The Low-level Spectrum of the $W_3$ String

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ABSTRACT

We investigate the spectrum of physical states in the $W_3$ string theory, up to level 2 for a multi-scalar string, and up to level 4 for the two-scalar string. The (open) $W_3$ string has a photon as its only massless state. By using screening charges to study the null physical states in the two-scalar $W_3$ string, we are able to learn about the gauge symmetries of the states in the multi-scalar $W_3$ string.

* Supported in part by the U.S. Department of Energy, under grant DE-FG05-91ER40633.

$ Supported in part by the Commission of the European Communities under Contract SC1*-CT91-0674.
1. Introduction

String theory arose as an attempt to explain hadronic physics. The first stage in this development was to deduce from spin-0 (tachyon) scattering the scattering amplitudes between any of the infinite number of particles present in the theory. In this process of factorisation, an infinite number of harmonic oscillators were introduced, and it was then that the Virasoro algebra first made its appearance, in connection with the physical states contained in the oscillator Fock space. Our present understanding of string theory gives a central rôle to the Virasoro algebra. Indeed, string theories are now constructed out of conformal blocks in such a way that their central charge add up to 26 (or 10 in the case of superstrings), so as to cancel the conformal anomaly of the ghosts (resp. superghosts).

Zamolodchikov introduced into conformal field theory a new two-dimensional algebra containing a spin two and a spin three current [1]. This is not a Lie algebra in the usual sense, but nevertheless it has been shown that it possesses a BRST charge that is nilpotent [2]. It was natural to suggest [3,4] that this $W_3$ algebra could be used to construct a string theory. To obtain a consistent string theory, one must cancel all local anomalies. In the work of [5,6], it was seen that this is the case if the matter carries a representation of the $W_3$ algebra with central charge $c = 100$. Unfortunately, unlike the case of the Virasoro algebra, one cannot in general simply add two commuting $W_3$ realisations together to form a third in order to build up a $c = 100$ realisation.

Realisations of the $W_N$ algebras with arbitrary central charge can be built from $N - 1$ scalar fields by means of a quantum generalisation of the Miura transformation [7]. These realisations may be generalised to include arbitrary numbers of scalar fields [8], thus providing a starting point for the construction of $W$-string theories. The original realisation of $W_3$ in [7] involved only two scalar fields; we refer to the corresponding string as the two-scalar $W_3$ string. The generalisation to $D + 2$ scalar fields leads to the $(D + 2)$-scalar $W_3$ string.

In [6,9,10,11], arguments were given towards the clarification of the $W$-string spectrum. In this paper, we start in section 2 by systematically finding the physical states in the ghost-vacuum sector of the $W_3$ string at low-lying levels by solving the physical-state conditions. In particular, for the two-scalar $W_3$ string we solve for all the levels up to and including level 4. For the $(D + 2)$-scalar $W_3$ string we solve for all states up to and including level 2. In section 3, we review a characteristic feature of these theories: the inner products of the physical states in the two-scalar realisation at all levels higher than 0 do not occur in conjugate pairs with respect to the inner-product momentum-conservation condition, and so must be treated as zero-norm states. The implications of this circumstance become more clear in section 4, where we show which of the physical states turn out to be BRST-trivial, finding agreement with the states assigned zero norm from the inner-product analysis. A general pattern for the null states emerges from our study, and we conjecture that this holds to all levels. For multi-scalar $W_3$ string realisations, the pattern found in the two-scalar string is also expected.
to generalise, with all non-null physical states in the ghost-vacuum sector being constructed solely from excitations not involving creation operators of the “frozen” coordinate that is allowed only discrete momentum values by the $W_3$ constraints. After reviewing the case of the single-scalar Virasoro string in section 5, in section 6 we show that all of the physical states at levels $N > 0$ of the two-scalar $W_3$ string, all of them null as seen earlier, may be written in terms of the action of screening operators on the vacuum state.

2. Physical States

The $W_3$ algebra [1] takes the form

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12}n(n^2 - 1)\delta_{n+m,0},$$
$$[L_n, W_m] = (2n - m)W_{n+m},$$
$$[W_n, W_m] = \frac{16}{22 + 5c}(n - m)\Lambda_{n+m}$$
$$+ (n - m)\left[\frac{1}{15}(n + m + 2)(n + m + 3) - \frac{1}{6}(n + 2)(m + 2)\right]L_{n+m}$$
$$+ \frac{c}{360}n(n^2 - 1)(n^2 - 4)\delta_{n+m,0},$$

where

$$\Lambda_n = \sum_m :L_{n+m}L_{-m}: - \frac{1}{20}\left[n^2 - 4 - \frac{5}{2}(1 - (-1)^m)\right]L_n$$

and the normal ordering means that $:L_pL_q := L_qL_p$ if $p > q$. There exists a well-known two-scalar realisation of the $W_3$ algebra. However, it was found in [8] that one could also have a many-scalar realisation. We define

$$T = -\frac{1}{2} (\partial\phi^{(2)})^2 - Q\partial^2\phi^{(2)} + \bar{T}$$
$$W = \frac{1}{3} (\partial\phi^{(2)})^3 + Q\partial\phi^{(2)}\partial^2\phi^{(2)} + \frac{1}{3}Q^2\partial^3\phi^{(2)} + 2\partial\phi^{(2)}\bar{T} + Q\partial\bar{T},$$

where

$$\bar{T} = -\frac{1}{2}\partial X^\mu\partial X^\nu\eta_{\mu\nu} - a_\mu\partial^2 X^\mu.$$ 

Here the scalar fields $X^\mu$ are $X^\mu = (X^\mu, \phi^{(1)})$ with $\mu = 0, 1, 2, \ldots, D - 1$, and $\eta^{\mu\nu} = (-1, 1, 1, \ldots, 1)$ is the flat $(D + 1)$-dimensional Minkowskian metric. $T$ and $W$ furnish a realisation of the $W_3$ algebra provided that $a_\mu a_\mu = \frac{1}{3}Q^2 - \frac{1}{12}D$. Note that to get the normalisation of equation (2.1), one should multiply $W_m$ by $\frac{24}{\sqrt{201}}$. This is unnecessary for the purposes of this paper, and for simplicity we use the normalisation in (2.4). The central charge of such a realisation is

$$c = 2 + D + 12Q^2 + 12a_\mu a_\mu.$$
These fields provide a suitable string theory if we cancel the conformal anomaly. Since the ghosts contribute \(-100\) to the central charge, we require that \(c = 100\) in (2.6) and so
\[
Q^2 = \frac{49}{8}, \quad a^2 \equiv a_{\hat{\mu}}a_{\hat{\mu}} = \frac{1}{12}(\frac{49}{2} - D) . \tag{2.7}
\]

A representation of the \( W_3 \) algebra in the language of Laurent modes is then given by
\[
L_n = \oint T(z) z^{-n-2} , \quad W_m = \oint W(z) z^{-n-3} , \tag{2.8}
\]
It will prove useful to have oscillator expressions for \( L_n \) and \( W_n \). We define
\[
i\partial \phi^{(2)} = \sum_n \alpha^{(2)}_n z^{-n-1} , \quad i\partial \hat{X}^{\hat{\mu}} = \sum_n \alpha^{\hat{\mu}}_n z^{-n-1} , \tag{2.9}
\]
whereupon
\[
L_n = \frac{1}{2} \sum_p : \alpha^{(2)}_p \alpha^{(2)}_{n-p} : - i(n+1)Q \alpha^{(2)}_n + \tilde{L}_n , \tag{2.10}
\]
and
\[
W_n = \frac{i}{3} \sum_{p,q} : \alpha^{(2)}_{n-p-q} \alpha^{(2)}_p \alpha^{(2)}_q : + Q \sum_p (p+1) : \alpha^{(2)}_{n-p} \alpha^{(2)}_p : - \frac{i}{3} Q^2 (n+1)(n+2) \alpha^{(2)}_n
\]
\[
- 2i \sum_p \alpha^{(2)}_{n-p} \tilde{L}_p - (n+2) \tilde{L}_n . \tag{2.11}
\]

Hermiticity of these generators requires that
\[
\alpha^{(2)}_n = \alpha^{(2)}_{-n} , \quad \alpha^{\hat{\mu}}_n = \alpha^{\hat{\mu}}_{-n} , \quad n \neq 0 , \quad \alpha^{(2)}_0 = \alpha^{(2)}_{-0} = \frac{1}{2} Q , \quad a^{\hat{\mu}}_0 = a^{\hat{\mu}}_{-0} = 2iQ . \tag{2.12}
\]

In this paper, we shall focus our attention on physical states that have the “standard” form \( \mid\text{phys}\rangle = \mid\psi\rangle \otimes \mid\text{gh}\rangle \), where \( \mid\psi\rangle \) is built purely from “matter” oscillators acting on a momentum state, and \( \mid\text{gh}\rangle \) is the ghost vacuum state. In terms of the \( b, c \) ghosts for spin 2, and the \( \beta, \gamma \) ghosts for spin 3, \( \mid\text{gh}\rangle \) takes the form \( c_1 \gamma_1 \gamma_2 \mid 0 \rangle \mid 9,12 \). BRST invariance of the state \( \mid\text{phys}\rangle \) then implies that for these “standard” states the physical-state conditions on \( \mid\psi\rangle \) for the \( W_3 \) string are [9,10]
\[
L_n \mid\psi\rangle = 0 = W_n \mid\psi\rangle , \quad n \geq 1 , \tag{2.13}
\]
\[
(L_0 - 4) \mid\psi\rangle = 0 = W_0 \mid\psi\rangle .
\]
The intercept of the $W_0$ generator is zero when $W(z)$ is a primary current with respect to the whole energy-momentum tensor $T(z)$. It suffices to solve the $L_0, L_1, L_2$ and $W_0$ constraints, since the remainder then follow by commutation. To this end, it is more convenient to rewrite $W_0$ using the other physical-state conditions. Consider a primary state $|\chi\rangle$, i.e. one that is annihilated by the positive Fourier modes of the currents $T(z)$ and $W(z)$, with $L_0|\chi\rangle = h|\chi\rangle$ and $W_0|\chi\rangle = \omega|\chi\rangle$. The relevant part of $W_0$ acting on the state takes the form

$$W_0 = \frac{i}{6}\left\{2\sum_{p,q}^{'} \alpha_0^{(2)} \alpha^{(2)}_p \alpha^{(2)}_q : + 6 \sum_{p>0}^{'} \sum_{q}^{'} \alpha_0^{(2)} \alpha^{(2)}_p \alpha^{(2)}_{p-q} : + f 
+ (\hat{\alpha}_0^{(2)})^2 [36\mathcal{N}^{(2)} + 12(4-h)] - 12 \sum_{p>0}^{'} \alpha_0^{(2)} \tilde{L}_p \right\}$$

(2.14)

where $f = \hat{\alpha}_0^{(2)} (8(\hat{\alpha}_0^{(2)})^2 + 1)$, and $\alpha_0^{(2)} = \alpha^{(2)}_0 - iQ$ is an hermitean operator ((2.14) corrects an expression given in [10]). $\mathcal{N}^{(2)}$ is the number operator for $\alpha^{(2)}$ oscillators, i.e.

$$\mathcal{N}^{(2)} = \sum_{n=1}^{\infty} \alpha_0^{(2)} \alpha_0^{(2)}$$

(2.15)

The primes on summation symbols in (2.14) means that they contain no zero-mode terms.

We can analyse the physical-state conditions at each level independently. The level number $N$ of a state is defined as the eigenvalue of the number operator

$$\mathcal{N}^{(2)} + \sum_{n>0} \eta_{\tilde{\nu}\tilde{\mu}} \alpha_0^{\tilde{\nu}} \alpha_0^{\tilde{\mu}}$$

(2.16)

acting on the state. A state with no oscillator excitations has level number 0. We shall systematically study the physical states up to and including level 4. However, we shall first make some general remarks on how to solve the physical-state conditions (2.13).

A state in the Fock space generated by $\alpha_0^{(2)}$ and $\alpha_0^{\tilde{\mu}}$ is of the form

$$|\alpha_0^{\tilde{\mu}}\rangle = \varepsilon_{\tilde{\mu}_1\ldots\tilde{\mu}_q} \alpha_0^{(2)} \alpha_0^{\tilde{\nu}_1} \alpha_0^{\tilde{\nu}_2} \ldots \alpha_0^{\tilde{\nu}_q} |p^{\tilde{\nu}_1},\ldots,p^{\tilde{\nu}_q}\rangle$$

(2.17)

where $|p^{\tilde{\nu}},\beta\rangle$ is a highest-weight state defined by

$$\alpha_0^{\tilde{\nu}} |p^{\tilde{\nu}},\beta\rangle = \beta |p^{\tilde{\nu}},\beta\rangle$$

$$\alpha_0^{(2)} |p^{\tilde{\nu}},\beta\rangle = 0 = \alpha_0^{(2)} \alpha_0^{\tilde{\nu}} |p^{\tilde{\nu}},\beta\rangle$$

(2.18)

and we refer to $\varepsilon_{\tilde{\mu}_1\ldots\tilde{\mu}_q}$ as polarisation tensors of the state. At level $N$, there are $C_N^{D+2}$ such possible states, with $C_N^{D+2}$ given by

$$\prod_{n=1}^{\infty} \frac{1}{(1 - x^n)(D+2)} = \sum_{N=0}^{\infty} C_N^{D+2} x^N$$

(2.19)
At level $N$, the $(L_0 - 4)\langle \psi \rangle = 0$ condition provides only one constraint, namely the mass-shell equation

$$\beta(\beta - 2iQ) + p^\mu(p_\mu - 2ia_{\hat{\mu}}) = 2(4 - N) .$$

The $L_1\langle \psi \rangle = 0$ and $L_2\langle \psi \rangle = 0$ conditions provide $C_{N-1}^{D+2}$ and $C_{N-2}^{D+2}$ constraint equations (not necessarily independent) respectively. The other non-trivial condition, $W_0\langle \psi \rangle = 0$, provides $C_N^{D+2}$ constraint equations. It is easier to study the $W_0$ constraint since $W_0$ acting on $\langle \psi \rangle$ will not change the level number of $\langle \psi \rangle$. Thus the number of equations from this constraint is the same as the number of polarisation-tensor components. The coefficients of the polarisation tensors in these equations are functions of the momenta $\beta$ and $p^\mu$ of the state on which the oscillators act. We may write $W_0\langle \psi \rangle = 0$ in the generic matrix form

$$A\varepsilon = 0 ,$$

where $A$ is a $C_N^{D+2} \times C_N^{D+2}$ matrix function of $\beta$ and $p^\mu$, since the numbers of equations and polarisation-tensor components are both equal to $C_N^{D+2}$. In (2.21), $\varepsilon$ is a column vector of the $C_N^{D+2}$ polarisation-tensor components. Clearly we have non-zero solutions only when $\det A = 0$. The operator $W_0$ however, which can be written in the form of equation (2.14), always preserves or lowers $N^{(2)}$, the level number for $\alpha_n^{(2)}$ oscillators. Therefore if we order the components of the column vector $\varepsilon$ such that the polarisation-tensor components $\varepsilon_1$ associated with $\alpha_n^{(2)}$ only come first, followed by the components $\varepsilon_2$ for mixed oscillator terms, and finally the components $\varepsilon_3$ for $\alpha_n^\hat{\mu}$ oscillators only at the bottom, then the matrix $A$ will be of the form

$$A = \begin{pmatrix} M_1 & 0 & 0 \\ * & M_2 & 0 \\ * & * & M_3 \end{pmatrix} ,$$

where $*$ indicates non-zero components whose precise form we do not need to know. Consequently, we have $\det A = \det M_1 \cdot \det M_2 \cdot \det M_3$, and so it will vanish if one or more of the sub-determinants vanishes.

Examining the form of $W_0$ in equation (2.14), we find that $M_1$, $M_2$ and $M_3$ do not depend on $p^\mu$ and thus they are functions of $\beta$ alone. This results from the fact that the term $-2i\sum_{p=1}^{\infty} \tilde{L}_{-p}\alpha_p^{(2)}$ lowers $N^{(2)}$ and so gives contributions only to the starred areas of $A$ in equation (2.22). Indeed when calculating $\det A$, we can drop this term completely. One can easily see that $M_3$ is of the form

$$(M_3)_{ij} = f\delta_{ij} ,$$

where $f$ is defined below eq. (2.14), and so $\det M_3 = (f)^{C_N^{D+1}}$.

As a consequence we find that the $W_0$ constraint implies that to have a physical state, it must be that the determinants of any of $M_1$, $M_2$ or $M_3$ vanishes. Each determinant
is a polynomial in $\beta$ that can be solved to find the allowed roots. Depending on which determinant vanishes, we have three cases:

**Case 1**, $\det M_1 = 0$.

A physical state that includes terms that involve *only* $\alpha_{-n}^{(2)}$ oscillators must have $\det M_1 = 0$, since $\det M_1 \neq 0$ implies that terms with only $\alpha_{-n}^{(2)}$ oscillators must be absent. The equation $\det M_1 = 0$ is a polynomial of degree $3C_1^N$ in $\beta$.

**Case 2**, $\det M_2 = 0$.

These states can have terms generated by $\alpha_{-n} \hat{\mu}$ and $\alpha_{-m}^{(2)}$, but with no terms that only involve $\alpha_{-m}^{(2)}$ alone, unless $\det M_1 = 0$ also. $\det M_2 = 0$ is a polynomial equation of degree $3(C_D^{D+2} - C_1^N - C_D^{D+1})$ in $\beta$.

**Case 3**, $\det M_3 = 0$.

This implies that $f = 0$, *i.e.* $\hat{\beta}(8\hat{\beta}^2 + 1) = 0$. The three allowed roots are $\beta = iQ, \frac{6}{7}iQ, \frac{8}{7}iQ$. Such physical states contain excitations only in the $\hat{\mu}$ directions and thus can be written in the form

$$\left| \psi \right\rangle_{\text{eff}} \otimes \left| \beta \right\rangle,$$

where $\left| \psi \right\rangle_{\text{eff}}$ is in the Fock space generated by $\alpha_{-n} \hat{\mu}$ alone and $\left| \beta \right\rangle$ is an eigenstate of $\alpha_0^{(2)}$ which is annihilated by the number operator $N^{(2)}$. The remaining $L_n$ physical-state conditions become [9,10,11]

$$\bar{L}_n \left| \psi \right\rangle_{\text{eff}} = 0 \quad n \geq 1,$$

$$\left( \bar{L}_0 - a_{\text{eff}} \right) \left| \psi \right\rangle_{\text{eff}} = 0,$$

where $a_{\text{eff}} = \frac{15}{16}$ if $\beta = iQ$, and $a_{\text{eff}} = 1$ if $\beta = \frac{6}{7}iQ$ or $\frac{8}{7}iQ$. We refer to states of the form (2.24) as “ordinary” states.

In all three cases, we find that the momentum component $\beta$ of physical states can take only a discrete set of values. This phenomenon of the freezing of the $\beta$ momentum was first observed in [6]. In the above analysis of $W_0 \left| \psi \right\rangle = 0$, we have assumed that $\left| \psi \right\rangle$ is a primary state of conformal dimension 4. Thus it remains to solve the $L_1 \left| \psi \right\rangle = L_2 \left| \psi \right\rangle = 0$ constraints in order to single out the physical solutions for the $\beta$ momentum from all the possible roots of $\det A = 0$. Given any allowed value of $\beta$, the values of $p^\mu$ are restricted by the mass-shell condition in (2.20). $L_1 \left| \psi \right\rangle = L_2 \left| \psi \right\rangle = 0$ can be used to solve the polarisation tensors of the physical states; in fact such solutions exist only for a limited set of $\beta$ values out of those allowed by $\det A = 0$. For the two-scalar $W_3$ string such constraints are much stronger. This is analogous to the one-scalar Virasoro string, where the number of physical states is significantly reduced.
We now start systematically to solve the physical state conditions for the \((D + 2)\)-scalar \(W_3\) string up to level 2, and the two-scalar \(W_3\) string up to level 4.

**Level 0**

The analysis of level 0 states is straightforward. Consider a state \(|p^\hat{\mu}, \beta\rangle\) defined in (2.18), which can also be viewed as a state of the form given in (2.24). One can easily see that all the physical states at this level are given by [6,10]

\[
|p^\hat{\mu}_1, iQ\rangle, \quad |p^\hat{\mu}_2, \frac{4}{7}iQ\rangle, \quad |p^\hat{\mu}_2, \frac{8}{7}iQ\rangle,
\]

subject to

\[
p^\hat{\mu}_1(p_1 - 2ia)_\hat{\mu} = \frac{15}{8}, \quad (\beta = iQ), \quad p^\hat{\mu}_2(p_2 - 2ia)_\hat{\mu} = 2, \quad (\beta = \frac{6}{7}iQ, \frac{8}{7}iQ).
\]

For the two-scalar \(W_3\) string, there is only one \(p^\hat{\mu}\), denoted by \(p\). In this case equation (2.27) yields six solutions [9];

\[
|\frac{6}{7}ia, iQ\rangle, \quad |\frac{9}{7}ia, iQ\rangle, \quad |\frac{6}{7}ia, \frac{6}{7}iQ\rangle,
\]

\[
|\frac{6}{7}ia, \frac{8}{7}iQ\rangle, \quad |\frac{8}{7}ia, \frac{6}{7}iQ\rangle, \quad |\frac{8}{7}ia, \frac{8}{7}iQ\rangle.
\]

**Level 1**

The most general form of the level-1 states is

\[
(\xi \alpha^{(2)}_\perp + \xi^\mu \alpha^\mu_\perp)|p^\hat{\mu}, \beta\rangle \equiv \mathcal{P}_1|p^\hat{\mu}, \beta\rangle,
\]

where \(\mathcal{P}_1\) denotes a generic level-1 excitation operator. After solving the physical-state conditions of equation (2.13), \(\mathcal{P}_1\) will be dependent on \(p^\hat{\mu}\) and \(\beta\); we shall denote it by \(\mathcal{P}_1(p^\hat{\mu}, \beta)\). Following the method for solving the physical-state conditions described above, we find that the level-1 physical states are [6]:

\[
\mathcal{P}_1(p^\hat{\mu}_1, \frac{10}{7}iQ)|p^\hat{\mu}_1, \frac{10}{7}iQ\rangle, \quad \mathcal{P}_1(p^\hat{\mu}_2, \frac{11}{7}iQ)|p^\hat{\mu}_2, \frac{11}{7}iQ\rangle,
\]

with

\[
p^\hat{\mu}_1(p_1 - 2ia)_\hat{\mu} = 1, \quad p^\hat{\mu}_2(p_2 - 2ia)_\hat{\mu} = \frac{15}{8}.
\]

The polarisation vectors in \(\mathcal{P}_1\) are given by

\[
\xi^\hat{\mu} = \frac{\xi p^\hat{\mu}}{iQ - 3\beta},
\]
and so we see that these states are actually scalars.

In addition, there are the “ordinary” (case 3) physical states

\[
\mathcal{P}_1(p_3^\mu, iQ)|p_3^\mu, iQ\rangle \quad \text{with} \quad p_3^\mu (p_3 - 2ia)_{\mu} = -\frac{1}{8},
\]

\[
\mathcal{P}_1(p_4^\mu, \frac{6}{7}iQ)|p_4^\mu, \frac{6}{7}iQ\rangle \quad \text{with} \quad p_4^\mu (p_4 - 2ia)_{\mu} = 0.
\]

(2.33)

The polarisation vectors for the states (2.33) obey

\[
\xi = 0, \quad (p_\mu - 2ia_\mu)\xi^\mu = 0.
\]

(2.34)

For the two-scalar \(W_3\) string we can determine \(p\), and we find at this level that there are a total of 6 physical states, whose polarisations \(\mathcal{P}_1\) follow from the above discussion. They are four case-1 states

\[
\mathcal{P}_1|\frac{12}{7}ia, \frac{10}{7}iQ\rangle, \quad \mathcal{P}_1|\frac{7}{2}ia, \frac{10}{7}iQ\rangle, \quad \mathcal{P}_1|\frac{7}{2}ia, \frac{11}{7}iQ\rangle, \quad \mathcal{P}_1|\frac{5}{7}ia, \frac{11}{7}iQ\rangle,
\]

(2.35)

together with two “ordinary” (case-3) states, for which \(p = 2ia\), owing to (2.34), is the only possibility. This excludes the states with \(\beta = iQ\). Thus the two ordinary states are

\[
\mathcal{P}_1|2ia, \frac{6}{7}iQ\rangle, \quad \mathcal{P}_1|2ia, \frac{8}{7}iQ\rangle.
\]

(2.36)

**Level 2**

The most general form of level-2 states is given by

\[
\mathcal{P}_2|p^\mu, \beta\rangle \equiv (\varepsilon \alpha^{(2)}_2 \alpha^{(2)}_1 + \varepsilon_\mu \alpha^{(2)}_1 \alpha^{(2)}_1 + \varepsilon_\mu \alpha^{(2)}_1 \alpha^{(2)}_1 + \xi \alpha^{(2)}_2 + \xi \mu \alpha^{(2)}_2)|p^\mu, \beta\rangle.
\]

(2.37)

We have the “ordinary” (case 3) physical states whose \(\beta\) values are, as for case-3 states at all levels, the same as those of the tachyon states (2.26), namely

\[
\mathcal{P}_2|p^\mu, iQ\rangle \quad \text{with} \quad p^\mu (p_\mu - 2ia_\mu) = -\frac{17}{8},
\]

(2.38)

and

\[
\mathcal{P}_2|p^\mu, \frac{6}{7}iQ\rangle \quad \text{with} \quad p^\mu (p_\mu - 2ia_\mu) = -2.
\]

(2.39)

The polarisation tensors of level-2 ordinary physical states all satisfy \(\varepsilon = \xi = \xi_\mu = 0\), together with

\[
(p^\mu - 2ia^\mu)\varepsilon_\mu_\nu + \xi_\mu_\nu = 0, \quad \varepsilon_\mu + 2(p_\mu - 3ia_\mu)\xi_\mu = 0.
\]

(2.40)
We also have a case-1 physical state

\[ P_2|p^\mu, \frac{12}{7}iQ\rangle \quad \text{with} \quad p^\mu(p_\mu - 2ia_\mu) = 1 , \quad (2.41) \]

whose polarisations are given by

\[
\begin{align*}
\epsilon_\mu &= \frac{8}{7}iQ\epsilon p_\mu , \\
\xi_\mu &= -\frac{2}{5}\epsilon(p_\mu - ia_\mu) , \\
\epsilon_{\tilde{\mu}\tilde{\nu}} &= \frac{1}{5}\epsilon(\eta_{\tilde{\mu}\tilde{\nu}} - 5p_{\tilde{\mu}}p_{\tilde{\nu}}) , \\
\xi &= -\frac{2}{7}iQ\epsilon . 
\end{align*}
\]

There are two case-2 physical states at level 2. Both have polarisations restricted by

\[
\epsilon = \xi = 0, \quad \epsilon_\mu = (4iQ - 3\beta)\xi_\mu \quad \epsilon_{\tilde{\mu}\tilde{\nu}} = \frac{1}{7}(\xi_{\tilde{\mu}}p_{\tilde{\nu}} + \xi_{\tilde{\nu}}p_{\tilde{\mu}}) , \quad (2.42)
\]

and can occur only for states with \( \beta = \frac{10}{7}iQ \) or \( \frac{11}{7}iQ \), i.e.

\[
\begin{align*}
P_2|p^\mu, \frac{10}{7}iQ\rangle \quad \text{with} \quad p^\mu(p_\mu - 2ia_\mu) = -1 , \\
P_2|p^\mu, \frac{11}{7}iQ\rangle \quad \text{with} \quad p^\mu(p_\mu - 2ia_\mu) = -\frac{1}{8} .
\end{align*}
\]

For the two-scalar \( W_3 \) string the number of solutions reduces significantly. There are a total of three physical states in this case; one ordinary (case 3) physical state \( P_2|\frac{17}{7}ia, iQ\rangle \), and two other (case 1) physical states \( P_2|\frac{12}{7}ia, \frac{12}{7}iQ\rangle \) and \( P_2|\frac{5}{7}ia, \frac{12}{7}iQ\rangle \).

**Level 3**

We carry out the analysis at level 3 only for the two-scalar \( W_3 \) string. The general states at this level have 10 independent excitation terms. We may write these as

\[
P_3 = \tau \alpha^{(2)} - \alpha^{(2)} - \alpha^{(2)} + \mu \alpha^{(2)} - \alpha^{(2)} - \sigma \alpha^{(2)} + \tau \alpha^{(1)} - \alpha^{(1)} - \alpha^{(1)} + \mu \alpha^{(1)} - \alpha^{(1)} - \alpha^{(1)} + \sigma \alpha^{(1)}
\]

\[+ \rho \alpha^{(2)} - \alpha^{(2)} - \alpha^{(2)} + \rho \alpha^{(2)} - \alpha^{(2)} - \alpha^{(2)} + \kappa \alpha^{(2)} - \alpha^{(2)} + \kappa \alpha^{(2)} - \alpha^{(2)} . \quad (2.45)\]

Enforcing the physical-state conditions in (2.13), one finds that there is no “ordinary” physical state. This situation is analogous to the one-scalar Virasoro string, where level-3 states cannot survive the restriction of the physical-state conditions. However, in the case of the two-scalar \( W_3 \) string, there are two case-2 physical states:

\[
P_3|\frac{17}{7}ia, \frac{11}{7}iQ\rangle \quad \text{and} \quad P_3|\frac{18}{7}ia, \frac{10}{7}iQ\rangle , \quad (2.46)
\]

where \( P_3 \) is given by (2.45) with

\[
\begin{align*}
\tau &= \mu = \sigma = \kappa = 0, \\
\bar{\rho} &= 1 , \\
\bar{\tau} &= \frac{12\rho}{f} , \\
\bar{\mu} &= \frac{12\beta(\beta - 2iQ)}{f} , \\
\bar{\sigma} &= -\frac{24(p - 2ia)}{f} , \\
\bar{\kappa} &= -(p - 2ia) .
\end{align*}
\]
where \( f = \beta \left( 8 \beta^2 + 1 \right) \), as well as two case-1 physical states:

\[
P_3\left| \frac{6}{7} i a, 2iQ \right\rangle \quad \text{and} \quad P_3\left| \frac{8}{7} i a, 2iQ \right\rangle ,
\]

where \( P_3 \) is given by (2.45) with

\[
\tau = \frac{3(p - 2ia)}{4iQ}, \quad \mu = -\frac{1}{2}(p - 2ia), \quad \sigma = -\frac{p - 2ia}{2iQ},
\]

\[
\bar{\tau} = -\frac{4iap}{49}, \quad \bar{\mu} = \frac{12ia}{49}, \quad \bar{\sigma} = \frac{-4ia(p - 2ia)}{49},
\]

\[
\rho = 1, \quad \bar{\rho} = -\frac{3p + 2ia}{4iQ}, \quad \kappa = \frac{a(p - 2ia)}{2Q}, \quad \bar{\kappa} = \frac{3}{2iQ}.
\]  

**Level 4**

Again, we consider only the two-scalar \( W_3 \) string at this level. There are 20 polarisation-tensor components at level 4, and the physical-state conditions give rise to 7 solutions. Three of these are case-1 states:

\[
P_4\left| \frac{5}{7} i a, \frac{15}{7} iQ \right\rangle , \quad P_4\left| \frac{9}{7} i a, \frac{15}{7} iQ \right\rangle \quad \text{and} \quad P_4\left| 0, 2iQ \right\rangle ;
\]

three are case-2 states:

\[
P_4\left| \frac{20}{7} i a, \frac{10}{7} iQ \right\rangle , \quad P_4\left| \frac{18}{7} i a, \frac{12}{7} iQ \right\rangle \quad \text{and} \quad P_4\left| 2ia, 2iQ \right\rangle ;
\]

and one is a case-3 ("ordinary") state:

\[
P_4\left| 3ia, iQ \right\rangle .
\]

The forms of the polarisation tensors are quite complicated here, and we shall not give them.

**3. Norms**

Having found the solutions to the physical state conditions, one must still ascertain which of these solutions truly represent physical states and which are just gauge artifacts, analogous to the longitudinal photons in QED. The surest way to do this is to consider which states are BRST cohomologically trivial and which are non-trivial, an issue to which we shall return in section 4. In ordinary gauge theories, one is accustomed to the fact that the gauge artifacts, or "spurious" states, are also of zero norm, as a result of the vanishing inner product of their polarisation tensors. The identification between spurious states and zero-norm physical states will be seen to persist in the case of the \( W_3 \) string, but the reason
for the zero norms of these states is different from that in ordinary gauge theories or in the critical bosonic string. The difference lies in the properties of the Fock space inner product and in the changed requirements of momentum conservation in the presence of background charges, and not in the properties of particular polarisation tensors.

Let us for simplicity consider at first a theory with only one species of oscillator \( \alpha_n \) that has a background charge \( Q \). As mentioned previously the hermiticity condition is \( \alpha_n^\dagger = \alpha_{-n} \) when \( n \neq 0 \), and \( \alpha_0^\dagger = \alpha_0 - 2iQ \). The notion of the adjoint corresponding to this hermiticity condition is the standard one of conformal field theory, in which the \( \text{SL}(2,\mathbb{C}) \)-invariant ket vector \( |\Omega\rangle \) is mapped into the \( \text{SL}(2,\mathbb{C}) \)-invariant bra vector \( \langle \Omega^*| \). The conditions of \( \text{SL}(2,\mathbb{C}) \) invariance require that \( |\Omega\rangle \) have a right-eigenvalue of the momentum operator \( \alpha_0 \) given by \( p_\Omega = 0 \), while the left-eigenvalue of \( \alpha_0 \) on \( \langle \Omega^*| \) is \( p_{\Omega^*} = 2iQ \). Given a ket \( |p\rangle \) with right-momentum \( p \), the conjugated bra \( \langle p^*| \) has right-momentum \( p^* + 2iQ \). Note that although the adjoint operation maps between vectors of different left- and right-momentum, it nonetheless satisfies the condition that for either purely imaginary \( p \) or for \( p \) of the form \( p = \text{real} + iQ \), the conformal eigenvalue \( h = \frac{1}{2}p(p - 2iQ) \) is the same real value for the conjugated bra and ket vectors. The eigenvalues \( p \) of \( \alpha_0 \), and even of the hermitean operator \( \hat{\alpha}_0 = \alpha_0 - iQ \), need not be real. We must, however, require that the eigenvalues of \( L_0 = \frac{1}{2}\alpha_0(\alpha_0 - 2iQ) + \cdots \) be real. This implies that \( p \) is either purely imaginary or \( p = iQ + \text{real} \). Thus, the conformal eigenvalue \( h \) is the same for conjugated bras and kets for all physically allowable values of the momentum.

From the above, given two momentum states, \( |p\rangle \) and \( |p'|\), then we have that

\[
\langle p'|\alpha_0|p\rangle = p\langle p'|p\rangle = (p'^* + 2iQ)\langle p'|p\rangle .
\] (3.1)

The momentum conservation law is then that \( \langle p'|p\rangle \) vanishes unless

\[
p = p'^* + 2iQ .
\] (3.2)

When the eigenvalues \( p \) of \( \alpha_0 \) are purely imaginary, the only value of \( p \) for which \( \langle p|p\rangle \) is non-vanishing is \( p = iQ \). If we have two states \( |p'| \) and \( |p\rangle \) for which \( p \neq iQ \) and \( p'^* = p - 2iQ \), then the inner-product is off-diagonal and so we will have one positive-norm and one negative-norm state.

An analogous situation arises in the Liouville theory formulation of the ordinary string in a non-critical spacetime dimension. It is well-known \cite{13,14} that the free-field realisation of Liouville theory actually has twice as many states as the true nonlinear Liouville theory itself. This happens because the presence of the Liouville interaction \( e^{\phi} \) has the effect of restricting the physical states to only one linear combination of the \( \phi \)-momentum eigenstates, where \( \phi \) is the Liouville mode, periodic in the spatial worldsheet coordinate as is appropriate for closed string theory. Thus, the true Liouville spectrum is obtained by a truncation of the spectrum of the free-field realisation. In the present case of the single free scalar, the
analogous truncation amounts to making a pairwise identification of the states with momenta $p$ and $p'$ for which $p^{*} = p - 2iQ$, i.e. one takes only the symmetric linear combination of the two states of the original free-field theory. In this truncation, the half of all the states having negative norm is set to zero, while the other half with positive norm is retained.

Although we shall not consider in detail the possibility of introducing interaction terms analogous to the Liouville interaction $e^{g_{\phi}}$, it is worth noting that such terms are in principle necessary in order to make inner product calculations such as (3.1) well-defined. In the case of the Liouville theory, this interaction is obtained from a cosmological term $\sim \int d^{2}\sigma \sqrt{\gamma}$ for the world-sheet metric $\gamma_{i\bar{j}}$ upon gauge fixing. The integrand of this term is of conformal weight one, and its presence dominates the integration over the zero-mode of $\phi$, rendering the integral convergent.

For the two-scalar realisation of the $W_3$ algebra, “screening charge” terms analogous to the cosmological term of Liouville theory have been found in [7,9], with integrands of the form $\exp(g e_{a}^{i} \phi_{i})$, where $e_{a}^{i}$, $a = 1, 2$ is the $a$'th simple root of $SU(3)$. A more careful discussion of the properties of inner products and norms in the $W_3$ string should properly include such terms, which are presumably responsible for the truncation of the free-field spectrum that excludes the negative-norm states.

For $W_3$ strings, we have two background charges, associated with $\alpha_n^{(2)}$ and $\alpha_n^{\hat{\mu}}$. We also find that the eigenvalues of $\alpha_0^{(2)}$ for physical states are always purely imaginary. In fact they are integer multiples of $\frac{1}{7}iQ$. At level 0, we find that physical states have momentum $\beta = iQ, \frac{6}{7}iQ, \frac{8}{7}iQ$. We thus identify the states with $\frac{6}{7}iQ$ and $\frac{8}{7}iQ$, and at the same time we make a similar identification of $p^{\mu}$ and $p^{\hat{\mu}} - 2ia^{\hat{\mu}}$. At this level we find, therefore, only two states instead of three. That we can match up the states at level 0 is a consequence of the fact that the zero-mode part of $L_0$ is symmetric under $\beta \rightarrow \beta - 2iQ$ and the zero-mode part of $W_0$ changes by a minus sign under this symmetry. The sign change is immaterial since the $W_0$ intercept is zero. However, the symmetry under $\beta \rightarrow \beta - 2iQ$ does not extend to the full $W_0$, nor even to $L_n$ when $n \neq 0$. Thus one cannot always match up the states at higher levels. An exception to this is provided by the “ordinary” states of case 3, where $\beta$ always takes the three values $iQ, \frac{6}{7}iQ$ and $\frac{8}{7}iQ$, regardless of the level number. The first is self-conjugate, and we can match up the latter two states, which are momentum conjugates of each other, and which have the same $\tilde{L}_0$ intercept of 1. For example, at level 1, aside from the ordinary states (case 3), we find states (2.30, 2.31) with $\beta = \frac{10}{7}iQ, \frac{11}{7}iQ$ that cannot be matched up. It follows that these states have zero scalar product with all other states, since momentum conservation cannot be satisfied in any two-point function containing them.

We shall show in the next section that these zero-norm states are also null, in the sense that they arise from $L_{-n}$’s and $W_{-n}$’s acting on highest-weight states. At level 2 we find ordinary states as well as states with $\beta = \frac{10}{7}iQ, \frac{11}{7}iQ$ and $\frac{12}{7}iQ$, and a similar conclusion is obtained. We also find similar results at levels 3 and 4, although there are subtleties at level 4 that we shall discuss at the end of the next section.
4. Null states

The physical states are by definition those that satisfy the on-shell conditions given by (2.13). Included amongst these states are those that have zero scalar product with all other physical states including themselves; such states are called null. Null states arise as a consequence of gauge invariance. The remaining non-null physical states describe the true on-shell degrees of freedom of the string theory. In a unitary theory these states must have positive norm. Some evidence for the unitarity of the physical states for $W_3$ and $W_N$ strings has been given in ref. [11]. We now wish to find which of the physical states found in the previous section are null and thus determine the true degrees of freedom of the $W_3$ string.

In a general string theory, null states can be written as generators of the string symmetry algebra acting on highest-weight states. For the $W_3$ string this means that a null state takes the form

$$\sum_{\{n_i\}} \sum_{\{m_j\}} L_{-n_1} L_{-n_2} \cdots L_{-n_p} W_{-m_1} W_{-m_2} \cdots W_{-m_q} |\Omega\rangle,$$

where $|\Omega\rangle$ is a highest-weight state, defined by

$$L_0 |\Omega\rangle = h |\Omega\rangle, \quad W_0 |\Omega\rangle = \omega |\Omega\rangle,$$

$$L_n |\Omega\rangle = W_n |\Omega\rangle = 0, \quad n \geq 1.$$

The level of the null state is $N_{\text{null}} = (\sum_{i=1}^p n_i + \sum_{j=1}^q m_j)$, and we can study null states level by level. Note that we distinguish between the total level number $N$, and $N_{\text{null}}$, since although $|\Omega\rangle$ is a highest-weight state, it can itself have a level number $N^\Omega$. Indeed we have $N = N_{\text{null}} + N^\Omega$. At a given level $N_{\text{null}}$, we write down all possible terms and then demand that the state be physical. This process, at times tedious, will generate a few operators which are capable of producing all null states. At level $N_{\text{null}} = 1$, we find the following two general null states

$$|N_1, \pm\rangle = (L_{-1} \pm \frac{2}{7} Q W_{-1}) |\Omega_1, \pm\rangle \equiv G_1^{\pm} |\Omega_1, \pm\rangle,$$

with $h = 3$ and $\omega = \mp \frac{2}{7} Q$. At level $N_{\text{null}} = 2$ we find

$$|N_2\rangle = (L_{-2} + \frac{3}{4} L_{-1}^2 - 2 W_{-1}^2) |\Omega_2\rangle \equiv G_2 |\Omega_2\rangle,$$

with $h = 2$ and $\omega = 0$. There exist two null states at level $N_{\text{null}} = 3$, but the corresponding $G_3$ operators can be shown to factorise, and be expressible, for example, as $G_1^\pm$ on highest-weight states:

$$|N_3, \pm\rangle = G_1^\pm (84 L_{-2} \pm 39 Q W_{-2} \mp 24 Q L_{-1} W_{-1} + \frac{147}{8} L_{-1}^2 W_{-1} + 21 W_{-1}^2) |\Omega_3, \pm\rangle \equiv G_3^\pm |\Omega_3, \pm\rangle,$$
where \( h = 1 \) and \( \omega = \pm 2Q \). At level 4, we find three null states, which can be expressed in terms of factorised operators as follows:

\[
\left| N_4, \pm \right> = G_2(48L_{-2} \pm \frac{100}{7}QW_{-2} + \frac{9}{2}L_{-1}^2 \pm \frac{240}{49}aQL_{-1}W_{-1} + 4W_{-1}^2)\left| \Omega_4, \pm \right> \equiv G_{4}^{\pm}\left| \Omega_4, \pm \right>,
\]

with \( h = 0, \omega = \mp\frac{20}{7}Q \), and

\[
\left| N_4, 0 \right> = G_1^{-}(12L_{-3} - 12L_{-1}L_{-2} - \frac{147}{8}L_{-1}^3 - 3W_{-1}W_{-2} + \frac{24}{7}QW_{-1}^3 - 63QL_{-1}W_{-2} - 21QL_{-1}^2W_{-1} + \frac{96}{7}QL_{-2}W_{-1} + 3L_{-1}W_{-1}^2)\left| \Omega_4, 0 \right> \equiv G_{4}^{0}\left| \Omega_4, 0 \right>,
\]

with \( h = 0 \) and \( \omega = 0 \). Thus all of these kinds of states, being null, need not be considered in the physical spectrum.

It is interesting to compare the null states of the usual Virasoro string with the ones of the \( W_3 \) string. In the case of the one-scalar Virasoro string, the null states occur only at levels \( N_{\text{null}} = 1, 2, 5, 7, 12, 15, \ldots \). There are only two fundamental null operators, namely \( L_{-1} \) and \( \bar{L}_{-2} = L_{-2} + \frac{3}{2}L_{-1}^2 \); all higher-level null operators can be composed as one or other of these two operators acting on highest-weight states. In the case of the two-scalar \( W_3 \) string, there exist null states at any level \( N_{\text{null}} \). To see this, note that the number of currents in the \( W_3 \) string is 2, namely the spin-2 current \( T \) and the spin-3 current \( W \). Thus all the physical states of the two-scalar \( W_3 \) string can be expressed, by a transformation of basis, as states of the form given in (4.1). (We assume for now that this transformation of basis is non-singular. See later, however.) Indeed all the excited physical states of the two-scalar \( W_3 \) string are null, just as all the excited physical states of the one-scalar Virasoro string are null.

To solve for the actual null states explicitly, we must solve the conditions on the highest-weight states such as \( \left| \Omega_1, \pm \right>, \left| \Omega_2 \right>, \left| \Omega_3, \pm \right>, \left| \Omega_4, \pm \right> \) and \( \left| \Omega_4, 0 \right> \). It can happen that at a given level there exists no solution; however, if a solution does exist it provides a null state for the \( W_3 \) string. We shall find the actual null states level by level using the processes described above. At level \( N = 0 \), clearly there are no null states, so all the physical states represent degrees of freedom of the system. Of course taking into account the identification of states with conjugate momenta, there are two, rather than three, degrees of freedom. At level \( N = 1 \), the null states can only arise from \( \left| N_1, \pm \right> \) acting on \( \left| \Omega_1, \pm \right> \) which has no excitations. Thus \( \left| \Omega_1, \pm \right> \) is of the form \( \left| \beta, p^\mu, \pm \right> \). Using the expression for \( W_0 \) in equation (2.14), with the appropriate \( h \) and \( \omega \) given below equation (4.3), we find that \( \left| \Omega_1, + \right> \) can only have the \( \beta \) values

\[
\beta = \frac{11}{7}iQ, \frac{6}{7}iQ, \frac{4}{7}iQ,
\]

while \( \left| \Omega_1, - \right> \) can only have \( \beta \) equal to

\[
\beta = \frac{10}{7}iQ, \frac{3}{7}iQ, \frac{8}{7}iQ.
\]
The actual null states are then given by

$$
|N_1, +\rangle \propto \left\{ \alpha^{(2)}_{-1} \left( 6 - 2\beta (\beta - \frac{10}{7}iQ) \right) + 2p_\mu (\beta - \frac{4}{7}iQ) \alpha^{\hat{\mu}}_{-1} \right\} |p^{\hat{\mu}}, \beta\rangle,
|N_1, -\rangle \propto \left\{ \alpha^{(2)}_{-1} \left( 6 - 2\beta (\beta - \frac{11}{7}iQ) \right) + 2p_\mu (\beta - \frac{4}{7}iQ) \alpha^{\hat{\mu}}_{-1} \right\} |p^{\hat{\mu}}, \beta\rangle.
$$

(4.10)

It is easy to check that the states for which $\beta = \frac{4}{7}iQ$ and $\beta = \frac{5}{7}iQ$ vanish. Substituting the remaining allowed values of $\beta$ of equations (4.8) and (4.9) into equation (4.10), we recover all the physical states of the two-scalar $W_3$ string given in equations (2.35) and (2.36). This explicitly verifies that all the states of the two-scalar $W_3$ string are null at this level. For the $(D + 2)$-scalar $W_3$ string we find that part of the states given in equation (2.33) with $\beta = \frac{4}{7}iQ$, $\frac{5}{7}iQ$ are null and so they are subject to the gauge invariance $\xi^{\hat{\mu}} \rightarrow \xi^{\hat{\mu}} + \Lambda p^{\hat{\mu}}$. This is analogous to the Virasoro string, since in this branch the effective intercept $a^{\text{eff}}$ is equal to 1; gauge invariance at this level implies the presence of massless states. All the states of equation (2.31) with $\beta = 10iQ$, $\frac{11}{7}iQ$ are null and therefore can be gauged away. Thus if we take into account the identification of the states with $\beta = \frac{9}{7}iQ$ and $\frac{8}{7}iQ$, and the constraints given by equation (2.34) and the above-mentioned gauge invariance, there exist $(D + 1) - 1 + (D + 1) - 2 = 2D - 1$ degrees of freedom at level one.

At level two, null states can arise either from $G^\pm_1$ acting on level-1 highest-weight states or from $G_2$ acting on level-0 states, with conformal weight and $W$ weight given below equations (4.3) and (4.4) respectively. Let us consider the latter first; applying $W_0$ of equation (2.14), with $h = 2$ and $\omega = 0$, we find that the only allowed values of $\beta$ are

$$
\beta = \frac{3}{7}iQ, iQ, \frac{12}{7}iQ.
$$

(4.11)

Evaluating $|N_2\rangle$ of equation (4.4), we find that it vanishes if $\beta = \frac{3}{7}iQ$. For $\beta = iQ$, it gives

$$
|N_2\rangle \propto \left\{ \alpha^{\hat{\mu}}_{-2}(7p_\mu + 3ia_\mu) + \alpha^{\hat{\mu}}_{-1} \alpha^{\hat{\nu}}_{-1}(4p_\mu p_\nu + \frac{3}{2}\eta_{\mu\nu}) \right\} |p^{\hat{\mu}}, iQ\rangle.
$$

(4.12)

For $\beta = \frac{12}{7}iQ$ the null states are

$$
|N_2\rangle \propto \left( 175\alpha^{(2)}_{-1}\alpha^{(2)}_{-1} + \frac{1400}{7}iQp_\mu \alpha^{(2)}_{-1} \alpha^{\hat{\mu}}_{-1} - \frac{296}{7}iQ\alpha^{(2)}_{-2}
- 70(p_\mu - ia_\mu)\alpha^{\hat{\nu}}_{-2} + 4\alpha^{\hat{\mu}}_{-1} \alpha^{\hat{\nu}}_{-1}(140p_\mu p_\nu - 33\eta_{\mu\nu}) \right) |p^{\hat{\mu}}, \frac{12}{7}iQ\rangle.
$$

(4.13)

The states of equation (4.13) can be used to gauge away the $\beta = \frac{12}{7}iQ$ state of equation (2.41). The null states given in (4.12) imply the gauge invariance

$$
\xi^{\hat{\mu}} \rightarrow \xi^{\hat{\mu}} + (7p_\mu + 3ia_\mu)\Lambda,
\epsilon^{\hat{\mu}\hat{\nu}} \rightarrow \epsilon^{\hat{\mu}\hat{\nu}} + (4p_\mu p_\nu + \frac{3}{2}\eta_{\mu\nu})\Lambda.
$$

(4.14)
of the “ordinary” states of equation (2.38). Thus talking into account the constraints of equation (2.40) and this gauge invariance, the states with $\beta = iQ$ have $(D+1)(D+2) - 2$ degrees of freedom.

Secondly, we look at the null states at level 2 arising from null-state operators $G_1^\pm$ acting on the level-1 highest-weight states $|\Omega_1, \pm \rangle$, which take the form

$$|\Omega_1, \pm \rangle = \xi_\mu \alpha_{-1}^\mu |p^\mu; \beta\rangle$$

provided that

$$(p_\mu - 2i a_\mu)\xi_\mu = 0.$$  

(4.16)

For the choice of “+” in (4.15), we have $\beta = \frac{3}{7} iQ, \frac{6}{7} iQ$ and $\frac{11}{7} iQ$; For the “−” choice, we have $\beta = \frac{3}{7} iQ, \frac{8}{7} iQ$ and $\frac{10}{7} iQ$. The null states $|N_1, \pm \rangle$ are then found to be of the form

$$|N_1, \pm \rangle \propto \left[\left(\beta(\beta - iQ) - p^\mu (p_\mu - 2ia_\mu) - 2 \pm \frac{1}{7} iQ\right)\alpha_{-1}^{(2)} \beta_1 \alpha_{-1}^\mu \right.$$

$$\left.\pm (-2\beta + iQ \pm \frac{1}{7} iQ)(p_\mu \xi_\nu \alpha_{-1}^\mu \alpha_{-1}^\nu + \xi_\mu \alpha_{-2}^\mu)\right] |p^\mu, \beta\rangle.$$  

(4.17)

For the “+” sign we find that the states with $\beta = \frac{3}{7} iQ$ vanish identically, while $\beta = \frac{6}{7} iQ$ and $\beta = \frac{11}{7} iQ$ lead respectively to the states

$$|N_1, + \rangle \propto (p_\mu \xi_\nu \alpha_{-1}^\mu \alpha_{-1}^\nu + \xi_\mu \alpha_{-2}^\mu) |p^\mu, \frac{6}{7} iQ\rangle,$$

$$|N_1, + \rangle \propto \left(13 \xi_\mu \alpha_{-1}^{(2)} \alpha_{-1}^\mu + 2iQ(p_\mu \xi_\nu \alpha_{-1}^\mu \alpha_{-1}^\nu + \xi_\mu \alpha_{-2}^\mu)\right) |p^\mu, \frac{11}{7} iQ\rangle.$$  

(4.18a)

Similarly, for the “−” sign we find that the states given in (4.17) vanish if $\beta = \frac{3}{7} iQ$, while they are non-zero for $\beta = \frac{8}{7} iQ$ and $\beta = \frac{10}{7} iQ$, which leads to the states

$$|N_1, - \rangle \propto (p_\mu \alpha_{-1}^\mu \xi_\nu \alpha_{-1}^\nu + \xi_\mu \alpha_{-2}^\mu) |p^\mu, \frac{8}{7} iQ\rangle,$$

$$|N_1, - \rangle \propto \left(7 \xi_\mu \alpha_{-1}^{(2)} \alpha_{-1}^\mu + 4iQp_\mu \alpha_{-1}^\mu \alpha_{-1}^\nu + \xi_\mu \alpha_{-2}^\mu\right) |p^\mu, \frac{10}{7} iQ\rangle.$$  

(4.18b)

The null states of equation (4.18b) and (4.19b) can be used to gauge away the physical states of equation (2.44). It follows from (2.40) that the null states of equation (4.18a) and (4.19a) imply the gauge invariance

$$\xi_\mu \rightarrow \xi_\mu + \Lambda_\mu \quad \varepsilon_{\mu \nu} \rightarrow \varepsilon_{\mu \nu} + \frac{1}{2}(p_\mu \Lambda_\nu + p_\nu \Lambda_\mu)$$  

(4.20)

of the physical states given in equation (2.39). Thus these states have $(D+1)(D+2)/2 - D - 1$ degrees of freedom, taking into account the identification, and the constraint equation (2.40), and that $(p - 2ia)_\mu \Lambda^\mu = 0$.

We note that all the degrees of freedom originate from states that are of ordinary (case 3) type, and that they give $(D + 1)(D + 2) - D - 3$ degrees of freedom in all at level 2. For
the two-scalar $W_3$ string no null state arises from $G^\pm_1$ acting on a level-1 highest-weight state $|\Omega_1, \pm\rangle$. However, one can obtain null states by acting with $G_2$ on the primary states $|p, \beta\rangle$ with $\beta = \frac{7}{2}iQ, iQ, \frac{12}{7}iQ$. When $\beta = \frac{7}{2}iQ$, $G_2$ annihilates $|p, \beta\rangle$, while for other two (2.44). Thus this verifies that all the states of two-scalar $W_3$ string at level 2 are null.

At levels 3 and 4, we only consider the two-scalar realisation. At level 3, it can be shown that all the physical states, namely of equation (2.46) and (2.48), can be rewritten as $G_3$ acting on primary states $|p, \beta\rangle$ with $\beta = \frac{11}{7}iQ, \frac{12}{7}iQ, 2iQ$. In other words, they can be written as $G_1^+$ acting on level-2 highest-weight states with $h = 3$ and $\omega = -\frac{2}{7}Q$. Thus the physical states at level 3 are all null for the two-scalar $W_3$ string. At level 4, a new feature emerges. The first and second states in (2.50) can be written in terms of $G_4^-$ in (4.6) acting on $h = 0, \omega = \frac{20}{7}Q$ highest-weight states. The first and second states in (2.51) can be written in terms of $G_4^+$ in (4.6) acting on $h = 0, \omega = -\frac{20}{7}Q$ highest-weight states. The third state in (2.51) can be written in terms of $G_0^+$ in (4.7) acting on $h = 0, \omega = 0$ highest-weight state. However, the remaining two level-4 states, namely the third state in (2.50) and the state in (2.52), cannot be written in terms of any $G$-type operator acting on tachyonic states. What has happened is that the tacit assumption that we made earlier in this section, namely that the transformation of basis from $\partial \varphi_1, \partial \varphi_2$ to $L_{-n}, W_{-n}$ is non-singular, has broken down in these cases. It is still the case that these two states are null however; we have explicitly checked that they can be written as BRST variations of some particular states. Unlike the ordinary string, where any BRST-trivial state can be written in terms of $L_{-n}$ descendants of some highest-weight state, for the $W_3$ string one can no longer write all BRST-trivial states in terms of $L_{-n}$ and $W_{-n}$ descendants of highest-weight states.

5. A toy model; the one-scalar string

In this section we consider a bosonic string consisting of one coordinate $\phi$ in the presence of a background charge. The energy-momentum tensor is given by $T = -\frac{1}{2}(\partial \phi)^2 - Q\partial^2 \phi$. To cancel the conformal anomaly, we require that $c = 26 = 1 + 12Q^2$, i.e. $Q^2 = \frac{25}{12}$. The physical states are given by the usual on-shell conditions

$$(L_0 - 1)|\psi\rangle = 0 \quad \text{and} \quad L_n|\psi\rangle = 0 \quad \text{with} \quad n \geq 1 . \quad (5.1)$$

We wish to find all the physical states of this one-dimensional string, and also to construct the states in terms of screening operators acting on the vacuum. This exercise proves to be a very useful model for the two-scalar $W_3$ string that will be considered in section 5. It is straightforward to analyse the lowest levels of the one-dimensional bosonic string. At level 0 we find two physical states,

$$|\frac{6}{5}iQ\rangle \quad \text{and} \quad |\frac{4}{5}iQ\rangle , \quad (5.2)$$
where $|p\rangle$ denotes a momentum eigenstate satisfying $\alpha_n|p\rangle = 0$ for $n \geq 1$, and $\alpha_0|p\rangle = p|p\rangle$. At level 1, we have only one physical state

$$\alpha_{-1}|2iQ\rangle. \quad (5.3)$$

At level 2 there is also only one physical state

$$\left(\frac{6}{5}iQ\alpha_{-1}^2 + \alpha_{-2}\right)|\frac{12}{5}iQ\rangle. \quad (5.4)$$

To find the physical states of the string at higher levels, it is useful to write a general state in the form

$$L_{-n_1}L_{-n_2} \cdots L_{-n_p}|\beta\rangle. \quad (5.5)$$

Note that we have replaced $\alpha_{-n}$ by $L_{-n}$ in generating states. For a generic state one can always make this change, since for a single-scalar realisation the $\alpha_{-n}$’s and $L_{-n}$’s are generically in one-to-one correspondence. However, for some very special momentum $\beta$ it may happen that the $L_{-n}$’s span only a subspace. In this context we note, for example, that $L_{-n}|\beta = 0\rangle$ contains no term linear in the oscillator $\alpha_{-n}$. For this particular section, however, we shall tacitly assume that for the on-shell momentum this possibility does not arise. Clearly all physical states, except the two tachyons of equation (5.2), will be null.

We recall [15,16] that a highest-weight state $|\chi\rangle$ of weight $\Delta$ will itself have a highest-weight state in its Verma module if

$$\Delta = \frac{c - 1}{24} + \frac{1}{4}(n\alpha_+ + m\alpha_-)^2, \quad n, m \in \mathbb{Z}_+, \quad (5.6)$$

where $c$ is the central charge and

$$\alpha_{\pm} = \frac{\sqrt{1 - c} \pm \sqrt{25 - c}}{\sqrt{24}}. \quad (5.7)$$

The highest-weight state occurs at level $N = nm$. Now for the bosonic string with $c = 26$, a null state has the form

$$L_{-n_1}L_{-n_2} \cdots L_{-n_p}|\Omega_N\rangle, \quad (5.8)$$

where $|\Omega_N\rangle$ satisfies $L_n|\Omega_N\rangle = 0$ for $n \geq 1$, and $(L_0 + N - 1)|\Omega_N\rangle = 0$ with $N = \sum_i n_i$. Such a states will be physical if there exist positive integers such that

$$\Delta_{n,m} + nm = 1, \quad i.e. \quad (3n - 2m)^2 = 1, \quad (5.9)$$

It is straightforward to show that the bosonic string will have such physical null states at levels

$$\frac{1}{2}n(3n \pm 1), \quad n = 1, 2, 5, \ldots. \quad (5.10)$$
The actual null states of the theory occur whenever the state $|\Omega\rangle$ admits a solution. Clearly, we can take $|\Omega_N\rangle = |\beta\rangle$, with $\beta(\beta - 2iQ) = 2(1 - N)$, to construct null states at the levels given in (5.10). However we shall also find other null states.

The above discussion holds for any bosonic string, since we used the abstract generators $L_{-n}$ to construct null physical states. For the one-scalar bosonic string we can similarly analyse when highest-weight states $|\Omega_N\rangle$ exist. For example, let us consider $|\Omega_1\rangle$, which can be written as $L_{-m_1}L_{-m_2} \cdots L_{-m_Q}|\Gamma_{1,M}\rangle$ where $M = \sum_i m_i$, if there exist $n, m \in \mathbb{Z}_+$ such that $\Delta_n + nm = 0$, i.e. $(3n - 2m)^2 = 25$. Analysing this result, we find that it leads to null physical states in the original bosonic string at levels

$$\frac{1}{2}n(3n \pm 1), \quad n = 2, 4, 6, \ldots.$$  \hspace{1cm} (5.11)

In fact, there are solutions of $|\Omega_N\rangle$ for arbitrary $N$, but they lead to null states at the same levels as those given by $|\Omega_1\rangle$. In principle one can look for null states within null states, but one finds no additional states. Putting the results of equations (5.10) and (5.11) together, we find that the one-scalar string has null states at levels

$$\frac{1}{2}n(3n \pm 1), \quad n \in \mathbb{Z}_+.$$  \hspace{1cm} (5.12)

It follows from the usual on-shell condition that the momentum $\beta$ is quantised for all physical states since the $\beta$ values are $\frac{1}{5}iQ$ multiplied by certain integers, i.e.

$$\beta = \left(5 \pm (6n \pm 1)\right)\frac{1}{5}iQ.$$  \hspace{1cm} (5.13)

This can be better understood by a screening-charge analysis that we shall discuss presently.

It is quite instructive to examine in detail the way in which the null physical states of the one-scalar string can be constructed by acting with $L_{-n}$’s on a highest-weight state. As is well known, if $|h\rangle$ is a highest-weight state with weight $h = 0$, i.e. $L_0|h\rangle = 0$, then it gives a highest-weight null state $L_{-1}|h\rangle$ that has weight 1. This is precisely a level-1 physical null state of string theory. In fact only if $h = 0$ does one get a highest-weight state by this means. At level 2, one finds that if $c = 26$ and $|h\rangle$ is a highest-weight state then the operator

$$\tilde{L}_{-2} = L_{-2} + \lambda L_{-1}^2$$  \hspace{1cm} (5.14)

will give a null highest-weight state $\tilde{L}_{-2}|h\rangle$ provided that either $h = -1$, in which case $\lambda = \frac{3}{2}$, or $h = -\frac{13}{8}$, in which case $\lambda = \frac{2}{3}$. The former case is relevant to string theory, since then the null state $\tilde{L}_{-2}|h\rangle$ will be highest weight with weight $-1 + 2 = 1$, thus satisfying the physical state conditions. In Appendix A we present results for the first twelve levels in string theory, i.e. we give the results for the operators $\tilde{L}_{-n}$ for $n = 1, 2, \ldots, 12$. 

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We now use the screening charges to construct explicitly the physical states of the one-dimensional string. The two screening charges are

\[ S_\pm = \oint e^{i\alpha_\pm \phi(z)} , \]  

(5.15)

where \( \alpha_+ = \frac{6}{5}iQ, \alpha_- = \frac{4}{5}iQ \). The two level-0 states of equation (4.2) are \( e^{i\alpha_\pm \phi(0)}|0\rangle \). Acting with \( S_\pm \) on a physical state of momentum \( p \) will be well defined if \( p \cdot \alpha_\pm \) is a negative-definite integer; such a state will also be physical. We find in this way the physical states

\[ (S_+)^n|\alpha_+\rangle , \quad n \in Z_+ , \]  

(5.16)

which have momentum \( (n + 1)\alpha_+ \) and occur at levels \( \frac{1}{2}n(3n + 1) \). We also find physical states at levels \( \frac{1}{2}n(3n - 1) \), namely

\[ (S_+)^n|\alpha_-\rangle , \quad n \in Z_+ , \]  

(5.17)

which have momenta \( n\alpha_+ + \alpha_- \). The only other well-defined states come from \( S_- \) acting on \( (S_+)^n|\alpha_+\rangle \), which, however, gives rise to the same physical states as those of equation (5.16). We note that we get precise agreement with the levels at which null states occur as predicted by the Kac formula, i.e. those of equation (5.12). This is perhaps not too surprising since the highest-weight states of the Verma module can be constructed from screening operators. The screening-charge analysis also gives us more understanding as to why the physical states of the one-dimensional string are null. As discussed in section 3, a physical state will have zero scalar product with all other physical states if there does not exist its “dual” partner in the physical spectrum. Since the momenta of the screening operators are positive, it follows that the momenta of the physical states will increase monotonically as the level number increases. Therefore no higher-level physical states can occur in the momentum-conjugate pairs necessary for a non-vanishing scalar product.

6. Screening operators and the two-scalar \( W_3 \) string

In section 2, we found all the physical states up to level 4 in the two-scalar \( W_3 \) string. In section 3, we showed that all these states above level zero are null. Here we shall construct these states in terms of screening operators acting on the vacuum.

The \( W_3 \) algebra possesses four screening charges [7]

\[ S_{i\pm} = \oint e^{i\alpha_\pm e_i \cdot \vec{\phi}(z)} , \quad i = 1, 2 , \]  

(6.1)
where \( e_i \) are the two simple roots of \( su(3) \), i.e., \( e_1 = (\sqrt{2}, 0) \), \( e_2 = (-\sqrt{\frac{1}{2}}, \sqrt{\frac{3}{2}}) \), and \( \alpha_\pm \) satisfy the equation \( \alpha_\pm^2 - i\alpha_0 \alpha_\pm - 1 = 0 \). Here \( \alpha_0 \) is a background charge parameter satisfying \( c = 2 + 24\alpha_0^2 \). For the \( W_3 \) string \( c = 100 \), and one finds \( \alpha_+ = 4i/\sqrt{12} \) and \( \alpha_- = 3i/\sqrt{12} \). Consequently,

\[
\begin{align*}
\alpha_+ e_1 &= (\frac{2}{7}ia, 0), & \alpha_- e_1 &= (\frac{6}{7}ia, 0), \\
\alpha_+ e_2 &= (-\frac{1}{7}ia, \frac{3}{7}iQ), & \alpha_- e_2 &= (-\frac{3}{7}ia, \frac{3}{7}iQ).
\end{align*}
\]

The screening operators of equation (6.1) commute with \( L_n \) and \( W_n \), as is most simply shown by utilising the quantum Miura construction of these operators based on \( su(3) \). A more lengthy calculation shows that these are the unique screening operators. Unlike the one-dimensional Virasoro string, where the screening operators are the same as the tachyon operators, the six level-0 states of the two-scalar \( W_3 \) string, given by equation (2.36), can be built from products of four screening operators acting on the vacuum, provided that one takes an appropriate limit \([9]\). There are a number of ways to achieve this result. One set of choices is

\[
\begin{align*}
\gamma_1 &= \alpha_+ e_2 + \alpha_- (e_2 + 2e_1) = (\frac{2}{7}ia, iQ), & \gamma_2 &= \alpha_+ (2e_1 + 2e_2) + \alpha_1 e_2 = (\frac{6}{7}ia, iQ), \\
\gamma_3 &= \alpha_- (2e_2 + 2e_1) = (\frac{6}{7}ia, \frac{6}{7}iQ), & \gamma_4 &= \alpha_+ e_1 + \alpha_- (e_1 + 2e_2) = (\frac{6}{7}ia, \frac{6}{7}iQ), \\
\gamma_5 &= \alpha_+ (e_1 + 2e_2) + \alpha_- e_1 = (\frac{6}{7}ia, \frac{8}{7}iQ), & \gamma_6 &= \alpha_+ (2e_1 + 2e_2) = (\frac{6}{7}ia, \frac{8}{7}iQ).
\end{align*}
\]

For \( \gamma_2 \), for example, we construct the corresponding state

\[
e^{i\alpha_- e_2 \cdot \vec{\phi}(z_1)} e^{i\alpha_+ e_2 \cdot \vec{\phi}(z_2)} e^{i\alpha_+ e_1 \cdot \vec{\phi}(z_3)} e^{i\alpha_+ e_1 \cdot \vec{\phi}(z_4)},
\]

set \( z_{12} = z_{23} = z_{34} = \epsilon \), where \( z_{ij} \equiv z_i - z_j \), and send \( \epsilon \to 0 \).

We now consider the action of one screening operator on the level-0 states. The operator \( S_{i\pm} \) acting on a state of momentum \( k \) is well defined if \( \alpha_{i\pm} k \cdot e^i \) is an integer. Such a state will be physical if the original state is physical. Acting with \( S_{i\pm} \) on the six tachyon states, we find four of the six level-1 states;

\[
\begin{align*}
S_{1+} |\gamma_3\rangle &= \alpha_+ e_1 \cdot \vec{\alpha}_{-1} |2ia, \frac{6}{7}iQ\rangle, & S_{2+} |\gamma_3\rangle &= \alpha_+ e_2 \cdot \vec{\alpha}_{-1} |2ia, \frac{10}{7}iQ\rangle, \\
S_{1+} |\gamma_5\rangle &= \alpha_+ e_1 \cdot \vec{\alpha}_{-1} |2ia, \frac{8}{7}iQ\rangle, & S_{2-} |\gamma_6\rangle &= \alpha_- e_2 \cdot \vec{\alpha}_{-1} |\frac{5}{7}ia, \frac{12}{7}iQ\rangle,
\end{align*}
\]

and two of the three level-2 states;

\[
\begin{align*}
S_{1+} |\gamma_2\rangle &= \oint \frac{1}{z_3} e^{i\alpha_+ e_1 \cdot \vec{\phi}(z)} |\frac{17}{7}ia, iQ\rangle, & S_{2+} |\gamma_5\rangle &= \oint \frac{1}{z_3} e^{i\alpha_+ e_2 \cdot \vec{\phi}(z)} |\frac{17}{7}ia, \frac{12}{7}iQ\rangle.
\end{align*}
\]

To get the additional states at levels one and two, we must act with two screening charges on the tachyon states. In this case, we shall examine when the action of two such operators is well defined. We consider the quantity

\[
\oint d\mathbf{z}_3 \oint d\mathbf{z}_2 : e^{i k_3 \cdot \vec{\phi}(z_3)} : e^{i k_2 \cdot \vec{\phi}(z_2)} : |\mathbf{p}\rangle.
\]

(6.7)
Using the above method, we find that
\[ a \text{ for all } a > 0. \]
We calculate the integral for \( z \) since the \( z_3 \) exponential comes first, we must have the contour with \( |z_3| > |z_2| \). We choose the contours so that the \( c_2 \) contour is always within the \( c_3 \) contour except at one point where the two contours touch. We now substitute \( z_2 = wz_3 \) and carry out the \( z_3 \) integral first. We note that \( |w| < 1 \) except when the contours touch, in which case it is equal to 1. Equation (5.8) then becomes
\[
\oint dw \oint dz_3 \left( \frac{k_3 \cdot k_2 + (k_3 + k_2) p + 1}{1 - w} \right) e^{ik_3 \cdot \phi(z_3) + ik_2 \cdot \phi(wz_3)} |p\rangle. \tag{6.9}
\]
This integral is well defined if \( k_3 \cdot k_2 + k_3 \cdot p + k_2 \cdot p \) is an integer. In this case the \( z_3 \) integral becomes analytic. For a general case of the action of \( n \) exponentials on a state of momentum \( p \), a similar argument implies that the integral is well defined if \( \sum_{i>0} k_i \cdot k_j + \sum_{j=1}^{n} k_j \cdot p \) is an integer.

Returning to the states of equation (5.9), and assuming that \( k_3 \cdot k_2 + k_3 \cdot p + k_2 \cdot p \) is indeed an integer, we can carry out the \( w \) integration, which has the generic form
\[
I(a, b) = \oint dw w^a(1 - w)^b. \tag{6.10}
\]
The contour of integration encloses the branch cut between 0 and 1 that results from \( a \) and \( b \) being non-integer. We can deform the contour to lie infinitesimally above and below the cut while remaining on the given Riemann sheet. However, the section below the cut is obtained by a \( z \rightarrow e^{i2\pi a} z \) rotation and we find that
\[
I(a, b) = (e^{2\pi ia} - 1) \frac{\Gamma(a + 1)\Gamma(b + 1)}{\Gamma(a + b + 2)} \tag{6.11}
\]
We calculate the integral for \( a > -1, b > -1 \), and then use analytic continuation to define it for all \( a \) and \( b \). We note that the phase \( e^{2\pi ia} \) is the same for all \( a \)'s that differ by an integer. Using the above method, we find that
\[
S_1S_2 |q_4\rangle = \oint dz_3 \oint dz_2 z_3^{-8/3}z_2^{-5/3}(z_3 - z_2)^{4/3} |q_4 + \alpha_+(e_1 + e_2)\rangle
\]
\[
= \oint dw w^{-8/3}(1 - w)^{4/3}(\alpha_+we_1 \cdot \alpha_1 + \alpha_+e_2 \cdot \alpha_1 - 1)|q_4 + \alpha_+(e_1 + e_2)\rangle
\]
\[
= \alpha_+(2e_2 - 5e_1) \cdot \alpha_1 - 1 |\frac{12}{5} ia, \frac{10}{5} iQ\rangle, \tag{6.12}
\]
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which is one of the remaining level-1 states. The other remaining level-1 state is constructed as

\[ S_{1-}S_{2-}|\gamma_5\rangle \propto \alpha_-(5e_2 - 2e_1) \cdot \bar{\alpha}_{-1}|\frac{9}{7}ia, \frac{11}{7}iQ\rangle . \]  

(6.13)

The remaining level-2 state with momentum \((\frac{12}{7}ia, \frac{12}{7}iQ)\) is given by

\[ S_{1+}S_{2+}|\gamma_6\rangle . \]  

(6.14)

At level three we find that all the four states can be constructed as a screening operator acting on the level-one states given in (6.12) and (6.13), or they can also be viewed as three screening operators acting on the corresponding tachyonic states. Those of equation (2.49) occur as

\[ S_{2+}\mathcal{P}_1|\frac{12}{7}ia, \frac{10}{7}iQ\rangle = S_{2+}S_{1+}S_{2+}|\gamma_4\rangle = \mathcal{P}_3|\frac{8}{7}ia, 2iQ\rangle , \]

\[ S_{2-}\mathcal{P}_1|\frac{9}{7}ia, \frac{11}{7}iQ\rangle = S_{2-}S_{1-}S_{2-}|\gamma_5\rangle = \mathcal{P}_3|\frac{6}{7}ia, 2iQ\rangle . \]  

(6.15)

Those of equation (2.47) are

\[ S_{1+}\mathcal{P}_1|\frac{9}{7}ia, \frac{11}{7}iQ\rangle = S_{1+}S_{1-}S_{2-}|\gamma_5\rangle = \mathcal{P}_3\frac{12}{7}ia, \frac{11}{7}iQ\rangle , \]

\[ S_{1-}\mathcal{P}_1|\frac{12}{7}ia, \frac{10}{7}iQ\rangle = S_{1-}S_{1+}S_{2+}|\gamma_4\rangle = \mathcal{P}_3\frac{12}{7}ia, \frac{10}{7}iQ\rangle . \]  

(6.16)

The reader may verify that the polarisations in the generic symbol \(\mathcal{P}_3\) are the same as given in equation (2.47) and (2.49). In fact, there are a number of ways to construct the physical states by using different screening operators acting on different states.

At level four, we can obtain the seven states listed in (2.50)-(2.52) as follows:

\[ S_{2+}S_{2-}S_{1-}|\gamma_5\rangle = \mathcal{P}_4|\frac{5}{7}ia, \frac{15}{7}iQ\rangle , \]

\[ S_{2+}S_{2+}S_{1+}|\gamma_2\rangle = \mathcal{P}_4|\frac{9}{7}ia, \frac{15}{7}iQ\rangle , \]

\[ S_{2-}S_{2-}|\gamma_5\rangle = \mathcal{P}_4|0, 2iQ\rangle , \]

\[ S_{2+}S_{1+}S_{1+}|\gamma_4\rangle = \mathcal{P}_4\frac{20}{7}ia, \frac{10}{7}iQ\rangle , \]

\[ S_{2+}S_{1+}S_{1-}|\gamma_6\rangle = \mathcal{P}_4\frac{18}{7}ia, \frac{12}{7}iQ\rangle , \]

\[ S_{2-}S_{2-}S_{1-}|\gamma_6\rangle = \mathcal{P}_4|2ia, 2iQ\rangle , \]

\[ S_{1+}S_{1+}|\gamma_1\rangle = \mathcal{P}_4|3ia, iQ\rangle . \]  

(6.17)

For a general level \(\ell\) in the two-scalar \(W_3\) string, we may consider building physical states by acting on any of the six tachyons, with momenta given in (6.3), with arbitrary integer powers of the four screening operators, whose momenta are given in (6.2). The condition given after eq. (6.9) is equivalent to the requirement that the conformal weight \(\Delta\) of the resulting exponential operator of the physical state should be an integer; in fact it is related to the level number by \(\ell = 4 - \Delta\). By requiring that \(\ell\) be an integer, we find that physical states built from screening operators can only arise at levels for which we may write

\[ \ell = \frac{1}{16}m^2 + \frac{1}{48}n^2 - \frac{1}{12} , \]  

(6.18)
where \( m \) and \( n \) are integers. This excludes the possibility of having such physical states at levels \( \ell = 7, 12, 17, 21, 22, 32, \ldots \). In fact, one arrives at exactly the same restriction (6.18) by simply requiring that the on-shell momentum of a level-\( \ell \) physical state should be such that its components \( p \) and \( \beta \) be integer multiples of \( \frac{1}{7}ia \) and \( \frac{1}{7}iQ \) respectively.

7. Conclusions

In this paper we have found the spectrum of the open \( W_3 \) string at low levels. For the case of the \((D+2)\)-dimensional \( W_3 \) string, we carried out the analysis up to and including level 2. For the case of the two-scalar \( W_3 \) string, we extended the analysis to level 4. This was done by solving the physical-state conditions at these levels and comparing the results with the null states. It was found that for the two-scalar \( W_3 \) string all the physical states except those at level 0 were null, and so at least up to level 4 the theory contains no degrees of freedom. For the \((D+2)\)-scalar \( W_3 \) string, however, it was found that any state that contained oscillators of the field \( \varphi^{(2)} \) was null. The physical states are contained in the remaining Fock space, \( \tilde{H} \) generated by the oscillators of the remaining \( D+1 \) fields subject to

\[
(\tilde{L}_0 - a^{\text{eff}}) |\psi\rangle = 0, \quad \tilde{L}_n |\psi\rangle = 0, \quad n \geq 1,
\]

where \( \tilde{L}_0 \) and \( \tilde{L}_n \) are the Virasoro generators restricted to the the oscillators in the \( D+1 \) fields, and \( |\psi\rangle \in \tilde{H} \). The intercept \( a^{\text{eff}} \) can take the values 1 or \( \frac{15}{16} \), leading to two sectors in the spectrum of physical states. There exist corresponding null states in each of these sectors, and the count of degrees of freedom is given above. For example, one finds at level zero two tachyons, one from each sector. At level 1, there are two vectors; one from the \( a^{\text{eff}} = 1 \) sector with \( D-1 \) degrees of freedom, and the other from the \( a^{\text{eff}} = \frac{15}{16} \) sector with \( D \) degrees of freedom. Although the presence of background charges obscures the interpretation of mass, one can take the above to mean that the open \( W_3 \) string describes only one massless state, which is a “photon.” It is shown that at any level the momentum in the \( \varphi^{(2)} \) direction obeys a polynomial equation and at the levels considered it has been found that the solutions are all integer multiples of \( \frac{1}{7}iQ \). It seems likely that all these features generalise to all levels.

It was then shown, for the two-scalar \( W_3 \) string, that all the physical states that were found could be described by the \( W_3 \) screening operators acting on the vacuum. This feature also seems likely to generalise to all levels. Although the closed \( W_3 \) string was not considered in this paper, it is clear that it shares many of the above features. The only massless states in this case will be the graviton, antisymmetric tensor, and scalar, just as for the usual closed bosonic string.
Note Added

After this work was completed, evidence for the existence of further physical states in the $W_3$ string has been found. As discussed in section 2, we have been concerned in this paper with physical states that have the “standard” form $|\psi\rangle \otimes |gh\rangle$, where $|\psi\rangle$ is built purely from matter oscillators acting on a momentum state, and $|gh\rangle$ is the ghost vacuum state. In [17], it was shown that in the two-scalar $W_3$ string there is a “ground-ring” structure of physical states of “non-standard” form, involving ghost as well as matter excitations. These are analogous to the discrete states of the ordinary two-dimensional string [18,19]. In fact, an example of a discrete state with non-standard ghost structure was found in the two-scalar $W_3$ string in [20]. In addition, it was found in [17] that there are continuous-momentum physical states involving ghost as well as matter excitations in the multi-scalar $W_3$ string too. These physical states have no analogue in ordinary string theory in more than two dimensions; their existence seems to be related to the fact that the gauge symmetries of the states in the $W_3$ string are insufficient to permit the introduction of a physical gauge like the light-cone gauge. Further evidence for the existence of new physical states was found in [21], where it was shown that there are poles in the four-point scattering amplitude for the $\beta = iQ$ tachyons in (2.26) that do not correspond to the masses of any of the physical states with “standard” ghost structure. The details of these new physical states will be discussed elsewhere [22].

ACKNOWLEDGMENTS

We are grateful to M. Freeman, K. Hornfeck, E. Sezgin and X.J. Wang for discussions. P.C.W. thanks the Newton Institute in Cambridge, and C.N.P. and P.C.W. thank the Chalmers Institute of Technology in Göteborg, for hospitality.
Appendix A

Here we collect together some results for the operators $\tilde{L}_{-n}$ for the Virasoro algebra that act on a highest-weight state $|h\rangle$ with weight $h$, and give a new highest-weight state $\tilde{L}_{-n}|h\rangle$, at level $n$ and having weight $h+n$. For each $n$ from 1 up to 12 we give the set of permitted $h$ values, and, for convenience, the corresponding weights $h+n$ of the resulting states. The allowed values of $h$ depend, in general, on the value $c$ of the central charge. We present results here for two values that are relevant for this paper, namely $c = 26$ for the usual bosonic string, and $c = 25\frac{1}{2}$ for the “effective” ordinary sector of the $W_3$ string.

At $c = 26$ we have:

\[
\begin{align*}
\tilde{L}_{-1} : \quad h &= 0 & \rightarrow & & 1 \\
\tilde{L}_{-2} : \quad h &= -1, -\frac{13}{8} & \rightarrow & & 1, \frac{3}{8} \\
\tilde{L}_{-3} : \quad h &= -4, -\frac{7}{3} & \rightarrow & & -1, \frac{2}{3} \\
\tilde{L}_{-4} : \quad h &= -4, -\frac{25}{8}, -\frac{57}{8} & \rightarrow & & 0, \frac{7}{8}, -\frac{25}{8} \\
\tilde{L}_{-5} : \quad h &= -4, -6, -11 & \rightarrow & & 1, -1, -6 \\
\tilde{L}_{-6} : \quad h &= -6, -\frac{25}{3}, -\frac{125}{8}, -\frac{419}{24} & \rightarrow & & 0, -\frac{7}{3}, -\frac{77}{8}, \frac{25}{24} \\
\tilde{L}_{-7} : \quad h &= -6, -11, -21 & \rightarrow & & 1, -4, -14 \\
\tilde{L}_{-8} : \quad h &= -14, -\frac{57}{8}, -\frac{77}{8}, -\frac{217}{8} & \rightarrow & & -6, \frac{7}{8}, -\frac{13}{8}, -\frac{153}{8} \\
\tilde{L}_{-9} : \quad h &= -34, -\frac{25}{3}, -\frac{52}{3} & \rightarrow & & -25, \frac{2}{3}, -\frac{25}{3} \\
\tilde{L}_{-10} : \quad h &= -11, -14, -21, -\frac{77}{8}, -\frac{333}{8} & \rightarrow & & -1, -4, -11, \frac{3}{8}, -\frac{253}{8} \\
\tilde{L}_{-11} : \quad h &= -11, -25, -50 & \rightarrow & & 0, -14, -39 \\
\tilde{L}_{-12} : \quad h &= -11, -\frac{88}{3}, -\frac{153}{8}, -\frac{473}{8}, -\frac{299}{24}, -\frac{299}{24} & \rightarrow & & 1, -\frac{52}{3}, -\frac{57}{8}, -\frac{377}{8}, -\frac{11}{24}, -\frac{11}{24}
\end{align*}
\]

We do not in general present the explicit forms for the $\tilde{L}_{-n}$ operators, since their structure is quite complicated. The first few are given below. Note that the coefficients of the various terms in these expressions will in general be $h$ dependent, as well as $c$ dependent. One can see from the above results that $\tilde{L}_{-1}, \tilde{L}_{-2}, \tilde{L}_{-5}, \tilde{L}_{-7}$ and $\tilde{L}_{-12}$ are relevant for string theory, since they can produce highest-weight states with conformal weight 1. One can also see that the following relevant factorisations occur:

\[
\begin{align*}
\tilde{L}_{-5}(-4) &= \tilde{L}_{-1}\tilde{L}_{-4} = \tilde{L}_{-2}\tilde{L}_{-3} \\
\tilde{L}_{-7}(-6) &= \tilde{L}_{-1}\tilde{L}_{-6} = \tilde{L}_{-2}\tilde{L}_{-5} \\
\tilde{L}_{-12}(-11) &= \tilde{L}_{-1}\tilde{L}_{-11} = \tilde{L}_{-2}\tilde{L}_{-10}.
\end{align*}
\]

Here, the conformal weight of the state on which the operator on the left-hand side acts is indicated in brackets. Presumably this structure of factorisations persists for all higher $\tilde{L}_{-n}$ operators.
The explicit forms of the first five $\tilde{L}_{-n}$ operators are as follows.

$$\tilde{L}_{-1} = L_{-1}$$  \hspace{1cm} (A.3)

gives a null state if and only if it acts on a highest-weight state of conformal weight $h = 0$, independent of the value of $c$. At level 2, we have

$$\tilde{L}_{-2} = L_{-2} - \frac{3}{2(2h + 1)} L_{-1}^2 ,$$  \hspace{1cm} (A.4)

for which $h$ must be a root of the equation

$$16h^2 - 10h + (2h + 1)c = 0 .$$  \hspace{1cm} (A.5)

At level 3,

$$\tilde{L}_{-3} = L_{-3} - \frac{2}{h} L_{-2} L_{-1} + \frac{1}{h(h + 1)} L_{-1}^3 ,$$  \hspace{1cm} (A.6)

where $h$ must be a root of

$$3h^2 - 7h + 2 + (h + 1)c = 0 .$$  \hspace{1cm} (A.7)

The level 4 operator has the form

$$\tilde{L}_{-4} = \lambda_1 L_{-4} + \lambda_2 L_{-3} L_{-1} + \lambda_3 L_{-2}^2 + \lambda_4 L_{-2} L_{-1}^2 + \lambda_5 L_{-1}^4 ,$$  \hspace{1cm} (A.8)

where the coefficients are given by

$$\lambda_1 = -\frac{4}{75} \left[ 16h^3 + (2c + 6)h^2 + (c + 50)h - 3c + 3 \right] \lambda_5 ,$$

$$\lambda_2 = \frac{2}{75} \left[ 16h^2 + (2c + 2)h + 3c + 12 \right] \lambda_5 ,$$

$$\lambda_3 = \frac{4}{45} \left[ 4h^2 + (-2c + 38)h - 3c + 3 \right] \lambda_5 ,$$

$$\lambda_4 = -\frac{4}{9} (2h + 3) \lambda_5 .$$  \hspace{1cm} (A.9)

The conformal weight of the state on which it acts must be a root of

$$(8h + c - 1) \left[ 16h^2 + (10c - 82)h + 15c + 66 \right] = 0 .$$  \hspace{1cm} (A.10)

Finally, at level 5 we have

$$\tilde{L}_{-5} = \mu_1 L_{-5} + \mu_2 L_{-4} L_{-1} + \mu_3 L_{-3} L_{-2} + \mu_4 L_{-3} L_{-2}^2 + \mu_5 L_{-2}^2 L_{-1} + \mu_6 L_{-2} L_{-1}^2 + \mu_7 L_{-1}^5 ,$$  \hspace{1cm} (A.11)

where the coefficients are given by

$$\mu_1 = \frac{1}{9} h \left[ 5h^3 + (c + 1)h^2 + (2c + 24)h + 12 \right] \mu_7 ,$$

$$\mu_2 = -\frac{1}{3} \left[ 5h^3 + (c + 1)h^2 + (2c + 24)h + 12 \right] \mu_7 ,$$

$$\mu_3 = -\frac{1}{3} h \left[ 7h^2 + (-c + 41)h - 2c + 18 \right] \mu_7 ,$$

$$\mu_4 = \frac{1}{3} \left[ 8h^2 + (c + 4)h + 2c + 12 \right] \mu_7 ,$$

$$\mu_5 = \frac{2}{9} \left[ 7h^2 + (-c + 41)h - 2c + 18 \right] \mu_7 ,$$

$$\mu_6 = -\frac{10}{3} (h + 2) \mu_7 .$$  \hspace{1cm} (A.12)
In this case, the conformal weight of the state on which the operator acts must satisfy the two equations
\[
\begin{align*}
    h(14h + c + 30) \left[ h^2 + (c - 9)h + 2c + 14 \right] &= 0 , \\
    (13h + 2c) \left[ h^2 + (c - 9)h + 2c + 14 \right] &= 0 .
\end{align*}
\] (A.13)

Similar, but progressively more complicated, results can be obtained at all higher levels too. In each case, by solving the conditions that follow from requiring that \( L_1 \bar{L}_{-n} |h \rangle = 0 \) and \( L_2 \bar{L}_{-n} |h \rangle = 0 \), one can determine the various coefficients in the expression for \( \bar{L}_{-n} \) and also the polynomial conditions relating \( c \) and \( h \). It is these values of \( h \) that are tabulated in eq. (A.1) for \( c = 26 \), and in eq. (A.14) below for \( c = 25\frac{1}{2} \).

Another case of interest here is when \( c = 25\frac{1}{2} \), since this is the value of the central charge for the “effective” Virasoro string theory of the case 3-states in section 3. The corresponding permitted values for \( h \) are
\[
\begin{align*}
    \bar{L}_{-1} & : \quad h = 0 \quad \rightarrow \quad 1 \\
    \bar{L}_{-2} & : \quad h = -\frac{3}{2}, \frac{3}{16} \quad \rightarrow \quad \frac{1}{2}, \frac{15}{16} \\
    \bar{L}_{-3} & : \quad h = \frac{3}{2}, -\frac{11}{3} \quad \rightarrow \quad \frac{1}{2}, -\frac{3}{2} \\
    \bar{L}_{-4} & : \quad h = -\frac{13}{2}, -\frac{49}{16}, -\frac{69}{16} \quad \rightarrow \quad -5, \frac{15}{2}, -\frac{5}{16} \\
    \bar{L}_{-5} & : \quad h = -10, -\frac{13}{2} \quad \rightarrow \quad -5, -\frac{3}{2} \\
    \bar{L}_{-6} & : \quad h = -5, -\frac{45}{6}, -\frac{145}{16}, -\frac{275}{48} \quad \rightarrow \quad 1, -\frac{49}{6}, -\frac{49}{16}, -\frac{13}{48} \\
    \bar{L}_{-7} & : \quad h = -12, -19, -\frac{13}{2} \quad \rightarrow \quad -5, -\frac{12}{5}, -\frac{1}{2} \\
    \bar{L}_{-8} & : \quad h = -\frac{49}{2}, -\frac{117}{16}, -\frac{145}{16}, -\frac{245}{16} \quad \rightarrow \quad -\frac{33}{2}, \frac{11}{16}, -\frac{17}{16}, -\frac{117}{16} \\
    \bar{L}_{-9} & : \quad h = -19, -\frac{92}{3}, -\frac{49}{6} \quad \rightarrow \quad -10, -\frac{65}{3}, \frac{5}{6} \\
    \bar{L}_{-10} & : \quad h = -10, -\frac{75}{2}, -\frac{145}{16}, -\frac{209}{16}, -\frac{369}{16} \quad \rightarrow \quad 0, -\frac{55}{2}, \frac{15}{16}, -\frac{49}{16}, -\frac{209}{16} \\
    \bar{L}_{-11} & : \quad h = -10, -45, -\frac{55}{2} \quad \rightarrow \quad 1, -34, -\frac{33}{2} \\
    \bar{L}_{-12} & : \quad h = -12, -\frac{319}{6}, -\frac{209}{16}, -\frac{517}{16}, -\frac{527}{48}, -\frac{851}{48} \quad \rightarrow \quad 0, -\frac{247}{6}, -\frac{17}{16}, -\frac{325}{48}, \frac{49}{48}, -\frac{275}{48}
\end{align*}
\] (A.14)

For the effective string theory, two values of the conformal weight are relevant, namely \( L_0^{\text{eff}} = 1 \) and \( L_0^{\text{eff}} = \frac{13}{16} \). For the first of these, we see that the following factorisation occurs:
\[
\bar{L}_{-11}(-10) = \bar{L}_{-11} \bar{L}_{-10} = \bar{L}_{-6} \bar{L}_{-5} .
\] (A.15)

For \( L_0^{\text{eff}} = \frac{13}{16} \), we see the factorisation
\[
\bar{L}_{-10}(-\frac{145}{16}) = \bar{L}_{-2} \bar{L}_{-8} = \bar{L}_{-4} \bar{L}_{-6} .
\] (A.16)

These factorisations illustrate what is presumably a generic pattern, namely that at \( c = 25\frac{1}{2} \) the operators that give conformal-weight 1 effective physical null states can all be factorised two ways, as either \( \bar{L}_{-1} \) or \( \bar{L}_{-6} \) acting on lower-level operators; and the operators that give conformal-weight \( \frac{13}{16} \) physical null states can also all be factorised two ways, as either \( \bar{L}_{-2} \) or \( \bar{L}_{-4} \) acting on lower-level operators. For comparison, at \( c = 26 \), the operators
that give conformal-weight 1 physical null states can all be factorised two ways too, as either $\tilde{L}_{-1}$ or $\tilde{L}_{-2}$ acting on lower-order operators. The significance of the level-numbers of the two “fundamental” factorising operators in each case can be understood by looking at the level numbers at which physical null states occur in the relevant 1-scalar string or effective string. By using the screening-charge analysis described in section 5, one easily sees that at $c = 26$ such null states occur at levels $\ell = \frac{1}{6} n(n + 1) = 0, 1, 2, 5, 7, 12, 15, 22 \ldots$, where $n \pmod{3} = 0, 2$. A similar analysis at $c = 25\frac{1}{2}$ shows that null states in a 1-scalar effective string with $L_0^{\text{eff}} = 1$ occur at levels $\ell = \frac{1}{12} n(n + 1) = 0, 1, 6, 11, 13, 20, 35 \ldots$, where $n \pmod{12} = 0, 3, 8, 11$. In the case that $L_0^{\text{eff}} = 15\frac{1}{3}$, the null states occur at levels $\ell = \frac{1}{15} n(n + 1) = 0, 2, 4, 10, 14, 24, 30 \ldots$, where $n \pmod{3} = 0, 2$. Thus in all cases, the fundamental factorising operators occur at the lowest two non-zero level numbers. It is a fluke of ordinary string theory that these happen to be levels 1 and 2.
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