Observation of quasiperiodic dynamics in a one-dimensional quantum walk of single photons in space

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Abstract
We realize the quasi-periodic dynamics of a quantum walker over 2.5 quasi-periods by realizing the walker as a single photon passing through a quantum-walk optical-interferometer network. We introduce fully controllable polarization-independent phase shifters in each optical path to realize arbitrary site-dependent phase shifts, and employ large clear-aperture beam displacers, while maintaining high-visibility interference, to enable 10 quantum-walk steps to be reached. By varying the half-wave-plate setting, we control the quantum-coin bias thereby observing a transition from quasi-periodic dynamics to ballistic diffusion.

Keywords: quantum walk, quasi-periodic dynamics, position-dependent phase defect

1. Introduction

The quantum walk (QW) [1, 2] is a quantized version of the ubiquitous random walk (RW) used for describing diffusion [3], for probabilistic algorithms [4] to solve constraint satisfaction problems in computer science [5], for quantum transport in complex systems [6] and for
demonstrating intriguing nonlinear dynamical quantum phenomena [7–9]. In its basic formulation a walker moves on an integer lattice with its periodic spatial sites labelled by integers \( x \). This one-dimensional line is regarded as being arbitrarily long and the walker commences at the origin \( x = 0 \). The walker carries a two-sided coin with the two sides labeled \( c \in \{ 0, 1 \} \).

The evolution of the walker proceeds as follows: the walker flips the coin to obtain an outcome \( c \) and then makes one step in the positive or negative direction on the line if the coin flip yields the outcome 0 or 1, respectively. The randomness of the coin flip leads to a diffusion rate that increases as \( \sqrt{t} \) with \( t \) the time of evolution (which we treat as a non-negative integer incrementing by one unit per step from \( t = 0 \)); this square-root dependence is characteristic of diffusive spreading, with the ‘spread’ signifying the width of the position distribution \( P(x) \) at time \( t \).

The QW eliminates random evolution by trading the coin for a quantum two-level system, which, in our case, is the polarization state of a single photon: horizontal (H) or vertical (V). Furthermore, the QW employs unitary dynamics by which the walker’s position is entangled with the coin state. When the evolution is complete, the coin state is ignored, which mathematically corresponds to tracing out this degree of freedom.

For this evolution, the walker’s position distribution then spreads proportional to \( t \) rather than to \( \sqrt{t} \). The proportionality of the position spreading to \( t \) is reminiscent of constant-velocity deterministic motion, and is thus known as ‘ballistic transport’. QWs are widely studied experimentally and theoretically because of their applications to quantum algorithms [3] and quantum transport [6] and also for the intriguing foundational studies of quantum phenomena such as defect-induced localization [10, 11] and quantum resonances [12, 13].

2. Experimental implementation of quasiperiodic QW

2.1. QW with position-dependent phase defects

Mathematically each evolutionary ‘step’ given by unitary operator \( U \) on the joint walker-coin system is achieved by repeating two sequential unitary operations. The first is the unitary analogue of the coin flip achieved by transforming the coin states \( |c = 0\rangle \) and \( |c = 1\rangle \) to \( \cos \theta |0\rangle + \sin \theta |1\rangle \) and \( \sin \theta |0\rangle - \cos \theta |1\rangle \), respectively, with \( \theta \) the coin-bias parameter. The second (entangling) operation translates the walker’s position dependent on the coin state

\[
\sum_{c=0}^{1} \sum_{x=-\infty}^{\infty} e^{i \phi(x)} \left| x + (-1)^x \right\rangle \left\langle x \right| \otimes \left| c \right\rangle \left\langle c \right|.
\]

Typically \( \phi(x) \equiv 0 \) in theory [14–18] and experimentally [19–27], but controllably varying the phase enables the realizations of remarkable quantum-walk properties such as quantum recurrences in diffusion, which arise in the ‘harmonic case’

\[
\phi(x) = 2\pi xy
\]

for \( y = q/p \) with \( q, p \) co-prime integers. For irrational \( y \), the walker’s diffusion is suppressed forever, which is termed ‘dynamical localization’ [7]. Our experiment focuses on the complementary ‘harmonic case’ of rational \( y \).
2.2. Implementing QW in a photonic system

The walker is manifested experimentally as a single photon, and the lattice as a set of allowed paths for the photon. Pairs of 800 nm photons are generated via type-I spontaneous parametric down-conversion (SPDC) in two 0.5 mm thick nonlinear-barium-borate (BBO) crystals cut at 29.41°, which are pumped by a 100 mW 400 nm continuous-wave diode laser. Triggering on one photon prepares the other photon in the pair as a single-photon state. After spectral filtering with a full width at half maximum (FWHM) of 3 nm, individual down-converted photons are steered into the optical modes of the linear-optical network formed by a series of birefringent calcite beam displacers (BDs), half-wave plates (HWPs) and phase shifters (PSs). Output photons are detected using avalanche photo-diodes (APDs, 7 ns time window) with a dark count rate of less than $10^{-4}$ whose coincident signals—monitored using a commercially available counting logic—are used to postselect two single-photon events. The total coincident counts are around 20 000 over 40 s. The probability of creating more than one photon pair is less than $10^{-4}$ and can thus be neglected.

Optical interferometers comprising birefringent-crystal BDs, wave plates (WPs), and PSs serve as stable devices for simulating quantum information processes such as heralded coined QWs. The interferometer set-up is shown in figure 1(a). The 10-step QW is implemented with a single HWP, BD, and the PSs in each path signifying one possible site for the walker. The coin state corresponds to the polarization of the photon, which is produced by the SPDC creating correlated photon pairs at random times. One photon in the pair is the signal photon, which undergoes the QW, and the other is the trigger photon, which heralds the presence of the signal photon. Experimentally the QW dynamics are detected by photon coincidence measurements between the signal and the trigger photon.

Ten BDs, each with length 28.165 mm and a clear aperture of 33 mm × 15 mm, are placed in sequence and need to have their optical axes mutually aligned. Co-alignment ensures that beams split by one BD in the sequence yield maximum interference visibility after passing through an HWP and the next BD in the sequence. In our experiment, we attain interference visibility of 0.998 for each step, i.e. for each pair of sequential BDs.

Output photons from the interferometric network are coupled into a single-mode optical fibre and subsequently detected by an APD in coincidence with the trigger photon. We characterize the quality of the experimental QW by its 1-norm distance [2] from the simulated QW according to

$$\frac{1}{2} \sum_{x} |P_{\text{exp}}(x) - P_{\text{th}}(x)|$$

in our case this distance is 0.085 after 10 steps. The distance increases monotonically with each step number due to a lack of relative phase control between the multiple interferometers. This limited control is due to nonplanar optical surfaces. By placing 10 crystals in sequence, we are able to achieve 10 quantum-walk steps, thereby surpassing the previous limit of 8 quantum-walk steps in a similar interferometric set-up [27].

The symmetric initial coin state \((|0\rangle + i|1\rangle)/\sqrt{2}\) ensures a symmetric walker position distribution, and is prepared by sending the signal photon through a polarizing beam splitter followed by a HWP and subsequently by a quarter-wave plate (QWP). A birefringent BD steers a photon to two possible pathways 3 mm apart in a polarization-dependent way, which effects the coin-state-dependent walker translation but without incorporating the position-dependent phase function.

We introduce a PS in every path between every pair of BDs and thereby obtain full phase-controlled coin-state-dependent walker translation. Microscope slides (MSs) with a certain
effective thickness, shown in figure 1(b) are introduced as PSs into the interferometric network after first completing the alignment described above and ensuring maximum interference and small distance between the simulated and empirical walker distributions. The MSs are inserted and aligned to recover the case of zero phase shift $\phi = 0$ for all locations, $x$. This microscope-slide alignment corresponds to each slide being in the plane perpendicular to the beams.

After achieving this alignment, each MS can be adjusted to an effective thickness in order to impart a controllable phase shift, independent of all other PSs. This effective thickness is achieved by rotating the slide out of the plane perpendicular to the beams. Tilting the slide is a viable alternative, but is not as stable for the times required to gather the data.

The instability of the MSs can be seen from its setting in the holder shown in figure 1(b). Due to the nature of the apparatus, rotating the slide out of the plane perpendicular to the beams is a better way to realize a PS with controllable effective thickness. Although tilting MSs to

**Figure 1.** Experimental scheme for a 10-step QW with a position-dependent phase function. (a) Single photons are created via spontaneous parametric down-conversion in two $\beta$ barium borate (BBO) crystals. One photon in the pair is detected to herald the other photon, which is injected into the optical network. Arbitrary initial coin states are prepared by a polarizing beam splitter (PBS) and half-wave (HWP) and quarter-wave plates (QWP). The phase shifters (PS) are placed in separate transverse spatial modes, and optical compensators (OC) compensate their resultant temporal delays. The coincident detection of trigger and heralded photons at avalanche photodiodes (APDs) yields data for the QW. (b) The PS is realized by the microscope slide (MS) with adjustable effective thickness. This effective thickness is achieved by rotating the slide out of the plane perpendicular to the beams. (c) Quasi-energy spectra for the standard and quasi-periodic QWs, respectively.
achieve an additional phase is viable, the tilted slide would be easily influenced by the optical-table vibrations as well as by gravity.

As an example, we explain how to align the slides for the second quantum-walk step. In this case, there are two PSs because there are two longitudinal spatial modes after the first step. We rotate the two PSs separately, then gather the photon-count data. These data are compared to the theoretical simulation. If the data are not satisfactory with respect to the 1-norm distance of the walker distribution, we discard the data, adjust the PSs and repeat. The PSs are realized by MSs with an adjustable effective thickness in our experiment. A set of parameters for given polarization-independent phase shifts is achieved through post-selected processing. The MSs are adjusted carefully until the required additional phase shifts are achieved, which can be characterized by the 1-norm distance of the position distribution. When the data agree well with the theory, we retain the walker distribution results and then add the third step of the walker by adding another three PSs within the three longitudinal spatial modes, respectively, a HWP and BD, and repeat this procedure.

Our chief technical innovation in addition to reaching 10 steps is the development of fully controllable polarization-independent PSs that can be inserted into each optical path of a quantum-walk interferometer. These controllable PSs greatly increase the versatility of quantum-walk optical-interferometer networks as they enable generalized QWs with arbitrary phase functions $\phi(x)$. We insert PSs into each optical path and adaptively calibrate and adjust the phases through sequential tests until the alignments achieve the desired $\phi(x)$ within acceptable tolerances.

The effectiveness of these controllable PSs is demonstrated here by realizing quasi-periodic dynamics in a QW for the first time. Our quasi-periodic evolution has been achieved over 2.5 quasi-periods, which suffices to see an unambiguous experimental signature revealing the quasi-period.

Quasi-periodic dynamics arise through symmetry breaking due to the imposition of the position-dependent phase shift. This position-dependent phase shift leads to an effective periodic potential. For $q$ co-prime to $p$, the potential wells behave effectively as a family of $N/p$ clusters, and we henceforth let $p$ be even. For the quasi-energy spectrum being the argument of the eigenvalues of $U$, the clustering of wells leads to quasi-energy bands comprising $p$ quasi-energy levels, in each of which only one is doubly degenerate.

We show the quasi-energy spectra in figure 1(c) for $y = q/p$ with the standard quantum-walk case $q = 0$ shown on the left and the $q \neq 0$ case depicted on the right. Since we use a two-sided coin, we have two bands of Bloch levels, each consisting of $N$ spectral lines that are equally spaced for $q = 0$, and the width of each band is proportional to the tunneling amplitude $d = \cos \theta$. On the other hand, for even $p$ and $q \neq 0$, the spectral lines collect into $p$ Bloch bands with $2p - 1$ lines per band, and the bandwidth is proportional to the effective tunneling amplitude $d_{\text{eff}} := d^{p/2}$. Hence, $U^p$ is close to the identity, thereby ensuring a quasi-revival of the initial state after every $p$ step [7].

The quasi-energies $E_{\ell mn}$ of the quantum walker navigating a lattice with position-dependent phase shifts are given by $\arg(\lambda_{\ell mn})$ with $\lambda_{\ell mn}$ the eigenvalues of the unitary operator $U$ and $\ell = 0, 1, \ldots, N/p - 1$, $m = 0, 1$, and $n = 0, 1, \ldots, p - 1$ [7]. For even $p$, the eigenvalues are [7]
\[ \lambda_{\ell mn} = e^{2\pi i p} \left( 2ir_r \left( (-1)^{m} \sqrt{1 - r_r^2} - ir_r \right)^{\ell p} - 1 \right), \]  
for \( r_r = d^{p/2} \sin \left( \frac{\pi p \ell}{N} \right). \)

Based on equation (3), a degeneracy emerges for even \( p \), and the resulting system has \( p \) quasi-energy bands in total. Each of these quasi-energy bands comprises \( 2p \) levels, with two of them, namely \( E_{000} \) and \( E_{010} \), being degenerate. For example, in our case with \( p = 4 \), the resulting system has four energy bands, with each band comprising 7 quasi-energy levels and the width of the band proportional to \( d^2 \).

For small \( d \), the bands are narrow and the system is almost harmonic, so quasi-periodic dynamics are expected. For large \( d \), non-harmonic effects should emerge, and consequently a transition from quasi-periodic behaviour to ballistic spreading is expected.

We can see the quasi-periodicity arising in our theoretical simulation and from the experimental data depicted in figure 2 for \( y = 1/4 \), i.e. by setting \( q = 1 \) and \( p = 4 \). In figure 2(a), we see the experimental data for the walker distribution at each step \( t = 0, 1, 2, ..., 10 \) with \( \theta = \pi/3 \). The distribution narrows almost to the initial spike at \( x = 0 \) for \( t = 4 \) and \( t = 8 \), but has additional support for \( x \neq 0 \); i.e. the width of the position distribution has increased. In contrast, for the standard QW corresponding to any integer-valued \( y \), the probability distribution spreads monotonically with a width proportional to \( t \), as seen in the previous 6-step standard-quantum-walk experiment [2].

The theoretical versus experimental results can be seen in figure 2(b) for \( t = 8 \), which shows their close agreement and also the degree of spreading after two periods of quasi-periodic revival of the position distribution. Figures 2(a), (b) suggest a quasi-periodic revival of the position distribution with period \( \tau = 4 \), and figures 2(c), (d) enable quantitative analysis to determine \( \tau \) for various values of \( \theta \). The variance and recurrence probabilities \( P_r(x = 0) \) are shown in figures 2(c), (d), respectively. Each plot shows an unambiguous period of \( \tau = 4 \), as expected from our choice of \( p = 4 \), thereby confirming our numerical simulation and our experiment.

The coin bias \( \theta \) enables tuning between two extremal modes of behaviour. One extremal mode is the adiabatic limit corresponding to a truly periodically varying walker position distribution, with a tendency of trapping at the origin. At the other extreme, we have the diabatic limit of no recurrences with just the standard QW, with the manifestation of ballistic spreading. The coin bias is a convenient and effective control of diabaticity versus adiabaticity because \( \theta \) can set the position spreading rate without modifying the nature of the position distribution.

Experimentally, the coin bias is controlled by adjusting the HWP angle, and experimental results for the walker position distribution of the walker are shown in figure 3(a) at the first quasi-period \( \tau = 4 \) for various choices of \( \theta \). Increased narrowing of the distribution for increased \( \theta \) is commensurate with the expected transition from diabatic to adiabatic behaviour. The time-dependent variance for the theory and experiment for various \( \theta \) in figure 3(b) further confirm the diabatic-to-adiabatic transition, especially when showing the vanishing of quasi-periodicity in the diabatic limit, yielding ballistic spreading instead.
3. Quantum-to-classical transition

Now we have established that our experiment shows quasi-periodic dynamics and a full transition from adiabatic to diabatic behaviour by controlling the coin bias, we proceed to show the quantum-to-classical transition from quasi-periodic to diffusive dynamics through controlling the decoherence in the optical interferometer network. This decoherence is controlled by tilting the BDs. A nonzero relative angle $\Delta \theta$ between two successive BDs, as shown in figure 4(a), leads to a spatiotemporal mode mismatch. This mismatch leads to dephasing between the relative paths and therefore manifests the decoherence of the QW.

We choose $\Delta \theta = 9.75^\circ$ to realize the full decoherence of the QW [2], and show in figures 4(b)–(d) the results for the walker position distribution and normalized variance in the coherent and decoherent cases for $t = 4$. Thus, we can see the dramatic transition from quasi-
periodic dynamics to ordinary diffusive dynamics through controlled decoherence on the scale of one quasi-period.

4. Conclusion

In summary, we have developed a versatile optical quantum network with single-photon inputs that can simulate QWs with arbitrary position-dependent phase shifts. We present the quasi-periodic dynamics of a quantum walker and clear signatures of adiabatic versus diabatic behaviour, as well as quasi-periodic-to-diffusive dynamics, by controlling the decoherence. Our results show a new realm of quantum-walk phenomenon, which is especially interesting in the context of quantum chaos and Bloch oscillations. In addition, our interferometer phase-shift control provides a valuable new tool for exploring QWs with various potentials.

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Figure 4. The transition from a quasi-periodic QW to fully decohered QW. (a) A relative angle $\Delta \theta$ between two beam deflectors reduces the recombined photon’s spatial mode overlap, which introduces controllable dephasing. (b) Measured position distribution at the first quasi-period with coin bias $\theta = \pi/4$, position-dependent phase function $\phi(x) = 2\pi x/4$ and a symmetric initial coin state. (c) Position distribution of a fully decohered QW. The blue and green bars show the experimental data and theoretical predictions, respectively. (d) Normalized position variances of quasi-periodic QW (blue boxes) and fully decohered QW (red triangles) up to 4 steps, compared with the theoretical predictions (solid lines). The error bars are smaller than the symbol size.

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