INTEGRAL REGION CHOICE PROBLEMS ON LINK DIAGRAMS

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Abstract. Shimizu introduced a region crossing change unknotting operation for knot diagrams. As extensions, two integral region choice problems were proposed and the existences of solutions of the problems were shown for all non-trivial knot diagrams by Ahara and Suzuki, and Harada. We relate both integral region choice problems with an Alexander numbering for regions of a link diagram, and give alternative proofs of the existences of solutions for knot diagrams. We also discuss the problems on link diagrams. For each of the problems on the diagram of a two-component link, we give a necessary and sufficient condition that there exists a solution.

1. Introduction

A link is a closed 1-manifold smoothly embedded in the 3-space $\mathbb{R}^3$ or in the 3-sphere $S^3$ and a knot is a link with one component. A link in the 3-space is presented as the natural projection image on the 2-plane $\mathbb{R}^2$ where the singular points are transverse double points with over/under information. This presentation is called a link diagram or a diagram of the link. A diagram of a link in the 3-sphere $S^3 = \mathbb{R}^3 \cup \{\infty\}$ is given on the 2-sphere $S^2 = \mathbb{R}^2 \cup \{\infty\}$ similarly. For each link diagram, a connected component of the complement of the projection image on $\mathbb{R}^2$ or $S^2$ is called a region.

In [10], Shimizu defined a region crossing change at a region for a diagram to be the crossing change at all the crossings on the boundary of the region as an unknotting operation for a knot diagram, which was proposed by Kengo Kishimoto. For example in Figure 1, the left diagram is changed to the right diagram, choosing the region marked with $*$ as illustrated on the middle and changing the three crossings on the boundary of the marked region. In [3, 4], Cheng and Gao gave a necessary and sufficient condition that a region crossing change is an unknotting operation on a link diagram.

Figure 1. An example of a region crossing change.

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It is known that a region crossing change can be interpreted as follows. We call a diagram ignored over/under information a projection. Let each crossing of the given projection be equipped with a score 0 or 1 modulo 2. We choose a region of the projection. Then the scores of all the crossings on its boundary are increased by 1 modulo 2. For example, the region crossing change illustrated on Figure 1 is interpreted as Figure 2. Shimizu showed that the scores of all the crossings on any knot diagram become 0 by some choices of regions. Cheng and Gao induced a $\mathbb{Z}_2$-homomorphism from region crossing changes on link diagrams. In [6, 7], Hashizume studied structures of their $\mathbb{Z}_2$-homomorphism.

As an extension of a region crossing change to an integral range, Ahara and Suzuki proposed an integral region choice problem and showed the existence of a solution of this problem for all knot projections in [4]. Let each crossing of the given projection be equipped with an integral score. We choose a region of the projection and assign an integer $u$ to it. Then the scores of all the crossings on its boundary are increased by $u$. For example in Figure 3, the scores of the crossings on the left projection are changed to the right, assigning integers to regions as the middle projection; $1 \mapsto 1 + 0 + 2 + (-1) + (-2) = 0$, $-1 \mapsto -1 + 0 + 0 + (-1) + 2 = 0$, $3 \mapsto 3 + 0 + 0 + (-2) + (-1) = 0$, and $2 \mapsto 2 + 0 + 0 + 0 + (-2) = 0$. Ahara and Suzuki showed that the scores of all the crossings on any knot projection become 0 by some choices of regions and some assignments of the integers to them. We shall call their problem a definite integral region choice problem. In Section 3, we state their result exactly.

By an argument similar to that due to Ahara and Suzuki, Harada showed in his master thesis [5] that there exists a solution of an alternating integral region choice problem for all knot diagrams, which was suggested by Yasuyoshi Yonezawa. Let each crossing of the given diagram be equipped with an integral score. We choose a region of the projection and assign an integer $u$ to it. Then the score of each crossing on its boundary is changed as follows. If the region touches the crossing at the corner contained by the underpass and the overpass counterclockwise, that
means $\bigtimes$ or $\bigcirc$, the score of the crossing is increased by $u$. If the region touches the crossing at the corner contained by the underpass and the overpass clockwise, that means $\bigtimes$ or $\bigcirc$, the score of the crossing is decreased by $u$. For example in Figure 4, the scores of the crossings on the left diagram are changed to the right, assigning integers to regions as the middle diagram: 1 $\mapsto$ $1 + 0 - (-2) + 0 - (-1) - 2 = 0$, $-1$ $\mapsto$ $-1 + 0 - 0 + (-1) - (-2) = 0$, $3$ $\mapsto$ $3 - 0 + 0 - 2 + (-1) = 0$, and $2$ $\mapsto$ $2 + 0 - 0 + 0 - 2 = 0$. Harada showed that the scores of all the crossings on any knot diagram become 0 by some choices of regions and some assignments of the integers to them. We shall call this proposed problem as another extension of a region crossing change an alternating integral region choice problem. In Section 4 we state his result exactly.

**Figure 4.** An example of an alternating integral region choice problem.

In [1, 5], Ahara, Suzuki and Harada reduced the above integral region choice problems to systems of linear equations, as explained in Section 3 and 4 in this article, and they showed the existences of solutions for non-trivial knot diagrams. We show that an Alexander numbering for regions of a link diagram is a solution of the system of homogeneous linear equations reduced from an alternating integral region choice problem in Section 5. By this result, we give alternative proofs of the existences of solutions of both alternating and definite integral region choice problems for all non-trivial knot diagrams in Section 6 and 7.

In [10], Shimizu used checkerboard colorings to regions of knot diagrams for showing that a region crossing change is an unknotting operation. Cheng and Gao [4], and Hashizume [6, 7] also used checkerboard colorings for discussing region crossing changes on link diagrams. An Alexander numbering is an integral extension of a checkerboard coloring, as mentioned in Section 2. In this article, we use Alexander numberings to discuss the integral region choice problems on link diagrams, which are integral extensions of region crossing changes.

In Section 8 and 9, we determine the ranks for the coefficient matrices of the systems of linear equations reduced from the integral region choice problems, applying the arguments in the original proofs of the solvability of integral region choice problems on knot diagrams in [1, 5] to link diagrams. Then we obtain an extension of the result about the incidence matrix due to Cheng and Gao [4].

In Section 10 we give a basis of the space of solutions of the system of homogeneous linear equations reduced from each of integral region choice problems on link diagrams. In Section 11 we give necessary and sufficient conditions that there exist solutions of integral region choice problems on the connected diagram of a two-component link. These results are extensions of some of the results about region crossing changes on link diagrams due to Cheng and Gao [4], and Hashizume [6, 7].

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2. Preliminary

By the Jordan curve theorem, any short arc without a crossing on a link diagram lies on the intersection of just two boundaries of regions. Each crossing is touched by at most four regions. If the number of the regions touching the fixed crossing is less than four, it must be three and the pair of the corners of the same region touching the crossing are not adjacent each other around the crossing. This fact is also shown from the Jordan curve theorem. In this case, such a crossing is called a reducible crossing. If a link diagram have a reducible crossing, it is called a reducible diagram. Otherwise, it is called an irreducible diagram.

Lemma 2.1 (cf. [1, 7]). Let $D$ be a link diagram or projection. If $D$ has $d$ connected components and $n$ crossings, then it has $n + d + 1$ regions.

Proof. It is shown by the Euler formula. \qed

On an oriented link diagram $D$, we say that we splice at a crossing $x$ if we change the diagram $D$ around the crossing $x$, $\bigvee$ or $\bigwedge$, to $\bigvee$ and obtain the new link diagram $D_x$. This local move between oriented link diagrams is called a splicing or smoothing at the crossing $x$. The change from $D_x$ to $D$ is called an unsplicing at $x$. In this article, the local moves $\bigvee$ to $\bigvee$ and $\bigwedge$ to $\bigwedge$ among oriented link projections are also called a splicing and an unsplicing respectively.

In [2], Alexander assigned an integer index to each region of an oriented link diagram or projection, so that for any oriented arc on the link diagram, an index of the left region adjacent to the arc is larger that of the right by one. Such an index is called an Alexander index, and this assignment of the indexes is called an Alexander indexing or an Alexander numbering. In [8], Kauffman also defined an Alexander indexing for an oriented link projection and show that there exist an Alexander indexing for any projection, though an index of the right region is assigned larger than that of the left by one for any oriented arc on the link projection.

It is known that we can shade regions for any link projection so that each two regions adjacent by an arc on the projection are shaded and unshaded, and such shading is call a checkerboard coloring. For any oriented link diagram or projection, if we shade only the regions assigned odd number by an Alexander numbering, then we obtain a checkerboard coloring. If we reverse the orientation of some link components fixing a region and its index, we obtain a new Alexander numbering and the same checkerboard coloring. In this article, we shall call an Alexander numbering modulo 2 a checkerboard coloring.

3. A definite integral region choice problem

Let $D$ be a link diagram or projection with $d$ connected components and $n$ crossings $x_1, \cdots, x_n, n \geq 1$. We note that $d$ is not greater than the number of the link components. Let $R_1, \cdots, R_{n+d+1}$ be the regions of $D$. In [1], Ahara and Suzuki induced two region choice matrices $A_{d1}(D)$ and $A_{d2}(D)$ with $n$ rows and $n + d + 1$ columns as follows, where they denoted them by $A_1(D)$ and $A_2(D)$. We determine each element $a_{ij}^{(d1)}$ by

$$a_{ij}^{(d1)} = \begin{cases} 1 & \text{if } x_i \in \partial R_j, \\ 0 & \text{if } x_i \notin \partial R_j. \end{cases}$$
The region choice matrix of the single counting rule for \( D \) is the matrix \( A_{d1}(D) \) with the element \( a_{ij}^{(d1)} \) on the \( i \)-th row and the \( j \)-th column. We determine each element \( a_{ij}^{(d2)} \) by

\[
a_{ij}^{(d2)} = \begin{cases} 
2 & \text{if } R_j \text{ touches } x_i \text{ twice,} \\
0 & \text{otherwise.}
\end{cases}
\]

The region choice matrix of the double counting rule for \( D \) is the matrix \( A_{d2}(D) \) with the element \( a_{ij}^{(d2)} \) on the \( i \)-th row and the \( j \)-th column. We shall call these two region choice matrices by the definite region choice matrices.

Using the definite region choice matrices, the definite integral region choice problem and the existence of solutions for it are stated as follows.

**Theorem 3.1 ([1]).** Let \( D \) be a knot diagram or projection with \( n \) crossings \( x_1, \ldots, x_n \), \( n \geq 1 \). Let \( R_1, \ldots, R_{n+2} \) be the regions of \( D \).

1. Let \( A_{d1}(D) \) be the definite region choice matrix of the single counting rule for \( D \). For any \( c \in \mathbb{Z}^n \), there exists a solution \( u \in \mathbb{Z}^{n+2} \) such that \( A_{d1}(D)u + c = 0 \).
2. Let \( A_{d2}(D) \) be the definite region choice matrix of the double counting rule for \( D \). For any \( c \in \mathbb{Z}^n \), there exists a solution \( u \in \mathbb{Z}^{n+2} \) such that \( A_{d2}(D)u + c = 0 \).

**Example 3.2.** Let \( D \) be the knot projection given in Figure 3. Under certain orders of crossings and regions, we have

\[
A_{d1}(D) = A_{d2}(D) = \begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}.
\]

Figure 3 implies the equation

\[
A_{d1}(D) \begin{pmatrix}
2 \\
-1 \\
-2 \\
0 \\
0 \\
0
\end{pmatrix} + \begin{pmatrix}
1 \\
-1 \\
3 \\
2 \\
0 \\
0
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
0 \\
0
\end{pmatrix}
\]

holds for \( i = 1, 2 \).

If we transpose the incidence matrices induced by Cheng and Gao [4] and Hashizume [5], it is same as the definite region choice matrix of the single counting rule modulo 2 up to permutations of rows and columns.

4. AN ALTERNATING INTEGRAL REGION CHOICE PROBLEM

Let \( D \) be a link diagram with \( d \) connected components and \( n \) crossings \( x_1, \ldots, x_n \), \( n \geq 1 \). Let \( R_1, \ldots, R_{n+d+1} \) be the regions of \( D \). In [5], Harada induced two region choice matrices \( A_{a1}(D) \) and \( A_{a2}(D) \) with \( n \) rows and \( n+d+1 \) columns as follows, where he denoted them by \( B_1(D) \) and \( B_2(D) \).

We determine each elements \( a_{ij}^{(a1)} \) as follows. We define \( a_{ij}^{(a1)} = 1 \), if the region \( R_j \) touches the crossing \( x_i \) at the corner contained by the underpass and the overpass counterclockwise, that means \( \bigcirc \) or \( \bigcirc \). We define \( a_{ij}^{(a1)} = -1 \), if \( R_j \) touches \( x_i \) at the corner contained by the underpass and the overpass clockwise, that means \( \bigcirc \) or \( \bigcirc \). If \( x_i \) does not lie on \( \partial R_j \), we define \( a_{ij}^{(a1)} = 0 \). The alternating region choice
matrix of the single counting rule for $D$ is the matrix $A_{a1}(D)$ with the element $a_{ij}^{(a1)}$ on the $i$-th row and the $j$-th column. We determine each element $a_{ij}^{(a2)}$ by

$$a_{ij}^{(a2)} = \begin{cases} 2a_{ij}^{(a1)} & \text{if $R_j$ touches $x_i$ twice as } \bigstar \bigstar \\ a_{ij}^{(a1)} & \text{otherwise} \end{cases}$$

The alternating region choice matrix of the double counting rule for $D$ is the matrix $A_{a2}(D)$ with the element $a_{ij}^{(a2)}$ on the $i$-th row and the $j$-th column.

We compare the definitions of $a_{ij}^{(d1)}$, $a_{ij}^{(d2)}$, $a_{ij}^{(a1)}$, $a_{ij}^{(a2)}$ on Table 1, where the region $R_j$ includes the corners marked with $\bigstar$ but does not include the unmarked corners around the crossing $x_i$.

| $x_i$ and $R_j$ | $\bigstar \bigstar$ or $\bigstar \bigstar$ | $\bigstar$ or $\bigstar$ | $\bigstar$ | otherwise |
|-----------------|-----------------|-----------------|-----------------|-----------------|
| $a_{ij}^{(d1)}$ | 1               | 1               | 1               | 1               | 0               |
| $a_{ij}^{(d2)}$ | 2               | 1               | 1               | 2               | 0               |
| $a_{ij}^{(a1)}$ | 1               | 1               | -1              | -1              | 0               |
| $a_{ij}^{(a2)}$ | 2               | 1               | -1              | -2              | 0               |

Table 1. $a_{ij}^{(d1)}$, $a_{ij}^{(d2)}$, $a_{ij}^{(a1)}$, $a_{ij}^{(a2)}$.

Using the alternating region choice matrices, the alternating integral region choice problem and the existence of solutions for it are stated as follows.

**Theorem 4.1** ([5]). Let $D$ be a knot diagram with $n$ crossings $x_1, \cdots, x_n$, $n \geq 1$. Let $R_1, \cdots, R_{n+2}$ be the regions of $D$.

1. Let $A_{a1}(D)$ be the alternating region choice matrix of the single counting rule for $D$. For any $c \in \mathbb{Z}^n$, there exists a solution $u \in \mathbb{Z}^{n+2}$ such that $A_{a1}(D)u + c = 0$.

2. Let $A_{a2}(D)$ be the alternating region choice matrix of the double counting rule for $D$. For any $c \in \mathbb{Z}^n$, there exists a solution $u \in \mathbb{Z}^{n+2}$ such that $A_{a2}(D)u + c = 0$.

**Example 4.2.** Let $D$ be the knot diagram given in Figure 4. Under certain orders of crossings and regions, we have

$$A_{a1}(D) = A_{a2}(D) = \begin{pmatrix} -1 & 1 & -1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & -1 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 & -1 & 1 \end{pmatrix}.$$ 

Figure 4 implies the equation

$$A_{a1}(D) \begin{pmatrix} -2 \\ -1 \\ 2 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

holds for $i = 1, 2$.

**Remark 4.3.** If we transpose the incidence matrix induced by Cheng and Gao [4] and Hashizume [6], it is same as the alternating region choice matrix of the single counting rule modulo 2 up to permutations of rows and columns.
Remark 4.4. In this article, we reverse signs of the elements in the alternating region choice matrices defined by Harada [5], since our alternating region choice matrix of the double counting rule coincides with the Alexander matrix defined in [2] if we substitute 1 for the variable. In [8], Kauffman illustrated the definition of the Alexander matrix as a crossing with labeled corners □. In his terms, our alternating region choice matrix and the definite region choice matrix of the double counting rule are denoted by □ and □ respectively. In [9], Kawauchi indicated that the transposed incidence matrix is same as the Alexander matrix substituted 1 modulo 2, and that the solvability of the original region choice problem on knot diagrams is induced by the fact the Alexander polynomial substituted 1 becomes 1 for any knot. This fact also implies that Theorem 4.1 (2).

We give more examples to compare definite and alternating region choice matrices of the single counting rule and of the double counting rule.

Example 4.5. Let $D$ be the link diagram given as the split sum of the knot diagram with only one crossing such that one region touches all crossings twice. The diagram $D$ represents a trivial $l$-component link. Under certain orders of crossings and regions, we obtain

$$A_{d1}(D) = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & \ldots & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & 0 & 0 & 0 & 0 & \ldots & 1 & 1 \end{pmatrix},$$

$$A_{d2}(D) = \begin{pmatrix} 2 & 1 & 1 & 0 & 0 & 0 & \ldots & 0 \\ 2 & 0 & 0 & 1 & 1 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 2 & 0 & 0 & 0 & 0 & \ldots & 1 & 1 \end{pmatrix},$$

and

$$A_{a1}(D) = \begin{pmatrix} -\varepsilon_1 & \varepsilon_1 & \varepsilon_1 & 0 & 0 & 0 & \ldots & 0 \\ -\varepsilon_2 & 0 & 0 & \varepsilon_2 & \varepsilon_2 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ -\varepsilon_l & 0 & 0 & 0 & 0 & \ldots & \varepsilon_l & \varepsilon_l \end{pmatrix},$$

$$A_{a2}(D) = \begin{pmatrix} -2\varepsilon_1 & \varepsilon_1 & \varepsilon_1 & 0 & 0 & 0 & \ldots & 0 \\ -2\varepsilon_2 & 0 & 0 & \varepsilon_2 & \varepsilon_2 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ -2\varepsilon_l & 0 & 0 & 0 & 0 & \ldots & \varepsilon_l & \varepsilon_l \end{pmatrix},$$

where $\varepsilon_i = 1$ if the $i$-th crossing is positive □, $\varepsilon_i = -1$ if it is negative □. Each of these matrices has $l$ rows and $2l + 1$ columns.

Example 4.6. Let $D$ be the link diagram given as the split sum of the knot diagram with only one crossing and the $l - 1$ copies of the trivial knot diagram □. The diagram $D$ represents a trivial $l$-component link. On $D$ with certain orders of regions, we obtain

$$A_{d1}(D) = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & \ldots & 0 \end{pmatrix},$$

$$A_{d2}(D) = \begin{pmatrix} 2 & 1 & 1 & 0 & 0 & 0 & \ldots & 0 \end{pmatrix},$$

and

$$A_{a1}(D) = \begin{pmatrix} -\varepsilon & \varepsilon & \varepsilon & 0 & 0 & 0 & \ldots & 0 \end{pmatrix},$$

$$A_{a2}(D) = \begin{pmatrix} -2\varepsilon & \varepsilon & \varepsilon & 0 & 0 & 0 & \ldots & 0 \end{pmatrix},$$

where $\varepsilon_i = 1$ if the $i$-th crossing is positive □, $\varepsilon_i = -1$ if it is negative □. Each of these matrices has $l$ rows and $2l + 1$ columns.
where $\varepsilon = 1$ if the crossing is positive, otherwise $\varepsilon = -1$, and the number of 0 appearing on each matrix is $l - 1$.

5. Kernel solutions from Alexander numberings

Let $D$ be a link diagram with $d$ connected components and $n$ crossings $x_1, \cdots, x_n$, $n \geq 1$. Let $R_1, \cdots, R_{n+d+1}$ be the regions of $D$. Let all crossings be equipped with 0. Then the integral region choice problems induce $\mathbb{Z}$-homomorphisms. We denote by $\Phi_d(D) : \mathbb{Z}^{n+d+1} \to \mathbb{Z}^n$ and $\Phi_a(D) : \mathbb{Z}^{n+d+1} \to \mathbb{Z}^n$ the induced homomorphisms with representation matrices $A_d(D)$ and $A_a(D)$ respectively, $i = 1, 2$. We call a vector $u \in \mathbb{Z}^{n+d+1}$ with $A_d(D)u = 0$ (resp. $A_a(D)u = 0$) a kernel solution for the definite region choice matrix of the single (resp. double) counting rule, similarly to that defined to knot projections in [1]. We call a vector $u \in \mathbb{Z}^{n+d+1}$ with $A_a(D)u = 0$ (resp. $A_a(D)u = 0$) a kernel solution for the alternating region choice matrix of the single (resp. double) counting rule, similarly to that defined to knot diagrams in [5].

**Lemma 5.1.** On any link diagram with at least one crossing, an Alexander numbering for an arbitrary orientation gives a kernel solution for an alternating region choice matrix of the double counting rule.

**Proof.** On the given oriented link diagram $D$, we fix an Alexander numbering for it. We take an arbitrary crossing $x$ of $D$. We may assume that $x$ lies as $\vartriangle$ or $\bigstar$ in $D$. We suppose that the index of the right region of $x$ is $p \in \mathbb{Z}$. Then the index of the left region of $x$ is $p + 2$ and the rest regions touching $x$ are $p + 1$. We have $p - (p + 1) + (p + 2) - (p - 1) = 0$ and $-p + (p + 1) - (p + 2) + (p - 1) = 0$. Then the alternating region choice obtained from the Alexander numbering does not change the scores of the crossings. \qed

Let $D$ be an oriented link diagram with ordered link components, and $D_i$ be a sub-diagram of $D$ representing $i$-th link component, $i = 1, \cdots, l$. We fix a sub-diagram $D_i$. We ignore the diagrams of link components other $D_i$, and take an Alexander numbering. Each region $R$ of the diagram $D$ is a subset of one region $S$ of the diagram $D_i$. Let $a_S$ be the integer assigned to $S$ by this Alexander numbering. We assign the integer $a_S$ to the region $R$ and denote it by $u_R$. We call this assignment of the integers to the region $\{u_R\}$ a componentwise Alexander numbering associated with $D_i$. Figure 5 gives an example of a pair of componentwise Alexander numberings on a 2-component link diagram.

**Figure 5.** Componentwise Alexander numberings.

**Lemma 5.2.** On any oriented link diagram with at least one crossing, each componentwise Alexander numbering gives a kernel solution for an alternating region choice matrix of the double counting rule.
Proof. Let $D$ be an oriented link diagram with at least one crossing and ordered link components, and $D_i$ be a sub-diagram of $D$ representing $i$-th link component, $i = 1, \ldots, l$. We fix a sub-diagram $D_i$. We take a componentwise Alexander numbering associated with $D_i$. Let $q$ be a crossing of $D$ and we denote the four corners touching $q$ by $C_q^1, C_q^2, C_q^3, C_q^4$ clockwise, and the regions on $D$ including $C_q^i$ by $R_q^i$, $j = 1, 2, 3, 4$. If $q$ is a crossing of $D_i$, the regions $R_q^1, R_q^2, R_q^3, R_q^4$ are assigned integers $r_1, r_2, r_3, r_4$ with $r_1 - r_2 + r_3 - r_4 = 0$ by Lemma 5.1. If $q$ is a crossing of an arc of $D_i$ and an arc of other components, we may assume that $R_q^1$ and $R_q^2$ are subsets of a region $S$ of the diagram $D_i$, and that $R_q^3$ and $R_q^4$ are subsets of a region $S'$ of the diagram $D_i$. Then the regions $R_q^1, R_q^2, R_q^3, R_q^4$ are assigned integers $r_1, r_1, r_3, r_3$ with $r_1 - r_3 = \pm 1$, and we have $r_2 - r_1 + r_3 - r_4 = 0$. If $q$ is a crossing not included in $D_i$, the regions $R_q^1, R_q^2, R_q^3, R_q^4$ are subsets of a region of the diagram $D_i$. Then they are assigned same integer $r_1$, and we have $r_1 - r_1 + r_3 - r_4 = 0$.

Therefore the componentwise Alexander numbering associated with $D_i$ becomes a kernel solution for the alternating integral region choice problem of double counting rule.

We can obtain kernel solutions for the definite region choice matrix from kernel solutions for the alternating region choice matrix and a fixed checkerboard coloring.

**Lemma 5.3.** For a given link diagram, we fix a checkerboard coloring. We take a kernel solution for an alternating region choice matrix of the double counting rule. For each region $R$, let $c_R$ and $u_R$ be the integers assigned by the checkerboard coloring and the kernel solution respectively. Assigning the integer $(-1)^c_R u_R$ to each region $R$, we obtain a kernel solution for a definite region choice matrix of the double counting rule.

Proof. For a crossing $x$ of the diagram, We denote the four corners touching $x$ by $C_1, C_2, C_3, C_4$ clockwise, and the regions including $C_j$ by $R_j$, $j = 1, 2, 3, 4$. Then we have $\pm(u_{R_1} - u_{R_2} + u_{R_3} - u_{R_4}) = 0$. We may assume $c_{R_1} = 0$. Then the equalities $c_{R_2} = 1, c_{R_3} = 0, c_{R_4} = 1$ hold. Hence we have

\[
\begin{align*}
(-1)^{c_{R_1}} u_{R_1} + (-1)^{c_{R_2}} u_{R_2} + (-1)^{c_{R_3}} u_{R_3} + (-1)^{c_{R_4}} u_{R_4} \\
= u_{R_1} - u_{R_2} + u_{R_3} - u_{R_4} \\
= 0.
\end{align*}
\]

**Lemma 5.4.** For a given link diagram, we fix a checkerboard coloring. We take a kernel solution for an alternating region choice matrix of the single counting rule. For each region $R$, let $c_R$ and $u_R$ be the integers assigned by the checkerboard coloring and the kernel solution respectively. Assigning the integer $(-1)^c_R u_R$ to each region $R$, we obtain a kernel solution for a definite region choice matrix of the single counting rule.

Proof. For a crossing $x$ of the diagram, We denote the four corners touching $x$ by $C_1, C_2, C_3, C_4$ clockwise, and the regions including $C_j$ by $R_j$, $j = 1, 2, 3, 4$. We may assume $c_{R_1} = 0$. Then the equalities $c_{R_2} = 1, c_{R_3} = 0, c_{R_4} = 1$ hold.

If $x$ is not reducible, then $R_j$'s are different each other and we have $\pm(u_{R_1} - u_{R_2} + u_{R_3} - u_{R_4}) = 0$. Hence we have

\[
\begin{align*}
(-1)^{c_{R_1}} u_{R_1} + (-1)^{c_{R_2}} u_{R_2} + (-1)^{c_{R_3}} u_{R_3} + (-1)^{c_{R_4}} u_{R_4} \\
= u_{R_1} - u_{R_2} + u_{R_3} - u_{R_4} \\
= 0.
\end{align*}
\]
We suppose that \( x \) is reducible. Then there exists just one pair of \( R_i \)'s coinciding each other. If \( R_1 \) coincides with \( R_3 \), the equality \( \pm (u_{R_1} - u_{R_2} - u_{R_4}) = 0 \) holds. Hence we have
\[
(-1)^{c_{R_1}}u_{R_1} + (-1)^{c_{R_2}}u_{R_2} + (-1)^{c_{R_3}}u_{R_4} = u_{R_1} - u_{R_2} - u_{R_4} = 0.
\]
Otherwise, \( R_2 \) coincides with \( R_4 \) and we have \( \pm (u_{R_1} - u_{R_2} + u_{R_3}) = 0 \). Hence we have
\[
(-1)^{c_{R_1}}u_{R_1} + (-1)^{c_{R_2}}u_{R_2} + (-1)^{c_{R_3}}u_{R_3} = u_{R_1} - u_{R_2} + u_{R_3} = 0.
\]

Similarly, we can obtain kernel solutions for the alternating region choice matrix from kernel solutions for the definite region choice matrix and a fixed checkerboard coloring.

**Lemma 5.5.** For a given link diagram, we fix a checkerboard coloring. We take a kernel solution for a definite region choice matrix of the double (resp. single) counting rule. For each region \( R \), let \( c_R \) and \( u_R \) be the integers assigned by the checkerboard coloring and the kernel solution respectively. Assigning the integer \( (-1)^{c_{R_1}}u_{R_1} \) to each region \( R \), we obtain a kernel solution for an alternating region choice matrix of the double (resp. single) counting rule. □

6. Solutions of the alternating integral region choice problem on knot diagrams

In this section, we give an alternative proof of Theorem 4.1.

First, we observe the alternating integral region choice problem of the double counting rule.

**Lemma 6.1.** Let \( D \) be a link diagram with \( n \) crossings, \( n \geq 1 \). We fix an arc \( \gamma \) in the link diagram \( D \), and let \( R \) and \( R' \) be two regions which are the both sides of the arc \( \gamma \). Then there exists a kernel solution \( u \) for \( A_{a_2}(D) \) such that the components of \( u \) corresponding to \( R \) and \( R' \) are 0 and 1 respectively.

**Proof.** By Lemma 5.1 or 5.2 there exists a kernel solution \( u \) for \( A_{a_2}(D) \) such that the components of \( u \) corresponding to \( R \) and \( R' \) are 0 and \( \pm 1 \) respectively. If \( R' \) is assigned \( -1 \), we multiply all components of \( u \) by \(-1\). □

Figure 6 gives an example of a link diagram with a kernel solution for an alternating region choice matrix such that two regions adjacent to the arc \( \gamma \) are assigned 0 and 1. This kernel solution is obtained from an Alexander numbering.

![Figure 6. A kernel solution for an alternating region choice matrix.](image-url)
Remark 6.2. In [5], Harada proved Lemma 6.1 for a knot diagram, showing that Reidemeister moves and crossing changes preserve the existence of the kernel solution, and that the knot diagram with only one crossing has a kernel solution. His argument is similar to that due to Ahara and Suzuki [1] for Lemma 7.1.

The following theorem also has been proved by Harada [5] for a knot diagram using Lemma 6.1. We give a proof using Lemma 5.2 instead.

Theorem 6.3. Let $D$ be a link diagram with $d$ connected components and $n \geq 1$. We take a crossing $x$ of $D$ of arcs in same link component. There exist $v_x \in \mathbb{Z}^{n+d+1}$ such that any components of $A_{a2}(D)v_x$ are 0 but the component of $A_{a2}(D)v_x$ to $x$ is 1.

Proof. The argument is similar to that for knot diagrams due to Harada [5]. We orient $D$ arbitrarily. We splice $D$ at $x$. On Figure 7, this splicing is illustrated as the transformation from top left to bottom left. The sub-diagram $D^0_x$ of the link component including $x$ splits to the diagrams of two link components $D^1_x$ and $D^2_x$. We note $D^1_x$ and $D^2_x$ may intersects each other as link projections. Let $\gamma_i$ be an oriented arc in $D^i_x$ appearing after the splice at $x$ for each $i = 1, 2$. We may assume that $\gamma_1$ lies on the left of $\gamma_2$. For the diagram $(D \setminus D^0_x) \cup D^1_x \cup D^2_x$, we take the componentwise Alexander numbering associated with $D^1_x$ such that the right and left regions of $\gamma_1$ are assigned 0 and 1 respectively. We denote this assignment of the indexes by $u'$. On Figure 7, $u'$ is illustrated on bottom right. By Lemma 5.2, $u'$ gives a kernel solution of $A_{a2}((D \setminus D^0_x) \cup D^1_x \cup D^2_x)$ if the spliced diagram has at least one crossing. We unsplice $(D \setminus D^0_x) \cup D^1_x \cup D^2_x$ to $D$ at $x$. Let $\varepsilon = 1$ if $x$ is a positive crossing, otherwise $\varepsilon = -1$. We assign the same integers to all regions of $D$ as the components of $\varepsilon u'$, where the integer assigned to the region between $\gamma_1$ and $\gamma_2$ is assigned to the two regions splitting at $x$. On Figure 7, this unsplicing is illustrated as the transformation from bottom right to top right, where the crossing $x$ is negative and we have $\varepsilon = -1$. Then we obtain the desired $v_x \in \mathbb{Z}^{n+d+1}$. □
Theorem 6.3 implies Theorem 4.1 (2), that is the existence of a solution of an alternating integral region choice problem of the double counting rule for a knot diagram, by the same argument as that due to Harada \[5\].

**Proof of Theorem 4.1 (2).** Applying Theorem 6.3 for each crossing \(x_i\), there exist \(v_i \in \mathbb{Z}^{n+d+1}\) such that any components of \(A_{a_2}(D)v_i\) are 0 but the \(i\)-th component of \(A_{a_2}(D)v_i\) is 1, \(i = 1, 2, \ldots, n\). Let \(c_i\) be the \(i\)-th component of \(c\). If we take \(u = -\sum_{i=1}^{n} c_i v_i\), then we have \(A_{a_2}(D)u + c = 0\). \(\square\)

Next, we observe the alternating integral region choice problem of the single counting rule. The following lemma has been proved by Harada \[5\] for knot diagrams.

**Lemma 6.4.** Let \(D\) be a link diagram with \(n\) crossings, \(n \geq 1\). We fix an arc \(\gamma\) in the link diagram \(D\), and let \(R\) and \(R'\) be two regions which are the both sides of the arc \(\gamma\). We take two arbitrary integers \(a\) and \(b\). Then there exists a kernel solution \(u\) for \(A_{a_1}(D)\) such that the components of \(u\) corresponding to \(R\) and \(R'\) are \(a\) and \(b\) respectively.

**Proof.** The argument is same as that for knot diagrams due to Harada \[5\]. His argument is similar to that due to Ahara and Suzuki \[1\] for Lemma 7.4. We use an induction on the number of reducible crossings.

If the given link diagram \(D\) is irreducible, the matrices \(A_{a_1}(D)\) and \(A_{a_2}(D)\) coincide. We apply Lemma 6.3 to the pairs \(R, R'\) and \(R', R\) in order to a kernel solutions \(u'\) and \(u''\) respectively. Then the components of \(u'\) corresponding to \(R\) and \(R'\) are 0 and 1 respectively, and the components of \(u''\) corresponding to \(R\) and \(R'\) are 1 and 0 respectively. Therefore \(u = au'' + bu'\) is the desired kernel solution for \(A_{a_1}(D)\) on the irreducible diagram \(D\).

**Figure 8.** Splicing at \(y\) and obtaining a kernel solution.

We assume that there exists a desired kernel solution if the number of reducible crossings is less than \(k\). We suppose that the link diagram \(D\) has \(k\) reducible crossings. We take a reducible crossing \(y\), and orient \(D\) arbitrarily. We splice \(D\) at \(y\). The diagram \(D\) splits to the disjoint link diagrams \(D^1_y\) and \(D^2_y\). Each of them has less reducible crossings than \(k\). We may assume that the given arc \(\gamma\) lies on \(D^1_y\). On Figure 8 we obtain the middle diagram splicing the reducible crossing \(y\) of
the left diagram, where we omit over/under information for $y$ and the orientation of $D$. Let $\gamma_i$ be an arc in $D_y^i$ appearing after the splice at $y$ for each $i = 1, 2$. We denote the region between the arcs $\gamma_1$ and $\gamma_2$ by $R_y^0$, and another region adjacent to $\gamma_j$ by $R_y^j$, $j = 1, 2$. We ignore $D_y^2$. If $D_y^1$ has no crossing, we assign $a$ and $b$ to the regions including $R_y^0$ and $R_y^1$ respectively, and 0 to other regions of $D_y^1$ where we may assign arbitrary integers. If $D_y^1$ has at least one crossing, we apply the assumption of induction to $D_y^1$ and $\gamma$. Then we obtain a kernel solution for $A_{a1}(D_y^1)$ whose components corresponding to the regions including $R_y^0$ and $R_y^1$ are $a$ and $b$ respectively. Let $c \in \mathbb{Z}$ assigned to $R_y^0$ and $d \in \mathbb{Z}$ to the region of $D_y^1$ including $R_y^0$. On the middle of Figure 8 we write $c$ and $d$, though we omit $\gamma_i$ and $R_y^j$. We assign the integer $-c + d$ to the region $R_y^2$ on $D_y^1 \cup D_y^2$ as the middle of Figure 8. We ignore $D_y^2$. If the diagram $D_y^1$ has no crossing, we assign 0 to the regions of $D_y^1$ including neither $R_y^0$ nor $R_y^2$, though we may assign arbitrary integers. Otherwise we apply the assumption of the induction to $D_y^2$ and $\gamma_2$, then we obtain a kernel solution for $A_{a1}(D_y^2)$ whose components corresponding to $R_y^0$ and the region including $R_y^2$ are $-c + d$ and $d$ respectively. Let $\tilde{R}$ be a region of the diagram $D_y^1 \cup D_y^2$. If $\tilde{R}$ is $R_y^0$, we assign $d$ to $\tilde{R} = R_y^0$. Otherwise $\tilde{R}$ coincides with one of regions of $D_y^1$ or $D_y^2$, then we assign to $\tilde{R}$ same integer as the region of $D_y^1$ or $D_y^2$. Therefore we obtain a kernel solution $u'$ of $A_{a1}(D_y^1 \cup D_y^2)$. We unsplice $D_y^1 \cup D_y^2$ at $y$. We assign the same integer as either $u'$ to all regions of $D_y$, where the region touching $y$ twice is assigned $d$, as illustrated on the right of Figure 8. Then we obtain the desired kernel solution for $A_{a1}(D)$ since we have $c - d + (-c + d) = 0$. □

The following lemma also has been proved by Harada [5] for knot diagrams. He proved it as a corollary to Lemma 6.4: the region $R_y^2$ in the proof of Lemma 6.4 is assigned $-c + d + \varepsilon$ instead of $-c + d$, where $\varepsilon = 1$ if $y$ is positive, otherwise $\varepsilon = -1$. We give an alternative proof.

**Lemma 6.5.** Let $D$ be a link diagram with $d$ connected components and $n$ crossings, $n \geq 1$. Let $y$ be a reducible crossing of $D$. There exist $v_y \in \mathbb{Z}^{n+d+1}$ such that any components of $A_{a1}(D)v_y$ are 0 but the component of $A_{a1}(D)v_y$ to $y$ is 1.

![Figure 9. Obtaining $v_y$ for the reducible and positive crossing $y$.](image-url)
Proof. We orient $D$ arbitrarily. We splice $D$ at the reducible crossing $y$. The diagram $D$ splits to the two disjoint link diagrams $D^1_y$ and $D^2_y$. Let $\gamma_i$ be an arc in $D^i_y$ appearing after the splice at $y$ for each $i = 1, 2$. We denote the region between the arcs $\gamma_1$ and $\gamma_2$ by $R^0_y$, and another region adjacent to $\gamma_j$ by $R^1_y$, $R^2_y$, $j = 1, 2$. We assign integers $0, 0, 1$ to $R^0_y$, $R^1_y$, $R^2_y$ respectively, as illustrated on the middle of Figure 9 or 10. If $D^2_y$ has at least one crossing, we apply Lemma 6.4 to $D^2_y$ and $\gamma_2$, in order to obtain a kernel solution for $A_{a1}(D^2_y)$ such that the region including $R^0_y$ is assigned 0 and $R^2_y$ is assigned 1. If $D^2_y$ has no crossing, we assign 0 to the regions of $D^2_y$ including neither $R^0_y$ nor $R^2_y$, though we may assign arbitrary integers. We assign 0 to all regions of $D^1_y$: it gives the trivial kernel solution for $A_{a1}(D^1_y)$ if $D^1_y$ has at least one crossing. Each of the other regions of $D^1_y \cup D^2_y$ than $R^0_y$, $R^1_y$, $R^2_y$ is a region of $D^1_y$ or $D^2_y$, then it has been assigned an integer. Therefore we obtain a kernel solution $u'$ for $A_{a1}(D^1_y \cup D^2_y)$ such that the components of $u'$ corresponding to $R^0_y$, $R^1_y$, $R^2_y$ are 0, 0, 1 respectively. Let $\varepsilon = 1$ if $y$ is a positive crossing, otherwise $\varepsilon = -1$. We unsplice at $y$ and assign the same integers to all regions of $D$ as $\varepsilon u'$, where the region touching $y$ twice is assigned 0. On the right of Figure 9 the crossing $y$ is positive and regions of $D$ are assigned the components of $u'$. On the right of Figure 10 the crossing $y$ is negative and regions of $D$ are assigned the components of $-u'$. Then we obtain the desired $v_y \in \mathbb{Z}^{n+1}$.

Combining Lemma 6.5 with Theorem 4.3 (2), we obtain the proof of Theorem 4.1 (1), that is the existence of a solution of an alternating integral region choice problem of the single counting rule for a knot diagram, by the same argument as that due to Harada [5].

Proof of Theorem 4.1 (1). If the given knot diagram $D$ is irreducible, then we have $A_{a1}(D) = A_{a2}(D)$ and a solution of the double counting rule, which has been obtained, is also a solution of the single counting rule.

We suppose that the knot diagram $D$ has at least one reducible crossing. For a region $R_j$, let $X_j$ be the set of reducible crossings touched by $R_j$ twice, $j = 1, \ldots, n + 2$. A set $X_j$ might be empty. Applying Lemma 6.5 for each reducible crossing $y \in X_j$, we obtain $v_y \in \mathbb{Z}^{n+2}$ such that any components of $A_{a1}(D)v_y$ are 0 but the component of $A_{a1}(D)v_y$ to $y$ is 1. We take $r_j \in \mathbb{Z}^{n+2}$ such that any components of $r_j$ are 0 but the $j$-th component is 1. Choosing $R_j$ once corresponds to solving the problem for $X_j$. By taking $v_y \oplus r_j$ and $v_y \ominus r_j$, respectively, we are able to choose the crossing $y$.
with \(r_j\). By the definitions of the alternating region choice matrices, we have

\[
A_{a2}(D)r_j - A_{a1}(D)r_j = \sum_{y \in \mathcal{X}_j} A_{a1}(D)v_y.
\]

Applying Theorem 4.1 (2), which has been proved, we obtain a solution of double counting rule, \(w \in \mathbb{Z}^{n+2}\) with \(A_{a2}(D)w + c = 0\). Let \(w_j\) be the \(j\)-th component of \(w\), \(j = 1, \ldots, n + 2\). We note \(w = \sum_j w_j r_j\). We take \(u = w + \sum_j w_j \sum_{y \in \mathcal{X}_j} v_y\). Then we obtain the desired components of Lemma 6.4, there exists a kernel solution \(u\) of \(A\) of \(u\). Applying Lemma 6.5, we obtain

\[
A_{a1}(D)u = A_{a1}(D)w + \sum_j w_j \sum_{y \in \mathcal{X}_j} A_{a1}(D)v_y
\]

\[
= \sum_j w_j A_{a2}(D)r_j
\]

\[
= A_{a2}(D)w
\]

\[
= -c.
\]

Therefore we have \(A_{a1}(D)u + c = 0\).

\[\square\]

Remark 6.6. In Section 8 we prove that the first and second results of Theorem 4.1 are equivalent.

The proof of Lemma 6.5 for knot diagrams due to Harada 5 implies the following fact. We give an alternative proof.

Lemma 6.7. Let \(D\) be a link diagram with \(d\) connected components and \(n\) crossings, \(n \geq 1\). Let \(y\) be a reducible crossing of \(D\). We fix an arc \(\gamma\) in the link diagram \(D\), and let \(R\) and \(R'\) be two regions which are the both sides of the arc \(\gamma\). We take two arbitrary integers \(a\) and \(b\). There exist \(v_y \in \mathbb{Z}^{n+d+1}\) such that any components of \(A_{a1}(D)v_y\) are 0 but the component of \(A_{a1}(D)v_y\) to \(y\) is 1, and such that the components of \(v_y\) corresponding to \(R\) and \(R'\) are \(a\) and \(b\) respectively.

Proof. Applying Lemma 6.5 we obtain \(v_y' \in \mathbb{Z}^{n+d+1}\) such that any components of \(A_{a1}(D)v_y'\) are 0 but the component of \(A_{a1}(D)v_y'\) to \(y\) is 1. We denote the components of \(v_y\) corresponding to \(R\) and \(R'\) by \(a'\) and \(b'\) respectively. Applying Lemma 6.5 there exists a kernel solution \(u\) for \(A_{a1}(D)\) such that the components of \(u\) corresponding to \(R\) and \(R'\) are \(a - a'\) and \(b - b'\) respectively. Let \(v_y = u + v_y'\). Then we obtain the desired \(v_y\).

\[\square\]

We note that we use Lemma 6.5 but does not use Lemma 6.7 to prove Theorem 4.1 (1) in this article.

Remark 6.8. In 10, Shimizu took checkerboard colorings to show that a region crossing change is an unknotting operation on a knot diagram. In the above argument for Theorem 4.1 we take Alexander numberings instead of checkerboard colorings. Then an extension of her argument is given.

7. Solutions of the Definite Integral Region Choice Problem on Knot Diagrams

In this section, we give an alternative proof of Theorem 4.1. First, we observe the definite integral region choice problem of the double counting rule.
Lemma 7.1. Let $D$ be a link diagram or projection with $n$ crossings, $n \geq 1$. We fix an arc $\gamma$ in the link diagram $D$, and let $R$ and $R'$ be two regions which are the both sides of the arc $\gamma$. Then there exists a kernel solution for $A_{d_2}(D)$ such that the components of $u$ corresponding to $R$ and $R'$ are 0 and 1 respectively.

Proof. If $D$ is a link projection, we arbitrarily add over/under information to crossings. Then we may assume that $D$ is a link diagram. By Lemma 6.1, there exists a kernel solution $\tilde{u}$ for $A_{a_2}(D)$ such that the components of $\tilde{u}$ corresponding to $R$ and $R'$ are 0 and 1 respectively. We fix a checkerboard coloring such that the region $R'$ is assigned 0. Applying Lemma 5.3 to $\tilde{u}$, we obtain a kernel solution $u$ for $A_{d_2}(D)$ such that the components of $u$ corresponding to $R$ and $R'$ are 0 and 1 respectively. $\square$

Figure 11 gives an example of a link diagram with a kernel solution for a definite region choice matrix such that two regions adjacent to the arc $\gamma$ are assigned 0 and 1. This kernel solution is obtained from Figure 6 applying Lemma 5.3.

Remark 7.2. In [1], Ahara and Suzuki proved Lemma 7.1 for a knot diagram, showing that Reidemeister moves preserve the existence of the kernel solution, and that the knot diagram with only one crossing has a kernel solution.

The following theorem also has been proved by Ahara and Suzuki [1] for a knot diagram splicing at the given crossing and applying Lemma 7.1. We give an alternative proof below.

Theorem 7.3. Let $D$ be a link diagram or projection with $d$ connected components and $n$ crossings, $n \geq 1$. We take a crossing $x$ of $D$ of arcs in same link component. There exist $v_x \in \mathbb{Z}^{n+d+1}$ such that any components of $A_{d_2}(D)v_x$ are 0 but the component of $A_{d_2}(D)v_x$ to $x$ is 1.

Proof. If $D$ is a link projection, we arbitrarily add over/under information to crossings. Then we may assume that $D$ is a link diagram. By Theorem 6.3, there exist $w_x \in \mathbb{Z}^{n+d+1}$ such that any components of $A_{a_2}(D)w_x$ are 0 but the component of $A_{a_2}(D)w_x$ to $x$ is 1. For each region $R$, let $w_R$ be the component of $w_x$ to $R$. We denote the four corners touching $x$ by $C_1, C_2, C_3, C_4$ clockwise, and the regions including $C_j$ by $R_j$, $j = 1, 2, 3, 4$. Then we have $w_{R_1} - w_{R_3} + w_{R_3} - w_{R_4} = 1$. We fix the checkerboard coloring such that the region $R_1$ is assigned 0. For each region $R$, let $c_R$ be the integer assigned by this checkerboard coloring. Let $v_x$ be the vector in $\mathbb{Z}^{n+d+1}$ such that the component to $R$ is $(-1)^{c_R}w_R$. By similar argument to the proof of Lemma 5.3, it is shown that any components of $A_{a_2}(D)v_x$ are 0 but the component of $A_{a_2}(D)v_x$ to $x$ is 1. Figure 12 gives an example of the above process from $w_x$ to $v_x$, where $w_x$ is a solution illustrated on top right of Figure 7. $\square$
In the above proof, we do not use Lemma 7.1 immediately. Theorem 7.3 implies Theorem 3.1 (2), that is the existence of a solution of a definite integral region choice problem of the double counting rule for a knot diagram, by the same argument as that due to Ahara and Suzuki [1].

Proof of Theorem 3.1 (2). Applying Theorem 7.3 for each crossing $x_i$, there exist $v_i \in \mathbb{Z}_n^{d+1}$ such that any components of $A_{d2}(D)v_i$ are 0 but the $i$-th component of $A_{d2}(D)v_i$ to $x_i$ is 1, $i = 1, 2, \ldots, n$. Let $c_i$ be the $i$-th component of $c$. If we take $u = -\sum_{i=1}^n c_i v_i$, then we have $A_{d2}(D)u + c = 0$. □

Next, we observe the definite integral region choice problem of the single counting rule. The following lemma has been proved by Ahara and Suzuki [1] for knot diagrams. We give an alternative proof.

Lemma 7.4. Let $D$ be a link diagram or projection with $n$ crossings, $n \geq 1$. We fix an arc $\gamma$ in the link diagram $D$, and let $R$ and $R'$ be two regions which are the both sides of the arc $\gamma$. We take two arbitrary integers $a$ and $b$. Then there exists a kernel solution $u$ for $A_{d1}(D)$ such that the components of $u$ corresponding to $R$ and $R'$ are $a$ and $b$ respectively.

Proof. If $D$ is a link projection, we arbitrarily add over/under information to crossings. Then we may assume that $D$ is a link diagram. Fix a checkerboard coloring of $D$. Let $c_R$ and $c_{R'}$ be the assigned integers to $R$ and $R'$ respectively by the fixed checkerboard coloring. By Lemma 6.4, there exists a kernel solution $w_x$ such that the components of $w_x$ to $x_i$ are $(-1)^{s_x} a$ and $(-1)^{s_x} b$ respectively. We apply Lemma 5.4 to $w_x$. Then we obtain a kernel solution $u$ for $A_{d1}(D)$ such that the components of $u$ corresponding to $R$ and $R'$ are $a$ and $b$ respectively. □

The following lemma also has been proved by Ahara and Suzuki [1] for knot diagrams. They proved it as a corollary to Lemma 7.4. We give an alternative proof.

Lemma 7.5. Let $D$ be a link diagram or projection with $d$ connected components and $n$ crossings, $n \geq 1$. Let $y$ be a reducible crossing of $D$. There exist $v_y \in \mathbb{Z}_n^{d+1}$ such that any components of $A_{d1}(D)v_y$ are 0 but the component of $A_{d1}(D)v_y$ to $y$ is 1.

Proof. If $D$ is a link projection, we arbitrarily add over/under information to crossings. Then we may assume that $D$ is a link diagram. By Lemma 6.5, there exist $w_y \in \mathbb{Z}_n^{d+1}$ such that any components of $A_{d1}(D)w_y$ are 0 but the component of $A_{d1}(D)w_y$ to $y$ is 1. For each region $R$, let $w_R$ be the component of $w_y$ to $R$. □

![Figure 12. Finding $v_x$ such that any components of $A_{d2}(D)v_x$ are 0 but the component of $A_{d2}(D)v_x$ to $x$ is 1.](image-url)
We denote the four corners touching \( x \) by \( C_1, C_2, C_3, C_4 \) clockwise, and the regions including \( C_j \) by \( R_j, j = 1, 2, 3, 4 \). We may assume that \( R_2 \) coincides with \( R_1 \). Then we have \( w_{R_1} - w_{R_2} - w_{R_3} = \pm 1 \). If \( w_{R_1} - w_{R_2} - w_{R_4} = 1 \), that is the crossing \( y \) is negative, we fix the checkerboard coloring such that the region \( R_1 \) is assigned 0 . Otherwise, we fix the checkerboard coloring such that the region \( R_1 \) is assigned 1 . For each region \( R \), let \( c_R \) be the integer assigned by the fixed checkerboard coloring. Let \( \nu_y \) be the vector in \( \mathbb{Z}^{n+2} \) such that the component to \( R \) is \( (-1)^{c_R} w_R \). By similar argument to the proof of Lemma 5.4 it is shown that any components of \( A_{d1}(D)\nu_y \) are 0 but the component of \( A_{d1}(D)\nu_y \) to \( y \) is 1 .

Combining Lemma 7.5 with Theorem 3.1 (2), we obtain the proof of Theorem 3.1 (1), that is the existence of a solution of a definite integral region choice problem of the single counting rule for a knot diagram, by the same argument as that due to Ahara and Suzuki [1].

**Proof of Theorem 3.1 (1).** If the given knot diagram or projection \( D \) is irreducible, then we have \( A_{d1}(D) = A_{d2}(D) \) and a solution of the double counting rule, which has been obtained, is also a solution of the single counting rule.

We suppose that the knot diagram or projection \( D \) has at least one reducible crossing. For a region \( R_j \), let \( X_j \) be the set of reducible crossings touched by \( R_j \) twice, \( j = 1, \cdots , n + 2 \). A set \( X_j \) might be empty. Applying Lemma 7.5 for each reducible crossing \( y \in X_j \), we obtain \( \nu_y \in \mathbb{Z}^{n+2} \) such that any components of \( A_{d1}(D)\nu_y \) are 0 but the component of \( A_{d1}(D)\nu_y \) to \( y \) is 1 . We take \( r_j \in \mathbb{Z}^{n+2} \) such that any components of \( r_j \) are 0 but the \( j \)-th component is 1 . Choosing \( R_j \) once corresponds with \( r_j \). By the definitions of the definite region choice matrices, we have

\[
A_{d2}(D)r_j - A_{d1}(D)r_j = \sum_{y \in X_j} A_{d1}(D)\nu_y.
\]

Applying Theorem 3.1 (2), which has been proved, we obtain a solution of double counting rule, \( w \in \mathbb{Z}^{n+2} \) with \( A_{d2}(D)w + c = 0 \). Let \( w_j \) be the \( j \)-th component of \( w \), \( j = 1, \cdots , n + 2 \). We note \( w = \sum_{j} w_j r_j \). We take \( u = w + \sum_{j} w_j \sum_{y \in X_j} \nu_y \).

Then we have

\[
A_{d1}(D)u = A_{d1}(D)w + \sum_{j} w_j \sum_{y \in X_j} A_{d1}(D)\nu_y
\]

\[
= \sum_{j} w_j \left( A_{d1}(D)r_j + \sum_{y \in X_j} A_{d1}(D)\nu_y \right)
\]

\[
= \sum_{j} w_j A_{d2}(D)r_j
\]

\[
= A_{d2}(D)w
\]

\[
= -c.
\]

Therefore we have \( A_{d1}(D)u + c = 0 \). □

**Remark 7.6.** In Section 9 we prove that the first and second results of Theorem 3.1 are equivalent.

The proof of Lemma 7.5 for knot diagrams due to Ahara and Suzuki [1] implies the following fact. We give an alternative proof.

**Lemma 7.7.** Let \( D \) be a link diagram or projection with \( d \) connected components and \( n \) crossings, \( n \geq 1 \). Let \( y \) be a reducible crossing of \( D \). We fix an arc \( \gamma \) in the link diagram \( D \), and let \( R \) and \( R' \) be two regions which are the both sides of the arc \( \gamma \) and the crossing \( y \).
We take two arbitrary integers \( a \) and \( b \). There exist \( v_y \in \mathbb{Z}^{n+d+1} \) such that any components of \( A_d(D)v_y \) are 0 but the component of \( A_d(D)v_y \) to \( y \) is 1, and such that the components of \( v_y \) corresponding to \( R \) and \( R' \) are \( a \) and \( b \) respectively.

Proof. Applying Lemma 7.5, we obtain \( v'_y \in \mathbb{Z}^{n+d+1} \) such that any components of \( A_d(D)v'_y \) are 0 but the component of \( A_d(D)v'_y \) to \( y \) is 1. We denote the components of \( v'_y \) corresponding to \( R \) and \( R' \) by \( a' \) and \( b' \) respectively. Applying Lemma 7.4, there exists a kernel solution \( u \) for \( A_d(D) \) such that the components of \( u \) corresponding to \( R \) and \( R' \) are \( a-a' \) and \( b-b' \) respectively. Let \( v_y = u + v'_y \). Then we obtain the desired \( v_y \).

We note that we use Lemma 7.5 but does not use Lemma 7.7 to prove Theorem 3.1 (1) in this article.

8. Region choice matrices of the double counting rule

From now on, we change link projections to link diagrams adding over/under information to crossings arbitrarily.

Applying the arguments in the original proofs of Theorem 3.1 and 4.1 in [1, 5] to link diagrams, the ranks of the definite and alternating region choice matrices are determined. In this section, we show that on the double counting rule.

Theorem 8.1. Let \( D \) be a diagram of an \( l \)-component link. We assume that \( D \) has \( d \) connected components and \( n \) crossings, \( n \geq 1 \). Then each rank of the definite and alternating region choice matrices of the double counting rule, \( A_{d2}(D) \) and \( A_{a2}(D) \), is \( n + d - l \), and each rank of the \( \mathbb{Z} \)-submodules \( \{ u \in \mathbb{Z}^{n+d+1} \mid A_{d2}(D)u = 0 \} \) and \( \{ u \in \mathbb{Z}^{n+d+1} \mid A_{a2}(D)u = 0 \} \) is \( l + 1 \).

If we transpose the incidence matrix induced by Cheng and Gao [4] and Hashizume [6], it is same as the definite region choice matrix of the single counting rule up to permutations of rows and columns. This transposed matrix also coincides with the alternating region choice matrix of the single counting rule modulo 2 up to permutations of rows and columns. For irreducible diagrams, Theorem 8.1 is an extension of their result on the rank of the incidence matrices.

It is well known that we can make any link diagram into a diagram of a trivial link after some crossing changes, and that some Reidemeister moves can transform any pair of non-trivial diagrams of a trivial link each other. We prove Theorem 8.1 using a similar argument to that in Appendix A of the article written by Ahara and Suzuki [1]. That means it is similar to the proof of the invariance for the Alexander polynomial in [2].

Lemma 8.2. If we change a crossing of a link diagram admitting integral region choices, the changed diagram also admit it, the definite region choice matrix of double counting rule is preserved, and the row concerning this crossing is multiplied by \(-1\) for the alternating region choice matrix of the double counting rule.

Proof. It is cleared by the definitions of region choice matrices.

Remark 8.3. In [5], Harada proved that crossings changes preserve kernel solutions for alternating region choice matrices of the double counting rule on any knot diagrams. Lemma 8.2 implies that his claim holds on any link diagrams.

Lemma 8.4. A Reidemeister move I between link diagrams admitting integral region choices preserves the rank of the \( \mathbb{Z} \)-submodules of kernel solutions for definite and alternating region choice matrices of the double counting rule.
Figure 13. Reidemeister moves I among the diagrams $D_+$, $D$, and $D_-$.

Proof. (cf. [1]) Let the middle of Figure 13 be an arc of a link diagram $D$ admitting an integral region choice, and we denote by $D_+$ and $D_-$ the obtained diagrams from $D$ by a Reidemeister move I at the arc as illustrated on the right side and the left side of Figure 13 respectively. We may order the regions of $D$ such that the upper and lower regions on the middle of Figure 13 are ordered 1 and 2 respectively. We denote the definite and the alternating region choice matrices of the double counting rule for $D$ by

$$A_{d2}(D) = (a_d \ b_d \ P_2), \quad A_{a2}(D) = (a_a \ b_a \ P_a).$$

Inserting the row and the column corresponding to the added crossing and the added region respectively, we obtain the matrices for $D_+$ and $D_-,

$$A_{d2}(D_+) = A_{d2}(D_-) = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & a_d & b_d & P_2 \end{pmatrix},$$

$$A_{a2}(D_+) = \begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & a_a & b_a & P_a \end{pmatrix}, \quad A_{a2}(D_-) = \begin{pmatrix} -1 & 2 & -1 & 0 \\ 0 & a_a & b_a & P_a \end{pmatrix}.$$}

Each of the matrices for $D_+$ and $D_-$ has one more column than $D$, and the equalities $\text{rank}A_{d2}(D_+) = \text{rank}A_{d2}(D) + 1$ and $\text{rank}A_{a2}(D_-) = \text{rank}A_{a2}(D) + 1$ hold. Then the $\mathbb{Z}$-submodules of the kernel solutions for these matrices have same rank. □

Lemma 8.5. A Reidemeister move II between link diagrams admitting integral region choices preserves the rank of the $\mathbb{Z}$-submodules of kernel solutions for definite and alternating region choice matrices of the double counting rule.

Figure 14. A Reidemeister move II between the diagrams $D$ and $D'$.

Proof. (cf. [1]) Let the left side of Figure 14 be two arcs of a link diagram $D$ admitting an integral region choice, and we denote by $D'$ the obtained diagram from $D$ by a Reidemeister move II around the arcs as illustrated on the right side of Figure 14.

If the Reidemeister move II does not change the number of the connected components of the diagram, and the regions appearing around the move are different each other, then we may order the regions of $D$ such that the bottom, middle and top regions of the left side of Figure 14 are ordered 1, 2 and 3 respectively. We denote the definite and the alternating region choice matrices of the double counting rule for $D$ by

$$A_{d2}(D) = (a_d \ b_d \ c_d \ P_d), \quad A_{a2}(D) = (a_a \ b_a \ c_a \ P_a).$$
Inserting the rows and the column corresponding to the added crossings and the added region respectively, we obtain the matrices for $D'$,

$$A_{d2}(D') = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & O \\ 1 & 1 & 0 & 1 & 1 \\ 0 & a_d & b_d' & b_d'' & c_d & P_d \end{pmatrix},$$

where $b_d' + b_d'' = b_d$, and

$$A_{a2}(D') = \begin{pmatrix} 1 & -1 & 1 & 0 & -1 & O \\ -1 & 1 & 0 & -1 & 1 \\ 0 & a_a & b_a' & b_a'' & c_a & P_a \end{pmatrix},$$

where $b_a' + b_a'' = b_a$. Each of the matrices for $D'$ has two more columns than $D$. Adding the fourth column to the third column, and taking the first column off the second, third and fifth columns on $A_{d2}(D')$, we obtain the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & O \\ 1 & 0 & 0 & 1 & 0 \\ 0 & a_d & b_d' & b_d'' & c_d & P_d \end{pmatrix},$$

and the equality $\text{rank}A_{d2}(D') = \text{rank}A_{d2}(D) + 2$. Adding the fourth column to the third column, and the first column to the second and fifth columns, and taking the first column multiplied by 2 off the second column on $A_{a2}(D')$, we obtain the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & O \\ -1 & 0 & 0 & -1 & 0 \\ 0 & a_a & b_a' & b_a'' & c_a & P_a \end{pmatrix},$$

and the equality $\text{rank}A_{a2}(D') = \text{rank}A_{a2}(D) + 2$. Then the $\mathbb{Z}$-submodules of the kernel solutions for these matrices have same rank.

If the Reidemeister move II does not change the number of the connected components of the diagram, and the top and bottom regions coincide, then we may order the regions of $D$ such that the upper and middle regions of the left side of Figure 14 are ordered 1, and 2 respectively. We denote the definite and the alternating region choice matrices of the double counting rule for $D$ by

$$A_{d2}(D) = (a_d \ b_d \ P_d), \quad A_{a2}(D) = (a_a \ b_a \ P_a).$$

Inserting the rows and the column corresponding to the added crossings and the added region respectively, we obtain the matrices for $D'$,

$$A_{d2}(D') = \begin{pmatrix} 1 & 2 & 1 & 0 & O \\ 1 & 2 & 0 & 1 \\ 0 & a_d & b_d' & b_d'' & P_d \end{pmatrix},$$

where $b_d' + b_d'' = b_d$, and

$$A_{a2}(D') = \begin{pmatrix} 1 & -2 & 1 & 0 & O \\ -1 & 2 & 0 & -1 \\ 0 & a_a & b_a' & b_a'' & P_a \end{pmatrix},$$

where $b_a' + b_a'' = b_a$. Each of the matrices for $D'$ has two more columns than $D$. Adding the fourth column to the third column, taking the first column off the third column, and taking the first column multiplied by 2 off the second column on $A_{d2}(D')$, we obtain the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & O \\ 1 & 0 & 0 & 1 \\ 0 & a_d & b_d' & b_d'' & P_d \end{pmatrix},$$
and the equality $\text{rank} A_{a2}(D) = \text{rank} A_{a2}(D') + 2$. Adding the fourth column to the third column, the first column multiplied by 2 to the second column, and taking the first column off the third column on $A_{a2}(D')$, we obtain the matrix
\[
\begin{pmatrix}
1 & 0 & 0 & 0 & O \\
-1 & 0 & 0 & -1 \\
0 & a_d & b_d & b_d' & P_d
\end{pmatrix},
\]
and the equality $\text{rank} A_{a2}(D') = \text{rank} A_{a2}(D) + 2$. Then the $\mathbb{Z}$-submodules of the kernel solutions for these matrices have same rank.

If the Reidemeister move II changes the number of the connected components of the diagram, then we may order the regions of $D$ such that the bottom, middle and top regions of the left side of Figure 14 are ordered 1, 2 and 3 respectively. We denote the definite and the alternating region choice matrices of the double counting rule for $D$ by
\[
A_{d2}(D) = (a_d \ b_d \ c_d \ P_d), \quad A_{a2}(D) = (a_a \ b_a \ c_a \ P_a).
\]
Inserting the rows and the column corresponding to the added crossings and the added region respectively, we obtain the matrices for $D'$,
\[
A_{d2}(D') = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
0 & a_d & b_d & c_d & P_d
\end{pmatrix}, \quad A_{a2}(D') = \begin{pmatrix}
1 & -1 & 1 & -1 & 1 \\
-1 & 1 & -1 & 1 \\
0 & a_a & b_a & c_a & P_a
\end{pmatrix}.
\]
Each of the matrices for $D'$ has one more column than $D$, and the equalities $\text{rank} A_{d2}(D') = \text{rank} A_{d2}(D) + 1$ and $\text{rank} A_{a2}(D') = \text{rank} A_{a2}(D) + 1$ hold. Then the $\mathbb{Z}$-submodules of the kernel solutions for these matrices have same rank.

**Lemma 8.6.** A Reidemeister moves III between link diagrams admitting integral region choices preserves the ranks of the $\mathbb{Z}$-submodules of kernel solutions for definite and alternating region choice matrices of the double counting rule.

**Figure 15.** A Reidemeister move III between the diagrams $D_\sigma$ and $D_\Delta$.

**Proof.** (cf. [2]) Let the left side of Figure 15 be a neighborhood of a triangle region on a link diagram $D_\sigma$ admitting an integral region choice, and we denote by $D_\Delta$ the obtained diagram from $D_\sigma$ by a Reidemeister move III around the triangle region as illustrated on the right side of Figure 15.

If regions appearing around the move are different each other, then we may order the regions of $D_\sigma$ such that the triangle region on Figure 15 is ordered 1 and that other six regions around are ordered 2, 3, 4, 5, 6, 7 clockwise from top left. The definite and the alternating region choice matrices of the double counting rule for $D_\sigma$ are
\[
A_{d2}(D_\sigma) = \begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & a_d & b_d & c_d & d_d & e_d & f_d & P_d
\end{pmatrix},
\]
After multiplying the top three rows on \( A \) column by \(-D\) the triangle of where the upper left crossing, the upper right crossing, and the lower crossing of the triangle of \( D\) are ordered 1, 2, and 3. We order the crossings of \( D\) such that the lower right crossing, the lower left crossing, and the upper crossing are ordered 1, 2, and 3, and that the others are ordered as \( D\). Then we obtain the matrices for \( D\),

\[
A_{d_2}(D) = \begin{pmatrix}
-1 & -1 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & -1 & 1 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 & -1 & 1 & -1 \\
0 & a_a & b_a & c_a & d_a & e_a & f_a & P_a \\
\end{pmatrix},
\]

where the upper left crossing, the upper right crossing, and the lower crossing of the triangle of \( D\) are ordered 1, 2, and 3.

We order the crossings of \( D\) such that the lower right crossing, the lower left crossing, and the upper crossing are ordered 1, 2, and 3, and that the others are ordered as \( D\). Then we obtain the matrices for \( D\),

\[
A_{d_2}(D) = \begin{pmatrix}
1 & 0 & 0 & 1 & 1 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & a_a & b_a & c_a & d_a & e_a & f_a & P_a \\
\end{pmatrix},
\]

After multiplying the top three rows on \( A\) column by \(-1\), if we add the first column multiplied by \(-a\) we obtain

\[
A_{d_2}(D) = \begin{pmatrix}
1 & 0 & 0 & 1 & 1 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & a_a & b_a & c_a & d_a & e_a & f_a & P_a \\
\end{pmatrix},
\]

\[
A_{d_2}(D) = \begin{pmatrix}
1 & 0 & 0 & 1 & 1 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & a_a & b_a & c_a & d_a & e_a & f_a & P_a \\
\end{pmatrix},
\]

After multiplying the top three rows on \( A\) column by \(-1\), if we add the first column multiplied by \(-a\) we obtain

\[
A_{d_2}(D) = \begin{pmatrix}
1 & 0 & 0 & 1 & 1 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & a_a & b_a & c_a & d_a & e_a & f_a & P_a \\
\end{pmatrix},
\]

After multiplying the top three rows on \( A\) column by \(-1\), if we add the first column multiplied by \(-a\) we obtain

\[
A_{d_2}(D) = \begin{pmatrix}
1 & 0 & 0 & 1 & 1 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & a_a & b_a & c_a & d_a & e_a & f_a & P_a \\
\end{pmatrix},
\]

After multiplying the top three rows on \( A\) column by \(-1\), if we add the first column multiplied by \(-a\) we obtain

\[
A_{d_2}(D) = \begin{pmatrix}
1 & 0 & 0 & 1 & 1 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & a_a & b_a & c_a & d_a & e_a & f_a & P_a \\
\end{pmatrix},
\]

After multiplying the top three rows on \( A\) column by \(-1\), if we add the first column multiplied by \(-a\) we obtain

\[
A_{d_2}(D) = \begin{pmatrix}
1 & 0 & 0 & 1 & 1 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & a_a & b_a & c_a & d_a & e_a & f_a & P_a \\
\end{pmatrix},
\]

After multiplying the top three rows on \( A\) column by \(-1\), if we add the first column multiplied by \(-a\) we obtain

\[
A_{d_2}(D) = \begin{pmatrix}
1 & 0 & 0 & 1 & 1 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & a_a & b_a & c_a & d_a & e_a & f_a & P_a \\
\end{pmatrix},
\]

After multiplying the top three rows on \( A\) column by \(-1\), if we add the first column multiplied by \(-a\) we obtain

\[
A_{d_2}(D) = \begin{pmatrix}
1 & 0 & 0 & 1 & 1 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & a_a & b_a & c_a & d_a & e_a & f_a & P_a \\
\end{pmatrix},
\]

After multiplying the top three rows on \( A\) column by \(-1\), if we add the first column multiplied by \(-a\) we obtain

\[
A_{d_2}(D) = \begin{pmatrix}
1 & 0 & 0 & 1 & 1 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & a_a & b_a & c_a & d_a & e_a & f_a & P_a \\
\end{pmatrix},
\]

After multiplying the top three rows on \( A\) column by \(-1\), if we add the first column multiplied by \(-a\) we obtain

\[
A_{d_2}(D) = \begin{pmatrix}
1 & 0 & 0 & 1 & 1 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & a_a & b_a & c_a & d_a & e_a & f_a & P_a \\
\end{pmatrix},
\]
the first column multiplied by 2 to the second column, then we obtain \( A_{d_2}(D_\Delta) \). The matrices \( A_{d_2}(D_\tau) \) and \( A_{d_2}(D_\Delta) \) have same size and same rank, then the \( \mathbb{Z} \)-submodules of the kernel solutions for the matrices have same rank. Adding the first column to the third, fourth, and sixth columns, and taking the first column off the fifth column and the first column multiplied by 2 off the second column, we obtain \( A_{a_2}(D_\tau) \) from \( A_{a_2}(D_\tau) \). The matrices \( A_{a_2}(D_\tau) \) and \( A_{a_2}(D_\Delta) \) have same size and same rank, then the \( \mathbb{Z} \)-submodules of the kernel solutions for these matrices have same rank.

If the top left, top right, and the bottom middle regions of the left side of Figure 15 coincide on \( D_\tau \), then this and the other regions appearing around the move are different each other. We may order the regions of \( D_\tau \) such that the triangle region on Figure 15 is ordered 1 and that other four regions around are ordered 2, 3, 4, 5 clockwise from top left. The definite and the alternating region choice matrices of the double counting rule for \( D_\tau \) are

\[
A_{d_2}(D_\tau) = \begin{pmatrix}
1 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 & 0 & O \\
1 & 1 & 0 & 1 & 1 \\
0 & a_d & b_d & d_d & f_d & P_d \end{pmatrix},
\]

\[
A_{a_2}(D_\tau) = \begin{pmatrix}
-1 & -1 & 1 & 0 & 1 \\
1 & 1 & -1 & -1 & 0 & O \\
1 & 1 & 0 & -1 & -1 \\
0 & a_a & b_a & d_a & f_a & P_a \end{pmatrix}.
\]

Then we obtain the matrices for \( D_\Delta \),

\[
A_{d_2}(D_\Delta) = \begin{pmatrix}
1 & 2 & 0 & 1 & 0 \\
1 & 2 & 0 & 0 & 1 & O \\
1 & 2 & 1 & 0 & 0 \\
0 & a_d & b_d & d_d & f_d & P_d \end{pmatrix},
\]

\[
A_{a_2}(D_\Delta) = \begin{pmatrix}
-1 & 2 & 0 & -1 & 0 \\
1 & -2 & 0 & 0 & 1 & O \\
1 & -2 & 1 & 0 & 0 \\
0 & a_a & b_a & d_a & f_a & P_a \end{pmatrix}.
\]

After multiplying the top three rows on \( A_{d_2}(D_\tau) \) by \(-1\) and multiplying the first column by \(-1\), if we add the first column to the third, fourth and fifth columns, and the first column multiplied by 3 to the second column, we obtain \( A_{a_2}(D_\Delta) \). The matrices \( A_{d_2}(D_\tau) \) and \( A_{d_2}(D_\Delta) \) have same size and same rank, then the \( \mathbb{Z} \)-submodules of the kernel solutions for the matrices have same rank. Adding the first column to the third, fourth, and fifth columns, and taking the first column multiplied by 3 off the second columns, we obtain \( A_{a_2}(D_\Delta) \) from \( A_{a_2}(D_\tau) \). The matrices \( A_{a_2}(D_\tau) \) and \( A_{a_2}(D_\Delta) \) have same size and same rank, then the \( \mathbb{Z} \)-submodules of the kernel solutions for these matrices have same rank.

If the top left and top right regions of the left side of Figure 15 coincide on \( D_\tau \), and if the bottom left and bottom right regions also coincide on \( D_\tau \), we may order the regions of \( D_\tau \) such that the triangle region on Figure 15 is ordered 1 and that other four regions around are ordered 2, 3, 4, 5 clockwise from top left. The definite and the alternating region choice matrices of the double counting rule for \( D_\tau \) are

\[
A_{d_2}(D_\tau) = \begin{pmatrix}
1 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 0 & O \\
1 & 0 & 0 & 2 & 1 \\
0 & a_d & b_d & d_d & e_d & P_d \end{pmatrix},
\]
\[
A_{a_2}(D_\Sigma) = \begin{pmatrix}
-1 & -1 & 1 & 1 & 0 \\
1 & 1 & -1 & -1 & 0 \\
1 & 0 & 0 & -2 & 1 \\
0 & a_a & b_a & d_a & e_a & P_a \\
\end{pmatrix}.
\]

Then we obtain the matrices for \( D_\Delta \),

\[
A_{d_2}(D_\Delta) = \begin{pmatrix}
1 & 1 & 0 & 1 & 1 & O \\
1 & 1 & 0 & 1 & 1 \\
1 & 2 & 1 & 0 & 0 \\
0 & a_d & b_d & d_d & e_d & P_d \\
\end{pmatrix},
\]

\[
A_{a_2}(D_\Delta) = \begin{pmatrix}
-1 & 1 & 0 & -1 & 1 \\
1 & -1 & 0 & 1 & -1 \\
1 & -2 & 1 & 0 & 0 \\
0 & a_a & b_a & d_a & e_a & P_a \\
\end{pmatrix}.
\]

After multiplying the top three rows on \( A_{d_2}(D_\Sigma) \) by \(-1\) and multiplying the first column by \(-1\), if we add the first column to the third and fifth columns, and the first column multiplied by \(-2\) to the second and fourth columns, then we obtain \( A_{a_2}(D_\Delta) \). The matrices \( A_{d_2}(D_\Sigma) \) and \( A_{d_2}(D_\Delta) \) have same size and same rank, then the \( \mathbb{Z} \)-submodules of the kernel solutions for the matrices have same rank. Adding the first column multiplied by \(-2\), \(1\), \(-2\) and \(-1\) to the second, third, fourth, and fifth columns respectively, we obtain \( A_{a_2}(D_\Sigma) \) from \( A_{a_2}(D_\Delta) \). The matrices \( A_{a_2}(D_\Sigma) \) and \( A_{a_2}(D_\Delta) \) have same size and same rank, then the \( \mathbb{Z} \)-submodules of the kernel solutions for these matrices have same rank.

The proof for the other cases are given by the similar arguments as above, or reduced to the one of the above cases applying Lemma 8.2. □

**Proof of Theorem 8.1.** It is well known that we can make any link diagram into a diagram of a trivial link after some crossing changes, and that some Reidemeister moves can transform the non-trivial diagram of a trivial link to the split sum of the knot diagram with only one crossing and the \( l-1 \) copies of the trivial knot diagram, which is given in Example 4.6. On this obtained diagram \( D_0 \) with certain orders of crossings and regions, we obtain

\[
A_{d_2}(D_0) = \begin{pmatrix} 2 & 1 & 1 & 0 & 0 & 0 & \ldots & 0 \end{pmatrix},
\]

and

\[
A_{a_2}(D_0) = \begin{pmatrix} -2\varepsilon & \varepsilon & \varepsilon & 0 & 0 & 0 & \ldots & 0 \end{pmatrix},
\]

where \( \varepsilon = 1 \) if the crossing is positive, otherwise \( \varepsilon = -1 \), and the number of 0 appearing on each matrix is \( l-1 \). The both matrices have \( l+2 \) columns. Their ranks are equal to \( 1 = 1 + l - l \). Then the desired claim holds for \( D_0 \). By Lemma 8.2, 8.4, 8.5, and 8.6, crossing changes and Reidemeister moves preserve this claim. Then this theorem is proved. □

**Remark 8.7.** As commented in Remark 6.2 and 7.2 Ahara and Suzuki [1] and Harada [5] showed that Reidemeister moves and crossing changes preserve the existence of the kernel solution, and that the knot diagram with only one crossing has a kernel solution, to prove Lemma 6.1 and 7.1 for a knot diagram. Their arguments are extended to that for the proof of Theorem 8.1, though Harada did not describe how alternating region choice matrices are affected.
9. Region choice matrices of the single counting rule

Before determining the rank of region choice matrices of the single counting rule, we observe the images of the homomorphisms \( \Phi_d(D), \Phi_{a1}(D), \Phi_{a2}(D) \) defined in Section 4. The arguments in the proofs of Theorem 3.1 (1) and 4.1 (1) given in Section 7 and 8 imply the following result.

**Lemma 9.1.** Let \( D \) be a diagram of an \( l \)-component link. We assume that \( D \) has \( d \) connected components and \( n \) crossings, \( n \geq 1 \). We take \( c \in \mathbb{Z}^n \).

1. If there exists a solution \( w \in \mathbb{Z}^{n+d+1} \) such that \( A_{d2}(D)w + c = 0 \), then there exists a solution \( u \in \mathbb{Z}^{n+d+1} \) such that \( A_{d1}(D)u + c = 0 \).
2. If there exists a solution \( w \in \mathbb{Z}^{n+d+1} \) such that \( A_{a2}(D)w + c = 0 \), then there exists a solution \( u \in \mathbb{Z}^{n+d+1} \) such that \( A_{a1}(D)u + c = 0 \).

By a similar argument, the converse of Lemma 9.1 is shown as follows.

**Lemma 9.2.** Let \( D \) be a diagram of an \( l \)-component link. We assume that \( D \) has \( d \) connected components and \( n \) crossings \( x_1, \ldots, x_n \), \( n \geq 1 \). Let \( R_1, \ldots, R_{n+d+1} \) be the regions of \( D \). We take \( c \in \mathbb{Z}^n \).

1. If there exists a solution \( u \in \mathbb{Z}^{n+d+1} \) such that \( A_{d1}(D)u + c = 0 \), then there exists a solution \( w \in \mathbb{Z}^{n+d+1} \) such that \( A_{d2}(D)w + c = 0 \).
2. If there exists a solution \( u \in \mathbb{Z}^{n+d+1} \) such that \( A_{a1}(D)u + c = 0 \), then there exists a solution \( w \in \mathbb{Z}^{n+d+1} \) such that \( A_{a2}(D)w + c = 0 \).

**Proof.** If the given diagram \( D \) is irreducible, then we have \( A_{d1}(D) = A_{d2}(D) \) and \( A_{a1}(D) = A_{a2}(D) \).

We suppose that the diagram \( D \) has at least one reducible crossing. For a region \( R_j \), let \( X_j \) be the set of reducible crossings touched by \( R_j \) twice, \( j = 1, \ldots, n+d+1 \). A set \( X_j \) might be empty. We take \( r_j \in \mathbb{Z}^{n+d+1} \) such that any components of \( r_j \) are 0 but the \( j \)-th component is 1. Choosing \( R_j \) once corresponds with \( r_j \).

1. We suppose that \( A_{d1}(D)u + c = 0 \). Let \( u_j \) be the \( j \)-th component of \( u \), \( j = 1, \ldots, n+d+1 \). Applying Theorem 7.3 for each reducible crossing \( y \in X_j \), we obtain \( v_y' \in \mathbb{Z}^{n+d+1} \) such that any components of \( A_{d2}(D)v_y' \) are 0 but the component of \( A_{d2}(D)v_y' \) to \( y \) is 1. By the definitions of the definite region choice matrices, we have
   \[
   A_{d2}(D)r_j - A_{d1}(D)r_j = \sum_{y \in X_j} A_{d2}(D)v_y'.
   \]

   We note \( u = \sum_j u_j r_j \). We take \( w = u - \sum_j u_j \sum_{y \in X_j} v_y' \). Then we have
   \[
   A_{d2}(D)w = A_{d2}(D)u - \sum_j u_j \sum_{y \in X_j} A_{d2}(D)v_y' = \sum_j u_j \left( A_{d2}(D)r_j - \sum_{y \in X_j} A_{d2}(D)v_y' \right) = \sum_j u_j A_{d1}(D)r_j = A_{d1}(D)u = -c.
   \]

Therefore we have \( A_{d2}(D)w + c = 0 \).
(2) We suppose that 
\[ A_{a_1}(D)u + c = 0. \]
Let \( u_j \) be the \( j \)-th component of \( u \), \( j = 1, \cdots , n + d + 1 \). Applying Theorem 6.3 for each reducible crossing \( y \in X_j \), we obtain \( v'_y \in \mathbb{Z}^{n+d+1} \) such that any components of \( A_{a_2}(D)v'_y \) are 0 but the component of \( A_{a_2}(D)v'_y \) to \( y \) is 1. By the definitions of the definite region choice matrices, we have

\[ A_{a_2}(D)r_j - A_{a_1}(D)r_j = \sum_{y \in X_j} A_{a_2}(D)v'_y. \]

We note \( u = \sum_j u_j r_j \). We take \( w = u - \sum_j u_j \sum_{y \in X_j} v'_y \). Then we have

\[
A_{a_2}(D)w = A_{a_2}(D)u - \sum_j u_j \sum_{y \in X_j} A_{a_2}(D)v'_y
\]

\[
= \sum_j u_j \left( A_{a_2}(D)r_j - \sum_{y \in X_j} A_{a_2}(D)v'_y \right)
\]

\[
= \sum_j u_j A_{a_1}(D)r_j
\]

\[
= A_{a_1}(D)u
\]

\[
= -c.
\]

Therefore we have \( A_{a_2}(D)w + c = 0. \)

\[ \square \]

By Lemma 9.1 and 9.2, we obtain the following result.

**Theorem 9.3.** Let \( D \) be a link diagram. We assume that \( D \) has at least one crossing.

1. The image of the homomorphism \( \Phi_{d_1}(D) \) coincides with the image of the homomorphism \( \Phi_{d_2}(D) \).
2. The image of the homomorphism \( \Phi_{a_1}(D) \) coincides with the image of the homomorphism \( \Phi_{a_2}(D) \).

\[ \square \]

The above theorem implies that the existence of a solution of the single counting rule coincides with that of the double counting rule for each of the both integral region choice problem. Particularly, the first and second results of Theorem 3.1 are equivalent, and the first and second results of Theorem 4.1 are equivalent.

By Theorem 8.1, 9.3, and the homomorphism theorem, we obtain the following theorem.

**Theorem 9.4.** Let \( D \) be a diagram of an \( l \)-component link. We assume that \( D \) has \( d \) connected components and \( n \) crossings, \( n \geq 1 \). Then each rank of the definite and alternating region choice matrices of the single counting rule, \( A_{d_1}(D) \) and \( A_{a_1}(D) \), is \( n + d - l \), and each rank of the \( \mathbb{Z} \)-submodules \( \{ u \in \mathbb{Z}^{n+d+1} | A_{d_1}(D)u = 0 \} \) and \( \{ u \in \mathbb{Z}^{n+d+1} | A_{a_1}(D)u = 0 \} \) is \( l + 1 \).

\[ \square \]

In [4], Chen and Gao determined the \( \mathbb{Z}_2 \)-rank of the incidence matrix for connected link diagrams. In [6], Hashizume generalized their result to disconnected diagrams and connected diagrams, and determined the rank, which she called the \( \mathbb{Z}_2 \)-dimension, of the \( \mathbb{Z}_2 \)-submodule of kernel solutions for the homomorphism induced from region crossing changes on diagrams. The ranks obtained in Theorem 8.1 and 9.3 are same values as their ranks. If we transpose their incidence matrix, it coincides with the definite region choice matrix of the single counting rule up to
permutations of rows and columns. This transposed matrix also coincides with the alternating region choice matrix of the single counting rule modulo 2 up to permutations of rows and columns. Hence Theorem 8.1 and 9.4 are integral extensions of their results.

10. Standard kernel solutions of the double counting rule

In [6, 7], Hashizume studied structures of the \( \mathbb{Z}_2 \)-homomorphism induced by region crossing changes on link diagrams. Particularly, she gave a basis of the kernel of the homomorphism on an irreducible link diagram. In this section, we observe the kernels of the \( \mathbb{Z}_2 \)-homomorphisms \( \Phi_a \) and \( \Phi_d \) given in Section 5.

**Lemma 10.1.** On any link diagram with at least one crossing, assigning a same integer to all regions gives a kernel solution for an alternating region choice matrix of the double counting rule.

**Proof.** Any crossing is touched by four corners of regions, and we have \( p - p + p - p = 0 \) for any integer \( p \). \( \square \)

We denote by \( u_\infty \) the kernel solution assigning 1 to all regions, as illustrated in Figure 16.

![Figure 16. The kernel solution \( u_\infty \).](image)

Let \( D \) be an oriented link diagram with ordered link components, and \( D_i \) be a sub-diagram of \( D \) representing \( i \)-th link component, \( i = 1, \cdots, l \). If the oriented link diagram \( D \) is a diagram on the plane \( \mathbb{R}^2 \), we may denote the unbounded region by \( R_\infty(D) \). If \( D \) is a diagram on the sphere \( S^2 = \mathbb{R}^2 \cup \{\infty\} \), we may denote the region including the infinite point by \( R_\infty(D) \). For each sub-diagram \( D_i \), we denote by \( u_i \), the kernel solution obtained by Lemma 5.2 from a componentwise Alexander numbering associated with \( D_i \) such that the region \( R_\infty(D) \) is assigned 0. We shall call \( u_i \) the standard kernel solution associated with \( D_i \). Figure 17 same as Figure 5 gives examples of standard kernel solutions on the same link diagram as Figure 16 and 10. The kernel solution given by an Alexander numbering for \( D \) is equal to \( ru_\infty + \sum_{i=1}^{l} u_i \) where the region \( R_\infty(D) \) is assigned the integer \( r \) in the Alexander numbering.

**Theorem 10.2.** Let \( D \) be an oriented link diagram with \( l \) ordered link components and at least one crossing, and \( R_\infty = R_\infty(D) \) be the above region. The set of the above kernel solutions \( u_1, \cdots, u_l \), and \( u_\infty \) is a basis of the kernel of the homomorphism induced by the alternating integral region choice problem of double counting rule.
Proof. For the linear independence of \( u_1, \cdots, u_l, u_{\infty} \), it is sufficient to prove that the standard kernel solutions are linearly independent, since the region \( R_{\infty} \) is assigned 1 by \( u_{\infty} \) and 0 by \( u_i \), \( i = 1, \cdots, l \). We use an induction on \( l \). If \( D \) is a knot diagram, the standard kernel solution associated with \( D \) has at least one component equal to 1 or \(-1\). Then it is linearly independent. We assume that \( l \geq 2 \) and that the standard kernel solutions are linearly independent on any oriented link diagram with less components than \( l \). Let \( D \) be an oriented link diagram with \( l \) components. We take a point \( p \) on \( \partial R_{\infty}(D) \) except crossings. We may assume that \( p \) lies on \( l \)-th link component diagram \( D_l \). We suppose \( \sum_{i=1}^{l} n_i u_i = 0 \), \( n_i \in \mathbb{Z} \). Let \( R_p \) be a region of \( D \) with \( p \in \partial R_p \) and \( R_p \neq R_{\infty} \). The region \( R_p \) is assigned 1 or \(-1\) by \( u_i \) and 0 by each of the other standard kernel solutions. Then we have \( n_l = 0 \) and \( \sum_{i=1}^{l-1} n_i u_i = 0 \). On the diagram obtained from \( D \) ignoring \( D_l \), the standard kernel solutions are linearly independent by the assumption of the induction. Then \( n_1, \cdots, n_{l-1} \) should be 0. Hence the standard kernel solutions on \( D \) are linearly independent.

Let \( x \) be a kernel solution of the homomorphism \( \Phi_{u2}(D) \). By Theorem 8.1, the rank of the kernel of the homomorphism \( \Phi_{u2}(D) \) is \( l + 1 \). Then \( x, u_1, \cdots, u_l, u_{\infty} \) are linearly dependent, since \( u_1, \cdots, u_l, u_{\infty} \) are linearly independent. Hence \( x = \sum_{i=1}^{l} q_i u_i + q_{\infty} u_{\infty} \) holds for certain rational numbers \( q_1, \cdots, q_l, q_{\infty} \). We show that \( q_1, \cdots, q_l, q_{\infty} \) are integers. The region \( R_{\infty} \) is assigned 1 by \( u_{\infty} \) and 0 by \( u_i \), \( i = 1, \cdots, l \). Hence \( q_{\infty} \) is an integer since the components of \( x \) are integers. Then it is sufficient to show that \( q_1, \cdots, q_l \) are integers if the all components of \( \sum_{i=1}^{l} q_i u_i \) are integers. We use an induction on \( l \). If \( D \) is a knot diagram, the standard kernel solution associated with \( D \) has at least one component equal to 1 or \(-1\). Then \( q_1 \in \mathbb{Z} \) holds. We assume that \( l \geq 2 \) and that the desired claim holds for any oriented link diagram with less components than \( l \). Let \( D \) be an oriented link diagram with \( l \) components. The above region \( R_p \) is assigned 1 or \(-1\) by \( u_i \) and 0 by each of the other standard kernel solutions. Hence \( q_l \in \mathbb{Z} \) holds and all components of \( \sum_{i=1}^{l-1} q_i u_i \) are integers. We apply the assumption of the induction to the diagram obtained from \( D \) ignoring \( D_l \). Then we have \( q_1, \cdots, q_{l-1} \in \mathbb{Z} \).

Therefore \( x \) is a linear combination of \( u_1, \cdots, u_l, u_{\infty} \) over \( \mathbb{Z} \). Then the set of the kernel solutions \( u_1, \cdots, u_l, u_{\infty} \) is a basis of the kernel of the homomorphism \( \Phi_{u2}(D) \). \( \square \)
For the above link diagram $D$, we fix a checkerboard coloring. We apply Lemma 5.3 to the above basis $u_1, \ldots, u_l, u_\infty$. Then we obtain kernel solutions of the definite integral region choice problem of double counting rule. We denote them by $\bar{u}_1, \ldots, \bar{u}_l, \bar{u}_\infty$.

**Theorem 10.3.** Let $D$ be an oriented link diagram with $l$ ordered link components and at least one crossing, and $R_\infty = R_\infty(D)$ be the above region. The set of the above kernel solutions $\bar{u}_1, \ldots, \bar{u}_l, \bar{u}_\infty$ is a basis of the kernel of the homomorphism induced by the definite integral region choice problem of double counting rule.

**Proof.** The linear independence of $\bar{u}_1, \ldots, \bar{u}_l, \bar{u}_\infty$ is shown by the similar argument to that for $u_1, \ldots, u_l, u_\infty$ in the proof of Theorem 10.2. Let $y$ be a kernel solution of the homomorphism $\Phi_{d_2}(D)$. By Theorem 8.1, the rank of the kernel of the homomorphism $\Phi_{d_2}(D)$ is $l + 1$. Then $y, \bar{u}_1, \ldots, \bar{u}_l, \bar{u}_\infty$ are linearly dependent, since $\bar{u}_1, \ldots, \bar{u}_l, \bar{u}_\infty$ are linearly independent. By the similar argument to that for a kernel solution $x$ of $\Phi_{a_2}(D)$ in the proof of Theorem 10.2, $y$ is a linear combination of $\bar{u}_1, \ldots, \bar{u}_l, \bar{u}_\infty$ over $\mathbb{Z}$. Therefore the set of the kernel solutions $\bar{u}_1, \ldots, \bar{u}_l, \bar{u}_\infty$ is a basis of the kernel of the homomorphism $\Phi_{d_2}(D)$. □

Figure 18 gives an example of the basis obtained by Theorem 10.3.

![Figure 18. The basis $\bar{u}_1$, $\bar{u}_2$, $\bar{u}_\infty$.](image)

Theorem 10.2 and 10.3 are extensions of the result due to Hashizume [6] on a region crossing change. Her basis of the kernel of $\mathbb{Z}_2$-homomorphism induced by region crossing changes is same as the basis given in Theorem 10.2 and the basis given in Theorem 10.3 modulo 2.

### 11. Images of homomorphisms induced by integral region choice problems

Let $D$ be a link diagram with at least one crossing. For each $i = 1, 2$, the system of linear equations $A_{d_i}(D)u + c = 0$ (resp. $A_{a_i}(D)u + c = 0$) is solvable if and only if $c$ lies in the image of the homomorphism $\Phi_{d_i}(D)$ (resp. $\Phi_{a_i}(D)$). In this section, we discuss about the images of the homomorphisms induced by integral region choice problems.

By Theorem 3.1 and 4.1, the homomorphisms $\Phi_{d_1}(D), \Phi_{d_2}(D), \Phi_{a_1}(D), \Phi_{a_2}(D)$ defined in Section 5 are surjective if $D$ is a knot diagram. Otherwise they are not surjective in general by Theorem 8.1 and 9.4.

In [4], Cheng and Gao proved that a region crossing change on a 2-component link diagram is an unknotting operation if and only if the linking number is even, showing that changing two crossings of different components on a 2-component link diagram is represented by certain region crossing changes.

For example, the canonical diagram of $(2, 4)$-torus link changes to a diagram of the trivial link by one region choice. Otherwise there exists an equipment of integers to the crossings on this diagram such that the definite and alternating integral region
choice problem does not have any solution. On the canonical diagram of \((2, 4)\)-torus link with certain orders of crossings and regions, the definite region choice matrix is

\[
A_d = \begin{pmatrix}
1 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 \\
\end{pmatrix}
\]

and the alternating region choice matrix is

\[
A_a = \begin{pmatrix}
-1 & 1 & 1 & -1 & 0 & 0 \\
0 & 1 & 1 & -1 & -1 & 0 \\
0 & 1 & 1 & 0 & -1 & -1 \\
-1 & 1 & 1 & 0 & 0 & -1 \\
\end{pmatrix}.
\]

The system of linear equations \(A_du + c = 0\) has a solution \(u \in \mathbb{Z}^6\) if and only if \(c_1 - c_2 + c_3 - c_4 = 0\) holds where \(c_i\) is the \(i\)-th component of \(c \in \mathbb{Z}^4\). The system of linear equations \(A_a u + c = 0\) is solvable if and only if \(c_1 - c_2 + c_3 - c_4 = 0\) holds. Then in the case \((c_1, c_2, c_3, c_4) = (1, 0, 0, -1)\), any \(u \in \mathbb{Z}^6\) does not hold \(A_d u + c = 0\) or \(A_a u + c = 0\), though we have

\[
A_d \begin{pmatrix}
1 \\
0 \\
0 \\
0 \\
0 \\
\end{pmatrix} = \begin{pmatrix}
1 \\
0 \\
0 \\
0 \\
1 \\
\end{pmatrix} = -\begin{pmatrix}
1 \\
0 \\
0 \\
0 \\
0 \\
\end{pmatrix} \in \mathbb{Z}_2^4, \quad A_a \begin{pmatrix}
1 \\
0 \\
0 \\
0 \\
0 \\
\end{pmatrix} = \begin{pmatrix}
-1 \\
0 \\
0 \\
0 \\
-1 \\
\end{pmatrix} = -\begin{pmatrix}
1 \\
0 \\
0 \\
0 \\
-1 \\
\end{pmatrix} \in \mathbb{Z}_2^4.
\]

In [7], Hashizume gave a generating system of the image of the \(\mathbb{Z}_2\)-homomorphism induced by region crossing changes on a link diagram. Their results include the following results.

**Lemma 11.1 ([8 7]).** Let \(D\) be a connected diagram of two-component link with \(n\) crossings. We take two distinct crossings \(x, y\) of \(D\) of arcs in different link components. There exist \(v_{xy} \in \mathbb{Z}_2^{n+2}\) such that any components of \(A_d(D)v_{xy} \in \mathbb{Z}_2^n\) are 0 but the components of \(A_a(D)v_{xy} \in \mathbb{Z}_2^n\) to \(x\) and \(y\) are 1.

**Theorem 11.2 ([7]).** Let \(D\) be a connected diagram of two-component link with \(n\) crossings. The image of the \(\mathbb{Z}_2\)-homomorphism induced by region crossing changes is generated by the elements in \(\mathbb{Z}_2^n\) of the following two types:

1. any components are 0 but the only one component corresponding to a crossing in same link component is 1;
2. any components are 0 but the only two components corresponding to two distinct crossings of distinct link components are 1.

We note that a connected diagram of two-component link has at least two crossings because of the Jordan curve theorem.

We extend Lemma 11.1 to the alternating integral region choice problem as follows, where we define \(\varepsilon_x = 1\) for a positive crossing \(x\) and \(\varepsilon_x = -1\) for a negative crossing \(x\).

**Lemma 11.3.** Let \(D = D_1 \cup D_2\) be a connected diagram of two-component oriented link with \(n\) crossings, where \(D_1\) and \(D_2\) are sub-diagram of \(D\) representing the first and the second components respectively. We take two distinct crossings \(x, y\) of \(D_1\) and \(D_2\). We suppose that \(D_2\) crosses \(D_1\) from the right to the left at \(x\).
(1) If $D_2$ crosses $D_1$ from the left to the right at $y$, then there exist $v_{xy} \in \mathbb{Z}^{n+2}$ such that any components of $A_2(D)v_{xy} \in \mathbb{Z}^n$ are 0 but the components of $A_2(D)v_{xy} \in \mathbb{Z}^n$ to $x$ and $y$ are $\varepsilon_x$ and $\varepsilon_y$.

(2) If $D_2$ crosses $D_1$ from the right to the left at $y$, then there exist $v_{xy} \in \mathbb{Z}^{n+2}$ such that any components of $A_2(D)v_{xy} \in \mathbb{Z}^n$ are 0 but the components of $A_2(D)v_{xy} \in \mathbb{Z}^n$ to $x$ and $y$ are $\varepsilon_x$ and $-\varepsilon_y$.

Figure 19. Obtaining $v_{xy}$ in the two cases.

Proof. We splice at $x$. Let $\gamma_1$ and $\gamma_2$ be oriented arcs appearing after the splice at $x$ on the obtained diagram. We suppose that $\gamma_1$ lies on the left of $\gamma_2$. We splice at $y$. We obtain a new diagram of a two-component link as illustrated on the middle of Figure 19, where the cases (1) and (2) are described in the upper and lower rows respectively. We denote the sub-diagram of the link component including the arc $\gamma_i$ by $D_{xy}^i$, $i = 1, 2$. For the diagram $D_{xy}^1 \cup D_{xy}^2$, we take the componentwise Alexander numbering associated with $D_{xy}^1$ such that the right and left regions of $\gamma_1$ are assigned 0 and 1 respectively. We denote by $a$ the integer assigned to the right region of two oriented arcs which appear after the splice at $y$. In the case (1), $D_{xy}^1$ includes the left of these two arcs, then the region between the arcs is assigned $a$, and the left region is assigned $a + 1$. In the case (2), $D_{xy}^1$ includes the right of these two arcs, then the right regions adjacent to the left arc are assigned $a + 1$. By Lemma 5.2, this numbering gives a kernel solution for $A_2(D_{xy}^1 \cup D_{xy}^2)$ if $D_{xy}^1 \cup D_{xy}^2$ has at least one crossing. We unsplice $D_{xy}^1 \cup D_{xy}^2$ at $x$ and $y$. We assign the same integers to all regions of $D$ as $D_{xy}^1 \cup D_{xy}^2$, where the two regions splitting at $x$ are assigned 0, and where the two regions splitting at $y$ are assigned $a$ and $a + 1$ in the case (1) and (2) respectively, as illustrated on the right of Figure 19. Then the obtained assignment of integers to regions is the desired $v_{xy} \in \mathbb{Z}^{n+2}$ in the both cases. 

By Lemma 9.1 (2) and 11.3, we obtain the following result.

Corollary 11.4. Let $D = D_1 \cup D_2$ be a connected diagram of two-component oriented link with $n$ crossings, where $D_1$ and $D_2$ are sub-diagram of $D$ representing the first and the second components respectively. We take two distinct crossings $x, y$ of $D_1$ and $D_2$. We suppose that $D_2$ crosses $D_1$ from the right to the left at $x$. 

By Lemma 9.1 (2) and 11.3, we obtain the following result.

Corollary 11.4. Let $D = D_1 \cup D_2$ be a connected diagram of two-component oriented link with $n$ crossings, where $D_1$ and $D_2$ are sub-diagram of $D$ representing the first and the second components respectively. We take two distinct crossings $x, y$ of $D_1$ and $D_2$. We suppose that $D_2$ crosses $D_1$ from the right to the left at $x$. 

By Lemma 9.1 (2) and 11.3, we obtain the following result.
Theorem 11.6. Let $D = D_1 \cup D_2$ be a connected diagram of two-component oriented link with $n$ crossings, where $D_1$ and $D_2$ are sub-diagram of $D$ representing the first and the second components respectively. We suppose that there exist three distinct crossings $x, y, z$ of $D_1$ and $D_2$, and that $D_2$ crosses $D_1$ from the right to the left at $x$. Let $v_{xy}, v_{xz} \in \mathbb{Z}^n$ be obtained by Lemma 11.3.

(1) If $D_2$ crosses $D_1$ from the left to the right at $y$, then there exist $v_{xy} \in \mathbb{Z}^{n+2}$ such that any components of $A_{a_1}(D)v_{xy} \in \mathbb{Z}^n$ are 0 but the components of $A_{a_2}(D)v_{xy} \in \mathbb{Z}^n$ to $x$ and $y$ are $\varepsilon_x$ and $\varepsilon_y$.

(2) If $D_2$ crosses $D_1$ from the right to the left at $y$, then there exist $v_{xy} \in \mathbb{Z}^{n+2}$ such that any components of $A_{a_1}(D)v_{xy} \in \mathbb{Z}^n$ are 0 but the components of $A_{a_2}(D)v_{xy} \in \mathbb{Z}^n$ to $x$ and $y$ are $\varepsilon_x$ and $-\varepsilon_y$.

□

We note that Lemma 11.3 is a modulo 2 reduction of Corollary 11.2.

To construct a generating system of the image of $\Phi_{a_2}(D)$ for a connected diagram of a 2-component link $D = D_1 \cup D_2$ extending Theorem 11.2, we do not need all pairs of the distinct crossings of $D_1$ and $D_2$, since we obtain the following result by easy calculation.

Corollary 11.5. Let $D = D_1 \cup D_2$ be a connected diagram of two-component oriented link with $n$ crossings, where $D_1$ and $D_2$ are sub-diagram of $D$ representing the first and the second components respectively. We suppose that there exist three distinct crossings $x, y, z$ of $D_1$ and $D_2$, and that $D_2$ crosses $D_1$ from the right to the left at $x$. Let $v_{xy}, v_{xz} \in \mathbb{Z}^n$ be obtained by Lemma 11.3.

(1) If $D_2$ crosses $D_1$ from the left to the right at $y$ and $z$, then any components of $A_{a_2}(D)(v_{xy} - v_{xz})$ are 0 but the components of $A_{a_2}(D)(v_{xy} - v_{xz})$ to $y$ and $z$ are $\varepsilon_y$ and $-\varepsilon_z$ respectively.

(2) If $D_2$ crosses $D_1$ from the left to the right at $y$ and from the right to $z$, then any components of $A_{a_2}(D)(v_{xy} - v_{xz})$ are 0 but the components of $A_{a_2}(D)(v_{xy} - v_{xz})$ to $y$ and $z$ are $\varepsilon_y$ and $\varepsilon_z$ respectively.

(3) If $D_2$ crosses $D_1$ from the right to the left at $y$ and $z$, then any components of $A_{a_2}(D)(v_{xz} - v_{xy})$ are 0 but the components of $A_{a_2}(D)(v_{xz} - v_{xy})$ to $y$ and $z$ are $\varepsilon_y$ and $-\varepsilon_z$ respectively.

□

We note that the above $v_{xy} - v_{xz}$ or $v_{xz} - v_{xy}$ are not equal to $v_{yz}$ obtained in the proof of Lemma 11.3 generally.

We obtain a basis of the image of the homomorphism of the alternating integral region choice problem as belows.

Theorem 11.6. Let $D = D_1 \cup D_2$ be a connected diagram of two-component oriented link with $n$ crossings $x_1, \ldots, x_n$, where $D_1$ and $D_2$ are sub-diagram of $D$ representing the first and the second components respectively. We suppose that each of $x_1, \ldots, x_k$ ($k < n$) is a crossing in $D_1$ or a crossing in $D_2$, $x_{k+1}, \ldots, x_n$ are crossings of $D_1$ and $D_2$, and $D_2$ crosses $D_1$ from the right to the left at $x_n$.

We take $e_1, \ldots, e_{n-1} \in \mathbb{Z}^n$ as follows:

(1) for $i = 1, \ldots, k$, let $e_i$ be the element of $\mathbb{Z}^n$ such that the $i$-th component is 1 and the others are 0;

(2) for $i = k + 1, \ldots, n - 1$, let $e_i$ be the element of $\mathbb{Z}^n$ such that the $n$-th component is $\varepsilon_x$ and the $i$-th component is $\varepsilon_{x_i}$ (resp. $-\varepsilon_{x_i}$) if $D_2$ crosses $D_1$ at $x_i$ from the left to the right (resp. from the right to the left), and that the others are 0.

Then the set of $e_1, \ldots, e_{n-1}$ is a basis of the image of the homomorphism induced by the alternating integral region choice problem. Therefore the systems of linear equations $A_{a_1}(D_1 \cup D_2)u + c = 0$ and $A_{a_2}(D_1 \cup D_2)w + c = 0$ have solutions $u, w \in \mathbb{Z}^{n+2}$ if and only if $c \in \mathbb{Z}^n$ is a linear combination of $e_1, \ldots, e_{n-1}$.

Proof. By Theorem 6.3 and Lemma 11.3, $e_1, \ldots, e_{n-1}$ are elements in the image of the homomorphism $\Phi_{a_2}(D_1 \cup D_2)$. They are linearly independent by the construction. Let $c$ be an element of the image of the homomorphism $\Phi_{a_2}(D_1 \cup D_2)$.
By Theorem 8.1, the rank of the image of the homomorphism $\Phi_{d2}(D_1 \cup D_2)$ is $n + 1 - 2 = n - 1$. Hence $c, e_1, \ldots, e_{n-1}$ are linearly dependent since $e_1, \ldots, e_{n-1}$ are linearly independent. By the construction of $e_1, \ldots, e_{n-1}$, it is shown that $c$ is a linear combination of $e_1, \ldots, e_{n-1}$ over $\mathbb{Z}$. Then the set of $e_1, \ldots, e_{n-1}$ is a basis of the image of the homomorphism $\Phi_{d2}(D_1 \cup D_2)$.

By Theorem 9.3 (2), the image of the homomorphism $\Phi_{a1}(D_1 \cup D_2)$ coincides with the image of the homomorphism $\Phi_{a2}(D_1 \cup D_2)$.

We note that the modulo 2 reduction of Theorem 11.6 implies Theorem 11.2.

From the above basis, we obtain a basis of the image of the definite integral region choice problem as belows. Let $D = D_1 \cup D_2$ be a connected diagram of two-component oriented link with $n$ crossings $x_1, \ldots, x_n$. Let $R_1, \ldots, R_{n+2}$ be the regions of $D$. We suppose that each of $x_1, \ldots, x_k$ $(k < n)$ is a crossing in $D_1$ or a crossing in $D_2$, $x_{k+1}, \ldots, x_n$ are crossings of $D_1$ and $D_2$, and $D_2$ crosses $D_1$ from the right to the left at $x_n$. For each of $e_1, \ldots, e_{n-1}$ obtained by Theorem 11.6, there exists a solution $v_1 \in \mathbb{Z}^{n+2}$ of $A_{d2}(D)v_1 = e_1$. We take the checkerboard coloring such that the left and right regions of both oriented arcs crossing at $x_n$ are assigned 0 as shown or . For each $i = 1, \ldots, k$, there exists a solution $v_i \in \mathbb{Z}^{n+2}$ of $A_{d2}(D)v_i = e_i$ by Theorem 7.3. For each $i = k + 1, \ldots, n - 1$, we multiply the $j$-th component of $v_i$ by $-1$ if the region $R_j$ is assigned the checkerboard index 1. We denote by $v_i$ the obtained element of $\mathbb{Z}^{n+2}$ from $v_i$. Let $v_i = A_{d2}(D)v_i$. Then the $n$-th component of $v_i$ becomes 1 or $-1$, and the others are 0.

**Theorem 11.7.** The set of the above $e_1, \ldots, e_k, e_{k+1}, \ldots, e_{n-1}$ is a basis of the image of the homomorphism induced by the definite integral region choice problem on the connected diagram of two-component link $D = D_1 \cup D_2$. Therefore the systems of linear equations $A_{d1}(D_1 \cup D_2)u + c = 0$ and $A_{d2}(D_1 \cup D_2)w + c = 0$ have solutions $u, w \in \mathbb{Z}^{n+2}$ if and only if $c \in \mathbb{Z}^n$ is a linear combination of $e_1, \ldots, e_k, e_{k+1}, \ldots, e_{n-1}$.

**Proof.** By the construction, $e_1, \ldots, e_k, e_{k+1}, \ldots, e_{n-1}$ are elements in the image of the homomorphism $\Phi_{d2}(D_1 \cup D_2)$ and linearly independent. Let $c$ be an element of the image of the homomorphism $\Phi_{d2}(D_1 \cup D_2)$. By Theorem 8.1, the rank of the image of the homomorphism $\Phi_{d2}(D_1 \cup D_2)$ is $n + 1 - 2 = n - 1$. Hence $c, e_1, \ldots, e_k, e_{k+1}, \ldots, e_{n-1}$ are linearly dependent since $e_1, \ldots, e_k, e_{k+1}, \ldots, e_{n-1}$ are linearly independent. By the construction of $e_1, \ldots, e_k, e_{k+1}, \ldots, e_{n-1}$, it is shown that $c$ is a linear combination of $e_1, \ldots, e_k, e_{k+1}, \ldots, e_{n-1}$ over $\mathbb{Z}$. Then the set of $e_1, \ldots, e_k, e_{k+1}, \ldots, e_{n-1}$ is a basis of the image of the homomorphism $\Phi_{d2}(D_1 \cup D_2)$.

By Theorem 9.3 (1), the image of the homomorphism $\Phi_{d1}(D_1 \cup D_2)$ coincides with the image of the homomorphism $\Phi_{d2}(D_1 \cup D_2)$.

**Remark 11.8.** In [7], Hashizume gave a generating system of the image of the $\mathbb{Z}_2$-homomorphism induced by region crossing changes for a link diagram with arbitrary number of link components. Her generating system includes that in Theorem 11.2.

For each of the $\mathbb{Z}$-homomorphisms induced by integral region choice problems on a link diagram with arbitrary number of link components, we are finding a basis of the image.

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