A CLASSIFICATION OF COMBINATORIAL TYPES OF DISCRIMINANTAL ARRANGEMENTS

SO YAMAGATA

Abstract. Manin and Schechtman introduced a family of arrangements of hyperplanes generalizing classical braid arrangements, which they called the discriminantal arrangements. Athanasiadis proved a conjecture by Bayer and Brandt providing a full description of the combinatorics of discriminantal arrangements in the case of very generic arrangements. Libgober and Settepanella described a sufficient geometric condition for given arrangements to be non-very generic in terms of the notion of dependency for a certain arrangement. Settepanella and the author generalized the notion of dependency introducing r-sets and $K_r$-vector sets, and provided a sufficient condition for non-very genericity but still not convenient to verify by hand. In this paper, we give a classification of the r-sets, and a more explicit and tractable condition for non-very genericity.

1. Introduction

A codimension one subspace in a vector space is called a hyperplane, and a finite set of hyperplanes is called an arrangement of hyperplanes or simply an arrangement. A basic example of an arrangement is the braid arrangement $Br(n) = \{H_{ij}\}_{1 \leq i < j \leq n}$, where $H_{ij} = \{(x_1, \ldots, x_n) \in \mathbb{C}^n \mid x_i = x_j, \ i \neq j\}$.

In 1989, Manin and Schechtman introduced the discriminantal arrangement as a generalization of the braid arrangement. In some literatures it is also called the Manin-Schechtman arrangement. In brief, the arrangement is defined as follows. For a fixed generic arrangement $\mathcal{A}^0 = \{H^0_1, \ldots, H^0_n\}$ consider the space $\mathcal{S}(\mathcal{A}^0)$ of all parallel translations of hyperplanes in $\mathcal{A}^0$, which is naturally isomorphic to $\mathbb{C}^n$. The subset of the space, which consists of all translations of hyperplanes failing to be general position is a hyperplane in $\mathcal{S}(\mathcal{A}^0) \cong \mathbb{C}^n$. The set of such hyperplanes is the discriminantal arrangement and denoted by $\mathcal{B}(n, k, \mathcal{A}^0), n, k \in \mathbb{N}$ for $k \geq 2$. In particular, $\mathcal{B}(n, 1) = \mathcal{B}(n, 1, \mathcal{A}^0)$ coincides with the classical braid arrangement.

The discriminantal arrangements relate various areas of mathematics such as vanishing of cohomology of bundles on toric varieties (see [18]), the higher braid groups and higher categorical perspective (see [8], [9], [10], [14], [11]), and higher Bruhat orders (see [7], [24]).

Athanasiadis [1] pointed out that Crapo [3] was doing pioneering work in which he introduced the geometry of circuits. In [3] he studied the matroid $M(n, k, \mathcal{C})$ of circuits of the configuration $\mathcal{C}$ of $n$ generic points in $\mathbb{R}^k$. The circuits of the matroid $M(n, k, \mathcal{C})$ are now the hyperplanes of $\mathcal{B}(n, k, \mathcal{A}^0)$, where $\mathcal{A}^0$ is an arrangement of $n$ hyperplanes in $\mathbb{R}^k$ orthogonal to the vectors joining the origin with the points of $\mathcal{C}$ (for further development see [4]).

Both Manin-Schechtman [16] and Crapo [3] were mainly interested in arrangements $\mathcal{B}(n, k, \mathcal{A}^0)$ for which the intersection lattice is constant when $\mathcal{A}^0$ varies within a Zariski open set $\mathcal{Z}$ in the space of generic arrangements of $n$ hyperplanes in $k$ dimensional space. Crapo showed that, in this case, the matroid $M(n, k)$ is isomorphic to the Dilworth completion of the k-th lower truncation of the Boolean algebra of rank $n$.

As for combinatorics of $\mathcal{B}(n, k, \mathcal{A}^0)$ for $\mathcal{A}^0$ to be very generic several results were given. In 1997, Bayer and Brandt (see [2]) conjectured a full description of the combinatorics of $\mathcal{B}(n, k, \mathcal{A}^0)$ when $\mathcal{A}^0$ belongs to $\mathcal{Z}$, and it is proved by Athanasiadis [1] in 1999. Following [1] (more precisely Bayer and Brandt), we call arrangements $\mathcal{A}^0$ in $\mathcal{Z}$ very generic, and non-very generic otherwise.

On the other hand, understanding the combinatorics of $\mathcal{B}(n, k, \mathcal{A}^0)$ for $\mathcal{A}^0$ to be non-very generic has still been incomplete. The first example of non-very generic arrangement was given by Crapo [3] in 1985. The arrangement consists of 6 generic lines in $\mathbb{R}^2$, which admits translations that are respectively sides and diagonals of a quadrilateral (see Figure [1]).

However, its non-very genericity did not get much attention at that time. In 1994, after the definition of discriminantal arrangement by Manin-Schechtman, Falk [6] constructed an arrangement of hyperplanes spanned by some generic points, and showed that the arrangement $\mathcal{B}(n, k, \mathcal{A}^0)$ realizes an adjoint of the matroid determined by the points. Using

2020 Mathematics Subject Classification. 52C35 05B35.
Key words and phrases. hyperplane arrangement, intersection lattice, braid arrangement, discriminantal arrangement.
the description he provided a second example of non-very generic arrangement. In this direction, Numata-Takemura [17] and Koizumi-Numata-Takemura [12] gave some computational results on the characteristic polynomials of the discriminantal arrangements.

In 2018, the first general results on non-very generic arrangements were provided. In [15], Libgober and Settepanella described a sufficient geometric condition for the arrangement $A_0$ to be non-very generic. This condition ensures that $B(n, k, A_0)$ admits codimension 2 strata of multiplicity 3, which do not exist in the very generic case. It is given in terms of the notion of dependency for the arrangement $A_\infty$ in $\mathbb{P}^{k-1}$ of hyperplanes $H_{\infty,1}, \ldots, H_{\infty,r}$, which are the intersections of projective closures of $H_1^\infty, \ldots, H_n^\infty \in A_0$ with the hyperplane at infinity. Their main result shows that $B(n, k, A_0), k > 1$ admits a codimension 2 stratum of multiplicity 3 if and only if $A_\infty$ is an arrangement in $\mathbb{P}^{k-1}$ admitting a restriction which is a dependent arrangement. This construction generalizes Falk’s example which corresponds to the case $n = 6, k = 3$ and which has been object of study in two subsequent papers by Sawada, Settepanella and the author [20], [21].

In 2021, Settepanella and the author [22] generalized the dependency condition given in [15], providing a sufficient condition for the existence of non-very generic intersections in rank $r \geq 2$, i.e., intersections which do not exist in $B(n, k, A_0), A_0 \in Z$ in terms of $r$-sets and $K_T$-vector sets. More recently in 2022, Settepanella and the author [23] gave a linear condition for non-very genericity as a continuation of [22]. Some related works have also been continued. In [5], Das-Palezzato-Settepanella provided examples of the classification of special configurations of points in the $k$-dimensional space relating to the combinatorics of $B(n, 2, A_0)$ and $B(n, 3, A_0)$ for $A_0$ to be non-very generic. In [19], Saito-Settepanella gave a characterization and a classification of few non-very generic arrangements in low dimensional space. In particular, they classified the combinatorics of $B(6, 3, A_0)$ over commutative field of characteristic 0.
The purpose of this paper is to classify su and intersecting types, which are purely combinatorial descriptions. In Section 5 we give a sufficient condition to have \(K_T\)-vector sets. This paper is organized as follows.

In Section 2 we recall basic definitions of discriminantal arrangements, \(r\)-sets and \(K_T\)-vector sets. We also recall a sufficient condition for non-very genericity following [23]. In Section 3 we give examples of constructions of non-very generic arrangements, which sets the stage for the later sections. In Section 4 we classify \(r\)-sets into non-intersecting and intersecting types, which are purely combinatorial descriptions. In Section 5 we give a sufficient condition to have the \(K_T\)-vector sets in the case of \(r\)-sets of non-intersecting type. Moreover we define a special class of \(r\)-sets of intersecting type, which we call the good \(r\)-partition, and give a sufficient condition to have the \(K_T\)-vector sets. This constitutes the first classification of \(r\)-sets of intersecting type.

2. Preliminaries

2.1. A hyperplane arrangement and discriminantal arrangement. Let us consider an arrangement of hyperplanes in \(\mathbb{C}^k\), i.e., a finite set of hyperplanes in \(\mathbb{C}^k\). For a linear hyperplane \(H^0\) define its translation by \(H^t = H^0 + at\), where \(a\) is a normal vector to \(H^t\), and \(t \in \mathbb{C}\). We denote an arrangement of linear hyperplanes by \(\mathcal{A}^0 = \{H^0_1, \ldots, H^0_n\}\) and its translation by \(\mathcal{A}^t = \{H^t_1, \ldots, H^t_n\}\), where \(H^t_i\) is a translation of \(H^0_i\) throughout this paper. We say that an arrangement of hyperplanes is generic if for all \(J \subset [n], |J| = k\) normal vectors \(a_{\alpha_J}\) to \(H^0_i, i \in J\) are linearly independent. Hyperplanes \(H^t_i, i = 1, \ldots, n\) are said to be in general position if the following two conditions are satisfied:

- For \(1 \leq m \leq k\), the intersection of any \(m\) hyperplanes has dimension \(k - m\).
- For \(m > k\), the intersection of any \(m\) hyperplanes is empty.

Let \(\mathcal{A}^0 = \{H^0_1, \ldots, H^0_n\}\) be a generic arrangement in \(\mathbb{C}^k\), \(k < n\). The space of parallel translations \(\mathcal{S}(\mathcal{A}^0)\) (or simply \(\mathcal{S}\) when dependence on \(H^0_i\) is clear or not essential) is the space of \(n\)-tuples of translations \(H^0_1, \ldots, H^0_n\) such that either \(H^0_i \cap H^0_j = \emptyset\) or \(H^0_i = H^0_j\) for \(i = 1, \ldots, n\).

We can identify \(\mathcal{S}\) with \(n\)-dimensional affine space \(\mathbb{C}^n\) in such a way that \((H^0_1, \ldots, H^0_n)\) corresponds to the origin. In particular, an ordering of hyperplanes in \(\mathcal{A}^0\) determines the coordinate system in \(\mathcal{S}\) (see [15]).

For a fixed generic arrangement \(\mathcal{A}^0\), consider the closed subset of \(\mathcal{S}\) formed by those collections which fail to form a general position. This subset of \(\mathcal{S}\) is a union of hyperplanes \(D_L \subset \mathcal{S}\) (see [16]). Each hyperplane \(D_L\) corresponds to a sub-index \(L = \{l_1, \ldots, l_{k+1}\} \subset [n]\), and it consists of \(n\)-tuples of translations of hyperplanes \(H^0_{l_1}, \ldots, H^0_{l_{k+1}}\) in which translations of \(H^0_{l_{i+1}}, \ldots, H^0_{l_{k+1}}\) fail to form a general position. The arrangement \(\mathcal{B}(n,k,\mathcal{A}^0)\) of hyperplanes \(D_L\) is called discriminantal arrangement and has been introduced by Manin and Schechtman in [16]. Although they defined the discriminantal arrangement starting from a general position arrangement instead of its centrally translated one, we adopt the latter for convenience.

2.2. (Non) very generic arrangements and \(r\)-simple intersections. It is well known (see among others [3, 16]) that there exists an open Zariski set \(\mathcal{Z}\) in the space of generic arrangements of \(n\) hyperplanes in \(\mathbb{C}^k\) such that the intersection lattice of the discriminantal arrangement \(\mathcal{B}(n,k,\mathcal{A}^0)\) is independent from the choice of the generic arrangement \(\mathcal{A}^0 \in \mathcal{Z}\). Bayer and Brandt [2] called the arrangements \(\mathcal{A}^0 \in \mathcal{Z}\) very generic and the ones \(\mathcal{A}^0 \notin \mathcal{Z}\) non-very generic. The name very generic comes from the fact that in this case the cardinality of the intersection lattice of \(\mathcal{B}(n,k,\mathcal{A}^0)\) is the largest possible for all generic arrangement of \(n\) hyperplanes in \(\mathbb{C}^k\).

In [3], Crapo showed that the intersection lattice of \(\mathcal{B}(n,k,\mathcal{A}^0)\) for very generic arrangement \(\mathcal{A}^0\) is isomorphic to the Dilworth completion \(D_k(B_n)\) of a \(k\)-times lower-truncated Boolean algebra. In [1] Athanasiadis gave a more precise description that the intersection lattice of \(\mathcal{B}(n,k,\mathcal{A}^0)\) for very generic arrangement \(\mathcal{A}^0\) is isomorphic to the lattice \(P(n,k)\) defined as follows. \(P(n,k)\) is the collection of all sets of the form \(\{S_1, \ldots, S_r\}\), where \(S_i \subset [n]\), \(|S_i| \geq k + 1\) such that

\[
| \bigcup_{i \in I} S_i | > k + \sum_{i \in I} (|S_i| - k)
\]

for all \(I \subset [r], |I| \geq 2\). The order on \(P(n,k)\) is given by letting \(\{S_1, \ldots, S_r\} < \{T_1, \ldots, T_r\}\) if for each \(1 \leq i \leq r\) there exists \(1 \leq j \leq r\) such that \(S_i \subset T_j\). The isomorphism was first conjectured by Bayer-Brandt in [2].

In [13], Libgober-Settepanella gave a full description of rank 2 elements of the intersection lattice of \(\mathcal{B}(n,k,\mathcal{A}^0)\). Based on their result, Sawada-Settepanella and the author [20, 21] showed that hyperplanes in the non-very generic arrangements give rise to special configurations such as the Pappus or Hesse configuration. In [13], Kumar showed
that certain line arrangements in the plane always give rise to non-very generic arrangements. In [5], Das-Palezzato-Settepanella provided examples of the classification of special configurations of points in the $k$-dimensional space relating to the combinatorics of $\mathcal{B}(n, 2, \mathcal{A}^0)$ and $\mathcal{B}(n, 3, \mathcal{A}^0)$ for $\mathcal{A}^0$ to be non-very generic. In [19], Saito-Settepanella gave a characterization and a classification of few non-very generic arrangements in low dimensional space. In particular, they classified the combinatorics of $\mathcal{B}(6, 3, \mathcal{A}^0)$ over commutative field of characteristic 0. More recently, Settepanella and the author [22], [23] provided a geometric and algebraic conditions for arrangement $\mathcal{A}^0$ to be non-very generic, and some examples of non-very generic arrangements.

In general, it would be complicated to consider intersections $\bigcap_{i=1}^{r} D_{s_i}, |S| > k$ with $\bigcap_{i \in \mathcal{I}} D_{s_i} \neq D_s$, where $D_S = \bigcap_{L \in \mathcal{S}, |L|=k+1} D_L, D_L \in \mathcal{B}(n, k, \mathcal{A}^0)$ because there would be a lot of case separations on the cardinality of $S_t$. As a first step in [22] and [23] they introduced a simple intersection which we call $r$-simple for simplicity.

**Definition 2.1.** An element $X$ in the intersection lattice of the discriminantal arrangement $\mathcal{B}(n, k, \mathcal{A}^0)$ is called an $r$-simple if

$$X = \bigcap_{i=1}^{r} D_{l_i}, |L_i| = k + 1,$$

and for any $I \subseteq [r], |I| \geq 2$ and any $S \subset [n], |S| > k + 1$, it follows that

$$\bigcap_{i \in I} D_{s_i} \neq D_s,$$

where $D_S = \bigcap_{L \in \mathcal{S}, |L|=k+1} D_L, D_L \in \mathcal{B}(n, k, \mathcal{A}^0)$.

Though the $r$-simplicity is a central notion, there are no examples of arrangements giving rise to simple and non-simple intersections in the literatures. Let us see some examples of arrangements giving rise to 5-simple and non-5-simple intersections.

**Example 2.2** (5-simple intersection). Let $L_1 = \{1, 2, 3, 4\}, L_2 = \{1, 5, 6, 7\}, L_3 = \{2, 5, 8, 9\}, L_4 = \{3, 6, 8, 10\}$ and $L_5 = \{4, 7, 9, 10\}$ be subsets of $[10]$. Let $\mathcal{A}$ be a generic arrangement of 10 hyperplanes (planes) in $\mathbb{C}^3$ and $\mathcal{A}'$ be its translated one as in Figure 2. The intersection $X = \bigcap_{i=1}^{5} D_{l_i}$ consists of all translations of hyperplanes $H^p_i$ in such a way that $\bigcap_{j=1}^{5} D_{l_j} = \bigcap_{i=1}^{5} D_{l_i}$. It also satisfies that for any $I \subset [5], |I| \geq 2$ and any $S \subset [10], |S| > 4, \bigcap_{i \in I} D_{s_i} \neq D_s$, where $D_S = \bigcap_{L \in \mathcal{S}, |L|=4} D_L, D_L \in \mathcal{B}(10, 3, \mathcal{A}^0)$. Thus, $X$ is a 5-simple intersection.

Notice that since there is a correspondence

$$\mathcal{A}' \in D_{l_i} \Leftrightarrow P'_i = \bigcap_{p \in L_i} H^p_i \neq \emptyset,$$

the relation is equivalent to saying that if

$$P'_{ij} = \bigcap_{p \in L_{ij}} H^p_i \neq \emptyset, \quad j = 1, 2, 3, 4,$$

then

$$P'_{ij} = \bigcap_{p \in L_{ij}} H^p_i \neq \emptyset$$

for any $i_1, i_2, i_3, i_4, i_5 \in [5]$. Each $P'_{ij}$ is an intersection point of exactly four hyperplanes $H^p_i, p \in L_i$ (see Figure 2).

**Example 2.3** (Non 5-simple intersection). Let $L_1 = \{1, 2, 3\}, L_2 = \{1, 2, 5\}, L_3 = \{1, 4, 7\}, L_4 = \{3, 6, 7\} and L_5 = \{4, 5, 6\}$ be subsets of $[7]$. Let $\mathcal{A}$ be a generic arrangement of 7 hyperplanes (lines) in $\mathbb{C}^2$ and let $\mathcal{A}'$ be its translated one, as shown in Figure 3. Since there exists an intersection $X = D_{l_1} \cap D_{l_2} = D_{[1,2,3,5]}$, the element $X = \bigcap_{i=1}^{5} D_{l_i}$ is not a 5-simple intersection. Notice that the “multiple intersections” $P'_{1} = P'_{2}$ are intersections of not three but four hyperplanes, while $P'_{3}, P'_{4}, P'_{5}$ are intersections of exactly three hyperplanes.

In the rest of this paper we will focus on non-very generic arrangements such that $X = \bigcap_{i=1}^{r} D_{l_i}$ is an $r$-simple intersection for simplicity (so we have exactly $r$ intersection points $P'_i, i = 1, \ldots, r$ which are intersections of $k + 1$ hyperplanes indexed in $L_i, i = 1, \ldots, r$ in the translated arrangement $\mathcal{A}' \in \bigcap_{i=1}^{r} D_{l_i}$).
Figure 2. An arrangement $\mathcal{A}' \in X = \bigcap_{i=1}^{5} D_{L_i}$.

Figure 3. An arrangement $\mathcal{A}' \in X = \bigcap_{i=1}^{5} D_{L_i}$.

We call the number $r$ the \textit{multiplicity} of $X$. If $\mathcal{A}^0$ is very generic and satisfies the condition (1), then it follows that the subspaces $D_{L_i}, i = 1, \ldots, r$ intersect transversely (see Theorem 3.1 in [1]). The fact is equivalent to saying that since $\text{rank} D_{L_i} = 1$,

\begin{equation}
\text{rank} \bigcap_{i=1}^{r} D_{L_i} = \sum_{i=1}^{r} (|L_i| - k) = r.
\end{equation}

Thus, if the intersection lattice of the discriminantal arrangement $\mathcal{B}(n, k, \mathcal{A}^0)$ contains an $r$-simple intersection of rank strictly lower than $r$, that is a multiplicity of $X$, then $\mathcal{A}^0$ is non-very generic (see [22] for details).

2.3. A \textbf{sufficient condition for non-very genericity}. Following [22] and [23] let us recall a sufficient condition for arrangement $\mathcal{A}^0$ to be non-very generic in this subsection.

For a fixed set $\mathbb{T} = \{L_1, \ldots, L_r\}$ of subsets $L_i \subset [n], |L_i| = k + 1$ and any translated arrangement $\mathcal{A}' = \{H_{L_i}^r, \ldots, H_{L_i}^L\}$ of $\mathcal{A}^0$ we denote $P_i = \bigcap_{p \in L_i} H_p^r$ and $H_{i,j} = \bigcap_{p \in L_i \cap L_j} H_p^r$. Notice that $P_i$ is a point if and only if $\mathcal{A}' \in D_{L_i}$; it is empty otherwise.
A non-very generic arrangement $\mathcal{A}^0$ holds the property that if $\mathcal{A} \in \cap_{p=1}^{d} D_{t_p}$, then $\mathcal{A} \in \bigcap_{i=1}^{r} D_{t_i}$. In other words, if we translate hyperplanes of $\mathcal{A}^0$ in such a way that $r-1$ intersection points $P'_{i_j}$, $j = 1, \ldots, r-1$ appear, then the $r$-th intersection point $P'_{i_r}$ also appears automatically. To realize such property Settepanella and the author [22] introduced the following definitions.

**Definition 2.4** ($r$-set, [22]). If $T = \{L_1, \ldots, L_r\}$ satisfies the conditions

$$\bigcup_{i \in [r] \setminus I_{|I|=r-1}}^{r} L_i = \bigcup_{i \in I} L_i \quad \text{and} \quad L_i \cap L_j \neq \emptyset$$

for any subset $I \subset [r], |I| = r-1$ and any two indices $1 \leq i < j \leq r$, we call the set $T$ an $r$-set.

**Definition 2.5.** ($K_T$-translated, $K_T$-configuration, [22]). A translation $\mathcal{A} = \{H^0_1, \ldots, H^0_n\}$ of $\mathcal{A}^0$ is called $K_T$-translated if $P^*_i = \cap_{p \in L_i} H^*_p \neq \emptyset$ is the intersection of exactly $k+1$ hyperplanes indexed in $L_i$ for any $L_i \in T$.

For a $K_T$-translation $\mathcal{A}$ we call the complete graph having the points $P^*_i$, $i = 1, \ldots, r$ as vertices and vectors $P^*_iP^*_j \in \cap_{p \in L_i \cap L_j} H^*_p$, $1 \leq i < j \leq r$ as edges $K_T$-configuration and denote by $K_T(\mathcal{A}^0)$.

In [23], a linear condition for non-very genericity was provided in terms of the above definitions. Let us briefly trace the sketch.

Consider essentialization of the discriminantal arrangement $\mathrm{ess}(\mathcal{B}(n, k, \mathcal{A}))$ in $\mathbb{C}^{n-k} \cong \mathbb{S}/D_{|I|}$. Then, $\mathcal{A} \in \mathrm{ess}(\mathcal{B}(n, k, \mathcal{A}))$ uniquely corresponds to a translation $t \in \mathbb{C}^n/C \cong \mathbb{C}^{n-k}$, where $C = \{t \in \mathbb{C}^n \mid \mathcal{A} \text{ is central}\}$ and the following proposition holds.

**Proposition 2.6** (Proposition 3.1, [23]). Let $\mathcal{A}^0$ be a generic arrangement of $n$ hyperplanes in $\mathbb{C}^k$. Translations $\mathcal{A}^v, \ldots, \mathcal{A}^v$ are linearly independent vectors in $\mathbb{S}/D_{|I|} \cong \mathbb{C}^{n-k}$ if and only if $t_1, \ldots, t_d$ are linearly independent vectors in $\mathbb{C}^n/C \cong \mathbb{C}^{n-k}$.

Let $\mathcal{A}^v$ be $K_T$-translated and $P^*_i$ be the intersection $\cap_{p \in L_i} H^*_p$. Then, we can consider a unique family $\{v^*_i\}$ of vectors $v^*_i \in \cap_{p \in L_i \cap L_j} H^*_p$ such that $P^*_i + v^*_i = P^*_j$, and we have the following definition.

**Definition 2.7** (Definition 3.1, [23]). Let $T$ be an $r$-set and $\mathcal{A}$ be $K_T$-translated of generic arrangement $\mathcal{A}^0$. Fix a number $i_0 \in [r]$. We call the set of vectors $\{v^*_{i_0i}\}$ satisfying $P^*_{i_0} + v^*_{i_0i} = P^*_i$ for any $j(\neq i_0) \in [r]$ the $K_T$-vector sets.

**Remark 2.8.** Since we have $v^*_{i_0j} = v^*_{i_0j} - v^*_{i_0j}$ by definition, the set $\{v^*_{i_0j}\}_{j \neq i_0}$ is determined by its subset $\{v^*_{i_0i}\}_{i \neq 1}$. We consider the set $\{v^*_{i_0j}\}_{j \neq 1}$ instead of $\{v^*_{i_0j}\}_{j \neq 1}$ in the rest of this paper.

![Figure 4. $K_T$-configuration $K_T(\mathcal{A}^0)$ and its associated $K_T$-vector set.](image-url)
For (a) given $K_T$-vector set(s) we define two operations as follows.

\[ \{v_{i,j}^t\}_{i=2,...,r} + \{v_{i,j}^t\}_{i=2,...,r} := \{v_{i,j}^t + v_{i,j}^t\}_{i=2,...,r} \quad \text{(sum)}, \]
\[ a\{v_{i,j}^t\}_{i=2,...,r} := \{av_{i,j}^t\}_{i=2,...,r}, \quad a \in \mathbb{C} \quad \text{(multiplication)}. \]

With above notations and operations, we have the following definition.

**Definition 2.9** (Definition 4.2 [23]). For a fixed $r$-set $T$ we call the $d$ different $K_T$-vector sets $\{v_{i,j}^t\}_{i=2,...,r}$, $t = 1, \ldots, d$ linearly independent if for any $a_1, \ldots, a_d \in \mathbb{C}$ such that

\[ \sum_{t=1}^d a_t v_{i,j}^t = 0, \]
then $a_1 = \ldots = a_d = 0$.

With the notion of independent $K_T$-vector sets the criterion for non-very genericity is provided in [23]. The following theorem is a basic result for the rest of this paper.

**Theorem 2.10** (Theorem 4.5 [23]). Let $\mathcal{A}^0$ be a generic arrangement of $n$ hyperplanes in $\mathbb{C}^k$. If there exists an $r$-set $T = \{L_1, \ldots, L_r\}$ with $|\cup_{i=1}^r L_i| = m$ and rank $\bigcap_{i=1}^r L_i$ $H_p^0 = y$, which admits $m - y - k - r'$ independent $K_T$-vector sets for some $r' < r$, then $\mathcal{A}^0$ is non-very generic.

**Remark 2.11.** According to Theorem 2.10 if we find a certain number, say $d \in \mathbb{Z}_{>1}$ independent $K_T$-vector sets, they give rise to a non-very generic arrangement. Indeed we can define hyperplanes $H_i^0 \in \mathcal{A}^0 = \{H_i^0\}_{i=1,\ldots,m}$, $l \in L_i \cap L_j$ by $v_{i,j}^t \in H_i^0$, $t \in [d]$.

We close this section by giving a notation we will use throughout this paper.

**Notation 2.12.** For vectors $v_1, \ldots, v_m \in \mathbb{C}^k$ we denote by $(v_1, \ldots, v_m)$ a subspace spanned by $v_1, \ldots, v_m$. Notice that the vectors $v_1, \ldots, v_m$ need not necessarily be independent in this notation.

### 3. Motivating examples

Let us begin with Crapo’s example.

**Example 3.1** (Crapo’s example [3]). Let $T = \{L_1, L_2, L_3, L_4\}$ be a 4-set defined by $L_1 = \{1, 2, 3\}, L_2 = \{1, 4, 5\}, L_3 = \{2, 4, 6\}, L_4 = \{3, 5, 6\}$. Consider an arrangement $\mathcal{A}^0 = \{H_i^0\}_{i=1,\ldots,6}$ of lines in $\mathbb{C}^2$ which admits a $K_T$-translation $\mathcal{A}$ as Figure 5.

![Figure 5. Translation $\mathcal{A}^0$ consisting of four quadrilateral and two diagonal lines.](image-url)
Let assume the arrangement $\mathcal{A}^0$ admits $d$ $K_T$-translations that is, we can choose $d$ linearly independent $K_T$-vector sets $\{v_{1,t}^t, v_{1,3,t}^t, v_{1,4,t}^t\}$, $t = 1, \ldots, d$.

For $1 \leq a < b \leq 4$ we denote by

$$V_{a,b} = \{v_{a,b}^t \mid t = 1, \ldots, d\}$$

the vector space spanned by $v_{a,b}^t \in H_{a,b}^0 = \bigcap_{p \in L_a \cap L_b} H_p^0$. The following two claims hold.

### Claim 3.2.
The dimension of $V_{a,b}$ is one for any $a, b$.

**Proof.** Since for any $a, b$ there exists a vector $a_p \in V_{a,b}^p$, $p \in L_a \cap L_b$, we have $\dim V_{a,b}^p \geq 1 \iff \dim V_{a,b} \leq 1$. Since $v_{a,b}^t \neq 0$, and otherwise $H_{a,b}^0 = \emptyset$, we also have $\dim V_{a,b} \neq 0$. Thus, $\dim V_{a,b} = 1$ for any $a, b$.

### Claim 3.3.
Let $\{v_{1,2,t}^t, v_{1,3,t}^t, v_{1,4,t}^t\}$, $t = 1, \ldots, d$ be $K_T$-vector sets. The sets are linearly independent if and only if $d = 1$.

**Proof.** If $d = 1$, the set $\{v_{1,2}^1, v_{1,3}^1, v_{1,4}^1\}$ is obviously linearly independent. Let us show the converse. If $d > 1$, there should exist a scalar $k$ such that $v_{i,j}^t = k v_{i,j}^1$ for $t = 2, \ldots, d$ and any $i, j$, and otherwise $\dim V_{a,b} \geq 2$ for some $a, b \in [4]$. Thus, in this case we have that $H_{a,b}^0 = k \{v_{1,2}^1, v_{1,3}^1, v_{1,4}^1\}$ for any $t = 2, \ldots, d$; i.e., we have dependent $K_T$-vector sets. Thus, if $\{v_{1,2,t}^t, v_{1,3,t}^t, v_{1,4,t}^t\}$, $t = 1, \ldots, d$ are independent $K_T$-vector sets, then $d = 1$.

By Claim 3.3 it is sufficient to consider only the case $d = 1$. Since each hyperplane (line) $H_p^0$, $p \in L_a \cap L_b$ contains a vector $v_{a,b}^t$, where $1 \leq a < b \leq 4$ and $|L_a \cap L_b| = 1$, it follows that

$$\langle \alpha_{a,b} \rangle = \left( H_{a,b}^0 \right)^{\perp} = \sum_{p \in L_a \cap L_b} \left( H_p^0 \right)^{\perp} = V_{a,b}^t,$$

where $\{a,b\} = L_a \cap L_b$. Notice that the $K_T$-vector set $\{v_{1,2}, v_{1,3}, v_{1,4}\}$ satisfies the condition

$$\dim \sum_{a,b \in L} V_{a,b}^t = |I| \text{ for any } I \subset [4], 1 \leq |I| \leq 2,$$

and $v_{i,k} \notin \langle v_{i,j} \rangle$ for distinct $i, j, k$.

Conversely, let us consider a set of vectors $\{v_{1,2}, v_{1,3}, v_{1,4}\}$ satisfying the condition (6). Then, we can choose generic vectors $\alpha_i$, $i = 1, \ldots, 6$ from the orthogonal spaces $V_{a,b}^t$ and we obtain a generic arrangement $\mathcal{A}^0$. In particular, the arrangement admits a $K_T$-translation. That is we have intersection points $P_i = \bigcap_{L_i} H_p^i$, $i = 1, 2, 3, 4$, which are intersections of exactly 3 hyperplanes indexed in $L_i$, $i = 1, 2, 3, 4$. Thus, we have a $K_T$-vector set $\{v_{1,2}, v_{1,3}, v_{1,4}\}$. By Theorem 2.7 it follows that the arrangement $\mathcal{A}^0$ constructed from the $K_T$-vector set is non-very generic.

The following arrangement is constructed as a “high-dimensional” Crapo’s example in [23].

### Example 3.4 ($\mathcal{B}(12, 8, \mathcal{A}^0)$). Let $\mathbb{T} = \{L_1, L_2, L_3, L_4\}$ be a 4-set defined by $L_1 = [12] \setminus K_1$, where $K_1 = [10, 11, 12]$, $K_2 = [7, 8, 9]$, $K_3 = [4, 5, 6]$, and $K_4 = [1, 2, 3]$. Consider an arrangement $\mathcal{A}^0 = \{H_p^i\}_{i=1}^{12}$ of hyperplanes in $\mathbb{R}^8$ which admits $d \geq 1$ $K_T$-translations $\mathcal{A}^1$, $t = 1, \ldots, d$ as in Figure 5. Remark that unlike Example 3.1, each line in the figure does not represent a hyperplane but a subspace $H_{i,j}^t = \bigcap_{p \in L_i \cap L_j} H_p^i$, since $|L_i \cap L_j| = 6$ for all $1 \leq i < j \leq 4$.

Let $\{v_{1,2,t}^t, v_{1,3,t}^t, v_{1,4,t}^t\}$ be their associated $K_T$-vector sets. For $t = 1, 2, 3, 4$ denote by

$$V_{0,|I|}^t = \{v_{a,b}^t \mid a, b \in [4] \setminus I, t = 1, \ldots, d\}$$

the vector space spanned by vectors $v_{a,b}^t$, $a, b \in [4] \setminus I$, $t = 1, \ldots, d$. Notice that we have

$$\bigcup_{p \in \mathcal{T} \setminus \{1\}} H_p^0 \supset V_{0,|I|}^t,$$

equivalently,

$$\langle \alpha_p \mid p \in \mathcal{T} \setminus \{1\} \rangle = \sum_{p \in \mathcal{T} \setminus \{1\}} \left( H_p^0 \right)^{\perp} \subseteq V_{0,|I|}^t.$$
Conversely, let assume there exist \(d\) particular, the arrangement admits a 4-set defined as in this example, we can see an explicit construction of the sets Then, the 4-vector sets satisfy the condition

\[
\dim \sum_{i \in I} V_{[4]\{i\}}^+ \begin{cases} \geq 3|I| & \text{for any } I \subset [4], 1 \leq |I| \leq 2, \\ \geq 8 & \text{for any } I \subset [4], |I| = 3. \end{cases}
\]

Conversely, let assume there exist \(d(\geq 1)\) sets of vectors \(\{v_{1,2}^{(t)}, v_{1,3}^{(t)}, v_{1,4}^{(t)}\}\) satisfying the condition \((\ref{eq:condition})\). Then, we can choose generic vectors \(a_l, l = 1, \ldots, 12\) from the orthogonal spaces \(V_{a,br}^+\) and we obtain a generic arrangement \(\mathcal{A}\). In particular, the arrangement admits \(K_2\)-translations \(\mathcal{A}'\), \(t = 1, \ldots, d\). That is we have intersection points \(P_i = \bigcap_{p \in \mathcal{A}_i} H_i^p\), \(i = 1, 2, 3, 4\) for each \(t = 1, \ldots, d\). Thus, the vectors \(\{v_{1,2}^{(t)}, v_{1,3}^{(t)}, v_{1,4}^{(t)}\}, t = 1, \ldots, d\) are \(K_2\)-vector sets. Moreover, if \(T\) is a 4-set defined as in this example, we can see an explicit construction of the sets \(\{v_{1,2}^{(t)}, v_{1,3}^{(t)}, v_{1,4}^{(t)}\}, t = 1, \ldots, d\) satisfying \((\ref{eq:condition})\). Let \(d_{l,i} = \dim \langle v_{i,j}^{(t)} | t = 1, \ldots, d \rangle\), \(i = 2, 3, 4\). The following proposition gives the way how to find the vectors \(\{v_{1,2}^{(t)}, v_{1,3}^{(t)}, v_{1,4}^{(t)}\}, t = 1, \ldots, d\).

**Proposition 3.5.** The sets \(\{v_{1,2}^{(t)}, v_{1,3}^{(t)}, v_{1,4}^{(t)}\}, t = 1, \ldots, d\) satisfy \((\ref{eq:condition})\) if and only if \(v_{1,a}^{(t)} \in \langle v_{1,a}^{(t)} | t = 1, \ldots, d \setminus \{l\} \rangle\) for any \(a \in \{2, 3, 4\}, l \in [d]\) and \(d_{l,i} \leq 2, d_{l,i} + d_{l,j} \leq 5\) for any \(i, j, k\).

**Proof.** First, let us prove that if the sets \(\{v_{1,2}^{(t)}, v_{1,3}^{(t)}, v_{1,4}^{(t)}\}, t = 1, \ldots, d\) satisfy \((\ref{eq:condition})\), then \(v_{1,a}^{(t)} \in \langle v_{1,a}^{(t)} | t = 1, \ldots, d \setminus \{l\} \rangle\) for any \(a \in \{2, 3, 4\}, l \in [d]\). We prove this in the case of \(a = 2\) by contradiction, assuming that there exists a vector \(v_{1,2}^{(t)} \in \langle v_{1,3}^{(t)}, v_{1,4}^{(t)} | t = 1, \ldots, d \rangle\). In this case we have \(v_{1,2}^{(t)} \in V_{[4]\{2\}}\).

By \((\ref{eq:condition})\) we have

\[
\dim \sum_{l=2}^{4} V_{[4]\{l\}}^+ = 8 \iff \dim \bigcap_{l=2}^{4} V_{[4]\{l\}}^+ = 0.
\]

On the other hand, since \(v_{1,2}^{(t)} \in V_{[4]\{l\}}, l = 3, 4\) and also \(v_{1,2}^{(t)} \in V_{[4]\{2\}}\) by assumption, it follows that \(v_{1,2}^{(t)} \in \bigcap_{l=2}^{4} V_{[4]\{l\}}^+\), which contradicts the fact that \(\dim \bigcap_{l=2}^{4} V_{[4]\{l\}}^+ = 0\). Thus, \(v_{1,2}^{(t)} \in \langle v_{1,2}^{(t)} | t = 1, \ldots, d \setminus \{l\} \rangle\). The analogous proofs follow for any \(a \in \{2, 3, 4\}\) and \(l \in [d]\).

Secondly, let us prove \(d_{l,k} \leq 2\) and \(d_{l,i} + d_{l,j} \leq 5\) for any \(i, j, k\). By the fact we have just proved we obtain

\[
\langle v_{1,2}^{(t)}, v_{1,3}^{(t)} | t = 1, \ldots, d \rangle \cap \langle v_{1,4}^{(t)} | t = 1, \ldots, d \rangle = \{0\}
\]

\[
\langle v_{1,2}^{(t)} | t = 1, \ldots, d \rangle \cap \langle v_{1,3}^{(t)} | t = 1, \ldots, d \rangle = \{0\}.
\]
On the other hand, we have
\begin{equation}
\dim \bigcup_{l \in I} V_{l}^{\perp} \geq 3|I| \iff \dim \bigcap_{l \in I} V_{l} \leq 8 - 3|I|
\end{equation}
for any $I \subset [4]$, $1 \leq |I| \leq 2$ by (9); thus we have $d_{1,k} \leq 2$ and $d_{1,i} + d_{1,j} \leq 5$ for any $i, j, k$.
Conversely, let us assume $v'_{1,a} \in \left\{ v_{1,a} \mid t = 1, \ldots, d \setminus \{l\} \right\}$ for any $a \in \{2, 3, 4\}$, $l \in [d]$ and $d_{1,k} \leq 2$, $d_{1,i} + d_{1,j} \leq 5$ for any $i, j, k$. In consideration of (12), if there exist the sets satisfying the assumptions, then the proof would be completed. For this reason it is sufficient to show that there exist such the sets.
Since $v'_{1,a} \in \left\{ v_{1,a} \mid t = 1, \ldots, d \setminus \{l\} \right\}$ for any $a \in \{2, 3, 4\}$, $l \in [d]$, we have (11). In particular, we have
\begin{equation}
\dim \left\{ v_{1,2}, v_{1,3}, v_{1,4} \mid t = 1, \ldots, d \right\}
= \dim \left\{ v_{1,2} \mid t = 1, \ldots, d \right\} + \dim \left\{ v_{1,3} \mid t = 1, \ldots, d \right\} + \dim \left\{ v_{1,4} \mid t = 1, \ldots, d \right\}
= d_{1,2} + d_{1,3} + d_{1,4}.
\end{equation}
To construct the sets $\left\{ v'_{1,2}, v'_{1,3}, v'_{1,4} \right\}$, $t = 1, \ldots, d$, we need to choose $3d$ vectors $v'_{1,i}, i = 2, 3, 4$, $t = 1, \ldots, d$ with $\dim \left\{ v'_{1,i} \mid t = 1, \ldots, d \right\} = d_{1,i}, i = 2, 3, 4$. In particular, it is sufficient to choose $d_{1,2} + d_{1,3} + d_{1,4}$ independent vectors in $\mathbb{C}^{8}$. By assumption we have $d_{1,k} \leq 2$ and $d_{1,i} + d_{1,j} \leq 5$ for any $i, j, k$. Notice that the second inequality automatically follows since the first one holds for any $k$. This implies that $d_{1,2} + d_{1,3} + d_{1,4} \leq 6 < 8$. Thus, the sets we expected actually exist.

\section{A classification of $r$-sets}

In this section we classify the $r$-sets $\mathcal{T} = \{L_{1}, \ldots, L_{r}\}$ into non-intersecting and intersecting types. Each $L_{i}$ is the set of indices of hyperplanes defining a hyperplane $D_{L_{i}}$ of the discriminant arrangement. Since we are considering arrangements in $\mathbb{C}^{k}$ and focusing on $r$-simple intersections, we assume $|L_{i}| = k + 1, i = 1, \ldots, r$.

\subsection{Non-intersecting type $r$-sets}

To begin with, let us consider $r$-sets for simple case that intersections of all three sets of an $r$-set are empty. More precisely, we give the following definition.

\begin{definition}
Let $r \geq 4$. We say that $r$-set $\mathcal{T} = \{L_{1}, \ldots, L_{r}\}$ is non-intersecting type if $L_{i} \cap L_{j} \cap L_{k} = \emptyset$ for any distinct $i, j, k$.
\end{definition}

Let us denote
\begin{equation}
L_{i} = \bigcup_{j \in [r] \setminus \{i\}} A_{i,j},
\end{equation}
where $A_{i,j} = L_{i} \cap L_{j} \subset [n]$ and $|A_{i,j}| = a_{i,j}, a_{i,j} \geq 1$.
Since $A_{1,j} = L_{j} \setminus \bigcup_{k \in [r] \setminus \{1\}, j} A_{1,j}$, we have
\begin{equation}
a_{1,j} = k + 1 - \sum_{l \in [r] \setminus \{1,j\}} a_{l,j}.
\end{equation}
By summing both sides of the formula (14) for $j \in [r] \setminus \{1\}$, we obtain
\begin{equation}
\sum_{j \in [r] \setminus \{1\}} a_{1,j} = \sum_{j \in [r] \setminus \{1\}} \left( k + 1 - \sum_{l \in [r] \setminus \{1,j\}} a_{l,j} \right)
= \sum_{j \in [r] \setminus \{1\}} (k + 1) - \sum_{j \in [r] \setminus \{1\}} \sum_{l \in [r] \setminus \{1,j\}} a_{l,j}.
\end{equation}
Thus, we have
\begin{equation}
k + 1 = (r - 1)(k + 1) - 2 \sum_{L_{i} \in [r] \setminus \{1\}, j} a_{i,j}.
\end{equation}
Equivalently,
\begin{equation}
\sum_{L_{i} \in [r] \setminus \{1\}, j} a_{i,j} = \frac{(r - 2)(k + 1)}{2}.
\end{equation}
Thus, in terms of \( r, k \) the number of hyperplanes can be written as

\[
\begin{align*}
\left\lvert \bigcup_{i=1}^{r} L_i \right\rvert &= \sum_{1 \leq i < j \leq r} a_{i,j} = \sum_{j \in [r] \setminus \{i\}} a_{i,j} + \sum_{l \in [r] \setminus \{i\}, l < i} a_{i,l} \\
&= (k + 1) + \frac{(r - 2)(k + 1)}{2} = \frac{r(k + 1)}{2}.
\end{align*}
\]

Since \( a_{i,j} \geq 1 \)

\[
\sum_{1 \leq i < j \leq r} a_{i,j} = \sum_{j \in [r] \setminus \{i\}} a_{i,j} + \sum_{l \in [r] \setminus \{i\}, l < i} a_{i,l} \\
\geq r - 1 + \frac{(r - 1)(r - 2)}{2} = \binom{r}{2}.
\]

Notice that since \( \frac{r(k + 1)}{2} \geq \binom{r}{2} \), we have

\[
k \geq r - 2.
\]

Once we determine the tuple \((a_{i,j})_{i,j}\), the \( r \)-set \( T = \{L_1, \ldots, L_r\} \) is uniquely determined up to renumbering of elements in \( \bigcup_{i=1}^{r} L_i \) or indices of the sets \( L_i \). Thus, to classify \( r \)-sets \( T \) of non-intersecting type, it is enough to determine a tuple \((a_{i,j})_{i,j}\) assuming (18) and (20). In particular, it is enough to determine \( a_{i,j} \) for \( 2 \leq i < j \leq r \), since once \( a_{i,j} \) for \( 2 \leq i < j \leq r \) are determined, the remaining ones \( a_{i,j}, j \in [r] \setminus \{i\} \) are automatically determined by (14).

In other words, \( r \)-sets of non-intersecting type correspond to decompositions into the sum

\[
\frac{(r - 2)(k + 1)}{2} = \sum_{l \in [r] \setminus \{i\}, l < i} a_{i,l}.
\]

Summarizing the above discussion, we obtain the following proposition.

**Proposition 4.2.** Let \( r \geq 4, k \geq r - 2 \) and \( T = \{L_1, \ldots, L_r\} \) be an \( r \)-set of non-intersecting type such that \( \left\lvert \bigcup_{i=1}^{r} L_i \right\rvert = \frac{r(k + 1)}{2} \). Then, the tuple \((a_{i,j})_{1 \leq i < j \leq r}\) one-to-one corresponds to the tuple \((a_{i,j})_{l \in [r] \setminus \{i\}, l < i}\), \( 1 < t \), which also corresponds to the decomposition into the sum

\[
\frac{(r - 2)(k + 1)}{2} = \sum_{l \in [r] \setminus \{i\}, l < i} a_{i,l}.
\]

Let us see examples of \( r \)-sets of non-intersecting type.

**Example 4.3.** (4-set of non-intersecting type). Let us consider \( r = 4 \). In this case we have a correspondence between tuples \((a_{i,j})_{2 \leq i < j \leq 4}\) and sum decompositions \( \frac{(r - 2)(k + 1)}{2} = k + 1 = a_{2,3} + a_{2,4} + a_{3,4} \), where \( k \geq 2 \). The following are examples for \( k = 2 \) and \( k = 3 \).

The case \( k = 2 \). There exists only one correspondence:

\[
3 = a_{2,3} + a_{2,4} + a_{3,4} = 1 + 1 + 1
\]

\[
\leftrightarrow (a_{1,2}, a_{1,3}, a_{1,4}, a_{2,3}, a_{2,4}, a_{3,4}) = (1, 1, 1, 1, 1, 1).
\]

In particular, we have a 4-set \( T = \{L_1, L_2, L_3, L_4\} \) with \( L_1 = \{1, 2, 3\}, L_2 = \{1, 4, 5\}, L_3 = \{2, 4, 6\}, L_4 = \{3, 5, 6\} \) for example. This 4-set is assumed in Example 5.7.

The case \( k = 3 \). The following is one example:

\[
4 = a_{2,3} + a_{2,4} + a_{3,4} = 2 + 1 + 1
\]

\[
\leftrightarrow (a_{1,2}, a_{1,3}, a_{1,4}, a_{2,3}, a_{2,4}, a_{3,4}) = (1, 1, 2, 1, 1, 1).
\]

In particular, we have \( T = \{L_1, L_2, L_3, L_4\} \) with \( L_1 = \{1, 2, 3, 4\}, L_2 = \{1, 5, 6, 7\}, L_3 = \{2, 5, 6, 8\}, L_4 = \{3, 4, 7, 8\} \) for example.

**Example 4.4.** (5-set of non-intersecting type). Let us consider \( r = 5 \). In this case we have correspondence between tuples \((a_{i,j})_{2 \leq i < j \leq 5}\) and sum decompositions \( \frac{(r - 2)(k + 1)}{2} = \frac{k(k + 1)}{2} = a_{2,3} + a_{2,4} + a_{2,5} + a_{3,4} + a_{3,5} + a_{4,5} \). Since the number \( \frac{3k + 1}{2} \) corresponds to the number of hyperplanes, \( k \) should be odd number with \( k \geq 3 \).

Let us see examples for \( k = 3 \) and \( k = 5 \).
The case \( k = 3 \). There exists only one correspondence:

\[
6 = a_{2,3} + a_{2,4} + a_{2,5} + a_{3,4} + a_{3,5} + a_{4,5} = 1 + 1 + 1 + 1
\]

\[
\Leftrightarrow (a_{1,2}, a_{1,3}, a_{1,4}, a_{1,5}, a_{2,3}, a_{2,4}, a_{2,5}, a_{3,4}, a_{3,5}, a_{4,5}) = (1, 1, 1, 1, 1, 1, 1).
\]

For example we have \( \mathbb{T} = \{L_1, L_2, L_3, L_4, L_5\} \) with \( L_1 = \{1, 2, 3, 4\}, L_2 = \{1, 5, 6, 7\}, L_3 = \{2, 5, 8, 9\}, L_4 = \{3, 6, 8, 10\}, L_5 = \{4, 7, 9, 10\} \). This 5-set is provided in Example 5.3 in [22].

The case \( k = 5 \). Following is one example:

\[
9 = a_{2,3} + a_{2,4} + a_{2,5} + a_{3,4} + a_{3,5} + a_{4,5} = 1 + 1 + 1 + 2 + 2
\]

\[
\Leftrightarrow (a_{1,2}, a_{1,3}, a_{1,4}, a_{1,5}, a_{2,3}, a_{2,4}, a_{2,5}, a_{3,4}, a_{3,5}, a_{4,5}) = (3, 1, 1, 1, 1, 1, 2, 2, 2).
\]

For example we have \( \mathbb{T} = \{L_1, L_2, L_3, L_4, L_5\} \) with \( L_1 = \{1, 2, 3, 4, 5, 6\}, L_2 = \{1, 2, 3, 7, 8, 9\}, L_3 = \{4, 7, 10, 11, 12, 13\}, L_4 = \{5, 8, 10, 11, 14, 15\}, L_5 = \{6, 9, 12, 13, 14, 15\} \).

4.2. **Intersecting type r-sets.** As the next, we consider more complicated case admitting some three sets of an r-set are not empty.

**Definition 4.5.** Let \( r \geq 3 \). We say that \( r \)-set \( \mathbb{T} = \{L_1, \ldots, L_r\} \) is an intersecting type if \( r = 3 \) and \( L_i \cap L_j \cap L_k = \emptyset \) or if \( r \geq 4 \) \( L_i \cap L_j \cap L_k \neq \emptyset \) for some distinct \( i, j, k \).

**Remark 4.6.** It is natural to classify 3-set with \( L_i \cap L_j \cap L_k = \emptyset \) into non-intersecting type, but we regard such 3-set as the trivial intersecting type for later convenience.

Let us denote

\[
L_i = \bigcup_{j \in [r] \setminus \{i\}} A_{i,j}
\]

where \( A_{i,j} = L_i \cap L_j \subset [n] \). For \( I \subset [r] \) with \( 2 \leq |I| \leq r - 1 \) we denote \( A_I = \cap_{i \in I} L_i \) and \( |A_I| = a_I \). We assume \( a_{\emptyset} = 0 \), and otherwise, by considering a restriction arrangement we obtain an arrangement with \( r \)-set \( \mathbb{T}' = \{L_i' \}_{i=1}^{r=r} \) such that \( \cap_{i=1}^{r=} L_i' = \emptyset \), i.e., \( a_{|r|} = 0 \), \( r' < r \). Since for \( I, J \subset [r] \) we should have \( |A_I| > |A_J| \) if \( |I| < |J| \), we also assume \( a_I > a_J \) if \( |I| < |J| \).

Let us denote \( I_i = \{ I \subset [r] \mid |I| = i, i \in I \} \).

Since

\[
L_i = \bigcup_{j \in [r] \setminus \{i\}} A_{i,j}
\]

we have

\[
k + 1 = \sum_{I=2}^{r-1} (-1)^I \sum_{I \in I_i} a_I = \sum_{I \in I_i} a_{I,j} + \sum_{I=3}^{r-1} (-1)^I \sum_{I \in I_i} a_I
\]

by set theoretic computation.

**Remark 4.7.** If \( a_I = 0 \) for any \( I \) with \( |I| \geq 3 \), we obtain the non-intersecting type r-sets explained in subsection 4.7.

Since \( A_{i,j} = L_i \setminus \bigcup_{I \in [r] \setminus \{i, j\}} A_{i,j} \), we have

\[
a_{1,j} = k + 1 - \sum_{I \in [r] \setminus \{i, j\}} a_{I,j} - \sum_{I=3}^{r-1} (-1)^I \sum_{I \in I_i} a_I
\]

By summing both sides of the formula (28) for all \( j \in [r] \setminus \{1\} \), we obtain

\[
\sum_{j \in [r] \setminus \{1\}} a_{1,j} = \sum_{j \in [r] \setminus \{1\}} \left( k + 1 - \sum_{I \in [r] \setminus \{1, j\}} a_{I,j} - \sum_{I=3}^{r-1} (-1)^I \sum_{I \in I_i} a_I \right)
\]

\[
= (r - 1)(k + 1) - \sum_{j \in [r] \setminus \{1\}} \sum_{I \in [r] \setminus \{1, j\}} a_{I,j} - \sum_{I=3}^{r-1} (-1)^I \sum_{I \in I_i} a_I.
\]
Thus, we have the following relation.

\[
    k + 1 - \sum_{i=3}^{r-1} (-1)^i \sum_{I \subseteq \mathcal{I}_j} a_I \\
    = (r - 1)(k + 1) - \sum_{j \in [r] \setminus \{1\}} \sum_{I \subseteq \mathcal{I}_j} a_{I,j} - \sum_{j \in [r] \setminus \{1\}} \sum_{i=3}^{r-1} (-1)^i \sum_{I \subseteq \mathcal{I}_j} a_I; 
\]

that is

\[
    (r - 2)(k + 1) = \\
    \sum_{j \in [r] \setminus \{1\}} \sum_{I \subseteq \mathcal{I}_j} a_{I,j} + \sum_{j \in [r] \setminus \{1\}} \sum_{i=3}^{r-1} (-1)^i \sum_{I \subseteq \mathcal{I}_j} a_I - \sum_{i=3}^{r-1} (-1)^i \sum_{I \subseteq \mathcal{I}_j} a_I \\
    = 2 \sum_{l \in [r] \setminus \{1\}, j \in \mathcal{L}} a_{I,j} + \sum_{j \in [r] \setminus \{1\}} (-1)^r \left( \sum_{I \subseteq \mathcal{I}_j} a_I - \sum_{I \subseteq \mathcal{I}_j} a_I \right). 
\]

In terms of \( r, k, \) and \( a_I, \) the number of hyperplanes can be written as

\[
    \left| \bigcup_{i=1}^{r} L_i \right| = \sum_{i=2}^{r-1} (-1)^i \sum_{j \in [r] \setminus \{1\}} \sum_{I \subseteq \mathcal{I}_j} a_I \\
    = \sum_{(j) \in \{1\}} a_{I,j} + \sum_{l \in [r] \setminus \{1\}} a_{I,l} + \sum_{j \in [r] \setminus \{1\}} (-1)^i \sum_{I \subseteq \mathcal{I}_j} a_I \\
    = (r - 1)(k + 1) - \sum_{l \in [r] \setminus \{1\}, j \in \mathcal{L}} a_{I,l} + \sum_{i=3}^{r-1} (-1)^i \sum_{I \subseteq \mathcal{I}_j} a_I. 
\]

Unlike non-intersecting case, computing the number of hyperplanes would be complicated because of the terms \( \sum a_{I,j} \) and \( \sum (-1)^i \sum a_I. \) It would be also difficult to give an explicit lower bound for \( k \) which is the dimension of the space. These facts give us the difficulty to give an explicit classification of general \( r \)-sets of intersecting type as in Proposition 4.2

Remark that once we fix numbers \( \{a_{I,j}, l \in [r] \setminus \{1\}, j \in \mathcal{L} \}, \) \( \{a_I, I \in [r] \setminus \{1\} \} \) satisfying (31), the remaining numbers \( \{a_I, I \in [r] \setminus \{1\} \} \) are uniquely determined by equation (28), and thus, the \( r \)-set \( \mathcal{T} = \{L_1, \ldots, L_r\} \) is uniquely determined up to renumbering of elements in \( \bigcup_{i=1}^{r} L_i \) or indices of the sets \( L_i. \)

Though it is difficult task to give an explicit classification of general intersecting \( r \)-sets as described in Section 4.1, it includes an important class of \( r \)-sets which already appeared in [15], [20], [21]. Let us see such examples.

**Example 4.8 (Good 3s-partition).** Let us consider \( r = 3. \) In this case we have \( \bigcup_{i=1}^{3} L_i \setminus \{ \} = \frac{3(3+1)}{2}. \) Since the number \( \frac{3(3+1)}{2} \) corresponds to the number of hyperplanes, \( k \) should be a positive odd number, so we can write it as \( k = 2s - 1. \)

Since in the case of \( s = 1 \) non-very generic arrangement does not appear (in this case we have braid arrangement with three hyperplanes \( \{H_{i,j} \}_{1 \leq i < j \leq 3} \), which is very generic), we have to assume \( s \geq 2. \)

We have \( |L_i| = k + 1 = 2s \) and \( |L_i \cap L_j| = s \) since there is only one correspondence

\[
    \frac{(3 - 2)(k + 1)}{2} = s = a_{2,3} \leftrightarrow (a_{1,2}, a_{1,3}, a_{2,3}) = (s, s, s). 
\]

**Remark** that the 3-set \( \mathcal{T} = \{L_1, L_2, L_3\} \) given in this example is the good 3s-partition first considered in [15] and developed in [20].

**Example 4.9 (4-set of intersecting type).** Equation (37) would be

\[
    18 = 2(a_{2,3} + a_{2,4} + a_{3,4}) - (a_{1,2,3} + a_{1,2,4} + a_{1,3,4}) - 3a_{2,3,4}. 
\]

Consider the following tuple

\[
    (a_{2,3}, a_{2,4}, a_{3,4}, a_{1,2,3}, a_{1,2,4}, a_{1,3,4}, a_{2,3,4}) = (6, 6, 3, 3, 3, 3). 
\]
Then the other corresponding numbers are
\[(a_{1,2}, a_{1,3}, a_{1,4}) = (3, 3, 3).\]

Then we have a 4-set \(T = \{L_1, L_2, L_3, L_4\}\) with \(L_1 = [12] \setminus \{10, 11, 12\}, L_2 = [12] \setminus \{7, 8, 9\}, L_3 = [12] \setminus \{4, 5, 6\}, L_4 = [12] \setminus \{1, 2, 3\}\) for example. Notice that this 4-set is assumed in Example 4.8 (see also [22]).

The following proposition is a generalization of Examples 4.8 and 4.9.

**Proposition 4.10.** Let us fix \(s \geq r - 1\). For subsets \(K_i \subset [rs]\) such that \(|K_i| = |K_j| = s\), \(K_i \cap K_j = \emptyset\) for any \(i, j\) and \(\bigcup_{i=1}^{r'} K_i = [rs]\) define \(L_i = [rs] \setminus K_i\). Then, the equation \((37)\) holds.

**Proof.** The proof is by direct computation. Since
\[(r - 2)(k + 1) = (r - 2)(r - 1)s\]
by (28), while
\[
(r - 2)(k + 1) = 2 \sum_{l \in [r]\setminus [1], l \neq r} a_{i,l} + \sum_{l=3}^{r' - 1} (-1)^{l} \left( \sum_{l \in [r]\setminus [1]} \sum_{l \neq l'} a_{i,l} - \sum_{l \in [r]\setminus [1]} a_{l,l'} \right)
\]
\[
= 2\left(\frac{r - 1}{2}\right)(r - 2)s + \sum_{l=3}^{r' - 1} (-1)^{l} \left( \sum_{l \in [r]\setminus [1]} a_{i,l} - \sum_{l \in [r]\setminus [1]} a_{l,l'} \right)
\]
Thus, the equation \((31)\) holds. \(\Box\)

**Remark 4.11.** For non-intersecting type \(T\), we gave an explicit classification as in Proposition 4.2, while for intersecting type \(T\) it would be complicated task to give such a classification. The r-set assumed in Proposition 4.10 would be the first classification for intersecting type r-set, which is one of the main topics in Section 5.

5. A SUFFICIENT CONDITIONS FOR NON-VERY GENERICITY

For a fixed \(d \in \mathbb{Z}\) consider an r-set \(T = \{L_1, \ldots, L_r\}\) and sets of vectors \(\{v_{i,t}\}_{i=2,\ldots,r}, t = 1, \ldots, d\) where each vector satisfies \(v_{i,j}' = v_{1,j} - v_{1,t}'\).

In this section let us give sufficient conditions for \(T\) to be non-very generic by giving conditions on the certain numbers of the sets \(\{v_{i,j}'\}_{i=2,\ldots,r}, t = 1, \ldots, d\) and showing they are sufficient conditions to be \(K_T\)-vector set. The conditions would be tractable and easy ones to check by hand.

**Notation 5.1.** We denote \(a_t = |\bigcap_{i \in I} L_i|\). If \(|I| = 2\), we also denote \(a_{i,j} = |L_i \cap L_j|\) for specifically.

The case of the r-set \(T\) being non-intersecting type. For \(1 \leq a < b \leq r\) let us denote by
\[
V_{a,b} = \{v_{a,b}' \mid t = 1, \ldots, d\}
\]
the vector space spanned by vectors \(v_{a,b}' \in H_{a,b}^0, t = 1, \ldots, d\). Since vectors \(v_{a,b}'\), \(t = 1, \ldots, d\) are contained in \(H_{a,b}^0\), for any \(a, b\), it follows that
\[
H_{a,b}^0 \supset V_{a,b},
\]
equivalently,
\[
\langle a_p \mid p \in L_a \cap L_b \rangle = \sum_{p \in L_a \cap L_b} (H_{a,b}^0)^\perp \subset V_{a,b}^\perp
\]
The following proposition holds.

**Proposition 5.2.** Let us fix \(n = \binom{d+k+1}{2}\) and \(T = \{H_{a,b}^0\}_{a=1,\ldots,n}\) be an arrangement in \(\mathbb{C}^k\) with normal vectors \(a_t\). Let \(T = \{L_1, \ldots, L_r\}\) be an r-set of non-intersecting type. If there exist \(d \geq n - k - r + 1\) sets of vectors \(\{v_{i,j}'\}_{i=2,\ldots,r}, t = 1, \ldots, d\) satisfying
\[
\dim \sum_{(i,j) \in D(t)} V_{i,j}^\perp \begin{cases} \geq \sum_{(i,j) \in D(t)} a_{i,j} \text{ if } \sum_{(i,j) \in D(t)} a_{i,j} < k, \text{ and} \\ = k \text{ if } \sum_{(i,j) \in D(t)} a_{i,j} \geq k, \end{cases}
\]
then $\mathcal{R}^0$ is non-very generic.

**Proof.** Let $\{v_{t,i}^t\}_{i=1,\ldots,n}$, $t = 1, \ldots, d$ satisfy (39). Then, there exist $\sum_{(i,j) \in D(t)} a_{i,j}$ independent vectors in $\sum_{(i,j) \in D(t)} V_{t,i}^j$ if $\sum_{(i,j) \in D(t)} a_{i,j} < k$, and $k$ independent vectors if $\sum_{(i,j) \in D(t)} a_{i,j} \geq k$. Thus, we can choose generic vectors $a_i$, $i = 1, \ldots, n$ from orthogonal spaces $V_{d,b}$, $1 \leq a < b \leq r$. That is we obtain a generic arrangement $\mathcal{R}^0$.

For each translation $\mathcal{R}$ of $\mathcal{R}^0$ consider $r - 1$ vectors $v_{t,i}^t$, $j \neq i$. Since $v_{t,i}^t \in H^t_{i,j}$, we have $\bigcap_{k \in \mathbb{Z}} H^t_{i,k} = \bigcap_{j \in 
} H^t_{i,j} \neq \emptyset$ which gives an intersection point $P_i$ of exactly $k + 1$ hyperplanes. Thus, the translation $\mathcal{R}$ is $K_T$-translated, i.e., the sets $\{v_{t,i}^t\}_{i=1,\ldots,n}$, $t = 1, \ldots, d$ are $K_T$-vector sets.

If the $K_T$-vector sets are linearly dependent, by replacing some vector $v_{t,i}^t$ with its multiple we can assume that the $K_T$-vector sets are independent a priori. Notice that the assumption $d \geq n - k - r + 1$ satisfies the one in Theorem 2.10. Therefore, Theorem 2.10 implies that if there exist $d \geq n - k - r + 1$ sets $\{v_{t,i}^t\}_{i=1,\ldots,n}$, $t = 1, \ldots, d$ of vectors $v_{t,i}^t$ satisfying (39), then $\mathcal{R}^0$ is non-very generic.

**Remark 5.3.** Let $\mathcal{R}^0$ be an arrangement of $n \left(\frac{r(k+1)}{2}\right)$ hyperplanes in $\mathbb{C}^k$, and $\mathbb{T} = \{L_1, \ldots, L_r\}$ be an $r$-set of non-intersecting type satisfying $\bigcup_{i=1}^r L_i \subset [n]$. Let $\mathcal{B}^0 \subset \mathcal{R}^0$ be a subarrangement consists of hyperplanes indexed in $\bigcup_{i=1}^r L_i$. If $\mathcal{B}^0$ is non-very generic, then $\mathcal{R}^0$ is non-very generic. Analogously, if there exists a restriction arrangement $(\mathcal{R}^0)^{\mathbb{T}} = \{H^0_i \cap D_{\mathbb{T}} \mid H^0 \in \mathcal{R}^0 \setminus \mathcal{B}^0\}$, $\mathcal{B}^0 = \left\{H_p \mid p \in \bigcup_{i=1}^r L_i \right\}$ with $d \geq \frac{r(k+1)}{2} - k + r + 1$ the sets $\{v_{t,i}^t\}_{i=1,\ldots,n}$, $t = 1, \ldots, d$ satisfying (39), then $\mathcal{R}^0$ is non-very generic.

By Proposition 5.2 and Remark 5.5 we have the following theorem in which we do not assume $n = \frac{r(k+1)}{2}$.

**Theorem 5.4.** Let $\mathcal{R}^0 = \{H^0_j\}_{j=1,\ldots,n}$ be an arrangement in $\mathbb{C}^k$ and $\mathbb{T} = \{L_1, \ldots, L_r\}$ be an $r$-set of non-intersecting type such that $\bigcup_{j=1}^n L_j = \frac{r(k+1)}{2}$. If there exists a restriction arrangement $(\mathcal{R}^0)^{\mathbb{T}}$, $\mathcal{B}^0 = \left\{H_p \mid p \in \bigcup_{i=1}^r L_i \right\}$ with $d \geq \frac{r(k+1)}{2} - k + r + 1$ the sets $\{v_{t,i}^t\}_{i=1,\ldots,n}$, $t = 1, \ldots, d$ satisfying (39), then $\mathcal{R}^0$ is non-very generic.

According to Theorem 5.4, if $\mathbb{T}$ is the non-intersecting $r$-set, then $\frac{r(k+1)}{2}$ is the minimum number of hyperplanes which give non-very generic arrangement. Thus, as a corollary of Theorem 5.4 we have the following.

**Corollary 5.5.** Let $\mathbb{T}$ be an $r$-set of non-intersecting type and $n < \frac{r(k+1)}{2}$. Any arrangement of $n$ hyperplanes is very generic.

The case of the $r$-set $\mathbb{T}$ being intersecting type. An example of a non-very generic arrangement with a 3-set of intersecting type first appeared in [6] and then was further developed in [15]. Based on work in [15], the authors defined a good 3-partition in [20] and showed that it gives rise to non-very generic arrangements. Let us introduce the good $rs$-partition as a generalization of the good 3s-partition and show that it gives rise to non-very generic arrangements. This would be the first classification of intersecting type $r$-sets.

**Definition 5.6.** Let us fix $s \geq r - 1$. For subsets $K_i \subset [rs]$ such that $|K_i| = |K_j| = s$, $K_i \cap K_j = \emptyset$ for any $i, j$ and $\bigcup_{i=1}^n K_i = [rs]$, define $L_s = [rs] \setminus K_i$. We call the set $\mathbb{T} = \{L_1, \ldots, L_r\}$ a good $rs$-partition.

Let $\mathcal{R}^0 = \{H_j^0\}_{j=1,\ldots,n}$ be an arrangement in $\mathbb{C}^k$ with normal vectors $a_i$, $i = 1, \ldots, n$ and $\mathbb{T} = \{L_1, \ldots, L_r\}$ be a good $rs$-partition with $L_i = [rs] \setminus K_i$, $K_i = \{(r-i)s + 1, \ldots, (r-i+1)s\}$, $i = 1, \ldots, r$. We assume $n = rs$ for a while in this section.

For $l = 1, \ldots, r$ denote by

$$V_{[r] \setminus [l]} = \left\{v_{a,b} \mid a, b \in [r] \setminus [l], t = 1, \ldots, d\right\}$$

the vector space spanned by vectors $v_{a,b}$, $a, b \in [r] \setminus [l]$, $t = 1, \ldots, d$. Since for fixed $l$ the vectors $v_{a,b}$ are contained in $H_a^b$, it follows that

$$\bigcap_{p \in \bigcup_{[r] \setminus [l]} L_s} H_p^0 \supset V_{[r] \setminus [l]}$$

and equivalently,

$$\left\{a_p \mid p \in \bigcap_{[r] \setminus [l]} L_s\right\} = \sum_{p \in \bigcup_{[r] \setminus [l]} L_s} (H_p^0)^{\perp} \subset V_{[r] \setminus [l]}.$$

The following proposition holds.
Proposition 5.7. Let $\mathcal{A}^0 = \{H^0\}_{i=1,...,r}$ be an arrangement in $\mathbb{C}^{(r-1)n-1}$ with normal vectors $\alpha_i$ and let $\mathcal{A} = \{L_1, \ldots, L_r\}$ be a good $rs$-partition. If there exist $d \geq s - r + 2$ sets of vectors $\{v^t_{1,i}\}_{i=2,...,r}$, $t = 1, \ldots, d$ satisfying

$$\dim \sum_{l \leq 4} V^{l}_{[4],[l]} \geq s|I| \quad \text{for any } I \subset [r], 1 \leq |I| \leq r - 2, \text{ and}$$

$$= (r - 1)s - 1 \quad \text{for any } I \subset [4], |I| = r - 1,$$

then $\mathcal{A}^0$ is non-very generic.

Proof. Let $\{v^t_{1,i}\}_{i=2,...,r}$, $t = 1, \ldots, d$ satisfy the condition (42). Then, by the similar observation in proof of Proposition 5.2, we can choose generic vectors $\alpha_l$, $l = 1, \ldots, n$ from the orthogonal spaces $V^{l}_{[4],[l]}$, $l = 1, \ldots, r$. That is we obtain a generic arrangement $\mathcal{A}^0$. Moreover, as similar as the observation in proof of Proposition 5.2 we can also see the translations $\mathcal{A}$, $t = 1, \ldots, d$ are $K_T$-translations, i.e., the sets $\{v^t_{1,i}\}_{i=2,...,r}$, $t = 1, \ldots, d$ are linearly independent $K_T$-vector sets. Notice that the condition $d \geq s - r + 2$ satisfies the assumption of Theorem 2.10 since $d \geq n - k - r + 1 = rs - (r - 1)s - 1 - r + 1 = s - r + 2$.

Therefore, Theorem 2.10 implies that if there exist $d \geq s - r + 2$ sets of vectors $\{v^t_{1,i}\}_{i=2,...,r}$, $t = 1, \ldots, d$ satisfying the condition (42), then $\mathcal{A}^0$ is non-very generic. □

Remark 5.8. Let $\mathcal{A}^0$ be an arrangement of $n(> rs)$ hyperplanes in $\mathbb{C}^k$, and $\mathcal{T} = \{L_1, \ldots, L_r\}$ be a good $rs$-partition satisfying $\bigcup_{i=1}^n L_i \subset [n]$. Let $\mathcal{B}^0 \subset \mathcal{A}^0$ be a subarrangement consists of hyperplanes indexed in $\bigcup_{i=1}^n L_i$. If $\mathcal{B}^0$ is non-very generic, then $\mathcal{A}^0$ is non-very generic. Analogously, if there exists a restriction arrangement $(\mathcal{A}^0)^{Y_{\mathcal{B}}^0} = \{H^0 \cap Y_{\mathcal{B}}^0 | H^0 \in \mathcal{A}^0 \setminus \mathcal{B}^0\}$, $\mathcal{B}^0 = \bigcap_{j \in \mathcal{B}} H$ of $\mathcal{A}^0$ which is non-very generic, then $\mathcal{A}^0$ is non-very generic.

By Proposition 5.7 and Remark 5.8 we have the following theorem in which we do not assume $n = rs$.

Theorem 5.9. Let $\mathcal{A}^0 = \{H^0\}_{i=1,...,r}$ be an arrangement in $\mathbb{C}^k$ and $\mathcal{T} = \{L_1, \ldots, L_r\}$ be a good $rs$-partition. If there exists a restriction $(\mathcal{A}^0)^{Y_{\mathcal{B}}^0}$, $\mathcal{B}^0 = \bigcap_{j \in \mathcal{B}} H$, $\mathcal{B}^0 = \{H^0 \cap \bigcup_{j \in \mathcal{B}} H \}$ with $d \geq s - r + 2$ the sets $\{v^t_{1,i}\}_{i=2,...,r}$, $t = 1, \ldots, d$ satisfying (42), then $\mathcal{A}^0$ is non-very generic.

As a generalization of Example 5.3 let us see how we find the sets $\{v^t_{1,i}\}_{i=2,...,r}$, $t = 1, \ldots, d$. By Remark 5.8 it suffices to consider arrangement $\mathcal{A}^0$ of $rs$ hyperplanes in $\mathbb{C}^{(r-1)n-1}$. Let $d_{1,l} = \dim \{v^t_{1,i} | t = 1, \ldots, d, i = 2, \ldots, r\}$. The following theorem gives an explicit way how to find $K_T$-vector sets when the $r$-set is the good $rs$-partition.

Theorem 5.10. the sets $\{v^t_{1,i}\}_{i=2,...,r}$, $t = 1, \ldots, d$ satisfy (42) if and only if $v^t_{1,a} \in \{v^t_{1,i} | t = 1, \ldots, d \setminus \{l\}\}$ for any $a \in \{2, \ldots, r\}$, $l \in [d]$ and $\sum_{a \in \{2, \ldots, r\}} d_{1,a} \leq (r - |l| - 1)s - 1$ for any $l \subset [r], 1 \leq |l| \leq r - 2$.

Proof. First, let us prove that if the sets $\{v^t_{1,i}\}_{i=2,...,r}$, $t = 1, \ldots, d$ satisfy (42), then $v^t_{1,a} \in \{v^t_{1,i} | t = 1, \ldots, d \setminus \{l\}\}$ for any $a \in \{2, \ldots, r\}$, $l \in [d]$. We prove this in the case of $a = 2$ by contradiction assuming that there exists a vector $v^t_{1,2}$ such that $v^t_{1,2} \notin \{v^t_{1,i} | i = 3, \ldots, r, t = 1, \ldots, d\}$. In this case we have $v^t_{1,2} \in V_{[r],[2]}$. By (42) we have

$$\dim \sum_{l \leq 2} V^{l}_{[r],[l]} = (r - 1)s - 1 \iff \dim \sum_{l \geq 2} V^{l}_{[r],[l]} = 0.$$
for any $I \subseteq \{r\}, 1 \leq |I| \leq r - 2$ by (42); thus we have $\sum_{i \inLe} d_{1,i} \leq (r - |I| - 1)s - 1$ for any $I \subseteq \{r\}, 1 \leq |I| \leq r - 2$.

Conversely, let assume $\sum_{i \inLe} d_{1,i} \leq (r - |I| - 1)s - 1$ for any $I \subseteq \{r\}, 1 \leq |I| \leq r - 2$. By considering (43), if there exist the sets satisfying the assumptions, then the proof would be completed. For this reason it is sufficient to show there exist such the sets.

Since $v_{t,i} \in \{v_{t,i}' \mid t = 1, \ldots, d \backslash \{l\}\}$ for any $a \in \{2, \ldots, r\}, l \in \{d\}$, we have (44). In particular, we have

$$\dim \langle v_{t,i}' \mid i = 2, \ldots, r, t = 1, \ldots, d \rangle = \sum_{i = 2}^{r} \dim \langle v_{t,i}' \mid t = 1, \ldots, d \rangle = \sum_{i = 2}^{r} d_{1,i}.$$ 

To construct the the sets $\{v_{t,i}' \mid t = 2, \ldots, r, \, t = 1, \ldots, d\}$ we need to choose $(r - 1)d$ vectors $v_{t,i}'$, $i = 2, \ldots, r, \, t = 1, \ldots, d$ with $\dim \langle v_{t,i}' \mid t = 1, \ldots, d \rangle = d_{1,i}, \, i = 2, \ldots, r$. In particular, it is sufficient to choose $\sum_{i = 2}^{r} d_{1,i}$ independent vectors in $\mathbb{C}^{(r - 1)s - 1}$.

By assumption we have $\sum_{i \inLe} d_{1,i} \leq (r - |I| - 1)s - 1$ for any $I \subseteq \{r\}, 1 \leq |I| \leq r - 2$.

Since the inequality

$$d_{1,i} \leq s - 1$$

holds for any $i$ when $|I| = r - 2$, and

$$\left(k - 1\right)d_{1,i} \leq \left(k - 1\right)(s - 1) \leq \left(k - 1\right)s - 1$$

holds for any $1 \leq r - k \leq r - 2$, we have $\sum_{i = 2}^{r} d_{1,i} \leq \left(r - 1\right)(s - 1) < \left(r - 1\right)s - 1$, and thus the the sets we expected actually exist. \hfill \Box

According to Theorem 5.10 if $T$ is the good $rs$-partition, then $rs$ is the minimum number of hyperplanes which give non-very generic arrangement. Thus, as a corollary of Theorem 5.10 we have following.

**Corollary 5.11.** Let $T$ be a good $rs$-partition and $n < rs$. Any arrangement of $n$ hyperplanes is very generic.

**Remark 5.12.** When $T$ is a general case of intersecting type, determining a sufficient condition for the sets $\{v_{t,i}' \mid t = 1, \ldots, d\}$ to be $K_T$-vector sets seems to be dependent on the manner in which tuples $(a_{t,i})_{t \inLe}$ are fixed. By this reason giving explicit conditions for the sets $\{v_{t,i}' \mid t = 1, \ldots, d\}$ to be $K_T$-vector sets is remain open problem, when $T$ is not good $rs$-partition but intersecting type.

**Acknowledgements.** The author would like to thank Anatoly Libgober for useful discussions and for pointing out Corollary 5.10 and Corollary 5.11 and Masahiko Yoshinaga for useful discussions. The author was supported by JSPS Research Fellowship for Young Scientists Grant Number 20J10012.

**References**

[1] C. A. Athanasiadis, The Largest Intersection Lattice of a Discriminantal Arrangement, Beitr. Algebra Geom., 40 (2), 283-289 (1999).

[2] M. M. Bayer, K. A. Brandt, Discriminantal arrangements, fiber polytopes and formality, J. Algebraic Combin., 6 (3), 229-246 (1997).

[3] H. Crapo, The combinatorial theory of structures. In: Matroid theory (Szeged, 1982), Colloq. Math. Soc. János Bolyai, 40, 107-213, North-Holland, Amsterdam (1985).

[4] H. Crapo and G. C. Rota, The resolving bracket. In: White, N.L. (ed.) Invariant Methods in Discrete and Computational Geometry: Proceedings of the Caracca Conference, 13-17 June, 1994, 197-222, Springer, Dordrecht (1995).

[5] P. Das, E. Palezzato, S. Settepanella, The generalized Sylvester’s and orchard problems via discriminantal arrangement, [arXiv:2201.03007](https://arxiv.org/abs/2201.03007) [math.CO].

[6] M. Falk, A note on discriminantal arrangements, Proc. Amer. Math. Soc., 122(4), 1221-1227 (1994).

[7] S. Felsner and G.M. Ziegler, Zonotopes associated with higher Bruhat orders, Discrete Math., 241, 301-312 (2001).

[8] M. M. Kapranov, V. A. Voevodsky, Combinatorial-Geometric Aspects of Polycategory Theory: Pasting Schemes and Higher Bruhat Orders (list of Results). In: International Category Theory Meeting (Bangor, 1989 and Cambridge, 1990), Cahiers Topologie Géom. Différentielle Catég., 32, 11-27 (1991).

[9] M. M. Kapranov, V. A. Voevodsky, The free $n$-category generated by a cube, oriented matroids and higher Bruhat orders, Funktional. Anal. Prilozhen., 25(1), 62-65 (1991).

[10] M. M. Kapranov, V. A. Voevodsky, Braided monoidal 2-categories and Manin-Schechtman higher braid groups, J. Pure Appl. Algebra, 92(3), 241-267 (1994).

[11] T. Kohno, Integrable connections related to Manin and Schechtman’s higher braid groups, Illinois J. Math., 34(2), 476-484 (1990).

[12] H. Koizumi, Y. Numata, A. Takemura, On intersection lattices of hyperplane arrangements generated by generic points, Ann. Comb., 16(4), 789-813 (2012).

[13] C. P. A. Kumar, On very generic discriminantal arrangements (2020), [arXiv:2011.05327](https://arxiv.org/abs/2011.05327).

[14] R. Lawrence, A Presentation for Manin and Schechtman’s Higher Braid Groups, MSRI Preprint # 04129-91.
[15] A. Libgober and S. Settepanella, Strata of discriminantal arrangements, J. Singul., 18, 440-454 (2018).
[16] Y. I. Manin and V. V. Schechtman, Arrangements of hyperplanes, higher braid groups and higher bruhat orders, Adv. Stud. Pure Math., 17, 289-308 (1989).
[17] Y. Numata, A. Takemura, On computation of the characteristic polynomials of the discriminantal arrangements and the arrangements generated by generic points. In: Harmony of Gröbner Bases and the Modern Industrial Society, World Scientific, 228-252 (2012).
[18] M. Perling, Divisorial cohomology vanishing on toric varieties, Doc. Math., 16, 209-251 (2011).
[19] T. Saito, S. Settepanella, Small examples of discriminantal arrangements associated to non-very generic arrangements, arXiv:2201.03007 [math.CO].
[20] S. Sawada, S. Settepanella and S. Yamagata, Discriminantal arrangement, 3 times 3 minors of Plücker matrix and hypersurfaces in Grassmannian $Gr(3, n)$, C. R. Math. Acad. Sci. Paris, 355(11), 1111-1120 (2017).
[21] S. Sawada, S. Settepanella and S. Yamagata, Pappus’s theorem in Grassmannian $Gr(3, C^n)$, Ars Math. Contemp., 16(1), 257-276 (2019).
[22] S. Settepanella, S. Yamagata, On the non-very generic intersections in discriminantal arrangements, C. R. Math., 360, 1027-1038 (2022).
[23] S. Settepanella, S. Yamagata, A linear condition for non-very generic discriminantal arrangements, arXiv:2205.04664 [math.CO].
[24] G.M. Ziegler, Higher Bruhat orders and cyclic hyperplane arrangements, Topology 32(2), 259-279 (1993).

Department of Mathematics, Hokkaido University, Japan.
Email address: so.yamagata@math.sci.hokudai.ac.jp