Geometric multigrid for an implicit-time immersed boundary method

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Abstract The immersed boundary (IB) method is an approach to fluid-structure interaction that uses Lagrangian variables to describe the deformations and resulting forces of the structure and Eulerian variables to describe the motion and forces of the fluid. Explicit time stepping schemes for the IB method require solvers only for Eulerian equations, for which fast Cartesian grid solution methods are available. Such methods are relatively straightforward to develop and are widely used in practice but often require very small time steps to maintain stability. Implicit-time IB methods permit the stable use of large time steps, but efficient implementations of such methods require significantly more complex solvers that effectively treat both Lagrangian and Eulerian variables simultaneously. Several different approaches to solving the coupled Lagrangian-Eulerian equations have been proposed, but a complete understanding of this problem is still emerging. This paper presents a geometric multigrid method for an implicit-time discretization of the IB equations. This multigrid scheme uses a generalization of box relaxation that is shown to handle problems in which the physical stiffness of the structure is very large. Numerical examples are provided to illustrate the effectiveness and efficiency of the algorithms described herein. These tests show that using multigrid as a preconditioner for a Krylov
method yields improvements in both robustness and efficiency as compared to using multigrid as a solver. They also demonstrate that with a time step 100–1000 times larger than that permitted by an explicit IB method, the multigrid-preconditioned implicit IB method is approximately 50–200 times more efficient than the explicit method.

**Keywords** Fluid-structure interaction · Immersed boundary method · Krylov methods · Multigrid solvers · Multigrid preconditioners

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1 Introduction

The immersed boundary (IB) method [29] was introduced by Peskin [28] to solve problems of fluid-structure interaction in which an elastic structure is immersed in a viscous incompressible fluid. This method was developed to simulate the dynamics of heart valves, but it has subsequently been applied to diverse problems in biofluid dynamics, and it is finding increasing use in other engineering problems [21]. The IB formulation of such problems uses an Eulerian description of the momentum, viscosity, and incompressibility of the fluid-structure system, and it uses a Lagrangian description of the deformation of the immersed structure and forces generated by these deformations. The Eulerian equations are approximated on a Cartesian grid, and the Lagrangian equations are approximated on a curvilinear mesh. Interaction between Eulerian and Lagrangian variables is through integral equations with delta function kernels. When discretized, the IB method uses a regularized version of the delta function to mediate Lagrangian-Eulerian coupling. A key feature of the method is that it does not require conforming discretizations of the fluid and structure; instead, the curvilinear mesh is free to cut through the background Cartesian grid in an arbitrary manner. Consequently, IB simulations do not require dynamic grid generation, even for problems involving very large structural deformations.

Typical implementations of the IB method adopt a fractional step approach to time stepping. In the simplest version of such a scheme, the Eulerian velocity and pressure fields are updated for a fixed configuration of the immersed structure, and then the position of the Lagrangian structure is updated from the newly computed velocity field. This approach effectively decouples the Eulerian and Lagrangian equations, and solvers are needed only for the Eulerian equations (i.e., the incompressible Stokes or Navier-Stokes equations), for which fast Cartesian grid solution methods are available. However, because this fractional step approach yields an explicit time stepping method for the structural dynamics problem, maintaining stability requires time steps that are small enough to resolve all of the elastic modes of the discrete equations. In many applications, these elastic time scales are well below the physical time scales of interest. Even for relatively simple elasticity models, the largest stable time step size scales like $\Delta t = O(\Delta s^2)$, in which $\Delta s$ is the Lagrangian mesh spacing. For problems involving bending-resistant elastic elements, the largest stable time step