ON THE MODULARITY OF REDUCIBLE MOD $l$ GALOIS REPRESENTATIONS

NICOLAS BILLEREY $\S$ AND RICARDO MENARES $\dag$

Abstract. We prove that every odd semisimple reducible (2-dimensional) mod $l$ Galois representation arises from a cuspidal eigenform. In addition, we investigate the possible different types (level, weight, character) of such a modular form. When the representation is the direct sum of the trivial character and a power of the mod $l$ cyclotomic character, we are able to characterize the primes that can arise as levels of the associated newforms. As an application, we determine a new explicit lower bound for the highest degree among the fields of coefficients of newforms of trivial Nebentypus and prime level. The bound is valid in a subset of the primes with natural (lower) density at least one half.

Introduction

Let $l$ be a rational prime number. In this paper we are interested in (2-dimensional) mod $l$ Galois representations, that is, continuous homomorphisms

$$\rho : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_2(\mathbb{F}_l).$$

Work of Deligne, extending earlier results by Eichler and Shimura, shows that such representations arise naturally in the theory of modular forms. More precisely, let us take $\overline{\mathbb{Q}}$ to be the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$ and fix a place $v$ of $\mathbb{Q}$ over $l$. We denote by $a \mapsto \overline{a}$ the reduction map mod. $v$ from the ring of integers $\mathbb{Z}$ of $\mathbb{Q}$ to the residue field $\mathbb{F}_l$. Let us moreover denote by $S_k(\Gamma_1(N))$ the $\mathbb{C}$-vector space of cuspidal modular forms of weight $k \geq 2$ for $\Gamma_1(N)$. Then, attached to any form $f \in S_k(\Gamma_1(N))$ which is an eigenform for the Hecke operators $\{T_p\}_{p \mid N}$ with corresponding set of eigenvalues $\{a_p\}_{p \mid N}$, there is a Dirichlet character $\chi$ of modulus $N$ and an odd semisimple Galois representation

$$\rho_f : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_2(\mathbb{F}_l),$$

unique up to isomorphism, which is unramified outside $Nl$ and satisfies for every prime $p \nmid Nl$

$$\begin{cases} 
\text{tr}(\rho_f(\text{Frob}_p)) = \overline{a_p} \\
\text{det}(\rho_f(\text{Frob}_p)) = \chi(p)p^{k-1}
\end{cases}$$

where $\text{Frob}_p$ denotes a Frobenius element at $p$ in $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

A natural problem is then to understand which mod $l$ Galois representations $\rho$ are modular, meaning that $\rho$ is isomorphic to some $\rho_f$ as above. We
shall also say in this case that $\rho$ arises from the cuspidal eigenform $f$. This question was first considered by Serre in the seventies and led him to his famous modularity conjecture in 1987 ([Ser87]) which (in its weak form) asserts that every odd, irreducible Galois representation $\rho : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}_2(\mathbb{F}_l)$ is modular. Serre’s conjecture is by now a theorem of Khare-Wintenberger ([KW09a, KW09b]).

In this paper we take an alternative point of view by considering the case of reducible representations. We first prove the analogue of the modularity statement in this context.

**Theorem 0.1.** Every odd representation which is the direct sum of two characters arises from a cuspidal eigenform.

The proof relies on explicit computations with Eisenstein series and their reductions modulo suitable prime numbers. Given a representation $\rho$ as in Theorem 0.1, we are able to specify a type (level, weight, character) of a Hecke (cuspidal) eigenform giving rise to it (cf. Section 2). The representation $\rho$ is then modular of infinitely many different types and by analogy with the irreducible case we can consider several questions regarding optimization and characterization of these possible types.

In sections 3 and 4 we discuss some of these questions in the case of the representation $1 \oplus \chi_l^{k-1}$, where $k$ is an even integer $\geq 2$ and $\chi_l$ denotes the mod $l$ cyclotomic character. This case is rather general, as every modular reducible representation arising from a newform with squarefree level and trivial Nebentypus is necessarily of this form (see §3.2). For $k = 2$, a well-known result of Mazur ([Maz77, Prop. (5.12)]) says that $1 \oplus \chi_l$ arises from a weight-2 newform of prime level $N$ and trivial Nebentypus if and only if $l$ divides the numerator of $(N - 1)/12$. We extend Mazur’s result to higher weights in the following sense.

**Theorem 0.2.** Let $k$ be an even integer $\geq 4$ and assume $l > k + 1$. Then the representation $1 \oplus \chi_l^{k-1}$ arises from a weight-$k$ newform of prime level $N$ and trivial Nebentypus if and only if at least one of the following conditions holds:

1. $N^k \equiv 1 \pmod{l}$
2. $N^{k-2} \equiv 1 \pmod{l}$ and $l$ divides the numerator of $B_k/k$, where $B_k$ denotes the $k$-th Bernoulli number.

We stress the fact that this statement considers not only eigenforms, but newforms (in Mazur’s statement this distinction is unnecessary because there are no oldforms of weight two and prime level). Our proof of the reverse implication proceeds by first constructing, using explicit computations with Eisenstein series, a cuspidal eigenform satisfying the necessary congruences. We then use a theorem of Diamond in [Dia91] to show that we can take the eigenform to be a newform of level $N$.

In §4.3 we discuss how the hypothesis of Theorem 0.2 relate to the classical level-raising condition. When $k \geq 4$ is even, we conjecture that the level-raising condition at a Steinberg prime $p$ is actually sufficient for the representation $1 \oplus \chi_l^{k-1}$ to arise from a newform of level divisible by $p$. This leads to the following conjectural description of the set of squarefree levels of weight-$k$ newforms with trivial Nebentypus that give rise to $1 \oplus \chi_l^{k-1}$.
Conjecture 0.3. Let $k$ be an even integer $\geq 4$ and assume $l > k + 1$. Then the representation $1 \oplus \chi^{k-1}_l$ arises from a weight-$k$ newform of squarefree level $N$ and trivial Nebentypus if and only if at least one of the following conditions holds:

1. we have $(p^k - 1)(p^{k-2} - 1) \equiv 0 \pmod{l}$ for every prime number $p$ dividing $N$ and there exists a prime divisor $p_0$ of $N$ such that $p_0^k \equiv 1 \pmod{l}$;
2. we have $p^{k-2} \equiv 1 \pmod{l}$ for every prime number $p$ dividing $N$ and $l$ divides the numerator of $B_k/k$, where $B_k$ denotes the $k$-th Bernoulli number.

We are able to show the direct implication in this conjecture (see Theorem 4.1). Concerning the reverse implication, we prove a weaker statement (see Theorem 4.3). On the other hand, Theorem 0.2 settles the conjecture in the case where $N$ is prime. The case $N = 1$ follows in one direction from a result of Ribet ([Rib75, Lem. 5.2]) and from Deligne-Serre’s lifting lemma ([DS74, Lem. 6.11]) in the other (cf. Corollary 4.4 and Remark 4.5).

In the case $k = 2$, the characterization of the possible squarefree levels $N$ attached to $1 \oplus \chi_l$ seems to be more delicate. Let $r$ be the number of prime factors of $N$. When $r = 1$, Mazur’s theorem solves the problem. A theorem of Ribet ([Rib10]) settles the case $r = 2$. His results have been extended to the case of $r = 3$ in the recent Ph. D. thesis by his student H. Yoo. Each one of these works show that the statement of Conjecture 0.3 is false when $k = 2$.

To a normalized eigenform $f \in S_k(\Gamma_0(N))^{\text{new}}$ with $q$-expansion at infinity

$$f = \sum_{n \geq 1} a_n(f)q^n, \quad q = e^{2\pi iz},$$

we attach the number field $K_f := \mathbb{Q}(a_n(f) : (n, N) = 1)$. Put

$$d_k^{\text{new}}(N) := \max\{[K_f : \mathbb{Q}] : f \in S_k(\Gamma_0(N))^{\text{new}}, \text{ normalized Hecke eigenform}\}.$$

A Theorem of Royer ([Roy00]) implies that

$$(0.1) \quad d_k^{\text{new}}(N) \gg_k \sqrt{\log \log N}, \quad N \to \infty, \quad N \text{ prime}^1.$$

As an application of our results, in the spirit of [DJUR11], we exploit the congruences given by Theorem 0.2 to obtain a lower bound for $d_k^{\text{new}}(N)$. As a new ingredient, we use a theorem of Goldfeld ([Gol69]) on prime numbers $p$ for which $p - 1$ has a large prime factor. We then manage to obtain a bound which is better than (0.1) but is only valid in a restricted class of prime numbers.

**Theorem 0.4.** There exists an explicit set of primes $\mathcal{P}$ of (natural) lower density at least $\frac{1}{2}$ with the property that, for every even integer $k \geq 2$, there exists a constant $c_k > 0$ such that the inequality

$$d_k^{\text{new}}(N) \geq c_k \log N$$

$^1$Royer’s theorem holds for arbitrary levels $N$ with a constant depending on a fixed prime not dividing $N$. It is however stated only in the case $k = 2$ in loc. cit. but the proof undoubtedly extends to weights $\geq 2$. 
holds for all \( N \in \mathcal{P} \) with \( N \geq (k+1)^2 \). The constant \( c_k \) can be taken as
\[
c_k = \left( 4 \log(1 + 2^{(k-1)/2}) \right)^{-1}.
\]

If we assume the truth of Conjecture 0.3, then it is possible to extend the validity of the above bound to appropriate squarefree integers (cf. Theorem 5.3).

In the spirit of Maeda’s conjecture, P. Tsaknias has proposed a conjectural lower bound for \( d_k^{\text{new}}(N) \) for \( N \) fixed and varying \( k \) [T13]. His conjecture implies that there exists a constant \( c > 0 \) such that, for all prime numbers \( N \), there is an integer \( k(N) \) such that \( d_k^{\text{new}}(N) > cN \) for all \( k \geq k(N) \). Further numerical data, that Tsaknias has generously shared with us, suggests that \( k(N) \) is a bounded function of \( N \). If this were true, then \( d_k^{\text{new}}(N) \) would grow linearly with \( N \) if \( k \) is fixed.

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1. Preliminaries on Eisenstein series

In this section we recall some classical definitions and compute the constant term of the \( q \)-expansion at various cusps of some specific Eisenstein series that will be used in the sequel.

1.1. Gauss sums and Bernoulli Numbers. Let \( \psi: (\mathbb{Z}/c\mathbb{Z})^\times \rightarrow \mathbb{C}^\times \) be a primitive Dirichlet character of modulus \( c \geq 1 \). The Gauss sum attached to \( \psi \) is defined by
\[
W(\psi) = \sum_{n=1}^{c} \psi(n)e^{2i\pi n/c}
\]
and the Bernoulli numbers \( (B_{m,\psi})_{m \geq 1} \) by
\[
\sum_{n=1}^{c} \psi(n)\frac{te^{nt}}{e^{nt}-1} = \sum_{m \geq 0} B_{m,\psi} \frac{t^m}{m!}.
\]

In particular, if \( \psi \) is the trivial character (of modulus 1), then \( B_{m,\psi} \) is the classical Bernoulli number \( B_m \), except when \( m = 1 \) in which case we have \( B_{1,\psi} = -B_1 = 1/2 \).

The Bernoulli numbers are related to certain special values of the \( L \)-function \( L(s,\psi) \) attached to \( \psi \). More precisely, we have the following proposition ([Was97, Ch. 4]).

**Proposition 1.1.** Let \( m \geq 2 \) be an integer such that \( \psi(-1) = (-1)^m \).

Then, we have that
\[
L(m, \psi) = -W(\psi) \frac{C_m}{c^m} \cdot \frac{B_{m,\psi}}{2m} \neq 0, \quad \text{where} \quad C_m = \frac{(-2i\pi)^m}{(m-1)!}
\]
and \( \overline{\psi} \) means the complex conjugate of \( \psi \).
1.2. Constant term computations. For $A \in \mathbb{R}$, let us denote by $\alpha_A$ the operator acting on complex valued functions $f$ on the upper half-plane $h$ by

$$\alpha_A(f)(z) = f(Az).$$

1.2.1. General computations. Let $k \geq 3$ be an integer and $\varepsilon_0 : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ be a primitive Dirichlet character of modulus $N \geq 1$ such that $\varepsilon_0(-1) = (-1)^k$. We denote by $E_{k, \varepsilon_0}^1$ the Eisenstein series in $\mathcal{M}_k(\Gamma_1(N), \varepsilon_0)$ given by the following $q$-expansion:

$$E_{k, \varepsilon_0}^1(z) = -\frac{B_{k, \varepsilon_0}}{2k} + \sum_{n \geq 1} \left( \sum_{m|n} \varepsilon_0(m)m^{k-1} \right) q^n. \tag{1.2}$$

Note that when $N = 1$, $E_{k, \varepsilon_0}^1 = E_{k, 1}^1$ is nothing but the classical level 1 Eisenstein series of weight $k$:

$$E_k = -\frac{B_k}{2k} + \sum_{n \geq 1} \sigma_{k-1}(n)q^n.$$

The main goal of this paragraph is to compute the constant term of the $q$-expansion of $(\alpha_M E_{k, \varepsilon_0}^1)|_{k\gamma}$ where $M \geq 1$ is an integer coprime to $N$, $\gamma \in \text{SL}_2(\mathbb{Z})$ and the notation $|_k$ means the classical slash operator on modular forms.

**Proposition 1.2.** Let $\varepsilon_0$ and $k$ as above. Let $M$ be an integer $\geq 1$ coprime to $N$ and $\gamma = \begin{pmatrix} u & \beta \\ v & \delta \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$. The constant term of the $q$-expansion of $(\alpha_M E_{k, \varepsilon_0}^1)|_{k\gamma}$ is

$$\left\{ \begin{array}{ll} 0 & \text{if } N \nmid \frac{v}{r} \\ \frac{-\overline{\varepsilon_0}(M')\varepsilon_0(\delta)}{M'k} B_{k, \varepsilon_0} \frac{2k}{2k} \neq 0 & \text{otherwise} \end{array} \right.$$ where $r = \gcd(v, M)$ and $M' = M/r$.

**Proof.** With the notations of §1.1 put

$$G = \frac{2C_k W(\overline{\varepsilon_0})}{\gamma_k} E_{k, \varepsilon_0}^1.$$ Recall also from [DS05, Ch. 4] that

$$G = \sum_{j=0}^{N-1} \frac{\varepsilon_0(j) G_k^{(0,j)}}{\overline{\varepsilon_0}(j, N)},$$ where the bar over $(0, j)$ means reduction modulo $N$ (while $\overline{\varepsilon_0}$ means the complex conjugate of $\varepsilon_0$ as in §1.1) and

$$G_k^{(0,j)}(\tau) = \sum_{\substack{(c,d) \in \mathbb{Z}^2 \setminus \{(0,0)\} \\mod N \atop (c,d) \equiv (0,j)}} \frac{1}{(c\tau + d)^k}.$$

Therefore for any $0 \leq j \leq N - 1$ coprime to $N$, we have

\[
\left( \alpha_M G_k^{(0,j)} \right)_{k \gamma} = \sum_{(c,d) \in \mathbb{Z}^2 \setminus \{(0,0)\}}^{(c,d) \equiv (0,j) \pmod{N}} \frac{1}{((cM + d)\tau + cM\beta + d\delta)^k}.
\]

Hence its constant term is given by

\[
\Upsilon_j = \sum_{(c,d) \in \mathbb{Z}^2 \setminus \{(0,0)\}}^{(c,d) \equiv (0,j) \pmod{N}} \frac{1}{(cM\beta + d\delta)^k}.
\]

Let $r = \gcd(v, M)$. Then $cMu + dv = 0$ if and only if $cM'u + dv' = 0$ where $M' = M/r$ and $v' = v/r$.

If $u = 0$, then $\Upsilon_j = 0$ (for any $j$) unless $N = 1$ in which case $j = 0$ and

\[
\Upsilon_0 = \sum_{c \in \mathbb{Z}} \frac{1}{(cM\beta)^k} = 2 \frac{1}{M^k} \zeta(k)
\]

since in this case $\beta = \pm 1$ and $k$ is even. Therefore, according to Prop. 1.1, the constant term $\Upsilon$ of $(\alpha_M G)_{k \gamma}$ is 0 if $N > 1$ (and thus $N \nmid v = \pm 1$) and is $-\frac{C_k B_k}{M^k k}$ when $N = 1$. Hence the result in this case.

Assume now $u \neq 0$. Given $d \in \mathbb{Z}$, $d \equiv j \pmod{N}$, the following conditions are then equivalent:

(1) there exists $c \in \mathbb{Z}$, $c \equiv 0 \pmod{N}$ such that $cMu + dv = 0$;
(2) we have $N \mid v'$ and $M'u \mid d$.

Indeed, if the first condition is satisfied, we have $cM'u + dv' = 0$ and thus $M'u \mid dv'$. But $M'u$ and $v'$ are coprime hence $M'u \mid d$. Besides, $0 = cM'u + dv' \equiv dv' \pmod{N}$ and since $d \equiv j \pmod{N}$ and $\gcd(j, N) = 1$, we get that $N \mid v'$.

On the other hand, if the second condition holds, put $c = -\frac{d}{M'u} v' = -\frac{d}{M'u} v$. Then $c \in \mathbb{Z}$ satisfies $cMu + dv = 0$ and since $N \mid v'$ by assumption, we get $c \equiv 0 \pmod{N}$.

Moreover if these equivalent conditions are satisfied, then we have

\[
cM\beta + d\delta = \frac{1}{u}(cu\beta + du\delta) = \frac{1}{u}(du\delta - dv\beta) = \frac{d}{u}.
\]
Therefore the constant term $\Upsilon$ of $(\alpha_M G)|_{k, \gamma}$ is 0 when $N \nmid v'$ and is otherwise given by

$$\Upsilon = \sum_{j=0}^{N-1} \frac{\varepsilon_0(j) \Upsilon_j}{\text{gcd}(j, N) = 1} = \sum_{j=0}^{N-1} \frac{\varepsilon_0(j)}{\text{gcd}(j, N) = 1} \sum_{d \equiv j (\text{mod } N)} \left( \frac{u}{d} \right)^k$$

$$= \sum_{j=0}^{N-1} \frac{\varepsilon_0(M'u)}{M^k} \sum_{d \equiv j (\text{mod } N)} \frac{1}{(M'd)^k}$$

$$= \frac{\varepsilon_0(M'u)}{M^k} \sum_{d \equiv j (\text{mod } N)} \frac{\varepsilon_0(d)}{d^k}$$

$$= 2 \frac{\varepsilon_0(M'u)}{M^k} \frac{B_{k, \varepsilon_0} \prod_{p | N} (1 - \delta_p \text{Id})}{N^k}$$

as $\varepsilon_0(-1) = (-1)^k$. Besides, since $N \nmid v$ and $u\delta - v\beta = 1$, we have $\varepsilon_0(u) = \varepsilon_0(\delta)$. Using the Prop. 1.1 of §1.1, we therefore find that in this case the constant term $\Upsilon$ of $(\alpha_M G)|_{k, \gamma}$ is non-zero and given by

$$\Upsilon = -\frac{2C_k W(\varepsilon_0) \varepsilon_0(M') \varepsilon_0(\delta) B_{k, \varepsilon_0}}{M^k 2k}.$$
Proof. The Eisenstein series $E_k$ is a normalized Hecke eigenform at level 1 with eigenvalue $1 + p^{k-1}$ for each prime number $p$. Let $p$ be a prime not dividing $N$. Then $T_p$ commutes with $\alpha_N$ and we get $T_pE = (1 + p^{k-1})E_k$. Let now $p$ be a prime dividing $N$. We have

$U_p(U_p - \delta_p\operatorname{Id})\alpha_N E_k = (U_p^2 - \delta_p U_p)\alpha_N E_k$

$= U_p\alpha_{N/p} E_k - \delta_p\alpha_{N/p}\alpha_N E_k$

$= (1 + p^{k-1})\alpha_{N/p} E_k - p^{k-1}\alpha_N E_k - \delta_p\alpha_{N/p} E_k$

$= \frac{p^{k-1}}{\delta_p} \alpha_{N/p} E_k - p^{k-1}\alpha_N E_k$

$= \frac{p^{k-1}}{\delta_p} (\alpha_{N/p} E_k - \delta_p\alpha_N E_k)$

$= \frac{p^{k-1}}{\delta_p} (U_p - \delta_p\operatorname{Id})\alpha_N E_k$.

Hence the result in this case as well.

In expanded form, we have

$E = \sum_{M \mid N} (-1)^{|M|} \delta_M \alpha_M E_k$,

where $|M|$ is the number of prime divisors of $M$ and

$\delta_M = \prod_{p \mid M} \delta_p$.

Let $s$ be a cusp of $X_0(N)$ which is $\Gamma_0(N)$-equivalent to $1/v$ where $v \mid N$. Let $M$ be a divisor of $N$. Then, according to Proposition 1.2 the constant term of the Fourier expansion of $\alpha_M E_k$ at the cusp $s$ is

$-\frac{B_k}{2} \left( \frac{\gcd(v,M)}{M} \right)^k$.

Therefore the constant term of the Fourier expansion of $E$ at the cusp $1/v$ is

$-\frac{B_k}{2} \sum_{M \mid N} (-1)^{|M|} \delta_M \left( \frac{\gcd(v,M)}{M} \right)^k = -\frac{B_k}{2} \prod_{p \mid N} \left( 1 - \delta_p \left( \frac{\gcd(p,v)}{p} \right)^k \right)$

as claimed. \(\square\)

2. Modularity of odd reducible semisimple representations

In this section we prove Thm. 0.1 of the Introduction, thus establishing the modularity of every odd reducible semisimple Galois representation. Let

$\rho : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_2(\mathbb{F}_l)$

be such a representation and fix a place $v$ of $\overline{\mathbb{Q}}$ over $l$.

Assume first that $\rho \simeq 1 \oplus \varepsilon \chi_l^b$ where $\varepsilon$ is unramified at $l$ and $0 \leq b \leq l-2$. The oddness condition here means $\varepsilon(-1) = (-1)^{b+1}$. We now define two integers $N \geq 1$ and $k \geq 2$ and a character $\varepsilon_0$. 

Let us denote by $N$ the Artin conductor of $\varepsilon$. It is coprime to $l$ by assumption. Moreover if we denote by $\varepsilon_0$ the Teichmüller lift (with respect to $v$) of $\varepsilon$ we may identify it with a primitive Dirichlet character of conductor $N$. It satisfies $\varepsilon_0(-1) = (-1)^{k+1}$ unless $l = 2$, in which case we have $\varepsilon_0(-1) = 1$.

We define a ‘weight’ $k$ attached to $1 \oplus \varepsilon \chi^b_l$ as follows:

$$k = \begin{cases} 
4 & \text{if } l = 2 \text{ and } b = 0 \\
l & \text{if } b = 0 \text{ and } l \geq 3 \\
l + 1 & \text{if } b = 1 \\
b + 1 & \text{if } b \geq 2.
\end{cases}$$

Note that $\varepsilon_0(-1) = (-1)^k$ and $k - 1 \equiv b \pmod{l - 1}$. Hence $\rho \simeq 1 \oplus \varepsilon \chi^{k-1}_l$.

**Remark 2.1.** Serre’s recipe (see [Ser87, Eq. (2.3.2)]) for the weight of such a representation is (with Edixhoven’s notation, [Edi92])

$$k_p = \begin{cases} 
l & \text{if } b = 0 \\
b + 1 & \text{if } b \geq 1
\end{cases}$$

while Edixhoven’s definition gives $k(\rho) = b + 1$ (loc. cit.). Our definition is motivated by the fact that we want to avoid working with Eisenstein series of weight 1 or 2 in the proof of Thm. 2.2 below.

Let $\lambda$ be the prime ideal induced by our fixed place $v$ in the ring of integers of the number field generated by the values of $\varepsilon_0$. We can now state a precise version of a special case of Thm. 0.1.

**Theorem 2.2.** Let $(N, k, \varepsilon_0)$ as above. Then, the representation $1 \oplus \varepsilon \chi^b_l$ arises from an eigenform in $S_k(\Gamma_1(Np), \varepsilon_0)$ for every prime number $p \nmid Nl$ such that $\lambda$ divides the non-zero algebraic number $B_{k, \varepsilon_0}/2k(\varepsilon_0(p)p^k - 1)$.

**Proof.** Let $p$ be a prime number not dividing $Nl$. We consider

$$E = E_k^{1, \varepsilon_0} - \alpha_p E_k^{1, \varepsilon_0}$$

where $E_k^{1, \varepsilon_0}$ is the Eisenstein series defined in Eq. (1.2) and compute the constant term, say $a_{\gamma}$, of the $q$-expansion of $E|_{k, \gamma}$ for any $\gamma = \begin{pmatrix} u & \beta \\ v & \delta \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$. By construction we have $\varepsilon_0(-1) = (-1)^k$ and thus Prop. 1.2 gives:

$$a_{\gamma} = \begin{cases} 
0 & \text{if either } N \nmid v \text{ or } Np \nmid v; \\
-\varepsilon_0(\delta) \frac{B_{k, \varepsilon_0}}{2k} \left(1 - \frac{\varepsilon_0(p)}{p^k}\right) & \text{if } N \mid v \text{ and } p \nmid v.
\end{cases}$$

Therefore, under the assumption that $\lambda$ divides $B_{k, \varepsilon_0}/2k(\varepsilon_0(p)p^k - 1)$, the reduction $F$ of $E$ modulo $\lambda$ is a cuspidal eigenform with coefficients in $\mathbb{F}_l$ and eigenvalues $1 + \varepsilon(q)q^{k-1}$ for every prime $q \nmid Np$. According to [DS74, Lem. 6.11] we can find a form $f \in S_k(\Gamma_1(Np), \varepsilon_0)$ which is an eigenform for the Hecke operators $\{T_q\}_{q \nmid Np}$ with corresponding eigenvalues $\{a_q\}_{q \nmid Np}$ satisfying (for $q \neq l$)

$$a_q \equiv 1 + \varepsilon(q)q^{k-1} \pmod{\mathcal{L}}$$

for some prime ideal $\mathcal{L}$ over $\lambda$. Hence the representation $1 \oplus \varepsilon \chi^b_l = 1 \oplus \varepsilon \chi^{k-1}_l$ arises from an eigenform in $S_k(\Gamma_1(Np), \varepsilon_0)$ as claimed. $\square$
**Proof of Theorem 0.1:** let $\rho$ be an odd representation which is the direct sum of two characters, say $\rho = \nu \oplus \nu'$. We thus have $(\nu \nu')(-1) = -1$. Put $\mu = \nu^{-1} \nu'$ and write $\mu = \varepsilon \chi_b^l$ where $\varepsilon$ is unramified at $l$ and $0 \leq b \leq l - 2$. It satisfies $\mu(-1) = -1$ (or, equivalently $\varepsilon(-1) = (-1)^{b+1}$). Let $N$, $k$ and $\varepsilon_0$ be the invariants as defined above attached to the representation $1 \oplus \mu = 1 \oplus \varepsilon \chi_b^l$. By the previous theorem the representation $1 \oplus \mu$ arises from a Hecke eigenform $f$ in $S_k(\Gamma_1(Np), \varepsilon_0)$ for infinitely many primes $p \nmid NL$.

Let us now consider some characteristic 0 lift $\nu_0$ of $\varepsilon$ with finite image. It may be identified with some primitive Dirichlet character of modulus $r$, say. Put $g = f \otimes \nu_0$. By a result of Shimura [Shi94, Prop. 3.64], we have that $g$ belongs to $S_k(\Gamma_1(\text{lcm}(Np,r^2), rN))$, $\varepsilon_0 \nu_0^2)$. Moreover $g$ is an eigenform for the Hecke operators outside $Npr$ and its attached mod $l$ Galois representation is isomorphic to $\rho_f \otimes \nu \simeq \rho$. We thus have proved Thm. 0.1.

### 3. On the different types attached to a modular mod $l$ Galois representation

#### 3.1. Quick review of the irreducible case

Let $\rho : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_2(\mathbb{F}_{l^i})$ be an irreducible odd mod $l$ Galois representation. Assume for simplicity $l > 3$ and $k_\rho = k(\rho) \geq 2$. Here, $k_\rho$ and $k(\rho)$ denote the weights attached to $\rho$ by Serre and Edixhoven, respectively (cf. Remark 2.1). Assume moreover that $\rho$ arises from a Hecke eigenform of weight $k$, level $N$ and character $\varepsilon$ with $N$ coprime to $l$ (such an assumption is now known to hold thanks to Khare-Wintenberger’s proof of Serre’s modularity conjecture). In this subsection we briefly recall some known results about possible weight, level and character of a modular form giving rise to $\rho$.

Under these assumptions, Edixhoven has proved ([Edi92, Thm. 4.5]) that $\rho$ arises from a Hecke eigenform of weight $k_\rho$, level $N$ and character $\varepsilon$ and does not arise from any form of weight $< k_\rho$ and prime-to-$l$ level.

On the other hand, a theorem of Carayol ([Car89, Prop. 3]) ensures that $\rho$ arises from a Hecke eigenform of weight $k$, level $N$ and character $\varepsilon'$ for any character $\varepsilon'$ having the same reduction as $\varepsilon$.

Finally, regarding the level, Carayol ([Car89]) and Livné ([Liv89]) independently proved that Serre’s conductor $N(\rho)$ is minimal (for divisibility) among all possible prime-to-$l$ levels of modular forms giving rise to $\rho$. Ribet’s famous level-lowering theorem ([Rib90, Thm. 1.1]) shows that $\rho$ indeed arises from a Hecke eigenform of this ‘optimal’ level. In the other direction, Diamond and Taylor soon thereafter ([DT94]) completely described the set of ‘non-optimal’ levels attached to $\rho$, that is, prime-to-$l$ integers $M$ such that $\rho$ arises from a weight-$k$ newform of level $M$.

#### 3.2. Reducible modular Galois representations

Let $f$ be a newform of weight $k \geq 2$, squarefree level $N \geq 1$ and trivial Nebentypus. For a prime $l$, we denote by $\rho_f$ its attached mod $l$ Galois representation. We claim that if $l \geq k - 1$ and $\rho_f$ is reducible, then it is isomorphic to $1 \oplus \chi_l^{k-1}$. Indeed, $\rho_f$ is the direct sum of two characters $\nu_1$ and $\nu_2$, each of them we can decompose as a product of a character $\varepsilon_i$ ($i = 1, 2$), which is unramified at $l$, and some power of the cyclotomic character. The squarefreeness assumption implies that the characters $\varepsilon_i$ are also unramified at primes dividing $N$ and hence
trivial. Finally the local description of the representation $\rho_f$ at $l$ ([Edi92, Thm. 2.5-2.6]) shows that under the assumption $l \geq k - 1$, the exponents of the cyclotomic characters are respectively 0 and $k - 1$.

3.3. **Weight optimization for $1 \oplus \chi^b_l$.** In view of the previous discussion, we now focus our attention on the representation $\rho = 1 \oplus \chi^b_l$ and briefly discuss the weight optimization problem for this representation by analogy with Edixhoven’s result on the irreducible case mentioned in §3.1. Let us assume as before that $l > 3$ and that $k_{\rho} = k(\rho) \geq 2$. In particular, $b$ is odd (by the oddness condition) and we have

$$k = \begin{cases} 
    l + 1 & \text{if } b = 1 \\
    b + 1 & \text{if } b \geq 3
\end{cases}$$

Therefore, if $b \geq 3$, Theorem 2.2 shows that $1 \oplus \chi^b_l$ arises from an eigenform of prime-to-$l$ level and ‘optimal’ weight $b + 1 = k(\rho)$.

The case $b = 1$ is slightly more involved. Theorem 2.2 asserts that $1 \oplus \chi_l$ arises from an eigenform in $S_{l+1}(\Gamma_0(p))$, with $p$ prime, if $p \equiv \pm 1 \pmod{l}$. Indeed, we have that

$$\frac{B_{l+1}}{2(l+1)} = \frac{B_2}{4} \equiv \frac{1}{24} \not\equiv 0 \pmod{l}$$

by Kummer’s congruences and $p^{l+1} - 1 \equiv p^2 - 1 \pmod{l}$ by Fermat’s little theorem.

Hence, a natural question is the following: if a prime number $p \equiv \pm 1 \pmod{l}$ mod $l$ is given, is it possible to lower the weight so that $1 \oplus \chi^b_l$ arises from a Hecke eigenform in $S_b(\Gamma_0(pM))$ with $(M,l) = 1$?

This question turns out to have a positive answer. If $p \equiv 1 \pmod{l}$, Mazur’s result mentioned in the Introduction implies that the representation $1 \oplus \chi_l$ arises from a Hecke eigenform in $S_2(\Gamma_0(p))$ (recall that we assume $l > 3$). If $p \equiv -1 \pmod{l}$, a recent result of Ribet (see the notes [Rib10] on his homepage) implies that $1 \oplus \chi_l$ arises from a Hecke eigenform in $S_2(\Gamma_0(pq))$ for any prime number $q \neq p,l$. While it helps for the weight optimization problem, Ribet’s work is actually primarily concerned with the level-raising problem which we study more carefully in the next section.

In a different direction, a theorem of Ash and Stevens ([AS86, Thm. 3.5]) allows us to take the weight equal to 2 by increasing the level by $l$.

4. **Non-optimal levels of $1 \oplus \chi^k_l$**

In this section we focus our attention on the level-raising problem for odd representations of the form $1 \oplus \chi^b_l$ with $0 \leq b \leq l - 2$. Contrary to what happens in the irreducible case (as recalled in §§3.1 and 3.3), $1 \oplus \chi^b_l$ does not always arise in optimal weight $b + 1$ from a Hecke eigenform of level one (its optimal level). Under the common assumption $l > b + 2$, we will describe (Cor. 4.4) how exactly it happens.

Following a terminology introduced by Diamond and Taylor ([DT94]) we shall call non-optimal level for $1 \oplus \chi^b_l$ any integer $N$ such that $1 \oplus \chi^b_l$ arises from a newform in $S_{b+1}(\Gamma_0(N))$.

The first result about non-optimal levels for this representation occurs in the case $b = 1$ and is due to Mazur. Namely he proved in [Maz77, Prop. 5.12]...
that $1 \oplus \chi_l$ arises from a weight-2 newform in $S_2(\Gamma_0(N))$ with $N$ prime if and only if $l$ divides the numerator of $(N - 1)/12$. Starting from this result, Ribet considers the problem of determining the set of squarefree non-optimal (signed) levels for $1 \oplus \chi_l$. He proves several results under the simplifying assumption $l > 3(= b + 2)$, and notably gives a complete answer in the case of composite levels that are products of two distinct primes coprime to $l$. His results are presented in a series of talks he gave between 2008 and 2010 (see the slides on his homepage, especially [Rib10]) and were very recently extended by his student Hwajong Yoo in his Ph.D. thesis (Spring 2013).

In this section we consider this problem for $b \geq 3$ (recall that $b$ is odd). We assume as Ribet $l > b + 2$ and write $b = k - 1$ so that $k$ is the (optimal) weight attached to $1 \oplus \chi_l^k$ as defined in Sec. 2. In this context we propose a conjecture (conjecture 0.3 of the Introduction) about the set of non-optimal squarefree levels. We can prove that our conjecture is actually equivalent to saying that the classical (necessary) level-raising condition at a Steinberg prime is sufficient (see §4.3). In §4.2 we combine Theorems 4.1 and 4.3 with a result of Diamond ([Dia91]) to prove the prime-level case of the conjecture. Using magma ([BCP97]), we have computationally checked the validity of the conjecture for fixed weights and levels in various ranges. In particular it holds true for $k = 4$ and $N < 5000$ and for $6 \leq k < 32$ and $N < 50$.

4.1. Necessary conditions. In this paragraph we prove the following statement which corresponds to the direct implication in conjecture 0.3.

**Theorem 4.1.** Let $k$ be an even integer and $N$ be a non-negative squarefree integer. Assume $k \geq 4$, $l > k + 1$ and $l \nmid N$. If the representation $1 \oplus \chi_l^{k-1}$ arises from a weight-$k$ newform of level $N$ and trivial Nebentypus, then at least one of the following assertions holds:

1. we have $(p^k - 1)(p^{k-2} - 1) \equiv 0 \pmod{l}$ for every prime number $p$ dividing $N$ and there exists a prime divisor $p_0$ of $N$ such that $p_0 \equiv 1 \pmod{l}$;
2. we have $p^{k-2} \equiv 1 \pmod{l}$ for every prime number $p$ dividing $N$ and $l$ divides the numerator of $B_k/k$.

The proof splits into two steps. We first deduce some weaker conditions from the local description of modular representations at Steinberg primes. In a slightly different form, this was already done in a joint paper by Dieulefait and the first author ([BD12]), but we briefly repeat the argument here for the sake of conciseness. We then strengthen these conditions using some (new) computations about Eisenstein series from §1.2.2 to obtain Theorem 4.1.

Let $k, l$ and $N$ as in the theorem. Assume that $1 \oplus \chi_l^{k-1}$ arises from some newform $f$ of weight $k$ and level $\Gamma_0(N)$. Let $p$ be a prime dividing $N$. By a theorem of Langlands, the restriction of $\rho_f$ to a decomposition group $D_p$ at $p$ is $\mu \chi_l^{k/2} \oplus \mu \chi_l^{k/2-1}$ where $\mu$ is the (at most) quadratic unramified character that maps a Frobenius at $p$ on $a_p(f)/p^{k/2-1}$. Therefore we have the following equality between sets of characters of $D_p$:

$$\{1, \chi_l^{k-1}\} = \{\mu \chi_l^{k/2}, \mu \chi_l^{k/2-1}\}.$$  (1)

Assume that $1 = \mu \chi_l^{k/2}$. Then, in particular $p^k \equiv 1 \pmod{l}$.
(2) Assume that \(1 = \mu \chi_{k/2}^{1} - 1\). Then, \(a_p(f) \equiv 1 \pmod{\lambda}\) and in particular, \(p^k - 2 \equiv 1 \pmod{l}\).

Let us now assume that the first assertion of the theorem is not satisfied. According to the above discussion we have

\[ p^k - 2 \equiv 1 \pmod{l}; \quad p^k \not\equiv 1 \pmod{l} \quad \text{and} \quad a_p(f) \equiv 1 \pmod{\lambda} \]

for every prime \(p\) dividing \(N\) and we must show that \(l\) divides the numerator of \(B_k/k\). This will be achieved using a careful study of the constant term of the Fourier expansion at various cusps of a specific Eisenstein series which we now introduce.

Let us consider the Eisenstein series \(E\) as in §1.2.2 with parameters \(\delta_p = p^{k-1}\) for every \(p \mid N\). Then by assumption and proposition 1.3, we have:

\[ a_p(f) \equiv a_p(E) \pmod{\lambda}, \quad \text{for all primes } p \neq l. \]

Besides, both \(E\) and \(f\) are normalized Hecke eigenforms. Therefore, we get

\[ a_n(f) \equiv a_n(E) \pmod{\lambda}, \quad \text{for all prime-to-} l \text{ integers } n. \]

Applying the \(\Theta\) operator (see [Ser73]) to the reductions mod. \(l\) \(\overline{f}\) and \(\overline{E}\) of \(f\) and \(E\) respectively then yields to an equality \(\Theta(\overline{f}) = \Theta(\overline{E})\). But this \(\theta\) operator is injective under the assumption \(l > k + 1\) (loc. cit. cor. 3 p. 326) and thus \(\overline{E}\) is cuspidal. In particular, \(l\) divides the numerator of the constant term of \(E\) at the cusp \(\infty\). By proposition 1.3, this means that \(l\) divides the numerator of \(\frac{B_k}{2k} \prod_{p \mid N}(1 - p^{k-1})\). Since \(p^{k-1} \equiv 1 \pmod{l}\) would imply \(p^k \equiv 1 \pmod{l}\) (as \(p^{k-2} \equiv 1 \pmod{l}\)), contrary to the hypotheses, we get the desired result. This ends the proof of Theorem 4.1.

**Remark 4.2.** According to Proposition 1.3, the vanishing modulo \(l\) of the constant terms of \(E\) at the other cusps of \(X_0(N)\) does not give additional information.

4.2. **Weaker converse statement and the prime level case.** In what follows we present a weaker statement in the direction of the reverse implication of Conjecture 0.3. We will finish this paragraph with a proof of Theorem 0.2.

**Theorem 4.3.** Let \(N\) be a positive squarefree integer. Let \(k\) be an even integer \(\geq 4\) and assume \(l > k + 1\). Assume that at least one of the following conditions holds:

1. we have \((p^k - 1)(p^{k-2} - 1) \equiv 0 \pmod{l}\) for every prime number \(p\) dividing \(N\) and there exists a prime divisor \(p_0\) of \(N\) such that \(p_0^k \equiv 1 \pmod{l}\);
2. we have \(p^{k-2} \equiv 1 \pmod{l}\) for every prime number \(p\) dividing \(N\) and \(l\) divides the numerator of \(B_k/k\).

Then the representation \(1 \oplus \chi_{l}^{k-1}\) arises from a weight-\(k\) eigenform of level \(N\) and trivial Nebentypus.
Proof. Assume that either condition of the theorem is satisfied and let us consider the Eisenstein series $E$ of §1.2.2 with the following choice of parameters:

$$
\delta_p = \begin{cases} 
1 & \text{if } p^k \equiv 1 \pmod{l} \\
 p^{k-1} & \text{otherwise}
\end{cases}
$$

Recall that in expanded form we have

$$
E = \sum_{M|N} (-1)^{|M|} \delta_M \alpha_M E_k,
$$

where $|M|$ is the number of prime divisors of $M$ and

$$
\delta_M = \prod_{p|M} \delta_p.
$$

According to the assumptions ($l > k+1$, $l \nmid N$) and the Van Staudt-Clausen theorem, the series $E$ has $l$-integral rational Fourier coefficients at $\infty$. Let us denote by $F$ its reduction modulo $l$. It is a well-defined modular form over $\mathbb{F}_l$.

We now prove that $F$ is actually cuspidal. By proposition 1.3, this is clear under the second assumption (as in particular, $l$ divides the numerator of $B_k/2k$). Else, if it is not satisfied, then there exists a prime divisor $p_0$ of $N$ such that $\delta_{p_0} = 1$. Let $s$ be a cusp of $X_0(N)$. It is $\Gamma_0(N)$-equivalent to some cusp of the form $1/v$ with $v | N$ and $1 \leq v \leq N$. Then

$$
1 - \delta_{p_0} \left( \frac{\gcd(p_0, v)}{p_0} \right)^k = \begin{cases} 
0 & \text{if } \gcd(p_0, v) = p_0 \\
1 - p_0^{-k} & \text{otherwise}
\end{cases}
$$

is congruent to 0 modulo $l$. Hence the result by proposition 1.3.

As it is already the case for $E$, the cuspidal form $F$ is a Hecke eigenform at level $N$. Therefore according to the Deligne-Serre lifting lemma, there exist a finite extension $K/\mathbb{Q}_l$ with ring of integers $\mathcal{O}$ and uniformizer $\mathcal{L}$ and a normalized Hecke eigenform $f \in S_k(\Gamma_0(N); \mathcal{O})$ with system of eigenvalues $\{c_p\}_p$ where $p$ runs over the primes, such that

$$
c_p \equiv 1 + p^{k-1} \pmod{\mathcal{L}} \text{ if } p \nmid N \text{ and } c_p \equiv p^{k-1}/\delta_p \pmod{\mathcal{L}} \text{ otherwise.}
$$

Moreover $f$ is a classical modular form (as its Fourier coefficients are roots of the characteristic polynomials of the Hecke operators).

By a direct combination of Theorems 4.1 and 4.3 we get the level 1 case of Conjecture 0.3:

Corollary 4.4. Let $k$ be an even integer $\geq 4$ and $l$ be a prime $> k+1$. Then the representation $1 \oplus \chi_l^{k-1}$ arises from a weight-$k$ eigenform of level 1 if and only if $l$ divides the numerator of $B_k/k$.

Remark 4.5. This result was implicit in the literature (see for instance [Rib75, Lem. 5.2] and [Gha02, Prop. 1] for the ‘only if’ and ‘if’ directions, respectively).

Proof of Theorem 0.2: the direct implication is a particular case of Theorem 4.1.

Now we prove the reverse implication. By Theorem 4.3, we have that $1 \oplus \chi_l^{k-1}$ arises from an eigenform $f_0 \in S_k(\Gamma_0(N))$. If $f_0$ is a newform, then
we are done. Hence, in what follows we will assume that $f_0$ is an oldform. We denote by $f$ its associated (normalized) level 1 eigenform. By a standard application of the Chebotarev density theorem, the mod $l$ representations $\rho_f$ and $\rho_{f_0}$ are isomorphic.

Let $K$ be the number field spanned by the Fourier coefficients of $f$. Since $\rho_{f_0}$ is isomorphic to $1 \oplus \chi_l^{k-1}$ and $l \nmid N$, we have that there is an integral prime ideal $\lambda \subset O_K$ above $l$ such that $a_N(f) \equiv 1 + N^{k-1} \mod \lambda$. We claim that

$$a_N(f)^2 \equiv N^{k-2}(1 + N)^2 \mod \lambda. \hspace{1cm} (4.4)$$

Indeed,

$$a_N(f) \equiv 1 + N^{k-1} \equiv \begin{cases} 1 + N^{-1} \mod \lambda & \text{if } N^k \equiv 1 \mod l \\ 1 + N \mod \lambda & \text{if } N^{k-2} \equiv 1 \mod l. \end{cases}$$

Since

$$N^{k-2}(1 + N)^2 \equiv \begin{cases} (1 + N^{-1})^2 \mod l & \text{if } N^k \equiv 1 \mod l \\ (1 + N)^2 \mod l & \text{if } N^{k-2} \equiv 1 \mod l \end{cases}$$

this proves the claim.

Relation (4.4) allows us to use a theorem of Diamond ([Dia91, Thm.1]²), to ensure that there exists a normalized newform $f_1 \in S_k(\Gamma_0(N))_{\text{new}}$ with eigenvalues in a finite extension $K'/K$ and an ideal $\lambda' \subset O_{K'}$ above $\lambda$ such that $a_p(f) \equiv a_p(f_1) \mod \lambda'$ for all primes $p \nmid Nl$.

Then, $\rho_{f_1}$ is isomorphic to $1 \oplus \chi_l^{k-1}$, concluding the proof.

4.3. Relationship with the level-raising condition. Let

$$\rho : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\mathbb{F}_l)$$

be an odd semisimple mod $l$ Galois representation of weight $k \geq 2$ and Serre’s level $N(\rho)$. We shall say that $\rho$ satisfies the level-raising condition at a prime $p \nmid Nl$ if

$$(\text{tr}\rho(\text{Frob}_p))^2 = p^{k-2}(1 + p)^2 \mod \mathbb{F}_l,$$

where $\text{Frob}_p$ denotes a Frobenius element at $p$ in $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Such a condition is satisfied if the representation $\rho$ arises from a newform in $S_k(\Gamma_0(Np))$ with $(N,p) = 1$. In particular, it is a necessary condition to raise the level of modular representation from $N$ to $Np$. In their paper [DT94], Diamond and Taylor prove that this is also sufficient when $\rho$ is assumed to be irreducible.

In the special case of the representation $1 \oplus \chi_l^{k-1}$, with even $k \geq 2$, the level-raising condition at a prime $p \neq l$ is merely

$$p^k - 1)(p^{k-2} - 1) \equiv 0 \mod l. \hspace{1cm} (4.5)$$

If $k = 2$, the congruence (4.5) is automatically fulfilled for every $p$, even though there are primes $p$ such that $1 \oplus \chi_l$ is not modular of level $p$. However, we believe that the case $k \geq 4$ is different and by analogy with the irreducible case, we propose the following conjecture:

²Note that Diamond’s $(N,l,p)$ in loc. cit. is $(1,N,l)$ in our notation.
Conjecture 4.6. Let $k \geq 4$ be an even integer and $l$ be a prime $> k + 1$. Assume that $1 \oplus \chi_{l}^{k-1}$ arises from a newform in $S_k(\Gamma_0(N))$ with $N$ squarefree and coprime to $l$ and that $p \nmid Nl$ is a prime number at which $1 \oplus \chi_{l}^{k-1}$ satisfies the level raising condition, namely $(p^k - 1)(p^k - 2 - 1) \equiv 0 \pmod{l}$. Then, the representation $1 \oplus \chi_{l}^{k-1}$ arises from a newform in $S_k(\Gamma_0(Np))$.

Using theorems 4.1 and 4.3 above we now prove the following result.

Proposition 4.7. Conjectures 0.3 and 4.6 are equivalent.

Proof. Assume Conjecture 0.3 and the hypothesis of Conjecture 4.6. Then Theorem 4.1 ensures that the squarefree integer $Np$ satisfies the hypothesis of Conjecture 0.3, thus proving Conjecture 4.6.

Assume conversely Conjecture 4.6 and let us show that Conjecture 0.3 holds. The direct implication therein corresponds to Theorem 4.1. Let us now prove the reverse implication. Let $N$ be a prime-to-$l$ squarefree integer that satisfies at least one of the conditions in the statement of Conj. 0.3. According to Theorem 4.3, $1 \oplus \chi_{l}^{k-1}$ arises from a newform in $S_k(\Gamma_0(M))$ for some integer $M \mid N$. If $M \neq N$, we want to show that we can now raise the level from $M$ to $N$. Let $p$ be a prime dividing $N/M$. Then one clearly has $(p^k - 1)(p^{k-2} - 1) \equiv 0 \pmod{l}$ and Conjecture 4.6 shows that $1 \oplus \chi_{l}^{k-1}$ arises from a newform in $S_k(\Gamma_0(pM))$. If $pM = N$, we are done. Otherwise we can repeat this process until we reach $N$. Hence, we have proved Conj. 0.3. □

Remark 4.8. As for irreducible representations, we deduce from Conj. 0.3 that if $1 \oplus \chi_{l}^{k-1}$ arises from newforms in $S_k(\Gamma_0(M))$ and in $S_k(\Gamma_0(N))$ for squarefree integers $M \mid N$, then it also arises from a newform in $S_k(\Gamma_0(N'))$ for any intermediate level $M \mid N' \mid N$. As noticed by Ribet, such a result is false for $k = 2$. The representation $1 \oplus \chi_5$ arises for instance in levels 11 and 66 but neither in level 22 nor in level 33.

5. LOWER BOUND FOR THE HIGHEST DEGREE OF THE COEFFICIENT FIELD OF NEWFORMS

In this section we prove Theorem 0.4. For a nonzero integer $m$, let $P^+(m)$ be the largest prime factor of $m$. Let $\mathcal{P}$ be the set of prime numbers $N$ such that $P^+(N - 1) > N^{1/2}$. That is, for every $N \in \mathcal{P}$, there exists a prime $l$ with

$$N \equiv 1 \pmod{l} \text{ and } l > N^{1/2}. \quad (5.6)$$

Lemma 5.1. The set $\mathcal{P}$ has natural lower density at least $1/2$.

Proof. Let

$$N(x) = |\{p \text{ prime such that } p \leq x \text{ and } P^+(p - 1) > \sqrt{x}\}|.$$

A theorem of Goldfeld ([Gold69], under Theorem 1) ensures that

$$N(x) \geq \frac{x}{2 \log x} + O\left(\frac{x \log \log x}{(\log x)^2}\right), \quad \text{as } x \to \infty. \quad (5.7)$$

Let $S(x) = |\{N \in \mathcal{P} : N \leq x\}|$. Since $S(x) \geq N(x)$, using (5.7) we obtain

$$\liminf_{x \to \infty} \frac{S(x)}{(x/\log x)} \geq \frac{1}{2}.$$
Proof of Theorem 0.4: take $N \in \mathcal{P}$ and a prime $l$ as in (5.6). Assume $N \geq (k+1)^2$. Then $N^k \equiv 1 \mod l$ and $l > k+1$. Hence, Theorem 0.2 (resp. Mazur’s Theorem [Maz77, Prop. (5.12)]) ensures that $1 \oplus \chi^{k-1}$ arises from a (normalized) newform $f$ of trivial Nebentypus, level $N$ and weight $k$ if $k \geq 4$ (resp. $k = 2$). Put $K = K_f$ and $d = [K_f : \mathbb{Q}]$. Take a prime ideal $\lambda | l$ such that

$$a_p \equiv 1 + p^{k-1} \mod \lambda, \quad \text{for all primes } p \nmid Nl, \quad a_p := a_p(f).$$

Deligne’s bound implies that, for every archimedean place $\tau$ of $K$, we have

$$|\tau(a_2)| \leq 2 \cdot 2^{(k-1)/2}.$$  

Since $l | N_{K/Q}(b)$, we conclude that $l \leq (1 + 2^{(k-1)/2})^{2d}$, implying

$$d_{\text{new}}^k(N) \geq d \geq \frac{\log l}{2 \log(1 + 2^{(k-1)/2})} \geq \frac{\log N}{4 \log(1 + 2^{(k-1)/2})}.$$  

This ends the proof of Thm. 0.4.

Remarks 5.2.

(1) The basic idea of using $a_2$ comes from the proof of a weaker statement in weight 2 by Dieulefait, Jimenez Urroz and Ribet ([Djur11, §2]). We are able to obtain a more general (resp. stronger) result because of our Thm. 0.2, that generalizes Mazur’s theorem to higher weight (resp. the information on primes $p$ with large prime factors of $p-1$ given by Goldfeld’s theorem).

(2) É. Fouvry mentioned to us that the correct (conjectural) density of the set $\mathcal{P}$ should be in $= 0.693...$

(3) It is conjectured that for any $\varepsilon > 0$, the set prime numbers $p$ such that $P^+(p-1) \geq p^{1-\varepsilon}$ has a positive lower density $\kappa(\varepsilon) > 0$. Thus, Goldfeld’s result establishes $\kappa(1/2) \geq 1/2$. For the purposes of Theorem 0.4, progress in this difficult problem would improve on the value of the constant $c_k$. However, such improvements would not change the fact that our method produces a $c_k$ tending to zero with $k$. Also, it is expected that the value of $\kappa(\varepsilon)$ decreases as $\varepsilon$ tends to zero, thus shrinking the set of primes for which our bound is valid. This is why we use Goldfeld’s result even though other authors have established the conjecture for smaller values of $\varepsilon$ (cf. [Fou85], [BH96]).

On the other hand, any value of $\varepsilon$ bigger than $1/2$ for which one could prove $\kappa(\varepsilon) > 1/2$ would enlarge the set of primes for which our bound is valid (at the expense of a small loss in the constant $c_k$), thus improving Theorem 0.4 in an interesting way.

(4) If we assume Conjecture 0.3, it is possible to show an analogous lower bound for $d_{\text{new}}^k(N)$ for $N$ in an appropriate family of squarefree
integers. Let $r$ be a non-negative integer. Put

$$N_r = \left\{ N \in \mathbb{N} : N = p_1 p_2 \cdots p_r, \omega(N) = r, P^+ \left( \gcd_{1 \leq i \leq r} (p_i - 1) \right) > N^\frac{1}{2r} \right\},$$

where $\omega(m)$ is number of different prime factors of the integer $m$ and $p_1, \ldots, p_r$ denote primes. It is shown in [LPM] that, as $x \to \infty$, we have:

$$x^{\frac{1}{r} + \frac{1}{2r}} \ll_r \left\{ \{ N \in N_r : N \leq x \} \right\} \ll_r \frac{x^{\frac{1}{r} + \frac{1}{2r}} (\log \log x)^{r-1}}{(\log x)^2}.$$  

These estimates show that the set $N_r$ is infinite and that, if $r \geq 2$, this set has density zero when regarded as a subset of squarefree numbers with exactly $r$ prime divisors.

Mimicking the argument given above when $N$ is prime, it is easy to show the following

**Theorem 5.3.** Assume Conjecture 0.3. Then, for every integer $r \geq 2$, we have that

$$d_{k, new}^N(N) \gg_k \frac{1}{r} \log N, \quad N \to \infty, \quad N \in N_r.$$  

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**Laboratoire de Mathématiques, Université Blaise Pascal Clermont-Ferrand 2, Campus universitaire des Cézeaux, 63177 Aubière Cedex, France**

*E-mail address: Nicolas.Billerey@math.univ-bpclermont.fr*

**Instituto de Matemáticas, Pontificia Universidad Católica de Valparaíso, Blanco Viel 596, Cerro Barón, Valparaíso, Chile**

*E-mail address: ricardo.menares@ucv.cl*