Voted Kernel Regularization

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Abstract. This paper presents an algorithm, Voted Kernel Regularization, that provides the flexibility of using potentially very complex kernel functions such as predictors based on much higher-degree polynomial kernels, while benefitting from strong learning guarantees. The success of our algorithm arises from derived bounds that suggest a new regularization penalty in terms of the Rademacher complexities of the corresponding families of kernel maps. In a series of experiments we demonstrate the improved performance of our algorithm as compared to baselines. Furthermore, the algorithm enjoys several favorable properties. The optimization problem is convex, it allows for learning with non-PDS kernels, and the solutions are highly sparse, resulting in improved classification speed and memory requirements.

1. Introduction

The hypothesis returned by learning algorithms such as SVMs [Cortes and Vapnik, 1995] and other algorithms for which the representer theorem holds is a linear combination of functions $K(x, \cdot)$, where $K$ is the kernel function used and $x$ is a training sample. The generalization guarantees for SVMs depend on the sample size and the margin, but also on the complexity of the kernel function $K$ used, measured by its trace [Koltchinskii and Panchenko, 2002].

These guarantees suggest that, for a moderate margin, learning with very complex kernels, such as sums of polynomial kernels of degree up to some large $d$ may lead to overfitting, which frequently is observed empirically. Thus, in practice, simpler kernels are typically used, that is small $d$s for sums of polynomial kernels. On the other hand, to achieve a sufficiently high performance in challenging learning tasks, it may be necessary to augment a linear combination of such functions $K(x, \cdot)$ with a function $K'(x, \cdot)$, where $K'$ is possibly a substantially more complex kernel, such as a polynomial kernel of degree $d' \gg d$. This flexibility is not available when using SVMs or other learning algorithms such as kernel Perceptron [Aizerman et al., 1964, Rosenblatt, 1958] with the same solution form: either a complex kernel function $K'$ is used and then there is risk of overfitting, or a potentially too simple kernel $K$ is used limiting the performance that could be achieved in some tasks.

This paper presents an algorithm, Voted Kernel Regularization, that precisely provides the flexibility of using potentially very complex kernel functions such as predictors based on much higher-degree polynomial kernels, while benefitting from strong learning guarantees. In a series of experiments we demonstrate the improved performance of our algorithm.

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We present data-dependent learning bounds for this algorithm that are expressed in terms of the Rademacher complexities of the reproducing kernel Hilbert spaces (RKHS) of the kernel functions used. These results are based on the framework of Voted Risk Minimization originally introduced by Cortes et al. [2014] for ensemble methods. We further extend these results using local Rademacher complexity analysis to show that faster convergence rates are possible when the spectrum of the kernel matrix is controlled. The success of our algorithm arises from these bounds that suggest a new regularization penalty in terms of the Rademacher complexities of the corresponding families of kernel maps. Therefore, it becomes crucial to have a good estimate of these complexity measures. We provide a thorough theoretical analysis of these complexities for several commonly used kernel classes.

Besides the improved performance and the theoretical guarantees Voted Kernel Regularization admits a number of additional favorable properties. Our formulation leads to a convex optimization problem that can be solved either via Linear Programming or using Coordinate Descent. Voted Kernel Regularization does not require the kernel functions to be positive-definite or even symmetric. This enables the use of much richer families of kernel functions. In particular, some standard distances known not to be PSD such as the edit-distance and many others can be used with this algorithm.

Yet another advantage of our algorithm is that it produces highly sparse solutions providing greater efficiency and less memory needs. In that respect, Voted Kernel Regularization is similar to so-called norm-1 SVM [Vapnik, 1998, Zhu et al., 2003] and Any-Norm-SVM [Dekel and Singer, 2007] which all use a norm-penalty to reduce the number of support vectors. However, to the best of our knowledge these regularization terms on their own has not led to performance improvement over regular SVMs [Zhu et al., 2003, Dekel and Singer, 2007]. In contrast, our experimental results show that Voted Kernel Regularization algorithm can outperform both regular SVM and norm-1 SVM, and at the same time significantly reduce the number of support vectors. In other work hybrid regularization schemes are combined to obtain a performance improvement [Zou, 2007]. Possibly this technique could be applied to our Voted Kernel Regularization algorithm as well resulting in additional performance improvements.

Somewhat related algorithms are learning kernels or multiple kernel learning and has been extensively investigated over the last decade by both algorithmic and theoretical studies [Lanckriet et al., 2004, Argyriou et al., 2005, 2006, Srebro and Ben-David, 2006, Lewis et al., 2006, Zien and Ong, 2007, Micchelli and Pontil, 2005, Jebara, 2004, Bach, 2008, Ong et al., 2005, Ying and Campbell, 2009, Cortes et al., 2010]. In learning kernels, training data is used to select a single kernel out of the family of convex combinations of \( p \) base kernels and to learn a predictor based on just one kernel. In contrast in Voted SVM, every training point can be thought of as representing a different kernel. Another related approach is Ensemble SVM [Cortes et al., 2011], where a predictor for each base kernel is used and these predictors are combined in to define a single predictor, these two tasks being performed in a single stage or in two subsequent stages. The algorithm where the task is performed in a single stage bears the most resemblance with our Voted Kernel Regularization. However the regularization is different and most importantly not capacity-dependent.

The rest of the paper is organized as follows. Some preliminary definitions and notation are introduced in Section 2. The Voted Kernel Regularization algorithm is presented in Section 3 and in Section 4 we provide strong data-dependent learning guarantees for this algorithm showing that it is possible to learn with highly complex kernel classes and yet not overfit. In Section 4, we also prove local complexity bounds that detail how faster convergence rates are possible provided that the spectrum of the kernel matrix is controlled. Section 5 discusses the implementation of the Voted
Kernel Regularization algorithm including optimization procedures and analysis of Rademacher complexities. We conclude with experimental results in Section 6.

2. PRELIMINARIES

Let \( \mathcal{X} \) denote the input space. We consider the familiar supervised learning scenario. We assume that training and test points are drawn i.i.d. according to some distribution \( \mathcal{D} \) over \( \mathcal{X} \times \{ -1, +1 \} \) and denote by \( S = ((x_1, y_1), \ldots, (x_m, y_m)) \) a training sample of size \( m \) drawn according to \( \mathcal{D}^m \).

Let \( \rho > 0 \). For a function \( f \) taking values in \( \mathbb{R} \), we denote by \( R(f) \) its binary classification error, by \( \hat{R}_S(f) \) its empirical error, and by \( \hat{R}_{\rho}(f) \) its empirical margin error for the sample \( S \):

\[
R(f) = \mathbb{E}_{(x, y) \sim \mathcal{D}} [1_{y \cdot f(x) \leq 0}], \quad \hat{R}_S(f) = \mathbb{E}_{(x, y) \sim S} [1_{y \cdot f(x) \leq 0}], \quad \text{and} \quad \hat{R}_{\rho}(f) = \mathbb{E}_{(x, y) \sim S} [1_{y \cdot f(x) \leq \rho}],
\]

where the notation \( (x, y) \sim S \) indicates that \( (x, y) \) is drawn according to the empirical distribution defined by \( S \). We will denote by \( \hat{\mathcal{R}}_S(H) \) the empirical Rademacher complexity of a hypothesis set \( H \) on the set \( S \) of functions mapping \( \mathcal{X} \) to \( \mathbb{R} \), and by \( \mathcal{R}_m(H) \) the Rademacher complexity [Koltchinskii and Panchenko, 2002, Bartlett and Mendelson, 2002]:

\[
\hat{\mathcal{R}}_S(H) = \frac{1}{m} \mathbb{E} \left[ \sup_{h \in H} \sum_{i=1}^m \sigma_i h(x_i) \right], \quad \mathcal{R}_m(H) = \mathbb{E}_{S \sim \mathcal{D}^m} \left[ \hat{\mathcal{R}}_S(H) \right],
\]

where the random variables \( \sigma_i \) are independent and uniformly distributed over \( \{ -1, +1 \} \).

3. THE VOTED KERNEL REGULARIZATION ALGORITHM

In this section, we introduce the Voted Kernel Regularization algorithm. Let \( K_1, \ldots, K_p \) be \( p \) positive semi-definite (PSD) kernel functions with \( \kappa_k = \sup_{x \in \mathcal{X}} \sqrt{K_k(x, x)} \) for all \( k \in [1, p] \). We consider \( p \) corresponding families of functions mapping from \( \mathcal{X} \) to \( \mathbb{R} \), \( H_1, \ldots, H_p \), defined by \( H_k = \{ x \mapsto \pm K_k(x, x') : x' \in \mathcal{X} \} \), where the sign accounts for two possible ways of classifying a point \( x' \in \mathcal{X} \). The general form of a hypothesis \( f \) returned by the algorithm is the following:

\[
f = \sum_{j=1}^m \sum_{k=1}^p \alpha_{k,j} K_k(\cdot, x_j),
\]

where \( \alpha_{k,j} \in \mathbb{R} \) for all \( j \) and \( k \). Thus, \( f \) is a linear combination of hypotheses in \( H_k \)'s. This form with many \( \alpha \)'s per point is distinctly different from that of learning kernels with only one \( \alpha \) per point. Since the families \( H_k \) are symmetric, this linear combination can be made a non-negative combination. Our algorithm consists of minimizing the Hinge loss on the training sample, as with SVMs, but with a different regularization term that tends to penalize hypotheses drawn from more complex \( H_k \)'s more than those selected from simpler ones and to minimize the norm-1 of the coefficients \( \alpha_{k,j} \). Let \( r_k \) denote the empirical Rademacher complexity of \( H_k \): \( r_k = \hat{\mathcal{R}}_S(H_k) \). Then, the following is the objective function of Voted Kernel Regularization:

\[
F(\alpha) = \frac{1}{m} \max_{i=1}^m \left( 0, 1 - y_i \alpha_{k,j} \sum_{j=1}^m r_k + \sum_{j=1}^m r_k + (\lambda r_k + \beta) |\alpha_{k,j}| \right),
\]

where \( \lambda \geq 0 \) and \( \beta \geq 0 \) are parameters of the algorithm. We will adopt the notation \( \Lambda_k = \lambda r_k + \beta \) to simplify the presentation in what follows.

Note that the objective function \( F \) is convex: the Hinge loss is convex thus its composition with an affine function is also convex, which shows that the first term is convex; the second term is
convex as the absolute value terms with non-negative coefficients; and $F$ is convex as the sum of these two convex terms. Thus, the optimization problem admits a global minimum. Voted Kernel Regularization returns the function $f$ defined by (3) with coefficients $\alpha = (\alpha_{k,j})_{k,j}$ minimizing $F$.

This formulation admits several benefits. First, it enables us to learn with very complex hypothesis sets and yet not overfit, thanks to Rademacher complexity-based penalties assigned to coefficients associated to different $H_k$s. We will see later that the algorithm thereby defined benefits from strong learning guarantees. Notice further that the penalties assigned are data-dependent, which is a key feature of the algorithm. Second, observe that the objective function (6) does not require the kernels $K_k$ to be positive-definite or even symmetric. Function $F$ is convex regardless of the kernel properties. This is a significant benefit of the algorithm which enables to extend its use beyond what algorithms such as SVMs require. In particular, some standard distances known not to be PSD such as the edit-distance and many others could be used with this algorithm. Another advantage of this algorithm compared to standard SVM and other $\ell_2$-regularized methods is that $\ell_1$-norm regularization used for Voted Kernel Regularization leads sparse solutions. The solution $\alpha$ is typically sparse, which significantly reduces prediction time and the memory needs.

Note that hypotheses $h \in H_k$ are defined by $h(x) = K_k(x, x')$ where $x'$ is an arbitrary element of the input space $X$. However, our objective only includes those $x_j$ that belong to the observed sample. We show that in the case of a PDS kernel, there is no loss of generality in that as we now show. Indeed, observe that for $x' \in X$ we can write $\Phi_k(x') = w + w^\perp$, where $\Phi_k$ is a feature map associated with the kernel $K_k$ and where $w$ lies in the span of $\Phi_k(x_1), \ldots, \Phi_k(x_m)$ and $w^\perp$ is in orthogonal compliment of this subspace. Therefore, for any sample point $x_i$

$$K_k(x_i, x') = \langle \Phi(x_i), \Phi(x') \rangle_{H_k} = \langle \Phi(x_i), w \rangle_{H_k} + \langle \Phi(x_i), w^\perp \rangle_{H_k}$$

$$= \sum_{j=1}^{m} \beta_j \langle \Phi(x_i), \Phi(x_j) \rangle_{H_k} = \sum_{j=1}^{m} \beta_j K_k(x_i, x_j),$$

which leads to objective (1). Note that since selecting $-K_k(\cdot, x_j)$ with weight $\alpha_{k,j}$ is equivalent to selecting $K_k(\cdot, x_j)$ with $-\alpha_{k,j}$, which accounts for the absolute value on the $\alpha_{k,j}$s in the regularization term.

The Voted Kernel Regularization algorithm has some connections with other algorithms previously described in the literature. In the absence of any regularization, that is $\lambda = 0$ and $\beta = 0$, it reduces to the minimization of the Hinge loss and is therefore of course close to the SVM algorithm [Cortes and Vapnik, 1995]. For $\lambda = 0$, that is when discarding our regularization based on the different complexity of the hypothesis sets, the algorithm coincides with an algorithm originally described by Vapnik [1998][pp. 426-427], later by several other authors starting with [Zhu et al., 2003], and sometimes referred to as the norm-1 SVM.

4. Learning Guarantees

In this section, we provide strong data-dependent learning guarantees for the Voted Kernel Regularization algorithm.

Let $F$ denote $\text{conv}(\bigcup_{k=1}^{p} H_k)$, that is the family of functions $f$ of the form $f = \sum_{t=1}^{T} \alpha_t h_t$, where $\alpha = (\alpha_1, \ldots, \alpha_T)$ is in the simplex $\Delta$ and where, for each $t \in [1, T]$, $H_{k_t}$ denotes the hypothesis set containing $h_t$, for some $k_t \in [1, p]$. Then, the following learning guarantee holds for all $f \in F$ [Cortes et al., 2014].

**Theorem 1.** Assume $p > 1$. Fix $\rho > 0$. Then, for any $\delta > 0$, with probability at least $1 - \delta$ over the choice of a sample $S$ of size $m$ drawn i.i.d. according to $D^m$, the following inequality holds for
all \( f = \sum_{t=1}^{T} \alpha_t h_t \in \mathcal{F} \):

\[
R(f) \leq \hat{R}_{S,\rho}(f) + \frac{4}{\rho} \sum_{t=1}^{T} \alpha_t \mathcal{R}_m(H_{k_t}) + \frac{2}{\rho} \sqrt{\frac{\log p}{m}} + \sqrt{\frac{4}{\rho^2} \log \left( \frac{\rho^2 m}{\log p} \right)} \frac{\log p}{m} + \frac{\log 2}{2m}.
\]

Thus, \( R(f) \leq \hat{R}_{S,\rho}(f) + \frac{4}{\rho} \sum_{t=1}^{T} \alpha_t \mathcal{R}_m(H_{k_t}) + O \left( \sqrt{\frac{\log p}{m} \log \left( \frac{\rho^2 m}{\log p} \right)} \right) \).

Theorem 1 can be used to derive VKR objective and we provide full details of this derivation in Appendix B. Furthermore, the results of Theorem 1 can further be improved using local Rademacher complexity analysis showing that faster rates of convergence are possible.

**Theorem 2.** Assume \( p > 1 \). Fix \( \rho > 0 \). Then, for any \( \delta > 0 \), with probability at least \( 1 - \delta \) over the choice of a sample \( S \) of size \( m \) drawn i.i.d. according to \( \mathcal{D}^m \), the following inequality holds for all \( f = \sum_{t=1}^{T} \alpha_t h_t \in \mathcal{F} \) for any \( K > 1 \):

\[
R(f) = \frac{K}{K-1} \hat{R}_{S,\rho}(f) \leq 6K \frac{1}{\rho} \sum_{t=1}^{T} \alpha_t \mathcal{R}_m(H_{k_t}) + 40 \frac{K \log p}{\rho^2 m} + 5K \frac{\log 2}{m} + 5K \left( \frac{8}{\rho^2} \frac{\log(1 + \frac{K}{K-1})m}{40K \log p} \right) \frac{\log p}{m}.
\]

Thus, for \( K = 2 \), \( R(f) \leq 2\hat{R}_{S,\rho}(f) + \frac{12}{\rho} \sum_{t=1}^{T} \alpha_t \mathcal{R}_m(H_{k_t}) + O \left( \frac{\log p}{\rho^2 m} \log \left( \frac{\rho m}{\log p} \right) + \frac{\log 2}{m} \right) \).

The proof of this result is given in Appendix A. Note that \( O(\log m / \sqrt{m}) \) in Theorem 1 is replaced with \( O(\log m / m) \) in Theorem 2. For full hypothesis classes \( H_k \), \( \mathcal{R}_m(H_k) \) may be on the order of \( O(1/\sqrt{m}) \) and will dominate the bound. However, if we use localized classes \( H_k(r) = \{ h \in H_k : E[h^2] < r \} \) then for certain values of \( r^* \) local Rademacher complexities \( \mathcal{R}_m(H_k(r^*)) \in O(1/m) \) leading to even stronger learning guarantees. Furthermore, this result leads to an extension of Voted Kernel Regularization objective:

\[
F(\alpha) = \frac{1}{m} \sum_{i=1}^{m} \max \left( 0, 1 - y_i y_j \sum_{j=1}^{p} \sum_{k=1}^{m} \alpha_{k,j} K_k(x_i, x_j) \right) + \sum_{j=1}^{p} \sum_{k=1}^{m} (\lambda \mathcal{R}_m(H_k(s)) + \beta) |\alpha_{k,j}|,
\]

which is optimized over \( \alpha \) and parameter \( s \) is set via cross-validation. In Section 5.3, we provide an explicit expression for the local Rademacher complexities of PDS kernel functions.

### 5. Optimization Solutions

In this section, we propose two different algorithmic approaches to solve the optimization problem (1): a linear programming (LP) and a coordinate descent (CD) approach.

#### 5.1. Linear Programming (LP) formulation

This section presents a linear programming approach for solving the Voted Kernel Regularization optimization problem (1). Observe that by introducing slack variables \( \xi \) the optimization can be equivalently written as follows:

\[
\min_{\alpha, \xi} \frac{1}{m} \sum_{i=1}^{m} \xi_i + \sum_{j=1}^{p} \sum_{k=1}^{m} \Lambda_k |\alpha_{k,j}| \quad \text{s.t.} \quad \xi_i \geq 1 - \sum_{j=1}^{p} \sum_{k=1}^{m} \alpha_{k,j} y_i y_j K_k(x_i, x_j), \forall i \in [1, m].
\]
Next, we introduce new variables $\alpha_{k,j}^+ \geq 0$ and $\alpha_{k,j}^- \geq 0$ such that $\alpha_{k,j} = \alpha_{k,j}^+ - \alpha_{k,j}^-$. Then, for any $k$ and $j$, $|\alpha_{k,j}|$ can be rewritten as $|\alpha_{k,j}| \leq \alpha_{k,j}^+ + \alpha_{k,j}^-$. The optimization problem is therefore equivalent to the following:

$$\min_{\alpha^+ \geq 0, \alpha^- \geq 0, \xi} \frac{1}{m} \sum_{i=1}^{m} \xi_i + \sum_{j=1}^{m} \sum_{k=1}^{p} \Lambda_k (\alpha_{k,j}^+ + \alpha_{k,j}^-)$$

s.t. $\xi_i \geq 1 - \sum_{j=1}^{m} \sum_{k=1}^{p} (\alpha_{k,j}^+ - \alpha_{k,j}^-)y_i y_j K_k(x_i, x_j), \forall i \in [1, m],$

since conversely, a solution with $\alpha_{k,j} = \alpha_{k,j}^+ - \alpha_{k,j}^-$ verifies the condition $\alpha_{k,j}^+ = 0$ or $\alpha_{k,j}^- = 0$ for any $k$ and $j$, thus $\alpha_{k,j} = \alpha_{k,j}^+$ when $\alpha_{k,j} \geq 0$ and $\alpha_{k,j} = \alpha_{k,j}^-$ when $\alpha_{k,j} \leq 0$. This is because if $\delta = \min(\alpha_{k,j}^+, \alpha_{k,j}^-) > 0$, then replacing $\alpha_{k,j}^+$ with $\alpha_{k,j}^+ - \delta$ and $\alpha_{k,j}^-$ with $\alpha_{k,j}^- - \delta$ would not affect $\alpha_{k,j}^+ - \alpha_{k,j}^-$ but would reduce $\alpha_{k,j}^+ + \alpha_{k,j}^-$. Note that the resulting optimization problem is an LP problem since the objective function is linear in both $\xi_i$s and $\alpha^+, \alpha^-$, and since the constraints are affine. There is a battery of well-established methods to solve this LP problem including interior-point methods and the simplex algorithm. An additional advantage of this formulation of the Voted Kernel Regularization algorithm is that there is a large number of generic software packages for solving LPs making the Voted Kernel Regularization algorithm easier to implement.

5.2. Coordinate Descent (CD) formulation. An alternative approach for solving the Voted Kernel Regularization optimization problem (1) consists of using a coordinate descent method. The advantage of such a formulation over the LP formulation is that there is no need to explicitly store the whole vector of $\alpha$s but rather only non-zero entries. This enables learning with very large number of base hypotheses including scenarios in which the number of base hypotheses is infinite. The full description of the algorithm is given in Appendix C.

5.3. Complexity penalties. An additional benefit of the learning bounds presented in Section 4 is that they are data-dependent. They are based on the Rademacher complexity $r_k$S of the base hypothesis sets $H_k$, which in some cases can be well estimated from the training sample. Our formulation directly inherits this advantage. However, in certain cases computing or estimating complexities $r_1, \ldots, r_j$ may be costly. In this section, we discuss various upper bounds on these complexities that be can used in practice for efficient implementation of the Voted Kernel Regularization algorithm.

Note that the hypothesis set $H_k = \{x \mapsto \pm K_k(x, x') : x' \in \mathcal{X}\}$ is of course distinct from the RKHS $\mathcal{H}_k$ of the kernel $K_k$. Thus, we cannot use the known upper bound on $\hat{\mathcal{R}}(\mathcal{H}_k)$ to bound $\hat{\mathcal{R}}(H_k)$. Nevertheless our proof of the upper bound is similar and leads to a similar upper bound.

**Lemma 3.** Let $K_k$ be the kernel matrix of the PDS kernel function $K_k$ for the sample $S$ and let $\kappa_k = \sup_{x \in \mathcal{X}} \sqrt{K_k(x, x)}$. Then, the following inequality holds:

$$\hat{\mathcal{R}}(H_k) \leq \frac{\kappa_k \sqrt{\text{Tr}[K_k]}}{m}.$$ 

We present the full proof of this result in Appendix A. Observe that the expression given by the lemma can be precomputed and used as the parameter $r_k$ of the optimization procedure.

The upper bound just derived is not fine enough to distinguish between different normalized kernels since for any normalized kernel $K_k$, $\kappa = 1$ and $\text{Tr}[K_k] = m$. In that case, finer bounds in
Thus, in view of the lemma, we can use

Lemma 4. Let \( K_k \) be a polynomial kernel of degree \( k \). Then, the empirical Rademacher complexity of \( H_k \) can be upper bounded as \( \mathcal{R}_m(H_k) \leq 12 \kappa_k^2 \sqrt{\frac{d_k m}{r}} \).

The proof of this result is in Appendix A Thus, in view of the lemma, we can use \( r_k = \kappa_k^2 \sqrt{d_k} \) as a complexity penalty in the formulation of the Voted Kernel Regularization algorithm with polynomial kernels, with \( d_k \) given by the expression (3).

6. Experiments

We experimented with several benchmark datasets from the UCI repository, specifically breastcancer, climate, diabetes, german (numeric), ionosphere, musk, ocr49, phishing, retinopathy, vertebral and waveform01. Here, ocr49 refers to the subset of the OCR dataset with classes 4 and 9, and similarly waveform01 refers to the subset of waveform dataset with classes 0 and 1. More details on all the datasets are given in Table 2 in Appendix D.

Our experiments compared Voted Kernel Regularization to regular SVM, that we refer to as \( L_2 \)-SVM, and to norm-1 SVM, called \( L_1 \)-SVM. In all of our experiments, we used lp_solve, an off-the-shelf LP solver, to solve the Voted Kernel Regularization and \( L_1 \)-SVM optimization problems. For \( L_2 \)-SVM, we used LibSVM.

In each of the experiments, we used standard 5-fold cross-validation for performance evaluation and model selection. In particular, each dataset was randomly partitioned into 5 folds, and each algorithm was run 5 times, with a different assignment of folds to the training set, validation set...
and test set for each run. Specifically, for each $i \in \{0, \ldots, 4\}$, fold $i$ was used for testing, fold $i + 1 \mod 5$ was used for validation, and the remaining folds were used for training. For each setting of the parameters, we computed the average validation error across the 5 folds, and selected the parameter setting with minimum average validation error. The average error across the 5 folds was then computed for this particular parameter setting.

In the first set of experiments we used polynomial kernels of the form $K_k(x, y) = (x^T y + 1)^k$. We report the results in Table 6. For Voted Kernel Regularization, we optimized over $\lambda \in \{10^{-i}: i = 0, \ldots, 6\}$ and $\beta \in \{10^{-i}: i = 0, \ldots, 6\}$ The family of kernel functions $H_k$ for $k \in [1, 10]$ was chosen to be the set of polynomial kernels of degree $k$. In our experiments we compared the bounds of both Lemma 3 and Lemma 4 used as an estimate of the Rademacher complexity. For $L_1$-SVM, we cross-validated over degrees in range 1 through 10 and $\beta$ in the same range as for Voted Kernel Regularization. Cross-validation for $L_2$-SVM was also done over the degree and regularization parameter $C \in \{10^i: i = -4, \ldots, 7\}$.

On 5 out of 11 datasets Voted Kernel Regularization outperformed $L_2$-SVM and $L_1$-SVM with a considerable improvement on 3 data sets. On the rest of the datasets, there was no statistical difference between these algorithms. Note that our results are also consistent with previous studies that indicated that $L_1$-SVM and $L_2$-SVM often have comparable performance. Observe that solutions obtained by Voted Kernel Regularization are often up to 10 times sparser than those of $L_2$-SVM. In other words, Voted Kernel Regularization has a benefit of sparse solutions and often

Table 1. Experimental results with Voted Kernel Regularization and polynomial kernels. VKRc refers to the algorithm obtained by using Lemma 3 as complexity measure, while VKR2 refers to the algorithm obtained by using Lemma 4. Indicated in boldface are results where the errors obtained are statistically significant at a confidence level of 5%. In italics are results that are better at 10% level.

| Dataset  | Error (%) | Number of support vectors |
|----------|-----------|---------------------------|
|          | L2 SVM    | L1 SVM | VKR2 | VKRc | L2 SVM | L1 SVM | VKR2 | VKRc |
|          | Mean (Stdev) | Mean (Stdev) | Mean (Stdev) | Mean (Stdev) | Mean (Stdev) | Mean (Stdev) | Mean (Stdev) | Mean (Stdev) |
| ocr49    | 5.05 (0.65) | 3.50 (0.85) | 2.70 (0.97) | 3.50 (0.85) | 449.8 (3.6) | 140.0 (3.6) | 6.8 (1.3) | 164.6 (9.5) |
| phishing | 4.64 (1.38) | 4.11 (0.71) | 3.62 (0.44) | 3.87 (0.80) | 221.4 (15.1) | 188.8 (7.5) | 73.0 (3.2) | 251.8 (4.0) |
| waveform01 | 8.38 (0.63) | 8.47 (0.52) | 8.41 (0.97) | 8.57 (0.58) | 415.6 (8.1) | 13.6 (1.3) | 18.4 (1.5) | 30.2 (2.3) |
| breastcancer | 11.45 (0.74) | 12.60 (2.88) | 11.73 (2.73) | 11.30 (1.31) | 83.8 (10.9) | 46.4 (2.4) | 66.6 (3.9) | 29.4 (1.9) |
| german | 23.00 (3.00) | 22.40 (2.58) | 24.10 (2.99) | 24.20 (2.61) | 357.2 (16.7) | 25.0 (2.2) | 30.2 (1.4) | 30.2 (2.3) |
| ionosphere | 6.54 (0.30) | 7.12 (3.18) | 4.27 (2.00) | 3.99 (2.12) | 152.0 (5.5) | 73.8 (4.9) | 43.6 (2.9) | 30.6 (1.8) |
| pima | 31.90 (1.17) | 30.85 (1.54) | 31.77 (2.68) | 30.73 (1.46) | 330.0 (6.6) | 26.4 (0.6) | 33.8 (3.6) | 40.6 (1.1) |
| musk | 15.34 (2.23) | 15.55 (1.49) | 10.71 (1.13) | 9.33 (1.39) | 251.8 (12.4) | 115.4 (4.5) | 125.6 (8.0) | 108.0 (5.2) |
| retinopathy | 24.58 (2.28) | 24.85 (2.65) | 23.46 (2.08) | 24.06 (2.43) | 648.2 (21.3) | 42.6 (6.7) | 43.6 (8.0) | 48.0 (5.2) |
| climate | 5.19 (2.41) | 5.93 (2.83) | 5.56 (2.85) | 6.30 (2.89) | 66.0 (4.6) | 19.0 (0.0) | 51.0 (6.7) | 18.6 (0.9) |
| vertebral | 17.74 (6.35) | 18.06 (5.51) | 17.10 (7.27) | 17.10 (6.99) | 75.4 (4.0) | 4.4 (0.6) | 9.6 (1.1) | 8.2 (1.3) |

VOTED KERNEL REGULARIZATION
an improved performance, which provides strong empirical evidence in the support of our formulation. In a second set of experiments we used families of Gaussian kernels based on distinct values of the parameter $\gamma \in \{10^i : i = -6, \ldots, 0\}$. We used the bound of Lemma 3 as an estimate of the Rademacher complexity. In our cross-validation we used the same range for $\lambda$ and $\beta$ parameters of Voted Kernel Regularization and $L_1$-SVM algorithms. For $L_2$-SVM we increased the range of the regularization parameter: $C \in \{10^i : i = -4, \ldots, 7\}$. The results of our experiments are comparable to the results with polynomial kernels, however, improvements obtained by Voted Kernel Regularization are not always as significant in this case. The sparseness of the solutions are comparable to those observed with polynomial kernels.

7. Conclusion

In this paper we presented a new support vector algorithm - Voted Kernel Regularization. Our algorithm benefits from strong data-dependent learning guarantees that enable learning with highly complex feature maps and yet not overfit. We further improved these learning guarantees using local complexity analysis leading to an extension of Voted Kernel Regularization algorithm. The key ingredient of our algorithm is a new regularization term that makes use of the Rademacher complexities of different families of kernel functions used by the Voted Kernel Regularization algorithm. We provide a thorough analysis of several different alternatives that can be used for this approximation. We also provide two practical implementations of our algorithm based on linear programming and coordinate descent. Finally, we presented results of extensive experiments that show that our algorithm always finds solutions that are much sparse than those of the other support vector algorithms and at the same time often outperforms other formulations.
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APPENDIX A. PROOFS OF LEARNING GUARANTEES

**Theorem 2.** Assume $p > 1$. Fix $\rho > 0$. Then, for any $\delta > 0$, with probability at least $1 - \delta$ over the choice of a sample $S$ of size $m$ drawn i.i.d. according to $D_m$, the following inequality holds for all $f = \sum_{t=1}^{T} \alpha_t h_t \in \mathcal{F}$ for any $K > 1$:

$$
R(f) - \frac{K}{K-1} \hat{R}_{S, \rho}(f) \leq 6K \frac{1}{\rho} \sum_{t=1}^{T} \alpha_t \mathcal{R}_m(H_{k_t}) + 40 \frac{K \log p}{\rho^2} + 5K \frac{\log^2 \frac{2}{\delta}}{m} + 5K \left[ \frac{8}{\rho^2} \log \left( \frac{\rho^2 (1 + K)}{40K \log p} \right) \right] \frac{\log p}{m}.
$$

Thus, for $K = 2$, $R(f) \leq 2 \hat{R}_{S, \rho}(f) + \frac{12}{\rho} \sum_{t=1}^{T} \alpha_t \mathcal{R}_m(H_{k_t}) + O \left( \frac{\log p}{\rho^2 m} \log \left( \frac{\rho m}{\rho^2} \right) + \frac{\log \frac{1}{\delta}}{m} \right)$.

**Proof.** For a fixed $h = (h_1, \ldots, h_T)$, any $\alpha \in \Delta$ defines a distribution over $\{h_1, \ldots, h_T\}$. Sampling from $\{h_1, \ldots, h_T\}$ according to $\alpha$ and averaging leads to functions $g$ of the form $g = \frac{1}{n} \sum_{t=1}^{T} n_t h_t$ for some $n = (n_1, \ldots, n_T)$, with $\sum_{t=1}^{T} m_t = n$, and $h_t \in H_{k_t}$.

For any $N = (N_1, \ldots, N_p)$ with $|N| = n$, we consider the family of functions

$$
G_{\mathcal{F}, N} = \left\{ \frac{1}{n} \sum_{k=1}^{p} \sum_{j=1}^{N_k} h_{k,j} | \forall (k, j) \in [p] \times [N_k], h_{k,j} \in H_k \right\},
$$

and the union of all such families $G_{\mathcal{F}, n} = \bigcup_{|N| = n} G_{\mathcal{F}, N}$. Fix $\rho > 0$. We define a class $\Phi \circ G_{\mathcal{F}, N} = \{ \Phi_v(g) : g \in G_{\mathcal{F}, N} \}$ and $G_r = G_{\Phi \circ G_{\mathcal{F}, N}, r} = \{ r \ell_g / \max(r, \mathbb{E}[\ell_g]) : \ell_g \in \Phi \circ G_{\mathcal{F}, N} \}$ for $r$ to be chosen later. Observe that for $v_g \in G_{\Phi \circ G_{\mathcal{F}, N, r}}$, $\mathbb{Var}[v_g] \leq r$. Indeed, if $r > \mathbb{E}[\ell_g]$ then $v_g = \ell_g$. Otherwise, $\mathbb{Var}[v_g] = r^2 \mathbb{Var}[\ell_g] / \left(\mathbb{E}[\ell_g]\right)^2 \leq r \left(\mathbb{E}[\ell_g^2]\right) / \mathbb{E}[\ell_g] \leq r$.

By Theorem 2.1 in Bartlett et al. [2005], for any $\delta > 0$ with probability at least $1 - \delta$, for any $0 < \beta < 1$,

$$
V \leq 2(1 + \beta) \mathcal{R}_m(G_{\Phi \circ G_{\mathcal{F}, N, r}}) + \sqrt{2r \frac{\log \frac{1}{\delta}}{m} + \left( \frac{1}{3} + \frac{1}{\beta} \right) \frac{\log \frac{1}{\delta}}{m}},
$$

where $V = \sup_{v \in G_r} \left[ \mathbb{E}[v] - \mathbb{E}_n[v] \right]$ and $\beta$ is a free parameter. Next we observe that if $\mathcal{R}_m(G_{\mathcal{F}, N, r}) \leq \mathcal{R}_m(\{ \alpha \ell_g : g \in \Phi \circ G_{\mathcal{F}, N}, \alpha \in [0, 1] \}) = \mathcal{R}_m(\Phi \circ G_{\mathcal{F}, N})$. Therefore, using Talagrand’s contraction lemma and convexity we have that $\mathcal{R}_m(G_{\Phi \circ G_{\mathcal{F}, N, r}}) \leq \frac{1}{\rho} \sum_{k=1}^{p} \frac{N_k}{n} \mathcal{R}_m(H_k)$. It follows that for any $\delta > 0$ with probability at least $1 - \delta$, for all $0 < \beta < 1$,

$$
V \leq 2(1 + \beta) \frac{1}{\rho} \frac{1}{n} \sum_{k=1}^{p} \frac{N_k}{n} \mathcal{R}_m(H_k) + \sqrt{2r \frac{\log \frac{1}{\delta}}{m} + \left( \frac{1}{3} + \frac{1}{\beta} \right) \frac{\log \frac{1}{\delta}}{m}}.
$$

Since there are at most $p^n$ possible $p$-tuples $N$ with $|N| = n$, by the union bound, for any $\delta > 0$, with probability at least $1 - \delta$,

$$
V \leq 2(1 + \beta) \frac{1}{\rho} \frac{1}{n} \sum_{k=1}^{p} \frac{N_k}{n} \mathcal{R}_m(H_k) + \sqrt{r \frac{\log \frac{p^n}{\delta}}{m} + \left( \frac{1}{3} + \frac{1}{\beta} \right) \frac{\log \frac{p^n}{\delta}}{m}}.
$$
Thus, with probability at least $1 - \delta$, for all functions $g = \frac{1}{n} \sum_{t=1}^{T} n_t h_t$ with $h_t \in H_{k_t}$, the following inequality holds

$$V \leq 2(1 + \beta) \frac{1}{\rho} \sum_{t=1}^{T} \frac{n_t}{n} \mathcal{R}_m(H_{k_t}) + \sqrt{\frac{r \log \frac{p^m}{m}}{m}} + \left(\frac{1}{3} + \frac{1}{\beta}\right) \frac{\log \frac{p^m}{m}}{m}.$$ 

Taking the expectation with respect to $\alpha$ and using $E_{\alpha}[n_t/n] = \alpha_t$, we obtain that for any $\delta > 0$, with probability at least $1 - \delta$, for all $h$, we can write

$$E_{\alpha}[V] \leq 2(1 + \beta) \frac{1}{\rho} \sum_{t=1}^{T} \alpha_t \mathcal{R}_m(H_{k_t}) + \sqrt{\frac{r \log \frac{p^m}{\delta}}{m}} + \left(\frac{1}{3} + \frac{1}{\beta}\right) \frac{\log \frac{p^m}{\delta}}{m}.$$ 

We now show that $r$ can be chosen in such a way that $E_{\alpha}[V] \leq r/K$. The right hand side of the above bound is of the form $A\sqrt{r} + B$. Note that solution of $r/K = C + A\sqrt{r}$ is bounded by $K^2 A^2 + 2KC$ and hence by Lemma 5 in [Bartlett et al., 2002] the following bound holds

$$E_{\alpha}[R_{\rho/2}(g)] \leq \frac{K}{K - 1} \hat{R}_{S,\rho}(g) \leq 4K(1 + \beta) \frac{1}{\rho} \sum_{t=1}^{T} \alpha_t \mathcal{R}_m(H_{k_t}) + \left(2K^2 + 2K\left(\frac{1}{3} + \frac{1}{\beta}\right)\right) \frac{\log \frac{1}{\delta}}{m}.$$ 

Set $\beta = 1/2$, then we have that

$$E_{\alpha}[R_{\rho/2}(g)] - \frac{K}{K - 1} \hat{R}_{S,\rho}(g) \leq 6K \frac{1}{\rho} \sum_{t=1}^{T} \alpha_t \mathcal{R}_m(H_{k_t}) + 5K \frac{\log \frac{1}{\delta}}{m}.$$ 

Then, for any $\delta_n > 1$, with probability at least $1 - \delta_n$,

$$E_{\alpha}[R_{\rho/2}(g)] - \frac{K}{K - 1} \hat{R}_{S,\rho}(g) \leq 6K \frac{1}{\rho} \sum_{t=1}^{T} \alpha_t \mathcal{R}_m(H_{k_t}) + 5K \frac{\log \frac{1}{\delta_n}}{m}.$$ 

Choose $\delta_n = \frac{\delta}{2^p \rho \delta}$ for some $\delta > 0$, then for $p \geq 2$, $\sum_{n \geq 1} \delta_n = \frac{\delta}{2(1 - 1/p)} \leq \delta$. Thus, for any $\delta > 0$ and any $n \geq 1$, with probability at least $1 - \delta$, the following holds for all $h$:

$$E_{\alpha}[R_{\rho/2}(g)] - \frac{K}{K - 1} \hat{R}_{S,\rho}(g) \leq 6K \frac{1}{\rho} \sum_{t=1}^{T} \alpha_t \mathcal{R}_m(H_{k_t}) + 5K \frac{\log \frac{2^{p-1}}{\delta}}{m}. \quad (4)$$

Now, for any $f = \sum_{t=1}^{T} \alpha_t h_t \in \mathcal{F}$ and any $g = \frac{1}{n} \sum_{t=1}^{T} n_t h_t$, we can upper bound $R(f) = \Pr_{(x,y) \sim \mathcal{D}}[yf(x) \leq 0]$, the generalization error of $f$, as follows:

$$R(f) = \Pr_{(x,y) \sim \mathcal{D}}[yf(x) - yg(x) + yg(x) \leq 0] \leq \Pr[yf(x) - yg(x) < -\rho/2] + \Pr[yg(x) \leq \rho/2]$$

$$= \Pr[yf(x) - yg(x) < -\rho/2] + R_{\rho/2}(g).$$

We can also write

$$\hat{R}_\rho(g) = \hat{R}_{S,\rho}(g - f + f) \leq \hat{\Pr}[yg(x) - yf(x) < -\rho/2] + \hat{R}_{S,3\rho/2}(f).$$

Combining these inequalities yields

$$\Pr_{(x,y) \sim \mathcal{D}}[yf(x) \leq 0] - \frac{K}{K - 1} \hat{R}_{S,3\rho/2}(f) \leq \Pr[yf(x) - yg(x) < -\rho/2]$$

$$+ \frac{K}{K - 1} \hat{\Pr}[yg(x) - yf(x) < -\rho/2] + R_{\rho/2}(g) - \frac{K}{K - 1} \hat{R}_{S,\rho}(g).$$
Taking the expectation with respect to $\alpha$ yields
\[
R(f) - \widehat{R}_{S,3\rho/2}(f) \leq \frac{E}{x \sim D, \alpha} [1_{yf(x)-yg(x)<-\rho/2}]
\]
\[
+ \frac{K}{K-1} \frac{E}{x \sim D, \alpha} [1_{yg(x)-yf(x)<-\rho/2}] + E[R_{\rho/2}(g) - \frac{K}{K-1}\widehat{R}_{S,\rho}(g)].
\]

Since $f = E[\alpha]$, by Hoeffding’s inequality, for any $x$,
\[
E[1_{yf(x)-yg(x)<-\rho/2}] = \Pr[\alpha] = \frac{\rho}{2} \leq e^{-\frac{n\rho^2}{8}}
\]
\[
E[1_{yg(x)-yf(x)<-\rho/2}] = \Pr[\alpha] = \frac{\rho}{2} \leq e^{-\frac{n\rho^2}{8}}.
\]
Thus, for any fixed $f \in F$, we can write
\[
R(f) - \widehat{R}_{S,3\rho/2}(f) \leq \left(1 + \frac{K}{K-1}\right)e^{-n\rho^2/8} + E[R_{\rho/2}(g) - \frac{K}{K-1}\widehat{R}_{S,\rho}(g)].
\]
Thus, the following inequality holds:
\[
\sup_{f \in F} (R(f) - \frac{K}{K-1}\widehat{R}_{S,\rho}(f)) \leq \left(1 + \frac{K}{K-1}\right)e^{-n\rho^2/8} + \sup_{f \in F} E[R_{\rho/2}(g) - \frac{K}{K-1}\widehat{R}_{S,\rho/2}(g)].
\]
Therefore, in view of (4), for any $\delta > 0$ and any $n \geq 1$, with probability at least $1 - \delta$, the following holds for all $f \in F$:
\[
R(f) - \frac{K}{K-1}\widehat{R}_{S,\rho}(f) \leq \left(1 + \frac{K}{K-1}\right)e^{-n\rho^2/8} + 6K\frac{1}{\rho}\sum_{t=1}^{T} \alpha_t \varrho_m(H_{k_t}) + 5K\frac{\log 2\rho_{2n-1}}{m}.
\]
To conclude the proof we optimize over $n$, $f$: $n \mapsto v_1e^{-nu} + v_2n$, which leads to $n = (1/u) \log(uv_2/v_1)$. Therefore, we set
\[
n = \left\lceil \frac{8}{\rho^2} \log \frac{\rho^2(1 + \frac{K}{K-1})m}{40K \log p} \right\rceil
\]
to obtain that the following bound
\[
R(f) - \frac{K}{K-1}\widehat{R}_{S,\rho}(f) \leq 6K\frac{1}{\rho}\sum_{t=1}^{T} \alpha_t \varrho_m(H_{k_t})
\]
\[
+ 40\frac{K \log p}{\rho^2 m} + 5K\left\lceil \frac{8}{\rho^2} \log \frac{\rho^2(1 + \frac{K}{K-1})m}{40K \log p} \right\rceil \log p.
\]
Thus, taking $K = 2$, simply yields
\[
R(f) \leq 2\widehat{R}_{S,\rho}(f) + \frac{12}{\rho}\sum_{t=1}^{T} \alpha_t \varrho_m(H_{k_t}) + O\left(\frac{\log p}{\rho^2 m} \log \left(\frac{\rho m}{\log p}\right) + \frac{\log \frac{1}{\delta}}{m}\right)
\]
and the proof is complete.

\[\square\]

**Lemma 3.** Let $K_{k}$ be the kernel matrix of the kernel function $K_{k}$ for the sample $S$ and let $\kappa_{k} = \sup_{x \in X} \sqrt{K_{k}(x, x)}$. Then, the following inequality holds:
\[
\widehat{R}_{S}(H_{k}) \leq \kappa_{k} \sqrt{\frac{\text{Tr}[K_{k}]}{m}}.
\]
Proof. $\hat{\mathcal{R}}_S(H_k)$ can be upper bounded as follows using the Cauchy-Schwarz inequality:

$$\hat{\mathcal{R}}_S(H_k) = \frac{1}{m} \mathbb{E} \left[ \sup_{x', x'' \in (-1, 1)} \sum_{i=1}^m \sigma_i s K_k(x_i, x'') \right] = \frac{1}{m} \mathbb{E} \left[ \sup_{x' \in \mathcal{X}} \left| \sum_{i=1}^m \sigma_i s K_k(x_i, x') \right| \right]$$

where we used in the last line Jensen’s inequality.

By the proof of Lemma 3, we can write

$$\hat{\mathcal{R}}_S(H_k) = \frac{k}{m} \mathbb{E} \left[ \left\| \sum_{i=1}^m \sigma_i \Phi_k(x_i) \right\|_{\mathcal{G}_k} \right] \leq \frac{k}{m} \left( \mathbb{E} \left[ \sum_{i=1}^m \left\| \sigma_i \Phi_k(x_i) \right\|_{\mathcal{G}_k} \right] \right)^{1/2} = \frac{k}{m} \sqrt{\text{Tr}[K_k]},$$

where we used in the last line Jensen’s inequality.

Lemma 4. Let $K_k$ be a polynomial kernel of degree $k$. Then, the empirical Rademacher complexity of $H_k$ can be upper bounded as follows:

$$\hat{\mathcal{R}}_S(H_k) \leq 12k^2 \sqrt{\frac{\pi d_k}{m}}.$$  

Proof. By the proof of Lemma 3, we can write

$$\hat{\mathcal{R}}_S(H_k) = \frac{k}{m} \mathbb{E} \left[ \left\| \sum_{i=1}^m \sigma_i \Phi_k(x_i) \right\|_{\mathcal{G}_k} \right] = 2k \hat{\mathcal{R}}_S(H_k),$$

where $H^1_k$ is the family of linear functions $H^1_k = \{ w \mapsto w \cdot \Phi_k(x) : \|w\|_{\mathcal{G}_k} \leq \frac{1}{2\kappa_k} \}$.

By Dudley’s formula [Dudley, 1989], we can write

$$\hat{\mathcal{R}}_S(H_k) \leq 12 \int_0^\infty \sqrt{\log N(\epsilon, H^1_k, L_2(\hat{\mathcal{D}}))} \frac{d\epsilon}{m},$$

where $\hat{\mathcal{D}}$ is the empirical distribution. Since $H^1_k$ can be viewed as a subset of a $d_k$-dimensional linear space and since $|w \cdot \Phi_k(x)| \leq \frac{1}{2}$ for all $x \in \mathcal{X}$ and $w \in H^1_k$, we have $\log N(\epsilon, H^1_k, L_2(\hat{\mathcal{D}})) \leq \log \left( \frac{1}{\epsilon} \right)^{d_k}$. Thus, we can write

$$\hat{\mathcal{R}}_S(H_k) \leq 12 \int_0^1 \sqrt{\frac{d_k \log \frac{1}{\epsilon}}{m}} d\epsilon = 12 \sqrt{\frac{d_k}{m}} \int_0^1 \sqrt{\frac{1}{\epsilon}} d\epsilon = 12 \sqrt{\frac{d_k \sqrt{\frac{2}{m}}}{2}},$$

which completes the proof.

Appendix B. Optimization Problem

This section provides the derivation for VKR optimization problem. We will assume that $H_1, \ldots, H_p$ are $p$ families of functions with increasing Rademacher complexities $\mathcal{R}_m(H_k)$, $k \in [1, p]$, and, for any hypothesis $h \in \bigcup_{k=1}^p H_k$, denote by $d(h)$ the index of the hypothesis set it belongs to, that is $h \in H_{d(h)}$. The bound of Theorem 1 holds uniformly for all $p > 0$ and functions $f \in \text{conv}(\bigcup_{k=1}^p H_k)$ at the price of an additional term that is in $O\left(\sqrt{\frac{\log \log \frac{2}{m}}{m}}\right)$. The condition $\sum_{t=1}^T \alpha_t = 1$ of Theorem 1 can be relaxed to $\sum_{t=1}^T \alpha_t \leq 1$. To see this, use for example a null
hypothesis \((h_t = 0 \text{ for some } t)\). Since the last term of the bound does not depend on \(\alpha\), it suggests selecting \(\alpha\) to minimize

\[
G(\alpha) = \frac{1}{m} \sum_{i=1}^{m} y_i \sum_{t=i}^{T} \alpha_t h_t(x_i) + \frac{4}{\rho} \sum_{t=1}^{T} \alpha_t r_t,
\]

where \(r_t = \mathcal{R}_m(H_d(h_t))\). Since for any \(\rho > 0\), \(f\) and \(f/\rho\) admit the same generalization error, we can instead search for \(\alpha \geq 0\) with \(\sum_{t=1}^{T} \alpha_t \leq 1/\rho\) which leads to

\[
\min_{\alpha \geq 0} \frac{1}{m} \sum_{i=1}^{m} y_i \sum_{t=i}^{T} \alpha_t h_t(x_i) + 4 \sum_{t=1}^{T} \alpha_t r_t \quad \text{s.t.} \quad \sum_{t=1}^{T} \alpha_t \leq \frac{1}{\rho}.
\]

The first term of the objective is not a convex function of \(\alpha\) and its minimization is known to be computationally hard. Thus, we will consider instead a convex upper bound based on the Hinge loss: let \(\Phi(-u) = \max(0, 1 - u)\), then \(1 - u \leq \Phi(-u)\). Using this upper bound yields the following convex optimization problem:

\[
\min_{\alpha \geq 0} \frac{1}{m} \sum_{i=1}^{m} \Phi\left(1 - y_i \sum_{t=i}^{T} \alpha_t h_t(x_i)\right) + \lambda \sum_{t=1}^{T} \alpha_t r_t \quad \text{s.t.} \quad \sum_{t=1}^{T} \alpha_t \leq \frac{1}{\rho},
\]  

(5)

where we introduced a parameter \(\lambda \geq 0\) controlling the balance between the magnitude of the values taken by function \(\Phi\) and the second term. Introducing a Lagrange variable \(\beta \geq 0\) associated to the constraint in (5), the problem can be equivalently written as

\[
\min_{\alpha \geq 0} \frac{1}{m} \sum_{i=1}^{m} \Phi\left(1 - y_i \sum_{t=1}^{T} \alpha_t h_t(x_i)\right) + \sum_{t=1}^{T} (\lambda r_t + \beta) \alpha_t.
\]

Here, \(\beta\) is a parameter that can be freely selected by the algorithm since any choice of its value is equivalent to a choice of \(\rho\) in (5). Let \((h_{k,j})_{k,j}\) be the set of distinct base functions \(x \mapsto K_k(\cdot, x_j)\).

Then, the problem can be rewritten as \(F\) be the objective function based on that collection:

\[
\min_{\alpha \geq 0} \frac{1}{m} \sum_{i=1}^{m} \Phi\left(1 - y_i \sum_{j=1}^{N} \alpha_j h_j(x_i)\right) + \sum_{j=1}^{N} \lambda_j \alpha_j,
\]  

(6)

with \(\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{R}^N\) and \(\Lambda_j = \lambda r_j + \beta\), for all \(j \in [1, N]\). This coincides precisely with the optimization problem \(\min_{\alpha \geq 0} F(\alpha)\) defining Voted Kernel Regularization. Since the problem was derived by minimizing a Hinge loss upper bound on the generalization bound, this shows that the solution returned by Voted Kernel Regularization benefits from the strong data-dependent learning guarantees of Theorem 1.

**APPENDIX C. COORDINATE DESCENT (CD) FORMULATION**

An alternative approach for solving the Voted Kernel Regularization optimization problem (1) consists of using a coordinate descent method. A coordinate descent method proceeds in rounds. At each round, it maintains a parameter vector \(\alpha\). Let \(\alpha_t = (\alpha_{t,k,j})_{k,j}\) denote the vector obtained after \(t \geq 1\) iterations and let \(\alpha_0 = 0\). Let \(e_{k,j}\) denote the unit vector in direction \((k, j)\) in \(\mathbb{R}^{p \times m}\). Then, the direction \(e_{k,j}\) and the step \(\eta\) selected at the \(t\)th round are those minimizing \(F(\alpha_{t-1} + \eta e_{k,j})\).
VOTED KERNEL REGULARIZATION

Table 2. Dataset statistics.

| Data set      | Examples | Features |
|---------------|----------|----------|
| breastcancer  | 699      | 9        |
| climate       | 540      | 18       |
| diabetes      | 768      | 8        |
| german        | 1000     | 24       |
| ionosphere    | 351      | 34       |
| musk          | 476      | 166      |
| ocr49         | 2000     | 196      |
| phishing      | 2456     | 30       |
| retinopathy   | 1151     | 19       |
| vertebral     | 310      | 6        |
| waveform01    | 3304     | 21       |

\[ F(\alpha) = \frac{1}{m} \sum_{i=1}^{m} \max \left( 0, 1 - y_i f_{t-1} - y_i y_j \eta K_k(x_i, x_j) \right) + \sum_{j=1}^{m} \sum_{k=1}^{p} \Lambda_k |\alpha_{t-1,j,k}| + \Lambda_k |\eta + \alpha_{t-1,k,j}|, \]

where \( f_{t-1} = \sum_{j=1}^{m} \sum_{k=1}^{p} \alpha_{t-1,j,k} y_j K_k(\cdot, x_j) \). To find the best descent direction, a coordinate descent method computes the sub-gradient in the direction \((k, j)\) for each \((k, j) \in [1, p] \times [1, m]\).

The sub-gradient is given by

\[
\delta F(\alpha_{t-1}, e_{j}) = \begin{cases} 
\frac{1}{m} \sum_{i=1}^{m} \phi_{t,j,k,i} + \text{sgn}(\alpha_{t-1,k,j}) \Lambda_k & \text{if } \alpha_{t-1,k,j} \neq 0 \\
0 & \text{else if } \left| \frac{1}{m} \sum_{i=1}^{m} \phi_{t,j,k,i} \right| \leq \Lambda_k \\
\frac{1}{m} \sum_{i=1}^{m} \phi_{t,j,k,i} - \text{sgn} \left( \frac{1}{m} \sum_{i=1}^{m} \phi_{t,j,k,i} \right) \Lambda_k & \text{otherwise.}
\end{cases}
\]

where \( \phi_{t,j,k,i} = -y_i K_k(x_i, x_j) \) if \( \sum_{k=1}^{p} \sum_{j=1}^{m} \alpha_{t-1,k,j} y_j K(x_i, x_j) < 1 \) and 0 otherwise. Once the optimal direction \( e_{k,j} \) is determined, the step size \( \eta_t \) can be found using a line search or other numerical methods.

The advantage of the coordinate descent formulation over the LP formulation is that there is no need to explicitly store the whole vector of \( \alpha \)s but rather only no-zero entries. This enables learning with very large number of base hypotheses including scenarios in which the number of base hypotheses is infinite.

Appendix D. Dataset Statistics

The dataset statistics are provided in Table 2