Abstract

The phonon spectrum of Coulomb lattice in neutron star crusts above the neutron drip density is affected by the interaction with the ambient neutron Fermi-liquid. For the values of the neutron-phonon coupling constant in the range $0.1 \leq \lambda \leq 1$ an appreciable renormalization of the phonon spectrum occurs which can lead to a lattice instability manifested in an exponential growth of the density fluctuations. The BCS phonon exchange mechanism of superconductivity leads to neutron pairing with a gap in the neutron excitation spectrum comparable to that due to the direct nuclear interaction.

1 Introduction

Below the melting temperature $T_m \simeq Z^2 e^2/100 r_i \sim 10^9 - 10^{10}$ K, where $r_i$ is the average interion spacing, $Z$ is the ion charge, the ionic component of plasma in neutron star crusts is arranged in a Coulomb lattice. The screening of the electrostatic potential of the lattice by electrons is ineffective since the ratio $\lambda_D/r_i < 1$, where $\lambda_D$ is the Debye screening radius, and, therefore, electrons are distributed almost uniformly. Above the neutron drip density $4 \times 10^{13}$ g cm$^{-3}$ the intervening space between the clusters is filled by a neutron fluid which goes over to a superfluid state below the critical temperature of the order $T_c \sim 10^9$ K; (for a review of the early work see, e.g., ref. [1]; recent progress is summarized in ref. [2]). Although initial studies of the collective effects in the crusts were focused either on the properties of the highly compressed solid matter or the unbound neutron fluid at subnuclear densities (with an exception of the equation of state where phase equilibrium conditions are imposed) this separation is justified only when the interaction between these components of the crusts is weak; this is not, however, always the case. The purpose of this work is to continue the discussion of the interaction between the neutrons in the continuum and the excitations of the crustal lattice, i.e. phonons, set up in an earlier work [3].
Figure 1: The phonon Dyson equation and the vertex equation for coupled system of fermions (solid lines) and phonon (wavy lines). The fermion Dyson equation is given by the block in Fig. 3. The thick lines correspond to the full propagators, while the thin lines to the free ones.

2 The Phonon Spectrum

The self-consistent coupled fermion-phonon problem is shown in terms of Feynman diagrams in Fig. 1 and 3. In this section we shall give the finite temperature solution of the Dyson equation for phonons which are coupled to fermions of arbitrary relativism; in the present case the neutrons are only mildly relativistic (if at all) while the electrons are ultrarelativistic. The problem can be solved for an arbitrary coupling strength since the perturbation series converge rapidly with respect to another parameter - ratio of the phonon to the fermion energies: as a result one may consider only to the lowest order term in the integral equation for the vertex (Migdal theorem). The sequence in which the Dyson equations for the fermions and phonons should be solved is fixed by the fact that the phonons affect only a narrow range of energies in the fermion propagator (of the order the Debye frequency $\omega_D$), which implies that the polarization function can be evaluated with free fermion propagators:

$$\gamma^0 G^0(\vec{p}, i\omega_\nu) = \frac{\Lambda_+(\vec{p})}{i\omega_\nu - (\epsilon_p - \mu)}; \quad \Lambda_+(\vec{p}) = \frac{1}{2} \left\{ 1 - \frac{\vec{\alpha} \cdot \vec{p}}{\epsilon_p} + \frac{\gamma_0 m}{\epsilon_p} \right\},$$

where $\epsilon(p) = \sqrt{\vec{p}^2 + m^2}$, $\vec{\alpha} = \gamma_0 \vec{\gamma}$ and we have kept only the positive energy states from the outset; (to excite an excitation at the top of the Fermi sea and an anti-particle at rest one needs to overcome an energy barrier of the order of the Fermi energy $\epsilon_F$, however $\omega \leq \omega_D \ll \epsilon_F$). The Matsubara frequencies assume discrete odd integer values $\omega_\nu = (2\nu + 1)/\beta$ where $\nu = 0, \pm 1, \pm 2, \ldots$ and $\beta$ is the inverse temperature; the other notations have their usual meaning.

We find the retarded polarization function depicted in Fig. 1 by performing the frequency summation over the product of the fermion propagators and an analytical continuation to the real axis:

$$\Pi^{(R)}(\vec{k}, \omega) = \mathcal{M}_k^2 \int \frac{d^3p}{(2\pi)^3} \frac{f(\epsilon_{p+k} - \mu) - f(\epsilon_p - \mu)}{\omega - \epsilon_{p+k} + \epsilon_p + i\delta} \left\{ 1 - \frac{\vec{p} \cdot (\vec{p} + \vec{k})}{\epsilon_p \epsilon_{p+k}} + \frac{m^2}{\epsilon_p \epsilon_{p+k}} \right\},$$
where $\mathcal{M}_k$ is the fermion-phonon coupling matrix element and $f(x) = [1 + \exp(\beta x)]^{-1}$ is the Fermi distribution function. The scalar polarization function above corresponds to particle–hole excitation for arbitrary degree of relativism of the system and finite-temperatures; for applications to the neutron subsystem one may take the non-relativistic limit. Fig. 2 shows the real part (more precisely $1 + \text{Re}\Pi(k, \omega)$) and the imaginary part of the polarization function as a function of the momentum transfer for $\beta^{-1} = 0$ and a typical ratio $c_s/v_F = 0.1$, where $c_s$ and $v_F$ are the sound and Fermi velocities, respectively. The first quantity gives the renormalization of the phonon frequencies $[\omega(k)/\omega_0]^2$ (where $\omega_0$ is the unperturbed phonon frequency) which are treated in the Debye model. The eigen modes of the system vanish first in the long-wave limit where, with increasing coupling constant, the curves cross zero. The disappearance of the real solutions to the dispersion relation indicates instability of the system with respect to the density fluctuations, as can be seen by inspecting the proper solutions which exhibit exponentially growing amplitude of phonon mode.
If this is the case, one may conclude that the starting lattice structure does not correspond to the true minimum of the energy of the system (in other words is not the ‘true vacuum’). Available estimates of the neutron-phonon coupling constants are, however, unreliable for two reasons: first, preliminary considerations [3], based on a fit to the elastic neutron-nucleus scattering cross-sections, does not include the medium effects (which most likely will reduce the effective cross-section) and, in addition, the substantial imaginary part of the optical potential responsible for the absorptive processes should be included in the estimates. Second, the compositions of the crustal matter due to different authors (see refs. in [2, 3]) imply coupling constants from marginal up to of the order unity at \( n_0/3 \), where \( n_0 \) is the nuclear saturation density, and consequently, very different assessments about the role of the neutron-phonon interaction.

3 Phonon Exchange Interaction and Neutron Pairing

The net two-body interaction between neutrons in the continuum comprises the nuclear component (which is dominated by the attractive \(^1S_0\) channel at densities of interest) and the component due to the phonon exchange. Since the latter interaction is attractive as well, the BCS pairing for some values of neutron-phonon coupling constant might be influenced by the phonon-exchange mechanism. The respective gap equations, which include the retardation of the effective interaction, emerge as a solution to the set of diagrams in Fig. 3. We find (for simplicity the \( \beta^{-1} = 0 \) limit is given below)

\[
\Delta(\omega) z(\omega) = \int_0^\infty d\xi \ [K_+(\omega, \xi) + \mathcal{W}] \text{Re} \frac{\Delta(\xi)}{\sqrt{\xi^2 - \Delta(\xi)^2}}
\]

\[
\omega [1 - z(\omega)] = \int_0^\infty d\xi \ K_-(\omega, \xi) \text{Re} \frac{\xi}{\sqrt{\xi^2 - \Delta(\xi)^2}}
\] (2)
where $z(\omega)$ is the wave function renormalization and the effective interactions via phonon exchange and direct nuclear force are defined respectively

$$
K_{\pm}(\omega, \xi) = \frac{1}{(2\pi)^2 v_F} \int_0^{2p_F} \mathcal{M}_k^2 \left[ \frac{1}{\xi + \omega + \omega(k) + i\delta} \pm \frac{1}{\xi - \omega + \omega(k) - i\delta} \right] k \, dk
$$

$$
\mathcal{W} = \frac{1}{(2\pi)^2 v_F} \int_0^{2p_F} |V(k)|k \, dk.
$$

Here the nuclear interaction is given by the time-local interaction $V(k)$; note that a pairing description based on the pion exchange model would allow for the effects of retardation, pion mode softening, condensation etc. The formal structure of equations (2) coincides with the Eliashberg equations for metallic superconductor in the presence of the repulsive Coulomb force between the electrons. For the values $\omega_D = 0.7 \text{ MeV}$ and $\lambda = 0.45$ (which correspond to the density $0.3n_0$, for the composition of Arponen, see [3]) and $\mathcal{W} = 0$, eqs. (2) predict a value $\Delta = 0.03 \text{ MeV}$, which is basically less, but comparable to the gap found neglecting the neutron-phonon coupling.

To conclude, if in a certain region of neutron star crusts the neutron-phonon coupling is not negligible (i.e. $\lambda \leq 1$), a number of novel effects emerge, which are potentially important for the correct description of phenomena in neutron star crusts. The crucial problem is whether the calculations of the coupling constants beyond the simple estimates would lead to significant values of the neutron-phonon coupling constant. In addition one may need a better understanding of the sources of the discrepancies between different models of neutron star crust composition and a reassessment based on the progress achieved in the many-body theory of nuclear matter in recent years.

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**References**

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