Cocompact Proper CAT(0) Spaces

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Abstract

This paper is about geometric and topological properties of a proper CAT(0) space \( X \) which is cocompact - i.e. which has a compact generating domain with respect to the full isometry group. It is shown that geodesic segments in \( X \) can “almost” be extended to geodesic rays. A basic ingredient of the proof of this geometric statement is the topological theorem that there is a top dimension \( d \) in which the compactly supported integral cohomology of \( X \) is non-zero. It is also proved that the boundary-at-infinity of \( X \) (with the cone topology) has Lebesgue covering dimension \( d - 1 \). It is not assumed that there is any cocompact discrete subgroup of the isometry group of \( X \); however, a corollary for that case is that “the dimension of the boundary” is a quasi-isometry invariant of CAT(0) groups. (By contrast, it is known that the topological type of the boundary is not unique for a CAT(0) group.)

1 Statement of Theorems

A CAT(0) space is a geodesic metric space \((X, d_X)\) whose geodesic triangles are “no fatter than” the corresponding comparison triangles in the Euclidean plane. A general reference for facts about CAT(0) spaces used here is [4]. We will usually suppress \( d_X \) referring just to \( X \). Such a space \( X \) is proper if all closed balls are compact, and is cocompact if there is a compact generating domain for the full isometry group of \( X \), i.e. there is a compact set \( C \subset X \) such that the sets \( \{h(C) \mid h \text{ is an isometry of } X\} \) cover \( X \). In particular, a proper CAT(0) space \( X \) has a compact boundary, \( \partial_X \), namely the set of asymptoty classes

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of geodesic rays in $X$ with the “cone topology”. Equivalently, picking a base point $p \in X$ one can regard $\partial_\infty X$ as the set of geodesic rays starting at $p$ endowed with the compact-open topology.

The boundary of a proper CAT(0) space can be infinite-dimensional (see Example 4 below) but there is the following theorem of E. L. Swenson:

**Theorem 0.** ([18]; Theorem 12). *If the proper CAT(0) space $X$ is cocompact then $\partial_\infty X$ has finite (Lebesgue covering) dimension.*

In this paper we sharpen Swenson’s theorem by showing that cocompactness implies more. As usual $H^*_c(X)$ denotes the integral cohomology of $X$ with compact supports. For cocompact $X$ we prove (i) that there is a highest dimension $d$ for which $H^d_c(X)$ is non-zero, and (ii) that $\partial_\infty X$ has dimension $d - 1$. While this is of interest in its own right, our initial motivation came from a wish to prove that every cocompact non-compact proper CAT(0) space is almost geodesically complete (Theorem 5, below). By previous work of the second-named author [14] this follows from (i). We know of no geometric proof of this geometric statement, i.e. a proof of almost geodesic completeness for proper cocompact CAT(0) spaces, which does not use algebraic topology.

Before stating our results in detail we need some definitions.

Two metric spaces have the same bounded homotopy type if there are maps (i.e. continuous functions) from each to the other such that either composition is homotopic to the appropriate identity map by a homotopy which only moves points by a bounded amount, i.e. there is a number $s \geq 0$ such that the image of $\text{point} \times [0, 1]$ under either of the homotopies has diameter at most $s$. When that happens the maps in question are called bounded homotopy equivalences. If the metric spaces are proper (i.e all closed balls are compact) then these must be proper; moreover, “having the same bounded homotopy type” implies “having the same proper homotopy type” and a bounded homotopy equivalence is a proper homotopy equivalence. In particular, $H^*_c$ is a bounded homotopy invariant.

When $K$ is a countable locally finite simplicial complex, $|K|$ will be understood to carry the unit metric unless we say otherwise: by this we
mean that each simplex is isometric to a standard Euclidean simplex whose edges have length 1, and the distance between points of $|K|$ is the greatest lower bound of the lengths of all piecewise linear paths, each piece in a simplex, joining them. Thus $|K|$ is a proper geodesic metric space.

**Theorem 1.** Let $X$ be a cocompact proper $CAT(0)$ space. (i) There is a finite-dimensional countable locally finite simplicial complex $K$ such that $X$ and $|K|$ have the same bounded homotopy type. (ii) There is a number $d$ such that $H_c^d(X)$ is non-trivial while $H_c^i(X)$ is trivial for all $i > d$.

A version of Theorem 1 appears in [14] but with the additional hypothesis that for some cocompact group of isometries $\Gamma$ the orbits of the $\Gamma$-action are discrete; see footnote on page 209 of [4]. One of the points of the present paper is to remove such discreteness hypotheses.

It is well known (see for example [7], [8] or [9]) that $H^n_c(X)$ is isomorphic to the reduced integral $(n - 1)$-dimensional Čech cohomology of $\partial_\infty X$. Thus we have:

**Corollary 2.** $\partial_\infty X$ has non-trivial reduced integral Čech cohomology in dimension $d - 1$ and trivial integral Čech cohomology in all higher dimensions.

**Example 3.** Let $Q$ denote the Hilbert Cube in Hilbert space, i.e. the set of sequences of real numbers whose $n$th entry lies in the closed interval $[2^{-n}, 2^n]$, with metric coming from the Hilbert space norm. Hilbert space is obviously $CAT(0)$, so $Q$, being a convex subset, is also $CAT(0)$. The boundary of $\mathbb{R}$ is homeomorphic to $S^0$, but the boundary of $\mathbb{R} \times Q$ is also homeomorphic to $S^0$, a simple illustration of how infinite-dimensional $CAT(0)$ spaces can have finite-dimensional boundaries. Similarly $\partial \mathbb{R}^n \times Q$ is an $n - 1$-sphere. These are cocompact examples with $d = n$.

**Example 4.** The proper $CAT(0)$ spaces $\mathbb{R}_{\geq 0} \times \mathbb{R}^n$ and “the open cone on $Q$” (made into a $CAT(0)$ space in the way described in [4] II.3.14) have contractible boundaries homeomorphic to $B^n$ and $Q$ respectively so Corollary 2 implies that they are not cocompact. (Indeed, by Theorem 1(ii), the underlying topological spaces do not admit proper cocompact
CAT(0) metrics because in this case the cohomology with compact supports is trivial.)

Our main application of Theorem 1 is to geometry. A CAT(0) metric space $X$ is almost geodesically complete if there is a number $r \geq 0$ such that for any points $a$ and $b$ of $X$ there is a geodesic ray $\gamma : [0, \infty) \to X$ with $\gamma(a) = 0$ whose image meets the open ball about $b$ of radius $r$. The term “almost extendible” is also used. Obviously, the spaces in Example 3 have this property while those in Example 4 do not. An example which has this property even though not every geodesic segment can be extended to a geodesic ray is the graph consisting of $\mathbb{R}$ together with, for each $n \in \mathbb{Z}$, a copy of the closed unit interval glued at its 0-point to $n \in \mathbb{R}$.

Theorem A of [14] says that if there were a non-compact cocompact proper CAT(0) space $X$ which is not almost geodesically complete then $H^*_c(X)$ would have to be trivial. Combining this with Theorem 1 we get:

**Theorem 5.** Every non-compact cocompact proper CAT(0) space is almost geodesically complete.

As explained above, this was proved in [14] under the additional hypothesis that there exists a cocompact group of isometries with discrete orbits.

We remark that Theorem H of [1] requires almost geodesic completeness as a hypothesis on the non-compact cocompact proper CAT(0) space under consideration. Theorem 5 shows that this hypothesis is redundant.

Corollary 2 gives $d - 1$ as the sharp upper bound for non-vanishing Čech cohomology, but we can prove a better statement:

**Theorem 6.** Let $X$ be a cocompact proper CAT(0) space. Then the (Lebesgue covering) dimension of $\partial_{\infty} X$ is $d - 1$, where $d$ is as in Theorem 1.

We remark that Theorem C of [13] might be considered to be an analog of our Theorem 6 for $\partial_{\infty} X$ when it is equipped with the Tits metric topology.
A CAT(0) group is a group $\Gamma$ which can act geometrically (i.e. properly discontinuously and cocompactly by isometries) on some proper CAT(0) space $X$. Then $H^*_c(X)$ is isomorphic to $H^*(\Gamma, \mathbb{Z}\Gamma)$ (see Exercise VIII.7.4 of [5]), so the number $d$ in Theorem 1 depends only on $\Gamma$, hence Theorem 6 implies that the dimension of $\partial_{\infty}X$ depends only on $\Gamma$. This is of interest because Croke and Kleiner [6] have given examples to show that the homeomorphism type of $\partial_{\infty}X$ is not an invariant of $\Gamma$. Indeed, if $\Gamma_1$ and $\Gamma_2$ are quasi-isometric CAT(0) groups then $H^*(\Gamma_1, \mathbb{Z}\Gamma_1)$ and $H^*(\Gamma_2, \mathbb{Z}\Gamma_2)$ are isomorphic [10], so we have:

**Corollary 7.** The Lebesgue covering dimension of the boundary of proper CAT(0) spaces on which CAT(0) groups act geometrically is a quasi-isometry invariant (of CAT(0) groups).

Of course, if $\Gamma$ acts cocompactly as covering transformations on $X$ then $\Gamma$ has finite cohomological dimension so the number $d$ of Theorem 1 exists for group theoretic reasons, and in that case Corollary 7 follows from the proof of Corollary 1.4 of [3], as is pointed out in [2].

2 Proofs.

We are to prove Theorems 1 and 6.

**Proof of Theorem 1(i).** Let $E \subset X$ be maximal with respect to the property that if $x, y \in E$ and $x \neq y$, then $d_X(x, y) \geq 1$. The family $U = \{B_X(x, 1) \mid x \in E\}$ is an open cover of $X$, where $B_X(x, 1)$ denotes the open ball of radius 1. Let $K$ be the nerve of this cover. Then $K$ is clearly countable and it is locally finite because the cover $U$ is star finite (i.e. each member of $U$ meets only finitely many others).

Suppose $K$ is not finite-dimensional. Then for all natural numbers $m$ we would have $\dim K \geq m$. Thus for each $m$ there would be points $x_m^0, \ldots, x_m^m \in E$ such that $1 \leq d_X(x_m^i, x_m^j) < 2$ when $i \neq j$. Let $C$ be a compact generating domain for $X$. For each $m$ there is an isometry $h_m$ such that $h_m(x_m^0) \in C$. Then for each $i$ the point $h_m(x_m^i)$ lies in $C_1$, the 1-neighborhood of $C$. By induction, we can then pick subsequences of $N$, each a subsequence of its predecessor, so that for each $n$ the sequence $(h_{n+k}(x_{n+k}^n))$ converges to some point $y_n \in C_1$. The resulting sequence
(yn) consists of points which are pairwise at least 1 apart while all lying in the compact set C₁, a contradiction.

Because X is CAT(0) and U consists of open balls, it follows that all non-empty intersections of members of U are contractible. It is well known that this implies that X is homotopy equivalent to |K| (the weak topology coincides with the unit metric topology since K is locally finite), and indeed the proof of this fact in §5 of [19] shows that, with the metric we have chosen, the mutually homotopy inverse maps in both directions given in that proof are bounded homotopy equivalences which are bounded homotopy inverse to one another. (Indeed, this also follows from the proof of Lemma 7A.15 on page 129 of [4]). q.e.d.

When Z is a metric space B_Z(p, r) denotes the open ball of radius r and center p ∈ Z. We say that Z is uniformly contractible if for every r > 0 there is s > 0 such that for every p ∈ Z, B_Z(p, r) contracts in B_Z(p, s). We say H^i_c(Z) is uniformly trivial if Z has the following property:

For every r > 0 there is s > 0 such that whenever a i-cocycle z has compact support contained in a ball B_Z(p, r), then z cobounds a cochain whose compact support lies in B_Z(p, s).

Recall that when U ⊂ V ⊂ Z with U and V both open in Z then the inclusion map i : U → V induces a homomorphism i_* : H^*_c(U) → H^*_c(V); see Remark 26.2 of [11].

Proposition 2.1. Let X be a proper cocompact CAT(0) space. If H^i_c(X) is trivial, then H^i_c(X) is uniformly trivial.

Proof. Let r > 0 and let x ∈ X.

Claim. There is r' ≥ r (depending only on r) such that i_* (H^i_c(B_X(x, r))) is a finitely generated subgroup of H^i_c(B_X(x, r')), where i : B_X(x, r) → B_X(x, r') is the inclusion.

To prove the Claim, let K be as in the proof of Theorem 1(i) and let f : X → |K|, g : |K| → X be such that gf is bounded homotopic to the identity map. Then there is a compact set C containing B_X(x, r) such that, for any cocycle z with compact support lying in B_X(x, r), z
and $f^*g^*z$ are compactly cohomologous in $C$.

Let $L$ be a finite subcomplex of $K$ such that:

\[(\ast) \quad g^{-1}(C) \subset \text{int } |L|.
\]

Choose $r'$ such that $g(|L|) \subset B_X(x, r')$. It follows that $C \subset g(\text{int } |L|) \subset B_X(x, r')$. Hence, if $z$ is a cocycle with compact support contained in $B_X(x, r)$, then $z$ is compactly cohomologous to $(gf)^*z = f^*g^*z$ in $B_X(x, r')$. By (\ast) $v := g^*z$ is a cocycle with compact support contained in $\text{int } |L|$. Consequently every cocycle with compact support contained in $B_X(x, r)$ is compactly cohomologous in $B_X(x, r')$ to a cocycle of the form $f^*v$ where $v$ is a cocycle with compact support contained in $\text{int } |L|$. Since $H^i_c(\text{int } |L|)$ is finitely generated we conclude that $f^*H^i_c(\text{int } |L|)$ is also finitely generated. The Claim follows.

Now let $r'$ be as in the Claim and let $z_1, \ldots, z_l$ be compactly supported cocycles representing a set of generators of $\iota^*(H^i_c(B_X(x, r)))$. Since we are assuming $H^i_c(X) = 0$, there is $s = s(r)$ such that $z_1, \ldots, z_l$ compactly cobound in $B_X(x, s)$. Since $X$ is cocompact it follows easily that, given $r$, a number $s$ independent of $x$ exists with the required property. q.e.d.

**Remark.** By Theorem 1(i) we have $H^i_c(X) = 0$, for $i > \text{dim } K$, hence the number $s$ can be chosen to be independent of $i$.

**Proof of Theorem 1(ii).** We will give enough information for the reader to understand the proof but we will omit some details when they are the same as corresponding steps in the proof of Theorem B of [14]. (Theorem B of [14] is the special case of Theorem 1 mentioned in §1.)

By Part (i) it is enough to prove that $H^*_c(X)$ is non-trivial. Suppose that $H^*_c(X) = 0$. It follows then from Proposition 2.1 that each $H^i_c(X)$ is uniformly trivial. Let $K$ be as in the proof of Part (i). Since $X$ is uniformly contractible and $K$ is bounded homotopy equivalent to $X$, it follows that $K$ has the following two properties:

1. $|K|$ is uniformly contractible.
2. $H^*_c(|K|)$ is uniformly trivial (with constants independent of the dimension - see Remark after the proof of Proposition 2.1).
To obtain a contradiction we now proceed as in the proof of Proposition B of [14]. Embed $|K|$ properly, in some $\mathbb{R}^n$ by an embedding which is affine on each simplex. Let $T$ be a triangulation of $\mathbb{R}^n$ such that there is a full subcomplex $J$ of $T$ with $|K| = |J|$ and $J$ is a subdivision of $K$. Denote the unit metric on $|K|$ by $d_K$. As mentioned in [14], we can extend this proper metric to a proper piecewise flat metric $d$ on $\mathbb{R}^n = |T|$. Indeed, we can assume, perhaps after a subdivision of $T$ away from $K$, that $\text{mesh}(T) \leq 1$ and that every simplex of $|J|$ is convex in $|T|$.

Let $M = |N(T', J)|$ where $T'$ is a first derived subdivision of $T$ away from $J$, and $N(., .)$ denotes the simplicial neighborhood. Then $M$ is a regular PL neighborhood of $|J|$ in $\mathbb{R}^n$ [15], so $M$ is an $n$-manifold with boundary $\partial M$ and $M$ is bounded homotopy equivalent to $|K|$. Denote by $d_M$ the intrinsic metric on $M$ induced from $\mathbb{R}^n$ with metric $d$; i.e. the distance between points of $M$ is the greatest lower bound of the $d$-lengths of PL paths in $M$ joining them. Since $|J| \subset M$ we have $d_M|J| \leq d_K$.

As in [14], we can choose $M$ close to $|J|$ to get $d_K$ close to $d_M|J|$:

Claim 1. We can choose $M$ so that $d_K \leq (d_M|J|) + 1$.

The proof is the same as the proof of Claim 1 in the proof of Proposition B of [14].

Let $c_t : M \to |K|$ be the canonical deformation retraction (see section 4 of [14]). As in [14] we also have

(i) $d_M(x, c_t(x)) \leq 1$ for every $t \in [0, 1]$, and

(ii) for every $x \in M$ there is $w \in \partial M$ with $d_M(x, w) \leq 1$.

Since $|K|$ is bounded homotopy equivalent to $M$ it follows from Properties 1 and 2 above that $M$ has the properties:

1’. $M$ is uniformly contractible.

2’. $H_c^i(M)$ is uniformly trivial.

Since $M$ is contractible and $H_c^i(M) = H_c^i(|K|) = 0$, for all $i$, using the same argument as in the beginning of the proof of Proposi-
tion B in [14], we can arrange everything up to this point so that the pair \((M, \partial M)\) is PL-homeomorphic to the pair \((\mathbb{R}^n_+, \mathbb{R}^{n-1})\) where \(\mathbb{R}^n_+ = \{x = (x_1, ..., x_n) \in \mathbb{R}^n : x_1 \geq 0\}\). (See also Remark 2.2 below.)

Now, instead of Claims 2 and 3 of [14] (because here we do not have a \(\Gamma\) action on \(K\)) we have the following:

**Claim 2.** \(\partial M\) is uniformly acyclic, i.e. given \(b > 0\), there is an \(a_b\) such that, for any \(x \in \partial M\), the homomorphism \(H_*(B_{\partial M}(x, b)) \to H_*(B_{\partial M}(x, a_b))\), induced by the inclusion \(B_{\partial M}(x, b) \hookrightarrow B_{\partial M}(x, a_b)\), is zero. (Here we consider \(\partial M\) with the metric \(d_M|_{\partial M}\).)

In other words, every cycle in \(\partial M\) supported in a ball of radius \(b\) (with respect to the metric \(d_M\)) bounds (in \(\partial M\)) in a ball of radius \(a_b\).

**Proof of Claim 2.** Let \(z\) be a cycle supported in some ball \(B_{\partial M}(x, b)\). By Property 1', \(M\) is uniformly contractible, therefore \(z\) bounds in some ball \(B_M(x, b')\), where \(b'\) depends only on \(b\). In symbols we have \(z = \partial v\), where \(v\) is a chain in \(B_M(x, b')\). Since \(\partial v = z\) and \(z\) is a cycle in \(\partial M\), we have that \(v\) is a relative cycle in \((M, \partial M)\), hence it is Poincaré dual to a cocycle in \(M\) that vanishes outside \(B_M(x, b')\).

By Property 2', \(H^*_c(M)\) vanishes uniformly. Hence, there is a \(b''\) such that the cocycle dual to \(v\) compactly cobounds in \(B_M(x, b'')\), again by Poincaré Duality. Thus there is a chain \(w\) in \(B_M(x, b'')\) such that \(\partial w = v + y\), where \(y\) is some chain in \(\partial M \cap B_M(x, b'')\). Then \(0 = \partial \partial w = \partial (v + y) = \partial v + \partial y\) and it follows that \(z = \partial v = \partial (-y)\). That is, \(z\) bounds in \(\partial M \cap B_M(x, b'')\). Take \(a_b = b''\). This completes the proof of Claim 2.

**Claim 3.** There is a number \(a \geq 0\) such that the following holds. Let \(P\) be any subset of \(M\) and let \(v\) be a relative \(i\)-cycle of \((M, \partial M)\) supported in \(P\) where \(i \leq n - 1\) (hence \(\partial v\) is a cycle in \(\partial M \cap P\)). Then there is a chain \(u\) in \(\partial M \cap N_a(P)\), with \(\partial v = \partial u\). (Here \(N_a(P)\) denotes the \(a\)-neighborhood of the set \(P\).)

The proof of Claim 3 is by induction on the dimension of the simplices of the cycle. It uses (i), (ii) and Claim 2 and is similar to the proof of Claim 4 in the proof of Proposition B of [14].
Finally, to obtain a contradiction we use an argument similar to the one used at the end of the proof of Proposition B of [14]:

Let \( N = a + 2 \), where \( a \) is the constant from claim 3, and choose a base point \( x_0 \in \partial M \). Recall that we can assume that \((M, \partial M)\) is PL-homeomorphic to \((\mathbb{R}^n_+, \mathbb{R}^{n-1})\). Recall also that \((M, d_M)\) is a proper metric space. Let \( A \) denote the closed ball in \( M \) with center \( x_0 \) and radius \( N \). Since \( A \) is compact, there is a PL-\((n-1)\)-ball \( \tilde{D} \subset \partial M \), such that \( A \cap \partial M \subset \text{int} ~ \tilde{D} \). Let \( \tilde{\rho} : S^{n-2} \to \partial M \) be a PL-embedding such that \( \tilde{\rho}(S) = \partial(\tilde{D}) \). Since \( x_0 \in \text{int} \tilde{D} \) we have that \( \tilde{\rho} \) represents a cycle \( z \) which does not bound in \( \partial M \setminus \{x_0\} \). Note that the cycle \( z \) bounds in \( M \setminus A \). Consequently, by Claim 3, there is a chain \( u \) in \( \partial M \cap N_a(M \setminus A) \) with \( \partial u = \partial v = z \). But \( \partial M \cap N_a(M \setminus A) \subset \partial M \setminus \{x_0\} \), which implies that \( z \) bounds in \( \partial M \setminus \{x_0\} \). This is a contradiction. q.e.d.

**Remark 2.2.** There is a step in the proof where (by referring to [14]) we use Stallings’ Theorem ([17]) that a contractible PL manifold of high dimension \( d \) which is simply connected at infinity is PL homeomorphic to \( \mathbb{R}^d \). And to apply that, some tricks are used in [14] to get \( \partial M \) to be simply connected at infinity. Stallings’ Theorem requires knowledge of engulfing, something invented in order to prove the high-dimensional Poincaré Conjecture, and certainly more complicated than what ought to be necessary here. Examination of the above proof, however, shows that all we actually need is that the pro-homology at the end of \( \partial M \) in dimension \( n - 2 \) is stably \( \mathbb{Z} \), something which follows from Poincaré Duality. The interested reader can supply the details of this; for background, see, for example, [7].

We denote the Alexander-Spanier cohomology of the pair \((Z, A)\) by \( \hat{H}^n(Z, A) \).

For the proof of Theorem 6, we need a Lemma:

**Lemma 2.3.** Let \( Z \) be a compact metric space and \( A \) a closed subset of \( Z \). Assume that \( \hat{H}^n(Z, A) \neq 0 \) and that \( \hat{H}^{n+1}(Z, B) = 0 \) for every closed subset \( B \) of \( Z \). Then there is a sequence of open balls \( \{B_k\}_{k \geq k_0} \) in \( Z \setminus A \), \( B_k \) of radius \( 1/k \), such that the homomorphisms \( \hat{H}^n(Z, Z \setminus B_k) \to \hat{H}^n(Z, A) \) are non-zero. Moreover, we can choose the \( B_k \)’s to satisfy \( d_Z(B_k, A) \geq \delta \), for some \( \delta > 0 \).

**Proof.** Since \( \hat{H}^n(Z, A) = \hat{H}^n_c(Z \setminus A) \) (see 6.6.11 of [16]), elements
can be represented by cocycles with compact support in $Z \setminus A$. Hence we can find a closed set $A'$, such that $A \subset \text{int } A'$ and the morphism $\hat{H}^n(Z, A') \to \hat{H}^n(Z, A)$ is non-zero. Let $k_0$ be such that $2/k_0 < d_Z(Z \setminus A', A)$.

Let $U_1, \ldots, U_j$ be a finite cover of $Z \setminus A'$ by balls of radius $1/k_0$. Write $U = U_1$ and $V = \bigcup_{i=2}^j U_i$. Note that $A \subset Z \setminus (U \cup V) \subset A'$. We have the following diagram of Mayer-Vietoris sequences:

$$
\begin{array}{cccc}
\hat{H}^n(Z, Z \setminus U) \oplus \hat{H}^n(Z, Z \setminus V) & \to & \hat{H}^n(Z, Z \setminus (U \cup V)) & \to & \hat{H}^{n+1}(Z, Z \setminus (U \cap V)) \\
\downarrow & & \downarrow & & \downarrow \\
\hat{H}^n(Z, A) \oplus \hat{H}^n(Z, A) & \to & \hat{H}^n(Z, A) & \to & \hat{H}^{n+1}(Z, A)
\end{array}
$$

where the last group of the first row is zero, by hypothesis. Since the non-zero morphism $\hat{H}^n(Z, A') \to \hat{H}^n(Z, A)$ factors through $\hat{H}^n(Z, Z \setminus (U \cup V))$, the middle vertical morphism in the diagram is non-zero. It follows from the diagram that (at least) one of the morphisms $\hat{H}^n(Z, Z \setminus U) \to \hat{H}^n(Z, A)$ and $\hat{H}^n(Z, Z \setminus V) \to \hat{H}^n(Z, A)$ is non-zero. If the first one is non-zero take $B_{k_0} = U$. Otherwise write $U = U_2$ and $V = \bigcup_{i=3}^j U_i$, and repeat the process (using the fact that the latter morphism is non-zero). Eventually, we will find an $i$ such that the morphism $\hat{H}^n(Z, Z \setminus U_i) \to \hat{H}^n(Z, A)$ is non-zero. Take $B_{k_0} = U_i$. To find $B_{k_0+1}$ take a finite cover $U_1, \ldots, U_j$ of $B_{k_0}$ by balls of radius $1/k_0 + 1$ and repeat the process. In this way we obtain a sequence of balls $B_k$. Note that $d_Z(B_k, A) \geq d_Z(A, Z \setminus A') - 1/k_0 > 0$. Take $\delta = d_Z(A, Z \setminus A') - 1/k_0$. q.e.d.

We recall some facts of dimension theory; see [12] for details. Let $Z$ be a compact metric space. The (Lebesgue covering) dimension of $Z$, $\text{dim } Z$, is $\leq n$ if every open cover of $Z$ has a refinement whose nerve has dimension at most $n$; one writes $\text{dim } Z = n$ if, in addition, $\text{dim } Z$ is not $\leq n - 1$. If there is no such $n$ then $\text{dim } Z$ is infinite. The $\mathbb{Z}$-cohomological dimension of $Z$, $\text{dim}_Z Z$, is $n$ if $\hat{H}^n(Z, A) \neq 0$ for some closed subset $A$ of $Z$ while for all $k > n$ and all closed subsets $B$ of $Z$, $\hat{H}^k(Z, B) = 0$. If there is no such $n$ then $\text{dim}_Z Z$ is infinite. Traditionally, here, $\hat{H}^n$ refers to Čech cohomology, but that is canonically isomorphic to Alexander-Spanier cohomology; at least when $\text{dim } Z$ is finite, as it is in our case.
- see page 342 of [16]. There is always the inequality \( \text{dim}_Z Z \leq \text{dim} Z \), and equality holds when \( \text{dim} Z \) is finite.

**Proof of Theorem 6.**

We write \( \bar{X} = X \cup \partial_\infty X \). By Corollary 2, \( \text{dim} \partial_\infty X \geq d - 1 \). We are to prove equality. By Theorem 0 we know that \( \text{dim} \partial_\infty X \) is finite so we may use the cohomological definition of dimension. Suppose that \( \text{dim} Z \partial_\infty X \) is \( n \geq d \). Then there is a closed set \( A \) such that \( \tilde{H}^n(\partial_\infty X, A) \neq 0 \). Let \( B_k \) and \( \delta > 0 \) be as in Lemma 2.3 above (taking \( Z = \partial_\infty X \) with a metric \( d_{\partial_\infty X} \) that induces the cone topology on \( \partial_\infty X \)). Moreover, we may assume that the balls \( B_k \) converge to some point \( \gamma \notin A \).

Fix a base point \( x_0 \in X \). For any set \( G \subset \partial_\infty X \), the cone \( CG \) of \( G \) is the union of all rays emanating from \( x_0 \) and ending in \( G \). Using geodesic retraction along rays emanating from \( x_0 \) we can find a closed set \( D \subset \bar{X} \) such that:

(i) \( D \cap \partial_\infty X = A \).

(ii) \( CA \subset D \) and \( D \setminus A \) lies in the 1-neighborhood (in \( X \)) of \( CA \setminus A \).

(iii) \( D \) is a strong deformation retract of \( \bar{X} \).

By Theorem 1 \( H^{n+1}_c(X) \) is trivial, hence uniformly trivial by Proposition 2.1, so there is a number \( s \) such that every \((n+1)\)-cocycle \( y \) with compact support of diameter less than 1 cobounds a cochain whose compact support lies in the \( s \)-neighborhood of the support of \( y \).

Since \( \gamma \notin A \) we can find a \( x_1 \) in \([x_0, \gamma]\), such that the ball \( B_X(x_1, s+2) \) does not intersect \( D \). Write \( a = d_X(x_0, x_1) \). Since \( B_k \to \gamma \) there is a \( k' \) such that every geodesic ray in \( \mathcal{C}B_{k'} \) intersects \( B_X(x_1, 1) \cap S_X(x_0, a) \) where \( S_X(x_0, a) \) denotes the sphere of radius \( a \) centered at \( x_0 \). Write \( B' = B_{k'} \). By Lemma 2.3, \( \tilde{H}^n(\partial_\infty X, \partial_\infty X \setminus B') \to \tilde{H}^n(\partial_\infty X, A) \) is non-zero. Let \( \{z\} \in \tilde{H}^n(\partial_\infty X, \partial_\infty X \setminus B') \) be such that its image in \( \tilde{H}^n(\partial_\infty X, A) \) is non-zero, where \( z \) denotes a cocycle with compact support lying in \( B' \). (Here, as at the beginning of the proof of Lemma 2.3, we are identifying a relative Čech cohomology group with the compactly supported cohomology of the complement - we will use this convention again below.) Using a geodesic deformation retraction we can find a closed set \( E \subset \bar{X} \) such that:
(i) $E \cap \partial_\infty X = \partial_\infty X \setminus B'$.

(ii) $C(\partial_\infty X \setminus B') \cup B_X(x_0, a) \subset E$.

From the exact sequence of the triple $(\bar{X}, \partial_\infty X \cup D, D)$ and the fact that $\check{H}^*(\bar{X}, D) = 0$ we conclude that $\check{H}^n(\partial_\infty X \cup D, D) \rightarrow \check{H}^{n+1}(\bar{X}, \partial_\infty X \cup D)$ is an isomorphism. By considering also the exact sequence of the triple $(\bar{X}, \partial_\infty X \cup E, E)$ and the inclusion $(\bar{X}, \partial_\infty X \cup D, D) \hookrightarrow (\bar{X}, \partial_\infty X \cup E, E)$ we get the following commutative diagram:

\[
\begin{array}{ccc}
\check{H}^n(\partial_\infty X, \partial_\infty X \setminus B') & \cong & \check{H}^n(\partial_\infty X \cup E, E) \rightarrow \check{H}^{n+1}(\bar{X}, \partial_\infty X \cup E) \\
\downarrow & & \downarrow \\
\check{H}^n(\partial_\infty X, A) & \cong & \check{H}^n(\partial_\infty X \cup D, D) \cong \check{H}^{n+1}(\bar{X}, \partial_\infty X \cup D)
\end{array}
\]

where the isomorphisms on the left are given by excision - see 6.6.5 of [16]. Let $\{z\}'$ be the image of $\{z\}$ in $\check{H}^{n+1}(\bar{X}, \partial_\infty X \cup E)$. Then the image of $\{z\}'$ in $\check{H}^{n+1}(\bar{X}, \partial_\infty X \cup D) \cong \check{H}^{n+1}(X, D \setminus A)$ is non-zero. We regard $z'$ as a cocycle compactly supported outside $\partial_\infty X \cup E$ and we obtain a contradiction by showing that $z'$ compactly cobounds outside $D \setminus A$ as follows: using a proper radial contraction along rays emanating from $x_0$ we find a cochain $c$ representing $z'$ with compact support lying in $B_X(x_1, 1)$. This $z'$ cobounds in $B_X(x_1, s + 1)$ - a contradiction since $B_X(x_1, s + 1)$ is disjoint from $D$. q.e.d.

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