Exact Ground States of Frustrated Quantum Spin Systems Consisting of Spin-Dimer Units

Toshiya Hikihara,1 Takashi Tonomura,2,3 Kiyomi Okamoto,4 and Tōru Sakai5,6

1Faculty of Science and Technology, Gunma University, Kiryu, Gunma 376-8515, Japan
2Professor Emeritus, Kobe University, Kobe 657-8501, Japan
3Department of Physical Science, Osaka Prefecture University, Sakai, Osaka 599-8531, Japan
4College of Engineering, Shibaura Institute of Technology, Saitama 337-8570, Japan
5Graduate School of Material Science, University of Hyogo, Hyogo 678-1297, Japan
6National Institutes for Quantum and Radiological Science and Technology (QST), SPring-8, Hyogo 679-5148, Japan

We study frustrated quantum spin systems consisting of dimers of spin-1/2 spins. We derive several models that have the exact ground state of the form of the direct product of dimer states. The ground states realized include the product state of dimer singlets, that of dimer triplets with zero magnetization, those of dimer-spin-nematic states, and those of a mixture of the dimer states. Pseudo spin-1/2 operators emerging in each dimer are also introduced.

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I. INTRODUCTION

Frustrated magnetism has been one of the central issues in condensed-matter physics for several decades. In frustrated magnets, a competition among interactions leads to a massive degeneracy in the ground-state manifold and provides a good opportunity for perturbations such as the quantum fluctuation to realize an unconventional ground state. Studies searching for such an unconventional ground state in frustrated quantum magnets have been performed intensively and succeeded in identifying exotic ground states, e.g., the quantum spin-liquid in a kagome antiferromagnet,2,3 the vector-chirality state in the quantum magnets in a zigzag ladder,4–12, and the spin-multipolar state in low-dimensional frustrated ferromagnets.13–16

Despite the efforts made for many years, studying frustrated quantum magnets is still a challenging task. This is mainly because many powerful theoretical tools for investigating quantum spin systems are not applicable to the problem. For instance, the mean-field approximation is not justified in investigating unconventional states without a classical long-range order. The quantum Monte-Carlo method breaks down when applied to frustrated systems because of the notorious negative-sign problem. Therefore, accurate results, especially exact ones, for the frustrated quantum magnets are highly desirable.

A famous example of frustrated quantum spin models with the exact ground state is the Majumdar–Ghosh model.17,18 The model has the form of a zigzag spin ladder and is constructed as a sum of projection operators. It was then shown that the model has the product states of singlet pairs of nearest-neighboring spins as the ground states with a finite excitation gap. Exact ground states with the product of the form of the direct product of dimer states. The ground states realized include the product state of dimer singlets, that of dimer triplets with zero magnetization, those of dimer-spin-nematic states, and those of a mixture of the dimer states. Pseudo spin-1/2 operators emerging in each dimer are also introduced.

The interdimer Hamiltonian consists of the XXZ exchange couplings,

\[ \mathcal{H}_{\text{inter}} = \sum_{\langle j,j' \rangle} h_{\text{inter}}(j,j') = \sum_{\langle j,j' \rangle} \left[ J_{11}(S_{1,j}, S_{1,j'})_{\alpha} + J_{22}(S_{2,j}, S_{2,j'})_{\alpha} \right. 
+ \left. J_{12}(S_{1,j}, S_{2,j'})_{\alpha} + J_{21}(S_{2,j}, S_{1,j'})_{\alpha} \right], \tag{2} \]

where \( S_{n,j} = (S_{n,j}^{x}, S_{n,j}^{y}, S_{n,j}^{z}) \) (\( n = 1, 2 \)) are spin-1/2 operators in the \( j \)th dimer and \( (S_{n,j}, S_{n',j'})_{\alpha} \) represents the XXZ anisotropic exchange coupling, i.e.,

\[ (S_{n,j}, S_{n',j'})_{\alpha} = S_{n,j}^{x} S_{n',j'}^{x} + S_{n,j}^{y} S_{n',j'}^{y} + \Delta S_{n,j}^{z} S_{n',j'}^{z}. \tag{3} \]

A schematic picture of the interdimer Hamiltonian is shown in Fig. 1(a). We note that the exchange constants \( J_{11}, J_{22}, J_{12}, \) and \( J_{21} \) are different in general, while we consider the case where the anisotropy parameter \( \Delta \) is the same for all the interdimer couplings. The sum \( \sum_{\langle j,j' \rangle} \) in Eq. (2) is taken for all the bonds connected by the interdimer exchange couplings. [See Figs. 1(b) and 1(c) for example.] The structure of the lattice composed of the interdimer exchanges is basically arbitrary.

![Image](https://example.com/image.png)

FIG. 1: (Color online) (a) Schematic picture of interdimer exchange couplings. The rectangles and circles represent dimer unit states and spin-1/2 spins, respectively. The solid lines represent interdimer exchange couplings. (b) System in one-dimensional lattice. (c) System in two-dimensional square lattice.
a major part of our results is valid for any lattice in any dimension, while for some results, we require that the lattice is bipartite.

For the intradimer Hamiltonian, we consider the XXZ and further anisotropic exchange coupling30, the Dzyaloshinskii–Moriya (DM) coupling with the DM vector in the z-direction, and the Zeeman terms of uniform and staggered fields. The intradimer Hamiltonian is then given by

\[ H_{\text{intra}} = \sum_j h_{\text{intra}}(j) = \sum_j \left[ h_{\text{XXZ}}(j) + h_{\text{uni}}(j) + h_{\text{DM}}(j) \right] + h_{\text{stg}}(j), \]

where \( h_{\text{XXZ}}(j) \) is the Hamiltonian term, \( h_{\text{uni}}(j) \) is the uniform field term, and \( h_{\text{stg}}(j) \) is the staggered field term.

The product state is the ground state of the intradimer Hamiltonian, which is the state in which the sum of the individual spin states is zero. The total spin of the system is thus zero.

\[ \sum_j S_j^z = 0 \]

where \( S_j^z \) is the z-component of the spin operator.

The spin-nematic states are eigenstates of the intradimer Hamiltonian, and they are defined as

\[ | n \rangle = \frac{1}{\sqrt{2}} (| ↑↑ \rangle + | ↓↓ \rangle) \]

and the Zeeman terms of uniform and staggered fields. The interdimer Hamiltonian is then given by

\[ H_{\text{inter}} = \sum_j h_{\text{inter}}(j) = \sum_j \left[ h_{\text{XXZ}}(j) + h_{\text{uni}}(j) + h_{\text{DM}}(j) \right] + h_{\text{stg}}(j), \]

where \( h_{\text{XXZ}}(j) \) is the Hamiltonian term, \( h_{\text{uni}}(j) \) is the uniform field term, and \( h_{\text{stg}}(j) \) is the staggered field term.

The sum of the individual spin states is zero.

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The spin-nematic states are eigenstates of the interdimer Hamiltonian, and they are defined as

\[ | n \rangle = \frac{1}{\sqrt{2}} (| ↑↑ \rangle + | ↓↓ \rangle) \]

II. Pseudospin operators and product states

In this section, we introduce two pseudo spin-1/2 operators defined in each dimer unit, which are used in the following discussion. The wave functions that have the form of the direct product of dimer states and will be considered as candidates of the exact ground state are also defined.

Let us focus on the spins \( S_{1,j} \) and \( S_{2,j} \) in a dimer unit. We construct two operators \( T_{1,j} \) and \( T_{2,j} \) defined by

\[ T_{1,j} = \frac{1}{2} (S_{1,j} - S_{2,j}) \quad T_{2,j} = S_{1,j} S_{2,j} \]

One can easily find that these operators obey the following commutation relations,

\[ [T_{1,j}, T_{1,j}'] = i \epsilon_{\alpha \beta \gamma} T_{1,j}^{\alpha} T_{1,j'}^{\beta} \]

where \( \alpha, \beta, \gamma \) are x, y, z, and \( T_{1,j} = (T_{1,j}^x + i T_{1,j}^y)/2, T_{1,j} = (T_{1,j}^x - i T_{1,j}^y)/(2i) \), and \( \epsilon_{\alpha \beta \gamma} \) is the Levi-Civita symbol. The operators \( T_{1,j} \) and \( T_{2,j} \) thus satisfy the usual commutation relations of spin operators and commute with each other.

Furthermore, actions of the operator \( T_{1,j} \) on the dimer states are found as

\[ T_{1,j} | \uparrow \uparrow \rangle = | \uparrow \downarrow \rangle \]

Therefore, \( T_{1,j} \) behaves as a pseudo spin-1/2 operator in the subspace of \( \{ \uparrow \downarrow \} \), and \( T_{2,j} \) is the operator on the dimer states of \( \{ \uparrow \downarrow \} \). The state is \( | \uparrow \downarrow \rangle \) and \( | \downarrow \uparrow \rangle \) correspond to the states \( | T_{1,j} \rangle = | \uparrow \downarrow \rangle \) and \( | T_{2,j} \rangle = | \uparrow \downarrow \rangle \), respectively.

Next, we consider a unitary transformation for dimer states,
In this section, we show our main result that model (1) in some parameter regions has an exact ground state of the form of the direct product of dimer states. The strategy used to prove the result is as follows.

(i) We first focus on the interdimer Hamiltonian (2) in a certain parameter region and show that the Hamiltonian has the product states of the dimer states Eqs. (19) and (20) with some constraints on the phases \( \{ \theta_j, \chi_j, \varphi_j, \zeta_j \} \) as eigenstates with zero eigenvalue.

(ii) We show that the product states with additional constraints on the phases are the eigenstates of the interdimer Hamiltonian (3) considered. At this stage, the product states obtained turn out to be eigenstates of the whole Hamiltonian (1).

(iii) Finally, we specify the parameter region of the interdimer Hamiltonian, which lowers the eigenenergy of one of the eigenstates obtained in (ii) and make it be the ground state of the whole Hamiltonian.

A. Interdimer Hamiltonian

Let us consider the interdimer exchange Hamiltonian \( h_{\text{inter}}(j, j') \) [Eq. (23)] between the dimers \( (j, j') \). Here, we rewrite the Hamiltonian as

\[
 h_{\text{inter}}(j, j') = \sum_{\epsilon, \epsilon' = \pm} J_{\epsilon \epsilon'} \left[ h_{\epsilon \epsilon'}(j, j') + \Delta_{\epsilon \epsilon'}^{\text{sing}}(j, j') \right],
\]

with

\[
 h_{\epsilon \epsilon'}(j, j') = \left( S_{1,j}^z + \epsilon S_{2,j}^z \right) \left( S_{1,j'}^z + \epsilon S_{2,j'}^z \right)
 + \left( S_{1,j}^{\pm} + \epsilon S_{2,j}^{\pm} \right) \left( S_{1,j'}^{\pm} + \epsilon S_{2,j'}^{\pm} \right),
\]
The original coupling constants \( \{J_{11}, J_{12}, J_{21}, J_{22}\} \) are related to \( J_{\text{en}} \) as

\[
\begin{align*}
J_{11} &= J_{++} + J_{+} + J_{--} + J_{--}, \\
J_{12} &= J_{++} - J_{+} + J_{--} - J_{--}, \\
J_{21} &= J_{++} + J_{+} - J_{--} - J_{--}, \\
J_{22} &= J_{++} - J_{+} - J_{--} + J_{--}.
\end{align*}
\] (32)

We consider the coupling terms \( h^{XY}_{en}(j, j') \) and \( h^{\text{Ising}}_{en}(j, j') \) acting on the following four product states of the two dimers, \(|\psi(\theta, \chi, \phi_{j, j'})\rangle\rangle\), \(|\psi(\phi_{j}, \chi_{j})\rangle\rangle\) for the Ising terms, the calculation is simple since \( h^{\text{Ising}}_{en}(j, j') \)’s are expressed in terms of \( z \) components of the pseudospin operators such as \( \phi_{j, j'} \) and \( \psi_{j} \) are independent of the position \( \theta \) and \( \chi \). Therefore, if the relation \( J_{--} = J_{+} \) holds, the product state,

\[
\prod_j |\psi(\phi_{j}, \chi_{j})\rangle\rangle,
\] (35)

with the uniform phases \( \theta_j = \theta \) and \( \chi_j = \chi \) is also an eigenstate of model (33) with zero eigenvalue for arbitrary \( \theta \) and \( \chi \).

Next, we include the intradimer Hamiltonian \( \mathcal{H}_{\text{intra}} \) [Eq. (34)] in the argument. Here, we consider the case where the coupling constants in the intradimer Hamiltonian are uniform, i.e., \( J_{d}(j) = J_d \) and \( \Delta_{d}(j) = \Delta_d \), and so on. As discussed in Sect. III the local intradimer Hamiltonian \( \mathcal{H}_{\text{intra}}(j) \) has eigenstates \(|\psi(\theta_0, \chi_0)\rangle\rangle\), \(|\psi(\theta_0 + \pi, \chi_0)\rangle\rangle\), \(|\phi(\phi_0, \chi_0)\rangle\rangle\), \(|\phi(\phi_0 + \pi, \chi_0)\rangle\rangle\), where the phases \( \theta_0, \chi_0, \phi_0 \), and \( \chi_0 \) are determined by the coupling constants. The phases are independent of the position \( j \) since the coupling constants are uniform. Combining this result with the one for the interdimer Hamiltonian discussed above, we find that the two product states \( \prod_j |\phi(\phi_{j}, \chi_{j})\rangle\rangle \) with \( \phi_j = \phi_0, \phi_0 + \pi \) and \( \chi_j = \chi_0 \) are eigenstates of the whole Hamiltonian \( \mathcal{H}_{\text{inter}} + \mathcal{H}_{\text{intra}} \). Furthermore, if \( J_{++} = J_{--} \), the other two product states \( \prod_j |\psi(\theta_{j}, \chi_{j})\rangle\rangle \) with \( \theta_j = \theta_0, \theta_0 + \pi \) and \( \chi_j = \chi_0 \) are also eigenstates of the whole Hamiltonian.

We note that when the eigenstates \(|\psi(\theta_0, \chi_0)\rangle\rangle\) and \(|\psi(\theta_0 + \pi, \chi_0)\rangle\rangle\) are degenerate, the phases \( \theta_0 \) and \( \chi_0 \) are not fixed. For example, if the intradimer Hamiltonian contains only the XXZ exchange term, the eigenstates \(|\phi(\phi_0, \chi_0)\rangle\rangle\) and \(|\phi(\phi_0 + \pi, \chi_0)\rangle\rangle\) of the local intradimer Hamiltonian \( h_{\text{intra}}(j) \) are degenerate regardless of the values of \( J_d \) and \( \Delta_d \). If this is the case, the phases \( \phi_0 \) and \( \chi_0 \) are not fixed, and the product state \( \prod_j |\psi(\phi_0, \chi_0)\rangle\rangle \) with arbitrary \( \phi_0 \) and \( \chi_0 \) is an eigenstate of the whole Hamiltonian. Such degenerate eigenstates were found in the model in a one-dimensional lattice\(^{22}\).

It can be proven in the following way that the eigenstates obtained above become the ground states of the whole Hamiltonian in some parameter regions. For instance, we consider the case of the interdimer Hamiltonian (33) with \( J_{++} = J_{--} \) [Fig. 2b] and the product state \( \prod_j |\psi(\theta_0, \chi_0)\rangle\rangle \). To prove that the state can be the ground state, it is convenient to consider the interdimer Hamiltonian first. Let us assume that \(|\psi(\theta_0, \chi_0)\rangle\rangle\) is the lowest-energy eigenstate of the local intradimer Hamiltonian \( h_{\text{intra}}(j) \) and is not degenerate to the other three eigenstates. In this case, the ground state of the interdimer Hamiltonian \( \mathcal{H}_{\text{intra}} \) for the whole system is the product state \( \prod_j |\psi(\theta_0, \chi_0)\rangle\rangle \) and there is a finite energy gap \( E_{\text{gap}} \).
to the first excited states [see Fig. 3(a)]. The ground state is unique while the first excited states are massively degenerate (\( N \)-fold or more, where \( N \) is the number of dimer units in the system). When the interdimer Hamiltonian \( H_{\text{inter}} \) is included, the ground state as well as its energy is unchanged since the state is an eigenstate of \( H_{\text{inter}} \) with zero eigenvalue. On the other hand, the excited states are modified by the interdimer Hamiltonian and the manifold of the first-excited states forms an energy band. The band width should be of the order of the energy scale of the interdimer Hamiltonian. Therefore, if the energy gap \( E_{\text{gap}} \) is larger than a critical value, which has the same order as the energy scale of the interdimer Hamiltonian, the product state \( \prod_j |\psi(\theta_j, \chi_j)\rangle \) remains as the ground state of the whole Hamiltonian. We thereby obtain the exact ground state. We note that such an exact ground state was found in the case of a one-dimensional lattice in Refs. [28] and [29]. In these studies, the Hamiltonian \( H_1^\text{int} \) with \( J_{+,} = J_{-} \) [Eq. (3)] with \( J_{11} = -J_{22} \) and \( J_{21} = J_{12} = 0 \), see Fig. 2(b)] was considered for the interdimer Hamiltonian, while the intradimer Hamiltonian \( H_0^\text{int} \) is assumed to contain the XXZ exchange term only \( [K_d = D_d = H_d^\text{int} = H_d^\text{tot} = 0] \). It was found\(^{28} \) that the model with \( J_{11} = -J_{22} = 1, \Delta = \Delta_d = 1, \) and \( J_d > 1.134461 \) has the product state of the dimer singlets, 

\[
|\text{DS}\rangle = \prod_j |s_j\rangle, \tag{36}
\]

\[
|s_j\rangle = \left| \psi\left(\frac{\pi}{2}, 0\right) \right)_j = \frac{1}{\sqrt{2}} \left( |\uparrow\downarrow\rangle_j + |\downarrow\uparrow\rangle_j \right), \tag{37}
\]

as the exact ground state. It was also reported\(^{29} \) that the model with \( J_{11} = -J_{22} = 0.2, \Delta = 1, J_d = -1, \) and \( 0 \leq \Delta_d \leq 0.83 \) has the product state of the dimer triplets with zero magnetization, 

\[
|\text{DT}_0\rangle = \prod_j |0_j\rangle, \tag{38}
\]

\[
|0_j\rangle = \left| \psi\left(\frac{\pi}{2}, 0\right) \right)_j = \frac{1}{\sqrt{2}} \left( |\uparrow\downarrow\rangle_j + |\downarrow\uparrow\rangle_j \right), \tag{39}
\]

as the exact ground state.

Finally, we discuss the case where the local intradimer Hamiltonian \( h_{\text{intra}}(j) \) has degenerate ground states, choosing, as an example, the case where the states \( |\psi(\varphi_0, \zeta_0)\rangle \) and \( |\psi(\varphi_0 + \pi, \zeta_0)\rangle \) are the doubly degenerate ground states of \( h_{\text{intra}}(j) \). (Here, the values of \( \varphi_0 \) and \( \zeta_0 \) can be taken arbitrarily.) This indeed occurs when \( h_{\text{intra}}(j) \) contains only the XXZ exchange term with \( J_d < 0 \) and \( \Delta_d > 1 \) (i.e., the exchange coupling is ferromagnetic and has the Ising anisotropy). In this case, the intradimer Hamiltonian \( H_{\text{intra}} \) has \( 2^N \)-fold degenerate ground states; the Hilbert space of the ground-state manifold can be expanded by the product states \( \prod_j |\psi(\varphi_j, \zeta_j)\rangle_j \) with \( \varphi_j \) taking one of the two values \( \{\varphi_0, \varphi_0 + \pi\} \) arbitrarily. When the interdimer Hamiltonian is included, this \( 2^N \)-fold degeneracy of the ground states is lifted: Although two out of the degenerate ground states of \( H_{\text{intra}} \), \( \prod_j |\psi(\varphi_0, \zeta_0)\rangle_j \) and \( \prod_j |\psi(\varphi_0 + \pi, \zeta_0)\rangle_j \) (the product states with uniform \( \varphi_j \)), remain the eigenstates, the other states are mixed by the interdimer Hamiltonian and form the energy band [see Fig. 3(b)]. As a result, the ground state of the whole Hamiltonian is not a simple product of dimer states but becomes a many-body entangled state. We note that, if the anisotropic exchange coupling \( h_{\text{intra}}(j) \) is present in the intradimer Hamiltonian, the ground state of \( H_{\text{intra}} \) becomes unique. Then, if the energy gap to the first excited states is sufficiently large, a product state, 

\[
|\text{DN}(\varphi_0, \zeta_0)\rangle = \prod_j |\psi(\varphi_0, \zeta_0)\rangle_j, \tag{40}
\]

where \( \varphi_0 \) and \( \zeta_0 \) are fixed according to \( H_{\text{intra}} \), becomes the
the three parameter regions for the interdimer Hamiltonian. We also note that the spin-nematic state try breaking but due to the explicit anisotropy in the Hamiltonian, we call the state a spin-nematic state. It should be noticed that $H_{\text{intra}}$ and the intradimer Hamiltonian $H_{\text{int}}$ are of the form,\[ (\text{DN}(\phi_0, \zeta_0)|S^+_{i,j}S^+_{j,-j}S^-_{-j,-j}S^-_{i,-j} |\text{DN}(\phi_0, \zeta_0)) = \frac{1}{4} \sin^2 \phi_0, \] we call the state a spin-nematic state. It should be noticed that this spin-nematic state is not a result of a spontaneous symmetry breaking but due to the explicit anisotropy in the Hamiltonian. We also note that the spin-nematic state $|\text{DN}(\phi_0, \zeta_0)\rangle$ can be regarded as a “ferromagnetic” state of the pseudospin $\mathbf{T}_{2,j}$ pointing to the direction Eq. (22) with $\phi_j = \phi_0$ and $\zeta_j = \zeta_0$.

**C. Example II of Exact Ground States**

We discuss another example of the model with an exact ground state. The model considered consists of the interdimer Hamiltonian of the form,

$$H_{\text{int}} = \sum_{(j,j')} \left[ J_{++}(j,j') + \Delta h_{++}(j,j') \right] + \tilde{J}_{--} \left[ h_{XY}(j,j') + \Delta h_{XY}(j,j') \right],$$

and the intradimer Hamiltonian $H_{\text{intra}}$ [Eq. (4)]. We consider the three parameter regions for the interdimer Hamiltonian:

(a) $\tilde{J}_{++} = 0$: In this case, the interdimer Hamiltonian is given by Eq. (2) with $J_{11} = J_{22} = -J_{12} = -J_{21} = \tilde{J}_{--}$ [Fig. 4(a)].

(b) $\Delta = 0$: The exchange couplings in the interdimer Hamiltonian are of the $XY$-type and the exchange coupling constants in Eq. (2) obey the relation $J_{11} = J_{22} = J_{12} = J_{21} = 0$ [Fig. 4(b)].

(c) $\tilde{J}_{++} = \tilde{J}_{--}$ and $\Delta = 0$: The interdimer exchanges are of the $XY$-type and the coupling constants in Eq. (2) obey $J_{11} = J_{22} = \tilde{J}_{++} + \tilde{J}_{--}$ and $J_{12} = J_{21} = 0$ [Fig. 4(c)].

This is a special case of the model (b) above.

We also assume that the lattice is bipartite and the coupling constants in the intradimer Hamiltonian take one of two values depending on the sublattice A or B, i.e., $J_{d,j}(j) = J_{d,A}(j \in \Lambda), J_{d,B}(j \in B)$, and so on.

We can find the parameter region of the model with an exact ground state in the same manner as described in the previous section. From the results in Table II it follows that in all of the three cases listed above, the interdimer Hamiltonian has the product state,

$$\prod_{j \in A} |\phi(\phi_0, \zeta_0)\rangle \prod_{j \in B} |\phi(-\phi_0, \zeta_0)\rangle,$$

with arbitrary $\phi$ and $\zeta$ as an eigenstate with zero eigenvalue. Then, if the local intradimer Hamiltonian $H_{\text{intra}}(j)$ in sublattices A and B has respectively the states $|\phi(\phi_0, \zeta_0)\rangle$ and $|\phi(-\phi_0, \zeta_0)\rangle$ with certain $\phi_0$ and $\zeta_0$ as an eigenstate, the product state \[\{\phi(\phi_0, \zeta_0)\}_j \prod_{j \in B} |\phi(-\phi_0, \zeta_0)\rangle\] with $\phi = \phi_0$ and $\zeta = \zeta_0$ is an eigenstate of the whole Hamiltonian. We note that such staggered $\phi_j = \pm \phi_0$ and uniform $\zeta_j = \zeta_0$ can be realized by taking the coupling constant of the anisotropic exchange terms $h_{\text{int}}(j)$ in the staggered way, $K_{d,j}(j) = K_{d,A} < 0$ ($j \in \Lambda), K_{d,B} > 0$ ($j \in B$) and setting $\mu(0) = H_{d}(j) = 0$ in the intradimer Hamiltonian. In this case, the product state,

$$\prod_{j \in A} |\phi\left(\frac{\pi}{2}, 0\right)\rangle \prod_{j \in B} |\phi\left(-\frac{\pi}{2}, 0\right)\rangle_{j_{2,j}},$$

is the eigenstate of the whole Hamiltonian. Finally, if the product state \[\{\phi(\phi_0, \zeta)\}_j \prod_{j \in B} |\phi(-\phi_0, \zeta_0)\rangle\] with $\phi = \phi_0$ and $\zeta = \zeta_0$ is the ground state of the interdimer Hamiltonian $H_{\text{intra}}$ with a sufficiently large excitation gap, the state becomes the exact ground state of the whole Hamiltonian $H_{\text{intra}} + H_{\text{intra}}$. Since this ground state exhibits a staggered long-range order of the spin-nematic operator $S^+_{i,j}S^+_{j,-j}$, we call the state the antiferro-spin-nematic state.
We also see in Table II that the interdimer Hamiltonian in case (c) mentioned above has the product state,

\[ \prod_{j \in A} |\psi(\theta_j, \chi_j)\rangle \prod_{j \in B} |\psi(-\theta_j, \chi_j)\rangle, \]

(45)

with arbitrary \( \theta \) and \( \chi \) as an eigenstate with zero eigenvalue. Then, if the interdimer Hamiltonian has the product state (45) with \( \theta = \theta_0 \) and \( \chi = \chi_0 \) as the ground state with a sufficiently large excitation gap, the product state becomes the exact ground state of the whole Hamiltonian. The staggered \( \theta_j = \pm \theta_0 \) and uniform \( \chi_j = \chi_0 \) of the ground state can be realized in a rather simple way: If the interdimer Hamiltonian contains only the XXZ exchange terms with \( J_d(j) = J_{dA} > 0 \) (antiferromagnetic) for \( j \in A \) and \( J_d(j) = J_{dB} < 0 \) and \( 0 < \Delta_{dA} < \Delta_{dB} < 1 \) (ferromagnetic and XY-like anisotropic) for \( j \in B \), the product state,

\[ \prod_{j \in A} |\psi(\frac{\pi}{2}, 0)\rangle \prod_{j \in B} |\psi(-\frac{\pi}{2}, 0)\rangle \]

\[ = \prod_{j \in A} \frac{1}{\sqrt{2}} \left( |\uparrow \downarrow\rangle_j - |\downarrow \uparrow\rangle_j \right) \prod_{j \in B} \frac{1}{\sqrt{2}} \left( |\uparrow \downarrow\rangle_j + |\downarrow \uparrow\rangle_j \right) \]

\[ = \prod_{j \in A} |s\rangle_j \prod_{j \in B} |l_0\rangle_j, \]

(46)

is the ground state of the interdimer Hamiltonian. This product state becomes the exact ground state of the whole Hamiltonian if the interdimer exchange constants \( J_{dA} \) and \( J_{dB} \) are sufficiently large. We thereby find the model with the exact ground state in which the dimer-singlet state and the dimer-triplet state with zero magnetization are arranged in a staggered fashion. We note that when \( \Delta_{dA} = 0 \) (i.e., not only the interdimer exchange couplings but also the intradimer ones are of the XY type), the model and the ground state [Eq. (46)] considered here are connected to the models and the states [Eqs. (36) and (39)] discussed in the previous section through unitary transformations of the spin rotation.

D. Other Examples

In addition to the cases discussed in the preceding sections, we see in Table II many cases where the outcome of \( h_{XX}^{XY}(j, j') \) acting on the two-dimer product state is zero. Namely,

\[ h_{XX}^{XY}(j, j')(f_j f'_j) = 0, \]

(47)

when \( \epsilon (\epsilon') = + \) and \( (f_j f'_j) = |s\rangle_j (f_j f'_j) = |s\rangle_j \), or \( \epsilon (\epsilon') = - \) and \( (f_j f'_j) = |l_0\rangle_j (f_j f'_j) = |l_0\rangle_j \). This comes from the fact that \( S_{1,j}^z + S_{2,j}^z + S_{1,j}^- S_{2,j}^+ \) acting on \( |s\rangle_j (|l_0\rangle_j) \) yields zero,

\[ (S_{1,j}^z + S_{2,j}^z)|s\rangle_j = (S_{1,j}^- S_{2,j}^+)|l_0\rangle_j = 0. \]

(48)

A simple example of the application of Eq. (47) can be found in the model consisting of the interdimer Hamiltonian,

\[ \mathcal{H}_{\text{inter}} = \sum_{(j, j')} \tilde{J}_{++} \left[ h_{XX}^{XY}(j, j') + \Delta h_{\text{sing}}^{XY}(j, j') \right], \]

(49)

and the interdimer Hamiltonian including only the XXZ exchange couplings. For this model, the product state of the dimer-singlet states, \( |S_1,0\rangle \left[ \text{Eq. (36)} \right] \), is the exact eigenstate of \( \mathcal{H}_{\text{inter}} \) with zero eigenvalue and becomes the exact ground state of the whole Hamiltonian if the XXZ exchange couplings in the interdimer Hamiltonian are antiferromagnetic and sufficiently strong. We note that this mechanism to realize the dimer-singlet-product ground state can be understood from the viewpoint that \( (S_{1,j} + S_{2,j})^2 \) for each dimer unit is a good quantum number in the model. This type of the exact ground state has been reported for various frustrated spin models.

Equation (47) can be used to derive many other models with an exact ground state. Let us consider, for instance, the interdimer Hamiltonian,

\[ \mathcal{H}_{\text{inter}} = \sum_{(j, j')} \tilde{J}_{--} \left[ h_{XX}^{XY}(j, j') + \Delta h_{\text{sing}}^{XY}(j, j') \right], \]

(50)

in a bipartite lattice. It is then found that the product state,

\[ \prod_{j \in A} |l_0\rangle_j \prod_{j \in B} |\varphi(\epsilon_j, \zeta_j)\rangle_j, \]

(51)

is the eigenstate of \( \mathcal{H}_{\text{inter}} \) with zero eigenvalue. Therefore, if the local interdimer Hamiltonian \( h_{\text{inter}}(j) \) in sublattice A has \( |l_0\rangle_j \) as an eigenstate and \( h_{\text{inter}}(j) \) in sublattice B does the spin-nematic state \( |\varphi(\varphi_0, \zeta_0)\rangle_j \) (with certain \( \varphi_0 \) and \( \zeta_0 \)), the product state \( \prod_{j \in A} |l_0\rangle_j \prod_{j \in B} |\varphi(\varphi_0, \zeta_0)\rangle_j \) is the eigenstate of the whole Hamiltonian. Furthermore, if the product state is the ground state of the interdimer Hamiltonian with a sufficiently large excitation gap, the product state becomes the exact ground state of the whole Hamiltonian. In such a manner, we can construct several models with an exact ground state written as a direct product of the dimer-singlet state \( |s\rangle_j \), the dimer-triplet state with zero magnetization \( |l_0\rangle_j \), and the spin-nematic state \( |\varphi(\varphi_0, \zeta_0)\rangle_j \).

E. Short Summary of Models and Exact Ground States

Here, we summarize the models and their exact ground states discussed in the preceding sections. The interdimer Hamiltonians of the models considered and the corresponding ground states are as follows:

(I) The XXZ exchange Hamiltonian \( \Sigma_3 \) with uniform phases \( \varphi_j = \varphi \) and \( \zeta_j = \zeta \) is a candidate of the exact ground state of the whole Hamiltonian.

(I') The XXZ exchange Hamiltonian \( \Sigma_1 \) with \( J_{+-} = J_{-+} \), shown schematically in Fig. 2(b): This interdimer Hamiltonian has the product state \( \Sigma_3 \) with \( \theta_j = \theta \) and \( \chi_j = \chi \) as a candidate of the ground state.

(II) The XXZ exchange Hamiltonian \( \Sigma_2 \) with \( J_{++} = 0 \) and the XY \((\Delta = 0)\) exchange Hamiltonian \( \Sigma_2 \) in a bipartite
lattice: The Hamiltonians are schematically shown in Figs. 4(a) - 4(c). For these interdimer Hamiltonians, the product of the dimer-spin-nematic states with staggered phases, Eq. (43), is a candidate of the ground state.

(II') The XY ($\Delta = 0$) exchange Hamiltonian (42) with $J_{+} = -J_{-}$, shown in Fig. 4(c), in a biparticle lattice: This interdimer Hamiltonian has the product state (45) as a candidate of the ground state.

(III) An exchange Hamiltonian in which operators acting on the $j$th dimer are expressed in terms of only $S_{\uparrow}^{x} + S_{\downarrow}^{x}$ or of only $S_{\uparrow}^{z} - S_{\downarrow}^{z}$: With an adequate inclusion of the XY and Ising terms according to Table I, the interdimer Hamiltonian has a product state of the dimer singlet $|s\rangle$, and the dimer triplet with zero magnetization, $|l_{0}\rangle$, with a configuration of $|s\rangle$, $|l_{0}\rangle$ corresponding to the Hamiltonian, as a candidate of the ground state.

Then, if the candidate state is a unique ground state of the intradimer Hamiltonian (4) with a sufficiently large excitation gap, the state is the exact ground state of the whole system composed of the inter- and intradimer Hamiltonians, $\mathcal{H}_{\text{inter}} + \mathcal{H}_{\text{intra}}$. We note that the Hamiltonian of type (I) with the dimer-singlet ground state [DS] [Eq. (34)] and the dimer-triplet ground state [DT] [Eq. (35)], in a one-dimensional lattice was investigated in Refs. 28 and 29, while the Hamiltonian of type (III), especially the one that is written in terms of $S_{\uparrow}^{x} + S_{\downarrow}^{x}$ and has the ground state including dimer singlets $|s\rangle$, was studied in the literature 20, 24, 27.

We emphasize that models that can be proven by our scheme to have an exact ground state are not limited to the ones mentioned above. For instance, adding the term $\sum_{(j, j')} h_{\text{Ising}}^{\prime}(j, j')$ to the interdimer Hamiltonian of type (I) above does not change the conclusion since the term acting on the ground state considered gives zero. An inhomogeneity in coupling constants $J_{\text{eff}}$ of the interdimer Hamiltonians also does not affect the conclusion. One can thus construct a variety of models with an exact ground state using Table I [Eq. (36)], that of the dimer-triplet states with zero magnetization [Eq. (35)], those of the dimer-spin-nematic states [Eq. (40)], and various products with a two-sublattice structure. We have also introduced two operators $T_{1,j}$ and $T_{2,j}$ [Eqs. (10) and (11)]: The operator $T_{1,j}$ ($T_{2,j}$) acts in the subspace $|\uparrow \uparrow \rangle_{j}, |\downarrow \downarrow \rangle_{j}$ ($|\uparrow \uparrow \rangle_{j}, |\downarrow \downarrow \rangle_{j}$) as a $\sigma/2$ operator, while it is zero in the subspace $|\uparrow \uparrow \rangle_{j}, |\downarrow \downarrow \rangle_{j}$ ($|\uparrow \uparrow \rangle_{j}, |\downarrow \downarrow \rangle_{j}$).

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Appendix A

The cases where the interdimer exchange terms $h_{XY}^{\prime}(j, j')$ and $h_{\text{Ising}}^{\prime}(j, j')$ acting on a two-dimer product state give zero, which are summarized in Table I can be divided into the following three groups.

First, the Ising terms $h_{\text{Ising}}^{\prime}(j, j')$ can be written in terms of $T_{1,k}^{+}$ and $T_{2,k}^{+}$ ($k = j, j'$) as mentioned in Sect. IIIA. Therefore, when $h_{\text{Ising}}^{\prime}(j, j')$ including $T_{1,k}^{+}[T_{2,k}^{+}]$ acts on $|\phi(\varphi_{j}, \zeta_{j})\rangle_{k}$, the outcome is zero. These cases are listed as "0" in Table I.

Second, it follows from Eq. (35) that the XY terms $h_{XY}^{\prime}(j, j')$ including the factor $(S_{1,k}^{+} + S_{2,k}^{+})[S_{1,k}^{+} - S_{2,k}^{+}]$ yield zero when acting on the dimer-singlet state $|s\rangle_{k}$ [the dimer-triplet state with zero magnetization, $|l_{0}\rangle_{k}$]. These cases are listed in Table II as "|s\rangle_{j} = |s\rangle_{j}'", "$|s\rangle_{j} = |l_{0}\rangle_{j}'", and so on.

Third, there are other nontrivial cases where the XY terms $h_{XY}^{\prime}(j, j')$ yield zero. For instance, the outcome of $h_{XY}^{\prime}(j, j')$ acting on $|\phi(\varphi_{j}, \zeta_{j})\rangle_{j}|\phi(\varphi_{j}, \zeta_{j})\rangle_{j'}$ is given by

$$h_{XY}^{\prime}(j, j')(\phi(\varphi_{j}, \zeta_{j}); \phi(\varphi_{j}, \zeta_{j}))(\phi(\varphi_{j}, \zeta_{j}))(\phi(\varphi_{j}, \zeta_{j})); \phi(\varphi_{j}, \zeta_{j})_{j'}$$

$$= -i 2 \left[ \cos \left( \frac{\zeta_{j} - \zeta_{j}'}{2} \right) \sin \left( \frac{\varphi_{j} - \varphi_{j}'}{2} \right) \right]$$

$$+ i \sin \left( \frac{\zeta_{j} - \zeta_{j}'}{2} \right) \sin \left( \frac{\varphi_{j} + \varphi_{j}'}{2} \right)$$

$$\times (|\uparrow \uparrow \rangle_{j} + |\downarrow \downarrow \rangle_{j}) (|\uparrow \uparrow \rangle_{j'} - |\downarrow \downarrow \rangle_{j'}) \right).$$

(A1)

This resultant state becomes zero if $\varphi_{j} = \varphi_{j}$ and $\zeta_{j} = \zeta_{j}'$. We note that the state (A1) is zero also in the case of $\varphi_{j} = -\varphi_{j}$ and $\zeta_{j} = \zeta_{j}' + \pi$. In our argument, we consider only the case of $\varphi_{j} = \varphi_{j}$ and $\zeta_{j} = \zeta_{j}'$ since these two cases give the same state $|\phi(\varphi_{j}, \zeta_{j}); \phi(\varphi_{j}, \zeta_{j})\rangle_{j}$ up to an overall factor. In addition, the state (A1) becomes zero for arbitrary $\zeta_{j}$ and $\zeta_{j}'$ if $\varphi_{j} = \varphi_{j} = 0$ or $\varphi_{j} = \varphi_{j} = \pi$. We ignore these cases of $\varphi_{j} = \varphi_{j} = 0$ and $\varphi_{j} = \varphi_{j} = \pi$ in our argument as they correspond to the trivial states $|\phi(\varphi_{j}, \zeta_{j}); \phi(\varphi_{j}, \zeta_{j}')\rangle_{j} = |\uparrow \uparrow \rangle_{j} |\uparrow \uparrow \rangle_{j'}$ and $|\downarrow \downarrow \rangle_{j} |\downarrow \downarrow \rangle_{j'}$, respectively.

IV. SUMMARY

In summary, we have studied frustrated quantum spin systems consisting of spin-dimer units, Eq. (1). We have shown that the systems in certain parameter regions have an exact ground state written in the form of the direct product of dimer states. In the argument, we first specified the interdimer Hamiltonian which has the product state considered as an eigenstate with zero eigenvalue, and showed that the state can be the eigenstate of a certain interdimer Hamiltonian simultaneously. We then showed that the eigenstate becomes an exact ground state of the whole Hamiltonian (the sum of the inter- and intradimer Hamiltonians) when the coupling parameters in the intradimer Hamiltonian are selected appropriately. In such a way, we have found several models, each of which has the exact ground state of the form of the product of dimer states, including the product of the dimer-singlet states.
In a similar way, one can find that the outcome is zero for the cases denoted in Table I as “\(\theta_j = \theta'_j\) and \(\chi_j = \chi'_j\)”, “\(\varphi_j = \varphi'_j\) and \(\zeta_j = \zeta'_j\)”, “\(\theta_j = -\theta'_j\) and \(\chi_j = \chi'_j\)”, and “\(\varphi_j = -\varphi'_j\) and \(\zeta_j = \zeta'_j\)”. The zero states obtained in these cases stem from perfect destructive interferences among the interdimer exchange processes.