Gauge-invariant operators of open bosonic string field theory in the low-energy limit

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Abstract

In the AdS/CFT correspondence we consider correlation functions of gauge-invariant operators on the gauge theory side, which we obtain in the low-energy limit of the open string sector. To investigate this low-energy limit we consider the action of open bosonic string field theory including source terms for gauge-invariant operators and classically integrate out massive fields to obtain the effective action for massless fields. While the gauge-invariant operators depend linearly on the open string field and do not resemble the corresponding operators such as the energy-momentum tensor in the low-energy limit, we find that nonlinear dependence is generated in the process of integrating out massive fields. We also find that the gauge transformation is modified in such a way that the effective action and the modified gauge transformation can be written in terms of the same set of multi-string products which satisfy weak A∞ relations, and we present explicit expressions for the multi-string products.
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The AdS/CFT correspondence [1] can be thought of as providing a nonperturbative definition of closed string theory in terms of a gauge theory without containing gravity. In the standard explanation [2] of the AdS/CFT correspondence we consider the theory on D-branes and take the low-energy limit. The gauge theory which provides a nonperturbative definition of closed string theory is obtained from the low-energy limit of the open string sector.

A key ingredient for proving the AdS/CFT correspondence is the equivalence of string theory with holes in the world-sheet and string theory on a different background without holes in the world-sheet. While such equivalence has been shown in the topological string [3], it is still challenging to prove it in the physical string. Even if we assume this equivalence, it is difficult to see gravity directly from the gauge theory, and one reason for this is that the world-sheet picture is gone after taking the low-energy limit. This motivates us to consider open string field theory as a theory on D-branes before taking the low-energy limit.

In the AdS/CFT correspondence we consider correlation functions of gauge-invariant operators on the gauge theory side. We are therefore interested in correlation functions of gauge-invariant operators of open string field theory in this context. While it is in general difficult to construct gauge-invariant operators in string field theory, gauge-invariant operators for open bosonic string field theory [7] have been constructed in [8, 9]. If we consider the theory on N D-branes, it was shown that the 1/N expansion of correlation functions of gauge-invariant operators has an interpretation as the closed-string genus expansion under the assumption of the equivalence between string theory with holes in the world-sheet and string theory without holes [10]. Thus to consider open string field theory as a theory before taking the low-energy limit can be a promising way for proving the AdS/CFT correspondence because we can keep track of the world-sheet picture in the limit.

Of course, we need to extend the discussion to open superstring field theory, as quantization of open bosonic string field theory is formal because of tachyons in generic backgrounds. Recently, there have been significant developments in constructing actions of open superstring field theory including the Ramond sector [11, 12, 13, 14], and we consider that it is time to

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1 See, for example, [4, 5, 6] for developments in this direction.
study open superstring field theory in this context. On the other hand, it would be also useful to consider gauge-invariant operators of open bosonic string field theory in the noncritical string or in the topological string where tachyons are absent. For example, it was shown in [15] that the Kontsevich model [16] can be realized as open bosonic string field theory in the noncritical string, and open bosonic string field theory on topological A-branes is equivalent to Chern-Simons theory [17]. See also [18] for recent discussion on holography in the B-model topological string. It would be also interesting to consider D-instanton contributions in two-dimensional string theory [19, 20, 21, 22, 23] from the viewpoint of open bosonic string field theory with gauge-invariant operators. With the extension to open superstring field theory in mind, let us begin with the discussion in open bosonic string field theory.

Construction of the low-energy effective action of string field theory was discussed by Sen [24] for closed superstring field theory. The string field projected onto the massless sector is used to describe the low-energy effective action, and it was shown that the gauge invariance of the low-energy effective action inherited from that of the original theory. The same strategy can be applied to open bosonic string field theory, and in this paper we consider the action including source terms for the gauge-invariant operators and use the string field projected onto the massless sector to study gauge-invariant operators in the low energy. In the case of open bosonic string field theory, however, we can integrate out massive fields only classically because, as we mentioned before, the existence of tachyons in the open string and in the closed string renders the quantization formal. Furthermore, we also integrate out the open-string tachyon and construct the effective action in terms of massless fields, although the tachyon plays an important role in the low energy. Actually, we can also use the string field projected onto the tachyonic and massless sectors to construct the effective action in terms of massless and tachyonic fields, and, as we will see, the structure of the effective action does not depend on the detail of the projection. At any rate, this issue is an artifact of the bosonic theory and we are interested in stable backgrounds of the superstring where tachyons are absent.

One puzzling feature regarding gauge-invariant operators of open bosonic string field theory in our context is that they depend linearly on the open string field. For example, the energy-momentum tensor is a typical example of the gauge-invariant operators we consider in the AdS/CFT correspondence, but the gauge-invariant operators of open bosonic string field theory do not resemble familiar energy-momentum tensors. In this paper we show that nonlinear dependence on the open string field is generated in the process of integrating out massive fields. Although our discussion is in the bosonic theory and classical, the mechanism of generating nonlinear dependence can be easily understood in terms of Feynman diagrams, and we expect that the same mechanism will work in the quantum theory of the superstring.

While the generation of nonlinear dependence of the gauge-invariant operators on the open string field is what we expected, an unexpected feature regarding the effective action is that
the gauge transformation is modified when we include the source terms, and gauge invariance requires terms which are nonlinear with respect to the sources. There are some similarities between terms in the effective action and terms in the gauge transformation, and we find that the effective action and the modified gauge transformation can be written in terms of the same set of multi-string products which satisfy weak $A_\infty$ relations\(^2\). With hindsight, the original action including the source terms has a weak $A_\infty$ structure in a rather trivial fashion, and the weak $A_\infty$ structure of the effective action can be understood as being inherited from that of the original action in the process of integrating out massive fields in accord with the general consideration by Sen [24].

An advantage of considering open string field theory based on the star product is that expressions for terms in the effective action are simpler and more explicit compared to closed string field theory. However, expressions for terms in the effective action become rather lengthy at higher orders even in open bosonic string field theory based on the star product. The weak $A_\infty$ structure provides us with analytic control over terms in the effective action, and in this paper we present explicit expressions for the multi-string products to all orders.

The rest of the paper is organized as follows. We begin with presenting the action and the gauge transformation of open bosonic string field theory in section 2, and we explain the gauge-invariant operators of open bosonic string field theory in section 3. We discuss the effective action of open bosonic string field theory for massless fields in section 4 and then we incorporate the gauge-invariant operators into the discussion of the effective action for massless fields in section 5. We expand the effective action and the gauge transformation in powers of the sources and in powers of the fields, and we explicitly construct terms at lower orders. In section 6 we present the basics of the weak $A_\infty$ structure, and we show that the terms of the effective action and the gauge transformation constructed in section 5 can be expressed using multi-string products satisfying weak $A_\infty$ relations. In section 7 we explain the coalgebra representation of the weak $A_\infty$ structure, and we make use of this coalgebra representation to construct the multi-string products to all orders in section 8. Section 9 is devoted to conclusions and discussion.

When the draft of this paper was almost complete, we were informed of independent work addressing the same problem by Erbin, Maccaferri, Schnabl and Vosmera [31], which is arranged to appear simultaneously with ours.

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\(^2\) The $A_\infty$ algebra consists of a set of multi-string products satisfying $A_\infty$ relations and plays an important role in open string field theory [25, 26, 27, 28, 29, 30]. When zero-string products are incorporated, it is referred to as the weak $A_\infty$ algebra or the curved $A_\infty$ algebra.
2 Open bosonic string field theory

In this section we present the action and the gauge transformation of open bosonic string field theory \[7\]. The open string field \( \Psi \) is a Grassmann-odd state of ghost number 1 in the Hilbert space of the boundary conformal field theory describing the open string background we consider, which consists of the matter sector and the \( bc \) ghost sector. The action of the free theory is given by

\[
S = -\frac{1}{2} \langle \Psi, Q\Psi \rangle ,
\]

(2.1)

where \( Q \) is the BRST operator which is Grassmann odd and \( \langle A_1, A_2 \rangle \) is the BPZ inner product of a pair of states \( A_1 \) and \( A_2 \). Three basic properties of the BPZ inner product and the BRST operator \( Q \) are

\[
\langle A_1, A_2 \rangle = (-1)^{A_1A_2}\langle A_2, A_1 \rangle , \quad Q^2 = 0 , \quad \langle QA_1, A_2 \rangle = -(-1)^{A_1}\langle A_1, QA_2 \rangle .
\]

(2.2)

Here and in what follows, a state in the exponent of \(-1\) represents its Grassmann parity: it is 0 mod 2 for a Grassmann-even state and 1 mod 2 for a Grassmann-odd state. The action of the free theory is invariant under the gauge transformation given by

\[
\delta_\Lambda \Psi = Q\Lambda ,
\]

(2.3)

where the gauge parameter \( \Lambda \) is a Grassmann-even state of ghost number 0.

The interacting theory by Witten \[7\] was constructed by introducing the star product \( A_1 * A_2 \) defined for a pair of states \( A_1 \) and \( A_2 \). The Grassmann parity of \( A_1 * A_2 \) is \( \epsilon(A_1) + \epsilon(A_2) \) mod 2, where \( \epsilon(A_i) \) is the Grassmann parity of \( A_i \) mod 2 for \( i = 1, 2 \). Three important properties involving the star product are

\[
\langle A_1, A_2 * A_3 \rangle = \langle A_1 * A_2, A_3 \rangle , \quad Q(A_1 * A_2) = QA_1 * A_2 + (-1)^{A_1} A_1 * QA_2 ,
\]

(2.4)

\[
(A_1 * A_2) * A_3 = A_1 * (A_2 * A_3) .
\]

The action of the interacting theory is given by

\[
S = -\frac{1}{2} \langle \Psi, Q\Psi \rangle - \frac{g}{3} \langle \Psi, \Psi * \Psi \rangle ,
\]

(2.5)

where \( g \) is the open string coupling constant. Using (2.2) and (2.4), we can show that this action is invariant under the gauge transformation given by

\[
\delta_\Lambda \Psi = Q\Lambda + g \left( \Psi * \Lambda - \Lambda * \Psi \right) .
\]

(2.6)


3 Gauge-invariant operators

It is in general difficult to construct gauge-invariant operators in string field theory, but a class of gauge-invariant operators have been constructed in open bosonic string field theory and we can define a gauge-invariant operator for each on-shell closed string vertex operator \[8, 9\].

We label on-shell closed string vertex operators of ghost number 2 as \( V_\alpha \), where the collective label \( \alpha \) generically contains both continuous and discrete variables. The gauge-invariant operator \( A_{V_\alpha}[\Psi] \) for \( V_\alpha \) is defined by the following correlation function on the upper half-plane:

\[
A_{V_\alpha}[\Psi] = \langle f_{\text{mid}} \circ V_\alpha(0) f_I \circ \Psi(0) \rangle_{\text{UHP}},
\]

where \( \Psi(0) \) is the operator corresponding to the state \( \Psi \) in the state-operator correspondence. Here and in what follows we denote the operator mapped from \( \phi(0) \) at \( \xi = 0 \) in the local coordinate \( \xi \) under a conformal transformation \( f(\xi) \) by \( f \circ \phi(0) \) both for boundary and bulk operators. The conformal transformation \( f_I(\xi) \) associated with the identity string field is given by

\[
f_I(\xi) = \tan \left( 2 \arctan \xi \right) = \frac{2\xi}{1 - \xi^2}
\]

and \( f_{\text{mid}}(\xi) \) is the translation to the open-string midpoint:

\[
f_{\text{mid}}(\xi) = \xi + i.
\]

We denote the closed string state corresponding to \( V_\alpha \) in the state-operator correspondence by \( \Phi_\alpha \), which is annihilated by the BRST operator:

\[
Q \Phi_\alpha = 0.
\]

Then the gauge-invariant operator \( A_{V_\alpha}[\Psi] \) can be expressed as a BPZ inner product with \( \Phi_\alpha \):

\[
A_{V_\alpha}[\Psi] = \langle \Phi_\alpha, D(\Psi) \rangle,
\]

where the closed string field \( D(A) \) constructed from an open string field \( A \) is defined in terms of the BPZ inner product with an arbitrary closed string field \( B \) as

\[
\langle B, D(A) \rangle = \langle f_{\text{mid}} \circ B(0) f_I \circ A(0) \rangle_{\text{UHP}},
\]

where \( A(0) \) is the open string field corresponding to \( A \) and \( B(0) \) is the closed string field corresponding to \( B \) in the state-operator correspondence. Since the gauge-invariant operator \( A_{V_\alpha}[\Psi] \) is linear in \( \Psi \), it can also be expressed in terms of a BPZ inner product with \( \Psi \) as

\[
A_{V_\alpha}[\Psi] = \langle J(\Phi_\alpha), \Psi \rangle,
\]
where the open string field \( J(B) \) constructed from a closed string field \( B \) is defined in terms of the BPZ inner product with an arbitrary open string field \( A \) as

\[
\langle J(B), A \rangle = \langle f_{\text{mid}} \circ B(0) f_I \circ A(0) \rangle_{\text{UHP}}.
\] (3.8)

The Grassmann parity of \( J(B) \) is the same as that of \( B \mod 2 \). Two important properties associated with \( J(B) \) are as follows:

\[
QJ(B) = J(QB),
\]

\[
J(B) \ast A = (-1)^{AB} A \ast J(B),
\] (3.9)

where \( A \) is an arbitrary open string field. We can use these properties to show the gauge invariance of \( A_{V_\alpha}[\Psi] \). The gauge variation of \( A_{V_\alpha}[\Psi] \) vanishes because

\[
\delta_\lambda A_{V_\alpha}[\Psi] = A_{V_\alpha}[Q\Lambda + g(\Psi \ast \Lambda - \Lambda \ast \Psi)]
\]

\[
= \langle J(\Phi_\alpha), Q\Lambda \rangle + g \langle J(\Phi_\alpha), \Psi \ast \Lambda \rangle - g \langle J(\Phi_\alpha), \Lambda \ast \Psi \rangle
\]

\[
= - \langle QJ(\Phi_\alpha), \Lambda \rangle + g \langle J(\Phi_\alpha) \ast \Psi, \Lambda \rangle - g \langle \Psi \ast J(\Phi_\alpha), \Lambda \rangle
\]

\[
= - \langle J(Q\Phi_\alpha), \Lambda \rangle + g \langle J(\Phi_\alpha) \ast \Psi, \Lambda \rangle - g \langle J(\Phi_\alpha) \ast \Psi, \Lambda \rangle = 0,
\] (3.10)

where in the last step we used \( Q\Phi_\alpha = 0 \).

We introduce the source \( G_\alpha \) for \( A_{V_\alpha}[\Psi] \). The source term \( S_{\text{source}} \) added to the action is given by

\[
S_{\text{source}} = \sum_\alpha G_\alpha A_{V_\alpha}[\Psi],
\] (3.11)

where the summation over \( \alpha \) should be understood to include integrals for continuous variables. The source term can be written in terms of \( D(\Psi) \) as

\[
S_{\text{source}} = \sum_\alpha G_\alpha \langle \Phi_\alpha, D(\Psi) \rangle
\]

(3.12)
or in terms of \( J(\Phi_\alpha) \) as

\[
S_{\text{source}} = \sum_\alpha G_\alpha \langle J(\Phi_\alpha), \Psi \rangle.
\] (3.13)

On-shell closed string states can be incorporated into a single closed string field \( \Phi \) as

\[
\Phi = \sum_\alpha G_\alpha \Phi_\alpha.
\] (3.14)

The on-shell closed string field \( \Phi \) is a Grassmann-even state of ghost number 2 and is annihilated by the BRST operator:

\[
Q\Phi = 0.
\] (3.15)
Then the source term $S_{\text{source}}$ can be expressed as

$$S_{\text{source}} = \langle \Phi, D(\Psi) \rangle$$  \hspace{1cm} (3.16)

or as

$$S_{\text{source}} = \langle J(\Phi), \Psi \rangle.$$ \hspace{1cm} (3.17)

In the rest of the paper we will use this form of the source term.

The action with the source term is given by

$$S = -\frac{1}{2} \langle \Psi, Q \Psi \rangle - \frac{g}{3} \langle \Psi, \Psi \ast \Psi \rangle + \frac{\kappa}{g} \langle J(\Phi), \Psi \rangle,$$ \hspace{1cm} (3.18)

where we introduced the parameter $\kappa$ to count the power of the sources. We can show that the action with the source term (3.18) is invariant under the gauge transformation (2.6) using the following properties of $J(\Phi)$:

$$QJ(\Phi) = 0,$$
$$J(\Phi) \ast A = A \ast J(\Phi)$$ \hspace{1cm} (3.19)

for any open string field $A$.

4 Open string field theory in the low-energy limit

We are interested in the low-energy limit of correlation functions of gauge-invariant operators in open string field theory. Let us first discuss the low-energy limit of open bosonic string field theory following the approach developed in [24] without introducing gauge-invariant operators.

The low-energy limit of closed superstring field theory discussed in [24] is described by the string field projected onto the massless sector. The action for the massless fields corresponds to the effective action obtained by integrating out massive fields. The idea can be applied to open string field theory at least in the classical theory, which corresponds to classically integrating out massive fields. In the case of open bosonic string field theory the low-energy limit is formal because of the existence of the tachyon in the spectrum. Since this is an artifact of the bosonic string and is absent in the superstring, we ignore this issue and we classically integrate out the tachyon field in addition to massive fields. Alternatively, we can also use the string field projected onto the tachyonic and massless sectors to construct the effective action in terms of massless and tachyonic fields, and, as we will see, the structure of the effective action does not depend on the detail of the projection.

An open string field $\Psi$ for massless fields is annihilated by $L_0 - \alpha' p^2$,

$$( L_0 - \alpha' p^2 ) \Psi = 0,$$ \hspace{1cm} (4.1)
where $L_0$ is the zero mode of the energy-momentum tensor, $p_\mu$ is the spacetime momentum operator, and $p^2 = p_\mu p^\mu$. We denote the projection operator onto the massless sector by $P$. The string field $\Psi$ for massless fields satisfies

$$P\Psi = \Psi.$$  \hfill (4.2)

The projection operator $P$ has the following properties:

$$P^2 = P, \quad PQ = QP, \quad \langle A_1, PA_2 \rangle = \langle PA_1, A_2 \rangle$$  \hfill (4.3)

for any pair of states $A_1$ and $A_2$.

Let us consider open bosonic string field theory in terms of this projected string field. The kinetic term of the theory is given by

$$S = -\frac{1}{2} \langle \Psi, Q\Psi \rangle,$$

where the string field $\Psi$ satisfies the condition

$$P\Psi = \Psi.$$  \hfill (4.5)

This kinetic term is invariant under the gauge transformation given by

$$\delta_\Lambda \Psi = Q\Lambda,$$

where the gauge parameter $\Lambda$ satisfies the condition

$$PL = \Lambda.$$  \hfill (4.7)

Let us add the cubic interaction to this kinetic term:

$$S = -\frac{1}{2} \langle \Psi, Q\Psi \rangle - \frac{g}{3} \langle \Psi, \Psi * \Psi \rangle.$$  \hfill (4.8)

The variation of the action under the gauge transformation

$$\delta_\Lambda \Psi = Q\Lambda + gP (\Psi * \Lambda - \Lambda * \Psi)$$  \hfill (4.9)

vanishes at $O(g)$, but it is nonvanishing at $O(g^2)$. Following the approach developed in [24], let us add the quartic term obtained by classically integrating out massive and tachyonic fields:

$$S = -\frac{1}{2} \langle \Psi, Q\Psi \rangle - \frac{g}{3} \langle \Psi, \Psi * \Psi \rangle + \frac{g^2}{2} \langle \Psi * \Psi, \frac{b_0}{L_0} (1 - P) (\Psi * \Psi) \rangle + O(g^3),$$  \hfill (4.10)

where $b_0$ is the zero mode of the $b$ ghost. See figure [1] The propagator for massive and tachyonic
fields appears frequently in what follows, and we denote it by $h$:

$$h \equiv \frac{b_0}{L_0} (1 - P).$$  \hfill (4.11)

It is Grassmann odd and is BPZ even:

$$\langle A_1, h A_2 \rangle = (-1)^{A_1} \langle h A_1, A_2 \rangle \quad (4.12)$$

for any pair of states $A_1$ and $A_2$. The important property of $h$ is

$$Q h + h Q = 1 - P. \quad (4.13)$$

We can show that the action is invariant up to $O(g^2)$ under the gauge transformation given by

$$\begin{align*}
\delta_\Lambda \Psi &= Q \Lambda + g P \left( \Psi \ast \Lambda - \Lambda \ast \Psi \right) \\
&\quad + g^2 P \left[ -h \left( \Psi \ast \Lambda \right) \ast \Lambda + h \left( \Psi \ast \Lambda \right) \ast \Psi - h \left( \Lambda \ast \Psi \right) \ast \Psi \\
&\quad - \Lambda \ast h \left( \Psi \ast \Lambda \right) + \Psi \ast h \left( \Lambda \ast \Psi \right) + \Lambda \ast h \left( \Psi \ast \Psi \right) \right] + O(g^3). \quad (4.14)
\end{align*}$$

5 Open string field theory with the source term in the low-energy limit

5.1 Modification of the gauge transformation

Let us now incorporate gauge-invariant operators by adding the source term in the low-energy limit. We denote the action in the low-energy limit before introducing the source term by $S^{(0)}$ and we expand it in $g$ as follows:

$$S^{(0)} = S_2^{(0)} + g S_3^{(0)} + g^2 S_4^{(0)} + O(g^3), \quad (5.1)$$
where
\[ S_2^{(0)} = -\frac{1}{2} \langle \Psi, Q \Psi \rangle, \quad S_3^{(0)} = -\frac{1}{3} \langle \Psi, \Psi * \Psi \rangle, \quad S_4^{(0)} = \frac{1}{2} \langle \Psi * \Psi, h (\Psi * \Psi) \rangle. \] (5.2)

We denote the gauge transformation by \( \delta^{(0)} \Psi \), and we also expand it in \( g \) as follows:
\[ \delta^{(0)} \Psi = \delta_0^{(0)} \Psi + g \delta_1^{(0)} \Psi + g^2 \delta_2^{(0)} \Psi + O(g^3), \] (5.3)

where
\[ \delta_0^{(0)} \Psi = Q \Lambda, \quad \delta_1^{(0)} \Psi = P (\Psi * \Lambda - \Lambda * \Psi) \]
\[ \delta_2^{(0)} \Psi = P \left[ -h (\Psi * \Psi) * \Lambda + h (\Psi * \Lambda) * \Psi - h (\Lambda * \Psi) * \Psi \right. \]
\[ \left. - \Psi * h (\Psi * \Lambda) + \Psi * h (\Lambda * \Psi) + \Lambda * h (\Psi * \Psi) \right]. \] (5.4)

We add the coupling
\[ \frac{\kappa}{g} \langle J(\Phi), \Psi \rangle \] (5.5)

to the action \( S^{(0)} \), where \( \Psi \) satisfies the condition \( P \Psi = \Psi \). It is easy to see that this coupling is invariant under \( \delta_0^{(0)} \Psi \),
\[ \delta_0^{(0)} \left[ \frac{\kappa}{g} \langle J(\Phi), \Psi \rangle \right] = 0, \] (5.6)

but it is not invariant when we include the correction \( \delta_1^{(0)} \Psi \):
\[ g \delta_1^{(0)} \left[ \frac{\kappa}{g} \langle J(\Phi), \Psi \rangle \right] \neq 0. \] (5.7)

Based on the insight we learned from the approach developed in [24], we expect that the gauge invariance can be recovered if we include terms which are obtained from the source term and the cubic vertex by classically integrating out massive and tachyonic fields. To leading order in \( g \), the following term is generated:
\[ -\kappa \langle J(\Phi), h (\Psi * \Psi) \rangle. \] (5.8)

Note that this term depends nonlinearly on \( \Psi \). See figure 2. Let us calculate the variation under the gauge transformation
\[ \delta^{(0)} \left[ \frac{\kappa}{g} \langle J(\Phi), \Psi \rangle - \kappa \langle J(\Phi), h (\Psi * \Psi) \rangle + O(g) \right], \] (5.9)

which can be expanded in \( g \) as
\[ \frac{1}{g} \left[ \kappa \delta_0^{(0)} \langle J(\Phi), \Psi \rangle \right] + \left[ \kappa \delta_1^{(0)} \langle J(\Phi), \Psi \rangle - \kappa \delta_0^{(0)} \langle J(\Phi), h (\Psi * \Psi) \rangle \right] + O(g). \] (5.10)
Figure 2: Nonlinear coupling to the gauge-invariant operators is generated in the effective action from the source term and the cubic vertex by integrating out massive and tachyonic fields. The black dot denotes the source, and the propagator for massive and tachyonic fields is shaded in the figure.

We find that the variation at $O(\kappa^0)$ is still nonvanishing,

$$\kappa \delta_1^{(0)} \langle J(\Phi), \Psi \rangle - \kappa \delta_0^{(0)} \langle J(\Phi), h (\Psi \ast \Psi) \rangle \neq 0, \quad (5.11)$$

but this can be written as

$$\kappa \delta_1^{(0)} \langle J(\Phi), \Psi \rangle - \kappa \delta_0^{(0)} \langle J(\Phi), h (\Psi \ast \Psi) \rangle = \kappa \delta_0^{(1)} \left[ \frac{1}{2} \langle \Psi, Q \Psi \rangle \right] \quad (5.12)$$

with $\delta_0^{(1)} \Psi$ given by

$$\delta_0^{(1)} \Psi = P \left[ h J(\Phi) \ast \Lambda - \Lambda \ast h J(\Phi) \right]. \quad (5.13)$$

This means that the gauge invariance can be restored at this order if we modify the gauge transformation at $O(\kappa)$.

As we expected, the gauge invariance was recovered by adding the term (5.8). This term is nonlinear in $\Psi$, and we have learned that such nonlinear terms can be generated by integrating out massive fields. This has resolved the puzzle we mentioned in section 1. On the other hand, what we did not expect was that the gauge transformation is modified when we include the source term. Let us explore more about this structure by investigating higher-order terms in the action.

### 5.2 Expansion in $\kappa$ and $g$

Since we have learned that the gauge transformation is modified when we include the coupling to the gauge-invariant operators, it is convenient to expand the action and the gauge transformation in $\kappa$ to discuss the gauge invariance systematically. We expand the action $S$ and the gauge transformation $\delta_{\Lambda} \Psi$ as

$$S = S^{(0)} + \kappa S^{(1)} + \kappa^2 S^{(2)} + \kappa^3 S^{(3)} + O(\kappa^4), \quad (5.14)$$

$$\delta_{\Lambda} \Psi = \delta^{(0)} \Psi + \kappa \delta^{(1)} \Psi + \kappa^2 \delta^{(2)} \Psi + O(\kappa^3), \quad (5.15)$$
Then the condition $\delta_{\Delta}S = 0$ for the gauge invariance can be expanded as
\[ \delta_{\Delta}S = \delta^{(0)}S^{(0)} + \kappa \left[ \delta^{(0)}S^{(1)} + \delta^{(1)}S^{(0)} \right] + \kappa^2 \left[ \delta^{(0)}S^{(2)} + \delta^{(1)}S^{(1)} + \delta^{(2)}S^{(0)} \right] + \kappa^3 \left[ \delta^{(0)}S^{(3)} + \delta^{(1)}S^{(2)} + \delta^{(2)}S^{(1)} + \delta^{(3)}S^{(0)} \right] + O(\kappa^4) = 0. \] (5.16)

### 5.2.1 Construction at $O(\kappa)$

We further expand $S^{(1)}$ and $\delta^{(1)}\Psi$ in $g$ as
\[ S^{(1)} = \frac{1}{g} S_1^{(1)} + S_2^{(1)} + g S_3^{(1)} + O(g^2), \] (5.17)
\[ \delta^{(1)}\Psi = \delta_0^{(1)}\Psi + g \delta_1^{(1)}\Psi + O(g^2), \] (5.18)
where $S_1^{(1)}$ and $S_2^{(1)}$ are given by
\[ S_1^{(1)} = \langle J(\Phi), \Psi \rangle, \quad S_2^{(1)} = -\langle J(\Phi), h(\Psi \ast \Psi) \rangle. \] (5.19)

Then the condition
\[ \delta^{(0)}S^{(1)} + \delta^{(1)}S^{(0)} = 0 \] (5.20)
for the gauge invariance at $O(\kappa)$ can be expanded in $g$ as
\[ \frac{1}{g} \delta_0^{(0)}S_1^{(1)} + \left[ \delta_1^{(0)}S_1^{(1)} + \delta_0^{(0)}S_2^{(1)} + \delta_0^{(1)}S_2^{(0)} \right] + g \left[ \delta_2^{(0)}S_1^{(1)} + \delta_0^{(0)}S_2^{(1)} + \delta_0^{(1)}S_2^{(0)} + \delta_1^{(1)}S_2^{(0)} + \delta_0^{(1)}S_3^{(0)} \right] + O(g^2) = 0. \] (5.21)

We have so far shown that
\[ \delta_0^{(0)}S_1^{(1)} = 0, \quad \delta_0^{(0)}S_1^{(1)} + \delta_0^{(0)}S_2^{(1)} + \delta_0^{(1)}S_2^{(0)} = 0, \] (5.22)
and the condition at the next order in $g$ is
\[ \delta_2^{(0)}S_1^{(1)} + \delta_0^{(0)}S_2^{(1)} + \delta_0^{(0)}S_3^{(1)} + \delta_1^{(1)}S_2^{(0)} + \delta_0^{(1)}S_3^{(0)} = 0. \] (5.23)

As before, we infer $S_3^{(1)}$ based on Feynman diagrams as
\[ S_3^{(1)} = \langle J(\Phi), h(\Psi \ast \Psi \ast \Psi) \rangle + \langle J(\Phi), h(\Psi \ast h(\Psi \ast \Psi)) \rangle, \] (5.24)
and we indeed find that the condition (5.23) can be satisfied by choosing $\delta_1^{(1)}\Psi$ as
\[ \delta_1^{(1)}\Psi = P \left[ h(\Psi \ast \Lambda) \ast h J(\Phi) - h(\Lambda \ast \Psi) \ast h J(\Phi) - h J(\Phi) \ast h(\Psi \ast \Lambda) \right] + h J(\Phi) \ast h(\Lambda \ast \Psi) - \Psi \ast h(h J(\Phi) \ast \Lambda) + \Psi \ast h(\Lambda \ast h J(\Phi)) + h (h J(\Phi) \ast \Lambda) \ast \Psi + \Lambda \ast h(h J(\Phi) \ast \Psi) + \Lambda \ast h(h J(\Phi) \ast \Psi) \ast \Lambda - h(\Psi \ast h J(\Phi)) \ast \Lambda \right]. \] (5.25)

See figure 3 for the two terms in $S_3^{(1)}$. 

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Figure 3: Illustration of the two terms in $S_3^{(1)}$. Propagators for massive and tachyonic fields are shaded in the figure.

### 5.2.2 Construction at $O(\kappa^2)$

Let us next consider the gauge invariance at $O(\kappa^2)$. The condition at this order is

$$\delta^{(0)} S^{(2)} + \delta^{(1)} S^{(1)} + \delta^{(2)} S^{(0)} = 0 .$$  \hspace{1cm} (5.26)

We expand $S^{(2)}$ and $\delta^{(2)} \Psi$ in $g$ as

$$S^{(2)} = \frac{1}{g} S_1^{(2)} + S_2^{(2)} + O(g) ,$$  \hspace{1cm} (5.27)

$$\delta^{(2)} \Psi = \delta_0^{(2)} \Psi + O(g) ,$$  \hspace{1cm} (5.28)

and the condition (5.26) can be expanded as

$$\delta^{(0)} S^{(2)} + \delta^{(1)} S^{(1)} + \delta^{(2)} S^{(0)}$$

$$= \frac{1}{g} \left[ \delta_0^{(0)} S_1^{(2)} + \delta_0^{(1)} S_1^{(1)} \right] + \left[ \delta_0^{(0)} S_1^{(2)} + \delta_0^{(0)} S_2^{(2)} + \delta_1^{(1)} S_1^{(1)} + \delta_0^{(1)} S_2^{(1)} + \delta_0^{(2)} S_2^{(0)} \right] + O(g) = 0 .$$  \hspace{1cm} (5.29)

We find that $\delta_0^{(1)} S_1^{(1)}$ is nonvanishing,

$$\delta_0^{(1)} S_1^{(1)} \neq 0 ,$$  \hspace{1cm} (5.30)

but the condition

$$\delta_0^{(0)} S_1^{(2)} + \delta_0^{(1)} S_1^{(1)} = 0$$  \hspace{1cm} (5.31)

can be satisfied with $S_1^{(2)}$ given by

$$S_1^{(2)} = - \langle J(\Phi) , h ( h J(\Phi) * \Psi ) \rangle$$  \hspace{1cm} (5.32)

in accord with the consideration based on Feynman diagrams.

At the next order in $g$, we learn from known terms in the effective action and the gauge transformation that

$$\delta_1^{(0)} S_1^{(2)} + \delta_1^{(1)} S_1^{(1)} + \delta_0^{(1)} S_2^{(1)} \neq 0 .$$  \hspace{1cm} (5.33)
By explicit calculations we find that the condition
\[ \delta_1^{(0)} S_1^{(2)} + \delta_0^{(0)} S_2^{(2)} + \delta_1^{(1)} S_1^{(1)} + \delta_0^{(1)} S_2^{(1)} + \delta_0^{(2)} S_2^{(0)} = 0 \] (5.34)
is satisfied with \( S_2^{(2)} \) and \( \delta_0^{(2)} \Psi \) given by
\[ S_2^{(2)} = \langle h J(\Phi) \ast h J(\Phi) , h (\Psi \ast \Psi) \rangle + \langle h J(\Phi) \ast \Psi , h (\Psi \ast h J(\Phi)) \rangle \\
+ \frac{1}{2} \langle h J(\Phi) \ast \Psi , h (h J(\Phi) \ast \Psi) \rangle + \frac{1}{2} \langle \Psi \ast h J(\Phi) , h (\Psi \ast h J(\Phi)) \rangle \] (5.35)
and
\[ \delta_0^{(2)} \Psi = P [ \Lambda \ast h (h J(\Phi) \ast h J(\Phi)) - h (h J(\Phi) \ast h J(\Phi)) \ast \Lambda \\
- h (\Lambda \ast h J(\Phi)) \ast h J(\Phi) + h J(\Phi) \ast h (\Lambda \ast h J(\Phi)) \\
- h J(\Phi) \ast h (h J(\Phi) \ast \Lambda) + h (h J(\Phi) \ast \Lambda) \ast h J(\Phi) ] . \] (5.36)

5.2.3 Construction at \( O(\kappa^3) \)

The condition for the gauge invariance at \( O(\kappa^3) \) is
\[ \delta^{(0)} S^{(3)} + \delta^{(1)} S^{(2)} + \delta^{(2)} S^{(1)} + \delta^{(3)} S^{(0)} = 0 . \] (5.37)
We expand \( S^{(3)} \) in \( g \) as
\[ S^{(3)} = \frac{1}{g} S_1^{(3)} + O(g^0) . \] (5.38)
Since \( \delta^{(3)} \Psi \) is of \( O(g^0) \),
\[ \delta^{(3)} \Psi = O(g^0) , \] (5.39)
the condition \([5.37]\) can be expanded as
\[ \delta^{(0)} S_1^{(3)} + \delta^{(1)} S_1^{(2)} + \delta^{(2)} S_1^{(1)} + \delta^{(3)} S_1^{(0)} = \frac{1}{g} \left[ \delta_0^{(0)} S_1^{(3)} + \delta_1^{(1)} S_1^{(2)} + \delta_0^{(2)} S_1^{(1)} \right] + O(g^0) = 0 . \] (5.40)
From known terms in the effective action and the gauge transformation we find that
\[ \delta_1^{(1)} S_1^{(2)} + \delta_0^{(2)} S_1^{(1)} \neq 0 . \] (5.41)
We determine \( S_1^{(3)} \) such that the condition
\[ \delta_0^{(0)} S_1^{(3)} + \delta_0^{(1)} S_1^{(2)} + \delta_0^{(2)} S_1^{(1)} = 0 \] (5.42)
is satisfied. Our result is as follows:
\[ S_1^{(3)} = \langle h J(\Phi) \ast h J(\Phi) , h (\Psi \ast h J(\Phi)) \rangle + \langle h J(\Phi) \ast h J(\Phi) , h (h J(\Phi) \ast \Psi) \rangle . \] (5.43)
5.3 Structure

To summarize, we can organize the terms in the action we have constructed as follows:

\[
S = S_2^{(0)} + g S_3^{(0)} + g^2 S_4^{(0)} + O(g^3) \\
+ \kappa \left[ \frac{1}{g} S_1^{(1)} + S_2^{(1)} + g S_3^{(1)} + O(g^2) \right] \\
+ \kappa^2 \left[ \frac{1}{g} S_1^{(2)} + S_2^{(2)} + O(g) \right] \\
+ \kappa^3 \left[ \frac{1}{g} S_1^{(3)} + O(g^0) \right] \\
+ O(\kappa^4). \tag{5.44}
\]

We can also organize the terms in the gauge transformation as follows:

\[
\delta_A \Psi = \delta_0^{(0)} \Psi + g \delta_1^{(0)} \Psi + g^2 \delta_2^{(0)} \Psi + O(g^3) \\
+ \kappa \left[ \delta_0^{(1)} \Psi + g \delta_1^{(1)} \Psi + O(g^2) \right] \\
+ \kappa^2 \left[ \delta_0^{(2)} \Psi + O(g) \right] \\
+ O(\kappa^3). \tag{5.45}
\]

When we compare \( S_m^{(n)} \) and \( \delta_{m-2}^{(n)} \Psi \) with \( m \geq 2 \), we recognize some similarities. In fact, we translate the result in \([24]\) into our setup to learn that the action \( S^{(0)} \) before introducing the source term and the gauge transformation \( \delta^{(0)} \Psi \) can be written in terms of the same set of multi-string products and this is reflected, for example, in a similarity between \( S_4^{(0)} \) and \( \delta_2^{(0)} \Psi \).

We will find in the following sections that the action \( S \) after introducing the source term and the gauge transformation \( \delta_A \Psi \) are also written in terms of the same set of multi-string products. This explains, for example, the similarity between \( S_2^{(1)} \) and \( \delta_0^{(1)} \Psi \).

The multi-string products before introducing the source term satisfy a set of relations called \( A_\infty \) relations. We say that an action has an \( A_\infty \) structure when it is written in terms of multi-string products which satisfy the \( A_\infty \) relations. While the expressions for the action and the gauge transformation become very complicated at higher orders, there is an efficient way to describe the \( A_\infty \) structure which provides us with control over all-order expressions.

The multi-string products after introducing the source term contain a zero-string product, and we will show that they satisfy a set of relations called weak \( A_\infty \) relations. We say that an action has a weak \( A_\infty \) structure when it is written in terms of multi-string products which satisfy the weak \( A_\infty \) relations. There is also an efficient way to describe the weak \( A_\infty \) structure, and we will use it to construct the action and the gauge transformation after introducing the source term to all orders.
6 Weak $A_\infty$ structure

In this section we will show that the action and the gauge transformation constructed in the preceding section can be expressed in terms of the same set of multi-string products which satisfy the weak $A_\infty$ relations. While the $A_\infty$ structure plays an important role in recent research on open string field theory, the weak $A_\infty$ structure is less familiar. We therefore begin with the basics of the $A_\infty$ structure and then motivate the generalization to the weak $A_\infty$ structure.

6.1 $A_\infty$ structure up to quartic interactions

Let us consider an action of the form 

$$S = \frac{1}{2} \langle \Psi, Q \Psi \rangle - \frac{g}{3} \langle \Psi, V_2(\Psi, \Psi) \rangle - \frac{g^2}{4} \langle \Psi, V_3(\Psi, \Psi, \Psi) \rangle + O(g^3), \quad (6.1)$$

where $V_2(A_1, A_2)$ defined for a pair of string fields $A_1$ and $A_2$ is a two-string product and $V_3(A_1, A_2, A_3)$ defined for three string fields $A_1$, $A_2$, and $A_3$ is a three-string product. The Grassmann parity of $V_2(A_1, A_2)$ is $\epsilon(A_1) + \epsilon(A_2) \mod 2$ and the Grassmann parity of $V_3(A_1, A_2, A_3)$ is $\epsilon(A_1) + \epsilon(A_2) + \epsilon(A_3) + 1 \mod 2$, where $\epsilon(A_i)$ is the Grassmann parity of $A_i \mod 2$ for $i = 1, 2, 3$.

These string products are assumed to have the following cyclic properties:

$$\langle A_1, V_2(A_2, A_3) \rangle = (-1)^{A_1(A_2+A_3)} \langle A_2, V_2(A_3, A_1) \rangle, \quad (6.2)$$

$$\langle A_1, V_3(A_2, A_3, A_4) \rangle = - (-1)^{A_1(A_2+A_3+A_4)} \langle A_2, V_3(A_3, A_4, A_1) \rangle. \quad (6.3)$$

These are equivalently described as

$$\langle A_1, V_2(A_2, A_3) \rangle = \langle V_2(A_1, A_2), A_3 \rangle, \quad (6.4)$$

$$\langle A_1, V_3(A_2, A_3, A_4) \rangle = - (-1)^{A_1} \langle V_3(A_1, A_2, A_3), A_4 \rangle. \quad (6.5)$$

The variation of the action (6.1) is then given by

$$\delta S = - \langle \delta \Psi, Q \Psi \rangle - g \langle \delta \Psi, V_2(\Psi, \Psi) \rangle - g^2 \langle \delta \Psi, V_3(\Psi, \Psi, \Psi) \rangle + O(g^3). \quad (6.6)$$

Under the gauge transformation given by

$$\delta \Lambda \Psi = Q \Lambda + g (V_2(\Psi, \Lambda) - V_2(\Lambda, \Psi)) + g^2 (V_3(\Psi, \Psi, \Lambda) - V_3(\Psi, \Lambda, \Psi) + V_3(\Lambda, \Psi, \Psi)) + O(g^3), \quad (6.7)$$

the action (6.1) is invariant up to $O(g^3)$,

$$\delta \Lambda S = O(g^3), \quad (6.8)$$
if the following relations hold:

\begin{align}
Q^2 &= 0, \quad (6.9) \\
Q V_2(A_1, A_2) - V_2(QA_1, A_2) - (-1)^{A_1} V_2(A_1, QA_2) &= 0, \quad (6.10) \\
Q V_3(A_1, A_2, A_3) + V_3(QA_1, A_2, A_3) + (-1)^{A_1} V_3(A_1, QA_2, A_3) + (-1)^{A_1 + A_2} V_3(A_1, A_2, QA_3) \\
&- V_2(V_2(A_1, A_2), A_3) + V_2(A_1, V_2(A_2, A_3)) = 0. \quad (6.11)
\end{align}

These relations can be extended to all orders in $g$, and they are called $A_\infty$ relations. As we mentioned before, we say that an action has an $A_\infty$ structure when it is written in terms of multi-string products which satisfy the $A_\infty$ relations.

### 6.2 $A_\infty$ structure in the low energy

The action (2.5) of open bosonic string field theory has an $A_\infty$ structure up to $O(g^3)$ under the identification

\begin{align}
V_2(A_1, A_2) &= A_1 \ast A_2, \quad V_3(A_1, A_2, A_3) = 0. \quad (6.12)
\end{align}

This two-string product has the cyclic property (6.4), and the $A_\infty$ relations (6.10) and (6.11) are satisfied with the vanishing three-string product. In fact, the action (2.5) has an $A_\infty$ structure to all orders, as we will see later.

The observation in [24] for the gauge invariance of the low-energy effective action in the context of closed string field theory can be translated into our setup, and it implies that the action $S^{(0)}$ in the low energy (5.1) also has an $A_\infty$ structure. In fact, the action $S^{(0)}$ up to $O(g^3)$ has an $A_\infty$ structure under the identification

\begin{align}
V_2(A_1, A_2) &= P (A_1 \ast A_2), \quad (6.13) \\
V_3(A_1, A_2, A_3) &= (-1)^{A_1} P (A_1 \ast h (A_2 \ast A_3)) - P (h (A_1 \ast A_2) \ast A_3). \quad (6.14)
\end{align}

These string products have the cyclic properties (6.4) and (6.5) for string fields projected onto the massless sector satisfying

\begin{align}
PA_i &= A_i \quad (6.15)
\end{align}

for $i = 1, 2, 3$. We can also confirm that the $A_\infty$ relations (6.10) and (6.11) are satisfied.

### 6.3 The BRST operator as a one-string product

As a step to the generalization to the weak $A_\infty$ structure, let us slightly change the notation for the $A_\infty$ structure. The action of the BRST operator can be regarded as a one-string product. We define the one-string product $V_1(A_1)$ for a string field $A_1$ by

\begin{align}
V_1(A_1) &= QA_1. \quad (6.16)
\end{align}
The Grassmann parity of $V_1(A_1)$ is then $\epsilon(A_1) + 1 \mod 2$. The BRST operator is BPZ odd,

$$\langle QA_1, A_2 \rangle = -(-1)^{A_1} \langle A_1, QA_2 \rangle, \quad (6.17)$$

and this is translated into the following cyclic property of the one-string product:

$$\langle A_1, V_1(A_2) \rangle = -(-1)^{A_1 + A_2 + A_1 A_2} \langle A_2, V_1(A_1) \rangle, \quad (6.18)$$

which is equivalently described as

$$\langle A_1, V_1(A_2) \rangle = -(-1)^{A_1} \langle V_1(A_1), A_2 \rangle. \quad (6.19)$$

The action $(6.1)$ is then written as

$$S = -\frac{1}{2} \langle \Psi, V_1(\Psi) \rangle - \frac{g}{3} \langle \Psi, V_2(\Psi, \Psi) \rangle - \frac{g^2}{4} \langle \Psi, V_3(\Psi, \Psi, \Psi) \rangle + O(g^3), \quad (6.20)$$

and the variation of the action $(6.1)$ is

$$\delta S = -\langle \delta \Psi, V_1(\Psi) \rangle - g \langle \delta \Psi, V_2(\Psi, \Psi) \rangle - g^2 \langle \delta \Psi, V_3(\Psi, \Psi, \Psi) \rangle + O(g^3). \quad (6.21)$$

The $A_\infty$ relations up to $O(g^3)$ can also be expressed as

$$V_1(V_1(A_1)) = 0, \quad (6.22)$$
$$V_1(V_2(A_1, A_2)) - V_2(V_1(A_1), A_2) - (-1)^{A_1} V_2(A_1, V_1(A_2)) = 0, \quad (6.23)$$
$$V_1(V_3(A_1, A_2, A_3)) + V_3(V_1(A_1), A_2, A_3) + (-1)^{A_1} V_3(A_1, V_1(A_2), A_3)$$
$$+ (-1)^{A_1 + A_2} V_3(A_1, A_2, V_1(A_3)) - V_2(V_2(A_1, A_2), A_3) + V_2(A_1, V_2(A_2, A_3)) = 0. \quad (6.24)$$

### 6.4 Action with linear terms

Let us now consider an action including a term linear in $\Psi$. We write it in the following form:

$$S = -\frac{1}{g} \langle \Psi, V_0 \rangle - \frac{1}{2} \langle \Psi, V_1(\Psi) \rangle - \frac{g}{3} \langle \Psi, V_2(\Psi, \Psi) \rangle - \frac{g^2}{4} \langle \Psi, V_3(\Psi, \Psi, \Psi) \rangle$$
$$- \frac{g^3}{5} \langle \Psi, V_4(\Psi, \Psi, \Psi, \Psi) \rangle + O(g^4), \quad (6.25)$$

where $V_0$ is a Grassmann-even string field of ghost number 2 and $V_4(A_1, A_2, A_3, A_4)$ is a four-string product defined for four string fields $A_1$, $A_2$, $A_3$, and $A_4$. The Grassmann parity of $V_4(A_1, A_2, A_3, A_4) = \epsilon(A_1) + \epsilon(A_2) + \epsilon(A_3) + \epsilon(A_4) \mod 2$, where $\epsilon(A_i)$ is the Grassmann parity of $A_i \mod 2$ for $i = 1, 2, 3, 4$. The four-string product is assumed to have the following cyclic property:

$$\langle A_1, V_4(A_2, A_3, A_4, A_5) \rangle = (-1)^{A_1 + A_2 + A_3 + A_4 + A_5} \langle A_2, V_4(A_3, A_4, A_5, A_1) \rangle, \quad (6.26)$$
These relations can be extended to all orders in $g$ we mentioned before, we say that an action has a weak structure when the action is written in terms of the multi-string products which satisfy the weak relations. As we mentioned before, we say that an action has a weak $A_\infty$ structure when the action is written in terms of the multi-string products which satisfy the weak $A_\infty$ relations.

Note that the relation (6.32) implies that the one-string product does not square to zero when $V_2(V_0, A_1) \neq V_2(A_1, V_0)$. This indicates that the one-string product is in general different from the BRST operator.

We can think of $V_0$ as a zero-string product. Since it is a Grassmann-even string field, we have

$$\langle A_1, V_0 \rangle = \langle V_0, A_1 \rangle$$

(6.28)

for any string field $A_1$, and this can be regarded as the cyclic property of the zero string product. The variation of the action is given by

$$\delta S = -\frac{1}{g} \langle \delta \Psi, V_0 \rangle - \langle \delta \Psi, V_1(\Psi) \rangle - g \langle \delta \Psi, V_2(\Psi, \Psi) \rangle - g^2 \langle \delta \Psi, V_3(\Psi, \Psi, \Psi) \rangle - g^3 \langle \delta \Psi, V_4(\Psi, \Psi, \Psi, \Psi) \rangle + O(g^4).$$

(6.29)

We can show that this action is invariant up to $O(g^3)$ under the gauge transformation given by

$$\delta \Lambda \Psi = V_1(\Lambda) + g(V_2(\Psi, \Lambda) - V_2(\Lambda, \Psi)) + g^2(V_3(\Psi, \Lambda) - V_3(\Lambda, \Psi)) + V_3(\Lambda, \Psi) + g^3(V_4(\Psi, \Lambda, \Psi) + V_4(\Lambda, \Psi)) + O(g^4)$$

(6.30)

if the multi-string products $V_0, V_1, V_2, V_3,$ and $V_4$ satisfy the following relations:

$$V_1(V_0) = 0,$$

(6.31)

$$V_1(V_1(A_1)) - V_2(V_0, A_1) + V_2(A_1, V_0) = 0,$$

(6.32)

$$V_1(V_2(A_1, A_2)) - V_2(V_1(A_1), A_2) - (-1)^{A_1}V_2(A_1, V_1(A_2)) + V_3(V_0, A_1, A_2) - V_3(A_1, V_0, A_2) + V_3(A_1, A_2, V_0) = 0,$$

(6.33)

$$V_1(V_3(A_1, A_2, A_3)) + V_3(V_1(A_1), A_2, A_3) + (-1)^{A_1}V_3(A_1, V_1(A_2), A_3) + (-1)^{A_1 + A_2}V_3(A_1, A_2, V_1(A_3)) - V_2(V_2(A_1, A_2), A_3) + V_2(A_1, V_2(A_2, A_3)) - V_4(V_0, A_1, A_2, A_3) + V_4(A_1, V_0, A_2, A_3) - V_4(A_1, A_2, V_0, A_3) + V_4(A_1, A_2, A_3, V_0) = 0.$$

(6.34)

These relations can be extended to all orders in $g$, and they are called weak $A_\infty$ relations. We have extended to all orders in $g$, and they are called weak $A_\infty$ relations. As we mentioned before, we say that an action has a weak $A_\infty$ structure when the action is written in terms of the multi-string products which satisfy the weak $A_\infty$ relations.
6.5 Weak $A_{\infty}$ structure in the low energy

The action (3.18) has a weak $A_{\infty}$ structure up to $O(g^2)$ under the identification

$$V_0 = - \kappa J(\Phi), \quad V_1(A_1) = QA_1, \quad V_2(A_1, A_2) = A_1 \ast A_2,$$
$$V_3(A_1, A_2, A_3) = 0, \quad V_4(A_1, A_2, A_3, A_4) = 0. \quad (6.35)$$

In fact, the action (3.18) has a weak $A_{\infty}$ structure to all orders, as we will see later.

One of the main observations in this paper is that the action of open bosonic string field theory with the source term in the low-energy limit also has a weak $A_{\infty}$ structure. The action and the gauge transformation we constructed in section 5 can be expressed in terms of the same set of multi-string products. For example, the term $S_2^{(1)}$ in the action and $\delta_0^{(1)} \Psi$ in the gauge transformation given by

$$S_2^{(1)} = - \langle J(\Phi), h(\Psi \ast \Psi) \rangle, \quad \delta_0^{(1)} \Psi = P \left[ h J(\Phi) \ast \Lambda \ast \Lambda \ast h J(\Phi) \right] \quad (6.36)$$

can be written as

$$S_2^{(1)} = - \frac{1}{2} \langle \Psi, V_1^{(1)}(\Psi) \rangle, \quad \delta_0^{(1)} \Psi = V_1^{(1)}(\Lambda) \quad (6.37)$$
in terms of the one-string product $V_1^{(1)}(A_1)$ given by

$$V_1^{(1)}(A_1) = P \left[ h J(\Phi) \ast A_1 - (-1)^{A_1} A_1 \ast h J(\Phi) \right]. \quad (6.38)$$

All the terms in the action and the gauge transformation we constructed in section 5 can be written using the following multi-string products $V_0$, $V_1$, $V_2$, and $V_3$. The zero-string product $V_0$ is given by

$$V_0 = \kappa V_0^{(1)} + \kappa^2 V_0^{(2)} + \kappa^3 V_0^{(3)} + O(\kappa^4), \quad (6.39)$$

where

$$V_0^{(1)} = - P J(\Phi), \quad (6.40)$$

$$V_0^{(2)} = P \left[ h J(\Phi) \ast h J(\Phi) \right], \quad (6.41)$$

$$V_0^{(3)} = - P \left[ h J(\Phi) \ast (h J(\Phi) \ast h J(\Phi)) + h (h J(\Phi) \ast h J(\Phi)) \ast h J(\Phi) \right]. \quad (6.42)$$

The one-string product $V_1$ is given by

$$V_1(A_1) = V_1^{(0)}(A_1) + \kappa V_1^{(1)}(A_1) + \kappa^2 V_1^{(2)}(A_1) + O(\kappa^3), \quad (6.43)$$

where

$$V_1^{(0)}(A_1) = QA_1, \quad (6.44)$$

$$V_1^{(1)}(A_1) = P \left[ h J(\Phi) \ast A_1 - (-1)^{A_1} A_1 \ast h J(\Phi) \right], \quad (6.45)$$

$$V_1^{(2)}(A_1) = P \left[ (-1)^{A_1} A_1 \ast h (h J(\Phi) \ast h J(\Phi)) - h (h J(\Phi) \ast h J(\Phi)) \ast A_1 \right.$$

$$- h (A_1 \ast h J(\Phi)) \ast h J(\Phi) + (-1)^{A_1} h J(\Phi) \ast h (A_1 \ast h J(\Phi)) \right.$$

$$- h J(\Phi) \ast h (h J(\Phi) \ast A_1) + (-1)^{A_1} h (h J(\Phi) \ast A_1) \ast h J(\Phi) \right]. \quad (6.46)$$
The two-string product $V_2$ is given by

$$V_2(A_1, A_2) = V_2^{(0)}(A_1, A_2) + \kappa V_2^{(1)}(A_1, A_2) + O(\kappa^2),$$  \hspace{1cm} (6.47)

where

$$V_2^{(0)}(A_1, A_2) = P \left[ A_1 * A_2 \right],$$

$$V_2^{(1)}(A_1, A_2) = P \left[ -h ( h J(\Phi) * A_1 ) * A_2 + (-1)^{A_1} h ( A_1 * h J(\Phi) ) * A_2 - (-1)^{A_1} h ( A_1 * A_2 ) \right].$$  \hspace{1cm} (6.48)

The three-string product $V_3$ is given by

$$V_3(A_1, A_2, A_3) = V_3^{(0)}(A_1, A_2, A_3) + O(\kappa),$$  \hspace{1cm} (6.50)

where

$$V_3^{(0)}(A_1, A_2, A_3) = P \left[ (-1)^{A_1} A_1 * h ( A_2 * A_3 ) - h ( A_1 * A_2 ) * A_3 \right].$$  \hspace{1cm} (6.51)

For the four-string product $V_4$, we do not need its explicit form, but obviously it of $O(\kappa^0)$:

$$V_4(A_1, A_2, A_3, A_4) = O(\kappa^0).$$  \hspace{1cm} (6.52)

Thus the four-string product is of $O(\kappa)$ when one of the four string fields $A_1, A_2, A_3,$ and $A_4$ is $V_0$. We can show that the weak $A_\infty$ relations (6.31), (6.32), (6.33) and (6.34) are satisfied with these multi-string products up to the following orders:

$$V_1(V_0) = O(\kappa^4),$$

$$V_1(V_1(A_1)) - V_2(V_0, A_1) + V_2(A_1, V_0) = O(\kappa^3),$$

$$V_1(V_2(A_1, A_2)) - V_2(V_1(A_1), A_2) - (-1)^{A_1} V_2(A_1, V_1(A_2)) + V_3(V_0, A_1, A_2) - V_3(A_1, V_0, A_2) + V_3(A_1, A_2, V_0) = O(\kappa^2),$$

$$V_1(V_3(A_1, A_2, A_3)) + V_3(V_1(A_1), A_2, A_3) + (-1)^{A_1} V_3(A_1, V_1(A_2), A_3) + (-1)^{A_1 + A_2} V_3(A_1, A_2, V_1(A_3)) - V_2(V_2(A_1, A_2), A_3) + V_2(A_1, V_2(A_2, A_3)) + V_4(V_0, A_1, A_2, A_3) - V_4(A_1, V_0, A_2, A_3) + V_4(A_1, A_2, V_0, A_3) + V_4(A_1, A_2, A_3, V_0) = O(\kappa).$$  \hspace{1cm} (6.54)

We can also confirm that the cyclic properties

$$\langle A_1, V_0^{(1)} \rangle = \langle V_0^{(1)}, A_1 \rangle,$$

$$\langle A_1, V_0^{(2)} \rangle = \langle V_0^{(2)}, A_1 \rangle,$$

$$\langle A_1, V_0^{(3)} \rangle = \langle V_0^{(3)}, A_1 \rangle,$$

$$\langle A_1, V_1^{(1)}(A_2) \rangle = - (-1)^{A_1} \langle V_1^{(1)}(A_1), A_2 \rangle,$$

$$\langle A_1, V_1^{(2)}(A_2) \rangle = - (-1)^{A_1} \langle V_1^{(2)}(A_1), A_2 \rangle,$$

$$\langle A_1, V_2^{(1)}(A_2, A_3) \rangle = \langle V_2^{(1)}(A_1, A_2), A_3 \rangle.$$  \hspace{1cm} (6.57)
hold for string fields projected onto the massless sector satisfying

\[ PA_i = A_i \]  \hspace{1cm} (6.58) 

with \( i = 1, 2, 3 \).

7 Coalgebra representation

Explicit expressions for the weak \( A_\infty \) relations are complicated at higher orders. The fourth relation (6.34) is already lengthy, and we may hesitate to write down the fifth relation explicitly. Fortunately, there is an efficient way called coalgebra representation to describe the weak \( A_\infty \) structure. While it is a straightforward generalization of the description for the \( A_\infty \) structure which has been used frequently in recent research on open string field theory, we seldom find an explicit description for the weak \( A_\infty \) structure in the literature so we describe it in this section. We will simplify the description of the weak \( A_\infty \) structure in three steps. Each of the first three subsections corresponds to one of the three steps.

Another ingredient which is necessary in our context is the projection onto a subspace of the Hilbert space. While it is intuitively understood in [24] using Feynman diagrams, one advantage of considering open string field theory based on the star product compared to closed string field theory is that various expressions can be represented very explicitly. In §7.4 we present an algebraic framework to incorporate the projection.

7.1 Degree

The first step to simplify the description of the weak \( A_\infty \) structure is to introduce degree for a string field \( A \) denoted by \( \text{deg}(A) \). It is defined by

\[ \text{deg}(A) = \epsilon(A) + 1 \mod 2, \]  \hspace{1cm} (7.1) 

where \( \epsilon(A) \) is the Grassmann parity of \( A \). We define \( \omega(A_1, A_2) \), \( M_0(M_1(A_1), M_2(A_1, A_2), M_3(A_1, A_2, A_3), \) and \( M_4(A_1, A_2, A_3, A_4) \) by

\[ \omega(A_1, A_2) = (-1)^{\text{deg}(A_1)} \langle A_1, A_2 \rangle, \]

\[ M_0 = \mathcal{V}_0, \]

\[ M_1(A_1) = \mathcal{V}_1(A_1), \]

\[ M_2(A_1, A_2) = (-1)^{\text{deg}(A_1)} \mathcal{V}_2(A_1, A_2), \]

\[ M_3(A_1, A_2, A_3) = (-1)^{\text{deg}(A_2)} \mathcal{V}_3(A_1, A_2, A_3), \]

\[ M_4(A_1, A_2, A_3, A_4) = (-1)^{\text{deg}(A_1)+\text{deg}(A_3)} \mathcal{V}_4(A_1, A_2, A_3, A_4). \]  \hspace{1cm} (7.2) 

\[ \text{deg}(A) \] The coalgebra representation of the \( A_\infty \) structure is explained in detail, for example, in appendix A of [32]. We mostly follow the conventions used in this appendix.
The inner product $\omega(A_1, A_2)$ is graded antisymmetric with respect to degree:

$$
\omega(A_1, A_2) = - (-1)^{\text{deg}(A_1)\text{deg}(A_2)} \omega(A_2, A_1).
$$

(7.3)

We have

$$
de\text{deg}(M_0) = 1,
$$
$$
de\text{deg}(M_1(A_1)) = \text{deg}(A_1) + 1,
$$
$$
de\text{deg}(M_2(A_1, A_2)) = \text{deg}(A_1) + \text{deg}(A_2) + 1,
$$
$$
de\text{deg}(M_3(A_1, A_2, A_3)) = \text{deg}(A_1) + \text{deg}(A_2) + \text{deg}(A_3) + 1,
$$
$$
de\text{deg}(M_4(A_1, A_2, A_3, A_4)) = \text{deg}(A_1) + \text{deg}(A_2) + \text{deg}(A_3) + \text{deg}(A_4) + 1,
$$

and we say that $M_0$, $M_1$, $M_2$, $M_3$, and $M_4$ are degree odd. The cyclic properties of the multi-string products are translated into

$$
\omega(A_1, M_0) = - (-1)^{\text{deg}(A_1)} \omega(M_0, A_1),
$$
$$
\omega(A_1, M_1(A_2)) = - (-1)^{\text{deg}(A_1)} \omega(M_1(A_1), A_2),
$$
$$
\omega(A_1, M_2(A_2, A_3)) = - (-1)^{\text{deg}(A_1)} \omega(M_2(A_1, A_2), A_3),
$$
$$
\omega(A_1, M_3(A_2, A_3, A_4)) = - (-1)^{\text{deg}(A_1)} \omega(M_3(A_1, A_2, A_3), A_4),
$$
$$
\omega(A_1, M_4(A_2, A_3, A_4, A_5)) = - (-1)^{\text{deg}(A_1)} \omega(M_4(A_1, A_2, A_3, A_4), A_5),
$$

(7.5)

and the weak $A_\infty$ relations (6.31), (6.32), (6.33), and (6.34) are written as

$$
M_1(M_0) = 0,
$$
$$
M_1(M_1(A_1)) + M_2(M_0, A_1) + (-1)^{\text{deg}(A_1)} M_2(A_1, M_0) = 0,
$$
$$
M_1(M_2(A_1, A_2)) + M_2(M_1(A_1), A_2) + (-1)^{\text{deg}(A_1)} M_2(A_1, M_1(A_2))
$$
$$
+ M_3(M_0, A_1, A_2) + (-1)^{\text{deg}(A_1)} M_3(A_1, M_0, A_2) + (-1)^{\text{deg}(A_1)+\text{deg}(A_2)} M_3(A_1, A_2, M_0) = 0,
$$
$$
M_1(M_3(A_1, A_2, A_3)) + M_3(M_1(A_1), A_2, A_3)
$$
$$
+ (-1)^{\text{deg}(A_1)} M_3(A_1, M_1(A_2), A_3) + (-1)^{\text{deg}(A_1)+\text{deg}(A_2)} M_3(A_1, A_2, M_1(A_3))
$$
$$
+ M_2(M_2(A_1, A_2), A_3) + (-1)^{\text{deg}(A_1)} M_2(A_1, M_2(A_2, A_3)) + M_4(M_0, A_1, A_2, A_3)
$$
$$
+ (-1)^{\text{deg}(A_1)} M_4(A_1, M_0, A_2, A_3) + (-1)^{\text{deg}(A_1)+\text{deg}(A_2)} M_4(A_1, A_2, M_0, A_3)
$$
$$
+ (-1)^{\text{deg}(A_1)+\text{deg}(A_2)+\text{deg}(A_3)} M_4(A_1, A_2, A_3, M_0) = 0.
$$

(7.9)

The three important string products in this paper are the BRST operator as a one-string product, the star product as a two-string product, and the string field $J(\Phi)$ as a zero-string product. The BRST operator is a one-string product of degree odd because

$$
\text{deg}(QA) = \text{deg}(A) + 1.
$$

(7.10)
for any string field $A$. The cyclic property of the BRST operator is expressed as
\[ \omega(A_1, QA_2) = -(-1)^{\deg(A_1)} \omega(QA_1, A_2). \] (7.11)

Associated with the star product, we define $m_2(A_1, A_2)$ by
\[ m_2(A_1, A_2) = (-1)^{\deg(A_1)} A_1 \ast A_2. \] (7.12)

The two-string product $m_2$ is degree odd,
\[ \deg(m_2(A_1, A_2)) = \deg(A_1) + \deg(A_2) + 1, \] (7.13)

and it has the following cyclic property:
\[ \omega(A_1, m_2(A_2, A_3)) = -(-1)^{\deg(A_1)} \omega(m_2(A_1, A_2), A_3). \] (7.14)

We define the zero-string product $w_0$ by
\[ w_0 = -J(\Phi) \] (7.15)

with the closed string field $\Phi$ annihilated by the BRST operator. It is degree odd,
\[ \deg(w_0) = 1, \] (7.16)

and the cyclic property of $w_0$ can be represented as
\[ \omega(A_1, w_0) = -(-1)^{\deg(A_1)} \omega(w_0, A_1) \] (7.17)

for any string field $A_1$. The basic relations involving the BRST operator, the star product, and $J(\Phi)$ given by
\[ Q^2 = 0, \]
\[ Q(A_1 \ast A_2) = QA_1 \ast A_2 + (-1)^{A_1} A_1 \ast QA_2, \]
\[ (A_1 \ast A_2) \ast A_3 = A_1 \ast (A_2 \ast A_3), \]
\[ QJ(\Phi) = 0, \]
\[ J(\Phi) \ast A_1 = A_1 \ast J(\Phi) \]

for any string fields $A_1$, $A_2$, and $A_3$ are expressed in terms of $Q$, $m_2$, and $w_0$ as
\[ Q^2 = 0, \]
\[ Qm_2(A_1, A_2) + m_2(QA_1, A_2) + (-1)^{\deg(A_1)} m_2(A_1, QA_2) = 0, \]
\[ m_2(m_2(A_1, A_2), A_3) + (-1)^{\deg(A_1)} m_2(A_1, m_2(A_2, A_3)) = 0, \]
\[ Qw_0 = 0, \]
\[ m_2(w_0, A_1) + (-1)^{\deg(A_1)} m_2(A_1, w_0) = 0. \] (7.19)
7.2 Tensor products of the Hilbert space

The second step to simplify the description of the $A_\infty$ structure is to use operators acting on tensor products of the Hilbert space $\mathcal{H}$. We denote the tensor product of $n$ copies of the Hilbert space $\mathcal{H}$ by $\mathcal{H}^\otimes n$. For an $n$-string product $c_n(A_1, A_2, \ldots, A_n)$ we define a corresponding operator $c_n$ which maps $\mathcal{H}^\otimes n$ into $\mathcal{H}$ by

$$c_n (A_1 \otimes A_2 \otimes \ldots \otimes A_n) \equiv c_n(A_1, A_2, \ldots, A_n).$$

We use the same symbol $c_n$ to denote this operator, and to distinguish the operator from the $n$-string product we call it $n$-string operator.

Let us use examples to demonstrate how the introduction of $n$-string operators helps simplify the description of the weak $A_\infty$ structure. The first example is the relation

$$Q m_2(A_1, A_2) + m_2(Q A_1, A_2) + (-1)^{\text{deg}(A_1)} m_2(A_1, Q A_2) = 0.$$  \hspace{1cm} (7.21)

Using $Q$ as a one-string operator and $m_2$ as a two-string operator, this relation can be expressed as

$$Q m_2(A_1 \otimes A_2) + m_2(Q A_1 \otimes A_2) + (-1)^{\text{deg}(A_1)} m_2(A_1 \otimes Q A_2) = 0.$$ \hspace{1cm} (7.22)

We denote the identity map as a one-string operator from $\mathcal{H}$ to $\mathcal{H}$ by $\mathbb{1}$,

$$\mathbb{1}(A) = A$$ \hspace{1cm} (7.23)

for any string field $A$, and we write $Q A_1 \otimes A_2$ and $(-1)^{\text{deg}(A_1)} A_1 \otimes Q A_2$ as

$$Q A_1 \otimes A_2 = (Q \otimes \mathbb{1})(A_1 \otimes A_2),$$

$$(-1)^{\text{deg}(A_1)} A_1 \otimes Q A_2 = (\mathbb{1} \otimes Q)(A_1 \otimes A_2),$$ \hspace{1cm} (7.24)

where in the second equation the sign factor $(-1)^{\text{deg}(A_1)}$ is canceled on the right-hand side when the degree-odd operator $Q$ passes through $A_1$. We then have

$$Q m_2(A_1 \otimes A_2) + m_2(Q A_1 \otimes A_2) + (-1)^{\text{deg}(A_1)} m_2(A_1 \otimes Q A_2)$$

$$= Q m_2(A_1 \otimes A_2) + m_2(Q \otimes \mathbb{1})(A_1 \otimes A_2) + m_2(\mathbb{1} \otimes Q)(A_1 \otimes A_2)$$

$$= (Q m_2 + m_2(Q \otimes \mathbb{1}) + m_2(\mathbb{1} \otimes Q))(A_1 \otimes A_2),$$

and the relation (7.21) can be expressed without using $A_1$ and $A_2$ as

$$Q m_2 + m_2(Q \otimes \mathbb{1}) + m_2(\mathbb{1} \otimes Q) = 0.$$ \hspace{1cm} (7.26)

Similarly, the left-hand side of the relation

$$m_2(m_2(A_1, A_2), A_3) + (-1)^{\text{deg}(A_1)} m_2(A_1, m_2(A_2, A_3)) = 0$$ \hspace{1cm} (7.27)
can be written as
\begin{align}
m_2(m_2(A_1, A_2), A_3) + (-1)^{\deg(A_1)} m_2(A_1, m_2(A_2, A_3)) \\
= m_2(m_2(A_1 \otimes A_2) \otimes A_3) + (-1)^{\deg(A_1)} m_2(A_1 \otimes m_2(A_2 \otimes A_3)) \\
= m_2(m_2 \otimes I + I \otimes m_2) (A_1 \otimes A_2 \otimes A_3),
\end{align}
(7.28)
so the relation (7.27) can be expressed without using $A_1$, $A_2$, and $A_3$ as
\begin{align}
m_2( m_2 \otimes I + I \otimes m_2) = 0. \tag{7.29}
\end{align}

We also introduce the vector space for the zero-string space denoted by $H^{\otimes 0}$. It is a one-dimensional vector space given by multiplying a single basis vector $1$ by complex numbers. The vector $1$ satisfies
\begin{align}
1 \otimes A = A, \quad A \otimes 1 = A \tag{7.30}
\end{align}
for any string field $A$. For a zero-string product $c_0$ we define a corresponding operator $c_0$ which maps $H^{\otimes 0}$ into $H$ by
\begin{align}
c_0 1 \equiv c_0, \tag{7.31}
\end{align}
where $c_0$ on the right-hand side is the zero-string product and $c_0$ on the left-hand side is the zero-string operator which maps $H^{\otimes 0}$ into $H$. The relation $Q w_0 = 0$ for the zero-string product $w_0$ is translated into $Q w_0 1 = 0$ for the zero-string operator $w_0$, and thus we can express it without using $1$ as the relation
\begin{align}
Q w_0 = 0 \tag{7.32}
\end{align}
for the zero-string operator $w_0$. The left-hand side of the relation
\begin{align}
m_2(w_0, A_1) + (-1)^{\deg(A_1)} m_2(A_1, w_0) = 0 \tag{7.33}
\end{align}
can be written as
\begin{align}
m_2(w_0, A_1) + (-1)^{\deg(A_1)} m_2(A_1, w_0) \\
= m_2( w_0 \otimes A_1 ) + (-1)^{\deg(A_1)} m_2( A_1 \otimes w_0 ) \\
= m_2( w_0 \otimes I + I \otimes w_0 ) ( A_1 ), \tag{7.34}
\end{align}
so the relation (7.33) can be expressed without using $A_1$ as
\begin{align}
m_2( w_0 \otimes I + I \otimes w_0 ) = 0. \tag{7.35}
\end{align}

To summarize, the relations (7.19) in terms of multi-string products are written as the
relations

\[ Q^2 = 0, \quad (7.36) \]
\[ Q m_2 + m_2 ( Q \otimes I + I \otimes Q ) = 0, \quad (7.37) \]
\[ m_2 ( m_2 \otimes I + I \otimes m_2 ) = 0, \quad (7.38) \]
\[ Q w_0 = 0, \quad (7.39) \]
\[ m_2 ( w_0 \otimes I + I \otimes w_0 ) = 0 \quad (7.40) \]

in terms of multi-string operators. We no longer need string fields to express these relations, and all the sign factors in (7.19) are now gone.

For the BPZ inner product, we define \( \langle \omega | \) which maps \( \mathcal{H}^{\otimes 2} \) into a complex number by

\[ \langle \omega | A_1 \otimes A_2 \equiv \omega (A_1, A_2) \quad (7.41) \]

for a pair of string fields \( A_1 \) and \( A_2 \). The left-hand side of the relation

\[ \omega(A_1, QA_2) + (-1)^{\text{deg}(A_1)} \omega(QA_1, A_2) = 0 \quad (7.42) \]

can be written as

\[
\begin{align*}
\omega(A_1, QA_2) + (-1)^{\text{deg}(A_1)} \omega(QA_1, A_2) \\
= \langle \omega | A_1 \otimes QA_2 + (-1)^{\text{deg}(A_1)} \langle \omega | QA_1 \otimes A_2 \\
= \langle \omega | (I \otimes Q + Q \otimes I) (A_1 \otimes A_2),
\end{align*}
\]

so the cyclic property of the BRST operator can be expressed without using \( A_1 \) and \( A_2 \) as

\[ \langle \omega | (I \otimes Q + Q \otimes I) = 0. \quad (7.44) \]

Similarly, the cyclic properties of \( m_2 \) and \( w_0 \) in (7.14) and (7.17), respectively, can be written as

\[
\begin{align*}
\langle \omega | (I \otimes m_2 + m_2 \otimes I) &= 0, \\
\langle \omega | (I \otimes w_0 + w_0 \otimes I) &= 0. 
\end{align*}
\]

The operators \( P \) and \( h \) are BPZ even:

\[
\begin{align*}
\langle A_1, PA_2 \rangle &= \langle PA_1, A_2 \rangle, \\
\langle A_1, h A_2 \rangle &= (-1)^{A_1} \langle h A_1, A_2 \rangle 
\end{align*}
\]

for any pair of states \( A_1 \) and \( A_2 \). These relations can be expressed using \( \langle \omega | \) as

\[
\begin{align*}
\langle \omega | I \otimes P &= \langle \omega | P \otimes I, \\
\langle \omega | I \otimes h &= \langle \omega | h \otimes I. 
\end{align*}
\]
Finally, let us use the $n$-string operator $M_n$ to simplify the description of the weak $A_\infty$ structure. The weak $A_\infty$ relations are written as

\[
M_1 M_0 = 0,
M_1 M_1 + M_2 (M_0 \otimes I + I \otimes M_0) = 0,
M_1 M_2 + M_2 (M_1 \otimes I + I \otimes M_1) + M_3 (M_0 \otimes I \otimes I + I \otimes M_0 \otimes I + I \otimes I \otimes M_0) = 0,
\]

and the cyclic properties are

\[
\langle \omega \mid (I \otimes M_0 + M_0 \otimes I) = 0,
\langle \omega \mid (I \otimes M_1 + M_1 \otimes I) = 0,
\langle \omega \mid (I \otimes M_2 + M_2 \otimes I) = 0,
\langle \omega \mid (I \otimes M_3 + M_3 \otimes I) = 0,
\langle \omega \mid (I \otimes M_4 + M_4 \otimes I) = 0.
\]

\[
(7.48)
\]

\[
(7.49)
\]

7.3 Coderivations

To describe the weak $A_\infty$ structure to all orders, it is convenient to consider linear operators acting on the vector space $T\mathcal{H}$ defined by

\[
T\mathcal{H} = \mathcal{H} \otimes \mathcal{H} \oplus \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H} \oplus \ldots
\]

(7.50)

We denote the projection operator onto $\mathcal{H} \otimes n$ by $\pi_n$. For an $n$-string operator $c_n$, we define an associated operator $c_n$ acting on $T\mathcal{H}$ as follows. The action on the sector $\mathcal{H} \otimes m$ vanishes when $m < n$:

\[
c_n \pi_m = 0 \text{ for } m < n.
\]

(7.51)

The action on the sector $\mathcal{H} \otimes n$ is given by the $n$-string operator $c_n$:

\[
c_n \pi_n = c_n \pi_n.
\]

(7.52)

The action on the sector $\mathcal{H} \otimes n+1$ is given by

\[
c_n \pi_{n+1} = (c_n \otimes I + I \otimes c_n) \pi_{n+1}.
\]

(7.53)

The action on the sector $\mathcal{H} \otimes m$ for $m > n + 1$ is given by

\[
c_n \pi_m = \left( c_n \otimes I^{\otimes (m-n)} + \sum_{k=1}^{m-n-1} I^{\otimes k} \otimes c_n \otimes I^{\otimes (m-n-k)} + I^{\otimes (m-n)} \otimes c_n \right) \pi_m \text{ for } m > n + 1.
\]

(7.54)
The degree of $c_n$ is defined to be the same as that of $c_n$. An operator acting on $TH$ of this form is called a codereivation.

Coderivations can be characterized using a linear operation called coproduct which maps $TH$ to a tensor product of two copies of $TH$ denoted as $TH \otimes TH$. We use the symbol $\otimes'$ to distinguish the tensor product of $TH$ from the tensor product $\otimes$ within $TH$. We denote the coproduct by $\triangle$, and its actions on $1, A_1, A_1 \otimes A_2, A_1 \otimes A_2 \otimes A_3$ are given by

\[
\triangle 1 = 1 \otimes' 1, \\
\triangle A_1 = 1 \otimes' A_1 + A_1 \otimes' 1, \\
\triangle (A_1 \otimes A_2) = 1 \otimes' (A_1 \otimes A_2) + A_1 \otimes' A_2 + (A_1 \otimes A_2) \otimes' 1, \\
\triangle (A_1 \otimes A_2 \otimes A_3) = 1 \otimes' (A_1 \otimes A_2 \otimes A_3) + A_1 \otimes' (A_2 \otimes A_3) + (A_1 \otimes A_2) \otimes' A_3 + (A_1 \otimes A_2 \otimes A_3) \otimes' 1. 
\]

(7.56)

The action of the coproduct on $A_1 \otimes A_2 \otimes \ldots \otimes A_n$ for $n > 1$ is as follows:

\[
\triangle (A_1 \otimes A_2 \otimes \ldots \otimes A_n) = 1 \otimes' (A_1 \otimes A_2 \otimes \ldots \otimes A_n) \\
+ \sum_{k=1}^{n-1} (A_1 \otimes A_2 \otimes \ldots \otimes A_k) \otimes' (A_{k+1} \otimes A_{k+2} \otimes \ldots \otimes A_n) \\
+ (A_1 \otimes A_2 \otimes \ldots \otimes A_n) \otimes' 1. 
\]

(7.57)

Using the coproduct, a codereivation is defined in the following way. A linear operator $a$ on $TH$ is a codereivation when it satisfies

\[
\triangle a = (a \otimes' I + I \otimes' a) \triangle, 
\]

(7.58)

where $I$ is the identity operator on $TH$. It is clear from this definition that the sum of two coderivations is a codereivation. In general, a codereivation is associated with a linear combination of multi-string operators, and the multi-string operators can be extracted from the codereivation $a$ by decomposing $\pi_1 a$ as

\[
\pi_1 a = \sum_{n=0}^{\infty} a_n \pi_n 
\]

(7.59)

where $a_n$ is an $n$-string operator. When $a$ is a codereivation and $\pi_1 a$ vanishes, we can show that $a$ itself vanishes.

Let us show that the commutator of two coderivations graded with respect to degree is a codereivation. We denote the commutator of two operators $A$ and $B$ on $TH$ graded with respect to degree by

\[
[A, B] = AB - (-1)^{\deg(A)\deg(B)} BA, 
\]

(7.60)
where $\deg(O)$ is the degree of the operator $O$. When $a$ and $b$ are coderivations, we find
\[
\Delta [a, b] = (a \otimes' I + I \otimes' a) (b \otimes' I + I \otimes' b) \Delta \\
- (-1)^{\deg(a) \deg(b)} (b \otimes' I + I \otimes' b) (a \otimes' I + I \otimes' a) \Delta \\
= ([a, b] \otimes' I + I \otimes' [a, b]) \Delta.
\]
(7.61)

The graded commutator $[a, b]$ is therefore a coderivation.

Let us present a few examples of coderivations. The action of the coderivation $Q$ associated with the BRST operator $Q$ as a one-string operator of degree odd is given by
\[
Q 1 = 0, \\
Q A_1 = Q A_1, \\
Q (A_1 \otimes A_2) = Q A_1 \otimes A_2 + (-1)^{\deg(A_1)} A_1 \otimes Q A_2, \\
Q (A_1 \otimes A_2 \otimes A_3) = Q A_1 \otimes A_2 \otimes A_3 + (-1)^{\deg(A_1)} A_1 \otimes Q A_2 \otimes A_3 \\
+ (-1)^{\deg(A_1) + \deg(A_2)} A_1 \otimes A_2 \otimes Q A_3, \\
\vdots
\]
(7.62)

The graded commutator $[Q, Q]$ is a coderivation, and it follows from $Q^2 = 0$ that $\pi_1 [Q, Q]$ vanishes:
\[
\pi_1 [Q, Q] = 2 \pi_1 Q^2 = 2 Q \pi_1 Q = 2 Q^2 \pi_1 = 0.
\]
(7.63)

Therefore, the coderivation $[Q, Q]$ vanishes:
\[
[Q, Q] = 0.
\]
(7.64)

The action of the coderivation $m_2$ associated with the two-string operator $m_2$ of degree odd is given by
\[
m_2 1 = 0, \\
m_2 A_1 = 0, \\
m_2 (A_1 \otimes A_2) = m_2 (A_1, A_2), \\
m_2 (A_1 \otimes A_2 \otimes A_3) = m_2 (A_1, A_2) \otimes A_3 + (-1)^{\deg(A_1)} A_1 \otimes m_2 (A_2, A_3), \\
m_2 (A_1 \otimes A_2 \otimes A_3 \otimes A_4) = m_2 (A_1, A_2) \otimes A_3 \otimes A_4 + (-1)^{\deg(A_1)} A_1 \otimes m_2 (A_2, A_3) \otimes A_4 \\
+ (-1)^{\deg(A_1) + \deg(A_2)} A_1 \otimes A_2 \otimes m_2 (A_3, A_4), \\
\vdots
\]
(7.65)

Let us calculate $\pi_1 [Q, m_2]$. We find
\[
\pi_1 [Q, m_2] = \pi_1 Q m_2 + \pi_1 m_2 Q = Q \pi_1 m_2 + m_2 \pi_2 Q \\
= Q m_2 \pi_2 + m_2 (Q \otimes I + I \otimes Q) \pi_2 = (Q m_2 + m_2 (Q \otimes I + I \otimes Q)) \pi_2 = 0,
\]
(7.66)
where we used (7.37). Therefore, the graded commutator \([Q, m_2]\) vanishes:
\[ [Q, m_2] = 0. \] (7.67)

Let us next calculate \(\pi_1 [m_2, m_2]\). We find
\[ \pi_1 [m_2, m_2] = 2\pi_1 m_2^2 = 2m_2 \pi_2 m_2 = 2m_2 (m_2 \otimes 1 + 1 \otimes m_2) \pi_3 = 0, \] (7.68)
where we used (7.38). Therefore, the graded commutator \([m_2, m_2]\) vanishes:
\[ [m_2, m_2] = 0. \] (7.69)

The action of the coderivation \(w_0\) associated with the zero-string operator \(w_0\) of degree odd is given by
\[
\begin{align*}
\pi_1 [w_0, w_0] &= 2\pi_1 w_0^2 = 2w_0 \pi_0 w_0 = 0. \tag{7.71}
\end{align*}
\]

Let us calculate \(\pi_1 [Q, w_0]\). We find
\[
\begin{align*}
\pi_1 [Q, w_0] &= \pi_1 Q w_0 + \pi_1 w_0 Q = Q \pi_1 w_0 + w_0 \pi_0 Q = Q w_0 \pi_0 = 0, \tag{7.73}
\end{align*}
\]
where we used (7.39). Therefore, the graded commutator \([Q, w_0]\) vanishes:
\[ [Q, w_0] = 0. \] (7.74)

Let us also calculate \(\pi_1 [m_2, w_0]\). We find
\[
\begin{align*}
\pi_1 [m_2, w_0] &= \pi_1 m_2 w_0 + \pi_1 w_0 m_2 = m_2 \pi_2 w_0 + w_0 \pi_0 m_2 \\
&= m_2 (w_0 \otimes 1 + 1 \otimes w_0) \pi_1 = 0, \tag{7.75}
\end{align*}
\]
where we used (7.40). Therefore, the graded commutator \([m_2, w_0]\) vanishes:

\[
[m_2, w_0] = 0. \tag{7.76}
\]

The cyclic properties of \(Q, m_2\), and \(w_0\) can be described in terms of the corresponding coderivations \(Q, m_2,\) and \(w_0\) as follows:

\[
\langle \omega | \pi_2 Q = 0, \quad \langle \omega | \pi_2 m_2 = 0, \quad \langle \omega | \pi_2 w_0 = 0. \tag{7.77}
\]

In general, we say that a coderivation \(a\) is cyclic when it satisfies

\[
\langle \omega | \pi_2 a = 0. \tag{7.78}
\]

All of the coderivations \(Q, m_2,\) and \(w_0\) are cyclic.

Let us now suppose that a linear operator \(M\) on \(T \mathcal{H}\) is a coderivation of degree odd and satisfies

\[
[M, M] = 0. \tag{7.79}
\]

We decompose \(\pi_1 M\) as

\[
\pi_1 M = \sum_{n=0}^{\infty} M_n \pi_n \tag{7.80}
\]

where \(M_n\) is an \(n\)-string operator. It follows from (7.79) that

\[
\pi_1 [M, M] \pi_n = 0 \tag{7.81}
\]

for any non-negative integer \(n\). Using the decomposition (7.80), we find

\[
\pi_1 [M, M] \pi_n = 2 \pi_1 M^2 \pi_n = 2 \sum_{k=0}^{\infty} M_k \pi_k M \pi_n. \tag{7.82}
\]

Since \(M\) is a coderivation, \(\pi_k M \pi_n\) vanishes when \(k = 0\) and \(k > n + 1\). We therefore have

\[
\sum_{k=1}^{n+1} M_k \pi_k M \pi_n = 0 \tag{7.83}
\]

for any non-negative integer \(n\). We find that the four relations in (7.48) precisely correspond to this equation with \(n = 0, 1, 2, 3\). In fact, this enables us to describe the weak \(A_\infty\) relations to all orders. When a degree-odd coderivation \(M\) satisfies (7.79), the set of multi-string operators \(\{M_0, M_1, M_2 \ldots\}\) satisfy the weak \(A_\infty\) relations.

Let us further suppose that the coderivation \(M\) is cyclic:

\[
\langle \omega | \pi_2 M = 0. \tag{7.84}
\]
We then have
\[ \langle \omega \mid \pi_2 M \pi_n = 0 \] (7.85)
for any non-negative integer \( n \). Since \( M \) is a coderivation, \( \pi_2 M \pi_n \) vanishes when \( n = 0 \). For \( n > 0 \), we have
\[ \pi_2 M \pi_n = (I \otimes M_{n-1} + M_{n-1} \otimes I) \pi_n . \] (7.86)
Therefore, the five relations in (7.49) precisely correspond to (7.85) with \( n = 1, 2, 3, 4, 5 \). In fact, this enables us to describe the cyclic properties of the multi-string operators \( \{ M_0, M_1, M_2 \ldots \} \) to all orders. When \( M \) is cyclic, the multi-string operators \( \{ M_0, M_1, M_2 \ldots \} \) have appropriate cyclic properties.

To summarize, when a linear operator \( M \) on \( T \mathcal{H} \) is degree odd and satisfies
\[ \triangle M = (M \otimes I + I \otimes M)' \triangle, \] (7.87)
\[ [M, M] = 0, \] (7.88)
\[ \langle \omega \mid \pi_2 M = 0, \] (7.89)
the corresponding multi-string operators \( \{ M_0, M_1, M_2 \ldots \} \) satisfy the weak \( A_\infty \) relations, and they have the cyclic property given by
\[ \langle \omega \mid (M_n \otimes I + I \otimes M_n) = 0. \] (7.90)

For the action of open bosonic string field theory without introducing the source term for the gauge-invariant operators, the \( A_\infty \) structure can be described in terms of \( M \) given by
\[ M = Q + m_2 . \] (7.91)
Since both of \( Q \) and \( m_2 \) are cyclic coderivations of degree odd, it is clear that \( M \) is a cyclic coderivation of degree odd. It follows from
\[ [Q, Q] = 0, \quad [Q, m_2] = 0, \quad [m_2, m_2] = 0 \] (7.92)
that the condition (7.88) is satisfied. The multi-string operators from \( M \) are
\[ M_0 = 0, \quad M_1 = Q, \quad M_2 = m_2, \quad M_n = 0 \quad \text{for} \quad n > 2 . \] (7.93)

For the action of open bosonic string field theory including the source term for the gauge-invariant operators, the weak \( A_\infty \) structure can be described in terms of \( M \) given by
\[ M = Q + m_2 + \kappa w_0 . \] (7.94)
Since $Q$, $m_2$, and $w_0$ are cyclic coderivations of degree odd, it is clear that $M$ is a cyclic coderivation of degree odd. It follows from (7.92) and

$$[w_0, w_0] = 0, \quad [Q, w_0] = 0, \quad [m_2, w_0] = 0$$  \hspace{1cm} (7.95)$$

that the condition (7.88) is satisfied. The multi-string operators from $M$ are

$$M_0 = \kappa w_0, \quad M_1 = Q, \quad M_2 = m_2, \quad M_n = 0 \text{ for } n > 2.$$  \hspace{1cm} (7.96)$$

### 7.4 Projection to the massless sector

We describe the low-energy limit of open bosonic string field theory in terms of string fields projected onto the massless sector. We denote the subspace of $\mathcal{H}$ for the massless sector by $\tilde{\mathcal{H}}$. A string field $A$ belongs to $\tilde{\mathcal{H}}$ if

$$PA = A$$  \hspace{1cm} (7.97)$$

is satisfied.

We are interested in an $n$-string product $c_n(A_1, A_2, \ldots, A_n)$ satisfying the condition

$$Pc_n(A_1, A_2, \ldots, A_n) = c_n(A_1, A_2, \ldots, A_n),$$  \hspace{1cm} (7.98)$$

where all of the string fields $A_1, A_2, \ldots, A_{n-1}$, and $A_n$ are in $\tilde{\mathcal{H}}$. When we write such an $n$-string product $c_n(A_1, A_2, \ldots, A_n)$ as

$$c_n(A_1, A_2, \ldots, A_n) = c_n(A_1 \otimes A_2 \otimes \ldots \otimes A_n),$$  \hspace{1cm} (7.99)$$

we require that the $n$-string operator $c_n$ be a map from $\tilde{\mathcal{H}} \otimes^n$ to $\tilde{\mathcal{H}}$, where $\tilde{\mathcal{H}} \otimes^n$ is the tensor product of $n$ copies of $\tilde{\mathcal{H}}$. Then the $n$-string operator $c_n$ satisfies

$$Pc_n P \otimes^n = c_n,$$  \hspace{1cm} (7.100)$$

where

$$P \otimes^n = \underbrace{P \otimes P \otimes \ldots \otimes P}_n.$$  \hspace{1cm} (7.101)$$

We also define the vector space $T\tilde{\mathcal{H}}$ by

$$T\tilde{\mathcal{H}} = \tilde{\mathcal{H}} \otimes^0 \oplus \tilde{\mathcal{H}} \oplus \tilde{\mathcal{H}} \otimes^2 \oplus \tilde{\mathcal{H}} \otimes^3 \oplus \ldots,$$  \hspace{1cm} (7.102)$$

where

$$\tilde{\mathcal{H}} \otimes^0 = \mathcal{H} \otimes^0.$$  \hspace{1cm} (7.103)$$
We denote the projection operator onto $T\hat{\mathcal{H}}$ by $P$. The action of $P$ is given by

\[
P 1 = 1, \\
P A_1 = PA_1, \\
P (A_1 \otimes A_2) = PA_1 \otimes PA_2, \\
P (A_1 \otimes A_2 \otimes A_3) = PA_1 \otimes PA_2 \otimes PA_3, \\
\vdots
\] (7.104)

It is degree even, and it has the following property:

\[
P^2 = P, \quad \triangle P = (P \otimes' P) \triangle .
\] (7.105)

An operator $A$ which maps from $T\hat{\mathcal{H}}$ into $T\hat{\mathcal{H}}$ satisfies

\[
P A P = A .
\] (7.106)

For an $n$-string operator $c_n$ which satisfies (7.100), we define an associated operator $c_n$ which is a coderivation on $T\hat{\mathcal{H}}$ as follows. The action on the sector $\hat{\mathcal{H}}^\otimes m$ vanishes when $m < n$:

\[
c_n \pi_m = 0 \quad \text{for} \quad m < n.
\] (7.107)

The action on the sector $\hat{\mathcal{H}}^\otimes n$ is given by the $n$-string operator $c_n$:

\[
c_n \pi_n = c_n \pi_n .
\] (7.108)

The action on the sector $\hat{\mathcal{H}}^\otimes n+1$ is given by

\[
c_n \pi_{n+1} = (c_n \otimes P + P \otimes c_n) \pi_{n+1} .
\] (7.109)

The action on the sector $\hat{\mathcal{H}}^\otimes m$ for $m > n + 1$ is given by

\[
c_n \pi_m = \left( c_n \otimes P^{\otimes(m-n)} + \sum_{k=1}^{m-n-1} P^{\otimes k} \otimes c_n \otimes P^{\otimes(m-n-k)} + P^{\otimes(m-n)} \otimes c_n \right) \pi_m \quad \text{for} \quad m > n + 1.
\] (7.110)

The degree of $c_n$ is defined to be the same as that of $c_n$. The operator $c_n$ has the following properties:

\[
P c_n P = c_n, \quad \triangle c_n = (c_n \otimes' P + P \otimes' c_n) \triangle .
\] (7.111)

We are interested in an $n$-string product $c_n(A_1, A_2, \ldots, A_n)$ of degree odd which has the following cyclic property:

\[
\omega(A_1, c_n(A_2, A_3, \ldots, A_{n+1})) = -(-1)^{\deg(A_1)} \omega(c_n(A_1, A_2, \ldots, A_n), A_{n+1})
\] (7.112)
for $A_1, A_2, \ldots A_{n+1}$ in $\mathcal{H}$. This cyclic property is described in terms of the corresponding $n$-string operator $c_n$ as

$$\langle \omega | (c_n \otimes P + P \otimes c_n) = 0$$

(7.113)

and in terms of $c_n$ as

$$\langle \omega | \pi_2 c_n = 0.$$  

(7.114)

Suppose that a linear operator $M$ on $\mathcal{H}$ is degree odd and satisfies

$$\mathcal{P} M \mathcal{P} = M,$$  

(7.115)

$$\triangle M = (M \otimes' \mathcal{P} + \mathcal{P} \otimes' M) \triangle,$$  

(7.116)

$$[M, M] = 0,$$  

(7.117)

$$\langle \omega | \pi_2 M = 0.$$  

(7.118)

We decompose $\pi_1 M$ as

$$\pi_1 M = \sum_{n=0}^{\infty} M_n \pi_n,$$  

(7.119)

where $M_n$ is an $n$-string operator. Then the multi-string operators $\{M_0, M_1, M_2, \ldots\}$ satisfy the weak $A_\infty$ relations, and they have the cyclic property given by

$$\langle \omega | (M_n \otimes P + P \otimes M_n) = 0.$$  

(7.120)

## 8 Construction to all orders

In this section we use the coalgebra representation of the weak $A_\infty$ structure explained in the preceding section to construct multi-string operators satisfying the weak $A_\infty$ relations to all orders in $g$ and $\kappa$. We first explain the construction for open bosonic string field theory without the source term in §8.1. This is not a new result, and the construction was used in [33], for example, in a different context. Once we fully understand the construction in this case, the generalization to include the source term is rather straightforward. We therefore explain the construction without the source term in detail. We then present the construction including the source term in §8.2.

### 8.1 Open bosonic string field theory without the source term in the low-energy limit

#### 8.1.1 Multi-string products

For the effective action of open bosonic string field theory without the source term, we have presented $V_2(A_1, A_2)$ in (6.13) and $V_3(A_1, A_2, A_3)$ in (6.14). The two-string product $M^{(0)}_2(A_1, A_2)$
from $V_2(A_1, A_2)$ and the three-string product $M_3^{(0)}(A_1, A_2, A_3)$ from $V_3(A_1, A_2, A_3)$ under the identification in (7.2) together with the one-string product $M_1^{(0)}(A_1)$ are

$$M_1^{(0)}(A_1) = QA_1, \quad (8.1)$$

$$M_2^{(0)}(A_1, A_2) = P m_2(A_1, A_2), \quad (8.2)$$

$$M_3^{(0)}(A_1, A_2, A_3) = -P m_2(h m_2(A_1, A_2), A_3) - P m_2(A_1, h m_2(A_2, A_3)). \quad (8.3)$$

We require the corresponding $n$-string operator $M_n^{(0)}$ to satisfy

$$P M_n^{(0)} P^\otimes n = M_n^{(0)}. \quad (8.4)$$

Since $PA_1 = A_1$, $P^2 = P$, and $Q P = P Q$, we write $M_1^{(0)}(A_1)$ as

$$M_1^{(0)}(A_1) = Q A_1 = Q P A_1 = Q P^2 A_1 = P Q P A_1 \quad (8.5)$$

and the one-string operator $M_1^{(0)}$ is given by

$$M_1^{(0)} = P Q P. \quad (8.6)$$

While $P$ on either side of $Q$ is sufficient because $P$ and $Q$ commute, we write $P$ on each side of $Q$ to make it manifest that the condition (8.4) is satisfied. For $M_2^{(0)}(A_1, A_2)$, we write it as

$$M_2^{(0)}(A_1, A_2) = P m_2(A_1, A_2) = P m_2(PA_1, PA_2) = P m_2(PA_1 \otimes PA_2) = P m_2(P \otimes P)(A_1 \otimes A_2), \quad (8.7)$$

and the two-string operator $M_2^{(0)}$ is given by

$$M_2^{(0)} = P m_2(P \otimes P). \quad (8.8)$$

For $M_3^{(0)}(A_1, A_2, A_3)$, we write it as

$$M_3^{(0)}(A_1, A_2, A_3) = -P m_2(h m_2(A_1, A_2), A_3) - P m_2(A_1, h m_2(A_2, A_3)) = -P m_2(h m_2(PA_1, PA_2), PA_3) - P m_2(PA_1, h m_2(PA_2, PA_3)) = -P m_2(h m_2 \otimes P + P \otimes h m_2)(P \otimes P \otimes P)(A_1 \otimes A_2 \otimes A_3), \quad (8.9)$$

and the three-string operator $M_3^{(0)}$ is given by

$$M_3^{(0)} = -P m_2(h m_2 \otimes P + P \otimes h m_2)(P \otimes P \otimes P) . \quad (8.10)$$

Our goal is thus to construct a linear operator $M^{(0)}$ on $\mathcal{H}$ of degree odd which satisfies

$$P M^{(0)} P = M^{(0)}, \quad (8.11)$$

$$\Delta M^{(0)} = (M^{(0)} \otimes' P + P \otimes' M^{(0)}) \Delta, \quad (8.12)$$

$$[M^{(0)}, M^{(0)}] = 0, \quad (8.13)$$

$$\langle \omega | \pi_2 M^{(0)} \rangle = 0 \quad (8.14)$$

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with \( M_0^{(0)} \), \( M_1^{(0)} \), \( M_2^{(0)} \), and \( M_3^{(0)} \) given by

\[
    \begin{align*}
        M_0^{(0)} &= 0, \\
        M_1^{(0)} &= P Q P, \\
        M_2^{(0)} &= P m_2 \left( P \otimes P \right), \\
        M_3^{(0)} &= -P m_2 \left( h m_2 \otimes P + P \otimes h m_2 \right) \left( P \otimes P \otimes P \right),
    \end{align*}
\]

where the \( n \)-string operator \( M_n^{(0)} \) is defined in terms of the decomposition

\[
    \pi_1 M^{(0)} = \sum_{n=0}^{\infty} M_n^{(0)} \pi_n. \tag{8.19}
\]

### 8.1.2 Expression to all orders

In constructing \( M^{(0)} \), we decompose it as

\[
    M^{(0)} = \sum_{n=1}^{\infty} M_n^{(0)}, \tag{8.20}
\]

where \( M_n^{(0)} \) satisfies

\[
    \begin{align*}
        P M_n^{(0)} P &= M_n^{(0)}, \\
        \Delta M_n^{(0)} &= \left( M_n^{(0)} \otimes' P + P \otimes' M_n^{(0)} \right) \Delta, \\
        \langle \omega | \pi_2 M_n^{(0)} &= 0
    \end{align*}
\]

with

\[
    \pi_1 M_n^{(0)} = M_n^{(0)} \pi_n. \tag{8.24}
\]

The condition \((8.13)\) is also decomposed as

\[
    \sum_{m=1}^{n} \left[ M_m^{(0)}, M_{n-m+1}^{(0)} \right] = 0 \tag{8.25}
\]

for \( n = 1, 2, 3, \ldots \).

Let us begin with \( M_1^{(0)} \). Since \( M_1^{(0)} = P Q P \), we simply replace \( P \) and \( Q \) with \( P \) and \( Q \), respectively, to obtain

\[
    M_1^{(0)} = P Q P. \tag{8.26}
\]

This satisfies the condition \((8.24)\) for \( n = 1 \):

\[
    \pi_1 M_1^{(0)} = \pi_1 P Q P = P Q P \pi_1 = M_1^{(0)} \pi_1. \tag{8.27}
\]
The condition (8.21) for \( n = 1 \) is satisfied because \( P^2 = P \). We can confirm that the condition (8.22) for \( n = 1 \) is satisfied in the following way:

\[
\triangle P Q P = (P \otimes' P) \triangle Q P = (P \otimes' P) (Q \otimes' I + I \otimes' Q) \triangle P = (P \otimes' P) (Q \otimes' I + I \otimes' Q) (P \otimes' P) \triangle = (P Q P \otimes' P + P \otimes' P Q P) \triangle. \tag{8.28}
\]

Let us next confirm that the condition (8.23) for \( n = 1 \) is satisfied. It follows from \( P Q = Q P \) that

\[
[Q, P] = 0, \tag{8.29}
\]

so we find

\[
\langle \omega | \pi_2 P Q P = \langle \omega | \pi_2 Q P^2 = 0, \tag{8.30}
\]

where we used \( \langle \omega | \pi_2 Q = 0 \). The condition (8.25) for \( n = 1 \) is as follows:

\[
[P Q P, P Q P] = 0. \tag{8.31}
\]

While it is easy to calculate this commutator directly, we will encounter similar commutators in what follows, so let us first prove the identity

\[
[P Q P, P A P] = P [Q, A] P \tag{8.32}
\]

for any operator \( A \) on \( T\mathcal{H} \). Using \( [Q, P] = 0 \) and \( P^2 = P \), this identity can be shown as follows:

\[
[P Q P, P A P] = P Q P A P - (-1)^{\deg(A)} P A P Q P = P Q A P - (-1)^{\deg(A)} P A Q P = P [Q, A] P. \tag{8.33}
\]

The condition (8.25) for \( n = 1 \) thus follows from \( [Q, Q] = 0 \):

\[
[P Q P, P Q P] = P [Q, Q] P = 0. \tag{8.34}
\]

Let us next consider \( M_2^{(0)} \). Since \( M_2^{(0)} = P m_2 (P \otimes P) \), we use \( P \) and \( m_2 \) to construct \( M_2^{(0)} \) as follows:

\[
M_2^{(0)} = P m_2 P. \tag{8.35}
\]

This satisfies the condition (8.24) for \( n = 2 \):

\[
\pi_1 M_2^{(0)} = \pi_1 P m_2 P = P m_2 (P \otimes P) \pi_2 = M_2^{(0)} \pi_2. \tag{8.36}
\]

The condition (8.21) for \( n = 2 \) is satisfied because \( P^2 = P \). We can confirm that the condition (8.22) for \( n = 2 \) is satisfied in the following way:

\[
\triangle P m_2 P = (P \otimes' P) \triangle m_2 P = (P \otimes' P) (m_2 \otimes' I + I \otimes' m_2) \triangle P = (P \otimes' P) (m_2 \otimes' I + I \otimes' m_2) (P \otimes' P) \triangle = (P m_2 P \otimes' P + P \otimes' P m_2 P) \triangle. \tag{8.37}
\]
Let us also confirm that the condition (8.23) for $n = 2$ is satisfied. It follows from $P^2 = P$ and $\langle \omega | I \otimes P = \langle \omega | P \otimes I \rangle$ that
\begin{equation}
\langle \omega | (P \otimes P) = \langle \omega | (I \otimes P) (P \otimes I) = \langle \omega | (P \otimes I) (P \otimes I) = \langle \omega | (P \otimes I).
\end{equation}
(8.38)

Similarly, we have
\begin{equation}
\langle \omega | (P \otimes P) = \langle \omega | (P \otimes I) (I \otimes P) = \langle \omega | (P \otimes P) (I \otimes P) = \langle \omega | (I \otimes P).
\end{equation}
(8.39)

Using these relations, we can confirm that the condition (8.23) for $n = 2$ is satisfied in the following way:
\begin{equation}
\langle \omega | \pi_2 P m_2 P = \langle \omega | (P \otimes P) (m_2 \otimes I) (P \otimes P \otimes P) \xi_3 + \langle \omega | (P \otimes P) (I \otimes m_2) (P \otimes P \otimes P) \xi_3
\end{equation}
\begin{equation}
= \langle \omega | (I \otimes P) (m_2 \otimes I) (P \otimes P \otimes P) \xi_3 + \langle \omega | (P \otimes P) (I \otimes m_2) (P \otimes P \otimes P) \xi_3
\end{equation}
\begin{equation}
= \langle \omega | (m_2 \otimes I + I \otimes m_2) (P \otimes P \otimes P) \xi_3 = 0,
\end{equation}
(8.40)

where in the last step we used the cyclic property of $m_2$ in (7.45). The condition (8.25) for $n = 2$ is as follows:
\begin{equation}
[P Q P, P m_2 P] = 0.
\end{equation}
(8.41)

Using the identity (8.32), this follows from $[Q, m_2] = 0$:
\begin{equation}
[P Q P, P m_2 P] = P [Q, m_2] P = 0.
\end{equation}
(8.42)

The three-string operator $M_3^{(0)}$ contains $h$ defined by
\begin{equation}
h = \frac{b_0}{L_0} (1 - P).
\end{equation}
(8.43)

Regarding $h$, we have so far used the relation
\begin{equation}
Q h + h Q = 1 - P
\end{equation}
and the fact that $h$ is BPZ even:
\begin{equation}
\langle \omega | I \otimes h = \langle \omega | h \otimes I.
\end{equation}
(8.45)

In what follows we also use the following properties of $h$:
\begin{equation}
h P = 0, \quad P h = 0, \quad h^2 = 0.
\end{equation}
(8.46)

In constructing $M_3^{(0)}$, the crucial ingredient is the operator $h$ introduced in [34]. It is a linear operator on $\mathcal{TH}$ of degree odd and is defined as follows. The action on the sector $\mathcal{H}^{\otimes 0}$ vanishes:
\begin{equation}
h \pi_0 = 0.
\end{equation}
(8.47)

\footnote{We thank Hiroaki Matsunaga for helpful discussions on the operator $h$ used in [33].}
For the sectors $\mathcal{H}$ and $\mathcal{H}^\otimes 2$, its actions are as follows:

\[ h_\pi_1 = h \pi_1, \quad (8.48) \]

\[ h_\pi_2 = (h \otimes P + I \otimes h) \pi_2. \quad (8.49) \]

The action on the sector $\mathcal{H}^\otimes n$ for $n > 2$ is given by

\[ h_\pi_n = \left( h \otimes P^\otimes (n-1) + \sum_{m=1}^{n-2} I^\otimes m \otimes h \otimes P^\otimes (n-m-1) + I^\otimes (n-1) \otimes h \right) \pi_n \quad \text{for } n > 2. \quad (8.50) \]

Note that the operator $P$ appears in the definition of $h$. Some examples for the action of $h$ are

\[ h_1 = 0, \]

\[ h A_1 = h A_1, \]

\[ h (A_1 \otimes A_2) = h A_1 \otimes PA_2 + (-1)^{\deg(A_1)} A_1 \otimes h A_2, \]

\[ h (A_1 \otimes A_2 \otimes A_3) = h A_1 \otimes PA_2 \otimes PA_3 + (-1)^{\deg(A_1)} A_1 \otimes h A_2 \otimes PA_3 \]

\[ + (-1)^{\deg(A_1) + \deg(A_2)} A_1 \otimes A_2 \otimes h A_3, \]

\[ h (A_1 \otimes A_2 \otimes A_3 \otimes A_4) = h A_1 \otimes PA_2 \otimes PA_3 \otimes PA_4 + (-1)^{\deg(A_1)} A_1 \otimes h A_2 \otimes PA_3 \otimes PA_4 \]

\[ + (-1)^{\deg(A_1) + \deg(A_2)} A_1 \otimes A_2 \otimes h A_3 \otimes PA_4 \]

\[ + (-1)^{\deg(A_1) + \deg(A_2) + \deg(A_3)} A_1 \otimes A_2 \otimes A_3 \otimes h A_4, \]

\[ \vdots \]

\[ (8.51) \]

The operator $h$ is characterized by

\[ \pi_1 h = h \pi_1 \quad (8.52) \]

and

\[ \Delta h = (h \otimes P + I \otimes h) \Delta, \quad (8.53) \]

and it has the following properties:

\[ [Q, h] = I - P, \quad h P = 0, \quad P h = 0, \quad h^2 = 0. \quad (8.54) \]

We then claim that $M_3^{(0)}$ is given by

\[ M_3^{(0)} = - P m_2 h m_2 P. \quad (8.55) \]

Let us first calculate $\pi_1 M_3^{(0)}$ in order to confirm that the condition $[8.24]$ for $n = 3$ is satisfied. We find

\[ \pi_1 M_3^{(0)} = - P m_2 (h \otimes P + I \otimes h) (m_2 \otimes I + I \otimes m_2) (P \otimes P \otimes P) \pi_3 \]

\[ = - P m_2 (h m_2 \otimes P + P \otimes h m_2) (P \otimes P \otimes P) \pi_3 \]

\[ = - P m_2 (h m_2 \otimes P + P \otimes h m_2) (P \otimes P \otimes P) \pi_3 = M_3^{(0)} \pi_3, \quad (8.56) \]
where we used \( hP = 0 \) and \( P^2 = P \). The condition (8.21) for \( n = 3 \) is satisfied because \( P^2 = P \). The condition (8.25) for \( n = 3 \) is written as

\[
\begin{align*}
[M_1^{(0)}, M_3^{(0)}] &= - M_2^{(0)} M_2^{(0)}. \tag{8.57}
\end{align*}
\]

Since

\[
\begin{align*}
[M_1^{(0)}, M_3^{(0)}] &= -[PQP, Pm_2h m_2P] = -P[Q, m_2h m_2]P \\
&= P m_2 [Q, h] m_2 P = P m_2 (I - P) m_2 P \\
&= P m_2 m_2 P - P m_2 P m_2 P = - (P m_2 P) (P m_2 P),
\end{align*}
\]

where we used \([Q, m_2] = 0\), \([Q, h] = I - P\), and \( m_2^2 = 0\), the condition (8.25) for \( n = 3 \) is satisfied. In fact, it is easy to generalize this to construct \( M_n^{(0)} \) such that (8.25) is satisfied. We define \( M_n^{(0)} \) for \( n > 3 \) by

\[
M_n^{(0)} = (-1)^n P m_2 (h m_2)^{n-2} P. \tag{8.59}
\]

The condition (8.25) for \( n > 2 \) is written as

\[
[M_1^{(0)}, M_n^{(0)}] = - \sum_{m=2}^{n-1} M_m^{(0)} M_{n-m+1}^{(0)}. \tag{8.60}
\]

Since

\[
\begin{align*}
[M_1^{(0)}, M_n^{(0)}] &= (-1)^n [PQP, P m_2 (h m_2)^{n-2} P] = (-1)^n P[Q, m_2 (h m_2)^{n-2}]P \\
&= - (-1)^n \sum_{m=0}^{n-3} P m_2 (h m_2)^m (I - P) m_2 (h m_2)^{n-m-3} P \\
&= - \sum_{m=0}^{n-3} (-1)^{m+2} (P m_2 (h m_2)^m P) (-1)^{n-m-1} (P m_2 (h m_2)^{n-m-3} P),
\end{align*}
\]

the condition (8.25) for \( n > 2 \) is satisfied. While the condition (8.21) is also satisfied, it is not obvious whether the conditions (8.22) and (8.23) are satisfied for \( n > 2 \). We will show that they are satisfied in \((8.1.3)\) and \((8.1.4)\).

Before we move on to consider these conditions, let us prove (8.13) without decomposing \( M^{(0)} \). After summing \( M_n^{(0)} \) over \( n \), \( M^{(0)} \) is given by

\[
M^{(0)} = PQP + P m_2 \frac{1}{I + h m_2} P, \tag{8.62}
\]

where

\[
\frac{1}{I + h m_2} \equiv I + \sum_{n=1}^{\infty} (-1)^n (h m_2)^n. \tag{8.63}
\]
It follows from this definition that

\[(I + hm_2) \frac{1}{I + hm_2} = I.\]  

(8.64)

The condition (8.11) is satisfied because \(P^2 = P\). Let us calculate \([M^{(0)}, M^{(0)}]\). We find

\[\begin{align*}
[M^{(0)}, M^{(0)}] &= [PQP, PQP] + 2 \left[ PQP, P \frac{m_2}{I + hm_2} P \right] + 2 \left( \frac{P m_2}{I + hm_2} P \right)^2 \\
&= 2P \left[ Q, \frac{m_2}{I + hm_2} \right] P + 2 \left( \frac{P m_2}{I + hm_2} P \right)^2 \\
&= -2P m_2 \left[ Q, \frac{1}{I + hm_2} \right] P + 2 \left( \frac{P m_2}{I + hm_2} P \right)^2.
\end{align*}\]  

(8.65)

Since

\[m_2 \left[ Q, \frac{1}{I + hm_2} \right] = -m_2 \frac{1}{I + hm_2} \left[ Q, I + hm_2 \right] \frac{1}{I + hm_2},\]  

(8.66)

where we used \(m_2^2 = 0\), we conclude that

\[\begin{align*}
[M^{(0)}, M^{(0)}] &= 0.
\end{align*}\]  

(8.67)

### 8.1.3 Coderivation

Let us show that \(M_n^{(0)}\) for \(n > 2\) given by

\[M_n^{(0)} = (-1)^n P m_2 (hm_2)^{n-2} P\]  

(8.68)

is a coderivation on the projected space:

\[\triangle M_n^{(0)} = (M_n^{(0)} \otimes' P + P \otimes' M_n^{(0)}) \triangle.\]  

(8.69)

Since

\[\begin{align*}
\triangle P &= (P \otimes' P) \triangle, \\
\triangle m_2 &= (m_2 \otimes' I + I \otimes' m_2) \triangle, \\
\triangle h &= (h \otimes' P + I \otimes' h) \triangle,
\end{align*}\]  

(8.70-8.72)

we have

\[\begin{align*}
\triangle hm_2 P &= (hm_2 P \otimes' P + P \otimes' hm_2 P) \triangle, \\
\triangle (hm_2)^2 P &= ((hm_2)^2 P \otimes' P + h m_2 P \otimes' h m_2 P + P \otimes' (hm_2)^2 P) \triangle.
\end{align*}\]  

(8.73-8.74)
where we used $h P = 0$, $P h = 0$, and $h^2 = 0$. Suppose that

$$\triangle (h m_2)^n P = \sum_{m=0}^{n} ((h m_2)^{n-m} P \otimes' (h m_2)^m P) \triangle. \quad (8.75)$$

We can then show that

$$\triangle h m_2 (h m_2)^n P = \sum_{m=0}^{n} ((h m_2)^{n-m} P \otimes' (h m_2)^m P) \triangle + \sum_{m=0}^{n+1} ((h m_2)^{n+1-m} P \otimes' (h m_2)^m P) \triangle. \quad (8.76)$$

As the assumption (8.75) holds for $n = 0, 1, \text{and} 2$, this proves by induction that the relation (8.75) holds for any non-negative integer $n$. We then find that

$$\triangle P m_2 (h m_2)^n P = \sum_{m=0}^{n} (P m_2 (h m_2)^{n-m} P \otimes P (h m_2)^m P) \triangle + \sum_{m=0}^{n} (P (h m_2)^{n-m} P \otimes P m_2 (h m_2)^m P) \triangle \quad (8.77)$$

This completes the proof that $M_n^{(0)}$ for $n > 1$ is a coderivation on the projected space.

It is useful to prove that the condition (8.12) is satisfied without decomposing $M^{(0)}$. We define $f^{(0)}$ by

$$f^{(0)} \equiv \frac{1}{I + h m_2} P. \quad (8.78)$$

It follows from (8.75) that

$$\triangle f^{(0)} = (f^{(0)} \otimes' f^{(0)}) \triangle. \quad (8.79)$$

We also find that

$$P f^{(0)} = P, \quad (8.80)$$

which follows from $P h = 0$ and $P^2 = P$. Then we can write $M^{(0)}$ as

$$M^{(0)} = P Q P + P m_2 f^{(0)}, \quad (8.81)$$

and we find

$$\triangle M^{(0)} = (P Q P \otimes' P + P \otimes' P Q P) \triangle + (P \otimes' P) (m_2 \otimes' I + I \otimes' m_2) (f^{(0)} \otimes' f^{(0)}) \triangle = (P Q P \otimes' P + P \otimes' P Q P) \triangle + (P m_2 f^{(0)} \otimes' P + P \otimes' P m_2 f^{(0)}) \triangle, \quad (8.82)$$

where we used $P f^{(0)} = P$. We have thus shown that the condition (8.12) is satisfied.
8.1.4 Cyclic property

Let us next prove that the condition (8.14) is satisfied. For this purpose, it is convenient to introduce a linear operation which maps $\mathcal{T}\mathcal{H} \otimes' \mathcal{T}\mathcal{H}$ to $\mathcal{T}\mathcal{H}$. We denote this operation by $\triangledown$, and it acts by simply replacing $\otimes'$ with $\otimes$. For example, its action on $A_1 \otimes A_2 \otimes' A_3 \otimes A_4 \otimes A_5$ is given by

$$\triangledown (A_1 \otimes A_2 \otimes' A_3 \otimes A_4 \otimes A_5) = A_1 \otimes A_2 \otimes A_3 \otimes A_4 \otimes A_5. \quad (8.83)$$

The main identity we will use in what follows is

$$\pi_{m+n} = \triangledown (\pi_m \otimes' \pi_n) \Delta. \quad (8.84)$$

It is easy to see how this identity works. Let us consider $\triangledown (\pi_2 \otimes' \pi_1) \Delta$ as an example. Its action on the sector $\mathcal{H}^{\otimes 2}$ vanishes:

$$\triangledown (\pi_2 \otimes' \pi_1) \Delta (A_1 \otimes A_2) = \triangledown (\pi_2 \otimes' \pi_1) (1 \otimes A_1 \otimes A_2 + A_1 \otimes' A_2 + A_1 \otimes A_2 \otimes' 1) = 0, \quad (8.85)$$

as no terms in $\Delta (A_1 \otimes A_2)$ survive the projection $\pi_2 \otimes' \pi_1$. It is clear that the action of $\triangledown (\pi_2 \otimes' \pi_1) \Delta$ on $\mathcal{H}^{\otimes n}$ vanishes unless $n = 3$. Its action on the sector $\mathcal{H}^{\otimes 3}$ is given by

$$\triangledown (\pi_2 \otimes' \pi_1) \Delta (A_1 \otimes A_2 \otimes A_3)$$

$$= \triangledown (\pi_2 \otimes' \pi_1) (1 \otimes' A_1 \otimes A_2 \otimes A_3 + A_1 \otimes' A_2 \otimes A_3 + A_1 \otimes A_2 \otimes' A_3 + A_1 \otimes A_2 \otimes A_3 \otimes' 1)$$

$$= \triangledown (A_1 \otimes A_2 \otimes' A_3) = A_1 \otimes A_2 \otimes A_3. \quad (8.86)$$

We see that only one term in $\Delta (A_1 \otimes A_2 \otimes A_3)$ survives the projection $\pi_2 \otimes' \pi_1$, and we convince ourselves that $\triangledown (\pi_2 \otimes' \pi_1) \Delta$ is equivalent to $\pi_3$.

The other identity we will use is

$$\triangledown (a_m \pi_m \otimes' b_n \pi_n) = (a_m \otimes b_n) \triangledown (\pi_m \otimes' \pi_n), \quad (8.87)$$

where $a_m$ is an $m$-string operator and $b_n$ is an $n$-string operator. It is again easy to see how this identity works. Consider the action of $\triangledown (m_2 \pi_2 \otimes' P \pi_1)$ on $A_1 \otimes A_2 \otimes' A_3$ as an example. We find

$$\triangledown (m_2 \pi_2 \otimes' P \pi_1) (A_1 \otimes A_2 \otimes' A_3) = \triangledown (m_2 (A_1 \otimes A_2) \otimes' P A_3) = m_2 (A_1 \otimes A_2) \otimes P A_3. \quad (8.88)$$

On the other hand, the action of $(m_2 \otimes P) \triangledown (\pi_2 \otimes' \pi_1)$ on $A_1 \otimes A_2 \otimes' A_3$ is given by

$$(m_2 \otimes P) \triangledown (\pi_2 \otimes' \pi_1) (A_1 \otimes A_2 \otimes' A_3) = (m_2 \otimes P) \triangledown (A_1 \otimes A_2 \otimes' A_3)$$

$$= (m_2 \otimes P) (A_1 \otimes A_2 \otimes A_3) = m_2 (A_1 \otimes A_2) \otimes P A_3. \quad (8.89)$$

We thus conclude that

$$\triangledown (m_2 \pi_2 \otimes' P \pi_1) = (m_2 \otimes P) \triangledown (\pi_2 \otimes' \pi_1). \quad (8.90)$$
The generalization to different multi-string operators is straightforward.

Having finished necessary preparations, let us prove that the condition (8.14) is satisfied. As we have seen that \( \langle \omega | \pi_2 P Q P \rangle \) vanishes, we have

\[
\langle \omega | \pi_2 M^{(0)} \rangle = \langle \omega | \pi_2 P m_2 f^{(0)} \rangle .
\]  

(8.91)

Using the identity

\[
\pi_2 = \nabla (\pi_1 \otimes \pi_1) \triangle,
\]

(8.92)

we find

\[
\langle \omega | \pi_2 P m_2 f^{(0)} \rangle = \langle \omega | \nabla (\pi_1 \otimes \pi_1) \triangle P m_2 f^{(0)} \rangle
\]

\[
= \langle \omega | \nabla (\pi_1 \otimes \pi_1) (P \otimes P) (m_2 \otimes I) (f^{(0)} \otimes f^{(0)}) \rangle \triangle
\]

\[
+ \langle \omega | \nabla (\pi_1 \otimes \pi_1) (P \otimes P) (I \otimes m_2) (f^{(0)} \otimes f^{(0)}) \rangle \triangle.
\]

(8.93)

We can write \( \langle \omega | \nabla (\pi_1 \otimes \pi_1) (P \otimes P) \rangle \) using (8.39) as

\[
\langle \omega | \nabla (\pi_1 \otimes \pi_1) (P \otimes P) \rangle = \langle \omega | \nabla (P \pi_1 \otimes P \pi_1) \rangle = \langle \omega | (P \otimes P) \nabla (\pi_1 \otimes \pi_1) \rangle
\]

\[
= \langle \omega | (I \otimes P) \nabla (\pi_1 \otimes \pi_1) \rangle = \langle \omega | \nabla (\pi_1 \otimes \pi_1) \rangle (I \otimes P)
\]

(8.94)

or using (8.38) as

\[
\langle \omega | \nabla (\pi_1 \otimes \pi_1) (P \otimes P) \rangle = \langle \omega | \nabla (P \pi_1 \otimes P \pi_1) \rangle = \langle \omega | (P \otimes P) \nabla (\pi_1 \otimes \pi_1) \rangle
\]

\[
= \langle \omega | (P \otimes I) \nabla (\pi_1 \otimes \pi_1) \rangle = \langle \omega | \nabla (\pi_1 \otimes \pi_1) \rangle (P \otimes I)
\]

(8.95)

to find

\[
\langle \omega | \nabla (\pi_1 \otimes \pi_1) (P \otimes P) (m_2 \otimes I) (f^{(0)} \otimes f^{(0)}) \rangle \triangle
\]

\[
+ \langle \omega | \nabla (\pi_1 \otimes \pi_1) (P \otimes P) (I \otimes m_2) (f^{(0)} \otimes f^{(0)}) \rangle \triangle
\]

\[
= \langle \omega | \nabla (\pi_1 \otimes \pi_1) (I \otimes P) (m_2 \otimes I) (f^{(0)} \otimes f^{(0)}) \rangle \triangle
\]

\[
+ \langle \omega | \nabla (\pi_1 \otimes \pi_1) (P \otimes I) (I \otimes m_2) (f^{(0)} \otimes f^{(0)}) \rangle \triangle
\]

\[
= \langle \omega | \nabla (\pi_1 m_2 f^{(0)} \otimes \pi_1 P) \rangle \triangle + \langle \omega | \nabla (\pi_1 P \otimes \pi_1 m_2 f^{(0)}) \rangle \triangle,
\]

where we used \( P f^{(0)} = P \).

So far we have reduced \( \langle \omega | \pi_2 M^{(0)} \rangle \) as follows:

\[
\langle \omega | \pi_2 M^{(0)} \rangle = \langle \omega | \nabla (\pi_1 m_2 f^{(0)} \otimes \pi_1 P) \rangle \triangle + \langle \omega | \nabla (\pi_1 P \otimes \pi_1 m_2 f^{(0)}) \rangle \triangle.
\]

(8.97)

It follows from (8.64) that

\[
(I + h m_2) f^{(0)} = P,
\]

(8.98)

and we write \( P \) as

\[
P = f^{(0)} + h m_2 f^{(0)}
\]

(8.99)
to find
\[
\langle \omega | \nabla ( \pi_1 m_2 f^{(0)} \otimes' \pi_1 P ) \triangle \nabla ( \pi_1 m_2 f^{(0)} \otimes' \pi_1 h m_2 f^{(0)} ) \rangle = \langle \omega | \nabla ( \pi_1 m_2 f^{(0)} \otimes' \pi_1 h m_2 f^{(0)} ) \rangle.
\] (8.100)

The first term on the right-hand side can be transformed as follows:
\[
\langle \omega | \nabla ( \pi_1 m_2 \otimes' \pi_1 ) \triangle f^{(0)} = \langle \omega | \nabla ( m_2 \pi_2 \otimes' \pi_1 ) \triangle f^{(0)} = \langle \omega | ( m_2 \otimes I ) \pi_3 f^{(0)} \rangle.\] (8.101)

The second term on the right-hand side can be transformed as follows:
\[
\langle \omega | \nabla ( \pi_1 m_2 \otimes' \pi_1 h m_2 ) \triangle f^{(0)} = - \langle \omega | \nabla ( \pi_1 \otimes' h \pi_1 ) ( m_2 \otimes' m_2 ) \triangle f^{(0)} = - \langle \omega | ( I \otimes h ) \nabla ( \pi_1 \otimes' \pi_1 ) ( m_2 \otimes' m_2 ) \triangle f^{(0)} \rangle.\] (8.102)

Similarly, we have
\[
\langle \omega | \nabla ( \pi_1 P \otimes' \pi_1 m_2 f^{(0)} ) \rangle = \langle \omega | ( I \otimes m_2 ) \pi_3 f^{(0)} + \langle \omega | ( h \otimes I ) \nabla ( \pi_1 \otimes' \pi_1 ) ( m_2 \otimes' m_2 ) \triangle f^{(0)} \rangle.\] (8.103)

We therefore find
\[
\langle \omega | \pi_2 M^{(0)} = \langle \omega | \nabla ( \pi_1 m_2 f^{(0)} \otimes' \pi_1 P ) \rangle + \langle \omega | \nabla ( \pi_1 P \otimes' \pi_1 m_2 f^{(0)} ) \rangle = \langle \omega | ( m_2 \otimes I + I \otimes m_2 ) \pi_3 f^{(0)} \\
- \langle \omega | ( I \otimes h - h \otimes I ) \nabla ( \pi_1 \otimes' \pi_1 ) ( m_2 \otimes' m_2 ) \triangle f^{(0)} \rangle = 0,\] (8.104)

where we used
\[
\langle \omega | ( m_2 \otimes I + I \otimes m_2 ) = 0, \]
\[
\langle \omega | I \otimes h = \langle \omega | h \otimes I.\] (8.105)

This completes the proof that the condition (8.14) is satisfied.

8.2 Open bosonic string field theory with the source term in the low-energy limit

We have explained the construction of $M^{(0)}$ for the effective action of open bosonic string field theory without the source term. In this subsection we incorporate the source term into the construction.
8.2.1 Multi-string products

Let us first summarize the results in section 6. The zero-string product $M_0$ is related to $V_0$ as

$$M_0 = V_0.$$ \hfill (8.106)

We expand $M_0$ in $\kappa$ as follows:

$$M_0 = \sum_{n=1}^{\infty} \kappa^n M_0^{(n)},$$ \hfill (8.107)

where $M_0^{(1)}$, $M_0^{(2)}$, and $M_0^{(3)}$ are

$$M_0^{(1)} = P w_0,$$ \hfill (8.108)
$$M_0^{(2)} = P m_2(h w_0, h w_0),$$ \hfill (8.109)
$$M_0^{(3)} = P m_2(h m_2(h w_0, h w_0), h w_0) + P m_2(h w_0, h m_2(h w_0, h w_0)).$$ \hfill (8.110)

The one-string product $M_1(A_1)$ is related to $V_1(A_1)$ as

$$M_1(A_1) = V_1(A_1).$$ \hfill (8.111)

We expand $M_1(A_1)$ in $\kappa$ as follows:

$$M_1(A_1) = \sum_{n=0}^{\infty} \kappa^n M_1^{(n)}(A_1),$$ \hfill (8.112)

where $M_1^{(0)}(A_1)$, $M_1^{(1)}(A_1)$, and $M_1^{(2)}(A_1)$ are

$$M_1^{(0)}(A_1) = QA_1,$$ \hfill (8.113)
$$M_1^{(1)}(A_1) = -P m_2(h w_0, A_1) - P m_2(A_1, h w_0),$$ \hfill (8.114)
$$M_1^{(2)}(A_1) = -P m_2(h m_2(h w_0, A_1), h w_0) - P m_2(h m_2(h w_0, A_1), h w_0)$$
$$- P m_2(h m_2(h w_0, A_1), h w_0) - P m_2(A_1, h m_2(h w_0, h w_0))$$
$$- P m_2(h w_0, h m_2(A_1, h w_0)) - P m_2(h w_0, h m_2(h w_0, A_1)).$$ \hfill (8.115)

The two-string product $M_2(A_1, A_2)$ is related to $V_2(A_1, A_2)$ as

$$M_2(A_1, A_2) = (-1)^{\deg(A_1)} V_2(A_1, A_2).$$ \hfill (8.116)

We expand $M_2(A_1, A_2)$ in $\kappa$ as follows:

$$M_2(A_1, A_2) = \sum_{n=0}^{\infty} \kappa^n M_2^{(n)}(A_1, A_2),$$ \hfill (8.117)
where $M_2^{(0)}(A_1, A_2)$ and $M_2^{(1)}(A_1, A_2)$ are

$$
M_2^{(0)}(A_1, A_2) = P m_2(A_1, A_2),
$$
(8.118)

$$
M_2^{(1)}(A_1, A_2) = P m_2(h \, m_2(h \, w_0, A_1), A_2) + P m_2(h \, m_2(A_1, h \, w_0), A_2)
+ P m_2(h \, m_2(A_1, A_2), h \, w_0) + P m_2(h \, w_0, h \, m_2(A_1, A_2))
+ P m_2(A_1, h \, m_2(h \, w_0, A_2)) + P m_2(A_1, h \, m_2(A_2, h \, w_0)).
$$
(8.119)

The three-string product $M_3(A_1, A_2, A_3)$ is related to $V_3(A_1, A_2, A_3)$ as

$$
M_3(A_1, A_2, A_3) = (-1)^{\text{deg}(A_2)} V_3(A_1, A_2, A_3).
$$
(8.120)

We expand $M_3(A_1, A_2, A_3)$ in $\kappa$ as follows:

$$
M_3(A_1, A_2, A_3) = \sum_{n=0}^{\infty} \kappa^n M_3^{(n)}(A_1, A_2, A_3),
$$
(8.121)

where $M_3^{(0)}(A_1, A_2, A_3)$ is

$$
M_3^{(0)}(A_1, A_2, A_3) = -P m_2(h \, m_2(A_1, A_2), A_3) - P m_2(A_1, h \, m_2(A_2, A_3)).
$$
(8.122)

8.2.2 Expression to all orders

Our goal is to construct a linear operator $M$ on $T\mathcal{H}$ of degree odd which satisfies

$$
P M P = M,
$$
(8.123)

$$
\Delta M = (M \otimes' P + P \otimes' M) \Delta,
$$
(8.124)

$$
[M, M] = 0,
$$
(8.125)

$$
\langle \omega | \pi_2 M = 0
$$
(8.126)

and reduces to $M^{(0)}$ when $\kappa = 0$,

$$
M \big|_{\kappa=0} = M^{(0)},
$$
(8.127)

with $M_0$ given by

$$
M_0 = \kappa P \, w_0 + O(\kappa^2),
$$
(8.128)

where the $n$-string operator $M_n$ is defined in terms of the decomposition

$$
\pi_1 M = \sum_{n=0}^{\infty} M_n \pi_n.
$$
(8.129)

Actually, the construction of $M^{(0)}$ immediately generalizes to that of $M$. We define $M$ by

$$
M = P Q P + P (m_2 + \kappa \, w_0) \frac{1}{1 + h (m_2 + \kappa \, w_0)} P,
$$
(8.130)
where
\[
\frac{1}{1 + h (m_2 + \kappa w_0)} \equiv I + \sum_{n=1}^{\infty} (-1)^n (h (m_2 + \kappa w_0))^n.
\] (8.131)

It follows from this definition that
\[
\frac{1}{1 + h (m_2 + \kappa w_0)} = I.
\] (8.132)

The condition (8.123) is satisfied because \(P^2 = P\). It is also obvious that \(M\) reduces to \(M^{(0)}\) at \(\kappa = 0\).

Let us next prove that the condition (8.125) is satisfied. We define \(n\) by
\[
n \equiv m_2 + \kappa w_0.
\] (8.133)

Since \(n\) is a sum of two coderivations of degree odd, it is also a coderivation of degree odd. It has the following properties:
\[
[Q, n] = 0, \quad [n, n] = 0,
\] (8.134)

which follow from \([Q, m_2] = 0, [Q, w_0] = 0, [m_2, m_2] = 0, [m_2, w_0] = 0, [w_0, m_2] = 0\). If we compare \(M\) and \(M^{(0)}\), we see that \(M\) is obtained from \(M^{(0)}\) by replacing \(m_2\) with \(n\). In the proof that \([M^{(0)}, M^{(0)}] = 0\), the properties we used regarding the coderivation \(m_2\) of degree odd were \([Q, m_2] = 0\) and \([m_2, m_2] = 0\). Since the coderivation \(n\) of degree odd has the properties \([Q, n] = 0\) and \([n, n] = 0\), we can replace \(m_2\) with \(n\) in the proof that \([M^{(0)}, M^{(0)}] = 0\) to conclude that the condition \([M, M] = 0\) is satisfied.

### 8.2.3 Coderivation

We now move on to the proof that the condition (8.124) is satisfied. We define \(f\) by
\[
f \equiv \frac{1}{1 + h (m_2 + \kappa w_0)} P = \frac{1}{1 + h n} P,
\] (8.135)

and we write \(M\) as
\[
M = P Q P + P n f.
\] (8.136)

When we proved in (8.1) that \(f^{(0)}\) given by
\[
f^{(0)} = \frac{1}{1 + h m_2} P
\] (8.137)

satisfies the relation
\[
\triangle f^{(0)} = (f^{(0)} \otimes f^{(0)}) \triangle,
\] (8.138)

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the only property we used regarding \( m_2 \) in \( f^{(0)} \) is that it is a coderivation of degree odd. Since \( f \) is obtained from \( f^{(0)} \) by replacing \( m_2 \) with \( n \) and \( n \) is a coderivation of degree odd, we conclude that
\[
\Delta f = (f \otimes f) \Delta. \tag{8.139}
\]
We then find that
\[
\Delta M = (\mathbb{P} \mathbb{Q} \mathbb{P} \otimes' \mathbb{P} + \mathbb{P} \otimes' \mathbb{P} \mathbb{Q} \mathbb{P}) \Delta + (\mathbb{P} \mathbb{n} \mathbb{f} \otimes' \mathbb{P} \mathbb{f} + \mathbb{P} \mathbb{f} \otimes' \mathbb{P} \mathbb{n} \mathbb{f}) \Delta. \tag{8.140}
\]
It follows from \( \mathbb{P} h = 0 \) and \( \mathbb{P}^2 = \mathbb{P} \) that
\[
\mathbb{P} f = \mathbb{P}, \tag{8.141}
\]
and thus we conclude that the condition \(8.124\) is satisfied.

### 8.2.4 Cyclic property

Finally, let us prove that the condition \(8.126\) is satisfied. When we reduced \( \langle \omega \mid \pi_2 M^{(0)} \rangle \) as
\[
\langle \omega \mid \pi_2 M^{(0)} \rangle = \langle \omega \mid \nabla (\pi_1 m_2 f^{(0)} \otimes' \pi_1 \mathbb{P}) \rangle \Delta + \langle \omega \mid \nabla (\pi_1 \mathbb{P} \otimes' \pi_1 m_2 f^{(0)}) \rangle \Delta \tag{8.142}
\]
in \(8.11\) the only property we used regarding \( m_2 \) in \( M^{(0)} \) is that it is a coderivation of degree odd. Since \( M \) is obtained from \( M^{(0)} \) by replacing \( m_2 \) with \( n \) and \( n \) is a coderivation of degree odd, we conclude that \( \langle \omega \mid \pi_2 M \rangle \) can be reduced as
\[
\langle \omega \mid \pi_2 M \rangle = \langle \omega \mid \nabla (\pi_1 n f \otimes' \pi_1 \mathbb{P}) \rangle \Delta + \langle \omega \mid \nabla (\pi_1 \mathbb{P} \otimes' \pi_1 n f) \rangle \Delta, \tag{8.143}
\]
where \( f \) on the right-hand side is obtained from \( f^{(0)} \) by replacing \( m_2 \) with \( n \). It follows from \(8.132\) that
\[
(I + h n) f = \mathbb{P}, \tag{8.144}
\]
and we write \( \mathbb{P} \) as
\[
\mathbb{P} = f + h n f \tag{8.145}
\]
to find
\[
\langle \omega \mid \nabla (\pi_1 n f \otimes' \pi_1 \mathbb{P}) \rangle \Delta
= \langle \omega \mid \nabla (\pi_1 n f \otimes' \pi_1 f) \rangle \Delta + \langle \omega \mid \nabla (\pi_1 n f \otimes' \pi_1 h n f) \rangle \Delta \tag{8.146}
= \langle \omega \mid \nabla (\pi_1 n \otimes' \pi_1) \Delta f \rangle + \langle \omega \mid \nabla (\pi_1 n \otimes' \pi_1 h n) \Delta f \rangle.
\]
The first term on the right-hand side can be transformed as follows:
\[
\langle \omega \mid \nabla (\pi_1 n \otimes' \pi_1) \rangle \Delta f = \langle \omega \mid \nabla (m_2 \pi_2 \otimes' \pi_1) \rangle \Delta f + \kappa \langle \omega \mid \nabla (w_0 \pi_0 \otimes' \pi_1) \rangle \Delta f
= \langle \omega \mid (m_2 \otimes \mathbb{I}) \Delta f + \kappa \langle \omega \mid (w_0 \otimes \mathbb{I}) \Delta f \rangle \tag{8.147}
= \langle \omega \mid (m_2 \otimes \mathbb{I}) \Delta f + \kappa \langle \omega \mid (w_0 \otimes \mathbb{I}) \pi_1 f \rangle.
\]
The second term on the right-hand side can be transformed as follows:

\[
\langle \omega | \nabla ( \pi_1 n \otimes \pi_1 h n ) \rangle \delta f = - \langle \omega | \nabla ( \pi_1 \otimes' h \pi_1 ) ( n \otimes' n ) \rangle \delta f \\
= - \langle \omega | ( I \otimes h ) \nabla ( \pi_1 \otimes' \pi_1 ) ( n \otimes' n ) \rangle \delta f .
\]

Similarly, we have

\[
\langle \omega | \nabla ( \pi_1 P \otimes' \pi_1 n f ) \rangle \delta = \langle \omega | ( I \otimes m_2 ) \pi_3 f + \kappa \langle \omega | ( I \otimes w_0 ) \pi_1 f \\
+ \langle \omega | ( h \otimes I ) \nabla ( \pi_1 \otimes' \pi_1 ) ( n \otimes' n ) \rangle \delta f .
\]

We therefore find

\[
\langle \omega | \pi_2 M = \langle \omega | \nabla ( \pi_1 n f \otimes' \pi_1 P ) \rangle \delta + \langle \omega | \nabla ( \pi_1 P \otimes' \pi_1 n f ) \rangle \delta \\
= \langle \omega | ( m_2 \otimes I + I \otimes m_2 ) \pi_3 f + \kappa \langle \omega | ( w_0 \otimes I + I \otimes w_0 ) \pi_1 f \\
- \langle \omega | ( I \otimes h - h \otimes I ) \nabla ( \pi_1 \otimes' \pi_1 ) ( n \otimes' n ) \rangle \delta f \\
= 0 ,
\]

where we used

\[
\langle \omega | ( m_2 \otimes I + I \otimes m_2 ) = 0 , \\
\langle \omega | ( w_0 \otimes I + I \otimes w_0 ) = 0 , \\
\langle \omega | I \otimes h = \langle \omega | h \otimes I .
\]

This completes the proof that the condition (8.126) is satisfied.

8.2.5 Reproducing multi-string products

We have constructed \( M \) which satisfies the conditions (8.123), (8.124), (8.125), and (8.126) and reduces to \( M^{(0)} \) when \( \kappa = 0 \). Let us extract \( M_n \) from \( M \) to confirm that the expressions for the multi-string products in §8.2.1 are reproduced.

The \( n \)-string operator \( M_n \) can be obtained from the decomposition of \( \pi_1 M \). We begin the decomposition as follows:

\[
\pi_1 M = \pi_1 P Q P + \pi_1 P n f = P Q P \pi_1 + P m_2 \pi_2 f + \kappa P w_0 \pi_0 f \\
= \kappa P w_0 \pi_0 + P Q P \pi_1 + P m_2 \pi_2 f ,
\]

where we used

\[
\pi_0 f = \pi_0 .
\]

The calculation of \( \pi_2 f \) can be reduced to that of \( \pi_1 f \) as follows:

\[
\pi_2 f = \nabla ( \pi_1 \otimes' \pi_1 ) \delta f = \nabla ( \pi_1 f \otimes' \pi_1 f ) \delta ,
\]
where we used the identity (8.84) with $n = 1$ and $m = 1$. We decompose $\pi_1 f$ as

$$\pi_1 f = \sum_{n=0}^{\infty} f_n \pi_n,$$

(8.155)

where $f_n$ is an $n$-string operator of degree odd, and $\pi_2 f$ can be expressed in terms of $f_n$ as

$$\pi_2 f = \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} \nabla (f_k \pi_k \otimes' f_{\ell} \pi_{\ell}) \triangle = \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} (f_k \otimes f_{\ell}) \nabla (\pi_k \otimes' \pi_{\ell}) \triangle$$

$$= \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} (f_k \otimes f_{\ell}) \pi_{k+\ell} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (f_m \otimes f_{n-m}) \pi_n.$$

(8.156)

Since

$$\pi_1 M = \kappa P w_0 \pi_0 + PQ \pi_1 + \sum_{n=0}^{\infty} \sum_{m=0}^{n} P m_2 (f_m \otimes f_{n-m}) \pi_n,$$

(8.157)

the $n$-string operator $M_n$ is given by

$$M_0 = \kappa P w_0 + P m_2 (f_0 \otimes f_0),$$

(8.158)

$$M_1 = PQ P + P m_2 (f_0 \otimes f_1) + P m_2 (f_1 \otimes f_0),$$

(8.159)

$$M_n = \sum_{m=0}^{n} P m_2 (f_m \otimes f_{n-m}) \quad \text{for} \quad n > 1.$$

(8.160)

The calculation of $M_n$ has been reduced to that of $f_n$. It follows from (8.141) that

$$\pi_1 f + \pi_1 h m_2 f + \kappa \pi_1 h w_0 f - \pi_1 P = 0.$$

(8.161)

Since

$$\pi_1 f + \pi_1 h m_2 f + \kappa \pi_1 h w_0 f - \pi_1 P = \sum_{n=0}^{\infty} f_n \pi_n + \sum_{n=0}^{\infty} \sum_{m=0}^{n} h m_2 (f_m \otimes f_{n-m}) \pi_n + \kappa h w_0 \pi_0 - P \pi_1$$

$$= \left[ f_0 + h m_2 (f_0 \otimes f_0) + \kappa h w_0 \right] \pi_0$$

$$+ \left[ f_1 + h m_2 (f_0 \otimes f_1) + h m_2 (f_1 \otimes f_0) - P \right] \pi_1$$

$$+ \sum_{n=2}^{\infty} \left[ f_n + \sum_{m=0}^{n} h m_2 (f_m \otimes f_{n-m}) \right] \pi_n,$$

(8.162)

we obtain the following relations:

$$f_0 + h m_2 (f_0 \otimes f_0) + \kappa h w_0 = 0,$$

(8.163)

$$f_1 + h m_2 (f_0 \otimes f_1) + h m_2 (f_1 \otimes f_0) - P = 0,$$

(8.164)

$$f_n + \sum_{m=0}^{n} h m_2 (f_m \otimes f_{n-m}) = 0 \quad \text{for} \quad n > 1.$$

(8.165)
We expand $M_n$ and $f_n$ in $\kappa$ as

$$M_n = \sum_{m=0}^{\infty} \kappa^m M_n^{(m)}, \quad f_n = \sum_{m=0}^{\infty} \kappa^m f_n^{(m)},$$

and let us explain how we recursively determine $M_n^{(m)}$ and $f_n^{(m)}$.

The zero-string product $f_0$ vanishes at $\kappa = 0$. We thus have

$$f_0^{(0)} = 0.$$ 

Then the relation (8.163) recursively determines $f_0^{(m)}$. We find

$$f_0^{(1)} = - h w_0,$$

$$f_0^{(2)} = - h m_2 (h w_0 \otimes h w_0).$$

Then $M_0^{(0)}$, $M_0^{(1)}$, $M_0^{(2)}$, and $M_0^{(3)}$ are given by

$$M_0^{(0)} = 0,$$

$$M_0^{(1)} = P w_0,$$

$$M_0^{(2)} = P m_2 (h w_0 \otimes h w_0),$$

$$M_0^{(3)} = P m_2 (h m_2 (h w_0 \otimes h w_0) \otimes h w_0) + P m_2 (h w_0 \otimes h m_2 (h w_0 \otimes h w_0)).$$

We have confirmed that $M_0^{(1)}$, $M_0^{(2)}$, and $M_0^{(3)}$ in §8.2.1 are reproduced.

The relation (8.164) recursively determines $f_1^{(m)}$. We find

$$f_1^{(0)} = P,$$

$$f_1^{(1)} = h m_2 (h w_0 \otimes P) + h m_2 (P \otimes h w_0).$$

Then $M_1^{(0)}$, $M_1^{(1)}$, and $M_1^{(2)}$ are given by

$$M_1^{(0)} = P Q P,$$

$$M_1^{(1)} = - P m_2 (h w_0 \otimes P) - P m_2 (P \otimes h w_0),$$

$$M_1^{(2)} = - P m_2 (h m_2 (h w_0 \otimes h w_0) \otimes P) - P m_2 (P \otimes h m_2 (h w_0 \otimes h w_0))$$

$$- P m_2 (h w_0 \otimes h m_2 (h w_0 \otimes P)) - P m_2 (h w_0 \otimes h m_2 (P \otimes h w_0))$$

$$- P m_2 (h m_2 (h w_0 \otimes P) \otimes h w_0) - P m_2 (h m_2 (P \otimes h w_0) \otimes h w_0).$$

We have confirmed that $M_1^{(1)}$ and $M_1^{(2)}$ in §8.2.1 are reproduced.

The relation (8.165) for $n = 2$ recursively determines $f_2^{(m)}$. We find

$$f_2^{(0)} = - h m_2 (P \otimes P).$$

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Then $M_2^{(0)}$ and $M_2^{(1)}$ are given by

$$
M_2^{(0)} = P m_2 (P \otimes P),
$$

(8.180)

$$
M_2^{(1)} = P m_2 (h w_0 \otimes h m_2 (P \otimes P)) + P m_2 (P \otimes h m_2 (h w_0 \otimes P)) + P m_2 (P \otimes h m_2 (P \otimes h w_0)) 
+ P m_2 (h m_2 (h w_0 \otimes P) \otimes P) + P m_2 (h m_2 (P \otimes h w_0) \otimes P),
$$

(8.181)

and $M_3^{(0)}$ is given by

$$
M_3^{(0)} = - P m_2 (P \otimes h m_2 (P \otimes P)) - P m_2 (h m_2 (P \otimes P) \otimes P).
$$

(8.182)

We have confirmed that $M_2^{(1)}$ in §8.2.1 is reproduced.

9 Conclusions and discussion

In this paper we discussed gauge-invariant operators of open bosonic string field theory in the low energy. We added source terms for gauge-invariant operators to the action and derived the effective action for massless fields obtained by classically integrating out massive and tachyonic fields. While the gauge-invariant operators depend linearly on the open string field and do not resemble the corresponding operators such as the energy-momentum tensor in the low-energy limit, we find that nonlinear dependence is generated in the process of integrating out massive and tachyonic fields. We also find that the gauge transformation is modified in such a way that the effective action and the modified gauge transformation can be written in terms of the same set of multi-string products satisfying weak $A_\infty$ relations, and we presented explicit expressions for the multi-string products. The effective action is in general highly complicated, but relatively compact expressions are possible for open bosonic string field theory based on the star product compared to closed string field theory, and the technology associated with the weak $A_\infty$ structure provides us with analytic control over our all-order expressions for the multi-string products.

Our discussion is motivated by the low-energy limit in the context of the AdS/CFT correspondence, and we are interested in the low-energy region compared to the scale determined by $\alpha'$ of the effective action for massless fields. After taking this low-energy limit, the theory will be invariant under the ordinary gauge transformation. Invariance under the ordinary gauge transformation requires familiar constraints, and, for example, the $\alpha'$ expansion of the effective action for the gauge field of the open string takes the form of a linear combination of gauge-invariant terms. However, invariance under the ordinary gauge transformation does not constrain the coefficients in front of such gauge-invariant terms. On the other hand, the effective action with an $A_\infty$ structure does not take the form of a linear combination of gauge-invariant
terms, and constraints from invariance under the gauge transformation associated with the $A_{\infty}$ structure have a more dynamical flavor. Furthermore, the insight we obtained from the analysis of this paper is that correlation functions of gauge-invariant operators are similarly constrained from a weak $A_{\infty}$ structure before strictly taking the low-energy limit. Since the weak $A_{\infty}$ structure is closely related to the world-sheet picture, we hope that the dynamics of gauge-invariant operators dictated by the weak $A_{\infty}$ structure will help us reveal quantum gravity from the open string sector.

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