SUMMARY

In this paper we outline the framework of mathematical statistics with which one may study the properties of galaxy distance estimators. We describe, within this framework, how one may formulate the problem of distance estimation as a Bayesian inference problem, and highlight the crucial question of how one incorporates prior information in this approach. We contrast the Bayesian approach with the classical ‘frequentist’ treatment of parameter estimation, and illustrate – with the simple example of estimating the distance to a single galaxy in a redshift survey – how one can obtain a significantly different result in the two cases. We also examine some examples of a Bayesian treatment of distance estimation – involving the definition of Malmquist corrections – which have been applied in recent literature, and discuss the validity of the assumptions on which such treatments have been based.

1 INTRODUCTION

Recently, the estimation of galaxy distances has assumed great importance in cosmology. The analysis of large-scale galaxy redshift surveys, used in conjunction with redshift-independent galaxy distance estimates, can place powerful constraints on the values of the cosmological parameters $H_0$ and $\Omega_0$ (c.f. Hendry, 1992b; Dekel, 1994), and in principle can allow one to test several of the hypotheses – including the form of the initial spectrum of density perturbations, the role of gravity in the growth of structure and the clustering properties of dark matter – on which current theories for the formation of large scale structure in the universe are largely based. Various methods have been developed to reconstruct the density and three-dimensional peculiar velocity field from galaxy redshift and redshift-distance surveys (c.f. Dekel et al, 1990, 1993; Simmons, Newsam & Hendry, 1995; Rauzy, Lachieze-Rey & Henriksen, 1994), based upon the ansatz that the peculiar
velocity field is a potential field – an idea first developed in the POTENT reconstruction method (Bertschinger & Dekel, 1989). At the same time, new statistical methods of analysing surveys which consist of redshifts alone have been developed (c.f. Lahav et al, 1994; Fisher et al, 1994; Heavens & Taylor, 1995) based upon the description of the large scale density and velocity field in terms of sets of orthogonal functions. One of the biggest current challenges in this field is to combine in an optimal fashion the results of these two different methods of analysis, in order to place stronger constraints on cosmological models and the values of cosmological parameters – a subject which would merit an entire article in itself. In this article we will focus instead only on those issues which concern the former group of reconstruction methods – i.e. where one attempts to obtain redshift-independent distance estimates to galaxies.

Attempts to map the large scale structure of the universe from redshift-independent galaxy distance estimates have not been without controversy. For many years considerable debate has been generated over the precise nature, or indeed the very existence, of galaxy concentrations such as the ‘Great Attractor’ in the direction of Hydra and Centaurus, for example (Lynden-Bell et al, 1988; Dressler & Faber, 1990; Mathewson, Ford & Buchhorn, 1992; Federspiel, Sandage & Tammann, 1994). A significant factor fuelling this controversy has been disagreement not so much over the astrophysical problems of determining ‘good’ galaxy distance indicators (although this has undoubtedly played a part also) but rather disagreement over the equally fundamental question of what statistical methods one should adopt to analyse the galaxy data. In this paper we attempt to clarify and place in the open some of the different statistical approaches which have been adopted in this field of cosmological research, and to discuss – within the framework of mathematical statistics – the different underlying philosophies upon which (often implicitly) they are based. Our discussion should be viewed as a general introduction to the problem, suitable for a reader previously unfamiliar both with the relevant astronomical details of measuring galaxy distances and with the basic theory of probability and statistics upon which the topic is founded. References to more detailed articles, covering both the astronomical and statistical aspects of the problem, will be given wherever appropriate.

The measurement of the distance of a galaxy, is an example of an inference problem: i.e. one cannot measure the distance directly but must infer it from the measurement of some other physical characteristic, such as the apparent visual magnitude or angular diameter. If one knew precisely the absolute magnitude or intrinsic diameter of the galaxy then one could immediately arrive at an exact determination of the galaxy distance. In early studies of the large scale distribution and motion of galaxies (c.f. Rubin et al, 1976; Sandage & Tammann, 1975a,b) the approach was simply to assume a priori some fiducial value for this absolute magnitude or diameter and thus infer galaxy distances on that basis. In practice, however, not all galaxies have the same absolute magnitude or diameter and so the inference is statistical in nature. Shortly after these early studies significant progress was made with the identification of empirical relationships between absolute magnitude and diameter and other, distance-independent but directly measurable, physical quantities such as velocity dispersion or colour (c.f. Faber & Jackson, 1976; Tully & Fisher, 1977; Visvanathan & Sandage, 1977). The Tully-Fisher relation, for example, essentially expresses a power law relationship between the luminosity and the rotation velocity – as measured from e.g. the 21cm neutral hydrogen radio emission – of spiral galaxies. Thus one measures the 21cm line width of neutral hydrogen for a given spiral
galaxy, applies the Tully-Fisher relation to infer the absolute magnitude of the galaxy, and then infers the galaxy distance from its observed apparent magnitude.

In the past decade the Tully-Fisher, and other similar, relations have been further refined and placed upon a firmer theoretical footing, (Pierce & Tully, 1988; Salucci, Frenk & Persic, 1993; Hendry et al, 1995) but they still contain a significant degree of intrinsic scatter and so do not provide an exact determination of absolute magnitude or diameter. Hence, the galaxy distance inferred from such a relation is still inherently statistical. In the language of mathematical statistics, the intrinsic scatter of the relation means that we can construct only an estimator of the galaxy distance, and that distance estimator will itself be subject to error. More formally, the distance estimator is a random variable with a definite distribution function, or equivalently probability density function (pdf), and a fortiori mean and variance.

Unfortunately there is no unique way to construct distance estimators. One can make a choice of distance estimator which has certain desirable properties, the most obvious being that its distribution should have a small ‘spread’, or variance; on average over many realisations the estimator should give the true distance of the galaxy; and the estimator should use all of the information about the galaxy distance available in the data. These rather loosely defined properties have their corresponding rigorous definitions in the statistical literature, and these are referred to as efficiency, unbiasedness and sufficiency respectively.

One should remark that when measurement errors and intrinsic variability are small in the physical system which one is modelling, then the adoption of a broad class of different statistical methods – or even different statistical philosophies – in testing models from observational data will usually make little difference to one’s conclusions. Large discrepancies in the conclusions reached by various authors in the literature concerning the estimated distances of galaxies and clusters therefore arise primarily because of large intrinsic uncertainties inherent to the data. In other words, galaxy distance indicators are noisy, with typical distance errors from, e.g., the Tully-Fisher relation of around 20% or larger to individual galaxies. It is this fact which makes the question of how one approaches the problem of choosing the ‘best’ galaxy distance estimator a non-trivial, and an extremely important, one. The typical size of distance errors has led many cosmologists to attempt to incorporate prior information on the distribution of galaxy distances when defining distance estimators, with the aim of reducing the uncertainty in the final estimate. All examples of this approach can be traced back to what is termed in the statistical literature as a Bayesian treatment of the problem of distance estimation, although references in the cosmology and astronomy literature have often not explicitly used the term ‘Bayesian’, nor indeed used wholly orthodox Bayesian methods, in their description of the problem. There are indeed some difficulties with this approach. One the one hand there are philosophical and methodological problems that have long been recognised and debated by statisticians (c.f. Kendall & Stuart, 1963; Mood & Graybill, 1974; von Mises, 1957; Feigelson & Babul, 1992) which go to the root definitions and concepts in the theory of probability. On the other, there is often no clear-cut way of deciding upon the nature of prior information one can justifiably use. This paper is not the appropriate place to discuss either of these questions in any great depth. We would like to emphasise here, however, the principle employed in Bayesian inference problems in the general statistics
literature: that results which depend heavily on the choice of prior information should be treated with caution.

Whilst the problem of galaxy distance estimation raises certain statistical issues which are somewhat unique to astronomy – in particular the important role of observational selection effects and the modelling of the physical processes underlying the various distance relations which are applied to galaxies – the fundamental concepts are precisely the same as one finds in the general statistical literature on inference problems and estimation. It seems sensible, therefore, for cosmologists to make full use of the ‘machinery’ – the definitions, notation and general results – developed by statisticians for tackling such problems. In this paper, as in our earlier papers on this subject, we shall attempt to adhere to this practice.

The structure of this paper is as follows. In section 2 we discuss in more detail the nature of distance estimators, placing our discussion in the rigorous context of mathematical statistics and introducing the appropriate notation and conventions. We go on to discuss the role of prior information, to explain the concepts of a ‘Bayesian’ approach to estimation problems, and to examine the relationship between Bayesian and more orthodox or ‘frequentist’ approaches. We show, by means of the simple example of estimating the distances to galaxies in a single catalogue, how a Bayesian and frequentist approach will yield different results. In section 3 we discuss the various galaxy distance estimators which have been used in recent literature, drawing particular attention to the statistical ‘philosophy’ (i.e Bayesian or frequentist) upon which they are based, the validity of the assumptions inherent in their definition, and the extent to which they can be regarded as ‘good’ estimators – in the sense of e.g. unbiasedness, efficiency and sufficiency, as introduced above. Finally we discuss the practical outcomes of using these different estimators for determining distances to individual galaxies and clusters and in the analysis of the peculiar velocity and density field by, e.g., the POTENT based methods mentioned above.

2 STATISTICAL PROPERTIES OF DISTANCE ESTIMATORS

One of the purposes of this section is to clarify our notation and statistical approach for the benefit of the reader previously unacquainted with the general statistics literature. In the interests of brevity we shall present here only the essential ideas and omit unnecessary detail, perhaps at the risk of appearing simplistic. A more thorough, and wholly rigorous, treatment of the mathematical foundations of parameter estimation can be found in a large number of textbooks on probability and statistics (c.f. Hoel, 1962; Kendall & Stuart, 1963; Mood & Graybill, 1974; Hogg & Craig, 1978)

What Is an Estimator?

In rough terms, an estimator of some unknown parameter is a rule based on statistical data – i.e. a random sample drawn from some underlying population – for estimating the value of that parameter. If the parameter of interest is \( q \) then we shall write \( \hat{q} \) to denote an estimator of \( q \), following the standard statistical convention. Note that \( \hat{q} \) is written in bold face to indicate the fact that it is a random or statistical variable (since it is a function of data which are themselves statistical variables), again in keeping with the standard practice in the literature. One cannot, of course, expect \( \hat{q} \) to take on the
true value of \( q \), \( q_0 \) say, for every set of statistical data, but we would regard an estimator as ‘good’ if it tends to yield the value \( q_0 \) ‘on average’, or ‘in the long run’ – rather vague statements which can be quantified in terms of the bias and loss function associated with the estimator chosen, as we discuss below.

By way of an illustrative example, a simple galaxy distance estimator could be constructed only from the observed apparent magnitude of a galaxy (c.f. Hendry & Simmons, 1990; Hendry, 1992a). Thus we may write

\[
m - M = 5 \log r + 25
\]  

(1)

where \( r \) is the true distance, measured in Mpc, and \( m \) and \( M \) denote the apparent and absolute magnitude of the galaxy respectively. Of course the actual distance of the galaxy can only be obtained if there is no error on the measured value of \( m \) and if \( M \) is known. We can estimate \( r \), however, by making some assumption about the value of \( M \) (for simplicity we shall ignore any error on \( m \) in this discussion) and solving for \( r \) in equation (1). Suppose we take the value of \( M \) to be the mean value of absolute magnitude, \( M_0 \) say, for the underlying population of all galaxies of a certain Hubble type. We thus obtain an estimator of log distance, viz

\[
\hat{\log} r = 0.2(m - M_0 - 25)
\]  

(2)

Here the hat indicates an estimator. If we consider that the galaxy has been randomly selected from an imaginary population of galaxies all at the same distance, but with different absolute magnitudes, then \( \hat{\log} r \) must be considered to be a statistical variable, as noted previously. The statistical properties of \( \hat{\log} r \) depend on the galaxy luminosity function and on the selection function which determines whether a galaxy will or will not be observed at true distance, \( r \). It follows from equations (1) and (2) that we may write

\[
\hat{\log} r = \log r + 0.2(M - M_0)
\]  

(3)

For brevity we shall in future refer to \( \log r \) as \( w \), and \( \hat{\log} r \) as \( \hat{w} \).

In general, the underlying pdf before selection for \( M \) is not known. This pdf is usually assumed to be independent of position and is just the luminosity function (LF) of \( M \), written \( \Psi(M) \). The distribution, \( \Psi_{\text{obs}}(M|r) \), of \( M \) for observable galaxies at actual distance, \( r \), will depend upon the selection function and indeed also on \( r \) (although for simplicity we assume here no dependence on direction). Once this pdf, \( \Psi_{\text{obs}}(M|r) \), is given the pdf of any function of the random variable, \( M \), may be determined. In particular the pdf of \( \hat{w} \) defined by equation (3) may easily be found. Note that while \( \hat{w} \) itself does not depend on \( w \), the pdf of \( \hat{w} \) does depend upon the true value of the parameter, as one might expect.

Biased and Unbiased Estimators

The mean, or expected, value of a random variable associated with a galaxy may be taken with respect to either the observable or the intrinsic galaxy distribution. We shall almost invariably consider the expectation with respect to the observable distribution in this paper. Thus the estimator of log distance, \( \hat{w} \), is defined to be unbiased if

\[
E(\hat{w}|w) = w
\]  

(4)
where the expectation value of any function, \( f(M) \), of \( M \) is defined as

\[
E[f(M)|w] = \int f(M)\Psi(M|w)\,dM
\]  

(5)

The bias, \( B(w) \), is defined as

\[
B(w) = E[\hat{w}|w] - w
\]  

(6)

When a galaxy survey is subject to a selection limit on apparent magnitude, the estimator of log distance given by equation (2) is biased for all true log distances. Moreover, simply replacing the mean absolute magnitude, \( M_0 \), of the underlying population by some fiducially corrected value, \( M_0 + c \), where \( c \) is a constant, cannot eradicate this bias (c.f. Hendry & Simmons, 1990). One can apply an iterative procedure – effectively adding a non-constant correction to \( M_0 \) – which considerably reduces the bias of \( \hat{w} \), although this procedure does not converge to an unbiased estimator for all log distance (Hendry, 1992a). It has been shown, however, (c.f. Schechter, 1980; Hendry & Simmons, 1994) that in the case of a relation of Tully-Fisher type – where one has an additional observable correlated with absolute magnitude – if the second observable is free from selection effects then one can define an estimator which is unbiased at all true log distances. We return to this issue in section 3.

Minimum Variance and Efficient Estimators

There are obvious advantages in using unbiased estimators: in particular, for large samples – e.g. when one is estimating the distance to a rich cluster of galaxies – the mean estimated distance for the sample will also be unbiased, and of course will have decreasing variance as the sample size increases. Furthermore, if we are interested in, say, the distribution of actual distances of a catalogue of galaxies, the histogram of estimated distances can be readily deconvolved to yield an estimate of this underlying distribution of true distances. For biased estimators this would be more difficult (c.f. Eddington, 1913; Newsam, Simmons & Hendry, 1994, 1995). Similarly, in model fitting problems – the simplest of which in the present context is e.g. the determination of the Hubble constant – we can expect parameter estimation to be much easier if we begin with unbiased estimators. Unbiasedness is not the only criterion for choosing an estimator, however. It is also natural to desire the estimator to have a small variance. The variance, \( V(w) \), of an estimator is defined as

\[
V(w) = E[(\hat{w} - w)^2|w]
\]  

(7)

In practice one finds that there is a trade-off between small variance and small bias, in the sense that if you reduce one then you increase the other. The Cramer-Rao inequality places a lower bound on the variance for both biased and unbiased estimators (c.f. Hogg and Craig, 1978; Hendry, 1992a; Gould, 1995; Zaccheo et al, 1995), and an efficient estimator is one which attains that lower bound – i.e. which is a minimum variance estimator.

In choosing an estimator it is also usually convenient to introduce a loss function, which essentially quantifies the ‘loss’, or cost, of making an incorrect estimate of a parameter. An obvious loss function to consider is

\[
L(\hat{w}, w) = (\hat{w} - w)^2
\]  

(8)
A good estimator should yield low values of the expected loss for a large range of values of the parameter \( w \). This expected loss is called the **risk**, i.e.

\[
R(w) = E[L(\hat{w}, w)]
\]  

(9)

Note that for an unbiased estimator the risk and variance are identical, but for a biased estimator the risk is always strictly greater than the variance. Thus, if one has an estimator with small variance but large bias, this would still result in an estimator of large risk – indicating that risk is often the more meaningful quantity in comparing estimators.

In general the bias, variance and risk of an estimator are related by the following simple expression

\[
R(w) = V(w) + [B(w)]^2
\]  

(10)

**Sufficiency**

In estimating the distance of a galaxy one does not generally adopt an estimator of the simple form of equation (2), which is a function only of apparent magnitude, but rather makes use of a distance indicator such as the Tully-Fisher relation which depends upon the the strong correlation between absolute magnitude and some other distance-independent, directly measurable, observable. Since the underlying physical relationship in an indicator of this type is unlikely to depend upon only two variables, one could in principle construct a distance estimator as a function of an arbitrary number of observables, or **statistics**. The bias and risk of such an estimator would depend, of course, upon how well correlated were the observables. In Hendry & Simmons (1994) the general case of estimators formed from three correlated observables is formulated, and in Kanbur & Hendry (1995) a specific example is considered where the addition of a *fourth* observable – the maximum apparent magnitude – to the period, mean luminosity, colour relation for Cepheid variable stars does indeed result in a distance estimator of significantly smaller variance and risk.

A obvious general question to ask, then, is whether there exists a function, say \( \hat{w}(x_1, ..., x_n) \), of a set of observables, \( x_1, ..., x_n \), which ‘contains’ all of the information about the true value of \( w \). Such a function is known as a **sufficient statistic** – and hence would define a sufficient estimator – for \( w \), and so should be preferred over another estimator without this property. The property of sufficiency can be given a more rigorous mathematical definition in terms of the joint pdf of \( x_1, ..., x_n \) and \( \hat{w} \) (c.f. Mood & Graybill, 1974). Suppose that \( \hat{w}_*(x_1, ..., x_n) \) is another statistic based on the observables, \( x_1, ..., x_n \), which is not a function of \( \hat{w} \). Then \( \hat{w} \) is defined to be sufficient if, for any such \( \hat{w}_* \), the conditional distribution of \( \hat{w}_* \), given \( \hat{w} \) does *not* depend on the true parameter value, \( w \).

This definition essentially states that once the value of the sufficient statistic has been specified, one cannot find any other statistic based on the same set of observables which gives any further information about the true value of \( w \). In a sense, \( \hat{w} \) ‘exhausts’ all the information about \( w \) that is contained in the observed values of \( x_1, ..., x_n \).

**Bayes’ Estimators**

So far we have said nothing about the incorporation of prior information in the estimation of galaxy distances. Bayesian approaches attempt to do precisely this.
A fully fledged Bayesian approach would regard $w$ – in the above notation – not as a parameter, but as a statistical variable. The probability (more commonly referred to as the \textit{likelihood}) of this variable taking any given value would be determined by what is known as its \textit{prior distribution}: prior, that is, to the data that we presently have at hand. In the cosmological setting, therefore, $w$ – the log distance of a galaxy – would be taken to have a prior distribution before the apparent magnitude or diameter or line width of this galaxy were measured. This prior distribution would be based on previous information about the distribution of galaxies as a whole – or even preconceptions about this distribution, such as the assumption that the spatial distribution of galaxies be uniform. In this case one has to modify the orthodox frequentist view of probability as a ‘limit’ of relative frequencies and adopt instead a view of probability as a measure of one’s state of knowledge about a random variable.

The \textit{posterior} distribution for $w$, once the data for a particular galaxy has been taken into account, is then obtained by applying Bayes’ theorem. Suppose one’s distance estimator is a function of two variables, $m$ and $P$ – denoting for example apparent magnitude and log rotation velocity for the Tully Fisher relation. Bayes’ theorem states that

$$p(m, P|w)p(w) = p(w|m, P)p(m, P)$$

Taking $p(m, P)$ to be a constant, one obtains the posterior distribution for $w$, viz

$$p(w|m, P) = C p(m, P|w)p(w)$$

where $p(w)$ is the prior, $C$ is a normalisation constant and $p(m, P|w)$ is the conditional probability of $m, P$ given $w$.

This approach in itself does not give an estimator of $w$, which is a statistical variable and not strictly speaking a parameter in the Bayesian context, but rather it gives a posterior pdf for $w$ from which one may define a \textit{Bayes’ estimator} (c.f. Mood & Graybill, 1974) in the following way. A Bayes’ estimator, $\hat{w}_{\text{bayes}}$, minimises the risk, $R(w)$ averaged over the prior distribution, $p(w)$ for $w$. Thus for a Bayes estimator the integral

$$\int R(w) p(w) \, dw$$

is a minimum. It can be shown that a Bayes’ estimator in fact minimises the loss function averaged over the distribution for $w$ conditional on the observed data. Explicitly it minimises

$$\int L(\hat{w}_{\text{bayes}}, w)p(w|\text{data}) \, dw$$

from which $\hat{w}_{\text{bayes}}$ can be found.

It is instructive to consider a simple example where we are estimating the log distance, $w$, of a galaxy. Let us assume that we have already an unbiased (in the sense of equation 6) ‘raw’ estimator, $\hat{w}$, based on some distance indicator, which we shall for expediency take to be normally distributed about the true log distance $w$ with variance $\sigma^2$. Thus the conditional distribution for $\hat{w}$ given $w$ is

$$p(\hat{w}|w) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2\sigma^2}(\hat{w} - w)^2\right]$$
Let us assume, however, that the galaxy is randomly selected from some underlying population with true log distance $w$ distributed normally about some mean value, $w_c$, and variance $\sigma^2_c$ – where the subscript $c$ refers to the catalogue from which the galaxy is drawn. This normal distribution is taken to be the prior, so in the above notation

$$p(w) = \frac{1}{\sqrt{2\pi}\sigma_c} \exp\left[-\frac{1}{2}\left(\frac{w - w_c}{\sigma_c}\right)^2\right]$$  \hspace{1cm} (16)$$

It is now straightforward to show that the conditional distribution for $w$ given the value of $\hat{w}$ is normally distributed with mean, $w_B$ and variance $\sigma^2_B$ given by

$$w_B = \frac{\hat{w} + \beta w_c}{1 + \beta}$$  \hspace{1cm} (17)$$

and

$$\sigma^2_B = \frac{\sigma^2}{1 + \beta}$$  \hspace{1cm} (18)$$

where $\beta = \sigma^2/\sigma^2_c$, from which it follows that

$$\hat{w}_{\text{bayes}} = \frac{\hat{w} + \beta w_c}{1 + \beta}$$  \hspace{1cm} (19)$$

The interpretation of this result is very straightforward. If the variance of the indicator is much smaller than the population variance of the normal distribution of true log distance for observable galaxies then $\beta \approx 0$ and one obtains essentially the ‘raw’ log distance estimator, $\hat{w}$, suggested by the indicator. If, on the other hand, the indicator provides very poor information about the distance of the galaxy then $\beta$ is very large, and the Bayes estimator yields approximately $w_c$, the mean true log distance of the observable galaxies in the catalogue. This simple example demonstrates that, provided the scatter in one’s distance indicator is sufficiently small, one obtains essentially the same estimator irrespective of whether one adopts a Bayesian approach or not – and the estimator is thus largely insensitive to the prior information. The role of the prior becomes increasingly important, however – and the difference between a Bayesian and frequentist approach becomes more apparent – as the scatter in the distance indicator increases.

One can regard equation (19) as defining a correction to $\hat{w}$ based on the prior information – in this case that the underlying population of true log distance is normally distributed. Corrections of this type have come to be known in the cosmology literature as Malmquist corrections, and in the context of mapping large scale structure they were initially applied assuming the distribution of galaxies to be spatially homogeneous (c.f. Lynden-Bell et al, 1988; Dekel et al, 1993) – just as the distribution of stars had been assumed homogeneous in the original analytical treatments of Malmquist (1920, 1922) and Eddington (1913). Recently, however, attempts have been made to apply more general, inhomogeneous, Malmquist corrections which address the fact that the galaxy distribution displays small-scale clustering (c.f. Landy & Szalay, 1992; Hudson, 1994; Dekel, 1994; Newsam, Simmons & Hendry, 1995; Hudson et al, 1995; Freudling et al, 1995). We briefly consider some important technical problems regarding the application of inhomogeneous Malmquist corrections in section 3. It is worth noting here, however, that an entirely frequentist approach to distance estimation has the advantage that the definition of an unbiased estimator is completely independent of the underlying galaxy true number density – and hence is unaffected by arguments about the form of prior distribution which one should adopt.
Most redshift-independent methods of estimating galaxy distances which have featured in the recent cosmological literature are based upon secondary distance indicators – which require to be calibrated using a sample of galaxies in, e.g., a nearby cluster, the distance of which is already known. Notable exceptions to this have been the recent extension to beyond the Local Group of the extragalactic distance scale measured from Cepheid variables and the application of the expanding photosphere method (EPM) to determine the distances of type II supernovae (SN). Both Cepheids and type II SN are examples of primary distance indicators which can be calibrated either locally – within our own galaxy – or from theoretical considerations. For a discussion of the physical basis for these indicators the reader is referred to, e.g., Kirschner & Kwan (1974), Eastman & Kirschner (1989), Jacoby et al (1992) and references therein. Both indicators have a small intrinsic dispersion (∼10 − 15% to individual objects) and are thus considerably less susceptible to the problems which arise in the definition of Malmquist corrections and sensitivity to the choice of prior information (essentially because the β, the ratio of the estimator variance to the variance of the underlying population, is small). This property of course makes both Cepheids and type II SN well suited to the estimation of the Hubble constant – either directly or in combination with other secondary indicators such as type Ia SN (c.f. Saha et al, 1994; 1995). Indeed, the high estimates of H₀ reported in Freedman et al (1994) and Pierce et al (1994), based on the distance of Cepheids in Virgo cluster galaxies, and those of Schmidt et al (1992, 1994) based on the EPM distances of type II SN beyond the Local Supercluster (and thus less adversely affected by peculiar velocities), provide a compelling argument in favour of a value of H₀ = 60 km s⁻¹ Mpc⁻¹ – despite the difficulties of reconciling these results with astrophysical estimates of the age of the galactic disc (c.f. van den Bergh 1995; Chamcham & Hendry, 1995).

Of the secondary distance indicators currently in widespread use, only two are thought to be sufficiently accurate to make the question of how to best use prior information essentially unimportant: these are surface brightness fluctuations (SBF) and the luminosity–light curve shape relation for type Ia SN. The former distance indicator, SBF, was pioneered by Tonry & Schneider (1988) and is based upon the fact that the fluctuations – due to the discreteness of individual stars – in surface brightness across the CCD image of a nearby elliptical galaxy will be larger than those for a more distant galaxy. The physical basis of SBF and the details of its calibration are described in Jacoby et al (1992). Relative distances of a typical accuracy of 5% have been derived to a sample of several hundred ellipticals out to a redshift of around 6000 km s⁻¹ using this indicator (Dressler, 1994).

Type Ia SN have long been recognised as useful ‘standard candles’ since they are observable to very large distances and have a luminosity function which is well described a Gaussian distribution of dispersion around 0.5 mag. (Sandage & Tammann, 1993; Hamuy et al, 1995). In Vaughan et al (1995), it is argued that the pre-selection of SN based on a colour criterion reduces this dispersion to ∼0.3 mag., which – although a significant improvement – still represents a typical percentage distance error of around 15% to an individual galaxy. In Riess, Press & Kirshner (1995a,b) however, the shape of the SN light curve is used to more tightly constrain the peak luminosity and leads to a typical relative distance error of only 5% – small enough to render Malmquist corrections largely
unimportant. This method has been used both to estimate the Hubble constant and to
determine the bulk flow motion of the Local Group on a scale of $\sim 7000 \text{ km s}^{-1}$, yielding a
motion which is consistent with the COBE measurement of the dipole anisotropy in the mi-
crowave background radiation, but inconsistent with the dipole motion reported by Lauer & Postman (1994), based on the redshifts of Abell clusters at distance of $8000 - 11000 \text{ km s}^{-1}$.

The vast majority of recent analyses of the peculiar velocity and density fields, and
the estimation of the density parameter $\Omega_0$ using redshift-independent distance indicators,
have been carried out primarily with the Tully-Fisher (TF) and $D_n - \sigma$ distance indicator
relations for spirals and ellipticals respectively. As we remarked above, the TF relation
essentially expresses a power law relationship between the luminosity and rotation veloc-
ity for spiral galaxies; the $D_n - \sigma$ relation similarly expresses a power law relationship
between the central velocity dispersion and isophotal diameter of early-type galaxies (c.f.
Jacoby et al, 1992). Although the number of galaxy distances estimated by these two
relations currently stands at over 4000 (around a factor of ten larger than the number of
distance estimates from SBF and SN distance indicators), and continues to grow rapidly
each year, both the TF and $D_n - \sigma$ relations are considerably more noisy – with disper-
sions of around 20% to individual galaxies. It is for this reason that the issue of how – or indeed if – one should make use of prior information in the definition of ‘optimal’
estimators continues to be regarded as of crucial importance when interpreting the results
of applying these distance indicators to analyse redshift surveys.

Both the TF and $D_n - \sigma$ relations are usually calibrated by performing a linear re-
grression on a calibrating sample of galaxies whose distances are otherwise known. It
is instructive to consider this calibration procedure in more detail, in order to illustrate
some of the statistical pitfalls which may arise, for the generic example of the TF relation.
As before, we denote the log rotation velocity by $P$ and let $\hat{M}$ denote the estimator of
absolute magnitude which one derives from the TF relation, from which one may derive
the corresponding ‘raw’ estimator of log distance, $\hat{w}$, from equation (3) in the obvious
way. Thus, we obtain from the calibration a linear relationship between $\hat{M}$ and $P$,

$$\hat{M} = \alpha P + \beta$$

(20)

where $\alpha$ and $\beta$ are constants. The choice of which linear regression is most appropriate
is non-trivial when one’s survey is subject to observational selection effects. We can
demonstrate this with the following simple example. Suppose that the intrinsic joint
distribution of absolute magnitude and log(rotation velocity) is a bivariate normal. Figure
1 shows schematically the scatter in the TF relation in this case, for a calibrating sample
which is free from selection effects – e.g. a nearby cluster. (More precisely, the ellipse
shown is an isocontour probability region enclosing a given confidence region for $M$ and $P$).
The solid and dotted lines show the linear relationship obtained by regressing rotation
velocities on magnitudes and magnitudes on line widths respectively. Thus the dotted
line is defined as the expected value of $M$ at given $P$, while the solid line is defined as the
expected value of $P$ at given $M$. Since in practice one wishes to infer the value of $M$ from
the measured value of $P$, the $M$ on $P$ regression has been referred to in the literature as
defining the ‘direct’ or ‘forward’ TF relation, while using the $P$ on $M$ regression defines
the ‘inverse’ TF relation. For the bivariate normal case the equations of the direct and
inverse regression lines are as follows:

\[
E(M|P) = M_0 + \frac{\rho \sigma_M}{\sigma_P} (P - P_0)
\]  
\[
E(P|M) = P_0 + \frac{\rho \sigma_P}{\sigma_M} (M - M_0)
\]

where \(M_0\), \(P_0\), \(\sigma_M\), \(\sigma_P\) and \(\rho\) denote the means, dispersions and correlation coefficient of the bivariate normal distribution of \(M\) and \(P\). Both regression lines can be written in the form of equation (20), thus defining \(\hat{M}\) as a function of \(P\), although of course the constants \(\alpha\) and \(\beta\) are different in each case. Moreover the definition of \(\hat{M}\) is subtly different in each case. For the direct regression \(\hat{M}\) is identified as the mean absolute magnitude at the observed log line width. For the inverse regression on the other hand \(\hat{M}\) is defined such that the observed log line width is equal to its expected value when \(M = \hat{M}\). Consequently, as is apparent from their slopes, the direct and inverse regression lines give rise to markedly different distance estimators, although it is straightforward to show that in the absence of selection effects both estimators are unbiased, in the sense defined in equation (4), above.

The situation is very different when we include the effects of observational selection, however. This is illustrated in Figure 2, which shows the scatter in the TF relation in a calibrating sample subject to a sharp cut-off in absolute magnitude – as would be the case in e.g. a distant cluster observed in an apparent magnitude-limited survey. We can see that in this case the slope of the direct regression of \(M\) on \(P\) is substantially changed from that in the nearby cluster – indeed the direct regression is no longer linear at all. This means that if one calibrates the TF relation in the nearby cluster using the direct regression and then applies this relation to the more distant cluster, one will systematically underestimate its distance, since the expected value of \(M\) given \(P\) in the distant cluster is systematically brighter than that in the nearby cluster as fainter galaxies progressively ‘fade out’ due to the magnitude limit. The corresponding ‘direct’, or ‘\(M\) on \(P\)’, log distance estimator will therefore be negatively biased.
In an important paper Schechter (1980) observed that the slope of the inverse regression line is unchanged, irrespective of the completeness of one’s sample, provided that the selection effects are in magnitude only. We can see that this observation is valid in the simple case considered in Figure 2. In other words the expected value of $P$ given $M$ is unaffected by the selection effects and, therefore, defines an unbiased log distance estimator. In Hendry & Simmons (1994), Schechter’s result is derived within the rigorous framework of mathematical statistics, and the assumptions upon which it is based are generalised. In particular it is shown that the inverse TF log distance estimator is gaussian and unbiased at all true log distances provided only that the conditional distribution of $P$ given $M$ is Gaussian, that $E(P|M)$ is a linear function of $M$, and that the sample is not subject to selection on rotation velocity. Moreover, since the inverse log distance estimator is Gaussian it will also automatically be a sufficient and efficient estimator, as defined in section 2. In Hendry (1992a) it was also shown that when there is no selection on rotation velocity then the inverse log distance estimator is the only unbiased estimator which is a linear function of log rotation velocity and apparent magnitude. In particular, the ‘orthogonal’ (c.f. Giraud 1987), ‘bisector’ (c.f. Pierce & Tully 1988) and ‘mean’ (c.f. Mould et al 1993) regression lines also give rise to estimators which are biased at all true log distances in this case. A similar conclusion was also reached in Triay, Lachieze-Rey & Rauzy (1994).

The unbiased properties of the inverse TF relation have led to its use in defining a ‘raw’ distance estimator in a number of different recent analyses of the peculiar velocity field, including Newsam, Simmons & Hendry (1995), Freudling et al (1995), Nusser & Dekel (1994), Shaya, Tully & Pierce (1992) and Shaya, Tully & Peebles (1995). Its acceptance has been far from universal, however. Part of the reason for this is that, of course, in practice it is not the case that galaxy samples are free from selection effects on rotation velocity. In fact, it is commonly the case that redshift surveys are first selected on the basis of either apparent diameter or B-band apparent magnitude, or both, while the TF photometry is then carried out in the near infra-red, or I-band. This leads to a considerably more complex selection function, as modelled in Sodre & Lahav (1993), which in general renders all linear regressions biased. Essentially this problem arises because...
diameter, I-band and B-band magnitude and rotation velocity are mutually correlated variables, so that the selection on B-band magnitude and angular diameter ‘pollutes’ the joint distribution of I-band magnitude and rotation velocity in the TF relation – thus effectively rendering the assumptions inherent in deriving the unbiasedness of the inverse TF relation no longer valid (c.f. Hendry & Simmons 1994; Willick 1994).

One can determine the correct slope and zero point of the ‘direct’ TF relation from a cluster subject to observational selection effects by the application of straightforward iterative procedure – thus solving what has been termed in the literature as the ‘calibration problem’ (c.f. Willick 1994; Hendry et al, 1995). It is important to recognise, however, that the corresponding ‘raw’ log distance estimator will still be biased, in the sense of equation (4), at all true distances if applied to a galaxy survey subject to magnitude selection effects. This is because the joint distribution of absolute magnitude and log rotation velocity for observable galaxies will not be equal to the intrinsic joint distribution.

Why has the use of the ‘direct’ TF relation in recent literature continued to be widespread? To understand the reason for this we must first note that most recent analyses of galaxy distances and peculiar velocities have been carried out within a Bayesian framework, thus involving the application of Malmquist corrections to the ‘raw’ log distance estimator. The motivation for adopting a Bayesian approach (even if the Bayesian nature of the problem has not always been explicitly acknowledged by authors!) comes about from the way in which galaxy distance estimates and redshifts have been combined in the majority of analyses. In both the early ‘toy’ parametric velocity field models of e.g. Lynden-Bell et al (1988), Dressler & Faber (1990), and the more sophisticated reconstruction methods such as POTENT (c.f. Dekel et al 1990), essentially galaxies are binned and grouped together and assigned radial peculiar velocity estimates on the basis of their estimated distance. The galaxy’s actual distance could be radically different, and will depend on the true spatial distribution of galaxies and the exact nature of the survey selection function. Clearly galaxies which have small estimated distance are more likely to have been scattered down from larger true distances, since a volume element of fixed solid angle increases in size with true distance; close to the limit of the survey volume, however, this might no longer be the case, as galaxies scattered from larger true distances might be too faint to be included in the redshift survey. By requiring that on average the actual radial coordinate of the galaxy be equal to its estimated distance, one would also ensure that on average the correct peculiar velocity would be ascribed to that galaxy’s apparent position. The estimator which satisfies this condition can be defined following the Bayesian approach outlined for the simple illustrative example of section 2, and it is straightforward to show that such a ‘Malmquist corrected’ distance estimator, \( \hat{r}_{\text{bayes}} \), satisfies the equation

\[
\hat{r}_{\text{bayes}} = C \int \text{dexp}(\hat{w}) p(\hat{w}|w) p(w) \, dw
\]  

(23)

where \( C \) is a normalisation constant.

The key point about equation (23) is that – as before – the Bayesian distance estimator depends upon the prior distribution of true log distance, \( p(w) \). There has been no consensus in the literature on which prior one should adopt. As we mentioned in section 2, in Lynden-Bell et al (1988) the prior is assumed to correspond to a homogeneous distribution of galaxies – thus defining homogeneous Malmquist corrections which are a
function only of distance. In Landy & Szalay (1992), on the other hand, a more general correction is derived by first estimating \( p(w) \) from a spline fit to the histogram of log distance estimates for the galaxies in the survey, thus in principle taking into account inhomogeneities in the galaxy distribution. Due to the sparseness of surveys, however, it is usually necessary to average the distribution of galaxies over large solid angles, if not all, of the sky. Therefore, the effects of clustering may still go largely unaccounted for in the Landy & Szalay prescription (c.f. Newsam et al, 1994). In other recent analyses (c.f. Hudson 1994; Hudson et al 1995; Dekel 1994; Willick 1994; Freudling et al 1995) a different method is proposed for obtaining the prior distribution – by reconstructing the density field of optical or IRAS-selected galaxies based on redshifts alone, assuming linear or mildly non-linear theory to adequately describe the gravitational collapse of structure – smoothed on a scale of the order of 10 Mpc.

In all of the above analyses the Malmquist corrections are derived assuming that the conditional distribution of the ‘raw’ log distance estimator, \( p(\hat{w}|w) \), is normally distributed at all true log distances. As shown in Hendry & Simmons (1994), this assumption is invalid when the ‘raw’ estimator is derived from the ‘Direct’ TF relation. Thus, the formula of Landy & Szalay will result in an incorrect Malmquist correction due to the bias of the ‘Direct’ TF log distance estimator. In general, if the prior distribution of true log distance is inferred from the observed distribution of log distance estimates, then one must apply the formula of Landy & Szalay using the ‘Inverse’ TF estimator – which we have seen is normally distributed and unbiased at all true log distances, subject to the conditions specified above and in Hendry & Simmons (1994). A similar conclusion is reached in Teerikorpi (1993), Feast (1994) and Freudling et al (1995).

It is further shown in Hendry & Simmons (1994) that the use of the ‘Direct’ TF relation as the raw log distance estimator in defining general Malmquist corrections can only be justified if the prior distribution in equation (23) corresponds to the intrinsic distribution of true log distance. As a special case of this result, note that the homogeneous Malmquist correction of Lynden-Bell et al (1988) applied to the ‘Direct’ TF estimator will therefore be valid provided that the intrinsic distribution of galaxies is homogeneous. In a similar way, the inhomogeneous corrections derived in Hudson et al (1995), Dekel (1994) and Freudling et al (1995), will be valid provided the density field reconstructed from optical or IRAS-selected surveys corresponds to the intrinsic distribution of true log distance for the TF galaxies – in other words that the selection function of the redshift survey has been adequately corrected for, and the redshift survey faithfully traces the same underlying population as the galaxies to which the TF relation is being applied.

4 CONCLUSIONS

In this paper we have set out to describe – within the framework of mathematical statistics – some of the properties of ‘optimal’ galaxy distance estimators, including unbiasedness, sufficiency and efficiency. We have shown that the intrinsic scatter of indicators such as the Tully-Fisher and \( D_n - \sigma \) relations is sufficiently large that the question of which statistical philosophy one should adopt in the analysis of redshift surveys is far from trivial. In particular we have seen that one may formulate the problem of galaxy distance estimation as a Bayesian inference problem – essentially the approach which has been...
adopted implicitly in the literature in defining Malmquist-corrected distance estimators—but that there is no general agreement over the issue of how one should then best make use of prior information on the distribution of true galaxy distances. In particular, a failure to adequately understand the properties of the ‘raw’ galaxy distance estimator used can lead to the definition of invalid Malmquist corrections, as was the case in e.g. Landy & Szalay (1992). In Newsam, Simmons & Hendry (1995) we show that the use of such invalid corrections can frequently be worse than applying no corrections at all. A similar conclusion was reported in Freudling et al (1995), where it was shown that a number of biases may have gone unresolved in earlier attempts to incorporate prior information in the definition of distance estimators.

In reality, the issue of defining an ‘optimal’ galaxy distance estimator is only the first part of the story. In applying the POTENT procedure, for example, whether or not a distance estimator is biased is not the crucial question; what is important is to construct an unbiased smoothed peculiar velocity field. Although there appears some justification as to why this procedure requires the application of an essentially Bayesian approach, the Malmquist corrections which this approach entails are strictly only valid if galaxies are not too sparse, the gradient of the velocity field is not too large, and the effective radius of the window function used to smooth the data is not too wide. In Newsam, Simmons & Hendry (1995) a Monte-Carlo procedure, involving the generation of large numbers of ‘mock’ redshift surveys, is devised and implemented with the purpose of eliminating all biases from the POTENT-recovered velocity and density fields—not only those associated with the scatter of the distance indicators. A similar algorithm may be adopted for other reconstruction methods, and has the distinct advantage of being easily adapted to more general (and more realistic!) selection functions and distance indicators—involve, e.g., correlations between three or more observables where a wholly analytic treatment can often be intractable (c.f. Hendry & Simmons, 1994). A very similar Monte-Carlo approach has been adopted in Freudling et al (1995). These papers serve as an important reminder that the question of galaxy distance estimation cannot be regarded in isolation: ultimately the choice of which distance estimator is ‘optimal’ depends on the context in which the distance estimator is being used.

It is perhaps worthwhile to end on a positive note. The use of redshift independent galaxy distance indicators in conjunction with redshift surveys has opened up an exciting—and highly productive—‘industry’ in cosmology during the past decade or so. Although the statistical problems arising from the large intrinsic scatter of these indicators are considerable, the mathematical machinery briefly sketched in this paper equips us with the necessary tools to address important issues such as their sensitivity to the choice of prior information. Moreover, the significant recent advances made in developing and applying more accurate distance indicators, such as surface brightness fluctuations and the supernova light curve shape method, offer some further cause for optimism: perhaps within the next decade we will be able to map the large scale structure of the local universe with sufficient accuracy that the question of whether one should adopt a Bayesian approach to the analysis—and how in detail it should be implemented—will no longer be important.

Putting this another way, in the general statistics literature on Bayesian inference, when one’s results are sensitive to the choice of prior information, one is usually advised to go out in search of better data. Fortunately for those cosmologists measuring galaxy
distances, it appears that such data are indeed on their way!

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Nearby cluster

$E(MP)$
$E(PM)$

absolute magnitude

log(velocity)
