Two Examples of Convex-Programming-Based High-Dimensional Econometric Estimators

Zhan Gao and Zhentao Shi
The Chinese University of Hong Kong

June 28, 2018

Abstract

Economists specify high-dimensional models to address heterogeneity in empirical studies with complex big data. Estimation of these models calls for optimization techniques to handle a large number of parameters. Convex problems can be effectively executed in modern statistical programming languages. We complement Koenker and Mizera (2014)’s work on numerical implementation of convex optimization, with focus on high-dimensional econometric estimators. In particular, we replicate the simulation exercises in Su, Shi, and Phillips (2016) and Shi (2016) to show the robust performance of convex optimization cross platforms. Combining R and the convex solver MOSEK achieves faster speed and equivalent accuracy as in the original papers. The convenience and reliability of convex optimization in R make it easy to turn new ideas into prototypes.

Key words: big data, convex optimization, high-dimensional model, numerical solver
JEL code: C13, C55, C61, C87

Zhan Gao: gaozhan.cuhk@gmail.com. Zhentao Shi (corresponding author): zhentao.shi@cuhk.edu.hk. Address: Department of Economics, 912 Esther Lee Building, the Chinese University of Hong Kong, Sha Tin, New Territories, Hong Kong SAR, China. Tel: (852) 3943-1432. Fax (852) 2603-5805. We thank Roger Koenker for inspiration and hospitality during the second author’s visit to University of Illinois.
1 Introduction

Equipped with tremendous growth of computing power over the last few decades, econometricians endeavor to tackle high-dimensional real world problems that we could hardly have imagined before. Along with the development of modern asymptotic theory, computation has gradually ascended onto the central stage. Today, discussion of numerical algorithms is essential for new econometric procedures.

Optimization is at the heart of estimation, and convex optimization is the best understood category. Convex problems are ubiquitous in econometric textbooks. The least square problem is convex, and the classical normal regression is also convex after straightforward reparametrization. Given a linear single-index form, the Logit or Probit binary regression, the Poisson regression and the regressions with a censored or truncated normal distributions are all convex. Another prominent example is the quantile regression (Koenker and Bassett, 1978), motivated from its robustness to non-Gaussian errors and outlier contamination.

With the advent of big data, practitioners attempt to build general models that involve hundreds or even more parameters in the hope to capture complex heterogeneity in empirical economic studies. Convex optimization techniques lay out the foundation of estimating these high-dimensional models. Recent years witnesses Bajari, Nekipelov, Ryan, and Yang (2015), Gu and Koenker (2017) and Doudchenko and Imbens (2016), to name a few, exploring new territories by taking advantage of convexity.

To facilitate practical implementation, Koenker and Mizera (2014) summarize the operation in R by MOSEK via Rmosek to solve linear programming, conic quadratic programming, quadratic programming, etc. R is open-source software, MOSEK is a proprietary convex optimization solver but offers free academic license, and Rmosek is the R interface that communicates with MOSEK. MOSEK specializes in convex problems with reliable performance, and is competitive in high-dimensional problems.

This note complements Koenker and Mizera (2014)'s work. We replicate by Rmosek two examples of high-dimensional estimators, namely Su, Shi, and Phillips (2016)'s classifier-Lasso (C-Lasso) and Shi (2016)'s relaxed empirical likelihood (REL). These exercises highlight two points. Firstly, the R environment is robust in numerical accuracy for high-dimensional convex optimization and has computational speed gain via Rmosek. Secondly, we showcase the ease of creating new econometric estimators—often no more than a few lines of code—by the code snippets in the Appendix. Such convenience lowers the cost of turning an idea into a prototype, and enables researchers to glean valuable insights about their archetypes by experimenting new possibilities. All code in this note is hosted at https://github.com/zhentaoshi/convex_prog_in_econometrics.

2 Classifier-Lasso

It is common practice to assume in linear fixed-effect panel data models that the cross-sectional units are heterogeneous in terms of the time-invariant individual intercept, while they all share the same slope coefficient. This pooling assumption can be tested and is often rejected in real-world applications. In recent years panel data group structure has been attracting attention. Bonhomme and Manresa (2015) allow group structure in the intercept and use the $k$-means algorithm for classification. When the slope coefficients exhibit group structure, Su, Shi, and Phillips (2016) propose Classifier-Lasso (C-Lasso) to identify the latent group pattern.
We illustrate the penalized least square (PLS), a simple special case of C-Lasso. Given a tuning parameter $\lambda$ and the number of groups $K$, PLS is defined as the solution to

$$\min_{\beta, (\alpha_k)_{k=1}^K} \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T (y_{it} - x'_{it}\beta_i)^2 + \frac{\lambda}{n} \sum_{i=1}^n \prod_{k=1}^K \|\beta_i - \alpha_k\|_2$$

where $\beta = (\beta_i)_{i=1}^n$. The additive-multiplicative penalty pushes the individual slope coefficients $\beta_i$ in the same group toward a common coefficient $\alpha_k$. This is not a convex problem, but the optimization with the additive-multiplicative penalty can be approximated by an iterative algorithm, as is explained in the Supplement of Su, Shi, and Phillips (2016, Section S3.1). Procedures based on such an iteration have been successfully applied to Su and Ju (2017), Su and Lu (2017) and Su, Wang, and Jin (2017). The iterative algorithm initiates at the within-group estimator, which is consistent when $T$ is large. In the $k$-th sub-step of the $r$-th iteration, $(\beta, \alpha_k)$ is chosen to minimize

$$\min_{\beta, \alpha_k} \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T (y_{it} - x'_{it}\beta_i)^2 + \frac{\lambda}{n} \sum_{i=1}^n \|\beta_i - \alpha_k\|_2 \gamma_i$$

(1)

where $\gamma_i = \prod_{k=1}^{K-1} \|\beta^{(r,k)}_i - \alpha_k^{(r)}\|_2 \prod_{k=k+1}^K \|\beta^{(r-1,k)}_i - \alpha_k^{(r-1)}\|_2$. The iteration proceeds until the $K$-convex problem numerically converges.

Given the multiplier $\gamma_i$, the above optimization problem is convex in $(\beta, \alpha_k)$ and the structure is very close to Lasso. Though the R packages lars and glmnet packages can carry out the standard Lasso, however, it is not straightforward how to modify these functions to accommodate (1), where $\alpha_k$ is also an unknown parameter to be optimized. A quick review of Koenker and Mizera (2014) approach to Lasso will be helpful.

Example 1 (Lasso). The standard Lasso problem is

$$\min_{\beta} \frac{1}{n} \|y - X\beta\|_2^2 + \lambda \|\beta\|_1$$

where $y \in \mathbb{R}^n$ and $X \in \mathbb{R}^{n \times p}$ are observed data, $\lambda$ is the tuning parameter and $\beta \in \mathbb{R}^p$ is the parameter of interests. However, Rmosek does not accept the $l_1$ norm. To overcome the difficulty, Koenker and Mizera (2014) introduce new parameters to transform the $l_1$-penalized problem into a conic optimization problem that Rmosek recognizes. We first deal with $\|\beta\|_1$. The $p \times 1$ vector $\beta$ can be decomposed into a positive part $\beta^+ = (\max \{0, \beta_j\})_{j=1}^p$ and a negative part $\beta^- = (\max \{0, -\beta_j\})_{j=1}^p$, so that $\beta = \beta^+ - \beta^-$ and $\|\beta\|_1 = e'\beta^+ + e'\beta^-$, where $e$ is the $p \times 1$ vector with all elements equal to 1. Next, we transform the $l_2$-norm $\|y - X\beta\|_2^2$ to a second-order conic constraint. Consider a minimization problem with $\|v\|_2^2$ in the objective function. We can use a new parameter $t$ to replace it and add a conic constraint $\|v\|_2^2 \leq t$, which is equivalent to $\|(v, \frac{t}{2})\|_2 \leq \frac{t+1}{2}$. Thus we obtain a standard conic constraint $\|(v, s)\|_2 \leq r$.

---

1 The profile log-likelihood function $Q_{1, n T} (\beta) = \sum_{i=1}^n \sum_{t=1}^T \psi (w_{it}, \beta_i, \mu_0 (\beta_i))$ for nonlinear models can be reformulated into a separable form, while penalized GMM (PGM1) can be handled under the same optimization framework as PLS. They are discussed in Appendix C.1 and Appendix C.2, respectively.
where \( s = \frac{t-1}{2} \) and \( r = \frac{t+1}{2} \). We rewrite the Lasso problem as

\[
\min_{\theta} \lambda \left( e' \beta^+ + e' \beta^- \right) + \frac{t}{n}
\]

\[
s.t. \quad v = y - X (\beta^+ - \beta^-), \quad \|(v, s)\|_2 \leq r, \quad s = \frac{t - 1}{2}, \quad r = \frac{t + 1}{2}
\]

where \( \theta = (\beta^+, \beta^-, v, t, s, r) \). This problem is of the standard form of second-order conic programming and hence can be executed in \texttt{Rmosek}.

 Applying the techniques in the Lasso formulation, we can transform the \( l_2 \)-norm terms in (1) and formulate the problem into a conic programming:

\[
\min_{\alpha_k, \theta} \sum_{i=1}^{n} \left( \left( \frac{1}{n \gamma_i} \right) t_i + \left( \frac{\lambda}{n} \gamma_i \right) w_i \right)
\]

\[
s.t. \quad x_i \beta_i + v_i = y_i, \quad \beta_i - \mu_i = 0, \quad s_i - \frac{1}{2} t_i = -\frac{1}{2}, \quad r_i - \frac{1}{2} t_i = \frac{1}{2},
\]

\[
\| (v_i, s_i) \|_2 \leq r_i, \quad \| \mu_i \|_2 \leq w_i, \quad t_i \geq 0, \quad \text{for all} \ i = 1, 2, \ldots, n
\]

where \( \theta = \{ \beta_i, v_i, \mu_i, s_i, r_i, t_i, w_i \}_{i=1}^{n} \). The convexity is manifest when we write the problem in matrix form, as is displayed in Appendix B.

### 2.1 Replication

We replicate the simulation studies in Su, Shi, and Phillips (2016, Section 4) in R via \texttt{Rmosek} and compare the performance of different numerical optimization approaches. Su, Shi, and Phillips (2016) conduct their numerical work in \texttt{MATLAB} via \texttt{CVX} (Grant and Boyd, 2014). \texttt{CVX} is a \texttt{MATLAB} add-on package for \textit{disciplined convex optimization} (Grant, Boyd, and Ye, 2006, DCP). It provides an interface to communicate with commercial or open-source solvers. In the R environment, the de facto solver is \texttt{optimx} (Nash and Varadhan, 2011); another option is the interface \texttt{nloptr} (Ypma, 2017) that hooks optimization solver \texttt{NLopt} (Johnson, 2017). They are general-purpose optimization solvers not tailored for convexity. Most recently, Fu, Balasubramanian, and Boyd (2017) are actively developing \texttt{CVXR}, \texttt{CVX}'s counterpart in R. At this stage, it is integrated with the open-source solver \texttt{ECOS} (Domahidi, Chu, and Boyd, 2013).\(^2\)

We follow the DGP 1 in Su, Shi, and Phillips (2016, Section 4). Table 1 reports, under various combinations of the cross-sectional units \( n \) and the time length \( T \), the root-mean-square error (RMSE) of \( \hat{\alpha}_1 \) and the probability of correct group classification (correct ratio). The DGP, simulation settings and the indicators are detailed in Appendix A.1. The numerical results by \texttt{Rmosek} are very close to those in Su, Shi, and Phillips (2016, p.2240) by \texttt{CVX}. It demonstrates the robustness of the numerical performance of C-Lasso across different computing platforms.

\(^2\)In the latest version (Version 0.99), \texttt{CVXR} supports \texttt{MOSEK} by sending the problem to \texttt{MOSEK} in the \texttt{Python} environment. In our experiment, large-scale problems like the C-Lasso cause errors in the communication between \texttt{R} and \texttt{Python}. In addition, \texttt{CVXR} with \texttt{MOSEK} currently cannot incorporate problems with nonlinear objective functions and hence cannot be used for REL in Section 3.
Table 1: Classification and Point Estimation of $\alpha_1$: Replication of Su et al. (2016)’s DGP 1

| $(n, T)$  | (100, 15) | (100, 25) | (100, 50) | (200, 15) | (200, 25) | (200, 50) |
|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| RMSE      |           |           |           |           |           |           |
| Rmosek    | 0.0624    | 0.0377    | 0.0253    | 0.0430    | 0.0278    | 0.0178    |
| nloptr    | 0.6451    | 0.6116    | 0.4586    | 0.6956    | 0.5055    | 0.7210    |
| CVXR      | 0.0624    | 0.0377    | 0.0253    | 0.0430    | 0.0278    | 0.0178    |
| CVX       | 0.0679    | 0.0364    | 0.0234    | 0.0466    | 0.0298    | 0.0181    |
| Results in Su et al. (2016) | 0.0594 | 0.0384 | 0.0249 | 0.0432 | 0.0272 | 0.0174 |
| Correct Ratio | | | | | | |
| Rmosek    | 0.8971    | 0.9665    | 0.9971    | 0.9043    | 0.9661    | 0.9966    |
| nloptr    | 0.5860    | 0.6363    | 0.7200    | 0.5820    | 0.7255    | 0.5790    |
| CVXR      | 0.8973    | 0.9665    | 0.9971    | 0.9043    | 0.9661    | 0.9966    |
| CVX       | 0.9033    | 0.9660    | 0.9983    | 0.8948    | 0.9617    | 0.9965    |
| Results in Su et al. (2016) | 0.8935 | 0.9674 | 0.9964 | 0.8987 | 0.9661 | 0.9966 |
| Running Time (in minute) | | | | | | |
| Rmosek    | 29.82     | 16.87     | 14.36     | 40.21     | 26.40     | 21.24     |
| nloptr    | 1010.51   | 2069.50   | 11072.48  | 5676.92   | 11486.22  | 19710.59  |
| CVXR      | 16541.03  | 9143.16   | 7463.23   | 32892.43  | 21057.10  | 16396.31  |
| CVX       | 104.40    | 62.37     | 51.65     | 108.20    | 64.99     | 51.28     |

We also implement the simulation by CVXR and nloptr. As is clear in Table 1, Rmosek and CVXR yield identical results up to rounding errors, while nloptr fails to attain an accurate solution in most cases. Moreover, if we implement it in MATLAB via CVX, the results are largely similar but small difference is observed due to the computing environment.

Practitioners may need to try out different specifications for robustness check in real applications. Without fast optimization solvers, computational cost can become a bottleneck. On the same computing platform of 16-core Intel(R) Xeon(R) CPU E5-2640 v3 @ 2.60GHz, each case is executed in a single thread and we record the running time in the lower panel in Table 1. Rmosek significantly outperforms all alternatives. nloptr and CVXR are hundreds or even thousands times solver than Rmosek. The huge gap in computation cost illustrates the advantages of MOSEK over open-source solvers like Nlopt or ECOS. CVX in MATLAB is about 2 to 4 times slower than Rmosek. Although CVX is also powered by MOSEK, the DCP system takes time to check the convexity of the input problem and automate the formulation. DCP is useful when we are uncertain about the convexity and solvability of a problem. However, for problems that are mathematically verified to be convex, directly calling MOSEK saves much computational time.

3 Relaxed Empirical Likelihood

Besides the regression setting in Section 2, convex programming is also useful in structural econometric estimation. Consider the models with a “true” parameter $\beta_0$ satisfying the unconditional moment condition $E [g(Z_i, \beta_0)] = \mathbf{0}_m$, where $\{Z_i\}_{i=1}^n$ is the observed data, $\beta \in \mathcal{B} \subset \mathbb{R}^D$
is a finite dimensional vector in the parameter space $\mathcal{B}$, and $g$ is an $\mathbb{R}^m$-valued moment function. GMM (Hansen, 1982) and empirical likelihood (EL) (Owen, 1988; Qin and Lawless, 1994) are two workhorses dealing with moment restriction models. In particular, EL solves

$$
\max_{\beta \in \mathcal{B}, \pi \in \Delta_n} \sum_{i=1}^{n} \log \pi_i \quad \text{s.t.} \quad \sum_{i=1}^{n} \pi_i g(Z_i, \beta) = 0_m
$$

where $\Delta_n = \{ \pi \in [0, 1]^n : \sum_{i=1}^{n} \pi_i = 1 \}$ is the $n$-dimensional probability simplex. However, neither GMM nor EL can be used to estimate a model with more moment equalities than observations, i.e. $m > n$. To make the optimization feasible, Shi (2016) relaxes the equality restriction $\sum_{i=1}^{n} \pi_i g_i(\beta) = 0_m$ in EL. REL is defined as the solution to

$$
\max_{\beta \in \mathcal{B}} \max_{\pi \in \Delta_n^\lambda(\beta)} \sum_{i=1}^{n} \log \pi_i
$$

where

$$
\Delta_n^\lambda(\beta) = \left\{ \pi \in \Delta_n : \left| \sum_{i=1}^{n} \pi_i h_{ij}(\beta) \right| \leq \lambda, j = 1, 2, \ldots, m \right\}
$$

is a relaxed simplex, $\lambda \geq 0$ is a tuning parameter, $h_{ij}(\beta) = g_j(Z_i, \beta) / \hat{\sigma}_j(\beta)$, $g_j(Z_i, \beta)$ is the $j$-th component of $g(Z_i, \beta)$, and $\hat{\sigma}_j(\beta)$ is the sample standard deviation of $\{g_j(Z_i, \beta)\}_{i=1}^{n}$. The formulation of REL is inspired by Dantzig selector (Candes and Tao, 2007).

**Example 2** (Dantzig selector). Similar to Lasso, Dantzig selector also produces a sparse solution to the linear regression model. Dantzig selector can be written as

$$
\min_{\beta} \| \beta \|_1 \quad \text{s.t.} \quad \| X' (y - X \beta) \|_{\infty} \leq \lambda,
$$

where $\lambda$ is a tuning parameter. We can immediately reformulate it as a linear programming problem

$$
\min_{\beta^+, \beta^-} \epsilon' \beta^+ + \epsilon' \beta^- \quad \text{s.t.} \quad X'y - \lambda \epsilon \leq (X'X) (\beta^+ - \beta^-) \leq X'y + \lambda \epsilon \quad \beta^+, \beta^- \geq 0.
$$

It is readily solvable using the R package `quantreg` (Koenker, 2017).

Dantzig selector slacks the sup-norm of the first-order condition for optimality. REL borrows the idea to estimate a finite-dimensional parameter in a structural economic model defined by many moment equalities. Comparing to Dantzig selector, REL uses a nonlinear objective function. It is still convex (in minus likelihood) but `quantreg` that deals with linear programming problems is no longer applicable.

Similar to standard EL, REL’s optimization involves an inner loop and an outer loop. The outer loop for $\beta$ is a general low-dimensional nonlinear optimization, which can be solved by Newton-type methods. With the linear constraints and the logarithm objective, the inner
loop is convex in $\pi = (\pi_i)_{i=1}^n$. For each $\beta$, the inner problem can be formulated as a separable convex optimization problem in the matrix form

$$\max_{\pi} \sum_{i=1}^n \log \pi_i$$

s.t.
$$\begin{bmatrix}
1 \\
\vdots \\
-\lambda
\end{bmatrix} \leq \begin{bmatrix}
h_{11}(\beta) & h_{21}(\beta) & \cdots & h_{n1}(\beta) \\
\vdots & \vdots & \ddots & \vdots \\
h_{1m}(\beta) & h_{2m}(\beta) & \cdots & h_{nm}(\beta)
\end{bmatrix} \begin{bmatrix}
\pi_1 \\
\pi_2 \\
\vdots \\
\pi_n
\end{bmatrix} \leq \begin{bmatrix}
1 \\
\vdots \\
\lambda
\end{bmatrix}$$

$$0 \leq \pi_i \leq 1, \text{ for each } i = 1, 2, \ldots, n$$

and it is readily solvable in Rmosek by translating the mathematical expression into computer code.

3.1 Replication

We follow the simulation design in Shi (2016, Section 4), which is described in Appendix A.2. Table 2 reports the bias and RMSE of the estimation of $\hat{\beta}_1$, implemented purely in R with the inner loop by Rmosek and the outer loop by nloptr. The results are close to those in Shi (2016), where the code is written in MATLAB with the outer loop handled by the function fmincon and the inner loop by CVX solved by MOSEK.

Table 2: Estimation of $\beta_1$ in linear IV model with REL: Replication of Shi (2016)

| $(n, m)$     | Replication |     | Original Results |     |
|-------------|------------|-----|------------------|-----|
|             | Bias       | RMSE| Bias             | RMSE|
| (120, 80)   | -0.020     | 0.135| -0.004           | 0.113|
| (120, 160)  | -0.018     | 0.162| -0.012           | 0.143|
| (240, 80)   | -0.004     | 0.078| -0.006           | 0.071|
| (240, 160)  | -0.008     | 0.093| -0.009           | 0.077|

We also experiment with other numerical alternatives. Since the scale of the optimization problems here is much smaller than C-Lasso, the inner loop can be correctly solved by Rmosek, CVXR, CVX in MATLAB, or even nloptr. These four methods produce virtually identical inner loop results up to rounding errors. This finding confirms the robustness of the R environment in high-dimensional optimization. The difference in Table 2, therefore, is attributed to the outer loop between the function nloptr in R and the function fmincon in MATLAB.

To evaluate the computational cost, we record the time spent in the inner loop. With 100 sets of data generated by the same DGP for each sample size, we fix $\beta = (0.9, 0.9)$ and only numerically solve the inner loop. Since four approaches have virtually identical inner loop results, we only report the running time of each method in Table 3. Although CVXR and nloptr are able to correctly solve the problem thanks to its small scale, Rmosek remains 4 to 30 times faster than these alternatives. We conjecture that bigger speed gain would be observed in a problem of larger scale.
Table 3: Running time of REL’s inner loop (in second)

|    | (n, m) | (120, 80) | (120, 160) | (240, 80) | (240, 160) |
|----|--------|-----------|------------|-----------|------------|
| Rmosek | 2.995 | 4.378 | 10.510 | 17.206 |
| nloptr | 64.904 | 117.533 | 115.738 | 226.661 |
| CVXR | 31.241 | 43.909 | 42.435 | 136.007 |
| CVX | 41.441 | 54.095 | 65.846 | 88.982 |

4 Conclusion

In this note, we demonstrate numerical implementation via Rmosek of two examples of high-dimensional econometric estimators. The convenience and reliability of high-dimensional convex optimization in R will open new possibilities to create estimation procedures. In the era of big data, we are looking forward to witnessing more algorithms blossoming and flourishing along with theoretical research of high-dimensional models.

References

BAJARI, P., D. NEKIPELOV, S. P. RYAN, AND M. YANG (2015): “Machine Learning Methods for Demand Estimation,” American Economic Review, 105(5), 481.

BONHOMME, S., AND E. MANRESA (2015): “Grouped patterns of heterogeneity in panel data,” Econometrica, 83(3), 1147–1184.

Candes, E., AND T. Tao (2007): “The Dantzig selector: Statistical estimation when p is much larger than n,” The Annals of Statistics, 35(6), 2313–2351.

DOMAHIDI, A., E. CHU, AND S. BOYD (2013): “ECOS: An SOCP solver for embedded systems,” 2013 European Control Conference (ECC), pp. 3071–3076.

DOUDCHENKO, N., AND G. W. IMBENS (2016): “Balancing, regression, difference-in-differences and synthetic control methods: A synthesis,” Discussion paper, National Bureau of Economic Research No.22791.

FU, A., N. BALASUBRAMANIAN, AND S. BOYD (2017): “CVXR: An R Package for Disciplined Convex Optimization,” Working paper.

GRANT, M., AND S. BOYD (2014): “CVX: Matlab Software for Disciplined Convex Programming, version 2.1,” http://cvxr.com/cvx.

GRANT, M., S. BOYD, AND Y. YE (2006): “Disciplined convex programming,” in Global optimization, pp. 155–210. Springer.

GU, J., AND R. KOENKER (2017): “Empirical Bayesball remixed: Empirical Bayes methods for longitudinal data,” Journal of Applied Econometrics, 32(3), 575–599.
Hansen, L. P. (1982): “Large sample properties of generalized method of moments estimators,” *Econometrica*, 50(4), 1029–1054.

Johnson, S. G. (2017): “The NLopt nonlinear-optimization package.”

Koenker, R. (2017): “quantreg: Quantile Regression R package version 5.33,” [https://cran.r-project.org/web/packages/quantreg/index.html](https://cran.r-project.org/web/packages/quantreg/index.html).

Koenker, R., and G. Bassett (1978): “Regression quantiles,” *Econometrica*, 46, 33–50.

Koenker, R., and I. Mizera (2014): “Convex Optimization in R,” *Journal of Statistical Software*, 60(5), 1–23.

Nash, J. C., and R. Varadhan (2011): “Unifying Optimization Algorithms to Aid Software System Users: optimx for R,” *Journal of Statistical Software*, 43(9), 1–14.

Owen, A. B. (1988): “Empirical Likelihood Ratio Confidence Intervals for a Single Functional,” *Biometrika*, 75(2), 237–249.

Qin, J., and J. Lawless (1994): “Empirical likelihood and general estimating equations,” *The Annals of Statistics*, 22(1), 300–325.

Shi, Z. (2016): “Econometric estimation with high-dimensional moment equalities,” *Journal of Econometrics*, 195(1), 104–119.

Su, L., and G. Ju (2017): “Identifying latent grouped patterns in panel data models with interactive fixed effects,” *Journal of Econometrics*, forthcoming.

Su, L., and X. Lu (2017): “Determining the number of groups in latent panel structures with an application to income and democracy,” *Quantitative Economics*, 8(3), 729–760.

Su, L., Z. Shi, and P. C. Phillips (2016): “Identifying latent structures in panel data,” *Econometrica*, 84(6), 2215–2264.

Su, L., X. Wang, and S. Jin (2017): “Sieve estimation of time-varying panel data models with latent structures,” *Journal of Business & Economic Statistics*, 0(0), 1–16.

Ypma, J. (2017): “nloptr: R interface to NLopt R package version 1.0.4,” [https://cran.r-project.org/web/packages/nloptr/index.html](https://cran.r-project.org/web/packages/nloptr/index.html).
Appendix
(To be published online only)

A Data Generating Process

For completeness of the note, in this section we detail the DGPs and simulation design.

A.1 C-Lasso

We follow the linear static panel data DGP (DGP 1) in Su, Shi, and Phillips (2016, p.2237) and apply PLS. The observations are drawn from three groups with the proportion $n_1 : n_2 : n_3 = 0.3 : 0.3 : 0.4$. The observed data $(y_{it}, x_{it})$ are generated from

$$x_{it} = (0.2\mu_{i1} + 0.2\mu_{i2} + e_{it1}, 0.2\mu_{i1} + e_{it2})$$
$$y_{it} = \beta_i' x_{it} + \mu_{i} + \epsilon_{it},$$

where $\mu_{i1}, e_{it1}, e_{it2} \sim \text{i.i.d.} \mathcal{N}(0, 1)$. The true coefficients are $(0.4, 1.6), (1, 1), (1.6, 0.4)$ for the three groups, respectively. In the implementation, the C-Lasso tuning parameter is specified as $\lambda = \frac{1}{2}\hat{\sigma}_Y^{-2} T^{-\frac{3}{4}}$, where $\hat{\sigma}_Y$ is the sample variance of demeaned dependent variable. Given the number of groups, we run the simulation for $R = 500$ replications and report the RMSE of the estimation of $\alpha_1$ and the probability of correct classification (correct ratio) in Table 1, where

$$\text{RMSE} (\hat{\beta}_1) = \sqrt{\frac{1}{R} \sum_{r=1}^{R} \left( \frac{1}{n} \sum_{i=1}^{n} \left( \hat{\alpha}_{k,1}^{(r)} - \alpha_{k,1}^0 \right) \right)^2}$$

$$\text{Correct Ratio} = \frac{1}{R} \sum_{r=1}^{R} \left( \frac{1}{n} \sum_{i=1}^{n} \mathbf{1} (\hat{g}_i^{(r)} = g_i^0) \right),$$

where $\hat{g}_i^{(r)}$ and $g_i^{(0)}$ are the estimated and the true group identity of the $i$’s individual, respectively, and $\mathbf{1} (\cdot)$ is the indicator function.

A.2 REL

We follow the data generating process in Shi (2016, Section 4.1) that features the linear IV model with many IVs. The observed data $\{y_i\}_{i=1}^n$ are generated by the structural equation

$$y_i = (x_{i1}, x_{i2}) \beta + e_i^0$$

where $\beta = (1, 1)'$, $x_i = (x_{i1}, x_{i2})$ are endogenous variables that are generated by $x_{i1} = 0.5z_{i1} + 0.5z_{i2} + e_{i1}$ and $x_{i2} = 0.5z_{i3} + 0.5z_{i4} + e_{i2}$, respectively, $e_i^0$ is the structural error, and $(e_{i1}, e_{i2})$ are reduced-form errors. The observed data contains $m$ IVs $\{z_{ij}\}_{j=1}^m$ orthogonal to $e_i^0$ but the information that which one is relevant is unknown. We generate $\{z_{ij}\}_{j=1}^m \sim \text{i.i.d.} \mathcal{N}(0, 1)$
and \( (e_0^0, e_1^1, e_2^2) \sim n \left( (0), \begin{pmatrix} 0.25 & 0.15 & 0.15 \\ 0.15 & 0.25 & 0 \\ 0.15 & 0 & 0.25 \end{pmatrix} \right) \). The endogeneity comes from the correlation among all error terms. The orthogonality yields the moment restrictions \( \mathbb{E}[z_i(y_i - x_i \beta)] = 0 \), which can be used to estimate \( \beta \) with REL. We run \( R = 500 \) replications and report bias and RMSE for \( \beta_1 \) as

\[
\text{Bias} = \frac{1}{R} \sum_{r=1}^{R} (\hat{\beta}_1 - \beta_1), \quad \text{RMSE} = \sqrt{\frac{1}{R} \sum_{r=1}^{R} (\hat{\beta}_1 - \beta_1)^2}.
\]

### B Code Snippets

In this section, we provide several code snippets to demonstrate the key formulation steps. All code in this note is hosted at [https://github.com/zhentaoshi/convex_prog_in_econometrics](https://github.com/zhentaoshi/convex_prog_in_econometrics).

We start with Lasso. In matrix notation, the Lasso problem is

\[
\min_{\theta} \lambda (e' \beta^+ + e' \beta^-) + \frac{t}{n}
\]

s.t. \[
\begin{bmatrix}
X & -X & I_n \\
0_{2\times(n+2p)} & \begin{pmatrix} 0_{n\times3} & 1 & 0 \\
-\frac{1}{2} & 0 & 1 \end{pmatrix}
\end{bmatrix} \theta = \begin{bmatrix} y \\
-\frac{1}{n} \end{bmatrix}, \quad \|(v, s)\|_2 \leq r, \quad \beta^+, \beta^- \geq 0
\]

where the inequality for a vector is taken elementwisely. The following annotated R code snippet implements the matrix form.

```r
P = list(sense = "min")

# Linear coefficients in objective
P$c = c(rep(lambda, 2*p), rep(0, n), 1/n, 0, 0)

# The matrix in linear constraints
A = as.matrix.csr(X)
A = cbind(A, -A, as.numeric("matrix.diag.csr"), as.matrix.csr(0, n, 3))
A = rbind(A, cbind(as.matrix.csr(0, 2, 2*p + n),
                   as.matrix.csr(c(-.5, -.5, 1, 0, 0, 1), 2, 3)))
P$A = as(A, "CsparseMatrix")

# Right-hand side of linear constraints
P$bc = rbind(c(y, -0.5, 0.5), c(y, -0.5, 0.5))

# Constraints on variables
P$bx = rbind(c(rep(0, 2*p), rep(-Inf, n), rep(0, 3)), c(rep(Inf, 2*p+n+3)))

# Conic constraints
P$cones = matrix(list("QUAD", c(n+2*p+3, (2*p+1):(2*p+n), n+2*p+2)), 2, 1)
rownames(P$cones) = c("type", "sub")

result = mosek(P, opts = list(verb))
```
We then take a step further to C-Lasso. The convexity is manifest when we write the problem in matrix form

$$\min_{\alpha_k, \theta} \left( \frac{1}{nT} \right) e^t + \left( \frac{\lambda}{n} \right) \gamma'w$$

s.t. \( t_i \geq 0, \| (\nu_i, s_i) \|_2 \leq r_i, \| \mu_i \|_2 \leq w_i, \) for all \( i = 1, 2, \ldots, n \)

$$\begin{bmatrix}
\text{diag}(X_1, \ldots, X_n) & I_{Tn} & 0 & 0 & 0 \\
I_{np} & 0 & -I_{np} & 0 & -1_n \otimes I_p \\
0 & I_2 \otimes I_n & -\frac{1}{2} I_2 \otimes I_n & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\theta \\
\alpha_k
\end{bmatrix} =
\begin{bmatrix}
y \\
0_{np} \\
-\frac{1}{2} e_n \\
\frac{1}{2} e_n
\end{bmatrix}$$

Though more tedious than Lasso, the construction of the large matrix in the linear constraints is straightforward. The formulation of the conic constraints is illustrated in the following chunk of code.

```
CC = list()

# locate the variables related
bench = N*(2*p + TT) + p

for(i in 1:N){
    # find index of each variable
    s.i = bench + i
    r.i = bench + N + i
    nu.i = (N*p + (i-1)*TT + 1):(N*p + i*TT)
    w.i = bench + 3*N + i
    mu.i = (N*(TT+p) + (i-1)*p + 1):(N*(TT+p) + i*p)
    CC = cbind(CC, list("QUAD", c(r.i, nu.i, s.i)),
               list("QUAD", c(w.i, mu.i)))
}
P$cones = CC
rownames(prob$cones) = c("type", "sub")
```

The penalty \( \gamma_i \) can be coded as follows.

```
pen.generate = function(b, a, N, p, K, kk){

    # Output arg: gamma
    # Input args:
    # b, a (estimate of last iteration)
    # kk (current focused group)
```
Regarding REL, it involves nonlinear logarithm terms in the objective. The objective of the separable convex problem can be formulated as follows.

NUMOPRO = n
opro = matrix(list(), nrow = 5, ncol = NUMOPRO)
rownames(opro) = c("type", "j", "f", "g", "h")
for(i in 1:n){
    opro[ , i] = list("LOG", i, 1.0, 1.0, 0)
}
P$scopt = list(opro = opro)

C Additional Examples of C-Lasso

In this section, we formulate the nonlinear Lasso and the penalized GMM (PGMM).

C.1 Nonlinear Lasso

In microeconometrics, it is common to see exponential, logarithm or power terms in objective functions. When the problem involves these nonlinear functions, we formulate the problem as a separable convex optimization problem. For example, the penalized Poison maximum likelihood estimator is defined as

$$\min_{\beta} -\frac{1}{n} \sum_{i=1}^{n} (y_i x_i' \beta - \exp (x_i' \beta)) + \lambda \| \beta \|_1$$

where $y \in \mathbb{R}^n$ and $X \in \mathbb{R}^{n \times p}$ are observed data, $\lambda$ is the tuning parameter and $\beta \in \mathbb{R}^p$ is the parameter of interests. This optimization problem involves the component $\exp \left( \sum_{j=1}^{p} x_{ij} \beta_j \right)$, which is non-separable. Define $v_i = x_i' \beta$, and the objective becomes

$$\min_{v, \beta} -\frac{1}{n} \sum_{i=1}^{n} (y_i v_i - \exp (v_i)) + \lambda \| \beta \|_1$$
We apply the same transformation as in Lasso to deal with the $l_1$-norm. The original optimization problem can be transformed to

$$\min_{v, \beta^+, \beta^-} -\frac{1}{n} \sum_{i=1}^{n} (y_i v_i - \exp(v_i)) + \lambda (\beta^+ + \beta^-)$$

s.t. $v_i = x'_i (\beta^+ - \beta^-)$ for each $i = 1, 2, 3, \ldots, n$, $\beta^+, \beta^- \geq 0$

In matrix form,

$$\min_\theta \begin{bmatrix} -y' & \lambda e' & \lambda e' \end{bmatrix} \theta + \frac{1}{n} \sum_{i=1}^{n} \exp(v_i)$$

s.t. $\begin{bmatrix} I_n & -X X \end{bmatrix} \theta = 0$, $\beta^+, \beta^- \geq 0$

where $\theta = (v, \beta^+, \beta^-)$. The following code snippet displays the formulation of these exponential terms.

```r
NUMOPRO = n
opro = matrix(list(), nrow = 5, ncol = NUMOPRO)
rownames(opro) = c("type", "j", "f", "g", "h")
for(i in 1:n){
    opro[,i] = list("EXP", i, 1/n, 1.0, 0)
}
P$scopt = list(opro=opro)
```

Now that we are able to deal with nonlinear Lasso, it is straightforward to extend it to penalized profile likelihood (PPL) in Su, Shi, and Phillips (2016).

### C.2 Penalized GMM

We consider the linear panel data model with latent group structures and endogeneity. After first-difference, we have

$$\Delta y_{it} = \beta'_i \Delta x_{it} + \Delta \varepsilon_{it}$$

Let $z_{it}$, of dimension $m \times 1$, $m \geq p$, be instrumental variables for $\Delta x_{it}$. The penalized GMM estimator is defined as the solution $(\beta, \alpha)$ to

$$\min_{\beta, \alpha} \frac{1}{n T^2} \sum_{i=1}^{n} \left| \left| W_i^{1/2} z_i (\Delta y_i - \Delta x_i \beta_i) \right| \right|_2^2 + \frac{\lambda}{n} \sum_{i=1}^{n} \prod_{k=1}^{K} \| \beta_i - \alpha_k \|_2$$

where $W_i$ is an $m \times m$ positive-definite symmetric weighting matrix. It is easy to see that the PGMM problem can be formulated as

$$\min_{\beta, \alpha} \frac{1}{n T^2} \sum_{i=1}^{n} \left| \left| \tilde{y}_i - \tilde{x}_i \beta_i \right| \right|_2^2 + \frac{\lambda}{n} \sum_{i=1}^{n} \prod_{k=1}^{K} \| \beta_i - \alpha_k \|_2$$

14
by the transformations \( \tilde{y}_i = W_i^{\frac{1}{2}} z_i \Delta y_i \) and \( \tilde{x}_i = W_i^{\frac{1}{2}} z_i \Delta x_i \). The following iterative algorithm is essentially the same as PLS and can be carried out as in Section 2.

D Software Installation

The installation of the Rmosek package requires successful installation of MOSEK. For Windows users, Rtools is also required. Once the prerequisites are satisfied, Rmosek can be installed by a command similar to the following one:

```r
install.packages("Rmosek", type="source", INSTALL_opts="--no-multiarch", repos="http://download.mosek.com/R/8")
```

For more details, readers can refer to the official installation manual at https://docs.mosek.com/8.1/rmosek/install-interface.html.

CVXR is now available on CRAN and can be installed as a standard R packages. The default solver ECOS is installed along with CVXR. To use MOSEK in CVXR, we will need Python and the R package reticulate. Details can be found at https://cvxr.rbind.io/post/examples/cvxr_using-other-solvers/