A New Approach to Generalized Fractional Derivatives

Udita N. Katugampola

Department Of Mathematics, Delaware State University, Dover DE 19901, USA

Abstract

The purpose of this paper is to present a new fractional derivative which generalizes the familiar Riemann-Liouville fractional derivative and Hadamard fractional derivative to a single form, which when a parameter is fixed at different values, produces the above derivatives as special cases. Example is given to show the results.

Keywords: Generalized fractional derivative, Riemann-Liouville fractional derivative, Erdélyi-Kober operator, Hadamard fractional derivative

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1. Introduction

The history of Fractional Calculus (FC) goes back to seventeenth century, when in 1695 the derivative of order $\alpha = 1/2$ was described by Leibnitz. Since then, the new theory turned out to be very attractive to mathematicians as well as biologists, economists, engineers and physicists. Several books were written on the theories and developments of FC [12, 14, 17, 15, 16]. In [17] Samko et al. provide a comprehensive study of the subject. Various type of fractional derivatives were studied: Riemann-Liouville, Caputo, Hadamard, Erdélyi-Kober, Grunwald-Letnikov and Riesz are just a few to name.

In fractional calculus, the fractional derivatives are defined via a fractional integral [14, 17, 15, 16]. According to the literature, the Riemann-Liouville fractional derivative (RLFD), hence Riemann-Liouville fractional integral have played major roles in FC [17]. The Caputo fractional derivative has also been defined via Riemann-Liouville fractional integral. Butzer, et al., investigate properties of Hadamard fractional integral and derivative [1, 2, 3, 11, 12, 14, 17]. In [2], they obtain the Mellin transform of the Hadamard integral and differential operators. Many of those results are summarized in [12] and [17].

In [7], author introduced a new fractional integral, which generalizes the Riemann-Liouville and the Hadamard integrals into a single form. In the present work we obtain a new fractional derivative which generalises the two derivatives in question.
In this section we give definitions and some properties of fractional integrals and fractional derivatives of various types. More detailed explanation can be found in the book by Samko et al [17].

The Riemann-Liouville fractional integrals (RLFI) \( I^\alpha_{a+} f \) and \( I^\alpha_{b-} f \) of order \( \alpha \in \mathbb{C}(\text{Re}(\alpha) > 0) \) are defined by [17],

\[
(I^\alpha_{a+} f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - \tau)^{\alpha-1} f(\tau) \, d\tau \quad ; \quad x > a.
\] (1)

and

\[
(I^\alpha_{b-} f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (\tau - x)^{\alpha-1} f(\tau) \, d\tau \quad ; \quad x < b.
\] (2)

respectively. Here \( \Gamma(\alpha) \) is the Gamma function. These integrals are called the left-sided and right-sided fractional integrals. When \( \alpha = n \in \mathbb{N} \), the integrals \( 1 \) and \( 2 \) coincide with the \( n \)-fold integrals [12, chap.2]. The Riemann-Liouville fractional derivatives (RLFD) \( D^\alpha_{a+} f \) and \( D^\alpha_{b-} f \) of order \( \alpha \in \mathbb{C} \), \( \text{Re}(\alpha) \geq 0 \) are defined by [17],

\[
(D^\alpha_{a+} f)(x) = \left( \frac{d}{dx} \right)^n \left( I^{\alpha-n}_{a+} f \right)(x) \quad ; \quad x > a,
\] (3)

and

\[
(D^\alpha_{b-} f)(x) = \left( - \frac{d}{dx} \right)^n \left( I^{\alpha-n}_{b-} f \right)(x) \quad ; \quad x < b,
\] (4)

respectively, where \( n = \lfloor \text{Re}(\alpha) \rfloor \). For simplicity, from this point onwards we consider only left-sided integrals and derivative. The interested reader can find more detailed information about right-sided integrals and derivatives in the references.

Erdélyi - Kober type fractional integral and differential operators are defined by [12, 17, 13],

\[
(I^\alpha_{a+;\rho,\eta} f)(x) = \frac{\rho x^{\rho(\alpha+\eta)}}{\Gamma(\alpha)} \int_a^x \frac{\tau^{\eta \rho - 1} f(\tau)}{(x^\rho - \tau^\rho)^{1-\alpha}} \, d\tau \quad ; \quad x > a, \text{Re}(\alpha) > 0.
\] (5)

\[
(D^\alpha_{a+;\rho,\eta} f)(x) = x^{-\eta \rho} \left( \frac{d}{dx} \right)^n \left( I^{\alpha-n}_{a+;\rho,\eta} f \right)(x)
\] (6)

for \( x > a, \text{Re}(\alpha) \geq 0, \rho > 0 \). When \( \rho = 2, a = 0 \), the operators are called Erdélyi-Kober operators. When \( \rho = 1, a = 0 \), they are called Kober-Erdélyi or Kober operators [12, p.105].

The next important type is the Hadamard Fractional integral introduced by J. Hadamard [6, 12, 17], and given by,

\[
I^\alpha_{a+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left( \frac{\tau}{x} \right)^{\alpha-1} f(\tau) \frac{d\tau}{\tau} \quad ; \quad \text{Re}(\alpha) > 0, x > a \geq 0.
\] (7)

while the Hadamard fractional derivative of order \( \alpha \in \mathbb{C}, \text{Re}(\alpha) > 0 \) is given by,

\[
D^\alpha_{a+} f(x) = \frac{1}{\Gamma(n - \alpha)} \left( \frac{d}{dx} \right)^n \int_a^x \left( \log \frac{x}{\tau} \right)^{n-\alpha+1} f(\tau) \frac{d\tau}{\tau} \quad ; \quad x > a \geq 0.
\] (8)

where \( n = \lfloor \text{Re}(\alpha) \rfloor \).

In [7], the author introduces a generalization to the Riemann-Liouville and Hadamard fractional integral and also provided existence results and semigroup properties. In the same reference, author introduces a generalised fractional derivative, which does not possess the inverse property. In this paper, we describe a new fractional derivative, which generalizes the Riemann-Liouville and Hadamard fractional derivatives. Notice also that this new derivative possesses the inverse property [Theorem 2.6].
2. Generalization of the fractional integration and differentiation

As in [9], consider the space $X^p_\alpha(a,b)$ ($c \in \mathbb{R}$, $1 \leq p \leq \infty$) of those complex-valued Lebesgue measurable functions $f$ on $[a,b]$ for which $\|f\|_{X^p_\alpha} < \infty$, where the norm is defined by

$$\|f\|_{X^p_\alpha} = \left( \int_a^b |f(t)|^p \frac{dt}{t} \right)^{1/p} < \infty \quad (1 \leq p < \infty, \ c \in \mathbb{R}) \quad (9)$$

and for the case $p = \infty$

$$\|f\|_{X^\infty_\alpha} = \text{ess sup}_{a \leq t \leq b} |f(t)| \quad (c \in \mathbb{R}). \quad (10)$$

In particular, when $c = 1/p$ ($1 \leq p \leq \infty$), the space $X^p_\alpha$ coincides with the classical $L^p(a,b)$-space with

$$\|f\|_p = \left( \int_a^b |f(t)|^p dt \right)^{1/p} < \infty \quad (1 \leq p < \infty), \quad (11)$$

$$\|f\|_\infty = \text{ess sup}_{a \leq t \leq b} |f(t)| \quad (c \in \mathbb{R}). \quad (12)$$

We start with the definitions introduced in [9] with a slight modification in the notations. Let $\Omega = [a,b]$ ($-\infty < a < b < \infty$) be a finite interval on the real axis $\mathbb{R}$.

The generalized fractional integral $\mathcal{I}^\alpha_{\alpha^\rho}$ of order $\alpha \in \mathbb{C}$ ($\text{Re}(\alpha) > 0$) of $f \in X^p_\alpha(a,b)$ is defined by

$$\mathcal{I}^\alpha_{\alpha^\rho} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{x^\alpha f(t)}{(x-t)^{1-\alpha}} dt \quad (13)$$

for $x > a$ and $\text{Re}(\alpha) > 0$. This integral is called the left-sided fractional integral. Similarly we can define the right-sided fractional integral $\mathcal{I}^\alpha_{\alpha^\rho}$ by

$$\mathcal{I}^\alpha_{\alpha^\rho} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{x^\alpha f(t)}{(t-x)^{1-\alpha}} dt \quad (14)$$

for $x < b$ and $\text{Re}(\alpha) > 0$. These are the fractional generalizations of the $n$-fold integrals of the form

$$\int_a^x \int_a^t \cdots \int_a^t f(t_1)dt_1 \cdots dt_n$$

and

$$\int_b^x \int_b^t \cdots \int_b^t f(t_1)dt_1 \cdots dt_n$$

for $n \in \mathbb{N}$, respectively.

**Remark 2.1.** When $b = \infty$, the generalized fractional integral is called a Liouville-type integral.

Now consider generalized fractional derivatives given below,

**Definition 2.2.** (Generalized fractional derivatives)

Let $\alpha \in \mathbb{C}$, $\text{Re}(\alpha) \geq 0$, $n = \lfloor \text{Re}(\alpha) \rfloor$ and $\rho > 0$. The generalized fractional derivatives, corresponding to the generalized fractional integrals (13) and (14), are defined, for $0 \leq a < x < b \leq \infty$, by

$$\mathcal{D}^\alpha_{\alpha^\rho} f(x) = \left( x^\alpha \frac{d}{dx} \right)^n \mathcal{I}^\alpha_{\alpha^\rho} f(x)$$

$$= \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)} \left( x^\alpha \frac{d}{dx} \right)^n \int_a^x \frac{x^\alpha f(t)}{(x-t)^{1-\alpha-n+1}} dt \quad (15)$$
and
\[
(\rho D_{a+}^{\alpha} f)(x) = \left( -x^{1-\rho} \frac{d}{dx} \right)^n \left( \rho D_{a+}^{\alpha-\rho} f \right)(x) \\
= \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)} \left( -x^{1-\rho} \frac{d}{dx} \right)^n \int_{x}^{\infty} \frac{\tau^{\alpha-1} f(\tau)}{(\tau^{\rho} - x^{\rho})^{n+1}} \, d\tau
\]  
(16)
if the integrals exist.

When \( b = \infty \), the generalized fractional derivative is called a Liouville-type derivative. Next we give existence results for generalised fractional derivatives.

**Theorem 2.3.** Generalized fractional derivatives (15) and (16) exist finitely.

*Proof.* This is clear from Theorem 3.1 of \([7]\) and the fact that generalized fractional integrals (13) and (14) have \( n \)-time differentiable kernel.

The following theorem gives the relations of generalized fractional derivatives to that of Riemann-Liouville and Hadamard. For simplicity we give only the left-sided versions here.

**Theorem 2.4.** Let \( \alpha \in \mathbb{C}, \text{Re}(\alpha) \geq 0, n = \lfloor \text{Re}(\alpha) \rfloor, \) and \( \rho > 0 \). Then, for \( x > a \),

1. \[
\lim_{\rho \to 1} (\rho D_{a+}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_{a}^{x} (x - \tau)^{\alpha-1} f(\tau) \, d\tau,
\]  
(17)
2. \[
\lim_{\rho \to 0^+} (\rho D_{a+}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_{a}^{x} \left( \log \frac{x}{\tau} \right)^{\alpha-1} f(\tau) \, d\tau,
\]  
(18)
3. \[
\lim_{\rho \to 1} (\rho D_{a+}^{\alpha} f)(x) = \frac{d}{dx} \frac{1}{\Gamma(\alpha-n+1)} \int_{a}^{x} f(\tau) (x - \tau)^{\alpha-n+1} \, d\tau,
\]  
(19)
4. \[
\lim_{\rho \to 0^+} (\rho D_{a+}^{\alpha} f)(x) = \frac{d}{dx} \frac{1}{\Gamma(\alpha)} \int_{a}^{x} \left( \log \frac{x}{\tau} \right)^{\alpha-n+1} f(\tau) \, d\tau.
\]  
(20)

*Proof.* (17) and (19) follow from direct substitution, while (18) and (20) follow from L’Hospital rule. Similar results for right-sided integrals and derivatives also exist and can be proved similarly.

**Remark 2.5.** Note that the equations (17) and (19) are related to Riemann-Liouville operators, while equations (18) and (20) are related to Hadamard operators.

Next is the inverse property.

**Theorem 2.6.** Let \( 0 < \alpha < 1, \) and \( f(x) \) be continuous. Then, for \( a > 0, \rho > 0, \)

\[
(\rho D_{a+}^{\alpha} \rho I_{a+}^{\alpha} f)(x) = f(x).
\]  
(21)

*Proof.* We prove this using Fubini’s theorem and Dirichlet technique. From direct integration,

\[
(\rho D_{a+}^{\alpha} \rho I_{a+}^{\alpha} f)(x) = \frac{\rho}{\Gamma(1-\alpha-\rho)} \int_{a}^{x} \left( x^{1-\rho} \frac{d}{dx} \right)^n \frac{\tau^{\alpha-1} f(\tau)}{(\tau^{\rho} - x^{\rho})^{n+1}} \, d\tau \, d\tau,
\]

\[
= \frac{\rho}{\Gamma(1-\alpha)\Gamma(\alpha)} \int_{a}^{x} \left( x^{1-\rho} \frac{d}{dx} \right)^n f(s) s^{\alpha-1} \int_{s}^{\infty} \frac{\tau^{\alpha-1} f(\tau)}{(\tau^{\rho} - s^{\rho})^{n+1}} \, d\tau \, ds,
\]

\[
= \frac{\rho}{\Gamma(1-\alpha)\Gamma(\alpha)} \int_{a}^{x} \left( x^{1-\rho} \frac{d}{dx} \right)^n f(s) s^{\alpha-1} ds \, \left( \frac{\Gamma(1-\alpha)\Gamma(\alpha)}{\rho} \right)
\]

\[= f(x).\]
Notice also the use of Beta function in the proof.

Compositions between the operators of generalized fractional differentiation and generalized fractional integration are given by the following theorem.

**Theorem 2.7.** Let \( \alpha, \beta \in \mathbb{C} \) be such that \( 0 < \text{Re}(\alpha) < \text{Re}(\beta) < 1 \). If \( 0 < a < b < \infty \) and \( 1 \leq p \leq \infty \), then, for \( f \in \mathcal{L}^p(a, b) \), \( \rho > 0 \),

\[ \rho \mathcal{D}_a^\alpha \rho \mathcal{I}_b^\beta f = \rho \mathcal{I}_b^{\beta-\alpha} \rho \mathcal{D}_a^\alpha f \] \quad \text{and} \quad \rho \mathcal{D}_b^\alpha \rho \mathcal{I}_a^\beta f = \rho \mathcal{I}_a^{\beta-\alpha} \rho \mathcal{D}_b^\alpha f. \]

**Proof.** Proof is very much similar to that of Theorem 2.6. Notice that we use the property \( \Gamma(x+1) = x \Gamma(x) \) of the Gamma function here, which is not used in the proof of the previous theorem.

**Remark 2.8.** The author suggests the interested reader to refer Property 2.27 in [12] for similar properties of the Hadamard fractional operator.

**Remark 2.9.** The Caputo generalized fractional derivative can be defined via the above generalised Riemann-Liouville fractional derivative as follows. If \( \rho \mathcal{D}_a^\alpha \) is the Caputo type differential operator, then [12],

\[ \rho \mathcal{D}_a^\alpha f(x) = \left( \rho \mathcal{D}_a^\alpha \left[ f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (t-a)^k \right] \right)(x) \]

and

\[ \rho \mathcal{D}_b^\alpha f(x) = \left( \rho \mathcal{D}_b^\alpha \left[ f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (t-a)^k \right] \right)(x) \]

respectively, where \( n = \lceil \text{Re}(\alpha) \rceil \).

Now we consider an example to illustrate the results. We find the generalized fractional derivative of the function \( f(x) = x^\nu \), where \( \nu \in \mathbb{R} \). The formula (15) yields

\[ \rho \mathcal{D}_0^\alpha x^\nu = \frac{\rho^\nu}{\Gamma(1-\alpha)} \left( x^{1-\rho} \frac{d}{dx} \right) \int_0^x \frac{u^{\rho-1} t^\nu}{(x^\rho - u^\rho)^\alpha} \, dt \]

To evaluate the inner integral, use the substitution \( u = t^\rho / x^\rho \) to obtain,

\[ \int_0^x \frac{t^{\rho-1}}{(x^\rho - u^\rho)^\alpha} \, dt = \frac{x^{\nu+\rho(1-\alpha)}}{\rho} \int_0^1 \frac{u^{\nu+\rho(1-\alpha)}}{(1-u)^\alpha} \, du \]

\[ = \frac{x^{\nu+\rho(1-\alpha)}}{\rho} B(1-\alpha, 1+\frac{\nu}{\rho}) \]

where \( B(\ldots) \) is the Beta function. Thus, we obtain,

\[ \rho \mathcal{D}_0^\alpha x^\nu = \frac{\Gamma(1+\frac{\nu}{\rho}(\rho-1))}{\Gamma(1+\frac{\nu}{\rho}-\alpha)} x^{\nu-\alpha \rho} \] (23)
for \( \rho > 0 \), after using the properties of the Beta function \([12]\) and the relation \( \Gamma(z + 1) = z \Gamma(z) \). When \( \rho = 1 \) we obtain the Riemann-Liouville fractional derivative of the power function given by \([12, 14, 17]\),

\[
1 D_0^\alpha x^\nu = \frac{\Gamma(1 + \nu)}{\Gamma(1 + \nu - \alpha)} x^{\nu - \alpha} 
\]  

(24)

This agrees well with the standard results obtained for Riemann-Liouville fractional derivative \([3]\). Interestingly enough, for \( \alpha = 1, \rho = 1 \), we obtain \( 1 D_0^1 x^\nu = \nu x^{\nu - 1} \), as one would expect.

To compare the results, we plot (23) for several values of \( \rho \). We also consider different values of \( \nu \) to see the effect on the degree of the power function. The results are summarized below in Figure 1 and Figure 2.

![Graphs of generalized derivatives for \( \nu = 1.0 \)](image)

- (a) \( \nu = 1.0 \)
- (b) \( \nu = 2.0 \)
- (c) \( \nu = 0.5 \)
- (d) \( \nu = 1.5 \)

Figure 1: Generalized fractional derivative of the power function \( x^\nu \) for \( \rho = 0.4, 1.0, 1.4 \) and \( \nu = 1.0, 2.0, 0.5, 1.5 \)
Figure 1 summarizes the comparison results for \( \rho \) and \( \nu \), while Figure 2 summarizes the comparison results for different values of \( \alpha \) and \( \nu \). It is apparent from the figure that the effect of changing the parameters is more visible for different values of \( \alpha \), which is related to the fractional effect of the derivative.

**Figure 2**: Generalized fractional derivative of the power function \( x^{\nu} \) for \( \alpha = 0.1, 0.5, 0.9 \) and \( \nu = 0.5, 1.0, 1.5 \)

**Remark 2.11**. In many cases we only considered the left-sided derivatives and integrals. But the right-sided derivatives can be treated similarly.

### 3. Conclusion

We conclude the paper with the following open problem.

**Problem 3.1**. Investigate the existence of an exact formula without using the generalized functions (i.e., Gauss-Hypergeometric, H-Functions, Meijer’s G-functions and etc) for the left-sided generalized fractional derivative of the power function, \((x - c)^{\omega}\) with \(\omega \in \mathbb{R}\), that is, evaluate the following integral,

\[
^{\rho}D_{a+}^{\alpha}(x - c)^{\omega} = \frac{\rho^{\rho-n+1}}{\Gamma(n - \alpha)} \left( x^{1-\rho} \frac{d}{dx} \right)^n \int_a^x \frac{\rho^{-1}(t - c)^{\omega}}{(x^\rho - t^\rho)^{\alpha-n+1}} dt, \tag{25}
\]

for \(x > a \geq c\), \(a, c \in \mathbb{R}\), \(\alpha \in \mathbb{C}\), \(n = [Re(\alpha)]\) and \(\rho > 0\).

According to Figure 1 and Figure 2, we notice that the characteristics of the fractional derivative is highly affected by the value of \(\rho\), thus it provides a new direction for control applications.
The paper presents a new fractional differentiation, which generalizes the Riemann-Liouville and Hadamard fractional derivatives into a single form, which when a parameter is fixed at different values, produces the above derivatives as special cases. Example is given to compare results.

In a future project, we will derive formulae for the Mellin, Laplace and Fourier transforms for the generalized fractional operators. We already know that we can deduce Hadamard and Riemann-Liouville operators for the special cases of $\rho$. We want to further investigate the effect on the new parameter $\rho$.

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