Superintegrable Cases of Four Dimensional Dynamical Systems

Oğul Esen\textsuperscript{a}, Anindya Ghose Choudhury\textsuperscript{b,3}, Partha Guha\textsuperscript{3}, Hasan Gümral\textsuperscript{4,5}

\textsuperscript{a}Department of Mathematics, Yeditepe University, 34755 Ataşehir, Istanbul, Turkey,
\textsuperscript{b}Department of Physics, Surendranath College, 24/2 Mahatma Gandhi Road, Calcutta 700009, India,
\textsuperscript{c}SN Bose National Centre for Basic Sciences, JD Block, Sector III, Salt Lake, Kolkata 700098, India,
\textsuperscript{d}Australian College of Kuwait, West Mishref, Kuwait

Abstract: Degenerate tri-Hamiltonian structures of Shivamoggi and generalized Raychaudhuri equations are exhibited. For certain specific values of the parameters, it is shown that hyperchaotic Lü and Qi systems are superintegrable and admit tri-Hamiltonian structures.

Mathematics Classification (2010) 34C14, 34C20.

Key words: First integrals, Darboux polynomials, Jacobi’s last multiplier, 4D Poisson structures, tri-Hamiltonian structures, Shivamoggi equations, generalized Raychaudhuri equations, Lü system and Qi system.

1 Introduction

Higher dimensional differential systems provide a setting and scope for studies of diverse physical phenomena together with their associated mathematical properties. In the case of integrable systems, for even dimensions, one can investigate the existence of symplectic structures; often in case of odd dimensions there exists what are known as Poisson structures displaying a range of mathematical possibilities emerging from the satisfaction of the Jacobi identity. Again from the dynamical point of view higher dimensional systems form a basis for occurrence of chaos which has become an intense field of study for a number of years now. Many applications of such higher dimensional differential systems have been found in the fields of engineering, biology, finance, medicine, climate studies etc. A major hurdle in the analysis of higher-order ODEs arises from the limited number of techniques at our disposal for their analysis. A majority of such systems, which have applications in real life situations, are nonlinear in character and until the advent of computational tools were beyond the realm of concrete analysis by conventional methods. This, however, should not be construed to undermine the power and utility of analytical investigations of such systems, since they often provide a basis for launching more focussed numerical investigations specially when dealing with practical problems.

The analysis of higher-order ODEs is greatly aided by the existence of first integrals since these allow for a reduction of the order of the system through elimination of one or more variables. The determination of first integrals is in itself a difficult task requiring, at times, substantial guile as well as some amount of luck often in equal measure. While for planar systems of ODEs there are some semi algorithmic methods depending on the nature of the functions that are involved, the same cannot be said for non-planar systems, thereby further complicating their analysis. Nevertheless one can often adapt certain tools and techniques tailored for planar systems and employ them for the analysis of non-planar systems.

The first nontrivial dimension for both the occurrence of chaos and non canonical Hamiltonian formalism is the three dimensions where, the foremost example of a chaotic systems, namely, the Lorenz system \cite{lorenz}, has been observed to admit non canonical bi-Hamiltonian structure for certain values of parameters involved, see \cite{guha1}. This structure has been extended to integrable cases of Lotka-Volterra, May-Leonard, Lü system, Chen system etc. as well as to the Kermack-McKendrick model and some of its generalizations describing the spread of epidemics \cite{guha2, guha3, guha4, guha5, guha6}. Important properties of three dimensional Poisson structures surrounding all these globally integrable bi-Hamiltonian systems are that the Jacobi identity is a scalar equation and is invariant under multiplication with arbitrary function.

\footnotetext{1oesen@yeditepe.edu.tr}
\footnotetext{2aghosechoudhury@gmail.com}
\footnotetext{3partha@bose.res.in}
\footnotetext{4h.gumral@ack.edu.kw}
\footnotetext{5On leave of absence from Department of Mathematics, Yeditepe University, Istanbul.}
1.1 Content of the work

The purpose of the present work is to extend the integrability analysis to four dimensions and to investigate the behavior of dynamical systems which are known to be chaotic for generic values of their physical parameters. This is essentially a non trivial problem with difficulties arising both from integrability analysis and chaotic behavior. On one side, we have the Jacobi identity which does not yield a single scalar equation, unlike the corresponding situation for three-dimensional systems [12, 14, 15]. On the other side, the characterization of chaotic behavior of systems may be more complicated. In four dimensions, one may encounter a chaotic system with more than one positive Lyapunov exponent. These are usually defined as hyperchaotic systems.

This work consists of two main sections. In the following section, we shall provide a brief summary of the notion of integrability as enunciated by Darboux. We shall recall method of Jacobi’s last multiplier for planar systems. Poisson structures and superintegrability will be reviewed. Bi-Hamiltonian structures of superintegrable three dimensional dynamical systems will be presented [15]. As an example, Lorentz model will be studied. Lorentz model has motivational importance for this study since it is chaotic but, for some certain values of the parameters, it is bi-Hamiltonian. This section will be ended by presenting a construction which enables to write tri-Hamiltonian structure of a four dimensional superintegrable dynamical system [9].

In section 3, we shall exhibit superintegrable structures and tri-Hamiltonian formulations of four dynamical systems in dimension four. The first dynamical system is Shivamoggi equations governing a 4D magneto-hydrodynamical system [34]. The second one is generalized Raychaudhuri equations defining geodesic flows on surface of a deformable media [11, 32]. Both of the Shivamoggi and generalized Raychaudhuri equations are autonomous and have time-independent first integrals. Existence of three time-independent integrals enables us to construct tri-Hamiltonian structures of these systems by simply following the general procedure presented in subsection (2.5). The third example is Lü system [5] and the fourth one is Qi system [30]. They are autonomous and hyperchaotic. We shall start with two time-dependent first integrals for these systems. After a change of variables, the systems will be nonautonomous whereas the integrals become time-independent. In the intersection of level surfaces of two time-independent first integrals, there will be a reduced autonomous dynamics in two dimensions. We shall apply method of Jacobi’s last multiplier for planar systems (c.f. subsection (2.2)) to obtain a Hamiltonian function of the reduced dynamics. The Hamiltonian functions for both of the reduced (Lü and Qi) dynamics turn out to be time-dependent. We shall change the time parameter and put some conditions on the other parameters in order to obtain an additional time-independent first integral. Obtaining three time-independent first integrals, we shall present tri-Hamiltonian structures of Lü and Qi systems. Thus we shall be able to show that even though Lü and Qi systems are hyperchaotic, for certain values of the parameters, they are superintegrable and admit tri-Hamiltonian structures.

2 Integrals, Superintegrability and Multi-Hamiltonian Systems

2.1 The first and second integrals

A first integral of a system of ODE’s

\[ \dot{x}^a = X^a(t, x^0, x^1, ..., x^{n-1}), \quad a = 0, 1, ..., n-1, \]

is a differentiable function \( I(t, x^0, x^1, ..., x^{n-1}) \) that retains a constant value on any integral curve of the system, that is, its derivative with respect to time \( t \) vanishes on the solution curves of the system [11], see [10, 29]. After introducing a vector field

\[ X(t, x^0, x^1, ..., x^{n-1}) = X^a(t, x^0, x^1, ..., x^{n-1}) \partial_{x^a} + \partial_t, \]

a first integral can be defined by the condition \( X(I) = 0 \). When the coefficient functions \( X^a \) in [11] have no explicit dependence on the time \( t \), the system is called autonomous. In this case, a first integral may still contain \( t \) explicitly.

A second integral is a \( C^1 \) function \( J \) satisfying the identity

\[ X(J) = \vartheta J. \]

Here, \( \vartheta \) is a real valued function called cofactor of the second integral. Polynomial second integrals for (polynomial) vector fields, called Darboux polynomials, simplify the determination of the first integrals.
When we have two relatively prime Darboux polynomials $P_1$ and $P_2$, having a common cofactor, of a (polynomial) vector field $X$, then the fraction $P_1/P_2$ is a rational first integral of $X$. The inverse of this statement is also true, that is if we have a rational first integral $P_1/P_2$ of a vector field $X$, then $P_1$ and $P_2$ are Darboux polynomials for $X$. For planar (polynomial) vector fields, when we have a certain number of relatively prime irreducible Darboux polynomials, not necessarily having a common cofactor, it is possible to write first integrals from the Darboux polynomials \[6, 7, 18, 33\]. The generalization of this theorem for arbitrary dimensions is still open.

In \[31, 33\], a semi-algorithm, called Prelle-Singer method, is presented for the determinations of elementary first integrals for planar systems. For higher dimensional systems, Darboux polynomials are useful, though sometimes the use of a specific ansatz or a polynomial in one variable (of particular degree) with coefficients depending on the remaining variables remains the only option. One may also use a variant of the Prelle-Singer/Darboux method to derive what are called quasi-rational first integrals \[21\].

### 2.2 Method of Jacobi’s last multiplier

The Jacobi’s last multiplier is a useful tool for deriving an additional first integral for a system of $n$ first-order ODEs when $n - 2$ first integrals of the system are known. It allows us to determine the Lagrangian/Hamiltonian of a planar system in many cases \[10, 16, 17, 37\]. In recent years a number of articles have dealt with this particular aspect \[11, 22, 23\].

For the case $n = 2$, Jacobi’s last multiplier lets us find a transformation $(x, y, t) \rightarrow (Q, P, t)$ such that the system

\[
\dot{x} = f(x, y, t), \quad \dot{y} = g(x, y, t).
\]

may be expressed in the form of Hamilton’s equations

\[
\dot{Q} = \partial_y H, \quad \dot{P} = -\partial_Q H.
\]

The method goes as follows. Wedge product of the Cartan one-forms $\theta^x = dx - f(x, y, t)dt$ and $\theta^y = dy - g(x, y, t)dt$ is

\[
\omega = \theta^x \wedge \theta^y.
\]

The Hamiltonian formulation in the extended phase space $\mathbb{R}^2 \times \mathbb{R}^2$ is characterized by the Poincaré-Cartan one-form $\Theta = PdQ - Hdt$ from which we may define a closed two-form $\Omega$ by minus the exterior derivative of $\Theta$, that is,

\[
\Omega = dQ \wedge dP + dH \wedge dt.
\]

It follows that, the two-form $\omega$ in Eq.\[5\] is proportional to $\Omega$ with

\[
M\omega = \Omega
\]

for some function $M$, called Jacobi Last Multiplier. Since $\Omega$ is closed, $M\omega$ must be closed. This leads to the following differential equation

\[
\partial_t M + \partial_x (Mf) + \partial_y (Mg) = 0
\]

determining $M$. After solving equation \[8\], the canonical coordinates can be obtained by substituting $M$ into Eq.\[7\]. To determine the Hamiltonian function $H$, there are two possible cases depending whether $\partial_t M$ vanishes or not.

If the multiplier $M$ does not depend on the time $t$ then the first term on the left hand side of Eq.\[8\] drops. In this case, from the equality $M\omega = \Omega$, we arrive at

\[
M(fdy - gdx) = dH,
\]

which relates $M$ with the Hamiltonian function $H$.

When the multiplier $M$ depends on time explicitly, we introduce two auxiliary functions $\phi$ and $\psi$ such that the structure of Eq.\[9\] is preserved, that is, for a pair of functions $\psi$ and $\phi$ we have

\[
M((f - \psi)dy - (g - \phi)dx) = dH + \partial dt,
\]
for a real valued function \( \vartheta \) of \((Q,P,t)\). This occurs if the condition
\[
\partial_x(M(f - \vartheta)) + \partial_y(M(g - \phi)) = 0
\]
is satisfied. By adding and subtracting two-form \( M \psi dy \wedge dt - M \phi dx \wedge dt \) into Eq.(7), we obtain
\[
M \omega = M \theta x \wedge \theta y = M dx \wedge dy + M (f dy - g dx) \wedge dt
\]
where the first two-form in the last line is the symplectic two-form
\[
M(dx - \psi dt) \wedge (dy - \phi dt) = dQ \wedge dP,
\]
and the latter can be obtained by taking the exterior product of both sides of Eq.(10) by \( dt \). Note that, substitutions of the auxiliary functions \( \psi \) and \( \phi \), and the multiplier \( M \) into Eq.(10) will enable us to find the Hamiltonian function whereas substitutions into Eq.(12) will determine the canonical coordinates.

### 2.3 Poisson structures and superintegrability

A Poisson structure on an \( n \)-dimensional space is a skew-symmetric bracket \( \{ , \} \) on the space of real-valued smooth functions satisfying the Leibnitz and the Jacobi identities \cite{19, 24, 29, 36}. We define the Poisson bracket of two functions \( F \) and \( H \) by
\[
\{ F, H \} = \nabla F \cdot N \nabla H,
\]
where \( \nabla F \) and \( \nabla H \) are gradients of the functions \( F \) and \( H \) respectively, and \( N \) is Poisson matrix. The Poisson bracket \eqref{poisson_bracket} automatically satisfies the Leibnitz identity. In a local chart \((x^a)\), the Jacobi identity takes the form
\[
N^{a[b} \partial_{x^a} N^{c]d} = 0,
\]
where \( N^{ab} \) are components of the Poisson matrix \( N \) and, \([ \ ]\) refers anti-symmetrization. The Jacobi identity \eqref{ji} is trivially satisfied in two dimensions, it gives a scalar equation in three dimensions and, it results in four equations in four dimensions.

A dynamical system is Hamiltonian if it can be written as
\[
\dot{x} = N \nabla H
\]
for \( H \) being a real valued function, called Hamiltonian function, and \( N \) being a Poisson matrix. For autonomous Hamiltonian systems, the Hamiltonian function is conserved, that is it is also a first integral of the system. It is also possible to find a Hamiltonian formulation of a nonautonomous system, but in this case the Hamiltonian is not a constant of motion, hence not a first integral of the system \cite{1}.

A Hamiltonian system in \( n \) dimensions is called maximally superintegrable if there are \( n - 1 \) first integrals. Existence of \( n - 1 \) integrals lets one to reduce the systems of differential equations to one quadrature. For the case of three dimensions, two first integrals are required for maximal superintegrability whereas for four dimensions, three first integrals are needed.

### 2.4 Superintegrability, bi-Hamiltonian systems and Lorentz example

A dynamical system is bi-Hamiltonian if it admits two different Hamiltonian structures
\[
\dot{x} = N^{(1)} \nabla H_2 = N^{(2)} \nabla H_1
\]
such that any linear pencil \( N^{(1)} + cN^{(2)} \) of Poisson matrices satisfies the Jacobi identity \cite{2, 3, 20, 29}. In this case, one can generate recursively enough constants of motion to ensure integrability \cite{8, 28}. In \cite{15}, it is shown that, Hamiltonian structures of dynamical systems in three dimensions always come in compatible pairs to form a bi-Hamiltonian structure with Poisson matrices
\[
N^{(i)ab} = -\epsilon^{ij} \epsilon^{abc} \partial_{x^c} H_j, \quad a, b, c = 1, 2, 3, \quad i, j = 1, 2,
\]
where $\epsilon^{ij}$ and $\epsilon^{abc}$ are completely antisymmetric tensors of rank two and three, respectively. In the rest of this subsection, we, particularly, focus on Lorentz system. Although it is chaotic, for some certain values of its parameters, it is superintegrable and bi-Hamiltonian [15]. In that sense, Lorentz system has a motivational importance for this paper.

The Lorenz model [25] is a three component dynamical system

$$\dot{x} = \sigma(y - x), \quad \dot{y} = \rho x - xz - y, \quad \dot{z} = -\beta z + xy \tag{16}$$

where $\sigma$ and $\rho$ are the Prandtl and Rayleigh numbers, respectively, and $\beta$ is another dimensionless number, the aspect ratio. It exhibits chaotic behavior for most values of these parameters. However, it admits bi-Hamiltonian structure in two limits which can, most conveniently, be characterized by the Rayleigh number $\rho$, namely $\rho = 0$ and $\rho \to \infty$.

The case $\rho = 0$, $\sigma = 1/2$, $\beta = 1$ is known to admit two time-dependent conserved quantities [38, 39, 40, 41]. The transformation

$$x = \frac{1}{2} \bar{t} u, \quad y = \frac{1}{4} \bar{t}^2 v, \quad z = \frac{1}{4} \bar{t}^2 w, \quad t = -\log(\bar{t}^2/4) \tag{17}$$

of dynamical variables and time brings the Lorenz system to the form

$$u' = \frac{1}{2} v, \quad v' = -uw, \quad w' = uv \tag{18}$$

with prime denoting differentiation with respect to $\bar{t}$. In terms of the new dynamical variables we find that

$$H_1 = w - u^2, \quad H_2 = v^2 + w^2, \tag{19}$$

are conserved. The Hamiltonian structure functions are given by

$$N^{(1)} = \frac{1}{4} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & -2u \\ 0 & 2u & 0 \end{pmatrix}, \quad N^{(2)} = \frac{1}{2} \begin{pmatrix} 0 & -w & v \\ w & 0 & 0 \\ -v & 0 & 0 \end{pmatrix}. \tag{20}$$

The second completely integrable case of the Lorenz system is the conservative limit [39, 42] obtained through the scaling of Eq. (16) with $t \to \epsilon t$, $x \to \frac{1}{\epsilon} x$, $y \to \frac{1}{\sigma \epsilon^2} y$, $z \to \frac{1}{\sigma \epsilon^2} z$, $\epsilon = \frac{1}{(\sigma \rho)^{1/2}} \tag{21}$

and taking the limit $\epsilon \to 0$. This brings Eq. (16) to the form

$$\dot{x} = y, \quad \dot{y} = -xz + x, \quad \dot{z} = xy \tag{22}$$

possessing two first integrals

$$H_1 = \frac{1}{2}(y^2 + z^2 - x^2), \quad H_2 = \frac{1}{2}v^2 - z. \tag{23}$$

The Poisson structures are defined by the matrices

$$N^{(1)} = \begin{pmatrix} 0 & z & -y \\ -z & 0 & -x \\ y & x & 0 \end{pmatrix}, \quad N^{(2)} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & -x \\ 0 & x & 0 \end{pmatrix} \tag{24}$$

which are compatible.

We have shown that, in two extreme cases Lorentz system is completely integrable and admits bi-Hamiltonian structure. In [15], it was shown that, these two cases are the same, hence they can be recognized as an example of a Nambu system [26] because the equations of motion take the form

$$\dot{x}^i = \epsilon^{ijk} \partial_j H_1 \partial_k H_2 \tag{25}$$

where $\epsilon^{ijk}$ is the completely skew Levi-Civita tensor.
2.5 Superintegrability and tri-Hamiltonian systems in four dimensions

In [9], a construction of multi-Hamiltonian formalism of maximal superintegrable systems is presented. For the case of four dimensions, this construction results with a tri-Hamiltonian system

\[ \dot{x} = \partial N^{(1)} \nabla H_1 = \partial N^{(2)} \nabla H_2 = \partial N^{(3)} \nabla H_3 \]  

(26)

consisting of three compatible Poisson matrices \( N_1, N_2 \) and \( N_3 \) with linearly independent Hamiltonians \( H_1, H_2 \) and \( H_3 \), respectively. Here, \( \nabla \) denotes the gradient vector in four dimension. The real valued function \( \partial \) in Eq. (26) is called the conformal factor. In general, by multiplying a Poisson matrix \( N \) with a real valued function \( \partial \), we obtain a new Poisson matrix \( \partial N \) in two and three dimensions. In dimension four, \( \partial N \) is a Poisson matrix if \( N \) is degenerate, that is the rank of \( N \) fails to be full. We shall refer this as conformal invariance.

A Poisson matrix can be written as

\[
N = \begin{pmatrix}
0 & -U^1 & -U^2 & -U^3 \\
U^1 & 0 & -V^3 & V^2 \\
U^2 & V^3 & 0 & -V^1 \\
U^3 & -V^2 & V^1 & 0
\end{pmatrix}
\]

which enables us to define three-component vector functions \( U = (U^1, U^2, U^3) \) and \( V = (V^1, V^2, V^3) \) of four variables \((u, x, y, z) = (u, \mathbf{x})\). Accordingly, we denote the gradient vector as follows

\[ \nabla = (\partial_u, \nabla) = (\partial_u, \partial_x, \partial_y, \partial_z), \]

where \( \nabla = (\partial_x, \partial_y, \partial_z) \). The Jacobi identity in Eq. (14) gives four equations that can be divided into one scalar equation and one vector equation

\[
\partial_u (U \cdot V) = V \cdot (\partial_u U - \nabla \times V),
\]

(27)

\[
\nabla (U \cdot V) = V (\nabla \cdot U) - U \times (\partial_u U - \nabla \times V).
\]

(28)

Note that, the left-hand sides of Eqs. (27) and (28) vanish for degenerate \((U \cdot V = 0)\) matrices.

Let \( H_1 \) and \( H_2 \) be two time-independent first integrals, we define three-component vector functions

\[
U = \nabla H_1 \times \nabla H_2, \quad V = \partial_u H_1 \nabla H_2 - \partial_u H_2 \nabla H_1.
\]

(29)

Note that, the vector functions \( U \) and \( V \) in Eq. (29) are orthogonal which implies \( N \) is degenerate for arbitrary \( C^1 \) functions \( H_1 \) and \( H_2 \). For Jacobi identity expressed in Eqs. (27) and (28), we compute

\[
\nabla \cdot U = \nabla \cdot (\nabla H_1 \times \nabla H_2) = \nabla \cdot (\nabla \times H_1 \nabla H_2) = 0
\]

\[
\partial_u U - \nabla \times V = \partial_u (\nabla H_1 \times \nabla H_2) - \nabla \times (\partial_u H_1 \nabla H_2 - \partial_u H_2 \nabla H_1) = 0.
\]

(30)

For a given four component dynamical vector field \( X = (X^u, X^x) \), the Hamilton’s equations (15) take the particular form

\[
\dot{u} = X^u = -(\nabla H_1 \times \nabla H_2) \cdot \nabla H,
\]

\[
\dot{x} = X^x = (\nabla H_1 \times \nabla H_2) \partial_x H + (\nabla H_2 \times \nabla H) \partial_u H_1 + (\nabla H \times \nabla H_1) \partial_u H_2.
\]

One can obtain the conservation equations

\[
X^u \partial_u H + X \cdot \nabla H = X^u \partial_u H_1 + X \cdot \nabla H_1 = X^u \partial_u H_2 + X \cdot \nabla H_2 = 0
\]

(31)

for Hamiltonian function \( H \) as well as for functions \( H_1 \) and \( H_2 \) defining the Poisson structure. The conservation laws (31) for \( H_1 \) and \( H_2 \) show that these functions are Casimirs of the Poisson matrix \( N \) they constructed, that is \( N \nabla H_i = 0 \) for \( i = 1, 2 \).
By interchanging cyclically the roles of the functions \((H_1, H_2, H = H_3)\) one obtains tri-Hamiltonian structure of the system. In a compact notation, the coefficients of Poisson matrices are in form
\[
N^{(i)ab} = -\epsilon^{ijk} \epsilon^{abcd} \partial_x H_j \partial_y H_k, \quad a, b, c, d = 0, 1, 2, 3, \quad i, j, k = 1, 2, 3, \tag{32}
\]
where \(\epsilon^{ijk}\) and \(\epsilon^{abcd}\) are completely antisymmetric tensors of rank three and four, respectively. In the vector notation, let \((U_1, V_1), (U_2, V_2)\) and \((U_3, V_3)\) be three-component vector functions of three Poisson matrices \(N^{(1)}, N^{(2)}\) and \(N^{(3)}\) given by
\[
\begin{align*}
U_1 &= \nabla H_2 \times \nabla H_3, & V_1 &= \partial_u H_2 \nabla H_3 - \partial_u H_3 \nabla H_2, \\
U_2 &= \nabla H_3 \times \nabla H_1, & V_2 &= \partial_u H_3 \nabla H_1 - \partial_u H_1 \nabla H_3, \\
U_3 &= \nabla H_1 \times \nabla H_2, & V_3 &= \partial_u H_1 \nabla H_2 - \partial_u H_2 \nabla H_1,
\end{align*}
\tag{33}
\]
respectively. It is straightforward to verify that the conditions
\[
\nabla \cdot U_i = 0, \quad \frac{\partial U_i}{\partial u} - \nabla \times V_i = 0 \tag{34}
\]
are satisfied to guarantee Jacobi identities for all Poisson matrices \(N^{(i)}, i = 1, 2, 3\). It is also straightforward to see that
\[
\Lambda_{ij} = U_i \cdot V_j + U_j \cdot V_i = 0 \tag{35}
\]
which shows that all three Hamiltonian structures are mutually compatible. Thus, Poisson structures for superintegrable systems in dimension four always form compatible pairs.

3 Examples

3.1 Shivamoggi equations

Shivamoggi equations are arising in the context of four dimensional magnetohydrodynamics and are given by
\[
\begin{align*}
\dot{u} &= -uy, \quad \dot{x} = zy, \quad \dot{y} = zx - u^2, \quad \dot{z} = xy,
\end{align*}
\tag{36}
\]
see [13, 34]. The first integrals of this system of equations are
\[
H_1 = x^2 - z^2, \quad H_2 = z^2 + u^2 - y^2, \quad H_3 = u(z + x). \tag{37}
\]
From Eq. (33), we identify the vectors \(U_i\) and \(V_i\) of Poisson matrices \(N^{(i)}\) for the Hamiltonian functions \(H_1, H_2\) and \(H_3, \ i = 1, 2, 3\), we find
\[
\begin{align*}
U_1 &= 2u (-y, z, y), & V_1 &= 2 \left(u^2, y(x + z), u^2 - z(x + z)\right), \\
U_2 &= 2 (x + z) (0, u, 0), & V_2 &= (x, 0, -z), \\
U_3 &= -4 (yz, zx, xy), & V_3 &= 4u (x, 0, -z),
\end{align*}
\tag{38}
\]
respectively. Note that, all of these three Poisson matrices are degenerate, since \(U_i \cdot V_i = 0\) holds for all \(i = 1, 2, 3\). The equations of motion can be written as
\[
X = \partial N^{(1)} \nabla H_1 = \partial N^{(2)} \nabla H_2 = \partial N^{(3)} \nabla H_3, \quad \vartheta = -\frac{1}{4(x + z)}
\]
up to multiplication with a conformal factor \(\vartheta\) for all three.

The quadratic system (36) may be related to certain four dimensional Lie algebra by defining a linear Poisson structure [24]. Using the first two first integrals, it is possible to define the Hamiltonian
\[
H = H_1 - H_2 = x^2 + y^2 - 2z^2 - u^2, \tag{39}
\]
for the system (36) satisfying \(\dot{X} = N \nabla H\). Here, \(N\) is a Poisson matrix defined by \(U = (0, y, 0)\) while \(V = (-x, 0, -2u)\). Note that, \(U \cdot V = 0\) so that the this Poisson structure is also degenerate.
3.2 Generalized Raychaudhuri equations

The generalized Raychaudhuri equations are a set of coupled first-order ODEs related to geodesic flows on surface of a deformable media [32]. In the case of a two dimensional curved surface of constant curvature they give rise to the following set of equations

\[ \dot{x} + \frac{1}{2}x^2 + \alpha x + 2(y^2 + z^2 - u^2) + 2\beta = 0, \]
\[ \dot{y} + (\alpha + x)y + \gamma = 0, \]
\[ \dot{z} + (\alpha + x)z + \delta = 0, \]
\[ \dot{u} + (\alpha + x)u = 0 \]  

upon use of the exact solutions of geodesic equations. The vector field

\[ X = -\left(\frac{1}{2}x^2 + 2(y^2 + z^2 - u^2)\right) \partial_x - xy \partial_y - xz \partial_z - xu \partial_u \]  

is associated to a particular case obtained by setting all the four parameters \( \alpha = \beta = \gamma = \delta = 0 \). This system admits the following Darboux polynomials

\[ J_1 = y, \quad J_2 = z, \quad J_3 = u, \quad J_4 = y^2 + z^2 - u^2 - \frac{1}{4}x^2 \]

together with cofactors

\[ \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = -x, \]

respectively, see [33, 35]. This is the case where the Darboux polynomials have common cofactor \(-x\), hence we have several first integrals obtained by the fractions of the Darboux polynomials. For example,

\[ H_1 = \frac{z}{u}, \quad H_2 = \frac{y}{z}, \quad H_3 = \frac{1}{u} \left( y^2 + z^2 - u^2 - \frac{1}{4}x^2 \right), \quad H_4 = \frac{1}{y} \left( y^2 + z^2 - u^2 - \frac{1}{4}x^2 \right). \]

The three-component vector functions defined in Eqs. (33) for Poisson matrices corresponding to the Hamiltonian functions \( H_1, H_2, H_3 \) are

\[ U_1 = \frac{-2}{uz^2} \left( 4 \left( y^2 + z^2 \right), xy, xz \right), \quad V_1 = \frac{4}{z^2} \left( y^2 + z^2 + u^2 - \frac{1}{4}x^2 \right) \left( 0, -z, y \right), \]
\[ U_2 = \frac{2}{u^2} \left( 4y, x, 0 \right), \quad V_2 = \frac{2}{u^3} \left( xz, -4yz, 2u^3 \left( y^2 + z^2 + u^2 - \frac{1}{4}x^2 \right) - 4z^2 \right), \]
\[ U_3 = \frac{1}{uz} \left( -1, 0, 0 \right), \quad V_3 = \frac{1}{z^2 u^2} \left( 0, -z, y \right), \]

respectively. The equations of motion can be written as

\[ X = \vartheta N^{(1)} \nabla H_1 = \vartheta N^{(2)} \nabla H_2 = \vartheta N^{(3)} \nabla H_3, \quad \vartheta = -\frac{1}{2} z u^3 \]

where the conformal factors \( \vartheta \) are the same for all three.

It may be instructive to start with two Hamiltonian functions \( H_1 \) and \( H_2 \) in Eq. (42), and obtain \( H_3 \) using the method of Jacobi’s last multiplier. On the common level sets of \( H_1 = \kappa \) and \( H_2 = \tau \), the dynamics generated by the vector field \( \nabla \) reduces to the dynamics

\[ \dot{x} = -\frac{1}{2} x^2 + \mu z^2, \quad \dot{z} = -xz, \]

where we have eliminated \( y = \tau z \) and \( u = z/\kappa \) in favour of \( x \) and \( z \), and \( \mu = 2/\kappa^2 - 2\tau^2 - 2 \) is a constant for dynamics of (43). By solving the defining PDE in Eq. (3) for the case of the reduced dynamics in Eq. (43), we find the multiplier \( M = 1/z^2 \). This enables us to determine the Hamiltonian function

\[ H(x, z) = \frac{x^2}{2z} + \mu z. \]
The canonical coordinates are $Q = x$ and $P = -1/z$, hence the symplectic two form is $\Omega = (1/z^2) \, dx \wedge dz$. Substitutions of $\mu$, $\kappa$ and $\tau$ into the Hamiltonian function $H$ in Eq. (44) result a scalar multiple of $H_3$ in Eq. (42) as expected.

Corresponding Poisson matrix for the fourth integral $H_4$ in Eq. (42) may be in the form

$$N = lN^{(1)} + mN^{(2)} + nN^{(3)}$$

for arbitrary functions $l, m$ and $n$. We find that these functions are subjected to the condition

$$-\frac{8}{z^3} (lH_2H_3 + mH_1H_3 + nH_1H_2) = 1.$$ 

### 3.3 Hyperchaotic Lü system

We consider a hyperchaotic system of four first-order ODEs obtained from the Lü system\([5]\) by adding an additional variable. The set of equations is

\[
\begin{align*}
\dot{u} &= \delta u + xz, \\
\dot{x} &= \alpha(y-x) + u, \\
\dot{y} &= \gamma y - xz, \\
\dot{z} &= -\beta z + xy,
\end{align*}
\] (45)

where $\alpha$, $\beta$, $\gamma$, and $\delta$ are real constant parameters. When the parameters $\gamma = -\beta = \delta$, we obtain two time-dependent first integrals

$$I_1 = e^{-\gamma t}(y + u), \quad I_2 = e^{-2\gamma t}(y^2 + z^2).$$ (46)

We change the variables $(u, x, y, z)$ with $(s, p, q, r)$ according to

$$s = ue^{-\gamma t}, \quad p = xe^{\alpha t}, \quad q = ye^{-\gamma t}, \quad r = ze^{-\gamma t}.$$ (47)

In this new coordinates, the Lü system becomes nonautonomous

\[
\begin{align*}
\dot{s} &= rp e^{-\alpha t}, \\
\dot{p} &= (\alpha q + s)e^{(\alpha + \gamma)t}, \\
\dot{q} &= -rp e^{-\alpha t}, \\
\dot{r} &= qpe^{-\alpha t},
\end{align*}
\] (48)

whereas the first integrals $I_1$ and $I_2$ in Eq. (47) become time-independent

$$H_1(s, p, q, r) = q + s, \quad H_2(s, p, q, r) = q^2 + r^2.$$ (49)

On the intersection of the level hypersurfaces $H_1 = \kappa$ and $H_2 = \tau$, the set of equations (48) reduces to a nonautonomous planar system

\[
\begin{align*}
\dot{p} &= (\kappa + (\alpha - 1)q)e^{(\alpha + \gamma)t}, \\
\dot{q} &= -p \left( r - q^2 \right) \frac{3}{2} e^{-\alpha t},
\end{align*}
\] (50)

where we have eliminated $s = (\kappa - q)$ and $r = \sqrt{\tau - q^2}$ in favor of $p$ and $q$. It is straightforward to verify that the Jacobi’s last multiplier for the system (50) is

$$M = (\tau - q^2)^{-\frac{3}{2}}.$$ 

Consequently, we write a Hamiltonian function

$$H(q, p) = e^{(\alpha + \gamma)t} \left[ \kappa \text{arcsin} \left( \frac{q}{\sqrt{\tau}} \right) - (\alpha - 1)\sqrt{\tau - q^2} \right] + \frac{1}{2}p^2 e^{-\alpha t}.$$ (51)

In terms of the canonical variables

$$Q = \text{arcsin} \left( \frac{q}{\sqrt{\tau}} \right), \quad P = p$$ (52)

nonautonomous system (50) takes the form

\[
\begin{align*}
\dot{Q} &= Pe^{-\alpha t}, \\
\dot{P} &= -e^{(\alpha + \gamma)t} \left( \kappa + (\alpha - 1)\sqrt{\tau} \sin Q \right)
\end{align*}
\] (53)
The hyperchaotic Qi system is given by the following set equations

\[ 3.4 \text{ Hyperchaotic Qi system} \]

of mutually compatible Poisson structures \( N \). Equation (50) becomes autonomous

\[ \frac{dH}{dt} = 0 \]

so that, we write the system (52) in form of Hamilton’s equations \( \dot{Q} = \partial H / \partial P \) and \( \dot{P} = -\partial H / \partial Q \). The system (53) is nonautonomous, hence the time-dependent Hamiltonian \( H \) in (54) is not an integral invariant of the motion because its total derivative \( dH/dt \) with respect to time \( t \) equals to \( \partial H / \partial t \).

If we substitute \( \kappa \) and \( \tau \) with their expressions in terms of the variables \( q, r \) and \( s \), then \( H \) in Eq.(51) has the form

\[ H(s, p, q, r) = e^{(\alpha + \gamma)t} \left[ (q + s) \arcsin \left( \frac{q}{\sqrt{q^2 + r^2}} \right) - (\alpha - 1) r \right] + \frac{1}{2} p^2 e^{-\alpha t}, \]

which is the Hamiltonian function of the nonautonomous system (50) with a degenerate Poisson matrix having three-component vectors \( U = (0, -r, 0) \) and \( V = (r, 0, q) \). Since the Lü system (48) is the nonautonomous the Hamiltonian \( H \) in Eq.(55) is not an integral invariant of the Lü system.

To have a time-independent first integral of the Lü system (48), we choose \( \gamma = -2\alpha \) and define a new time variable \( \hat{t} = e^{-\alpha t}/a \). This enables us to write the system in an autonomous form

\[ p' = \alpha q + s, \quad q' = -rp, \quad r' = qp, \quad s' = rp \]

where prime denotes derivative with respect to the time parameter \( \hat{t} \). In this case, the reduced dynamics in Eq.(50) becomes autonomous

\[ p' = \kappa + (\alpha - 1)q, \quad q' = -p\sqrt{\tau - q^2}. \]

Integrating this we find that

\[ H = \frac{1}{2} p^2 + \kappa \arcsin \left( \frac{q}{\sqrt{\tau}} \right) - (\alpha - 1) \sqrt{\tau - q^2} \]

is the Hamiltonian of the system (57) and it is conserved. Substituting \( H_1 = \kappa \) and \( H_2 = \tau \) from Eq.(49), we arrive at the third autonomous conserved quantity

\[ H_3(s, p, q, r) = \frac{1}{2} p^2 + (q + s) \arcsin \left( \frac{q}{\sqrt{q^2 + r^2}} \right) - (\alpha - 1) r, \]

of the Lü system.

Note that, we started with Lü system in Eqs.(45) and showed that when the parameters are satisfying the relations \( \gamma = -\beta = \delta = -2\alpha \), the system has three time-independent first integrals, namely \( H_1, H_2 \) in Eq.(49) and \( H_3 \) in Eq.(58). Now, we follow the definitions in Eqs.(33) in order to obtain the three-component vector functions

\[ U_1 = (2\alpha q + s + r \arcsin(\frac{q}{\sqrt{q^2 + r^2}}), -rp, qp), \quad V_1 = -2 \arcsin(\frac{q}{\sqrt{q^2 + r^2}})(0, q, r), \]

\[ U_2 = (\alpha - 1 + (q + s) \frac{q}{q^2 + r^2}, 0, p), \quad V_2 = (\alpha - 1 + (q + s) \frac{r}{q^2 + r^2}, \alpha - 1 + (q + s) \frac{q}{q^2 + r^2}), \]

\[ U_3 = 2r(1, 0, 0), \quad V_3 = 2(0, q, r), \]

of mutually compatible Poisson structures \( N^{(1)}, N^{(2)} \) and \( N^{(3)} \), respectively. The common conformal factor is \(-1/2\).

### 3.4 Hyperchaotic Qi system

The hyperchaotic Qi system is given by the following set equations

\[ \dot{u} = -\delta u + \lambda z + xy, \]
\[ \dot{x} = \alpha(y - x) + yz \]
\[ \dot{y} = \beta(x + y) - xz \]
\[ \dot{z} = -\gamma z - eu + xy \]

(59)
where $\alpha, \beta, \gamma, \epsilon, \delta$ and $\lambda$ are real constants called the parameters. When the parameters satisfy the constraints

$$\alpha + \beta = 0, \quad \gamma + \epsilon + \lambda = \delta,$$

(60)

Qi system admits the following time-dependent first integrals

$$I_1 = (z - u)e^{(\gamma + \lambda)t}, \quad I_2 = (x^2 + y^2)e^{2\alpha t}.\tag{61}$$

We introduce a transformation $(u, x, y, z) \to (s, p, q, r)$ given by

$$s = ue^{(\gamma + \lambda)t}, \quad q = ye^{\alpha t}, \quad p = xe^{\alpha t}, \quad r = ze^{(\gamma + \lambda)t}.\tag{62}$$

In the new coordinates, the system assumes the form

$$\dot{s} = \lambda r - es + pqe^{(\gamma + \lambda - 2\alpha)t}, \quad \dot{q} = p(\beta - re^{-(\gamma + \lambda)t})$$
$$\dot{r} = q(re^{-(\gamma + \lambda)t} - \beta)$$
$$\dot{\phi} = \lambda r - es + pqe^{(\gamma + \lambda - 2\alpha)t}$$

while the first integrals $I_1$ and $I_2$ in Eq.(61) becomes autonomous

$$H_1 = r - s, \quad H_2 = p^2 + q^2.\tag{64}$$

Note that, time derivatives $\dot{r}$ and $\dot{s}$ are the same, this is a manifestation of functional form of the first integral $H_1 = r - s$. On the intersection of the level surfaces of $H_1 = \kappa$ and $H_2 = \tau$ the non-autonomous system reduces to the planar system

$$\dot{r} = \epsilon\kappa + (\lambda - \epsilon) r + q\sqrt{\tau - q^2}e^{(\gamma + \lambda - 2\alpha)t}, \quad \dot{q} = \sqrt{\tau - q^2}(\beta - re^{-(\gamma + \lambda)t})$$

(65)

upon elimination of the variables $p$ and $s$ in favor of $q$ and $r$. The latter system of planar ODEs admits a Jacobi last multiplier given by

$$M = \frac{e^{(\epsilon - \lambda)t}}{\sqrt{\tau - q^2}}.\tag{66}$$

The multiplier $M$ is time-dependent, hence we introduce two auxiliary functions

$$\psi(q, r, t) = (\lambda - \epsilon)r, \quad \phi(q, r, t) = \beta\sqrt{\tau - q^2},\tag{67}$$

satisfying the identity in Eq.(11). To find Hamiltonian of the reduced system in Eq.(65), we recall the defining equation (10), hence compute the Hamiltonian function as

$$H(q, r) = e^{(\epsilon - \lambda)t}\left(e^{-(\gamma + \lambda)t}\frac{r^2}{2} + \epsilon\kappa\arcsin\left(\frac{q}{\sqrt{\tau}}\right) + \frac{q^2}{2}e^{(\lambda + \gamma - 2\alpha)t}\right).\tag{68}$$

Note that, the system (65) is nonautonomous, hence $H$ is not a constant of motion. Using Eq.(12), we arrive at the canonical coordinates

$$Q = e^{(\epsilon - \lambda)t}r, \quad P = \arcsin\left(\frac{q}{\sqrt{\tau}}\right) - \beta t.$$

In the canonical coordinates, the planar system (65) takes the form

$$\dot{Q} = e^{(\epsilon - \lambda)t}\epsilon\kappa + \frac{r}{2}\sin(2(P + \beta t))e^{(\epsilon + \gamma - 2\alpha)t}, \quad \dot{P} = -Qe^{-(\epsilon + \gamma)t},\tag{69}$$

whereas the Hamiltonian for the planar system (68) assumes the form

$$H(Q, P) = e^{-(\gamma + \alpha)t}\frac{Q^2}{2} + \epsilon\kappa e^{(\epsilon - \lambda)t}(P + \beta t) + \frac{\tau e^{(\epsilon + \gamma - 2\alpha)t}}{2}\sin^2(P + \beta t).\tag{70}$$
So that, the canonical system (69) can be recasted as \( \dot{Q} = \partial H/\partial P \) and \( \dot{P} = -\partial H/\partial Q \). Note that, the Hamiltonian function in Eq. (70) is not conserved, that is \( dH/dt = \partial H/\partial t \), because the system (69) is nonautonomous. If we substitute \( \kappa \) and \( \tau \) with their expressions in terms of the variables \( q, r \) and \( s \), then the Hamiltonian in (68) becomes time-dependent function

\[
H(q, p, r, s) = e^{(\epsilon - \lambda)t} \left( e^{-\left(\delta + \lambda\right)t/2} + \epsilon \left( r - s \right) \arcsin \left( \frac{q}{\sqrt{q^2 + p^2}} \right) + \frac{q^2}{2} e^{(\lambda + \gamma - 2\alpha)t} \right). \tag{71}
\]

In order to have a time-independent integral of the Qi system we take \( \lambda = -\gamma \) and \( \alpha = 0 \) and change time variable \( t \) with \( \bar{t} = \frac{1}{\epsilon - \lambda} e^{\epsilon - \lambda t} \). In these circumstances, the reduced planar ODE system in Eq. (65) takes the form

\[
r' = \frac{1}{\epsilon - \lambda} \left( \frac{\epsilon \kappa}{\epsilon - \lambda} + q\sqrt{\tau - q^2} - r \right), \quad q' = - \frac{r}{\sqrt{\tau - q^2}}, \tag{72}
\]

where prime denotes derivative with respect to the time parameter \( \bar{t} \). In this case, the Jacobi’s last multiplier becomes \( M = \bar{t}/\sqrt{\tau - q^2} \) and the auxiliary functions turn out to be \( \psi = -r/\bar{t} \) and \( \phi = 0 \). Using Eq. (10), we compute

\[
H(r, q) = \frac{1}{\epsilon - \lambda} \left( \epsilon \kappa \arcsin \left( \frac{q}{\sqrt{\tau}} \right) + \frac{q^2}{2} + \frac{r^2}{2} \right),
\]

which does not depend on time explicitly and hence after the substitution of the first two integrals \( \kappa = r - s \) and \( \tau = q^2 + p^2 \) we arrive at the new time-independent integral

\[
H_3(q, p, r, s) = \frac{1}{\epsilon - \lambda} \left( \epsilon (r - s) \arcsin \left( \frac{q}{\sqrt{q^2 + p^2}} \right) + \frac{q^2}{2} + \frac{r^2}{2} \right). \tag{73}
\]

of the Qi system.

We thus have obtained three time-independent integrals of the Qi system, namely \( H_1, H_2 \) in Eq. (64) and \( H_3 \) in Eq. (73), with the parameters satisfying the conditions \( \lambda = -\gamma, \delta = \epsilon \) and \( \alpha, \beta = 0 \). Using the definitions in Eq. (63), the tri-Hamiltonian structure of the Qi system is obtained as follows. For the Hamiltonian function \( H_1 = r - s \), the three-component vectors of the corresponding Poisson structure \( N^{(1)} \) are given by

\[
U_1 = \frac{2}{\epsilon - \lambda} (ep \arcsin(\frac{q}{\sqrt{q^2 + p^2}}) + pr), -e\arcsin(\frac{q}{\sqrt{q^2 + p^2}}) - qr, -\epsilon (r - s) - pq)
\]

\[
V_1 = \frac{2e}{\epsilon - \lambda} \arcsin(\frac{q}{\sqrt{q^2 + p^2}})(q, p, 0).
\]

For the Hamiltonian function \( H_2 = q^2 + p^2 \), the three-component vectors of \( N^{(2)} \) are

\[
U_2 = \frac{1}{\epsilon - \lambda} (-\epsilon (r - s) \frac{q}{p^2 + q^2}, -\epsilon (r - s) \frac{p}{q^2 + p^2} - q, 0)
\]

\[
V_2 = \frac{1}{\epsilon - \lambda} (\epsilon (r - s) \frac{p}{q^2 + p^2} + q, -\epsilon (r - s) \frac{q}{p^2 + q^2}, r).
\]

Finally, for the Hamiltonian \( H_3 \) in Eq. (73), we have

\[
U_3 = (-2p, 2q, 0), \quad V_3 = (-2q, -2p, 0),
\]

which may be related to four dimensional Lie algebras [24].

4 Discussion and outlook

In this paper, after presenting some technical details about superintegrability and Poisson structures in four dimensions, we have studied superintegrable and tri-Hamiltonian structures of a set of four dimensional dynamical systems, namely Shivamoggi and generalized Raychaudhuri equations, hyperchaotic Lü and Qi systems. Generically hyperchaotic attractors are not expected to be integrable, but in this paper, we have showed that, for some particular values of their parameters, hyperchaotic Lü and Qi systems exhibit not only integrable but superintegrable properties admitting tri-Hamiltonian formulations.
Acknowledgement

We gratefully acknowledge support from Professor G. Rangarajan and National Mathematics Initiative programme at IISC Mathematics Department where the work was started. HG thanks Yeditepe University for travel support.

References

[1] Abraham, R., Marsden, J. E., & Marsden, J. E. (1978). Foundations of mechanics. Reading, Massachusetts: Benjamin/Cummings Publishing Company.
[2] Abadoğlu, E., Gümrü, H. (2009). Bi-Hamiltonian structure in Frenet–Serret frame. Physica D: Nonlinear Phenomena, 238(5), 526-530.
[3] Blaszak, M., Wojciechowski, S. (1989). Bi-Hamiltonian dynamical systems related to low-dimensional Lie algebras. Physica A: Statistical Mechanics and its Applications, 155(3), 545-564.
[4] del Castillo, G. T. (2009). The Hamiltonian description of a second-order ODE. Journal of Physics A: Mathematical and Theoretical, 42(26), 265202.
[5] Chen, A., Lu, J., Lü, J., Yu, S. (2006). Generating hyperchaotic Lü attractor via state feedback control. Physica A: Statistical Mechanics and its Applications, 364, 103-110.
[6] Darboux G. (1878). Mémoire sur les équations différentielles algébriques du premier ordre et du premier degré., Bull. Sci. Math. (2) 2 (1878), 60-96, 123-144, 151-200.
[7] Dumortier, F., Llibre, J., & Artés, J. C. (2006). Qualitative theory of planar differential systems. Berlin: Springer.
[8] Fernandes, R. L. (1994). Completely integrable bi-Hamiltonian systems. Journal of Dynamics and Differential Equations, 6(1), 53-69.
[9] Gonera, C., Nutku, Y. (2001). Super-integrable Calogero-type systems admit maximal number of Poisson structures. Physics Letters A, 285(5), 301-306.
[10] Goriely, A. (2001). Integrability and nonintegrability of dynamical systems (Vol. 19). World Scientific.
[11] Choudhury, A. G., Guha, P., Khanra, B. (2009). Determination of elementary first integrals of a generalized Raychaudhuri equation by the Darboux integrability method. Journal of Mathematical Physics, 50(10), 102502.
[12] Guha, P., Choudhury, A. G. (2010). On Planar and Non-planar Isochronous Systems and Poisson Structures. International Journal of Geometric Methods in Modern Physics, 7(07), 1115-1131.
[13] Guha, P., Choudhury, A. G., First integrals and Hamiltonian structure for a system of ordinary differential equations occurring in magnetohydrodynamics, AIP Conference Proceedings 1582 (2014), 116-123.
[14] Gümrü H. Existence of Hamiltonian structure in 3D, Adv. Dyn. Syst. Appl. 5 (2010), no. 2, 159-171.
[15] Gümrü H., Y. Nutku, "Poisson structure of dynamical systems with three degrees of freedom", J. Math. Phys. 34 (1993) 5691-5723.
[16] Jacobi C.G.J. Sul principio dell’ultimo moltiplicatore, e suo uso come nuovo principio generale di meccanica, Giornale Arcadico di Scienze, Lettere ed Arti 99 (1844), 129-146.
[17] Jacobi C.G.J. Theoria novi multiplicatoris systemati aequationum differentialium vulgarium applicandi, J. Reine Angew. Math 27 (1844), 199-268, Ibid 29(1845), 213-279 and 333-376. Astrophys. Journal, 342, 635-638, (1989).
[18] Juanolou J.P. Équations de Pfaff algébriques. Lecture Notes in Mathematics, 708, Springer, Berlin, 1979. v+255 pp.
[19] Lieberman P, Marle C.-M. Symplectic Geometry and Analytical Mechanics, D. Reidel Publishing Company, 1987.
[20] Magri, F., Morosi, C. (1984). A geometrical characterization of integrable Hamiltonian systems through the theory of Poisson-Nijenhuis manifolds (Quaderno 19-1984, Univ. of Milan).
[21] Man, Y. K. (1994). First integrals of autonomous systems of differential equations and the Prelle-Singer procedure. Journal of Physics A: Mathematical and General, 27(10), L329.
[22] Nucci, M. C. (2005). Jacobi last multiplier and Lie symmetries: a novel application of an old relationship. Journal of Nonlinear Mathematical Physics, 12(2), 284-304.
[23] Nucci M.C., Leach P.G.L., Nucci, M. C., Leach, P. G. L. (2008). Jacobi’s last multiplier and Lagrangians for multidimensional systems. Journal of Mathematical Physics, 49(7), 073517.
[24] Laurent-Gengoux, C., Pichereau, A., & Vanhaecke, P. (2012). Poisson structures (Vol. 347). Springer Science & Business Media.
[25] Lorenz, E. N. (1963). Deterministic nonperiodic flow. Journal of the atmospheric sciences, 20(2), 130-141.
[26] Nambu, Y. (1973). Generalized hamiltonian dynamics. Physical Review D, 7(8), 2405.
[27] Nutku, Y. (1990). Hamiltonian structure of the Lotka-Volterra equations. Physics Letters A, 145(1), 27-28.
[28] Olver, P. J. (1990). Canonical forms and integrability of bi-Hamiltonian systems. Physics Letters A, 148(3), 177-187.
[29] Olver, P. J. (2000). Applications of Lie groups to differential equations (Vol. 107). Springer Science and Business Media.
[30] Qi, G., van Wyk, M. A., van Wyk, B. J., Chen, G. (2009). A new hyperchaotic system and its circuit implementation. Chaos, Solitons & Fractals, 40(5), 2544-2549.
[31] Prelle, M. J., Singer, M. F. (1983). Elementary first integrals of differential equations. Transactions of the American Mathematical Society, 279(1), 215-229.
[32] Raychaudhuri, A. (1955). Relativistic cosmology. I. Physical Review, 98(4), 1123.
[33] Singer, M. F. (1992). Liouvillian first integrals of differential equations. Transactions of the American Mathematical Society, 333(2), 673-688.
[34] Shivamoggi, B. K. (1999). Current-sheet formation near a hyperbolic magnetic neutral line in the presence of a plasma flow with a uniform shear-strain rate: An exact solution. Physics Letters A, 258(2), 131-134.
[35] Valls, C. (2011). Darbouxian integrals for generalized Raychaudhuri equations. Journal of Mathematical Physics, 52(3), 2703.
[36] Weinstein, A. (1983). The local structure of Poisson manifolds. Journal of differential geometry, 18(3), 523-557.
[37] Whittaker, E. T. (1970). A treatise on the analytical dynamics of particles and rigid bodies: with an introduction to the problem of three bodies. CUP Archive.
[38] Segur H. (1980). Lectures at the International School "Enrico Fermi", Varenna, Italy, unpublished.
[39] Tabor, M., Weiss, J. (1981). Analytic structure of the Lorenz system. Physical Review A, 24(4), 2157.
[40] Kus, M. (1983). Integrals of motion for the Lorenz system. Journal of Physics A: Mathematical and General, 16(18), L689.

[41] Sen, T., Tabor, M. (1990). Lie symmetries of the Lorenz model. Physica D: Nonlinear Phenomena, 44(3), 313-339.

[42] Steeb, W. H., Euler, N. (1988). Nonlinear evolution equations and Painlevé test (pp. 114-114). Singapore: World Scientific.