Heat-Kernel Asymptotics of Locally Symmetric Spaces of Rank One and Chern–Simons Invariants

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The asymptotic expansion of the heat kernel associated with Laplace operators is considered for general irreducible rank-one locally symmetric spaces. Invariants of the Chern–Simons theory of irreducible $U(n)$--flat connections on real compact hyperbolic 3-manifolds are derived.

1. Introduction

The semiclassical approximation for the Chern–Simons partition function may be expressed by the asymptotics which leads to a series of $C^\infty$--invariants associated with triplets $(X; F; \xi)$ with $X$ a smooth homology 3--sphere, $F$ a homology class of framings of $X$, and $\xi$ an acyclic conjugacy class of orthogonal representations of the fundamental group $\pi_1(X)$ [1]. In addition the cohomology $H(X; Ad\xi)$ of $X$ with respect to the local system related to $Ad\xi$ vanishes. In dimension three there are two important topological quantum field theories of cohomological type, namely topological $SU(2)$ gauge theory of flat connection and a version of the Seiberg-Witten theory. The twisted $\mathcal{N} = 4$ SUSY $SU(2)$ pure gauge theory (version of the Donaldson-Witten theory) describes the Casson invariant [2] while Seiberg-Witten theory is a 3d twisted version of $\mathcal{N} = 4$ SUSY $U(1)$ gauge theory with matter multiplet [3, 4]. Both theories can be derived from 4d $\mathcal{N} = 2$ SUSY $SU(2)$ gauge theory corresponding via twist to Donaldson-Witten theory. It would be interesting and natural to investigate dual description of the $\mathcal{N} = 2$ theory in low-energy limit. It could provide formulation of invariants of four-manifolds involving elements of the Chern–Simons invariants. In this paper we turn into Chern–Simons invariants related to locally symmetric spaces.

The invariant $W_{CS}(X; k)$ associated with the Chern–Simons functional $CS(A^{(j)})$ has to all orders in $k^{-1} = \hbar/2\pi$ ($k \in \mathbb{Z}$) an asymptotic stationary phase approximation of the form [5]

$$W_{CS}(X; k) = \sum_j W_0^{(j)}(X; k) \times \exp \sqrt{-1}k \left( CS(A^{(j)}) + \sum_{n=2}^{\infty} CS_n(A^{(j)})k^{-n} \right),$$

where $CS_n(A^{(j)})$ are the n-loop quantum corrections of flat connection $A^{(j)}$ coming from the n-loop 1-particle irreducible Feynman diagrams.

The partition function of quadratic functional (one-loop expansion) $W_0(X; k)$ can be written in the form [6, 7]

$$W_0(X; k) = \left( \frac{\hbar}{2} \right)^{\zeta(0,|D|)/2} e^{\left(-\sqrt{-1}\pi/4\right)\eta(0,\mathcal{D})} \times \left[T^{(2)}_{an}(X)\right]^{1/2} \frac{1}{\text{Vol}(X)^{-\dim H^0(\nabla)/2}}.$$  \hspace{1cm} (1.2)

The holomorphic function

$$\eta(s, \mathcal{D}) \overset{\text{def}}{=} \sum_{\lambda \in \text{Spec } \mathcal{D}\backslash\{0\}} \text{sgn}(\lambda)|\lambda|^{-s} = \text{Tr} \left( \mathcal{D} \left( \mathcal{D}^2 \right)^{-(s+1)/2} \right),$$  \hspace{1cm} (1.3)

is well defined for all $\Re s \gg 0$ (in Eq. (1.3) the sum has to be taken over all the spectrum $\lambda$) and extends to a meromorphic function on
Indeed, from the asymptotic behaviour of the heat kernel of the Dirac operator $\mathcal{D}$ at $t = 0$, $\text{Tr} \left( e^{-t\mathcal{D}^2} \right) = \mathcal{O}(t^{1/2})$ [10], it follows that $\eta(s, \mathcal{D})$ admits a meromorphic extension to the whole $s-$plane, with at most simple poles at $s = (\dim X - q)/(\text{ord} \mathcal{D})$ ($q \in \mathbb{Z}_+$) and locally computable residues. It has been established that the point $s = 0$ is not a pole, which makes it possible to define the eta invariant of $\mathcal{D}$ by $\eta(0, \mathcal{D})$. One can attach the eta invariant to any operator of Dirac type on a compact Riemannian manifold of odd dimension. Dirac operators on even dimensional manifolds have symmetric spectra and, therefore, trivial eta invariants. As far as dimensional manifolds have symmetric spectra of odd dimension. Dirac operators on even dimensional manifolds have symmetric spectra and, therefore, trivial eta invariants. As far as the zeta function $\zeta(0, |\mathcal{D}|)$ is concerned in Eq. (1.2), we recall that there exist $\varepsilon, \delta > 0$ such that for $0 < t < \delta$ the heat kernel expansion for self-adjoint Laplace operators $\mathcal{L}_p$ (acting on the space of $p-$forms) is given by

$$\text{Tr} \left( e^{-t\mathcal{L}_p} \right) = \sum_{0 \leq \ell \leq t_0} A_\ell(\mathcal{L}_p) t^{-\ell} + \mathcal{O}(t^{\varepsilon}).$$

We shall calculate the heat kernel coefficients $A_\ell$ for locally symmetric spaces of rank one in the next sections. One can show that the zeta function $\zeta(s, |\mathcal{D}|)$ is well-defined and analytic for $\Re s > 0$ and can be continued to a meromorphic function on $\mathbb{C}$, regular at $s = 0$. Moreover (see Refs. [11, 12]),

$$\zeta(0, |\mathcal{D}|) = \sum_{p=0} (-1)^p \mathcal{A}_p(\mathcal{L}_p) - \dim H^p(R(S)).$$

(1.5)

where $R(S)$ is the resolvent of the quadratic functional $S$ (a chain of linear maps). $\zeta(0, |\mathcal{D}|)$ can be expressed in terms of the dimensions of the cohomology spaces of $\mathcal{D}$. Indeed, for all $p$, $A_0(\mathcal{L}_p) = 0$, because we are dealing with odd-dimensional manifold without boundary. Since $H^p(R(S)_{\mathcal{D}}) = H^{m-p}(\nabla)$ (by virtue of Poincaré duality), $m = (\dim X - 1)/2$, it follows that

$$\zeta(0, |\mathcal{D}|) = \sum_{p=0} (-1)^p \dim H^p(R(S))$$

(1.6)

The Ray-Singer norm $|| \cdot ||_{RS}$ on the determinant line $\det H(X; \xi)$ is defined by [4]

$$|| \cdot ||_{RS} \overset{\text{def}}{=} \prod_{p=0}^{\dim X} \exp \left( -\frac{d}{ds} \zeta(s, \mathcal{L}_p)|_{s=0} \right) (-1)^{p/2}. \quad (1.7)$$

For a closed connected orientable smooth manifold of odd dimension and for Euler structure $\eta \in \text{Eul}(X)$ the Ray-Singer norm of its cohomological torsion $T^{(2)}_{\text{an}}(X; \eta) = T^{(2)}_{\text{an}}(X) \in \det H(X; \xi)$ is equal to the positive square root of the absolute value of the monodromy of $\xi$ along the characteristic class $c(\eta) \in H^1(X)$ [11]: $|| T^{(2)}_{\text{an}}(X) ||_{RS} = |\det c(\eta)|^{1/2}$. In the special case where the flat bundle $\xi$ is acyclic ($H^p(X; \xi) = 0$) we have

$$T^{(2)}_{\text{an}}(X)^2 = |\det c(\eta)|$$

$$\times \prod_{p=0}^{\dim X} \exp \left( -\frac{d}{ds} \zeta(s, \mathcal{L}_p)|_{s=0} \right) (-1)^{p+1}.$$ \quad (1.8)

This note is an extension of previous papers [11, 12, 13, 14, 15, 16, 17, 18]. Our aim is to evaluate the semiclassical partition function, weighted by $\exp[\sqrt{\text{Vol} \text{CS}(A)}]$. We shall do this analysis using the spectral properties of elliptic operators acting on locally symmetric spaces of rank one.

2. Asymptotics of the heat kernel on rank one locally symmetric spaces

In Refs. [13, 20, 21], Miatello studies the case of a closed locally symmetric rank one manifold $X$, using the representation theory of the group of isometries of $X$. We consider the same case, but we use the spectral zeta function of $X$. By our approach we determine the expansion coefficients explicitly, given the results of Ref. [22]. We shall be working with an irreducible rank one symmetric space $M = G/K$ of non-compact type. Thus $G$ will be a connected non-compact simple split rank one Lie group with finite centre and $K \subset G$ will be a maximal compact subgroup [12].
Let $\Gamma \subset G$ be a discrete, co-compact torsion free subgroup. Then $X = X_\Gamma = \Gamma \backslash M$ is a compact Riemannian manifold with fundamental group $\Gamma$, i.e. $X$ is a compact locally symmetric space. Given a finite-dimensional unitary representation $\chi$ of $\Gamma$ there is the corresponding vector bundle $V_\chi \to X$ over $X$ given by $V_\chi = \Gamma \backslash (M \otimes F_\chi)$, where $F_\chi$ (the fibre of $V_\chi$) is the representation space of $\chi$ and where $\Gamma$ acts on $M \otimes F_\chi$ by the rule $\gamma \cdot (m, f) = (\gamma \cdot m, (\gamma \cdot \chi) f)$ for $(\gamma, m, f) \in (\Gamma \otimes M \otimes F_\chi)$. Let $\Sigma_\Gamma$ be the Laplace-Beltrami operator of $X$ acting on smooth sections of $V_\chi$; we obtain $\Sigma_\Gamma$ by projecting the Laplace-Beltrami operator of $M$ (which is $G$-invariant and thus $\Gamma$-invariant) to $X$.

The spectral zeta function $\zeta(s; \Sigma_\Gamma) \equiv \zeta_\Gamma(s; \chi)$ of $X_\Gamma$ of Minakshisundaram-Pleijel type [24] is a holomorphic function on the domain $\Re s > d/2$, where $d$ is the dimension of $M$, and by general principles $\zeta_\Gamma(s; \chi)$ admits a meromorphic continuation to the full complex plane $\mathbb{C}$. However since the manifold $X_\Gamma$ is quite special it is desirable to have the meromorphic continuation of $\zeta_\Gamma(s; \chi)$ in an explicit form, for example in terms of the structure of $G$ and $\Gamma$. Using the Selberg trace formula and the $K$-spherical harmonic analysis of $G$, such a form has been obtained in [22] also see Refs. [23, 20]. To state these results we introduce further notation.

Up to local isomorphism we can represent $M = G/K$ by the following quotients:

$$
M = \begin{bmatrix}
SO_1(n, 1)/SO(n) & (I) \\
SU(n, 1)/U(n) & (II) \\
SP(n, 1)/(SP(n) \otimes SP(1)) & (III) \\
F_{4(-20)}/Spin(9) & (IV)
\end{bmatrix}
$$

(2.1)

where $d = n, 2n, 4n, 16$ and $\rho_0$ is given by $\rho_0 = (n - 1)/2, n, 2n + 1, 11$ respectively in the cases (I) to (IV). For details on these matters the reader may consult [24], and also the Appendix in [23]. The spherical harmonic analysis on $M$ is controlled by Harish-Chandra’s Plancherel density $\mu(r)$, a function on the real numbers $\mathbb{R}$, computed by Miatello [14, 21, 22], and others, in the rank one case we are considering.

2.1. The heat kernel coefficients

The object of interest is the heat kernel $\omega_\Gamma(t; \chi)$ defined for $t > 0$ by

$$
\omega_\Gamma(t; \chi) = \sum_{j=0}^{\infty} n_j(\chi) e^{-\lambda_j(\chi)t},
$$

(2.2)

where $n_j$ is the (finite) multiplicity of the eigenvalue $\lambda_j$. If $h_1$ is the fundamental solution of the heat equation on $M$, then $h_1$ and $\omega_\Gamma(t; \chi)$ are related by the Selberg trace formula (cf. [22])

$$
\omega_\Gamma(t; \chi) = \chi(1) \text{Vol}(\Gamma \backslash G) h_1(1) + \theta_\Gamma(t; \chi),
$$

(2.3)

where we denote by Vol$(\Gamma \backslash G)$ the $G$-invariant volume of $\Gamma \backslash G$ induced by Haar measure on $G$, the theta function $\theta_\Gamma(t; \chi)$ is given by Eq. (4.18) of [22] (for $b = 0$ there) and where

$$
h_1(1) = \frac{1}{4\pi} e^{-\rho_0^2 t} \int_{\mathbb{R}} e^{-r^2} \mu(r) dr.
$$

(2.4)

We shall not need the result (2.2). Our goal is to compute explicitly all coefficients $A_k = A_k(\Gamma; \chi)$ in the asymptotic expansion

$$
\omega_\Gamma(t; \chi) \simeq (4\pi t)^{-d/2} \sum_{k=0}^{\infty} A_k t^k,
$$

(2.5)

$\zeta_\Gamma(s; \chi)$ and $\omega_\Gamma(t; \chi)$ are related by the Mellin transform:

$$
\zeta_\Gamma(s; \chi) = \frac{\mathcal{M}[\omega_\Gamma](s)}{\Gamma(s)} = \frac{1}{\Gamma(s)} \int_0^\infty \omega_\Gamma(t; \chi) t^{s-1} dt, \quad \text{for } \Re s > \frac{d}{2}.
$$

(2.6)

Moreover one knows by abstract generalities (cf. [24, 27] for example) that the coefficients $A_k$ are related to residues and special values of $\zeta_\Gamma(s; \chi)$. We obtain the following main result.

**Theorem 2.1** (Ref. [14]). *The heat kernel $\omega_\Gamma(t; \chi)$ in (2.2) admits an asymptotic expansion (2.5). For all $G$ except $G = SO_1(\ell, 1), SU(q, 1)$ with $\ell$ odd and $q$ even, and for $0 \leq k \leq d/2 - 1$,

$$
A_k(\Gamma, \chi) = (4\pi)^{\frac{d}{2} - 1} \chi(1) \text{Vol}(\Gamma \backslash G) C_G \pi
$$

where $C_G$ is the volume of a fundamental domain for $\Gamma$ in $G$. For $\ell = 1$ (for $\ell$ even, $q$ odd but $q$ even, $\ell$ odd; when $G$ is of rank one, $G$ is necessarily $SO_1(\ell, 1)$ or $SU(q, 1)$; and when $G$ is of rank higher than one, $G = SO_1(\ell, 1)$ or $SU(q, 1)$,

$$
A_k(\Gamma, \chi) = (4\pi)^{\frac{d}{2} - 1} \chi(1) \text{Vol}(\Gamma \backslash G) C_G \pi
$$

The object of interest is the heat kernel $\omega_\Gamma(t; \chi)$ defined for $t > 0$ by

$$
\omega_\Gamma(t; \chi) = \sum_{j=0}^{\infty} n_j(\chi) e^{-\lambda_j(\chi)t},
$$

(2.2)
\[
\sum_{\ell=0}^{k} \frac{(-\rho_0)^{k-\ell}}{(k-\ell)!} \left\lfloor \frac{d}{2} - (\ell + 1) \right\rfloor \! a_{2\ell} \! (2^{\ell + 1}), \quad (2.7)
\]
while for \( n = 0, 1, 2, \ldots \) we have

\[
A_{\frac{d}{2}+n}(\Gamma, \chi) = (-1)^n (4\pi)^{\frac{d}{2}-1} \chi(1) \text{Vol}(\Gamma \backslash G) C_G \pi \times \left[ \sum_{j=0}^{\frac{d}{2}-1} (-1)^{j+1} \rho_0^2(n+1+j) j! a_{2j} \right] + 2 \sum_{j=0}^{\frac{d}{2}-1} (-1)^{j+1} \rho_0^2(n+1+j) j! a_{2j} \right] \times \sum_{j=0}^{\frac{d}{2}-1} (-1)^{j+1} \rho_0^2(n+1+j) j! a_{2j} \right] \times \frac{(2n-\ell)}{(n-\ell)!} \beta_{r+1}(j) a_{2j} \right] . \quad (2.8)
\]

Here \( \beta_r(j) \) \((r \in \mathbb{Z}_+)\) is given by

\[
\beta_r(j) \begin{cases} 
2^{1-2(r+j)} - 1 \quad \text{if } G = SO_0(1, \ell) \text{ with } \ell \text{ even}, \\
\frac{\pi}{2} \quad \text{if } G = SU(q, 1) \text{ with } q \text{ odd}, \\
\end{cases} \quad (2.9)
\]

\( B_r \) is the \( r \)-th Bernoulli number,

\[ a(G) \begin{cases} 
\pi & \text{if } G = SO_0(1, \ell) \text{ with } \ell \text{ even}, \\
\frac{\pi}{2} & \text{if } G = SU(q, 1) \text{ with } q \text{ odd}, \\
\end{cases} \quad (2.9)
\]

and \( a_{2j}, C_G \) are some constants (\( C_G \) depending on \( G \)). For \( G = SO_1(2n+1, 1), \) \( k = 0, 1, 2, \ldots \)

\[
A_k(\Gamma, \chi) = \pi(4\pi)^{n-\frac{d}{2}} \chi(1) \text{Vol}(\Gamma \backslash G) C_G \times \sum_{\ell=0}^{\min(k,n)} (-n^2)^k(-\ell)(n-\ell+\frac{1}{2}) a_{2(n-\ell)}. \quad (2.11)
\]

2.2. The case \( G = SU(q, 1) \)

We considered the Minakshisundaram-Pleijel coefficients \( A_k(X_{\Gamma}) \) for all compact rank one space forms \( X_{\Gamma} \) (up to local isomorphism) with one exception - i.e. the case \( X_{\Gamma} = \Gamma \backslash G/K \) with \( G = SU(q, 1) \). We assume \( G = SU(q, 1) \) where now \( q \geq 2 \) is even. The meromorphic structure of \( \zeta_r(s; \chi) \) in this case differs essentially from the case of odd \( q \) in its non-singular terms - not the singular terms of \( \zeta_r(s; \chi) \) where information on poles is determined \([4]\). We now define \( \beta_r(j) \) by

\[
\beta_r(j) = \frac{(-1)^j 2^{2(r+j)} B_{2(r+j)}}{2(r+j)(r+j)!}. \quad (2.12)
\]

At this point the earlier discussions apply and we may conclude the following.

**Theorem 2.2.** Formulae (2.7) and (2.8) also hold for \( G = SU(q, 1) \) with \( q \geq 2 \) even, where \( d/2 = q = \rho_0 \), provided that in formula (2.8) definition (2.9) for \( \beta_r(j) \) is replaced by definition (2.12).

3. Real compact hyperbolic manifolds

In this section we consider briefly the Freed trace formula which is useful for calculation of the partition function of quadratic functional defined on real hyperbolic space. The heat kernel coefficients for \( p \)-forms on hyperbolic space can be found in the paper of F.L. Williams in this volume.

As before \( X_{\Gamma} = \Gamma \backslash G/K \) is a compact manifold, \( G = SO_1(n, 1) \) \((n \in \mathbb{Z}_+)\) and \( K = SO(n) \). The corresponding symmetric space of non-compact type is the real hyperbolic space \( \mathbb{H}^n \) of sectional curvature \(-1\). Its compact dual space is the unit \( n \)-sphere.

3.1. Fried’s trace formula

Let \( a_0, n_0 \) denote the Lie algebras of \( A, N \) in an Iwasawa decomposition \( G = KAN \). Since the rank of \( G \) is one, \( \dim a_0 = 1 \) by definition, say \( a_0 = \mathbb{R}H_0 \) for a suitable basis vector \( H_0 \). One can normalize the choice of \( H_0 \) by \( \beta(H_0) = 1 \), where \( \beta : a_0 \to \mathbb{R} \) is the positive root which defines \( n_0 \); for more detail see Ref. \([24]\). Since \( \Gamma \) is torsion free, each \( \gamma \in \Gamma - \{1\} \) can be represented uniquely as some power of a primitive element \( \delta : \gamma = \delta^j(\gamma) \) where \( j(\gamma) \geq 1 \) is an integer and \( \delta \) cannot be written as \( \gamma^j \) for \( j \geq 1 \) an integer. Taking \( \gamma \in \Gamma, \gamma \neq 1 \), one can find \( t_{\gamma} > 0 \) and \( m_{\gamma} \in \mathbb{M} \begin{cases} m_{\gamma} \in K | m_{\gamma}a = am_{\gamma}, va \in A \end{cases} \) such that \( \gamma \) is \( G \) conjugate to \( m_{\gamma} \exp(t_{\gamma}H_0) \), namely for some \( g \in G, g \gamma g^{-1} = m_{\gamma} \exp(t_{\gamma}H_0) \). Besides
let $\chi_\sigma(m) = \text{trace}(\sigma(m))$ be the character of $\sigma$, for $\sigma$ a finite-dimensional representation of $\mathfrak{m}$.

For $0 \leq p \leq d - 1$ the Fried trace formula holds \cite{28}:

$$\text{Tr} \left( e^{-tE^{(p)}} \right) = I^{(p)}(t, b^{(p)}) + I^{(p-1)}(t, b^{(p-1)})$$

$$+ H^{(p)}(t, b^{(p)}) + H^{(p-1)}(t, b^{(p-1)}),$$

where

$$I^{(p)}(t, b^{(p)}) \overset{def}{=} \frac{\chi(1) \text{Vol}(\Gamma \backslash G)}{4\pi}$$

$$\times \int_{\mathbb{R}} \mu_{\sigma}(r) e^{-t(r^2 + b^{(p)} + (\rho_0 - p)^2)} dr,$$

$$H^{(p)}(t, b^{(p)}) \overset{def}{=} \frac{1}{\sqrt{4\pi t}} \sum_{\gamma \in C_r - \{1\}} \frac{\chi(\gamma)}{J(\gamma)} \gamma C(\gamma)$$

$$\times \chi_{\sigma_p}(m_\gamma) \exp \left\{ - \left[ b^{(p)} t + (\rho_0 - p)^2 t + \frac{t^2}{4t} \right] \right\},$$

$$\rho_0 = (d - 1)/2, b^{(p)}$$ are some constants, and the function $C(\gamma), \gamma \in \Gamma$, defined on $\Gamma - \{1\}$ by

$$C(\gamma) \overset{def}{=} e^{-\rho_0 t} |\det_{m_\gamma} (\text{Ad}(m_\gamma e^{t\gamma} H_0) - 1)|^{-1}.$$  \hfill (3.3)

For $\text{Ad}$ denoting the adjoint representation of $G$ on its complexified Lie algebra, one can compute $t_\gamma$ as follows \cite{29}.

$$e^{t_\gamma} = \max\{|c| | c = \text{an eigenvalue of } \text{Ad}(\gamma)\}.$$  \hfill (3.5)

Here $C$ is a complete set of representatives in $\Gamma$ of its conjugacy classes; Haar measure on $G$ is suitably normalized. For $p = 0$ (i.e. for smooth functions or smooth vector bundle sections) the measure $\mu(r) = \mu_0(r)$ corresponds to the trivial representation of $\mathfrak{m}$. For $p \geq 1$ there is a measure $\mu_{\sigma_p}(r)$ corresponding to a general irreducible representation $\sigma$ of $\mathfrak{m}$. Let $\sigma_p$ be the standard representation of $\mathfrak{m} = SO(d - 1)$ on $\Lambda^p \mathbb{C}^{d-1}$. If $d = 2n$ is even then $\sigma_p (0 \leq p \leq d - 1)$ is always irreducible; if $d = 2n + 1$ then every $\sigma_p$ is irreducible except for $p = (d - 1)/2 = n$, in which case $\sigma_n$ is the direct sum of two spin-(1/2) representations $\sigma^\pm$ : $\sigma_n = \sigma^+ \oplus \sigma^-$. For $p = n$ the representation $\tau_n$ of $K = SO(2n)$ on $\Lambda^n \mathbb{C}^{2n}$ is not irreducible, $\tau_n = \tau_n^+ \oplus \tau_n^-$ is the direct sum of spin-(1/2) representations.

3.2. The Harish-Chandra Plancherel measure

We should note that the reason for the pair of terms \{I^{(p)}, I^{(p-1)}\}, \{H^{(p)}, H^{(p-1)}\} in the trace formula Eq. (3.1) is that $\tau_p$ satisfies $\tau_p|_{\mathfrak{m}} = \sigma_p \oplus \sigma_{p-1}$. Then we have

$$\mu_{\sigma_p}(r) = C^{(p)}(d) P(r, d) \times \left\{ \begin{array}{ll}
\tanh(\pi r), & d = 2n, \\
1, & d = 2n + 1
\end{array} \right.$$  \hfill (3.6)

$$= C^{(p)}(d) \times \left\{ \begin{array}{ll}
\sum_{\ell=0}^{d/2-1} a_{2\ell}^{(p)}(d) r^{2\ell+1} \tanh(\pi r), & d = 2n + 1, \\
\sum_{\ell=0}^{(d-1)/2} a_{2\ell}^{(p)}(d) r^{2\ell}, & d = 2n
\end{array} \right.$$  \hfill (3.7)

where the $P(r, d)$ are even polynomials (with suitable coefficients $a_{2\ell}^{(p)}(d)$) of degree $d - 1$ for $G \neq SO_1(2n + 1, 1)$, and of degree $d = 2n + 1$ for $G = SO_1(2n + 1, 1)$ \cite{30, 31, 26}.

3.3. Case of the trivial representation

For $p = 0$ we take $I^{(-1)} = H^{(-1)} = 0$. Since $\sigma_0$ is the trivial representation one has $\chi_{\sigma_0}(m_\gamma) = 1$. In this case formula (2.3) reduces exactly to the trace formula for $p = 0$ \cite{29, 32}:

$$\omega_0^{(0)}(t, b^{(0)}) = \frac{\chi(1) \text{Vol}(\Gamma \backslash G)}{4\pi}$$

$$\times \int_{\mathbb{R}} \mu_{\sigma_0}(r) e^{-r^2 + b^{(0)} + \rho_0^2 t^2} dr + H^{(0)}(t, b^{(0)}),$$  \hfill (3.8)

where $\rho_0$ is associated with the positive restricted (real) roots of $G$ (with multiplicity) with respect to a nilpotent factor $N$ of $G$ in an Iwasawa decomposition $G = KAN$. The function $H^{(0)}(t, b^{(0)})$ has the form

$$H^{(0)}(t, b^{(0)}) = \frac{1}{\sqrt{4\pi t}} \sum_{\gamma \in C_r - \{1\}} \chi(\gamma) L_0 j(\gamma)^{-1}$$

$$\times C(\gamma) e^{-[b^{(0)} t + \rho_0^2 t^2 + t^2]/(4t)},$$  \hfill (3.9)
3.4. Case of zero modes

It can be shown [22] that the Mellin transform of \( H^{(0)}(t, 0) \) (\( b^{(0)} = 0 \), i.e. the zero modes case)

\[
\mathcal{S}^{(0)}(s) \overset{\text{def}}{=} \int_0^\infty H^{(0)}(t, 0) t^{s-1} dt,
\]

is a holomorphic function on the domain \( \Re s < 0 \). Then using the result of Refs. [31–33] one can obtain on \( \Re s < 0 \),

\[
\mathcal{S}^{(0)}(s) = \frac{\sin(\pi s)}{\pi} \Gamma(s)
\]

\[
\times \int_0^\infty \psi_t(t + 2\rho_0)(2\rho_0t + t^2)^{-s} dt.
\]

Here \( \psi_t(s; \chi) \equiv d(\log Z_\Gamma(s; \chi))/ds \), and \( Z_\Gamma(s; \chi) \) is a meromorphic suitably normalized Selberg zeta function [33, 34, 35, 36, 31].

4. The index theorem and the contribution to the partition function

For any representation \( \chi : \Gamma \to U(n) \) one can construct a vector bundle \( \mathbb{E}_\chi \) over a certain 4-manifold \( Y \) with boundary \( \partial Y = X \) which is an extension of a flat vector bundle \( \mathbb{E}_\chi \) over \( X \). Let \( \tilde{X} \) be any extension of a flat connection \( A_\chi \) corresponding to \( \chi \). The index theorem of Atiyah–Patodi–Singer for the twisted Dirac operator \( D_{\tilde{X}} \) [37, 38, 39] is given by

\[
\text{Index}\left(D_{\tilde{X}}\right) = \int_Y \text{ch}(\mathbb{E}_\chi) \hat{A}(Y)
\]

\[
- \frac{1}{2} \eta(0, \mathcal{D}_\chi) + h(0, \mathcal{D}_\chi), \quad (4.1)
\]

where ch(\( \mathbb{E}_\chi \)) and \( \hat{A}(Y) \) are the Chern character and \( \hat{A} \)- genus respectively, \( \hat{A} = 1 - p_1(Y)/24, p_1(Y) \) is the 1-st Pontryagin class, \( h(0, \mathcal{D}_\chi) \) is the dimension of the space of harmonic spinors on \( X \) (\( h(0, \mathcal{D}_\chi) = \dimker \mathcal{D}_\chi \) is the multiplicity of the 0-eigenvalue of \( \mathcal{D}_\chi \) acting on \( X \); \( \mathcal{D}_\chi \) is a Dirac operator on \( X \) acting on spinors with coefficients in \( \chi \).

The Chern–Simons invariants of \( X \) can be derived from Eq. (4.1). Indeed we have (see for detail Refs. [13, 14, 17, 18]):

\[
CS(\chi) \equiv \frac{1}{2} \left( \dim \chi \eta(0, \mathcal{D}) - \eta(0, \mathcal{D}_\chi) \right) \mod(\mathbb{Z}/2).
\]

\[
(4.2)
\]

A remarkable formula relating \( \eta(s, \mathcal{D}) \), to the closed geodesics on \( X = \Gamma \backslash \mathbb{H}^3 \) has been derived in [40, 41]. More explicitly the following function can be defined, initially for \( \Re(s^2) \gg 0 \), by the formula

\[
\log Z(s, \mathcal{D}) \overset{\text{def}}{=} \sum_{[\gamma] \in E_1(\Gamma)} (-1)^q \frac{L(\gamma, \mathcal{D})}{|\det(F - P_h(\gamma))|^{1/2}} e^{-st(\gamma)},
\]

where \( E_1(\Gamma) \) is the set of those conjugacy classes \( [\gamma] \) for which \( X_\gamma \) has the property that the Euclidean de Rham factor of \( X_\gamma \) is 1-dimensional (\( X \) is a simply connected cover of \( X \) which is a symmetric space of noncompact type), the number \( q \) is half the dimension of the fibre of the central bundle \( C(TX) \) over \( X_\gamma \), and \( L(\gamma, \mathcal{D}) \) is the Lefschetz number (see Ref. [11]). Furthermore \( \log Z(s, \mathcal{D}) \) has a meromorphic continuation to \( \mathbb{C} \) given by the identity

\[
\log Z(s, \mathcal{D}) = \log \det\left( \frac{\mathcal{D} - \sqrt{-1}s}{\mathcal{D} + \sqrt{-1}s} \right) + \sqrt{-1}\pi \eta(s, \mathcal{D}),
\]

\[
(4.4)
\]

where \( s \in \sqrt{-1}(\text{Spec}(\mathcal{D}) - \{0\}) \), and \( Z(s, \mathcal{D}) \) satisfies the functional equation

\[
Z(s, \mathcal{D})Z(-s, \mathcal{D}) = e^{2\pi i \eta(s, \mathcal{D})}.
\]

\[
(4.5)
\]

Let now \( \chi : \Gamma \to U(F) \) be a unitary representation of \( \Gamma \) on \( F \). The Hermitian vector bundle \( \mathbb{F} = \tilde{X} \times_{\Gamma} F \) over \( X \) inherits a flat connection from the trivial connection on \( \tilde{X} \times F \). We specialize to the case of locally homogeneous Dirac operators \( \mathcal{D} : C^\infty(X, \mathbb{E}) \to C^\infty(X, \mathbb{E}) \) in order to construct a generalized operator \( \mathcal{O}_\chi \), acting on spinors with coefficients in \( \chi \). If \( \mathcal{D} : C^\infty(X, V) \to C^\infty(X, V) \) is a differential operator acting on the sections of the vector bundle \( V \), then \( \mathcal{D} \) extends canonically to a differential operator \( \mathcal{D}_\chi : C^\infty(X, V \otimes F) \to C^\infty(X, V \otimes F) \).
uniquely characterized by the property that \( D_\chi \) is locally isomorphic to \( D \otimes \ldots \otimes D \) (dim \( F \) times) \([11]\).

One can repeat the arguments to construct a twisted zeta function \( \tilde{Z}(s, D_\chi) \). There exists a zeta function \( \tilde{Z}(s, D_\chi) \), meromorphic on \( \mathbb{C} \), given for \( \Re(s^2) > 0 \) by the formula

\[
\log \tilde{Z}(s, D_\chi) \overset{def}{=} \sum_{[\gamma] \in E_1(\Gamma)} (-1)^{g} \text{Tr} \chi(\gamma) \frac{L(\gamma, D)}{|\det(I - P_h(\gamma))|^{1/2} m(\gamma)} \cdot e^{-st(\gamma)},
\]

moreover one has \( \log \tilde{Z}(0, D_\chi) = \pi \sqrt{-1} \eta(0, D_\chi) \).

It follows that

\[ \tilde{Z}(0, D_\chi) = \tilde{Z}(0, D)^{\dim \chi} e^{-2\pi \sqrt{-1} C S(\chi)}, \]

and eventually the Chern–Simons functional takes the form

\[
CS(\chi) \equiv \frac{1}{2\pi \sqrt{-1}} \log \left[ \frac{\tilde{Z}(0, D)^{\dim \chi}}{\tilde{Z}(0, D_\chi)} \right] \pmod{\mathbb{Z}/2}.
\]

The classical factor becomes

\[
\exp \left[ \sqrt{-1} k CS(\chi) \right] = \left[ \frac{\tilde{Z}(0, D)^{\dim \chi}}{\tilde{Z}(0, D_\chi)} \right]^k \times \exp[2\pi \sqrt{-1} \tilde{h}(\pmod{\mathbb{Z}/2})].
\]

**4.1. The Ray-Singer norm**

For odd-dimensional manifold the Ray-Singer norm is a topological invariant: it does not depend on the choice of metric on \( X \) and \( \xi \), used in the construction. But for even-dimensional \( X \) this is not the case \([42]\). For real hyperbolic manifolds of the form \( \mathbb{H}^3 \) the dependence of the \( L^2 \)–analytic torsion \((1.8)\) on zeta functions can be expressed in terms of Selberg functions \( Z(1; \chi) \). In the presence of non-vanishing Betti numbers \( b_i \equiv b_i(X) = \text{rank}_\mathbb{Z} H_i(X; \mathbb{Z}) \) we have \([11, 14]\)

\[
[T^{(2)}_{an} X]^2 = \frac{(b_1 - b_0)! |Z_{b_0}(1; \chi)|^2}{[b_0]!^2 Z_{b_0}^{(b_1 - b_0)}(1; \chi)} \times \exp \left( -\frac{1}{3\pi} \text{Vol}(\Gamma \setminus G) \right).
\]

There is a class of compact sufficiently large hyperbolic manifolds which admit arbitrary large value of \( b_1(X) \). Sufficiently large manifold contains a surface \( \Sigma \) whereas \( \pi_1(\Sigma) \) is finite and \( \pi_1(\Sigma) \subset \pi_1(X) \). In general, hyperbolic manifolds have not been completely classified and therefore a systematic computation is not yet possible. However it is not the case for sufficiently large manifolds \([43]\), which give an essential contribution to the torsion \((4.10)\).

Finally we note that formulae \((4.8), (4.9), (1.2), (1.5) \) and \((4.10)\) give the value of the asymptotics of the Chern–Simons invariant in the one-loop expansion. The invariant involves the \( L^2 \)–analytic torsion on a hyperbolic 3-manifold, which can be expressed by means of the Selberg zeta function \( Z(s; \chi) \) and Shintani function \( \tilde{Z}(0, D_\chi) \), associated with the eta invariant of Atiyah–Patodi–Singer.

**REFERENCES**

1. S. Axelrod and I.M. Singer, J. Diff. Geom. 39 (1994) 173.
2. M. Blau and G. Thompson, Commun. Math. Phys. 152 (1993) 41.
3. N. Seiberg and E. Witten, Nucl. Phys. B 426 (1994) 19.
4. E. Witten, Math. Res. Lett. 1 (1994) 769.
5. L. Rozansky and E. Witten, Selecta Math. 3 (1997) 401.
6. D.H. Adams and S. Sen, "Partition Function of a Quadratic Functional and Semiclassical Approximation for Witten’s 3-Manifold Invariant", hep-th/9503095.
7. D.H. Adams, Phys. Lett. B 417 (1998) 53.
8. J.-M. Bismut and D.S. Freed, Commun. Math. Phys. 107 (1986) 103.
9. D. Ray and I. Singer, Adv. Math. 7 (1971) 145.
10. M. Farber and V. Turaev, "Poincare’-Reidemeister Metric, Euler Structures, and Torsion", math.DG/9803137.
11. A.A. Bytsenko, L. Vanzo and S. Zerbini, Nucl. Phys. B 505 (1997) 641.
12. A.A. Bytsenko and F.L. Williams, J. Math. Phys. 39 (1997) 1075.
13. A.A. Bytsenko, A.E. Gonçalves and F.L. Williams, Mod. Phys. Lett. A 13 (1998) 99.
14. A.A. Bytsenko, A.E. Gonçalves and W. da Cruz, Mod. Phys. Lett. A 13 (1998) 2453.
15. A.A. Bytsenko and F.L. Williams, J. Phys. A 32 (1999) 5773.
16. A.A. Bytsenko, L. Vanzo and S. Zerbini, Phys. Lett. B 459 (1999) 535.
17. A.A. Bytsenko, A.E. Gonçalves and F.L. Williams, Mod. Phys. Lett. A 15 (2000) 1031.
18. A.A. Bytsenko, A.E. Gonçalves and F.L. Williams, “Chern-Simons Invariants of Closed Hyperbolic 3-Manifolds” in “Mathematical Methods in Physics”, Eds. A.A. Bytsenko and F.L. Williams, Proceedings of the 1999 Londrina Winter School, World Sci., Singapore (2000).
19. R. Miatello, “The Minakshisundaram-Pleijel Coefficients for the Vector-Valued Heat Kernel on Compact Locally Symmetric Spaces of Negative Curvature”, PhD Thesis, Rutgers University, (1976) 1-26.
20. R. Miatello, Manuscripta Math. 29 (1979) 249.
21. R. Miatello, Trans. Am. Math. Soc. 260 (1980) 1.
22. F. Williams, Pacific J. of Math. 182 (1998) 137.
23. S. Helgason, “Differential Geometry and Symmetric Spaces”, Pure and Applied Math. Ser. 12, Academic Press (1962).
24. S. Minakshisundaram and A. Pleijel, Canadian J. Math. 1 (1949) 242.
25. B. Randol, Trans. Am. Math. Soc. 201 (1975) 241.
26. F.L. Williams, JMP 38 (1997) 796.
27. A. Voros, Commun. Math. Phys. 110 (1987) 439.
28. D. Fried, Invent. Math. 84 (1986) 523.
29. N. Wallach, Bull. Am. Math. Soc. 82 (1976) 171.
30. E. Elizalde, S.D. Odintsov, A. Romeo, A.A. Bytsenko and S. Zerbini, “Zeta Regularization with Applications”, World Scientific, Singapore (1994).
31. A. A. Bytsenko, G. Cognola, L. Vanzo and S. Zerbini, Phys. Rep. 266 (1996) 1.
32. F. Williams, "Lectures on the Spectrum of $L^2(\Gamma \backslash G)$", Pitman Research Notes in Math. 242, Longman House Pub. (1990).
33. A. Selberg, J. India Math. Soc. 20 (1956) 47.
34. D. Fried, Ann. Sci. Ecole Norm. Sup. 10 (1977) 133.
35. R. Gangolli and G. Warner, Nagoya Math. J. 78 (1980) 1.
36. F. Williams, "Some Zeta Functions Attached to $\Gamma \backslash G/K$", in New Developments in Lie Theory and Their Applications, Edited by J. Tirao and N. Wallach, Birkhäuser Progress in Math. Ser. 105 (1992) 163.
37. M.F. Atiyah, V.K. Patodi and I.M. Singer, Math. Proc. Camb. Phil. Soc. 77 (1975) 43.
38. M.F. Atiyah, V.K. Patodi and I.M. Singer, Math. Proc. Camb. Phil. Soc. 78 (1975) 405.
39. M.F. Atiyah, V.K. Patodi and I.M. Singer, Math. Proc. Camb. Phil. Soc. 79 (1976) 71.
40. J.J. Millson, Ann. Math. 108 (1978) 1.
41. H. Moscovici and R.J. Stanton, Invent. Math. 95 (1989) 629.
42. J.-M. Bismut and W. Zhang, “An Extension of a Theorem by Cheeger and Müller”, Astérisque 205 (1992).
43. W. Haken, Acta Math. 105 (1961) 245.