Nonrenormalizability of the nonprojectable Hořava theory

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Abstract

We perform an analysis of the ultraviolet divergences of the quantum non-projectable Hořava gravity. We work the quantum field theory directly in the Hamiltonian formalism provided by the Batalin-Fradkin-Vilkovisky quantization. In this way the second-class constraints can be incorporated to the quantization. A known local gauge-fixing condition leads to a local canonical Lagrangian. Although the canonical fields acquire regular propagators, irregular propagators persist for the field associated to the measure of the second-class constraints. These lead to irregular one-loops exhibiting subdivergences along the integration of the frequency that cannot be cancelled. This shows that the theory is non-renormalizable. Even nonprojectable models with higher anisotropy cannot be finite due to the same effect.
1 Introduction

The Hořava theory [1] is a proposal for quantum gravity. It may be formulated as a quantum field theory with perturbative interactions. In this sense, the theory has the remarkable feature of being power-counting renormalizable and ghost-free.\(^1\) Its main characteristic is the breaking of the local Lorentz symmetry, due to the definition of a foliation of spacelike hypersurfaces along a given direction of time with absolute physical meaning. This enables the definition of an anisotropic Lagrangian that is power-counting renormalizable, thanks to the higher order in spatial derivatives. Thus, in this proposal the breaking of the local Lorentz symmetry is the price to pay for obtaining the power-counting renormalizability, simultaneously with the elimination of ghost-like modes, which are known to be present in relativistic higher curvature theories [3]. The underlying gauge symmetry to write the Lagrangian is given by the group of foliation-preserving diffeomorphisms. This symmetry is compatible with two independent formulations of the theory: the projectable and nonprojectable cases. In the projectable case one of the field variables, the lapse function, is restricted to be a function only of time, whereas in the nonprojectable case it can be a general function of time and space. The nonprojectable case has field equations closer to general relativity, hence it can be more interesting from a physical point of view.

Due to its interest as a possible theory for quantum gravity, several studies have been devoted to analyze the consistency of the theory, both at classical and quantum levels. An essential extension of the Lagrangian of the nonprojectable case was found in Ref. [2], incorporating more terms that are compatible with the foliation-preserving diffeomorphisms. These terms improve the dynamics of the classical theory. The consistency of the classical Hamiltonian formulation for this version has been shown in Refs. [4, 5, 6]. On the projectable side, the Hamiltonian analysis (for the effective action) was carried out in Ref. [7].

One of the highlighting aspects of the dynamics of the nonprojectable case is the presence of second-class constraints. Their origin can be regarded qualitatively as a consequence of the reduced symmetry. As it is well known, general relativity has first-class constraints that are the generators of general diffeomorphisms over the spacetime. An analogue of the Hamiltonian constraint is still present in the Hořava theory, but it acquires a second-class behavior since there is no generator of a gauge symmetry associated to the transformation of time with space. The other second-class constraint is a formal statement: the vanishing of the conjugate momentum of the lapse function. In the projectable case the symmetry is equally reduced, but the simplification on the lapse function render the Hamiltonian constraint to be a global (integrated) constraint, hence effectively there is no local second-class constraint that affects the functional quantum modes.

The main task to do about the self-consistency of the Hořava theory is to study its renormalization beyond the power-counting criterium. In the case of the projectable theory, its renormalizability has been shown in Ref. [8]. Despite the similarities, the presence of the second-class constraints on the side of the nonprojectable version constitutes an important technical difference between the two cases. Thus, although the quantization of the projectable case has been focused essentially by following the procedure of gauge theories,\(^2\)

\(^1\)Actually, there are regions of the space of coupling constants where ghosts are present, see, for example, [2].
this is not so direct in the case of the nonprojectable theory. Other studies concerning the renormalization and renormalization group of the projectable Hořava theory can be found, among others, in Refs. [9, 10, 11, 12, 13, 14, 15, 16]. On the nonprojectable case, an early study about the effect of loops on the coupling to relativistic matter fields was performed in Ref. [17]. A computation of the heat kernel was focused in [18].

An essential step in the proof of renormalization of the projectable case consists in obtaining regular propagators for all quantum fields [8]. The definition of regular propagators was introduced in studies of renormalizability of nonrelativistic gauge theories [19], and implies that at the ultraviolet they acquire suitable scalings both in frequency and spatial momentum. Once the regularity of all propagators is achieved, the complete renormalization is proved in a similar way as relativistic gauge theories. The gauge-fixing condition chosen for the quantization plays a central role in obtaining the regular propagators for all quantum fields. Remarkably, it turns out that it must be nonlocal. This analysis is based on the Lagrangian formalism under which the classical theory is originally formulated, as usual in the quantization of gauge theories. In Ref. [8] it was noticed that a reformulation of the same theory in terms of a Hamiltonian formalism may lead to a local canonical Lagrangian with also regular propagators. This issue is rather nondirect, since the nonlocal gauge-fixing condition is also a noncanonical condition. In Refs. [20, 21] we showed that the Batalin-Fradkin-Vilkovisky (BFV) quantization may be used in Hořava theory to conciliate these two aspects: the Hamiltonian approach and the noncanonical gauge-fixing conditions. Previously, two of us focused the quantization of the nonprojectable case using a canonical gauge, the transverse gauge [22]. In this case the quantization is performed in the original phase space using the measures for the first- and second-class constraints [23, 24]. In Ref. [21] we showed that when the gauge-fixing condition is introduced with a local operator in the BFV quantization of the projectable case, then the integration on momenta of its linearized version reproduces the renormalizable quantum Lagrangian. In this way the connection between the two approaches, BFV and Lagrangian quantization, is established.

Once the BFV quantization of the Hořava theory has been established, in this paper we utilize this scheme of quantization to study loop divergences in the nonprojectable Hořava theory, with the aim of elucidate its renormalizability. For definiteness, we take the $2 + 1$ theory, but we discuss that the approach and the results apply automatically to the $3 + 1$ theory. Our approach consists on analyzing the quantum theory directly on the Hamiltonian formalism, deriving Feynman rules for all the canonical and auxiliary fields (Lagrange multipliers, ghosts). We want to stay as close as possible to the quantization of the projectable case, hence we use the previously commented gauge-fixing condition. Indeed, we find that most of the quantum fields get regular propagators.

The BFV formalism introduces an extension of the phase space, including canonical ghost fields, which we call the BFV ghosts. In this way the first-class constraints and the gauge fixing are not imposed in the measure of the path integral. Instead, the second-class constraints are imposed in the standard way as part of the measure. Technically, this measure can be promoted to the quantum canonical Lagrangian with the help of auxiliary fields. Thus, one may develop Feynman digramatics for the theory, including the auxiliary variables. We undertake this analysis emphasizing on the impossibility of getting regular propagators for all fields, and the consequent impossibility of cancelling the associated divergences in loops.
by local counterterms. We discuss about the influence that the gauge-fixing condition has on the irregular propagators. We finally observe that the problem of the divergences affects to the models with higher degree of anisotropy, which are believed to be finite.

2 Quantization in the Hamiltonian formalism

Here we present a summary of the BFV quantization of the nonprojectable Hořava theory on a foliation of 2 + 1 dimensions, with a specific gauge-fixing condition. Further details about the procedure can be found in Refs. [20, 21]. The BFV quantization of systems with constraints was developed in the papers [25, 26, 27], starting the study with bosonic systems with first-class constraints only and ending with general systems with bosonic and fermionic constraints that can be of first and second class. This scheme of quantization is based on a Hamiltonian formalism. It allows us to introduce a more general class of gauge-fixing conditions, in particular the noncanonical gauges.

We start with the definition of the classical theory. Given the foliation with absolute physical meaning, the action of the nonprojectable theory is

$$ S = \int dt d^2 x \sqrt{gN} \left( K_{ij} K^{ij} - \lambda K^2 - V \right), \quad (2.1) $$

where the independent fields are the Arnowitt-Deser-Misner variables $N(t, \vec{x})$, $N_i(t, \vec{x})$ and $g_{ij}(t, \vec{x})$. The kinetic terms of the Lagrangian are defined in terms of the extrinsic curvature tensor,

$$ K_{ij} = \frac{1}{2N} \left( \dot{g}_{ij} - 2 \nabla (N N_j) \right), \quad K = g^{ij} K_{ij}. \quad (2.2) $$

$V$ is usually called the potential. To have an anisotropic and power-counting renormalizable theory in 2 + 1 dimensions, $V$ must be of order $z = 2$. In this case, the complete potential is [28]

$$ V = -\beta R - \alpha a^2 + \alpha_1 R^2 + \alpha_2 a^4 + \alpha_3 R a^2 + \alpha_4 a^2 \nabla_i a^i + \alpha_5 R \nabla_i a^i + \alpha_6 \nabla^i a^i \nabla_i a_j + \alpha_7 (\nabla_i a^i)^2, \quad (2.3) $$

where

$$ a_i = \frac{\partial_i N}{N}, \quad (2.4) $$

is a vector that enters in the Lagrangian in a way compatible with the gauge symmetry [2]. $\nabla_i$ and $R$ are the covariant derivative and the Ricci scalar of the spatial metric $g_{ij}$. The coupling constants of the theory are $\lambda, \beta, \alpha, \alpha_1, ..., \alpha_7$. We use shorthands for frequent combinations of these coupling constants,

$$ \sigma = \frac{1 - \lambda}{1 - 2\lambda}, \quad \bar{\sigma} = \frac{\lambda}{1 - 2\lambda}, \quad \alpha_{67} = \alpha_6 + \alpha_7. \quad (2.5) $$

After the Legendre transformation, the primary classical Hamiltonian results

$$ H_0 = \int d^2 x \sqrt{gN} \left( \pi^{ij} \pi_{ij} + \bar{\sigma} \pi^2 + V \right). \quad (2.6) $$
The canonically conjugate pairs are \((g_{ij}, \pi^{ij})\) and \((N, P_N)\). The unique first-class constraint the theory has is

\[ \mathcal{H}_i = -2g_{ij} \nabla_k \pi^{kj} = 0. \]  

(2.7)

The second-class constraints are

\[ \theta_1 = \sqrt{g}N \left( \frac{\pi^{ij}\pi_{ij}}{g} + \frac{\sigma^2}{g} + \mathcal{V} \right) + \sqrt{g} \left( 2\alpha \nabla_i (Na^i) - 4\alpha_2 \nabla_i (Na^2a^i) - 2\alpha_3 \nabla_i (NRa^i) \right) + \alpha_4 \left( \nabla^2 (Na^2) - 2\nabla_i (Na^i \nabla_j a^j) \right) + \alpha_5 \nabla^2 (NR) + 2\alpha_6 \nabla^i \nabla^j (N \nabla_j a_i) + 2\alpha_7 \nabla^2 (N \nabla_i a_i) \right) = 0, \]  

(2.8)

\[ \theta_2 = P_N = 0. \]  

(2.9)

Following Dirac’s procedure, constraint \(\theta_2\) can be regarded as a primary constraint since in the Lagrangian there is no time derivative of \(N\), whereas \(\theta_1\) is the consequence of preserving \(\theta_2\). An important feature is that the primary Hamiltonian (2.6) can be written as the integral of one of the second-class constraints,

\[ H_0 = \int d^2x \theta_1. \]  

(2.10)

In the classical canonical formulation the shift vector \(N^i\) is regarded as the Lagrange multiplier associated to the first-class constraint (2.7).

The BFV quantization of the nonprojectable Horava theory requires to extend the phase space by adding the canonical pairs \((N^i, \pi_i), (C^i, \bar{P}_i)\) and \((\bar{C}_i, \mathcal{P}^i)\). The last two pairs are the BFV ghosts. The definition of the BFV path integral, under a given gauge-fixing condition, is

\[ Z = \int \mathcal{D}V e^{iS}, \]  

(2.11)

where the measure and the action are given, respectively, by

\[ \mathcal{D}V = \mathcal{D}g_{ij} \mathcal{D}\pi^{ij} \mathcal{D}N \mathcal{D}P_N \mathcal{D}N^k \mathcal{D}\pi^k \mathcal{D}C^i \mathcal{D}\bar{P}_i \mathcal{D}\bar{C}_i \mathcal{D}\mathcal{P}^i \times \delta(\theta_1) \delta(\theta_2) \sqrt{\det\{\theta_{\rho}, \theta_{\eta}\}}, \]  

(2.12)

\[ S = \int dt d^2x \left( \pi^{ij} \dot{g}_{ij} + P_N \dot{N} + \pi_i \dot{N}^i + \bar{P}_i \dot{C}^i + \mathcal{P}^i \dot{\bar{C}}_i - \mathcal{H}_\Psi \right). \]  

(2.13)

The quantum gauge-fixed Hamiltonian density is

\[ \mathcal{H}_\Psi = \mathcal{H}_0 + \{\Psi, \Omega\}_D, \]  

(2.14)

where \(\Omega\) is the generator of the BRST symmetry, \(\Psi\) is a gauge-fixing fermionic function and \(\{\, , \}_D\) indicates Dirac brackets. A theorem developed in the original formulation of the BFV quantization [25, 26, 27] ensures that the above path integral is independent of the choice of the function \(\Psi\). The BRST generator for the nonprojectable Hořava theory becomes

\[ \Omega = \mathcal{H}_i C^i + \pi_i \mathcal{P}^i - \frac{1}{2} U_{ik}^j C^i C^j \bar{P}_k. \]  

(2.15)
The factor \( \delta(\theta_1)\delta(\theta_2)\sqrt{\det\{\theta_p,\theta_q\}} \) in (2.12) is the measure associated to the second-class constraints. In the present theory, due to the algebra of constraints, it turns out that

\[
\sqrt{\det\{\theta_p,\theta_q\}} = \det\{\theta_1,\theta_2\}.
\]

(2.16)

The variable \( P_N \) may be integrated directly by using the delta \( \delta(\theta_2) \) in the measure, such that \( P_N \) is eliminated from the Lagrangian (the bracket \( \{\theta_1,\theta_2\} \) in (2.16) does not depend on \( P_N \)). The other delta, \( \delta(\theta_1) \), may be promoted to the quantum Lagrangian by means of a Lagrange multiplier,

\[
\delta(\theta_1) = \int \mathcal{D}A \exp \left( i \int dt d^2 x A \theta_1 \right).
\]

(2.17)

We may also promote the factor (2.16) to the quantum Lagrangian by means of a pair of fermionic ghosts, \( \eta, \bar{\eta} \),

\[
\det\{\theta_1,\theta_2\} = \int D\bar{\eta} D\eta \exp \left( i \int dt d^2 x \bar{\eta}\{\theta_1,\theta_2\} \eta \right).
\]

(2.18)

Thus, \( A \) and \( \eta, \bar{\eta} \) enter as additional fields in the quantum theory.

The power-counting criterium is on the origin of the anisotropic character of the theory, which is parameterized by the value \( z = d = 2 \). Time and space are declared to have dimensions

\[
[t] = -z, \quad [x^i] = -1.
\]

(2.19)

The scalings of the quantum fields are

\[
[g_{ij}] = 0, \quad [\pi^{ij}] = d = 2, \\
[N] = 0, \quad [\pi_s] = 1 + d - z = 1, \\
[N^i] = z - 1 = 1, \quad [\pi_i] = 1 + d - z = 1, \\
[C^i] = \bar{C}_i = (d - z)/2 = 0, \quad [\bar{P}^i] = [\bar{P}_i] = (d + z)/2 = 2, \\
[A] = z - d = 0, \quad [\bar{\eta}] = [\eta] = (z - d)/2 = 0.
\]

(2.20)

The aim in the BFV formulation is to use a gauge-fixing condition and an associated fermionic function of the general forms

\[
\dot{N}^i - \chi^i = 0, \\
\Psi = \bar{P}_i N^i + \bar{C}_i \chi^i.
\]

(2.21)

(2.22)

Equation (2.21) expresses the noncanonical nature of the gauge fixing; it depends on time derivatives. This kind of gauge-fixing condition was originally introduced in the BFV quantization to make compatible the Hamiltonian formalism with the relativistic gauges (specially for unitarity). It turns out that, since the functional part \( \chi^i \) is left free, one may use this form of gauge fixing both for relativistic and nonrelativistic gauges. By using (2.15) and (2.22), and assuming that \( \chi^i \) does not depend on the BFV ghosts, the gauge-fixed Hamiltonian (2.14) takes the form

\[
\mathcal{H}_\Psi = \mathcal{H}_0 + \mathcal{H}_i N^i + \bar{P}_i P^i - \bar{P}_i \left( N^j \partial_j C^i + N^i \partial_j C^j \right) + \pi_i \chi^i \\
+ \bar{C}_i \{\chi^i, \mathcal{H}_j\}_D C^j + \bar{C}_i \frac{\delta \chi^i}{\delta N^j} P^j.
\]

(2.23)
Now we specify the gauge-fixing condition. To this end, we move to perturbative variables, defined by

\[ g_{ij} = \delta_{ij} + h_{ij}, \quad \pi^{ij} = p^{ij}, \quad N = 1 + n, \quad N^i = n^i. \tag{2.24} \]

On the rest of quantum fields we keep the original notation, considering them as perturbative variables. We introduce the gauge-fixing condition in the form

\[ \chi^i = \mathcal{D}^{ij}\pi_j + \Gamma^i, \tag{2.25} \]

where

\[ \mathcal{D}^{ij} = \delta_{ij} \Delta + \kappa \partial_{ij}, \tag{2.26} \]

\[ \Gamma^i = -2\Delta \partial_j h_{ij} + 2\lambda (1 + \kappa) \Delta \partial_i h - 2\kappa \partial_{ijk} h_{jk}. \tag{2.27} \]

\(\kappa\) is an arbitrary constant. We use the shorthand \(\partial_{ij...k} = \partial_i \partial_j \cdots \partial_k\). It is known [21] that by using this local gauge in the projectable theory, after the integration on momenta, the nonlocal quantum Lagrangian of Ref. [8] is recovered. The coefficients in the gauge-fixing are chosen to simplify the resulting propagators of the quantum fields. Other frequent combinations of constants that arise are

\[ \tilde{\kappa} = \kappa + 1, \quad \rho_1 = 2(1 - \lambda)\tilde{\kappa}, \quad \rho_2 = 4\alpha_1 - \frac{\alpha_5^2}{\alpha_6^7}. \tag{2.28} \]

Under the gauge-fixing condition (2.25), the path integral takes the form

\[ Z = \int Dh_{ij} Dp^{ij} Dn^i D\pi_j D\mathcal{C}^i D\mathcal{P}_i D\mathcal{C}^i D\mathcal{P}^i D\mathcal{A} D\bar{\eta} D\eta \]

\[ \exp \left[ i \int dt d^2x \left( p^{ij} \dot{h}_{ij} + \pi_i \dot{n}^i + \mathcal{P}_i \dot{C}^i + C_i \dot{\mathcal{P}}^i - \mathcal{H}_0 - \mathcal{H}_i n^i - \mathcal{P}_i \mathcal{P}^i + \mathcal{P}_i (n^j \partial_j C^i) \right. \right. \]

\[ \left. + n^i \partial_j C^{ij} - \pi_i \mathcal{D}^{ij}\pi_j + 2\pi_i (\Delta \partial_j h_{ij} - \lambda \tilde{\kappa} \Delta \partial_i h + \kappa \partial_{ijk} h_{jk}) \right. \]

\[ - \tilde{C}_i \{\mathcal{C}^i, \mathcal{H}_j\} C^j + \mathcal{A} \theta_1 + \tilde{\eta} \{\theta_1, \theta_2\} \eta \right]. \tag{2.29} \]

Some terms in this Lagrangian must be expanded in perturbations. Here we show them at quadratic order, in the appendix we show their expressions at cubic order,

\[ \mathcal{H}_0 = p^{ij} p_{ij} + \bar{\sigma} \sigma^2 + \beta \left( \frac{1}{4} h^2 \Delta h - \frac{1}{4} h_{ij} \partial_{ij} h_{ij} + \frac{1}{2} h_{ij} \partial_{ik} h_{kj} + n \Delta h - n \partial_{ij} h_{ij} - \frac{1}{2} h \partial_{ij} h_{ij} \right) \]

\[ + \alpha n \Delta n + \alpha_1 \left( h^2 \Delta h + h_{ij} \partial_{ij} h_{ij} - 2 h \Delta \partial_{ij} h_{ij} \right) - \alpha_5 n (\Delta^2 h - \Delta \partial_{ij} h_{ij}) + \alpha_6^7 n \Delta^2 n, \]

\[ \mathcal{H}_i n^i = -2 n^i \partial_j p^{ij}, \]

\[ \tilde{C}_i \{\mathcal{C}^i, \mathcal{H}_j\} C^j = -2 \tilde{C}_i \left( \delta_{ij} \Delta^2 + (\rho_1 - 1) \Delta \partial_{ij} \right) C_j, \]

\[ \tilde{\eta} \{\theta_1, \theta_2\} \eta = 2 \alpha_6^7 \tilde{\eta} \Delta^2 \eta, \]

\[ \mathcal{A} \theta_1^{(1)} = \mathcal{A} (\beta (\Delta h - \partial_{ij} h_{ij}) + 2 \alpha \Delta n - \alpha_5 \Delta (\Delta h - \partial_{ij} h_{ij}) + 2 \alpha_6^7 \Delta^2 n). \]  

(2.30)
This quantum formulation of the nonprojectable Hořava theory has a BRST symmetry when the theory is evaluated on the surface of the phase space defined by the second-class constraints. By definition, the measure of the second-class constraints becomes 1 evaluated on this surface. Therefore, the field variables $\mathcal{A}$ and $\eta, \bar{\eta}$ are not relevant for this symmetry since they are associated precisely to this measure (indeed, they are not canonical variables).

3 Loops in diagrams

We may define Feynman rules in the framework of the BFV formalism. The first step is to obtain the propagators of the quantum fields (canonical fields, canonical BFV ghosts and the fields $\mathcal{A}$ and $\eta, \bar{\eta}$). A clean way to achieve this is by performing a transverse–longitudinal decomposition on all vector and tensor fields in the second-order Lagrangian, finding the propagators by inversion, and then reconstructing the vector and tensor propagators \[8\]. We use the two-dimensional transverse–longitudinal decomposition for tensors

\begin{align}
  h_{ij} &= \left( \delta_{ij} - \frac{\partial_{ij}}{\Delta} \right) h^T + \partial_{(i} h_{j)}^L + \frac{\partial_{ij}}{\Delta} h^L, \\
  p^{ij} &= \left( \delta_{ij} - \frac{\partial_{ij}}{\Delta} \right) p^T + \partial_{(i} p_{j)}^L + \frac{\partial_{ij}}{\Delta} p^L, \\
  \partial_{i} h_{i}^T &= \partial_{i} p_{i}^L = 0,
\end{align}

and vectors

\begin{align}
  n^i &= n_i^T + \partial_{i} n^L, \quad \pi_i = \pi_i^T + \partial_{i} \pi^L, \quad \partial_{i} n_i^T = \partial_{i} \pi_i^T = 0.
\end{align}

The four BFV ghost fields are also decomposed as in (3.2). From now on we focus on the high energy regime, hence we send the low order coupling constants $\beta$ and $\alpha$ to zero. The canonical quantum Lagrangian at second order becomes

\begin{align}
  \mathcal{L}^{(2)} &= \mathcal{P}_i^T \dot{C}_i^T - \mathcal{P}_i^L \Delta C_i^L + \dot{C}_i^T \dot{P}_i^T - \dot{C}_k^L \Delta P_k^T - \mathcal{P}_i^T P_i^T + \dot{P}_i^L \Delta P_i^L + 2 \dot{C}_k^T \Delta^2 C_k^T \\
  &\quad - 2\rho_1 \Delta^3 C_i^L + p^T \left( h^T - 2\sigma p^L \right) - \frac{1}{2} p_i^L \Delta h_i^L - \sigma (p^T)^2 + \frac{1}{2} p_i^L \Delta p_i^L \\
  &\quad + \pi_i (p_i^T)^2 + p_i^L \Delta \pi_i - 2 p_i^L \Delta n_i^L - \alpha_1 h^T \Delta^2 h^T + \alpha_5 n \Delta^2 h^T \\
  &\quad - \alpha_{67} n \Delta^2 n + \pi_k^T \dot{\pi}_k^T - \pi_k^T \Delta \pi_i - \pi_k^T \Delta \pi_k + \rho_1 \pi_i \Delta^2 \pi_i + \pi_k^T \Delta^2 \pi_k \\
  &\quad + 2\lambda \bar{\pi} \Delta^2 h^T - \rho_1 \pi^L \Delta^2 h^T + \mathcal{A} (-\alpha_5 \Delta^2 h^T + 2\alpha_{67} \Delta^2 n) + 2\alpha_{67} \bar{\eta} \Delta^2 \eta.
\end{align}

We present all the nonzero propagators derived from the quantum Lagrangian (3.3).\footnote{In the complete canonical Lagrangian (2.29) there are no vertices for the canonical momenta $\pi_i$ and $\mathcal{P}_i$, hence these fields do not arise in connected diagrams. This could be changed by a different gauge-fixing condition, but this is not relevant for the present study.}
Fourier space, after a Wick rotation, they are

\[
\langle \mathcal{P}_i \mathcal{P}_j \rangle = 4k^4 \left( P_{ij} T_3 + 2 \rho_1 \hat{k}_i \hat{k}_j T_2 \right), \\
\langle \mathcal{P}_i \mathcal{C}_j \rangle = - \langle \mathcal{P}_i \mathcal{C}_j \rangle = \omega S_{ij}, \quad \langle \mathcal{C}_i \mathcal{C}_j \rangle = - S_{ij}, \\
\langle p^{ij}_p \rangle = \rho_2 k^4 P_{ij} T_1, \\
\langle h_{ij} h_{kl} \rangle = 2Q_{ijkl} \left( 1 - 2k^4 T_3 \right) T_3 + 4 \left( \sigma P_{ij} P_{kl} + \hat{\sigma} \hat{k}_i \hat{k}_j P_{kl} + \hat{\sigma} \hat{k}_k \hat{k}_l P_{ij} \right) T_1 + 4\sigma Q \hat{k}_i \hat{k}_j \hat{k}_k \hat{k}_l T_1 T_2, \\
\langle h_{ij} \hat{p}^{kl} \rangle = \omega \left[ Q_{ijkl} T_3 + 2P_{ij} P_{kl} T_1 + 2\sigma \left( \rho_2 + 2\rho_1 - 4\lambda \kappa \right) k^2 \hat{k}_i \hat{k}_j P_{kl} T_1 T_2 + 2\hat{k}_i \hat{k}_j \hat{k}_k \hat{k}_l T_2 \right], \\
\langle n^i h_{jk} \rangle = 16i\omega k^3 \left( P_{ij} \hat{k}_j T_3^2 + \hat{k}_i \hat{k}_j \hat{k}_k T_2^2 \right), \\
\langle n^i \hat{p}^{jk} \rangle = 2ik^3 \left[ 2P_{ij} \hat{k}_k T_3 + \hat{k}_i \left( \rho_1 \hat{k}_j \hat{k}_k - 2\lambda \kappa P_{jk} \right) T_2 \right], \\
\langle n^i n^j \rangle = 4k^2 P_{ij} T_3 (1 - 4k^4 T_3) - 4\kappa \omega^2 \left[ \omega^2 - \left( \rho_1 + \sigma \rho_2 \left( 2\rho_1 + 1 \right) k^4 \right) k^2 \hat{k}_i \hat{k}_j T_1 T_2 \right], \\
\langle h_{ij} n \rangle = \frac{2\sigma_5}{\sigma_67} \left( \sigma P_{ij} + \hat{\sigma} \hat{k}_i \hat{k}_j \right) T_1, \quad \langle p^{ij} n \rangle = \frac{\alpha_5}{\alpha_67} \omega P_{ij} T_1, \quad \langle n^i \pi_j \rangle = \omega S_{ij}, \\
\langle \pi_i h_{jk} \rangle = - 4i \left( P_{ij} \hat{k}_k T_2 + \hat{k}_i \hat{k}_j \hat{k}_k T_3 \right), \quad \langle nn \rangle = \frac{\alpha_2^2 \sigma}{\alpha_67} T_1, \\
\langle AA \rangle = \langle An \rangle = - \langle \bar{\eta} \eta \rangle = \frac{1}{\alpha_67 k^4},
\]

(3.4)

where

\[
\hat{k}_i = \frac{k_i}{k}, \quad P_{ij} = \delta_{ij} - \hat{k}_i \hat{k}_j, \\
Q_{ijkl} = \hat{k}_i \hat{k}_k P_{jl} + \hat{k}_j \hat{k}_l P_{ik} + \hat{k}_i \hat{k}_l P_{jk} + \hat{k}_j \hat{k}_k P_{il}, \\
Q = \omega^4 + \left( 12\lambda \kappa - 2(1 + 2\lambda)\rho_1 - \rho_2 \right) \frac{\omega^2 k^4}{1 - \lambda} + 4\kappa \left( \rho_2 + 4\lambda^2 \kappa \right) k^8, \\
T_1 = \left( \omega^2 - \rho_2 \sigma k^4 \right)^{-1}, \quad T_2 = \left( \omega^2 + 2\rho_1 k^4 \right)^{-1}, \\
T_3 = \left( \omega^2 + 2k^4 \right)^{-1}, \quad S_{ij} = 2P_{ij} T_3 + 2\hat{k}_i \hat{k}_j T_2.
\]

(3.5)

The definition of regular propagators was introduced in Ref. [19] to study the renormalizability of nonrelativistic gauge theories. This was used in the proof of renormalization of the projectable Hořava theory [8], whose definition we take here. Consider a propagator between two fields that have scaling dimensions \( r_1 \) and \( r_2 \). It is regular if it is given by the sum of terms of the form

\[
P(\omega, k^4) \\
D(\omega, k^4),
\]

where \( D \) is the product

\[
D = \prod_{m=1}^{M} \left( A_m \omega^2 + B_m k^2d + \cdots \right), \quad A_m, B_m > 0,
\]

(3.7)
and $P(\omega, k^i)$ is a polynomial of maximal scaling degree less or equal to $r_1 + r_2 + 2(M - 1)d$. All the propagators in the list (3.4) satisfy this condition of regularity, except for the three propagators $\langle AA \rangle$, $\langle An \rangle$ and $\langle \bar{\eta}n \rangle$, which are independent of $\omega$. The three irregular propagators involve the variables $A, \eta, \bar{\eta}$, and these are fields associated to the measure of the second-class constraints, as we have indicated.

**Superficial degree of divergence** To define the superficial degree of divergence $D_{\text{div}}$ of each diagram we consider the different scaling between space and time. In the $2 + 1$-dimensional case we have

$$k^i \rightarrow bk^i, \quad \omega \rightarrow b^2 \omega .$$

Given a diagram with $L$ loops, its superficial degree of divergence is given by

$$D_{\text{div}} = 4L + \sum_r w_r I_r + \sum_s d_s V_s - X ,$$

where $I_r$ and $V_s$ denote the number of propagators and vertices of type $r$ and $s$ respectively, $w_r$ is the scaling for each propagator at the ultraviolet and $d_s$ is the number of spatial derivatives of each type of vertex. The factor $X$ counts the number of spatial derivatives that may exist in the external legs of the diagram. We remark that the BFV quantization is based on a Hamiltonian formalism, as a consequence there are no time derivatives in the vertices extracted from the canonical Lagrangian. Hence, we do not need to consider time derivatives on the external legs. By inspection of the propagators (3.4) and vertices in (2.29), and using the topological identity $L - 1 = \sum I_r - \sum V_s$, we obtain

$$D_{\text{div}} = 4 + 4I_{pp} + 2I_{pn} + 2I_{NN} + 2I_{hp} + I_{hN} + 3I_{pN} + 2I_{\bar{\eta}C}$$

$$- 4V_{App} - 4V_{pp[n]} - 4V_{hp[\bar{n}]} - 3V_{Np[h]} - 3V_{\bar{\eta}C[N]} - X .$$

The brackets indicate that the vertex has multiple legs for that field (index $N$ stands for the shift vector). For a given field, we can match the number of internal and external lines that contain it with the number of vertices that have legs for it. By doing so for some of the fields, we obtain the identities

$$2I_{pp} + I_{pn} + I_{hp} + I_{pN} + E_p = 2V_{App} + V_{Np[h]} + 2V_{pp[n]} + 2V_{hp[\bar{n}]} ,$$

$$2I_{NN} + I_{N\bar{\eta}} + I_{pN} + E_N = V_{Np[h]} + V_{\bar{\eta}C[N]} ,$$

$$I_{\bar{\eta}p} + I_{\bar{\eta}C} = V_{\bar{\eta}C[N]} .$$

Finally, by substituting these relations in (3.10), we obtain

$$D_{\text{div}} = 4 - E_N - 2E_p - X .$$

Therefore, the diagrams with maximal superficial degree of divergence are those with no external legs for the $N^i$ and $p_{ij}$ fields, and carrying no spatial derivatives on their external legs. These diagrams have a superficial degree of divergence of 4, which is the order of the

---

3Actually, the condition of regular propagator in (3.7) requires the coefficients in the denominators to be strictly positive. In the present theory this is satisfied by the regular propagators if $\rho_1 > 0$ and $\sigma \rho_2 < 0$. 

---

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bare Lagrangian of the $2+1$ nonprojectable Hořava theory. This result is in agreement with the original power-counting renormalizability of the theory.

We comment that in the list of propagators (3.4) there are several scalings on $\omega$, in particular the irregular propagators which do not depend on $\omega$. This is the critical issue for the divergences in loops. Since the theory is nonrelativistic, it makes sense to analyze the degree of divergence of the integration on $\omega$ separately, which we denote by $D_{\text{div}}(\omega)$. For a given diagram, it is given by

$$D_{\text{div}}(\omega) = 2L + \sum_r w_r(\omega) I_r ,$$  

(3.13)

where $w_r(\omega)$ denotes the scaling that each propagator has in $\omega$. By inspection on the propagators in (3.4), we find

$$D_{\text{div}}(\omega) = 2 + 2(I_{AA} + I_{An} + I_{\bar{\eta}n}) - 2(I_{hh} + I_{nn} + I_{pp} + I_{hn} + I_{NN}$$

$$+ 2I_{hN} + I_{pN} + I_{\bar{C}C}) - 2 \sum_s V_s .$$  

(3.14)

This confirms that irregular propagators are the only ingredients that increase the divergence in the $\omega$ direction.

**Unavoidable irregular propagators** We have found irregular propagators for the fields $A$ and $\eta, \bar{\eta}$. Now we discuss how the irregularity of these propagators is an unavoidable consequence of the structure of the second-class constraints, which in particular affects the variable $n$.

We denote all fields of the theory by $\Psi_A$. The second-order Lagrangian (3.3) can be written in matrix form,

$$L^{(2)} = \frac{1}{2} \Psi_A M_{AB} \Psi_B , \quad M_{AB} = \frac{\delta^2 L^{(2)}}{\delta \Psi_A \delta \Psi_B} .$$  

(3.15)

Each entry of propagators is given by $\langle \Psi_A \Psi_B \rangle = [M^{-1}]_{AB}$. Let us start with the propagators involving $A$. We denote by $\hat{\Psi}_A$ the subset of $\Psi_A$ that excludes $A$. The key point is the fact that constraint $\theta_1$ is equivalent to the derivative

$$\theta_1 = \frac{\delta H_0}{\delta n} ,$$  

(3.16)

along with the dependence that the canonical Lagrangian has on $n$. Indeed, all the dependence that the quadratic canonical Lagrangian (3.3) has on the fields $n$ and $A$ comes from the structure

$$\mathcal{L} = H^{(2)}_0 + A \frac{\delta H^{(2)}_0}{\delta n} .$$  

(3.17)

From this it is straightforward to obtain the relations

$$\frac{\delta^2 \mathcal{L}}{\delta A \delta A} = 0 ,$$  

(3.18)

$$\frac{\delta^2 \mathcal{L}}{\delta \hat{\Psi}_A \delta A} = \frac{\delta^2 \mathcal{L}}{\delta \hat{\Psi}_A \delta n} .$$  

(3.19)
The determinant of $M$ is given by 

$$\det M = \epsilon_{ABC\ldots D} \left( \frac{\delta^2 L}{\delta \Psi_A \delta n} \right) \left( \frac{\delta^2 L}{\delta \Psi_B \delta A} \right) W_{C\ldots D},$$

where the sector 

$$W_{C\ldots D} = \left( \frac{\delta^2 L}{\delta \Psi_C \delta n} \right) \cdots \left( \frac{\delta^2 L}{\delta \Psi_D \delta n} \right)$$

covers all columns of $M$ that are not $n$ or $A$ (the ordering of variables in the determinant is irrelevant for our purposes). The only terms that survive in (3.20) are those for which the index $A$ takes the row $A = A$. This is so because when $B = A$ the terms cancels due to (3.18), and when the row $A$ does not fall in $A, B$ we can use (3.19) in the factors 

$$\left( \frac{\delta^2 L}{\delta \hat{\Psi}_A \delta n} \right) \left( \frac{\delta^2 L}{\delta \hat{\Psi}_B \delta A} \right) = \left( \frac{\delta^2 L}{\delta \hat{\Psi}_A \delta n} \right) \left( \frac{\delta^2 L}{\delta \hat{\Psi}_B \delta n} \right),$$

but this is symmetric in $\hat{A}, \hat{B}$. Hence, we have 

$$\det M = \epsilon_{\hat{A}\hat{B}\hat{C}\ldots \hat{D}} \left( \frac{\delta^2 L}{\delta \hat{A} \delta n} \right) \left( \frac{\delta^2 L}{\delta \hat{B} \delta A} \right) W_{\hat{C}\ldots \hat{D}} \left( \frac{\delta^2 L}{\delta \hat{C} \delta n} \right) \cdots \left( \frac{\delta^2 L}{\delta \hat{D} \delta n} \right)$$

In the second line we have used (3.19), and in the third line we have identified the minor of $M$ in the $AA$ position. Thus, the determinant of $M$ is proportional to its $AA$-minor. Due to this, the entry $[M^{-1}]_{AA}$ lacks any contribution coming from the kinetic terms of the Lagrangian, which are the only terms that bring dependence on $\omega$ (indeed, $[M^{-1}]_{AA}$ lacks the contribution of most terms of the Lagrangian). The only remainder in this entry is the coefficient, which, in Fourier transform, is 

$$\frac{\delta^2 L}{\delta n \delta n} = \alpha_{67}k^4.$$

Therefore, we have the propagator 

$$\langle AA \rangle = \frac{1}{\alpha_{67}k^4},$$

as shown in (3.4).

The other propagator involving the $A$ field is $\langle An \rangle$. We compute $\text{Minor}(M)_{An}$ by eliminating row $A$ and column $n$ from the determinant (3.20), 

$$\text{Minor}(M)_{An} = \epsilon_{\hat{B}\hat{C}\ldots \hat{D}} \left( \frac{\delta^2 L}{\delta \hat{B} \delta A} \right) W_{\hat{C}\ldots \hat{D}} \left( \frac{\delta^2 L}{\delta \hat{D} \delta n} \right)$$

$$= \epsilon_{\hat{B}\hat{C}\ldots \hat{D}} \left( \frac{\delta^2 L}{\delta \hat{B} \delta n} \right) W_{\hat{C}\ldots \hat{D}} = \text{Minor}(M)_{AA},$$

(3.26)
hence

\[ \langle nA \rangle = \langle AA \rangle = \frac{1}{\alpha_{67} k^4}. \] (3.27)

Any other propagator involving \( A \) is zero. Let \( \phi \) denote any field that is not \( A \) or \( n \). The minor of \( A\phi \) is computed by eliminating row \( A \) and column \( \phi \),

\[
\text{Minor}(M)_{A\phi} = \epsilon_{\hat{A}\hat{B}...\hat{D}} \left( \frac{\delta^2 L}{\delta \hat{A} \delta n} \right) \left( \frac{\delta^2 L}{\delta \hat{B} \delta A} \right) W^{*}_{...\hat{D}},
\] (3.28)

where \( W^{*}_{...\hat{D}} \) is \( W_{\hat{C}...\hat{D}} \) with the factor \( \frac{\delta^2 L}{\delta \hat{C} \delta \phi} \) eliminated. The derivatives in the last line are symmetric in \( \hat{A}\hat{B} \). Hence we have \( \langle A\phi \rangle = 0 \).

A natural question is whether a different gauge-fixing condition could change the situation for the propagators of the \( A \) field. In order to this happens, the structure (3.17) should be modified. Remarkably, by introducing a functional dependence of the gauge-fixing condition \( \chi^i \) on \( n \), which is allowed by the BFV formalism, the structure (3.17) is altered since the term \( \pi_i \chi^i \) in (2.23) generates additional quadratic terms that depend on \( n \). There are other places where dependence on \( n \) is generated by the gauge-fixing, but they are of higher order in perturbations and hence do not contribute to the propagators. It turns out that the new terms coming from \( \pi_i \chi^i \) do not modify the propagators \( \langle AA \rangle \) and \( \langle An \rangle \), which can be checked by direct computations. This is the only possibility for the choice of the gauge-fixing condition could alter the irregular propagators, but at the end it does not happen.

The other sector where irregular propagators arise is on the ghosts \( \eta, \bar{\eta} \), which are also associated to the measure of the second-class constraints. In this case the result is more obvious, since the sector \( \eta, \bar{\eta} \) forms an isolated block of the \( M \) matrix. The only entry where \( \eta \) and \( \bar{\eta} \) arise is in the term \( \bar{\eta}\{\theta_1, \theta_2\} \eta \) of (2.29), such that the \( \langle \bar{\eta}\eta \rangle \) propagator is directly the inverse of this entry of \( M \). In turn, in the linearized theory, this factor becomes

\[
\{\theta_1, \theta_2\} = \frac{\delta \theta_1}{\delta n} = \frac{\delta \mathcal{H}_0^{(2)}}{\delta n \delta n} = \frac{\delta \mathcal{L}^{(2)}}{\delta n \delta n}.
\] (3.29)

In the last equality we have used (3.17). Thus, this factor is the same as (3.24), and the propagator for the ghosts is

\[
\langle \bar{\eta}\eta \rangle = -\frac{1}{\alpha_{67} k^4}.
\] (3.30)

Therefore, we have again an irregular propagator that is a consequence of the structure of the second-class constraints.

We stress the origin of the terms that depend on \( \eta, \bar{\eta} \), which is the factor \( \sqrt{\det \{\theta_p, \theta_q\}} \) of the measure of the second-class constraints. Although we have been able to incorporate this factor to the Lagrangian, its form is robust in two senses: first, the gauge-fixing condition represented by the choice of \( \chi^i \) plays no role in this factor, hence one cannot modify it by a different gauge. Second, there cannot be time derivatives in this factor since the measure is
defined strictly in terms of constraints, which by definition do not depend on time derivatives. Moreover, the ghosts \( \eta, \bar{\eta} \) are not canonical variables (unlike the BFV ghosts), hence there is no place for time derivatives of them coming directly from kinetic terms.\(^4\)

**Absence of nested irregular loops**  Our concern here are the loops whose all internal lines are irregular propagators, which we call irregular loops for short. We want to show that nested irregular loops are not possible. To clarify, we illustrate the idea of nested loops in figure 1.

![Figure 1: An example of nested loops. The integrals over the two loops are not independent](image)

Since the Lagrangian (2.29) is linear in \( A \) and bilinear in \( \eta \bar{\eta} \), the only possibility is to have vertices with a single \( A \) leg and vertices with a leg for \( \eta \) and a leg for \( \bar{\eta} \), independently of the number of legs they have for the other fields. Given a diagram or subdiagram, let \( V_A \) denote the number of vertices having the \( A \) leg, and \( V_{\eta\bar{\eta}} \) the same for the \( \eta \bar{\eta} \) ghosts. Since these are the only vertices involving the \( A, \eta, \bar{\eta} \) fields, these numbers are related to the corresponding lines according to

\[
\begin{align*}
V_A &= 2I_{AA} + I_{An} + E_A, \\
V_{\eta\bar{\eta}} &= I_{\eta\bar{\eta}} + 2E_{\eta},
\end{align*}
\]

(3.31)

where \( E_A \) is the number of external \( A \) legs and \( E_{\eta} \) the number of external \( \eta \) legs (which must be the same of the external \( \bar{\eta} \) legs). We have left open the possibility of having external legs for the ghosts with the aim of applying the analysis for subdiagrams. In the general case, we have a diagram or subdiagram with \( L \) loops, \( I_R + I_Q \) internal lines, where \( I_R \) are the regular propagators and \( I_Q \) the irregular ones, \( V_A \) and \( V_{\eta\bar{\eta}} \) vertices, and the rest are the \( V \) vertices. The topological identity is

\[
L - 1 = I_R + I_Q - (V_A + V_{\eta\bar{\eta}} + V).
\]

(3.32)

Now we consider a diagram whose all internal lines are irregular propagators. In this case we have \( I_R = V = 0 \), since it has no regular propagators and the vertices \( V \) are not allowed since they have no legs to join with the irregular propagators. The lines corresponding to irregular propagators are \( I_Q = I_{AA} + I_{An} + I_{\eta\bar{\eta}} \). By substituting this and (3.31) in (3.32), we obtain

\[
L - 1 = -(I_{AA} + E_A + 2E_{\eta}).
\]

(3.33)

\(^4\)All these features apply also to the terms involving the \( A \) field, but since it couples to canonical fields in the linearized theory, the rest of the Lagrangian is relevant to determine its propagators.
But this is only possible, besides the tree level, for \( L = 1 \) and \( I_{AA} = E_A = E_\eta = 0 \). Therefore, diagrams or subdiagrams with all internal lines given by irregular propagators can be formed only at one loop, only with the propagators \( \langle A \rangle \) or \( \langle \eta \bar{\eta} \rangle \) and necessarily with external legs corresponding to regular lines. A corollary is that nested irregular loops cannot be formed since they would be, at least, of two loops. This is what we wanted to prove.

Multiple irregular loops can arise in the form of several irregular one-loop subdiagrams, but these subdiagrams must be connected by regular propagators \( (E_A = E_\eta = 0) \). Hence they cannot be nested. We illustrate this kind of multiple irregular one-loop subdiagrams in figure 2.

\[ \text{Figure 2: A diagram with multiple irregular one-loops. The irregular propagators are represented generically by dotted lines.} \]

Since the irregular loops cannot be nested, they factorize in the computations of diagrams, that is, their respective integrals are independent and multiply each other. Another possibility is that irregular loops can be nested with other loops with some irregular lines, but not all of their lines. We illustrate an example where the \( \langle AA \rangle \) propagator is used as additional irregular line in figure 3.

\[ \text{Figure 3: Nested loops with some irregular lines. Only the inner loop is an irregular loop.} \]

In this case the regular propagators arising in the outer loop render the integration in \( \omega \) finite for that loop (we discuss this below). Therefore, the degree of divergence on \( \omega \) of this diagram is the same of the inner irregular loop.

Another soundly feature of the irregular one-loop diagrams is that they can be formed with multiple \( \langle A \rangle \) or \( \langle \eta \bar{\eta} \rangle \) propagators, but not with mixing between them. This is just because there is no propagator connecting \( A,n \) legs with \( \eta, \bar{\eta} \) legs. Hence there is no way two close a loop when one attempts to join a vertex that has the \( A,n \) legs with a vertex that has the \( \eta, \bar{\eta} \) legs.

The divergence of the irregular one-loops  The irregular propagators have all the same form, hence we can give a general evaluation of irregular one-loops formed with them. These
one-loops lead to the integral

\[ \cdots \int d\omega \int d^2k E(k^i; p^i, \ldots) , \]

where \( E(k^i; p^i, \ldots) \) is some expression of the spatial momentum \( k^i \) circulating along the loop and the external momenta \( p^i, \ldots \). The ellipsis at the left is the rest of the diagram that does not enter in the integration over the loop. In the integrand there is no dependence on \( \omega \) because the irregular loops do not depend on it and the vertices in this formalism have no time derivatives. With respect to the integration on \( k^i \), the scalings of the irregular propagators and vertices may lead to a cancellable divergence, if any. This is in agreement with the power-counting renormalizability. But the integration on \( \omega \) leads to a linear divergence that multiplies the whole loop and hence multiplies the whole diagram. Therefore, the divergences on \( \omega \) coming from the irregular one-loops are not renormalizable.

The irregular one-loops are the only loops that exhibits this behavior in their integration on \( \omega \). Indeed, it is enough that a loop has at least one regular propagator to avoid any divergence coming directly from the \( \omega \) direction. We may check this starting with the worst case, which is the case of loops that have a single regular propagator of the lower negative scaling in \( \omega, \omega^{-1} \). These propagators are, for example, \( \langle h_{ij} p^k \rangle \) and \( \langle p^i n \rangle \). They involve a single canonical momentum. Since the only dependence the integrand has in \( \omega \) comes from this propagator, one might face a logarithmic divergence in \( \omega \). But these propagators are odd in \( \omega \). Hence, loops with a single propagator of this kind are automatically zero due to the integration in the entire domain of \( \omega \). The next level are the loops that have a single regular propagator with scaling \( \omega^{-2} \), or two of the previous propagators forming an even integrand, and yielding also a total scaling of \( \omega^{-2} \). The integration in \( \omega \) does not lead to divergences in these cases.

**Some cancellations between irregular one-loops** Motivated by the particularities of the irregular one-loops, one may wonder whether there can be cancellations between the loops formed by the \( \langle A n \rangle \) and \( \langle \eta \bar{\eta} \rangle \) propagators, specially because \( \eta, \bar{\eta} \) are ghost fields. We present a computation on this using three-leg vertices that involve the fields \( A, \eta, \bar{\eta} \) and \( n \). Cancellations indeed occur between the poles of the spatial integration, but not for the finite terms. Hence, these cancellations are not enough to remove the divergence that comes from the integration in \( \omega \).

In the appendix we show the expansion of the Lagrangian at cubic order in all variables. Here we only require the relevant sectors of the canonical Lagrangian (2.29) for these fields at cubic order. They are

\[
\begin{align*}
\theta^{(2)}_1 &= \gamma_8 \partial_i j n \partial_j i n + \gamma_9 \Delta n \Delta n - 4\alpha_{67} \Delta \partial_i n \partial_j i n , \\
\{ \theta_1, \theta_2 \}^{(1)} &= 2\alpha_{67} \Delta^2 n + 4\alpha_{67} \Delta \partial_i n \partial_i + \gamma_{10} \partial_j i n \partial_j i n + \gamma_{11} \Delta n \Delta + 4\alpha_{67} \partial_i n \Delta \partial_i .
\end{align*}
\]

where \( \gamma_8 = 2\alpha_4 - \alpha_6 - 4\alpha_7, \gamma_9 = \alpha_7 - 2\alpha_6 - 2\alpha_4, \gamma_{10} = 2(2\alpha_4 + 3\alpha_6) \) and \( \gamma_{11} = 2(3\alpha_7 - 2\alpha_4) \). The following three-leg vertices can be extracted from these terms,

\[
\gamma_8 (p^i k^i)^2 + \gamma_9 p^2 k^2 + 4\alpha_{67} p^i k^i (p^2 + k^2) ,
\]

(3.37)
\[ = 2\alpha_{67} p^4 + \gamma_{10} (p^i k^i)^2 + \gamma_{11} p^2 k^2 - 4\alpha_{67} p^i k^i (p^2 + k^2) . \] (3.38)

With these vertices we may form the following irregular one-loop diagrams of two legs. First, a diagram with the \( \langle A_n \rangle \) propagator,

\[
\begin{align*}
= i \frac{\omega}{\alpha_{67}^2} \int \frac{d\omega}{2\pi} \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^4(k-p)^4} \left[ \gamma_8 (p^i k^i)^2 + \gamma_9 p^2 k^2 + 4\alpha_{67} p^i k^i (p^2 + k^2) \right] \\
\times \left[ \gamma_8 (p^i (k^i - p^i))^2 + \gamma_9 p^2 ((k^i - p^i)^2 - 4\alpha_{67} p^i (k^i - p^i)(p^2 + (k^j - p^j)^2) \right] .
\end{align*}
\] (3.39)

We focus the integration on \( k^i \), using dimensional regularization. After expanding the numerator, the only term that gives a pole is a term proportional to \( \alpha_{67}^2 \). The rest of terms are finite in the integration on \( k^i \). Then, in this case the total integration is given by

\[ = i \Lambda_\omega p^2 \left( -\frac{4}{\pi^2 \epsilon} + \text{finite terms} \right) . \] (3.40)

The second irregular loop is formed with the \( \langle \bar{\eta} \eta \rangle \) propagator,

\[
\begin{align*}
= i \frac{\omega}{\alpha_{67}^2} \int \frac{d\omega}{2\pi} \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^4(k-p)^4} \left[ 2\alpha_{67} p^4 + \gamma_{10} (p^i k^i)^2 + \gamma_{11} p^2 k^2 - 4\alpha_{67} p^i k^i (p^2 + k^2) \right] \\
\times \left[ 2\alpha_{67} p^4 + \gamma_{10} (p^i (k^i - p^i))^2 + \gamma_{11} p^2 ((k^i - p^i)^2 - 4\alpha_{67} p^i (k^i - p^i)(p^2 + (k^j - p^j)^2) \right] .
\end{align*}
\] (3.41)

It turns out that the term containing the pole in the spatial integration is also proportional to \( \alpha_{67}^2 \),

\[ = i \Lambda_\omega p^2 \left( \frac{4}{\pi^2 \epsilon} + \text{finite terms} \right) . \] (3.42)

Thus, the spatial divergences cancel each other between these two diagrams. However, this does not happen for the other terms in the spatial integration, it can be seen in the above integrals that they are different.

## 4 Reduced phase space

One way to avoid dealing with second-class constraints in the quantum theory is to solve them. It is remarkable that, under certain assumptions, one can give the solution of the \( \theta_1 \)
constraint and still arrive at a local canonical Lagrangian. We show this in the following, by considering the linear theory with the dominant modes at the ultraviolet regime. Keeping only the coupling constants of the $z = 2$ terms, the linearized constraint becomes

$$\theta_1^{(1)} = -\alpha_5 \Delta^2 h^T + 2\alpha_6 \Delta^2 n = 0.$$  \hfill (4.1)

With suitable boundary conditions, we may solve it for $n$,

$$n = \frac{\alpha_5}{2\alpha_6} h^T.$$  \hfill (4.2)

Thus, $n$ can be integrated out in the linearized theory, substituting (4.2) everywhere in the path integral. By definition, the measure of the second-class constraint becomes equal to 1 evaluated on the solution. The path integral (2.29) becomes

$$Z = \int \mathcal{D}h_{ij} \mathcal{D}p^{ij} \mathcal{D}n^i \mathcal{D}p_i \mathcal{D}C_i \mathcal{D}P_i \exp \left[ i \int dt d^2x \left( \mathcal{P}_i^{T} \mathcal{C}_i^{T} - \mathcal{P}_i L \Delta \mathcal{C}_i L + \mathcal{P}_i^{L} \mathcal{C}_i^{L} - \rho_1 \pi_1 L \Delta \mathcal{P}_i L \Delta \pi_1 L + 2\lambda \mathcal{P}_i T \Delta^2 \mathcal{C}_i L + 2\rho_2 \pi_2 L \Delta^2 h^T - \rho_1 \pi_1 L \Delta^2 h^L \right) \right].$$

Therefore, under these conditions one obtains a local canonical quantum Lagrangian, which is of quadratic order. Regular propagators can be obtained for all quantum modes in this case. Despite this, the locality of the quantum Lagrangian is broken in two ways. The first one is when one includes interactions, such that higher orders in perturbations must be considered. In this case the solution of the $\theta_1$ constraints is required at second order and higher, leading to nonlocal solution for $n$. Since this affects the vertices where $n$ arises, these vertices get nonlocal expressions. The second one is when one recovers the terms of lower order in derivatives. This is again an obstruction to get a local solution for $n$, leading to a nonlocal Lagrangian.

## 5 3+1 dimensions and models with higher anisotropy

The relationship between the measure of the second-class constraints and the irregular propagators can be translated to the case of a foliation of 3 + 1 dimensions, with the same conclusion about the impossibility of cancellation of divergences in $\omega$ with local counter-terms. Thanks to the previous analysis, we may discuss this on very general grounds, without necessity of entering in detailed computations (we presented the BFV quantization of the 3 + 1 nonprojectable theory in [21]).

In 3 + 1 dimensions the Lagrangian and the classical primary Hamiltonian are of $z = 3$ order, that is, they have terms up to sixth order in spatial derivatives. Constraint $\theta_1$ is still a derivative of $H_0$, getting the $z = 3$ order. The gauge-fixing condition must be adapted
to the dimension and the anisotropic scaling, but it remains as a functional of $N_i$, $\pi_i$ and $h_{ij}$. Once the measure of the second-class constraints has been incorporated, the dependence of the quantum canonical Lagrangian on $n$ and $A$ keeps the structure (3.17). Therefore, all the analysis leading to (3.23), (3.26) and (3.28) holds in the same way. The field $A$ acquires irregular propagators, with a scaling $k^{-6}$ on spatial momentum. In the case of the ghosts $\eta, \bar{\eta}$, the result (3.29) is also kept. It is again a consequence of the structure of the second-class constraints. Thus, the same conclusion is valid for these ghosts, they acquire unavoidably irregular propagators. These facts lead to the result that the propagators of the fields $A$ and $\eta, \bar{\eta}$ form irregular one-loops that have the multiplicative subdivergence in $\omega$. Therefore, the nonprojectable $3+1$ theory is nonrenormalizable.

Another case of interest is given by the Hořava models that are potentially finite. A characteristic feature of the Hořava theory is that one may increase the order of spatial derivatives in the Lagrangian. Power-counting arguments suggest that for a potential of order $z > d$ the theory can be rendered finite [1]. However, this is not the case for the nonprojectable version, due again to the second-class constraints. The reasoning is quite similar to the passing from the $2+1$ to the $3+1$ theory. First, the second-class constraints are still present in $z > d$ models. They are qualitatively the same of the $z = d$ theory. Constraint $\theta_1$ increases its order in spatial derivatives, according to the parameter $z$. Consequently, the structure (3.17) is kept, since the dependence on $n$ comes from the same terms and $\theta_1$ is still a derivative of $H_0$. The only change is in the increasing of order in spatial derivatives in $H_0$. Thus, the increasing of anisotropy is reflected on the irregular propagators as an increasing on their (negative) scaling on $k^2$. This is likely going to render finite the integration on spatial momenta over loops, as the power-counting suggests, considering that a similar increasing of the $k^2$ scaling occurs in the sector of regular propagators. But the irregular propagators remains independent of $\omega$, hence the divergence along this direction is a multiplicative factor of the diagrams and cannot be cancelled. Therefore, the apparent finiteness of the $z > d$ models is broken down in the nonprojectable case.

6 Conclusions

From the analysis of divergences in the BFV quantization of the nonprojectable Hořava theory we conclude that the theory is nonrenormalizable. This is a Hamiltonian formalism that allows us to pose a local canonical Lagrangian with a suitable gauge-fixing condition and the second-class constraints. We have shown that there are irregular one-loops producing multiplicative subdivergences in the direction of the frequency that cannot be cancelled. These loops are formed with irregular propagators of fields associated to the measure of the second-class constraints. We emphasize that indeed there is a local gauge-fixing condition yielding regular propagators for the canonical fields, but not for these fields. The second-class constraints have a structure that leads unavoidably to the irregular propagators. We have discussed that changing the gauge-fixing condition has no influence on them.

We have found some particular features for the irregular loops. Remarkably, only isolated irregular one-loops can be formed. They cannot be nested. There are only two kinds of irregular loops, one for an auxiliary field and the other one for ghosts. We have found
that some spatial poles are cancelled between these loops. Although these features are encouraging, at the end they are not enough to remove the subdivergences along the direction of the frequency.

The second-class constraints are present in the nonprojectable theory and hence must be considered in the process of quantization. Thus, the nonprojectable Hořava theory is a case in which the second-class constraints spoil the argument of power-counting renormalizability. Moreover, the models with higher degree of anisotropy, which are believed to be finite, in the nonprojectable case exhibit the same subdivergences due to the irregular loops. Hence they are not finite.

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A Cubic order

For the sake of simplicity, here we show only the $z = 2$ terms.

$$
\mathcal{H}^{(3)}_0 = 2p^k h_{ki} p^{ij} + 2\bar{\sigma} p^{ij} h_{ij} + \left( n - \frac{1}{2} h \right) \left( p^{ij} p^{ij} + \bar{\sigma} p^2 \right) + \alpha_1 \left[ \left( n + \frac{1}{2} h \right) (R^{(1)})^2 
- 2R^{(1)} h_{kl} P^{(1)}_{kl} - 2R^{(1)} R^{(2)} \right] + \alpha_3 \left( \partial_{ij} h_{ij} \partial_{kn} \partial_{ln} - \Delta h_{kn} \partial_{ln} \right) + \alpha_4 \partial_{i} n \partial_{j} n \Delta n
+ \alpha_5 \left[ R^{(1)} \left( \frac{h}{2} \Delta n - n \partial_{kn} \partial_{kn} + h_{kl} \partial_{kl} n + \chi^{l}_{kk} \partial_{ln} \right) + \Delta n R^{(2)} 
- \frac{1}{2} \Delta n h_{ij} \left( \partial_{kl} h_{kj} - \partial_{ij} h - \Delta h_{ij} + \partial_{kj} h_{kl} \right) \right] - \alpha_6 \left( n \partial_{ij} n \partial_{ij} n + 2 \partial_{ij} n \partial_{i} n \partial_{j} n \right) 
- \frac{h}{2} \partial_{ij} n \partial_{ij} n + 2 \partial_{ij} n \partial_{i} n \chi_{ij}^{m} + 2 h_{ij} \partial_{i} n \partial_{j} n \right) - \alpha_7 \left( n \Delta n \Delta n - \frac{h}{2} \Delta n \Delta n 
+ 2 \Delta n \partial_{j} n \partial_{j} n + 2 \Delta n \partial_{k} n \partial_{jl} \chi^{k}_{jl} + 2 h_{jl} \Delta n \partial_{j} n \right), \tag{A.1}
$$

where

$$
\chi^{i}_{kl} = \partial_{k} h_{ii} - \frac{1}{2} \partial_{i} h_{kl}, \tag{A.2}
$$

$$
R^{(1)}_{ij} = \frac{1}{2} \partial_{i} \partial_{k} h_{kj} + \frac{1}{2} \partial_{j} \partial_{k} h_{ki} - \frac{1}{2} \partial_{i} \partial_{j} h - \frac{1}{2} \Delta h_{ij}, \tag{A.3}
$$

$$
R^{(2)} = - \partial_{k} h_{kl} \partial_{i} h_{i} + \partial_{k} h_{kl} \partial_{i} h + \partial_{k} h_{kl} \partial_{i} h + \partial_{k} h_{kl} \partial_{i} h 
+ \frac{3}{4} \partial_{i} h_{kl} \partial_{k} h_{kl} + h_{kl} \Delta h_{kl} - \frac{1}{4} \partial_{i} h_{kl} h - \frac{1}{2} \partial_{j} h_{ik} \partial_{k} h_{ji}. \tag{A.4}
$$
Other expansions are

\[ \theta_1^{(2)} = p^j p^j + \sigma p^2 + (-\alpha_6 + 2\alpha_4 - 4\alpha_7) \partial_{ij} n \partial_{ij} n + (\alpha_7 - 2\alpha_4 - 2\alpha_6) \Delta n \Delta n + \alpha_1 \left( \partial_{ij} h_{ij} \partial_{kl} h_{kl} - 2\Delta h \partial_{ij} h_{ij} + \Delta h \Delta h \right) + (2\alpha_3 - 2\alpha_5) \left( \Delta n \Delta h - \Delta n \partial_{ij} h_{ij} \right) - \partial_k n \partial_{ik} h_{ij} \right) + (2\alpha_3 - 2\alpha_5 + \alpha_6 \gamma) \partial_k n \Delta \partial_k h + (-2\alpha_6 - 4\alpha_7) \partial_i h_{ij} \partial_j \Delta n - 2\alpha_6 \left( \Delta h_{kl} \partial_{kl} n + 2\partial_h h_{ij} \partial_{ij} n + \alpha_6 \left( h \Delta^2 n + 2\partial_k h \Delta \partial_k n + 2\partial_{ij} n \partial_{ij} h - 4\partial_k h_{ik} \partial_j n \right) - 2\partial_i n \Delta \partial_k h_{ik} - 4\Delta \partial_k n \partial_k n - 4ij \Delta \partial_{ij} n \right) + \alpha_5 \left( \frac{1}{2} h \Delta \partial_i h_{ij} + \frac{1}{2} \partial_k h \partial_{ki} h_{ij} \right) + 2h_{ij} \partial_{ij} \Delta h + \Delta \partial_k h_{kl} \partial_i h + 2\partial_k h_{kl} \partial_{ij} h + 2\partial_k h_{ij} \partial_{kj} h_{kl} - \partial_{ij} h_{ij} \partial_{kj} h_{kl} + h_{ij} \Delta h \Delta h \right) - 4\partial_j h_{kl} \partial_{jkl} h_{ii} + \frac{7}{2} \Delta \partial_h h_{kl} \partial_i h_{kl} + \frac{3}{2} \partial_{jkl} h_{kl} \partial_{ij} h_{kl} + \Delta h_{ij} \Delta h_{ij} + h_{ij} \Delta^2 h_{ij} - \Delta \partial_{jkl} h_{kl} \partial_{jkl} h_{ii} - 2\partial_{ij} h_{kl} \partial_{jkl} h_{ii} + 2\partial_h h_{kl} \partial_k h_{ij} - \frac{1}{2} h \Delta^2 h_{ij} - \partial_k h \partial_k \Delta h - \frac{1}{2} \partial_{ij} h \partial_{ij} h - n \Delta^2 h + n \Delta \partial_{ij} h_{ij} + \Delta h_{ij} \partial_{ij} h - 2\partial_k h_{kl} \partial_i h_{ii} - 2\partial_j h_{kl} \partial_{ij} h_{ii} - \Delta h_{kl} \partial_{kl} h_{ii} + 2\partial_i h_{ij} \partial_j \Delta h \right) \] (A.5)

\[
\det\{\theta_1, \theta_2\}^{(1)} = 2\alpha_6 \left( \Delta^2 n + 2\Delta \partial_i n \partial_i + 2\partial_k n \Delta \partial_k + \partial_i h \Delta \partial_i + \partial_i h \Delta \partial_i + 2h \Delta^2 - 2\partial_{kl} h_{ij} \partial_k - 3\partial_i h_{ij} \Delta \partial_j - 2h_{ij} \Delta \partial_{ij} \right) - 2(\alpha_6 + 2\alpha_7) \left( 2\Delta \partial_i h_{ij} \partial_j + 4\partial_k h_{ij} \partial_{kj} + \Delta h_{kl} \partial_k + 2\partial_k h_{ij} \partial_{kl} \right) - \alpha_5 \Delta^2 h + 2(2\alpha_4 + 3\alpha_6) \partial_j n \partial_{ij} + 2(3\alpha_7 - 2\alpha_4) \Delta n \Delta + (2\alpha_3 - 2\alpha_5 + \alpha_6 \gamma) \Delta \partial_i h \partial_i + 2(\alpha_3 - \alpha_5 + \alpha_6 \gamma) \Delta \partial_{ij} h_{ij} + (2\alpha_5 - 3\alpha_5 + \alpha_6 \gamma) \partial_{ij} h_{ij} \Delta . \] (A.6)

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