SURFACE SINGULARITIES DOMINATED BY SMOOTH VARIETIES

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ABSTRACT. We give a version in characteristic $p > 0$ of Mumford’s theorem characterizing a smooth complex germ of surface $(X, x)$ by the triviality of the topological fundamental group of $U = X \setminus \{x\}$.

1. Introduction

Let $(X, x)$ be a 2-dimensional normal complex analytic germ. Let $U = X \setminus \{x\}$. Mumford [12] showed the celebrated theorem

Theorem 1.1 (Mumford). $(X, x)$ is smooth if and only if the topological fundamental group of $U$ is trivial.

This is a remarkable theorem which connects a topological notion to a scheme-theoretic one. His theorem has been a bit refined by Flenner [7] who showed that in fact, the conclusion remains true if one replaces the topological by the étale fundamental group of $U$, that is by its profinite completion. Then one can replace the analytic germ by a complete or henselian germ over an algebraically closed field $k$ of characteristic 0.

If $k$ is an algebraically closed field $k$ of characteristic $p > 0$, Mumford himself observed that the theorem is no longer true. As an example, while in characteristic 0, the singularity $z^2 + xy$ is the quotient of $\hat{A}^2$, the completion of $A^2$ at the origin, by the group $\mathbb{Z}/2$ acting via diag($-1, -1$), in characteristic 2, it is the quotient of $\hat{A}^2$ by $\mu_2 = \text{Spec} k[t]/(t^2 - 1)$ acting via diag$(t, t)$. Thus $\pi^\text{et}(U) = \pi^\text{et}(\hat{A}^2 \setminus \{0\}) = 0$, yet $z^2 + xy$ is not smooth.

Artin asked in [3] whether, if $\pi^\text{et}(U)$ is finite, there is always a finite morphism $\hat{A}^2 \to X$. He shows this if $(X, x)$ is a rational double point loc.cit..

The purpose of this note is to give an answer to a similar question where one replaces the étale fundamental group by the Nori one. Strictly speaking, Nori in [13, Chapter II] defined his fundamental group-scheme for irreducible reduced schemes endowed with a rational point. But as $U$ has no rational point, one has to modify a tiny bit Nori’s construction to make it work. This is done in subsection 2.2. While the étale fundamental group of $X$ is trivial, Nori’s one
isn’t. So the right notion of Nori’s fundamental group is a relative one denoted by $\pi_{\text{loc}}(U, X, x)$ (see Lemma 2.25). Roughly speaking, it measures the torsors on $U$ under a finite flat $k$-group-scheme $G$ which do not come by restriction from a torsor on $X$. We show (Theorem 2.4) that if $\pi_{\text{loc}}^N(U, X, x)$ is finite, then $(X, x)$ is a rational singularity, and if $\pi_{\text{loc}}^N(U, X, x) = 0$, then there is a finite morphism $f : \hat{K}^2 \to X$.

This note relies on discussions the authors had during the Christmas break 2009/10 in Ivry. They have been written down by Hélène in the night when Eckart died, as a despaired sign of love.

2. Local Nori Fundamental Groupscheme

2.1. Nori’s construction. Let $U$ be a scheme defined over a field $k$, endowed with a rational point $u \in U(k)$. In [13, Chapter II] Nori constructed the fundamental group-scheme $\pi^N(U, u)$. Let $C(U, u)$ be the following category. The objects are triples $(h : V \to U, G, v)$ where $G$ is a finite $k$-group-scheme, $h$ is a $G$-principal bundle and $v \in V(k)$ with $h(v) = u$. Recall [13, Chapter I, 2.2] that a $G$-principal bundle $h : V \to U$ is a flat morphism, together with a group action $G \times_k V \to V$ such that $V \times_k G \xrightarrow{(1, h)} V \times_U V$ is an isomorphism. Then $\text{Hom}((h_1 : V_1 \to U, G_1, v_1), (h_2 : V_2 \to U, G_2, v_2))$ consists of the $U$-morphisms $f : V_1 \to V_2$ which are compatible with the principal bundle structure.

The objects of the ind-category $C^{\text{ind}}(U, u)$ associated to $C(U, u)$ are triples $(h : V \to U, G, v)$ where $G = \lim_{\alpha} G_\alpha$ is a prosystem of finite $k$-group-schemes $G_\alpha$, $h = \lim_{\leftarrow \alpha} h_\alpha, h_\alpha : V_\alpha \to U$, is a pro-$G$-principal bundle and $v = \lim_{\leftarrow \alpha} v_\alpha \in Y(k)$ is a pro-point with $h(v) = u$. The morphisms are the ind-morphisms $V_1 \to V_2$ over $U$ which are compatible with the principal bundle structure and such that $f(v_1) = v_2$.

Then $(U, u)$ has a fundamental group-scheme $\pi^N(U, u)$, which is then a $k$-profinite group-scheme, if by definition [13, Chapter II, Definition 1] there is a $(h : W \to U, \pi^N(U, u), w) \in C^{\text{ind}}(U, u)$ with the property that for any $(h : V \to U, G, v) \in C^{\text{ind}}(U, u)$, there is a unique map $(h : W \to U, \pi^N(U, u), w) \to (h : V \to U, G, v)$ in $C^{\text{ind}}(U, u)$.

Nori shows [13, Chapter II, Lemma 1] that if $G_1, G_2, G_0$ are three finite $k$-group-schemes, $h_i : V_i \to U$ are $G_i$-principal bundles, and $f_i : V_i \to V_0, i = 1, 2$ are principal bundle $U$-morphisms, then $V_1 \times_{V_0} V_2 \to Z$ is a principal bundle under $G_1 \times_{G_0} G_2$, where $Z \subset U$ is a closed subscheme (no reference to the base point here). Then he shows that $(U, u)$ has a fundamental group-scheme if and only if $Z = U$ for all $(h_i : V_i \to U, G_i, y_i), f_i \in C(U, u)$ and he concludes [13, Chapter II, Proposition 2] that if $U$ is reduced and irreducible, then $(U, u)$ has a fundamental group-scheme.

2.2. Local Nori fundamental group-scheme. Let $k$ be a field, let $A$ be a complete normal local $k$-algebra with maximal ideal $m$ and residue field $k$. We
define $X = \text{Spec} \ A$ and $U = X \setminus \{x\}$, where $x \in X(k)$ is the rational point associated to $m$. So in particular, $U(k) = \emptyset$, and we have to slightly modify Nori’s construction to define the group-scheme of $U$.

Let $G$ be a finite $k$-group-scheme, and let $h : V \to U$ be a $G$-principal bundle. Recall from [15, Corollaire 6.3.2, Proposition 6.3.4] that the integral closure $\tilde{h} : Y \to X$ of $h$ is the unique extension $\tilde{h} : Y \to X$ of $h$ such that $Y = \text{Spec} \ B$, $B$ is the integral closure of $A$ in $j_*h_*\mathcal{O}_V$, where $j : U \to X$ is the open embedding. Then $\tilde{h}$ is finite. In particular, if $h_i : V_i \to U$ are principal bundles under the finite $k$-group-schemes $G_i$, and $f : V_1 \to V_2$ is a $U$-morphism which respects the principal bundle structures, then it extends uniquely to a $X$-morphism $\tilde{f} : Y_1 \to Y_2$, which is then finite. We can now mimic Nori’s construction.

**Definition 2.1.** The objects of the category $\mathcal{C}_{\text{loc}}(U,x)$ are triples $(h : V \to U, G, y)$ where $G$ is a finite $k$-group-scheme, $y \in Y(k)$ with $\tilde{h}(y) = x$, where $\tilde{h} : Y \to X$ is the integral closure of $h$. The morphisms $\text{Hom}((h_1 : V_1 \to U, G_1, y_1) \to (h_2 : V_2 \to U, G_2, y_2))$ consist of $U$-morphisms $f : V_1 \to V_2$ which respect the principal bundle structure and such that $\tilde{f}(y_1) = y_2$.

The objects of the ind-category $\mathcal{C}_{\text{ind}}^{\text{loc}}(U,x)$ associated to $\mathcal{C}_{\text{loc}}(U,x)$ are triples $(h : V \to U, G, y)$ where $G = \lim_{\leftarrow \alpha} G_\alpha$ is a pro-system of finite $k$-group-schemes, $h = \lim_{\leftarrow \alpha} h_\alpha, h_\alpha : V_\alpha \to U$, is a pro-$G$-principal bundle, and $y = \lim_{\leftarrow \alpha} y_\alpha \in \lim_{\leftarrow \alpha} Y_\alpha(k)$ is a pro-point in the integral closure of $V_\alpha$ mapping to $x$.

One says that $(U,x)$ has a local fundamental group-scheme $\pi^N_{\text{loc}}(U,x)$, which is then a $k$-profinite group-scheme, if there is a $(\mathfrak{h} : W \to U, \pi^N_{\text{loc}}(U,x), z) \in \mathcal{C}_{\text{loc}}^{\text{ind}}(U,x)$ with the property that for any $(h : V \to U, G, y) \in \mathcal{C}_{\text{loc}}^{\text{ind}}(U,x)$, there is a unique map $(\mathfrak{h} : W \to U, \pi^N_{\text{loc}}(U,x), z) \to (h : V \to U, G, y)$ in $\mathcal{C}_{\text{loc}}^{\text{ind}}(U,x)$.

**Proposition 2.2.** If $X$ is reduced and irreducible, then $(U,x)$ has a local fundamental group-scheme $\pi^N_{\text{loc}}(U,x)$.

**Proof.** As explained above, the condition on $X$ implies that if $f_i : (h_i : V_i \to U, G_i, y_i) \to (h_0 : V_0 \to U, G_0, y_0)$ is a morphism in $\mathcal{C}_{\text{loc}}(U,x)$, then $(V_1 \times_{V_0} V_2 \to U, G_1 \times_{G_0} G_2, y_1 \times_{y_0} y_2) \in \mathcal{C}_{\text{loc}}(U,x)$, so as in [15, Chapter II,p.87], the prosystem $\lim_{\leftarrow \alpha} (h_\alpha : V_\alpha \to U, G_\alpha, y_\alpha)$ over all objects $(h_\alpha : V_\alpha \to U, G_\alpha, y_\alpha)$ of $\mathcal{C}_{\text{loc}}(U,x)$ is well defined. So $\pi^N_{\text{loc}}(U,x) = \lim_{\leftarrow \alpha} G_\alpha$. □

There is a restriction functor $\rho : \mathcal{C}(X,x) \to \mathcal{C}_{\text{loc}}(U,x)$ which sends $(h : Y \to X, G, y)$ to its restriction $(h_U : Y \times_X U \to U, G, y)$, as the integral closure of $X$ in $Y \times_X U$ is $Y$. This defines the $k$-group-scheme homomorphism

$$\rho_* : \pi^N_{\text{loc}}(U,x) \to \pi^N(X,x).$$

**Proposition 2.3.** The homomorphism $\rho$ is faithfully flat.

**Proof.** Faithful flatness of $\rho$ means that if $(h : Y \to X, G, y) \in \mathcal{C}(X,x)$ is such that $(Y_U \to, G, y) \to (U, \{1\}, x)$ factors through $(\ell : V \to U, H, y) \in \mathcal{C}_{\text{loc}}(U,x)$, where
Y_U = Y \times_X U$, then necessarily $(\ell : V \to U, H, y) = \rho(\ell_X : Z \to X, H, y)$ for some $(\ell_X : Z \to X, H, y) \in \mathcal{C}(X, x)$. Let $K = \text{Ker}(G \to H)$. Since $K$ is a $k$-subgroup-scheme of $G$, it acts on $Y$. We define $Z$ to be $Y/K$. By definition, $Z_U = V$. The compositum $h : Y \to Z \to X$ is a $G$-torsor. The embedding $Y \times_Z Y \subset Y \times_X Y$ is closed, and while restricted to $U$, it is described as $Y_U \times_k K \subset Y_U \times_k G$. Thus $Y \times_Z Y$ contains the closure of $Y_U \times_k K$ in $Y \times_k G$, that is $Y \times_k K$. Thus $Y \times_k K$ consists of connected components of $Y \times_Z Y$ and moreover, if there is another connected component, it lies in $\{y\} \times_Z Y = \text{Spec } k$. Thus $Y \times_Z Y \cong_k Y \times_k K$ and $Y \to Z$ is a $K$-torsor. This finishes the proof.

We denote by $\pi^\text{ab}(U, x)$ the étale proquotient of $\pi^\text{et}(U, x)$. From now on, we assume $k = \bar{k}$. Then $\pi^\text{et}(U, x)$ is identified with $\pi^\text{et}(U, \eta)$ where $\eta \to U$ is a geometric generic point and $\pi^\text{et}(U, \eta)$ is Grothendieck’s étale fundamental group. The étale proquotient of $\pi^N(X, x)$ is identified with Grothendieck’s fundamental group based at $x$, and is trivial by Hensel’s lemma, as $A$ is complete. If $\ell$ is a prime number (including $p$), we denote by $\pi^\text{et,ab,}\ell(U, x)$ the maximal pro-$\ell$-abelian quotient of $\pi^\text{et}(U, x)$.

**Definition 2.4.** One defines $\pi^\text{et,loc}(U, x) = \text{Ker}(\pi^\text{et}(U, x) \xrightarrow{\ell} \pi^N(X, x))$.

From the discussion, we see

**Lemma 2.5.** The compositum $\pi^\text{et,loc}(U, x) \to \pi^\text{et}(U, x)$ is surjective. In particular, if $\pi^\text{et,loc}(U, x)$ is a finite $k$-group-scheme, $\pi^\text{et}(U, x)$ is a finite group.

3. **Construction and Elementary Properties of the Picard Scheme for Surface Singularities**

Let $k$ be a field, perfect if of characteristic $p > 0$, let $A$ be a complete normal local $k$-algebra with maximal ideal $m$, $X = \text{Spec } A$ and $U = X \setminus \{x\}$, where $x \in X(k)$ is the rational point associated to $m$. In [16] Exposé XIII,Section 5 Grothendieck initiated the construction of a pro-system of locally algebraic $k$-group-schemes $G_n$ and a canonical isomomorphism $G(k) = \text{Pic}(U)$ with $G(k) = \varprojlim_n G_n(k)$. This construction is performed in [11] (see overview in [9] p. 273) and relies on Mumford’s basic idea [12] Section 2 to use a desingularization of $X$, if it exists, so in characteristic 0 or if $\dim_k X \leq 2$ if $k$ has characteristic $p > 0$. We now summarize the construction and the elementary properties under the assumptions

1) $X$ is normal
2) $\dim_k X = 2$.

Let $\sigma : \tilde{X} \to X$ be a desingularization such that $\sigma^{-1}(x)_{\text{red}} = \bigcup_i D_i$ is a strict normal crossings divisor and all components $D_i$ are $k$-rational. There is linear combination $D = \sum_i m_i D_i$ with all $m_i \geq 1$ such that $\mathcal{O}_X(-D)$ is relatively ample. We define $\tilde{X}_n$ to be scheme $\bigcup_i D_i$ with structure sheaf $\mathcal{O}_{\tilde{X}}/\mathcal{O}_{\tilde{X}}(-(n+1)D)$, so
\( \tilde{X}_0 = D \), and we also define \( D_{\text{red}} \) with structure sheaf \( \mathcal{O}_{\tilde{X}}/\mathcal{O}_{\tilde{X}}(-\sum_i D_i) \). Then the functors \( \mathcal{P}ic(\tilde{X}_n/k) \) and \( \mathcal{P}ic(D_{\text{red}}/k) \), taken as a Zariski, an étale or a fppf functor, are representable by locally algebraic \( k \)-group-schemes \( \mathcal{P}ic(\tilde{X}_n/k) \) and \( \mathcal{P}ic(D_{\text{red}}/k) \), so \( \mathcal{P}ic(\tilde{X}_n) = \mathcal{P}ic(\tilde{X}_n/k)(k) \), \( \mathcal{P}ic(D_{\text{red}}) = \mathcal{P}ic(D_{\text{red}}/k)(k) \) (see [9 p. 273], [11, Theorem 1.2]). On the other hand, for all \( n \geq 0 \), and all \( k \)-algebras \( R \), one has \( \mathcal{P}ic(\tilde{X}_n \otimes_k R) = H^1(\tilde{X}_n \otimes_k R, \mathcal{O}^*) \). As the relative dimension of \( \sigma \) is 1, this implies that the transition homomorphisms \( \mathcal{P}ic(\tilde{X}_{n+1}) \to \mathcal{P}ic(\tilde{X}_n) \to \mathcal{P}ic(\tilde{X}_0) \to \mathcal{P}ic(D_{\text{red}}) \) are all surjective, and that \( \text{Ker}(\mathcal{P}ic(\tilde{X}_{n+1}) \to \mathcal{P}ic(\tilde{X}_n)) = H^1(\tilde{X}_0, \mathcal{O}_{\tilde{X}_0}(-(n+1)D)) \). Since \(-D\) is a relatively ample divisor on \( \tilde{X} \), there is a \( n_0 \geq 0 \) such that the transition homomorphisms \( \mathcal{P}ic(\tilde{X}_n) \to \mathcal{P}ic(\tilde{X}_{n_0}) \) are all constant for \( n \geq n_0 \). Since the 1-component \( \mathcal{P}ic^0(D_{\text{red}}) \) of \( \mathcal{P}ic(D_{\text{red}}) \) is a semi-abelian variety, so in particular smooth, and the fibers \( \mathcal{P}ic(\tilde{X}_n) \to \mathcal{P}ic(D_{\text{red}}) \) are affine [14 p. 9, Corollaire], \( \mathcal{P}ic(\tilde{X}_{n_0}) \) is smooth. One defines

\[
(3.1) \quad \mathcal{P}ic(\tilde{X}) = \mathcal{P}ic(\tilde{X}_{n_0}).
\]

It is thus a locally algebraic smooth \( k \)-group-scheme. It is an extension of \( \oplus_i \mathbb{Z}[D_i] \) by its 1-component. Its 1-component \( \mathcal{P}ic^0(\tilde{X}) \subset \mathcal{P}ic(\tilde{X}) \) is an extension of a semi-abelian variety by smooth, connected commutative unipotent algebraic group over \( k \).

Let \( \langle D \rangle \subset \mathcal{P}ic(\tilde{X}) \) be the subgroup-scheme spanned by those divisors with support in \( D \). (In fact, \( \langle D \rangle \) injects into \( \mathcal{P}ic(D_{\text{red}}) \) via the surjection \( \mathcal{P}ic(\tilde{X}) \to \mathcal{P}ic(D_{\text{red}}) \)). It is a discrete subgroup-scheme. One sets

\[
(3.2) \quad \mathcal{P}ic(U) = \mathcal{P}ic(\tilde{X})/\langle D \rangle.
\]

The Zariski tangent space at 1 is

\[
(3.3) \quad H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) = H^1(\tilde{X}_n, \mathcal{O}_{\tilde{X}_n}) = \text{Ker}(\mathcal{P}ic(\tilde{X}_n[\epsilon]) \to \mathcal{P}ic(\tilde{X}_n))
\]

for \( n \geq n_0 \), where \( \tilde{X}_n[\epsilon] := \tilde{X}_n \times_k k[\epsilon]/(\epsilon^2) \). Since \( \mathcal{P}ic(\tilde{X}) \) is smooth,

\[
(3.4) \quad \dim_k H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) = \dim \mathcal{P}ic^0(\tilde{X}) = \mathcal{P}ic^0(U).
\]

The last equality comes from the fact that \( \langle D \rangle \subset \mathcal{P}ic(\tilde{X}) \) is a discrete étale subgroup.

Recall that the surface singularity \((X, x)\) is said to be rational if \( H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) = 0 \). The definition does not depend on the choice of the resolution \( \sigma : \tilde{X} \to X \) of singularities of \((X, x)\).

One has

**Lemma 3.1.** The following conditions are equivalent.

1. The surface singularity \((X, x)\) is rational.
2. \( \mathcal{P}ic^0(\tilde{X}) = 0 \).
3. \( \mathcal{P}ic(U) \) is finite.
Proof. The equivalence of 1) and 2) is given by \((3.3)\). As \(\langle D \rangle \subset \text{Pic}(\tilde{X})\) is discrete, the definition \((3.2)\) shows that 3) implies 2). Vice-versa, assume 2) holds. Then \(\text{Pic}(\tilde{X})\) is a discrete group of finite type. Let \(L \in \text{Pic}(\tilde{X})\). Since the intersection matrix \((D_i \cdot D_j)\) is negative definite (but not necessarily unimodular), there is a \(m \in \mathbb{N} \setminus \{0\}\) such that \(L^{\circ m} \in \langle D \rangle \subset \text{Pic}(\tilde{X})\). Thus any \(L \in \text{Pic}(\tilde{X})\) has finite order in \(\text{Pic}(U)\). Since \(\text{Pic}(\tilde{X})\) is of finite type, this shows 3).

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4. The Theorems

Throughout this section, we assume \(k\) to be a field, perfect if of characteristic \(p > 0\), \(A\) to be a complete normal local \(k\)-algebra with maximal ideal \(m\), of Krull dimension 2 over \(k\). We set \(X = \text{Spec} A\), \(U = X \setminus \{x\}\), where \(x \in X(k)\) is the rational point associated to \(m\). We say \((X, x)\) is a surface singularity over \(k\).

We denote by \(\sigma : \tilde{X} \to X\) a desingularization such that \(\sigma^{-1}(x)_{\text{red}} = \cup_i D_i\) is a strict normal crossings divisor. We define \(H^i(Z, \mathbb{Z}_\ell(1)) := \lim H^i(Z, \mu_{\ell^n})\) for a \(k\)-scheme \(Z\).

**Theorem 4.1.** Let \((X, x)\) be a surface singularity over an algebraically closed field \(k\). The following conditions are equivalent

1) \(H^1(\tilde{X}, \mathbb{Z}_\ell(1)) = 0\).
2) \(H^1(\tilde{U}, \mathbb{Z}_\ell(1)) = 0\).
3) There is a prime number \(\ell\), different from \(p\) if \(\text{char}(k) = p > 0\), such that \(\pi_{\text{et}, \text{ab}, \ell}(U, x)\) is finite.
4) For all prime numbers \(\ell\), \(\pi_{\text{et}, \text{ab}, \ell}(U, x)\) is finite and if \(\text{char}(k) = p > 0\), then \(\pi_{\text{et}, \text{ab}, \ell}(U, x) = 0\).
5) \(\text{Pic}^0(\tilde{X}) = \text{Pic}^0(U)\) is a smooth, connected commutative unipotent algebraic group-scheme over \(k\).
6) \(D\) is a tree of \(\mathbb{P}^1\)s.
7) \(\text{Pic}^0(D_{\text{red}}) = 0\).

**Proof.** We first make general remarks. For any surface singularity, one has the localization sequence

\[
(4.1) \quad H^1(\tilde{X}, \mathbb{Z}_\ell(1)) \to H^1(U, \mathbb{Z}_\ell(1)) \to H^2_{\text{red}}(\tilde{X}, \mathbb{Z}_\ell(1)) \to H^2(\tilde{X}, \mathbb{Z}_\ell(1)) \to H^3(U, \mathbb{Z}_\ell(1)) \to H^3_{\text{red}}(\tilde{X}, \mathbb{Z}_\ell(1)) \to H^3(\tilde{X}, \mathbb{Z}_\ell(1)).
\]

By purity \([8, \text{Theorem 2.1.1}]\), the restriction map \(H^1(\tilde{X}, \mathbb{Z}_\ell(1)) \to H^1(U, \mathbb{Z}_\ell(1))\) is injective, and \(H^2_{\text{red}}(\tilde{X}, \mathbb{Z}_\ell(1)) \cong \oplus_i \mathbb{Z}_\ell \cdot [D_i]\). By base change, \(H^i(\tilde{X}, \mathbb{Z}_\ell(1)) = H^i(D_{\text{red}}, \mathbb{Z}_\ell(1))\). Thus this group is 0 for \(i \geq 3\), equal to \(\oplus_i \mathbb{Z}_\ell \cdot [D_i]\) for \(i = 2\), and equal to \(\text{Pic}(D_{\text{red}})[\ell]\) for \(i = 1\). In fact, since \(H^2(D_{\text{red}}, \mathbb{Z}_\ell(1))\) is torsion free, one has \(\text{Pic}(D_{\text{red}})[\ell] = \text{Pic}^0(D_{\text{red}})[\ell]\), where \(\mathbb{Z}_\ell\) means of degree 0 on each component \(D_i\). Furthermore, by definition, the map \(\oplus_i \mathbb{Z}_\ell \cdot [D_i] \to \oplus_i \mathbb{Z}_\ell \cdot [D_i]\) is defined by \([D_i] \mapsto \oplus_j \deg \mathcal{O}_{D_j}(D_i)\). Since the intersection matrix is definite, the map is injective,
with finite torsion cokernel $\mathcal{T}$. (This cokernel is 0 if and only if the intersection matrix is unimodular). Again by purity, $H^3_{D_{\text{red}}}(\tilde{X}, \mathbb{Z}_\ell(1)) \subset \oplus_i H^1(D_i, \mathbb{Z}_\ell)$ where $D_i^0 = D_i \setminus \bigcup_{j \neq i} D_i \cap D_j$. In particular, $H^3_{D_{\text{red}}}(\tilde{X}, \mathbb{Z}_\ell(1))$ is torsion free. So we extract from \eqref{eq:1} for any surface singularity the relations

$$H^1(\tilde{X}, \mathbb{Z}_\ell(1)) \to H^1(U, \mathbb{Z}_\ell(1)) = \text{Pic}(D_{\text{red}})[\ell] = \text{Pic}^0(D_{\text{red}})[\ell]$$

and an exact sequence

$$0 \to \mathcal{T} \to H^2(U, \mathbb{Z}_\ell(1)) \to H^3_{D_{\text{red}}}(\tilde{X}, \mathbb{Z}_\ell(1)) \to 0$$

with finite $\mathcal{T}$ and torsion free $H^3_{D_{\text{red}}}(\tilde{X}, \mathbb{Z}_\ell(1))$. As $\text{Pic}^0(D_{\text{red}})$ is a semiabelian variety, we see that \eqref{eq:2} implies that 1), 2) and 7) are equivalent conditions.

From the exact sequence

$$1 \to \mathcal{O}_{D_{\text{red}}}^0 \to \oplus_i \mathcal{O}_{D_i}^0 \to \oplus_{i < j} k_{D_i \cap D_j}^0 \to 1$$

one has that 6) and 7) are equivalent. Furthermore, from the structure of $\text{Pic}(\tilde{X})$ explained in section 3, one has that 5) is equivalent to 7).

We show that 2) is equivalent to 3). The condition 2) implies that $H^1(U, \mu_{p^n}) \subset \mathcal{T}$ for all $n \geq 0$, thus there are finitely many $\mu_{p^n}$ torsors on $U$. This shows 2) implies 3). On the other hand, if $\text{Pic}^0(D_{\text{red}})$ is not trivial, then $\text{Pic}(D_{\text{red}})[\ell]$ contains $\mathbb{Z}_\ell$. Thus $H^1(U, \mathbb{Z}_\ell(1))$ contains $\mathbb{Z}_\ell$ as well by \eqref{eq:2}. Thus 3) implies 2).

Since obviously 4) implies 3), it remains to see that 3) implies 4). We assume 3). For any commutative finite $k$-group-scheme $G$, with Cartier dual $G' = \text{Hom}(G, \mathbb{G}_m)$, one has the exact sequence

$$0 \to H^1(X, G') \to H^1(U, G') \to \text{Hom}(G, \text{Pic}(U)) \to 0.$$ 

(See \cite[III, Théorème 4.1]{[5]} and \cite[III, Corollaire 4.9]{[5]} for the 0 on the right, which we will use only on the proof of Theorem \ref{thm:4.3} as $k = k$). We apply it for $G = \mathbb{Z}/p^n$ for some $n \in \mathbb{N} \setminus \{0, 1\}$. Since $\text{Pic}(U)$ is an extension of a discrete (étale) group by $\text{Pic}^0(U)$ which is a product of $\mathbb{G}_m$'s by 5), one has $\text{Hom}(\mu_{p^n}, \text{Pic}(U)) = 0$. On the other hand, $A \xrightarrow{x \mapsto (x^{p^n} - x)} A$ is surjective, as $A$ is complete. Thus $H^1(U, \mathbb{Z}/p^n) = H^1(X, \mathbb{Z}/p^n) = 0$. This shows that 3) implies 4) and finishes the proof of the theorem.

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**Theorem 4.2.** Let $(X, x)$ be a surface singularity over an algebraically closed field $k$.

1) If $\pi^N_{\text{loc}}(U, x)$ is a finite group-scheme, $(X, x)$ is a rational singularity, in particular the dualizing sheaf $\omega_U$ has finite order.

2) If in addition, the order of $\omega_U$ is prime to $p$, then there is $(h : V \to U, \pi^N(U, x), y) \in C_{\text{loc}}(U, x)$ such that the surface singularity $(Y, y)$ of the integral closure $\tilde{h} : Y \to X$ is a rational double point.

3) If $\pi^N_{\text{loc}}(U, x) = 0$, then $(X, x)$ is a rational double point.
Proof. We show 1). If \( \pi_{\text{loc}}^N(U, X, x) \) is a finite group-scheme, then, by Lemma 2.5 the condition 3) of Theorem 4.1 is fulfilled, thus \( \text{Pic}^0(\tilde{X}) = \text{Pic}^0(U) \) is a product of \( \mathbb{G}_a \)'s. We apply (4.5) to \( G = \mathbb{Z}/p^n \). If \( \text{Pic}^0(U) \) is not trivial, then \( \text{Hom}(\mathbb{Z}/p^n, \text{Pic}(U)) \neq 0 \) for all \( n \geq 0 \). Thus \( U \) admits nontrivial \( \mu_{p^n} \)-torsors for all \( n \geq 1 \), which do not come from \( X \). This contradicts the finiteness of \( \pi_{\text{loc}}^N(U, X, x) \).

Thus \( \text{Pic}^0(U) = \text{Pic}^0(\tilde{X}) = 0 \). We apply Lemma 3.1 to finish conclude that \((X, x)\) is a rational singularity. Again by Lemma 3.1 all line bundles on \( U \), in particular the dualizing sheaf \( \omega_U \) of \( U \), is torsion. This proves 1).

We show 2). So there is a \( M \in \mathbb{N} \setminus \{0\} \) such that \( \omega_U^M \cong \mathcal{O}_U \). Choosing such a trivialization yields an \( \mathcal{O}_U \)-algebra structure on \( \mathbb{A} = \bigoplus M^{-1} \omega_U \), and thus a flat nontrivial \( \mu_M \)-torsor \( h : V = \text{Spec} \mathcal{O}_U \mathbb{A} \to U \). Since \( (M, p) = 1 \), \( h \) is étale, thus \((Y, y)\) is normal. In fact one has \( Y = \text{Spec} \mathcal{O}_X \mathcal{B} \) where \( \mathcal{B} \) is the \( \mathcal{O}_X \)-algebra \( j_* \mathcal{A} = j_* \mathcal{O}_X \mathbb{A} \). By duality theory, \( h_\ast \omega_Y = \text{Hom}_{\mathcal{O}_X}(h_\ast \mathcal{O}_Y, \omega_X) \cong \mathcal{O}_X \). Let \( y \in Y \) be the closed point of \( Y \). Thus \((Y, y)\) is a Gorenstein normal surface singularity. On the other hand, since \( h \) is a \( \mu_M \)-torsor, one has \( \pi^N(V, y) \subset \pi^N(U, x) \), thus \( \pi_{\text{loc}}^N(V, Y, y) \subset \pi_{\text{loc}}^N(U, X, x) \), and therefore is a finite \( k \)-group-scheme. Thus by 1) it is a rational singularity. Thus \( (Y, y) \) is a Gorenstein rational singularity, thus is a rational double point ([6]).

Now 3) follows directly from 2) as \( \omega_U \) has then order 1.

□

We now refer to [3, Section 3] for the notation, and we go to Artin’s list [3, Section 4/5] to conclude using Theorem 4.2 3):

**Corollary 4.3.** If \( \pi_{\text{loc}}^N(U, X, x) = 0 \), then \( X \) admits a finite morphism \( f : \mathbb{A}^2 \to X \). The morphism \( f \) is the identity (i.e. \((X, x)\) is smooth) except possibly in the cases:

1) \( \text{char}(k) = 2, E_8\)^1, \( E_8^3 \)
2) \( \text{char}(k) = 3, E_8\)^1

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