ON AN INTERPOLATION PROBLEM IN THE CLASS OF FUNCTIONS OF EXPONENTIAL TYPE IN A HALF-PLANE

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Abstract. Solvability conditions for interpolation problem \( f(n) = d_n, \ n \in \mathbb{N} \) in the class of entire functions satisfying the condition \(|f(z)| \leq e^{\pi|\text{Im} z|+o(|z|)}, z \to \infty \) are well known. In the presented paper we study the interpolation problem \( f(\lambda n) = d_n \) in the class of exponential type functions in the half-plane. We find sufficient solvability conditions for the considerate problem. In particular, a sufficient part of Carleson’s interpolation theorem is generalized and an analogue of a classic interpolation condition is found in the form

\[
\sum_{j=k}^{\infty} \text{Re} \left( -\xi_j \frac{\lambda_k^2 - 1}{\lambda_k + \lambda_j^2} \right) \leq c_3, \quad \xi_j := \frac{\text{Re} \lambda_j}{1 + |\lambda_j|^2}.
\]

The necessity of sufficient conditions is also discussed. The results are applied to studying a problem on splitting and searching an analogue of the identity \( 2 \cos z = \exp(-iz) + \exp(iz) \) for each function of exponential type in the half-plane. We prove that each holomorphic in the right-hand half-plane function \( f \) obeying the , estimate \(|f(z)| \leq O(\exp(\sigma|\text{Im} z|))\) can be represented in the form \( f = f_1 + f_2 \) and the functions \( f_1 \) and \( f_2 \) holomorphic in the right-hand half-plane satisfy conditions

\[
|f_1(z)| \leq O(\exp(|z|h_-(\varphi))) \quad \text{and} \quad |f_2(z)| \leq O(\exp(|z|h_+(\varphi))),
\]

where \( \sigma \in [0; +\infty), \ z = re^{i\varphi}, \)

\[
h_+(\varphi) = \begin{cases} 
\sigma|\sin \varphi|, & \varphi \in \left[0; \frac{\pi}{2}\right], \\
0, & \varphi \in \left[-\frac{\pi}{2}; 0\right],
\end{cases} \quad h_-(\varphi) = \begin{cases} 
0, & \varphi \in \left[0; \frac{\pi}{2}\right], \\
\sigma|\sin \varphi|, & \varphi \in \left[-\frac{\pi}{2}; 0\right].
\end{cases}
\]

The paper uses methods works by L. Carleson, P. Jones, K. Kazaryan, K. Malyutin and other mathematicians.

Keywords: holomorphic functions of exponential type in the half-plane, interpolation, splitting of holomorphic functions

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1. INTRODUCTION

It is known that for each sequence \( d = (d_n) \in l^{\infty} \) there exists an entire function \( f \) such that \[ f(n) = d_n, \quad n \in \mathbb{N}, \]

\[ |f(z)| \leq e^{\pi|\text{Im} z|+o(|z|)} , \quad z \to \infty. \]

In (1.2), “\( o(|z|) \)” can not be omitted \[1\], \[2\]. Our aim is to prove the following statement.
Theorem 1. For each sequence \((d_n) \in l^\infty\) there exists a holomorphic in the half-plane \(C_+ = \{z : \text{Re } z > 0\}\) function \(f\) such that (1.1) holds and
\[
|f(z)| \leq c_1 e^{\pi |\text{Im } z|}, \quad z \in C_+.
\] (1.3)

Hereinafter \(c_j\) stand for positive constants.

We let \(h \in \mathcal{C}[-\pi/2; \pi/2], \sigma \in [0; +\infty), h_0(\varphi) = \sigma |\sin \varphi|,\)
\[
h_+(\varphi) = \begin{cases} 
\sigma |\sin \varphi|, & \varphi \in \left[0; \frac{\pi}{2}\right], \\
0, & \varphi \in \left[-\frac{\pi}{2}; 0\right],
\end{cases}
\]
\[
h_-(\varphi) = \begin{cases} 
0, & \varphi \in \left[0; \frac{\pi}{2}\right], \\
\sigma |\sin \varphi|, & \varphi \in \left[-\frac{\pi}{2}; 0\right].
\end{cases}
\]

and let \(H^\infty(C_+; h)\) be the space of functions \(f\) holomorphic in \(C_+\) obeying
\[
\|f\| := \sup \{|f(z)| e^{-r h(\varphi)} : z = x + iy = re^{i\varphi} \in C_+\} < +\infty.
\]

We employ Theorem 1 and its modifications for proving the following statement.

Theorem 2. Let \(\sigma \in [0; +\infty)\). Then each function \(f \in H^\infty(C_+; h_0)\) is represented as
\[
f = f_1 + f_2, \quad f_1 \in H^\infty(C_+; h_-), \quad f_2 \in H^\infty(C_+; h_+).
\] (1.4)

The problem on splitting (1.4), which is an analogue of the identity \(\cos \sigma z = \frac{1}{2} e^{i\sigma z} + \frac{1}{2} e^{-i\sigma z}\), arises in seeking analogue of Paley-Wiener theorem for some weighted spaces and studying some convolution type equations (see [3, 4]). It was studied in works by V.M. Dilnyi [5, 6]. However, positive resolving is known mostly for spaces defined by \(L_2\)-metric. For the space \(H^\infty(C_+; h_0)\), the issue remained open. Theorem 2 positively resolves this. A more complicated and important similar question for the space of exponential type in the half-plane defined by \(L_1\)-metric remains open.

Let \(\lambda = (\lambda_n) = ([\lambda_n |e^{i\varphi_n}|)\) be an arbitrary sequence of different complex numbers in the complex half-plane \(C_+, l^\infty(h; \lambda)\) be the space of sequences \(d\), for which
\[
\|d\| := \sup \{|d_n| e^{-|\lambda_n| h(\varphi_n)} : n \in \mathbb{N}\} < +\infty.
\]

Let
\[
S(r) := \sum_{1 < |\lambda_k| \leq r} \left( \frac{1}{|\lambda_k|^2} - \frac{1}{r^2} \right) \text{Re } \lambda_k.
\]

Various interpolation problems in the classes of functions holomorphic in the half-plane were considered in many works, see [7–9] and the references therein. However, the solvability criteria of the interpolation problem
\[
f(\lambda_n) = d_n, \quad n \in \mathbb{N},
\] (1.5)
in the class \(H^\infty(C_+; h_0)\) is not known.

We employ some ideas from [7–9] and obtain the above formulated theorems on the base of the following statement, which in fact contains a sufficient part of the interpolation Carleson theorem; its elementary proof for the half-plane was provided, for instance, in [9].
Theorem 3. Let \( (\lambda_k) \) be a sequence of different complex numbers in the half-plane \( \mathbb{C}_+ = \{z : \text{Re} \, z > 0\} \) such that
\[
\sum_{|\lambda_k| \leq 1} \text{Re} \lambda_k < +\infty, \tag{1.6}
\]
\[
\sup \left\{ S(r) - \frac{\sigma}{\pi} \ln r : r \in [1; +\infty) \right\} < +\infty, \tag{1.7}
\]
\[
\sum_{j=k}^{\infty} \text{Re} \left( -\xi_j \frac{\lambda_j^2 - 1}{\lambda_j + \lambda_j} \right) \leq c_3, \quad \xi_j := \frac{\text{Re} \lambda_j}{1 + |\lambda_j|^2}. \tag{1.8}
\]
Moreover, let the sequence \( (\lambda_k) \) is a subsequence of zeroes of a holomorphic in \( \mathbb{C}_+ \) function \( \Omega \) such that
\[
\left| \frac{\Omega(z) (z + \lambda_k)}{(z - \lambda_k) \text{Re} \lambda_k \Omega''(\lambda_k)} \right| \leq c_0 e^{r h_0(\varphi)} e^{-|\lambda_k| h_0(\varphi_n)}, \quad z = x + iy = r e^{i \varphi} \in \mathbb{C}_+, \quad k \in \mathbb{N}. \tag{1.9}
\]
Then for each sequence \( d \in l^\infty(h_0; \lambda) \) there exists a function \( f \in H^\infty(\mathbb{C}_+; h_0) \) satisfying condition (1.5).

Remark 1. If \( \sigma = 0 \), then conditions (1.6) and (1.7) are equivalent to the condition
\[
\sum_{j=1}^{\infty} \text{Re} \lambda_j \frac{1}{1 + |\lambda_j|^2} < +\infty,
\]
and if \( \Omega(z) = B(z) \) is the Blaschke product for \( \mathbb{C}_+ \), then condition (1.9) is equivalent to the Carleson condition
\[
\inf \left\{ \prod_{k=1, k \neq n}^{\infty} \frac{\lambda_n - \lambda_k}{\lambda_n + \lambda_k} : n \in \mathbb{N} \right\} \geq \delta > 0,
\]
while the latter implies (1.8), see, for instance, [9]. The issue on necessity of conditions (1.8) and (1.9) remains for us open. Some comments on this issues are given in the end of the paper.

2. Proof of Theorem 3

Let \( s_0(t) = \sum_{1 < |\lambda_k| \leq t} \text{Re} \lambda_k \). Since
\[
\frac{1}{t^2} \leq \frac{4}{3} \left( \frac{1}{t^2} - \frac{1}{4s^2} \right) \quad \text{as} \quad |t| \leq |s|,
\]
then
\[
s_0(r) \leq r^2 \sum_{1 < |\lambda_k| \leq r} \frac{\text{Re} \lambda_k}{|\lambda_k|^2} \leq r^2 \frac{4}{3} \sum_{1 < |\lambda_k| \leq r} \left( \frac{1}{|\lambda_k|^2} - \frac{1}{(2r)^2} \right) \text{Re} \lambda_k \leq r^2 \frac{4}{3} \sum_{1 < |\lambda_k| \leq 2r} \left( \frac{1}{|\lambda_k|^2} - \frac{1}{(2r)^2} \right) \text{Re} \lambda_k = \frac{4}{3} r^2 S(2r).
\]
This is why conditions (1.6) and (1.7) implies the convergence of the series \( \sum_{j=1}^{\infty} \left( \frac{\text{Re} \lambda_j}{1 + |\lambda_j|^2} \right)^2 \) and \( \sum_{j=1}^{\infty} \text{Re} \lambda_j \left( 1 + |\lambda_j|^2 \right)^{-3/2} \). Therefore, \( \xi_j \to 0 \). This is why, as in [7–9], in the proof of Theorem 3
we can assume that the sequence \((\xi_j)\) is non-increasing. Let

\[
\Psi_j(z) = -\xi_j \frac{z^2 - 1}{z + \lambda_j} \quad \text{and} \quad F_k(z) = \exp \left( -\sum_{j=k}^{\infty} \Psi_j(z) \right).
\]

The latter series converges uniformly on compact sets in \(\mathbb{C}_+\). Let us show that the sought function is

\[
f(z) = \sum_{k=1}^{\infty} d_k \frac{\Omega(z) \left( z + \lambda_k \right)}{(z - \lambda_k)\Omega'(\lambda_k)2\text{Re}\lambda_k} \left( 1 + z \right) \frac{2\text{Re}\lambda_k}{z + \lambda_k} \frac{\left( 1 + z \right) \frac{2\text{Re}\lambda_k}{z + \lambda_k} e^{\xi_k \lambda_k} F_k(z)}{e^{\xi_k z} F_k(\lambda_k)}.
\]

Indeed,

\[
\frac{z^2 - 1}{z + \lambda_j} = \frac{z - 1 + z\lambda_j}{z + \lambda_j}, \quad \text{Re} \frac{1 + z\lambda_j}{z + \lambda_j} = \frac{(1 + |z|^2) \text{Re} \lambda_j + (1 + |\lambda_j|^2) \text{Re} z}{|z + \lambda_j|^2}
\]

and

\[
\text{Re} \Psi_j(z) = \frac{(1 + |z|^2) \text{Re}^2 \lambda_j}{(1 + |\lambda_j|^2) |z + \lambda_j|^2} + \frac{\text{Re} \lambda_j \text{Re} z}{|z + \lambda_j|^2} - \xi_j \text{Re} z.
\]

Hence,

\[
|F_k(z)| \leq \exp \left( \sum_{j=k}^{\infty} \left( -\frac{(1 + |z|^2) \text{Re}^2 \lambda_j}{(1 + |\lambda_j|^2) |z + \lambda_j|^2} + \xi_j \text{Re} z \right) \right)
\]

\[
\leq \exp (\xi_k \text{Re} z) \exp \left( \sum_{j=k}^{\infty} \left( -\frac{(1 + |z|^2) \text{Re}^2 \lambda_j}{(1 + |\lambda_j|^2) |z + \lambda_j|^2} \right) \right).
\]

Moreover, see [9],

\[
\left| \frac{2\text{Re}\lambda_j}{1 + \lambda_j} \frac{z + 1}{z + \lambda_j} \right|^2 \leq 4 \frac{\text{Re}\lambda_j}{(1 + |\lambda_j|^2) |z + \lambda_j|^2} \left( (|z|^2 + 1) \text{Re} \lambda_j + (1 + |\lambda_j|^2) \text{Re} z \right)
\]

\[
= 4 \text{Re} \frac{\lambda_j}{1 + |\lambda_j|^2} \frac{1 + z\lambda_j}{z + \lambda_j}.
\]

In addition, according condition (1.8),

\[
|F_k(\lambda_k)| = \exp \left( -\sum_{j=k}^{\infty} \text{Re} \left( -\xi_j \frac{\lambda_j}{\lambda_k + \lambda_j} + \xi_j \frac{1}{\lambda_k + \lambda_j} \right) \right) \geq c_2.
\]

Therefore,

\[
d_k \frac{\Omega(z) \left( z + \lambda_k \right)}{(z - \lambda_k)\Omega'(\lambda_k)2\text{Re}\lambda_k} \left( 1 + z \right) \frac{2\text{Re}\lambda_k}{z + \lambda_k} \frac{\left( 1 + z \right) \frac{2\text{Re}\lambda_k}{z + \lambda_k} e^{\xi_k \lambda_k} F_k(z)}{e^{\xi_k z} F_k(\lambda_k)} \leq c_3 \left| \frac{1 + z}{1 + \lambda_k} \right|^2 \frac{2\text{Re}\lambda_k}{z + \lambda_k} \frac{\left( 1 + z \right) \frac{2\text{Re}\lambda_k}{z + \lambda_k} e^{\xi_k \lambda_k} F_k(z)}{e^{\xi_k z} F_k(\lambda_k)} \leq c_4 \frac{\text{Re}\lambda_k}{1 + |\lambda_k|^2} \text{Re} \frac{1 + z\lambda_k}{z + \lambda_k} \exp \left( \sum_{j=k}^{\infty} \left( \text{Re} \lambda_k \frac{1}{1 + |\lambda_j|^2} \text{Re} \frac{1 + z\lambda_k}{z + \lambda_k} \right) \right).
\]

Since

\[
\sum_{k=1}^{\infty} |a_k| \exp \left( -\sum_{j=k}^{\infty} |a_j| \right) < 1,
\]

we arrive at the statement of Theorem 3.
3. Proof of Theorem 1

**Lemma 3.1.** Let \( \sigma \in [0; +\infty) \), the function \( \Omega \in H^\infty(C_+; h_0) \) has the zeroes at the points \( \lambda_k \in C_+ \),
\[
\Omega_k(z) = \frac{\Omega(z)(z + \lambda_k)}{z - \lambda_k}, \quad \tau_k = \frac{\delta_k}{1 + \sqrt{1 + \delta_k^2}},
\]
where \( \delta_k = 1 \) if \( \Re \lambda_k < 1 \) or if \( \sigma = 0 \), and \( \delta_k = (\Re \lambda_k)^{-1} \) if \( \sigma > 0 \) and \( \Re \lambda_k \geq 1 \). Then
\[
|\Omega_k(z)| \leq \frac{c_2}{\tau_k} \exp(\sigma|y|)
\]
as \( z \in C_+, k \in \mathbb{N} \).

**Proof.** Since \( \tau_k \in (0; 1) \) and \( \delta_k = \frac{2\tau_k}{1-\tau_k} \), the circles
\[
U_k := \{ \varsigma \in \mathbb{C} : \left| \frac{\varsigma - \lambda_k}{\varsigma + \lambda_k} \right| < \tau_k \}
\]
are contained in \( C_+ \). Then
\[
|\Omega_k(z)| \leq \frac{|\Omega(z)|}{\tau_k} \leq \frac{c_1}{\tau_k} \exp(\sigma|y|) \quad \text{if} \quad \left| \frac{z - \lambda_k}{z + \lambda_k} \right| \geq \tau_k.
\]
If \( \left| \frac{z - \lambda_k}{z + \lambda_k} \right| < \tau_k \), then by the maximum principle we obtain
\[
|\Omega_k(z)| \leq \max \left\{ \frac{c_1 e^{\sigma|\Im \varsigma|}}{\tau_k} : \left| \frac{\varsigma - \lambda_k}{\varsigma + \lambda_k} \right| = \tau_k \right\} \leq \frac{1}{\tau_k} e^{\sigma|y|+2\sigma\delta_k \Re \lambda_k}.
\]
Since \( \sigma \delta_k \Re \lambda_k \leq \sigma \), this completes the proof. \( \square \)

We note that
\[
\tau_k \geq \frac{1}{3} \Re \lambda_k
\]
if \( \sigma > 0 \) and \( \Re \lambda_k \geq 1 \). Therefore, the proven lemma implies that the sequence \( \lambda = (k) \) satisfies all assumptions of Theorem 3 for \( \sigma = \pi \), and at that, we can take \( \Omega(z) = \sin \pi z \). In addition, \( l^\infty \subset l^\infty(h_0; \lambda) \) if \( \lambda = (k) \). This is why Theorem 1 follows Theorem 3.

4. Proof of Theorem 2

**Lemma 4.1.** Let \( (\lambda_k) \) be a sequence of different complex numbers in the half-plane \( C_+ = \{ z : \Re z > 0 \} \) such that inequalities (1.6), (1.8) hold and
\[
\sup \left\{ S(r) - \frac{\sigma}{2\pi} \ln r : r \in [1; +\infty) \right\} < +\infty.
\]
Let also \( (\lambda_k) \) be a subsequence of zeroes of a holomorphic in \( C_+ \) function \( \Omega \) such that
\[
\left| \frac{\Omega(z)(z + \lambda_k)}{(z - \lambda_k)\Re \lambda_k \Omega'(\lambda_k)} \right| \leq c_0 e^{r h(\varphi)} e^{-|\lambda_k| h(-\varphi_n)}, \quad z = x + iy = re^{i\varphi} \in C_+, \quad k \in \mathbb{N}.
\]
Then for each sequence \( (d_k) \in l^\infty(h_+; \lambda) \) there exists a holomorphic in \( C_+ \) function \( f \in H^\infty(C_+; h_+) \) satisfying condition (1.5).

The proof of this lemma reproduces literally the proof of Theorem 3.
Lemma 4.2. Let \( \sigma \in [0; +\infty) \), a function \( \Omega \in H^\infty(C_+; h_+) \) has zeroes at the points \( \lambda_k \in C_+ \) and
\[
\Omega_k(z) = \frac{\Omega(z)(z + \lambda_k)}{z - \lambda_k}.
\]
Then
\[
|\Omega_k(z)| \leq \frac{c_2}{\tau_k} \exp \left( rh_+(\varphi) \right) \quad \text{as} \quad z = x + iy = re^{i\varphi} \in C_+.
\]

The proof of this lemma is similar to the proof of Lemma 3.1.

We proceed to proving Theorem 2. We assume that \( \sigma = \pi \). Let \( \Omega(z) = e^{-i\frac{\pi}{2}(z-1)} \sin \frac{\pi}{2} (z - 1) \).

This functions has zeroes in \( C_+ \) at the points \( \lambda_k = 2k-1, k \in \mathbb{N} \), and \( \Omega \in H^\infty(C_+; h_+) \). At that, \( |\Omega'(\lambda_k)| = \pi/2 \), and according Lemma 4.2, the sequence \( \lambda_k = 2k-1 \) satisfies all assumptions of Lemma 4.1. Let \( d_k = f(\lambda_k) \). Then \( (d_k) \in l^\infty(h_+; \lambda) \). Hence, according Lemma 4.1, there exists a function \( f_0 \in H^\infty(C_+; h_+) \) such that \( f_0(\lambda_k) = f(\lambda_k), k \in \mathbb{N} \). Let \( \tilde{f}(z) = \frac{f(z) - f_0(z)}{\Omega(z)} \). Since [10]
\[
\left| \sin \frac{\pi}{2} (z - 1) \right| \geq c_0 \exp \left( \frac{\pi}{2} |\text{Im} z| \right)
\]
outside the circles \( |z - \lambda_k| \leq \varepsilon \) and therefore, outside these circles the estimate
\[
|\tilde{f}(z)| \leq c_5 \exp (rh_-(\varphi)), \quad z = x + iy = re^{i\varphi},
\]
holds true. Now by the maximum principle we infer that \( \tilde{f} \in H^\infty(C_+; h_-) \). Moreover,
\[
f(z) = \tilde{f}(z) \Omega(z) + f_0(z) = \frac{1}{2i} \tilde{f}(z) + f_0(z) - \frac{1}{2i} e^{-i\pi z} \tilde{f}(z).
\]

Since \( f_1(z) := \frac{1}{2i} \tilde{f}(z) \in H^\infty(C_+; h_-) \) and \( f_2(z) := f_0(z) - \frac{1}{2i} e^{-i\pi z} \tilde{f}(z) \in H^\infty(C_+; h_+), \) this completes the proof of Theorem 2.

5. Addenda and Remarks

Conditions (1.6) and (1.7) are necessary for the statement of Theorem 3. Indeed, let \( Q(z) = f(z) e^{-\frac{\lambda_k}{z + \lambda_k}} \), where \( f \in H^\infty(C_+; h_0) \) is a function such that \( f(\lambda_k) = 1 \) and \( f(\lambda_k) = 0 \) if \( k \neq 1 \). Then \( Q \in H^\infty(C_+; h_0) \) and \( (\lambda_k) \) is a sequence of zeroes of the function \( Q \). This is why, by the generalized Carleman formula [11] we obtain (1.6) and (1.7) [12]. If the sequence \( (\lambda_k) \) satisfies conditions (1.6) and (1.7), then [10] there exists a function \( \tilde{f} \in H^\infty(C_+; h_0), \) for which this is a sequence of its zeroes. Each function \( f \in H^\infty(C_+; h_0), f \neq 0, \) is represented as [11]
\[
f(z) = e^{ia_0 + a_1z} \tilde{B}(z) \tilde{T}(z), \quad (5.1)
\]
where \( a_0 \in \mathbb{R} \) and \( a_1 \in \mathbb{R} \) are constants,
\[
Q_1(t; z) = \frac{(tz + i)^2}{(1 + t^2)^2(t + iz)},
\]
\[
\tilde{T}(z) = \exp \left\{ \frac{1}{\pi i} \right. \left. \int_{-\infty}^{+\infty} Q_1(t; z) \ln |f_0(it)| dt + dh(t) \right\}, \quad \tilde{B}(z) = \prod_{j=1}^{\infty} W_j(z),
\]
\[f_0(it) = f(it)\] are angular boundary values of \( f(z) \) on \( \partial C_+, h(t) \) is a non-increasing function (a singular boundary function of the function \( f \)), whose derivative vanishes everywhere,
\[
W_j(z) = \frac{z - \lambda_j}{z + \lambda_j} \quad \text{as} \quad |\lambda_j| \leq 1, \quad W_j(z) = \frac{1 - \frac{z}{\lambda_j}}{1 + \frac{z}{\lambda_j}} \exp \left( \frac{z}{\lambda_j} + \frac{z}{\lambda_j} \right) \quad \text{as} \quad |\lambda_j| > 1.
\]

In [13], the following statement was proved.
Proposition 1. If \( f \in H^\infty(C_+; h_0) \) and \( f \not\equiv 0 \), then \( 1_a \) \( \log |f_0| \in L^1_{\text{loc}}(i\mathbb{R}) \), \( 2_a \) \( f_0(iy) \exp (-\sigma|y|) \in L^\infty(\mathbb{R}) \), \( 1_b \) \( \sup \{ K(r) : r \in [1; +\infty) \} < +\infty \), and (1.6) holds, where

\[
K(r) := K_Z(r) + K_S(r) + K_B(r), \quad K_Z(r) := 2 \sum_{1 < |\lambda_k| \leq r} \left( \frac{1}{|\lambda_k|^2} - \frac{1}{r^2} \right) \text{Re} \lambda_k,
\]

\[
K_S(r) := -\frac{1}{\pi} \int_{1 \leq |t| \leq r} \left( \frac{1}{|t|^2} - \frac{1}{r^2} \right) dh(t),
\]

\[
K_B(r) := -\frac{1}{\pi} \int_{1 \leq |t| \leq r} \left( \frac{1}{|t|^2} - \frac{1}{r^2} \right) \log |f_0(it)| \, dt.
\]

Vice versa, if the sequence \((\lambda_k)\) of the points in the half-plane \( C_+ \), a function \( f_0 : i\mathbb{R} \to \mathbb{C} \) and a non-increasing function \( h : \mathbb{R} \to \mathbb{R} \), whose derivative vanishes almost everywhere are such that conditions \( 1_a \), \( 2_a \), \( 1_b \) and (1.6) hold, then the function \( f \) defined by identity (5.1) is holomorphic in \( C_+ \) and satisfies the estimates \(|f(z)| \leq c_1 \exp (\sigma|y| + c_1 x)\). At that, if in the product \( B(z) \) we omit some of the factors, the above estimate remains true and the constant \( c_1 \) does not increase.

Employing this statement and some ideas from the proof of necessary part of Carleson interpolation theorem (see [14]), we confirm that each of the following conditions

\[
\prod_{j \in \mathbb{N}, j \neq k} |W_j(\lambda_k)| \geq c_3 \exp (-c_3 \text{Re} \lambda_k), \quad k \in \mathbb{N},
\]

\[
\sum_{j \in \mathbb{N}, j \neq k} \left( \frac{2 \text{Re} \lambda_k \text{Re} \lambda_j}{|\lambda_k + \lambda_j|^2} - \frac{2 \text{Re} \lambda_k \text{Re} \lambda_j}{1 + |\lambda_j|^2} \right) \leq c_4 \text{Re} \lambda_k, \quad k \in \mathbb{N},
\]

is necessary for the solvability of interpolation problem (1.5) in the class \( H^\infty(C_+; h_0) \) for each sequence \( d \in l^\infty(h_0; \lambda) \). However, we fail in trying to prove the necessity of conditions (1.8) and (1.9). In view of this, it is useful to mention the inequality

\[
\sum_{j=k}^{\infty} \text{Re} \left( -\xi_j \frac{\lambda_j^2 - 1}{\lambda_k + \lambda_j} \right) = \sum_{j=k}^{\infty} \left( \xi_j \frac{(1 + |\lambda_k|^2) \text{Re} \lambda_j + (1 + |\lambda_j|^2) \text{Re} \lambda_k}{|\lambda_k + \lambda_j|^2} - \xi_j \text{Re} \lambda_k \right) \leq \sum_{j=k}^{\infty} \left( \frac{2 \text{Re} \lambda_k \text{Re} \lambda_j}{|\lambda_k + \lambda_j|^2} - \frac{\text{Re} \lambda_k \text{Re} \lambda_j}{1 + |\lambda_j|^2} \right).
\]

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