INTEGRAL CURVATURE BOUNDS AND BOUNDED DIAMETER WITH BAKRY–EMERY RICCI TENSOR

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ABSTRACT. For Riemannian manifolds with a smooth measure \((M, g, e^{-f}dv_g)\), we prove a generalized Myers compactness theorem when Bakry–Emery Ricci tensor is bounded from below and \(f\) is bounded.

1. INTRODUCTION

One of the most fundamental results in Riemannian geometry is the Myers theorem, which states that if a complete Riemannian manifold \((M, g)\) satisfies \(\text{Ric} \geq (n-1)H\) with \(H > 0\), then \(M\) is compact and \(\text{diam}(M) \leq \frac{\pi}{\sqrt{H}}\). Here, \(\text{Ric}\) is the Ricci curvature of the metric \(g\). This theorem has been generalized through different approaches (see [1], [2], [6], and [8]), one of which is the effort of Wei and Wylie, who proved the theorem for manifolds with a positive lower Bakry-Emery Ricci curvature bound in [7]. A Bakry-Emery Ricci tensor is defined as

\[ \text{Ric}_f = \text{Ric} + \text{Hess} \, f, \]

where \(f\) is a smooth function on \(M\) and \(\text{Hess} \, f\) is the hessian of \(f\). Where \(|f| \leq k\), they proved that

\[ \text{diam}(M) \leq \frac{\pi}{\sqrt{H}} + \frac{4k}{(n-1)\sqrt{H}}. \]

Several works have attempted to generalize this result (for example, see [4], [5], [9], and [10]), including that of Sprouse which can be summarized in the following three theorems.

**Theorem 1.1** ([6]). Let \((M, g)\) be a compact Riemannian manifold of nonnegative Ricci curvature. Then, for any \(\delta > 0\), there exists \(\epsilon = \epsilon(n, \delta)\) such that if

\[ \frac{1}{\text{vol}(M)} \int_M ((n-1) - \text{Ric}_-)_+ \, dv_g < \epsilon(n, \delta), \]

then \(\text{diam}(M) < \pi + \delta\).

Here, \(dv_g\) is the Riemannian volume density on \(M\), \(\text{Ric}_-\) is the lowest eigenvalue of the Ricci tensor \(\text{Ric}(x)\), and \(h_+(x) = \max\{h(x), 0\}\) for an arbitrary function \(h\) on \(M\). For \(\text{Ric} \geq (n-1)k\) with \(k \leq 0\), these generalizations are attained.

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Theorem 1.2 (6). Let \((M, g)\) be a complete Riemannian manifold with \(\text{Ric} \geq (n-1)k\), \(k \leq 0\). Then, for any \(R, \delta > 0\), there exists \(\epsilon = \epsilon(n, k, R, \delta)\) such that if
\[
\sup_x \frac{1}{\text{vol}(B(x, R))} \int_{B(x, R)} ((n - 1) - \text{Ric}_-)_+ dv_g < \epsilon(n, k, R, \delta),
\]
then \((M, g)\) is compact and \(\text{diam}(M) < \pi + \delta\).

Theorem 1.3 (6). Let \((M, g)\) be a complete Riemannian manifold with \(\text{Ric} \geq (n-1)k\), \(k \leq 0\). Then, for any \(R > 0\), there exists \(\tilde{\epsilon} = \tilde{\epsilon}(n, k, R)\) such that if
\[
\sup_x \frac{1}{\text{vol}(B(x, R))} \int_{B(x, R)} ((n - 1) - \text{Ric}_-)_+ dv_g < \tilde{\epsilon}(n, k, R),
\]
then the universal cover of \(M\) is compact, and hence, \(\pi_1(M)\) is finite.

We will generalize these results to the Bakry-Emery Ricci tensor bounded from below. Let \((M, g, e^{-f} dv_g)\) be a smooth metric measure space, where \(M\) is a complete \(n\)-dimensional Riemannian manifold with metric \(g\). Likewise, let \(\text{Ric}_f(x)\) denote the lowest eigenvalue of the Bakry-Emery Ricci tensor \(\text{Ric}_f(x)\). Then, we prove the following theorem.

Theorem 1.4. Let \((M, g, e^{-f} dv_g)\) be a compact \(n\)-dimensional Riemannian manifold with \(\text{Ric}_f \geq 0\) and \(|f| \leq k\). Then, for any \(\delta > 0\), there exists \(\epsilon = \epsilon(n + 4k, \delta)\) such that if
\[
\frac{1}{\text{vol}(M)} \int_M ((n - 1) - \text{Ric}_f)_+ e^{-f} dv_g < \epsilon(n + 4k, \delta),
\]
then \(\text{diam}(M) < \pi + \delta\).

Given that \((M, g)\) is noncompact or does not exhibit a nonnegative Bakry-Emery Ricci curvature, averaging the bad part of \(\text{Ric}_f\) over metric ball, as in (6) yields a similar result as follows.

Theorem 1.5. Let \((M, g, e^{-f} dv_g)\) be a complete \(n\)-dimensional Riemannian manifold with \(\text{Ric}_f \geq (n-1)H\), \(H < 0\), and \(|f| \leq k\). Then, for any \(R, \delta > 0\), there exists \(\epsilon = \epsilon(n + 4k, H, R, \delta)\) such that if
\[
\sup_x \frac{1}{\text{vol}(B(x, R))} \int_{B(x, R)} ((n - 1) - \text{Ric}_f)_+ e^{-f} dv_g < \epsilon(n + 4k, H, R, \delta),
\]
then \(M\) is compact and \(\text{diam}(M) < \pi + \delta\).

Finally, we could obtain the result for the fundamental group of \(M\), which is stated as follows.

Theorem 1.6. Let \((M, g, e^{-f} dv_g)\) be a complete \(n\)-dimensional Riemannian manifold with \(\text{Ric}_f \geq (n-1)H\), \(H < 0\), and \(|f| \leq k\). Then for any \(R > 0\), there exists \(\hat{\epsilon} = \hat{\epsilon}(n + 4k, H, R)\) such that if
\[
\sup_x \frac{1}{\text{vol}(B(x, R))} \int_{B(x, R)} ((n - 1) - \text{Ric}_f)_+ e^{-f} dv_g < \hat{\epsilon}(n + 4k, H, R),
\]
then the universal cover of \(M\) is compact, and hence, \(\pi_1(M)\) is finite.
2. Proof of Theorem 1.4

Let \((M, g, e^{-f} dv_g)\) be a smooth metric measure space, where \((M, g)\) is a complete \(n\)-dimensional Riemannian manifold. Let \(A_1, A_2, W\) be open subsets of \(M\) such that \(A_1, A_2 \subset W\), and all minimal geodesics \(\gamma_{x,y}\) from \(x \in A_1\) to \(y \in A_2\) lie in \(W\).

We will use the estimate of Cheeger and Colding for Bakry-Emery Ricci tensor (3, Proposition 2.3); thus, for a nonnegative integrable function on \(M\),

\[
\int_{A_1 \times A_2} \int_{\gamma_{x,y}} h(\gamma(s)) ds(e^{-f} dv_g)^2 \leq C(n + 4k, H, R)(\text{diam}(A_2) \text{vol}_f(A_1) + \text{diam}(A_1) \text{vol}_f(A_2))
\]

\[
\times \int_W h e^{-f} dv_g,
\]

then,

\[
C(n + 4k, H, R) = \sup_{0 < s < u} \frac{A^{n+4k}_H(s)}{A^{n+4k}_H(u)},
\]

and

\[
R \geq \sup \{d(x,y) \mid (x, y) \in (A_1 \times A_2)\},
\]

where \(A^{n+4k}_H(r)\) denotes the area element on \(\partial B(r)\) in \(M^{n+4k}_H\), the simply connected model space of dimension \(n + 4k\) with constant curvature \(H\). Because \(H = 0\), we denote \(C(n + 4k, H, R)\) by \(C(n + 4k)\).

Applying \(s H_H(r)\) as a solution to

\[
s H_H'' + H s H_H = 0
\]

, \(s H_H(0) = 0\) and \(s H_H'(0) = 1\) are satisfied. Moreover, if \(H = 0\), then

\[
m^n_H = (n - 1) \frac{s H_H'}{s H_H}
\]

with the solution \(s H_H(r) = r\).

By the mean curvature comparison (3.15) in the proof of Theorem 1.1 in [7], we have

\[
s H_H^2(r) m_f(r) \leq s H_H^2(r) m_H(r) - f(r)(s H_H^2(r))' + \int_0^r f(t)(s H_H^2(t))' dt.
\]

Thus,

\[
m_f(r) \leq m_H(r) + \frac{4k}{s H_H(r)},
\]

implying that

\[
m_f(r) \leq m_H(r) \left(1 + \frac{4k}{n - 1}\right) = m^{n+4k}_H(r).
\]

Therefore,

\[
\frac{\text{vol}_f(B(p, R))}{\text{vol}_f(B(p, r))} \leq \frac{\text{vol}^{n+4k}_H(R)}{\text{vol}^{n+4k}_H(r)}.
\]
Now, let \( p, q \in M \) such that \( d(p, q) = \text{diam}(M) = D \), \( r > 0 \), \( A_1 = B(p, r) \), and \( A_2 = B(q, r) \). Applying the inequality (2.1),

\[
\int_{A_1 \times A_2} (n-1) - \text{Ric}_f^- ds e^{-f} dv_g \leq C(n+4k)(2r \text{vol}_f(A_1) + 2r \text{vol}_f(A_2)) \int_M (n-1) - \text{Ric}_f^- e^{-f} dv_g.
\]

Consequently, let \( \text{vol}_H^{n+4k}(r) \) be the volume of the radius \( r \)-ball in \( M_H^{n+4k} \), the simply connected model space of dimension \( n+4k \) with constant curvature \( H \). Then, we have

\[
\inf_{(x, y) \in A_1 \times A_2} ((n-1) - \text{Ric}_f^-) ds \\
\leq 2r C(n+4k)(\frac{1}{\text{vol}_f(A_1)} + \frac{1}{\text{vol}_f(A_2)}) \int_M ((n-1) - \text{Ric}_f^-) e^{-f} dv_g \\
\leq 4r C(n+4k) \frac{\text{vol}_H^{n+4k}(D)}{\text{vol}_H^{n+4k}(r)} \frac{1}{\text{vol}_f(M)} \int_M ((n-1) - \text{Ric}_f^-) e^{-f} dv_g,
\]

where the last inequality follows from (2.3).

Note that if \( H = 0 \), the volume element \( v(r) = r^{n+4k-1} \), which gives

\[
\text{vol}_H^{n+4k}(r) = \int_{S^{n+4k-1}} ds n^{n+4k-1} \int_0^r t^{n+4k-1} dt,
\]

to obtain

\[
\frac{\text{vol}_H^{n+4k}(D)}{\text{vol}_H^{n+4k}(r)} = \frac{D^{n+4k}}{r^{n+4k}}.
\]

Therefore,

\[
\inf_{(x, y) \in (A_1 \times A_2)} ((n-1) - \text{Ric}_f^-) ds \\
\leq 4r C(n+4k) \frac{D^{n+4k}}{r^{n+4k}} \frac{1}{\text{vol}_f(M)} \int_M ((n-1) - \text{Ric}_f^-) e^{-f} dv_g.
\]

Now, we can find a minimizing unit speed geodesic \( \gamma \) from \( x \in A_1 \) to \( y \in A_2 \) of length \( L = d(x, y) \). Let \( \{ E_1, \cdots, E_n = \gamma' \} \) be a parallel orthonormal frame along \( \gamma \) and a smooth function \( b \in C^\infty([0, L]) \) such that \( b(0) = b(L) = 0 \); then, by the second variation of \( \gamma \), we have

\[
\sum_{i=1}^{n-1} I(bE_i, bE_i) = \int_0^L (b')^2(n-1) dt - \int_0^L b^2 \text{Ric}_f(\gamma', \gamma') dt \\
+ \int_0^L b^2 \text{Hess}(f)(\gamma', \gamma') dt.
\]

Likewise, note that

\[
\int_0^L b^2 \text{Hess}(f)(\gamma', \gamma') dt = \int_0^L b^2 \frac{d}{dt} (\nabla f, \gamma') dt,
\]
If we set the function \( b \) such that,

\[
\int_0^L b^2 \frac{d}{dt} (\nabla f, \gamma') dt = \int_0^L \left( -2bb' \frac{d}{dt} (f(\gamma(t))) + \frac{d}{dt} (b^2 (\nabla f, \gamma')) \right) dt
\]

\[
= \int_0^L \left( 2f \frac{d}{dt} (bb') - 2 \frac{d}{dt} (fb') + \frac{d}{dt} (b^2 (\nabla f, \gamma')) \right) dt \leq 0,
\]

such that,

\[
\sum_{i=1}^{n-1} I(bE_i, bE_i) \leq \int_0^L (b')^2 (n-1) dt - \int_0^L b^2 Ric(\gamma', \gamma') dt.
\]

If we set the function \( b \) as \( b(t) = \sin \left( \frac{\pi t}{L} \right) \), then we obtain \((b'(t))^2 = \frac{\pi^2}{L^2} \cos^2 \left( \frac{\pi t}{L} \right)\) and \(b^2(t) = \sin^2 \left( \frac{\pi t}{L} \right)\). Thus,

\[
\sum_{i=1}^{n-1} I(bE_i, bE_i) \leq (n-1) \int_0^L \frac{\pi^2}{L^2} \cos^2 \left( \frac{\pi t}{L} \right) dt - \int_0^L \sin^2 \left( \frac{\pi t}{L} \right) Ric(\gamma', \gamma') dt,
\]

\[
= (n-1) \frac{\pi^2}{L^2} \int_0^L \cos^2 \left( \frac{\pi t}{L} \right) dt - (n-1) \int_0^L \sin^2 \left( \frac{\pi t}{L} \right) dt
\]

\[
+ \int_0^L \sin^2 \left( \frac{\pi t}{L} \right) \left((n-1) - Ric(\gamma', \gamma')\right) dt,
\]

\[
= -\frac{(n-1)L}{2} \left(1 - \frac{\pi^2}{L^2}\right) + \int_0^L \sin^2 \left( \frac{\pi t}{L} \right) \left((n-1) - Ric(\gamma', \gamma')\right) dt
\]

\[
\leq -\frac{(n-1)L}{2} \left(1 - \frac{\pi^2}{L^2}\right) + \int_0^L (n-1) - Ric_{\gamma} dt.
\]

By the inequality (2.4),

\[
\sum_{i=1}^{n-1} I(bE_i, bE_i) \leq -\frac{(n-1)L}{2} \left(1 - \frac{\pi^2}{L^2}\right) + 4rC(n + 4k) \frac{D^{n+4k} \frac{1}{vol(M)}}{\frac{1}{N} \frac{1}{D^{n+4k-1} \frac{1}{vol(M)}}} \int_M (n-1) - Ric_{\gamma} e^{-f} dv_g.
\]

Now, let \( r = \frac{D}{N} \), and choose \( N = N(\delta) \) such that

\[
\frac{1}{1 - \frac{\pi}{L}} < \frac{\pi + \delta}{\pi + \frac{\delta}{2}},
\]

by the triangle inequality,

\[
L = d(x, y) \geq d(p, q) - d(p, x) - d(y, q) = D \left(1 - \frac{2}{N}\right).
\]

Thus,

\[
\sum_{i=1}^{n-1} I(bE_i, bE_i) \leq -\frac{(n-1)L}{2} \left(1 - \frac{\pi^2}{L^2}\right) + 4C(n + 4k) \frac{L}{1 - \frac{\pi}{2N} \frac{1}{vol(M)}} \int_M (n-1) - Ric_{\gamma} e^{-f} dv_g.
\]
Setting
\[ \epsilon = \frac{(n-1)(1-\frac{\pi}{2})}{8C(n+4k)N^{n+4k-1}}(1-\frac{\pi^2}{(\pi + \frac{\delta}{2})^2}), \]
and if
\[ \frac{1}{\text{vol}_f(M)} \int_M ((n-1) - \text{Ric}_f) + e^{-f} dv_g < \epsilon, \]
then,
\[ \sum_{i=1}^{n-1} I(bE_i, bE_i) < -\frac{(n-1)L}{2} \left( 1 - \frac{\pi^2}{L^2} \right) + \frac{(n-1)L}{2} \left( 1 - \frac{\pi^2}{(\pi + \frac{\delta}{2})^2} \right). \]
Because \( \gamma \) is a minimal geodesic such that
\[ \sum_{i=1}^{n-1} I(bE_i, bE_i) \geq 0, \]
Then by \((2.7)\), we obtain
\[ \frac{(n-1)L}{2} \left( 1 - \frac{\pi^2}{(\pi + \frac{\delta}{2})^2} \right) \geq \frac{(n-1)L}{2} \left( 1 - \frac{\pi^2}{L^2} \right). \]
This inequality gives
\[ L \leq \pi + \frac{\delta}{2}. \]
Finally, by the inequality \((2.5), (2.6), \) and \((2.8)\), we have
\[ \text{diam}(M) = D < \pi + \delta. \]
This completes the proof.

3. Proof of Theorem 1.5 and 1.6

We will prove Theorem 1.5 in this section following the same setting for Theorem 1.4. Let \( \gamma \) be a minimizing unit speed geodesic from \( x \in A_1 \) to \( y \in A_2 \) of length \( L = d(x, y) \). Likewise, let \( \{E_1, \ldots, E_n = \gamma'\} \) be a parallel orthonormal frame along \( \gamma \) and a smooth function \( b \in C^\infty([0, L]) \) such that \( b(0) = b(L) = 0 \). For the proof, we need the following result.

Lemma 3.1. Let \( (M, g, e^{-f} dv_g) \) be a complete Riemannian manifold with \( \text{Ric}_f \geq (n-1)H, H < 0, \) and \( |f| \leq k \). Then, for any fixed \( R > \pi \), there exists \( \epsilon = \epsilon(n + 4k, H, R, \delta) \) such that if
\[ \frac{1}{\text{vol}_f(B(p, R))} \int_{B(p, R)} ((n-1) - \text{Ric}_f) + e^{-f} dv_g < \epsilon(n + 4k, H, R, \delta) \]
for some \( B(p, R) \subset M \), then \( M = B(p, R) \subset B(p, \pi + \delta) \).

Proof of Lemma 3.1. Let \( sn_H(r) \) be the solution to
\[ sn_H'' + H sn_H = 0 \]
such that \( sn_H(0) = 0 \) and \( sn_H'(0) = 1 \), then
\[ m_H^n = (n-1) \frac{sn_H'}{sn_H}. \]
When $H < 0$, this solution is given by $sn_H(r) = \frac{1}{\sqrt{H}} \sinh(\sqrt{H}r)$. By the inequality \(2.2\),

$$m_f(r) \leq m_H(r) + \frac{4k sn_H(r)}{sn_H(r)} = (n + 4k - 1) \frac{sn_H(r)}{sn_H(r)} = m_H^{n+4k}(r),$$

thus,

\[
(3.1) \quad \frac{\text{vol}_f(B(p, R))}{\text{vol}_f(B(p, r))} \leq \frac{\text{vol}_f^{n+4k}(R)}{\text{vol}_f^{n+4k}(r)}.
\]

If we set $p \in M$ and $W = B(p, R)$, then $q$ will be any point in $W$ satisfying $\pi + 4r < d(p, q) < R - 3r$, where $0 < r < \frac{1}{8}(R - \pi)$ is to be determined, and $A_1 = B(p, r), A_2 = B(q, r)$. Thus, by (3.1)

\[
\int_{\gamma_{x,y}} ((n - 1) - \text{Ric}_f_-)_+ ds \leq 2rC(n + 4k, H, R) \left( \frac{1}{\text{vol}_f(B(p, r))} + \frac{1}{\text{vol}_f(B(q, r))} \right) 
\times \int_{B(p, R)} ((n - 1) - \text{Ric}_f_-)_+ e^{-f} dv_g,
\]

\[
\leq 2rC(n + 4k, H, R) \left( \frac{\text{vol}_f^{n+4k}(R)}{\text{vol}_f^{n+4k}(r)} \frac{1}{\text{vol}_f(B(p, R))} + \frac{\text{vol}_f^{n+4k}(2R)}{\text{vol}_f^{n+4k}(r)} \int_{B(p, R)} ((n - 1) - \text{Ric}_f_-)_+ dv_g \right),
\]

\[
\leq 2rC(n + 4k, H, R) \left( \frac{\text{vol}_f^{n+4k}(R) + \text{vol}_f^{n+4k}(2R)}{\text{vol}_f^{n+4k}(r)} \right) 
\times \frac{1}{\text{vol}_f(B(p, R))} \int_{B(p, R)} ((n - 1) - \text{Ric}_f_-)_+ e^{-f} dv_g.
\]

Let $r = \frac{1}{8}\delta$, where $\delta < \frac{1}{8}(R - \pi)$. Then

\[
\int_{\gamma_{x,y}} ((n - 1) - \text{Ric}_f_-)_+ ds \leq \frac{1}{2} \delta C(n + 4k, H, R) \left( \frac{\text{vol}_f^{n+4k}(R) + \text{vol}_f^{n+4k}(2R)}{\text{vol}_f^{n+4k}(\frac{1}{8}\delta)} \right) 
\times \frac{1}{\text{vol}_f(B(p, R))} \int_{B(p, R)} ((n - 1) - \text{Ric}_f_-)_+ e^{-f} dv_g.
\]

By the second variation of $\gamma$,

\[
\sum_{i=1}^{n-1} I(bE_i, bE_i) \leq \frac{n-1}{2}L^{L_2} + \int_{0}^{L} ((n - 1) - \text{Ric}_f_-)_+ ds,
\]

\[
\leq \frac{n-1}{2}L^{L_2} + \frac{1}{2} \delta C(n + 4k, H, R) \left( \frac{\text{vol}_f^{n+4k}(R)}{\text{vol}_f^{n+4k}(\frac{1}{8}\delta)} \right) 
\times \frac{1}{\text{vol}_f(B(p, R))} \int_{B(p, R)} ((n - 1) - \text{Ric}_f_-)_+ e^{-f} dv_g.
\]
Setting

\[ \epsilon = \frac{(n-1)(\pi + \frac{1}{2}\delta)}{\delta C(n+4k, H, R)} \left( 1 - \frac{\pi^2}{(\pi + \frac{1}{2}\delta)^2} \right) \frac{\text{vol}_{H}^{n+4k}(\frac{1}{2}\delta)}{\text{vol}_{H}^{n+4k}(R) + \text{vol}_{H}^{n+4k}(2R)}, \]

we obtain

\[ \sum_{i=1}^{n-1} I(bE_i, bE_i) < -\frac{(n-1)L}{2} \left( 1 - \frac{\pi^2}{L^2} \right) + \frac{(n-1)(\pi + \frac{1}{2}\delta)}{2} \left( 1 - \frac{\pi^2}{(\pi + \frac{1}{2}\delta)^2} \right). \]

Moreover, by the minimality of \( \gamma \), we have

\[ \frac{(n-1)(\pi + \frac{1}{2}\delta)}{2} \left( 1 - \frac{\pi^2}{(\pi + \frac{1}{2}\delta)^2} \right) \geq \frac{(n-1)L}{2} \left( 1 - \frac{\pi^2}{L^2} \right), \]

implying that

\[ L \leq \pi + \frac{1}{2}\delta. \]

By the triangle inequality,

\[ (3.2) \quad d(p, q) \leq \pi + \delta. \]

We assumed that \( \pi + 4r < d(p, q) < R - 3r \), or \( \pi + \delta < d(p, q) < R - 3r \). However, by (3.2), no geodesic starting from \( p \) of a length greater than \( \pi + \delta \) can be length minimizing, which implies that \( B(p, R) \subset B(p, \pi + \delta) \). If \( R \) goes to infinity, then \( B(p, R) \) tends to \( M \). Hence, we may conclude that \( M = B(p, R) \subset B(p, \pi + \delta) \). \( \square \)

Now we can prove Theorem 1.5.

**Proof of Theorem 1.5.** Note that Lemma 3.1 shows Theorem 1.5 for \( R > \pi \). Thus, it suffices to prove the case when \( R \leq \pi \).

Let \( R' > \pi \) be fixed. Then for any \( R \leq \pi \), there exists \( N = N(H, R, R') > 0 \), such that any \( R' \)-ball in \( M \) can be covered by \( N \) or fewer \( R \)-balls, \( B(x_i, R), 1 \leq i \leq N \). Subsequently,

\[
\frac{1}{\text{vol}_f(B(z, R'))} \int_{B(z, R')} ((n-1) - \text{Ric}_f)_+ e^{-f} dv_g \\
\leq N(H, R, R') \frac{1}{\text{vol}_f(B(z, R'))} \sup_{x_i} \int_{B(x_i, R)} ((n-1) - \text{Ric}_f)_+ e^{-f} dv_g, \\
\leq N(H, R, R') \frac{\text{vol}_H^{n+4k}(R + R')}{\text{vol}_H^{n+4k}(R')} \frac{1}{\text{vol}_f(B(z, R + R'))} \\
\times \sup_{x_i} \int_{B(x_i, R)} ((n-1) - \text{Ric}_f)_+ e^{-f} dv_g, \\
\leq N(H, R, R') \frac{\text{vol}_H^{n+4k}(R + R')}{\text{vol}_H^{n+4k}(R')} \sup_{x_i} \frac{1}{\text{vol}_f(B(x_i, R))} \\
\times \int_{B(x_i, R)} ((n-1) - \text{Ric}_f)_+ e^{-f} dv_g.
\]
Hence, we can conclude that
\[
\sup_x \frac{1}{\text{vol}_f(B(x, R'))} \int_{B(x, R')} \left( (n - 1) - \text{Ric}_f - e^{-f} \right) dv_g \leq N(H, R, R') \frac{\text{vol}_H^{n+4k}(R + R') \text{vol}_H^{n+4k}(R')}{\text{vol}_H^{n+4k}(R')} \times \sup_x \frac{1}{\text{vol}_f(B(x, R))} \int_{B(x, R)} \left( (n - 1) - \text{Ric}_f - e^{-f} \right) dv_g.
\]

Finally, let us prove Theorem 1.6.

**Proof of Theorem 1.6.** Let \((\bar{M}, \bar{g})\) be a Riemannian universal cover of \((M, g)\). Because the inequality
\[
\sup_x \frac{1}{\text{vol}_f(B(x, R))} \int_{B(x, R)} \left( (n - 1) - \text{Ric}_f - e^{-f} \right) dv_g < \bar{\epsilon} (n + 4k, H, R)
\]
holds on \((M, g)\), the same inequality holds on \((\bar{M}, \bar{g})\).

Based on this, it is easy to see that Theorem 1.5 with \(\delta = 0\) also holds. When \(R \leq \pi\), we just need to follow the proof of Theorem 1.5 moreover, when \(R > \pi\), setting \(\pi < d(p, q) < R\) with \(r = \frac{\pi}{4}\), we can prove that \(\text{diam}(\bar{M}) \leq \pi\). Hence, we can conclude that \(\bar{M}\) is compact, implying that the fundamental group \(\pi_1(M)\) is finite.

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