Explicit Pseudo-Kähler Metrics on Flag Manifolds

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November 29, 2022

Abstract The coadjoint orbits of compact Lie groups each carry a canonical (positive definite) Kähler structure, famously used to realize the group's irreducible representations in holomorphic sections of appropriate line bundles (Borel-Weil theorem). Less studied are the (indefinite) invariant pseudo-Kähler structures they also admit, which can be used to realize the same representations in higher cohomology of the sections (Bott's theorem). Using "eigenflag" embeddings, we give a very explicit description of these metrics in the case of the unitary group. As a byproduct we show that $U_n/(U_{n_1} \times \cdots \times U_{n_k})$ has exactly $k!$ invariant complex structures, a count which seems to have hitherto escaped attention.

Mathematics Subject Classification (2010) 14M15 · 17B08 · 32M10 · 32Q15 · 53C50

Keywords Coadjoint orbit · Unitary group · Pseudo-Kähler manifold · Homogeneous complex manifold · Flag manifold

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1 Introduction

One of A. Borel’s early claims to fame was his discovery of a complex structure on the quotient $G/T$ of a compact Lie group by its maximal torus [B53, §29]:

On the off chance, let us point out a property common to the $G/T$ and certain torsion free homogeneous spaces of $U(n)$ (cf. §31, No. 1), but without knowing whether it is related to the topological question which occupies us;

$G/T$ admits a complex manifold structure invariant under the homeomorphisms from $G$.

His original argument was that $G/T$ coincides with the quotient of the complexified group by what we now call a Borel subgroup,

$$G/T \cong G(C)/B, \tag{1}$$

and the pointer to §31 referred to

$$U_n/(U_{n_1} \times \cdots \times U_{n_k}), \quad \left(\sum n_i = n\right) \tag{2}$$

which he described as the manifold “of flags” whose generating element consists of $k-1$ nested subspaces of $C^n$. Soon after, H. C. Wang showed that the property of admitting (finitely many) invariant complex structures is characteristic of homogeneous spaces of the form

$$G/C(S) \tag{3}$$

with $G$ compact and $C(S)$ the centralizer of a torus $S \subset G$ [W54]; Borel observed that these spaces admit invariant Kähler metrics [B54], and with Hirzebruch, gave the theory its classic exposition [B58, §§12–13].

Notably, this contains a section 13.7 Number of invariant complex structures which actually only gives the count in the two extreme cases where $\dim(S) = 1$ or $\text{rank}(G)$, i.e. (in (2)) $k = 2$ or $n$.

The purpose of this paper is to fill this gap and give as explicit a description as possible of all invariant complex structures in the case of the unitary group. (Extension to other types is a seemingly intricate problem.) An ulterior motive is to get a concrete handle on Bott-Borel-Weil modules for various purposes in representation theory; for example, our results should afford a constructive proof of Bott’s theorem in the spirit of [T14].

Paper organization and terminology

The multi-faceted nature of flag manifolds has led different authors to different choices of a working definition, with different connotations. For instance (1) singles out a complex structure, and all of (1–3) a base point, foreign to
the nested subspaces definition. In this paper we follow [B87] and work with

\[ \text{a coadjoint orbit } X \text{ of the compact Lie group } G (= U_n) \] (4)

(§2). This is base-point free but singles out a symplectic form \( \omega \) (of Kirillov-Kostant-Souriau).\(^1\) In that setting \( X \) turns out to have a preferred complex structure \( I \) and attendant Kähler metric \( \omega(I \cdot, \cdot) \) (§3), and classifying other invariant complex structures \( J \) is tantamount to classifying compatible pseudo-Kähler metrics \( \omega(J \cdot, \cdot) \) (§4). Embeddings into products of Grassmannians then allow us to give for these the very explicit formulas we are after (§5).

We note that Theorems 1 to 3 are well-known and indeed easily generalized to any compact connected \( G \) in (4): see [B82, Exerc. 4.8 and 6.13]. For notational simplicity and uniformity, we resisted the temptation to present them in more generality than Theorems 4 to 6, which we emphatically do not know how to extend beyond type A.

As recently pointed out by G. Nawratil [N17], the idea of nested subspaces and the name flag itself originate with R. de Saussure (in Esperanto! Fig. 1). They were used only sporadically, mainly by associates of projective geometer H. de Vries [W11], [W36, p. 15], [F49, p. 22], [F69, p. 415], until A. Borel revived them in his Thesis.

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\(^1\) Thus the historical order is reversed, in which one finds first complex structures, then metrics (Fubini-Study [S05], Cartan-Ehresmann [C29,E34]), and only finally symplectic forms.
2 The coadjoint orbits

2.1 The unitary group and its complexification

Throughout this paper $G$ will denote the group $U_n = \{ g \in G(\mathbb{C}) : \overline{g} = g^{-1} \}$ of unitary matrices in $G(\mathbb{C}) = GL_n(\mathbb{C})$, and $\overline{m}$ will always mean the adjoint (a.k.a. complex conjugate transpose) of any row, column or matrix $m$. The Lie algebra $g(\mathbb{C}) = gl_n(\mathbb{C})$ splits as the sum $g \oplus ig$ of the skew-adjoint and the self-adjoint:

$$
\begin{align*}
\mathfrak{g} &= u_n = \{ Z \in g(\mathbb{C}) : \overline{Z} + Z = 0 \}, \\
ig &= iu_n = \{ Z \in g(\mathbb{C}) : Z = \overline{Z} \}
\end{align*}
$$

where $i = \sqrt{-1}$.

2.2 The trace form and duality

We write $\langle \cdot, \cdot \rangle$ for the symmetric, complex bilinear form $g(\mathbb{C}) \times g(\mathbb{C}) \to \mathbb{C}$ defined by

$$
\langle A, B \rangle = -\operatorname{Trace}(AB).
$$

This form is $G(\mathbb{C})$-invariant, i.e. it satisfies $\langle \operatorname{Ad}_g A, \operatorname{Ad}_g B \rangle = \langle A, B \rangle$ and infinitesimally

$$
\langle \text{ad}_Z A, B \rangle + \langle A, \text{ad}_Z B \rangle = 0
$$

where $\operatorname{Ad}_g A = gA\overline{g}^{-1}$ and $\text{ad}_Z A = [Z, A]$. These formulas hold for all $g \in G(\mathbb{C})$ and $A, B, Z \in g(\mathbb{C})$, and we have

**Proposition 1** The restriction $\langle \cdot, \cdot \rangle_{\mathfrak{g} \times \mathfrak{g}}$ is real-valued, real bilinear, $G$-invariant and positive definite. This allows us to identify $g^*$ with $ig$ (and hence $g$ with $ig^*$) so that duality and the coadjoint action read, for $(g, x, Z) \in G \times g^* \times g$,

$$
\langle x, Z \rangle := \langle ix, Z \rangle, \quad g(x) = gzg^{-1}, \quad Z(x) = [Z, x].
$$

(The restriction $\langle \cdot, \cdot \rangle_{ig \times ig}$ has the same properties, except it is negative definite.) \(\square\)

2.3 The orbits

A coadjoint orbit is an orbit $X$ of the action (8) of $G$ on $g^* = ig$, or in other words, a conjugacy class of self-adjoint matrices. Since such matrices have real eigenvalues and an orthonormal basis of eigenvectors, we have
Proposition 2 Each orbit meets, exactly once, the dominant Weyl chamber

\[ D = \left\{ \lambda = \begin{pmatrix} \lambda_{s_1} \\ \vdots \\ \lambda_{s_k} \end{pmatrix} \in g^* : \lambda_{s_1} > \lambda_{s_2} > \cdots > \lambda_{s_k} \right\} \]  

consisting of nonincreasing real diagonal matrices. \hfill \square

Here \( \lambda_{s_i} \) denotes the scalar matrix \( \lambda_{s_i} I \) of a certain size \( |s_i| \), i.e. we are lumping equal eigenvalues together; while the map \( i \mapsto \lambda_i \) is nominally \( \{1, \ldots, n\} \to \mathbb{R} \), it is constant on the members of a partition

\[ S = \{s_1, \ldots, s_k\} \]  

of \( \{1, \ldots, n\} \) into consecutive segments whose cardinalities are the \( |s_i| \); hence it induces a map \( S \to \mathbb{R} \) which we write again \( s \mapsto \lambda_s \).

Example For \( \lambda \) as in (36) below, \( S = \{\{1\}, \{2, 3\}, \{4\}\} \).

2.4 The stabilizer and its center

Proposition 3 Under the coadjoint action (8), the stabilizer \( G_\lambda \) of \( \lambda \) in (9) equals

\[ H = \begin{pmatrix} U_{|s_1|} \\ \vdots \\ U_{|s_k|} \end{pmatrix} \cong \prod_{s_i} U_{|s_i|}. \]  

This subgroup is also the centralizer of its center \( S \cong U_{1} \times \cdots \times U_{1} \) (\( k \) factors). When we move to another point \( x = g(\lambda) \) in the coadjoint orbit \( \overline{X} = G(\lambda) \), the stabilizer and its center become \( G_x = gHg^{-1} \) and \( gSg^{-1} \). \hfill \square

We note that \( S \subset T \subset H \), where \( T \) is the maximal torus of all diagonal matrices in \( G \), and equality holds when all \( |s_i| = 1 \) (nondegenerate eigenvalues). Again the trace form (6) allows us to identify \( (s^*, t^*, h^*) \) with \( (i\bar{s}, i\bar{t}, i\bar{h}) \); under this identification, the projections

\[ h^* \rightarrow t^* \xrightarrow{\text{avg}} s^* \]  

(12)
consist in taking the diagonal part, resp. the block average
\[
\text{avg} \begin{pmatrix} \mu_{s_1} & \cdots & \mu_{s_k} \\ \vdots & \ddots & \vdots \\ \mu_{s_k} & \cdots & \mu_{s_1} \end{pmatrix} = \begin{pmatrix} \frac{\text{Trace}(\mu_{s_1})}{|s_1|} & \cdots & \frac{\text{Trace}(\mu_{s_k})}{|s_k|} \end{pmatrix}.
\] (13)

2.5 The tangent space \( T_xX \) and its complexification

Under the identifications of Proposition 1, the last formula in (8) says that the tangent space \( T_xX = \mathfrak{g}(x) \) to \( X \) at \( x \) is the image of the map
\[
\text{ad}_{ix} = [ix, \cdot] : \mathfrak{i}g \to \mathfrak{i}g.
\] (14)

As this image sits in the real part of \( \mathfrak{g}(\mathbb{C}) = \mathfrak{g}^* \oplus \mathfrak{i}g^* \), we can complexify it “in place” as
\[
T_xX \oplus iT_xX = [ix, i\mathfrak{g} \oplus \mathfrak{g}] \subset \mathfrak{g}^* \oplus \mathfrak{i}g^*.
\] (15)

And as (14) is skew-adjoint (see (7)), its image is the orthogonal of its kernel \( i\mathfrak{g}_x \) relative to \( \langle \cdot, \cdot \rangle_{\mathfrak{g} \times \mathfrak{i}g} \), i.e. we have

**Proposition 4**
\[
T_xX = \mathfrak{i}g_x^\perp \quad \text{and in particular} \quad T_x\lambda = \mathfrak{i}h_x^\perp. \quad \square
\] (16)

**Remark 1** When \( G \) is \( U_2 \) (or \( SU_2 \), or \( SO_3 \)), coadjoint orbits are just 2-spheres. Then (16) is the statement that the tangent space at a point is the orthogonal to the axis of rotations around that point (Fig. 2). Counterclockwise rotation by 90° provides one of the complex structures we are about to describe.
3 The canonical complex structure

Let $X = G(\lambda)$ be the coadjoint orbit with dominant element $\lambda$, as in (9). The restriction of $\text{ad}_{x}$ (14) to its tangent space (16) has kernel $i g_x \cap i g_x^\perp = \{0\}$, hence is a (still skew-adjoint) linear bijection we shall denote

$$A_x : T_x X \to T_x X.$$ (17)

Recall that $T_x X$ carries the Kirillov-Kostant-Souriau (KKS) 2-form $\omega$, defined by $\omega(Z(x), Z'(x)) = \langle Z(x), Z'(x) \rangle = \langle x, [Z', Z] \rangle$.

**Theorem 1** The KKS 2-form of $X$ is given by

$$\omega(\delta x, \delta' x) = \langle \delta x, A_x^{-1} \delta' x \rangle.$$ (18)

Moreover the formulas

$$I_x = |A_x|^{-1} A_x, \quad g(\delta x, \delta' x) = -\langle \delta x, |A_x|^{-1} \delta' x \rangle,$$ (19)

where $|A_x| = \sqrt{-A_x^2}$, make $\omega$ part of a $G$-invariant Kähler structure $(I, g, \omega)$:

(a) $I$ is an (integrable) complex structure,
(b) $g$ is a positive definite metric,
(c) we have $\omega(\cdot, \cdot) = g(\cdot, I \cdot)$ and $g(\cdot, \cdot) = \omega(I \cdot, \cdot)$.

**Proof** Fix $\delta x, \delta' x \in T_x X$ and put $iZ = A_x^{-1} \delta' x \in i g_x$. Then (17, 14, 8) give

$$\delta' x = A_x A_x^{-1} \delta' x = [ix, iZ] = Z(x),$$ (20)

whence the definition of the KKS 2-form and (8) give us (18):

$$\omega(\delta x, \delta' x) = \langle \delta x, Z \rangle = \langle \delta x, iZ \rangle = \langle \delta x, A_x^{-1} \delta' x \rangle.$$ (21)

Next we note that $|A_x|$ and $I_x$ are the (commuting) positive definite and unitary part of the polar decomposition of $A_x$. So they depend smoothly on $A_x$ [S70, 6.70] and $I_x$, being again skew-adjoint, is an almost complex structure: $I_x^2 = -|A_x|^2 I_x = -I$. Now (c) is clear by plugging $A_x = |A_x| I_x$ into (18, 19), and so is (b) since $\langle \cdot, \cdot \rangle_{|A_x|^2 I_x}$ is negative definite. There remains to see (a). For a $G$-invariant $I$, such as ours is by construction, this is equivalent to either of

(22) sections of the bundle of $+i$-eigenspaces of $I$ in $TX \oplus iTX$ are closed under Lie bracket (Frobenius-Newlander-Nirenberg [N57]);
(23) at $x = \lambda$, the preimage of the $+i$-eigenspace of $I_\lambda$ under the infinitesimal action $g(\mathbb{C}) \to T_x X \oplus iT_x X$ (15) is a Lie subalgebra (Frölicher [F55, §20]).

We prove (23). First observe that if $u$ and $v$ are eigenvectors of $x \in X$ for eigenvalues $\lambda_r$ and $\lambda_s$, then the matrix $uv$ is an eigenvector of $\text{ad}_x$ for eigenvalue $\lambda_r - \lambda_s$:

$$[x, uv] = xu - ux = (\lambda_r - \lambda_s)uv.$$  \hspace{1cm} (24)

It follows that $\text{ad}_{ix}$ is diagonalizable with spectrum $\Delta = \{i(\lambda_r - \lambda_s) : r, s \in S\}$, and so is $A_x$ with spectrum $\Delta \setminus \{0\}$. And indeed $A_x$ explicitly is “diagonal” with eigenvectors the elementary matrices $E_{ij} = e_i e_j$: in more detail, writing tangent vectors $V \in T_{x} X = iT_{x} X$ (16) as self-adjoint matrices with blocks $V_{r|s}$ in the shape (11), formula (24) gives

$$A_x \begin{pmatrix} V_{p|q} & V_{p|r} \\ V_{q|p} & V_{q|r} \\ V_{r|p} & V_{r|q} \end{pmatrix} = i \begin{pmatrix} (\lambda_p - \lambda_q) V_{p|q} & (\lambda_p - \lambda_r) V_{p|r} \\ (\lambda_q - \lambda_p) V_{q|p} & (\lambda_q - \lambda_r) V_{q|r} \\ (\lambda_r - \lambda_p) V_{r|p} & (\lambda_r - \lambda_q) V_{r|q} \end{pmatrix},$$  \hspace{1cm} (25)

(the general pattern should be clear, though we only write out the case where the partition (10) is into 3 segments $p$, $q$, $r$). Hence we obtain by definition of $|A_\lambda|$ and $I_\lambda$ that the latter is the same as (25) with each $i(\lambda_a - \lambda_b)$ divided by its modulus, i.e.

$$I_\lambda \begin{pmatrix} V_{p|q} & V_{p|r} \\ V_{q|p} & V_{q|r} \\ V_{r|p} & V_{r|q} \end{pmatrix} = \begin{pmatrix} i V_{p|q} & i V_{p|r} \\ -i V_{q|p} & i V_{q|r} \\ -i V_{r|p} & -i V_{r|q} \end{pmatrix}.$$  \hspace{1cm} (26)

Thus we see that the $+i$-eigenvectors of $I_\lambda$, and likewise their preimages under (14) or $\text{ad}_x$, are the block upper triangular matrices — hence a Lie subalgebra in $g(\mathbb{C})$. \hfill $\Box$

**Remark 2** We could have shortened the proof by using the fact that, given (b) and (c), (a) is equivalent to $d\omega = 0$. But this is a “delicate” fact [B87, 2.29], whereas (26) is both easy and useful for the sequel. We note also that Theorem 6 will independently reprove (a) from knowing it on Grassmannians (28).

**Remark 3** Using the diagonalizability (24) and Lagrange interpolation [H71, §6.7] one can give an explicit formula for $I_x$ at any point, viz.

$$I_x = \sum_{\delta \in \Delta \setminus \{0\}} i \text{sign}(\delta) E_\delta \quad \text{with} \quad E_\delta = \prod_{\varepsilon \in \Delta \setminus \{0, \delta\}} \frac{\text{ad}_x - \varepsilon}{(\delta - \varepsilon)},$$  \hspace{1cm} (27)
which confirms e.g. the G-invariance and smoothness (indeed algebraicity) of \( I \). Unfortunately this formula seems rather less enlightening than (19).

**Remark 4** The idea of using the polar decomposition to produce ("tamed") almost complex structures occurs in a general context in [W77, p. 8]; its application to obtain this one seems new. Other, less direct descriptions of \( I \) are found in [S54, §2], [B54, §4], [B58, 14.6], [G82, p. 522], [B87, 8.34], [V87, 5.8].

### 3.1 The case of Grassmannians

Let \( \text{Gr}_m \) be the Grassmannian of complex \( m \)-planes in \( \mathbb{C}^n \), each identified with the self-adjoint projector \( x \) upon it, i.e.

\[
\text{Gr}_m = \{ x \in \mathfrak{g}^* : x^2 = x, \text{Trace}(x) = m \} = G \left( \begin{array}{cc} 1_m & 0 \\ 0 & 0_{n-m} \end{array} \right). \tag{28}
\]

Its dominant element \( \sigma_m \) is the highest weight of the fundamental \( G \)-module \( \wedge^m \mathbb{C}^n \).

**Proposition 5** In this case we have \( |A_x| = \frac{1}{2} \) so that the canonical structure of \( \text{Gr}_m \) is simply

\[
\begin{align*}
I \delta x & = [ix, \delta x] \tag{29a} \\
g(\delta x, \delta' x) & = \text{Trace}(\delta x \delta' x) \tag{29b} \\
\omega(\delta x, \delta' x) & = \text{Trace}(\delta x \mathbf{1} \delta' x). \tag{29c}
\end{align*}
\]

**Proof** Deriving and reusing the relations \( x = x^2 = x^3 \) gives \( \delta x = \delta z x + x. \delta x = \delta z x + x. \delta x + x. \delta x \). This implies \( x. \delta x = 0 \) and \( -A_2^2 \delta x = [x, [x, \delta z x]] = x. \delta z - 2x. \delta x x + \delta z x = \delta z x \). So \( -A_2^2 \) and hence its square root are the identity. \( \square \)

**Remark 5** The Hermitian metric \( g + i\omega \) in (29) can be seen as Kähler reduction of the flat metric \( (v, v') := 2 \text{Trace}(\overline{v} v') \) on \( \mathbb{C}^n \times \mathbb{C}^m \cong \text{Hom}(\mathbb{C}^n, \mathbb{C}^m) \). Indeed \( U_m \) acts there by \( a(v) = va^{-1} \), preserving \( \Omega = \text{Im}(,\cdot) \) with moment map \( \psi(v) = -\overline{v} v \), and (29) obtains on passing to the quotient \( \text{Gr}_m = \psi^{-1}(\mathcal{J})/U_m \) [G73, §V.5], [T06, p. 240]. E.g. for \( m = 1 \) one recovers the **Fubini-Study** metric.
metric on projective space, i.e. [S05, §5]

$$2 \left[ \frac{(\delta v, \delta' v)}{\|v\|^2} - \frac{(\delta v, v)(v, \delta' v)}{\|v\|^4} \right] \quad \text{on} \quad \text{Gr}_1 = \left\{ x = \frac{v(v, \cdot)}{\|v\|^2} : v \in \mathbb{C}^n \setminus \{0\} \right\}.$$  

(30)

Formulas (29) are emblematic of the explicitness we’d like to have in general.

4 The invariant complex structures classified: $k!$ parabolic subalgebras

In this section we review the classification of complex structures which results from the principle: a $G$-invariant structure $J$ on $X = G(\lambda) = G/H$ amounts to an $H$-invariant $J_\lambda \in \text{End}(T_\lambda X)$, squaring to $-1$. The results are well-known except perhaps Theorem 4.

4.1 The decomposition of the isotropy representation

Let $g_{r|s}(\mathbb{C})$ denote, for segments $r \neq s$ in the partition $S$ (10), the matrices (25) whose blocks all vanish except perhaps $V_{r|s}$, i.e.

$$g_{r|s}(\mathbb{C}) = \left\{ Z \in g(\mathbb{C}) : Z_{ij} = 0 \text{ for } (i, j) \notin r \times s \right\},$$  

(31)

and $X_{r|s}$ (resp. $iX_{r|s}$) the intersection of $g$ (resp. $ig$) with $g_{r|s}(\mathbb{C}) \oplus g_{s|r}(\mathbb{C})$.

**Theorem 2** The isotypic decomposition of the isotropy representation of $H = G_\lambda$ in the complexified tangent space (15) at $\lambda$ into inequivalent irreducibles is

$$T_\lambda X \oplus iT_\lambda X = \bigoplus_{r \neq s \text{ in } S} g_{r|s}(\mathbb{C}).$$  

(32)

Consequently,

(a) Every $G$-invariant almost complex structure $J$ on $X = G(\lambda)$ is obtained by flipping the sign of 1 (and hence $g$) on some summands in $T_\lambda X = \bigoplus_{r < s} iX_{r|s}$.

(b) As $g$ coincides with $-1_{|r|\cdot|s|} \langle \cdot, \cdot \rangle$ on $iX_{r|s}$, each such flip affects its signature by turning a block of $|r|\cdot|s|$ pluses into minuses.

(c) If $S$ has $k$ segments, then $X$ admits $2^{k(k-1)/2}$ different $G$-invariant almost complex structures.

**Proof** Using the notation of (11) and (25), one checks without trouble that the isotropy action of $h = \text{diag}(u_1, \ldots, u_k) \in H$ takes block $V_{r|s}$ of $V \in T_\lambda X$ to

$$h(V_{r|s}) = u_r V_{r|s} u_s^{-1}.$$  

(33)
So the \( g_{r|s}(C) \) are \( H \)-invariant and the representation on each factors through the natural representation of \( U_{|r|} \times U_{|s|} \) on \( \text{Hom}(C^{[s]}, C^{[r]}) \cong C^{[r]} \otimes C^{[s]} \). As these are irreducible and different for different pairs \((r, s)\), we obtain (32). Now \( J_\lambda \) is determined by its \( \pm i \)-eigenspaces

\[
T^\pm_\lambda X = \text{Im}(J_\lambda \pm i) = \text{Ker}(J_\lambda \mp i)
\]  

(34)

which are (complex conjugate) \( H \)-invariant subspaces of (32), hence are each the sum of some \( g_{r|s}(C) \) [B12, Prop. 4.4d] — one per pair \((g_{r|s}(C), g_{s|r}(C))\). So they can only differ from those of \( I_\lambda \) (26) by the indicated sign flips, and we obtain (a, b, c). \( \square \)

4.2 The invariant complex structures

There remains to characterize which of the almost complex structures of Theorem 2 are integrable.

**Theorem 3** We have

\[
\left[ g_{p|q}(C), g_{r|s}(C) \right] = \begin{cases} 
0 & \text{if } s \neq p; q \neq r \\
g_{p|q}(C) & \text{if } s \neq p; q = r \\
g_{r|q}(C) & \text{if } s = p; q \neq r \\
(g_{p|p}(C) + g_{q|q}(C)) \cap \mathfrak{sl}_n(C) & \text{if } s = p; q = r.
\end{cases}
\]  

(35)

Consequently,

(a) An almost complex structure \( J \) obtained as in Theorem 2a is integrable iff it respects the *Chasles rule*: if the sign is flipped on \( iX_{r|s} \) and \( iX_{s|t} \) \((r < s < t)\), then it is also flipped on \( iX_{r|t} \).

(b) Such is the case iff the preimage of \( T^+_\lambda X \) (see (34)) under the infinitesimal action (23) is a *parabolic subalgebra* \( p \) of \( g(C) \), containing \( h(C) \).

**Proof** Relations (35) follow from noting that \( g_{r|s}(C) \) is the span of elementary matrices \( E_{ij} = e_i e_j \) for \((i, j) \in r \times s\), and computing \([e_i e_j, e_k e_l]\). Next (a) translates condition (23) that the preimage in (b) be a subalgebra; and (b) translates, via [B75, Déf. VIII.3.2], the observation made after (34) that each \( E_{ij} \) not in \( h(C) \) is in either \( T^+_\lambda X \) or \( T^-\lambda X \). \( \square \)

**Remark 6** Versions of Theorems 2 and/or 3 valid for any compact \( G \) can be found in [K10, A03, A98, A97, V90, A86, B82, S69, B58]. (The latter didn’t have the benefit of the *parabolic* terminology introduced in [G60], but instead called *roots of* \( J \) the roots \( \alpha_{ij} = E_{ii} - E_{jj} \in t^* \) whose root space \( CE_{ij} \) is in \( T^+_\lambda X \).)
4.3 Parabolic subalgebras with a given Levi component

Theorem 3b reduces the classification of invariant complex structures $J$ on $X$ to describing the set $\mathcal{P}(h)$ of parabolic subalgebras $p$ of $\mathfrak{g}(\mathbb{C})$ whose Levi component is $h(\mathbb{C})$ (see (11); this set is discussed in e.g. [A81, p. 8], [D11, §5]). We claim:

**Theorem 4** $\mathcal{P}(h)$ is in natural bijection with the symmetric group $\mathfrak{S}_k$.

**Proof** We describe the construction of the bijection in general and illustrate it on the case where $\lambda$ in (9) has $k = 3$ eigenvalues with multiplicities 1, 2, 1, say $\lambda_1 > \lambda_{2,3} > \lambda_4$:

$$
\lambda = \begin{pmatrix}
\lambda_1 & & \\
\lambda_{2,3} & & \\
& \lambda_4 & \\
& & \lambda_1
\end{pmatrix}.
$$

(36)

Let a permutation $\pi \in \mathfrak{S}_k$ be given. Regard it as acting on the $k$ letters $\lambda_{s_1}, \ldots, \lambda_{s_k}$ and rearrange the blocks of (9) accordingly, obtaining here e.g.

$$
\lambda' = \begin{pmatrix}
\lambda_{2,3} & & \\
& \lambda_4 & \\
& & \lambda_1
\end{pmatrix}.
$$

(37)

Next, form the $n \times n$ matrix $\Xi$ whose columns are the standard basis vectors in the order that indices appear in $\lambda'$: in our case

$$
\Xi = (e_2 \ e_3 \ e_4 \ e_1) = \begin{pmatrix}
1 & & \\
& 1 & \\
& & 1
\end{pmatrix}.
$$

(38)

This is by construction a (“uniform block”) permutation matrix $\Xi \in \mathfrak{S}_n$ such that $\Xi \lambda' \Xi^{-1} = \lambda$ [A08,T61]. Now let $p$ be the $\Xi$-conjugate of block upper triangular matrices of shape (37), i.e. (with both $\cdot$s and $+$s denoting arbitrary entries)

$$
p := \Xi \begin{pmatrix}
\cdot & \cdot & + & + \\
\cdot & + & \cdot & + \\
+ & + & \cdot & + \\
+ & + & + & +
\end{pmatrix} \Xi^{-1} = \begin{pmatrix}
\cdot & \cdot & + & + \\
\cdot & + & \cdot & + \\
+ & + & \cdot & + \\
+ & + & + & +
\end{pmatrix}.
$$

(39)

This is clearly a subalgebra of the form required by Theorem 3, i.e. obtained by sign flips from the block upper triangular decoration of (36) (see (26)).
Conversely, let \( p \in \mathcal{P}(h) \) be given — e.g. the one in (39). It is a parabolic containing \( h(C) \) (11), with half all off-diagonal blocks marked \(+\) after Theorem 2a. Now collapse all blocks to size \( 1 \times 1 \): \( p \) becomes a Borel \( b \subset \mathfrak{gl}_k \) containing the diagonals. By [C57, Cor. 3], \( b \) is conjugate to the upper triangular Borel by a unique permutation matrix \( \pi \in \mathfrak{S}_k \), which is the one we attach to \( p \).

One checks without trouble that the maps \( \pi \mapsto p \) and \( p \mapsto \pi \) thus defined are each other’s inverse. \( \square \)

Remark 7 The cases \( k = 2 \) and \( k = n \) of Theorem 4 are due to Borel and Hirzebruch, who observed that all \( J \)'s are then related by the action of complex conjugation (\( k = 2 \)) or the Weyl group (\( k = n \)) [B58, 13.8], [B82, Exerc. 4.8e]. But in general our bijection does not arise from a geometrical action of \( \mathfrak{S}_k \) on \( X = G/H \). In fact, as stated in [B58, p. 506] and detailed in [N84, p. 44], any diffeomorphism transforming one invariant complex structure into another must come from the natural action, \( a(gH) = a(g)H \), of some \( a \) belonging to the stabilizer of \( H \) in the automorphism group

\[
\text{Aut}(G) = Z_2 \rtimes \text{Int}(G). \tag{40}
\]

Here \( \text{Int}(G) \) is inner automorphisms and \( Z_2 \) is the effect of complex conjugation; see [B82, Exerc. 4.3], [S01, Thm 1.5]. As \( Z_2 \) preserves \( H \), and \( \text{Int}(g) \) preserves \( H \) iff \( g \) is in the normalizer \( N_G(H) \), and \( \text{Int}(h) \) (\( h \in H \)) preserves any \( G \)-invariant \( J \), we see that things boil down to an action of

\[
Z_2 \rtimes (N_G(H)/H). \tag{41}
\]

The Weyl-like quotient \( N_G(H)/H \) is computed in [M11, Cor. 12.11] and isomorphic to the subgroup of those \( \sigma \in \mathfrak{S}_n \) that send each segment of the partition (10) to a same-sized segment, modulo the \( \sigma \) that take each segment to itself. When all segments have different sizes, that is trivial and so (41) is far from able to account for all \( |\mathfrak{S}_k| = k! \) structures.

Remark 8 Extending Theorem 4 to compact groups of other types seems challenging, which may explain its apparent absence from the literature. The role of \( \mathfrak{S}_k \) should presumably be taken over by a putative Weyl “group” of either the quotient systems of [L04, 12.18] or the T-root systems of [A86,A97,A98,A03,K10] (their \( T \) is our \( S \) from (3, 12)). One would also need to generalize the rather mysterious (to us) map \( \pi \mapsto \pi \).

4.4 Example: The adjoint variety

The orbit with dominant element (36) we have used as a running example is the adjoint variety \( U_4/(U_1 \times U_2 \times U_3) \), studied in [B58, 13.9], [B61,K98,L02,
Table 1 The adjoint variety's 6 complex structures

| permutation $\pi \in S_3$ | permutation $\pi \in S_4$ | parabolic subalgebra $p \in P(h)$ | signature of $\omega(J, \cdot)$ (over $\mathbb{C}$) |
|---------------------------|---------------------------|---------------------------|---------------------------|
| $\lambda_1 \lambda_2, \lambda_4$ | $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ | $\begin{pmatrix} + & + & + \\ \cdot & \cdot & + \\ + & + & + \end{pmatrix}$ | $(5, 0)$ |
| $\lambda_4 \lambda_2, \lambda_1$ | $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ | $\begin{pmatrix} + & + & + \\ \cdot & \cdot & + \\ + & + & + \end{pmatrix}$ | $(0, 5)$ |
| $\lambda_1 \lambda_4 \lambda_2$ | $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ | $\begin{pmatrix} + & + & + \\ \cdot & \cdot & + \\ + & + & + \end{pmatrix}$ | $(3, 2)$ |
| $\lambda_4 \lambda_1 \lambda_2$ | $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ | $\begin{pmatrix} + & + & + \\ \cdot & \cdot & + \\ + & + & + \end{pmatrix}$ | $(2, 3)$ |
| $\lambda_2, \lambda_3 \lambda_1$ | $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ | $\begin{pmatrix} + & + & + \\ \cdot & \cdot & + \\ + & + & + \end{pmatrix}$ | $(3, 2)$ |
| $\lambda_2, \lambda_3 \lambda_4$ | $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ | $\begin{pmatrix} + & + & + \\ \cdot & \cdot & + \\ + & + & + \end{pmatrix}$ | $(2, 3)$ |

[72x394]H05]. Table 1 traces the construction of the entire bijection $\pi \mapsto p$ in this case. Note how

- Of all $2^{3-1}/2 = 8$ possible sign flips on the top right matrix, the two not reached are precisely those failing the Chasles rule (Theorem 3a):

$$\begin{pmatrix} + & + & + \\ \cdot & \cdot & + \\ + & + & + \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} + & + & + \\ \cdot & \cdot & + \\ + & + & + \end{pmatrix}.$$ (42)

- Because (36) has same-sized blocks, (41) is a four-group $\mathbb{Z}_2 \times \mathbb{Z}_2$; it has two orbits on $P(h)$: the first two rows of the table, and the other four.
- Signatures can be read off as (number of $+$ above, number of $+$ below) the diagonal; this transparently recovers the algorithm of [Y14, §4].
Remark 9 A dominant $\lambda$ with multiplicities 1, 1, 2 can of course lead also to metrics of signature (4, 1) or (1, 4) on the “same” manifold $U_4/(U_1 \times U_1 \times U_2)$: signature depends not only on $J$ but also on the chosen $\omega$ (or coadjoint orbit).

5 The invariant complex structures realized: $k!$ eigenflag embeddings

Theorem 3 only spells out complex structures by giving the effect of $J$ at the base point $\lambda$. At any other point $x = g(\lambda)$, computation of $J_x = g J g^{-1}$ requires use of some $g \in G$, on whose nonuniqueness the outcome is known not to depend. Our goal below is a more tangible picture where $J_x$ can be explicit in terms of $x$ alone, as in (19, 27, 29a). We freely use the notation introduced in (9–10, 28–29).

5.1 Maps to products of Grassmannians

A first idea is to note that spectral decomposition expresses each $x \in X$ as a linear combination of eigenprojectors, $E_s \in \text{Gr}_{|s|}$, belonging to the (fixed) eigenvalues $\lambda_s$:

$$x = \sum_{s \in S} \lambda_s E_s,$$

where

$$E_s = \prod_{r \in S \setminus \{s\}} \frac{(x - \lambda_r)}{(\lambda_s - \lambda_r)}$$

(Lagrange interpolation [H71, §6.7]). So sending $x$ to $y = (E_s)_{s \in S}$ embeds $X$ equivariantly as a submanifold $Y$ of a product $\prod_{s \in S} \text{Gr}_{|s|}$ of Grassmannians (28), hopefully pulling product structures back to useful ones on $X$. Alas, Theorem 5 below dashes this hope: $Y$ isn’t a complex submanifold of the product, so there is no complex structure to transport back. Fortunately, the same Theorem will also indicate the way out.

To state it, note that the $E_s$ are just a small part of $x$’s spectral measure $A \mapsto E_A$ which maps subsets of $S$ (or alternatively, of the spectrum $\{\lambda_s : s \in S\}$) to projectors

$$E_A = \sum_{s \in A} E_s \in \text{Gr}_{||A||}, \quad ||A|| := \sum_{s \in A} |s|,$$

with the property that $E_A \cap B = E_A E_B$ (so the $E_A$ all commute). Thus, not only the singletons but any subfamily $A \subset 2^S$ gives rise to a $G$-equivariant map, $x \mapsto (E_A)_{A \in A}$, from $X$ to a product of Grassmannians.

**Theorem 5** The image $Y$ of this map is a complex submanifold of $\prod_{A \in A} \text{Gr}_{||A||}$ (for the product complex structure) iff $A$ is totally ordered by inclusion.
Proof First note that as $G$ is transitive on $X$, the map’s equivariance (visible on (43)) ensures that $Y$ is an orbit of a smooth group action, hence as always an (“initial”) submanifold [H12, Prop. 10.1.14].

Assume that $A \subseteq 2^S$ is totally ordered by inclusion. Then a tuple $(E_A)_{A \in A}$ in $\prod_{A \in A} \text{Gr}_{\|A\|}$ is a member $y \in Y$ iff it satisfies

$$E_B E_A = E_A$$

for all pairs $A \subseteq B$ in $A$ (the reverse order follows by taking adjoints); and a tangent vector $\delta y = (\delta E_A)_{A \in A}$ is in $T_y Y$ iff we also have the derived relation

$$\delta E_B. E_A + E_B. \delta E_A = \delta E_A.$$  

Assume (46). Multiplying it on the left by $E_B$ gives $E_B. \delta E_A + E_B. \delta E_A = 0$ and hence

$$I \delta E_B. E_A + E_B. I \delta E_A = [iE_B, \delta E_A]E_A + [iE_B, \delta E_A]$$

$$= -i \delta E_B. E_A + iE_B. E_A - iE_B. \delta E_A. E_A$$

$$= iE_A. \delta E_A - i(\delta E_B. E_A + E_B. \delta E_A). E_A$$

$$= iE_A. \delta E_A - i \delta E_A. E_A$$

$$= [iE_A, \delta E_A]$$

$$= I \delta E_A.$$  

Thus we see that $I \delta y$ also satisfies (46). This confirms that the product complex structure preserves $T_y Y$.

Conversely, assume that $A$ is not totally ordered. So there are $A, B \in A$ such that $A \not\subset B$ and $B \not\subset A$. Pick $r \in A \setminus B$ and $s \in B \setminus A$ and nonzero eigenvectors $u, v \in \mathbb{C}^n$ for eigenvalues $\lambda_r, \lambda_s$ of $x$; thus we have

$$E_A u = u, \quad E_A v = 0, \quad E_B u = 0, \quad E_B v = v.$$  

Now put $Z = u \n - v \n \in \mathfrak{g}$ and consider the image $\delta y \in T_y Y$ of $\delta x := [Z, x] \in T_x X$. By equivariance and (48), its components in $T_{E_A \text{Gr}_{\|A\|}}$ and $T_{E_B \text{Gr}_{\|B\|}}$ are respectively

$$\delta E_A = [Z, E_A] = [u \n - v \n, E_A] = -u \n - v \n,$$

$$\delta E_B = [Z, E_B] = [u \n - v \n, E_B] = u \n + v \n.$$  

They (of course) satisfy the relation $[\delta E_A, E_B] + [E_A, \delta E_B] = 0$ which any tangent vector to $Y$ must, as one sees by deriving $[E_A, E_B] = 0$. On the other
hand, we claim that $I\delta E_A$ and $I\delta E_B$ fail that relation. Indeed (29a) gives

\[ I\delta E_A = [iE_A, \delta E_A] = i(v\bar{u} - u\bar{v}) = iZ, \]
\[ I\delta E_B = [iE_B, \delta E_B] = i(v\bar{u} - u\bar{v}) = iZ, \]

whence (using (49))

\[ [I\delta E_A, E_B] + [E_A, I\delta E_B] = [iZ, E_B - E_A] = i(\delta E_B - \delta E_A) = 2i(u\bar{v} + v\bar{u}) \neq 0. \]

Thus the product complex structure fails to preserve $T_yY$, as claimed. \( \Box \)

5.2 The eigenflag embeddings

Choosing \( A = \{s_{i(1)}, \ldots, s_{i(k)}\} : i = 1, \ldots, k \) in Theorem 5, we obtain our main result which provides

- for \( \pi = 1 \), an independent reconstruction of the Kähler structure (18, 19);
- for other \( \pi \in S_k \), explicit models of \( X \) with every pseudo-Kähler structure:

Theorem 6 Let \( \pi \in S_k \) give rise to complex structure \( J \) and metric \( h = \omega(J \cdot, \cdot) \) (Theorems 3, 4) and write \( A_i = \{s_{i(1)}, \ldots, s_{i(k)}\} \) where \( \{s_1, \ldots, s_k\} \) is the partition (10). Then the coadjoint orbit \( X \) with pseudo-Kähler structure \((J, h, \omega)\) is isomorphic to the orbit \( Y \) of \( (\varpi_{|A_i|})_{i=1}^k \) in \( \prod_{i=1}^k \text{Gr}_{|A_i|} \) endowed with the product complex structure and the metric and 2-form

\[ \sum_{i=1}^k (\lambda_{s_{i(1)}} - \lambda_{s_{i+1(1)}}) g_{|A_i|}, \quad \sum_{i=1}^k (\lambda_{s_{i(1)}} - \lambda_{s_{i+1(1)}}) \omega_{|A_i|}, \] (52)

where \( (\text{Gr}_{m_i}, l_m, g_m, \omega_m) \) is the Grassmannian (28, 29) and we set \( \lambda_{s_{(k+1)}} = 0 \).

The (moment) map from \( Y \) to \( X \) and inverse map from \( X \) to \( Y \) are respectively

\[ (y_{|A_i|})_{i=1}^k \mapsto \sum_{i=1}^k (\lambda_{s_{i(1)}} - \lambda_{s_{i+1(1)}}) y_{|A_i|} \quad \text{and} \quad x \mapsto (E_{A_i})_{i=1}^k, \] (53)

Proof Formula (52) defines on the product \( P = \prod_{i=1}^k \text{Gr}_{|A_i|} \) a 2-form which is clearly symplectic and \( G \)-invariant with moment map given by (53). Its restriction to \( Y \) is \textit{a priori} presymplectic with moment map \( \Phi \) still given by
Equivariance ensures that $\Phi$ maps $Y$ onto a coadjoint orbit, which is $X$ since summation by parts gives $\sum_{i=1}^k (\lambda_{s(i)} - \lambda_{s(i-1)}) \sigma|_{A_i} = \lambda_{s(0)} \sigma|_{A_0} + \sum_{i=2}^k \lambda_{s(i)} (\sigma|_{A_i} - \sigma|_{A_{i-1}}) = \lambda'$ (37).

An easy dimension count, or indeed the explicit inverse in (53), then shows that $\Phi$ is a diffeomorphism $Y \to X$ which is symplectic by [S70, 11.17]. There remains to see that the derivative of $\Phi$ maps (the $+i$-eigenspace of) the product complex structure at $\sigma = (\sigma|_{A_i})_{i=1}^k$ to (the $+i$-eigenspace of) $J$ at the base point $\lambda'$. But this boils down to the observation that linear combination takes the block upper triangular matrices in $T^+_\sigma \text{Gr}_m$ to block upper triangular matrices in $T^+_{\lambda'} X$ (39).

Remark 10 It seems natural to refer to $\omega$ as an eigenflag of the corresponding matrix $x$. Thus we have as many “eigenflag embeddings” of $X$ as there are orderings of its eigenvalues, and each induces a different complex structure. Note that by the observation made before (45), $Y$ is algebraic in $\prod_{i=1}^k \text{Gr}_{|A_i|}$ with equations $y_{\mid A_i} \omega_{|A_i|} = y_{|A_i|}$ ($i = 1, \ldots, k - 2$).

5.3 Example: The adjoint variety (continued)

Table 2 details all embeddings when $X$ is the adjoint variety (§4.4) with $\lambda = \text{diag}(1, 0, 0, -1)$; the singleton $\text{Gr}_4 \{1\}$ could of course be mostly omitted from the notation. Taking the last row as an example, the signature $(2, 3)$ metric is $h(\delta y, \delta' y) = \text{Trace}(\delta y \delta' y) - 2 \text{Trace}(\delta y \delta y) + 1$ (54) and gives $\omega(\cdot, \cdot) = h(\cdot, J \cdot)$ with the product complex structure $J \delta y = \left[ \begin{array}{c} \delta y_2 \\ \delta y_3 \\ \delta y_4 \end{array} \right]$.

Table 2 The adjoint variety’s 6 eigenflag embeddings

| permutation $\pi \in S_3$ | base point $\lambda' \in X \cap t^*$ | manifold $Y$ | moment map: $y \mapsto$ |
|-------------------------|---------------------------------|-------------|-----------------------|
| 1, 2, 3                 | $\left( \begin{array}{cc} 0 & 1 \\ \frac{1}{2} & -1 \end{array} \right)$ | $y = \left( \begin{array}{c} y_1 \\ y_2 \\ y_3 \end{array} \right) \in \text{Gr}_1 \times \text{Gr}_3 \times \text{Gr}_4$ : $y_3 y_1 = y_1$, $y_4 y_2 = y_2$ | $y_0 + y_0 - y_4$ |
| 3, 2, 1                 | $\left( \begin{array}{cc} \frac{1}{2} & 1 \\ 0 & -1 \end{array} \right)$ | | $-y_0 + y_0 + y_4$ |
| 1, 3, 2                 | $\left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right)$ | $y = \left( \begin{array}{c} y_1 \\ y_2 \\ y_3 \end{array} \right) \in \text{Gr}_1 \times \text{Gr}_2 \times \text{Gr}_4$ : $y_2 y_1 = y_1$, $y_4 y_3 = y_3$ | $2y_1 - y_0$ |
| 3, 1, 2                 | $\left( \begin{array}{cc} \frac{1}{2} & 0 \\ 0 & -1 \end{array} \right)$ | | $-2y_1 + y_2$ |
| 2, 1, 3                 | $\left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right)$ | $y = \left( \begin{array}{c} y_1 \\ y_2 \\ y_3 \end{array} \right) \in \text{Gr}_2 \times \text{Gr}_3 \times \text{Gr}_4$ : $y_2 y_3 = y_3$, $y_4 y_2 = y_2$ | $-y_0 + 2y_2 - y_4$ |
| 2, 3, 1                 | $\left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right)$ | | $y_0 - 2y_2 + y_4$ |

Acknowledgements We wish to thank Arnaud Beauville, Ivan Penkov, Jacqueline Rey-Glardon, Loren Spice and Alan Weinstein for very helpful indications.
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