Jealousy-freeness and other common properties in Fair Division of Mixed Manna

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Abstract
We consider a fair division setting where indivisible items are allocated to agents. Each agent in the setting has strictly negative, zero or strictly positive utility for each item. We, thus, make a distinction between items that are good for some agents and bad for other agents (i.e. mixed), good for everyone (i.e. goods) or bad for everyone (i.e. bads). For this model, we study axiomatic concepts of allocations such as jealousy-freeness up to one item, envy-freeness up to one item and Pareto-optimality. We obtain many new possibility and impossibility results in regard to combinations of these properties. We also investigate new computational tasks related to such combinations. Thus, we advance the state-of-the-art in fair division of mixed manna.

1 Introduction
Consider a Foodbank problem where donated food is given out to people in need. Such shelters exist in many countries around the world. Although some people would gladly accept any “free-of-charge” food, other people might find some of it undesirable. Thus, people may like some food items and dislike others. Also, consider a paper assignment problem where scientific papers are matched to reviewers. Although reviewers tend to bid for papers from their own field of expertise, they might as well have to bid for papers from other areas. Thus, they receive papers they like and also often papers they dislike. Both of these problems can be modeled mathematically as resource allocations of indivisible items to agents.

Resource allocation of indivisible items lies on the intersection of fields such as social choice theory, computer science and algorithmic economics. Though a large body of work is devoted to the case when the items are goods (e.g. [Brams and Taylor, 1996] [Moulin, 2003] [Hugo, 1948] [Young, 1995]), there is a rapidly growing interest in the case of mixed manna (e.g. [Aleksandrov and Walsh, 2019] [Aziz et al., 2019b] [Caragiannis et al., 2012] [Sandomirskiy and Segal-Halevi, 2019]). In a mixed manna, each item can be classified as mixed (i.e. some agents strictly like it and other agents strictly dislike it), good (i.e. all agents weakly like it and some agents strictly like it) or bad (i.e. all agents weakly dislike it and some agents strictly dislike it).

An allocation of the manna gives to each agent some different bundle of items. A common task in resource allocation is to compute an allocation that minimizes the inequalities between the agents’ utilities for their bundles. A central axiomatic property that encodes this objective is equitability. An allocation is equitable if all agents derive exactly the same utility from their bundles (i.e. perfect equitability). Unfortunately, such allocations might not exist even with two agents and one item. In response, we might consider approximate versions of equitability. For example, we might require that pairwise perfect equitability is restored whenever one item is moved across the agents’ bundles. However, it might also not be possible to achieve this version simply because it requires that the agents’ altered utilities become perfectly equal.

We receive an inspiration from the work of Gourvès et al. [2014] in order to relax the “perfect equitability” requirement. They proposed an axiomatic property such as jealousy-freeness which does precisely this. An agent is jealous of another agent in a given allocation if the utility of the former agent for their own bundle is strictly lower than the utility of the latter agent for their own bundle. Otherwise, the former agent is jealousy-free of the latter agent. For any pair of agents, it is always the case that one of them is jealousy-free of the other one. However, a given allocation is jealousy-free if no agent is jealous of any other agent. Thus, such an allocation is also equitable. On the plus side, this means that there is zero inequality in it. On the minus side, it also means that such an allocation might not exist.

In this paper, we analyse four new relaxations of jealousy-freeness (i.e. JF1, JF10, JFX and JFX0) for allocations of the manna. They insist on achieving jealousy-freeness by decreasing the utilities of jealousy-free agents in “up to one item” fashion. For example, an allocation is JF1 if each agent is jealousy-free of any other agent or, otherwise, an agent who is jealous becomes jealousy-free of another agent, after some non-zero valued bad from the jealous agent’s bundle is added to the other agent’s bundle or some non-zero valued good is removed from the other agent’s bundle. JFX strengthens these requirements to each bad in the jealous agent’s bundle and each good in the other agent’s bundle. Furthermore, JF10 and JFX relax the “non-zero valued” requirements imposed by JF1 and JFX, respectively. We will shortly observe the following relations between these properties.

\[ JFX_0 \Rightarrow JFX \Rightarrow JF10 \Rightarrow JF1 \]
We also investigate how these properties interact with common efficiency and fairness criteria such as Pareto-optimality (PO) and envy-freeness up to one item (EF1, EFX and EFX₀). PO ensures that we cannot re-distribute items among agents’ bundles so that we make every agent weakly happier and some agent strictly happier. EF1 and EFX for our model are from [Aziz et al., 2019a]. For example, EF1 requires that an agent’s envy for another agent’s bundle is eliminated by removing some item from these agents’ bundles. EFX strengthens EF1 to any non-zero valued item in these bundles, increasing the envy agent’s utility or decreasing the other agent’s utility. EFX₀ extends envy-freeness up to any (possibly zero valued) good from [Plaut and Roughgarden, 2018] to any (possibly zero valued) bad. These fairness properties obey the following well-known pattern.

\[ \text{EFX₀} \Rightarrow \text{EFX} \Rightarrow \text{EF1} \]

## 2 Our contributions

We highlight in this section our results. Although we emphasise in our work on possibility and impossibility results, we report some computational results as well. Table 1 contains all results. Figures 1 and 2 depict most of them and some existing results. Some possibility results rely on exponential-time solutions such as leximin and leximin++. Others rely on polynomial-time solutions such as Algorithm 1. In particular, our findings give answers to the following question regarding combinations of JF1, EF1 and PO.

**Question:** When does it exist an allocation that satisfies a given combination of JF1, EF1 and PO?

We first report our results for the new properties JF1, JF1₀, JFX and JFX₀.

- We start by proving that JF1 allocations might not exist in problems with mixed items. (Proposition 1). The related decision problem is \( \text{NP}-\text{hard} \) (Theorem 1).
- By comparison, JFX₀ allocations might not exist in problems without mixed items (Proposition 2).
- Further, we show that each leximin++ allocation in such problems is JFX (Theorem 2).
- We also show that Algorithm 1 returns a JF1₀ allocation in such problems (Theorem 3).

We further summarize our results for combinations, including PO but excluding EF1, EFX and EFX₀.

- The leximin solution satisfies PO (Remark 1).
- JF1 and PO might be incompatible in problems with goods and bads (Proposition 3). The related decision problem is \( \text{NP}-\text{hard} \) (Theorem 4).
- Nevertheless, each leximin allocation of pure goods (i.e. all agents strictly like them) and bads is JFX and PO (Corollary 1).

We lastly list our results for other combinations, including also EF1, EFX and EFX₀. Some of these add solely to the literature of EF1, EFX, EFX₀ and PO.

![Table 1: Results for \( n \geq 2 \) agents: √-possible, ×-not possible, ⋆-normalised additive utilities, ⋆⋆-normalised general utilities.](image-url)

![Figure 1: Combinations for \( n \geq 2 \) agents: the generalized envy-free algorithm from [Aziz et al., 2019a], leximin from [Dubins and Spanier, 1961], leximin++ from [Plaut and Roughgarden, 2018], Algorithm 1 from our work.](image-url)
• For the case of 2 agents, EFX_0 allocations might not exist in problems without mixed items (Proposition 2).
• On the contrary, each leximin allocation is EFX and PO with normalised additive utilities for any manna (Theorem 5), and JFX for pure goods and bads (Corollary 2).
• We also show that each leximin++ allocation is JFX and EFX with normalised additive utilities for goods and bads (Theorem 6).
• We additionally prove that JF1 and EF1 are incompatible in two contexts with 2 agents (Propositions 4-5) and one context with 3 agents (Proposition 6).

Although leximin and leximin++ are exponential-time solutions, they often come with nice axiomatic guarantees. For this reason, we feel that their computability would not be problematic in settings with few items Bliem et al., 2016].

3 Related work

Jealousy-freeness up to one item relates to near jealousy-freeness for matroids from Gourvès et al., 2014]. In fact, JFX_0 and JFX coincide with this notion in their setting. Hence, their near jealousy-freeness algorithm returns JFX_0 allocations in problems with additive utilities for pure goods. By comparison, we prove that such allocations stop to exist as soon as we add bads to the problem. Other works of matroids are Gourvès et al., 2013a and Gourvès et al., 2013b. They also consider only positive-valued utility functions whereas we consider real-valued utility functions.

Jealousy-freeness up to one item appears to be an ideal complement to minimizing inequality. To elaborate further on this, let us consider the popular Gini index Gini, 1912. Aleksandrov and Walsh [Aleksandrov et al., 2019] showed that each allocation that minimizes the Gini index in some problems with pure goods might not share any PO guarantees. In contrast, we prove that JFX and PO allocations always exist in problems with pure goods and bads. Other works of inequalities are Endriss, 2013 and Schneckenburger et al., 2017]. However, they consider inequality measures for allocations of goods whereas we consider inequality properties for allocations of goods, bads and mixed items.

JF1_0 and JFX relate to existing approximations of equitability. Freeman et al., 2019] proposed two such notions in the context of goods (i.e. EQ1 and EQX). They proved that the leximin solution is EQX and PO in problems with pure goods. They also studied EQ1 and EQX in the context of bads Freeman et al., 2020]. However, they discovered that this solution is no longer EQ1 and no other allocation is EQX and PO. In response, they proposed two notions of equitability up to one duplicated item (i.e. DEQ1 and DEQX). Each of them is compatible with PO. Notably, EQ1, EQX for goods and DEQ1, DEQX for bads require like our properties that jealousy-freeness is restored by diminishing the jealousy-free agents’ utilities. As a result, JF1 and JFX degenerate respectively to EQ1, EQX for goods and DEQ1, DEQX for bads.

Jealousy-freeness up to one item does not relate much to EF1. For example, Aziz et al., 2019a] gave the generalized envy-graph algorithm for computing EF1 allocations. At the same time, we show that JF1 allocations might not exist in problems with mixed items and prove that the related computational problem is NP-hard [Bezákova and Dani, 2005; Dobzinski and Vondrak, 2013]. This solution has received less attention than the leximin solution (e.g. Rawls, 1971; Sen, 1976; Sen, 1977). The leximin solution is PO. However, we give problems with goods where it may fail JF1 whereas the leximin++ solution is JFX even with bads but it may not be PO. Computing the leximin and leximin++ solutions relates to computing max-min fair allocations which is NP-hard Bezákova and Dani, 2005 and Dobzinski and Vondrak, 2013. Other related works are listed in Freeman et al., 2019, Freeman et al., 2020 and Aziz et al., 2019a. However, our results do not follow from existing results.

Figure 2: Combinations for 2 agents and norm. additive utilities.
4 Formal preliminaries

In this section, we define more formally the model of fair division of mixed manna, the aforementioned axiomatic properties for allocations in this model as well as the leximin and leximin+ solutions.

4.1 Model

We consider a set \([n] = \{1, \ldots, n\}\) of \(n \in \mathbb{N}_{\geq 2}\) agents and a set \([m] = \{1, \ldots, m\}\) of \(m \in \mathbb{N}_{\geq 1}\) indivisible items. We let each \(a \in [n]\) have some utility function \(u_a : 2^{[m]} \to \mathbb{R}\). Thus, they assign some bundle utility \(u_a(M)\) to each bundle \(M \subseteq [m]\). We write \(u_a(o)\) for \(u_a(\{o\})\). We say that \(u_a(M)\) is additive iff \(u_a(M) = \sum_{o \in M} u_a(o)\). Otherwise, we say that it is general. We sometimes consider normalised utilities.

In this case, we suppose that \(u_a(\emptyset) = 0\) and \(u_a([m]) = c\) holds for each \(a \in [n]\) and some \(c \in \mathbb{R}\).

With additive utilities, the set of items \([m]\) can be partitioned into mixed items, goods and bads. Respectively, we write \([m]^+ = \{o \in [m] : \exists a \in [n] : u_a(o) > 0, \exists b \in [n] : u_b(o) < 0\}\), \([m]^0 = \{o \in [m] : \forall a \in [n] : u_a(o) \geq 0, \exists b \in [n] : u_b(o) > 0\}\) and \([m]^− = \{o \in [m] : \forall a \in [n] : u_a(o) \leq 0, \exists b \in [n] : u_b(o) < 0\}\) for the sets of these items. We refer to an item \(o\) from \([m]^+\) as a pure good if \(\forall a \in [n] : u_a(o) > 0\). Also, we refer to an item \(o\) from \([m]^−\) as a pure bad if \(\forall a \in [n] : u_a(o) < 0\).

With additive utilities, the type of item (i.e. mixed, good or bad) does not depend on the distribution of bundles to agents but only on the combination of the agents’ cardinal utilities for the item. With general utilities, we cannot do this partitioning because now whether a given item is mixed, good or bad depends on the distribution of bundles to agents and the combination of the agents’ marginal utilities for the item with respect to this bundle.

Example 1. Suppose there are 2 agents and 4 items, say \(a, b, c\) and \(d\). Further, consider the following utilities for some of the bundles of these items.

| bundle | utility | bundle | utility |
|--------|---------|--------|---------|
| \(\{b\}\) | 1       | \(\{c\}\) | 3       |
| \(\{a, b\}\) | 2       | \(\{a, c\}\) | 2       |
| \(\{b, d\}\) | 2       | \(\{c, d\}\) | 2       |
| \(\{a, b, d\}\) | 1.5     | \(\{a, c, d\}\) | 4       |

We can observe that an item can be good in one allocation and bad in another one. To see this, pick \(A = (\{a, b\}, \{c, d\}\) and \(B = (\{a, c\}, \{b, d\}\) and focus on item \(a\). This item is pure good in \(A\) because \(u_1(A_1) - u_1(A_1 \setminus \{a\}) = 1 > 0\) and \(u_2(A_2 \cup \{a\}) - u_2(A_2) = 2 > 0\) hold. However, item \(a\) is pure bad in \(B\) because \(u_1(B_1) - u_1(B_1 \setminus \{a\}) = 0 < 0\) and \(u_2(B_2 \cup \{a\}) - u_2(B_2) = -0.5 < 0\) hold.

In this case, let us consider agent \(a \in [n]\), item \(o \in [m]\) and bundle \(M \subseteq [m] \setminus \{o\}\). We say that \(o\) is good for \(a\) with respect to \(M\) if \(u_a(M \cup \{o\}) \geq u_a(M)\). We refer to \(o\) as pure good whenever \(u_a(M \cup \{o\}) > u_a(M)\). Similarly, we say that \(o\) is bad for \(a\) with respect to \(M\) if \(u_a(M \cup \{o\}) \leq u_a(M)\). We refer to \(o\) as pure bad whenever \(u_a(M \cup \{o\}) < u_a(M)\). Further, let us consider another agent \(b \in [n]\) and another bundle \(N \subseteq [m] \setminus (M \cup \{o\})\). Thus, we say that \(o\) mixed if \(u_a(M \cup \{o\}) > u_a(M)\) and \(u_b(N \cup \{o\}) < u_b(N)\).

We pay a special attention in our work to three types of problems. In a problem with mixed items, there is an allocation, an item and two agents such that one of the agents’ marginal utilities for the item in the allocation is strictly positive and the other one is strictly negative. In a problem without mixed items, all agents reach a consensus on whether a given item is good or bad in a given allocation. In a problem with pure goods and bads, the agents’ marginal utilities for a given item in a given allocation are either all strictly positive or all weakly negative.

4.2 Axiomatic properties

An (complete) allocation \(A = (A_1, \ldots, A_n)\) is such that (1) \(A_a\) is the set of items allocated to agent \(a \in [n]\), (2) \(\cup_{a \in [n]} A_a = [m]\) and (3) \(A_a \cap A_b = \emptyset\) for each \(a, b \in [n]\) with \(a \neq b\). We consider several properties for allocations.

Jealousy-freeness up to one item Let us consider an allocation and a pair of agents, say \(1\) and \(2\). One of them is jealousy-free of the other one, say \(2\). Our approximations of jealousy-freeness rely on the idea of decreasing the utility of the agent who is jealousy-free, i.e. \(2\)’s utility.

Thus, agent \(1\) is JF1 of agent \(2\) whenever \(1\)’s utility is at least as much as \(2\)’s utility, after taking a non-zero valued bad from \(1\)’s bundle and adding it to \(2\)’s bundle or removing a non-zero valued good from \(2\)’s bundle.

Definition 1. (JF1) An allocation \(A\) is jealousy-free up to some non-zero valued item if, \(\forall a, b \in [n], u_a(A_a) \geq u_b(A_b), \exists o \in A_a s.t. u_a(A_a) < u_a(A_a \setminus \{o\})\) : \(u_a(A_a) \geq u_b(A_b \cup \{o\})\) or \(\exists o \in A_b s.t. u_b(A_b) > u_b(A_b \setminus \{o\})\) : \(u_a(A_a) \geq u_b(A_b \cup \{o\})\).

Agent \(1\) is JFX of agent \(2\) whenever the above requirements hold for any bad in \(1\)’s bundle, strictly increasing \(1\)’s utility, and any good in \(2\)’s bundle, strictly decreasing \(2\)’s utility.

Definition 2. (JFX) An allocation \(A\) is jealousy-free up to any non-zero-valued item if, \(\forall a, b \in [n], (1) \forall o \in A_a s.t. u_a(A_a) < u_a(A_a \setminus \{o\})\) : \(u_a(A_a) \geq u_b(A_b \cup \{o\})\) and (2) \(\forall o \in A_b s.t. u_b(A_b) > u_b(A_b \setminus \{o\})\) : \(u_a(A_a) \geq u_b(A_b \setminus \{o\})\).

JF1 is a strictly weaker concept than JFX. Indeed, there are problems where a JF1 allocation might violate JFX. Interestingly, this can be observed even in problems where agents have the same utility for each item.

Example 2. Let us consider a problem with 2 agents and 3 pure bads, subject to the utilities in the below matrix.

|        | \(a\) | \(b\) |
|--------|------|------|
| agent 1| -1   | -2   |
| agent 2| -1   | -2   |

The allocation \(A = (\{a, c\}, \{b\})\) is such that \(u_1(A_1) = -4 < -3 = u_2(A_2 \cup \{a\})\) and \(u_1(A_1) = -4 > -5 = u_2(A_2 \cup \{c\})\) hold. Hence, \(A\) is JF1 but not JFX.

By comparison, an allocation that satisfies JFX is clearly JF1. This follows directly by these concepts’ definitions.

Freeman et al. [2019] considered a stronger variant of JF1 for problems with goods, not-imposing the non-zero marginal requirements. We generalize this concept to our setting in a similar fashion.
Definition 3. (JF1) An allocation $A$ is jealousy-free up to some item if, $\forall a, b \in [n]$, $w_a(A_b) \geq u_b(A_b)$, $\exists o \in A_a$: $u_a(A_a) \geq u_b(A_b \cup \{o\})$ or $\not\exists o \in A_b$: $w_a(A_a) \geq u_a(A_a \setminus \{o\})$.

Freeman et al. [2020] also defined similarly a stronger notion than JFX for problems with bads. We generalize this concept to our setting by relaxing the non-zero marginal requirements and refer to it as JFX0.

Definition 4. (JFX0) An allocation $A$ is jealousy-free up to any item if, $\forall a, b \in [n]$, $w_a(A_b) \geq u_b(A_b)$ and (2) $w_a(A_a) \geq u_b(A_b \cup \{o\})$ for each $o \in A_b$ s.t. $w_a(A_a) \geq u_b(A_b \setminus \{o\})$ for each $a \in A_b$ s.t. $u_a(A_a \setminus \{o\}) \geq u_b(A_b \setminus \{o\})$.

A JFX0 allocation is also JFX. Moreover, JFX is stronger than JF1 and JF1 is stronger than JF0. These relations follow directly by the definitions of these concepts.

Envy-freeness up to one item

Envy-freeness up to one item requires that an agent’s envy for another’s bundle is eliminated by removing an item from the bundles of these agents. Two notions for our model that are based on this idea are EF1 and EFX [Aziz et al., 2019a].

Definition 5. (EF1) An allocation $A$ is envy-free up to some item if, $\forall a, b \in [n]$, $w_a(A_b) \geq u_b(A_b)$ or $\not\exists o \in A_a \cup A_b$ s.t. $u_a(A_a \cup \{o\}) \geq u_b(A_b \cup \{o\})$.

Definition 6. (EFX) An allocation $A$ is envy-free up to any non-zero valued item if, $\forall a, b \in [n]$, (1) $\forall o \in A_a$ s.t. $u_a(A_a) < u_a(A_a \setminus \{o\})$: $u_a(A_a \setminus \{o\}) \geq u_b(A_b \setminus \{o\})$ and (2) $\forall o \in A_b$ s.t. $u_b(A_b) > u_b(A_b \setminus \{o\})$: $u_b(A_b \setminus \{o\}) \geq u_a(A_a \setminus \{o\})$.

Plaut and Roughgarden [2018] considered a variant of EFX for goods where, for any given pair of agents, the removed item may be valued with zero utility by the envy agent. Kyropoulou et al. [2019] referred to this one as EFX0. We adapt this property to our model by relaxing the non-zero marginal requirements in the definition of EFX.

Definition 7. (EFX0) An allocation $A$ is envy-free up to any item if, $\forall a, b \in [n]$, (1) $u_a(A_a \setminus \{o\}) \geq u_a(A_a)$ for each $o \in A_b$ s.t. $u_a(A_a) \leq u_a(A_a \setminus \{o\})$ and (2) $u_a(A_a) \geq u_a(A_a \setminus \{o\})$ for each $o \in A_a$ s.t. $u_a(A_a) \geq u_a(A_a \setminus \{o\})$.

An allocation that is EFX0 further satisfies EFX. Also, EFX is stronger than EF1. It is well-known that the opposite relations might not hold.

Pareto-optimality

Vilfredo Pareto had proposed its optimality a long time ago in his seminal work [Pareto, 1897]. We next define it formally for allocations in our model.

Definition 8. (PO) An allocation $A$ is Pareto-optimal if there is no allocation $B$ that Pareto-improves $A$, i.e. $\forall a \in [n]$: $u_a(A_a) \geq u_a(B_a)$ and $\exists b \in [n]$: $u_b(B_b) > u_b(A_b)$.

4.3 Leximin and leximin++

Plaut and Roughgarden [2018] implemented one operator for comparing allocations: $\succeq$. This operator induces a total order between allocations. Thus, an leximin allcation is a maximal element under this order. Such an allocation maximizes the minimum utility of any agent, subject to which the second minimum utility is maximized, and so on. For this reason, each leximin allocation is trivially PO.

Remark 1. In fair division of mixed manna with general utilities, the leximin solution is PO.

Plaut and Roughgarden [2018] further proposed another total operator for comparing allocations: $\succ$. They refer to the maximal elements under it as leximin++ allocations. Such an allocation maximizes the minimum utility, then maximizes the size of the bundle of an agent with minimum utility, before it maximizes the second minimum utility and the size of the second minimum utility bundle, and so on. Unfortunately, leximin++ allocations might generally not be PO.

Freeman et al. [2020] noted that there might multiple leximin allocations in some problems. Perhaps, the most relevant to us is that this observation holds for leximin++ allocations as well (see Example 2).

5 Common assumptions

An allocation in a problem with non-normalised utilities is envy-free up to one item or Pareto-optimal if and only if it is envy-free up to one item or Pareto-optimal in the corresponding problem with normalised utilities. This might not be true for a concept such as jealousy-freeness up to one item because normalisation reduces the agents’ total utilities.

Example 3. Let us consider the first problem with 4 goods and 2 agents. The second problem is its normalised version.

|   | a | b | c | d |
|---|---|---|---|---|
| Agent 1 | 1 | 1 | 1 | 1 |
| Agent 2 | 1 | 0 | 0 | 0 |

The only JF1 and PO allocation with normalised utilities is $A = \{(b, c, d), \{a\}\}$: agent 2 gets utility 1 and agent 1 gets utility $\frac{3}{4}$. However, $A$ falsifies JF1 with non-normalised utilities: agent 1 gets 1 and agent 2 gets 3.

Additionally, an allocation that ignores agents with zero utilities for items is envy-free up to one item or Pareto-optimal wrt the ignored agents. Again, this may not be true for jealousy-freeness up to one item. In fact, the only way to achieve this concept in some problems might be to give items to such agents.

Example 4. Let us consider the below problem with 4 goods and 2 agents, having 0/1 utilities for the items.

|   | a | b | c | d |
|---|---|---|---|---|
| Agent 1 | 1 | 1 | 1 | 1 |
| Agent 2 | 0 | 0 | 0 | 0 |

If we ignore agent 2, allocating all items to agent 1 is EF1 and PO but it is not even JF1 because agent 2 is not JF1 of agent 1. Otherwise, allocating one item to agent 1 and three items to agent 2 is JF1 but it is clearly neither EF1 nor PO.

We conclude that common assumptions such as normalised utilities and ignoring agents with zero utilities might be too strong in our study. For this reason, we do not make any of them throughout our work unless we explicitly mention it.
6  JF1 with mixed items

Consider again the Foodbank problem from the beginning of the paper. In this particular context, JF1 would somehow minimize the inequalities between the people’s levels of satisfaction with the received food. Unfortunately, there are such settings where none of the allocations is JF1 because some agent like items that another dislike. Thus, the utility levels of such agents in each allocation diverge from each other.

Proposition 1. There are problems with 2 agents and normalised additive utilities for 2 mixed items and 1 bad, in which no allocation is JF1.

Proof. Let us consider a problem with 2 mixed items, 1 bad and 2 agents, having the following normalised utilities.

|   | a     | b     |
|---|-------|-------|
| 1 | 1     | -4    |
| 2 | -1    | 0     |

Assume that a JF1 allocation exist in this problem. We let A denote such an allocation. We derive a contradiction.

Case 1: Let \((a, b) \cap A_1 \geq 2\) hold. If c \(\in A_2\), then \(u_1(A_1) = 2\) whereas \(u_2(A_2) = 0\). Hence, A cannot be JF1 because \(u_2(A_2) < 1 \leq u_1(A_1)\) for each \(a, b \cap A_1\). If c \(\in A_1\), then \(u_1(A_1) = 2\) whereas \(u_2(A_2) = 0\). Now, A cannot be JF1 as well because of \(u_2(c) = 0\) and, therefore, \(u_1(A_1) < 0 \leq u_2(A_2) = u_2(A_2 \cup \{c\})\).

Case 2: Let \([(a, b) \cap A_1 < 2\) hold. If c \(\in A_1\), then \(u_1(A_1) \leq -3\) whereas \(u_2(A_2) \geq 2\). But, then \(u_1(A_1) < 2 \leq u_2(A_2) = u_2(A_2 \cup \{c\})\) holds. If c \(\not\in A_2\), then \(u_1(A_1) \geq 0\) whereas \(u_2(A_2) \leq -1\). Now, \(u_2(A_2) < 0 = u_1(A_1 \{o\})\) for each \(o \in \{a, b\} \cap A_1\) and \(u_2(A_2) < 1 \leq u_1(A_1 \cup \{o\})\) for each \(o \in \{a, b\} \cap A_2\).

This result compares favorably against an axiomatic property such as envy-freeness up to some item in the sense that EF1 allocations exist in each problem [Aziz et al., 2019]. In response to this axiomatic result, we study the following computational question related to JF1 allocations.

We next present the polynomial-time reduction from X3C to POSSIBLEJF1. Let \(X = \{x_1, \ldots, x_q\}\), \(q \geq 2\) and \(C = \{C_1, \ldots, C_Q\}\). The set of agents is \([Q+1]\). The set of items is \(\{x_1, \ldots, x_q, y_1^1, y_1^2, y_1^3, \ldots, y_{Q_{-q+1}}, y_{Q_{-q+1}}^2, y_{Q_{-q+1}}^3, \ldots, z\}\). Let \(M \in \mathbb{N}_{\geq 2}\) be such that \(M \geq 3Q - 3q + 7\) holds. The utilities of agents for items are:

- for agent \(a \in [Q]: u_a(x_i) = 1\) if \(x_i \in C_a\) and else \(-M\), \(u_a(y_i^1) = 1\) and \(u_a(z) = 0\),
- for agent \((Q+1): u_{Q+1}(x_i) = 0\), \(u_{Q+1}(y_i^1) = 1\) and \(u_{Q+1}(z) = -(3q - 3)^3 + 3\).

We note that the agents’ utilities for the set of items are normalised and sum up to \((-3q - 3)M + 3 + 3(Q - q + 1)\).

We next prove that there is an exact cover by 3-sets in the instance of X3C iff there is a JF1 allocation in the instance of POSSIBLEJF1.

Let \(C'\) be an exact cover for \(X\). Hence, \(|C'| = q\) and \(C_a \cap C_b = \emptyset\) for each \(C_a, C_b \in C'\) with \(C_a \neq C_b\). Wlog, let \(C' = \{C_1, \ldots, C_q\}\). We construct the following allocation:

- Assume that a JF1 allocation exist in this problem. We let \(A\) denote such an allocation. We derive a contradiction.
- Case 1: Let \((a, b) \cap A_1 \geq 2\) hold. If c \(\in A_2\), then \(u_1(A_1) = 2\) whereas \(u_2(A_2) = 0\). Hence, A cannot be JF1 because \(u_2(A_2) < 1 \leq u_1(A_1)\) for each \(a, b \cap A_1\). If c \(\in A_1\), then \(u_1(A_1) = 2\) whereas \(u_2(A_2) = 0\). Now, A cannot be JF1 as well because of \(u_2(c) = 0\) and, therefore, \(u_1(A_1) < 0 \leq u_2(A_2) = u_2(A_2 \cup \{c\})\).
- Case 2: Let \([(a, b) \cap A_1 < 2\) hold. If c \(\in A_1\), then \(u_1(A_1) \leq -3\) whereas \(u_2(A_2) \geq 2\). But, then \(u_1(A_1) < 2 \leq u_2(A_2) = u_2(A_2 \cup \{c\})\) holds. If c \(\not\in A_2\), then \(u_1(A_1) \geq 0\) whereas \(u_2(A_2) \leq -1\). Now, \(u_2(A_2) < 0 = u_1(A_1 \{o\})\) for each \(o \in \{a, b\} \cap A_1\) and \(u_2(A_2) < 1 \leq u_1(A_1 \cup \{o\})\) for each \(o \in \{a, b\} \cap A_2\).

This result compares favorably against an axiomatic property such as envy-freeness up to some item in the sense that EF1 allocations exist in each problem [Aziz et al., 2019]. In response to this axiomatic result, we study the following computational question related to JF1 allocations.

We next present the polynomial-time reduction from X3C to POSSIBLEJF1 and PO allocation.

X3C Data: a set \(X = \{x_1, \ldots, x_q\}\) for some \(q \in \mathbb{N}_{\geq 2}\) and a collection \(C = \{C\} \subseteq X, |C| = 3\).

Result: is there \(C' \subseteq C\) s.t. \(\cup C' \subseteq X'\)?
As the number of $xs$ and $ys$ is $3(Q + 1)$, it follows that each $e \in [Q]$ receives 3 of these items. Also, agent $Q + 1$ receive 3 of the $ys$ because they have zero utilities for the $xs$. Consequently, $g$ agents from $[Q]$ receive the $xs$ and $Q - g + 1$ agents from $[Q + 1]$ receive the $ys$ in $A$. This implies that $l = q, N = \{a_1, \ldots, a_g\}$ and $Q + 1 \notin N$ hold. We note that $A_f \cap A_p = \emptyset$ holds for each $f, q \in N$ with $f \neq q$. We can now construct an exact cover by 3-sets for $X$ by uniting the bundles of the agents in $N: \forall n \in N A_h = X$. ∗

It follows by this result that checking whether JF10, JFX or JFX0 (and PO) allocations exist is intractable. Indeed, supposing that there is a polynomial-time algorithm for such allocations would lead to a contradiction with our complexity result unless $P = NP$.

7 JFX0 and EFX0 without mixed items

The presence of mixed items in the problem may make it impossible to achieve even the weakest concept JF1. By comparison, the strongest concepts JFX0 and EFX0 might be violated by any allocation in the problem even if we remove the mixed items. This follows because some moved items in some allocations are valued with zero marginal utilities.

**Proposition 2.** There are problems with 2 agents and normalised additive utilities for 1 pure good and 2 bads, in which no allocation is JFX0 or EFX0.

**Proof.** Let us consider 2 agents, 1 pure good and 2 bads. We let both agents like the good but dislike different bads.

| agent 1 | a | b | c |
|---------|---|---|---|
| agent 1 | 2 | -1 | 0 |
| agent 2 | 2 | 0 | -1 |

By the symmetry of the agents’ utilities for item a, we consider only four allocations: $A = \{(a, b), (c)\}$, $B = \{(a, c), (b)\}$, $C = \{(b, c), (a)\}$, $D = \{(a, b, c), 0\}$.

We argue that none of these allocations is JFX0 or EFX0. To see this for EFX0, we give one violation of this property for each allocation: (1) $u_2(A_2 \setminus \{c\}) = 0 < 2 = u_2(A_1)$, (2) $u_2(B_2 \setminus \{b\}) = 0 < 1 = u_2(B_1)$, (3) $u_1(C_1 \setminus \{b\}) = 0 < 2 = u_1(C_2)$ and (4) $u_2(D_2) = 0 < 1 = u_2(D_1 \setminus \{b\})$.

We next give all violations of JFX0 for each allocation: (1) $u_2(A_2) = -1 < 1 = u_1(A_1 \cup \{c\})$, (2) $u_2(B_2) = 0 < 2 = u_1(B_1 \cup \{c\})$, (3) $u_1(C_1) = -1 < 2 = u_2(C_2 \cup \{b\})$, (4) $u_1(C_1) = -1 < 1 = u_2(C_2 \cup \{c\})$ and (4) $u_2(D_2) = 0 < 1 = u_1(D_1 \setminus \{b\})$.

This result diverges from existing results. For example, EFX0 allocations exist in problems with 2 agents and goods [Plaut and Roughgarden, 2018]. Also, JFX0 allocations exist in problems with any number of agents and pure goods [Gouvervès et al., 2014]. It follows that adding bads to such problems breaks these results.

8 JFX without mixed items

JFX0 coincides with JFX whenever the problem contains pure goods and pure bads. This is not true in problems with goods and bads. In this case, JFX0 allocations may not exist whereas JFX allocations are guaranteed to exist. For example, the lexicimin++ solution is JFX.

**Theorem 2.** In fair division of goods and bads with general utilities, the lexicimin++ solution satisfies JFX.

**Proof.** Let $A$ be an lexicimin++ allocation. Suppose that $A$ is not JFX for a pair of agents $a, b \in [n]$ with $a \neq b$. That is, $u_a(A_a) < u_b(A_b)$. Also, (1) $u_a(A_a) < u_b(A_b \cup \{o\})$ holds for some $o \in A_a$ with $u_a(A_a) < u_a(A_b \setminus \{o\})$ or (2) $u_a(A_a) < u_b(A_b \setminus \{o\})$ holds for some $o \in A_b$ with $u_a(A_b) > u_b(A_b \setminus \{o\})$. Wlog, let $u_1(A_1) \leq \ldots \leq u_n(A_n)$ denote the utility order induced by $A$. We let $k = \max \{i \in [n] | u_i(A_i) \leq u_i(A_a)\}$. We note $a \leq k$ and $k < b$. We consider two cases.

Case 1: Let (1) hold for bad $o \in A_a$. Let us move item $o$ from the bundle $A_a$ to the bundle $A_b$. We let $C$ denote this new allocation. That is, $C_a = A_a \setminus \{o\}$, $C_b = A_b \cup \{o\}$ and $C_c = A_c$ for each $c \in [n] \setminus \{a, b\}$. We argue that $C \succ^{++} A$.

We note that $C_c = A_c$ holds for each $c \in [k] \setminus \{a\}$. We show $u_a(C_a) > u_k(A_k)$ where $a_k$ is the $k$th agent in the utility order induced by $C$. If this agent is $a$, then $u_a(C_a) = u_a(A_a \setminus \{o\}) > u_A(A_a) = u_k(A_k)$ by (1). If this agent is $b$, then $u_b(C_b) = u_b(A_b \cup \{o\}) > u_A(A_a) = u_k(A_k)$ by (1). Otherwise, $u_a(C_a) = u_{a+1}(A_{a+1}) > u_k(A_k)$ by the definition of $k$. It follows in each case that $u_d(A_d) \geq u_k(C_k) = u_k(A_k)$ holds for each agent $d \in [n] \setminus \{[k] \setminus \{a\} \cup \{a_k\}\}$. Therefore, $A$ cannot be lexicimin++. This is a contradiction.

Case 2: Let (2) hold for good $o \in A_b$. Let us move only item $o$ from $A_b$ to $A_a$. We let $B$ denote this allocation. $B_a = A_a \cup \{o\}$, $B_b = A_b \setminus \{o\}$ and $B_c = A_c$ for each $c \in [n] \setminus \{a, b\}$. Similarly as for $C$ in the first case, we next argue that $B \succ^{++} A$ holds.

As item $o$ is good, it follows $u_a(B_a) \geq u_a(A_a)$. If $u_a(B_a) = u_a(A_a)$, then $B_c = A_c$ holds for each $c \in a$, $|B_a| = |A_a| + 1$ and $B_d = A_d$ for each $d \in [a, k]$. Moreover, it follows that $u_e(C_e) > u_k(A_k) = u_k(A_a)$ holds for each agent $e \in [n] \setminus \{k\}$, including for agent $b$ by (2). As $\succ^{++}$ maximizes the bundle size as a secondary objective, it follows that $B$ is strictly larger than $A$ under $\succ^{++}$. Again, $A$ cannot be lexicimin++. If $u_a(B_a) > u_a(A_a)$, then we reach a contradiction in a similar way as in the first case. ∗

This characterization result is tight. Indeed, if we relaxed JFX to JFX0 in it, then we would derive an impossibility result by Proposition 2.

9 JF10 without mixed items

The lexicimin++ solution is JFX and it can be computed in $O(n^m)$ time. This might be fine for small $m$. However, $m$ can be much larger than $n$ in practice. For this reason, we may wish to return an allocation that satisfies the weaker concept JF10. Surprisingly, we can do this in $O(mn)$ time by using Algorithm [1]. We next describe this algorithm.

Basically, Algorithm [1] allocates the items one-by-one to agents in an arbitrary order. If the current item is a pure good, then it goes to an agent with minimum utility. If it is a pure bad, then it goes to an agent with maximum utility, supposing the item is given to them. Otherwise, it goes to an agent who has zero utility for it.
The key idea behind the inductive proof of the correctness of the algorithm relies on the fact that it differentiates between pure goods, pure bads and indifferent items. Thus, it is guaranteed that if the partial allocation at a given round were not JF1, then the partial allocation at the previous round would also violate JF1.

Algorithm 1 Jealousy-freeness up to some item

1: procedure JF10_ALLOCATION([n], [m], (u_a)_a)
2: ∀a ∈ [n]: A_a ← ∅
3: for t = 1 : m do
4: if ∀c ∈ [n]: u_c(A_c ∪ {t}) > u_c(A_c) then
5: a ← arg min_b∈[n] u_b(A_b)
6: else if ∀c ∈ [n]: u_c(A_c ∪ {t}) < u_c(A_c) then
7: a ← arg max_b∈[n] u_b(A_b ∪ {t})
8: else
9: a ← arg{b ∈ [n]|u_b(A_b ∪ {t}) = u_b(A_b)}
10: A_a ← A_a ∪ {t}
11: return A

Theorem 3. In fair division of goods and bads with general utilities, Algorithm 1 returns an JF10 allocation.

Proof. Algorithm 1 gives the items in rounds 1 to m. We let A' denote the allocation constructed up to round t. We will prove that each A' is JF10 by induction on t.

In the base case when t = 1, the proof is trivial. In the hypothesis, let us assume that A'−1 is JF10. In the step case, Algorithm 1 allocates item t. Wlog, let agent 1 receive t. Thus, the bundle of each other agent in A' is the same as in A'−1. Hence, two agents a, b ∈ [n] \ {1} with a ≠ b are JF10 of each other in A' by the hypothesis. For this reason, we next consider three remaining cases.

Case 1: If u_1(A') = u_1(A'−1), then agent 1’s utility in A' is equal to their utility in A'−1. Hence, the allocation A' remains JF10.

Case 2: If u_1(A') < u_1(A'−1), then t must be a pure bad. By the hypothesis, it follows that each other agent is JF10 of agent 1 in A' simply because 1’s utility in A' is strictly lower than 1’s utility in A'−1. Consequently, we only prove that agent 1 is JF10 of each agent a ∈ [n] \ {1}. If this were not the case for some a ∈ [n] \ {1}, then u_1(A'−1) < u_a(A_a ∪ {t}) would hold. This would imply u_1(A'−1) < u_a(A_a ∪ {t}) because of A'−1 = A'−1 ∪ {t}, A_a = A_a−1 and contradict the fact that u_1(A'−1) ≥ u_a(A_a−1 ∪ {t}).

Case 3: If u_1(A') > u_1(A'−1), then t must be a pure good. As agent 1 is JF10 of any other agent in A'−1, they remain JF10 of them in A' simply because 1’s utility in A' is strictly greater than 1’s utility in A'−1. Hence, we only prove that each agent a ∈ [n] \ {1} is JF10 of agent 1. As agent 1 gets item t, it must be that u_a(A_a−1) ≥ u_1(A'−1) holds. Also, u_a(A_a) = u_a(A_a−1) and u_1(A'−1) = u_1(A_1 \ {t}). Hence, u_a(A_a) ≥ u_1(A_1 \ {t}).

We further consider combinations of jealousy-freeness up to one item and Pareto-optimality.

10 JF1, JFX and PO

Freeman et al. [2019] gave a problem with 3 agents and normalised additive and binary (i.e. 0/1) utilities, where none of the allocations satisfies simultaneously JF10 and PO. This incompatibility might further hold in our setting even with just 2 agents.

Proposition 3. There are problems with 2 agents and normalised additive utilities for 2 goods and 2 bads, where no PO allocation is JF1.

Proof. Let us consider the below problem with 2 agents, 2 goods and 2 bads. We note that the agents’ utilities are normalised.

| agent 1 | a | b | c | d |
|---|---|---|---|---|
| agent 1 | 1 | 1 | −5 | 0 |
| agent 2 | 0 | 0 | −3 | −3 |

There is only one PO allocation (leximin). This one gives items a, b, c to agent 1 and item d to agent 2. Let A denote this allocation. We have u_1(A_1) = 2, u_2(A_2) = 0 and u_2(A_2) < 1 = u_1(A_1 \ {c}) for each o ∈ A_1 with u_1(o) > 0. Hence, the allocation A violates JF1. ∗

This result reveals the technical difference between the leximin and leximin++ solutions. The former one is PO but may not be JF1 because it could give to an agent high utility whilst the latter one is JFX but may not be PO because it could give to an agent an item for which they have zero utility.

Freeman et al. [2019] proved that deciding whether JF10 and PO allocations exist in problems with non-normalised and non-binary utilities is NP-hard. We strengthen this hardness result to the weaker combination of JF1 and PO in problems with normalised and binary utilities.

Theorem 4. In fair division of goods and normalised additive and 0/1 utilities, Possible(JF1 ∧ PO) is NP-hard.

Proof. Let us consider the reduction in Theorem 4. Suppose that we remove agent (Q + 1) and items y_{Q−q+1}, y_{Q−q+1}', y_{Q−q+1}^3, z. Further, suppose that we substitute each −M with 0. This transformation gives us a problem with Q agents, 3Q goods and normalised 0/1 utilities. Let C denote an exact cover for X. We can construct an allocation A_C as in the proof of Theorem 4. It follows that A_C is JF1. This allocation is also PO because each agent receive items valued with 1. Let there be an JF1 and PO allocation A. By PO, it must be the case that each agent receive items valued with 1. Hence, the sum of agents’ utilities in A is equal to 3Q. This is only possible whenever each agent get utility 3. We can construct an exact cover C_A for X as in the proof of Theorem 4. The result follows. ∗

The impossibility result differs from the existence of EF1 and PO allocations in the case of 0/1 utilities. In fact, such allocations can be computed in polynomial time by the online algorithm BALANCED LIKE from [Aleksandrov et al., 2015]. Benade et al. [2018] made this observation.

The result further breaks whenever there are just pure items in the problem (i.e. non-zero marginal utilities). In fact, an allocation in such a problem is leximin++ if it is leximin. By Remark 1 and Theorem 2, it follows that the leximin solution is JFX and the leximin++ solution is PO.
We can safely extend these guarantees to problems where agents are indifferent for some bads. Indeed, the leximin solution is PO and, for this reason, it gives each such bad to some agent who has zero marginal utility for it. These decisions are optimal from a JFX perspective as well.

**Corollary 1.** In fair division of pure goods and bads with general utilities, the leximin solution satisfies JFX and PO.

This result is tight by Proposition 2. This also follows by an existing result of Freeman et al. [2020] who observed that JFX₀ and PO might not be attainable in problems with bads.

### 11 JF1, JFX and EF1, EFX

JF1 and EF1 might be unachievable in problems where some agents have zero total utility for the items (see Example 4). For this reason, we assume in this section that there are no such agents. We present indeed some positive results under this common assumption.

#### 11.1 The case of 2 agents

JF1 and EF1 might as well be violated by each allocation in problems with non-normalised utilities for bads. This is because JF1 may bias the allocation towards agents with the greatest total utility for bads. Thus, such an allocation may give all bads to a single agent. As a result, it could easily falsify an axiomatic property such as EF1.

**Proposition 4.** There are problems with 2 agents and additive utilities for 2 pure bads, where no JF1 allocation is EF1.

**Proof.** The proof is in terms of the below counter-example. The agents’ utilities are clearly not normalised, one with a total of −2 and the other one with a total of −6.

|      | a  | b  |
|------|----|----|
| agent 1 | −1 | −1 |
| agent 2 | −3 | −3 |

To achieve JF1, we argue that we should give both items to agent 1. Otherwise, agent 2 would get disutility of at least −3 but be still jealous up to one item of agent 1 because shifting one item from 2’s bundle to 1’s bundle would make 1’s disutility at most −2. Well, let us then give the items to agent 1. Clearly, this violates EF1 because agent 1 envies agent 2 even after removing any item from their 1’s bundle. ♦

It follows that the leximin and leximin++ solutions might violate EF1 in some problems with non-normalised utilities. At the same time, it seems to us that normalisation occurs often in practice. For example, some web-applications on Spliddit ask agents to share a fixed total (i.e., normalised) utility for items [Caragiannis et al., 2016].

As a response, we will prove shortly that the impossibility result breaks in two contexts with normalised additive utilities: (1) JFX, EFX and PO for pure goods and bads; (2) JFX and EFX for goods and bads. However, we first give another strong result. Namely, EFX and PO are always attainable in problems with such utilities for arbitrary items.

**Theorem 5.** In fair division of mixed manna with 2 agents and normalised additive utilities, the leximin solution satisfies EFX and PO.

**Proof.** Let A be an leximin allocation. By Remark 1, A is PO. Suppose that A is not EFX. Wlog, let agent 1 be not EFX of agent 2. Hence, it must be the case that (1) \( u_1(A_1 \cup \{o\}) < u_1(A_2) \) holds for some \( o \in A_1 \) with \( u_1(o) < 0 \) or (2) \( u_1(A_1) < u_1(A_2 \setminus \{o\}) \) holds for some \( o \in A_2 \) with \( u_1(o) > 0 \). We consider two cases.

If (1) holds for \( o \in A_1 \) with \( u_1(o) < 0 \), then \( u_2(o) < 0 \) by the PO of A. Let us consider bundles \( S_1 = A_1 \setminus \{o\} \) and \( S_2 = A_2 \cup \{o\} \) in this case.

If (2) holds for \( o \in A_2 \) with \( u_1(o) > 0 \), then \( u_2(o) > 0 \) by the PO of A. Let us consider bundles \( S_1 = A_1 \cup \{o\} \) and \( S_2 = A_2 \setminus \{o\} \) in this case.

We construct an allocation \( B \) and show that the minimum utility in \( B \) is greater than the minimum utility in \( A \), reaching a contradiction with the leximin-optimality of \( A \). Let

\[
B_1 = \arg\min_{S \in \{S_1, S_2\}} u_2(S),
\]

\[
B_2 = \arg\max_{S \in \{S_1, S_2\}} u_2(S).
\]

By construction, \( u_2(B_2) \geq u_2(B_1) \) holds in \( B \). Moreover, \( u_1(S_1) > u_1(A_1) \) and \( u_1(S_2) > u_1(A_1) \) hold in each of the cases (1) and (2). These inequalities follow because agent 1’s utilities for the moved item are non-zero and agent 1 is not EFX of agent 2. We conclude \( u_1(B_1) > u_1(A_1) \).

We have \( u_1(A_1) + u_1(A_2) = c \) and \( u_2(A_1) + u_2(A_2) = c \) for some \( c \in \mathbb{R} \) by the fact that the agents’ utilities are normalised and additive. As \( u_1(A_1) < u_1(A_2) \), it follows \( u_1(A_1) < c/2 \). By the PO of A, \( u_2(A_2) > u_2(A_1) \). Hence, \( u_2(A_1) < c/2 \) and \( u_2(A_2) > c/2 \). Further, as \( u_2(B_2) \geq u_2(B_1) \) and \( u_2(B_1) + u_2(B_2) = c \), it follows \( u_2(B_2) \geq c/2 \).

We are ready to derive the aforementioned contradiction:

\[
\min\{u_1(A_1), u_2(A_2)\} = u_1(A_1) < \min\{u_1(B_1), c/2\} \leq \min\{u_1(B_1), u_2(B_2)\}. \]

These follow because of \( u_2(A_2) > c/2 \), \( u_1(A_1) < c/2 \), \( u_1(A_1) < u_1(B_1) \) and \( u_2(B_2) \geq c/2 \). The result follows. ♦

This result provides stronger guarantees than some existing results. For example, the MNW solution is guaranteed to be EF1 and PO in problems with goods [Caragiannis et al., 2016]. On the other hand, it may violate EFX or PO in problems with mixed items (see [Aleksandrov and Walsh, 2019]) where the leximin solution remains EFX and PO.

**Theorem 5** holds for problems with normalised additive utilities. By comparison, EFX and PO may be incompatible in problems with normalised general utilities for pure goods (i.e., with non-zero marginal utilities for goods) [Plaut and Roughgarden, 2018]. As a consequence, **Theorem 5** breaks with general utilities.

By Corollary 1, the leximin solution is JFX in problems with normalised additive utilities for pure goods and bads. Furthermore, both the leximin and leximin++ solutions induce now the same utilities but their distributions of indifferent bads may differ. By **Theorem 5**, the leximin++ solution is EFX and PO in such problems.

**Corollary 2.** In fair division of pure goods and bads with 2 agents and normalised additive utilities, the leximin solution satisfies JFX, EFX and PO.
By Proposition 3, we cannot extend this result to problems where the agents are indifferent for goods. However, Plaut and Roughgarden [2018] argued that each such good (or even a mixed item in our view) could be given to the agent who values it positively prior to the allocation.

Otherwise, Proposition 3 implies that no PO allocation can satisfy both EF1 and JF1, including the leximin solution. Nevertheless, we might wish to drop PO and achieve only EFX and JFX. Surprisingly, we can do this in a special case. For this purpose, we now need the leximin++ solution.

**Theorem 6.** In fair division of goods and bads with 2 agents and normalised additive utilities, the leximin++ solution is JFX and EFX.

**Proof.** Let $A$ denote an leximin++ allocation. By Theorem 2, JFX follows. We next prove EFX. Suppose that agent 1 is not EFX of agent 2. Consequently, $u_1(A_1) < u_2(A_2)$. Further, as the utilities of agent 1 are normalised, it follows that $u_1(A_1) < c/2$ holds for some $c \in \mathbb{R}$. We construct a new allocation $B$ as in Theorem 5.

We have $u_1(B_1) > u_1(A_1)$. We also show that $u_2(B_2) < u_1(A_1)$ holds. The argument for it is by contradiction. Let $u_2(B_2) < u_1(A_1)$ hold. By the definition of $B$, $u_2(B_2) \geq u_2(B_1)$. Therefore, $u_2(B_2) < c/2$ and $u_2(B_1) < c/2$ follow because $u_1(A_1) < c/2$. This is in conflict with $u_2(B_1) + u_2(B_2) = c$. Hence, $u_2(B_2) > u_1(A_1)$.

Let $k = \min\{u_1(A_1), u_2(A_2)\}$. We derive $u_1(B_1) > k$ and $u_2(B_2) > k$. The minimum utility in $B$ is strictly greater than the minimum utility in $A$. As an leximin++ allocation maximizes this utility before the bundle size, it follows that $B$ is strictly larger than $A$ under $\succ^{++}$. This means that agent $A$ cannot be leximin++. We reached a contradiction. \hfill \Box

The leximin and leximin++ solutions are both intractable. We can however compute in $O(mn)$ time an EFX allocation in problems with additive utilities. For this purpose, we can use the “cut-and-choose” protocol from [Plaut and Roughgarden, 2018] with the EFX algorithm for identical utilities from [Aleksandrov and Walsh, 2019] as a sub-routine.

With general utilities, there are problems where no allocation satisfies both JF1 and EF1, even under the assumption that the agents’ utilities for the set of items are the same. Hence, Theorem 6 breaks. That is, the leximin++ or even leximin solution may no longer satisfy even EFX if we move to normalised general problems with pure goods.

**Proposition 5.** There are problems with 2 agents and normalised general utilities for 4 pure goods, where no JF1 allocation is EF1.

**Proof.** Let us consider 2 agents and a set of goods $\{m\}$, where $m \geq 4$. Define each agent’s utilities as follows: (1) $u_1(S) = |S|$ for each $S \subseteq [m]$ and (2) $u_2(S) = \epsilon |S|$ for each $S$ such that $S \subseteq [m]$ and $u_2([m]) = m$, where $\epsilon \in (0, 1/\epsilon)$. We note that the agents’ utilities are normalised: $u_1(\emptyset) = u_2(\emptyset) = 0$ and $u_1([m]) = u_2([m]) = m$.

If agent 1 received at least two items, then their utility would be at least 2. But, then agent 2 get at most $(m - 2)$ items and, hence, their utility is at most $\epsilon(m - 2)$. This is strictly lower than 1 because of $\epsilon < 1/\epsilon$. As a result, agent 2 is jealous of agent 1 even after the removal of any item from 1’s bundle. Hence, such an allocation cannot be JF1.

To achieve JF1, agent 1 should receive at most one item. If they received no item, then they would not be JF1 of agent 2. Hence, they receive one item and their utility is 1. However, agent 2 now receive $(m - 1)$ items and agent 1’s utility for 2’s bundle without one item is $(m - 2)$. This is at least 2 because of $m \geq 4$. Hence, such an allocation cannot be EF1. \hfill \Box

By Proposition 2, the possibility results break whenever we relax JFX to JFX$_0$ or EFX to EFX$_0$. Hence, they are tight. This is also in-line with an impossibility result for EFX$^0$ and PO allocations of goods [Plaut and Roughgarden, 2018].

**11.2 The case of $n \geq 3$ or fewer items**

In the case of 2 agents, normalisation plays a crucial role in achieving JFX and EFX in problems with goods and bads. However, normalisation may not help us whenever there are more agents in the problem. This result complements an existing impossibility result for JF1 and EFX allocations in problems with 3 agents and non-normalised utilities for 7 pure goods [Freeman et al., 2019].

**Proposition 6.** There are problems with 3 agents and normalised additive utilities for 3 pure bads, where no JF1 allocation is EFX.

**Proof.** The proof uses a simple counter-problem with 3 agents whose utilities for 3 pure bads sum up to $-30$.

|     | a  | b  | c  |
|-----|----|----|----|
| agent 1 | -28 | -1 | -1 |
| agent 2 | -24 | -3 | -3 |
| agent 3 | -16 | -7 | -7 |

To achieve EF1, it is easy to see that each agent should get exactly one item. Otherwise, an agent with at least two items would not be EF1 of an agent with zero items. Hence, there are 6 EF1 allocations.

For each allocation of $a$, there are two symmetrical EF1 allocations giving $b$ and $c$ to different agents. For this reason, let us consider only 3 EF1 allocations: $A = \{(a), \{b\}, \{c\}\}$, $B = \{(b), \{a\}, \{c\}\}$ and $C = \{(c), \{b\}, \{a\}\}$.

We simply show that each of these allocations violates JF1: (1) $u_1(A_1) = -28 < -27 = u_2(A_2 \cup \{a\})$, (2) $u_2(B_2) = -24 < -23 = u_3(B_3 \cup \{a\})$ and (3) $u_2(C_2) = -3 < -2 = u_3(C_3 \cup \{b\})$. The result follows. \hfill \Box

This result suggests that we might wish to achieve JF1 and EF1 in isolation from each other. For JF1, in problems without mixed items, we can use Algorithm 1. For EF1 in problems with any items, we can use the generalized envy-graph algorithm from [Aziz et al., 2019].

The case of pure bads contrasts axiomatically and computationally with the case of pure goods and additive utilities. In this case, JFX, EFX and PO allocations exist. For example, the allocation that maximizes the sum of agents’ utilities for at most one pure good satisfies these three properties. Such an allocation can be computed in $O(n^3)$ time by using a matching procedure such as the hungarian method [Kuhn, 1955].

In contrast, let us consider a problem with normalised general utilities for at most $n$ goods or bads. Giving the goods (bads) to different agents is JFX and EFX (EFX). Such an allocation can be computed in $O(n)$ time. We submit it as an open question whether JFX or EFX allocations can be computed efficiently in the case of at most $n$ goods and bads.
12 Discussion

We could attempt to tackle the impossibility results by relaxing further JF1 or EF1. For example, two natural approximations of EF1 for goods are EF2 from [Bilo et al., 2019] and PROP1 from [Conitzer et al., 2017]. We can construct though problems as in Propositions 5 and 6 where neither EF2 nor PROP1 is compatible with JF1. Alternatively, we can approximate JF1 by relaxing the “up to one item” constraints to “up to two items”. Say, we call this property JF2. We can then again give similar problems where JF2 is incompatible with EF1. These observations suggest that further relaxations of EF2 and JF2 also might not interact in some problems.

Even more, Algorithm 1 for JF1 allocations remind us of the popular envy-graph algorithm for EF1 allocations [Lipton et al., 2004]. Both algorithms allocate the items one-by-one in some order. However, the envy-graph algorithm lets agents exchange bundles of some previous items at each step. By comparison, a notable advantage of Algorithm 1 is that it makes the decision for the current item without re-allocating any of the previous items or using information about any of the next items. Hence, it can be adapted to work in online environments such as the one in [Aleksandrov et al., 2015] by inputting the items to it one-by-one.

13 Conclusions

We considered a fair division setting where agents assign utilities to bundles of indivisible items in a mixed manna. For this model, we studied combinations of properties for concepts such as jealousy-freeness up to one item, envy-freeness up to one item and Pareto-optimality. We obtained many possibility and impossibility results for such combinations. We also studied computational tasks related to these combinations: some of them exhibit exponential-time algorithms (unless P = NP) and some others admit polynomial-time algorithms. We summarized all our results in Table 1.

Our work opens up many questions for future work. For example, how well perform the proposed solutions in simulations? Also, the lexicmin++ solution is unfortunately intractable even when the agents’ utilities are specified in unary. For this reason, we believe that it is natural to ask whether there is a pseudo-polynomial-time algorithm for JFX allocations in this case? Further, the lexicmin++ solution is EFX in problems with goods and general but identical utilities. It is, therefore, also pertinent to ask if these guarantees extend to problems with mixed manna?

References

[Aleksandrov and Walsh, 2019] Martin Aleksandrov and Toby Walsh. Greedy algorithms for fair division of mixed manna. CoRR, abs/1911.11005, 2019.

[Aleksandrov et al., 2015] Martin Aleksandrov, Haris Aziz, Serge Gaspers, and Toby Walsh. Online fair division: analysing a food bank problem. In Proceedings of the Twenty-Fourth International Joint Conference on Artificial Intelligence, IJCAI 2015, Buenos Aires, Argentina, July 25-31, 2015, pages 2540–2546, 2015.

[Aleksandrov et al., 2019] Martin Aleksandrov, Cunjing Ge, and Toby Walsh. Fair division minimizing inequality. In Progress in Artificial Intelligence, 19th EPIA Conference on Artificial Intelligence, EPIA 2019, Vila Real, Portugal, September 3-6, 2019, Proceedings, Part II, pages 593–605, 2019.

[Aziz et al., 2019a] Haris Aziz, Ioannis Caragiannis, Ayumi Igarashi, and Toby Walsh. Fair allocation of indivisible goods and chores. In Proceedings of the Twenty-Eighth International Joint Conference on Artificial Intelligence, IJCAI-19, pages 53–59. International Joint Conferences on Artificial Intelligence Organization, 7 2019.

[Aziz et al., 2019b] Haris Aziz, Hervé Moulin, and Fedor Sandomirsky. A polynomial-time algorithm for computing a Pareto optimal and almost proportional allocation. CoRR, abs/1909.00740, 2019.

[Benade et al., 2018] Gerdus Benade, Aleksandr M. Kazachkov, Ariel D. Procaccia, and Christos-Alexandros Psomas. How to make envy vanish over time. In Proceedings of the 2018 ACM Conference on Economics and Computation, EC ’18, pages 593–610, New York, NY, USA, 2018. ACM.

[Bezáková and Dani, 2005] Ivona Bezáková and Varsha Dani. Allocating indivisible goods. Association for Computing Machinery (ACM) SIGecom Exchanges, 5(3):11–18, April 2005.

[Bilo et al., 2019] Vittorio Bilò, Ioannis Caragiannis, Michele Flammini, Ayumi Igarashi, Gianpiero Monaco, Dominik Peters, Cosimo Vinci, and William S. Zwicker. Almost envy-free allocations with connected bundles. In Avrim Blum, editor, 10th Innovations in Theoretical Computer Science Conference, ITCS 2019, January 10-12, 2019, San Diego, California, USA, volume 124 of LIPIcs, pages 14:1–14:21. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2019.

[Bliem et al., 2016] Bernhard Bliem, Robert Bredereck, and Rolf Niedermeier. Complexity of efficient and envy-free resource allocation: Few agents, resources, or utility levels. In Proceedings of the Twenty-Fifth International Joint Conference on Artificial Intelligence, New York, NY, USA, 9-15 July 2016, pages 102–108, 2016.

[Brams and Taylor, 1996] Steven J. Brams and Alan D. Taylor. Fair Division - from Cake-cutting to Dispute Resolution. Cambridge University Press, 1996.

[Caragiannis et al., 2012] Ioannis Caragiannis, Christos Kaklamanis, Panagiotis Kanellopoulos, and Maria Kyropoulou. The efficiency of fair division. Theory of Computing Systems, 50(4):589–610, May 2012.

[Caragiannis et al., 2016] Ioannis Caragiannis, David Kurokawa, Hervé Moulin, Ariel D. Procaccia, Nisarg Shah, and Junxing Wang. The unreasonable fairness of maximum Nash welfare. In Proceedings of the 2016 Association for Computing Machinery ACM Conference on EC ’16, Maastricht, The Netherlands, July 24-28, 2016, pages 305–322, 2016.
[Conitzer et al., 2017] Vincent Conitzer, Rupert Freeman, and Nisarg Shah. Fair public decision making. In Proceedings of the 2017 ACM Conference on Economics and Computation, EC ’17, pages 629–646, New York, NY, USA, 2017. ACM.

[Dobzinski and Vondrák, 2013] Shahar Dobzinski and Jan Vondrák. Communication complexity of combinatorial auctions with submodular valuations. In Proceedings of the Twenty-fourth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA ’13, pages 1205–1215, Philadelphia, PA, USA, 2013. Society for Industrial and Applied Mathematics.

[Dubins and Spanier, 1961] L. E. Dubins and E. H. Spanier. How to cut a cake fairly. The American Mathematical Monthly, 68(1P1):1–17, 1961.

[Endriss, 2013] Ulle Endriss. Reduction of economic inequality in combinatorial domains. In Maria L. Gini, Onn Shehory, Takayuki Ito, and Catholijn M. Jonker, editors, International conference on Autonomous Agents and Multi-Agent Systems, AAMAS ’13, pages 175–182. IFAAMAS, 2013.

[Freeman et al., 2019] Rupert Freeman, Sujoy Sikdar, Rohit Vaish, and Lirong Xia. Equitable allocations of indivisible goods. In Proceedings of the Twenty-Eighth International Joint Conference on Artificial Intelligence, IJCAI-19, pages 280–286. International Joint Conferences on Artificial Intelligence Organization, 2019.

[Freeman et al., 2020] Rupert Freeman, Sujoy Sikdar, Rohit Vaish, and Lirong Xia. Equitable allocations of indivisible chores. CoRR, abs/2002.11504, 2020.

[Garey and Johnson, 1979] M. R. Garey and David S. Johnson. Computers and Intractability: A Guide to the Theory of NP-Completeness. W. H. Freeman, 1979.

[Gini, 1912] Corrado Gini. Variabilità e mutabilità. C. Cuppini, Bologna, 1912.

[Gourvès et al., 2013a] Laurent Gourvès, Jérôme Monnot, and Lydia Tlilane. A matroid approach to the worst case allocation of indivisible goods. In Francesca Rossi, editor, IJCAI 2013, Proceedings of the 23rd International Joint Conference on Artificial Intelligence, Beijing, China, August 3-9, 2013, pages 136–142. IJCAI/AAAI, 2013.

[Gourvès et al., 2013b] Laurent Gourvès, Jérôme Monnot, and Lydia Tlilane. A protocol for cutting matroids like cakes. In Yiling Chen and Nicole Immorlica, editors, Web and Internet Economics - 9th International Conference, WINE 2013, Cambridge, MA, USA, December 11-14, 2013, Proceedings, volume 8289 of Lecture Notes in Computer Science, pages 216–229. Springer, 2013.

[Gourvès et al., 2014] Laurent Gourvès, Jérôme Monnot, and Lydia Tlilane. Near fairness in matroids. In ECAI 2014 - 21st European Conference on Artificial Intelligence, Prague, Czech Republic - Including Prestigious Applications of Intelligent Systems (PAIS 2014), August 18-22, pages 393–398, 2014.

[Hugo, 1948] Steinhaus Hugo. The problem of fair division. Econometrica, 16:101–104, 1948.

[Kuhn, 1955] Harold W. Kuhn. The hungarian method for the assignment problem. Naval Research Logistics Quarterly, 2:83–97, 1955.

[Kyropoulou et al., 2019] Maria Kyropoulou, Warut Saksompong, and Alexandros Voudouris. Almost envy-freeness in group resource allocation. In Proceedings of the Twenty-Eighth International Joint Conference on Artificial Intelligence, IJCAI-19, pages 400–406, International Joint Conferences on Artificial Intelligence Organization, 08 2019.

[Lipton et al., 2004] Richard J. Lipton, Evangelos Markakis, Elchanan Mossel, and Amin Saberi. On approximately fair allocations of indivisible goods. In Proceedings of the 5th ACM Conference on Electronic Commerce, New York, USA, May 17-20, 2004, pages 125–131, 2004.

[Moulin, 2003] Hervé Moulin. Fair Division and Collective Welfare. MIT Press, 2003.

[Pareto, 1897] Vilfredo Pareto. Cours d’Économie politique. Professeur à l’Université de Lausanne. Vol. I. Pp. 430. 1896. Vol. II. Pp. 426. 1897. Lausanne: F Rouge, 1897.

[Plaut and Roughgarden, 2018] Benjamin Plaut and Tim Roughgarden. Almost envy-freeness with general valuations. In Proceedings of the Twenty-Ninth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2018, New Orleans, LA, USA, January 7-10, 2018, pages 2584–2603, 2018.

[Rawls, 1971] John Rawls. A Theory of Justice. Belknap Press of Harvard University Press, Cambridge, Massachusetts, 1 edition, 1971.

[Sandomirskiy and Segal-Halevi, 2019] Fedor Sandomirskiy and Erel Segal-Halevi. Fair division with minimal sharing. CoRR, abs/1908.01669, 2019.

[Schneckenburger et al., 2017] Sebastian Schneckenburger, Britta Dorn, and Ulle Endriss. The Atkinson inequality index in multiagent resource allocation. In Kate Larson, Michael Winikoff, Sanmay Das, and Edmund H. Durfee, editors, Proceedings of the 16th Conference on Autonomous Agents and MultiAgent Systems, AAMAS 2017, São Paulo, Brazil, May 8-12, 2017, pages 272–280. ACM, 2017.

[Sen, 1976] Amartya Sen. Welfare inequalities and rawlsian axiomatics. Theory and Decision, 7(4):243–262, Oct 1976.

[Sen, 1977] Amartya Sen. Social choice theory: A re-examination. Econometrica, 45(1):53–89, 1977.

[Young, 1995] H. Peyton Young. Equity - in theory and practice. Princeton University Press, 1995.