REGULARITY, MATCHINGS AND CAMERON-WALKER GRAPHS

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ABSTRACT. Let $G$ be a simple graph and let $\nu(G)$ be the matching number of $G$. It is well-known that $\text{reg } I(G) \leq \nu(G) + 1$. In this paper we show that $\text{reg } I(G) = \nu(G) + 1$ if and only if every connected component of $G$ is either a pentagon or a Cameron-Walker graph.

INTRODUCTION

Let $G$ be a graph with vertex set $\{1, \ldots, n\}$, and let $R := k[x_1, \ldots, x_n]$ be the polynomial ring over a field $k$. We associated to $G$ an ideal in $R$

$I(G) = (x_ix_j \mid \{i, j\} \text{ is an edge of } G)$

which is called the edge ideal of $G$.

Castelnuovo-Mumford regularity of a homogeneous ideal $I$ in $R$, denoted by $\text{reg}(I)$, is an important algebraic invariant which measures the complexity of the ideal $I$. Finding bounds for the regularity of $I(G)$ in terms of combinatorial data of $G$ is an active research program in combinatorial commutative algebra in recent years (see [6] and references therein).

Throughout the paper we assume that $G$ has at least one edge unless otherwise stated. Let $\nu_0(G)$ be the induced matching number of $G$. Katzman [9] showed that

(1) $\text{reg}(I(G)) \geq \nu_0(G) + 1.$

There are many classes of graphs $G$ for which the equality occurs (see [11] Theorem 4.12 for the survey).

For upper bounds, Hà and Van Tuyl [7] obtained

(2) $\text{reg}(I(G)) \leq \nu(G) + 1$

where $\nu(G)$ is the matching number of $G$. This bound is improved by Woodroofe [10] as follows. A graph $G$ is chordal if every induced cycle in $G$ has length 3, and is co-chordal if the complement graph $G^c$ of $G$ is chordal. The co-chordal cover number, denoted co-chord($G$), is the minimum number of co-chordal subgraphs required to cover the edges of $G$. Then,

$\text{reg}(I(G)) \leq \text{co-chord}(G) + 1.$

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In the paper we interested in graph-theoretically classifying $G$ such that the equality occurs in each bound above. More precisely,

**Problem:** Classify graph-theoretically graphs $G$ such that

1. $\text{reg}(I(G)) = \nu_0(G) + 1$.
2. $\text{reg}(I(G)) = \nu(G) + 1$.
3. $\text{reg}(I(G)) = \text{cochord}(G) + 1$.

It is worth mentioning that there is a graph $G$ (see Example [12]) such that the equality $\text{reg}(G) = \nu_0(G) + 1$ (resp. $\text{reg}(G) = \text{cochord}(G) + 1$) is dependent on the characteristic of the field $k$. Thus we cannot solve Problems 1 and 3 without taking into account the characteristic of the based field.

The main result of the paper is to settle Problem 2. Note that this problem is asked in [1]. At first sight when $\nu_0(G) = \nu(G)$, we have $\text{reg}(I(G)) = \nu(G) + 1$ by Inequalities (1) and (2). The graph $G$ satisfies $\nu(G) = \nu_0(G)$ is called a Cameron-Walker graph (after Hibi et al. [8]), which is classified in [2, 8] as follows.

**Theorem 1.** ([2, Theorem 1] or [8, p. 258]) A connected graph $G$ is Cameron-Walker if and only if it is one of the following graphs (see Figure 1):

- (1) a star;
- (2) a star triangle;
- (3) a graph consisting of a connected bipartite graph with a bipartition partition $(X, Y)$ such that there is at least one leaf edge attached to each vertex $x \in X$ and that there may be possibly some pendant triangles attached to each vertex $y \in Y$.

![Figure 1. Three kinds of connected Cameron-Walker graphs](image)

Recall that a pentagon is a cycle of length 5. Then, the main result of the paper is the following.

**Theorem 11** Let $G$ be a graph. Then, $\text{reg}(I(G)) = \nu(G) + 1$ if and only if each connected component of $G$ is either a pentagon or a Cameron-Walker graph.
Our paper is structured as follows. In Sect. 1, we collect notations and terminology used in the paper, and recall a few auxiliary results. In Sect. 2, we settle Problem 2.

1. Preliminaries

Let $k$ be a field, and let $R := k[x_1, \ldots, x_n]$ be a standard graded polynomial ring of $n$ variables over $k$. The object of our work is the Castelnuovo-Mumford regularity of graded modules and ideals over $R$. This invariant can be defined via the minimal free resolution. Let $M$ be a finitely generated graded nonzero $R$-module and let

$$0 \rightarrow \bigoplus_{j \in \mathbb{Z}} R(-j)^{\beta_{p,j}(M)} \rightarrow \cdots \rightarrow \bigoplus_{j \in \mathbb{Z}} R(-j)^{\beta_{0,j}(M)} \rightarrow 0$$

be the minimal free resolution of $M$. Then,

$$\text{reg}(M) = \max\{j - i \mid \beta_{i,j}(M) \neq 0\}.$$ 

Let $G$ be a finite simple graph. We use the symbols $V(G)$ and $E(G)$ to denote the vertex set and the edge set of $G$, respectively. The algebra-combinatorics framework used in this paper is described via the edge ideal construction. Assume that $V(G) = \{1, \ldots, n\}$. The edge ideal of $G$ is define by

$$I(G) = (x_i x_j \mid \{i, j\} \in E(G)) \subset R.$$ 

For simplicity, in the sequel, we write $\text{reg}(G)$ to means:

1. If $G$ has at least one edge, then $\text{reg}(G) = \text{reg}(I(G))$.
2. If $G$ is totally disconnected, then $\text{reg}(G) = 1$.
3. If $G$ is empty, i.e. $V(G) = \emptyset$, then $\text{reg}(G) = 0$.

The complementary graph $G^c$ of $G$ is the graph whose vertex set is again $V(G)$ and whose edges are the non-edges of $G$. A graph $G$ is called chordal if each cycle of length at least 4 has a chord. We recall the following result of Fröberg.

**Lemma 2.** ([4, Theorem 1]) $\text{reg}(G) = 1$ if and only if $G^c$ is chordal.

A matching in a graph $G$ is a subgraph consisting of pairwise disjoint edges. If the subgraph is an induced subgraph, the matching is an induced matching. A matching of $G$ is maximal if it is maximal with respect to inclusion. The matching number of $G$, denoted $\nu(G)$, is the size of a maximum matching; that is, the maximum number of pairwise disjoint edges; the minimum cardinality of the maximal matchings of $G$ is the minimum matching number of $G$ and is denoted by $\text{min-match}(G)$; and the induced matching number of $G$, denoted by $\nu_0(G)$, is the size of a maximum induced matching. It follows from [10, Theorem 2] that:

**Lemma 3.** $\text{reg}(G) \leq \text{min-match}(G) + 1$.

When there is no confusion, the edge $\{u, v\}$ of $G$ we simply write $uv$. For a vertex $u$ in a graph $G$, let $N_G(u) := \{v \in V(G) \mid uv \in E\}$ be the set of neighbors of $u$, and set $N_G[u] := N_G(u) \cup \{u\}$. An edge $e$ is incident to a vertex $u$ if $u \in e$. The degree of a vertex $u \in V(G)$, denoted by $\deg_G(u)$, is the number of edges incident to $u$. 

3
For an edge $e$ in a graph $G$, define $G \setminus e$ to be the subgraph of $G$ with the edge $e$ deleted (but its vertices remained). For a subset $W \subseteq V(G)$ of the vertices in $G$, define $G[W]$ be the induced subgraph of $G$ on $W$ and $G \setminus W$ to be the subgraph of $G$ with the vertices in $W$ (and their incident edges) deleted. When $W = \{u\}$ consists of a single vertex, we write $G \setminus u$ stands for $G \setminus \{u\}$. Define $G_u := G \setminus N_G[u]$. If $e = \{u, v\}$, then $G_e$ to be the subgraph $G \setminus (N_G[u] \cup N_G[v])$ of $G$.

In the study of the regularity of edge ideals, the following lemmas enable us to do induction on the number of vertices and edges.

**Lemma 4.** ([6] Lemma 3.1) Let $H$ be an induced subgraph of $G$. Then,
\[
\text{reg}(H) \leq \text{reg}(G).
\]

**Lemma 5.** ([3] Lemma 2.10) Let $x$ be a vertex of a graph $G$. Then,
\[
\text{reg}(G) \in \{\text{reg}(G \setminus x), \text{reg}(G_x) + 1\}.
\]

**Lemma 6.** ([6] Theorem 3.5) Let $e$ be an edge of $G$. Then,
\[
\text{reg}(G) \leq \{\text{reg}(G \setminus e), \text{reg}(G_e) + 1\}.
\]

### 2. Prove the main result

In this section we classify graphs $G$ that satisfy $\text{reg}(G) = \nu(G) + 1$. The following lemma shows that it suffices to consider connected graphs.

**Lemma 7.** Let $G$ be a graph with connected components $G_1, \ldots, G_s$. Then,
\[
\begin{align*}
(1) & \quad \text{reg}(G) = \sum_{i=1}^{s} (\text{reg}(G_i) - 1) + 1; \\
(2) & \quad \nu(G) = \sum_{i=1}^{s} \nu(G_i); \\
(3) & \quad \nu_0(G) = \sum_{i=1}^{s} \nu_0(G_i).
\end{align*}
\]

**Proof.** (1) follows from [1] Corollary 3.10; (2) and (3) are obvious. \hfill \square

**Lemma 8.** Let $G$ be a $C_5$-free graph with $\text{reg}(G) = \nu(G) + 1$. Then, $\nu(G) = \nu_0(G)$.

**Proof.** We prove by induction on $|V(G)|$. If $|V(G)| = 1$, then $G$ is just one point, and then $\nu(G) = \nu_0(G) = 0$.

Assume that $|V(G)| > 1$. If $\nu(G) = 1$, then $\nu_0(G) = 1$, and the lemma follows.

Assume that $\nu(G) \geq 2$. By Lemma 7 we may assume that $G$ is connected. If $G$ has a vertex $v$ such that $\text{reg}(G) = \text{reg}(G \setminus v)$. By Lemma 3 we have
\[
\nu(G) + 1 = \text{reg}(G) = \text{reg}(G \setminus v) \leq \nu(G \setminus v) + 1,
\]

hence $\nu(G) \leq \nu(G \setminus v)$. The converse inequality $\nu(G) \geq \nu(G \setminus v)$ holds since $G \setminus v$ is an induced subgraph of $G$. Thus, $\nu(G \setminus v) = \nu(G)$. It follows that $\text{reg}(G \setminus v) = \nu(G \setminus v) + 1$.

By the induction hypothesis, we have $\nu(G \setminus v) = \nu_0(G \setminus v)$. Since $G \setminus v$ is an induced subgraph of $G$, $\nu_0(G \setminus v) \leq \nu_0(G)$. Hence, $\nu_0(G) = \nu(G)$, and the lemma holds.

Therefore, by Lemmas 4 and 5 we may assume that $\text{reg}(G_v) = \text{reg}(G) - 1 = \nu(G)$ for every $v$. 

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Let $v$ be a vertex of minimal degree of $G$ and let $x$ be a neighbor of $v$ in $G$. Since $x$ is not an isolated vertex of $G$, we have $\nu(G_x) \leq \nu(G) - 1$. Together with equality $\text{reg}(G_x) = \nu(G)$, it yields $\text{reg}(G_x) \geq \nu(G_x) + 1$. Together with Lemma 5, we obtain $\text{reg}(G_x) = \nu(G_x) - 1$ and $\nu(G) = \nu(G_x) + 1$.

By the induction hypothesis we have $\nu(G_x) = \nu_0(G_x)$. Let $m = \nu_0(G_x)$ and $\{e_1, \ldots, e_m\}$ be an induced matching of $G_x$. Then, $\{xv, e_1, \ldots, e_m\}$ is a maximal matching of $G$. Note that $x$ is not incident to $e_i$ for every $i$.

Let $S$ be the set of vertices of $G$ which are different from $x$, $v$, and all vertices of $e_i$ for $i = 1, \ldots, m$. Then, $S$ is an independent set of $G$ because $\{xv, e_1, \ldots, e_m\}$ is a maximal matching of $G$.

Assume that $v$ is incident to $e_i$ for some $i$. Without loss of generality we may assume that $i = 1$. Let $e_1 = yz$ and assume that $v$ is adjacent with $y$.

If $v$ is adjacent with $z$. Let $H := G \setminus \{x, v, y, z\}$. Observe that $\{e_2, \ldots, e_m\}$ is a maximal matching of $H$, so $\text{reg}(H) \leq m$ by Lemma 3. On the other hand, since $G_v$ is an induced subgraph of $H$, by Lemma 4 we have $\text{reg}(G_v) \leq \text{reg}(H) \leq m$. Therefore, $\text{reg}(G) = \text{reg}(G_v) + 1 \leq m + 1 < \nu(G) + 1$, a contradiction.

Thus, $v$ is not adjacent with $z$ (see Figure 2). Note that $x$ and $y$ are two neighbors of $v$ so that $\deg_G(v) \geq 2$. Since $\deg_G(z) \geq \deg_G(v) \geq 2$, it follows that $z$ must be incident with some vertex in $S$, say $t$. Then, $t$ is not adjacent with $x$. Because if $t$ is adjacent with $x$, then $G$ would have a cycle $xvyzt$ of length 5, a contradiction. Similarly, $x$ has a neighbor in $S$, say $s$, which is not adjacent with $z$. In particular, $s \neq t$. But then we would have $\{sx, vy, zt, e_2, \ldots, e_m\}$ is a matching of $G$, so $\nu(G) \geq m + 2$, a contradiction.

Therefore, $v$ is not incident to any $e_i$. Then, $\{xv, e_1, \ldots, e_m\}$ is an induced matching of $G$. Since $\nu(G) = m + 1$, it follows that $\nu_0(G) = m + 1 = \nu(G)$, and the proof of the lemma is complete. \hfill $\square$

**Lemma 9.** Let $G$ be a graph with $\text{reg}(G) = \nu(G) + 1$. If $e$ is an edge of $G$ lying in the middle of a simple path of length 3 in $G$, then

$$
\text{reg}(G) = \nu(G) = \nu(G \setminus e) = \text{reg}(G \setminus e).
$$

*Proof.* Assume that $e = uv$ and $G$ has a simple path $xuv$. Let $H := G \setminus \{x, u, v, y\}$. We have $G_{uv}$ is an induced subgraph of $H$, so $\text{reg}(G_{uv}) \leq \text{reg}(H)$. If $M$ is a matching of $H$, then $M \cup \{xu, vy\}$ is a matching of $G$. It follows that $\nu(H) \leq \nu(G) - 2$. Therefore,
Lemma 10. Let \( G \) be a connected graph which contains a cycle \( C_5 \) of length 5. If \( \text{reg}(G) = \nu(G) + 1 \), then \( G \) is just \( C_5 \).

Proof. For simplicity, let \( \gamma(G) := |V(G)| + |E(G)| \). Since \( C_5 \) is a subgraph of \( G \), \( \gamma(G) \geq 10 \).

We will prove the lemma by induction on \( \gamma(G) \). If \( \gamma(G) = 10 \), then \( G \) is just the cycle \( C_5 \), and the lemma holds.

Assume that \( \gamma(G) \geq 10 \). If \( V(G) = V(C_5) \), then \( G \) is a pentagon with some chords. It follows that \( G^c \) is a chordal graph, so \( \text{reg}(G) = 2 \) by Lemma 2. On the other hand, \( \nu(G) = 2 \). It implies \( \text{reg}(G) < \nu(G) + 1 \), a contradiction.

Therefore, \( |V(G)| \neq V(C_5) \). We first prove that \( G \) has only one cycle. Indeed, if \( G \) has another cycle \( C \neq C_5 \). Since \( G \) is connected, it has an edge of \( C \), say \( e \), such that

1. \( e \) is not in \( C_5 \);
2. \( e \) is in the middle of a simple path of length 3 in \( G \).

By Lemma 9 we have \( \text{reg}(G \setminus e) = \nu(G \setminus e) \). Note that \( G \setminus e \) is connected and has the cycle \( C_5 \) of length 5. Since \( \gamma(G \setminus e) = \gamma(G) - 1 \), by the induction hypothesis we have \( G \setminus e \) must be \( C_5 \), a contradiction. Thus, \( G \) has only cycle which is just \( C_5 \).

Now let \( uv \) be an edge of \( C_5 \) and \( H := G \setminus uv \). Then, \( H \) is a connected graph without cycles, so it is a tree. By Claim 1 we have

\[ \text{reg}(G) = \nu(G) + 1 = \nu(H) + 1 = \text{reg}(H), \]

so \( H \) is a Cameron-Walker graph by Lemma 8.

Since \( G = H + uv \) and \( G \) has a cycle of length 5, \( H \) is not a star. Together with Theorem 1, we conclude that \( H \) is a bipartite graph with bipartition \( (X, Y) \) such that every vertex \( x \) in \( X \) is adjacent to some leaves in \( Y \). Let \( m = \nu(G) \). Then, we have \( m = \nu(H) = |X| \).

Again, because \( G = H + uv \) and \( G \) has an odd cycle, we have \( u \) and \( v \) both are in \( X \) or both are in \( Y \). If \( u, v \in X \), then \( G_u \) is an induce subgraph of \( G \setminus \{u, v\} \). In this case, \( \nu(G_u) \leq \nu(G \setminus \{u, v\}) = |X| - 2 = \nu(G) - 2 \), and therefore \( \text{reg}(G_u) \leq \nu(G_u) + 1 \leq \nu(G) - 1 \). Similarly, \( \nu(G \setminus u) = |X| - 1 = \nu(G) - 1 \) and \( \text{reg}(G \setminus u) \leq \nu(G \setminus u) + 1 = \nu(G) \).

By Lemma 8, we get \( \text{reg}(G) \leq \nu(G) \), a contradiction.

Therefore, \( u, v \in Y \). We may assume that the cycle \( C_5 \) is \( suvtws \). Then, \( s, t \in X \) and \( w \in Y \). Let \( X = \{s, t, x_3, \ldots, x_m\} \). Let \( y_3, \ldots, y_m \in Y \) are leaves of \( G \) such that \( x_i y_i \in E(G) \) for \( i = 3, \ldots, m \).

We consider two possible cases:

Case 1: \( \deg_G(w) = 2 \). Since \( G \) is connected and \( V(G) \neq V(C_5) \), there is a vertex \( z \in Y \setminus \{u, v, w\} \) such that \( z \) is adjacent with \( s \) or \( t \). We may assume that \( z \) is adjacent
with \( t \) (see Figure 3). Observe that \( z \notin \{y_3, \ldots, y_m\} \), so that
\[
\{zt, ws, uv, x_3y_3, \ldots, x_my_m\}
\]
is a matching of \( G \). Consequently, \( \nu(G) \geq m + 1 \), a contradiction.

**Figure 3.**

**Case 2:** \( \text{deg}_G(w) > 2 \). We first prove that \( u \) and \( v \) are two leaves of \( H \). Indeed, if it is not the case, we may assume that \( v \) is not a leaf. Let \( z \in Y \) be a leaf of \( H \) that is a neighbor of \( t \). It is obvious that \( z \notin \{u, v, w\} \). By the same argument as in the case 1 we obtain a contradiction that \( \nu(G) \geq m + 1 \). Thus, \( u \) and \( v \) are two leaves of \( G \).

We next prove that \( \text{deg}_G(s) = \text{deg}_G(t) = 2 \). Indeed, if it is not the case, we may assume that \( \text{deg}_G(t) > 2 \). Since \( \text{deg}_H(t) = \text{deg}_G(t) > 2 \) and \( u \) is a leaf of \( H \), \( t \) has a neighbor in \( z \in Y \setminus \{u, v, w\} \). Again by the same argument as in the case 1 we obtain a contradiction that \( \nu(G) \geq m + 1 \). Thus, \( \text{deg}_G(s) = \text{deg}_G(t) = 2 \).

**Figure 4.**

Since \( \text{deg}_G(w) > 2 \), \( w \) is adjacent with some vertices in \( \{x_3, \ldots, x_m\} \) (see Figure 4). We may assume that \( w \) is adjacent with \( x_3, \ldots, x_p \) and not adjacent with \( x_{p+1}, \ldots, x_m \) for \( 3 \leq p \leq m \). Let \( G' \) be the graph obtained from \( G \) by deleting \( p - 3 \) edges \( wx_4, \ldots, wx_p \). By applying successively Lemma \( \ref{lem:reg} \) we have
\[
\text{reg}(G') = \nu(G') + 1 = \nu(G) + 1 = \text{reg}(G).
\]

Similarly, if we let \( G'' \) be the graph obtained from \( G' \) by deleting all edges of the form \( x_3y \) where \( y \in Y \setminus \{w\} \) and \( y \) is not a leaf of \( G \), then
\[
\text{reg}(G'') = \nu(G'') + 1 = \nu(G') + 1 = \text{reg}(G')
\]
and then
\[
\text{reg}(G'') = \nu(G'') + 1 = \nu(G) + 1 = \text{reg}(G).
\]
Let $S$ be all leaves of $G''$ that are adjacent with $x_3$. Let $G_1 := G[\{s, u, v, t, w, x_3\} \cup S]$ and $G_2 := G \setminus (\{s, u, v, t, w, x_3\} \cup S)$. Then, $G'' = G_1 \sqcup G_2$, therefore

$$\text{reg}(G'') = \text{reg}(G_1) + \text{reg}(G_2) - 1.$$ 

Since $G_2$ is a Cameron-Walker graph by Theorem 1, $\text{reg}(G_2) = \nu(G_2) + 1 = m - 2$. Now we compute $\text{reg}(G_1)$. Observe that $G_1 \setminus w$ consists of two connected components that are a path of length 4 and a star with center $x_3$, so $\text{reg}(G_1 \setminus w) = 3$; and $(G_1)_w$ consists of an edge and the set $S$ of isolated vertices, so $\text{reg}((G_1)_w)) = 2$. By Lemma 6 we get $\text{reg}(G_2) = 3$. Therefore,

$$\text{reg}(G'') = \text{reg}(G_1) + \text{reg}(G_2) - 1 = 3 + (m - 2) - 1 = m = \nu(G).$$

This contradicts the fact that $\text{reg}(G'') = \nu(G) + 1$.

In summary, we must have $V(G) = V(C_5)$, thus $G = C_5$ as we have seen in the beginning of the proof, and thus the lemma follows. \hfill \Box

We are now in position to prove the main result of the paper.

**Theorem 11.** Let $G$ be a graph. Then, $\text{reg}(I(G)) = \nu(G) + 1$ if and only if each connected component of $G$ is either a pentagon or a Cameron-Walker graph.

**Proof.** By Lemma 7 we may assume that $G$ is connected. If $G$ is $C_5$-free, then it is a Cameron-Walker graph by Lemma 8. If $G$ has a cycle of length 5, say $C_5$, it is just $C_5$ by Lemma 10, as required. \hfill \Box

We conclude the paper with an example to show that $\text{reg}(G) = \nu_0(G) + 1$ (resp. $\text{reg}(G) = \text{cochord}(G) + 1$), in general, depends not only the structure of $G$ but also the characteristic of the based field $k$.

**Example 12.** Let $G$ be the graph $G_2$ in [9, Appendix A], depicted in Figure 5.

![Figure 5](image_url)

Then, Macaulay 2 (see [5]) computations show that:
(1) If char(k) $\neq 2$, then reg(G) = 3.
(2) If char(k) = 2, then reg(G) = 4.

We now claim that $\nu_0(G) = 2$ and cochord(G) = 3. Indeed, take k to be a field with char(k) = 0 so that reg(G) = 3. From [9, Lemma 2.2], we obtain $\nu_0(G) \leq reg(G) - 1 = 2$. Observe that $\{(1,4), (3,10)\}$ is an induced matching of G, so $\nu_0(G) \geq 2$. It follows that $\nu_0(G) = 2$. Next, take k to be a field with char(k) = 2 so that reg(G) = 4. By Lemma [10, Theorem 1], cochord(G) $\geq$ reg(G) - 1 = 3. On the other hand, we have three co-chordal subgraphs $G_1$, $G_2$ and $G_3$ of G that cover the edges of G; where we color the edges of $G_1$ by red, the edges of $G_2$ by green, and the edges of $G_3$ by blue. Hence, cochord(G) $\leq$ 3, and hence cochord(G) = 3, as claimed.

Thus,
(1) reg(G) = $\nu_0(G) + 1$ if and only if char(k) $\neq 2$.
(2) reg(G) = cochord(G) + 1 if and only if char(k) = 2.

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