CHARACTERIZING NON-SEPARABLE CONVEX SETS
HOMEOMORPHIC TO $\ell_2^2(\kappa)$ OR $\Gamma^\omega \times \ell_2^2(\kappa)$

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Abstract. For an infinite cardinal $\kappa$ let $\ell_2(\kappa)$ be the linear hull of the standard orthonormal base of the Hilbert space $\ell_2(\kappa)$ of density $\kappa$. We prove that a non-separable convex subset $X$ of density $\kappa = \text{dens}(X)$ in a locally convex linear metric space if homeomorphic to the space

- $\ell_2^f(\kappa)$ if and only if $X$ can be written as countable union of finite-dimensional locally compact subspaces;
- $\Gamma^\omega \times \ell_2^f(\kappa)$ if and only if $X$ contains a topological copy of the Hilbert cube $\Gamma^\omega$ and $X$ can be written as a countable union of locally compact subspaces.

For an infinite cardinal $\kappa$

\begin{align*}
\ell_2(\kappa) &= \{(x_i)_{i \in \kappa} \in \mathbb{R}^\kappa : \sum_{i \in \kappa} |x_i|^2 < +\infty\} \\
\ell_2^f(\kappa) &= \{(x_i)_{i \in \kappa} \in \ell_2(\kappa) : \{|i \in \kappa : x_i \neq 0\} < \infty\}
\end{align*}

be the standard Hilbert space of density $\kappa$, endowed with the norm $\|(x_i)_{i \in \kappa}\| = \left(\sum_{i \in \kappa} |x_i|^2\right)^{1/2}$ and be the linear hull of the standard orthonormal basis in $\ell_2(\kappa)$. If $\kappa = \omega$ is the smallest infinite cardinal, then the spaces $\ell_2(\omega)$ and $\ell_2^f(\omega)$ are denoted by $\ell_2$ and $\ell_2^f$, respectively. By $\mathbb{I} = [0, 1]$ we shall denote the closed unit interval and by $\mathbb{I}^\omega$ the Hilbert cube. A closed subset $A$ of a topological space $X$ is called a $Z$-set in $X$ if the set $\{f \in C(\mathbb{I}^\omega, X) : f(\mathbb{I}^\omega) \cap A = \emptyset\}$ is dense in the function space $C(\mathbb{I}^\omega, X)$ endowed with the compact-open topology. Let $A \subset B$ be two subsets in a linear space $L$. We shall say that $A$ has infinite codimension in $B$ if the linear hull $\text{lin}(A)$ has infinite codimension in the linear hull $\text{lin}(B)$ of the set $B$.

The following characterization of convex sets homeomorphic to $\ell_2^f$ or $\Gamma^\omega \times \ell_2^f$ is obtained due to combined efforts of T. Dobrowolski [6], D. Curtis, T. Dobrowolski, J. Mogilski [5], and T. Banakh [1] (see also Theorem 5.3.12 and 5.3.2 in [3]).

Theorem 1. A convex subset $X$ of a locally convex linear metric space is homeomorphic to the space

- $\ell_2^f$ if and only if $X$ is infinite-dimensional and $X$ can be written as a countable union of finite-dimensional compact sets;
- $\Gamma^\omega \times \ell_2^f$ if $X$ can be written as a countable union of compact $Z$-sets and $X$ contains a subset $Q$ which is homeomorphic to the Hilbert cube and has infinite dimension in $X$.

In this paper we shall prove a non-separable counterpart of Theorem 1. For a topological space $X$ its density $\text{dens}(X)$ is the smallest cardinality $|D|$ of a dense subset $D \subset X$.

Theorem 2. A non-separable convex subset $X$ of density $\kappa = \text{dens}(X)$ in a locally convex linear metric space is homeomorphic to the space

- $\ell_2^f(\kappa)$ if and only if $X$ can be written as a countable union of finite-dimensional locally compact spaces;
- $\Gamma^\omega \times \ell_2^f(\kappa)$ if and only if $X$ contains a subspace homeomorphic to the Hilbert cube $\Gamma^\omega$ and $X$ can be written as a countable union of locally compact spaces.

In fact, Theorem 2 follows from a more general Theorem 3 characterizing pairs of convex sets homeomorphic to the pairs $(\ell_2(\kappa), \ell_2^f(\kappa))$ or $(\Gamma^\omega \times \ell_2(\kappa), \Gamma^\omega \times \ell_2^f(\kappa))$. We say that for topological spaces $A \subset X$ and $B \subset Y$ the pairs $(X, A)$ and $(Y, B)$ are homeomorphic if there is a homeomorphism $h : X \to Y$ such that $h(A) = B$. By a Fréchet space we understand a locally convex linear complete metric space.

1991 Mathematics Subject Classification. 57N17, 52A07.
Key words and phrases. Convex set, pre-Hilbert space, homeomorphic.
Theorem 3. Let $X$ be a non-separable convex set of density $\kappa = \text{dens}(X)$ in a Fréchet space $L$ and $\bar{X}$ be the closure of $X$ in $L$. The pair $(\bar{X}, X)$ is homeomorphic to the pair

1. $(\ell_2(\kappa), \ell_2^I(\kappa))$ if and only if $X$ can be written as a countable union of finite-dimensional locally compact spaces;

2. $(\ell^w \times \ell_2(\kappa), \ell^w \times \ell_2^I(\kappa))$ if and only if $X$ contains a topological copy of the Hilbert cube $\ell^w$ and $X$ can be written as a countable union of locally compact spaces.

Theorem 4 will be derived from the following two results, first of which is due to T. Banakh and R. Cauty [2].

Theorem 4 (Banakh, Cauty). Each non-separable closed convex set $X$ in a Fréchet space is homeomorphic to the Hilbert space $\ell_2(\kappa)$ of density $\kappa = \text{dens}(X)$.

The other result used in the proof of Theorem 4 is a topological characterization of the pairs $(\ell_2(\kappa), \ell_2^I(\kappa))$ and $(\ell^w \times \ell_2(\kappa), \ell^w \times \ell_2^I(\kappa))$ due to J. West [9] (see also [8] and [7]).

Theorem 5 (West). A pair $(X, Y)$ of topological spaces $Y \subset X$ is homeomorphic to the pair $(\ell^w \times \ell_2(\kappa), \ell^w \times \ell_2^I(\kappa))$ (resp. $(\ell_2(\kappa), \ell_2^I(\kappa))$) for an infinite cardinal $\kappa$ if and only if

1. the space $X$ is homeomorphic to $\ell_2(\kappa)$;
2. the space $Y$ can be written as a countable union of (finite-dimensional) locally compact spaces, and
3. the space $Y$ absorbs (finite-dimensional) compact subsets of $X$ in the sense that for each compact (finite-dimensional) subset $K \subset X$, a compact subset $B \subset K \cap Y$, and an open cover $U$ of $X$ there is a topological embedding $h : K \to Y$ such that $h|B = \text{id}|B$ and $h$ is $U$-near to the identity embedding $\text{id} : K \to X$.

Given a cover $U$ of a topological space $X$, we say that two maps $f, g : Z \to X$ are $U$-near and denote this writing $(f, g) \prec U$ if for each point $z \in Z$ the doubleton $\{f(z), g(z)\}$ is contained in some set $U \in U$.

Proof of Theorem 4. The “only if” part in the both statements of Theorem 4 are trivial. To prove the “if” parts, assume that $X$ is a non-separable convex subset of a Fréchet space $L$ and let $\bar{X}$ be the closure of $X$ in $L$. By Theorem 4 the space $\bar{X}$ is homeomorphic to the Hilbert space $\ell_2(\kappa)$ of density $\kappa = \text{dens}(\bar{X}) = \text{dens}(X)$. Now we consider two cases:

1) Assume that the convex set $X$ can be written as a countable union of finite-dimensional locally compact spaces. By Theorem 5 the homeomorphism of the pairs $(\bar{X}, X)$ and $(\ell_2(\kappa), \ell_2^I(\kappa))$ will follow as soon as we check that the set $X$ absorbs finite-dimensional compact subsets of $\bar{X}$. Fix a finite-dimensional compact subset $K \subset \bar{X}$, a compact subset $B \subset K \cap X$, and an open cover $U$ of $\bar{X}$. By the density of $X$ in $\bar{X}$ and the separability of $K$, there is a separable convex subset $Y \subset X$ of $X$ such that $B \subset Y$ and $K \subset \bar{Y}$. Moreover, using the fact that $X$ is not separable, we can choose $Y$ so that the closure $\bar{Y}$ is not locally compact. By Theorem 4.4 of [5], the pair $(\bar{Y}, \bar{Y} \cap X)$ is homeomorphic to the pair $(\ell_2, \ell_2^I)$, and by Theorem 5 the set $\bar{Y} \cap X$ absorbs finite-dimensional compact subsets of $\bar{Y}$. Consequently, for the finite-dimensional compact subset $K \subset \bar{Y} \subset X$ there is a topological embedding $f : K \to \bar{Y} \cap X \subset X$ such that $f|B = \text{id}|B$ and $f$ is $U$-near to the identity embedding $\text{id} : K \to X$. This means that $X$ absorbs finite-dimensional compact subsets of $\bar{X}$. By Theorem 4 the pair $(\bar{X}, X)$ is homeomorphic to $(\ell_2(\kappa), \ell_2^I(\kappa))$.

2) Next, assume that $X$ contains a subspace $A \subset X$ homeomorphic to the Hilbert cube and $X$ can be written as a countable union of locally compact subsets. By Theorem 5 the homeomorphism of the pairs $(\bar{X}, X)$ and $(\ell^w \times \ell_2(\kappa), \ell^w \times \ell_2^I(\kappa))$ will follow as soon as we check that the set $X$ absorbs compact subsets of $\bar{X}$. Fix a compact subset $K \subset \bar{X}$, a compact subset $B \subset K \cap X$, and an open cover $U$ of $\bar{X}$. By the density of $X$ in $\bar{X}$ and the separability of the compact set $K \cup A$, we can find a separable convex subset $Y \subset X$ of $X$ such that $A \cup B \subset Y$ and $K \subset \bar{Y}$. We can assume that $Y = \bar{Y} \cap X$. Since $X$ is not separable, the compact set $A$ has infinite codimension in $X$. So we can choose $Y$ to be so large that $A$ has infinite codimension in $Y$ and $\bar{Y}$ is not locally compact. By Proposition 3.1 of [5], each closed locally compact subset of $\bar{Y}$ is a $Z$-set in $\bar{Y}$. It follows that the separable convex set $Y = \bar{Y} \cap X$ is a countable union of compact $Z$-sets. Since the topological copy $A$ of the Hilbert cube has infinite codimension in $Y$, the convex set $Y$ is homeomorphic to $\ell^w \times \ell_2^I$ by Theorem 112. By the Uniqueness Theorem for Skeletoids [4, Theorem 2.1], the pair $(\bar{Y}, Y)$ is homeomorphic to the pair $(\ell^w \times \ell_2, \ell^w \times \ell_2^I)$ and by Theorem 5 the space $Y$ absorbs compact subsets of $\bar{Y}$. In particular,
for the compact subset $K \subset \bar{Y}$ there is a topological embedding $f : K \to Y \subset X$ such that $f|B = \text{id}|B$ and $(f, \text{id}) \prec \mathcal{U}$. This means that $X$ absorbs compact subsets of $\bar{X}$ and the pair $(\bar{X}, X)$ is homeomorphic to the pair $(I^\omega \times \ell_2(\kappa), I^\omega \times \ell_2(\kappa))$ according to Theorem 5. This completes the proof of Theorem 3.

We do not know if the condition on $Q$ to have infinite codimension in $X$ in Theorem 12) can be omitted.

**Problem 1.** Assume that a subset $A$ of a Fréchet space is homeomorphic to the Hilbert cube $I^\omega$. Does $A$ contains a subset $B$, which is homeomorphic to the Hilbert cube and has infinite codimension in $A$?

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