Legendrian Submanifold Path Geometry

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1 Introduction

In [Ch1], Chern gives a generalization of projective geometry by considering foliations on the Grassman bundle of $p$-planes $Gr(p, R^n) \to R^n$ by $p$-dimensional submanifolds that are integrals of the canonical contact differential system. The equivalence method yields an $\mathfrak{sl}(n + 1, R)$-valued Cartan connection whose curvature captures the geometry of such foliation. In the flat case, the space of leaves of the foliation is a second order homogeneous space [Br2].

[Ch2] deals with the geometry of the foliation of $Z^4 \to Y^3$, where $Z$ is the bundle of Legendrian line elements over a contact threefold $Y$, by canonical lifts of Legendrian curves, or equivalently, the geometry of 3-parameter families of curves in the plane. An $\mathfrak{sp}(2, R)$-valued Cartan connection plays the role of projective connection.

A generalization of [Ch2] to 4-parameter family of curves in the plane leads to a geometric realization of some exotic holonomies in dimension four [Br1].

In this paper, we generalize [Ch2] to higher dimensions. Let $Z \to Y^{2n+1}$ be the bundle of Legendrian $n$-planes over a contact manifold $Y$. We consider a foliation of $Z$ by canonical lifts of Legendrian submanifolds, which we call a Legendrian submanifold path geometry. Note that a path in this case is a Legendrian $n$-fold. The equivalence method provides an $\mathfrak{sp}(n + 1, R)$-valued Cartan connection form that captures the geometry of such foliation. In the flat case, the space $X$ of leaves of the foliation is again a second order homogeneous space. The prolonged structure equation of this second order homogeneous space is in turn that of $Sp(n + 1, R)$, which explains the appearance of a $\mathfrak{sp}(n + 1, R)$-valued Cartan connection form. In fact, we may consider a contact manifold $Y$ endowed with a Legendrian submanifold path geometry structure as a union of infinitesimal homogeneous spaces

$$Sp(n + 1, R) \to RP^{2n+1}$$

connected by the Cartan connection mentioned above. As a by product, this gives a geometric realization of the Lie algebra $\mathfrak{sp}(n + 1, R)$ as the symmetry vector fields of a family of Legendrian submanifolds in a $(2n + 1)$-dimensional contact manifold.
After a short analysis of the structure equations associated to a general Legendrian submanifold path geometry, two special cases are considered. The first case is characterized by having a well defined conformal class of symmetric \((n+1)\) differentials on the space of leaves of the foliation \(X\). The vanishing of this symmetric differential represents a necessary condition for the contact of neighboring Legendrian leaves. A double fibration naturally arises, and we give a dual interpretation of the contact manifold \(Y\) in terms of \(X\). The \(G\)-structure induced on \(X\) gives an example of a classical non-metric irreducible holonomy \(GL(n+1, R)\) with representation on \(\text{sym}^2(R^{n+1})\).

It is well known that the normal projective connection uniquely associated to a torsion free affine connection captures the geometry of paths defined by the geodesics of the affine connection. The second case is a direct generalization of this to Legendrian submanifold path geometry. We consider a Legendrian connection on the contact hyperplane vector bundle over \(Y\) whose geodesic Legendrian submanifolds give rise to a Legendrian submanifold path geometry. There exists a unique normal symplectic connection associated to a Legendrian connection such that any other Legendrian connection with the equivalent Legendrian submanifold path geometry is a section of the normal symplectic connection. An analysis of the normal symplectic connection shows in fact the family of geodesic isotropic \(k\)-folds for \(1 \leq k \leq n - 1\) with respect to a Legendrian connection is also an invariant of the normal symplectic connection.

\(RP^{2n+1}\), as a quotient space of \(Sp(n+1, R)\), carries a Legendrian submanifold path geometry, which is flat. For a nonflat example with symmetry, consider a hypersurface \(M^n\) in the \((n+1)\)-dimensional space form \(\bar{M}^{n+1}_c\), \(c = 1, 0,\) or \(-1\), without any extrinsic symmetry. The images of \(M\) under the motion by \(\text{Iso}(\bar{M}^{n+1}_c)\), when lifted to \(Gr(n, \bar{M}^{n+1}_c)\), generates an \(N = \frac{1}{2}(n+1)(n+2)\)-parameter family of Legendrian submanifolds. Since \(\text{Iso}(\bar{M}^{n+1}_c)\) does not arise as a subgroup of \(Sp(n+1, R)\), it is not equivalent to the flat example.

Similar constructions are likely to work for other (irreducible) second order homogeneous spaces \([Br2]\). For instance, in the holomorphic category for simplicity, a manifold with \(CO(V)\) structure has, as its dual, a manifold with \(GL(W)\) structure with representation \(\Lambda^2(W)\) where \(V\) and \(W\) are vector spaces of suitable dimensions. The corresponding Cartan connection form would be \(\mathfrak{so}(m, \mathbb{C})\)-valued for suitable \(m\). Two exceptional cases, \(\mathbb{C}^*\text{Spin}(10, \mathbb{C})\) on \(S_+\) and \(\mathbb{C}^*E_6^\mathbb{C}\) on \(\mathbb{C}^{27}\), would yield Cartan connection forms with values in \(\mathfrak{e}_6^\mathbb{C}\) and \(\mathfrak{e}_7^\mathbb{C}\), respectively, for the associated geometries.

All the arguments remain valid when we replace real and smooth by complex and holomorphic. In fact, a real Legendrian submanifold path geometry can be considered as a split real form of a complex one. In analogy with the geometry of real hypersurfaces in \(\mathbb{C}^n\) considered as a real form of a complex hypersurface path geometry via Segre families \([Fa] [ChM]\), it would be interesting to study other possible real forms of complex Legendrian submanifold path geometry.

We shall agree that all the Latin indices \(i, j, k\) run from 1 to \(n\), and, for simplicity, that \(n \geq 2\).

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2 Legendrian Submanifold Path Geometry

2.1 Definition

Let $Y$ be a $(2n + 1)$-dimensional manifold with a contact structure, i.e., a differential 1-form $\theta_0$ defined up to multiplication by a nonzero function with the property

$$\theta_0 \wedge (d\theta_0)^n \neq 0. \quad (1)$$

The differential system on $Y$ locally generated by $\theta_0$ and $d\theta_0$ is called a contact differential system. It is known that at each point of $Y$, a subspace of the tangent space of $Y$ that is integral to the contact differential system is of dimension at most $n$ and such a subspace is called a Legendrian $n$-plane. An integral $n$-dimensional submanifold is similarly called a Legendrian $n$-fold or Legendrian submanifold. Let $Z \to Y$ be the associated bundle of Legendrian $n$-planes, which we may regard as the first prolongation of the contact differential system on $Y$. The theorem of Darboux on the local normal form for the contact structures provides local coordinates $\{x^i, u, p_i, p_{ij}\}$ on $Z$ such that $\{x^i, u, p_i\}$ is a local coordinate system on $Y$ and so that the first prolongation of the contact system to $Z$ is generated by

$$\theta_0 = du - \sum_{k=1}^{n} p_k \, dx^k$$
$$\theta_i = dp_i - \sum_{k=1}^{n} p_{ik} \, dx^k, \quad i = 1,..,n.$$ 

This means that an $n$-dimensional integral submanifold of the differential system generated by

$$\{ \theta_0, \theta_1, .., \theta_n, d\theta_1, .., d\theta_n \}$$

on which $dx^1 \wedge .. \wedge dx^n \neq 0$ is the canonical lift of a Legendrian $n$-fold in $Y$. In fact, given any point in a Legendrian submanifold in $Y$, there exists a local coordinate system $\{x^i, u, p_i\}$ of $Y$, in the neighborhood of the given point, such that the given Legendrian submanifold is defined by the equations $\{u = 0, p^1 = 0, .., p^n = 0\}$, the contact form $\theta_0$ is a multiple of

$$du - \sum_{k=1}^{n} p_k \, dx^k,$$

and $dx^1 \wedge .. \wedge dx^n \neq 0$ on the given Legendrian submanifold, at least in a neighborhood of the given point.

The problem we are interested in is a local geometry of the foliation of $Z$, necessarily transversal to the fibers of the projection $Z \to Y$, by $n$-dimensional submanifolds that are lifts of Legendrian $n$-folds in $Y$. In terms of $Y$, this means to each Legendrian $n$-plane (possibly only those in an open set of the set of Legendrian $n$-planes) at a point, there exists a unique Legendrian submanifold of this family tangent to the given Legendrian $n$-plane. Let $X$ denote
the space of the leaves of the foliation, which we always assume to be a nice \( N = \frac{1}{2}(n+2)(n+1) \)
-dimensional manifold.

We could also describe this geometry as the geometry of a nondegenerate \( N \)-parameter family of Legendrian submanifolds in \( Y \). Here, an \( N \)-parameter family of Legendrian submanifolds is said to be non degenerate if the following is true: Let \( X^N \) be the parameter space and \( l : X \times \mathbb{R}^n \to Y \) the \( N \)-parameter family of Legendrian immersions of \( \mathbb{R}^n \) to \( Y \). We require that the associated lift \( \hat{l} : X \times \mathbb{R}^n \to Z \) be a (local) diffeomorphism. Roughly, this means that the union of the lifts of the \( N \)-parameter family of Legendrian submanifolds to \( Z \) fills out an open set in \( Z \).

**Definition** Legendrian Submanifold Path Geometry

Let \( Z \to Y^{2n+1} \) be the bundle of Legendrian \( n \)-planes of a contact manifold \( Y \). Legendrian submanifold path geometry is the geometry of foliations on \( Z \), up to the diffeomorphism of \( Z \) induced from the contact transformation of \( Y \), whose leaves are the canonical lifts of Legendrian submanifolds in \( Y \). Locally, this is equivalent to the geometry of a non degenerate \( N = \frac{1}{2}(n+1)(n+2) \)-parameter family of Legendrian submanifolds in \( Y \) up to contact transformations.

### 2.2 Structure Equations

In this section, we give a brief analysis of the structure equation for general Legendrian submanifold path geometry.

Let \( \mathcal{I} \) be the Frobenius system on \( Z \) describing the given Legendrian submanifold path geometry. The local normal form theorem above shows that \( \mathcal{I} \) is locally generated by the following 1-forms.

\[
\theta_0 = du - \sum_k p_k \, dx^k \\
\theta_i = dp_i - \sum_k p_{ik} \, dx^k, \quad i = 1, \ldots n \\
\Theta_{ij} = dp_{ij} - \sum_k F_{ijk} \, dx^k, \quad i, j = 1, \ldots n,
\]

where \( \{ x^i, u, p_i, p_{ij} \} \) is the local coordinate system mentioned above and \( \{ F_{ijk} \} \) is a set of functions locally defined on \( Z \). These differential forms form a subset of a coframe \( \{ \omega^i = dx^i, \theta_0, \theta_i, \Theta_{ij} \} \) of \( Z \), defined up to the diffeomorphisms of \( Z \) induced from the contact transformations of \( Y \), with

\[
\mathcal{I} = \{ \theta_0, \theta_i, \Theta_{ij} \}
\]
and

\begin{align*}
    d\theta_0 &\equiv -\sum_k \theta_k \wedge \omega^k \mod \theta_0 \quad (4) \\
    d\theta_i &\equiv -\sum_k \Theta_{ik} \wedge \omega^k \mod \theta_0, \theta \\
    d\Theta_{ij} &\equiv 0 \mod \theta_0, \theta, \Theta,
\end{align*}

where \( \mod \theta \) means \( \mod \theta_1, \ldots, \theta_n \), and similarly for \( \mod \Theta \).

Given such a differential system \( \mathcal{I} \) on \( Z \), the set of all coframes \( \{ \omega^i, \theta_0, \theta_i, \Theta_{ij} \} \) on \( Z \) satisfying (3) and (4) form a \( H_1 \subset GL(N+n,R) \) bundle \( F_1 \) over \( Z \), where \( H_1 \) is the subgroup whose induced action on \( T^*Z \) preserves (3) and (4). Equivalently, the principal right \( GL(N+n,R) \) coframe bundle can be reduced to a \( H_1 \) subbundle via \( \mathcal{I} \). It can be easily shown the right action of \( H_1 \) on the tautological forms of \( F_1 \), which are by definition the restriction to \( F_1 \) of the \( GL(N+n,R) \) equivariant \( R^{N+n} \)-valued tautological 1-form on the principal \( GL(N+n,R) \) bundle, is as follows.

\begin{align*}
    \theta_0^* &= \lambda \theta_0 \\
    \theta^* &= A(\theta + b \theta_0) \\
    \Theta^* &= \frac{1}{\lambda} A(\Theta + e \theta_0 + \sum_i A_i \theta_i) A' \\
    \omega^* &\equiv \lambda A^{-1}(\omega + c \theta) \mod \theta_0,
\end{align*}

where we denote

\begin{align*}
    \theta^t &= (\theta_1, \ldots, \theta_n) \\
    \Theta &= (\Theta_{ij}) = (\Theta_{ji}) \\
    \omega^t &= (\omega^1, \ldots, \omega^n),
\end{align*}

and \( \lambda \neq 0, A \in GL(n,R), A_i, e, c \in \text{sym}^2(n,R), \) and \( b \in R^n \). Here \( \text{sym}^2(n,R) \) denotes the set of \( n \) by \( n \) symmetric matrices. We used the same notation \( \omega^i, \theta_0, \theta_i, \Theta_{ij} \) as above to denote the corresponding tautological forms.

Thus on \( F_1 \), we have the following structure equations.

\begin{align*}
    d\theta_0 &= -\rho \wedge \theta_0 - \theta^t \wedge \omega + T_{00} \wedge \theta_0 \\
    d\theta &= -\beta \wedge \theta_0 - \alpha \wedge \theta - \Theta \wedge \omega + \sum_k T_{11}^k \wedge \theta_k + T_{10} \wedge \theta_0 \\
    d\Theta &= -\epsilon \wedge \theta_0 - \sum_k \pi^k \wedge \theta_k - (\alpha - \frac{\rho}{2}) \wedge \Theta + \Theta \wedge (\alpha - \frac{\rho}{2})^t \\
    &\quad + \sum_k T_{21}^k \wedge \theta_k + T_{20} \wedge \theta_0 + T_0 \\
    d\omega &= -\mu \wedge \theta_0 - \gamma \wedge \theta + (\alpha^t - \rho) \wedge \omega + T_{01}^t \wedge \theta_0 + T_{11}.
\end{align*}
Here $\rho$ is a scalar 1-form, $\alpha$ is a $gl(n, R)$-valued 1-form, $\pi^k, \epsilon, \gamma$ are symmetric $gl(n, R)$-valued 1-forms, and $\beta, \mu$ are $R^n$ (column) valued 1-forms. These are the pseudo connection forms on $F_1 \to Z$, and $T_{00}, T_{11}^k, T_{10}, T_{21}^k, T_{20}, T_{0}, T_{1}^0, T_{11}^{11}$ represent the torsion of this pseudo connection. The pseudo connection forms are not uniquely defined. By modifying the pseudo connection forms, we may reduce the torsion to the following simple form.

**Proposition 1** There exists a pseudo connection on $F_1$ for which the structure equation takes the following form.

$$
\begin{align*}
d\theta_0 &= -\rho \wedge \theta_0 - \theta^t \wedge \omega \\
d\theta &= -\beta \wedge \theta_0 - \alpha \wedge \theta - \Theta \wedge \omega \\
d\Theta &= -\epsilon \wedge \theta_0 - \sum_k \pi^k \wedge \theta_k - (\alpha - \frac{\rho}{2}) \wedge \Theta + (\alpha - \frac{\rho}{2})^t + T_0 \\
d\omega &= -\mu \wedge \theta_0 - \gamma \wedge \theta + (\alpha^t - \rho) \wedge \omega,
\end{align*}
$$

where $T_0 = \sum_{ij} \Theta_{ij} \wedge \tau^{ij}$ with each $\tau^{ij} = \tau^{ji}$ being symmetric $gl(n, R)$-valued 1-form satisfying $\tau^{ij} \wedge \omega = 0$.

**Proof.** First, by modifying $\rho$ and $\beta$, we can absorb $T_{00}$ and $T_{10}$. Also, the second equation in (4) implies that we can arrange $T_{11}^k$’s to be 0 by modifying $\alpha$. Thus all the torsion terms in $d\theta_0$ and $d\theta$ can be absorbed, which we assume from now on.

We modify $\mu$ to absorb $T_{0}^1$ and arrange $T_{11}^{11}$ to be quadratic in $\{\omega, \theta, \Theta\}$. Now, $d(d\theta_0) \equiv 0 \mod \theta_0$ gives $\theta^t \wedge T_{11}^{11} = 0$, which means $T_{11}^{11}$ is of the form

$$T_{11}^{11} = (h_{ij}) \wedge \theta$$

with $(h_{ij})$ being a $gl(n, R)$-valued 1 form in $\omega, \theta, \Theta$, which is not uniquely defined. Now, it is easily verified that by modifying the first row or column of $(\gamma_{ij}) = (\gamma_{ji})$ and the representation $(h_{ij})$, we may have $h_{1j} = 0$ and $h_{ij} = 0$ for $j = 1, \ldots, n$ and $(h_{ij}) = 0 \mod \omega, \theta_2, \theta_3, \ldots, \theta_n, \Theta$. Hence by induction, we can absorb all of $T_{11}^{11}$ using $\gamma$.

Finally, we modify $\pi^k$ and $\epsilon$ to absorb $T_{21}^k, T_{20}$ and arrange $T_0$ to be quadratic in $\{\omega, \Theta\}$. But $d(d\theta) \equiv 0 \mod \theta_0, \theta$ gives

$$T_0 \wedge \omega = 0,$$

which implies $T_0$ cannot have any quadratic terms in $\Theta$, and since $\mathcal{I}$ is Frobenius, it cannot have any quadratic terms in $\omega$ either. □

The torsion $T_0$, as it stands, is not an invariant of the foliation. In fact the structure group $H_1$ acts on $T_0$. However, rather than continuing the analysis of equivalence problem directly, we examine a special case of a foliation motivated by [Ch2], namely, that of quadric hypersurfaces in $R^{n+1}$. 

6
3 Second Order Developables for Quadric Hypersurfaces in $R^{n+1}$

The local normal form (2) for $I$ on $Z$ shows, at least locally, we can identify $Z \rightarrow Y$ with $J^2(R^n, R^{n+1}) \rightarrow J^1(R^n, R^{n+1})$ and regard the geometry of the foliation as the geometry of an $N$-parameter family of hypersurfaces in $R^{n+1}$ up to contact transformations. In this section, we take the simple example of $I$ on $J^2(R^n, R^{n+1}) \sim Z$ defining the quadric hypersurfaces in $R^{n+1}$, $u = a_0 + \sum_i a_i x_i^i + \sum_{ij} \frac{1}{2} a_{ij} x_i^i x_j^j$ (7)

$p_i = a_i + \sum_j a_{ij} x_j^i$

$p_{ij} = a_{ij}$

where $a_{ij} = a_{ji}$ and $\{u, x^i, p_i, p_{ij}\}$ is a local coordinate system of $Z$ introduced earlier. Given the explicit form of solutions, we may regard $\{a_0, a_i, a_{ij}\}$ as a local coordinate system on the space of solutions of $I$. Also, it is easy to see that this family of submanifolds in $J^1(R^n, R^{n+1}) \cong Y$ is non degenerate in the sense discussed earlier.

Consider a hypersurface $S$ in $R^{n+1}$ defined as the graph of a function $f$ in $n$ variables

$u = f(x_1, .. x_n)$.

At each point of $S$, there exists a quadric hypersurface of the form (7) that osculates the given hypersurface $S$ up to second order. The set of all such quadric hypersurfaces along $S$ generically form an $n$-parameter family of solutions to $I$, or an $n$-dimensional submanifold $S$ in the space of solutions. Conversely we may consider the original hypersurface $S$ as the second order developable of the family $S$.

From the construction, each quadric hypersurface in the family $S$ has the point of contact with the given $S$, $\{u, x^1, .. x^n\}$, at which

$\theta_0 = du - \sum_i p_i dx^i = da_0 + \sum_i x^i da_i + \sum_{ij} \frac{1}{2} x^j x^k da_{jk} = 0$

$\theta_i = dp_i - \sum_j p_{ij} dx^j = da_i + \sum_j x^j da_{ij} = 0$.

Equivalently

$$\begin{pmatrix} 2 da_0 & da_t \\ da & dA \end{pmatrix} \begin{pmatrix} 1 \\ X \end{pmatrix} = 0,$$

where $da_t = (da_1, .. da_n)$, $(dA)_{ij} = da_{ij}$, and $X^t = (x^1, .. x^n)$, which is now considered as a vector valued function on $S$. In other words, $S$, as an $n$-dimensional submanifold in the space
of solutions \( \{ a_0, a_i, a_{ij} \} \), is not only a null submanifold with respect to the symmetric \((n + 1)\) differential

\[
\det \begin{pmatrix} 2da_0 & da^t \\ da & dA \end{pmatrix},
\]

but in fact it is a *singular* null submanifold, meaning the matrix valued 1-form above has a null vector as in (8).

Conversely, suppose \( S \) is an \( n \)-dimensional singular null submanifold in the space of quadric hypersurfaces in \( R^{n+1} \). Generically, there is a vector valued function \((1, V^t)\) along \( S \) with \( V^t = (v^1, \ldots, v^n) \) such that

\[
\begin{pmatrix} 2da_0 & da^t \\ da & dA \end{pmatrix} \begin{pmatrix} 1 \\ V \end{pmatrix} = 0,
\]

and \( dv^1 \wedge dv^2 \wedge \ldots \wedge dv^n \neq 0 \) on \( S \). From the argument above, it is clear that the formula (7) with \( x^i \) replaced by \( v^i \) describes a hypersurface in \( R^{n+1} \) that is the second order developable to the given family of hyperquadrics \( S \).

Thus, at least in this flat example, the vanishing of the \((n + 1)\) symmetric differential

\[
\det \begin{pmatrix} 2da_0 & da^t \\ da & dA \end{pmatrix},
\]

is a necessary condition for the contact of the neighboring Legendrian submanifolds. In case \( n = 1 \), it is also sufficient. We mention that for general *nonflat* family of Legendrian \( n \)-folds with \( n \geq 2 \), the condition of contact of the neighboring submanifolds may not be as simple as this, as is discussed in [Ch2].
4. G structure on the Space of Solutions

4.1 Contact of neighboring Legendrian Leaves

The flat example considered above suggests the special class of the differential system \( I \) on \( Z \) for which the conformal class of symmetric \((n+1)\) differentials, the vanishing of which represents a necessary condition for the contact of the neighboring Legendrian submanifolds, is well defined on the space of the leaves of the foliation \( X \). In fact, [Ch2] shows in case \( n = 1 \), the vanishing of a single relative invariant associated to \( I \) is both necessary and sufficient condition for a conformal class of quadratic differential to be well defined on the space of solutions.

A higher dimensional analogue of this result exists and can be described as follows. We continue to use the notation adopted in Section 3.

Proposition 2 Let \( F_1 \to Z \to Y \) be the bundle associated to a differential system \( I \) with a pseudo connection such that the structure equation (6) is true. The conformal class of the symmetric \((n+1)\) differential

\[
\det \begin{pmatrix} 2\theta_0 & \theta^t \\ \theta & \Theta \end{pmatrix}
\]

is well defined on the space of solutions if the bundle \( F_1 \to Z \) admits a reduction to a subbundle \( F \subset F_1 \) on which

\[
T_0 = 0,
\]

\[
\pi^k_{ij} \equiv \frac{1}{2}(\delta_{ik}\beta_j + \delta_{jk}\beta_i) \mod \theta_0, \theta, \Theta,
\]

\[
\epsilon \equiv 0 \mod \theta_0, \theta, \Theta.
\]

The structure equations on \( F \) become

\[
d\theta_0 = -\rho \wedge \theta_0 - \theta^t \wedge \omega
\]

\[
d\theta = -\beta \wedge \theta_0 - \alpha \wedge \theta - \Theta \wedge \omega
\]

\[
d\Theta = -\frac{1}{2}(\beta \wedge \theta^t - \theta \wedge \beta^t) - (\alpha - \rho) \wedge \Theta + \Theta \wedge (\alpha - \rho)^t + T
\]

\[
d\omega = -\mu \wedge \theta_0 - \gamma \wedge \theta + (\alpha^t - \rho) \wedge \omega,
\]

where \( T \) is quadratic in \( \{\theta_0, \theta, \Theta\} \) with \( T \equiv 0 \mod \theta_0, \theta, \Theta. \)

Before we begin the proof, we wish to give a interpretation of the reduction procedure in local coordinates as in (5). Once we get the structure equation (6) starting from the representation (2) of \( I \), the torsion \( T_0 \) is an expression involving \( \{x^i, u, p_i, p_{ij}\} \), \( \{F_{ijk}\} \) and their derivatives, and the group variables \( \{\lambda, A, b, e, A_0, A_i, \ldots\} \).

First, the reduction \( T_0 = 0 \) means we must be able to solve the equation \( T_0 = 0 \) by expressing \( A_i \) in terms of \( \{x^i, u, p_i, p_{ij}\} \), \( \{F_{ijk}\} \) and their derivatives, and \( b \). Once we impose this relation back into the structure equation, we have

\[
d\Theta = -\epsilon \wedge \theta_0 - \frac{1}{2}(\beta \wedge \theta^t - \theta \wedge \beta^t) - (\alpha - \rho) \wedge \Theta + \Theta \wedge (\alpha - \rho)^t + T'_0
\]
where \( T'_0 \) is a new torsion term.

The second reduction \( \pi^k_{ij} \equiv \frac{1}{2}(\delta_{ik}\beta_j+\delta_{jk}\beta_i) \pmod{\theta_0, \theta, \Theta} \) means we must be able to express \( e \) in terms of \( \{x^i, u_i, p_i, p_{ij}\}, \{F_{ijk}\} \) and their derivatives such that \( T'_0 \) does not have any terms of the form \( \theta_i \wedge \omega^j \). Finally, the third reduction \( \epsilon \equiv 0 \pmod{\theta_0, \theta, \Theta} \) means that when we impose the expression for \( e \) obtained as above back into the structure equations, the resulting torsion \( T \) should not have any terms involving \( \omega_i \)'s.

**Proof of the Proposition** Note that the structure equation on \( F \) can be rewritten in a matrix form

\[
d\begin{pmatrix} 2\theta_0 & \theta^t \\ \theta & \Theta \end{pmatrix} = -\begin{pmatrix} \frac{1}{2}\rho & -\omega^t \\ \frac{1}{2}\beta & \alpha - \frac{1}{2}\rho \end{pmatrix} \wedge \begin{pmatrix} 2\theta_0 & \theta^t \\ \theta & \Theta \end{pmatrix} + \begin{pmatrix} 2\theta_0 & \theta^t \\ \theta & \Theta \end{pmatrix} \wedge \begin{pmatrix} \frac{1}{2}\rho & \frac{1}{2}\beta^t \\ -\omega & \alpha^t - \frac{1}{2}\rho \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & T \end{pmatrix},
\]

with

\[d\omega = -\mu \wedge \theta_0 - \gamma \wedge \theta + (\alpha^t - \rho) \wedge \omega.\]

Let \( G \subseteq GL(N, R) \) be the subgroup corresponding to the representation of \( GL(n+1, R) \) on the space of symmetric quadratic differentials on \( R^{n+1} \) or on the space of \( (n+1) \) by \( (n+1) \) symmetric matrices. The equation above, when \( \theta_0, \theta, \) and \( \Theta \) are interpreted as a basis of semibasic 1-forms, is the equation of structure of an ordinary \( G \) bundle over \( X \). From the representation, it is clear that the \( G \) structure on \( X \) induces a conformal class of symmetric \( (n+1) \) differential \( (9) \) on \( X \).

\( \square \)

**Theorem 1** Consider a Legendrian submanifold path geometry on \( Z \rightarrow Y \) defined by a Frobenius system \( \mathcal{I} \) on \( Z \). If the original \( H_1 \subset GL(N+n, R) \) bundle \( F_1 \) on \( Z \) admits a reduction to a subbundle \( F \) on which the conditions \( (10) \) are satisfied, the bundle

\[ F \rightarrow X \]

via \( F \rightarrow Z \rightarrow X \) is an ordinary \( G \subset GL(N, R) \) structure on \( X \) with \( (11) \) as a structure equation. A generic \( n \)-dimensional singular null submanifold in \( X \) with respect to \( (9) \) corresponds to an \( n \)-parameter family of solutions of \( \mathcal{I} \) that admits a developable Legendrian submanifold in \( Y \).

Note that the notion of an \( n \)-plane in the tangent space of \( X \) being singular null with respect to the matrix valued 1-form \( (9) \) is equivariant under the action of \( G \). It is likely the conditions \( (10) \) in the proposition is also necessary for the symmetric differential to be well defined on the space of solutions \( X \).
4.2 Dual Description

In the present case, \( X \) inherits a \( G \) structure from being the space of Legendrian leaves. We briefly discuss what could possibly be a description dual to this. Let \( X \) be an \( N = \frac{1}{2}(n+1)(n+2) \)-dimensional manifold with a \( G \) structure with the structure equation (11) defined on the associated bundle \( F_G \to X \). Define an \( m = \frac{1}{2}n(n+1) \)-plane \( \mathcal{N}_p \subset T_pX \) to be totally null if it corresponds, under the identification of \( T_pX \) with the space of quadratic differentials on \( R^{n+1} \) via the \( G \) structure, to a subspace

\[
\mathcal{N}_p \cong \{ Q \in \text{sym}^2(R^{n+1^*}) \mid v \cdot Q = 0 \text{ for some nonzero } v \in R^{n+1} \}.
\]

Let \( \mathcal{N} \to X \) be the bundle of totally null \( m \)-planes with the associated projection \( F_G \to \mathcal{N} \). The structure group \( G \) acts transitively on the set of totally null \( m \)-planes, and from (11), we may take the differential system on \( F_G \) generated by

\[
\{ \theta_0, \theta, d\theta \}
\]

as the pullback of the differential system \( J_0 \) on \( \mathcal{N} \) whose solutions are totally null submanifolds.

Consider a Frobenius differential system \( J \supset J_0 \) on \( \mathcal{N} \) whose pull back to \( F_G \) is generated by

\[
\{ \theta_0, \theta, \omega \}.
\]

Equation (11) shows that the differential system above can indeed be pushed down to \( \mathcal{N} \). The definition of the bundle \( \mathcal{N} \) and the second equation in (11) also shows that \( \omega \), considered as a pseudo connection form of the bundle \( \mathcal{N} \to X \), measures the rate of change of the tangent \( m \)-planes along a totally null submanifolds. The differential system \( J \) on \( \mathcal{N} \) thus describes the canonical lifts to \( \mathcal{N} \) of geodesic totally null submanifolds in \( X \).

It is now clear from (11) that the space of leaves of the foliation of \( \mathcal{N} \) by geodesic totally null submanifolds, \( Y \), inherits a contact structure with an associated Legendrian submanifold path geometry. In fact equations (11) asserts that we can identify \( \mathcal{N} \) with the bundle \( Z \) of Legendrian \( n \)-planes over \( Y \).

\[
\begin{array}{ccc}
Z & \cong & \mathcal{N} \\
\pi_1 \downarrow & & \downarrow \pi_2 \\
Y & \to & X
\end{array}
\]

Note the fibers of the bundle \( Z \to Y \) project under \( \pi_2 \) to geodesic totally null submanifolds in \( X \) and the fibers of the bundle \( \mathcal{N} \to X \) in turn project under \( \pi_1 \) to Legendrian submanifolds in \( Y \) that give rise to the Legendrian submanifold path geometry.

We mention that for an arbitrary \( G \) structure on an \( N \)-dimensional manifold, the differential system \( J \) describing geodesic totally null \( m \)-folds is in general not Frobenius. The Frobenius condition would force a single irreducible piece of the torsion tensor associated to the \( G \) structure to vanish.
5 Equivalence Problem

In this section, we continue the analysis of the equivalence problem for the class of Legendrian submanifold path geometry discussed in section 4. The underlying idea that guides us through the reduction procedure is the construction of an $\mathfrak{sl}(3,\mathbb{R})$-valued projective connection associated to the path geometry on a surface [Ca1] and its generalization demonstrated by Chern in [Ch1] and [ChM].

We start with the structure equations (11) on $F$,

\begin{align*}
    d\theta_0 &= -\rho \wedge \theta_0 - \theta^t \wedge \omega \\
    d\theta &= -\beta \wedge \theta_0 - \alpha \wedge \theta - \Theta \wedge \omega \\
    d\Theta &= -\frac{1}{2} (\beta \wedge \theta^t - \theta \wedge \beta^t) - (\alpha - \frac{\rho}{2}) \wedge \Theta + \Theta \wedge (\alpha - \frac{\rho}{2})^t + T \\
    dw &= -\mu \wedge \theta_0 - \gamma \wedge \theta - (\alpha^t - \rho) \wedge \omega,
\end{align*}

where $T = (T_{ij})$ is quadratic in $\{\theta_0, \theta, \Theta\}$ with $T \equiv 0 \mod \theta_0, \theta$. Explicitly, we write

\begin{align*}
    T_{ij} &= T_{ji} = \sum_k T_{ij}^k \theta_k \wedge \theta_k + \sum_{kl} T_{ijkl} \theta_k \wedge \theta_l \wedge \Theta_{lm} + \sum_{klm} T_{ijklm} \theta_k \wedge \Theta_{lm}
\end{align*}

with $T_{ij,kl} = T_{ijkl}$, $T_{ij}^k = -T_{ij}^{kl}$, and $T_{ijklm}^k = T_{ijklm}^{kl}$. Note that the pseudoconnection forms $\rho, \alpha, \beta$ in (12) are determined up to the change

\begin{align*}
    \rho^* &= \rho + p\theta_0 \\
    \alpha^*_i &= \alpha_i + c^i_j \theta_0 + \sum_k c^i_{jk} \theta_k \\
    \beta^*_i &= \beta_i + c^i_j \theta_0 + \sum_k c^i_{jk} \theta_k,
\end{align*}

where $p, c^i, c^i_j, c^i_{jk} = c^i_{kj}$ are independent variables. Following the procedure of the method of equivalence [Ga], we shall determine the coefficients $c^i, c^i_j, c^i_{jk} = c^i_{kj}$ by imposing conditions on the torsion $T$.

Applying the transformation (13) to the structure equation (12), we find

\begin{align*}
    T_{ij}^{k*} &= T_{ij}^k - \frac{1}{2} (c^i_j \delta_{jk} + c^i_k \delta_{ik}) \\
    T_{ij}^{kl*} &= T_{ij}^{kl} - \frac{1}{2} (c^i_k \delta_{jl} - c^i_j \delta_{jk}) - \frac{1}{2} (c^i_k \delta_{il} - c^i_l \delta_{ik}) \\
    T_{ij,kl}^{*} &= T_{ij,kl} - \frac{1}{2} (c^i_k \delta_{jl} + c^i_j \delta_{jk}) - \frac{1}{2} (c^i_k \delta_{il} + c^i_l \delta_{ik}) \\
        &\quad + \frac{1}{2} p (\delta_{uk} \delta_{jl} + \delta_{ul} \delta_{jk}) \\
    T_{ij,lm}^{k*} &= T_{ij,lm}^k - \frac{1}{2} (c^i_{kl} \delta_{jm} + c^i_{km} \delta_{jl}) - \frac{1}{2} (c^i_{kl} \delta_{im} + c^i_{km} \delta_{il}).
\end{align*}
Upon contraction, it becomes (no summation convention)

\[ T_{ii}^* = T_{ii}^i - c_i \]
\[ T_{ii}^{ki} = T_{ii}^{ki}^i - c_k, \quad i \neq k \]
\[ T_{ii,ii}^* = T_{ii,ii}^i + p - 2c_i^i \]
\[ T_{ii,im}^{k*} = T_{ii,im}^{k*} - c_{km}, \quad i \neq m \]
\[ T_{ii,ii}^{k*} = T_{ii,ii}^{k*} - 2c_{ki}. \]

Hence, \( c_i, c_j^i, c_{jk}^i = c_{kj}^i \) can be determined so as to achieve (no summation convention)

\[ T_{ii}^i = 0 \] (14)
\[ T_{ii}^{ki} = 0 \]
\[ T_{ii,ii} = 0 \]
\[ T_{ii,im}^{k} + T_{ii,ik}^{m} = 0, \quad i \neq m \quad \text{and} \quad i \neq k \]
\[ T_{ii,ii}^{k} = 0, \quad i, m, k = 1, \ldots, n, \]

which we assume from now on. The admissible transformations of the pseudo connection forms preserving the equations (12) and the symmetry (14) of the torsion \( T \) now become

\[ \rho^* = \rho + p\theta_0 \] (15)
\[ \alpha_{ji}^* = \alpha_{ji}^i + \delta_{ij}^i \frac{1}{2} p\theta_0 \]
\[ \beta^i = \beta^i + \frac{1}{2} p\theta_i \]
\[ \mu^i = \mu^i + h^i\theta_0 + \sum_k h_{ik}\theta_k - \frac{1}{2} p\omega^i \]
\[ \gamma_{ij}^* = \gamma_{ij}^* \]
\[ = \gamma_{ij} + h_{ij}\theta_0. \]

with new independent variables \( h^i \) and \( h_{ij} = h_{ji} \).

Consider the bundle

\[ B_1 \rightarrow F \]

whose fiber at each point of \( F \) is the space of pseudo connection forms \( \rho, \alpha, \beta, \mu, \gamma \) for which the equations (12) and (14) are satisfied. Equations (15) then give explicit formulas, in terms of the parameters \( p, h^i, h_{ij} = h_{ji} \), for the tautological forms on \( B_1 \), which exist by the definition of the bundle \( B_1 \). We drop * and use \( \rho, \alpha, \beta, \mu, \gamma \) to denote the corresponding tautological forms.
Set
\[
\begin{align*}
    d\rho &= -\beta^t \land \omega - \theta^t \land \mu + \theta_0 \land \psi + \Omega_{\rho} \\
    d\beta &= \rho \land \beta - \alpha \land \beta - \Theta \land \mu + \frac{1}{2}\theta \land \psi + \Omega_{\beta} \\
    d\alpha &= -\alpha \land \alpha - \frac{1}{2}(\beta^t \land \omega - \beta \land \omega^t) - \frac{1}{2}(\theta^t \land \mu + \theta \land \mu^t) \\
        &\quad - \Theta \land \gamma + \frac{1}{2}\theta_0 \land \psi + \Omega_{\alpha}
\end{align*}
\]
where \( \psi = -dp \). Exterior derivatives of the first two equations in (12) then give structure equations for the tautological forms \( \rho, \alpha, \beta \),
\[
\begin{align*}
    \Omega_{\rho} \land \theta_0 &= 0 \\
    \Omega_{\beta} \land \theta_0 + \Omega_{\alpha} \land \theta + T \land \omega &= 0.
\end{align*}
\]
From the first equation in (17), we may arrange that
\[
    d\rho = -\beta^t \land \omega - \theta^t \land \mu + \theta_0 \land \psi. \tag{18}
\]
by modifying \( \psi \) if necessary. Note that at this stage, the equations (12), (14) and (18) determine \( \psi \) up to the change
\[
\psi^* = \psi + t \theta_0 - \sum_k h^k \theta_k. \tag{19}
\]
where \( t \) is a new variable. Put
\[
    \Omega^i_{\beta} \equiv \sum_j P^i_{j} \theta_0 \land \theta_j + \sum_{jk} P^i_{jk} \theta_0 \land \Theta_{jk} + \sum_{jk} P^i_{jk} \theta_j \land \theta_k + \sum_{klm} P^i_{klm} \theta_k \land \Theta_{lm}
\]
mod \( \omega, \rho, \alpha, \beta \), where \( \Omega^i_{\beta} = (\Omega^1_{\beta}, \ldots, \Omega^n_{\beta}) \), and \( P^i_{jk} = P^i_{kj} \), \( P^i_{jk} = -P^i_{kj} \), \( P^i_{k,lm} = P^i_{k,ml} \).

Applying the transformation (15) and (19) to (16), we get
\[
\begin{align*}
    P^i_{j} &= P^i_{j} - (\frac{1}{4}p^2 - \frac{1}{2}t)\delta_{ij} \\
    P^i_{jk} &= P^i_{jk} + \frac{1}{2}(\delta_{ij}h^k + \delta_{ik}h^j) \\
    P^i_{k,lm} &= P^i_{k,lm} - \frac{1}{2}(\delta_{im}h_{lk} + \delta_{il}h_{mk}).
\end{align*}
\]
The contraction of the above gives (no summation convention)
\[
\begin{align*}
    P^i_{i} &= P^i_{i} - (\frac{1}{4}p^2 - \frac{1}{2}t) \\
    P^i_{ii} &= P^i_{ii} + h^i \\
    P^i_{k,ii} + P^k_{i,kk} &= P^i_{k,ii} + P^k_{i,kk} - 2h_{ik}.
\end{align*}
\]
In terms of the bundle $B_1$, the above computations imply that there exists a subbundle $B \subset B_1$ on which (no summation convention)

$$
\sum_{i=1}^{n} P_i^i = 0,
$$

$$
P_{ii}^i = 0
$$

$$
P_{k,ii}^i + P_{i,kk}^k = 0, \quad i, k = 1, \ldots, n.
$$

(20)

In fact, (12), (14), (18) and (20) determine the pseudo connection forms $\rho, \alpha, \beta, \mu, \gamma$ and $\psi$ up to the change

$$
\rho^* = \rho + p\theta_0
$$

$$
\alpha_j^* = \alpha_j^i + \delta_{ij} \frac{1}{2} p\theta_0
$$

$$
\beta^* = \beta^i + \frac{1}{2} p\theta_0
$$

$$
\mu^* = \mu^i - \frac{1}{2} p\omega_i
$$

$$
\gamma_{ij}^* = \gamma^i_{ji}
$$

$$
= \gamma_{ij}
$$

$$
\psi^* = \psi + \frac{1}{2} p^2 \theta_0.
$$

Note that $p$ is the fiber variable of the bundle $B \to F$.

Now, the differential forms

$$
\{\theta_0, \theta, \Theta, \omega, \rho, \beta, \alpha, \psi, \mu, \gamma\}
$$

are invariantly defined and form a basis of 1-forms on $B$. Set

$$
d\psi = \rho \wedge \psi - (\beta^i \wedge \mu + \mu^i \wedge \beta) + \Omega \psi
$$

$$
d\mu = -\frac{1}{2} \psi \wedge \omega + \alpha^i \wedge \mu - \gamma \wedge \beta + \Omega \mu
$$

$$
d\gamma = -\rho \wedge \gamma + \frac{1}{2} (\mu \wedge \omega^i - \omega \wedge \mu^i) + (\alpha^i \wedge \gamma - \gamma \wedge \alpha) + \Omega \gamma.
$$

Then the structure equations so far can be written as

$$
d\Phi = -\Phi \wedge \Phi + \Omega
$$

(22)

where $\Phi$ is the $\mathfrak{sp}(n + 1, R) \subset \mathfrak{sl}(2n + 2, R)$ -valued 1-form

$$
\Phi = \left( \begin{array}{cc} \phi & \pi \\ \eta & -\phi^t \end{array} \right)
$$

(23)
with
\[ \eta = \begin{pmatrix} 2\theta_0 & \theta^t \\
\theta & \Theta \end{pmatrix}, \quad \phi = \begin{pmatrix} -\frac{1}{2}\rho & -\frac{1}{2}\beta^t \\
\omega & -(\alpha^t - \frac{1}{2}\rho) \end{pmatrix}, \quad \pi = \begin{pmatrix} -\frac{1}{4}\psi & \frac{1}{2}\mu^t \\
\frac{1}{2}\mu & \gamma \end{pmatrix}, \] (24)

and the \(\mathfrak{sp}(n+1, R)\)-valued curvature form
\[ \Omega = \begin{pmatrix} \Omega_\phi & \Omega_\pi \\
\Omega_\eta & -\Omega_\phi \end{pmatrix} \] (25)

with
\[ \Omega_\eta = \begin{pmatrix} 0 & 0 \\
0 & T \end{pmatrix}, \quad \Omega_\phi = \begin{pmatrix} 0 & -\frac{1}{2}\Omega_\phi^\beta \\
0 & -\Omega_\alpha^t \end{pmatrix}, \quad \Omega_\pi = \begin{pmatrix} -\frac{1}{4}\Omega_\psi & \frac{1}{2}\Omega_\mu^t \\
\frac{1}{2}\mu & \Omega_\gamma \end{pmatrix}. \] (26)

Exterior differentiation of the second and the last equation in (12) and (18) gives the following algebraic equations satisfied by the curvature form.
\[ \Omega_\beta \wedge \theta_0 + \Omega_\alpha \wedge \theta + T \wedge \omega = 0, \] (27)
\[ \Omega_\mu \wedge \theta_0 + \Omega_\gamma \wedge \theta + \Omega_\alpha^t \wedge \omega = 0, \]
\[ \Omega_\psi \wedge \theta_0 - \Omega_\mu^t \wedge \theta + \Omega_\beta \wedge \omega = 0. \]

In particular,
\[ \Omega \equiv 0 \mod \theta_0, \theta, \omega. \] (28)

Along with the structure equation (22), this implies that each fiber of the bundle \(B \rightarrow Y\) via \(B \rightarrow F \rightarrow Z \rightarrow Y\) has the structure of the Lie group \(P_1\), where \(P_1 \subset \text{Sp}(n+1, R)\) is the stabilizer of a line in \(R^{2n+2}\) under the standard representation of \(\text{Sp}(n+1, R)\).

We also mention that if \(T = 0\), the \(G\)-structure induced on \(X\) is torsion free, which by a result in [Br2] implies \(\Omega = 0\). For this reason, we call \(T\) the primary invariant of the Legendrian path geometry under consideration.

**Theorem 2** Given a Legendrian submanifold path geometry on \(Z \rightarrow Y^{2n+1}\) whose associated bundle \(F_1\) admits a reduction to a subbundle \(F \rightarrow F_1\) with the structure equations (12), there exists a \(P_1 \subset \text{Sp}(n+1, R)\) bundle \(B \rightarrow Y\) and a \(\mathfrak{sp}(n+1, R)\)-valued 1-form \(\Phi\) on \(B\) given by (23) with structure equations (22), or (12), (16), (18) and (21). The components of the curvature form \(\Omega\) in (25) and (26) satisfy (27). At each point of \(B\), \(\Phi\) induces an isomorphism of the tangent space of \(B\) with \(\mathfrak{sp}(n+1, R)\). Two such Legendrian submanifold path geometries are equivalent if and only if their associated bundles and the pseudoconnections are isomorphic.

A result of Cartan [Ca2] implies the Legendrian submanifold path geometries with the maximal dimension of symmetry vector fields are those for which the coefficients of the curvature form \(\Omega\) are all constants, the simplest being the case when \(\Omega = 0\). The homogeneous Legendrian path geometry realizing this flat structure equation with the full group of symmetry \(\text{Sp}(n+1, R)\) will be examined in section 7.
6 Normal Symplectic Connection

It is well known in projective geometry that to every torsion free affine connection on a manifold there exists a unique normal projective connection whose paths coincide with the set of geodesics of the given affine connection [Ch3]. Moreover, the sets of geodesics of two torsion free affine connections induce an equivalent path geometry if and only if their associated normal projective connections are equivalent. The normal projective connection associated to a torsion free affine connection thus captures the geometry of path defined by the set of geodesics of the affine connection.

The purpose of this section is to generalize this idea to Legendrian submanifold path geometry and to discuss a Legendrian connection on a contact manifold that plays the role of a torsion free affine connection and induces an associated Legendrian submanifold path geometry. Since a path in Legendrian submanifold path geometry is an \( n \geq 2 \)-dimensional submanifold, the integrability condition becomes nontrivial in this case.

We use \( \theta_0 \) to denote a (local) generator of the contact structure on \( Y \) or its pull back to the frame or other bundles over \( Y \).

6.1 Legendrian Connection

Let \( P \to Y^{2n+1} \) be the contact hyperplane vector bundle of fiber dimension \( 2n \).

\[
P = \{ (p, v) \mid p \in Y, \ v \in T_pY \text{ with } \theta_0(v) = 0 \}
\]

Since
\[
d(f \theta_0) \equiv f \, d\theta_0, \mod \theta_0, \ f \in C^\infty(Y)
\]
\[(d\theta_0)^n \neq 0 \mod \theta_0,
\]
the restriction of \( d\theta_0 \) induces a conformal symplectic structure on each fiber. A basis \( \{A_i, B_i\} \) of a contact hyperplane is called a conformal symplectic frame if
\[
d\theta_0(A_i, A_j) = 0, \quad (29)
\]
\[
d\theta_0(B_i, B_j) = 0,
\]
\[
d\theta_0(A_i, B_j) = c \delta_{ij}, \quad c \neq 0, \quad i, j = 1, \ldots, n.
\]

Let \( D \) be a conformal symplectic connection on the vector bundle \( P \). The infinitesimal displacement of a conformal symplectic frame field is given by
\[
D (A, B) = (A, B) \Psi,
\]
with \( A = (A_1, \ldots, A_n) \), \( B = (B_1, \ldots, B_n) \), and \( \Psi \) is a \( \mathfrak{sp}(n, R) \)-valued connection form, i.e.,
\[
\Psi = \begin{pmatrix}
\alpha + \rho I & \gamma \\
\Theta & -\alpha^t + \rho I
\end{pmatrix}
\]

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where $\gamma$, $\Theta$ are symmetric $gl(n, R)$-valued 1-forms, $\alpha$ is a $gl(n, R)$-valued 1-form, and $\rho$ is a scalar 1-form. Under the change of the frame field

$$(A^*, B^*) = (A, B) \, g,$$

(31)

where $g$ is a $CSp(n, R)$-valued function, we have

$$D(A^*, B^*) = (A^*, B^*) \, \Psi^*, $$

with

$$\Psi^* = g^{-1}dg + g^{-1}\Psi \, g.$$  

(32)

The curvature form of the connection is defined by

$$\Omega_{\Psi} = d\Psi + \Psi \wedge \Psi.$$  

Differentiating (32), we have

$$\Omega_{\Psi^*} = g^{-1}\Omega_{\Psi} \, g.$$  

(33)

Take a local conformal symplectic frame field $\{A_i, B_i\}$. The identity of Cartan

$$d\theta_0(V_1, V_2) = V_1\theta_0(V_2) - V_2\theta_0(V_1) - \theta_0([V_1, V_2])$$

together with (29) gives

$$\theta_0([A_i, A_j]) = 0,$$

$$\theta_0([B_i, B_j]) = 0,$$

$$\theta_0([A_i, B_j]) = c \delta_{ij}, \quad c \neq 0, \quad i, j = 1, \ldots, n.$$  

(34)

Thus, if $V_1, V_2$ are vector fields tangent to the contact hyperplane fields at each point spanning a isotropic plane field with respect to the conformal symplectic structure, then $[V_1, V_2]$ also lies in the contact hyperplane. A connection $D$ is called isotropic torsion free if

$$D_{V_1}V_2 - D_{V_2}V_1 = [V_1, V_2]$$

whenever the plane field generated by $V_1, V_2$ is isotropic with respect to the conformal symplectic structure on the contact hyperplane. We assume the connection $D$ to be isotropic torsion free from now on.

Let $F_P \to Y$ be the bundle of conformal symplectic frames, which fits into the diagram

$$\begin{array}{c}
F_P \\
\downarrow \\
Z \\
\downarrow \\
Y
\end{array}$$
where $Z$ is the bundle of Legendrian $n$-planes. The projection map $F_P \rightarrow Z$ is given by

$$(A_1, \ldots, A_n, B_1, \ldots, B_n) \rightarrow A_1 \wedge A_2 \wedge \ldots, A_n \in Z$$

From the definition, there exists a set of tautological forms $\{\omega^i, \theta_i\}$, $i = 1, \ldots, n$, conformal symplectic coframe on $F_P$, defined up to adding multiples of $\theta_0$. The equation

$$d\theta_0 \equiv -\sum_i \theta_i \wedge \omega^i \mod \theta_0$$

in turn determines $\theta_0$ uniquely on $F_P$.

The isotropic torsion free condition on $D$ implies the following structure equation on $F_P$ dual to (30).

$$d\left(\frac{\omega}{\theta}\right) \equiv -\Psi \wedge \left(\frac{\omega}{\theta}\right) + \sum_i t_i \theta_i \wedge \omega^i \mod \theta_0$$

where $\omega^t = (\omega^1, \ldots, \omega^n)$, $\theta^t = (\theta^1, \ldots, \theta^n)$, and $t_i$'s are functions on $F_P$. We call a isotropic torsion free connection $D$ torsion free if we can modify $\omega$ and $\theta$ by adding multiples of $\theta_0$ to arrange

$$d\left(\frac{\omega}{\theta}\right) = -\Psi \wedge \left(\frac{\omega}{\theta}\right) + \theta_0 \wedge \Gamma$$

with $\Gamma \equiv 0 \mod \omega, \theta$. Equation (37) determines the tautological form $\{\omega, \theta\}$ uniquely on $F_P$.

Torsion free connections also admit the following equivalent but more geometric description, which is a direct consequence of (37).

**Definition** Torsion free connection

A conformal symplectic connection $D$ on the vector bundle $P \rightarrow Y$ is called torsion free if there exists a line field in $Y$ transversal to the contact hyperplane field such that for any vector fields $V_1, V_2$ tangent to the contact hyperplane field,

$$D_{V_1} V_2 - D_{V_2} V_1 = p([V_1, V_2])$$

where $p : TY \rightarrow P$ is the projection map induced from the given line field.

Of the torsion free connections, we consider the ones that give rise to a Legendrian submanifold path geometry on $Y$. The natural analogue to the geodesics of the torsion free affine connection would be the geodesic Legendrian submanifolds, i.e., the Legendrian submanifolds whose tangent planes are parallel along the submanifold with respect to the given connection. In terms of (37), they are the solutions to the differential system $\mathcal{I}$ on $Z$ whose pull back to $F_P$ is locally generated by

$$\{\theta_0, \theta, \Theta\}.$$
The structure equation (33) shows the differential system (38) can indeed be pushed down to $Z$.

**Definition** Legendrian connection

A torsion free connection on the contact hyperplane vector bundle $P \to Y$ is called *Legendrian* if the differential system $\mathcal{I}$ describing the geodesic Legendrian submanifolds is Frobenius on $Z$, or equivalently if the differential system (38) is Frobenius on $F_P$.

Set

$$\begin{pmatrix} \Omega_{\alpha+\rho} & \Omega_\gamma \\ \Omega_\Theta & -\Omega_{\alpha+\rho}^{\ell} \end{pmatrix} = d\Psi + \Psi \wedge \Psi
\begin{pmatrix} \Omega_\Theta \end{pmatrix} = \Omega_\Psi$$

Since the curvature form is quadratic in $\{\theta_0, \theta, \omega\}$, the Frobenius condition corresponds to

$$\Omega_\Theta \equiv 0 \mod \theta_0, \theta.$$  \hspace{1cm} (39)

We assume the connection to be Legendrian from now on and study the consequences of the equation (39).

Let $V$ be the standard $2n$-dimensional representation of $Sp(n, R)$. Consider $S^2(V) \otimes \Lambda^2(V)$ with the irreducible decomposition, [FH],

$$\Lambda^2(V) = \Gamma_{010} \oplus R$$

$$S^2(V) \otimes \Gamma_{010} = \Gamma_{21} \oplus \Gamma_{20} \oplus \Gamma_{01} \hspace{1cm} \text{if } n = 2$$

$$= \Gamma_{2100} \oplus \Gamma_{1010} \oplus \Gamma_{200} \oplus \Gamma_{010} \hspace{1cm} \text{if } n \geq 2.$$  \hspace{1cm} (40)

From (33) and (37), $\Omega \mod \theta_0$ can be considered as a $Sp(n, R)$ equivariant $(S^2(V) \oplus R) \otimes \Lambda^2(V)$ -valued function on $F_P$. Equation (40) shows that in order for the equation (39) to be satisfied on $F_P$, the curvature must be of the form

$$\Omega \equiv R \sum_{i=1}^n \theta_i \wedge \omega^i \mod \theta_0,$$  \hspace{1cm} (41)

where $R$ is a $csp(n, R)$ -valued function on $F_P$.

Exterior differentiation of (37) using (35) and (41) now gives

$$\sum_i \theta_i \wedge \omega_i \wedge \left( \Gamma + R \left( \begin{array}{c} \omega \\ \theta \end{array} \right) \right) \equiv 0 \mod \theta_0.$$  \hspace{1cm} (42)

Since $n \geq 2$ and $\Gamma \equiv 0 \mod \theta, \omega$, we get

$$\Gamma = -R \left( \begin{array}{c} \omega \\ \theta \end{array} \right).$$  \hspace{1cm} (42)
Under the change (31),
\[ \sum \theta_i \wedge \omega^i = \det(g) \sum \theta_i^\ast \wedge \omega^i. \]

From (35) and (37), we get
\[ d\theta_0 = -2\rho \wedge \theta_0 - \sum \theta_i \wedge \omega^i + \theta_0 \wedge \sigma \] (43)
with \( \sigma \) linear in \( \theta, \omega \). Differentiating (43) and reducing \( \mod \theta_0 \) yields,
\[ \sum \theta_i \wedge \omega^i \wedge \sigma \equiv 0 \mod \theta_0. \]
Thus \( \sigma = 0 \) and we have
\[ d\theta_0 = -2\rho \wedge \theta_0 - \sum \theta_i \wedge \omega^i. \] (44)

By taking the exterior derivative of (44),
\[ d \text{tr}(\Psi) = 2n d(\rho) \equiv \text{tr}(R) \sum \theta_i \wedge \omega^i \mod \theta_0. \] (45)

**Proposition 3** Let \( Y^{2n+1} \) be a contact manifold with the contact line bundle \( L \subset T^*Y \), contact hyperplane bundle \( P \to Y \), and its associated conformal symplectic frame bundle \( F_P \). The equation (35) determines a section of the projection \( L \oplus F_P \to F_P \) and thus gives a well defined 1-form \( \theta_0 \) on \( F_P \), which becomes a (local) generator of the contact structure on \( Y \) when pulled back by a section of \( F_P \to Y \). A Legendrian connection on \( P \) induces a set of tautological forms \( \{ \theta, \omega \} \) with the structure equations
\[ d\theta_0 = -2\rho \wedge \theta_0 - \sum \theta_i \wedge \omega^i \] (46)
\[ d \left( \frac{\omega}{\theta} \right) = -\Psi \wedge \left( \frac{\omega}{\theta} \right) - \theta_0 \wedge R \left( \frac{\omega}{\theta} \right) \]
\[ d\Psi \equiv -\Psi \wedge \psi + R \sum \theta_i \wedge \omega^i \mod \theta_0. \]
where \( R \) is a \( \text{csp}(n, R) \)-valued equivariant function on \( F_P \). A Legendrian submanifold path geometry is defined by the set of geodesic Legendrian submanifold with respect to the given Legendrian connection, which, from the definition (40), is an \( N = \frac{1}{2}(n + 1)(n + 2) \)-parameter family of Legendrian submanifolds in \( Y \).

Note that (45) agrees with the trace of the third equation in (46).
6.2 Normal Symplectic Connection

A projective connection on a manifold can be considered as a family of torsion free affine connections all of which induce the same path geometry. In this section we wish to prove an analogous result for Legendrian connections.

**Theorem 3** Let $P \to Y^{2n+1}$ be the contact hyperplane bundle, and $F_P$ the associated conformal symplectic frame bundle. Let $D$ be a Legendrian connection on $P \to Y$. There exists a bundle $B \to Y$

\[
\begin{array}{c}
B \\
\downarrow \\
F_P \\
\leftarrow \\
Y
\end{array}
\]

and an $\mathfrak{sp}(n+1, R)$-valued 1-form $\Phi$, the normal symplectic connection form, on $B$ with the following property.

1. The bundle $B \to Y$ has the structure of a principal $P_1$ bundle, where $P_1 \subset \text{Sp}(n+1, R)$ is the stabilizer of a line in $R^{2n+2}$ under the standard representation of $\text{Sp}(n+1, R)$.

2. At each point $u \in B$, $\Phi$ induces an isomorphism between $T_u B$ and $\mathfrak{sp}(n+1, R)$.

3. Under the right action by an element $g \in P_1$,

\[\Phi_{ug} = \text{Ad}_{g^{-1}} \Phi_u.\]

4. A Legendrian connection $D'$ induces the same Legendrian path geometry as $D$ if and only if $D'$ arises from a section of the projection $B \to F_P$.

The meaning of a Legendrian connection arising from a section of the projection $B \to F_P$ will become clear once the bundle $B$ is constructed.

Given a Legendrian connection $D$ with the structure equation (46), we consider the space of Legendrian connections that induce the same Legendrian submanifold path geometry equivalent as that of $D$. First of all, the deformation of the transversal line field is expressed by

\[
\begin{aligned}
\omega^* &= \omega + p \theta_0 \\
\theta^* &= \theta + q \theta_0
\end{aligned}
\]

where $(p, q) = (p_1, \ldots, p_n, q_1, \ldots, p_n)$ is a $R^{2n}$-valued equivariant function on $F_P$. In order to keep the equation (44), we must have

\[
\rho^* = \rho + \frac{1}{2} (q^t \omega - p^t \theta) + s \theta_0
\]

where $s$ is a scalar equivariant function on $F_P$.

Equation (47) and (48) effect the second equation in (46). Equating (46) mod $\theta_0$, we have

\[
\Psi^* = \Psi + \frac{1}{2} \begin{pmatrix} p \\ q \end{pmatrix} (\theta^t, -\omega^t) + \frac{1}{2} \begin{pmatrix} \omega \\ \theta \end{pmatrix} (q^t, -p^t) + x + A \theta_0
\]  

\[22\]
where $\Psi = \Psi + \rho I$, $A$ is a $\mathfrak{sp}(n,R)$-valued equivariant function on $F_P$, and $x$ is in the kernel of the map $\mathfrak{sp}(n,C) \otimes V^* \to V \otimes \bigwedge^2 V^*$. Here $V$ is the standard $2n$-dimensional representation of $\mathfrak{sp}(n,C)$. The kernel is isomorphic to the irreducible representation $S^3(V)$ of $Sp(n,R)$ and, since the ideal (38) describing the Legendrian submanifold paths must be preserved, $x$ must be 0.

Let $B_1 \to F_P$ be the bundle of all such point connections with the explicit parametrization (47), (48), and (49). We drop * and use the same notation to denote the tautological forms on $B_1$. Taking the exterior derivative of (47), (48), and (49), a computation shows that we can choose $A, \pi_0, \phi_0, \pi^0_0$ so as to achieve

$$
\begin{align*}
    d\theta_0 &= -2\rho \wedge \theta_0 - \sum_i \theta_1 \wedge \omega^i \\
    d\left(\frac{\omega}{\theta}\right) &= -(\Psi + \rho) \wedge \left(\frac{\omega}{\theta}\right) - \theta_0 \wedge \left(\frac{\pi_0}{-\phi_0}\right) \\
    d\Psi &\equiv -\Psi \wedge \Psi + \frac{1}{2} \left(\begin{array}{c}
        \pi_0 \\
        -\phi_0
    \end{array}\right) \wedge (\theta^t, -\omega^t) + \frac{1}{2} \left(\begin{array}{c}
        \omega \\
        -\theta
    \end{array}\right) \wedge (\phi^t_0, \pi^t_0) \\
    \text{mod } \theta_0 \\
    dp &= \frac{1}{2} (\theta^t \wedge \pi_0 + \omega^t \wedge \phi_0) + 2\pi^0_0 \wedge \theta_0.
\end{align*}
$$

where

$$
\begin{align*}
    \pi_0 &\equiv dp \\
    \phi_0 &\equiv -dq \\
    \pi^0_0 &\equiv \frac{1}{2} ds \mod \rho, \Psi, \theta_0, \theta, \omega.
\end{align*}
$$

Let $B \subset B_1$ be the subbundle on which the set of equations (50) hold. The 1-forms $\{\theta_0, \theta, \omega, \Psi, \rho\}$ are now uniquely defined on $B$, because of the fact that the only solution $a \in gl(n,R)$ to the equation

$$
a \omega \wedge \theta^t + \omega \wedge \theta^t a = 0
$$

is $a = 0$. The admissible change of $\pi_0, \phi_0, \pi^0_0$ preserving (50) now becomes

$$
\begin{align*}
    \pi_0^* &= \pi_0 + 2y \theta_0 \\
    \phi_0^* &= \phi_0 + 2z \theta_0 \\
    \pi^0_0 &= \pi^0_0 - \frac{1}{2} (\theta^t y + \omega^t z) + t \theta_0
\end{align*}
$$

where $(y, z) \in R^{2n}$ and $t \in R$ are new variables.

Let us write

$$
\begin{align*}
    d\Psi &= -\Psi \wedge \Psi + \frac{1}{2} \left(\begin{array}{c}
        \pi_0 \\
        -\phi_0
    \end{array}\right) \wedge (\theta^t, -\omega^t) + \frac{1}{2} \left(\begin{array}{c}
        \omega \\
        -\theta
    \end{array}\right) \wedge (\phi^t_0, \pi^t_0) + \theta_0 \wedge \gamma
\end{align*}
$$

23
with $\Upsilon \equiv 0 \mod \theta, \omega$. Under the representation of $Sp(n, R)$, $\Upsilon$ takes values in $S^2(V) \otimes V$ with the irreducible decomposition, [FH],

$$S^2(V) \otimes V = S^3(V) \oplus V \oplus \Gamma_{110..0}.$$  

Hence, we can absorb the $V$ piece by modifying $\pi_0, \phi_0$, and we have

$$\Upsilon \subset S^3(V) \oplus \Gamma_{110..0} \subset S^2(V) \otimes V.$$  

(51)

This condition in turn determines $\phi_0$, and $\pi_0$ uniquely.

Put

$$d\phi_0 = (\rho + \alpha^t) \wedge \phi_0 + 2\pi^0_0 \wedge \theta + \Theta \wedge \pi_0 + \Omega_{\phi_0}$$
$$d\pi_0 = (\rho - \alpha) \wedge \pi_0 - 2\pi^0_0 \wedge \omega + \gamma \wedge \phi_0 + \Omega_{\pi_0}$$  

(52)

In order to determine $\pi^0_0$, we write

$$\Omega^i_{\phi_0} \equiv \sum_j K^i_j \theta_0 \wedge \theta_j \mod \omega, \rho, \Theta, \alpha, \gamma,$$

where $\Omega^i_{\pi_0} = (\Omega^1_{\pi_0},.. \Omega^n_{\pi_0})$. The 1-form $\pi^0_0$ is uniquely determined by imposing

$$\sum_i K^i_i = 0.$$  

(53)

Now, the structure equations (50), (51), (52) and (53) uniquely determine $\{\theta_0, \theta, \omega, \Psi, \rho, \pi_0, \phi_0, \pi^0_0\}$, and they form a basis of 1-forms on $B$.

Put

$$d\pi^0_0 = 2\rho \wedge \pi^0_0 - \frac{1}{2} \phi^t_{\phi_0} \wedge \pi_0 + \Omega_{\pi^0_0}.$$  

Then the structure equations so far can be written as

$$d\Phi = -\Phi \wedge \Phi + \Omega$$  

(54)

where $\Phi$ is a $\mathfrak{sp}(n+1, R) \subset \mathfrak{sl}(2n+2, R)$-valued 1-form

$$\Phi = \begin{pmatrix} \phi & \pi \\ \eta & -\phi^t \end{pmatrix}$$  

(55)

with

$$\eta = \begin{pmatrix} 2\theta & \theta^t \\ \theta & \Theta \end{pmatrix}, \quad \phi = \begin{pmatrix} -\rho & -\frac{1}{2} \phi^t_{\phi_0} \\ \omega & \alpha \end{pmatrix}, \quad \pi = \begin{pmatrix} \pi^0_0 & \frac{1}{2} \pi^t_0 \\ -\frac{1}{2} \pi_0 & \gamma \end{pmatrix},$$  

(56)

and the curvature form

$$\Omega = \begin{pmatrix} \Omega_{\phi} & \Omega_{\pi} \\ \Omega_{\eta} & -\Omega^t_{\phi} \end{pmatrix}$$  

(57)
with
\[
\Omega_\eta = \begin{pmatrix} 0 & 0 \\ 0 & \Omega_\Theta \end{pmatrix}, \quad \Omega_\phi = \begin{pmatrix} 0 & -\frac{1}{2}\Omega_t^\phi \\ 0 & \Omega_\alpha \end{pmatrix}, \quad \Omega_\pi = \begin{pmatrix} \Omega_{\pi_0} & -\frac{1}{2}\Omega_t^\pi \\ -\frac{1}{2}\Omega_{\pi_0} & \Omega_\gamma \end{pmatrix}.
\] (58)

Note that
\[
\begin{pmatrix} \Omega_\alpha & \Omega_\gamma \\ \Omega_\Theta & -\Omega_\alpha \\ \end{pmatrix} = \theta_0 \wedge \Upsilon
\]
where \( \Upsilon \) satisfies (51).

It is easily verified \( \Upsilon = 0 \) implies \( \Omega = 0 \). We call \( \Upsilon \) the primary invariant of the Legendrian submanifold path geometry arising from a Legendrian connection. A curvature form \( \Omega \) for which the set of forms \( \Upsilon \) and \( \Omega_\phi \) satisfy (51) and (53) is called normal.

The exterior differentiation of the last equation in (50) gives the following algebraic equations satisfied by the curvature forms.
\[
\Omega_t^\phi \wedge \omega + \Omega_t^\pi \wedge \theta - 4\Omega_{\pi_0} \wedge \theta_0 = 0.
\]
In particular, we have
\[
\Omega \equiv 0 \mod \theta_0, \theta, \omega,
\]
which explains 1. in Theorem 3. The rest of the statements in Theorem 3 are also easily verified.

### 6.3 Legendrian Flag

From equation (46) or (50), consider the differential ideal
\[
\mathcal{I}_k = \mathcal{I} \cup \{ \omega^{k+1}, \ldots, \omega^n \} \cup \{ \alpha^j_i | k + 1 \leq i \leq n \text{ and } 1 \leq j \leq k \}
\]
for \( 1 \leq k \leq n - 1 \),
\[
\mathcal{I}_n = \mathcal{I}.
\]

It is easy to check that each \( \mathcal{I}_k \) is Frobenius on \( B \). From (46), \( k \)-dimensional integral manifolds to \( \mathcal{I}_k \) correspond to geodesic isotropic \( k \)-folds in \( Y^{2n+1} \) with respect to the given Legendrian connection, which from (50) are invariantly defined independent of the choice of Legendrian connection of a normal symplectic connection. Since the structure group acts transitively on the set of isotropic \( k \)-planes, \( \mathcal{I}_k \) defines a unique geodesic isotropic \( k \)-fold tangent to each \( k \)-plane in a contact hyperplane of \( Y \) that is isotropic with respect to the induced conformal symplectic structure.

**Corollary 1** Consider a Legendrian connection on a contact manifold \( Y^{2n+1} \). To each isotropic \( k \)-plane at a point of \( Y \), there exists a unique geodesic isotropic \( k \)-fold tangent to the given \( k \)-plane. The family of geodesic isotropic submanifold paths defined by \( \mathcal{I}_k \) for \( 1 \leq k \leq n \) is in fact an invariant of the associated normal symplectic connection.
Note that equation (54), when restricted to a geodesic isotropic \( k \)-fold \( \Sigma_k \subset Y \), gives rise to a bundle

\[ \mathcal{B}_k \to \Sigma_k \]

with a \( \text{Gl}(k + 1, R) \) -valued Cartan connection form \( \phi_k \) given by the upper left hand corner \((k + 1) \times (k + 1)\) submatrix of \( \phi \). From (50) and (58), each fiber of this bundle has the structure of a Lie subgroup \( P_{k+1} \subset \text{Gl}(k + 1, R) \), where \( P_{k+1} \) is isomorphic to the fiber of the bundle

\[ \text{Gl}(k + 1, R) \to \mathbb{R}P^k. \]
7 Examples

7.1 Flat Example

Among the Legendrian submanifold path geometry is the simplest is that of hyperquadrics in $\mathbb{R}^{n+1}$, the flat example discussed earlier. It is easily verified this is the case when the curvature form

$$\Omega = 0$$

in (22). The structure equation then suggests that the Legendrian submanifold path geometry of hyperquadrics in $\mathbb{R}^{n+1}$ is locally equivalent to the canonical homogeneous Legendrian submanifold path geometry on $\mathbb{RP}^{2n+1}$.

Let $x^A, y^A, 0 \leq A \leq n$, be the coordinates in $\mathbb{R}^{2n+2}$ with the symplectic form

$$\omega = \sum_A dx^A \wedge dy^A. \tag{59}$$

Let $Q = \text{Lag}(n+1, \mathbb{R}^{2n+2})$ be the space of Lagrangian $(n+1)$-planes in $\mathbb{R}^{2n+2}$. The group of linear transformations $Sp(n+1, \mathbb{R}) \subset SL(2n+2, \mathbb{R})$ that preserves $\omega$ then acts transitively on $Q$ and the space of lines in $\mathbb{R}^{2n+2}$, $\mathbb{RP}^{2n+1}$, respectively.

In fact, it acts transitively via the product action on the incidence correspondence

$$I = \{(l, E) \in \mathbb{RP}^{2n+1} \times Q \mid l \subset E\}$$

of dimension $(2n+1) + \frac{1}{2} n(n+1)$. The spaces $I$, $\mathbb{RP}^{2n+1}$ and $Q$, each of which is a homogeneous space of $Sp(n+1, \mathbb{R})$, fit into the following diagram.

$$\begin{array}{cccc}
Sp(n+1, \mathbb{R}) & \downarrow & I = Sp(n+1, \mathbb{R})/P_1 \cap P_2 \\
\pi_1 \searrow & & \pi_2 \\
Sp(n+1, \mathbb{R})/P_1 = \mathbb{RP}^{2n+1} & & Q = Sp(n+1, \mathbb{R})/P_2
\end{array}$$

Here, we may choose $P_1 \subset Sp(n+1, \mathbb{R})$ to be the stabilizer of the line $l = \{(x^0, x^i = 0, y^A = 0)\}$ and $P_2 \subset Sp(n+1, \mathbb{R})$ to be the stabilizer of the Lagrangian $(n+1)$-plane $E = \{(x^A, y^A = 0)\}$. Note that the fibers of the projections $\pi_1, \pi_2$ are $\text{Lag}(n, \mathbb{R}^n)$ and $\mathbb{RP}^n$ respectively.

The contact structure on $\mathbb{RP}^{2n+1}$ can now be described as follows. Take $l \in \mathbb{RP}^{2n+1}$. For a generator $v \in \mathbb{R}^{2n+2}$ of $l$, consider the 1-form

$$v \perp \omega. \tag{60}$$

Since the generator $v$ is defined up to a nonzero scalar multiple, the 1-form $v \perp \omega$ is also well defined on $T_l \mathbb{RP}^{2n+1}$ up to a nonzero multiple. From (59), it is easy to see this is a contact
structure as in (1). Also it follows from the construction that the fiber of the bundle \( I \to RP^{2n+1} \) is the bundle of Legendrian \( n \)-planes over \( RP^{2n+1} \) with respect to the given contact structure.

The description of the contact structure above naturally induces a Legendrian submanifold path geometry on \( RP^{2n+1} \). Take \( l \in RP^{2n+1} \) and a Legendrian \( n \)-plane \( E_0 \subset T_l RP^{2n+1} \). Then there is a unique Lagrangian \((n + 1)\)-plane \( E \subset T_l RP^{2n+1} \) such that its image in \( RP^{2n+1} \) under the projection \( R^{2n+2} - \{0\} \to RP^{2n+1} \) is a \( n \)-dimensional Legendrian submanifold of \( RP^{2n+1} \) passing through \( l \) with \( E_0 \) as its tangent space.

We wish to show that this path geometry is locally equivalent to the geometry of the hyperquadrics in \( R^{n+1} \). Consider a local coordinates parametrization of \( RP^{2n+1} \) by

\[
\begin{pmatrix}
X^0 \\
X^i \\
Y^0 \\
Y^i
\end{pmatrix} = \begin{pmatrix}
1 \\
x^i \\
2u - x^i p^j \\
p^j
\end{pmatrix}.
\]

The contact structure is then locally generated by

\[
\sum_A X^A dY^A - Y^A dX^A = 2(du - \sum_i p^i dx^i).
\]

Under these coordinates, the hyperquadrics

\[
u = a_0 + \sum_i a_i x^i + \sum_{ij} \frac{1}{2} a_{ij} x^i x^j
\]

\[p_i = a_i + \sum_j a_{ij} x^j\]

with \( a_{ij} = a_{ji} \) correspond to

\[
Y^0 = 2a_0 X^0 + \sum_i a_i X^i
\]

\[
Y^i = a_i X^0 + \sum_j a_{ij} X^j,
\]

which are the equations in \( R^{2n+2} \) that define Lagrangian \((n + 1)\)-planes.

### 7.2 Example with Symmetry

Let \( M^n \) be a hypersurface in a space of constant curvature \( \tilde{M}^{n+1} \), \( c = 1, 0, \text{ or } -1 \), without any continuous extrinsic symmetry, i.e., the subgroup of motions \( \tilde{I}(M) \subset \text{Iso}(\tilde{M}^{n+1}) \) that preserves \( M \) is at most discrete. Let \( Y^{2n+1} = Gr(n, \tilde{M}^{n+1}) \to \tilde{M}^{n+1} \) be the bundle of hyperplane elements, which has a natural contact structure. Consider the \( N = \dim \text{Iso}(\tilde{M}^{n+1}) = \frac{1}{2}(n + 1)(n + 2) \)-parameter family of hypersurfaces in \( \tilde{M}^{n+1} \) generated by the action of \( \text{Iso}(\tilde{M}^{n+1}) \) on \( M \). The canonical lifts of this family to \( Y \) will in general be an \( N \)-parameter family of Legendrian
submanifolds, nondegenerate in the sense discussed in Section 2 if the second fundamental form of \( M \) is nondegenerate with distinct and functionally independent eigenvalues.

In order to see that this Legendrian submanifold path geometry is not flat, it would suffice to show the nonexistence of injective homomorphisms

\[
\text{Iso}(\bar{M}_c^{n+1}) \to \text{Sp}(n+1, R).
\]

for \( n \geq 1 \).

**Lemma 1** There does not exist an injective homomorphism

\[
\text{Iso}(\bar{M}_c^{n+1}) \to \text{Sp}(n+1, R).
\]

for \( n \geq 1 \).

**Proof.** Let’s take the case \( c = 1 \),

\[
SO(n+1) \to \text{Sp}(n, R).
\]

with \( n \geq 4 \). The cases \( n = 2, 3 \) and \( c = -1, 0 \) follow from similar arguments. Also, we may assume the image of \( SO(n+1) \) lies in the maximal compact subgroup \( U(n) \subset \text{Sp}(n, R) \), in fact in \( SU(n) \subset U(n) \).

The group \( SO(n+1) \) has, as the first two representations of minimal dimensions, the standard representation \( V \) of dimension \( (n+1) \), and the adjoint representation of dimension \( \frac{1}{2}n(n+1) > 2n \) for \( n \geq 4 \). An injective homomorphism \( SO(n+1) \to SU(n) \) induces a faithful representation of \( SO(n+1) \) of dimension \( 2n, W^{2n} \). From the injectivity and the inequality above, \( W \) must contain a \( V^{n+1} \) piece. Since \( SO(n+1) \subset SU(n) \), it also preserves the complex conjugate \( J(V) \), which is impossible for dimension reasons. \( \square \)

The invariants of the Legendrian submanifold path geometry thus obtained in general are expressed in terms of fourth order information on the original hypersurface.
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