A New Cyber-Secure Countermeasure for LTI systems under DoS attacks

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Abstract

This paper presents a new counter-measure to mitigate denial-of-service cyber-attacks in linear time-invariant (LTI) systems. We first design a sparse linear quadratic regulator (LQR) optimal controller for a given LTI plant and evaluate the priority of the feedback communication links in terms of the loss of closed-loop performance when the corresponding block of the feedback gain matrix is removed. An attacker may know about this priority ordering, and thereby attack the links with the highest priority. To prevent this, we present a message rerouting strategy by which the states that are scheduled to be transmitted through the high priority links can be rerouted through lower priority ones in case the former get attacked. Since the attacked link is not available for service, and the states of the low priority links can no longer be accommodated either, we run a structured $H_2$ control algorithm to determine the post-attack optimal feedback gains. We illustrate various aspects of the proposed algorithms by simulations.

I. INTRODUCTION

Security of cyber-physical systems has drawn a significant research attention in recent times. Due to notable instances of cyber-attacks such as WannaCry [1], NotPetya [2], and Ukrainian blackout [3], there have been increased interests to design countermeasures to mitigate attacks for cyber-physical systems. A significant part of the existing work focuses on centralized systems (see e.g., [4], [5]) while the recent results in [6], [7] rely on distributed algorithms. Prompted by these considerations, this paper presents a novel sparse optimization based countermeasure to alleviate cyber-attacks for a general class of LTI systems.

System theoretic approaches to tackle a class of cyber-attacks namely Denial of Service (DoS in short) have recently been investigated in [8, 9, 10]. Given a LTI system, [8] presents a novel analysis methodology to maintain the closed loop stability under DoS attacks, while [9] unveils a similar analysis for its nonlinear counterpart. Given a class of complex networks, [10] analyzes the consensus property of self-triggered agents in the presence of DoS attacks. The analysis documented in [10] introduces the notion of persistence-of-communication and characterizes DoS frequency and duration to attain consensus under DoS attacks.

Substantial studies have been undertaken on the analysis of consensus/synchronization behavior under DoS attacks, see, for example [11, 12, 13] and the references therein. In the context of multi-agent systems, a group of agents is said to reach consensus/synchronization when all individuals converge towards a common value (consensus)/state (synchronization). Consensus analysis over unreliable networks has been motivated by the seminal contribution from [11], in which, the authors consider a linear consensus model in the presence of misbehaving agents whose behavior diverge from the nominal consensus evolution. Given a complex network having misbehaving agents, [11] illustrates the problem of ensuring consensus under non-colluding and Byzantine attacks. Given a class of general LTI systems, consensus under DoS attacks is analyzed for undirected [12] and directed [13] topology. The results in [12, 13] hypothesize sufficient conditions to ensure asymptotic consensus and also characterize the frequency and the duration of the DoS interval. However, the above analysis primarily relies on static graphs illustrating an idealistic setup.

Recently game-theoretic results are employed in conjunction with distributed optimization to tackle the security problems for large-scale cyber-physical networks. In the game-theoretic setup, the notion of interdependent security games has recently been explored to compute optimal and strategic security investments by multiple defenders, for example see [14, 15]. In [14, 15], the authors consider each defender is responsible for the security of multiple assets, in which the inter-dependencies among the assets captured by an interdependency graph. The authors redesign the problem of computing the optimal defense allocation by a single defender as a convex optimization problem and establish the existence of a pure Nash equilibrium of the game between multiple defenders. Given a networked control system, [16] investigates a slightly different problem, where the authors reformulate a general-sum, two-player, mixed strategy game between an attacker and a defender. The authors of [16] exploit the nonlinear programming paradigm to analyze the dependence of a Mixed Strategy Nash Equilibrium on the relative budgets of the players and preserve important network nodes to attain a desirable LQR performance.

Summary of Contributions: In light of the aforementioned works, in this paper we present a new approach for mitigating DoS attacks by using ideas from sparse optimal control. Given a LTI system defined over a network of
$N \geq 2$ nodes, we first design a sparse linear quadratic regulator (LQR) optimal controller using $l_1$-sparsity promotion techniques, proposed in [17]. The LQR control law is given as $u(t) = K x(t)$. The non-zero blocks $K_{ij}$ of the sparse matrix $K$ indicate the existence of communication links between nodes $i$ and $j$, carrying state $x_i(t)$ to controller $u_j(t)$. We carry out an offline analysis to evaluate the priority of these feedback communication links in terms of how much loss is incurred in the closed-loop LQR objective function when any block $K_{ij}$ is removed. We assume that an attacker may also know about this priority ordering, and thereby is most likely to attack the links with highest priority so that he/she can cause maximum damage to the closed-loop response. To prevent this, we present a message rerouting strategy by which the states that are scheduled to be transmitted through the high priority links can now be quickly rerouted through lower priority ones in case the former becomes dysfunctional from a DoS attack. We present algorithms that capture various practical issues related to the size of the rerouted state vector versus the volume of the low-priority link. One must note that following the re-routing, the attacked link is not available for service, and the states of the low priority links can no longer be accommodated for communication either. Thus, retaining the same control gains for the rest of the states may result in a severely sub-optimal closed-loop performance. We, therefore, finally run a structured $H_2$ control algorithm, proposed in [13] to determine the post-attack optimal feedback gains. We illustrate various aspects of the proposed algorithms by simulations.

The rest of the paper is organized as follows: In Section IV we formulate the problem, while in Section III we document a preparatory note on the sparsity promoting optimal control problem. We present our proposed rerouting algorithm in Section IV. Finally in Section V we provide an academic example to verify our contribution.

**Notation:** We denote the set of real and natural numbers by $\mathbb{R}$ and $\mathbb{N}$ respectively. $I$ is a matrix with all its entries equal to one, while $I_k$ symbolizes the identity matrix of dimension $k$. $0_{pq}$ denotes a zero matrix of dimension $p \times q$. Given a matrix $M \in \mathbb{R}^{pq}$, $||M||_F$ defines the ‘Frobenius norm’ of $M$, and $M(j,:) \in \mathbb{R}^{1 \times q}$ presents the $j^{th}$ row of $M$, with $j \in \{1, 2, \cdots, p\}$. For a square matrix $N \in \mathbb{R}^{n \times p}$, trace($N$) is calculated as trace($N$) := $\sum_{i=1}^{n} n_i$. Given two matrices $A \in \mathbb{R}^{pq}$ and $B \in \mathbb{R}^{nx}$, $A \otimes B \in \mathbb{R}^{pqn}$ defines their kronecker product. Similarly, for two matrices $C, D \in \mathbb{R}^{m \times n}$, the standard hadamard product is calculated as $C \circ D \in \mathbb{R}^{m \times n}$.

**II. Problem Setup**

Let us consider the continuous-time LTI system as

$$\begin{align*}
\dot{x}(t) &= A x(t) + Bu(t) + Wd(t), \quad x(0) = x_0, \\
y(t) &= C x(t) + Du(t),
\end{align*}$$

(1)

where $x(t) = \begin{bmatrix} x_1(t), \cdots, x_N(t) \end{bmatrix}^\top \in \mathbb{R}^n$ is the overall state vector, with $x_j(t) \in \mathbb{R}^{n_j \times 1}$ is the state corresponding to node $j \in \{1, 2, \cdots, N\}$, and $n = \sum_{j=1}^{N} n_j$. $u(t) = \begin{bmatrix} u_1^\top(t), \cdots, u_N^\top(t) \end{bmatrix}^\top \in \mathbb{R}^m$ is the overall control input with $u_j(t) \in \mathbb{R}^{m_j \times 1}$ is the control input of node $j$, and $m = \sum_{j=1}^{N} m_j$. $d(t) \in \mathbb{R}^q$ is the disturbance, and $y(t) \in \mathbb{R}^p$ is the system output. All the matrices documented above i.e., $A \in \mathbb{R}^{p \times n}$, $B \in \mathbb{R}^{p \times m}$ and $W \in \mathbb{R}^{n \times q}$ are with appropriate dimensions. In addition, the matrices $C \in \mathbb{R}^{p \times n}$ and $D \in \mathbb{R}^{p \times n}$ are defined as $C := \begin{bmatrix} Q^{1/2} & 0 \end{bmatrix}$ and $D := \begin{bmatrix} 0 & R^{1/2} \end{bmatrix}$, where $Q = Q^\top \geq 0$ and $R = R^\top > 0$ are the state and the control performance weights. We assume, the matrix pair $(A, B)$ is stabilizable and $(A, Q^{1/2})$ is detectable.

We consider a state feedback control input

$$u(t) = K x(t),$$

(2)

where $K \in \mathbb{R}^{m \times n}$ is the feedback gain matrix. In the vector form, (2) can be expressed as

$$\begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_N(t) \end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} & \cdots & K_{1N} \\ K_{21} & K_{22} & \cdots & K_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ K_{N1} & K_{N2} & \cdots & K_{NN} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_N(t) \end{bmatrix},$$

(3)

where the sub-matrix $K_{ij} \in \mathbb{R}^{m_j \times n_i}$ represents a communication link that delivers a block of information of state $j \in \{1, \cdots, N\}$ to control $i \in \{1, \cdots, N\}$. For simplicity, for $i = j$, the feedback links are referred to as local links, while for $i \neq j$ as communication links.

Notice that, given a LTI system [11], an optimal gain $K \in \mathbb{R}^{m \times n}$ can be designed employing the LQR strategy [19]. When $K$ is optimal, every $K_{ij}$ and $K_{ii}$, in general, are non-zero sub-matrices, implying that the communication
network required for exchanging the states is a dense graph i.e., communication links from every state to every control input. Such dense graphs can result in high communication costs. Therefore, to reduce the cost, we impose an additional structural constraint $\Omega$ on the structure of $K$ as follows \[18\]

$$
\begin{align*}
\min_K J(K) \\
\text{subject to } K \in \Omega,
\end{align*}
$$

where the cost function $J(K)$ is defined as

$$J(K) := \begin{cases} 
\text{trace} (W^T P(K) W) & \text{if } K \text{ is stabilizing} \\
+\infty & \text{otherwise}.
\end{cases}$$

The $P(K) \in \mathbb{R}^{m \times n}$ matrix stated above, denotes the closed-loop observability Gramian and presented as

$$P(K) := \int_0^\infty e^{(A-BK)^\top} \sigma (Q + K^\top RK) e^{(A-BK)\sigma} \, d\sigma,$$

which is obtained by solving the Lyapunov equation

$$(A - BK)^\top P + P(A - BK) = -(Q + K^\top RK). \quad (6)$$

Given a system $[11]$ and $[2]$, let $K_1 \in \Omega$ be the solution of $[11]$, in which, a communication link $K_{ij} \in \mathbb{R}^{m \times n}$ delivers a block of information state $j$ to control $i$. We assume, during the closed loop operation, an attacker attacks either a communication link or a local link of $K_1$ to destabilize the system. For instance, killing a communication/local link say $K_{ij}^r$, containing $r > 0$, number of messages equivalently signifies $K_{ij}^r$ is zeroed out. Let $K_2 \in \mathbb{R}^{m \times n}$ be the post-attack feedback gain in which $K_{ij}^r = 0_{ij}$. Therefore, after the attack, we focus on addressing the objective of:

Problem 2.1: finding some communication space in $K_2$ to reroute the attacked $r$ messages. In particular, we need to determine ‘$r$ spots’ in the off-diagonal and the diagonal blocks of $K_2$ such that the resultant closed loop system is stable, and it minimizes the cost function $J(K_2)$.

III. TECHNICAL PRELIMINARIES

In the following, we briefly review the sparsity promoting optimal control problem. The readers are encouraged to see $[17]$ for details and further references.

A. Sparsity-Promoting Optimal Control Problems

The optimization problem $[11]$ solely relies on the structure of the communication graph. Hence, in the following, we characterize the optimization setup, in which the sparsity of the feedback gain directly subsumed into the objective function as

$$
\begin{align*}
\min_K J(K) + \beta \text{card}(K),
\end{align*}
$$

where, $\text{card} : \mathbb{R}^{m \times n} \to \mathbb{N}$ is the cardinality function i.e., the number of nonzero elements of a matrix (denoted as $\text{nnz}(\cdot)$), and is defined as:

$$
\text{card}(K) := \sum_{i,j=1}^N \text{nnz}(K_{ij}). \quad (8)
$$

In addition, $\beta \in [0, \infty)$ is a scalar gain, and a large value of $\beta$ leads to a sparser $K$.

Since the objective function $[17]$ is non-convex due to the cardinality function, therefore it is typically replaced by the weighted $l_1$ norm of the optimization variables $[17]$. Given a feedback gain $K \in \mathbb{R}^{m \times n}$, the weighted $l_1$ norm is represented as

$$
\text{card}(K) = \sum_{i,j} G_{ij} \|K_{ij}\|_F,
$$

where $G_{ij}$ are the non-negative weights, defined as $G_{ij} := 1/ (\|K_{ij}\|_F + \varepsilon)$ with $0 < \varepsilon \ll 1$. Therefore, the minimization problem $[17]$ can further be approximated as

$$
\begin{align*}
\min_K J(K) + \beta \sum_{i,j} G_{ij} \|K_{ij}\|_F ,
\end{align*}
$$

where, $J(K)$ is the square of the closed loop $\mathcal{H}_2$ norm and $\sum_{i,j} G_{ij} \|K_{ij}\|_F$ is the sparsity promoting penalty function.
IV. PROPOSED REROUTING ALGORITHM

Given a continuous-time system (1), this section presents a rerouting algorithm to mitigate the cyber-attacks for LTI systems. Our strategy can typically be categorized in three steps: i) link prioritization ranking algorithm, ii) rerouting algorithm, and iii) structured $\mathcal{H}_2$ algorithm.

Remark 4.1: As indicated earlier, we consider $K$ in (2) to be partitioned into $r_1$ non-zero block matrices. Each block $K_{ij}$ is associated with a communication link carrying state $x_i(t)$ from the $i^{th}$ node to controller $u_j(t)$ at the $j^{th}$ node. When an attacker deactivates a link, the entire block $K_{ij}$ is zeroed out. Let us assume the block $K_{ij}$ contains $r$ number of messages. Hence, as defined in Problem 2.1, we need “r spots” in the lower priority control channels to reroute the attacked messages. Now, let us consider a scenario, in which, we do not have a lower priority control channels to reroute the attacked messages. Now, let us consider a scenario, in which, we do not have a lower priority control channels to reroute the entire message.

Step 1: Link prioritization ranking algorithm: Given a LTI system (1) with the feedback gain $K \in \mathbb{R}^{m \times n}$, this step evaluates the priority of the feedback communication links $K_{ij}$. It typically carries out an offline analysis technique to evaluate the priority of the feedback communication links in terms of how much loss is incurred in the closed-loop LQR objective function when any block $K_{ij}$ is removed. We initialize a small value of $\beta \in \mathbb{R}$ denoted as $\beta_{\text{initial}}$. It results a centralized LQR gain $K = K_o \in \mathbb{R}^{m \times n}$, and starts minimizing (10). As stated earlier, the solution of (10) becomes sparser as $\beta$ increases; hence we obtain different footprints of the sparsity pattern of $K$ by varying $\beta$. Therefore, from these different sparsity patterns of $K$, we accumulate the knowledge of the priority of each control blocks. Following this, we construct a matrix $N_{\text{initial}} : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^{r_1 \times r_2}$ in which the control blocks are placed based on their priority. In the sequel, we denote $r_1 \in \mathbb{N}$ as the total number of non-zero control blocks present in $K$, while $r_2 \in \mathbb{N}$ defines the ‘size’ of each block i.e. the amount of information stored in each block.

**Algorithm 1** (Offline) link prioritization ranking algorithm

1. **Input data:** $A, B, W, Q, R$ from (1), and $\beta_{\text{initial}}$.
2. **Output data:** $K$ and $N_{\text{initial}}(K)$.
3. Initialize $\beta_1 = \beta_{\text{initial}}, N_o(\cdot) = 0_{r_1,r_2}$.
4. **for** $j = \beta_1, \beta_2, \beta_3, \cdots$ with, $\beta_1 < \beta_2 < \beta_3 < \cdots$ **do**
5. Minimize (10) and obtain the total number of nonzero control blocks in $K$ at $\beta_j$.
6. Determine the control blocks which are vanished to sparsify $K$ at $\beta_j$ w.r.t $\beta_{j-1}$.
7. Assign the vanished control blocks in $N_o(\cdot)$, with the priority pertaining to $\beta_j$.
8. **end for**
9. **Return** $N_{\text{initial}}(K) = N_o(\cdot)$.

The functionality of Algorithm 1 is illustrated in Fig. 1.

![Diagram](image)

Fig. 1: A pictorial representation of Algorithm 1 with $\beta_1 < \beta_2 < \beta_3$.

Remark 4.2: The link prioritization ranking algorithm stated above, is performed offline, and it typically considers the class of problems in which the attacked communication links do not vary with time. For instance, let us assume $S(t')$ is the set of communication channels which are attacked by the attacker at time $t' \geq 0$, then the link prioritization ranking algorithm considers $S(t) = S(t')$ for all $t \geq t'$.
Step 2: Rerouting algorithm: Given a continuous-time LTI system \( \mathbf{A}, \mathbf{B} \) and \( \mathbf{C} \). Step-1 generates \( N_{\text{initial}}(K) \in \mathbb{R}^{n \times r \times 2} \) accumulating all the non-zero control blocks of \( K \in \mathbb{R}^{m \times n} \) and assigns priority to each of them. In addition, let \( q : \mathbb{R}^{n \times r \times 2} \to \mathbb{R}^{r} \) be the priority vector, which hold the priority information of the control blocks stacked in \( N_{\text{initial}}(K) \). We assume the attacker knows \( N_{\text{initial}}(K) \) and \( q(N_{\text{initial}}(K)) \in \mathbb{R}^{r} \) and based on this information, it attacks communication links to destabilize the overall closed-loop system. Let \( p_{\text{attack}}(\cdot) \in \mathbb{R}^{r} \) be the attack index, and it contains information about the attacked control blocks/communication links, in which, the \( j^{th} \) row denoted as \( p_{\text{attack}}(j) \), defined as:

\[
p_{\text{attack}}(j) := \begin{cases} 1 & \text{if } N_{\text{initial}}(j,:) \text{ is under attack}, \\ 0 & \text{if } N_{\text{initial}}(j,:) \text{ is not attacked.} \end{cases}
\]

To design a countermeasure, we assume the defender knows \( N_{\text{initial}}(K) \), \( q(\cdot) \) and \( p_{\text{attack}}(\cdot) \). In the following, we present three variants of the rerouting algorithms described in Algorithm 2–4.

Let us remark, the first rerouting algorithm given in Algorithm 2, based on the following assumption:

Assumption 4.1: All the sub-matrices \( K_{ij} \) of \( K \), are with the same dimension.

Given a feedback gain matrix \( K \), the above assumption illustrates that all the block matrices \( K_{ij} \in \mathbb{R}^{m \times n} \) are required to construct \( K \), having same dimension. Additionally, it also signifies that all the state vectors in \( \mathbf{R}^{n} \) have same dimension, i.e., \( x(j) \in \mathbb{R}^{n} \) where, \( n_{1} = n_{2} = \cdots = n \) for all, \( j \in \{1, 2, \cdots, N\} \).

Algorithm 2 Rerouting algorithm: all the sub-matrices have same dimensions

1. Input data: \( N_{\text{initial}}(K), q(\cdot), \) and \( p_{\text{attack}}(\cdot) \).
2. Output data: \( N_{\text{final}}(K) \).
3. Initialize \( \mathbb{R}^{n \times r \times 2} \ni N(\cdot) := N_{\text{initial}}(K), \)
4. Determine the attacked links from \( p_{\text{attack}}(\cdot) \in \mathbb{R}^{r} \) and the corresponding priority of the attacked links are stored in \( \alpha(\cdot) \in \mathbb{R}^{r} \), where \( r_{3} \leq r_{1} \).
5. Sort \( \alpha(\cdot) \) in descending order and denote as \( a1(\cdot) \in \mathbb{R}^{r} \).
6. if \( (r_{3} = 0) \) then
7. STOP. No attack has occurred.
8. else if \( (r_{3} > r_{1}/2) \) then
9. STOP. Countermeasure can not be implemented.
10. else
11. for \( (j = 1, 2, \cdots, r_{3}) \) do
12. if \( (p_{\text{attack}}(j) = 1) \) \& \( (a1(j) < q(j + 1)) \) then
13. Lower priority communication space is unavailable, discard the attacked control block information by \( N(a1(j),:) = 0_{1 \times r_{3}} \),
14. else
15. Lower priority communication space is available, reroute high priority information by \( N(j,:), = 0_{1 \times r_{3}} \),
16. end if
17. end for
18. end if
19. Return \( N_{\text{final}}(K) = N(\cdot) \).

We present the following example to demonstrate Algorithm 2:

Example 1: Consider a continuous-time LTI system \( \mathbf{A}, \mathbf{B} \) with \( (A, B) \in \mathbb{R}^{8 \times 8} \times \mathbb{R}^{8 \times 4} \). Let \( K \in \Omega_{1} \) be the solution of \( \Omega_{1} \). The structural constraint \( \Omega_{1} \in \mathbb{R}^{4 \times 8} \) is given as

\[
\Omega_{1} = \begin{bmatrix}
* & 0 & * & * \\
0 & * & 0 & * \\
0 & * & 0 & 0 \\
* & * & 0 & *
\end{bmatrix},
\]

1In this work, the notion of rerouting signifies to give up some lower priority communication links to sustain an attacked one.
Given a feedback gain $K$ stated above, we employ algorithm 1 to evaluate the priorities of the non-zero blocks $K_{ij}$. The priorities corresponding to the non-zero blocks $K_{ij}$ are represented as $q(\cdot) = \{3, 1, 0, 7, 9, 3, 2\}$. Notice that, the defender reroutes the attacked information with priority level $q(\cdot) = 8$ and $q(\cdot) = 7$ via the lower priority control channels with priority $q(\cdot) = 1$ and $q(\cdot) = 2$ respectively (see Step 15 in Algorithm 2). Likewise, an alternative communication link to redirect the attacked information of the control block with $q(\cdot) = 4$ will be the control block with $q(\cdot) = 3$. However, the priority of the control block with $q(\cdot) = 4$ is higher than the priority of the control block with $q(\cdot) = 3$. Hence the defender discards the information corresponding to $q(\cdot) = 3$ (see Step 13 in Algorithm 2); see Fig. 2 for an illustration.

Fig. 2: (a) Communication topology of the control gain $K \in \Omega_1$ given in (11), where (●) defines the links which are attacked, (b) post-attack communication graph in which (●) denotes the links whose data safely rerouted through lower priority blocks. The dotted line signifies lower priority communication links which are sacrificed to reroute the attacked information. Here each communication link carries 2 unit of information, while the priority of each link denoted as $q(\cdot)$, are specified over it.

It is noteworthy that Algorithm 2 is designed based on Assumption 4.1 which implies each control block contains the same amount of information. In the subsequent algorithms, we relax Assumption 4.1 and consider a more generalized setup, in which, each control block contains different amount of information. Hence in the sequel, we present two variants of the rerouting algorithms described in Algorithm 3 and 4, in which Algorithm 3 presents a rerouting strategy for a single link attack, while Algorithm 4 elucidates multiple links attack.

Remark 4.3: As stated in Step 1, given an $N_{\text{initial}}(\cdot) \in \mathbb{R}^{r_1 \times r_2}$, $r_1$ denotes the total number of the non-zero control blocks present in the gain matrix $K$, while $r_2$ defines the size of each block. However, Algorithm 3 and 4 consider a $K$ matrix, in which, each control blocks are with different sizes. Let, $\bar{r}_1, \bar{r}_2, \cdots, \bar{r}_n$ with $n \in \mathbb{N}$, be the sizes of the control blocks present in $K$, then in the sequel, we select $r_2 := \max \{\bar{r}_1, \bar{r}_2, \cdots, \bar{r}_n\}$.

Subsequently in the Algorithm 3, $r_{\text{attack}}$ denotes the single link which is attacked.

The null matrix $[0 \ 0]$ represented as the dotted line in Fig. 2 signifies the control blocks which are scarified to reroute the attacked information.
Algorithm 3 Rerouting algorithm: non-zero control blocks of different sizes with a single link attack

1: **Input data:** $N_{\text{initial}}(K)$, $r_{\text{attack}}$,
2: **Output data:** $N_{\text{final}}(K)$,
3: **Initialization** $\mathbb{R}^{r_1 \times r_2} \ni N(\cdot) := N_{\text{initial}}(K)$,
4: Define $s(\cdot) \in \mathbb{R}^r$, contains the amount of information accumulated by each control block,
5: if ($r_{\text{attack}} = 1$) then
6: Set $N(r_{\text{attack}} ; :) = 0_{1,r_2}$,
7: else
8: Set $c_2 = s(r_{\text{attack}})$ and $c_1 = 0$,
9: Calculate the number of available (not attacked) communication spaces (denoted as $c_1 \in \mathbb{R}$) to reroute $c_2 \in \mathbb{R}$ attacked messages,
10: if ($c_1 < c_2$) then
11: Insufficient lower priority communication spaces are available; set $N(r_{\text{attack}} ; :) = 0_{1,r_2}$,
12: else
13: Set $k = 0$,
14: while ($c_2 > 0$) do
15: Update $k = k + 1$ and reroute $c_2$ information via low priority control channel $N(k ; :) = 0_{1,r_2}$,
16: Update $c_2 = c_2 - s(k)$,
17: end while
18: end if
19: end if
20: Return $N_{\text{final}}(K) = N(\cdot)$.

We present a variant of Algorithm 3 with multiple links attack, as follows:

Algorithm 4 Rerouting algorithm: non-zero control blocks with different sizes with multiple links attack

1: **Input data:** $N_{\text{initial}}(K)$, $p_{\text{attack}}(\cdot)$,
2: **Output data:** $N_{\text{final}}(K)$,
3: **Initialization** see Step 3 of Algorithm 3.
4: Set $b_1 = b_2 = 0$,
5: Calculate $s(\cdot) \in \mathbb{R}^r$ given in Step 4 of Algorithm 3,
6: Calculate $a'(\cdot), a1(\cdot) \in \mathbb{R}^r$ given in Step 4–5 of Algorithm 2,
7: Determine the location of the attacked control block with the highest priority and denote as $a'(\text{end}) \in \mathbb{R}$,
8: Calculate the total number of attacked information (determined as $b_1 \in \mathbb{R}$) and the available communication space to reroute the attacked $b_1$ information (determined as $b_2 \in \mathbb{R}$) upto $a'(\text{end})$,
9: if ($b_1 = 0$) then
10: STOP. No attack has happened.
11: else if ($b_2 = 0$) & ($r_3 < r_1/2$) then
12: for ($j = 1, \ldots, r_3$) do
13: Set $N(j ;) = 0_{1,r_2}$,
14: end for
15: else if ($b_1 > b_2$) & ($r_3 \geq r_1/2$) then
16: STOP. Countermeasure can not be designed.
17: else
18: for ($j = 1 \cdot r_3$) do
19: Set $c_2 = s(a1(j))$ and $c_1 = 0$,
20: Go to Step 8 – 19 of Algorithm 3,
21: end for
22: end if
23: Return $N_{\text{final}}(K) = N(\cdot)$.

Remark 4.4: Notice that, in the algorithmic setup given in Algorithm 3 and 4, we recall the class of gain matrices
in which, each communication link (say $K_{ij}$) carries different amounts of information corresponding to state $x_i(t)$ from the $i^{th}$ node to controller $u_j(t)$ at the $j^{th}$ node. Under these circumstances, a single link attack describes a scenario, where the attacker attacks only one communication link, while in a multiple links attack, the attacker kills several (more than one) communication channels. However, Algorithm 4 is capable enough to tackle multiple links attack problem; it can also be implemented for a single link attack case as well. In other words, Algorithm 3 is considered a simpler version of Algorithm 4 and provides a much-needed foundation to design Algorithm 4.

**Example 2**: This example illustrates the functionality of Algorithm 4. We revisit the continuous time LTI system \((A, B) \in \mathbb{R}^{14 \times 14} \times \mathbb{R}^{14 \times 4}\). Let $K \in \Omega_2$ be the solution of (4), where $\Omega_2$ is structured as

$$
\Omega_2 = \begin{bmatrix}
\star & \star & 0 & 0 \\
0 & 0 & \star & 0 \\
\star & 0 & 0 & \star \\
\star & 0 & 0 & 0
\end{bmatrix}.
$$

(12)

\(*\) and $\star$ denote non-zero control blocks having dimensions $1 \times 2$ and $1 \times 4$ respectively. In other words, it contains 2 and 4 unit of information. The feedback gain $K$ is evaluated as

$$
K = \begin{bmatrix}
2 & 1 & 3 & 7 & 5 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 3 & 1 & 3 & 6 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 & 2 & 6 & 4 & 0 \\
3 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
$$

(13)

The priority vector $q(\cdot)$ of the $K$ matrix in (13) is given as $q(\cdot) = 1 \mapsto \begin{bmatrix} 2 & 1 \end{bmatrix}$, $q(\cdot) = 2 \mapsto \begin{bmatrix} 1 & 5 \end{bmatrix}$, $q(\cdot) = 3 \mapsto \begin{bmatrix} 3 & 5 \end{bmatrix}$, $q(\cdot) = 4 \mapsto \begin{bmatrix} 3 & 7 & 5 & 8 \end{bmatrix}$, $q(\cdot) = 5 \mapsto \begin{bmatrix} 3 & 1 & 3 & 6 \end{bmatrix}$, $q(\cdot) = 6 \mapsto \begin{bmatrix} 7 & 2 & 6 & 4 \end{bmatrix}$. We assume the attacker attacks the control blocks pertaining to the priority of $q(\cdot) = 5$. Then, exploiting Algorithm 4, we obtain post-attack $N_{\text{final}}(K)$ as, $q(\cdot) = 1 \mapsto \begin{bmatrix} 0 & 0 \end{bmatrix}$, $q(\cdot) = 2 \mapsto \begin{bmatrix} 0 & 0 \end{bmatrix}$, $q(\cdot) = 3 \mapsto \begin{bmatrix} 3 & 5 \end{bmatrix}$, $q(\cdot) = 4 \mapsto \begin{bmatrix} 3 & 7 & 5 & 8 \end{bmatrix}$, $q(\cdot) = 5 \mapsto \begin{bmatrix} 3 & 1 & 3 & 6 \end{bmatrix}$, $q(\cdot) = 6 \mapsto \begin{bmatrix} 7 & 2 & 6 & 4 \end{bmatrix}$. Notice that, as stated in Remark 4.1, the attacked information with the priority level $q(\cdot) = 5$ contains 4 unit of information and the information is rerouted through the communication links with the priority level $q(\cdot) = 1$ and $q(\cdot) = 3$ respectively (2 unit each). An equivalent pictorial representation is presented in Fig. 3.

![Fig. 3: (a) Communication topology of $K$ in $\Omega_2$ given in (13), where $\bullet$ signifies the attacked link, (b) post-attack communication graph where $\bullet$ denotes the link whose data are safely rerouted through lower priority blocks. The dotted line denotes lower priority communication links which are sacrificed to reroute the attacked information, while, the nomenclature stated above the communication links e.g., $s(\cdot)$ and $q(\cdot)$, specifies the amount of information communicated over that links and the priority of the link respectively.](https://example.com/image)

**Remark 4.5**: The rerouting algorithms proposed in Algorithm 2, 3 and 4, are general enough to tackle cyber-attacks for any class of structured gain $K \in \mathbb{R}^{m \times n}$ with large values of $m, n \in \mathbb{N}$. However, we adhere to the smaller dimensions of $K$ in example 1 and 2 in order not to blur the message of our work.

**Step 3: Structured $\mathcal{H}_2$ algorithm**: The structured $\mathcal{H}_2$ algorithm collects the post-attack $N_{\text{final}}(K) \in \mathbb{R}^{n \times r_2}$ from Step-2. $N_{\text{final}}(K)$ contains the information corresponding to the zero and non-zero control blocks, and allows us through dotted lines.

3The null matrix $\begin{bmatrix} 0 & 0 \end{bmatrix}$ symbolizes the control blocks which are scarified to reroute the attacked information. It is illustrated in Fig. 3 by dotted lines.
to construct the post-attack structural constraint Ω (say ˆΩ). Then our objective boils down to solve the following optimization problem

\[
\begin{align*}
\text{minimize} & \quad J(K) \\
\text{subject to} & \quad K \circ I_Ω^c = 0,
\end{align*}
\]

where \( I_Ω^c = 1 - I_Ω \), and \( I_Ω \) is the structural identity of the subspace ˆΩ with its \( i^{th} \) entry as

\[
[I_Ω]_{ij} := \begin{cases} 
1 & \text{if } K_{ij} \text{ is a free variable} \\
0 & \text{if } K_{ij} = 0 \text{ is required.}
\end{cases}
\]

The augmented Lagrangian for (14) is evaluated as

\[
L_γ(K, Λ) = J(K) + \text{trace}(Λ^T(K \circ I_Ω^c)) + \frac{γ}{2} \| K \circ I_Ω^c \|_F^2,
\]

in which, \( \frac{γ}{2} \| K \circ I_Ω^c \|_F^2 \) is introduced to locally convexify the Lagrangian. (see [18], [20] for the details). In (15) the penalty weight \( γ \) is a positive scalar, while, \( Λ \in \mathbb{R}^{m \times n} \) is the Lagrange multiplier.

\begin{algorithm}
\textbf{Algorithm 5} Structured \( \mathcal{H}_2 \) algorithm
\begin{enumerate}
\item \textit{Initialize} \( Λ^0 = 0, \text{ and } γ^0 > 0 \)
\item \textbf{for} \( (i = 0, 1, \cdots) \) \textbf{do}
\item \textbf{for} fixed \( Λ^i \), minimize \((15)\) with respect to unstructured \( K \)
\item Update \( Λ^{i+1} = Λ^i + γ(K \circ I_Ω^c) \)
\item Update \( γ^{i+1} = αγ^i \) with \( α > 1 \)
\item \textbf{until}: the stopping criteria \( \| K^i \circ I_Ω^c \| < ϵ \)
\item \textbf{end for}
\end{enumerate}
\end{algorithm}

The optimization paradigm given in (14), always yields a post-attack stabilizing control gain \( K \); see [17] Remark 3 for details.

\textbf{Remark 4.6}: It is noteworthy that Step 1 of our proposed algorithm is mainly carried out offline, while Step 2 and 3 are performed online after an attack occurs in the network. Step 1 can be executed at any time before the attack, and it doesn’t depend on the characteristics of the attack model.

\section{V. Simulation Results}

In this section, we present an academic example to illustrate the functionality of our proposed algorithm. We revisit a coupled dynamical system with \( N = 10 \) sub-systems, as

\[
\dot{x}(t) = \sum_{j=1}^{N} A_{ij} x_i(t) + B_{ii} u_i(t) + W_i d_i(t), \quad \forall i \in \{1, \cdots, N\},
\]

where, \( x_i(t) \in \mathbb{R}^2 \) is the state vector of the \( i^{th} \) sub-system, while \( d_i(t) \in \mathbb{R}^2 \), and \( u_i(t) \in \mathbb{R} \) are the corresponding disturbance and the control input. The overall system dynamics evolves as

\[
\dot{x}(t) = Ax(t) + Bu(t) + Wd(t),
\]

where the drift matrix \( A \in \mathbb{R}^{20 \times 20} \) is structured as

\[
A = \begin{bmatrix}
A_{11} & A_{12} & \cdots & A_{1N} \\
A_{21} & A_{22} & \cdots & A_{2N} \\
\vdots & \vdots & \ddots & \vdots \\
A_{N1} & A_{N2} & \cdots & A_{NN}
\end{bmatrix}.
\]

We consider \( A \) to be a random matrix, in which, all the eigenvalues lie in the open left half of the complex plane, while the dominant eigenvalue is placed relatively close to the origin. In addition, we calculate the overall \( B \in \mathbb{R}^{20 \times 10} \) and \( W \in \mathbb{R}^{20 \times 20} \) given in (17), as

\[
B = B_{ii} \otimes I_{10}, \quad D = D_{ii} \otimes I_{10}, \quad \forall i \in \{1, 2, \cdots, N\}
\]
where

\[ B_{ii} = \begin{bmatrix} 10 \\ 0 \end{bmatrix}, \quad W_{ii} = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}. \]

We select the design parameters as \( Q = I_{20} \) and \( R = 10I_{10} \) respectively. Given the system dynamics (17), we consider the control law \( u(t) = Kx(t) \), where \( K \in \mathbb{R}^{10 \times 20} \) is the feedback gain matrix. The gain matrix \( K \) is designed based on the sparsity promoting algorithm (10). First, we start minimizing (10) for a small initial value of \( \beta \). We consider the block sparsity structure with the weighted sum of the Frobenius norm given in (9), as the sparsity-promoting penalty function and set \( G_{ij} = 1/\left(\|K_{ij}\|_F + \varepsilon\right) \) with \( \varepsilon = 10^{-3} \). As stated earlier, the small (initial) value of \( \beta \) yields a centralized LQR gain \( K_c \in \mathbb{R}^{10 \times 20} \), depicted in Fig. 4 (a). Then, we perform an offline analysis based on Algorithm 1 to collect all the non-zero control blocks of \( K \in \mathbb{R}^{10 \times 20} \) and prioritize each of them. We do the prioritization in terms of the loss of closed-loop performance when the corresponding block of the feedback gain matrix is removed. We assume, an anonymous attacker has the knowledge of the priority of each control blocks of \( K \) and attacks a set of communication links (denoted in red in Fig. 4 (a)) to yield a poor closed loop response. Finally, an application of the re-routing algorithms evaluates the post-attack sparsity structure illustrated in Fig. 4 (b).

![Fig. 4: (a) The sparsity pattern of the centralized LQR gain \( K_c \in \mathbb{R}^{10 \times 20} \). The (•) denotes the attacked control blocks. The square box denotes the amount of information accumulated by each control block. (b) illustrates the post-attack sparsity pattern after applying the rerouting algorithm given in Section IV.](image)

| TABLE I: Comparison between the pre and post-attack \( J(K) \) values |
|-----------------|-----------------|-----------------|-----------------|
| \( J(K) \)      | Before attack   | Immediate after attack | After reroute   |
| Fig. 4          | 0.47            | 0.48             | 0.475          |
| Fig. 5          | 0.46            | 0.47             | 0.467          |

Notice that, initially the centralized LQR gain \( K = K_c \) has 50 control blocks, where each control block carries 4 unit of information (placed inside the rectangle shown in Fig. 4 (a), (b)). The attacker attacks 22 control blocks (denoted in red in Fig. 4 (a)) which are successfully rerouted via rest of the 28 control blocks shown in 4 (b), as expected.

Next, we simulate another numerical setup illustrated in Fig. 5 (a), (b). We assume the centralized LQR gain \( K_c \in \mathbb{R}^{10 \times 20} \) partitioned into 100 control blocks, in which, each control block contains 2 unit of information. Notice that, the attacker attacks 48 control blocks (denoted in red in Fig. 5 (a)). Then, by exploiting the rerouting
algorithm the attacked information are safely rerouted via remaining unaffected (Lower priority) 52 control blocks; see Fig. 5 (b).

![Figure 5](image-url)

**Fig. 5:** (a) The sparsity pattern of the centralized LQR gains \( K_c \in \mathbb{R}^{10 \times 20} \). The (●) denotes the control blocks which are attacked by the attacker. The square box signifies amount of information carried by each control block. In (b), we illustrate the post-attack sparsity pattern after applying rerouting algorithm given in Section IV.

In Table I we present a comparative study of the \( J(K) \) values corresponding to both the simulations stated above.

**Remark 5.1:** Notice that, the figures presented above in particular, Fig. 4 (b), 5 (b), illustrate the post-attack structural constraint, in which, we employ the structured \( \mathcal{H}_2 \) algorithm given in Algorithm 5, to evaluate the post-attack feedback gains and its corresponding \( J(K) \) values. The post-attack (after rerouting) \( J(K) \) values are documented in Table I however, we exclude the post-attack feedback gain matrices due to the space limitation.

**Remark 5.2:** In Table I, it seems that not a lot of control performance is lost when the DoS attack is happening, and also not a lot of performance is gained when the proposed countermeasure is applied to the attacked systems. This is occurred due to the choice of the system matrix \( A \) given in Eq. (17). Notice that, \( A \) is a random matrix in which each element lies between [0, 1]. A larger value of the system matrix also alter the control performances. However, as stated earlier, we adhere to simple examples to validate our contributions.

**VI. Concluding remarks**

Given a continuous-time LTI system, this work highlights a new algorithmic strategy to alleviate cyber-attacks for LTI systems. We assume the anonymous attacker has the information corresponding to the structured feedback gain matrix and its priority of its control channels. Then based on the attack model, we develop a rerouting strategy, in which, after an attack, the higher priority communication data are rerouted through lower priority control channels. The priority of the control channels are assigned employing sparse optimization methods with sparsity promoting penalty functions. The rerouting functions allow us to determine the post-attack structural constraint, in which an application of the structured \( \mathcal{H}_2 \) algorithm evaluates the post-attack feedback gain. Future work involves analyzing the complexity of the proposed algorithms.

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