PATHS ARE GENERICALLY REALISABLE

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Abstract. We show that every 0-1 multiplicity matrix for a simple graph \( G \) is generically realisable for \( G \). In particular, every multiplicity matrix for a path is generically realisable. We use this result to provide several families of joins of graphs that are realisable by a matrix with only two distinct eigenvalues.

1. Introduction

Let \( G \) be a simple graph with vertex set \( V(G) = \{1, \ldots, n\} \) and edge set \( E(G) \), and consider \( S(G) \), the set of all real symmetric \( n \times n \) matrices \( A = (a(i, j)) \) such that, for \( i \neq j \), \( a(i, j) \neq 0 \) if and only if \( \{i, j\} \in E(G) \), with no restriction on the diagonal entries of \( A \). The inverse eigenvalue problem for graphs (IEP-G) seeks to characterise all possible spectra \( \sigma(A) = \{\lambda_1, \ldots, \lambda_n\} \) of matrices \( A \in S(G) \). The IEP-G is solved for a few families of graphs (complete graphs, paths, generalized stars \[13\], cycles \[9, 10\], generalized barbell graphs \[18\]) and for graphs of order at most five \[5\]; the problem remains open in many other cases.

The closely related ordered multiplicity inverse eigenvalue problem for graphs seeks to characterise all possible ordered multiplicities of eigenvalues of matrices in \( S(G) \), i.e., to characterise the ordered lists of nonnegative integers \( (m_1, \ldots, m_r) \) for which there exists a matrix \( A \in S(G) \) and \( \lambda_1 < \ldots < \lambda_r \), such that \( \sigma(A) = \{\lambda_1^{(m_1)}, \ldots, \lambda_r^{(m_r)}\} \), where \( \lambda_i^{(m_i)} \) denotes \( m_i \) copies of \( \lambda_i \). We call such an ordered list \( (m_1, \ldots, m_r) \) an ordered multiplicity vector of \( G \). The ordered multiplicity inverse eigenvalue problem has been resolved for all connected graphs of order at most six \[4\]. For some specific families of connected graphs, several ordered multiplicity vectors have been determined (see e.g. \[1, 5, 18\]). Moreover, Monfared and Shader proved the following theorem in \[19\], showing that \((1,1,\ldots,1)\) is an ordered multiplicity vector of any connected graph \( G \), which can be realised with a nowhere-zero eigenbasis, that is, a basis of eigenvectors, each containing no zero entry.

Theorem 1.1. \[19\] Theorem 4.3] For any connected graph \( G \) on \( n \) vertices and distinct real numbers \( \lambda_1, \ldots, \lambda_n \), there exists \( A \in S(G) \) with

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spectrum $\sigma(A) = \{\lambda_1, \ldots, \lambda_n\}$ and an eigenbasis consisting of nowhere-zero vectors.

The present work draws motivation from the above result, in the context of multiplicity matrices of disconnected graphs. Such multiplicity matrices were introduced in [17] as a generalisation of ordered multiplicity vectors of connected graphs, and are defined as follows. Let $G$ be a graph with $k$ connected components $G_1, \ldots, G_k$, and $r, k \in \mathbb{N}$. An $r \times k$ matrix $V$ with non-negative integer entries is said to be a multiplicity matrix for $G$ if for $1 \leq i \leq k$, the $i$th column of $V$ is an ordered multiplicity list realised by a matrix in $S(G_i)$. Note that a trivial necessary condition for $V$ to be a multiplicity matrix of $G$ is that the orders of the connected components of $G$ are the same as the column sums of $V$; we abbreviate this by saying that $V$ fits $G$.

A matrix is said to be nowhere-zero if each of its entries is nonzero. Let $\sigma = \{\lambda_1, \ldots, \lambda_n\}$ be a multiset of real numbers, and let $D \in \mathbb{R}^{n \times n}$ be a diagonal matrix with diagonal elements $\lambda_i$ (in some order). For a connected graph $G$ we say that a multiset $\sigma$ is realisable for $G$ if $\sigma = \sigma(A)$ for some $A \in S(G)$; we say $\sigma$ is generically realisable for $G$ if, moreover, for any finite set $Y \subseteq \mathbb{R}^n \setminus \{0\}$ there is an orthogonal matrix $U$ so that $UDU^\top \in S(G)$, where $D$ is a diagonal matrix with spectrum $\sigma$ and $Uy$ is nowhere-zero for all $y \in Y$. (Note that this condition is stronger than the realisability of $\sigma$ in $S(G)$ with a nowhere-zero eigenbasis: consider $Y = \{e_1, \ldots, e_n\}$.) An ordered multiplicity vector $m = (m_1, \ldots, m_r) \in \mathbb{N}^r$ is spectrally arbitrary for $G$ if for any real numbers $\lambda_1 < \cdots < \lambda_r$, the multiset $\sigma = \{\lambda_1^{(m_1)}, \ldots, \lambda_r^{(m_r)}\}$ is realisable for $G$. Further, we say $m$ is generically realisable for $G$ if it is spectrally arbitrary for $G$, and every assignment of eigenvalues results in a generically realisable multiset for $G$, as defined above. If every column of a multiplicity matrix $V$ for a graph $G$ is generically realisable for the corresponding connected component of $G$, then we say that $V$ is generically realisable for $G$.

The methods used in [5,6] to resolve the IEP-G and the ordered multiplicity IEP-G for small graphs make essential use of strong properties. In particular, a symmetric $n \times n$ matrix $A$ is said to have the strong spectral property (the SSP) if the zero matrix $X = 0$ is the only symmetric matrix satisfying $AX =XA$ and $A \circ X = I_n \circ X = 0$, where $\circ$ denotes the Hadamard product. The SSP was first defined in [6]. One of its key features is the following perturbation result.

**Theorem 1.2.** [6, Theorem 10] Let $G'$ be a spanning subgraph of a graph $G$. If $A' \in S(G')$ is a matrix with the SSP, then for any $\varepsilon > 0$ there is a matrix $A \in S(G)$ with the SSP such that $A$ and $A'$ have the same eigenvalues and $\|A - A'\| < \varepsilon$.

We call a matrix a 0-1 matrix if each of its entries is either 0 or 1. Our main result is Theorem 2.5 below, in which we apply the SSP to
show that every 0-1 multiplicity matrix which fits a graph $G$ is generically realisable for $G$. In particular, this implies that every multiplicity matrix which fits a path $P_n$ is generically realisable for $P_n$. In the terminology of [17], we say $P_n$ is generically realisable.

We provide some applications in Section 3. In particular, we are interested in finding examples of joins of graphs that allow a small number of distinct eigenvalues. The minimum number of distinct eigenvalues of a graph

$$ q(G) = \min \{ q(A) : A \in S(G) \}, $$

where $q(A)$ denotes the number of distinct eigenvalues of a square matrix $A$, is one of the parameters motivated by IEP-G. The study of $q(G)$ was initiated by Leal-Duarte and Johnson in [15], and it has been broadly studied since then (see e.g., [2,3,6–8,16,17,19]). Monfared and Shader [19, Theorem 5.2] proved that $q(G \lor H) = 2$ if $G$ and $H$ are connected graphs with $|G| = |H|$. A consequence of our result is the following generalisation (Theorem 3.4 below): if $G$ and $H$ are arbitrary graphs, each with $k$ connected components $G_1, \ldots, G_k$ and $H_1, \ldots, H_k$, so that $|G_i| - |H_i| \leq 2$ for each $i$, then we still have $q(G \lor H) \leq 2$.

In this paper we use the following notation. For an integer $n$, let us denote $[n] = \{1, 2, \ldots, n\}$, and we also write $k + [n] = \{k + 1, k + 2, \ldots, k + n\}$. Column vectors are typically written using boldface; for example, $1_n$ denotes the column vector of ones in $\mathbb{R}^n$, and $e_i := (0, \ldots, 0, 1, 0, \ldots, 0)^\top \in \mathbb{R}^n$ is the vector with the 1 in the $i$th entry and zeros elsewhere. The $k \times k$ identity matrix is $I_k$ is the $k \times k$ identity matrix, $0_{m \times n}$ is the zero $m \times n$ matrix, and we also write $0_m := 0_{m \times m}$ and $0_m := 0_{m \times 1}$. Where the context allows, we may omit these subscripts altogether.

For a vector $x \in \mathbb{R}^n$ let us denote by $x(i) \in \mathbb{R}^{n-1}$ the vector $x$ with its $i$th component removed, and for a matrix $A$ let $A(i)$ denote its principal submatrix with the $i$th row and column of $A$ removed.

All graphs $G = (V(G), E(G))$ considered are simple undirected graphs with non-empty vertex sets $V(G)$. The order of $G$ is $|G| = |V(G)|$ and we often assume (without loss of generality) that $V(G) = \{ \{1, 2 \ldots, |V(G)|\} \}$. For a connected graph $G$, the distance between a pair of vertices is the number of edges of the shortest path between them, and the diameter $\text{diam}(G)$ is the largest distance between any pair of vertices. A subgraph $H$ is a spanning subgraph of $G$ if $V(H) = V(G)$. The join $G \lor H$ of two graphs $G$ and $H$ is the disjoint union $G \cup H$ together with all the possible edges joining the vertices in $G$ to the vertices in $H$. We abbreviate the disjoint graph union of $k$ copies of the same graph $G$ by $kG := G \cup \cdots \cup G$. For a subgraph $H$ of $G$ and a matrix $A \in S(G)$, we define a matrix $A[H]$ as the principal submatrix of $A$ whose rows and columns are the vertices of a subgraph $H$. We write $P_n$, $C_n$ and $K_n$.
for the path, the cycle and the complete graph on $n$ vertices, respectively, and we denote the complete bipartite graph on two disjoint sets of cardinalities $m$ and $n$ by $K_{m,n} := mK_1 \cap nK_1$.

2. Generic realisability of 0-1 matrices

In this section we will prove that any 0-1 multiplicity matrix that fits a graph $G$ is generically realisable for $G$.

Throughout this section, we fix $m \in \mathbb{N}$, real numbers $\lambda_1 < \lambda_2 < \cdots < \lambda_m$, and the diagonal matrix $\Lambda := \text{diag}(\lambda_1, \ldots, \lambda_m)$. Further, let $\mathcal{E}_\Lambda$ denote the smooth manifold of all $m \times m$ symmetric matrices with eigenvalues $\lambda_1, \ldots, \lambda_m$. This manifold was studied in [6], where a special case of the following lemma was proven (for the case the $f(\{i, j\}) = 1$ for all $i, j$). In fact, a nearly identical argument yields the more general result we require. We give the details for completeness.

Lemma 2.1. Let $G$ be a connected graph of order $m$. Given a function $f : E(G) \to \mathbb{N}$ and $t \in \mathbb{R}$, consider the family of manifolds given by

$$\mathcal{M}_{f,G}(t) := \{A = (a(i, j)) \in S(G) : a(i, j) = t f(\{i, j\})\} \text{ for } \{i, j\} \in E(G)\}.$$

For every $\varepsilon > 0$, there exists $t_0 > 0$ and a matrix $A \in \mathcal{M}_{f,G}(t_0) \cap \mathcal{E}_\Lambda$ with the SSP so that $\|A - \Lambda\| < \varepsilon$.

Proof. In this proof we use definitions and notation from [6]. In particular, $N_{M,X}$ denotes the normal space to a smooth submanifold $M$ of the $m \times m$ matrices at some $X \in M$, and two such manifolds $M$ and $M'$ intersect transversally at $X \in M \cap M'$ if $N_{M,X} \cap N_{M',X} = \{0\}$.

Let $M(t) := \mathcal{M}_{f,G}(t)$ be the smooth family of manifolds of $m \times m$ symmetric matrices defined in [1]. Note that $M(0)$ is the set of diagonal matrices. By [6] we have $N_{E_{\lambda, \Lambda}} = \{\alpha I_m : \alpha \in \mathbb{R}\}$ and $N_{M(0), \Lambda} = \{X : X \circ I_m = 0\}$. Since these two normal spaces have trivial intersection, the manifolds $M(0)$ and $E_{\lambda}$ intersect transversally at $\Lambda$.

By [6, Theorem 3] there exists $r > 0$ and a continuous function $F : (-r, r) \to E_{\lambda}$ such that $F(0) = \Lambda$ and for $t \in (-r, r)$, the manifolds $E_{\lambda}$ and $M(t)$ intersect transversally at $F(t)$. Hence, for any $\varepsilon > 0$, for sufficiently small $t_0 > 0$, the matrix $A := F(t_0)$ has $\|A - \Lambda\| < \varepsilon$ and $A \in \mathcal{M}(t_0) \cap E_{\lambda}$. To see that $A$ has the SSP, we have to prove that $S(G)$ and $E_{\lambda}$ intersect transversally at $A$ (see [6, page 11]). Since $M(t_0) \subseteq S(G)$, it follows that $N_{S(G), A} \subseteq N_{M(t_0), A}$. Further, since $E_{\lambda}$ and $M(t_0)$ intersect transversally, we have $N_{S(G), A} \cap N_{E_{\lambda}, A} \subseteq N_{M(t_0), A} \cap N_{E_{\lambda}, A} = \{0\}$, proving that $S(G)$ and $E_{\lambda}$ intersect transversally at $A$. \qed

Lemma 2.2. For $n \in \mathbb{N}$, let $A_n \in \mathcal{E}_\Lambda$ and let $U_n$ be an orthogonal matrix with non-negative diagonal entries satisfying $U_n^\top A_n U_n = \Lambda$. If $A_n \to \Lambda$ as $n \to \infty$, then $U_n \to I_n$ as $n \to \infty$. 
**Proof.** For each $k \in [m]$, let $p_k$ be a polynomial with $p_k(\lambda_\ell) = \delta_{k,\ell}$ for $\ell \in [m]$. Then $p_k(A_n)$ is the orthogonal projection onto the $\lambda_k$-eigenspace of $A_n$. Hence

$$p_k(A_n) = p_k(U_n A_n U_n^\top) = U_n p_k(\Lambda) U_n^\top = (U_n e_k)(U_n e_k)^\top = p_k(\Lambda) = e_k e_k^\top.$$  

In particular, $(U_n e_k)_k(U_n e_k)_\ell = p_k(A_n)_{k,\ell} = \delta_{k,\ell}$. Since $(U_n e_k)_k \geq 0$, this implies that $(U_n e_k)_k \to 1$, which implies in turn that $(U_n e_k)_\ell \to 0$ if $\ell \neq k$. Hence, $U_n e_k \to e_k$, proving $U_n \to I_n$. □

In the previous lemma, we showed that the off-diagonal elements of $U_n$ decay to zero. We now show that when we also have $A_n \in M_{f,G}(t_n)$ where $t_n \to 0$, it is sometimes possible to precisely determine the rate of decay of off-diagonal elements of $U_n$.

**Lemma 2.3.** Let $G$ be a tree of order $m$. Given $g : E(G) \to \mathbb{N}$, let $N_0 \in \mathbb{N}$ with $N_0 > \max\{g(e) : e \in E(G)\}$ diam$(G)$ and let $f : E(G) \to \mathbb{N}$, $f(e) := N_0 + g(e)$.

For $i, j \in V(G)$, let $P(i, j)$ be the subgraph of $G$ consisting of the shortest path from $i$ to $j$, $c(i, j) := \prod_{k \in V(P(i, j) \setminus \{i, j\})} (\lambda_j - \lambda_k)^{-1}$, and $s(i, j) := \sum_{e \in E(P(i, j))} f(e)$. (In particular, $c(i, j) = 0$, $c(j, j) := 1$ and $s(i, j) := 0$.)

Suppose $t_n > 0$ with $t_n \to 0$, and $A_n \in M_{f,G}(t_n) \cap \mathcal{E}_{\Lambda}$ with $A_n \to \Lambda$ as $n \to \infty$. For $n \in \mathbb{N}$, let $U_n = (u_n(i, j))$ be an orthogonal matrix with non-negative diagonal entries so that $U_n^\top A_n U_n = \Lambda$. Then $\frac{u_n(i, j)}{s(i, j)} \to c(i, j)$ as $n \to \infty$, for all $i, j \in V(G)$.

**Proof.** Assuming $V(G) = [m]$ and $A_n = (a_n(i, j)) \in M_{f,G}(t_n)$, it follows that $a_n(i, j) = t_n^{f(i, j)}$ for each $(i, j) \in E(G)$. Since $0 < t_n \to 0$ and $A_n \to \Lambda$, we may assume (by taking $n$ sufficiently large) that $0 < t_n \to 0$ and for $i \neq j$, $a_n(i, i) - \lambda_j$ is bounded away from zero.

Fix $j_0 \in \mathbb{N}$ and consider the vector $U_n e_{j_0} = (u_n(i, j_0))_{i \in [m]}$. This is a normalised eigenvector of $A_n$ with eigenvalue $\lambda_{j_0}$. Equivalently, for $i \in [m]$ we have

$$\lambda_{j_0} - a_n(i, i))u_n(i, j_0) = \sum_{k \in N_G(i)} t_n^{f(i, k)} u_n(k, j_0)$$

where $N_G(i)$ is the set of neighbours of $i$ in $G$.

For $i \in V(G)$, let $d(i, j_0) := |E(P(i, j_0))| \geq 0$ denote the distance in $G$ from $i$ to $j_0$. We claim that for $0 \leq x \leq \text{diam}(G)$:

(a) if $i \in V(G)$ with $d(i, j_0) > x$, then $\frac{a_n(i, j_0)}{t_n^{d(i, j_0)}} \to 0$ as $n \to \infty$; and

(b) if $i \in V(G)$ with $d(i, j_0) = x$, then $\frac{a_n(i, j_0)}{t_n^{d(i, j_0)}} \to c(i, j_0)$ as $n \to \infty$.

We will establish this claim by induction on $x$.

Since $A_n$ converges to $\Lambda$, the $j_0$-th column of $U_n$ converges to $e_{j_0}$ by Lemma 2.2. This implies that the claim holds for $x = 0$.

Now assume inductively that the claim holds for all $x$ with $0 \leq x \leq x_0 < \text{diam}(G)$. We proceed to prove that it holds for $x = x_0 + 1$ as well.
First consider claim (a) and suppose \( i \in V(G) \) with \( d(i, j_0) > x_0 + 1 \). Rearranging (2), we obtain:
\[
\frac{u_n(i, j_0)}{t_n^{(x_0+1)N_0}} = \frac{1}{\lambda_{j_0} - a_n(i, i)} \sum_{k \in N_G(i)} \frac{u_n(k, j_0)}{t_n^{(x_0+1)N_0-f([i,k])}}.
\]
For any \( e \in E(G) \), we have \( f(e) > N_0 \), so \((x_0+1)N_0 - f(e) < x_0N_0\), and:
\[
\left| \frac{u_n(i, j_0)}{t_n^{(x_0+1)N_0}} \right| \leq \frac{1}{|\lambda_{j_0} - a_n(i, i)|} \sum_{k \in N_G(i)} \left| \frac{u_n(k, j_0)}{t_n^{(x_0+1)N_0-f([i,k])}} \right| \leq \frac{1}{|\lambda_{j_0} - a_n(i, i)|} \sum_{k \in N_G(i)} \frac{u_n(k, j_0)}{t_n^{x_0N_0}}.
\]
Since \( d(i, j_0) > x_0 + 1 \), we have \( d(k, j_0) > x_0 \) for all \( k \in N_G(i) \), hence by part (a) of the claim for \( x = x_0 \), each term in the sum above converges to 0. Since we also have \( \lambda_{j_0} - a_n(i, i) \to \lambda_{j_0} - \lambda_i \neq 0 \), part (a) of the claim holds for \( x = x_0 + 1 \). This proves (a) for all \( x \) with \( 0 \leq x \leq \text{diam}(G) \).

Now consider part (b). Suppose \( d(i, j_0) = x_0 + 1 \). Since \( G \) is a tree, there exists precisely one \( k_0 \in N_G(i) \) with \( d(k_0, j_0) = x_0 \), and by the definition of \( s \), we have \( s(i, j_0) = f([i,k_0]) + s(k_0, j_0) \). We have \( i \neq j_0 \), so \( \lambda_{j_0} - a_n(i, i) \neq 0 \) and we can rearrange (2) as follows:
\[
(3) \quad \frac{u_n(i, j_0)}{t_n^{(i,j_0)}} = \frac{1}{\lambda_{j_0} - a_n(i, i)} \left( \frac{u_n(k_0, j_0)}{t_n^{(k_0,j_0)}} + \sum_{k \in N_G(i) \setminus \{k_0\}} \frac{u_n(k, j_0)}{t_n^{x_0N_0-f([i,k])}} \right).
\]
For every \( k \in N_G(i) \setminus \{k_0\} \) we have:
\[
s(i, j_0) - f([i,k]) = \left( \sum_{e \in E(P(i, j_0))} f(e) \right) - f([i,k]) = (x_0 + 1)N_0 + \left( \sum_{e \in E(P(i, j_0))} g(e) \right) - N_0 - g([i,k]) < x_0N_0 + (x_0 + 1) \max\{g(e) : e \in E(G)\} < (x_0 + 1)N_0.
\]
Hence, by part (a) of the claim, all the terms under the sum in (3) converge to zero. Taking limits and using part (b) of the claim shows that \( \frac{u_n(i, j_0)}{t_n^{(i,j_0)}} \to (\lambda_{j_0} - \lambda_i)^{-1}c(k_0, j_0) = c(i, j_0) \).

**Theorem 2.4.** Let \( G \) be a connected graph of order \( m \), and choose a finite set \( \mathcal{Y} \subset \mathbb{R}^m \setminus \{0\} \). There exists a matrix \( A \in S(G) \cap \mathcal{E}_\lambda \) with the SSP and an orthogonal matrix \( U \) with \( U^*AU = \Lambda \) so that \( UV \) is nowhere-zero for all \( \mathbf{v} \in \mathcal{Y} \).

**Proof.** First we prove the statement for trees, so let \( G' \) be a tree. Let us assume that the statement does not hold. Choose \( g : E(G') \to \mathbb{N} \) and \( f := N_0 + g \) as in Lemma 2.3 so that the corresponding function \( s : V(G) \times V(G) \to \mathbb{N} \) satisfies \( s(i, j) = s(k, \ell) \iff \{i, j\} = \{k, \ell\} \).
(This may be guaranteed by a suitable choice of \( g \).) Further, let \( A_n \in \mathcal{M}_{fG'}(t_n) \cap \mathcal{E}_A \) also be as in Lemma 2.3 with the SSP (which we can guarantee by Lemma 2.1) and let \( U_n \) be orthogonal matrices with nonnegative diagonal so that \( (U_n v)_k = 0 \) for every \( n \in \mathbb{N} \). Let \( u_n = (u_n(k,1) \ u_n(k,2) \ldots \ u_n(k,m)) \) denote the \( k \)-th row of \( U_n \), hence we are assuming \( u_n^\top v = 0 \) for all \( n \in \mathbb{N} \). We aim to arrive at the contradiction by proving \( v = 0 \). From Lemma 2.3 we know that \( U_n \to I \), hence \( v_k = 0 \). Let \( i_1, i_2, \ldots, i_{m-1} \in [m] \setminus \{k\} \) be such that \( s(k, i_1) < s(k, i_2) < \cdots < s(k, i_{m-1}) \), and \( \ell \in [m-1] \) be minimal with \( v_{\ell} \neq 0 \). Now:

\[
0 = \frac{1}{t_n^{s(k, i_\ell)}} u_n^\top v = \sum_{r=\ell}^{m-1} \frac{u_n(k, i_r)}{t_n^{s(k, i_r)}} v_{i_r}.
\]

By Lemma 2.3 we know that \( \frac{u_n(k, i_r)}{t_n^{s(k, i_r)}} \) converges to a nonzero constant and \( \frac{u_n(k, i_\ell)}{t_n^{s(k, i_\ell)}} \) converges to zero for \( r > \ell \). Hence \( v_{i_\ell} = 0 \), a contradiction.

Now, let \( G \) be a connected graph, and \( G' \) be a spanning tree of \( G \). Since the claim holds for \( G' \), there exists \( A' \in S(G') \) with the SSP so that \( U'^\top A' U' = \Lambda \), \( U'^\top U' = I_m \), and \( U' v = 0 \) for all \( v \in \mathcal{Y} \). Since \( A' \) has the SSP, by Theorem 1.2 there exists \( A \in S(G) \) with the SSP arbitrarily close to \( A' \) with the same eigenvalues as \( A' \). Since the eigenvalues are distinct, the \( \lambda_k \) eigenspaces are close for \( A \) and \( A' \), so for \( A \) and \( A' \) sufficiently close, there is an orthogonal matrix \( U \) diagonalising \( A \) which is sufficiently close to \( U' \) to ensure that \( U v = 0 \) is also nowhere zero for all \( v \in \mathcal{Y} \).

**Theorem 2.5.** If \( G \) is any graph, then every 0-1 multiplicity matrix which fits \( G \) is generically realisable for \( G \).

**Proof.** Given such a matrix \( V \), label the connected components \( G_1, \ldots, G_k \) of \( G \) so that the multiplicity vector \( Ve_i \) fits \( G_i \). Every entry of \( Ve_i \) is 0 or 1, so by Theorem 2.4 \( Ve_i \) is generically realisable for \( G_i \). Hence, \( V \) is generically realisable for \( G \).

If every multiplicity matrix for a graph \( G \) is generically realisable for \( G \), then we say that \( G \) is generically realisable. This is a strong requirement for a graph; in particular, it implies that \( G \) is spectrally arbitrary for every multiplicity matrix that can be realised by \( G \). In fact, the only families of generically realisable graphs previously known are unions of complete graphs [17]. We can now extend this to include paths.

**Corollary 2.6.** (a) The path \( P_n \) is a generically realisable graph, for every \( n \in \mathbb{N} \).
If every connected component of a graph $G$ is either a complete graph or a path, then $G$ is generically realisable.

Proof. The maximum multiplicity of an eigenvalue of a path is one [11], so every multiplicity vector for $P_n$ is a 0-1 vector, hence it is generically realisable by Theorem 2.5. Complete graphs are also generically realisable [17], so the second assertion follows immediately. □

3. Applications

In [17] the authors developed an approach to study $q(G)$, where $G$ is the join of two graphs. We now give some applications of preceding results to this problem.

First we briefly review the set up from [17]. For a matrix $X$ with at least 3 rows, we write $\tilde{X}$ for the matrix obtained by deleting the first row and the last row of $X$. Let $S^{m \times n}$ denote the set of $m \times n$ matrices with entries in a set $S$, and let $\mathbb{N}_0$ be the set of non-negative integers. Given $r, k, \ell \in \mathbb{N}$ with $r \geq 3$, two matrices $V \in \mathbb{N}_0^{r \times k}$ and $W \in \mathbb{N}_0^{r \times \ell}$ are said to be compatible if $\tilde{V}1_k = \tilde{W}1_\ell$ and $\tilde{V}^\top W$ is nowhere-zero. We say that two graphs $G, H$ have compatible multiplicity matrices if there exist compatible matrices $V, W$ where $V$ is a multiplicity matrix for $G$ and $W$ is a multiplicity matrix for $H$.

Which graphs $G$ and $H$ have $q(G \lor H) = 2$? The answer is closely related to the existence of compatible multiplicity matrices.

**Theorem 3.1.** [17, Theorems 2.5 and 2.14] Let $G$ and $H$ be two graphs. If $q(G \lor H) = 2$, then $G$ and $H$ necessarily have compatible multiplicity matrices.

Moreover, if there exist compatible generically realisable multiplicity matrices $V$ and $W$ for $G$ and $H$, then $q(G \lor H) = 2$.

By Corollary 2.6, in the case of unions of paths or complete graphs we can strengthen this to a necessary and sufficient condition.

**Corollary 3.2.** If $G$ and $H$ are unions of paths or complete graphs, then $q(G \lor H) = 2$ if and only if $G$ and $H$ have a pair of compatible multiplicity matrices.

For general graphs, by combining Theorem 2.5 and Theorem 3.1 we obtain the following purely combinatorial sufficient condition for $q(G \lor H)$ to be 2.

**Corollary 3.3.** Let $G$ and $H$ be two graphs. If there exist compatible 0-1 matrices $V$ and $W$ that fit $G$ and $H$, respectively, then $q(G \lor H) = 2$.

In general, deciding whether there exist compatible 0-1 matrices with prescribed column sums seems to be a difficult combinatorial question, which we plan to examine further in upcoming work.

Monfared and Shader [19, Theorem 5.2] proved that $q(G \lor H) = 2$ if $G$ and $H$ are connected graphs with $|G| = |H|$. Using Corollary 3.3 it
is now simple to generalise their result to the case $|G| - |H| \leq 2$, and also to pairs of disconnected graphs with equal numbers of connected components.

**Theorem 3.4.** Let $k \in \mathbb{N}$, and for $i \in [k]$, let $G_i$ and $H_i$ be connected graphs with $|G_i| - |H_i| \leq 2$. Then

$$q\left( \bigcup_{i=1}^{k} G_i \vee \bigcup_{i=1}^{k} H_i \right) = 2.$$ 

Moreover, if $m_i + 2 \geq |H_i|$ for all $i \in [k]$, then

$$q\left( \bigcup_{i=1}^{k} K_{m_i} \vee \bigcup_{i=1}^{k} H_i \right) = 2.$$

**Proof.** Let $p_i := \min\{|G_i|, |H_i|\}$ for $i \in [k]$, $p := \max\{p_i; i \in [k]\}$ and $q_i := \sum_{i \in [p]} e_i \in \{0, 1\}^p$. Let $E := (q_1 \ q_2 \ldots \ q_k) \in \{0, 1\}^{p \times k}$ and consider

$$V := \begin{pmatrix} v_1 \\ E \\ v_{p+2} \end{pmatrix} \in \{0, 1\}^{(p+2) \times k} \quad \text{and} \quad W := \begin{pmatrix} w_1 \\ E \\ w_{p+2} \end{pmatrix} \in \{0, 1\}^{(p+2) \times k},$$

where $v_1, v_{p+2}, w_1, w_{p+2} \in \{0, 1\}^{1 \times k}$ are chosen so that $1^T_{p+2} V e_i = |G_i|$, and $1^T_{p+2} W e_i = |H_i|$ for all $i \in [k]$. (The existence of such vectors is assured by the condition $|G_i| - |H_i| \leq 2$.) Since $\overline{V} = \overline{W} = E$ and the first row of $E$ is nowhere-zero, $V$ and $W$ are compatible 0-1 matrices fitting $\bigcup_{i=1}^{k} G_i$ and $\bigcup_{i=1}^{k} H_i$, respectively, hence $q(\bigcup_{i=1}^{k} G_i \vee \bigcup_{i=1}^{k} H_i) = 2$ by Corollary 3.3.

If $G = \bigcup_{i=1}^{k} K_{m_i}$, then the assumption $m_i \geq |H_i| - 2$ for all $i \in [k]$ implies the existence of compatible matrices $V$ and $W$ as in (4), except that we don’t require the first and the last row of $V$ to be 0-1 vectors, i.e., $v_1, v_{p+2} \in \mathbb{N}_0^k$. By [17] and Theorem 2.5, $V$ and $W$ are generically realisable multiplicity matrices for $G$ and $H$, respectively, so $q = 2$ by Theorem 3.1. \hfill \Box

For connected graphs $G$ and $H$ we have just seen that $q(G \vee H) = 2$ if $|G| - |H| \leq 2$. We now show that we generally cannot relax this inequality.

**Example 3.5.** Suppose that $G = P_n$ and $H$ is a connected graph with $|H| \leq n$. Let us prove that in this case the condition $|G| - |H| \leq 2$, i.e., $|H| \geq n - 2$, is also necessary for $q(P_n \vee H) = 2$.

If $q(P_n \vee H) = 2$, then by Theorem 3.1 there exist compatible multiplicity matrices $V \in \mathbb{N}_0^{(r+2) \times 1}$ and $W \in \mathbb{N}_0^{(r+2) \times 1}$ for $P_n$ and $H$, respectively. The maximum multiplicity of a path is 1, so $V \in \{0, 1\}^{(r+2) \times 1}$. By compatibility, we have $\overline{V} = \overline{W}$ and without loss of generality, we
can delete any zeros in these column vectors to reduce to the case
\[
V = \begin{pmatrix} v_1 \\ 1_r \\ v_{r+2} \end{pmatrix} \quad \text{and} \quad W = \begin{pmatrix} w_1 \\ 1_r \\ w_{r+2} \end{pmatrix},
\]
where \(v_1, v_{r+2} \in \{0, 1\}\) and \(w_1, w_{r+2} \in \mathbb{N}_0\). Since these matrices fit \(G\) and \(H\), we have \(r = n - (v_1 + v_{r+2}) \geq n - 2\) and \(|H| \geq r \geq n - 2\), as required.

Recall that if \(T\) is a tree and \(A \in S(T)\), then the extreme eigenvalues of \(A\) have multiplicity one. Hence, by a similar argument to that of the previous paragraph, we have \(q(T_1 \vee T_2) = 2 \iff |T_1| - |T_2| \leq 2\) for any trees \(T_1, T_2\).

We now turn to another class of applications of Theorem 2.5 in which we obtain certain achievable spectra of partial joins. We require additional terminology given below.

Recall that if \(G\) is a graph and \(W \subseteq V(G)\), then the vertex boundary of \(W\) in \(G\) is the set of all vertices in \(V(G) \setminus W\) which are joined to some vertex of \(W\) by an edge of \(G\).

Suppose that \(G_1\) and \(G_2\) are two disjoint graphs and let \(V_i \subseteq V(G_i)\) for \(i = 1, 2\). The partial join of graphs \(G_1\) and \(G_2\) is the graph \((G, V) = (G_1, V_1) \vee (G_2, V_2)\) formed from \(G_1 \cup G_2\) by joining each vertex of \(V_1\) to each vertex of \(V_2\). If \(V_2 = V(G_2)\), then we write \((G_1, V_1) \vee G_2 \coloneqq (G_1, V_1) \vee (G_2, V(G_2))\).

Suppose \(G\) and \(G'\) are graphs so that \(G\) is obtained from \(G'\) by deleting \(s\) vertices and adding arbitrary number of edges (in particular, \(|G'| = |G| + s\)). If a matrix \(A \in S(G)\) has the SSP, then by the Minor Monotonicity Theorem [5, Theorem 6.13] there exist a matrix \(A' \in S(G')\) with the SSP, such that \(\sigma(A') = \sigma(A) \cup \{\mu_1, \ldots, \mu_s\}\), where \(\mu_i \neq \mu_j\) for distinct \(i, j \in [s]\). In the next result we provide a statement of a similar flavour, that does not depend on the SPP. The construction given the Lemma was first developed in the context of the nonnegative inverse eigenvalue problem [14, 20].

**Lemma 3.6.** Let \(G\) be a graph, \(V(G) = V_1 \cup V_2\) a partition of \(V(G)\), and \(X \subseteq V_1\) the vertex boundary of \(V_2\) in \(G\). Suppose \(H\) is any connected graph with \(|H| \geq |V_2|\) and consider the graph \(G' \coloneqq (G[V_1], X) \vee H\).

Let \(M \in S(G)\) and suppose that \(\sigma'\) is a multiset of real numbers so that \(\sigma(M[V_2]) \cup \sigma'\) is generically realisable for \(H\). Then there exists \(N \in S(G')\) with spectrum \(\sigma(N) = \sigma(M) \cup \sigma'\).

**Proof.** Let us denote \(m_i := |V_i|\) for \(i = 1, 2\) and \(k := |X|\). Assume without loss of generality that \(V(G) = [m_1 + m_2], V_1 = [m_1]\) and \(X = m_1 - k + [k]\).

We have \(M = \begin{pmatrix} A & B \\ B^t & C \end{pmatrix} \in S(G)\), with \(A = M[V_1], C = M[V_2]\), and \(\sigma(C) = \{\lambda_1, \ldots, \lambda_m\}\). Let \(Q \in \mathbb{R}^{m_1 \times m_1}\) be an orthogonal matrix, such that \(Q^t C Q = \Lambda_0 := \text{diag}(\lambda_1, \ldots, \lambda_m)\). Write \(\sigma' = \{\mu_1, \ldots, \mu_t\}\), and
let $\Lambda' := \text{diag}(\mu_1, \ldots, \mu_t)$. Moreover, let $N' = M \oplus \Lambda'$ with eigenvalues $\sigma(M) \cup \sigma'$.

Since $X = m_1 - k + [k]$ is the vertex boundary of $V_2$ in $G$, we have $B^\top = (0_{m_2 \times (m_1-k)} \ B_0^\top) \in \mathbb{R}^{m_2 \times m_1}$, where no column of $B_0^\top \in \mathbb{R}^{m_2 \times k}$ is zero. Let us denote the set of columns of $Q^\top B_0^\top$ by $\mathcal{X} \subset \mathbb{R}^{m_2}$, and let $\mathcal{Y} \subset \mathbb{R}^{m_2+t}$ denote the set of vectors obtained from elements of $\mathcal{X}$ by appending $t$ zeros. Then $|\mathcal{X}| = |\mathcal{Y}| = k$. Since $Q$ is orthogonal, all vectors in $\mathcal{Y}$ are nonzero.

Since $\sigma(M[V_2]) \cup \sigma'$ is generically realisable for $H$, there exist a matrix $C' \in S(H)$ with $\sigma(C') = \sigma(C) \cup \sigma'$, and an orthogonal matrix $U \in \mathbb{R}^{(m_2+t) \times (m_2+t)}$, such that $U^\top C' U = \Lambda_0 \oplus \Lambda'$ and $U y$ is a nowhere-zero vector for all $y \in \mathcal{Y}$.

Let

$$U' := I_{m_1} \oplus (U(Q^\top \oplus I_t))$$

and

$$N := U' N' U'^\top = \begin{pmatrix} A & Z \\ Z^\top & C' \end{pmatrix},$$

where

$$Z^\top = \begin{pmatrix} 0_{(m_2+t) \times (m_1-k)} & U \begin{pmatrix} Q^\top B_0^\top \\ 0_{t \times k} \end{pmatrix} \end{pmatrix}.$$ 

Since $U y$ is a nowhere-zero vector for $y \in \mathcal{Y}$, the matrix $U \begin{pmatrix} Q^\top B_0^\top \\ 0_{t \times k} \end{pmatrix}$ is nowhere-zero. Hence, $N \in S(G')$, and $\sigma(N) = \sigma(N') = \sigma(M) \cup \sigma'$ as desired.

Lemma [3.6] can be used to build explicit examples or to provide more general results. Implementation is faced with two issues: first we need some information on $\sigma(M[V_2])$, and second, we need to prove generic realisability of $\sigma(M[V_2]) \cup \sigma'$ for $H$. In our first application, we rely on Theorem 2.5 for generic realisably.

**Theorem 3.7.** Let $G$ be a graph, $V(G) = V_1 \cup V_2$ a partition of $V(G)$, and $X \subseteq V_1$ be the vertex boundary of $V_2$ in $G$. Suppose $H$ is any connected graph with $|H| \geq |V_2|$ and consider the graph $G' := (G[V_1], X) \lor H$.

If there exists $M \in S(G)$ so that $M[V_2]$ has distinct eigenvalues $\lambda_1, \ldots, \lambda_{|V_2|}$, then there exists $N \in S(G')$ with spectrum $\sigma(N) = \sigma(M) \cup \{\mu_1, \ldots, \mu_t\}$, where $t = |H| - |V_2|$ and $\mu_1, \ldots, \mu_t$ are any distinct real numbers satisfying $\mu_i \neq \lambda_j$ for all $i \in \{1, \ldots, t\}, j \in \{1, \ldots, |V_2|\}$.

Theorem 3.7 allows us, in some sense, to replace a submatrix with distinct eigenvalues with a matrix corresponding to an arbitrary connected graph of equal or bigger size. Note, the technical requirement that $M[V_2]$ has distinct eigenvalues is automatically satisfied when $G[V_2]$ is a path. This observation allows us to improve an upper bound for the number of distinct eigenvalues of joins of graphs that is given.
Lemma 3.9. IEP-G for cycles is known to have the following solution. since every induced connected subgraph of a cycle is a path, and the \( G \) is taken to be a cycle. Cycles are a nice family to use for this purpose.

Fix Example 3.10. \( S \in \{ \text{fied. } \) For example, the eigenvalues \( \mu \) values of \( H \) in \([7, \text{Section 4.2}]\), where it is proved that for connected graphs \( G \) and \( H \), \( q(G \lor H) \leq q(G \lor P_n) + |H| - n \).

In particular, if \( G \) and \( H \) are both connected, then \( q(G \lor H) \leq \max \{2, |G| - |H|\} \).

**Proof.** The first part follows easily from Theorem 3.7. Assume now that both \( G \) and \( H \) are connected. If \( |H| - |G| \leq 2 \), then \( q(G \lor H) = 2 \) by Theorem 3.4. To cover the case \( |H| \geq |G| + 3 \), we note that Theorem 3.4 implies \( q(G \lor P_{|G|+2}) = 2 \) and hence we have \( q(G \lor H) \leq 2 + |H| - (|G| + 2) = |H| - |G| \). By symmetry the statement follows.

Next we explore applications of Lemma 3.6 in the special case where \( G \) is taken to be a cycle. Cycles are a nice family to use for this purpose, since every induced connected subgraph of a cycle is a path, and the IEP-G for cycles is known to have the following solution.

**Lemma 3.9.** [10, Theorem 3.3] Let \( \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \) be a list of real numbers, \( n \geq 3 \). Then \( \{\lambda_1, \ldots, \lambda_n\} \) is the set of eigenvalues of a matrix in \( S(C_n) \) if and only if

\[
\lambda_1 \leq \lambda_2 < \lambda_3 \leq \lambda_4 < \ldots \quad \text{or} \quad \lambda_1 < \lambda_2 \leq \lambda_3 < \lambda_4 \leq \ldots .
\]

**Example 3.10.** Fix \( n \in \mathbb{N} \) and let \( X \) be the set of two degree one vertices of \( P_n \). Given a connected graph \( H \), consider \( J := (P_n, X) \lor H \). We claim that if \( s \in \mathbb{N}_0 \) and \( 2s \leq |J| = n + |H| \) then \( \{2^{(s)}, 1^{(|J| - 2s)}\} \) is an unordered multiplicity list for some matrix in \( S(J) \). Let \( G = C_{n+|H|} \). By Lemma 3.9, there is a matrix in \( S(G) \) with unordered multiplicity list \( \{2^{(s)}, 1^{(|J| - 2s)}\} \). Partition the vertices of \( G \) as \( V_1 \cup V_2 \) so that \( P_n = G[V_1] \) and \( P_{|H|} = G[V_2] \), and apply Theorem 3.7.

![Figure 1](image)

**Figure 1.** Sketch of the partial join \((P_7, X) \lor H\), where pendant vertices \( X \) of \( P_7 \) are coloured grey. The edges of \( H \) are not drawn.

Note that Lemma 3.6 allows us to increase the multiplicites of eigenvalues of \( M \), provided the technical conditions of the lemma are satisfied. For example, the eigenvalues \( \{\mu_1, \ldots, \mu_t\} \) that we add in Lemma 3.6
can be chosen to agree with eigenvalues in $\sigma(M)$, hence increasing their multiplicity, provided they are not also eigenvalues of $M[V_2]$. When this condition is not satisfied, Theorem 2.3 cannot be applied to assure generic realisability. Examples of both eventualities are given in the two examples below. In the first example we exhibit ordered multiplicity lists that achieve $q(G)$, where $G$ is a wheel graph of order $4m - 1$.

Example 3.11. Let us apply Lemma 3.6 to $G = C_{2m}$, $X = V_1 = \{2m\}$ and $V_2 = [2m - 1]$. Choose any distinct real numbers $\lambda_1, \ldots, \lambda_m$. By Lemma 3.9 there exists a matrix $M \in S(C_{2m})$ with

$$\sigma(M) = \{\lambda_1^{(2)}, \lambda_2^{(2)}, \ldots, \lambda_m^{(2)}\}.$$  

Let $B = M(2m) \in S(P_{2m-1})$ be the leading principal $(2m - 1) \times (2m - 1)$ submatrix of $M$. By interlacing and (11) we have

$$\sigma(B) = \{\lambda_1, \mu_1, \lambda_2, \mu_2, \ldots, \mu_{m-1}, \lambda_m\},$$

$\mu_i \in (\lambda_i, \lambda_{i+1})$ for $i \in [m - 1]$.

Let $H = C_{4m-2}$ and let $\sigma' = \sigma(B)$. By Lemma 3.9 there exists $A \in S(C_{4m-2})$ with $\sigma(A) = \sigma(B) \cup \sigma' = \{\lambda_1^{(2)}, \mu_1^{(2)}, \lambda_2^{(2)}, \mu_2^{(2)}, \ldots, \mu_{m-1}^{(2)}, \lambda_m^{(2)}\}$. Let us prove that we can choose a nowhere-zero eigenbasis for $A$. Every eigenspace $V$ of $A$ is spanned by two linearly independent vectors $v = (v_i)$ and $w = (w_i)$. If $v_i = w_i = 0$ for some $i \in [4m - 2]$, then $v(i)$ and $w(i)$ are linearly independent eigenvectors of $A(i) \in S(P_{4m-3})$ corresponding to the same eigenvalue, which contradicts the fact every matrix corresponding to a path has simple eigenvalues, (11). Hence for each $i$ either $v_i \neq 0$ or $w_i \neq 0$, so we can choose their linear combinations to be nowhere-zero and hence we can choose a nowhere-zero eigenbasis for $A$.

Since $A$ has a nowhere-zero eigenbasis and no simple eigenvalues, the proof of (17) Proposition 3.4 shows that $\sigma(A) = \sigma(B) \cup \sigma'$ is generically realisable for $C_{4m-2}$. By Lemma 3.6 there exists a matrix $N \in S(K_1 \vee C_{4m-2})$ with spectrum

$$\sigma(N) \cup \sigma' = \{\lambda_1^{(3)}, \mu_1^{(3)}, \lambda_2^{(3)}, \mu_2, \ldots, \mu_{m-1}, \lambda_m^{(3)}\}.$$  

Hence the ordered multiplicity list $(3, 1, 3, 1, \ldots, 3)$ is realisable for $K_1 \vee C_{4m-2}$, which are also known as wheel graphs.

Note that Lemma 3.9 prohibits odd number of eigenvalues between any two double eigenvalues of a matrix corresponding to a cycle. Hence by interlacing a matrix corresponding to $K_1 \vee C_{4m-2}$ cannot have an even number (including zero) of eigenvalues between any two triple eigenvalues. Therefore with the above construction we have found a matrix corresponding to $K_1 \vee C_{4m-2}$ with the maximal multiplicity of an eigenvalue and the minimal number of distinct eigenvalues. In particular, $q(K_1 \vee C_{4m-2}) = 2m - 1$.

Example 3.12. If $m \geq 2$ and $H$ is any connected graph with $3m - 2$ vertices, then the multiplicity list $\{3^{(m)}\}$ is spectrally arbitrary for
$K_2 \lor H$ and, in particular, $q(K_2 \lor H) \leq m$. To see this, take $G = C_{2m}$, $X = V_1 = \{2m, 2m-1\}$ and $V_2 = \{2m-2\}$. For any $\lambda_1 < \cdots < \lambda_m$, let $M \in S(C_{2m})$ with $\sigma(M) = \{\lambda_1^{(2)}, \lambda_2^{(2)}, \ldots, \lambda_m^{(2)}\}$. Then $\sigma(M(2m)) = \{\lambda_1, \mu_1, \lambda_2, \mu_2, \ldots, \mu_{m-1}, \lambda_m\}$, where $\mu_i \in (\lambda_i, \lambda_{i+1})$ for $i \in [m-1]$ (since a path has only simple eigenvalues). Now, $M[V_2]$ is a principal submatrix of $M(2m)$, so by [12] Problem 4.3.P17, the eigenvalues of $M[V_2]$ strictly interlace those of $M(2m)$. In particular, the eigenvalues of $M[V_2]$ do not intersect $\sigma' = \{\lambda_1, \ldots, \lambda_m\}$. Applying Theorem 3.7 we see that the spectrum $\sigma(M) \cup \sigma' = \{\lambda_1^{(3)}, \lambda_2^{(3)}, \ldots, \lambda_m^{(3)}\}$ is realised by a matrix in $S(K_2 \lor H)$, as required.

**Remark 3.13.** In particular, by interlacing, the maximal multiplicity of an eigenvalue of a matrix corresponding to $P_n \lor K_2$ is equal to 3 and hence $q(P_{3m-2} \lor K_2) = m$. This result complements [7, Example 4.5], where they proved that $q(P_n \lor K_1) = \lceil \frac{n+1}{2} \rceil$. It would be interesting to determine $q(P_n \lor K_i)$ in general.

In Theorem 3.7 we can make use of graphs for which IEP-G is solved (or better understood) to construct new realisable multiplicity lists for partial joins; we illustrate this idea also for generalised stars [13].

**Example 3.14.** Let $k \in \mathbb{N}$ and $H$ an arbitrary connected graph. If $m = \{m_1, m_2, \ldots, m_\ell, 1^{(n)}\}$, such that $m_i \geq 2$, $\ell \leq n$, and $\sum_{i=\ell} m_i = |H| - n + k + 1 \leq k + \ell$, then $m$ is an unordered multiplicity list for $K_1 \lor (3K_1 \cup H)$.

![Figure 2. Sketch of the partial join $K_1 \lor (3K_1 \cup H)$, where the high degree vertex $v$ is coloured grey. The edges of $H$ are not drawn.](image)

To prove this, let $G = GS_{1,\ldots,1,|H|}$ be a generalised star with $k$ arm lengths equal to 1 and one arm length equal to $|H|$, $G[V_1] = K_{1,k}$, $G[V_2] = P_{|H|}$, and let $v \in V(G)$ be the high degree vertex of $G$. By [13] Theorems 14 and 15] any matrix in $S(G)$ has the unordered multiplicity list $m$, such that $m_1, \ldots, m_\ell$ is majorized by $(k + 1, 1^{|H| - 1})$, hence $\sum_{i=\ell} m_i \leq k + \ell$. By Theorem 3.7 there exists a matrix in $(K_{1,k}, \{v\}) \lor H = K_1 \lor (kK_1 \lor H)$ with the same eigenvalues as a matrix in $S(G)$. 
PATHS ARE GENERICALLY REALISABLE

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