Transpiling Quantum Circuits using the Pentagon Equation

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We consider the application of the pentagon equation in the context of quantum circuit compression. We show that if solutions to the pentagon equation are found, one can transpile a circuit involving non-Heisenberg-type interactions to a circuit involving only Heisenberg-type interactions while, in parallel, reducing the depth of a circuit. In this context, we consider a model of non-local two-qubit operations of Zhang et. al. (the A gate), and show that for certain parameters it is a solution of the pentagon equation.

I. INTRODUCTION

Quantum technologies and quantum computing in particular are advancing in an unprecedented pace. As a result, the capabilities of quantum computers are constantly increasing wherein the first sings of “quantum supremacy” or “quantum advantage” hint that it might not take as long as originally thought to adopt them in a large scale. However, the current generation of quantum computers, often termed the Noisy Intermediate-Scale Quantum (NISQ) computers, suffer from various limitations which make them quite hard to exploit for useful applications across various domains such as finance, logistics, chemistry or material science. A key limitation is known as the (effective) depth of a quantum circuit. In the quantum circuit model, one applies a sequence of unitary transformations \( U \) on quantum states. The number of subsequent applications of unitary transformations on (up to) two qubits at a time, is known as the depth of the circuit. The depth of the circuit is limited by the ratio of coherence time and gate time, as well as the fidelity of the two-qubit gates. The limit on the depth of a quantum circuit limits what algorithms can be implemented, and the quality of the output, despite the flourishing of various error mitigation techniques. While Fault Tolerant Error Corrected (FTEC) quantum computers will be more resilient to these limitations, one may envision that even there, it will be preferable to reduce the number of gates. Circuit transpiling techniques, which can reduce circuit depth, will be essential in both the NISQ and FTEC eras of quantum computing. The transpiling of quantum circuits starting from a potentially large gate set to a particular hardware-specific “native” gate set, can be quite challenging, esp. in architectures with limited connectivity. Limited connectivity is another key limitation of the current quantum computers and several challenges must be overcome to fully exploit these powerful machines. Notice that, due to the limited connectivity, two-qubit gates often cannot be readily applied and the states of the corresponding qubits must be swapped with those of their neighbors until the states reside on qubits where a two qubit gate is supported. SWAP gates are, however, often expensive. For instance in CNOT-based native gate sets, they are often implemented using three CNOT gates. As a result, reducing the number of SWAP gates is desirable, but computing the minimum number of SWAP gates required in a given circuit is an NP-Hard problem. Most techniques for transpiling quantum circuits are multi-pass heuristics, rather than exact\cite{10}. First, an initial set of transformations is used to translate the quantum circuit to a “native” gate set. Second, a heuristic mapping of logical qubits to physical qubits is suggested. Third, standard “circuit compression” transformations\cite{11,12} are applied in a Knuth-Bendix fashion. Ref.\cite{11} introduced the first five templates and many more followed\cite{12,16} e.g.. We refer to\cite{17,15} for more in-depth surveys and to\cite{19,20} for recent experimental comparisons.

![initial (with SWAPs) vs. compressed](Image)

FIG. 1. The schematic of a circuit compression for gates \( T \) that satisfy the pentagon equation. In the LHS there is a non-local 2-qubit gate that involves SWAP gates.

Our work is inspired by Ref.\cite{16}. There, the authors approached the problem of quantum circuit compression, in the framework of the Hamiltonian evolution of the 1D Heisenberg model, using the Yang-Baxter Equation (YBE)\cite{21,25}. (See\cite{26} for a modern survey of YBE.) Specifically, they map the YBE to parametrized quantum circuits composed of 2-qubit gates and find the necessary conditions on these parameters such that the gates permuted in a certain way can be compressed by adding the parameter values of two sequential gates on the same two qubits. In this context, solutions of the YBE have...
the potential to significantly reduce the depth of a local circuit.

In this note, we elaborate upon the approach of Ref. [16] by compressing the depth of quantum circuits using lesser known tools from integrability theory, specifically the so-called “pentagon equation”. However, our method yields a surprise on the potential applications on quantum circuits, a form of duality.

First, different to Ref. [16], we show the necessary conditions required to compress the depth of quantum circuits that contain 2-qubit non-local interactions, that is non-nearest-neighbor long-range interactions, that usually are implemented by running first SWAP gates (see Figs. 1 and 2). This can achieve circuit compression under certain conditions. The schematic of the compression is shown in Fig. 1. In a more general context, our method obtains a dual picture where circuits with non-local interactions can be expressed as circuits with local ones as long or vice-versa. Our approach depends on the so-called “duality” to be solutions of the so-called pentagon equation [27], Eq. (2), and we focus on the implementation using the evolution operator of the 1D Heisenberg model [28] as well as the so-called A gate [29].

This note is organized as follows: In Section II we collect all the algebraic preliminaries we need for the rest of the paper and fix some notation. In Section III we collect all the necessary material for understanding the 1D Heisenberg model. In Section IV we introduce the pentagon equation and conclude with a digression to Yang-Baxter equation. In Section V we present our results: we discuss initially the form of duality we obtain from the application of the pentagon equation. Then, we explain how to transpile quantum circuits and conclude with the necessary conditions for the evolution operator of the 1D Heisenberg model and the A-gate to satisfy the pentagon equation.

![Diagram of quantum circuit](image)

**FIG. 2.** To implement the non-local interaction $H_{i,i+2}$ on the $i$-th and $i+2$-th qubits, SWAP gates must be implemented.

## II. ALGEBRAIC SETUP

Throughout the paper, we will denote a finite dimensional vector space over the complex numbers by $V$ with the usual tensor product of vector spaces $\otimes$. The identity map from $V$ to $V$ will be denoted $\text{id}_V$ and the twist map $\tau_{V,V} : V \otimes V \rightarrow V \otimes V$ reads $x \otimes y \mapsto y \otimes x$. A linear map $f : U \rightarrow V$, between two vector spaces $U$ and $V$, has an associated matrix with respect a basis of $U$ and $V$. By abusing notation, we will denote the linear map and its associated matrix by the same letter.

The Pauli matrices $\sigma_a$, $a \in \{x, y, z\}$ which are the generators of the Lie algebra $\mathfrak{su}(2)$ of the Lie group $SU(2)$ read

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

where $i$ is the imaginary unit. It is worth mentioning here that $\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = 1$. Based on this fact, one computes

$$\exp(i \theta \sigma_a) = \cos(\theta) \mathbb{1} + i \sin(\theta) \sigma_a,$$

for $a = x, y, z$ and $\theta \in \mathbb{R}$. Precisely, the matrix exponentials of $\sigma_x$, $\sigma_y$ and $\sigma_z$ give rise to the rotation operators $R_x(\theta)$, $R_y(\theta)$ and $R_z(\theta)$ respectively which read

$$R_x(\theta) = \begin{pmatrix} \cos(\theta/2) & -i \sin(\theta/2) \\ -i \sin(\theta/2) & \cos(\theta/2) \end{pmatrix},$$

$$R_y(\theta) = \begin{pmatrix} \cos(\theta/2) & \sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix},$$

$$R_z(\theta) = \begin{pmatrix} e^{-i \theta/2} & 0 \\ 0 & e^{i \theta/2} \end{pmatrix}.$$

### Standard Gates

A quantum circuit is a series of unitary transformations, called gates, which act on qubits. Gates which will appear in this note are: the Hadamard gate $H$ and the phase shift gate $S$ which are applied to single qubits and read respectively

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}.$$

The controlled NOT gate, denoted by $\text{CNOT}$, and the SWAP which are applied to 2-qubits and read respectively

$$\text{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \text{SWAP} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Note that the inverse of the SWAP matrix is equal to itself, symbolically holds $\text{SWAP}^{-1} = \text{SWAP}$.

### The A and B gates

While there are many standard gates, it would be convenient to reason about all non-local two-qubit opera-
tions at the same time, while considering as few parameters as possible. To do so, Zhang et al. \cite{29,30} introduced the three-parameter gate $A$, where each choice of the three parameters corresponds to a local equivalence class of two-qubit gates. In particular, the gate $A$ is defined as the product of the unitary matrices $XX(c_1) := e^{i \frac{1}{2} \sigma_x \otimes \sigma_x}$, $YY(c_2) := e^{i \frac{3}{2} \sigma_y \otimes \sigma_y}$ and $ZZ(c_3) := e^{i \frac{1}{2} \sigma_z \otimes \sigma_z}$ where

$$XX(c_1) = \begin{pmatrix}
\cos \frac{c_3}{2} & 0 & 0 & i \sin \frac{c_3}{2} \\
0 & \cos \frac{c_3}{2} & i \sin \frac{c_3}{2} & 0 \\
i \sin \frac{c_3}{2} & 0 & \cos \frac{c_3}{2} & 0 \\
0 & i \sin \frac{c_3}{2} & \cos \frac{c_3}{2} & 0
\end{pmatrix},$$

$$YY(c_2) = \begin{pmatrix}
\cos \frac{c_3}{2} & 0 & 0 & -i \sin \frac{c_3}{2} \\
0 & \cos \frac{c_3}{2} & i \sin \frac{c_3}{2} & 0 \\
i \sin \frac{c_3}{2} & 0 & \cos \frac{c_3}{2} & 0 \\
0 & i \sin \frac{c_3}{2} & \cos \frac{c_3}{2} & 0
\end{pmatrix},$$

$$ZZ(c_3) = \begin{pmatrix}
i e^{\frac{i c_3}{2}} & 0 & 0 & 0 \\
0 & e^{-\frac{i c_3}{2}} & 0 & 0 \\
0 & 0 & e^{\frac{i c_3}{2}} & 0 \\
0 & 0 & 0 & e^{-\frac{i c_3}{2}}
\end{pmatrix}.$$ 

The analytic form of the gate $A$ is displayed in Fig. 3. See also \cite{29,30} for the elegant geometry thereof.

Associated with the gate $A$, but not as important as the $A$ gate for our considerations, is the $B$ gate which reads $B = e^{i \frac{1}{2} \sigma_y \otimes \sigma_y} \cdot e^{i \frac{1}{2} \sigma_z \otimes \sigma_z}$. The $B$ gate is related to the $A$ gate as $A \sim BUB$, for some $U \in U(2) \times U(2)$ (see \cite{30} page 2]).

### III. THE 1D HEISENBERG MODEL

The Heisenberg model \cite{31} is a spin model of ferromagnetism on a lattice where the coupling energy $J$ between nearest neighbor lattice sites is positive and parallel alignment of local spins is favorable. The variables in the Heisenberg model are subject to a continuous internal symmetry which once broken to $Z_2$ it yields the Ising model which is quite commonly used in the context of Variational Quantum Algorithms (VQAs) \cite{32}. Both the Heisenberg and the Ising models are of particular interest in the theory of quantum integrability precisely due to their integrable nature; they satisfy certain equations for which one can find analytic solutions at any value of the coupling energy $J$ and the thermodynamic limit $N \to \infty$, where $N$ is the number of lattice sites or spins.

Our work is motivated by Ref. \cite{16} which we summarize briefly. In this context, we are interested especially for the 1D Heisenberg model with $N = 2$. This is a case of particular interest since it is the only scenario where the components of the model’s Hamiltonian commute.

The Heisenberg Hamiltonian is defined as

$$\hat{H} = - \sum_a J_a (\sigma^a_1 \otimes \sigma^a_2)$$

where $a \in \{x, y, z\}$, $\sigma^a$ is the Pauli matrix at the direction of $a$, $J_a$ is the coupling constant or interaction strength. The evolution operator of the Schrödinger equation is $e^{i \hat{H}t/\hbar}$ is displayed in Fig. 4.

Notice that $\gamma = \theta_x - \theta_y$ and $\theta_x + \theta_y = \theta_x + \theta_y$ in Fig. 4 is based on a calculation of the individual components:

$$e^{i J_x (\sigma^y_1 \otimes \sigma^y_2)/\hbar} = \begin{pmatrix}
\cos(\theta_x) & 0 & \sin(\theta_x) \\
0 & \cos(\theta_x) & i \sin(\theta_x) \\
i \sin(\theta_x) & 0 & \cos(\theta_x)
\end{pmatrix},$$

$$e^{i J_y (\sigma^y_1 \otimes \sigma^y_2)/\hbar} = \begin{pmatrix}
\cos(\theta_y) & 0 & 0 \\
0 & \cos(\theta_y) & \sin(\theta_y) \\
0 & \sin(\theta_y) & \cos(\theta_y)
\end{pmatrix},$$

$$e^{i J_z (\sigma^z_1 \otimes \sigma^z_2)/\hbar} = \begin{pmatrix}
e^{i \theta_z} & 0 & 0 \\
0 & e^{i - \theta_z} & 0 \\
0 & 0 & e^{i \theta_z}
\end{pmatrix}.$$ 

**Remark 1.** One obtains the matrix $e^{i \hat{H}t/\hbar}$ from the $A$ gate by setting $c_1 = 2(\theta_x - \theta_y)$, $c_2 = 2(\theta_x + \theta_y)$ and $c_3 = 2\theta_z$ in matrix $A$.

The evolution operator of the 1D Heisenberg model as a quantum circuit appears in Fig. 5.

### IV. PENTAGON EQUATION

The pentagon equation is an equation, which belongs in an infinite family of equations called polygon equations, and is similar to the Yang-Baxter equation. It appears in many branches of mathematics, such as representation theory and topological field theory \cite{33}, conformal field theory \cite{34}, Teichmüller theory \cite{35}, Hopf algebras, quantum dilogarithm \cite{36,37}, and operator theory, to name a few. See \cite{38} and the rich bibliography. The pentagon equation has found applications in the context of topological quantum computing previously, see \cite{39} and \cite{40} Ch. 4), as well as in the context of symmetries of non-local matrix product operators \cite{41}.

In this section, we define the pentagon equation. Then, we make a digression on Yang-Baxter equation, pointing on what the two equations differ and on what the two equations look similar.

Let $V$ be a finite dimensional and let $T: V \otimes V \to V \otimes V$ be a linear map. Define the maps

$$T_{12}, T_{23}, T_{13}: V \otimes V \otimes V \to V \otimes V \otimes V$$

by the formulae

$$T_{12} := T \otimes \text{id}_V,$$

$$T_{23} := \text{id}_V \otimes T,$$

$$T_{13} := (\text{id}_V \otimes \tau_{V,V})^{-1} \circ (T \otimes \text{id}_V) \circ (\text{id}_V \otimes \tau_{V,V}).$$
The pentagon equation is defined by

\[ T_{23}T_{12} = T_{12}T_{13}T_{23}. \]  

(2)

A solution of the pentagon equation is a linear operator \( T \) which satisfies Eq. (2). In the bibliography, solutions of the pentagon equations are called fusion operators. Finding solutions of the pentagon equation, otherwise fusion operators, is a challenging process and an active area of research.

**Remark 2.** One way to produce fusion operators, i.e., solutions of the pentagon equation, is from a bialgebra. A bialgebra \( B \) is a vector space with an algebra structure and a compatible coalgebra structure. Denote by \( m \) the product of the algebra and by \( \theta_x + \theta_y \) the coproduct of the coalgebra. Then, the composite map \( T := (\text{id} \otimes m) \circ (\delta \otimes \text{id}) \) is a fusion operator, see [27, Proposition 1.2] for more details. See also [27] for a more physical oriented approach.

One of our interests is to make use of the pentagon equation and compress quantum circuit, that means to reduce the number of gates. For this purpose, we depict the pentagon equation [2] as a quantum circuit as shown in Fig. 6.

**Digression: Yang-Baxter equation**

In this section, we will highlight the connection between the pentagon equation, which is the main topic of this paper with the rather famous Yang Baxter equation.

The Yang-Baxter equation has significant applications in the theory of 2D integrable systems and the theory of quantum groups, see [23,26] and also has been used in the context of quantum computing, see [43].

For a finite dimensional vector space \( V \) and a linear map \( R: V \otimes V \rightarrow V \otimes V \) the Yang-Baxter equation is defined by

\[ (R \otimes \text{id})(\text{id} \otimes R)(R \otimes \text{id}) = (\text{id} \otimes R)(R \otimes \text{id})(\text{id} \otimes R) \]  

(3)

or equivalently by

\[ R_{12}R_{23}R_{12} = R_{23}R_{12}R_{23}. \]  

(4)

where the subscripts indicate which tensor factors are being utilized. The quantum circuit implementation of the YBE [4] is depicted in Fig. 7.

Among the solutions of the Yang-Baxter equation are the identity map \( id: V \otimes V \rightarrow V \otimes V \) and the transposition map \( \tau: V \otimes V \rightarrow V \otimes V \) which reads \( \tau(v_1 \otimes v_2) := v_2 \otimes v_1 \). In Ref. [43], the authors showed that the Bell basis change matrix consists of a solution of the Yang-Baxter equation and, moreover, it is a universal gate for quantum computing, among others. Other solutions may be obtained as universal \( R \) elements of quasi triangular Hopf algebras [43].

**Theorem 3.** (Foklore) Let \( R: V \otimes V \rightarrow V \otimes V \) be a linear map. Then \( R \) is a solution of the Yang-Baxter equation if and only if \( R' := \tau V \circ R \) satisfies the equation

\[ R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}. \]  

(5)

The circuit description of the equation (5) is shown in Fig. 8.
Remark 4. The pentagon equation (2) is Eq. (5) with the term $R_{13}$ omitted on the right-hand side.

There is an analogous result to Theorem 3 for the pentagon equation.

Theorem 5. (Street [27]) Let $T: V \otimes V \rightarrow V \otimes V$ be a linear map. Then, $T$ is a fusion operator if and only if $T' := \tau_{V,V} \circ T$ satisfies the 3-cocycle condition

\[(T' \otimes \text{id}_V) \circ (\text{id}_V \otimes \tau_{V,V}) \circ (T' \otimes \text{id}_V) = (\text{id}_A \otimes T') \circ (T' \otimes \text{id}_V) \circ (\text{id}_V \otimes T') \quad (6)\]

The circuit description of the 3-cocycle is shown in Fig. 9.

Reading this circuit from right to left is more instructive for reasons that will become apparent in the next section. Naturally, the easiest way to implement this type of circuit, should a fusion operator $T$ exist, is to use a SWAP gate. As a matter of fact, in what follows we assume that quantum wire twist is indeed implemented using SWAP gates. However, as mentioned in Fig. 6 in certain architectures, for example in the case of analog neutral atom approaches, it is possible to directly implement such a twist when indeed one is interested in the next-to-nearest-neighbor interactions, just as shown above.

V. RESULTS

In this section, we combine the pentagon equation and the $A$ gate and the evolution operator of the 1D Heisenberg model to the compression of quantum circuits. Initially, we discuss how the pentagon equation is understood as a quantum circuit offering a “local/non-local” form of duality. Then, we show that the $A$ gate and the evolution operator of the 1D Heisenberg model for $N = 2$ are both solutions of the pentagon equation subject to a number of constraining equations.

Local/non-local duality

Our first result is an observation of the nature of the solutions of the pentagon equation and the relevance they can have in the context of quantum circuits.

Purely from the diagrammatic definition of the pentagon equation, Fig. 6, and especially the 3-cocycle condition, Fig. 9, it is possible to realize that should 2-qubit fusion operators of interest be found, the corresponding circuits can be viewed in two dual ways. Before we proceed, we stress again that “Heisenberg interactions” refers only to nearest-neighbor interactions, and we interchangeably use it with the phrase “local interactions”. Similarly, as mentioned in the introduction, non-nearest neighbor (long-range) interactions refer to (2-qubit interactions) non-adjacent qubits.

Let us consider one point of view first: one can reduce the number of non-Heisenberg-type interactions, effectively reducing the corresponding subcircuit depth. However, this requires the ability to enforce the corresponding entangling interactions $T$ which are solutions of the pentagon equation. Currently, for most quantum computing architectures with limited interqubit connectivity, implementation of non-local interactions requires the applica-
Theorem 6. Let $T$ be a unitary gate and a quantum circuit which contains the following component as a sub-circuit:

![Diagram of a quantum circuit with a T gate and SWAP gates]

If $T$ satisfies the pentagon equation (2), then the above component can be replaced by

![Diagram of a single T gate]

Proof. It is straightforward from the assumption that $T$ satisfies the pentagon equation (2).

Theorem 6 provides a way to transpile a quantum circuit assuming that the unitary gate $T$ is a fusion operator. The $A$-gate satisfies the pentagon equation and hence Theorem 6 can be applied for transpiling a quantum circuit for specific values of its coefficients of its coefficients $c_1$, $c_2$ and $c_3$. Precisely, we have the following.

Proposition 7. The $A$-gate, as defined in Eq. (11), satisfies the pentagon equation (2) if and only if the equations of Fig. 11 are satisfied.

Proof. Substituting the $A$ gate into the pentagon equation and equating both sides one arrives at solving the system of equations in Fig. 11.

Solving the equations of Fig. 11 one obtains $c_1 = c_2 = 0$ and $c_3 = -2\pi k$ where $k \in \mathbb{Z}$. This yields the triple eigenvalue of 1 in the corresponding unitary. The convex hull of the eigenvalues will thus not include the origin, which does not make it possible to reapply the proof technique of Corollary 2 in Ref. [39] to show that the parameter values yield a perfect entangler. Therefore, the $A$ gate is a fusion operator albeit of limited use.

Based on the relation of the matrix $A$ and the evolution operator of the 1D Heisenberg model, cf. Remark (1), we have the following.

Corollary 8. The evolution operator (11) of the 1D Heisenberg model for $N = 2$ is a solution of the Pentagon equation (2) if and only if the equations in Fig. 11 are satisfied. That is the case, when $\theta_z = \theta_y = 0$ and $\theta_z = -k\pi$ for $k \in \mathbb{Z}$.

Proof. Substituting the evolution operator of the 1D Heisenberg model into the pentagon equation and equating both sides, one arrives at solving the system of equations of Fig. 11. Solving them, we obtain the values for the coefficients $\theta_z$, $\theta_y$ and $\theta_z$.

VI. SUMMARY AND CONCLUSION

In this note, we investigate under which conditions the $A$ gate and the unitary $e^{i\hat{H}}$, where $\hat{H}$ is the Hamiltonian of the 1D Heisenberg model, satisfy the pentagon equation (2) in the context of circuit compression. By carefully analyzing the pentagon equation, we were able to show that a quantum circuit composed of 2-qubit interactions that include non-local interactions can be compressed to a local interactions only quantum circuit if the corresponding gates are fusion operators, that is, for a particular set of parameters that satisfy the constraining equations.

It is natural to wonder if it is possible to implement non-local, at least next-to-nearest-neighbor, interactions in the current NISQ architectures. As implied in several occasions in the preceding content, there are indeed cases where this is possible, especially in cold-atomic architectures based on the Rydberg blockade [17] or trapped...
ions [8], although it is not straightforward to see how this would scale to more than a few such interactions since it is highly non-trivial to find and control the optimal pulses that would implement such interactions while eliminating possible cross-talk effects. (This challenge is often referred to as “crowding”.) In that sense finding fusion operators that could perform the reduction as described previously would be highly desired. Furthermore, in future iterations of NISQ devices and early fault-tolerant (EFT) devices, there is potential for accurate implementation of next-to-nearest-neighbor interactions required for demanding quantum algorithms, and we hope that finding interesting fusion operators and other such local/non-local dualities could have a substantial impact by reducing the corresponding number of SWAP gates when direct non-nearest neighbor interactions are not possible.

Generally, finding solutions that satisfy the pentagon equation, fusion operators, and transpiling quantum circuits in the approach presented above, is highly non-trivial. While we only considered the pentagon equation in the context of the $A$ gate and the evolution operator of the 1D Heisenberg model, coming up with the “least trivial solution,” one can argue that more solutions exist and can actually perform circuit reduction. Such solutions can potentially arise from other integrable systems.

Another approach to finding fusion operators would be to follow the mathematical recipe given in Remark 2 for the group ring $\mathbb{C}[U(4)]$ of the Lie group $U(4)$. To be more precise, $\mathbb{C}[U(4)]$ is the set of all linear combinations of finitely many elements of $U(4)$ with coefficients in $\mathbb{C}$ and has a bialgebra structure. The product and the coproduct of $\mathbb{C}[U(4)]$ define a fusion operator. We plan to expand this direction in a future note.

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\begin{align*}
    e^{2i\theta_z}\cos^2(\theta_x - \theta_y) - e^{i\theta_z}\cos(\theta_x - \theta_y) & \left( e^{2i\theta_z}\cos^2(\theta_x - \theta_y) - e^{2i\theta_z}\sin^2(\theta_x - \theta_y) \right) = 0 \\
    i e^{2i\theta_z}\sin(\theta_x - \theta_y)\cos(\theta_x - \theta_y) & \left( e^{2i\theta_z}\cos^2(\theta_x - \theta_y) - e^{2i\theta_z}\sin^2(\theta_x - \theta_y) \right) = 0 \\
    i e^{2i\theta_z}\sin(\theta_x - \theta_y)\cos(\theta_x - \theta_y) - 2i e^{3i\theta_z}\sin(\theta_x - \theta_y)\cos^2(\theta_x - \theta_y) & = 0 \\
    2e^{3i\theta_z}\sin^2(\theta_x - \theta_y)\cos(\theta_x - \theta_y) & = 0 \\
    \cos(\theta_x - \theta_y)\cos(\theta_x + \theta_y) - e^{-i\theta_z}\cos(\theta_x + \theta_y) & \left( e^{2i\theta_z}\cos^2(\theta_x - \theta_y) - e^{2i\theta_z}\sin^2(\theta_x - \theta_y) \right) = 0 \\
    i \sin(\theta_x + \theta_y)\cos(\theta_x - \theta_y) - i e^{-i\theta_z}\sin(\theta_x + \theta_y) & \left( e^{2i\theta_z}\cos^2(\theta_x - \theta_y) - e^{2i\theta_z}\sin^2(\theta_x - \theta_y) \right) = 0 \\
    i \sin(\theta_x - \theta_y)\cos(\theta_x + \theta_y) & \left( e^{-2i\theta_z}\cos^2(\theta_x + \theta_y) - e^{-2i\theta_z}\sin^2(\theta_x + \theta_y) \right) = 0 \\
    i \sin(\theta_x + \theta_y) \cos(\theta_x - \theta_y) & \left( e^{-2i\theta_z}\cos^2(\theta_x + \theta_y) - e^{-2i\theta_z}\sin^2(\theta_x + \theta_y) \right) = 0 \\
    -\sin(\theta_x - \theta_y) \sin(\theta_x + \theta_y) & \left( e^{-2i\theta_z}\cos^2(\theta_x + \theta_y) - e^{-2i\theta_z}\sin^2(\theta_x + \theta_y) \right) = 0 \\
    i e^{-2i\theta_z}\sin(\theta_x + \theta_y) & \left( e^{-2i\theta_z}\cos^2(\theta_x + \theta_y) - e^{-2i\theta_z}\sin^2(\theta_x + \theta_y) \right) = 0 \\
    i e^{-2i\theta_z}\sin(\theta_x + \theta_y) & \left( e^{-2i\theta_z}\cos^2(\theta_x + \theta_y) - e^{-2i\theta_z}\sin^2(\theta_x + \theta_y) \right) = 0 \\
    2e^{-3i\theta_z}\sin^2(\theta_x + \theta_y) & = 0
\end{align*}
\begin{align*}
e^{i\xi_2} \cos^2 \left( \frac{\pi}{2} (c_1 - c_2) \right) & - e^{i\xi_3} \cos \left( \frac{\pi}{2} (c_1 - c_2) \right) \left( e^{i\xi_3} \cos^2 \left( \frac{\pi}{2} (c_1 - c_2) \right) - e^{i\xi_3} \sin^2 \left( \frac{\pi}{2} (c_1 - c_2) \right) \right) = 0 \\
i e^{i\xi_3} \sin \left( \frac{\pi}{2} (c_1 - c_2) \right) \cos \left( \frac{\pi}{2} (c_1 - c_2) \right) & - i e^{i\xi_2} \sin \left( \frac{\pi}{2} (c_1 - c_2) \right) \left( e^{i\xi_3} \cos^2 \left( \frac{\pi}{2} (c_1 - c_2) \right) - e^{i\xi_3} \sin^2 \left( \frac{\pi}{2} (c_1 - c_2) \right) \right) = 0 \\
i e^{i\xi_3} \sin \left( \frac{\pi}{2} (c_1 - c_2) \right) \cos \left( \frac{\pi}{2} (c_1 - c_2) \right) & - 2 i e^{\frac{i\pi}{2}} \sin \left( \frac{\pi}{2} (c_1 - c_2) \right) \cos \left( \frac{\pi}{2} (c_1 - c_2) \right) = 0 \\
2 e^{\frac{2i\pi}{2}} \sin^2 \left( \frac{\pi}{2} (c_1 - c_2) \right) \cos \left( \frac{\pi}{2} (c_1 - c_2) \right) & - e^{i\xi_3} \sin^2 \left( \frac{\pi}{2} (c_1 - c_2) \right) = 0 \\
\cos \left( \frac{\pi}{2} (c_1 - c_2) \right) \cos \left( \frac{\pi}{2} (c_1 + c_2) \right) & - e^{-\frac{i\pi}{2}} \cos \left( \frac{\pi}{2} (c_1 + c_2) \right) \left( e^{i\xi_3} \cos^2 \left( \frac{\pi}{2} (c_1 - c_2) \right) - e^{i\xi_3} \sin^2 \left( \frac{\pi}{2} (c_1 - c_2) \right) \right) = 0 \\
i \sin \left( \frac{\pi}{2} (c_1 + c_2) \right) \cos \left( \frac{\pi}{2} (c_1 + c_2) \right) & - i e^{-\frac{i\pi}{2}} \cos \left( \frac{\pi}{2} (c_1 + c_2) \right) \left( e^{i\xi_3} \cos^2 \left( \frac{\pi}{2} (c_1 - c_2) \right) - e^{i\xi_3} \sin^2 \left( \frac{\pi}{2} (c_1 - c_2) \right) \right) = 0 \\
- \sin \left( \frac{\pi}{2} (c_1 - c_2) \right) \sin \left( \frac{\pi}{2} (c_1 + c_2) \right) + 2 e^{\frac{i\pi}{2}} \sin \left( \frac{\pi}{2} (c_1 - c_2) \right) \sin \left( \frac{\pi}{2} (c_1 + c_2) \right) \cos \left( \frac{\pi}{2} (c_1 - c_2) \right) = 0 \\
\cos \left( \frac{\pi}{2} (c_1 - c_2) \right) \sin \left( \frac{\pi}{2} (c_1 + c_2) \right) & - e^{i\xi_3} \cos \left( \frac{\pi}{2} (c_1 + c_2) \right) \left( e^{-i\xi_3} \cos^2 \left( \frac{\pi}{2} (c_1 - c_2) \right) - e^{-i\xi_3} \sin^2 \left( \frac{\pi}{2} (c_1 - c_2) \right) \right) = 0 \\
i \sin \left( \frac{\pi}{2} (c_1 + c_2) \right) \sin \left( \frac{\pi}{2} (c_1 - c_2) \right) & - i e^{-\frac{i\pi}{2}} \sin \left( \frac{\pi}{2} (c_1 - c_2) \right) \left( e^{-i\xi_3} \cos^2 \left( \frac{\pi}{2} (c_1 + c_2) \right) - e^{-i\xi_3} \sin^2 \left( \frac{\pi}{2} (c_1 + c_2) \right) \right) = 0 \\
- \sin \left( \frac{\pi}{2} (c_1 + c_2) \right) \sin \left( \frac{\pi}{2} (c_1 - c_2) \right) + 2 e^{-\frac{i\pi}{2}} \sin \left( \frac{\pi}{2} (c_1 - c_2) \right) \sin \left( \frac{\pi}{2} (c_1 + c_2) \right) \cos \left( \frac{\pi}{2} (c_1 + c_2) \right) = 0 \\
e^{-i\xi_3} \cos^2 \left( \frac{\pi}{2} (c_1 + c_2) \right) & - e^{-\frac{i\pi}{2}} (i \xi_3) \cos \left( \frac{\pi}{2} (c_1 + c_2) \right) \left( e^{-i\xi_3} \cos^2 \left( \frac{\pi}{2} (c_1 + c_2) \right) - e^{-i\xi_3} \sin^2 \left( \frac{\pi}{2} (c_1 + c_2) \right) \right) = 0 \\
i e^{-i\xi_3} \sin \left( \frac{\pi}{2} (c_1 + c_2) \right) \cos \left( \frac{\pi}{2} (c_1 + c_2) \right) & - i e^{-\frac{i\pi}{2}} (i \xi_3) \sin \left( \frac{\pi}{2} (c_1 + c_2) \right) \left( e^{-i\xi_3} \cos^2 \left( \frac{\pi}{2} (c_1 + c_2) \right) - e^{-i\xi_3} \sin^2 \left( \frac{\pi}{2} (c_1 + c_2) \right) \right) = 0 \\
- i e^{-i\xi_3} \sin \left( \frac{\pi}{2} (c_1 + c_2) \right) \cos \left( \frac{\pi}{2} (c_1 + c_2) \right) & - 2 i e^{-\frac{i\pi}{2}} (3i \xi_3) \sin \left( \frac{\pi}{2} (c_1 + c_2) \right) \cos \left( \frac{\pi}{2} (c_1 + c_2) \right) = 0 \\
2 e^{-\frac{i\pi}{2}} (3i \xi_3) \sin^2 \left( \frac{\pi}{2} (c_1 + c_2) \right) & \cos \left( \frac{\pi}{2} (c_1 + c_2) \right) - e^{-i\xi_3} \sin^2 \left( \frac{\pi}{2} (c_1 + c_2) \right) = 0
\end{align*}

FIG. 11. The equations to be satisfied by the $A$ gate of the fusion operator of Fig. 6

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