Compactification of Tensionless String Theories

Ori J. Ganor

origa@puhep1.princeton.edu

Department of Physics, Jadwin Hall, Princeton University
Princeton, NJ 08544, U. S. A.

Abstract

We study compactifications of the $N = 2$ 6D tensionless string on various complex two-folds down to two-dimensions. In the IR limit they become non-trivial conformal field theories in 2D. Using results of Vafa and Witten on the partition functions of twisted Super-Yang-Mills theories, we can study the resulting CFT. We also discuss the contribution of instantons made by wrapping strings on 2-cycles of the complex two-fold.
1 Introduction

It has been suggested almost two years ago [1] that S-duality of $\mathcal{N} = 4$ Yang-Mills theory can be better understood as originating in a higher dimensional theory. It was then shown in [2] how the $\tau \to -\frac{1}{\tau}$ duality of the pure $U(1)$ gauge theory on a compact 4-manifold is natural when the $U(1)$ theory is realized as a compactification of an anti-self-dual free 2-form field $B_{\mu\nu}^{(-)}$ from 6D to 4D on a $T^2$ whose complex structure is identified with the complex coupling constant of $U(1)$.

A year ago, a new type of quantum theory, which is probably not a field theory, was discovered in [3]. It was defined as the low-energy of type-IIB compactified on K3 with a 2-cycle that becomes very small. The resulting low-energy description in 6D is a $\mathcal{N} = 2$ theory of a massless tensor multiplet consisting of an anti-self-dual 2-form $B_{\mu\nu}^{(-)}$ and 5 scalars (and their super-partners). The spectrum also includes a light BPS string which couples to $B_{\mu\nu}^{(-)}$. Since the string is much lighter than the compactification scale the theory can be considered on its own without taking into account gravity and the other type-IIB modes. The tension of the string is controlled by the VEVs of the scalars and when the VEVs become zero the theory becomes scale invariant.

Theories with similar features have been encountered in various other constructions. In [4] the world-volume theory of two parallel 5-branes in M-theory that get very close was shown to contain the same kind of objects. The connection between that construction and the previous one was made explicit in [5]. The examples with $\mathcal{N} = 1$ in 6D are richer. In [6] it was argued that the phase discovered in [7] of a 5-brane in the bulk of M-theory on roughly $S^1/Z_2 \times K3$ can contain light strings with an $E_8$ current algebra on them. An extensive description of $\mathcal{N} = 1$ tensionless string theories that arise at phase transition points of heterotic vacua on $K_3$ was given in [8]. More recently, examples of such theories with light stringy objects have been given in 4D [9, 10]. Their possible role in exciting new phenomena in 4D [9, 10] and (of tensile strings) in black-hole physics [11, 12, 13] has been suggested.

All these constructions [1] can be described in the framework of F-theory [16, 8, 9, 17, 18].\footnote{The construction of [10] can probably also be realized as an orbifold of M-theory on $T^7$ with intersecting 5-branes in the bulk, which is probably dual to a theory on the same moduli space of type-IIB on a Calabi-Yau}
as type-IIB vacua on a base manifold with a 2-cycle that shrinks to zero.

In this paper we will study the (probably) simplest of those new theories, i.e. the one with $\mathcal{N} = 2$ in 6D.

For lack of a better name, we will refer to it as $S$-theory. We will use that term for both the “tensionless” theory as well as the tensile theory which differ from one another by the values of the VEVs of the tensor-multiplet.

The question that we wish to ask is “what happens to $S$-theory upon compactification?”.

Compactifying on $T^2$ was argued \cite{3} to give an alternative construction of $\mathcal{N} = 4$ $SU(2)$ Super-Yang-Mills. The charged W-bosons, monopoles and dyons of SYM become the strings of $S$-theory wrapped on 1-cycles of the torus $T^2$.

In the present paper we will discuss compactification on a 4-manifold $X_4$. In order to study a large class of $X_4$-s and to preserve part of the supersymmetry we will have to perform a twist. The resulting theories are non-trivial 2D CFTs in the IR limit. Those turn out to have $(0,2)$ SUSY for general complex manifolds, $(0,4)$ for Kähler manifolds and $(0,8)$ for K3.

We will extract information about the theories in three ways:

1. 2D gravitational anomalies.
2. The partition functions.
3. An instanton calculation.

A calculation of the partition function has been made possible due to the work of Vafa and Witten on twisted $\mathcal{N} = 4$ Yang-Mills \cite{1}. They found that in many cases the partition function of the 4D theory on a 4-manifold in terms of the Yang-Mills coupling constant $\tau = \frac{i}{\beta} + \frac{\theta}{2\pi}$ equals the partition function of a simple rational conformal field theory where $\tau$ now is the complex structure of the 2D torus. This surprising connection between Yang-Mills instantons and RCFTs has been interpreted recently for the case of K3 as the manifestation of type-IIA on K3 and heterotic on $T^4$ duality which replaces 4-branes with elementary heterotic string excitations \cite{19}. In this paper, however, we will follow the original suggestion of \cite{1} (and that of \cite{2}) to identify these RCFTs with the left-moving part after compactification
Recent developments in string theory \[20, 21\] allow us to give a low-energy description of the fields on the tensile strings of S-theory and to estimate the contribution of instantons made by wrapping the strings on 2-cycles of $X_4$ as in $[22]$. We will calculate the condition for such an instanton to contribute in the IR limit.

One of the questions we wish to explore is to what extent can S-theory be described as a tensor multiplet together with strings. In particular, is it possible to describe the compactification by first reducing the free tensor multiplet and then adding the contribution of the strings?

We do not have a systematic way to implement that procedure but for the case of K3 we will compare the result to a $\mathbb{Z}_2$ orbifold of a free tensor multiplet and check that there are no obvious instanton corrections. In the comparison we will find a discrepancy that we were not fully able to explain.

The paper is organized as follows:

1. Section (2): Discussion of the setting for the compactification and the twist.
2. Section (3): The anomaly matching conditions that allow us to derive the difference between left moving and right moving central charges in 2D.
3. Section (4): The free field computation – ignoring the contribution of the strings.
4. Section (5): We calculate the anomaly condition following $[22]$ for an instanton to contribute to the effective IR action.
5. Section (6): Application to various $X_4$-s.
6. Section (7): Discussion.

\section{Compactification}

We wish to learn more about S-theory (“tensionless strings”) by compactifying it to 2D. Compactifying the scale invariant S-theory on a 4-manifold $X_4$ of finite size we expect to find in 2D a theory whose scale is related to the size of $X_4$. We note that in 2D the would-be VEVs of scalars are better interpreted as target space coordinates so the 5 scalars of
the 6D tensor multiplet (i.e. the super-partners of $B^{(-)}_{\mu\nu}$) have “fluctuating VEVs” after compactification. Since the VEVs of the 5 scalars determine the tension of the string of S-theory we see that the compactification “probes” the tensile strings as well. In other words, whether we compactify the tensile or tensionless S-theory we end-up with one and the same theory in 2D.

The next step would be to take the IR limit of the 2D theory, or put differently – to take the size of $X_4$ to zero. In order to extract information on the resulting theory, we need some kind of “topological rigidity”. This is given by the assumption that we can “twist” the supersymmetry in the compact directions and if we use a special twist we have the extra advantage of making contact with the results of Vafa and Witten on twisted $\mathcal{N} = 4$ Yang-Mills.

2.1 Twisting

Although it seems unlikely that 6D tensionless strings are described by a field-theory, it was suggested by E. Witten that the theory might have local currents like the energy-momentum tensor. In that case, we can consider twisting the theory to a topological one. In fact, the compactified tensionless string theory can be realized as type-IIB compactified on an 8-manifold with a 4-manifold $X_4$ of $A_1$ singularities. The tensionless string theory becomes *automatically* twisted in this setting [23, 24].

The R-symmetry group of the $\mathcal{N} = 2$ theory in 6D is $Sp(2)$ and the SUSY charges are in the $\mathbf{4}$ of $Sp(2)$. Since the R-symmetry is part of the super-conformal group we expect it to be a good symmetry [25].

The procedure of twisting embeds (part of) the R-symmetry group in the Lorenz group. We will take a subgroup $SU(2) \subset Sp(2)$ of the R-symmetry. Under $SU(2)_R$ the super charges transform as $2(\mathbf{2})$.

Let us compactify on $T^2 \times X_4$, where $X_4$ is a complex manifold of complex dimension 2. The 6D $\mathcal{N} = 2$ charges are in the $\mathbf{4} \otimes \mathbf{4}$ of $Sp(2)_R \times SO(5,1)$. Decomposing under $SO(1,1) \times SO(4) \subset SO(5,1)$ we have $\mathbf{4} = \mathbf{2}_+ \oplus \mathbf{2}'_+$. we find that the super-charges are in

$$ (\mathbf{2}_R \oplus \mathbf{2}_R) \otimes (\mathbf{2}_+ \oplus \mathbf{2}'_-) $$

(1)
The twist that was used in [1] replaces both $2_R$'s with the $2$ of the Lorenz group $SO(4)$ [1]. This leaves us with two covariantly constant super-charges on $X_4$ which would in general give $(0, 2)$ SUSY in 2D.

### 2.2 The amount of SUSY in 2D

In fact for Kähler manifolds the amount of SUSY is larger. The super-charges are in the $3 + 1$ of $SO(4)$. The $3$ will correspond to anti-self-dual 2-forms on $X_4$. Every such 2-form that is in addition covariantly constant will give rise to 2 additional super-charges in 2D. The Kähler class is always anti-self-dual and covariantly constant, so for Kähler manifolds we always have $(0, 4)$ SUSY in 2D. If in addition, $X_4$ is a K3 we have 2 more covariantly constant anti-self-dual 2-forms corresponding to the nowhere zero sections of the trivial canonical bundle. Thus, in this case we find $(0, 8)$ SUSY in 2D.

### 2.3 Non trivial CFTs in the IR limit.

The IR limit of the 2D theory is some $(0, 2)$ CFT. We will now claim that it is not trivial and in fact its partition function is known.

Writing the Hilbert space of the 2D CFT that we obtain in the form

$$\mathcal{H} = \sum_{\alpha} L_{\alpha} \otimes R_{\alpha}$$

where $R_{\alpha}$ is a right super-Verma module and $L_{\alpha}$ is a left Verma module with no supersymmetry, the partition function is

$$Z(q, \bar{q}) = \text{tr}_\mathcal{H}\{(-)^{F_L + F_R} q^{L_0} \bar{q}^{\bar{L}_0}\} = \sum_{\alpha'} \chi_{\alpha'}(q)$$

where $\alpha'$ are those $\alpha$-s with a super-symmetric vacuum for $R_{\alpha'}$ and $\chi_\alpha(q)$ is the character of the left moving $L_{\alpha}$ module (with $(-)^F$ if it has fermions). This is very similar to the elliptic genus of $\mathcal{N} = 2$ theories [27].

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[2] For 4D SYM there is also another twist that was explored in [26], but I do not see how it can be implemented in our setting.
Now we can make contact with the results of [1]. They expressed the partition function of twisted $\mathcal{N} = 4$ Super-Yang-Mills on a 4-manifold as a sum of characters of 2D CFTs. It is tempting to identify these characters with the characters $\chi_\alpha(q)$.

We can calculate the “functional integral” of tensionless strings on $T^2 \times X_4$\(^3\)

\[
\begin{align*}
\text{S-theory} & \quad \frac{T^2 \times X_4}{T^2} \\
\text{S-theory} & \quad \frac{T^2}{T^2} \rightarrow_{IR} \mathcal{N} = 4 \text{ Yang-Mills (5)}
\end{align*}
\]

in two ways. In the first way we use

\[
\frac{\text{S-theory}}{T^2} \rightarrow_{IR} \mathcal{N} = 4 \text{ Yang-Mills (5)}
\]

and then use the results of [1] for $\mathcal{N} = 4$ twisted Yang-Mills compactified on $X_4$. In the second way we use

\[
\frac{\text{Twisted S-theory}}{X_4} = \text{CFT + (massive)} (6)
\]

Here all the massive states in 2D are in $\mathcal{N} = 2$ multiplets (as a result of the twisting) so that only the $\mathcal{N} = 2$ vacuum contributes to the partition function – which is then multiplied by the character of the right-moving CFT. It is important that this result is independent of the ratio of sizes of $X_4$ and $T^2$ because each of the two ways of calculating is in a different limit. Vafa and Witten [1] expressed the twisted $\mathcal{N} = 4$ path integral as a sum of characters of 2D CFTs. We see now that it is very plausible that those CFTs are related to the left moving $L_\alpha$-s.

In the rest of the paper we will try to get more information on the full CFT as follows:

- We will calculate $c_L - c_R$, the difference of central charges on the left and on the right.
- Starting with the tensile theory, we will wrap a non-critical string around a 2-cycle of the manifold to obtain an instanton and check the condition for it to contribute to the low-energy.

\(^3\)Although we have no known Feynman path integral, the association of a numerical value to every manifold, or more precisely a section over the moduli space of the 6D manifolds is more general and can be constructed from a Hilbert space approach.
2.4 A note on type-IIB compactifications to 2D

It was shown in [28] that type-IIA on $K3 \times K3$ has a $B_{\mu \nu}$ tadpole because of interactions of the form $\int B \wedge Y_8$ where $Y_8$ is an 8-form made of the curvature tensor. It was further explained in [29] that in order to cancel the resulting $B$ charge one needs to add 24 elementary strings reduced on $K3 \times K3$ and filling the entire uncompactified 2D.

What is the manifestation of that for type-IIB on $K3 \times K3$ (which is the theory we are interested in because type-IIB on a singular K3 and more generally type-IIB on an 8-manifold with a 4-manifold of $A_1$ singularities contains S-theory)?

Compactifying further on $S^1$ to $0 + 1$ dimensions type-IIA and type-IIB are T-dual. $B_{\mu \nu}$ becomes the winding number of elementary type-IIA strings and the $B_{\mu \nu}$ tadpole becomes a shift of 24 units in the winding-number charge of the vacuum. In type-IIB we therefore expect to find a shift of 24 units in the momentum of the vacuum around $S^1$. This is measured by a shift in $L_0 - \bar{L}_0$ and indeed the 2D reduction of the self-dual 4-form $B_4$ of type-IIB gives rise to chiral bosons each having a shift of $(-\frac{1}{24})$ in $L_0$. Just like the $B \wedge Y_8$ term the $(-\frac{1}{24})$ shift is also a one-loop effect.

A similar phenomenon happens when we compactify S-theory on K3. We will find in section (4) that there are chiral bosons in 2D which bring about a shift of $24(-\frac{1}{24}) = -1$ in $L_0 - \bar{L}_0$.

3 Anomaly conditions

We can determine $c_L - c_R$ by an argument similar to ‘t Hooft anomaly matching. Perturbing the theory to a tensile string, only the tensor multiplet remains in the IR. However, the 2D gravitational anomaly $c_L - c_R$ should stay the same.

Let us see what the tensile theory would give. The $B^{(-)}_{\mu \nu}$ is unaffected by the twist and gives $b^+$ left-moving scalars and $b^-$ right-moving scalars, where $(b^-) b^+$ is the number of (anti-) self-dual two-forms on $X_4$. For Kähler manifolds, those are given by

$$ (b^+, b^-) = (b^{1,1} - 1, 1 + 2b^{0,2}) $$

---

I am grateful to E. Witten for pointing this out to me.
Before the twist, the fermions are of the form $\lambda^{\alpha I}$ with $\alpha$ a spinor index in $4$ of $SO(5,1)$ and $I$ an index in the $4$ of the R-symmetry group $Sp(2)$.

The spinor index decomposes under $SO(1,1) \times SO(4) \subset SO(1,5)$:

$$4 = 2_+ \oplus 2'_-.$$  \hspace{1cm} (8)

The next step is to restrict to the subgroup $SU(2)_R \subset Sp(2)$ of the R-symmetry. The $4$ of $Sp(2)$ becomes $2(2)$. Next we identify $SU(2)_R$ with the subgroup of $SO(4)$. This converts the $\lambda$ field into two fields each transforming in

$$(2 \otimes 2)_+ + (2 \otimes 2')_- = 3_+ + 1_+ + 4.$$  \hspace{1cm} (9)

Here $3$ is the anti-self-dual tensors of $SO(4)$. The two fields are real.

The 5 scalars are in the $5$ of $Sp(2)$ which decomposes as

$$5 = 3 + 2(1)$$  \hspace{1cm} (10)

under $SU(2) \subset Sp(2)$. The $3$ would give $2b^{2,0} + 1$ bosons. Altogether we find $2b^{2,0} + 3$ left moving and $2b^{2,0} + 3$ right moving bosons.

To sum up:

| 6D field | Left | Right |
|----------|------|-------|
| $B_{MN}^{(-)}$ | $b^{1,1} - 1$ bosons | $1 + 2b^{2,0}$ bosons |
| $\lambda$ | $2b^{2,0} + 3$ bosons | $2(2b^{2,0} + 2)$ fermions |
| $\phi$ | $2b^{2,0} + 3$ bosons | $2b^{2,0} + 3$ bosons |

**Table (1):** The fields of the 6D tensor multiplet after the twist.

Thus

$$c_L - c_R = b^{1,1} - 4b^{2,0} - 4.$$  \hspace{1cm} (11)

For K3 we find $c_L - c_R = 12$.

If we go further and try to guess from the results of [1] that $c_L(K3) = 24$ we find $c_R = 12$. Moreover, according to [1] we might guess that blowing up a point on K3 adds $\Delta c_L = +1$. Substituting we find the conjecture that $c_R = 12$ is true for K3 and blow-ups of K3.
4 Free field computation

Let us suppose, just for a moment, that the strings of S-theory are very tensile and we can neglect them altogether. This leaves us with only the free fields of the tensor multiplet in 6D. We recall from section (2) that in 2D all tensions are probed because scalar VEVs fluctuate – but we take the free field case as a starting point.

The results of the twist operation are given in table (1).

We see, for example, that for K3 the free field computation gives 24 free left moving bosons and 8 free right moving bosons with (0,8) free fermion super-partners.

The partition function of [1] for K3 is given by (18). Let us go back to (3) and see if this is what we get entirely from free-fields.

We will discuss the oscillator part and its relation to 24 free left-moving bosons later, but let us first see what the compactification radii of the 24 bosons are and why in (18) the momenta of the bosons (i.e. $p_L$) do not contribute.

The scalars $\phi_i$ are uncompactified and so have a matching condition $p_L = p_R$.

The radii of the modes from the anti-self-dual 2-form $B_{\mu\nu}^{(-)}$ have been determined in [4]. The requirement is that the 3-form field strength obtained from $B_{\mu\nu}^{(-)}$ gives an integer multiple of $2\pi$ when integrated on 3-cycles. Here for a 3-cycle we take $S^1$ (compactifying the spatial direction of 2D) times a 2-cycle in $H^2(X_4, \mathbb{Z})$. It is interesting to note that in [2] this requirement was necessary to obtain compact $U(1)$ in 4D, but now we really need this requirement already in 6D because there are physical objects whose $B_{\mu\nu}^{(-)}$ charge is quantized!

Thus, the left moving and right moving bosons that come from $B_{\mu\nu}^{(-)}$ live on a lattice of signature $(b^{1,1} - 1, 2b^{2,0} + 1)$ that corresponds to the embedding of self-dual and anti-self-dual 2-forms inside $H^2(X_4, \mathbb{Z})$ [2].

The fermions are in the R-sector, so it seems at first sight that there are no states in short $\mathcal{N} = 2$ multiplets. However, there is an extra $\mathbb{Z}_2$ that we didn’t yet take into account.
The extra $Z_2$ gauge symmetry

To complete the free field computation we need to orbifold by an extra $Z_2$ that reverses the sign of all the fields in the tensor multiplet:

$$Z_2 : B_{\mu\nu}^{(-)} \leftrightarrow -B_{\mu\nu}^{(-)}, \quad \phi^i \leftrightarrow -\phi^i, \quad \lambda \leftrightarrow -\lambda \quad (12)$$

The necessity to include this $Z_2$ is best seen from the description of [4] of the theory of two 5-branes that are close. The tensor multiplet that couples to the strings of S-theory is the difference between the two free low-energy fields on each 5-brane. The $Z_2$ global symmetry corresponds to interchange of the two 5-branes. After compactification to 4D it is related to the Weyl group of broken $SU(2)$ $\mathcal{N} = 4$ Yang-Mills and after compactification to 2D, as in our case, the $Z_2$ becomes a gauge symmetry and we need to orbifold by it.

Now we can see what are the short multiplets that enter into (3). Since the $Z_2$ acts as $(-)^F_R$ on the fermion zero modes, states with $p_R = 0$ and no right moving oscillators will contribute to (3).

The restriction on the $\phi$-s sets $p_L = p_R = 0$.

For generic K3-s the $H^{1,1}$ cohomology 2-forms give non-integer numbers when integrated on 2-cycles (because the $H^{2,0}$ cohomology mixes with $H^{1,1}$ to form $H^2(\mathbf{Z})$) and in fact for generic K3-s no combination of 19 self-dual 2-forms is in $H^2(\mathbf{Z})$. Thus $p_R = 0$ forces $p_L = 0$ for the $B_{\mu\nu}^{(-)}$ modes as well.

For a blow-up of K3, on the other hand, the exceptional divisor corresponds to an anti-self-dual integral (1,1) class and thus the mode that comes from it is compactified on a radius of $1/\sqrt{2}$. The $\sqrt{2}$ normalization is because, in the picture of [4], the strings are charged under the difference of two normalized $B_{\mu\nu}^{(-)}$-s, or in the construction of [3], the 2-cycle that shrinks to zero has self-intersection $(-2)$ and not $(-1)$. in general the radius is $1/\sqrt{2|\langle C \cdot C \rangle|}$ for an extra cycle $C$.

Let us calculate the resulting partition function for K3. In the untwisted sector there are 8 zero modes of the fermions on which the $Z_2$ acts like the fermion number. States with non-zero $(p_L, p_R)$ will be matched with $(-p_L, -p_R)$ and those two states have different fermion numbers so their total contribution cancels. States with zero momenta but an even number of left-moving oscillators will have to have an even number of zero modes and come
with a (+) sign while states with odd number have an odd number of zero modes and come with a minus sign. Altogether we find

\[
\frac{2^3}{q \prod_{n=1}^{\infty} (1 + q^n)^24} = \frac{2^3 \eta(q)^{24}}{\eta(q^2)^{24}} \tag{13}
\]

As for the twisted sectors, all oscillators have $\frac{1}{2}$-integer modes including the fermions. Thus there are no zero modes but there are $2^{19+3}$ fixed points and the twisted sector gives

\[
2^{19+3} \left( \frac{1}{2q^{1/2} \prod_{n=1}^{\infty} (1 - q^{n-1/2})^{24}} - \frac{1}{2q^{1/2} \prod_{n=1}^{\infty} (1 + q^{n-1/2})^{24}} \right) = 2^{21} \frac{\eta(q)^{24}}{\eta(q^{1/2})^{24}} + 2^{21} \frac{\eta(q)^{24}}{\eta(-q^{1/2})^{24}} \tag{14}
\]

Defining

\[
G(q) = \frac{1}{\eta^{24}} \tag{15}
\]

we find altogether

\[
\mathcal{Z} = \frac{1}{G(q)} (2^3 G(q^2) + 2^{21} G(q^{1/2}) + 2^{21} G(-q^{1/2})) \tag{16}
\]

We note that this is not completely modular invariant. For modular invariance we should have given a different weight to the twisted sectors since the modular invariant combination is

\[
\mathcal{Z} = \frac{1}{G(q)} (2^3 G(q^2) + 2^{15} G(q^{1/2}) + 2^{15} G(-q^{1/2})) \tag{17}
\]

This is the time to recall the results of Vafa and Witten [1]:

\[
G(q) = \frac{1}{\eta^{24}}
\]

\[
\mathcal{Z}_{SU(2)}(q = e^{i\tau}) = \frac{1}{8} G(q^2) + \frac{1}{4} G(q^{1/2}) + \frac{1}{4} G(-q^{1/2})
\]

\[
\mathcal{Z}_{SO(3)}(q = e^{i\tau}) = \frac{1}{4} G(q^2) + 2^{21} G(q^{1/2}) + 2^{10} G(-q^{1/2})
\]

\[
\tag{18}
\]

There are several differences between (18) and (16). First there is the overall factor of $G(q)$ that is missing in (16) and then the relative coefficient between the twisted and untwisted sectors in (16) does not agree with the coefficients of $G(q^{1/2})$ and $G(-q^{1/2})$ in either of the two formulas (18).
Before we try to interpret the discrepancy between (18) and (16) we have to ask two questions:

- Is S-theory on $T^2$ identical to $\mathcal{N} = 4$ $SU(2)$ Yang-Mills or are there topological restrictions on the gauge bundle? In particular, should we pick $SU(2)$ gauge bundles or $SO(3)$ or some combination of both?
- Do we expect the free field result above to be a good starting point for obtaining the complete low-energy in 2D?

We start with the first question. The authors of [1] calculated the modular transformation properties of their results (18) and found

$$Z_{SU(2)}(-\frac{1}{\tau}) = 2^{-12} \tau^{-12} Z_{SO(3)}$$

So we must start by asking whether it is possible that S-theory on $T^2 \times K3$ is not modular invariant. For 2D CFTs a modular anomaly usually appears when we do not have a (manifestly covariant) path-integral formula for the partition function and have to resort to a trace in the Hilbert space (for example a chiral boson away from the self-dual radius). In this case, the partition function is not a number but a section of a non-trivial line-bundle over the moduli space of Riemann-surfaces. Thus, we could accept the modular anomaly of S-theory if there is no (manifestly covariant) path-integral for it only some unknown Hilbert space. This is not so surprising since $B_{\mu\nu}^{(-)}$ is chiral.

The next problem is the discrepancy in the relative coefficients of $G(\pm q^{1/2})$ and $G(q^2)$. The answer to that seems to be that we do not trust the twisted sectors of the $Z_2$ orbifold. The fixed points are localized near the region where the 5 scalars of the tensor multiplet are zero. This is the region where the strings of S-theory are tensionless and the free field approximation is the farthest from being trustworthy.

Nevertheless, it is amusing to note that all that is missing seems to be a coefficient in the correct weight for the twisted sector in order to get the $Z_{SU(2)}$ result.

As for the question which gauge group we take, $SO(3)$ or $SU(2)$, it seems that the answer lies in the twisted sectors on which we have no information.

One possible answer might be that we actually get both $SU(2)$ and $SO(3)$ since we can move from one to the other by a “change of coordinates” on the patch of the moduli space.
We just want to note one more thing concerning the $2^{-19}$ factor of $G(\pm q^{1/2})$. The sum $G(q^{1/2}) + G(-q^{1/2})$ appeared in [1] as the contribution of gauge bundles of odd instanton numbers. The $SU(2)$ result for even instanton numbers is [1]:

$$Z_{SU(2)}^{(\text{even})} = \frac{1}{8} G(q^2)$$

(20)

and for odd instanton numbers [1]:

$$Z_{SU(2)}^{(\text{odd})} = \frac{1}{4} G(q^{1/2}) + \frac{1}{4} G(-q^{1/2})$$

(21)

We also note that from the above formula and the behaviour [1]:

$$Z \sim \frac{1}{q^2}$$

(22)

we can read off the momentum of the vacuum $L_0 - \bar{L}_0 = -2$ if this is indeed the partition function of a CFT.

Finally we wish to understand where the missing overall factor of $G(q)$ could possibly come from.

4.1 The effect of the strings

The “spectrum” of S-theory contains the tensor multiplet and the strings. The free field computation neglects the strings and now we wish to discuss what they can do.

To start with, let us note a few possibilities regarding the applicability of the free fields as a starting point.

First it might be that the free fields account for all the low-energy fields but there are generated interaction terms. In the next section we will check the instanton interaction terms and find that, for K3, instantons made by a string wrapping an analytic 2-cycle do not contribute to the low-energy. The other possibility is that the free fields (and their possible generated interactions) are indeed there but there are more low energy fields. This is the case, for example, when $X_4$ has non-trivial $\pi_1$ and the tensionless strings can wind around 1-cycles. We note that for compactification on $\mathbf{T}^2$ there are no modes of the “tensionless strings” other than those from $\pi_1(\mathbf{T}^2)$ that give rise to massless fields. In our case $\pi_1(K3) = 0$ so we do not expect more fields from such a mechanism either. The third possibility is of course that the free field orbifold is a bad starting point altogether.
Let us recall what the strings are capable of doing:

- They can wrap on 2-cycles of the K3 to produce instantons in 2D. This will be the subject of the next section.
- They can wrap on 1-cycles to produce more low-energy fields in 2D, but there are no topologically non-trivial 1-cycles on K3.
- They cannot “wrap” on 0-cycles of K3, i.e. be reduced on K3 and stretch on the entire 2D because the $B^{(-)}_{\mu\nu}$ charge will have nowhere to go.

None of these options seems to explain the missing $1/\eta^{24}$ factor, but let us speculate. It seems that what we need are states that are points in K3 and carry after compactification one unit of $L_0$ each. Then for $n$ such states the moduli space will be $K3^n/S_n$ and quantization as in [19] will give the partition function of 24 bosons. Such states cannot be the strings of S-theory because of $B^{(-)}_{\mu\nu}$ charge conservation and because they would then interact with the fields $\phi_i$ which set the tension.

## 5 Instanton calculation

In 2D scalars have no VEVs. Because correlators do not fall off at large distances, flat directions are interpreted as target space coordinates and not as a moduli space. Thus, the field $\sum_1^5 \phi_i^2$ that specifies the tension of the string does not have a VEV in 2D but fluctuates over the entire region. Nevertheless, we can think of a region of field-space for which the tension is large. In that region the leading low-energy action will be the free field theory with the fields found above and we intend to look for possible instanton corrections.

There are, however, (at least) three possible flaws in the plan. First, we have to be sure that there are no low-energy fields other than those from the free-field computation. In particular, we have to restrict to manifolds with $\pi_1(X_4) = 0$ since other-wise the strings that would wrap 1-cycles in $X_4$ would give new low-energy fields. Second, the twisted sectors of the $\mathbb{Z}_2$ orbifold from the previous section are localized near the small tension regime and that sector might be qualitatively different in the full theory. Bearing that in mind we will proceed.
What is the effect of the non-critical string?

A 6D string can wrap around a 2-cycle of $X_4$. Such an instanton can correct the effective action.

The situation is quite similar to that of [22] where a 5-brane that is wrapped on a 6-cycle in M-theory compactification to 3D could produce a super-potential.

There are however two important differences between that case and ours. First the geometry in our case is simpler because dimensions are lower and second the supersymmetry structure is different.

It was argued in [22] that only special 6-cycles (with arithmetic genus equal to 1) can create a super-potential. In four dimensions, to find whether a super-potential can be generated one has to count the number of zero modes of the instanton configuration that are in the 2 of the Lorenz group $SO(3,1)$ and subtract from it the number of fermionic zero modes in the other representation $2'$. An F-term can be generated if this difference is 2.

In 2D things are slightly different. We have to count the difference between the number of fermionic zero modes in the $+\frac{1}{2}$ of $SO(1,1)$ and in the $-\frac{1}{2}$ of $SO(1,1)$ (i.e. left moving and right-moving). However, whereas in 4D one can combine two 2-s to a scalar (and form an F-term Yukawa coupling), in 2D unequal numbers of left and right moving fermions can only be coupled to derivatives of scalars. Such terms will be irrelevant in the IR. In fact, an unequal number of left-moving and right-moving zero modes would seem to give a non Lorenz invariant contribution to the action.

There is, however, another contribution to the chirality in 2D. The instanton term for a string that wraps on a 2-cycle $D \subset X_4$ behaves like:

$$e^{-A_D V^{-1/2} \sqrt{\sum_{i=1}^5 \phi_i^2} e^{-i \int_D B^{(-)}}}$$

The first exponent is the tension of the string in terms of the scalars ($A_D$ is the area of $D$ and $V$ is the volume of $X_4$). The second term is the contribution of the interaction with the 2-form. It is expressible as

$$B^{(-)} = \sum_k \phi^k_L(x_0, x_1) \omega^k_-(x_2, \ldots, x_5) + \sum_{k'} \phi^{k'}_R(x_0, x_1) \omega^{k'}_+(x_2, \ldots, x_5)$$

where $(\omega^+) \omega^-$ are (anti) self-dual 2-forms on $X_4$ and $(\phi_L) \phi_R$ are (left) right-moving bosons.
The chirality (i.e. $L_0 - \bar{L}_0$) of the extra term $e^{-i \int_D B^{(-)}}$ is given by:

$$\frac{1}{2} \sum_k \left( \int_D \omega_k^{(-)} \right)^2 - \frac{1}{2} \sum_{k'} \left( \int_D \omega_{k'}^{(+)} \right)^2 = (D \cdot D)$$

(25)

where $D \cdot D$ is the self-intersection of $D$. (The $\sqrt{2}$ normalization is because, in the picture of $\mathbf{4}$, the strings are charged under the difference of two normalized $B^{(-)}_{\mu\nu}$’s, or in the construction of $\mathbf{4}$, the 2-cycle that shrinks to zero has self-intersection $(-2)$ and not $(-1)$.)

We will see in the next subsection that $-2(D \cdot D)$ is exactly the difference between right moving and left-moving zero modes. So, counting $\frac{1}{2}$ for each fermion shows that the total $L_0 - \bar{L}_0$ is zero, as it should.

5.1 The low-energy fields on the string

We will consider the case of a string wrapped on an analytic 2-cycle $D \subset X_4$.

Let us determine first the low-energy fields on the non-critical string after the twist. The effective low-energy on the (tensile) anti-self-dual string is the dimensional reduction of $\mathcal{N} = 1$ 6D $U(1)$ SYM [21] since it is derived from a wrapped 3-brane in type-IIB. Under

$$Sp(2)_R \times SO(4) \times SO(1,1),$$

(26)

where the $SO(4)$ is the transverse Lorenz group and $SO(1,1)$ is the world-sheet, the fields transform as follows:

| 2D field | Representation |
|----------|----------------|
| $A_a$    | $(\mathbf{1}, \mathbf{1})_{+1} \oplus (\mathbf{1}, \mathbf{1})_{-1}$ |
| $\lambda$ | $(\mathbf{4}, \mathbf{2})_{+\frac{1}{2}}$ |
| $\phi$   | $(\mathbf{1}, \mathbf{4})_0$ |

where $A_a$ is the world-sheet gauge field, $\lambda$ are the fermions and $\phi$ represent transverse oscillations of the world-sheet.

5.2 Wrapping and Twisting.

Now we embed the string in the compact 4 dimensions. Let’s call the non-compact directions 0, 1. The string is in the compact 2, 3 directions and the other compact directions are 4, 5.
What are the quantum numbers under

\[ SO(2)_N \times SO(2)_K \times SO(1, 1) \]  

(27)

where \( SO(2)_N \) is rotations in the 4, 5 directions, \( SO(2)_K \) is in 2, 3 and \( SO(1, 1) \) is in 0, 1?

The twist replaces the 4 of \( Sp(2) \) with 2(2) of \( SO(4) \) in the 2, 3, 4, 5 directions, so

\[ \mathbf{4} \longrightarrow 2\left(\frac{1}{2}, 0, \frac{1}{2}, 0\right) \oplus 2\left(-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, 0\right) \]  

(28)

Also, the 2 of \( SO(4) \) becomes

\[ \mathbf{2} \longrightarrow \left(+\frac{1}{2}, 0, \frac{1}{2}\right) \oplus \left(-\frac{1}{2}, 0, -\frac{1}{2}\right) \]  

(29)

(recalling that now the four directions orthogonal to the string are 0, 1, 4, 5). and the \(+\frac{1}{2}\) in the table above becomes

\[ (, +\frac{1}{2}) \longrightarrow (0, +\frac{1}{2}, 0) \]  

(30)

The twist replaces \( \lambda \) with anti-commuting world-sheet fields in the

\[ \mathbf{4} \otimes \mathbf{2} \otimes (\cdots)_{+\frac{1}{2}} \longrightarrow 2(+1, +1, +\frac{1}{2}) \oplus 2(0, +1, -\frac{1}{2}) \oplus 2(0, 0, +\frac{1}{2}) \oplus 2(-1, 0, -\frac{1}{2}) \]  

(31)

Now we wrap around a 2-cycle \( D \). Let the canonical bundle of \( D \) be \( \mathcal{K} \) and the normal bundle of \( D \) in \( X_4 \) be \( \mathcal{N} \). So, in terms of bundles we have right-moving fermions in

\[ 2\mathcal{K} \otimes \mathcal{N} + 2\mathcal{O} \]  

(32)

and left-movers in

\[ 2\mathcal{K} \oplus 2\mathcal{N}^{-1} \]  

(33)

where \( \mathcal{O} \) is the trivial bundle. We see that we get two generic right-moving zero modes as we should. Those are the super-partners of translations \([22]\). We also see that for K3, \( \mathcal{K} \otimes \mathcal{N} \) is trivial and we get two more generic zero modes.

The difference between left and right moving zero modes is

\[ 2\{h^0(\mathcal{K} \otimes \mathcal{N}) - h^0(\mathcal{K}) - h^0(\mathcal{N}^{-1})\} \]  

(34)

Using the Riemann-Roch theorem this becomes

\[ 2(c_1(\mathcal{N}) - 1) = -2(D \cdot D) - 2 \]  

(35)

If we add to this the 2 generic zero modes from \( \mathcal{O} \) we find that this cancels the chirality of \( e^{-\int_D B^{(-)}} \) exactly.
5.3 Conditions on 2-cycles

If $-2(D \cdot D) - 2 < 0$ it means that there are more left-moving zero modes than right-moving ones (other than the 2 generic ones from $O$). Since there are no left moving fermions the instanton term will vanish in this case. If $-2(D \cdot D) - 2 = 0$ the resulting instanton term looks like

$$e^{-A_D V^{-1/2}} \sum_{i=1}^{5} \phi_i^2 \left( e^{-i(\alpha \phi_R + \beta \phi_L)} \psi_R^1 \psi_R^2 \right)$$

(36)

where $\psi^i_R$ are right moving fermions and $\phi_R$ and $\phi_L$ are (right and left) chiral bosons. $\alpha, \beta$ are some coefficients. If $\alpha$ vanishes, we find (by a naive count of dimensions) an operator of dimension $(L_0 = 1, \bar{L}_0 = 1)$ in the large $\phi_i$ region. If $\alpha$ does not vanish or if $-2(D \cdot D) - 2 > 0$ in which case there are more right moving fermions, the instanton term will be irrelevant in the IR. The condition that we found is thus $D \cdot D = -1$. It also means that the cycle $D$ is isolated since $D \cdot D$ is also the first Chern class of the normal bundle.

On a 2-cycle $D$ inside a K3, $\mathcal{N}_D = -K_D$ and the anomaly condition (35) implies $2g - 2 = c_1(K_D) = -1$ which is impossible. So no cycle meets the conditions and it is plausible to expect no instanton correction. An analytic cycle $D \subset \mathbb{CP}^2$ always has positive intersection and doesn’t meet the condition either. An exceptional divisor from a blown-up point, on the other hand, is the generic example for a cycle that does meet the condition.

It is a bit puzzling however that this is the case because the results of [1] for a blow-up are essentially free-field results. On the other hand the result for $\mathbb{CP}^2$ of [1] have a holomorphic anomaly, i.e. are not a function of $q$ but of $q$ and $\bar{q}$. It seems that an instanton might cause such an effect because an irregular behaviour in the spectrum of $\phi_i$ but we found that (at least for analytic cycles) the instanton term vanishes.

Finally we note that compactification on $T^2 \times \Sigma_g$, a torus times a Riemann surface $\Sigma_g$ of genus $g$, can be studied from the results of [30, 31]. They discussed compactification of partially twisted $\mathcal{N} = 4$ on $\Sigma_g$ with non-trivial ‘t Hooft flux and found a sigma model with Hyper-Kähler target space $\mathcal{M}_{6g-6}$ of solutions to a Hitchin system of equations.

This is an example with $\pi_1(X_4) \neq 0$ where the extra fields come from strings wound on the $2g$ cycles of $\Sigma_g$.

Compactifications of S-theory on $\Sigma_g$ might also be studied in a setting similar to that of [22] of 5-branes wrapped on $\mathbb{R}^{3,1} \times \Sigma_g$. 

18
6 Discussion

Out of the new quantum theories M, F and S the latter which is probably the simplest was the least explored. Like M-theory and F-theory, the description of S-theory in its uncompactified form is unknown but compactifications of S-theory can be described in a more conventional way in the low-energy.

In this paper we were interested in the compactification down to 2D. The identification of S-theory on $T^2$ as $SU(2) \mathcal{N} = 4$ SYM and the results of Vafa and Witten on twisted SYM partition functions made it possible to study the 2D IR limit. Furthermore, the low-energy description of the tensile strings of S-theory, which was derived from the D-brane description and the techniques developed in [22] enabled us to study the contribution of strings wrapped on 2-cycles to the 2D low-energy.

We compared the results of Vafa and Witten for K3 to the contribution of (a $\mathbb{Z}_2$ orbifold of) the 6D tensor multiplet and found a missing $1/\eta^{24}$ factor which remained a puzzle.

We also found that anomaly conditions like in [22] seem to prevent an instanton contribution from wrapped strings for K3.

This paper dealt with what might be called type-II S-theory (i.e. $\mathcal{N} = 2$ in 6D). There are other types as well. The one related to small $E_8$ instantons might be called heterotic S-theory and it is interesting to compactify that theory as well. This time we have to specify an $E_8$ gauge field background in addition to the 4-manifold. It seems also natural to ask what are orbifolds of S-theory? Are orientifolds possible? Is there a type-I S-theory as well?

Finally we come to the hardest question: what did we learn about the microscopic description of S-theory. Even from the compactification on $T^2$ we see that the result can be described with only a finite number of fields in the low energy. This is a bit mysterious from the point of view of tensionless strings with an infinite number of states becoming massless (see e.g. [33]). Although this is not exactly a contradiction because most of the string spectrum has width of order $1/\lambda_{st}$ and now $\lambda_{st} \sim 1$ but still it is hard to see how a conventional world-sheet approach could reproduce that. The $\mathcal{N} = 2$ strings [34, 35] which have recently been suggested to play a fundamental role [37, 38] seem to exhibit a better behaviour and in particular do not have the unwanted tower of states but their critical
dimension is not 6 and it is still hard to see how any world-sheet approximation could be a good starting point when \( \lambda_{st} \sim 1 \). As was recalled in [38] the Green-Schwarz action is classically defined in \( D = 6 \). There is a good S-theory reason for that! It is the low-energy description of a tensile string of S-theory. However a low-energy approximation cannot be extrapolated to a microscopic theory [39].

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References

[1] C. Vafa and E. Witten, “A Strong Coupling Test of S-Duality”, Nucl. Phys. B431 (1994) 3
[2] E. Verlinde, “Global Aspects of Electric-Magnetic Duality”, hep-th/9506011
[3] E. Witten, “Some Comments on String Dynamics”, contributed to Strings ’95, hep-th/9507121
[4] A. Strominger, “Open p-Branes”, hep-th/9512059
[5] E. Witten, “Five-Branes and M-Theory On an Orbifold”, Nucl. Phys. B463 (1996) 383
[6] O.J. Ganor and A. Hanany, “Small \( E_8 \) Instantons and Tensionless Non-Critical Strings”, hep-th/9602120
[7] M.J. Duff, R. Minasian and E. Witten, “Evidence For Heterotic / Heterotic Duality”, hep-th/9601036
[8] N. Seiberg and E. Witten, “Comments On String Dynamics In Six-Dimensions”, hep-th/9603003
[9] E. Witten, “Phase Transitions In M-Theory And F-Theory”, hep-th/9603150
[10] A. Hanany and I.R. Klebanov, “On Tensionless Strings in 3 + 1 Dimensions”, hep-th/9606136
[11] R. Dijkgraaf, E. Verlinde and H. Verlinde, “BPS Spectrum of the Five-Brane and Black Hole Entropy”, hep-th/9603126

[12] R. Dijkgraaf, E. Verlinde and H. Verlinde, “BPS Quantization of the Five-Brane”, hep-th/9604055

[13] J. Maldacena, “Statistical Entropy of Near Extremal Five-Branes”, hep-th/9605016

[14] R. Dijkgraaf, E. Verlinde and H. Verlinde, “Counting Dyons in \( N = 4 \) String Theory”, hep-th/9607026

[15] R. Gopakumar and S. Mukhi, “Orbifold and Orientifold Compactifications of F-Theory and M-Theory to Six and Four Dimensions”, hep-th/9607057

[16] C. Vafa, “Evidence For F-Theory”, hep-th/96020

[17] D.R. Morrison and C. Vafa, “Compactifications Of F-Theory On Calabi-Yau Threefolds - I”, hep-th/9602114

[18] D.R. Morrison and C. Vafa, “Compactifications Of F-Theory On Calabi-Yau Threefolds - II”, hep-th/9603161

[19] C. Vafa, “Instantons on D-branes”, hep-th/9512078

[20] J. Polchinski, “Dirichlet-Branes and Ramond-Ramond Charges”, ITP preprint, hep-th/9510017

[21] E. Witten, “Bound States of Strings and p-Branes”, IASSNS-HEP-95-83, hep-th/9510135

[22] E. Witten, “Non-perturbative Super-potentials In String Theory”, hep-th/9604030

[23] M. Bershadsky, C. Vafa, V. Sadov, “D-Branes And Topological Field Theories” HUP-95-A047, hep-th/9511222

[24] S. Katz, D.R. Morrison and M.R. Plesser, “Enhanced Gauge Symmetry In Type II String Vacua”, hep-th/9601108

[25] E. Witten, private communication.

[26] N. Marcus, “The Other Topological Twisting Of \( N = 4 \) Yang-Mills”, Nucl. Phys. B452 (1995) 331
[27] E. Witten, “On The Landau-Ginzburg Description Of N = 2 Minimal Models”, hep-th/9304026

[28] E. Witten and C. Vafa, “A One-Loop Test Of String Duality”, hep-th/9505053

[29] S. Sethi, C. Vafa and E. Witten, “Constraints On Low-Dimensional String Compactifications”, hep-th/96061

[30] M. Bershadsky, A. Johansen, V. Sadov and C. Vafa, “Topological Reduction of 4D SYM to 2D Sigma Models”, Nucl. Phys. B448 (1995) 166

[31] J.A. Harvey and G. Moore and A. Strominger, “Reducing S-Duality to T-Duality”, Phys. Rev. D52 (1995) 7161

[32] A. Klemm, W.Lerche, P. Mayr, C. Vafa and N. Warner, “Self-dual Strings and N = 2 Super-symmetric Field Theory”, hep-th/9604034

[33] F. Lizzi and G. Sparano, “Statistical Mechanics Of Null-Strings”, Nucl. Phys. B232 (199) 311 and refs. therin

[34] H. Ooguri and C. Vafa, “Geometry of N = 2 Strings”, Nucl. Phys. B361 (1991) 469

[35] H. Ooguri and C. Vafa, “N = 2 Heterotic Strings”, Nucl. Phys. B367 (1991) 83

[36] D. Kutasov, E. Martinec and M. O’Loughlin, “Vacua of M-theory and N = 2 Strings”, hep-th/9603116

[37] D. Kutasov and E. Martinec, “New Principles For String / Membrane Unification”, hep-th/9602049

[38] J.H. Schwarz, “Self-Dual Super-string in Six Dimensions”, hep-th/9604171

[39] E. Witten, Lectures on string duality, Summer 1995, Princeton