Isoperimetric inequalities of the fourth order Neumann eigenvalues

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Abstract
In this paper, we obtain some isoperimetric inequalities for the first \((n-1)\) eigenvalues of the fourth order Neumann Laplacian on bounded domains in an \(n\)-dimensional Euclidean space. Our result supports strongly the conjecture of Chasman.

Keywords: Eigenvalues; Neumann problem; Isoperimetric inequality

1 Introduction
Letting \(\Omega\) be a bounded domain with a smooth boundary \(\partial \Omega\) in the Euclidean space \(\mathbb{R}^n\), we consider the Neumann problem of the Laplacian \(\Delta\) as follows:

\[
\begin{align*}
\Delta u &= \mu u, \quad \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} &= 0, \quad \text{on } \partial \Omega,
\end{align*}
\]

where \(\nu\) is the outward unit normal to the boundary. It is well known that the free membrane problem (1.1) has a discrete spectrum consisting of a sequence

\[
0 = \mu_0 < \mu_1 \leq \mu_2 \leq \cdots \rightarrow +\infty.
\]

When \(\Omega\) is a bounded domain in \(\mathbb{R}^2\), Szegö [6] proved the following classical isoperimetric inequality:

\[
\mu_1(\Omega) \leq \mu_1(B_{12}),
\]

where \(B_{12}\) is the ball of same volume as \(\Omega\). Weinberger [11] generalized this result to \(n\)-dimensions. Ashbaugh and Benguria [2] extended the Szegö–Weinberger inequality (1.2) to the bounded domains in hyperbolic space and a hemisphere. On the other hand, Ashbaugh and Benguria [1] conjectured that

\[
\sum_{i=1}^{n} \frac{1}{\mu_i(\Omega)} \geq \frac{n}{\mu_1(B_{12})}, \quad \text{with equality if and only if } \Omega \text{ is a ball},
\]
where $\mu_i(\Omega)$ is the $i$th Neumann eigenvalue on $\Omega$, $\mu_1(B_{\Omega})$ is the first nonzero Neumann eigenvalue on $B_{\Omega}$. In [10], Wang and Xia proved an isoperimetric inequality for the sums of the reciprocals of the first $(n-1)$ nonzero eigenvalues of the Neumann Laplacian on bounded domains in $\mathbb{R}^n$ as follows:

$$\sum_{i=1}^{n-1} \frac{1}{\mu_i(\Omega)} \geq \frac{n-1}{\mu_1(B_{\Omega})},$$  \hspace{1cm} (1.4)  

which means the Ashbaugh–Benguria’s conjecture is true for the first $(n-1)$ nonzero eigenvalues of the Neumann Laplacian on bounded domains in $\mathbb{R}^n$. So (1.4) supports the above conjectures of Ashbaugh and Benguria. On the other hand, Benguria, et al. [3] proved a result which is similar to (1.4) for the first $(n-1)$ nontrivial Neumann eigenvalues on domains in a hemisphere of $S^n$. Moreover, some works on eigenvalues are related to the spectra of matrix operators and can be seen in [7–9].

Let $\Delta$ and $\Delta_{\partial}$ be the Laplace–Beltrami operators on $\Omega$ and $\partial\Omega$, respectively. Let $\nabla$ and $\nabla_{\partial}$ be the gradient operators on $\Omega$ and $\partial\Omega$, respectively. Consider the following Neumann eigenvalue problem of the bi-harmonic operator:

$$\begin{cases}
\Delta^2 u - \tau \Delta u = \Lambda u & \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega, \\
\tau \frac{\partial u}{\partial \nu} - \text{div}_{\partial\Omega}(\nabla^2 u(\nu)) - \frac{\partial \Lambda u}{\partial \nu} = 0, & \text{on } \partial\Omega,
\end{cases} \hspace{1cm} (1.5)$$

where $\tau \geq 0$ and $\sigma$ are two constants, $\text{div}_{\partial\Omega}$ denotes the tangential divergence operator on $\partial\Omega$, and $\nabla^2 u$ is the Hessian of $u$, $\nu$ is the outward unit normal to the boundary. In this setting, problem (1.5) has a discrete spectrum, and all eigenvalues in the discrete spectrum can be listed nondecreasingly as follows:

$$0 = \Lambda_0 < \Lambda_1 \leq \Lambda_2 \leq \cdots \uparrow +\infty.$$  

By the Rayleigh–Ritz characterization, the $(k+1)$th eigenvalue of (1.5) can be given as follows (see, e.g., [5]):

$$\Lambda_{k+1} = \inf_{u \in H^2(\Omega)} \left\{ Q[u] = \frac{\int_{\Omega} \|
abla^2 u\|^2 + \tau \|
abla u\|^2 \,dx}{\int_{\Omega} u^2 \,dx} \mid \int_{\partial\Omega} uu_j = 0, j = 1, \ldots, k \right\}. \hspace{1cm} (1.6)$$

Letting $B_{\Omega}$ be the ball of same volume as $\Omega$, Chasman [5] proved the following isoperimetric inequality:

$$\Lambda_1(\Omega) \leq \Lambda_1(B_{\Omega}), \hspace{1cm} \text{with equality if and only if } \Omega \text{ is a ball.}$$

Chasman [5] also conjectured that

$$\sum_{i=1}^{n} \frac{1}{\Lambda_i(\Omega)} \geq \frac{n}{\Lambda_1(B_{\Omega})}, \hspace{1cm} \text{with equality if and only if } \Omega \text{ is a ball.} \hspace{1cm} (1.7)$$

In this paper, we prove an isoperimetric inequality for the sums of the reciprocals of the first $(n-1)$ nonzero eigenvalues of the fourth Neumann Laplacian which supports the
Chasman’s conjecture, actually, we get
\[
\sum_{i=1}^{n-1} \frac{1}{\Lambda_i(\Omega)} \geq \frac{n-1}{\Lambda_1(B_\Omega)},
\]
with equality if and only if \( \Omega \) is a ball. (1.8)

In [4], Buoso et al. proved a quantitative isoperimetric inequality for the fundamental tone of problem (1.5) as follows:
\[
\Lambda_1(\Omega) \leq \left(1 - \eta_{n,\tau,|\Omega|} A^2(\Omega)\right) \Lambda_1(B_\Omega),
\] (1.9)
where \( \eta_{n,\tau,|\Omega|} > 0 \), and \( A(\Omega) \) is the so-called Fraenkel asymmetry of the domain \( \Omega \in \mathbb{R}^n \), which is defined by:
\[
A(\Omega) := \inf \left\{ \frac{|\Omega \Delta B_\Omega|}{|\Omega|}\right\},
\]
where \( B_\Omega \) is the ball of same volume as \( \Omega \) and \( \Omega \Delta B_\Omega \) is the symmetric difference of \( \Omega \) and \( B_\Omega \). In what follows, we generalize (1.9) to the sum of the first \( n-1 \) eigenvalues, and we get
\[
\frac{1}{n-1} \sum_{i=1}^{n-1} \Lambda_i(\Omega) \leq \left(1 - \eta_{n,\tau,|\Omega|} A^2(\Omega)\right) \Lambda_1(B_\Omega).
\] (1.10)

2 Preliminaries
In this section, we recall some notations and results, more details can be seen in [4, 5].

Let \( j_1, i_1 \) be the ultraspherical and modified ultraspherical Bessel functions of the first kind and order 1, respectively; \( j_1, i_1 \) can be expressed by the standard Bessel and modified Bessel functions of the first kind \( J_\nu, I_\nu \) as follows:
\[
j_1(z) = z^{1-n/2} J_{n/2}(z), \quad i_1(z) = z^{1-n/2} I_{n/2}(z).
\]

Let \( B \) be the unit ball in \( \mathbb{R}^n \) centered at the origin and \( \omega_n \) be the Lebesgue measure \( |B| \) of \( B \), and let \( \lambda_1(B) \) be the first eigenvalue of problem (1.5) on unit ball \( B \). For \( \tau > 0 \), \( a, b \) are positive constants satisfying the conditions \( a^2 b^2 = \lambda_1(B) \) and \( b^2 - a^2 = \tau \). Set
\[
R(r) = j_1(ar) + \gamma i_1(br), \quad \gamma = \frac{-a^2 j_1''(ar)}{b^2 i_1''(br)} > 0.
\]
Then we define the function \( \rho : [0, +\infty) \rightarrow [0, +\infty) \) as
\[
\rho(r) = \begin{cases} 
R(r), & r \in [0, 1), \\
R(1) + (r - 1)R'(1), & r \in [1, +\infty).
\end{cases}
\]
Let \( u_i : \mathbb{R}^n \rightarrow \mathbb{R} \) be defined by
\[
u_i(x) := \rho(|x|) \frac{X_i}{|x|}, \quad \text{for } i = 1, \ldots, n.
\] (2.1)
The functions $u_i|_B$ are, in fact, the eigenfunctions associated with the eigenvalues $\lambda_1(B)$ of problem (1.5) on unit ball $B$. We know that $\lambda_1(B)$ has multiplicity and $u_i$ satisfy

$$\sum_{i=1}^{n} |u_i|^2 = \rho^2(|x|), \quad (2.2)$$

$$\sum_{i=1}^{n} |\nabla u_i|^2 = \frac{n-1}{|x|^2} \rho(|x|)^2 + (\rho'(|x|))^2, \quad (2.3)$$

$$\sum_{i=1}^{n} |\nabla^2 u_i|^2 = (\rho''(|x|))^2 + \frac{3(n-1)}{|x|^4} [\rho(|x|) - |x|\rho'(|x|)]^2. \quad (2.4)$$

Define $N[\rho] = \sum_{i=1}^{n} (|\nabla^2 u_i|^2 + \tau |\nabla u_i|^2)$. Then $\rho$ and $N[\rho]$ satisfy the following properties which given in [4, 5].

**Lemma 2.1** Function $\rho$ and $N[\rho]$ satisfy the following properties:

1. $\rho''(r) < 0$ for all $r \geq 0$, therefore $\rho'$ is nonincreasing.
2. $\rho(r) - r \rho'(r) \geq 0$, with equality holding only for $r = 0$.
3. The function $\rho^2(r)$ is strictly increasing.
4. The function $\rho^2(r)/r^2$ is decreasing.
5. The function $3(\rho(r) - r \rho'(r))^2/r^4 + \tau \rho^2(r)/r^2$ is decreasing.
6. $N[\rho(r_1)] > N[\rho(r_2)]$ for any $r_1 \in [0, 1), r_2 \in [1, +\infty)$.
7. For all $r \geq 0$, we have

$$N[\rho(r)] = (\rho''(r))^2 + \frac{3(n-1)(\rho(r) - r \rho'(r))^2}{r^4} + \tau (n-1) \frac{\rho^2(r)}{r^2} + \tau (\rho'(r))^2. \quad (2.5)$$

8. For all $r \geq 1$, $N[\rho(r)]$ is decreasing.

We introduce the notation of a partially monotonic function. A function $F$ is partially monotonic on $\Omega$ if it satisfies

$$F(x) > F(y), \quad \text{for all } x \in \Omega \text{ and } y \notin \Omega. \quad (2.5)$$

It is seen that $N[\rho(r)]$ is a partially monotonic function from Lemma 2.1.

**Lemma 2.2** For any radial function $F(r(x))$ that satisfies the partially monotonicity condition on $B_\Omega$,

$$\int_{\Omega} F \, dx \leq \int_{B_\Omega} F \, dx \quad (2.6)$$

with equality if and only if $\Omega = B_\Omega$. For any strictly increasing radial function $F(r(x))$,

$$\int_{\Omega} F \, dx \geq \int_{B_\Omega} F \, dx \quad (2.7)$$

with equality if and only if $\Omega = B_\Omega$. 

Lemma 2.3 For all $s > 0$, we have

$$\Lambda_i(\tau, \Omega) = s^4 \Lambda_i(s^{-2} \tau, s\Omega), \quad i = 1, \ldots, n,$$  

(2.8)

where $s\Omega = \{ x \in \mathbb{R}^n : x/s \in \Omega \}$ for $s > 0$.

Proof For any $u \in H^2(\Omega)$ with $u \neq 0$ and

$$\int_{\Omega} u \, dx = \int_{\Omega} uu_1 \, dx = \cdots = \int_{\Omega} uu_{i-1} \, dx = 0, \quad i = 1, \ldots, n,$$

let $\tilde{u}(x) = u(x/s)$, then $\tilde{u}$ is a valid trial function on $s\Omega$ and so

$$Q_{s^{-2}\tau, s\Omega}[\tilde{u}] = \frac{\int_{s\Omega} (|\nabla^2 \tilde{u}|^2 + s^{-2}\tau |\nabla \tilde{u}|^2) \, dx}{\int_{s\Omega} u^2 \, dx} = \frac{\int_{s\Omega} (s^{-2}|\nabla^2 u|(x/s)|^2 + s^{-2}\tau |s^{-1}(\nabla u)(x/s)|^2) \, dx}{\int_{s\Omega} u(x/s)^2 \, dx}$$

$$= s^{-4n} \frac{\int_{B} (|\nabla^2 u|^2 + \tau |\nabla u|^2) \, dy}{s^n \int_{B} u^2 \, dy} \quad \text{(substituting } y = x/s)$$

$$= s^{-4} Q_{\tau, \Omega}[u].$$  

(2.9)

The lemma follows from (1.6). □

3 Proofs of the main results

In this section, we give the proofs of the main results of this paper.

Theorem 3.1 Let $\Omega$ be a bounded domain in an $n$-dimensional Euclidean space $\mathbb{R}^n$ and let $B_{\Omega}$ be the ball of same volume as $\Omega$, then the first $(n-1)$ eigenvalues of (1.5) in $\Omega$ satisfy

$$\sum_{i=1}^{n-1} \frac{1}{\Lambda_i(\Omega)} \geq \frac{n-1}{\Lambda_1(B_{\Omega})},$$  

(3.1)

with equality if and only if $\Omega$ is a ball.

Proof Assume that the volume of $\Omega$ is equal to that of the unit ball $B$. Letting $\varphi_i = \frac{\rho(r)x_i}{r}$, we know that

$$\int_{\Omega} \varphi_i(r) \, dx = 0, \quad \text{for } i = 1, \ldots, n,$$

which means $\varphi_i$ is perpendicular to $u_0 = 1/\sqrt{|\Omega|}$, which is the first eigenfunction of (1.5). Letting $\{u_i\}_{i=0}^{\infty}$ be an orthonormal set of eigenfunctions of (1.5) on $\Omega$, next we will show that there exists new coordinate functions $\{x'_i\}_{i=1}^{n}$ such that

$$\int_{\Omega} \frac{\rho(r)x'_i}{r} u_i \, dx = 0,$$  

(3.2)

for $j = 1, \ldots, i - 1$ and $i = 2, \ldots, n$. To see this, we define an $n \times n$ matrix $A = (a_{ij})$, where

$$a_{ij} = \int_{\Omega} \varphi_i u_j \, dx = \int_{\Omega} \frac{\rho(r)}{r^2} x_i x_j \, dx, \quad \text{for } i, j = 1, 2, \ldots, n.$$  

Using the orthogonalization of Gram
and Schmidt (QR-factorization theorem), we know that there exist an upper-triangular matrix \( T = (T_{ij}) \) and an orthogonal matrix \( B = (b_{ij}) \) such that \( T = UQ \), i.e.,

\[
T_{ij} = \sum_{k=1}^{n} b_{ik} a_{kj} = \int \sum_{k=1}^{n} \frac{\rho(r)}{r} b_{ik} x_k u_j \, dx = 0, \quad 1 \leq j < i \leq n.
\]

Letting \( x'_i = \sum_{k=1}^{n} b_{ik} x_k, \ i = 1, \ldots, n \), we get (3.2). Since \( B = (b_{ij}) \) is an orthogonal matrix, \( \{x'_i\}_{i=1}^{n} \) is also a set of coordinate functions. Therefore, denoting \( x'_i, \ i = 1 \ldots, n \) still by \( x_i, \ i = 1 \ldots, n \), and \( \varphi_i = \frac{x'_i}{r} x_i \), we have

\[
\varphi_i \neq 0 \quad \text{and} \quad \int_{\Omega} \varphi_i \, dx = \int_{\Omega} \varphi_i x_1 \, dx = \cdots = \int_{\Omega} \varphi_i u_{i-1} \, dx = 0, \quad i = 1, \ldots, n.
\]

It follows from the Rayleigh–Ritz inequality that

\[
\Lambda_i(\Omega) \int_{\Omega} \varphi_i^2 \, dx \leq \int_{\Omega} (|\nabla^2 \varphi_i|^2 + \tau |\nabla \varphi_i|^2) \, dx, \quad i = 1, \ldots, n, \tag{3.3}
\]

which implies that

\[
\int_{\Omega} \varphi_i^2 \, dx \leq \frac{1}{\Lambda_i(\Omega)} \int_{\Omega} (|\nabla^2 \varphi_i|^2 + \tau |\nabla \varphi_i|^2) \, dx, \quad i = 1, \ldots, n. \tag{3.4}
\]

Summing over \( i \) from 1 to \( n \), we have

\[
\sum_{i=1}^{n} \int_{\Omega} \varphi_i^2 \, dx \leq \sum_{i=1}^{n} \frac{1}{\Lambda_i(\Omega)} \int_{\Omega} (|\nabla^2 \varphi_i|^2 + \tau |\nabla \varphi_i|^2) \, dx. \tag{3.5}
\]

Since \( \sum_{i=1}^{n} |\nabla^2 \varphi_i|^2 = (\rho''')^2 + \frac{3(n-1)}{n} (\rho'' - \rho')^2 \), for any point \( p \in \Omega \), by a transformation of coordinates if necessary, we have \( |\nabla^2 \varphi_i|^2 \leq \frac{(\rho'')^2}{n-1} + \frac{3}{n} (\rho' - \rho')^2, \ i = 1, \ldots, n \). Then

\[
\sum_{i=1}^{n} \frac{1}{\Lambda_i(\Omega)} |\nabla^2 \varphi_i|^2
\]

\[
= \sum_{i=1}^{n-1} \frac{1}{\Lambda_i(\Omega)} |\nabla^2 \varphi_i|^2 + \frac{1}{\Lambda_n(\Omega)} |\nabla^2 \varphi_i|^2
\]

\[
= \sum_{i=1}^{n-1} \frac{1}{\Lambda_i(\Omega)} |\nabla^2 \varphi_i|^2 + \frac{1}{\Lambda_n(\Omega)} \left( \frac{(\rho'')^2 + 3(n-1)}{r^4} (\rho'' - \rho')^2 - \sum_{j=1}^{n-1} |\nabla^2 \varphi_i|^2 \right)
\]

\[
\leq \sum_{i=1}^{n-1} \frac{1}{\Lambda_i(\Omega)} |\nabla^2 \varphi_i|^2 + \sum_{j=1}^{n-1} \frac{1}{\Lambda_j(\Omega)} \left( \frac{(\rho'')^2 + 3(n-1)}{r^4} (\rho'' - \rho')^2 - |\nabla^2 \varphi_i|^2 \right)
\]

\[
= \frac{1}{n-1} \left( \frac{(\rho'')^2 + 3(n-1)}{r^4} (\rho'' - \rho')^2 \right) \sum_{i=1}^{n-1} \frac{1}{\Lambda_i(\Omega)}.
\]
Similarly, we have

\[
\sum_{i=1}^{n} \frac{1}{\Lambda_i(\Omega)} |\nabla \phi_i|^2 = \sum_{i=1}^{n-1} \frac{1}{\Lambda_i(\Omega)} |\nabla \phi_i|^2 + \frac{1}{\Lambda_n(\Omega)} |\nabla \phi_i|^2
\]

\[
= \sum_{i=1}^{n-1} \frac{1}{\Lambda_i(\Omega)} |\nabla \phi_i|^2 + \frac{1}{\Lambda_n(\Omega)} \left( \frac{n-1}{r^2} \rho^2 + (\rho')^2 - \sum_{j=1}^{n-1} |\nabla \phi_j|^2 \right)
\]
\[
\leq \sum_{i=1}^{n-1} \frac{1}{\Lambda_i(\Omega)} |\nabla \phi_i|^2 + \sum_{j=1}^{n-1} \frac{1}{\Lambda_j(\Omega)} \left( \frac{n-1}{r^2} \rho^2 + (\rho')^2 - \sum_{i=1}^{n-1} |\nabla \phi_i|^2 \right)
\]
\[
= \frac{1}{n-1} \left( \frac{n-1}{r^2} \rho^2 + (\rho')^2 \right) \sum_{i=1}^{n-1} \frac{1}{\Lambda_i(\Omega)}. \quad (3.7)
\]

On the other hand,

\[
\sum_{i=1}^{n} |\phi_i|^2 = \rho^2. \quad (3.8)
\]

Substituting (3.6)–(3.8) into (3.5), we have

\[
\frac{1}{n-1} \sum_{i=1}^{n-1} \frac{1}{\Lambda_i(\Omega)} \geq \frac{\int_{\Omega} \rho^2 \, dx}{\int_{\Omega} (\rho')^2 + \frac{2(n-1)}{r^2} (\rho - \rho')^2 + \tau \left( \frac{n-1}{r^2} \rho^2 + (\rho')^2 \right) \, dx}
\]
\[
= \frac{\int_{\Omega} \rho^2 \, dx}{\int_{\Omega} N[\rho] \, dx} \geq \frac{\int_{B_\Omega} \rho^2 \, dx}{\int_{B_\Omega} N[\rho] \, dx} = \frac{1}{\Lambda_1(B_\Omega)}. \quad (3.9)
\]

the last step is deduced by Lemma 2.2. If the equality holds, then equality holds in (3.9), which implies \( \Omega \) must be a unit ball. By Lemma 2.3, for any domain \( \Omega \) in \( \mathbb{R}^n \), we get

\[
\frac{1}{n-1} \sum_{i=1}^{n-1} \frac{1}{\Lambda_i(\Omega)} \geq \frac{1}{\Lambda_1(B_\Omega)}. \quad (3.10)
\]

This completes the proof of Theorem 3.1. \( \square \)

**Theorem 3.2** Let \( \Omega \) be a bounded domain in an \( n \)-dimensional Euclidean space \( \mathbb{R}^n \) and let \( B_\Omega \) be the ball of same volume as \( \Omega \), then the first \((n-1)\) eigenvalues of (1.5) in \( \Omega \) satisfy

\[
\frac{1}{n-1} \sum_{i=1}^{n-1} \Lambda_i(\Omega) \leq (1 - \eta_{n,t;\Omega} A^2(\Omega)) \Lambda_1(B_\Omega). \quad (3.11)
\]

**Proof** Case I. \( \Omega \) is a bounded domain in \( \mathbb{R}^n \) of class \( C^1 \) with the same measure as the unit ball \( B \). By a similar argument as in the proof of Theorem 3.1, we have

\[
\Lambda_i(\Omega) \int_{\Omega} \phi_i^2 \, dx \leq \int_{\Omega} \left( |\nabla^2 \phi_i|^2 + \tau |\nabla \phi_i|^2 \right) \, dx, \quad i = 1, \ldots, n. \quad (3.12)
\]
Summing over $i$ from 1 to $n$, we have

$$
\sum_{i=1}^{n} \Lambda_i(\Omega) \int_{\Omega} \phi_i^2 \, dx \leq \sum_{i=1}^{n} \int_{\Omega} \left( |\nabla \phi_i|^2 + \tau |\nabla \psi_i|^2 \right) \, dx = \int_{\Omega} \mathcal{N}[\rho] \, dx.
$$

(3.13)

Since $\sum_{i=1}^{n} \phi_i^2 = \rho^2$, for any point $p \in \Omega$, by a transformation of coordinates if necessary, we have $\phi_i^2 \leq \frac{\rho^2}{n-1}$, $i = 1, \ldots, n$. Then

$$
\sum_{i=1}^{n} \Lambda_i(\Omega) \phi_i^2 = \sum_{i=1}^{n-1} \Lambda_i(\Omega) \phi_i^2 + \Lambda_n(\Omega) \phi_n^2
$$

$$
= \sum_{i=1}^{n-1} \Lambda_i(\Omega) \phi_i^2 + \Lambda_n(\Omega) \left( \rho^2 - \sum_{j=1}^{n-1} \phi_j^2 \right)
$$

$$
\geq \sum_{i=1}^{n-1} \Lambda_i(\Omega) \phi_i^2 + \sum_{j=1}^{n-1} \Lambda_j \left( \frac{\rho^2}{n-1} - \phi_j^2 \right)
$$

$$
= \sum_{i=1}^{n-1} \Lambda_i \frac{\rho^2}{n-1}.
$$

(3.14)

Substituting (3.13) into (3.14), we have

$$
\frac{1}{n-1} \sum_{i=1}^{n-1} \Lambda_i(\Omega) \leq \frac{\int_{\Omega} \mathcal{N}[\rho] \, dx}{\int_{\Omega} \rho^2 \, dx}.
$$

(3.15)

On the other hand, we have

$$
\Lambda_1(B) \int_{\Omega} \rho^2 \, dx - \frac{1}{n-1} \sum_{i=1}^{n-1} \Lambda_i(\Omega) \int_{\Omega} \rho^2 \, dx \geq \int_{B} \mathcal{N}[\rho] \, dx - \int_{\Omega} \mathcal{N}[\rho] \, dx.
$$

(3.17)

From equation (16) in [4], we know that

$$
\Lambda_1(B) \int_{\Omega} \rho^2 \, dx - \Lambda_1(\Omega) \int_{\Omega} \rho^2 \, dx \leq C^{(1)}_{n,\tau} (\Lambda_1(B) - \Lambda_1(\Omega)),
$$

where $C^{(1)}_{n,\tau} = n \omega_n \int_0^1 \rho^2(r) \rho^{n-1} \, dr$. Then we have

$$
\Lambda_1(B) \int_{\Omega} \rho^2 \, dx - \frac{1}{n-1} \sum_{i=1}^{n-1} \Lambda_i(\Omega) \int_{\Omega} \rho^2 \, dx \leq C^{(1)}_{n,\tau} \left( \Lambda_1(B) - \frac{1}{n-1} \sum_{i=1}^{n-1} \Lambda_i(\Omega) \right).
$$

(3.18)

From (15) and (20) in [4], we know that

$$
\Lambda_1(B) \int_{\Omega} \rho^2 \, dx - \Lambda_1(\Omega) \int_{\Omega} \rho^2 \, dx \geq \int_{B \cap B_1} \mathcal{N}(\rho) \, dx - \int_{B \setminus B_1} \mathcal{N}(\rho) \, dx,
$$
and
\[
\int_{B_1} N(\rho) \, dx - \int_{B_2} N(\rho) \, dx = C^{(2)}_{n, \tau} a^2,
\]
where \(B_1\) and \(B_2\) are two balls centered at the origin with radii \(r_1, r_2\) such that \(|\Omega \cap B| = |B_1| = \omega_n r_1^n\) and \(|\Omega/B| = |B_2/B| = \omega_n (r_2^n - 1)\). Then we have
\[
\int_B N[\rho] \, dx - \int_{\Omega} N[\rho] \, dx \geq C^{(2)}_{n, \tau} |\Omega B| / |\Omega|,
\]
where \(C^{(2)}_{n, \tau} = n \omega_n ((3 + \tau)(R(1) - R'(1))^2 + 2\tau R'(1)(R(1) - R'(1))) c_n\).

Combining (3.18) and (3.19), we have
\[
\Lambda_1(B) - \frac{1}{n - 1} \sum_{i=1}^{n-1} \Lambda_i(\Omega) \geq \frac{C^{(2)}_{n, \tau}}{C^{(1)}_{n, \tau}} A^2(\Omega),
\]
which implies that
\[
\frac{1}{n - 1} \sum_{i=1}^{n-1} \Lambda_i(\Omega) \leq \Lambda_1(B) \left(1 - \frac{C^{(2)}_{n, \tau}}{C^{(1)}_{n, \tau}} A^2(\Omega)\right).
\]

Case 2. \(\Omega\) is the generic domain in \(\mathbb{R}^n\) of class \(C^1\). Since
\[
\Lambda_i(\tau, \Omega) = s^4 A_i(s^{-2} \tau, s\Omega), \quad i = 1, \ldots, n,
\]
for all \(s > 0\). Taking \(s = (\omega_n / |\Omega|)^{1/2}\), for any domain \(\Omega\) in \(\mathbb{R}^n\) of class \(C^1\), we infer from (3.21) that
\[
\frac{1}{n - 1} \sum_{i=1}^{n-1} \Lambda_i(\tau, \Omega) = s^4 \frac{1}{n - 1} \sum_{i=1}^{n-1} \Lambda_i(s^{-2} \tau, s\Omega) \\
\leq s^4 A_1(s^{-2} \tau, B) \left(1 - \frac{C^{(2)}_{n, s^{-2} \tau}}{A_1(s^{-2} \tau, B) C^{(1)}_{n, s^{-2} \tau}} A^2(s\Omega)\right) \\
= A_1(s^{-2} \tau, B) \left(1 - \frac{C^{(2)}_{n, s^{-2} \tau}}{A_1(s^{-2} \tau, B) C^{(1)}_{n, s^{-2} \tau}} A^2(\Omega)\right).
\]

Setting \(\eta_{n, \tau, |\Omega|} = \frac{C^{(2)}_{n, s^{-2} \tau}}{A_1(s^{-2} \tau, B) C^{(1)}_{n, s^{-2} \tau}}\), we have (1.10). This completes the proof of Theorem 3.2. \(\square\)

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