ASYMPTOTICS OF SIGNED BERNOULLI
CONVOLUTIONS SCALED BY MULTINACCI NUMBERS

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Abstract. We study the signed Bernoulli convolution
\[ \nu_\beta^{(n)} = \ast_{j=1}^n \left( \frac{1}{2} \delta_{\beta^{-j}} - \frac{1}{2} \delta_{-\beta^{-j}} \right), \quad n \geq 1 \]
where \( \beta > 1 \) satisfies
\[ \beta^m = \beta^{m-1} + \cdots + \beta + 1 \]
for some integer \( m \geq 2 \). When \( m \) is odd, we show that the variation
\( |\nu_\beta^{(n)}| \) coincides the unsigned Bernoulli convolution
\[ \mu_\beta^{(n)} = \ast_{j=1}^n \left( \frac{1}{2} \delta_{\beta^{-j}} + \frac{1}{2} \delta_{-\beta^{-j}} \right). \]
When \( m \) is even, we obtain the exact asymptotic of the total variation
\( ||\nu_\beta^{(n)}|| \) as \( n \to \infty \).

1. Introduction

In this paper we initiate the study of the signed Bernoulli convolution
\[ \nu_\beta^{(n)} = \ast_{j=1}^n \left( \frac{1}{2} \delta_{\beta^{-j}} - \frac{1}{2} \delta_{-\beta^{-j}} \right), \quad n \geq 1 \]
where \( \beta > 1 \) and \( \delta_x \) denotes the Dirac measure at \( x \in \mathbb{R} \). Equivalently, \( \nu_\beta^{(n)} \)
is defined inductively by letting
\[ \begin{cases} 
\nu_\beta^{(0)} = \delta_0, \\
\nu_\beta^{(n)} = \nu_\beta^{(n-1)} \ast \left( \frac{1}{2} \delta_{\beta^{-n}} - \frac{1}{2} \delta_{-\beta^{-n}} \right), \quad n \geq 1.
\end{cases} \]
By expanding out the convolution in (1), we also have
\[ \nu_\beta^{(n)} = \frac{1}{2^n} \sum_{\varepsilon_j = \pm 1} \varepsilon_1 \cdots \varepsilon_n \delta_{\sum_{j=1}^n \varepsilon_j \beta^{-j}}, \quad n \geq 1. \]
The definition of \( \nu_\beta^{(n)} \) is related to that of the unsigned Bernoulli convolution
\[ \mu_\beta^{(n)} = \ast_{j=1}^n \left( \frac{1}{2} \delta_{\beta^{-j}} + \frac{1}{2} \delta_{-\beta^{-j}} \right). \]

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which converges weakly to a probability measure $\mu_\beta$ as $n \to \infty$. The measure $\mu_\beta$ is a classical subject of study. We refer the reader to [18] for more background, and to [20], [22], [7] for some recent results.

In contrast to $\mu_\beta^{(n)}$, the signed Bernoulli convolution $\nu_\beta^{(n)}$ converges weakly to the null measure for any $\beta > 1$. This can be seen by writing for $f \in C(\mathbb{R})$

$$\langle \nu_\beta^{(n)}, f \rangle = \frac{1}{2} \langle \nu_\beta^{(n-1)}, f \ast \delta_{-\beta^{-n}} - f \ast \delta_{\beta^{-n}} \rangle$$

(where $\langle \sigma, f \rangle$ denotes $\int f d\sigma$), noting that the last integral converges to zero as $\nu_\beta^{(n-1)}$ is supported in $[-(\beta - 1)^{-1}, (\beta - 1)^{-1}]$ and $\beta^{-n} \to 0$. Moreover, if there is cancellation in the expansion (3), or equivalently, if the total variation satisfies

$$\|\nu_\beta^{(n)}\| < 1$$

for some $n$, then by Young’s convolution inequality $\|\nu_\beta^{(n)}\|$ must decay at least exponentially as $n \to \infty$. It is then of interest to determine the exact rate of decay of $\|\nu_\beta^{(n)}\|$ in such situation.

In the present paper, we study the case where $\beta > 1$ satisfies

$$\beta^m = \beta^{m-1} + \cdots + \beta + 1$$

for some integer $m \geq 2$. Note that when $m = 2$, this corresponds to the golden ratio

$$\beta = \frac{1 + \sqrt{5}}{2} = 1.618033 \cdots.$$  

Our main result is the following.

**Theorem 1.** Let

$$a_n = 2^n \|\nu_\beta^{(n)}\|, \quad n \geq 0.$$  

Suppose $\beta$ satisfies (4) for an even integer $m$. Then

$$a_n = \begin{cases} 
2^n, & \text{if } n \leq m, \\
2a_{n-1} - 2a_{n-m} + 2a_{n-m-1}, & \text{if } n \geq m + 1.
\end{cases}$$

In particular, there exists a constant $C > 0$ such that

$$\|\nu_\beta^{(n)}\| \sim C \left(\frac{\lambda}{2}\right)^n, \quad \text{as } n \to \infty$$

where $\lambda \in (1, 2)$ is the only real root of the equation

$$x^{m+1} = 2x^m - 2x + 2.$$  

For instance, if $\beta$ is given by the golden ratio (5), Theorem 1 gives

$$\|\nu_\beta^{(n)}\| \sim C(0.771844 \cdots)^n, \quad \text{as } n \to \infty.$$  

On the other hand, if $\beta$ satisfies (4) for an odd integer $m$, then the total variation $\|\nu_\beta^{(n)}\|$ has no decay in $n$; in fact, $\|\nu_\beta^{(n)}\| \equiv 1$ in this case. This will be shown in Section 7.
The proof of Theorem 1 relies on analyzing the cancellation pattern in $\nu^{(n)}_\beta$ as $n$ increases. The presence of overlap (due to $\beta < 2$) is remedied by the fact that cancellation occurs whenever an overlap is formed. This allows us to identify $\{\nu^{(n)}_\beta\}_{n \geq 0}$ with a plane tree, based on which the recurrence relation (6) is derived. The situation becomes more involved if one considers more general $\beta$. However, we hope the analysis in this paper will provide a simple model for the study of more general situations. As an application, Theorem 1 can be used to derive nontrivial bounds on certain sine products; see Section 7.

Although not directly related, our study should be compared with that of the unsigned Bernoulli convolutions. We refer the reader to [1], [10], [11], [14], [15], [8], [16], [12], [21], [9], [19], [13], [3], [5], [4], and references therein.

The plan of the paper is as follows. In Section 2, we introduce some notation which will be used throughout the remainder of the paper. We also state at the end three main lemmas. In Sections 3, 4 and 5, respectively, we give the proofs of the three lemmas. In Section 6 we prove Theorem 1. In Section 7, we conclude the paper with some remarks.

2. Notation and definition

For $n \geq 1$, denote
\[ \mathcal{D}_n = \{\pm 1\}^n. \]
Suppose
\[ \varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \mathcal{D}_n. \]
We will denote
\[ x_\varepsilon := \sum_{j=1}^n \varepsilon_j \beta^{-j} \in \mathbb{R}. \]
With this notation, we can write
\[ \nu^{(n)}_\beta = \frac{1}{2^n} \sum_{\varepsilon \in \mathcal{D}_n} \varepsilon_1 \cdots \varepsilon_n \delta_{x_\varepsilon}, \ n \geq 1. \]
Note that some of the $x_\varepsilon$’s may coincide and cancel each other. This motivates the definition
\[ A_n := \text{supp}(\nu^{(n)}_\beta) \subset \mathbb{R}, \ n \geq 0. \]
Where supp(·) stands for the support of the measure. It will be crucial to understand the sets $A_n$. To this end, we introduce more notation.

Suppose $\varepsilon$ is as above and
\[ \varepsilon' = (\varepsilon'_1, \ldots, \varepsilon'_{n+1}) \in \mathcal{D}_{n+1}. \]
We write
\[ \varepsilon \rightarrow \varepsilon' \]
if
\[ \varepsilon_j = \varepsilon'_j, \ j = 1, \ldots, n, \]
in which case we call \( \varepsilon' \) a child of \( \varepsilon \) ([6, Section 3.2]). Note that \( \varepsilon' \) must be one of
\[
\varepsilon^- := (\varepsilon_1, \ldots, \varepsilon_n, -1), \\
\varepsilon^+ := (\varepsilon_1, \ldots, \varepsilon_n, +1).
\]

For convenience, we let
\[
D_0 = \{0\}, \quad x_0 = 0
\]
and set
\[
0 \rightarrow -1, \quad 0 \rightarrow +1.
\]

**Definition 1.** Using the above notation, we define inductively
\[
\begin{cases}
D_0^* = D_0, \\
D_{n+1}^* = \{ \varepsilon' \in D_{n+1} : \varepsilon \rightarrow \varepsilon' \text{ for some } \varepsilon \in D_n^* \text{ and } x_{\varepsilon'} \in A_{n+1} \}, \quad n \geq 0.
\end{cases}
\]

In general, the set \( D_n^* \) is represented by \( A_n \) with multiplicities. However, under the assumption of Theorem 1, it will be shown that the multiplicity is always one, namely, \( D_n^* \) is 'isomorphic' to \( A_n \) (see Lemma 2 below).

Note that, together with the relation "\( \rightarrow \)", the set \( T := \bigcup_{n \geq 0} D_n \) forms a directed rooted tree ([6, Section 3.2]). Moreover, each \( D_n \) is naturally equipped with the lexicographical order (with the convention \(-1 < +1\)). Let
\[
T^* := \bigcup_{n \geq 0} D_n^*.
\]

Then \( T^* \) can be thought of as obtained from pruning the full binary tree \( T \) defined above. As a subset of \( T \), \( T^* \) inherits the relation "\( \rightarrow \)" and itself becomes a directed rooted tree (with the same root 0); also, each \( D_n^* \) inherits the lexicographical order of \( D_n \).

For \( \varepsilon \in T \), we will denote by \( T(\varepsilon) \) the subtree ([6, Section 3.1]) of \( T \) rooted at \( \varepsilon \). Similarly, for \( \varepsilon^* \in T^* \), \( T^*(\varepsilon^*) \) denotes the subtree of \( T^* \) rooted at \( \varepsilon^* \). For \( n \geq 0 \), \( T^*(\varepsilon^*; n) \) denotes the \( n \)-th level ([6, Section 3.1]) of \( T^*(\varepsilon^*) \), with the convention \( T^*(\varepsilon^*; 0) = \{ \varepsilon^* \} \).

The proof of Theorem 1 will be based on the following three lemmas, whose proofs are given in Sections 3, 4 and 5 respectively.

From now on till the end of Section 6, unless otherwise stated, we will always assume that \( \beta \) satisfies (4) for an even integer \( m \), as assumed in Theorem 1.

**Lemma 1** (The first pruning).
\[
D_n^* = \begin{cases}
D_n, & \text{if } n \leq m, \\
D_n \setminus \{(-, +, \cdots, +), (+, -, \cdots, -)\}, & \text{if } n = m + 1.
\end{cases}
\]
Lemma 2 (Tree isomorphism). For any $n \geq 0$, the map

$$x_\ast : D_n^* \to A_n, \varepsilon_\ast \mapsto x_{\varepsilon_\ast}$$

is bijective and order-preserving, that is,

$$\varepsilon_\ast < \varepsilon'_\ast \Rightarrow x_{\varepsilon_\ast} < x_{\varepsilon'_\ast}.$$

Lemma 3 (Leaflessness). Each $\varepsilon_\ast \in T^*$ has at least one child in $T^*$.

3. Proof of Lemma 1

Lemma 1 follows momentarily from the following lemma.

Lemma 4. Suppose $\varepsilon_a, \varepsilon_b \in D_n$.

(i) If $n \leq m$, then

$$\varepsilon_a < \varepsilon_b \Rightarrow x_{\varepsilon_a} < x_{\varepsilon_b}.$$

(ii) If $n = m + 1$, then

$$\varepsilon_a < \varepsilon_b \Rightarrow x_{\varepsilon_a} \leq x_{\varepsilon_b}$$

and equality holds exactly when

$$\varepsilon_a = (-,+,\cdots,+) , \ \varepsilon_b = (+,-,\cdots,-).$$

Lemma 4 ⇒ Lemma 1. Combining (10) and part (i) of Lemma 4, we see that $|A_n| = 2^n$ holds when $n \leq m$. Therefore, by the definition of $D_n^*$,

$$D_n^* = D_n$$

for all $n \leq m$. If $n = m + 1$, then by part (ii) of Lemma 4, the map

$$x_\ast : D_n \setminus \{(-,+,\cdots,+)\} \to \mathbb{R}$$

is injective; moreover, since $m$ is even, in (10) the Dirac measure at $x_{(-,+,\cdots,+)}$ cancels that at $x_{(+,-,\cdots,-)}$ due to overlap and opposite signs of coefficients. From this it follows that

$$|A_n| = 2^n - 1$$

and

$$D_n^* = D_n \setminus \{(-,+,\cdots,+), (+,-,\cdots,-)\}.$$ 

This proves Lemma 1 assuming the truth of Lemma 4. □

It remains to prove Lemma 4. The proof is based on the following lemma. For convenience, we will write

$$\rho = \beta^{-1}.$$

Lemma 5. Suppose $\beta > 1$ satisfies (4) for some integer $m \geq 2$. Then for any $n \leq m$, we have

$$\rho - \sum_{j=2}^{n} \rho^j \geq \rho^{n+1}.$$
Proof. By (4), we have

\[ 1 - \rho - \cdots - \rho^m = 0. \]

Therefore, if \( n \leq m \), then

\[ 1 - \rho - \cdots - \rho^{n-1} = \rho^n + \cdots + \rho^m \geq \rho^n. \]

Multiplying both sides by \( \rho \), we obtain the desired bound. \( \square \)

We can now prove Lemma 4.

Proof of Lemma 4. Write

\[ \varepsilon_b - \varepsilon_a = 2(\eta_1, \cdots, \eta_n) \]

where each \( \eta_j \in \{\pm 1, 0\} \). Suppose \( \eta_{j_1} \) is the first nonzero component of \( (\eta_1, \cdots, \eta_n) \), then, since \( \varepsilon_a < \varepsilon_b \), we must have

\[ \eta_{j_1} = 1. \]

Correspondingly,

\[ x_{\varepsilon_b} - x_{\varepsilon_a} = 2 \sum_{j=j_1}^{n} \eta_j \rho^j \]

\[ = 2 \left( \rho^{j_1} + \sum_{j=j_1+1}^{n} \eta_j \rho^j \right). \]

(i) If \( n \leq m \), then by Lemma 5,

\[ \rho^{j_1} + \sum_{j=j_1+1}^{n} \eta_j \rho^j \geq \rho^{j_1} - \sum_{j=j_1+1}^{n} \rho^j \geq \rho^{n+1}. \]

Thus \( x_{\varepsilon_b} - x_{\varepsilon_a} > 0 \).

(ii) If \( n = m+1 \) and \( j_1 > 1 \), then the same argument shows that \( x_{\varepsilon_b} - x_{\varepsilon_a} > 0 \).

If \( n = m + 1 \) and \( j_1 = 1 \), then

\[ \rho^{j_1} + \sum_{j=j_1+1}^{n} \eta_j \rho^j = \rho + \sum_{j=2}^{m+1} \eta_j \rho^j \]
\[ \geq \rho - \sum_{j=2}^{m+1} \rho^j = 0. \]

Moreover, the inequality above is strict unless \( \eta_2 = \cdots = \eta_n = -1 \), which corresponds to the case

\[ \varepsilon_a = (-, +, \cdots, +), \quad \varepsilon_b = (+, -, \cdots, -). \]

Therefore, except for this case (where \( x_{\varepsilon_a} = x_{\varepsilon_b} \)) we always have \( x_{\varepsilon_a} < x_{\varepsilon_b} \). This completes the proof of Lemma 4. \( \square \)
4. Proof of Lemma 2

To prove Lemma 2, we will use the following.

**Lemma 6.** Suppose \( \beta > 1 \) satisfies (4) for some integer \( m \geq 2 \). Then for any \( n \geq 0 \), we have
\[
\sum_{j=n+1}^{\infty} \rho^j < 2\rho^n.
\]

**Proof.** It suffices to consider the case \( n = 0 \). We need to show
\[
\sum_{j=1}^{\infty} \rho^j = \frac{\rho}{1 - \rho} < 2.
\]
However, by (4),
\[
1 = \rho + \cdots + \rho^m = \frac{\rho}{1 - \rho}(1 - \rho^m).
\]
Therefore, it suffices to show \( \rho^m < 1/2 \), or equivalently, \( \beta^m > 2 \). But this follows immediately from (4) and the assumption \( \beta > 1 \). \( \square \)

Lemma 6 implies the following.

**Lemma 7.** For any \( n \geq 1 \) and \( \varepsilon_a^*, \varepsilon_b^* \in D^*_n \),
\[
\varepsilon_a^* \in \mathcal{T}^*(-), \; \varepsilon_b^* \in \mathcal{T}^*(+) \Rightarrow x_{\varepsilon_a^*}^+ \leq x_{\varepsilon_b^*}^-
\]
and equality holds if and only if \( n = m \) and
\[
\varepsilon_a^* = (-, +, \cdots, +), \; \varepsilon_b^* = (+, -, \cdots, -).
\]

**Proof.** By Lemma 1 and Lemma 4, the statement holds for \( n \leq m \). Suppose \( n \geq m + 1 \). Then there exist
\[
\varepsilon_{a_0}^* \in \mathcal{T}^*(-; m), \; \varepsilon_{b_0}^* \in \mathcal{T}^*(+; m)
\]
such that
\[
\varepsilon_{a_0}^*^+ \in \mathcal{T}(\varepsilon_{a_0}^*), \; \varepsilon_{b_0}^*^- \in \mathcal{T}(\varepsilon_{b_0}^*).
\]
It follows from (9) and Lemma 6 that
\[
|x_{\varepsilon_a^*}^+ - x_{\varepsilon_{a_0}^*}|, \; |x_{\varepsilon_b^*}^- - x_{\varepsilon_{b_0}^*}| < \sum_{j=m+2}^{\infty} \rho^j < 2\rho^{m+1}.
\]
On the other hand, by Lemma 1,
\[
(-, +, \cdots, +) \notin \mathcal{T}^*(-; m), \; (+, -, \cdots, -) \notin \mathcal{T}^*(+; m);
\]
hence
\[
x_{\varepsilon_{b_0}^*} - x_{\varepsilon_{a_0}^*} \geq x_{\varepsilon_{(+, -, \cdots, +)}^*} - x_{\varepsilon_{(-, +, -, \cdots, +)}^*}
\]
\[
= 2\left(\rho - \sum_{j=2}^{m} \rho^j\right) + 2\rho^{m+1}
\]
\[
= 4\rho^{m+1}.
\]
Combining these, we get
\[ x_{\epsilon^*_b - \epsilon^*_a} = (x_{\epsilon^*_b - \epsilon^*_a}) + (x_{\epsilon^*_b - \epsilon^*_a}) - (x_{\epsilon^*_b + \epsilon^*_a}) \]
\[ > -2\rho^{m+1} + 4\rho^{m+1} - 2\rho^{m+1} \]
\[ = 0. \]
This shows \( x_{\epsilon^*_b - \epsilon^*_a} < x_{\epsilon^*_b - \epsilon^*_a} \) whenever \( n \geq m + 1 \), and the proof is complete. \( \square \)

Based on Lemma 7, we can now prove:

**Lemma 8 (A separation property).** Suppose \( n \geq 1 \) and \( \epsilon^*_a, \epsilon^*_b \in \mathcal{D}_n^* \) satisfy
\[ \epsilon^*_a < \epsilon^*_b. \]
Then for any \( k \geq 0 \),
\[ \epsilon^*_a \in \mathcal{T}^*(\epsilon^*_a; k), \epsilon^*_b \in \mathcal{T}^*(\epsilon^*_b; k) \Rightarrow x_{\epsilon^*_a +} \leq x_{\epsilon^*_b -} \]
and strict inequality holds when \( k \geq m \).

**Proof.** The proof is by induction on \( n \). The case \( n = 1 \) follows directly from Lemma 7. Suppose the statement holds for \( 1, \ldots, n - 1 \). We now prove that it also holds for \( n \).

It suffices to consider the case where
\[ \epsilon^*_a = \epsilon^*_b - , \quad \epsilon^*_b = \epsilon^*_b + \]
for some \( \epsilon^*_a \in \mathcal{D}_{n-1}^* \). This is because otherwise we can consider the nearest common ancestor (say \( \epsilon^*_a \)) of \( \epsilon^*_a \) and \( \epsilon^*_b \), and apply the induction hypothesis to \( \epsilon^*_a - \) and \( \epsilon^*_b + \).

By the induction hypothesis, for any \( \epsilon^*_a, \epsilon^*_b \in \mathcal{D}_{n-1}^* \),
\[ \epsilon^*_a < \epsilon^*_b \Rightarrow x_{\epsilon^*_a +} \leq x_{\epsilon^*_b -} \]
and equality holds only if
\[ \epsilon^*_a + , \epsilon^*_b - \notin \mathcal{D}_n^*. \]
Since \( \epsilon^*_a = \epsilon^*_b - , \epsilon^*_b = \epsilon^*_b + \in \mathcal{D}_n^* \), after mapped by \( x \), \( \epsilon^*_a \) and \( \epsilon^*_b \) do not overlap with the (either left or right) children of the nodes in \( \mathcal{D}_{n-1}^* \setminus \{\epsilon^*_a\} \). Consequently, we have
\[ \epsilon^*_b - + \cdots + \in \mathcal{T}^*(\epsilon^*_b m), \]
\[ \epsilon^*_b - \cdots - \in \mathcal{T}^*(\epsilon^*_b m), \]
and by cancellation
\[ \epsilon^*_b - + \cdots + \notin \mathcal{T}^*(\epsilon^*_b m + 1), \]
\[ \epsilon^*_b - \cdots - \notin \mathcal{T}^*(\epsilon^*_b m + 1). \]
Therefore, by the same argument as in the proof of Lemma 7, we have, for any \( k \geq m \),
\[ \epsilon^*_a \in \mathcal{T}^*(\epsilon^*_a - k), \epsilon^*_b \in \mathcal{T}^*(\epsilon^*_b + k) \Rightarrow x_{\epsilon^*_a +} < x_{\epsilon^*_b -}. \]
The case \( k \leq m - 1 \) is obvious. This completes the proof. \( \square \)
Lemma 2 now follows immediately.

Proof of Lemma 2. By taking \( k = 0 \) in Lemma 8, we see that

\[ \varepsilon_a^* < \varepsilon_b^* \Rightarrow x_{\varepsilon_a^*} < x_{\varepsilon_b^*} \leq x_{\varepsilon_b^+} < x_{\varepsilon_b^-}. \]

In particular, the map

\[ x : \mathcal{D}_n^* \to A_n, \varepsilon^* \mapsto x_{\varepsilon^*} \]

is order-preserving (thus injective). Surjectivity of this map follows easily from (2) and induction on \( n \). \( \Box \)

As a corollary of Lemma 2, the following identities follow easily by induction.

Lemma 9. For any \( n \geq 1 \), we have

\[ \nu_{\beta}^{(n)} = \frac{1}{2^n} \sum_{\varepsilon \in \mathcal{D}_n^*} \varepsilon_1 \cdots \varepsilon_n \delta_{x_\varepsilon}. \]

In particular,

\[ \left\| \nu_{\beta}^{(n)} \right\| = \frac{|\mathcal{D}_n^*|}{2^n}, n \geq 0. \]

5. Proof of Lemma 3

To prove Lemma 3, we first show:

Lemma 10 (Diamond pattern). Suppose \( n \geq m \) and \( \varepsilon^* \in \mathcal{D}_n^* \). Then

\[ \varepsilon^* - \notin \mathcal{D}_n^+ \]

if and only if

\[ \varepsilon^* = \varepsilon_0^* + \cdots \]

for some \( \varepsilon_0^* \in \mathcal{D}_{n-m}^* \) with \( \varepsilon_0^* \pm \in \mathcal{D}_{n-m+1}^* \); similarly,

\[ \varepsilon^* + \notin \mathcal{D}_n^+ \]

if and only if

\[ \varepsilon^* = \varepsilon_0^* - \cdots + \]

for some \( \varepsilon_0^* \in \mathcal{D}_{n-m}^* \) with \( \varepsilon_0^* \pm \in \mathcal{D}_{n-m+1}^* \).

Proof. We prove only the first case where \( \varepsilon^* - \notin \mathcal{D}_n^+ \). The other case follows by symmetry. Also, the ‘if’ part is easy by the proof of Lemma 8. So we only need to prove the ‘only if’ part of the statement.

Suppose \( \varepsilon_0^* \in \mathcal{D}_{n-m}^* \) is the \( m \)-th ancestor of \( \varepsilon^* \), that is,

\[ \varepsilon^* \in \mathcal{T}^*(\varepsilon_0^*; m). \]

Then, by Lemma 8, \( \varepsilon^* - \) must be canceled within \( \mathcal{T}^*(\varepsilon_0^*) \) – more precisely, there must exist \( \varepsilon_*^* \in \mathcal{T}^*(\varepsilon_0^*; m) \) such that

\[ x_{\varepsilon_*^*+} = x_{\varepsilon_*^*-}. \]
However, by Lemma 1, this is impossible unless
\[ \varepsilon^* = \varepsilon^o + \cdots, \quad \varepsilon^* = \varepsilon^o - \cdots. \]
It follows also that
\[ \varepsilon^*_\pm \in \mathcal{D}^*_n - \mathcal{D}^*_m + 1. \]
This completes the proof. 

6. Proof of Theorem 1

We are now ready to prove Theorem 1. By Lemma 9,
\[ a_n = 2^n \| \nu^{(n)}_\beta \| = | \mathcal{D}^*_n |, \quad n \geq 0. \]
In particular, Lemma 1 gives
\[ (13) \quad a_n = 2^n, \quad n \leq m. \]
For \( n \geq m \), we have, by the proof of Lemma 8,
\[ a_{n+1} = 2a_n - 2b_n \]
where \( b_n \) denotes the number of pairs \( (\varepsilon^*_a, \varepsilon^*_b) \in \mathcal{D}^*_n \times \mathcal{D}^*_n \) such that
\[ x\varepsilon^*_a = x\varepsilon^*_b. \]
By Lemma 10, such pairs are in one-to-one correspondence with the nodes \( \varepsilon^*_n \in \mathcal{D}^*_{n-m} \) satisfying
\[ (14) \quad \varepsilon^*_\pm \in \mathcal{D}^*_{n-m+1}. \]
Note that each of these nodes contributes an increment of one from \( a_{n-m} \) to \( a_{n-m+1} \). On the other hand, by Lemma 3, all other nodes in \( \mathcal{D}^*_{n-m} \) have exactly one child in \( \mathcal{D}^*_{n-m+1} \), therefore contribute no increment from \( a_{n-m} \) to \( a_{n-m+1} \). It follows that the number of nodes satisfying (14) is given by \( a_{n-m+1} - a_{n-m} \), that is,
\[ b_n = a_{n-m+1} - a_{n-m}. \]
Combining, we obtain the desired recurrence relation
\[ (6) \quad a_{n+1} = 2a_n - 2a_{n-m+1} + 2a_{n-m}, \quad n \geq m. \]
This completes the proof of (6).

It remains to prove the asymptotic (7). For that we will need:
Lemma 11. Let $m \geq 2$ be an even integer. Then the equation
\begin{equation}
  z^{m+1} = 2z^m - 2z + 2
\end{equation}
has exactly one real root; moreover, the real root lies in the interval $(1, 2)$
and has the largest absolute value among all the roots of (15).

Proof. Let
\[ f(z) = z^{m+1} - 2z^m + 2z - 2. \]
By writing
\[ f(x) = x^m(x - 2) + 2(x - 1), \]
it is easy to see that $f(x) < 0$ when $x \leq 1$ and $f(x) > 0$ when $x \geq 2$. In
particular, $f$ has at least one zero in $(1, 2)$. Suppose $m \geq 4$. Then, since
\begin{align*}
  f'(x) &= (m + 1)x^m - 2mx^{m-1} + 2, \\
  f''(x) &= (m + 1)mx^{m-1} - 2m(m - 1)x^{m-2}
\end{align*}
are both negative at $x = 1$, to show that $f$ has only one zero in $(1, 2)$ it
suffices to show that $f$ has exactly one inflection point in $(1, 2)$. However,
this is clear as one can write
\[ f''(x) = mx^{m-2}(x - 2(m - 1)), \]
which changes sign only at $x = \frac{2(m - 1)}{m + 1}$. In particular, the zero must lie in
the interval
\[ \left( \frac{2(m - 1)}{m + 1}, 2 \right). \]
The case $m = 2$ is simpler, as $f'$ would be positive on $(1, 2)$ in this case.

Let $\lambda_m$ denote the real root of $f$, and let $z_1, \ldots, z_m$ be the complex roots
of $f$. It remains to show
\[ |z_j| < \lambda_m, \ j = 1, \ldots, m. \]
In fact, we will show
\begin{equation}
  |z_j| < \frac{3}{2}, \ j = 1, \ldots, m.
\end{equation}
This suffices because $\lambda_m$ is increasing in $m$ and $\lambda_2 = 1.543 \ldots$. To show
(16), we write
\[ f(z) = g(z) + h(z) \]
with
\[ g(z) = -2z^m, \ h(z) = z^{m+1} + 2z - 2. \]
Let $r \in (1, 2)$. Then on the circle $|z| = r$, we have
\[ |g(z)| = 2r^m, \ |h(z)| \leq r^{m+1} + 2r + 2. \]
In particular,
\[ |h(z)| < |g(z)|, \ |z| = r \]
holds provided
\[ r^{m+1} + 2r + 2 < 2r^m, \]
or equivalently,
\begin{equation}
\label{eq:17}
m^m > \frac{2r + 2}{2 - r}.
\end{equation}
Now fix \( r = \frac{3}{2} \) and let \( m \geq 6 \). It is easy to see that (17) is satisfied. By Rouché’s theorem, \( f(z) \) and \( g(z) \) have the same number \( (m) \) of zeros in the disk \( |z| < r \). On the other hand, since \( \lambda_m > \frac{3}{2} \), it follows that all the \( m \) complex roots of \( f \) are in the disk \( |z| < r \), that is, (16) holds.

By direct checking, (16) also holds when \( m = 2, 4 \). This completes the proof of the lemma. \( \square \)

Now denote by \( \lambda \) the real root of (15). Consider the generating function
\[
F(z) = \sum_{n=0}^{\infty} a_n z^n, \ |z| < \lambda^{-1}.
\]
By (6), it is easy to find
\[
F(z) = \frac{1 + 2 z^m}{1 - 2 z + 2 z^m - 2 z^{m+1}}.
\]
Notice that \( 1 + 2 z^m \neq 0 \) when \( z = \lambda^{-1} \). Combining this with Lemma 11, by [17, Theorem 10.8], it follows that there exists a constant \( C > 0 \) such that
\[
a_n \sim C \lambda^n, \quad n \to \infty.
\]
This proves (7) and the proof of Theorem 1 is complete.

7. Remarks

Remark 1. By (3), in order for
\[
||v^{(n)}_\beta|| < 1
\]
to hold, there must exist \((\varepsilon_1', \ldots, \varepsilon_n') , (\varepsilon_1'', \ldots, \varepsilon_n'') \in \{\pm 1\}^n\) such that
\begin{equation}
\label{eq:18}
\sum_{j=1}^{n} \varepsilon_j' \beta^{-j} = \sum_{j=1}^{n} \varepsilon_j'' \beta^{-j}
\end{equation}
and such that
\[
\varepsilon_1' \cdots \varepsilon_n' = -\varepsilon_1'' \cdots \varepsilon_n''.
\]
Rewriting (18), this means
\begin{equation}
\label{eq:19}
2 \sum_{j=1}^{n} \eta_j \beta^{-j} = 0
\end{equation}
holds for some \((\eta_1, \ldots, \eta_n) \in \{0, \pm 1\}^n\) satisfying
\[
|\eta_1| + \cdots + |\eta_n| \text{ is odd},
\]
or equivalently,
\[
\eta_1 + \cdots + \eta_n \text{ is odd}.
\]
It is easy to see that the above reasoning is invertible. Thus, after multiplying (19) by $\beta^n$, we obtain the following.

**Lemma 12.** $\|\nu_{(n)}^{(\beta)}\| < 1$ holds if and only if there exists a polynomial

$$p(x) = \sum_{j=0}^{n-1} \eta_j x^j$$

with $(\eta_0, \cdots, \eta_{n-1}) \in \{0, \pm 1\}^n$ such that $p(1)$ is odd and $p(\beta) = 0$.

Now suppose $\beta > 1$ satisfies (4) for some odd integer $m \geq 3$. Denote its minimal polynomial by

$$m(x) = x^m - x^{m-1} - \cdots - x - 1.$$  

If there is an integer-coefficient polynomial

$$p(x) = \sum_{j=0}^{n-1} \eta_j x^j$$

such that $p(\beta) = 0$, then, by the minimality of $m(x)$, there must exist $q(x) \in \mathbb{Z}[x]$, such that

$$p(x) = m(x)q(x).$$

In particular, we have

$$p(1) = m(1)q(1).$$

However, since $m(1) = -(m-1)$ is even, it follows that $p(1)$ must be even too. Thus, combining this with Lemma 12, we obtain the following.

**Proposition 1.** Suppose $\beta > 1$ satisfies (4) for an odd integer $m \geq 3$. Then

$$\|\nu_{(n)}^{(\beta)}\| = 1, \ n \geq 1.$$ 

**Remark 2.** By taking the Fourier transform of $\nu_{(n)}^{(\beta)}$, one obtains

$$F_n(\beta; \xi) := \prod_{j=1}^{n} \sin(2\pi \beta^{-j} \xi), \ \xi \in \mathbb{R}.$$ 

Since the Fourier transform satisfies $\|\hat{\nu}\|_{\infty} \leq \|\nu\|$, this provides an upper bound for $\|F_n(\beta; \cdot)\|_{\infty}$. In particular, when $\beta = \frac{1+\sqrt{5}}{2}$, Theorem 1 gives

$$\|F_n\|_{\infty} \leq C(0.771844 \cdots)^n.$$ 

Sharpness of this bound will be addressed in [2].
References

[1] J. C. Alexander and D. Zagier. The entropy of a certain infinitely convolved Bernoulli measure. *J. London Math. Soc. (2)*, 44(1):121–134, 1991.

[2] X. Chen and T.-Y. Hu. In preparation.

[3] A. H. Fan. Asymptotic behaviour of multiperiodic functions scaled by Pisot numbers. *J. Anal. Math.*, 86:271–287, 2002.

[4] D.-J. Feng. The limited Rademacher functions and Bernoulli convolutions associated with Pisot numbers. *Adv. Math.*, 195(1):24–101, 2005.

[5] P. J. Grabner, P. Kirschenhofer, and R. F. Tichy. Combinatorial and arithmetical properties of linear numeration systems. *Combinatorica*, 22(2):245–267, 2002. Special issue: Paul Erdős and his mathematics.

[6] J. L. Gross and J. Yellen. *Graph theory and its applications*. Discrete Mathematics and its Applications (Boca Raton). Chapman & Hall/CRC, Boca Raton, FL, second edition, 2006.

[7] K. G. Hare and N. Sidorov. A lower bound for the dimension of Bernoulli convolutions. *arXiv preprint*, 2016.

[8] T.-Y. Hu. The local dimensions of the Bernoulli convolution associated with the golden number. *Trans. Amer. Math. Soc.*, 349(7):2917–2940, 1997.

[9] S. P. Lalley. Random series in powers of algebraic integers: Hausdorff dimension of the limit distribution. *J. London Math. Soc. (2)*, 57(3):629–654, 1998.

[10] K.-S. Lau. Fractal measures and mean $p$-variations. *J. Funct. Anal.*, 108(2):427–457, 1992.

[11] K.-S. Lau. Dimension of a family of singular Bernoulli convolutions. *J. Funct. Anal.*, 116(2):335–358, 1993.

[12] K.-S. Lau and S.-M. Ngai. $L^q$-spectrum of the Bernoulli convolution associated with the golden ratio. *Studia Math.*, 131(3):225–251, 1998.

[13] K.-S. Lau and S.-M. Ngai. $L^q$-spectrum of Bernoulli convolutions associated with P. V. numbers. *Osaka J. Math.*, 36(4):993–1010, 1999.

[14] F. Ledrappier and A. Porzio. A dimension formula for Bernoulli convolutions. *J. Statist. Phys.*, 76(5-6):1307–1327, 1994.

[15] F. Ledrappier and A. Porzio. On the multifractal analysis of Bernoulli convolutions. II. Dimensions. *J. Statist. Phys.*, 82(1-2):397–420, 1996.

[16] S.-M. Ngai. A dimension result arising from the $L^q$-spectrum of a measure. *Proc. Amer. Math. Soc.*, 125(10):2943–2951, 1997.

[17] A. M. Odlyzko. Asymptotic enumeration methods. In *Handbook of combinatorics, Vol. 1, 2*, pages 1063–1229. Elsevier Sci. B. V., Amsterdam, 1995.

[18] Y. Peres, W. Schlag, and B. Solomyak. Sixty years of Bernoulli convolutions. In *Fractal geometry and stochastics, II (Greifswald/Koserow, 1998)*, volume 46 of *Progr. Probab.*, pages 39–65. Birkhäuser, Basel, 2000.

[19] A. Porzio. On the regularity of the multifractal spectrum of Bernoulli convolutions. *J. Statist. Phys.*, 91(1-2):17–29, 1998.

[20] P. Shmerkin. On the exceptional set for absolute continuity of Bernoulli convolutions. *Geom. Funct. Anal.*, 24(3):946–958, 2014.

[21] N. Sidorov and A. Vershik. Ergodic properties of the Erdős measure, the entropy of the golden shift, and related problems. *Monatsh. Math.*, 126(3):215–261, 1998.

[22] P. P. Varjú. Recent progress on Bernoulli convolutions. *arXiv preprint*, 2016.
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