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Finite-time stabilization of chaotic gyros based on a homogeneous supertwisting-like algorithm

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Abstract. This paper presents a finite-time stabilization scheme for nonlinear chaotic gyros. The scheme utilizes a supertwisting-like continuous control algorithm for the systems of dimension more than one with a Lipschitz disturbance. The algorithm yields finite-time convergence similar to that produces by discontinuous sliding mode control algorithms. To design the controller, the nonlinearities in the gyro are treated as a disturbance in the system. Thanks to the dissipativeness of chaotic systems, the nonlinearities also possess the Lipschitz property. Numerical results are provided to illustrate the effectiveness of the scheme.

1. Introduction

Gyros are important components in navigational, aeronautical and space systems. However, gyros can exhibit chaotic behavior, which will adversely affect the performance of the systems [1]. Recently, many authors have discussed the problem of controlling the chaotic gyros [2–4].

Chaotic control is challenging and has been extensively studied in the past decades. The study can be classified into chaos stabilization and chaos synchronization. Chaos stabilization attempts to eliminate the chaotic behavior of systems while chaos synchronization is to control a chaotic system so that it follows another chaotic system. Since Ott et al. [5] introduced the idea of controlling chaotic systems, a number of approaches have been proposed to control the chaotic systems [6–8]. Finite-time control has recently gained the attention of the chaos control community [9–11]. The main attractions of the finite-time control include faster convergence, robustness, and disturbance rejection.

Sliding mode control (SMC) has attracted interest in nonlinear control because of its numerous attractive properties. The properties include robustness against uncertainties and external disturbances, a fast dynamic response and a simple design. However, chattering phenomenon is one of the major obstacles for the implementation of SMC in a wide range of applications. A number of solutions have been proposed to eliminate or reduce chattering, including the boundary layer technique [12] and high-order SMC (HOSMC) methods [13, 14]. The supertwisting algorithm is one of effective HOSMC methods. HOSMC has recently attracted interest in chaos control [15–17].

In this paper, we examine the problems of stabilization of nonlinear chaotic gyros based on the HOSMC. To achieve this goal, a recently established homogeneous supertwisting-like modification of twisting algorithm [18] is employed. The algorithm is highly effective for the stabilization of both system state and its derivative in a finite time.
The rest of the paper is organized as follows. Preliminaries are presented in Section 2. Control design is described in Section 3. In Section 4, numerical results are given to illustrate the effectiveness of the control scheme. Finally, conclusions are drawn in the last section.

2. Preliminaries

2.1. Chaotic gyro

Consider the symmetric gyro mounted on a vibrating base as shown in Figure 1. Here, (X, Y, Z) and (x, y, z) are the reference and body frames, respectively. The dynamics of the gyro is described by z-x-z Euler angles: $\phi$ (precession), $\theta$ (nutation), $\psi$ (spin) and body rates: $\omega_x$, $\omega_y$, $\omega_z$. The rotational sequence of the Euler angles, starting by assuming (X, Y, Z) and (x, y, z) coincide, is listed as [19]:

1. Rotation about Z axis through angle $\phi$ to produce $(x', y', z')$ frame.
2. Rotation about $x'$ axis through angle $\theta$ to produce $(x'', y'', z'')$ frame.
3. Rotation about $z''$ axis through angle $\psi$ to produce (x, y, z) frame.

In the figure, $M_g$ is the gravity force, $l$ is the distance of the centre of gravity (CG) of the gyro from the origin, $\tau_m$ is the external torque applied to the gyro and $\bar{l} \sin(\omega t)$ is the motion of the vibrating base.

The kinetic energy of the gyro can be expressed as:

$$T = \frac{1}{2} I_1 (\omega_x^2 + \omega_y^2 + \omega_z^2) + \frac{1}{2} I_3 \omega_z^2$$

(1)

where $I_1$ and $I_3$ are the polar and equatorial moments of inertia of the gyro, respectively. Based on the z-x-z Euler angle transformation, the body rates can be expressed in terms of Euler rates $\dot{\phi}$, $\dot{\theta}$, $\dot{\psi}$ as [19]:

![Figure 1. A symmetric gyroscope mounted on a vibrating base.](image-url)
\[
\omega_x = \left( \dot{\phi} \sin \theta \sin \psi + \dot{\theta} \cos \psi \right), \\
\omega_y = \left( \dot{\phi} \sin \theta \cos \psi - \dot{\theta} \sin \psi \right), \\
\omega_z = \left( \dot{\phi} \cos \theta + \dot{\psi} \right).
\]

By substituting equations (2), (3) and (4) in equation (1), the kinetic energy becomes
\[
T = \frac{1}{2} I_1 \left( \dot{\phi}^2 \sin^2 \theta + \dot{\theta}^2 \right) + \frac{1}{2} I_3 \left( \dot{\phi} \cos \theta + \dot{\psi} \right)^2.
\]
The potential energy of the gyro can be expressed as:
\[
V = M \left( l + I \sin \omega \right) \cos \theta,
\]
Thus, the Lagrangian of the system can be expressed as [20]:
\[
L = T - V = \frac{1}{2} I_1 \left( \dot{\phi}^2 \sin^2 \theta + \dot{\theta}^2 \right) + \frac{1}{2} I_3 \left( \dot{\phi} \cos \theta + \dot{\psi} \right)^2 - M \left( l + I \sin \omega \right) \cos \theta.
\]
Since \( \phi \) and \( \psi \) do not appear explicitly in the Lagrangian, they are cyclic coordinates. Thus, the momentum integrals:
\[
P_\phi = \frac{\partial L}{\partial \dot{\phi}} = I_1 \dot{\phi} \sin^2 \theta + I_3 \left( \dot{\phi} \cos \theta + \dot{\psi} \right) \cos \theta, \\
P_\psi = \frac{\partial L}{\partial \dot{\psi}} = I_3 \left( \dot{\phi} \cos \theta + \dot{\psi} \right),
\]
are conserved. Then,
\[
I_1 \dot{\phi} \sin^2 \theta + I_3 \left( \dot{\phi} \cos \theta + \dot{\psi} \right) \cos \theta = \beta_\phi,
\]
\[
I_3 \left( \dot{\phi} \cos \theta + \dot{\psi} \right) = I_3 \dot{\psi} = \beta_\psi,
\]
where \( \beta_\phi \) and \( \beta_\psi \) are constants. From equations (10) and (11), the expressions for \( \dot{\phi} \) and \( \dot{\psi} \) can be written as:
\[
\dot{\phi} = \frac{\beta_\phi - \beta_\psi \cos \theta}{I_1 \sin^2 \theta},
\]
\[
\dot{\psi} = \frac{\beta_\psi}{I_3} - \cos \theta \left( \frac{\beta_\phi - \beta_\psi \cos \theta}{I_1 \sin^2 \theta} \right).
\]
The Routhian of the system is expressed as [20]:
\[
R = L - \beta_\phi \dot{\phi} - \beta_\psi \dot{\psi}.
\]
Substitute equations (7), (12) and (13) in equation (14) yields
\[
R = \frac{1}{2} I_1 \dot{\phi}^2 - \left[ \frac{(\beta_\phi - \beta_\psi \cos \theta)^2}{2I_1 \sin^2 \theta} + \frac{\beta_\psi^2}{2I_3} + M \left( l + I \sin \omega \right) \cos \theta \right].
\]
By assuming the torque \( \tau_m \) comprises the linear-plus-cubic dissipative torque \( \tau_d = -D_1 \dot{\theta} - D_2 \dot{\theta}^3 \) and the control torque \( \tau_c \), the equation of motion can be expressed in the coordinate of \( \theta \) as:
\[
\frac{d}{dt} \left( \frac{\partial R}{\partial \dot{\theta}} \right) - \frac{\partial R}{\partial \theta} = -D_1 \dot{\theta} - D_2 \dot{\theta}^3 + \tau_c
\]
where \( D_1 \) and \( D_2 \) are constants. Substituting (15) in (16) yields
\[ I_1 \ddot{\theta} + \frac{\beta_\theta - \beta_v \cos \theta}{I_1 \sin^3 \theta} \dot{\theta} - \beta_v \dot{\theta}^3 + \tau_c + M(\dot{\theta} - \dot{\theta}_0) = -D_1 \dot{\theta} - D_2 \dot{\theta}^3 + \tau_c \]  

(17)

Note that, from equations (10) and (11), \( \beta_\theta = \beta_v \) when \( \theta = 0 \). Since \( \beta_\theta \) and \( \beta_v \) are constants, the equality \( \beta_\theta = \beta_v \) is true for all \( \theta \). By using \( \beta_\theta = \beta_v \), equation (17) becomes

\[ \ddot{\theta} + \frac{\beta_v^2 (1 - \cos \theta)^2}{I_1^2 \sin^3 \theta} + \frac{D_1}{I_1} \dot{\theta} + \frac{D_2}{I_1} \dot{\theta}^3 - \frac{M}{I_1} \sin \theta = \frac{M}{I_1} \dot{\theta}_0 (\sin \alpha \theta)(\sin \theta) + \frac{1}{\alpha I_1} \tau_c. \]

(18)

By expressing equation (18) in a state equation form as in [1], we obtain:

\[ \begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= -\alpha \left( \frac{1 - \cos x_1}{\sin^3 x_1} \right) - c_1 x_2 - c_2 x_2^3 + \beta \sin x_1 + (f \sin \omega t)(\sin x_1) + u
\end{align*} \]

(19)

where \( x_1 = \theta \), \( x_2 = \dot{\theta} \) are the state variables, \( u = \tau_c / I_1 \) is the control input, \( \alpha = \beta_\theta / I_1 = I_0 \omega_0 / I_1 \), \( c_1 = D_1 / I_1 \), \( c_2 = D_2 / I_1 \), \( \beta = M \dot{\theta}_0 / I_1 \), and \( f = M I_1 / I_1 \) are positive constants. When \( 32 < f < 36 \), \( \alpha = 10 \), \( c_1 = 0.5 \), \( \beta = 1 \), \( c_2 = 0.05 \), and \( \omega = 2 \), the gyro with zero input becomes chaotic [1]. An example of the chaotic behaviour for \( f = 35.5 \) and the initial conditions \( (x_1(0), x_2(0)) = (1, -1) \) is shown in figure 2. Note that all trajectories are bounded since chaotic systems are dissipative.

2.2. Homogeneous supertwisting-like algorithm

Consider the dynamical system:

\[ \begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= \rho(t) + u
\end{align*} \]

(20)

where \( x_1 \) and \( x_2 \) are the state variables, \( u \) is the control input and \( \rho(t) \) is the disturbance. It is assumed that \( \rho(t) \) satisfies the Lipschitz condition with constant \( L \).

Figure 2. Chaotic behaviour: (a) Phase trajectory and (b) Time history
A homogeneous continuous supertwisting-like algorithm for systems of dimension more than one has recently been introduced in [18]. The algorithm generalizes the well-known supertwisting algorithm to higher dimensional systems. It yields finite-time convergence similar to that produced by discontinuous sliding mode control algorithms.

Based on the homogeneous continuous supertwisting-like algorithm [18], the control law for systems of dimension two with a Lipschitz disturbance is given as:

\[
    u = -\lambda_0 \left| \int_0^t x_1(s) ds \right|^{1/4} \text{sgn} \left( \int_0^t x_1(s) ds \right) - \lambda_1 |x_1(t)|^{1/3} \text{sgn}(x_1(t)) - \lambda_2 |x_2(t)|^{1/2} \text{sgn}(x_2(t)) - \alpha_c \left| \int_0^t \text{sgn}(x_2(s)) ds \right|
\]  

(21)

where \( \lambda_0, \lambda_1, \lambda_2 > 0 \) and \( \alpha_c > 0 \) are control parameters. The control law (21) yields finite-time convergence of both state variables \( x_1 \) and \( x_2 \) to the origin if the following conditions hold for the control parameters [18]:

\[
    \alpha_c > L \quad \text{and} \quad \lambda_2^2 > 2(c_L + L)^2 / (\lambda_2 - L).
\]

(22)

3. Control design

The control objective is to suppress the chaos governed by equation (19) by driving the state variables \( x_1 \) and \( x_2 \) to the origin in finite time. To achieve the objective, we define the control law as:

\[
    u = c_1 x_2 + c_2 x_2^3 + u_{st}.
\]

(23)

where \( u_{st} \) adopts the supertwisting-like control law (21).

Substituting the control law (23) into equation (19) yields

\[
    \dot{x}_1 = x_2 \\
    \dot{x}_2 = -\alpha_c^2 \frac{(1 - \cos x_1)^2}{\sin^3 x_1} + \beta \sin x_1 + (f \sin \omega t)(\sin x_1) + u_{st}.
\]

(24)

The system (24) can be written in the form (20) by defining:

\[
    \rho(t) = -\alpha_c^2 \frac{(1 - \cos x_1)^2}{\sin^3 x_1} + \beta \sin x_1 + (f \sin \omega t)(\sin x_1).
\]

(25)

In order to adopt the control law (21), \( \rho(t) \) must possess the Lipschitz property. A sufficient condition for a function to be Lipschitz is its derivative being bounded. Thus, we examine the derivative of \( \rho(t) \) instead.

By taking the derivative of \( \rho(t) \) with respect to time, it yields

\[
    \rho(t) = \alpha_c^2 \dot{x}_1 \left[ \frac{2 \cos x_1 - 1}{\sin^2 x_1} - \frac{3 \cos x_1}{\sin^4 x_1} \right] + \beta \dot{x}_1 \cos x_1 + \omega f \cos \omega t (\sin x_1) \\
    + f \dot{x}_1 (\sin \omega t) (\cos x_1) \\
    = \alpha_c^2 \dot{x}_2 \left[ \frac{2 \cos x_1 - 1}{\sin^2 x_1} - \frac{3 \cos x_1}{\sin^4 x_1} \right] + \beta \dot{x}_2 \cos x_1 + \omega f \cos \omega t (\sin x_1) \\
    + f \dot{x}_2 (\sin \omega t) (\cos x_1)
\]  

(26)

5
Since chaotic systems are dissipative, $x_1$ and $x_2$ are bounded. It is obvious that $\rho(t)$ can become unbounded only if \( \frac{2[\cos x_i - 1]}{\sin^2 x_i} \) and \( \frac{3\cos x_i [\cos x_i - 1]^2}{\sin^4 x_i} \) have a singularity. In this case, the singularity can occur only at $x_i = 0$. However, since $0 \leq x_i \leq 75$, respectively. By examining the value of
\[
\lim_{x_i \rightarrow 0} \left\{ \frac{2[\cos x_i - 1]}{\sin^2 x_i} \right\} = -1
\]
and
\[
\lim_{x_i \rightarrow 0} \left\{ \frac{3\cos x_i [\cos x_i - 1]^2}{\sin^4 x_i} \right\} = 0.75 ,
\]
$\rho(t)$ has no singularity. Thus, we can conclude that $\rho(t)$ is bounded (i.e., $\max |\rho(t)| \leq L$, $0 < L < \infty$). Therefore, $\rho(t)$ is Lipschitz with the constant $L$.

4. Numerical simulations

In this section, numerical simulations are utilized to illustrate the effectiveness of the supertwisting-like algorithm-based control scheme to suppress the chaos of chaotic gyroscopes. The parameters of the gyroscopes are set as follows: $a=10$, $b=1$, $c_1=0.5$, $c_2=0.05$, $\omega = 2$ and $f = 35.5$. The fourth-order Runge-Kutta method with a time step of 0.001 second was used in all simulations.

First, the Lipschitz constant $L$ is estimated. From equation (26), we obtain
\[
\max |\rho(t)| = \max \left\{ \alpha^2 x_2 \left[ \frac{2[\cos x_i - 1]}{\sin^2 x_i} \right] + \frac{3\cos x_i [\cos x_i - 1]^2}{\sin^4 x_i} \right\} + \beta \max |x_2| + \omega \max |\cos x_i|
\]
\[
\leq \alpha^2 \max |x_2| + \frac{3\cos x_i [\cos x_i - 1]^2}{\sin^4 x_i} \right\} + \beta \max |x_2| + \omega \max |\cos x_i|
\]
\[
\leq \alpha^2 \max |x_2| + \frac{3\cos x_i [\cos x_i - 1]^2}{\sin^4 x_i} \right\} + \beta \max |x_2| + \omega
\]
\[
= \alpha^2 X_2 R_{\max} + (\beta + \omega) X_2 + \omega
\] (27)

where $X_2$ and $R_{\max}$ are upper bounds of $|x_2|$ and $R(t) = \left[ \frac{2[\cos x_i - 1]}{\sin^2 x_i} \right] + \frac{3\cos x_i [\cos x_i - 1]^2}{\sin^4 x_i}$, respectively.

From figure 2(b), it can be found that $X_2 \leq 4$. By examining the value of $R(t)$ over $x_i \in (-\frac{\pi}{2}, \frac{\pi}{2})$ (see figure 3), it results in $R_{\max} \leq 2.1$. From equation (27), it yields $\max |\rho(t)| \leq 1057$. Here, the Lipschitz constant $L = 1100$ is chosen.
Next, the parameters $\alpha_c$, $\lambda_0$, $\lambda_1$, and $\lambda_2$ of $\mu_i$ are selected. Here, $\alpha_c = 1200$ and $\lambda_2 = 400$ were taken satisfying the condition (22) and $\lambda_0 = 200$ and $\lambda_1 = 100$ were chosen such that the desired transient response of the control system is achieved.

Simulation results are depicted in figure 4. The initial condition of the system was set as $(x_1(0), x_2(0)) = (1, -1)$. The controller was activated at time equal to 5 sec. The result shows that the state variables $x_1$ and $x_2$ converged to the origin in less than 0.5 sec after the controller was activated.

![Figure 3. Plot of $R(t)$ versus $x_1$.](image)

![Figure 4. State variables and control input of the controlled chaotic gyro.](image)
In addition, a comparison with a sliding mode controller based on the twisting algorithm (TA) [14, 15] was made. The twisting algorithm control law was given as:

\[ u(t) = -k_1 \text{sign}(x_1(t)) - k_2 \text{sign}(x_2(t)) \]  

(28)

where \( k_1, k_2 > 0 \) were control parameters. The parameters \( k_1 = 100, k_2 = 50 \) were chosen such that the control system had a similar response to the supertwisting-like algorithm-based control system (i.e., \( x_i \) and \( x_s \) converged to the origin within about 0.5 sec). A simulation result is depicted in figure 5.

From figures 4 and 5, both controllers (23) and (28) yield similar responses of the state variables, whereas the controller (23) provides markedly smaller chattering control signal. This illustrates the benefit of the supertwisting-like algorithm.

![Figure 5. State variables and control input of the controlled chaotic gyro using TA.](image)

5. Conclusions

In this paper, a finite time control scheme based on a recently developed homogeneous supertwisting-like continuous control algorithm is proposed for stabilization of chaotic gyro. Numerical simulations have illustrated that the proposed control scheme can be effectively applied to a chaos stabilization problem. By comparing with the twisting algorithm based control scheme, both control schemes provide similar control responses, whereas the proposed scheme results in smaller chattering control signal.

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