Logarithmic electroweak corrections to
\[ \text{e}^+\text{e}^- \rightarrow \nu_e \bar{\nu}_e W^+ W^- \]

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Abstract:
We consider W-boson scattering at high-energy e\textsuperscript{+}e\textsuperscript{-} colliders and study one-loop logarithmic electroweak corrections within the Standard Model assuming a light Higgs boson. We present explicit analytical results for W\textsuperscript{+}W\textsuperscript{-} \rightarrow W\textsuperscript{+}W\textsuperscript{-}. Using the equivalent vector-boson approximation, we have implemented these corrections into a Monte Carlo program for the process e\textsuperscript{+}e\textsuperscript{-} \rightarrow \nu_e \bar{\nu}_e W^+ W^- . The quality of the equivalent vector-boson approximation and of the logarithmic high-energy approximation for the electroweak corrections is discussed in detail. The impact of the radiative effects is quantitatively analysed. The corrections are negative and their size, typically of the order of 10\%, increases with energy reaching up to -20\% and -50\% at the ILC and CLIC, respectively.

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1 Introduction

One of the foremost open questions in particle physics concerns the mechanism of electroweak symmetry breaking. Depending on its realization in nature, the understanding of this subtle mechanism must be approached in different ways. If the Standard Model (SM) with a light Higgs boson is realized, the Higgs boson can be directly produced and its properties investigated. If the Higgs boson is heavy or absent, a complementary approach must be pursued. In this case, information can be extracted from the scattering of longitudinally polarized gauge bosons. In fact, at high energies, longitudinal vector bosons unveil their origin as Goldstone bosons and, by virtue of the equivalence theorem, reflect the dynamics of electroweak symmetry breaking.

Accordingly, vector-boson scattering (VBS) looks very much different in these different scenarios of electroweak symmetry breaking. In the SM with a light Higgs boson the gauge sector remains weakly interacting and the cross section can be reliably predicted within perturbation theory. In the alternative scenario, where a light Higgs boson is absent, perturbative unitarity is violated, and the longitudinal gauge bosons must become strongly interacting at high energies thus allowing for non-perturbative restoration of unitarity [1].

In the past years, a variety of models have been proposed to parametrize the strongly-interacting electroweak gauge sector, and to recover unitarity (see for instance Ref. [2] and references therein). The common prediction is an enhanced production of longitudinal gauge bosons. However, the phenomenological consequences as well as the new particles the various models provide can be sensibly different. They can be classified into two main groups [3]. In the most optimistic case, one could expect many new resonances at future colliders. In the less favorable scenario, the mass of any new particle could be much bigger than the energy scale probed at the planned accelerators. In this case, the indirect effect of such particles would only consist in a slight increase of the VBS event rate at high energy compared to the predictions of the SM with a light Higgs boson.

Several studies have been performed in order to estimate the possible reach of the future lepton and hadron colliders [2–8]. The answers strongly depend on energy and luminosity parameters. Also a good control of the SM background can prove essential, particularly for the less favorable case. In this respect, the scattering of W bosons within the SM with a light Higgs boson constitutes an irreducible background to any new-physics signal pointing towards a strongly-interacting VBS regime.

In this paper, we consider VBS within the SM with a light Higgs boson at the planned $e^+e^-$ colliders. For the International Linear Collider (ILC) [9] we consider a centre-of-mass (CM) energy $\sqrt{s} = 1$ TeV and for the Compact LInear Collider (CLIC) [10] $\sqrt{s} = 3$ TeV. More precisely, we focus on the production of $W$-boson pairs plus neutrinos in the reaction $e^+e^- \rightarrow \nu_\ell\bar{\nu}_\ell W^+W^-$. For the projected luminosity $L = 1$ ab$^{-1}$, the experimental collaborations will collect thousands of events coming from VBS in the high-energy domain, where a possible strongly-interacting regime of the weak gauge sector could manifest itself.

To match the envisaged statistical precision, the SM predictions have to be computed beyond lowest order. Indeed, in the very same high-energy region of interest, the electroweak radiative effects are enhanced by electroweak Sudakov logarithms [11–15], i.e. double and single logarithms of the ratio of the scattering energy over the vector-boson
mass (recent progress in the evaluation of electroweak Sudakov logarithms is discussed in Ref. [16]). At $\mathcal{O}(\alpha)$, these corrections can reach several tens of per cent, as confirmed by various analyses performed for different processes at lepton and hadron colliders [17–26]. Hence, in the case at hand, they have to be taken into account in order to search for small deviations between data and SM predictions that might appear as a signal of strongly interacting electroweak symmetry breaking.

So far, the process $e^+e^- \to \nu_e\bar{\nu}_eW^+W^-$ has been computed at Born level, and found promising for investigating electroweak symmetry breaking at high invariant masses of the produced W-boson pairs [7,27,28]. The aim of our work is to study the contributions of the electroweak Sudakov logarithms, which represent the dominant electroweak corrections at high energies.

For the calculation of the electroweak corrections we use the equivalent vector-boson approximation following the approach of Ref. [29]. Within this approximation we only consider virtual $\mathcal{O}(\alpha)$ corrections to the WW-scattering subprocess. These corrections are calculated using the method of Refs. [12,13] in the high-energy logarithmic approximation. As in Refs. [12,13], the virtual photonic corrections are split into a symmetric-electroweak and a purely electromagnetic part, which originate from above and below the electroweak scale, respectively. The former part is in practice obtained by setting the photon mass equal to $M_W$ in the photonic virtual corrections, and is infrared finite. The infrared singularities are contained in the purely-electromagnetic part and are cancelled when including soft-photon bremsstrahlung. Both the symmetric-electroweak and purely-electromagnetic parts are included in our analytical results. However the latter is omitted in our numerical studies since it strictly depends on the experimental setup.

The paper is organized as follows. In Sect. 2 we define the process and give the setup for the numerical evaluation. In Sect. 3 we describe the strategy of the calculation and discuss the quality of the used approximations. Numerical results are presented in Sect. 4, and Sect. 5 contains the summary. Explicit analytical results are listed in the appendices.

2 Process definition and numerical setup

We consider the production of a W-boson pair plus two neutrinos in electron–positron collisions:

$$e^+e^- \to \nu_e\bar{\nu}_eW^+W^-.$$  \hfill (2.1)

This process contains the VBS subprocess $W^+W^- \to W^+W^-$ generically described by the first Feynman diagram in Fig. 1. The WW-scattering signal is not the only contribution to the final state in (2.1). The irreducible background, exemplified by the second and third Feynman graphs in Fig. 1, is indeed sizeable and must be properly suppressed in order to enhance the VBS signal-to-background ratio. A set of appropriate kinematical cuts to be imposed is summarized below.

In our analyses, we use the input values [30]

$$M_W = 80.403 \text{ GeV}, \quad M_Z = 91.1876 \text{ GeV},$$

$$M_H = 120 \text{ GeV}, \quad m_t = 174.2 \text{ GeV},$$  \hfill (2.2)
Figure 1: Feynman diagrams contributing to the process $e^+e^- \rightarrow \nu_e \bar{\nu}_e W^+W^-$. The first graph on the left is a generic representation of the WW-scattering signal. The remaining two diagrams are examples of irreducible background.

for vector-boson, Higgs-boson, and top-quark masses. All other fermions are taken to be massless. The sine $s_W$ and cosine $c_W$ of the weak mixing angle are fixed by

$$c_W^2 = 1 - s_W^2 = \frac{M_W^2}{M_Z^2}. \quad (2.3)$$

Moreover, we adopt the so called $G_\mu$-scheme, which effectively includes higher-order contributions associated with the running of the electromagnetic coupling and the leading universal two-loop $m_t$-dependent corrections in the definition of $\alpha$. Using

$$G_\mu = 1.16637 \times 10^{-5} \text{GeV}^{-2}, \quad (2.4)$$

we have

$$\alpha = \sqrt{2} G_\mu M_W^2 s_W^2 / \pi = 1/132.38 \ldots \quad (2.5)$$

In the following sections, we present results for a CM energy $\sqrt{s} = 1 \text{TeV}$, which can be reached at the ILC, and for $\sqrt{s} = 3 \text{TeV}$ as is planned for CLIC. In both cases we assume an integrated luminosity $L = 1 \text{ab}^{-1}$.

Based on the study of Ref. [7], we have implemented a general set of cuts, proper for ILC and CLIC analyses, defined as follows. In the listed cuts, the numbers outside parentheses are used for $\sqrt{s} = 1 \text{TeV}$, those within parentheses for $\sqrt{s} = 3 \text{TeV}$.

- We require a W-boson transverse momentum $P_T(W^\pm) \geq 100(200) \text{GeV}$, since the production of longitudinal vector bosons, i.e. the VBS signal, is enhanced for large scattering angles and high energies. Moreover, this cut removes events dominated by $t$-channel photon exchange in subprocesses.

- We require $|\cos \theta(W^\pm)| \leq 0.8$, where $\theta(W^\pm)$ is the angle of the produced $W^\pm$ boson with respect to the incoming positron in the laboratory frame, since the VBS signal is characterized by central W-boson production. This cut also removes events dominated by $t$-channel photon exchange in subprocesses.
• We require $|y(W^+) - y(W^-)| \leq 2$, where $y(W^\pm)$ is the rapidity of the produced $W^\pm$ boson defined as $y = 0.5 \ln[(E + P_L)/(E - P_L)]$ and $E$ and $P_L$ are the $W$-boson energy and component of the momentum along the beam axis, respectively. This additional angular cut ensures the production of central $W$ bosons also in the CM frame of vector-boson scattering.

• We require a transverse momentum of the $W$-boson pair $P_T(WW) \geq 40(50)$ GeV. This cut suppresses the reducible background coming from the process $e^+e^- \rightarrow e^+e^-W^+W^-$, i.e. $\gamma\gamma$ fusion, when the two produced $e^\pm$ are emitted forward/backward.

• We require a neutrino-pair invariant mass $M(\nu_e\bar{\nu}_e) \geq 150(200)$ GeV. This cut removes $W^+W^-Z$ production events, in which the neutrinos come from the $Z$-boson decay (see the last graph in Fig. 1).

• We require a diboson invariant mass $M(WW) \geq 400(700)$ GeV. Selecting high diboson CM energies allows one to test a possible strongly-interacting regime of the electroweak gauge sector.

3 Strategy of the calculation

In this section, we describe the main ingredients of our calculation. We summarize the adopted approximations and discuss their domain of applicability.

We consider the process (2.1), with two on-shell $W$ bosons and two neutrinos in the final state. For this process, exact lowest-order matrix elements are employed in our Monte Carlo, simultaneously accounting for signal and irreducible background. We moreover use the complete four-particle phase space and exact kinematics.

Computing $\mathcal{O}(\alpha)$ electroweak corrections in leading-pole approximation, as in Refs. [31–33] and references therein, has revealed successful for analysing $WW$ physics at LEP2. A similar philosophy can be adopted for the incoming bosons in VBS at energies that are large compared to the gauge-boson masses, since this process is dominated by small invariant masses of these incoming bosons, which are thus relatively close to their mass shell, even though these invariant masses are actually negative.

We thus compute the $\mathcal{O}(\alpha)$ electroweak corrections to the process (2.1) in equivalent-vector-boson approximation (EVBA). As discussed in the introduction, we do not include real photonic corrections. For the virtual corrections we work in the logarithmic approximation and we restrict our calculation to the infrared-finite part coming from above the electroweak scale. These corrections correspond to the case where the photon has effectively the mass $M_W$ and are precisely defined in Ref. [12]. This approach is sensible since at high energies, the electroweak corrections are dominated by double and single logarithms of the ratio of the energy to the electroweak scale. Hence, keeping only the terms proportional to $\alpha \ln^2(\hat{s}/M_W^2)$ and $\alpha \ln(\hat{s}/M_W^2)$, where $\hat{s}$ is the CM energy of the VBS subprocess, provides the bulk of the radiative corrections.
3.1 Equivalent vector-boson approximation

In EVBA, the process $e^+e^- \rightarrow \nu_e \bar{\nu}_e W^+W^-$ is entirely described by the subset of Feynman diagrams generically represented by the first graph in Fig. 1. The approximation considers in fact only those contributions to the final state that come from the scattering of the two vector bosons emitted by the incoming particles. The goodness of the approximation thus relies on the assumption that such contributions are indeed the dominating ones in the considered kinematical domain. This hypothesis depends on the process at hand, and can only be checked against an exact computation. In this section, we discuss the reliability of the EVBA in describing the process $e^+e^- \rightarrow \nu_e \bar{\nu}_e W^+W^-$ and its validity domain.

In implementing the EVBA, we follow the approach of Ref. [29]. This method preserves the exact kinematics of the process, a very useful property for imposing realistic cuts.

We assign the following set of momenta $p_i$ and helicities $\lambda_i = 0, \pm 1$ to the particles involved in the process we are considering:

$$e^+(p_1, +) e^-(p_2, -) \rightarrow \nu_e(p_3, -) \bar{\nu}_e(p_4, +) W^+(p_5, \lambda_5) W^-(p_6, \lambda_6). \quad (3.1)$$

In EVBA only left-handed lepton chiralities are relevant, since the electrons and neutrinos couple always to $W$ bosons. Using the unitary gauge and writing the propagators of the two incoming $W$ bosons, emitted by the initial $e^\pm$ and exchanged in $t$ channel (see Fig. 1), as a sum over the vector-boson polarizations

$$\frac{1}{p^2 - M_W^2} \left( -g^{\mu\nu} + \frac{p^\mu p^\nu}{M_W^2} \right) = \sum_{\lambda = -1, 0, 1} \frac{\epsilon^\mu(p)\epsilon^\nu(p)}{p^2 - M_W^2}, \quad (3.2)$$

the exact amplitude corresponding to the first graph in Fig. 1 assumes the form

$$\mathcal{M}^{e^+e^- \rightarrow \nu_e \bar{\nu}_e W^+W^-}(p_1, p_2, p_3, p_4, p_5, p_6; \lambda_5, \lambda_6) = \frac{1}{q_+^2 - M_W^2} \frac{1}{q_-^2 - M_W^2} \times \sum_{\lambda_+ = -1, 0, 1} \mathcal{M}^{e^+ \rightarrow \nu_e W^+}(p_1, p_4, q_+; \lambda_+) \mathcal{M}^{e^- \rightarrow \nu_e W^-}(p_2, p_3, q_-; \lambda_-) \times \mathcal{M}^{W^+W^- \rightarrow W^+W^-}(q_+, q_-, p_5, p_6; \lambda_+, \lambda_-, \lambda_5, \lambda_6). \quad (3.3)$$

In EVBA the off-shell amplitude for WW scattering, $\mathcal{M}^{W^+W^- \rightarrow W^+W^-}$, is replaced by a suitably defined on-shell amplitude. In this way gauge invariance of this amplitude is ensured, and artifacts from using an incomplete off-shell amplitude are avoided. Modifications of the on-shell amplitude are necessary in order to describe the dependence of the off-shell amplitude on the off-shell masses $q_\pm^2$ to a satisfactory accuracy. We here follow the approach of Ref. [29] which describes the extrapolation to off-shell masses by simple proportionality factors for each incoming vector boson. These are chosen to be equal to 1 for transverse bosons. For each longitudinal $W^+$ or $W^-$ boson, the on-shell

\footnote{Instead, the exact matrix elements that we employ for our tree-level predictions receive (background) contributions also from right-handed leptons.}
Table 1: Exact lowest-order cross section (second column) as well as total cross section in EVBA (third column) and their difference in per cent of the exact result (fourth column). Kinematical cuts as in Sect. 2 are applied.

amplitude is multiplied by a factor $M_W/\sqrt{-q_+^2}$ in order to describe the singular behaviour $\epsilon_{\lambda=0}(q_{\pm}) \sim 1/\sqrt{-q_{\pm}^2}$ of the off-shell longitudinal polarization vectors. One can thus write

$$
\mathcal{M}_{\text{EVBA}}^{e^+e^-\to\nu\bar{\nu}W^+W^-}(p_1, p_2, p_3, p_4, p_5, p_6; \lambda_5, \lambda_6) = \frac{1}{q_+^2 - M_W^2} \frac{1}{q_-^2 - M_W^2} \sum_{\lambda_+, \lambda_- = -1, 0, 1} \mathcal{M}^{e^+\to\nu\bar{\nu}W^+}(p_1, p_4, q_+; \lambda_+) \mathcal{M}^{e^-\to\nu\bar{\nu}W^-}(p_2, p_3, p_6; \lambda_-, \lambda_5, \lambda_6) \times \left[ \frac{M_W}{\sqrt{-q_+^2}} \delta_{\lambda_+, 0} + \delta_{\lambda_+, \pm} \right] \left[ \frac{M_W}{\sqrt{-q_-^2}} \delta_{\lambda_-, 0} + \delta_{\lambda_-, \pm} \right],
$$

(3.4)

where $q_{\pm}^{on}$ are the on-shell projected momenta of the two incoming W bosons. The definition of the on-shell projection is given in Appendix A. Note that the W-boson momenta in the matrix elements $\mathcal{M}^{e^+\to\nu\bar{\nu}W^+}$ and $\mathcal{M}^{e^-\to\nu\bar{\nu}W^-}$ are not projected on shell.

In order to proceed, in Ref. [29] the squared amplitude was considered and the contributions of the matrix elements $\mathcal{M}^{e^+\to\nu\bar{\nu}W^+}$ and $\mathcal{M}^{e^-\to\nu\bar{\nu}W^-}$ were transformed into vector-boson luminosities. Instead, we work at the matrix element level and compute the three amplitudes on the right-hand side of (3.4) with the help of PHACT [34], a routine based on the helicity-amplitude method of Ref. [35].

In Table 1, we show the comparison between the EVBA and the exact lowest-order result for the total cross section. We select the kinematical domain where we expect new-physics effects related to strongly interacting vector bosons to be enhanced. Such a region, characterized by high diboson invariant masses and large scattering angles of the two produced W bosons, is selected via appropriate cuts described in detail in Sect. 2.

At $\sqrt{s} = 1\text{ TeV}$ and $\sqrt{s} = 3\text{ TeV}$ the accuracy of the EVBA for the total cross section, $\Delta_{\text{EVBA}} = (\sigma_{\text{exact}} - \sigma_{\text{EVBA}})/\sigma_{\text{exact}}$, amounts to about 20% and 1%, respectively.

The integrated cross section gives only a partial information on the goodness of the EVBA. In order to display more extensively the reliability of this approximation in the selected kinematical domain, in Figs. 2 and 3 we analyse distributions in both energy-like and angular-like variables. In particular, we consider four observables of interest:

- diboson invariant mass $M(WW)$,
Figure 2: Lowest-order distributions for $\sqrt{s} = 1$ TeV from the exact matrix elements and in EVBA: invariant mass of the diboson pair (upper left), transverse momentum of the diboson pair (upper right), rapidity of the diboson pair (lower left), transverse momentum of the produced W boson (lower right). The inset plots show the relative difference $\Delta_{\text{EVBA}}$ in per cent. Standard cuts are applied.

- diboson transverse momentum $P_T^{\text{(WW)}}$,
- diboson rapidity $y^{\text{(WW)}} = 0.5 \ln \left[ \frac{E^{\text{(WW)}} + P_T^{\text{(WW)}}}{E^{\text{(WW)}} - P_T^{\text{(WW)}}} \right]$,
- W-boson transverse momentum $P_T^{\text{(W)}} = P_T^{\text{(W}^+)}$.

The distributions in $P_T^{\text{(W}^-)}$ and $P_T^{\text{(W}^+)}$ are identical. We plot the lowest-order results of the EVBA and of the exact calculation for the two collider energies $\sqrt{s} = 1$ TeV and $\sqrt{s} = 3$ TeV in Fig. 2 and Fig. 3, respectively. The inset plots show the difference in per cent between the two results, i.e. $\Delta_{\text{EVBA}} = (d\sigma^{\text{exact}} - d\sigma^{\text{EVBA}})/d\sigma^{\text{exact}}$. For ILC and CLIC, in most of the regions that are not statistically irrelevant, the difference is below 25% and 20%, respectively. The increase for small $M^{\text{(WW)}}$ is not problematic, since the
radiative corrections are small in this region (see Sect. 4). As a remark, let us add that the agreement between EVBA and exact result reached in this analysis should not be taken as the best one can do. The quality of the approximation is sensibly cut-dependent. In this paper, following Ref. [7] we have adopted generic cuts and, in principle, the EVBA could be improved imposing more stringent constraints. The choice of the set-up depends, however, on the particular analysis to be performed.

In the following, we use the EVBA only for computing the $\mathcal{O}(\alpha)$ electroweak corrections. The lowest-order results are calculated exactly. The $\mathcal{O}(\alpha)$ inaccuracy associated with the EVBA can be estimated to be of the order of the product of the $\mathcal{O}(\alpha)$ corrections (computed in EVBA) times the inaccuracy of the EVBA at tree level. Since the radiative effects typically amount to 10–30%, the inaccuracy of the EVBA translates into a few-percent error for the full result. As shown in Sect. 4, apart for the case of very high $P_T(WW)$, the $\mathcal{O}(\alpha)$ uncertainty associated with the EVBA is always smaller than the expected statistical error at the ILC and CLIC.
3.2 Virtual electroweak corrections

In analogy with the leading-pole approximation for virtual corrections, the EVBA allows two types of radiative contributions, factorizable and non-factorizable ones. The former are those that can be associated to either the emission of one of the two incoming W bosons from the beam particles or the VBS subprocess. The latter are those connecting these subprocesses.

The non-factorizable corrections in leading-pole approximation consist only of photonic contributions. These corrections have been evaluated for W-boson pair production in $e^+e^-$ annihilation in Refs. [36, 37]. There it was found that all infrared and mass-singular logarithms cancel between virtual and real corrections and that the remaining effects are small. In EVBA non-factorizable corrections do not only result from photon exchange between the subprocesses of W-boson-production and WW scattering but also from analogous exchanges of massive gauge bosons. These contributions deserve further investigations. In this paper, we do not consider the non-factorizable corrections but restrict ourselves to the calculation of the factorizable contributions.

The virtual factorizable corrections are represented by the schematic diagram of Fig. 4, in which the big blobs contain all one-loop corrections to the incoming W-boson-production and on-shell WW-scattering subprocesses. The corresponding matrix element can be written as

$$
\delta M_{\text{virt, EVBA, fact}}^{e^+e^-\to \nu_\ell \bar{\nu}_\ell W^+_{\lambda^+}W^-_{\lambda^-}} = \frac{1}{q_+^2 - M_W^2} \frac{1}{q_-^2 - M_W^2} \times \sum_{\lambda^+, \lambda^-} \left\{ \delta M_{\text{virt}}^{e^+\to \nu_\ell W^+_{\lambda^+}} M_{\text{Born}}^{e^-\to \nu_\ell W^-_{\lambda^-}} M_{\text{Born, on}}^{W^+_{\lambda^+}W^-_{\lambda^-}\to W^+_{\lambda^+}W^-_{\lambda^-}} \right\}
$$
The produced gauge bosons should be energetic and emitted at sufficiently large angles with respect to the beam. This is precisely the kinematical region where effects due to a possible strongly interacting regime of the gauge sector are maximally enhanced. In this region, the accuracy of the logarithmic high-energy approximation is expected to be of the order of a few per cent. We can thus reasonably adopt this approximation at \( e^+e^- \) colliders with energy in the 1–3 TeV range, where the experimental error is at the few-per-cent level. Since the emission subprocesses of the two incoming \( W \) bosons involve no large energy variable (they peak at \( s \), \( q^2 \), and \( M_W^2 \)), the corresponding virtual corrections vanish in the logarithmic approximation. As a consequence, we do not consider the first two contributions on the right-hand side of (3.5) in the following. Moreover, for the \( W^+W^- 
leftrightarrow W^+W^- \) subprocess we take into account only the corrections to the matrix elements for the \( W \)-boson-production and on-shell \( WW \)-scattering subprocesses. The index ‘on’ indicates that on-shell vector-boson momenta are used to calculate these matrix elements.

We calculate the factorizable \( \mathcal{O}(\alpha) \) virtual corrections in logarithmic high-energy approximation including single and double enhanced logarithms, i.e. contributions proportional to \( \alpha \ln^2(s/M_W^2) \) and \( \alpha \ln(s/M_W^2) \), where \( s \) is the CM energy of the scattering subprocess. The logarithmic approximation yields the dominant corrections as long as CM energies and scattering angles are large. Pure angular-dependent logarithms of the form \( \alpha \ln^2(s/\hat{r}) \) and \( \alpha \ln(s/\hat{r}) \), with \( \hat{r} \) equal to the Mandelstam variables \( \hat{t} \) and \( \hat{u} \) of the \( WW \)-scattering subprocess, are not included. However, angular-dependent terms of the form \( \alpha \ln(s/\hat{r}) \ln(s/M_W^2) \) are taken into account. The validity of the results relies therefore on the assumption that all invariants are large compared with \( M_W^2 \) and approximately of the same size

\[
\hat{s} \sim |\hat{t}| \sim |\hat{u}| \gg M_W^2. \tag{3.6}
\]

This implies that the produced gauge bosons should be energetic and emitted at sufficiently large angles with respect to the beam. This is precisely the kinematical region where effects due to a possible strongly interacting regime of the gauge sector are maximally enhanced. In this region, the accuracy of the logarithmic high-energy approximation is expected to be of the order of a few per cent. We can thus reasonably adopt this approximation at \( e^+e^- \) colliders with energy in the 1–3 TeV range, where the experimental error is at the few-per-cent level. Since the emission subprocesses of the two incoming \( W \) bosons involve no large energy variable (they peak at \( s/\hat{r} \), \( q^2 \)), the corresponding virtual corrections vanish in the logarithmic approximation. As a consequence, we do not consider the first two contributions on the right-hand side of (3.5) in the following. Moreover, for the \( W^+W^- \to W^+W^- \) subprocess we take into account only the corrections to the dominating channels involving four transverse (TTTT) or two transverse and two longitudinal (LLTT, LTLT, TLTL) gauge bosons. The contributions of the channels with an odd number of longitudinally polarized \( W \) bosons are suppressed by \( M_W/\sqrt{s} \), and those of the channels LTTL and TLLT by \( M_W^2/\hat{s} \). Moreover, the configurations with two final-state longitudinal \( W \) bosons are numerically small within the SM with a light Higgs boson. As shown in Table 2, for \( M_H = 120 \) GeV, the cross section for the production of two longitudinal \( W \) bosons is suppressed by a factor 50 or 100 compared to the full result for \( \sqrt{s} = 1 \) TeV or 3 TeV, respectively.

The analytical expressions for the \( \mathcal{O}(\alpha) \) virtual corrections to \( WW \) scattering in the high-energy limit are given in Appendix B. The formulas are rather compact and easy to implement. In our default set-up, we have used the version with exact SU(2)-transformed

\[
\begin{align*}
+ M_{\text{Born}}^{e^+\rightarrow W^+_\lambda \nu_e} \delta M_{\text{virt}}^{e^-\rightarrow W^-\lambda \nu_e} & \cdot M_{\text{Born, on}}^{W^+_\lambda \nu_e \rightarrow W^+_\lambda \nu_e} \\
+ M_{\text{Born}}^{e^+\rightarrow W^+_\lambda \nu_e} \delta M_{\text{virt}}^{e^-\rightarrow W^-\lambda \nu_e} & \cdot M_{\text{Born, on}}^{W^+_\lambda \nu_e \rightarrow W^+_\lambda \nu_e} \\
\times \left[ \frac{M_W}{\sqrt{-q^2}} \delta_{\lambda+,0} + \delta_{\lambda+,\pm} \right] \left[ \frac{M_W}{\sqrt{-q^2}} \delta_{\lambda-,0} + \delta_{\lambda-,\pm} \right], \tag{3.5}
\end{align*}
\]

where \( \delta M_{\text{virt}}^{e^+\rightarrow W^+_\lambda \nu_e} \), \( \delta M_{\text{virt}}^{e^-\rightarrow W^-\lambda \nu_e} \), and \( \delta M_{\text{virt, on}}^{W^+_\lambda \nu_e \rightarrow W^+_\lambda \nu_e} \) denote the virtual corrections to the matrix elements for the \( W \)-boson-production and on-shell \( WW \)-scattering subprocesses. The index ‘on’ indicates that on-shell vector-boson momenta are used to calculate these matrix elements.
Table 2: Born cross section for the process \( e^+e^- \rightarrow \nu_e\bar{\nu}_e W^+_\lambda W^-_{\lambda'} \) and various polarizations \((\lambda, \lambda' = T, L)\) of the produced W bosons. Kinematical cuts as specified in Sect. 2 are applied.

| \( \sqrt{s} \) [TeV] | \( \sigma^{TT} \) [fb] | \( \sigma^{TL} \) [fb] | \( \sigma^{LT} \) [fb] | \( \sigma^{LL} \) [fb] | \( \sigma^{tot} \) [fb] |
|------------------------|------------------|------------------|------------------|------------------|------------------|
| 1                      | 0.500            | 0.0410           | 0.0410           | 0.0134           | 0.595            |
| 3                      | 3.111            | 0.1786           | 0.1786           | 0.0390           | 3.507            |

The terms, omitted in the logarithmic approximation might be, however, not negligible. Figure 5 shows indeed that the agreement between exact and approximate \( \mathcal{O}(\alpha) \) cross section is within a few per cent for the first two polarization configurations, while it goes up to about 13% for the third one. This discrepancy is unexpectedly large. It could be understood by performing a complete high-energy approximation including non-logarithmic terms. However, for the unpolarized cross section this discrepancy is still tolerable. The lowest-order cross section is in fact dominated by the \( TTTT \) configuration, and receives only a 10% contribution from \( TLTL \), as shown in Table 2. We can thus safely assume our \( \mathcal{O}(\alpha) \) inaccuracy to be of the order of a few per cent.
Figure 5: WW-scattering subprocess $W^+ W^- \rightarrow W^+ W^-$. Difference between complete $\mathcal{O}(\alpha)$ cross section ($\sigma^{\text{Full}}$) and $\mathcal{O}(\alpha)$ cross section in logarithmic high-energy approximation ($\sigma^{\text{HE}}$), normalized to the lowest-order result. In the legend, T and L refer to the transverse and longitudinal polarizations of the two incoming and two outgoing W bosons from left to right.

4 Numerical results

In this section, we illustrate the effect of the logarithmic electroweak corrections on the production of a $W^+ W^-$ pair plus missing energy at future $e^+ e^-$ colliders. We consider the process (2.1) in the numerical setup given in Sect. 2. We analyse the behaviour of the VBS included in (2.1) over the kinematical region characterized by large diboson invariant masses and large scattering angles of the produced W bosons. This is the domain where possible new-physics effects entering the VBS would be maximally enhanced.

We focus on the SM predictions for a light Higgs boson. As pointed out by Bagger et al. [4], in this case one would observe the production of mostly transversely polarized W bosons in the high WW invariant-mass region. The longitudinal spin configuration is in fact strongly suppressed in the presence of a light Higgs boson. For the specific case at hand, this is shown in Table 2 for $M_H = 120$ GeV and two possible setups.

In Table 3, we show the impact of the corrections on the total cross section for $e^+ e^- \rightarrow \nu_e \bar{\nu}_e W^+ W^-$ and compare it with the expected statistical error. The second column contains the lowest-order result. The third and fourth entries respectively display the $\mathcal{O}(\alpha)$-corrected cross section and the contribution of the one-loop corrections relative to the Born result, $\Delta_{\text{EW}} = (\sigma - \sigma_{\text{Born}})/\sigma_{\text{Born}}$. The fifth column provides an estimate of the
\[ \sigma(e^+e^- \rightarrow \nu_e \bar{\nu}_e W^+W^-) \]

| \( \sqrt{s} [\text{TeV}] \) | \( \sigma_{\text{Born}} \) [fb] | \( \sigma \) [fb] | \( \Delta_{\text{EW}} \) [%] | \( \Delta \) [%] | \( 1/\sqrt{L}\sigma_{\text{Born}} \) [%] |
|-----------------|-------------|-------------|-----------------|--------|-----------------|
| 1               | 0.595       | 0.556       | -6.7            | 1.3    | 4.1             |
| 3               | 3.507       | 2.897       | -17.4           | 0.2    | 1.7             |

Table 3: Total lowest-order cross section (second column) as well as total \( \mathcal{O}(\alpha) \) cross section (third column) and electroweak corrections in per cent of the lowest-order result (fourth column), including their uncertainty (fifth column). The last entry shows the statistical error for an integrated luminosity \( L = 1 \text{ ab}^{-1} \). Kinematical cuts as in Sect. 2 are applied.

uncertainty of the one-loop contributions due to the EVBA, \( \Delta = \Delta_{\text{EW}} \times \Delta_{\text{EVBA}} \), which is obtained combining the one-loop corrections with the uncertainty of the EVBA at Born level. For the considered process, the electroweak radiative effects are negative and of the order of \(-5\%\) to \(-20\%\). This has to be compared with the last column of Table 3, where we show an estimate of the statistical error based on an integrated luminosity \( L = 1 \text{ ab}^{-1} \). As can be seen, the electroweak corrections are quite important. Already comparable with the statistical uncertainty at the ILC, they further increase at higher energies giving rise to a 10\( \sigma \) effect at CLIC.

The influence of the \( \mathcal{O}(\alpha) \) corrections is highly dependent on the cuts imposed and the selected kinematical domain. Also, distributions can be differently affected by the radiative corrections. We illustrate this point for the sample variables defined and analysed in Sect. 3.1. The \( \mathcal{O}(\alpha) \) effects on these four observables at \( \sqrt{s} = 1 \text{ TeV} \) and 3 TeV are displayed in Fig. 6 and Fig. 7, respectively. The upper curves represent the lowest-order differential cross section, the lower ones the corresponding corrected result.

We first consider the case of \( \sqrt{s} = 1 \text{ TeV} \). In the left-upper plot of Fig. 6, we show the distribution in the diboson invariant mass. This variable, representing the CM energy of the VBS subprocess, gives direct access to the energy scale at which new physics could appear. In absence of a light Higgs boson, the SM VBS amplitudes would violate perturbative unitarity at high CM energies. In order to recover it, new physics should manifest itself at those scales. Hence, at future colliders it will be useful to analyse the diboson production (plus missing energy) at the highest possible \( M(WW) \) values. In this region, the \( \mathcal{O}(\alpha) \) corrections are enhanced. They can go up to \(-18\%\), as shown by the corresponding inset plot. The increase of the radiative effects with \( M(WW) \) is a typical effect of large logarithms of Sudakov type. The electroweak corrections should therefore be included to match the experimental accuracy.

A second variable of interest, \( P_T(WW) \), combines energy and angle information. This observable is expected to be sensitive to a strongly interacting VBS signal at low and intermediate values, say for \( P_T(WW) \) between 50 and 300 GeV [7]. In this region, the radiative effects are of order \(-5\%\) to \(-7\%\), thus smaller than in the case discussed above. Their size is, however, comparable with the statistical accuracy in the considered range.
The same conclusion holds for the pure angular-like variable, $y(WW)$, shown in the left-lower plot, which gets corrections in the range between $-7\%$ and $-10\%$. The transverse momentum $P_T(W)$ on the right-lower plot displays instead a behaviour analogous to $M(WW)$. Note that for all considered distributions the corrections reduce the tree-level SM predictions.

In Fig. 7, we show the same set of distributions as above for $\sqrt{s} = 3 \, \text{TeV}$ and the corresponding cuts specified in Sect. 2. As expected the impact of the radiative corrections increases with the collider energy since this allows for higher CM energies, which translate into higher diboson invariant masses and transverse momenta. This behaviour is well depicted in all four plots. For the distributions in $P_T(WW)$ and $y(WW)$ the corrections...
Figure 7: Distributions for 3 TeV in lowest order and including logarithmic corrections. Same conventions as in Fig. 6.

are in the range between $-10\%$ and $-20\%$, apart from the region of very high $P_T(WW)$ where statistics is small. For the $M(WW)$ distribution the electroweak corrections grow from $-10\%$ to $-50\%$ with increasing invariant mass and for the $P_T(W)$ distribution the effect is similar. For the distribution in $y(WW)$, the corrections are large at the maximum of the distribution at small $y(WW)$. For the other distributions, the cross sections get small where the corrections get large. Still, the $O(\alpha)$ corrections are statistically relevant. This is illustrated in Table 4, where the size of the electroweak corrections $\Delta_{EW}$ and the $O(\alpha)$ uncertainty $\Delta = \Delta_{EW} \times \Delta_{EVBA}$, which results from the EVBA, are compared with the statistical accuracy $\Delta_{stat}$ based on the projected luminosity $L = 1 \text{ ab}^{-1}$. Dividing the range of energy-like variables in 200 GeV intervals, we find that the influence of the $O(\alpha)$ corrections is not washed out by the binning. In the more statistically relevant bins, the radiative corrections ranging from $-10\%$ to $-30\%$ give rise to a 3–6$\sigma$ effect. In the tails of the distributions, the $O(\alpha)$ contributions can increase up to $-50\%$, still being bigger than the estimated experimental accuracy. We also observe that, apart from the region of very
Table 4: Impact of electroweak corrections on binned distributions at $\sqrt{s} = 3$ TeV. The first and sixth columns show the bin. For each variable, the four entries from left to right give the total number of events for a luminosity $L = 1$ ab$^{-1}$, the corresponding statistical accuracy, the size of the $O(\alpha)$ electroweak corrections relative to the Born result, and the estimated one-loop uncertainty due to the EVBA.

| bin | $M(WW)$ | bin | $P_T(WW)$ |
|-----|---------|-----|-----------|
| [GeV] | $N_{evt}$ | $\Delta_{stat} [%]$ | $\Delta_{EW} [%]$ | $\Delta [%]$ | [GeV] | $N_{evt}$ | $\Delta_{stat} [%]$ | $\Delta_{EW} [%]$ | $\Delta [%]$ |
| 700–900 | 1719 | 2.4 | -10.4 | 0.7 | 50–250 | 1467 | 2.6 | -15.9 | 0.7 |
| 900–1100 | 916 | 3.3 | -18.2 | 1.2 | 250–450 | 1119 | 3.0 | -16.6 | 2.0 |
| 1100–1300 | 446 | 4.7 | -25.1 | 1.8 | 450–650 | 550 | 4.3 | -17.0 | 0.5 |
| 1300–1500 | 216 | 6.8 | -30.4 | 0.7 | 650–850 | 249 | 6.3 | -21.4 | 6.5 |
| 1500–1700 | 106 | 9.7 | -36.8 | 0.8 | 850–1050 | 92 | 10.4 | -32.6 | 24.9 |
| 1700–1900 | 53 | 13.7 | -43.3 | 1.5 | 250–450 | 1119 | 2.6 | -15.9 | 0.7 |
| 1900–2100 | 26 | 19.4 | -49.5 | 2.2 | 450–650 | 550 | 4.3 | -17.0 | 0.5 |

In Table 5 we give analogous results for $\sqrt{s} = 1$ TeV. In this case, the radiative effects are comparable with the statistical accuracy. Their significance is strictly dependent on the binning.

In our analysis, we have not included any option on the polarization of the initial beams. In the process $e^+e^- \rightarrow \nu_\ell \bar{\nu}_e W^+ W^-$, the VBS signal is purely given by the $W$-boson scattering, as illustrated in the first Feynman diagram of Fig. 1. Only the right–left combination $e^+ e^- \rightarrow \nu_\ell \bar{\nu}_e W^+ W^-$ thus contributes to the signal. The irreducible background (see for instance the last diagram in Fig. 1) can receive instead contributions also from other helicity configurations. The possibility of selecting a given initial polarization has thus two advantages. In first place, it helps in suppressing the background. As a second benefit, it increases the statistics. Assuming a polarization efficiency of 80% and 60% for electron and positron, respectively, we would have in fact an increase of about a factor two in the high $P_T(WW)$, the uncertainty associated with the EVBA never exceeds the statistical error.
Table 5: Impact of electroweak corrections on binned distributions at $\sqrt{s} = 1$ TeV. Same conventions as in Table 4.

number of events at fixed luminosity. In this set-up, the relevance of the radiative effects would be further enhanced.

5 Conclusions

If the Higgs boson should be heavy or absent, the scattering of longitudinal electroweak gauge bosons can provide information on the mechanism of electroweak symmetry breaking. An irreducible background to signals of new physics in vector-boson scattering is provided by the Standard Model contribution for a light Higgs boson. In order to be able to disentangle possible small new-physics effects a precise knowledge of the latter is required.

We have studied electroweak radiative corrections to $W^+W^-$ scattering at high-energy $e^+e^-$ colliders. We have used the equivalent vector-boson approximation and calculated the factorizable one-loop corrections in the high-energy logarithmic approximation. Corrections to the splitting of the $W$ bosons from the incoming particles as well as non-factorizable corrections have not been taken into account. We have presented explicit analytical results for the logarithmic corrections to $W^+W^-$ together with the complete lowest-order matrix elements.

We have defined a set of cuts suitable for the analysis of the scattering of strongly interacting $W$ bosons and investigated our approximations within this setup. In kinematical regions that are statistically relevant and receive non-negligible corrections, we find that the effective vector-boson approximation agrees with the complete lowest-order
prediction within about 20% to 25%. The logarithmic approximation reproduces the complete one-loop corrections for the unpolarized cross section of $W^+W^- \rightarrow W^+W^-$ at the level of a few per cent. These approximations cause an uncertainty of our predictions at the level of a few per cent. This uncertainty is always comparable to the statistical error. In principle it can be further reduced by introducing appropriate cuts that improve the accuracy of the EVBA. But, in general, for a given set of cuts, a better precision can be achieved only by means of an exact calculation.

The size of the electroweak corrections depends strongly on the cuts and the considered observable. We have studied their effect on the total cross section and four physically interesting distributions. Within our set-up, the corrections to the total cross section amount to $-7\%$ and $-17\%$ for $\sqrt{s} = 1\text{ TeV}$ and $\sqrt{s} = 3\text{ TeV}$, respectively. For the distributions in the transverse momentum and the rapidity of the W-boson pair they are of similar size. In the distributions in the invariant mass of the W-boson pair and the transverse momentum of a W boson, the corrections are negative, of the order of 10\% and increase in magnitude with increasing energy. They can reach up to $-20\%$ and $-50\%$ for $\sqrt{s} = 1\text{ TeV}$ and $\sqrt{s} = 3\text{ TeV}$, respectively. In summary, the electroweak corrections reduce the Standard Model predictions by a sizeable amount that is comparable or larger than the expected statistical error. Therefore, they should be taken into account when searching for effects of a strongly interacting scalar sector. In fact, being negative, the corrections increase the sensitivity to this kind of effects, which typically appear as an enhancement of the WW-scattering cross section.

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**A On-shell projection**

In this section we give the explicit form of the on-shell projection of the two incoming W bosons which induce the VBS. When performing the projection, care should be taken that the on-shell projected W-boson momenta lie in the physical phase-space region. This can be ensured by fixing the angles of the two incoming bosons while performing their on-shell limit. We thus fix the direction of the incoming $W^+$ in the CM frame of the W-boson pair. In this way, the on-shell projected momenta can be written as

$$\hat{q}_\pm^{\text{on}} = \frac{\sqrt{s}}{2} \left( 1, \pm \beta \frac{\hat{q}_+}{|\hat{q}_+|} \right),$$

where $\sqrt{s}$ is the VBS CM energy, $\beta = \sqrt{1 - 4M_W^2/s}$, and $\hat{q}_+$ is the three-momentum of the incoming (off-shell ) $W^+$ boson in the CM frame of the W-boson pair. The on-shell momenta are then boosted to the laboratory frame.
B Logarithmic electroweak corrections

In this appendix, using the general results of Refs. [12, 14], we derive analytical formulas for the logarithmic electroweak corrections to the subprocess

$$W_{\lambda_1}^{-}(k_1)W_{\lambda_2}^{+}(k_2) \rightarrow W_{-\lambda_3}^{-}(-k_3)W_{-\lambda_4}^{+}(-k_4). \quad (B.1)$$

The momenta $k_3, k_4$ and the helicities $\lambda_3, \lambda_4$ of the final states are defined as incoming. The $2 \rightarrow 2$ process (B.1) is thus equivalent to the $4 \rightarrow 0$ process

$$W_{\lambda_1}^{-}(k_1)W_{\lambda_2}^{+}(k_2)W_{\lambda_3}^{-}(k_3)W_{\lambda_4}^{+}(k_4) \rightarrow 0. \quad (B.2)$$

This convention facilitates the application of the formalism of Refs. [12, 14], which is based on $n \rightarrow 0$ reactions. We consider the limit of high energies and large scattering angles, where all invariants are much larger than the electroweak scale,

$$|r_{ij}| = |(k_i + k_j)^2| \simeq |2k_ik_j| \gg M_W^2 \quad \text{for} \quad i \neq j. \quad (B.3)$$

In order to specify our conventions for the gauge-boson helicities, which we denote by $\lambda_i = 0, \pm 1$, we choose the CM frame. There, the gauge-boson momenta can be parametrized as

$$k_1^\mu = E(1, 0, 0, 1), \quad -k_3^\mu = E(1, \sin \vartheta, 0, \cos \vartheta),$$
$$k_2^\mu = E(1, 0, 0, -1), \quad -k_4^\mu = E(1, -\sin \vartheta, 0, -\cos \vartheta), \quad (B.4)$$

and for the Mandelstam variables we have $r_{12} = r_{34} = 4E^2$, $r_{13} = r_{24} = -r_{12}(1 - \cos \vartheta)/2$, $r_{23} = r_{14} = -r_{12}(1 + \cos \vartheta)/2$. Note that mass terms are systematically neglected in the high-energy limit. The polarization vectors for transverse gauge bosons ($\lambda_i = \tau_i = \pm 1$) read

$$\varepsilon^\mu(k_1, \tau_1) = \frac{1}{\sqrt{2}}(0, 1, \tau_1i, 0), \quad \varepsilon^\mu(-k_3, -\tau_3) = \frac{1}{\sqrt{2}}(0, \cos \vartheta, \tau_3i, -\sin \vartheta),$$
$$\varepsilon^\mu(k_2, \tau_2) = \frac{1}{\sqrt{2}}(0, -1, \tau_2i, 0), \quad \varepsilon^\mu(-k_4, -\tau_4) = \frac{1}{\sqrt{2}}(0, -\cos \vartheta, \tau_4i, \sin \vartheta). \quad (B.5)$$

Here and in the following, the symbol $\tau_i = \pm 1$ is used to denote the helicity of transversely polarized gauge bosons. Longitudinal gauge bosons ($\lambda_i = 0$) have to be related to corresponding would-be Goldstone bosons using the Goldstone Boson Equivalence Theorem (GBET) as discussed in Appendix C. There, we also introduce effective couplings for longitudinal gauge bosons, which permit to apply the general results of Ref. [12] directly to physical matrix elements.

The matrix element for the process (B.1) or, equivalently, (B.2) is denoted as

$$\mathcal{M}^{W_{-\lambda_1}^{-}W_{+\lambda_2}^{+}W_{-\lambda_3}^{-}W_{-\lambda_4}^{+}} \equiv \mathcal{M}^{W_{-\lambda_1}^{-}W_{+\lambda_2}^{+}W_{+\lambda_3}^{+}W_{-\lambda_4}^{+}}(r_{12}, r_{13}, r_{23}). \quad (B.6)$$

In the high-energy limit, we restrict ourselves to the matrix elements that are not mass-suppressed by factors of order $M_W/\sqrt{r_{12}}$. By means of the GBET, it can easily be seen that only the matrix elements involving helicity combinations with an even number of
longitudinally (L) and transversely (T) polarized gauge bosons are not suppressed. The reason is that all vertices involving an odd number of Goldstone bosons are suppressed by coupling factors proportional to masses. In addition, the matrix elements with helicities TL → LT and LT → TL are suppressed, i.e.

\[ \mathcal{M}^{W_1^+ W_2^+ W_3^+ W_4^-} = \mathcal{M}^{W_1^- W_2^- W_3^- W_4^+} = 0, \]  

up to terms of order \( M_W^2/r_{12} \). Therefore, in the following we restrict ourselves to the non-suppressed combinations

\[
\begin{align*}
\text{LL} \rightarrow \text{LL} & : W_0^- W_1^+ \rightarrow W_0^- W_1^+, \\
\text{TT} \rightarrow \text{LL} & : W_2^- W_3^+ \rightarrow W_2^- W_3^+, \\
\text{LL} \rightarrow \text{TT} & : W_0^- W_1^+ \rightarrow W_{-\tau_3}^- W_{-\tau_4}^+, \\
\text{TL} \rightarrow \text{LT} & : W_{-\tau_1}^- W_2^+ \rightarrow W_{-\tau_3}^- W_0^+, \\
\text{LT} \rightarrow \text{LT} & : W_0^- W_{-\tau_2}^+ \rightarrow W_0^- W_{-\tau_4}^+, \\
\text{TT} \rightarrow \text{TT} & : W_{-\tau_1}^- W_{-\tau_2}^+ \rightarrow W_{-\tau_3}^- W_{-\tau_4}^+.
\end{align*}
\]

The amplitudes for LL → TT, TL → TL, and LT → LT can be obtained from the amplitude TT → LL using the relations

\[
\begin{align*}
\mathcal{M}^{W_0^- W_1^+ W_2^+ W_3^- W_4^-} &= \mathcal{M}^{W_{-\tau_4}^- W_{-\tau_3}^+ W_0^+ W_1^-}, \\
\mathcal{M}^{W_{-\tau_1}^- W_0^+ W_2^+ W_3^- W_4^-} &= \mathcal{M}^{W_{-\tau_4}^- W_{-\tau_3}^+ W_0^+ W_1^-} \bigg|_{r_{12} \leftrightarrow r_{13}}, \\
\mathcal{M}^{W_0^- W_1^+ W_2^+ W_3^- W_4^-} &= \mathcal{M}^{W_{-\tau_4}^- W_{-\tau_3}^+ W_0^+ W_1^-} \bigg|_{r_{12} \leftrightarrow r_{13}},
\end{align*}
\]

which follow from crossing symmetry.

### B.1 Structure of the one-loop logarithmic corrections

In the following sections, our results for the one-loop logarithmic corrections are given either in explicit form, or as correction factors

\[ \delta_{W_{\lambda_1}^- W_{\lambda_2}^+ \rightarrow W_{-\lambda_3}^- W_{-\lambda_4}^+} = \frac{\delta \mathcal{M}_{W_{\lambda_1}^- W_{\lambda_2}^+ W_{\lambda_3}^+ W_{\lambda_4}^-}}{\mathcal{M}_{0}} \]  

relative to the Born matrix elements. Following Ref. [12], we split the logarithmic corrections (B.10) according to their origin as

\[ \delta = \delta^{\text{LSC}} + \delta^{\text{SSC}} + \delta^{\text{C}} + \delta^{\text{PR}}. \]

The double logarithms originating from soft-collinear gauge bosons are split into leading contributions \( \delta^{\text{LSC}} \) of the type \( \alpha \ln^2 (|r_{ij}|/M^2) \), and subleading contributions \( \delta^{\text{SSC}} \) of the type \( \alpha \ln (|r_{ij}|/M^2) \ln (|r_{ij}|/|r_{12}|) \), with \( r_{ij} = r_{13}, r_{23} \). These latter depend on ratios of Mandelstam variables, \( |r_{13}/r_{12}| = (1 - \cos \vartheta)/2, |r_{23}/r_{12}| = (1 + \cos \vartheta)/2 \), and thus on the scattering angle \( \vartheta \) between initial and final states. Purely angular-dependent logarithms
of the type $\alpha \ln^2(|r_{ij}/r_{12}|)$ and $\alpha \ln (|r_{ij}/r_{12}|)$ are neglected in our approximation. The part $\delta^C$ contains the single-logarithmic contributions from collinear (or soft) particles to loop diagrams and wave-function renormalization constants. Finally, $\delta^{PR}$ consists of the single logarithms that originate from parameter renormalization.

All these logarithmic contributions depend on various mass scales $M=M_W, M_Z, M_\gamma, M_H, m_t$. This mass dependence is separated from the energy dependence by writing

$$\ln \left( \frac{|r_{ij}|}{M^2} \right) = \ln \left( \frac{|r_{ij}|}{M_W^2} \right) - \ln \left( \frac{M^2}{M_W^2} \right),$$

i.e. we split all logarithms into a contribution with scale $M_W$ and a remaining part that depends on the ratio $M^2/M_W^2$. The logarithms of $m_t/M_W$ and $M_H/M_W$ can be found in Ref. [14]. Here we give all logarithms of $m_t/M_W$ as well as the universal logarithms of $M_H/M_W$, i.e. those that originate from parameter renormalization and collinear singularities. Also the subleading logarithms of the type $\alpha \ln (|r_{12}|/M_W^2) \ln (M_Z^2/M_W^2)$ are included.

In Ref. [12], the virtual electromagnetic corrections have been regularized by an infinitesimal photon mass $M_\gamma = \lambda$ and split as in (B.12) into a contribution corresponding to a heavy photon ($\lambda = M_W$) and a remaining part which originates from the mass gap $\lambda \ll M_W$ in the gauge sector. The heavy-photon contribution has been combined with the weak corrections resulting into the so-called symmetric-electroweak part of the corrections. The remaining mass-gap contribution has been isolated into the infrared-divergent logarithms $L_{em}(s, \lambda^2, m_k^2)$, $l(M_W^2, \lambda^2)$, $l_{em}(m_k^2)$ which appear in Eqs. (3.7), (3.8), (3.10), (3.12), (4.6), (4.7), (4.10) and (4.33) of Ref. [12]. These logarithms contain only contributions from virtual photons.

In Appendix E we provide simple substitutions that permit to generalize the results of Ref. [12] to semi-inclusive $2 \to 2$ processes, by including the soft-photon bremsstrahlung corrections. The resulting logarithms, defined in (E.1), (E.2), and (E.3), are infrared finite and depend on the soft-photon cutoff $\Delta E$.

**Notation**

The coefficients of the various logarithms are expressed in terms of the eigenvalues $I^{V^a}_{\psi}$, or of the matrix components $I^{V^a}_{\psi\psi'}$, of the generators$^2$

$$I^A = -Q = -\frac{Y}{2} - T^3, \quad I^Z = -\frac{S_W Y}{2} + \frac{C_W Y}{s_W} T^3, \quad I^{W^\pm} = \frac{T^1 \pm i T^2}{\sqrt{2} s_W},$$

where $c_w^2 = 1 - s_w^2 = M_W^2/M_Z^2$. Another group-theoretical object that often appears in our results is the electroweak Casimir operator

$$C^{ew} = \sum_{V^a=A,Z,W^\pm} I^{V^a} I^{\bar{V}^a} = \frac{1}{c_w^2} \left( \frac{Y}{2} \right)^2 + \frac{1}{s_w^2} T(T + 1),$$

where $T$ represents the total isospin.

$^2$A detailed list of the gauge-group generators and of related quantities that are used in the following can be found in App. B of Ref. [12] and App. B of Ref. [14].

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B.2 Leading soft–collinear corrections

Below we list the angular-independent leading soft–collinear (LSC) corrections for various polarizations of the gauge bosons. These results are obtained from Eqs. (3.6) and (3.7) of Ref. [12] and depend on the eigenvalues of the electroweak Casimir operator

\[ C^\text{ew}_\Phi = \frac{1 + 2c^2_W}{4s^2_Wc^2_W}, \quad C^\text{ew}_W = \frac{2}{s^2_W}, \tag{B.15} \]

as well as on the squared Z-boson couplings

\[ (I^Z_{W\pm})^2 = \frac{c^2_W}{s^2_W}, \quad (I^Z_{\phi\pm})^2 = \frac{(c^2_W - s^2_W)^2}{4s^2_Wc^2_W}. \tag{B.16} \]

The explicit expressions for the electromagnetic logarithms \( L^\text{EM}(M^2_W) \), which contain contributions from both virtual and real soft photons, are given in (E.1).

Purely longitudinal polarizations

\[
\delta^\text{LSC}_{W^+_0W^-_0\rightarrow W^-_0W^+_0} = -\frac{\alpha}{2\pi} \left[ C^\text{ew}_\Phi \ln^2 \left( \frac{|r_{12}|}{M^2_W} \right) - 2 \left( I^Z_{W\pm} \right)^2 \ln \left( \frac{M^2_Z}{M^2_W} \right) \ln \left( \frac{|r_{12}|}{M^2_W} \right) \right] \\
- 2L^\text{EM}(M^2_W). \tag{B.17} \]

Mixed polarizations

\[
\delta^\text{LSC}_{W^+_{\tau_1}W^-_{\tau_2}\rightarrow W^-_{\tau_2}W^+_{\tau_1}} = \delta^\text{LSC}_{W^-_{\tau_1}W^-_{\tau_2}W^+_{\tau_3}W^-_{\tau_4}} = \delta^\text{LSC}_{W^-_{\tau_1}W^+_{\tau_2}W^-_{\tau_3}W^+_{\tau_4}} = \delta^\text{LSC}_{W^+_{\tau_1}W^-_{\tau_2}W^-_{\tau_3}W^+_{\tau_4}} = \\
-\frac{\alpha}{4\pi} \left[ (C^\text{ew}_W + C^\text{ew}_\Phi) \ln^2 \left( \frac{|r_{12}|}{M^2_W} \right) \right] \\
- 2 \left( I^Z_{W\pm} \right)^2 \ln \left( \frac{M^2_Z}{M^2_W} \right) \ln \left( \frac{|r_{12}|}{M^2_W} \right) \right] - 2L^\text{EM}(M^2_W). \tag{B.18} \]

Purely transverse polarizations

\[
\delta^\text{LSC}_{W^+_{\tau_1}W^-_{\tau_2}W^-_{\tau_3}W^+_{\tau_4}} = -\frac{\alpha}{2\pi} \left[ C^\text{ew}_W \ln^2 \left( \frac{|r_{12}|}{M^2_W} \right) - 2 \left( I^Z_{W\pm} \right)^2 \ln \left( \frac{M^2_Z}{M^2_W} \right) \ln \left( \frac{|r_{12}|}{M^2_W} \right) \right] \\
- 2L^\text{EM}(M^2_W). \tag{B.19} \]

B.3 Subleading soft–collinear corrections

The angular-dependent subleading soft–collinear (SSC) corrections to the \( 2 \rightarrow 2 \) processes (B.8) are obtained by applying the formula (3.12) of Ref. [12] [see also (C.2)] to the corresponding \( 4 \rightarrow 0 \) processes which result from reversing the outgoing particles by a crossing transformation, i.e. the charges of the outgoing states have to be reversed as in (B.6). The corrections to matrix elements involving longitudinal gauge bosons are
The contribution (B.20) of soft neutral gauge bosons gives purely longitudinal polarizations for \( TT \rightarrow \gamma \gamma \) invariant with respect to simultaneous exchange of the polarization \( s \) (B.8). Note that, owing to crossing symmetry, both contributions (B.20) and (B.21) are evaluated in the high-energy limit in Appendix D.

The SSC corrections originating from soft neutral gauge bosons \( N = A, Z \) result in

\[
\sum_{N=A,Z} \delta^{N,\text{SSC}}_{W_{\lambda_1}^+ W_{\lambda_2}^+ W_{\lambda_3}^- W_{\lambda_4}^-} = \frac{\alpha}{2\pi} \ln \left( \frac{|r_{12}|}{M_W^2} \right) \sum_{N=A,Z} \left[ (I_{W_{\lambda_1}^N}^N I_{W_{\lambda_3}^N}^N + I_{W_{\lambda_2}^N}^N I_{W_{\lambda_4}^N}^N) \ln \left( \frac{|r_{13}|}{|r_{12}|} \right) \right] + \ln \left( \frac{|r_{23}|}{|r_{12}|} \right) \text{EM}^{\text{SSC}},
\]

(B.20)

where the couplings \( I_{W_{\lambda_1}^N} \) for longitudinal gauge bosons are given in (C.13), whereas for transverse gauge bosons \( I_{W_{\lambda_1}^T} = \mp 1 \) and \( I_{W_{\lambda_1}^T} = \pm c_w/s_w \). The electromagnetic logarithms \( \text{EM}^{\text{SSC}} \) are defined in (E.2).

Soft virtual W bosons can be exchanged only in the \( r_{13} \) channel, and yield

\[
\sum_{V=W^\pm} \delta^{V,\text{SSC}} M^{W_{\lambda_1}^T W_{\lambda_2}^T W_{\lambda_3}^- W_{\lambda_4}^-} = \frac{\alpha}{2\pi} \ln \left( \frac{|r_{12}|}{M_W^2} \right) \ln \left( \frac{|r_{13}|}{|r_{12}|} \right) \times \left[ \sum_{N_{\lambda_1}} \sum_{N'_{\lambda_3}} J_{N_{\lambda_1}}^- J_{N'_{\lambda_3}}^+ M_{0}^{W_{\lambda_1}^T W_{\lambda_2}^T W_{\lambda_3}^- W_{\lambda_4}^-} + \sum_{N_{\lambda_2}} \sum_{N'_{\lambda_4}} J_{N_{\lambda_2}}^- J_{N'_{\lambda_4}}^+ M_{0}^{W_{\lambda_1}^T W_{\lambda_2}^T W_{\lambda_3}^- W_{\lambda_4}^-} \right],
\]

(B.21)

where the sums over \( N_{\lambda} \) and \( N_{\lambda}' \) depend on the polarization \( \lambda \). In the case \( \lambda = 0 \), they run over \( N_0 = H, Z_0 \), whereas for \( \lambda = \tau = \pm 1 \) they run over \( N_{\tau} = A, Z_{\tau} \). The corresponding couplings \( J_{H}^\pm, J_{Z_0}^\pm \) and \( J_{A, Z_{\tau}}^\pm \) are given in (C.15) and (C.16), respectively. The SU(2)-transformed Born matrix elements that appear on the right-hand side of (B.21) are evaluated in the high-energy limit in Appendix D.

In the following we list the explicit results corresponding to all helicity combinations (B.8). Note that, owing to crossing symmetry, both contributions (B.20) and (B.21) are invariant with respect to simultaneous exchange of the polarizations \( \lambda_1 \leftrightarrow \lambda_4, \lambda_2 \leftrightarrow \lambda_3 \), i.e.

\[
\sum_{N=A,Z} \delta^{N,\text{SSC}}_{W_{\lambda_1}^T W_{\lambda_2}^T W_{\lambda_3}^- W_{\lambda_4}^-} = \sum_{N=A,Z} \delta^{N,\text{SSC}}_{W_{\lambda_2}^T W_{\lambda_1}^N W_{\lambda_3}^- W_{\lambda_4}^-}, \quad \sum_{V=W^\pm} \delta^{V,\text{SSC}} M^{W_{\lambda_1}^T W_{\lambda_2}^T W_{\lambda_3}^- W_{\lambda_4}^-} = \sum_{V=W^\pm} \delta^{V,\text{SSC}} M^{W_{\lambda_2}^T W_{\lambda_1}^N W_{\lambda_3}^- W_{\lambda_4}^-}. \quad \text{(B.22)}
\]

Therefore, the corrections for \( LL \rightarrow TT \) and \( LT \rightarrow LT \) can easily be obtained from those for \( TT \rightarrow LL \) and \( TL \rightarrow TL \), respectively.

**Purely longitudinal polarizations**

The contribution (B.20) of soft neutral gauge bosons gives

\[
\sum_{N=A,Z} \delta^{N,\text{SSC}}_{W_0^+ W_0^+ W_0^+ W_0^+} = -\frac{\alpha}{4\pi s_w^2 c_w^2} \ln \left( \frac{|r_{12}|}{M_W^2} \right) \ln \left( \frac{|r_{13}|}{|r_{23}|} \right) - 4 \ln \left( \frac{|r_{13}|}{|r_{23}|} \right) \text{EM}^{\text{SSC}}.
\]

(B.23)
The contribution (B.21) of soft W bosons yields
\[
\sum_{V=W^\pm} \delta^V_{W_0^- W_0^+ W_0^- W_0^+} = \\
= \frac{\alpha}{2\pi} \ln \left( \frac{|r_{12}|}{M_W^2} \right) \ln \left( \frac{|r_{13}|}{r_{12}} \right) \sum_{S,S'=H,Z_0} J^+_{S^1} J^+_{S'^2} \left[ \mathcal{M}_0^{0W_0^+ W_0^-} + \mathcal{M}_0^{W_0^- S W_0^+ S} \right] \\
= \frac{\alpha}{8\pi s^2_W} \ln \left( \frac{|r_{12}|}{M_W^2} \right) \ln \left( \frac{|r_{13}|}{|r_{12}|} \right) \left\{ \left[ \mathcal{M}_0^{H W_0^+ H W_0^-} + \mathcal{M}_0^{H W_0^- H W_0^+} \right] - (H \rightarrow Z_0) \right\} \\
- \left[ \left( \mathcal{M}_0^{H W_0^+ Z_0 W_0^-} + \mathcal{M}_0^{W_0^- Z_0 W_0^+ H} \right) - (H \leftrightarrow Z_0) \right].
\]

Using our expressions (D.7), (D.8) for the SU(2)-transformed Born matrix elements in the high-energy limit, and dividing by the Born matrix element (D.6) we obtain
\[
\sum_{V=W^\pm} \delta^V_{W_0^- W_0^+ W_0^- W_0^+} = \left( \frac{\alpha}{2\pi s^2_W} \right) \ln \left( \frac{|r_{12}|}{M_W^2} \right) \ln \left( \frac{|r_{13}|}{|r_{12}|} \right) \left[ \lambda_H + \frac{e^2}{4s^2_W c^2_W} \left( \frac{r_{13} - r_{23}}{r_{12}} + \frac{r_{12} - r_{23}}{r_{13}} \right) \right]^{-1},
\]

where \(\lambda_H\) is the scalar self coupling defined in (D.5).

**Mixed polarizations: TT \rightarrow LL and LL \rightarrow TT**

For the TT \rightarrow LL configuration, the contribution (B.20) of soft neutral gauge bosons gives
\[
\sum_{N=A,Z} \delta^N_{W^+_1 W^+_2 W_0^- W_0^+} = \frac{\alpha}{2\pi s^2_W} \ln \left( \frac{|r_{12}|}{M_W^2} \right) \ln \left( \frac{|r_{13}|}{|r_{23}|} \right) - 4 \ln \left( \frac{|r_{13}|}{|r_{23}|} \right) l_{EM}^{SSC}.
\]

Soft virtual W bosons (B.21) yield
\[
\sum_{V=W^\pm} \delta^V_{W^-_{1} W^+_2 W_0^- W_0^+} = \\
= \frac{\alpha}{2\pi} \ln \left( \frac{|r_{12}|}{M_W^2} \right) \ln \left( \frac{|r_{13}|}{r_{12}} \right) \sum_{N=A,Z} \sum_{S=H,Z_0} J^+_N J^+_S \mathcal{M}_0^{N_{1} W^+_2 W_0^-} + \mathcal{M}_0^{W^-_{1} N_{2} W_0^+ S} \\
= \frac{\alpha}{4\pi s_W} \ln \left( \frac{|r_{12}|}{M_W^2} \right) \ln \left( \frac{|r_{13}|}{|r_{12}|} \right) \left\{ \left[ \mathcal{M}_0^{A_{1} W^+_2 H W_0^-} + \mathcal{M}_0^{W^-_{1} A_{2} W_0^+ H} \right. \right. \\
+ \left. \left. \mathcal{M}_0^{A_{1} W^+_2 Z_0 W_0^-} + \mathcal{M}_0^{W^-_{1} A_{2} W_0^+ Z_0} \right] - \frac{c_W}{s_W} [A \rightarrow Z] \right\}.
\]

Using the Born amplitudes (D.11), (D.12) in the high-energy limit we obtain the relative correction
\[
\sum_{V=W^\pm} \delta^V_{W^+_1 W^+_2 W_0^- W_0^+} = \frac{\alpha}{2\pi s^2_W} \left( \frac{r_{13}}{r_{23}} - 1 \right) \ln \left( \frac{|r_{12}|}{M_W^2} \right) \ln \left( \frac{|r_{13}|}{|r_{12}|} \right).
\]

These results can directly be extended to the LL \rightarrow TT configuration using (B.22).
Mixed polarizations: TL → TL and LT → LT

For the TL → TL configuration, the contribution (B.20) of soft neutral gauge bosons gives

\[
\sum_{N=A,Z} \delta^{N,SSC}_{W_N^+ W_0^-} = -\frac{\alpha}{2\pi s_W^2} \ln \left( \frac{|r_{12}|}{M_W^2} \right) \ln \left( \frac{|r_{13}|}{r_{23}} \right) + \frac{1}{4c_W^2} \ln \left( \frac{|r_{13}|}{|r_{12}|} \right) - 4 \ln \left( \frac{|r_{13}|}{|r_{23}|} \right) \rho_{EM}^{SSC}. \tag{B.29}
\]

Soft virtual W bosons (B.21) yield

\[
\sum_{V=W^\pm} \delta^{V,SSC}_{W_N^+ W_0^-} = \frac{\alpha}{2\pi} \ln \left( \frac{|r_{12}|}{M_W^2} \right) \ln \left( \frac{|r_{13}|}{r_{12}} \right) \\
\times \left[ \sum_{N=A,Z} \sum_{N'=A,Z} J_{N'}^+ J_{N}^- \mathcal{M}^{N_{r_0} W^+_{r_3} W^-_{r_0}}_0 + \sum_{S=H,Z_0} \sum_{S'=H,Z_0} J_{S'}^+ J_{S}^- \mathcal{M}^{W^+_N W^+_N W^-_{r_0}}_0 \right] \\
= \frac{\alpha}{2\pi} \ln \left( \frac{|r_{12}|}{M_W^2} \right) \ln \left( \frac{|r_{13}|}{r_{12}} \right) \left\{ -\mathcal{M}_0^{A r_1 W^+_{r_0} A r_2 W^-_{r_0}} - \frac{c_W^2}{s_W^2} \mathcal{M}_0^{Z r_1 W^+_{r_0} Z r_3 W^-_{r_0}} \right. \\
+ \left. \frac{c_W}{s_W} \left[ \mathcal{M}_0^{A r_1 W^+_{r_0} Z r_3 W^-_{r_0}} + (A \leftrightarrow Z) \right] + \frac{1}{4s_W^2} \left( \mathcal{M}_0^{W^+_N H W^+_N H} + \mathcal{M}_0^{W^+_N H W^+_N Z_0} \right) \right\}. \tag{B.30}
\]

Using the Born amplitudes (D.16), (D.17) in the high-energy limit we obtain the relative correction

\[
\sum_{V=W^\pm} \delta^{V,SSC}_{W_N^+ W_0^-} = -\frac{\alpha}{2\pi s_W^2} \left( \frac{r_{12}}{r_{23}} + \frac{1}{2} \right) \ln \left( \frac{|r_{12}|}{M_W^2} \right) \ln \left( \frac{|r_{13}|}{r_{23}} \right). \tag{B.31}
\]

These results can directly be extended to the LT → LT configuration using (B.22).

Purely transverse polarizations

The contribution (B.20) of soft neutral gauge bosons gives

\[
\sum_{N=A,Z} \delta^{N,SSC}_{W_N^+ W_0^-} = -\frac{\alpha}{\pi s_W^2} \ln \left( \frac{|r_{12}|}{M_W^2} \right) \ln \left( \frac{|r_{13}|}{r_{23}} \right) - 4 \ln \left( \frac{|r_{13}|}{r_{23}} \right) \rho_{EM}^{SSC}. \tag{B.32}
\]

Soft virtual W bosons (B.21) yield

\[
\sum_{V=W^\pm} \delta^{V,SSC}_{W_N^+ W_0^-} = \frac{\alpha}{2\pi} \ln \left( \frac{|r_{12}|}{M_W^2} \right) \ln \left( \frac{|r_{13}|}{r_{12}} \right) \sum_{N,N'=A,Z} J_{N'}^+ J_{N}^- \left[ \mathcal{M}_0^{N_{r_1} W^+_{r_2} N_{r_3} W^-_{r_4}} + \mathcal{M}_0^{W^+_N N_{r_0} W^+_N W^-_{r_0}} \right] \\
= \frac{\alpha}{2\pi} \ln \left( \frac{|r_{12}|}{M_W^2} \right) \ln \left( \frac{|r_{13}|}{r_{12}} \right) \left\{ -\left( \mathcal{M}_0^{A r_1 W^+_{r_2} A r_3 W^-_{r_4}} + \mathcal{M}_0^{W^+_N A r_2 W^+_N W^-_{r_4}} \right) \right\}.
\]
where following general formula, which only depends on the numbers \( n \) in Ref. [14]. The resulting corrections to all processes (B.8) can be expressed by the longitudinal polarized W bosons, respectively. Here we have included also the logarithms of the single-logarithms originating from collinear singularities associated to external transverse or longitudinal W bosons can be obtained from Eqs. (4.10) and (4.33) in Ref. [12]. The renormalization of the parameters \( \beta \) and \( \lambda \) of the one-loop coefficient of the SU(2) \( \beta \)-function reads \( b^\text{ew}_W = 19/(6s^2_W) \), and the electromagnetic contributions \( l^\text{EM}(M^2_W) \) are defined in (E.3).

**B.4 Single logarithms from collinear singularities**

The single-logarithms originating from collinear singularities associated to external transverse or longitudinal W bosons can be obtained from Eqs. (4.10) and (4.33) in Ref. [12]. Here we have included also the logarithms of \( m_t/M_W \) and \( M_H/M_W \), which can be found in Ref. [14]. The resulting corrections to all processes (B.8) can be expressed by the following general formula, which only depends on the numbers \( n_T, n_L \) of transversely and longitudinally polarized W bosons, respectively.

\[
\delta_{W^+_0 \rightarrow W^-_0} = n_T \delta_{W^+_T} + n_L \delta_{W^+_L},
\]

where

\[
\delta_{W^+_T} = \frac{\alpha}{4\pi} \left\{ \frac{1}{2} b^\text{ew}_W \ln \left( \frac{|r_{12}|}{M^2_W} \right) + \frac{1}{2} b^\text{ew}_W \ln \left( \frac{m^2_t}{M^2_W} \right) + \frac{1}{2} b^\text{ew}_W \ln \left( \frac{M^2_H}{M^2_W} \right) \right\} + l^\text{EM}(M^2_W),
\]

\[
\delta_{W^+_L} = \frac{\alpha}{4\pi} \left\{ 2 b^\text{ew}_W \ln \left( \frac{|r_{12}|}{M^2_W} \right) - \frac{3}{4} b^\text{ew}_W \ln \left( \frac{m^2_t}{M^2_W} \right) + \frac{1}{8} b^\text{ew}_W \ln \left( \frac{M^2_H}{M^2_W} \right) \right\} + l^\text{EM}(M^2_W),
\]

the one-loop coefficient of the SU(2) \( \beta \)-function reads \( b^\text{ew}_W = 19/(6s^2_W) \), and the electromagnetic contributions \( l^\text{EM}(M^2_W) \) are defined in (E.3).

**B.5 Single logarithms from parameter renormalization**

**Purely longitudinal polarizations**

The renormalization of the parameters \( g = e^2/(4s^2_Wc^2_W) \) and \( \lambda_H \) (D.5) in the Born amplitude (D.6) gives rise to the relative correction

\[
\delta_{W^-_0 \rightarrow W^+_0} = \frac{\delta g}{g} + \left( \frac{\delta \lambda_H}{\lambda_H} - \frac{\delta g}{g} \right) \frac{\lambda_H}{\lambda_H + gA},
\]

where

\[
A = \frac{r_{13} - r_{23}}{r_{12}} + \frac{r_{12} - r_{23}}{r_{13}} = 2 \left( \frac{r^2_{23}}{r_{12}r_{13}} - 1 \right)
\]

(B.33)

In the high-energy limit, using (D.25) we obtain the relative correction factor

\[
\sum_{V=W^\pm} \delta_{W^-_1 W^+_2 \rightarrow W^-_3 W^+_4} = \frac{\alpha}{\pi s^2_W} \frac{r_{13}}{r_{23}} \ln \left( \frac{|r_{12}|}{M^2_W} \right) \ln \left( \frac{|r_{13}|}{|r_{12}|} \right).
\]

(B.34)
in the high-energy limit, and the ’t Hooft scale of dimensional regularization has to be set to \( \mu^2 = r_{12} \) in the counterterms [12, 13]. In the on-shell scheme (including the tadpole contributions in the renormalization of \( \lambda_H \)) these read [14]

\[
\left. \frac{\delta g}{g} \right|_{\mu^2=r_{12}} = \frac{\alpha}{4\pi} \left\{ \begin{array}{l}
- b_{WW}^{ew} \ln \left( \frac{r_{12}}{M_W^2} \right) + \frac{c_w^2 - s_w^2}{s_w^2 c_w^2} \left[ \frac{5}{6} \ln \left( \frac{M_H^2}{M_W^2} \right) \\
- \frac{9 + 6 s_w^2 - 32 s_w^4}{18 s_w^4} \ln \left( \frac{m_t^2}{M_W^2} \right) \right] \right\} + \Delta \alpha(M_W^2), \\
\left. \frac{\delta \lambda_H}{\lambda_H} \right|_{\mu^2=r_{12}} = \Delta \alpha(M_W^2) + \frac{\alpha}{4\pi} \left\{ \begin{array}{l}
3 \left( \frac{M_H^2}{M_W^2} \right) \left[ \frac{1}{2} - \frac{1}{c_w^2} \right] \ln \left( \frac{r_{12}}{M_W^2} \right) \\
+ \frac{3}{2 s_w^2} \left[ - \frac{3}{2} \frac{M_H^2}{M_W^2} \left( 2 + \frac{1}{c_w^2} \right) + \frac{10}{9} - \frac{M_H^2}{M_W^2} \right] \ln \left( \frac{M_H^2}{M_W^2} \right) \\
- \left[ \frac{N_c}{2 s_w^2} \frac{M_W^2}{M_H^2} \left( 1 - \frac{2 m_t^2}{M_H^2} \right) + \frac{9 - 12 s_w^2 - 32 s_w^4}{18 s_w^4} \ln \left( \frac{m_t^2}{M_W^2} \right) \right],
\end{array} \right\}
\]

where \( b_{WW}^{ew} = s_w^2 b_B^{ew} + c_w^2 b_W^{ew} = (19 - 38 s_w^2 - 22 s_w^4)/(6 s_w^2 c_w^2) \), with the U(1) and SU(2) one-loop \( \beta \)-function coefficients \( b_B^{ew} \) and \( b_W^{ew} \) defined in Ref. [12], and \( \Delta \alpha(M_W^2) \) represents the running of the electromagnetic coupling constant from the scale 0 to \( M_W \). Within the \( G_\mu \) scheme, these effects of the running are already included in the definition of \( \alpha_{G_\mu} \). Thus, the \( \Delta \alpha(M_W^2) \)-terms appearing in (B.39)–(B.41) were not included in our implementation.

Mixed polarizations

In this case, the Born matrix element is proportional to the squared SU(2) coupling \( g_2^2 = e^2/s_w^2 \). As a result, the parameter renormalization yields

\[
\left. \delta_{W_{r_1} W_{r_2} \rightarrow W_0 W_0}^{PR} \right|_{\mu^2=r_{12}} = \delta_{W_0 W_0 \rightarrow W_{r_3} W_{r_4}}^{PR} = \delta_{W_{r_1} W_0 \rightarrow W_{r_3} W_0}^{PR} = \delta_{W_0 W_{r_2} \rightarrow W_0 W_{r_4}}^{PR} = \frac{\alpha}{4\pi} \left[ - b_{WW}^{ew} \ln \left( \frac{r_{12}}{M_W^2} \right) + \frac{5}{6 s_w^2} \ln \left( \frac{M_H^2}{M_W^2} \right) \right] + \Delta \alpha(M_W^2),
\]

with \( b_{WW}^{ew} = 19/(6 s_w^2) \). Note that the contribution \( b_{WW}^{ew} \ln (n_{12}/M_W^2) \) from parameter renormalization cancels a corresponding contribution \( b_{WW}^{ew} \ln (n_{12}/M_W^2) \) in (B.35), which is associated to the two transverse gauge bosons (\( n_T = 2 \)). This cancellation is analogous to the one observed in Eq. (A.11) in Ref. [12].

Purely transverse polarizations

Also in this case, the Born matrix element is proportional to the squared SU(2) coupling \( g_2^2 \) and we have

\[
\left. \delta_{W_{r_1} W_{r_2} \rightarrow W_{r_3} W_0}^{PR} \right|_{\mu^2=r_{12}} = \frac{\alpha}{4\pi} \left[ - b_{WW}^{ew} \ln \left( \frac{r_{12}}{M_W^2} \right) + \frac{5}{6 s_w^2} \ln \left( \frac{M_H^2}{M_W^2} \right) \right]
\]

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\[-\frac{9 + 6 s_W^2 - 32 s_W^4}{18 s_W^4} \ln \left( \frac{m_t^2}{M_W^2} \right) + \Delta \alpha (M_W^2). \quad (B.41)\]

In this case the contributions \( b_W^W \ln \left( |r_{12}|/M_W^2 \right) \) from parameter renormalization cancel only two of the four \( (n_T = 4) \) contributions \( b_W^W \ln \left( |r_{12}|/M_W^2 \right) \) in (B.35).

## C Longitudinal gauge bosons

In order to apply the general formulas of Ref. [12] to matrix elements involving longitudinal gauge bosons \( V_0^a = W_0^\pm, Z_0 \), these have to be transformed first into corresponding matrix elements involving would-be Goldstone bosons \( \Phi_a = \phi^\pm, \chi \) by means of the Goldstone-Boson Equivalence Theorem (GBET) in its naive lowest-order form\(^3\),

\[ \mathcal{M}^{\phi_i_1 \cdots V_0^a \cdots \phi_i_n} = i^{1-Q_{V^a}} \mathcal{M}^{\phi_i_1 \cdots \Phi_a \cdots \phi_i_n}, \quad (C.1) \]

where \( \phi_i_1, \ldots \phi_i_n \) represent arbitrary particles with arbitrary polarizations and \( Q_{V^a} \) is the gauge-boson charge, i.e. \( Q_{W^\pm} = \pm 1 \) and \( Q_Z = 0 \).

Particular care must be taken of the angular-dependent subleading soft-collinear corrections (see Eqs. (3.9)–(3.12) of Ref. [12])

\[ \delta^{SSC} \mathcal{M}^{\phi_i_1 \cdots \phi_i_n} = \frac{\alpha}{2 \pi} \sum_{k=1}^{n} \sum_{l<k} \sum_{V^a=A,Z,W^\pm} \ln \left( \frac{|r_{12}|}{M^2} \right) \ln \left( \frac{|r_{kl}|}{|r_{12}|} \right) I_{V^a}^{\phi_i_1 \phi_i_l} I_{V^a}^{\phi_i_k \phi_i_l} \mathcal{M}_0^{\phi_i_1 \cdots \phi_i_l \cdots \phi_i_k \cdots \phi_i_n}, \quad (C.2) \]

which involve non-abelian couplings \( I_{\phi_i_1 \phi_i_l}^{V^a} \) that are in general non-diagonal and lead therefore to SU(2)-transformed Born matrix elements with \( \phi_i' \neq \phi_i \) on the right-hand side of (C.2). For processes involving longitudinal gauge bosons and Higgs bosons, this formula has to be applied to corresponding processes involving would-be Goldstone bosons and Higgs bosons. Then the unphysical matrix elements on the left- and right-hand sides of (C.2) have to be transformed into physical matrix elements by means of the GBET. In doing this, the factors \( i^{1-Q_{V^a}} \) originating from the GBET (C.1) must be carefully taken into account, since the would-be Goldstone bosons and Higgs bosons appearing on the left- and right-hand side of (C.2) (and thus the corresponding GBET factors) can be different.

In order to avoid these complications related to the explicit use of the GBET, in the following we introduce effective couplings for longitudinal gauge bosons and Higgs bosons, which permit to apply (C.2) directly to physical matrix elements. To this end the factors \( i^{1-Q_{V^a}} \) from the GBET are combined with the gauge couplings for would-be Goldstone bosons and Higgs bosons, resulting into the effective couplings

\[ I_{V_0^a V_0^a}^{V^a} = (-i)^{1-Q_{V^a}} I_{\Phi_a\Phi_a}^{V^a} i^{1-Q_{V^a}}, \quad I_{H V_0^a}^{V^a} = I_{H \Phi_a}^{V^a} i^{1-Q_{V^a}}, \quad I_{V_0^a H}^{V^a} = (-i)^{1-Q_{V^a}} I_{\Phi_a H}^{V^a}, \quad (C.3) \]

\(^3\)The relevant quantum corrections to the GBET, which contribute to the collinear single-logarithms are already taken into account into the corresponding corrections factors (4.33) in Ref. [12].
where \( V^c = A, Z, W^\pm \), and \( \Phi_a = \phi^\pm, \chi \) are the would-be Goldstone bosons corresponding to \( V_0^a = W_0^\pm, Z_0 \). We observe that the relations (C.1) and (C.3) can be regarded as the result of a reparametrization of the scalar sector through the unitary transformation

\[
\Phi_a = i^{1-Q_V} V_0^a. \quad \text{(C.4)}
\]

If one performs this transformation directly at the level of the Lagrangian one can express the Feynman rules in terms of the longitudinal gauge-boson fields. This simplifies the calculation of matrix elements for longitudinal gauge bosons in the high-energy limit (see Appendix D). The Feynman rules for the scalar and gauge interactions of the fields \( S_i = W_0^\pm, Z_0, H \) and their propagators in the 't Hooft–Feynman gauge read

\[
S_i(p_1) \rightarrow S_{i_1}(p_1) \quad = ie I_{S_{i_1} S_{i_2}}^{V_{a_1}} (p_2 - p_1)_{\mu_1}, \quad \text{(C.5)}
\]

\[
S_i(p_2) \rightarrow S_{i_2}(p_2) \quad = i e^2 g_{\mu_1 \mu_2} \{ I^{V_{a_1}}, I^{V_{a_2}} \}_{S_{i_1} S_{i_2}}, \quad \text{(C.6)}
\]

\[
S_i \rightarrow S_{i_3} \quad = -i \frac{\lambda_H}{2} \left[ \delta_{S_{i_1} S_{i_2}} \delta_{S_{i_3} S_{i_4}} + (2 \leftrightarrow 3) + (2 \leftrightarrow 4) \right], \quad \text{(C.7)}
\]

\[
S_{i_1}(p) \rightarrow S_{i_2}(-p) \quad = \frac{i \delta_{S_{i_1} S_{i_2}}}{p^2 - M_{S_i}^2}. \quad \text{(C.8)}
\]

Here, all fields are incoming, the curly brackets represent an anticommutator and products of couplings have to be understood as

\[
(I^{V_{a_1}} I^{V_{a_2}})_{S_i S_j} = \sum_{S_k = W_0^\pm, Z_0, H} I_{S_i S_k}^{V_{a_1}} I_{S_k S_j}^{V_{a_2}}. \quad \text{(C.9)}
\]

The Feynman rules (C.5)–(C.8) are closely analogous to those in Ref. [14]. However here, as a result of the unitary transformation (C.4), the hermitian conjugation of the fields \( V_0^a = Z_0, W_0^\pm \) generates a minus sign,

\[
V_0^{a+} = -\bar{V}_0^a. \quad \text{(C.10)}
\]
where $V_0^a = (-i)^{(1-Q_a)}\Phi_0^+$ or, equivalently, $\bar{W}_0^\pm = W_0^\mp$ and $\bar{Z}_0 = Z_0$. As a consequence, the coupling matrices and the Kronecker symbols that involve hermitian conjugate fields in (C.5)–(C.8) have to be understood as

$$I_{S_i^+ S_j^-}^{V^a} = - I_{S_i S_j}^{V^a}, \quad \delta_{S_i^+ S_j^-} = - \delta_{S_i S_j} \quad (C.11)$$

for $S_i = W_0^\pm, Z_0$. Instead, $H^+ = \bar{H} = H$ for the Higgs field, and

$$I_{H^+ S_j}^{V^a} = I_{HS_j}^{V^a}, \quad \delta_{H^+ S_j} = \delta_{HS_j}. \quad (C.12)$$

Below we list the explicit expression for all non-vanishing effective couplings (C.3), which can be easily derived from Eqs. (B.21)–(B.23) in Ref. [12]. For neutral gauge bosons ($N = A, Z$) we have

$$I_N^{W_0^\pm W_0^-} = \delta_{\sigma \sigma'} I_N^{W_0^\sigma}, \quad \text{with} \quad I_N^{W_0^\sigma} = - \sigma, \quad I_N^{W_0^-} = \sigma 1 - 2 s_w^2, \quad (C.13)$$

and

$$I_N^{Z_0 H} = I_N^{Z_0} = \delta_{NZ} \frac{1}{2 s_w c_w}. \quad (C.14)$$

For charged gauge bosons coupling to $S = Z_0, H$ we obtain,

$$I_{W_0^\pm W_0^-}^{W_0^-} = I_{SW_0^\sigma}^{W_0^-} = - \delta_{\sigma \sigma'} J_{S_0}^{\sigma}, \quad J_{S_0}^\sigma = \sigma \frac{1}{2 s_w}, \quad J_H^\sigma = - \frac{1}{2 s_w}. \quad (C.15)$$

Note that the couplings (C.15) are analogous to the couplings for transverse gauge bosons defined in (B.26) of Ref. [12], which can also be written as

$$I_{W_0^\pm N_T}^{W_0^-} = I_{N_T W_0^-}^{W_0^-} = - \delta_{\sigma \sigma'} J_{N_T}^{\sigma}, \quad J_A^\sigma = - \sigma, \quad J_Z^\sigma = \sigma \frac{c_w}{s_w}. \quad (C.16)$$

**D Born matrix elements in the high-energy limit**

As input for the evaluation of the angular-dependent subleading corrections (B.21), which originate from exchange of soft-collinear $W$ bosons, we need the SU(2)-transformed Born matrix elements that appear on the right-hand side of (B.21). The needed amplitudes are evaluated in the following in high-energy approximation, i.e. omitting mass-suppressed terms. For longitudinal gauge bosons or Higgs bosons, which are denoted as

$$S_i = W_0^\pm, Z_0, H, \quad \bar{S}_i = W_0^\mp, Z_0, H, \quad (D.1)$$

we use the couplings introduced in Appendix C. Transverse gauge bosons are denoted as

$$V_\tau^a = A_\tau, Z_\tau, W_\tau^\pm, \quad \bar{V}_\tau^a = A_\tau, Z_\tau, W_\tau^\mp, \quad (D.2)$$

with $\tau = \pm 1$. 
Purely longitudinal polarizations: LL → LL

For a generic process

\[ S_{i_1}(k_1)S_{i_2}(k_2) \rightarrow \bar{S}_{i_3}(-k_3)\bar{S}_{i_4}(-k_4), \] (D.3)

in the high-energy limit we obtain the amplitude

\[
\mathcal{M}_{0}^{S_{i_1}S_{i_2}S_{i_3}S_{i_4}} = \left( e^2 \frac{r_{13} - r_{23}}{r_{12}} \sum_{V^a = A,Z,W} I_{V^a}^{S_{i_1}S_{i_2}} I_{V^a}^{S_{i_3}S_{i_4}} - \frac{\lambda_H}{2} \delta_{S_{i_1}S_{i_2}} \delta_{S_{i_3}S_{i_4}} \right) + (2 \leftrightarrow 3) + (1 \leftrightarrow 3),
\] (D.4)

where

\[
\lambda_H = \frac{e^2 M_H^2}{2 s^2_W M_W^2},
\] (D.5)

is the scalar self-coupling.

The Born matrix element for W-boson scattering reads

\[
\mathcal{M}_{0}^{W^{-}_0W^{+}_0W^{-}_0W^{-}_0} = -\left[ \lambda_H + \frac{e^2}{4 s^2_W c^2_W} \left( \frac{r_{13} - r_{23}}{r_{12}} + \frac{r_{12} - r_{23}}{r_{13}} \right) \right].
\] (D.6)

The SU(2)-transformed Born matrix elements involving two equal neutral states, \( S = H \) or \( Z_0 \), give

\[
\mathcal{M}_{0}^{SW^{+}_0SW^{-}_0} = \mathcal{M}_{0}^{W^{-}_0SW^{+}_0S} = \epsilon_S \left[ \frac{\lambda_H}{2} + \frac{e^2}{4 s^2_W} \left( \frac{r_{13} - r_{23}}{r_{12}} + \frac{r_{13} - r_{12}}{r_{23}} \right) \right],
\] (D.7)

with \( \epsilon_H = 1 \) and \( \epsilon_{Z_0} = -1 \). For the case of different neutral states, \( S \neq S' \) with \( S, S' = H \) or \( Z_0 \), we have

\[
\mathcal{M}_{0}^{SW^{+}_0S'W^{-}_0} = \mathcal{M}_{0}^{W^{-}_0S'W^{+}_0S} = -\epsilon_S e^2 \left[ \frac{1}{4 s^2_W} \left( \frac{r_{13} - r_{23}}{r_{12}} - \frac{r_{13} - r_{12}}{r_{23}} \right) + \frac{s^2_W - c^2_W}{4 s^2_W} \frac{r_{12} - r_{23}}{r_{13}} \right].
\] (D.8)

Mixed polarizations: TT → LL and LL → TT

For a generic TT → LL process,

\[ V_{\tau_1}^{\alpha_1}(k_1)V_{\tau_2}^{\alpha_2}(k_2) \rightarrow \bar{S}_{i_3}^{\bar{\alpha}_3}(-k_3)\bar{S}_{i_4}^{\bar{\alpha}_4}(-k_4), \] (D.9)

in the high-energy limit we obtain the amplitude

\[
\mathcal{M}_{0}^{V_{\tau_1}^{\alpha_1}V_{\tau_2}^{\alpha_2}S_{i_3}S_{i_4}} = 2e^2(1 - \delta_{\tau_1\tau_2}) \left[ \frac{r_{23}}{r_{12}} \left( I^{\alpha_1} I^{\alpha_2} \right) S_{i_3}^{ \bar{\alpha}_3} S_{i_4}^{ \bar{\alpha}_4} + \frac{r_{13}}{r_{12}} \left( I^{\alpha_2} I^{\alpha_1} \right) S_{i_3}^{ \bar{\alpha}_3} S_{i_4}^{ \bar{\alpha}_4} \right].
\] (D.10)

Inserting the explicit values of the couplings we obtain

\[
\mathcal{M}_{0}^{W^{-}_0W^{+}_0W^{-}_0W^{-}_0} = -\frac{e^2}{s^2_W}(1 - \delta_{\tau_1\tau_2}) \frac{r_{23}}{r_{12}},
\] (D.11)
and the SU(2)-transformed amplitudes

\[ M_0^{N_r W^+_r W_0^-} = -M_0^{W^+_r N_r W_0^-} = -M_0^{N_r W^+_r Z_0 W_0^-} = -M_0^{W^+_r N_r W_0^+ Z_0} = \frac{e^2}{s_W^2} (1 - \delta_{\gamma \gamma}) \frac{r_{23}}{r_{12}} D_N, \]

with \( N_{\gamma} = A_{\gamma}, Z_{\gamma} \), and

\[ D_A = \frac{r_{13}}{r_{23}}, \quad D_Z = \frac{1}{2s_W c_W} - \frac{r_{13} 1 - 2s_W^2}{r_{23} 2s_W c_W}. \] (D.13)

Corresponding amplitudes for LL \( \rightarrow \) TT processes are directly obtained using

\[ M_0^{S_1 S_2 V_{\alpha 1}^a V_{\alpha 2}^a S_3 S_4} = M_0^{V_{\alpha 1}^a V_{\alpha 2}^a S_1 S_2 S_3 S_4}. \]

**Mixed polarizations: TL \( \rightarrow \) TL and LT \( \rightarrow \) LT**

For a generic TL \( \rightarrow \) TL process

\[ V_{\tau 1}^{a_1} (k_1) S_{12} (k_2) \rightarrow V_{-\gamma 3}^{a_3} (-k_3) S_{4}^{a_4} (-k_4), \] (D.14)

in the high-energy limit we obtain the amplitude

\[ M_0^{V_{\tau 1}^{a_1} S_{12} V_{\gamma 3}^{a_3} S_{4}^{a_4}} = 2e^2 (1 - \delta_{\gamma \gamma}) \left[ \frac{r_{23}}{r_{13}} \left( I^{V_{\alpha 1}^a} I^{V_{\gamma 3}^{a_3}} \right) S_{12}^{a_3} S_{4}^{a_4} + \frac{r_{12}}{r_{13}} \left( I^{V_{\gamma 3}^{a_3}} I^{V_{\alpha 1}^a} \right) S_{12}^a S_{4}^{a_4} \right]. \] (D.15)

Inserting the explicit values of the couplings we obtain the Born amplitude

\[ M_0^{W^+_r W_0^- W_3^+ W_0^-} = -\frac{e^2}{s_W^2} (1 - \delta_{\gamma \gamma}) \frac{r_{23}}{r_{13}}, \] (D.16)

and the SU(2)-transformed amplitudes

\[ M_0^{N_r W^+_r N'_r W_0^-} = 2e^2 (1 - \delta_{\gamma \gamma}) I^{N_r W^+_r} I^{N'_r W_0^-}, \]
\[ M_0^{W^+_r W_0^- Z_0 W_3^+} = -M_0^{W^+_r Z_0 W_0^- Z_0} = -\frac{e^2}{2s_W^2} (1 - \delta_{\gamma \gamma}), \]
\[ M_0^{W^+_r W_0^- Z_0} = -M_0^{W^+_r Z_0 W_3^+} = \frac{e^2}{2s_W^2} (1 - \delta_{\gamma \gamma}) \frac{r_{12} - r_{23}}{r_{13}}, \] (D.17)

with \( N_{\gamma} = A_{\gamma}, Z_{\gamma} \), \( I^{A_{\gamma}}_{W^+_r} = -1 \) and \( I^{Z_{\gamma}}_{W^+_r} = (1 - 2s_W^2)/(2s_W c_W) \).

Corresponding amplitudes for LT \( \rightarrow \) LT processes are directly obtained using

\[ M_0^{S_1^{a_1} S_2^{a_2} V_{\alpha 3}^a S_3^{a_3} V_{\alpha 4}^a S_4} = M_0^{V_{\alpha 1}^{a_1} S_1^{a_2} V_{\alpha 3}^a S_2^{a_2} S_3^{a_3} S_4^{a_4}}. \]

**Purely transverse polarizations: TT \( \rightarrow \) TT**

For a generic process

\[ V_{\tau 1}^{a_1} (k_1) V_{\tau 2}^{a_2} (k_2) \rightarrow V_{-\gamma 3}^{a_3} (-k_3) V_{-\gamma 4}^{a_4} (-k_4), \] (D.18)
in the high-energy limit we obtain the amplitude

\[ \mathcal{M}_0^{V_1^{a_1} V_2^{a_2} V_3^{a_3} V_4^{a_4}} = \frac{2e^2}{s_W^2} \left[ \left( \delta_{V_1 V_2}^{SU(2)} \delta_{V_3 V_4}^{SU(2)} - \delta_{V_1 V_3}^{SU(2)} \delta_{V_2 V_4}^{SU(2)} \right) A(\vec{\tau}, \{r_{ij}\}) \\
+ \left( \delta_{V_1 V_2}^{SU(2)} \delta_{V_3 V_4}^{SU(2)} - \delta_{V_1 V_4}^{SU(2)} \delta_{V_2 V_3}^{SU(2)} \right) B(\vec{\tau}, \{r_{ij}\}) \right], \]

(D.19)

where the matrix \( \delta^{SU(2)} \) is defined in Appendix B.2 of Ref. [14], and has the non-vanishing components \( \delta_{AA}^{SU(2)} = s_w^2, \delta_{ZZ}^{SU(2)} = c_w^2, \delta_{AZ}^{SU(2)} = -c_w s_w, \delta_{W^+ W^-}^{SU(2)} = 1 \).

The functions \( A, B \) depend on the invariants \( r_{ij} \) and the polarizations \( \vec{\tau} = (\tau_1, \tau_2, \tau_3, \tau_4) \). The only combinations of polarizations that yield non-vanishing contributions are

\[ \vec{\tau}_a = (\sigma, -\sigma, -\sigma, -\sigma), \quad \vec{\tau}_b = (\sigma, -\sigma, \sigma, -\sigma), \quad \vec{\tau}_c = (-\sigma, \sigma, \sigma, -\sigma) \]

(D.20)

with \( \sigma = \pm \), and we have

\[ A(\vec{\tau}_a, \{r_{ij}\}) = -\frac{r_{12}}{r_{23}}, \quad A(\vec{\tau}_b, \{r_{ij}\}) = -\frac{r_{13}}{r_{12}r_{23}}, \quad A(\vec{\tau}_c, \{r_{ij}\}) = -\frac{r_{23}}{r_{12}}, \]

(D.21)

and

\[ B(\vec{\tau}_a, \{r_{ij}\}) = -\frac{r_{12}}{r_{13}}, \quad B(\vec{\tau}_b, \{r_{ij}\}) = -\frac{r_{13}}{r_{12}}, \quad B(\vec{\tau}_c, \{r_{ij}\}) = -\frac{r_{23}}{r_{12}r_{13}}, \]

(D.22)

whereas if \( \vec{\tau} \neq \vec{\tau}_a, \vec{\tau}_b, \vec{\tau}_c \) then \( A(\vec{\tau}, \{r_{ij}\}) = B(\vec{\tau}, \{r_{ij}\}) = 0 \). Inserting the explicit values of the couplings we obtain

\[ \mathcal{M}_0^{W_1^{a_1} W_2^{a_2} W_3^{a_3} W_4^{a_4}} = \frac{2e^2}{s_W^2} B(\vec{\tau}, \{r_{ij}\}), \]

(D.23)

and the SU(2)-transformed matrix elements

\[ \mathcal{M}_0^{N_{V_1}^{a_1} N_{V_2}^{a_2} N_{V_3}^{a_3} N_{V_4}^{a_4}} = \mathcal{M}_0^{W_{V_1}^{a_1} W_{V_2}^{a_2} W_{V_3}^{a_3} W_{V_4}^{a_4}} = \frac{2e^2}{s_W^2} \delta^{SU(2)}_{N'N} A(\vec{\tau}, \{r_{ij}\}), \]

(D.24)

for \( N, N' = A, Z \). We note that the ratio between the Born amplitudes (D.24) and (D.23),

\[ \frac{\mathcal{M}_0^{N_{V_1}^{a_1} N_{V_2}^{a_2} N_{V_3}^{a_3} N_{V_4}^{a_4}}}{\mathcal{M}_0^{W_{V_1}^{a_1} W_{V_2}^{a_2} W_{V_3}^{a_3} W_{V_4}^{a_4}}} = \frac{\mathcal{M}_0^{W_{V_1}^{a_1} W_{V_2}^{a_2} W_{V_3}^{a_3} W_{V_4}^{a_4}}}{\mathcal{M}_0^{W_{V_1}^{a_1} W_{V_2}^{a_2} W_{V_3}^{a_3} W_{V_4}^{a_4}}} = -\delta^{SU(2)}_{N'N} A(\vec{\tau}, \{r_{ij}\}) = -\delta^{SU(2)}_{N'N} \frac{r_{13}}{r_{23}}, \]

(D.25)

is independent of the polarizations.

E Electromagnetic virtual and real contributions

In this appendix we provide simple substitutions that permit to generalize the results of Ref. [12] to semi-inclusive \( 2 \rightarrow 2 \) processes, by including the soft-photon bremsstrahlung corrections\(^4\). These substitutions concern the infrared-divergent logarithms

\(^4\)The soft bremsstrahlung corrections to squared matrix elements factorize into the squared Born matrix elements times correction factors. These latter have been divided by 2 and combined with the virtual correction factors to (non-squared) matrix elements given in Ref. [12].
Leads $(s, \lambda^2, m_k^2)$, $l(M_W^2, \lambda^2)$, $l^{em}(m_k^2)$ that appear in Eqs. (3.7), (3.8), (3.10), (3.12), (4.6), (4.7), (4.10) and (4.33) of Ref. [12] and have to be replaced with the logarithms $L^{EM}(m_k^2)$, $l^{em}_{SSC}$, and $l^{EM}(m_k^2)$, defined in the following.

These results are valid for arbitrary $2 \rightarrow 2$ processes in the CM frame, with a soft-photon cut-off $\Delta E$. The contributions from virtual photons (superscript ‘em’) and real bremsstrahlung (superscript ‘brems’) as well as their sum (superscript ‘EM’) are given separately and split into leading-, subleading-soft-collinear and collinear (or soft) parts.

**Leading soft-collinear contributions**

The terms $L^{em}(s, \lambda^2, m_k^2)$, which are defined in Eq. (3.8) of Ref. [12] and contribute to Eq. (3.7) of Ref. [12] have to be substituted by $L^{EM}(m_k^2) = L^{em}(m_k^2) + L^{brems}(m_k^2)$, with

$$ L^{em}(m_k^2) = \frac{\alpha}{4\pi} \left\{ 2 \ln \left( \frac{|r_{12}|}{m_k^2} \right) \ln \left( \frac{M_W^2}{\lambda^2} \right) - \ln^2 \left( \frac{M_W^2}{m_k^2} \right) \right\}, $$

$$ L^{brems}(m_k^2) = \frac{\alpha}{4\pi} \left\{ \ln \left( \frac{|r_{12}|}{m_k^2} \right) \left[ 2 \ln \left( \frac{\lambda^2}{4\Delta E^2} \right) + \ln \left( \frac{|r_{12}|}{m_k^2} \right) \right] \right\}, $$

$$ L^{EM}(m_k^2) = \frac{\alpha}{4\pi} \left\{ -\ln^2 \left( \frac{|r_{12}|}{M_W^2} \right) + 2 \ln \left( \frac{|r_{12}|}{4\Delta E^2} \right) \ln \left( \frac{|r_{12}|}{m_k^2} \right) \right\}. \quad \text{(E.1)} $$

**Subleading soft-collinear contributions**

The terms $l(M_W^2, \lambda^2)$ in Eqs. (3.10) and (3.12) of Ref. [12] have to be substituted by $l^{em}_{SSC} = l^{em}_{SSC} + l^{brems}_{SSC}$, with

$$ l^{em}_{SSC} = \frac{\alpha}{4\pi} \ln \left( \frac{M_W^2}{\lambda^2} \right), \quad l^{brems}_{SSC} = \frac{\alpha}{4\pi} \ln \left( \frac{\lambda^2}{4\Delta E^2} \right), \quad l^{EM}_{SSC} = \frac{\alpha}{4\pi} \ln \left( \frac{M_W^2}{4\Delta E^2} \right). \quad \text{(E.2)} $$

**Collinear and soft single logarithms**

The terms $l^{em}(m_k^2)$, which are defined in Eq. (4.7) of Ref. [12] and contribute to Eqs. (4.6), (4.10) and (4.33) of Ref. [12], have to be substituted by $l^{EM}(m_k^2) = l^{em}(m_k^2) + l^{brems}(m_k^2)$, with

$$ l^{em}(m_k^2) = \frac{\alpha}{4\pi} \left\{ \ln \left( \frac{M_W^2}{\lambda^2} \right) + \frac{1}{2} \ln \left( \frac{M_W^2}{m_k^2} \right) \right\}, $$

$$ l^{brems}(m_k^2) = \frac{\alpha}{4\pi} \left\{ \ln \left( \frac{\lambda^2}{4\Delta E^2} \right) + \ln \left( \frac{|r_{12}|}{m_k^2} \right) \right\}, $$

$$ l^{EM}(m_k^2) = \frac{\alpha}{4\pi} \left\{ \ln \left( \frac{|r_{12}|}{4\Delta E^2} \right) + \frac{3}{2} \ln \left( \frac{M_W^2}{m_k^2} \right) \right\}. \quad \text{(E.3)} $$

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