A no-go theorem for the $n$-twistor description of a massive particle

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Abstract

It is proved that the $n$-twistor expression of a particle’s four-momentum vector reduces, by a unitary transformation, to the two-twistor expression for a massive particle or the one-twistor expression for a massless particle. Therefore the genuine $n$-twistor description of a massive particle in four-dimensional Minkowski space fails for the case $n \geq 3$.

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I. INTRODUCTION

About 40 years ago, attempts to describe massive particles and their associated internal symmetries were made by Penrose, Perjés, and Hughston within the framework of twistor theory [1–6]. To describe a massive particle in four-dimensional Minkowski space, $M$, they introduced two or more [i.e., $n(\in \mathbb{N} + 1)$] independent twistors and their dual twistors

$$Z^A_i = (\omega^i_\alpha, \pi^i_{\dot{\alpha}}), \quad \bar{Z}^i_A = (\bar{\pi}^i_{\dot{\alpha}}, \bar{\omega}^{i\dot{\alpha}}) \quad (1.1)$$

($A = 0, 1, 2, 3; \alpha = 0, 1; \dot{\alpha} = 0, 1$) distinguished by the index $i$ ($i = 1, 2, \ldots n$). Here, $\bar{\pi}^i_{\dot{\alpha}}$ and $\bar{\omega}^{i\dot{\alpha}}$ denote the complex conjugates of the two-component spinors $\pi^i_{\dot{\alpha}}$ and $\omega^i_\alpha$, respectively:

$$\bar{\pi}^i_{\dot{\alpha}} := \pi^i_{\dot{\alpha}}, \quad \bar{\omega}^{i\dot{\alpha}} := \omega^i_\alpha.$$  

(1.2)

The spinors $\omega^i_\alpha$ and $\pi^i_{\dot{\alpha}}$ are related by

$$\omega^i_\alpha = i z^{\alpha\dot{\alpha}} \pi^i_{\dot{\alpha}},$$

where $z^{\alpha\dot{\alpha}}$ are coordinates of a point in complexified Minkowski space, $\mathbb{C}M$. It was shown in Refs. [1–6] that the massive particle system described by $n$ twistors possesses the internal symmetry specified by an inhomogeneous extension of $SU(n)$, denoted by $ISU(n)$. Penrose, Perjés, and Hughston proposed the idea of identifying the $SU(2)$ [or $ISU(2)$] symmetry in the two-twistor system with the symmetry for leptons, and the $SU(3)$ [or $ISU(3)$] symmetry in the three-twistor system with the symmetry for hadrons.

Long after Penrose, Perjés, and Hughston made their attempts, Lagrangian mechanics of a massive spinning particle in $M$ has been formulated in terms of twistors, and it has been studied until quite recently [7–17]. In almost all these studies [7–15, 17], only the two-twistor description of a massive particle is conventionally adopted without clarifying the reason why the $n(\geq 3)$-twistor description is not employed. Under such circumstances, Routh and Townsend showed that only the two-twistor formulation can successfully describe a massive particle in $M$ [16]. (See also the note added in Sec. 3.)

As can be seen in Refs. [1–6], the $n$-twistor expression of a particle’s four-momentum vector is given by

$$p_{\alpha\dot{\alpha}} = \sum_{i=1}^{n} \bar{\pi}^i_{\dot{\alpha}} \pi^i_{\dot{\alpha}} = \bar{\pi}^i_{\dot{\alpha}} \pi^i_{\dot{\alpha}}.$$  

(1.3)

(The expression (1.3) has recently been exploited to realize massive representations of the Poincaré algebra [18]. In this study, the massive representations are actually considered only
in the case $n = 2$.) By using Eq. (1.3), the squared mass $m^2 = p_{\alpha\dot{\alpha}}p^{\alpha\dot{\alpha}}$ can be written as

$$m^2 = \bar{\pi}_i^\alpha \pi_i^{\dot{\alpha}} \bar{\pi}_{j\dot{\alpha}} \pi_j^\alpha,$$  \hspace{1cm} (1.4)

where $\bar{\pi}^{i\alpha} := \epsilon^{\alpha\beta} \bar{\pi}_\beta$ and $\pi_i^{\dot{\alpha}} := \epsilon_{\dot{\alpha}\dot{\beta}} \pi_i^{\dot{\beta}}$ ($\epsilon^{01} = \epsilon^{0\dot{1}} = 1$). In Lagrangian mechanics mentioned above, Eq. (1.4) with $n = 2$, or its equivalent expression

$$\epsilon_{ij} \pi_i^{\dot{\alpha}} \pi_j^{\dot{\alpha}} - \sqrt{2} me^{i\varphi} = 0,$$  \hspace{1cm} (1.5a)

$$\epsilon_{ij} \bar{\pi}_i^\alpha \bar{\pi}_j^{\dot{\alpha}} - \sqrt{2} me^{-i\varphi} = 0$$  \hspace{1cm} (1.5b)

($\epsilon^{12} = \epsilon_{12} = 1$) with a real parameter $\varphi$ [13, 14, 17], is incorporated into a generalization of the Shirafuji action [19] with the aid of Lagrange multipliers.

Now, we present the following theorem:

**Theorem:** The $n$-twistor expression given in Eq. (1.3) reduces to the two-twistor expression

$$p_{\alpha\dot{\alpha}} = \bar{\pi}_1^\alpha \pi_{1\dot{\alpha}} + \bar{\pi}_2^\alpha \pi_{2\dot{\alpha}}, \hspace{1cm} \bar{\pi}_{1\dot{\alpha}} \bar{\pi}_2^{\dot{\alpha}} \neq 0$$  \hspace{1cm} (1.6)

or the one-twistor expression

$$p_{\alpha\dot{\alpha}} = r \bar{\pi}_1^\alpha \pi_{1\dot{\alpha}},$$  \hspace{1cm} (1.7)

where $\pi_{i\dot{\alpha}} := U_i^{j\dot{\alpha}} \pi_j^\alpha$, $\bar{\pi}_1^\alpha := \bar{\pi}_i^\alpha U_{i1}^\dagger$, and $r$ is a real constant. Here, $U_i^{j\dot{\alpha}}$ are the entries of an $n \times n$ unitary matrix $U$.

Hence the $n$-twistor system eventually turns out to be a two-twistor system representing a massive particle or a one-twistor system representing a massless particle. The purpose of the present paper is to prove this theorem. The theorem leads to the fact that the genuine $n(\geq 3)$-twistor description of a massive particle in $\textbf{M}$ fails owing to the reduction from Eq. (1.3) to Eq. (1.6) or Eq. (1.7) caused by a unitary transformation. For this reason, the above-mentioned idea for the $SU(3)$ [or $\text{ISU}(3)$] symmetry cannot be accepted. In this sense, the theorem given here can be said to be a no-go theorem. Also, the theorem justifies the fact that in the context of a four-dimensional Minkowski background, only the two-twistor description (i.e., the case $n = 2$) has been considered in Lagrangian mechanics of a massive spinning particle formulated in terms of twistors.

This paper is organized as follows. In Sec. II, we prove the theorem using a lemma provided there. Section III is devoted to remarks.
II. A PROOF OF THE THEOREM

We first provide a lemma necessary to prove the theorem.

**Lemma:** Let $A$ be an $n \times n$ complex antisymmetric matrix, satisfying $A^T = -A$. Then $A$ can be transformed into its normal form, $\tilde{A}$, according to

$$\tilde{A} = UAU^T,$$  \hspace{1cm} (2.1)

where $U$ is an $n \times n$ unitary matrix, satisfying $U^* = U^{-1}$. If $n$ is even, then the normal form $\tilde{A}$ is given by

$$\tilde{A} = \begin{pmatrix} 0 & \sqrt{a_1} & 0 & 0 & \cdots & 0 & 0 \\ -\sqrt{a_1} & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \sqrt{a_2} & \cdots & 0 & 0 \\ 0 & 0 & -\sqrt{a_2} & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & \ddots & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & 0 & \cdots & 0 & \sqrt{a_{n/2}} \\ 0 & 0 & 0 & 0 & \cdots & -\sqrt{a_{n/2}} & 0 \end{pmatrix},$$  \hspace{1cm} (2.2)

and if $n$ is odd, then $\tilde{A}$ is given by

$$\tilde{A} = \begin{pmatrix} 0 & \sqrt{a_1} & 0 & 0 & \cdots & 0 & 0 & 0 \\ -\sqrt{a_1} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{a_2} & \cdots & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{a_2} & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & \ddots & \ddots & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & 0 & \cdots & 0 & \sqrt{a_{(n-1)/2}} & 0 \\ 0 & 0 & 0 & 0 & \cdots & -\sqrt{a_{(n-1)/2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}.$$  \hspace{1cm} (2.3)

Here, $a_1, a_2, \ldots, a_{n/2}$ [or $a_{(n-1)/2}$] are eigenvalues of the Hermitian matrix $AA^*$; hence it follows that these eigenvalues are non-negative real numbers.

In this paper, we do not give the proof of this lemma, because the proof can be seen in Refs. [20–23].
Proof of the theorem: In order to prove the theorem, let us consider the $n \times n$ complex matrix $\Pi$ consisting of the elements

$$\Pi_{ij} := \pi_{ia} \pi_{j\alpha}^\alpha. \quad (2.4)$$

Because $\pi_{ia} \pi_{j\alpha}^\alpha = -\pi_{i\alpha}^\alpha \pi_{j\alpha}$ holds, $\Pi$ turns out to be antisymmetric. According to the lemma, the matrix $\Pi$ can be transformed into its normal form

$$\tilde{\Pi} = U \Pi U^T \quad (2.5)$$

by means of an appropriate $n \times n$ unitary matrix $U = (U_{ij})$. Expressions corresponding to Eqs. (2.2) and (2.3) are concisely given by

$$\tilde{\Pi}_{2r-1,j} = \delta_{2r,j} \tilde{\Pi}_{2r-1,2r}, \quad (2.6a)$$
$$\tilde{\Pi}_{2r,j} = \delta_{2r,j+1} \tilde{\Pi}_{2r,2r-1}, \quad (2.6b)$$
$$r = 1, 2, \ldots, n/2, \quad \text{for } n \text{ even},$$
$$r = 1, 2, \ldots, (n+1)/2, \quad \text{for } n \text{ odd},$$

where $\tilde{\Pi}_{2r-1,2r} = -\tilde{\Pi}_{2r,2r-1}$ is the square root of an eigenvalue of $\Pi \Pi^\dagger$, so that it is a non-negative real number.

Substituting Eq. (2.4) into Eq. (2.5), we can express the elements of $\tilde{\Pi}$ as

$$\tilde{\Pi}_{ij} = \bar{\pi}_{ia} \pi_{j\alpha}^\alpha, \quad (2.7)$$

with the two-component spinor

$$\bar{\pi}_{ia} := U_{i}^{j} \pi_{j\alpha}. \quad (2.8)$$

Since $U$ is unitary and hence invertible, Eq. (2.8) can be inversely solved as $\pi_{ia} = U_{i}^{j} \pi_{j\alpha}$. Substituting this and its complex conjugate into Eq. (1.3), we have

$$p_{\alpha\dot{\alpha}} = \sum_{i=1}^{n} \bar{\pi}_{i\alpha} \pi_{i\alpha} \equiv \bar{\pi}_{i\alpha} \pi_{i\alpha}. \quad (2.9)$$

From Eqs. (2.6a) and (2.7), we see $\bar{\pi}_{1\alpha} \pi_{k\alpha}^\alpha = 0$ ($k = 3, 4, \ldots, n$). This implies that $\bar{\pi}_{k\alpha}$ ($k = 3, 4, \ldots, n$) is proportional to $\bar{\pi}_{1\alpha}$, i.e.,

$$\bar{\pi}_{k\alpha} = \rho_{k1} \bar{\pi}_{1\alpha}, \quad \rho_{k1} \in \mathbb{C}. \quad (2.10)$$
Substituting Eq. (2.10) into Eq. (2.7) and noting the property \( \pi_{i\alpha} \pi_{i}^\alpha = 0 \) (no sum with respect to \( i \)), we obtain

\[
\tilde{\Pi}_{kl} = \tilde{\pi}_{k\alpha} \tilde{\pi}_{l}^\alpha = \rho_{k1} \rho_{l1} \tilde{\pi}_{1\alpha} \tilde{\pi}_{1}^\alpha = 0, \quad k, l = 3, 4, \ldots, n. \tag{2.11}
\]

By using Eq. (2.10), \( \tilde{\Pi}_{2k} = \tilde{\pi}_{2\alpha} \tilde{\pi}_{k}^\alpha \) (\( k = 3, 4, \ldots, n \)) can be written as

\[
\tilde{\Pi}_{2k} = \rho_{k1} \tilde{\pi}_{2\alpha} \tilde{\pi}_{1}^\alpha = \rho_{k1} \tilde{\Pi}_{21}. \tag{2.12}
\]

Equations (2.6b) and (2.12) together give

\[
\rho_{k1} \tilde{\Pi}_{21} = 0, \quad k = 3, 4, \ldots, n, \tag{2.13}
\]

with which we consider the following two cases: (a) \( \tilde{\Pi}_{21} \neq 0 \) and (b) \( \tilde{\Pi}_{21} = 0 \).

A. Case (a)

If \( \tilde{\Pi}_{21} \neq 0 \), then it follows that \( \rho_{k1} = 0 \) for any arbitrary \( k \). Accordingly, Eq. (2.10) becomes

\[
\tilde{\pi}_{k\alpha} = 0, \quad k = 3, 4, \ldots, n. \tag{2.14}
\]

Substituting Eq. (2.14) into Eq. (2.9), we have

\[
p_{\alpha\dot{\alpha}} = \tilde{\pi}_{1\alpha} \tilde{\pi}_{1}^\alpha + \tilde{\pi}_{2\alpha} \tilde{\pi}_{2}^\alpha. \tag{2.15}
\]

Here, \( \tilde{\pi}_{1\alpha} \tilde{\pi}_{2}^\alpha = -\tilde{\Pi}_{21} \neq 0 \) has already been assumed.

B. Case (b)

In case (b), each of the \( \rho_{k1} \) does not need to vanish. Combining \( \tilde{\Pi}_{21} = 0 \), Eq. (2.11), and the \( \tilde{\Pi}_{1k} = \tilde{\Pi}_{2k} = 0 \) (\( k = 3, 4, \ldots, n \)) included in Eq. (2.6) together, we ultimately have \( \tilde{\Pi}_{ij} = 0 \) or, equivalently, \( \tilde{\pi}_{i\alpha} \tilde{\pi}_{j}^\alpha = 0 \). This leads to

\[
\tilde{\pi}_{i\alpha} = \rho_{i1} \tilde{\pi}_{1\alpha}, \quad \rho_{i1} \in \mathbb{C}, \quad \rho_{11} = 1. \tag{2.16}
\]

Substituting Eq. (2.16) into Eq. (2.9), we obtain

\[
p_{\alpha\dot{\alpha}} = r \tilde{\pi}_{1\alpha} \tilde{\pi}_{1}^\alpha, \quad r := \sum_{i=1}^{n} |\rho_{i1}|^2 \in \mathbb{R}. \tag{2.17}
\]
We have indeed derived Eqs. (1.6) and (1.7) by investigating the two cases (a) and (b). Thus the proof of the theorem is complete. ■

In case (a), the squared mass \( m^2 = p_\alpha p^{\alpha} \) becomes \( m^2 = 2|\bar{H}_{21}|^2 \neq 0 \) and it follows that the corresponding particle is massive, while, in case (b), the squared mass becomes \( m^2 = r^2|\bar{H}_{11}|^2 = 0 \) and it follows that the corresponding particle is massless.

Now we consider the new twistor \( \bar{Z}_i^A := U_{ij} Z_j^A = (\bar{\omega}_i^\alpha, \bar{\pi}_i^\alpha) \), with \( \bar{\omega}_i^\alpha := iz^{\alpha\dot{\alpha}} \bar{\pi}_i^\alpha \). In case (a), Eq. (2.14) gives \( \bar{Z}_k^A = 0 (k = 3, 4, \ldots , n) \), and we see that the \( n \)-twistor system turns out to be the two-twistor system described by \((\bar{Z}_1^A, \bar{Z}_2^A)\). In case (b), Eq. (2.16) gives \( \bar{Z}_i^A = \rho_i \bar{Z}_i^A \), and we see that all the twistors \( \bar{Z}_i^A \) correspond to a single projective twistor defined as the proportionality class \( [\bar{Z}_i^A] := \{ \rho \bar{Z}_1^A | \rho \in \mathbb{C} \setminus \{0\} \} \). Hence, in case (b), the \( n \)-twistor system is described by \( [\bar{Z}_1^A] \) and turns out to be essentially a one-twistor system. (This statement is consistent with the expression in Eq. (2.17).) As is well known in twistor theory, a projective twistor precisely specifies the configuration of a massless particle. From this fact, we see again that in case (b), the \( n \)-twistor system represents a massless particle.

III. REMARKS

It is now clear that in case (a), the \( SU(n) \) [or \( ISU(n) \)] symmetry of the \( n \)-twistor system reduces to the \( SU(2) \) [or \( ISU(2) \)] symmetry of a two-twistor system for a massive particle, while, in case (b), the \( SU(n) \) [or \( ISU(n) \)] symmetry is realized in a one-twistor system for a massless particle. Hence, the \( SU(n) \) [or \( ISU(n) \)] symmetry in the case \( n \geq 3 \) cannot be identified with the internal symmetry of a massive physical system consisting of, e.g., hadrons. For this reason, the idea proposed by Penrose, Perjés, and Hughston fails in the case \( n \geq 3 \). Of course, there still remains a possibility that the \( SU(n) \) [or \( ISU(n) \)] symmetry will be identified with the internal symmetry of a massless system.

As can be seen from the theorem proved above, the case \( n = 2 \) is the only case in which a massive particle in \( \mathbf{M} \) can be described. In fact, Lagrangian mechanics of a massive spinning particle in \( \mathbf{M} \) has been successfully formulated in Refs. [7–17] by using two twistors. In the respective formulations, the \( SU(2) \) symmetry in two-twistor system is maintained in the action of a massive spinning particle. It was shown in Ref. [17] that this \( SU(2) \) symmetry is a symmetry realized in the particle-antiparticle doublet, not in the lepton doublet consisting.
of, e.g., the electron and the electron-neutrino. Therefore the idea proposed by Penrose, Perjés, and Hughston is not valid in the present Lagrangian mechanics based on twistor theory. In addition to the $SU(2)$ symmetry, the action possesses the $U(1)$ symmetry due to a common phase transformation of two twistors. It was pointed out that this symmetry is identified as a symmetry leading to chirality conservation [17].

Note added: After completing an earlier version of this paper, the authors were informed that the same result concerning values of $n$ was obtained by Routh and Townsend using an inequality [16]. Because this inequality is derived by considering the phase space dimension and appropriate constraints including the spin-shell constraints, the result being obtained may be understood to be depending on a specific model. In contrast, our proof of the theorem is purely linear algebraic and is independent of Lagrangian mechanics.

The authors were also informed that the twistor description of a massive particle in anti-de Sitter space has been performed with the use of more than one twistor [24–26]. In this context, it would be of considerable interest to clarify the necessary number of twistors by taking a similar linear algebraic approach. We hope to address this issue in the future.

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