Quantum deformations of the restriction of $GL_{mn}(\mathbb{C})$-modules to $GL_m(\mathbb{C}) \times GL_n(\mathbb{C})$

Dedicated to Sri Ramakrishna

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Abstract

In this paper, we consider the restriction of finite dimensional $GL_{mn}(\mathbb{C})$-modules to the subgroup $GL_m(\mathbb{C}) \times GL_n(\mathbb{C})$. In particular, for a Weyl module $V_\lambda^{(X)}$ of $U_q(gl_{mn})$ we construct a representation $W_\lambda$ of $U_q(gl_m) \otimes U_q(gl_n)$ such that at $q = 1$, the restriction of $V_\lambda(\mathbb{C}^{mn})$ to $U_1(gl_m) \otimes U_1(gl_n)$ matches its action on $W_\lambda$ at $q = 1$. Thus $W_\lambda$ is a $q$-deformation of the module $V_\lambda$. This is achieved by first constructing a $U_q(gl_m) \otimes U_q(gl_n)$-module $\wedge^k$, a $q$-deformation of the simple $GL_{mn}(\mathbb{C})$-module $\wedge^k(\mathbb{C}^{mn})$. We also construct the bi-crystal basis for $\wedge^k$ and show that it consists of signed subsets. Next, we develop $U_q(gl_m) \otimes U_q(gl_n)$-equivariant maps $\psi_{a,b} : \wedge^{a+1} \otimes \wedge^{b-1} \to \wedge^{a} \otimes \wedge^{b}$. This is used as the building block to construct the general $W_\lambda$.

1 Introduction

$GL_N(\mathbb{C})$ will denote the general linear group of invertible $N \times N$ complex matrices, and $gl_N(\mathbb{C})$ its Lie algebra. Consider the group $GL_m(\mathbb{C}) \times GL_n(\mathbb{C})$ acting on $X$, the space of $m \times n$-matrices with complex entries, as follows:

$$(a,b) \cdot x \to a \cdot x \cdot b^T$$

where $a \in GL_m(\mathbb{C})$, $b \in GL_n(\mathbb{C})$ and $x \in X$. Via this action, we have a homomorphism

$$\phi : GL_m(\mathbb{C}) \times GL_n(\mathbb{C}) \to GL_{mn}(\mathbb{C})$$

For a Weyl module $V_\lambda(X)$, via $\phi$, we have:

$$V_\lambda(X) = \oplus_{a,\beta} n^\lambda_{a,\beta} V_a(\mathbb{C}^m) \otimes V_\beta(\mathbb{C}^n)$$

The numbers $n^\lambda_{a,\beta}$ and its properties are of abiding interest. Even the simplest question of when is $n^\lambda_{a,\beta} > 0$ remains unanswered.

Our own motivation comes from the outstanding problem of P vs. NP, and other computational complexity questions in theoretical computer science (see [16]). More specifically, we look at the geometric-invariant-theoretic approach to the problem, as proposed in [13, 14]. In this approach, the

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general subgroup restriction problem, i.e., analysing an irreducible representation of a group $G$ when restricted to a subgroup $H \subseteq G$, is an important step. An approach to the problem was presented in [15], via the dual notion of FRT-algebras (see, e.g., [11]); more on this later.

A useful tool in the analysis of representations of the linear groups $GL_N(\mathbb{C})$ (henceforth, just $GL_N$), has been the quantizations $U_q(gl_N)$ of the enveloping algebra of the Lie algebra $gl_N(\mathbb{C})$, see [11] [12] [13] [14] [10] [12]. The representation theory of $U_q(gl_N)$ mimics that of $GL_N$ and has contributed significantly to the understanding of the diagonal embedding $GL_N \rightarrow GL_N \times GL_N$, i.e., in the tensor product of Weyl modules. This is achieved by the Hopf $\Delta : U_q(gl_N) \rightarrow U_q(gl_N) \otimes U_q(gl_N)$, a $q$-deformation of the diagonal embedding. However, there seems to be no quantization of $\phi : GL_m \times GL_n \rightarrow GL_{mn}$, i.e., an algebra map (also $\phi$) $U_q(gl_m) \otimes U_q(gl_n) \rightarrow U_q(gl_{mn})$; perhaps none exists [4].

On the other hand, we may separately construct embeddings $U_q(gl_m) \rightarrow U_q(gl_{mn})$ and $U_q(gl_n) \rightarrow U_q(gl_{mn})$ which correspond to $\phi$ at $q = 1$. However, the images $(U_q(gl_m))$ and $(U_q(gl_n))$ do not commute within $U_q(gl_{mn})$. This prevents the standard $U_q(gl_{mn})$-module $V_\lambda(\mathbb{C}^{mn})$ from becoming a $U_q(gl_m) \otimes U_q(gl_n)$-module.

This paper constructs a $U_q(gl_m) \otimes U_q(gl_n)$-module $W_\lambda$ with the following properties.

- $W_\lambda$ has a weight structure which matches that of $V_\lambda(\mathbb{C}^{mn})$. Further, there is a weight-preserving bijection $W_\lambda \rightarrow V_\lambda(\mathbb{C}^{mn})$.

- The action of $U_q(gl_m) \otimes U_q(gl_n)$ on $W_\lambda$ at $q = 1$ matches the action of $U_1(gl_m) \otimes U_1(gl_n)$ via the embedding $\phi : U_1(gl_m) \times U_1(gl_n) \rightarrow U_1(gl_{mn})$ on $V_\lambda(\mathbb{C}^{mn})$.

This construction is done in three steps. We first construct $U_q(gl_m) \otimes U_q(gl_n)$-modules $W_\lambda$ when $V_\lambda = \wedge^k(\mathbb{C}^{mn})$, i.e., $\lambda$ is a single column shape. Next, we construct $U_q(gl_m) \otimes U_q(gl_n)$-equivariant maps

$$\psi_{a,b} : \wedge^{a+1}(\mathbb{C}^{mn}) \otimes \wedge^{b-1}(\mathbb{C}^{mn}) \rightarrow \wedge^a(\mathbb{C}^{mn}) \otimes \wedge^b(\mathbb{C}^{mn})$$

whose co-kernel is $W_\lambda$ when $\lambda$ has two columns. Finally, the above map gives us straightening relations which yield the construction of general $W_\lambda$. Both, the construction of $\wedge^k(\mathbb{C}^{mn})$ and the map $\psi_{a,b}$ are deformations of the usual $U_1(gl_{mn})$-structures, at $q = 1$.

We use the standard model for $U_q(gl_n)$ and its modules consisting of semi-standard young tableau, see, e.g., [8]. Thus a basis for $V_\lambda(\mathbb{C}^{mn})$ is identified with $SS(\lambda, mn)$, i.e., semi-standard tableau of shape $\lambda$ with entries in $[mn]$.

In Section 2 we set up notation and then construct the $U_q(gl_m) \otimes U_q(gl_n)$-modules $\wedge^k$. In the next section, we construct the abstract module $W_\lambda$ for general $\lambda$. Section 4 proves some elementary properties of $U_q(gl_m) \otimes U_q(gl_n)$-modules in the chosen basis parametrized by column tableaus. This is used for an explicit construction of $\psi_{a,b}$. In Section 5 we revert back to $\wedge^k(\mathbb{C}^{mn})$ and prove that signed column-tableaus do indeed form a bi-crystal basis for the $U_q(gl_m) \otimes U_q(gl_n)$-action thus validating the construction in 2.

The construction in this paper has many similarities with that in [15]. Indeed, our construction of the basic subspaces $\wedge^2(\mathbb{C}^{mn})$ and $Sym^2(\mathbb{C}^{mn})$ of $\mathbb{C}^{mn} \otimes \mathbb{C}^{mn}$ is identical to that in [15]. There, these subspaces are used to construct the $R$-matrix and the dual algebra $GL_q(\mathbb{C}^{mn})$ and maps $GL_q(\mathbb{C}^{mn}) \rightarrow GL_q(\mathbb{C}^{mn}) \otimes GL_q(\mathbb{C}^{mn})$. The representation theory of $GL_q(\mathbb{C}^{mn})$ does not quite match that of the standard $GL_q(\mathbb{C}^{mn})$, and hence the construction of $V_\lambda(\mathbb{C}^{mn})$ must follow a different route. Our construction starts with the same $R$-matrix but bypasses the construction of $GL_q(\mathbb{C}^{mn})$ to arrive directly at a $GL_q(\mathbb{C}^{mn}) \otimes GL_q(\mathbb{C}^{mn})$-structure for $\wedge^k(\mathbb{C}^m)$. As in [15], we have the “compactness” observation, see Proposition 33. However, many other structures of [15] are as yet missing.

2 The $U_q(gl_m) \otimes U_q(gl_n)$ structure for $\wedge^k(\mathbb{C}^{mn})$

To begin, we lift almost verbatim, the initial parts of Section 2 of [8]. $U_q(gl_N)$ is the associative algebra over $\mathbb{C}(q)$ generated by the $4N - 2$ symbols $e_i, f_i, i = 1, \ldots, N - 1$ and $q^e, q^{-e}, i = 1, \ldots, N$ subject
to the relations:

\[ q^{e_i}q^{-e_i} = q^{-e_i}q^{e_i} = 1, \quad [q^{e_i}, q^{e_j}] = 0 \]

\[ q^{e_i}e_jq^{-e_i} = \begin{cases} qe_j & \text{for } i = j \\ q^{-1}e_j & \text{for } i = j + 1 \\ e_j & \text{otherwise} \end{cases} \]

\[ q^{e_i}f_jq^{-e_i} = \begin{cases} -q^{-1}f_j & \text{for } i = j \\ qf_j & \text{for } i = j + 1 \\ f_j & \text{otherwise} \end{cases} \]

\[ [e_i, f_j] = \delta_{ij} \frac{q^{e_i}q^{-e_i+1} - q^{-e_i}q^{e_i+1}}{q - q^{-1}} \]

\[ e_i^2 - (q + q^{-1})e_i + q^{-1} = 0 \]

The subalgebra generated by \( e_i, f_i \) and

\[ q^{h_i} = q^{e_i}q^{e_i+1}, \quad q^{-h_i} = q^{-e_i}q^{e_i+1} \quad \text{for } i = 1, \ldots, N - 1 \]

is denoted by \( U_q(sl_N) \).

The \( U_q(gl_N) \) module \( V_{1^k} \) (henceforth \( \wedge^k(\mathbb{C}^N) \)) is an \( \binom{N}{k} \)-dimensional \( C(q) \)-vector space with basis \( \{v_c\} \) indexed by the subsets \( c \) of \([N]\) with \( k \) elements, i.e., by Young Tableau of shape \( 1^k \) with entries in \([N]\). The action of \( U_q(gl_N) \) on this basis is given by

\[ q^{e_i}v_c = \begin{cases} v_c & \text{if } i \notin c \\ qv_c & \text{otherwise} \end{cases} \]

\[ e_iv_c = \begin{cases} 0 & \text{if } i + 1 \notin c \text{ or } i \in c \\ v_d & \text{otherwise, where } d = c - \{i + 1\} + \{i\} \end{cases} \]

\[ f_iv_c = \begin{cases} 0 & \text{if } i + 1 \in c \text{ or } i \notin c \\ v_d & \text{otherwise, where } d = c - \{i\} + \{i + 1\} \end{cases} \]

In order to construct more interesting modules, we use the tensor product operation. Given two \( U_q(gl_N) \)-modules \( M, L \), we can define a \( U_q(gl_N) \)-structure on \( M \otimes L \) by putting

\[ q^{e_i}(u \otimes v) = q^{e_i}u \otimes q^{e_i}v \]

\[ e_i(u \otimes v) = e_iu \otimes v + q^{-h_i}u \otimes e_i v \]

\[ f_i(u \otimes v) = f_iu \otimes q^{h_i}v + u \otimes f_i v \]

Indeed, the Hopf map \( \Delta : U_q(gl_N) \to U_q(gl_N) \otimes U_q(gl_N) \):

\[ \Delta q^{e_i} = q^{e_i} \otimes q^{e_i}, \quad \Delta e_i = e_i \otimes 1 + q^{-h_i} \otimes e_i, \quad \Delta f_i = f_i \otimes q^{h_i} + 1 \otimes f_i \]

is an algebra homomorphism and makes \( U_q(gl_N) \) into a bialgebra.

## 2.1 Some basic lemmas

We consider the \( U_q(gl_{mn}) \)-module \( \wedge^p(\mathbb{C}^{mn}) \), i.e., the homomorphism \( U_q(gl_{mn}) \to End_{\mathbb{C}(q)}(\wedge^p(\mathbb{C}^{mn})) \).

We gather together some lemmas on this particular action.

**Lemma 1** On the module \( \wedge^p(\mathbb{C}^{mn}) \), we have:

- \( e_i^2 = 0 \) for all \( i \).
Lemma 2 Let $\sigma = [\sigma_1, \ldots, \sigma_n]$ integers such that the set $\{\sigma_1, \ldots, \sigma_n\} = \{1, \ldots, n\}$. Then, on the module $\wedge^p(\mathbb{C}^n)$, for the monomial $e_\sigma = e_{\sigma_1} \cdots e_{\sigma_n}$ there exists positive integers $k_1, \ldots, k_n$ such that

$$e_\sigma = e_{n-k_n+1} e_{n-k_n+2} \cdots e_{n-k_n-1+1} e_{n-k_n-1+2} \cdots e_{n-k_n} \cdots e_1 e_2 \cdots e_k$$

An important property of the re-ordering is that either (i) the position of $e_i$ is to the left of position of $e_{i-1}$ or (ii) is immediately to the right.

Example 3 We may verify that:

$$e_2 e_6 e_7 e_3 e_5 e_1 e_4 = e_6 e_7 e_5 e_2 e_3 e_4 e_1$$

with $k_1 = 1, k_2 = 3, k_3 = 1, k_4 = 2$.

Corollary 4 Let $\sigma$ be a permutation on the set $\{i, \ldots, j\}$ then for the action on $\wedge^p(\mathbb{C}^n)$ we have:

- if $k < i - 1$ or $k > j + 1$ then $e_k e_\sigma = e_\sigma e_k$.
- if $i \leq k \leq j$ then $e_k e_\sigma = e_\sigma e_k = 0$.
- if $k < i$ or $k > j$ then $f_k e_\sigma = e_\sigma f_k$.

For $i < j$, let $E_{i,j}$ denote the term $[e_i, [e_{i+1}, \ldots [e_{j-1}, e_j]]]$ and $F_{i,j}$ denote $[[[f_j, f_{j-1}], \ldots, f_i]]$.

Lemma 5

$$E_{i,j}(v_c) = \begin{cases} (-1)^{|c\cap[i+1,j]|} v_d & \text{if } j+1 \notin c \text{ and } i \notin c, \text{ where } d = c - \{j+1\} + \{i\} \\ 0 & \text{otherwise} \end{cases}$$

$$F_{i,j}(v_c) = \begin{cases} (-1)^{|c\cap[i+1,j]|} v_d & \text{if } j+1 \notin c \text{ and } i \in c, \text{ where } d = c - \{i\} + \{j+1\} \\ 0 & \text{otherwise} \end{cases}$$

Proof: We provide a detailed proof for $E_{i,j}$. The proof for $F_{i,j}$ is similar.

We prove this by induction on $j - i$. The base case is when $j - i = 0$. Here, with the convention that $E_{i,i} = e_i$, the lemma follows from the definition of the operator $e_i$.

For the inductive case (i.e. $i < j$), consider $E_{i,j} = [e_i, E_{i+1,j}] = e_i E_{i+1,j} - E_{i+1,j} e_i$. Thus,

$$E_{i,j}(v_c) = e_i E_{i+1,j}(v_c) - E_{i+1,j} e_i(v_c)$$

Suppose that $E_{i+1,j}(v_c) = 0$, so the first-term in the above expression is zero. Then, by the induction hypothesis, either $j + 1 \notin c$ or $i + 1 \in c$.

If $i + 1 \notin c$, then $j + 1 \notin c$. Note that in this case, $e_i(v_c) = 0$. Thus, $E_{i,j}(v_c) = 0$ and $j + 1 \notin c$.

If $j + 1 \in c$, then $i + 1 \in c$. In this case, if $i \in c$, then $e_i(v_c) = 0$ and thus, $E_{i,j}(v_c) = 0$ and $i \in c$. Therefore, we assume that $i \notin c$ along with $j + 1 \in c$ and $i + 1 \in c$. So, we have

$$e_i(v_c) = v_d \text{ where } d = c - \{i+1\} + \{i\}$$

As, $j + 1 \in d$ and $i + 1 \notin d$, by induction hypothesis,

$$E_{i+1,j}(v_d) = (-1)^{|d\cap[i+2,j]|} v_e \text{ where } e = d - \{j+1\} + \{i+1\}$$
The last equation follows from the fact that $i + 1 \in c$ and $d = c - \{i + 1\} + \{i\}$. Also, observe that $e = c - \{j + 1\} + \{i\}$.

Now, we consider the case when $E_{i+1,j}(v_c) \neq 0$. Then, by induction, we have that $j + 1 \in c$ and $i + 1 \not\in c$. Further,

$$E_{i+1,j}(v_c) = (-1)^{|e[i+2,j]|}v_d$$

where $d = c - \{j + 1\} + \{i + 1\}$

Note that, as $i + 1 \not\in c$, $e_i(v_c) = 0$. Thus, in this case,

$$E_{i,j}(v_c) = e_iE_{i+1,j}(v_c) - E_{i+1,j}e_i(v_c)$$

$$= e_i((-1)^{|e[i+2,j]|}v_d)$$

$$= (-1)^{|e[i+2,j]|}e_i(v_d)$$

$$= (-1)^{|e[i+1,j]|}e_i(v_d)$$

The last equality follows from the observation that $i + 1 \not\in c$.

If $i \in c$, then $i \in d$ as well and $e_i(v_d) = 0$, consequently $E_{i,j}(v_c) = 0$ as expected.

If $i \not\in c$, then $i \not\in d$ as well. As $i + 1 \in d$, we have

$$E_{i,j}(v_c) = (-1)^{|e[i+1,j]|}e_i(v_d) = (-1)^{|e[i+1,j]|}v_c$$

where $e = d - \{i + 1\} + \{i\} = c - \{j + 1\} + \{i\}$.

Q.E.D.

**Lemma 6** For $i, j, i', j'$, on $\wedge^k(\mathbb{C}^{mn})$ we have:

(i) $[E_{i,j}, E_{i',j'}] = 0$ unless either $j' + 1 = i$ or $j + 1 = i'$.

(ii) $[F_{i,j}, E_{i',j'}] = 0$ unless either $j' = j$ or $i' = i$.

(iii) $E_{i,j}E_{i',j'} = E_{i',j'}E_{i,j} = 0$ if $i = i'$ or $j = j'$.

(iv) $F_{i,j}E_{i',j'} = E_{i',j'}F_{i,j} = 0$ if $j + 1 = i'$ or $i = j' + 1$.

### 2.2 Commuting actions on $\wedge^k(\mathbb{C}^{mn})$

We are now ready to define two actions, that of $U_q(gl_m)$ and $U_q(gl_n)$ on $\wedge^p(\mathbb{C}^{mn})$. This will consist of some special elements $(E^L_i, F^L_i, q^L_i)$ and $(E^R_i, F^R_i, q^R_i)$ which will implement the action of $U_q(sl_m)$ and $U_q(sl_n)$, respectively.

We consider the free $\mathbb{Z}$-module $E = \oplus_{i=1}^{mn} \mathbb{Z} \epsilon_i$ and define an inner product by extending $<\epsilon_i, \epsilon_j> = \delta_{i,j}$. Define $\kappa_{i,j} \in E$ as $\epsilon_i - \epsilon_j$.

We note that:

**Lemma 7** For $\alpha \in E$, we have:

- $e_j q^\alpha = q^{<\alpha, \kappa_{j+1,j}>} q^\alpha e_j$.
- $f_j q^\alpha = q^{<\alpha, \kappa_{j,j+1}>} q^\alpha f_j$.
- $E_{i,j} q^\alpha = q^{<\alpha, \kappa_{j+1,i}>} q^\alpha E_{i,j}$.
Next, we define the left operators using:

\[ B^k_i = \sum_{j=0}^{k-2} -h_{jm+i} \]
\[ A^k_i = \sum_{j=k}^{n-1} h_{jm+i} \]

We define the map \( \phi_L : U_q(gl_m) \to U_q(gl_{mn}) \) as:

\[ q^k_i = \phi_L(q^i) = \prod_{j=0}^{n-1} q^{h_{jm+i}} \]
\[ E^L_i = \phi_L(e_i) = e_i + q^{-h_i} e_{m+i} + \ldots + (q^{-h_{i}}) e_{(n-1)m+i} \]
\[ F^L_i = \phi_L(f_i) = (\prod_{j=0}^{n-1} q^{h_{jm+i}}) f_i + \ldots + q^{h(n-1)m+i} f_{(n-2)m+i} + f_{(n-1)m+i} \]

**Proposition 8** The map \( \phi_L : U_q(gl_m) \to U_q(gl_{mn}) \) is an algebra homomorphism.

**Proof:** The embedding of \( \phi_L : U_q(gl_m) \to U_q(gl_{mn}) \) actually comes from:

\[ U_q(gl_m) \xrightarrow{\Delta} U_q(gl_m) \otimes \ldots U_q(gl_m) \to U_q(gl_{mn}) \]

where (i) there are \( n \) copies in the tensor-product, and (ii) \( \Delta \) is the \( n \)-way Hopf. This verifies that \( \phi_L \) is an algebra map.

We define the right operators:

**Definition 9**

\[ b^k_i = \sum_{j=i+1}^{m} \epsilon_{km+j} - \sum_{j=i+1}^{m} \epsilon_{(k-1)m+j} \]
\[ a^k_i = \sum_{j=1}^{m} \epsilon_{i-k-1+m} - \sum_{j=1}^{m} \epsilon_{km+j} \]

We define the “map” \( \phi_R : U_q(gl_m) \to U_q(gl_{mn}) \) as:

\[ \phi_R(q^k_i) = \prod_{i=1}^{m} q^{\epsilon_{(k-1)m+i}} \]
\[ \phi_R(E^R_k) = \sum_{j=1}^{m} q^{h_j} E_{(k-1)m+i, km+i-1} \]
\[ \phi_R(F^R_k) = \sum_{j=1}^{m} q^{h_j} F_{(k-1)m+i, km+i-1} \]
\[ \phi_R(h^k_i) = \sum_{j=1}^{m} \epsilon_{(k-1)m+i} - \epsilon_{km+i} \]

**Remark:** \( \phi_R \) serves merely to identify a set of elements in \( U_q(gl_{mn}) \) corresponding to the generators of \( U_q(gl_m) \). Thus, while \( \phi_L : U_q(gl_m) \to U_q(gl_{mn}) \) is an algebra homomorphism, the corresponding statement for \( U_q(gl_n) \) is not. However, as we will show that the composites:

\[ U_q(gl_m) \xrightarrow{\phi_L} U_q(gl_{mn}) \xrightarrow{\text{End}_{\mathbb{C}(q)}(\wedge^p(\mathbb{C}^m))} \]
\[ U_q(gl_n) \xrightarrow{\phi_R} U_q(gl_{mn}) \xrightarrow{\text{End}_{\mathbb{C}(q)}(\wedge^p(\mathbb{C}^m))} \]

are commuting algebra homomorphisms making \( \wedge^p(\mathbb{C}^m) \) into a \( U_q(gl_m) \otimes U_q(gl_n) \)-module.

We will identify \( \mathbb{C}^mn \) as \( \mathbb{C}^m \otimes \mathbb{C}^n \) arranging the typical element in an \( m \times n \) array, reading column-wise from left to right, and within each column from top to bottom (see below). In this notation, see Fig. 1 for individual terms of the left operators and Fig. 2 for the right operators.

\[
\begin{array}{cccc}
1 & 6 & 11 & 16 \\
2 & 7 & 12 & 17 \\
3 & 8 & 13 & 18 \\
4 & 9 & 14 & 19 \\
5 & 10 & 15 & 20 \\
\end{array}
\]
2.3 Proofs

For an operator $O = q^\mu E_{i,j}$ (where $\mu \in \mathbb{E}$ is arbitrary) let us define $\kappa(O) = \epsilon_{j+1} - \epsilon_i$ and for the operator $O = q^\mu F_{i,j}$, we define $\kappa(O)$ as $\epsilon_i - \epsilon_{j+1}$. We extend this notation so that $E_{i,i} = e_i$ (with $\kappa(E_{i,i}) = \epsilon_{i+1} - \epsilon_i$) and $F_{j,j} = f_j$ (with $\kappa(F_{j,j}) = \epsilon_j - \epsilon_{j+1}$).

We define $\mathcal{L}$ and $\mathcal{R}$ as two sets of operators:

$$\mathcal{L} = \{q^{B_i} e_{(k-1)m+i}, q^{A_i} f_{(k-1)m+i} | 1 \leq i \leq m-1, 1 \leq k \leq n\}$$

$$\mathcal{R} = \{q^{B_i} E_{(k-1)m+i,km+i-1}, q^{A_i} F_{(k-1)m+i,km+i-1} | 1 \leq i \leq m, 1 \leq k \leq n-1\}$$

Notice that we may write $E^L_i = \sum_p l_{ip}$ and $E^R_i = \sum_j r_{kj}$ where $l_{ip} \in \mathcal{L}$ and $r_{kj} \in \mathcal{R}$. Whence $[E^L_i, E^R_k]$ is expressible as lie-brackets of elements of $\mathcal{L}$ and $\mathcal{R}$. Of course, we wish to show that $[E^L_i, E^R_k]$ and its three cousins are actually zero.

**Lemma 10** For any $L \in \mathcal{L}$ and any $R \in \mathcal{R}$ if $\langle \kappa(L), \kappa(R) \rangle \geq 0$ then $[L, R] = 0$.

**Proof:** We first take the case when $\langle \kappa(L), \kappa(R) \rangle = 0$. We take for example $L = q^{B_i} e_{(k'-1)m+i'}$ and $R = q^{A_i} F_{(k-1)m+i,km+i-1}$. The condition $\langle \kappa(L), \kappa(R) \rangle = 0$ implies (see Figs. 1, 2) that

$$F_{(k-1)m+i,km+i-1} q^{B_i} = q^{B_i} F_{(k-1)m+i,km+i-1}$$

Whence

$$[L, R] = q^{B_i} + q^{A_i} [e_{(k'-1)m+i'}, e_{(k-1)m+i,km+i-1}] = 0$$

where the last equality follows from Lemma 3(ii).

For the case with $\langle \kappa(L), \kappa(R) \rangle = 1$, Lemma 6 parts (iii),(iv), immediately implies an even stronger claim. Q.E.D.

Thus the only non-commuting $(L, R)$ pairs are shown in Fig. 3.
By lemma 10, for the purpose of showing commutation we may as well assume that \( n = m = 2 \).

The following argument assumes \( n = 2 \) but retains \( m \) for notational convenience. In other words, we have:

\[
\begin{align*}
E_i & = e_i + q^{-h_i} e_{m+i} \\
F_i & = q^{h_i} f_1 + f_{m+i}
\end{align*}
\]

For \( i = 1, \ldots, m \) define \( \beta_i, \alpha_i \in \mathbb{E} \) as

\[
\begin{align*}
\beta_i &= \sum_{j=i+1}^{m} \epsilon_{m+j} - \sum_{j=i+1}^{m} \epsilon_{j} \\
\alpha_i &= \sum_{j=1}^{i} \epsilon_{j} - \sum_{j=1}^{m} \epsilon_{m+j}
\end{align*}
\]

Next, define

\[
\begin{align*}
E^R &= \sum_{i=1}^{m} q^{\beta_i} E_{i,m+i-1} \\
F^R &= \sum_{i=1}^{m} q^{\alpha_i} F_{i,m+i-1} \\
h^R &= \sum_{i=1}^{m} \epsilon_{i} - \epsilon_{m+i}
\end{align*}
\]

Note that \( E_1^R = E^R, F_1^R = F^R \) and \( h_1^R = h^R \).

**Lemma 11** For \( 1 \leq i \leq m-1 \),

- \([e_i, q^{\beta_{i+1}} E_{i+1,m+i}] = q^{\beta_{i+1}} E_{i,m+i}\).
- \([q^{-h_i} e_{m+i}, q^{\beta_i} E_{i,m+i-1}] = q^{\beta_i} e_{i+1} E_{i+1,m+i-1} - q^{\beta_i} e_{i+1} E_{i+1,m+i} + q^{\beta_i} e_{i+1} E_{i+1,m+i} - q^{\beta_i} e_{i+1} E_{i+1,m+i} e_{i}\).

**Proof:** We prove the first assertion below. We start with analyzing

\[
\begin{align*}
[e_i, q^{\beta_{i+1}} E_{i+1,m+i}] &= e_i q^{\beta_{i+1}} E_{i+1,m+i} - q^{\beta_{i+1}} E_{i+1,m+i} \epsilon_{i} \\
&= q^{\beta_{i+1} - h_i} e_i E_{i+1,m+i} - q^{\beta_{i+1} + 1} E_{i+1,m+i} \epsilon_{i}
\end{align*}
\]

A small calculation shows that \( < \beta_{i+1}, -h_i > = 0 \). Therefore,

\[
\begin{align*}
[e_i, q^{\beta_{i+1}} E_{i+1,m+i}] &= q^{\beta_{i+1}} (e_i E_{i+1,m+i} - E_{i+1,m+i} \epsilon_{i}) \\
&= q^{\beta_{i+1}} E_{i,m+i}
\end{align*}
\]

Figure 3: The Eight Non-Commuting Terms
Now, we turn to the second claim. Towards this, we expand \([q^{-h_i}, e_{m+i}, q^\beta_i E_{i,m+i-1}]\) as

\[
\begin{align*}
q^{-h_i} e_{m+i} q^\beta_i E_{i,m+i} - q^\beta_i E_{i,m+i-1} &- q^{-h_i} e_{m+i} \\
q^{-h_i} q^{\beta_i} E_{i,m+i-1} &- q^\beta_i q^{-h_i} e_{m+i} E_{i,m+i-1} - q^\beta_i q^{\beta_i} E_{i,m+i-1}
\end{align*}
\]

We observe that \(\beta_i, -h_{m+i} > 1\) and \(<\beta_i, \kappa_{m+i,i} >= 1\). Therefore,

\[
\begin{align*}
q^{-h_i} e_{m+i}, q^\beta_i E_{i,m+i-1} &= q^{\beta_i-h_i}(q e_{m+i} E_{i,m+i-1} - q E_{i,m+i-1} e_{m+i}) \\
q^\beta_i - q^{-h_i} &\{e_{m+i}, E_{i,m+i-1}\}
\end{align*}
\]

Q.E.D.

**Lemma 12** \([E^L_i, E^R] = 0\)

**Proof:**

\[
\begin{align*}
[E^L_i, E^R] &= [e_i + q^{-h_i} e_{m+i}, \sum_{j=1}^{m} q^\beta_j E_{j,m+j-1}] \\
&= [e_i, q^\beta_i E_{i+1,m+i}] + [q^{-h_i} e_{m+i}, q^\beta_i E_{i,m+i-1}] \\
&= q^\beta_i E_{i,m+i} + q h_i (e_{m+i} E_{i,m+i-1} - E_{i,m+i-1} e_{m+i})
\end{align*}
\]

As \(\beta_i = \beta_{i+1} + \epsilon_{m+i-1} - \epsilon_i\), \(\beta_i - h_i = \beta_{i+1} + \epsilon_{m+i-1} - \epsilon_i = \beta_{i+1} + \kappa_{m+i+1,i}\).

\[
[E^L_i, E^R] = q^{\beta_i+1} (E_{i,m+i} + q q^{\kappa_{m+i+1,i}} (e_{m+i} E_{i,m+i-1} - E_{i,m+i-1} e_{m+i}))
\]

Now we evaluate the outer bracket at \(v_c\). So, we are looking at \((*)\)

\[
E_{i,m+i}(v_c) + q q^{\kappa_{m+i+1,i}} (e_{m+i} E_{i,m+i-1}(v_c) - E_{i,m+i-1} e_{m+i}(v_c))
\]

If \(m+i+1 \notin c\), then all the three terms in the above expression evaluate to 0. The middle term certainly evaluates to 0 after the application of \(e_{m+i}\) even if \(E_{i,m+i-1}(v_c) \neq 0\).

Similarly, if \(i \in c\), then all the three terms evaluate to 0.

So, henceforth, we work with the assumption that \(m+i+1 \in c\) and \(i \notin c\).

Now, we consider the case where \(m+i \in c\). In this case, with \(c_1 = c - \{m+i+1\} + \{i\}\) and \(c_2 = c - \{m+i\} + \{i\}\), \((*)\) evaluates to

\[
\begin{align*}
&= (-1)^{|c||i+1,m+i||} v_{c_1} + q q^{\kappa_{m+i+1,i}} e_{m+i} \left((-1)^{|c||i+1,m+i-1||} v_{c_2}\right) \\
&= (-1)^{|c||i+1,m+i-1||} \left(-v_{c_1} + q q^{\kappa_{m+i+1,i}} v_{c_1}\right) \\
&= (-1)^{|c||i+1,m+i-1||} \left(-v_{c_1} + q q^{\kappa_{m+i+1,i}} v_{c_1}\right) \\
&= 0
\end{align*}
\]

Now, we consider the remaining case where \(m+i \notin c\). In this case, with the notation \(c_1 = c - \{m+i+1\} + \{i\}\) and \(c_2 = c - \{m+i\} + \{i\}\), \((*)\) evaluates to

\[
\begin{align*}
&= (-1)^{|c||i+1,m+i||} v_{c_1} - q q^{\kappa_{m+i+1,i}} E_{i,m+i-1}(v_{c_2}) \\
&= (-1)^{|c||i+1,m+i||} v_{c_1} - q q^{\kappa_{m+i+1,i}} \left((-1)^{|c||i+1,m+i-1||} v_{c_1}\right) \\
&= (-1)^{|c||i+1,m+i-1||} \left(v_{c_1} - q q^{\kappa_{m+i+1,i}} v_{c_1}\right) \\
&= (-1)^{|c||i+1,m+i-1||} \left(v_{c_1} - q q^{\kappa_{m+i+1,i}} v_{c_1}\right) \\
&= 0
\end{align*}
\]

Q.E.D.

**Lemma 13** For \(1 \leq i \leq m-1\),

- \([f_i q^{h_{m+i}}, q^\beta_i E_{i,m+i-1}] = q^{h_{m+i}+\beta_i}[f_i, E_{i,m+i-1}]\).
- \([f_{m+i}, q^{\beta_{i+1}} E_{i+1,m+i}] = q^{\beta_{i+1}}[f_{m+i}, E_{i+1,m+i}]\).
Proof: We start by proving the first claim.

\[
\begin{align*}
[f_i q^{h_{m+i}}, q^\beta E_{i, m+i-1}] & = f_i q^{h_{m+i}} q^\beta E_{i, m+i-1} - q^\beta E_{i, m+i-1} f_i q^{h_{m+i}} \\
q_i q^{h_{m+i}} q^\beta E_{i, m+i-1} & = q^{h_{m+i}} q^\beta E_{i, m+i-1} f_i q^{h_{m+i}} \\
q^\beta E_{i, m+i-1} q_i q^{h_{m+i}} & = q^\beta E_{i, m+i-1} q_i q^{h_{m+i}} f_i \\
q_i q^\beta E_{i, m+i-1} & = q^\beta E_{i, m+i-1} q_i q^{h_{m+i}} f_i
\end{align*}
\]

Thus,

\[
\begin{align*}
[f_i q^{h_{m+i}}, q^\beta E_{i, m+i-1}] & = q^\beta (f_i E_{i, m+i-1} - E_{i, m+i-1} f_i) \\
& = q^\beta (f_i E_{i, m+i-1} - E_{i, m+i-1} f_i)
\end{align*}
\]

Now, we turn to the second claim.

\[
\begin{align*}
[f_{m+i}, q^\beta E_{i+1, m+i}] & = f_{m+i} q^\beta E_{i+1, m+i} - q^\beta E_{i+1, m+i} f_{m+i} \\
f_{m+i} q^\beta E_{i+1, m+i} & = q^\beta f_{m+i} E_{i+1, m+i} \\
\end{align*}
\]

Thus,

\[
\begin{align*}
[f_{m+i}, q^\beta E_{i+1, m+i}] & = q^\beta f_{m+i} E_{i+1, m+i} - q^\beta E_{i+1, m+i} f_{m+i} \\
& = q^\beta [f_{m+i}, E_{i+1, m+i}]
\end{align*}
\]

Lemma 14 \([F_i^L, E^R] = 0\)

Proof:

\[
\begin{align*}
[F_i^L, E^R] & = [f_i q^{h_{m+i}}, f_{m+i}, \sum_{j=1}^{m} q^\beta E_{j, m+j-1}] \\
& = [f_i q^{h_{m+i}}, q^\beta E_{i, m+i-1} + f_{m+i}, q^\beta E_{i+1, m+i}] \\
& = q^\beta [f_i, E_{i, m+i-1}] + q^\beta [f_{m+i}, E_{i+1, m+i}]
\end{align*}
\]

As \(\beta_i = \beta_{i+1} + \epsilon_{m+i+1} - \epsilon_{i+1}, \beta_i + h_{m+i} = \beta_{i+1} + \epsilon_{m+i} - \epsilon_{i+1} = \beta_{i+1} + \kappa_{m+i, i+1}\).

\[
\begin{align*}
[E_i^L, E^R] & = q^\beta (q q^{k_{m+i, i+1}} [f_i, E_{i, m+i-1}] + [f_{m+i}, E_{i+1, m+i}])
\end{align*}
\]

Now we evaluate the outer bracket at \(v_c\). So, we are looking at (*)&

\[
q q^{k_{m+i, i+1}} (f_i E_{i, m+i-1}(v_c) - E_{i, m+i-1} f_i(v_c)) + f_{m+i} E_{i+1, m+i}(v_c) - E_{i+1, m+i} f_{m+i}(v_c)
\]

If \(m + i \not\in c\), then all the four terms in the above expression evaluate to 0. Similarly, if \(i + 1 \in c\), then all the four terms evaluate to 0.

So, henceforth, we work with the assumption that \(m + i \in c\) and \(i + 1 \not\in c\).

Now, we consider the case where \(i \in c\). In this case, the first term evaluates to 0. If we further assume that \(m + i + 1 \not\in c\), then the third term also evaluates to 0. Overall, with \(c_1 = c - \{i\} + \{i + 1\}\) and \(c_2 = c - \{m + i\} + \{m + i + 1\}\), (*)& evaluates to

\[
* = -qq^{k_{m+i, i+1}} E_{i, m+i-1}(v_{c_1}) - E_{i+1, m+i}(v_{c_2})
\]

With the notation \(d = c - \{m + i\} + \{i + 1\}\),

\[
* = -qq^{k_{m+i, i+1}}(-1)^{c_1 \cap \{i+1\} + 1} v_d - (-1)^{c_2 \cap \{i+1\} + 1} v_d
\]

\[
= (-1)^{c_1 \cap \{i+1\} + 1} (q q^{k_{m+i, i+1}} v_d - v_d)
\]

\[
= (-1)^{c_1 \cap \{i+1\} + 1} \left(q \frac{v_d - v_d}{q}\right)
\]

\[
= 0
\]
Now we work with the assumptions \( i \in c \) and \( m + i + 1 \in c \) and evaluate \((*)\). With these assumptions, the first and the last term of \((*)\) evaluate to 0. Here, with \( c_1 = c - \{i\} + \{i + 1\} \), \( c_2 = c - \{m + i + 1\} + \{i + 1\} \) and \( d = c - \{m + i\} + \{i + 1\} \), \((*)\) evaluates to

\[
\begin{align*}
* &= -qq^{v_{c_1}(i)}E_{i,m+i+1}(\alpha_{c_1}) + f_{m+i}((-1)|v_{c_2}|v_{c_2}) \\
  &= -qq^{c_{m+i+1}}i_{c_1}(-(1)|v_{c_1}|v_{c_1}) + (-1)|v_{c_1}|v_{c_1} \\
  &= (-1)|v_{c_1}|v_{c_1} - (qq^{c_{m+i+1}}i_{c_1} - v_{c_2}) \\
  &= (-1)|v_{c_1}|v_{c_1} - (\frac{1}{q}v_{c_1} - v_{c_2}) \\
  &= 0
\end{align*}
\]

Now we consider the case with \( i \notin c \). In this case, the second term in \((*)\) evaluates to 0. As before, if we further assume that \( m + i + 1 \notin c \), then the third term also evaluates to 0. Overall, with \( c_1 = c - \{m + i\} + \{i\} \), \( c_2 = c - \{m + i\} + \{i + 1\} \) and \( d = c - \{m + i\} + \{i + 1\} \), \((*)\) evaluates to

\[
\begin{align*}
* &= qq^{c_{m+i+1}}f_{i}((-1)|v_{c_1}|v_{c_1}) + f_{m+i}((-1)|v_{c_2}|v_{c_2}) \\
  &= qq^{c_{m+i+1}}i_{c_1}(-(1)|v_{c_1}|v_{c_1}) + (-1)|v_{c_1}|v_{c_1} \\
  &= (-1)|v_{c_1}|v_{c_1} - (qq^{c_{m+i+1}}i_{c_1} - v_{c_2}) \\
  &= (-1)|v_{c_1}|v_{c_1} - (\frac{1}{q}v_{c_1} - v_{c_2}) \\
  &= 0
\end{align*}
\]

Q.E.D.

We have shown that \([E_L^i, E^R_R] = 0\) and \([F_L^i, E^R_R] = 0\). One can similarly show that \([E_L^i, F^R_R] = 0\). We now prepare towards proving \([F_R, E_R] = (q^{-h_R} - q^{h_R})/(q - q^{-1})\).

**Lemma 15** For \( i \neq j \), we have:

\[
[q^{\alpha_{i}}E_{i,m+i+1}, q^{\beta_{j}}E_{j,m+j-1}] = 0
\]

**Proof:**

\[
[q^{\alpha_{i}}E_{i,m+i+1}, q^{\beta_{j}}E_{j,m+j-1}] = q^{\alpha_{i}}E_{i,m+i+1}q^{\beta_{j}}E_{j,m+j-1} - q^{\beta_{j}}E_{j,m+j-1}q^{\alpha_{i}}E_{i,m+i+1}
\]

\[
= q^{\alpha_{i}+\beta_{j}}(q^{\beta_{j}(i)}-\beta_{j}(m+i))E_{i,m+i}E_{j,m+j-1} - q^{\alpha_{i}(m+j)-\alpha_{j}(j)}E_{j,m+j-1}E_{i,m+i+1}
\]

\[
= q^{\alpha_{i}+\beta_{j}}[F_{i,m+i+1}, E_{j,m+j-1}]
\]

for an appropriate integer \( a \) depending on the whether \( i \leq j \) or not. Now, the only material case for \( v_c \) is when \( i, m + j \in c \) and \( j, m + i \notin c \). We may then verify that \([F_{i,m+i+1}, E_{j,m+j-1}]v_c = 0\). Q.E.D.

**Lemma 16** For \( 1 \leq i \leq m \),

\[
[q^{\alpha_{i}}E_{i,m+i-1}, q^{\beta_{j}}E_{i,m+i-1}] = q^{\alpha_{i}+\beta_{j}}[F_{i,m+i-1}, E_{i,m+i-1}]
\]

\[
[q^{\alpha_{i}}E_{i,m+i-1}, q^{\beta_{j}}E_{i,m+i-1}] = q^{\alpha_{i}}E_{i,m+i-1}q^{\beta_{j}}E_{i,m+i-1} - q^{\beta_{j}}E_{i,m+i-1}q^{\alpha_{i}}E_{i,m+i-1}
\]

\[
= q^{\alpha_{i}+\beta_{j}}(q^{\beta_{j}(i)}-\beta_{j}(m+i))E_{i,m+i}E_{i,m+i-1} - q^{\alpha_{i}(m+i)-\alpha_{j}(i)}E_{i,m+i}E_{i,m+i-1}
\]

\[
= q^{\alpha_{i}+\beta_{j}}[F_{i,m+i-1}, E_{i,m+i-1}]
\]

This proves the lemma. Q.E.D.

Define \( \delta_j = \epsilon_j - \epsilon_{m+j} \) and let \( v_c \in \Lambda^p(C^{mn}) \).
Lemma 17

\[(q - q^{-1})[F_{i,m+i-1}, E_{i,m+i-1}]v_c = (q^{-\delta_i} - q^{\delta_i})v_c\]

Proof: If both \(i, m + i \in c\) or both \(i, m + i \notin c\) then the equality clearly holds. Now if \(i \in c, m + i \notin c\) then \(q^{\delta_i}v_c = qv_c\) and we have:

\[(q - q^{-1})[F_{i,m+i-1}, E_{i,m+i-1}]v_c = (q - q^{-1})(-v_c) = (q^{-\delta_i} - q^{\delta_i})v_c\]

On the other hand, if \(i \notin c, m + i \in c\), then \(q^{\delta_i}v_c = q^{-1}v_c\) and we have:

\[(q - q^{-1})[F_{i,m+i-1}, E_{i,m+i-1}]v_c = (q - q^{-1})(v_c) = (q^{-\delta_i} - q^{\delta_i})v_c\]

This proves the lemma.

We now prove:

Proposition 18 Let \(h_R = \sum_{i=1}^{m} \epsilon_i - \epsilon_{m+i}\) then

\[[F^R, E^R] = \frac{q^{-h_R} - q^{h_R}}{q - q^{-1}}\]

Proof: By the above lemmas, we have:

\[[F^R, E^R] = \sum_{i=1}^{m} q^{\alpha_i + \beta_i}[F_{i,m+i-1}, E_{i,m+i-1}]\]

Whence

\[(q - q^{-1})[F^R, E^R]v_c = \sum_{i=1}^{m} (q^{-\delta_i} - q^{\delta_i})q^{\alpha_i + \beta_i}v_c = \sum_{i=1}^{m} q^{\alpha_i + \beta_i - \delta_i}v_c - q^{\alpha_i + \beta_i + \delta_i}v_c\]

Now

\[\alpha_i + \beta_i = \left(\sum_{j=1}^{i-1} \delta_j\right) - \left(\sum_{j=i+1}^{m} \delta_j\right)\]

and thus

\[\alpha_i + \beta_i - \delta_i = \alpha_{i-1} + \beta_{i-1} + \delta_{i-1} = \left(\sum_{j=1}^{i-1} \delta_j\right) - \left(\sum_{j=i+1}^{m} \delta_j\right)\]

Consequently

\[(q - q^{-1})[F^R, E^R]v_c = \sum_{i=1}^{m} (q^{-\delta_i} - q^{\delta_i})q^{\alpha_i + \beta_i}v_c = \sum_{i=1}^{m} q^{\alpha_i + \beta_i - \delta_i}v_c - q^{\alpha_m + \beta_m + \delta_m}v_c = (q^{-h_R} - q^{h_R})v_c\]

This proves the proposition. Q.E.D.

We next prove the braid identity.

Definition 19 For \(i = 1, \ldots, m\) define \(\beta_i, \alpha_i \in \mathbb{E}\) as

\[\beta_i = \sum_{j=i+1}^{m} \epsilon_{m+j} - \sum_{j=i+1}^{m} \epsilon_j\]
\[\beta_i^* = \sum_{j=i+1}^{m} \epsilon_{2m+j} - \sum_{j=i+1}^{m} \epsilon_{m+j}\]

Next, define

\[E^{R} = \sum_{i=1}^{m} q^{\beta_i} E_{i,m+i-1}\]
\[E^{*R} = \sum_{i=1}^{m} q^{\beta_i^*} E_{m+i,2m+i-1}\]
Note that $E^* = E^{R}$. We will show that:

$$(E^R)^2E^* - (q + q^{-1})E^RE^*E^R + E^*(E^R)^2 = 0$$

We define $g_i = q^{β_i}E_{i,m+i-1}$ and $g_j^* = q^{β_j}E_{m+j,2m+j-1}$.

**Lemma 20** For distinct $i,j,k \in [m]$ and on $\land^p(\mathbb{C}^m)$, we have that

$$(g_ig_j + g_jg_i)g_k^* - (q + q^{-1})(g_ig_k^*g_j + g_jg_k^*g_i) + g_k^*(g_ig_j + g_jg_i) = 0$$

**Proof:** Let us prove this in several cases. In all cases, we will use:

$$E_{i,m+i-1}q^β_j = \begin{cases} q^2q^{β_j}E_{i,m+i-1} & \text{if } i > j \\ q^{β_j}E_{i,m+i-1} & \text{if } i \leq j \end{cases}$$

$$E_{i,m+i-1}q^β_j = \begin{cases} q^{-1}q^{β_j}E_{i,m+i-1} & \text{if } i > j \\ q^{β_j}E_{i,m+i-1} & \text{if } i \leq j \end{cases}$$

$$E_{m+i,2m+i-1}q^β_j = \begin{cases} q^{-1}q^{β_j}E_{m+i,2m+i-1} & \text{if } i > j \\ q^{β_j}E_{m+i,2m+i-1} & \text{if } i \leq j \end{cases}$$

We first consider the case $i < j < k$ and $v_c$ such that $v = E_{i,m+i-1}E_{j,j+m-1}E_{m+k,2m+k-1}v_c$, where, by Lemma 20, the sequence of the operators does not matter. Note further that $g_ig_jg_k^*(v_c) = v^* = q^β.v$. We suppress the factor $q^β$ uniformly in this proof and in the next lemma as well. We see that:

$$(g_ig_j + g_jg_i)g_k^*v_c = (1 + q^2)E_{i,m+i-1}E_{j,j+m-1}E_{m+k,2m+k-1}v_c$$

$$g_k^*(g_ig_j + g_jg_i)v_c = (q^{-2} + q^2)E_{i,m+i-1}E_{j,j+m-1}E_{m+k,2m+k-1}v_c$$

$$g_k^*(g_ig_j + g_jg_i)v_c = (q^{-1} + q^{-1})E_{i,m+i-1}E_{j,j+m-1}E_{m+k,2m+k-1}v_c$$

This proves the assertion for $i < j < k$.

Next, let us consider $i < k < j$:

$$(g_ig_j + g_jg_i)g_k^*v_c = (q^{-1} + q)v$$

$$g_k^*(g_ig_j + g_jg_i)v_c = (q^{-1} + q)v$$

$$(g_ig_j + g_jg_i)g_k^*v_c = 2v$$

This proves the assertion for $i < k < j$.

Next, let us consider $k < i < j$:

$$(g_ig_j + g_jg_i)g_k^*v_c = (1 + q^{-2})v$$

$$g_k^*(g_ig_j + g_jg_i)v_c = (1 + q^2)v$$

$$(g_ig_j + g_jg_i)g_k^*v_c = (q + q^{-1})v$$

This proves the assertion for $k < i < j$ and completes the proof of the lemma. Q.E.D.

**Lemma 21** For distinct $i,j \in [m]$ and on $\land^p(\mathbb{C}^m)$, we have that

$$(g_ig_j + g_jg_i)g_i^* - (q + q^{-1})(g_ig_i^*g_j + g_jg_i^*g_i) + g_i^*(g_ig_j + g_jg_i) = 0$$

**Proof:** There are two cases to consider, viz., $g_ig_i^*v_c = 0$ and $g_i^*g_iv_c = 0$. Let us consider the first case, i.e., $g_ig_i^*v_c = 0$, in which case we need to show:

$$-(q + q^{-1})g_jg_i^*g_i + g_i^*(g_ig_j + g_jg_i) = 0$$
Let \( v \) be such that \( E_{m+i,2m+i-1} E_{i,m+i-1} E_{j,m+j-1} v_c = v \) (see comment in proof of Lemma 20). We see that for \( j > i \):

\[
\begin{align*}
g_i^*(g_i g_j + g_j g_i) v_c &= (1 + q^2)v \\
g_j g_i^* g_i v_c &= qv
\end{align*}
\]

This proves the lemma for \( j > i \). Next, for \( j < i \), with \( v = E_{j,m+j-1} E_{m+i,2m+i-1} E_{i,m+i-1} v_c \) and we have:

\[
\begin{align*}
g_i^*(g_i g_j + g_j g_i) v_c &= (q + q^{-1})v \\
g_j g_i^* g_i v_c &= v
\end{align*}
\]

This proves the case when \( g_i g_j^* v_c = 0 \). The other case is similarly proved. Q.E.D.

**Proposition 22** For \( E^R = E_1^R \) and \( E^*R = E_2^R \), we have:

\[
(E^R)^2 E^*R - (q + q^{-1}) E^R E^*R E^R + E^*R (E^R)^2 = 0
\]

**Proof:** Let

\[
B = (E^R)^2 E^*R - (q + q^{-1}) E^R E^*R E^R + E^*R (E^R)^2
\]

For a given \( v_c \), we look at \( B \cdot v_c \) and classify the result by the \( U_q(gl_{mn}) \) weight. We see that the allowed weights are \( wt(v_c) - \kappa_{m+i,i} - \kappa_{m+j,j} - \kappa_{m+k,k} \) for various \( i, j, k \). Further, we see that:

\[
\begin{align*}
E^R &= \sum_{i=1}^m g_i \\
E^*R &= \sum_{i=1}^m g_i^*
\end{align*}
\]

is a separation of \( E^R \) and \( E^*R \) by \( U_q(gl_{mn}) \)-weights. Therefore showing \( B \cdot v_c = 0 \) amounts to various cases on \( i, j, k \). The main cases are settled by Lemmas 20 21. Other cases are easier. Q.E.D.

**Proposition 23** The map \( \phi_R : U_q(gl_m) \to \text{End}_{\mathbb{C}(q)}(\wedge^p \mathbb{C}^{mn}) \) is an algebra homomorphism. At \( q = 1 \), \( \phi_R \) factorizes through \( U_q(gl_{mn}) \), i.e.,

\[
\phi_R(1) : U_1(gl_m) \to U_1(gl_{mn}) \to \text{End}_{\mathbb{C}}(\wedge^p \mathbb{C}^{mn})
\]

The proof is obvious. The family \( \{ E^R_k, F^R_k, q^k \} \) satisfy all the properties for \( U_q(gl_m) \). Also note that at \( q = 1 \), \( \phi_R(1) \) reduces to the standard injection which commutes with \( \phi_L(1) \).

## 3 The module \( V_\lambda \)

We have thus seen the algebra maps \( \phi_L : U_q(gl_m) \to U_q(gl_{mn}) \to \text{End}_{\mathbb{C}(q)}(\wedge^k \mathbb{C}^{mn}) \) and \( \phi_R : U_q(gl_m) \to \text{End}_{\mathbb{C}(q)}(\wedge^k \mathbb{C}^{mn}) \). Since the two actions commute, this converts \( \wedge^k \mathbb{C}^{mn} \) into a \( U_q(gl_m) \otimes U_q(gl_n) \)-module. Also note that at \( q = 1 \), we have the factorization:

\[
\begin{align*}
\phi_L(1) : U_1(gl_m) &\to U_1(gl_{mn}) \to \text{End}_{\mathbb{C}}(\wedge^k \mathbb{C}^{mn}) \\
\phi_R(1) : U_1(gl_m) &\to U_1(gl_{mn}) \to \text{End}_{\mathbb{C}}(\wedge^k \mathbb{C}^{mn})
\end{align*}
\]

**Proposition 24** The actions \( \phi_L, \phi_R \) convert \( \wedge^k \mathbb{C}^{mn} \) into a \( U_q(gl_m) \otimes U_q(gl_n) \) module. Furthermore, at \( q = 1 \) this matches the restriction of the \( U_1(gl_{mn}) \) action on \( \wedge^k \mathbb{C}^{mn} \) to \( U_1(gl_m) \otimes U_1(gl_n) \).

Since, both \( U_q(gl_m) \) and \( U_q(gl_n) \) are Hopf-algebras, we see that if \( M, N \) are \( U_q(gl_m) \otimes U_q(gl_n) \)-modules then so is \( M \otimes N \). The action of \( U_q(gl_m) \) on \( M \otimes N \) defined by

\[
\Phi_L : U_q(gl_m) \to U_q(gl_m) \otimes U_q(gl_m) \to U_q(gl_{mn}) \to U_q(gl_{mn}) \to \text{End}_{\mathbb{C}(q)}(M \otimes N)
\]

In the case \( M, N \) are \( U_q(gl_{mn}) \)-modules, we also have:

\[
\Phi'_L : U_q(gl_m) \to U_q(gl_{mn}) \to U_q(gl_{mn}) \to \text{End}_{\mathbb{C}(q)}(M \otimes N)
\]
We may similarly define $\Phi_R$

$$\Phi_R : U_q(gl_n) \xrightarrow{\Delta} U_q(gl_n) \otimes U_q(gl_n) \xrightarrow{\phi_R \otimes \phi_R} \text{End}_C(\mathbb{M} \otimes \mathbb{N})$$

Again, if $M, N$ are $U_q(gl_{mn})$-modules, we have at $q = 1$:

$$\Phi'_R(1) : U_1(gl_n) \xrightarrow{\phi_1(1)} U_1(gl_m) \xrightarrow{\Delta} U_1(gl_m) \otimes U_1(gl_m) \xrightarrow{} \text{End}_C(M \otimes \mathbb{N})$$

**Proposition 25**

- If $M, N$ are $U_q(gl_m) \otimes U_q(gl_n)$-modules then so is $M \otimes N$, interpreted as $U_q(gl_m) \otimes U_q(gl_n)$ module through $\Phi_L$ and $\Phi_R$.

- The maps $\Phi_L = \Phi'_L$ and $\Phi_R = \Phi'_R$ when $q = 1$. Thus $\Phi_L$ and $\Phi_R$ are deformations of the action of $U_q(gl_m)$ restricted to $U_q(gl_m) \otimes U_q(gl_m)$.

The proof of the first part is obvious. For the second part notice that for $q = 1$ both $\phi_L$ and $\phi_R$ match the classical injections (algebra homomorphisms) of $U_q(gl_m)$ (or $U_q(gl_n)$) into $U_q(gl_{mn})$.

Unless otherwise stated, for $U_q(gl_{mn})$-modules $M, N$, the $U_q(gl_m)$ and $U_q(gl_n)$ structure on $M \otimes N$ will be that arising from $\Phi_L$ and $\Phi_R$.

**Lemma 26** For the module $\Lambda^k(C^{mn})$ as a $U_q(gl_m) \otimes U_q(gl_n)$-module, we have:

$$\Lambda^k(C^{mn}) = \sum_\lambda V_\lambda(C^m) \otimes V_\lambda(C^n)$$

where $|\lambda| = k$.

The proof is clear by setting $q = 1$. Q.E.D.

Next, for a $U_1(gl_m)$-module $V_1$ and the standard embedding $U_1(gl_m) \otimes U_1(gl_n)$, let

$$V_\lambda(C^{mn}) = \oplus_{a, \beta} n_{a, \beta}^\lambda V_a(C^m) \otimes V_\beta(C^n)$$

**Lemma 27** For $a, b \in \mathbb{Z}$, consider $\Lambda^{a+1}(C^{mn}) \otimes \Lambda^{b-1}(C^{mn})$ and $\Lambda^a(C^{mn}) \otimes \Lambda^b(C^{mn})$ as $U_q(gl_m) \otimes U_q(gl_n)$-modules. Then there exists an $U_q(gl_m) \otimes U_q(gl_n)$-equivariant injection $\psi_{a,b}$:

$$\psi_{a,b} : \Lambda^{a+1}(C^{mn}) \otimes \Lambda^{b-1}(C^{mn}) \rightarrow \Lambda^a(C^{mn}) \otimes \Lambda^b(C^{mn})$$

If $\lambda$ is the shape of two columns sized $a$ and $b$ then the co-kernel $\text{cok}(\psi_{a,b})$ may be written as:

$$\text{cok}(\psi_{a,b}) = \oplus_{a, \beta} n_{a, \beta}^\lambda V_a(C^m) \otimes V_\beta(C^n)$$

**Proof:** For $q = 1$ the above map is a classical construction (see, e.g., [5]). This implies that for general $q$, the multiplicity of the $U_q(gl_m) \otimes U_q(gl_n)$-module $V_{\alpha}(C^m) \otimes V_{\beta}(C^n)$ in $\Lambda^{a+1}(C^{mn}) \otimes \Lambda^{b-1}(C^{mn})$ does not exceed that in $\Lambda^a(C^{mn}) \otimes \Lambda^b(C^{mn})$. Whence a suitable $\psi_{a,b}$ may be constructed respecting the isotypical components of both modules. The second assertion now follows. Q.E.D.

We now proceed to construct the $U_q(gl_m) \otimes U_q(gl_n)$ module $W_\lambda$. Let $\lambda' = [\mu_1, \ldots, \mu_k]$, i.e., $\lambda$ has $k$-columns of length $\mu_1, \ldots, \mu_k$. Let $C^k$ the the collection of all columns of size $k$ with strictly increasing entries from the set $[mn]$. For $a \geq b$ and $c \in C^a$ and $c' \in C^b$, we say that $c \leq c'$ if for all $1 \leq i \leq a$, we have $c(i) \leq c'(i)$. A basis for $W_\lambda$ will be the set $SS(\lambda, mn)$, i.e., semi-standard tableau of shape $\lambda$ with entries in $[mn]$. We interpret this basis as $X^\lambda \subseteq Z^\lambda = \prod_c C^{\mu_i}$. In other words,

$$X^\lambda = \{[c_1, \ldots, c_k] | c_i \in C^{\mu_i}, c_i \leq c_{i+1} \}$$

We call $X^\lambda$ as **standard** and $Y^\lambda = Z^\lambda - X^\lambda$ as non-standard. We represent $\Lambda^p(C^{mn})$ as in [8], with the basis $C^p$ and construct $M = \otimes_i \Lambda^\mu_i(C^{mn})$ with the basis $Z^\lambda$. Note that $M$ is a $U_q(gl_m) \otimes U_q(gl_n)$-module.
The structure of \( \psi \) follows from the straightening relations imposed by the maps \( \psi_q \). The construction of Lemma 27. In this section we will construct a family of maps:

Let us fix the basis \( B \).

4.1 Normal bases

For the action of \( E^L \) and \( E^R \) as short-form for \( E^L \) and \( E^R \). We have the EF-Lemma:

Lemma 28 There is a \( U_q(gl_m) \otimes U_q(gl_n) \)-submodule \( N \subseteq M \) such that

- \( \dim(N) = |Y^\lambda| = d \) and
- if \( b_i = \sum_{t \in \mathbb{Z}^\lambda} \gamma_i^t \cdot t \) (for \( i = 1, \ldots, d \)) is a basis for \( N \) then the \( d \times d \)-matrix \( D = (\gamma_i^t)_{i=1,\ldots,d} \) is invertible.

Proof: This again reduces to a choice of \( \psi_{a,b} \) for various \( a, b \). We know that for \( q = 1 \), the above lemma follows from the straightening relations imposed by the maps \( \psi_{a,b} \). Whence, for general \( q \), there must exist an open set of such maps \( \psi_{a,b} \).

Proposition 29 There is a \( U_q(gl_m) \otimes U_q(gl_n) \)-module \( W_\lambda \) and a basis \( w_t \) for \( t \in X^\lambda \) such that:

- For \( q = 1 \), the module is isomorphic to \( V_\lambda(\mathbb{C}^m) \) treated as a \( U_1(gl_m) \otimes U_1(gl_n) \)-module with the vectors \( w_t \) as \( U_1(gl_{mn}) \) weight vectors.
- For general \( q \), \( w_t \) continue to be \( U_q(gl_m) \otimes U_q(gl_n) \) weight vectors.

Proof: The desired module is \( M/N \).

4 The construction of \( \psi_{a,b} \)

The structure of \( W_\lambda \) depends intrinsically on the “straightening relations” \( \psi_{a,b} \) (for various \( a, b \)) of Lemma 27. In this section we will construct a family of maps:

\[
\psi_{a,b} : \wedge^{a+1} \otimes \wedge^{b-1} \rightarrow \wedge^a \otimes \wedge^b
\]

These maps will have the following important properties:

- \( \psi_{a,b} \) will be \( U_q(gl_m) \otimes U_q(gl_n) \)-equivariant, and
- at \( q = 1 \), they will also be \( U_1(gl_{mn}) \)-equivariant and will match the standard resolution.

This is done in three steps:

- First, the construction of equivariant maps \( \psi_a : \wedge^{a+1} \rightarrow \wedge^a \otimes \wedge^1 \) and \( \psi'_a : \wedge^a+1 \rightarrow \wedge^1 \otimes \wedge^a \).
- Next, for a module map \( \mu : A \rightarrow B \), the construction of the “adjoint” \( \mu^* : B \rightarrow A \).
- Finally constructing \( \psi_{a,b} \) using \( \psi_a \) and \( \psi_b^* \).

We first begin with the adjoint.

4.1 Normal bases

Let us fix the basis \( B = \{ v_c | c \subseteq [mn], |c| = k \} \) as the basis of \( \wedge^k(\mathbb{C}^m) \). We define an inner product on \( \wedge^k(\mathbb{C}^m) \) as follows. For elements \( v_c, v_{c'} \in \wedge^k(\mathbb{C}^m) \), let \( \langle v_c, v_{c'} \rangle = \delta_{c,c'}. \) In other words, the inner product is chosen so that \( B \) are ortho-normal.

Abusing notation slightly, we denote, for example by \( \langle E^L_{i,c}, c' \rangle \) as short-form for \( \langle E^L_{i,c}, v_{c'} \rangle \). We have the EF-Lemma:

Lemma 30 For the action of \( U_q(gl_m) \) and \( U_q(gl_n) \) as above, on \( \wedge^k(\mathbb{C}^m) \) as above, we have:

\[
q^{-1} q^{h^L} (v_{c}) (E^L_{i,c}, c') = q q^{h^L} (v_{c}) (E^L_{i,c}, c') = \langle F^L_{i,c'}, c \rangle
\]

\[
q^{-1} q^{h^R} (v_{c}) (E^R_{i,c}, c') = q q^{h^R} (v_{c}) (E^R_{i,c}, c') = \langle F^R_{i,c'}, c \rangle
\]
Proof: We have:

\[ E^L_i(v_c) = (e_i + q^{-h}e_{m+i} + \ldots + (\prod_{j=0}^{n-2} q^{-h_jm+i})e_{(n-1)m+i})v_c \]

Now, by examining the \( gl_m \)-weights of \( c, c' \), exactly one of these terms will lead to \( v_{c'} \), and so

\[ \langle E^L_i, c' \rangle v_{c'} = (\prod_{j=0}^{k-1} q^{-h_jm+i})e_{(k+1)m+i})v_c = (\prod_{j=0}^{k} q^{-h_jm+i}(v_c)) \cdot v_{c'} \]

Now, we see that:

\[ F^L_i(v_{c'}) = (\prod_{j=0}^{n-1} q^{-h_jm+i})f_i + \ldots + q^{h(n-1)m+i}f_{(n-2)m+i} + q^{h(n-1)m+i}v_{c'} \]

It must be the \( f_{(k+1)m+i} \) term that led to \( v_{c'} \). Whence, we have:

\[ \langle F^L_i c', c \rangle = (\prod_{j=0}^{n-1} q^{-h_{j+1}m+i}) \cdot v_{c'} = (\prod_{j=0}^{n-1} q^{-h_jm+i})v_{c} \]

But since \( c, c' \) differ only in the entry \( (k+1)m+i \), we have

- \( q^{h(k+1)m+i}(v_c) = q^{-1} \) and \( q^{h(k+1)m+i}(v_{c'}) = q \).
- \( (\prod_{j=0}^{k} q^{-h_jm+i}(v_c)) = (\prod_{j=0}^{k} q^{-h_jm+i}(v_{c})) \)
- \( q^{h^L_i}(v_c) = \prod_{j=0}^{n-1} q^{h_jm+i}v_c \)

Finally,

\[ q q^{h^L_i}(v_c)\langle E^L_i, c, c' \rangle = q \prod_{j=0}^{n-1} q^{-h^L_jm+i}(v_c) \prod_{j=0}^{n-1} q^{-h_jm+i}(v_c) = \langle F^L_i c', c \rangle \]

Other assertions are similarly proved. Q.E.D.

Definition 31 Let \( A \) be a \( U_q(gl_m) \otimes U_q(gl_n) \) module, and let \( A = \{a_1, \ldots, a_r\} \) be a basis of \( A \) of weight vectors. Define an inner product \( \langle \cdot, \cdot \rangle \) on \( A \) making \( A \) orthogonal. We say that \( A \) is normal if the EF-lemma Lemma 30 holds (with \( a, a' \in A \) replacing \( c, c' \)).

Lemma 32 Let \( A, B \) be \( U_q(gl_m) \otimes U_q(gl_n) \)-modules such that \( A = \{a_1, \ldots, a_r\} \) and \( B = \{b_1, \ldots, b_s\} \) are normal bases for \( A \) and \( B \) respectively. Then \( A \otimes B \) is a normal basis for \( A \otimes B \) with the inner product \( \langle a \otimes b, a' \otimes b' \rangle = \delta_{a \otimes b, a' \otimes b'} \).

Proof: Let consider the element \( a \otimes b \), and the elements \( a' \otimes b \) and \( a \otimes b' \) such that \( a' \) appears in \( E^L_i a \) and \( b' \) appears in \( E^L_i b \).

We see that:

\[ q \cdot q^{h^L_i}(a \otimes b)\langle E^L_i(a \otimes b), a' \otimes b \rangle = q \cdot q^{h^L_i}(a \otimes b)\langle (E^L_i \otimes 1 + q^{-h^L_i} \otimes E^L_i)(a \otimes b), a' \otimes b \rangle = q \cdot q^{h^L_i}(a \otimes b)\langle E^L_i a, a' \rangle = q^{h^L_i}(b)[q \cdot q^{h^L_i}(a)\langle E^L_i a, a' \rangle] = q^{h^L_i}(b)\langle F^L_i a, a' \rangle \]
On the other hand, we have:

\[ \langle F_i^L (a' \otimes b), a \otimes b \rangle = \langle (F_i^L \otimes q^{h_i^L}) (a' \otimes b), a \otimes b \rangle = \langle (F_i^L \otimes q^{h_i^L}) (a' \otimes b), a \otimes b \rangle = q^{h_i^L} (b) \langle F_i^L a', a \rangle \]

Other cases are similar. Q.E.D.

Let \( \Xi \) be the \( \mathbb{Z} \)-submodule generated by \( e_i^L \) and \( e_j^R \). Let \( \chi \) be a \( \Xi \)-weight and let \( \chi' = \chi + h_i^L \). For a module \( A \) with a normal base \( A \), let \( A_\chi \) be the weight-space of weight \( \chi \). We see that \( E_i^L : A_\chi \rightarrow A_{\chi'} \), while \( F_i^L : A_{\chi'} \rightarrow A_\chi \). Let \( a_\chi \) be the column-vector of elements of \( A \) of weight \( \chi \). Let us define matrices \( E_A^A, F_A^A \) as:

\[ E_A^A a_{\chi'} = E_i^L a_{\chi} \quad F_A^A a_{\chi} = F_i^L a_{\chi'} \]

By the EF-lemma (i.e., Lemma[30]),

\[ q \cdot q^{\langle \chi, h_i^L \rangle} E_A^A = (F_A^A)^T \]

Now, let \( A \) and \( B \) be \( U_q(\mathfrak{gl}_m) \otimes U_q(\mathfrak{gl}_n) \) with normal bases \( A \) and \( B \) respectively. Let \( \mu : A \rightarrow B \) be an equivariant map and let \( \mu_\chi \) be a matrix such that:

\[ \mu a_\chi = \mu_\chi b_\chi \]

Equivariance implies:

\[ \mu \cdot E_i^L a_\chi = \mu \cdot E_A^A a_{\chi'} = E_A^A \mu_\chi b_{\chi'} \quad E_i^L \cdot \mu a_\chi = E_i^L \mu_\chi b_\chi = \mu_\chi E_B^B b_{\chi'} \]

Or in other words,

\[ E_A^A \mu_\chi = \mu_\chi E_B^B \quad F_A^A \mu_\chi = \mu_\chi F_B^B \]

Transposing the second equivariance condition, we get:

\[ (F_A^A \mu_\chi)^T = (\mu_\chi F_B^B)^T \]

We may simplify this as:

\[ \mu_\chi^T (F_A)^T = (F_B)^T \mu_\chi^T \]

and further:

\[ q \cdot q^{\langle \chi, h_i^L \rangle} \mu_\chi^T E_A = q \cdot q^{\langle \chi, h_i^L \rangle} E_B \mu_\chi^T \]

i.e., finally:

\[ \mu_\chi^T E_A = E_B \mu_\chi^T \]

We may similarly prove that

\[ \mu_\chi^T F_A = F_B \mu_\chi^T \]

Both these observations immediately imply:

**Proposition 33** Let \( \mu : A \rightarrow B \) be an equivariant map, and let \( \mu_\chi \) be defined as above. We construct the map \( \mu^* : B \rightarrow A \) as follows. Define \( \mu^* \) such that:

\[ \mu^* b_\chi = \mu_\chi^T a_\chi \]

Then \( \mu^* : B \rightarrow A \) is equivariant.
4.2 The Construction of $\psi_a$

In this section we construct the $U_q(gl_m) \otimes U_q(gl_n)$-equivariant maps

$$\psi_a : \Lambda^{a+1} \rightarrow \Lambda^a \otimes \Lambda^1$$

$$\psi'_a : \Lambda^{a+1} \rightarrow \Lambda^1 \otimes \Lambda^a$$

Note that $\Lambda^1 = \mathbb{C}^{mn} = \mathbb{C}^m \otimes \mathbb{C}^n$. For convenience, we identify $[mn]$ with $[m] \times [n]$. Under this identification, an element $(i, j) \in [m] \times [n]$ maps to the element $m \ast (j - 1) + i$.

In this notation, the natural basis for the representation $\Lambda^k = \Lambda^k(\mathbb{C}^{mn})$ is parametrized by subsets of $[m] \times [n]$ with $k$ elements.

Recall that, as a $U_q(gl_m) \otimes U_q(gl_n)$-module, we have

$$\Lambda^k(\mathbb{C}^{mn}) = \sum_{\lambda} V_\lambda(\mathbb{C}^m) \otimes V_{\lambda'}(\mathbb{C}^n)$$

where $|\lambda| = k$. Further, $\lambda$ has at most $m$ parts and $\lambda'$ has at most $n$ parts, that is, the shape $\lambda$ fits inside the $m \times n$ rectangle.

For a shape $\lambda = (\lambda_1, \ldots, \lambda_m)$ with $\lambda' = (\lambda_1', \ldots, \lambda_n')$, consider the subset $c_\lambda \subset [mn]$ defined as:

$$c_\lambda = \{ 1, m+1, \ldots, m \ast (\lambda_1 - 1) + 1, 2m+1, \ldots, m \ast (\lambda_2 - 1) + 2, \ldots, m, 2m, \ldots, m \ast (\lambda_m - 1) \}$$

Equivalently,

$$c_\lambda = \{ 1, 2, \ldots, \lambda_1', m+1, m+2, \ldots, m+\lambda_2', \ldots, m \ast (n-1) + 1, m \ast (n-1) + 2, \ldots, m \ast (n-1) + \lambda_n' \}$$

Under the identification of $[mn]$ with $[m] \times [n]$, we have

$$c_\lambda = \{(i, j) \mid 1 \leq i \leq \lambda_1', 1 \leq j \leq \lambda_1 \}$$

We slightly abuse the notation and write $(i, j) \in \lambda$ as a short-form for $(i, j) \in c_\lambda$.

With this notation, we have the following important lemma:

**Lemma 34** Consider the $U_q(gl_m) \otimes U_q(gl_n)$-module $\Lambda^k(\mathbb{C}^{mn})$. For a shape $\lambda$ which fits in the $m \times n$ rectangle with $|\lambda| = k$, the weight vector $v_{c_\lambda} \in \Lambda^k$ is the highest $U_q(gl_m) \otimes U_q(gl_n)$-weight vector of weight $(\lambda, \lambda')$.

**Proof:** The lemma follows from the observation that $E_i^k(v_{c_\lambda}) = E_j^k(v_{c_\lambda}) = 0$ for all $i, j$. Q.E.D.

Now we turn our attention to the construction of the $U_q(gl_m) \otimes U_q(gl_n)$-equivariant map

$$\psi_a : \Lambda^{a+1} \rightarrow \Lambda^a \otimes \Lambda^1$$

As a $U_q(gl_m) \otimes U_q(gl_n)$-module, we have the following decomposition

$$\Lambda^{a+1} = \sum_{\lambda, |\lambda| = a+1} V_\lambda(\mathbb{C}^m) \otimes V_{\lambda'}(\mathbb{C}^n)$$

Moreover, $v_{c_\lambda}$ is the highest-weight vector for the $U_q(gl_m) \otimes U_q(gl_n)$-submodule $V_\lambda(\mathbb{C}^m) \otimes V_{\lambda'}(\mathbb{C}^n)$ of $\Lambda^{a+1}$. 

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Thus, in order to construct the $U_q(\mathfrak{gl}_m) \otimes U_q(\mathfrak{gl}_n)$-equivariant map $\psi_a$, we need to simply define the images $\psi_a(v_{c_\lambda})$ inside $\Lambda^a \otimes \Lambda^1$. Moreover the vector $\psi_a(v_{c_\lambda})$ should be a highest-weight vector of weight $(\lambda, \lambda')$. Note that, unlike $\Lambda^{a+1}$, $\Lambda^a \otimes \Lambda^1$ is not multiplicity-free. Below, we outline the construction of a highest-weight vector (upto scalar multiple) $v_\lambda$ of weight $(\lambda, \lambda')$ inside $\Lambda^a \otimes \Lambda^1$.

We begin with some notation. As before, fix a shape $\lambda$ which fits in the $m \times n$ rectangle with $|\lambda| = a + 1$. Write $\lambda = (\lambda_1, \ldots, \lambda_m)$ with $\lambda' = (\lambda_1', \ldots, \lambda_n')$ and 

$$c_\lambda = \{(i, j) \mid 1 \leq i \leq \lambda'_j, 1 \leq j \leq \lambda_i\}$$

For $(i, j) \in \lambda$, we set 

$$t_{i,j} = v_{c_\lambda - (i,j)} \in \Lambda^a$$

$$\chi_{i,j} = v_{(i,j)} \in \Lambda^1$$

In other words, $t_{i,j}$ is the vector in $\Lambda^a$ corresponding to the subset obtained from the subset $c_\lambda$ by removing the element $(i, j) \in \lambda$. Further, $\chi_{i,j}$ is the vector in $\Lambda^1$ corresponding to the singleton set containing the element $(i, j)$. Below, we abuse notations and denote by $t_{i,j}$ and $\chi_{i,j}$ also the subsets that correspond to these vectors.

**Lemma 35** For $(i, j) \in \lambda$, $1 \leq k < m$, $1 \leq l < n$,

- $E^L_k(t_{i,j}) = 0$ if $i \neq k$.
- $E^L_k(t_{i,j}) = t_{i+1,j}$ if $(i+1, j) \in \lambda$ and $0$ otherwise.
- $\chi_{i,j} = v_{(i,j)} \in \Lambda^1$
- $E^R(t_{i,j}) = 0$ if $j \neq l$.
- $E^R(t_{i,j}) = (-1)^{\lambda'_j - 1}q^{\lambda'_j - 1}t_{i,j+1}$ if $(i, j+1) \in \lambda$ and $0$ otherwise.
- $E^R(t_{i,j}) = (-1)^{\lambda'_j - 1}q^{\lambda'_j - 1}t_{i,j+1}$ if $(i, j+1) \in \lambda$ and $0$ otherwise.

**Proof:** Let $k \neq i$ and consider $E^L_k(t_{i,j})$. Note that, for all $j'$, if $(k+1, j') \in t_{i,j}$, then $(k, j') \in t_{i,j}$. Thus, by definition of $E^L_k$, we have $E^L_k(t_{i,j}) = 0$.

Now consider $E^L_k(t_{i,j})$. Note that $(i, j) \notin t_{i,j}$. If $(i+1, j) \in \lambda$, then $(i+1, j) \in t_{i,j}$. Further, for all $j' < j$, if $(i+1, j') \in t_{i,j}$ then $(i, j') \in t_{i,j}$. Thus, by definition $E^L_k(t_{i,j})$ operates only at the position $(i+1, j)$ if $(i+1, j) \in \lambda$ and produces the subset $t_{i+1,j}$.

Now we assume that $(i+1, j) \in \lambda$, and evaluate $q^{-h^L_i}(t_{i+1,j})$. Note that, except for $(i+1, j)$, $(i+1, j') \in t_{i+1,j}$ for $1 \leq j' \leq \lambda_{i+1}$. Also, for $j' > \lambda_{i+1}$, $(i+1, j') \notin t_{i+1,j}$. Thus $q^{c_{i+1}}(t_{i+1,j}) = q^{\lambda_{i+1}-1}$. Similarly, $q^{h^L_i}(t_{i+1,j}) = q^{\lambda_{i+1}}$. Therefore, 

$$q^{-h^L_i}(t_{i+1,j}) = q^{\lambda_{i+1}-\lambda_i}t_{i+1,j}$$

It is easy to that $E^R(t_{i,j}) = 0$ if $j \neq l$. So, we turn our attention to $E^R(t_{i,j})$. Note that, for $i'$ such that $\lambda'_{i+1} < i' \leq \lambda'_j$, $(i', j) \in t_{i,j}$ and $(i', j+1) \notin t_{i,j}$. For other values of $i'$ except $i$, either both or none of $(i', j)$ and $(i', j+1)$ belong to $t_{i,j}$. Therefore, as expected, $E^R(t_{i,j})$ operates only at the position $(i, j+1)$ if $(i, j+1) \in \lambda$. Further, by definition of $E^R$, if $(i, j+1) \in \lambda$, we have 

$$E^R(t_{i,j}) = (-1)^{\lambda'_j}q^{\lambda'_j-1}t_{i,j+1}$$

The sign $(-1)^{\lambda'_j-1}$ results from the fact that exactly $\lambda'_j - 1$ elements of $[mn]$ strictly in the range from $(i, j)$ to $(i, j+1)$ belong to $t_{i,j}$.

We skip the proof for the last assertion as it follows from a similar reasoning applied earlier for the left $E$-operator. Q.E.D.
Lemma 36  For \((i, j) \in \lambda\),

- \[
E_i^L(t_{i,j} \otimes \chi_{i,j}) = \begin{cases} 
  t_{i+1,j} \otimes \chi_{i,j} & \text{if } (i+1, j) \in \lambda \\
  0 & \text{otherwise}
\end{cases}
\]

- If \((i+1, j) \in \lambda\), then
  \[
  E_i^L(t_{i+1,j} \otimes \chi_{i+1,j}) = q^{\lambda_i+1-\lambda_i-1}t_{i+1,j} \otimes \chi_{i,j}
  \]

- \[
E_j^R(t_{i,j} \otimes \chi_{i,j}) = \begin{cases} 
  (-1)^{j_i-1}q^{\chi_j+1-\chi_j}t_{i,j+1} \otimes \chi_{i,j} & \text{if } (i, j+1) \in \lambda \\
  0 & \text{otherwise}
\end{cases}
\]

- If \((i, j+1) \in \lambda\), then
  \[
  E_j^R(t_{i,j+1} \otimes \chi_{i,j+1}) = q^{\lambda_j+1-\lambda_j-1}t_{i,j+1} \otimes \chi_{i,j}
  \]

- For remaining \(1 \leq k < m\) and \(1 \leq l < n\), \(E_k^L(t_{i,j} \otimes \chi_{i,j}) = E_k^R(t_{i,j} \otimes \chi_{i,j}) = 0\).

Proof: For the first assertion, consider
\[
E_i^L(t_{i,j} \otimes \chi_{i,j}) = E_i^L(t_{i,j}) \otimes \chi_{i,j} + q^{-h_i^L}(t_{i,j}) \otimes E_i^L(\chi_{i,j})
\]
As \((i+1, j) \not\in \chi_{i,j}\), \(E_i^L(\chi_{i,j}) = 0\). Therefore, the claim follows from the previous lemma.

For the second assertion, let us assume that \((i+1, j) \in \lambda\). Then
\[
E_i^L(t_{i+1,j} \otimes \chi_{i+1,j}) = E_i^L(t_{i+1,j}) \otimes \chi_{i+1,j} + q^{-h_i^L}(t_{i+1,j}) \otimes E_i^L(\chi_{i+1,j})
\]
Note that, from the previous lemma \(E_i^L(t_{i+1,j}) = 0\). Also, \(E_i^L(\chi_{i+1,j}) = \chi_{i,j}\). Again, using the previous lemma, we have
\[
E_i^L(t_{i+1,j} \otimes \chi_{i+1,j}) = q^{\lambda_i+1-\lambda_i-1}t_{i+1,j} \otimes \chi_{i,j}
\]
The third and fourth assertions are proved in a similar fashion. Q.E.D.

Lemma 37  Let \(v_\lambda \in \wedge^a \otimes \wedge^1\) be defined as follows:
\[
v_\lambda = \sum_{(k,l) \in \lambda} \alpha_{k,l}t_{k,l} \otimes \chi_{k,l}
\]
where
\[
\alpha_{k,l} = (-1)^{\chi_k+\chi_l-1+k}q^{k+l-\lambda_k}
\]
Then \(v_\lambda\) is a highest-weight vector of weight \((\lambda, \lambda')\).

Proof: It is clear that \(v_\lambda\) is a weight vector of weight \((\lambda, \lambda')\). Below, we show that it is a highest-weight vector by checking that \(E_i^L(v_\lambda) = E_j^R(v_\lambda) = 0\) for all \(i, j\).

Towards this, by previous lemma, we have
\[
E_i^L(v_\lambda) = \sum_{(k,l) \in \lambda} \alpha_{k,l}E_i^L(t_{k,l} \otimes \chi_{k,l})
= \sum_{(i,j) \in \lambda} \alpha_{i,j}E_i^L(t_{i,j} \otimes \chi_{i,j}) + \sum_{(i+1,j) \in \lambda} \alpha_{i+1,j}E_i^L(t_{i+1,j} \otimes \chi_{i+1,j})
= \sum_{(i,j) \in \lambda} \left(\alpha_{i,j}E_i^L(t_{i,j} \otimes \chi_{i,j}) + \alpha_{i+1,j}E_i^L(t_{i+1,j} \otimes \chi_{i+1,j})\right)
\]
For \(l\) such that both \((i, l)\) and \((i+1, l)\) are in \(\lambda\), from previous lemma, we have
\[
E_i^L(t_{i,l} \otimes \chi_{i,l}) = t_{i+1,l} \otimes \chi_{i+l}
E_i^L(t_{i+1,l} \otimes \chi_{i+1,l}) = q^{\lambda_{i+1}-\lambda_i-1}t_{i+1,l} \otimes \chi_{i+l}
\]
Therefore, the coefficient of \( t_{i+1,l} \otimes \chi_{i,l} \) in \( E^L_i(\nu_\lambda) \) is
\[
\begin{align*}
E^L_i(\chi_{i,j} \otimes t_{i,j}) &= \begin{cases}
q^{-1} \chi_{i,j} \otimes t_{i+1,j} & \text{if } (i+1,j) \in \lambda \\
0 & \text{otherwise}
\end{cases} \\
\text{If } (i+1,j) \in \lambda, \text{ then } E^L_i(\chi_{i+1,j} \otimes t_{i+1,j}) &= \chi_{i,j} \otimes t_{i+1,j} \\
E^R_j(\chi_{i,j} \otimes t_{i,j}) &= \begin{cases}
(-1)^{i_j+1} q^{\lambda_{j+1}+\lambda_{j}} q^{\lambda_{j+1}+\lambda_{j}} & \text{if } (i,j+1) \in \lambda \\
0 & \text{otherwise}
\end{cases} \\
\text{If } (i,j+1) \in \lambda, \text{ then } E^R_j(\chi_{i+1,j} \otimes t_{i+1,j}) &= \chi_{i,j} \otimes t_{i+1,j+1} \\
\text{For remaining } 1 \leq k < m \text{ and } 1 \leq l < n, \text{ } E^L_k(\chi_{i,j} \otimes t_{i,j}) = E^R_l(\chi_{i,j} \otimes t_{i,j}) = 0.
\end{align*}
\]

\textbf{Lemma 38} For \((i, j) \in \lambda,\)

\begin{itemize}
\item \[ E^L_i(\chi_{i,j} \otimes t_{i,j}) = \begin{cases}
q^{-1} \chi_{i,j} \otimes t_{i+1,j} & \text{if } (i+1,j) \in \lambda \\
0 & \text{otherwise}
\end{cases} \]
\item \[ E^R_j(\chi_{i,j} \otimes t_{i,j}) = \begin{cases}
(-1)^{i_j+1} q^{\lambda_{j+1}+\lambda_{j}} q^{\lambda_{j+1}+\lambda_{j}} & \text{if } (i,j+1) \in \lambda \\
0 & \text{otherwise}
\end{cases} \]
\item \[ E^R_j(\chi_{i,j} \otimes t_{i,j}) = \chi_{i,j} \otimes t_{i+1,j+1} \]
\item \[ E^R_j(\chi_{i,j} \otimes t_{i,j}) = E^R_j(\chi_{i,j} \otimes t_{i,j}) = 0. \]
\end{itemize}

\textbf{Proof:} For the first assertion, consider
\[
E^L_i(\chi_{i,j} \otimes t_{i,j}) = E^L_i(\chi_{i,j}) \otimes t_{i,j} + q^{-h_i^L(\chi_{i,j})} \otimes E^L_i(t_{i,j})
\]
As \((i+1,j) \not\in \chi_{i,j}, \text{ } E^L_i(\chi_{i,j}) = 0. \text{ Further, } q^{-h_i^L(\chi_{i,j})} = q^{-1} \chi_{i,j}. \text{ Therefore, the claim follows.}
\]

For the second assertion, let us assume that \((i+1,j) \not\in \lambda. \text{ Then}
\[
E^L_i(\chi_{i+1,j} \otimes t_{i+1,j}) = E^L_i(\chi_{i+1,j}) \otimes t_{i+1,j} + q^{-h_i^L(\chi_{i+1,j})} \otimes E^L_i(t_{i+1,j})
\]
Note that, \(E^L_i(t_{i+1,j}) = 0. \text{ Also, } E^L_i(\chi_{i+1,j}) = \chi_{i,j}. \text{ Therefore, we have}
\[
E^L_i(\chi_{i+1,j} \otimes t_{i+1,j}) = \chi_{i,j} \otimes t_{i+1,j}
\]

\textbf{Proof:} For the first assertion, consider
\[
E^L_i(\chi_{i,j} \otimes t_{i,j}) = E^L_i(\chi_{i,j}) \otimes t_{i,j} + q^{-h_i^L(\chi_{i,j})} \otimes E^L_i(t_{i,j})
\]
As \((i+1,j) \not\in \chi_{i,j}, \text{ } E^L_i(\chi_{i,j}) = 0. \text{ Further, } q^{-h_i^L(\chi_{i,j})} = q^{-1} \chi_{i,j}. \text{ Therefore, the claim follows.}
\]

For the second assertion, let us assume that \((i+1,j) \not\in \lambda. \text{ Then}
\[
E^L_i(\chi_{i+1,j} \otimes t_{i+1,j}) = E^L_i(\chi_{i+1,j}) \otimes t_{i+1,j} + q^{-h_i^L(\chi_{i+1,j})} \otimes E^L_i(t_{i+1,j})
\]
Note that, \(E^L_i(t_{i+1,j}) = 0. \text{ Also, } E^L_i(\chi_{i+1,j}) = \chi_{i,j}. \text{ Therefore, we have}
\[
E^L_i(\chi_{i+1,j} \otimes t_{i+1,j}) = \chi_{i,j} \otimes t_{i+1,j}
\]
For the third assertion, consider
\[ E^R_j(\chi_{i,j} \otimes t_{i,j}) = E^R_j(\chi_{i,j}) \otimes t_{i,j} + q^{-h^R_j}(\chi_{i,j}) \otimes E^R_j(t_{i,j}) \]
Recall that, we have
\[ E^R_j(t_{i,j}) = (-1)^{\lambda'_i+1-\lambda'_j} t_{i,j+1} \text{ if } (i,j) \in \lambda \text{ and } 0 \text{ otherwise} \]
Therefore, the claim follows.

For the fourth claim, we assume \((i,j) \in \lambda\). Then
\[ E^R_j(\chi_{i,j+1} \otimes t_{i,j+1}) = E^R_j(\chi_{i,j+1}) \otimes t_{i,j+1} + q^{-h^R_j}(\chi_{i,j+1}) \otimes E^R_j(t_{i,j+1}) = \chi_{i,j} \otimes t_{i,j+1} \]
The last claim can be easily proved. Q.E.D.

**Lemma 39** Let \(v_\lambda \in \wedge^1 \otimes \wedge^a\) be defined as follows:
\[ v_\lambda = \sum_{(k,l) \in \lambda} \beta_{k,l} \chi_{k,l} \otimes t_{k,l} \]
where
\[ \beta_{k,l} = (-1)^{\lambda'_i+...+\lambda'_{i-1}+k} q^{\lambda'_i-k-l} \]
Then \(v_\lambda\) is a highest-weight vector of weight \((\lambda, \lambda')\).

**Proof:** Clearly, \(v_\lambda\) is a weight-vector of weight \((\lambda, \lambda')\). We now check that \(E^L_i(v_\lambda) = 0\) for all \(i\). As expected, this finally reduces to checking if the following expression, coefficient of \(\chi_{i,l} \otimes t_{i+1,l}\) in \(E^L_i(v_\lambda)\), is zero. Towards this, consider
\[
= q^{-1} \beta_{i,l} + \beta_{i+1,l} \\
= q^{-1} (-1)^{\lambda'_i+...+\lambda'_{i-1}+i} q^{\lambda'_i-i-l} + (-1)^{\lambda'_i+...+\lambda'_{i-1}+i+1} q^{\lambda'_i-i-1-l} \\
= 0
\]
Similarly, to check if \(E^R_j(v_\lambda) = 0\), we need to check if the following expression, coefficient of \(\chi_{k,j} \otimes t_{k,j+1}\) in \(E^R_j(v_\lambda)\), is zero. Towards this, consider
\[
= \beta_{k,j} (-1)^{\lambda'_j-1} q^{\lambda'_{j+1}-\lambda'_j-1} + \beta_{k,j+1} \\
= (-1)^{\lambda'_j+...+\lambda'_{j-1}+k} q^{\lambda'_j-k-j} (-1)^{\lambda'_j-k-j} q^{\lambda'_{j+1}-\lambda'_j-1} + (-1)^{\lambda'_j+...+\lambda'_{j+k}} q^{\lambda'_{j+1}-k-j-1} \\
= 0
\]
Thus, we have verified that \(E^L_i(v_\lambda) = E^R_j(v_\lambda) = 0\) for all \(i, j\). This shows that \(v_\lambda\) is a highest-weight vector. Q.E.D.

Now we are ready to define the \(U_q(gl_m) \otimes U_q(gl_n)\)-equivariant map
\[ \psi'_a : \wedge^{a+1} \rightarrow \wedge^1 \otimes \wedge^a \]
As expected, this is done by simply setting \(\psi'_a(v_{\lambda'}) = v_{\lambda'}\) and taking the unique \(U_q(gl_m) \otimes U_q(gl_n)\)-equivariant extension. Also, as before, this extension matches the classical \(U_q(gl_{mn})\)-equivariant construction at \(q = 1\).

Note that \(\wedge^{a+1}\) and \(\wedge^1 \otimes \wedge^a\) have normal bases. Whence, by Prop. 33 there is the \(U_q(gl_m) \otimes U_q(gl_n)\)-equivariant map:
\[ \psi'^*_a : \wedge^1 \otimes \wedge^a \rightarrow \wedge^{a+1} \]
Finally, we construct \(\psi_{a,b}\) as follows:
\[ \psi_{a,b} : \wedge^{a+1} \otimes \wedge^{b-1} \rightarrow \wedge^a \otimes \wedge^1 \otimes \wedge^{b-1} \rightarrow \wedge^a \otimes \wedge^{b-1} \rightarrow \wedge^a \otimes \wedge^b \]
5 The crystal basis for $\wedge^K$

In this section we examine the crystal structure (see [9][10]) of the $U_q(\mathfrak{gl}_m) \otimes U_q(\mathfrak{gl}_n)$-module $\wedge^K(\mathbb{C}^{m \times n})$. We show that there is a sign function $\text{sign}^*$ on $K$-subsets of $\lfloor mn \rfloor$ such that the collection $\mathcal{B}^* = \{\text{sign}^* \cdot v_c \}_c$ is a crystal basis for $\wedge^K$.

We identify $\lfloor mn \rfloor$ with $[m] \times [n]$ and also order the elements as follows:

$$(1,1) \prec (2,1) \prec \ldots (m,1) \prec (1,2) \prec \ldots (m-1,n) \prec (m,n)$$

In other words $(i,j) \prec (i',j')$ iff either $j < j'$ or $j = j'$ with $i < i'$. For $(i,j) \prec (i',j')$, we denote by $[(i,j),(i',j')]$ as the indices between $(i,j)$ and $(i',j')$ including both $(i,j)$ and $(i',j')$.

Recall that (cf. Section 2), as a $\mathbb{C}(q)$-vector space, $\wedge^K(\mathbb{C}^{mn})$ is generated by the basis vectors $\mathcal{B} = \{v_c | c \subseteq \lfloor mn \rfloor, |c| = K\}$. Let us fix an index $i$ and look at the sub-algebra $U^L_i$ of $U_q(\mathfrak{gl}_m)$ generated by $E^L_i, F^L_i$ and $h^L_i$. We define the standard $U_q(\mathfrak{sl}_2)$ generated by symbols $e, f, h$ satisfying the following equations:

$$q^h q^{-h} = 1, \quad q^h eq^{-h} = q^2 e, \quad q^h f q^{-h} = q^{-2} f, \quad e f - f e = \frac{q^h - q^{-h}}{q - q^{-1}}$$

We use the Hopf $\Delta$:

$$\Delta q^h = q^h \otimes q^h, \Delta e = e \otimes 1 + q^{-h} \otimes e, \Delta f = f \otimes q^h + 1 \otimes f$$

In other words, they satisfy exactly the same relations that $e^L_i, f^L_i, h^L_i$ satisfy, including the Hopf. Clearly, $U^L_i$ is isomorphic to $U_q(\mathfrak{sl}_2)$ as algebras and we denote this isomorphism by $L : U^L_i \rightarrow U_q(\mathfrak{sl}_2)$.

We construct the $U_q(\mathfrak{sl}_2)$-module $\mathbb{C}^2$ with basis $x_1, x_2$ with the action:

$$ex_2 = x_1, ex_1 = 0, fx_2 = 0, fx_1 = x_2, q^h x_1 = qx_1, q^h x_2 = q^{-1} x_2$$

With the Hopf $\Delta$ above, $M = \otimes_{i=1}^N \mathbb{C}^2$ is a $U_q(\mathfrak{sl}_2)$-module with the basis $\mathcal{S} = \{y_1 \otimes \ldots \otimes y_N | y_i \in \{x_1, x_2\}\}$, and with the action:

$$e(y_1 \otimes \ldots y_N) = \sum_{j} (\prod_{k=1}^{j-1} q^{-h}(y_k)) \cdot y_1 \otimes \ldots y_{j-1} \otimes e(y_j) \otimes y_{j+1} \otimes \ldots \otimes y_N$$

A similar expression may be written for the action of $f$.

Let us identify $\lfloor mn \rfloor$ with $[m] \times [n]$ and define the signature $\sigma^L_i(c)$, for $c \subseteq \lfloor mn \rfloor$. Towards this, we define

$$I(c) = \{1 \leq j \leq n | \text{ both } (i,j), (i+1,j) \in c\}$$

$$J(c) = \{1 \leq j \leq n | \text{ both } (i,j), (i+1,j) \not\in c\}$$

$$S(c) = \{(i',j') \in c | i' \neq i \text{ and } i' \neq i + 1\}$$

The signature $\sigma^L_i(c)$ is the tuple $(I(c), J(c), S(c))$.

Next, for a $\sigma = (I,J,S)$, we define the vector space $V^L_{\sigma,i}$ as the $\mathbb{C}(q)$-span of all elements

$$\mathcal{B}^L_{\sigma,i} = \{v_c | \sigma^L_i(c) = \sigma\}$$

Let $N = n - |I| - |J|$ and let $M = \otimes^N \mathbb{C}^2$ be the $U_q(\mathfrak{sl}_2)$-module as above.

We prove the following:

**Proposition 40** Given $\sigma = (I_\tau, J_\tau, S_\tau)$ as above,

(i) $V^L_{\sigma,i}$ is a $U^L_i$-invariant subspace.
(ii) The $U_q(sl_2)$ module $M$ is isomorphic to the $U_{\lambda}^L$-module $V_{\sigma,i}^L$ via the isomorphism $\iota_L$ above.

**Proof:** For any $v_c \in \mathcal{B}_{\sigma,i}^L$, if $E_i^L(v_c) = \sum \alpha(c') \cdot v_{c'}$, then it is clear that $v_{c'} \in \mathcal{B}_{\sigma,i}^L$ as well. The same holds for $F_i^L$ and $h_i^L$. This proves (i) above. For (ii), first note that

$$E_i^L = \sum_{j} \prod_{k=1}^{j-1} q^{-h(k-1)m+i} e(j-1)m+i$$

which matches the Hopf $\Delta$ of $U_q(sl_2)$. Next, if $j \in I(c) \cup J(c)$ then the index $j$ is irrelevant to the action of $E_i^L$ on $v_c$, whence in the restriction to $V_{\sigma,i}^L$, the indices in $I_\sigma \cup J_\sigma$ do not play a role.

Next, note that $|\mathcal{B}_{\sigma,i}^L| = 2^N$. Assume for simplicity that $I_\sigma \cup J_\sigma = \{N+1, \ldots, n\}$. Indeed, we may set up a $U_q(sl_2)$-module isomorphism $\iota_L$ by setting

$$\iota_L(v_c) = y_1 \otimes \ldots \otimes y_N$$

such that $y_k = \begin{cases} x_1 & \text{iff } (i,k) \in c \\ x_2 & \text{otherwise} \end{cases}$

One may verify that $\iota_L : V_{\sigma,i}^L \rightarrow M$ is indeed equivariant via $L$. Q.E.D.

**Proposition 41** The elements $\mathcal{B}$ is a crystal basis for $\wedge^K(\mathbb{C}^m_n)$ for the action of $U_q(gl_m)$.

**Proof:** This is obtained by first noting that $\mathcal{S}$ is indeed a crystal basis for $M$, see [9], for example. Next, the equivariance of $\iota_L$ shows that for $v_c \in \mathcal{B}_{\sigma,i}^L$,

$$\widetilde{E}_i^L(v_c) = \iota_L^{-1}(\widetilde{e}(\iota_L(v_c)))$$

This proves that $\mathcal{B}_{\sigma,i}^L$ is indeed a crystal basis for $V_{\sigma,i}^L$. Next, by applying Proposition 40 for all $i$ and all $\sigma$, we see that $\{\mathcal{B}_{\sigma,i}^L \}_{i,\sigma}$ together cover $\mathcal{B}$. Q.E.D.

We now move to the trickier $U_q(gl_n)$-action. Let us denote by $\epsilon_{i,j}$ the weight $\epsilon_{(j-1)m+i}$ and $h_{i,j} = \epsilon_{i,j} - \epsilon_{i,j+1}$. There are two sources of complications.

- The operator $E^R_k$ may be re-written as:

$$E^R_k = \sum_{i} \prod_{a=i+1}^{m} (q^{-h_{a,k}}) E_{(k-1)m+i,km+i-1} = \sum_{i} E_{(k-1)m+i,km+i-1} \prod_{a=i+1}^{m} q^{-h_{a,k}}$$

Thus, the Hopf works from the “right”.

- For a general $v_c$, if $E_{(k-1)m+i,km+i-1}v_c$ is non-zero then it is $\pm v_d$, where $v_d = v_c - (i,k+1) + (i,k)$ where the sign is $(-1)^M$ where $M$ is the number of elements in $c \cap [(i+1,k), \ldots, (i-1,k+1)]$.

To fix the sign, we first define an “intermediate global” sign as follows. For a set $c \subset [m] \times [n]$, we define $c^* \subset [m] \times [n]$ as that obtained by moving the elements of $c$ to the right, as far as they can go (see Example 40). Note that $F^R_k(c^*) = 0$ for all $k$ and thus $c^*$ is one of the lowest weight vectors in $\wedge^K(\mathbb{C}^m_n)$. For an $(i,j) \in c$, let $(i,j^*)$ be its final position in $c^*$. We may define $j^*$ explicitly as $n - \{(j'|(i,j') \in c, j' > j)\}$. Next, we define for $(i,j) \in c$,

$$S_{i,j}(c) = \{(i',j') \in c \mid (i,j) \prec (i',j') \prec (i',j^*) \prec (i,j^*)\}$$

$$n_{ij} = |S_{i,j}(c)|$$

Setting $N_c = \sum_{(i,j) \in c} n_{ij}$ we finally define:

$$\text{sign}(c) = (-1)^{N_c}$$

$$\text{sign}(d/c) = \text{sign}(d)/\text{sign}(c)$$
Lemma 42 Let \( v_c \in \mathcal{B}^R_{\sigma,k} \) be such that \( E_{(k-1)m+i,km+i-1}v_c \neq 0 \) then
\[
E_{(k-1)m+i,km+i-1}v_c = \text{sign}(d/c)v_d
\]
where \( v_d = v_c - (i, k + 1) + (i, k) \).

Proof: It is clear that \( e^* = d^* \) and thus for \((i, k + 1) \in e \) and \((i, k) \in d \), let \((i, k^*) \) be the final position of both \((i, k + 1) \in e \) and \((i, k) \in d \). For \((i, k + 1) \prec (i', j') \) or \((i', j') \prec (i, k) \) we have (i) \( S_{i',j'}(c) = S_{i',j'}(d) \) and (ii) \((i', j') \in S_{i,k+1}(c) \) iff \((i', j') \in S_{i,k}(d) \).

Next, it is clear that (i) \( S_{i,k}(d) \supseteq S_{i,k+1}(c) \), and (ii) for \((i, k) \prec (i', j') \prec (i, k + 1) \), \( S_{i',j'}(d) \subseteq S_{i',j'}(c) \) and in fact, \( S_{i',j'}(c) - S_{i',j'}(d) \) can atmost be the element \((i, k + 1) \).

Now let us look at \( S_{i,k}(d) - S_{i,k+1}(c) \). These contain all \((i', j') \in e \) such that
\[
(i, k) \prec (i', j') \prec (i, k + 1) \prec (i', j^*) \prec (i, k^*)
\]
On the other hand, for \((i', j') \in e \) such that \((i, k) \prec (i', j') \prec (i, k + 1) \), which are not counted above, it must be that \((i, k^*) \prec (i', j^*) \) in which case, \( S_{i',j'}(c) = S_{i',j'}(d) \cup \{(i, k + 1)\} \).

In short, for every \((i', j') \in e \) such that \((i, k) \prec (i', j') \prec (i, k + 1) \) either it contributes to an increment in \( S_{i,k}(d) \) or \( S_{i,k+1}(c) \) or a decrement in \( S_{i',j'}(d) \) over \( S_{i',j'}(c) \). Of course, the two cases are exclusive.

Thus we have \( \text{sign}(d)/\text{sign}(c) = (-1)^M \) where \( M \) is exactly the number of elements in \( e \cap [(i + 1, k), \ldots, (i - 1, k + 1)] \). Q.E.D.

Next, we define a new Hopf \( \Delta' \) on \( U_q(sl_2) \) as
\[
\Delta' q^h = q^h \otimes q^h, \Delta' e = 1 \otimes e + e \otimes q^{-h}, \Delta' f = q^h \otimes f + f \otimes 1
\]
We denote by \( M' \), the \( U_q(sl_2) \)-module \( \otimes \mathbb{C}^2 \) via the Hopf \( \Delta' \) and with the basis \( S = \{y_1 \otimes \cdots \otimes y_N | y_i \in \{x_1, x_2\} \} \). Under \( \Delta' \) we have:
\[
e(y_1 \otimes \cdots y_N) = \sum_j (\prod_{k=j+1}^{N} q^{-h}(y_k)) \cdot y_1 \otimes \cdots \otimes y_{j-1} \otimes e(y_j) \otimes y_{j+1} \otimes \cdots \otimes y_N
\]

We denote by \( U^R_k \) the algebra generated by \( E^R_k, F^R_k, H^R_k \) and let \( R : U^R_k \rightarrow U_q(sl_2) \) be the natural isomorphism.

As before, we define \( \sigma^R_k(c) \) analogously as
\[
I(c) = \{1 \leq i \leq m \mid \text{both } (i, k), (i, k + 1) \in c \}
J(c) = \{1 \leq i \leq m \mid \text{both } (i, k), (i, k + 1) \not\in c \}
S(c) = \{(i', k') \in e \mid k' \neq k \text{ and } k' \neq k + 1 \}
\]
Next, for a \( \sigma = (I, J, S) \), we define the vector space \( \mathcal{B}^R_{\sigma,k} \) as the \( \mathbb{C}(q) \)-span of all elements
\[
\mathcal{B}^R_{\sigma,k} = \{v_c \mid \sigma^R_k(c) = \sigma \}
\]
Again, as before, let \( N = n - |I| - |J| \). Let us also assume, for simplicity that \( I \cup J = \{N+1, \ldots, m\} \).

Proposition 43 Given \( \sigma \) as above,

(i) \( \mathcal{V}^R_{\sigma,k} \) is a \( U^R_k \)-invariant subspace.

(ii) The \( U_q(sl_2) \) module \( M' \) is isomorphic to the \( U^R_k \)-module \( \mathcal{V}^R_{\sigma,k} \) via the isomorphism \( R \) above.
Proof: Part (i) above is obvious. For (ii), note that
\[ E^R_k = \sum_i E_{(k-1)m+i,km+i-1}^{(m)} \left( \prod_{a=i+1}^m q^{h_{a,k}} \right) \]
which matches the Hopf \( \Delta' \) of \( U_q(sl_2) \). Again, if \( j \in I(c) \cup J(c) \) then the index \( j \) is irrelevant to the action of \( E^R_k \) on \( v_c \), whence in the restriction to \( V^R_{\sigma,k} \), the indices in \( I \cup J \) do not play a role.

Next, note that \( |B^R_{\sigma,k}| = 2^N \). Recall that, we have assumed that \( I \cup J = \{N+1, \ldots, m\} \). Indeed, we may set up a \( U_q(sl_2) \)-module isomorphism \( \iota_R \) by setting
\[ \iota_R(v_c) = \text{sign}(c) \cdot y_1 \otimes \ldots \otimes y_N \text{ such that } y_i = \begin{cases} x_1 \text{ iff } (i, k) \in c \\ x_2 \text{ otherwise} \end{cases} \]
One may verify (using Lemma 42) that \( \iota_R : V^R_{\sigma,k} \rightarrow M' \) is indeed equivariant via \( R \). Q.E.D.

Proposition 44 Let \( B' = \{ \text{sign}(b) \cdot v_b | b \in B \} \) be “signed” elements. Then the elements \( B' \) is a crystal basis for \( \wedge^K (\mathbb{C}^{mn}) \) for the action of \( U_q(gl_n) \). In other words \( E^R_k (v_c) = \pm v_d \cup 0 \).

Proof: Let \( B^R_{\sigma,k} \) be the “signed” elements of \( B^R_{\sigma,k} \). We first note that \( S \) continues to be a crystal basis for \( M' \). Next, the equivariance of \( \iota_R \) shows that for \( v_c \in B^R_{\sigma,k} \),
\[ E^R_k (v_c) = \iota_R^{-1}(\iota_R(v_c)) \]
This proves that \( B^R_{\sigma,k} \) is indeed a crystal basis for \( V^R_{\sigma,k} \).

Thus, keeping in mind that the signs are allotted by our global \text{sign}-function and, by considering all \( \sigma \) and all \( k \), we obtain the assertion. Q.E.D.

We now define our final global sign \( \text{sign}^*(b) \) as follows. Firstly, let \( S = \{ b \mid F^L_i v_b = F^R_k v_b = 0 \} \). These are the lowest weight vectors for both the left and the right action. We see that:
- For any \( b \in S \), we have \( b^* = b \).
- If \( wt_i(b) \) denotes the cardinality of the set \( \{(i, k) | (i, k) \in b\} \), then \( wt_1(b) \leq \ldots \leq wt_m(b) \).

We define \( \text{sign}^*(b) = \text{sign}(b) \) for all \( b \) such that \( b^* \in S \). Next, for a \( c \) such that \( c^* \not\in S \), we inductively (by \( wt_i \) above) define \( \text{sign}^*(c) = \text{sign}(F^L_i v_c) \) where \( F^L_i (v_c) \neq 0 \). By the commutativity of \( F^L_i \) with \( F^R_k \), we see that \( \text{sign}^*(c) \) is well defined over all \( K \)-subsets of \( [m] \times [n] \).

Let \( v^*_c = \text{sign}^*(b) \cdot v_b \) and let \( B^* = \{ v^*_c | v_b \in B \} \).

Proposition 45 The elements \( B^* \) is a crystal basis for \( \wedge^K (\mathbb{C}^{mn}) \) for the action of both \( U_q(gl_n) \) and \( U_q(gl_m) \). In other words \( E^R_k (v^*_c) \in B^* \cup 0 \) and \( E^L_i (v^*_c) \in B^* \cup 0 \).

Proof: The proof follows from the commutativity condition and the well-defined-ness of \( \text{sign}^* \). Q.E.D.

Example 46 Let us consider \( \wedge^2 (\mathbb{C}^{2 \times 2}) \) whose six elements, their matrix notation, and signs are given below:
For a $b \subseteq [m] \times [n]$ define the **left word** $LW(b)$ as the $i$-indices of all elements $(i, k) \in b$, read bottom to top within a column, reading the columns left to right. Similarly, define the **right word** $RW(b)$ as the $k$-indices of all elements $(i, k) \in b$, read right to left within a row, reading the rows from bottom to top. For a word $w$, let $rs(w)$ be the Robinson-Schenstead tableau associated with $w$, when read from left to right. Define the **left tableau** $LT(b) = rs(LW(b))$ and the **right tableau** as $RT(b) = rs(RW(b))$.

**Example 47** Let $m = 3$ and $n = 4$ and let $b = \{1, 3, 5, 6, 9, 10\}$.

For semi-standard tableau, recall the crystal operators $\sim e^T_i, \sim f^T_i$, see for example, [9]. These crystal operators may be connected to our crystal operators via the following proposition. This obtains the result in [2].

**Proposition 48** For any $v^*_b \in B^*$ the crystal basis for $\wedge^K(\mathbb{C}^{mn})$ as above, we have:

- If $E^L_i (v^*_b) = v^*_c$ then $\sim e^T_i (LT(b)) = LT(c)$.
- If $E^R_k (v^*_b) = v^*_c$ then $\sim e^T_k (RT(b)) = RT(c)$.

A similar assertion holds for the $\sim F$-operators.

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