Dynamical Gauge Field induced by the Berry Phase Mechanism

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Abstract

Some part of the local gauge symmetries in the low energy region, say, lower than GUT or the Planck energy can be an induced symmetry describable with the holonomy fields associated with a topologically non-trivial structure of partially compactified space.

In the case where a six dimensional space is compactified by the Kaluza-Klein mechanism into a product of the four dimensional Minkowski space $M_4$ and a two dimensional Riemann surface with the genus $g$, $\Sigma_g$, we show that, in a limit where the compactification mass scale is sent to infinity, a model lagrangian with a U(1) gauge symmetry produces the dynamical gauge fields in $M_4$ with a product of $g$ U(1)'s symmetry, i.e., $U(1) \times \cdots \times U(1)$. These fields are induced by a Berry phase mechanism, not by the Kaluza-Klein. The dynamical degrees of freedom of the induced fields are shown to come from the holonomies, or the solenoid potentials, associated with the cycles of $\Sigma_g$. The production mechanism of kinetic energy terms for the induced fields are discussed in detail.

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1 Introduction

The gauge principle is one of the most fundamental theoretical principles in modern physics. In most of theories in particle physics one usually assumes a high gauge invariance in the Planck or GUT energy region and then breaks the symmetry into some lower one with the assumption of the Higgs mechanism. It is therefore considered natural that the symmetry of system becomes less symmetric as the energy decreases.

An extreme example of this type can be observed in the Kaluza-Klein theory[1]. The basic idea of the theory is to begin with a general diffeomorphism invariance in a higher dimensional space-time, then one derives some gauge symmetries in the four dimensional Minkowski space as a subgroup of the diffeomorphism.

In a previous paper[2], however, one of the authors pointed out that an opposite mechanism is possible, namely, the gauge symmetry or a part of gauge symmetry in low energy physics is able to be dynamically induced rather than given as a required principle. As a consequence, the gauge symmetry in low energy region could be higher than that in high energy region.

The purpose of this paper is to refine the idea of the dynamical gauge theory proposed in the previous paper and make it clear what sort of geometrical mechanism is working in the generation of gauge symmetry.

We begin with a U(1) gauge theory interacting with a fermion and the gravity in 6-dimensions, then assume that the space is compactified into the 4-dimensional Minkowski space $M_4$ times a 2-dimensional Riemann surface with genus $g$. We do not ask the detail of the compactification mechanism but assume that the Kaluza-Klein mechanism (KKM) has worked there. As a matter of fact a certain number of components of the metric fields in 6-dimensions are converted into the gauge potentials in $M_4$ by KKM, but we are not concerned with them. What we concern is another set of gauge fields converted from the
holonomies whose mechanism will be discussed in detail in the text.

In many cases the Berry phase potential have attracted physicist’s attention and some have tried to employ them as dynamical fields [3], but the difficult feature was how to generate the kinetic energy terms for them. In some non-linear or four-fermi interaction models the kinetic energy terms were generated by Feynman graph summations and the renormalizability was always dubious[4]. In our discussions therefore a central point is how to generate the kinetic energy terms for the induced gauge fields.

The dynamical degrees of freedom of the induced gauge fields are shown to originate in the solenoid potentials[3], or the holonomies[3] which are associated with the independent cycles of $\Sigma_g$. The kinetic energy terms for them are generated from those for the compactified 2-dimensional components of the vector potential prepared in the original lagrangian. The mechanism of generating $g$ sets of new kinetic energy terms are discussed in §4 and §5.

The basic mechanism of the induction is able to be best demonstrated in the case of the compactified space being $\Sigma_1$, i.e., a torus. In §2 we discuss the case of $\Sigma_1$ in detail. Although in this case the induced gauge field decouples from the matter field, the analysis shows why the gauge field is decoupled and what sort of modification is needed to get a non-trivial gauge field coupling with matter.

A simple but nontrivial model is shown to be the case where the compactified Riemann surface has the genus larger than 1. This is discussed in §5.

The last section is devoted to the discussions and further outlooks.

2 A U(1) Model

We study a model of U(1) gauge theory in 6 dimensions interacting with a fermion and gravity, and assume that the space-time is compactified into a four-dimensional Minkowski
space $M_4$ times a torus $\Sigma_1$ (a Riemann surface with genus 1). The compactification is assumed to be caused by the Kaluza-Klein mechanism. The relevant part of the lagrangian besides the gravity term is

$$L = \overline{\Psi} i \Gamma^A (\partial_A + ig' V_A) \Psi - \frac{1}{4} F_{AB}^2 + \cdots$$  \hspace{1cm} (2.1)$$

where $A$ and $B$ run from 0 to 5 and $\Gamma$'s are Dirac matrices defined by

$$\Gamma^A = \{ \Gamma^\mu = \gamma^\mu \otimes 1 \text{ for } \mu = 0, 1, 2, 3, \text{ and } \}$$

$$\Gamma^{\alpha+3} = \gamma^5 \otimes i \tau^\alpha \text{ for } \alpha = 1, 2$$  \hspace{1cm} (2.2)$$

in which the first factors of the direct products refer to the Minkowski space $M_4$ and the second, the Pauli matrices, to the compactified space $\Sigma_1$. A vector in $\Sigma_1$ is denoted with indices $\alpha (=1, 2)$, or frequently with a vector notation like $\vec{\tau} = (\tau^1, \tau^2)$. The two periods of the torus are both taken to be a constant value $l$. The typical size of the torus is characterized by the Planck length $l \equiv M^{-1}_P$. We further assume that the fermion $\Psi$ stays at the ground state level of the compactified modes with the wave function $u_a(x, \vec{y})$ ($a = 1, 2$), hence the full fields are represented as

$$\Psi(x, \vec{y}) = \frac{1}{l} \psi(x) \otimes u(x, \vec{y})$$  \hspace{1cm} (2.3)$$

$$V_A(x, \vec{y}) = \frac{1}{l} \{ V_\mu(x), V_\alpha(x, \vec{y}) \}$$  \hspace{1cm} (2.4)$$

where $\psi(x)$ stands for a Dirac spinor in $M_4$ and $\vec{y}$ for the coordinate vector in $\Sigma_1$. The constant $l^{-1}$ is multiplied to adjust the dimension. The wave function $u$ is normalized as

$$< u^\dagger u > \equiv \frac{1}{l^2} \int u^\dagger u \ d^2 y = 1.$$  \hspace{1cm} (2.5)$$
Substituting (2.3) and (2.4) into (2.1) and integrating over the compactified space coordinates, one obtains the effective lagrangian in the Minkowski space

\[
\mathcal{L} = \langle L \rangle = \bar{\psi}i\gamma^\mu[\partial_\mu + igV_\mu + iA_\mu]\psi - \frac{1}{4}F_{\mu\nu}^2 - \frac{1}{2}F_{\mu\alpha}^2 > - \frac{1}{4}F_{\alpha\beta}^2 > + i\bar{\psi}\gamma^5\psi \langle u^\dagger[i\vec{\tau} \cdot (\vec{\partial} + ig\vec{V})]u \rangle \tag{2.6}
\]

where \(A_\mu\) is the induced potential and defined by

\[
A_\mu = -i \langle u^\dagger \partial_\mu u \rangle \tag{2.7}
\]

and \(g = g'l^{-1}\). The vector \(A_\mu\) is well-known as Berry’s potential[3]. The lagrangian (2.6) now has an extra artificial gauge invariance under

\[
\psi(x) \rightarrow \psi(x)e^{i\theta(x)} \\
u(x) \rightarrow e^{-i\theta(x)}u .
\]

In \(\Sigma_1\), we look for such a classical solution for \(\vec{V}\) that provides a constant magnetic field perpendicular to the torus plane. The solution is

\[
\vec{V}_C = (By_2, 0). \tag{2.8}
\]

Taking account of the small (long wave) fluctuations around \(\vec{V}_C\) we choose the potential in \(\Sigma_1\) as

\[
\vec{V}(x, \vec{y}) = [v_1(x) + y_2B]\vec{e}_1 + v_2(x)\vec{e}_2. \tag{2.9}
\]

For this choice the third and fourth terms in \(\mathcal{L}\) become \(u\)-independent and the equation of motion for \(u\) can be derived from the stationary condition of (2.6) with the normalization (2.5) as

\[
Hu \equiv -i\vec{\tau} \cdot \left(\vec{\partial} + ig\vec{V}(x, \vec{y})\right)u = \epsilon u, \tag{2.10}
\]

4
where we have assumed that the interaction energy of $A_\mu$ with $\psi$ is small compared with the Planck energy as will be discussed below. The problem is now reduced to the well known Landau motion on a torus in a uniform magnetic field $B$. As one sees below (2.10) is exactly solvable and the fermion states are labeled by the Landau level number $n$.

Before considering the detail we give a comment on the last term in (2.6). If the eigenvalue $\epsilon$ of (2.10) is happened to be non-zero, the lagrangian acquires a mass term for $\psi$, because $\bar{\psi}\gamma^5\psi$ can be converted into $\bar{\psi}\psi$ by the chiral transformation $\psi \to \exp(i\frac{\pi}{4}\gamma^5)\psi$. The magnitude of the mass must be of order of the Planck mass because this is the only massive parameter involved in this model. To get low energy physics only the possibility is that the system must provide the zero-eigenvalue in (2.10). This is possible as one see below.

For the potential (2.9), the equation (2.10) is known to have solutions under the flux quantization (Dirac) condition

$$\Phi \equiv -Bl^2 = -\frac{2\pi}{g}N , \quad N = 0,1,2,\cdots$$

(2.11)

The eigenvalues are independent of the function $v(x)$, which is called the solenoid potential or the holonomy in mathematics. The wave function $u(\vec{y})$ on a torus is defined on its universal covering space. The uniqueness of wave function on the universal covering space provides us with the Dirac condition. The two wave functions $u(\vec{y})$ and $u(\vec{y}')$ at equivalent positions are usually different by phase factors, i.e., the solenoid potentials or the holonomies, which can depend on the external coordinates $x$ in $M_4$. The $x$-dependence of the solenoid potential plays the crucial role in the next section. The solutions are given for a fixed $N$ as follows

$$\epsilon_{\pm n} = \pm \sqrt{2gBn} , \quad n = 0,1,2,\cdots$$

(2.12)
\[ u_0^{(l)} = e^{-ig\vec{v} \cdot \vec{y}} \begin{pmatrix} \phi_0^{(l)} \\ 0 \end{pmatrix} \quad \text{for } \epsilon_0 = 0 \quad (2.13) \]

\[ u_{\pm n}^{(l)} = \frac{e^{-ig\vec{v} \cdot \vec{y}}}{\sqrt{2}} \begin{pmatrix} \phi_n^{(l)} \\ \mp i\phi_{n-1}^{(l)} \end{pmatrix} \quad \text{for } \epsilon_{\pm n} (n \neq 0). \quad (2.14) \]

and

\[ < u_{n}^{(l) \dagger} u_{n'}^{(l')} > = \delta_{nn'}\delta_{ll'} \quad (2.15) \]

where \( \phi_n^{(l)} (l = 0, 1, 2, \ldots, N - 1) \) are \( N \) degenerate eigenfunctions of the Landau problem for particles on a torus, whose explicit forms are found in ref.\[5\].

Before closing this section a comment is in order. As is seen from the Dirac condition (2.11) and (2.12), the eigenvalue is of order of the Planck mass and the non-zero eigenstates are not directly related with the low energy physics in the Minkowski space. Nevertheless, the excited states with \( n \neq 0 \) play an essential role in generating the induced fields as seen below.

### 3 Local Field Strength

Let us observe whether the Berry’s potential (2.7) has any dynamical degrees of freedom. In this and the next sections we assume \( N = 1 \).

Since the \( x \)-dependence of the spinor function \( u(x) \) occurs through only the solenoid potential \( \vec{v}(x) \), we can write \( A_\mu(x) \) as

\[ A_\mu(x) = \partial_\mu v^\alpha(x) A_\alpha(x) \quad (3.1) \]

where

\[ A_\alpha(x) = -i < u_0^{\dagger} \frac{\partial}{\partial u_0} u_0 > \equiv -i < u_0^{\dagger} \partial_\alpha u_0 >. \quad (3.2) \]

The induced field strength is then

\[ G_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) \]

\[ = \partial_\mu v^\alpha(x) \partial_\nu v^\beta(x) \hat{G}_{\alpha\beta} \quad (3.3) \]
with
\[ \hat{G}_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha \]
\[ = -i\{ <\partial_\alpha u_0^\dagger\partial_\beta u_0 > -(\alpha \leftrightarrow \beta) \}. \quad (3.4) \]

The calculation of \( \hat{G}_{\alpha\beta} \) is parallel to the Berry method\[8\], i.e., by substituting the complete set in the right hand side of (3.4), one obtains
\[ \hat{G}_{\alpha\beta} = -i \sum_{n \neq 0} \left[ <\partial_\alpha u_0^\dagger u_n > <u_n^\dagger \partial_\beta u_0 > -(\alpha \leftrightarrow \beta) \right] \]
\[ = -i \sum_{n \neq 0} \frac{<u_0^\dagger \partial_\alpha Hu_n > <u_n^\dagger \partial_\beta Hu_0 > -(\alpha \leftrightarrow \beta)}{(\epsilon_0 - \epsilon_n)^2} \quad (3.5) \]

where to get the last equality we have used
\[ <u_m^\dagger \partial_\alpha u_n > = \frac{<u_m^\dagger \partial_\alpha Hu_n >}{\epsilon_n - \epsilon_m} \quad (3.6) \]
which is easily derivable by operating \( \partial_\alpha \) on both sides of (2.10) and taking the matrix elements. The matrix elements of the numerator are given as
\[ <u_{\pm n}^\dagger \partial_\alpha Hu_o > = \left\{ \begin{array}{l} \pm i \frac{g}{\sqrt{2}} \delta_{n,1} \quad (\alpha = 1) \\ \mp \frac{g}{\sqrt{2}} \delta_{n,1} \quad (\alpha = 2) \end{array} \right. \quad \text{for } n > 0. \quad (3.7) \]

All transitions to the levels \( n \geq 2 \) are prohibited. Thus we get the exact result
\[ \hat{G}_{12} = -\hat{G}_{21} = \frac{g}{B}. \quad (3.8) \]

The field strength in \( x \)-space is therefore
\[ G_{\mu\nu} = \frac{g}{B}(\partial_\mu v_1(x)\partial_\nu v_2(x) - \partial_\nu v_1(x)\partial_\mu v_2(x)). \quad (3.9) \]

For the sake of confirmation we directly calculated the formula (3.4) with the use of explicit ground state wave function \( u_0 \), and obtained the same results.

As one sees in (3.9) the gauge field strength \( G_{\mu\nu}(x) \) has two independent local degrees of freedom \( v_1(x) \) and \( v_2(x) \), which are necessary and sufficient for an abelian field. It
may be instructive, at this point, to emphasize the role of the solenoid potential $\vec{v}(x)$. In the $v$-space the field strength is simply a constant as in (3.8). The role of the solenoid potential is to convert the constant induced field in $v$-space to the local field in $x$-space. When the vector potential $A_\mu(x)$ is line integrated along a closed curve $C$, $\vec{v}(x)$ draw a closed curve $C'$ in $v$-space on which the constant field $B$ is perpendicularly being applied. The flux picked up on the $v$-space is the flux in the $x$-space, which are now $x$-dependent (fig.1).

4 Effective Lagrangian in 4D

Now we argue the generation of the kinetic energy term for the induced field $G_{\mu\nu}(x)$. For our choice of classical solution (2.8) and (2.9), the fourth term $< F_{\alpha\beta}^2 >$ in the effective lagrangian (2.6) is simply a constant. It makes a contribution to the cosmological constant, and we disregard it here.

The third term is explicitly written as

$$< F_{\mu\alpha}^2 > = (\partial_\mu v_1(x))^2 + (\partial_\mu v_2(x))^2. \quad (4.1)$$

If (2.6) is regarded as the lagrangian for the independent field $\vec{v}(x)$, (4.1) is the kinetic energy term for the scalar fields $\vec{v}(x)$ and the interactions of them are taken place via $A_\mu$, which is a complicated functional of $\vec{v}$. We now want to write all of these terms in terms of $A_\mu$.

To do this we first suppose that a quantized theory exists with $\vec{v}$ fields. Then the local product of $v$-fields, say, such as

$$G^2_{\mu\nu}(x) = \frac{g^2}{B^2}(\partial_\mu v_1(x)\partial_\nu v_2(x) - \partial_\nu v_1(x)\partial_\mu v_2(x))^2 \quad (4.2)$$

may not be well defined unless any regularization is introduced. In our case one should remind the theory has a natural cut off $M_P$, the Planck mass, through the Kaluza-Klein
mechanism assumed at the beginning. In our Born-Oppenheimer approximation the four
dimensional coordinates $x^\mu$ are regarded as slow parameters, while the internal coordinates
$\vec{y}$ as fast parameters. The vacuum expectation values of the local product of two or more
operators in the $x$-space then may be regularized with the Planck mass parameter. For
instance, one can assume for local limit of operator products

$$\lim_{\Delta \to \nu = 1/M_P} < 0|\partial_\mu v^\alpha(x)\partial_\nu v^\beta(x + \Delta)|0 > \approx \frac{1}{4} M_P^4 \delta^{\alpha\beta} \eta_{\mu\nu}$$

(4.3)

where the Lorentz invariance in 4D space and the rotational invariance in the compactified
space have been assumed, and $M_P^4$ comes from the dimensional argument.

If (4.3) is used, the composite operator (4.2) is able to be decomposed into a sum of
normal ordered products,

$$G_{\mu\nu}(x) \approx \frac{g^2}{B^2} : (\partial_\mu v_1 \partial_\nu v_2 - \partial_\nu v_1 \partial_\mu v_2)^2 :$$

$$+ \frac{3 M_P^4}{2 B^2} : (\partial_\mu v_1)^2 + (\partial_\nu v_2)^2 : + \frac{3 M_P^8}{2 B^2}.$$  (4.4)

Now, let us take the limit of $M_P \to \infty$ provided that the dimensionless combination
$M_P^2 / B \approx \left(\frac{2\pi}{g} N\right)^{-1}$ fixed finite. Then the first term in the r.h.s. of (4.4) vanishes, and the
second term remains finite. The third is a large constant but should be absorbed into the
cosmological term. Comparing the result with (4.1) we can conclude

$$G_{\mu\nu}(x) = 2e^2 < F_{\mu\nu}^2 >$$

(4.5)

where $e$ is a dimensionless constant.

The effective lagrangian of our system in $M_4$, therefore, is expressed as follows

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^2 - \frac{1}{4} G_{\mu\nu}^2 + \bar{\psi} i \gamma^\mu [\partial_\mu + ie A_\mu + ig V_\mu] \psi.$$  (4.6)

In this simple model, however, the induced vector field decouples from the matter. Namely,
if one redefines fields as
\[
A^{(0)}_\mu = \frac{1}{\sqrt{e^2 + g^2}}(eA_\mu - gV_\mu)
\]
\[
A^{(1)}_\mu = \frac{1}{\sqrt{e^2 + g^2}}(gA_\mu + eV_\mu),
\] (4.7)
the lagrangian takes the form
\[
\mathcal{L} = -\frac{1}{4}F^{(0)2}_{\mu\nu} - \frac{1}{4}F^{(1)2}_{\mu\nu} + \bar{\psi}i\gamma^\mu(\partial_\mu + ifA^{(1)}_\mu)\psi
\] (4.8)
where \( f = \sqrt{e^2 + g^2} \).

At this point a couple of comments must be added on (4.8). The first is on the number of degrees of freedom for the induced potential \( A_\mu \). We have mentioned that the dynamical degrees of freedom are two if counted in terms of \( \vec{v} \). When the theory have been expressed with \( A_\mu \), extra two freedoms are implicitly added. The original theory is therefore equivalent to the gauge fixed version of the new lagrangian (4.8). The second is on the Jacobian factor coming from the variable change \((v_1, v_2) \rightarrow A_\mu\). The transformation is non-singular at \( \vec{v} = 0 \) because \( A_\mu \) vanishes as \( \vec{v} \) approaches zero as seen in (3.1). The term coming from Jacobian therefore may be expressed in a series of gauge invariant and local products of various fields such as
\[
aG^2_{\mu\nu} + \frac{b}{M_P}G_{\mu\nu}G_{\nu\rho}G_{\rho\mu} + \cdots,
\] (4.9)
where \( a, b, \cdots \) are dimensionless constants. As one takes a limit of \( M_P \) going to infinity, the second term and those having higher dimensions are negligible. The first term is able to be renormalized to the second term in (4.6), hence the effective action still takes the form of (4.8).

As we mentioned above, however, the \( \Sigma_1 \) model provides us with the decoupled gauge field model as seen in (4.8). The way out of the decoupling is given in the next section.
5 Interacting Gauge Models

In the previous section we derived an induced gauge field, which turned out to be decoupled from matter. The model, however, is instructive because it shows the geometrical mechanism of generating the local gauge field. It even suggests us how one is able to escape the decoupling trouble.

Let first observe why it decouples. In the previous model we chose a single ground state for the internal Hamiltonian (2.10) and adopted a simple product for the ground state

\[ Ψ = ψ(x)u(x, \vec{y}). \]  

(5.1)

The original theory is invariant under the U(1) gauge transformation under which

\[ Ψ \rightarrow e^{iχ}Ψ. \]  

(5.2)

The artificial gauge transformation associated (5.1) is

\[
\begin{aligned}
    ψ &\rightarrow ψ(x)e^{iθ(x)} \\
u &\rightarrow e^{-iθ(x)}u(x, \vec{y}).
\end{aligned}
\]  

(5.3)

Under these situation for the field \( ψ \), however, the theory is invariant even under

\[ ψ \rightarrow ψ(x)e^{i(θ+χ)} \]

namely, the artificial gauge can be absorbed into the original \( U(1) \) gauge transformation. This is the reason of decoupling.

A way out of this problem is therefore to introduce a degenerate set of ground states for the internal Hamiltonian (2.10). Let \( u^a (a = 1, 2, \cdots, n) \) be zero-energy eigenstates of \( H \). Then the ground state should be expressed as

\[ Ψ = \sum_{a=1}^{n} ψ^a(x)u^a(x, \vec{y}). \]  

(5.4)
This has an $n \times n$ unitary matrix $U(x)$ invariance

$$
u(x) \rightarrow U(x)u(x)$$

$$\psi(x) \rightarrow \psi(x)U^{-1}(x).$$

(5.5)

Then the $U(1)$ part of $U(x)$ is again absorbed into the original $U(1)$ gauge, but the other $\text{SU}(n)$ parts remain as new degrees of freedom.

The simplest model might be constructed even for the $\Sigma_1$ compactification. Choose the magnetic field $B$ stronger so that the degeneracy of zero-energy ground states is $N \geq 2$. For (5.4) we are able to introduce $N \times N$ vector potential

$$A^{ab}_\mu = -i < u^a \partial_\mu u^b > .$$

(5.6)

We have, however, found that this is essentially equivalent to the previous model because an explicit construction shows

$$A^{ab}_\mu = \delta^{ab} A_\mu, \quad A^1_\mu(x) = A^2_\mu(x) = \cdots = A^N_\mu(x) \equiv A_\mu(x).$$

(5.7)

The reason comes from the fact that, even if one chooses the magnetic field $B$ stronger, the number of independent solenoid potentials are still two, hence no way to produce more than single vector potential.

To get more solenoid potentials, we choose the compactified surface $\Sigma_g$, a Riemann surface with genus $g \geq 2$. The independent number of solenoid potentials, or holonomies, is $2g$ hence $g$ independent vector potentials $A^{(i)}_\mu$ are expected.

Next question is whether the massless spinors for (2.10) exist. For $g \geq 2$ if one so chooses the magnetic field $B$ that the curvature term is canceled, massless spinor solutions do exist. The Dirac index theorem

\[ \text{index } \mathcal{N} = n_+ - n_- = \frac{1}{2\pi} \int B \neq 0 \]
guarantees at least \((n_+ - n_-)\) massless stable solutions, where \(n_\pm\) represents the number of chiral spinors with chirality \(\pm1\). The number of massless spinors is chosen arbitrarily large by adjusting the strength \(B\) as discussed in ref. [10].

In the following we prepare \(g\) zero-mass spinors

\[
u^a \quad (a = 1, 2, \cdots, g)
\]

for the Riemann surface of genus \(g\). Even if one chooses larger numbers, the induced gauge potential may not be independent because the independent number of solenoid potentials are restricted as discussed above. In fact, even if more than \((5.8)\) are chosen not all of them are independent. We choose therefore the \(g\) diagonal components

\[
A^a_\mu(x) \equiv -i < u^{*a} \partial_\mu u^a >
\]

which are supposed to be independent. The independence of them are understood by the choice of cycles as a canonical way as shown in fig.2.

Since, for this choice, the vector potentials \((5.9)\) are all abelian, the field strengths are presented by the formula as \((3.9)\) for each superscript \(a\).

The crucial point in the generalized model is how to generate \(g\) set of kinetic energy terms out of \(< F_{\mu\alpha}^2 >\).

Theory of Riemann surfaces[3] tells us that one can always construct \(g\) sets of harmonic 1-form basis \(\omega_i\) and anti-harmonic basis \(\bar{\omega}_i\) which satisfy

\[
\int_{a_i} \omega_j(\vec{y}) = \delta_{ij} \\
\int_{b_i} \omega_j(\vec{y}) = \Omega_{ij} \quad (i, j = 1, 2, \cdots, g)
\]

where \((a_i, b_i)\) represent canonical cycles and \(\Omega_{ij}\) is the period \((g \times g)\) matrix, which is symmetric and has positive imaginary parts.
Now, as the vector potential on $\Sigma_g$, we introduce the following 1-form

$$V(x, \vec{y}) = v^{(0)}(\vec{y}) + \xi(x, \vec{y})$$  \hspace{1cm} (5.11)

where the 1-form potential $v^{(0)}(\vec{y})$ provides a constant magnetic flux $B$ perpendicular to the surface, and

$$\xi = \frac{g}{\sum_{i=1}^{g}} v^{(i)}(x) \omega_i, \quad \bar{\xi} = \frac{g}{\sum_{i=1}^{g}} \overline{v^{(i)}}(x) \overline{\omega_i}$$  \hspace{1cm} (5.12)

which are curl free in $\vec{y}$ space.

Then the kinetic energy term $< F_{\mu \alpha}^2 >$ on $\Sigma_g$ is given by

$$< F_{\mu \alpha}^2 > \sim \frac{i}{2} \int \partial_{\mu} \xi \wedge \partial_{\mu} \bar{\xi} = \text{Im}\Omega_{ij} \partial_{\mu} \overline{v}^{(i)}(x) \partial_{\mu} v^{(j)}(x).$$  \hspace{1cm} (5.13)

The first term $v^{(0)}$ in (5.11) makes no contribution because of its $x$-independence. Owing to the positivity of the period matrix $\text{Im}\Omega$, one can diagonalize (5.13) by some linear transformation and gets

$$< F_{\mu \alpha}^2 > \sim \sum_{i=1}^{g} [(\partial_{\mu} \tilde{v}^{(i)})^2 + (\partial_{\mu} \tilde{\nu}^{(i)})^2]$$  \hspace{1cm} (5.14)

where $\tilde{v}$ are linear combinations of $v$'s. The relation (5.14) guarantees the generation of a set of kinetic energy terms for the induced potentials (5.9).

As in the case of $\Sigma_1$, we have an original $\text{U}(1)$ vector field, and we have generated a set of abelian vector fields. One of these fields decouples from the fermions as before, and others couple. After some orthogonalizations for the gauge potentials we finally obtain the following lagrangian in the limit of $M_P \to \infty$,

$$\mathcal{L} = - \sum_{i=0}^{g} (F_{\mu \nu}^{(i)})^2 + \sum_{i=1}^{g} \bar{\psi}^{(i)} i\gamma^\mu [\partial_{\mu} + ig T_i V^{(i)}] \psi^{(i)}$$  \hspace{1cm} (5.15)

where $T_i$ ($i = 1, 2, \cdots, g - 1$) are $g \times g$ matrices which are traceless, diagonal and mutually orthogonal, and $T_g$ is a unit matrix.
The induced gauge theory has generated a set of new quantum numbers for fermions which couple with $g$-independent currents.

6 Comments and Conclusion

We have demonstrated a new mechanism of induced gauge theory. We start off with a higher dimensional space and assume that a part of the space is compactified in a topologically non-trivial way by the Kaluza-Klein mechanism. In the limit that the compactification scale ratio, say, the Planck mass divided by the observable mass scale is sent to infinity, two kinds of local gauge symmetries are expected. The first is the well-known Kaluza-Klein gauge field, which is induced from the compactified space components of the metric tensor of original space-time. The second is our gauge fields discussed in the text. The fields are generated by the local holonomies associated with the cycles of topologically non-trivial compactified space. The mechanism of gauge generation is the Berry phase effect. If the quantum states in the compact space produce vanishing mass for particles in the four dimensional Minkowski space $M_4$, the quantum states in $M_4$ are described by the form (5.4). The Berry phase effect then picks up the $x$-dependent holonomy fields and provides a set of local gauge fields.

Although we demonstrated an induction of abelian gauge fields, it will be straightforward to induce non-abelian gauge if one chooses a compactified space with non-abelian holonomies. This problem will be discussed in a future work.

In closing our paper we give some comments. The first is about a possibility of generating a new massive scale in the low energy physics. In our arguments we have disregarded the interactions between fermion modes ($u$-fields) and assumed that all energy levels are degenerated in the compactified space. If the interactions among $u$-fields are existed, it may be possible to introduce another scale factor due to the interaction energy.
and generate mass splittings among fermions.

Final comment is about the gauge symmetries of our world. Once people devoted some time to the study of the Kaluza-Klein theory to associate all gauge symmetries to the structure of compactified space. However, in so far as other mechanisms are shown to be possible for generation of gauge symmetries, the problem must be reconsidered. One can expect richer gauge structures from a simpler geometry.

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Figure Captions

**Fig.1** Mechanism of picking up the local field $F_{\mu\nu}(x)$.

**Fig.2** A canonical holonomy basis in $\Sigma_g$. 
This figure "fig1-1.png" is available in "png" format from:

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