Dynamics of Teleparallel Dark Energy

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ABSTRACT

Recently, Geng et al. proposed to allow a non-minimal coupling between quintessence and gravity in the framework of teleparallel gravity, motivated by the similar one in the framework of General Relativity (GR). They found that this non-minimally coupled quintessence in the framework of teleparallel gravity has a richer structure, and named it “teleparallel dark energy”. In the present work, we note that there might be a deep and unknown connection between teleparallel dark energy and Elko spinor dark energy. Motivated by this observation and the previous results of Elko spinor dark energy, we try to study the dynamics of teleparallel dark energy. We find that there exist only some dark-energy-dominated de Sitter attractors. Unfortunately, no scaling attractor has been found, even when we allow the possible interaction between teleparallel dark energy and matter. However, we note that $w$ at the critical points is in agreement with observations (in particular, the fact that $w = -1$ independently of $\xi$ is a great advantage).

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I. INTRODUCTION

Since the striking discovery of the current accelerated expansion in 1998, it has been one of the most active fields in modern cosmology [1,3]. Besides the cosmological constant, the simplest candidate of dark energy is the well-known quintessence, which is described by a canonical scalar field $\phi$ in the framework of General Relativity (GR). The relevant action reads [1]

$$S = \int d^4x \sqrt{-g} \left[ \frac{R}{2\kappa^2} + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right] + S_m,$$

where $R$ is the Ricci scalar; $S_m$ is the matter action; $\kappa^2 \equiv 8\pi G$; we use the metric signature convention $(+, -, -, -)$ throughout. Considering a spatially flat Friedmann-Robertson-Walker (FRW) universe and a homogeneous scalar field $\phi$, the corresponding Friedmann equation and Raychaudhuri equation are given by

$$H^2 = \frac{\kappa^2}{3} \rho_{\text{tot}} = \frac{\kappa^2}{3} (\rho_\phi + \rho_m),$$

$$\dot{H} = -\frac{\kappa^2}{2} (p_{\text{tot}} + \rho_{\text{tot}}) = -\frac{\kappa^2}{2} (p_\phi + \rho_\phi + p_m + \rho_m),$$

where $H \equiv \dot{a}/a$ is the Hubble parameter; $a$ is the scale factor; a dot denotes the derivatives with respect to cosmic time $t$; $p_m$ and $\rho_m$ are the pressure and energy density of background matter, respectively. In this work, we assume that the background matter is described by a perfect fluid with barotropic equation-of-state parameter (EoS), namely

$$p_m = w_m \rho_m \equiv (\gamma - 1) \rho_m,$$

where the so-called barotropic index $\gamma$ is a positive constant. In particular, $\gamma = 1$ and $4/3$ correspond to dust matter and radiation, respectively. As is well known, the pressure and energy density of quintessence are given by

$$p_\phi = \frac{1}{2} \dot{\phi}^2 - V(\phi), \quad \rho_\phi = \frac{1}{2} \dot{\phi}^2 + V(\phi),$$

where $V(\phi)$ is the potential. The quintessence has been extensively discussed in the literature, and we refer to e.g. [1] for some comprehensive reviews.

As is well known, in the literature one can generalize quintessence by including a non-minimal coupling between quintessence and gravity. The relevant action reads [1, 4, 5]

$$S = \int d^4x \sqrt{-g} \left[ \frac{R}{2\kappa^2} + \frac{1}{2} (\partial_\mu \phi \partial^\mu \phi + \xi R \phi^2) - V(\phi) \right] + S_m,$$

where $\xi$ is a constant measuring the non-minimal coupling. In this case, the corresponding Friedmann equation and Raychaudhuri equation are the same as Eqs. (2) and (3), while the effective pressure and energy density of the non-minimally coupled quintessence (sometimes called “extended quintessence” in the literature) are changed to [1, 4, 5]

$$p_\phi = \frac{1}{2} (1 + 4\xi) \dot{\phi}^2 - V + 2\xi (1 + 6\xi) \ddot{H} \phi^2 - 2\xi H \dot{\phi}^2 - 2\xi \phi V_{,\phi} + 3\xi (1 + 8\xi) H^2 \phi^2,$$

$$\rho_\phi = \frac{1}{2} \dot{\phi}^2 + V - 6\xi H \phi \dot{\phi} - 3\xi H^2 \phi^2,$$

where $V_{,\phi} \equiv dV/d\phi$, and we have used the equation of motion

$$\ddot{\phi} + 3H \dot{\phi} - \xi R \phi + V_{,\phi} = 0,$$

which is equivalent to the energy conservation equation $\dot{\rho}_\phi + 3H (\rho_\phi + p_\phi) = 0$ in fact (it is worth noting that $R = 6(\dot{H} + 2H^2)$ in a spatially flat FRW universe). We refer to e.g. [1, 4, 5] for details.
Recently, the so-called teleparallel gravity originally proposed by Einstein [6, 7] and its generalization, namely the so-called $f(T)$ theory [8, 9], attracted much attention in the community. In teleparallel gravity, the Weitzenböck connection is used, rather than the Levi-Civita connection which is used in GR. Following [6–9], here we briefly review the key ingredients of teleparallel gravity. The orthonormal tetrad components $e_i(x^\mu)$ relate to the metric through

$$g_{\mu\nu} = \eta_{ij} e_i^\mu e_j^\nu ,$$  

where Latin $i$, $j$ are indices running over 0, 1, 2, 3 for the tangent space of the manifold, and Greek $\mu$, $\nu$ are the coordinate indices on the manifold, also running over 0, 1, 2, 3. In teleparallel gravity, the relevant action is

$$S = \frac{1}{2\kappa^2} \int d^4x |e| T + S_m ,$$  

where $|e| = \det (e_i^\mu) = \sqrt{-g}$. The torsion scalar $T$ is given by

$$T \equiv S^r_{\mu\nu} T^\rho_{\mu\nu} ,$$

$$T^\rho_{\mu\nu} \equiv e_i^\rho (\partial_\mu e_i^\nu - \partial_\nu e_i^\mu) ,$$

$$K^\mu\nu\rho \equiv -\frac{1}{2} (T^{\mu\nu}_{\rho} - T^{\mu\rho}_{\nu} - T^{\rho\mu}_{\nu}) ,$$

$$S^{\rho\mu\nu} \equiv \frac{1}{2} (K^{\mu\nu}_{\rho} + \delta^{\rho}_{\mu} T^{\theta\nu}_{\theta} - \delta^{\rho}_{\nu} T^{\theta\mu}_{\theta}) .$$

For a spatially flat FRW universe, it is easy to find that

$$T = -6H^2 .$$

So, one can use $T$ and $H$ interchangeably. As is well known, the FRW universe described by action (11) is completely equivalent to a matter-dominated universe in the framework of GR, and hence cannot be accelerated. In the literature, there are two ways out. In analogy to $f(R)$ theory, the first approach is to generalize teleparallel gravity to $f(T)$ theory by modifying action (11) to

$$S = \frac{1}{2\kappa^2} \int d^4x |e| f(T) + S_m ,$$

where $f(T)$ is a function of the torsion scalar $T$. Recently, $f(T)$ theory attracted much attention in the community, and we refer to e.g. [8–14, 33] for some relevant works. Obviously, the second approach is to directly add dark energy into teleparallel gravity. Of course, the simplest candidate of dark energy is still quintessence, and the relevant action is given by

$$S = \int d^4x |e| \left[ T \frac{2\kappa^2}{2} + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right] + S_m .$$

However, one can easily find that dark energy in the framework of teleparallel gravity is completely identical to the one in the framework of GR, and hence there is nothing new. Very recently, motivated by the similar one in the framework of GR, Geng et al. [15] proposed to modify action (15) by including a non-minimal coupling between quintessence and gravity in the framework of teleparallel gravity, namely

$$S = \int d^4x |e| \left[ T \frac{2\kappa^2}{2} + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \xi T \phi^2 - V(\phi) \right] + S_m .$$

They found that this non-minimally coupled quintessence in the framework of teleparallel gravity has a richer structure, and named it “teleparallel dark energy” [15]. The corresponding Friedmann equation
and Raychaudhuri equation are the same as Eqs. (2) and (3), while the effective pressure and energy density of teleparallel dark energy are given by \[15\]

\[
p_\phi = \frac{1}{2} \dot{\phi}^2 - 4 \xi H \phi \dot{\phi} + \xi \left( 3 H^2 + 2 \dot{H} \right) \phi^2, \tag{20}
\]

\[
\rho_\phi = \frac{1}{2} \dot{\phi}^2 + V(\phi) - 4 \xi H^2 \phi^2. \tag{21}
\]

The equation of motion reads

\[
\ddot{\phi} + 3 H \dot{\phi} - \xi T \phi + V_{,\phi} = 0, \tag{22}
\]

which is equivalent to the energy conservation equation \(\dot{\rho}_\phi + 3 H (\rho_\phi + p_\phi) = 0\) in fact. Obviously, from Eqs. (20) and (21), the EoS of teleparallel dark energy \(w_\phi = p_\phi/\rho_\phi\) can cross the phantom divide \(w = -1\).

In the present work, we are interested in teleparallel dark energy because we note that it is reminiscent of the so-called Elko spinor dark energy \[16–18\]. The Elko spinor was originally proposed by Ahluwalia and Grumiller \[19\], which is a spin one half field with mass dimension one. Unlike the standard fields which obey \((\text{CPT})^2 = 1\), the Elko spinor is a non-standard spinor according to the Wigner classification \[20\] and obeys the unusual property \((\text{CPT})^2 = -1\) instead. In fact, the Elko spinor fields (together with Majorana spinor fields) belong to a wider class of spinor fields, i.e., the so-called flagpole spinor fields, according to the Lounesto general classification of all spinor fields \[21, 34\]. The effective pressure and energy density of Elko spinor dark energy are given by \[17, 18\]

\[
p_\phi = \frac{1}{2} \dot{\phi}^2 - V(\phi) - \frac{3}{8} H^2 \phi^2 - \frac{1}{4} H \phi^2 - \frac{1}{2} H \phi \dot{\phi}, \tag{23}
\]

\[
\rho_\phi = \frac{1}{2} \dot{\phi}^2 + V(\phi) + \frac{3}{8} H^2 \phi^2. \tag{24}
\]

We strikingly find that when \(\xi = -1/8\), Eqs. (20) and (21) become identical to Eqs. (23) and (24). If it is not an accident, this notable observation might hint a deep and unknown connection between teleparallel dark energy and Elko spinor dark energy. In particular, one might consider the deep relation between spinor and torsion. However, this is out of the scope of the present work, and we leave it as an open question. Here, we instead focus on another issue concerning the cosmological coincidence problem. As is shown in \[18, 22\] by using the dynamical system method, Elko spinor dark energy is plagued with the cosmological coincidence problem. Noting the aforementioned connection between teleparallel dark energy and Elko spinor dark energy, it is very natural to ask whether or not the cosmological coincidence problem could be alleviated in teleparallel dark energy which has an extra free model parameter \(\xi\). This is our main goal of the present work.

Noting that if \(\xi = 0\), teleparallel dark energy reduces to the ordinary quintessence (which is a very trivial case), we assume \(\xi \neq 0\) throughout this work.

II. DYNAMICAL SYSTEM OF TELEPARALLEL DARK ENERGY

As is well known, the observational data tell us that we are living in an epoch in which the dark energy density and the matter energy density are comparable \[1\]. However, since the densities of dark energy and matter scale differently with the expansion of our universe, there should be some kinds of fine-tunings. This is the well-known cosmological coincidence problem \[1\]. Usually, this problem can be alleviated in most dark energy models via the method of scaling solution. In fact, since the nature of both dark energy and dark matter is still unknown, there is no physical argument to exclude the possible interaction between them. On the contrary, some observational evidences of the interaction in dark sector have been found recently (see e.g. \[23, 24\]). If there is a possible interaction between dark energy and matter, their evolution equations could be rewritten as a dynamical system \[25\] (see also e.g. \[26–30\]). There might be some scaling attractors in this dynamical system, and both the fractional densities of dark energy and matter are non-vanishing constants over there. The universe will eventually enter these scaling attractors regardless of the initial conditions, and hence the cosmological coincidence problem could be alleviated without fine-tunings. This method works fairly well in most dark energy...
models (especially the scalar field models). However, in a few of dark energy models this method fails because there is no scaling attractor being found. As mentioned above, Elko spinor dark energy model is an example of failures [18, 22]. In the present work, we hope that teleparallel dark energy could avoid this fate with the help of the extra free model parameter \( \xi \), although there is a deep connection between teleparallel dark energy and Elko spinor dark energy as mentioned above.

To be general, we assume that teleparallel dark energy and background matter interact through a coupling term \( Q \), according to

\[
\dot{\rho}_\phi + 3H (\rho_\phi + p_\phi) = -Q, \tag{25}
\]
\[
\dot{\rho}_m + 3H (\rho_m + p_m) = Q, \tag{26}
\]

which preserves the total energy conservation equation \( \dot{\rho}_{\text{tot}} + 3H (\rho_{\text{tot}} + p_{\text{tot}}) = 0 \). Obviously, \( Q = 0 \) means that there is no interaction between teleparallel dark energy and background matter. Also to be general, we assume that the background matter could be characterized by Eq. (4). Following e.g. [25–30], we introduce the following dimensionless variables

\[
x \equiv \frac{\kappa \dot{\phi}}{\sqrt{6} H}, \quad y \equiv \frac{\kappa \sqrt{V}}{\sqrt{3} H}, \quad u \equiv \kappa \phi, \quad v \equiv \frac{\kappa \sqrt{\rho_m}}{\sqrt{3} H}. \tag{27}
\]

So, the Friedmann equation (2) can be recast as

\[
x^2 + y^2 - \xi u^2 + v^2 = 1. \tag{28}
\]

From the Raychaudhuri equation (3) and Eqs. (4), (20), (21), we have

\[
s \equiv -\frac{\dot{H}}{H^2} = 3x^2 - \xi su^2 + 2\sqrt{6} \xi xu + \frac{3}{2} \gamma v^2, \tag{29}
\]

in which \( s \) appears in both sides. From Eq. (29), it is easy to find that

\[
s = \left( 3x^2 + 2\sqrt{6} \xi xu + \frac{3}{2} \gamma v^2 \right) (1 + \xi u^2)^{-1}. \tag{30}
\]

With the help of Eqs. (2), (3), (20) and (21), the evolution equations (25) and (26) can be rewritten as a dynamical system, namely

\[
x' = (s - 3)x - \sqrt{6} \xi u - \frac{\kappa V_\phi}{\sqrt{6} H^2} - Q_1, \tag{31}
\]
\[
y' = sy + \frac{x}{\sqrt{2} H} \sqrt{\frac{\dot{\phi}}{V_\phi}}, \tag{32}
\]
\[
u' = \sqrt{6} x, \tag{33}
\]
\[
v' = \left( s - \frac{3}{2} \gamma \right) v + Q_2. \tag{34}
\]

where a prime denotes derivative with respect to the so-called e-folding time \( N \equiv \ln a \), and

\[
Q_1 \equiv \frac{\kappa Q}{\sqrt{6} H^2}, \quad Q_2 \equiv \frac{vQ}{2H \rho_m}. \tag{35}
\]

On the other hand, the fractional energy densities \( \Omega_i \equiv (\kappa^2 \rho_i)/(3H^2) \) of teleparallel dark energy and background matter are given by

\[
\Omega_\phi = x^2 + y^2 - \xi u^2, \quad \Omega_m = v^2. \tag{36}
\]

The EoS of teleparallel dark energy reads

\[
w_\phi \equiv \frac{p_\phi}{\rho_\phi} = \frac{x^2 - y^2 + \xi u^2 - \frac{3}{2} \xi su^2 + 4\sqrt{2} \xi xu}{x^2 + y^2 - \xi u^2}. \tag{37}
\]
On the other hand, the total EoS is given by

\[ w_{\text{tot}} \equiv p_{\text{tot}} \rho_{\text{tot}} = \Omega_\phi w_\phi + \Omega_m w_m = x^2 - y^2 + \xi u^2 - \frac{2}{3} \xi su^2 + 4 \sqrt{\frac{2}{3}} \xi xu + (\gamma - 1) v^2. \]  

(38)

Eqs. (31)–(34) could be an autonomous system when the potential \( V(\phi) \) and the interaction term \( Q \) are chosen to be suitable forms. In fact, we will consider the models with an exponential and power-law potential in the following sections. In each model with different potential, we consider four cases with various interaction forms between teleparallel dark energy and background matter. The first case is the one without interaction, i.e., \( Q = 0 \). The other three cases are taken as the most familiar interaction terms extensively considered in the literature (see e.g. [26–30]), namely

- Case (I) \( Q = 0 \),
- Case (II) \( Q = \alpha \kappa \rho_m \dot{\phi} \),
- Case (III) \( Q = 3 \beta H \rho_{\text{tot}} \),
- Case (IV) \( Q = 3 \eta H \rho_m \),

where \( \alpha, \beta \) and \( \eta \) are all dimensionless constants.

| Label | Critical Point \((\bar{x}, \bar{y}, \bar{u}, \bar{v})\) | Existence |
|-------|---------------------------------|-----------|
| E.I.1m | \( 0, \sqrt{\frac{2}{3}} \xi u + \sqrt{\frac{3}{2}} \lambda y^2 - Q_1 \) | \( \xi \geq \lambda^2 \) |
| E.I.1p | \( 0, \sqrt{\frac{2}{3}} \xi u + \sqrt{\frac{3}{2}} \lambda y^2 - Q_1 \) | \( \xi \geq \lambda^2 \) or \( \xi < 0 \) |
| E.I.2 | 0, 0, 0, 1 | always |

TABLE I: Critical points for the autonomous system (40)–(43) and their corresponding existence conditions, for Case (I) \( Q = 0 \). See text for details.

III. TELEPARALLEL DARK ENERGY WITH AN EXPONENTIAL POTENTIAL

At first, we consider teleparallel dark energy with an exponential potential

\[ V(\phi) = V_0 e^{-\lambda \kappa \phi}, \]

(39)

where \( \lambda \) is a dimensionless constant. In this case, Eqs. (31)–(34) become

\[ x' = (s - 3) x - \sqrt{6} \xi u + \sqrt{\frac{3}{2}} \lambda y^2 - Q_1, \]

(40)

\[ y' = sy - \sqrt{\frac{3}{2}} \lambda xy, \]

(41)

\[ u' = \sqrt{6} x, \]

(42)

\[ v' = \left( s - \frac{3}{2} \right) v + Q_1. \]

(43)

If \( Q \) is given, we can obtain the critical points \((\bar{x}, \bar{y}, \bar{u}, \bar{v})\) of the above autonomous system by imposing the conditions \( \dot{x}' = \dot{y}' = \dot{u}' = \dot{v}' = 0 \). Of course, they are subject to the Friedmann constraint \( (28) \), i.e., \( x^2 + y^2 - \xi u^2 + v^2 = 1 \). On the other hand, by definitions in Eq. (27), \( \bar{x}, \bar{y}, \bar{u}, \bar{v} \) should be real, and \( \bar{y} \geq 0, \bar{v} \geq 0 \) are required.
For Case (I) $Q = 0$, the corresponding $Q_1 = 0$ and $Q_2 = 0$. In this case, there are three critical points, and we present these critical points and their corresponding existence conditions in Table I From Eqs. (30), (36), (37) and (38), we find that at Points (E.I.1m) and (E.I.1p), $\Omega_1 = 1$, $\Omega_m = 0$, $w_0 = -1$ and $w_{tot} = -1$, namely, they are both dark-energy-dominated de Sitter solutions. On the other hand, Point (E.I.2) is a matter-dominated solution. Therefore, there is no scaling solution in Case (I) $Q = 0$.

For Case (II) $Q = \alpha \kappa \rho_m \dot{\phi}$, the corresponding $Q_1 = \sqrt{2} \alpha \nu x^2$ and $Q_2 = \sqrt{2} \alpha x v$. In this case, there are four critical points, and we present these critical points and their corresponding existence conditions in Table I. From Eqs. (30), (36), (37) and (38), we find that at Points (E.II.1m) and (E.II.1p), $\Omega_1 = 1$, $\Omega_m = 0$, $w_0 = -1$ and $w_{tot} = -1$, namely, they are both dark-energy-dominated de Sitter solutions. On the other hand, Points (E.II.1m) and (E.II.1p) are both scaling solutions. Thus, they can give the hope to alleviate the cosmological coincidence problem. However, their stabilities are required in order to be attractors which are necessary to this end (see the discussions below).

For Case (III) $Q = 3 \beta H \rho_{tot}$, the corresponding $Q_1 = \frac{3}{2} \beta x^{-1}$ and $Q_2 = \frac{3}{2} \beta v^{-1}$. On the other hand, for Case (IV) $Q = 3 \eta H \rho_m$, the corresponding $Q_1 = \frac{3}{2} \eta x^{-1} v^2$ and $Q_2 = \frac{3}{2} \eta v$. Unfortunately, we find that there is no critical point in these two cases.

| Label      | Critical Point $(\bar{x}, \bar{y}, \bar{u}, \bar{v})$ | Existence |
|------------|------------------------------------------------------|-----------|
| E.II.1m    | $[\bar{x} + \sqrt{\xi (\xi - \alpha^2)}] (\alpha \xi)^{-1}$, $[2 \bar{x} - \sqrt{\xi (\xi - \alpha^2)}] \alpha^{-2}$ | $\xi \geq \alpha^2$ |
| E.II.1p    | $[\bar{x} - \sqrt{\xi (\xi - \alpha^2)}] (\alpha \xi)^{-1}$, $[2 \bar{x} + \sqrt{\xi (\xi - \alpha^2)}] \alpha^{-2}$ | $\xi \geq \alpha^2$ or $\xi < 0$ |
| E.II.2m    | $\sqrt{2} [\bar{x} - \sqrt{\xi (\xi - \lambda^2)}] \lambda^{-2}$, $[\xi - \sqrt{\xi (\xi - \lambda^2)}] (\lambda \xi)^{-1}$, 0 | $\xi \geq \lambda^2$ |
| E.II.2p    | $\sqrt{2} [\bar{x} + \sqrt{\xi (\xi - \lambda^2)}] \lambda^{-2}$, $[\xi + \sqrt{\xi (\xi - \lambda^2)}] (\lambda \xi)^{-1}$, 0 | $\xi \geq \lambda^2$ or $\xi < 0$ |

**TABLE II**: The same as in Table I except for Case (II) $Q = \alpha \kappa \rho_m \dot{\phi}$.

To study the stability of the critical points for the autonomous system Eqs. (40)–(43), we substitute linear perturbations $x \to \bar{x} + \delta x$, $y \to \bar{y} + \delta y$, $u \to \bar{u} + \delta u$, and $v \to \bar{v} + \delta v$ about the critical point $(\bar{x}, \bar{y}, \bar{u}, \bar{v})$ into the autonomous system Eqs. (40)–(43) and linearize them. Because of the Friedmann constraint (28), there are only three independent evolution equations, namely

$$
\delta x' = (\bar{s} - 3) \delta x + \bar{x} \delta s - \sqrt{6} \xi \delta u + \sqrt{6} \lambda \bar{y} \delta y - \delta Q_1,
$$

$$
\delta y' = \bar{s} \delta y + \bar{y} \delta s - \left(\frac{3}{2}\right) \lambda (\bar{y} \delta y + \bar{y} \delta x),
$$

$$
\delta u' = \sqrt{6} \delta x,
$$

where

$$
\bar{s} = \left[3 \bar{x}^2 + 2 \sqrt{6} \xi \bar{x} \bar{u} + \frac{3}{2} (1 - \bar{x}^2 - \bar{y}^2 + \xi \bar{u}^2) \right] (1 + \xi \bar{u}^2)^{-1},
$$

$$
\delta s = \left[-2 \xi \bar{s} \delta u + 6 \bar{x} \delta x + 2 \sqrt{6} \xi (\bar{x} \delta u + \bar{u} \delta x) + 3 \gamma (\xi \bar{u} \delta u - \bar{x} \delta x - \bar{y} \delta y) \right] (1 + \xi \bar{u}^2)^{-1},
$$

and $\delta Q_1$ is the linear perturbation coming from $Q_1$. The three eigenvalues of the coefficient matrix of Eqs. (44)–(46) determine the stability of the critical point.

For Case (I) $Q = 0$, the corresponding $\delta Q_1 = 0$. The three eigenvalues for Point (E.I.1m) are the three roots of equation (in which $r$ is the unknown quantity)

$$
(r + 3 \gamma) \left[6 \lambda^2 \xi + \left[-\xi + \sqrt{\xi (\xi - \lambda^2)} \times [6 \xi - r (3 + r)] \right] \right] = 0.
$$
Obviously, the first eigenvalue is $-3\gamma$. The other two eigenvalues are complicated and hence we do not give their explicit expressions here. Similarly, the three eigenvalues for Point (E.I.1p) are the three roots of equation (in which $r$ is the unknown quantity)

$$\left(r + 3\gamma\right) \left\{6\lambda^2\xi - \left[\xi + \sqrt{\xi(\xi - \lambda^2)}\right] \times \left[6\xi - r(3 + r)\right]\right\} = 0.$$  

(50)

Again, the first eigenvalue is $-3\gamma$. The other two eigenvalues are complicated and hence we do not give their explicit expressions here. Unfortunately, we find that if Point (E.I.1p) exists (under condition $\xi \geq \lambda^2$ or $\xi < 0$), it is unstable. Finally, the three eigenvalues for Point (E.I.2) are given by

$$\left\{\frac{3\gamma}{2}, \frac{1}{4} \left[6 + 3\gamma - \sqrt{9(\gamma - 2)^2 - 96\xi}\right], \frac{1}{4} \left[6 + 3\gamma + \sqrt{9(\gamma - 2)^2 - 96\xi}\right]\right\}.$$  

(51)

Because the first eigenvalue $3\gamma/2$ is positive, Point (E.I.2) is unstable. For Case (II) $Q = \alpha\kappa\rho_m \dot{\phi}$, the corresponding $\delta Q_1 = \sqrt{6}\alpha(\xi \ddot{u} - \ddot{x}\dot{y} - \ddot{y}\dot{y})$. The three eigenvalues for Point (E.II.1m) are

$$\left\{\frac{3\gamma}{2}, \frac{1}{4} \left(-6 + 3\gamma - \sigma_-\right), \frac{1}{4} \left(-6 + 3\gamma + \sigma_-\right)\right\},$$  

(52)

where

$$\sigma_- \equiv \sqrt{9(\gamma - 2)^2 - 96\xi(\xi - \alpha^2)}.$$  

(53)

The three eigenvalues for Point (E.II.1p) are

$$\left\{\frac{3\gamma}{2}, \frac{1}{4} \left(-6 + 3\gamma - \sigma_+\right), \frac{1}{4} \left(-6 + 3\gamma + \sigma_+\right)\right\},$$  

(54)

where

$$\sigma_+ \equiv \sqrt{9(\gamma - 2)^2 + 96\xi(\xi - \alpha^2)}.$$  

(55)

Since their first eigenvalue $3\gamma/2$ is positive, Points (E.II.1m) and (E.II.1p) are both unstable. Therefore, although they are scaling solutions, however, they are not attractors and hence cannot alleviate the cosmological coincidence problem. On the other hand, the three eigenvalues for Point (E.II.2m) are the three roots of Eq. (49). So, if Point (E.II.2m) exists (under condition $\xi \geq \lambda^2$), it is stable. Similarly, the three eigenvalues for Point (E.II.2p) are the three roots of Eq. (50). So, if Point (E.II.2p) exists (under condition $\xi \geq \lambda^2$ or $\xi < 0$), it is unstable.

Since in both Case (III) $Q = 3\beta H \rho_m$ and Case (IV) $Q = 3\eta H \rho_m$, there is no critical point, we need not perform the stability analysis for them.

So, for teleparallel dark energy with an exponential potential, in Case (I) $Q = 0$ there is only one attractor (E.I.1m) which is a dark-energy-dominated de Sitter solution, and in Case (II) $Q = \alpha\kappa\rho_m \dot{\phi}$ there is only one attractor (E.II.2m) which is also a dark-energy-dominated de Sitter solution. No scaling attractor has been found unfortunately.

**IV. TELEPARALLEL DARK ENERGY WITH A POWER-LAW POTENTIAL**

Due to the failure in the models with an exponential potential, we turn to teleparallel dark energy with a power-law potential

$$V(\phi) = V_0 (\kappa \phi)^n,$$  

(56)
where $n$ is a dimensionless constant. In this case, Eqs. (31)–(34) become

$$x' = (s - 3)x - \sqrt{6}\xi u - \sqrt{\frac{3}{2}} ny^2 u^{-1} - Q_1,$$

(57)

$$y' = sy + \sqrt{\frac{3}{2}} nxyu^{-1},$$

(58)

$$u' = \sqrt{6}x,$$

(59)

$$v' = \left(s - \frac{3}{2}\right)v + Q_2.$$  

(60)

If $Q$ is given, we can obtain the critical points $(\bar{x}, \bar{y}, \bar{u}, \bar{v})$ of the above autonomous system by imposing the conditions $\bar{x}' = \bar{y}' = \bar{u}' = \bar{v}' = 0$. Of course, they are subject to the Friedmann constraint (28), i.e., $\bar{x}^2 + \bar{y}^2 - \xi \bar{u}^2 + \bar{v}^2 = 1$. On the other hand, by definitions in Eq. (27), $\bar{x}$, $\bar{y}$, $\bar{u}$, $\bar{v}$ should be real, and $\bar{y} \geq 0$, $\bar{v} \geq 0$ are required.

| Label | Critical Point $(\bar{x}, \bar{y}, \bar{u}, \bar{v})$ | Existence |
|-------|-------------------------------------------------|-----------|
| P.I.1m | $0, \sqrt{\frac{2}{n+2}}, -\sqrt{\frac{n}{(n+2)}}, 0$ | $(a) \xi < 0$ and $n \geq 0$ or $(b) \xi > 0$ and $-2 < n \leq 0$ |
| P.I.1p | $0, \sqrt{\frac{2}{n+2}}, \sqrt{\frac{n}{(n+2)}}, 0$ | $(a) \xi < 0$ and $n \geq 0$ or $(b) \xi > 0$ and $-2 < n \leq 0$ |

TABLE III: Critical points for the autonomous system (57)–(60) and their corresponding existence conditions, for Case (I) $Q = 0$. See text for details.

For Case (I) $Q = 0$, the corresponding $Q_1 = 0$ and $Q_2 = 0$. In this case, there are two critical points, and we present these critical points and their corresponding existence conditions in Table III. From Eqs. (31), (36), (37) and (38), we find that at Points (P.I.1m) and (P.I.1p), $\Omega_\phi = 1$, $\Omega_m = 0$, $w_\phi = -1$ and $w_{tot} = -1$, namely, they are both dark-energy-dominated de Sitter solutions. Therefore, there is no scaling solution in Case (I) $Q = 0$. For Case (II) $Q = \alpha \kappa \rho_m \dot{\phi}$, the corresponding $Q_1 = \sqrt{\frac{2}{3}} \alpha v^2$ and $Q_2 = \sqrt{\frac{2}{3}} \alpha x v$. In this case, there are four critical points, and we present these critical points and their corresponding existence conditions in Table IV. From Eqs. (31), (36), (37) and (38), we find that at Points (P.II.1m) and (P.II.1p), $\Omega_\phi = 1$, $\Omega_m = 0$, $w_\phi = -1$ and $w_{tot} = -1$, namely, they are both dark-energy-dominated de Sitter solutions. On the other hand, Points (P.II.1m) and (P.II.1p) are both scaling solutions. So, they can give the hope to alleviate the cosmological coincidence problem. However,
their stabilities are required in order to be attractors which are necessary to this end (see the discussions below). For Case (III) \( Q = 3\beta H\rho_{\text{tot}} \), the corresponding \( Q_1 = \frac{2}{3}x^{-1} \) and \( Q_2 = \frac{1}{3}\beta v^{-1} \). On the other hand, for Case (IV) \( Q = 3\eta H\rho_m \), the corresponding \( Q_1 = \frac{3}{2}x^{-1}v^2 \) and \( Q_2 = \frac{3}{2}\eta v \). Unfortunately, we find that there is no critical point in these two cases.

To study the stability of the critical points for the autonomous system Eqs. (57)—(60), we substitute linear perturbations \( x \rightarrow \bar{x} + \delta x \), \( y \rightarrow \bar{y} + \delta y \), \( u \rightarrow \bar{u} + \delta u \), and \( v \rightarrow \bar{v} + \delta v \) about the critical point \((\bar{x}, \bar{y}, \bar{u}, \bar{v})\) into the autonomous system Eqs. (57)—(60) and linearize them. Because of the Friedmann constraint (28), there are only three independent evolution equations, namely

\[
\delta x' = (\bar{s} - 3)\delta x + \bar{x}\delta s - \sqrt{\bar{\alpha}\bar{\xi}}\delta u - \sqrt{\frac{3}{2}}n\bar{y}\bar{u}^{-1}(2\delta y - \bar{y}\bar{u}^{-1}\delta u) - \delta Q_1, \tag{61}
\]

\[
\delta y' = \bar{s}\delta y + \bar{y}\delta s + \sqrt{\frac{3}{2}}n\bar{u}^{-1}(\bar{x}\delta y + \bar{y}\delta x - \bar{x}\bar{y}\bar{u}^{-1}\delta u), \tag{62}
\]

\[
\delta u' = \sqrt{6}\delta x, \tag{63}
\]

where \( \delta Q_1 \) is the linear perturbation coming from \( Q_1 \), and \( s, \delta s \) are given in Eqs. (47) and (48). The three eigenvalues of the coefficient matrix of Eqs. (61)—(63) determine the stability of the critical point.

For Case (I) \( Q = 0 \), the corresponding \( \delta Q_1 = 0 \). The three eigenvalues for both Points (P.I.1m) and (P.I.1p) are given by

\[
\left\{ -3\gamma, \frac{1}{2}\left[-3 + \sqrt{9 - 24\xi(n + 2)}\right], \frac{1}{2}\left[-3 + \sqrt{9 - 24\xi(n + 2)}\right] \right\}. \tag{64}
\]

So, both Points (P.I.1m) and (P.I.1p) exist and are stable under condition \( \xi > 0 \) and \(-2 < n \leq 0 \).

For Case (II) \( Q = \alpha\kappa\rho_m\phi \), the corresponding \( \delta Q_1 = \sqrt{6}\alpha(\bar{x}\delta u - \bar{x}\delta x - \bar{y}\delta y) \). The three eigenvalues for Points (P.II.1m) and (P.II.1p) are given by Eq. (59) and (61), respectively. Since their first eigenvalue \( 3\gamma/2 \) is positive, Points (P.II.1m) and (P.II.1p) are both unstable. So, although they are scaling solutions, however, they are not attractors and hence cannot alleviate the cosmological coincidence problem. On the other hand, the three eigenvalues for both Points (P.II.2m) and (P.II.2p) are given by Eq. (64). So, they exist and are stable under condition \( \xi > 0 \) and \(-2 < n \leq 0 \).

Since in both Case (III) \( Q = 3\beta H\rho_{\text{tot}} \) and Case (IV) \( Q = 3\eta H\rho_m \), there is no critical point, we need not perform the stability analysis for them.

So, for teleparallel dark energy with a power-law potential, in Case (I) \( Q = 0 \) there are two attractors (P.I.1m) and (P.I.1p) which are both dark-energy-dominated de Sitter solutions, and in Case (II) \( Q = \alpha\kappa\rho_m\phi \) there are two attractors (P.II.2m) and (P.II.2p) which are also dark-energy-dominated de Sitter solutions. Again, no scaling attractor has been found unfortunately.

V. CONCLUDING REMARKS

Recently, Geng et al. [15] proposed to allow a non-minimal coupling between quintessence and gravity in the framework of teleparallel gravity, motivated by the similar one in the framework of GR. They found that this non-minimally coupled quintessence in the framework of teleparallel gravity has a richer structure, and named it “teleparallel dark energy” [15]. In the present work, we note that there might be a deep and unknown connection between teleparallel dark energy and Elko spinor dark energy. Motivated by this observation and the previous results of Elko spinor dark energy [15, 22], we try to study the dynamics of teleparallel dark energy. We find that there exist only some dark-energy-dominated de Sitter attractors. Unfortunately, no scaling attractor has been found, even when we allow the possible interaction between teleparallel dark energy and matter. However, we note that \( w \) at the critical points is in agreement with observations (in particular, the fact that \( w = -1 \) independently of \( \xi \) is a great advantage).

Some remarks are in order. Firstly, in the present work we have chosen some particular potentials \( V(\phi) \) and interaction forms \( Q \). So, it is still possible to find some suitable and delicate potentials \( V(\phi) \) and interaction forms \( Q \) to obtain the scaling attractors of the most general dynamical system (51)—(54), and hence the hope to alleviate the cosmological coincidence problem still exists, although this is a fairly hard
task (see e.g. [22]). Of course, there also might be other smart methods, different from the usual method used in most of dark energy models, to alleviate the cosmological coincidence problem (see e.g. [31]). Secondly, we would like to briefly discuss the most general case. From the most general Eq. (33), we have $\bar{x} = 0$ at the critical points. From Eq. (28), it is easy to see that $1 + \xi \bar{u}^2 = y^2 + \bar{v}^2 \geq 0$. Therefore, from Eq. (30), we find that $\bar{s} > 0$ when $\bar{v} \neq 0$ (i.e., $\Omega_m \neq 0$, scaling solution), whereas $\bar{s} = 0$ when $\bar{v} = 0$ (i.e., $\Omega_m = 0$, dark-energy-dominated solution). From Eq. (37), we have $\bar{w}_\phi = -1 - (2/3) \xi \bar{s} \bar{u}^2 / (y^2 - \xi \bar{u}^2)$. From Eq. (36), $y^2 - \xi \bar{u}^2 = \Omega_\phi \geq 0$. Therefore, in the case of $\bar{s} = 0$ when $\bar{v} = 0$ (i.e., $\Omega_m = 0$, dark-energy-dominated solution), we always have $\bar{w}_\phi = -1$ and then $\bar{w}_{\text{tot}} = -1$, regardless of $\xi$. In the case of $\bar{s} > 0$ when $\bar{v} \neq 0$ (i.e., $\Omega_m \neq 0$, scaling solution), if $\xi \leq 0$ we have $\bar{w}_\phi \geq -1$, and if $\xi > 0$ we obtain $\bar{w}_\phi < -1$. Therefore, teleparallel dark energy being quintessence-like ($w_\phi > -1$) or phantom-like ($w_\phi < -1$) at the scaling attractors (if any) heavily depends on the sign of $\xi$. If $\xi \leq 0$, our universe can avoid the fate of big rip. Thirdly, strictly speaking, for $\xi > 0$ the phase space is not compact, namely one could also investigate possible critical points at infinity (i.e., when $\phi$ and $\dot{\phi}$ diverge) [35]. Since this issue has been discussed in detail by Xu et al. [36] (which appeared in arXiv after our submission), we do not consider it any more. Finally, as mentioned by Geng et al. [15], teleparallel dark energy has some interesting features, and we consider that this model still deserves further investigations.

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