Existence of Periodic Solutions for a Class of Second Order Ordinary Differential Equations

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Abstract We provide sufficient conditions for the existence of a periodic solution for a class of second order differential equations of the form \( \ddot{x} + g(x) = \varepsilon f(t, x, \dot{x}, \varepsilon) \), where \( \varepsilon \) is a small parameter.

Keywords Periodic orbit · Second-order differential equation · Averaging theory

Mathematics Subject Classification (2010) 37G15 · 37C80 · 37C30

1 Introduction and Statement of the Results

The second order differential equations of the form
\[
\ddot{x} + g(x) = \varepsilon f(t, x, \dot{x}, \varepsilon),
\]
have been studied by many authors because they have many applications, see for instance [2, 3, 5, 8–12, 15, 17, 19]. Two of the main families studied are the Duffing equations see [6, 7], and the forced pendulum, see the nice survey [14] and the references quoted therein.

The aim of this work is to study periodic solutions of the second order differential equation
\[
\ddot{x} + g(x) = \mu^{2n+1} p(t) + \mu^{4n+1} q(t, x, \dot{x}, \mu), \tag{1}
\]
where \( n \) is a positive integer, \( \mu \) is a small parameter, and the functions
\[
g(x) = x + x^{2n+1}(b + x h(x)),
\]
and \(h(x)\) are smooth, \(b \neq 0\), \(p(t)\) and \(q(t, x, y, \mu)\) are smooth and periodic with period \(2\pi\) in the variable \(t\).

Let \(\Gamma(x)\) be the Gamma function, see for more details [1], and let \(\alpha\) and \(\beta\) be the first Fourier coefficients of the periodic function \(p(t)\), i.e.

\[
\alpha = \frac{1}{\pi} \int_0^{2\pi} p(t) \cos t \, dt, \quad \beta = \frac{1}{\pi} \int_0^{2\pi} p(t) \sin t \, dt.
\]

Now our main result is the following.

**Theorem 1** If \(\alpha \beta \neq 0\) then for \(\mu \neq 0\) sufficiently small the differential equation (1) has a \(2\pi\)-periodic solution \(x(t, \mu)\) such that

\[
x(0, \mu) = \pi \frac{\Gamma(n + 2)}{2b \Gamma(n + \frac{3}{2})} \alpha \left(\frac{\beta^2}{\alpha^2} + 1\right)^{-n} + O(\mu^{2n}).
\]

Theorem 1 is proved in Sect. 3, where we use the averaging theory for computing periodic solutions, see Sect. 2 for a summary of the results on this theory that we shall need.

### 2 The Averaging Theory

We want to study the \(T\)-periodic solutions of the periodic differential systems of the form

\[
x' = F_0(t, x) + \varepsilon F_1(t, x) + o(\varepsilon), \tag{2}
\]

with \(\varepsilon > 0\) sufficiently small, where \(F_0, F_1 : \mathbb{R} \times \Omega \to \mathbb{R}^n\) and \(F_2 : \mathbb{R} \times \Omega \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n\) are \(C^2\) functions, \(T\)-periodic in the variable \(t\), and \(\Omega\) is an open subset of \(\mathbb{R}^n\). Let \(x(t, z, \varepsilon)\) be the solution of the differential system (2) such that \(x(0, z, 0) = z\). Suppose that the unperturbed system

\[
x' = F_0(t, x), \tag{3}
\]

has an open set \(V\) with \(\overline{V} \subset \Omega\) such that for each \(z \in \overline{V}\), \(x(t, z, 0)\) is \(T\)-periodic.

Let \(y\) be an \(n \times n\) matrix, and consider the first order variational equation

\[
y' = D_x F_0(t, x(t, z, 0)) y, \tag{4}
\]

of the unperturbed system (3) on the periodic solution \(x(t, z, 0)\). Let \(M_z(t)\) be the fundamental matrix of the linear differential system (4) with periodic coefficients such that \(M_z(0)\) is the \(n \times n\) identity matrix.

**Theorem 2** Consider the function \(F : \overline{V} \to \mathbb{R}^n\)

\[
f(z) = \int_0^T M_z^{-1}(t) F_1(t, x(t, z, 0)) \, dt. \tag{5}
\]

If there exists \(\alpha \in V\) with \(f(\alpha) = 0\) and

\[
\det\left(\left(\frac{df}{dz}\right)(\alpha)\right) \neq 0, \tag{6}
\]

then there exists a \(T\)-periodic solution \(x(t, \varepsilon)\) of system (2) such that \(x(0, \varepsilon) = \alpha + O(\varepsilon)\).

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The existence of the periodic solution of Theorem 2 is due to Malkin [13] and Roseau [16], for a shorter and easier proof see [4]. The proof for the stability follows in a similar way to the proof of Theorem 11.6 of [18].

### 3 Proof of Theorem 1

The differential equation of second order (1) can be written as the first order differential system

\[
\begin{align*}
\dot{x} &= y, \\
\dot{y} &= -x - x^{2n+1}(b + xh(x)) + \mu^{2n+1} p(t) + \mu^{4n+1} q(t, x, y, \mu). 
\end{align*}
\]

(7)

In order to apply the averaging theory described in Sect. 2 to this differential system we do the scaling \( x \to \mu x \) and \( y \to \mu y \). Hence the differential system (7) becomes

\[
\begin{align*}
\dot{x} &= y, \\
\dot{y} &= -x + \mu^{2n}(-bx^{2n+1} + p(t)) \\
&\quad + \mu^{2n+1} x^{2n+2} h(\mu x) + \mu^{4n} q^*(t, \mu x, \mu y, \mu).
\end{align*}
\]

(8)

This system is written into the normal form (2) for applying the averaging theory described in Sect. 2, where

\[
\begin{align*}
x &= (x, y), \\
\varepsilon &= \mu^{2n}, \\
F_0(x) &= (y, -x), \\
F_1(t, x) &= (0, -bx^{2n+1} + p(t)), \\
o(\varepsilon) &= \mu^{2n+1} x^{2n+2} h(\mu x) + \mu^{4n} q^*(t, \mu x, \mu y, \mu).
\end{align*}
\]

(9)

From Sect. 2 the solution \( x(t, z, 0) = (x(t, z, 0), y(t, z, 0)) \) of system (8) with \( \varepsilon = 0 \) satisfies \( x(0, z, 0) = z = (x_0, y_0) \), and consequently

\[
\begin{align*}
x(t, z, 0) &= x_0 \cos t + y_0 \sin t, \\
y(t, z, 0) &= -x_0 \sin t + y_0 \cos t.
\end{align*}
\]

The fundamental matrix \( M_z(t) = M(t) \) of the first order variational equation (4) satisfying (9) is

\[
M(t) = \begin{pmatrix}
\cos t & \sin t \\
-\sin t & \cos t
\end{pmatrix}.
\]

According with Theorem 2 in order to compute the \( 2\pi \)-periodic solutions of the differential system (8) we must compute the integral

\[
\begin{align*}
f(z) &= \begin{pmatrix}
\tilde{f}_1(x_0, y_0) \\
\tilde{f}_2(x_0, y_0)
\end{pmatrix} \\
&= \int_0^{2\pi} M^{-1}(t) F_1(t, x(t, z, 0)) \, dt \\
&= \begin{pmatrix}
b \int_0^{2\pi} \sin t (x_0 \cos t + y_0 \sin t)^{2n+1} \, dt - \int_0^{2\pi} p(t) \sin t \, dt \\
-b \int_0^{2\pi} \cos t (x_0 \cos t + y_0 \sin t)^{2n+1} \, dt + \int_0^{2\pi} p(t) \cos t \, dt
\end{pmatrix}.
\end{align*}
\]
Doing induction with respect to $n$ it is not difficult to show that
\[
\int_0^{2\pi} \sin t (x_0 \cos t + y_0 \sin t)^{2n+1} dt = \frac{2\sqrt{\pi} \Gamma\left(\frac{3}{2} + n\right)}{\Gamma(2 + n)} y_0 (x_0^2 + y_0^2)^n,
\]
\[
\int_0^{2\pi} \cos t (x_0 \cos t + y_0 \sin t)^{2n+1} dt = \frac{2\sqrt{\pi} \Gamma\left(\frac{3}{2} + n\right)}{\Gamma(2 + n)} x_0 (x_0^2 + y_0^2)^n.
\]
Therefore we must solve the system
\[
\begin{pmatrix}
    f_1(x_0, y_0) \\
    f_2(x_0, y_0)
\end{pmatrix}
= \begin{pmatrix}
    \frac{2\sqrt{\pi} b \Gamma\left(\frac{3}{2} + n\right)}{\Gamma(2 + n)} y_0 (x_0^2 + y_0^2)^n - \pi \beta_1 \\
    -\frac{2\sqrt{\pi} b \Gamma\left(\frac{3}{2} + n\right)}{\Gamma(2 + n)} x_0 (x_0^2 + y_0^2)^n + \pi \alpha_1
\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]
This system has a unique solution
\[
\begin{pmatrix}
    x_0^* \\
    y_0^*
\end{pmatrix} = \pi^{-\frac{1}{2n+2}} \left( \frac{\Gamma(n+2)}{2b \Gamma(n+\frac{3}{2})} \alpha \left( \frac{\alpha^2}{\beta^2} + 1 \right)^{\frac{1}{2n+1}} - \frac{1}{2^n} \right) \left( \beta \left( \frac{\beta^2}{\alpha^2} + 1 \right)^\frac{1}{2n+1} - \frac{1}{2^n} \right) \left( \frac{\alpha^2}{\beta^2} + 1 \right)^\frac{1}{2n+1}.
\]

The determinant (6) of the Jacobian matrix $Df(x_0^*, y_0^*)$ is
\[
\det(Df(x_0^*, y_0^*)) = 4 \pi^{\frac{1}{2n+1}} (2n+1) \pi^{\frac{2n+1}{2n+2}} \left( \frac{\Gamma(n+2)}{b \Gamma(n+\frac{3}{2})} \right)^{\frac{1}{2n+1}} \times \left( \left( \beta \left( \frac{\beta^2}{\alpha^2} + 1 \right)^\frac{1}{2n+1} - \frac{1}{2^n} \right) \left( \alpha \left( \frac{\alpha^2}{\beta^2} + 1 \right)^\frac{1}{2n+1} - \frac{1}{2^n} \right) \right)^{2n},
\]
and by assumptions it is positive because $\alpha \beta b \neq 0$.

In summary all the assumptions of Theorem 2 hold and consequently from Theorem 2 it follows Theorem 1.

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