Chapter 1
Interval Linear Algebra and Computational Complexity

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Abstract This work connects two mathematical fields – computational complexity and interval linear algebra. It introduces the basic topics of interval linear algebra – regularity and singularity, full column rank, solving a linear system, deciding solvability of a linear system, computing inverse matrix, eigenvalues, checking positive (semi)definiteness or stability. We discuss these problems and relations between them from the view of computational complexity. Many problems in interval linear algebra are intractable, hence we emphasize subclasses of these problems that are easily solvable or decidable. The aim of this work is to provide a basic insight into this field and to provide materials for further reading and research.

1.1 Introduction

The purpose of this work is to emphasize relations between the two mathematical fields - interval linear algebra and computational complexity. This is not a pioneer work. Variety of relations between interval problems and computational complexity is covered by many papers. There are also few monographs that are devoted to this topic [4, 22, 46]. Some questions may arise in mind while reading the previous works. Among all, it is the question about the equivalence of the notions NP-hardness and co-NP-hardness. Some authors use these notions as synonyms. Some
distinguish between them. Another questions that may arise touches the representation and reducibility of interval problems in a given computational model. We would like to shed more light (not only) on these issues.

Many well-known problems of classical linear algebra become intractable when we introduce intervals into matrices and vectors. However, not everything is lost. There are many interesting sub-classes of problems that behave well. We would like to point out these feasible cases, since they are interesting either from the theoretical or the computational point of view.

Our work does not aspire to replace the classical monographs or handbooks. It lacks many of their details that are cited in the text. Nevertheless, it collects even some recent results that are missing in the monographs. It also provides links and reductions between the various areas of interval linear algebra. It provides a necessary and compact introduction to computational complexity and interval linear algebra. Then it considers complexity and feasibility of various well-known linear algebraic tasks when considered with interval structures – regularity and singularity, full column rank, solving a linear system, deciding solvability of a linear system, computing inverse matrix, eigenvalues, checking positive (semi)definiteness or stability.

We hope this paper should help newcomers to this area to improve her/his orientation in the field or professionals to provide a signpost to more deeper literature.

1.2 Interval linear algebra – part I

Interval linear algebra is a mathematical field developed from classical linear algebra. The only difference is, that we do not work with real numbers but with real closed intervals

\[ a = [a, \overline{a}], \]

where \( a \leq \overline{a}. \) The set of all closed real intervals is denoted \( \mathbb{I} \mathbb{R} \) (the set of all closed rational intervals is denoted \( \mathbb{I} \mathbb{Q} \)) We can use intervals for many reasons – in applications we sometimes do not know some parameters precisely, that is why, we rather use intervals of possible values; some real numbers are problematic (e.g., \( \pi, \sqrt{2}, \ldots \)) because it is not easy to represent them precisely, that is why, we can represent them with rigorous intervals containing them etc. With interval we can define arithmetic (there are more possible definitions, we chose one of the most basic ones).

**Definition 1.** Let us have two intervals \( x = [x, \overline{x}] \) and \( y = [y, \overline{y}] \). The arithmetical operations \( +, *, -, / \) are defined as follows

\[
\begin{align*}
  x + y & = [\underline{x} + \underline{y}, \overline{x} + \overline{y}], \\
  x - y & = [\underline{x} - \overline{y}, \overline{x} - \underline{y}], \\
  x \times y & = [\min(S), \max(S)], \text{ where } S = \{ xy, \underline{x}y, \overline{x}y, \overline{x}\underline{y} \}, \\
  x / y & = x \times (1/y), \quad \text{where } 1/y = [1/\overline{y}, 1/\underline{y}], \ 0 \notin y.
\end{align*}
\]
Hence, we can use intervals instead of real numbers in formulas. However, we have to be careful. If there is a multiple occurrence of the same interval in a formula, the interval arithmetic does see them as two different intervals and we get an overestimation in the resulting interval. For example, let us have \( x = [-2, 1] \) and functions \( f_1(x) = x^3 \) and \( f_2(x) = x \times x \). Then we get

\[
\begin{align*}
  f_1(x) &= f_1([-2, 1]) = [-2, 1]^2 = [0, 4], \\
  f_2(x) &= f_2([-2, 1]) = [-2, 1] \times [-2, 1] = [-2, 4].
\end{align*}
\]

In the first case we see the optimal result, in the second case we see overestimation. That is why, the form of our mathematical expression matters. However, we know the cases when the resulting interval is optimal [28].

**Theorem 1.** Applying interval arithmetic on expressions in which all variables occur only once gives the optimal resulting interval.

Using intervals we can build larger structures. In the interval linear algebra the main notion is an interval matrix. It is defined as follows:

\[
A = \{ A \mid A \leq \overline{A} \leq \underline{A} \},
\]

where \( \underline{A}, \overline{A} \) are real \( m \times n \) matrices called lower and upper bound and the relation \( \leq \) is always understood componentwise. In another words, it is a matrix with coefficients formed by real closed intervals. In the following text, we will denote every interval structure in boldface. Since an interval vector is a special case of an interval matrix, we define it similarly. We can see that if all intervals in the structures are degenerate, i.e., \( \underline{A} = \overline{A} \), we get a classical linear algebra. Therefore, interval linear algebra is actually a generalization of the previous one.

Another way to define an interval matrix is using its midpoint matrix \( A_c \) and its radius matrix \( \Delta \geq 0 \) as

\[
A = [A_c - \Delta, A_c + \Delta].
\]

In the following text we automatically suppose that \( A_c, \Delta \) represent corresponding midpoint and radius matrix of \( A \), and \( b_c, \delta \) represent corresponding midpoint and radius vector of \( b \). When we talk about a general square matrix we automatically assume that it is of size \( n \).

We mention some special structures that we will use quite often. The identity matrix is denoted \( I \), the matrix containing only ones \( E \) and the vector containing only ones \( e \). Another useful matrix is \( D_y = \text{diag}(y_1, \ldots, y_n) \) a matrix with the vector \( y \) as the main diagonal. We often need to describe some properties of interval structures vectors consisting of only \( \pm 1 \). We denote the set of all \( n \)-dimensional \( \pm 1 \) vectors as \( Y_n \). A useful concept is a matrix \( A_{yz} \) defined as

\[
A_{yz} = A_c - D_y \Delta D_z,
\]

for some given \( y, z \in Y_n \). Every its coefficient on the positon \( (i, j) \) is an upper or a lower bound of \( A_{ij} \) depending on the sign of \( y_i \cdot z_j \). We will sometimes need to check spectral radius of a real matrix \( A \), we denote it \( \rho(A) \).
Many definitions have an intuitive generalization for interval linear algebra:

An interval matrix \( A \) has a property \( \mathcal{P} \) if every \( A \in A \) has the property \( \mathcal{P} \).

This applies to stability, full column rank, inverse nonnegativity, diagonally dominant matrices, M-matrix and H-matrix property, among others.

Many problems in interval linear algebra are very difficult to be computed exactly (with resulting intervals of tightest possible bounds). That is why we inspect the possibility of approximation of these bounds. There are many types of approximation. There are several kinds of errors when we approximate a number \( a \) – the absolute, relative \([6]\) and inverse relative \([21]\) approximation errors.

**Definition 2.** An algorithm computes \( a \) with absolute approximation error \( \varepsilon \) if it computes \( a^0 \) such that \( a^0 \in [a - \varepsilon, a + \varepsilon] \).

An algorithm computes \( a \) with relative approximation error \( \varepsilon \) if it computes \( a^0 \) such that \( a^0 \in (1 + [-\varepsilon, \varepsilon])a \).

An algorithm computes \( a \) with inverse relative approximation error \( \varepsilon \) if it computes \( a^0 \) such that \( a \in (1 + [-\varepsilon, \varepsilon])a^0 \).

At the end we mention a very useful theorem that we will use very often in this text. It originally comes from the area of numerical mathematics \([23]\).

**Theorem 2 (Oettli-Prager).** Let us have an interval matrix and vector \( A, b \). For a real vector \( x \in \mathbb{R}^n \) it holds \( Ax = b \) for some \( A \in A, b \in b \) if and only if

\[
|Ax - b| \leq \Delta |x| + \delta.
\]

This was just a brief introduction to interval analysis. Interval linear algebra has many important applications – system verification, model checking, handling uncertain data. For a huge variety of applications see, e.g., \([16, 17, 18]\). For more information or applications in nonlinear mathematics see \([25]\).

### 1.3 Complexity theory background

Now, we take a small break and dig deeper into the area of computational complexity. With that in mind we return back to interval linear algebra and introduce some well-known issues from the viewpoint of computational complexity.

#### 1.3.1 Binary encoding and size of an instance

For complexity-theoretic classification of interval-theoretic problems, it is a standard to use the Turing computation model. We assume that an instance of a computational problem is formalized as a bit-string, i.e., a finite 0-1 sequence. Thus we
cannot work with real-valued instances; instead we usually restrict ourselves to rational numbers expressed as fractions \( \pm \frac{q}{r} \) with \( q, r \in \mathbb{N} \) written down in binary in the coprime form. Then, the size of a rational number \( \pm \frac{q}{r} \) is understood as the number of bits necessary to write down the sign and both \( q \) and \( r \) (to be precise, one should also take care of delimiters). If an instance of a problem consists of multiple rational numbers \( A = (a_1, \ldots, a_n) \) (e.g., when the input is a vector or a matrix), we define \( \text{size}(A) = \sum_{i=1}^{n} \text{size}(a_i) \).

In interval-theoretic problems, inputs of algorithms are usually interval numbers, vectors or matrices. When we say that an algorithm is to process an \( m \times n \) interval matrix \( A \), we understand that the algorithm is given the pair \( (A \in \mathbb{Q}^{m \times n}, \overline{A} \in \mathbb{Q}^{m \times n}) \) and that the size of the input is \( L := \text{size}(A) + \text{size}(\overline{A}) \). Whenever we speak about complexity of such algorithm, we mean a function \( \phi(L) \) counting the number of steps of the corresponding Turing machine as a function of the bit-size \( L \) of the input \( (A, \overline{A}) \).

Although the literature focuses mainly on the Turing model (and here we also do so), it is challenging to investigate the behavior of interval-theoretic problems in other computational models, such as the Blum-Shub-Smale (BSS) model for real-valued computing [2] or the quantum model [1].

### 1.3.2 Functional problems and decision problems

Formally, a functional problem \( F \) is a total (defined for each input) function \( F : \{0,1\}^* \to \{0,1\}^* \), where \( \{0,1\}^* \) is the set of all finite bit-strings. A decision problem (or YES/NO problem) \( A \) is a total function \( A : \{0,1\}^* \to \{0,1\} \).

If there exists a Turing machine computing \( A(x) \) for every \( x \in \{0,1\}^* \), we say that the problem \( A \) (either decision or functional) is recursive.

It is well known that many decision problems in mathematics are nonrecursive; e.g., deciding whether a given formula is provable in Zermelo-Fraenkel Set Theory is nonrecursive by the famous Gödel Incompleteness Theorem. Fortunately, a majority of decision problems in interval linear algebra are recursive. Such problems can usually be written down as arithmetic formulas (i.e., quantified formulas containing natural number constants, arithmetical operations plus, times, relations equals, less than or equal to and propositional connectives). Such formulas are decidable (over the reals) by Tarski’s Quantifier Elimination Method [31, 32, 33].

- **Example A:** Regularity of an interval matrix. Each matrix \( A \in A \) is nonsingular iff \( (\forall A)[A \leq A \leq \overline{A} \to \det(A) \neq 0] \). This formula is arithmetical since \( \det(\cdot) \) is a polynomial, and thus it is expressible in terms of plus, times.

- **Example B:** Is a given \( \lambda \in \mathbb{Q} \) the largest eigenvalue of some symmetric \( A \in A \)?

   This question can be written down as \( (\exists A)[A = A^T \land A \leq A \leq \overline{A} \land (\exists x \neq 0)[Ax = \lambda x] \land (\forall \lambda')[(\exists x' \neq 0)[Ax' = \lambda' x'] \to \lambda' \leq \lambda]) \).

Although Quantifier Elimination proves recursivity, it is a highly inefficient method from the practical viewpoint — the computation time can be doubly exponential in
general. In spite of this, for many problems, reduction to Quantifier Elimination is the only (and thus “the best”) known algorithmic result.

1.3.3 Weak and strong polynomiality

It is a usual convention to say that a problem $A$ is “efficiently” solvable if it is solvable in polynomial time, i.e., in at most $p(L)$ steps of the corresponding Turing machine, where $p$ is a polynomial and $L$ is the size of the input. The class of efficiently solvable decision problems is denoted by $P$.

Taking a more detailed viewpoint, this is a definition of polynomial-time solvability in the weak sense. In our context, we are usually processing a family $a_1, \ldots, a_n$ of rational numbers, where $L = \sum_{i=1}^{n} \text{size}(a_i)$, performing arithmetical operations $+, -, \times, \div, \leq$ with them. The definition of (weak) polynomiality implies that an algorithm can perform at most $p_1(L)$ arithmetical operations with numbers of size at most $p_2(L)$ during its computation, where $p_1, p_2$ are polynomials.

If a polynomial-time algorithm satisfies the stronger property that it performs at most $p_1(n)$ arithmetical operations with numbers of size at most $p_2(L)$ during its computation, we say that it is strongly polynomial. The difference is whether we can bound the number of arithmetical operations only by a polynomial in $L$, or by a polynomial in $n$.

Example. Given a rational $A$ and $b$, the question $(\exists x)[Ax = b]$ can be decided in strongly polynomial time (although it is nontrivial to implement the Gaussian elimination to yield a strongly polynomial algorithm). On the contrary, the question $(\exists x)[Ax \leq b]$ (which is a form of linear programming) is known to be solvable in weakly polynomial time only and it is a major open question whether a strongly polynomial algorithm exists (this is Smale’s Ninth Millenium Problem, see [52]).

The main message of the previous example is: whenever an interval-algebraic problem is solvable in polynomial time and requires linear programming (which is a frequent case), it is only a weakly polynomial result. This is why the rare cases, when interval-algebraic problems are solvable in strongly polynomial time, are of special interest.

1.3.4 NP, coNP

Recall that NP is the class of decision problems $A$ with the following property: there is a polynomial $p$ and a decision problem $B(x,y)$, solvable in time polynomial in $\text{size}(x) + \text{size}(y)$, such that, for any instance $x \in \{0,1\}^*$,

$$A(x) = 1 \text{ iff } (\exists y \in \{0,1\}^*) \underbrace{\text{size}(y) \leq p(\text{size}(x))}_{(\star)} \text{ and } B(x,y) = 1.$$ (1.1)
The string y is called witness for the $\exists$-quantifier, or also witness of the fact that $A(x) = 1$. The algorithm for $B(x,y)$ is called verifier. For short, we often write $A(x) = (\exists y)B(x,y)$, showing that $A$ results from the $\exists$-quantification of the efficiently decidable question $B$ (and the quantifier ranges over strings of polynomially bounded size). Observe that the question $(\exists y)B(x,y)$ need not be decidable in polynomial time (in fact, this is the open problem “$P = NP$”), since the quantification range is exponential in $size(x)$.

A lot of $\exists$-problems from various areas of mathematics are in NP: “does a given boolean formula $x$ have a satisfying assignment $y$?”, “does a given graph $x$ have 3-coloring $y$?”, “does a given system $x = Ax \leq b$ have an integral solution $y$?”, and many others.

The class coNP is characterized by replacement of the quantifier in (1.1):

$$A(x) = 1 \iff (\forall y \in \{0,1\}^n)\ size(y) \leq p(size(x)) \rightarrow B(x,y) = 1.$$ 

It is easily seen that the class coNP is formed of complements of NP-problems, and vice versa. (Recall that a decision problem $A$ is a 0-1 function; its complement is defined as $coA = 1 - A$.)

The prominent example of a coNP-question is deciding whether a boolean formula is a tautology, or in other words, “given a boolean formula $x$, is it true that every assignment $y$ makes it true?”.

It is easy to see again that deciding a coNP-question can take exponential time since the $\forall$-quantifier ranges over a set exponentially large in $size(x)$.

**Example.** Interval linear algebra is not an exception: a lot of $\exists$-questions belong to NP, but we should be careful a bit. As an example, consider the problem SINGULARITY: given $A \in \mathbb{IQ}^{n \times n}$, $\exists A \in A$ which is singular? We could expect that SINGULARITY $\in$ NP since the positive answer can be certified by the $\exists$-witness $A_0 = a particular singular matrix in A$. Indeed, the natural verifier $B(A,A_0)$, checking whether $A_0 \in A$ and $A_0$ is singular, works in polynomial time. But a problem is hidden in the condition $(\star)$ in (1.1). To be fully correct, we would have to prove: there exists a polynomial $p$ such that whenever $A$ contains a singular matrix, then it also contains a rational singular matrix $A_0$ such that $size(A_0) \leq p(L)$, where $L = size(A) + size(\overline{A})$. Direct proofs of such properties are “uncomfortable”. But we can proceed in a more elegant way, using Theorem[2]

$$\exists A \in A \ s.t. \ A \text{ is singular}$$

$$\iff \exists A \in A, \exists x \neq 0 \ s.t. \ Ax = 0$$

$$\iff \exists x \neq 0 \ s.t. \ -\Delta \leq Ax \leq \Delta,$$

$$\iff \exists x \in \{\pm1\}^n \ s.t. \ \Delta D x \leq Ax \leq \Delta D x, \ D x \geq 0, \ e^T D x \geq 1. \ (1.2)$$

Given $s \in \{\pm1\}^n$, the relation (1) can be checked in polynomial time by linear programming. Thus, we can define the verifier $B(A,s)$ as the algorithm checking the validity of (1). In fact, we have reformulated the $\exists$-question, “is there a singular
A ∈ A?” into an equivalent ∃-question, “is there a sign vector \( s \in \{\pm 1\}^n \) s.t. \((\dagger)\) holds true?”, and now \( \text{size}(s) \leq L \) is obvious.

The method of (1.2) is known as orthant decomposition since it reduces the problem to inspection of orthants \( D_s x \geq 0 \), for every \( s \in \{\pm 1\}^n \), and the work in each orthant is “easy” (here, the work in an orthant amounts to a single linear program). Many properties with interval data are described by sufficient and necessary conditions that use orthant decomposition.

We can also immediately see that REGULARITY = \( \text{cosingularity} \) (“given \( A \), is every \( A \in A \) nonsingular?”) belongs to coNP.

### 1.3.5 Decision problems: NP-, coNP-completeness

A decision problem \( A \) is reducible to a decision problem \( B \) (denoted \( A \leq B \)) if there exists a polynomial-time computable function \( g : \{0,1\}^* \to \{0,1\}^* \), called reduction, such that for every \( x \in \{0,1\}^* \) we have

\[
A(x) = B(g(x)).
\]

Said informally, any algorithm for \( B \) can also be used for solving \( A \): given an instance \( x \) of \( A \), we can efficiently “translate” it into an instance \( g(x) \) of the problem \( B \) and run the method deciding \( B(g(x)) \), yielding the correct answer to \( A(x) \). Thus, any decision method for \( B \) is also a valid method for \( A \), if we admit the polynomial time for computation of the reduction \( g \). In this sense we can say that if \( A \leq B \), then \( B \) “as hard as \( A \), or harder”. If both \( A \leq B \) and \( B \leq A \), then problems \( A, B \) are called polynomially equivalent.

The relation \( \leq \) induces a partial ordering on classes of polynomially equivalent problems in NP (called NP-degrees) and this ordering can be shown to have a maximum element. The problems in the maximum class are called NP-complete problems. And similarly, coNP has a class of coNP-complete problems. They are complementary: a problem \( A \) is NP-complete iff its complement is coNP-complete.

Let \( \mathcal{X} \in \{\text{NP}, \text{coNP}\} \). If a problem \( B \) is \( \mathcal{X} \)-complete, any method for it can be understood as a universal method for any problem \( A \in \mathcal{X} \), modulo polynomial time needed for computing the reduction. Indeed, since \( B \) is the maximum element, we have \( A \leq B \) for any \( A \in \mathcal{X} \). It is generally believed that \( \mathcal{X} \) contains problems which are not efficiently decidable. In NP, boolean satisfiability is a prominent example; in coNP, it is the tautology problem. Then, by \( \leq \)-maximality, no \( \mathcal{X} \)-complete problem is efficiently decidable. This shows why a proof of \( \mathcal{X} \)-completeness of a newly studied problem is often understood as proof of its computational intractability.

**Remark.** From a practical perspective, a proof of NP- or coNP-completeness is the same bad news, telling us that “nothing better than superpolynomial-time algorithms can be expected”. But formally we must distinguish between NP- and co-NP completeness because it is believed that NP-complete problems are not polynomi-
ally equivalent with coNP-complete problems. (This is the “NP =? coNP” open problem).

**NP- and coNP-complete problems in interval analysis.** A survey of such problems forms the core of this paper. An important example of an NP-complete problem is SINGULARITY of an interval matrix \( A \). Its complement, REGULARITY, is thus coNP-complete.

When we know that \( B \) is \( \mathcal{X} \)-complete and we prove \( B \leq C \) for a problem \( C \in \mathcal{X} \), then \( C \) is also \( \mathcal{X} \)-complete. This is the method behind all \( \mathcal{X} \)-completeness proofs of this paper. For example, let EIGENVALUE be the problem “given a square interval matrix \( A \) and a number \( \lambda \), decide whether \( \lambda \) is an eigenvalue of some \( A \in A \)”. It is easy to prove SINGULARITY \( \leq \) EIGENVALUE; indeed, if we are to decide whether there is a singular matrix \( A \in A \), it suffices to use the reduction \( g : A \mapsto (A, \lambda = 0) \). The proof of EIGENVALUE \( \in \) NP can be derived from the orthant decomposition method; this proves that EIGENVALUE is an NP-complete problem.

### 1.3.6 Decision problems: NP-, coNP-hardness

We restrict ourselves to NP-hard problems; the reasoning for coNP-hard problems is analogous.

In the previous section we spoke about NP-complete problems as the \( \leq \)-maximum elements in NP. But our reasoning can be more general. We can work on the entire class of decision problems, including those outside NP. We say that a decision problem \( H \), not necessarily in NP, satisfying \( C \leq H \) for an NP-complete problem \( C \), is **NP-hard**. Clearly: NP-complete problems are exactly those NP-hard problems which are in NP. But we might encounter a problem \( H \) for which we do not have the proof \( H \in \) NP, but still it might be possible to prove \( C \leq H \). Then the bad news for practice is again the same, that the problem \( H \) is computationally intractable. (But we might possibly need even worse computation time than for NP-problems; recall that all problems in NP can be solved in exponential time, not worse.)

To summarize: a proof that a decision problem is NP-hard is a weaker theoretical result than a proof that a decision problem is NP-complete; it leads to an immediate research problem to inspect why it is difficult to prove the presence in NP. Usually, the reason is that it is not easy (or impossible at all) to write down the \( \exists \)-definition; recall the example (1.2), where the proof of presence in NP required the aid of Theorem 2.

**Remark.** If we are unsuccessful in placing the problem in NP or coNP, being unable to write down the \( \exists \) or \( \forall \)-definition, it might be appropriate to place the problem \( H \) into higher levels of the Polynomial Time Hierarchy, or even higher, such as the PSPACE-level; for details see [1], Chapter 5.
1.3.7 Functional problems: efficient solvability and NP-hardness

Functional problems are problems of computing values of general functions, in contrast to decision problems where we expect only YES/NO answers. We also want to classify functional problems from the complexity-theoretic perspective, whether they are “efficiently solvable”, or “intractable”, as we did with decision problems. Efficient solvability of a functional problem is again generally understood as polynomial-time computability. To define NP-hardness, we need the following notion of reduction: a decision problem \( A \) is reducible to a functional problem \( F \), if there exist functions \( g : \{0, 1\}^* \rightarrow \{0, 1\}^* \) and \( h : \{0, 1\}^* \rightarrow \{0, 1\}_0 \), both computable in polynomial time, such that
\[
A(x) = h(F(g(x))) \quad \text{for every} \quad x \in \{0, 1\}^*. \tag{1.4}
\]

The role of \( g \) is analogous to \( 1.3 \): it translates an instance \( x \) of \( A \) into an instance \( g(x) \) of \( F \). What is new here is the function \( h \). Since \( F \) is a functional problem, the value \( F(g(x)) \) can be an arbitrary bitstring (say, a binary representation of a rational number); then we need another efficiently computable function \( h \) translating the value \( F(g(x)) \) into a 1-0 value giving the YES/NO answer to \( A(x) \). A trivial example: deciding regularity of a rational matrix (decision problem \( A \)) is reducible to the computation of rank (functional problem \( F \)). It suffices to define \( g(A) = A \) and \( h(\zeta) = 1 \) \(-\) \( \min \{n - \zeta, 1\} \).

Now, a functional problem \( F \) is NP-hard if there is an NP-hard decision problem reducible to \( F \). For example, the functional problem of counting the number of ones in the truth-table of a given boolean formula is NP-hard since this information allows us to decide whether or not the formula is satisfiable.

Remark. It is not necessary to distinguish between NP-hardness and coNP-hardness for functional problems. We could also try to define coNP-hardness of a functional problem \( G \) in terms of reducibility of a coNP-hard decision problem \( C \) to \( G \) via \( 1.4 \). But this is superfluous because here NP-hardness and coNP-hardness coincide. Indeed, if we can reduce a coNP-hard problem \( C \) to a functional problem \( G \) via \( (g, h) \), then we can also reduce the NP-hard problem \( coC \) to \( G \) via \( (g, 1 - h) \). Thus, in case of functional problems, we speak about NP-hardness only.

1.3.8 More general reductions: do we indeed have to distinguish between NP-hardness and coNP-hardness of decision problems?

In literature, the notions of NP-hardness and coNP-hardness are sometimes used quite freely even for decision problems. Sometimes we can read that a decision problem is “NP-hard”, even if it would qualify as a coNP-hard problem under our definition based on the reduction \( 1.3 \). This is nothing serious as far as we are aware. It depends how the author understands the notion of a reduction between
two decision problems. We have used the \textit{many-one reduction} (1.3), known also as \textit{Karp reduction}, between two decision problems. This is a standard in complexity-theoretic literature.

However, one could use a more general reduction between two decision problems $A, B$. For example, taking inspiration from (1.3), we could define $"A \leq' B"$ \textit{iff} $A(x) = h(B(g(x)))$ for some polynomial-time computable functions $g, h$. Then the notions of $\leq'$-NP-hardness and $\leq'$-coNP-hardness coincide and need not be distinguished.

(Observe that $h$ must be a function from $\{0, 1\}$ to $\{0, 1\}$ and there are only two such nonconstant functions: $h_1(\xi) = \xi$ and $h_2(\xi) = 1 - \xi$. If we admit only $h_1$, we get the many-one reduction; if we admit also the negation $h_2$, we have a generalized reduction under which a problem is NP-hard \textit{iff} it is coNP-hard. Thus: the notions of NP-hardness and coNP-hardness based on many-one reductions do not coincide just because many-one reductions do not admit the negation of the output of $B(g(x))$.

To be fully precise, one should always say “a problem $A$ is $\mathcal{Z}$-hard w.r.t. a particular reduction $\leq$". For example, in the previous sections we spoke about $\mathcal{Z}$-hard problems for $\mathcal{Z} = \{\text{NP, coNP}\}$ w.r.t. the many-one reduction (1.3). If another author uses $\mathcal{Z}$-hardness w.r.t. $\leq'$ (e.g., because (s)he considers the ban of negation as too restrictive in her/his context), then (s)he need not distinguish between NP-hardness and coNP-hardness.

For the sake of completeness, we conclude that in literature we can meet the notions of hardness w.r.t. various types of reductions.

\textbf{Logspace-computable reduction:} $A \leq_{\text{log}} B$ \textit{iff} there is a function $g$ computable in memory of size $O(\log \text{size}(x))$, such that $A(x) = B(g(x))$ for every $x$. (This reduction in weaker than (1.3) since every logspace-computable function is also computable in polynomial time.)

\textbf{Truth-table reduction:} $A \leq_{tt} B$ \textit{iff} there is a finite number of polynomial-time computable functions $g_1, \ldots, g_k : \{0, 1\}^* \rightarrow \{0, 1\}^*$ and a “truth-table” function $h : \{0, 1\}^k \rightarrow \{0, 1\}$ such that $A(x) = h(B(g_1(x)), \ldots, B(g_k(x)))$. This reduction is a generalization of $\leq'$; indeed, $\leq'$ is a restricted truth-table reduction with a two-line truth table. Under $\leq_{tt}$, to decide $A(x)$ one can compute $k$ instances of $B$ from which the boolean expression $h$ combines the result $A(x)$.

\textbf{Turing reduction (or Cook reduction):} $A \leq_T B$ \textit{iff} there is a polynomial-time algorithm (Turing machine) $Q$, equipped with a subroutine (an algorithm, \textit{oracle}) computing $B$, and the entire computation of $B$ is counted as a single step of $Q$. This is the most general type of reduction: when deciding $A(x)$, the reduction allows for a polynomial number of computations of $B(y)$ with $\text{size}(y)$ polynomially bounded in $\text{size}(x)$, and the results can be combined in an arbitrary way; the only limitation is that the overall number of steps is polynomial in $\text{size}(x)$, assuming that one computation of $B(y)$ is at the unit cost.

The above mentioned reductions can be ordered in the sequence according to their generality: $A \leq_{\text{log}} B \Rightarrow A \leq B \Rightarrow A \leq' B \Rightarrow A \leq_{tt} B \Rightarrow A \leq_T B$, where $"\Rightarrow"$ means “implies”. We know that NP-hardness and coNP-hardness coincide for $\leq'$, and thus also for the generalizations $\leq_{tt}, \leq_T$. 

1.3.9 A reduction-free definition of hardness

For practical purposes, when we do not want to play with properties of particular reductions, we can define the notion of a “hard” problem $H$ (either decision of functional) intuitively as a problem fulfilling this implication: if $H$ is decidable/solvable in polynomial time, then $P = NP$. This is usually satisfactory for the practical understanding of the notion of computational hardness. (Under this definition: if $P = NP$, then every decision problem is hard; and if $P \neq NP$, then the class of hard decision problems is exactly the class of decision problems not decidable in polynomial time, including all NP-hard and coNP-hard decision problems.)

Even if we accept this definition and do not speak about reductions explicitly, all hardness proofs (at least implicitly) contain some kinds of reductions of previously known hard problems to the newly studied ones.

1.4 Interval linear algebra – part II

In the following sections we will deal with various problems in interval linear algebra. There are many interesting topics that are unfortunately beyond the scope of this work. We will at least point out some of them in section 1.4.10. We chose basic topics from introductory courses to linear algebra – regularity and singularity of a matrix, full column rank, solving and solvability of a system of linear equations, matrix inverse, determinant, eigenvalues and eigenvectors, positive (semi)definiteness and stability. The next chapters will offer a great disappointment and also a great challenge, since implanting intervals into a classical linear algebra makes solving most of the problems intractable. That is why, we look for solving relaxed problems, special feasible subclasses of problems or for sufficient conditions checkable in polynomial time. Interval linear algebra still offers many open problems and a lot of place for further research. At the end of each section we present a summary of problems and their complexity. If we only know that a problem is weakly polynomial yet, we just write that it belongs to the class $P$. When complexity of a problem is not known to our best knowledge (or it is an open problem), we mark it with question mark.

1.4.1 Regularity and singularity

Deciding regularity and singularity of an interval matrix is an important task in linear algebra. The definition of interval regularity (and singularity) is intuitive.

**Definition 3.** A square interval matrix $A$ is regular if every $A \in A$ is nonsingular. Otherwise, $A$ is called singular.
Considering complexity we can find in the literature the following theorem giving NP-completeness result even for the simple case.

**Theorem 3.** Deciding whether an interval matrix \( A = [A - E, A + E] \) is singular for some nonnegative symmetric positive definite rational matrix \( A \) is NP-complete.

We can prove NP-hardness of this decision problem. Moreover, we get NP-completeness since we know that a singular \( A \) in this form mentioned in the theorem must contain a singular matrix

\[
A - \frac{zz^T}{z^T A^{-1} z},
\]

for some \( z \in \{\pm 1\}^n \) which is a polynomial witness and the above mentioned matrix is checkable in polynomial time (e.g., by Gaussian elimination). This implies that deciding singularity of a general interval matrix is NP-hard. However, in the section 1.3.4 we saw the construction of a polynomial witness \( z \in \{\pm 1\}^n \) certifying that an interval matrix is singular. Hence, we get that checking singularity of a general interval matrix is NP-complete. Clearly, checking regularity as the complement problem to singularity is coNP-complete.

The sufficient and necessary conditions for checking regularity are of exponential nature. In \([44]\) you can see 40 of them. For example, we can use the classical definition of matrix regularity (a matrix \( A \) is regular if the system \( Ax = 0 \) has only trivial solution) and combine it with Oettli-Prager theorem. We get that an interval matrix is regular if and only if the inequality

\[
|A_c x| \leq \Delta |x|,
\]

has only trivial solution.

Fortunately, there are some sufficient conditions that are computable in polynomial time. It is advantageous to have more conditions, because some of them may suit better to a certain class of matrices or limits of our software tools. Here we present three sufficient conditions for checking regularity and three sufficient conditions for checking singularity.

**Theorem 4 (Sufficient conditions for regularity).** An interval matrix \( A = [A_c - \Delta, A_c + \Delta] \) is regular if at least one of the following conditions holds

1. \( \rho(|A_c^{-1}| \Delta) < 1 \) \([40]\),
2. \( \sigma_{\text{max}}(\Delta) < \sigma_{\text{min}}(A_c) \) \([48]\),
3. \( A_c^T \Delta^{-1} A_c = \| \Delta^T \Delta \| I \) is positive definite for some consistent matrix norm \( \| \cdot \| \) \([34]\).

**Theorem 5 (Sufficient conditions for singularity).** An interval matrix \( A = [A_c - \Delta, A_c + \Delta] \) is singular if at least one of the following conditions holds

1. \( \max_j(|A_{c,j}^{-1}| \Delta)_{jj} \geq 1 \) \([45]\),
2. \( (\Delta - |A_c|)^{-1} \geq 0 \) \([40]\),
3. \( \Delta^T \Delta - A_c^T A_c \) is positive semidefinite \([34]\).
In the above two theorems, the first condition in the triplet is among the most frequently used sufficient conditions. You can find more sufficient conditions for regularity and singularity in [34].

We can also take a look at the classes of interval matrices that are immediately regular. These are, for example, diagonally dominant matrices [51], M-matrices and H-matrices [28]. There properties are checkable in polynomial time.

Summary.

| Problem          | Complexity   |
|------------------|--------------|
| Is \( A \) regular? | coNP-complete |
| Is \( A \) singular? | NP-complete  |

### 1.4.2 Full column rank

The definition of the full column rank is natural.

**Definition 4.** An \( m \times n \) interval matrix \( A \) has **full column rank** if every \( A \in A \) has full column rank (i.e., it has rank \( n \)).

Deciding whether an interval matrix has full column rank is connected to checking regularity. If an interval matrix \( A \) of size \( m \times n, m \geq n \), contains a regular submatrix of size \( n \), then obviously \( A \) has a full column rank. What is surprising is that the implication does not hold conversely (in contrast to real matrices). The interval matrix by Irene Sharaya (see [51]) might serve as a counterexample.

\[
A = \begin{pmatrix}
1 & [0,1] \\
-1 & [0,1] \\
[-1,1] & 1
\end{pmatrix}.
\]

It has full column rank, but contains no regular submatrix of size 2.

For square matrices, checking regularity can be polynomially reduced to checking full column rank (we just check the matrix \( A \)), but the converse is not so easy. Therefore, checking full column rank is coNP-hard. Finding a polynomial certificate for an interval matrix not having full column rank can be done by orthant decomposition similarly as in the case of singularity. That is why, checking full column rank is coNP-complete.

Again, fortunately, we have some sufficient conditions that are computable in polynomial time.

**Theorem 6.** Let \( A = [A_c - \Delta, A_c + \Delta] \) be an \( m \times n \) interval matrix. This matrix has full column rank if at least one of the following conditions holds

1. \( A_c \) has full column rank and \( \rho([A_c^+|\Delta]) < 1 \), [46].
2. $\sigma_{\text{max}}(\Delta) < \sigma_{\text{min}}(A_c)$ [51].

The symbol $\dagger$ stands for Moore-Penrose inverse. The first condition is mentioned implicitly in [46], however the explicit proof can be found in [51]. Notice that the second sufficient condition is the same as the sufficient condition for checking regularity. Many problems can be transformed to checking full column rank – e.g., deciding whether a given interval linear system is solvable, deciding whether a solution set of an interval linear system is bounded.

Summary.

| Problem                           | Complexity   |
|----------------------------------|--------------|
| Does $A$ have full column rank?  | coNP-complete|

1.4.3 Solving a system of linear equations

To be brief the title of this section contained the word “solving”. Nevertheless, this notion could be a little misleading. Let us explain what do we mean by solving a system of interval linear equations (or interval linear system for short). The solution set of an interval linear system is defined as follows.

Definition 5. Let $Ax = b$, where $A$ is an $m \times n$ interval matrix and $b$ is an $m$-dimensional right-hand side vector. Then by a solution set $\Sigma$ we mean

$$\Sigma = \{x \mid Ax = b \text{ for some } A \in A, \ b \in b\}.$$  

We could imagine it as a collection of all solutions of all crisp real systems contained within the bounds of an interval system. Unfortunately, this set is of quite a complex shape. For its description we can use the already mentioned Oettli-Prager Theorem [2]. A vector $x \in \mathbb{R}^n$ is a solution of $Ax = b$ (i.e., $x \in \Sigma$) if and only if $x$ satisfies

$$|A_c x - b_c| \leq \Delta |x| + \delta.$$

We can see that checking whether a vector $y$ is a solution of $Ax = b$ is strongly polynomial (we just check the inequality for $y$).

Oettli-Prager theorem implies that the set $\Sigma$ is generally non-convex but convex in each orthant (for graphical examples of possible shapes of the solution set see e.g., [25, 13, 27]). That is why, we usually approximate this set by an $n$-dimensional box (aligned with axes) containing $\Sigma$. Notice that we can view an $n$-dimensional interval vector as an $n$-dimensional box aligned with axes.

Definition 6. An $n$-dimensional interval vector $x$ is called an interval enclosure of $\Sigma$ if $\Sigma \subseteq x$. If it is the tightest possible enclosure w.r.t. inclusion (there is no interval box $y$ such that $\Sigma \subseteq y \subsetneq x$), we call $x$ the interval hull.
By solving an interval linear system we understand computing any enclosure \( x \) of its solution set \( \Sigma \). To be brief, we call that \( x \) an enclosure (or the hull) of \( Ax = b \). The notion of enclosure is quite intuitive because we are not always able to compute the interval hull. In [22] we can see that computing the exact hull of \( Ax = b \) is NP-hard.

An interval \( a = [a - \Delta, a + \Delta] \) is absolutely \( \delta \)-narrow if \( \Delta \leq \delta \) and relatively \( \delta \)-narrow if \( \Delta \leq \delta \cdot |a| \). The problem is still NP-hard even if we limit widths of intervals of a matrix in a system with some \( \delta > 0 \) [22]. We can summarize it in the following theorem.

**Theorem 7.** For every \( \delta > 0 \), the problem of computing the hull of \( Ax = b \), where \( a_{ij}, b \) are both absolutely and relatively \( \delta \)-narrow is NP-hard.

Unfortunately, even computing various \( \varepsilon \)-approximations of the hull components is an NP-hard problem [22].

**Theorem 8.** For a given \( \varepsilon > 0 \) computing the relative and absolute \( \varepsilon \)-approximation of the hull (its components) of \( Ax = b \) are NP-hard problems.

That is why, we are usually looking for enclosures, not the hull. Of course, the tighter enclosure the better. For computing enclosures of square systems, there have been various methods developed. Some of them extend the traditional algorithms for the real systems, such as the Gaussian elimination, Jacobi or Gauss-Seidel method [25, 28]. Some of them were designed specifically for interval systems; see for instance [8, 12, 20, 25, 28, 4] among many others.

**Overdetermined systems.** For an overdetermined system (where \( A \) is an \( m \times n \) matrix with \( m > n \)) the situation is slightly more difficult. Many people automatically think of solving overdetermined systems via least squares, i.e.,

**Definition 7.**

\[
\Sigma^{lsq} = \{ x \mid ATAx = ATb \text{ for some } A \in A, b \in b \}. 
\]

Obviously, \( \Sigma^{lsq} \) is not the same set as \( \Sigma \). Nevertheless, it is not difficult to see that \( \Sigma \subseteq \Sigma^{lsq} \). Hence, we can use methods for computing least squares for enclosing \( \Sigma \) [27]. The problem of computing the interval hull of \( \Sigma^{lsq} \) is NP-hard, since when \( A \) is square and regular, then \( \Sigma^{lsq} = \Sigma \) and computing the exact hull of \( \Sigma \) is NP-hard even for \( A \) regular [4].

If we primarily focus on enclosing just \( \Sigma \) there is a variety of methods – modified Gaussian elimination for overdetermined systems [7], method developed by Rohn [41], Popova [30], or a method using square subsystems [14].

We can try to identify some classes of systems with exact hull computation algorithms that run in polynomial time. If we restrict the right hand side \( b \) to contain only degenerate intervals, we have \( Ax = b \). Then, this problem is still NP-hard [22]. If we, however, restricts the matrix to be consisting only of degenerate intervals and we have a system \( Ax = b \), then, computing exact bounds of the solution set is polynomial, since it can be rewritten as a linear program.
However, even if we allow at most one nondegenerate interval coefficient in each equation, the problem becomes again NP-hard, since an arbitrary interval linear system can be rewritten in this form [22].

**Structured systems.** We can also explore band and sparse matrices.

**Definition 8.** A matrix \( A \) is a \( w \)-band matrix if \( a_{ij} = 0 \) for \( |i - j| \geq w \).

Band matrices with \( d = 1 \) are diagonal and computing the hull is clearly strongly polynomial. For \( d = 2 \) (tridiagonal matrix) it is an open problem. And for \( d \geq 3 \) it is again NP-hard. We inspected the case of bidiagonal matrices. The result is to our best knowledge new.

**Theorem 9.** For a bidiagonal matrix (the matrix with only the main diagonal and an arbitrary neighbouring diagonal) computing the exact hull of \( Ax = b \) is strongly polynomial.

**Proof.** Without the loss of generality let us suppose that the matrix \( A \) consists of the main diagonal and the one beyond it. By the forward substitution, we have \( x_1 = \frac{b_1}{a_{11}} \) and

\[
x_i = \frac{b_i - a_{i,i-1}x_{i-1}}{a_{ii}}, \quad i = 2, \ldots, n.
\]

By induction, \( x_{i-1} \) is optimally computed with no use of interval coefficients of the \( i \)th equations. Since an evaluation in interval arithmetic is optimal in the case there are no multiple occurrences of variables (Theorem 1), \( x_i \) is optimal as well.

**Definition 9.** A matrix \( A \) is a \( d \)-sparse matrix if in each row \( i \) at most \( d \) elements \( a_{ij} \neq 0 \).

For sparse matrices with \( d = 1 \) computing the hull is clearly strongly polynomial. For \( d \geq 2 \) it is again NP-hard [22]. Nevertheless, if we combine \( w \)-band matrix with system coefficient bounds coming from a given finite set of rational numbers, then we have a polynomial algorithm for computing the hull [22].

If an interval system \( Ax = b \) is in a certain form the hull can be computed in polynomial time using some already introduced algorithms. If the matrix \( A \) has full column rank and \( A_r \) is a diagonal matrix with positive entries, then Hansen-Bliek-Rohn prescription for enclosure gives the exact hull [4]. If \( A \) is an M-matrix, then Gauss-Seidel iteration method converges to the exact hull [28]. And if \( A \) is an M-matrix and \( b \) is nonnegative then the interval version of Gaussian elimination yields the exact hull [28].

In this section we silently supposed that the solution set \( \Sigma \) is bounded. This is not always the case. Many mentioned methods can not deal with an unbounded solution set. That is why we usually need to check for boundedness. However, it is an coNP-complete problem since it is identical with checking the full column rank of the interval matrix \( A \).

**Remark.** A natural generalization of an interval linear system is by incorporating linear dependencies. That is, we have a family of linear systems.
\[ A(p)x = b(p), \quad p \in p, \quad (1.5) \]

where \( A(p) = \sum_{k=1}^{K} A^k p_k \) and \( b(p) = \sum_{k=1}^{K} b^k p_k \). Here, \( p \) is a vector of parameters varying in \( p \). Since this concept generalizes the standard interval systems, many related problems are intractable. We point out one particular efficiently solvable problem. Given \( x \in \mathbb{R}^n \), deciding whether it is a solution of a standard interval system \( Ax = b \) is strongly polynomial. For systems with linear dependencies, the problem still stays polynomial, but we can show weak polynomiality only; this is achieved by rewriting (1.5) as a linear program.

Summary.

| Problem | Complexity |
|--------|-----------|
| Is \( x \) a solution of \( Ax = b \)? | strongly P |
| Computing an enclosure of \( Ax = b \) | P |
| Computing the hull of \( Ax = b \) | NP-hard |
| Computing the hull of \( Ax = b \) | NP-hard |
| Computing the hull of \( Ax = b \), where \( A \) is regular | NP-hard |
| Computing the hull of \( Ax = b \), where \( A \) is \( M \)-matrix | P |
| Computing the hull of \( Ax = b \), where \( A \) is diagonal | strongly P |
| Computing the hull of \( Ax = b \), where \( A \) is bidiagonal | strongly P |
| Computing the hull of \( Ax = b \), where \( A \) is tridiagonal | ? |
| Computing the hull of \( Ax = b \), where \( A \) is 3-band | NP-hard |
| Computing the hull of \( Ax = b \), where \( A \) is 1-sparse | strongly P |
| Computing the hull of \( Ax = b \), where \( A \) is 2-sparse | NP-hard |
| Computing the exact least squares hull of \( Ax = b \) | NP-hard |
| Is \( \Sigma \) bounded? | coNP-complete |

1.4.4 Matrix inverse

Computation of a matrix inverse is usually avoided in applications. Nonetheless, we chose to mention this topic, since it holds a worthy place in interval linear algebra theory. An interval inverse matrix is defined as follows.

**Definition 10.** Let us have a square regular interval matrix \( A \). We define its interval inverse matrix as \( A^{-1} = [\underline{B}, \overline{B}] \), where \( \underline{B} = \min \{ A^{-1}, A \in A \} \) and \( \overline{B} = \max \{ A^{-1}, A \in A \} \), where the min and max is understood componentwise.
As usual, the inverse matrix can be computed using knowledge of inverses of boundary matrices $A_{yz}$ \[37\].

**Theorem 10.** Let $A$ be regular. Then its inverse $A^{-1} = [\underline{B}, \overline{B}]$ is described by

$$\underline{B} = \min_{y,z \in \mathbb{F}_n} A_{yz}^{-1},$$

$$\overline{B} = \max_{y,z \in \mathbb{F}_n} A_{yz}^{-1},$$

where the min and max is understood componentwise.

The maximum and minimum bound of each component of the interval inverse is attained at one of the inverse of $2^n$ boundary matrices. No wonder, it can be proved that generally computing exact inverse matrix is NP-hard \[3\].

When $A_c = I$, we can compute the exact inverse in polynomial time according to the next theorem \[45\].

**Theorem 11.** Let $A$ be a regular interval matrix with $A_c = I$. Let $M = (I - \Delta)^{-1}$. Then its inverse $A^{-1} = [\underline{B}, \overline{B}]$ is described by

$$\underline{B} = -M + D_k,$$

$$\overline{B} = -M,$$

where $k_j = \frac{2m_{jj}^2}{\sum_{j=1}^{n} m_{jj}^2}$ for $j = 1, \ldots, n$, with $m_{jj}$ being diagonal elements of $M$.

There also exists a formula for the exact matrix inverse if all intervals have uniform widths, i.e., $A = [A_c - \alpha E, A_c + \alpha E]$ \[47\].

If we wish to only compute an enclosure $B$ of the matrix inverse we can use any method for computing enclosures of interval linear systems. We get the $i$-th column of $B$ by solving the systems $Ax = e_i$, where $e_i$ is $i$-th column of the identity matrix of order $n$.

As we mentioned, computing the exact interval inverse is NP-hard. We close this section with a surprising result on inverse nonnegativity ($A^{-1} \geq 0$ for every $A \in \mathbb{A}$).

It was first proved in slightly different form in \[23\]. For this form see \[28\]. It implies that checking inverse nonnegativity and also computing the exact interval inverse of an inverse nonnegative matrix is strongly polynomial.

**Theorem 12.** If $A, \overline{A}$ are regular and $A^{-1}, \overline{A}^{-1} \geq 0$ then $A$ is regular and

$$A^{-1} = [A^{-1}, \overline{A}^{-1}] \geq 0.$$

**Summary.**

| Problem                               | Complexity  |
|---------------------------------------|-------------|
| Computing the exact inverse of $A$    | NP-hard     |
| Is $A$ inverse nonnegative?           | strongly P  |
| Computing the exact inverse of inverse nonnegative $A$ | strongly P  |
1.4.5 Solvability of a linear system

Of course, before solving a linear system we might want to know, whether it is actually solvable. Considering solvability we should distinguish between two types of solvability.

**Definition 11.** An interval linear system \( Ax = b \) is (weakly) solvable if some system \( Ax = b \), where \( A \in A \), \( b \in b \) is solvable.

In another words, its solution set \( \Sigma \) is not empty. Otherwise, we call the system **insolvable**.

**Definition 12.** An interval linear system \( Ax = b \) is strongly solvable if every system \( Ax = b \), where \( A \in A \), \( b \in b \) is solvable.

The first definition is interesting for model checking. The second for system verification and automated proofs.

Checking whether an interval systems is solvable is an NP-hard problem [22]. The sign coordinates of the orthant containing the solution can serve as a polynomial witness and existence of a solution can be verified by linear programming, hence this problem is NP-complete and checking unsolvability coNP-complete. The problem of deciding strong solvability is coNP-complete. It can be reformulated as checking insolvability of a certain linear system using the well known Farkas lemma, e.g., [43].

Sometimes, we look only for nonnegative solutions – **nonnegative solvability**. Checking whether an interval linear system has a nonnegative solution is weakly polynomial. We know the orthant in which the solution should lie. Therefore, we can get rid of the absolute values in Oettli-Prager theorem and apply linear programming. However, checking whether a system is nonnegative strongly solvable is still coNP-complete [4]. We summarize the results in the following table.

**Theorem 13.** Checking various types of solvability of \( Ax = b \) is of the following complexity.

|                      | weak     | strong   |
|----------------------|----------|----------|
| solvability          | NP-complete | coNP-complete |
| nonnegative solvability | \( P \) | coNP-complete |

It is easy to see that an interval linear system \( Ax = b \) is insolvable if the matrix \([A \ b]\) has full column rank. That is why, we can use sufficient conditions for full column rank to check insolvability. Moreover, we can also use methods for computing enclosures. If we have some enclosure \( x \), then clearly a system \( Ax = b \) is unsolvable if \( Ax \cap b = \emptyset \). Many enclosure algorithms enable detection of insolvability. Generally speaking, they work in iterative stages and when we intersect enclosures of the solution set from the two subsequent stages and get an empty set, we know for
sure that the system is insolvable. These methods are, for example, Gaussian elimination \cite{7}, Jacobi method \cite{25}, Gauss-Seidel method \cite{25}, subsquares method \cite{14}.

**Linear inequalities.** Just for comparison, considering systems of interval linear inequalities, the problems of checking various types of solvability become much easier. The results are resumed in the following table \cite{4}.

**Theorem 14.** Checking various types of solvability of $Ax \leq b$ is of the following complexity.

|                    | weak          | strong        |
|--------------------|---------------|---------------|
| solvability        | NP-complete   | $P$           |
| nonnegative solvability | $P$         | $P$           |

We also would like to mention an interesting nontrivial property of strong solvability of systems of interval linear inequalities. When a system $Ax \leq b$ is strongly solvable (i.e., every $Ax \leq b$ has a solution), then there exists a solution $x$ satisfying $Ax \leq b$ for every $A \in A$ and $b \in b$ \cite{4}.

**∀∃-solutions.** Let us come back to interval linear systems. The traditional concept of a solution (Definition \ref{def:13}) employs existential quantifiers: $x$ is a solution if $\exists A \in A$, $\exists b \in b : Ax = b$. Nevertheless, in some applications, another quantification makes sense, too. In particular, $\forall \exists$ quantification was deeply studied \cite{50}. For illustration of complexity of such solution, we will focus on two concepts of solutions – tolerance \cite{4} and control solution \cite{4, 49}.

**Definition 13.**
A vector $x$ is a tolerance solution of $Ax = b$ if $\forall A \in A$, $\exists b \in b : Ax = b$.

A vector $x$ is a control solution of $Ax = b$ if $\forall b \in b$, $\exists A \in A : Ax = b$.

Notice that a tolerance solution can equivalently be characterized as $\{Ax \mid A \in A\} \subseteq b$ and a control solution as $b \subseteq \{Ax \mid A \in A\}$.

Both solutions can be described by a slight modification of Oettli-Prager theorem (one sign change in Oettli-Prager formula) \cite{4}.

**Theorem 15.** Let us have a system $Ax = b$, then $x$ is

- a tolerance solution if it satisfies $|A_c x - b_c| \leq -\Delta |x| + \delta$.
- a control solution if it satisfies $|A_c x - b_c| \leq \Delta |x| - \delta$.

In case of tolerance solution this change makes checking whether a systems has this kind of solution decidable in weakly polynomial time. In the case of control solution the decision problem stays NP-complete. The same complexity holds for a problem of deciding whether an interval linear systems has a tolerance (polynomial) or control solution (NP-complete) \cite{22}.
### Summary

| Problem | Complexity |
|---------|------------|
| Is $Ax = b$ solvable? | NP-complete |
| Is $Ax = b$ strongly solvable? | coNP-complete |
| Is $Ax = b$ nonnegative solvable? | P |
| Is $Ax = b$ nonnegative strongly solvable? | coNP-complete |
| Is $Ax \leq b$ solvable? | NP-complete |
| Is $Ax \leq b$ strongly solvable? | P |
| Is $Ax \leq b$ nonnegative solvable? | P |
| Is $Ax \leq b$ nonnegative strongly solvable? | P |
| Is $x$ a tolerance solution of $Ax = b$? | P |
| Is $x$ a control solution of $Ax = b$? | NP-complete |
| Does $Ax = b$ have a tolerance solution? | P |
| Does $Ax = b$ have a control solution? | NP-complete |

### 1.4.6 Determinant

Determinants of interval matrices are not often studied. However, we included this section for completeness.

**Definition 14.** A determinant of $A$ is defined as $\text{det}(A) = [d, \overline{d}]$, where

\[
\begin{align*}
    d &= \min \{ \text{det}(A) \mid A \in \mathcal{A} \}, \\
    \overline{d} &= \max \{ \text{det}(A) \mid A \in \mathcal{A} \}.
\end{align*}
\]

Its bounds can be computed from $2^n^2$ boundary matrices $A_{ij} \in \{A_{ij}, \overline{A}_{ij}\}$ for $i, j = 1, \ldots, n$. We have the following theoretical result [40].

**Theorem 16.** Computing interval determinant of $A = [A - E, A + E]$, where $A$ is rational nonnegative is NP-hard.

It is intractable even in this simplified case. For interesting relations to eigenvalues and singularity see [40].

### Summary

| Problem | Complexity |
|---------|------------|
| Computing $\text{det}(A)$ | NP-hard |
| Computing $\overline{\text{det}(A)}$ | NP-hard |
1.4.7 Eigenvalues

First, we briefly start with general matrices, then we continue with the symmetric case. Checking singularity of $A$ can be polynomially reduced to checking whether 0 is an eigenvalue of some matrix $A \in \mathbb{A}$. As we saw in section 1.3.5 checking whether $\lambda$ is an eigenvalue of some matrix $A \in \mathbb{A}$ is NP-complete problem. Surprisingly, checking for eigenvectors can be done efficiently [36]. It is strongly polynomial.

How it is with Perron theory? An interval matrix $A \in \mathbb{IR}^{n \times n}$ is nonnegative irreducible if every $A \in \mathbb{A}$ is nonnegative irreducible. For Perron vectors (positive vectors corresponding to the dominant eigenvalues), we have the following result [42].

**Theorem 17.** Let $A$ be nonnegative irreducible. Then the problem of deciding whether $x$ is a Perron eigenvector of some matrix in $\mathbb{A}$ is strongly polynomial.

For the sake of simplicity we mentioned only some results considering eigenvalues of a general matrix $A$. We will go into more detail with symmetric matrices, where their eigenvalues are real.

**Definition 15.** Let $A \in \mathbb{IR}^{n \times n}$ with $\Delta$. $A_c$ symmetric. Then the corresponding symmetric interval matrix is defined as a subset of symmetric matrices in $\mathbb{A}$, that is,

$$A^S := \{ A \in \mathbb{A} : A = A^T \}.$$  

For a symmetric $A \in \mathbb{R}^{n \times n}$, we use $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ for its smallest and largest eigenvalue, respectively. For a symmetric interval matrix, we define the smallest and largest eigenvalues respectively as

$$\lambda_{\min}(A^S) := \min \{ \lambda_{\min}(A) : A \in A^S \},$$

$$\lambda_{\max}(A^S) := \max \{ \lambda_{\max}(A) : A \in A^S \}.$$  

Even if we consider the symmetric case some problems remain intractable [22, 40]. We are yet able to prove the hardness results, since it is difficult to find a proper polynomial witness.

**Theorem 18.** On a class of problems with $A_c \in \mathbb{Q}^{n \times n}$ symmetric positive definite and entrywise nonnegative, and $\Delta = E$, the following problems are intractable

- checking whether 0 is an eigenvalue of some matrix $A \in A^S$ is NP-hard,
- checking $\lambda_{\max}(A^S) \in (\underline{a}, \overline{a})$ for a given open interval $(\underline{a}, \overline{a})$ is coNP-hard.

However, there are some known subclasses for which the eigenvalue range or at least one of the extremal eigenvalues can be determined efficiently [10]:

- If $A_c$ is essentially non-negative, i.e., $(A_c)_{ij} \geq 0 \forall i \neq j$, then $\lambda_{\max}(A^S) = \lambda_{\max}(\overline{A})$.
- If $\Delta$ is diagonal, then $\lambda_{\min}(A^S) = \lambda_{\min}(\overline{A})$ and $\lambda_{\max}(A^S) = \lambda_{\max}(\overline{A})$. 
In contrast to the extremal eigenvalues \( \lambda_{\min}(A^S) \) and \( \lambda_{\max}(A^S) \), the largest of the minimal eigenvalues and the smallest of the largest eigenvalues,

\[
\max\{\lambda_{\min}(A) : A \in A^S\}, \\
\min\{\lambda_{\max}(A) : A \in A^S\},
\]
can be computed with an arbitrary precision in polynomial time by using semidefinite programming [15]. As in the general case, checking whether a given vector \( 0 \neq x \in \mathbb{R}^n \) is an eigenvector of some matrix in \( A^S \) is a polynomial time problem. Nevertheless, strong polynomiality has not been proved yet.

We already know that computing exact bounds on many problems with interval data is intractable. Since we can do no better, we can inspect the hardness of various approximations of their solutions. While doing this we use the following assumption: Throughout this section, we consider a computational model, in which the exact eigenvalues of rational symmetric matrices are polynomially computable.

The table below from [10] summarizes the main results. We use the symbol \( \infty \) in case there is no finite approximation factor with polynomial complexity.

**Theorem 19.** Approximating the extremal eigenvalues of \( A^S \) is of the following complexity.

|                  | abs. error | rel. error | inverse rel. error |
|------------------|------------|------------|--------------------|
| NP-hard with error | any        | < 1        | 1                  |
| polynomial with error | \( \infty \) | 1          | 2                  |

The table below gives analogous results for the specific case of approximating \( \lambda_{\max}(A^S) \) when \( A_c \) is positive semi-definite.

**Theorem 20.** Approximating the extremal eigenvalues of \( A^S \) with \( A_c \) rational positive semi-definite is of the following complexity.

|                  | abs. error | rel. error | inverse rel. error |
|------------------|------------|------------|--------------------|
| NP-hard with error | any        | \( 1/(32n^4) \) | \( 1/(32n^4) \)   |
| polynomial with error | \( \infty \) | \( 1/3 \)    | \( 1/3 \)          |

The tables sums up the generalized idea behind several theorems on computing extremal eigenvalues. For more information and formal details see [10].

At the end of this subsection we mention spectral radius.

**Definition 16.** Let \( A \in \mathbb{R}^{n \times n} \), we define the range of spectral radius naturally as

\[
\rho(A) = \{\rho(A) : A \in A\}.
\]
Notice that $\rho(A)$ is a compact real interval due to continuity of eigenvalues. Similarly we define spectral radius for $A^S$.

Complexity of computing $\rho(A)$ is an open problem (as Schur stability is; see Section 1.4.9), and, to the best of our knowledge, complexity of computing $\rho(A)$ has not been investigated yet.

Anyway, the following gives polynomially solvable subclasses:

- If $A \geq 0$, then $\rho(A) = [\rho(A), \rho(\overline{A})]$.
- If $A$ is diagonal, then $\rho(A) = [\max_i \min_{a \in \alpha_i} |a|, \max_i \{ |a_{ii}|, |\overline{a}_{ii}| \}]$.

### Summary

| Problem | Complexity |
|---------|------------|
| Is $\lambda$ eigenvalue of some $A \in A$? | NP-complete |
| Is $x$ eigenvector of some $A \in A$? | strongly P |
| Is $x$ Perron vector of nonnegative irreducible $A$? | strongly P |
| Is 0 eigenvalue of some $A \in A^S$? | NP-hard |
| Is $x$ eigenvector of some $A \in A^S$? | P |
| Does $\lambda_{\text{max}}(A^S)$ belong to a given open interval? | coNP-hard |
| Computing $\rho(A)$ | ? |
| Computing $\rho(A)$ | ? |
| Computing exact bounds on $\rho(A)$ with $A$ nonnegative | strongly P |
| Computing exact bounds on $\rho(A)$ with $A$ diagonal | strongly P |

### 1.4.8 Positive definiteness and semidefiniteness

We should not leave out mentioning the positive definiteness and semidefiniteness. Here without the loss of the generality symmetric matrices are of the only interest. We distinguish between weak and strong definiteness.

**Definition 17.** A symmetric interval matrix $A^S$ is weakly positive (semi)definite if some $A \in A^S$ is positive (semi)definite.

**Definition 18.** A symmetric interval matrix $A^S$ is strongly positive (semi)definite if every $A \in A^S$ is positive (semi)definite.

Checking positive definiteness \[38\] and semidefiniteness \[26\] are both coNP-hard according to the two following theorems.

**Theorem 21.** Checking strong positive semidefiniteness of $A^S$ is co-NP-hard on a class of problems with $A_e \in \mathbb{Q}^{n \times n}$ symmetric positive definite and entrywise nonnegative, and $\Delta = E$. 
Theorem 22. Checking strong positive definiteness of $A^S$ is co-NP-hard on a class of problems with $A_c \in \mathbb{Q}^{n \times n}$ symmetric positive definite and entrywise nonnegative, and $\Delta = E$.

Considering positive definiteness, we have some sufficient conditions that can be checked polynomially [39].

Theorem 23. An interval matrix $A^S$ is strongly positive definite if at least one of the following condition holds

- $\lambda_n(A_c) > \rho(\Delta)$,
- $A_c$ is positive definite and $\rho(||(A_c)^{-1}|\Delta|| < 1$.

The second condition can be reformulated as $A^S$ being regular and $A_c$ positive definite. If the first condition holds with $\geq$ then $A^S$ is strongly positive semidefinite.

In contrast to checking strong positive definiteness, weak positive definiteness can be checked in polynomial time by using semidefinite programming [15]; this polynomial result holds also for a more general class of symmetric interval matrices with linear dependencies [11]. For positive semidefiniteness it needn’t be the case since semidefinite programming methods work only with some given accuracy.

Summary.

| Problem                          | Complexity |
|----------------------------------|------------|
| Is $A^S$ strongly positive definite? | coNP-hard  |
| Is $A^S$ strongly positive semidefinite? | coNP-hard  |
| Is $A^S$ weakly positive definite? | P          |
| Is $A^S$ weakly positive semidefinite? | ?          |

1.4.9 Stability

The last section is dedicated to an important and more practical problem – deciding a stability of a matrix. There are many types of stabilities. For illustration, we chose two of them – Hurwitz and Schur.

Definition 19. An interval matrix $A$ is Hurwitz stable if every $A \in A$ is Hurwitz stable (i.e., all eigenvalues have negative real parts).

Similarly, we define Hurwitz stability for symmetric interval matrices. Due to their relation to positive definiteness ($A^S$ is Hurwitz stable if $-A^S$ is positive definite) we could presume that the problem is coNP-hard. It is so, even if we limit ourselves to a special case [38].
**Theorem 24.** Checking Hurwitz stability of a symmetric interval matrix $A^S$ is coNP-hard on a class of problems with $A_c \in \mathbb{Q}^{n \times n}$ symmetric Hurwitz stable and entrywise nonpositive, and $\Delta = E$.

For general matrices, coNP-hardness holds as well. The problem is still coNP-hard even if we limit the number of interval coefficients in our matrix [26].

**Theorem 25.** Checking Hurwitz stability of $A$ is co-NP-hard on a class of interval matrices with intervals in the last row and column only.

Likewise, as for checking regularity, also checking Hurwitz stability of $A$ can not be done by checking stability of matrices of type $A_{yz}$ (for reductions of other properties see [5]). On the other hand, it can be checked in this way for $A^S$. For more discussion and historical context see [22] or [46]. As sufficient conditions we can use conditions for positive definiteness applied to $-A$. For more sufficient conditions see e.g., [24].

**Definition 20.** An interval matrix $A$ is **Schur stable** if every $A \in A$ is Schur stable (i.e., $\rho(A) < 1$).

In a similar way, we define Schur stability for symmetric interval matrices. For general interval matrices, complexity of checking Schur stability is an open problem, however, for the symmetric case the problem is intractable [38].

**Theorem 26.** Checking Schur stability of $A^S$ is coNP-hard on a class of problems with $A_c \in \mathbb{Q}^{n \times n}$ symmetric Schur stable and offdiagonal entries nonpositive, and $\Delta = E$.

**Summary.**

| Problem             | Complexity   |
|---------------------|--------------|
| Is $A$ Hurwitz stable? | coNP-hard    |
| Is $A^S$ Hurwitz stable? | coNP-hard    |
| Is $A$ Schur stable? | ?            |
| Is $A^S$ Schur stable? | coNP-hard    |

1.4.10 **Further topics**

Due to the limited space, we had to omit many interesting topics. We touched only briefly the complexity issues of interval linear inequalities, but there are more results; see, e.g., [4, 9]. We did not discussed complexity of computing the range of polynomials over intervals [22], too. In short, we mention two particular problems:

- **Matrix power.** Computing the exact bounds on second power of the matrix $A^2$ is strongly polynomial (just by evaluating by interval arithmetic), but computing the cube $A^3$ turns out to be NP-hard [19].
• **Matrix norm.** Computing the range of \( \|A\| \) when \( A \in \mathbb{A} \) is a trivial task for vector \( \ell_p \)-norms applied on matrices (including Frobenius norm or maximum norm) or for induced 1- and \( \infty \)-norms. On the other hand, determining the largest value of the spectral norm \( \|A\|_2 \) (the largest singular value) subject to \( A \in \mathbb{A} \) is NP-hard \([20]\).

### 1.5 Summary

In this work we explored the fundamental problems of interval linear algebra. Our goal was to:

- provide a basic introduction to interval linear algebra
- answer elementary computational complexity questions linked with interval linear algebra
- discuss the computational complexity of the basic problems
- explain the relations between these problems
- mention relaxations or special classes of these problems that are easily decidable or there exist polynomial algorithms solving them
- provide a basis for further reading and research

At this place we also would like to apologize to those whose results are not mentioned in this work. There are many great achievements, however this work can unfortunately consume only limited amount of space. We provide links to the literature, where you can find much more of them.

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### References

1. S. Arora and B. Barak. *Computational complexity: a modern approach*. Cambridge University Press, 2009.
2. L. Blum, F. Cucker, M. Shub, and S. Smale. *Complexity and real computation*. Springer Science & Business Media, 2012.
3. G. E. Coxson. Computing exact bounds on elements of an inverse interval matrix is NP-hard. *Reliable Computing*, 5(2):137–142, 1999.
4. M. Fiedler, J. Nedoma, J. Ramík, J. Rohn, and K. Zimmermann. *Linear optimization problems with inexact data*. Springer, 2006.
5. J. Garloff, M. Adm, and J. Titi. A survey of classes of matrices possessing the interval property and related properties. *Reliab. Comput.*, 22:1–10, 2016.
6. G. H. Golub and C. F. Van Loan. *Matrix computations*. Johns Hopkins University Press, Baltimore, 3rd edition, 1996.
7. E. Hansen and G. Walster. Solving overdetermined systems of interval linear equations. *Reliable Computing*, 12(3):239–243, 2006.
8. M. Hladík. New operator and method for solving real preconditioned interval linear equations. *SIAM J. Numer. Anal.*, 52(1):194–206, 2014.
9. M. Hladík. AE solutions and AE solvability to general interval linear systems. *Linear Algebra Appl.*, 465(0):221–238, 2015.
10. M. Hladík. Complexity issues for the symmetric interval eigenvalue problem. *Open Math.*, 13(1):157–164, 2015.
11. M. Hladík. Positive semidefiniteness and positive definiteness of a linear parametric interval matrix. to appear in a Springer book series, 2016.
12. M. Hladík and J. Horáček. A shaving method for interval linear systems of equations. In R. Wyrzykowski et al., editor, *Parallel Processing and Applied Mathematics*, volume 8385 of LNCS, pages 573–581. Springer, 2014.
13. J. Horáček and M. Hladík. Computing enclosures of overdetermined interval linear systems. *Reliable Computing*, 19:143, 2013.
14. J. Horáček and M. Hladík. Subsquares approach – a simple scheme for solving overdetermined interval linear systems. In R. Wyrzykowski et al., editor, *Parallel Processing and Applied Mathematics*, volume 8385 of LNCS, pages 613–622. Springer, 2014.
15. L. Jaulin and D. Henrion. Contracting optimally an interval matrix without losing any positive semi-definite matrix is a tractable problem. *Reliab. Comput.*, 11(1):1–17, 2005.
16. L. Jaulin, M. Kieffer, O. Didrit, and E. Walter. Applied interval analysis. *With examples in parameter and state estimation, robust control and robotics*. Springer, London, 2001.
17. R. Kearfott. Interval computations: Introduction, uses, and resources. *Euromath Bulletin*, 2(1):95–112, 1996.
18. R. Kearfott and V. Kreinovich, editors. *Applications of interval computations*. Kluwer, Dordrecht, 1996.
19. O. Kosheleva, V. Kreinovich, G. Mayer, and H. Nguyen. Computing the cube of an interval matrix is NP-hard. In *Proceedings of the ACM Symposium on Applied Computing*, volume 2, pages 1449–1453, 2005.
20. R. Krawczyk. Newton-algorithmen zur bestimmung von nullstellen mit fehlerschranken. *Computing*, 4(3):187–201, 1969.
21. V. Kreinovich. How to define relative approximation error of an interval estimate: A proposal. *Appl. Math. Sci.*, 7(5):211–216, 2013.
22. V. Kreinovich, A. V. Lakeyev, J. Rohn, and P. Kahl. *Computational complexity and feasibility of data processing and interval computations*. Kluwer, Dordrecht, 1998.
23. J. Kuttler. A fourth-order finite-difference approximation for the fixed membrane eigenproblem. *Mathematics of Computation*, 25(114):237–256, 1971.
24. M. Mansour. Robust stability of interval matrices. In *Proceedings of the 28th IEEE Conference on Decision and Control*, volume 1, pages 46–51. Tampa, Florida, 1989.
25. R. Moore, R. Kearfott, and M. Cloud. *Introduction to interval analysis*. Society for Industrial Mathematics, 2009.
26. A. Nemirovskii. Several NP-hard problems arising in robust stability analysis. *Math. Control Signals Syst.*, 6(2):99–105, 1993.
27. A. Neumaier. Linear interval equations. In *Interval Mathematics 1985*, pages 109–120. Springer, 1986.
28. A. Neumaier. Interval methods for systems of equations, volume 37. Cambridge University Press, 1990.
29. W. Oettli and W. Prager. Compatibility of approximate solution of linear equations with given error bounds for coefficients and right-hand sides. *Numerische Mathematik*, 6(1):405–409, 1964.
30. E. D. Popova. Improved solution enclosures for over- and underdetermined interval linear systems. In I. Lirkov et al., editor, *Large-Scale Scientific Computing*, volume 3743 of LNCS, pages 305–312, 2006.
31. J. Renegar. On the computational complexity and geometry of the first-order theory of the reals. Part I: Introduction. Preliminaries. The geometry of semi-algebraic sets. The decision problem for the existential theory of the reals. *J. Symb. Comput.*, 13(3):255–299, 1992.
32. J. Renegar. On the computational complexity and geometry of the first-order theory of the reals. Part II: The general decision problem. Preliminaries for quantifier elimination. *J. Symb. Comput.*, 13(3):301–327, 1992.
33. J. Renegar. On the computational complexity and geometry of the first-order theory of the reals. Part III: Quantifier elimination. *J. Symb. Comput.*, 13(3):329–352, 1992.
34. G. Rex and J. Rohn. Sufficient conditions for regularity and singularity of interval matrices. *SIAM Journal on Matrix Analysis and Applications*, 20(2):437–445, 1998.
35. J. Rohn. Systems of linear interval equations. *Linear algebra and its applications*, 126:39–78, 1989.
36. J. Rohn. Interval matrices: Singularity and real eigenvalues. *SIAM journal on matrix analysis and applications*, 14(1):82–91, 1993.
37. J. Rohn. Inverse interval matrix. *SIAM Journal on Numerical Analysis*, 30(3):864–870, 1993.
38. J. Rohn. Checking positive definiteness or stability of symmetric interval matrices is NP-hard. *Commentat. Math. Univ. Carol.*, 35(4):795–797, 1994.
39. J. Rohn. Positive definiteness and stability of interval matrices. *SIAM Journal on Matrix Analysis and Applications*, 15(1):175–184, 1994.
40. J. Rohn. Checking properties of interval matrices. Technical Report 686, Institute of Computer Science, Academy of Sciences of the Czech Republic, Prague, 1996.
41. J. Rohn. Enclosing solutions of overdetermined systems of linear interval equations. *Reliable Computing*, 2(2):167–171, 1996.
42. J. Rohn. Perron vectors of an irreducible nonnegative interval matrix. *Linear Multilinear Algebra*, 54(6):399–404, 2006.
43. J. Rohn. Solvability of systems of interval linear equations and inequalities. pages 35–77, 2006.
44. J. Rohn. Forty necessary and sufficient conditions for regularity of interval matrices: A survey. *Electronic Journal of Linear Algebra*, 18(500-512):2, 2009.
45. J. Rohn. Explicit inverse of an interval matrix with unit midpoint. *Electronic Journal of Linear Algebra*, 22(1):8, 2011.
46. J. Rohn. A handbook of results on interval linear problems. Technical Report 1163, Institute of Computer Science, Academy of Sciences of the Czech Republic, Prague, 2012.
47. J. Rohn and R. Farhadsefat. Inverse interval matrix: a survey. *Electronic Journal of Linear Algebra*, 22(1):46, 2011.
48. S. M. Rump. Verification methods for dense and sparse systems of equations. In J. Herzberger, editor, *Topics in Validated Computations*, Studies in Computational Mathematics, pages 63–136, 1994.
49. S. P. Shary. On controlled solution set of interval algebraic systems. *Interval Computations*, 6(6), 1992.
50. S. P. Shary. A new technique in systems analysis under interval uncertainty and ambiguity. *Reliab. Comput.*, 8(5):321–418, 2002.
51. S. P. Shary. On full-rank interval matrices. *Numerical Analysis and Applications*, 7(3):241–254, 2014.
52. S. Smale. Mathematical problems for the next century. *Mathematical Intelligencer*, 20:7–15, 1998.