Long Range Scattering and Modified Wave Operators for some Hartree Type Equations*

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Abstract

We study the theory of scattering for a class of Hartree type equations with long range interactions in space dimension \( n \geq 3 \), including Hartree equations with potential \( V(x) = \lambda |x|^{-\gamma} \) with \( \gamma < 1 \). For \( 1/2 < \gamma < 1 \) we prove the existence of modified wave operators with no size restriction on the data and we determine the asymptotic behaviour in time of solutions in the range of the wave operators.

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1 Introduction

In this paper, we study the theory of scattering and more precisely the existence of modified wave operators for a class of long range Hartree type equations

\[ i\partial_t u + \frac{1}{2}\Delta u = g(|u|^2)u \] (1.1)

where \( u \) is a complex function defined in space time \( IR^{n+1} \), \( \Delta \) is the Laplacian in \( IR^n \), and

\[ g(|u|^2) = \lambda t^{-\gamma} \omega^{-n} |u|^2 \] (1.2)

with \( \omega = (-\Delta)^{1/2}, \lambda \in IR, 0 < \gamma < 1 \) and \( 0 < \mu < n \). The operator \( \omega^{-n} \) can also be represented by the convolution in \( x \)

\[ \omega^{-n} f = C_{n,\mu} |x|^{-\mu} * f \] (1.3)

so that (1.2) is a Hartree type interaction with potential \( V(x) = C|x|^{-\mu} \). The more standard Hartree equation corresponds to the case \( \gamma = \mu \). In that case, the nonlinearity \( g(|u|^2) \) becomes

\[ g(|u|^2) = V * |u|^2 = \lambda |x|^{-\gamma} * |u|^2 \] (1.4)

with a suitable redefinition of \( \lambda \).

A large amount of work has been devoted to the theory of scattering for the Hartree equation (1.1) with nonlinearity (1.4) as well as with similar nonlinearities with more general potentials \( V \) [8, 9, 11-17, 19-21, 25]. As in the case of the linear Schrödinger equation, one must distinguish the short range case, corresponding to \( \gamma > 1 \), from the long range case corresponding to \( \gamma \leq 1 \). Fairly satisfactory results exist in the short range case. In particular it is known that the (ordinary) wave operators exist in suitable function spaces for \( \gamma > 1 \) [25]. Furthermore for repulsive interactions, namely for \( \lambda \geq 0 \), it is known that all solutions in suitable spaces admit asymptotic states in \( L^2 \) for \( \gamma > 1 \), and that asymptotic completeness holds in suitable spaces for \( \gamma > 4/3 \) [20]. In the long range case \( \gamma \leq 1 \), the ordinary wave operators are known not to exist in any reasonable sense [20], and one expects that they should be replaced by modified wave operators including a suitable phase in their definition, as is the case for the linear Schrödinger equation. A well developed theory of long range scattering exists for the latter. See for instance
for a recent and comprehensive treatment and for an extensive bibliography. In contrast with that situation, only preliminary results are available for the Hartree equation (and even less results for the more difficult nonlinear Schrödinger (NLS) equation), all of them restricted to the case of small solutions. On the one hand, the existence of modified wave operators has been proved in the critical case $\gamma = 1$ for small solutions [8]. On the other hand, it has been shown recently, first in the critical case $\gamma = 1$ [12, 15] and then in the whole range $\gamma < 1$ [11, 13, 14] that the global solutions of the Hartree equation (1.1) (1.3) with small initial data exhibit an asymptotic behaviour as $t \to \pm \infty$ of the expected scattering type characterized by scattering states $u_{\pm}$ and including suitable phase factors that are typical of long range scattering. In particular, in the framework of scattering theory, the results of [11, 13, 14] are just the property of asymptotic completeness for small data.

In the simpler case of short range interactions, it is a fact of experience that for the Hartree equation as well as for other equations such that the NLS equation or the nonlinear wave (NLW) equation, asymptotic completeness for small data can be proved together with and by the same method as the existence of the wave operators with no size restriction on the data. The results of [11, 13, 14] therefore suggest that the same method that has been used there to prove asymptotic completeness for small data can be used to prove the existence of suitably modified wave operators, again with no size restriction on the data. It is the purpose of the present paper to explore that possibility in the case of the Hartree type equation (1.1) (1.2).

The special choice of nonlinearity (1.2) has been dictated by the following reasons. In one direction, at the present preliminary stage of long range nonlinear scattering theory, it seems more appropriate to look for the basic facts on a rather specific example rather than on a general class of nonlinearities of the type (1.4) with $V$ replaced by some general function satisfying suitable smoothness and decay conditions. In the opposite direction, the strictly Hartree interaction (1.4) has the drawback that one single parameter $\gamma$ serves two unrelated purposes. On the one hand, it characterizes the long distance behaviour of the interaction, leading in particular to the distinction between short range and long range cases. On the other hand, it characterizes the local regularity of the interaction, and as a consequence the local regularity that is required for the solutions. In order to avoid that confusion, we consider the time dependent interaction (1.2) with two independent parameters $\gamma$ and $\mu$. With that choice
it turns out that $\gamma$ again characterizes the long distance or equivalently long time behaviour, whereas $\mu$ characterizes the local regularity. Finally the nonlinearity (1.2) is covariant under dilations, which is an important simplifying property.

The main result of the present paper is the existence of modified wave operators for the equation (1.1) (1.2), together with a description of the asymptotic behaviour in time of solutions in the ranges of those operators, with no size restriction on the data, in suitable spaces and for suitable values of $\gamma$ and $\mu$. The method is an extension of the energy method used in [11, 13, 14], and uses in particular the equations introduced in [13] to study the asymptotic behaviour of small solutions. The spaces of initial data, namely in the present case of asymptotic states, are Sobolev spaces of finite order similar to those used in [14]. The parameter $\mu$ characterizing the regularity of the interactions has to satisfy the condition $\mu \leq n-2$ and $\mu < 2$. The condition $\mu \leq n-2$ is the really important one and is needed for the treatment of the problem in a neighborhood of infinity in time. It restricts the theory to space dimension $n \geq 3$. That condition and consequently the restriction $n \geq 3$ could most probably be relaxed at the cost of using more complicated spaces such as those used in [11, 13]. The condition $\mu < 2$ is imposed in order to make contact with the available treatments of the equation (1.1) at finite times [1, 18, 25]. It cannot be avoided in the attractive case $\lambda \leq 0$, whereas it can most probably be relaxed to $\mu < \min(4, n)$ in the repulsive case $\lambda \geq 0$. The parameter $\gamma$ characterizing the long distance/long time behaviour of the interaction will have to satisfy $1/2 < \gamma < 1$. The critical case $\gamma = 1$ can be treated easily by the same methods, but is excluded here in order to simplify the exposition, since it would require different formulas involving $\text{ln} t$ instead of $t^{1-\gamma}$. The case $\gamma \leq 1/2$ can be treated by the methods of this paper, but it is more complicated and requires a more careful analysis of the asymptotic behaviour of the solutions of (1.1). It will be deferred to a subsequent paper.

The construction of the modified wave operators is too complicated to allow for a more precise statement of the results at the present stage. That construction will be described in heuristic terms in Section 2 below. It involves in particular the study of an auxiliary system of equations involving a new function $w$ and a phase $\varphi$ instead of the original function $u$ and the construction of local wave operators in a neighborhood of infinity for that system. After collecting some notation and a number of preliminary estimates in Section 3, we shall study the
local Cauchy problem at finite times for the auxiliary system in Section 4. We shall then study
the Cauchy problem at infinity and the asymptotic behaviour of solutions for the auxiliary
system in Sections 5 and 6. In particular we shall essentially construct local wave operators
at infinity for that system. We shall then come back from the auxiliary system to the original
equation (1.1) for $u$ and construct the wave operators for the latter in Section 7, where the final
result will be stated in Proposition 7.5. A more detailed description of the technical sections 3 - 7 will be given at the end of Section 2. The reader who wants to get quickly to the heart of
the matter is invited to read Section 2, to skip most of Section 3 except for the notation at the
beginning and for the definition of admissibility (Definition 3.1) to look at Proposition 4.1 and
skip its proof, and to proceed to Section 5 where the main construction starts.

We conclude this section with some general notation which will be used freely throughout
this paper. We denote by $\| \cdot \|_r$ the norm in $L^r \equiv L^r(\mathbb{R}^n)$. For $1 \leq r \leq \infty$, we define
$\delta(r) = n/2 - n/r$. For any interval $I$ and any Banach space $X$, we denote by $\mathcal{C}(I, X)$ the space
of strongly continuous functions from $I$ to $X$ and by $L^\infty(I, X)$ (resp. $L^\infty_{loc}(I, X)$) the space of
measurable essentially bounded (resp. locally essentially bounded) functions from $I$ to $X$. For
real numbers $a$ and $b$, we use the notation $a \lor b = \text{Max}(a, b)$, $a \land b = \text{Min}(a, b)$ and $[a] = \text{integral part of } a$. Finally if $(p \cdot q)$ is the numbering of a double inequality, we denote by $(p \cdot qL)$ and
$(p \cdot qR)$ the left hand and right hand inequality of $(p \cdot q)$ respectively.

Additional notation will be given at the beginning of Section 3.

2 Heuristics.

In this section, we discuss in heuristic terms the construction of the modified wave operators
for the equation (1.1), as it will be performed below in this paper. The discussion applies to any
Schrödinger like equation of the type (1.1) where $\tilde{g}(|u|^2)$ is real and depends on $u$ only through
$|u|$. In addition $\tilde{g}$ may also at this stage depend on $x$ and $t$. For instance $\tilde{g}$ can be independent
of $u$ and depend only on $(x, t)$, thereby leading for (1.1) to a linear Schrödinger equation with
time dependent potential ; $\tilde{g}$ can be given by (1.4) or (1.2), thereby leading to the Hartree or
Hartree-type equation considered here ; $\tilde{g}$ can be a local function of $|u|^2$ and possibly $t$ such as

$$\tilde{g}(|u|^2) = \lambda t^{\mu-\gamma} |u|^{2\mu/n}$$

(2.1)
thereby leading to a NLS equation with power nonlinearity, and with the parameters $\gamma$ and $\mu$ playing the same role as in (1.2). Whatever the case, we assume that the Cauchy problem for (1.1) is globally well posed at finite time, namely that for any $t_0 \in IR$ and any $u_0$ in a suitable space, (1.1) has a unique solution $u$ with $u(t_0) = u_0$, defined for all $t$ in $IR$ and depending continuously on $u_0$ in suitable norms.

The problem addressed by scattering theory is first of all that of classifying the possible asymptotic behaviour of the global solutions of (1.1) by relating them to a set of model functions with suitably chosen and preferably simple asymptotic behaviour. The first question to be considered is then the following one. For each function $v$ of the previous set, construct a solution $u$ of (1.1) such that $u(t)$ behaves as $v(t)$ when $t \to +\infty$, for instance in the sense that $u(t) - v(t)$ tends to zero when $t \to +\infty$ in suitable norms. A similar question can be asked for $t \to -\infty$. From now on we restrict our attention to the case of positive times.

A natural method to attack the previous question is the following one. Let $v$ be a fixed model function. Take $t_0 > 0$, $t_0$ large. Using the wellposedness of (1.1) for finite initial time, define $u_{t_0}$ as the solution of the Cauchy problem for (1.1) with initial data $u_{t_0}(t_0) = v(t_0)$ at time $t_0$. For fixed $v$, take now the limit $t_0 \to \infty$. In favourable cases, $u_{t_0}$ will tend to a limiting solution $u_{\infty}$ of (1.1) answering the previous question.

Of special interest is the case where the set of model functions $v$ is the set of solutions of an evolution problem which is globally well posed (and preferably simpler than (1.1)). In that case, the set of model functions $v$ can be characterized by its initial data at some prescribed time $T$, whereas the solutions $u = u_{\infty}$ as constructed above can also be characterized by their values $u(T)$ at time $T$. The map $v \to u_{\infty}$ classifying (part of) the solutions of (1.1) through their asymptotic behaviour is then equivalent to the simpler map $\Omega_+: v(T) \to u(T)$ relating the values of $v$ and $u$ at time $T$. That map is the wave operator (for positive time).

At this stage there is some symmetry between the original evolution for $u$ and the model evolution for $v$. In fact let $u$ be a solution of (1.1) constructed by the previous limiting process. Then $v$ can be recovered from $u$ as follows. Take $t_0 > 0$, $t_0$ large. Define $v_{t_0}$ as the solution of the Cauchy problem for the model evolution with initial data $v_{t_0}(t_0) = u(t_0)$ at time $t_0$. For fixed $u$, take now the limit $t_0 \to \infty$. In general $v_{t_0}$ will then tend to the original function $v$ from which $u$ was constructed, and the limiting process will provide additional information on
the asymptotic behaviour of $u$.

In the short range case, the previous scheme is implemented by taking for $v$ the solutions of the equation

$$i\partial_t v + (1/2)\Delta v = 0$$

(2.2)

hereafter referred to as the free equation. The generic solution of that equation is $v(t) = U(t)u_+$ where $U(t)$ is the unitary group

$$U(t) = \exp(i(t/2)\Delta) .$$

(2.3)

It is then natural to take $T = 0$. The initial data $u_+$ for $v$ is called the asymptotic state for the solution $u = u_\infty$ of (1.1) constructed by the method described above, and that method yields the ordinary wave operator $\Omega_+ : u_+ \to u(0)$. Note also that the asymptotic closeness of $u$ and $v$ as $t \to \infty$ can be better expressed in terms of the function $\tilde{u}(t) = U(-t)u(t)$ than in terms of $u$ itself. In fact that function is expected to tend to $u_+$ in suitable norms, whereas $u(t)$ and $v(t)$ separately tend weakly to zero in any reasonable sense.

In the long range case, it is known that the previous ordinary wave operators fail to exist, which means that the previous set of $v$’s, namely the set of solutions of the free evolution, is badly chosen. A better set of model functions $v$ is then obtained by modifying the previous ones by a suitable phase. That modification can be done in several ways and uses some additional structure of $U(t)$. In fact $U(t)$ can be written as

$$U(t) = M(t) D(t) F M(t)$$

(2.4)

where $M(t)$ denotes the operator of multiplication by the function, also denoted $M(t)$,

$$M(t) = \exp(ix^2/2t) ,$$

(2.5)

$F$ is the Fourier transform and $D(t)$ is the dilation operator defined by

$$D(t) f(x) = (it)^{-n/2} f(x/t) .$$

(2.6)

Let now $\varphi_0 = \varphi_0(x,t)$ be a real function of space and time, to be chosen later, let $z_0(x,t) = \exp(-i\varphi_0(x,t))$, let $\varphi_0(t)$ and $z_0(t)$ be the operators of multiplication by the function $\varphi_0(x,t)$ and $z_0(x,t)$ respectively and let $\phi_0(t)$ and $Z_0(t)$ be the operators

$$\phi_0(t) = \varphi_0(-i\nabla, t) = F^\ast \varphi_0(t) F ,$$

and
\[ Z_0(t) = z_0(-i\nabla, t) = F^*z_0(t) F \quad . \] (2.7)

In what follows, we shall sometimes omit the time dependence of the various operators when no confusion is likely to arise.

Instead of the free evolution \( v(t) = U(t)u_+ \) one can now consider the following three modified free evolutions:

\[ v_1(t) = U(t) Z_0(t) u_+ = U(t) F^* z_0(t) w_+ \quad , \] (2.8)

where \( w_+ = Fu_+ \),

\[ \begin{align*}
  v_2(t) &= U(t) M^*(t) Z_0(t) u_+ = M(t) D(t) F Z_0(t) u_+ \\
  &= M(t) D(t) z_0(t) w_+ \\
  &= M(t) (D(t) z_0(t) D^*(t)) D(t) w_+ \\
\end{align*} \] (2.9)

\[ \begin{align*}
  v_3(t) &= U(t) M^*(t) Z_0(t) M(t)u_+ = M(t) D(t) F Z_0(t) M(t) u_+ \\
  &= M(t) D(t) z_0(t) U^*(1/t) w_+ \\
  &= M(t) (D(t) z_0(t) D^*(t)) D(t) U^*(1/t) w_+ \\
\end{align*} \] (2.10)

where we have used the fact that

\[ F M(t) F^* = U^*(1/t) \quad . \] (2.11)

Note that \( D(t)z_0(t)D(t)^* \) is the operator of multiplication by \( z_0(x/t, t) \) so that the modification due to \( \varphi_0 \) appears only as an overall phase factor in \( v_2 \) and \( v_3 \).

Most of the literature on long range scattering for the linear Schrödinger equation makes use of \( v_1 \). The function \( v_2 \) has been introduced in [28] and further used in [4] in the linear case. It has been introduced independently in [26] and used in [3, 28] in the nonlinear case. The function \( v_3 \) is mentioned but not really used in [8].

In the short range case with \( v(t) = U(t)u_+ \), it was hinted that the function \( \tilde{u}(t) = U(-t)u(t) \) was a better object of study than \( u(t) \). Similarly, in the long range case, it will be useful to introduce new functions which will be better suited than \( u \) for comparison with \( v_i(t) \). Furthermore, it will be useful to express those functions as suitable combinations of a phase factor \( z_i(t) = \exp[-i\varphi_i(t)] \) and of an amplitude \( \tilde{u}_i(t) \) in such a way that the comparison of \( u \) with \( v_i \) can be reduced to the facts that asymptotically \( \varphi_i(t) \) behaves as \( \varphi_0(t) \) and \( \tilde{u}_i(t) \) tends to \( u_+ \) or...
equivalently $w_i(t)$ tends to $w_+$, where $w_i(t) = F\tilde{u}_i(t)$ for $i = 1, 2, 3$. This is done through the following definitions, to be compared with (2.8) (2.9) (2.10):

$u(t) = U(t) Z_1(t) \tilde{u}_1(t) = U(t) F^* z_1(t) w_1(t)$ ,  

(2.12)  

$u(t) = U(t) M^*(t) Z_2(t) \tilde{u}_2(t) = M(t) D(t) z_2(t) w_2(t)$ ,  

(2.13)  

$u(t) = U(t) M^*(t) Z_3(t) M(t) \tilde{u}_3(t) = M(t) D(t) F Z_3(t) M(t) \tilde{u}_3(t)$  

= $M(t) D(t) z_3(t) U^*(1/t) w_3(t) .  

(2.14)  

The study of the asymptotic behaviour of small solutions of the equation (1.1) has been performed in [12] [15] by using $(w_1, \varphi_1)$ and in [11] [13] [14] by using essentially $(w_2, \varphi_2)$. We now explain the construction of the wave operators performed in the present paper. For technical reasons, that construction will use the variables $(w_3, \varphi_3)$, but we first explain it on the example of $(w_2, \varphi_2)$ because the algebra is slightly simpler. We shall then indicate the necessary modifications needed to switch to $(w_3, \varphi_3)$.

Instead of trying to construct directly the wave operators for $u$, we first try to construct wave operators for $(w_2, \varphi_2)$ by using the general method given at the beginning of this section. The equation (1.1) is equivalent to the following equation for $z_2 w_2$

$$\left(i\partial_t + (2t^2)^{-1}\Delta - D^*\tilde{g}D\right)(z_2 w_2) = 0$$  

(2.15)  

as can be seen by an elementary computation. Note also that

$$|u(t)| = |D(t) w_2(t)|$$  

by (2.13), so that

$$\tilde{g} \equiv \tilde{g}(|u|^2) = \tilde{g}(|Dw_2|^2)$$  

(2.17)  

and $\tilde{g}$ depends only on $w_2$ (actually only on $|w_2|$), but not on $\varphi_2$. Expanding the derivatives in (2.15), we obtain the equivalent form [13]

$$\left\{i\partial_t + (2t^2)^{-1}\Delta - i(2t^2)^{-1} (2\nabla \varphi_2 \cdot \nabla + (\Delta \varphi_2))\right\} w_2$$  

$$+ \left(\partial_t \varphi_2 - (2t^2)^{-1}|\nabla \varphi_2|^2 - D^*\tilde{g}D\right) w_2 = 0$$  

(2.18)  

We are now in the situation of a gauge theory. The equation of evolution (2.15) and therefore also (2.18) is invariant under the transformation $(w_2, \varphi_2) \rightarrow (w_2 \exp(i\sigma), \varphi_2 + \sigma)$, where $\sigma$ is
an arbitrary function of space time, and the original gauge invariant equation (2.15) is not sufficient to provide evolution equations for the two gauge dependent quantities \((w_2, \varphi_2)\). At this point, we arbitrarily add a gauge condition, which will serve as a second evolution equation, and replace (2.18) by

\[
\begin{cases}
  i\partial_t + (2t^2)^{-1}\Delta - i(2t^2)^{-1}(2\nabla \varphi_2 \cdot \nabla + (\Delta \varphi_2)) \right) w_2 = 0 \\
  \partial_t \varphi_2 = (2t^2)^{-1}|\nabla \varphi_2|^2 + D^* \tilde{g} D
\end{cases}
\]

(2.19) (2.20)

where the second equation, namely the gauge condition, is one of the Hamilton-Jacobi (HJ) equations for the classical system associated with (1.1) \([3, 4]\). The situation here is similar to that occurring for the Maxwell equations where for instance one can impose the Lorentz gauge condition \(\partial_{\mu} A^\mu = 0\) and use it as an evolution for \(A_0\) in order to reduce the gauge freedom in the study of the Cauchy problem.

We have now replaced the original evolution (1.1) by the system (2.19) (2.20) and we try to study the asymptotic behaviour of its solutions and to construct wave operators for it by the same method that we intended to use for (1.1). In contrast with (1.1) however, the Cauchy problem for (2.19) (2.20) cannot be expected to be globally well posed. It turns out however, and that is sufficient for our purposes, that this problem is locally well posed in a neighborhood of infinity in time. Roughly speaking for given initial data of arbitrary size, the Cauchy problem is well posed for initial time \(t_0\) in some interval \([T, \infty)\) for some sufficiently large \(T\) depending on the size of the data. As a consequence, we shall be able to construct only local wave operators for (2.19) (2.20) in a neighborhood of infinity. Wave operators for (1.1) will then be obtained from those by switching back to \(u\) and using the global wellposedness of (1.1) for finite times.

In order to construct the local wave operators for (2.19) (2.20), we need to choose a set of model functions playing the role of \(v\), preferably defined through a model evolution. Keeping in mind that \(u\) is represented by (2.13) and should be asymptotic to \(v_2\) defined by (2.9), preferably with \(w_2(t)\) tending to \(w_+\) and \(\varphi_0(t)\) asymptotic to \(\varphi_2(t)\), we define the model evolution for a pair \((w_0, \varphi_0)\) corresponding to \((w_2, \varphi_2)\) by

\[
\begin{cases}
  \partial_t w_0 = 0 \\
  \partial_t \varphi_0 = (2t^2)^{-1}|\nabla \varphi_0|^2 + D^* \tilde{g} (|Dw_0|^2) D
\end{cases}
\]

(2.21)

The first equation is immediately solved by \(w_0(t) = w_+\), thereby leading to the form (2.9) of \(v_2\)
where now the phase $\varphi_0$ should be a solution of the equation

$$
\partial_t \varphi_0 = (2t^2)^{-1} |\nabla \varphi_0|^2 + D^* \tilde{g}(|Dw_+|^2)D .
$$

As in the case of the system (2.19) (2.20), the Cauchy problem for (2.22) is well posed only in a neighborhood of infinity in time, but in general not globally in time. Although (2.22) is not the model evolution that we shall use later on, we use it to continue the heuristic discussion.

The local wave operators at infinity for the system (2.19) (2.20) as compared with (2.22) are now constructed by the general method described at the beginning of this section. Let $\Gamma_0 = (w_+, \varphi_0)$ be a solution of (2.22), defined in some interval $[T, \infty)$ with $T$ sufficiently large, and depending on the initial data $\varphi_0(T)$. Let $t_0 > T$ and let $\Gamma_{t_0} = (w_{2,t_0}, \varphi_{2,t_0})$ be the solution of (2.19) (2.20) with initial data $(w_{2,t_0}(t_0) = w_+, \varphi_{2,t_0}(t_0) = \varphi_0(t_0))$ at time $t_0$. Under suitable assumptions, $\Gamma_{t_0}$ will be defined in the interval $[T, \infty)$ and will converge to a well defined limit $\Gamma_\infty = \Gamma$ when $t_0 \to \infty$ for fixed $\Gamma_0$. This could provide a basis for the definition of local wave operators at infinity for the system (2.19) (2.20), although the fact that $T$ depends on the size of the data would cause some difficulties, but we are actually interested in wave operators for $u$, and from the previous construction we keep only the map $\Gamma_0 \to \Gamma$. Reconstructing $u$ from $\Gamma = (w_2, \varphi_2)$ by the use of (2.13), we obtain a map $(w_+, \varphi_0(T)) \to u$ where $u$ is a local solution of the equation (1.1) in a neighborhood of infinity, namely defined in $[T, \infty)$, which behaves asymptotically as $v_2$ when $t \to \infty$ in a sense that is expressed by the relation between $(w_+, \varphi_0) = \Gamma_0$ and $(w_2, \varphi_2) = \Gamma$ at infinity, as it follows from the previous construction. We can finally complete the construction of $u$ by solving the Cauchy problem for (1.1) with initial data $u(T)$ at time $T$ obtained from the previous step down to time $t = 1$ and define accordingly a map $(w_+, \varphi_0(T)) \to u(1)$, which is a reasonable candidate for the wave operator for (1.1).

The map $(w_+, \varphi_0(T)) \to u$ however is not yet satisfactory for two reasons. Firstly $u$ depends on too many data. We want $u$ to depend only on $w_+$ and not in addition on an arbitrary initial condition for $\varphi_0$. That defect is easily remedied, as in linear long range scattering, by imposing arbitrarily some initial condition for $\varphi_0$. For instance, given $w_+$, one could choose in some preassigned way some $T = T(w_+)$ sufficiently large for all subsequent constructions to be possible, and choose for instance $\varphi_0(T(w_+)) = 0$. Secondly, because of gauge invariance, the map $(w_+, \varphi_0(T)) \to u$ has no chance of being injective, since different $(w_+, \varphi_0(T))$ can very
well produce different but gauge equivalent \((w_2, \varphi_2)\), thereby leading to the same \(u\). Fixing arbitrarily the initial condition for \(\varphi_0\) certainly will improve the injectivity, but at the risk of restricting the set of \(u\) obtained by the previous construction, namely the range of the wave operator, and making it dependent on that initial condition. We now show that in principle, this should not happen, and that fixing the initial condition for \(\varphi_0\) exactly removes the gauge freedom and ensures the injectivity of the map \(\Gamma_0 \rightarrow u\) without restricting its range. For that purpose we have to consider in more detail the gauge covariance of the map \(\Gamma_0 \rightarrow u\) without restricting its range. For that purpose we have to consider in more detail the gauge covariance of the map \(\Gamma_0 \rightarrow u\) without restricting its range. 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The map $\Gamma_0 \to \Gamma$ as we shall construct it later will be such that $\varphi_0(t) - \varphi_2(t) \to 0$ and $w_2(t) \to w_+$ as $t \to \infty$. Consequently if we define the gauge transformation $G_{\sigma}$ on the solutions of the model evolution (2.22) by $G_{\sigma}(w_+, \varphi_0) = (w'_+, \varphi'_0)$ with $\lim_{t \to \infty} \varphi'_0 - \varphi_0(t) = \sigma$ and $w'_+ = w_+ e^{i\sigma}$, then the map $\Gamma_0 \to \Gamma$ is gauge covariant in the sense that the image of $G_{\sigma}\Gamma_0$ is $G_{\sigma}\Gamma$ if $\Gamma$ is the image of $\Gamma_0$.

From the previous discussion it follows that imposing an initial condition on $\varphi_0$ exactly fixes the gauge, thereby ensuring the injectivity of the map $\Gamma_0 \to u$ without restricting its range. Actually in practice that picture is clouded by the fact that all the constructions involved produce a small loss of regularity which prevents a complete proof of the previous statements. On the other hand the basic construction $\Gamma_0 \to \Gamma$ also produces a similar loss of regularity, and it turns out that the former is hidden by the latter, so that an entirely satisfactory discussion of gauge invariance can be given at the level of regularity of the construction $\Gamma_0 \to \Gamma$.

The previous heuristic discussion was based on the system (2.19) (2.20) for $(w_2, \varphi_2)$ and the model evolution (2.22) for $\varphi_0$. For technical reasons however, we shall use different equations. The first two reasons are rooted in the basic construction of the method, namely the construction of the solution $\Gamma_{t_0}$ of the full evolution coinciding at $t_0$ with a given solution $\Gamma_0$ of the model evolution, and the third one is connected with gauge invariance.

(1) We shall take $w_+ \in H^k$, where $H^k$ is the standard Sobolev space, and look for $w_2$ as a continuous function of time with values in $H^{k'}$ for some $k' \leq k$. In the construction $\Gamma_0 \to \Gamma_{t_0}$, the term $\nabla \varphi_2 \cdot \nabla w_2$ in (2.19) produces a loss of one derivative, which seems difficult to avoid. The term $\Delta w_2$ on the other hand produces a loss of two derivatives but that loss is easily avoided by switching from $(w_2, \varphi_2)$ to $(w_3, \varphi_3)$ and we shall therefore use $(w_3, \varphi_3)$ instead of $(w_2, \varphi_2)$. The equations for $(w_3, \varphi_3)$ could be obtained easily from the equation for $u$, but it is simpler to deduce them from the system (2.19) (2.20). We impose $\varphi_2 = \varphi_3 = \varphi$ and $w_3(t) = U(1/t)w_2(t) = w(t)$, which is consistent with (2.13) (2.14). The resulting system for $(w, \varphi)$ is then

$$
\begin{align*}
\partial_t w &= (2t^2)^{-1}U(1/t)(2\nabla \varphi \cdot \nabla + (\Delta \varphi))U^*(1/t)w \\
\partial_t \varphi &= (2t^2)^{-1} |\nabla \varphi|^2 + D^*\tilde{g}(|DU^*(1/t)w|^2)D.
\end{align*}
$$

(2.25) (2.26)

Correspondingly, the model evolution for $(w_0, \varphi_0)$ will replace (2.25) by $\partial_t w = 0$, so as to
produce model functions of the type (2.10), thereby leaving for $\varphi_0$ the equation

$$\partial_t \varphi_0 = (2t^2)^{-1} |\nabla \varphi_0|^2 + D^* \tilde{g} \left(|DU^*(1/t)w_+|^2\right) D .$$

(2.27)

Since all the estimates on $w$ will be made in spaces $H^k$ where $U(1/t)$ is unitary, the explicit occurrence of that operator in (2.25)-(2.27) will not make any difference in those estimates.

(2) In this paper we restrict our attention to the case $\gamma > 1/2$. It is well known in linear long range scattering theory that under that condition the correcting phase $\varphi_0$ need not be a solution of the full HJ equation (2.27) and can be chosen simply according to the Dollard prescription, namely as a solution of the simpler equation

$$\partial_t \varphi_0 = D^* \tilde{g} \left(|DU^*(1/t)w_+|^2\right) D .$$

(2.28)

We shall partly work with (2.28) instead of (2.27) in what follows. This produces a number of simplifications in all the questions involving only $\Gamma_0$. In particular the Cauchy problem for (2.28) is trivially solved globally by a simple integration, thereby allowing in particular for imposing an initial condition for $\varphi_0$ at a fixed time independent of $w_+$.

Whereas the term $|\nabla \varphi_0|^2$ can be omitted from (2.27) for $\gamma > 1/2$, fiddling with the term $|\nabla \varphi|^2$ in (2.26) may not be harmless. For instance shifting that term from (2.26) to (2.25) would produce additional restrictions on $\gamma$ and require at least $\gamma > 2/3$ in the construction $\Gamma_0 \rightarrow \Gamma$.

(3) The model evolution (2.27) and its simplified version (2.28) are best suited to study the asymptotic behaviour of the system (2.25) (2.26), which we have chosen in order to minimize the loss of regularity. On the other hand they are ill suited for a study of gauge invariance, which plays an important role in the reconstruction of $u$. In particular a change $w_+ \rightarrow w_+ e^{i\sigma}$ produces a nontrivial change in the $\tilde{g}$ term in (2.27), whereas no such change occurs in (2.22). As a consequence, we cannot avoid using also (2.22), or rather its simplified version

$$\partial_t \varphi_0 = D^* \tilde{g} (|Dw_+|^2) D$$

(2.29)

obtained as previously by omitting the $|\nabla \varphi_0|^2$ term. We shall therefore use both (2.29), which allows for a simple discussion of gauge invariance and for a cleaner construction of $u$, and (2.28), which yields better asymptotic approximations.
The last technical modification is independent of the previous choices.

(4) The right-hand sides of (2.25) (2.26) (2.27) (2.28) (2.29) contain \( \varphi \) and \( \varphi_0 \) only through their gradients, and the construction of \( \Gamma \) can be discussed entirely in terms of those variables. Only for the discussion of gauge invariance and for the reconstruction of \( u \) are \( \varphi_0 \) and \( \varphi \) themselves needed. We therefore introduce the \( IR^m \) valued functions \( s = \nabla \varphi \) and \( s_0 = \nabla \varphi_0 \) and replace the basic equations (2.26)-(2.29) by their gradients. Using the fact that

\[
\partial_i |\nabla \varphi|^2 = \sum_j (\partial_i \partial_j \varphi)(\partial_j \varphi) = \sum_j (\partial_j \varphi)(\partial_j \partial_i \varphi) = (\nabla \varphi \cdot \nabla) \partial_i \varphi
\]

we obtain

\[
\begin{cases}
\partial_t w = (2t^2)^{-1}U(1/t)(2s \cdot \nabla + (\nabla \cdot s))U^*(1/t)w \\
\partial_t s = t^{-2}(s \cdot \nabla)s + \nabla D^* \tilde{g}(\|DU^*(1/t)w\|^2)D
\end{cases}
\]

and either

\[
\begin{align*}
\partial_t s_0 &= t^{-2}(s_0 \cdot \nabla)s_0 + \nabla D^* \tilde{g}(\|DU^*(1/t)w_+\|^2)D \\
\partial_t s_0 &= \nabla D^* \tilde{g}(\|DU^*(1/t)w_+\|^2)D
\end{align*}
\]

or

\[
\partial_t s_0 = \nabla D^* \tilde{g}(\|Dw_+\|^2)D
\]

The phase \( \varphi \) itself will then be recovered from \( s \) through the use of (2.26) as follows. The equation (2.31) is an Euler like equation for \( s \) and implies that the vorticity \( \omega = \nabla \times s \) remains zero for all time if it is zero for some initial time \( t_0 \), namely with an initial condition \( s(t_0) = \nabla \varphi_0(0) \). In fact \( \omega \) satisfies the linear equation

\[
\partial_t \omega = t^{-2} \left( (s \cdot \nabla)\omega + A\omega + \omega A^T \right)
\]

where \( A \) is the matrix with entries \( A_{ij} = \partial_j s_i \), and that equation implies the result just mentioned through the Gronwall inequality for sufficiently regular \( s \). It follows then from (2.26) and (2.31) that

\[
\partial_t (s - \nabla \varphi) = t^{-2}(s \times \omega) = 0
\]

and therefore \( s - \nabla \varphi \) vanishes for all \( t \) if it vanishes for some \( t_0 \).

We are now in a position to describe in more detail the contents of the technical part of this paper, Sections 3-7. In Section 3, we introduce some notation, define the relevant function.
spaces needed to study the system (2.30) (2.31), and we derive a number of estimates which are used throughout the paper. In Section 4, we study the Cauchy problem at finite times for the system (2.30) (2.31), and we prove that that problem is locally well posed (Proposition 4.1). In Section 5, we study the local Cauchy problem at infinity for the same system, and construct local wave operators for it as compared with the model equation (2.33). We first solve the Cauchy problem in a neighborhood of infinity for finite but large $t_0$ (Proposition 5.1) and derive a uniqueness result for given asymptotic behaviour (Proposition 5.2). We then prove the existence of asymptotic states for solutions $\Gamma = (w, s)$ thereby obtained, in the following sense: firstly $w(t)$ has a limit $w_+$ when $t \to \infty$ (Proposition 5.3). Secondly the solution $\Gamma_{0,t_0}$ of (2.33) which coincides with $\Gamma$ at time $t_0$ satisfies estimates uniform in $t_0$ (Proposition 5.4) and has a limit $\Gamma_0$ when $t_0 \to \infty$ which is asymptotic to $\Gamma$ (Proposition 5.5). We then turn to the converse construction, which is that of the local wave operators at infinity. For a fixed solution $\Gamma_0$ of (2.33), we construct a solution $\Gamma_{t_0}$ of the system (2.30) (2.31) which coincides with $\Gamma_0$ at $t_0$ and we estimate it uniformly in $t_0$ (Proposition 5.6). We then prove that when $t_0 \to \infty$, $\Gamma_{t_0}$ has a limit $\Gamma$ which is asymptotic to $\Gamma_0$ in the same sense as in Proposition 5.5 (Proposition 5.7). We conclude that section with some comments on the possible use of other equations, and in particular on the modifications required to use the more complicated equation (2.32) instead of (2.33). In Section 6, we perform exactly the same analysis of the system (2.30) (2.31) at infinity, now however compared with the model equation (2.34), which yields less precise asymptotics, but which is better suited for the study of gauge invariance and the reconstruction of $u$. Propositions 6.1-6.4 are the exact analogues of Propositions 5.4-5.7 with (2.33) replaced by (2.34) and their proofs rely to a large extent on those of the latter.

Finally in Section 7 we exploit the results of Sections 5 and 6, esp. Propositions 5.7 and 6.4, to construct the wave operators for the equation (1.1) and to describe the asymptotic behaviour of solutions in their range. We first supplement the constructions of Sections 5 and 6 with the appropriate definitions in order to recover $\varphi$ and $\varphi_0$ from $s$ and $s_0$, both at finite times and in their correspondence at infinity as it follows from Propositions 5.5, 5.7, 6.2 and 6.4. We then prove that the local wave operator at infinity for the system (2.30) (2.31) as compared with (2.34) defined through Proposition 6.4 in Definition 7.1 is gauge covariant in the sense of Definitions 7.2 and 7.3 in the best form that can be expected with the available regularity.
(Propositions 7.2 and 7.3). With the help of some information on the Cauchy problem for (1.1) at finite time (Proposition 7.1), we then define the wave operator $\Omega : u_+ \rightarrow u$ (Definition 7.4), we prove that it is injective and has the expected range (Proposition 7.4). We then collect all the available information on $\Omega$ and on solutions of (1.1) in its range in Proposition 7.5, which contains the main results of this paper. Finally, by using some information on the global Cauchy problem at finite time for (1.1) (Proposition 7.6), we define the usual wave operator $\Omega_1 : u_+ \rightarrow u(1)$ (Definition 7.5).

A question that we leave unsettled in this paper is that of the intertwining property of the wave operator. That property can be stated in terms of $\Omega$ as the fact that for $t$ sufficiently large and $\tau \geq 0$,

$$\left( \Omega(U(\tau)u_+) \right)(t) = \left( \Omega(u_+) \right)(t + \tau).$$

(2.36)

That property is an asymptotic form of time translation invariance, and unfortunately that invariance has been severely broken by the change of variables from $u$ to $(w, \varphi)$ in (2.14). Therefore the method used in this paper is ill suited for a study of the intertwining property and we leave that question open here.

We finally remark that the basic equations (2.19) (2.20) from [13], of which we used the modified from (2.25) (2.26), are very similar to the equations used in [7] [10] to study the classical limit $\hbar \rightarrow 0$ of the nonlinear Schrödinger equation

$$i\hbar \partial_t u = -\frac{1}{2} \hbar^2 \Delta u + \tilde{g}(|u|^2)u.$$ 

This comes as no surprise, since the latter are also obtained by separating $u$ into an amplitude and a phase, $u = w \exp(-i\varphi/\hbar)$. Accordingly, the same energy methods can be applied to the small $\hbar$ problem [10] and to the large time problem.

### 3 Notation and preliminary estimates.

In this section we introduce some additional notation and we collect a number of estimates which will be used throughout this paper. We first define

$$g_0(w_1, w_2) = \lambda \Re \omega^{\mu-n} w_1 \bar{w}_2, \quad (3.1)$$

$$g(w_1, w_2) = g_0(U^*(1/t)w_1, U^*(1/t)w_2). \quad (3.2)$$
In particular
\[ g(0)(w_1, w_1) - g(0)(w_2, w_2) = g(0)(w_-, w_+) \]
where \( w_\pm = w_1 \pm w_2 \). By using the definition (1.2) and (2.6), we rewrite the nonlinearity in (2.26) as
\[ D^*\tilde{g} \left( |DU^*(1/t)w|^2 \right) D = t^{-\gamma} g(w, w) . \] (3.3)
The nonlinearities in (2.31)-(2.34) can be rewritten in a similar way, so that the basic equations (2.30)-(2.34) become respectively
\[
\begin{align*}
\partial_t w &= (2t^2)^{-1} U(1/t)(2s \cdot \nabla + (\nabla \cdot s)) U^*(1/t)w \\
\partial_t s &= t^{-2}(s \cdot \nabla)s + t^{-\gamma} \nabla g(w, w) \\
\partial_t s_0 &= t^{-2}(s_0 \cdot \nabla)s_0 + t^{-\gamma} \nabla g(w_+, w_+) \\
\partial_t s_0 &= t^{-\gamma} \nabla g_0(w_+, w_+) .
\end{align*}
\] (2.30) \equiv (3.4) (2.31) \sim (3.5) (2.32) \sim (3.6) (2.33) \sim (3.7) (2.34) \sim (3.8)
We next introduce the function spaces where we shall solve the basic equations (3.4) (3.5). We denote multi-indices by greek letters \( \alpha, \beta, \cdots \), their lengths by \( |\alpha|, |\beta|, \cdots \), and nonnegative integers by \( j, k, \ell, \cdots \). For any function \( u \) and any function space norm \( \| \cdot \| \), we define
\[ \| \partial^j u \| = \sum_{\alpha : |\alpha| = j} \| \partial^\alpha u \| . \]
We shall use Sobolev spaces of integer order \( H^k_r \) defined for \( 1 \leq r \leq \infty \) by
\[ H^k_r = \left\{ u : \| u; H^k_r \| \equiv \sum_{0 \leq j \leq k} \| \partial^j u \|_r < \infty \right\} \]
and the associated homogeneous spaces \( \hat{H}^k_r \) with norm
\[ \| u; \hat{H}^k_r \| = \| \partial^k u \|_r . \]
The subscript \( r \) will be omitted if \( r = 2 \).

We first recall the well known Sobolev-Gagliardo-Nirenberg inequalities [6, 22, 23].
Lemma 3.1. Let $1 \leq p, q, r \leq \infty$. Let $j$ and $k$ be nonnegative integers with $j < k$. If $p = \infty$, assume that $k - j > n/r$. Let $\sigma$ satisfy $j/k \leq \sigma \leq 1$ and

$$n/p - j = (1 - \sigma)n/q + \sigma(n/r - k).$$

Then the following inequality holds for any function $u \in L^q$:

$$\| \partial^j u \|_p \leq C \| u \|_{q}^{1-\sigma} \| \partial^k u \|_r^\sigma$$

except for $p < \infty$, $j = 0$ and $q = \infty$. In the latter case, for any function $u \in L^\infty$ there exists a constant $c$ depending on $u$ such that

$$\| u - c \|_p \leq C \| u - c \|_{\infty}^{1-\sigma} \| \partial^k u \|_r^\sigma.$$ 

Remark 3.1. The statement as just given may differ from the usual ones (see for instance [3]) by unnecessarily excluding a few trivial cases.

We shall use extensively the following spaces. Let $\ell_0 = [n/2]$ and define $r_0$ by $\delta(r_0) = \ell_0$ so that $r_0 = 2n$ for $n$ odd and $r_0 = \infty$ for $n$ even. Let $k$ and $\ell$ be nonnegative integers with $\ell \geq \ell_0 - 1$. Let $I \subset IR^+$ be an interval. We shall look for $w$ as a complex valued function in spaces $L^\infty_{loc}(I, H^k)$ or $C(I, H^k)$ and for $s$ as an $IR^n$ or $\Phi^n$ vector valued function in spaces $L^\infty_{loc}(I, X^\ell)$ or $C(I, X^\ell)$ where

$$X^\ell = L^{r_0} \cap \dot{H}^{\ell_0} \cap \dot{H}^{\ell+1}.$$

(3.9)

For $n$ odd, it follows from Lemma 3.1 that for $s \in X^\ell$,

$$\| s \|_{r_0} \equiv \| s \|_{2n} \leq C \| s; \dot{H}^{\ell_0} \| \equiv C \| s; \dot{H}^{(n-1)/2} \|$$

so that the $L^{r_0}$ norm will not need to be estimated separately. The space $L^{r_0}$ has been included in the definition of $X^\ell$ only in order to make Lemma 3.1 applicable by eliminating arbitrary polynomials of degree $\ell_0 - 1$ which are not seen by the other norms. For $\ell \geq \ell_0$, the inclusion $X^\ell \subset L^\infty$ holds. In fact, by Lemma 3.1 again

$$\| s \|_{\infty} \leq C \left( \| s \|_{r_0} \| s; \dot{H}^{\ell_0+1} \| \right)^{1/2} \leq C \left( \| s; \dot{H}^{\ell_0} \| \| s; \dot{H}^{\ell_0+1} \| \right)^{1/2}$$

(3.10)

so that also the $L^\infty$ norm will not need to be estimated either.
For $n$ even, the $L^r_0$ norm, namely the $L^\infty$ norm, is not controlled by the $\dot{H}^r_0$ norm, namely by the $\dot{H}^{n/2}$ norm, and will require separate estimates.

The spaces $X^\ell$ obviously satisfy the embedding $X^\ell' \subset X^\ell$ for $\ell' \geq \ell$.

Because of the presence of $L^r_0$ in the definition of $X^\ell$, one can replace $\dot{H}^\ell_0$ in that definition by

$$K^\ell_0 = \{ u : u \in \dot{H}^\ell_0 \text{ and } <x>^{-(n+1)/2} u \in L^2 \}$$

which is a Hilbert space, and similarly one can replace $\dot{H}^{\ell+1}$ by $K^{\ell+1}$. As a consequence, $X^\ell$ is the intersection of the duals of (compatible) Banach spaces and is therefore itself the dual of a Banach space.

We shall use systematically the short hand notation

$$|w|_k = \| w; H^k \|, \quad |s|_\ell = \| s; X^\ell \|$$

and the meaning of the symbol $|a|_b$ will always be made unambiguous by the fact that the pair $(a, b)$ contains either the pair $(w, k)$ or the pair $(s, \ell)$.

We shall need estimates of the solutions (in the sense of distributions) of transport diffusion equations of the form

$$\partial_t u = \eta \Delta u + \nabla \cdot (uv) + h$$

where $u$ and $h$ are complex valued functions and $v$ a $\mathbb{C}^n$ vector valued function defined in space time.

**Lemma 3.2.** Let $2 \leq p \leq \infty$, let $I$ be an open interval. Let $u$, $h \in L^p_{loc}(I, L^p)$ and $v \in L^\infty_{loc}(I, L^\infty)$ with $\partial v \in L^\infty_{loc}(I, L^\infty)$ satisfy the equation (3.12) for some $\eta \geq 0$, for all $t \in I$. Then for almost all $t_1$, $t_2 \in I$, with $t_1 \leq t_2$, the following estimate holds:

$$\|u(t_2)\|_p \leq \|u(t_1)\|_p + \int_{t_1}^{t_2} dt' \left\{ p^{-1} \| \nabla \cdot v(t') \|_\infty \| u(t') \|_p \\
+C_0 \| \partial v(t') \|_\infty \| u(t') \|_p + \| h(t') \|_p \right\}$$

for some absolute constant $C_0$. If $\eta = 0$, a similar estimate holds for $t_1 \geq t_2$. 

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Proof. The formal computation leading to (3.13) is given in Appendix A. The actual proof is obtained by following the methods in [5].

Note that the estimate (3.13) is independent of $\eta$. In subsequent applications of Lemma 3.2, we shall for brevity state the results thereby obtained in the shorter differential form corresponding to

$$
\partial_t \| u \|_p \leq p^{-1} \| \nabla \cdot v(t) \|_\infty \| u(t) \|_p + C_0 \| \partial v(t) \|_\infty \| u(t) \|_p + \| h(t) \|_p .
$$

(3.14)

We next give some preliminary estimates. They involve functions called $s$ and $w$ since that is suggestive of subsequent applications, but at the present stage it is irrelevant whether those functions are real or complex, scalar or vector valued.

**Lemma 3.3.** Let $\alpha$ and $\beta$ be multi-indices with $\beta \leq \alpha$ and let $\ell = |\alpha|$. Then the following estimate holds :

$$
\| \partial^\beta s_1 \partial^{\alpha-\beta} s_2 \|_2 \leq C \| s_1; L^\infty \cap \dot{H}^\ell \| \| s_2; L^\infty \cap \dot{H}^\ell \| .
$$

(3.15)

Proof. We estimate by the Hölder inequality

$$
\| \partial^\beta s_1 \partial^{\alpha-\beta} s_2 \|_2 \leq \| \partial^\beta s_1 \|_{r_1} \| \partial^{\alpha-\beta} s_2 \|_{r_2}
$$

with $1/r_1 + 1/r_2 = 1/2$.

For $\beta = \alpha$, we take $r_1 = 2$, $r_2 = \infty$. For $\beta = 0$, we take $r_1 = \infty$ and $r_2 = 2$. For $0 < \beta < \alpha$, we apply Lemma 3.1 and obtain

$$
\cdots \leq C \| s_1 \|_{\infty}^{1-\sigma_1} \| s_1; \dot{H}^\ell \|^{\sigma_1} \| s_2 \|_{\infty}^{1-\sigma_2} \| s_2; \dot{H}^\ell \|^{\sigma_2}
$$

(3.16)

with

$$
\sigma_1(n/2 - \ell) = n/r_1 - |\beta| ,
$$

$$
\sigma_2(n/2 - \ell) = n/r_2 - \ell + |\beta| ,
$$

$$
|\beta|/\ell \leq \sigma_1 \leq 1 , \quad 1 - |\beta|/\ell \leq \sigma_2 \leq 1 .
$$
The equalities imply $\sigma_1 + \sigma_2 = 1$, while the inequalities imply $\sigma_1 + \sigma_2 \geq 1$. As a consequence, all of them are satisfied for the unique choice

$$
2/r_1 = \sigma_1 = |\beta|/\ell, \quad 2/r_2 = \sigma_2 = 1 - |\beta|/\ell.
$$

Then (3.15) follows from (3.16).

\[\square\]

**Lemma 3.4.** Let $\alpha$ and $\beta$ be multi-indices with $\beta \leq \alpha$, $|\alpha| \leq k$, $|\beta| \leq \ell$ and $\ell > n/2$. Then the following estimate holds:

$$
\| \partial^\beta s \partial^{\alpha-\beta} w \|_2 \leq C \| s; L^\infty \cap \dot{H}^\ell \| \| w \|_k.
$$

(3.17)

**Proof.** It is sufficient to consider the case $\alpha = \beta$. The general case with $\alpha \geq \beta$ will then follow therefrom by replacing $w$ by $\partial^{\alpha-\beta} w$. We estimate by the Hölder inequality

$$
\| \partial^\beta s \partial^\gamma w \|_2 \leq \| \partial^\beta s \|_{r_1} \| w \|_{r_2}
$$

with $1/r_1 + 1/r_2 = 1/2$.

For $\beta = 0$, we take $r_1 = \infty$, $r_2 = 2$. For $|\beta| = \ell$, we take $r_1 = 2$, $r_2 = \infty$ and use the fact that in that case

$$
\| w \|_\infty \leq C \| w; H^\ell \| = C \| w; H^\ell \|
$$

since $\ell > n/2$. For $0 < |\beta| < \ell$, we apply Lemma 3.1 and obtain

$$
\cdots \leq C \| s \|_{\infty}^{1-\sigma} \| \partial^\gamma s \|_{2}^{\sigma} \| w \|_{r_2}
$$

(3.18) with $\sigma(n/2 - \ell) = n/r_1 - |\beta|$ and $|\beta|/\ell \leq \sigma \leq 1$. We then choose $\sigma = |\beta|/\ell$ so that $2/r_2 = 1 - |\beta|/\ell$ and $\delta_2 = \delta(r_2) = n|\beta|/2\ell < |\beta|$ since $\ell > n/2$, so that by Lemma 3.1 again $\| w \|_{r_2} \leq \| w; H^{|\beta|} \|$, which together with (3.18), implies (3.17) in the special case $\alpha = \beta$.

\[\square\]

**Lemma 3.5.** Let $\varphi$ be a real function with $s = \nabla \varphi \in L^\infty \cap \dot{H}^\ell$ for some $\ell > n/2$ and let $k \leq \ell + 1$. Then the following estimate holds:

$$
\left| e^{-i\varphi} w \right|_k \leq C \left( 1 + \| \nabla \varphi; L^\infty \cap \dot{H}^\ell \| \right)^k \| w \|_k.
$$

(3.19)
Let in addition \( \varphi \in L^\infty \). Then the following estimate holds:

\[
\left| (e^{-i\varphi} - 1)w \right|_k \leq C \left( \| \varphi \|_\infty + \| \nabla \varphi; L^\infty \cap \dot{H}^\ell \| \left( 1 + \| \nabla \varphi; L^\infty \cap \dot{H}^\ell \| \right)^{k-1} \right) |w|_k .
\] (3.20)

**Proof.** For any multi-indices \( \alpha, \beta \) with \( \beta \leq \alpha \) and \( 1 \leq |\beta| \leq |\alpha| \leq k \), one has to estimate the \( L^2 \) norm of

\[
\partial^\beta (e^{-i\varphi}) \partial^{\alpha - \beta} w = e^{-i\varphi} \sum_{\{\beta_i\}} (-i)^m C_{\{\beta_i\}} \left( \prod_{1 \leq i \leq m} \partial^{\beta_i} \varphi \right) \partial^{\alpha - \beta} w
\] (3.21)

where the sum runs over all possible decompositions \( \beta = \beta_1 + \cdots + \beta_m \) of \( \beta \) as the sum of \( m \geq 1 \) multi-indices. We have omitted the terms with \( \beta = 0 \) which trivially satisfy the required estimate with obvious assumptions on \( \varphi \). We estimate each term in the RHS of (3.21) by Lemma 3.4 applied with \( s = \nabla \varphi, \beta \leq \beta_1, |\beta| = |\beta_1| - 1, \alpha = \beta \) and \( w \) replaced by the product of the last \( m \) factors in (3.21). We obtain

\[
\| \partial^\beta (e^{-i\varphi}) \partial^{\alpha - \beta} w \|_2 \leq C \| \nabla \varphi; L^\infty \cap \dot{H}^\ell \| \sum_{\{\beta'_i, \alpha'_i\}} \left( \prod_{2 \leq i \leq m} \partial^{\beta'_i} \varphi \right) \partial^{\alpha'} w \|_2
\]

where \( \beta'_i \) (\( 2 \leq i \leq m \)) and \( \alpha' \) are multi-indices obtained by distributing at most \( |\beta_1| - 1 \) derivatives on \( \beta_2, \ldots, \beta_m, \alpha - \beta \). In particular, in the nontrivial case \( m \geq 2 \), one has \( |\beta'_i| \leq |\beta| + |\beta_1| - 1 \leq |\beta| - 1 \leq |\alpha| - 1 \leq k - 1 \leq \ell \). One can then iterate the process, thereby extracting the \( m \)-th power of \( \| \nabla \varphi; L^\infty \cap \dot{H}^\ell \| \) and obtaining for each \( m \geq 1 \) a contribution

\[
\| \nabla \varphi; L^\infty \cap \dot{H}^\ell \|^m |w|_{k-m} .
\]

Taking the sum over \( m \) and adding the contribution of the term \( m = 0 \) yields (3.19) (3.20).

\[\square\]

The next estimate will be needed to estimate the nonlinearity \( \tilde{g}(|u|^2) \).

**Lemma 3.6.** Let \( n \geq 2, 0 < \mu < n, \ell, k_1 \) and \( k_2 \) be nonnegative integers with \( k_1 \leq k_2 \), and

\[
n/2 < \ell + \mu \leq (n/2 + k_1 + k_2) \wedge (n + k_1) ,
\] (3.22)

and in addition \( k_2 > n/2 \) if \( \ell + \mu = n + k_1 \). Then the following estimate holds:

\[
\| \partial^\ell \omega^{\mu-n}(w_1 w_2) \|_2 \leq C |w_1|_{k_1} |w_2|_{k_2} .
\] (3.23)
Proof. Let $m = [n - \mu] \land \ell$ and define $r$ by

$$\delta \equiv \delta(r) = n - \mu - m$$

so that $0 \leq \delta(r) < n/2$ if $\ell + \mu \leq n$ by (3.22L), while $0 \leq \delta(r) < 1$ if $\ell + \mu \geq n$. By the Hardy-Littlewood Sobolev (HLS) inequality ([22] p. 116) if $m < n - \mu$ and by inspection if $m = n - \mu$, we estimate

$$\| \partial^\ell \omega^{\mu-n}(w_1 w_2) \|_2 = \| \partial^m \omega^{\mu-n} \partial^{\ell-m}(w_1 w_2) \|_2$$

$$\leq C \| \omega^{-\delta} \partial^{\ell-m}(w_1 w_2) \|_2 \leq C \| \partial^{\ell-m}(w_1 w_2) \|_\bar{r}$$

(3.24)

where $1/r + 1/\bar{r} = 1$ and $\bar{r} > 1$. By the Leibnitz formula and the Hölder inequality, we continue (3.24) as

$$\cdots \leq C \sum_{0 \leq j \leq \ell - m} \| \partial^j w_1 \|_{r_1} \| \partial^{\ell-m-j} w_2 \|_{r_2}$$

(3.25)

where $\delta_i \equiv \delta(r_i)$, $i = 1, 2$, have to satisfy

$$\delta_1 + \delta_2 = n/2 - \delta = \mu + m - n/2$$

(3.26)

One can then continue (3.25) as

$$\cdots \leq C |w_1|_{k_1} |w_2|_{k_2}$$

(3.27)

provided $\ell - m \leq k_1$ and provided for each $j$ one can choose $\delta_1$ and $\delta_2$ satisfying (3.26) and

$$\begin{cases} 
0 \leq \delta_1 \leq k_1 - j \\
0 \leq \delta_2 \leq k_2 - (\ell - m - j)
\end{cases}$$

(3.28)

with the RHS inequality being strict if the corresponding $\delta_i$ is equal to $n/2$. The condition $\ell - m \leq k_1$ is equivalent to $\ell \leq k_1 + [n - \mu]$ and therefore to $\ell \leq k_1 + n - \mu$ since $\ell$ and $k_1$ are integers, and follows therefore from (3.22). The compatibility of (3.26) (3.28) in $\delta_i$ reduces to

$$\mu + m - n/2 \leq k_1 + k_2 - \ell + m ,$$

again a consequence of (3.22). Finally the only possible exceptional case corresponds to $\delta = 0$, $j = \ell - m = k_1$, namely $\ell + \mu = n + k_1$, and requires $k_2 > n/2$. 

$\Box$
We shall use Lemma 3.6 through the following corollary. We recall that \( \ell_0 = \lceil n/2 \rceil \).

**Corollary 3.1.** Let \( 0 < \mu < n \) and let \( k, \ell \) be nonnegative integers satisfying \( \ell > n/2 \) and

\[
\ell + 2 + \mu \leq (n/2 + 2k) \wedge (n + k)
\]

and in addition \( k > n/2 \) if \( \ell + 2 + \mu = n + k \). Then the following estimates hold :

\[
\| \partial^{\ell+1} \omega^{\mu-n} w_1 w_2 \|_2 + \| \partial^\ell \omega^{\mu-n} w_1 w_2 \|_2 \leq C |w_1|_{k-2} |w_2|_k ,
\]

(3.30)

\[
\| \partial^{\ell+1} \omega^{\mu-n} w_1 w_2 \|_2 \leq C |w_1|_{k-1} |w_2|_k ,
\]

(3.31)

\[
\| \partial^{\ell+2} \omega^{\mu-n} w_1 w_2 \|_2 \leq C |w_1|_k |w_2|_k ,
\]

(3.32)

\[
\| \partial^{\ell+3} \omega^{\mu-n} |w|^2 \|_2 \leq C |w|_k |w|_{k+1} ,
\]

(3.33)

\[
\| \partial \omega^{\mu-n} w_1 w_2 \|_\infty \leq C |w_1|_{k-1} |w_2|_k ,
\]

(3.34)

\[
\| \omega^{\mu-n} w_1 w_2 \|_\infty \leq C |w_1|_{k-1} |w_2|_{k-1} .
\]

(3.35)

If \( n \) is even, assume in addition that the inequality (3.29) with \( \ell \) replaced by \( n/2 + 1 \) (which is the lowest allowed value) is strict, namely

\[
n/2 + 3 + \mu < (n/2 + 2k) \wedge (n + k) .
\]

(3.36)

Then the following estimate holds :

\[
\| \partial \omega^{\mu-n} w_1 w_2 \|_\infty \leq C |w_1|_{k-2} |w_2|_k .
\]

(3.37)

**Proof.** The estimates (3.30)-(3.32) are direct applications of Lemma 3.6 with \( \ell \) replaced respectively by \( \ell_0 + 1 \) and \( \ell \), by \( \ell + 1 \) and by \( \ell + 2 \), while \( (k_1, k_2) \) are replaced respectively by \( (k-2, k) \), by \( (k-1, k) \) and by \( (k, k) \). The condition (3.22L) follows from \( n/2 < \ell_0 + 1 \leq \ell \), while the condition (3.22R) follows from, actually reduces to (3.29) in all three cases. The estimate (3.33) follows from (3.32) applied to \((\bar{w}, \partial \bar{w})\) and to \((w, \partial w)\).

In order to prove (3.34) (3.35) and (3.37), we note that by (1.3) \( \omega^{\mu-n} w_1 w_2 \) and \( \partial \omega^{\mu-n} w_1 w_2 \) belong to \( L^r_+ + L^r_- \) for \( w_1, w_2 \in H^1 \) with \( n/r_\pm = \mu \pm \eta \). One can then estimate by Lemma 3.1

\[
\| \omega^{\mu-n} w_1 w_2 \|_\infty \leq C \| \omega^{\mu-n} w_1 w_2 \|_{n/\varepsilon}^{1/2} \| \partial \omega^{\mu-n} w_1 w_2 \|_{n/(1-\varepsilon)}^{1/2}
\]
which by the Hardy-Littlewood-Sobolev inequality implies
\[
\| \omega^{\mu-n} w_1 w_2 \|_\infty \leq C \| \omega^{\mu-n/2-\varepsilon} w_1 w_2 \|_2^{1/2} \| \omega^{\mu-n/2+\varepsilon} w_1 w_2 \|_2^{1/2} \tag{3.38}
\]
for any \(\varepsilon, 0 < \varepsilon \leq 1\). Similarly one can write the estimate
\[
\| \partial \omega^{\mu-n} w_1 w_2 \|_\infty \leq C \| \partial \omega^{\mu-n/2-\varepsilon} w_1 w_2 \|_2^{1/2} \| \partial \omega^{\mu-n/2+\varepsilon} w_1 w_2 \|_2^{1/2} \tag{3.39}
\]
We rewrite each one of the four norms in the RHS of (3.38) and (3.39) as \(\| \partial^{m_1} \omega^{\mu-m_0-n/2\pm\varepsilon} w_1 w_2 \|_2\) with \(m_1 = m_0\) in (3.38) and \(m_1 = 1 + m_0\) in (3.39), the values of \(m_0\) to be chosen later. The proof is now achieved by repeated application of Lemma 3.6 after identification of the relevant variables, (3.35) being implied by (3.38) and (3.34) and (3.37) by (3.39). The quantity \(\varepsilon\) will always be taken sufficiently small.

For \(n\) even we choose \(m_0 = n/2\). We apply (3.23) with \((\ell, \mu, k_1, k_2)\) replaced by \((n/2, \mu \pm \varepsilon, k-1, k-1)\) to (3.38) and by \((n/2+1, \mu \pm \varepsilon, k-1, k)\) to (3.39), thereby obtaining (3.35) and (3.34) respectively. Similarly we prove (3.37) by replacing \((\ell, \mu, k_1, k_2)\) by \((n/2 + 1, \mu \pm \varepsilon, k-2, k)\).

The condition (3.22L) is obviously satisfied while (3.22R) follows from (3.29) in the first two cases and from (3.36) in the last one.

For \(n\) odd we choose \(m_0 = (n+1)/2\) if \(1/2 + \varepsilon < \mu < n\) and \(m_0 = (n-1)/2\) if \(0 < \mu \leq 1/2 + \varepsilon\). In the first case we apply (3.23) with \((\ell, \mu, k_1, k_2)\) replaced by \(((n+1)/2, \mu - 1/2 \pm \varepsilon, k-1, k-1)\) to (3.38) and by \(((n+3)/2, \mu - 1/2 \pm \varepsilon, k-1, k)\) to (3.39), thereby obtaining (3.35) and (3.34) respectively. In the second case we apply (3.23) with \((\ell, \mu, k_1, k_2)\) replaced by \(((n-1)/2, \mu + 1/2 \pm \varepsilon, k-1, k-1)\) to (3.38) and by \(((n+1)/2, \mu + 1/2 \pm \varepsilon, k-1, k)\) to (3.39), thereby obtaining (3.35) and (3.34) respectively. In both cases (3.22L) is obviously satisfied while (3.22R) follows from (3.29).

We now introduce the following definition

**Definition 3.1.** A pair of nonnegative integers \((k, \ell)\) will be called admissible if it satisfies \(k \leq \ell, \ell > n/2\) and (3.29) and in addition \(k > n/2\) if \(\ell + 2 + \mu = n + k\), and (3.36) if \(n\) is even.
Admissible pairs exist only if \( \mu \leq n - 2 \). For \( \mu = n - 2 \), admissible pairs are pairs \((k, \ell)\) such that \( k = \ell > n/2 \). Admissible pairs always have \( k > 1 + \mu/2 \) and therefore \( k \geq 2 \). If \((k, \ell)\) is an admissible pair, so is \((k + j, \ell + j)\) for any positive integer \( j \). For \( n = 3, \mu = 1 \), the pair \((k, \ell) = (2, 2)\) is admissible.

We next derive a number of a priori estimates for solutions of the basic equations (3.4) (3.5), to be understood in the distribution sense. In the rest of this section, we assume \( n \geq 3 \), \( 0 < \mu \leq n - 2 \), and we denote by \((k, \ell)\) an admissible pair. For solutions of (3.4) (3.5), we shall derive a number of estimates similar to (3.13) in Lemma 3.2, and we shall state them in the shorter differential form corresponding to (3.14).

**Lemma 3.7.** Let \( 0 < \mu \leq n - 2 \) and let \((k, \ell)\) be an admissible pair. Let \( I \subset \mathbb{R}^+ \) be an open interval, and let \((w, s)\) be a solution of (3.4) (3.5) in \( L^\infty_{\text{loc}}(I, H^k \oplus X^\ell) \). Then \((w, s)\) satisfies the following estimates:

\[
\begin{align*}
|\partial_t|w|_k &\leq C t^{-2} \| \partial s; L^\infty \cap \dot{H}^\ell \| |w|_k \leq C t^{-2}|s|_\ell |w|_k , \quad (3.40) \\
|\partial_t|s; \dot{H}^{\ell+1} \| &\leq C t^{-2} \| \partial s; L^\infty \cap \dot{H}^\ell \| + C t^{-\gamma}|w|_k^2 , \quad (3.41) \\
|\partial_t|s; \dot{H}^{\ell_0} \| &\leq C t^{-2} \| s; L^\infty \cap \dot{H}^{\ell_0} \| \| \partial s; L^\infty \cap \dot{H}^{\ell_0} \| + C t^{-\gamma}|w|_{k-2} |w|_k , \quad (3.42) \\
|\partial_t|s|_\infty &\leq C t^{-2} \| s|_\infty \| \| \partial s|_\infty \| + C t^{-\gamma}|w|_{k-1} |w|_k , \quad (3.43) \\
|\partial_t|s|_\ell &\leq C t^{-2}|s|_\ell^2 + C t^{-\gamma}|w|_k^2 . \quad (3.44)
\end{align*}
\]

**Proof.** (3.40L). For any multi-index \( \alpha \) with \( |\alpha| \leq k \), we obtain from (3.4)

\[
\partial_t \partial^\alpha w = (2t^2)^{-1}U(1/t) \sum_{\beta \leq \alpha} \left( 2\partial^\beta s \cdot \nabla + (\partial^\beta \nabla \cdot s) \right) U^*(1/t) \partial^{\alpha-\beta} w . \quad (3.45)
\]

We now use a minor extension of Lemma 3.2 with \( p = 2 \), taking advantage of the unitarity of \( U(1/t) \) in \( L^2 \), with \( u = U^*(1/t)\partial^\alpha w, v = Ct^{-2}s, \) and \( h \) the sum of all terms not containing \( \nabla \partial^\alpha w \), thereby obtaining

\[
|\partial_t| \partial^\alpha w |_2 \leq Ct^{-2} \sum_{0 \leq j \leq k} \| \partial^{j+1} s \partial^{k-j} w |_2 , \quad (3.46)
\]

from which (3.40L) follows by Lemma 3.4 applied with \( s \) replaced by \( \partial s \).
For any multi-index $\alpha$ with $|\alpha| = \ell + 1$, we obtain from (3.5)

$$\partial_t \partial^\alpha s = t^{-2} \sum_{\beta \leq \alpha} (\partial^\beta s \cdot \nabla) \partial^{\alpha-\beta} s + t^{-\gamma} \partial^\alpha \nabla g(w, w) . \quad (3.47)$$

We now use Lemma 3.2 with $p = 2$, $u = \partial^\alpha s$, $v = t^{-2} s$, and $h$ the sum of all terms with $|\beta| \geq 1$ and of the contribution of $g$. The terms quadratic in $s$ all have at least one derivative on each $s$ and are estimated in $L^2$ by Lemma 3.3 with $s_1$ and $s_2$ replaced by $\partial s$, while the term containing $g$ is estimated by (3.32) of Corollary 3.1, thereby yielding (3.41).

(3.42). For any multi-index $\alpha$ with $|\alpha| = \ell_0$, we obtain again (3.47) from (3.5). We estimate the terms quadratic in $s$ directly by Lemma 3.3 with $s_1$ and $s_2$ replaced by $s$ and $\nabla s$ respectively, and with $\ell$ replaced by $\ell_0$. We estimate the contribution of $g$ by (3.30) of Corollary 3.1. This yields (3.42).

(3.43). The estimate of the terms quadratic in $s$ is obvious and that of $g$ follows from (3.34) of Corollary 3.1.

(3.40R) and (3.44). The norms of $s$ appearing in the LHS of (3.41)-(3.43) are precisely the norms defining $X^\ell$. The norms of $s$ appearing in the middle member of (3.40) and in the RHS of (3.41)-(3.43) are again norms in the definition of $X^\ell$, with the exception of $\| \partial s \|_\infty$. However by Lemma 3.1

$$\| \partial s \|_\infty \leq C \| s \|_\infty^{1-\sigma} \| \partial^{\ell+1} s \|_2^\sigma \leq C |s| K_{\ell} \quad (3.48)$$

with $1/(\ell + 1) < \sigma = 1/(\ell + 1 - n/2) < 1$ since $\ell > n/2$. This yields (3.40R) and (3.44) from (3.40L) and (3.41)-(3.43) respectively.

The next result is a linear estimate of higher norms needed for regularity.

**Lemma 3.8.** Let $0 < \mu \leq n - 2$ and let $(k, \ell)$ be an admissible pair. Let $I \subset IR^+$ be an open interval and let $(w, s)$ be a solution of (3.4) (3.5) in $L^\infty_{loc}(I, H^{k+1} \oplus X^{\ell+1})$. Then $(w, s)$ satisfies
the following estimates:

\[
\left| \partial_t \| w; \dot{H}^{k+1} \| \right| \leq C t^{-2} \left\{ \| \partial s \|_{\infty} \| w; \dot{H}^{k+1} \| + \| \partial^2 s; L^\infty \cap \dot{H}^\ell \| \| w_k \| \right\} ,
\]

(3.49)

\[
\left| \partial_t w \right|_{k+1} \leq C t^{-2} (|s|_{\ell} \| w \|_{k+1} + |s|_{\ell+1} \| w \|_k) ,
\]

(3.50)

\[
\left| \partial_t \| s; \dot{H}^{\ell+2} \| \right| \leq C t^{-2} \| \partial s; L^\infty \cap \dot{H}^\ell \| \| \partial^2 s; L^\infty \cap \dot{H}^\ell \| + C t^{-\gamma} \| w_k \| \| w \|_{k+1} ,
\]

(3.51)

\[
\left| \partial_t s \right|_{\ell+1} \leq C t^{-2} |s|_{\ell} |s|_{\ell+1} + C t^{-\gamma} \| w_k \| \| w \|_{k+1} .
\]

(3.52)

**Proof.** The proof is very similar to that of Lemma 3.7 and will be sketched briefly.

(3.49). We take the \(L^2\) norm of (3.45) now with \(|\alpha| = k+1\), we apply (the same minor extension of) Lemma 3.2 with \(p = 2\) and end up with (3.46) with \(k\) replaced by \(k + 1\). We separate the term \(j = 0\) and estimate the remaining terms by Lemma 3.4 with \(s\) replaced by \(\partial^2 s\). This yields (3.49).

(3.51). We take the \(L^2\) norm of (3.47), now with \(|\alpha| = \ell + 2\), we apply again Lemma 3.2 followed by Lemma 3.3 with \(s_1\) and \(s_2\) replaced by \(\partial s\) and \(\partial^2 s\) respectively, in order to estimate the terms quadratic in \(s\), and we estimate the term containing \(g\) by (3.33) of Corollary 3.1. This yields (3.51).

(3.50) and (3.52) follow immediately from (3.40) (3.49) and from (3.44) (3.51) respectively, from (3.48) and from the fact that by Lemma 3.1

\[
\| \partial^2 s \|_{\infty} \leq C \| \partial s \|_{\infty}^{1-\sigma} \| \partial^{\ell+2} s \|_2^{\sigma} \leq C |s|_{\ell+1}
\]

(3.53)

with the same \(\sigma\) as in (3.48).

\(\square\)

We shall also need some estimates for the difference of two solutions of (3.4) (3.5).

**Lemma 3.9.** Let \(0 < \mu \leq n - 2\) and let \((k, \ell)\) be an admissible pair. Let \(I \subset \mathbb{R}^+\) be an open interval and let \((w_1, s_1)\) and \((w_2, s_2)\) be two solutions of (3.4) (3.5) in \(L^\infty_{loc}(I, H^k \oplus X^\ell)\). Let
\(w_\pm = w_1 \pm w_2\) and \(s_\pm = s_1 \pm s_2\). Then the following estimates hold:

\[
\left| \partial_t |w_-|_{k-1} \right| \leq C t^{-2} \left\{ \| \partial s_+; L^\infty \cap \dot{H}^\ell \| \ |w_-|_{k-1} + \| s_-; L^\infty \cap \dot{H}^\ell \| \ |w_+|_{k} \right\} \\
\leq C t^{-2} \left\{ |s_+|_\ell \ |w_-|_{k-1} + |s_-|_{\ell-1} \ |w_+|_{k} \right\} ,
\]

\[
\left| \partial_t \| s_-; \dot{H}^\ell \| \right| \leq C t^{-2} \left\{ \| s_-; L^\infty \cap \dot{H}^\ell \| \| \partial s_+; L^\infty \cap \dot{H}^\ell \| \ + C t^{-\gamma} \ |w_-|_{k-1} \ |w_+|_{k} \right\} ,
\]

\[
\left| \partial_t \| s_- \|_\infty \right| \leq C t^{-2} \| s_- \|_\infty \| \partial s_+ \|_\infty \ + C t^{-\gamma} \ |w_-|_{k-1} \ |w_+|_{k} \right\} ,
\]

\[
\left| \partial_t |s_-|_{\ell-1} \right| \leq C t^{-2} |s_-|_{\ell-1} |s_+|_\ell + C t^{-\gamma} \ |w_-|_{k-1} \ |w_+|_{k} \right\} .
\]

**Proof.** The proof is again very similar to that of Lemma 3.7 and will be sketched briefly.

(3.54). Taking the difference of (3.4) for \(w_1\) and \(w_2\) and applying \(\partial^\alpha\) with \(|\alpha| \leq k - 1\) yields

\[
\partial_t \partial^\alpha w_- = (4t^2)^{-1} U(1/t) \sum_{\beta \leq \alpha} \left\{ \left( 2 \partial^\beta s_+ \cdot \nabla + (\partial^\beta \nabla \cdot s_+) \right) U^*(1/t) \partial^{\alpha-\beta} w_- \right. \\
+ \left. \left( 2 \partial^\beta s_- \cdot \nabla + (\partial^\beta \nabla \cdot s_-) \right) U^*(1/t) \partial^{\alpha-\beta} w_+ \right\} .
\]

We apply (the same minor extension of) Lemma 3.2 with \(p = 2\), \(u = \partial^\alpha w_-\) and \(v = (2t^2)^{-1} s_+\), followed by an application of Lemma 3.4 with \((w, s, k, \ell)\) replaced by \((w_-, \partial s_+, k - 1, \ell)\) for the terms with \((w_-, s_+)\) and by \((w_+, s_-, k, \ell)\) for the terms with \((w_+, s_-)\). This yields (3.54L), while (3.54R) follows from the definition of \(X^\ell\) and from (3.48).

(3.55). Taking the difference of (3.5) for \(s_1\) and \(s_2\) and applying \(\partial^\alpha\) with \(|\alpha| = \ell\) yields

\[
\partial_t \partial^\alpha s_- = (2t^2)^{-1} \sum_{\beta \leq \alpha} \left\{ (\partial^\beta s_+ \cdot \nabla) \partial^{\alpha-\beta} s_- + (\partial^\beta s_- \cdot \nabla) \partial^{\alpha-\beta} s_+ \right\} + t^{-\gamma} \partial^\alpha \nabla g(w_-, w_+) \right\} .
\]

We apply Lemma 3.2 with \(p = 2\), \(u = \partial^\alpha s_-\), \(v = (2t^2)^{-1} s_+\), we estimate the terms quadratic in \(s\) by Lemma 3.3 with \(s_1\) and \(s_2\) replaced by \(s_-\) and \(\partial s_+\), and the contribution of \(g\) by (3.31) of Corollary 3.1. This yields (3.55).

(3.56). We proceed in the same way, applying Lemma 3.2 with \(p = \infty\) to (3.59) with \(\alpha = 0\), \(u = s_-\) and \(v = (2t^2)^{-1} s_+\), we estimate the terms quadratic in \(s\) in the obvious way and the contribution of \(g\) by (3.34) of Corollary 3.1 This yields (3.56).
Finally (3.57) follows from the definition of \( X^\ell \), from (3.55) and its analogue with \( \ell \) replaced by \( \ell_0 \), from (3.56) and from (3.48).

\[ \square \]

**Lemma 3.10.** Let \( 0 < \mu \leq n - 2 \) and let \((k, \ell)\) be an admissible pair. Let \( I \subset \mathbb{R}^+ \) be an open interval and let \((w_1, s_1)\) and \((w_2, s_2)\) be two solutions of (3.4) (3.5) in \( L^\infty_{loc}(I, H^{k+1} \oplus X^{\ell+1}) \) and \( L^\infty_{loc}(I, H^k \oplus X^\ell) \) respectively. Let \( w_\pm = w_1 \pm w_2 \) and \( s_\pm = s_1 \pm s_2 \). Then the following estimates hold:

\[
|\partial_t|w_-|k| \leq Ct^{-2}\left\{ \| s_2; L^\infty \cap \dot{H}^\ell \| w_-|k| + \| \partial s_-; L^\infty \cap \dot{H}^\ell \| w_1|k| + \| s_- \|_\infty \| w_1; \dot{H}^{k+1} \| \right\}
\]

\[
\leq Ct^{-2}\left\{ |s_2|_\ell \ |w_-|_k | + |s_-|_\ell \ |w_1|_k | + \| s_- \|_\infty \ |w_1|_{k+1} \right\},
\]

(3.60)

\[
|\partial_t s_-; \dot{H}^{\ell+1} \| \leq Ct^{-2}\left\{ \| \partial s_-; L^\infty \cap \dot{H}^\ell \| \sum_{i=1,2} \| \partial s_i; L^\infty \cap \dot{H}^\ell \|
\]

\[ + \| s_- \|_\infty \| s_1; \dot{H}^{\ell+2} \| \right\} + Ct^{-\gamma} |w_-|_k |w_+|_k ,
\]

(3.61)

\[
|\partial_t s_-|_\ell \leq Ct^{-2}\left\{ |s_-|_\ell \ (|s_1|_\ell + |s_2|_\ell) + \| s_- \|_\infty \ |s_1|_{\ell+1} \right\} + Ct^{-\gamma} |w_-|_k |w_+|_k .
\]

(3.62)

**Proof.** The proof is again very similar to that of Lemma 3.7 and will be sketched briefly.

(3.60). We rewrite the difference of (3.4) for \( w_1 \) and \( w_2 \) as follows

\[
\partial_t w_- = (2t^2)^{-1}U(1/t)\left\{ (2s_2 \cdot \nabla + (\nabla \cdot s_2))U^*(1/t)w_- + (2s_- \cdot \nabla + (\nabla \cdot s_-))U^*(1/t)w_1 \right\}.
\]

(3.63)

We apply \( \partial^\alpha \) to (3.63) with \( |\alpha| \leq k \), use (the same minor extension of) Lemma 3.2 with \( p = 2 \), \( u = \partial^\alpha w_- \) and \( v = (2t^2)^{-1}s_2 \), estimate all the resulting terms by Lemma 3.4 with \((w, s)\) replaced by \((w_-, \partial s_2)\) or by \((w_1, \partial s_-)\), except for the terms \( s_- \cdot \nabla \partial^\alpha U^*(1/t)w_1 \) which we estimate directly in an obvious way. This yields (3.60L), while (3.60R) follows from the definition of \( X^\ell \) and from (3.48).

(3.61). We rewrite the difference of (3.5) for \( s_1 \) and \( s_2 \) similarly as

\[
\partial_t s_- = t^{-2}((s_2 \cdot \nabla)s_- + (s_- \cdot \nabla)s_1) + t^{-\gamma} g(w_-, w_+) .
\]

(3.64)

We apply \( \partial^\alpha \) to (3.64) with \( |\alpha| = \ell + 1 \), we use Lemma 3.2 with \( p = 2 \), \( u = \partial^\alpha s_- \) and \( v = t^{-2}s_2 \), we estimate all the resulting terms quadratic in \( s \) by Lemma 3.3 with \((s_1, s_2)\) replaced by
(∂s₂, ∂s₋) or by (∂s₋, ∂s₁), except for the terms $(s_- \cdot \nabla)\partial^\alpha s_1$ which we estimate directly in an obvious way. The contribution of $g$ is estimated by (3.32) of Corollary 3.1. This yields (3.61).

Finally (3.62) follows from the definition of $X^\ell$, from (3.57), (3.61) and from (3.48).

\[ \square \]

Lemma 3.11. Let $0 < \mu \leq n - 2$ and let $(k, \ell)$ be an admissible pair. Let $I \subset IR^+$ be an open interval and let $(w_1, s_1)$ and $(w_2, s_2)$ be two solutions of (3.4) (3.5) in $L^\infty_{loc}(I, H^k \oplus X^\ell)$. Let $w_\pm = w_1 \pm w_2$ and $s_\pm = s_1 \pm s_2$. Then the following estimates hold:

\[
\begin{align*}
|\partial_t|w_{-}|_{k-2} & \leq Ct^{-2}\left\{ |s_+|_\ell |w_-|_{k-2} + |s_-|_{\ell-2} |w_+|_k\right\}, \quad (3.65) \\
|\partial_t|s_{-}|_{\ell-2} & \leq Ct^{-2}|s_-|_{\ell-2} |s_+|_\ell + Ct^{\gamma}|w_-|_{k-2} |w_+|_k. \quad (3.66)
\end{align*}
\]

Proof. The proof is similar to that of Lemma 3.9.

(3.65). We apply (the same minor extension of) Lemma 3.2 with $p = 2$, $u = \partial^\alpha w_-$ and $v = (2t^2)^{-1}s_+$ to (3.58) with now $|\alpha| \leq k-2$. Applying Lemma 3.4 to the terms containing $(w_-, s_+)$ and omitting the irrelevant operator $U^s(1/t)$ for brevity in the terms containing $(w_+, s_-)$, we obtain

\[
|\partial_t|w_{-}|_{k-2} \leq Ct^{-2}\left\{ || s_+; L^\infty \cap \dot{H}^{\ell} || |w_-|_{k-2} + || s_- w_+; H^{k-1} || \right\} . \quad (3.67)
\]

We now distinguish two cases.

If $\ell > n/2 + 1$, we estimate the last norm in (3.67) by Lemma 3.4 with $(w, s, k, \ell)$ replaced by $(w_+, s_-, k-1, \ell-1)$ so that

\[
\begin{align*}
|| s_- w_+; H^{k-1} || & \leq C || s_-; L^\infty \cap \dot{H}^{\ell-1} || |w_+|_{k-1} \\
& \leq C||s_-|_{\ell-2} |w_+|_k.
\end{align*}
\]

If $n/2 < \ell \leq n/2 + 1$, namely for the lowest admissible value \( \ell = \ell_0 + 1 \), we estimate directly

\[
\begin{align*}
|| s_- w_+; H^{k-1} || & \leq C \sum_{j + j' \leq k-1} || \partial^j s_- \partial^{j'} w_+ ||_2 \\
& \leq C \sum_{j + j' \leq k-1} || \partial^j s_- ||_{r_1} || \partial^{j'} w_+ ||_{r_2} \quad (3.69)
\end{align*}
\]

with

\[
0 \leq j \leq k-1 \leq \ell - 1 \leq n/2,
\]
\[
\begin{cases}
0 \leq \delta(r_1) = \ell - 1 - j \leq n/2 - j , \\
0 \leq \delta(r_2) = n/2 - \ell + 1 + j.
\end{cases}
\] (3.70)

By Lemma 3.1, we then estimate
\[
\| \partial^j s_- \|_{r_1} \leq C \| \partial^{\ell-1} s_- \|_2 = C \| s_-; \dot{H}^{\ell_0} \|
\] (3.71)

with the only exception of the case of even \( n, \ell = n/2 + 1 \) and \( j = 0 \) where \( r_1 = \infty \) and that norm reduces to \( \| s_- \|_\infty \), so that in all cases
\[
\| \partial^j s_- \|_{r_1} \leq C |s_-|_{\ell-2} .
\] (3.72)

On the other hand
\[
\delta(r_2) + j' \leq n/2 - \ell + 1 + k - 1 < k
\]

and therefore
\[
\| \partial^{j'} w_+ \|_{r_2} \leq C |w_+|_{k} .
\] (3.73)

Substituting (3.72) and (3.73) into (3.69), substituting either the result thereof or (3.68) into (3.67) and using (3.48) yields (3.65).

(3.66). We apply Lemma 3.2 with \( p = 2, u = \partial^\alpha s_-, v = (2t^2)^{-1}s_+ \) to (3.59) with \( |\alpha| = \ell - 1 \) and obtain
\[
\left| \partial_\ell \right| \| \partial^{\ell-1} s_- \|_2 \leq Ct^{-2} \| \partial^{\ell-1}((\partial s_+ s_-)) \|_2 + Ct^{-\gamma} \| \partial^\ell g(w_-,w_+) \|_2 .
\] (3.74)

We distinguish again two cases.

If \( \ell > n/2 + 1 \), we apply Lemma 3.3 with \( (s_1, s_2, \ell) \) replaced by \( (\partial s_+, s_-, \ell - 1) \) to estimate
\[
\| \partial^{\ell-1}((\partial s_+ s_-)) \|_2 \leq C \| \partial s_+; L^\infty \cap \dot{H}^{\ell-1} \| \| s_-; L^\infty \cap \dot{H}^{\ell-1} \|
\]
\[
\leq C |s_+|_{\ell-1} |s_-|_{\ell-2} .
\] (3.75)

If \( n/2 < \ell \leq n/2 + 1 \), we estimate directly
\[
\| \partial^{\ell-1}((\partial s_+ s_-)) \|_2 \leq C \sum_{j \leq \ell-1} \| \partial^j s_- \|_{r_1} \| \partial^{\ell-j} s_+ \|_{r_2}
\] (3.76)

with \( r_1 \) and \( r_2 \) again given by (3.70), so that (3.72) holds as before, while \( \delta(r_2) + \ell - j = n/2 + 1 \) so that
\[
\| \partial^{\ell-j} s_+ \|_{r_2} \leq C |s_+|_{\ell} .
\] (3.77)
Substituting (3.72) (3.77) into (3.76), and either the result thereof or (3.75) into (3.74) and estimating the contribution of \( g \) by (3.30) of Corollary 3.1 yields
\[
\left| \partial_t \| \partial^{\ell-1} s_\ell \|_2 \right| \leq Ct^{-2} |s_\ell_{|\ell-2} |s_+|_\ell + Ct^{-\gamma} |w_-|_{k-2} |w_+|_k .
\] (3.78)

In the case of even \( n \), we estimate in addition
\[
\left| \partial_t \| s_- \|_\infty \right| \leq Ct^{-2} \| s_- \|_\infty \| \partial s_+ \|_\infty + Ct^{-\gamma} |w_-|_{k-2} |w_+|_k
\] (3.79)
in the same way as in the proof of (3.56), but estimating now the contribution of \( g \) by (3.37) instead of (3.34) of Corollary 3.1.

Collecting (3.78), its analogue with \( \ell \) replaced by \( \ell_0 + 1 \), and in addition (3.79) for even \( n \) yields (3.66).

□

**Remark 3.2.** In Lemmas 3.7-3.11, not all the properties in the definition of admissibility for \((k, \ell)\) are used in every single estimate. The condition \( \ell > n/2 \) is used in many places. However the condition \( k \leq \ell \) is used only in the estimates of \( w \) or \( w_- \) from (3.4), but not in the estimates of \( s \) or \( s_- \) from (3.5). Conversely the condition (3.29) is used only in the estimates of \( s \) or \( s_- \) from (3.5), but not in the estimates of \( w \) or \( w_- \) from (3.4). Furthermore that condition is used only through the estimates (3.30)-(3.35) of Corollary 3.1, so that (3.29) could be replaced by (3.30)-(3.35) in the definition of admissibility, thereby opening the possibility of treating more general nonlinearities \( \tilde{g} \) than simply (1.2). Finally the condition (3.36) for \( n \) even is used only through (3.37) in Lemma 3.11.

**4 The Cauchy problem at finite times.**

In this section, we study the Cauchy problem for the basic system (3.4) (3.5) at finite times. We use the basic spaces \( H^k \) and \( X^\ell \) defined at the beginning of Section 3, as well as the notation (3.11) for the norm in those spaces. Admissible pairs \((k, \ell)\) are defined in Definition 3.1. We prove that the Cauchy problem for the system (3.4) (3.5) is locally well posed in \( IR^+ \setminus \{0\} \) for positive initial time \( t_0 \) and initial data in \( H^k \oplus X^\ell \) for admissible \((k, \ell)\). A similar result holds in \( IR^- \setminus \{0\} \). We make no effort to study the situation as \( t \to 0 \), since that is of no interest
for later purposes, and since the system (3.4) (3.5) is singular at \( t = 0 \) anyway because of the choice (1.2) of \( g_0 \) and of the change of variables (2.14) from \( u \) to \((w,s)\). The main result can be stated as follows.

**Proposition 4.1.** Let \( \gamma > 0 \), \( n \geq 3 \) and \( 0 < \mu \leq n - 2 \). Let \((k,\ell)\) be an admissible pair. Let \( t_0 > 0 \). Then for any \((w_0,s_0) \in H^k \oplus X^\ell\), there exist \( T_\pm \) with \( 0 \leq T_- < t_0 < T_+ \leq \infty \) such that:

1. The system (3.4) (3.5) has a unique solution \((w,s) \in C(I,H^k \oplus X^\ell)\) with \((w,s)(t_0) = (w_0,s_0)\), where \( I = (T_-,T_+)\). If \( T_- > 0 \) (resp. \( T_+ < \infty \)), then \(|w(t)|_k + |s(t)|_\ell \to \infty \) when \( t \) decreases to \( T_- \) (resp. increases to \( T_+ \)).

2. If \((w_0,s_0) \in H^{k'} \oplus X^{\ell'}\) for some admissible pair \((k',\ell')\) with \( k' \geq k \) and \( \ell' \geq \ell \), then \((w,s) \in C(I,H^{k'} \oplus X^{\ell'})\).

3. For any compact subinterval \( J \subset I \), the map \((w_0,s_0) \to (w,s)\) is continuous from \( H^{k-1} \oplus X^{\ell-1}\) to \( L^\infty(J,H^{k-1} \oplus X^{\ell-1})\) uniformly on the bounded sets of \( H^k \oplus X^\ell\), and is pointwise continuous from \( H^k \oplus X^\ell\) to \( L^\infty(J,H^k \oplus X^\ell)\).

**Proof.** Most of the proof proceeds by standard arguments, and we shall mainly concentrate on those which are not. We concentrate on the case of increasing time, namely \( t \geq t_0 \). The case of decreasing time \( t \leq t_0 \) can be treated in the same way, possibly after changing \( t \) to \(1/t\) and \( s \) to \(-s\), thereby transforming (3.4) (3.5) into the system

\[
\partial_t w = (1/2)U(t)(2s \cdot \nabla + (\nabla \cdot s))U(-t)w
\]

\[
\partial_t s = (s \cdot \nabla)s + t^{\gamma-2} \nabla g(w,w)
\]

and considering that system for increasing time.

The (negative) powers of \( t \) in the coefficients of (3.4) (3.5) are bounded on compact subintervals of \([t_0,\infty)\), actually bounded on \([t_0,\infty)\), and play no role in the present problem. We omit them for brevity.

The proof proceeds in several steps.
Step 1. We introduce a parabolic regularization and consider the system

\[ \partial_tw = \eta \Delta w + U(1/t)(s \cdot \nabla + (1/2)(\nabla \cdot s))U^*(1/t)w \equiv \eta \Delta w + F(w, s) \] (4.3)

\[ \partial ts = \eta \Delta s + (s \cdot \nabla)s + \nabla g(w, w) \equiv \eta \Delta s + G(w, s) \] (4.4)

with \( \eta > 0 \). The Cauchy problem for the system (4.3) (4.4) can be recast in the integral form

\[ \begin{pmatrix} w \\ s \end{pmatrix}(t) = V_\eta(t - t_0) \begin{pmatrix} w_0 \\ s_0 \end{pmatrix} + \int_{t_0}^{t} dt' \ V_\eta(t - t') \begin{pmatrix} F(w, s) \\ G(w, s) \end{pmatrix}(t') \] (4.5)

where \( V_\eta(t) = \exp(\eta t \Delta) \). The operator \( V_\eta(t) \) is a contraction in \( H^k \oplus X^\ell \), while the operator \( \nabla V_\eta(t) \) satisfies the bound

\[ \| \nabla V_\eta(t); \mathcal{L}(H^k \oplus X^\ell) \| \leq C(\eta t)^{-1/2} \] (4.6)

From these facts and from estimates on \( F, G \) similar to, but simpler than, those in Lemma 3.7, it follows by a standard contraction argument that there exists \( T > 0 \) depending only on \( \eta \) and on \(|w_0|_k + |s_0|_\ell\) such that the system (4.5) has a unique solution \((w_\eta, s_\eta) \in C([t_0, t_0 + T], H^k \oplus X^\ell)\).

Step 2. Estimates uniform in \( \eta \). We estimate \((w_\eta, s_\eta)\) by Lemma 3.7, taking into account the fact that by Lemma 3.2, the term \( \eta \Delta w \) in (4.3) (4.4) does not contribute to the estimates. Let

\[ y(t) = |w_\eta(t)|_k, \quad z(t) = |s_\eta(t)|_\ell \] (4.7)

We obtain from Lemma 3.7 (with the powers of \( t \) omitted)

\[ \begin{cases} \partial_t y \leq Cyz \\ \partial_t z \leq C(y^2 + z^2) \end{cases} \]

and by integration, with \( y_0 = y(t_0), z_0 = z(t_0) \),

\[ y(t) + z(t) \leq (y_0 + z_0) (1 - C(t - t_0)(y_0 + z_0))^{-1} \leq 2(y_0 + z_0) \]

for \( 2C(t - t_0)(y_0 + z_0) \leq 1 \), so that for some \( T \) depending only on \(|w_0|_k + |s_0|_\ell\), \((w_\eta(t), s_\eta(t))\) is estimated a priori in \( C([t_0, t_0 + T], H^k \oplus X^\ell) \) uniformly in \( \eta \). By a standard globalisation argument, the solution constructed in Step 1 can be extended to that new interval, now independent of \( \eta \).
Step 3. Limit \( \eta \to 0 \). Let \( I_0 = [t_0, t_0 + T] \) be the interval, independent of \( \eta \), obtained in Step 2. We now prove that \((w_\eta, s_\eta)\) converges in norm in \( L^\infty(I_0, H^{k-1} \oplus X^{\ell-1}) \). We know already that \((w_\eta, s_\eta)\) is estimated uniformly in \( \eta \) according to

\[
\| w_\eta; L^\infty(I_0, H^k) \| \leq a \quad , \\
\| s_\eta; L^\infty(I_0, X^\ell) \| \leq b \quad .
\]

Let now \( \eta_1, \eta_2 > 0 \) and let \( w_i = w_{\eta_i}, i = 1, 2 \). We estimate the difference \((w_1 - w_2, s_1 - s_2)\) by Lemma 3.9, except for the contribution of the terms coming from \((\eta \Delta w, \eta \Delta s)\) in (4.3) (4.4), which are estimated directly as follows. For \(|\alpha| \leq k - 1\), we estimate

\[
\partial_t \| \partial^\alpha (w_1 - w_2) \|_2^2 \leq 2 \text{Re} < \partial^\alpha (w_1 - w_2), \eta_1 \partial^\alpha \Delta w_1 - \eta_2 \partial^\alpha \Delta w_2 > + \text{other terms} \\
\leq 4a^2 (\eta_1 + \eta_2) + \text{other terms}
\]

where the other terms are those coming from Lemma 3.9. The contribution of the terms \( \eta \Delta s \) from (4.4) to the estimate of \( s_1 - s_2 \) are treated in the same way. Defining now

\[
y(t) = |(w_1 - w_2)(t)|_{k-1} \quad , \quad z(t) = |(s_1 - s_2)(t)|_{\ell-1} \quad ,
\]
and combining the previous estimates with Lemma 3.9, we obtain

\[
\begin{cases}
\partial_t y^2 \leq C(a^2 (\eta_1 + \eta_2) + by^2 + ayz) \\
\partial_t z^2 \leq C(b^2 (\eta_1 + \eta_2) + bz^2 + ayz)
\end{cases}
\]

for \( t \in I_0 \), which together with \( y(t_0) = 0, z(t_0) = 0 \), implies that \( y \) and \( z \) tend to zero uniformly in \( I_0 \) when \( \eta_1, \eta_2 \to 0 \) by Gronwall’s Lemma. As a consequence \((w_\eta, s_\eta)\) converges to a limit \((w, s)\) in norm in \((C \cap L^\infty)(I_0, H^{k-1} \oplus X^{\ell-1})\). Clearly, \((w, s)\) is a solution of the system (3.4)(3.5) with the appropriate initial data. Furthermore, since \((w_\eta, s_\eta)\) is uniformly bounded in \( L^\infty(I_0, H^k \oplus X^\ell) \), it follows from a standard compactness argument that \((w, s)\) belongs to that space with the same bound and that \((w_\eta, s_\eta)\) converges to \((w, s)\) in that space in the weak-* sense.

Step 4. Uniqueness. That step is independent of the previous ones and could equally well have been made at the very beginning. Actually uniqueness in \( L^\infty(\cdot, H^k \oplus X^\ell) \) follows immediately
from Lemma 3.9 and from Gronwall’s Lemma.

**Step 5. Regularity.** It follows immediately from Lemmas 3.7 and 3.8 and from Gronwall’s Lemma that a solution \((w, s) \in L^\infty(I, H^k \oplus X^\ell)\) with initial data \((w, s)(t_0) \in H^{k+1} \oplus X^{\ell+1}\) belongs to \(L^\infty(I, H^{k+1} \oplus X^{\ell+1})\) for the same interval \(I\). A similar regularity with general \((k', \ell')\) follows by iteration.

Using the previous five steps and standard arguments, one can then prove most of Proposition 4.1, with the only restriction that Parts (1) (3) hold only with \(C(\cdot, H^k \oplus X^\ell)\) replaced by \(C(\cdot, H^{k-1} \oplus X^{\ell-1}) \cap L^\infty_{\text{loc}}(\cdot, H^k \oplus X^\ell)\), with continuity in Part (3) being in norm in the former space and in the weak-\(*\) sense in the latter, while Part (2) holds only with a similar restriction.

We now turn to the proof of the missing continuities, which is more delicate. We follow a method used in [1]. We first derive an additional estimate for the difference of two solutions of (3.4) (3.5) in \(L^\infty(I, H^k \oplus X^\ell)\) for some interval \(I\). For brevity we introduce the short hand notation \(y = (w, s), y_0 = (w_0, s_0)\) and for two solutions \(y_i = (w_i, s_i), y_{0i} = (w_{0i}, s_{0i}), i = 1, 2, y_- = y_1 - y_2\) and \(y_{0-} = y_{01} - y_{02}\). Furthermore for \((k, \ell)\) an admissible pair and \(\theta\) an integer, \(-2 \leq \theta \leq 1\), we denote \(j + \theta = (k + \theta, \ell + \theta)\) and \(Y^{j+\theta} = H^{k+\theta} \oplus X^{\ell+\theta}\). Let now \(y_i \in L^\infty(I, Y^j), i = 1, 2,\) be two solutions of (3.4) (3.5) with initial data \(y_{0i}\) at time \(t_0 \in I\) for some compact interval \(I\), satisfying the estimate

\[
\| y_i; L^\infty(I, Y^j) \| \leq a < \infty , \quad i = 1, 2 . \tag{4.12}
\]

Assume furthermore that \(y_{01} \in Y^{j+1}\), so that by Step 5 \(y_1 \in L^\infty(I, Y^{j+1})\). We now estimate \(y_-\) in \(Y^j\) by Lemma 3.10. From (3.60) (3.62) we obtain

\[
\partial_t |y_-|_j \leq C(a |y_-|_{j+1} + \| s_- \|_{\infty} |y_1|_{j+1}) . \tag{4.13}
\]

By Lemma 3.8, esp. (3.50) (3.52), we estimate

\[
\partial_t |y_1|_{j+1} \leq Ca |y_1|_{j+1}
\]

and therefore by Gronwall’s Lemma, for all \(t \in I\),

\[
|y_1|_{j+1} \leq C(a, |I|) |y_{01}|_{j+1} . \tag{4.14}
\]
We next estimate for $\ell > n/2$, possibly by using (3.10),
\[
\begin{cases}
\|s_{-}\|_{\infty} \leq C|s_{-}|_{\ell-2} & \text{for } \ell \geq n/2 + 1, \\
\|s_{-}\|_{\infty} \leq C|s_{-}|_{\ell-2}^{1/2}|s_{-}|_{\ell-1}^{1/2} & \text{for } n \text{ odd, } \ell = (n + 1)/2.
\end{cases}
\]
We now estimate
\[
\partial_t|y_{-}|_{j-\theta} \leq Ca|y_{-}|_{j-\theta}
\]
with $\theta = 1$, by Lemma 3.9, esp. (3.54) (3.57), and with $\theta = 2$ by Lemma 3.11, esp. (3.65) (3.66), so that by Gronwall’s Lemma again, for $\theta = 1, 2$ and for all $t \in I$
\[
|y_{-}|_{j-\theta} \leq C(a, |I|)|y_{0-}|_{j-\theta}
\]
and therefore by (4.15), for all $t \in I$,
\[
\|s_{-}\|_{\infty} \leq C(a, |I|)\|y_{0-}\|_{b}
\]
where
\[
\begin{cases}
\|y_{0-}\|_{b} = |y_{0-}|_{j-2} & \text{for } \ell \geq n/2 + 1, \\
\|y_{0-}\|_{b} = |y_{0-}|_{j-2}^{1/2}|y_{0-}|_{j-1}^{1/2} & \text{for } n \text{ odd, } \ell = (n + 1)/2.
\end{cases}
\]
Substituting (4.14) and (4.17) into (4.13) and applying Gronwall’s Lemma again, we obtain for all $t \in I$
\[
|y_{-}|_{j} \leq C(a, |I|)(|y_{0-}|_{j+1}|y_{0-}|_{b} |y_{01}|_{j+1})
\]
We now come back to the proof of the missing continuities, which will make an essential use of the estimate (4.19).

**Step 6.** Continuity of the solutions in $H^k \oplus X^\ell$. Let $I \subset \mathbb{R}^+ \setminus \{0\}$ be a compact interval and let $(w, s) \in C(I, H^{k-1} \oplus X^{\ell-1}) \cap L^{\infty}(I, H^{k} \oplus X^{\ell})$ be solution of the system (3.4) (3.5) with initial data $(w_0, s_0) \in H^k \oplus X^{\ell}$ at some time $t_0 \in I$. We shall prove that $(w, s) \in C(I, H^k \oplus X^{\ell})$. We use the short hand notation $y, y_0, Y^j, \text{etc.}$ introduced above. We introduce a regularisation defined as follows. We choose a function $\psi_1 \in \mathcal{S}(\mathbb{R}^n)$ such that $\int dx \psi_1(x) = 1$ and such that $|\xi|^{-m}(\hat{\psi}_1(\xi) - 1)|_{\xi=0} = 0$ for $m = 1, 2$, we define $\psi_{\varepsilon}(x) = \varepsilon^{-m}\psi_1(x/\varepsilon)$ so that $\hat{\psi}_{\varepsilon}(\xi) = \hat{\psi}_1(\varepsilon\xi)$ and we define the regularisation by $f \rightarrow f_{\varepsilon} = \psi_{\varepsilon} \ast f$ for all $f \in \mathcal{S}'$. Clearly the regularisation is a bounded operator with norm at most $\|\psi_1\|_1$ and tends strongly to the unit operator when $\varepsilon \rightarrow 0$ in $Y^j$ for all relevant $j$. 

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We now regularize the initial data $y_0$ to $y_{0\varepsilon}$. By the previous steps, $y_{0\varepsilon}$ generates a solution $y_\varepsilon$ of (3.4) (3.5). For $\varepsilon$ sufficiently small, and possibly after a small restriction of $I$, that solution can be assumed to be in $L^\infty(I,Y^j)$ for the same interval $I$ as $y$ and to be bounded there uniformly in $\varepsilon$, namely

$$\| y; L^\infty(I,Y^j) \| \lor \| y_\varepsilon; L^\infty(I,Y^j) \| \leq a < \infty .$$

(4.20)

Furthermore, since $y_{0\varepsilon} \in Y^{j+1}$, by regularity (Step 5), $y_\varepsilon \in C(I,Y^j) \cap L^\infty(I,Y^{j+1})$. In order to prove that $y \in C(I,Y^j)$ it is therefore sufficient to prove that $y_\varepsilon$ converges to $y$ in norm in $L^\infty(I,Y^j)$. For that purpose we apply the estimate (4.19) with $y_1 = y_\varepsilon$, $y_2 = y$, $y_- = y_\varepsilon - y$. Now for all $f$

$$\| \partial f_\varepsilon \|_2 = \| \partial(f_\varepsilon * f) \|_2 \leq \| \partial f_\varepsilon \|_1 \| f \|_2 \leq \varepsilon^{-1} \| \partial f_1 \|_1 \| f \|_2 ,$$

(4.21)

$$\| f_\varepsilon - f \|_2 = \| f_\varepsilon * f - f \|_2 = \| (\hat{\psi}_\varepsilon(\xi) - 1) \hat{f}(\xi) \|_2 \leq \varepsilon^\theta \| |\xi|^{-\theta} (\hat{\psi}_\varepsilon(\xi) - 1) \|_\infty \| f; \dot{H}^{\theta} \|$$

(4.22)

for $\theta = 1, 2$. Furthermore for $n$ even and $s \in L^\infty \cap \dot{H}^{n/2} \cap \dot{H}^{n/2+\theta}$ so that $|\xi|^{n/2}s(\xi) \in L^2$, and for $\theta = 1, 2$,

$$\| \hat{\psi}_\varepsilon * s - s \|_\infty \leq \| (\hat{\psi}_\varepsilon - 1)\hat{s} \|_1 \leq \varepsilon^\theta \| |\xi|^{-n/2-\theta} (\hat{\psi}_1(\xi) - 1) \|_2 \| s; \dot{H}^{n/2+\theta} \| .$$

(4.23)

It follows from (4.21)-(4.23) that for $\theta = 1, 2$,

$$\left\{ \begin{array}{c}
|y_{0\varepsilon}|_{j+1} \leq C \varepsilon^{-1}|y_0|_j , \\
|y_0 - |y_\varepsilon|_{j-\theta} \leq C \varepsilon^\theta|y_0|_j .
\end{array} \right.$$  

(4.24)

Substituting the estimate (4.24) into (4.18) (4.19) and using the fact that $|y_0-|_j \to 0$ when $\varepsilon \to 0$ shows that $|y_-|_j$ tends to zero when $\varepsilon \to 0$ uniformly for $t \in I$, which completes the proof.

**Step 7.** Continuity with respect to initial data in $C(\cdot, H^k \oplus X^\ell)$. From Steps 1-5 and general arguments, it follows only that the solution $(w,s)$ so far constructed is norm continuous in $C(\cdot, H^{k-1} \oplus X^{\ell-1})$ and weak-* continuous in $L^\infty(\cdot, H^k \oplus X^\ell)$ as a function of the initial data $(w_0, s_0) \in H^k \oplus X^\ell$. We now prove strong continuity in $C(\cdot, H^k \oplus X^\ell)$. We use again the short hand notation $y$, $y_0$, $Y^j$, etc. introduced above. Let $I \subset IR^+ \setminus \{0\}$ be a compact interval and
let $y \in (C \cap L^\infty)(I, Y^j)$ be a fixed solution of the system (3.4) (3.5) with initial data $y_0 \in Y^j$ at some time $t_0 \in I$. Let $y'_0$ be an initial data in a small neighborhood of $y_0$ in $Y^j$ and let $y'$ be the solution of (3.4) (3.5) thereby generated. We also consider the regularized initial data $y_{0\varepsilon}$ and $y'_{0\varepsilon}$, and the solutions $y_\varepsilon$ and $y'_\varepsilon$ thereby generated. By taking $y'_0$ sufficiently close to $y_0$ and $\varepsilon$ sufficiently small and possibly after a small restriction of $I$, we can assume that $y'$ and $y_\varepsilon$, $y'_\varepsilon$ are in $L^\infty(I, Y^j)$ for the same interval $I$ as $y$, and are bounded there uniformly in $y'_0$ and in $\varepsilon$ so that both (4.20) and its analogue with $y$ replaced by $y'$ hold. For all $t \in I$, we estimate

$$|y - y'|_j \leq |y_\varepsilon - y|_j + |y_\varepsilon - y'_\varepsilon|_j + |y'_\varepsilon - y'|_j$$

(4.25)

and we estimate the three norms in the RHS by (4.19) with $(y_1, y_2)$ replaced by $(y_\varepsilon, y)$, $(y_\varepsilon, y'_\varepsilon)$ and $(y'_\varepsilon, y')$ respectively. Using in addition the first inequality in (4.24), we obtain

$$|y - y'|_j \leq C(a, |I|)\left(|y_\varepsilon - y_0|_j + |y_\varepsilon - y'_\varepsilon|_j + |y'_\varepsilon - y_0|_j + a\varepsilon^{-1}\left(\|y_\varepsilon - y_0\|_b + \|y_\varepsilon - y'_\varepsilon\|_b + \|y'_\varepsilon - y_0\|_b\right)\right).$$

(4.26)

We now estimate

$$|y_{0\varepsilon} - y'_{0\varepsilon}|_j \leq C|y_0 - y'_0|_j$$

$$|y'_{0\varepsilon} - y'_0|_j \leq |y'_{0\varepsilon} - y_{0\varepsilon}|_j + |y_{0\varepsilon} - y_0|_j + |y_0 - y'_0|_j \leq |y_{0\varepsilon} - y_0|_j + C|y'_0 - y_0|_j$$

and similarly

$$\|y_{0\varepsilon} - y'_{0\varepsilon}\|_b \leq C\|y_0 - y'_0\|_b \leq C|y_0 - y'_0|_j$$

$$\|y'_{0\varepsilon} - y'_0\|_b \leq \|y_{0\varepsilon} - y_0\|_b + C\|y'_0 - y_0\|_b \leq \|y_{0\varepsilon} - y_0\|_b + C|y'_0 - y_0|_j.$$

Substituting those estimates into (4.26) yields

$$|y - y'|_j \leq C(a, |I|)\left(|y_{0\varepsilon} - y_0|_j + a\varepsilon^{-1}\|y_{0\varepsilon} - y_0\|_b + (1 + a\varepsilon^{-1})|y_0 - y'_0|_j\right)$$

(4.27)

which can be made to tend to zero uniformly for $t \in I$ by letting $y'_0$ tend to $y_0$ in $Y^j$ and letting $\varepsilon$ tend to zero, in that order.
Remark 4.1. Whereas the map $y_0 \to y$ is uniformly continuous from $Y^j$ to $(C \cap L^\infty)(\cdot, Y^j)$ on the bounded sets of $Y^j$, it is only pointwise continuous from $Y^j$ to $(C \cap L^\infty)(\cdot, Y^j)$. In fact Step 7 is performed for fixed $y_0$, and does not yield an estimate of $\|y - y'; L^\infty(\cdot, Y^j)\|$ in terms of $|y_0 - y_0'|$. This is a standard situation in that kind of problems.

5 The auxiliary system at infinite time. Existence and asymptotics I.

In this section we study the existence of solutions in a neighborhood of infinity in time for the auxiliary system

$$\begin{align*}
\partial_t w &= (2t^2)^{-1} U(1/t)(2s \cdot \nabla + (\nabla \cdot s))U^*(1/t)w \\
\partial_t s &= t^{-2}(s \cdot \nabla)s + t^{-\gamma} \nabla g(w, w) \\
\end{align*}$$

(2.30) $\equiv$ (3.4) $\equiv$ (5.1)

(2.31) $\sim$ (3.5) $\equiv$ (5.2)

where $g$ is defined by (3.1) (3.2), and we study the asymptotic behaviour in time of those solutions by essentially constructing local wave operators at infinity for the system (5.1) (5.2) as compared with the auxiliary free equation

$$\partial_t s_0 = t^{-\gamma} \nabla g(w_+, w_+) .$$

(2.33) $\sim$ (3.7) $\equiv$ (5.3)

The general solution of (5.3) can be written as

$$s_0(t) = s_0(1) + \int_1^t dt' t'^{-\gamma} \nabla g(w_+, w_+) .$$

(5.4)

Since from (5.2) and (5.4) the functions $s(t)$ and $s_0(t)$ are expected to increase as $t^{1-\gamma}$, we define the functions

$$\bar{s}(t) = t^{\gamma-1} s(t) , \quad \bar{s}_0(t) = t^{\gamma-1} s_0(t)$$

(5.5)

which are expected to be bounded in time. We shall use those functions throughout this section. It follows from Corollary 3.1 that for admissible $(k, \ell)$ and for $(w_+, s_0(1)) \in H^k \oplus X^\ell$, $\bar{s}_0(t) \in (C \cap L^\infty)([1, \infty), X^\ell)$ and that $s_0(t)$ satisfies the estimate

$$\| \bar{s}_0; L^\infty([1, \infty); X^\ell) \| \leq |s_0(1)|_{\ell} + C(1 - \gamma)^{-1} |w_+|^2_k .$$

We shall use the basic spaces $H^k$ and $X^\ell$ defined at the beginning of Section 3, as well as the notation (3.11) for the norms in those spaces. We recall that admissible pairs $(k, \ell)$ are defined in Definition 3.1. In all this section, we assume that $n \geq 3$ and $0 < \mu \leq n - 2$. The
letter $C$ in subsequent estimates will denote various constants depending on $n$, $\mu$ and possibly on an admissible pair $(k, \ell)$. On the other hand we shall keep the dependence of the estimates on $\gamma$ sufficiently explicit for the constants $C$ to be uniform in $\gamma$ in the range of $\gamma$ where the estimates are stated. For instance a factor $\gamma^{-1}$ will be kept explicitly in estimates valid for all $\gamma > 0$, but will be included in $C$ for estimates valid for $\gamma > 1/2$.

In the first three propositions, we study the existence of solutions of the system (5.1) (5.2) defined in a neighborhood of infinity and some of their asymptotic properties. All those results hold for all $\gamma > 0$, but we restrict our attention to $0 < \gamma < 1$ in order to simplify the exposition, as explained in the introduction. Most of those results are consequences of Proposition 4.1 and of a priori estimates where we now carefully keep track of the time dependence. We begin with the existence of solutions defined in a neighborhood of infinity in time.

**Proposition 5.1.** Let $0 < \gamma < 1$ and let $(k, \ell)$ be an admissible pair. Let $(w_0, \tilde{s}_0) \in H^k \oplus X^\ell$ and define $y_0 = |w_0|_k$ and $\tilde{z}_0 = |	ilde{s}_0|_\ell$. Then there exists $T_0 < \infty$, depending on $y_0$, $\tilde{z}_0$, such that for all $t_0 \geq T_0$, there exists $T \leq t_0$, depending on $y_0$, $\tilde{z}_0$ and $t_0$, such that the system (5.1) (5.2) with initial data $w(t_0) = w_0$, $s(t_0) = \tilde{s}_0 t_0^{1-\gamma}$, has a unique solution $(w, s)$ such that $(w, \tilde{s}) \in (C \cap L^\infty)([T, \infty), H^k \oplus X^\ell)$. One can take

$$\gamma T_0^\gamma = C \left( \tilde{z}_0 + (1 - \gamma)^{-1} y_0^2 \right),$$

$$T = \gamma T_0^\gamma t_0^{1-\gamma},$$

and the solution $(w, s)$ is estimated by

$$|w|_k \leq 2y_0,$$

$$|s|_\ell \leq \left( 2\tilde{z}_0 + C (1 - \gamma)^{-1} y_0^2 \right) (t_0 \lor t)^{1-\gamma}$$

for all $t \geq T$.

**Proof.** The result follows from Proposition 4.1 and standard globalisation arguments, provided we can derive (5.8) (5.9) as a priori estimates under the assumptions made on $t_0$ and $t$.

Let $(w, s)$ be the maximal solution of (5.1) (5.2) with the appropriate initial condition at $t_0$.
obtained by Proposition 4.1 and define \( y = |w|_k \) and \( z = |s|_\ell \). By Lemma 3.7, \( y \) and \( z \) satisfy

\[
\begin{cases}
|\partial_t y| \leq C \, t^{-2} \, yz \\
|\partial_t z| \leq C \, t^{-2} \, z^2 + C t^{-\gamma} \, y^2
\end{cases}
\tag{5.10}
\]

and we estimate \( y \) and \( z \) from those inequalities, taking \( C = 1 \) for the rest of the proof. We distinguish two cases.

**Case \( t \geq t_0 \).** Let \( \bar{t} > t_0 \) and define \( Y = Y(\bar{t}) = \| y; L^\infty([t_0, \bar{t}]) \| \) and \( Z = Z(\bar{t}) = \| t^{-1} z; L^\infty([t_0, \bar{t}]) \| \). Then for all \( t \in [t_0, \bar{t}] \)

\[
\begin{cases}
\partial_t y \leq t^{-1-\gamma} \, Y \, Z \\
\partial_t z \leq t^{-2\gamma} \, Z^2 + t^{-\gamma} \, Y^2
\end{cases}
\tag{5.11}
\]

and therefore by integration with the appropriate initial condition at \( t_0 \)

\[
\begin{cases}
Y \leq y_0 + \gamma^{-1} \, t_0^{-\gamma} \, Y \, Z \\
Z \leq z_0 + t_0^{-\gamma} \, Z^2 + (1 - \gamma)^{-1} \, Y^2
\end{cases}
\tag{5.12}
\]

where we have used the fact that the function \( f(\gamma) = \int_{t_0}^t \, dt' \, t'^{-2\gamma} \) is logarithmically convex in \( \gamma \) and therefore satisfies

\[
f(\gamma) \leq f(0)^{1-\gamma} \, f(1)^{\gamma} = (t - t_0)(t_0 t)^{-\gamma} \leq t_0^{-\gamma} \, t^{1-\gamma} .
\tag{5.13}
\]

Now (5.12) defines a closed subset \( R \) of \( IR^+ \times IR^+ \) in the \((Y, Z)\) variables, containing the point \((y_0, z_0)\), and \((Y, Z)\) is a continuous function of \( \bar{t} \) starting from that point for \( \bar{t} = t_0 \). If we can find an open region \( R_1 \) of \( IR^+ \times IR^+ \) containing \((y_0, z_0)\) and such that \( \overline{R \cap R_1} \subset R_1 \), then \((Y, Z)\) will remain in \( R \cap R_1 \) for all time, because \( R \cap R_1 = \overline{R \cap R_1} \) is both open and closed in \( R \). We take for \( R_1 \) the strip \( R_1 = \{(Y, Z) : Z < (1/2)\gamma t_0^\gamma \} \), so that in \( \overline{R \cap R_1} \)

\[
\begin{cases}
Y \leq 2y_0 \\
Z \leq 2z_0 + 2(1 - \gamma)^{-1} \, Y^2 \leq 2z_0 + 8(1 - \gamma)^{-1} \, y_0^2
\end{cases}
\tag{5.14}
\]

and the condition \( \overline{R \cap R_1} \subset R_1 \) is ensured by

\[
\gamma t_0^\gamma > 4z_0 + 16(1 - \gamma)^{-1} \, y_0^2 .
\tag{5.15}
\]

**Case \( t \leq t_0 \).** Let \( \bar{t} < t_0 \) and define \( Y = Y(\bar{t}) = \| y; L^\infty([\bar{t}, t_0]) \| \) and \( Z = Z(\bar{t}) = \| z; L^\infty([\bar{t}, t_0]) \| \). Then for all \( t \in [\bar{t}, t_0] \)

\[
\begin{cases}
-\partial_t y \leq t^{-2} \, Y \, Z \\
-\partial_t z \leq t^{-2} \, Z^2 + t^{-\gamma} \, Y^2
\end{cases}
\tag{5.16}
\]
and therefore by integration with the appropriate initial condition at \( t_0 \)

\[
\begin{aligned}
Y &\leq y_0 + t^{-1} Y Z \\
Z &\leq (\bar{z}_0 + (1 - \gamma)^{-1} Y^2) t_0^{1-\gamma} + t^{-1} Z^2
\end{aligned}
\tag{5.17}
\]

so that for \( Z < t/2 \)

\[
\begin{aligned}
Y &< 2y_0 \\
Z &< 2 (\bar{z}_0 + (1 - \gamma)^{-1} Y^2) t_0^{1-\gamma} < (2\bar{z}_0 + 8(1 - \gamma)^{-1} y_0^2) t_0^{1-\gamma}
\end{aligned}
\tag{5.18}
\]

which ensures the condition \( Z < t/2 \) provided

\[
t > \left(4\bar{z}_0 + 16(1 - \gamma)^{-1} y_0^2\right) t_0^{1-\gamma}.
\tag{5.19}
\]

Putting back constants at appropriate places, we obtain (5.8) (5.9) from (5.14) (5.18), while (5.15) (5.19) allow for the choice (5.6) (5.7).

\[\square\]

**Remark 5.1.** The weakness of Proposition 5.1 is immediately apparent on (5.7) (5.9). In fact, for the construction of wave operators, we want to solve the Cauchy problem for (5.1) (5.2) for an initial time \( t_0 \to \infty \). However when \( t_0 \to \infty \), the interval \([T, \infty)\) where the solution is defined disappears, while the estimate for \(|s|_\ell\) blows up. This defect will be remedied in Proposition 5.6 below, but only for \( \gamma > 1/2 \). For the next results, we shall need the following estimate.

**Lemma 5.1.** Let \( 0 < \gamma < 1 \), let \( a > 0, b > 0, t_0 > 0 \) and let \( y, z \) be nonnegative continuous functions satisfying \( y(t_0) = y_0, z(t_0) = z_0 \), and

\[
\begin{aligned}
|\partial_t y| &\leq t^{-1-\gamma} by + t^{-2} az \\
|\partial_t z| &\leq t^{-1-\gamma} bz + t^{-\gamma} ay
\end{aligned}
\tag{5.20}
\]

Define \( \bar{y}, \bar{z} \) by

\[
(y, z) = (\bar{y}, \bar{z}) \exp\left[b\gamma^{-1}|t^{-\gamma} - t_0^{-\gamma}|\right].
\tag{5.21}
\]

Then, for \( \gamma(t_0^\gamma \wedge t^\gamma) \geq 2a^2 \), the following estimates hold:

\[
\begin{aligned}
\bar{y} &\leq 2\left(y_0 + a z_0 t_0^{-1}\right) \\
\bar{z} &\leq z_0 + 2(1 - \gamma)^{-1} a \left(y_0 + a z_0 t_0^{-1}\right) t^{1-\gamma}
\end{aligned}
\tag{5.22}
\]

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for $t \geq t_0$, and

$$
\begin{align*}
\bar{y} &\leq y_0 + 2a \left( z_0 + (1 - \gamma)^{-1} a y_0 t_0^{1-\gamma} \right) t^{-1} \\
\bar{z} &\leq 2 \left( z_0 + (1 - \gamma)^{-1} a y_0 t_0^{1-\gamma} \right)
\end{align*}
$$

(5.23)

for $t \leq t_0$.

**Proof.** The inequalities (5.20) have been written in differential form, but should be understood in integrated form. Changing the variables from $(y, z)$ to $(\bar{y}, \bar{z})$ yields

$$
\begin{align*}
\pm \partial_t \bar{y} &\leq t^{-2}a \bar{z} \\
\pm \partial_t \bar{z} &\leq t^{-\gamma}a \bar{y}
\end{align*}
$$

(5.24)

for $t > t_0$, and in integrated form

$$
\begin{align*}
\bar{y} &\leq y_0 + a \left| \int_{t_0}^t dt' t'^{-2} \bar{z}(t') \right| \\
\bar{z} &\leq z_0 + a \left| \int_{t_0}^t dt' t'^{-\gamma} \bar{y}(t') \right|
\end{align*}
$$

(5.25)

Substituting the second component of (5.25) into the first one yields an inequality for $\bar{y}$ alone:

$$
\bar{y} \leq y_0 + a z_0 | t^{-1} - t_0^{-1} | + a^2 \left| \int_{t_0}^t dt' t'^{-\gamma} \left( t'^{-1} - t^{-1} \right) \bar{y}(t') \right| .
$$

(5.26)

We shall estimate $\bar{y}$ by (5.26) and estimate $\bar{z}$ by substituting the result into the second component of (5.25). (One could equally well obtain an equation for $\bar{z}$ alone similar to (5.26) and estimate $\bar{z}$ directly therefrom). We distinguish the cases $t > t_0$.

**Case** $t \geq t_0$. Let $\bar{t} > t_0$. We define $Y = \| \bar{y}; L^\infty([t_0, \bar{t}]) \|$. It then follows from (5.26) that

$$
Y \leq y_0 + a z_0 t_0^{-1} + a^2 \gamma^{-1} t_0^{-\gamma} Y
$$

and therefore

$$
Y \leq 2 \left( y_0 + a z_0 t_0^{-1} \right)
$$

for $\gamma t_0^\gamma \geq 2a^2$, while by (5.25)

$$
\bar{z} \leq z_0 + a(1 - \gamma)^{-1} t^{1-\gamma} Y
$$

which immediately yields (5.22).
Case $t \leq t_0$. Let $\bar{t} < t_0$. We define $Y = \| t(\bar{y} - y_0); L^\infty([\bar{t}, t_0]) \|$. It then follows from (5.26) that

$$\bar{y} \leq y_0 + a z_0 t^{-1} + a^2 y_0 t^{-1} (1 - \gamma)^{-1} t_0^{1 - \gamma} t + a^2 Y t^{1 - \gamma} t^{-\gamma}$$

so that

$$Y \leq a z_0 + a^2 y_0 (1 - \gamma)^{-1} t_0^{1 - \gamma} + a^2 \gamma^{-1} t^{-\gamma} Y$$

and therefore

$$Y \leq 2a (z_0 + a y_0 (1 - \gamma)^{-1} t_0^{1 - \gamma})$$

for $\gamma t^{\gamma} \geq 2a^2$. Substituting that result into the second component of (5.25) yields

$$z \leq z_0 + a y_0 (1 - \gamma)^{-1} t_0^{1 - \gamma} + a \gamma^{-1} t^{-\gamma} Y$$

$$\leq 2 (z_0 + a y_0 (1 - \gamma)^{-1} t_0^{1 - \gamma})$$

by using again the condition $\gamma t^{\gamma} \geq 2a^2$. The previous estimates immediately yield (5.23).

**Remark 5.2.** Since (5.20) is a linear system, it is clear that one could obtain estimates of $y$ and $z$ valid for all $t_0 > 0$, $t > 0$, but in view of Proposition 5.1, we are not interested in estimates for small $t_0$, $t$. The apparent lower restrictions on $t_0$, $t$ come from the fact that the ansatz $Y$ for $\bar{y}$ is inadequate for small $t$.

As a first consequence of Lemma 5.1, we obtain a uniqueness result at infinity for the system (5.1) (5.2).

**Proposition 5.2.** Let $0 < \gamma < 1$ and let $(k, \ell)$ be an admissible pair. Let $(w_i, s_i), i = 1, 2$ be two solutions of the system (5.1) (5.2) such that $(w_i, \bar{s}_i) \in L^\infty([T, \infty), H^k \oplus X^\ell)$ for some $T > 0$ and such that $|w_1 - w_2|_{k-1} t^{1-\gamma}$ and $|s_1 - s_2|_{\ell-1}$ tend to zero when $t \to \infty$. Then $(w_1, s_1) = (w_2, s_2)$.

**Proof.** Define

$$a = \max_i \| w_i; L^\infty([T, \infty), H^k) \| \quad , \quad b = \max_i \| \bar{s}_i; L^\infty([T, \infty), X^\ell) \| \quad , \quad (5.27)$$

$$y = |w_1 - w_2|_{k-1} \quad , \quad z = |s_1 - s_2|_{\ell-1} \quad . \quad (5.28)$$
Then by Lemma 3.9, \( y \) and \( z \) satisfy the inequalities
\[
\begin{align*}
|\partial_t y| &\leq C t^{-1-\gamma} by + C t^{-2} az \\
|\partial_t z| &\leq C t^{-1-\gamma} bz + C t^{-\gamma} ay
\end{align*}
\]
and therefore (up to constants), the estimates (5.21) (5.23) for all \( T \leq t \leq t_0, t \) and \( t_0 \) sufficiently large. Taking the limit \( t_0 \to \infty \) in (5.23) for fixed \( t \) shows that \( y(t) = 0, z(t) = 0 \).

\[\Box\]

We now begin the study of the asymptotic behaviour of solutions of the system (5.1) (5.2) by showing that for the solutions obtained in Proposition 5.1 (actually for slightly more general solutions, if any) \( w(t) \) tends to a limit when \( t \to \infty \).

**Proposition 5.3.** Let \( 0 < \gamma < 1 \) and let \((k, \ell)\) satisfy \( k \leq \ell + 1 \) and \( \ell > n/2 \). Let \((w, s)\) be a solution of the system (5.1) (5.2) such that \((w, \tilde{s}) \in (C \cap L^\infty)([T, \infty), H^k \oplus X^\ell)\) for some \( T > 0 \). Let
\[
a = \| w; L^\infty([T, \infty), H^k) \| \quad \text{and} \quad b = \| \tilde{s}; L^\infty([T, \infty), X^\ell) \|.
\]

Then there exists \( w_+ \in H^k \) such that \( w(t) \) tends to \( w_+ \) strongly in \( H^{k-1} \) and weakly in \( H^k \) when \( t \to \infty \). Furthermore the following estimates hold:
\[
|w_+|_k \leq a \quad (5.31)
\]
\[
|w(t_0) - w(t)|_{k-1} \leq C ab \gamma^{-1} (t_0 \wedge t)^{-\gamma} \quad (5.32)
\]
\[
|w(t) - w_+|_{k-1} \leq C ab \gamma^{-1} t^{-\gamma} \quad (5.33)
\]
for \( t_0, t \) large enough, namely \( \gamma(t_0 \wedge t)^\gamma \geq Cb \) or \( \gamma t^\gamma \geq Cb \).

**Proof.** Let \( T \leq t_0 \leq t \) and \( w_0 = w(t_0) \). By exactly the same method as in Lemma 3.9 (see esp. (3.54)) we obtain from (5.1)
\[
\partial_t |w - w_0|_{k-1} \leq C t^{-2} |s|_\ell (|w - w_0|_{k-1} + |w_0|_k) \\
\leq C t^{-1-\gamma} b (|w - w_0|_{k-1} + a)
\]
and by integration between \( t_0 \) and \( t \)
\[
|w - w_0|_{k-1} \leq a \left( \exp \left( C b \gamma^{-1} t_0^{-\gamma} \right) - 1 \right) \\
\leq C(e - 1)ab \gamma^{-1} t_0^{-\gamma}
\]
for \( \gamma t_0^\gamma \geq Cb \). This proves (5.32), from which it follows that \( w(t) \) has a limit \( w_+ \in H^{k-1} \) such that (5.33) holds. Since in addition \( w(t) \) is uniformly bounded in \( H^k \), it follows by a standard compactness argument that \( w_+ \in H^k \), that \( w_+ \) satisfies (5.31) and that \( w(t) \) tends weakly to \( w_+ \) in \( H^k \).

\( \square \)

**Remark 5.3.** Note that the condition on \((k, \ell)\) in Proposition 5.3 is weaker than admissibility, since in particular we do not use (5.2) and therefore do not require estimates of \( g \).

We now turn to the proof of existence of asymptotic states for the solutions of (5.1) (5.2) constructed in Proposition 5.1. As explained in Section 2, we want to perform the following construction on such a solution. We take \( t_0 \geq T \) and we consider the solution \((w_0, s_0, t_0)\) of the free equation (5.3) which coincides with \((w, s)\) at time \( t_0 \). That solution is defined by \( w_0 = w(t_0) \) and
\[
s_{0,t_0}(t) = s(t_0) + \int_{t_0}^{t} dt' t'^{-\gamma} \nabla g(w_0, w_0) \tag{5.34}
\]
or equivalently, by (5.2)
\[
s_{0,t_0}(t) = s(t) - \int_{t_0}^{t} dt' \left\{ t'^{-2}(s \cdot \nabla)s + t'^{-\gamma}(\nabla g(w, w) - \nabla g(w_0, w_0)) \right\} \tag{5.35}
\]

We want to prove that \((w_0, s_{0,t_0})\) has a limit when \( t_0 \to \infty \). Proposition 5.3 already provides us with the limit \( w_+ = \lim w(t_0) \), and it remains only to be proved that also \( s_{0,t_0} \) has a limit. Taking formally the limit \( t_0 \to \infty \) in (5.35) leads us to expect that that limit should be defined by
\[
s_0(t) = s(t) + \int_{t}^{\infty} dt' \left\{ t'^{-2}(s \cdot \nabla)s + t'^{-\gamma}(\nabla g(w, w) - \nabla g(w_0, w_0)) \right\} \tag{5.36}
\]

From Corollary 3.1 and from (5.34), it follows that for \((w, \bar{s}) \in (C \cap L^\infty)([T, \infty), H^{k} \oplus X^\ell)\), also \((w_0, \bar{s}_{0,t_0})\) belongs to the same space, actually with constant \( w_0 \), and by (5.30), satisfies the
estimate

\[ |s_{0,t_0}(t)|_{\ell} \leq b\, t_0^{1-\gamma} + C\, a^2(1 - \gamma)^{-1}(t_0 \lor t)^{1-\gamma}. \quad (5.37) \]

That estimate however is not bounded uniformly in \( t_0 \), and is therefore useless to take the limit \( t_0 \to \infty \). In order to proceed further, we shall need stronger assumptions, and in particular \( \gamma > 1/2 \). Under that condition and by using (5.35), we can indeed obtain estimates on \( s_{0,t_0} \) that are uniform in \( t_0 \).

**Proposition 5.4.** Let \( 1/2 < \gamma < 1 \) and let \( (k, \ell) \) be an admissible pair. Let \((w, s)\) be a solution of the system (5.1) (5.2) such that \((w, \tilde{s}) \in (C \cap L^\infty)([T, \infty), H^k \oplus X^\ell)\) for some \( T \geq 1 \) and define \( a \) and \( b \) by (5.30). Let \( t_0 \geq T \), let \( w_0 = w(t_0) \) and define \( s_{0,t_0}(t) \) by (5.34). Then \( s_{0,t_0}(t) \) satisfies the estimates

\[ |s_{0,t_0}(t) - s(t)|_{\ell-1} \leq C \left( b^2 + (1 - \gamma)^{-1}a^2b \right)\, t_0^{-\gamma}\, t^{1-\gamma} \quad (5.38) \]

for \( t \geq t_0 \) and \( t_0^\gamma \geq Cb \),

\[ |s_{0,t_0}(t) - s(t)|_{\ell-1} \leq C(2\gamma - 1)^{-1} \left( b^2 + a^2b \right)\, t^{1-2\gamma} \quad (5.39) \]

for \( T \leq t \leq t_0 \) and \( t^\gamma \geq Cb \).

Assume in addition that \((1 - \gamma)t_0^\gamma \geq Ca^2 \) and \((2\gamma - 1)t^\gamma \geq C(a^2 + b)\). Then \( s_{0,t_0} \) satisfies the estimate

\[ |s_{0,t_0}(t)|_{\ell-1} \leq C\, b\, t^{1-\gamma}. \quad (5.40) \]

**Proof.** By the same method as in Lemma 3.9 (see esp. (3.57)), we estimate the integrand in (5.35) by

\[ |\{\cdot\}|_{\ell-1} \leq C\, t^{-2}\, |s|_{\ell-1} \, |s|_{\ell} + C\, t^{-\gamma}\, |w - w_0|_{k-1} \, |w + w_0|_k \quad (5.41) \]

which by (5.30) and Proposition 5.3, esp. (5.32), can be continued as

\[ \cdots \leq C\, b^2\, t^{-2\gamma} + C\, a^2b\, t^{-\gamma}\, (t \land t_0)^{-\gamma} \quad (5.42) \]

for \((t \land t_0)^\gamma \geq Cb\). Integrating (5.42) between \( t_0 \) and \( t \) and using (5.13) if \( t \geq t_0 \) yields (5.38) for \( t \geq t_0 \) and (5.39) for \( t \leq t_0 \).
Finally (5.40) follows from (5.30), from (5.38) (5.39) and from the additional conditions on $t, t_0$.  

We next prove that $s_0(t)$ is actually well defined by (5.36) and is the limit of $s_{0,t_0}(t)$ when $t_0 \to \infty$.

**Proposition 5.5.** Let $1/2 < \gamma < 1$ and let $(k, \ell)$ be an admissible pair. Let $(w, s)$ be a solution of the system (5.1) (5.2) such that $(w, \tilde{s}) \in (C \cap L^\infty)([T, \infty), H^k \oplus X^\ell)$ for some $T \geq 1$ and define $a$ and $b$ by (5.30). Let $w_+$ be the limit of $w(t)$ when $t \to \infty$ obtained in Proposition 5.3. Then

1. The integral in (5.36) is absolutely convergent in $X^{\ell-1}$ and defines a solution $s_0$ of the equation (5.3) such that $\tilde{s}_0 \in (C \cap L^\infty)([1, \infty), X^{\ell-1})$ and such that the following estimate holds for $t \geq T, \ t^\gamma \geq Cb$:

$$|s_0(t) - s(t)|_{\ell-1} \leq C(2\gamma - 1)^{-1} \left( b^2 + a^2b \right) t^{1-2\gamma}.$$  

(5.43)  

If in addition $(2\gamma - 1)t^\gamma \geq C(b + a^2)$, the following estimate holds:

$$|s_0(t)|_{\ell-1} \leq C b t^{1-\gamma}.$$  

(5.44)  

2. The function $s_{0,t_0}$ defined by (5.34) converges to $s_0$ in norm in $X^{\ell-1}$ when $t_0 \to \infty$ for $t \geq T, t^\gamma \geq Cb$, uniformly in compact intervals, and the following estimate holds for $t \wedge t_0 \geq T, (t \wedge t_0)^\gamma \geq Cb$:

$$|s_{0,t_0}(t) - s_0(t)|_{\ell-1} \leq C \left( (1-\gamma)^{-1} + (2\gamma - 1)^{-1} \right) \left( b^2 + a^2b \right) t_0^{-\gamma}(t \vee t_0)^{1-\gamma}.$$  

(5.45)  

**Proof.** Part (1). By the same method as in the proof of Proposition 5.4, we estimate the integrand in (5.36) by (5.41) with $w_0$ replaced by $w_+$ and therefore by (5.30) (5.33)

$$|\{\cdot\}|_{\ell-1} \leq C \left( b^2 + a^2b \right) t^{-2\gamma}$$  

(5.46)  

which proves the convergence of the integral in $X^{\ell-1}$ and yields the estimate (5.43). Finally (5.44) follows from (5.30) (5.43) and from the additional condition on $t$.  

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Part (2). For $t \geq t_0$, we estimate

$$|s_{0,t_0}(t) - s_0(t)|_{\ell-1} \leq |s_{0,t_0}(t) - s(t)|_{\ell-1} + |s_0(t) - s(t)|_{\ell-1}. \quad (5.47)$$

We estimate the first norm in the RHS by (5.38) and the second norm by (5.43). This yields (5.45) for $t \geq t_0$.

For $t \leq t_0$, we start from

$$s_{0,t_0}(t) - s_0(t) = s(t_0) - s_0(t_0) + \int_t^{t_0} dt' \ t'^{-\gamma} (\nabla g(w_+, w_+) - \nabla g(w(t), w(t_0))). \quad (5.48)$$

We estimate the first difference in the RHS by (5.43) with $t = t_0$ and the integrand by Corollary 3.1 and (5.33) as

$$|\nabla g(w_+, w_+) - \nabla g(w(t_0), w(t_0))|_{\ell-1} \leq C |w(t_0) - w_+|_{k-1} |w(t_0) + w_+|_k \leq C a^2 b t_0^{-\gamma} \quad (5.49)$$

which after integration yields (5.45) for $t \leq t_0$.

The convergence of $s_{0,t_0}$ to $s_0$ in the norms indicated follows from (5.45).

\[\square\]

**Remark 5.4.** Note that in Proposition 5.5 we are losing one degree of regularity in $s$, namely a solution $(w, s)$ in $H^k \oplus X^\ell$ has an asymptotic free solution $(w_+, s_0)$ in $H^k \oplus X^{\ell-1}$ only. We have no uniform estimate in $t_0$ of $s_{0,t_0}$ in $X^\ell$, the best we have being (5.37) and we are therefore unable to assert that the limiting $s_0$ remains in $X^\ell$.

We now turn to the main and more difficult question of existence of the wave operator. The construction is now the converse of that performed in Propositions 5.3-5.5. We consider a fixed solution $(w_+, s_0)$ of (5.3), we construct a solution $(w_{t_0}, s_{t_0})$ of (5.1) (5.2) which coincides with $(w_+, s_0)$ at time $t_0$, and we take the limit of that solution when $t_0 \to \infty$. Here again we shall encounter a loss of derivative, now more severe than in the previous case. In particular we shall need to start with a free solution in $H^{k+1} \oplus X^{\ell+1}$ for admissible $(k, \ell)$ and end up with a solution $(w, s)$ which is only in $H^k \oplus X^\ell$. The crucial step is the construction of $(w_{t_0}, s_{t_0})$ and will be performed by an extension of the energy method used in Proposition 5.1.
Proposition 5.6. Let $1/2 < \gamma < 1$ and let $(k, \ell)$ be an admissible pair. Let $(w_+, s_0(1)) \in H^{k+1} \oplus X^{\ell+1}$, define $s_0$ by (5.4) and let

$$a = |w_+|_{k+1}, \quad b = \|s_0; L^\infty([1, \infty), X^{\ell+1})\|.$$  

Then, there exist $T_0$ and $T$, $1 \leq T_0$, $T < \infty$, depending only on $(\gamma, a, b)$, such that for all $t_0 \geq T_0 \lor T$, the system (5.1) (5.2) with initial data $w(t_0) = w_+$, $s(t_0) = s_0(t_0)$, has a unique solution $(w_{t_0}, s_{t_0})$ such that $(w_{t_0}, \tilde{s}_{t_0}) \in (C \cap L^\infty)([T, \infty), H^k \oplus X^\ell)$. One can take

$$T_0^\gamma = C \left(b + (1 - \gamma)^{-1}a^2\right)$$

$$T^\gamma = C(2\gamma - 1)^{-1} (b + a^2)$$

and the solution satisfies the estimates

$$\left\{\begin{array}{l}
|w_{t_0}(t) - w_+|_k \leq C ab t_0^{-\gamma} \\
|s_{t_0}(t) - s_0(t)|_\ell \leq C (b^2 + (1 - \gamma)^{-1}a^2b) t_0^{-\gamma} t^{1-\gamma}
\end{array}\right.$$  

for $t \geq t_0$,

$$\left\{\begin{array}{l}
|w_{t_0}(t) - w_+|_k \leq C b t^{-\gamma} \\
|s_{t_0}(t) - s_0(t)|_\ell \leq C(2\gamma - 1)^{-1}(b^2 + a^2b) t^{1-2\gamma}
\end{array}\right.$$  

for $T \leq t \leq t_0$, and

$$|w_{t_0}(t)|_k \leq C a, \quad |s_{t_0}(t)|_\ell \leq C b t^{1-\gamma}$$

for all $t \geq T$.

Proof. The result follows from Proposition 4.1 and standard globalisation arguments provided we can derive (5.53)-(5.54) as a priori estimates under the assumptions of the proposition. Let $(w_{t_0}, s_{t_0})$ be the maximal solution of (5.1) (5.2) with the appropriate initial condition at $t_0$ and define $y = |w_{t_0} - w_+|_k$ and $z = |s_{t_0} - s_0|_\ell$. We rewrite (5.1) and the difference between (5.2) (5.3) as

$$\left\{\begin{array}{l}
\partial_t w_{t_0} - w_+ = (2t^2)^{-1}U(1/t) (2s_{t_0} \cdot \nabla + (\nabla \cdot s_{t_0})) U^*(1/t) w_{t_0} \\
\partial_t s_{t_0} - s_0 = t^{-2}(s_{t_0} \cdot \nabla) s_{t_0} + t^{-\gamma} (\nabla g(w_{t_0}, w_{t_0}) - \nabla g(w_+, w_+))
\end{array}\right.$$  

From (5.56) (5.57), by exactly the same method as in Lemma 3.8, we obtain

$$\left\{\begin{array}{l}
|\partial_t w_{t_0} - w_+|_k \leq Ct^{-2}|s_{t_0}|_\ell (|w_{t_0} - w_+|_k + |w_+|_{k+1}) \\
|\partial_t s_{t_0} - s_0|_\ell \leq Ct^{-2}|s_{t_0}|_\ell (|s_{t_0} - s_0|_\ell + |s_0|_{\ell+1}) + Ct^{-\gamma}|w_{t_0} - w_+|_k |w_{t_0} + w_+|_k
\end{array}\right.$$  

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and therefore by (5.50)
\[
\begin{cases}
|\partial_t y| \leq C t^{-2}(z + b t^{1-\gamma})(y + a) \\
|\partial_t z| \leq C t^{-2}(z + b t^{1-\gamma})^2 + Ct^{-\gamma} y(y + a)
\end{cases}
\] (5.59)

We estimate \( y \) and \( z \) from (5.59), taking \( C = 1 \) for the rest of the proof. We distinguish again two cases.

**Case \( t \geq t_0 \).** Let \( \bar{t} > t_0 \) and define \( Y = Y(\bar{t}) = \| y; L^\infty([t_0, \bar{t}]) \| \) and \( Z = Z(\bar{t}) = \| t^{\gamma-1}z; L^\infty([t_0, \bar{t}]) \| \). Then for all \( t \in [t_0, \bar{t}] \)
\[
\begin{cases}
\partial_t y \leq t^{-1-\gamma}(Z + b)(Y + a) \\
\partial_t z \leq t^{-2\gamma}(Z + b)^2 + t^{-\gamma}Y(Y + a)
\end{cases}
\] (5.60)

and therefore by integration with the appropriate initial condition at \( t_0 \)
\[
\begin{cases}
Y \leq 2t_0^{-\gamma}(Z + b)(Y + a) \\
Z \leq t_0^{-\gamma}(Z(Z + 2b) + b^2) + (1 - \gamma)^{-1}Y(Y + a)
\end{cases}
\] (5.61)

where we have used the condition \( \gamma > 1/2 \) and (5.13) again.

As in the proof of Proposition 5.1, we impose an additional condition
\[
2t_0^{-\gamma}(Z + 2b) < \lambda \leq 1/2
\] (5.62)

and we obtain from (5.61)
\[
\begin{cases}
Y \leq (1 - \lambda)^{-1}\lambda a \\
Z \leq (1 - \lambda)^{-1}b^2t_0^{-\gamma} + (1 - \lambda)^{-3}(1 - \gamma)^{-1}\lambda a^2
\end{cases}
\] (5.63)

We now choose \( \lambda = 8bt_0^{-\gamma} \) so that (5.62L) reduces to \( Z < 2b \), and we obtain from (5.63)
\[
\begin{cases}
Y \leq 16ab t_0^{-\gamma} \\
Z \leq (2b^2 + 64(1 - \gamma)^{-1}a^2b) t_0^{-\gamma}
\end{cases}
\] (5.64)

The condition \( Z < 2b \) is ensured by (5.51) for \( t_0 \geq T_0 \), and the estimate (5.53) follows from (5.64).

**Case \( t \leq t_0 \).** Let \( \bar{t} \leq t_0 \) and define \( Y = Y(\bar{t}) = \| t^\gamma y; L^\infty([\bar{t}, t_0]) \| \) and \( Z = Z(\bar{t}) = \| t^{2\gamma-1}z; L^\infty([\bar{t}, t_0]) \| \). Then for all \( t \in [\bar{t}, t_0] \)
\[
\begin{cases}
-\partial_t y \leq t^{-1-\gamma}(Zt^{-\gamma} + b)(Yt^{-\gamma} + a) \\
-\partial_t z \leq t^{-2\gamma}(Zt^{-\gamma} + b)^2 + t^{-2\gamma}Y(Yt^{-\gamma} + a)
\end{cases}
\] (5.65)
and therefore by integration with the appropriate initial condition at $t_0$

$$\begin{align*}
y &\leq 2abt^{-\gamma} + (aZ + bY)t^{-2\gamma} + YZt^{-3\gamma} \\
z &\leq (2\gamma - 1)^{-1}(b^2 + aY)t^{1-2\gamma} + 2(Y^2 + 2bZ)t^{1-3\gamma} + Z^2t^{1-4\gamma}
\end{align*}$$

(5.66)

where we have used the condition $\gamma > 1/2$, so that

$$\begin{align*}
Y &\leq 2ab + aZt^{-\gamma} + (b + Zt^{-\gamma})Yt^{-\gamma} \\
Z &\leq (2\gamma - 1)^{-1}(b^2 + aY) + 2Y^2t^{-\gamma} + (4b + Zt^{-\gamma})Zt^{-\gamma}.
\end{align*}$$

(5.67)

We impose the additional conditions $Zt^{-\gamma} < 2b$ and $t^\gamma \geq 12b$ and we obtain from (5.67)

$$\begin{align*}
Y &\leq 8ab \\
Z &\leq 2((2\gamma - 1)^{-1}(b^2 + aY) + (6b)^{-1}Y^2)
\end{align*}$$

(5.68)

which yields (5.54), while the conditions $Zt^{-\gamma} < 2b$ and $t^\gamma \geq 12b$ are ensured by (5.52).

Finally (5.55) follows from (5.50)-(5.54).

$\Box$

**Remark 5.5.** The major improvement of Proposition 5.6 over Proposition 5.1, which has been achieved by estimating the difference of $(w, s)$ and $(w_+, s_0)$ instead of estimating $(w, s)$ alone, is that now $(w, s)$ is defined in a time interval $[T, \infty)$ which is independent of $t_0$, and is estimated in that interval uniformly with respect to $t_0$. The condition $\gamma > 1/2$ plays a crucial role in obtaining that improvement (cf Remark 5.1 above).

**Remark 5.6.** One may wonder whether the loss of regularity from $(k + 1, \ell + 1)$ to $(k, \ell)$ when constructing $(w_{t_0}, s_{t_0})$ from $(w_+, s_0)$ is unavoidable. Actually $(w_{t_0}(t_0), s_{t_0}(t_0)) \in H^{k+1} \oplus X^{\ell+1}$ and by regularity (Proposition 4.1 part (2)), $(w_{t_0}, s_{t_0}) \in C([T, \infty), H^{k+1} \oplus X^{\ell+1})$. One can then estimate $y = |w_{t_0}|_{k+1}$ and $z = |s_{t_0}|_{\ell+1}$. By Lemma 3.8 and by (5.1) (5.2) (5.55), $y$ and $z$ satisfy the system (5.29) and therefore are estimated up to constants by (5.22) (5.23). However under natural assumptions $y_0 = O(1)$ and $z_0 = O(t_0^{1-\gamma})$, the estimate (5.23), which is the one really relevant for large $t_0$, is not uniform in $t_0$, so that the estimate is lost when one takes the limit $t_0 \to \infty$, which is what we shall do next. As a consequence the regularity at the level of $(k + 1, \ell + 1)$ is also lost in that limit, and we have therefore made no effort to keep track of it at the stage of Proposition 5.6.
We can now take the limit $t_0 \to \infty$ of the solution $(w_{t_0}, s_{t_0})$ constructed in Proposition 5.6 for fixed $(w_+, s_0)$.

**Proposition 5.7.** Let $1/2 < \gamma < 1$ and let $(k, \ell)$ be an admissible pair.

(1) Let $(w_+, s_0(1)) \in H^{k+1} \oplus X^{\ell+1}$ and define $s_0$, $a$, $b$, by (5.4) (5.50). Then there exists $T$, $1 \leq T < \infty$, depending only on $(\gamma, a, b)$ and a unique solution $(w, s)$ of the system (5.1) (5.2) such that $(w, \tilde{s}) \in (C \cap L^\infty)([T, \infty), H^k \oplus X^\ell)$, satisfying (5.36) and such that the following estimates hold for all $t \geq T$:

\[
\begin{align*}
|w(t) - w_+|_k &\leq C \alpha b t^{-\gamma} \\
|s(t) - s_0(t)|_\ell &\leq C(2\gamma - 1)^{-1}(b^2 + a^2 b)t^{1-2\gamma},
\end{align*}
\]

(5.69)

\[
\begin{align*}
|w(t)|_k &\leq C a, \quad |s(t)|_\ell \leq C b t^{1-\gamma}.
\end{align*}
\]

(5.70)

One can take

\[
T^\gamma = C(2\gamma - 1)^{-1}(b + a^2).
\]

(5.71)

(2) Let $(w_{t_0}, s_{t_0})$ be the solution of the system (5.1) (5.2) constructed in Proposition 5.6 for $t_0 \geq T_0 \lor T$ and in particular such that $(w_{t_0}, \tilde{s}_{t_0}) \in (C \cap L^\infty)([T, \infty), H^k \oplus X^\ell)$. Then $(w_{t_0}, s_{t_0})$ converges to $(w, s)$ in norm in $L^\infty(J, H^{k-1} \oplus X^{\ell-1})$ and in the weak-* sense in $L^\infty(J, H^k \oplus X^\ell)$ for any compact interval $J \subset [T, \infty)$, and in the weak-* sense in $H^k \oplus X^\ell$ pointwise in $t$.

(3) The map $(w_+, s_0(1)) \mapsto (w, s)$ defined in Part (1) is continuous on the bounded sets of $H^{k+1} \oplus X^{\ell+1}$ from the norm topology of $(w_+, s_0(1))$ in $H^{k-1} \oplus X^{\ell-1}$ to the norm topology of $(w, s)$ in $L^\infty(J, H^{k-1} \oplus X^{\ell-1})$ and to the weak-* topology in $L^\infty(J, H^k \oplus X^\ell)$ for any compact interval $J \subset [T, \infty)$, and to the weak-* topology in $H^k \oplus X^\ell$ pointwise in $t$.

**Proof.** Parts (1) (2) will follow from the convergence of $(w_{t_0}, s_{t_0})$ when $t_0 \to \infty$ in the topologies stated in Part (2). We recall that $(w_{t_0}, s_{t_0})$ satisfies the estimates (5.53) (5.54) which we rewrite more briefly as

\[
\begin{align*}
[w_{t_0} - w_+]_k &\leq M_1(t \land t_0)^{-\gamma} \\
|s_{t_0} - s_0|_\ell &\leq M_2(t \land t_0)^{-\gamma} t^{1-\gamma}
\end{align*}
\]

(5.72)

where $M_1$, $M_2$ depend on $(\gamma, a, b)$ and can be read from (5.53) (5.54), and satisfies the estimate (5.55).
Let now $T_0 \vee T \leq t_0 \leq t_1$. From (5.72) it follows that for all $t \geq t_0$

\[
\begin{cases}
|w_{t_0} - w_{t_1}|_k \leq 2M_1 t_0^{-\gamma} \\
|s_{t_0} - s_{t_1}|_\ell \leq 2M_2 t_0^{-\gamma} t^{1-\gamma}.
\end{cases}
\tag{5.73}
\]

We now estimate $(w_{t_0} - w_{t_1}, s_{t_0} - s_{t_1})$ in $H^{k-1} \oplus X^{\ell-1}$ for $t \leq t_0$. Let

\[
y = |w_{t_0} - w_{t_1}|_{k-1}, \quad z = |s_{t_0} - s_{t_1}|_{\ell-1}.
\tag{5.74}
\]

From Lemma 3.9 and from (5.55), it follows that $y$ and $z$ satisfy the system (5.29). Integrating that system for $t \leq t_0$ with initial data at $t_0$ estimated by (5.73), we obtain from Lemma 5.1, esp. (5.23)

\[
\begin{cases}
y(t) \leq M \left( t_0^{-\gamma} + t_0^{-2\gamma} t^{-1} \right) \\
z(t) \leq M t_0^{-2\gamma}
\end{cases}
\tag{5.75}
\]

for some $M$ depending on $(\gamma, a, b)$ and for $T \leq t \leq t_0$. From (5.75) it follows that there exists $(w, s) \in \mathcal{C}([T, \infty), H^{k-1} \oplus X^{\ell-1})$ such that $(w_{t_0}, s_{t_0})$ converges to $(w, s)$ in $L^\infty(J, H^{k-1} \oplus X^{\ell-1})$ for all compact intervals $J \subset [T, \infty)$. From that convergence, from (5.54) (5.55) and from standard compactness arguments, it follows that $(w, s) \in (L^\infty \cap \mathcal{C}^{\infty})([T, \infty), H^k \oplus X^\ell)$, that $(w, s)$ satisfies the estimates (5.69) (5.70) for all $t \geq T$, and that $(w_{t_0}, s_{t_0})$ converges to $(w, s)$ in the other topologies considered in Part (2). Furthermore, by the local result of Proposition 4.1, part (1), $(w, s) \in \mathcal{C}([T, \infty), H^k \oplus X^\ell)$. Obviously $(w, s)$ is a solution of (5.1) (5.2). We now prove that $(w, s)$ satisfies (5.36). Let $s_0'(t)$ be the RHS of (5.36). By Proposition 5.5, part (1), $s_0'(t)$ is well defined and satisfies the analogue of (5.43). Furthermore $s_0'(t)$ satisfies (5.3) so that $s_0(t) - s_0'(t)$ is constant in time. By (5.43) for $s_0'(t)$ and (5.69), that constant is zero, namely $s_0'(t) = s_0(t)$. This proves (5.36).

From the uniqueness result of Proposition 5.2, it follows that $(w, s)$ is unique under the condition (5.69). This completes the proof of Parts (1) and (2).

**Part (3).** Let $(w_+, s_0(1))$ and $(w'_+, s'_0(1))$ belong to $H^{k+1} \oplus X^{\ell+1}$, define $s_0$ and $s'_0$ by (5.4) and its analogue, assume that

\[
|w_+|_{k+1} \vee |w'_+|_{k+1} \leq a
\]

\[
\| t^{\gamma-1} s_0; L^\infty([1, \infty), X^{\ell+1}) \| \vee \| t^{\gamma-1} s'_0; L^\infty([1, \infty), X^{\ell+1}) \| \leq b .
\]

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let \((w, s)\) and \((w', s')\) be the associated solutions of the system (5.1) \((5.2)\) constructed in Part (1), satisfying (5.69) and its analogue with the same \((a, b)\) for \(t \geq T\) with the same \(T\) defined by (5.71). We take \((w'_+, s'_0(1))\) close to \((w_+, s_0(1))\) in \(H^{k-1} \oplus X^{\ell-1}\) in the sense that for some small \(\varepsilon, \varepsilon_0 > 0\)

\[
|w_+ - w'_+|_{k-1} \leq \varepsilon a
\]  

(5.76)

\[
|s_0(1) - s'_0(1)|_{\ell-1} \leq \varepsilon_0 b
\]  

(5.77)

and therefore by (5.4) and estimates from Corollary 3.1

\[
|s_0 - s'_0|_{\ell-1} \leq \varepsilon_0 b + C(1 - \gamma)^{-1} a^2 \varepsilon t^{1-\gamma} .
\]  

(5.78)

From (5.69) and its analogue for \((w', s')\) and from (5.76) (5.78), it follows that for all \(t \geq T\)

\[
\begin{cases}
|w - w'|_{k-1} \leq \varepsilon a + C \varepsilon b t^{-\gamma} \\
|s - s'|_{\ell-1} \leq \varepsilon_0 b + C(1 - \gamma)^{-1} a^2 \varepsilon t^{1-\gamma} + C(2\gamma - 1)^{-1}(b^2 + a^2 b)t^{1-2\gamma} .
\end{cases}
\]  

(5.79)

We now define \(t_0\) by \(t_0^{-\gamma} = \varepsilon b^{-1}\) so that for \(t \geq t_0\),

\[
|w - w'|_{k-1} \leq C \varepsilon a
\]  

(5.80)

\[
|s - s'|_{\ell-1} \leq \varepsilon_0 b + C [(1 - \gamma)^{-1} a^2 + (2\gamma - 1)^{-1}(b + a^2)] \varepsilon t^{1-\gamma}
\]  

and in particular

\[
|s(t_0) - s'(t_0)|_{\ell-1} \leq \varepsilon_0 b + M \varepsilon^{(2\gamma - 1)/\gamma}
\]  

(5.81)

for some \(M\) depending on \((\gamma, a, b)\). We now estimate \((w - w', s - s')\) in \(H^{k-1} \oplus X^{\ell-1}\) for \(t \leq t_0\). Let

\[
y = |w - w'|_{k-1} , \quad z = |s - s'|_{\ell-1} .
\]

From Lemma 3.9, it follows that \(y\) and \(z\) satisfy the system (5.29). Integrating that system for \(t \leq t_0\) with initial data at \(t_0\) estimated by (5.80) (5.81), we obtain from Lemma 5.1, esp. (5.23),

\[
\begin{cases}
y(t) \leq C \varepsilon a + \left( C \varepsilon_0 b + M \varepsilon^{(2\gamma - 1)/\gamma} \right) at^{-1} \\
z(t) \leq C \varepsilon_0 b + M \varepsilon^{(2\gamma - 1)/\gamma}
\end{cases}
\]  

(5.82)

for some \(M\) depending on \((\gamma, a, b)\) and for all \(t, T \leq t \leq t_0\). This implies the continuity of \((w, s)\) as a function of \((w_+, s_0(1))\) in the norm topology of \(L^\infty(J, H^{k-1} \oplus X^{\ell-1})\) for all compact
intervals $J \subset [T, \infty)$. The other continuities follow therefrom and from the boundedness of $(w, \bar{s})$ in $L^\infty([T, \infty), H^k \oplus X^\ell)$ by standard compactness arguments.

\[ \square \]

**Remark 5.7.** By analogy with Proposition 4.1, part (3), one expects the map $(w_+, s_0(1)) \to (w, s)$ to be also continuous from the norm topology in $H^k \oplus X^\ell$ to the norm topology in $L^\infty(J, H^k \oplus X^\ell)$ for compact $J \subset [T, \infty)$. A proof of that fact would require a combination of Steps 6 and 7 in the proof of Proposition 4.1 with the proof just given of Proposition 5.7, part (3) with however estimates at the level $(k, \ell)$ instead of $(k-1, \ell-1)$. However, the coupling between $t_0$ and $\varepsilon$ in the latter has the effect that, when the time dependence is taken into account in the former, the resulting dependence of the estimates on $\varepsilon$ or on $t_0$ is not sufficiently good to establish the result without additional assumptions on $\gamma$, namely without assuming $\gamma$ sufficiently close to 1. Since the argument is rather complicated for a result of restricted validity, we refrain from pushing it any further.

In Propositions 5.5 and 5.7, we have defined two maps $(w, s) \to (w_+, s_0)$ and $(w_+, s_0) \to (w, s)$ between solutions $(w, s)$ of the system (5.1) (5.2) and solutions of the auxiliary free equation (5.3) defined in a neighborhood of infinity in time. Both of these maps suffer from the loss of one derivative. Nevertheless, they are inverse of each other (and in particular injective) whenever they can be applied successively. We state that fact in the following proposition.

**Proposition 5.8.** Let $1/2 < \gamma < 1$ and let $(k, \ell)$ be an admissible pair.

1. Let $(w_+, s_0)$ be a solution of (5.3) such that $(w_+, \bar{s}_0) \in (C \cap L^\infty)([1, \infty), H^{k+1} \oplus X^{\ell+1})$ or equivalently defined by (5.4) with $(w_+, s_0(1)) \in H^{k+1} \oplus X^{\ell+1}$. Let $(w, s)$ be the solution of (5.1) (5.2) defined in Proposition 5.7, part (1), and let $(w'_+, s'_0)$ be the solution of (5.3) defined from $(w, s)$ in Propositions 5.3 and 5.5. Then $w'_+ = w_+$ and $s'_0 = s_0$.

2. Let $(w, s)$ be a solution of (5.1) (5.2) such that $(w, \bar{s}) \in (C \cap L^\infty)([T, \infty), H^{k+2} \oplus X^{\ell+2})$ for some $T$, $1 \leq T < \infty$. Let $(w_+, s_0)$ be the solution of (5.3) defined in Propositions 5.3 and 5.5, so that $(w_+, \bar{s}_0) \in (C \cap L^\infty)([1, \infty), H^{k+1} \oplus X^{\ell+1})$ and let $(w', s')$ be the solution of (5.1) (5.2) defined from $(w_+, s_0)$ in Proposition 5.7, part (1), so that $(w', \bar{s}') \in (C \cap L^\infty)([T', \infty), H^k \oplus X^\ell)$
for some $T', 1 \leq T' < \infty$. Then $w' = w$ and $s' = s$ for $t \geq T \lor T'$.

**Proof. Part (1).** From Proposition 5.7, esp. (5.69) we obtain

$$|w - w_+|_k \leq M t^{-\gamma}, \quad |s - s_0|_\ell \leq M t^{1-2\gamma}$$

for some $M$ depending on $(w_+, s_0)$. From Propositions 5.3 and 5.5, esp. (5.33) and (5.43), we obtain similarly

$$|w - w'_+|_{k-1} \leq M t^{-\gamma}, \quad |s - s'_0|_{\ell-1} \leq M t^{1-2\gamma}.$$  

Taking the limit $t \to \infty$ shows that $w'_+ = w_+$, so that by (5.4) $s'_0 - s_0$ is constant in time, and therefore zero.

**Part (2).** From Propositions 5.3 and 5.5, we obtain in the same way

$$|w - w_+|_{k+1} \leq M t^{-\gamma}, \quad |s - s_0|_{\ell+1} \leq M t^{1-2\gamma}.$$  

From Proposition 5.7, we then obtain

$$|w' - w_+|_k \leq M t^{-\gamma}, \quad |s' - s_0|_\ell \leq M t^{1-2\gamma}$$

so that

$$|w - w'|_k \leq M t^{-\gamma}, \quad |s - s'|_\ell \leq M t^{1-2\gamma}$$

for $t \geq T \lor T'$. The result then follows from Proposition 5.2.

\[ \square \]

We conclude this section with a brief discussion of the other systems of equations that can be used instead of (5.1) (5.2) (5.3) and on their drawbacks as compared with the latter. That discussion was briefly sketched at the end of Section 2 and can now be resumed at a more technical level.

(i) Instead of the system (5.1) (5.2) corresponding to (2.25) (2.26) and to the choice $w = w_3$, we could have used a system corresponding to (2.19) (2.20) and to the choice $w = w_2$, with an additional term $(2t^2)^{-1}\Delta w$ in the equation for $w$. That term would make no difference in the energy estimate (3.40) of Lemma 3.7 and therefore in Proposition 5.1. However it would produce
an additional term $Ct^{-2}|w_+|_{k+2}$ in the energy estimate (5.58) for $|w - w_+|_k$, and therefore a loss of two derivatives instead of one on $w$ in the crucial Proposition 5.6.

(ii) Instead of the free auxiliary equation (5.3) corresponding to (3.7), we could have used the more accurate HJ equation corresponding to (3.6). Since however we need the assumption $\gamma > 1/2$ in a crucial way in order to obtain estimates uniform in $t_0$ for $t \leq t_0$, both in Proposition 5.4 (see esp. (5.39)) and in Proposition 5.6 (see esp. (5.54)) using (3.6) instead of (3.7) would not produce any improvement of the results. On the other hand it would make the treatment of $s_0$ more complicated, since the equation (3.6) is hardly simpler than the system (3.4) (3.5). In particular, instead of the explicit solution (5.4), we would need a proposition similar to Proposition 5.1 in order to solve (3.6) for $s_0$, with a result valid for large time only. Similarly, in order to estimate $s_0(t)$ uniformly in $t_0$ for $t \leq t_0$ and to take the limit $t_0 \to \infty$ so as to prove the existence of asymptotic states, we would have to replace the relatively simple Propositions 5.4 and 5.5 by more complicated ones of the same degree of complication as needed to prove the existence of wave operators, namely Propositions 5.6 and 5.7. Finally it would no longer be possible, in the definition of the wave operators, to characterize the asymptotic solution $(w_+, s_0)$ by an initial condition at a fixed time $t = 1$ independent of $w_+$.

6 The auxiliary system at infinite time. Asymptotics II.

In this section, we perform a construction similar to that of Section 5, and we essentially construct local wave operators at infinity for the auxiliary system (5.1) (5.2), now however compared with the auxiliary free equation (3.8) which is better suited than (3.7) = (5.3) for the study of gauge invariance, as is explained in Section 2. We recall that $g_0$ and $g$ are defined by (3.1) (3.2). Most of the results of this section will be obtained from those of Section 5 and will involve a comparison of solutions of (5.3) and (3.8). In order to make that comparison more transparent, we shall use exclusively $g_0$ and refrain from using $g$ in this and the next section. For brevity we shall also use the short hand notation $g_0(w) = g_0(w, w)$ for the diagonal restriction of $g_0$. With that notation

$$g(w, w) = g_0(U^*(1/t)w).$$

In order to distinguish solutions of (3.8) from those of (5.3) considered in the previous
section, we shall use the notation \( s_{02} \) for the former, the additional subscript 2 referring to the fact that the nonlinearity in (3.8) is that of the free auxiliary equation (2.22) naturally associated with the system (2.19) (2.20) for \((w_2, \varphi_2)\). With the previous notation, the equation (3.8) is rewritten as

\[
\partial_t s_{02} = t^{-\gamma} \nabla g_0(w_+)
\]

and is trivially solved by

\[
s_{02}(t) = s_{02}(1) + (1 - \gamma)^{-1} (t^{1-\gamma} - 1) \nabla g_0(w_+)
\]

(6.2)

to be compared with the general solution (5.4) of (5.3). It follows from (6.2) and Corollary 3.1 that for admissible \((k, \ell)\) and for \((w_+, s_{02}(1)) \in H^k \oplus X^\ell\), \(\bar{s}_{02} \equiv t^{\gamma-1} s_{02} \in (C \cap L^\infty)([1, \infty), X^\ell)\) and

\[
\| \bar{s}_{02}; L^\infty([1, \infty), X^\ell) \| \leq |s_{02}(1)|_\ell + C(1 - \gamma)^{-1} |w_+|_k^2 .
\]

We first compare solutions of (5.3) and (6.1) which coincide in a suitable sense at some time \(t_0\).

**Lemma 6.1.** Let \(1/2 < \gamma < 1\) and let \((k, \ell)\) be an admissible pair. Let \(w_0 \in H^k\). Let \(1 \leq t_0 \leq \infty\). Let \(s_0\) be a solution of (5.3) with \(w_+ = w_0\) and let \(s_{02}\) be a solution of (6.1) with \(w_+ = U^*(1/t_0)w_0\) such that \(s_0(t_0) = s_{02}(t_0)\), so that

\[
s_0(t) - s_{02}(t) = \int_{t_0}^t dt' t'^{-\gamma} (\nabla g_0(U^*(1/t')w_0) - \nabla g_0(U^*(1/t_0)w_0)) .
\]

(For \(t_0 = \infty\), coincidence at \(t_0\) is defined by (6.3) and justified by the estimates to follow).

Then

1. The following estimates hold:

\[
|s_0(t) - s_{02}(t)|_{\ell-1} \leq C(1 - \gamma)^{-1} |w_0|_k^2 t_0^{-1/2} t^{1-\gamma} \quad (t \geq t_0) \tag{6.4}
\]

\[
|s_0(t) - s_{02}(t)|_{\ell-1} \leq C(2\gamma - 1)^{-1} |w_0|_k^2 t_0^{1/2 - \gamma} \quad (t \leq t_0) . \tag{6.5}
\]

2. Assume in addition that \(w_0 \in H^{k+1}\). Then the following estimates hold:

\[
|s_0(t) - s_{02}(t)|_{\ell+1} \leq C(1 - \gamma)^{-1} |w_0|_{k+1}^2 t_0^{-1/2} t^{1-\gamma} \quad (t \geq t_0) \tag{6.6}
\]

\[
|s_0(t) - s_{02}(t)|_{\ell+1} \leq C(2\gamma - 1)^{-1} |w_0|_{k+1}^2 t_0^{1/2 - \gamma} \quad (t \leq t_0) . \tag{6.7}
\]
Proof. Part (1). From (6.3) and from Corollary 3.1, we obtain

$$|s_0(t) - s_{02}(t)|_{\ell-1} \leq C \int_{t_0}^{t} dt' t'^{-\gamma} |(U^*(1/t') - U^*(1/t_0)) w_0|_{k-1} |w_0|_k.$$  \hfill (6.8)

Now

$$|(U^*(1/t') - U^*(1/t_0)) w_0|_{k-1} \leq |1/t' - 1/t_0|^{1/2} |w_0|_k.$$  \hfill (6.9)

Substituting (6.9) into (6.8) and integrating over time yields (6.4) (6.5).

Part (2). We estimate similarly by (6.3) and Corollary 3.1

$$|s_0(t) - s_{02}(t)|_{\ell+1} \leq C \int_{t_0}^{t} dt' t'^{-\gamma} |(U^*(1/t') - U^*(1/t_0)) w_0|_k |w_0|_{k+1}$$  \hfill (6.10)

from which (6.6) (6.7) follow as previously.

We now follow step by step the constructions performed in Section 5 with $s_0$, leading to the existence of asymptotic states (Propositions 5.4 and 5.5) and to the existence of local wave operators at infinity (Propositions 5.6 and 5.7). We first consider a fixed solution $(w, s)$ of the system (5.1) (5.2) as constructed in Proposition 5.1 and we look for a solution of (6.1) which is asymptotic to $s(t)$ at infinity. For that purpose we take some $t_0$ large enough and we define the solution of (6.1) which coincides with $s(t)$ at $t_0$ by

$$s_{02,t_0}(t) = s(t_0) + \int_{t_0}^{t} dt' t'^{-\gamma} \nabla g_0(U^*(1/t_0) w_0)$$  \hfill (6.11)

or equivalently

$$s_{02,t_0}(t) = s(t) - \int_{t_0}^{t} dt' \left\{ t'^{-2}(s \cdot \nabla) s + t'^{-\gamma} (\nabla g_0(U^*(1/t') w) - \nabla g_0(U^*(1/t_0) w_0)) \right\}$$  \hfill (6.12)

with $w_0 = w(t_0)$ (compare with (5.34) (5.35)). As in Section 5, $s_{02,t_0}(t)$ satisfies the analogue of the estimate (5.37), which is not uniform in $t_0$, and the first task is to obtain an estimate uniform in $t_0$. This is done in the following proposition, which is the analogue of Proposition 5.4.

**Proposition 6.1.** Let $1/2 < \gamma < 1$ and let $(k, \ell)$ be an admissible pair. Let $(w, s)$ be a solution of the system (5.1) (5.2) such that $(w, \tilde{s}) \in (C \cap L^\infty)([T, \infty), H^k \oplus X^\ell)$ for some $T > 0$ and...
define a and b by (5.30). Let \( t_0 \geq T \) and \( w_0 = w(t_0) \). Then \( s_{0,t_0} \) defined by (6.11) satisfies the estimates

\[
|s_{0,t_0}(t) - s(t)|_{\ell-1} \leq C \left( (b^2 + (1 - \gamma)^{-1}a^2b)t_0^{-\gamma} + (1 - \gamma)^{-1}a^2t_0^{-1/2} \right) t^{1-\gamma}
\]

(6.13) for \( t \geq t_0 \) and \( t_0^0 \geq Cb \),

\[
|s_{0,t_0}(t) - s(t)|_{\ell-1} \leq C(2\gamma - 1)^{-1} \left( (b^2 + a^2b)t^{1-2\gamma} + a^2 t^{1/2-\gamma} \right)
\]

(6.14) for \( T \leq t \leq t_0 \) and \( t^\gamma \geq Cb \).

Assume in addition that \( (1 - \gamma)t_0^0 \geq Ca^2 \), \( (1 - \gamma)t_0^{1/2} \geq C a^2 b^{-1} \), \( (2\gamma - 1)t^\gamma \geq C(a^2 + b) \) and \( (2\gamma - 1)t^{1/2} \geq C a^2 b^{-1} \). Then the following estimate holds :

\[
|s_{0,t_0}(t)|_{\ell-1} \leq C b t^{1-\gamma}
\]

(6.15)

**Proof.** The result follows from Proposition 5.4 and Lemma 6.1, part (1), applied to \( s_{0,t_0} \) and \( s_{0,t_0} \) defined by (6.11) and (5.34).

\[\square\]

We now want to prove that \( s_{0,t_0} \) has a limit when \( t_0 \to \infty \). Following the method of Section 5 would lead us to define the limiting function \( s_0 \) by taking the formal limit \( t_0 \to \infty \) in (6.12), namely

\[
\begin{aligned}
s_0(t) &= s(t) + \int_t^\infty dt' \left( t'^{-2}(s \cdot \nabla)s + t'^{-\gamma} (\nabla g_0(U^*(1/t')w) - \nabla g_0(w_+)) \right)
\end{aligned}
\]

(6.16)

where \( w_+ \) is the limit of \( w(t) \) as \( t \to \infty \) obtained in Proposition 5.3. It is however simpler to take advantage of the results of Section 5, esp. Proposition 5.5 and to define \( s_0(t) \) in terms of \( s_0(t) \) obtained in the latter and defined by (5.36), namely to define \( s_0(t) \) by

\[
\begin{aligned}
s_0(t) &= s_0(t) + \int_t^\infty dt' \ t'^{-\gamma} (\nabla g_0(U^*(1/t')w_+ - \nabla g_0(w_+))
\end{aligned}
\]

(6.17)

The following proposition is the analogue for \( s_0 \) of Proposition 5.5.

**Proposition 6.2.** Let \( 1/2 < \gamma < 1 \) and let \((k, \ell)\) be an admissible pair. Let \((w, s)\) be a solution of the system (5.1) (5.2) such that \((w, \bar{s}) \in (C \cap L^\infty)([T, \infty), H^k \oplus X^\ell)\) for some \( T \geq 1 \) and
define $a$ and $b$ by (5.30). Let $w_+$ be the limit of $w(t)$ when $t \to \infty$ obtained in Proposition 5.3.

Then

(1) The integral in (6.16) is absolutely convergent in $X^{t-1}$ and defines a solution $s_{02}$ of the equation (6.1) such that $\bar{s}_{02} \in (C \cap L^\infty)([1, \infty), X^{t-1})$ and such that the following estimate holds for $t \geq T$, $t^{\gamma} \geq Cb$ :

$$|s_{02}(t) - s(t)|_{t-1} \leq C(2\gamma - 1)^{-1} \left((b^2 + a^2b)t^{1-2\gamma} + a^2 t^{1/2-\gamma}\right). \quad (6.18)$$

If in addition $(2\gamma - 1)t^{\gamma} \geq C(b + a^2)$ and $(2\gamma - 1)t^{1/2} \geq Ca^2b^{-1}$, the following estimate holds :

$$|s_{02}(t)|_{t-1} \leq C b t^{1-\gamma}. \quad (6.19)$$

(2) The function $s_{02,t_0}$ defined by (6.11) converges to $s_{02}$ in norm in $X^{t-1}$ when $t_0 \to \infty$ for $t \geq T$, $t^{\gamma} \geq Cb$, uniformly in compact intervals, and the following estimate holds for $t \wedge t_0 \geq T, (t \wedge t_0)^{\gamma} \geq Cb$ :

$$|s_{02,t_0}(t) - s_{02}(t)|_{t-1} \leq C \left((1 - \gamma)^{-1} + (2\gamma - 1)^{-1}\right) \left((b^2 + a^2b)t_0^{-\gamma} + a^2 t_0^{-1/2}\right) (t \vee t_0)^{1-\gamma}. \quad (6.20)$$

**Proof.** Part (1). Let $s_0(t)$ be defined by (5.36) supplemented by Proposition 5.5, part (1) and define $s_{02}(t)$ by (6.17). The result now follows from Proposition 5.5, part (1) and Lemma 6.1, part (1). In particular (6.18) follows from (5.43) and (6.5).

**Part (2).** For $t \geq t_0$, we estimate

$$|s_{02,t_0}(t) - s_{02}(t)|_{t-1} \leq |s_{02,t_0}(t) - s(t)|_{t-1} + |s_{02}(t) - s(t)|_{t-1}$$

and we estimate the first norm in the RHS by (6.13) and the second norm by (6.18). This yields (6.20) for $t \geq t_0$.

For $t \leq t_0$, we obtain from (6.11) and (6.1)

$$s_{02,t_0}(t) - s_{02}(t) = s(t_0) - s_{02}(t_0) + \int_t^{t_0} dt' t'^{-\gamma} \left(\nabla g_0(w_+) - \nabla g_0(U^*(1/t_0)w(t_0))\right). \quad (6.21)$$

We estimate the first difference in the RHS by (6.18) with $t = t_0$, and the integral by the same method as in the proof of Lemma 6.1, part (1), and by the use of (5.33) and (6.9) with $t' = \infty$, so that

$$\left|\int_t^{t_0} dt' \cdot \cdot \cdot \right|_{t-1} \leq C(1 - \gamma)^{-1} \left(a^2 b t_0^{1-2\gamma} + a^2 t_0^{1/2-\gamma}\right)$$

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which completes the proof of (6.20) for $t \leq t_0$.

The convergence stated in Part (2) follows from the estimate (6.20).

\[ \square \]

**Remark 6.1.** As announced in Section 2, the approximation of $s$ by $s_{02}$ for solutions $(w, s)$ of the system (5.1) (5.2) is not as good as the approximation by $s_0$ obtained in Section 5. This can be seen on (6.18) where the term $a^2 t^{1/2-\gamma}$ is dominant for large $t$ as compared with the term $(b^2 + a^2 b)t^{1-2\gamma}$ obtained from (5.43).

We now turn to the converse construction, namely to the construction of a solution $(w, s)$ of the system (5.1) (5.2) which is asymptotic to a fixed solution $(w_+, s_{02})$ of (6.1), defined by (6.2). The first step consists in constructing a solution $(w, s)$ which coincides with $(w_+, s_0)$ at some time $t_0$. The next result is the analogue of Proposition 5.6.

**Proposition 6.3.** Let $1/2 < \gamma < 1$ and let $(k, \ell)$ be an admissible pair. Let $(w_+, s_{02}(1)) \in H^{k+1} \oplus X^{\ell+1}$, define $s_{02}$ by (6.2) and let

\[ a = \lvert w_+ \rvert_{k+1} \quad , \quad b = \lVert \tilde{s}_{02}; L^\infty([1, \infty), X^{\ell+1}) \rVert \quad . \]  

(6.22)

Then, there exist $T_0$ and $T$, $1 \leq T_0$, $T < \infty$, depending only on $(\gamma, a, b)$, such that for all $t_0 \geq T_0 \lor T$, the system (5.1) (5.2) with initial data $w(t_0) = U(1/t_0)w_+$, $s(t_0) = s_{02}(t_0)$, has a unique solution $(w_{t_0}, s_{t_0})$ such that $(w_{t_0}, \tilde{s}_{t_0}) \in (C \cap L^\infty)([T, \infty), H^k \oplus X^\ell)$. One can take

\[ T_0 = C \left\{ (b + (1 - \gamma)^{-1} a^2)^{1/\gamma} \lor (1 - \gamma)^{-2} a^4 b^{-2} \right\} \]  

(6.23)

\[ T = C(2\gamma - 1)^{-2} \left( (b + a^2)^{1/\gamma} \lor a^4 b^{-2} \right) \]  

(6.24)

and the solution satisfies the estimates

\[ \left\{ \begin{array}{l} \lvert w_{t_0}(t) - w_+ \rvert_k \leq C \left( a b t_0^{-\gamma} + a t_0^{-1/2} \right) \\ \lvert s_{t_0}(t) - s_{02}(t) \rvert_\ell \leq C \left( (b^2 + (1 - \gamma)^{-1} a^2 b) t_0^{-\gamma} + (1 - \gamma)^{-1} a^2 t_0^{-1/2} \right) t_1^{-\gamma} \end{array} \right. \]  

(6.25)

for $t \geq t_0$, and

\[ \left\{ \begin{array}{l} \lvert w_{t_0}(t) - w_+ \rvert_k \leq C \left( a b t^{-\gamma} + a t_0^{-1/2} \right) \\ \lvert s_{t_0}(t) - s_{02}(t) \rvert_\ell \leq C(2\gamma - 1)^{-1} \left( (b^2 + a^2 b) t^{1-2\gamma} + a^2 t^{1/2-\gamma} \right) \end{array} \right. \]  

(6.26)
for $T \leq t \leq t_0$, and
\[ |w_{t_0}(t)|_k \leq Ca, \quad |s_{t_0}(t)|_\ell \leq C b t^{1-\gamma} \]  \hspace{1cm} (6.27)
for all $t \geq T$.

**Proof.** Let $w_0 = U(1/t_0)w_+$ and define $s_0(t)$ by (6.3) so that $s_0(t)$ solves (5.3) and satisfies $s_0(t_0) = s_{02}(t_0)$. By Lemma 6.1, part (2), $s_0 \in C([1,\infty),X^{\ell+1})$ and $s_0$ satisfies
\[ \|\tilde{s}_0; L^\infty([T,\infty),X^{\ell+1}) \| \leq C b \]  \hspace{1cm} (6.28)
provided $(1 - \gamma)_0^{1/2} \geq Ca^2b^{-1}$ and $(2\gamma - 1)T^{1/2} \geq Ca^2b^{-1}$ which follow from (6.23) (6.24) for $t_0 \geq T_0$. We now apply Proposition 5.6 with $s_0$ just defined. Let $(w_{t_0},s_{t_0})$ be the solution of the system (5.1) (5.2) thereby obtained under the conditions (5.5) (5.52) which also follow from (6.23) (6.24). That solution satisfies the required initial condition $w_{t_0}(t_0) = w_0$ and $s_{t_0}(t_0) = s_0(t_0) = s_{02}(t_0)$. Furthermore it satisfies the estimates (5.53) (5.54) (5.55) with however $w_+$ replaced by $w_0$. The estimates (6.25) (6.26) (6.27) follow from the previous ones, from Lemma 6.1, part (2) and from the estimate
\[ |w_0 - w_+|_k = |(U(1/t_0) - 1)w_+|_k \leq t_0^{-1/2} |w_+|_{k+1} \]  \hspace{1cm} (6.29)
\[ \square \]

We now take the limit $t_0 \to \infty$ of the solution $(w_{t_0},s_{t_0})$ constructed in Proposition 6.3 for fixed $(w_+,s_{02})$. The next result is the analogue of Proposition 5.7.

**Proposition 6.4.** Let $1/2 < \gamma < 1$ and let $(k,\ell)$ be an admissible pair.

(1) Let $(w_+,s_{02}(1)) \in H^{k+1} \oplus X^{\ell+1}$ and define $s_{02}, a, b$ by (6.2) (6.22). Then there exists $T, 1 \leq T < \infty$, depending only on $(\gamma,a,b)$ and a unique solution $(w,s)$ of the system (5.1) (5.2) such that $(w,\bar{s}) \in (C \cap L^\infty)([T,\infty),H^k \oplus X^\ell)$ and such that the following estimates hold for all $t \geq T$:
\[ \begin{cases} 
|w(t) - w_+|_k \leq C ab t^{-\gamma} , \\
|s(t) - s_{02}(t)|_\ell \leq C(2\gamma - 1)^{-1} \left( (b^2 + a^2 b) t^{1-2\gamma} + a^2 t^{1/2-\gamma} \right) , 
\end{cases} \]  \hspace{1cm} (6.30)
\[ |w(t)|_k \leq C a \quad , \quad |s(t)|_\ell \leq C b \ t^{1-\gamma} . \quad (6.31) \]

One can take
\[ T = C(2\gamma - 1)^{-2} \left( (b + a^2)^{1/\gamma} \vee a^4 \ b^{-2} \right) . \quad (6.32) \]

(2) Let \((w_{t_0}, s_{t_0})\) be the solution of the system (5.1) (5.2) constructed in Proposition 6.3 for \(t_0 \geq T_0 \vee T\) and in particular such that \((w_{t_0}, s_{t_0}) \in (C \cap L^\infty)([T, \infty), H^k \oplus X^\ell)\). Then \((w_{t_0}, s_{t_0})\) converges to \((w, s)\) in norm in \(L^\infty(J, H^{k-1} \oplus X^{\ell-1})\) and in the weak-\(*\) sense in \(L^\infty(J, H^k \oplus X^\ell)\) for any compact interval \(J \subset [T, \infty)\), and in the weak-\(*\) sense in \(H^k \oplus X^\ell\) pointwise in \(t\).

(3) The map \((w_+, s_{02}(1)) \mapsto (w, s)\) defined in Part (1) is continuous on the bounded sets of \(H^{k+1} \oplus X^{\ell+1}\) from the norm topology of \((w_+, s_{02}(1))\) in \(H^{k-1} \oplus X^{\ell-1}\) to the norm topology of \((w, s)\) in \(L^\infty(J, H^{k-1} \oplus X^{\ell-1})\) and to the weak-\(*\) topology in \(L^\infty(J, H^k \oplus X^\ell)\) for any compact interval \(J \subset [T, \infty)\), and to the weak-\(*\) topology in \(H^k \oplus X^\ell\) pointwise in \(t\).

**Proof.** Part (1). Take now (6.17) as the definition of \(s_0\) in terms of \(s_{02}\), so that \(s_0\) satisfies (6.28) with \(T\) given by (6.32) by Lemma 6.1, part (2). Let \((w, s)\) be the solution of the system (5.1) (5.2) constructed in Proposition 5.7. Then \((w, s)\) satisfies the properties stated in Part (1). In particular (6.30) follows from (5.69) and from (6.7). Uniqueness of \((w, s)\) follows from Proposition 5.2.

Part (2). We estimate \(y = |w_{t_0}(t) - w(t)|_{k-1}\) and \(z = |s_{t_0}(t) - s(t)|_{\ell-1}\) in the same way as in the proof of Proposition 5.7, part (2) by using Lemma 3.9 and Lemma 5.1, with initial conditions \(y(t_0)\) and \(z(t_0)\) estimated by (6.26) and (6.30) at time \(t_0\). The rest of the proof is identical with that of Proposition 5.7, part (2).

Part (3). The proof is almost identical with that of Proposition 5.7, part (3) with \(s_0\) replaced everywhere by \(s_{02}\). The only difference is the appearance of an additional term \((2\gamma - 1)^{-1} a^2 t^{1/2 - \gamma}\) in the RHS of the second inequality in (5.79), coming from (6.30). This leads to the choice \(t_0^{-1/2} = \varepsilon\) instead of \(t_0^{-\gamma} = \varepsilon b^{-1}\) so that in (5.81) (5.82) the factor \(\varepsilon^{(2\gamma - 1)/\gamma}\) is replaced by \(\varepsilon^{2\gamma - 1}\). \(\Box\)
Proposition 5.8 applies mutatis mutandis to the map \((w_+, s_{02}(1)) \rightarrow (w, s)\) because the map \((w_+, s_{02}(1)) \rightarrow (w_+, s_{02}(1))\) is bijective since it is defined by the explicit formula (6.17) and exactly preserves the relevant regularity by Lemma 6.1.

7 Wave operators and asymptotics for \(u\).

In this section we complete the construction of the wave operators for the equation (1.1) and we derive asymptotic properties of solutions in their range. The construction relies in an essential way on those of Sections 5 and 6, esp. Proposition 6.4, and will require a discussion of the gauge invariance of those constructions. The first task is to supplement them with the determination of the phase \(\varphi\), which appears so far only through its gradient \(s\) (see (2.26) (2.31)). Actually the treatment in Sections 4-6 only involved the variable \(s\), and did not even assume that \(s\) was a gradient.

We recall that \(g_0, g\) are defined by (3.1) (3.2) and we continue to use the short hand notation \(g_0(w, w) = g_0(w)\) so that
\[
g(w, w) = g_0(U^*(1/t)w) .
\]
We are interested in solving the system (2.25) (2.26) which we rewrite as
\[
\partial_t w = (2t^2)^{-1}U(1/t)(2\nabla \varphi \cdot \nabla + (\Delta \varphi))U^*(1/t)w \quad (2.25) \equiv (7.1)
\]
\[
\partial_t \varphi = (2t^2)^{-1}|\nabla \varphi|^2 + t^{-\gamma} g_0(U^*(1/t)w) . \quad (2.26) \sim (7.2)
\]
In analogy with (5.5), we shall use the notation \(\tilde{\varphi} = t^{\gamma-1} \varphi\). The relevant spaces for the phase \(\varphi\) are the spaces \(Y^\ell\) defined by
\[
Y^\ell = \left\{ \varphi : \varphi \in L^\infty \text{ and } \nabla \varphi \in X^\ell \right\} . \quad (7.3)
\]
We first consider the Cauchy problem with finite initial time \(t_0\) for the system (7.1) (7.2) with initial data \((w_0, \varphi_0) \in H^k \oplus Y^\ell\). That problem is solved by first solving the Cauchy problem for the system (5.1) (5.2) with initial data \((w_0, \nabla \varphi_0)\) at time \(t_0\) for \((w, s)\) by Proposition 4.1 or 5.1 and then recovering \(\varphi\) from \((w, s)\) by \(\varphi(t_0) = \varphi_0\) and
\[
\varphi(t) = \varphi(t_0) + \int_{t_0}^{t} dt' \left\{ (2t^2)^{-1}|s|^2 + t'^{-\gamma} g_0(U^*(1/t')w) \right\} . \quad (7.4)
\]
As mentioned in Section 2, it follows from (5.2) that the vorticity \( \omega = \nabla \times s \) satisfies the equation (2.35). Furthermore \( \omega(t_0) = 0 \). Under the available regularity properties of \( s \), it follows from Lemma 3.2 with \( p = \infty, u = \omega, v = s \) and from Gronwall’s Lemma that \( \omega(t) = 0 \) for all \( t \). On the other hand, from (5.2) and (7.2), it follows that

\[
(s - \nabla \varphi)(t) = \int_{t_0}^{t} dt' t'^{-2}(s \times \omega)(t')
\]

where \( (s \times \omega)_i = \sum_j s_j \omega_{ji} \), and therefore \( s(t) = \nabla \varphi(t) \) for all \( t \) for which \( (w, s) \) is defined since \( \omega = 0 \).

If \( (w, \varphi) \) is a solution of the system (7.1) (7.2) as obtained from Proposition 5.1 and from the previous argument, it follows from that proposition and from Corollary 3.1 that \( (w, \tilde{\varphi}) \in (C \cap L^\infty)([T, \infty), H^k \oplus Y^\ell) \).

We next consider the Cauchy problem with infinite initial time covered by Propositions 5.5 and 5.7. There we have established a correspondence between solutions \( (w, s) \) of (5.1) (5.2) and solutions \( (w_+, s_0) \) of (5.3), and we now extend it to a correspondence between solutions \( (w, \varphi) \) of (7.1) (7.2) and solutions \( (w_+, \varphi_0) \) of the auxiliary free equation

\[
\partial_t \varphi_0 = t^{-\gamma} g_0(U^*(1/t)w_+) .
\]

The general solution of (7.6) can be written as

\[
\varphi_0(t) = \varphi_0(1) + \int_1^t dt' t'^{-\gamma} g_0(U^*(1/t')w_+)
\]

and if \( s_0(t) \) and \( \varphi_0(t) \) are defined by (5.4) (7.7) with \( s_0(t) = \nabla \varphi_0(t) \) for one \( t \) (for instance \( t = 1 \)), the same relation holds for all \( t \). We supplement the correspondence established in Propositions 5.5 and 5.7 with the relation

\[
\varphi(t) = \varphi_0(t) - \int_t^\infty dt' \left\{ (2t'^2)^{-1}|s|^2 + t'^{-\gamma}(g_0(U^*(1/t')w) - g_0(U^*(1/t')w_+)) \right\}
\]

which will be used to define \( \varphi(t) \) and/or \( \varphi_0(t) \) in terms of each other. From (5.36) and (7.8) it follows that

\[
(s - \nabla \varphi)(t) - (s_0 - \nabla \varphi_0)(t) = -\int_t^\infty dt' t'^{-2}(s \times \omega)(t') .
\]
Define \((w_+, s_0)\) by Proposition 5.3, by (5.36) and by Proposition 5.5, part (1), and define \(\varphi_0\) by (7.8). Then it follows from (7.9) that \(s_0(t) = \nabla \varphi_0(t)\) for all \(t\).

Conversely, corresponding to the situation of Proposition 5.7, let \((w_+, \varphi_0(1)) \in H^{k+1} \oplus Y^{l+1},\) define \(\varphi_0(t)\) by (7.7) and \(s_0(t)\) by \(s_0(t) = \nabla \varphi_0(t)\), define \((w, s)\) by Proposition 5.7, part (1) and define \(\varphi(t)\) by (7.8). Let \((w_{t_0}, s_{t_0})\) be defined by Proposition 5.6. From the finite initial time results it follows that \(\omega_{t_0}(t) \equiv \nabla \times s_{t_0}(t) = 0\) for all \(t\) and \(t_0\), while by Proposition 5.7, part (2) \(\omega_{t_0}\) converges to \(\omega\) in \(X^{l-2}\) uniformly in compact intervals, so that \(\omega = 0\). It then follows again from (7.9) that \(s(t) = \nabla \varphi(t)\) for all \(t\). This proves that the correspondence established in Propositions 5.5 and 5.7 extends to a correspondence between solutions \((w, \varphi)\) of (7.1) (7.2) and solutions \((w_+, \varphi_0)\) of (7.6) preserving the relations \(s = \nabla \varphi; s_0 = \nabla \varphi_0\).

The same discussion applies mutatis mutandis to the situation of Propositions 6.2 and 6.4 and allows for an extension of the correspondence between solutions \((w, s)\) of (5.1) (5.2) and solutions \((w_+, s_{02})\) of (6.1) established there to a correspondence between solutions \((w, \varphi)\) of (7.1) (7.2) and solutions \((w_+, \varphi_{02})\) of the equation

\[
\partial_t \varphi_{02} = t^{-\gamma} g_0(w_+) \tag{2.29} \sim (7.10)
\]

which preserves the relations \(s = \nabla \varphi; s_{02} = \nabla \varphi_{02}\). The relation (7.8) between \(\varphi(t)\) and \(\varphi_0(t)\) is replaced by the relation

\[
\varphi(t) = \varphi_{02}(t) - \int_t^\infty dt' \left\{ (2t'^{-1}|s|^2 + t'^{-\gamma} (g_0(U^*(1/t')w) - g_0(w_+)) \right\} \tag{7.11}
\]

between \(\varphi(t)\) and \(\varphi_{02}(t)\), so that \(\varphi_0(t)\) and \(\varphi_{02}(t)\) are related by

\[
\varphi_0(t) = \varphi_{02}(t) - \int_t^\infty dt' t'^{-\gamma} (g_0(U^*(1/t')w_+) - g_0(w_+)) \tag{7.12}
\]

in agreement with (6.16) and (6.17).

The equation (7.10) is trivially solved by

\[
\varphi_{02}(t) = \varphi_{02}(1) + (1 - \gamma)^{-1} (t^{1-\gamma} - 1) g_0(w_+) \tag{7.13}
\]

We can now embark on the explicit construction of wave operators, and we first construct a wave operator \(W\) for the auxiliary system (5.1) (5.2). This could be based either on Proposition 5.7 or on Proposition 6.4. We choose the latter since as mentioned before it is better suited
than the former for the discussion of gauge invariance to be performed next.

**Definition 7.1.** We define $W$ as the map

$$W : (w_+, \varphi_{02}(1)) \rightarrow (w, \varphi)$$

from $H^{k+1} \oplus Y^{\ell+1}$ to the set of $(w, \varphi)$ such that $(w, \tilde{\varphi}) \in (C \cap L^\infty)([T, \infty), H^k \oplus Y^\ell)$ for some $T, 1 \leq T < \infty$, as follows. Define $\varphi_{02}(t)$ by (7.13) and $s_{02}(t)$ by $s_{02}(t) = \nabla \varphi_{02}(t)$. Define $(w, s)$ by Proposition 6.4, part (1) and finally define $\varphi$ by (7.11), so that $s = \nabla \varphi$ by the previous discussion. Then $W$ is well defined by (7.14) as a map between the spaces indicated.

Before defining the wave operators for $u$, we now study the gauge invariance of $W$, which plays an important role in justifying that definition, as was explained in Section 2. For that purpose we need some information on the Cauchy problem for the equation (1.1) at finite times. In addition to the operators $M = M(t)$ and $D = D(t)$ defined by (2.5) (2.6), we introduce the operator

$$J = J(t) = x + it \nabla,$$

the generator of Galilei transformations. The operators $M, D, J$ satisfy the commutation relation

$$i M D \nabla = J M D.$$

For any interval $I \subset [1, \infty)$ and any nonnegative integer, we define the space

$$\mathcal{X}^k(I) = \left\{ u : D^* M^* u \in \mathcal{C}(I, H^k) \right\}$$

$$= \left\{ u : < J(t) >^k u \in \mathcal{C}(I, L^2) \right\},$$

where $< \lambda > = (1 + \lambda^2)^{1/2}$ for any real number or self-adjoint operator $\lambda$ and where the second equality follows from (7.16). Then

**Proposition 7.1.** Let $k$ be a positive integer and let $0 < \mu < n \wedge 2k$. Then the Cauchy problem for the equation (1.1) with initial data $u(t_0) = u_0$ such that $< J(t_0) >^k u_0 \in L^2$ at some initial time $t_0 \geq 1$ is locally well posed in $\mathcal{X}^k(\cdot)$, namely

(1) There exists $T > 0$ such that (1.1) has a unique solution with initial data $u(t_0) = u_0$ in $\mathcal{X}^k([1 \vee (t_0 - T), t_0 + T])$. 

For any interval \( I, t_0 \in I \subset [1, \infty) \), (1.1) with initial data \( u(t_0) = u_0 \) has at most one solution in \( \mathcal{X}^k(I) \).

The solution of Part (1) depends continuously on \( u_0 \) in the norms considered there.

**Proof.** The proof is obtained by minor variations of the corresponding results in [9]. The differences come from the factor \( t^{\mu - \gamma} \) in (1.2), which is irrelevant for the present problem, and from the replacement of \( \mathcal{C}(I, H^k) \) by \( \mathcal{X}^k(I) \). That replacement is made possible by the properties of the operator \( J(t) \) and especially the commutation relation (7.16), which implies that \( J(t) \) behaves as a derivative on gauge invariant functions. See for instance [2].

\[ \square \]

In the study of gauge invariance for \( W \) we shall actually need only the uniqueness statement, Part (2) of Proposition 7.1.

We recall that in the transition from the system (5.1) (5.2) to the equation (1.1), \( u \) should be defined by (2.14) with \((w, \varphi) = (w_3, \varphi_3)\) and accordingly we define the map

\[ \Phi : (w, \varphi) \rightarrow u = MD \exp[-i\varphi] U^*(1/t)w. \tag{7.18} \]

That map satisfies the following property.

**Lemma 7.1.** The map \( \Phi \) defined by (7.18) is bounded and continuous from \( \mathcal{C}(I, H^k \oplus Y^\ell) \) to \( \mathcal{X}^k(I) \) for any admissible pair \((k, \ell)\) and any interval \( I \subset [1, \infty) \).

**Proof.** An immediate consequence of Lemma 3.5.

\[ \square \]

We can now make the following definition.

**Definition 7.2.** Let \((k, \ell)\) be an admissible pair and let \((w, \varphi)\) and \((w', \varphi')\) be two solutions of the system (7.1) (7.2) in \( \mathcal{C}(I, H^k \oplus Y^\ell) \) for some interval \( I \subset [1, \infty) \). We say that \((w, \varphi)\) and \((w', \varphi')\) are gauge equivalent if they give rise to the same \( u \), namely \( \Phi((w, \varphi)) = \Phi((w', \varphi')) \).
equivalently if
\[ \exp[-i\varphi(t)] U^*(1/t) \, w(t) = \exp[-i\varphi'(t)] U^*(1/t) \, w'(t) \] (7.19)
for all \( t \in I \).

A sufficient condition for gauge equivalence is given by the following Lemma.

**Lemma 7.2.** Let \((k, \ell)\) be an admissible pair and let \((w, \varphi)\) and \((w', \varphi')\) be two solutions of the system (7.1) (7.2) in \( C(I, H^k \oplus Y^\ell) \). In order that \((w, \varphi)\) and \((w', \varphi')\) be gauge equivalent, it is sufficient that (7.19) holds for one \( t \in I \).

**Proof.** An immediate consequence of Lemma 7.1, of Proposition 7.1, part (2), and of the fact that \((k, \ell)\) admissible implies \( k > 1 + \mu/2 \).

\[ \square \]

The gauge covariance properties of \( W \) will be expressed by the following two propositions.

**Proposition 7.2.** Let \( 0 < \gamma < 1 \) and let \((k, \ell)\) be an admissible pair. Let \((w, \varphi)\) and \((w', \varphi')\) be two solutions of the system (7.1) (7.2) such that \((w, \bar{\varphi}), (w', \bar{\varphi}') \in (C \cap L^\infty)([T, \infty), H^k \oplus Y^\ell)\) for some \( T \geq 1 \), and assume that \((w, \varphi)\) and \((w', \varphi')\) are gauge equivalent. Then

(1) There exists \( \sigma \in Y^{\ell-1} \) such that \( \varphi'(t) - \varphi(t) \) converges to \( \sigma \) strongly in \( Y^{\ell-2} \) and in the weak-* sense in \( Y^{\ell-1} \). The following estimates holds :

\[ \| \varphi'(t) - \varphi(t) - \sigma; Y^{\ell-2} \| \leq A t^{-\gamma} \] (7.20)

for some constant \( A \) depending on \( T \) and on the norms of \( \bar{\varphi}, \bar{\varphi}' \) in \( L^\infty(\cdot, Y^\ell) \), with the exception of the case \( n \) even, \( \ell = n/2 + 1 \) where the \( L^\infty \) norm of \( \nabla \varphi \) satisfies only

\[ \| \nabla \varphi'(t) - \nabla \varphi(t) - \nabla \sigma \|_\infty \leq A t^{-\gamma/2} \] . (7.21)

(2) Assume in addition that \( \gamma > 1/2 \). Then \( \varphi'(t) - \varphi(t) \) converges to \( \sigma \) in norm in \( Y^{\ell-1} \) and the following estimate holds :

\[ \| \varphi'(t) - \varphi(t) - \sigma; Y^{\ell-1} \| \leq A t^{1-2\gamma} \] . (7.22)
Furthermore \( w'_+ = w_+ \exp[i \sigma] \) where \( w_+ \), \( w'_+ \) are the limits of \( w(t) \), \( w'(t) \) as \( t \to \infty \) obtained in Proposition 5.3.

(3) Assume in addition that \( \gamma > 1/2 \) and that \((w, \varphi), (w', \varphi') \in \mathcal{R}(W)\). Then \( \varphi'_{02}(t) = \varphi_{02}(t) + \sigma \) for all \( t \geq 1 \). In particular \( \sigma \in Y^\ell+1 \).

Proof. Part (1). Define \( \varphi_- = \varphi' - \varphi \), \( s = \nabla \varphi \), \( s' = \nabla \varphi' \), \( s_\pm = s' \pm s \), and

\[ b = \| \tilde{s}; L^\infty([T, \infty), X^\ell) \| \vee \| \tilde{s}'; L^\infty([T, \infty), X^\ell) \| \quad . \]  

(7.23)

From (5.2) and gauge equivalence it follows that

\[ \partial_t s_- = t^{-2} ((s_- \cdot \nabla) s_+ + (s_+ \cdot \nabla) s_-) \quad (7.24) \]

and therefore by Lemma 3.9

\[ \partial_t |s_-|_{\ell-1} \leq C t^{-2} |s_-|_{\ell-1} |s_+|_{\ell} \leq C b t^{-1-\gamma} |s_-|_{\ell-1} \]

so that by Gronwall’s Lemma, for all \( t \geq T \),

\[ |s_-(t)|_{\ell-1} \leq |s_-(T)|_{\ell-1} \exp(C b \gamma^{-1} T^{-\gamma}) \equiv A_0 \quad , \]

namely

\[ \| s_-; L^\infty([T, \infty), X^{\ell-1}) \| \leq A_0 \quad . \]  

(7.25)

¿From (7.2) and gauge equivalence, it follows that

\[ \partial_t \varphi_- = (2t^2)^{-1} (s_- \cdot s_+) \]

and therefore by (7.23) (7.25) for any \( t \geq t_0 \geq T \)

\[ \| \varphi_-(t) - \varphi_-(t_0) \|_\infty \leq \int_{t_0}^t dt' b A_0 t'^{-1-\gamma} \leq b A_0 \gamma^{-1} t_0^{-\gamma} \quad . \]  

(7.26)

This implies that \( \varphi_-(t) \) converges in norm in \( L^\infty \) to some \( \sigma \in L^\infty \) and that

\[ \| \varphi_-(t) - \sigma \|_\infty \leq b A_0 \gamma^{-1} t^{-\gamma} \quad . \]  

(7.27)

which is the part of (7.20) involving \( \varphi_- \) itself (and not \( s_- \) only). From the uniform estimate (7.25) and standard compactness arguments, it follows that \( \nabla \sigma \in X^{\ell-1} \), that \( |\nabla \sigma|_{\ell-1} \leq A_0 \).
and that \( s_- \) converges to \( \nabla \sigma \) in the weak-* sense in \( X^{\ell-1} \), which together with (7.27) implies weak-* convergence of \( \varphi_- \) to \( \sigma \) in \( Y^{\ell-1} \).

We finally prove the strong convergence of \( s_- \) to \( \nabla \sigma \) in \( X^{\ell-2} \). From (7.24) which we rewrite as

\[
\partial_t (s_- - \nabla \sigma) = t^{-2} \{ ((s_- - \nabla \sigma) \cdot \nabla) s_+ + (s_+ \cdot \nabla) (s_- - \nabla \sigma) + (\nabla \sigma \cdot \nabla) s_+ + (s_+ \cdot \nabla) \nabla \sigma \}
\]

and by the same estimates as in the proof of Lemma 3.11, we obtain

\[
\partial_t |s_- - \nabla \sigma|_{\ell-2} \leq C t^{-2} \{ |s_- - \nabla \sigma|_{\ell-2} |s_+|_\ell + |\nabla \sigma|_{\ell-1} |s_+|_\ell \} \leq C b t^{-1-\gamma} (|s_- - \nabla \sigma|_{\ell-2} + A_0)
\]

with the only exception of the case \( n \) even, \( \ell = n/2 + 1 \), where the \( L^\infty \) norm of \( s_- - \nabla \sigma \) which occurs in the \( X^{\ell-2} \) norm, is not estimated as in (7.29) because \( \| \partial^2 \sigma \|_\infty \) is not controlled by \( |\nabla \sigma|_{\ell-1} \). From (7.29) we obtain by Gronwall’s Lemma

\[
|s_- (t) - \nabla \sigma|_{\ell-2} \leq A_0 \{ \exp(C b \gamma^{-1} t^{-\gamma}) - 1 \}
\]

which together with (7.27) completes the proof of (7.20).

For \( n \) even, \( \ell = n/2 + 1 \), we estimate simply

\[
\| s_- (t) - \nabla \sigma \|_\infty \leq C \| \varphi_- (t) - \sigma \|_\infty^{1/2} \| s_- (t) - \nabla \sigma; \dot{H}^{n/2+1} \|^{1/2}
\]

and (7.21) follows from (7.27) and from the \( \dot{H}^{n/2} \) part of (7.30).

**Part (2).** Let \( t \geq t_0 \geq T, s_- (t) = s_- \) and \( s_- (t_0) = s_0 \). We rewrite (7.24) as

\[
\partial_t (s_- - s_0) = t^{-2} \{ ((s_- - s_0) \cdot \nabla) s_+ + (s_+ \cdot \nabla) (s_- - s_0) + (s_0 \cdot \nabla) s_+ + (s_+ \cdot \nabla) s_0 \} .
\]

By Lemma 3.9, we estimate

\[
\partial_t |s_- - s_0|_{\ell-1} \leq C t^{-2} \{ |s_- - s_0|_{\ell-1} |s_+|_\ell + |s_-|_{\ell-1} |s_+|_\ell + |s_+|_{\ell-1} |s_0|_\ell \} \leq C t^{-2} (|s_- - s_0|_{\ell-1} + |s_0|_\ell) |s_+|_\ell \leq C b t^{-1-\gamma} (|s_- - s_0|_{\ell-1} + b t_0^{1-\gamma})
\]

and therefore by Gronwall’s Lemma

\[
|s_- (t) - s_- (t_0)|_{\ell-1} \leq b t_0^{1-\gamma} \{ \exp(C b \gamma^{-1} t_0^{-\gamma}) - 1 \} .
\]
This yields a separate proof of the convergence of \( s_-(t) \) in \( X^{\ell-1} \), together with the estimate
\[
|s_-(t) - \nabla \sigma|_{\ell-1} \leq b \ t^{1-\gamma} \left( \exp(C \ b \ \gamma^{-1} \ t^{-\gamma}) - 1 \right),
\]
which together with (7.27) completes the proof of (7.22).

We now prove that \( w'_+ = w_+ \exp(i\sigma) \). For that purpose we estimate
\[
|w'_+ - w_+ e^{i\sigma}|_{k-1} \leq |w'_+ - U^*(1/t)w'(t)|_{k-1} + |U^*(1/t)w'(t) - \exp[i\varphi_-(t)] \ U^*(1/t)w(t)|_{k-1}
+ \left| (\exp[i(\varphi_-(t) - \sigma)] - 1) \exp(i\sigma) \ U^*(1/t)w(t) \right|_{k-1} + \left| \exp(i\sigma) \ (U^*(1/t)w(t) - w_+) \right|_{k-1}.
\]

We estimate the first norm in the RHS by Proposition 5.3, esp. (5.33) and by (6.9) as
\[
|w'_+ - U^*(1/t)w'(t)|_{k-1} \leq A (t^{-\gamma} + t^{-1/2}) \ .
\]

The second norm in the RHS of (7.31) is zero by gauge equivalence. The third norm is estimated by Lemma 3.5 as
\[
| \cdot | \leq C \left( \| \varphi_+ - \sigma \|_\infty + |\nabla (\varphi_+ - \sigma)|_{\ell-1} (1 + |\nabla (\varphi_+ - \sigma)|_{\ell-1})^{k-2} \right) (1 + |\nabla \sigma|_{\ell-1})^{k-1} |w|_{k-1}
\]
and the last norm is estimated by Lemma 3.5 again followed by the analogue of (7.32) for \( w \) as
\[
| \cdot | \leq (1 + |\nabla \sigma|_{\ell-1})^{k-1} \ A \left( t^{-\gamma} + t^{-1/2} \right) \ .
\]

Collecting (7.32) (7.33) (7.34) and using (7.25) and (7.22) shows that the RHS of (7.31) tends to zero when \( t \to \infty \) and therefore that the LHS is zero since it is time independent.

**Part (3).** We recall that in the situation of Proposition 6.4 and of the definition of \( W, \varphi(t) \) and \( \varphi_{02}(t) \) are related by (7.11), and by the estimates (3.35) (6.30) (6.31)
\[
\| \varphi(t) - \varphi_{02}(t) \|_\infty \leq A \ t^{1/2-\gamma}
\]
for some \( A \) depending only on \( a, b \) defined by (6.22) and for \( t \) sufficiently large. Similarly
\[
\| \varphi'(t) - \varphi'_{02}(t) \|_\infty \leq A \ t^{1/2-\gamma} \ .
\]

It follows then from (7.35) (7.36) and (7.20) (or (7.22)) that
\[
\| \varphi'_{02}(t) - \varphi_{02}(t) - \sigma \|_\infty \to 0 \ \text{when} \ t \to \infty \ .
\]
On the other hand from (7.10) (or (7.13)) and from the condition $w'_+ = w_+ \exp(i\sigma)$ it follows that $\varphi'_{02}(t) - \varphi_{02}(t)$ is constant in time. Therefore $\varphi'_{02}(t) - \varphi_{02}(t) = \sigma$ for all $t$.

\[\square\]

Proposition 7.2 prompts us to make the following definition of gauge equivalence for asymptotic states.

**Definition 7.3.** Two pairs $(w_+, \varphi_{02}(1))$ and $(w'_+, \varphi'_{02}(1))$ are gauge equivalent if there exists a real function $\sigma = \sigma(x)$ such that $w'_+ = w_+ \exp(i\sigma)$ and $\varphi'_{02}(1) - \varphi_{02}(1) = \sigma$.

Two gauge equivalent pairs generate two solutions $(w_+, \varphi_{02})$ and $(w'_+, \varphi'_{02})$ of (7.10) such that $\varphi'_{02}(t) - \varphi_{02}(t) = \sigma$ for all $t \geq 1$. Those two solutions will also be said to be gauge equivalent.

In Definition 7.3, we have not specified the regularity of $(w_+, \varphi_{02}(1))$ and $(w'_+, \varphi'_{02}(1))$. This can be done easily, depending on the needs, and possibly with the help of Lemma 3.5.

With the previous definition, Proposition 7.2 should be understood to mean that two gauge equivalent solutions of the system (7.1) (7.2) in $\mathcal{R}(W)$ are images of two gauge equivalent solutions of (7.10). The next proposition states that conversely two gauge equivalent solutions of (7.10) have gauge equivalent images under $W$.

**Proposition 7.3.** Let $1/2 < \gamma < 1$ and let $(k, \ell)$ be an admissible pair. Let $(w_+, \varphi_{02}(1))$, $(w'_+, \varphi'_{02}(1)) \in H^{k+1} \oplus Y^{\ell+1}$ be gauge equivalent, and let $(w, \varphi)$, $(w', \varphi')$ be their images under $W$. Then $(w, \varphi)$ and $(w', \varphi')$ are gauge equivalent, and $\lim_{t \to \infty} \varphi'(t) - \varphi(t) = \sigma \equiv \varphi'_{02}(1) - \varphi_{02}(1)$.

**Proof.** Let $t_0$ be sufficiently large and let $(w_{t_0}, \varphi_{t_0})$ and $(w'_{t_0}, \varphi'_{t_0})$ be the solutions of the system (7.1) (7.2) constructed by Proposition 6.3 supplemented with (7.4) with the appropriate initial conditions at $t_0$, namely

$$\varphi_{t_0}(t_0) = \varphi_{02}(t_0), \quad \varphi'_{t_0}(t_0) = \varphi'_{02}(t_0).$$

(7.37)
From (7.37) and from the initial conditions

\[ w_{t_0}(t_0) = U(1/t_0)w_+ , \quad w'_{t_0}(t_0) = U(1/t_0)w'_+ \]

imposed in Proposition 6.3 it follows that

\[ \exp[-i\varphi_{t_0}(t_0)] U^*(1/t_0) w_{t_0}(t_0) = \exp[-i\varphi'_{t_0}(t_0)] U^*(1/t_0) w'_{t_0}(t_0) \]

and therefore by Lemma 7.2, \((w_{t_0}, \varphi_{t_0})\) and \((w'_{t_0}, \varphi'_{t_0})\) are gauge equivalent, namely satisfy

\[ \exp[-i\varphi_{t_0}(t)] U^*(1/t) w_{t_0}(t) = \exp[-i\varphi'_{t_0}(t)] U^*(1/t) w'_{t_0}(t) \tag{7.38} \]

for all \(t\) for which they are defined.

We now take the limit \(t_0 \rightarrow \infty\) for fixed \(t\) in (7.38). By Proposition 6.4, part (2) supplemented with similar estimates on \(\varphi_{t_0}\) and \(\varphi'_{t_0}\), for fixed \(t\), \((w_{t_0}, \varphi_{t_0})\) and \((w'_{t_0}, \varphi'_{t_0})\) converge respectively to \((w, \varphi)\) and \((w', \varphi')\) in norm in \(H^{k-1} \oplus Y^{t-1}\). By an easy application of Lemma 3.5, it follows therefrom that one can take the limit \(t_0 \rightarrow \infty\) in (7.38), thereby obtaining (7.19), so that \((w, \varphi)\) and \((w', \varphi')\) are gauge equivalent.

The last statement of Proposition 7.3 is a repetition of Proposition 7.2, part (3).

\[ \square \]

We can now define the wave operator for \(u\). We recall from the heuristic discussion in Section 2 that we want to exploit the operator \(W\) defined in Definition 7.1, reconstruct \(u\) through the map \(\Phi\) defined by (7.18), and eliminate the arbitrariness in \(\varphi_{02}\) by fixing some initial condition for it, namely \(\varphi_{02}(1) = 0\), thereby purporting to ensure the injectivity of the wave operator without restricting its range. That program is implemented by the following definition and proposition.

**Definition 7.4.** We define the wave operator \(\Omega\) as the map

\[ \Omega : u_+ \rightarrow u = (\Phi \circ W)(Fu_+, 0) \tag{7.39} \]

from \(FH^{k+1}\) to \(X^k([T, \infty))\) for some \(T, 1 \leq T < \infty\), where \(k\) is the first element of an admissible pair, and \(W, \Phi\) are defined by Definition 7.1 and by (7.18).
The fact that $\Omega$ acts between the spaces indicated follows from Proposition 6.4 and from Lemma 7.1. The value of $T$ depends on $u_+$ and can be taken according to (6.32) with $b = C(1 - \gamma)^{-1}a^2$, namely
\[
T = C \left( (1 - \gamma)^{-2} + (2\gamma - 1)^{-2} \right) (|Fu_+|_{k+1} \vee 1)^{2/\gamma} . 
\] (7.40)

**Proposition 7.4.**

(1) The map $\Omega$ is injective.

(2) $\mathcal{R}(\Omega) = \mathcal{R}(\Phi \circ W)$.

**Proof.** Part (1). Let $u = \Omega(u_+) = \Omega(u'_+) \quad \text{and let} \quad (w, \varphi) = W(Fu_+,0), \quad (w', \varphi') = W(Fu'_+,0), \quad \text{so that} \quad u = \Phi(w, \varphi) = \Phi(w', \varphi') \quad \text{and therefore} \quad u_+ = u'_+.$

Part (2). Let $u = (\Phi \circ W)(w_+, \varphi_02(1)) \in \mathcal{R}(\Phi \circ W)$. Then by Proposition 7.3, also $u = (\Phi \circ W)(w'_+,0)$ where $w'_+ = w_+ \exp[-i\varphi_02(1)]$, and $w'_+ \in H^{k+1}$ by Lemma 3.5. Therefore $u = \Omega(F^*w'_+) \in \mathcal{R}(\Omega)$.

Note in particular that Proposition 7.4 part (2) means that we have not restricted the range of the wave operators from $W$ to $\Omega$ by arbitrarily imposing $\varphi_02(1) = 0$.

We now collect all the available information on the solutions of the original equation (1.1) so far constructed, namely existence through the previous definition of $\Omega$, some partial form of uniqueness coming from Proposition 5.2, and asymptotic decay estimates coming from Propositions 5.7 and 6.4. In order to state the result we need the phases $\varphi_02(t)$ and $\varphi_0(t)$ defined now by (7.13) (7.12) with $w_+ = Fu_+$ and $\varphi_02(1) = 0$, namely
\[
\varphi_02(t) = (1 - \gamma)^{-1} (t^{1-\gamma} - 1) g_0(Fu_+) \quad \text{(7.41)} 
\]
\[
\varphi_0(t) = \varphi_02(t) - \int_t^\infty \! dt' t'^{-\gamma} \left( g_0(FMu_+) - g_0(Fu_+) \right) . \quad \text{(7.42)} 
\]
We recall that $M = M(t)$ is defined by (2.5) and satisfies (2.11). The main result of this paper can now be stated as follows.

**Proposition 7.5.** Let $n \geq 3$, $0 < \mu \leq n - 2$, $1/2 < \gamma < 1$ and let $(k, \ell)$ be an admissible pair. Let $u_+ \in FH^{k+1}$ and define $\varphi_0(t)$ and $\varphi(t)$ by (7.41) and (7.42). Then

1. There exists a unique solution $u \in X^k([T, \infty))$ of the equation (1.1) which can be represented as

$$u = MD \exp(-i\varphi)U^*(1/t)w$$

(7.43)

where $(w, \varphi)$ is a solution of the system (7.1) (7.2) such that $(w, \tilde{\varphi}) \in (C \cap L^\infty)([T, \infty), H^k \oplus Y^\ell)$ and such that

$$|w(t) - F u_+|_{k-1} t^{1-\gamma} \to 0$$

(7.44)

$$\|\varphi(t) - \varphi_0(t); Y^{\ell-1}\| \to 0$$

(7.45)

when $t \to \infty$. The time $T$ depends on $\gamma$ and $u_+$ and can be taken in the form (7.40).

2. The solution $u$ is obtained as $u = \Omega(u_+)$ where the map $\Omega$ is defined in Definition 7.4. The map $\Omega$ is injective.

3. The map $\Omega$ is continuous on the bounded sets of $FH^{k+1}$ from the norm topology in $FH^{k-1}$ for $u_+$ to the norm topology in $X^{k-1}(J)$ and to the weak-* topology in $X^k(J)$ for $u$ for any compact interval $J \subset [T, \infty)$, and to the weak topology in $FH^k$ pointwise in $t$.

4. The solution $u$ satisfies the following estimates for $t \geq T$:

$$\|\langle J(t) \rangle^k (\exp[i\varphi_0(t, x/t)]u(t) - M(t) D(t) Fu_+)\|_2 \leq A(2\gamma - 1)^{-1} t^{1/2-\gamma} ,$$

(7.46)

$$\|\langle J(t) \rangle^k (\exp[i\varphi_0(t, x/t)]u(t) - U(t) u_+)\|_2 \leq A(2\gamma - 1)^{-1} t^{1-2\gamma}$$

(7.47)

for some constant $A$ depending on $\gamma$ and $u_+$ and bounded in $\gamma$ for fixed $u_+$ and $\gamma$ away from 1.

**Proof.** Parts (1) and (2). All the results except uniqueness follow from Proposition 6.4 supplemented with the reconstruction of $\varphi$ and from the subsequent definition of $\Omega$. In particular (7.43) is essentially (7.18) and the injectivity of $\Omega$ is Proposition 7.4, part (1).

Uniqueness is an immediate consequence of Proposition 5.2, given the asymptotic behaviour of $(w, \varphi)$ that follows from Proposition 6.4.
Part (3) follows from Proposition 6.4, part (3) and from Lemma 7.1.

Part 4. From Proposition 5.7, part (1), esp. (5.69) and from Proposition 6.4, part (1), esp. (6.30), supplemented by similar estimates on $\varphi - \varphi_0$ and on $\varphi - \varphi_{02}$ easily obtained from (7.8) (7.11), it follows that

$$|w(t) - Fu_+|_k \leq A_0 t^{-\gamma} \quad (7.48)$$

$$\|\varphi(t) - \varphi_0(t); Y^\ell\| \leq A_0 (2\gamma - 1)^{-1} t^{1-2\gamma} \quad (7.49)$$

$$\|\varphi(t) - \varphi_{02}(t); Y^\ell\| \leq A_0 (2\gamma - 1)^{-1} t^{1/2-\gamma} \quad (7.50)$$

for some constant $A_0$ of the type stated for $A$. From the definition (7.15) of $J$, from the commutation relation (7.16), from (7.43) and from Lemma 3.5, we obtain

$$\|<J(t) >_k (\exp[i\varphi_{02}(t, x/t)]u(t) - M D F u_+)\|_2 = \|\exp[i(\varphi_{02} - \varphi)]U^*(1/t)w - Fu_+|_k$$

$$\leq \left\{\|\varphi_{02} - \varphi\|_{\infty} + |\nabla(\varphi_{02} - \varphi)|_{\ell-1} (1 + |\nabla(\varphi_{02} - \varphi)|_{\ell-1})^{k-1}\right\} |w|_k$$

$$+ |w - Fu_+|_k + |(U^*(1/t) - 1)Fu_+|_k \quad (7.51)$$

which yields immediately (7.46) by the use of (7.48) (7.50) and (6.9). Similarly

$$\|<J(t) >_k (\exp[i\varphi_0(t, x/t)]u(t) - U(t) u_+)\|_2 = \|\exp[i(\varphi_0 - \varphi)]U^*(1/t)w - U^*(1/t) Fu_+|_k$$

$$\leq \left\{\|\varphi_0 - \varphi\|_{\infty} + |\nabla(\varphi_0 - \varphi)|_{\ell-1} (1 + |\nabla(\varphi_0 - \varphi)|_{\ell-1})^{k-1}\right\} |w|_k + |w - Fu_+|_k \quad (7.52)$$

from which (7.47) follows by the use of (7.48) (7.49).

□

Remark 7.1. The uniqueness statement in Proposition 7.5 is rather restrictive because it requires the representation of $u$ by (7.43). It would be more satisfactory to have uniqueness under assumptions bearing directly on $u$, for instance under (7.46). However (7.46) seems insufficient to derive the asymptotic conditions on $w$ and $\varphi$ separately which are required in Proposition 5.2.

Remark 7.2. The estimate (7.46) states that $u$ behaves asymptotically as expected, namely

$$u(t) \sim \exp[-i\varphi_{02}(t, x/t) + ix^2(2t)^{-1}] (it)^{-n/2} (Fu_+(x/t)) \quad . \quad (7.53)$$
In order to have the strongest possible statement however, one has to shift the phase factor to \( u \) before taking derivatives as contained in \( < J(t) >^k \). On the other hand, this is not necessary if one wants only asymptotic estimates in \( L^r \). In fact one has the following corollary.

**Corollary 7.1.** Let \( u \) be the solution of (1.1) obtained in Proposition 7.5. Let \( r \) satisfy \( 0 \leq \delta(r) \leq k \wedge n/2, \delta(r) < n/2 \) if \( k = n/2 \). Then \( u \) satisfies the following estimates

\[
\| u(t) - \exp[-i\varphi_0(t, x/t)]M D F u_+ \|_r \leq A(2\gamma - 1) t^{-\delta(r)/2} - \frac{1}{2} t^{1 - \gamma},
\]

(7.54)

\[
\| u(t) - \exp[-i\varphi_0(t, x/t)]U(t) u_+ \|_r \leq A(2\gamma - 1) t^{-\delta(r)+1 - 2\gamma}.
\]

(7.55)

**Proof.** An immediate consequence of (7.46) (7.47) and of the inequality

\[
\| f \|_r = t^{-\delta(r)} \| D^* M^* f \|_r \leq C t^{-\delta(r)} \| < \nabla >^k D^* M^* f \|_2
\]

\[
= C t^{-\delta(r)} \| < J(t) >^k f \|_2
\]

which follows from Lemma 3.1 and from the commutation relation (7.16).

\( \square \)

**Remark 7.3.** As already mentioned before, the phase \( \varphi_0 \) used in Proposition 5.7 produces a better asymptotic approximation of \( u \) than the simpler phase \( \varphi_{02} \) used in Proposition 6.4. This shows up in the estimate behaving as \( t^{1-2\gamma} \) in the RHS of (7.47) as compared with \( t^{1/2-\gamma} \) in the RHS of (7.46). In the same spirit, we have compared \( u \) with the solution \( U(t)u_+ \) of the free Schrödinger equation in (7.47), and with the standard asymptotic form \( MDF u_+ \) thereof in (7.46), obtained in dropping the second \( M \) in \( U = MDFM \). The difference between the two is

\[
|F(M(t) - 1)u_+|_k \leq t^{-1/2} |Fu_+|_{k+1}
\]

and shows up if one uses the latter, in the form of the last norm in (7.51). Since \( t^{-1/2} < t^{1/2-\gamma} \) for \( \gamma < 1 \), that term is smaller than the other terms in the RHS of (7.51), thereby preserving the estimate (7.46). This justifies the use of the simple explicit form \( MDF u_+ \) in that case. On the other hand \( t^{-1/2} > t^{1-2\gamma} \) for \( \gamma > 3/4 \), and we have therefore preferred to keep the more precise \( U(t)u_+ \) in (7.47) in order to preserve the \( t^{1-2\gamma} \) decay in the RHS for all \( \gamma < 1 \).

So far we have constructed local solutions of the equation (1.1) in a neighborhood of infinity associated with given asymptotic states \( u_+ \) and defined the local wave operator at infinity.
Ω. In order to complete the construction of the standard wave operators, it remains only to extend the previous solutions from a neighborhood of infinity by using the results on the global Cauchy problem at finite times. This can be done with the help of the following result which is essentially contained in \[25\].

**Proposition 7.6.** Let $k$ be a positive integer and let $0 < \mu < 2$. Then the Cauchy problem for the equation (1.1) with initial data $u(t_0) = u_0$ such that $<J(t_0)>^k u_0 \in L^2$ at some initial time $t_0 \geq 1$ is globally well posed in $X^k([1, \infty))$, namely the local solutions of Proposition 7.1 can be extended to $[1, \infty)$.

**Proof.** A minor variation of Proposition 2.1 part (1) in \[25\].

\[\square\]

We can now define the standard wave operator $\Omega_1$ for the equation (1.1).

**Definition 7.5.** Under the assumptions of Proposition 7.5 supplemented with $\mu < 2$, we define the wave operator $\Omega_1$ as the map $\Omega_1 : u_+ \to u(1)$ where $u$ is the solution of the equation (1.1) obtained by continuing $\Omega(u_+)$ down to $t = 1$ with the help of Proposition 7.6.

From Propositions 7.5 and 7.6 it follows that $\Omega_1$ is an injective map from $FH^{k+1}$ to the space

$$K^k = \{ u : \exp(-ix^2/2)u \in H^k \}$$

and that $\Omega_1$ satisfies continuity properties easily obtained from Proposition 7.5, part (3). Since all the interesting information is already contained in that proposition, we refrain from a more formal statement.
Appendix A

Sketch of the proof of Lemma 3.2

We discuss only the case where $p < \infty$, the case $p = \infty$ being obtainable from the previous one by a limiting procedure. The proof proceeds by first writing a regularized equation for a suitable approximant of the function $|u|^p$, then by proving an analogue of the estimate (3.13) for that function and finally by removing the regularisation. The method used is known and applied in [5] to the equation (3.12) with $\eta = 0$. For this reason, even though the assumptions on $u$ and $v$ used there are slightly different from those of Proposition 3.2, we refer to [5] for estimating the second and third term in the RHS of (3.12). Here below we continue with the analysis of the first ($\eta$ dependent) term.

Let $\varphi \in C_0^\infty(\mathbb{R}^n)$ denote a regularizing sequence of functions of the space variable, namely a sequence converging to the $\delta$ function. Let $\phi$ denote the operator of convolution with $\varphi$, i.e. $\phi f = \varphi * f$, let $\varepsilon > 0$, let $u_1 \equiv \phi * u$, and let $w \equiv (|u_1|^2 + \varepsilon^2)^{1/2}$. The function $w$ is uniformly bounded away from zero, is $C^\infty$ in the space variable and satisfies

$$\lim_{\varphi \to \delta} \lim_{\varepsilon \to 0} \int w(t, x)^p dx = \| u(t)^p \|_p .$$

We compute $\partial_t w^p$ by using (3.12), thereby obtaining

$$\partial_t w^p = 2w^{p-2} \text{Re}(\bar{u}_1 \partial_t u_1)$$

$$= p \{ \eta J_1 + J_2 + J_3 \}$$

where

$$J_1 = w^{p-2} \text{Re} \bar{u}_1 \Delta u_1$$

$$J_2 = w^{p-2} \text{Re} \bar{u}_1 \phi \nabla(uv)$$

$$J_3 = w^{p-2} \text{Re} \bar{u}_1 \phi h .$$

The terms $J_2$ and $J_3$ are treated as in [5]. For the completion of the proof it is sufficient to show that the space integral of $J_1$ is negative, so that $J_1$ does not contribute to the estimates (3.13) (3.14). Repeated application of the Leibnitz rule yields

$$J_1 = -w^{p-2}|\nabla u_1|^2 + 1/2 w^{p-2} (\nabla \cdot \nabla |u_1|^2)$$

$$= -w^{p-2}|\nabla u_1|^2 - (1/4)(p-2)w^{p-4} (\nabla |u_1|^2)^2 + (1/2)\nabla \cdot \{ w^{p-2}\nabla |u_1|^2 \}$$

so that

$$J_1 \leq (1/2)\nabla \cdot \{ w^{p-2}\nabla |u_1|^2 \} .$$
which essentially implies that

\[ \int J_1 \, dx \leq 0. \]

Actually, in order to ensure integrability of \( J_1 \), the limit \( \varepsilon \to 0 \) should be taken before performing the integration over the space variables up to infinity.
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