The application of the improved Talbot’s inverse Laplace transformation method in solving the flow problem of porous materials

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Abstract. In this paper, we extend the fixed Talbot’s method to the complex-valued function in order to get the general solutions for the Biot’s consolidation in the physical domain. We derive a solution for the unsteady flow field of layered porous media with anisotropic permeability under a point fluid source. By a Laplace and two-dimensional (2D) Fourier transform, the continuity equation of the fluid can be solved, and the flow field can be expressed in an analytical form in the transformed domain. Using the boundary and interface condition, the flow field for general layered porous media can be solved in the transform domain. The actual solutions in the physical domain can be obtained by inverting the Laplace-2D Fourier transform. Numerical examples are given to demonstrate the validity of the extended fixed Talbot’s inverse Laplace transformation method, and its application in solving dynamic problems of porous materials.

1. Introduction
The flow problem of porous materials is important and has been studied considerably. In many seepage models in porous media, the deformation of solid matrix usually is ignored, and the solid affects the fluid only by the fluid equation of motion, such as Darcy’s law. These models focus on the fluid flow, and are called fluid mechanics in porous media. Some fundamental solutions for basic flow problems in porous media can be found from textbooks [1-2]. Biot’s consolidation theory [3-6] which can truly reflect the coupling effect between dissipation of pore water pressure in soil and soil skeleton deformations is widely studied by a large number of scientists. But the Biot’s consolidation equation is a partial differential equation coupled with pore water pressure and displacement, it is difficult to be solved. In layered porous materials under dynamic loading, the governing equations are usually partial derivative ones with both time and space, a combined Laplace–Fourier or Laplace-Hankel transform often used to solve the equations [7-11]. For the inverse Fourier transform, the Gauss numerical integral can be applied. For the inverse Laplace transform, it’s not easy to be obtained, and some numerical algorithms have been given (Talbot[13], Davies[14], Abate[15][16], Avdis[17], Li[18], Luisa D’Amore[19], et al.) . Among them, the Talbot algorithm which is based on cleverly deforming the contour in the Bromwich inversion integral, is often be used [7][8]. Abate et al. [15][16] gave a fixed Talbot’s algorithm, which is a considerable simplification, and easy to be used. When solving the
consolidation problem, however, the existing inverse Laplace transform method cannot be applied directly for the complex-valued function.

In the present paper, for simplification, first we ignore the solid part in Biot’s porous consolidation equations and get the continuity equation of the fluid. Then using the Laplace -2D Fourier transform method, we derive the solutions for the fluid field in porous materials in the transformed domain. The results in physical domain can be obtained numerically by the inverse of Fourier and Laplace transform. For the inverse Fourier transform, polar coordinates are used and the Gauss quadrature integral is adopted. For the inverse of the Laplace transform, we extend the fixed Talbot’s method to the complex-valued function. Numerical examples are given to demonstrate the validity and accuracy of the present method.

2. Problem description
In a fixed Cartesian coordinate system \((x_1, x_2, x_3)\), for an anisotropic solid, the generalized Darcy’s law of the fluid phase with general fluid permeability is

\[
q_i = -\frac{k_{ij}}{\phi C_i \mu} P_{ij}
\]

where \(k_{ij}\) is the permeability tensor, and \(\mu\) is the dynamic viscosity of the fluid. \(P\) is the fluid pressure, \(p_{ij} = \partial p / \partial x_j\), the gradient of the pore pressure in the \(i\) direction with \(i, j = 1, 2, 3\). The permeability is related to the direction, and \(k_{ij}\) is a symmetry two-tensor. \(\phi\) is the porosity, and \(C_i\) is the coefficient of compressibility, \(q_i\) the flow rate in the \(i\) direction.

Following the Biot’s consolidation model and theory [3-6], the basic governing equations of saturated poroelastic soil with anisotropic permeability can be expressed as follows

\[
B_{ijkl} u_{ijkl} + M_{ij} \frac{1}{M} P_{ij} = -f_i \tag{2a}
\]

\[
\frac{k_{ij}}{\mu} P_{ij} - \frac{1}{M} (\dot{\rho} - M_{ij} \dot{u}_{ij}) = -\chi \tag{2b}
\]

The equations of equilibrium indicate displacements \(u_k\) and fluid pressure \(P\) change under the action of a concentrated force \(f_i\) and fluid source \(\chi\). The dot above the letter indicates the derivative with respect to time, \(B_{ijkl} = c_{ijkl} - M_{ij} \frac{1}{M} M_{kl}\) is the elastic stiffness tensor of the corresponding region. The fourth-order tensor \(c_{ijkl}\) represents elastic properties of the porous aggregate. The symmetric tensor of elastic parameters \(M_{ij}\) represents elastic coupling between fluid and solid phases of the porous aggregate. The \(M\) measures the pressure exerted on fluid to saturate the porous solid frame.

Through observation, it can be found that only the derivative of equation (2b) with respect to time, namely only the equation (2b) needs to be Laplace transformation. To illustrate extend the fixed Talbot’s method to the complex-valued function, we will simplify the equation (2), ignore the solid part and only retain the fluid part. When subjected to a point fluid source \(q_i\) at \(X = (X_1, X_2, X_3)\) from in the time \(t = t_0\), the continuity equation [1] of the fluid phase is

\[
\frac{k_{ij}}{\mu \phi C_i} P_{ij}(x) - \dot{\rho}(x) = -q_j \delta(t, t_0) \delta(x, X) \tag{3}
\]

Here \(\delta(x, X)\) is the Dirac delta function, and \(x = (x_1, x_2, x_3)\).
3. General solutions in the transformed domain

In order to get a general solution, especially for layered materials, the Fourier-2D Laplace transform is used. The Laplace-Fourier transform with respect to \( t, x_1, x_2 \) [12] are defined as:

\[
\tilde{\mathcal{L}}(y_1, y_2, x_3, s) = \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} \tilde{f}(x_1, x_2, x_3, t) e^{v_x x_3} e^{-st} d\xi_2 d\xi_1
\]

\[
f(x_1, x_2, x_3, t) = \frac{1}{2\pi i} \int_{C} \int_{0}^{\infty} \int_{0}^{\infty} \tilde{f}(y_1, y_2, x_3, s) e^{v_x x_3} e^{-st} d\xi_2 d\xi_1
\]

where \( s, y_1, y_2 \) denote the Laplace-Fourier transform parameters with respect to \( t, x_1, x_2 \) respectively.

\( i = \sqrt{-1}, \alpha \) takes the summation from 1 to 2, and \( y_\alpha x_\alpha = y_1 x_1 + y_2 x_2 \).

Therefore in the transformed domain, the equilibrium equation (3) becomes

\[
-s\tilde{p} - \frac{k_{2\alpha}}{\mu\phi C_i} y_\beta \tilde{p} - \frac{1}{\mu\phi C_i} i(k_{3\alpha} + k_{3\beta}) y_\alpha \tilde{p}_{33} + \frac{k_{3\alpha}}{\mu\phi C_i} \tilde{p}_{33} = -q_s e^{i\eta x_\alpha} e^{-i\theta_0}
\]

where \( \alpha, \beta = 1, 2 \). Changing the variables \( (y_1, y_2) \) to the polar coordinates \( (\eta, \theta) \) as

\[
y_\alpha = \eta n_\alpha \quad \text{with} \quad n_1 = \cos \theta, \quad n_2 = \sin \theta
\]

Equation (5) is an ordinary differential equation in \( x_3 \). The general solution of it can be expressed as

\[
\tilde{p}(\theta, \eta, x_3) = f e^{-i\eta x_3}
\]

with \( \tau \) satisfying the following characteristic equation:

\[
Q + 2\tau R + \tau^2 \frac{k_{33}}{\mu\phi C_i} = 0
\]

with

\[
Q = \frac{k_{33}}{\mu\phi C_i} n_\alpha n_\beta + \frac{s}{\eta^2}, \quad R = \frac{k_{33}}{\mu\phi C_i} n_\alpha
\]

It can be proved that Equation (8) has a pair of roots \( \tau_1 \) and \( \tau_2 \) as

\[
\tau_1 = \frac{-R + i\Delta}{k_{33}/\mu\phi C_i}, \quad \tau_2 = \frac{-R - i\Delta}{k_{33}/\mu\phi C_i}, \quad \Delta = \sqrt{Q k_{33}/\mu\phi C_i - R^2}
\]

So the general solution has a form as

\[
\tilde{p}(\theta, \eta, x_3) = f_1 e^{-i\eta x_3} + f_2 e^{i\eta x_3}
\]

where \( f_1 \) and \( f_2 \) are determined by the boundary conditions. Applying the Laplace transform and the 2D Fourier transform to Equation (1) and substituting the solution Equation (11) of \( \tilde{p} \) into it, the flow rate in the transformed domain can be expressed as

\[
\tilde{q}_i = i\eta(g_1 f_1 e^{-i\eta x_3} + g_2 f_2 e^{i\eta x_3})
\]

with

\[
g_1 = \frac{k_{3\alpha}}{\mu\phi C_i} n_\alpha + \frac{k_{3\beta}}{\mu\phi C_i} \tau_1, \quad g_2 = \frac{k_{3\alpha}}{\mu\phi C_i} n_\alpha + \frac{k_{3\beta}}{\mu\phi C_i} \tau_2
\]

Thus equations (11) and (12) give the general expressions of the formation pressure and flow rate in the transformed domain. It is also worth mentioning that the general solutions (11) remain valid if \( f_1 \) and \( f_2 \) are multiplied by functions of variables \( (\theta, \eta, X) \), since \( f_1 \) and \( f_2 \) are functions of variables \( (\theta, \eta, X) \) to be determined. Such as for the convenience of calculation later on, the general solution (11) can be rewritten in a form as

\[
\tilde{p}(\theta, \eta, x_3) = \eta^{i}(f_1 e^{-i\eta (x_3 - X_3)} + f_2 e^{i\eta (x_3 - X_3)}) e^{i\eta n_\alpha}
\]
with $X_3$ being the $x_3$ coordinate of an interface or a point source. Correspondingly the transformed fluid equation (12) can be rewritten as

$$\tilde{q}_i = i(g_1f_1e^{-i\tau_{ij}(x_i)} + g_2f_2e^{-i\tau_{ij}(x_i)})e^{im_\theta X_\theta}$$

(15)

### 4. Fluid Green’s functions

The formation pressure and flux field in the porous material under to a unit point fluid source are called the Green’s functions.

#### 4.1. Fluid Green’s functions in the transformed domain

The anisotropic permeability homogeneous porous material is considered with the interface being at $x_i = X_3$ plane. We assume that the source point $X = (X_1, X_2, X_3)$ is at the interface. When the interface is a flow perfect interface, the formation pressure and the flow rate in the 3 direction satisfy the following conditions

$$p\bigg|_{x_i-X_3} = p\bigg|_{x_i-X_3}, \quad q_i\bigg|_{x_i-X_3} - q_i\bigg|_{x_i-X_3} = q_i\delta(t-t_0)\delta(x_1-X_1)\delta(x_2-X_2)$$

(16)

In the transform domain, the interface boundary conditions are

$$\tilde{p}\bigg|_{x_i-X_3} = \tilde{p}\bigg|_{x_i-X_3}, \quad \tilde{q}_i\bigg|_{x_i-X_3} - \tilde{q}_i\bigg|_{x_i-X_3} = q_i e^{-i\tau_\theta} e^{im_\theta X_\theta}$$

(17)

Furthermore, the condition that the solutions should vanish as $|x_i|$ approaches infinity, so $f_i = 0$ in the solution for a half space $x_3 > X_3$, and $f_i$ for a half space $x_3 < X_3$.

Substituting (14) and (15) into (17), we can obtain

$$f_i = f_j = f^\infty = \frac{q_i e^{-i\tau_\theta}}{2\Delta}$$

(18)

So the fluid Green’s functions in the transformed domain as

$$\tilde{p}^\infty(\theta, \eta, x_i) = \begin{cases} \frac{q_i e^{-i\tau_\theta}}{2\Delta} e^{-i\tau_{ij}(x_i)} e^{im_\theta X_\theta} , & x_3 > X_3 \\ \frac{q_i e^{-i\tau_\theta}}{2\Delta} e^{-i\tau_{ij}(x_i)} e^{im_\theta X_\theta} , & x_3 < X_3 \end{cases}$$

(19)

$$\tilde{q}_i^\infty(\theta, \eta, x_i) = \begin{cases} ig_i \frac{q_i e^{-i\tau_\theta}}{2\Delta} e^{-i\tau_{ij}(x_i)} e^{im_\theta X_\theta} , & x_3 > X_3 \\ ig_i \frac{q_i e^{-i\tau_\theta}}{2\Delta} e^{-i\tau_{ij}(x_i)} e^{im_\theta X_\theta} , & x_3 < X_3 \end{cases}$$

(20)

#### 4.2. Fluid Green’s functions in the physical domain

Applying the Laplace-2D Fourier inverse transform, the fluid Green’s functions in equations (19) and (20) give

$$p^\infty = \begin{cases} \frac{1}{8\pi^3} \int_{r>0} \int_{\tau<0} \int_{x>0} e^{ikr} e^{-ik_\eta x} \frac{q_i e^{-i\tau_{ij}(x_i)} e^{im_\theta X_\theta}}{2\Delta} d\eta d\theta dx_3 > X_3 \\ \frac{1}{8\pi^3} \int_{r>0} \int_{\tau<0} \int_{x<0} e^{ikr} e^{-ik_\eta x} \frac{q_i e^{-i\tau_{ij}(x_i)} e^{im_\theta X_\theta}}{2\Delta} d\eta d\theta dx_3 < X_3 \end{cases}$$

(21)
\[ q_i = \begin{cases} 
\frac{1}{8\pi^2} \int_{-\infty}^{\infty} e^{i\eta t} \left[ \int_{-\infty}^{\infty} e^{-i\eta n} \right] \frac{1}{\Delta} \left[ e^{i\eta(\tau_1(s_x-x))} + n_i(s_x-x) \right] d\eta d\theta & x_3 > X_3 \\
\frac{1}{8\pi^2} \int_{-\infty}^{\infty} e^{i\eta t} \left[ \int_{-\infty}^{\infty} e^{-i\eta n} \right] \frac{1}{\Delta} \left[ e^{i\eta(\tau_1(s_x-x))} + n_i(s_x-x) \right] d\eta d\theta & x_3 < X_3 
\end{cases} \] (22)

Since \( \tau_1 \) and \( \tau_\alpha \) are complex Numbers, the existing numerical inverse Laplace transform method does not apply to equations (21) and (22). We extend the fixed Talbot’s method [15][16] to the complex-valued function. The inversion formula to calculate \( f(t) \) numerically for complex-valued \( f \) is

\[ f(t) = \frac{1}{5t} \sum_{k=-N+1}^{N-1} \gamma_k f^*(\delta_k) \] (23)

where

\[ \delta_0 = \frac{2N}{5}, \quad \delta_k = \frac{2k\pi}{5} \left[ \cot\left(\frac{k\pi}{N}\right) + i \right], \quad -N < k < N, \quad s = \frac{\delta_k}{t} \]

\[ \gamma_0 = e^{\delta_0}, \quad \gamma_k = \left\{ 1 + i \left[ \frac{k\pi}{N} \right] \left[ 1 + \left[ \cot\left(\frac{k\pi}{N}\right) \right]^2 \right] - i \cot\left(\frac{k\pi}{N}\right) \right\} e^{\delta_k}, \quad -N < k < N \] (24)

5. Verification of the method

To verify the Green’s functions obtained by the present method, numerical examples are given with \( \phi = 0.157 \), \( C_i = 1.47 \times 10^{-4} MPa^{-1} \), \( \mu = 0.05 \) mPa-s, \( k = 0.05 \) D for the isotropic case and \( k_{11} = 0.04 \) D, \( k_{22} = 0.05 \) D, \( k_{33} = 0.06 \) D for the orthogonal anisotropy case. Green’s functions at field (x) under point flux source at (X), are presented in Table 1 and 2 for the case of isotropic and orthogonal anisotropic fluid permeability, and compared with the results by Chen [1]. It shows that the results by the present method are in good agreement with those known ones.

The present method can be used to calculate the Green’s functions with general anisotropic fluid permeability, in layered porous materials.

Table 1. Surface Green’s functions results in an isotropic at X (1, 2, 3), x (4, 5, 6)

| Time (s) | Present | Previous [1] | Present | Previous [1] |
|----------|---------|--------------|---------|--------------|
| 1        | 0.0003122047894 | 0.0003122047834 | 0.00046830718 | 0.00046830719 |
| 10       | 0.0001630211928 | 0.0001630211928 | 2.445317892e-05 | 2.445317891e-05 |
| 100      | 6.823772149e-06 | 6.823772148e-06 | 1.02356582e-07 | 1.02356589e-07 |
| 1000     | 2.2192312e-07   | 2.2192313e-07   | 3.32884e-10    | 3.32886e-10    |

Table 2. Green’s functions results in an orthotropic materials at X (0, 0, 0), x (1, 2, 3)

| Time (s) | \( p \) (Present) | \( p \) (Previous [1]) |
|----------|-------------------|-----------------------|
| 1        | 0.00164986557906  | 0.00164986557902     |
| 10       | 0.0001961204638   | 0.0001961204637      |
| 100      | 7.07995254e-06    | 7.07995256e-06       |
| 1000     | 2.2687209e-07     | 2.2687210e-07        |

6. Conclusions

The general solution for the unsteady flow field of porous media with anisotropic permeability under a point fluid source is given. The fixed Talbot’s method is extended to calculate the inverse Laplace transform numerically for the complex-valued function. The results obtained by the present method is good agreement with the existing analytic solutions for isotropic and orthotropic cases. The proposed method in this paper is stable and efficient to solve the inverse Laplace transform for a complex-valued function, especially in solving the flow problems in porous materials.
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