UNIVERSAL PROPERTIES OF SOME QUIVERS

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Abstract. In this paper, I characterize four particular classes of directed multigraphs, or quivers, as images under left and right adjoints to the natural vertex and edge functors. In particular, the following notions coincide.

1. Introduction

The primary theme of this paper is to characterize some constructions in the category of directed multigraphs, or quivers, as adjoint functor operations. As stated by Saunders Mac Lane in [7, p. vii], “Adjoint functors arise everywhere”, and the category of quivers is no different.

Specifically, Proposition 3.4 (resp. 3.6) demonstrates that the standard independent set of vertices (resp. edges) is a reflection along the vertex (resp. edge) functor. This shows that these examples of quivers arise naturally from the structure of the category, possess a particular initial universal property, and “parallel” each other as intuitively understood in graph theory. “Parallel” is used rather than “dual” since the latter has the technical meaning in category theory of reversing arrows.

Similarly, Proposition 3.9 (resp. 3.12) codifies the standard complete digraph (resp. bouquet of loops) on a set as a coreflection along the vertex (resp. edge) functor. These two classes of quivers also arise naturally from the structure of the category and possess a terminal universal property. Given the “parallel” nature of independent sets of vertices and edges, this result seems to indicate that complete digraphs and bouquets could be considered as “graph-theoretic duals” of each other.

Further, one could say that independent sets of vertices (resp. edges) are “category-theoretic dual” to complete digraphs (resp. bouquets of loops), having dual universal properties along the vertex (resp. edge) functor. This
notion is supported by the fact that the independent set is the complement to a complete digraph.

In this paper, Section 2 covers the essential definitions of the objects and morphisms that will be considered. Section 3 quickly reviews the definitions of the vertex and edge functors before characterizing the left and right adjoint functors to each. Most definitions will be followed with indicative examples. This paper does assume familiarity with the notions of category, functor, natural transformation, and adjoint functor. Those unfamiliar with these topics might be interested in [1], [2], or [7].

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2. Definitions

To begin this discussion, let \textbf{Set} denote the category of sets with functions. Recall the following definitions and conventions.

\textbf{Definition} (Quiver, [3, Definition 2.1], [6, p. 83]). A \textit{directed multigraph} or \textit{quiver} is a quadruple \((V, E, \sigma, \tau)\), where \(V, E \in \text{Ob(}\textbf{Set}\text{)}\) are sets, and \(\sigma, \tau \in \text{Set}(E, V)\) are functions. Elements of \(V\) are \textit{vertices}, and \(V\) the \textit{vertex set}. Elements of \(E\) are \textit{edges}, and \(E\) the \textit{edge set}. The function \(\sigma\) is the \textit{source map}, and \(\tau\) the \textit{target map}. For \(e \in E\), \(\sigma(e)\) is the \textit{source} of \(e\), and \(\tau(e)\) the \textit{target} of \(e\).

The usual way of representing a quiver is by plotting points for the vertices, and for each edge, drawing a directed arrow from the source to the target. This is illustrated in the diagram below.

\textit{Example} 2.1. Let \(V := \{0, 1\}\), \(E := \{e, f, g\}\), and \(\sigma, \tau : E \to V\) by

\[
\begin{align*}
\sigma(t) := & \begin{cases} 0, & t = e, \\
0, & t = f, \\
0, & t = g,
\end{cases} & \text{and} & \tau(t) := & \begin{cases} 0, & t = e, \\
1, & t = f, \\
1, & t = g.
\end{cases}
\end{align*}
\]

Then, this can be represented with the following diagram.

\[
\begin{array}{c}
\bullet_0 \\
\bullet_1
\end{array} \quad \xrightarrow{f} \quad \xrightarrow{g}
\]

Observe that this quiver has two edges originating at 0 and terminating at 1, as well as a \textit{loop} at 0, an edge which has the same source and target vertex.

Similarly, given any diagram like the one above, one can write a quadruple \((V, E, \sigma, \tau)\) as in the definition above. Thus, it is common for quiver to be defined in terms of diagrams in this way, implicitly defining the component sets and functions. Notice also that the sets \(V\) and \(E\) are not assumed to be finite.

The remainder of this paper will often have several quivers considered at once. For now, the sets and functions of a quiver \(G\) will be denoted \(G = (V_G, E_G, \sigma_G, \tau_G)\) to distinguish its component parts from other quivers.
being considered. In Section 3 two functors will be introduced that allow a more natural operational notation.

With the objects defined, morphisms are defined to preserve this structure.

**Definition (Quiver map, [3, Definition 2.4]).** Given quivers $G$ and $H$, a directed multigraph homomorphism or quiver homomorphism from $G$ to $H$ is a pair $(\phi_V, \phi_E)$, where $\phi_V \in \text{Set}(V_G, V_H)$ and $\phi_E \in \text{Set}(E_G, E_H)$ satisfy $\phi_V \circ \sigma_G = \sigma_H \circ \phi_E$ and $\phi_V \circ \tau_G = \tau_H \circ \phi_E$. The function $\phi_V$ is the vertex map, and $\phi_E$ the edge map.

Diagrammatically, this definition can be shown with the following two commutative squares.

\[ \begin{array}{ccc}
E_G & \xrightarrow{\phi_E} & E_H \\
\sigma_G \downarrow & & \downarrow \sigma_H \\
V_G & \xrightarrow{\phi_V} & V_H
\end{array} \quad \begin{array}{ccc}
E_G & \xrightarrow{\phi_E} & E_H \\
\tau_G \downarrow & & \downarrow \tau_H \\
V_G & \xrightarrow{\phi_V} & V_H
\end{array} \]

**Example 2.2.** Consider the following quivers.

\[
\begin{array}{c|c}
G & H \\
\begin{array}{c}
\circ \underline{0} \circ \\
\circ \underline{f} \circ \\
\circ \underline{g} \circ
\end{array} & \\
\begin{array}{c}
\circ \underline{h} \circ \\
\circ \underline{i} \circ
\end{array}
\end{array}
\]

Define $\phi_V : V_G \to V_H$ by $\phi_V(t) := 2$ and $\phi_E : E_G \to E_H$ by

$$\phi_E(t) := \begin{cases} h, & t = e, \\ i, & t = f, g. \end{cases}$$

A quick check shows that the pair $(\phi_V, \phi_E)$ is a quiver map from $G$ to $H$.

**Example 2.3 (Identity maps, [3, Example 2.12, #2]).** Given any quiver $G$, consider the identity maps $\text{id}_{V_G} \in \text{Set}(V_G, V_G)$ and $\text{id}_{E_G} \in \text{Set}(E_G, E_G)$. Then, $\text{id}_G := (\text{id}_{V_G}, \text{id}_{E_G})$ is a quiver map from $G$ to itself.

Like a quiver, a quiver map will be denoted $\phi = (\phi_V, \phi_E)$ to distinguish the vertex and edge maps from one another. This notation too will be eased by the material in Section 3.

Quiver homomorphisms have a natural composition.

**Proposition 2.4 (Composition, [3 Fact 2.5]).** Let $G, H, K$ be quivers, $\phi$ a quiver map from $G$ to $H$, and $\psi$ a quiver map from $H$ to $K$. Then, $\psi \circ \phi := (\psi_V \circ \phi_V, \psi_E \circ \phi_E)$ is a quiver map from $G$ to $K$.

With this composition, the category of study is now defined.

**Definition (Quiver category, [3 Example 2.12, #2]).** Let $\textbf{Quiv}$ stand for the following data:

- $\text{Ob}(\textbf{Quiv}) :=$ the class of all quivers,
Proposition 2.5. \textbf{Quiv} as defined above is a category.

The proof of this proposition is routine and will be omitted. Notice that while the construction of \textbf{Quiv} was in terms of \textbf{Set}, \textbf{Set} can be replaced by an arbitrary category \textbf{C}. One could call these constructs “directed multi-graphs or quivers on \textbf{C}”. While these more general notions are interesting, the current paper will continue to focus on the category \textbf{Quiv} defined above.

3. \textbf{The Vertex and Edge Functors}

This section considers two natural functors from \textbf{Quiv} to \textbf{Set}. In so doing, four important classes of quivers arise as reflections and coreflections along each functor.

The first is the association of the vertex set and vertex maps.

\textbf{Definition} (The Vertex Functor, \cite{5} p. 106). Let \( V \) denote the following associations:

- given \( G \in \text{Ob(Quiv)} \), \( V(G) := V_G \), the vertex set of \( G \),
- given \( G \xrightarrow{\phi} H \in \text{Quiv} \), \( V(\phi) := \phi_V \), the vertex map of \( \phi \).

\textbf{Proposition 3.1.} The associations \( V \) defined above compose a functor from \textbf{Quiv} to \textbf{Set}.

The proof of this proposition is routine and will be omitted. The second holds the same duty for the edge associations. The proof of functoriality is also routine.

\textbf{Definition} (The Edge Functor). Let \( E \) denote the following associations:

- given \( G \in \text{Ob(Quiv)} \), \( E(G) := E_G \), the edge set of \( G \),
- given \( G \xrightarrow{\phi} H \in \text{Quiv} \), \( E(\phi) := \phi_E \), the edge map of \( \phi \).

\textbf{Proposition 3.2.} The associations \( E \) defined above compose a functor from \textbf{Quiv} to \textbf{Set}.

Another way of thinking of these functors is as “projections” of \textbf{Quiv} to one of its component parts, its “vertex” copy of \textbf{Set} and its “edge” copy of \textbf{Set}. However, each has more structure, as both right adjoint and left adjoint to well-known constructions of quivers.

Note that this treatment uses the reflection description of left adjoint.

\textbf{Definition} (Reflection, \cite{2} p. 97). Given a functor \( F : \mathcal{D} \to \mathcal{C} \) and \( C \in \text{Ob(\mathcal{C})} \), a \textit{reflection} of \( C \) along \( F \) is an object \( R \in \text{Ob(\mathcal{D})} \) equipped with a morphism \( \eta \in \mathcal{C}(C, FR) \) such that for any \( D \in \text{Ob(\mathcal{D})} \) and \( \phi \in \mathcal{D}(C, FD) \), there is a unique \( \hat{\phi} \in \mathcal{C}(R, D) \) such that \( F\hat{\phi} \circ \eta = \phi \).
Example 3.3 (The Discrete Space). Let $\textbf{Top}$ be the category of topological spaces with continuous functions. There is a natural forgetful functor $F : \textbf{Top} \to \textbf{Set}$, where the topology is disregarded. Given a set $S$, let $D_S$ denote $S$ equipped with the discrete topology. Let $\xi_S := \text{id}_S$ be the identity map, regarded as $\xi_S \in \textbf{Set}(S, F(D_S))$. Then, one can show that $D_S$ equipped with $\xi_S$ is a reflection of $S$ along $F$.

It is shown in \cite{2} Theorem 3.1.5 that a functor admits a reflection for every object in its codomain if and only if the functor has a left adjoint.

In the case of the functor $V$, every object of $\textbf{Set}$ admits an empty quiver. The definition is analogous to that of “empty graph” given in \cite{4} p. 20.

Definition. Given a set $S$, let $0_S : \emptyset \to S$ be the empty function to $S$. The independent set of vertices or empty quiver on $S$ is

$$I_S := (S, \emptyset, 0_S, 0_S),$$

the quiver with vertex set $S$ and no edges. Here, $I_S$ is equipped with $\eta_S := \text{id}_S$, regarded as $\eta_S \in \textbf{Set}(S, V(I_S))$.

Proposition 3.4 (Empty Quiver Characterization). For a set $S$, $I_S$ equipped with $\eta_S$ is a reflection of $S$ along $V$.

Proof. Let $G$ be a quiver and $S \xrightarrow{\phi} V(G) \in \textbf{Set}$. Then, the following diagrams exist in the categories $\textbf{Quiv}$ and $\textbf{Set}$.

\[
\begin{array}{cccc}
\text{Quiv} & \text{Set} \\
\hline
G & S \xrightarrow{\phi} V(G) \\
& \eta_S \cong \\
I_S & V(I_S)
\end{array}
\]

Since $\emptyset$ is initial in $\textbf{Set}$, the following diagrams commute in $\textbf{Set}$.

\[
\begin{array}{cccc}
\emptyset & \xrightarrow{0_{E(G)}} E(G) & \emptyset & \xrightarrow{0_{E(G)}} E(G) \\
0_S & \xrightarrow{\sigma_G} S & 0_S & \xrightarrow{\tau_G} S \\
S & \xrightarrow{\phi} V(G) & S & \xrightarrow{\phi} V(G)
\end{array}
\]
Thus, \( \hat{\phi} := (\phi, 0_{E(G)}) \) is a quiver map from \( I_S \) to \( G \). Further,
\[
V(\hat{\phi}) \circ \eta_S = \phi \circ id_S = \phi.
\]

If \( I_S \xrightarrow{\varphi} G \in \text{Quiv} \) satisfies \( V(\varphi) \circ \eta_S = \phi \), then
\[
\phi = V(\varphi) \circ id_S = V(\varphi).
\]

Also, \( E(\varphi) \in \text{Set}(\emptyset, E(G)) \), meaning \( E(\varphi) = 0_{E(G)} \). Hence, \( \varphi = \hat{\phi} \), proving uniqueness.

\( \square \)

**Corollary 3.5.** There is a unique functor \( I : \text{Set} \rightarrow \text{Quiv} \) satisfying the following:

- \( I(S) = I_S \),
- \( \eta := (\eta_S)_{S \in \text{Ob}(\text{Set})} \) is a natural transformation from \( id_{\text{Set}} \) to \( V \circ I \),
- \( I \dashv V \).

In fact, \( V \circ I = id_{\text{Set}} \).

Similarly for the functor \( E \), every set admits an independent set of edges.

**Definition.** Given a set \( S \) and \( j = 0, 1 \), let \( \iota_j : S \rightarrow \{0, 1\} \times S \) by \( \iota_j(s) := (j, s) \) be the usual inclusions. The independent set of edges on \( S \) is the quiver
\[
M_S := (\{0, 1\} \times S, S, \iota_0, \iota_1).
\]

Here, \( M_S \) is equipped with \( \theta_S := id_S \), regarded as \( \theta_S \in \text{Set}(S, E(M_S)) \).

**Proposition 3.6** (Independent Set of Edges Characterization). For a set \( S \), \( M_S \) equipped with \( \theta_S \) is a reflection of \( S \) along \( E \).

**Proof.** Let \( G \) be a quiver and \( S \xrightarrow{\phi} E(G) \in \text{Set} \). Then, the following diagrams exist in the categories \( \text{Quiv} \) and \( \text{Set} \).

\[
\begin{array}{ccc}
\text{Quiv} & \xrightarrow{\phi} & \text{Set} \\
G & \xrightarrow{\theta_S} & E(M_S) \\
M_S & \xrightarrow{= \text{iso}} & E(M_S)
\end{array}
\]

Define \( \phi_V : \{0, 1\} \times S \rightarrow V(G) \) by
\[
\phi_V(j, s) := \begin{cases} 
(\sigma_G \circ \phi)(s), j = 0, \\
(\tau_G \circ \phi)(s), j = 1.
\end{cases}
\]

For \( s \in S \), note that
\[
(\phi_V \circ \iota_0)(s) = \phi_V(0, s) = (\sigma_G \circ \phi)(s)
\]
and
\[
(\phi_V \circ \iota_1)(s) = \phi_V(1, s) = (\tau_G \circ \phi)(s).
\]
Thus, $\hat{\phi} := (\phi_V, \phi)$ is a quiver map from $M_S$ to $G$. Further,
\[
E(\hat{\phi}) \circ \theta_S = \phi \circ id_S = \phi.
\]

If $M_S \xrightarrow{\varphi} G \in \text{Quiv}$ satisfies $E(\varphi) \circ \theta_S = \phi$, then
\[
\varphi = E(\varphi) \circ id_S = E(\varphi).
\]

Also, $V(\varphi) \circ \iota_0 = \sigma_G \circ E(\varphi) = \sigma_G \circ \phi$ and $V(\varphi) \circ \iota_1 = \tau_G \circ E(\varphi) = \tau_G \circ \phi$.

For $s \in S$,
\[
V(\varphi)(0, s) = (V(\varphi) \circ \iota_0)(s) = (\sigma_G \circ \phi)(s)
\]
and
\[
V(\varphi)(1, s) = (V(\varphi) \circ \iota_1)(s) = (\tau_G \circ \phi)(s).
\]
Therefore, $V(\varphi) = \phi_V$, meaning $\varphi = \hat{\phi}$, proving uniqueness.

\[\square\]

**Corollary 3.7.** There is a unique functor $M : \text{Set} \to \text{Quiv}$ satisfying the following:

- $M(S) = M_S$,
- $\theta := (\theta_S)_{S \in \text{Ob(Set)}}$ is a natural transformation from $id_{\text{Set}}$ to $E \circ M$,
- $M \dashv E$.

In fact, $E \circ M = id_{\text{Set}}$.

Thus, the functors $V$ and $E$ are right adjoints to $I$ and $M$, respectively. However, they are also left adjoints, admitting coreflections. Recall that a coreflection is precisely the dual notion to a reflection, obtained by reversing all morphisms in the definition.

For the functor $V$, every object of $\text{Set}$ admits a full quiver.

**Definition** ([4, p. 20]). Given a set $S$ and $j = 0, 1$, let $\pi_j : S^2 \to S$ by $\pi_1(s, t) := s$ and $\pi_2(s, t) := t$ be the usual projections. The (directed) complete graph or full quiver on $S$ is the quiver
\[
K_S := (S, S^2, \pi_1, \pi_2).
\]

Here, $K_S$ is equipped with $\zeta_S := id_S$, regarded as $\zeta_S \in \text{Set} \left( V(K_S), S \right)$.

**Example 3.8.** Consider the set $S := \{0, 1\}$. Then, $K_S$ is the quiver drawn below.

\[
\begin{array}{c}
(0,0) \bigcirc 0 \bigcirc 1 \bigcirc (1,1) \\
\end{array}
\]

**Proposition 3.9** (Full Quiver Characterization). For a set $S$, $K_S$ equipped with $\zeta_S$ is a coreflection of $S$ along $V$. 

Proof. Let \( G \) be a quiver and \( V(G) \xrightarrow{\phi} S \in \text{Set} \). Then, the following diagrams exist in the categories \( \text{Quiv} \) and \( \text{Set} \).

\[
\begin{array}{c|c}
\text{Quiv} & \text{Set} \\
\hline
G & S \\
& \xrightarrow{\phi} V(G) \\
& \zeta_S \cong \\
K_S & V(K_S)
\end{array}
\]

Define \( \phi_E : E(G) \to S^2 \) by

\[
\phi_E(e) := ((\phi \circ \sigma_G)(e), (\phi \circ \tau_G)(e)).
\]

For \( e \in E(G) \),

\[
(\pi_1 \circ \phi_E)(e) = \pi_1((\phi \circ \sigma_G)(e), (\phi \circ \tau_G)(e)) = (\phi \circ \sigma_G)(e)
\]

and

\[
(\pi_2 \circ \phi_E)(e) = \pi_2((\phi \circ \sigma_G)(e), (\phi \circ \tau_G)(e)) = (\phi \circ \tau_G)(e).
\]

Thus, \( \hat{\phi} := (\phi, \phi_E) \) is a quiver map from \( G \) to \( K_S \). Further,

\[
\zeta_S \circ V(\hat{\phi}) = \text{id}_S \circ \phi = \phi.
\]

If \( G \xrightarrow{\varphi} K_S \in \text{Quiv} \) satisfies \( \zeta_S \circ V(\varphi) = \phi \), then

\[
\phi = \text{id}_S \circ V(\varphi) = V(\varphi).
\]

Also, \( \pi_1 \circ E(\varphi) = V(\varphi) \circ \sigma_G = \phi \circ \sigma_G \) and \( \pi_2 \circ E(\varphi) = V(\varphi) \circ \tau_G = \phi \circ \tau_G \).

For \( e \in E(G) \),

\[
E(\varphi)(e) = ((\pi_1 \circ E(\varphi))(e), (\pi_2 \circ E(\varphi))(e)) = ((\phi \circ \sigma_G)(e), (\phi \circ \tau_G)(e)).
\]

Therefore, \( E(\varphi) = \phi_E \), meaning \( \varphi = \hat{\phi} \), proving uniqueness. \( \square \)

**Corollary 3.10.** There is a unique functor \( K : \text{Set} \to \text{Quiv} \) satisfying the following:

- \( K(S) = K_S \),
- \( \zeta := (\zeta_S)_{S \in \text{Ob}(\text{Set})} \) is a natural transformation from \( V \circ K \) to \( \text{id}_{\text{Set}} \),
- \( V \dashv K \).

In fact, \( V \circ K = \text{id}_{\text{Set}} \).

Likewise for the functor \( E \), every set admits a bouquet. The definition is analogous to that of “bouquet” given in [4, p. 20].

**Definition.** Given a set \( S \), let \( 1 := \{1\} \) and \( 1_S : S \to 1 \) be the constant function from \( S \). The (directed) bouquet on \( S \) is

\[
B_S := (1, S, 1_S, 1_S),
\]

the quiver with edge set \( S \) and one vertex. Here, \( B_S \) is equipped with \( \epsilon_S := \text{id}_S \), regarded as \( \epsilon_S \in \text{Set}(E(B_S), S) \).
Example 3.11. Consider the set $S = \{e, f, g, h\}$. Then, $B_S$ is the quiver drawn below.

![Quiver diagram]

Proposition 3.12 (Bouquet Characterization). For a set $S$, $B_S$ equipped with $\epsilon_S$ is a coreflection of $S$ along $E$.

Proof. Let $G$ be a quiver and $E(G) \xrightarrow{\phi} S \in \textbf{Set}$. Then, the following diagrams exist in the categories $\textbf{Quiv}$ and $\textbf{Set}$.

\[
\begin{array}{ccc}
\text{Quiv} & \to & \text{Set} \\
G & \xrightarrow{\phi} & E(G) \\
B_S & \xleftarrow{\epsilon_S} & E(B_S) \\
\end{array}
\]

Since $\mathbb{1}$ is terminal in $\textbf{Set}$, the following diagrams commute in $\textbf{Set}$.

\[
\begin{array}{ccc}
E(G) & \xrightarrow{\phi} & S \\
\sigma_G & \downarrow & \downarrow \tau_G \\
V(G)_{1_{V(G)}} & \xrightarrow{1_S} & V(G)_{1_{V(G)}} & \xrightarrow{1_S} \\
\end{array}
\]

Thus, $\hat{\phi} := (1_{V(G)}, \phi)$ is a quiver map from $G$ to $B_S$. Further,

$$\epsilon_S \circ E(\hat{\phi}) = \text{id}_S \circ \phi = \phi.$$ 

If $G \xrightarrow{\varphi} B_S \in \textbf{Quiv}$ satisfies $\epsilon_S \circ E(\varphi) = \phi$, then

$$\phi = \text{id}_S \circ E(\varphi) = E(\varphi).$$

Also, $V(\varphi) \in \textbf{Set}(V(G), \mathbb{1})$, meaning $V(\varphi) = 1_{V(G)}$. Hence, $\varphi = \hat{\phi}$, proving uniqueness. 

\[\square\]

Corollary 3.13. There is a unique functor $B : \textbf{Set} \to \textbf{Quiv}$ satisfying the following:

- $B(S) = B_S$,
- $\epsilon := (\epsilon_S)_{S \in \text{Ob}(\textbf{Set})}$ is a natural transformation from $E \circ B$ to $\text{id}_\textbf{Set}$,
- $E \dashv B$.

In fact, $E \circ B = \text{id}_\textbf{Set}$. 

These adjoint characterizations above show that the ideas of “independent set of vertices”, “independent set of edges”, “complete graph”, and “bouquet” arise naturally from the categorical structure of $\mathbf{Quiv}$. This reinforces that all these classes of quivers are fundamental to graph theory.

References

[1] Jiří Adámek, Horst Herrlich, and George E. Strecker. Abstract and concrete categories: the joy of cats. Repr. Theory Appl. Categ., (17):1–507, 2006. Reprint of the 1990 original [Wiley, New York; MR1051419].

[2] Francis Borceux. Handbook of categorical algebra. 1, volume 50 of Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 1994. Basic category theory.

[3] H. Ehrig, K. Ehrig, U. Prange, and G. Taentzer. Fundamentals of algebraic graph transformation. Monographs in Theoretical Computer Science. An EATCS Series. Springer-Verlag, Berlin, 2006.

[4] Jonathan L. Gross and Jay Yellen, editors. Handbook of graph theory. Discrete Mathematics and its Applications (Boca Raton). CRC Press, Boca Raton, FL, 2004.

[5] Frederick Hoffman, Ronald C. Mullin, Ralph G. Stanton, and K. Brooks Reid, editors. Proceedings of the sixteenth Southeastern international conference on combinatorics, graph theory and computing, Winnipeg, MB, 1985. Utilitas Mathematica Publishing Inc.

[6] Bernhard Keller. Cluster algebras, quiver representations and triangulated categories. In Triangulated categories, volume 375 of London Math. Soc. Lecture Note Ser., pages 76–160. Cambridge Univ. Press, Cambridge, 2010.

[7] Saunders Mac Lane. Categories for the working mathematician, volume 5 of Graduate Texts in Mathematics. Springer-Verlag, New York, second edition, 1998.