Abstract

Confidence intervals based on the central limit theorem (CLT) are a cornerstone of classical statistics. Despite being only asymptotically valid, they are ubiquitous because they permit statistical inference under very weak assumptions, and can often be applied to problems even when nonasymptotic inference is impossible. This paper introduces time-uniform analogues of such asymptotic confidence intervals. To elaborate, our methods take the form of confidence sequences (CS) — sequences of confidence intervals that are uniformly valid over time. CSs provide valid inference at arbitrary stopping times, incurring no penalties for “peeking” at the data, unlike classical confidence intervals which require the sample size to be fixed in advance. Existing CSs in the literature are nonasymptotic, and hence do not enjoy the aforementioned broad applicability of asymptotic confidence intervals. Our work bridges the gap by giving a definition for “asymptotic CSs”, and deriving a universal asymptotic CS that requires only weak CLT-like assumptions. While the CLT approximates the distribution of a sample average by that of a Gaussian at a fixed sample size, we use strong invariance principles (stemming from the seminal 1970s work of Komlós, Major, and Tusnády) to uniformly approximate the entire sample average process by an implicit Gaussian process. We demonstrate their utility by deriving nonparametric asymptotic CSs for the average treatment effect based on doubly robust estimators in observational studies, for which no nonasymptotic methods can exist even in the fixed-time regime (due to confounding bias). These enable doubly robust causal inference that can be continuously monitored and adaptively stopped.

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1 Introduction

The central limit theorem (CLT) is arguably the most widely used result in applied statistical inference, due to its ability to provide large-sample confidence intervals (CI) and \(p\)-values in a broad range of problems under weak assumptions. Examples include (a) nonparametric estimation of means, such as population proportions, (b) maximum likelihood and other M-estimation problems [50], and (c) modern semiparametric causal inference methodology involving inverse propensity score weighting and doubly robust estimation [35, 48, 20, 5]. Crucially, in some of these problems such as doubly robust estimation in observational studies, nonasymptotic inference is typically not possible, and hence the CLT yields asymptotic CIs for an otherwise unsolvable inference problem.

While the CLT makes efficient statistical inference possible in a broad array of problems, the resulting CIs are only valid at a prespecified sample size \(n\), invalidating any inference that occurs at data-dependent stopping times, for example under continuous monitoring. CIs that retain validity in sequential environments are known as confidence sequences (CS) [7, 32] and can be used to make decisions at arbitrary stopping times (e.g. while adaptively sampling, continuously peeking at the data, etc.). CSs are an inherently nonasymptotic notion, and thus essentially every published CS is nonasymptotic, including the recent state-of-the-art [16, 14, 60].

This paper presents a new notion: an “asymptotic confidence sequence”. For the familiar reader, this might at first seem almost paradoxical, like an oxymoron. Further, it is not obvious how to posit a definition that is simultaneously sensible and tractable, meaning it is possible to develop such asymptotic CSs (whatever it may mean). We believe that we have formulated the “right” definition, because we accompany it with a universality result that parallels the CLT — a universal asymptotic CS that is valid under moment assumptions that only slightly stronger than the CLT (requiring \(2 + \delta\) moments for some \(\delta > 0\)). This enables the construction of asymptotic CSs in a huge number of new situations where the distributional assumptions are weak enough to remain out of the reach of nonasymptotic techniques even in fixed-time settings. The width of this universal asymptotic CS scales with the variance of the data, just like the empirical variance t-statistic used in the CLT — such variance-adaptivity is only achievable for nonasymptotic methods in very specialized settings [60].

Before proceeding, let us first briefly review some notation and key facts about CSs.
1.1 Time-uniform confidence sequences (CSs)

Consider the problem of estimating the population mean $\mu = \mathbb{E}(Y_1)$ from a sequence of iid data $(Y_t)_{t=1}^{\infty} \equiv (Y_1, Y_2, \ldots)$ that are observed sequentially over time. A nonasymptotic $(1 - \alpha)$-CI for $\mu$ is a set $C_n \equiv \hat{C}(Y_1, \ldots, Y_n)$ with the property that

$$\forall n \in \mathbb{N}^+, \quad \mathbb{P}(\mu \in \hat{C}_n) \geq 1 - \alpha, \quad \text{or equivalently,} \quad \forall n \in \mathbb{N}^+, \quad \mathbb{P}(\mu \notin \hat{C}_n) \leq \alpha. \quad (1)$$

The coverage guarantee (1) of a CI is only valid at some prespecified sample size $n$, which must be decided in advance of seeing any data — pecking at the data in order to determine the sample size is a well known form of “$p$-hacking”. However, it is restrictive to fix $n$ beforehand, and even if clever sample size calculations are carried out based on prior knowledge, it is impossible to know a priori whether $n$ will be large enough to detect some signal of interest: after collecting the data, one may regret collecting too little data or collecting much more than would have been required.

CSs provide the flexibility to choose sample sizes data-adaptively while controlling the type-I error rate (see Fig. 1). Formally, a CS is a sequence of CIs $(C_t)_{t=1}^{\infty}$ such that

$$\mathbb{P}(\forall t \in \mathbb{N}^+, \mu \in C_t) \geq 1 - \alpha, \quad \text{or equivalently,} \quad \mathbb{P}(\exists t \in \mathbb{N}^+: \mu \notin \hat{C}_t) \leq \alpha. \quad (2)$$

The statements (1) and (2) look similar but are markedly different from the data analyst’s or experimenter’s perspective. In particular, employing a CS has the following implications:

(a) The CS can be (optionally) updated whenever new data become available;
(b) Experiments can be continuously monitored, adaptively stopped, or continued;
(c) The type-I error is controlled at all stopping times, including data-dependent times.

In fact, CSs may be equivalently defined as CIs that are valid at arbitrary stopping times, i.e.

$$\mathbb{P}(\mu \in \hat{C}_\tau) \geq 1 - \alpha \quad \text{for any stopping time } \tau. \quad (3)$$

A proof of the equivalence between (2) and (3) can be found in Howard et al. [16, Lemma 3].

As mentioned before, while nonparametric CSs have been developed for several problems, they have thus far been nonasymptotic. Nonasymptotic inference (CSs as well as CIs) require strong assumptions on the distribution of the data such as a parametric likelihood [58, 14] or known bounds on the random variables themselves [16, 60], on their moments [56], or on their moment generating functions [16].

These added distributional assumptions make existing CSs to be quite unlike CLT-based CIs which (a) are universal, meaning they take the same form — up to a change in influence functions — and are computed in the same way for most problems, and (b) are often applicable even when no nonasymptotic CI is known, such as in doubly robust inference of causal effects in observational studies. Our work bridges this gap, bringing properties (a) and (b) to the anytime-valid sequential regime by making one simple modification to the usual CIs and requiring a slightly stronger moment assumption ($2 + \delta$ moments, for $\delta > 0$, instead of 2 moments). Just as CLT-based CIs yield approximate inference for a wide variety of problems in fixed-$n$ settings, our paper yields the same for sequential settings.

1.2 Contributions and outline

The primary contributions of this paper are in rigorously defining “asymptotic confidence sequences” (Definition 2.1), and deriving explicit constructions thereof (Theorem 2.2), which are as easy to implement and apply as the CLT. Additionally, we develop a Lindeberg-type asymptotic CS that is able to capture time-varying means under martingale dependence (Section 2.4). In Section 3 we describe in detail how the asymptotic CSs of Section 2.1 enable anytime-valid doubly robust inference.

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1 We use overhead dots $\hat{C}_n$ to denote fixed-time (pointwise) CIs and overhead bars $\bar{C}_t$ to denote time-uniform CSs.
of causal effects in both randomized experiments and observational studies (Section 3). To be clear, we are not focused on deriving new doubly robust estimators; we simply demonstrate how doubly robust causal inference — a problem for which no known CSs exist — can now be tackled in fully sequential environments using the existing state-of-the-art estimators combined with our time-uniform bounds (Theorems 3.1 and 3.2). We illustrate the use of our CSs with a real observational data set by sequentially estimating the effects of fluid intake on 30-day mortality in sepsis patients. In Section 5, we discuss how the aforementioned techniques apply to functional estimation tasks more generally. In sum, this work greatly expands the scope of anytime-valid inference, and does so via technically nontrivial derivations.

# 2 Time-uniform central limit theory

We first define what it means for a sequence of intervals to form an asymptotic confidence sequence (AsympCS). Then, we derive a “universal” AsympCS\(^2\) in the sense that the AsympCS does not depend on any features of the distribution beyond its mean and variance. This universal AsympCS is fundamentally related to Gaussians — much like classical asymptotic confidence intervals based on the CLT — since it stems from Brownian motion approximations and so-called strong invariance principles. Finally, similar to CIs based on martingale CLTs, we derive a Lindeberg-type martingale AsympCS that can track a moving average of conditional means.

## 2.1 Defining asymptotic confidence sequences

As mentioned in Section 1, the literature on CSs has focused on the nonasymptotic regime where strong assumptions must be placed on the observed random variables, such as boundedness, a parametric likelihood, or upper bounds on their MGFs. On the other hand, in batch — i.e. fixed-time — statistical

\(^2\)We use the term universal in the same way that the CLT and the law of large numbers are considered universality results [43], as they describe macroscopic behaviors that are independent of most microscopic details of the system.
analyses, the CLT is routinely applied to obtain approximately valid CIs in large samples, as it requires weak finite moment assumptions and has a simple, universal closed-form expression. Here, we define and present “asymptotic confidence sequences” as time-uniform analogues of asymptotic CIs, making similarly weak moment assumptions and providing a universal closed-form boundary.

The term “asymptotic confidence sequence” may at first seem paradoxical. Indeed, ever since their introduction by Robbins and collaborators [7, 33, 24, 25], CSs have been defined nonasymptotically, satisfying the time-uniform guarantee in equation (2). So how could a bound be both time-uniform and asymptotically valid? We clarify this critical point soon, with an analogy to classical asymptotic CIs. Similar to asymptotic CIs, AsympCSs trade nonasymptotic guarantees for (a) simplicity and universality, and (b) the ability to tackle a much wider variety of problems, especially those for which there is no known nonasymptotic CS. Said differently, AsympCSs trade finite sample validity for versatility (exemplified in Section 3 with a particular emphasis on modern causal inference).

Indeed, there is a clear desire for time-uniform methods with CLT-like simplicity and versatility, especially in the context of causal inference. For example, Johari et al. [17, Section 4.3] use a Gaussian mixture sequential probability ratio test (SPRT) to conduct A/B tests (i.e. randomized experiments) for data coming from (non-Gaussian) exponential families and mentions that CLT approximations hold at large sample sizes. Similarly, Yu et al. [61] develop a mixture SPRT for causal effects in generalized linear models, where they say that their likelihood ratio forms an “approximate martingale”, meaning its conditional expectation is constant up to a factor of $\exp\{o_p(1)\}$. Moreover, Pace and Salvan [27] suggest using Robbins’ Gaussian mixture CS as a closed-form “approximate CS” and they demonstrate through simulations that the time-uniform coverage guarantee tends to hold in the asymptotic regime. However, all of these approaches justify time-uniform inference with $o_p(\cdot)$ approximations that only hold at a fixed, pre-specified sample size, and yet inferences are being carried out at data-dependent sample sizes. This section remedies the tension between fixed-$n$ approximations and time-uniform inference by defining AsympCSs such that Gaussian approximations must hold almost surely for all sample sizes simultaneously. The AsympCSs we define will also be valid in a wide range of nonparametric scenarios (beyond exponential families, parametric models, and so on).

To motivate the definition of an AsympCS that follows, let us briefly review the CLT in the batch (non-sequential) setting. Suppose $Y_1, \ldots, Y_n \sim P$ with mean $\mathbb{E}(Y_1) = \mu$ and variance $\text{var}(Y_1) = \sigma^2$. Then the standard CLT-based CI for $\mu$ takes the form

$$\hat{C}_n := \left( \hat{\mu}_n \pm \mathfrak{B}_n \right), \quad \text{where } \mathfrak{B}_n = \hat{\sigma}_n \cdot \frac{\Phi^{-1}(1 - \alpha/2)}{\sqrt{n}},$$

where $\hat{\mu}_n$ is the sample mean and $\Phi^{-1}(1 - \alpha/2)$ is the $(1 - \alpha/2)$-quantile of a standard Gaussian $N(0, 1)$ (e.g. for $\alpha = 0.05$, we have $\Phi^{-1}(0.975) \approx 1.96$). The classical notion of “asymptotic validity” is

$$\limsup_{n \to \infty} \mathbb{P}(\mu \notin \hat{C}_n) \leq \alpha.$$  

While the above is the standard definition of an asymptotic CI, we can arrive at a different definition by noting that a stronger statement can be made under the same conditions. Indeed, there exist iid standard Gaussians $Z_1, \ldots, Z_n \sim N(0, 1)$ such that

$$\frac{1}{n} \sum_{i=1}^{n} (Y_i - \mu)/\sigma = \frac{1}{n} \sum_{i=1}^{n} Z_i + o_p(1/\sqrt{n}).$$

Thus, one could define $(\hat{\mu}_n \pm \mathfrak{B}_n)$ to be an asymptotic $(1 - \alpha)$-CI if there exists some (unknown) nonasymptotic $(1 - \alpha)$-CI $(\hat{\mu}_n \pm \mathfrak{B}^*_n)$ such that $\mathfrak{B}^*_n - \mathfrak{B}_n = o_p(\mathfrak{B}_n)$, meaning that

$$\mathfrak{B}^*_n / \mathfrak{B}_n \overset{p}{\to} 1.$$  

$^3$Technically, writing (6) may require enriching the probability space so that both $Y$ and $Z$ can be defined (but without changing their laws). See Einmahl [10, Equation (1.2)] for a precise statement.
Statements of the form (6) are known as “couplings” and appear in the literature on strong approximations and invariance principles [10, 22, 23]. While justifications and derivations of CLT-based CIs like (4) are not typically presented using couplings, it is indeed the case that (6) implies the classical asymptotic validity guarantee (5). However, we deliberately highlight the alternative definition of asymptotic CIs because it ends up serving as a more natural starting point for defining asymptotic confidence sequences. In particular, we will define asymptotic CSs so that the approximation (7) holds uniformly over time, almost surely.

**Definition 2.1 (Asymptotic confidence sequences).** We say that \((\hat{\mu}_t \pm \mathcal{B}_t)_{t=1}^\infty\) is a two-sided \((1-\alpha)\)-asymptotic confidence sequence (AsympCS) for a parameter \(\mu\) if there exists a (typically unknown) two-sided nonasymptotic \((1-\alpha)\)-CS \((\hat{\mu}_t \pm \mathcal{B}_t^\alpha)\) for \(\mu\) such that

\[
\mathcal{B}_t^\alpha \mathcal{B}_t \xrightarrow{a.s.} 1. \tag{8}
\]

Furthermore, we say that \(\mathcal{B}_t\) has approximation rate \(r_t\) if \(\mathcal{B}_t^\alpha - \mathcal{B}_t = O_{a.s.}(r_t)\). We can similarly define one-sided upper (or lower) AsympCSs by replacing \(\pm\) by \(+\) (or \(-\)), respectively.

In words, Definition 2.1 says that an AsympCS \((\hat{\mu}_t \pm \mathcal{B}_t)_{t=1}^\infty\) is an arbitrarily precise approximation of some nonasymptotic CS \((\hat{\mu}_t \pm \mathcal{B}_t^\alpha)_{t=1}^\infty\) as \(t \to \infty\) — aiming to capture the intuition of having a time-uniform coverage guarantee that is valid for large samples.

It is important to note that alternate definitions to ours fail to be coherent in different ways. As one example, we could have hypothetically defined a sequence of intervals \(C_t(\alpha) = (\hat{\mu}_t \pm \mathcal{B}_t)_{t=1}^\infty\) to be a \((1-\alpha)\)-AsympCS if \(\limsup_{n \to \infty} \mathbb{P}(\exists t \geq n : \mu \notin C_t(\alpha)) \leq \alpha\), analogously to asymptotic CIs which satisfy \(\limsup_{n \to \infty} \mathbb{P}(\mu \notin C_n(\alpha)) \leq \alpha\). In words, we could have posited that if we just start peeking late enough, then the probability of eventual miscoverage would indeed be below \(\alpha\). Unfortunately, even for nonasymptotic CSs constructed at any level \(\alpha' \in (0,1)\), the former limit is zero, so this inequality would be vacuously true, regardless of \(\alpha'\), even if \(\alpha' \gg \alpha\). (Ideally, a nonasymptotic \((1-\alpha')\) CS should also be an \((1-\alpha')\)-AsympCS for any \(\alpha' \leq \alpha\), but not an asymptotic \((1-\alpha)\)-AsympCS for every \(\alpha\).)

**Remark 1 (Why almost surely??).** One may wonder why it is necessary to define AsympCSs so that \(\mathcal{B}_t^\alpha / \mathcal{B}_t \to 1\) almost surely (rather than in probability, for example). Since CSs are bounds that hold uniformly over time with high probability, convergence in probability \(\mathcal{B}_t^\alpha / \mathcal{B}_t = 1 + o_p(1)\) is not the right notion of convergence, as it only requires that the approximation term \(o_p(1)\) be small with high probability for sufficiently large fixed \(t\), but not for all \(t\) uniformly. It is natural to try to extend convergence in probability to time-uniform convergence with high probability — i.e.

\[
\sup_{k \geq t} \mathbb{P} \left( \mathcal{B}_k^\alpha / \mathcal{B}_t = 1 + o_p(1) \right) \quad \text{but it turns out (Appendix C.4) that this is equivalent to almost-sure convergence} \quad \mathcal{B}_t^\alpha / \mathcal{B}_t = 1 + o_{a.s.}(1).
\]

Going forward, we may omit “a.s.” from \(o_{a.s.}(\cdot)\) and \(O_{a.s.}(\cdot)\) and instead simply write \(o(\cdot)\) and \(O(\cdot)\), respectively to simplify notation. Now that we have defined AsympCSs as time-uniform analogues of asymptotic CIs, we will explicitly derive AsympCSs for the mean of iid random variables with \(2 + \delta\) finite moments.

### 2.2 A universal asymptotic confidence sequence for the mean

We now construct an explicit AsympCS for the mean of iid random variables by combining a variant of Robbins’ (nonasymptotic) Gaussian mixture boundary [32] with Komlós, Major, and Tusnády’s strong approximation theorems [22, 23]. This will serve as a time-uniform analogue of the CLT-based CI (4), but we require \(2 + \delta\) moments for \(\delta > 0\), rather than just 2.

**Theorem 2.2 (Gaussian mixture asymptotic confidence sequence).** Suppose \((Y_i)_{i=1}^\infty \sim \mathbb{P}\) is an infinite sequence of iid observations from a distribution \(\mathbb{P}\) with mean \(\mu\) and \(q > 2\) finite absolute moments. Let \(\hat{\mu}_t := \frac{1}{t} \sum_{i=1}^t Y_i\) be the sample mean, and \(\hat{\sigma}_t^2 := \frac{1}{t} \sum_{i=1}^t Y_i^2 - (\hat{\mu}_t)^2\) the sample variance based on the
first $t$ observations. Then, for any prespecified constant $\rho > 0$,

$$
\bar{C}_t^\rho = (\hat{\mu}_t \pm \bar{B}_t^\rho) := \left(\hat{\mu}_t \pm \hat{\sigma}_t \sqrt{\frac{2(t\rho^2 + 1)}{t^2 \rho^2 \log \left(\frac{\sqrt{t\rho^2 + 1}}{\alpha}\right)}}\right) \tag{9}
$$

forms a $(1 - \alpha)$-AsympCS for $\mu$ with approximation rate $r_t = o(\sqrt{\log \log t})$. Furthermore, if $q \geq 4$, then $r_t$ has a faster rate of $o\left((\log \log t/t)^{3/4}\right)$. In either case, $r_t = o_{a.s.}(\bar{B}_t^\rho) = o(\sqrt{\log \log t})$.

The proof in Appendix A.1 combines the strong approximation results due to Komlós et al. [22, 23] with Ville’s inequality for nonnegative supermartingales [54] applied to Gaussian mixture martingales. We can think of $\rho > 0$ as a user-chosen tuning parameter which dictates the time at which (9) is tightest, and we discuss how to easily tune this value in Appendix C.3. A one-sided analogue of (9) can be found in Appendix C.1.

While (9) may look visually similar to Robbins’ (sub)-Gaussian mixture CS [32] — written explicitly in Howard et al. [16, Eq. (14)] — it is worth pausing to reflect on how they are markedly different. Firstly, Robbins’ CS is a nonasymptotic bound that is only valid for $\sigma$-sub-Gaussian random variables, meaning $\mathbb{E}\exp\{\lambda Y_t\} \leq \exp(\sigma^2 \lambda^2/2)$ for some a priori known $\sigma > 0$, while Theorem 2.2 does not require the existence of a finite MGF at all (much less a known upper bound on it). Secondly, Robbins’ CS uses this known (possibly conservative) $\sigma$ in place of $\hat{\sigma}_t$ in (9), and thus it cannot adapt to an unknown variance, while (9) always scales with $\sqrt{\text{var}(Y_t)}$. In simpler terms, Theorem 2.2 is a time-uniform analogue of the CLT in the same way that Robbins’ CS is a time-uniform analogue of a sub-Gaussian concentration inequality (e.g. Hoeffding’s or Chernoff’s inequality [13, 15]).

It is important not to confuse Theorem 2.2 with a martingale CLT as the latter still gives fixed-time CIs in the spirit of the usual CLT but under different assumptions on the martingale difference sequence (however, we do present an analogue of Theorem 2.2 under martingale dependence in Theorem 2.3). Furthermore, notice that we could not have arrived at Theorem 2.2 using Donsker’s (weak) invariance principle, because Theorem 2.2 provides the explicit rate of convergence $r_t$, which relies on the (strong) invariance principle of Komlós et al. [22, 23]. Knowing this rate is crucial for deriving nonparametric AsympCSs for causal effects, a topic which we explore in Section 3.

### 2.3 An asymptotic confidence sequence with iterated logarithm rates

As a consequence of the law of the iterated logarithm, a confidence sequence for $\mu$ cannot have an asymptotic width smaller than $O(\sqrt{\log \log t/t})$. This is easy to see since

$$
\limsup_{t \to \infty} \frac{|\hat{\mu}_t - \mu|}{\sigma \sqrt{2t \log \log t}} = 1.
$$

This raises the question as to whether $\bar{C}_t^\rho$ can be improved so that the optimal asymptotic width of $O(\sqrt{\log \log t/t})$ is achieved. Indeed, we can use the bound in Howard et al. [16, Equation (2)] to derive such a confidence sequence, but as the authors discuss, mixture boundaries such as the one in Theorem 2.2 may be preferable in practice, because any bound that is tighter “later on” (asymptotically) must be looser “early on” (at practical sample sizes) due to the fact that all such bounds have a cumulative miscoverage probability $\leq \alpha$. Nevertheless, we present an AsympCS with an iterated logarithm rate here for completeness.

**Proposition 2.1** (Iterated logarithm asymptotic confidence sequences). Under the same conditions as Theorem 2.2,

$$
\bar{C}_t^\rho = (\hat{\mu}_t \pm \bar{B}_t^\rho) := \left(\hat{\mu}_t \pm \hat{\sigma}_t \cdot 1.7 \sqrt{\frac{\log \log (2t) + 0.72 \log(10.4/\alpha)}{t}}\right)
$$

forms a $(1 - \alpha)$-AsympCS for $\mu$ with the same rate $r_t$ as in Theorem 2.2.
We omit the proof of Proposition 2.1 as it proceeds in a similar fashion to that of Theorem 2.2 but using the stitched CS of Howard et al. [15, Eq. (2)] instead of Robbins’ Gaussian mixture CS [32]. Nevertheless, both Theorem 2.2 and Proposition 2.1 form AsympCSs for a single mean \( \mu \) of iid random variables. In the following section, however, we extend AsympCSs to the non-iid regime where means and variances can change over time.

### 2.4 Martingale asymptotic confidence sequences for time-varying means

All of the results thus far have focused on the situation where the observed random variables are independent and identically-distributed, as this is one of the most commonly-studied regimes in statistical inference. In practice, however, we may not wish to assume that means and variances remain constant over time, or that observations are independent of each other. Nevertheless, an analogue of Theorem 2.2 still holds for random variables with time-varying means and variances under martingale dependence. In this case, rather than the CS covering some fixed \( \mu \), it covers the average conditional mean thus far:

\[
\bar{\mu}_t := \frac{1}{t} \sum_{i=1}^{t} \mu_i
\]

to be made precise shortly.\(^4\)

Given the additional complexity introduced by considering time-varying conditional distributions, we will first explicitly spell out the assumptions required to achieve a time-varying analogue of Theorem 2.2. Note however, that these assumptions are no more restrictive, meaning that they reduce to the assumptions of Theorem 2.2 in the iid regime. Suppose \((Y_t)_{t=1}^{\infty}\) is a sequence of random variables with conditional means and variances given by \( \mu_t := \mathbb{E}(Y_t \mid Y_1^{t-1}) \) and \( \sigma^2_t := \text{var}(Y_t \mid Y_1^{t-1}) \), respectively. First, we require that the average conditional variance \( \bar{\sigma}_t^2 := \frac{1}{t} \sum_{i=1}^{t} \sigma_i^2 \) either does not vanish, or does so superlinearly; equivalently, we require that the cumulative conditional variance diverges.

![Figure 2](image_url)

Figure 2: A 90%-AsympCS for the time-varying mean \( \tilde{\mu}_t \) using Theorem 2.3 with \( \rho \) optimized for \( t^* = 500 \) based on the exact solution of Appendix C.3. Here, we have set \( \mu_t := \frac{1}{2}(1 - \sin(2 \log(e + 10t)) / \log(e + 0.01t)) \) to produce the sinusoidal behavior of \( \tilde{\mu}_t \). Notice that \( \tilde{C}_t \) uniformly captures \( \tilde{\mu}_t \), adapting to its non-stationarity.

**Assumption 1** (Cumulative variance diverges almost surely). For each \( t \geq 1 \), let \( \sigma^2_t := \text{var}(Y_t \mid Y_1^{t-1}) \)

\(^4\)Throughout this section, we use the overhead tilde (e.g. \( \tilde{\mu}_t, \tilde{\sigma}_t, \text{and } \tilde{C}_t \)) to emphasize that these quantities can change over time. For example, Fig. 2 explicitly displays means and CSs with sinusoidal behaviors resembling a tilde.
be the conditional variance of $Y_t$. Then,

$$V_t := \sum_{i=1}^{t} \sigma_i^2 \rightarrow \infty \text{ almost surely.}$$ (10)

Eq. (10) can also be interpreted as saying that the average conditional variance $\bar{\sigma}_t^2 := \frac{1}{t} \sum_{i=1}^{t} \sigma_i^2$ does not vanish faster than $1/t$ (if at all), meaning $\bar{\sigma}_t^2 = \omega_{a.s.}(1/t)$. For example, Assumption 1 would hold if $\bar{\sigma}_t^2 \xrightarrow{a.s.} \sigma_0^2$ for some $\sigma_0^2 > 0$ or in the iid case where $\sigma_1^2 = \sigma_2^2 = \cdots = \sigma_n^2$. Second, we require a Lindeberg-type uniform integrability condition on the tail behavior of $(Y_i)_{i=1}^{T}$.

**Assumption 2** (Lindeberg-type uniform integrability). For each $t \geq 1$, let $\sigma_t^2 := \text{var}(Y_t \mid Y_{t-1}^t)$ be the conditional variance of $Y_t$. Then there exists some $q > 2$ such that

$$\sum_{t=1}^{\infty} \frac{\mathbb{E} \left[ (Y_t - \mu_t)^2 \mathbb{1} \left( (Y_t - \mu_t)^2 > V_t^2 \right) \mid Y_{t-1}^t \right]}{V_t^2} < \infty \text{ almost surely.}$$ (11)

Notice that Eq. (11) is satisfied if all $q^{th}$ moments are almost surely uniformly bounded, meaning $1/K \leq \mathbb{E}|Y_t - \mu_t|^q < K$ a.s. for all $t \geq 1$ and for some constant $K > 0$, or more generally under a Lyapunov-type condition that states $\sum_{i=1}^{\infty} \mathbb{E}|Y_t - \mu_t|^q < \infty$ a.s. where $q > 2 + 2\delta$. Third and finally, we require a consistent variance estimator.

**Assumption 3** (Consistent variance estimation). Let $\hat{\sigma}_t^2$ be an estimator of $\sigma_t^2$ constructed using $Y_1, \ldots, Y_t$ such that

$$\hat{\sigma}_t^2 \xrightarrow{a.s.} \sigma_t^2.$$ (12)

Note that in the iid case, (12) would hold for the sample variance by the strong law of large numbers. Given Assumptions 1, 2, and 3, we have the following AsympCS for the time-varying conditional mean $\hat{\mu}_t := \frac{1}{t} \sum_{i=1}^{t} \mu_i$.

**Theorem 2.3** (Lindeberg-type martingale AsympCS). Let $(Y_i)_{i=1}^{\infty}$ be a sequence of random variables with conditional mean $\mu_t := \mathbb{E}(Y_t \mid Y_{t-1}^t)$ and conditional variance $\sigma_t^2 := \text{var}(Y_t \mid Y_{t-1}^t)$. Then under Assumptions 1, 2, and 3, we have that

$$\hat{C}_t \equiv (\hat{\mu}_t \pm \hat{\mathbb{B}}_t) := \left( \hat{\mu}_t \pm \sqrt{\frac{2(\hat{\sigma}_t^2 \rho^2 + 1)}{t^2 \rho^2} \log\left( \frac{\sqrt{\hat{\sigma}_t^2 \rho^2 + 1}}{\alpha} \right)} \right)$$ (13)

forms a $(1-\alpha)$-AsympCS for the running average conditional mean $\hat{\mu}_t := \frac{1}{t} \sum_{i=1}^{t} \mu_i$, with approximation rate $o(\sqrt{V_t \log V_t / t})$.

The proof of Theorem 2.3 can be found in Appendix A.2 and uses Strassen’s strong approximation result for random variables under martingale dependence [42, Thm. 4.4]. Fig. 2 illustrates what $\hat{C}_t$ may look like in practice. Note that when $(Y_i)_{i=1}^{\infty}$ are independent with $\mu_1 = \mu_2 = \cdots = \mu_\infty$, and $\sigma_1^2 = \sigma_2^2 = \cdots = \sigma_\infty^2$, it is nevertheless the case that $\hat{C}_t$ forms a $(1-\alpha)$-AsympCS for $\mu_\infty$ under the same assumptions as Theorem 2.2. In this sense, we can view $(\hat{C}_t)_{t=1}^{\infty}$ as “robust” to deviations from independence and stationarity. A one-sided analogue of Theorem 2.3 is presented in Proposition C.2 within Appendix C.1.

As an immediate corollary of Theorem 2.3, we have the following Lyapunov-type AsympCS under independent but non-identically distributed random variables.

---

1. One can verify that the Lyapunov-type condition is sufficient by using the identity $\mathbb{E}(|Y| \mathbb{1}(|Y| > y)) = y\mathbb{P}(|Y| > y) + \int_{y}^{\infty} \mathbb{P}(|Y| > a) da$.
2. Here, the term “robust” should not be interpreted in the same spirit as “doubly robust”, where the latter is specific to the discussions surrounding functional estimation and causal inference in Section 3.
Corollary 2.1 (Lyapunov-type AsympCS). Suppose \((Y_t)_{t=1}^\infty\) is a sequence of independent random variables with individual means and variances given by \(\mu_t := \mathbb{E}(Y_t)\) and \(\sigma_t^2 := \text{var}(Y_t)\), respectively. Suppose that (1) the cumulative variance diverges \(V_t := \sum_{i=1}^t \sigma_i^2 \to \infty\), (2), the moments do not vanish too quickly — meaning \(\sum_{i=1}^\infty \frac{\mathbb{E}|Y_i - \mu_i|^q}{\sigma_i^{2q}} < \infty\) for \(q > 2 + 2\delta\) — and (3) the variance is estimated consistently, \(\hat{\sigma}_t^2 / \sigma_t^2 \xrightarrow{d} 1\). Then \(\tilde{C}_t\) forms a \((1 - \alpha)\)-AsympCS for the running average mean \(\tilde{\mu}_t := \frac{1}{t} \sum_{i=1}^t \mu_i\).

In Section 3, we will employ Theorem 2.2 and Corollary 2.1 to illustrate how one can derive AsympCSs for treatment effects in sequential randomized experiments and observational studies.

2.5 Asymptotic confidence sequences using Robbins’ delayed start

As is clear from the definition of AsympCSs (Definition 2.1), virtually any boundary for Gaussian observations can be used to derive an AsympCS as long as an appropriate strong invariance principle can be applied under the given assumptions — indeed, Theorem 2.2, Proposition 2.1, Theorem 2.3, and Corollary 2.1 are all instantiations of this general phenomenon.

Another AsympCS that may be of interest to practitioners is one that leverages Robbins’ CS for means of Gaussian random variables with a delayed start time \([32, \text{Eq. (20)}]\). In a nutshell, Robbins calculated an upper bound on the probability that a centered Gaussian random walk would cross a particular boundary for any time \(t \geq m\) for some starting time \(m \geq 1\). That is, Robbins showed that for iid standard Gaussians \((G_i)_{i=1}^\infty\), letting \(S_t := \sum_{i=1}^t G_i\), and for any \(a > 0\),

\[
P\left(\exists t \geq m : |S_t| \geq \sqrt{t(a^2 + \log(t/m))}\right) \leq 2(1 - \Phi(a) + a\phi(a)),
\]

where \(\Phi\) and \(\phi\) are the cumulative distribution and probability density functions of a standard Gaussian, respectively. Consequently, if \(a\) is chosen so that the right-hand side of (14) is set to a given \(\alpha \in (0, 1)\), then \(\sqrt{t(a^2 + \log(t/m))}\) becomes a boundary for \(|S_t|\) with crossing probability \(\alpha\). Combining this fact with the definition of an AsympCS along with the strong invariance principles of Komlós et al. \([22, 23]\), Major \([26]\), we have the following corollary.

Corollary 2.2. Given iid observations \((Y_t)_{t=1}^\infty\) with mean \(\mu\) and variance \(\sigma^2\), we have that under the same conditions as Theorem 2.2,

\[
(L_t, U_t) := \begin{cases} 
(-\infty, \infty) & \text{if } t < m \\
\left(\tilde{\mu}_t \pm \tilde{\sigma}_t \sqrt{\varphi(a^2 + \log(t/m)/t)}/t\right) & \text{if } t \geq m
\end{cases}
\]

forms a \((1 - \alpha)\)-AsympCS for \(\mu\), where \(a\) is chosen so that \(2(1 - \Phi(a) + a\phi(a)) = \alpha\).

It is easy to check that such a mapping above from \(\alpha\) to \(a\) always exists by continuity; indeed \(2(1 - \Phi(a) + a\phi(a))\) is continuous in \(a\) and takes values 1 and 0 at \(a = 0\) and \(a = \infty\) respectively. In words, as long as the AsympCS is treated as vacuous for all times \(t\) leading up to some prespecified delayed start time \(m\), the potentially tighter boundary given in (15) can be used to derive an AsympCS for the mean \(\mu\). The particular method in Corollary 2.2, in its essence, was independently discovered by Bibaut et al. \([1]\), who prove further properties about the implied test, which they term a “universal SPRT” and its downstream CS a “universal CS”. As previously alluded to, however, essentially any other (sub)-Gaussian boundary can be used in place of (15) due to the generality of Definition 2.1 and the versatility of the aforementioned strong approximation theorems.

3 Confidence sequences for the average treatment effect

Given the groundwork laid in Section 2.1, we now focus on time-uniform inference for causal effects — namely the average treatment effect — via AsympCSs. This will enable researchers to quantify
uncertainty for causal effects in fully sequential environments, where confidence sets from randomized experiments and observational studies can be continuously monitored as the data are being collected. However, obtaining AsympCSs for the average treatment effect is not as simple as applying Theorem 2.2 to some appropriately chosen random variable due to the presence of (potentially infinite-dimensional) nuisance parameters. Nevertheless, after introducing and carefully analyzing sequential sample-splitting and cross-fitting (Section 3.2), we will see that efficient time-uniform inference in these sequential settings is still possible.

To solidify the notation and problem setup, we expand on the prelude provided in Section 1. Suppose that we observe a (potentially infinite) sequence of independent and identically distributed (iid) variables $Z_1, Z_2, \ldots$ from a distribution $\mathbb{P}$, where $Z_t := (X_t, A_t, Y_t)$ denotes the $t^{th}$ subject’s triplet and

- $X_t \in \mathbb{R}^d$ is subject $t$’s measured baseline covariates,
- $A_t \in \{0, 1\}$ is the treatment that subject $t$ received, and
- $Y_t \in \mathbb{R}$ is subject $t$’s measured outcome after treatment.

Our target estimand is the average treatment effect (ATE) $\psi$ defined as

$$\psi := \mathbb{E}(Y^1 - Y^0),$$

where $Y^a$ is the counterfactual outcome for a randomly selected subject had they received treatment $a \in \{0, 1\}$. The ATE can be interpreted as the average population outcome if everyone were treated $\mathbb{E}(Y^1)$ versus if no one were treated $\mathbb{E}(Y^0)$. However, without further identifying assumptions, we cannot hope to estimate this counterfactual quantity using the observed data $(Z_t)_{t=1}^{\infty}$. Consider the following standard causal identifying assumptions, which we require for $a \in \{0, 1\}$.

(IA1): Consistency: $A = a \implies Y = Y^a$,

(IA2): No unmeasured confounding: $A \perp Y^a \mid X$, and

(IA3): Positivity: $\mathbb{P}(A = a \mid X) > 0$ almost surely.

The consistency assumption (IA1) can be thought of as stating that there is no interference between subjects, so that an individual’s counterfactual does not depend on the treatment of others (which, e.g. could be violated in a vaccine efficacy trial where a subject is protected by the fact that their friends received a vaccine). (IA2) effectively states that the treatment is as good as randomized within levels of the observed covariates, and (IA3) simply ensures that all subjects have a nonzero probability of receiving treatment $a \in \{0, 1\}$. Throughout the remainder of the paper, we assume (IA1). In Section 3.3, we will consider an experimental setting in which (IA2) and (IA3) hold by design, while in the observational case which we consider in Section 3.4, (IA2) and (IA3) will need to be assumed. It is well-known that if identifying assumptions (IA1)–(IA3) hold, then the average treatment effect $\psi$ can be written as

$$\psi = \mathbb{E}(Y \mid X, A = 1) - \mathbb{E}(Y \mid X, A = 0),$$

a purely statistical quantity for which we aim to derive sharp AsympCSs under nonparametric assumptions. To this end, we briefly review efficient estimators for $\psi$ and the sense in which they are optimal.

### 3.1 A brief review of efficient estimators

For a detailed account of efficient estimation in semiparametric models, we refer readers to Bickel et al. [2], van der Vaart [51], van der Laan and Robins [47], Tsiatis [45] and Kennedy [20], but we provide a brief overview of their fundamental relevance to estimation of the ATE here.
A central goal of semiparametric efficiency theory is to characterize the set of influence functions for a parameter (in our case, $\psi$). Of particular interest is finding the efficient influence function (EIF) as its variance acts as a semiparametric analogue of the Cramer-Rao lower bound, hence providing a benchmark for constructing optimal estimators (in an asymptotic local minimax sense). In the case of $\psi$, the (uncentered) EIF is given by

$$f(z) \equiv f(x, a, y) := \{\mu^1(x) - \mu^0(x)\} + \left(\frac{a}{\pi(x)} - \frac{1 - a}{1 - \pi(x)}\right)\{y - \mu^a(x)\}, \quad (16)$$

where $\mu^a(x) := \mathbb{E}(Y \mid X = x, A = a)$ is the regression function among those treated at level $a \in \{0, 1\}$ and $\pi(x) := \mathbb{P}(A = 1 \mid X = x)$ is the propensity score (i.e. probability of treatment) for an individual with covariates $x$. In particular, this means that no estimator of $\psi$ based on $t$ observations can have asymptotic mean squared error smaller than $\text{var}(f(Z))/t$ without imposing additional assumptions.

In a randomized experiment, the joint distribution of $(X, Y)$ is known but the conditional distribution of $A \mid X = x$ is known to be Bernoulli($\pi(x)$) by design. In this case, our statistical model for $Z$ is a proper semiparametric model, and hence there are infinitely many influence functions, all of which take the form,

$$\hat{f}(z) \equiv \hat{f}(x, a, y) := \{\hat{\mu}^1(x) - \hat{\mu}^0(x)\} + \left(\frac{a}{\hat{\pi}(x)} - \frac{1 - a}{1 - \hat{\pi}(x)}\right)\{y - \hat{\mu}^a(x)\}, \quad (17)$$

where $\hat{\mu}^a : \mathbb{R}^d \rightarrow \mathbb{R}$ is any function. However, when the joint distribution of $(X, A, Y)$ is left completely unspecified (such as in an observational study with unknown propensity scores), our statistical model for $\mathbb{P}$ is nonparametric, and hence there is only one influence function, the EIF given in (16).

Not only does the EIF $f(z)$ provide us with a benchmark against which to compare estimators, but it hints at the first step in deriving the most efficient estimator. Namely, $\frac{1}{T} \sum_{t=1}^{T} f(Z_t)$ is a consistent estimator for $\psi$ with asymptotic variance equal to the efficiency bound, $\text{var}(f)$ by construction. However, $f(Z)$ depends on possibly unknown nuisance functions $\eta := (\mu^1, \mu^0, \pi)$. A natural next step would be to simply estimate $\eta$ from the data $(Z_t)_{t=1}^{T}$. Crucially, it is possible to ensure that only a negligible amount of additional estimation error is incurred by replacing $\eta$ by a data-dependent estimate $\hat{\eta}_t$, the essential technique here being sample splitting and cross-fitting [34, 62, 5]. In the following section, we introduce sequential sample-splitting and cross-fitting, allowing the same types of analyses of $\hat{\eta}_t$ to be carried out but in fully sequential settings.

### 3.2 Sequential sample-splitting and cross-fitting

Following Robins et al. [34], Zheng and van der Laan [62], and Chernozhukov et al. [5], we employ sample-splitting to derive an estimate $\hat{f}$ of the influence function $f$ on a “training” sample, and evaluate $\hat{f}$ on values of $Z_t$ in an independent “evaluation” sample. Sample-splitting sidesteps complications introduced from “double-dipping” (i.e. using $Z_t$ to both construct $\hat{f}$ and evaluate $\hat{f}(Z_t)$) and greatly simplifies the analysis of the downstream estimator. However, the aforementioned authors employed sample-splitting in the batch (non-sequential) regime where one can simply randomly split the data into two halves. Given our sequential setup where data are continually observed in an online stream over time, we modify the sample-splitting procedure as follows. We will denote $\mathcal{D}_t^{\text{trn}}$ and $\mathcal{D}_t^{\text{eval}}$ as the “training” and “evaluation” sets, respectively. At time $t$, we assign $Z_t$ to either group with equal probability:

$$Z_t \in \begin{cases} \mathcal{D}_t^{\text{trn}} & \text{with probability } 1/2, \\ \mathcal{D}_t^{\text{eval}} & \text{otherwise.} \end{cases}$$

Note that at time $t + 1$, $Z_t$ is not re-randomized into either split — once $Z_t$ is randomly assigned to one of $\mathcal{D}_t^{\text{trn}}$ or $\mathcal{D}_t^{\text{eval}}$, it remains in that split for the remainder of the study. In this way, we can write $\mathcal{D}_t^{\text{trn}} = (Z_1^{\text{trn}}, Z_2^{\text{trn}}, \ldots)$ and $\mathcal{D}_t^{\text{eval}} = (Z_1^{\text{eval}}, Z_2^{\text{eval}}, \ldots)$ and think of these as independent, sequential
observations from a common distribution $P$. To keep track of how many subjects have been randomized to $D_\infty^{trn}$ and $D_\infty^{eval}$ at time $t$, define

$$T := |D_\infty^{eval}| \quad \text{and} \quad T' := |D_\infty^{trn}| = t - T,$$

where we have left the dependence on $t$ implicit.

$$\begin{align*}
T^{trn} & \rightarrow \left( \hat{\mu}_{T'}, \hat{\mu}_T^0, \hat{\pi}_{T'} \right) \\
T^{val} & \rightarrow \hat{\psi}_t := \frac{1}{T} \sum_{i=1}^{T} \hat{f}_{T'}(Z_i^{eval})
\end{align*}$$

Figure 3: A schematic illustrating sequential sample splitting. At each time step $t$, the new observation $Z_t$ is randomly assigned to $D_\infty^{trn}$ or $D_\infty^{eval}$ with equal probability $(1/2)$. Nuisance function estimators $(\hat{\mu}_{T'}, \hat{\mu}_T^0, \hat{\pi}_{T'})$ are constructed using $D_\infty^{trn}$ which then yield $\hat{f}_{T'}$. The sample-split estimator $\hat{\psi}_t^{split}$ is defined as the sample average $\frac{1}{T} \sum_{i=1}^{T} \hat{f}_{T'}(Z_i^{eval})$ where each $Z_i^{eval} \in D_\infty^{eval}$.

**Remark 2.** Strictly speaking, under the iid assumption, we do not need to randomly assign subjects to training and evaluation groups for the forthcoming results to hold (e.g. we could simply assign even-numbered subjects to $D_\infty^{trn}$ and odd-numbered subjects to $D_\infty^{eval}$). However, the analysis is not further complicated by this randomization, and it can be used to combat bias in treatment assignments when the iid assumption is violated [9].

### 3.2.1 The sequential sample-split estimators $(\hat{\psi}^{split}_t)_{t=1}^\infty$

After employing sequential sample-splitting, the sequence of sample-split estimators $(\hat{\psi}^{split}_t)_{t=1}^\infty$ for $\psi$ are given by

$$\hat{\psi}^{split}_t := \frac{1}{T} \sum_{i=1}^{T} \hat{f}_{T'}(Z_i^{eval}),$$

where $\hat{f}_{T'}$ is given by (16) with $\eta \equiv (\mu^1, \mu^0, \pi)$ replaced by $\hat{\eta}_{T'} \equiv (\hat{\mu}^1_{T'}, \hat{\mu}_T^0, \hat{\pi}_{T'})$ which is built solely from $D_\infty^{trn}$. The sample-splitting procedure for constructing $\hat{\psi}^{split}_t$ is summarized pictorially in Fig. 3. In the batch setting for a fixed sample size, (19) is often referred to as the “doubly robust” or “augmented inverse probability weighted” (AIPW) estimator [36, 37] and we adopt similar nomenclature here.

### 3.2.2 The sequential cross-fit estimators $(\hat{\psi}_t^\times)_{t=1}^\infty$

A commonly cited downside of sample-splitting is the loss in efficiency by using $T \approx t/2$ subjects instead of $t$ when evaluating the sample mean $\frac{1}{T} \sum_{i=1}^{T} \hat{f}_{T'}(Z_i^{eval})$. An easy fix is to cross-fit: swap the two samples, using the evaluation set $D_\infty^{eval}$ for training and the training set $D_\infty^{trn}$ for evaluation to
recover the full sample size of \( t = T + T' \) \([34, 62, 5]\). That is, construct \( \hat{f}_T \) solely from \( \mathcal{D}_i^\text{eval} \) and define the cross-fit estimator \( \hat{\psi}_T^\times \) as

\[
\hat{\psi}_T^\times := \frac{\sum_{t=1}^{T'} f_T(Z_i^\text{eval}) + \sum_{t=1}^{T'} f_T(Z_i^\text{trn})}{2},
\]

(20)

and the associated cross-fit variance estimate

\[
\sqrt{\text{var}}_T(f) := \frac{\sqrt{\text{var}}_T(\hat{f}_T) + \sqrt{\text{var}}_T(\hat{f}_T)}{2}.
\]

(21)

Note that (21) is simply the average of sample variances of the cross-fit pseudo-outcomes \( (\hat{f}_T(Z_i^\text{eval}))_{t=1}^{T'} \) and \( (\hat{f}_T(Z_i^\text{trn}))_{t=1}^{T'} \), respectively. All of the results that follow are stated in terms of the cross fit estimators \( (\hat{\psi}_T^\times)_{t=1}^{T'} \) but they can be amended to use \( (\hat{\psi}_T^{\text{plfit}})_{t=1}^{T'} \) instead. Using the assumptions laid out so far, we are ready to apply the confidence sequences of Section 2.1 to randomized sequential experiments.

3.3 Asymptotic confidence sequences in randomized experiments

Consider an experiment in which subjects are recruited sequentially and administered treatment in a randomized and controlled manner. In particular, suppose that samples are iid and a subject with covariates \( x \) has a known propensity score given by

\[
\pi(x) := \mathbb{P}(A = 1 \mid X = x).
\]

Consider the doubly robust cross-fit estimator \( \hat{\psi}_T^\times \) as given in (20) but with estimated propensity scores — \( \hat{\pi}_T(x) \) and \( \hat{\pi}_T(x) \) — replaced by their true values \( \pi(x) \), and with \( \hat{\mu}_T^a(x) \) and \( \hat{\mu}_T^a \) being possibly misspecified estimators for \( \mu^a \). That is, we will assume that \( \hat{\mu}_T^a \) converges to some function \( \bar{\mu}^a \), which need not coincide with \( \mu^a \). We are now ready to state the main result of this section.

**Theorem 3.1** (Confidence sequences for the ATE in randomized experiments). Let \( \hat{\psi}_T^\times \) be the doubly robust cross-fit estimator as in (20). Suppose \( \|\hat{\mu}_T^a(X) - \bar{\mu}^a(X)\|_{L^2(\mathbb{P})} = o(1) \) for each \( a \in \{0, 1\} \) where \( \bar{\mu}^a \) is some function (but need not be \( \mu^a \)), and hence \( \|\hat{f}_T - f\|_{L^2(\mathbb{P})} = o(1) \) for some influence function \( f \) of the form (17). Suppose that propensity scores are bounded away from 0 and 1, i.e. \( \pi(X) \in [\delta, 1 - \delta] \) for some \( \delta > 0 \), and suppose that the influence function \( f(Z) \) has at least four moments, \( \mathbb{E}|f(Z)|^4 < \infty \). Then for any prespecified constant \( \rho > 0 \),

\[
\hat{\psi}_T^\times \pm \sqrt{\sqrt{\text{var}}_T(f)} \cdot \sqrt{\frac{2(t\rho^2 + 1)}{t^2 \rho^2} \log \left( \frac{\sqrt{t\rho^2 + 1}}{\alpha} \right)}
\]

(22)

forms a \((1 - \alpha)\)-AsympCS for \( \psi \) with approximation rate \( o(\sqrt{\log \log t/t}) \).

The proof in Appendix A.3 combines an analysis of the almost-sure convergence of \( (\hat{\psi}_T^\times - \psi) \) with the AsympCS of Theorem 2.2. Notice that since \( \hat{\mu}_T^a \) is consistent for a function \( \bar{\mu}^a \), we have that \( \hat{f}_T \) is asymptotically equivalent to an influence function \( \bar{f} \) of the form (17). In practice, however, one must choose \( \hat{\mu}_T^a \). As alluded to at the beginning of Section 3, the best possible influence function is the EIF \( f(z) \) defined in (16), and thus it is natural to attempt to construct \( \hat{\mu}_T^a \) so that \( \|\hat{f}_T - f\|_{L^2(\mathbb{P})} = o(1) \). The resulting confidence sequences would inherit such optimality properties, a point which we discuss further in Appendix C.5.

Since \( \mu^a \) is an unknown regression function with a potentially complex structure, we cannot in general expect to estimate it with a simple parametric model. Instead, we suggest building \( \hat{\mu}_T \) as a weighted model average of several parametric and nonparametric machine learning algorithms. This technique of averaging several candidate models is commonly known as “stacking” \([3]\), “aggregation”
Confidence sequences for
the average treatment effect
Unadjusted
Parametric
Super Learner
ψ

Figure 4: Three 90%-AsympCSs for the average treatment effect in a simulated randomized experiment using different regression estimators. Notice that all three confidence sequences uniformly capture the average treatment effect ψ, but increasingly sophisticated models do so more efficiently, with the Super Learner greatly outperforming an unadjusted estimator. For more details on this simulation, see Appendix B.1.

[46], or “Super Learning” [49, 29, 30]. The candidate models can include both flexible machine learning methods (e.g. random forests [4], generalized additive models [12], etc.) as well as simpler parametric models and yet for large samples, the Super Learner will perform as well as the best weighted average of candidate models [47, 52, 49]. This advantage can be seen empirically in Fig. 4 where the true regression functions µ₀ and µ₁ are non-smooth, nonlinear functions of covariates x ∈ ℝ^d. See Appendix B.1 for more details on how this simulation was designed.

So far, flexible nonparametric regression techniques such as Super Learning have been used to build efficient estimators  \( \hat{\mu}_a \) of \( \mu^a \), \( a \in \{0, 1\} \), but were not required to derive valid confidence sequences for \( \psi \). In an observational setting where neither \( \mu^a(x) \) nor \( \pi(x) \) are known, flexible nuisance estimation will be an essential tool in ensuring that confidence sequences capture \( \psi \), as we will see in the following section.

3.4 Asymptotic confidence sequences in observational studies

We now consider a situation where identifying assumptions (IA2) and (IA3) do not hold by design, but must be assumed. This may occur in a purely observational sequential study, or in a randomized sequential experiment where subjects do not comply with their assigned treatments or have missing outcomes. In any case, it is well-known that (IA2) and (IA3) are untestable from the observed data, and we assume that they hold for the discussions that follow. As before, under (IA1)–(IA3), we have that

\[
\psi = E(Y^1 - Y^0) = E(E(Y \mid X, A = 1) - E(Y \mid X, A = 0)),
\]

which is the target parameter we aim to estimate. Since we no longer have knowledge of each subject’s propensity score \( \pi(x) := P(A \mid X = x) \) we must instead estimate \( \pi(x) \) in addition to the regression functions \( \mu^a(x) := E(Y \mid X = x, A = a) \) for each \( a \in \{0, 1\} \) under nonparametric conditions. Let \( \hat{\psi} \) denote the doubly robust cross-fit estimator (20) built using estimates of \( \pi \), \( \mu^1 \), and \( \mu^0 \). Then
the following theorem provides the conditions under which we can construct AsympCSs for \( \psi \) in observational studies.

![Figure 5: Three 90%-AsympCSs for the average treatment effect in an observational study using three different estimators. Unlike the randomized setup, only the nonparametric ensemble (Super Learner) is consistent, since parametric (and especially the unadjusted) estimators are misspecified. Not only is the doubly robust Super Learner confidence sequence converging to \( \psi \), but it is also the tightest of the three models at each time step. For more details on this simulation, see Appendix B.2.](image)

**Theorem 3.2** (Confidence sequence for the ATE in observational studies). Consider the same setup as Theorem 3.1 but with \( \pi(x) \) no longer known. Suppose that regression functions and propensity scores are consistently estimated in \( L^2(P) \) at a product rate of \( o(\sqrt{\log \log t/\log t}) \), meaning that

\[
\| \hat{\pi}_t - \pi \|_{L^2(P)} \leq o(\sqrt{\log \log t/\log t}).
\]

Moreover, suppose that \( \| \hat{f}_t - f \|_{L^2(P)} = o(1) \) where \( f \) is the efficient influence function (16) and that \( f(Z) \) has at least four finite moments \( \mathbb{E}|f(Z)|^4 < \infty \). Then for any prespecified constant \( \rho > 0 \),

\[
\hat{\psi}_t \pm \sqrt{\text{Var}(\hat{f}_t)} \cdot \sqrt{\frac{2(t\rho^2 + 1)}{t^2 \rho^2}} \log \left( \frac{\sqrt{t\rho^2 + 1}}{\alpha} \right)
\]

forms a \((1 - \alpha)\)-AsympCS for \( \psi \) with approximation rate \( o(\sqrt{\log \log t/\log t}) \).

The proof in Appendix A.3.2 proceeds similarly to the proof of Theorem 3.1 by combining Theorem 2.2 with an analysis of the almost-sure behavior of \((\hat{\psi}_t - \psi)\). Notice that the requirement that nuisance functions are estimated at a product rate of \( o(\sqrt{\log \log t/\log t}) \) rate that appears in the fixed-time doubly robust estimation literature. In fact, this requirement can be weakened to a product rate of \( o(1/\sqrt{t}) \) but we omit this derivation.

Unlike the experimental setting of Section 3.3, Theorem 3.2 requires that \( \hat{\mu}_t \) and \( \hat{\pi}_t \) consistently estimate \( \mu^a \) and \( \pi \), respectively. As a consequence, \( \hat{f}_t \) converges to the efficient influence function \( f \)
and thus \( \hat{\psi}_t \) not only consistently estimates \( \psi \) but also attains the nonparametric efficiency bound. Unadjusted estimators or those built using misspecified models may neither be efficient nor consistent (see Fig. 5).

3.5 Time-varying treatment effects

The results in Sections 3.3 and 3.4 considered the classical regime where the ATE \( \psi \) is a fixed functional that does not change over time. Consider a strict generalization where distributions — and hence individual treatment effects in particular — may change over time. In other words,

\[
\psi_t := \mathbb{E}\{Y_t^1 - Y_t^0\} = \mathbb{E}\{\mathbb{E}(Y_t | X_t, A_t = 1) - \mathbb{E}(Y_t | X_t, A_t = 0)\},
\]

where the equality \((*)\) holds under the usual causal identification assumptions (IA1)–(IA3). Despite the non-stationary and non-iid structure, it is nevertheless possible to derive CSs for the time-varying average treatment effect \( \hat{\psi}_t := \frac{1}{t} \sum_{i=1}^{t} \psi_i \) using the Lyapunov-type bounds of Corollary 2.1. However, given this more general and complex setup, the assumptions required are more subtle (but no more restrictive) than those for Theorems 3.1 and 3.2; as such, we explicitly describe their details here.

**Assumption 4** (Regression estimator is uniformly well-behaved in \( L_2(\mathbb{P}) \)). We assume that regression estimators \( \hat{\mu}_t^a(Z_i) \) converge to some function \( \bar{\mu}^a \) in \( L_2(\mathbb{P}) \) regardless of the distribution of \( Z_i \), i.e.

\[
\sup_{1 \leq i \leq \infty} \|\hat{\mu}_t^a(X_i) - \bar{\mu}^a(X_i)\|_{L_2(\mathbb{P})} = o(1)
\]

for each \( a \in \{0, 1\} \).

Assumption 4 simply requires that the regression estimator \( \hat{\mu}_t^a \) must converge to some function \( \bar{\mu}^a \), which need not coincide with true regression function \( \mu^a \). In the iid setting where \( X_1, X_2, \ldots \) all have the same distribution, we would simply drop the \( \sup_{1 \leq i \leq \infty} \), recovering the conditions for Theorems 3.1 and 3.2.

**Assumption 5** (Iterated logarithm convergence of average nuisance errors). Let \( \hat{\mu}_t^a \) be an estimator of the regression function \( \mu^a \), \( a \in \{0, 1\} \) and \( \hat{\pi}_t \) an estimator of the propensity score \( \pi \). We assume that the average bias shrinks at an LIL rate, i.e.

\[
\frac{1}{t} \sum_{i=1}^{t} \left\|\hat{\pi}_t(X_i) - \pi(X_i)\|_{L_2(\mathbb{P})} \right\|_{L_2(\mathbb{P})} \sum_{a=0}^{1} \|\hat{\mu}_t^a(X_i) - \mu^a(X_i)\|_{L_2(\mathbb{P})} = o\left(\sqrt{\frac{\log \log t}{t}}\right).
\]

Note that Assumption 5 would hold in two familiar scenarios. Firstly, in a randomized experiment (Theorem 3.3) where \( \hat{\pi}_t = \pi \) is known by design, we have that (24) is always zero, satisfying Assumption 5 trivially. Second, in an observational study (Theorem 3.4) where the product of errors \( \|\hat{\pi}_t(X_i) - \pi(X_i)\|_{L_2(\mathbb{P})}\|\hat{\mu}_t^a(X_i) - \mu^a(X_i)\|_{L_2(\mathbb{P})} \) vanishes at a rate of \( \sqrt{\log \log t} \), for each \( i \) and for both \( a \in \{0, 1\} \), we also have that their average product errors vanish at the same rate (24). With these assumptions in mind, let us summarize how time-varying treatment effects can be captured in randomized experiments.

**Theorem 3.3** (Confidence sequences for time-varying effects in randomized experiments). Suppose \( Z_1, Z_2, \ldots \) are independent triples \( Z_t := (X_t, A_t, Y_t) \) and consider the individual treatment effects given by \( \psi_t := \mathbb{E}\{\mathbb{E}(Y_t | X_t, A_t = 1) - \mathbb{E}(Y_t | X_t, A_t = 0)\} \). Suppose that regression estimators converge to some limit (Assumption 4). Assume that the treatment mechanism \( \pi(x) := \mathbb{P}(A_t | X_t = x) \) is known, and hence we have that Assumption 5 holds by design. Finally, suppose that the conditions of Corollary 2.1 hold, but with \( (Y_t)_{t=1}^{\infty} \) replaced by the influence functions \( \{f(Z_t)\}_{t=1}^{\infty} \). Then,

\[
\hat{\psi}_t \pm \sqrt{\frac{2(t^2 \text{var}_t(f) + 1)}{t^2 \rho^2} \log \left( \frac{\sqrt{t^2 \text{var}_t(f) + 1}}{\alpha} \right)}
\]

forms a \((1 - \alpha)\)-AsympCS for the cumulative average of individual treatment effects \( \tilde{\psi}_t := \frac{1}{t} \sum_{i=1}^{t} \psi_i \).
Figure 6: Three 90% AsympCSs for \( \tilde{\psi}_t \) constructed using various estimators via Theorem 3.3. Since this is a randomized experiment, all three CSs capture \( \tilde{\psi}_t \) uniformly over time with high probability. Similar to Fig. 4, however, the doubly robust estimator constructed via Super Learning greatly outperforms those based on parametric or unadjusted estimators.

The proof can be found in Appendix A.4. The important takeaway from Theorem 3.3 is that under some rather mild conditions on the moments of \((f(Z_t))_{t=1}^{\infty}\), it is possible to derive an AsympCS for a time-varying treatment effect \( \tilde{\psi}_t \) (see Fig. 6). Nevertheless, under the commonly considered regime where the treatment effect is constant \( \psi_1 = \psi_2 = \cdots = \psi \), we have that (25) forms a \((1 - \alpha)\)-AsympCS for \( \psi \). Let us now consider the extension to observational studies with unknown propensity scores.

**Theorem 3.4** (Confidence sequences for time-varying effects in observational studies). Consider the same setup as Theorem 3.3 but with \( \pi(x) \) no longer known. Suppose we have estimators \((\hat{\pi}_t, \hat{\mu}_t^1, \hat{\mu}_t^0)\) such that Assumption 5 holds. Finally, suppose that the conditions of Corollary 2.1 hold but with \((Y_t)_{t=1}^{\infty}\) replaced by the efficient influence functions \((f(Z_t))_{t=1}^{\infty}\). Then,

\[
\tilde{\psi}_t^* \pm \sqrt{\frac{2t(2\rho^2\text{var}_t(f) + 1)}{\rho^2}} \log \left( \frac{\sqrt{t\rho^2\text{var}_t(f) + 1}}{\alpha} \right)
\]

forms a \((1 - \alpha)\)-AsympCS for the cumulative average of individual treatment effects \( \tilde{\psi}_t := \frac{1}{t} \sum_{i=1}^{t} \psi_i \).

The proof can be found in Appendix A.4. Similar to Theorem 3.3, the important takeaway lies in the fact that it is possible to derive an AsympCS for the time-varying treatment effect \( \tilde{\psi}_t \), but that nevertheless reduces to an AsympCS for \( \psi \) in the familiar regime of a constant ATE \( \psi_1 = \psi_2 = \cdots = \psi \).

**Remark 3** (Avoiding sample splitting via martingale AsympCSs). The reader may wonder whether it is possible to simply plug in a *predictable* estimate of \( \hat{\mu}_t \) — i.e. so that \( \hat{\mu}_t \) only depends on \( Z_t^{-1} \) — and employ the Lindeberg-type martingale AsympCS of Theorem 2.3 in place of Corollary 2.1, thereby sidestepping the need for sequential sample splitting and cross-fitting altogether. Indeed, it is possible to derive such an analogue of Theorem 3.3 (and with more technical effort, of Theorem 3.4), but to avoid overloading the current paper, we leave this as a practically interesting extension for future work.
4 Application to the effects of IV fluid caps in sepsis patients

Let us now illustrate the use of doubly robust confidence sequences by sequentially estimating the effect of fluid-restrictive strategies on mortality in an observational study of real sepsis patients. We will use data from the Medical Information Mart for Intensive Care III (MIMIC-III), a freely available database consisting of health records associated with more than 45,000 critical care patients at the Beth Israel Deaconess Medical Center [19, 28]. The data are rich, containing demographics, vital signs, medications, and mortality, among other information collected over the span of 11 years.

Following Shahn et al. [40], we aim to estimate the effect of restricting intravenous (IV) fluids within 24 hours of intensive care unit (ICU) admission on 30-day mortality in sepsis patients. In particular, we considered patients at least 16 years of age satisfying the Sepsis-3 definition — i.e. those with a suspected infection and a Sequential Organ Failure Assessment (SOFA) score of at least 2 [41]. Sepsis-3 patients can be obtained from MIMIC-III using SQL scripts provided by Johnson and Pollard [18], but we provide detailed instructions for reproducing our data collection and analysis process on GitHub. This resulted in a total of 5231 sepsis patients, each of whom received out-of-hospital followup of at least 90 days.

We considered IV fluid intake within 24 hours of ICU admission \( L^{24h} \). To construct a binary treatment \( A \in \{ 0, 1 \} \), we dichotomized \( L^{24h} \) so that \( A_i = 1( L^{24h}_i \leq 6L ) \). The 30-day mortality \( Y \) was defined as 1 if the patient died within 30 days of hospital admission, and 0 otherwise. Baseline covariates \( X \) included for modelling consisted of the patients’ age and sex, whether they are diabetic, modified Elixhauser scores [53], and SOFA scores. We are interested in the causal estimand, 

\[
\psi := \mathbb{P}( Y \cdot L^{24h} \leq 6L = 1 ) - \mathbb{P}( Y \cdot L^{24h} > 6L = 1 ),
\]

which is the difference in average 30-day mortality that would be observed if all sepsis patients were randomly assigned an IV fluid level according to the lower truncated distribution \( \mathbb{P}( L^{24h} \leq l \mid l \leq 6L ) \) versus the upper truncated distribution \( \mathbb{P}( L^{24h} \leq l \mid l > 6L ) \) [8]. While this is technically a stochastic intervention effect, we have that under causal identification assumptions (IA1)–(IA3), \( \psi \) is identified as

\[
\hat{\psi} = \mathbb{E}( Y \mid X, A = 1 ) - \mathbb{E}( Y \mid X, A = 0 ),
\]

which is the same functional considered in the previous sections. Therefore, we can estimate \( \psi \) under the same assumptions and with the same techniques as Section 3.4. Similar to the simulations in Sections B.1 and B.2, we produced confidence sequences for \( \psi \) using unadjusted, parametric, and Super Learner estimators to demonstrate the impacts of different modelling choices on estimation (see Fig. 7).

Remark 4. These simple binary treatment and outcome variables were used so that the methods outlined in Section 3.4 are immediately applicable, but as we will discuss in Section 5, our confidence sequences may be used to sequentially estimate other causal functionals.

Our Super Learner-based confidence sequences cover the null treatment effect of 0 from the 1000th to the 5231st observed patient, and thus we cannot conclude with confidence whether 6L IV fluid caps have an effect on 30-day mortality in sepsis patients. Note that the Super Learner-based confidence sequences nearly drop below 0 after observing the 5231st patient’s outcome. If we were using fixed-time confidence intervals, we would need to resist the temptation to resume data collection (e.g. to see whether the null hypothesis \( H_0 : \psi = 0 \) could be rejected with a slightly larger sample size) as this would inflate type-I error rates (as seen in Fig. 1). On the other hand, our confidence sequences permit exactly this form of continued sampling.

\[ \text{github.com/WannabeSmith/drconfseq/tree/main/paper_plots/sepsis} \]
Figure 7: Three 90%-AsympCSs for the effect of capped IV fluid intake (defined as \( \leq 6 \) litres) on 30-day mortality using the same three estimators as those outlined for Fig. 5. Notice that an analysis using unadjusted estimators would conclude that the treatment effect is negative after observing fewer than 1500 patients.

5 Extensions to general functional estimation

The discussion thus far has been focused on deriving confidence sequences for the ATE in the context of causal inference. However, the tools presented in this paper are more generally applicable to any pathwise differentiable functional with positive semiparametric information bound. Here we list some prominent examples in causal inference:

- Stochastic interventions: \( E(Y^{A+\delta}) = \int E(Y \mid X = x, A = a + \delta)p(a \mid X = x) \, da \, dp(x) \);
- Complier-average effects: \( E(Y^1 - Y^0 \mid A^1 > A^0) = \frac{E[E(Y \mid X, R = 1) - E(Y \mid X, R = 0)]}{E[E(A \mid X, R = 1) - E(A \mid X, R = 0)]} \);
- Time-varying: \( E(Y^{S_s}) = \int \ldots \int E(Y \mid X_s, A_s = \pi_s) \prod_{s=1}^S dP(X_s \mid X_{s-1}, A_{s-1} = \pi_{s-1}) \);
- Mediation effects: \( E(Y^{am}) = E(E(Y \mid X, A = a, M = m)) \),

where \( R \) is an instrumental variable, \( M \) is a mediator, and the notation \( \hat{a}_s \) is shorthand for the tuple \((a_1, a_2, \ldots, a_s)\). Some examples outside of causal inference include

- Expected density: \( E\{p(X)\} \);
- Entropy: \( -E\{\log p(X)\} \);
- Expected conditional variance: \( E\{\text{var}(Y \mid X)\} \),

where \( p \) is the density of the random variable \( X \).

All of the aforementioned problems, including estimation of the ATE in Section 3 can be written in the following general form. Suppose \( Z_1, Z_2, \ldots \sim Q \) and let \( \theta(Q) \) be some functional (such as those listed above) of the distribution \( Q \). In the case of a finite sample size \( n \), \( \hat{\theta}_n \) is said to be an asymptotically linear estimator [45] for \( \theta \) if

\[
\hat{\theta}_n - \theta = \frac{1}{n} \sum_{i=1}^n \phi(Z_i) + o_P \left( \frac{1}{\sqrt{n}} \right),
\]
where \( \phi \) is the influence function of \( \hat{\theta}_n \). When the sample size is not fixed in advance, we may analogously say that \( \hat{\theta}_t \) is an *asymptotically linear time-uniform estimator* if instead,

\[
\hat{\theta}_t - \theta = \frac{1}{t} \sum_{i=1}^{t} \phi(Z_i) + o \left( \sqrt{\frac{\log \log t}{t}} \right),
\]

with \( \phi \) being the same influence function as before. For example, in the case of the ATE with \( (Z_t)_{t=1}^{T} \sim P \), we presented an efficient estimator \( \hat{\psi}_t \) which took the form,

\[
\hat{\psi}_t - \psi = \frac{1}{t} \sum_{i=1}^{t} (f(Z_i) - \psi) + o \left( \sqrt{\frac{\log \log t}{t}} \right),
\]

where \( f \) is the uncentered efficient influence function (EIF) defined in (16). In order to justify that the remainder term is indeed \( o \left( \sqrt{\log \log t/t} \right) \), we used sequential sample splitting and additional analysis in the randomized and observational settings (see the proofs in Sections A.3 and A.3.2 for more details). In general, as long as an estimator \( \hat{\theta}_t \) for \( \theta \) has the form (27), we may derive AsympCSs for \( \theta \) as a simple corollary of Theorem 2.2.

**Corollary 5.1.** Suppose \( \hat{\theta}_t \) is an asymptotically linear time-uniform estimator of \( \theta \) with influence function \( \phi \), that is, satisfying (27). Additionally, suppose that \( E|\phi(Z_i)|^q < \infty \) for some \( q > 2 \). Then,

\[
\hat{\theta}_t \pm \sqrt{\text{var}(\hat{\theta}_t)} \sqrt{\frac{2(t \rho^2 + 1)}{t^2 \rho^2}} \log \left( \frac{\sqrt{t \rho^2 + 1}}{\alpha} \right)
\]

forms a \((1 - \alpha)\)-AsympCS for \( \theta \).

If desired, the iterated logarithm boundary of Proposition 2.1 can be used here in place of Theorem 2.2. If computing \( \hat{\theta}_t \) additionally involves the estimation of a nuisance parameter \( \eta \) such as in Theorems 3.1 and 3.2, this must be handled carefully on a case-by-case basis where sequential sample splitting and cross fitting (Section 3.2) may be helpful, and higher moments on \( \phi(Z_i) \) may be needed.

6 Conclusion

This paper introduced the notion of an “asymptotic confidence sequence” as the time-uniform analogue of an asymptotic confidence interval based on the central limit theorem. We derived an explicit universal asymptotic confidence sequence for the mean from iid observations under weak moment assumptions by appealing to the strong invariance principles of Komlós et al. [22, 23] and Major [26]. These results were extended to the setting where observations’ distributions (including means and variances) can vary over time under martingale dependence, such that our confidence sequences capture a moving parameter — the running average of the conditional means so far. We then applied the aforementioned results to the problem of doubly robust sequential inference for the average treatment effect in both randomized experiments and observational studies under iid sampling. Finally, we showed how these causal applications remain valid in the non-iid setting where distributions change over time, in which case our confidence sequences capture a running average of individual treatment effects. The aforementioned results will enable researchers to continuously monitor sequential experiments — such as clinical trials and online A/B tests — as well as sequential observational studies even if treatment effects do not remain stationary over time.

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A Proofs of the main results

A.1 Proof of Theorem 2.2

We first introduce two lemmas which will later be used in the main proof of Theorem 2.2.

Lemma A.1 (Almost-sure approximation of the standard deviation under four moments). Suppose $(Y_t)_{t=1}^\infty \sim \mathbb{P}$ and let $\bar{\mu}_t = \frac{\sum_{i=1}^t Y_i}{t}$. Consider the sample standard deviation estimator for all $t \geq 2$,

$$\hat{\sigma}_t := \sqrt{\frac{\sum_{i=1}^t (Y_i - \bar{\mu}_t)^2}{t}}.$$

If $\mathbb{P}$ has a finite fourth moment, then

$$\sigma = \hat{\sigma}_t + O \left( \left( \frac{\log \log t}{t} \right)^{1/4} \right).$$

Proof. Define the partial sums,

$$S_t := \sum_{i=1}^t (Y_i - \mu), \quad S'_t := \sum_{i=1}^t [Y_i^2 - (\mu^2 + \sigma^2)].$$

Now, consider the quantity,

$$\frac{1}{t} S'_t - \left( \frac{1}{t} S_t \right)^2 = \frac{1}{t} \sum_{i=1}^t [Y_i^2 - (\mu^2 + \sigma^2)] - \left( \frac{1}{t} \sum_{i=1}^t (Y_i - \mu) \right)^2$$

$$= \frac{1}{t} \sum_{i=1}^t Y_i^2 - \mu^2 - \sigma^2 - \bar{Y}_t^2 + 2\bar{Y}_t \mu - \mu^2$$

$$= \hat{\sigma}_t^2 + (-\mu^2 - \sigma^2 + 2\bar{Y}_t \mu - \mu^2)$$

$$= \hat{\sigma}_t^2 - \sigma^2 + 2\mu(\bar{Y}_t - \mu)$$

$$= \hat{\sigma}_t^2 - \sigma^2 + \frac{2\mu}{t} S_t.$$

Therefore, we have by the law of the iterated logarithm (LIL),

$$\sigma^2 - \hat{\sigma}_t^2 = -\frac{1}{t} S'_t + \left( \frac{1}{t} S_t \right)^2 + \frac{2\mu}{t} S_t,$$

$$= O \left( \sqrt{\frac{\log \log t}{t}} \right) + O \left( \frac{\log \log t}{t} \right) + O \left( \sqrt{\frac{\log \log t}{t}} \right)$$

$$= O \left( \frac{\log \log t}{t} \right).$$

Finally, using the fact that $|\sqrt{a} - \sqrt{b}| \leq \sqrt{|a-b|}$ for all $a, b \geq 0$, we have that

$$\sigma - \hat{\sigma}_t = O \left( \left( \frac{\log \log t}{t} \right)^{1/4} \right),$$

completing the proof. \qed
Lemma A.2 (Strong Gaussian approximation of the sample average). Let \((Y_t)_{t=1}^\infty\) be an iid sequence of random variables with mean \(\mu\), variance \(\sigma^2\), and \(q > 2\) finite absolute moments. Suppose \(\hat{\sigma}_t\) is an almost-surely consistent estimator for \(\sigma\) (such as the sample standard deviation). Then (after sufficiently enriching the probability space), there exist iid Gaussian random variables \((G_t)_{t=1}^\infty\) such that

\[
\frac{1}{t} \sum_{i=1}^{t} (Y_i - \mu) = \frac{\hat{\sigma}_t}{t} \sum_{i=1}^{t} G_i + \varepsilon_t
\]

where \(\varepsilon_t = o(\sqrt{\log \log t/t})\) if \(q > 2\), and \(\varepsilon_t = O((\log \log t/t)^{3/4})\) if \(q \geq 4\).

**Proof.** First, by Komlós et al. and Major’s strong approximation theorems (KMT) [22, 23, 26] we have that (after sufficiently enriching the probability space) there exist iid Gaussian random variables \((G_t)_{t=1}^\infty\) such that

\[
\frac{1}{t} \sum_{i=1}^{t} (Y_i - \mu) = \frac{\sigma}{t} \sum_{i=1}^{t} G_i + \kappa_t
\]

with \(\kappa_t = O(\log t/t)\) if \(Y_1\) has a finite moment generating function, and \(\kappa_t = o(t^{1/q-1})\) if \(Y_1\) has \(q > 2\) finite absolute moments. Since \((G_t)_{t=1}^\infty\) have mean zero and unit variance, we have by the LIL that

\[
\frac{1}{t} \sum_{i=1}^{t} G_i = O\left( \frac{\log \log t}{t} \right). \tag{28}
\]

**Case I:** If \(Y_1\) has \(q > 2\) finite absolute moments, then by the strong law of large numbers, \(\hat{\sigma}_t \stackrel{a.s.}{\to} \sigma\). Combining this fact with with (28), we have

\[
\frac{1}{t} \sum_{i=1}^{t} (Y_i - \mu) = \frac{\hat{\sigma}_t}{t} \sum_{i=1}^{t} G_i + \kappa_t + o \left( \frac{\log \log t}{t} \right)
\]

\[
= \frac{\hat{\sigma}_t}{t} \sum_{i=1}^{t} G_i + \kappa_t + O \left( \frac{\log \log t}{t} \right) \tag{29}
\]

**Case II:** If \(Y_1\) has at least 4 moments, then by Lemma A.1,

\[
\hat{\sigma}_t - \sigma = O \left( (\log \log t/t)^{1/4} \right).
\]

Combining the above with (28) and (29), we have

\[
\frac{1}{t} \sum_{i=1}^{t} (Y_i - \mu) = \left( \frac{\hat{\sigma}_t}{t} + O \left( (\log \log t/t)^{1/4} \right) \right) \sum_{i=1}^{t} G_i + \kappa_t + O \left( \frac{\log \log t}{t} \right)^{3/4}
\]

\[
= \frac{\hat{\sigma}_t}{t} \sum_{i=1}^{t} G_i + \kappa_t + O \left( \left( \frac{\log \log t}{t} \right)^{3/4} \right),
\]

which completes the proof.

**Proof of the main theorem** The proof proceeds in 3 steps. First, we use the fact that for any martingale \(M_t(\lambda)\), we have that \(\int_{\mathbb{R}} M_t(\lambda) dF(\lambda)\) is also a martingale where \(F\) is any probability
distribution on $\mathbb{R}$ [15, 16]. We apply this fact to an exponential Gaussian martingale and use a Gaussian density $f(\lambda; 0, \rho^2)$ as the mixing distribution. Second, we apply Ville’s inequality [54] to this mixture exponential Gaussian martingale to obtain Robbins’ normal mixture confidence sequence [32]. Third, we use Lemma A.2 to approximate $\sum_{i=1}^t (Y_i - \mu)$ by a cumulative sum of Gaussian random variables and apply the results from steps 1 and 2.

**Proof.** **Step 1.** Let $(G_i)_{i=1}^\infty$ be a sequence of iid standard Gaussian random variables and define their cumulative sum $W_i := \sum_{i=1}^t G_i$. Write the exponential process for any $\lambda \in \mathbb{R}$,

$$M_t(\lambda) := \exp \{\lambda W_t - t\lambda^2/2\}.$$  

It is well-known that $M_t(\lambda)$ is a nonnegative martingale starting at $M_0 = 1$ with respect to the canonical filtration $(\mathcal{F}_t)_{t=0}^\infty$ where $\mathcal{F}_t := \sigma(X_t)$ is the sigma-field generated by $X_1, \ldots, X_t$ and $\mathcal{F}_0$ is the trivial sigma-field [32]. Moreover, for any probability distribution $F(\lambda)$ on $\mathbb{R}$, we also have that the mixture,

$$\int_{\lambda \in \mathbb{R}} M_t(\lambda) dF(\lambda)$$  

is a nonnegative martingale with initial value one with respect to the canonical filtration [32]. In particular, consider the Gaussian probability distribution function $f(\lambda; 0, \rho^2)$ with mean zero and variance $\rho^2 > 0$ as the mixing distribution. The resulting martingale can be written as

$$M_t := \int_{\lambda \in \mathbb{R}} \exp \left\{ \lambda W_t - \frac{t\lambda^2}{2} \right\} f(\lambda; 0, \rho^2) d\lambda$$

$$= \frac{1}{\sqrt{2\pi \rho^2}} \int_{\lambda} \exp \left\{ \lambda W_t - \frac{t\lambda^2}{2} \right\} \exp \left\{ -\frac{\lambda^2}{2\rho^2} \right\} d\lambda$$

$$= \frac{1}{\sqrt{2\pi \rho^2}} \int_{\lambda} \exp \left\{ \lambda W_t - \frac{\lambda^2(t\rho^2 + 1)}{2\rho^2} \right\} d\lambda$$

$$= \frac{1}{\sqrt{2\pi \rho^2}} \int_{\lambda} \exp \left\{ -a(\lambda - \frac{b}{2\lambda}) \right\} d\lambda$$

by setting $a := t\rho^2 + 1$ and $b := \rho^2 W_t$. Focusing on the integrand and completing the square, we have

$$\exp \left\{ \frac{-\lambda^2 + 2\lambda \frac{b}{a} + \left( \frac{b}{a} \right)^2 - \left( \frac{b}{a} \right)^2}{2\rho^2/a} \right\} = \exp \left\{ \frac{-(\lambda - \frac{b}{a})^2}{2\rho^2/a} + \frac{a \left( \frac{b}{a} \right)^2}{2\rho^2} \right\}$$

$$= \exp \left\{ \frac{-(\lambda - \frac{b}{a})^2}{2\rho^2/a} \right\} \exp \left\{ \frac{\frac{b^2}{2\rho^2/a}}{2\rho^2/a} \right\}.$$  

Plugging this back into the integral and multiplying the entire quantity by $\frac{a^{-1/2}}{\sqrt{a}}$, we finally get the closed-form expression of the mixture exponential Wiener process,

$$M_t := \frac{1}{\sqrt{2\pi \rho^2/a}} \int_{\lambda \in \mathbb{R}} \exp \left\{ -\frac{(\lambda - \frac{b}{a})^2}{2\rho^2/a} \right\} d\lambda \exp \left\{ \frac{\frac{b^2}{2\rho^2/a}}{\sqrt{a}} \right\}$$

$$= \exp \left\{ \frac{\rho^2 W_t^2}{2(\rho^2 + 1)} \right\} \exp \left\{ \frac{\rho^2}{\sqrt{a}} \right\} \exp \left\{ \frac{\rho^2}{\sqrt{a}} \right\}.$$  

$$= \frac{\rho^2_{W_t^2}}{\sqrt{t\rho^2 + 1}}.$$  

$$= \frac{\rho^2_{W_t^2}}{\sqrt{t\rho^2 + 1}}.$$  

(30)
Step 2. Since $M_t$ is a nonnegative martingale with initial value one, we have by Ville’s inequality [54] that
\[ P(\forall t \geq 1, \ M_t < 1/\alpha) \geq 1 - \alpha. \]
Writing this out explicitly for $M_t$ and solving for $W_t$ algebraically, we have that
\[ P \left( \forall t \geq 1, \ \frac{\rho^2 W_t^2}{2(t/\rho^2 + 1)} < \log(1/\alpha) + \log \left( \frac{\sqrt{t/\rho^2 + 1}}{\alpha} \right) \right) \geq 1 - \alpha. \]

Step 3. First, note that by the triangle inequality,
\[ \left| \frac{1}{t} \sum_{i=1}^{t} Y_i - \mu \right| \leq \left| \frac{1}{t} \sum_{i=1}^{t} (Y_i - \mu) - \tilde{\sigma}_t \sum_{i=1}^{t} G_i \right| + \tilde{\sigma}_t \left| \frac{1}{t} \sum_{i=1}^{t} G_i \right|, \]
and thus by Lemma A.2 and Step 2, we have with probability at least $(1 - \alpha)$,
\[ \forall t \geq 1, \ \left| \frac{1}{t} \sum_{i=1}^{t} Y_i - \mu \right| < \tilde{\sigma}_t \sqrt{\frac{2(t/\rho^2 + 1)}{t^2 \rho^2} \log \left( \frac{\sqrt{t/\rho^2 + 1}}{\alpha} \right)} + \varepsilon_t, \]
where $\varepsilon_t$ is defined as in Lemma A.2. This completes the proof. \hfill \Box

### A.2 Proof of Theorem 2.3

First, we present a lemma that is implicit in the proof of Strassen [42, Theorem 4.4].

**Lemma A.3** (Strong approximation under martingale dependence). Let $(Y_i)_{i=1}^\infty$ be a sequence of random variables with conditional means and variances given by $\mu_t = E(Y_t | Y_{t-1}^{t-1})$ and $\sigma_t^2 = \text{var}(Y_t | Y_{1}^{t-1})$, respectively. Let $V_t = \sum_{i=1}^{t} \sigma_i^2$ be the cumulative conditional variance process, and suppose that $V_t \to \infty$ almost surely. Furthermore, assume that the Lindeberg-type condition given in Eq. (11) holds. Then, on a potentially enriched probability space, there exist iid standard Gaussians $(G_t)_{t=1}^\infty$ such that
\[ \frac{1}{t} \sum_{i=1}^{t} (Y_i - \mu_i) - \frac{1}{t} \sum_{i=1}^{t} \sigma_i G_i = o \left( \frac{V_t^{3/8} \log V_t}{t} \right). \]  

**Proof.** The proof centrally relies on Strassen [42, Eq. 159] which states that on a potentially enriched probability space,
\[ \sum_{i=1}^{t} (Y_i - \mu_i) = \xi(V_t) + o(h(V_t)) \]  

where $\xi$ is a standard Brownian motion, and $h(v) = (vf(v))^{1/4} \log v$ for some $f(\cdot)$ so that $f(v)$ is increasing in $v$, but $f(v)/v$ is decreasing. For the purposes of this proof, we set $f(v) = v^{1/2}$, and hence $h(v) = v^{3/8} \log v$. Since a standard Brownian motion evaluated at $V_t = \sum_{i=1}^{t} \sigma_i^2$ is equal in distribution to the discrete time process $\sum_{i=1}^{t} \sigma_i G_i$ at each $t$, we have that
\[ \sum_{i=1}^{t} (Y_i - \mu_i) = \sum_{i=1}^{t} \sigma_i G_i + o \left( V_t^{3/8} \log V_t \right), \]  

which completes the proof after dividing both sides by $t$. \hfill \Box
With Lemma A.3 in mind, we can now prove the main result (Theorem 2.3).

**Proof of Theorem 2.3.** The proof proceeds in three steps. First, we use a similar technique to that of Theorem 2.2 to obtain a nonnegative martingale for dependent Gaussian observations with time-varying means and variances and a martingale dependence structure. Second, we combine Step 1 with Assumption 2 and Lemma A.3 to obtain an AsympCS in terms of the (unknown) time-varying conditional variances $\tilde{\sigma}_t^2$. Second and finally, we use Assumptions 1 and 3 to obtain the same AsympCS but with $\tilde{\sigma}_t^2$ replaced by the variance estimator $\hat{\sigma}_t^2$.

**Step 1: A Gaussian martingale for time-varying means and variances.** Let $(G_i)_{i=1}^\infty$ be a sequence of iid standard Gaussian random variables. Define the conditional variance $\sigma_t^2 := \text{var}(Y_t | Y_{1:t-1})$ and note that

$$\tilde{M}_t(\lambda) := \exp\left\{ \sum_{i=1}^t (\lambda \sigma_i G_i - \lambda^2 \sigma_i^2 / 2) \right\}$$

is a nonnegative martingale starting at one. Mixing over $\lambda$ with the probability density $dF(\lambda)$ of a mean-zero Gaussian with variance $\rho^2$ as in the proof of Theorem 2.2, we have that

$$\tilde{M}_t := \int_{\lambda \in \mathbb{R}} \tilde{M}_t(\lambda) dF(\lambda) = \exp\left\{ \frac{\rho^2 (\sum_{i=1}^t \sigma_i G_i)^2}{2(V_t \rho^2 + 1)} \right\} \cdot (V_t \rho^2 + 1)^{-1/2}$$

is also a martingale. By Ville’s inequality for nonnegative (super)martingales, we have that $\mathbb{P}(\exists t : \tilde{M}_t \geq 1/\alpha) \leq \alpha$ and hence with probability at least $(1 - \alpha)$,

$$\forall t \geq 1, \quad \frac{1}{t} \sum_{i=1}^t \sigma_i G_i < \sqrt{\frac{2(V_t \rho^2 + 1)}{t^2 \rho^2} \log \left( \frac{\sqrt{V_t \rho^2 + 1}}{\alpha} \right)}. \quad (34)$$

**Step 2: Strong approximation via Strassen [42].** By Lemma A.3, under Assumptions 1 and 2, we have that after enriching the probability space if needed, there exist iid standard Gaussians $(G_i)_{i=1}^\infty$ such that

$$\frac{1}{t} \sum_{i=1}^t (Y_i - \mu_i) = \frac{1}{t} \sum_{i=1}^t \sigma_i G_i + o \left( \frac{V_t^{3/8} \log V_t}{t} \right). \quad (35)$$

Therefore, with probability at least $(1 - \alpha)$,

$$\forall t \geq 1, \quad \frac{1}{t} \sum_{i=1}^t (Y_i - \mu_i) < \sqrt{\frac{2(V_t \rho^2 + 1)}{t^2 \rho^2} \log \left( \frac{\sqrt{V_t \rho^2 + 1}}{\alpha} \right) + o \left( \frac{(V_t)^{3/8} \log V_t}{t} \right)}. \quad (36)$$

In particular, by Assumption 1, we have that $(V_t)^{3/8} \log V_t = o \left( \sqrt{V_t \log V_t} \right)$ as $t \to \infty$, and hence

$$\left( \tilde{\mu}_t \pm \sqrt{\frac{2(V_t \rho^2 + 1)}{t^2 \rho^2} \log \left( \frac{\sqrt{V_t \rho^2 + 1}}{\alpha} \right)} \right)$$

forms a $(1 - \alpha)$-AsympCS for $\tilde{\mu}_t$. 

30
Step 3: Deriving an AsympCS in terms of the empirical variance $\hat{\sigma}^2_i$. Writing out the margin of (36) combined with the assumption $\hat{\sigma}^2_i - \hat{\sigma}^2_i = o(\hat{\sigma}^2_i)$ and recalling that $V_i = t\hat{\sigma}_i$, we have

$$\sqrt{\frac{2(V_i\rho^2 + 1)}{t^2\rho^2}} \log \left( \frac{\sqrt{V_i\rho^2 + 1}}{\alpha} \right) = \sqrt{\frac{2(t(\hat{\sigma}^2_i + o(\hat{\sigma}^2_i))\rho^2 + 1)}{t^2\rho^2}} \log \left( \frac{\sqrt{(t(\hat{\sigma}^2_i + o(\hat{\sigma}^2_i))\rho^2 + 1)}}{\alpha} \right)$$

$$= \sqrt{\frac{t(\hat{\sigma}^2_i + o(\hat{\sigma}^2_i))\rho^2 + 1}{t^2\rho^2}} \log \left( \frac{t(\hat{\sigma}^2_i + o(\hat{\sigma}^2_i))\rho^2 + 1}{\alpha^2} \right)$$

$$= \sqrt{\frac{t\hat{\sigma}^2_i\rho^2 + o(t\hat{\sigma}^2_i) + 1}{t^2\rho^2}} \log \left( \frac{t\hat{\sigma}^2_i\rho^2 + o(t\hat{\sigma}^2_i) + 1}{\alpha^2} \right)$$

$$= \sqrt{\frac{(t\hat{\sigma}^2_i\rho^2 + 1)/t}{t^2\rho^2}} \log \left( \frac{(t\hat{\sigma}^2_i\rho^2 + 1)/t}{\alpha^2} \right). \quad (37)$$

Focusing on the logarithmic factor, we have

$$\log \left( \frac{t\hat{\sigma}^2_i\rho^2 + o(t\hat{\sigma}^2_i) + 1}{\alpha^2} \right) = \log \left( \frac{1 + t\hat{\sigma}^2_i\rho^2}{\alpha^2} + o(t\hat{\sigma}^2_i) \right)$$

$$= \log \left( \frac{1 + t\hat{\sigma}^2_i\rho^2}{\alpha^2} \right) [1 + o(1)]$$

$$= \log \left( \frac{1 + t\hat{\sigma}^2_i\rho^2}{\alpha^2} \right) + \log (1 + o(1))$$

$$= \log \left( \frac{1 + t\hat{\sigma}^2_i\rho^2}{\alpha^2} \right) + o(1) \quad (38)$$

where the last line follows from the Taylor expansion $\log(1 + x) = x + o(1)$ for $|x| < 1$. Combining (37) and (38), we have that the margin of (36) can be written as

$$\sqrt{\frac{2(V_i\rho^2 + 1)}{t^2\rho^2}} \log \left( \frac{\sqrt{V_i\rho^2 + 1}}{\alpha} \right) = \sqrt{\frac{2(t\hat{\sigma}^2_i\rho^2 + 1)/t}{t^2\rho^2}} \log \left( \frac{1 + t\hat{\sigma}^2_i\rho^2}{\alpha^2} \right) + o(1)$$

$$= \sqrt{\frac{t\hat{\sigma}^2_i\rho^2 + 1}{t^2\rho^2}} \log \left( \frac{1 + t\hat{\sigma}^2_i\rho^2}{\alpha^2} + o(V_i/t^2) \right) + o(V_i/t^2)$$

$$= \sqrt{\frac{2(t\hat{\sigma}^2_i\rho^2 + 1)}{t^2\rho^2}} \log \left( \frac{\sqrt{1 + t\hat{\sigma}^2_i\rho^2}}{\alpha} \right) + o\left( V_i \log V_i/t^2 \right)$$

$$\leq \sqrt{\frac{2(t\hat{\sigma}^2_i\rho^2 + 1)}{t^2\rho^2}} \log \left( \frac{\sqrt{1 + t\hat{\sigma}^2_i\rho^2}}{\alpha} \right) + o \left( \frac{V_i \log V_i}{t} \right), \quad (39)$$

where the last inequality follows from $\sqrt{a + b} \leq \sqrt{a} + \sqrt{b}$ for $a, b \geq 0$. In particular, this means that

$$(\hat{\mu} \pm \hat{\mathcal{B}}) := \left( \hat{\mu} \pm \sqrt{\frac{2(t\hat{\sigma}^2_i\rho^2 + 1)}{t^2\rho^2}} \log \left( \frac{\sqrt{1 + t\hat{\sigma}^2_i\rho^2}}{\alpha} \right) + o \left( \frac{V_i \log V_i}{t} \right) \right)$$

forms a nonasymptotic $(1 - \alpha)$-CS for $\hat{\mu}_t$, meaning $\mathbb{P}\left( \exists t \geq 1 : \hat{\mu}_t \notin (\hat{\mu} \pm \hat{\mathcal{B}}) \right) \leq \alpha$. Combined with Assumption 3, we have that

$$(\hat{\mu} \pm \hat{\mathcal{B}}) := \left( \hat{\mu} \pm \sqrt{\frac{2(t\hat{\sigma}^2_i\rho^2 + 1)}{t^2\rho^2}} \log \left( \frac{\sqrt{1 + t\hat{\sigma}^2_i\rho^2}}{\alpha} \right) \right)$$
forms a $(1 - \alpha)$-AsympCS for $\hat{\mu}_t$ since $\hat{h}_t = \sqrt{\log \log t}$. This completes the proof.

\[\square\]

A.3 Proof of Theorems 3.1 and 3.2

In the proofs that follow, we will make extensive use of some convenient notation, namely the sample average operator $\mathbb{P}_t f(Z) \equiv \frac{1}{t} \sum_{i=1}^t f(Z_i)$ and the conditional expectation operator $\mathbb{E}(\hat{f}(Z) = \mathbb{E}(f(Z_i) \mid Z_1, \ldots, Z_n)$ where $Z_1, \ldots, Z_n$ are the data used to construct $\hat{f}$.

First, let us analyze the almost-sure behavior of the doubly robust estimator $\hat{\psi}_t$ for the average treatment effect $\psi$.

Lemma A.4 (Decomposition of $\hat{\psi}_t - \psi$). Let $\hat{\psi}_t := \mathbb{P}_t(\hat{f}_{T^{\text{trn}}}) = \frac{1}{T} \sum_{i=1}^T \hat{f}_{i}(Z_{i}^{\text{eval}})$ be a (possibly misspecified) estimator of $\psi := \mathbb{P}(f) = \mathbb{E}(f(Z^{\text{eval}}))$ based on $(Z_1^{\text{eval}}, \ldots, Z_T^{\text{eval}})$ where $\hat{f}_{i}$ can be any estimator built from $(Z_1^{\text{trn}}, \ldots, Z_T^{\text{trn}})$ and $f : Z \rightarrow \mathbb{R}$ any function. Furthermore, assume that there exists $\hat{f}$ such that $\|\hat{f}_{T^{\text{trn}}} - \hat{f}\|_{L_2(\mathbb{P})} \rightarrow 0$. In other words, $\hat{f}_{T^{\text{trn}}}$ is an estimator of $f$ but may instead converge to $\hat{f}$. Then we have the decomposition,

$$\hat{\psi}_t - \psi = \Gamma_{t}^{\text{SA}} + \Gamma_{t}^{\text{EP}} + \Gamma_{t}^{\text{B}}$$

where

$$\Gamma_{t}^{\text{SA}} := (\mathbb{P}_T - \mathbb{P})\hat{f}$$

is the centered sample average term,

$$\Gamma_{t}^{\text{EP}} := (\mathbb{P}_T - \mathbb{P})(\hat{f} - \bar{f})$$

is the empirical process term, and

$$\Gamma_{t}^{\text{B}} := \mathbb{P}(\bar{f} - f)$$

is the bias term.

Proof. By definition of the quantities involved, we decompose

$$\hat{\psi}_t - \psi = \mathbb{P}_T(\hat{f}_{T^{\text{trn}}}) - \mathbb{P}(f)$$

$$= (\mathbb{P}_T - \mathbb{P})(\hat{f}_{T^{\text{trn}}}) + \mathbb{P}(\hat{f}_{T^{\text{trn}}} - f)$$

$$= (\mathbb{P}_T - \mathbb{P})(\hat{f}_{T^{\text{trn}}} - \bar{f}) + \mathbb{P}(\bar{f} - f)$$

$$\Gamma_{t}^{\text{EP}} \quad \Gamma_{t}^{\text{SA}} \quad \Gamma_{t}^{\text{B}}$$

which completes the proof. \[\square\]

Now, let us analyze the almost-sure behaviour of the empirical process term $\Gamma_{t}^{\text{EP}}$ and the bias term $\Gamma_{t}^{\text{B}}$ to show that they vanish asymptotically at sufficiently fast rates. First, let us examine $\Gamma_{t}^{\text{EP}}$.

Lemma A.5 (Almost sure convergence of $\Gamma_{t}^{\text{EP}}$). Let $\mathbb{P}_T$ denote the empirical measure over $Z_{T^{\text{eval}}} := (Z_1^{\text{eval}}, \ldots, Z_T^{\text{eval}})$ and let $\hat{f}_{T^{\text{trn}}}(z)$ be any function estimated from a sample $D_{T^{\text{trn}}} = (Z_1^{\text{trn}}, Z_2^{\text{trn}}, \ldots, Z_T^{\text{trn}})$ which is independent of $\mathbb{D}_{T^{\text{eval}}}$. If $\hat{\mu}_t \in [\delta, 1 - \delta]$ almost surely, then,

$$\Gamma_{t}^{\text{EP}} := (\mathbb{P}_T - \mathbb{P})(\hat{f}_{T^{\text{trn}}} - \bar{f}) = O \left( \sqrt{\log \log t} \right)$$

In particular, if $\|\hat{\mu}_t - \bar{f}\|_{L_2(\mathbb{P})} = o(1)$ for each $a$, then we have that $\Gamma_{t}^{\text{EP}}$ almost-surely converges to 0 at a rate of $o(\sqrt{\log \log t})$, but possibly faster.

The proof proceeds in two steps. First, we use an argument from Kennedy et al. [21] and the law of the iterated logarithm to show $\Gamma_{t}^{\text{EP}} = O \left( \|\hat{f}_{T^{\text{trn}}} - \bar{f}\| \sqrt{\log \log t/\log t} \right)$. Second and finally, we upper bound $\|\hat{f}_{T^{\text{trn}}} - \bar{f}\|$ by $O \left( \sum_{a=0}^{1} \|\hat{\mu}_t - \bar{\mu}\| \right)$. 32
Therefore, we have that
\[ \mathbb{E} \left\{ \mathbb{P}(\hat{f}_T' - \bar{f}) \mid D_{\infty}^{\text{trn}}, S_{\infty}^{\text{trn}} \right\} = \mathbb{E}(\hat{f}_T' - \bar{f}) \mid D_{\infty}^{\text{trn}}, S_{\infty}^{\text{trn}} = \mathbb{P}(\hat{f}_T' - \bar{f}). \]

Now, we upper bound the conditional variance of a single summand,
\[ \text{var} \left\{ (1 - \mathbb{P})(\hat{f}_T' - \bar{f}) \mid D_{\infty}^{\text{trn}}, S_{\infty}^{\text{trn}} \right\} = \text{var} \left\{ (\hat{f}_T' - \bar{f}) \mid D_{\infty}^{\text{trn}}, S_{\infty}^{\text{trn}} \right\} \leq \|\hat{f}_T' - \bar{f}\|^2. \]

In particular, this means that
\[ \left( \frac{T(\mathbb{P}(\hat{f}_T') - \bar{f})}{\|\hat{f}_T' - \bar{f}\|} \mid D_{\infty}^{\text{trn}}, S_{\infty}^{\text{trn}} \right) \]

is a sum of iid random variables with conditional mean zero and conditional variance at most 1, and thus by the law of the iterated logarithm,
\[ \mathbb{P} \left( \lim_{t \to \infty} \frac{\pm \sqrt{T(\mathbb{P}(\hat{f}_T') - \bar{f})}}{\sqrt{2\log \log T}\|\hat{f}_T' - \bar{f}\|} \leq 1 \mid D_{\infty}^{\text{trn}}, S_{\infty}^{\text{trn}} \right) = 1. \]

Therefore, we have that
\[
\mathbb{P} \left( \frac{(\mathbb{P}(\hat{f}_T') - \bar{f})}{\|\hat{f}_T' - \bar{f}\|\sqrt{\log \log t/t}} = O(1) \mid D_{\infty}^{\text{trn}}, S_{\infty}^{\text{trn}} \right) = \mathbb{P} \left( \frac{\sqrt{T(\mathbb{P}(\hat{f}_T') - \bar{f})}}{\sqrt{\log \log t}\|\hat{f}_T' - \bar{f}\|} = O(1) \mid D_{\infty}^{\text{trn}}, S_{\infty}^{\text{trn}} \right) = \mathbb{P} \left( \lim_{t \to \infty} \frac{\|\hat{f}_T' - \bar{f}\|}{\|\hat{f}_T' - \bar{f}\|/\sqrt{\log \log t/t}} = O(1) \mid D_{\infty}^{\text{trn}}, S_{\infty}^{\text{trn}} \right) = 1. 
\]

Finally, by iterated expectation,
\[ \mathbb{P} \left( \frac{(\mathbb{P}(\hat{f}_T') - \bar{f})}{\|\hat{f}_T' - \bar{f}\|\sqrt{\log \log t/t}} = O(1) \right) = \mathbb{E} \left[ \mathbb{P} \left( \frac{(\mathbb{P}(\hat{f}_T') - \bar{f})}{\|\hat{f}_T' - \bar{f}\|\sqrt{\log \log t/t}} = O(1) \mid D_{\infty}^{\text{trn}}, S_{\infty}^{\text{trn}} \right) \right] = \mathbb{E}1 = 1, \]

which completes Step 1.

**Step 2.** Now, let us upper bound $\|\hat{f}_t - \bar{f}\|$ by $O \left( \sum_{n=0}^{\infty} \|\hat{\mu}^{(n)} - \bar{\mu}^{(n)}\| \right)$. To simplify the calculations which follow, define
\[ \hat{f}^1(Z_i) := \hat{\mu}^1(X_i) + \frac{A_i}{\pi(X_i)} \{ Y_i - \hat{\mu}^1(X_i) \} \quad \text{and} \quad \bar{f}^1(Z_i) := \bar{\mu}^1(X_i) + \frac{A_i}{\pi(X_i)} \{ Y_i - \bar{\mu}^1(X_i) \}. \]
Analogously define \( \hat{f}^0 \) and \( f^0 \) so that \( \hat{f} = \hat{f}^1 - \hat{f}^0 \) and \( f = f^1 - f^0 \). Writing out \( \| \hat{f}^1 - f^1 \| \),

\[
\| \hat{f}^1_t - f^1_t \| = \left\| \hat{\mu}^1_t + \frac{A_t}{\hat{\pi}_t} \{ Y - \hat{\mu}_t \} - \hat{\mu}^1_t - \frac{A_t}{\pi_t} \{ Y - \mu^1 \} \right\|
\]

\[
= \left\| \{ \hat{\mu}^1_t - \hat{\mu}_t \} \left\{ 1 - \frac{A}{\hat{\pi}_t} \right\} \right\|
\]

\[
\leq (i) \| \hat{\mu}^1_t - \hat{\mu}_t \| \left\| 1 - \frac{A}{\hat{\pi}_t} \right\|
\]

\[
\leq (ii) \frac{1}{\delta} \| \hat{\mu}^1_t - \hat{\mu}_t \| \left\| \hat{\pi}_t - A \right\|_1 \| \hat{\pi}_t - \frac{A}{\hat{\pi}_t} \| = O \left( \| \hat{\mu}^1_t - \hat{\mu}_t \| \right)
\]

where \((i)\) follows from Cauchy-Schwartz and \((ii)\) follows from the assumed bounds on \( \hat{\pi}(X) \). A similar story holds for \( \| \hat{f}^0_t - f^0_t \| \), and hence by the triangle inequality,

\[
\| \hat{f}_t - f \| = O \left( \sum_{a=0}^{1} \| \hat{\mu}^a_t - \hat{\mu}^a \| \right),
\]

which completes the proof. \( \square \)

Now, we examine the asymptotic almost-sure behaviour of the bias term, \( \Gamma_t^B \) by upper-bounding this term by a product of \( L_2(\hat{\pi}) \) estimation errors of nuisance functions.

**Lemma A.6** (Almost-surely bounding \( \Gamma_t^B \) by \( L_2(\hat{\pi}) \) errors of nuisance functions). Suppose \( \hat{\pi}_t \in [\delta, 1 - \delta] \) almost surely for some \( \delta > 0 \). Then

\[
\Gamma_t^B = O \left( \| \hat{\pi}_t - \pi \|_{L_2(\hat{\pi})} \left\{ \| \hat{\mu}^1_t - \mu^1 \|_{L_2(\hat{\pi})} + \| \hat{\mu}^0_t - \mu^0 \|_{L_2(\hat{\pi})} \right\} \right)
\]

This is an immediate consequence of the usual proof for \( O_p \) combined with the fact that expectations are real numbers, and thus stochastic boundedness is equivalent to almost-sure boundedness. For completeness, we recall this proof here as it is short and illustrative.

**Proof.** To simplify the calculations which follow, define

\[
\hat{f}^1(Z_i) := \hat{\mu}^1(X_i) + \frac{A_i}{\hat{\pi}(X_i)} \{ Y_i - \hat{\mu}^A(X_i) \} \quad \text{and} \quad f^1(Z_i) := \mu^1(X_i) + \frac{A_i}{\pi(X_i)} \{ Y_i - \mu^1(X_i) \}.
\]

Analogously define \( \hat{f}^0 \) and \( f^0 \) so that \( \hat{f} = \hat{f}^1 - \hat{f}^0 \) and \( f = f^1 - f^0 \). Therefore,

\[
\mathbb{P} \left( \hat{f}^1 - f^1 \right) = \mathbb{P} \left( \frac{A_t}{\hat{\pi}(X_t)} \{ Y - \hat{\mu}^A(X_t) \} + \hat{\mu}^1_t - \mu^1_t \right)
\]

\[
\leq (i) \mathbb{P} \left( \left( \frac{\hat{\pi}_t}{\pi} - 1 \right) \left( \hat{\mu}^1_t - \mu^1_t \right) \right)
\]

\[
\leq (ii) \frac{1}{\delta} \mathbb{P} \left( \left\| \hat{\pi}_t - \pi \right\| \left\| \hat{\mu}^1_t - \mu^1_t \right\| \right)
\]

\[
\leq (iii) \frac{1}{\delta} \left\| \hat{\pi}_t - \pi \right\|_{L_2(\hat{\pi})} \left\| \hat{\mu}^1_t - \mu^1_t \right\|_{L_2(\hat{\pi})},
\]

where \((i)\) and \((ii)\) follow by iterated expectation, \((iii)\) follows from the assumed bounds on \( \hat{\pi} \), and \((iv)\) by Cauchy-Schwarz. Similarly, we have

\[
\mathbb{P}(\hat{f}^0 - f^0) \leq \frac{1}{1 - \delta} \left\| \hat{\pi}_t - \pi \right\| \left\| \hat{\mu}^0_t - \mu^0_t \right\|.
\]

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Finally by the triangle inequality,
\[ \mathbb{P}(\hat{f} - f) = O\left(\|\hat{\pi} - \pi\| \sum_{a=0}^{1} \|\hat{\mu}^a - \mu^a\|\right), \]
which completes the proof.

\[ \Box \]

**Lemma A.7** (Almost-sure consistency of the influence function variance estimator). Suppose that \( \|\hat{f}_T - f\|_2 = o(1) \) and that \( f(Z) \) has a finite fourth moment. Then,
\[ \text{var}(\hat{f}_T) = \text{var}(\hat{f}) + o(1). \]

**Proof.** First, write
\[ \text{var}(\hat{f}_T) = \mathbb{E}(\hat{f}_T^2) - (\mathbb{E} \hat{f}_T)^2. \]

We will separately show that (i) and (ii) are \( o(1) \).

**Almost-sure convergence of (i)** Decompose (i) into sample average, empirical process, and bias terms:
\[ \mathbb{P}(\hat{f}^2 - f^2) = \underbrace{(\mathbb{P}(\hat{f}^2) - f^2)}_{\Gamma_i^{\text{EP}}} + \underbrace{(\mathbb{P}(\hat{f}) - \mathbb{P})\hat{f}^2}_{\Gamma_i^{\text{SA}}} + \underbrace{\mathbb{P}(\hat{f}^2 - f^2)}_{\Gamma_i^{\text{B}}} . \]

Since \( f(Z) \) has four finite moments, we have that \( f(Z)^2 \) has a variance. In particular, \( \Gamma_i^{\text{SA}} = o(1) \) by the strong law of large numbers, and \( \Gamma_i^{\text{EP}} = O\left(\|\hat{f}^2 - f^2\|_2 \sqrt{\log \log t/f}\right) \) by Step 1 of the proof of Lemma A.5. By our assumption that \( \|\hat{f} - f\|_2 = o(1) \), we have that \( \Gamma_i^{\text{EP}} = o(\sqrt{\log \log t/f}) \).

Now, let us upper-bound \( \Gamma_i^{\text{B}} \) by \( L_2(\mathbb{P}) \) norms:
\[ \Gamma_i^{\text{B}} = \mathbb{P}(\hat{f}^2 - f^2) \leq \|\hat{f}^2 - f^2\| \leq \|\hat{f} - f\| \|\hat{f} + f\| = o(1), \]
where the second inequality follows from Cauchy-Schwartz. Therefore, (i) = \( o(1) \).

**Almost-sure convergence of (ii)** Using the same analysis as above, we have that
\[ \mathbb{P}_T \hat{f} - \mathbb{P} \hat{f} = o(1), \]
or equivalently,
\[ \mathbb{P}_T \hat{f} \xrightarrow{a.s.} \mathbb{P} \hat{f}. \]

By the continuous mapping theorem,
\[ (\mathbb{P}_T f)^2 \xrightarrow{a.s.} (\mathbb{P} f)^2, \]
which completes the proof that (ii) = \( o(1) \). Therefore, \( \text{var}(\hat{f}_T) - \text{var}(\hat{f}) = o(1) \). \[ \Box \]
Proposition A.1 (General AsympCSs under sequential cross-fitting). Consider the cross-fit estimator as defined in (20):
\[ \hat{\psi}^\times_t := \frac{1}{t} \sum_{i=1}^{T_t} f_T(Z_i^{\text{eval}}) + \frac{1}{t} \sum_{i=1}^{T'_t} f_T(Z_i^{\text{trn}}), \]
and the cross-fit variance estimator as defined in (21):
\[ \text{var}_t(f) := \frac{\text{var}_T(\hat{f}_T) + \text{var}_{T'}(\hat{f}_{T'})}{2}. \]
Suppose that \( \Gamma^B_t \) and \( \Gamma^\text{EP}_t \) are both \( o(\sqrt{\log t/t}) \). Then,
\[ \hat{\psi}^\times_t \pm \sqrt{\text{var}_t(f)} \sqrt{\frac{2(t \rho^2 + 1)}{t^2 \rho^2}} \log \left( \frac{\sqrt{t \rho^2 + 1}}{\alpha} \right) \]
forms a \((1 - \alpha)\)-AsympCS for \( \psi \).

**Proof.** Writing out the centered cross-fit estimator \( \hat{\psi}^\times_t - \psi \) using the decomposition of Lemma A.4, we have
\[
\hat{\psi}^\times_t - \psi = \frac{1}{t} \sum_{i=1}^{T_t} \hat{f}_T(Z_i^{\text{eval}}) - \frac{1}{t} \sum_{i=1}^{T'_t} \hat{f}_T(Z_i^{\text{trn}}) - \psi \\
= \frac{1}{t} \sum_{i=1}^{T_t} (\hat{f}_T(Z_i^{\text{eval}}) - \psi) + \frac{1}{t} \sum_{i=1}^{T'_t} (\hat{f}_T(Z_i^{\text{trn}}) - \psi) \\
= \frac{1}{t} \left( T_t^{\text{SA}} + T_t^{\text{EP}} - T_t^{\text{trn}} \right) + \frac{1}{t} \left( T_t^{\text{trn}} \right) \\
= \frac{1}{t} \left( T_t^{\text{SA}} + T_t^{\text{EP}} + T_t^{\text{trn}} \right) + \frac{1}{t} \left( T_t^{\text{trn}} \right) \\
= \text{o}(\sqrt{\log t/t})
\]
where \( \Gamma_{t,\text{eval}} := \frac{1}{T_t} \sum_{i=1}^{T_t} \hat{f}_T(Z_i^{\text{eval}}) \) and \( \Gamma_{t,\text{trn}}^{\text{EP}} := \frac{1}{T_t} \sum_{i=1}^{T'_t} \hat{f}_T(Z_i^{\text{trn}}) \), and similarly for \( \Gamma_{t,\text{eval}}^{\text{SA}}, \Gamma_{t,\text{trn}}^{\text{B}}, \Gamma_{t,\text{eval}}^{\text{B}}, \) and \( \Gamma_{t,\text{trn}}^{\text{B}} \). Applying the proof of Theorem 2.2 (but with variance consistency \( \text{var}_t(f) \), \( \text{var}_t(\hat{f}) \) obtained via Lemma A.7), we have that
\[ \hat{\psi}_t^\times \pm \sqrt{\text{var}_t(f)} \sqrt{\frac{2(t \rho^2 + 1)}{t^2 \rho^2}} \log \left( \frac{\sqrt{t \rho^2 + 1}}{\alpha} \right) + \text{o}(\sqrt{\log t/t}) \]
forms a nonasymptotic \((1 - \alpha)\)-CS for \( \psi \). Consequently,
\[ \hat{\psi}_t^\times \pm \sqrt{\text{var}_t(f)} \sqrt{\frac{2(t \rho^2 + 1)}{t^2 \rho^2}} \log \left( \frac{\sqrt{t \rho^2 + 1}}{\alpha} \right) \]
forms a \((1 - \alpha)\)-AsympCS for \( \psi \) with rate \( \text{o}(\sqrt{\log t/t}) \) which completes the proof. \( \square \)

A.3.1 Proof of Theorem 3.1

**Proof.** When propensity scores are known, we have that \( \Gamma^B_t = 0 \) by Lemma A.6. By assumption, \( E[\tilde{\mu}_T(X) - \bar{\mu}(X)]^2 = o(1) \), and thus by Lemma A.5, \( \Gamma^\text{EP}_t = o(\sqrt{\log t/t}) \). Combining these conditions on \( \Gamma^B_t \) and \( \Gamma^\text{EP}_t \) with Proposition A.1, we obtain the desired result. This completes the proof of Theorem 3.1. \( \square \)
A.3.2 Proof of Theorem 3.2

Proof. By Lemmas A.5 and A.6 we have that both $\Gamma_t^B$ and $\Gamma_t^{EP}$ are $o(\sqrt{\log \log t})$. Applying Proposition A.1, we obtain the desired result, completing the proof of Theorem 3.2. \hfill \Box

A.4 Proof of Theorems 3.3 and 3.4

Lemma A.8 (Decomposition of $t\hat{\psi}_t^\times - t\tilde{\psi}_t$). Let $\hat{\psi}_t^\times$ be as in (20). Furthermore, assume that there exists $\hat{f}$ such that $\|\hat{f}_t - \hat{f}\|_{L^2(P)} \to 0$. In other words, $\hat{f}_t$ is an estimator of $f$ but may instead converge to $\bar{f}$. Then we have the decomposition,

$$t\hat{\psi}_t^\times - t\tilde{\psi}_t = \tilde{S}_t^{SA} + \tilde{S}_{t,\text{eval}}^{EP} + \tilde{S}_{t,\text{trn}}^{EP} + \tilde{S}_{t,\text{eval}}^{B} + \tilde{S}_{t,\text{trn}}^{B} \tag{41}$$

where

$$\tilde{S}_t^{SA} := \sum_{i=1}^{t} \left[ \hat{f}(Z_i) - \mathbb{P}(\bar{f}(Z_i)) \right],$$

$$\tilde{S}_{t,\text{eval}}^{EP} := \sum_{i=1}^{t} \left\{ \left[ \hat{f}_t(Z_i^{\text{eval}}) - \mathbb{P}(\hat{f}_t(Z_i^{\text{eval}})) \right] - \left[ \bar{f}(Z_i^{\text{eval}}) - \mathbb{P}(\bar{f}(Z_i^{\text{eval}})) \right] \right\},$$

$$\tilde{S}_{t,\text{trn}}^{EP} := \sum_{i=1}^{t'} \left\{ \left[ \hat{f}_t(Z_i^{\text{trn}}) - \mathbb{P}(\hat{f}_t(Z_i^{\text{trn}})) \right] - \left[ \bar{f}(Z_i^{\text{trn}}) - \mathbb{P}(\bar{f}(Z_i^{\text{trn}})) \right] \right\},$$

$$\tilde{S}_t^{B,\text{eval}} := \sum_{i=1}^{t} \mathbb{P}(\hat{f}_t(Z_i^{\text{eval}}) - f(Z_i^{\text{eval}})), \quad \text{and}$$

$$\tilde{S}_t^{B,\text{trn}} := \sum_{i=1}^{t'} \mathbb{P}(\hat{f}_t(Z_i^{\text{trn}}) - f(Z_i^{\text{trn}})).$$

Proof. First, note that $t\hat{\psi}_t^\times - t\tilde{\psi}_t$ can be written as

$$t\hat{\psi}_t^\times - t\tilde{\psi}_t = \sum_{i=1}^{t} \hat{f}_t(Z_i^{\text{eval}}) + \sum_{i=1}^{t'} \hat{f}_t(Z_i^{\text{trn}}) - \sum_{i=1}^{t} \mathbb{P}f(Z_i^{\text{eval}}) - \sum_{i=1}^{t'} \mathbb{P}f(Z_i^{\text{trn}})$$

$$= \sum_{i=1}^{t} \left[ \hat{f}_t(Z_i^{\text{eval}}) - \mathbb{P}f(Z_i^{\text{eval}}) \right] + \sum_{i=1}^{t'} \left[ \hat{f}_t(Z_i^{\text{trn}}) - \mathbb{P}f(Z_i^{\text{trn}}) \right].$$

We will handle each sum separately and then combine them to arrive at the final decomposition (41). Taking a closer look at (i) first, we have

$$(i) = \sum_{i=1}^{t} \left\{ \left[ \hat{f}_t(Z_i^{\text{eval}}) - \mathbb{P}(\hat{f}_t(Z_i^{\text{eval}})) \right] - \left[ \bar{f}(Z_i^{\text{eval}}) - \mathbb{P}(\bar{f}(Z_i^{\text{eval}})) \right] \right\}$$

$$+ \sum_{i=1}^{t} \mathbb{P} \left\{ \hat{f}_t(Z_i^{\text{eval}}) - f(Z_i^{\text{eval}}) \right\} + \sum_{i=1}^{t'} \left\{ \bar{f}(Z_i^{\text{eval}}) - \mathbb{P}(\bar{f}(Z_i^{\text{eval}})) \right\}. $$

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Similarly for (ii), we have

\[
(ii) = \sum_{i=1}^{T'} \left\{ \left[ \hat{f}_T(Z_i^{\text{trn}}) - \mathbb{P} (\hat{f}_T(Z_i^{\text{trn}})) \right] - \left[ \bar{f}(Z_i^{\text{trn}}) - \mathbb{P}(\bar{f}(Z_i^{\text{trn}})) \right] \right\} \\
+ \sum_{i=1}^{T'} \mathbb{P} \left\{ \hat{f}_T(Z_i^{\text{trn}}) - f(Z_i^{\text{trn}}) \right\} + \sum_{i=1}^{T'} \left\{ \bar{f}(Z_i^{\text{trn}}) - \mathbb{P}(\bar{f}(Z_i^{\text{trn}})) \right\}.
\]

Putting (i) and (ii) together, we have

\[
t_{\tilde{\psi}} - t_{\tilde{\psi}} = \sum_{i=1}^{T'} \left\{ \hat{f}(Z_i^{\text{eval}}) - \mathbb{P}(\hat{f}(Z_i^{\text{eval}})) \right\} + \sum_{i=1}^{T'} \left\{ \bar{f}(Z_i^{\text{trn}}) - \mathbb{P}(\bar{f}(Z_i^{\text{trn}})) \right\} + \tilde{S}_t^\text{EP} + \tilde{S}_t^\text{B}
\]

\[
= \sum_{i=1}^{t} \left\{ \hat{f}(Z_i) - \mathbb{P}(\hat{f}(Z_i)) \right\} + \tilde{S}_t^\text{EP} + \tilde{S}_t^\text{B},
\]

which completes the proof. \qed

Lemma A.9 (Almost sure behavior of \( S_t^\text{EP} \)). Suppose that there exists \( \delta > 0 \) such that \( \tilde{\psi}_t \in [\delta, 1 - \delta] \) almost surely for all \( t \). Then,

\[
\tilde{S}_t^\text{EP} = o \left( \left\{ \sum_{a=0}^{1} \sup_i \| \hat{\mu}_a^i (X_i) - \bar{\mu}^a (X_i) \|_{L_2(\mathbb{P})} \right\} \sqrt{t \log \log t} \right),
\]

where

Proof. We will show that the result (42) holds for each of \( \tilde{S}_t^\text{eval} \) and \( \tilde{S}_t^\text{trn} \), thereby yielding the same result for their sum \( \tilde{S}_t^\text{EP} \). The proof proceeds in two steps. First, we use an argument from Kennedy et al. [21] and the law of the iterated logarithm to bound \( \tilde{S}_t^\text{SA} \) in terms of \( \sup_i \| \hat{f}_T(Z_i) - \bar{f}(Z_i) \| \). Second and finally, we upper bound \( \sup_i \| \hat{f}_T(Z_i) - \bar{f}(Z_i) \| \) by \( O \left( \sum_{a=0}^{1} \sup_i \| \hat{\mu}_a^i (Z_i) - \bar{\mu}(Z_i) \| \right) \).

Step 1 Let us first consider \( \tilde{S}_t^\text{eval} \). Following the proof of Kennedy et al. [21, Lemma 2] and of Lemma A.5, note that conditional on \( D_t^\text{trn} := \{ Z_i^{\text{trn}} \}_{i=1}^T \) and \( S_t^\text{trn} := \{ 1(1 \in D_t^{\text{trn}}) \}_{i=1}^T \) the summands of \( \tilde{S}_t^\text{eval} \) have mean zero:

\[
\mathbb{P} \left\{ \left[ \hat{f}_T(Z_i^{\text{eval}}) - \bar{f}(Z_i^{\text{eval}}) \right] - \left[ \bar{f}(Z_i^{\text{eval}}) - \mathbb{P}(\bar{f}(Z_i^{\text{eval}})) \right] \mid D_t^\text{trn}, S_t^\text{trn} \right\} = 0.
\]

Similar to the proof of Lemma A.5, we upper bound the conditional variance of a single summand,

\[
\text{var} \left\{ (1 - \mathbb{P})(\hat{f}_{T'}(Z_i^{\text{eval}}) - \bar{f}(Z_i^{\text{eval}})) \mid D_t^\text{trn}, S_t^\text{trn} \right\} = \text{var} \left\{ (\hat{f}_{T'}(Z_i^{\text{eval}}) - \bar{f}(Z_i^{\text{eval}})) \mid D_t^\text{trn}, S_t^\text{trn} \right\} \leq \| \hat{f}_{T'}(Z_i^{\text{eval}}) - \bar{f}(Z_i^{\text{eval}}) \|_{L_2(\mathbb{P})}^2.
\]

Denote the following process \( u_{t, \text{eval}} \) as the supremum of the above with respect to \( i \in \{1, 2, \ldots \} \):

\[
u_{t, \text{eval}} := \sup_{1 \leq i \leq t} \| \hat{f}_{T'}(Z_i^{\text{eval}}) - \bar{f}(Z_i^{\text{eval}}) \|_{L_2(\mathbb{P})}.
\]
Then we can upper bound the following conditional probability

\[
\mathbb{P} \left( \limsup_{t \to \infty} \frac{\hat{S}_{\text{eval}}^t + \tilde{S}_{\text{eval}}^t}{\nu_{t, \text{eval}} \sqrt{2t \log \log t}} \leq 1 \mid \mathcal{D}_{\infty}^{\text{trn}}, \mathcal{S}_{\infty}^{\text{trn}} \right)
\]

\[
= \mathbb{P} \left( \limsup_{t \to \infty} \sum_{i=1}^{T} \frac{\pm \left\{ f_t(Z_i^{\text{eval}}) - P(f_t(Z_i^{\text{eval}})) \right\} - \left\{ f(Z_i^{\text{eval}}) - P(f(Z_i^{\text{eval}})) \right\}}{\nu_{t, \text{eval}} \sqrt{2t \log \log t}} \leq 1 \mid \mathcal{D}_{\infty}^{\text{trn}}, \mathcal{S}_{\infty}^{\text{trn}} \right)
\]

\[
\leq \mathbb{P} \left( \limsup_{t \to \infty} \sum_{i=1}^{T} \frac{\pm \left\{ f_t(Z_i^{\text{eval}}) - P(f_t(Z_i^{\text{eval}})) \right\} - \left\{ f(Z_i^{\text{eval}}) - P(f(Z_i^{\text{eval}})) \right\}}{\| f_t(Z_i^{\text{eval}}) - f(Z_i^{\text{eval}}) \|_{L_2(P)} \sqrt{2t \log \log t}} \leq 1 \mid \mathcal{D}_{\infty}^{\text{trn}}, \mathcal{S}_{\infty}^{\text{trn}} \right)
\]

\[
= \mathbb{P} \left( \limsup_{t \to \infty} \sum_{i=1}^{T} \frac{\pm \zeta_i}{\sqrt{2t \log \log t}} \leq 1 \mid \mathcal{D}_{\infty}^{\text{trn}}, \mathcal{S}_{\infty}^{\text{trn}} \right),
\] (43)

where \( \zeta_i \) are independent mean-zero random variables with variance at most one (conditional on \( \mathcal{D}_{\infty}^{\text{trn}}, \mathcal{S}_{\infty}^{\text{trn}} \)). By the law of the iterated logarithm, we have that (43) = 1. In particular, since this event happens with probability one conditionally, it also happens with probability one marginally. It follows that

\[
\tilde{S}_{\text{eval}}^t = O \left( \sup_i \| \hat{f}_t(Z_i) - \bar{f}(Z_i) \|_{L_2(P)} \sqrt{t \log \log t} \right).
\]

Applying the same technique to \( \tilde{S}_{\text{trn}}^{t, \text{eval}} \), we have that \( \tilde{S}_{t, \text{eval}}^t = O \left( \sup_i \| \hat{f}_t(Z_i) - \bar{f}(Z_i) \|_{L_2(P)} \sqrt{t \log \log t} \right) \), and hence

\[
\tilde{S}_{t}^{\text{EP}} = O \left( \sup_i \| \hat{f}_t(Z_i) - \bar{f}(Z_i) \|_{L_2(P)} \sqrt{t \log \log t} \right).
\] (44)

**Step 2** Now, following the same technique as Step 2 in the proof of Lemma A.5, we have that

\[
\| \hat{f}_t(Z_i) - \bar{f}(Z_i) \| = O \left( \sum_{a=0}^{1} \| \hat{\mu}_t^a(X_i) - \bar{\mu}_t^a(X_i) \| \right).
\] (45)

Combining (44) and (45), we have the desired result,

\[
\tilde{S}_{t}^{\text{EP}} = O \left( \left\{ \sum_{a=0}^{1} \sup_{1 \leq t \leq \infty} \| \hat{\mu}_t^a(X_i) - \bar{\mu}_t^a(X_i) \| \right\} \sqrt{t \log \log t} \right),
\]

which completes the proof.

Now, we examine the asymptotic almost-sure behaviour of the bias term, \( \Gamma_{\text{bias}}^t \) by upper-bounding this term by a product of \( L_2(P) \) estimation errors of nuisance functions.

**Lemma A.10** (Almost-sure behavior of \( \tilde{S}_t^B \)). Suppose \( \tilde{\pi}_t \in [\delta, 1 - \delta] \) for every \( t \) almost surely for some \( \delta > 0 \). Then,

\[
\tilde{S}_t^B = O \left( \sum_{i=1}^{t} \| \tilde{\pi}_t(X_i) - \pi(X_i) \|_{L_2(P)} \sum_{a=0}^{1} \| \hat{\mu}_t^a(X_i) - \bar{\mu}_t^a(X_i) \|_{L_2(P)} \right)
\]

The proof proceeds similarly to that of Lemma A.6 but with additional care given to the fact that observations are no longer iid.
Proof. Similar to the proof of Lemma A.9, we will first prove the result for $\hat{S}^B_{t,\text{eval}}$, and the proof proceeds similarly for $\hat{S}^B_{t,\text{trn}}$, thereby yielding the desired result for $\hat{S}^B_t = \hat{S}^B_{t,\text{eval}} + \hat{S}^B_{t,\text{trn}}$. Following the same technique as Lemma A.6, we have that

$$\mathbb{P}(\hat{f}_T(Z_i^{\text{eval}}) - f(Z_i^{\text{eval}})) = O\left(\|\hat{\pi}_T(X_i^{\text{eval}}) - \pi(X_i^{\text{eval}})\|_1 + \sum_{a=0}^1 \|\hat{\mu}_T^a(X_i^{\text{eval}}) - \mu^a(X_i^{\text{eval}})\|\right).$$

Putting the above term back into the sum $\hat{S}^B_{t,\text{eval}}$, we have

$$\hat{S}^B_{t,\text{eval}} := \sum_{i=1}^T \mathbb{P}(\hat{f}_T(Z_i^{\text{eval}}) - f(Z_i^{\text{eval}})) = O\left(\sum_{i=1}^T \|\hat{\pi}_T(X_i^{\text{eval}}) - \pi(X_i^{\text{eval}})\|_1 + \sum_{a=0}^1 \|\hat{\mu}_T^a(X_i^{\text{eval}}) - \mu^a(X_i^{\text{eval}})\|\right).$$

Using a similar argument to bound $\hat{S}^B_{t,\text{trn}}$ and putting these together, we have the following bound for $\hat{S}^B_t = \hat{S}^B_{t,\text{eval}} + \hat{S}^B_{t,\text{trn}}$,

$$\hat{S}^B_t = O\left(\sum_{i=1}^t \|\hat{\pi}_t(X_i) - \pi(X_i)\|_1 + \sum_{a=0}^1 \|\hat{\mu}_t^a(X_i) - \mu^a(X_i)\|\right), \quad (46)$$

which completes the proof. $\square$

Proposition A.2 (General AsympCSs for time-varying causal effects under sequential cross-fitting).

Consider the cross-fit estimator as defined in (20):

$$\hat{\psi}_t^\times := \sum_{i=1}^T f_T(Z_i^{\text{eval}}) + \sum_{i=1}^T f_T(Z_i^{\text{trn}}),$$

and suppose we have access to a variance estimator $\hat{\text{var}}_t(\hat{f})$ such that

$$\hat{\text{var}}_t(\hat{f}) - \text{var}(\hat{f}) = o(1).$$

Suppose that $\hat{S}^B_t$ and $\hat{S}^{\text{EP}}_t$ are both $o(\sqrt{T\log \log T})$, and that the conditions of Corollary 2.1 hold but with $(Y_i)_{i=1}^T$ replaced by $(\hat{f}(Z_i))_{i=1}^T$. Then,

$$\hat{\psi}_t^\times \pm \sqrt{\frac{2(t\rho^2 \hat{\text{var}}_t(\hat{f}) + 1)}{t^2 \rho^2} \log \left(\frac{\sqrt{t\rho^2 \hat{\text{var}}_t(\hat{f}) + 1}}{\alpha}\right)}$$

forms a $(1 - \alpha)$-AsympCS for $\hat{\psi}_t := \frac{1}{t} \sum_{i=1}^t \psi_i$.

Proof. Writing out the centered cross-fit estimator on the “sum scale” $t(\hat{\psi}_t^\times - \hat{\psi}_t)$ using the decomposition of Lemma A.8, we have

$$t(\hat{\psi}_t^\times - \hat{\psi}_t) = S^\text{SA} + \hat{S}^B_{t,\text{eval}} + \hat{S}^B_{t,\text{trn}} + \hat{S}^{\text{EP}}_{t,\text{eval}} + \hat{S}^{\text{EP}}_{t,\text{trn}}.$$

Therefore, we have that

$$\hat{\psi}_t - \hat{\psi}_t = \frac{1}{t} \sum_{i=1}^t (\hat{f}(Z_i) - \psi_i) + o(\sqrt{T\log t}).$$

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Applying Corollary 2.1 to \((\tilde{f}(Z_i))^\alpha\)_i above, we have that

\[
\tilde{C}_t^{(\alpha)} := \tilde{\psi}_t^{\alpha} \pm \sqrt{\frac{2(t \rho^2 \text{var}(\tilde{f}) + 1)}{t^2 \rho^2} \log \left( \frac{\sqrt{t \rho^2 \text{var}(\tilde{f}) + 1}}{\alpha} \right) + o(\sqrt{\log \log t/t})}
\]

forms a nonasymptotic \((1 - \alpha)\)-CS for \(\tilde{\psi}_t := \frac{1}{n} \sum_{i=1}^n \psi_i\), meaning \(\mathbb{P}\left( \exists t : \tilde{\psi}_t \notin \tilde{C}_t^{(\alpha)} \right) \leq \alpha\). Consequently,

\[
\tilde{\psi}_t^{\alpha} \pm \sqrt{\frac{2(t \rho^2 \text{var}(\tilde{f}) + 1)}{t^2 \rho^2} \log \left( \frac{\sqrt{t \rho^2 \text{var}(\tilde{f}) + 1}}{\alpha} \right)}
\]

forms a \((1 - \alpha)\)-AsympCS for \(\tilde{\psi}_t\), which completes the proof. \(\square\)

A.4.1 Proof of Theorem 3.3

Proof. By Lemma A.9 combined with Assumption 4, we have that \(\tilde{S}_t^{\text{EP}} = o(\sqrt{t \log \log t})\). In a randomized experiment, Assumption 5 holds by design, and thus by Lemma A.10, we have that \(\tilde{S}_t^{\text{B}} = o(\sqrt{t \log \log t})\). Invoking Proposition A.2, we obtain the desired result. \(\square\)

A.4.2 Proof of Theorem 3.4

Proof. Similar to the proof of Theorem 3.3, by Lemma A.9 combined with Assumption 4, we have that \(\tilde{S}_t^{\text{EP}} = o(\sqrt{t \log \log t})\). For observational studies, we assume that Assumption 5 holds, and thus by Lemma A.10, we have that \(\tilde{S}_t^{\text{B}} = o(\sqrt{t \log \log t})\). Invoking Proposition A.2, we obtain the desired result. \(\square\)

B Simulation details

B.1 Simulated randomized experiment (Fig. 4)

First, we describe the simulated randomized sequential experiment displayed in Fig. 4.

**Data-generating process** Consider a randomized experiment with \(n = 10^4\) subjects, each with 3 real-valued covariates. Generate \(10^4\) 3-tuples of said covariates \(X_1, X_2, \ldots \sim N_3(0, I_3)\) from a standard trivariate Gaussian. Randomly assign subjects to treatment or control groups with equal probability: \(A_1, \ldots, A_n \sim \text{Bernoulli}(1/2)\). Define the regression function,

\[
\mu^*(x_i) := 1 - x_{i,1}^2 - 2 \sin(x_{i,2}) + 3|x_{i,3}|, \tag{47}
\]

and the target parameter \(\psi := 1\) (which we will ensure is the average treatment effect by design). Finally, generate outcomes \(Y_1, \ldots, Y_n\) as

\[
Y_i := \mu^*(X_{i,1}, X_{i,2}, X_{i,3}) + \psi \cdot A_i + \epsilon_i,
\]

where \(\epsilon_i \sim t_5\) are drawn from a \(t\)-distribution with 5 degrees of freedom (we use this heavy-tailed distribution in an attempt to stress-test the finite fourth absolute moment condition of Theorems 3.1 and 3.2). We now describe the three models used to estimate \(\psi\) knowing the treatment assignment distribution of \(A_1, \ldots, A_n\) but without knowledge of \(\mu^*\) or the distribution of \(\epsilon_i\).
Estimators The unadjusted estimator \( \hat{\psi}_t \) used in this example is the simplest of the three and takes the form,

\[
\hat{\psi}_t := \frac{1}{t} \sum_{i=1}^{t} \left( \frac{A_i}{1/2} - \frac{1 - A_i}{1/2} \right) Y_i.
\] (48)

Since this estimator does not estimate the regression functions \( \mu^a \) for \( a = 0, 1 \), no sequential cross-fitting is needed. The other two estimators employ sequential cross-fitting as in Section 3.2 and take the form (20) but with various choices of \( \hat{\mu}_t^a \) and \( \hat{\mu}_t^0 \). Specifically, the “Parametric” estimator uses linear regression to construct \( \hat{\mu}_t^a \) and \( \hat{\mu}_t^0 \) which in this case is misspecified. The “Super Learner” estimator, uses a weighted ensemble of several machine learning algorithms. In this simulation, these consisted of adaptive regression splines, generalized additive models, generalized linear models with LASSO (\( \ell_1 \) regularization) and pairwise interactions, and random forests. The weights were chosen via cross-validation [49, 30]. We then applied Theorem 3.1 to obtain the confidence sequences displayed in Fig. 4.

B.2 Simulated observational study (Fig. 5)

Data-generating process The simulation scenario used to produce Fig. 5 is identical to the previous section but without complete Bernoulli randomization of treatments. Instead, each individual is assigned treatment with propensity score \( \pi(x_1, x_2, x_3) \), defined by

\[
\pi(x_1, x_2, x_3) := 0.2 + 0.6 \cdot \logit(\mu^*(x_1, x_2, x_3)),
\]

where \( \mu^* \) is the regression function defined in (47). A scale of 0.6 and a translation of 0.2 is applied to ensure that \( \pi(x_1, x_2, x_3) \in [0.2, 0.8] \) is bounded away from 0 and 1.

Estimators As before, the unadjusted estimator does not make use of sequential cross-fitting and is defined in (48), but uses the cumulative fraction of treated subjects as an estimate of the propensity score, \( \pi \). On the other hand, the “Parametric” and “Super Learner” estimators invoke sequential cross-fitting and take the form (20) where \( \hat{\mu}_t^a \) is constructed in the same way as in the experimental setup of the previous section. Since \( \pi(x) \) is unknown, it must now be estimated. The ‘Parametric’ estimator uses logistic regression to accomplish this, while the ‘Super Learner’ uses the same ensemble as in the previous section (appropriately modified for classification rather than regression). Invoking Theorem 3.2 yields the confidence sequences of Fig. 5.

C Additional discussions

C.1 One-sided asymptotic confidence sequences

In Sections 2.2 and 2.4, we derived universal two-sided AsympCSs for the means of independent random variables in the iid and time-varying settings, respectively. Here, we give analogous one-sided bounds for the aforementioned settings. First, let us derive a one-sided AsympCS for the mean of iid random variables analogous to Theorem 2.2.

Proposition C.1. Given the same setup as in Theorem 2.2, we have that

\[
\hat{\mu}_i - \hat{\sigma}_i \sqrt{\frac{2(t \rho^2 + 1)}{\rho^2 \mu^2} \log \left( 1 + \sqrt{\frac{t \rho^2 + 1}{2\alpha}} \right)}
\] (49)

forms a lower \((1 - \alpha)\)-AsympCS for \( \mu \) with the same rates as given in Theorem 2.2.
Notice that the $(1-\alpha)$-AsympCS of Proposition C.1 resembles the $(1-2\alpha)$-AsympCS of Theorem 2.2 but with an additional additive 1 inside the log. A similar phenomenon appears in the one- and two-sided sub-Gaussian CSs of Howard et al. [16]. Recall, however, that their bounds are nonasymptotic and require much stronger assumptions (and in particular are not applicable to the observational causal inference setup of this paper).

Similar to the relationship between iid (Theorem 2.2) and martingale (Theorem 2.3) two-sided AsympCSs, an analogue of Proposition C.1 can be derived under martingale dependence with time-varying means and variances.

**Proposition C.2.** Given the same setup and assumptions as in Theorem 2.3, we have that

$$
\tilde{\mu}_t = \sqrt{\frac{2(t\hat{\sigma}_t^2 \rho^2 + 1)}{(t\rho^2)}} \log \left( 1 + \frac{\sqrt{t\hat{\sigma}_t^2 \rho^2 + 1}}{2\alpha} \right) \tag{50}
$$

forms a lower $(1-\alpha)$-AsympCS for the time-varying average $\tilde{\mu}_t := \frac{1}{t} \sum_{i=1}^{t} \mu_i$.

We will first prove a lemma concerning one-sided boundaries for sums of independent Gaussian random variables, which in turn will be used to prove Propositions C.1 and C.2 shortly.

**Lemma C.1.** Suppose $(G_t)_{t=1}^{\infty} \sim N(0,1)$ is an iid sequence of standard Gaussian random variables. Then,

$$
P \left( \forall t \in \mathbb{N}, \tilde{\mu}_t \geq 1 \sum_{i=1}^{t} (\sigma_i G_i + \mu_i) - \sqrt{\frac{2(t\rho^2 \hat{\sigma}_t^2 + 1)}{(t\rho^2)}} \log \left( 1 + \frac{\sqrt{t\rho^2 \hat{\sigma}_t^2 + 1}}{2\alpha} \right) \right) \geq 1 - \alpha. \tag{51}
$$

In other words, $L^*_t$ forms a nonasymptotic lower $(1-\alpha)$-CS for $\tilde{\mu}_t$.

**Proof.** The proof begins similarly to that of Theorem 2.2 but with a modified mixing distribution, and proceeds in four steps. First, we derive a sub-Gaussian nonnegative supermartingale (NSM) indexed by a parameter $\lambda \in \mathbb{R}$ identical to that of Theorem 2.2. Second, we mix this NSM over $\lambda$ using a folded Gaussian density (rather than the classical Gaussian density used in the proof of Theorem 2.2), and justify why the resulting process is also an NSM. Third, we derive an implicit lower CS for $(\tilde{\mu}_t)_{t=1}^{\infty}$. Fourth and finally, we compute a closed-form lower bound for the implicit CS.

**Step 1: Constructing the $\lambda$-indexed NSM** Similar to the proof of Theorem 2.2, let $(G_t)_{t=1}^{\infty}$ be an infinite sequence of iid standard Gaussian random variables, and let $S_t := \sum_{i=1}^{t} \sigma_i G_i$. Then, we have that for any $\lambda \in \mathbb{R},$

$$
M_t(\lambda) := \exp \left\{ \lambda S_t - t\hat{\sigma}_t^2 \lambda^2 / 2 \right\}, \tag{52}
$$

forms an NSM with respect to the filtration given by $\mathcal{F}_t := \sigma(G_t^1)$.

**Step 2: Mixing over $\lambda \in (0, \infty)$ to obtain a mixture NSM** Let us now construct a one-sided sub-Gaussian mixture NSM. First, note that the mixture of an NSM with respect to a probability density is itself an NSM [32, 15] and is a simple consequence of Fubini’s theorem. For our purposes, we will consider the density of a folded Gaussian distribution with location zero and scale $\rho^2$. In particular, if $\Lambda \sim N(0, \rho^2)$, let $\Lambda_+ := |\Lambda|$ be the folded Gaussian. Then $\Lambda_+$ has a probability density function $f^-_{\rho^2}(\lambda)$ given by

$$
f^-_{\rho^2}(\lambda) := \mathbb{I}(\lambda > 0) \frac{2}{\sqrt{2\pi \rho^2}} \exp \left\{ -\frac{\lambda^2}{2\rho^2} \right\}. \tag{53}
$$
Note that $f_{\rho^2}^+$ is simply the density of a mean-zero Gaussian with variance $\rho^2$, but truncated from below by zero, and multiplied by two to ensure that $f_{\rho^2}^+(\lambda)$ integrates to one.

Then, since mixtures of NSMs are themselves NSMs, the process $(M_t)_{t \geq 0}$ given by

$$M_t := \int_{t} M_t(\lambda) f_{\rho^2}^+(\lambda) d\lambda$$

is an NSM. We will now find a closed-form expression for $M_t$. Some of the algebraic steps are the same as those in the proof of Theorem 2.2, but we repeat them here for completeness. Writing out the definition of $M_t$, we have

$$M_t := \int_{\lambda} \exp \{ \lambda S_t - t\sigma_t^2 \lambda^2 / 2 \} f_{\rho^2}^+(\lambda) d\lambda$$

$$= \int_{\lambda} \exp \{ \lambda S_t - t\sigma_t^2 \lambda^2 / 2 \} \frac{2}{\sqrt{2\pi\rho^2}} \exp \left\{ -\frac{\lambda^2}{2\rho^2} \right\} d\lambda$$

$$= \frac{2}{\sqrt{2\pi\rho^2}} \int_{\lambda} \exp \{ \lambda S_t - t\sigma_t^2 \lambda^2 / 2 \} \exp \left\{ -\frac{\lambda^2}{2\rho^2} \right\} d\lambda$$

$$= \frac{2}{\sqrt{2\pi\rho^2}} \int_{\lambda} \exp \left\{ \lambda S_t - \frac{\lambda^2(t\rho^2\sigma_t^2 + 1)}{2\rho^2} \right\} d\lambda$$

$$= \frac{2}{\sqrt{2\pi\rho^2}} \int_{\lambda} \exp \left\{ \frac{-\lambda^2(t\rho^2\sigma_t^2 + 1) + 2\lambda\rho^2 S_t}{2\rho^2} \right\} d\lambda$$

$$= \frac{2}{\sqrt{2\pi\rho^2}} \int_{\lambda} \exp \left\{ \frac{-a(\lambda^2 - \frac{b}{a} 2\lambda)}{2\rho^2} \right\} d\lambda,$$

where we have set $a := t\rho^2\sigma_t^2 + 1$ and $b := \rho^2 S_t$. Completing the square in $(\ast)$, we have that

$$\exp \left\{ \frac{-a(\lambda^2 - \frac{b}{a} 2\lambda)}{2\rho^2} \right\} = \exp \left\{ \frac{-\lambda^2 + 2\lambda \frac{b}{a} + \left( \frac{b}{a} \right)^2 - \left( \frac{b}{a} \right)^2}{2\rho^2/a} \right\}$$

$$= \exp \left\{ \frac{-\lambda^2}{2\rho^2/a} + \frac{a (b/a)^2}{2\rho^2} \right\}$$

$$= \exp \left\{ \frac{-\lambda^2}{2\rho^2/a} \exp \left\{ \frac{b^2}{2a\rho^2} \right\} \right\}.$$

Plugging this back into our derivation of $M_t$ and multiplying the entire quantity by $a^{-1/2}/a^{-1/2}$, we have

$$M_t = \frac{2}{\sqrt{2\pi\rho^2}} \int_{\lambda} \exp \left\{ \frac{-a(\lambda^2 + \frac{b}{a} 2\lambda)}{2\rho^2} \right\} d\lambda$$

$$= \frac{2}{\sqrt{2\pi\rho^2}} \int_{\lambda} \exp \left\{ \frac{-\lambda^2 + \frac{b}{a} 2\lambda}{2\rho^2/a} \right\} d\lambda$$

$$= \frac{2}{\sqrt{a}} \exp \left\{ \frac{b^2}{2a\rho^2} \right\} \int_{\lambda} \exp \left\{ \frac{-\lambda^2 + \frac{b}{a} 2\lambda}{2\rho^2/a} \right\} d\lambda.$$
Now, notice that \((**\)) = \(\mathbb{P}(N(b/a, \rho^2/a) \geq 0)\), which can be rewritten as \(\Phi(b/\rho \sqrt{a})\), where \(\Phi\) is the CDF of a standard Gaussian. Putting this all together and plugging in \(a = t\rho^2\bar{\sigma}_t^2 + 1\) and \(b = \rho^2S_t\), we have the following expression for \(M_t\),

\[
M_t = \frac{2}{\sqrt{\pi}} \exp \left\{ \frac{b^2}{2a\rho^2} \right\} \Phi \left( \frac{b}{\rho \sqrt{a}} \right) = \frac{2}{\sqrt{t\rho^2\bar{\sigma}_t^2 + 1}} \exp \left\{ \frac{\rho^4S_t^2}{2(t\rho^2\bar{\sigma}_t^2 + 1)\rho^2} \right\} \Phi \left( \frac{\rho^2S_t}{\sqrt{t\rho^2\bar{\sigma}_t^2 + 1}} \right).
\]

\[(55)\]

**Step 3: Deriving a \((1 - \alpha)\)-lower CS \((L_t')_t=1^\infty\) for \((\tilde{\mu}_t)_t=1^\infty\)**

Now that we have computed the mixture NSM \((M_t)_t=1^\infty\), we apply Ville’s inequality to it and “invert” a family of processes — one of which is \((M_t)_t=1^\infty\) — to obtain an implicit lower CS (we will further derive an explicit lower CS in Step 4).

First, let \((m_t)_t=1^\infty\) be an arbitrary real-valued process — i.e. not necessarily equal to \((\mu_t)_t=1^\infty\) — and define their running average \(\tilde{m}_t := \frac{1}{t} \sum_{i=1}^{t} m_i\). Define the partial sum process in terms of \((\tilde{m}_t)_t=1^\infty\),

\[
S_t(\tilde{m}_t) := S_t + t\tilde{\mu}_t - t\tilde{m}_t
\]

and the resulting nonnegative process,

\[
M_t(\tilde{m}_t) := \frac{2}{\sqrt{t\rho^2\bar{\sigma}_t^2 + 1}} \exp \left\{ \frac{\rho^2S_t(\tilde{m}_t)^2}{2(t\rho^2\bar{\sigma}_t^2 + 1)} \right\} \Phi \left( \frac{\rho S_t(\tilde{m}_t)}{\sqrt{t\rho^2\bar{\sigma}_t^2 + 1}} \right).
\]

\[(56)\]

Notice that if \(\tilde{m}_t = \tilde{\mu}_t\), then \(S_t(\tilde{\mu}_t) = S_t = \sum_{i=1}^{t} \sigma_i G_i\) and \(M_t(\tilde{\mu}_t) = M_t\) from Step 2. Importantly, \((M_t(\tilde{\mu}_t))_t=0^\infty\) is an NSM. Indeed, by Ville’s inequality, we have

\[
\mathbb{P}(\exists t : M_t(\tilde{\mu}_t) \geq 1/\alpha) \leq \alpha.
\]

\[(57)\]

We will now “invert” this family of processes to obtain an implicit lower boundary given by

\[
L'_t := \inf \{\tilde{\mu}_t : M_t(\tilde{\mu}_t) < 1/\alpha\},
\]

and justify that \((L'_t)_t=1^\infty\) is indeed a lower \((1 - \alpha)\)-CS for \(\tilde{\mu}_t\). Writing out the probability of miscoverage at any time \(t\), we have

\[
\mathbb{P}(\exists t : \tilde{\mu}_t < L'_t) = \mathbb{P} \left( \exists t : \tilde{\mu}_t < \inf_{\tilde{m}_t} \{M_t(\tilde{m}_t) < 1/\alpha\} \right) = \mathbb{P} \left( \exists t : M_t(\tilde{\mu}_t) \geq 1/\alpha \right) \leq \alpha,
\]

where the last line follows from Ville’s inequality applied to \((M_t(\tilde{\mu}_t))_t=0^\infty\). In particular, \(L'_t\) forms a \((1 - \alpha)\)-lower CS, meaning

\[
\mathbb{P}(\forall t : \tilde{\mu}_t \geq L'_t) = 1 - \alpha.
\]

**Step 4: Obtaining a closed-form lower bound \((\tilde{L}_t)_t=1^\infty\) for \((L'_t)_t=1^\infty\)**

The lower CS of Step 3 is simple to evaluate via line- or grid-searching, but a closed-form expression may be desirable in practice, and for this we can compute a sharp lower bound on \(L'_t\).

First, take notice of two key facts:

(a) When \(\tilde{m}_t = S_t/t + \tilde{\mu}_t\), we have that \(S_t(\tilde{m}_t) = 0\) and hence \(M_t(\tilde{m}_t) < 1\), and

45
(b) $S_t(\tilde{m}_t)$ is a strictly decreasing function of $\tilde{m}_t \leq S_t/t + \tilde{\mu}_t$, and hence so is $M_t(\tilde{m}_t)$.

Property (a) follows from the fact that $\Phi(0) = 1/2$, and that $\sqrt{\rho^2 \sigma_t^2 + 1} > 1$ for any $\rho > 0$. Property (b) follows from property (a) combined with the definitions of $S_t(\cdot)$,

$$S_t(\tilde{m}_t) := S_t + t\tilde{\mu}_t - t\tilde{m}_t$$

and of $M_t(\cdot)$,

$$M_t(\tilde{m}_t) := \frac{2}{\sqrt{\rho^2 \sigma_t^2 + 1}} \exp \left\{ \frac{\rho^2 S_t(\tilde{m}_t)^2}{2(\rho^2 \sigma_t^2 + 1)} \right\} \Phi \left( \frac{\rho S_t(\tilde{m}_t)}{\sqrt{\rho^2 \sigma_t^2 + 1}} \right),$$

In particular, by facts (a) and (b), the infimum in (58) must be attained when $S_t(\cdot) \geq 0$. That is,

$$S_t(L'_t) \geq 0. \quad (59)$$

Using (59) combined with the inequality $1 - \Phi(x) \leq \exp\{-x^2/2\}$ (a straightforward consequence of the Cramér-Chernoff technique), we have the following lower bound on $M_t(L'_t)$:

$$M_t(L'_t) = \frac{2}{\sqrt{\rho^2 \sigma_t^2 + 1}} \exp \left\{ \frac{\rho^2 S_t(L'_t)^2}{2(\rho^2 \sigma_t^2 + 1)} \right\} \Phi \left( \frac{\rho S_t(L'_t)}{\sqrt{\rho^2 \sigma_t^2 + 1}} \right)$$

$$\geq \frac{2}{\sqrt{\rho^2 \sigma_t^2 + 1}} \exp \left\{ \frac{\rho^2 S_t(L'_t)^2}{2(\rho^2 \sigma_t^2 + 1)} \right\} \left( 1 - \exp \left\{ -\frac{\rho^2 S_t(L'_t)^2}{2(\rho^2 \sigma_t^2 + 1)} \right\} \right)$$

$$= \frac{2}{\sqrt{\rho^2 \sigma_t^2 + 1}} \left( \exp \left\{ \frac{\rho^2 S_t(L'_t)^2}{2(\rho^2 \sigma_t^2 + 1)} \right\} - 1 \right)$$

$$= M_t(L'_t).$$

Finally, the above lower bound on $M_t(L'_t)$ implies that $1/\alpha \geq M_t(L'_t) \geq \tilde{M}_t(L'_t)$ which yields the following lower bound on $L'_t$:

$$\tilde{M}_t(L'_t) \leq 1/\alpha \iff \frac{2}{\sqrt{\rho^2 \sigma_t^2 + 1}} \left( \exp \left\{ \frac{\rho^2 S_t(L'_t)^2}{2(\rho^2 \sigma_t^2 + 1)} \right\} - 1 \right) \leq 1/\alpha$$

$$\iff \exp \left\{ \frac{\rho^2 S_t(L'_t)^2}{2(\rho^2 \sigma_t^2 + 1)} \right\} \leq 1 + \frac{\sqrt{\rho^2 \sigma_t^2 + 1}}{2\alpha}$$

$$\iff \frac{\rho^2 S_t(L'_t)^2}{2(\rho^2 \sigma_t^2 + 1)} \leq \log \left( 1 + \frac{\sqrt{\rho^2 \sigma_t^2 + 1}}{2\alpha} \right)$$

$$\iff S_t(L'_t) \leq \sqrt{\frac{2(\rho^2 \sigma_t^2 + 1)}{\rho^2} \log \left( 1 + \frac{\sqrt{\rho^2 \sigma_t^2 + 1}}{2\alpha} \right)}$$

$$\iff \sum_{i=1}^{t} \sigma_i G_i + t\tilde{\mu}_t - tL'_t \leq \sqrt{\frac{2(\rho^2 \sigma_t^2 + 1)}{\rho^2} \log \left( 1 + \frac{\sqrt{\rho^2 \sigma_t^2 + 1}}{2\alpha} \right)}$$

$$\iff tL'_t \geq \sum_{i=1}^{t} (\sigma_i G_i + \mu_i) - \sqrt{\frac{2(\rho^2 \sigma_t^2 + 1)}{\rho^2} \log \left( 1 + \frac{\sqrt{\rho^2 \sigma_t^2 + 1}}{2\alpha} \right)}$$

$$\iff L'_t \geq \frac{1}{t} \sum_{i=1}^{t} (\sigma_i G_i + \mu_i) - \sqrt{\frac{2(\rho^2 \sigma_t^2 + 1)}{\rho^2} \log \left( 1 + \frac{\sqrt{\rho^2 \sigma_t^2 + 1}}{2\alpha} \right)},$$
and hence \( \mathbb{P} ( \forall t \in \mathbb{N}, \; \tilde{\mu}_t \geq L_t^\ast ) \geq 1 - \alpha \).

\( \square \)

**Proof of Proposition C.1.** In this case, the data \((Y_i)_{i=1}^t\) are iid, and hence we will assume that \(\sigma_1 = \sigma_2 = \cdots = \sigma\) and \(\mu_1 = \mu_2 = \cdots = \mu\). First, notice that if we define \(\beta = \rho \sigma\), then we can write \(L_t^\ast\) as
\[
L_t^\ast := \frac{\sigma}{t} \sum_{i=1}^t (G_i + \mu) - \sigma \sqrt{\frac{2(t\beta^2 + 1)}{(t\beta)^2} \log \left( 1 + \frac{\sqrt{t\beta^2 + 1}}{2\alpha} \right)}.
\]
(60)

Now, by the strong approximations of KMT [22, 23, 26], we have that
\[
L_t^\ast = \frac{1}{t} \sum_{i=1}^t Y_i - \sigma \sqrt{\frac{2(t\beta^2 + 1)}{(t\beta)^2} \log \left( 1 + \frac{\sqrt{t\beta^2 + 1}}{2\alpha} \right)} + \varepsilon_t.
\]
(61)

where \(\varepsilon_t = O(\log t/t)\) if \(Y_i\) has a moment generating function, and \(\varepsilon_t = o(t^{1/2} - 1)\) if \(Y_i\) has \(q > 2\) finite absolute moments. Writing the above in terms of an empirical standard deviation \(\tilde{\sigma}_t\), we have by the proof of Lemma A.2 that
\[
L_t^\ast = \frac{1}{t} \sum_{i=1}^t Y_i - \tilde{\sigma}_t \sqrt{\frac{2(t\beta^2 + 1)}{(t\beta)^2} \log \left( 1 + \frac{\sqrt{t\beta^2 + 1}}{2\alpha} \right)} + \varepsilon_t.
\]
(62)

where \(\varepsilon_t = o \left( (\log \log t/t)^{3/4} \right)\) if \(Y_i\) has at least 4 absolute moments, and \(\varepsilon_t = o(\sqrt{\log \log t/t})\) otherwise. In either case, we have that
\[
\frac{1}{t} \sum_{i=1}^t Y_i - \tilde{\sigma}_t \sqrt{\frac{2(t\beta^2 + 1)}{(t\beta)^2} \log \left( 1 + \frac{\sqrt{t\beta^2 + 1}}{2\alpha} \right)}
\]
(63)
forms a lower \((1 - \alpha)\)-AsympCS for \(\mu\) with approximation rate \(\varepsilon_t\). This completes the proof.\(^8\)

\( \square \)

**Proof of Proposition C.2.** Similar to the proof of Theorem 2.3, Lemma A.3 yields the following strong invariance principle
\[
\sum_{i=1}^t Y_i \leq \sum_{i=1}^t \sigma_i (G_i + \mu_i) + o \left( V_t^{3/8} \log V_t \right).
\]
Therefore, with probability at least \((1 - \alpha)\),
\[
\forall t \geq 1, \; \hat{\mu}_t \geq \frac{1}{t} \sum_{i=1}^t Y_i - \sqrt{\frac{2(t\bar{\sigma}^2 \rho^2 + 1)}{(t\bar{\sigma})^2} \log \left( 1 + \frac{\sqrt{t\bar{\sigma}^2 \rho^2 + 1}}{2\alpha} \right)} + o \left( V_t^{3/8} \log V_t/t \right).
\]

In particular, we have that
\[
(\hat{\mu}_t \pm \sqrt{\frac{2(t\bar{\sigma}^2 \rho^2 + 1)}{(t\bar{\sigma}^2 \rho^2) \log \left( 1 + \frac{\sqrt{t\bar{\sigma}^2 \rho^2 + 1}}{2\alpha} \right)}})
\]
(64)

\(^8\)While we wrote this final bound in terms of \(\beta\), we leave the statement of the original result in terms of \(\rho\) to maintain consistency with other boundaries throughout the paper. The change from \(\beta\) to \(\rho\) and back is entirely cosmetic, and does not affect the interpretation of the final result.
forms a \((1 - \alpha)\)-AsympCS for \(\hat{\mu}_t\). The derivation of an analogous lower AsympCS in terms of the empirical variance \(\hat{\sigma}_t^2\) proceeds similarly to Step 3 of the proof of Theorem 2.3. In particular, we get that
\[
\hat{\mu}_t - \tilde{\mathcal{H}}_t^* := \hat{\mu}_t - \sqrt{\frac{2(\hat{\sigma}_t^2 \rho^2 + 1)}{t^2 \rho^2}} \log \left( 1 + \frac{\sqrt{\hat{\sigma}_t^2 \rho^2 + 1}}{2\alpha} \right) + o \left( \frac{\sqrt{V_t \log V_t}}{t} \right)
\]
forms a nonasymptotic \((1 - \alpha)\)-CS for \(\hat{\mu}_t\), meaning \(\mathbb{P} \left( \forall t \in \mathbb{N}, \hat{\mu}_t \geq \hat{\mu}_t - \tilde{\mathcal{H}}_t^* \right) \leq \alpha\). Combined with Assumption 3, we have that
\[
\hat{\mu}_t - \tilde{\mathcal{H}}_t := \hat{\mu}_t - \sqrt{\frac{2(\hat{\sigma}_t^2 \rho^2 + 1)}{t^2 \rho^2}} \log \left( 1 + \frac{\sqrt{\hat{\sigma}_t^2 \rho^2 + 1}}{2\alpha} \right)
\]
forms a \((1 - \alpha)\)-AsympCS for \(\hat{\mu}_t\) since \(\tilde{\mathcal{H}}_t = \sqrt{V_t \log V_t}/t\). This completes the proof. \(\square\)

C.2 Asymptotic \(\epsilon\)-processes and \(p\)-processes

In Section 2.1 we introduced AsympCSs as anytime-valid analogues of asymptotic CIs. Depending on the application at hand, however, a testing perspective (e.g. via \(p\)-values) may be sufficient or even preferable. In this section, we present “asymptotic \(\epsilon\)-processes” (Definition C.2) and demonstrate how they yield anytime-valid analogues of CLT-based \(p\)-values.

To begin, let \(\mathcal{P}_0\) be a set of distributions (the “null hypothesis”). Recall that a classical (non-sequential) \(p\)-value for \(\mathcal{P}_0\) is a random variable \(P^\ast\) that is superuniform for any element of \(\mathcal{P}_0\); that is,
\[
\sup_{\mathcal{P} \in \mathcal{P}_0} \mathbb{P}(P^\ast \leq \alpha) \leq \alpha \quad (65)
\]
for any prespecified error level \(\alpha\) \((0, 1)\). In the nonasymptotic sequential regime, a \(p\)-process \((P^\ast)_t\) for \(\mathcal{P}_0\) — sometimes referred to as an “anytime \(p\)-value” for \(\mathcal{P}_0\) — is a stochastic process satisfying (65) time-uniformly \([17, 16]\), meaning
\[
\sup_{\mathcal{P} \in \mathcal{P}_0} \mathbb{P}(\exists t \geq 1 : P^\ast_t \leq \alpha) \leq \alpha \quad (66)
\]
By far the most common method for deriving \(p\)-processes is by first deriving a so-called \(\epsilon\)-process — a stochastic process that is upper-bounded by a nonnegative supermartingale under the null \(\mathcal{P}_0\) [39, 11, 16]. Formally, we say that a process \((\hat{E}_t)_t\) is an \(\epsilon\)-process for \(\mathcal{P}_0\) if for every \(\mathcal{P} \in \mathcal{P}_0\), there exists a \(\mathbb{P}\)-nonnegative supermartingale with initial value (\(\mathbb{P}\)-NSM) \((M^\mathbb{P}_t)_{t=1}^\infty\) such that
\[
\forall t, \hat{E}_t \leq M^\mathbb{P}_t, \quad \mathbb{P}\text{-almost surely}. \quad (67)
\]
A \(p\)-process \((\bar{p}_t)_t\) satisfying (66) can be derived from \((\hat{E}_t)_t\) by simply setting \(\bar{p}_t := 1/\hat{E}_t\) for each \(t\). Indeed, \(\bar{p}_t \leq \alpha\) if and only if \(\hat{E}_t \geq 1/\alpha\), and hence
\[
\sup_{\mathcal{P} \in \mathcal{P}_0} \mathbb{P}(\exists t \geq 1 : \bar{p}_t \leq \alpha) = \sup_{\mathcal{P} \in \mathcal{P}_0} \mathbb{P}(\exists t \geq 1 : \hat{E}_t \geq 1/\alpha)
\]
\[
\leq \sup_{\mathcal{P} \in \mathcal{P}_0} \mathbb{P}(\exists t \geq 1 : M^\mathbb{P}_t \geq 1/\alpha) \leq \alpha.
\]
Note that the definition of an \(\epsilon\)-process given in (67) and the above time-uniform guarantee is nonasymptotic, but asymptotic batch \(p\)-values are ubiquitous in the statistical sciences. Nevertheless the literature still lacks formal definitions for asymptotic \(\epsilon\)-processes and asymptotic anytime \(p\)-values, so we propose such definitions here.

---

\(A\) process \((M^\mathbb{P}_t)_{t=1}^\infty\) is a \(\mathbb{P}\)-NSM if it is nonnegative \(\mathbb{P}\)-almost surely, and \(E_0(M_t) \leq M_{t-1}\) for every \(t \geq 1\).
Definition C.2. We say that $(\overline{E}_t)_{t=1}^\infty$ forms an asymptotic $e$-process (Asymp-$e$-proc) for the null $P_0 \subseteq P$ if there exists some nonasymptotic $e$-process $(\overline{E}_t)_{t=1}^\infty$ for $P_0$ — meaning it satisfies (67) — such that
\[
\frac{\log(\overline{E}_t^*)}{\log(\overline{E}_t)} \to 1 \quad \mathbb{P}\text{-almost surely for every } P \in P,
\]
In other words, the logarithmic behaviors of $\overline{E}_t^*$ and $\overline{E}_t$ coincide for all $t$ sufficiently large, $\mathbb{P}$-almost surely for any $P \in P$.

Definition C.2 serves as an analogue of Definition 2.1 but for $e$-processes. The use of the logarithm in the above definition may seem arbitrary but the natural way to compare the asymptotics of $e$-processes — and even their fixed-time counterparts, $e$-values — is on the logarithmic scale [11, 60]. Notice that even though a nonasymptotic $e$-process $(\overline{E}_t)_{t=1}^\infty$ need only satisfy (67) under the null $P_0$, we nevertheless require that the asymptotic $e$-process $(\overline{E}_t)_{t=1}^\infty$ satisfies (68) for every $P \in P$, not just for the null $P_0$. We define asymptotic $p$-processes similarly.

Definition C.3. We say that $(\overline{P}_t)_{t=1}^\infty$ forms an asymptotic $p$-process (Asymp-$p$-proc) for the null $P_0 \subseteq P$ if there exists some nonasymptotic $p$-process $(\overline{P}_t)_{t=1}^\infty$ for $P_0$ — meaning it satisfies (66) — such that
\[
\frac{\log(\overline{P}_t^*)}{\log(\overline{P}_t)} \to 1 \quad \mathbb{P}\text{-almost surely for every } P \in P,
\]
Similar to asymptotic $e$-processes in Definition C.2, the condition (69) requires that the logarithmic behaviors of $\overline{P}_t^*$ and $\overline{P}_t$ coincide for all $t$ sufficiently large, $\mathbb{P}$-almost surely for any $P \in P$.

It is easy to see that if $(E_t)_{t=1}^\infty$ is an Asymp-$e$-proc, then $(1/E_t)_{t=1}^\infty$ is an Asymp-$p$-proc. Given the above definitions, we will now derive an explicit Asymp-$e$-proc (and hence Asymp-$p$-proc) for nonparametric testing of means from iid data.

Proposition C.3 (Asymptotic sequential tests for the mean). Let $(Y_t)_{t=1}^\infty \sim P$ for some distribution $P$ and let $P_0$ be the set of distributions with mean $\mu_0$ and $q > 2$ finite absolute moments. For any $\mu \in \mathbb{R}$, let $S_t(\mu) := \hat{\sigma}_t^{-1} \sum_{i=1}^t (Y_i - \mu)$ where $\hat{\sigma}_t$ is the sample standard deviation based on $Y_1, \ldots, Y_t$. Then for any prespecified $p > 0$,
\[
\overline{E}_t(\mu_0) := \exp \left\{ \frac{\rho^2 S_t(\mu_0)^2}{2(t\rho^2 + 1)} \right\} (t\rho^2 + 1)^{-1/2}
\]
forms an Asymp-$e$-proc for $P_0$. Consequently, $\overline{P}_t(\mu_0) := 1/\overline{E}_t(\mu_0)$ forms an Asymp-$p$-proc for $P_0$.

Proof. By KMT [22, 23, 26], there exists (on a potentially enriched probability space) a sequence of iid standard Gaussians $(G_t)_{t=1}^\infty$ such that $\sigma^{-1} \sum_{i=1}^t Y_i - \sum_{i=1}^t G_i = o(t^{1/4})$, where $\sigma^2 = \text{var}(Y_1)$. By the proof of Theorem 2.2, we have that
\[
\overline{E}_t^* := \exp \left\{ \frac{\rho^2 \left( \sum_{i=1}^t G_i \right)^2}{2(t\rho^2 + 1)} \right\} (t\rho^2 + 1)^{-1/2}
\]
forms a nonnegative martingale starting at one (and hence forms an $e$-process). Using the above strong
KMT approximation, we have

\[
\log(\hat{E}_t^*) := \frac{\rho^2 \left( \sigma^{-1} \sum_{i=1}^t Y_i + o(t^{1/4}) \right)^2}{2(t\rho^2 + 1)} - \frac{1}{2} \log(t\rho^2 + 1)
\]

\[
= \frac{\rho^2 \left( \sigma^{-1} \sum_{i=1}^t Y_i \right)^2 + o(t^{1/4} \sqrt{t \log \log t}) + o(t^{2/4})}{2(t\rho^2 + 1)} - \frac{1}{2} (t\rho^2 + 1)
\]

\[
= \frac{\rho^2 \sigma^{-2} \left( \sum_{i=1}^t Y_i \right)^2 + o(t \log \log t) + o(t^{2/4})}{2(t\rho^2 + 1)} - \frac{1}{2} (t\rho^2 + 1)
\]

\[
= \frac{\rho^2 \left( \sigma_t^{-1} \sum_{i=1}^t Y_i \right)^2 + o(t \log \log t)}{2(t\rho^2 + 1)} - \frac{1}{2} (t\rho^2 + 1)
\]

\[
= \frac{\rho^2 S_t^2}{2(t\rho^2 + 1)} - \frac{1}{2} (t\rho^2 + 1) + o(\log \log t),
\]

where we have set \( S_t := \sigma_t^{-1} \sum_{i=1}^t Y_i \). Now, it remains to show that \( \log(\hat{E}_t^*)/\log(\hat{E}_t) \to 1 \) almost surely. Indeed, writing out the ratio of the log-processes \( \log(\hat{E}_t^*)/\log(\hat{E}_t) \) using the above derivation, we have that

\[
\frac{\log(\hat{E}_t^*)}{\log(\hat{E}_t)} = 1 + \frac{\rho^2 \left( \sigma_t^{-1} \sum_{i=1}^t Y_i \right)^2 /2(t\rho^2 + 1) - \frac{1}{2} (t\rho^2 + 1) + o(\log \log t)}{\rho^2 \left( \sigma_t^{-1} \sum_{i=1}^t Y_i \right)^2 /2(t\rho^2 + 1) - \frac{1}{2} (t\rho^2 + 1)}
\]

\[
= 1 + \frac{o(\log \log t)}{\rho^2 \left( \sigma_t^{-1} \sum_{i=1}^t Y_i \right)^2 /2(t\rho^2 + 1) - \frac{1}{2} (t\rho^2 + 1)} \tag{72}
\]

and notice that it suffices to show that \( \frac{\log(\hat{E}_t^*)}{\log(\hat{E}_t)} \to 1 \) almost surely. To this end, we will show that the absolute reciprocal of \( (*) \) diverges to \( \infty \), which will complete the proof. Indeed, notice that we can write \( (*)^{-1} \) as

\[
(*)^{-1} = \frac{\rho^2 \left( \sigma_t^{-1} \sum_{i=1}^t Y_i \right)^2 /2(t\rho^2 + 1) - \frac{1}{2} (t\rho^2 + 1)}{o(\log \log t)}
\]

\[
:= \omega \left( \frac{1}{\log \log t} \right) \left( \frac{\rho^2 \left( \sigma_t^{-1} \sum_{i=1}^t Y_i \right)^2 /2(t\rho^2 + 1) - \frac{1}{2} (t\rho^2 + 1)}{(i)} \right),
\]

where \((i) = O(\log \log t)\) by the law of the iterated logarithm and \((ii) \approx t\), and hence \((i) - (ii) \approx -t\). Consequently, we have that

\[
\|(*)^{-1}\| = \Omega \left( \frac{t}{\log \log t} \right) \to \infty, \quad \text{and hence}
\]

\[ (*) = o(1). \]

Combining the above with \( (72) \), we have the desired result

\[
\frac{\log(\hat{E}_t^*)}{\log(\hat{E}_t)} = 1 + o(1) \quad \text{almost surely},
\]

50
which completes the proof.

Remark 5. Given that \( e \)-processes and \( p \)-processes address the similar problem of time-uniform testing, it is natural to wonder about their relative advantages and disadvantages. Certainly \( p \)-values are more commonly used in statistical applications, but there are some practical and philosophical reasons why one may prefer to use \( e \)-processes directly (rather than just as tools to derive \( p \)-processes) due to the fact that they form \( e \)-values at arbitrary stopping times. That is, if \( (E_t)_{t=1}^{\infty} \) is an \( e \)-process, then

\[
E(E_\tau) \leq 1 \quad \text{for any stopping time } \tau.
\]  

(73)

By a simple application of Markov’s inequality, we have that

\[
P\{E_\tau \leq \alpha\} \geq \frac{1}{\alpha} \exp\{-\alpha\} - 1.
\]

(74)

Nevertheless, working with the \( e \)-value \( E_\tau \) directly makes it simple to combine evidence across several studies [11, 44, 55] or to control the false discovery rate under arbitrary dependence [57], both of which are less straightforward with \( p \)-values. Furthermore, \( e \)-values have received considerable attention for philosophical reasons including how they relate testing to betting [38] and connect frequentist and Bayesian notions of uncertainty [11, 59]. While the details of these advantages are well outside the scope of this paper, they are advantages that can now be thought about in the asymptotic regime.

C.3 Optimizing Robbins’ normal mixture for \((t, \alpha)\)

In this section, we outline how one can choose \( \rho \) to optimize the boundary \( \mathcal{B}_t \) in Theorem 2.2 for a specific time \( t^* \) and type-I error level \( \alpha \in (0, 1) \).\textsuperscript{10} We will outline both the (computationally inexpensive) exact solution, and the closed-form approximate solution. Note that the derivations that follow are essentially the same as those in Howard et al. [16, Section 3.5] but we repeat them here to keep our results self-contained.

The exact solution. Let \( W_{-1} \) be the lower branch of the Lambert \( W \) function [6]. Then we have that

\[
\argmin_{\rho > 0} \mathcal{B}_t(\alpha) = \sqrt{-W_{-1}(-\alpha^2 \exp\{-1\}) - 1}.
\]

(75)

Proof. Consider the boundary in Theorem 2.2 at time \( t \),

\[
\mathcal{B}_t(\alpha) := \sqrt{\frac{2(t\rho^2 + 1)}{t^2 \rho^2} \log\left(\frac{\sqrt{tp^2 + 1}}{\alpha}\right)}.
\]

Defining \( x := \rho^2 \) and after some simple algebra, notice that

\[
\argmin_{\rho > 0} \mathcal{B}_t(\alpha) = \sqrt{\argmin_{x > 0} f(x)},
\]

where

\[
f(x) := \frac{tx + 1}{t^2 x} \log\left(\frac{tx + 1}{\alpha^2}\right).
\]

Notice that \( \lim_{x \to 0} f(x) = \lim_{x \to \infty} f(x) = \infty \) and thus if we find that \( df/dx = 0 \) has exactly one positive solution, we know that it must be the minimizer of \( f \).

To that end, it is straightforward to show that

\[
\frac{df}{dx} = -\frac{1}{t^2 x^2} \log\left(\frac{tx + 1}{\alpha^2}\right) + \frac{1}{tx}.
\]

\textsuperscript{10}We will discuss choosing \( \rho \) for the two-sided AsympCS in Theorem 2.2 but for the one-sided AsympCSs of Appendix C.1, we suggest repeating the same argument but with \( \alpha \) replaced by \( 2\alpha \).
Setting the above to 0, we obtain
\[ \alpha^2 \exp \{tx\} = tx + 1, \]
which, after some algebra, can be rewritten as
\[ -\alpha^2 \exp \{-1\} = -(tx + 1) \exp \{- (tx + 1)\} \quad (75) \]
Notice that if we rewrite \( y := -(tx + 1) \), we have that \( y = W_{-1} \left( -\alpha^2 \exp \{-1\} \right) \) where \( W_{-1} \) is the lower branch of the Lambert \( W \) function. Furthermore, \( y = W_{-1}(z) \) only has a solution if \( z \geq -e^{-1} \), requiring that \( \alpha^2 \leq 1 \), which we have trivially by the definition of \( \alpha \in (0, 1) \). In summary, we have that
\[
\argmin_{\rho>0} \mathfrak{B}_t^*(\alpha) = \sqrt{-W_{-1} \left( -\alpha^2 \exp \{-1\} \right) - 1 \over t^*}. 
\]
This completes the proof. \( \square \)

An approximate solution We can derive a closed-form approximation to (74) by considering the Taylor series expansion to the Lambert \( W \) function [6],
\[ W_{-1}(z) = \log(-z) - \log(- \log(-z)) + o(1). \]
Replacing \( W_{-1}(z) \) by \( \log(-z) - \log(- \log(-z)) \) in (74), we obtain the following approximate solution,
\[
\rho'(t^*) := \sqrt{-2 \log \alpha + \log(-2 \log \alpha + 1) \over t^*}. \quad (76) 
\]
In practice, we find that using (76) over (74) has negligible downstream effects on the resulting CSs, but both are inexpensive to compute. Moreover, notice that \( \rho'(t^*) \) is quite similar to \( \sqrt{2 \log(1/\alpha) / t^*} \), which is precisely what one would choose when sharpening a sub-Gaussian confidence interval based on the Cramér-Chernoff technique for a fixed sample size \( t^* \).

C.4 Time-uniform convergence in probability is equivalent to almost sure convergence

In Theorems 2.2, 3.1, and 3.2, we justified the asymptotic validity of our confidence sequences by showing that the approximation error
\[ \varepsilon_t \overset{a.s.}{\longrightarrow} 0 \quad (77) \]
at a particular rate. At first glance, this may seem like a slightly stronger statement than required since we only need the approximation error \( \varepsilon_t \) to vanish \textit{time-uniformly in probability}:
\[ \sup_{k \geq t} |\varepsilon_k| \overset{L^p}{\longrightarrow} 0. \quad (78) \]
As it turns out, however, (77) and (78) are equivalent. This is not a new result, but we present a proof here for completeness.

**Proposition C.4.** Let \( (X_n)_{n=1}^\infty \) be a sequence of random variables. Then,
\[ X_n \overset{a.s.}{\longrightarrow} 0 \iff \sup_{k \geq n} |X_k| \overset{L^p}{\longrightarrow} 0. \]

**Proof.** First, we prove \( (\Rightarrow ) \). By the continuous mapping theorem, \( |X_n| \overset{a.s.}{\longrightarrow} 0 \). Therefore,
\[ 1 = \mathbb{P} \left( \lim_n |X_n| = 0 \right) \leq \mathbb{P} \left( \limsup_n |X_n| = 0 \right) \leq 1. \]
In other words, \( \sup_{k \geq n} |X_k| \xrightarrow{a.s.} 0 \), which implies \( \sup_{k \geq n} |X_k| \xrightarrow{p} 0 \).

Now, consider \( \leftarrow \). Suppose for the sake of contradiction that \( \mathbb{P} (\lim_n |X_n| = 0) < 1 \). Then with some probability \( \delta > 0 \), we have that \( \lim_n |X_n| \neq 0 \), meaning there exists some \( \epsilon > 0 \) such that \( |X_k| > \epsilon \) for some \( k \geq n \) no matter how large \( n \) is. In other words,

\[
\delta < \mathbb{P} \left( \lim_{n \to \infty} \sup_{k \geq n} |X_k| > \epsilon \right) \\
\leq \mathbb{P} \left( \sup_{k \geq n} |X_k| > \epsilon \right) \text{ for any } n \geq 1.
\]

In particular, \( \mathbb{P} (\sup_{k \geq n} |X_k| > \epsilon) \to 0 \), which is equivalent to saying \( \sup_{k \geq n} |X_k| \xrightarrow{p} 0 \), a contradiction. This completes the proof.

\[\square\]

### C.5 Unimprovability of AsympCSs using efficient influence functions

Consider the confidence sequence of Theorem 3.2,

\[
\frac{\hat{\psi}_i^N \pm \sqrt{\text{var}_t(f)}}{\text{var}_t(f)} \cdot \left[ \frac{2(t \rho^2 + 1)}{t^2 \rho^2} \log \left( \frac{\sqrt{t \rho^2 + 1}}{\alpha} \right) \right] \text{ with rate } o \left( \frac{\log \log t}{t} \right), \tag{79}
\]

It is natural to whether (79) can be tightened at all. In a certain sense, (79) inherits optimality from its three main components: (i) Robbins’ normal mixture boundary, (ii) the approximation error rate, and (iii) the estimated standard deviation \( \sqrt{\text{var}_t(f)} \) of the efficient influence function \( f \).

**Term (i)** Starting with the width, we have that in the case of iid Gaussian data \( G_1, G_2, \ldots \sim \mathcal{N}(\mu, \sigma^2) \), Robbins’ normal mixture confidence sequence [32] is obtained by first showing that

\[
M_t(\mu) := \exp \left\{ \frac{\rho^2 (\sum_{i=1}^t (G_i - \mu))^2}{2(t \rho^2 + 1)} \right\} (t \rho^2 + 1)^{-1/2}
\]

is a nonnegative martingale starting at one, and hence by Ville’s inequality [54],

\[
\mathbb{P}(\exists t \geq 1 : M_t(\mu) \geq 1/\alpha) \leq \alpha.
\]

The resulting confidence sequence \( \bar{C}_t^N \) at each time \( t \) is defined as the set of \( m \) such that \( M_t(m) < 1/\alpha \), i.e. \( \bar{C}_t^N := \{ m \in \mathbb{R} : M_t(m) < 1/\alpha \} \) and consequently,

\[
\mathbb{P}(\exists t \geq 1 : \mu \notin \bar{C}_t^N) = \mathbb{P}(\exists t \geq 1 : M_t(\mu) \geq 1/\alpha) \leq \alpha.
\]

This inequality is extremely tight, since Ville’s inequality almost holds with equality for nonnegative martingales. Technically, the paths of the martingale need to be continuous for equality to hold, which can only happen in continuous time (such as for a Wiener process). However, any deviation from equality only holds because of this “overshoot” and in practice, the error probability is almost exactly \( \alpha \). This means that the normal mixture confidence sequence \( \bar{C}_t^N \) cannot be uniformly tightened: any improvement for some times will necessarily result in looser bounds for others. For a precise characterization of this optimality for the (sub)-Gaussian case, see Howard et al. [16, Section 3.6], or Ramdas et al. [31] for a more general discussion of admissible confidence sequences.
**Term (ii)** The error incurred from almost-surely approximating a sample average \( \frac{1}{t} \sum_{i=1}^{t} f(Z_i) \) of influence functions by Gaussian random variables is a direct consequence of Komlós et al. [22, 23] and Major [26], and is unimprovable without additional assumptions. Further approximation errors result from using \( \text{var}(\hat{f}) \) to estimate \( \text{var}(f) \), where almost-sure law of the iterated logarithm rates appear, and are themselves unimprovable.

**Term (iii)** Using the approximations mentioned in (ii) permits the use of Robbins’ normal mixture confidence sequence in (i). However, a factor of \( \sqrt{\text{var}(\hat{f})} \) necessarily appears in front of the width as an estimate of the standard deviation \( \sqrt{\text{var}(f)} \) of the efficient influence function \( f \) discussed in Section 3. Importantly, \( \sqrt{\text{var}(f)} \) corresponds to the semiparametric efficiency bound, so that no estimator of \( \psi \) can have asymptotic mean squared error smaller than \( \text{var}(f(Z)) / t \) without imposing additional assumptions [51].