A New Conformal Invariant for four-Dimensional Hypersurfaces

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Abstract: A new conformally invariant energy for four-dimensional hypersurfaces is devised. It renders possible the study of a large class of curvature energies, and we show that their critical points are smooth. As corollaries, we obtain the regularity of the critical points of the four-dimensional analogues of the Willmore energy, of the $Q$-curvature energy, but also that Bach-flat hypersurfaces are smooth, along with relevant estimates.

I Introduction and Main Results

Let $\tilde{\Phi} : \Sigma^2 \rightarrow \mathbb{R}^3$ be an immersion, and let $h$, $h_0$, and $H$ be respectively its associated second fundamental form, traceless second fundamental form, and mean curvature. We suppose that $\tilde{\Phi}$ is smooth for the time being. The Willmore energy

$$\int_{\Sigma} H^2 d\text{vol}_g$$

being (nearly) conformally invariant has long been a subject of interest, and there would be too much work to appropriately report about it. Let us content ourselves with mentioning the numerous works on the Willmore energy initiated by Tristan Rivière in [Riv], where for the first time the conformal invariance of the problem is put to analytical use via Noether currents$^1$. The present work features an exposition with similar spirit.

As the Willmore energy is not the primary object – but the primary inspiration – of this paper, we mention only a few its relevant properties. Firstly, by the Gauss-Bonnet theorem, we have

$$\int_{\Sigma} H^2 d\text{vol}_g = \frac{1}{4} \int_{\Sigma} |h_0|^2 d\text{vol}_g + 2\pi \chi(\Sigma),$$

where $\chi(\Sigma)$ is the Euler characteristic. The energy appearing on the right-hand side is fully conformally invariant (translation, rotation, dilation, inversion) and it of course shares its critical points (named Willmore surfaces) with the Willmore energy itself. The critical points of the Willmore energy are known to satisfy a second-order nonlinear equation for $H$ but with the highest order being linear, namely

$$\Delta_g H - 2H|h_0|^2 = 0.$$ 

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$^1$The conservative fields discovered by Rivière were not originally identified as Noether currents; which was done in [Ber].
Minimal surfaces are examples.

In an attempt to realistically generalize the Willmore energy to four dimension, several authors (Guven [Guv], Robin-Graham-Reichert [RGR], Gover-Waldron [GW]) have arrived at the same answer from different routes, namely

\[ \mathcal{E}_A := \int_\Sigma (|\nabla H|^2 - H^2 |h|^2 - 7H^4) dvol_g. \]

The specific choice for these coefficients will be explained later.

The critical points of the energy \( \mathcal{E}_A \) satisfy a sixth-order nonlinear equation with linear leading-order term \( \Delta_g^2 H \). Moreover, minimal hypersurfaces are critical points. Unfortunately, \( \mathcal{E}_A \) is neither positive nor does it relate \textit{a priori} nicely to intrinsic geometry and to the natural conformal invariants given by

\[ \int_\Sigma \det_g h \, dvol_g, \quad \int_\Sigma |h_0|^4 \, dvol_g \quad \text{and} \quad \int_\Sigma \text{Tr}_g(h_0^4) \, dvol_g. \]

Using similar techniques as those originally devised by Guven [Guv] to obtain \( \mathcal{E}_A \), we derive a new conformally invariant energy:

**Theorem I.1** The energy

\[ \mathcal{E}_C := \int_\Sigma (|\nabla h|^2 - 6|h|^2 H^2 + 60H^4) \, dvol_g \]

is conformally invariant.

Naturally, any linear combination of

\[ \mathcal{E}_A \ , \ \mathcal{E}_C \ , \ \mathcal{E}_{\mu,\lambda,\sigma} := \int_\Sigma (\mu|h_0|^4 + \lambda \text{Tr}_g(h_0^4) + \sigma \det_g(h)) \, dvol_g \]

will also be conformally invariant. In [?], the authors favor one of these linear combinations, namely

\[ \mathcal{E}_{Wm} := 6\mathcal{E}_A + \mathcal{E}_{\mu,\lambda,\sigma}. \]

and argue it is the suitable Willmore energy in dimension four. As there are many pros and cons to each linear combination, favoring one over the others is mainly a question of purpose. Nevertheless, one linear combination appears particularly interesting as it relates nicely to \textit{intrinsic} geometry. We define

\[ \mathcal{E}_B := \frac{1}{3} (\mathcal{E}_C - 16\mathcal{E}_A) + \mathcal{E}_{-\frac{7}{3},0,1} \quad (I.1) \]

\[ \equiv \frac{1}{3} \int_\Sigma \left( |\nabla h|^2 - 16|\nabla H|^2 + 10H^2 |h|^2 - 52H^4 - \frac{9}{2} |h_0|^4 + 3\det_g h \right) \, dvol_g. \]

**Theorem I.2** \( \mathcal{E}_B \) is conformally invariant, positive, and intrinsic. It satisfies

\[ \mathcal{E}_B = \frac{1}{2} \int_\Sigma |W|^2 \, dvol_g, \]

where \( W \) denotes the Weyl tensor, which for an immersion satisfies

\[ \int_\Sigma |W|^2 \, dvol_g = \int_\Sigma \left( \frac{7}{3} |h_0|^4 - 4\text{Tr}_g h_0^4 \right) \, dvol_g \equiv \mathcal{E}_{\frac{7}{3},-4,0}. \]

\footnote{The well-known conformally invariant \( \int_\Sigma |W|^2 \), where \( W \) denotes the Weyl tensor, is a linear combination of the last two.}
This identification appears to be new, although it is of close kinship with the numerous works of Gover et al. (e.g. [BGW]). We see that Bach-flat submanifolds (i.e. the critical points of \( \int |W|^2 \)) are critical points of \( \mathcal{E}_B \). Furthermore, since \( \int Q \), where \( Q \) denotes \( Q \)-curvature\(^3\) shares the same critical points as \( \int |W|^2 \), it follows that both Bach-flat submanifolds and the critical points of \( Q \)-curvature are also critical points of \( \mathcal{E}_B \).

It must be noted that \( \mathcal{E}_B \) is an interesting character: although it seems to be neither conformally invariant, nor definite, nor intrinsic, it is actually all three. Moreover, even though it seems to be defined from first derivatives of the second fundamental, \( \mathcal{E}_B \) is in fact solely dependent on the (traceless) second fundamental form and not on derivatives. Finally, \( \mathcal{E}_B \) is zero if and only if our manifold is locally conformally flat (namely when the Weyl tensor vanishes identically).

Theorem \([2]\) relies on an unintuitive identity, namely:

\[
\int_{\Sigma} (|\nabla h|^2 - 16|\nabla H|^2) \, dvol_g = \int_{\Sigma} \left( H Tr_g h^3 + \frac{1}{4} |h|^4 \right) \, dvol_g.
\]

Thus, in particular

**Corollary I.1** Any hypersurface \( \Phi : \Sigma^4 \rightarrow \mathbb{R}^5 \) satisfies

\[
\int_{\Sigma} |\nabla h|^2 \, dvol_g = 16 \int_{\Sigma} |\nabla H|^2 \, dvol_g + \mathcal{O} \left( \int_{\Sigma} |h|^4 \, dvol_g \right).
\]

With this observation at hand, it follows that

**Corollary I.2** Any linear combination

\[ E := \tilde{\alpha}E_A + \tilde{\gamma}E_C + \mathcal{E}_{\mu,\tilde{\lambda},\tilde{\sigma}} \]

may be expressed as

\[ E := \alpha E_A + \mathcal{E}_{\mu,\lambda,\sigma} \]

with

\[ \alpha := \tilde{\alpha} + 16\tilde{\gamma} \quad, \quad \mu := \tilde{\mu} + \frac{7}{2} \tilde{\gamma} - \frac{9}{2} \quad, \quad \lambda := \tilde{\lambda} - 6\tilde{\gamma} \quad, \quad \sigma := \tilde{\sigma} + 3. \]

An analogous version of this corollary — from a different approach — may also be found in [BGW].

Let \( \{a_1, \ldots, a_7\} \) be real numbers. One easily verifies that

\[
\begin{align*}
& a_1 |\nabla h|^2 + a_2 |\nabla H|^2 + a_3 Tr_g h^4 + a_4 |h|^4 + a_5 H Tr_g h^3 + a_6 H^2 |h|^2 + a_7 H^4 \\
& = a_1 (|\nabla h|^2 - 16|\nabla H|^2) + (16a_1 + a_2) |\nabla H|^2 + b_1 \det_g h + b_2 Tr_g (h_0^4) + b_3 |h_0|^4 + b_4 H^2 |h|^2 + b_5 H^4 \\
& = a_1 \mathcal{E}_B + (16a_1 + a_2) |\nabla H|^2 + c_1 \det_g h + c_2 Tr_g (h_0^4) + c_3 |h_0|^4 + c_4 H^2 |h|^2 + c_5 H^4 \\
& = \alpha |\nabla H|^2 + \mathcal{E}_{\mu,\lambda,\sigma} + \beta H^2 |h|^2 + \gamma H^4, \quad (I.2)
\end{align*}
\]

\(^3\)cf. [CEOY].

\(^4\)the converse is false, as not all intrinsic variations on the metric descend from an extrinsic variation on the immersion. In other words, the critical points of \( \mathcal{E}_B \) need not be Bach-flat.
for some suitable constants \( \{b_j, c_j, \alpha, \beta, \gamma, \mu, \lambda, \sigma\} \). The analytical study of any energy of the type described on the left-hand side of (I.2) is thus reduced to the study of an energy of the type

\[
\int_\Sigma \left( \alpha |\nabla H|^2 + \beta H^2 |h|^2 + \gamma H^4 \right) \, d\text{vol}_g + \mathcal{E}_{\mu, \lambda, \sigma}.
\] (I.3)

As we have seen above, this energy is conformally invariant if and only if \((\alpha, \beta, \gamma) = k(1, -1, 7)\) for some \(k \in \mathbb{Z}\).

Depending upon \(\alpha\), the critical points of the energy given in (I.3) satisfy very different Euler-Lagrange equations. When \(\alpha \neq 0\), the leading-order term is \(\int \|\nabla H\|^2 \). One should thus not be surprised to discover that any critical point in that case satisfies a sixth-order equation whose leading-order is \(\Delta^2 g H\). In [BO], techniques were developed to analyse efficiently its regularity properties. In the present paper, we give a new proof of the main result in [BO] along with new estimates. On the other hand, when \(\alpha = 0\), the energy involves only zero-th order curvature terms. The contribution from the Gauss-Bonnet term \(\int \text{det}_g(h)\) is of course naught upon variation. Accordingly, there is a catastrophic loss of order in the Euler-Lagrange equation, and the leading-order term is now a second-order nonlinear operator. The regularity of its critical points cannot be obtained by the same methods as in the case \(\alpha \neq 0\). In order to thwart this imposing difficulty, I employ a technique I call “conformal redressing”. It amounts to appropriately modify the Noether currents of a given energy and bring them into the ones with better disposition (namely into those of \(\int |\nabla H|^2\)). This delicate procedure constitutes the heart of this paper.

We will prove:

**Theorem I.3** Let \(\vec{\Phi} : \Sigma^4 \hookrightarrow \mathbb{R}^5\) in \(W^{2,1} \cap W^{1,\infty}\) be a critical point of the energy

\[
\int_\Sigma \left( a_1 |\nabla h|^2 + a_2 |\nabla H|^2 + a_3 Tr_g h^4 + a_4 |h|^4 + a_5 H Tr_g h^3 + a_6 H^2 |h|^2 + a_7 H^4 \right) \, d\text{vol}_g ,
\]

for some constants \(\{a_j\}_{j=1,\ldots,7}\). Let \(B\) be a ball. Suppose that for some \(c_0 > 1\), it holds

\[
\frac{1}{c_0} < \|\nabla \vec{\Phi}\|_{L^{\infty}(B)} < c_0 \quad \text{and moreover that} \quad \|\nabla H\|_{L^2(B)} < \infty .
\]

There exists \(\varepsilon_0 > 0\) such that whenever

\[
\|h\|_{L^4(B)} < \varepsilon_0 ,
\]

then it holds

\[
\|\nabla H\|_{L^\infty(B_r)} \leq C \left( \|\nabla H\|_{L^2(B)} + \|h\|_{L^4(B)} \right) \quad \forall r \in (0, 1) ,
\]

where \(B_r\) is the rescaled version of \(B\) by a factor \(r\). The constant \(C\) depends on \(\|d\vec{\Phi}\|_{L^\infty(B)}\), \(\|h\|_{L^4(B)}\), and \(\|\nabla H\|_{L^2(B)}\).

The special case of \(\mathcal{E}_B\) is – I think – the most interesting one, for our argument in that case provides a new proof of the regularity of Bach-flat hypersurfaces. It is to be compared with the very interesting work of Tian and Viaclovsky [TV]. The present paper gives a purely codimension 1 proof. Nevertheless, I strongly suspect our results extend to higher codimension at the cost of a cumbersome notation and a bit of combinatorics. By the Nash Embedding Theorem, the regularity of analytically critical 4-dimensional Bach-flat manifolds follows, via a completely
extrinsic argument. Usually, when studying Bach-flat manifolds, the approach favored is intrinsic. This is one novelty of this paper.

A great deal of literature on four-dimensional manifolds is available to the community. Much too much for me to loyally render account of in this short introduction. The primary four-dimensional sources of inspiration of this paper are the works of Guven [Guv], Robin-Graham-Reichert [RGR], Gover-Waldron [GW]. The regularity of Bach-flat submanifolds came only as a corollary.

Many applications to our identification and techniques should now be at reach, analogous to those that Rivière et al. developed using the method of Noether currents in two dimensions for the Willmore energy.

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II A Conformally Invariant Energy

Let $\tilde{\Phi} : \Sigma^4 \hookrightarrow \mathbb{R}^5$ be an immersion. Let $h$, $h_0$, and $H$ be its associated second fundamental form, traceless second fundamental form, and mean curvature. We denote by $\tilde{n}$ the outward unit normal vector to $\tilde{\Phi}(\Sigma)$.

Using ideas developed by Guven in [Guv], we will show how to devise a conformally invariant energy of the form

$$E_C := \int_\Sigma |\nabla h|^2 d\text{vol}_g + \mathcal{E}(\alpha, \beta), \quad \text{with} \quad \mathcal{E}(\alpha, \beta) := \int_\Sigma (\alpha H^2|h|^2 + \beta H^4) d\text{vol}_g. \quad (II.5)$$

We will show that for suitable values of $\alpha$ and $\beta$, the energy $E_C$ is indeed conformally invariant. To that end, one first observes that $E_C$ is unchanged by translations, rotations, and dilations of the ambient space. Therefore, it suffices to verify that our energy is also preserved by special conformal transformations (to render account of the missing inversion). In order to do this, it is necessary to have an expression for the Noether current associated with translations. For a functional depending on the metric, on curvature, and on its derivatives, a systematic explicit way to do that was devised by M.M. M"uller [Mue]. We outline said procedure below.

II.1 M.M. M"uller’s Process

Let $\mathcal{F} = \mathcal{F}(g_{ab}, h_{ab}, \nabla_c h_{ab})$.

Put

$$K^{cab} := \frac{\partial \mathcal{F}}{\partial \nabla_c h_{ab}}, \quad \mathcal{F}^{ab} := -\nabla_c K^{cab} + \frac{\partial \mathcal{F}}{\partial h_{ab}} \quad \text{and} \quad G^{cab} := K^{cab} h_d^c - (K^{acd} + K^{cad}) h_d^b.$$ 

In addition, let

$$T^{ab} := -\mathcal{F} g^{ab} - 2 \frac{\partial \mathcal{F}}{\partial g_{ab}} + 2 \nabla_c G^{cab} - \mathcal{F}^{ac} h_c^b.$$ 

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The Noether current associated with translation for the energy \( \int_\Sigma \mathcal{F} \text{dvol}_g \) is then given by
\[
\vec{V}^a := T^{ab} \nabla_b \Phi - (\nabla_b \mathcal{F}^{ab}) \vec{n}.
\]
and it is conservative: \( \nabla_a \vec{V}^a = 0 \).

As explained in [Guv], the energy \( \int_\Sigma \mathcal{F} \text{dvol}_g \) is conformally invariant if and only if the following condition holds:
\[
\int (T_a^a \Phi + \mathcal{F}^a_a \vec{n}) \text{dvol}_g = 0.
\]

Lemma II.0 The energy \( \mathcal{E}(\alpha, \beta) \) defined in (II.5) has
\[
T_a^a = 0 \quad \text{and} \quad \mathcal{F}^a_a = 2\alpha H |h|^2 + 4(2\alpha + \beta)H^3. \tag{II.6}
\]

Proof. We will proceed in steps.

Claim 1. We have
\[
\frac{\partial H^2}{\partial g_{ab}} = -\frac{1}{2} H h^{ab}, \quad \frac{\partial H^2}{\partial h_{ab}} = \frac{1}{2} H g_{ab}, \quad \frac{\partial |h|^2}{\partial g_{ab}} = -2h^a_c h^{cb}, \quad \frac{\partial |h|^2}{\partial h_{ab}} = 2h^{ab}.
\]

Proof. One easily checks that
\[
\frac{\partial g^{cd}}{\partial g_{ab}} = -\frac{1}{2} (g^{ac} g^{bd} + g^{ad} g^{cb}),
\]
from which we obtain
\[
\frac{\partial H}{\partial g_{ab}} = \frac{\partial H}{\partial g^{cd}} \frac{\partial g^{cd}}{\partial g_{ab}} = -\frac{1}{8} h_{cd}(g^{ac} g^{bd} + g^{ad} g^{cb}) = \frac{1}{4} h^{ab},
\]
and the first announced identity easily ensues. The third identity is obtained analogously:
\[
\frac{\partial |h|^2}{\partial g_{ab}} = h_{cd} h^{ij} \frac{\partial (g^{ci} g^{dj})}{\partial g_{ab}} = -\frac{1}{2} h_{cd} h^c_i (g^{ad} g^{bij} + g^{aj} g^{db}) - \frac{1}{2} h_{cd} h^d_j (g^{ac} g^{bj} + g^{aj} g^{cb}) = -2h^a_c h^{bc}.
\]
The second and fourth identities are trivial.

Let next
\[
\mathcal{F} := \alpha H^2 |h|^2 + \beta H^4.
\]

Claim 2. It holds
\[
\frac{\partial \mathcal{F}}{\partial g_{ab}} = -\frac{\alpha}{2} H |h|^2 h^{ab} - 2\alpha H^2 h^a_c h^{cb} - \beta H^3 h^{ab}, \quad \frac{\partial \mathcal{F}}{\partial h_{ab}} = \frac{\alpha}{2} H |h|^2 g^{ab} + 2\alpha H^2 h^{ab} + \beta H^3 g^{ab}.
\]

Proof. Immediate from Claim 1.
Assembling together the pieces yields
\[ \mathcal{F}^{ab} = \frac{\alpha}{2} H |h|^2 g^{ab} + 2\alpha H^2 h^{ab} + \beta H^3 g^{ab} \]
and
\[ T^{ab} = -(\alpha H^2 |h|^2 + \beta H^4)g^{ab} + \frac{1}{2}(\alpha H |h|^2 + 2\beta H^3)h^{ab} + 2\alpha H^2 h^c_hh^{bc}, \]
from which the statement of the lemma follows.

We now carry out the same procedure for the energy \( \int_\Sigma |\nabla h|^2 d\vol_g \). We find first that
\[ K^{cab} = 2\nabla c_hh^{ab} \]
By the Codazzi-Mainardi identity, any two of \( \{a, b, c\} \) may be swapped without changing the result. Hence
\[ \mathcal{F}^{ab} = -2\Delta_g h^{ab} \quad \text{and} \quad G^{cab} = 2h^c_h\nabla^a h^{bd} - 4h^b_h\nabla^c h^{ad}. \]
Accordingly, we find
\[ \nabla_c G^{ca} = 8\nabla cd(h^d h^c H) - 2\Delta_g(8H^2 + |h|^2), \]
and thus
\[ T^a_a = -4|\nabla h|^2 - 2g_{ab}\frac{\partial |\nabla h|^2}{\partial g_{ab}} + 2\nabla c G^{ca} - \mathcal{F}^{ac}h^b_c \]
\[ = 2|\nabla h|^2 + 16\nabla cd(h^d H) - 4\Delta_g(8H^2 + |h|^2) + 2h_{ac}\Delta_g h^{ac} \]
\[ = 16\nabla cd(h^d H) - \Delta_g(32H^2 + 3|h|^2). \] (II.7)
Putting together (II.7) and the results of Lemma II.0, shows that for the compounded energy
\[ \int_\Sigma \left( H^2|\nabla h|^2 \right) d\vol_g, \]
we have
\[ T^a_a = 16\nabla cd(h^d H) - \Delta_g(32H^2 + 3|h|^2) \quad \text{and} \quad \mathcal{F}^a_a = -8\Delta_g H + 2\alpha H|h|^2 + 4(2\alpha + \beta)H^3, \]
thereby yielding
\[ \int (T^a_a\Phi + \mathcal{F}^a_a\vec{n}) d\vol_g = \int \left( 16H|h|^2 - 128H^3 - 12H|h|^2 - 8\Delta_g H + 2\alpha H|h|^2 + 4(2\alpha + \beta)H^3 \right)\vec{n} d\vol_g \]
\[ = -\int \left( 8\Delta_g H - 2(\alpha + 2)H|h|^2 - 4(2\alpha + \beta - 32)H^3 \right)\vec{n} d\vol_g. \] (II.8)

**Claim.** Choosing \( \alpha = -6 \) and \( \beta = 60 \) guarantees that
\[ \int (T^a_a\Phi + \mathcal{F}^a_a\vec{n}) d\vol_g = 0. \]

**Proof.** It suffices to show that
\[ \int (\Delta_g H + H|h|^2 - 8H^3)\vec{n} d\vol_g = 0. \]
To this end, we will employ an interesting generic identity obtained from computing the Noether current associated with translation for the energy $\int_{\Sigma} H^2 \, d\text{vol}_g$. This identity is quite remarkable and it plays a central role in our arguments. It was, as far as I know, first written down and successfully by Guven in a near-identical context [Guv].

**Lemma II.0** We have

$$\Delta_{\perp} \tilde{H} + |h|^2 \tilde{H} - 8H^2 \tilde{H} = \nabla_j \left( \nabla^j \tilde{H} - 2(H^2 g^{jk} - H h^{jk}) \nabla_k \tilde{\Phi} \right).$$

**Proof.** The identity may be checked by brute force, namely:

$$\nabla_j \left( \nabla^j \tilde{H} - 2(H^2 g^{jk} - H h^{jk}) \nabla_k \tilde{\Phi} \right) = \nabla_j \left( \pi_{\perp} \nabla^j \tilde{H} - (2H^2 g^{jk} - H h^{jk}) \nabla_k \tilde{\Phi} \right) = \Delta_{\perp} \tilde{H} + \pi_T \nabla_j \pi_{\perp} \nabla^j \tilde{H} - (8H^2 - |h|^2) \tilde{H} + h^{jk} \nabla_j H \nabla_k \tilde{\Phi} = \Delta_{\perp} \tilde{H} - (8H^2 - |h|^2) \tilde{H}. \tag{II.9}$$

The identity from Lemma [II.0] implies readily that

$$\int_{\Sigma} (\Delta_g H + |h|^2 H - 8H^3) \tilde{n} \, d\text{vol}_g = 0,$$

and in turn that

$$\mathcal{E}_C = \int_{\Sigma} (|\nabla h|^2 - 6H^2 |h|^2 + 60H^4) \, d\text{vol}_g$$

is conformally invariant. This proves Theorem [II.1]

**Remark II.1** The identity in Lemma [II.0] is obtained from using the invariance by translation of the energy $\int_{\Sigma} H^2 \, d\text{vol}_g$. Interestingly enough, the very same identity is obtained from using the invariances by translation and by rotation of any energy

$$\int_{\Sigma} (\sigma H^2 - |h|^2) \, d\text{vol}_g,$$

except when $\sigma = 16$ in which case the energy becomes

$$\int_{\Sigma} R \, d\text{vol}_g,$$

where $R$ denotes scalar curvature. In this very special case (i.e. when extrinsic becomes intrinsic), the identity obtained by variation simply yields the well-known Einstein equation

$$E^{ij} h_{ij} \tilde{n} = \nabla_j (E^{ij} \nabla_i \tilde{\Phi}), \tag{II.9}$$

where $E^{ij}$ denotes the divergence-free Einstein tensor. This reinforces me into thinking that the identity in Lemma [II.0] is indeed fundamental. For one thing, it gives an exact tensor whose divergence is a normal vector. Such tensors are decisive for our proof, as the “conformal redressing” of our Noether fields depends on their existence. The

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5Here $\tilde{H} := H \tilde{n}$, and $\Delta_{\perp} \tilde{H} := \pi_{\perp} \nabla_{\perp} \pi_{\perp} \nabla^j \tilde{H}$ is the Laplacian in the normal bundle.

6Denoting by $\pi_{\perp}$ and $\pi_T$ respectively the normal and tangential projections.

7This energy is known as the Einstein-Hilbert action.
exact same phenomenon is to be found in dimension two, while the Noether fields associated with $\int H^2$ are analytically tame, those of $\int |h_0|^2$ must be redressed in order to be brought into those $\int H^2$. This procedure amounts to finding a tensor field whose divergence is a normal vector. In dimension 2, this is easily found:

$$2H\vec{n} = \nabla_j \nabla^j \vec{\Phi}.$$  

In dimension four, we will redress our Noether fields using Lemma II.0 rather than the identity (II.9). Indeed, while the left-hand side in the former has $\Delta H$, that of the latter only is of the order of $|h|^3$, which is insufficient to carry out our argument.

Now that we have found $\mathcal{E}_C$, we may create $\mathcal{E}_B$:

$$\mathcal{E}_B := \frac{1}{3} (\mathcal{E}_C - 16\mathcal{E}_A) + \mathcal{E}_\frac{3}{2},0,1 =: \int_\Sigma E_B \, d\text{vol}_g,$$

where

$$E_B := \frac{1}{3} \left( |\nabla h|^2 - 16|\nabla H|^2 + 10H^2|h|^2 - 52H^4 - \frac{9}{2}|h_0|^4 + 3\det g h \right). \quad (\text{II.10})$$

II.2 An identity: proof of Theorem I.2

We have on one hand by Simon’s identity that

$$h^{ij}\nabla_{ij}H = \frac{1}{4} h^{ij} \Delta h_{ij} - H \text{Tr}_g h^3 + \frac{1}{4} |h|^4$$

$$= \frac{1}{8} \Delta_g |h|^2 - \frac{1}{4} |\nabla h|^2 - H \text{Tr}_g h^3 + \frac{1}{4} |h|^4. \quad (\text{II.11})$$

On the other hand, using the Codazzi-Mainardi identity, we find

$$h^{ij}\nabla_{ij}H = \nabla_{ij} (h^{ij}H - 4H^2 g^{ij}) + 2\Delta_g H^2 - 4|\nabla H|^2. \quad (\text{II.12})$$

Equating (II.11) and (II.12), we arrive at

$$\frac{1}{8} \Delta_g R = 4|\nabla H|^2 - \frac{1}{4} |\nabla h|^2 - H \text{Tr}_g h^3 + \frac{1}{4} |h|^4 - \nabla_{ij} (h^{ij}H - 4H^2 g^{ij}), \quad (\text{II.13})$$

where we have used that $R = 16H^2 - |h|^2$.

Next, we introduce the traceless second fundamental form $h_0 := h - Hg$. One readily verifies that

$$|h_0|^4 = |h|^4 - 8H^2|h|^2 + 16H^4 \quad \text{and} \quad \text{Tr}_g h_0^4 = \text{Tr}_g h^4 - 4H \text{Tr}_g h^3 + 6H^2|h|^2 - 12H^4. \quad (\text{II.14})$$

Using these and the well-known expression for the norm of the Ricci tensor for a submanifold, it holds

$$\text{Ric}^2 = \text{Tr}_g h^4 - 8H \text{Tr}_g h^3 + 16H^2|h|^2$$

$$= \text{Tr}_g h_0^4 - 4H \text{Tr}_g h^3 + 10H^2|h|^2 + 12H^4. \quad$$

As far as I know, the procedure in question has unfortunately not been written anywhere.
Accordingly, we find that

\[
\frac{1}{4}|h|^4 - H\text{Tr}_{g}h^3 = \frac{1}{4}|h_0|^4 + 2H^2|h|^2 - 8H^4 + \frac{1}{4}\text{Ric}^2 - \frac{1}{4}\text{Tr}_{g}h_0^4 - \frac{5}{2}H^2|h|^2 - 3H^4
\]

\[
= \frac{1}{4}|h_0|^4 - \frac{1}{4}\text{Tr}_{g}h_0^4 - \frac{1}{2}H^2|h|^2 - 7H^4 + \frac{1}{4}\text{Ric}^2. \tag{II.15}
\]

This expression may be further simplified owing to the fact that

\[
\text{Ric}^2 = \frac{1}{2}|W|^2 - 3\det_{g}h + \frac{1}{3}R^2
\]

\[
= \frac{1}{2}|W|^2 - 3\det_{g}h + \frac{1}{3}(256H^4 - 32H^2|h|^2 + |h|^4)
\]

\[
= \frac{1}{2}|W|^2 - 3\det_{g}h + 80H^4 - 8H^2|h|^2 + \frac{1}{3}|h_0|^4.
\]

where \(W\) is the Weyl tensor.

Introducing the latter into (II.15) yields

\[
\frac{1}{4}|h|^4 - H\text{Tr}_{g}h^3 = \frac{1}{8}|W|^2 - \frac{3}{4}\det_{g}h + \frac{1}{4}|h_0|^4 - \frac{1}{4}\text{Tr}_{g}h_0^4 - \frac{5}{2}H^2|h|^2 + 13H^4.
\]

An elementary computation gives

\[
|W|^2 = \frac{7}{3}|h_0|^4 - 4\text{Tr}_{g}h_0^4,
\]

so that

\[
\frac{1}{4}|h|^4 - H\text{Tr}_{g}h^3 = \frac{3}{16}|W|^2 - \frac{3}{4}\det_{g}h + \frac{9}{8}|h_0|^4 - \frac{5}{2}H^2|h|^2 + 13H^4. \tag{II.16}
\]

As a final step, we introduce (II.16) into (II.13) to arrive at

\[
\frac{1}{6}\Delta_{g}R = \frac{16}{3}\nabla H^2 - \frac{1}{3}|\nabla h|^2 - \frac{10}{3}H^2|h|^2 + \frac{52}{3}H^4 + \frac{3}{2}|h_0|^4
\]

\[
+ \frac{1}{2}|W|^2 - \det_{g}h - \frac{4}{3}\nabla_{ij}(h^{ij}H - 4H^2g^{ij})
\]

\[
= -E_{B} + \frac{1}{2}|W|^2 - \frac{4}{3}\nabla_{ij}(h^{ij}H - 4H^2g^{ij}), \tag{II.17}
\]

where we have used (II.10). From this, we obtain the following expression for the Q-curvature\(^9\)

\[
Q \equiv 6\det_{g}h - \frac{1}{4}|W|^2 - \frac{1}{6}\Delta_{g}R
\]

\[
= E_{B} + 6\det_{g}h - \frac{3}{4}|W|^2 + \frac{4}{3}\nabla_{ij}(h^{ij}H - 4H^2g^{ij}). \tag{II.18}
\]

This expression confirms the conformal invariance of the newly identified energy \(E_{B}\). In the locally conformally flat case, we have \(|W|^2 = 0\), and it is known that

\[
\int_{\Sigma}Qd\text{vol}_{g} = 8\pi^2\chi(\Sigma) \equiv 6\int_{\Sigma}\det_{g}(h)d\text{vol}_{g}.
\]

\(^9\)See [CEOY].
Accordingly, 
\[ \mathcal{E}_B = \int_{\Sigma} E_B \, d\text{vol}_g = 0. \]
becomes a topological invariant.

In general, note that \( \int_{\Sigma} |W|^2 \, d\text{vol}_g \) has the same critical points as those of \( \mathcal{E}_B \) since
\[ \mathcal{E}_B = \int_{\Sigma} \left( Q + \frac{3}{4} |W|^2 - 6 \text{det}_g h \right) \, d\text{vol}_g = \frac{1}{2} \int_{\Sigma} |W|^2 \, d\text{vol}_g. \]

It immediately follows that
\[ \mathcal{E}_B \geq 0. \]
The proof of Theorem I.2 is now complete.

### III Regularity Matters

#### III.1 First Estimates

Henceforth, the symbol \( \lesssim \) stands for an inequality up to a multiplicative constant involving no more than:
\[ \| \nabla H \|_{L^2(B)} , \| h \|_{L^4(B)} , \| \nabla \Phi \|_{L^\infty(B)}. \]

We open with an elementary observation, namely:
\[ \| H \|_{L^4(B_k)} \lesssim \| \nabla H \|_{L^2(B)} + k^a, \quad \text{for some } a > 0, \quad (\text{III.19}) \]
and where \( B_k \) is the rescaled version of the ball \( B \) with dilation factor \( k \in (0, 1) \). This estimate will come in handy in due time.

Let next
\[ \vec{X}^j := \nabla^j \vec{H} + 2H^j_h \nabla_k \Phi. \quad (\text{III.20}) \]
We have seen in Lemma II.0 that in general the field \( \vec{X} \) has the peculiar feature:
\[ \nabla_j \vec{X}^j = (\Delta H + |h|^2 H - 8H^3) \vec{n}. \]

Accordingly, dotting with \( \vec{h} := h \vec{n} \) on both sides yields:
\[ h\Delta H = \nabla_j (\vec{h} \cdot \vec{X}^j) - \vec{X}^j \cdot \nabla_j \vec{h} + O(|h|^4) \]
\[ = \nabla_j (h \nabla^j H) - \nabla_j h \nabla^j H + O(|\nabla h||h|^2 + |\nabla H||h|^2 + |h|^4). \]

From the latter, it follows that
\[ \| h \Delta H \|_{W^{-1,4/3}} \lesssim \| h \nabla H \|_{L^{4/3}} + (\| \nabla h \|_{L^2} + \| h \|_{L^4}) (\| \nabla H \|_{L^2} + \| h \|_{L^4}) \lesssim 1. \quad (\text{III.21}) \]
III.2 Noether’s Fields

M.M. Müller’s process described above is used to find the Noether field for translation associated with various energies of the type

$$\int_{\Sigma} F \, d\text{vol}_g$$

where $F = F(g_{ab}, h_{ab}, \nabla h_{ab})$.

It results in a vector field

$$\vec{V}^r := G^{sr} \nabla_s \Phi - \vec{n} \nabla_s F^{rs}.$$

Computations akin to those down in Section II.1 yield the following table:

| Energy | $G$ | $F$ |
|--------|-----|-----|
| $\int |\nabla H|^2$ | $-|\nabla H|^2 g + 2\nabla H \otimes \nabla H - \frac{1}{2} h \Delta H$ | $-\frac{1}{2} g^{ab} \Delta H$ |
| $\int H^2 |h|^2$ | $-H^2 |h|^2 g + 2H^2 h^2 + \frac{1}{2} |h|^2 H h$ | $2H^2 h + \frac{1}{2} |h|^2 H g$ |
| $\int H^4$ | $H^4 h_0$ | $H^4 g$ |
| $\int \text{Tr}_g(h_0^4) - \text{Tr}_g(h_0^4)g + 4h_0^4 + 4Hh_0^3 - h \text{Tr}_g(h_0^3)$ | $4h_0^3 - \text{Tr}_g(h_0^3)g$ |
| $\int |h_0|^4$ | $-|h_0|^4 g + 4|h_0|^2 h_0^2 + 4|h_0|^2 H h_0$ | $4|h_0|^2 h_0$ |

For the energy

$$\mathcal{E} := \int_{\Sigma} \left( \alpha \mathcal{E}_A + \beta H^2 |h|^2 + \gamma H^4 + \mu |h_0|^4 + \lambda \text{Tr}_g(h_0^4) \right) \, d\text{vol}_g,$$

we see that

$$\text{Tr}_g G = -\alpha \Delta H^2 \quad \text{and} \quad \text{Tr}_g F = -2\alpha \Delta H + \mathcal{O}(|h|^3).$$

The fact that the trace of $F$ loses its main contributor when $\alpha = 0$ will cause serious damage and it is the raison d’être of conformal redressing. Such redressing is not necessary when $\alpha = 1$ (see [BO]).

**Remark III.2** The Noether field associated with the ghost term $\int \text{det}_g(h)$ is identically vanishing. Indeed, the interested reader will verify that $G^{ab}_{\text{det}_g(h)} \equiv 0$ by the Cayley-Hamilton theorem, while $F^{ab}_{\text{det}_g(h)} \equiv 0$ by the Codazzi-Mainardi identity.

Had we chosen to compute the field $\vec{V}$ associated with $\mathcal{E}_B$ expressed with derivatives of the curvatures, we would have found the main contributor to it is in fact the Bach tensor, well-known to vanish identically for critical points of $\int |W|^2$, which is in essence the energy $\mathcal{E}_B$. Details of this fastidious computation are left to the reader.

Let us return to the matter of estimates. From the table above along with (III.21) and the embedding $W^{-1,4/3} \hookrightarrow W^{-2,2}$, we find that

$$\|\vec{V}\|_{L^1 \otimes W^{-2,2}(B)} \lesssim \left( \|\nabla H\|_{L^2(B)} + \|h\|_{L^4(B)} \right)^2 \lesssim 1.$$
Since our energy is invariant by translation, we automatically obtain from Noether’s theorem that
\[ d^* \vec{V} = \vec{0}. \]

The following decisive Bourgain-Brezis type result has a highly non-trivial proof amply outlined in Chapter 6 of Eduard Curca’s Ph.D. thesis “Inversion of the divergence and Hodge systems”, Institut Camille Jordan, Lyon, France (2020). The original predecessor of this result is due to Maz’ya in “Bourgain-Brezis type inequality with explicit constants” in Interpolation theory and applications, Contemp. Math. 445, AMS (2007).

**Proposition III.1** Let \( B \subset \mathbb{R}^4 \) be a ball and let \( V \in \Lambda^1 \otimes W^{-2,2}(B) \) with \( d^* V = 0 \). There exists \( L \in \Lambda^2 \otimes W^{-1,2}(B) \) such that
\[ d^* L = V , \quad dL = 0 , \quad \| L \|_{W^{-1,2}(B)} \lesssim \| V \|_{L^1 \otimes W^{-2,2}(B)}. \]

Applied to our situation, we deduce the existence of \( \tilde{L}_0 \in \Lambda^2(B, \Lambda^1(\mathbb{R}^5)) \) satisfying
\[ d^* \tilde{L}_0 = \vec{V} , \quad d\tilde{L}_0 = \vec{0} , \quad \| \tilde{L} \|_{W^{-1,2}(B)} \lesssim 1. \quad (III.23) \]

We must now venture in some obligatory notational guidelines. We fix a basis \( \{ e^i \}_{i=1,\ldots,N} \) of \( \mathbb{R}^N \) and an orientation. Suppose \( \mathbb{R}^N \) is equipped with a metric \( g \). In order to ease the notation, we will denote the differential elements \( e^1 \wedge \ldots \wedge e^k \) by \( e^{i_1 \ldots i_k} \). A \( k \)-form \( A \) is defined as
\[ A = \frac{1}{k!} \sum_{i_1 < \ldots < i_k} A_{[i_1 \ldots i_k]} e^{i_1 \ldots i_k} \]
where
\[ A_{[i_1 \ldots i_k]} = \frac{1}{k!} \delta^{j_1 \ldots j_k}_{i_1 \ldots i_k} A_{j_1 \ldots j_k}. \]

The wedge product is defined as follows. For \( A \in \Lambda^p(\mathbb{R}^n) \) and \( B \in \Lambda^q(\mathbb{R}^n) \), we have
\[ A \wedge B = \frac{1}{(p+q)!} \sum_{a_1 < \ldots < a_p < b_1 < \ldots < b_q} A_{[a_1 \ldots a_p]b_{b_1} \ldots b_q} e^{a_1 \ldots a_p b_1 \ldots b_q} \in \Lambda^{p+q}(\mathbb{R}^n). \]

We next define two important bilinear maps. \( \mathbb{L} : \Lambda^p(\mathbb{R}^n) \times \Lambda^q(\mathbb{R}^n) \to \Lambda^{p+q}(\mathbb{R}^n) \), known as the *interior multiplication* of multivectors, by
\[ \langle A \mathbb{L} B, C \rangle_g = \langle A, B \wedge C \rangle_g \]
where \( A \in \Lambda^p(\mathbb{R}^n) \), \( B \in \Lambda^q(\mathbb{R}^n) \) and \( C \in \Lambda^{p+q}(\mathbb{R}^n) \) with \( p \geq q \). If \( p = 1 = q \) then we have the usual dot product: \( A \mathbb{L} B = \langle A, B \rangle_g \).

The *bullet dot operator* \( \bullet : \Lambda^p(\mathbb{R}^n) \times \Lambda^q(\mathbb{R}^n) \to \Lambda^{p+q-2}(\mathbb{R}^n) \) is well known in geometric algebra as the *first-order contraction operator*. For a \( k \)-form \( A \) and a 1-form \( B \), we define
\[ A \bullet B := A \mathbb{L} B. \]

For a \( p \)-form \( B \) and a \( q \)-form \( C \), we have
\[ A \bullet (B \wedge C) := (A \bullet B) \wedge C + (-1)^{pq} (A \bullet C) \wedge B. \]

\[ ^{10} \text{see in particular Theorem 6.4 and the discussion following Remark 6.5 on page 129.} \]
Defining the Levi-Civita tensor \( \epsilon_{abcd} := |g|^{1/2} \text{sign} \begin{pmatrix} 1 & 2 & 3 & 4 \\ a & b & c & d \end{pmatrix} \) enables us to define the Hodge star operator. For \( A \in \Lambda^p \),
\[
\star A = \frac{1}{(4 - p)!} \sum_{i_0 < i_1 < \ldots < i_p} \epsilon_{i_0 \ldots i_4} \left( \frac{1}{p!} A^{[i_1 \ldots i_p]} \right) e^{i_p+1 \ldots i_4}.
\]

With this definition of \( \star \), we find
\[
\star \star A = (-1)^{p(4-p)} A.
\]

If \( A \) is a \( p \)-form, then its exterior derivative is
\[
dA = \frac{1}{(p + 1)!} \sum_{i_0 < i_1 < \ldots < i_p} \nabla_{[i_0} A_{i_1 \ldots i_p]} e^{i_1 \ldots i_p}.
\]

For \( A \in \Lambda^p(\mathbb{R}^N) \) and \( B \in \Lambda^q(\mathbb{R}^N) \), the Leibnitz rule reads
\[
d(A \wedge B) = dA \wedge B + (-1)^p A \wedge dB.
\]

We also define \( d^* := \star d \star \). With this definition, we find that
\[
d^* A = \frac{1}{(p - 1)!} \sum_{i_1, i_2 < \ldots < i_p} \nabla^{i_1} A^{[i_1 \ldots i_p]} e^{i_1 \ldots i_p}.
\]

In particular, for a \( p \)-form \( A \),
\[
\Delta A = d^* dA + dd^* A = \frac{1}{p!} \sum_{i_1 < \ldots < i_p} \Delta g A^{[i_1 \ldots i_p]} e^{i_1 \ldots i_p}.
\]

We will work within the context of multivector-valued forms. Our objects will be elements of \( \Lambda^p(\mathbb{R}^4, \Lambda^q(\mathbb{R}^5)) \). To distinguish operations taking place in the parameter space \((\mathbb{R}^4, \Phi^* g)\) from operations taking place in the ambient space \((\mathbb{R}^5, g_{\text{Euc}})\), we will superimpose the operators defined above with the convention that the one on the bottom acts on parameter space, while the one at the top acts on ambient space. Typically, consider the two-vector-valued two-form \( \vec{\eta} \in \Lambda^2(\mathbb{R}^4, \Lambda^2(\mathbb{R}^5)) \) whose components are given by
\[
\vec{\eta}_{ij} = \nabla_i \vec{\Phi} \wedge \nabla_j \vec{\Phi}.
\]

We will write more succinctly
\[
\vec{\eta} := d\vec{\Phi} \wedge d\vec{\Phi}.
\]

Similar quantities are defined following the same logic. Naturally, for multivector 0-forms, the bottom symbol will be omitted, while for 0-vector-valued forms, the symbol on the top instead will be omitted. This should cause no confusion.

It is now time to return to our situation at hand. We have found \( \vec{L}_0 \in \Lambda^2(B, \Lambda^1(\mathbb{R}^5)) \) with \( d\vec{L}_0 = 0 \) and \( d^* \vec{L}_0 = \vec{V} \). This implies that
\[
d^*(\vec{L}_0 \mathbf{L} \, d\vec{\Phi}) = d^* \vec{L}_0 \mathbf{L} \, d\vec{\Phi} = \vec{V} \mathbf{L} \, d\vec{\Phi} = \text{Tr}_g G = -\alpha \Delta H^2.
\]
Letting
\[ \vec{L} := \vec{L}_0 - \frac{2\alpha}{3} d(H^2 d\vec{\Phi}) \]
yields that \( d\vec{L} = 0 \), and
\[ d^*(\vec{L} \cdot d\vec{\Phi}) = 0. \]
Hence there exists a potential \( S \in \Lambda^2(B, \Lambda^0(\mathbb{R}^5)) \) such that
\[ d^* S = \vec{L} \cdot d\vec{\Phi}. \tag{III.24} \]
The following estimate ensues from (III.23), namely:
\[ \|d^* S\|_{W^{-1,2}(B)} \lesssim 1. \]
I have also subreptitiously used the fact \( \nabla_2 \vec{\Phi} \) is essentially \( h \), which lies in \( L^4 \), and again that \( L^{4/3} \) injects continuously into \( W^{-1,2} \).

As we are free to set \( dS = 0 \), it follows by Miranda’s classical result that
\[ \|S\|_{L^2(B)} \lesssim 1. \tag{III.25} \]

We next compute
\[ d^*(\vec{L}_0 \wedge d\vec{\Phi}) = d^* \vec{L}_0 \wedge d\vec{\Phi} = \vec{V} \wedge d\vec{\Phi} = \nabla_b F^{ab} \vec{n} \wedge \nabla_a \vec{\Phi} = \nabla_b (F^{ab} \vec{n} \wedge \nabla_a \vec{\Phi}) + F^{ab} h^c_b \nabla_c \vec{\Phi} \wedge \nabla_a \vec{\Phi}. \]
As \( F^{ab} h^c_b \) is symmetric in the indices \( a \) and \( c \), the last term on the right-hand side vanishes. Putting
\[ \vec{F} := F^{ab} \vec{n} \wedge \nabla_a \vec{\Phi}, \]
we find that
\[ d^*(\vec{L}_0 \wedge d\vec{\Phi} - \vec{F}) = 0. \]
Note that\(^{11}\)
\[ d(H^2 d\vec{\Phi}) \wedge d\vec{\Phi} = \frac{1}{2} \nabla_j H^2 \eta^{ij} = \frac{1}{2} H^2 d^* \vec{\eta} - \frac{1}{2} d^*(H^2 \vec{\eta}). \]
Hence now\(^{12}\)
\[ d^* \left( \vec{L} \wedge d\vec{\Phi} - \vec{F} - \frac{\alpha}{3} H^2 d^* \vec{\eta} \right) = 0. \tag{III.26} \]

We now introduce the first conformal redressing. Let \( \vec{X} \) be the vector-field defined in (III.20). We remark\(^{13}\) that
\[ d^* \vec{X} = : \vec{J} \quad \text{and} \quad d\vec{X} = \mathcal{O}(|h| |\nabla h|), \tag{III.27} \]
where \( \vec{J} \) is given by Lemma II.0
\[ \vec{J} = \vec{n} \Delta H + \mathcal{O}(|H| |\nabla h|^2). \tag{III.28} \]

\(^{11}\)Recall that \( \vec{\eta}^{ij} := \nabla^i \vec{\Phi} \wedge \nabla^j \vec{\Phi}. \)

\(^{12}\)Recall also that \( \vec{L} = \vec{L}_0 - (2\alpha/3) d(H^2 d\vec{\Phi}). \)

\(^{13}\)Indeed, by the Codazzi-Mainardi identity, we have \( \nabla^i \left( h^{jk} \nabla_k \vec{\Phi} \right) - \nabla^j \left( h^{ik} \nabla_k \vec{\Phi} \right) = 0 \), so that no term of the forms \( \mathcal{O}(|h|^3) \) or \( \mathcal{O}(|\nabla h|) \) remains.

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We compute next
\[
(d^*(\vec{X} \wedge \nabla \Phi))^i = \frac{1}{2} \nabla_j (\vec{X}^j \wedge \nabla^i \Phi - \vec{X}^i \wedge \nabla^j \Phi)
\]
\[
= \frac{1}{2} (d^* \vec{X} \wedge d\tilde{\Phi} + \frac{1}{2}(\vec{X}^j \vec{X} \wedge - \vec{X}^i \vec{X} \wedge \nabla_j \Phi - \frac{1}{2} \nabla^i (\vec{X}^j \wedge \nabla^j \Phi)) + O(||\vec{X}||h))
\]
\[
= \frac{1}{2} d\vec{f} \wedge d\tilde{\Phi} + \frac{1}{2} d\vec{X} \wedge d\tilde{\Phi} + \frac{1}{2} d\vec{f} + O(||h||\nabla H| + |H||h|)
\]
\[
= \frac{n}{2} \Delta H \wedge d\tilde{\Phi} + d\vec{f} + O(||h||\nabla H| + |H||h|)
\]
(III.29) (III.28)
\[
= \frac{n}{2} \Delta H \wedge d\tilde{\Phi} + d\vec{f} + \tilde{u}.
\]

where
\[
\vec{f} := \frac{1}{2} \nabla^i \vec{X} \wedge \nabla_i \Phi = \frac{1}{2} \nabla^i \vec{H} \wedge \nabla_i \Phi.
\]

For notational convenience, we denote the remainder by \(\tilde{u} \in \Lambda^2(\Lambda^2)\), so as to arrive at the identity
\[
d^*(\vec{X} \wedge \nabla \Phi) = \frac{n}{2} \Delta H \wedge d\tilde{\Phi} + d\vec{f} + \tilde{u}.
\]
(III.29)

Bringing (III.29) into (III.26) yields next
\[
d^*(\vec{X} \wedge \nabla \Phi) = \frac{n}{2} \Delta H \wedge d\tilde{\Phi} + d\vec{f} + \tilde{u}.
\]

Therefore there exists \(\vec{R} \in \Lambda^2(\Lambda^2)\) satisfying
\[
d^* \vec{R} = \vec{L} \vec{d} \Phi - \vec{F} - \frac{\alpha}{2} H^2 d^* \vec{\eta} + (1 - \alpha) \left( \frac{n}{2} \Delta H \wedge d\tilde{\Phi} + d\vec{f} + \tilde{u} \right)
\]
(III.30)
\[
= \vec{L} \vec{d} \Phi + \frac{n}{2} \Delta H \wedge d\tilde{\Phi} + (1 - \alpha) d\vec{f} + \tilde{u} + \tilde{u}_1,
\]

where, per the data given in the table at the beginning of this section, we have
\[
\tilde{u}_1 = (1 - \alpha) \tilde{u} + O(H|h|^2) - \left(4\mu h_0^2 h_0^{ij} + 4\lambda(h_0^3)^{ij} + \lambda(T_{g}(h_0^3))g \right) \vec{n} \wedge \nabla_i \vec{\Phi}.
\]

Since \(\tilde{u}_0 = O(||h||\nabla H| + |H||h|)\) we get that \(\tilde{u}_1 = O(||h||\nabla H| + |H||h|)\) as well, and therefore that
\[
||\tilde{u}_1||_{L^4(\Lambda^2(B_k))} \lesssim \left(||\nabla H||_{L^2(B_k)} + ||h||_{L^4(B_k)}\right)||h||_{L^4(B_k)}
\]
\[
\lesssim (\varepsilon_0 + k^\alpha)||\nabla H||_{L^2(B)}
\]
(III.31)

where I have used the \(\varepsilon\)-regularity hypothesis as well as the early estimate (III.19).

This nice estimate is indeed sufficient for our purposes, but it is actually not evident at this point, and one must now endure a technical observation that caused me great grief to spot.

**Lemma III.0** It holds:
\[
\vec{\eta} \vec{L} \tilde{u}_1 = \vec{\eta} \vec{L} \tilde{u}_0, \quad \vec{\eta} \vec{L} \tilde{u}_1 + 3\tilde{u}_1 = \vec{\eta} \vec{L} \tilde{u}_0 + 3\tilde{u}_0, \quad \tilde{u}_1 \vec{L} d\tilde{\Phi} = \tilde{u}_0 \vec{L} d\tilde{\Phi},
\]
so that by (III.31), we have
\[
||\vec{\eta} \vec{L} \tilde{u}_1||_{L^4(\Lambda^2(B_k))} + ||\vec{\eta} \vec{L} \tilde{u}_1 + 3\tilde{u}_1||_{L^4(\Lambda^2(B_k))} + ||\tilde{u}_1 \vec{L} d\tilde{\Phi}||_{L^4(\Lambda^2(B_k))} \lesssim (\varepsilon_0 + k^\alpha)||\nabla H||_{L^2(B)}.
\]

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Proof. It is clear that
\[ \vec{\eta} \cdot (\vec{n} \nabla \Phi) = 0 , \]
hence the first claim follows immediately. As for the second one, we painstakingly compute
\[ \vec{\eta}^{jk} \cdot (\vec{n} \land \nabla_i \Phi) = (\nabla^j \Phi \land \nabla^k \Phi) \cdot (\vec{n} \land \nabla_i \Phi) = -g^{k}_{i} \vec{n} \land \nabla^j \Phi + g^{j}_{i} \vec{n} \land \nabla^k \Phi . \] (III.32)
Since \( h_0 \) is traceless, we find:
\[ \vec{\eta}^{jk} \cdot (\vec{n} \land \nabla_i \Phi) (h_0^i)^j = - (\vec{n} \land \nabla^j \Phi) (h_0^k)^j . \]
Moreover,
\[ \vec{\eta}^{jk} \cdot (\vec{n} \land \nabla_i \Phi) (h_0^3)^j = - (\vec{n} \land \nabla^j \Phi) (h_0^3)^j + \text{Tr}_g (h_0^3) \vec{n} \land \nabla^k \Phi \]
and
\[ \vec{\eta}^{jk} \cdot (\vec{n} \land \nabla_j \Phi) = -3 \vec{n} \land \nabla_k \Phi . \]
Putting the last three equalities together easily yields the second announced identity. Remains only to establish the last one. This is done similarly:
\[ (\vec{n} \land \nabla_i \Phi) \cdot \nabla_j \Phi = -g_{ij} \vec{n} , \]
so that, as \( h_0 \) is traceless:
\[ (h_0)^{ij} (\vec{n} \land \nabla_i \Phi) \cdot \nabla_j \Phi = 0 . \]
Moreover,
\[ (h_0^3)^{ij} (\vec{n} \land \nabla_i \Phi) \cdot \nabla_j \Phi = - \text{Tr}_g (h_0^3) \vec{n} . \]
The last desired identity ensues effortlessly.

Let us return to our previous considerations. We have showed that
\[ d^* \vec{R} = \vec{L} \cdot d \Phi + \frac{n}{2} \Delta H \land d \Phi + (1 - \alpha) d \vec{f} + \vec{u}_1 , \] (III.33)
From (III.31) along with (III.23) and the fact that \( |d \vec{f}| \simeq |\nabla H| \), we conclude without difficulty that
\[ \| d^* \vec{R} \|_{W^{-1,2}(B)} \lesssim 1 . \]
As we are free to set \( d \vec{R} = \vec{0} \), it follows by Miranda’s classical result that
\[ \| \vec{R} \|_{L^2(B)} \lesssim 1 . \] (III.34)

The final two pieces of the jigsaw puzzle are granted by the fact that \( d \vec{L} = \vec{0} \). This implies that
\[ d^* (\vec{L} \cdot d \Phi) = * d \vec{L} \cdot d \Phi = 0 \]
and similarly
\[ d^* (\vec{L} \land d \Phi) = * d \vec{L} \land d \Phi = 0 . \]
Accordingly, there exists \( U \in \Lambda^2(B, \Lambda^0(\mathbb{R}^5)) \) and \( \vec{T} \in \Lambda^2(B, \Lambda^2(\mathbb{R}^5)) \) such that
\[ d^* U = * \vec{L} \cdot d \Phi \quad \text{and} \quad d^* \vec{T} = * \vec{L} \land d \Phi , \] (III.35)
with estimates following from (III.23), namely:
\[ \| d^* U \|_{W^{-1,2}(B)} + \| d^* \vec{T} \|_{W^{-1,2}(B)} \lesssim 1 . \]
From this and that we may choose \( dU = 0 \) and \( d\vec{T} = \vec{0} \), it follows by Miranda’s classical result that
\[ \| U \|_{L^2(B)} + \| \vec{T} \|_{L^2(B)} \lesssim 1 . \] (III.36)
III.3 Some Intriguing Identities

We will show in this section that the potentials \( \mathcal{S}, \vec{R}, U, \) and \( \vec{T} \) are linked by some peculiar expressions for which I have no physical explanation. I deem them “intriguing”. Let us start by letting

\[
A := \vec{L} \cdot d\Phi, \quad B := \ast \vec{L} \cdot d\Phi, \quad C := \vec{L} \wedge d\Phi, \quad \bar{D} := \ast \vec{L} \wedge d\Phi. \]

**Proposition III.2** Let \( \vec{L} \in \Lambda^2(\Lambda^1) \). The following identities hold

\[
\bar{\eta} \vec{L} \cdot C = A \quad \text{and} \quad \bar{C} = \frac{1}{6} \bar{\eta} \vec{L} \cdot \bar{C} + \frac{1}{2}(\ast \bar{D}) \vec{L} \cdot \bar{\eta} - \frac{1}{6} \bar{\eta} \vec{L} \cdot A - \frac{1}{2}(\ast B) \vec{L} \cdot \bar{\eta}. \]

**Proof.** Let us first compute:

\[
(\vec{L}^i_j \wedge \nabla^k \vec{\Phi}) \cdot (\nabla_a \vec{\Phi} \wedge \nabla_b \vec{\Phi}) = g_{bc} \vec{L}^i_j \wedge \nabla_a \vec{\Phi} + g_{ab} \vec{L}^i_j \wedge \nabla_b \vec{\Phi} + (\vec{L}^i_j \cdot \nabla_a \vec{\Phi}) \eta^k_b - (\vec{L}^i_j \cdot \nabla_b \vec{\Phi}) \eta^k_a. \quad (III.37)
\]

Accordingly, we find on one hand:

\[
\ast \bar{D} \cdot \bar{\eta} = (\vec{L} \wedge d\Phi) \vec{L} \cdot \bar{\eta} = \frac{1}{3} (\vec{L}^a \wedge \nabla^c \vec{\Phi} + \vec{L}^a \wedge \nabla^b \vec{\Phi} + \vec{L}^b \wedge \nabla^a \vec{\Phi}) \bullet (\nabla_a \vec{\Phi} \wedge \nabla_b \vec{\Phi}) = \frac{1}{3} (\vec{L}^a \wedge \nabla^c \vec{\Phi} + 2\vec{L}^a \wedge \nabla^b \vec{\Phi}) \bullet (\nabla_a \vec{\Phi} \wedge \nabla_b \vec{\Phi}) = \frac{1}{3} \left( 2\vec{L}^a_c \wedge \nabla_a \vec{\Phi} + 2(\vec{L}^a \cdot \nabla_a \vec{\Phi}) \eta^c_b + 6\vec{L}^a \wedge \nabla_a \vec{\Phi} - 2(\vec{L}^a \cdot \nabla_b \vec{\Phi}) \eta^b_a \right) = -\frac{4}{3} \bar{C} + \frac{2}{3} \bar{\eta} \vec{L} \cdot A + \frac{2}{3}(\vec{L}^a \cdot \nabla^b \vec{\Phi}) \bar{\eta}_{ab}. \quad (III.38)
\]

And on the other hand, we have

\[
\bar{\eta} \vec{L} \cdot \bar{C} = \bar{\eta} \vec{L} \cdot (\vec{L} \wedge d\Phi) = \bar{\eta} \vec{L} \cdot (\vec{L} \wedge \nabla^a \vec{\Phi}) = (\nabla^c \vec{\Phi} \wedge \nabla^b \vec{\Phi}) \bullet (\vec{L}^a \wedge \nabla^c \vec{\Phi}) = -g^{bc} \vec{L} \wedge \nabla_a \vec{\Phi} - (\vec{L} \cdot \nabla^c \vec{\Phi}) \eta^c_a + (\vec{L} \cdot \nabla^c \vec{\Phi}) \eta^c_a = \bar{C} + \bar{\eta} \vec{L} \cdot A + (\vec{L} \cdot \nabla^c \vec{\Phi}) \eta_{ac}. \quad (III.39)
\]

Finally, we compute

\[
\ast B \vec{L} \cdot \bar{\eta} = (\vec{L} \cdot d\Phi) \vec{L} \cdot \bar{\eta} = \frac{1}{3} (\vec{L}^a \cdot \nabla^c \vec{\Phi} + \vec{L}^a \cdot \nabla^b \vec{\Phi} + \vec{L}^b \cdot \nabla^c \vec{\Phi}) \eta_{ab} = \frac{1}{3} (\vec{L}^a \cdot \nabla^c \vec{\Phi} + 3\vec{L}^a \cdot \nabla^b \vec{\Phi}) \eta_{ab}. \quad (III.40)
\]
Combining together (III.38)-(III.40), we see that
\[ \vec{C} = -\frac{1}{3} \vec{C} + \eta \bigtriangleup A - *D^\bigtriangledown \eta - *B \bigtriangleup \eta . \]

This is the first announced identity. The second one is found by the same token, namely:
\[
\vec{C} = \eta \bigtriangleup \bigtriangleup \eta \cdot (\vec{L} \bigtriangleup \bigtriangleup d\vec{\Phi}) = \eta^{ab} \cdot (\vec{L}_{ac} \bigtriangleup \bigtriangleup \vec{\Phi})
\]
\[= \vec{L}^{ab} \cdot \nabla \vec{\Phi} \]
\[= A . \]

In Proposition [III.2], choose \( A = d^* S, B = d^* U, \vec{D} = d^* \vec{T}, \) and
\[ \vec{C} = d^* \vec{R} - \frac{\eta}{2} \Delta H \bigtriangleup d\vec{\Phi} + (\alpha - 1)d\vec{f} - \vec{u}_1 , \]
so as to satisfy (III.24), (III.33), and (III.35). Observing that
\[ \eta \bigtriangleup \bigtriangleup \eta \cdot (\vec{u} \bigtriangleup d\vec{\Phi}) = -3(\vec{u} \bigtriangleup \nabla \vec{\Phi}) \]
then yields the identity
\[ d^* \vec{R} = -\frac{1}{3} \eta \bigtriangleup \bigtriangleup \eta \bigtriangleup d^* \vec{R} + \eta \bigtriangleup \bigtriangleup d^* S - d(\bigtriangleup^* \vec{T}) \eta \bigtriangleup \bigtriangleup \eta \bigtriangleup \bigtriangleup d\vec{f} + (1 - \alpha)d\vec{f} \]
\[= -\frac{1}{3} \eta \bigtriangleup \bigtriangleup \eta \bigtriangleup \bigtriangleup d^* \vec{R} + \eta \bigtriangleup \bigtriangleup d^* S - *\eta \bigtriangleup \bigtriangleup d\vec{\Phi} - *\eta \bigtriangleup \bigtriangleup d^* U + \vec{u}_2 + \frac{1 - \alpha}{3} \eta \bigtriangleup \bigtriangleup d\vec{f} + (1 - \alpha)d\vec{f} , \]
(III.41)
where we have set
\[ \vec{u}_2 := \vec{u}_1 + \frac{1}{3} \eta \bigtriangleup \bigtriangleup \vec{u}_1 . \]

From Lemma [III.0] it holds immediately
\[ ||\vec{u}_2||_{L^4(B_k)} \lesssim (\varepsilon_0 + k^a)||\nabla H||_{L^2(B)} . \]
(III.42)

In exactly the same manner, using this time the first identity given in Proposition [III.2] yields
\[ d^* S = \eta \bigtriangleup \bigtriangleup d^* \vec{R} + u_3 + (\alpha - 1)\eta \bigtriangleup \bigtriangleup d\vec{f} , \]
(III.43)
where
\[ u_3 := \eta \bigtriangleup \bigtriangleup \vec{u}_1 , \]
satisfies according to Lemma [III.0]
\[ ||u_3||_{L^4(B_k)} \lesssim (\varepsilon_0 + k^a)||\nabla H||_{L^2(B)} . \]
(III.44)
III.4 Further Identities and Second Redressing

Using the fact that $S$ is a closed two-form, we find without difficulty that

$$
\tilde{\eta} \mathcal{L} d^* S = d^* (\tilde{\eta} \cdot S) + \frac{1}{2} d(\tilde{\eta}, S) + \mathcal{O}(|S| \|
abla \tilde{\eta}||)
$$

Similarly,

$$
\tilde{\eta} \mathcal{L} d^* \tilde{R} = d^* (\tilde{\eta} \cdot \tilde{R}) + \frac{1}{2} d(\tilde{\eta}, \tilde{R}) + \mathcal{O}(|\tilde{R}| \|
abla \tilde{\eta}||)
$$

as well as

$$
\star \tilde{\eta} \mathcal{L} d^* U = d^* (\star \tilde{\eta} \cdot U) + \frac{1}{2} d(\star \tilde{\eta}, U) + \mathcal{O}(|U| \|
abla \tilde{\eta}||)
$$

and

$$
\star \tilde{\eta} \mathcal{L} d^* \tilde{T} = d^* (\star \tilde{\eta} \cdot \tilde{T}) + \frac{1}{2} d(\star \tilde{\eta}, \tilde{T}) + \mathcal{O}(|\tilde{T}| \|
abla \tilde{\eta}||)
$$

Finally

$$
\tilde{\eta} \mathcal{L} d \tilde{f} = d^* (\tilde{\eta} \cdot \tilde{f}) + \mathcal{O}(|\tilde{f}| \|
abla \tilde{\eta}||) = \mathcal{O}(|\tilde{f}| \|
abla \tilde{\eta}||)
$$

and

$$
\tilde{\eta} \mathcal{L} d \tilde{f} = d^* (\tilde{\eta} \cdot \tilde{f}) + \mathcal{O}(|\tilde{f}| \|
abla \tilde{\eta}||).
$$

Bringing the latter six equalities into (III.41) gives now

$$
d^* \left( \tilde{R} + \frac{1}{3} \tilde{\eta} \cdot \tilde{R} - \tilde{\eta} \cdot S + (\star \tilde{\eta}) \cdot \tilde{T} + \star \tilde{\eta} \cdot U + \frac{1}{3} \tilde{\eta} \cdot \tilde{f} \right)
$$

$$
= \frac{1}{2} d\left( -\frac{1}{3} (\tilde{\eta} \cdot \tilde{R}) + (\tilde{\eta}, S) - (\star \tilde{\eta} \cdot \tilde{T}) - (\star \tilde{\eta}, U) + 2(1-\alpha)\tilde{f} \right) + u_4,
$$

where

$$
u_4 := u_2 + \mathcal{O}(\|
abla \tilde{\eta}|| |S| + |\tilde{R}| + |U| + |\tilde{T}||)
$$

so that by (III.42), along with (III.25), (III.34), and (III.36):

$$
\|u_4\|_{L^{4/3}(B_k)} \lesssim (\varepsilon_0 + k^2) \|\nabla H\|_{L^2(B)}.
$$

(III.46)

Letting now

$$
\tilde{P} := \frac{1}{3} \tilde{\eta} \cdot \tilde{R} - \tilde{\eta} \cdot S + (\star \tilde{\eta}) \cdot \tilde{T} + \star \tilde{\eta} \cdot U + \frac{1}{3} \tilde{\eta} \cdot \tilde{f}
$$

and

$$
\tilde{q} := -\frac{1}{3} (\tilde{\eta} \cdot \tilde{R}) + (\tilde{\eta}, S) - (\star \tilde{\eta} \cdot \tilde{T}) - (\star \tilde{\eta}, U) + 2(1-\alpha)\tilde{f}
$$

gives the equivalent equation

$$
d^*(\tilde{R} + \tilde{P}) = \frac{1}{2} d\tilde{q} + u_4.
$$

(III.47)

Our previous estimates (III.25), (III.34), and (III.36) yield

$$
\|\tilde{q}\|_{L^2(B)} \lesssim 1.
$$

(III.48)
Since $q$ is a scalar, it follows that
\[ \Delta q = -2d^* \bar{u}_4. \]
Such equations have been extensively studied. In particular, the estimates given in [Diff] imply
\[ \|d\bar{q}\|_{L^{4/3}(B_k)} \lesssim \|\bar{u}_4\|_{L^{4/3}(B_k)} + k^b \|\bar{q}\|_{L^2(B)} \lesssim \|\bar{u}_4\|_{L^{4/3}(B_k)} + k^b, \quad \text{for some } b > 0. \]
Put into (III.47) yields in turn that
\[ \|d^*(\bar{R} + \bar{P})\|_{L^{4/3}(B_k)} \lesssim \|\bar{u}_4\|_{L^{4/3}(B_k)} + k^b \]
\[ \lesssim (\varepsilon_0 + k^c) \|\nabla H\|_{L^2(B)} \quad \text{for some } c > 0. \quad \text{(III.49)} \]
This estimate will play a decisive role in the procedure that comes next. But first we need an important result that will prove there does exist a suitable redressing of our variables.

**Lemma III.0** The operator
\[ \bar{P} : \Lambda^1(\Lambda^1(\mathbb{R}^5)) \to \Lambda^2(\Lambda^2(\mathbb{R}^5)) \quad \text{defined by} \quad \bar{P} : \ell \mapsto \ell \wedge d\bar{\Phi} \]
is algebraically invertible. Moreover $\bar{P}^{-1}$ injects $L^2$ into itself continuously.

**Proof.** We have
\[ \bar{P}_{pq} := (\bar{P}(\bar{e}))_{pq} = \bar{e}_p \wedge \nabla_q \bar{\Phi} - \bar{e}_q \wedge \nabla_p \bar{\Phi}. \]
Let $\bar{n}$ be the normal vector to our hypersurface.
We compute
\[ \bar{P}_{pq} \cdot (\bar{n} \wedge \nabla^q \bar{\Phi}) = (\bar{e}_p \wedge \nabla_q \bar{\Phi} - \bar{e}_q \wedge \nabla_p \bar{\Phi}) \cdot (\bar{n} \wedge \nabla^q \bar{\Phi}) \]
\[ = 3\bar{n} \cdot \bar{e}_p, \]
so that
\[ \bar{n} \cdot \bar{e}_p = \frac{1}{3} \bar{P}_{pq} \cdot (\bar{n} \wedge \nabla^q \bar{\Phi}). \quad \text{(III.50)} \]
On the other hand, we have
\[ \bar{P}_{pq} \cdot (\nabla^p \bar{\Phi} \wedge \nabla^q \bar{\Phi}) = (\bar{e}_p \wedge \nabla_q \bar{\Phi} - \bar{e}_q \wedge \nabla_p \bar{\Phi}) \cdot (\nabla^p \bar{\Phi} \wedge \nabla^q \bar{\Phi}) \]
\[ = 6\bar{e}_p \cdot \nabla^p \bar{\Phi}, \]
so that
\[ \bar{e}_p \cdot \nabla^p \bar{\Phi} = \frac{1}{6} \bar{P}_{pq} \cdot (\nabla^p \bar{\Phi} \wedge \nabla^q \bar{\Phi}). \quad \text{(III.51)} \]
Finally, we check that
\[ \bar{P}_{pq} \cdot (\nabla^i \bar{\Phi} \wedge \nabla^q \bar{\Phi}) = (\bar{e}_p \wedge \nabla_q \bar{\Phi} - \bar{e}_q \wedge \nabla_p \bar{\Phi}) \cdot (\nabla^i \bar{\Phi} \wedge \nabla^q \bar{\Phi}) \]
\[ = 2\bar{e}_p \cdot \nabla^i \bar{\Phi} + \delta^i_p \bar{e}_q \cdot \nabla^q \bar{\Phi} \]
\[ \quad \text{(III.51)} \]
in divergence form and with uniformly elliptic coefficients in $W^{1,4} \cap L^\infty$. 

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16 in divergence form and with uniformly elliptic coefficients in $W^{1,4} \cap L^\infty$. 

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so that
\[ \vec{\ell}_p \cdot \nabla^i \Phi = \frac{1}{2} \vec{\omega}_{pq} \cdot (\nabla^i \Phi \wedge \nabla^q \Phi) - \frac{1}{12} \delta^i_p \vec{\omega}_{sq} \cdot (\nabla^s \Phi \wedge \nabla^q \Phi). \] (III.52)

According to (III.50) and (III.52), the 1-form \( \vec{\ell} \) can be totally recovered from \( \vec{\omega} \) via
\[ \vec{\ell}_p = \left[ \frac{1}{2} \vec{\omega}_{pq} \cdot (\nabla^i \Phi \wedge \nabla^q \Phi) - \frac{1}{12} \delta^i_p \vec{\omega}_{sq} \cdot (\nabla^s \Phi \wedge \nabla^q \Phi) \right] \nabla^i \Phi + \left[ \frac{1}{3} \vec{\omega}_{pq} \cdot (\vec{\eta} \wedge \nabla^q \Phi) \right] \vec{\eta}. \]

That \( \vec{\omega}^{-1} \) maps \( L^2 \) onto itself continuously is trivial.

Consider next
\[ \vec{\ell} := * \vec{\omega}^{-1}(* \vec{\omega}) \]

By definition, it satisfies
\[ d^* \vec{\ell} \overset{\wedge}{\bigwedge} d\Phi = d^* \vec{\omega}. \]

Accordingly, recalling (III.33), we find
\[ d^* \vec{R}_1 = \vec{L}_1 \overset{\wedge}{\bigwedge} d\Phi + \frac{1}{2} \Delta H \wedge d\Phi + (1 - \alpha) df + \vec{u}_1. \] (III.53) where
\[ \vec{R}_1 := \vec{R} + \vec{\omega} \quad \text{and} \quad \vec{L}_1 := \vec{L} + d^* \vec{\ell}. \]

From (III.49), it follows that
\[ ||d^* \vec{R}_1||_{L^{3/2}(B_k)} \lesssim (\varepsilon_0 + k^c) \| \nabla H \|_{L^2(B)}. \] (III.54)

We similarly redress \( S \) and set
\[ S_1 := S + \vec{\ell} \overset{\wedge}{\bigwedge} d\Phi \]

to discover that
\[ d^* S_1 = \vec{L}_1 \overset{\wedge}{\bigwedge} d\Phi. \] (III.55)

Since \( \vec{R}_1 \) and \( S_1 \) are linked by relations analogous to those linking \( \vec{R} \) and \( S \), we may invoke once more Proposition (III.2) to obtain the counterpart to (III.43):
\[ d^* S_1 = \vec{\eta} \overset{\wedge}{\bigwedge} d^* \vec{R}_1 + u_3 + (\alpha - 1) \vec{\eta} \overset{\wedge}{\bigwedge} df \]

With the help of the first identity in (III.45), we arrive at
\[ d^* S_1 = \vec{\eta} \overset{\wedge}{\bigwedge} d^* \vec{R}_1 + u_3 + \mathcal{O}(|| \nabla \vec{\eta} ||_{L^1(B)}) \]

Hence (III.54) and (III.44), along with our \( \varepsilon \)-regularity hypothesis (I.4) imply
\[ ||d^* S_1||_{L^{3/2}(B_k)} \lesssim (\varepsilon_0 + k^c) \| \nabla H \|_{L^2(B)}. \] (III.56)
III.5 Back to Geometry: the Return Equation

All of our efforts up to this point are about to pay off, but it has first to be made manifest that conformal invariance has been broken, and that a return to geometry is indeed possible. That is accomplished thanks to the following elementary – but illuminating – computations. We first observe that

\[
(\hat{L}_1 \mathcal{L} d\tilde{\Phi}) + (\hat{L}_1 \mathcal{L} d\tilde{\Phi}) \mathcal{L} d\tilde{\Phi} = (\hat{L}^{ab} \wedge \nabla_a \tilde{\Phi}) \mathcal{L} \nabla_b \tilde{\Phi} + (\hat{L}^{ab} \cdot \nabla_a \tilde{\Phi}) \nabla_b \tilde{\Phi}
\]

\[
= (\hat{L}^{ab} \cdot \nabla_b \tilde{\Phi}) \nabla_a \tilde{\Phi} + (\hat{L}^{ab} \cdot \nabla_a \tilde{\Phi}) \nabla_b \tilde{\Phi}
\]

\[
= \hat{0}.
\]

In addition, it holds

\[
\mathcal{L} (f d\tilde{\Phi}) = \nabla_i (\nabla^i H \cdot \nabla^j \tilde{\Phi}) \mathcal{L} \nabla^i \tilde{\Phi}
\]

\[
= (\nabla^i H \cdot \nabla^j \tilde{\Phi}) \nabla^i \tilde{\Phi} - \tilde{n} \Delta H + (\nabla^i H \cdot \nabla^j \tilde{\Phi}) \tilde{n}_{ij}
\]

\[
= \nabla^i (\nabla^j H \cdot \nabla^i \tilde{\Phi}) - 2\nabla^j H^2 - \tilde{n} \Delta H - H |h|^2
\]

\[
= \nabla^i (-H \tilde{n}_{ij} - 2H^2) - H |h|^2 - \tilde{n} \Delta H
\]

\[
= u_5 - \tilde{n} \Delta H,
\]

where, by the Codazzi-Mainardi identity,

\[
\begin{align*}
\|u_5\|_{L^4/3(B)} & \lesssim (\varepsilon_0 + k^c) \|\nabla H\|_{L^2(B)}.
\end{align*}
\]

Exactly as we proved (III.31), we obtain

\[
\|u_5\|_{L^4/3(B)} \lesssim (\varepsilon_0 + k^c) \|\nabla H\|_{L^2(B)}.
\]

Altogether, (III.54), (III.56), (III.57), and (III.58) now yield the Return Equation:

\[
\begin{align*}
d^{\star} \hat{R}_1 \mathcal{L} d\tilde{\Phi} + d^{\star} S_1 \mathcal{L} d\tilde{\Phi} &= (1 - \alpha) df \mathcal{L} d\tilde{\Phi} + \bar{u}_1 \mathcal{L} d\tilde{\Phi} + \frac{\Delta H}{2} (\bar{n} \wedge \nabla^i \tilde{\Phi}) \mathcal{L} \nabla^i \tilde{\Phi}
\end{align*}
\]

\[
= (1 - \alpha) u_5 + \bar{u}_1 \mathcal{L} d\tilde{\Phi} + \left(\alpha - \frac{1}{2}\right) \bar{n} \Delta H.
\]

Since without loss of generality\(^{17}\), we may choose \(\alpha \neq 1/2\). We then piece together (III.54), (III.56), (III.59), and Lemma III.0 to obtain from the latter the estimate

\[
\|\Delta H\|_{L^4/3(B)} \lesssim (\varepsilon_0 + k^c) \|\nabla H\|_{L^2(B)}.
\]

At this point in the analysis, the game is almost over, and it is routine to finish things. Let us see precisely how. Consider the operator in flat divergence form

\[
|g|^{1/2} \Delta u = \partial_i (|g|^{1/2} g^{ij} \partial_j u).
\]

By hypothesis on our metric, the operator on the right-hand is uniformly elliptic with bounded coefficients and in divergence form. As shown in [GrW], the corresponding Green function \(G\) satisfies

\[
\|G\|_{L^{2,\infty}(B)} + \|\nabla G\|_{L^{4/3,\infty}(B)} < \infty,
\]

\(^{17}\)via rescaling our energy, the only two cases are \(\alpha = 0\) or \(\alpha = 1\).
where the weak Marcinkiewicz space $L^{p,\infty}(B)$ is made of functions $f$ which satisfy

$$\sup_{\alpha > 0} \alpha^p \left| \left\{ x \in B : |f(x)| \geq \alpha \right\} \right| < \infty.$$  

The space $L^{p,\infty}$ is also a Lorentz space \[\text{Hun}\]. Formally, we have the convolution product

$$\nabla H = \nabla G \ast \left( |g|^{1/2} \Delta H \right)$$

with the estimate

$$\|\nabla H\|_{L^2(B_{x,2})} \leq \|\nabla H\|_{L^{2,4/3}(B_{x,2})} \leq \|\nabla G\|_{L^{4/3,\infty}(B_1)} \|g|^{1/2} \Delta H\|_{L^{4/3}(B_2)} + k^s,$$

for some $s > 0$. The first inequality is the embedding $L^{2,4/3} \hookrightarrow L^{2,2} \equiv L^2$. The second one is a classical singular integral estimate along with the convolution product rule

$$L^{4/3,\infty} \ast L^{4/3,\infty} \equiv L^{4/3,4/3} \ast L^{4/3,4/3} \hookrightarrow L^{2,4/3}.$$

Calling upon (III.61) and (III.60) yields thus

$$\|\nabla H\|_{L^2(B_{x,2})} \lesssim (\varepsilon_0 + k^q) \|\nabla H\|_{L^2(B)} \quad \text{for some } q > 0.$$

From standard controlled-growth arguments (see for instance Lemma 5.13 in \[\text{GM}\]), a Morrey-type decay arises:

$$\|\nabla H\|_{L^2(B_r)} \lesssim r^\beta \quad \text{for some } \beta > 0,$$

which in turn implies via (III.60) that

$$\|\Delta H\|_{L^{4/3}(B_r)} \lesssim r^\beta. \quad (\text{III.62})$$

Consider next the maximal function

$$\mathcal{M}_{3-\beta}[f] := \sup_{r > 0} r^{-1-\beta} \|f\|_{L^1(B_r)}.$$

By Jensen’s inequality, we have

$$\mathcal{M}_{3-\beta}[f] \lesssim \sup_{r > 0} r^{-\beta} \|f\|_{L^{4/3}(B_r)}.$$

Using the Morrey decay (III.62), we obtain

$$\|\mathcal{M}_{3-\beta}[|g|^{1/2} \Delta H]\|_{L^{\infty}(B_r)} \lesssim 1 \quad \forall \ r < 1. \quad (\text{III.63})$$

For a locally integrable function $f$ on $\mathbb{R}^n$ and $\alpha \in (0,4)$, the Riesz potential $\mathcal{I}_\alpha$ of order $\alpha$ of $f$ is defined by the convolution

$$(\mathcal{I}_\alpha \ast f)(x) := \int_{\mathbb{R}^4} f(y) |x - y|^{\alpha-n} dy.$$  

We will now use the following interesting result from \[\text{Ada}\].

**Proposition III.3** If $\alpha > 0$, $0 < \lambda \leq 4$, $1 < p < \lambda/\alpha$, $1 \leq q \leq \infty$, and $f \in L^p(\mathbb{R}^4)$ with $\mathcal{M}_{\lambda/p}[f] \in L^q(\Omega)$, $\Omega \subset \mathbb{R}^4$, then

$$\|\mathcal{I}_\alpha[f]\|_{L^r(\Omega)} \lesssim \|\mathcal{M}_{\lambda/p}[f]\|_{L^q(\Omega)}^{\alpha p/\lambda} \|f\|_{L^p(\Omega)}^{1-\alpha p/\lambda}$$

where $1/r = 1/p - \alpha/\lambda + (\alpha p)/(\lambda q)$.  

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Putting $\alpha = 1, q = \infty, p = 4/3, \lambda = 4(3-\beta)/3$ in Proposition III.3, we find

$$\|I_1[|g|^{1/2}\Delta H]\|_{L^s(B_r)} \lesssim \|M_{3-\beta}[|g|^{1/2}\Delta H]\|_{L^\infty(B_r)}^{1/(3-\beta)}\|g\|_{L^{1/(3-\beta)}(B_r)}^{(2-\beta)/(3-\beta)}. \quad \text{(III.64)}$$

where

$$s := \frac{4}{3}\left(\frac{3-\beta}{2-\beta}\right) \in \left(2, \frac{8}{3}\right).$$

Introducing (III.63) and (III.60) in (III.64) yields

$$\|I_1[|g|^{1/2}\Delta H]\|_{L^s(B_r)} \lesssim 1,$$

which gives, by standard elliptic estimates that we have for all $r < 1$

$$\|\nabla H\|_{L^s(B_r)} \lesssim 1.$$ 

As this holds for all $s \in (2, \frac{8}{3})$, we see that the integrability of $\nabla H$ has been improved, and the proof essentially ends now. One eventually reaches the following typical $\varepsilon$-regularity estimate

$$\|\nabla H\|_{L^\infty(B_r)} \leq \frac{C}{r^2} \left(\|\nabla H\|_{L^2(B)} + \|h\|_{L^4(B)}\right),$$

for some constant $C$ depending only on $\|d\Phi\|_{L^\infty(B)}$. The smoothness of $\Phi$ ensues with corresponding estimates.

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