ON A THEOREM OF CONANT-VOGTMANN

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ABSTRACT. We prove that the graph complex is a strong homotopy Lie (super) bialgebra.

1. INTRODUCTION AND DEFINITIONS

The graph complex was introduced by Kontsevich in [K1] and [K2]. In [CV1], Conant and Vogtmann constructed a new differential, a Lie bracket and a Lie cobracket on the graph complex. They proved that on the subspace of one-particle irreducible graphs, the Lie bracket and the Lie cobracket give a Lie bialgebra structure. In this paper, we prove that the whole graph complex is a strong homotopy Lie (super) bialgebra (cf. [M, §5], [G, §3.2]).

We shall work in the category of super vector spaces \( V = V_+ \oplus V_- \) over a field of characteristic 0. Denote by \( \Pi \) the parity-change functor \( (\Pi V)_\pm = V_{\mp} \).

By a graph, we mean a finite 1-dimensional CW-complex (not necessarily connected) such that the valency of each vertex is at least 3. An orientation or of a graph \( X \) is an ordering of its vertices and a choice of direction on each of its edges (cf. [CV2, Definition 2]). Switching the order of two vertices or reversing the direction of an edge changes the orientation from or to \(-or\). We shall identify an ordering of the vertices with a labeling of the vertices by \( 1, 2, \ldots \).

Let \( \mathfrak{G} \) be the vector space generated by graphs \( (X, or) \), modulo the equivalence relation \( (X, -or) = -(X, or) \). Observe that if \( X \) has an edge-loop, then it is equal to 0 in \( \mathfrak{G} \). Let \( \mathfrak{G}_+ \) (resp. \( \mathfrak{G}_- \)) be the subspace of \( \mathfrak{G} \) spanned by graphs with an even (resp. odd) number of vertices. Clearly, \( \mathfrak{G} = \mathfrak{G}_+ \oplus \mathfrak{G}_- \).

If \( X, Y \in \mathfrak{G} \), then define their product \( X \cdot Y \) to be the disjoint union of \( X \) and \( Y \). The orientation of \( X \cdot Y \) is obtained by adding the number of vertices of \( X \) to the labels of the vertices of \( Y \). The vector space \( \mathfrak{G} \) is a commutative super algebra by extending this product bilinearly to linear combinations of graphs.

Given an edge \( e \) of a graph \( X \in \mathfrak{G} \) such that \( e \) is not an edge-loop, we denote by \( X_e \) the graph obtained from \( X \) by contracting the edge \( e \). The induced orientation on \( X_e \) is defined as follows: if the source vertex of \( e \) is labeled 1 and the target vertex of \( e \) is labeled 2, then the vertex of \( X_e \) formed by the contracted edge \( e \) is labelled 1, and the labeling of other vertices are reduced by 1. If \( e \) is an edge-loop, then we set \( X_e \) to be 0.

Given a half-edge \( h \) of a graph \( X \in \mathfrak{G} \), we denote by \( v(h) \) the vertex of \( X \) attached to \( h \), and by \( \bar{h} \) the half-edge such that \( e(h) := h \cup \bar{h} \) is an edge of \( X \).
Let $h_1, h_2$ be two half-edges such that they belong to two distinct edges of $X$, and choose a representative for the orientation on $X$ so that $v(h_1)$ (resp. $v(h_2)$) is the source vertex of $e(h_1)$ (resp. $e(h_2)$). Assume that $e(h_1)$ and $e(h_2)$ are not edge-loops. Denote by $X(h_1, h_2)$ the graph obtained from $X$ by adding in an edge $e_1$ directed from $v(h_1)$ to $v(h_2)$, adding in an edge $e_2$ directed from $v(h_2)$ to $v(h_1)$, and deleting away the two edges $e(h_1), e(h_2)$. Denote by $h'_i$ ($i = 1, 2$) the half-edge of $X(h_1, h_2)$ such that $e_i = e(h'_i)$ and $e_i$ is directed from $h'_i$ to $\overline{h}'_i$; see following picture:

Finally, denote $X_{h_1, h_2} := X(h_1, h_2)e_1$. If $v$ is a vertex of $X(h_1, h_2)$ attached to $e_1$, we will also denote by $v$ the vertex of $X_{h_1, h_2}$ contracted from the edge $e_1$. (Thus, the vertex of $X_{h_1, h_2}$ contracted from $e_1$ may be denoted by more than one name.) Observe that $X_{h_2, h_1} = X_{h_1, h_2}$. If $e(h_1)$ or $e(h_2)$ is an edge-loop, then we set $X_{h_1, h_2}$ to be 0. (Our notation differs from the one in [CV1], where they write $X_{h_1, h_2}$ for our $X_{h_1, h_2}$.)

For positive integers $m, n \leq 2$, we define an odd linear map $\alpha_{m,n} : \mathfrak{g}^\otimes n \to \mathfrak{g}^\otimes m$ as follows:

\begin{equation}
\alpha_{m,n}(X_1 \otimes \cdots \otimes X_n) := \sum_{h_1, h_2} \sum_{Y_1 \otimes \cdots \otimes Y_m} Y_1 \otimes \cdots \otimes Y_m.
\end{equation}

In (1), the first summation is taken over all ordered pairs of half-edges $h_1, h_2$ of $X_1 \cdots X_n$ such that each graph $X_i$ contains $h_j$ for some $j$, and $h_1, h_2$ belong to distinct edges. The second summation is taken over all $Y_1 \otimes \cdots \otimes Y_m \in \mathfrak{g}^\otimes m$ such that

\begin{equation}
Y_1 \cdots Y_m = (X_1 \cdots X_n)h_1, h_2,
\end{equation}

and each $Y_i$ is a graph containing $v(h'_j)$ for some $j$. If $m$ and $n$ are positive integers such that $m > 2$ or $n > 2$, then we let $\alpha_{m,n} : \mathfrak{g}^\otimes n \to \mathfrak{g}^\otimes m$ be the zero map. The maps $\alpha_{1,1}, \alpha_{1,2},$ and $\alpha_{2,1}$ are, respectively, the differential, Lie bracket, and Lie cobracket constructed in [CV1], but we will not use these facts in the proof of our theorem below. The map $\alpha_{2,2}$ is new.
By a corolla, we mean a vertex with directed half-edges attached to it such that there is at least one incoming half-edge and at least one outgoing half-edge. Let $m, n$ be positive integers. We shall denote by $T(m, n)$ the set consisting of all “flowcharts” $T$ described as follows: $T$ is obtained from two corollas, called $s(T)$ and $t(T)$, by joining an outgoing half-edge of $s(T)$ to an incoming half-edge of $t(T)$, such that the resulting flowchart has $n$ inputs and $m$ outputs, and moreover there is a labeling of the inputs by $1, \ldots, n$ and a labeling of the outputs by $1, \ldots, m$. For example, the elements of $T(2, 2)$ are listed in Figure 1. If $v$ is a corolla, we write $i(v)$ (resp. $o(v)$) for the number of incoming (resp. outgoing) half-edges at $v$. For each flowchart $T \in T(m, n)$, we write

$$\alpha_{o(t(T)), i(t(T))} \circ T \alpha_{o(s(T)), i(s(T))} : G^\otimes n \to G^\otimes m$$

for the composition of $\alpha_{o(t(T)), i(t(T))}$ and $\alpha_{o(s(T)), i(s(T))}$ according to $T$. 

![Figure 1. Elements of $T(2, 2)$.](image-url)
Theorem 3. For each ordered pair of positive integers $m, n$, the map $\alpha_{m,n}$ is commutative and cocommutative, and the following identity holds:

\[
\sum_{T \in T(m,n)} \alpha_{o(t(T)),i(t(T))} \circ_T \alpha_{o(s(T)),i(s(T))} = 0.
\]

In other words, $\Pi G$ is a strong homotopy Lie (super) bialgebra.

We will prove Theorem 3 in section 2. In the appendix at the end of the paper, we give the definition of strong homotopy Lie bialgebra.

Note that the left hand side of (4) is zero if $m$ or $n$ is greater than 3 because $\alpha_{i,j} = 0$ if $i$ or $j$ is greater than 2. Taking $(m, n) = (1, 1), (1, 2), (1, 3), (2, 1), (3, 1)$ in (4), we obtain the results of [CV1] that $\alpha_{1,1}$ is a differential, $\alpha_{1,1}$ is a derivation with respect to $\alpha_{1,2}$, $\alpha_{1,2}$ satisfies the Jacobi identity, $\alpha_{1,1}$ is a coderivation with respect to $\alpha_{2,1}$, and $\alpha_{2,1}$ satisfies the coJacobi identity. Taking $(m, n) = (2, 2), (2, 3), (3, 2)$ and $(3, 3)$, we obtain new identities. In particular, taking $(m, n) = (2, 2)$, we deduce [CV1, Theorem 1] which states that $\Pi G^{1PI}$ is a Lie super bialgebra, where $G^{1PI} \subset G$ is the subspace spanned by one-particle irreducible graphs (i.e. connected graphs which remain connected after the removal of any edge). Indeed, it is plain that $\alpha_{2,2}(X_1 \otimes X_2) = 0$ if $X_1$ or $X_2$ is in $G^{1PI}$.

We remark that in [C], Conant constructed a family of strong homotopy Lie algebra structures and strong homotopy Lie coalgebra structures on graph complexes associated to cyclic operads. We do not know if these structures also form strong homotopy Lie bialgebras.

2. Proof of Theorem 3

The commutativity and cocommutativity of $\alpha_{m,n}$ are clear from (2). We have to prove that (4) holds. To this end, we shall pair the terms which appear in the left hand side of (4) such that the terms in a pair are negative of each other. Roughly speaking, the terms which come from two pairs of half-edges taken in different order have opposite signs. However, this is not always true, as we now illustrate:
In $X_{h_1, h_2}$:

In $(X_{h_1, h_2})_{h_3, h_4}$:

However, in this example, the edge $e(h_1)$ becomes an edge-loop in $X_{h_3, h_4}$ and hence $X_{h_3, h_4}$ is equal to zero. In below, we shall pair the term $(X_{h_1, h_2})_{h_3, h_4}$ with $(X_{h_1, h_2})_{h_3, h_4}$, which is equal to $-(X_{h_1, h_2})_{h_3, h_4}$ (the minus sign is due to orientation).

To make precise how the terms on the left hand side of (4) cancel, we shall define a set $F$ and an involution $\mu : F \to F$. The terms in the left hand side of (4) will be grouped according to $F$, and the terms corresponding to $f \in F$ will cancel with the terms corresponding to $\mu(f) \in F$.

Fix any $m, n \leq 3$ and graphs $X_1, \ldots, X_n \in \mathcal{G}$. We may assume that there is no edge-loop in $X_1 \cdots X_n$. Define $F$ to be the set consisting of all data

$$f := (h_1, h_2, h_3, h_4, U_1, \ldots, U_m)$$

where

- $h_1, h_2$ are half-edges of $X_1 \cdots X_n$ which belong to distinct edges;
- $e(h'_1), e(h'_2)$ are not edge-loops in $(X_1 \cdots X_n)\langle h_1, h_2 \rangle$;
\[ h_3, h_4 \text{ are half-edges of } (X_1 \cdots X_n)_{h_1, h_2} \text{ which belong to distinct edges;} \]
\[ e(h_3), e(h_4) \text{ are not edge-loops in } (X_1 \cdots X_n)_{h_1, h_2}, \text{ and } e(h'_3), e(h'_4) \]
\[ \text{are not edge-loops in } (X_1 \cdots X_n)_{h_1, h_2}(h_3, h_4); \]
\[ \text{defining } S_i (i = 1, \ldots, n) \text{ to be the set of all } j \in \{1, 2, 3, 4\} \text{ such that either} \]
\[ - X_i \text{ contains } h_j, \text{ or} \]
\[ - X_i \text{ contains } h_2, \text{ and } h_j = h'_2; \]
\[ - X_i \text{ contains } h_1, \text{ and } h_j = \overline{h'_2}, \]
\[ \text{we have: each } S_i \text{ is nonempty and } S_1 \cup \ldots \cup S_n = \{1, 2, 3, 4\} \text{ is a disjoint union;} \]
\[ \text{at least one } S_i \text{ which contains } 1 \text{ or } 2 \text{ also contains } 3 \text{ or } 4; \]
\[ \text{each } U_i \text{ is a nonempty subset of } \{1, 2, 3, 4\} \text{ and } U_1 \cup \ldots \cup U_m = \{1, 2, 3, 4\} \text{ is a disjoint union;} \]
\[ \text{at least one } U_i \text{ which contains } 3 \text{ or } 4 \text{ also contains } 1 \text{ or } 2; \]
\[ \text{if } v(h'_j), v(h'_k) \text{ are in the same connected component of} \]
\[ ((X_1 \cdots X_n)_{h_1, h_2})_{h_3, h_4} \]
\[ \text{and } U_i \text{ contains } j, \text{ then } U_i \text{ contains } k. \]

Given \( f \in F \) as above, there is a unique flowchart \( T_f \in T(m, n) \) such that the input of \( T_f \) labelled \( i \) goes into \( s(T_f) \) if and only if \( S_i \) contains 1 or 2, and the output of \( T_f \) labelled \( i \) goes out from \( t(T_f) \) if and only if \( U_i \) contains 3 or 4.

Let \( m, n \leq 3 \). Consider the left hand side of (4) applied to \( X_1 \otimes \cdots \otimes X_n \). For each \((h_1, h_2, h_3, h_4, U_1, \ldots, U_m) \in F\), there is a corresponding term
\[ \sum_{Y_1 \otimes \cdots \otimes Y_m} Y_1 \otimes \cdots \otimes Y_m \]
where the summation is taken over all \( Y_1 \otimes \cdots \otimes Y_m \in \mathfrak{S}^\otimes m \) such that
\[ Y_1 \cdots Y_m = ((X_1 \cdots X_n)_{h_1, h_2})_{h_3, h_4} \text{ and } Y_i \text{ is a graph containing } v(h'_j) \text{ if } U_i \text{ contains } j. \]

We define an involution \( \mu : F \rightarrow F \) by
\[ \mu(h_1, h_2, h_3, h_4, U_1, \ldots, U_m) := (h^\vee_1, h^\vee_2, h^\vee_3, h^\vee_4, U^\vee_1, \ldots, U^\vee_m), \]
\[ U^\vee_i := \{ j \mid \xi(j) \in U_i \}, \quad \xi : \{1, 2, 3, 4\} \rightarrow \{1, 2, 3, 4\}, \]
where \( h^\vee_1, h^\vee_2, h^\vee_3, h^\vee_4 \) and \( U^\vee_1, \ldots, U^\vee_m \) will be defined accordingly in each of the following cases:

Case (i), \( h_3, h_4 \not\subseteq e(h_2) \):
- \( \text{if } e(h_1) \text{ connects } v(h_3) \text{ with } v(h_4), \text{ then} \)
  \[(h^\vee_1, h^\vee_2, h^\vee_3, h^\vee_4) = (\overline{h_1}, \overline{h_2}, h_3, h_4), \quad \xi = \text{Id}; \]
- \( \text{if } e(h_2) \text{ connects } v(h_3) \text{ with } v(h_4), \text{ then} \)
  \[(h^\vee_1, h^\vee_2, h^\vee_3, h^\vee_4) = (h_1, h_2, h_3, h_4), \quad \xi = \text{Id}; \]
Case (ii), \( h_3 = h_2' \):
- if \( v(h_1) = v(h_2) \) and \( v(h_1) = v(h_4) \), then
  \[
  (h_1^y, h_2^y, h_3^y, h_4^y) = (h_1, h_4, h_2), \quad h_4^y = h_2'^y, \quad \xi = \text{Id};
  \]
- if \( v(h_2) = v(h_4) \) and \( v(h_1) = v(h_2) \), then
  \[
  (h_1^y, h_2^y, h_3^y) = (h_4, h_2, h_1), \quad h_4^y = h_2'^y, \quad \xi = \text{Id};
  \]
- if \( v(h_2) = v(h_4) \) and \( v(h_1) = v(h_2) \), then
  \[
  (h_1^y, h_2^y, h_3^y) = (h_1, h_2, h_4), \quad h_3^y = h_2'^y, \quad \xi = \text{Id};
  \]
- if \( v(h_2) = v(h_4) \), \( v(h_1) \neq v(h_2) \) and \( v(h_1) \neq v(h_4) \), then
  \[
  (h_1^y, h_2^y, h_3^y) = (h_4, h_2, h_1), \quad h_4^y = h_2'^y, \quad \xi = 1, \quad \xi(2) = 1, \quad \xi(3) = 3, \quad \xi(4) = 4;
  \]
- otherwise, let
  \[
  (h_1^y, h_2^y, h_3^y) = (h_2, h_4, h_1), \quad h_2^y = h_2'^y, \quad \xi(1) = 2, \quad \xi(2) = 4, \quad \xi(3) = 1, \quad \xi(4) = 4.
  \]

Case (iii), \( h_4 = h_2' \):
- if \( v(h_2) = v(h_3) \) and \( v(h_2) = v(h_1) \), then
  \[
  (h_1^y, h_2^y, h_3^y) = (h_3, h_2, h_1), \quad h_3^y = h_2'^y, \quad \xi = \text{Id};
  \]
- if \( v(h_3) = v(h_1) \) and \( v(h_2) = v(h_1) \), then
  \[
  (h_1^y, h_2^y, h_3^y) = (h_1, h_3, h_2), \quad h_3^y = h_2'^y, \quad \xi = \text{Id};
  \]
- if \( v(h_3) = v(h_1) \) and \( v(h_2) = v(h_3) \), then
  \[
  (h_1^y, h_2^y, h_3^y) = (h_1, h_2, h_3), \quad h_3^y = h_2'^y, \quad \xi = \text{Id};
  \]
- if \( v(h_3) = v(h_1) \), \( v(h_2) \neq v(h_1) \) and \( v(h_2) \neq v(h_3) \), then
  \[
  (h_1^y, h_2^y, h_3^y) = (h_2, h_3, h_1), \quad h_3^y = h_2'^y, \quad \xi(1) = 1, \quad \xi(2) = 3, \quad \xi(3) = 3, \quad \xi(4) = 4;
  \]
- otherwise, let
  \[
  (h_1^y, h_2^y, h_3^y) = (h_3, h_1, h_2), \quad h_3^y = h_2'^y, \quad \xi(1) = 3, \quad \xi(2) = 1, \quad \xi(3) = 1, \quad \xi(4) = 2.
  \]
Case (iv), \( h_3 = \overline{h_2} \):
\[
(\bar{h}_1, \bar{h}_2, \bar{h}_4, \bar{h}_3, U_1', \ldots, U_m') = \mu(h_1, h_2, \bar{h}_4, \bar{h}_3, U_1, \ldots, U_m)
\]
where the right hand side is defined by case (iii).

Case (v), \( h_4 = \overline{h_2} \):
\[
(\bar{h}_1, \bar{h}_2, \bar{h}_4, \bar{h}_3, U_1', \ldots, U_m') = \mu(h_1, h_2, \bar{h}_4, \bar{h}_3, U_1, \ldots, U_m)
\]
where the right hand side is defined by case (ii).

In the above, \( h'_2 \) is always taken to be in \((X_1 \cdots X_n)_{\bar{h}_1', \bar{h}_2} \). It is a straightforward check that \( \mu \) defines a pairing between elements of \( F \), and moreover,
\[
((X_1 \cdots X_n)_{h_1, h_2})_{h_3, h_4} = -((X_1 \cdots X_n)_{\bar{h}_1', \bar{h}_2'})_{h_3', h_4'}.
\]
Hence, the sum in (5) corresponding to \( f \in F \) is the negative of the sum in (5) corresponding to \( \mu(f) \). This completes the proof of Theorem 3.

**Appendix: Strong homotopy Lie bialgebras**

In this appendix, we give the definition of strong homotopy Lie (super) bialgebras.

A strong homotopy Lie bialgebra in the category of super vector spaces over a field \( k \) of characteristic 0 consists of the data of:

- a super vector space \( V = V_+ \oplus V_- \) over \( k \);
- an odd linear map \( \alpha_{m,n} : (\Pi V)^{\otimes n} \to (\Pi V)^{\otimes m} \) for each ordered pair of positive integers \( m, n \).

These data are required to satisfy the following conditions:

- \( \alpha_{m,n} \) is commutative and cocommutative for all \( m, n \);
- one has:
  \[
  \sum_{T \in T(m,n)} \alpha_{o(t(T)),i(t(T))} \circ_T \alpha_{o(s(T)),i(s(T))} = 0
  \]
  for all \( m, n \).

We remind that the set \( T(m, n) \) of flowcharts was defined in Section 1, and for each \( T \in T(m, n) \),
\[
\alpha_{o(t(T)),i(t(T))} \circ_T \alpha_{o(s(T)),i(s(T))} : (\Pi V)^{\otimes n} \to (\Pi V)^{\otimes m}
\]

is the composition of \( \alpha_{o(t(T)),i(t(T))} \) and \( \alpha_{o(s(T)),i(s(T))} \) according to \( T \).

The above definition comes from the theory of Koszul duality of dioperads, cf. [G]. The reader may refer to [M] for further details.

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