Rigid germs of finite morphisms of smooth surfaces and rational Belyi pairs

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Abstract. In the paper “On rigid germs of finite morphisms of smooth surfaces” (Sb. Math., 211:10 (2020), 1354–1381), we defined a map $\beta: R \to Bel$ from the set $R$ of equivalence classes of rigid germs of finite morphisms branched in germs of curves having ADE singularity types onto the set $Bel$ of rational Belyi pairs $f: \mathbb{P}^1 \to \mathbb{P}^1$, considered up to the action of $\text{PGL}(2, \mathbb{C})$. In this article the inverse images of this map are investigated in terms of monodromies of Belyi pairs.

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Introduction

In this article we continue our investigation of the properties of germs $F: (U, o') \to (V, o)$ of finite morphisms of smooth surfaces (in what follows, for short, the germs of covers) begun in [3] and [4]. In [3], the notion of deformation equivalence of germs of covers was introduced. A germ of cover $F: (U, o') \to (V, o)$ is called rigid if any germ of cover $F_1: (U_1, o'_1) \to (V, o)$ which is deformation equivalent to $F$ is equivalent to it, that is, in short, the covers $F$ and $F_1$ are different from each other by changes of coordinates in $(U, o')$ and $(V, o)$. In [4] it was proved that if the germ $(B, o) \subset (V, o)$ of the branch curve of a germ of cover $F: (U, o') \to (V, o)$ has one of the ADE singularity types, then $F$ is a rigid germ.

Let $R = \bigcup_{n \geq 1} R_{A_n} \cup \bigcup_{n \geq 4} R_{D_n} \cup \bigcup_{n \in \{6, 7, 8\}} R_{E_n}$ denote the set of rigid germs of covers branched in curve germs having the singularity types $A_n$, $n \geq 1$, $D_n$, $n \geq 4$, and $E_6$, $E_7$, $E_8$, respectively.

A germ of cover $F$ of degree $\deg F = d$ defines a homomorphism $F_*: \pi_1(V \setminus B, p) \to S_d$ (the monodromy of the germ $F$), where $S_d$ is the symmetric group acting on the fibre $F^{-1}(p)$. The group $G_F = \text{im} F_* \subset S_d$ is called the (local) monodromy group of $F$. Note that $G_F$ is a transitive subgroup of $S_d$. By the Grauert-Remmert-Riemann-Stein Theorem (see [7]), the monodromy homomorphism $F_*$ defines the cover $F$ uniquely up to equivalence.

Denote the set of rational Belyi pairs considered up to the action of the group $\text{PGL}(2, \mathbb{C})$ on $\mathbb{P}^1$ by $Bel$. A cover $f: \mathbb{P}^1 \to \mathbb{P}^1$, defined over the algebraic closure $\overline{\mathbb{Q}}$ of the field of rational numbers $\mathbb{Q}$, is called a Belyi pair if it is branched in at most

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three points, $\text{Bel} = \text{Bel}_2 \cup \text{Bel}_3$, where $\text{Bel}_2$ is the set of Belyi pairs branched in at most two points and the Belyi pairs $f \in \text{Bel}_3$ are branched in three points. We will assume below that $f \in \text{Bel}_2$ is given in nonhomogeneous coordinates by functions $z = ax^n, n \geq 1$, and its branch locus is $B_f = \{0, \infty\}$ (if $n \geq 2$), and the branch locus of $f \in \text{Bel}_3$ is $B_f = \{0, 1, \infty\}$.

In [4] a map $\beta: \mathcal{R} \to \text{Bel}$ was defined as follows. Let $F: (U, o') \to (V, o)$ be a germ of cover branched in a germ $(B, o) \subset (V, o)$ having one of the $ADE$ singularity types and let $\sigma: \tilde{V} \to V$ be a minimal sequence of $\sigma$-processes with centres at points such that $\sigma^{-1}(B)$ is a divisor with normal crossings (but if the singularity type of $B$ is $A_0$ or $A_1$, then $\sigma$ is the single $\sigma$-process with centre at $o$). Denote the exceptional curve of the last $\sigma$-process by $E \subset \tilde{V}$ and let $\tilde{F}: \tilde{U} \to \tilde{V}$ and $\tau: \tilde{U} \to U$ denote the two natural holomorphic maps from the normalization of the fibre product $\tilde{U} = U \times_V \tilde{V}$ of the holomorphic maps $F: (U, o') \to (V, o)$ and $\sigma: \tilde{V} \to (V, o)$. It is easy to show that $C = \tilde{F}^{-1}(E)$ is an irreducible rational curve and its restriction $f = \tilde{F}|_C: C \to \tilde{E}$ is branched in at most three points. By definition, the map $\beta$ sends $F \in \mathcal{R}$ to $f \in \text{Bel}$.

Similarly to the two-dimensional case, a cover $f \in \text{Bel}$ defines a homomorphism $f_*: \pi_1(\mathbb{P}^1 \setminus B_f, p) \to S_n$ (the monodromy of $f$), where $n = \deg f$. The image $G_f = \text{im } f_* \subset S_n$ is called the monodromy group of $f$. If $f \in \text{Bel}_2$, then $G_f = \mu_n \subset S_n$ is a cyclic group of order $n$.

The group $\pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}, p)$ is the free group generated by two simple loops $\gamma_0$ and $\gamma_1$ around the points 0 and 1 such that the loop $\gamma_\infty = \gamma_0 \gamma_1$ is the trivial element of $\pi_1(\mathbb{P}^1 \setminus \{0, 1\}, p)$. For $f \in \text{Bel}_3$ let

$$T_c(f) = \{c_i = (m_{1,i}, \ldots, m_{k,i}) \mid m_{1,i} + \ldots + m_{k,i} = \deg f, i \in \{0,1,\infty\}\}$$

denote the set of cycle types of permutations $f_*(\gamma_i)$. Then, by Hurwitz’s formula connecting the degree of $f: \mathbb{P}^1 \to \mathbb{P}^1$ and the orders of ramification at critical points of $f$, we have the following equality:

$$n + 2 = k_0 + k_1 + k_\infty.$$  \hspace{1cm} (1)

Conversely, if a transitive group $G \subset S_n$ is generated by two permutations $\sigma_0$ and $\sigma_1$ such that their cycle types and the cycle type of $\sigma_\infty = \sigma_0 \sigma_1$ satisfy (1) then there is a rational Belyi pair $f$ such that $f_*(\gamma_i) = \sigma_i$.

In [4] it was shown that for $F \in \mathcal{R}$ the covers $\tilde{F}$ and $F$ can be represented as compositions of two finite maps (see diagram (*) in § 2.1), $\tilde{F} = \tilde{H}_2 \circ \tilde{H}_1$ and $F = H_2 \circ H_1$, where $\tilde{H}_1: \tilde{U} \to \tilde{W}$ and $H_1: U \to W$ are cyclic covers (here $\tilde{W}$ and $W$ are normal surfaces) such that $H_1|_C: C \to \tilde{H}_1(C)$ is an isomorphism, and
Let $p_1 = (0, 1)$ and $p_2 = (1, 0)$ be two points such that \( \{ f(p_1), f(p_2) \} \cup B_f = \{0, 1, \infty\} \). Then a cover \( F: (U, o') \to (V, o) \) given by the functions
\[
u = h_1(z^{m_1}, w^{m_2}) \quad \text{and} \quad v = h_2(z^{m_1}, w^{m_2}),
\]
where \( m_1, m_2 \in \mathbb{N} \) are such that \( \gcd(m_1, m_2) = 1 \) and where \( m_1 > 1 \) and \( f(p_1) = 1 \) if \( f \in \text{Bel}_2 \), belongs to \( \mathcal{R}_{D_4} \).

Conversely, any \( F \in \mathcal{R}_{D_4} \) is equivalent to a cover given by functions of the form (2) and its image \( \beta(F) \) is \( f: (x_1, x_2) \mapsto (h(x_1, x_2) : h_2(x_2, x_2)) \).

A complete description of the sets \( \mathcal{R}_T \cap \beta^{-1}(\text{Bel}_2) \) is given in the following theorem.

**Theorem 2.** If \( F \in (\bigcup_{k=1}^{\infty} \mathcal{R}_{A_{2k}}) \cup \mathcal{R}_{E_6} \cup \mathcal{R}_{E_8} \), then \( \beta(F) \in \text{Bel}_3 \).

If \( \beta(F) = f \in \text{Bel}_2 \), where \( \deg f = n \), for \( F \in \mathcal{R} \setminus (\bigcup_{k=1}^{\infty} \mathcal{R}_{A_{2k}}) \cup \mathcal{R}_{E_6} \cup \mathcal{R}_{E_8} \), then \( F \) is equivalent to one of the following covers:

- \( F \in \mathcal{R}_{A_0} \): \( u = z^m, v = w, \) where \( m \geq 1 \) and \( n = 1 \);
- \( F \in \mathcal{R}_{A_1} \): \( u = z^{nm_1}, v = w^{nm_2}, \) where \( n \geq 1 \) and \( m_1 \geq m_2 \geq 1 \);
- \( F \in \mathcal{R}_{A_{2k+1}}, k \geq 1 \): \( u = (z^m + w^{m_0})^n, v = w, \) where \( n, m, m_0 > 1 \) and \( k + 1 = nm_0 \);
- \( F \in \mathcal{R}_{A_{2k+1}}, k \geq 1 \): \( u = z^{nm_1}, v = z^{m_1} + w^{m_2}, \) where \( m_1 \geq 1 \) and \( n, m_2 > 1 \);
- \( F \in \mathcal{R}_{D_{2k+3}}, k \geq 1 \): \( u = z^{m_1}, v = z^{m_1(2k+1)} + w^{m_2}, \) where \( m_1 \geq 1 \), \( m_2 > 1 \) and \( \gcd(2k + 1, m_2) = 1 \);
- \( F \in \mathcal{R}_{D_{2k+2}}, k \geq 2 \): \( u = z^{m_1}, v = (z^{m_1k_2} + w^{m_2})^n, \) where \( k = k_1k_2, n = n_1k_1 \geq 2, m_1, m_2 \geq 1 \) and \( \gcd(nm_2, k_2) = 1 \);
- \( F \in \mathcal{R}_{D_{2k+2}}, k \geq 2 \): \( u = (z^{m_1} - w^{m_2})^n, v = (z^{m_1} - \omega_j w^{m_2})^n, \) where \( n = n_1k_1 \geq 2, m_1, m_2 \geq 1 \) and \( \omega_j = \exp(2\pi ji/n) \) for \( j = 1, \ldots, n - 1 \);
- \( F \in \mathcal{R}_{D_4} : u = z^{m_1}, v = (z^{m_1} + w^{m_2})^n, \) where \( n \geq 2 \) and \( m_1, m_2 \geq 1 \);
- \( F \in \mathcal{R}_{D_4} : u = (z^{m_1} - w^{m_2})^n, v = (z^{m_1} - \omega_j w^{m_2})^n, \) where \( n \geq 2, m_1, m_2 \geq 1 \) and \( \omega_j = \exp(2\pi ji/n) \) for \( 1 \leq j \leq n - 1 \);
- \( F \in \mathcal{R}_{E_2} : u = z^{m_1}, v = z^{m_1} + w^{m_2}, \) where \( m_1 \geq 1 \) and \( m_2 > 1 \).

In all cases \( \gcd(m_1, m_2) = 1 \).

The proof of Theorem 2 is given in §4.
§ 1. Preliminary results

1.1. The fundamental groups. Let \((X, o)\) denote a germ of a normal surface and \((B, o) = \bigcup_{j=1}^{m} B_j\) a union of \(m \geq 0\) irreducible curve germs \((B_j, o) \subset (X, o)\).

Let \(\sigma : \tilde{X} \to (X, o)\) be the minimal resolution of the singularity of the pair \((X, B, o)\), that is, \(\tilde{X}\) is smooth and \(\tilde{B} = \sigma^{-1}(B)\) is a divisor with normal crossings in which each \((-1)\)-curve intersects at least three irreducible components of \(\sigma^{-1}(B)\). Below we assume that \(\sigma^{-1}(o) = \bigcup_{j=1}^{k} E_j\) is a union of rational curves and the dual graph of \(\sigma^{-1}(o)\) is a tree. Also, if this does not lead to misunderstanding, the proper inverse images \(\sigma^{-1}(B_j)\) of the irreducible curve germs \(B_j\) of \((B, o)\) will be marked with the same letter \(B_j\).

The dual weighted graph \(\Gamma(\tilde{B})\) of \(\tilde{B}\) is a tree having \(m + k\) vertices \(v_j\). The vertices \(v_j, j = 1, \ldots, m\), correspond to the curve germs \(B_j\) and their weights are \(w_j = 0\), the vertices \(v_{m+j}, j = 1, \ldots, k\), correspond to the curves \(E_j\) and their weights are \(w_{m+j} = -(E_j^2)\). For each pair of vertices \(v_i, v_j\) of \(\Gamma(\tilde{B})\) we define

\[
\delta_{i,j} = \begin{cases} 
1 & \text{if } v_i \text{ and } v_j \text{ are connected by an edge in } \Gamma(\tilde{B}), \\
0 & \text{if } v_i \text{ and } v_j \text{ are not connected by an edge in } \Gamma(\tilde{B}), \\
0 & \text{if } i = j.
\end{cases}
\]

Theorem 3, which follows, allows us to obtain a presentation of the fundamental group \(\pi_1(\tilde{X} \setminus \tilde{B})\) in terms of the graph \(\Gamma(\tilde{B})\). The proof of this theorem coincides almost word-for-word with the proof of a similar statement in [6] (see also [4]) and therefore we omit it.

**Theorem 3.** The group \(\pi_1(\tilde{X} \setminus \tilde{B})\) is generated by \(m + k\) elements of which \(b_1, \ldots, b_m\) are in one-to-one correspondence with the vertices \(v_1, \ldots, v_m\) of \(\Gamma(\tilde{B})\), and \(e_{m+1}, \ldots, e_{m+k}\) are in one-to-one correspondence with the vertices \(v_{m+1}, \ldots, v_{m+k}\), and that are subject to the following defining relations:

\[
e^{-w_{m+i}} b_1^{\delta_{1,m+i}} \ldots b_m^{\delta_{m,m+i}} e_{m+1}^{\delta_{m+1,m+i}} \ldots e_{m+k}^{\delta_{m+k,m+i}} = 1 \quad \text{for } i = 1, \ldots, k,
\]

\[
[b_j, e_{m+i}] = 1 \quad \text{if } \delta_{j,m+i} = 1,
\]

\[
[e_{m+i_1}, e_{m+i_2}] = 1 \quad \text{if } \delta_{m+i_1,m+i_2} = 1.
\]

**Remark 1.** The generators \(b_1, \ldots, b_m\) and \(e_{m+1}, \ldots, e_{m+k}\) of \(\pi_1(\tilde{X} \setminus \tilde{B})\) in Theorem 3 are presented by some simple loops around the curves corresponding to them (see [4]).

The following lemma is well known (see [5], for example).

**Lemma 1.** Let \((Y, o)\) be a germ of a smooth surface, \(\sigma : X \to (Y, o)\) the \(\sigma\)-process with centre at \(o\), and let \((C_1, o)\) and \((C_2, o)\) be two smooth curve germs in \((Y, o)\) meeting transversally at \(o\). Then \(\gamma_E = \gamma_1 \gamma_2\) in

\[
\pi_1(Y \setminus (C_1 \cup C_2)) \simeq \pi_1(X \setminus \sigma^{-1}(C_1 \cup C_2)),
\]

where \(\gamma_E\) is the element of \(\pi_1(X \setminus \sigma^{-1}(C_1 \cup C_2))\) represented by a simple loop around the exceptional curve \(E = \sigma^{-1}(o)\) and \(\gamma_j, j = 1, 2\), are the elements represented by simple loops around \(C_j\).
1.2. Graphs of resolution of singularities of ADE singularity types. Recall that curve germs \((B, o)\) having one of the ADE singularity types have the following equations (see [1]):

\[
\begin{align*}
\mathbf{A}_n & : \quad u^2 - v^{n+1} = 0, \ n \geq 0; \\
\mathbf{D}_n & : \quad v(u^2 - v^{n-2}) = 0, \ n \geq 4; \\
\mathbf{E}_6 & : \quad u^3 - v^4 = 0; \\
\mathbf{E}_7 & : \quad u(u^2 - v^3) = 0; \\
\mathbf{E}_8 & : \quad u^3 - v^5 = 0.
\end{align*}
\]

The graph \(\Gamma(B)\) of the curve germ \((B, o)\) of singularity type \(\mathbf{A}_{2k+1}, \ k \geq 0\), is depicted in Figure 1 (if \(k = 0\) then the weight of the vertex \(e_3\) is equal to \(-1\)).

\[\text{Figure 1}\]

The graph \(\Gamma(B)\) of the curve germ \((B, o)\) of singularity type \(\mathbf{A}_{2k}, \ k \geq 1\), is depicted in Figure 2.

\[\text{Figure 2}\]

The graph \(\Gamma(B)\) of the curve germ \((B, o)\) of singularity type \(\mathbf{D}_{2k+2}, \ k \geq 1\), is depicted in Figure 3.

\[\text{Figure 3}\]

The graph \(\Gamma(B)\) of the curve germ \((B, o)\) of singularity type \(\mathbf{D}_{2k+3}, \ k \geq 1\), is depicted in Figure 4.

The graph \(\Gamma(B)\) of the curve germ \((B, o)\) of singularity type \(\mathbf{E}_6\) is depicted in Figure 5.

The graph \(\Gamma(B)\) of the curve germ \((B, o)\) of singularity type \(\mathbf{E}_7\) is depicted in Figure 6.

The graph \(\Gamma(B)\) of the curve germ \((B, o)\) of singularity type \(\mathbf{E}_8\) is depicted in Figure 7.
Remark 2. Note that in all graphs \( \Gamma(\tilde{B}) \) of a curve germ \((B, o)\) of ADE singularity type (apart from the singularity types \(A_0\) and \(A_1\)) there is a single vertex \(e\) of valency 3 (we denote the corresponding curve by \(E\)) and this vertex has weight \(w = -1\).

**Proposition 1** (see [4], Corollary 1). Let \((B, o)\) be a curve germ having one of the ADE singularity types, let \(E \subset \sigma^{-1}(o) \subset \tilde{V}\) be the exceptional curve of the last blowup in the sequence of blowups \(\sigma: \tilde{V} \to V\) resolving the singular point of \((B, o)\), and let \(e\) be an element in \(\pi_{1}^{\text{loc}}(B, o)\) represented by a simple loop around \(E\). Then \(e\) belongs to the centre of \(\pi_{1}^{\text{loc}}(B, o) = \pi_1(V \setminus B) \simeq \pi_1(\tilde{V} \setminus \tilde{B})\).

**Proposition 2** (see [4], Proposition 1). Let \((B, o)\) be a curve germ having one of the ADE singularity types. If the singularity type of \((B, o)\) is not \(A_0\) or \(A_1\), then \(\pi_{1}^{\text{loc}}(B, o)\) is generated by \(e\) and the elements \(\gamma_1, \gamma_2, \gamma_3\) corresponding to the vertices of \(\Gamma(\tilde{B})\) connected by an edge to the vertex \(e\) (if the singularity type of \((B, o)\) is \(A_1\), then \(\pi_{1}^{\text{loc}}(B, o)\) is generated by \(b_1, b_2\) and \(e\)).

Below, if the singularity type of \((B, o)\) is \(A_{2n+1}\) or \(D_{2n+2}\), then we identify the element \(\gamma_1\) with \(e_{n+2}\) (see Figures 1 and 3); if the singularity type is \(A_{2n}\), then we identify \(\gamma_1\) with \(e_{n+1}\) and \(\gamma_2\) with \(e_{n+3}\) (see Figure 2); if the singularity type is \(D_{2n+3}\), then we identify \(\gamma_1\) with \(e_{n+2}\) and \(\gamma_2\) with \(e_{n+4}\) (see Figure 4); if the singularity type is \(E_6\), then we identify \(\gamma_1\) with \(e_2\) and \(\gamma_2\) with \(e_4\) (see Figure 5); if the singularity type is \(E_7\), then we identify \(\gamma_1\) with \(e_5\) and \(\gamma_2\) with \(e_3\) (see Figure 6); and if the singularity type is \(E_8\), then we identify \(\gamma_1\) with \(e_5\) and \(\gamma_2\) with \(e_3\) (see Figure 7).
Let \( \overline{B \setminus E} \) be the closure of \( \tilde{B} \setminus E \) in \( \tilde{V} \).

**Definition 1.** If the singularity type of \((B, o)\) is not \(A_0\) or \(A_1\), then \( \overline{B \setminus E} \) is a disjoint union of three chains of curves which we call *trails* of \( B \). Let \( B_j, j = 1, 2, 3 \), denote the trail containing the curve for which \( \gamma_j \) is represented by a loop around this curve. A trail is *exceptional* (completely exceptional) if it contains an exceptional curve of \( \sigma \), that is, one contracted by the map (if it contains only exceptional curves, respectively).

Let \( Z_e \) denote the subgroup of \( \pi^\text{loc}_1(B, o) \) generated by \( e \). The imbedding \( i_1: \tilde{V} \setminus \tilde{B} \hookrightarrow \tilde{V} \setminus (\tilde{B}_1 \cup \tilde{B}_2 \cup \tilde{B}_3) \) induces an epimorphism \( i^*_1: \pi_1(\tilde{V} \setminus \tilde{B}) \rightarrow \pi_1(\tilde{V} \setminus (\tilde{B}_1 \cup \tilde{B}_2 \cup \tilde{B}_3)) \) whose kernel is \( Z_e \). It easily follows from Theorem 3 that \( e = \gamma_1\gamma_2\gamma_3 \) and

\[
\pi_1(\tilde{V} \setminus (\tilde{B}_1 \cup \tilde{B}_2 \cup \tilde{B}_3)) \simeq (\langle \tilde{\gamma}_1 \rangle * \langle \tilde{\gamma}_2 \rangle * \langle \tilde{\gamma}_3 \rangle) / (\tilde{\gamma}_1\tilde{\gamma}_2\tilde{\gamma}_3)
\]

is a quotient group of the free product of three cyclic groups \( \langle \tilde{\gamma}_j \rangle \), \( j = 1, 2, 3 \), by the normal closure \( \langle \langle \tilde{\gamma}_1\tilde{\gamma}_2\tilde{\gamma}_3 \rangle \rangle \) of the cyclic subgroup generated by the product \( \tilde{\gamma}_1\tilde{\gamma}_2\tilde{\gamma}_3 \), where \( \tilde{\gamma}_j = i^*_1(\gamma_j) \). It follows from Theorem 3 that the group \( \langle \tilde{\gamma}_j \rangle \) is finite if and only if \( B_j \) is a union of exceptional trails \( E_l \).

It is easy to see that the imbedding \( i_2: E \setminus (\tilde{B}_1 \cup \tilde{B}_2 \cup \tilde{B}_3) \hookrightarrow \tilde{V} \setminus (\tilde{B}_1 \cup \tilde{B}_2 \cup \tilde{B}_3) \) induces an epimorphism

\[ i^*_2: \pi_1(E \setminus (\tilde{B}_1 \cup \tilde{B}_2 \cup \tilde{B}_3), p) \rightarrow \pi_1(\tilde{V} \setminus (\tilde{B}_1 \cup \tilde{B}_2 \cup \tilde{B}_3), p) \]

(4)

(here we assume that \( p \in E \setminus (\tilde{B}_1 \cup \tilde{B}_2 \cup \tilde{B}_3) \)).

**Remark 3.** Note that if the singularity type of \((B, o)\) is \( D_4 \) then \( i^*_2 \) is an isomorphism. Therefore, in this case we identify the groups \( \pi_1(E \setminus (\tilde{B}_1 \cup \tilde{B}_2 \cup \tilde{B}_3), p) \) and \( \pi_1(\tilde{V} \setminus (\tilde{B}_1 \cup \tilde{B}_2 \cup \tilde{B}_3), p) \).

Set \( P_j = E \cap \tilde{B}_j \) and let \( \overline{\gamma}_j \in \pi_1(E \setminus (\tilde{B}_1 \cup \tilde{B}_2 \cup \tilde{B}_3), p) \) denote a loop around \( P_j \) such that \( i^*_2(\overline{\gamma}_j) = \overline{\gamma}_j \).

**Definition 2.** If \( B_j \) is an exceptional trail, denote the *union of the exceptional curves* contained in \( B_j \) by \( \tilde{B}_j^0 \) and set \( \overline{\pi}_j := \pi_1(N_T \setminus \tilde{B}_j) \) and \( \overline{\pi}_j^0 := \pi_1(N_T \setminus \tilde{B}_j^0) \), where \( N_T \) is a sufficiently small tubular neighbourhood of \( \tilde{B}_j \).

**Remark 4.** It follows from Theorem 3 that \( \overline{\pi}_j \) and \( \overline{\pi}_j^0 \) are cyclic groups.

### 1.3. Cyclic quotients.

Let the cyclic group \( \mu_m \simeq \mathbb{Z}_m \) of order \( m \) act on a germ of a smooth surface \((U, o')\). Let \((W, o_1) \rightarrow (U, o')/\mu_m \) denote the quotient space and \( \xi: (U, o') \rightarrow (W, o_1) \) the quotient map. By Cartan’s lemma we can assume that \((U, o')\) is biholomorphic to the ball \( \mathbb{B}_2 = \{ (u_1, u_2) \in \mathbb{C}^2 \mid |u_1|^2 + |u_2|^2 < 1 \} \) and the action of a generator \( g \) of \( \mu_m \) is given by

\[ g: (u_1, u_2) \mapsto \left( \exp \left( \frac{2\pi p_1 i}{m} \right) u_1, \exp \left( \frac{2\pi p_2 i}{m} \right) u_2 \right), \]

where \( p_j, j = 1, 2 \), are some integers, \( 1 \leq p_j \leq m \), and \( \text{GCD}(m, p_1, p_2) = 1 \). Let \( m = m_1m_2m_0 \), \( p_1 = m_1t_1s \) and \( p_2 = m_2t_2s \), where

\[ \text{GCD}(m_1t_1, m_2t_2) = \text{GCD}(st_1, m_0) = \text{GCD}(st_2, m_0) = 1. \]
Then

\[ g^{m_1m_0} : (u_1, u_2) \mapsto \left( \exp \left( \frac{2\pi p_1 i}{m_2} \right) u_1, u_2 \right) \]
and

\[ g^{m_2m_0} : (u_1, u_2) \mapsto \left( u_1, \exp \left( \frac{2\pi p_2 i}{m_1} \right) u_2 \right) \]

and the subgroup \( \mu_{m_1m_2} \subset \mu_m \), generated by \( g^{m_1m_0} \) and \( g^{m_2m_0} \) is a cyclic group of order \( m_1m_2 \). The map \( \xi \) can be decomposed into a composition of two maps, \( \xi = \varphi \circ \vartheta_{m_1m_2} \), where \( \vartheta_{m_1m_2} : (U, o') \rightarrow (X, \tilde{\varnothing}) \) is the quotient map defined by the action of \( \mu_{m_1m_2} \) on \( (U, o') \) and \( \varphi : (X, \tilde{\varnothing}) \rightarrow (W, o_1) \) is the quotient map defined by the action of the quotient group \( \mu_m/\mu_{m_1m_2} \cong \mu_{m_0} \) of order \( m_0 \) on \( (X, \tilde{\varnothing}) \).

It is easy to see that \( (X, \tilde{\varnothing}) \) is a germ of a smooth surface,

\[ (X, \tilde{\varnothing}) \cong \mathbb{B}_2 = \{(x_1, x_2) \in \mathbb{C}^2 \mid |x_1|^2 + |x_2|^2 < 1 \}, \]

and the map \( \vartheta_{m_1m_2} \) is given by \( x_1 = u_1^{m_2} \) and \( x_2 = u_2^{m_1} \). The image \( \overline{g} \) in \( \mu_{m_0} \) of the generator \( g \in \mu_m \) acts on \( (X, \tilde{\varnothing}) \) as follows:

\[ \overline{g} : (x_1, x_2) \mapsto \left( \exp \left( \frac{2\pi st_1 i}{m_0} \right) x_1, \exp \left( \frac{2\pi st_2 i}{m_0} \right) x_2 \right). \]

There is an integer \( r \) such that \( rst_2 \equiv 1 \) mod \( m_0 \) and \( rst_1 \equiv q \) mod \( m_0 \), where \( 1 \leq m_0 \), since \( \gcd(st_j, m_0) = 1 \) for \( j = 1, 2 \). Therefore,

\[ \overline{g}^r : (x_1, x_2) \mapsto \left( \exp \left( \frac{2\pi qi}{m_0} \right) x_1, \exp \left( \frac{2\pi i}{m_0} \right) x_2 \right), \]

and it is easy to show that \( (W, o_1) \) is the normalization of the germ of singularity in \( \mathbb{B}_3 = \{(z_1, z_2, z_3) \in \mathbb{C}^3 \mid |z_1|^2 + |z_2|^2 + |z_3|^2 < 1 \} \) given by \( z_3^n = z_1 z_2^{n-q} \), where \( z_3 = x_1 x_2^{n-q} \), \( z_1 = x_1^{m_0} \) and \( x_2 = x_2^{m_0} \), that is, the germ \( (W, o_1) \) has the so-called Hirzebruch-Jung singularity type \( A_{m_0,q} \).

The map \( \vartheta_{m_1m_2} \) is branched in \( L_1 = \{x_1 = 0\} \) (if \( m_2 > 0 \)) and \( L_2 = \{x_2 = 0\} \) (if \( m_1 > 0 \)), and \( \varphi \) is unramified outside \( o_1 \) (in what follows we denote the map \( \varphi \) by \( \theta_{m_0,q} \)). Therefore, \( \pi_1(W \setminus o_1) \cong \mu_{m_0} \) and \( \theta_{m_0,q} : X \setminus \tilde{\varnothing} \rightarrow W \setminus o_1 \) is the universal unramified cover.

![Figure 8](image-url)

Let \( \tau : \widetilde{W} \rightarrow (W, o_1) \) be the minimal resolution of the singular point \( o_1 \subset W \). Let \( B_j = \tau^{-1}(\theta_{m_0,q}(L_j)) \), \( j = 1, 2 \), denote the proper inverse image of \( \theta_{m_0,q}(L_j) \) and let \( \tau^{-1}(o_1) = \bigcup_{j=1}^k E_j \). It is well known (for example, see [2], Ch. III, §5) that the \( E_j \) are rational curves and, up to renumbering the \( E_j \), the dual weighted graph \( \Gamma(\tilde{B}) \) of the curve \( \tilde{B} = (B_1 \cup B_2) \cup \bigcup_{j=1}^k E_j \) is a chain, that is, it has the form...
shown in Figure 8, where the \( \omega_j = -(E_j^2)_{\tilde{W}} \) satisfy the following equality:

\[
\frac{m_0}{q} = \omega_1 - \frac{1}{\omega_2} - \frac{1}{\omega_3} - \cdots - \frac{1}{\omega_k},
\]

\( \omega_j = \frac{[w_1, w_2, \ldots, w_k]}{\omega_k} \)  \( \text{def} \) (5).

Conversely, if \( \tau: \tilde{W} \to (W, o_1) \) is the minimal resolution of a normal singularity such that \( \tau^{-1}(o_1) = \bigcup_{j=1}^{k} E_j \) is a chain of rational curves (see Figure 8), then \((W, o_1)\) is a Hirzebruch-Jung singularity of type \( A_{m_0, q} \), where \( m_0 \) and \( q \) can be found using (5).

**Remark 5.** A representation of the singularity \((W, o_1)\) of type \( A_{m_0, q} \) as a cyclic quotient singularity is uniquely defined by the choice of the divisors \( B_1 \) and \( B_2 \) (see Figure 8) in \( \tilde{W} \) ([2], Ch. III, §5).

Note that if we renumber the curves \( E_j \) and the weights \( \omega_j \) as follows: \( E_j' := E_{k-j+1} \) and \( \omega_j' := \omega_{k-j+1} \), and substitute the new \( \omega_j' \) for the old \( \omega_j \) on the right-hand side of (5) then we obtain \( m_0/q' \) on the left-hand side of (5) with \( q' \) such that \( qq' \equiv 1 \) mod \( m_0 \) (see [2], Ch. III, §5). In particular, the singularity types \( A_{m_0, q} \) and \( A_{m_0, q'} \) are the same.

**Remark 6.** In the notation used in Theorem 1, it easily follows from Theorem 3 that the group \( \pi_1(\tilde{W} \setminus (B_1 \cup B_2 \cup (\bigcup_{j=1}^{k} E_j))) \) is generated by the elements \( b_1 \) and \( e_1 \), and \( \pi_1(\tilde{W} \setminus (B_2 \cup (\bigcup_{j=1}^{k} E_j))) \) is the free group \( \mathbb{F}_1 \) generated by \( e_1 \).

For a singularity \((W, o_1)\) of type \( A_{m_0, q} \) we have

\[
\pi_1(W \setminus \{o_1\}) = \pi_1(\tilde{W} \setminus \left( \bigcup_{j=1}^{k} E_j \right)) \simeq \mu_{m_0},
\]

since \( \theta_{m_0, q}: X \setminus \{\tilde{o}\} \to W \setminus \{o_1\} \) is the universal cover.

**Lemma 2.** If \( [\omega_1, \omega_2, \ldots, \omega_k] = [2, \ldots, 2], k \geq 1, \) then \((W, o_1)\) has the singularity type \( A_{k+1, k} \) and, in particular, \( \pi_1(W \setminus \{o_1\}) \simeq \mu_{k+1} \).

**Proof.** We have

\[
[2; 2, \ldots, 2] = \frac{k + 1}{k}.
\]

Note that the singularity types \( A_k \) and \( A_{k+1, k} \) are the same. The lemma is proved.

**Lemma 3.** If \( [\omega_1, \omega_2, \ldots, \omega_{k+1}] = [n, 2, \ldots, 2], k \geq 0, \) then \((W, o_1)\) has the singularity type \( A_{n(k+1) - k, k+1} \) and, in particular, \( \pi_1(W \setminus \{o_1\}) \simeq \mu_{n(k+1) - k} \).

**Proof.** We have

\[
[\omega_1; \omega_2, \ldots, \omega_{k+1}] = \omega_1 - \frac{1}{[\omega_2; \ldots, \omega_{k+1}]}.
\]

Therefore, \( [n; 2, \ldots, 2] = n - \frac{k}{k + 1} = \frac{n(k + 1) - k}{k + 1} \). The lemma is proved.
Let \( D_{r_1,r_2}^2 = \{(y_1,y_2) \in \mathbb{C}^2 \mid |y_1| < r_1, |y_2| < r_2\} \) denote a bidisc in \( \mathbb{C}^2 \), where \((r_1,r_2) \in \mathbb{R}_+^2 \). Let \( L_{y_j} = \{y_j = 0\} \subset D_{r_1,r_2}^2, j = 1,2, \) be the coordinate axes in \( D_{r_1,r_2}^2 \).

The following Lemma is a direct consequence of Theorem 5.1 in [2], Ch. III.

**Lemma 4.** Let \( Z \) be an irreducible germ of normal surface and let \( \xi: Z \rightarrow D_{(r_1,r_2)}^2(y_1,y_2) \) be a cyclic cover of degree \( n \) branched in \( L_{y_1} \cup L_{y_2} \). Then \( n = n_{12} n_3 \) for some \( n_1 \geq 1, n_2 \geq 1 \) and \( n_3 \geq 1 \) such that \( \text{GCD}(n_1,n_2) = 1 \) and \( \xi \) is ramified over \( L_j \) with multiplicity \( n_j n_3, j = 1,2, \) and if \( n_3 > 1 \) then the singularity type of \( Z \) over the origin \((0,0) \in D_{r_1,r_2}^2 \) is \( A_{n_3,q} \) for some \( q \), where \( \text{GCD}(n_3,q) = 1 \), and \( Z \) is a germ of a smooth surface if \( n_3 = 1 \).

Let a germ \((W,o_1)\) of normal surface have the singularity type \( A_{n,q} \), let \( \tau: \widetilde{W} \rightarrow (W,o_1) \) be the minimal resolution of the singular point \( o_1 \in W \), let \( \tau^{-1}(o_1) = \bigcup_{j=1}^k E_j \) be a chain of rational curves, \( (E_j)^2 = -\omega_j \), and let \( \widetilde{B} \subset \widetilde{W} \) be a curve whose dual graph \( \Gamma(\widetilde{B}) \) is shown in Figure 8.

Let \( m \) be a divisor of \( n \), \( n = mk \). Let \( \varphi_m: (X_m,\tilde{o}) \rightarrow (W,o_1) \) denote the cyclic cover of degree \( m \) defined by the natural epimorphism \( \varphi_m^*: \pi_1(W \setminus \{o_1\}) \cong \mu_n \rightarrow \mu_m \). The cover \( \varphi_m \) is unramified outside \( \tilde{o} \) and \((X_m,\tilde{o})\) is a normal variety having the singularity of type \( A_{k,q'} \) for some \( q' \) if \( k > 1 \), and \((X_m,\tilde{o})\) is a germ of a smooth surface if \( k = 1 \).

Consider the commutative diagram

\[
\begin{array}{ccc}
X_m & \xrightarrow{\varphi} & \widetilde{X}_m \\
\downarrow{\varphi_m} & & \downarrow{\varphi_m} \\
\widetilde{W} & \xrightarrow{\tau} & (W,o_1)
\end{array}
\]

in which \( \widetilde{X}_m \) is the normalization of the fibre product \( \widetilde{W} \times_{(W,o_1)} (X_m,\tilde{o}) \) and \( \varphi: \widetilde{X}_m \rightarrow \widetilde{X}_m \) is the minimal resolution of the singular points of \( \widetilde{X}_m \). It follows from Lemma 4 that \( \widetilde{X}_m \) can have singular points (and their singularity types are \( A_{m',q'} \) for some divisors \( m' \) of \( m \)) only over points of intersection of neighbouring exceptional curves \( E_j \) and \( E_{j+1} \) of \( \tau \).

Denote the composition of the maps \( \rho \) and \( \varphi \) by \( \psi := \rho \circ \varphi \). Note that \( \psi \) is a resolution of the singular point \( \tilde{o} \in X_m \). The map \( \psi \) can be decomposed into a composition of two maps, \( \psi = \varsigma \circ \sigma \), where \( \varsigma: \widetilde{X}_{m,\text{min}} \rightarrow (X_m,\tilde{o}) \) is the minimal resolution of the singular point \( \tilde{o} \) if \( m < n \) and \( \sigma: \widetilde{X}_m \rightarrow \widetilde{X}_{m,\text{min}} \) is a composition of \( \sigma \)-processes, \( \sigma = \sigma_1 \circ \cdots \circ \sigma_1 \) (\( \psi = \sigma \) if \( m = n \)).

Let \( \widetilde{B} = B_1 \cup B_2 \cup \bigcup_{j=1}^k E_j \subset \widetilde{W} \) be the union of curves and curve germs whose dual weighted graph is shown in Figure 8. We use the same letter \( C_j \) to denote the proper inverse image \( (\tau \circ \tilde{\varphi}_m)^{-1}(B_j) \) of the germ \( B_j, j = 1,2, \) and the proper inverse images \( (\sigma_1 \circ \cdots \circ \sigma_{l-s})^{-1}(C_j) \) for \( 1 \leq s \leq l \). Let \( \Delta_m(n,q) \) denote the number of \( \sigma \)-processes from the set \( \{\sigma_1, \ldots, \sigma_l\} \) which blowup a point in \( C_1 \); we call it the \( m \)th supplement for the singularity type \( A_{n,q} \).
Lemma 5. Let a germ \((W, o_1)\) have the singularity type \(A_{n,n-1}\), \(n = mk\), and let \(\varphi_m: (Z_m, \tilde{o}) \rightarrow (W, o_1)\) be the cyclic cover of degree \(m\) defined by the natural epimorphism \(\varphi_m*: \pi_1(W \setminus \{o_1\}) \approx \mu_m \rightarrow \mu_m\). Then

(i) the singularity type\(^1\) of \((Z_m, \tilde{o})\) is \(A_{k,k-1}\);

(ii) \(\Delta_m(n, n - 1) = m - 1\).

Proof. To prove (ii), consider the quadric \(Q = \mathbb{P}^1 \times \mathbb{P}^1\) and let \(S_1\) and \(S_2\) be two fibres of the projection \(\pi_2: Q \rightarrow \mathbb{P}^1\) onto the second factor and \(L\) a fibre of the projection \(\pi_1: Q \rightarrow \mathbb{P}^1\) onto the first factor. Consider the following diagram

\[
\begin{array}{ccc}
\bar{X}_m & \xrightarrow{\varrho} & \bar{X}_m \\
\varphi_m \downarrow & & \varphi_m \downarrow \\
\bar{Q} & \xrightarrow{\tau} & Q
\end{array}
\]

in which

1) \(\varphi_m\) is defined by the epimorphism \(\varphi_m*: \pi_1(Q \setminus (S_1 \cup S_2)) \approx \mathbb{Z} \rightarrow \mu_m \subset \mathbb{S}_m\);
2) \(\tau = \tau_1 \circ \cdots \circ \tau_n\) is the composition of \(n\) blowups of the point \(p = S_1 \cap L \in S_1\);
3) \(\bar{X}_m\) is the normalization of the fibre product \(\bar{Q} \times_Q X_m\);
4) \(\varrho\): \(\bar{X}_m \rightarrow \bar{X}_m\) is the minimal resolution of the singular points of \(\bar{X}_m\).

Let \(E_j \subset \bar{Q}, j = 1, \ldots, n - 1\), denote the proper inverse image of the exceptional curve of blowup \(\tau_j\), let \(B_1 \subset \bar{Q}\) be the proper inverse image of the fibre \(L\), and \(B_2 \subset \bar{Q}\) the exceptional curve of the blowup \(\tau_n\). We have

\[(B_1^2)_{\bar{Q}} = (B_2^2)_{\bar{Q}} = -1 \quad \text{and} \quad (E_j^2)_{\bar{Q}} = -2 \quad \text{for} \quad j = 1, \ldots, n - 1;\]

and the dual graph \(\Gamma(\tilde{B})\) of \(\tilde{B} = B_1 \cup B_2 \cup (\bigcup_{j=1}^{n-1} E_j)\) is shown in Figure 8 (in it, \(\omega_j = -2\) for \(j = 1, \ldots, n - 1\)). Therefore we can identify \(\bar{W}\) with a tubular neighbourhood of \(\bigcup_{j=1}^{n-1} E_j\) and \(\tilde{Z}_m\) with \(\varphi_m(\bar{W})\). Note that the dual weighted graph \(\Gamma(\bigcup_{j=1}^{n-1} E_j)\) of the curve \(\bigcup_{j=1}^{n-1} E_j\) is a central-symmetric graph, that is, the weights \(\omega_j\) of the vertices \(e_j\) satisfy the relation \(\omega_j = \omega_{n-j}\).

It is obvious that \(X_m \approx \mathbb{P}^1 \times \mathbb{P}^1\), where \(\varphi_m^{-1}(S_1)\) and \(\varphi_m^{-1}(S_2)\) are two fibres of the projection \(\pi_2: X_m \rightarrow \mathbb{P}^1\) onto the second factor and \(\varphi_m^{-1}(L) = F\) is a fibre of the projection \(\pi_1: X_m \rightarrow \mathbb{P}^1\) onto the first factor.

The fundamental group \(\pi_1(Q \setminus (S_1 \cup S_2)) \approx \pi_1(\bar{Q} \setminus (\tau^{-1}(S_1 \cup S_2) \cup B_2 \cup (\bigcup_{j=1}^{n-1} E_j)))\) is generated by the element \(\gamma\) represented by a simple loop around the curve \(S_1\). Denote the elements of \(\pi_1(\bar{Q} \setminus (\tau^{-1}(S_1 \cup S_2) \cup B_2 \cup (\bigcup_{j=1}^{n-1} E_j)))\) represented by simple loops around the \(E_j\) and \(B_2\) by \(e_j, j = 1, \ldots, n - 1, \text{and} b_2\), respectively. It follows from Lemma 1 that

\[e_j = \gamma^j \quad \text{for} \quad j = 1, \ldots, n - 1 \quad \text{and} \quad b_2 = \gamma^n.\]  

(10)

First, we consider the case \(m = n\). The element \(g_1 = \varphi_n*(\gamma)\) is a generator of \(\mu_n \subset \mathbb{S}_n\) and

\[\varphi_n*(e_j) = g_1^j.\]  

(11)

\(^1\)By definition, if \(k = 1\) and \((Z_m, \tilde{o})\) has the singularity type \(A_{1,0}\), then \(\tilde{o}\) is a smooth point of \(Z_m\).
in particular, \( \varphi_{n*}(e_{n-1}) = g_1^{n-1} := g_2 \) is a generator of \( \mu_n \) and \( \varphi_{n*}(b_2) = g_1^n = \text{id} \). Therefore \( \varphi_n \) is not branched in \( B_2 \) and \( \tilde{X}_n \) is smooth in a neighbourhood of \( \varphi^{-1}_n(B_1) \) and \( \varphi^{-1}_n(B_2) \). The restriction of \( \tilde{\varphi}_n \) to \( \tilde{Z}_n \subset \tilde{X}_n \) is defined by the monodromy homomorphism \( \varphi_{n*}: \pi_1(W \setminus (\bigcup_{j=1}^{n-1} E_j)) \to \mu_n \subset S_n \) sending \( e_j \) to \( g_1^j \).

It easily follows from Theorem 3 that \( \pi_1(W \setminus (\bigcup_{j=1}^{n-1} E_j)) \) is generated by \( e_{n-1} \) and the \( e_j = e_{n-j} \). Therefore, if we set \( e'_j = e_{n-j} \) then

\[
\varphi_{n*}(e_j') = g_j^2. \tag{12}
\]

The inverse image \((\varphi_n \circ g)^{-1}(\bigcup_{j=1}^{n-1} E_j) = \bigcup_{j=1}^{N} E_j \subset \tilde{X}_n \) is a chain of rational curves which can be contracted to a smooth point, the curves \( B_j = (\varphi_n \circ g)^{-1}(B_j) \), \( j = 1, 2 \), are rational curves,

\[
(\tilde{B}_j^2)_{\tilde{X}_n} = \deg \tilde{\varphi}_n \cdot (B_j^2)_{\tilde{X}_n} = -n \quad \text{and} \quad (\tilde{B}_1, \tilde{E}_1)_{\tilde{X}_n} = (\tilde{B}_2, \tilde{E}_N)_{\tilde{X}_n} = 1,
\]

and \((\varphi_n \circ \rho \circ g)^{-1}(L) = \overline{B}_1 \cup \overline{B}_2 \cup (\bigcup_{j=1}^{N} E_j) \) is a fibre of the ruled surface \( X_n \).

Using the central symmetry of the graph \( \Gamma(\bigcup_{j=1}^{n-1} E_j) \), and (11) and (12), it follows that the dual weighted graph \( \Gamma(\bigcup_{j=1}^{N} E_j) \) is also central-symmetric. Therefore, if \( \overline{E}_j \) is a \((-1)\)-curve, then \( \overline{E}_{N-j+1} \) is also a \((-1)\)-curve and the curves \( \overline{E}_j, \overline{E}_{N-j+1} \) can be simultaneously contracted to points. After successive contractions of all such pairs of \((-1)\)-curves and the contraction of the curve \( \overline{E}_{K+1} \) in the last step (it is easy to see that \( N = 2K + 1 \) must be an odd integer and the central curve \( \overline{E}_{K+1} \) is contracted in the last step of the contractions) we find that the images of \( \overline{B}_1 \) and \( \overline{B}_2 \) are \((-1)\)-curves, since the union of these images is a fibre of ruled structure. Therefore, \( \Delta_n(n, n-1) = n-1 \), since \( (B_1^2)_{X_n} = (B_2^2)_{X_n} = -n \).

Consider the case when \( m < n \). The element \( g_1 = \varphi_{m*}(\gamma) \) is a generator of \( \mu_m \subset S_m \). It follows from (10) that \( \varphi_{m*}(e_{j+lm}) = g_1^j \) for \( j = 1, \ldots, m-1, l = 0, \ldots, k-1 \) and \( \varphi_{m*}(e_{lm}) = \varphi_{m*}(b_2) = \text{id} \) for \( l = 1, \ldots, k-1 \). Therefore, \( \tilde{\varphi}_m \) is not branched in \( E_{lm} \), \( l = 1, \ldots, k-1 \), nor in \( B_1 \) and \( B_2 \). Hence \( \tilde{X}_m \) is smooth in a neighbourhood of \( \tilde{\varphi}_m^{-1}(B_1 \cup B_2 \cup (\bigcup_{l=1}^{k-1} E_{lm})) \) and

\[
(\tilde{\varphi}_m^{-1}(E_{lm}), \tilde{\varphi}_m^{-1}(E_{lm}))_{\tilde{X}_m} = -2m \quad \text{and} \quad (\tilde{\varphi}_m^{-1}(B_j), \tilde{\varphi}_m^{-1}(B_j))_{\tilde{X}_m} = -m
\]

for \( l = 1, \ldots, k-1 \) and \( j = 1, 2 \).

For each \( l = 0, \ldots, k-1 \) the union \( \bigcup_{j=1}^{m-1} \tilde{\varphi}_m^{-1}(E_{j+lm}) \) can be contracted to a smooth point and similarly to the case \( m = n \) it is easy to see that, first \( \Delta_m(n, n-1) = \Delta_m(m, m-1) = m-1 \) and, second, that after the contraction of the curves \( \bigcup_{j=0}^{k-1} \bigcup_{j=1}^{m-1} \tilde{\varphi}_m^{-1}(E_{j+lm}) \) the images of the curves \( \tilde{\varphi}_m^{-1}(E_{lm}) \) form a chain of \( k-1 \) \((-2)\)-curves, that is, the singularity type of \( (Z_m, \tilde{\varnothing}) \) is \( A_{k,k-1} \).

Lemma 5 is proved.

\[ \text{§ 2. A description of } \beta^{-1}(f): \text{ the general case} \]

2.1. Necessary conditions. In this section we use the notation from § 1.

Consider a rigid germ of cover \( F: (U, o') \to (V, o) \) branched in a germ \((B, o)\) having one of the ADE singularity types at the point \( o \), where \( \deg F = d \), and let
\[ F_*: \pi_1^{\text{loc}}(B, o) = \pi_1(V \setminus B, p) \to G_F \subset \mathbb{S}_d \] is its monodromy homomorphism. Recall that the symmetric group \( \mathbb{S}_d \) acts (from the right) on the fibre \( F^{-1}(p) = \{q_1, \ldots, q_d\} \) and the monodromy group \( G_F \) is a transitive subgroup of \( \mathbb{S}_d \). We let \( G^*_F \) denote the subgroup of \( G_F \) leaving the point \( q_1 \) fixed. Then the action of \( G_F \) on \( F^{-1}(p) \) can be identified with the action of \( G_F \) on the set of right cosets of the subgroup \( G^*_F \).

By Proposition 1 the cyclic group \( F_*(Z_e) \subset G_F \), generated by \( F_*(e) \), is a central subgroup of \( G_F \) and by Proposition 13 in [4] the group \( F_*(Z_e) \) acts on \( (U, o') \). Let \( H_1: (U, o') \to (W, o_1) = (U, o') / F_*(Z_e) \) denote the quotient map, \( \deg H_1 = m = |F_*(Z_e)| \), where \( (W, o_1) \) is a germ of a normal surface. By Proposition 13 in [4], there is a finite map \( H_2: (W, o_1) \to (V, o) \) such that \( F = H_2 \circ H_1 \), \( \deg H_2 = n = d/m \). The monodromy group \( G_{H_1} \subset \mathbb{S}_m \) of \( H_1 \) is isomorphic to \( F_*(Z_e) \) and by Remark 2 in [4] the monodromy group \( G_{H_2} \subset \mathbb{S}_n \) of \( H_2 \) is isomorphic to \( G_F / N \), where \( N \) is the maximal normal subgroup of \( G_F \) contained in \( G^*_F \times F_*(Z_e) \subset G_F \).

Let \( \tilde{W} \) denote the normalization of the fibre product \( \tilde{V} \times_{(V, o)} (W, o_1) \), where \( \sigma: \tilde{V} \to (V, o) \) is the minimal resolution of the singular point \( o \in V \) of the curve germ \( (B, o) \) (if \( (B, o) \) has a singularity of type \( A_0 \) or \( A_1 \) then \( \sigma \) consists of the single blowup) and let \( \tilde{H}_2: \tilde{W} \to \tilde{V} \) and \( \zeta: \tilde{W} \to (W, o_1) \) be the projections onto the factors. In addition, let \( \tilde{U} \) denote the normalization of the fibre product \( \tilde{W} \times_{(W, o_1)} (U, o') \) and let \( \tilde{H}_1: \tilde{U} \to \tilde{W} \) and \( \tau: \tilde{U} \to (U, o') \) be the projections onto the factors. The group \( G_{H_1} \) acts on \( \tilde{U} \), and \( \tilde{H}_1 \) is also the quotient map.

Let \( C = \tilde{H}_2^{-1}(E) \) denote the proper inverse image of the exceptional curve \( E \) of the last \( \sigma \)-process and let \( f: C \to E \), \( \deg f = \deg \tilde{H}_2 = n \), be the restriction of \( \tilde{H}_2 \) to \( C \) (by definition, \( f = \beta(F) \)). The group \( F_*(Z_e) \) acts on \( \tilde{U} \) and it is easy to see that \( \tilde{H}_1^{-1}(C) \) is an irreducible curve. Therefore, the curve \( C \) is contained in the branch locus of \( \tilde{H}_1 \) and \( \tilde{H}_1 \) is branched in \( C \) with multiplicity \( m = \deg \tilde{H}_1 \). For the same reason the branch locus of \( \tilde{H}_2 \) is contained in \( \tilde{B} \setminus E \), where \( \tilde{B} = \sigma^{-1}(B) \) is the inverse image of the germ \((B, o)\). The dual graph of \( \tilde{B} \) is depicted in one of Figures 1–7.

Let \( \tilde{B}^0 = \sigma^{-1}(o) \) and let \( \tilde{B}_j \subset \tilde{B} \), \( j = 1, 2, 3 \), be the trails of \( \tilde{B} \) (see Definition 1). It follows from Remark 4 and Lemma 4 that \( \tilde{W} \) can only have singular points (and they have singularity types \( A_{k', q'} \) for some divisors \( k' \) of \( n \)) over points of intersection of neighbouring irreducible components of trails of \( \tilde{B} \), since \( \tilde{H}_2 \) is not branched in \( E \). Let \( \varsigma_r: \tilde{W} \to \tilde{W} \) denote a resolution of the singular points of \( \tilde{W} \). Then \( \varsigma \circ \varsigma_r: \tilde{W} \to (W, o_1) \) is a resolution of the singular point \( o_1 \) of \( (W, o_1) \).

Let \( \varsigma_1: \tilde{W} \to \tilde{W}_m \) denote a holomorphic bimeromorphic map, where \( \tilde{W}_m \) is a smooth surface and \( \varsigma_1 \) contracts the maximum number of irreducible components belonging to \( (\tilde{H}_1 \circ \varsigma_r)^{-1}(\cup_{j=1}^3 \tilde{B}_j) \) to points. Then \( \varsigma \circ \varsigma_r = \varsigma_m \circ \varsigma_1 \), where \( \varsigma_m: \tilde{W}_m \to (W, o_1) \) is also a resolution of the singular point \( o_1 \) of \( (W, o_1) \).

Set \( \tilde{B}^0 = \varsigma_m^{-1}(o_1) \). The composition \( \tilde{H}_1 = \varsigma_m^{-1} \circ H_1: (U, o') \to \tilde{W}_m \) is a meromorphic map such that the finite cover \( \tilde{H}_1: U \setminus \{o'\} \to \tilde{W}_m \setminus \tilde{B}^0 \) is naturally isomorphic to the cover \( H_1: U \setminus \{o'\} \to W \setminus \{o_1\} \). Note that \( \overline{C} = \varsigma_1 \circ \varsigma_r^{-1}(C) \) is a component of \( \overline{B}^0 \).

The cover \( H_1 \) is branched in at most two irreducible curve germs in \((W, o_1)\) (see §1.3). Let \( \overline{B} = \overline{B}_1 \cup \overline{B}_2 \cup \overline{B}^0 \subset \tilde{W}_m \) denote the inverse image of these germs
(of course, one of the $\overline{B}_j$ or both can be empty). The dual graph of $\overline{B}$ is a chain similar to the one in Figure 8.

As a result, we have the following commutative diagram:

$$
\begin{array}{ccccccc}
\mathbb{P}^1 & \simeq & C & \subset & \tilde{W} & \xrightarrow{\varsigma} & \tilde{W}_m \\
\downarrow{f} & & \downarrow{\sigma} & & \downarrow{\varsigma} & & \downarrow{\varsigma_r} \\
\mathbb{P}^1 & \simeq & E & \subset & \tilde{V} & & V \\
\end{array}
$$

The maps $\tilde{H}_1$ and $H_1$ in diagram (*) are the quotient maps under the action of a cyclic group. Therefore, $H_1$ is a composition of two maps, $H_1 = \theta_{n',q} \circ \vartheta_{m_1,m_2}$ (see §1.3), where $m = n'm_1m_2$ and $\text{GCD}(m_1,m_2) = 1$, and hence we obtain the following commutative diagram

$$
\begin{array}{ccccccc}
\tilde{U} & \xrightarrow{\tau} & U \ni o' \\
\downarrow{\tilde{H}_1} & & \downarrow{\Pi_1} & & \downarrow{H_1} & & \\
W & \xrightarrow{\varsigma_1} & \tilde{W}_m \\
\downarrow{\varsigma} & & \downarrow{\varsigma_m} & & \downarrow{\varsigma} & & \\
W \ni o_1 & & & & & & W \ni o \\
\end{array}
$$

(13)

The monodromy homomorphism $f_*$ is the composition of two homomorphisms, $f_* = H_{2*} \circ i_{2*}$ (see (3) and (4)).

Remark 7. In view of Remark 3 we will identify the monodromy homomorphism $\tilde{H}_{2*}$ with the monodromy homomorphism $\beta(F)_* = f_*$ in the case when $(B,o)$ has the singularity type $D_4$.

The monodromy group $G_f$ of the Belyi pair $f$ is isomorphic to $G_{H_2} = G_F/N \subset S_n$, generated by $f_*(\gamma_j) \in S_n$, $j = 1, 2, 3$, where $\gamma_j \in \pi_1(E \setminus (\overline{B}_1 \cup \overline{B}_2 \cup \overline{B}_3), p)$ are the elements defined in §1.2. Let

$$_n^{T_c(H_2)} = T_c(f) = \{c_1,c_2,c_3\}, \quad c_j = (n_{j,1}, \ldots, n_{j,k_j}), \quad n = \sum_{i=1}^{k_j} n_{j,i},$$

be the set of cycle types of the permutations $\tilde{H}_{2*}(\tilde{\gamma}_j) = f_*(\gamma_j)$. For each $j = 1, 2, 3$ the inverse image $\tilde{H}_2^{-1}(\overline{B}_j)$ of a trail $\overline{B}_j$ is the disjoint union of $k_j$ connected components, $\tilde{H}_2^{-1}(\overline{B}_j) = \bigsqcup_{l=1}^{k_j} \overline{B}_{j,l}$. The properties of cyclic covers described in §1.3 imply the following contractibility condition:
\[ \bar{B}_{j,l} \cap \bar{H}^{-1}_2(\bar{B}_j^0) \text{ in } \bar{W} \text{ can be contracted to a nonsingular point if and only if the order } |\bar{\pi}_j^0| \text{ of the group } \bar{\pi}_j^0 \text{ is a divisor of } n_{j,l} \text{ when } \bar{B}_j \text{ is an exceptional trail, and } |\bar{\pi}_j^0| = n_{j,l} \text{ when } \bar{B}_j \text{ is a completely exceptional trail,} \]

since \( \bar{B}_j^0 \) can be contracted to a point of singularity type \( A_{|\bar{\pi}_j^0|,q} \). Note that, for the same reason, the \( n_{j,l} \) are divisors of \( |\bar{\pi}_j^0| \) if \( \bar{B}_j \) is a completely exceptional trail.

Let \( r_j(H_{2*}) \) denote the number of cycles in the permutation \( f_*(\gamma_j) \) whose lengths do not satisfy the contractibility condition if \( \bar{B}_j \) is an exceptional trail and set \( r_j(H_{2*}) = 0 \) if \( \bar{B}_j \) is not an exceptional trail. Then we obtain

\[
    r_1(\bar{H}_{2*}) + r_2(\bar{H}_{2*}) + r_3(\bar{H}_{2*}) \leq 2, \tag{14}
\]

since the dual graph of \( \bar{B} \) is a chain.

2.2. Sufficient conditions. Let a rational Belyi pair \( f: C \simeq \mathbb{P}^1 \to \mathbb{P}^1 \) of degree \( n \) branched at \( B_f \subset \{0,1,\infty\} \) be fixed, such that the cycle type of its monodromy is \( T(f) = \{c_1,c_2,c_3\} \). Recall that the set of rational Belyi pairs is considered up to actions of \( \text{PGL}(2,\mathbb{C}) \) on \( C \) and \( \mathbb{P}^1 \), and the ordered cycle type \( T(f) = \{c_1,c_2,c_3\} \) of \( f \) depends on the choice of the base in \( \pi_1(\mathbb{P}^1 \setminus \{0,1,\infty\}) \). Therefore we can arrange the cycle type \( T(f) \) so that the cycle type of \( f_*(\gamma_0) \) is \( c_1 \), the cycle type of \( f_*(\gamma_\infty) \) is \( c_2 \) and the cycle type of \( f_*(\gamma_1) \) is \( c_3 \), where \( \gamma_0,\gamma_1 \) and \( \gamma_\infty \) are elements of \( \pi_1(\mathbb{P}^1 \setminus \{0,1,\infty\}) \) represented by simple loops around \( 0,1 \) and \( \infty \), respectively, and such that \( \gamma_0^2 \gamma_1 \gamma_\infty = \text{id} \) in \( \pi_1(\mathbb{P}^1 \setminus \{0,1,\infty\}) \).

Definition 3. We say that a rational Belyi pair \( f \) has type:

- \( A_{2k+1}, k \geq 1 \text{, if } (k_1-r_1) \text{ lengths } n_{1,j} \text{ in the cycle type } c_1 = (n_{1,1}, \ldots, n_{1,k_1}) \text{ are equal to } k+1 \text{, } r_1 \leq 2, \text{ and the remaining } r_1 \text{ lengths are divisors of } k+1; \)

- \( A_{2k}, k \geq 1 \text{, if } (k_1-r_1) \text{ lengths } n_{1,j} \text{ in the cycle type } c_1 = (n_{1,1}, \ldots, n_{1,k_1}) \text{ are equal to } 2k+1, \text{ and the remaining } r_1 \text{ lengths are divisors of } 2k+1, (k_2-r_2) \text{ lengths } n_{2,j} \text{ in the cycle type } c_2 = (n_{2,1}, \ldots, n_{2,k_2}) \text{ are equal to } 2 \text{ and the remaining } r_2 \text{ lengths are equal to } 1, \text{ and } r_1 + r_2 \leq 2; \)

- \( D_{2k+2}, k \geq 2 \text{, if } (k_1-r_1) \text{ lengths } n_{1,j} \text{ in the cycle type } c_1 = (n_{1,1}, \ldots, n_{1,k_1}) \text{ are multiples of } k, r_1 \leq 2; \)

- \( D_{2k+3}, k \geq 1 \text{, if } (k_1-r_1) \text{ lengths } n_{1,j} \text{ in the cycle type } c_1 = (n_{1,1}, \ldots, n_{1,k_1}) \text{ are multiples of } 2k+1, (k_2-r_2) \text{ lengths } n_{2,j} \text{ in the cycle type } c_2 = (n_{2,1}, \ldots, n_{2,k_2}) \text{ are equal to } 2 \text{ and the remaining } r_2 \text{ lengths are equal to } 1 \text{ and } r_1 + r_2 \leq 2; \)

- \( E_6 \text{ if } (k_1-r_1) \text{ lengths } n_{1,j} \text{ in the cycle type } c_1 = (n_{1,1}, \ldots, n_{1,k_1}) \text{ are equal to } 4 \text{ and the remaining } r_1 \text{ lengths are equal to } 2 \text{ or } 1, (k_2-r_2) \text{ lengths } n_{2,j} \text{ in the cycle type } c_2 = (n_{2,1}, \ldots, n_{2,k_2}) \text{ are equal to } 3, \text{ the remaining } r_2 \text{ lengths are equal to } 1 \text{ and } r_1 + r_2 \leq 2; \)

- \( E_7 \text{ if } (k_1-r_1) \text{ lengths } n_{1,j} \text{ in the cycle type } c_1 = (n_{1,1}, \ldots, n_{1,k_1}) \text{ are even, } (k_2-r_2) \text{ lengths } n_{2,j} \text{ in the cycle type } c_2 = (n_{2,1}, \ldots, n_{2,k_2}) \text{ are equal to } 3, \text{ the remaining } r_2 \text{ lengths are equal to } 1 \text{ and } r_1 + r_2 \leq 2; \)

- \( E_8 \text{ if } (k_1-r_1) \text{ lengths } n_{1,j} \text{ in the cycle type } c_1 = (n_{1,1}, \ldots, n_{1,k_1}) \text{ are equal to } 3 \text{ and the remaining } r_1 \text{ lengths are equal to } 1, (k_2-r_2) \text{ lengths } n_{2,j} \text{ in the cycle type } c_2 = (n_{2,1}, \ldots, n_{2,k_2}) \text{ are equal to } 5, \text{ the remaining } r_2 \text{ lengths are equal to } 1 \text{ and } r_1 + r_2 \leq 2. \)
Remark 8. Note that a rational Belyi pair $f$ can have several ADE types. For example, if $f$ has type $\text{A}_{2k+1}$, $k \geq 1$, then it also has type $\text{D}_{2k+4}$. In addition, we will assume that any $f \in \text{Bel}$ has type $\text{D}_{4}$.

It follows from Lemmas 2 and 3, the contractibility condition and inequality (14) that a necessary condition for the branch curve $(B, o)$ of a cover $F \in \mathcal{R}_T \subset \mathcal{R} \setminus (\mathcal{R}_{\text{A}_0} \cup \mathcal{R}_{\text{A}_1})$ to belong to $\beta^{-1}(f), \deg f = n$, is that $f$ has type $T$.

If this necessary condition is met, then we can consider a monodromy homomorphism $H_{2*}: \pi^\text{loc}_1(B, o) \simeq \pi_1(\tilde{V} \setminus \tilde{B}) \to \mathcal{S}_n$ sending $\gamma_1$ to $f_*(\gamma_0)$, $\gamma_2$ to $f_*(\gamma_\infty)$, $\gamma_3$ to $f_*(\gamma_1)$ and $e$ to id. The homomorphism $H_{2*}$ defines finite coverings $H_2: (W, o_1) \to (V, o)$ and $\tilde{H}_2: \tilde{W} \to \tilde{V}$, where $\sigma: \tilde{V} \to (V, o)$ is the minimal resolution of the singular point of $(B, o)$. To the maps $H_2, \tilde{H}_2, \sigma$, and $\varsigma: \tilde{W} \to (W, o_1)$ we can add the bimeromorphic maps $\varsigma_r: \tilde{W} \to \tilde{W}, \varsigma_1: \tilde{W} \to W_m$ and $\varsigma_m: \tilde{W}_m \to (W, o_1)$.

As a result, we obtain the lower part of the diagram (*). Again, we let $C = \tilde{H}_2^{-1}(E)$ denote the proper inverse image of the exceptional curve $E$ of the last $\sigma$-process. The restriction of $\tilde{H}_2$ to $C$ obviously coincides with a Belyi pair $f: C \to E$, $\deg f = \deg \tilde{H}_2 = n$.

As above, $m_0$ denotes the order of the fundamental group $\pi_1 = \pi_1(W \setminus \{o_1\}) \simeq \pi_1(W_m \setminus \tilde{B}^0)$, where $\tilde{B}^0 = \varsigma_1^{-1}(o_1)$. We obtain the following commutative diagram

\[
\begin{array}{ccc}
\mathcal{U}_{f,\text{min}T} & \xrightarrow{\varsigma} & (U_{f,\text{min}T}, o_2) \\
\pi_f \downarrow & & \downarrow H_f \\
\mathcal{W}_m & \xrightarrow{\varsigma_m} & (W, o_1)
\end{array}
\]

in which $\overline{H}_f: \mathcal{U}_{f,\text{min}T} \to \mathcal{W}_m$ and $H_f: (U_{f,\text{min}T}, o_2) \to (W, o_1)$ are cyclic covers of degree $m_0$. $H_f: U_{f,\text{min}T} \setminus \{o_2\} \to W \setminus \{o_1\}$ is the universal cover, $\overline{H}_f$ is branched in $\overline{B}^0$, $\mathcal{U}_{f,\text{min}T}$ is a normal surface and $(U_{f,\text{min}T}, o_2)$ is a germ of a smooth surface, and $\varsigma$ is a bimeromorphic holomorphic map.

Let $\gamma_C \in \pi_1$ be the element represented by a simple loop around $\overline{C} = \varsigma_1 \circ \varsigma_1^{-1}(C)$. We call the condition:

the dual graph of $\overline{B}^0$ is a chain and $\gamma_C$ generates the group $\pi_1$,

the second necessary condition. If $f$ has type $T$ and the second necessary conditions is met for the germ $(B, o)$ having singularity type $T$, then $\overline{H}_f$ is branched in $\overline{C}$ with multiplicity $m_0$ and it is easy to see that the cover $F_{f,\text{min}T} := \overline{H}_f \circ \overline{H}_f: (U_{f,\text{min}T}, o_2) \to (V, o)$ of degree $nm_0$ belongs to $\beta^{-1}(f)$. The cover $F_{f,\text{min}T} \in \mathcal{R}_T$ will be called the minimal cover in $\beta^{-1}(f)$ of the rational Belyi pair $f$ of type $T$.

Let $\chi: \mathcal{U} \to U_{f,\text{min}T}$ denote the minimal resolution of the singular points of $U_{f,\text{min}T}$. Then $\varsigma \circ \chi: \mathcal{U} \to U_{f,\text{min}T}$ is a composition of $\sigma$-processes with centres at nonsingular points. Note that $(\varsigma \circ \chi)^{-1}(o_2)$ is a chain of exceptional curves of $\varsigma \circ \chi$. We let $\overline{C}_r = \chi^{-1}(C)$ denote the proper inverse image of $\overline{C}$.

Consider the curve germ $B$ as a divisor in $(V, o)$ and let $F_{f,\text{min}T}^*(B) = \sum r_j R_j$ be the inverse image of $B$, where the $R_j$ are its irreducible components. We let $S_1$ denote the set of pairs $(R_j, r_j)$ in which $R_j$ is a smooth germ for each $j$ and
where \(\gamma \overline{C} = \gamma_j^{a_j}\),

\[ \gamma \overline{C} = \gamma_j^{a_j}, \quad (16) \]

where \(\gamma \overline{C}\) is an element represented by a simple loop around \(\overline{C}\), and \(a_j\) can be computed using Lemma 1 step by step.

Set \(M_j = \{m \in \mathbb{N} \mid \text{GCD}(m, a_j) = 1\}\) for \((R_j, r_j) \in S_1\) and

\[ M_{j_1,j_2} = \{(m_1, m_2) \in \mathbb{N}^2 \mid \text{GCD}(m_1 m_2, m_1 a_{j_1} + m_2 a_{j_2}) = 1\} \]

for \((R_{j_1}, r_{j_1}), (R_{j_2}, r_{j_2})\) \in S_2\) and the \(a_j\) defined in (16) (note that if \((m_1, m_2) \in M_2\) then \(\text{GCD}(m_1, m_2) = 1\).

For \((R_{j_1}, r_{j_1}), (R_{j_2}, r_{j_2})\) \in S_2\) and \((R_{j_1}, r_{j_1}) \in S_1\) we choose coordinates \((y_1, y_2)\) in \(U_{f, \text{min}\{T\}}\) such that \(R_{j_1}\) is given by the equation \(y_l = 0\) for \(l = 1, 2\) (if \((R_{j_1}, r_{j_1}) \in S_1\) then \(R_{j_2}\) is any smooth curve germ meeting \(R_{j_1}\) transversally) and for each \((m_1, m_2) \in M_{j_1,j_2}\) (for each \(m_1 \in M_{j_1}\), and \(m_2 = 1\), respectively) consider the cyclic cover \(\vartheta_{m_1,m_2} : (U, o') \to (U_{f, \text{min}\{T\}}, o_2)\) given by \(x_1^{m_1} = y_1, x_2^{m_2} = y_2\). It is easy to see that \(F_{R_1,R_2,m_1,m_2} := F_f \circ \vartheta_{m_1,m_2} : (U, o') \to (V, o)\) of degree \(\deg F_{R_1,R_2,m_1,m_2} = n m_0 m_1 m_2\) also belongs to \(\beta^{-1}(f)\).

Let \(\text{Aut}(V, B, o) = \{g \in \text{Aut}(V) \mid g(B) = B, g(o) = o\}\) be the automorphism group of the triple \((V, B, o)\) and \(\text{Gal}(F_f) = \{g \in \text{Aut}(U_{f, \text{min}\{T\}}, o_2) \mid F_f \circ g = F_f\}\), the automorphism group of \((U_{f, \text{min}\{T\}}, o_2)\) over \((V, o)\). The group \(\text{Aut}(V, B, o) \times \text{Gal}(F_f)\) acts on the sets \(S_1\) and \(S_2\). Let \(\text{orb}_j(f)\) denote the number of orbits of the action of the group \(\text{Aut}(V, B, o) \times \text{Gal}(F_f)\) on \(S_j, j = 1, 2\).

The results obtained above finally give the following.

**Theorem 4.** Let \(T\) be one of the ADE singularity types, let the curve germ \((B, o)\) have the singularity of type \(T\) at \(o\), and let \(f \in \text{Bel}, \deg f = n\). Then the intersection \(R_T \cap \beta^{-1}(f)\) is nonempty if and only if \(f\) has the type \(T\) and \((B, o)\) satisfies the second necessary condition.

If \(R_T \cap \beta^{-1}(f) \neq \emptyset\) then \(R_T \cap \beta^{-1}(f)\) consists of the minimal cover \(F_{f, \text{min}\{T\}} : (U_{f, \text{min}\{T\}}, o_2) \to (V, o)\) of degree \(n m_0\) and of \((\text{orb}_1(f) + \text{orb}_2(f))\) infinite series of covers \(F_{R_1,R_2,m_1,m_2} : \deg F_{R_1,R_2,m_1,m_2} = n m_0 m_1 m_2\).

§ 3. Proof of Theorem 1

**Lemma 6.** Let \(f : C = \mathbb{P}^1 \to \mathbb{P}^1, \deg f = n > 1,\) be a morphism given by \(y_1 = h_1(x_1, x_2)\) and \(y_2 = h_2(x_2, x_2)\), where \(h_1(x_1, x_2)\) and \(h_2(x_2, x_2)\) are two coprime forms which are homogeneous in the variables \(x_1\) and \(x_2\). Then the ramification divisor \(R_f\) of \(f\) is given by the equation

\[ J_f(x_1, x_2) := \det \begin{pmatrix} \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} \\ \frac{\partial h_2}{\partial x_1} & \frac{\partial h_2}{\partial x_2} \end{pmatrix} = 0. \quad (17) \]
Proof. The projective line $C$ is covered by the four neighbourhoods

$$U_{i,j} = \{(x_1 : x_2) \in C \mid x_i \neq 0, h_j(x_1, x_2) \neq 0\}, \quad 1 \leq i, j \leq 2,$$

and $\tilde{x}_i = x_i/x^i$ is a coordinate in $U_{i,j}$, where $\{i, \tilde{i}\} = \{1, 2\}$. Similarly, $\mathbb{P}^1$ is covered by two affine lines $V_j = \{(y_1 : y_2) \in \mathbb{P}^1 \mid y_j \neq 0\}$, $j = 1, 2$, and $\tilde{y}_j = y_j/y^j$ is a coordinate in $V_j$. The morphism $f$ defines four rational functions $\tilde{y}_j = f_{i,j}(\tilde{x}_i)$ and it is obvious that the restriction of $R_f$ to $U_{i,j}$ is the sum of the critical points of $f_{i,j}$ counted with multiplicities. In particular, if $i = 2$ and $j = 2$ (the other cases are similar), then $R_f$ in $U_{2,2}$ is given by the equation $d\tilde{y}_1/d\tilde{x}_1 = 0$, where $\tilde{y}_1 = f_{2,2}(\tilde{x}_1) = h_1(\tilde{x}_1, 1)/h_2(\tilde{x}_1, 1)$. Therefore,

$$\frac{d\tilde{y}_1}{d\tilde{x}_1} = \frac{h'_{1x_1}(\tilde{x}_1, 1)h_2(\tilde{x}_1, 1) - h_1(\tilde{x}_1, 1)h'_{2x_1}(\tilde{x}_1, 1)}{h_2(\tilde{x}_1, 1)^2}. \quad (18)$$

It follows from Euler’s formula, $nh(x_1, x_2) = x_1h'_{x_1}(x_1, x_2) + x_2h'_{x_2}(x_1, x_2)$ for homogeneous forms $h(x_1, x_2)$ of degree $n$, that

$$h_i(\tilde{x}_1, 1) = \frac{1}{n}[\tilde{x}_1 h'_{i x_1}(\tilde{x}_1, 1) + h'_{i x_2}(\tilde{x}_1, 1)], \quad (19)$$

and applying (19) we see that the numerator on the right-hand side of (18) coincides with $\frac{1}{n}J_f(\tilde{x}_1, 1)$.

The lemma is proved.

Using direct calculations in nonhomogeneous coordinates defining the $\sigma$-processes with centres at points, it is easy to prove the following lemma.

Lemma 7. Let $F: (U, o') \rightarrow (V, o)$ be a finite cover given by

$$y_j = h_j(x_1, x_2) + \sum_{k=n_j+1}^{\infty} \sum_{m=0}^{k} a_{j,m} x_1^m x_2^{k-m}, \quad j = 1, 2, \quad (20)$$

where $h_j(x_1, x_2)$ are homogeneous forms of degree $n_j$ in the variables $x_1$ and $x_2$, and let $\tau: \tilde{U} \rightarrow U$ and $\sigma: \tilde{V} \rightarrow V$ be $\sigma$-processes with centres at $o'$ and $o$, $\tau^{-1}(o') = \tilde{E}$ and $\sigma^{-1}(o) = E$. Then

(i) $\sigma^{-1} \circ F \circ \tau: \tilde{U} \rightarrow \tilde{V}$ is a holomorphic map if and only if $h_1(x_1, x_2)$ and $h_2(x_1, x_2)$ are coprime forms;

(ii) $\sigma^{-1} \circ F \circ \tau(\tilde{E}) = E$ if and only if $n_1 = n_2$, and the forms $h_1(x_1, x_2)$ and $h_2(x_1, x_2)$ are linearly independent;

(iii) if $n_1 = n_2 := n$, and $h_1(x_1, x_2)$ and $h_2(x_1, x_2)$ are coprime forms, then

(iii) $\sigma^{-1} \circ F \circ \tau(\tilde{E}) = E$ of $\sigma^{-1} \circ F \circ \tau$ to $\tilde{E}$ is given by $y_1 = h_1(x_1, x_2)$ and $y_2 = h_2(x_1, x_2)$,

(iii) $\deg F = n^2$ and $\deg f = n$.

To prove Theorem 1 we use the definitions and notation introduced in the previous sections. We assume that the branch locus $(B, o)$ of a cover $F: (U, o') \rightarrow (V, o)$ is given by the equation $uw(u-v) = 0$. The cover $\tilde{H}_2$ (see diagram (2)) is branched only in the disjoint union $B_1 \sqcup B_2 \sqcup B_3$ of three smooth curves, the proper inverse
images of the irreducible branches of the curve $B$. Therefore, $\tilde{W}$ is a smooth surface (that is, $\tilde{W} = \tilde{W}_m$) and the restriction of $\tilde{H}_2$ to $C = \tilde{H}_2^{\perp}(E)$ is $\beta(F)$, where $\deg \beta(F) = \deg \tilde{H}_2 := n'$. Hence $(C^2)_{\tilde{W}} = -n'$ and $\varsigma: \tilde{W} \to (W, o_1)$ is the minimal resolution of the singular point $o_1 \in W$ of singularity type $\tilde{A}_{n',1}$ (that is, $q = 1$). In addition, in diagram (13) the map $\gamma$ is the blowup of the point $\tilde{o}$, $\sigma^{-1}(\tilde{o}) = \tilde{D}_{n,1}(C) = \tilde{E}$ and $(\tilde{E}^2)_X = -1$; $\tau$ is the blowup of the point $o_1$, $\tau^{-1}(o') = \tilde{D}_{m,1,2}(\tilde{E}) = \tilde{E}$ and $(\tilde{E}^2)_{\tilde{U}} = -1$. In particular, all the maps in (13) are holomorphic. Note also that the restriction of $\tilde{D}_{n,1}$ to $\tilde{E}$ is an isomorphism between $\tilde{E}$ and $C$. Therefore, we can identify the restriction of $F_{f,\min} := \tilde{H}_2 \circ \tilde{D}_{n,1}$ to $\tilde{E}$ with $\beta(F)$.

Let $F_{f,\min}: (X, \tilde{o}) \to (V, o)$ be given by equations (20). Then, by Lemma 7, $f = \beta(F)$ is given by homogeneous forms $y_1 = h_1(x_1, x_2)$ and $y_2 = h_2(x_2, x_2)$ of degree $n$. Therefore, $n' = n$. In addition, according to Remark 7 we can identify the monodromy homomorphism $\tilde{H}_{2*}$ with monodromy homomorphism $f_*$.

If $F: (U, o') \to (V, o)$ is given by (2), then $F_{f,\min}$ is given by the homogeneous forms $y_1 = h_1(x_1, x_2)$ and $y_2 = h_2(x_2, x_2)$. Therefore, first, we can regard $F_{f,\min}$ as the restriction to $\mathbb{B}_2 \subset \mathbb{C}^2$ of the morphism $F_{f,\min}: \mathbb{C}^2 \to \mathbb{C}^2$ defined by the same functions. Second, we can identify $C \simeq \mathbb{P}^1$ in the rational Belyi pair $f: C \to \mathbb{P}^1$ with the quotient space $\mathbb{C}^2/\{(x_1, x_2) \sim (\lambda x_1, \lambda x_2) \quad \text{for} \quad \lambda \neq 0\}$ and the line $\mathbb{P}^1$ with $\mathbb{C}^2/\{(y_1, y_2) \sim (\lambda y_1, \lambda y_2) \quad \text{for} \quad \lambda \neq 0\}$. The ramification divisor $R_{f,\min}$ of $F_{f,\min}$ is given by (17). Therefore, $R_{f,\min}$ is a sum of lines passing through the origin such that

$$R_{f,\min} = \{(x_1, x_2) \sim (\lambda x_1, \lambda x_2) \quad \text{for} \quad \lambda \neq 0\} = R_f,$$

and it follows from Lemma 6 that the branch locus $B_{f,\min}$ of $F_{f,\min}$ is given by the equation $uv(u - v) = 0$ if $f \in Bel_3$. Therefore the singularity type of $B_{f,\min}$ is $D_4$. If $f \in Bel_2$ then the branch locus $B_{f,\min}$ of $F_{f,\min}$ is given by equation $uv = 0$ and therefore the singularity type of $B_{f,\min}$ is $A_1$. But, in both cases the branch locus $B_F$ of $F$ is given by the equation $uv(u - v) = 0$, since the branch curve of $\vartheta_{m,1,2}$ is contained in the union of the two lines given by $x_1 = 0$ and $x_2 = 0$ and $\{f(p_1), f(p_2)\} \subset B_F = \{0, 1, \infty\}$ for $p_1 = (0, 1)$ and $p_2 = (1, 0) \subset C$.

Conversely, if $F \in \mathcal{R}_{D_4}$ is given by equations (20), then it follows from the above considerations that $f = \beta(F)$ is given by homogeneous forms $y_1 = h_1(x_1, x_2)$ and $y_2 = h_2(x_2, x_2)$. Consider a cover $F_{f,\min}': (X', \tilde{o}) \to (V, o)$ given by the same homogeneous forms $u = h_1(x_1, x_2)$ and $v = h_2(x_2, x_2)$ and consider diagram (13) for $F_{f,\min}$ in which $\vartheta_{m,1,2} = \vartheta_{1,1}$ and we denote the germs of surfaces $W, \tilde{W}, \tilde{X}$ and the maps $H_2, \tilde{H}_2$ and so on by the same letters with primes ($W', \tilde{H}_2'$ and so on). Then, by Lemmas 6 and 7, $f' = \beta(F_{f,\min}')$: $C' \to E$ is given by the same homogeneous forms $y_1 = h_1(x_1, x_2)$ and $y_2 = h_2(x_2, x_2)$.

According to Remark 7, the covers $\tilde{H}_2$ and $\tilde{H}_2'$ have the same monodromy homomorphism $f_* = f'_*$. Therefore, by the Grauert-Remmert-Riemann-Stein Theorem, there exist biholomorphic isomorphisms $\varphi: \tilde{W} \to \tilde{W}'$ and $\psi: W \to W'$ such that $\tilde{H}_2 = \tilde{H}_2' \circ \varphi$ and $H_2 = H_2' \circ \psi$, and we can identify $\tilde{W}$ with $\tilde{W}'$ and $W$ with $W'$ with the help of these isomorphisms. Therefore there are biholomorphic isomorphisms $\varphibar: \tilde{X} \to \tilde{X}'$ and $\psi: X \to X'$ such that $\varphi \circ \vartheta_{n,1} = \varphibar_{n,1} \circ \varphibar$ and
\[ \psi \circ \theta_{n,1} = \theta'_{n,1} \circ \widetilde{\psi}, \] since \( \tilde{\theta}_{n,1} = \theta_{n,1} : \tilde{X} \setminus \tilde{E} = X \setminus \tilde{o} \to \tilde{W} \setminus C = W \setminus o_1 \) and \( \tilde{\theta}'_{n,1} = \theta'_{n,1} : \tilde{X}' \setminus \tilde{E}' = X' \setminus \tilde{o}' \to \tilde{W}' \setminus C' = W' \setminus o'_1 \) are the universal unramified covers. Hence we can identify \( \tilde{X} \) with \( \tilde{X}' \) and \( X \) with \( X' \). As a result, we see that the covers \( F_{f,\text{min}} : X \to W \) and \( F'_{f,\text{min}} : X' \to W' \) are equivalent.

If, in \( \vartheta_{m_1,m_2} : (U,o') \to (X,\tilde{o}) \), either \( m_1 \) or \( m_2 \), or both, are greater than 1, then we can obtain the cyclic cover \( \vartheta'_{m_1,m_2} : (U',o'') \to (X',\tilde{o}') \) branched in the images of the branch curves of \( \vartheta_{m_1,m_2} \), under the holomorphic isomorphism \( \tilde{\psi} \). The covers \( \vartheta_{m_1,m_2} \) and \( \vartheta'_{m_1,m_2} \) are obviously equivalent. Therefore the compositions \( F = F_{f,\text{min}} \circ \vartheta_{m_1,m_2} \) and \( F' = F'_{f,\text{min}} \circ \vartheta'_{m_1,m_2} \) of equivalent covers are equivalent and it is easy to see that there is a coordinate change in \( (U',o'') \) such that the cover \( F' \) is given by equations (2).

§ 4. The proof of Theorem 2

4.1. Let \( f = \beta(F) \in \mathcal{B}cl_2 \), deg \( f = n \), for \( F \) branched in \( (B,o) \) and having one of the \( ADE \) singularity types. Without loss of generality we can assume that \( f \) is given by \( y = x^n \) in nonhomogeneous coordinates and its branch locus is \( B_f = \{ 0, \infty \} \subset \{ 0, 1, \infty \} \). Then in the general case, \( \tilde{H}_2 : \tilde{W} \to \tilde{V} \) is a cyclic cover branched in two of the three trails of \( B \), deg \( \tilde{H}_2 = n \), the cover \( \tilde{H}_1 = \tilde{\theta}_{n,q} \circ \tilde{\vartheta}_{m_1,m_2} : \tilde{U} \to \tilde{W} \) is also a cyclic cover, and \( \tilde{\vartheta}_{m_1,m_2} \) must be branched in at least one of the irreducible components of the inverse image of the third trail.

4.2. The cases \( \mathcal{R}_{A_0} \) and \( \mathcal{R}_{A_1} \). It is easy to show that if \( F \in (\mathcal{R}_{A_0} \cup \mathcal{R}_{A_1}) \cap \beta^{-1}(\mathcal{B}cl) \), deg \( \beta(F) = n \), then \( F \) is equivalent to one of the following germs of covers of degree \( n^2m_1m_2 \) given by \( u = z^{nm_1} \) and \( v = w^{nm_2} \) for some \( m_1 \geq m_2 \geq 1 \), GCD \( (m_1, m_2) = 1 \) (if \( F \in \mathcal{R}_{A_0} \), then \( n = m_2 = 1 \)).

4.3. The case \( \mathcal{R}_{A_{2k+1}} \), \( k \geq 1 \). Without loss of generality we can assume that \( (B,o) \) is given by the equation \( u(u - v^{k+1}) = 0 \) and \( u = 0 \) is the equation of the irreducible component \( B_1 \) of \( B \). The graph \( \Gamma(\tilde{B}) \) of the curve germ \( (B,o) \) of singularity type \( A_{2k+1} \), \( k \geq 1 \), is shown in Figure 1 (in this case \( E = E_{k+3} \)). We number the trails of \( B \) as follows: \( \tilde{B}_1 = B_1, \tilde{B}_2 = B_2, \) and \( \tilde{B}_3 = B_3 \). Then \( \tilde{H}_2 \) is branched in either \( B_1 \cup B_2 \) or \( B_1 \cup (\bigcup_{j=3}^{k+2} E_j) \), where \( l = 1 \) or 2.

We show that the first case is impossible. In fact, if \( \tilde{H}_2 \) is branched in \( B_1 \cup B_2 \), then \( n \) must be equal to 2, since \( \tilde{H}_2^{-1}(\bigcup_{j=3}^{k+2} E_j) \) must be a chain of rational curves satisfying the second necessary condition. But if \( n = 2 \) then \( \tilde{H}_1 \) must be branched only in \( \tilde{H}_2^{-1}(\bigcup_{j=3}^{k+2} E_j) = \bigcup_{j=1}^{k+2} \tilde{E}_j \), where \( \tilde{E}_{k+1} = \tilde{E}_{k+2}^{-1}(E_{k+3}) = C \). In this case \( \tilde{W} \) is a smooth surface and \( (\tilde{E}_j)^2_{\tilde{W}} = -2 \) for all \( j \). Therefore, by Lemma 2 the group \( \pi_1(N_T \setminus (\bigcup_{j=1}^{k+1} \tilde{E}_j)) \simeq \mu_{2k+2} \), where \( N_T \) is a small tubular neighbourhood of \( \bigcup_{j=1}^{k+1} \tilde{E}_j \). It follows from Theorem 3 that the group \( \pi_1(N_T \setminus (\bigcup_{j=1}^{k+1} \tilde{E}_j)) \) is generated by an element \( \gamma_{\tilde{E}_1} \) represented by a simple loop around \( \tilde{E}_1 \) and \( \gamma_C = \gamma_{\tilde{E}_1}^{k+1} \), where \( \gamma_C \) is an element represented by a simple loop around \( \tilde{E}_{k+1} \). Therefore, in this case the second necessary condition is not satisfied, since \( \gamma_C = \gamma_{\tilde{E}_1}^{k+1} \) does not generate the group \( \mu_{2k+2} \).
In the second case we can assume without loss of generality that $\tilde{H}_2$ is branched in $\tilde{B}_1 = B_1$ and $\tilde{B}_3 = \bigcup_{j=3}^{k+2} E_j$, and, in addition, that $n = \deg \tilde{H}_2$ is a divisor of $k + 1$, $k + 1 = nm_0$, since $\pi_3^0 \simeq \mu_{k+1}$.

Consider the surface $\tilde{W}_m$ and the curve $\tilde{B} \subset \tilde{W}_m$ (see diagram (*)). The inverse image $(H_2 \circ \varsigma_m)^{-1}(B_2) = \bigcup_{j=1}^n \tilde{B}_{2,j}$ is the disjoint union of $n$ curve germs each of which intersects $\tilde{C}$ transversally and $(H_2 \circ \varsigma_m)^{-1}(B_1) = B_{1,1}$ is an irreducible curve germ. By Lemma 5, there are three possibilities for the curve $\tilde{B}^0 \subset \tilde{W}_m$ (see the notation introduced in § 2.1). The first (when $m_0 > 1$) is when $\tilde{B}^0 = \tilde{C} \cup (\bigcup_{j=1}^{m_0-1} \tilde{E}_j)$, the curve $\tilde{B}_1$ is one of the irreducible components of $\tilde{B}^0 \circ \varsigma_m^{-1}(B_2)$ (say $\tilde{B}_{2,1}$), and $\tilde{B}_2 = \emptyset$. In this case we have $(\tilde{E}_j^2)_{\tilde{W}_m} = -2$ and $(\tilde{C}^2)_{\tilde{W}_m} = -1$. In the second case (when $m_0 = 1$) $\tilde{B}_1 = \tilde{B}_{1,1}$, $\tilde{B}_2$ is one of the irreducible components of $(H_2 \circ \varsigma_m)^{-1}(B_2)$, and $\tilde{B}^0 = \tilde{C}$ and $(\tilde{C}^2)_{\tilde{W}_m} = -1$. In the third case (when $m_0 = 1$) $\tilde{B}_1$ and $\tilde{B}_2$ are two of the irreducible components of $(H_2 \circ \varsigma_m)^{-1}(B_2)$, and $\tilde{B}^0 = \tilde{C}$ and $(\tilde{C}^2)_{\tilde{W}_m} = -1$.

In all these cases, $(W, o_1)$ is a germ of smooth surface and $H_2$ is given by $u = y_1^n$ and $v = y_2$, where $y_1$ and $y_2$ are coordinates in $(W, o_1)$. We have $H_2^{-1}(B_2) = \bigcup_{j=1}^n B_{2,j}$, where the $B_{2,j}$ are given by the equations $y_1 - \omega_3 y_2^{m_0} = 0$, $\omega_3 = \exp(2\pi ji/n)$. Making scalar coordinate changes in $(V, o)$ and $(W, o_1)$ if necessary, we can assume that one of irreducible components of $H_2^{-1}(B_2)$ included in the branch curve of $H_1$ is given by the equation $y_1 - y_2^{m_0} = 0$.

In the first case set $x_1 = y_1 - y_2^{m_0}$ and $x_2 = y_2$. Then it is easy to see that $H_1: (U, o') \rightarrow (W, o_1)$ is given by the functions $z^m = x_1$ and $w = x_2$, where $z, w$ are coordinates in $(U, o')$ and $m > 1$. In view of Lemma 1, the second necessary condition entails the equality $\gcd(m_0, m) = 1$. Therefore, $F: (U, o') \rightarrow (V, o)$ is given by $u = (z^m + w^{m_0})^n$ and $v = w$, where $n, m, m_0 > 1$ and $\gcd(m_0, m) = 1$.

In the second case set $x_1 = y_1$ and $x_2 = y_2 - y_1$. Then it is easy to see that $H_1: (U, o') \rightarrow (W, o_1)$ is given by the functions $z^{m_1} = x_1$ and $w^{m_2} = x_2$, where $z$ and $w$ are coordinates in $(U, o')$, $m_1 > 1, m_2 > 1$ and $\gcd(m_1, m_2) = 1$. Therefore, $F: (U, o') \rightarrow (V, o)$ is given by $u = z^{m_1}$ and $v = z^{m_1} + w^{m_2}$, where $m_1 > 1, n, m_2 > 1$ and $\gcd(m_1, m_2) = 1$.

In the third case the cover $H_1: (U, o') \rightarrow (W, o_1)$ is branched in the two curves given by $y_1 - y_2 = 0$ and $y_1 - \omega_j y_2 = 0$ for some $j$, $1 \leq j \leq n - 1$. We put $x_1 = (\omega_j - 1)^{-1}(y_1 - y_2)$ and $x_2 = (\omega_j - 1)^{-1}(y_1 - \omega_j y_2)$. Then $H_1: (W, o_1) \rightarrow (V, o)$ is given by the functions $z^{m_1} = x_1$ and $w^{m_2} = x_2$, where $z$ and $w$ are coordinates in $(U, o')$, $m_1, m_2 > 1$ and $\gcd(m_1, m_2) = 1$. Therefore, $F: (U, o') \rightarrow (V, o)$ is given by $u = (\omega_j z^{m_1} - w^{m_2})^n$ and $v = z^{m_1} - w^{m_2}$, where $n, m_1, m_2 > 1$, $\gcd(m_1, m_2) = 1$ and $\omega_j = \exp(2\pi ji/n)$, $1 \leq j \leq n - 1$. 


4.4. The case $\mathcal{R}_{A_{2k}}, k \geq 1$. The graph $\Gamma(\tilde{B})$ of the curve germ $(B, o)$ of singularity type $A_{2k}, k \geq 1$, is shown in Figure 2 (in this case $E = E_{k+2}$). We show that in this case $(\bigcup_{k=1}^{\infty} \mathcal{R}_{A_{2k}}) \cap \beta^{-1}(\text{Bel}_2) = \emptyset$. Indeed, assume that there is $F \in \mathcal{R}_{A_{2k}} \cap \beta^{-1}(\text{Bel}_2)$ for some $k \geq 1$. Then (see diagram (\star)) $\tilde{H}_2$ can be branched in either $\tilde{B}_1 \cup \tilde{B}_2$, or $\tilde{B}_1 \cup \tilde{B}_3$, or $\tilde{B}_2 \cup \tilde{B}_3$, where $\tilde{B}_1 = E_{k+3}$, $\tilde{B}_2 = E_{2k+1} \cup \cdots \cup E_{k+1}$ and $\tilde{B}_3 = B_1$.

By Lemmas 2 and 3, $\tilde{\pi}_1^0 \simeq \mu_2$ and $\tilde{\pi}_2 \simeq \mu_{2k+1}$. Therefore, $\tilde{H}_2$ cannot be branched in $\tilde{B}_1 \cup \tilde{B}_2$ since $\text{GCD}(2, 2k + 1) = 1$.

Assume that $\tilde{H}_2$ is branched in $\tilde{B}_2 \cup \tilde{B}_3$. Then $\deg \tilde{H}_2$ is a divisor of $2k + 1$ and hence the dual graph of $\tilde{H}_2^{-1}(E_{k+2} \cup E_{k+3})$ is not a tree, that is, the second necessary condition is not satisfied.

Assume that $\tilde{H}_2$ is branched in $\tilde{B}_1 \cup \tilde{B}_3$. Then $\deg \tilde{H}_2 = 2$ and the dual graph of $\tilde{H}_2^{-1}(\tilde{B}_2 \cup E_{k+2}) = \bigcup_{j=1}^{2k+1} \tilde{E}_j$ (here $\tilde{E}_{k+1} = \tilde{H}_2^{-1}(E) = C$) is a tree with weights $[2, \ldots, 2, 3, 2, 3, 2, \ldots, 2]$. Therefore $\overline{B}^0 = \bigcup_{j=1}^{2k+1} \tilde{E}_j \subset \overline{W}_m$ is a tree with the weights $[2, \ldots, 2, 3, 2, \ldots, 2]$, since $(\tilde{H}_2^{-1}(E_{k+2}), \tilde{H}_2^{-1}(E_{k+3}))_{\overline{W}} = -1$. It follows from Theorem 3 that $\gamma_{\overline{W}} = 1$ in $\pi_1(\overline{W}_m \setminus \overline{B}^0)$, that is, the second necessary condition is not satisfied in this case.

4.5. The case $\mathcal{R}_{D_{2k+3}}, k \geq 1$. The graph $\Gamma(\tilde{B})$ of the curve germ $(B, o)$ of singularity type $D_{2k+3}, k \geq 1$, is shown in Figure 4 (in this case $E = E_{k+3}$).

We can assume that $(B, o)$ is given by the equation $u(v^2 - u^{2k+1}) = 0$. The cyclic cover $\tilde{H}_2: \tilde{W} \rightarrow \tilde{V}$ (see diagram (\star)) can be branched in either $\tilde{B}_1 \cup \tilde{B}_2$, $\tilde{B}_1 \cup \tilde{B}_3$ or $\tilde{B}_2 \cup \tilde{B}_3$, where $\tilde{B}_1 = B_1 \cup E_3 \cup \cdots \cup E_{k+2}$, $\tilde{B}_2 = E_{k+4}$ and $\tilde{B}_3 = B_2$, but using the same arguments as in § 4.4 it is easy to show that $\tilde{H}_2$ cannot be branched in either $\tilde{B}_1 \cup \tilde{B}_3$ or $\tilde{B}_2 \cup \tilde{B}_3$.

By Remark 6 and Lemmas 2 and 3, $\tilde{\pi}_1$ is the infinite cyclic group generated by $e_{k+2}$ and $\tilde{\pi}_1^0 \simeq \mu_{2k+1}$, $\tilde{\pi}_2 \simeq \mu_2$. Therefore, if $\tilde{H}_2$ is branched in $\tilde{B}_1 \cup \tilde{B}_2$, then $\deg \tilde{H}_2 = 2$ and $\tilde{H}_2(e_{k+2})$ generates the monodromy group $G_{\tilde{H}_2} \simeq \mu_2$. Applying the representation of the group $\tilde{\pi}_1$ obtained with the help of Theorem 3, it is not difficult to show that $\overline{W}$ has singular points of type $A_1$ over points of intersection of the irreducible components of $\overline{B}$ and $\overline{B}^0 \subset \overline{W}_m$ is a chain of rational curves, $\overline{B}^0 = C \cup \bigcup_{j=1}^{2k} \tilde{E}_j$, and the weights of its dual graph are $[2, \ldots, 2, 1]$. Therefore $(W, o_1)$ is a germ of a smooth surface and $H_2: (W, o_1) \rightarrow (V, o)$ is given by the functions $y_1^2 = u$ and $y_2 = v$, where $y_1$ and $y_2$ are some coordinates in $(W, o_1)$. The inverse image $H_2^{-1}(B_2) = B_{2,1} \cup B_{2,2}$ is the union of two branches given by the equations $y_2 - y_1^{2k+1} = 0$ and $y_2 + y_1^{2k+1} = 0$. The cyclic cover $H_2: (U, o') \rightarrow (W, o_1)$ is branched in $H_2^{-1}(B_1)$, given by $y_1 = 0$ with multiplicity $m_1$, and in one of the irreducible components of $H_2^{-1}(B_2)$ with multiplicity $m_2$. Without loss of generality we can assume that $H_1$ is branched in $B_{2,1}$. Set $x_1 = y_1$ and $x_2 = y_2 - y_1^{2k+1}$. Then $H_1$ is given by the functions $z^{m_1} = x_1$ and $w^{m_2} = x_2$. Therefore, $F: (U, o') \rightarrow (V, o)$
is given by the functions
\[ u = z^{2m_1} \quad \text{and} \quad v = z^{m_1(2k+1)} + w^{m_2}, \]
where \( k, m_1 \geq 1, m_2 > 1 \) and \( \gcd(m_1, m_2) = 1 \). Note that \( \zeta_m : W_m \to (W, o_1) \) is a composition of \( 2k + 1 \) \( \sigma \)-processes blowing up the point \( o_1 \) and points lying in the proper inverse images of \( B_{2,2} \) and such that \( \overline{C} \) is the exceptional curve of the last blowup. Therefore, by Lemma 1 it follows from the second necessary condition that \( \gcd(m_2, 2k + 1) = 1 \).

4.6. The cases \( \mathcal{R}_{\mathbf{E}_6} \) and \( \mathcal{R}_{\mathbf{E}_8} \). The graphs \( \Gamma(\tilde{B}) \) of the curve germs \((B, o)\) of singularity types \( \mathbf{E}_6 \) and \( \mathbf{E}_8 \) are shown in Figures 5 and 7. We show that \( \mathcal{R}_{\mathbf{E}_6} \cap \beta^{-1}(\mathcal{B}l_{2}) = \emptyset \) (the proof that \( \mathcal{R}_{\mathbf{E}_8} \cap \beta^{-1}(\mathcal{B}l_{2}) = \emptyset \) is similar and therefore is omitted). Assume that there exists \( F \in \mathcal{R}_{\mathbf{E}_6} \cap \beta^{-1}(\mathcal{B}l_{2}) \). Then (see diagram (*)\) \( \tilde{H}_2 \) can be branched either in \( \tilde{B}_1 \cup \tilde{B}_2, \tilde{B}_1 \cup \tilde{B}_3 \) or \( \tilde{B}_2 \cup \tilde{B}_3 \), where \( \tilde{B}_1 = E_2, \tilde{B}_2 = E_4 \cup E_5, \) and \( \tilde{B}_3 = B_1 \).

It follows from Theorem 3 that \( \tilde{\pi}_1^{\mathbf{E}_6} \simeq \mu_4 \) and \( \tilde{\pi}_2 \simeq \mu_3 \). Therefore, \( \tilde{H}_2 \) cannot be branched in \( \tilde{B}_1 \cup \tilde{B}_2 \) since \( \gcd(4, 3) = 1 \).

If \( \tilde{H}_2 \) is branched in \( \tilde{B}_1 \cup \tilde{B}_3 \) or in \( \tilde{B}_2 \cup \tilde{B}_3 \), then it is easy to see that the dual graph of \( \overline{B}^0 \subset W_m \) is not a tree, that is, the second necessary condition is not satisfied.

4.7. The case \( \mathcal{R}_{\mathbf{E}_7} \). The graph \( \Gamma(\tilde{B}) \) of the curve germ \((B, o)\) of singularity type \( \mathbf{E}_7 \) given by \( u(v^2 - u^3) = 0 \) is shown in Figure 6 (in this case \( E = E_4 \)). We number the trails of \( \tilde{B} \) as follows: \( \tilde{B}_1 = B_1 \cup E_5 \), where \( B_1 \) is given by the equation \( u = 0, \tilde{B}_2 = E_3, \) and \( \tilde{B}_3 = B_2 \). Similarly to the cases considered above, it is easy to see that \( \tilde{H}_2 \) must be branched in \( B_1 \cup B_2 \) and \( \deg \tilde{H}_2 = 3 \), and then \( (W, o_1) \) is a germ of smooth surface and \( H_2 \) is given by \( u = y_1^3 \) and \( v = y_2 \). Then \( H_2^{-1}(B_2) = B_{2,1} \cup B_{2,2} \cup B_{2,3} \), where the \( B_{2,j} \subset W \) are given in coordinates \( y_1, y_2 \) in \((W, o_1)\) by the equations \( y_1 - \omega_j y_2^2 = 0, \omega_j = \exp(2\pi ji/3), j = 1, 2, 3 \). The cover \( H_1: (U, o') \to (W, o_1) \) is branched in one of the irreducible components of \( H_2^{-1}(B_2) \) with multiplicity \( m_2 > 1 \) and possibly also in \( H_2^{-1}(B_1) \). As above, without loss of generality we can assume that \( H_1: (U, o') \to (W, o_1) \) is branched in \( B_{2,3} \). Set \( x_1 = y_1 \) and \( x_2 = y_2 - y_1^2 \). Then \( H_1 \) is given by the functions \( z^{m_1} = x_1 \) and \( w^{m_2} = x_2 \), where \( z \) and \( w \) are coordinates in \((U, o')\) and \( \gcd(m_1, m_2) = 1 \). Therefore, \( F: (U, o') \to (V, o) \) is given by
\[ u = z^{3m_1} \quad \text{and} \quad v = z^{2m_1} + w^{m_2}, \]
where \( m_1 \geq 1, m_2 > 1, \) and \( \gcd(m_1, m_2) = 1 \).

4.8. The case \( \mathcal{R}_{\mathbf{D}_4} \). It easily follows from the proof of Theorem 1 that a germ of cover \( F \in \mathcal{R}_{\mathbf{D}_4} \cap \beta^{-1}(\mathcal{B}l_{2}) \) is equivalent to a cover given by one of the following pairs of functions:
\[ u = z^{m_1 n} \quad \text{and} \quad v = (z^{m_1} + w^{m_2})^n \quad (21) \]
or
\[ u = (z^{m_1} - w^{m_2})^n \quad \text{and} \quad v = (z^{m_1} - \omega_j w^{m_2})^n, \quad (22) \]
where \( n \geq 2, m_1, m_2 \geq 1, \) \( \gcd(m_1, m_2) = 1 \) and \( \omega_j = \exp(2\pi ji/n), 1 \leq j \leq n - 1 \).
4.9. The case $\mathcal{R}_{D_{2k+2}}$, $k \geq 2$. Without loss of generality we can assume that 
$(B, o)$ is given by the equation $uv(v - u^k) = 0$, where $u = 0$ is the equation of 
the irreducible component $B_1$ of $B$ and $v = 0$ is the equation of the irreducible 
component $B_2$. The graph $\Gamma(B)$ of the curve germ $(B, o)$ of singularity type $D_{2k+2}$, 
k $\geq 2$, is shown in Figure 3 (in this case $E = E_{k+3}$). We number the trails of $B$ 
as follows: $\tilde{B}_1 = B_1 \cup E_4 \cup \cdots \cup E_{k+2}, \tilde{B}_2 = B_2$ and $\tilde{B}_3 = B_3$. The cyclic cover 
$\tilde{H}_2: \tilde{W} \to \tilde{V}$ is branched either in $\tilde{B}_1 \cup \tilde{B}_2$ or $\tilde{B}_2 \cup \tilde{B}_3$ (the case when $\tilde{H}_2$ is 
branched in $\tilde{B}_1 \cup \tilde{B}_3$ is the same as when $\tilde{H}_2$ is branched in $\tilde{B}_1 \cup \tilde{B}_2$, since we can 
make a coordinate change in $(V, o)$).

As in §4.3, it is easy to show that the case when $\tilde{H}_2$ is branched in $\tilde{B}_2 \cup \tilde{B}_3$ is 
impossible.

Consider the case when the cyclic cover $\tilde{H}_2: \tilde{W} \to \tilde{V}$ (see diagram (*) ) is 
branched in $\tilde{B}_1 \cup \tilde{B}_2$, $\deg \tilde{H}_2 = n$. Let $n = n_1 k_1$ and $k = k_1 k_2$, where $k_1 = 
\gcd(n, k)$ and $\gcd(n, k_2) = 1$. The group $\pi_1$ is generated by the element $\gamma_{E_{k+2}}$ 
represented by a simple loop around the curve $E_{k+2}$. It follows from Theorem 3 that 
$\gamma_{B_1} = \gamma_{E_{k+2}}^{k_1}$, where $\gamma_{B_1}$ is an element in $\tilde{\pi}_1$ represented by a simple loop around 
the germ $B_1$. The monodromy group $G_{\tilde{H}_2} \simeq \mu_n \subset \mathbb{S}_n$ is generated by the element 
g = $\tilde{H}_2*(\gamma_{E_{k+2}})$. The element $\tilde{H}_2*(\gamma_{B_1}) = g^{-1}$ is also a generator of the group $\mu_n$, 
where $\gamma_{B_2}$ is an element in $\tilde{\pi}_1$ represented by a simple loop around the germ $B_2$, 
and $\tilde{H}_2*(\gamma_{B_1}) = g^k$ is an element of order $n_1$, Therefore $\tilde{H}_2$ is branched in $B_1$ with 
multiplicity $n_1$ and in $B_2$ with multiplicity $n$. As a result, we see that $H_2$ is also 
only branched in $B_1$ with multiplicity $n_1$ and in $B_2$ with multiplicity $n$.

In diagram (13) the cover $\theta_{n', q}: (X, o) \to (W, o_1)$ is branched only at the 
point $o_1$. Therefore, in some coordinates $(x_1, x_2)$ in $(X, o)$ the map $F_{f, \min} = 
H_2 \circ \theta_{n', q}$; $(X, o) \to (V, o)$ is given by the functions 

$$u = x_1^{n_1} \quad \text{and} \quad v = x_2^n,$$

since $(X, o)$ is a germ of a smooth surface and $H_2 \circ \theta_{n', q}$ is branched in the 
divisor with normal crossing $B_1 \cup B_2$. The inverse image $F_{f, \min}^{-1}(B_3) \cup \mathbb{S}_n$ is the 
union of $n$ smooth curves given by the equation $x_2^n - x_1^{n_1 k} = \prod_{j=1}^{n} (x_2 - \omega_j x_1^{k_2}) = 0$, 
where $\omega_j = \exp(2\pi ij/n)$, $j = 1, \ldots, n$.

The map $\vartheta_{m_1, m_2}:(U, o') \to (X, o)$ is branched in at most two irreducible curves, 
one of which belongs to $F_{f, \min}^{-1}(B_3)$. Without loss of generality we can assume that 
it is given by $y_2 := x_2 - x_1^{k_2} = 0$. If the branch locus of $\vartheta_{m_1, m_2}$ consists of two 
irreducible components, then the other is an irreducible component of the inverse 
image of either $B_1$, $B_2$ or $B_3$. Therefore we have three possibilities: the second 
irreducible component is either given by the equation $y_1 := x_1 = 0$ or $y_1 := x_2 = 0$ 
(if $k_2 = 1$), or $y_1 := x_2 - \omega_j x_1^{k_2} = 0$ for some $j = 1, \ldots, n - 1$ (if $k_2 = 1$), and 
$\vartheta_{m_1, m_2}$ is given by functions $y_1 = z^{m_1}$ and $y_2 = w^{m_2}$. Applying Lemma 1, the 
second necessary condition is equivalent to the condition that $\gcd(m_1, m_2) = \gcd(m_2, k_2) = 1$ in the first case, and $\gcd(m_1, m_2) = 1$ in the second and third cases. As a result, $F: (U, o') \to (V, o)$ is equivalent to one of the following covers 
given by the functions

$$u = z^{m_1 k_1} \quad \text{and} \quad v = (z^{m_1 k_2} + w^{m_2})^{n_1 k_1}.$$
in the first case;
\[ u = (z^{m_1} - w^{m_2})^{n_1} \quad \text{and} \quad v = z^{m_1 n_1 k_1} \]
in the second case;
\[ u = (z^{m_1} - w^{m_2})^{n_1} \quad \text{and} \quad v = (z^{m_1} - \omega_j w^{m_2})^{n_1 k_1} \]
in the third case, where \( k_1 k_2 \geq 2, \ n_1 k_1 \geq 2, \ m_1, m_2 \geq 1, \ GCD(m_1, m_2) = GCD(n m_2, k_2) = 1, \) and \( \omega_j = \exp(2\pi ji/n_1k_1), j = 1, \ldots, n_1 k_1 - 1. \)

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