On the number of edges of separated multigraphs

Andrew Suk (UC San Diego)

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Multigraph drawings

- Loops
- Multiple edges
Multigraph drawings

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Theorem (Euler)

Every $n$-vertex planar multigraph with edge multiplicity at most $m$ has at most $(3n - 6)m$ edges.
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Theorem

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Theorem

Let $G$ be an $n$-vertex multigraph with $e$ edges and edge multiplicity at most $m$. Then

$$\text{cr}(G) \geq \Omega \left( \frac{e^3}{m \cdot n^2} \right) - O(m^2 n).$$
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Question (Kaufmann) Can we improve this crossing lemma for multigraphs with no empty lenses?
Crossing lemma for multigraphs

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$$\text{cr}(G) \geq \Omega \left( \frac{e^3}{m \cdot n^2} \right) - O(m^2 n).$$

**Question** (Kaufmann) How many edges can there be in a multigraph with no empty lenses?
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Multigraph drawings

- No loops
- Multiple edges
- No two parallel edges cross
Multigraph drawings with no empty lenses

- No loops
- Multiple edges
- No two parallel edges cross
- No empty lenses
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\[
\binom{n}{2} \cdot (n - 1) = O(n^3).
\]
Multigraph drawings with no empty lenses

- No loops
- Multiple edges
- No two parallel edges cross
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Multigraph drawings with no empty lenses

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{i} {j}
Multigraph drawings with no empty lenses

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On the number of edges of separated multigraphs
Multigraph drawings with no empty lenses

- No loops
- Multiple edges
- No two parallel edges cross
- no empty lenses

Any two edges cross at most twice. $\Omega(n^3)$. 
Multigraph drawings with no empty lenses

- No loops
- Multiple edges
- No two parallel edges cross
- No empty lenses
- Two edges cross at most once.
Multigraph drawings with no empty lenses

- No loops
- Multiple edges
- No two parallel edges cross
- No empty lenses
- Two edges cross at most once. Trivial: $O(n^3)$
Multigraph drawings with no empty lenses

- No loops
- Multiple edges
- No two parallel edges cross
- no empty lenses
- Two edges cross at most once.
- Dependent edges are non-crossing

Theorem (Pach-Tóth, 2018)

The maximum number of edges in an n-vertex multigraph that can be drawn in the plane with the rules above is $O(n^2)$
Multigraph drawings with no empty lenses

- No loops
- Multiple edges
- No two parallel edges cross
- no empty lenses
- Two edges cross at most once. Trivial: $O(n^3)$
- Dependent edges are non-crossing
Multigraph drawings with no empty lenses

- No loops
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**Theorem (Pach-Tóth, 2018)**

*The maximum number of edges in an n-vertex multigraph that can be drawn in the plane with the rules above is $O(n^2)$.*
Multigraph drawings with no empty lenses

- No loops
- Multiple edges
- No two parallel edges cross
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- Two edges cross at most once.
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Theorem (Pach-Tóth, 2018)

The maximum number of edges in an n-vertex multigraph that can be drawn in the plane with the rules above is $O(n^2)$

Proof. Probabilistic Method + Thrackles
Concluding remarks

- Multiple edges
- No two parallel edges cross
- No empty lenses
- Two edges cross at most once (including dependent edges).

**Theorem (Fox-Pach-Suk)**

Every $n$-vertex multigraph that can be drawn in the plane with the properties described above has at most $O(n^2 \log n)$ edges.

**Corollary (Fox-Pach-Suk)**

Let $G$ be an $n$-vertex multigraph with $e$ edges that can be drawn in the plane with the properties described above. Then

$$cr(G) \geq \Omega \left( \frac{e^3}{n^2 \log n} \right) - O(n)$$
Concluding remarks

- Multiple edges
- No two parallel edges cross
- no empty lenses
- Two edges cross at most once (including dependent edges).

Theorem (Fox-Pach-Suk)

Every $n$-vertex multigraph that can be drawn in the plane with the properties described above has at most $O(n^2 \log n)$ edges.

Open problem: Is the log factor necessary?
Thank you!