Dynamical realizations of $\mathcal{N} = 1$ $l$-conformal Galilei superalgebra

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Abstract

Dynamical systems which are invariant under $\mathcal{N} = 1$ supersymmetric extension of the $l$-conformal Galilei algebra are constructed. These include a free $\mathcal{N} = 1$ superparticle which is governed by higher derivative equations of motion and an $\mathcal{N} = 1$ supersymmetric Pais-Uhlenbeck oscillator for a particular choice of its frequencies. A Niederer-like transformation which links the models is proposed.

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1. Introduction

In recent years nonrelativistic (super)conformal algebras have attracted considerable attention [1]-[31]. On the one hand, this interest originates from the current investigation of the nonrelativistic version of the AdS/CFT-correspondence [32, 33]. On the other hand, this research is motivated by the desire to construct new integrable models and explore novel correlations.

As is well-known, the Galilei algebra is relevant for physics in flat nonrelativistic spacetime. Conformal extension of the Galilei algebra is feasible and, moreover, it is not unique [34]-[36]. In general, conformally extended Galilei algebra involves \((2l + 1)\) vector generators where \(l\) is a positive integer or half-integer. Such an extension is called the \(l\)-conformal Galilei algebra [35]. The first two options \(l = \frac{1}{2}\) and \(l = 1\), which are known in the literature as the Schrödinger algebra [37] and the conformal Galilei algebra [1, 38], have been the focus of most studies [1]-[5], [11]-[13]. More recently, various aspects of \(l > 1\) conformal Galilei symmetry have been extensively investigated. These include the construction of dynamical realizations [18]-[24], [27], the study of admissible central and infinite dimensional extensions [14, 15, 31], the analysis of supersymmetric generalizations [15, 17, 26], the investigation of irreducible representations [16, 29] and the possible twist deformations [28].

The Galilei algebra can be obtained from the Poincaré algebra by the nonrelativistic contraction [39]. Likewise, the so-called Newton-Hooke algebra [40, 41] can be derived from the (anti) de Sitter algebra. A specific feature of the Newton-Hooke algebra is that its structure relations involve the nonrelativistic cosmological constant \(\Lambda = \mp \frac{1}{R^2}\), where \(R\) is the characteristic time which is proportional to the radius of the parent (anti) de Sitter spacetime [41]. The Newton-Hooke algebra also admits the \(l\)-conformal extension, which, however, is isomorphic to the \(l\)-conformal Galilei algebra [35, 36, 14]. The change of the basis in the \(l\)-conformal Galilei algebra

\[ K_{-1} \to K_{-1} \pm \frac{1}{R^2} K_1, \]

where \(K_{-1}\) is the generator of time translations and \(K_1\) is the generator of special conformal transformations, leads to the structure relations of the \(l\)-conformal Newton-Hooke algebra with negative (upper sign) or positive (lower sign) cosmological constant. By this reason it is customary to speak about realizations of one and the same algebra in a flat spacetime and in the Newton-Hooke spacetime [41].

Supersymmetric extensions of the \(l = \frac{1}{2}\)-conformal Galilei algebra have been studied in [42, 43, 5, 10]. In particular, such supersymmetry was revealed in a nonrelativistic spin-\(\frac{1}{2}\) particle, the nonrelativistic limit of the Chern-Simons matter systems, and quantum many-body mechanics. \(\mathcal{N}\)-extended version was systematically studied in [44]. More recently, in [15, 17, 26] various supersymmetric extensions of the \(l\)-conformal Galilei algebra were constructed for the case of arbitrary \(l\), but their dynamical realizations remain completely unexplored. The purpose of this work is to construct new dynamical realizations of \(\mathcal{N} = 1\) supersymmetric extension of the \(l\)-conformal Galilei algebra in the basis chosen in [17].

The paper is organized as follows. In Section 2 we recall the basic facts about the \(l\)-conformal Galilei algebra. In Section 3 and Section 4 we construct dynamical realizations of \(\mathcal{N} = 1\) \(l\)-conformal Galilei superalgebra in flat superspace and in Newton-Hooke superspace, respectively. In Section 5 we summarize our results and discuss possible further developments.
2. The $l$-conformal Galilei algebra

Let us recall the structure of the $l$-conformal Galilei algebra. Besides the generators $K_{-1}$ and $K_1$ mentioned above, it involves the generator of dilatations $K_0$, the chain of vector generators $C_i^{(n)}$ with $n = 0, 1, ..., 2l$, and the generators of spatial rotations $M_{ij}$. The structure relations read

\[ [K_p, K_m] = (m - p)K_{p+m}, \quad [K_p, C_i^{(n)}] = (n - (p + 1))C_i^{(p+n)}, \]
\[ [M_{ij}, C_k^{(n)}] = \delta_{ik}C_j^{(n)} - \delta_{jk}C_i^{(n)}, \quad [M_{ij}, M_{kl}] = \delta_{ik}M_{jl} + \delta_{jl}M_{ik} - \delta_{il}M_{jk} - \delta_{jk}M_{il}. \]  

(2)

The algebra admits the central extension whose form depends on whether $l$ is even or odd [14]

\[ [C_i^{(n)}, C_j^{(m)}] = (-1)^n n! m! \lambda_{ij} \delta_{n+m,2l} M, \]

(3)

where

\[ \lambda_{ij} = \begin{cases} \delta_{ij}, & i, j = 1, 2, ..., d, \quad \text{for half-integer } l; \\ \epsilon_{ij}, & i, j = 1, 2, \quad \text{for integer } l, \end{cases} \]

(4)

$\epsilon_{12} = 1$ and $M$ is the central charge. In dynamical realizations the central charges correspond to physical parameters of a systems. As we shall see below in the Sections 3 and 4, the central charges play the crucial role in constructing the dynamical realizations.

3. Dynamical realization of $\mathcal{N} = 1$ $l$-conformal Galilei superalgebra

Let us consider $\mathcal{N} = 1$ supersymmetric extension of the $l$-conformal Galilei algebra presented in [17]. In addition to the generators considered in the preceding section, it involves the supersymmetry generator $G_{-1/2}$, the generator of superconformal transformations $G_{1/2}$ and the fermionic partners $L_i^{(n)}$, with $n = 0, 1, ..., 2l - 1$, of the vector generators. Along with (2) the nonvanishing (anti)commutation relations of the superalgebra include

\[ \{G_r, G_s\} = 2iK_{r+s}, \quad [K_p, L_i^{(m)}] = (m - (l - 1/2)(p + 1))L_i^{(p+m)}, \]
\[ \{G_r, L_i^{(n)}\} = iC_i^{(n+r+1/2)}, \quad [G_r, C_i^{(n)}] = (n - 2l (r + 1/2)) L_i^{(r+n-1/2)}, \]
\[ [K_p, G_r] = (r - p/2) G_{n+r}, \quad [M_{ij}, L_k^{(n)}] = \delta_{ik}L_j^{(n)} - \delta_{jk}L_i^{(n)}. \]

(5)

The structure relations (2) and (5) are compatible with (3) only if the anticommutators of the fermionic vector generators are modified as follows [17]:

\[ \{L_i^{(n)}, L_j^{(m)}\} = i(-1)^n n! m! \lambda_{ij} \delta_{n+m,2l-1} M. \]

(6)

As the first step, let us check that all the scalar generators in the superalgebra, as well as space rotations, can be realized as quadratic combinations of the bosonic and fermionic vector generators. Indeed, choosing the ansatz for $K_n$\footnote{Throughout the work the summation over repeated spatial indices is understood.}

\[ K_n = \sum_{k,m=0}^{2l} \alpha(k, m; n) \lambda_{ij} C_i^{(k)} C_j^{(m)} + \sum_{k,m=0}^{2l-1} \beta(k, m; n) \lambda_{ij} L_i^{(k)} L_j^{(m)}, \]
one can unambiguously fix the constants $\alpha(k, m; n)$ and $\beta(k, m; n)$ by imposing the structure relations of the superalgebra. The explicit form of the generators obtained in this way is\(^{2}\)

\[
K_n = \sum_{k=0}^{2l} \frac{\alpha_k}{2} (k - l(n + 1)) \lambda_i C_i^{(2l-k)} C_j^{(k+n)} + \sum_{k=0}^{2l-1} \frac{\beta_k}{2} (k - (l - 1/2)(n + 1)) \lambda_i L_i^{(2l-1-k)} L_j^{(k+n)},
\]

\[
G_r = \sum_{k=1}^{2l} \alpha_k k \lambda_i C_i^{(2l-k+1/2)} L_j^{(k-1)}, \quad M_{ij} = \sum_{k=0}^{2l} \alpha_k C_i^{(2l-k)} C_j^{(k)} + \sum_{k=0}^{2l-1} \beta_k L_i^{(2l-k-1)} L_j^{(k)}. \tag{7}
\]

Note that for integer $l$ the generator of rotation reads

\[
M_{12} = - \sum_{k=0}^{2l} \frac{\alpha_k}{2} C_i^{(2l-k)} C_j^{(k)} - \sum_{k=0}^{2l-1} \frac{\beta_k}{2} L_i^{(2l-k-1)} L_j^{(k)}, \tag{8}
\]

where we denoted

\[
\alpha_k = \frac{(-1)^{2l+k}}{M k! (2l - k)!}, \quad \beta_k = \frac{i (-1)^{2l+k-1}}{M k! (2l - k - 1)!}.
\]

Thus if one succeeds in constructing a system with conserved vector charges obeying (3) and (6) with respect to some graded Poisson bracket, one can automatically produce additional integrals of motion by making use of (7) and (8). Together with $C_i^{(n)}$, $L_i^{(n)}$ they will obey the structure relations (2), (5) with respect to the same bracket.

Let us construct such a system by applying the method of nonlinear realizations \([45, 46]\) to the subalgebra formed by $K_{-1}$, $C_i^{(n)}$, $L_i^{(n)}$ and $M$. To this end, one starts with a generic subgroup element $e^{aK_{-1}} e^{\zeta_i^{(n)} C_i^{(n)}} e^{\psi_i^{(n)} L_i^{(n)}} e^{\chi M}$, where $a, \zeta_i^{(n)}, \psi_i^{(n)}, \chi$ are parameters, and considers the transformation of the space

\[
G = e^{t K_{-1}} e^{\zeta_i^{(n)} C_i^{(n)}} e^{\psi_i^{(n)} L_i^{(n)}} e^{\chi M}, \tag{9}
\]

parametrized by the coordinates $t, x_i^{(n)}, \psi_i^{(n)}, \varphi$, which is generated by the left multiplication with the subgroup element. It is assumed that $\zeta_i^{(n)}$ and $\psi_i^{(n)}$ anticommute with $L_i^{(n)}$ as well as with each other. The resulting infinitesimal coordinate transformations read

\[
\delta t = a, \quad \delta x_i^{(n)} = \sum_{k=n}^{2l} \frac{\lambda_i}{n!(k-n)!} t^{k-n} \zeta_i^{(k)}, \quad \delta \psi_i^{(n)} = \sum_{k=n}^{2l-1} \frac{\lambda_i}{n!(k-n)!} t^{k-n} \psi_i^{(k)} + \frac{2l}{2(k-n)!} t^{k-n} \zeta_i^{(k)} \lambda_{ij} x_j^{(2l-n)} + i \sum_{n=0}^{2l-1} \frac{\lambda_i}{2(k-n)!} t^{k-n} \zeta_i^{(k)} \lambda_{ij} \psi_j^{(2l-n-1)}. \tag{10}
\]

Then one constructs the Maurer-Cartan one-forms

\[
G^{-1} dG = \omega_K K_{-1} + \omega_i^{(n)} C_i^{(n)} + i \omega_i^{(n)} L_i^{(n)} + \omega_M M,
\]

\(^{2}\)For $\mathcal{N} = 2$ supersymmetric extensions of the $l$-conformal Galilei algebra the analogs of (7) and (8) are presented in \([17, 26]\).
where\(^3\)
\[ \omega_K = dt, \quad \omega_i^{(n)} = dx_i^{(n)} + (n+1)x_i^{(n+1)}dt, \quad \tilde{\omega}_i^{(n)} = d\psi_i^{(n)} + (n+1)\psi_i^{(n+1)}dt, \quad (11) \]
\[ \omega_M = d\varphi + \frac{\lambda_{ij}}{2} \left( \sum_{n=0}^{2l} (-1)^n \omega_i^{(n)} x_j^{(2l-n)} n!(2l-n)! + i \sum_{n=0}^{2l-1} (-1)^n \tilde{\omega}_i^{(n)} \psi_j^{(2l-n-1)} n!(2l-n-1)! \right), \]
which hold invariant under all the transformations (10).

In general, one can either reduce the number of degrees of freedom or obtain the dynamical equations of motion by setting some of the Maurer-Cartan one-forms to vanish \([47]\). Let us choose the restrictions
\[ \omega_i^{(n)} = 0, \quad \tilde{\omega}_i^{(n)} = 0, \quad (12) \]
which allow us to exclude all the vector variables except for \(x_i^{(0)} \equiv x_i\) and \(\psi_i^{(0)} \equiv \psi_i\). Then, taking \(t\) to be a temporal coordinate, from (12) one obtains the constraints
\[ x_i^{(n)} = \frac{(-1)^n}{n!} \frac{d^n x_i}{dt^n}, \quad \psi_i^{(n)} = \frac{(-1)^n}{n!} \frac{d^n \psi_i}{dt^n}, \quad (13) \]
as well as the dynamical equations of motion
\[ \frac{d^{2l+1} x_i}{dt^{2l+1}} = 0, \quad \frac{d^{2l} \psi_i}{dt^{2l}} = 0. \quad (14) \]

Note that the equations (14) can be derived from the action functional
\[ S = \int (-1)^{2l+1} \omega_M = \frac{1}{2} \int dt \lambda_{ij} \left( x_i \frac{d^{2l+1} x_i}{dt^{2l+1}} - i \psi_i \frac{d^{2l} \psi_i}{dt^{2l}} \right), \quad (15) \]
which is derived from \(\omega_M\) by taking into account the constraints (13). The model (15) is an \(\mathcal{N} = 1\) supersymmetric generalization of the free higher derivative particle studied in \([18,19,22]\).

In accord with (10), the action (15) is invariant under the transformations\(^4\)
\[ \delta t = a, \quad \delta x_i = \sum_{n=0}^{2l} \zeta_{i}^{(n)} t^n, \quad \delta \psi_i = \sum_{n=0}^{2l-1} \tilde{\zeta}_{i}^{(n)} t^n. \quad (16) \]

Then the Noether theorem yields the vector constants of the motion\(^5\)
\[ C_i^{(n)} = \lambda_{ij} \sum_{k=0}^{n} \frac{(-1)^{k+1} n!}{(n-k)!} t^{n-k} x_j^{(2l-k)}, \quad L_i^{(n)} = i \lambda_{ij} \sum_{k=0}^{n} \frac{(-1)^k n!}{(n-k)!} t^{n-k} \psi_j^{(2l-k-1)}, \quad (17) \]
as well as \(K_{-1}\)
\[ K_{-1} = \frac{1}{2} \lambda_{ij} \left( \sum_{k=1}^{2l} (-1)^{k+1} x_i^{(k)} x_j^{(2l-k+1)} + i \sum_{k=1}^{2l-1} (-1)^k \psi_i^{(k)} \psi_j^{(2l-k)} \right). \quad (18) \]

\(^3\)By definition, \(x_i^{(-1)} = x_i^{(2l+1)} = \psi_i^{(-1)} = \psi_i^{(2l)} = 0.\)
\(^4\)We put \(\zeta_{i}^{(n)} = (-1)^n \zeta_{i}^{(n)}, \tilde{\zeta}_{i}^{(n)} = (-1)^n \tilde{\zeta}_{i}^{(n)}\).
\(^5\)Here and in what follows the upper superscript in braces, which is attached to coordinates, denotes the number of time derivatives.
Note that the latter is related to (17) via (7).

Introducing the graded Poisson bracket\(^6\)

\[
[A, B] = \lambda_{ij} \sum_{n=0}^{2l} (-1)^n \frac{\partial A}{\partial x_i^{(n)}} \frac{\partial B}{\partial x_j^{(2l-n)}} + i \lambda_{ij} \sum_{n=0}^{2l-1} (-1)^{n+1} \frac{\delta A}{\partial \psi_i^{(n)}} \frac{\delta B}{\partial \psi_j^{(2l-n-1)}},
\]

it is straightforward to check that the integrals of motion (7) and (17) do obey the structure relations of the centrally extended \(\mathcal{N} = 1\) \(l\)-conformal Galilei superalgebra (2), (3), (5), (6) with \(M = 1\). When verifying the algebra, the following relations

\[
[x_i^{(n)}, x_j^{(m)}] = (-1)^n \delta_{n+m,2l} \lambda_{ij}, \quad [\psi_i^{(n)}, \psi_j^{(m)}] = i(-1)^{n+1} \delta_{n+m,2l-1} \lambda_{ij}
\]

prove to be helpful.

Concluding this section we display the infinitesimal symmetry transformations of the action (15) which correspond to the integrals of motion constructed above

\[
K_n: \quad \delta t = t^{n+1} a_n, \quad \delta x_i = l(n+1)t^n x_i a_n, \quad \delta \psi_i = (l - 1/2)(n + 1)t^n \psi_i a_n,
\]

\[
G_r: \quad \delta x_i = it^{r+1/2} \psi_i \alpha_r, \quad \delta \psi_i = (t^{r+1/2} x_i - 2l(r + 1/2)x_i) \alpha_r,
\]

\[
M_{ij}: \quad \delta x_i = w_{ij} x_j, \quad \delta \psi_i = w_{ij} \psi_j, \quad (w_{ij} = -w_{ji}).
\]

To summarize, a free superparticle obeying the higher derivative equations of motion (14) provides the simplest dynamical realization of \(\mathcal{N} = 1\) \(l\)-conformal Galilei superalgebra. Note that this result is in agreement with the previous studies in [18]-[19] and [22]-[23].

4. Dynamical realization of \(\mathcal{N} = 1\) \(l\)-conformal Newton-Hooke superalgebra

As is known, realizations of the \(l\)-conformal Galilei algebra in a flat spacetime and in the Newton-Hooke spacetime [35, 36] are related by the coordinate transformations which, for the case of a negative cosmological constant, reads [11, 14, 48]

\[
t' = R \tan(t/R), \quad x'_i(t') = x_i(t)/\cos^{2l}(t/R).
\]

Here the prime designates coordinates parametrizing flat spacetime.

Let us construct an analogous transformation which links the model (15) to its Newton-Hooke counterpart. To this end, we first note that the equations of motion for \(\psi_i\) in (14) can be formally obtained from the equations of motion for \(x_i\) by the substitution \(x_i \to \psi_i\), \(l \to l - 1/2\). Using this observation, one obtains the transformation for the odd variables

\[
\psi'_i(t') = \psi_i(t)/\cos^{2l-1}(t/R).
\]

Implementing the transformations (22) and (23) to (15) one derives the action functional

\[
S = \frac{1}{2} \int dt \left( x_i \prod_{k=1}^{l+\frac{1}{2}} \left( \frac{d^2}{dt^2} + \frac{(2k-1)^2}{R^2} \right) x_i - i\psi_i \prod_{k=1}^{l-\frac{1}{2}} \left( \frac{d^2}{dt^2} + \frac{(2k)^2}{R^2} \right) \psi_i \right).
\]

\(^6\)It should be noted that the equations \([x_i^{(n)}, K_{-1}] = x_i^{(n+1)}, [\psi_i^{(n)}, K_{-1}] = \psi_i^{(n+1)}\) hold.
which is valid for half-integer \( l \). For integer \( l \) one gets

\[
S = \frac{1}{2} \int dt \epsilon_{ij} \left( x_i \prod_{k=1}^{l} \left( \frac{d^2}{dt^2} + \frac{(2k)^2}{R^2} \right) \dot{x}_j - i \psi_i \prod_{k=1}^{l} \left( \frac{d^2}{dt^2} + \frac{(2k-1)^2}{R^2} \right) \psi_j \right). \tag{25}
\]

These actions describe \( \mathcal{N} = 1 \) supersymmetric extensions of the Pais-Uhlenbeck oscillator \([49]\) (for a review see \([50]\)) for a particular choice of its frequencies. The bosonic limit of (24) has been considered in detail in a recent work \([30]\) (see also \([25]\)). The form of the symmetry transformations which leave the actions (24) and (25) invariant, as well as the form of the associated integrals of motion, is readily obtained by applying (22), (30) to (16), (21), (17) and (7), respectively. In particular, from (17) one derives the transformations corresponding to the vector generators

\[
\delta x_i = \sum_{n=0}^{2l} \tilde{c}^{(n)}_{si} n \sin^n \frac{t}{R} \cos^{2l-n} \frac{t}{R}, \quad \delta \psi_i = \sum_{n=0}^{2l-1} \tilde{c}^{(n)}_{si} n \sin^n \frac{t}{R} \cos^{2l-n-1} \frac{t}{R}, \tag{26}
\]

and the integrals of motion associated to them\(^7\)

\[
C_i^{(n)} = \lambda_{ij} \sum_{k=0}^{l} \frac{(-1)^{n+k} n!}{(n-k)!} (d')^{n-k} (x')^{(2l-k)}, \quad L_i^{(n)} = i \lambda_{ij} \sum_{k=0}^{l} \frac{(-1)^{n+k} n!}{(n-k)!} (d')^{n-k} (\psi')^{(2l-k-1)}. \tag{27}
\]

With respect to the graded bracket

\[
[A, B] = \lambda_{ij} \sum_{n=0}^{2l} \frac{(-1)^n}{(n-k)!} \left( \frac{\partial A}{\partial (x_j')^{(n)}} \frac{\partial B}{\partial (x_j')} \right) + \lambda_{ij} \sum_{n=0}^{2l-1} \frac{(-1)^k n!}{(n-k)!} \left( \frac{\partial A}{\partial (\psi_j')^{(n)}} \frac{\partial B}{\partial (\psi_j')} \right), \tag{28}
\]

they obey the structure relations (3) and (6).

The remaining symmetries of the actions (24) and (25) read

\[
K_0 : \quad \delta t = \frac{R}{2} \sin \frac{2t}{R} a_0, \quad \delta x_i = l \cos \frac{2t}{R} x_i a_0, \quad \delta \psi_i = \left( l - \frac{1}{2} \right) \cos \frac{2t}{R} \psi_i a_0;
\]

\[
K_1 : \quad \delta t = R^2 \sin^2 \frac{t}{R} a_1, \quad \delta x_i = l R \sin \frac{2t}{R} x_i a_1, \quad \delta \psi_i = \left( l - \frac{1}{2} \right) R \sin \frac{2t}{R} \psi_i a_1;
\]

\[
G_{-\frac{1}{2}} : \quad \delta x_i = i \cos \frac{t}{R} \psi_i \alpha_{-\frac{1}{2}}, \quad \delta \psi_i = \left( \cos \frac{t}{R} \dot{x}_i + \frac{2l}{R} \sin \frac{t}{R} x_i \right) \alpha_{-\frac{1}{2}};
\]

\[
G_{\frac{1}{2}} : \quad \delta x_i = i R \sin \frac{t}{R} \psi_i \alpha_{\frac{1}{2}}, \quad \delta \psi_i = \left( R \sin \frac{t}{R} \dot{x}_i - 2l \cos \frac{t}{R} x_i \right) \alpha_{\frac{1}{2}}, \tag{29}
\]

while the transformations corresponding to the generators \( K_{-1} \) and \( M_{ij} \) are the same as in (21)\(^9\). Thus, the actions (24) and (25) are invariant under \( \mathcal{N} = 1 \) \( l \)-conformal Galilei superalgebra realized in Newton-Hooke superspace.

\(^7\)The existence of the relations similar to (7) for the bosonic limit of (24) was anticipated in \([51]\).

\(^8\)Eqs. (27) and (28) involve the derivatives of \( x' \) and \( \psi' \) with respect to \( t' \), i.e. \( \frac{d}{dt'} = \cos^2 \frac{t}{R} \frac{d}{dt} \).

\(^9\)In order to obtain the transformation corresponding to \( K_{-1} \) and the corresponding integral of motion, one has to make the linear change of the basis (1).
The case of a positive cosmological constant can be treated by implementing the formal the characteristic time $R \to iR$. In particular, for a half-integer $l$ the analogues of (22) and (23)

$$t' = R \tanh(t/R), \quad x'_i(t') = x_i(t)/\cosh^{2l}(t/R), \quad \psi'_i(t') = \psi_i(t)/\cosh^{2l-1}(t/R),$$

yield the action functional

$$S = \frac{1}{2} \int dt \left( x_i \prod_{k=1}^{l+\frac{1}{2}} \left( \frac{d^2}{dt^2} - \frac{(2k-1)^2}{R^2} \right) x_i - i\psi_i \prod_{k=1}^{l-\frac{1}{2}} \left( \frac{d^2}{dt^2} - \frac{(2k)^2}{R^2} \right) \dot{\psi}_i \right),$$

while for integer $l$ one gets

$$S = \frac{1}{2} \int dt \epsilon_{ij} \left( x_i \prod_{k=1}^{l} \left( \frac{d^2}{dt^2} - \frac{(2k)^2}{R^2} \right) \dot{x}_j - i\psi_i \prod_{k=1}^{l} \left( \frac{d^2}{dt^2} - \frac{(2k-1)^2}{R^2} \right) \dot{\psi}_j \right).$$

These actions can be viewed as describing $\mathcal{N} = 1$ supersymmetric extensions of the Pais-Uhlenbeck oscillator for the particular choice of imaginary frequencies.

It should be stressed that in the flat space limit $R \to \infty$ all the formulas corresponding to the realizations (24), (25), (31), (32) in the Newton-Hooke superspace correctly reproduce the respective expressions in a flat superspace.

5. Conclusion

To summarize, in this work we have constructed dynamical realizations of $\mathcal{N} = 1$ $l$-conformal Galilei superalgebra in a flat superspace and in the Newton-Hooke superspace. Coordinate transformations which link the realizations were found. The models describe a free higher derivative superparticle and an $\mathcal{N} = 1$ supersymmetric extension of the Pais-Uhlenbeck oscillator for the particular choice of its frequencies, respectively.

Turning to possible further developments, it is interesting to investigate whether an $\mathcal{N} = 1$ supersymmetric extension of the $l$-conformal Galilei algebra can be realized in systems without higher derivatives in the equations of motion. In this context it would be interesting to superextend the models constructed in [20], [25]. A generalization of the analysis in this paper to the case of $\mathcal{N} = 2$ $l$-conformal Galilei superalgebras [15, 17, 26] is worth studying as well.

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