Renormalization group for the internal space

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Abstract

The renormalization group method is a successive integration over the fluctuations which are ordered according to their length scale, a parameter in the external space. A different procedure is described, where the fluctuations are treated in a successive manner, as well, but their order is given by an internal space parameter, their amplitude. The differential version of the renormalization group equation is given which is the functional generalization of the Callan-Symanzik equation in one special case and resums the loop expansion in another one.

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I. INTRODUCTION

The renormalization group method is usually applied in two different manners. The original way is to provide an insight into the scale dependence of the coupling constants \cite{1}. Another, more recent use is to perform a partial resummation of the perturbation expansion by making an infinitesimal change of the cutoff in a time and using the functional formalism \cite{2}-\cite{7}. Both goals are realized by the blocking procedure, the successive elimination of the degrees of freedom which lie above the running ultraviolet cutoff. The resulting evolution equation yields the dependence of the coupling constants in the cutoff which is a scale parameter introduced in the space-time, the external space. A difficult problem in this program is the lack of explicit gauge invariance. Since the separation of the modes according to their length scales is not gauge invariant, the gauge symmetry is unavoidably lost during the blocking.

We propose here a different slicing procedure of the path integral, the gradual increase of the amplitude of the fluctuations. We introduce a cutoff in the internal space\cite{1} by constraining the amplitude of the fluctuating field and obtain the evolution equation by the gradual release of this constraint. Since the field variable is usually dimensional this procedure can be interpreted in terms of a scale dependence, as well. More precisely, the mass parameter of the lagrangian serves as a cutoff for both spaces thereby providing a dynamical correspondence between the scales in the internal and the external spaces. An important feature of this scheme is that the scale parameter, the mass in particular, is introduced by hand in the internal space. The length scale of the external space is, on the contrary, induced dynamically. Such a dynamically generated length scale in the external space appears more natural and avoids the problems mentioned above. In fact, having imposed the cutoff in the internal space the gradient expansion, being an approximation procedure in the external space, goes through without problem. The gauge invariance can be maintained, as well, the only requirement is that the suppression of the fluctuations must be achieved in a gauge independent fashion, a condition easy to comply with in the internal space. This latter improvement is somehow reminiscent of the achievement of the Higgs mechanism where a dynamically generated mass is produced without spoiling the gauge symmetry.

There are further advantages in making in the internal space the analogue procedure of the blocking. One is to get a new insight into the origin of the anomalous dimension. To see this we recall that the internal space has actually been used for the purpose of establishing the renormalization group. The renormalization conditions was imposed at a non-vanishing background field field in ref. \cite{8} providing the scale for the internal space. The resulting renormalization group function was computed in the one-loop level and found to be identical with the usual functions, resulting from the renormalization conditions introduced in the external space. This agreement between the scale dependence in the external and internal space is not generally present and the two-loops results should already yield a difference. In fact, the scales inferred from these two spaces are related by the terms of the lagrangian which mix the internal and the external spaces, the terms with space-time derivatives. The

\footnote{The internal and the external spaces stand for the space of the field amplitude and the space-time manifold, respectively.}
simplest of them is the wave function renormalization constant, $\phi_B(x) = Z^{1/2} \phi_R(x)$, which is trivial for a scalar model in the one-loop approximation. The anomalous dimension is nothing but the demonstration of the difference between the scale dependence in the internal and the external spaces.

Another bonus of the internal space renormalization group is the generalization of the Callan-Symanzik equation for the functional formalism. The Callan-Symanzik equation describes the dependence of the Green functions on the mass. Since the mass parameter controls the width of the peak in the wave functional around the vacuum configuration the mass can be thought as the simplest cutoff parameter in the internal space. The similarity of the renormalization group coefficient functions obtained by the bona fide renormalization group and the Callan-Symanzik equation demonstrates that the length scale of the external space induced by the mass can be identified by the cutoff. The functional evolution equation resulting from the infinitesimal change of the mass is the generalization of the Callan-Symanzik equation without relying on the multiplicative renormalization scheme. Such an extension, presented below can be used to study non-renormalizable models or effects close to the cutoff.

When the procedure in the internal space is performed by the help of the bare action the resulting flow provides a resummation of the loop expansion. This scheme is specially advantegous for models with inhomogeneous saddle points because the blocking does not distorts the tree level structure.

The renormalization group step can be expressed by means of the bare action or the effective action. The two schemes become equivalent in the infrared limit. This can be seen easily by remarking that the loop contributions are suppressed in the infrared limit, where the ultraviolet cutoff tends to zero with the number of degrees of freedom. Since the difference between the bare and the effective action comes from the loop contributions the gradient expansion produces the same bare and effective action in the vicinity of the infrared fixed point.

The different standard methods to obtain the renormalization group are briefly discussed in Section II. The internal space evolution equation, our new renormalization group scheme is introduced in Section III. Section IV contains the application of the general strategy of the internal space renormalization to obtain the generalization of the Callan-Symanzik equation. The possibility of resumming the loop expansion by solving the evolution equation is shown in Section V. The Section VI is for the summary. The appendices give details about the Legendre transformation and the gradient expansion.

II. RENORMALIZATION GROUP SCHEMES

In this Section we briefly recapitulate some methods the renormalization group equation can be obtained.

The traditional field theoretical methods for the renormalization group equation are based on the simplification offered by placing the ultraviolet cutoff far away from the scale of the observables. Such a separation of the scales removes the non-universal pieces of the renormalized action and the rather complicated blocking step can be simplified by retaining the renormalizable coupling constants only. The underlying formalism is the renormalized perturbation expansion, in particular the multiplicative renormalization scheme. The usual
perturbative proof of the renormalizability asserts that the renormalized field and the Green functions can be written in terms of the bare quantities as

\[ G_n(p_1,\cdots,p_n;g_R(\mu))_R = Z^{-\frac{n}{2}} \left( g_R;g_B;\frac{\Lambda}{\mu} \right) G_n(p_1,\cdots,p_n;g_B,\Lambda) \left( 1 + O\left( \frac{p^2}{\Lambda^2} \right) \right), \]

where \( \Lambda \) is the cutoff and the renormalized coupling constants are defined by some renormalization conditions imposed at \( p^2 = \mu^2 \). The evolution equation for the bare and the renormalized coupling constants result from the requirements

\[ \frac{d}{d\Lambda} G_R = Z^{\frac{n}{2}} G_B = 0, \]

\[ \frac{d}{d\mu} G_B = Z^{\frac{n}{2}} \frac{d}{d\mu} G_R = 0. \]

Note that the non-renormalizable operators can not be treated in this fashion because the \( O(p/\Lambda) \) contributions are neglected in (1). The renormalization of composite operators and the corresponding operator mixing requires the introduction of additional terms in the lagrangian. Another aspect of this shortcoming is that these methods are useful for the study of the ultraviolet scaling laws only. The study of the infrared scaling or models where there are several non-trivial scaling regimes [[10]] require the more powerful functional form, introduced below.

The third conventional procedure is the Callan-Symanzik equation which is based on the change of the bare mass parameter,

\[ \frac{d}{dm^2} G_B = \frac{d}{dm^2} Z^{\frac{n}{2}} G_R = Z^{\frac{n}{2}} Z_{\phi^2} G_R^{\text{comp}} \]

where \( Z_{\phi^2} \) is the renormalization constant for the composite operator \( \phi^2(x) \) and \( G^{\text{comp}} \) is the Green function with an additional insertion of \( \phi^2(p = 0) \). One can convert the mass dependence inferred from the Callan-Symanzik equation into the momentum dependence by means of dimensional analysis and the resulting expression is usually called a renormalization group equation.

The functional generalizations of the renormalization group method which are based on the infinitesimally small change of the cutoff allows up to follow the mixing of non-renormalizable operators, as well, and to trace the evolution close to the cutoff. Another advantage of these methods is that the renormalization group equation is either exact or holds in every order of the loop expansion.

The traditional blocking [[1]] in momentum space yields the Wegner-Houghton equation [[2]] which is based on sharp cutoff in the momentum space. Consider the action with cutoff \( k \) in the derivative expansion,

\[ S_k = \sum_{n=0}^{\infty} \int d^4x U_k^{(n)}[\phi(x),\partial_\mu] \]

where \( U_k^{(n)}[\phi(x),\partial_\mu] \) is an n-th order homogeneous polynom in the gradient operator \( \partial_\mu \) and a local functional of the field variable. We separate the quadratic part of the action in the field,
\[ S^{\text{quadr}} = \frac{1}{2} \int d^d x d^d y (\phi(x) - \phi_0(x)) G^{-1}(x, y; \phi_0)(\phi(y) - \phi_0(x)) \]  

(6)

where

\[ G^{-1}(x, y; \phi_0) = \frac{\delta^2 S[\phi]}{\delta \phi(x) \delta \phi(y)|_{\phi=\phi_0}} \]  

(7)

to organize the loop expansion. The blocking step consists of the elimination of the Fourier components of the field \( \phi(x) \) within the shell \( k - \Delta k < p < k \). Let us write

\[ \phi(x) = \tilde{\phi}(x) + \phi'(x) \]  

(8)

where the first and the second term in non-vanishing in the Fourier space for \( p < k - \Delta k \) and \( k - \Delta k < p < k \), respectively. The blocked action can be written as

\[ e^{-S_k - \Delta k[\tilde{\phi}]} = \int D[\phi'] e^{-S_k[\tilde{\phi}+\phi']} = e^{-S_k[\tilde{\phi}+\psi]} - \frac{1}{2} \text{tr} G^{-1}[\tilde{\phi}] + \ldots \]  

(9)

where \( \psi \) is the saddle point, the trace is within the functional space spanned by the plane waves \( k - \Delta k < p < k \), and the dots stand for the higher loop contributions. The density of the modes to be eliminated, \( \Omega_d \Delta k d^{d-1} \), \( \Omega_d \) being the solid angle, is chosen to be small. This makes the volume of the loop integral appearing in the computation of the trace small, too. One expects that the n-th loop contributions which contain an n-fold integration in this small volume will be suppressed as \( \Delta k \to 0 \) since the integrand is non-singular in the integration domain. A more careful argument shows that the small parameter is actually the ratio of the modes to be eliminated and left in the system \( \frac{\Delta k}{k} \), assuming that the amplitude of each Fourier mode is of the same order of magnitude. With this condition kept in mind the higher loop contributions are dropped yielding the evolution equation

\[ S_k[\tilde{\phi}] - S_k - \Delta k[\tilde{\phi}] = -\frac{1}{2} \text{tr} G^{-1}[\tilde{\phi}] \left( 1 + O\left( \frac{\Delta k}{k} \right) \right) \]  

(10)

for vanishing saddle point, \( \psi = 0 \). All we needed in deriving this equation was the availability of the loop-expansion for the blocking step (9). The simplification, the suppression of the higher loop contribution, results from the appearance of a new small parameter \( \Delta k/k \) and (10) is valid for weakly coupled models only.

We may call this the scheme the functional form of the bare renormalization group, (2), since the resulting renormalized trajectory is in the space of the cutoff theories. The truncation of the gradient expansion

\[ \text{It is enough to introduce an infrared cutoff and consider the elimination of the modes one-by-one when this latter condition fails to become convinced that the limit } \Delta k \to 0 \text{ can not always suppresses the higher loop contributions to the blocking relation. Such a problem shows up when there is a non-trivial saddle point to the blocking which enhances certain modes.} \]
\[
S_k = \int d^d x \left[ \frac{1}{2} (\partial_\mu \phi)^2 (x) + U_k (\phi (x)) \right]
\]

simplifies the functional equation (10) into a partial differential equation

\[
k \partial_k U_k (\rho) = - \frac{\Omega_d}{2 (2 \pi)^d} k^d \ln \left[ \left( k^2 + \partial_\rho^2 U_k (\rho) \right) \left( k^2 + \frac{1}{\rho} \partial_\rho U_k (\rho) \right)^{N-1} \right].
\]

(12)

for an \(N\)-component field of modulus \(\rho\).

The inclusion of the \(O(\partial^4)\) higher order terms in the action is problematic because the sharp momentum space cutoff generates non-local interactions \(\text{(11)}\).

A similar method was developed for the effective action \(\text{(3)} - \text{(5)}\) where one introduces the cutoff parameter \(k\) in the quadratic part of the action \(G^{-1} (p) \rightarrow G_k^{-1} (p)\), with

\[
G_k^{-1} (p) = f \left( \frac{p}{k} \right) G^{-1} (p),
\]

(13)

where \(f(\kappa)\) approaches 1 and \(\infty\) for \(\kappa = \infty\) and \(\kappa = 0\), respectively. We have hope to generate local, gradient expandable interactions only if \(f(\kappa)\) is an analytic function. Thus locality almost excludes the complete suppression of the modes, the choice \(f(\kappa) = \infty\) for \(\kappa > \kappa_0 < \infty\). The exception to this rule consists of the periodic, analytic functions,

\[
G_k^{-1} (p + K) = G_k^{-1} (p)
\]

(14)

where the components of the vector \(K\) are integer times \(k\). An example is the lattice regularization where the complete suppression of the modes outside the first Brillouin-zone can be achieved without generating non-local interactions. When the modes are suppressed gradually one characterizes the suppression functions by the condition that it changes the most within the interval \(1 - \epsilon < \kappa < 1 + \epsilon\). The limit \(\epsilon \rightarrow 0\) is called the sharp cutoff. The path integration consists of the elimination of the modes \(p > k\) and the evolution equation

\[
\partial_k \int D[\phi] e^{-S_k [\phi] + \int_x j x [\phi]} = - \int D[\phi] \partial_k S_k [\phi] e^{-S_k [\phi] + \int_x j x [\phi]}
\]

(15)

is obtained for the generator functional. This relation is finally converted into an equation for the effective action. Since the effective action for the modes \(p > k\) is followed in this manner, these methods appear the analogues of \(\text{(3)}\), the evolution equation for the running coupling constants.

The application of \(\text{(13)}\) is a safe non-perturbative step only for a differentiable suppression function \(f(\kappa)\). When the \(f(\kappa) = \infty\) for finite values of \(\kappa\) or when \(f(\kappa)\) has infinite derivative (say \(\epsilon \rightarrow 0\)) then \(\text{(13)}\) is invalid. Instead one has to consider the finite difference equation \(k \rightarrow k - \Delta k\) and the corresponding functional Taylor expansion brings back the assumption concerning the applicability of the loop-expansion. Thus \(\text{(13)}\) produces an exact relation for smoothly suppressed modes only.

The internal space renormalization group method presented in the next section generalises the third traditional way of introducing the renormalization group, the Callan-Symanzyk equation by controlling the amplitude of the fluctuations in the internal space only. The corresponding suppression function can be choose as smooth as possible, \(f(\kappa) = \lambda\), in order to suppress the higher order derivative terms generated by the blocking.
III. EVOLUTION EQUATION

Our goal is to obtain the effective action $\Gamma[\phi]$ of the Euclidean model defined by the action $S_B[\phi]$, by reducing the renormalization group strategy into an algorithm to solve the theory. The connection to the external scale dependence will be considered later. The usual Legendre transformation (c.f. Appendix A.) yields

$$e^{W[j]} = \int D[\phi] e^{-S_B[\phi]} + \int_x j_x \phi_x$$

and

$$W[j] + \Gamma[\phi] = \int_x j_x \phi_x = j \cdot \phi,$$

the source $j$ is supposed to be expressed in terms of

$$\phi_x = \frac{\delta W[j]}{\delta j_x}.$$  \hspace{1cm} (18)

A cutoff $\Lambda$ is assumed implicitly in the path integral and $S_B[\phi]$ stands for the bare, cutoff action.

A sharp cutoff could be introduced in the internal space as

$$Z_{\Phi} = \prod_x \Phi^\phi \int_{-\Phi}^\Phi d\phi_x e^{-S[\phi]},$$

but it leads to unwanted complications and will be replaced by a smooth cutoff as follows. We modify the bare action

$$S_B[\phi] \rightarrow S_{\lambda}[\phi] = \lambda S_s[\phi] + S_B[\phi]$$

in such a manner that the model with $\lambda \rightarrow \infty$ be soluble. This is achieved by requiring that the minimum $\phi_\infty$ of the local functional $S_s[\phi] \geq 0$, is nondegenerate. In fact, the path integral for $\lambda = \infty$ contains no fluctuations. The role of the new piece in the action is to suppress the fluctuations around $\phi_\infty$ for large value of $\lambda$ and render the model perturbative. The control of the amplitude of the fluctuations offered by the parameter $\lambda$ is considered as a smooth cutoff in the internal space. The difference between our strategy and the other methods is that the internal space cutoff appears multiplicatively in the action and influences all modes simultaneously while the external space cutoff is modifying the momentum dependence. In other words, it is the order the different fluctuations are treated as the cutoff is decreased what distinguishes the internal and the external space renormalization group schemes.

We plan to follow the $\lambda$ dependence of the effective action by integrating out the functional differential equation

$$\partial_\lambda \Gamma = \mathcal{F}_\lambda[\Gamma],$$

from the initial condition
\[
\Gamma_{\text{init}}[\phi] = \lambda_{\text{init}} S_s[\phi] + S_B[\phi],
\] (22)

imposed at \(\lambda_{\text{init}} \approx \infty\) to \(\lambda = 0\). (21) can be interpreted as a generalization of the Callan-Symanzik equation because both generate a one-parameter family of different theories organized according to the strength of the quantum fluctuations \(^3\). So long as the parameter \(\lambda\) introduces a renormalization scale, \(\mu(\lambda)\), the trajectory \(\Gamma_{\lambda(\mu)}[\phi]\) in the effective action space can be thought as a renormalized trajectory. Another way to interpret (21) is to consider its integration as a method which builds up the fluctuations of the model with \(\lambda = 0\) by summing up the effects of increasing the fluctuation strength infinitesimally, \(\lambda \to \lambda - \Delta \lambda\). Notice that the gauge invariance of the evolution equation (21) is obvious when \(S_s[\phi]\) is gauge invariant. The gradient expansion is compatible with (21) if the suppression is sufficiently smooth in the momentum space, i.e. \(S_s\) is a local functional.

The starting point to find \(F_{\lambda}[\Gamma]\) is the relation

\[
\partial_{\lambda} \Gamma[\phi] = -\partial_{\lambda} W[j] - \frac{\delta W[j]}{\delta j} \partial_{\lambda} j + \partial_{\lambda} j \cdot \phi = -\partial_{\lambda} W[j],
\] (23)

\(\lambda\) and \(\phi\) being the independent variables. This relation will be used together with

\[
\partial_{\lambda} W[j] = -e^{-W[j]} \int D[\phi] S_s[\phi] e^{-\lambda S_s[\phi] - S_B[\phi]} = -e^{-W[j]} S_s \left[ \frac{\delta}{\delta j} \right] e^{W[j]}. \] (24)

It is useful to perform the replacement

\[
\Gamma[\phi] \longrightarrow \lambda S_s[\phi] + \Gamma[\phi]
\] (25)

which results the evolution equation

\[
\partial_{\lambda} \Gamma[\phi] = e^{-W[j]} S_s \left[ \frac{\delta}{\delta j} \right] e^{W[j]} - S_s[\phi]. \] (26)

The next question is the choice of \(S_s[\phi]\). The simplest is to use a quadratic suppression term,

\[
S_s[\phi] = \frac{1}{2} \int_{x,y} \phi_x \mathcal{M}_{x,y} \phi_y = \frac{1}{2} \phi \cdot \mathcal{M} \cdot \phi. \] (27)

If a gauge symmetry should be preserved then such an \(S_s[\phi]\) is not acceptable and

\[
S_s[\phi] = S_B[\phi] \] (28)

is the most natural choice. The evolution equation (26) then sums up the loop expansion and produces the dependence in \(\bar{h}\). We return now to the case of a simple scalar field without local symmetry, (27). The corresponding evolution equation can be obtained from (26), and considering the relation (A4) between the functional derivatives of \(W[j]\) and \(\Gamma[\phi]\),

\(^3\) Note that the inverse mass is proportional to the amplitude of the fluctuations.
\[
\partial_\lambda \Gamma[\phi] = \frac{1}{2} \int_{x,y} M_{x,y} \left[ W_{x,y}^{(2)} + \phi_x \phi_y \right] - \frac{1}{2} \int_{x,y} \phi_x M_{x,y} \phi_y \\
= \frac{1}{2} \int_{x,y} M_{x,y} \left[ \Gamma^{(2)}_{x,y} + \lambda M_{x,y} \right]^{-1}
\]

(29)

where the functional derivatives are denoted by

\[
\Gamma^{(n)}_{x_1,\ldots,x_n} = \frac{\delta^n \Gamma[\phi]}{\delta \phi_{x_1} \cdots \delta \phi_{x_n}}.
\]

(30)

(29) reads in an operator notation

\[
\partial_\lambda \Gamma[\phi] = \frac{1}{2} \text{Tr} \left\{ \mathcal{M} \cdot \left[ \lambda \mathcal{M} + \Gamma^{(2)} \right]^{-1} \right\},
\]

(31)

We should bear in mind that \( \Gamma^{(n)}_{x_1,\ldots,x_n} \) remains a functional of the field \( \phi_x \).

It is illuminating to compare this result with the evolution equations presented in refs. 3-7 what one can obtain by means of (13) and (26),

\[
\partial_k \Gamma[\phi] = \frac{1}{2} \text{Tr} \left\{ \partial_k G^{-1}_{k} \cdot \left[ G^{-1}_{k} + \Gamma^{(2)} \right]^{-1} \right\}.
\]

(32)

The role of \( S_s \) is played here by the propagator \( G_{k}(p) \) which contains the external space scale parameter \( k \) to control the suppression of the fluctuations. The formal similarity with (29) reflects that the different schemes agree in ”turning on” the fluctuations in infinitesimal steps. But the internal space scheme operates with a suppression term which is regular in or even independent of the momentum.

The evolution equation can be converted into a more treatable form by the means of the gradient expansion,

\[
\Gamma[\phi] = \int_x \left\{ \frac{1}{2} Z_x (\partial_\mu \phi_x)^2 + U_x + O(\partial^4) \right\}
\]

(33)

where the notation \( f_x = f(\phi_x) \) was introduced. This ansatz gives

\[
\Gamma^{(1)}_{x_1} = -\frac{1}{2} Z^{(1)}_x x_1 (\partial_\mu \phi_x x_1)^2 - Z_{x_1} \Box \phi_{x_1} + U^{(1)}_{x_1}
\]

(34)

\[
\Gamma^{(2)}_{x_1,x_2} = -\frac{1}{2} \delta_{x_1,x_2} Z^{(2)}_{x_1} (\partial_\mu \phi_{x_1})^2 - \partial_\mu \delta_{x_1,x_2} Z^{(1)}_{x_1} \partial_\mu \phi_{x_1}
\]

\[
-\delta_{x_1,x_2} Z^{(1)}_{x_1} \Box \phi_{x_1} - \Box \delta_{x_1,x_2} Z_{x_1} + U^{(2)}_{x_1}
\]

where the \( f^{(n)}(\phi) = \partial_\phi^n f(\phi) \). Such an expansion is unsuitable for \( W[j] \) due to the strong non-locality of the propagator but might be more successful for the effective action where the one-particle irreducible structure and the removal of the propagator at the external legs of the contributing diagrams strongly reduce the non-local effects. The replacement of this ansatz into (26) gives (c.f. Appendix B.)

\[
\partial_\lambda U_\lambda(\phi) = \frac{1}{2} \int_{p} \frac{\mathcal{M}(p)}{\lambda \mathcal{M}(p) + Z_\lambda(\phi)p^2 + U^{(2)}_\lambda(\phi)}
\]

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\[ \partial_\lambda Z_\lambda(\phi) = \frac{1}{2} \int_p \mathcal{M}(p) \left[ -\frac{Z^{(2)}_\lambda(\phi)}{\left(\lambda \mathcal{M}(p) + Z_\lambda(\phi)p^2 + U^{(2)}_\lambda(\phi)\right)^2} \right. \\
+ 2Z^{(1)}_\lambda(\phi) \frac{2\left(Z^{(1)}_\lambda(\phi)p^2 + U^{(3)}_\lambda(\phi)\right) + Z^{(1)}_\lambda(\phi)p^2/d}{\left(\lambda \mathcal{M}(p) + Z_\lambda(\phi)p^2 + U^{(2)}_\lambda(\phi)\right)^3} \\
- \left. \left(\frac{Z^{(1)}_\lambda(\phi)p^2 + U^{(3)}_\lambda(\phi)}{\left(\lambda \mathcal{M}(p) + Z_\lambda(\phi)p^2 + U^{(2)}_\lambda(\phi)\right)^4}\right) \right) \\
- \frac{4}{d} \frac{Z^{(1)}_\lambda(\phi) \left(Z^{(1)}_\lambda(\phi)p^2 + U^{(3)}_\lambda(\phi)\right)}{\left(\lambda \mathcal{M}(p) + Z_\lambda(\phi)p^2 + U^{(2)}_\lambda(\phi)\right)^4} \left(\lambda \partial_\mu \mathcal{M}(p) + 2Z_\lambda(\phi)p^2\right) \\
+ \frac{2}{d} \frac{2\left(Z^{(1)}_\lambda(\phi)p^2 + U^{(3)}_\lambda(\phi)\right)^2}{\left(\lambda \mathcal{M}(p) + Z_\lambda(\phi)p^2 + U^{(2)}_\lambda(\phi)\right)^5} \right]\]

where \( f_p = \frac{d^dp}{(2\pi)^d} \) and we assumed that \( \partial_\mu \mathcal{M}(p) \) is proportional to \( p_\mu \). Since the order the fluctuations are treated is different for the internal and the external space methods, (35) is different, as well, from the evolution equations of [4]-[7]. The final solution, corresponding to \( \lambda = 0 \) in our case and \( k = 0 \) in the others, is the effective potential expressed in terms of the bare and the renormalized coupling constants, respectively.

### IV. MASS DEPENDENCE

The simplest choice is \( \lambda = m^2 \) with

\[ \mathcal{M}_{x,y} = \delta_{x,y} \tag{36} \]

which minimizes strength of the higher order derivative terms generated during the evolution by being a momentum independent suppression mechanism. The evolution equation is the functional differential renormalization group version of the Callan-Symanzik equation,

\[ \partial_{m^2} \Gamma[\phi] = \frac{1}{2} \text{Tr} \left[ m^2 \delta_{x,y} + \Gamma^{(2)}_{x,y} \right]^{-1}. \tag{37} \]

The projection of this functional equation onto the gradient expansion ansatz gives

\[
\begin{align*}
\partial_{m^2} U(\phi) &= \frac{1}{2} \int_p \frac{1}{Z(\phi)p^2 + m^2 + U^{(2)}(\phi)} \\
\partial_{m^2} Z(\phi) &= \frac{1}{2} \int_p \left[ -\frac{Z^{(2)}(\phi)}{(Z(\phi)p^2 + m^2 + U^{(2)}(\phi))^2} \\
&\quad + 2Z^{(1)}(\phi) \frac{p^2/dZ^{(1)}(\phi) + 2\left(Z^{(1)}(\phi)p^2 + U^{(3)}(\phi)\right)}{(Z(\phi)p^2 + m^2 + U^{(2)}(\phi))^3} \\
&\quad - 2Z(\phi) \frac{\left(Z^{(1)}(\phi)p^2 + U^{(3)}(\phi)\right)^2}{(Z(\phi)p^2 + m^2 + U^{(2)}(\phi))^4} \right]
\end{align*}
\]
\[ - \frac{8p^2}{d} Z(\phi) Z^{(1)}(\phi) \left( \frac{Z^{(1)}(\phi)p^2 + U^{(3)}(\phi)}{(Z(\phi)p^2 + m^2 + U^{(2)}(\phi))^4} \right) \]
\[ + \frac{8p^2}{d} Z^2(\phi) \left( \frac{Z^{(1)}(\phi)p^2 + U^{(3)}(\phi)}{(Z(\phi)p^2 + m^2 + U^{(2)}(\phi))^3} \right)^2 \]  

\[(38)\]

It is important to bear in mind that we are dealing here with a well regulated theory and that the procedure described here does not aim at removing the external space cutoff \( \Lambda \) which remains an important external parameter. In the usual, external space renormalization schemes the blocking provides the cutoff and the regulator for the models. In our case the cutoff in the internal space can not replace the usual external space cutoff \( \Lambda \) and this latter must properly be implemented from the very beginning, c.f. the remark after (18). In this case the loop integrals occurring in the evolution equations are finite.

Let us now simplify the differential equation for \( U(\phi) \) and \( Z(\phi) \) by integrating over \( p \) in (38) with sharp momentum cutoff \( \Lambda \) in four dimensions,

\[
\partial_{m^2} U(\phi) = \frac{1}{32\pi^2 Z(\phi)} \left[ \Lambda^2 - \frac{m^2 + U^{(2)}(\phi)}{Z(\phi)} \ln \left( 1 + \frac{Z(\phi)\Lambda^2}{m^2 + U^{(2)}(\phi)} \right) \right] \\
\partial_{m^2} Z(\phi) = \frac{1}{32\pi^2 Z(\phi)} \left[ \frac{1}{Z^2(\phi)} \left( \frac{5}{2} \left( Z^{(1)}(\phi) \right)^2 - Z(\phi)Z^{(2)}(\phi) \right) \ln \left( 1 + \frac{Z(\phi)\Lambda^2}{m^2 + U^{(2)}(\phi)} \right) \right] \\
+ \frac{1}{Z^2(\phi)} \left( Z(\phi)Z^{(2)}(\phi) - \frac{43}{12} \left( Z^{(1)}(\phi) \right)^2 \right) \\
+ \frac{1}{Z(\phi)} \frac{Z^{(1)}(\phi)U^{(3)}(\phi)}{m^2 + U^{(2)}(\phi)} - \frac{1}{6} \left( \frac{U^{(3)}(\phi)}{m^2 + U^{(2)}(\phi)} \right)^2 \right]  \\
\] 

\[(39)\]

In the approximation \( Z = 1 \) we obtain

\[
\partial_{m^2} U(\phi) = -\frac{m^2 + U^{(2)}(\phi)}{32\pi^2} \ln \left( 1 + \frac{\Lambda^2}{m^2 + U^{(2)}(\phi)} \right)  \\
\] 

\[(40)\]

after removing a field independent term. In order to simplify the scaling relations we consider the regime \( m^2 \gg U^{(2)} \), where

\[
\partial_{m^2} U(\phi) = -\frac{1}{32\pi^2} \ln \left( \frac{m^2 + \Lambda^2}{m^2} \right) U^{(2)}(\phi)  \\
\] 

\[(41)\]

The asymptotic scaling formula of the bare renormalization group (12) for \( d = 4 \) in the same regime is

\[ \text{Paragraph on asymptotic scaling formula} \]

\[ ^4 \text{The sharp momentum space cutoff generates nonlocal interactions. Since these nonlocal contributions} \]

\[ \text{come from the surface terms of the loop integrals they are suppressed in a renormalizable} \]

\[ \text{theory when } \Lambda \text{ is kept large. Thus the gradient expansion anzatz can be justified} \]
\[
k \partial_k U(\phi) = -\frac{k^2}{16\pi^2} U^{(2)}(\phi). \tag{42}
\]

The evolutions (11) and (12) agree up to an overall constant in the scale parameter if

\[
\frac{dk^2}{dm^2} = \ln \left( \frac{m^2 + \Lambda^2}{m^2} \right) \tag{43}
\]

can be considered as constant, i.e. \( m^2 \) and \( k \) independent. We obtain in this manner the usual justification of calling the Callan-Symanzik equation a renormalization group method, the equivalence of the scales in the internal and the external spaces in the ultraviolet scaling regime \( m^2 \gg U^{(2)} \) up to non-universal contributions. The equivalence of the scales and the elimination of the non-universal contributions requires that the cutoff should be far above the mass, \( m^2 \ll \Lambda^2 \).

The non-vanishing anomalous dimension sets up the relation between the internal and external space scaling. In fact, when \( Z \neq 1 \) the relation (13) becomes field dependent according to the first equation of (39). It is worthwhile comparing what (39) gives in the asymptotical regime \( m^2 \gg U^{(2)} \),

\[
\partial_m^2 Z_m(\phi) = -\frac{1}{32\pi^2 Z_k^3(\phi)} \ln \left( \frac{Z_m(\phi)\Lambda^2 + m^2}{m^2} \right) \left[ Z_m(\phi) Z_m^{(2)}(\phi) - \frac{5}{2} (Z_m^{(1)}(\phi))^2 \right] \tag{44}
\]

with the prediction of the Wegner-Houghton equation. A possible attempt to save the gradient expansion with sharp cutoff for the latter is the following: The contributions to the coefficient functions of the gradient, such as \( Z_k(\phi) \), come from taking the derivative of the loop integral, the trace in the second equation of (9), with respect to the momentum of the infrared background field \( \tilde{\phi}(x) \). There are two kind of contributions, one which comes form the derivative of the integrand, another from the external momentum dependence of the limit of the integration. It is easy to verify that the \( \epsilon \)-dependent non-local contributions come form the second types only [5]. Thus one may consider the approximation where these contributions are simply neglected, assuming a cancellation mechanism between the successive blocking steps. The result is, for \( k^2 \gg U^{(2)}_k(\phi) \), c.f. Appendix C,

\[
k \partial_k Z_k(\phi) = -\frac{k^2}{32\pi^2 Z_k^2(\phi)} \left[ 2Z_k(\phi) Z_k^{(2)}(\phi) - \frac{5}{2} (Z_k^{(1)}(\phi))^2 \right]. \tag{45}
\]

The formal similarity between the two different schemes, (44) and (45), can be considered as a measure of the cancellation of the non-local terms evoked above.

The beta-functions of the coupling constants \( g_n \) and \( z_m \) introduced as

\[
U(\phi) = \sum_n \frac{g_n}{n!} \phi^n, \quad Z(\phi) = \sum_n \frac{z_n}{n!} \phi^n \tag{46}
\]

are of the form

\[
\beta_n = m^2 \partial_m^2 g_n = C_d m^2 \frac{\partial^n}{\partial \phi^n} \int_0^{\Lambda^2} dy y^{d-1} \frac{1}{Z(\phi)y + m^2 + U^{(2)}(\phi)}
\]
\[ \gamma_n = m^2 \partial_m z_n \]
\[ = C_d m^2 \frac{\partial^n}{\partial \phi^n} \int_0^\Lambda^2 dy y^{\frac{d}{2} - 1} \left[ \frac{Z^{(2)}(\phi)}{(Z(\phi) y + m^2 + U^{(2)}(\phi))^2} \right. \]
\[ + 2Z^{(1)}(\phi) \frac{y/dZ^{(1)}(\phi)}{(Z(\phi) y + m^2 + U^{(2)}(\phi))^3} - \frac{2Z(\phi)}{(Z(\phi) y + m^2 + U^{(2)}(\phi))^2} \]
\[ \left. + \frac{8y}{d} Z^{(1)}(\phi) \frac{Z^{(1)}(\phi) y + U^{(3)}(\phi)}{(Z(\phi) y + m^2 + U^{(2)}(\phi))^4} + \frac{8y}{d} Z^2(\phi) \frac{Z^{(1)}(\phi) y + U^{(3)}(\phi)}{(Z(\phi) y + m^2 + U^{(2)}(\phi))^5} \right] \]
\[ (47) \]

with \( C_d = \Omega_d/2(2\pi)^d \). The integration over \( y \) produces simple expressions for \( \beta_n \) and \( \gamma_n \) in terms of the coupling constants \( g_m \) and \( z_m \). The simultaneous integration of this set of equations produces the solution of the evolution equation (38).

It is instructive to consider the solution in the independent mode approximation where the \( m^2 \) dependence is ignored in the integrals, \( U(\phi) = U_B(\phi) \) and \( Z(\phi) = 1 \). We get

\[ U_{\text{eff}}(\phi) = U_B(\phi) + \frac{1}{2} M^2 \int p \frac{1}{p^2 + m^2 + U^{(2)}_B(\phi)} \]
\[ = U_B(\phi) + \frac{1}{2} \int p \ln[p^2 + U^{(2)}_B(\phi)] + O(M^{-2}), \]

which reproduces the usual one-loop effective potential for \( M \gg \Lambda \). For the kinetic term, the integration of (39) in the same approximation leads to

\[ Z_{\text{eff}}(\phi) = 1 - \frac{1}{192\pi^2} \int_0^\Lambda^2 dm^2 \frac{\left( U^{(3)}_B(\phi) \right)^2}{\left( m^2 + U^{(2)}_B(\phi) \right)^2} \]
\[ = 1 + \frac{1}{192\pi^2} \frac{\left( U^{(3)}_B(\phi) \right)^2}{U^{(2)}_B(\phi)} + O(M^{-2}), \]
\[ (48) \]

for \( d = 4 \) which reproduces the one-loop solution found in [14]. The agreement between the independent mode approximation to the internal-space renormalization group equation and the one-loop solution is expected not only because the right hand side of (29) is \( O(\bar{h}) \) but because the one-loop contribution to the gamma function is universal, scheme independent. But this agreement does not hold beyond \( O(\bar{h}) \) as indicated by the incompatibility of (44) and (45).

**V. \( \bar{h} \) DEPENDENCE**

It may happen that the quadratic suppression is not well suited to a problem. In the case \( S_B[\phi] \) possesses local symmetries which should be preserved then the choice (28) is more appropriate. The application of our procedure for a gauge model can for example be based on the choice
\[ S_B[A] = -\frac{1}{4g_B^2} \int dx F^a_{\mu \nu} F^{a \mu \nu} + S_{gf}[A], \]
\[ S_s[A] = -\frac{1}{4g_B^2} \int dx F^a_{\mu \nu} F^{a \mu \nu}, \] (49)
where \( S_{gf} \) contains the gauge fixing terms and on the application of a gauge invariant regularization scheme. As mentioned after eq. (28) we need a regulator to start with in order to follow the dependence on the amplitude of the fluctuations. One may use lattice, analytic (asymptotically free models) or Pauli-Villars (QED) regulator to render (26) well defined. The explicit gauge invariance of \( S_s[A] \) which was achieved by suppressing the gauge covariant field strength instead of the gauge field itself makes obvious the independence of the resulting flow for the gauge invariant part of the action from the choice of the gauge, \( S_{gf} \). When a non-trivial saddle point appears in the blocking step then it may develop a discontinuous evolution. The choice (28) makes the saddle point approximatively "renormalization group invariant".

We present the evolution equation for the \( \phi^4 \) model with quartic suppression,

\[ S_B[\phi] = S_s[\phi] = \int_x \left[ \frac{1}{2} (\partial_\mu \phi_x)^2 + \frac{g_2}{2} S + \frac{g_3}{3!} \phi_x^3 + \frac{g_4}{4!} \phi_x^4 \right]. \] (50)

The similarity of this scheme with the loop expansion suggests the replacement

\[ \frac{1}{\hbar} = 1 + \lambda = 1 + \frac{1}{g}, \] (51)

which yields the evolution equation

\[ \partial_g \Gamma[\phi] = -\frac{1}{g^2} e^{-W[j]} S_s \left[ \frac{\delta}{\delta j} \right] e^{W[j]} + \frac{1}{g^2} S_s[\phi]. \] (52)

The integration of the evolution equation from \( g_{in} = 0 \) to \( g_{fin} = \infty \) coresponds to the resummation of the loop expansion, i.e. the integration between \( h_{in} = 0 \) and \( h_{fin} = 1 \).

The gradient expansion anstatz (33) with \( Z = 1 \) gives

\[ \partial_g U(\phi) = -\frac{1}{g^2} \left\{ \frac{1}{2} \int_p (p^2 + g_2) G(p) \right. \]
\[ + \frac{g_3}{3!} \left[ 3 \phi \int_p G(p) - \frac{1}{p_1 p_2} G(p_1) G(p_2) G(-p_1 - p_2) \left( U^{(3)}(\phi) + g^{-1}(g_3 + g_4 \phi) \right) \right] \]
\[ + \frac{g_4}{4!} \left[ 3 \left( \int_p G(p) \right)^2 + 6 \phi^2 \int_p G(p) \right. \]
\[ - \frac{1}{p_1 p_2 p_3} G(p_1) G(p_2) G(p_3) G(-p_1 - p_2 - p_3) \left( U^{(4)}(\phi) + g^{-1} g_4 \right) \]
\[ - 3 \frac{1}{p_1 p_2 p_3} G(p_1) G(p_2) G(p_3) G(-p_1 - p_2) G(-p_1 - p_2 - p_3) \]
\[ \times \left( U^{(3)}(\phi) + g^{-1}(g_3 + g_4 \phi) \right)^2 \]
\[ - 4 \phi \int_p G(p_1) G(p_2) G(-p_1 - p_2) \left( U^{(3)}(\phi) + g^{-1}(g_3 + g_4 \phi) \right) \right\}, \] (53)
where we used the fact that the Fourier transform of the 1PI amplitude for \( n \geq 3 \) and \( Z = 1 \) is

\[
\int_{x_1, \ldots, x_n} e^{i(p_1 \cdot x_1 + \cdots + p_n \cdot x_n)} \Gamma^{(n)}(x_1, \ldots, x_n) = (2\pi)^d \delta(p_1 + \cdots + p_n) U^{(n)}(\phi).
\] (54)

The propagator in the presence of the homogeneous background field \( \phi \) is given by

\[
G(p) = \left[ p^2 + U^{(2)}(\phi) + g^{-1}(p^2 + g_2 + g_3 \phi + \frac{g_4}{2} \phi^2) \right]^{-1}.
\] (55)

Since the momentum dependence in the right hand side of (53) is explicit and simple the one, two and three loop integrals can be carried out easily by means of the standard methods. The successive derivatives of the resulting expression with respect to \( \phi \) yield the renormalization group coefficient functions.

The use of internal space renormalization described in this section shows that this method can be generalized to any kind of action \( S_s \) and not only to a quadratic suppression term, as shown in the previous sections.

VI. SUMMARY

The strategy of the renormalization group is developed further in this paper. Instead of following the evolution of the coupling constants corresponding to the same physics our renormalization group flow sweeps through models with different dynamics. The parameter of the flow is the scale of the quantum or thermal fluctuations. The result is an exact functional differential equation for the effective action. As a special case the functional generalization of the Callan-Symanzik equation is recovered. A different choice of the "blocking transformation" allows us to control \( \hat{h} \) and the resulting flow amounts to the resummation of the loop expansion.

Our scheme can be considered as a renormalization group method in the internal space. The similarity of the renormalization group flow in the external and the internal space is shown for the local potential, the zero momentum piece of the effective action. The difference between the two schemes is the source of the anomalous dimension and it appears in the momentum dependent parts of the effective action.

The novel feature of the method is its manifest gauge invariance. This is achieved by the possibility of characterizing the modes in the internal space while the gauge transformations are carried out in the external space.

The integration of the internal space evolution equation provides an algorithm to solve models in a manner similar to the traditional renormalization group method. The only truncation is done in the gradient expansion of the effective action, in the Taylor expansion of the 1PI functions in the momentum. We believe that this procedure is an interesting alternative to the stochastic solution method of the lattice regulated models. The drawback is that it is rather cumbersome, thought possible in principle, to increase the precision in the momentum and going to higher orders in the gradient expansion. The advantage is that it can be cast in infinite, continuous space-time equipped with Minkowski metric.
Note added in proof: After this work has been completed we learned that a method presented for gauge models in ref. [15] is similar to ours in the case of mass dependence (section IV). [15] gives a loop expanded solution of the exact equation, whereas our solution is built in the framework of the derivative expansion. Finally, our approach can be generalized to any kind of suppression action $S_s$ which is compatible with the symmetries as shown in section V.

APPENDIX A: LEGENDRE TRANSFORMATION

We collect in this Appendix the relations between the derivatives of the generator functional $W[j]$ and $\Gamma[\phi]$ used in obtaining the evolution equations for $\Gamma$.

We start with the definitions

$$W[j] + \Gamma[\phi] + \lambda S_s[\phi] = j \cdot \phi,$$

and

$$\dot{\phi}_x = W_x^{(1)}.$$  \hfill (A2)

The first derivative of $\Gamma$ gives the inversion of (A2),

$$\Gamma_{x_1,x_2}^{(1)} = j_x - \lambda S_s^{(1)}.$$  \hfill (A3)

The second derivative is related to the propagator $W_{x_1,x_2}^{(2)} = G_{x_1,x_2}$

$$\Gamma_{x_1,x_2}^{(2)} = \frac{\delta j_{x_1}}{\delta \phi_{x_2}} - \lambda S_s^{(2)} = G_{x_1,x_2}^{-1} - \lambda S_s^{(2)}.$$  \hfill (A4)

The third derivative is obtained by differentiating (A4),

$$\Gamma_{x_1,x_2,x_3}^{(3)} = - \int_{y_1,y_2,y_3} G_{x_1,y_1} G_{x_2,y_2} G_{x_3,y_3} W_{y_1,y_2,y_3}^{(3)} - \lambda S_s^{(3)}.$$  \hfill (A5)

The inverted form of this equation is

$$W_{x_1,x_2,x_3}^{(3)} = - \int_{y_1,y_2,y_3} G_{x_1,y_1} G_{x_2,y_2} G_{x_3,y_3} \left( \Gamma_{y_1,y_2,y_3}^{(3)} + \lambda S_s^{(3)} \right).$$  \hfill (A6)

The further derivation gives

$$\Gamma_{x_1,x_2,x_3,x_4}^{(4)} = \int_{y_1,y_2,y_3,y_4} \left[ G_{x_1,y_1} G_{x_2,y_2} G_{x_3,y_3} G_{x_4,y_4} W_{y_1,y_2,y_3,y_4}^{(4)} ight.\
+ G_{x_1,y_1} G_{x_2,y_2} W_{y_1,y_2,z_1}^{(3)} G_{x_3,z_1} G_{x_4,y_4} W_{y_1,y_2,y_3}^{(3)} G_{x_3,y_3} G_{x_4,y_4} \\+ G_{x_1,y_1} G_{x_2,y_2} W_{y_1,y_2,z_1}^{(3)} G_{x_3,z_1} G_{x_4,y_4} W_{y_1,y_2,y_3}^{(3)} G_{x_3,y_3} G_{x_4,y_4} \\
+ G_{x_1,y_1} G_{x_2,y_2} W_{y_1,y_4,z_1}^{(3)} G_{x_3,z_1} G_{x_4,y_4} W_{y_1,y_2,y_3}^{(3)} G_{x_3,y_3} G_{x_4,y_4} \\
+ \left. \lambda S_s^{(4)} \right].$$  \hfill (A7)
Its inversion expresses the four point connected Green function in terms of the 1PI amplitudes,

\[
W_{x_1,x_2,x_3,x_4}^{(4)} = \int_{y_1,y_2,y_3,y_4,z_1,z_2} \left[ G_{x_1,y_1} G_{x_2,y_2} G_{x_3,y_3} G_{x_4,y_4} \left( \Gamma_{y_1,y_2,y_3,y_4}^{(4)} + \lambda S_{y_1,y_2,y_3,y_4}^{(4)} \right) - G_{x_1,y_1} G_{x_2,y_2} \left( \Gamma_{y_1,y_2,z_1,z_2}^{(3)} + \lambda S_{y_1,y_2,z_1,z_2}^{(3)} \right) G_{z_1,z_2} \left( \Gamma_{x_2,y_3,y_4}^{(3)} + \lambda S_{x_2,y_3,y_4}^{(3)} \right) G_{x_3,y_3} G_{x_4,y_4} 
- G_{x_3,y_3} G_{x_2,y_2} \left( \Gamma_{y_1,y_2,z_1,z_2}^{(3)} + \lambda S_{y_1,y_2,z_1,z_2}^{(3)} \right) G_{z_1,z_2} \left( \Gamma_{x_2,y_3,y_4}^{(3)} + \lambda S_{x_2,y_3,y_4}^{(3)} \right) G_{x_1,y_1} G_{x_4,y_4} 
- G_{x_1,y_1} G_{x_4,y_4} \left( \Gamma_{y_1,y_4,z_1,z_2}^{(3)} + \lambda S_{y_1,y_4,z_1,z_2}^{(3)} \right) G_{z_1,z_2} \left( \Gamma_{x_4,y_3,y_2}^{(3)} + \lambda S_{x_4,y_3,y_2}^{(3)} \right) G_{x_3,y_3} G_{x_2,y_2} \right].
\]

**APPENDIX B: EVOLUTION EQUATION IN THE INTERNAL SPACE**

We give here some details on the computation of (B5). To get the evolution equation of the potential part of the gradient expansion \( \bar{\phi} = \phi_0 \) in (29). But to distinguish the kinetic contribution from the potential one, a non-homogeneous field \( \phi(x) = \phi_0 + \eta(x) \) is needed, as well. Let \( k \) be the momentum where the field \( \eta \) is non-vanishing. Then the effective action can be written as

\[
\Gamma[\phi] = V_d U_\lambda(\phi_0) + \frac{1}{2} \int_q \bar{\eta}(q) \eta(-q) \left( Z_\lambda(\phi_0) q^2 + U_\lambda^{(2)}(\phi_0) \right) + O(\bar{\eta}^3, k^4)
\]

where \( V_d \) is the spatial volume. Thus we need the second derivative of the effective action in (29) up to the second order in \( \eta \) to identify the different contributions. The terms independent of \( \bar{\eta} \) give the equation for \( U_\lambda \) and the ones proportional to \( k^2 \bar{\eta}^2 \) the equation for \( Z_\lambda \). The contributions proportional to \( \bar{\eta}^2 \) but independent of \( k \) yield an equation for \( U_\lambda^{(2)} \) which must be consistent with the equation for \( U_\lambda \). The result is

\[
\Gamma_{p_1,p_2}^{(2)} = \left[ Z_\lambda(\phi_0) p_1^2 + U_\lambda^{(2)}(\phi_0) \right] \delta(p_1 + p_2) + \int_q \bar{\eta}(q) \left[ Z_\lambda^{(1)}(\phi_0)(p_1^2 + q^2 + q p_1) + U_\lambda^{(3)}(\phi_0) \right] \delta(p_1 + p_2 + q) + \frac{1}{2} \int_{q_1,q_2} \bar{\eta}(q_1) \bar{\eta}(q_2) \left[ Z_\lambda^{(2)}(\phi_0)(p_1^2 + 2q_1^2 + q_1 p_1 + 2q_1 p_1) + U_\lambda^{(4)}(\phi_0) \right] \delta(p_1 + p_2 + q_1 + q_2) + O(\bar{\eta}^3, k^4)
\]

Finally one computes the inverse of the operator \( \lambda \mathcal{M}_{p_1,p_2} + \Gamma_{p_1,p_2}^{(2)} \) and expands it in powers of \( \bar{\eta} \) and \( k \). The trace over \( p_1 \) and \( p_2 \) needs the computations of terms like

\[
\text{Tr} \{ (p_1 q_1)(p_2 q_2) F(p_1, p_2) \delta(p_1 + p_2 + q_1 + q_2) \} = \frac{q_1^2}{d} \delta(q_1 + q_2) \int_p p^2 F(p, -p)
\]

and they lead to (B3). The consistency with the equation for \( U_\lambda^{(2)} \) is satisfied.
APPENDIX C: EVOLUTION EQUATION IN THE EXTERNAL SPACE

The evolution for the blocking in the momentum space is given by the Wegner-Houghton equation (14). In order to obtain its simplified version in the gradient expansion one needs the trace of the logarithm of $\Gamma_{p_1,p_2}^{(2)}$. This time the trace has to be computed for $|p_1|$ and $|p_2|$ in the shell $[k - \delta k, k]$. This implies that $|p_1 + q|$ has to be in the shell, as well, because $\Gamma_{p_1,p_2}^{(2)}$ contains $\delta(p_1 + p_2 + q)$. This constrain implies the appearance of $\sqrt{q^2}$ in the equation for $k\partial_k Z_k(\phi)$. The terms containing $\sqrt{q^2}$ are non-local and spoil the gradient expansion. $k\partial_k Z_k(\phi)$ is proportional to the second derivative of the trace in (14) with respect the momentum $q$ of the infrared background field, $\tilde{\phi}$. The trace can be written as a momentum internal over the shell $[k - \delta k, k]$. There are contributions from the dependence on $q$ of the integrand and the limit of integration. It is worthwhile noting that the later contains all non-local terms.

There are two ways to rid the non-local contributions when the model is solved by the loop expansion, i.e. by means of loop integrals for momenta $0 \leq p \leq \Lambda$. One is to use lattice regularization where the periodicity in the Brillouin zone cancel the $q$ dependence of the domain of the integration. Another way to eliminate the non-local terms is to remove the cutoff. Since the non-local contributions represent surface terms they vanish as $\Lambda \to \infty$.

One may furthermore speculate that some of the non-local terms cancel between the consecutive steps of the blocking $k \to k - \Delta k$ for a suitable choice of the cutoff function $f(k)$ in the propagator $G_k^{-1}(p) = f(p/k)G_k^{-1}(p)$. Ignoring simply the non-local terms the identification of the coefficients of the different powers of the gradient in the two sides of (14) leads to

\[
\begin{align*}
    k\partial_k U_k(\phi_0) &= -\frac{\hbar\Omega_0 k^d}{2(2\pi)^d} \ln \left( \frac{Z_k(\phi_0)k^2 + U_k^{(2)}(\phi_0)}{Z_k(0)k^2 + U_k^{(2)}(0)} \right) \\
    k\partial_k Z_k(\phi_0) &= -\frac{\hbar\Omega_0 k^d}{2(2\pi)^d} \left( \frac{Z_k^{(2)}(\phi_0)}{Z_k(\phi_0)k^2 + U_k^{(2)}(\phi_0)} - 2Z_k^{(1)}(\phi_0) \frac{Z_k^{(1)}(\phi_0)k^2 + U_k^{(3)}(\phi_0)}{(Z_k(\phi_0)k^2 + U_k^{(2)}(\phi_0))^2} \right) \\
    &- \frac{k^2}{d} \frac{(Z_k^{(1)}(\phi_0))^2}{(Z_k(\phi_0)k^2 + U_k^{(2)}(\phi_0))^2} + \frac{4k^2}{d} Z_k^{(1)}(\phi_0) Z_k^{(1)}(\phi_0) \frac{Z_k^{(1)}(\phi_0)k^2 + U_k^{(3)}(\phi_0)}{(Z_k(\phi_0)k^2 + U_k^{(2)}(\phi_0))^3} \\
    &\quad + Z_k^{(1)}(\phi_0) \frac{(Z_k^{(1)}(\phi_0)k^2 + U_k^{(3)}(\phi_0))^2}{(Z_k(\phi_0)k^2 + U_k^{(2)}(\phi_0))^3} - \frac{4k^2}{d} Z_k^{(2)}(\phi_0) \frac{(Z_k^{(1)}(\phi_0)k^2 + U_k^{(3)}(\phi_0))^2}{(Z_k(\phi_0)k^2 + U_k^{(2)}(\phi_0))^4}.
\end{align*}
\]

When $k^2 \gg U_k^{(2)}(\phi)$ this gives (43) in dimension $d = 4$. 

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