Mellin Transforms of the Generalized Fractional Integrals and Derivatives

Udita N. Katugampola

Department Of Mathematics, Delaware State University, Dover DE 19901, USA

Abstract

The purpose of this paper is to present the Mellin transforms of the fractional integrals and derivatives, which generalize the Riemann-Liouville and Hadamard fractional integrals and derivatives, respectively. We also obtain an interesting result, which combines the fractional derivatives with Stirling numbers of the second kind and Lah numbers.

Keywords:
Generalized fractional derivative, Riemann-Liouville fractional derivative, Hadamard fractional derivative, Mellin’s transform, $\delta$ - derivative, Umbral calculus, Stirling numbers of the 2nd kind, Lah numbers

2008 MSC: 26A33

1. Introduction

Fractional calculus is the generalization of the classical calculus to arbitrary orders. The history of Fractional Calculus (FC) goes back to seventeenth century, when in 1695 the derivative of order $\alpha = 1/2$ was described by Leibnitz. Since then, the new theory turned out to be very attractive to mathematicians as well as biologists, economists, engineers and physicists. Several books were written on the theories and developments of FC [18, 21, 31, 22, 23]. In [31] Samko et al, provide a comprehensive study of the subject. Several different derivatives were studied: Riemann-Liouville, Caputo, Hadamard, Grunwald-Letnikov and Riesz are just a few to name.

In fractional calculus, the fractional derivatives are defined via a fractional integrals [21, 31, 22, 23]. According to the literature, the Riemann-Liouville fractional derivative(RLFC), hence Riemann-Liouville fractional integral have played major roles in FC [31]. Caputo fractional derivative has also been defined via Riemann-Liouville fractional integral. Butzer, et al., investigate properties of Hadamard fractional integral and derivative [3, 4, 5, 15, 17, 18, 21, 31]. In [5], they obtain the Mellin transform of the Hadamard integral and differential operators. Many of those results are summarized in [18] and [31].

In [13], the author introduces a new fractional integral, which generalizes Riemann-Liouville and Hadamard integrals into a single form. In [12, 14, the author introduces new fractional deriva-
tives which generalize the two derivatives in question. The present work is devoted to the Mellin Transforms of the generalized fractional operators developed in [13] and [14].

The paper is organized in the following order. In the next section we give definitions and some properties of fractional integrals and fractional derivatives of various types. We develop Mellin transforms of the generalized fractional integrals and derivatives in the following section. Finally we investigate the relationship between the Mellin transform operator and Stirling numbers of the 2nd kind.

The Riemann-Liouville fractional integrals (RLFI) $I_{a+}^\alpha f$ and $I_{b-}^\alpha f$ of order $\alpha \in \mathbb{C}(\text{Re}(\alpha) > 0)$ are defined by [31],

$$I_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-\tau)^{\alpha-1} f(\tau) d\tau \quad ; x > a.$$  (1)

and

$$I_{b-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (\tau-x)^{\alpha-1} f(\tau) d\tau \quad ; x < b.$$  (2)

respectively. Here $\Gamma(\alpha)$ is the Gamma function. These integrals are called the left-sided and right-sided fractional integrals respectively. When $\alpha = n \in \mathbb{N}$, the integrals (1) and (2) coincide with the $n$-fold integrals [18, chap.2]. The Riemann-Liouville fractional derivatives (RLFD) $D_{a+}^\alpha f$ and $D_{b-}^\alpha f$ of order $\alpha \in \mathbb{C}, \text{Re}(\alpha) \geq 0$ are defined by [31],

$$D_{a+}^\alpha f(x) = \left( \frac{d}{dx} \right)^n \left( I_{a+}^{n-\alpha} f \right)(x) \quad x > a,$$  (3)

and

$$D_{b-}^\alpha f(x) = \left( - \frac{d}{dx} \right)^n \left( I_{b-}^{n-\alpha} f \right)(x) \quad x < b,$$  (4)

respectively, where $n = \lceil \text{Re}(\alpha) \rceil$. For simplicity, from this point onwards we consider only the left-sided integrals and derivatives. The interested reader can find more detailed information about right-sided integrals and derivatives in the references.

The next important type is the Hadamard Fractional integral introduced by J. Hadamard [10, 18, 31], and given by,

$$I_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left( \log \frac{x}{\tau} \right)^{\alpha-1} f(\tau) \frac{d\tau}{\tau} \quad ; \text{Re}(\alpha) > 0, x > a \geq 0.$$  (5)

while Hadamard fractional derivative of order $\alpha \in \mathbb{C}, \text{Re}(\alpha) > 0$ is given by,

$$D_{a+}^n f(x) = \frac{1}{\Gamma(n-\alpha)} \left( \frac{d}{dx} \right)^n \int_a^x \left( \log \frac{x}{\tau} \right)^{n-\alpha+1} f(\tau) \frac{d\tau}{\tau} \quad ; x > a \geq 0.$$  (6)

where $n = \lceil \text{Re}(\alpha) \rceil$.

In [13], the author introduces a generalization to the Riemann-Liouville and Hadamard fractional integral and also provided existence results and semigroup properties. In [14], author introduces a generalization to the Riemann-Liouville and Hadamard fractional derivatives. This new operators have been defined on the following space;
1.1. Generalized fractional integration and differentiation

As in [15], consider the space \( X^{\rho}_{c}(a, b) \) \((c \in \mathbb{R}, 1 \leq p \leq \infty)\) of those complex-valued Lebesgue measurable functions \( f \) on \([a, b]\) for which \( \|f\|_{X^{\rho}_{c}} < \infty \), where the norm is defined by

\[
\|f\|_{X^{\rho}_{c}} = \left( \int_{a}^{b} |f(t)|^{p} \frac{dt}{t} \right)^{1/p} < \infty \quad (1 \leq p < \infty, c \in \mathbb{R})
\]

and for the case \( p = \infty \)

\[
\|f\|_{X^{\rho}_{\infty}} = \text{ess sup}_{a \leq t \leq b} |f(t)| \quad (c \in \mathbb{R}).
\]

In particular, when \( c = 1/p \) \((1 \leq p \leq \infty)\), the space \( X^{\rho}_{c} \) coincides with the classical \( L^{p}(a, b)\)-space with

\[
\|f\|_{p} = \left( \int_{a}^{b} |f(t)|^{p} \, dt \right)^{1/p} < \infty \quad (1 \leq p < \infty),
\]

\[
\|f\|_{\infty} = \text{ess sup}_{a \leq t \leq b} |f(t)| \quad (c \in \mathbb{R}).
\]

Here we give the generalized definitions of the fractional integrals introduced in [13] with a slight modification in the notations.

**Definition 1.1.** (Generalized Fractional Integrals)

Let \( \Omega = [a, b] \) \((-\infty < a < b < \infty)\) be a finite interval on the real axis \( \mathbb{R} \). The generalized *left-sided* fractional integral \( \mathcal{I}^{\alpha}_{a+} f \) of order \( \alpha \in \mathbb{C} \) \((\text{Re}(\alpha) > 0)\) of \( f \in X^{\rho}_{c}(a, b) \) is defined by

\[
(\mathcal{I}^{\alpha}_{a+} f)(x) = \frac{\rho_{1-\alpha}}{\Gamma(\alpha)} \int_{a}^{x} \frac{\tau^{\rho-1} f(\tau)}{(\tau^{\rho} - x^{\rho})^{1-\alpha}} \, d\tau
\]

for \( x > a \) and \( \text{Re}(\alpha) > 0 \). The *right-sided* generalized fractional integral \( \mathcal{I}^{\alpha}_{b-} f \) is by

\[
(\mathcal{I}^{\alpha}_{b-} f)(x) = \frac{\rho_{1-\alpha}}{\Gamma(\alpha)} \int_{x}^{b} \frac{\tau^{\rho-1} f(\tau)}{(\tau^{\rho} - x^{\rho})^{1-\alpha}} \, d\tau
\]

for \( x < b \) and \( \text{Re}(\alpha) > 0 \). When \( b = \infty \), the generalized fractional integral is called a Liouville-type integral.

Now consider the generalized fractional derivatives introduced in [12, 14].

**Definition 1.2.** (Generalized Fractional Derivatives)

Let \( \alpha \in \mathbb{C}, \text{Re}(\alpha) \geq 0, n = \lfloor \text{Re}(\alpha) \rfloor + 1 \) and \( \rho > 0 \). The generalized fractional derivatives, corresponding to the generalized fractional integrals (11) and (12), are defined, for \( 0 \leq a < x < b \leq \infty \), by

\[
(\mathcal{D}^{\alpha}_{a+} f)(x) = \frac{\rho_{1-n+1}}{\Gamma(\alpha)} \left( x^{1-\rho} \frac{d}{dx} \right)^{n} \left( \mathcal{I}^{n-\alpha}_{a+} f \right)(x)
\]

\[
= \frac{\rho_{1-\alpha}}{\Gamma(\alpha)} \left( x^{1-\rho} \frac{d}{dx} \right)^{n} \int_{a}^{x} \frac{\tau^{\rho-1} f(\tau)}{(\tau^{\rho} - x^{\rho})^{1-\alpha}} \, d\tau
\]

and

\[
(\mathcal{D}^{\alpha}_{b-} f)(x) = \frac{\rho_{1-n+1}}{\Gamma(\alpha)} \left( - x^{1-\rho} \frac{d}{dx} \right)^{n} \left( \mathcal{I}^{n-\alpha}_{b-} f \right)(x)
\]

\[
= \frac{\rho_{1-\alpha}}{\Gamma(\alpha)} \left( - x^{1-\rho} \frac{d}{dx} \right)^{n} \int_{x}^{b} \frac{\tau^{\rho-1} f(\tau)}{(\tau^{\rho} - x^{\rho})^{1-\alpha}} \, d\tau
\]
if the integrals exist. When $b = \infty$, the generalized fractional derivative is called a Liouville-type derivative.

Next we give several important theorems without proofs. Interested reader can find them in [14].

**Theorem 1.3.** Generalized fractional derivatives ([13] and [14]) exist finitely.

The following theorem gives the relations of generalized fractional derivatives to that of Riemann-Liouville and Hadamard. For simplicity we give only the left-sided versions here.

**Theorem 1.4.** Let $\alpha \in \mathbb{C}, \Re(\alpha) \geq 0$, $n = \lfloor \Re(\alpha) \rfloor$ and $\rho > 0$. Then, for $x > a$,

1. $\lim_{\rho \to 1} (\rho^\alpha D^\rho_a f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - \tau)^{\alpha - 1} f(\tau) d\tau$, \hspace{1cm} (15)
2. $\lim_{\rho \to 0} (\rho^\alpha D^\rho_a f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left( \frac{1}{\tau} \right)^{\alpha - 1} f(\tau) \frac{d\tau}{\tau}$, \hspace{1cm} (16)
3. $\lim_{\rho \to 1} (\rho^\alpha D^\rho_b f)(x) = \frac{d}{dx} \frac{1}{\Gamma(\alpha)} \int_a^x \left( \frac{1}{\tau} \right)^{\alpha - 1} f(\tau) \frac{d\tau}{\tau}$, \hspace{1cm} (17)
4. $\lim_{\rho \to 0} (\rho^\alpha D^\rho_b f)(x) = \frac{1}{\Gamma(\alpha)} \left( \frac{d}{dx} \right)^n \int_a^x \left( \frac{1}{\tau} \right)^{\alpha - n} f(\tau) \frac{d\tau}{\tau}$, \hspace{1cm} (18)

**Remark 1.5.** Note that the equations (15) and (17) are related to Riemann-Liouville operators, while equations (16) and (19) are related to Hadamard operators.

Next is the inverse property.

**Theorem 1.6.** Let $0 < \alpha < 1$, and $f(x)$ be continuous. Then, for $a > 0$, $\rho > 0$,

\[ \left( \rho^\alpha D^\rho_a \rho^\alpha I^\alpha_a \right) f(x) = f(x). \] \hspace{1cm} (19)

Compositions between the operators of generalized fractional differentiation and generalized fractional integration are given by the following theorem.

**Theorem 1.7.** Let $\alpha, \beta \in \mathbb{C}$ be such that $0 < \Re(\alpha) < \Re(\beta) < 1$. If $0 < a < b < \infty$ and $1 \leq \rho \leq \infty$, then, for $f \in L^\rho(a, b)$, $\rho > 0$,

\[ \rho^\alpha D^\rho_a \rho^\beta I^\beta_a f = \rho^\beta I^\beta_a \rho^\alpha D^\rho_a f \quad \text{and} \quad \rho^\alpha D^\rho_b \rho^\beta I^\beta_b f = \rho^\beta I^\beta_b \rho^\alpha D^\rho_b f. \]

**Example 1.8.** For example, we give the generalized fractional derivative ([13]) of the power function and show the behavior for different values of $\rho$, which is the parameter used to control characteristics of the derivative. For simplicity assume $\alpha \in \mathbb{R}^+$, $0 < \alpha < 1$ and $a = 0$. As derived in [14], the generalized derivative of the function $f(x) = x^n$, where $n \in \mathbb{R}$ is given by

\[ \rho^\alpha D^\rho_0 x^n = \frac{\Gamma(1 + \frac{n}{\rho})}{\Gamma(1 + \frac{n}{\rho} - \alpha)} x^{n - \alpha}. \] \hspace{1cm} (20)

for $\rho > 0$. When $\rho = 1$ we obtain the Riemann-Liouville fractional derivative of the power function given by [18, 21, 31],

\[ 1^\alpha D^\rho_0 x^n = \frac{\Gamma(1 + \frac{n}{\rho})}{\Gamma(1 + \frac{n}{\rho} - \alpha)} x^{n - \alpha}. \] \hspace{1cm} (21)
This agrees well with the standard results obtained for Riemann-Liouville fractional derivative (3). Interestingly enough, for $\alpha = 1, \rho = 1$, we obtain $^1D_0^1 x^\nu = \nu x^{\nu-1}$, as one would expect.

To compare results, we plot (20) for several values of $\rho, \nu$ and $\alpha$. The results are summaries in Figure 1. It is apparent from the figure that the effect of changing the parameters is more visible for different values of $\alpha$, which is related to the fractional effect of the derivative.

2. Mellin Transforms of generalized fractional operators

According to Flajolet et al [8], Hjalmar Mellin (1854-1933) gave his name to the Mellin transform that associates to a function $f(x)$ defined over the positive reals the complex function $Mf$ [20]. It is closely related to the Laplace and Fourier transforms.

---

1H. Mellin was a Finnish and his advisor was Gosta Mittag-Leffler, a Swedish mathematician. Later he also worked with Kurt Weierstrass [13].
We start by recalling the important properties of the Mellin transform. The domain of definition is an open strip, \(<a, b>\), say, of complex numbers \(s = \sigma + it\) such that \(0 \leq a < \sigma < b\). Here we adopt the definitions and properties mentioned in [8] with some minor modifications to the notations.

**Definition 2.1.** (Mellin transform) Let \(f(x)\) be locally Lebesgue integrable over \((0, \infty)\). The *Mellin transform* of \(f(x)\) is defined by

\[
M[f](s) = \int_0^\infty x^{s-1} f(x) \, dx.
\]

The largest open strip \(<a, b>\) in which the integral converges is called the **fundamental strip**.

Following theorem will be of great importance in applications.

**Theorem 2.2** ([8], Theorem 1). Let \(f(x)\) be a function whose transform admits the fundamental strip \(<a, b>\). Let \(p\) be a nonzero real number, and \(\mu, \nu\) be positive reals. Then,

1. \(M\left[ \sum_x \lambda_x f(\mu x) \right](s) = \left( \sum_x \frac{\lambda_x}{\mu^s} \right) M[f](s), \quad I \text{ finite, } \lambda_x > 0, \ s \in <a, b>\)
2. \(M[x^i f(x)](s) = M[f](s + \nu) \quad s \in <a, b>\)
3. \(M[f(x^i)](s) = \frac{1}{\rho} M[f]\left(\frac{s}{\rho}\right), \quad s \in \rho a, \rho b>\)
4. \(M\left[ \frac{d}{dx} f(x) \right](s) = (1 - s) M[f](s - 1)\)
5. \(M\left[ x \frac{d}{dx} f(x) \right](s) = -s M[f](s)\)
6. \(M\left[ \int_0^\infty f(t) dt \right](s) = -\frac{1}{s} M[f](s + 1)\)

Next we cite inversion theorem for Mellin transform.

**Theorem 2.3** ([8], Theorem 2). Let \(f(x)\) be integrable with fundamental strip \(<a, b>\). If \(c\) is such that \(a < c < b\), and \(M[f](c + it)\) integrable, then the equality,

\[
\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} M[f](s) x^{-s} \, ds = f(x)
\]

holds almost everywhere. Moreover, if \(f(x)\) is continuous, then equality holds everywhere on \((0, \infty)\).

Following is the first most important result of the paper, i.e., the Mellin transforms of the generalized fractional integrals. We give both the **left-sided** and **right-sided** versions of the results.

**Lemma 2.4.** Let \(\alpha \in \mathbb{C}, \text{Re}(\alpha) > 0, \text{and } \rho > 0.\) Then,

\[
M(\mathcal{I}_a \alpha^\rho f)(s) = \frac{\Gamma(1 - \frac{\alpha}{\rho} - \alpha)}{\Gamma(1 - \frac{\alpha}{\rho}) \rho^\alpha} M[f(s + \alpha)], \quad \text{Re}(s/\rho + \alpha) < 1, \ x > a, \tag{22}
\]

\[
M(\mathcal{I}_{b-a} \alpha^\rho f)(s) = \frac{\Gamma\left(\frac{s}{\rho} + \alpha\right)}{\Gamma\left(\frac{s}{\rho} + \alpha\right) \rho^\alpha} M[f(s + \alpha)], \quad \text{Re}(s/\rho) > 0, \ x < b, \tag{23}
\]

for \(f \in X^\rho_{-\alpha}(\mathbb{R}^+), \) if \(M[f(s + \alpha)]\) exists for \(s \in \mathbb{C}\).
Proof. We again use Fubinis theorem and Dirichlet technique here. By definition \(2.1\) and \(11\), we have
\[
M(\rho^a \Gamma_a f)(s) = \int_0^\infty x^{s-1} \frac{\rho^{1-a}}{\Gamma(a)} \int_0^x (x^\rho - t^\rho)^{a-1} t^{a-1} f(t) \, dt \, dx,
\]
\[
= \frac{\rho^{1-a}}{\Gamma(a)} \int_0^\infty t^{s-1} f(t) \int_0^t x^{s-1} (x^\rho - t^\rho)^{a-1} x \, dx \, dt
\]
\[
= \frac{\rho^{1-a}}{\Gamma(a)} \int_0^\infty t^{s-1} f(t) \int_0^t u^{\frac{s}{\rho} - (1 - a)} (1 - a) \, du \, dt
\]
\[
= \frac{\rho^{1-a} \Gamma(1 - \frac{s}{\rho} - a)}{\Gamma(1 - \frac{s}{\rho})} \int_0^\infty t^{s-1} f(t)
\]
\[
= \frac{\Gamma(1 - \frac{s}{\rho} - a)}{\Gamma(1 - \frac{s}{\rho})} \rho^a M[f](s + a \rho) \quad \text{for } \text{Re}(s + a \rho) < 1.
\]

after using the change of variable \(u = (\tau/\chi)^\rho\), and properties of the Beta function. This proves \(22\). The proof of \(23\) is similar. \(\square\)

Remark 2.5. The two transforms above confirm Lemma 2.15 of [18] for \(\rho = 1\). For the case, when \(\rho \to 0^+\), consider the quotient expansion of two Gamma functions at infinity given by [7].

\[
\frac{\Gamma(z + a)}{\Gamma(z + b)} = \varepsilon^{a-b} \left[1 + O\left(\frac{1}{z}\right)\right] \quad |\text{arg}(z + a)| < \pi; \quad |z| \to \infty.
\]

(24)

Then, it is clear that,
\[
\lim_{\rho \to 0^+} (\rho^a \Gamma_a f)(s) = \lim_{\rho \to 0^+} \frac{\Gamma(1 - \frac{s}{\rho} - a)}{\Gamma(1 - \frac{s}{\rho})} \rho^a M[f](s + a \rho)
\]
\[
= (-s)^{-a} M[f](s) \quad \text{Re}(s) < 0,
\]

which confirms Lemma 2.38(a) of [18].

To prove the next result we use the Mellin transform of \(m\)-th derivative of a \(m\)-times differentiable function given by,

Lemma 2.6 ([18], p.21). Let \(\varphi \in C^m(\mathbb{R}^+)\), the Mellin transforms \(M[\varphi(t)](s - m)\) and \(M[D^m \varphi(t)](s)\) exist, and \(\lim_{t \to 0^+} [t^{s-k-1} \varphi^{(m-k-1)}(t)]\) and \(\lim_{t \to +\infty} [t^{s-k-1} \varphi^{(m-k-1)}(t)]\) are finite for \(k = 0, 1, \ldots, m - 1, \quad m \in \mathbb{N}\), then
\[
M[D^m \varphi(t)](s) = \frac{\Gamma(1 + m - s)}{\Gamma(1 - s)} M[\varphi](s - m)
\]
\[
+ \sum_{k=0}^{m-1} \frac{\Gamma(1 + k - s)}{\Gamma(1 - s)} [x^{s-k-1} \varphi^{(m-k-1)}(s)]_{k=0}^\infty
\]
\[
M[D^m \varphi(t)](s) = (-1)^m \frac{\Gamma(s)}{\Gamma(s - m)} M[\varphi](s - m)
\]
\[
+ \sum_{k=0}^{m-1} (-1)^k \frac{\Gamma(s)}{\Gamma(s - k)} [x^{s-k-1} \varphi^{(m-k-1)}(s)]_{k=0}^\infty
\]
The next result is the Mellin transforms of the generalized fractional derivatives. For simplicity we consider only the case $0 < a < 1$ here. In this case, $n = [a] = 1$.

**Theorem 2.7.** Let $\alpha \in \mathbb{C}$, $Re(\alpha) > 0$, $s \in \mathbb{C}$, $\rho > 0$ and $f(x) \in X^{1-a}_{a^{-}}(\mathbb{R}^{+})$. Also assume $f(x)$ satisfies the following conditions:

$$\lim_{x \to 0^{+}} x^{-\rho} \left( p I_{a^{+}}^{1-a} f \right)(x) = \lim_{x \to 0^{-}} x^{-\rho} \left( p I_{a^{-}}^{1-a} f \right)(x) = 0,$$

and

$$\lim_{x \to 0^{+}} x^{-\rho} \left( p I_{a^{+}}^{1-a} f \right)(x) = \lim_{x \to 0^{-}} x^{-\rho} \left( p I_{a^{-}}^{1-a} f \right)(x) = 0,$$

respectively. Then,

$$M(p D_{a^{+}}^{\rho} f)(s) = \frac{\rho^\rho}{\Gamma(1 - \frac{s}{\rho} + \alpha) \Gamma(1 - \frac{s}{\rho})} M[f](s - \alpha \rho), \quad Re(s/\rho) < 1, \quad s > a \geq 0,$$

and

$$M(p D_{a^{-}}^{\rho} f)(s) = \frac{\rho^\rho \Gamma(\frac{s}{\rho} - \alpha)}{\Gamma(\frac{s}{\rho} - \alpha)} M[f](s - \alpha \rho), \quad Re(s/\rho) > 0, \quad s < b \leq \infty,$$

respectively.

**Proof.** By definition (2.1) and (13), we have

$$M(p D_{a^{+}}^{\rho} f)(s) = \frac{\rho^\rho}{\Gamma(1 - \frac{s}{\rho} + \alpha) \Gamma(1 - \frac{s}{\rho})} \int_{0}^{\infty} x^{s-1} \left( x^\rho \frac{d}{dx} \right) \int_{0}^{x} \frac{x^{\rho-1}}{(x^\rho - \tau^\rho)^{\alpha + 1}} d\tau \ dx,$$

for $Re(s/\rho) < 1$, using (25) with $m = 1$ and (11). This establishes (27). The proof of (28) is similar.

**Remark 2.8.** The Mellin transforms of Riemann-Liouville and Hadamard fractional derivatives follow easily, i.e., the transforms above confirm Lemma 2.16 of [18] for $\rho = 1$, $b = \infty$ and Lemma 2.59 of [18] for the case when $\rho \to 0^{+}$ in view of (24).

**Remark 2.9.** For $Re(\alpha) > 1$, we need to replace $x^{1-\rho} \frac{d}{dx}$ in (29) by $(x^{1-\rho} \frac{d}{dx})^{[a]}$. In this case the equation becomes very complicated and introduces several interesting combinatorial problems. There is a well-developed branch of mathematics called Umbral Calculus [29, 30], which consider the case when $k \in \mathbb{N}$. Bucchianico and Lobe [2] provide an extensive survey of the subject. We discuss some of them here referring interested reader to the works of Rota, Robinson and Roman [24, 26, 27, 28, 30]. First consider the generalized $\delta_k$-derivative defined by

**Definition 2.10.** Let $k \in \mathbb{R}$. The generalized $\delta_k$-differential operator is defined by

$$\delta_k := x^k \frac{d}{dx}$$

(30)
The $n^{th}$ degree $\delta_k$-derivative operator has interesting properties. When $k = 1$ it generates the Stirling numbers of the Second kind, $S(n, k)$ [6], Sloane’s A008278 [32], while $k = 2$ generates the unsigned Lah numbers, $L(n, k)$, Sloane’s A008297 [32]. They have been well-studied and appear frequently in literature of combinatorics [1]. These can also be looked at as the number of ways to place $j$ non-attacking rooks on a Ferrer’s-board with certain properties based on $k$ [11].

Now, let $\delta_{k,j}^n, j = 1, 2, 3, \ldots, n; n \in \mathbb{N}, k \in \mathbb{R}$ be the $j^{th}$ coefficient of the expansion of $\delta_k^n$, in a basis, $\mathcal{B} = \left\{ x^{n(k-1)+1} \frac{d}{dx} x^{n(k-1)+2} \frac{d^2}{dx^2}, \ldots, x^{nk} \frac{d^n}{dx^n} \right\}$, i.e.,

$$\left(\frac{d^n}{dx^n}\right)^n \delta_{k,j}^n x^{n(k-1)+1} \frac{d}{dx} x^{n(k-1)+2} \frac{d^2}{dx^2} + \cdots + \delta_{kn} x^{nk} \frac{d^n}{dx^n}$$  \hspace{1cm} (31)

Figure 2 lists the cases for $k = 1$ and $k = 2$ in a triangular setting.

|   |   |   |
|---|---|---|
| 1 | 1 | 1 |
| 1 | 3 | 1 |
| 1 | 7 | 6 | 1 |
| 1 | 15 | 25 | 10 | 1 |
| 1 | 1 | 1 | 2 | 1 |
| 1 | 24 | 36 | 12 | 1 |
| 1 | 20 | 120 | 240 | 120 | 20 | 1 |

Figure 2: Coefficients of $n^{th}$ degree generalized $\delta$-derivative

The Stirling numbers of the second kind, $S(n, k)$ and unsigned Lah numbers, $L(n, k)$ are given by [1]

$$S(n, k) = \frac{1}{k!} \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^n : 0 \leq k \leq n \in \mathbb{N} \cup \{0\},$$  \hspace{1cm} (32)

and

$$L(n, k) = \binom{n-1}{k-1} \frac{n!}{k!} : 0 \leq k \leq n \in \mathbb{N} \cup \{0\},$$  \hspace{1cm} (33)

respectively.

A closed formula exists for $k = 2$, and asymptotic formula exists for $k = 1$ [8, 24]. Johnson [11] used analytical methods and Goldman [9] used algebraic methods to derive generating functions for $k = 1, 2$ given by

$$z(n, k) = \begin{cases} \exp \left\{ \frac{e^x - 1}{x} \right\} & \text{if } k = 1, \\ \exp \left\{ \frac{1}{x} [1 - (k-1)xy]^{1/(k-1)} - \frac{1}{x} \right\} & \text{if } k = 2. \end{cases}$$  \hspace{1cm} (34)

Remark 2.11. It would be very interesting if one could classify all or a large class of such $\delta_k$-sequences. This could be a direction for future research. For the moment we can name, $S(n, k)$ as $\delta_1$-type and $L(n, k)$ as $\delta_2$-type, and etc. We can even consider non-integer or irrational values of $k \in \mathbb{R}$. To our knowledge this would lead to a new classification or at least a new method to
represent some familiar sequences in combinatorial theory. Following would be the most general form to consider in such a research,

\[ \delta_{k,l} := x^k \left( \frac{d}{dx} \right)^l \]  

for \( k \in \mathbb{R} \) and \( l \in \mathbb{R} \).

**Remark 2.12.** In many cases we only considered *left-sided* derivatives and integrals. But *right-sided* operators can be treated similarly.

### 3. Conclusion

According to the Figure 1 we notice that the characteristics of the fractional derivative is highly affected by the value of \( \rho \), thus it provides a new direction for control applications.

The paper presents the Mellin transforms of the generalized fractional integrals and derivatives, which generalize the Riemann-Liouville and Hadamard fractional operators. In a future project, we will derive formulae for the Laplace and Fourier transforms for the generalized fractional operators. We already know that we can deduce the Hadamard and Riemann-Liouville operators for the special cases of \( \rho \). We want to further investigate the effect on the new parameter \( \rho \). We will also classify the \( \delta_{k,l} \) sequences and study the properties of generalized fractional derivatives. Those results will appear elsewhere.

**Acknowledgement.** Author thanks Philip Feinsilver, Department of Mathematics at Southern Illinois University, for pointing out interesting results related to the umbral calculus, Darin B. Johnson, Department of Mathematics at Delaware State University, for pointing out interesting connections to some literature in combinatorial theory and Jerzy Kocik, Department of Mathematics at Southern Illinois University for valuable comments and suggestions.

**References**

1. Bóna, M., *Introduction to Enumerative Combinatorics*, 1st Ed, McGraw-Hill, New York, 2007.
2. Buchheim, A.D. and Loeb D., *A Selected Survey of Umbral Calculus*, Elec. J. Combin., 3: Dynamical Surveys Section, #DS3 (1995).
3. Butzer, P. L., Kilbas, A. A., and Trujillo, J.J., *Compositions of Hadamard-type fractional integration operators and the semigroup property*, Journal of Mathematical Analysis and Applications, 269, (2002), 387-400.
4. Butzer, P. L., Kilbas, A. A., and Trujillo, J.J., *Fractional calculus in the Mellin setting and Hadamard-type fractional integrals*, Journal of Mathematical Analysis and Applications, 269, (2002), 1-27.
5. Butzer, P. L., Kilbas, A. A., and Trujillo, J.J., *Mellin transform analysis and integration by parts for Hadamard-type fractional integrals*, Journal of Mathematical Analysis and Applications, 270, (2002), 1-15.
6. Butzer, P. L., Kilbas, A. A., and Trujillo, J.J., *Stirling Functions of the Second Kind in the Setting of Difference and Fractional Calculus*, Numerical Functional Analysis and Optimization, 24(7&8), (2003), 673-711.
7. Erdélyi, A., Magnus, W., Oberhettinger, F., Tricomi, F.G., *Higher Transcendental Functions*, Vol 1. I-III, Krieger Pub., Melbourne, Florida, 1981.
8. Flajolet, P., Gourdon, X., Dumas, P., *Mellin transforms and asymptotics: Harmonic sums*, Theoretical Computer Science, 144(1-2), (1995), 3-58.
9. Goldman, J., Joschi, J., White, D., *Rook Theory I, Rook equivalence on Ferrers boards*, Proc. Amer. Math. Soc, 52, (1975), 485-492.
10. Hadamard, J., *Essai sur l'étude des fonctions données par leur développement de Taylor*, Journal of pure and applied mathematics, 4(8), (1892), 101-186.
[11] Johnson, D. B., *Topics in Probabilistic Combinatorics*, Ph.D. Dissertation, Southern Illinois University, Carbondale. MathSciNet (2009).
[12] Katugampola, U.N., *On Generalized Fractional Integrals and Derivatives*, Ph.D. Dissertation, Southern Illinois University, Carbondale, USA. August 2011.
[13] Katugampola, U.N., *New approach to a generalized fractional integral*, Applied Mathematics and Computation, 218(3), (2011), 860-865.
[14] Katugampola, U.N., *New approach to generalized fractional derivatives*, Submitted to Computers & Mathematics with Applications, arXiv:1106.0965v3.
[15] Kilbas, A., *Hadamard-type Fractional Calculus*, Journal of Korean Mathematical Society, 38(6), (2001), 1191-1204.
[16] Killbas, A. A., and Saigo, M., *H-Transforms: Theory and Applications*, Chapman & Hall/CRC, New York, 2004.
[17] Kilbas, A. A., and Trujillo, J.J., *Hadamard-type integrals as G-transforms*, Integral Transforms and Special Functions, 14(5), (2003), 413-427.
[18] Kilbas, A. A., Srivastava, H.M., and Trujillo, J.J., *Theory and Applications of Fractional Differential Equations*, Elsevier B.V., Amsterdam, Netherlands, 2006.
[19] Kiryakova, V., *Generalized fractional calculus and applications*, John Wiley & Sons Inc., New York, 1994.
[20] Lindelöf, E., Robert Hjalmar Mellin, *Advances in Mathematics*, 61(i-vi), (1933), (H. Mellin’s note’s and bibliography).
[21] Podlubny, I., *Fractional Differential Equations*, Academic Press, San Diego, California-U.S.A., 1999.
[22] Miller, K.S. and Ross, B., *An introduction to the fractional calculus and fractional differential equations*, Wiley, New York, 1993.
[23] Oldham, K. B. and Spanier, J., *The fractional calculus*, Academec Press, New York, 1974.
[24] Robbins, H., *A remark on Stirling’s formula*, Amer. Math. Monthly, 62, (1955), 26-29.
[25] Robinson, T.J., *Formal calculus and umbral calculus*, Elec. J. Combin., 17 (2010), #R95.
[26] Roman, S., *The theory of the umbral calculus I*, J. Math. Anal. Appl., 87(1), (1982), 58-115.
[27] Roman, S., *The theory of the umbral calculus II*, J. Math. Anal. Appl., 89(1), (1982), 290-311.
[28] Roman, S., *The theory of the umbral calculus III*, J. Math. Anal. Appl., 95(2), (1983), 528-563.
[29] Rota, G.-C., *Finite Operator Calculus*, Acaademic Press, New York, 1975.
[30] Rota, G.-C. and Roman, S., *The Umbral Calculus*, Adv. in Math. 27, (1978), 95-188.
[31] Samko, S.G., Kilbas, A.A., and Marichev, O.I., *Fractional Integrals and Derivatives. Theory and Applications*, Gordon and Breach, Yverdon et alibi, 1993.
[32] Sloane, N., *The On-line Encyclopedia of Integer Sequences*, [http://oeis.org/], (2011).
[33] The MacTutor History of Mathematics Archive, [http://www-history.mcs.st-andrews.ac.uk/], (2011).