Short Proofs of Linear Growth of Quantum Circuit Complexity

Zhi Li

1Perimeter Institute for Theoretical Physics, Waterloo, Ontario N2L 2Y5, Canada

The complexity of a quantum gate, defined as the minimal number of elementary gates to build it, is an important concept in quantum information and computation. It is shown recently that the complexity of quantum gates built from random quantum circuits almost surely grows linearly with the number of building blocks. In this article, we provide two short proofs of this fact. We also discuss a discrete version of quantum circuit complexity growth.

I. INTRODUCTION

The complexity of a quantum gate, defined as the minimal number of elementary gates to build it, is an important concept in quantum information and computation. It is shown recently that the complexity of quantum gates built from random quantum circuits almost surely grows linearly with the number of building blocks.

In this article, we provide two short proofs of this fact. We also have

\[ U^B(k_0) \subseteq U^B(k_0 + 1) \subseteq U^B(k_0) \Rightarrow U^B(k_0) = U^B(k_0 + 1), \]

and

\[ U^B(k_0 + 2) = U^B(1)U^B(k_0 + 1) \subseteq U^B(1)U^B(k_0) \]

\[ \subseteq U^B(1)U^B(k_0) = U^B(k_0 + 1). \]

Iteratively, we get

\[ U^B(k_0) = U^B(k_0 + 1) = U^B(k_0 + 2) = \cdots = U^B(K). \]

This contradicts the definition of \( K \).

This means that, for \( \forall k < K \), there exists a \( g \in U^B(k + 1) \) such that \( g \notin U^B(k) \). By definition of the Zariski topology, there exist a polynomial \( p \) on \( SU(2^n) \) such that \( p = 0 \) on \( U^B(k) \) but \( p(g) \neq 0 \). By definition of \( U^B(k + 1) \),

\[ \exists x \in SU(4)^{2k+1|B|}, p(F_{k+1}(x)) \neq 0. \]

where \( F_{k+1} \) is the construction map in Eq. (1), \( p \circ F_{k+1} \) being a nonzero polynomial on \( SU(4)^{2k+1|B|} \), its zero point set must have zero measure. Hence for almost all \( x \), \( p(F_{k+1}(x)) \neq 0 \) and therefore \( F_{k+1}(x) \notin U^B(k) \). In other words, almost all \( n \)-qubit gates generated with \( k+1 \) blocks cannot be generated using \( k \) blocks.

Remark: This proof can be modified to the state complexity growth by considering the space of states after \( k \) blocks.

II. FIRST PROOF

We denote the space of \( n \)-qubit gates after \( k \) blocks as \( U^B(k) \). It is the image of a construction map

\[ F_k : SU(4)^{k|B|} \rightarrow SU(2^n). \]

\( U^B(k) \), as a semialgebraic set, can be decomposed as disjoint union of smooth manifolds, and has a well-behaved dimension theory \([4, 5]\). Denote \( d^B(k) \) to be its dimension.

Define \( K \) to be the minimal integer such that \( U^B(K) = SU(2^n) \). Here the closure is taken in the Zariski topology (regarding \( SU(2^n) \) as an affine variety in \( \mathbb{R}^{2^{2n+1}} \)). The (algebraic or geometric) dimension of \( U^B(k) \) is the same as the dimension of \( U^B(k) \) \([3]\). By counting degrees of freedom,

\[ d^B(k) \leq 15k|B|. \]

Hence \( k \) is at least exponential in \( n \).

We claim that \( U^B(k + 1) \notin U^B(k) \) for \( \forall k < K \). If not, assuming \( U^B(k_0 + 1) \subseteq U^B(k_0) \) for some \( k_0 < K \), we have

\[ U^B(k_0) \subseteq U^B(k_0 + 1) \subseteq U^B(k_0) \Rightarrow U^B(k_0) = U^B(k_0 + 1), \]

and

\[ U^B(k_0 + 2) = U^B(1)U^B(k_0 + 1) \subseteq U^B(1)U^B(k_0) \]

\[ \subseteq U^B(1)U^B(k_0) = U^B(k_0 + 1). \]

Iteratively, we get

\[ U^B(k_0) = U^B(k_0 + 1) = U^B(k_0 + 2) = \cdots = U^B(K). \]

This contradicts the definition of \( K \).

This means that, for \( \forall k < K \), there exists a \( g \in U^B(k + 1) \) such that \( g \notin U^B(k) \). By definition of the Zariski topology, there exist a polynomial \( p \) on \( SU(2^n) \) such that \( p = 0 \) on \( U^B(k) \) but \( p(g) \neq 0 \). By definition of \( U^B(k + 1) \),

\[ \exists x \in SU(4)^{2k+1|B|}, p(F_{k+1}(x)) \neq 0. \]

where \( F_{k+1} \) is the construction map in Eq. (1), \( p \circ F_{k+1} \) being a nonzero polynomial on \( SU(4)^{2k+1|B|} \), its zero point set must have zero measure. Hence for almost all \( x \), \( p(F_{k+1}(x)) \neq 0 \) and therefore \( F_{k+1}(x) \notin U^B(k) \). In other words, almost all \( n \)-qubit gates generated with \( k+1 \) blocks cannot be generated using \( k \) blocks.

Remark: This proof can be modified to the state complexity growth by considering the space of states after \( k \) blocks.

III. SECOND PROOF

This proof uses the observation in Ref. [4] that the dimension \( d^B(k) \) is a proxy for the complexity.

We note the following facts:

- \( d^B(k) \) is non-decreasing in \( k \);
- A \((k_1 + k_2)\)-block circuit can be decomposed as two circuits with \( k_1 \) and \( k_2 \) blocks, hence

\[ d^B(k_1 + k_2) \leq d^B(k_1) + d^B(k_2); \]

where \( X \sqcup Y = \bigcup_{x \in X} xY = \bigcup_{x \in X} xY \subseteq \bigcup_{x \in X} XY = XY \).

1 The second \( \subseteq \) is valid for all subsets \( X, Y \) in a topological group.
• $O(2^n)$ two-qubits gates are enough to synthesize all $n$-qubit gates \cite{3}. Therefore, as long as $\mathcal{U}(k)$ contains all two-qubit gates,

$$d^B(2^{2n+c_1}) \geq \dim \mathbb{SU}(2^n) = 2^{2n} - 1.$$  \hspace{1cm} (8)

Here $c_1$ (and later $c_2$ and $c$) is a constant.

From these facts we can deduce a lower bound of $d^B(k)$ for all $k \leq 2^n$:

$$d^B(k) \geq d^B(2^r) \geq \frac{d^B(2^{r+1})}{2} \geq \cdots \geq \frac{d^B(2^{2n+c_1})}{2^{2n+c_1-r}} > c_2 k.$$  \hspace{1cm} (9)

Here $r = \lfloor \log_2 k \rfloor$.

On the other hand, define $\mathcal{U}(s)$ as the space of $s$-qubit gates built from arbitrary $s$ 2-qubit gates in all possible structures. Denote $d(s)$ to be its dimension. Similar to Eq. \ref{eq:2}, we know $d(s) \leq 15s$. Then we have:

$$d^B(k) > 15 \frac{c_2 k}{15} \geq d(\lfloor ck \rfloor).$$  \hspace{1cm} (10)

This means, at step $k$, those “short-cut” gates with no larger than $\lfloor ck \rfloor$ gate complexity is subdimensional in $\mathcal{U}(k)$.

Let us decompose $\mathbb{SU}(4) = R \cup R^c$, where $R$ is the set of regular points where $F$ has maximal rank; its complement $R^c$ is the set of critical points. Then $F_k^{-1}(\mathcal{U}(\lfloor ck \rfloor))$ can be decomposed as

$$(F_k^{-1}(\mathcal{U}(\lfloor ck \rfloor)) \cap R^c) \cup (F_k^{-1}(\mathcal{U}(\lfloor ck \rfloor)) \cap R).$$  \hspace{1cm} (11)

$R^c$ has measure 0 as a subvariety, since the rank can be characterized by minor determinants which are polynomial functions. The second term also has measure 0 since locally they are (unions of) subdimensional manifolds. Therefore, the preimage of $\mathcal{U}(\lfloor ck \rfloor)$ has measure 0.

In other words, almost all gates at step $k$ have greater than $\lfloor ck \rfloor$ gate complexity.

**Remark:** For fact Eq. \ref{eq:3} to hold, $|B| = O(n)$ is enough. For example, we can use some SWAP gates to convert a gate acting on qubits $i$ and $j$ to a gate acting on qubits 1 and 2.

\section{IV. DISCUSSIONS}

Besides synthesizing $n$-qubit gates by 2-qubit gates drawn from $\mathbb{SU}(2)$, one can also draw 2-qubit gates from a discrete, universal gate set (for example, Clifford and $T$). In this setting, there is also a version of quantum circuit complexity growth.

More precisely, denote $g_k$ as the resulting gate by randomly applying some gates from a fixed universal gate set $S$ (assuming $S = S^{-1}$ without loss of generality) for $k$ times: $g_k = h_1 h_2 \cdots h_k$, $h_i \in S$. Define the complexity $C(g_k)$ as the minimal number of gates to build $g_k$ from gates in $S$. Then there exist a constant $c > 0$ such that

$$\lim_{k \to \infty} \frac{C(g_k)}{k} = c \text{ almost surely.}$$  \hspace{1cm} (12)

Note that different from the continuous version, the discrete complexity grows forever.

This statement is a known mathematical fact from the study of random walks on groups. A proof sketch is as follows (see Ref. \cite{14} for a survey on terminologies used here). The complexity satisfies a subadditive relation similar to Eq. \ref{eq:7}:

$$C(g_{k_1 + k_2}) \leq C(g_{k_1}) + C(g_{k_2}).$$  \hspace{1cm} (13)

By Kingman’s subadditive ergodic theorem, $\lim_{k \to \infty} \frac{C(g_k)}{k} = c$ almost surely, where $c$ is a circuit independent constant ($c \geq 0$ for now). Similarly, $\lim_{k \to \infty} \Pr(\rho = g_{2k}) = \rho$ exists. It can be proved that $\rho < 1$ implies $c > 0$. By Kesten’s theorem, $\rho < 1$ if and only if $\mathbb{S}$, the subgroup generated by $S$, is nonamenable. In our setting, $\langle S \rangle$ is a dense subgroup of $\mathbb{SU}(2^n)$, so by Tits alternative and Ref. \cite{15} (or Ref. \cite{16}) it is indeed nonamenable.

It must be emphasized that the above-defined complexities (both the continuous and the discrete version) are exact complexities. The case of the approximate complexity, where gate synthesis can be approximate yet the complexity is still conjectured to grow linearly, is a more difficult question since it requires understanding how gates distribute on $\mathbb{SU}(2^n)$ \cite{10}.

\section{ACKNOWLEDGMENTS}

The author thanks Timothy H. Hsieh, Cheng-Ju Lin, Zi-Wen Liu, and Beni Yoshida for helpful discussions. Research at Perimeter Institute is supported in part by the Government of Canada through the Department of Innovation, Science and Economic Development and by the Province of Ontario through the Ministry of Colleges and Universities.

\begin{thebibliography}{99}

\bibitem{1} Ethan Bernstein and Umesh Vazirani, \textit{“Quantum complexity theory,” SIAM Journal on Computing \textbf{26}, 1411–1473 (1997)}
\end{thebibliography}
[2] Douglas Stanford and Leonard Susskind, “Complexity and shock wave geometries,” Phys. Rev. D 90, 126007 (2014).
[3] Adam R. Brown and Leonard Susskind, “Second law of quantum complexity,” Phys. Rev. D 97, 086015 (2018).
[4] Jonas Haferkamp, Philippe Faist, Naga BT Kothakonda, Jens Eisert, and Nicole Yunger Halpern, “Linear growth of quantum circuit complexity,” Nature Physics, 1–5 (2022).
[5] Jacek Bochnak, Michel Coste, and Marie-Françoise Roy, Real algebraic geometry, Vol. 36 (Springer Science & Business Media, 2013).
[6] Juha J. Vartiainen, Mikko Möttönen, and Martti M. Salomaa, “Efficient decomposition of quantum gates,”
[7] Tianyi Zheng, “Asymptotic behaviors of random walks on countable groups,” (2021).
[8] Emmanuel Breuillard and Tsachik Gelander, “On dense free subgroups of Lie groups,” Journal of Algebra 261, 448–467 (2003).
[9] Joseph Max Rosenblatt, “Invariant measures and growth conditions,” Transactions of the American Mathematical Society 193, 33–53 (1974).
[10] Fernando G.S.L. Brandão, Wissam Chemissany, Nicholas Hunter-Jones, Richard Kueng, and John Preskill, “Models of quantum complexity growth,” PRX Quantum 2, 030316 (2021).