coherent states with \( |0\rangle \) the vacuum state.

The first-order moments of a state \( \rho \) constitute the displacement vector, defined as \( d = \langle \hat{r}\rangle = \text{Tr}(\hat{r}\rho) \), while the second moments make up the covariance matrix \( V \) whose elements are given by
\[ V_{ij} = 2\text{Cov}[\hat{r}_i, \hat{r}_j] = \langle \{\hat{r}_i, \hat{r}_j\} \rangle - 2\langle \hat{r}_i \rangle \langle \hat{r}_j \rangle \]
where \( \{\cdot, \cdot\} \) represents the anticommutator.

A Gaussian state is fully characterized by its displacement vector and covariance matrix. Its characteristic function is a Gaussian:
\[ \chi^G(\xi) = e^{-\frac{1}{2}\xi^T\Omega\xi}e^{-i\xi^Tz}, \]
with
\[ \Omega = \bigoplus_{j=1}^n \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \]
Here, for all \( 1 \leq i \leq n, \xi_i^T = (\xi_{i1}, \xi_{i2}) \in \mathbb{R}^2 \) and \( \xi^T = (\xi_1^T, \ldots, \xi_n^T) \in \mathbb{R}^{2n} \). Also, we define
\[ z_j = \xi_{j1} + i \xi_{j2}, \]
and \( z = (z_1, \ldots, z_n) \in \mathbb{C}^n \) and we will write, with the usual abuse of notation: \( \chi^G(z) = \chi^G(\xi) \).

The Wigner function \( W^G(\alpha) \) of a Gaussian state is also a Gaussian. See Appendix C for the explicit expression.
where \(a^\dagger(c)\) is given by Eq. (5), \(c \in \mathbb{C}^n\) and \(\sum_j |c_j|^2 = 1\). In general, a photon-added state is then defined as
\[
\rho_+ = N_+ a^\dagger(c) \rho a(c) \quad \text{with} \quad N_+ = (\text{Tr} \left[ a^\dagger(c) \rho a(c) \right])^{-1},
\]
(13)
where \(\rho\) is the initial or mother state to which a photon is added. Similarly, the photon-subtracted state is defined as
\[
\rho_- = N_- a(c) \rho a^\dagger(c) \quad \text{with} \quad N_- = (\text{Tr} \left[ a(c) \rho a^\dagger(c) \right])^{-1}.
\]
(14)

Note that
\[
\text{Tr} \left[ a^\dagger(c) \rho a(c) \right] = \text{Tr} \left[ a(c) \rho a^\dagger(c) \right] + 1 \geq 1,
\]
so that \(0 < N_+ \leq 1\). However, \(\text{Tr} \left[ a(c) \rho a^\dagger(c) \right]\) can vanish, in which case \(a(c) \rho a^\dagger(c) = 0\) so that \(\rho_-\) is not defined. We will come back to this point below, but for now we assume \(N_- < +\infty\).

We write \(\chi\) for the characteristic function of \(\rho_+\). Its expression is obtained by a short and straightforward computation and we find:
\[
\chi_\pm(z) = -N_\pm \left[ c \cdot \left( \partial \pm \frac{z}{2} \right) \right] \left[ c \cdot \left( \partial \mp \frac{z}{2} \right) \right] \chi(z)
\]
(15)
where \(\chi(z)\) is the characteristic function of the state \(\rho\). To see this, we note first that the displacement operator can be written as
\[
D(z) = e^{a^\dagger(z)} e^{-a(z)} e^{-|z|^2/2}
\]
or equivalently, as
\[
D(z) = e^{-a(z)} e^{a^\dagger(z)} e^{-|z|^2/2}.
\]

Consequently
\[
\partial_z D(z) = \left( a^\dagger_j - \frac{\bar{z}_j}{2} \right) D(z) = D(z) \left( a_j + \frac{z_j}{2} \right),
\]
\[
\partial_{\bar{z}} D(z) = -D(z) \left( a_j + \frac{z_j}{2} \right) = -\left( a_j - \frac{\bar{z}_j}{2} \right) D(z).
\]
Hence, for all \(c \in \mathbb{C}^n\), a short computation shows that
\[
-\left[ c \cdot \partial_z - \frac{c \cdot z}{2} \right] \left[ c \cdot \partial_{\bar{z}} - \frac{c \cdot \bar{z}}{2} \right] D(z) = a(c) D(z) a^\dagger(c).
\]
This implies Eq. (15) for \(\chi_+\). The proof for \(\chi_-\) is similar.

It is clear from Eq. (15) that, when adding \(m\) photons, one needs to apply \(m\) times the operator \(-\left[ c \cdot \partial_z - \frac{c \cdot z}{2} \right] \left[ c \cdot \partial_{\bar{z}} - \frac{c \cdot \bar{z}}{2} \right]\) and to normalize the result.

To compute the Wigner function \(W_\pm(\alpha)\) of \(\rho_\pm\) it now suffices to compute the Fourier transform of \(\chi_\pm(z)\) [see Eq. (8)]. Details of the calculation can be found in Appendix A. We obtain
\[
W_\pm(\alpha) = N_\pm \left[ \frac{c \cdot \left( \partial_\alpha \mp \alpha \right)}{2} \right] \left[ \frac{c \cdot \left( \partial_\alpha \mp \alpha \right)}{2} \right] W(\alpha).
\]
(16)

Clearly then, if the characteristic function \(\chi\) (or Wigner function \(W\)) of \(\rho\) is known, the characteristic/Wigner function of an arbitrary photon-added/subtracted state can be straightforwardly computed. We illustrate this in the following paragraph for Gaussian states.

### 3.2 Photon-added/subtracted Gaussian states

We suppose now that \(\rho = \rho_G\) is Gaussian. The computation in Eq. (15) then reduces to elementary algebra, using (11). The details are given in Appendix B and the result is
\[
\chi_\pm(z) = N_\pm \left( \frac{1}{2} \bar{m}_e^T V m_e \pm \frac{1}{2} \beta_\pm m_e \bar{m}_e^T \beta_\pm \right) \chi^G(z).
\]
(17)

Here the covariance matrix \(V\) is the one of the Gaussian mother state,
\[
\beta_\pm = \frac{1}{2} (\Omega V \Omega \mp 1) U^\dagger Z + i \Omega d,
\]
(18)

the matrix \(U\) is given by \([41]\]
\[
U = \bigoplus_{j=1}^n u \quad \text{where} \quad u = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix},
\]
and
\[
Z = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} = \sqrt{2} U \xi, \quad m_e = U^\dagger \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}.
\]
(19)

Note that
\[
\bar{m}_e^T V m_e = 2 \text{Cov}[a^\dagger(c), a(c)].
\]

With analogous calculations (see Appendix C), one also finds the Wigner function of a photon-added/subtracted Gaussian state. The resulting expressions are similar with the difference that they involve the inverse of the covariance matrix \(V\). One finds \(^1\)
\[
W_\pm(r) = N_\pm \left( M_\pm(V, c) + \lambda_\pm m_e \bar{m}_e^T \lambda_\pm \right) W^G(r)
\]
(20)

\(^1\)With the usual abuse of notation, we write: \(W(\alpha) = W(r)\).
where 
\[ \lambda_\pm = [(V^{-1} \pm I) r - V^{-1} d] \in \mathbb{R}^{2n}, \]
and \( M_\pm(V,c) \in \mathbb{R} \) is independent of \( r \) and given by
\[ M_\pm(V,c) = \mp \frac{1}{2} - \frac{1}{2} m_c^T V^{-1} m_c. \tag{21} \]

Let us note that in [42, 43] expressions for the characteristic and Wigner functions of photon-added/subtracted Gaussian states were derived through a rather involved computation of the truncated correlation functions of the states, which then need to be summed. Our derivation here, starting as it does from the general expressions in Eq. (15) and (16), is elementary and the results are straightforwardly expressed in terms of the displacement vector \( d \) and of the (inverse of the) covariance matrix \( V \) of the Gaussian mother state. We use them now to analyze the nonclassicality and Wigner positivity of the photon-added/subtracted Gaussian states.

### 3.3 Wigner positivity of photon-added/subtracted Gaussian states

Recall that a state is said to be Wigner positive if its Wigner function is everywhere nonnegative. Using (20)-(21) one easily characterizes the Wigner positive photon-added/subtracted states as follows. Note that \( m_c m_c^T \) is a rank one projector in \( \mathbb{C}^{2n} \). It is therefore nonnegative, so that the term \( \lambda_\pm^T m_c m_c^T \lambda_\pm \) is nonnegative as well. It vanishes when \( \lambda_\pm \) is perpendicular to \( m_c \). Hence the photon-added/subtracted Gaussian state is Wigner positive if and only if
\[ M_\pm(V,c) \geq 0. \tag{22} \]

For photon-added Gaussian states, Eq. (22) is never satisfied as is clear from Eq. (21) and as was already remarked in [42, 43]. As a result, they always have some negativity in their Wigner functions and are therefore always nonclassical, even if the initial Gaussian mother state was itself classical.

Photon-subtracted states, on the other hand, are Wigner-positive if and only if
\[ m_c^T V^{-1} m_c \leq 1. \]

This straightforward condition therefore identifies a family of Wigner-positive states indexed by \( V \) and by \( c \) which is of interest in particular because a complete characterization of all Wigner positive states is not known [44]. In the case only a single mode is present \((n = 1)\), this criterion becomes particularly simple since then
\[ m_c^T V^{-1} m_c = \frac{1}{2} \text{Tr} V^{-1} = \frac{1}{2} \frac{\text{Tr} \sqrt{V}}{\det V}. \]

We shall identify all one-mode Wigner positive nonclassical photon-subtracted Gaussian states in Section 4.3.

### 3.4 (Non)Classicality of photon-subtracted Gaussian states

It is well known that photon-subtraction transforms a classical state into a classical state. We recall the argument. Suppose \( \rho \) is classical and let \( P(z) \) be its \( P \)-function, which is nonnegative. Then it follows directly from Eq. (9) that \( N_-(|z|^2 P(z), \zeta) \) is still nonnegative, is the \( P \)-function of \( \rho_-. \). In addition, photon subtraction can make a nonclassical state classical: \( \rho_1 |1 \rangle \langle 1 | = 0 \) is an example. In other words, photon subtraction does not generally preserve the nonclassicality of states. We show here that, nevertheless, photon subtraction always transforms a Gaussian nonclassical state into a nonclassical state. This is the content of Proposition (i) below. It generalizes an observation made in [45] where it is remarked that, under photon subtraction, a single mode squeezed vacuum state remains nonclassical for all values of squeezing \( r > 0 \). Our result holds for all nonclassical Gaussian multi-mode states.

We first identify those \( c \in \mathbb{C}^n \) with \( \zeta \cdot c = 1 \), and \( \rho_G \) for which \( a(c) \rho_G a(c) = 0 \); for such \( c \) and \( \rho_G \) photon subtraction therefore does not lead to a state. The result is stated in the following Lemma.

**Lemma.** Let \( \rho^G \) be a Gaussian state with covariance matrix \( V \) and displacement vector \( d \), and let \( c \in \mathbb{C}^n \). Then \( a(c) \rho^G a(c) = 0 \) if and only if \( m_c \in \text{Ker}(V - I) \) and \( m_c^T d = 0 \).

When \( V = I \), the Gaussian state is in fact a coherent state \(|z \rangle \). In that case the first condition of the Lemma is satisfied for all \( c \in \mathbb{C}^n \) and the second condition reads \( \zeta \cdot z = 0 \). In other words, one has
\[ a(c)|z \rangle = 0 \Leftrightarrow \zeta \cdot z = 0. \tag{23} \]

Of course, this particular case follows immediately from the well-known identity
\[ a(c)|z \rangle = (\zeta \cdot z)|z \rangle, \]
which is in turn a direct consequence of Eq. (7). When there is only one mode, then Eq. (23) can only be satisfied if \( |z \rangle = |0 \rangle \). With several modes, on the other hand it does occur for nonzero \( z \). The Lemma treats the case of a general Gaussian state and the proof, which uses Eq. (17)-(18), is slightly more involved.

**Proof.** Set \( \rho_- := a(c) \rho_G a(c) = 0 \) if and only if \( \tilde{\chi}_-(z) = 0 \) for all \( z \in \mathbb{C}^n \), where \( \chi_- \) is the characteristic function of \( \rho_- \). From Eq. (17)-(18), it is given by
\[ \tilde{\chi}_-(z) = \left( \frac{1}{2} m_c^T V m_c + \frac{1}{2} - \beta_\pm^T m_c m_c^T \beta_\pm \right) \chi^G(z). \]

For this to vanish, the polynomial factor preceding the exponential factor \( \chi^G \) must vanish for all \( z \in \mathbb{C}^n \). Let
\( \mathbf{v} \in \mathbb{R}^{2n} \) be an eigenvector of \( V \) with eigenvalue \( \lambda \neq 1 \). Then define, for all \( \mu \in \mathbb{R} \),
\[
Z(\mu) = \mu U^\top \mathbf{v}.
\]
(24)

Then
\[
\beta^T \mathbf{m}_c = \frac{\mu}{2}(\lambda - 1)(\Omega \mathbf{v})^T \mathbf{m}_c + i(\Omega d)^T \mathbf{m}_c.
\]
(25)

It follows that
\[
\chi_-(Z(\mu)) = p(\mu)\chi(G(\mu)),
\]
(26)

where \( p(\mu) \) is a polynomial of degree two. This polynomial vanishes identically if and only if it has vanishing coefficients. One readily checks this is equivalent to
\[
(\Omega \mathbf{v})^T \mathbf{m}_c = 0,
\]
(27)
\[
\frac{1}{2}(\mathbf{m}_c^T(V - \mathbb{I}) \mathbf{m}_c) + |(\Omega d)^T \mathbf{m}_c|^2 = 0.
\]
(28)

Since \( \Omega^T \mathbf{m}_c = i\mathbf{m}_c \), the first of these two conditions is equivalent to \( \mathbf{v}^T \mathbf{m}_c = 0 \). Since this needs to hold for all eigenvectors of \( V \) with eigenvalue \( \lambda \neq 1 \), it follows that \( \mathbf{m}_c \in \text{Ker}(V - \mathbb{I}) \). Hence the first term in Eq. (28) vanishes and so does therefore the second one. This concludes the proof. \( \square \)

We are now ready to fully characterize the classical and hence the nonclassical photon-subtracted Gaussian states.

**Proposition.** Let \( \rho^G \) be a Gaussian state. Let \( \mathbf{c} \in \mathbb{C}^n \) and suppose \( a(\mathbf{c})\rho^G a(\mathbf{c})^\dagger \neq 0 \). Then:

(i) \( \rho_- \) is classical/nonclassical if and only if \( \rho^G \) is classical/nonclassical.

(ii) \( \rho_- \) is classical if and only if \( V - \mathbb{I} \geq 0 \).

Proposition (i) and (ii) are equivalent since it is well known that the classicality of a Gaussian state is equivalent to \( V \geq \mathbb{I} \) [41]. Proposition (i) asserts that, whereas it is true that photon subtraction cannot produce a nonclassical state from a classical one, it is also true that it does never transform a nonclassical Gaussian state into a classical one. We will show in the next section that it can in fact considerably increase the degree of nonclassicality of a given Gaussian state.

**Proof.** In view of the previous comment, it is sufficient to prove that if \( \rho_- \) is classical then \( V \geq \mathbb{I} \). For that purpose, we use the fact that, if \( \rho_- \) is classical, then the Fourier transform of the \( P \)-function, which is known to be given by \( e^{i\xi} \chi_-(\xi) \) [46], is a bounded function. Using Eq. (11) and (17) this implies
\[
|e^{i\xi} \chi_-(\xi)| = N(-\frac{1}{2} \mathbf{m}_c^T V \mathbf{m}_c - \frac{1}{2} - \beta^T \mathbf{m}_c \mathbf{m}_c^T \beta_-)
\times e^{-\frac{1}{2} \xi^T(\Omega - \mathbb{I})\Omega^T \xi}
\]
(29)
is bounded. Suppose it is not true that \( V \geq \mathbb{I} \). Then there exists \( \mathbf{v} \in \mathbb{R}^{2n}, \mathbf{v} \cdot \mathbf{v} = 1 \), and \( 0 \leq \gamma < 1 \) so that \( V \mathbf{v} = \gamma \mathbf{v} \). For such \( \mathbf{v} \), we define \( Z(\mu) \) as in Eq. (24) and hence \( \xi(\mu) = \frac{1}{\sqrt{2}} U^\dagger Z(\mu) = \mu \frac{1}{\sqrt{2}} \mathbf{v}^T \mathbf{v} \). The exponential factor in (29) then grows without bound for large \( \mu \). Hence \( e^{i\xi} \chi_-(\xi) \) can be bounded only if the polynomial prefactor \( p(\mu) \) in Eq. (26) vanishes identically. This in turn is equivalent to Eq. (27)-(28). Since Eq. (27) holds for all eigenvectors of \( V \) with eigenvalue strictly less than 1, it follows that \( \mathbf{m}_c \) belongs to the nonnegative spectral subspace of \( V - \mathbb{I} \). Eq. (28) then implies that \( \mathbf{m}_c \) in fact belongs to the kernel of \( V - \mathbb{I} \). And, in addition, that \( d \) is perpendicular to \( \mathbf{m}_c \). By the Lemma, this in turn implies that \( a(\mathbf{c})\rho^G a(\mathbf{c}) = 0 \), which is a contradiction. In conclusion, \( V - \mathbb{I} \geq 0 \). \( \square \)

We now use the characteristic function of photon-added/subtracted Gaussian states to further study their nonclassicality quantitatively.

## 4 Nonclassicality of photon-added and subtracted Gaussian states: examples

### 4.1 The single mode case: generalities

As announced in the introduction, we will use the quadrature coherence scale (QCS) defined in Eq. (1) to evaluate the nonclassicality of the states under study. There exist a number of other expressions for the QCS of a state, in particular
\[
C^2(\rho) = \frac{|||\chi(\xi)||^2}{n||\chi(\xi)||^2}.
\]
(30)

Here \( \chi(\xi) \) designates the characteristic function of the state [see Eq. (3)] and \( || \cdot ||_2 \) the \( L^2 \)-norm. This allows one, for example, to compute the QCS of an \( n \)-mode Gaussian state \( \rho^G \) [38] very simply in terms of its covariance matrix \( V \):
\[
C^2(\rho^G) = \frac{1}{2n} \text{Tr} V^{-1}.
\]
(31)

When there is only one mode, one can take, without loss of generality \( c = 1 \) and one finds
\[
C^2(\rho^G) = \frac{1}{2} \text{Tr} V^{-1} = \frac{1}{2} \mathbf{m}_c^T V^{-1} \mathbf{m}_c,
\]
\[
M_\pm(V,c) = \pm \frac{1}{2} - \frac{1}{2} \mathbf{m}_c^T V^{-1} \mathbf{m}_c
\]
(32)

This establishes a direct link between the negativity of the Wigner function of the photon-added/subtracted state and the QCS of the Gaussian mother state. We already showed that photon-subtracted states are nonclassical if and only if \( \rho_G \) is nonclassical. From the above one sees in addition that their Wigner function has some negativity if and only if the Gaussian mother state is strongly nonclassical, meaning that \( C^2(\rho_G) > 1 \).
\section{Photon-added squeezed thermal states}

We focus here on adding a photon to a general single mode Gaussian state. The degree of nonclassicality of such states has previously been investigated for the two particular cases of photon added thermal states (see \cite{16, 47, 48}) and photon added squeezed vacuum states (see \cite{42, 43, 49, 50}). We consider here the general case of all squeezed thermal (SqTh) states. Since the QCS is invariant under phase space translations, we may limit ourselves to centered Gaussians. We will see that the de-Gaussification process of photon-adding can lead to a considerable percentage gain in nonclassicality. At high squeezing and high temperature, however, nonclassicality may be lost in the process. We use Eq. (30) to compute the QCS of photon added/subtracted single mode Gaussian states.

A SqTh state is defined as $\rho_{\text{SqTh}} = S\rho_{\text{Th}}S^\dagger$ where

$$\rho_{\text{Th}} = (1-q) \sum_n q^n |n\rangle\langle n|$$

is a thermal state of temperature\footnote{The actual temperature is given by $T$ with $q = e^{-\hbar c T}$, $q$ is also related to the mean photon number $\langle n \rangle$ as $q = \frac{\langle n \rangle}{\langle n \rangle^2}$.} $q$ and $S = e^{\frac{1}{2} (z^2 - z^2)}$ is the squeezing operator with $z = re^{i\phi}$. The rotational invariance of the QCS implies we can restrict ourselves to the case where $\phi = 0$. The covariance matrix of these states is

$$V_{\text{SqTh}} = \begin{pmatrix} 1 + q & (1-q) \cosh(2r) \\ (1-q) \cosh(2r) & 1 - q \end{pmatrix}$$

and their characteristic function is

$$\chi_{\text{SqTh}}(z) = e^{-\frac{1}{2} \log^2 \left( e^{2r} \xi_1^2 + e^{-2r} \xi_2^2 \right)},$$

where we recall $z = \xi_1 + i\xi_2$. Their QCS, computed with Eq. (31), is then equal to

$$C_{\text{SqTh}}(q,r) = \frac{1-q}{1+q} \cosh(2r).$$

Adding one photon to a squeezed thermal state gives

$$\rho_{\text{SqTh}+} = \mathcal{N}_{\text{SqTh}+} a^\dagger \rho_{\text{SqTh}} a$$

where $\mathcal{N}_{\text{SqTh}+} = 2 \left( 1 + \frac{1+q}{1-q} \cosh(2r) \right)^{-1}$. The characteristic function can then be computed with Eq. (17)

$$\chi_{\text{SqTh}+}(z) = \chi_{\text{SqTh}}(z) \left( \frac{2q |z|^2}{(1-q^2) \cosh 2r + (1-q)^2} + \frac{q + 1}{q - 1} \left( e^{2r} \xi_1^2 + e^{-2r} \xi_2^2 \right) + 1 \right)$$

and the QCS with Eq. (30). The result is explicit (it can be found in Appendix D, but is not very instructive).

From Fig. 1, one sees that $C_{\text{SqTh}+}(q,r)$ is non-decreasing in $r$ and non-increasing in $q$: $\partial_q C_{\text{SqTh}+}(q,r) \leq 0$ and $\partial_r C_{\text{SqTh}+} \geq 0$. The line $C_{\text{SqTh}+}(q,r) = 1$ separates the strongly nonclassical states from the weakly nonclassical ones. As a point of comparison, we plotted in orange the level line $C_{\text{Th}+}(q,r) = 1$. One notices that photon-addition has considerably enlarged the region of strongly nonclassical states. Note however that some states that were originally strongly nonclassical have, due to the photon-addition process, lost QCS: these are the states above the orange and below the red line in the figure. More generally, as we will see below, whereas photon addition tends to increase nonclassicality, it can at high temperatures and squeezing lead to a loss of nonclassicality. We now analyze these phenomena quantitatively.

Fig. 2 shows the contour plot in the $(q, r)$-plane of the relative gain $R_{\text{SqTh}+}(q,r)$ [see Eq. (2)] of the nonclassicality obtained with the addition of a photon. We concentrate on the region above the (red) contour line $C_{\text{SqTh}+}(q,r) = 1$, where the resulting states are strongly nonclassical. The blue sub region corresponds to the situation where the gain is positive. The relative gain reaches its maximal value (200\%) for the squeezed vacuum states, independently of the squeezing $r$. It de-
As a result, there is a large region in the parameter space where the relative gain \( R_{\text{SqTh}} \) is positive and the QCS \( C_{\text{SqTh}} \geq 1 \) so that the photon-added squeezed thermal state is strongly non classical.

The level curves have vertical asymptotes, reflecting the fact that, at large \( r \), the gain is independent of the squeezing. One finds readily, for all \( q \) and \( r \) (see Appendix D) that

\[
R_{\text{SqTh}}(q, r) \geq R_{\text{SqTh}}(q, +\infty) = 2 - 12q - \frac{q^2 + 1}{q^4 + 10q^2 + 1}.
\]

As a result, there is a large region in the parameter space \((q, r)\) where the relative gain is sizeable. For example, when \( q \leq 0.2, \) it is larger than 20\% for all values of \( r \). For low \( q \) and values of \( r \) in the range \( 1 \leq r \leq 2 \), it is as high as 175-200\%.

Now consider the region I above both the blue \( R_{\text{SqTh}}(q, r) = 0 \) and red \( C_{\text{SqTh}}(q, r) = 1 \) level lines. While the resulting non-Gaussian state is still strongly nonclassical, it has actually lost some nonclassicality in the process. In other words, in this region, the de-Gaussification comes at the cost of a loss in nonclassicality.

Finally, in region II, the photon added state gains some nonclassicality, but remains close to the set of classical states and is only weakly nonclassical.

We end this subsection with a brief discussion of the Wigner negative volume [22], which is a different and much used witness of nonclassicality, that we shall denote by \( N_W(\rho) \): it is defined as the absolute value of the integral of the Wigner function over the area where the latter is negative. The Wigner function of the SqTh+ state can be readily computed with (20) (see Appendix E). One easily sees that it is negative inside an ellipse centered at the origin where it reaches its minimal value. Contrary to \( C_{\text{SqTh}}(r, q) \), an analytical expression for \( N_W(\rho_{\text{SqTh}}) \) is not readily obtained but the result of a numerical computation is shown in Appendix E. One observes that, at fixed temperature, increased squeezing actually leads to a decrease in the Wigner negative volume. This is in sharp contrast to the increase in the QCS of the photon-added state with growing squeezing, as observed in Fig. 1. We further remark that at large squeezing, the Wigner negative volume of these states – as well as the negative value at the origin – saturate to a finite value that decreases with increasing temperature \( q \). These observations taken together indicate that the Wigner negative volume fails to capture the underlying growing nonclassicality coming from the squeezing. This is of course not surprising, since the Gaussian mother states have no negativity at all.

### 4.3 Photon-subtracted squeezed thermal states

We start again with a Gaussian \( \text{SqTh} \) state, but this time we remove one photon to obtain a photon-subtracted squeezed thermal (\( \text{SqTh}^- \)) state. The characteristic function is given by

\[
\chi_{\text{SqTh}^-}(z) = \chi_{\text{SqTh}}(z) \left( \frac{2g|z|^2}{(1-q^2) \cosh 2r - (1-q)^2} + \frac{q + 1}{q - 1} \left( e^{2r\xi_1^2} + e^{-2r\xi_2^2} + 1 \right) \right). \tag{36}
\]

which allow us to compute the QCS. We again obtain an explicit, but not very instructive, expression (see Appendix D), which can be simplified for a SqV- or a Th-state as

\[
C_{\text{SqV}}^2 = 3 \cosh(2r) \quad C_{\text{Th}}^2 = -\frac{2(1-q)}{q^2 + 1} + \frac{6}{q + 1} - 3.
\]

The general case is plotted in Fig. 3. One sees again that \( C_{\text{SqTh}^-}(q, r) \) is non-decreasing in \( r \) and non-increasing in \( q \). There is one major difference with the photon-added case discussed in the previous section. Recall first from Proposition (i) and (ii) that the line (dotted purple in Fig. 3)

\[
r = \frac{1}{2} \ln \left( \frac{1 + q}{1 - q} \right)
\]

separates the classical SqTh states from the nonclassical ones and also the classical \( \text{SqTh}^- \) states from the nonclassical ones. Next, one notices that the red dashed line \( C_{\text{SqTh}^-}(q, r) = 1 \), which separates the strongly nonclassical photon-subtracted Gaussian states from the weakly nonclassical ones, coincides with the level line \( C_{\text{SqTh}} = 1 \) which separates similarly the weakly nonclassical Gaussian states from the strongly nonclassical ones. These observations show that under photon subtraction weakly nonclassical Gaussian states remain weakly nonclassical and strongly nonclassical Gaussian states stay strongly nonclassical. These results extend the known fact that under photon subtraction, classical states stay classical.
Figure 3: Level lines of the QCS $C_{SqTh}^2(q,r)$ of photon-subtracted squeezed thermal states in function of the temperature $q$ and the squeezing $r$. In dashed red the line $C_{SqTh}^2(q,r) = C_{SqTh}^2(q,0) = 1$, and in dotted purple, the line $r = \frac{1}{2} \ln \left( \frac{1+q}{1-q} \right)$ below which the SqTh and SqTh- states are classical. The region delimited by the dotted purple and dashed red lines corresponds to weakly nonclassical states, for which the Wigner function is positive. Above the dashed red line, both types of states are strongly nonclassical and the SqTh- states have Wigner negativity.

Finally, we point out that Eq. (32) implies the photon-subtracted state is Wigner positive if and only if the values of $q$ and $r$ are such that $C_{SqTh}^2(q,r) \leq 1$. In other words, the SqTh- states show Wigner negativity if and only if they are strongly nonclassical.

Note that adding a photon to or removing a photon from a squeezed vacuum state (with $r \neq 0$) gives the exact same state. Indeed using the relations

$$S^\dagger(z) a^\dagger S(z) = a^\dagger \cosh r - a e^{-i\phi} \sinh r$$
$$S^\dagger(z) a S(z) = a \cosh r - a^\dagger e^{i\phi} \sinh r$$ (37)

we have

$$a^\dagger |SqV\rangle = a^\dagger S |0\rangle = S(a^\dagger \cosh r - a e^{-i\phi} \sinh r)|0\rangle \propto S |1\rangle$$
$$a |SqV\rangle = a S |0\rangle = S(a \cosh r - a^\dagger e^{i\phi} \sinh r)|0\rangle \propto S |1\rangle$$

As we can see, once they are normalized, SqV+ and SqV− states are identical. This is of course no longer true for SqTh± states.

We plotted in Fig. 4 the relative gain $R_{SqTh-}$ of the photon subtracted squeezed thermal state, restricting ourselves again to the region where $R_{SqTh-} \geq 0$ (to the left of the two blue curves) and $C_{SqTh}^2 \geq 1$ (above the red dashed curve).

Like in the photon-addition case, subtracting a photon can enhance nonclassicality, as already mentioned in [42, 43, 51]. The relative gain reaches its maximal value of 200% for the SqV- state and is then independent of the squeezing. Contrary to what happens for photon addition, it now increases with increasing $r$; but it still decreases with increasing $q$. As for photon addition, the level curves of the relative gain have vertical asymptotes meaning that at large $r$ the gain is independent of the squeezing. This now upper bounds the relative gain as

$$R_{SqTh-}(q,r) \leq R_{SqTh-}(+\infty,q) = 2 - \frac{12q (q^2 + 1)}{q^4 + 10q^2 + 1}. \quad (38)$$

This means that to obtain a sizable gain the state cannot be too noisy. Typically, one can achieve a gain of at least 20% provided that $1 < r < 2$ and $q$ stays below 0.15. This is not very different from the values obtained for photon addition in the same parameter range.

In fact, by noticing that $R_{SqTh+}(+\infty,q) = R_{SqTh-}(+\infty,q)$, we see that photon addition or subtraction has a similar effect on sufficiently squeezed Gaussian states. In this regime, we therefore find

$$C_{SqTh\pm}^2(r,q) \approx \left( 3 - \frac{12q (q^2 + 1)}{q^4 + 10q^2 + 1} \right) \frac{1-q}{1+q} \cosh(2r).$$

Still, Eqs. (38) and (35) show that for all values of $q$ and $r$, $R_{SqTh+}(r,q) \geq R_{SqTh-}(r,q)$ so that photon addition systematically leads to a somewhat stronger nonclassicality gain.

4.4 Photon added two-mode squeezed thermal states 2SqTh+

In this section we briefly illustrate how the nonclassicality of multi-mode photon added/subtracted states can be analyzed using their QCS and the explicit formulas we proved for their characteristic functions. We limit ourselves to photon-addition for two-mode states. We
Gaussian states was elaborated upon in [42]. The generated in [38]; the entanglement of pure photon-added between nonclassicality and entanglement was investigated. It may be when \( c \neq 0 \) or \( c = 0 \), the negativity of the nonclassicality gain can be obtained in both cases.

We finally point out that the method to compute the characteristic function of one photon-added/subtracted state we introduce here can easily be generalized to the case of multi photon addition/subtraction and can provide a useful tool for further studies of various features of such states. For example, another important feature of a general state \( \rho \) is its non-Gaussianity, defined to be the distance between \( \rho \) and the Gaussian state \( \rho_G \) with the same first and second moments as \( \rho \). One way to measure this distance is to use the Hilbert-Schmidt

\[ \text{distance} = \sqrt{\text{tr}((\rho - \rho_G)^2)} \]

5 Conclusion/Discussion

In this paper we have provided simple general expressions for the Wigner and characteristic functions of photon-added/subtracted states. These have allowed us to perform both a qualitative and a quantitative study of the nonclassicality of photon-added/subtracted Gaussian n-mode states through the use of their quadrature coherence scale, a recently introduced nonclassicality measure. Our focus has been on the relative gain of nonclassicality that can be realized through the photon addition/subtraction process applied to such states. It is well known that photon addition transforms any Gaussian state into a nonclassical state with a nonpositive Wigner function. On the other hand, we show that photon-subtracted Gaussian states are nonclassical if and only if the Gaussian mother state is nonclassical.

We finally remark that, when \( r_1 \neq r_2 \), the relative nonclassicality gain, and hence the nonclassicality of the photon-added Gaussian state may be optimized by values of \( c_1 \neq 0, 1 \), as illustrated in the right panel of Fig. 5. It is easily checked that, on the contrary, the negativity of the Wigner function at the origin is always optimized either at \( c_1 = 0 \) or \( c_1 = 1 \).
distance between $\rho$ and $\rho_Q$ [52], which can be directly expressed in term of the characteristic function. An interesting extension of this work would be to compute the non-Gaussianity of the photon-added/subtracted states and to compare it with the relative gain of nonclassicality analyzed in this paper.

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Appendix A  Proof of Eq. (16)

We focus on the photon-addition case. Calculations are similar in the photon-subtraction case. The Wigner function is defined as

$$W_+(\alpha) = \frac{1}{\pi^{2n}} \int \chi_+(z)e^{(\bar{z} \cdot \alpha - z \cdot \bar{\alpha})} d^{2n}z$$

where $\chi_+$ is given by Eq. (15). Computing each term of the integral, we have:

$$\frac{1}{\pi^{2n}} \int (\bar{c} \cdot \partial \bar{z})(c \cdot \partial z)\chi(z)e^{(\bar{z} \cdot \alpha - z \cdot \bar{\alpha})} d^{2n}z = \sum_{ij} \bar{c}_i c_j \frac{1}{\pi^{2n}} \int (\partial_{\bar{z}_i} \partial_{z_j} \chi(z)) e^{(\bar{z} \cdot \alpha - z \cdot \bar{\alpha})} d^{2n}z$$

$$= -\sum_{ij} \bar{c}_i c_j \frac{1}{\pi^{2n}} \int (\partial_{\bar{z}_j} \chi(z)) \alpha_i e^{(\bar{z} \cdot \alpha - z \cdot \bar{\alpha})} d^{2n}z$$

$$= -\sum_{ij} \bar{c}_i c_j \alpha_i \bar{\alpha}_j \frac{1}{\pi^{2n}} \int \chi(z)e^{(\bar{z} \cdot \alpha - z \cdot \bar{\alpha})} d^{2n}z$$

$$= -|c \cdot \bar{\alpha}|^2 W(\alpha),$$

where we used an integration by parts. Here, $W(\alpha)$ is the Wigner function of the initial state. The other terms in the integral are:

$$\frac{1}{\pi^{2n}} \int (c \cdot \bar{z})(\bar{c} \cdot z)\chi(z)e^{(\bar{z} \cdot \alpha - z \cdot \bar{\alpha})} d^{2n}z = \sum_{ij} \bar{c}_i c_j \frac{1}{\pi^{2n}} \int \bar{z}_i (\partial_{\bar{z}_j} \chi(z)) e^{(\bar{z} \cdot \alpha - z \cdot \bar{\alpha})} d^{2n}z$$

$$= -\sum_{ij} \bar{c}_i c_j \frac{1}{\pi^{2n}} \left( \int \bar{z}_i \alpha_j \chi(z) e^{(\bar{z} \cdot \alpha - z \cdot \bar{\alpha})} d^{2n}z + \delta_{ij} \int \chi(z) e^{(\bar{z} \cdot \alpha - z \cdot \bar{\alpha})} d^{2n}z \right)$$

$$= -\sum_{ij} \bar{c}_i c_j \left( \frac{1}{\pi^{2n}} \alpha_j \bar{\alpha}_i \int \chi(z) e^{(\bar{z} \cdot \alpha - z \cdot \bar{\alpha})} d^{2n}z + \delta_{ij} W(\alpha) \right)$$

$$= -(c \cdot \bar{\alpha})(c \cdot \partial \alpha) W(\alpha) - W(\alpha),$$

and similarly

$$\frac{1}{\pi^{2n}} \int (c \cdot \bar{z})(\bar{c} \cdot z)\chi(z)e^{(\bar{z} \cdot \alpha - z \cdot \bar{\alpha})} d^{2n}z = -(c \cdot \bar{\alpha})(\bar{c} \cdot \partial \alpha) W(\alpha) - W(\alpha),$$

$$\frac{1}{\pi^{2n}} \int (c \cdot \bar{z})(\bar{c} \cdot z)\chi(z)e^{(\bar{z} \cdot \alpha - z \cdot \bar{\alpha})} d^{2n}z = -(c \cdot \partial \alpha)(\bar{c} \cdot \partial \alpha) W(\alpha).$$

Putting everything together, we obtain the Wigner function of the photon added state:

$$W_+(\alpha) = N_+ \left( -\frac{1}{2} + |c \cdot \bar{\alpha}|^2 - \frac{1}{2}(c \cdot \alpha)(c \cdot \partial \alpha) - \frac{1}{2}(\bar{c} \cdot \alpha)(\bar{c} \cdot \partial \alpha) + \frac{1}{4}(c \cdot \partial \alpha)(\bar{c} \cdot \partial \alpha) \right) W(\alpha)$$

$$= N_+ \left[ c \cdot \left( \frac{\partial \alpha}{2} - \alpha \right) \right] \left[ \bar{c} \cdot \left( \frac{\partial \alpha}{2} - \alpha \right) \right] W(\alpha)$$
Appendix B  Characteristic function of a photon-added/subtracted Gaussian state (proof of Eq. (17))

We will compute the characteristic function of a general multi-mode photon-added/subtracted Gaussian state using Eq. (15). This involves taking derivatives of the Gaussian characteristic function (11)

\[ \chi^G(\xi) = e^{K^G(\xi)}, \quad \text{with} \quad K^G(\xi) = -\frac{1}{2} \xi^T \Omega \Omega^T \xi - i \sqrt{2} (\Omega d)^T \xi \quad \text{and} \quad \Omega = \bigoplus_{j=1}^{n} \Omega_j, \quad \Omega_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \]

with respect to the complex variables \( z_j = \xi_1 + \sqrt{2} i \xi_2 \in \mathbb{C}, \ z_j \). For that purpose, we first recall the expression of \( K^G \) in terms of these variables. We define \( \nu = \frac{1}{\sqrt{2}} (1 \ -i)^T \) so that \( \xi_\ell = \frac{1}{\sqrt{2}} (z_\ell \nu + \bar{z}_\ell \bar{\nu}) \). Since \( \Omega_2 \nu = -i \nu \) and \( \Omega_2 \bar{\nu} = i \bar{\nu} \), we find

\[ \Omega_2 \xi_\ell = i \frac{1}{\sqrt{2}} u^T \Omega_2 (z_\ell \bar{\nu} + \bar{z}_\ell \nu), \quad \text{where} \quad u = \left( \bar{\nu}^T \nu^T \right) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}. \]

Introducing the unitary matrix \( U = \bigoplus_j u \), and defining \( A = UVU^T \), we find that \( A \) is the matrix of the covariances of the creation and annihilation operators:

\[
\begin{pmatrix}
\tilde{A}_{11} & \tilde{A}_{12} & \cdots & \tilde{A}_{1n} \\
\tilde{A}_{21} & \tilde{A}_{22} & \cdots & \tilde{A}_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
\tilde{A}_{n1} & \tilde{A}_{n2} & \cdots & \tilde{A}_{nn}
\end{pmatrix}
\quad \text{with} \quad \tilde{A}_{ij} = 2 \begin{pmatrix} \text{Cov}[a_i, a_j] & \text{Cov}[a_i, a_j^\dagger] \\ \text{Cov}[a_i^\dagger, a_j] & \text{Cov}[a_i^\dagger, a_j^\dagger] \end{pmatrix} = u \tilde{V}^{ij} u^T.
\]

Here \( \tilde{V}^{ij} \) is the two-by-two submatrix of the covariance matrix \( V \) defined by

\[
\tilde{V}^{ij} = \begin{pmatrix} V_{2i-1,2j-1} & V_{2i-1,2j} \\ V_{2i,2j-1} & V_{2i,2j} \end{pmatrix} = \begin{pmatrix} \text{Cov}[\hat{x}_i, \hat{x}_j] & \text{Cov}[\hat{x}_i, \hat{p}_j] \\ \text{Cov}[\hat{p}_i, \hat{x}_j] & \text{Cov}[\hat{p}_i, \hat{p}_j] \end{pmatrix}.
\]

One then finds

\[
\frac{1}{2} \xi^T \Omega \Omega^T \xi = \frac{1}{2} \sum_{kl} \xi_k^T (\Omega_2 \tilde{V}^{kl} \Omega_2) \xi_l
\]

\[
= -\frac{1}{4} \sum_{kl} (z_k \bar{z}_k) \Omega_2^T \tilde{A}^{kl} \Omega_2 \begin{pmatrix} z_l \\ \bar{z}_l \end{pmatrix}
\]

and

\[
i \sqrt{2} (\Omega d)^T \xi = \sum_k d_k^T u^T \Omega_2 \begin{pmatrix} z_k \\ \bar{z}_k \end{pmatrix}.
\]

Using that

\[
d_k^T u^T = d_k^T (\nu \ \bar{\nu}) = (\langle a_k \rangle \ \langle a_k^\dagger \rangle),
\]

this leads to

\[
K^G(Z) = \frac{1}{4} Z^T (\Omega^T A \Omega) Z - \Delta^T \Omega Z,
\]

where

\[
Z = \begin{pmatrix} z_1 \\ \bar{z}_1 \\ \vdots \\ z_n \\ \bar{z}_n \end{pmatrix}, \quad \text{and} \quad \Delta = \begin{pmatrix} \langle a_1 \rangle \\ \langle a_1^\dagger \rangle \\ \vdots \\ \langle a_n \rangle \\ \langle a_n^\dagger \rangle \end{pmatrix} = Ud.
\]
It is now easy to take the derivatives along $z_k$ and $\bar{z}_k$ and we obtain:

$$c \cdot \partial_z \chi^G(z) = (\text{Cov}[a_1^\dagger(c), a_1^\dagger(z) - a(z)] + \langle a_1^\dagger(c) \rangle) \chi^G(z),$$

$$\overline{c} \cdot \partial_{\bar{z}} \chi^G(z) = - \left( \text{Cov}[a_1^\dagger(c), a_1^\dagger(z) - a(z)] + \langle a_1^\dagger(c) \rangle \right) \chi^G(z),$$

$$(\overline{c} \cdot \partial_{\bar{z}})(c \cdot \partial_z) \chi^G(z) = - \left( \text{Cov}[a_1^\dagger(c), a_1^\dagger(z) - a(z)] + \langle a_1^\dagger(c) \rangle \right) \left( \text{Cov}[a_1(c), a_1^\dagger(z) - a(z)] + \langle a_1(c) \rangle \right) \chi^G(z) - \text{Cov}[a_1^\dagger(c), a_1(c)] \chi^G(z).$$

According to Eq. (15), the characteristic function of the photon added/subtracted state is given by

$$\chi_{\pm}(z) = N_{\pm} \left[ \frac{\chi^G(z)}{2} - (\overline{c} \cdot \partial_{\bar{z}})(c \cdot \partial_z) \chi^G(z) + \frac{c \cdot \bar{z}}{2} (\overline{c} \cdot \partial_{\bar{z}}) \bar{z} \chi^G(z) + \frac{\bar{z} \cdot c}{2} (c \cdot \partial_z) \chi^G(z) - \frac{|\bar{z} \cdot c|^2}{4} \chi^G(z) \right].$$

Hence we obtain

$$\chi_{\pm}(z) = N_{\pm} \left( \text{Cov}[a_1^\dagger(c), a_1(c)] \pm \frac{1}{2} - \left( c \cdot \gamma_{\pm} \right) \left( \bar{c} \cdot \delta_{\pm} \right) \right) \chi^G(z),$$

with

$$(\gamma_{\pm})_k = \text{Cov}[a_k^\dagger, a_k(z) - a(z)] \mp \frac{1}{2} \bar{z}_k + \langle a_k^\dagger \rangle \quad \text{and} \quad (\delta_{\pm})_k = - \text{Cov}[a_k, a_k(z) - a(z)] \mp \frac{1}{2} z_k - \langle a_k \rangle.$$

This expression can be further simplified as follows. Note that $\mathcal{U} U^\dagger = \bigoplus_{j=1}^n \sigma_x$ with $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $d^T = (\langle a_1 \rangle \quad \langle a_1^\dagger \rangle \quad \cdots \quad \langle a_n \rangle \quad \langle a_n^\dagger \rangle \quad \overline{U})$. Using $U^T \Omega U = -i \Omega$ and the unitarity of $U$, one then finds

$$(\gamma_1 \quad \delta_1 \quad \cdots \quad \gamma_n \quad \delta_n)^T = \frac{1}{2} \left( \Omega^T A \Omega Z \mp \overline{U} U^\dagger Z \right) - \Omega U d$$

$$= \mathcal{U} \left( \frac{1}{2} \left( \Omega \Omega \mp I \right) U^\dagger Z + i \Omega d \right).$$

Recalling from Eq. (19) that, for all $c \in \mathbb{C}^n$,

$$m_c = \mathcal{U}^T \begin{pmatrix} c_1 \\ 0 \\ \vdots \\ c_n \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} c_1 \\ -i c_1 \\ \vdots \\ c_n \\ -i c_n \end{pmatrix} \in \mathbb{C}^{2n},$$

we have

$$m_c^T U^\dagger (0 \quad \bar{c}_1 \quad \cdots \quad 0 \quad \bar{c}_n) \quad \text{Cov}[a_1^\dagger(c), a_1(c)] = \frac{1}{2} m_c^T V m_c.$$
Appendix C  Wigner function of a photon added/subtracted state (proof of Eq. (20))

To derive the expression in Eq. (20), we proceed similarly. Note first that, using Eq. (8), one readily computes the well-known Wigner function of a Gaussian state with characteristic function $\chi^G$. It reads

$$W^G(\alpha) = \frac{2^n}{\pi^n \sqrt{\det V}} \exp\{ -Y^T A^{-1} Y \}$$

where $Y = (\alpha_1 - \langle a_1 \rangle, \bar{\alpha}_1 - \langle a_1^\dagger \rangle, \ldots, \alpha_n - \langle a_n \rangle, \bar{\alpha}_n - \langle a_n^\dagger \rangle) \in \mathbb{C}^{2n}$. One then readily computes the $\alpha_k$ and $\bar{\alpha}_k$ derivatives of $W^G(\alpha)$. Inserting them in Eq. (16), using the definition of $m_c$ in Eq. (19) and

$$A^{-1} = U V^{-1} U^\dagger,$$

one obtains the Wigner function of the photon-added/subtracted state:

$$W_\pm(\alpha) = N_\pm \left( (c \cdot \mu_\pm)(\bar{c} \cdot \eta_\pm) + M_\pm(V, c) \right) W^G(\alpha)$$

where $M_\pm(V, c) \in \mathbb{R}$ is independent of $\alpha$ and given by

$$M_\pm(V, c) = \mp \frac{1}{2} - \frac{1}{2} (c_1 \ 0 \ \ldots \ c_n \ 0) A^{-1} \begin{pmatrix} 0 \\ \bar{c}_1 \\ \vdots \\ 0 \end{pmatrix} = \mp \frac{1}{2} - \frac{1}{2} \overline{m}_c V^{-1} m_c,$$

and where the vectors $\mu_\pm, \eta_\pm \in \mathbb{C}^n$ are defined by

$$\begin{pmatrix} \mu_1 \\ \eta_1 \\ \mu_2 \\ \eta_2 \\ \vdots \\ \mu_n \\ \eta_n \end{pmatrix}_\pm = U (V^{-1} \pm I) U^\dagger - A^{-1} \begin{pmatrix} (a_1) \\ (\bar{a}_1) \\ (a_2) \\ (\bar{a}_2) \\ \vdots \\ (a_n) \\ (\bar{a}_n) \end{pmatrix} = U (V^{-1} \pm I) r - UV^{-1} d.$$

Here we used the fact that the vector of quadratures $r \in \mathbb{R}^{2n}$ and the vector of displacement $d \in \mathbb{R}^2$ can be written as $r^T = (\alpha_1 \ \bar{\alpha}_1 \ \ldots \ \alpha_n \ \bar{\alpha}_n) U$ and $d^T = (\langle a_1 \rangle \ \langle a_1^\dagger \rangle \ \ldots \ \langle a_n \rangle \ \langle a_n^\dagger \rangle) U$. Using $U^T \Omega U = -i \Omega$, the term $(c \cdot \mu_\pm)(\bar{c} \cdot \eta_\pm)$ can be rewritten as follows:

$$\begin{pmatrix} (c \cdot \mu_\pm)(\bar{c} \cdot \eta_\pm) \\ 0 \end{pmatrix} = U (V^{-1} \pm I) U^\dagger \begin{pmatrix} c_1 \\ \bar{c}_1 \end{pmatrix} = U (V^{-1} \pm I) U^\dagger \begin{pmatrix} (V^{-1} \pm I) r - V^{-1} d \end{pmatrix}$$

$$= \begin{pmatrix} r^T (V^{-1} \pm I) - d^T V^{-1} \end{pmatrix} U^\dagger \begin{pmatrix} c_1 \\ \bar{c}_1 \end{pmatrix}$$

$$= \begin{pmatrix} r^T (V^{-1} \pm I) - d^T V^{-1} \end{pmatrix} m_c \overline{m}_c \begin{pmatrix} (V^{-1} \pm I) r - V^{-1} d \end{pmatrix}.$$
Introducing
\[ \lambda_\pm = (V^{-1} + i) r - V^{-1} d \in \mathbb{R}^{2n}, \]
this yields
\[ W_\pm(r) = N_\pm \left( M_\pm(V, c) + \lambda_\pm^T \mathbf{m}_c^T \lambda_\pm \right) W^G(r) \]
which is Eq. (20).

**Appendix D  QCS of the SqTh+ and SqTh- states**

With the characteristic function (34) and Eq. (30) we find the value of the QCS of the SqTh+ state :
\[ \mathcal{C}_{\text{SqTh}^+}^2(q,r) = \frac{(1 - q)/(1 + q)}{2(1 - q^4) \cosh 2r + 2(1 + q^2)^2 + (q^4 + 10q^2 + 1) \sinh^2 2r} \]
\[ \times [-8q(q^2 - 1) + 3(q^4 - 4q^3 + 10q^2 - 4q + 1) \cosh 2r + 6(q - 1)^2 (1 - q^2) \cosh 2r \]
\[ + (3q^4 + 8q^3 - 26q^2 + 8q + 3) \cosh 2r] \]

One then readily computes
\[ \lim_{r \to +\infty} \mathcal{R}_{\text{SqTh}^+}(q, r) = 2 - 12q \frac{q^2 + 1}{q^4 + 10q^2 + 1}. \]

Similarly, with the characteristic function (36) and Eq. (30) we find the value of the QCS of the SqTh- state :
\[ \mathcal{C}_{\text{SqTh}^-}^2(q,r) = \frac{(1 - q)/(1 + q)}{2 \sqrt{q + 1} (4(q^4 - 1) \cosh(2r) + 3q^4 - 2q^2 + (q^4 + 10q^2 + 1) \cosh(4r) + 3)} \]
\[ \times [12(q + 1)(q - 1)^3 \cosh(4r) + (21q^4 - 4q^3 - 14q^2 - 4q + 21) \cosh(2r) \]
\[ + 3(1 - 4q + 10q^2 - 4q^3 + 4q^4) \cosh(6r) + 4(q + 1)(3q^2 + 2q + 3)(q - 1)] \]

and
\[ \lim_{r \to +\infty} \mathcal{R}_{\text{SqTh}^-}(q, r) = 2 - 12q \frac{q^2 + 1}{q^4 + 10q^2 + 1} = \lim_{r \to +\infty} \mathcal{R}_{\text{SqTh}^+}(q, r). \]

**Appendix E  Wigner negative volume of the SqTh+ state**

The Wigner negative volume [22], denoted by \( N_W(\rho) \) is defined as the absolute value of the integral of the Wigner function over the area where the latter is negative. The Wigner function of the SqTh+ state is computed with (20) and we obtain
\[ W_{\text{SqTh}^+}(x,p) = \frac{4(q - 1)^2}{\pi ((q + 1)^2 \cosh(2r) + 1 - q^2)} \exp \left( \frac{(q - 1) \left( e^{2r} x^2 + e^{-2r} p^2 \right)}{q + 1} \right) \]
\[ \times \left( e^{2r} \frac{1 - q}{1 + q} + 1 \right)^2 x^2 + \left( 1 + \frac{1 - q}{1 + q} e^{-2r} \right)^2 p^2 - \frac{1 - q}{1 + q} \cosh(2r) - 1 \]

We easily see that the Wigner function of a SqTh+ state is negative inside the ellipse
\[ \left( \frac{(1 - q)e^{2r}}{q + 1} + 1 \right)^2 x^2 + \left( \frac{(1 - q)e^{-2r}}{q + 1} + 1 \right)^2 p^2 = 1 - \frac{(q - 1) \cosh(2r)}{q + 1}. \]

The semi-major and semi-minor axes are given by
\[ \kappa_x = \frac{e^{-r} \sqrt{(q + 1)^2 - (q^2 - 1) \cosh(2r)}}{\sqrt{2} \cosh(r - q \sinh r)} \]
\[ \kappa_p = \frac{e^r \sqrt{(q + 1)(-q - 1) \cosh(2r) + q + 1}}{\sqrt{2} (q \sinh r + \cosh r)} \]
and the Wigner function reaches its minimal value at the origin:

\[ W_{SqTh}(0) = \frac{2(q - 1)^2((q - 1) \cosh(2r) - q - 1)}{\pi(q + 1)^2((q + 1) \cosh(2r) - q + 1)} \]

On Fig. 6, we computed \( N_W(\rho_{SqTh}) \), the Wigner negative volume of the SqTh state. We see that it increases when \( q \) decreases (behavior similar to QCS), and \( r \) decreases (behavior opposite to QCS). At large squeezing, the Wigner negative volume of these states saturates to a value that decreases with increasing temperature \( q \). Indeed, when \( r \to \infty, \kappa_x \to 0 \) and \( \kappa_p \to \infty \), and the volume can be approximated as

\[ N_W(\rho_{SqTh})(+\infty, q) \approx \frac{19(1 - q)^3}{64e^{1/4}(q + 1)^3}. \]

This shows, as explained in the main text, that the Wigner negative volume does not capture their underlying nonclassicality coming from the squeezing. This is of course not surprising, since the Gaussian mother states have no negativity at all.

![Figure 6: Level lines of the Wigner negative volume \( N_W(\rho_{SqTh}) \) in function of the temperature \( q \) and the squeezing \( r \).](image)

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