Stability analysis of flow and heat transfer over a permeable stretching/shrinking sheet with internal heat generation and viscous dissipation

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Abstract. In this study, a steady two-dimensional boundary layer flow and heat transfer in an incompressible viscous fluid over a stretching/shrinking sheet with suction, internal heat generation and viscous dissipation is studied. The governing equations are first reduced to the nonlinear ordinary differential equations using a set of similarity variables. The obtained equations are then solved numerically using the bvp4c function in MATLAB. It is found that the nonlinear ordinary differential equations have dual (first and second) solutions in a certain range of the suction parameter. Stability analysis is employed to test the stability of the dual solutions. The results indicate that the first solution is stable while the second solution is unstable.

1. Introduction

Over the past few years, the research involving flow and heat transfer regarding to a stretching/shrinking sheet is important in the manufacturing industry. For example, manufacturing rubber sheets and plastic, metal and polymer working processes, paper production, etc. Preliminary work on the viscous flow past a stretching sheet was undertaken by Crane [1]. Later, Gupta and Gupta [2] continued this study to a permeable stretching sheet. On the contrary, Miklavčič and Wang [3] were the first who studied two-dimensional viscous flow by considering the shrinking sheet with suction and obtained that the solutions are not unique. Since then, a number of papers have been published to study the flow and heat transfer over a stretching/shrinking sheet with suction in a viscous fluids or non-Newtonian fluids such as Arifin et al. [4], Bhattacharyya [5], Roşca and Pop [6] and Naganthran and Nazar [7], among others.

This study is mainly focus on extending the research by Vajravelu and Hadjinicolau [8] with consideration of the existence of the dual (first and second) solutions in a specific range of the suction parameter. To this end, few studies have been made to find the dual solutions (see [9]-[14] for more details). In the present study, the stability of the dual (first and second) solutions was determined by
performs the stability analysis. Some figures are plotted as well as the characteristics and effects of the suction parameter are discussed further.

2. Governing equations
Consider the two-dimensional flow over a stretching/shrinking sheet in an incompressible viscous fluid with suction, internal heat generation/absorption and viscous dissipation. It is assumed that the velocity and temperature of the stretching/shrinking sheet is in the form

\[ u = ax, \quad T = T_x + bx^2l^{-2}, \]

respectively. Here, \( a \) and \( b \) are constants and \( l \) is the characteristic length. The equations of the present model are as follows (see [8]):

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (1) \]

\[ \frac{u \partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2}, \quad (2) \]

\[ \frac{u \partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{1}{\rho C_p} \left[ k \frac{\partial^2 T}{\partial y^2} + \mu \left( \frac{\partial u}{\partial y} \right)^2 + Q_o(T - T_x) \right], \quad (3) \]

where \( u \) and \( v \) are the velocity components corresponding to the \( x \)- and \( y \)-directions, respectively, \( C_p \) denotes the specific heat at constant pressure, \( \nu \) denotes the kinematic viscosity, \( k \) denotes the thermal conductivity, \( \mu \) denotes the dynamic viscosity, \( T \) denotes the fluid temperature, \( T_\infty \) denotes the ambient temperature, \( \rho \) denotes the fluid density and \( Q_0 \) is the volumetric rate of heat generation. The boundary conditions are

\[ u = cu_w, \quad v = v_0, \quad T = T_w \quad \text{at} \quad y = 0, \]

\[ u \to 0, \quad T \to T_\infty \quad \text{as} \quad y \to \infty. \quad (4) \]

Here \( c \) denotes the stretching/shrinking parameter with \( c = 1 \) for stretching sheet and \( c = -1 \) for shrinking sheet, respectively, \( v_0 \) is the mass velocity with \( v_0 > 0 \) for injection and \( v_0 < 0 \) for suction.

We introduce the similarity variables as follows:

\[ u = ax f(\eta), \quad v = -(av)^{1/2} f(\eta), \]

\[ \theta(\eta) = (T - T_\infty)/(T_w - T_\infty), \quad \eta = y(a\nu)^{1/2}. \quad (5) \]

By applying (5), equation (1) is identically satisfied while equations (2) and (3) reduce to

\[ f'''' + f f'' - (f')^2 = 0, \quad (6) \]

\[ \frac{1}{\Pr} \theta'' + (\alpha - 2f') \theta + f \theta' + Ec \left( f'' \right)^2 = 0. \quad (7) \]

Corresponding boundary conditions are

\[ f(\eta) = s, \quad f'(\eta) = c, \quad \theta(\eta) = 1 \quad \text{at} \quad \eta = 0, \]

\[ f'(\eta) \to 0, \quad \theta(\eta) \to 0 \quad \text{as} \quad \eta \to \infty. \quad (8) \]

Here \( \Pr = \mu C_p k^{-1} \) represents the Prandtl number, \( Ec = a^2 l^2 b^1 C_p^{-1} \) denotes the Eckert number, \( \alpha = Q_o a^{-1} \rho^{-1} C_p^{-1} \) represents the heat source/sink parameter and \( s = -v_0(a\nu)^{-1/2} \) denotes the mass transfer parameter with \( s < 0 \) for injection and \( s > 0 \) for suction.

The important physical quantities are the skin friction coefficient \( C_f \) and the local Nusselt number \( Nu_s \) that can be defined as follows:

\[ C_f = \frac{\tau_w}{\rho u_w^2}, \quad Nu_s = \frac{\chi q_s}{k(T_w - T_\infty)}. \quad (9) \]
where $q_w$ and $\tau_w$ represent the wall heat flux and the wall shear stress, respectively, as below

$$\tau_w = \mu \left( \frac{\partial u}{\partial y} \right)_{y=0}, \quad q_w = -k \left( \frac{\partial T}{\partial y} \right)_{y=0}. \tag{10}$$

Using (5), we obtain

$$\text{Re}^{1/2} C_f = f''(0), \quad \text{Re}^{1/2} \text{Nu}_w = -\theta'(0), \tag{11}$$

where $\text{Re}_w = u_w x_w v^{-1}$ denotes the local Reynolds number.

### 3. Stability analysis

For the purpose of stability analysis (Merkin [9] and Weidman et al. [10]), unsteady equations have been considered. Equation (1) holds while equations (2) and (3) are substituted as follows:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2}, \tag{12}$$

$$\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{1}{\rho C_p} \left[ k \frac{\partial^2 T}{\partial y^2} + \mu \left( \frac{\partial u}{\partial y} \right)^2 + Q_o(T - T_w) \right], \tag{13}$$

where $t$ denotes the time. Following [9] and [10], we use the new similarity variables, as follow:

$$u = ax f(\eta, \tau), \quad v = -(a/\nu)^{1/2} f(\eta, \tau), \quad \theta(\eta, \tau) = (T - T_w)/(T_u - T_w), \quad \eta = (a/\nu)^{1/2} y, \quad \tau = at. \tag{14}$$

Using (14), equations (12) and (13) take the form

$$f'''' + f f'' - \left( f' \right)^2 - f' = 0, \tag{15}$$

$$\frac{1}{\text{Pr}} \theta'' + \left( \alpha - 2 f' \right) \theta + f \theta + \text{Ec} \left( f'' \right)^2 - \theta = 0, \tag{16}$$

and the boundary conditions are as follows:

$$f(\eta, \tau) = s, \quad f'(\eta, \tau) = c, \quad \theta(\eta, \tau) = 1 \text{ at } \eta = 0, \quad f(\eta, \tau) \to 0, \quad \theta(\eta, \tau) \to 0 \text{ as } \eta \to \infty. \tag{17}$$

Following Weidman et al. [10], to determine the stability of $f(\eta) = f_0(\eta)$ as well as $\theta(\eta) = \theta_0(\eta)$, which satisfy the equations (6) and (7) subject to the boundary equations (8), we express

$$f(\eta, \tau) = f_0(\eta) + e^{\gamma \tau} F(\eta, \tau), \quad \theta(\eta, \tau) = \theta_0(\eta) + e^{\gamma \tau} G(\eta, \tau). \tag{18}$$

Here $F(\eta, \tau)$ and $G(\eta, \tau)$ are functions and $\gamma$ is an unknown eigenvalue. Substituting (18) into equations (15)-(16) and the use of equations (6) and (7), we obtained the following equations:

$$F''' + f_0 F'' + f_0' F - 2f_0' - \gamma F''' - F' - F = 0, \tag{19}$$

$$\frac{1}{\text{Pr}} G'' + f_0 G' + (\alpha - 2 f_0' + \gamma) G + 2 \text{Ec} f_0'' F'' - 2 \theta_0 F' + \theta_0' F - G = 0, \tag{20}$$

subject to the boundary conditions

$$F(\eta, \tau) = 0, \quad F'(\eta, \tau) = 0, \quad G(\eta, \tau) = 0 \text{ at } \eta = 0, \quad F(\eta, \tau) \to 0, \quad G(\eta, \tau) \to 0 \text{ as } \eta \to \infty. \tag{21}$$

As noted by Weidman et al. [10], we test the stability of $f_0(\eta)$ and $\theta_0(\eta)$ by setting $\tau = 0$, and hence the solutions $F = F_0(\eta)$ and $G = G_0(\eta)$ in equations (19) and (20) are used to identify initial decay or growth of the solution (18). Equations (19) and (20) become
subject to the boundary conditions

\[ F_0(\eta) = 0, \quad F_0'(\eta) = 0, \quad G(\eta) = 0 \quad \text{at} \quad \eta = 0, \]

\[ F_0'(\eta) \rightarrow 0, \quad G_0(\eta) \rightarrow 0 \quad \text{as} \quad \eta \rightarrow \infty. \]

To establish whether the flow is stable or unstable, if \( \gamma_1 > 0 \), then there is an initial decay of disturbances and proof that the flow is stable. Meanwhile, if \( \gamma_1 < 0 \), then there is an initial growth of disturbances and hence corresponds to unstable flow. Here, \( \gamma_1 \) is the smallest eigenvalue. Harris et al. [11] suggested that the range of the possible eigenvalues can be investigated by relaxing an appropriate boundary condition. Thus, we relax the condition \( F_0'(\eta) \rightarrow 0 \) as \( \eta \rightarrow \infty \) and solve the system of linearized problem (22) and (24) along with the new condition \( F_0''(0) = 1 \).

4. Results and discussion

Numerical solutions for equations (6) and (7) with the boundary conditions (8), have been obtained for several values of the parameters \( s, c, \alpha \) and \( Ec \) using the bvp4c function in MATLAB. We compared our results for the values of \(-\theta'(0)\) with those of Mohammadein and Gorla [15]. Table 1 shows the comparison of \(-\theta'(0)\) and it is found to be in excellent agreement. Therefore, this is assuring that our results are correct and accurate.

| \( \alpha \) | \( Ec \) | Mohammadein and Gorla [15] | Present study |
|---|---|---|---|
| -0.1 | 0 | 1.19261 | 1.19261 |
| 0.01 | | 1.18969 | 1.18969 |
| 0.5 | | 1.04617 | 1.04617 |
| 0 | 0 | 1.09995 | 1.09995 |
| 0.01 | | 1.09690 | 1.09690 |
| 0.5 | | 0.94737 | 0.94737 |

| \( s \) | First solution | Second solution |
|---|---|---|
| 2.1 | 0.4958 | -0.2732 |
| 2.5 | 2.3593 | -0.5619 |
| 3 | 3.1349 | -0.6671 |

The variations of the skin friction coefficient, \( f''(0) \) as well as the local Nusselt number, \(-\theta'(0)\) with \( s \) when \( Pr = 1, \alpha = -0.1, Ec = 0.5 \) and \( c = -1 \) (shrinking case) are presented in figures 1 and 2, respectively. Interestingly, we can see that dual (first and second) solutions exist for \( s > 2 \), and no solution exists for \( s < 2 \). The study uses stability analysis in order to test the stability of the dual solutions. Following this, we find the smallest eigenvalue \( \gamma_1 \). Table 2 provides the smallest eigenvalues \( \gamma_1 \) for several values of \( s \) when \( c = -1, Pr = 1, \alpha = -0.1 \) and \( Ec = 0.5 \). It is apparent from this table that \( \gamma_1 > 0 \) for the first solution and \( \gamma_1 < 0 \) for the second solution. Hence the first solution is stable, while the second solution is unstable. It is important to highlight that figure 1 displays the...
values of $f''(0)$ for the first solution which are usually higher than the second solution. Conversely, figure 2 reveals that the values of $-\theta'(0)$ for the first solution are lower than the second solution when $s > 2.27$, while the values of $-\theta'(0)$ for the first solution are higher than the second solution when $2 < s < 2.27$.

Figure 1. Variation of $f''(0)$ with $s$ when $Pr = 1$, $\alpha = -0.1$, $Ec = 0.5$ and $c = -1$.

Figure 2. Variation of $-\theta'(0)$ with $s$ when $Pr = 1$, $\alpha = -0.1$, $Ec = 0.5$ and $c = -1$.

Figure 3. Velocity profiles for various values of $s$ when $Pr = 1$, $\alpha = -0.1$, $Ec = 0.5$ and $c = -1$. 
Finally, figures 3 and 4 illustrate the velocity profiles $f'(\eta)$ and temperature profiles $\theta(\eta)$ against $\eta$ for different values of $s$ when $Pr = 1$, $\alpha = -0.1$, $Ec = 0.5$ and $c = -1$. It can be seen that the second solutions for these profiles display a larger boundary layer thickness compared to the first solutions. The enhancement of the value of $s$ increases the velocity profiles for first solution and decreases for second solution, while it decreases the temperature profiles for both first and second solutions.

5. Conclusions
We have numerically investigated the problem of a steady flow and heat transfer in a viscous fluid over a stretching/shrinking surface with suction, internal heat generation/absorption and viscous dissipation. The governing boundary layer equations were solved numerically by utilizing the bvp4c function in MATLAB. Dual (first and second) solutions were obtained in a specific range of the suction parameter. The stability analysis was performed in order to test the stability of the dual solutions. The results showed that the first solution is in stable state, while the second solution is in unstable state.

Acknowledgement
The first author would like to express appreciation to the Universiti Pertahanan Nasional Malaysia for supporting the research work.

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