Subnormal solutions of non-homogeneous periodic ODEs, Special functions and related polynomials

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Abstract. This paper offers a new and complete description of subnormal solutions of certain non-homogeneous second order periodic linear differential equations first studied by Gundersen and Steinbart in 1994. We have established a previously unknown relation that the general solutions (i.e., whether subnormal or not) of the DEs can be solved explicitly in terms of classical special functions, namely the Bessel, Lommel and Struve functions, which are important because of their numerous physical applications. In particular, we show that the subnormal solutions are written explicitly in terms of the degenerate Lommel functions \( S_{\mu, \nu}(\zeta) \) and several classical special polynomials related to the Bessel functions. In fact, we solve an equivalent problem in special functions that each branch of the Lommel function \( S_{\mu, \nu}(\zeta) \) degenerates if and only if \( S_{\mu, \nu}(e^z) \) has finite order of growth in \( \mathbb{C} \). We achieve this goal by proving new properties and identities for these functions. A number of semi-classical quantization-type results are obtained as consequences. Thus our results not only recover and extend the result of Gundersen and Steinbart [16], but the new identities and properties found for the Lommel functions are of independent interest in a wider context.

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1. Introduction and main results

The growth of entire solutions of the equation

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(1.1) \[ f'' + P(e^z)f' + Q(e^z)f = R_1(e^z) + R_2(e^{-z}), \]

where \( P(\zeta), Q(\zeta), R_1(\zeta) \) and \( R_2(\zeta) \) are polynomials in \( \zeta \) and that \( P(\zeta) \) and \( Q(\zeta) \) are not both constant, were considered in [13], [15], [16], [17] and [19]. It was shown that certain subnormal solutions can be written in the form \( f(z) = e^{dz}S(e^z) \) where \( S(\zeta) \) is a polynomial and \( d \) is a constant. The same problem when the equation (1.1) is homogeneous was considered, for examples, in [1], [2], [3] and [12].

In this paper, we exhibit a previously unknown relation that the solutions of a subcase of the equation (1.1) when \( \deg P < \deg Q \leq 1 \) can be solved in terms of the sum of the Bessel functions and the Lommel function \( S_{\mu, \nu}(\zeta) \). In the most general consideration the existence of the subnormal solutions of this important subclass of (1.1) is equivalent to the degeneration of the \( S_{\mu, \nu}(\zeta)/\zeta^{\mu-1} \) into a polynomial in \( \zeta \) and \( 1/\zeta \). In several specialized considerations, classical special polynomials related to the Bessel functions such as the Struve functions, the Neumann polynomials, the Gegenbauer polynomials and the Schl"afli polynomials [37], §9.1-9.3 and §10.4 are needed in order to describe the subnormal solutions.

The Lommel functions \( S_{\mu, \nu}(\zeta) \) have numerous applications in, for examples, electromagnetic scattering in a multilayered medium [5], [6], thermal inflation [22], one-dimensional stochastic model with branching and coagulation reactions [24], oscillatory limited compressible fluid flow [30], computation of toroidal shells and propeller blades [34] and strain gradient elasticity theory for antiplane shear cracks [35], etc. In particular, its special case, when \( \mu = \nu \), the Struve function \( H_\nu(\zeta) \) also occurs in many applications [37], pp. 328–338. See for examples, in the theory of loud speakers [23] and in the theory of light [36], chap. 7.

An entire solution \( f(z) \) of (1.1) is called subnormal if either

(1.2) \[ \lim_{r \to +\infty} \frac{\log \log M(r, f)}{r} = 0 \quad \text{or} \quad \lim_{r \to +\infty} \frac{\log T(r, f)}{r} = 0 \]

holds. Here \( M(r, f) = \max_{|z| \leq r} |f(z)| \) denotes the usual maximum modulus of the entire function \( f(z) \) and \( T(r, f) \) is the Nevanlinna characteristics of \( f(z) \). We denote the the order of a meromorphic function \( f(z) \) by \( \sigma(f) = \lim_{r \to +\infty} \log \log M(r, f) / \log r = \lim_{r \to +\infty} \log T(r, f) / \log r \). We refer the reader to [18] or [19] for the details.

In [16], Gundersen and Steinbart proved

**Theorem A** Suppose that \( \deg P < \deg Q \) in the non-homogeneous differential equation (1.1) and that (1.1) admits a subnormal solution \( f(z) \). Then \( f(z) \) must have the form

(1.3) \[ f(z) = S_1(e^z) + S_2(e^{-z}), \]

where \( S_1(\zeta) \) and \( S_2(\zeta) \) are polynomials in \( \zeta \).

Gundersen and Steinbart also considered the cases when \( \deg P > \deg Q \) and \( \deg P = \deg Q \), respectively, and obtained subnormal solutions similar to (1.3) under the same assumption that the equation (1.1) admits a subnormal solution. In this paper we shall only consider the case \( \deg P < \deg Q \) and we refer the reader to [16] for other details.
Remark 1.1 We note that Theorem A and the other results obtained by Gundersen and Steinbart [19] are generalizations to those of Wittich’s [39] for the periodic homogeneous equation

\[(1.4) \quad f'' + P(e^z)f' + Q(e^z)f = 0.\]

Wittich showed that each subnormal solution \(f(z)\) to this equation admits a representation of the form \(f(z) = e^{dz}S(e^z)\), where \(S(\zeta)\) is a polynomial in \(\zeta\) and \(d\) is a constant.

Ismail and one of the authors showed in [8], Remark 1.11 (see also [7]) that the subnormal solutions of a subclass of homogeneous differential equations of (1.4), first considered by Frei in [12] and then by Bank and Laine in [2], was in fact an important diatomic molecule model in Wave (quantum) mechanics proposed by P. M. Morse in a landmark paper [25] in 1929, and it can be solved explicitly in terms of a class of confluent hypergeometric functions – the Coulomb Wave functions and a class of orthogonal polynomials – the Bessel polynomials [8]. The study appears to be the first of its kind concerning subnormal solutions of special cases of (1.4) and special functions. We continue the study in this paper and to show that special functions solutions also exist for a subclass of (1.1) when \(\deg P < \deg Q\). We are able to solve the equation (1.1) by finding explicit solutions in terms of several classes of classical special functions, whether the solution is subnormal or not, and from which necessary and sufficient conditions for the existence of subnormal solutions can be derived as special cases of our main results. These characterization results can be considered as a semi-classical quantization-type results for non-homogeneous equations. This matter will be further discussed in §6.

We shall first state our main results on subnormal solutions, followed by the main result on the growth of \(S_{\mu, \nu}(e^z)\) which is the crux of the paper.

**Theorem 1.2** Let \(\sigma, \nu\) be arbitrary complex constants with \(\sigma\) non-zero, and let \(f(z)\) be an entire solution to the differential equation

\[(1.5) \quad f'' + (e^{2z} - \nu^2)f = \sigma e^{(\mu+1)z}.\]

(a) Then the general solution \(f(z)\) to (1.5) is given by

\[(1.6) \quad f(z) = AJ_{\nu}(e^z) + BY_{\nu}(e^z) + \sigma S_{\mu, \nu}(e^z),\]

where \(A\) and \(B\) are constants.

(b) The function \(f(z)\) given in (1.6) is subnormal if and only if \(A = B = 0\), and either

\[\mu + \nu = 2p + 1 \quad \text{or} \quad \mu - \nu = 2p + 1\]

holds for a non-negative integer \(p\) and

\[(1.7) \quad S_{\mu, \nu}(\zeta) = \zeta^{\mu-1} \left[ \sum_{k=0}^{p} \frac{(-1)^k c_k}{\zeta^{2k}} \right],\]
where the coefficients \( c_k, k = 0, 1, \ldots, p \), are defined by

\[
(1.8) \quad c_0 = 1, \quad c_k = \prod_{m=1}^{k} [(\mu - 2m + 1)^2 - \nu^2].
\]

The Lommel function \( S_{\mu, \nu}(\zeta) \) (see §3.1) with respect to parameters \( \mu, \nu \) appears to be first studied by Lommel [21] in 1876. It is a particular solution of the non-homogeneous Bessel differential equation

\[
(1.9) \quad \zeta^2 y''(\zeta) + \zeta y'(\zeta) + (\zeta^2 - \nu^2) y(\zeta) = \zeta^{\mu+1}.
\]

The functions \( J_\nu(\zeta) \) and \( Y_\nu(\zeta) \) in (1.6) are the standard Bessel functions of the first and second kinds respectively. They are two linearly independent solutions of the corresponding homogeneous differential equation of (1.9).

When \( \mu = \nu \), we have the following special case:

**Corollary 1.3** Let \( f(z) \) be an entire solution to the differential equation

\[
(1.10) \quad f'' + (e^{2z} - \nu^2)f = \sigma e^{(\nu+1)z} = \frac{e^{(\nu+1)z}}{2^{\nu-1}\sqrt{\pi}\Gamma(\nu + \frac{1}{2})}.
\]

(a) Then the general solution \( f(z) \) to (1.10) is given by

\[
(1.11) \quad f(z) = AJ_\nu(e^z) + BY_\nu(e^z) + K_{\nu}(e^z),
\]

where \( A \) and \( B \) are constants.

(b) The function \( f(z) \) given in (1.11) is subnormal if and only if \( A = B = 0 \), \( \nu = p + \frac{1}{2} \) for a non-negative integer \( p \) and

\[
K_{p+\frac{1}{2}}(\zeta) = \frac{\zeta^{p+\frac{1}{2}}}{2^{p+\frac{1}{2}}\sqrt{\pi}\Gamma(p+1)} \left[ \sum_{k=0}^{p} \frac{(-1)^k c_k}{\zeta^{2k}} \right],
\]

where each of the coefficient \( c_k \) is defined in (1.8).

The function \( K_{\nu}(\zeta) \) that appears in (1.11) above is related to the Struve function of order \( \nu \), \( H_\nu(\zeta) \), which is a particular solution of the differential equation (B.4). Detailed relations amongst \( K_{\nu}(\zeta) \), \( H_\nu(\zeta) \) and \( S_{\nu, \nu}(\zeta) \) will be given in §B.5. The Struve function was first studied by Struve in 1882 (see [37], §10.4). We shall derive analytic continuation formulae for the Lommel functions. Then the corresponding formula for the Struve function follows as a corollary which is stated in §B.5. Besides, we also prove that \( H_\nu(e^z) \) is of infinite order of growth (and in fact, not subnormal) for any choice of \( \nu \) as a corollary of Proposition 4.1.

We shall obtain the Theorem 1.2 as a special case of the following more general result. We first introduce a set of more general coefficients.

Suppose that \( n \) is a positive integer and \( A, B, \nu, L, M, N, \sigma, \sigma_i, \mu_j \) are complex numbers such that \( L, M \) are non-zero and at least one of \( \sigma_j, j = 1, 2, \ldots, n \), being non-zero. We also let
Theorem 1.4 Let \( f(z) \) be an entire solution to the differential equation

\[
f'' + 2N f' + \left[ L^2 M^2 e^{2Mz} + (N^2 - \nu^2 M^2) \right] f = \sum_{j=1}^{n} \sigma_j L^{\mu_j+1} M^2 e^{[M(\mu_j+1)-N]z}.\]

(1.13)

(a) Then the general solution \( f(z) \) to (1.13) is given by

\[
f(z) = e^{-Nz} \left[ AJ_\nu(L e^{Mz}) + BY_\nu(L e^{Mz}) + \sum_{j=1}^{n} \sigma_j S_{\mu_j, \nu}(L e^{Mz}) \right].\]

(1.14)

(b) If all the \( \text{Re}(\mu_j) \) are distinct, then the function \( f(z) \) given in (1.14) is subnormal if and only if \( A = B = 0 \) and for each non-zero \( \sigma_j \), we have either

\[
\mu_j + \nu = 2p_j + 1 \quad \text{or} \quad \mu_j - \nu = 2p_j + 1,
\]

where \( p_j \) is a non-negative integer and

\[
S_{\mu_j, \nu}(\zeta) = \zeta^{\mu_j-1} \left[ \sum_{k=0}^{p_j} \frac{(-1)^k c_k}{\zeta^{2k}} \right],
\]

(1.16)

for \( j \in \{1, 2, \ldots, n\} \), and where each coefficient \( c_k, j \) is defined in (1.12).

Remark 1.5 We note that for all values of \( \mu_j \) and \( \nu \), \( J_\nu(L e^{Mz}) \), \( Y_\nu(L e^{Mz}) \) and \( S_{\mu_j, \nu}(L e^{Mz}) \) are entire functions in the complex \( z \)-plane. Hence it is a single-valued function and so is independent of the branches of \( S_{\mu_j, \nu}(\zeta) \).

Remark 1.6 If we choose \( L = 2, M = \frac{1}{2} \) in Theorem 1.4 and let

\[
P(\zeta) = 2N, \quad Q(\zeta) = \zeta + \left( N^2 - \frac{\nu^2}{4} \right), \quad R(\zeta) = \sum_{j=1}^{n} \sigma_j 2^{\mu_j-1} \frac{\zeta^{\frac{1}{2}(\mu_j+1)-N}},
\]

then \( P(\zeta) \) and \( Q(\zeta) \) are polynomials in \( \zeta \) in (1.1) with \( \deg P < \deg Q \). But we note that \( R(\zeta) \) so chosen may not necessary be a polynomial in \( \zeta \). It follows from Theorem 1.4 that any subnormal solution \( f(z) \) given by (1.14) has the form (1.3). Thus our results (1.14) and (1.16) generalize the Theorem A and give explicit formulae of the subnormal solutions of (1.3) in this particular case.
Unlike the method used in [16] which was based on Nevanlinna’s value distribution theory, our method is different, which is based on special functions, their asymptotic expansions, and the analytic continuation formulae for $S_{\mu, \nu}(\zeta)$ ($\S\, 3.2$ and $\S\, 3.3$). A crucial step in our proof is to apply the Lommel transformation ($\S\, 2$) to transform the equation (1.13) into the equation

\[
\zeta^2 y''(\zeta) + \zeta y'(\zeta) + (\zeta^2 - \nu^2) y(\zeta) = \sum_{j=1}^{n} \sigma_j \zeta^{\mu_j + 1}.
\]

We recall from the basic differential equations theory that the general solution of (1.17) is the sum of complementary functions, which are the Bessel functions, and a particular integral where each of these particular integrals satisfies (1.17). Since a particular integral is generally not uniquely determined, so the novelty here is to apply standard asymptotic expansion theory (see [26]) to each branch of $S_{\mu, \nu}(\zeta)$ in order to identify the ones that we need are precisely the particular integral whose modulus decrease to zero and degenerate when $|\zeta| \to +\infty$, except perhaps on the negative real axis of the $\zeta$-plane. It turns out that these are precisely the classical Lommel function $S_{\mu, \nu}(\zeta)$. We then use the inverse Lommel transformation to (1.17) and to recover the results for (1.13).

The crux of the matter lies in the proof of Theorem 1.4, which establishes the fact that the function $S_{\mu, \nu}(e^{Mz})$ is subnormal if and only if either $\mu + \nu$ or $\mu - \nu$ is equal to an odd positive integer.

**Theorem 1.7** Let $S_{\mu, \nu}(\zeta)$ be a Lommel function of an arbitrary branch. Then the entire function $S_{\mu, \nu}(e^{\zeta})$ is of finite order of growth if and only if either $\mu + \nu$ or $\mu - \nu$ is an odd positive integer $2p + 1$ for some non-negative integer $p$. In particular, the entire function $S_{\mu, \nu}(e^{\zeta})$ degenerates into the form (1.7) with $\zeta = e^{\zeta}$.

Thus, Theorem 1.4 characterizes the subnormal solutions found by Gundersen and Steinbart in [16] to be those that correspond to the vanishing of the complementary functions and the reduction of particular integrals (Lommel’s functions) to polynomials in $\zeta$ and $1/\zeta$.

This paper is organized as follows. We introduce the Lommel transformation in $\S\, 2$, and show how to apply it to the equation (1.13) to prove Theorem 1.4(a). In $\S\, 3$, the definitions of the Lommel functions are given. In particular, we derive several new analytic continuation formulae for the Lommel function $S_{\mu, \nu}(\zeta)$ in terms of the Bessel functions of the third kind (i.e., Hankel functions) in the independent variable and also in terms of the Chebyshev polynomials of the second kind $U_m(\cos \zeta)$ but in the parameters. In $\S\, 4$, by applying the asymptotic expansions of $H^{(1)}_{\nu}(\zeta)$, $H^{(2)}_{\nu}(\zeta)$ and $S_{\mu, \nu}(\zeta)$, we can show that several entire functions are not subnormal. The crux of the matter in the proof of Theorem 1.4 that utilizes Theorem 1.7 is to show that the function $S_{\mu, \nu}(e^{Mz})$ is subnormal if and only if either $\mu + \nu$ or $\mu - \nu$ must be an odd positive integer. Although the $S_{\mu, \nu}(e^{Mz})$ is single-valued, the $S_{\mu, \nu}(\zeta)$ so defined is multi-valued in general. The nature of the problem forces us to consider our problem for the Lommel functions for all branches. Unfortunately, we have found that the literature on the analytic continuation of the Lommel functions is inadequate, so that we have derived these new formulae in $\S\, 3$. The details of the proof of Theorem 1.4 is also given in $\S\, 4$. In $\S\, 5$, we prove Theorem 1.7 and a consequence of it will be given. In $\S\, 6$, we establish some analogues of now classical ‘Quantization-type’ theorems (see e.g. [19], Theorem
5.22) for non-homogeneous equations as a corollary to Theorem 1.4. These theorems can be regarded as semi-classical quantization-type results from quantum mechanics viewpoint. They are followed by corresponding examples. It is here that we identify the polynomials in (1.3) correspond to a number of special polynomials: Neumann’s polynomials, Gegenbauer’s polynomials and Schlöfli’s polynomials. The details of these polynomials and Struve’s functions are given in Appendix B.

The analytic continuation formulae and asymptotic expansions of Bessel functions are listed in Appendix A.

2. The Lommel transformations and a proof of Theorem 1.4(a)

Lommel investigated transformations that involve Bessel equations \[20\] in 1868. Our standard references are \[37\], §4.31 and \[11\], p. 13. We mentioned that the same transformations were also considered independently by K. Pearson (see \[37\], p. 98) in 1880. Lommel considered the transformation

\[
(2.1) \quad \zeta = \alpha x^\beta, \quad y(\zeta) = x^\gamma u(x),
\]

where \(x\) and \(u(x)\) are the new independent and dependent variables respectively, \(\alpha, \beta \in \mathbb{C} \setminus \{0\}\) and \(\gamma \in \mathbb{C}\).

We apply this transformation to equation (1.17). It is straightforward to verify that the function \(u\) satisfies the equation

\[
(2.2) \quad x^2 u''(x) + (2\gamma + 1)x u'(x) + \left[\alpha^2 \beta^2 x^{2\beta} + (\gamma^2 - \nu^2 \beta^2)\right] u(x) = \sum_{j=1}^{n} \sigma_j \alpha^{\mu_j+1} \beta^2 x^{\beta(\mu_j+1)-\gamma}
\]

which has \(x^{-\gamma} y(\alpha x^\beta)\) as its general solution. Following the idea in \[5\], we now apply a further change of variable by

\[
x = e^{mz}, \quad f(z) = u(x)
\]

to (2.2), then we have

\[
f'' + 2\gamma m f' + m^2 \left[\alpha^2 \beta^2 e^{2m\beta z} + (\gamma^2 - \nu^2 \beta^2)\right] f = m^2 \sum_{j=1}^{n} \sigma_j \alpha^{\mu_j+1} \beta^2 e^{m[\beta(\mu_j+1)-\gamma]z}.
\]

Choosing \(m = 1\) in (2.3) and then replacing \(\alpha, \beta\) and \(\gamma\) by \(L, M\) and \(N\) respectively, yields (1.13). As we have noted in §1 that the general solution of (1.9) is given by a combination of the Bessel functions of first and second kinds and the Lommel function \(S_{\mu, \nu}(\zeta)\) (see \[11\], §7.5.5), hence the general solution to (1.17) is

\[
y(\zeta) = AJ_{\nu}(\zeta) + BY_{\nu}(\zeta) + \sum_{j=1}^{n} \sigma_j S_{\mu_j, \nu}(\zeta).
\]

Then the general solution of the differential equation (1.13) is therefore given by (1.14). This proves Theorem 1.4(a).
3. New formulae to the Lommel functions

The major part of the proof of our main theorem consists of studying the growth of the composite function $S_{\mu, \nu}(e^z)$ in the complex $z$-plane. However, the growth of $S_{\mu, \nu}(e^z)$ as a subnormal solution must be independent of the different branches of $S_{\mu, \nu}(\zeta)$, so one needs to consider its growth in all such branches. We first note that the Lommel functions have a rather complicated definition with respect to different subscripts (in four different cases) even in the principal branch of $\zeta$. Since we cannot find such analytic continuation formulae for the Lommel functions in the literature in general, and the formulae with respect to the different singular subscripts in particular (i.e., $\mu \pm \nu$ equals an odd negative integer), we shall derive these new continuation formulae in this section. Due to the complicated nature of the Lommel functions with respect to different subscripts in the principal branch, so we make no apology to list some known properties in §3.3 below before we derive new analytic continuation formulae in §3.2 and §3.3. Here our main reference for the Lommel functions are Watson [37], §10.7-10.75 and Lommel [21].

3.1. The definitions of Lommel's functions $s_{\mu, \nu}(\zeta)$ and $S_{\mu, \nu}(\zeta)$. Suppose that $\mu$ and $\nu$ are complex numbers such that none of $\mu + \nu$ and $\mu - \nu$ is an odd negative integer. Standard variation of parameters method applied to the equation (1.9) yields a particular solution

$$s_{\mu, \nu}(\zeta) = \frac{\pi}{2} \left[ Y_\nu(\zeta) \int_0^\zeta t^\mu J_\nu(t) \, dt - J_\nu(\zeta) \int_0^\zeta t^\mu Y_\nu(t) \, dt \right],$$

which gives (see [37], §10.7) raise to the following expansion

$$s_{\mu, \nu}(\zeta) = \sum_{m=0}^{+\infty} \frac{(-1)^m \zeta^{\mu+1+2m}}{[(\mu + 1)^2 - \nu^2][(\mu + 3)^2 - \nu^2] \cdots [(\mu + 2m + 1)^2 - \nu^2]} \frac{\zeta^{\mu+1}}{(\mu - \nu + 1)(\mu + \nu + 1)} \cdot _1 F_2 \left( 1; \frac{1}{2}; \frac{1}{2} \mu - \frac{1}{2} \nu + \frac{3}{2} \cdot \frac{1}{2} \mu + \frac{1}{2} \nu + \frac{3}{2}; \frac{1}{4} \zeta^2 \right).$$

The above series, which begins with the term $\zeta^{\mu+1}$, is convergent for all $\zeta$, provided that none of the parameters $\mu + \nu$ and $\mu - \nu$ is allowed to be an odd negative integer (see [37], §10.7). Otherwise the second and third arguments in the $_1 F_2$ in (3.1) would be meaningless. This explains the restriction on $\mu \pm \nu$ given above (see [37], §10.7). Following Watson, we define another particular solution $S_{\mu, \nu}(\zeta)$ for the equation (1.9), which is also called a Lommel function. It is related to the $s_{\mu, \nu}(\zeta)$ by the following formulae

$$(3.2)\quad S_{\mu, \nu}(\zeta) := s_{\mu, \nu}(\zeta) + \frac{K}{\sin \nu \pi} \left[ J_{-\nu}(\zeta) \cos \left( \frac{\mu - \nu}{2} \pi \right) - J_\nu(\zeta) \cos \left( \frac{\mu + \nu}{2} \pi \right) \right],$$

$$(3.3)\quad := s_{\mu, \nu}(\zeta) + K \left[ J_{\nu}(\zeta) \sin \left( \frac{\mu - \nu}{2} \pi \right) - Y_\nu(\zeta) \cos \left( \frac{\mu - \nu}{2} \pi \right) \right],$$

where

$$K = 2^\mu \Gamma \left( \frac{1}{2} \mu - \frac{1}{2} \nu + \frac{1}{2} \right) \Gamma \left( \frac{1}{2} \mu + \frac{1}{2} \nu + \frac{1}{2} \right),$$
\(\mu, \nu \in \mathbb{C}\). The first definition (3.2) holds in all cases of \(\mu, \nu\) except when \(\nu\) is an integer. The equivalent form (3.3) holds even when \(\nu\) is an integer. So we shall adopt the second form (3.3) as the general definition for the function \(S_{\mu, \nu}(\zeta)\) [37], p. 347.

This second Lommel function \(S_{\mu, \nu}(\zeta)\) so defined has the advantage that it is still meaningful even when either \(\mu + \nu\) or \(\mu - \nu\) is an odd negative integer (see below), while \(s_{\mu, \nu}(\zeta)\) remains undefined in (3.3) for either of these parameter values. Since our solution to the main Theorems will involve the Lommel functions valid for all complex subscripts, so we shall seek a way to define \(S_{\mu, \nu}(\zeta)\) when either \(\mu + \nu\) or \(\mu - \nu\) is an odd negative integer. Since one easily see from the formulae (3.1), (3.2) and (3.4) that \(s_{\mu, \nu}(\zeta)\) is an even function of \(\nu\), it would be sufficient to derive a formula for \(S_{\mu, \nu}(\zeta)\) when \(\mu - \nu\) is an odd negative integer and the way we define it is shown below.

It is known that both the \(s_{\mu, \nu}(\zeta)\) and \(S_{\mu, \nu}(\zeta)\) satisfy the same recurrence relation [37], §10.72 (1) and (6):

\[
S_{\mu+2, \nu}(\zeta) = \zeta^{\mu+1} - [(\mu + 1)^2 - \nu^2]S_{\mu, \nu}(\zeta).
\]

In particular, when \(\mu \pm \nu\) is an odd negative integer, then one may use this recurrence relation repeatedly to define \(S_{\mu, \nu}(\zeta)\). Indeed letting \(\mu = \nu - 2p - 1\) and applying (3.6) as in [37], §10.73 (1) repeatedly yields

\[
S_{\nu-2p-1, \nu}(\zeta) := \sum_{m=0}^{p-1} \frac{(-1)^m \zeta^{-2p+2m}}{2^{m+2}(p)^m(\nu - p)_m} + \frac{(-1)^p S_{\nu-1, \nu}(\zeta)}{2^p p!(1 - \nu)_p},
\]

where \(p\) is a positive integer. Thus the formula (3.6) indicates that it remains to define the \(S_{\nu-1, \nu}(\zeta)\). We distinguish three cases as follows:

(i) If \(-\nu \not\in \{0, 1, 2, \ldots\}\), then we apply L’Hospital’s theorem once to the relation (3.5) and obtain (3.7) that

\[
S_{\nu-1, \nu}(\zeta) = \frac{\zeta^\nu}{4} \sum_{m=0}^{+\infty} \frac{(-1)^m (\frac{\zeta}{2})^{2m}}{m!\Gamma(\nu + m + 1)} [2\log \zeta - A(m)]
- 2^{\nu-2} \pi \Gamma(\nu) Y_\nu(\zeta),
\]

where \(A(m) = 2\log 2 + \psi(\nu + m + 1) + \psi(m + 1)\) and \(\psi\) is the digamma function. For our later applications, let us rewrite the formula (3.7) as

\[
\begin{align*}
S_{\nu-1, \nu}(\zeta) &= \Gamma(\nu) \left[ 2^{\nu-1} J_\nu(\zeta) \log \zeta - 2^{\nu-2} \pi Y_\nu(\zeta) \\
&\quad - \frac{\zeta^\nu}{4} \sum_{m=0}^{+\infty} \frac{(-1)^m (\frac{\zeta}{2})^{2m} A(m)}{m!\Gamma(\nu + m + 1)} \right].
\end{align*}
\]

(ii) If \(\nu = 0\), then we apply L’Hospital’s Theorem twice to the formula

\[
S_{\mu, 0}(\zeta) = \frac{\zeta^{\mu+1} - S_{\mu+2, 0}(\zeta)}{((\mu + 1)^2}.
\]
to obtain (3.7, §10.73 (4))

\[
S_{-1,0}(\zeta) = \frac{1}{2} \sum_{m=0}^{+\infty} \frac{(-1)^m (\frac{1}{2}\zeta)^{2m}}{(m!)^2} \left[ \log \frac{\zeta}{2} - \psi(m + 1) \right]^2 - \frac{1}{2} \psi'(m + 1) + \frac{\pi^2}{4}.
\]

(3.9)

Again for the ease of our applications later, let us rewrite the formula (3.9) as

\[
S_{-1,0}(\zeta) = \frac{1}{2} J_0(\zeta)(\log \zeta)^2 + \frac{1}{2} \sum_{m=0}^{+\infty} \frac{(-1)^m (\frac{1}{2}\zeta)^{2m} B(m)}{(m!)^2} \left( \log 2 + \psi(m + 1) \right) \log \zeta,
\]

(3.10)

where \(B(m) = \left[ \log 2 + \psi(m + 1) \right]^2 - \frac{1}{2} \psi'(m + 1) + \frac{\pi^2}{4}\).

(iii) If \(\nu = -n\) for a positive integer \(n\), then we can apply the formula (see 3.7, §10.72 (7))

\[
S'_\mu, \nu(\zeta) + \frac{\nu}{\zeta} S_{\mu, \nu}(\zeta) = (\mu + \nu - 1)S_{\mu-1, \nu-1}(\zeta)
\]

(3.11)

to obtain the formula (21, Eqn. (XIX), p. 439)

\[
S_{-n-1, -n}(\zeta) = \frac{(-1)^n \zeta^n}{n!} \cdot \frac{d^n}{d(\zeta^n)^n} (S_{-1,0}(\zeta)).
\]

(3.12)

Hence the function \(S_{-n-2p-1, -n}(\zeta)\) can be defined from (3.6), (3.10) and (3.12).

Since the \(S_{-1,0}(\zeta)\) is a solution of the differential equation (1.9) with \(\mu = -1\) and \(\nu = 0\), we deduce from the formula (3.12) that

\[
S_{-n-1, -n}(\zeta) = \frac{(-1)^n \zeta^{-n}}{2^{n+1}} \left[ A_n(\zeta) + B_n(\zeta)S_{-1,0}(\zeta) + \zeta C_n(\zeta)S'_{-1,0}(\zeta) \right],
\]

(3.13)

where \(A_n(\zeta), B_n(\zeta)\) and \(C_n(\zeta)\) are polynomials in \(\zeta\) of degree at most \(n\) such that \(A_1(\zeta) = B_1(\zeta) \equiv 0, C_1(\zeta) \equiv 1\), and when \(n \geq 2\), that they satisfy the following recurrence relations:

\[
A_n(\zeta) = -2(n-1)A_{n-1}(\zeta) + \zeta A'_{n-1}(\zeta) + C_{n-1}(\zeta),
\]

(3.14)

\[
B_n(\zeta) = -2(n-1)B_{n-1}(\zeta) + \zeta B'_{n-1}(\zeta) - \zeta^2 C_{n-1}(\zeta),
\]

\[
C_n(\zeta) = -2(n-1)C_{n-1}(\zeta) + B_{n-1}(\zeta) + \zeta C'_{n-1}(\zeta).
\]

Remark 3.1 We remark about a property of the formula (3.13) that will be used in the proof of Proposition 4.4 as follows: Let \(P(1/\zeta) = a_0 + a_1/\zeta + \cdots + a_n/\zeta^n\) be a
polynomial in $1/\zeta$ of degree $n$. Then we can show by the method of comparing coefficients that $P(1/\zeta)$ does not satisfy the differential equation \[1.10\] when $\mu = -n - 1$ and $\nu = -n$. This shows that we cannot have $B_n(\zeta)S_{-1,0}(\zeta) + \zeta C_n(\zeta)S'_{-1,0}(\zeta) \equiv 0$ for any positive integer $n$ in \[3.10\].

**Remark 3.2** Since it is not straightforward to derive an analytic continuation formula from the formula \[3.10\], we need to further simplify it. In fact, we have

\[
\pi Y_0(\zeta) = 2 \left[ J_0(\zeta) \log \frac{\zeta}{2} - \sum_{m=0}^{+\infty} \frac{(-1)^m (\frac{\zeta}{2})^{2m}}{(m!)^2} \psi(m + 1) \right]
\]

(see [37], Eqn. (2), p. 60 and Eqn. (2), p. 64), so the equation \[3.10\] can be further written as

\[
S_{-1,0}(\zeta) = -\frac{1}{2} J_0(\zeta) (\log \zeta)^2 + \frac{\pi}{2} Y_0(\zeta) \log \zeta + \frac{1}{2} \sum_{m=0}^{+\infty} \frac{(-1)^m (\frac{\zeta}{2})^{2m} B(m)}{(m!)^2}.
\]

We are ready to derive new analytic continuation formulae for the Lommel functions first with respect to regular subscripts in §3.2 (Lemma 3.3 and Theorem 3.4), and then with respect to the singular subscripts in §3.3 (Lemmas 3.5 to 3.10).

3.2. **New analytic continuation formula for $S_{\mu, \nu}(\zeta)$ when none of $\mu \pm \nu$ is an odd negative integer.** The proof of the Theorem 1.4(b) relies heavily on the fact that the function $S_{\mu, \nu}(e^z)$ so defined is independent of a particular branch of $S_{\mu, \nu}(\zeta)$ under consideration. We thus need to derive analytic continuation formulae for all such branches of the function. We first derive the continuation formula with a full proof in Lemma 3.3 below.

**Lemma 3.3** We have

\[
S_{\mu, \nu}(\zeta e^{-\pi i}) = -e^{-\mu \pi i} S_{\mu, \nu}(\zeta) + K_+ H_{\nu}^{(1)}(\zeta),
\]

where $K_+ = Ki[1 + e^{(-\mu + \nu)\pi i}] \cos\left(\frac{\mu + \nu}{2}\pi\right)$, and $K$ is given by \[3.4\].

**Proof.** Let $m \in \mathbb{Z}$, then it is easy to check that

\[
s_{\mu, \nu}(\zeta e^{m \pi i}) = (-1)^m e^{\mu m \pi i} s_{\mu, \nu}(\zeta)
\]

holds. Let $K$ be given by \[3.4\]. It follows from \[3.3\], \[3.10\], \[3.1\] and \[3.2\] with $m = -1$ that

\[
S_{\mu, \nu}(\zeta e^{-\pi i}) = s_{\mu, \nu}(\zeta e^{-\pi i}) + K \left[ \sin\left(\frac{\mu - \nu}{2}\pi\right) J_\nu(\zeta e^{-\pi i}) - \cos\left(\frac{\mu - \nu}{2}\pi\right) Y_\nu(\zeta e^{-\pi i}) \right]
\]

\[
= -e^{-\mu \pi i} s_{\mu, \nu}(\zeta) + K \times \left\{ e^{-\nu \pi i} \sin\left(\frac{\mu - \nu}{2}\pi\right) J_\nu(\zeta) - \cos\left(\frac{\mu - \nu}{2}\pi\right) [e^{\nu \pi i} Y_\nu(\zeta) - 2i \cos(\nu \pi) J_\nu(\zeta)] \right\}.
\]
We now substitute for $s_{\mu, \nu}(\zeta)$ in terms of $S_{\mu, \nu}(\zeta)$ from (3.3) in the above equation to yield an analytic continuation formula so that

\begin{equation}
S_{\mu, \nu}(\zeta e^{-\pi i}) = -e^{-\mu \pi i} S_{\mu, \nu}(\zeta) + K \left\{ \left( e^{-\mu \pi i} + e^{-\nu \pi i} \right) \sin \left( \frac{\mu - \nu}{2} \pi \right) J_{\nu}(\zeta) \right. \\
+ 2i \cos \left( \frac{\mu - \nu}{2} \pi \right) \cos(\nu \pi) J_{\nu}(\zeta) \} \\
- K \left( e^{-\mu \pi i} + e^{\nu \pi i} \right) \cos \left( \frac{\mu - \nu}{2} \pi \right) Y_{\nu}(\zeta).
\end{equation}

Replacing the Bessel functions of the first and second kinds in (3.17) by the Hankel functions (A.3) yields

\begin{equation}
S_{\mu, \nu}(\zeta e^{-\pi i}) = -e^{-\mu \pi i} S_{\mu, \nu}(\zeta) + K_+ H_{\nu}^{(1)}(\zeta) + K_- H_{\nu}^{(2)}(\zeta),
\end{equation}

where

\begin{align*}
K_+ &= K \left[ \left( e^{-\mu \pi i} + e^{-\nu \pi i} \right) \sin \left( \frac{\mu - \nu}{2} \pi \right) + 2i \cos \left( \frac{\mu - \nu}{2} \pi \right) \cos(\nu \pi) \\
&\pm i \left( e^{-\mu \pi i} + e^{\nu \pi i} \right) \cos \left( \frac{\mu - \nu}{2} \pi \right) \right].
\end{align*}

Substituting $\cos \theta = \frac{e^{\theta i} + e^{-\theta i}}{2}$ and $\sin \theta = \frac{e^{\theta i} - e^{-\theta i}}{2i}$ into the above equations yields

\begin{align*}
K_+ &= K i [1 + e^{(-\mu + \nu) \pi i}] \cos \left( \frac{\mu + \nu}{2} \pi \right) \quad \text{and} \quad K_- = 0.
\end{align*}

Let us write for each integer $m$ that

\begin{equation}
U_{m-1}(\cos \zeta) := \sin m\zeta \sin \zeta
\end{equation}

which is the Chebyshev polynomials of the second kind, see [11]. It is a polynomial of $\cos \zeta$ of degree $(m - 1)$. We note the elementary facts that

$U_0(\cos \nu \pi) = 1$, $U_1(0) = 0$, $U_{m-1}(1) = \lim_{\zeta \to 0} \frac{\sin m\zeta}{\sin \zeta} = m$

and

\begin{equation}
U_{m-1}(\cos k \pi) = \lim_{\nu \to k} \frac{\sin m\nu \pi}{\sin \nu \pi} = m(-1)^{(m-1)}
\end{equation}

hold for each integer $k$. Thus we have the following analytic continuation formula:

**Theorem 3.4** Let $m$ be a non-zero integer, and $\mu \pm \nu \neq 2p + 1$ for any integer $p$. 
(a) We have

\[
S_{\mu, \nu}(\zeta e^{-m\pi i}) = (-1)^m e^{-m\pi i} S_{\mu, \nu}(\zeta) + K_+ [P_m(\cos \nu \pi, e^{-\mu \pi i}) H^{(1)}_{\nu}(\zeta) + e^{-\nu \pi i} Q_m(\cos \nu \pi, e^{-\mu \pi i}) H^{(2)}_{\nu}(\zeta)],
\]

where $K_+$ is given in the Lemma \[3.3\]. $P_m(\cos \nu \pi, e^{-\mu \pi i})$ and $Q_m(\cos \nu \pi, e^{-\mu \pi i})$ are rational functions of $\cos \nu \pi$ and $e^{-\mu \pi i}$ given by

\[
P_m(\cos \nu \pi, e^{-\mu \pi i}) = \frac{U_{m-1}(\cos \nu \pi) + e^{-\mu \pi i} U_m(\cos \nu \pi) + (-1)^{m+1} e^{-(m+1)\mu \pi i}}{[1 + e^{-(\mu+\nu)\pi i]}[1 + e^{-(\mu-\nu)\pi i]}},
\]

and

\[
Q_m(\cos \nu \pi, e^{-\mu \pi i}) = \frac{U_{m-2}(\cos \nu \pi) + e^{-\mu \pi i} U_{m-1}(\cos \nu \pi) + (-1)^m e^{-m\mu \pi i}}{[1 + e^{-(\mu+\nu)\pi i]}[1 + e^{-(\mu-\nu)\pi i]}},
\]

where $U_m(\cos \nu \pi)$ is given by \[3.18\].

(b) Furthermore, the coefficients $P_m(\cos \nu \pi, e^{-\mu \pi i})$ and $Q_m(\cos \nu \pi, e^{-\mu \pi i})$ are not identically zero simultaneously for all $\mu, \nu$ and all non-zero integers $m$.

Before proving Theorem \[3.4\], we need the following relations which can be derived easily from the definition \[3.18\]: For any integer $m$, we have

\[
U_{-m}(\cos \nu \pi) = -U_{m-2}(\cos \nu \pi),
\]

\[
U_{-m-2}(\cos \nu \pi) = -U_m(\cos \nu \pi),
\]

\[
U_{m-1}(\cos \nu \pi) + U_m(\cos \nu \pi)U_{-m}(\cos \nu \pi) = 1.
\]

Proof of Theorem \[3.4\]. In fact, we first prove the claim that the formula \[3.20\] holds with the expressions $P_m(\cos \nu \pi, e^{-\mu \pi i})$ and $Q_m(\cos \nu \pi, e^{-\mu \pi i})$ given by

\[
P_m(\cos \nu \pi, e^{-\mu \pi i}) = \begin{cases} 
\sum_{j=0}^{m} (-1)^j e^{-j\mu \pi i} U_{m-j-1}(\cos \nu \pi), & \text{if } m > 0; \\
(-1)^m e^{-m\mu \pi i} [P_m(\cos \nu \pi, e^{-\mu \pi i}) U_{m}(\cos \nu \pi) - Q_m(\cos \nu \pi, e^{-\mu \pi i}) U_{m-1}(\cos \nu \pi)], & \text{if } m < 0,
\end{cases}
\]

and

\[
Q_m(\cos \nu \pi, e^{-\mu \pi i})
\]
We observe that the polynomial (3.18) satisfies the relation

\[
U(\nu \pi) \equiv 1 \quad \text{and} \quad Q(\cos \nu \pi, e^{-\mu \pi i}) \equiv 0
\]

as given by (3.21) and (3.25) respectively, which are trivial rational functions in cos \( \nu \pi \) and \( e^{-\mu \pi i} \).

We note that the analytic continuation formulae (A.6) and (A.7) for \( H^{(1)}(\zeta) \) and \( H^{(2)}(\zeta) \) can be rewritten as

\[
\begin{align*}
H^{(1)}(\zeta e^{\mu \pi i}) &= U_m(\cos \nu \pi) H^{(1)}(\zeta) - e^{-\nu \pi i} U_{m-1}(\cos \nu \pi) H^{(2)}(\zeta), \\
H^{(2)}(\zeta e^{\mu \pi i}) &= U_m(\cos \nu \pi) H^{(2)}(\zeta) + e^{-\nu \pi i} U_{m-1}(\cos \nu \pi) H^{(1)}(\zeta).
\end{align*}
\]

Let us suppose that the formula (3.26) holds for \( m = k \), where \( k \in \mathbb{N} \), i.e.,

\[
S_{\mu, \nu}(\xi e^{-k \pi i}) = (-1)^k e^{-k \mu \pi i} S_{\mu, \nu}(\xi) + K_+ [P_k(\cos \nu \pi, e^{-\mu \pi i}) H^{(1)}(\zeta)] + e^{-\nu \pi i} Q_k(\cos \nu \pi, e^{-\mu \pi i}) H^{(2)}(\zeta).
\]

We observe that the polynomial (3.18) satisfies the relation

\[
U_m(\cos \zeta) = -U_{m-1}(\cos \zeta)
\]

for any integer \( m \). Thus we have by Lemma 3.3, (3.20) and the relation (5.28) that,

\[
S_{\mu, \nu}(\xi e^{-(k+1) \pi i}) = -e^{-\mu \pi i} S_{\mu, \nu}(\xi e^{-k \pi i}) + K_+ [P_k(\cos \nu \pi, e^{-\mu \pi i}) H^{(1)}(\zeta)] + e^{-\nu \pi i} Q_k(\cos \nu \pi, e^{-\mu \pi i}) H^{(2)}(\zeta)
\]

\[
= (-1)^k e^{-k \mu \pi i} S_{\mu, \nu}(\xi) + K_+ [P_k(\cos \nu \pi, e^{-\mu \pi i}) H^{(1)}(\zeta)] + e^{-\nu \pi i} Q_k(\cos \nu \pi, e^{-\mu \pi i}) H^{(2)}(\zeta)
\]

where \( P_{k+1}(\cos \nu \pi, e^{-\mu \pi i}) \) and \( Q_{k+1}(\cos \nu \pi, e^{-\mu \pi i}) \) are expressions matching exactly the formulae (3.24) and (3.25) \( (m > 0) \) respectively. We conclude, by induction, that the formula (3.20) holds for all positive integers \( m \).

The relation (3.28) is obtained by applying an inductive argument to \( Q_{k+1}(\cos \nu \pi, e^{-\mu \pi i}) \) via \( P_k(\cos \nu \pi, e^{-\mu \pi i}) \) and \( Q_k(\cos \nu \pi, e^{-\mu \pi i}) \).
For a negative integer $m$, $-m$ must be positive and then we apply the formula (3.20) for positive $-m$,

\[
S_{\mu, \nu}(\zeta) = (-1)^m e^{-m\mu\pi i} \left\{S_{\mu, \nu}(\zeta e^{m\pi i}) - K_+ \left[ P_{-m}(\cos \nu \pi, e^{-\mu\pi i}) H^{(1)}_{\nu}(\zeta) \right. \right. \\
+ e^{-\nu\pi i} Q_{-m}(\cos \nu \pi, e^{-\mu\pi i}) H^{(2)}_{\nu}(\zeta) \left. \right\}.
\]

Then we replace $\zeta$ by $\zeta e^{-m\pi i}$ in the formula (3.29) to get

\[
\text{(3.30)} \quad S_{\mu, \nu}(\zeta e^{-m\pi i}) = (-1)^m e^{-m\mu\pi i} \left\{S_{\mu, \nu}(\zeta) - K_+ \left[ P_{-m}(\cos \nu \pi, e^{-\mu\pi i}) \times \right. \right. \\
H^{(1)}_{\nu}(\zeta e^{-m\pi i}) + e^{-\nu\pi i} Q_{-m}(\cos \nu \pi, e^{-\mu\pi i}) H^{(2)}_{\nu}(\zeta e^{-m\pi i}) \right\}.
\]

Thus the desired results for (3.24) and (3.25) ($m < 0$) follows from the formula (3.30) and the continuation formulae (3.26) and (3.27) with $\zeta$ replaced by $\zeta e^{-m\pi i}$. This proves our claim.

Now we show that the two formulae (3.24) and (3.25) can be reduced to the formulae (3.21) and (3.22) respectively. We note that the expression of $P_m(\cos \nu \pi, e^{-\mu\pi i})$ when $m > 0$ can be simplified as follows:

\[
P_m(\cos \nu \pi, e^{-\mu\pi i})
\]

\[
= \sum_{j=0}^{m} (-1)^j e^{-j\mu\pi i} U_{m-j-1}(\cos \nu \pi)
\]

\[
= \frac{1}{2i \sin \nu \pi} \sum_{j=0}^{m} (-1)^j e^{-j\mu\pi i} \left[ e^{(m-j)\nu\pi i} - e^{-(m-j)\nu\pi i} \right]
\]

\[
= \frac{1}{2i \sin \nu \pi} \sum_{j=0}^{m} (-1)^j e^{-j(\mu+\nu)\pi i} - \frac{e^{m\nu\pi i}}{2i \sin \nu \pi} \sum_{j=0}^{m} (-1)^j e^{-j(\mu-\nu)\pi i}
\]

\[
= \frac{e^{m\nu\pi i}}{2i \sin \nu \pi} \left[ 1 - (-1)^{m+1} e^{-(m+1)(\mu+\nu)\pi i} \right] - \frac{e^{-m\nu\pi i}}{2i \sin \nu \pi} \left[ 1 - (-1)^{m+1} e^{-(m+1)(\mu-\nu)\pi i} \right]
\]

\[
= \frac{1}{2i \sin \nu \pi} e^{m\nu\pi i} \left[ 1 + e^{-(\mu+\nu)\pi i} \right] \left[ 1 + e^{-(\mu-\nu)\pi i} \right]
\]

\[
\times \left[ e^{2m\nu\pi i} \left[ 1 + e^{-(\mu-\nu)\pi i} \right] \left[ 1 + (-1)^m e^{-(m+1)(\mu+\nu)\pi i} \right]
\]

\[
- \left[ 1 + e^{-(\mu+\nu)\pi i} \right] \left[ 1 + (-1)^m e^{-(m+1)(\mu-\nu)\pi i} \right] \right].
\]

When we expand the products in the numerator in the equation (3.31), the term $(-1)^m e^{m\nu\pi i} e^{-(m+2)\mu\pi i}$ is eliminated and the remaining terms can be simplified to

\[
e^{m\nu\pi i} \left( e^{m\nu\pi i} - e^{-m\nu\pi i} \right) + e^{-\mu\pi i} e^{m\nu\pi i} \left[ e^{(m+1)\nu\pi i} - e^{-(m+1)\nu\pi i} \right]
\]

\[
+ (-1)^m e^{-(m+1)\mu\pi i} e^{m\nu\pi i} \left( e^{-\nu\pi i} - e^{\nu\pi i} \right)
\]

\[
= e^{m\nu\pi i} \cdot 2i \sin \nu \pi + e^{-\mu\pi i} e^{m\nu\pi i} \cdot 2i \sin (m+1)\nu \pi
\]
\[-(1)^m e^{-(m+1)\mu\pi i} e^{m\nu\pi i} \cdot 2i \sin \nu \pi \]
\[= 2i e^{m\nu\pi i} \left[ \sin m\nu \pi + e^{-\mu\pi i} \sin(m+1)\nu \pi - (1)^m e^{-(m+1)\mu\pi i} \sin \nu \pi \right],\]

and thus the equation (3.31) yields the formula (3.21) for all positive integers \(m\).

Since the formulae (3.24) and (3.25) are connected by
\[Q_m(\cos \nu \pi, e^{-\mu\pi i}) = -e^{\mu\pi i} \left[ P_m(\cos \nu \pi, e^{-\mu\pi i}) - U_{m-1}(\cos \nu \pi) \right],\]

so when \(m > 0\), the expression (3.32) can be written, after applying the (3.21), as
\[Q_m(\cos \nu \pi, e^{-\mu\pi i}) = \frac{1}{1 + e^{-(\mu+\nu)\pi i}} \left[ e^{\mu\pi i} + e^{-\nu\pi i} \right] \times \left[ (e^{\nu\pi i} + e^{-\nu\pi i}) U_{m-2}(\cos \nu \pi) - U_m(\cos \nu \pi) \right.
\[+ e^{-\mu\pi i} U_{m-1}(\cos \nu \pi) + (1)^m e^{-m\mu\pi i} \left. \right].\]

Now it follows from the definition (3.18) and the compound angle formulae for sine function that the first two terms in the numerator in the above equation is exactly \(U_{m-2}(\cos \nu \pi)\), thus proving that the formula (3.22) holds when \(m > 0\).

However, when \(m\) is a negative integer, \(-m\) is a positive integer. We substitute \(-m\) into the formulae (3.21) and (3.22) respectively. Substituting the resulting \(P_{-m}(\cos \nu \pi, e^{-\mu\pi i})\) and \(Q_{-m}(\cos \nu \pi, e^{-\mu\pi i})\) into the expression (3.21), we obtain
\[P_m(\cos \nu \pi, e^{-\mu\pi i})
\[= (1)^{-m+1} e^{-m\mu\pi i} \left[ 1 + e^{-(\mu+\nu)\pi i} \right] \times \left[ U_{m-1}(\cos \nu \pi) U_m(\cos \nu \pi) - U_{-m-2}(\cos \nu \pi) U_{m-1}(\cos \nu \pi) \right.
\[+ e^{-\mu\pi i} U_{-m}(\cos \nu \pi) U_m(\cos \nu \pi) - e^{-\mu\pi i} U_{m-1}(\cos \nu \pi) U_{m-1}(\cos \nu \pi) \right.
\[+ (1)^{-m+1} e^{(m+1)\mu\pi i} U_{m-1}(\cos \nu \pi) \left. \right] \times \left[ (e^{\nu\pi i} + e^{-\nu\pi i}) U_m(\cos \nu \pi) + (1)^{m+1} e^{m\mu\pi i} U_{m-1}(\cos \nu \pi) \right].\]

The identities in (3.23) imply that the first two terms and the following two terms in the numerator in the above equation vanish identically and equal to \(e^{-\mu\pi i}\), respectively. But this is exactly the formula (3.21) in the case \(m < 0\). The validity of (3.22), when \(m < 0\), can be obtained similarly. This completes the proof of (a).

In order to prove (b), we note that the result of (a) implies that the expression (3.32) holds for all non-zero integers \(m\). Thus it is easy to deduce from (3.32) that it is impossible for \(P_m(\cos \nu \pi, e^{-\mu\pi i})\) and \(Q_m(\cos \nu \pi, e^{-\mu\pi i})\) to be identically zero simultaneously for all non-zero integers \(m\), thus completing the proof of the Theorem. □

We note that when \(\mu \pm \nu = 2p + 1\) where \(p\) is a non-negative integer, then the constant \(K_t = 0\). In fact, according to the Lemma 3.3 it is easy to see that the continuation formula is given trivially in Remark 3.11.
3.3. New analytic continuation formulae for $S_{\mu, \nu}(\zeta)$ when either $\mu + \nu$ or $\mu - \nu$ is an odd negative integer. We recall that in this case that the $s_{\mu, \nu}(\zeta)$ is undefined, so we cannot apply the definition (3.3). Instead, we shall use the formulae (3.3), (3.13) and (3.15) to obtain analytic continuation formulae for $S_{\nu-1, \nu}(\zeta)$ and then for $S_{\nu-2p-1, \nu}(\zeta)$, where $p$ is a non-negative integer. We shall only deal with the case $\mu - \nu = -2p - 1$ in the following argument, while the remaining case $\mu + \nu = -2p - 1$ can be dealt with similarly with the fact that $S_{\mu, \nu}(\zeta)$ is an even function of $\nu$.

**Lemma 3.5** If $-\nu \not\in \{0, 1, 2, \ldots\}$, then for each integer $m$,

\begin{equation}
S_{\nu-1, \nu}(\zeta e^{-m\pi i}) = e^{-m\pi i}S_{\nu-1, \nu}(\zeta) + K'_+ H^{(1)}_\nu(\zeta) + K'_- H^{(2)}_\nu(\zeta),
\end{equation}

where

\begin{equation}
K'_\pm = 2\nu^{-2}\pi e^{-m\pi i}H(\nu) \left[U_{m-1}(\cos \nu \pi)e^{(m\pm 1)\nu \pi i} - m\right].
\end{equation}

**Proof.** Since $-\nu \not\in \{0, 1, 2, \ldots\}$, it follows from the formulae (3.8), (A.1–A.2) that

\begin{align*}
S_{\nu-1, \nu}(\zeta e^{-m\pi i}) &= 2\nu^{-1}\Gamma(\nu)J_\nu(\zeta e^{-m\pi i})\log(\zeta e^{-m\pi i}) - 2\nu^{-2}\pi \Gamma(\nu)Y_\nu(\zeta e^{-m\pi i}) \\
&= -\frac{1}{4}(e^{-m\pi i})^2\Gamma(\nu)\sum_{k=0}^{\infty} \frac{(-1)^k (e^{-m\pi i})^{2k}A(k)}{k!\Gamma(\nu + k + 1)} \\
&= 2\nu^{-1}\Gamma(\nu)(\log \zeta - m\pi i)e^{-m\pi i}J_\nu(\zeta) \\
&- 2\nu^{-2}\pi \Gamma(\nu)[e^{m\pi i}Y_\nu(\zeta) - 2i \sin(\nu \pi) \cot(\nu \pi)J_\nu(\zeta)] \\
&+ e^{-m\pi i}[S_{\nu-1, \nu}(\zeta) - 2\nu^{-1}\Gamma(\nu)J_\nu(\zeta) \log \zeta + 2\nu^{-2}\pi \Gamma(\nu)Y_\nu(\zeta)] \\
&= e^{-m\pi i}S_{\nu-1, \nu}(\zeta) + 2\nu^{-2}\pi \Gamma(\nu)e^{-m\pi i}Y_\nu(\zeta) \\
&- 2\nu^{-1}\pi m \Gamma(\nu)e^{-m\pi i}J_\nu(\zeta) \\
&- 2\nu^{-2}\pi \Gamma(\nu)[e^{m\pi i}Y_\nu(\zeta) - 2i \sin(\nu \pi) \cot(\nu \pi)J_\nu(\zeta)] \\
&= e^{-m\pi i}S_{\nu-1, \nu}(\zeta) - 2\nu^{-1}\pi \Gamma(\nu)\sin(\nu \pi)Y_\nu(\zeta) \\
&+ 2\nu^{-2}\pi \Gamma(\nu)[\sin(\nu \pi) \cot(\nu \pi) - me^{-m\pi i}]J_\nu(\zeta) \\
&= e^{-m\pi i}S_{\nu-1, \nu}(\zeta) + K'_+ H^{(1)}_\nu(\zeta) + K'_- H^{(2)}_\nu(\zeta),
\end{align*}

where $m$ is an integer and

\[K'_\pm = 2\nu^{-2}\pi \Gamma(\nu)\left\{ \pm \sin(\nu \pi) + i\left[ \sin(\nu \pi) \cot(\nu \pi) - me^{-m\pi i} \right] \right\}.\]

It is a routine verification that the above expression for $K'_\pm$ can be reduced to our desired form (3.3), and thus proving the formula (3.3).

It remains to substitute the above formula for $S_{\nu-1, \nu}(\zeta)$ into (3.3) to obtain a continuation formula of $S_{\nu-2p-1, \nu}(\zeta)$ when $-\nu \not\in \{0, 1, 2, \ldots\}$:

**Lemma 3.6** If $-\nu \not\in \{0, 1, 2, \ldots\}$, then for each integer $m$,

\begin{align*}
S_{\nu-2p-1, \nu}(\zeta e^{-m\pi i}) &= e^{-m\pi i}S_{\nu-2p-1, \nu}(\zeta) + \frac{(-1)^p K'_+}{2\pi \rho(1 - \nu \rho)} H^{(1)}_\nu(\zeta)
\end{align*}
the term of every polynomial
Lemma 3.7

3.35

ν

Lemma 3.8

(3.35)

\[ \nu = 0 \]

in the following results.

Hence the formula (3.35) follows from (A.3).

Suppose that
Lemma 3.9

This is easily obtained by substituting
Proof. We have

\[ S_{-1,0}(\zeta e^{-m\pi i}) = S_{-1,0}(\zeta) + K_n'' H_0^{(1)}(\zeta) + K_n'' H_0^{(2)}(\zeta), \]

where

\[ K_n'' = -\frac{m \pi^2 (m+1)}{4}. \]

Proof. We have \( Y_0(\zeta e^{-m\pi i}) = Y_0(\zeta) - 2m_iJ_0(\zeta) \) from the equation (A.2), so (3.13) gives

\[
S_{-1,0}(\zeta e^{-m\pi i}) = \frac{1}{2} J_0(\zeta)(\log \zeta - m\pi i)^2 + \frac{\pi}{2}[Y_0(\zeta) - 2m_iJ_0(\zeta)](\log \zeta - m\pi i)
+ \frac{1}{2} \sum_{k=0}^{\infty} (-1)^k (\frac{1}{2}\zeta)^{2k} B(k)
+ \frac{m^2 \pi^2}{2} J_0(\zeta) - \frac{m \pi^2}{2} iY_0(\zeta).
\]

Hence the formula (3.35) follows from (A.3).

Lemma 3.8

If \( \nu = 0 \), then for any integer \( m \),

\[ S_{-2p-1,0}(\zeta e^{-m\pi i}) = S_{-2p-1,0}(\zeta) + \frac{(-1)^p K_n''}{2^{2p}(p!)^2} H_0^{(1)}(\zeta) + \frac{(-1)^p K_n''}{2^{2p}(p!)^2} H_0^{(2)}(\zeta). \]

Proof. This is easily obtained by substituting \( \zeta e^{-m\pi i} \) into the equation (3.9) with \( \nu = 0 \) and applying (3.35).

Finally, the case when \( -\nu \in \mathbb{N} \) is now considered in the next two results.

Lemma 3.9

Suppose that \( m \) is any integer. We define \( \delta_m = 1 + (-1)^{m-1} \) and for every polynomial \( P_n(\zeta) \) of degree \( n \), we define \( \tilde{P}_n(\zeta) \) to be the polynomial containing the term of \( P_n(\zeta) \) with odd powers in \( \zeta \) and \( \overline{P}_n(\zeta) := P_n(\zeta) - \delta_m \tilde{P}_n(\zeta) \). If \( \nu = -n \) for a positive integer \( n \), then we have

\[ S_{-2p-1,0}(\zeta e^{-m\pi i}) = S_{-2p-1,0}(\zeta) + \frac{(-1)^p K_n''}{2^{2p}(p!)^2} H_0^{(1)}(\zeta) + \frac{(-1)^p K_n''}{2^{2p}(p!)^2} H_0^{(2)}(\zeta). \]

Proof. This is easily obtained by substituting \( \zeta e^{-m\pi i} \) into the equation (3.9) with \( \nu = 0 \) and applying (3.35).

Finally, the case when \( -\nu \in \mathbb{N} \) is now considered in the next two results.
Proof. Differentiating (3.35) and applying (A.8) yields

\[(3.37) \frac{d}{d\zeta}S_{-1, 0}(\zeta e^{-m\pi i}) = S'_{-1, 0}(\zeta) - K''H_{1}^{(1)}(\zeta) - K''H_{1}^{(2)}(\zeta).\]

We note that it can be derived from the definition easily that for every polynomial \(P_n(\zeta)\) of degree \(n\) and every integer \(m\), we must have

\[(3.38) P_n(\zeta e^{-m\pi i}) = P_n(\zeta) - \delta_m \hat{P}_n(\zeta) = P_n(\zeta).\]

Hence it can be seen without difficulty that the formula (3.36) follows from the formulae (3.13), (3.35), (3.37) and (3.38).

\[\square\]

Now we can substitute the formula (3.36) into the formula (3.6) to obtain a continuation formula of

\[S_{-n-2p-1, -n}(\zeta):\]

Lemma 3.10 If \(\nu = -n\) for a positive integer \(n\), then for any integer \(m\),

\[(3.39) S_{-n-2p-1, -n}(\zeta e^{-m\pi i}) = (-1)^{mn}S_{-n-2p-1, -n}(\zeta) + \frac{(-1)^{(m+1)n+p}}{2^{p+1}n!(1+n)p} \zeta^{-n}\times\]

\[
\left\{- \delta_m [\hat{A}_n(\zeta) + \hat{B}_n(\zeta)S_{-1, 0}(\zeta) + \zeta \hat{C}_n(\zeta)S'_{-1, 0}(\zeta)]
+ \hat{T}_n(\zeta)[K''H_0^{(1)}(\zeta) + K''H_1^{(2)}(\zeta)]
- \zeta \hat{C}_n(\zeta)[K''H_0^{(1)}(\zeta) + K''H_1^{(2)}(\zeta)]\right\}.
\]

3.4. An asymptotic expansion of \(S_{\mu, \nu}(\zeta)\). It is known that when \(\mu \pm \nu\) are not odd positive integers, then \(S_{\mu, \nu}(\zeta)\) has the asymptotic expansion

\[(3.40) S_{\mu, \nu}(\zeta) = \zeta^{\mu-1} \left[ \sum_{k=0}^{p} \frac{(-1)^k c_k}{\zeta^{2k}} \right] + O(\zeta^{\mu-2p-2})\]

for large \(|\zeta|\) and \(|\arg \zeta| < \pi\), where \(p\) is a non-negative integer and the numbers \(c_k\) are the coefficients defined in (1.8). See also [27, §10.75].

Remark 3.11 It is clear that (3.40) is a series in descending powers of \(\zeta\) starting from the term \(\zeta^{\mu-1}\) and (3.40) terminates if one of the numbers \(\mu \pm \nu\) is an odd positive integer. In particular, if \(\mu = \nu = 2p + 1\) for some non-negative integer \(p\), then we have \(K_+ = 0\) in the analytic continuation formula (3.20) and thus, in this degenerate case, the formula (3.20) becomes

\[S_{2p+1, \nu}(\zeta e^{-m\pi i}) = e^{-m\nu \pi i}S_{2p+1, \nu}(\zeta)\]

for every integer \(m\) and \(|\arg \zeta| < \pi\).
3.5. **Linear independence of Lommel’s functions.** We next discuss the linear independence of the Lommel functions \( S_{\mu_j, \nu}(\zeta) \).

**Lemma 3.12** Suppose \( n \geq 2 \), and \( \mu_j \) and \( \nu \) be complex numbers such that \( \text{Re}(\mu_j) \) are all distinct for \( j = 1, 2, \ldots, n \). Then the Lommel functions

\[
S_{\mu_1, \nu}(\zeta), S_{\mu_2, \nu}(\zeta), \ldots, S_{\mu_n, \nu}(\zeta)
\]

are linearly independent.

**Proof.** Let us now assume that the Lommel functions \( S_{\mu_j, \nu}(\zeta), j = 1, 2, \ldots, n \), to be linearly dependent. Then there exist constants \( C_j \) not all zero such that

\[
\sum_{j=1}^{n} C_j S_{\mu_j, \nu}(\zeta) = 0.
\]

We may assume, without loss of generality, that none of the constants \( C_j \) is zero, and that \( \text{Re}(\mu_1) < \text{Re}(\mu_2) < \cdots < \text{Re}(\mu_n) \).

We substitute the asymptotic expansions (3.40) of the Lommel functions into (3.41) and consider only the leading terms there. We deduce

\[
-C_n S_{\mu_n, \nu}(\zeta) = \sum_{j=1}^{n-1} C_j S_{\mu_j, \nu}(\zeta),
\]

and

\[
|C_n||\zeta|^{|\text{Re}(\mu_n)|-1}e^{-\text{Im}(\mu_n)\arg\zeta} \leq \sum_{j=1}^{n-1} |C_j||\zeta|^{|\text{Re}(\mu_j)|-1}e^{-\text{Im}(\mu_j)\arg\zeta},
\]

\[
|C_n||\zeta|^{|\text{Re}(\mu_n)|-1}e^{-\text{Im}(\mu_n)\arg\zeta} \leq (n-1)|C_{n-1}||\zeta|^{|\text{Re}(\mu_{n-1})|1}e^{-\text{Im}(\mu)\arg\zeta},
\]

\[
|C_n|e^{-\text{Im}(\mu_n)\arg\zeta} \leq (n-1)|C_{n-1}||\zeta|^{|\text{Re}(\mu_{n-1})|1}e^{-\text{Im}(\mu)\arg\zeta},
\]

for sufficiently large \( \zeta \) and \( |\arg\zeta| < \pi \), where \( \text{Im}(\mu) = \min\{\text{Im}(\mu_1), \ldots, \text{Im}(\mu_{n-1})\} \). Since \( \text{Re}(\mu_{n-1}) - \text{Re}(\mu_n) < 0 \), the right hand side of the last inequality approaches zero as \( \zeta \to \infty \) in \( |\arg\zeta| < \pi \) which is a contradiction. Hence, our desired result follows.

4. **Proof of Theorem 1.4(b)**

The general form of the solution of (1.13) was already derived in §2.

Let \( y(\zeta) \) be the general solution of (1.14). We shall recall from Theorem 1.4 that \( A, B, L, M, \sigma_j \) and \( \mu_j \) are complex constants, \( L \) and \( M \) are non-zero and at least one of \( \sigma_j \), \( j = 1, 2, \ldots, n \), being non-zero. Thus it follows from the definitions (A.3) that we may rewrite the general solution (2.4) of the equation (1.17) in terms of Hankel’s functions in the form

\[
y(\zeta) = CH^{(1)}_{\nu}(\zeta) + DH^{(2)}_{\nu}(\zeta) + \sum_{j=1}^{n} \sigma_j S_{\mu_j, \nu}(\zeta),
\]
Lemma 4.2

Theorem (4.3) in the two lemmas below.

Proof of Proposition 4.1.

We shall estimate the growth of each of the functions in (4.2) and only if \( C = D = 0 \). Thus the general solution \( f(z) = e^{-Nz}g(Le^{Mz}) \) of (1.13) assumes the form

\[
(4.2) \quad f(z) = e^{-Nz} \left[ CH_\nu^{(1)}(Le^{Mz}) + De^{-Nz}H_\nu^{(2)}(Le^{Mz}) + \sum_{j=1}^{n} \sigma_j S_{\mu_j, \nu}(Le^{Mz}) \right].
\]

To prove Theorem (4.2), we must first show that \( f(z) \) is subnormal, then we have \( C = D = 0 \). The proof of this depends on the following result:

Proposition 4.1 Suppose \( C \) and \( D \) are complex numbers such that \((C, D) \neq (0, 0)\). Then there exists a sequence of complex numbers \( \{z_n\} = \{r_n e^{i\theta_n}\} \) such that \( |z_n| = r_n \to +\infty \) as \( n \to +\infty \), \(-\pi < \arg(Le^{Mz_n}) < \pi\) for all positive integers \( n \) and that the entire function

\[
G(z) = CH_\nu^{(1)}(Le^{Mz}) + DH_\nu^{(2)}(Le^{Mz}) + \sum_{j=1}^{n} \sigma_j S_{\mu_j, \nu}(Le^{Mz})
\]

satisfies the estimate

\[
M(r_n, G) \geq (2^{1/2}-1)R_n^{-1/2}
\]

\[
(4.4) \quad \times \begin{cases} |Ce^{-iz_\nu}|e^{R_n} \left( 1 - \frac{D}{C}e^{i\omega \pi} \right) e^{-2R_n} + o(e^{R_n}), & \text{if } C \neq 0; \\ |De^{iz_\nu}|e^{R_n} + o(e^{R_n}), & \text{if } C = 0; \end{cases}
\]

where \( R_n = \frac{|L|}{\sqrt{2}} \frac{\mu_n}{\nu} \). In particular, the entire function \( \text{4.2} \) is not subnormal.

Proof of Proposition 4.1. We shall estimate the growth of each of the functions in the \( \text{4.3} \) in the two lemmas below.

Lemma 4.2 Suppose \( C \) and \( D \) are complex numbers such that \((C, D) \neq (0, 0)\). Then there exists a sequence \( \{z_n\} = \{r_n e^{i\theta_n}\} \) such that \( r_n \to +\infty \) as \( n \to +\infty \), \(-\pi < \arg(Le^{Mz_n}) < \pi\), where \( \theta_n \) is fixed for all positive integers \( n \), and that the entire function

\[
F(z) = CH_\nu^{(1)}(Le^{Mz}) + DH_\nu^{(2)}(Le^{Mz})
\]

satisfies the estimate \( \text{4.4} \) with \( G \) replaced by \( F \). Hence we have \( \sigma(F) = +\infty \) and the \( F \) is not subnormal from the definition \( \text{1.2} \).

Proof of Lemma 4.2. We consider the growth of \( \text{4.5} \) in the principal branch of \( H_\nu^{(1)}(\zeta) \) and \( H_\nu^{(2)}(\zeta) \).\footnote{The principal branch of \( H_\nu^{(1)}(\zeta) \) and \( H_\nu^{(2)}(\zeta) \) is assumed to be \(-\pi < \arg \zeta < \pi\).} Let
where \(a, b \in (-\pi, \pi]\) and \(r_n \to +\infty\) as \(n \to +\infty\). The idea of proof is to choose suitable sequences \(\{\theta_n\}\) and \(\{r_n\}\) (and hence \(\{z_n\}\)), so that we can apply the asymptotic expansions (A.4) and (A.5) simultaneously to estimate explicitly the growths of \(H_{\nu}^{(1)}(Le^{Mz_n})\) and \(H_{\nu}^{(2)}(Le^{Mz_n})\).

We consider the sequence

\[
\{z_n\} = \{r_n e^{i\theta_n}\},
\]

where \(\theta_n = \frac{\pi}{4} - b\) for all \(n \in \mathbb{N}\) and

\[
r_n = \begin{cases} 
\sqrt{2} & \left(\frac{2\pi}{|M|} - \frac{\pi}{4} - a\right), \quad \text{if } C \neq 0; \\
\sqrt{2} & \left(\frac{2\pi}{|M|} + \frac{\pi}{4} - a\right), \quad \text{if } C = 0.
\end{cases}
\]

Hence a routine computation yields

\[
Le^{Mz_n} = |L| \exp \left\{ |M| r_n \cos(b + \theta_n) + i \left[ a + |M| r_n \sin(b + \theta_n) \right] \right\}
\]

\[
= |L| \exp \left\{ \frac{|M|r_n}{\sqrt{2}} + i \left( a + \frac{|M|r_n}{\sqrt{2}} \right) \right\}
\]

\[
= |L|e^{\frac{2i\pi r_n}{|M|}}e^{(2n\pi x + 2)}
\]

\[
:= \begin{cases} 
R_n(1 - i), \quad \text{if } C \neq 0; \\
R_n(1 + i), \quad \text{if } C = 0;
\end{cases}
\]

and it is easy to see that \(-\pi < \arg(Le^{Mz_n}) < \pi\) for each positive integer \(n\). Now we can apply the asymptotic expansions (A.4) and (A.5) for sufficiently large \(n\) and (4.8) to obtain the following estimates:

\[
F(z_n) = CH_{\nu}^{(1)}(Le^{Mz_n}) + DH_{\nu}^{(2)}(Le^{Mz_n})
\]

\[
= \left( \frac{2}{\pi R_n(1 - i)} \right)^{1/2} \left\{ \frac{C(1 - i)}{\sqrt{2}} e^{-i\frac{2\pi}{R_n(\frac{1 - i}{2})}} e^{iR_n(\frac{1 + i}{2})} \right\}
\]

\[
\times \left[ 1 + \frac{D(1 + i)}{C(1 - i)} e^{2\pi i R_n(1 - i)} + o(e^{-R_n + i R_n}) \right],
\]

\[
(4.9) = \begin{cases} 
\left( \frac{2}{\pi R_n(1 + i)} \right)^{1/2} D(1 + i) e^{i\frac{2\pi}{R_n(\frac{1 + i}{2})}} e^{-iR_n(1 + i)} + o(e^{R_n - i R_n}), \quad \text{if } C \neq 0; \\
\left( \frac{2}{\pi R_n(1 - i)} \right)^{1/2} D(1 - i) e^{i\frac{2\pi}{R_n(\frac{1 - i}{2})}} e^{-iR_n(1 - i)} + o(e^{R_n - i R_n}), \quad \text{if } C = 0.
\end{cases}
\]

Hence
\[ M(r_n, F) \geq |F(z_n)| \]
\[ \geq (2^{1/2} \pi^{-1} r_n^{-1})^{\frac{1}{2}} \]
\[ \times \begin{cases} |C e^{-i \frac{\pi}{4}}| e^{R_n} \left( 1 - \frac{D}{C} e^{i \frac{\pi}{4}} \right) e^{-2R_n} + o(e^{R_n}), & \text{if } C \neq 0; \\
|D e^{-i \frac{\pi}{4}}| e^{R_n} + o(e^{R_n}), & \text{if } C = 0. \end{cases} \]

(4.10)

Hence the estimate (4.10) is our desired result. Clearly the (4.10) also implies that \( \sigma(F) = +\infty \).

The same estimate (4.10) also shows that, as \( n \to +\infty \)
\[ \frac{\log \log e^{R_n}}{r_n} = \frac{\log |L| \sqrt{2} + |M| r_n \sqrt{2}}{r_n} \to |M| \sqrt{2} \neq 0. \]

It follows from the definition (1.2) that \( F(z) \) is not subnormal. This completes the proof of Lemma 4.2. \( \square \)

We next estimate the growth of the Lommel function \( S_{\mu, \nu}(Le^{Mz}) \) on the same sequence \( \{z_n\} \) defined in Lemma 4.2.

**Lemma 4.3** Let \( S_{\mu, \nu}(\cdot) \) be the principal branch of the Lommel function, where \( \mu \pm \nu \) are not odd positive integers, and \( L, M \) are non-zero constants. Then on the sequence (4.6) defined in Lemma 4.2, we have
\[ |S_{\mu, \nu}(Le^{Mz_n})| \sim \left| (Le^{Mz_n})^{\mu-1} \right| = \left| (L e^{\frac{Mz_n}{\sqrt{2}}})^{\text{Re}(\mu)-1} \times e^{\epsilon \pi \text{Im}(\mu)} \right|, \]

where the value of \( \epsilon = \pm 1 \) depends on the sequence (4.7) chosen.

**Proof of Lemma 4.3** It is clear that \( -\pi < \arg(Le^{Mz_n}) < \pi \) for the sequence (4.6) (see also (4.8)). Thus, by choosing \( p = 0 \) in the asymptotic expansion (3.10), we obtain that
\[ |S_{\mu, \nu}(Le^{Mz_n})| \sim \left| (Le^{Mz_n})^{\mu-1} \right| 
= \left| (Le^{Mz_n})^{\text{Re}(\mu)-1} \cdot \left( Le^{Mz_n} \right)^{\text{Im}(\mu)} \right| 
= \left| (L e^{\frac{Mz_n}{\sqrt{2}}})^{\text{Re}(\mu)-1} \cdot \left( Le^{Mz_n} \right)^{\text{Im}(\mu)} \right|. \]

It is clear from (4.8) that
\[ \left| \left( Le^{Mz_n} \right)^{\text{Im}(\mu)} \right| = \left| e^{\text{Im}(\mu) \log|R_n(1+i)|} \right| = \left| e^{-\text{Im}(\mu) \arg(1+i)} \right| = e^{\epsilon \pi \text{Im}(\mu)}, \]

where the value of \( \epsilon = \pm 1 \) depends on the sequence (4.7) such that \( \epsilon = +1 \) if \( C \neq 0 \) and \( \epsilon = -1 \) otherwise. Thus we complete the proof of Lemma 4.3. \( \square \)
We can now prove Proposition 4.1. Let the right hand side of (4.3) be in the principal branch of \( H^{(1)}_{\nu}(\zeta) \), \( H^{(2)}_{\nu}(\zeta) \) and \( S_{\mu_j, \nu}(\zeta) \) for \( j = 1, 2, \ldots, n \). It is obvious that

\[
|G(z)| \geq |F(z)| - \sum_{j=1}^{n}|\sigma_j S_{\mu_j, \nu}(Le^{Mz})|.
\]

If in addition that we let \( z = z_n \) be the sequence (4.6), then the estimates in Lemmata 4.2 and 4.3 clearly imply that \( G(z) \) satisfies the estimate (4.4) for all sufficiently large \( n \) in the principal branch of the functions \( H^{(1)}_{\nu}(\zeta) \), \( H^{(2)}_{\nu}(\zeta) \) and \( S_{\mu_j, \nu}(\zeta) \).

By the similar argument as in the proof of Lemma 4.2, we know that \( G(z) \) is not subnormal. Since we have

\[
f(z) = e^{-Nz} \left[ F(z) + \sum_{j=1}^{n}\sigma_j S_{\mu_j, \nu}(Le^{Mz}) \right] = e^{-Nz}G(z)
\]

and the function \( e^{-Nz} \) is clearly subnormal, the function \( f(z) \) is not subnormal too.

We now continue the proof of Theorem 1.4(b). Since \( f \) is subnormal, so according to the above analysis we must have \( C = D = 0 \). That is, \( f(z) \) must reduce to the following form:

\[
(4.11) \quad f(z) = \sum_{j=1}^{n}\sigma_j e^{-Nz}S_{\mu_j, \nu}(Le^{Mz}).
\]

Since Lemma 3.12 shows that the Lommel functions \( S_{\mu_j, \nu}(\zeta) \), \( j = 1, 2, \ldots, n \), are linearly independent over \( \mathbb{C} \) and not all \( \sigma_i \) are zero, the solution (4.11) is clearly not identically zero.

To complete the proof of Theorem 1.4(b), we also need to prove that when \( \sigma_j \) is non-zero, \( \mu_j \) and \( \nu \) must satisfy either

\[
(4.12) \quad \cos\left(\frac{\mu_j + \nu}{2}\pi\right) = 0 \quad \text{or} \quad 1 + e^{-(\mu_j + \nu)\pi i} = 0,
\]

where \( j \in \{1, 2, \ldots, n\} \). To do so we will need Lemma 4.3 and the following result.

**Proposition 4.4** Let \( p \) be a non-negative integer and let \( \nu \) be an arbitrary complex number such that if \( \nu \) is an integer, then it is not greater than \( p \). Then the entire function \( S_{\nu-2p-1, \nu}(Le^{Mz}) \) is not subnormal.

**Proof of Proposition 4.4** We assume that \( S_{\nu-2p-1, \nu}(Le^{Mz}) \) is subnormal. We recall from the beginning of §3 that its growth must be independent of the different branches of \( S_{\nu-2p-1, \nu}(\zeta) \). Let us distinguish three cases:

---

3The principal branch of each \( S_{\mu_j, \nu}(\zeta) \), \( j = 1, 2, \ldots, n \), is assumed to be \(-\pi < \arg \zeta < \pi\).

4By the first paragraph of §3.3, the Proposition 4.3 is also valid when \( \mu + \nu \) is an odd negative integer.
(i) \(-\nu \not\in \{0, 1, 2, \ldots\}\). It follows from this branches-argument and the Lemma 3.6 that

\[
S_{-2p-1, \nu}(Le^{Mz}e^{-m\pi i}) = e^{-m\nu\pi i}S_{-2p-1, \nu}(Le^{Mz}) \\
+ \frac{(-1)^p K_+'}{2^{2p}p!(1 - \nu)_p} H_1^{(1)}(Le^{Mz}) \\
+ \frac{(-1)^p K_-'}{2^{2p}p!(1 - \nu)_p} H_2^{(2)}(Le^{Mz}),
\]

where \(K_+'\) are given by (3.34), is required to be subnormal for each integer \(m\). We note that the Lommel and Hankel functions on the right side of (4.13) are in their principal branch. Let

\[
G(z) := S_{-2p-1, 0}(Le^{Mz})
\]

in the Proposition 4.1. Then we deduce that this \(G(z)\) also satisfies the estimate (4.4) on the sequence (4.6). Thus it is not subnormal unless \(\frac{K_+'}{(1 - \nu)_p} = 0\) and from which we deduce

\[
\sin(m\nu\pi) = 0 \quad \text{for each integer } m.
\]

If \(\nu = n \in \{1, 2, \ldots, p\}\), then equation (4.14) implies

\[
0 = \lim_{\nu \to n} \frac{\sin(m\nu\pi)}{(1 - \nu)_p} = \frac{m\pi(-1)^{mn}}{-(1-n)(2-n) \cdots (-1)(1)(p-n)}
\]

which is valid only when \(m = 0\), a contradiction. Therefore we must have \(\nu \not\in \{1, 2, \ldots, p\}\) and equation (4.14) implies that \(\sin(m\nu\pi) = 0\) for every integer \(m\). Thus \(\nu\) is an integer and \(\nu = n \geq p + 1\), a contradiction to the assumption. Hence we have \(\frac{K_+''}{(1 - \nu)_p} \neq 0\) and so \(S_{-2p-1, \nu}(Le^{Mz})\) is not subnormal in this case by the Proposition 4.1.

We now consider the second case.

(ii) If \(\nu = n = 0\), then the independence of the branches means that we need to apply the analytic continuation formula in Lemma 5.8 in our consideration instead. Thus,

\[
G(z) := S_{-2p-1, 0}(Le^{Mz}e^{-m\pi i}) \\
= S_{-2p-1, 0}(Le^{Mz}) + \frac{(-1)^p K_+''}{2^{2p}(p!)^2} H_1^{(1)}(Le^{Mz}) \\
+ \frac{(-1)^p K_-''}{2^{2p}(p!)^2} H_2^{(2)}(Le^{Mz})
\]

satisfies the estimate (4.4) on the sequence (4.6), hence it is not subnormal for any integer \(m\). Again the Proposition 4.1 asserts that \(S_{-2p-1, 0}(Le^{Mz})\) is not subnormal unless \(K_+'' = 0\) in (4.15), which implies that \(m = 0\). A contradiction to a free choice of \(m\). Hence we have \(K_+'' \neq 0\) and so \(S_{-2p-1, 0}(Le^{Mz})\) is not subnormal.

The final case is treated as follows:
(iii) Suppose that \( \nu = -k \) for a positive integer \( k \). However, the Proposition 4.1 is not applicable. Instead, we shall note from Lemma 3.10 that if \( -\pi < \text{arg}(Le^{M_z}) < \pi \), then for each integer \( m \),

\[
G(z) := S_{-k-2p-1, -k}(Le^{M_z}e^{-2m\pi i}) = S_{-k-2p-1, -k}(Le^{M_z}) + \frac{(-1)^{k+p}}{2^{k-2p}k!p!(1+k)p}(Le^{M_z})^{-k}H(z),
\]

where the entire function \( H(z) \) is defined by

\[
H(z) := B_k(Le^{M_z})[K''_0H''_0(1)(Le^{M_z}) + K''_0H''_0(2)(Le^{M_z})] - Le^{M_z}C_k(Le^{M_z})[K''_1H''_1(1)(Le^{M_z}) + K''_1H''_1(2)(Le^{M_z})].
\]

Now we shall obtain an estimate of the growth of \( H(z) \) by following the idea of proof of the Proposition 4.1. We first define the polynomials \( D^+_k(\zeta) := B_k(\zeta) \pm iC_k(\zeta) \). Then we must have \( D^+_k(\zeta) \neq 0 \) for \( B_k(\zeta) \equiv C_k(\zeta) \equiv 0 \) otherwise, which is not permitted by the remark following (3.14). We further redefine the sequences (1.6), (1.7) and (1.8) with the constant \( C \) replaced by the polynomial \( D'^+_k(\zeta) \). We let \( d^+_k = \deg D'^+_k(\zeta) \). Here \( 0 \leq d^+_k \leq k + 1 \). Thus it follows from the asymptotic expansions (A.3), (A.6) and (A.8) all on the sequence (1.6) that (4.17) becomes

\[
H(z_n) = \left( \frac{2}{\pi e^{M_z}n} \right)^\frac{1}{2} \left\{ D^+_k(Le^{M_zn})K''_0e^{-i\zeta_0}e^{Le^{M_zn}} \left[ 1 + O((Le^{M_z})^{-1}) \right] + D^+_k(Le^{M_zn})K''_0e^{i\zeta_0}e^{Le^{M_zn}} \left[ 1 + O((Le^{M_z})^{-1}) \right] \right\}
= \left( \frac{2}{\pi(1+i)} \right)^\frac{1}{2} R_n^{-\frac{1}{2}} \left\{ D^+_k(R_n(1+i))K''_0e^{-i\zeta_0}e^{iR_n(1+i)} \left[ 1 + O((R_n(1+i))^{-1}) \right] + D^+_k(R_n(1+i))K''_0e^{i\zeta_0}e^{-iR_n(1+i)} \left[ 1 + O((R_n(1+i))^{-1}) \right] \right\}
= \left\{ \begin{aligned}
&\left( \frac{2}{\pi(1-i)} \right)^\frac{1}{2} R_n^{-\frac{1}{2}} D^+_k(R_n(1-i))K''_0e^{-i\zeta_0}e^{iR_n(1-i)} \times \left[ 1 + O((R_n(1-i))^{-1}) \right], & \text{if } D^+_k \neq 0; \\
&\left( \frac{2}{\pi(1+i)} \right)^\frac{1}{2} R_n^{-\frac{1}{2}} D^+_k(R_n(1+i))K''_0e^{i\zeta_0}e^{-iR_n(1+i)} \times \left[ 1 + O((R_n(1+i))^{-1}) \right], & \text{if } D^+_k \equiv 0.
\end{aligned} \right.
\]

Hence

\[
|H(z_n)| \geq (2^{1/2}e^{-iR_n^{-1}}R_n^{-1})^{\frac{1}{2}} \times \left\{ \begin{aligned}
&|D^+_k(R_n(1-i))K''_0e^{R_n}| - |D^+_k(R_n(1-i))K''_0e^{-R_n} + o(R_n^{d^+_k}e^{R_n})|, & \text{if } D^+_k \neq 0; \\
&|D^+_k(R_n(1+i))K''_0e^{R_n} + o(R_n^{d^+_k}e^{R_n})|, & \text{if } D^+_k \equiv 0;
\end{aligned} \right.
= (2^{1/2}e^{-iR_n^{-1}}R_n^{-1})^{\frac{1}{2}}.
Lemma 4.3 and equation (4.13), (4.15) or (4.16) in the proof of the Proposition contradicts our assumption that similarly and applying the property that each $S_{\mu}$ the case of Theorem 3.4 we rewrite the solution (4.11) as

(4.18)

$$
\begin{align*}
|D_k^+(R_n(1-i))K_n^\mu|e^{R_n} & \left(1 - \frac{|D_k^-(R_n(1-i))K_n^\mu|}{D_k^+(R_n(1-i))K_n^\mu}e^{-2R_n}\right) \\
+ o(R^d_n e^{R_n}), & \text{if } D_k^+ \neq 0; \\
|D_k^-(R_n(1+i))K_n^\mu|e^{R_n} + o(R^d_n e^{R_n}), & \text{if } D_k^+ \equiv 0.
\end{align*}
$$

Therefore it follows from the estimate (1.18) and Lemma 4.3 that the entire function $G(z) = S_{-k-2p-1,-k}(Le^Mz e^{-2m\pi i})$ is not subnormal unless $K^\mu_{\pm} = 0$ in equation (1.16) which implies that $m = 0$, a contradiction to the free choice of $m$ again. Hence we have $K^\mu_{\pm} \neq 0$ and so $S_{-k-2p-1,-k}(Le^Mz)$ is not subnormal. This completes the proof of the Proposition.

\[\square\]

We may now continue the proof of the Theorem 1.4 (b).

To prove the result that $\mu_j$ and $\nu$ must satisfy either one of the equations in (4.12) when $\sigma_j \neq 0$, where $j \in \{1, 2, \ldots, n\}$, we recall from the Remark 1.5 that $S_{\mu_j,\nu}(Le^Mz)$ are entire functions in the $z$-plane and that each $S_{\mu_j,\nu}(Le^Mz)$ ($j = 1, 2, \ldots, n$) is independent of the branches of $S_{\mu_j,\nu}(z)$. This fact allows us to do the following: Let $j$ be an element of the set $\{1, 2, \ldots, n\}$ such that $\sigma_j \neq 0$. For such a fixed $j$, we rewrite the solution (4.11) as

(4.19) 

$$
f(z) = \sigma_j e^{-Nz}S_{\mu_j,\nu}(Le^Mz e^{-m\pi i}) + \sum_{k \neq j} n \sigma_k e^{-Nz}S_{\mu_k,\nu}(Le^Mz),
$$

where the function $S_{\mu_j,\nu}(z)$ is in the branch $-(m+1)\pi < \arg \zeta < -(m-1)\pi$ and the other Lommel functions $S_{\mu,\nu}(\zeta), \ldots, S_{\mu_j-1,\nu}(\zeta), S_{\mu_j+1,\nu}(\zeta), \ldots, S_{\mu,\nu}(\zeta)$ are in the principal branch $-\pi < \arg \zeta < \pi$ and $m$ is an arbitrary but otherwise fixed non-zero integer.

Remark 4.5 We note again that in the following discussion that we only consider the case $\mu_j - \nu = -2p_j - 1$. The other case $\mu_j + \nu = -2p_j - 1$ can be dealt with similarly and applying the property that each $S_{\mu_j,\nu}(\zeta)$ is even in $\nu$.

If $\mu_j - \nu = -2p_j - 1$ for some non-negative integer $p_j$, then it follows from the Lemma 4.3 and equation (4.12) or (4.13) in the proof of the Proposition that $f(z)$ satisfies the estimate (1.2) or (1.19) on the sequence (1.6), thus it contradicts our assumption that $f(z)$ is subnormal. Hence $\mu_j - \nu$, and then $\mu_j + \nu$ by Remark 4.3 cannot be an odd negative integer.

Now we can apply the analytic continuation formula in the Theorem 3.4 with this fixed integer $m$ to get

(4.20) 

$$
S_{\mu_j,\nu}(Le^Mz e^{-m\pi i}) = K_+P_m(\cos \nu \pi, e^{-\mu_j \pi i})H^{(1)}_\nu(Le^Mz) \\
+ K_- e^{-\nu \pi i}Q_m(\cos \nu \pi, e^{-\mu_j \pi i})H^{(2)}_\nu(Le^Mz) \\
+ (-1)^m e^{-\mu_j \pi i}S_{\mu_j,\nu}(Le^Mz),
$$
where $P_m(\cos \nu \pi, e^{-\mu_j \pi i})$ and $Q_m(\cos \nu \pi, e^{-\mu_j \pi i})$ are polynomials as defined in the Theorem 3.4. Then expressions (4.19) and (4.20) give

\[
(4.21) \quad f(z) = K_+ \sigma_j e^{-Nz} P_m(\cos \nu \pi, e^{-\mu_j \pi i}) H^{(1)}_{\nu}(Le^{Mz}) \\
+ K_+ \sigma_j e^{-Nz} e^{-\nu \pi i} Q_m(\cos \nu \pi, e^{-\mu_j \pi i}) H^{(2)}_{\nu}(Le^{Mz}) \\
+ (-1)^m \sigma_j e^{-m\mu_j \pi i} - Nz S_{\mu_j, \nu}(Le^{Mz}) + \sum_{k=1, k \neq j}^{n} \sigma_k e^{-Nz} S_{\mu_k, \nu}(Le^{Mz}).
\]

If either of the coefficients of $H^{(1)}_{\nu}(Le^{Mz})$ and $H^{(2)}_{\nu}(Le^{Mz})$ in the (4.21) is non-zero, then the Proposition 4.1 implies that the entire function $f(z)$ cannot be subnormal, which is impossible. Thus we must have

\[
(4.22) \quad K_+ P_m(\cos \nu \pi, e^{-\mu_j \pi i}) = 0 = K_+ e^{-\nu \pi i} Q_m(\cos \nu \pi, e^{-\mu_j \pi i}).
\]

Now we are ready to solve the equations (4.12), we recall again that the growth of $S_{\mu_j, \nu}(Le^{Mz})$ must be independent of branches which is equivalent to equations (4.22) hold for each integer $m$. It is clear from Theorem 3.4(b) that $P_m(\cos \nu \pi, e^{-\mu_j \pi i})$ and $Q_m(\cos \nu \pi, e^{-\mu_j \pi i})$ cannot be both identically zero with respect to each non-zero integer $m$, this yields from (4.22) that the condition $K_+ = 0$ holds, i.e., when $\sigma_j \neq 0,$

\[
\cos(\mu_j + \nu \pi) = 0 \quad \text{or} \quad 1 + e^{(-\mu_j + \nu) \pi} = 0,
\]

where $j \in \{1, 2, \ldots, n\}.$

We can now complete the proof of Theorem 1.4(b) by considering each equation above individually and obtain the conclusions: either

\[
(1.7) \quad \mu_j + \nu = 2p_j + 1 \quad \text{or} \quad \mu_j - \nu = 2p_j + 1
\]

below for non-negative integers $p_j$ when $\sigma_j \neq 0$, where $j \in \{1, 2, \ldots, n\}.$

Case (i): Suppose that $\cos(\frac{\mu_j + \nu \pi}{2}) = 0$. This equation gives

\[
\mu_j + \nu = 2p_j + 1
\]

for some integer $p_j$. By the paragraph following Remark 3.11, $p_j$ must be a non-negative integer and then the Remark 3.11 implies that the expansion of $S_{\mu_j, \nu}(Le^{Mz})$ terminates and $S_{\mu_j, \nu}(Le^{Mz})/(Le^{Mz})^{\mu_j - 1}$ becomes a polynomial in $Le^{Mz}$ and $1/Le^{Mz}$, as asserted in (1.16).

Case (ii): Suppose that $1 + e^{(-\mu_j + \nu) \pi} = 0$. That is,

\[
\mu_j - \nu = -2p_j - 1
\]

for some integer $p_j$. Since we have shown that $\mu_j - \nu$ cannot be an odd negative integer, it follows that $p_j$ must be negative and so $\mu_j - \nu$ is an odd positive integer. Hence the Remark 3.11 again implies that $S_{\mu_j, \nu}(Le^{Mz})$
terminates and \( S_{\mu_j, \nu}(L^eM^z)/(L^eM^z)^{\mu_j-1} \) becomes a polynomial in \( L^eM^z \) and \( 1/L^eM^z \), as stated in (1.10).

We recall that \( j \) is an arbitrary element in the set \( \{1, 2, \ldots, n\} \) such that \( \sigma_j \neq 0 \). Thus the above argument is valid for each such \( j \) and hence we have the necessary part of the Theorem.

Conversely, suppose \( A = B = 0 \) for \( f(z) \) in (1.14) and either \( \mu_j + \nu = 2p_j + 1 \) or \( \mu_j - \nu = 2p_j + 1 \), \( p_j \) is a non-negative integer, where \( j \in \{1, 2, \ldots, n\} \) with \( \sigma_j \neq 0 \). Then clearly, (according to the Remark 4.5) each \( S_{\mu_j, \nu}(L^eM^z)/(L^eM^z)^{\mu_j-1} \) is a polynomial in \( L^eM^z \) and/or \( 1/L^eM^z \). Hence \( f(z) \) is clearly subnormal. This proves the converse part and so completes the proof of Theorem 1.4(b). \( \square \)

5. Proof of Theorem 1.7 and a consequence

The proof is a direct consequence of the proof to the Theorem 1.4 given in §4. In fact, the Proposition 4.1 asserts that

\[
G(z) = \sum_{j=1}^{n} \sigma_j S_{\mu_j, \nu}(L^eM^z)
\]

whenever \( G(z) \) is of finite order of growth. The argument in remaining proof of the Theorem 1.4 that \( \text{Re}(\mu_j) \) are distinct certainly applies when we have only a single \( S_{\mu, \nu}(L^eM^z) \). Thus we must have either

\[
\mu - \nu = 2p + 1 \quad \text{or} \quad \mu + \nu = 2p + 1
\]

for a non-negative integer \( p \). Conversely, we suppose that \( \mu - \nu = 2p + 1 \). Then the Remark 5.11 gives that

\[
S_{\nu+2p+1, \nu}(L^eM^z)/(L^eM^z)^{\nu+2p}
\]

is the composition of a polynomial and the exponential, and hence it is of finite order of growth. Since the entire function \( (L^eM^z)^{\nu+2p} \) is certainly of finite order of growth, we have the result that the function \( S_{\nu+2p+1, \nu}(L^eM^z) \) is also of finite order of growth. The case when \( \mu + \nu = 2p + 1 \) now follows easily from Remark 4.5. This completes the proof of Theorem 1.7. \( \square \)

Moreover, it follows from (B.5) and (B.6) that for every integer \( m \),

\[
H_{\nu}(e^{m\pi i}e^{-m\pi i}) = (-1)^m e^{-m\pi i} H_{\nu}(e^z)
\]

\[
= (-1)^m e^{-m\pi i} \left[ Y_\nu(e^z) + \frac{2^{1-\nu}}{\sqrt{\pi \Gamma(\nu + \frac{1}{2})}} S_{\nu, \nu}(e^z) \right]
\]

\[
= (-1)^m e^{-m\pi i} \left\{ \frac{1}{2i} \left[ H_{\nu}^{(1)}(e^z) - H_{\nu}^{(2)}(e^z) \right] + \frac{2^{1-\nu}}{\sqrt{\pi \Gamma(\nu + \frac{1}{2})}} S_{\nu, \nu}(e^z) \right\}.
\]

Then the Proposition 4.11 immediately implies the new result:

**Corollary 5.1** Let \( \nu \) be an arbitrary complex number. The composition of the Struve function (irrespective of branches) and the exponential function (which is an entire function) \( H_{\nu}(e^z) \) is of infinite order of growth. In particular, it is not subnormal.
Let \( f(z) \) be an entire function of order \( \sigma \), where \( 0 < \sigma < 1 \). Then it is easy to check by the definitions that the entire function \( g(z) = e^{f(z)} \) is subnormal and has an infinite order of growth. Then Corollary 5.1 shows that the \( H_\nu(e^z) \) grows faster than subnormal solutions.

6. Quantization-type results and examples

Ismail and one of the authors strengthened [8] (announced in [7]) earlier results of Bank, Laine and Langley [2], [3] (see also [33]) that an entire solution \( f \) of either the equation

\[
y'' + e^z y = K y
\]

or

\[
y'' + \left( -\frac{1}{4}e^{-2z} + \frac{1}{2}e^{-z} \right)y = K y
\]

can be solved in terms of Bessel functions and Coulomb Wave functions respectively. Besides, we identify two classes of classical orthogonal polynomials (Bessel and generalized Bessel polynomials respectively) in the explicit representation of solutions under the assumption \( \lambda(f) = \lim_{r \to +\infty} \log n(r)/\log r < +\infty \) (boundary condition).

This also results in a complete determination of the eigenvalues and eigenfunctions of the equations. This is known as the semi-classical quantization in quantum mechanics. Both equations have important physical applications. For example, the Eqn. (6.1) is derived as a reduction of a non-linear Schrödinger equation in a recent study of Benjamin-Feir instability phenomena in deep water in [29], while the second Eqn. (6.2) is a standard classical diatomic model in quantum mechanics introduced by P. M. Morse in 1929 [25]. However, the equation (6.2) also appears as a basic model in the recent \( \mathcal{PT} \)-symmetric quantum mechanics research [40] (see also [4]).

We now consider special cases of the Theorem 1.4 so that the equations (1.13) exhibits a kind of semi-classical quantization phenomenon that usually only applies to homogeneous equations. In particular, these equations admits classical polynomials solutions (Neumann’s polynomials, Gegenbauer’s polynomials, Schlöffen’s polynomials and Struve’s functions) that are related to special functions when the first derivative term in (1.13) is zero.

Suppose that \( L = 2, M = \frac{1}{2}, N = 0 \) and \( n = 1 \) in Theorem 1.4 so that the differential equation (1.13) becomes

\[
f'' + (e^z - K)f = \sigma 2^{\mu - 1}e^{\frac{\nu}{2}(\mu + 1)z},
\]

where \( K = \frac{\nu^2}{4} \).

**Theorem 6.1** Then, in each of the cases below, we have the necessary and sufficient condition on \( K \) that depends on the non-negative integer \( p \) so that the equation

\[
5 \text{ See [31, pp. 1-4] for a historical background of the Morse potential.}
\]
admits a subnormal solution. Furthermore, the forms of the subnormal solutions are given explicitly in Table 1.

Table 1. Special cases of (6.3).

| Cases | Corresponding K | Subnormal solutions |
|-------|-----------------|---------------------|
| (1)   | µ = 1           | p²                  |
|       |                 | 2σe²O₂p(2e²)         |
| (2)   | µ = 0           | (2p + 1)²/4         |
|       |                 | 2σ/(2p + 1)²O₂p+1(2e²) |
| (3)   | µ = -1          | (p + 1)²            |
|       |                 | σ/(4p + 1)S₂p+2(2e²) |
| (4)   | µ = ν           | (2p + 1)²/16        |
|       |                 | σ²p⁻¹/₂√πp! [H_p+½(2e²) − Y_p+½(2e²)] |

Here O₂p(ζ) and O₂p+1(ζ) are the Neumann polynomials of degrees 2p and 2p + 1 respectively; S₂p(ζ) is the Schléfli polynomial and H₁₂(ζ) is the Struve function. Their background information will be given in Appendix B.

**Proof of Theorem 6.1.** We only prove the first case and the other cases can be dealt with similarly. We note that O₂p(ζ) is actually a polynomial in 1/ζ and that the degree of each individual term is odd, so we see that 2σe²O₂p(2e²) indeed has the form e²ζS(ζ) where S(ζ) is a polynomial as asserted in (1.3). According to Theorem 1.4, f(z) is subnormal if and only if µ ± ν = 2p + 1, i.e., ν² = 4p², where p is a non-negative integer and thus

\[ K = \frac{ν²}{4} = p². \]

It follows from (1.1) that

\[ f(z) = σS_{1, 2p}(2e²) = 2σe²O₂p(2e²). \]

**Remark 6.2** In fact, we also have

\[ f(z) = σS_{1, 2p}(2e²) = σe²A_{2p, 0}(2e²), \]

where the A_{2p, 0}(ζ) is Gegenbauer’s polynomial which will be discussed in §B.2.

**Example 6.3** (Even Neumann’s polynomial) Take p = 1 in Theorem 6.1 Case (1), then we have

\[ f(z) = σS_{1, 2}(2e²) = 2σe²O₂(2e²) = σ + σe⁻² \]

which is a subnormal solution of the equation

\[ f'' + (e² - 1)f = σe^z. \]
Example 6.4 (Odd Neumann’s polynomial) Take \( p = 1 \) in Theorem 6.1, Case (2), then we have
\[
f(z) = \sigma S_{0, 3}(2e^{\frac{z}{2}}) = \frac{2\sigma}{3} e^{\frac{z}{2}} O_3(2e^{\frac{z}{2}}) = \frac{\sigma}{2} e^{-\frac{z}{2}} + \sigma e^{-\frac{3z}{2}}
\]
which is a subnormal solution of the equation
\[
f'' + \left( e^z - \frac{9}{4} \right) f = \frac{\sigma}{2} e^{\frac{z}{2}}.
\]

Example 6.5 (Schläfli’s polynomial) Take \( p = 1 \) in Theorem 6.1, Case (3), then we have
\[
f(z) = \sigma S_{-1, 4}(2e^{\frac{z}{2}}) = \frac{\sigma}{8} S_4(2e^{\frac{z}{2}}) = \frac{\sigma}{4} e^{-z} + \frac{3\sigma}{4} e^{-2z}
\]
which is a subnormal solution of the equation
\[
f'' + (e^z - 4) f = \frac{\sigma}{4}.
\]

Example 6.6 (Struve’s function) Take \( p = 1 \) in Theorem 6.1, Case (4), then we have
\[
f(z) = \sigma S_{\frac{3}{4}, \frac{3}{8}}(2e^{\frac{z}{2}}) = \sigma \sqrt{2\pi} \left[ H_{\frac{3}{4}}(2e^{\frac{z}{2}}) - Y_{\frac{3}{8}}(2e^{\frac{z}{2}}) \right] = \sigma \sqrt{2\pi} \left( 1 + \frac{1}{2e^{z/2}} \right)
\]
which is a subnormal solution of the equation
\[
f'' + \left( e^z - \frac{9}{16} \right) f = \sigma \sqrt{2\pi} e^{\frac{z}{2}}.
\]

Remark 6.7 We note that the differential equations in Examples 6.4 and 6.6 are not in the form (1.1) and therefore their corresponding solutions are not in the form (1.3). These give explicit examples to Remark 1.6.

7. Concluding remarks

Semi-explicit representations of special entire solutions for homogeneous second order linear periodic differential equations were obtained by complex analysts using tools from Nevanlinna theory starting from the 1970s (see for example [12], [39], [2]). The main idea has an origin in the Picard\'Borel exceptional value for entire functions and the theories develop quite independent from the classical approaches to Hill’s equations and Mathieu equations ([38]). Many of these equations are related to models in quantum mechanics ([25]) and it is found that the determination of their (discrete) eigenvalues (i.e., their spectrum) can be described by using exponent of convergence of the zeros of solutions, a quantity that is related to Borel exceptional value in classical function theory ([2], [8]). Moreover, one of the authors and Ismail identified an important class of orthogonal polynomials: generalized Bessel polynomials appear in the representation of the special solutions to (1.5) and (1.6) in [38] along with the determination of their eigenvalues. This paper extends, on the one hand, that one can use the Lommel functions and related polynomials (to Bessel functions) to describe the subnormal (special) solutions of (1.5)
first considered by Gundersen and Steinbart [10], and on the other hand, shows that a kind of semi-classical quantization for non-homogeneous equations also exists for [15], [19] and the equations in the Theorem 6.1. In addition, we obtain a number of new analytic continuation formulae for the Lommel functions, and a new property for the Lommel functions (Theorem 1.7). Although the Lommel functions have numerous physical applications as mentioned in the Introduction, to the best of the authors’ knowledge, only few papers have been written to investigate their mathematical properties in the past decades. See for examples [9], [10], [32], [27], [28] and [14], and the references therein.

Although we generally do not have a simple quantum mechanical interpretation for non-homogeneous equations like the equation (1.5), its homogeneous counterpart and the Lommel functions themselves have numerous applications in various branches of physical applications. So it is hoped the results in this paper will be of interest for others in due course.

**Appendix A. Preliminaries on Bessel functions**

Let \( m \) be an integer. We record here the following analytic continuation formulae for the Bessel functions [37], §3.62:

\[
\begin{align*}
J_\nu(\zeta e^{m\pi i}) &= e^{m\nu\pi i}J_\nu(\zeta), \\
Y_\nu(\zeta e^{m\pi i}) &= e^{-m\nu\pi i}Y_\nu(\zeta) + 2i \sin(m\nu\pi \cot(\nu\pi))J_\nu(\zeta).
\end{align*}
\]

We recall the *Bessel functions of the third kind of order* \( \nu \) [37], §3.6 are given by

\[
H^{(1)}_\nu(\zeta) = J_\nu(\zeta) + iY_\nu(\zeta), \quad H^{(2)}_\nu(\zeta) = J_\nu(\zeta) - iY_\nu(\zeta).
\]

They are also called the *Hankel functions of order* \( \nu \) of the first and second kinds.

The asymptotic expansions of \( H^{(1)}_\nu(\zeta) \) and \( H^{(2)}_\nu(\zeta) \) are also recorded as follows:

\[
\begin{align*}
\left( \frac{\pi}{2} \right)^{\frac{1}{2}} H^{(1)}_\nu(\zeta) &= e^{i(\zeta - \frac{\nu\pi}{2} + \frac{\pi}{4})} \left[ \sum_{k=0}^{p-1} \frac{(\frac{1}{2} - \nu)_k (\frac{1}{2} + \nu)_k}{k!(2i\zeta)^k} + R^{(1)}_p(\zeta) \right] \\
\left( \frac{\pi}{2} \right)^{\frac{1}{2}} H^{(2)}_\nu(\zeta) &= e^{-i(\zeta - \frac{\nu\pi}{2} - \frac{\pi}{4})} \left[ \sum_{k=0}^{p-1} \frac{(\frac{1}{2} - \nu)_k (\frac{1}{2} + \nu)_k}{k!(-2i\zeta)^k} + R^{(2)}_p(\zeta) \right]
\end{align*}
\]

where \( R^{(1)}_p(\zeta) = O(\zeta^{-p}) \) in \( -\pi < \arg \zeta < 2\pi \);

\[
\begin{align*}
\left( \frac{\pi}{2} \right)^{\frac{1}{2}} H^{(1)}_\nu(\zeta) &= e^{i(\zeta - \frac{\nu\pi}{2} + \frac{\pi}{4})} \left[ \sum_{k=0}^{p-1} \frac{(\frac{1}{2} - \nu)_k (\frac{1}{2} + \nu)_k}{k!(2i\zeta)^k} + R^{(1)}_p(\zeta) \right]
\end{align*}
\]

where \( R^{(2)}_p(\zeta) = O(\zeta^{-p}) \) in \( -2\pi < \arg \zeta < \pi \). See [37], §7.2. As a result, we see that the asymptotic expansions [A.4] and [A.5] are valid *simultaneously* in the range \( -\pi < \arg \zeta < \pi \).

Now we deduce from [A.1] and [A.2] the analytic continuation formulae for \( H^{(1)}_\nu(\zeta) \) and \( H^{(2)}_\nu(\zeta) \) [37], §3.62:

\[
\begin{align*}
H^{(1)}_\nu(\zeta e^{m\pi i}) &= \frac{\sin(1 - m\nu\pi)}{\sin \nu\pi} H^{(1)}_\nu(\zeta) - \frac{e^{-\nu\pi i} \sin m\nu\pi}{\sin \nu\pi} H^{(2)}_\nu(\zeta), \\
H^{(2)}_\nu(\zeta e^{m\pi i}) &= \frac{e^{\nu\pi i} \sin m\nu\pi}{\sin \nu\pi} H^{(1)}_\nu(\zeta) + \frac{\sin(m + 1)\nu\pi}{\sin \nu\pi} H^{(2)}_\nu(\zeta).
\end{align*}
\]
We note that the right hand sides of (A.6) and (A.7) are the principal branch of the Hankel functions.

Finally, we record the following derivative formulae for the Hankel functions [37], §3.6:

\[
\frac{d}{d\zeta} H_0^{(1)}(\zeta) = -H_1^{(1)}(\zeta), \quad \frac{d}{d\zeta} H_0^{(2)}(\zeta) = -H_1^{(2)}(\zeta).
\]

**Appendix B. Special polynomials and functions**

**B.1. Neumann’s polynomials.** The Neumann polynomials \(O_n(\zeta)\) are defined as the coefficients in the expansion of \(1/\zeta - z\) in terms of Bessel functions \(J_j(z)\) with \(j \geq 0\), see [37], chap. IX and the connection between the Lommel function \(S_{\mu, \nu}(\zeta)\) and Neumann’s polynomials is given by [37], §9.1 and §10.74:

\[
O_n(\zeta) = \frac{1}{2} \sum_{m=0}^{n} \frac{n! (\frac{1}{2} n + \frac{1}{2} m) \cos^{2} \left( \frac{1}{2} (m \pm n) \pi \right)}{\Gamma \left( \frac{1}{2} n - \frac{1}{2} m + 1 \right)} \left( \frac{\zeta}{2} \right)^{-m-1}
\]

\[
= \begin{cases} 
S_{1, 2p}(\zeta), & \text{if } n = 2p; \\
\zeta S_{0, 2p+1}(\zeta), & \text{if } n = 2p + 1.
\end{cases}
\]

**B.2. Gegenbauer’s polynomials.** The Gegenbauer polynomials \(A_{n, \nu}(\zeta)\) appear in the problem of expansion of \(\frac{1}{t - \zeta}\). The formulae connecting Lommel’s function \(S_{\mu, \nu}(\zeta)\) and Gegenbauer’s polynomial \(A_{n, \nu}(\zeta)\) are given by

\[
A_{n, \nu}(\zeta) = \frac{2^{\nu+n} \zeta^{n+1}}{\zeta^{n+1}} \sum_{m=0}^{n} \frac{\Gamma(\nu + n - m)}{m!} \left( \frac{\zeta}{2} \right)^{2m} \cdot \frac{\nu + 2p}{\nu + 1} S_{1-\nu, \nu+2p}(\zeta), \quad \text{if } n = 2p;
\]

\[
= \begin{cases} 
\frac{2^{\nu+1} \Gamma(\nu + p + 1)}{p!} \cdot \frac{\nu + 2p + 1}{\zeta^{1-\nu}} S_{-\nu, \nu+2p+1}(\zeta), & \text{if } n = 2p + 1.
\end{cases}
\]

See [37], §9.2 and §10.74 for details.

**B.3. Schlöfli’s polynomials.** The polynomial is defined by \(S_0(\zeta) = 0\) and

\[
\frac{1}{2} p S_p(\zeta) = \zeta O_p(\zeta) - \cos^{2} \left( \frac{p\pi}{2} \right)
\]

for each positive integer \(p \geq 1\). The polynomials are mainly introduced because they have greater simplicity over Neumann’s polynomials [37], §9.3. The explicit formulae are given by

\[
S_n(\zeta) = \begin{cases} 
\sum_{m=1}^{p} \frac{(p + m - 1)!}{(p - m)!} \left( \frac{\zeta}{2} \right)^{-2m} & , \text{if } n = 2p; \\
\sum_{m=1}^{p} \frac{(p + m)!}{(p - m)!} \left( \frac{\zeta}{2} \right)^{-2m-1} & , \text{if } n = 2p + 1.
\end{cases}
\]
B.4. Lommel’s functions. We briefly discuss the nature of the Lommel function $S_{\mu, \nu}(\zeta)$ that we encountered. We start with an equation of more general form

$$w'' + \left[ \sum_{j=0}^{+\infty} f_j x^j \right] w' + \left[ \sum_{j=0}^{+\infty} g_j x^j \right] w = z^\alpha \left[ \sum_{j=0}^{+\infty} p_j x^j \right],$$

in $|x| > a$ for some $a > 0$. The solution to this equation is given by

$$w(x) = Aw_1(x) + Bw_2(x) + W(x),$$

where $A, B \in \mathbb{C}$. The asymptotic expansions of the complementary functions $w_1$ and $w_2$ are considered in [26], chap. 7. It can be shown ([26], chap. 7) that there exists a particular integral $W$ so that its asymptotic expansion is given by

$$(B.3) \quad W_n(x) = x^\alpha \left[ \sum_{j=0}^{n-1} a_j x^j + O\left(\frac{1}{x^n}\right) \right],$$

as $x \to \infty$ in an unbounded region that depends on the coefficients $f_0$ and $g_0$. We note that there is no claim that such a particular integral is unique. We refer the reader to [26] for the details.

The Lommel function $s_{\mu, \nu}(\zeta)$ given by (3.1) is a $1F_2$ function. The $S_{\mu, \nu}(\zeta)$ defined by (3.2) and (3.3) matches the $W(x)$ mentioned above, as clearly indicated by its asymptotic expansion in (3.40). Thus the Lommel functions contribute to the subnormal solutions that concerns us in this paper.

B.5. Struve’s functions. If we take $n = 1$, we have $\sigma_1 = [2^{\nu-1} \sqrt{\pi} \Gamma(\nu + \frac{1}{2})]^{-1}$ and $\mu = \nu$ in equation (1.17), then $\sigma_1 s_{\nu, \nu}(\zeta)$ is the particular solution of it. Furthermore, it is known that the Struve function $H_\nu(\zeta)$ of order $\nu$ [11], pp. 37-39, also satisfies the same differential equation

$$(B.4) \quad \zeta^2 y''(\zeta) + \zeta y'(\zeta) + (\zeta^2 - \nu^2)y(\zeta) = \sigma_1 \zeta^{\nu+1}.$$

Following Olver’s notation [26], chap. 7, we call

$$K_\nu(\zeta) = H_\nu(\zeta) - Y_\nu(\zeta)$$

and it follows from (3.2) that

$$(B.5) \quad H_\nu(\zeta) = \frac{2^{1-\nu}}{\sqrt{\pi} \Gamma(\nu + \frac{1}{2})} s_{\nu, \nu}(\zeta) = Y_\nu(\zeta) + \frac{2^{1-\nu}}{\sqrt{\pi} \Gamma(\nu + \frac{1}{2})} S_{\nu, \nu}(\zeta).$$

By (B.5), the continuation formula for the Struve function can be determined immediately. In fact, when $\nu \neq n - \frac{1}{2}$ for any non-negative integer $n$, then it is noted easily from (3.10) and the first half of (B.5) that
\[ H_\nu(\zeta e^{-m\pi i}) = (-1)^m e^{-m\nu\pi i} H_\nu(\zeta), \]

where \( m \) is any integer. When \( \nu = -n - \frac{1}{2} \) for a non-negative integer \( n \), then \( 2\nu = -2n - 1 \) is an odd negative integer and so \( s_{\nu,\nu}(\zeta) \) is undefined in this case but the second half of (B.5) gives \( H_{-n-\frac{1}{2}}(\zeta) = Y_{-n-\frac{1}{2}}(\zeta) = (-1)^n J_n(\zeta) \). Then it follows from (A.1) that the analytic continuation formula (B.6) is also valid at \( \nu = -n - \frac{1}{2} \). (We shall note that (B.6) was already given in [37], §10.41 (5).)

It is shown [26], chap. 7 that the function \( K_\nu(\zeta) \) defined above is the unique particular integral of the Struve equation that satisfies (B.3). The expansion terminates if and only if \( \nu \) is half of an odd positive integer. Example 6.6 corresponds to this situation. So the general solution for the equation (B.4) can be written in terms of Bessel functions and \( K_\nu(\zeta) \).

We remark that we can prove a special case directly on (B.4) by appealing to the analytic continuation formula [26], chap. 7, Ex. 15.4

\[ K_\nu(\zeta e^{-\pi i}) = -e^{-\nu\pi i} K_\nu(\zeta) + 2i \cos(\nu\pi) H_\nu^{(1)}(\zeta), \]

which is a special case of Lemma 3.3.

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Note added in Proof of Theorem 3.4 When the authors revised the paper, we realized that the following simple relation holds for \( P_m(\cos \nu \pi, e^{-\mu \pi i}) \) and \( Q_m(\cos \nu \pi, e^{-\mu \pi i}) \) in the Theorem 3.4:

\[ Q_m(\cos \nu \pi, e^{-\mu \pi i}) = P_{m-1}(\cos \nu \pi, e^{-\mu \pi i}), \]

for a non-zero integer \( m \). This simplifies the statement of the Theorem.

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