On the maximal ideal space of even quasicontinuous functions on the unit circle

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Abstract

Let $PQC$ stand for the set of all piecewise quasicontinuous functions on the unit circle, i.e., the smallest closed subalgebra of $L^\infty(T)$ which contains the classes of all piecewise continuous function $PC$ and all quasicontinuous functions $QC = (C + H^\infty) \cap (C + \overline{H^\infty})$. We analyze the fibers of the maximal ideal spaces $M(PQC)$ and $M(QC)$ over maximal ideals from $M(\tilde{QC})$, where $\tilde{QC}$ stands for the $C^*$-algebra of all even quasicontinuous functions. The maximal ideal space $M(\tilde{QC})$ is described and partitioned into various subsets corresponding to different descriptions of the fibers.

1 Introduction

Let $L^\infty(T)$ stand for the $C^*$-algebra of all (complex-valued) Lebesgue measurable and essentially bounded functions on the unit circle $T = \{ t \in \mathbb{C} : |t| = 1 \}$, let $C(T)$ stand for the class of all continuous functions on $T$, and let $PC$ stand for the set of all piecewise continuous functions on $T$, i.e., all functions $f : T \to \mathbb{C}$ such that the one-sided limits $f(\tau \pm 0) = \lim_{\varepsilon \to +0} f(\tau e^{\pm i \varepsilon})$ exist at each $\tau \in T$. The class of quasicontinuous functions is defined by

$$QC = (C + H^\infty) \cap (C + \overline{H^\infty}),$$

where $H^\infty$ stands for the Hardy space consisting of all $f \in L^\infty(T)$ such that its Fourier coefficients $f_n = \frac{1}{2\pi} \int_0^{2\pi} f(e^{ix}) e^{-inx} \, dx$ vanish for all $n < 0$. The space $\overline{H^\infty}$ is the Hardy space of all functions $f \in L^\infty(T)$ such that $f_n = 0$ for all $n > 0$.

The Toeplitz and Hankel operators $T(a)$ and $H(a)$ with $a \in L^\infty(T)$ acting on $\ell^2(\mathbb{Z}_+)$ are defined by the infinite matrices

$$T(a) = (a_{j-k})_{j,k=0}^\infty, \quad H(a) = (a_{j+k+1})_{j,k=0}^\infty.$$
Quasicontinuous functions arise in connection with Hankel operators. Indeed, it is known that both $H(a)$ and $H(\bar{a})$ are compact if and only if $a \in QC$ (see, e.g., [1, Theorem 2.54]). Here, and what follows, $\tilde{a}(t) := a(t^{-1})$, $t \in \mathbb{T}$.

Sarason [9], generalizing earlier work of Gohberg/Krupnik [6] and Douglas [2], established necessary and sufficient conditions for Toeplitz operators $T(a)$ with $a \in PQC$ to be Fredholm. This result is based on two ingredients. Firstly, due to Widom’s formula $T(\tilde{a}b) = T(a)T(b) + H(a)H(\tilde{b})$, Toeplitz operators $T(a)$ with $a \in QC$ commute with other Toeplitz operators $T(b)$, $b \in L^\infty(\mathbb{T})$, modulo compact operators. Hence $C^*$-algebras generated by Toeplitz operators can be localized over $QC$. Secondly, in case of the $C^*$-algebra generated by Toeplitz operators $T(a)$ with $a \in PQC$, the local quotient algebras arising from the localization allow an explicit description, which is facilitated by the characterization of the fibers of the maximal ideal space $M(PQC)$ over maximal ideals $\xi \in M(QC)$. These underlying results were also developed by Sarason [8, 9], and we are going to recall them in what follows.

Let $\mathfrak{A}$ be a commutative $C^*$-algebra, and let $\mathfrak{B}$ be a $C^*$-subalgebra such that both contain the same unit element. Then there is a natural continuous map between the maximal ideal spaces,

$\pi : M(\mathfrak{A}) \to M(\mathfrak{B})$, \quad \alpha \mapsto \alpha|_{\mathfrak{B}}$

defined via the restriction. For $\beta \in M(\mathfrak{B})$ introduce

$M_\beta(\mathfrak{A}) = \{ \alpha \in M(\mathfrak{A}) : \alpha|_{\mathfrak{B}} = \beta \} = \pi^{-1}(\beta),$

which is called the fiber of $M(\mathfrak{A})$ over $\beta$. The fibers $M_\beta(\mathfrak{A})$ are compact subsets of $M(\mathfrak{A})$, and $M(\mathfrak{A})$ is the disjoint union of all $M_\beta(\mathfrak{A})$. Because $\mathfrak{A}$ and $\mathfrak{B}$ are $C^*$-algebras, $\pi$ is surjective, and therefore each fiber $M_\beta(\mathfrak{A})$ is non-empty (see, e.g., [1, Sect. 1.27]).

Corresponding to the embeddings between the $C^*$-algebras $C(\mathbb{T})$, $QC$, $PC$, and $PQC$, which are depicted in first diagram below, there are natural maps between the maximal ideal spaces shown in the second diagram:

$\begin{array}{ccc}
PQC & \overset{\cong}{\longrightarrow} & QC \\
\uparrow & & \uparrow \\
PC & \overset{\cong}{\longrightarrow} & C(\mathbb{T}) \\
\downarrow & & \downarrow \\
M(PQC) & \overset{\cong}{\longrightarrow} & M(QC) \\
\downarrow & & \downarrow \\
M(PC) \cong \mathbb{T} \times \{+1, -1\} & \overset{\cong}{\longrightarrow} & M(C(\mathbb{T})) \cong \mathbb{T}
\end{array}$

Therein the identification of $y \in M(PC)$ with $(\tau, \sigma) \in \mathbb{T} \times \{+1, -1\}$ is made through $y(f) = f(\tau \pm \sigma)$ for $\sigma = \pm 1$, $f \in PC$.

Let $M_\tau(QC)$ stand for the fiber of $M(QC)$ over $\tau \in \mathbb{T}$, i.e.,

$M_\tau(QC) = \{ \xi \in M(QC) : \xi(f) = f(\tau) \text{ for all } f \in C(\mathbb{T}) \},$

and define

$M_\tau^\pm(QC) = \left\{ \xi \in M(QC) : \xi(f) = 0 \text{ whenever } \limsup_{t \to \tau \pm 0} |f(t)| = 0 \text{ and } f \in QC \right\}.$

Both $M_\tau^+(QC)$ and $M_\tau^-(QC)$ are closed subsets of $M_\tau(QC)$. Sarason introduced another subset $M_0^\tau(QC)$ (to be defined in (2.3) below) and established the following result (see [9], or [1, Prop. 3.34]).
Proposition 1.1 Let $\tau \in \mathbb{T}$. Then

$$M_\tau^0(QC) = M_\tau^+(QC) \cap M_\tau^-(QC), \quad M_\tau^+(QC) \cup M_\tau^-(QC) = M_\tau(QC). \quad (1.1)$$

The previous definitions and observations are necessary to analyze the fibers of $M(PQC)$ over $\xi \in M(QC)$. In view of the second diagram above, for given $z \in M(PQC)$ we can define the restrictions $\xi = z|_{QC}$, $z|_{C(T)} \cong \tau \in \mathbb{T}$, and $y = z|_{PC} \cong (\tau, \sigma) \in \mathbb{T} \times \{+1, -1\}$. Note that $\xi \in M_\tau(QC)$. Consequently, one has a natural map

$$z \in M(PQC) \mapsto (\xi, \sigma) \in M(QC) \times \{+1, -1\}. \quad (1.2)$$

This map is injective because $PQC$ is generated by $PC$ and $QC$. Therefore, $M(PQC)$ can be identified with a subset of $M(QC) \times \{+1, -1\}$. With this identification, the fibers $M_\xi(PQC) = \{ z \in M(PQC) : z|_{QC} = \xi \}$ are given as follows (see [8], or [1, Thm. 3.36]).

Theorem 1.2 Let $\xi \in M_\tau(QC)$, $\tau \in \mathbb{T}$. Then

(a) $M_\xi(PQC) = \{ (\xi, +1) \}$ for $\xi \in M_\tau^+(QC) \setminus M_\tau^0(QC)$;

(b) $M_\xi(PQC) = \{ (\xi, -1) \}$ for $\xi \in M_\tau^-(QC) \setminus M_\tau^0(QC)$;

(c) $M_\xi(PQC) = \{ (\xi, +1), (\xi, -1) \}$ for $\xi \in M_\tau^0(QC)$.

In order to describe the content of this paper, let us consider what happens if one wants to develop a Fredholm theory for operators from the $C^*$-algebra generated by Toeplitz and Hankel operators with $PQC$-symbols [10]. In this situation, one cannot use localization over $QC$ because the commutativity property fails. However, one can localize over the $C^*$-algebra of all even quasicontinuous functions. Indeed, due to the identity $H(ab) = T(a)H(b) + H(a)T(b)$, any $T(a)$ with $a \in \widehat{QC}$ commutes with any $H(b)$, $b \in L^\infty(\mathbb{T})$, modulo compact operators. When faced with the problem of identifying the local quotient algebras, it is necessary to understand the fibers of $M(PQC)$ over $\eta \in M(QC)$. This is what this paper is about.

When $\widehat{QC}$ and the $C^*$-algebra $\tilde{C}(\mathbb{T})$ of all even continuous functions are added to the picture, one arrives at the following diagrams:

As before, the diagram on the left shows the embeddings of the $C^*$-algebras, and the one on the right displays the corresponding (surjective) mappings between the maximal ideal spaces. Here $\mathbb{T}_+ = \{ t \in \mathbb{T} : \text{Im}(t) > 0 \}$ and $\overline{\mathbb{T}}_+ = \mathbb{T}_+ \cup \{+1, -1\}$. The map $\Psi'$ is defined
in such a way that the pre-image of \( \tau \in \mathbb{T}_+ \) equals the set \( \{ \tau, \bar{\tau} \} \), which consists of either one or two points.

Recall that Theorem 1.2 describes the fibers of \( M(PQC) \) over \( \xi \in M(QC) \). Hence if we want to understand the fibers of \( M(PQC) \) over \( \eta \in M(\tilde{QC}) \), it is sufficient to analyze the fibers of \( M(QC) \) over \( \eta \in M(\tilde{QC}) \). Let

\[
\Psi : M(QC) \to M(\tilde{QC}), \quad \xi \mapsto \hat{\xi} := \xi|_{\tilde{QC}},
\]

be the (surjective) map shown in the previous diagram. For \( \eta \in M(\tilde{QC}) \) define

\[
M^n(QC) = \{ \xi \in M(QC) : \hat{\xi} = \eta \},
\]

the fiber of \( M(QC) \) over \( \eta \). Let us also define the fibers of \( M(\tilde{QC}) \) over \( \tau \in \mathbb{T}_+ \),

\[
M_\tau(\tilde{QC}) = \{ \eta \in M(\tilde{QC}) : \eta(f) = f(\tau) \text{ for all } f \in \tilde{C}(\mathbb{T}) \}.
\]

Notice that we have the disjoint unions

\[
M(QC) = \bigcup_{\eta \in M(\tilde{QC})} M^n(QC), \quad M(QC) = \bigcup_{\tau \in \mathbb{T}} M_\tau(QC), \quad M(\tilde{QC}) = \bigcup_{\tau \in \mathbb{T}_+} M_\tau(\tilde{QC}).
\]

Furthermore, it is easy to see that \( \Psi \) maps

\[
M_\tau(QC) \cup M_\tau(\tilde{QC})
\]

onto \( M_\tau(\tilde{QC}) \) for each \( \tau \in \mathbb{T}_+ \).

The main results of this paper concern the description of the fibers \( M^n(QC) \) and the decomposition of \( M_\tau(\tilde{QC}) \) into disjoint sets, analogous to the decomposition of \( M_\tau(QC) \) into the disjoint union of

\[
M^0_\tau(QC), \quad M^+_\tau(QC) \setminus M^0_\tau(QC), \quad \text{and} \quad M^-_\tau(QC) \setminus M^0_\tau(QC)
\]

(see Proposition 1.1). This will be done in Section 3. In Section 2 we establish auxiliary results. In Section 4 we describe the fibers \( M^n(PQC) \) of \( M(PQC) \) over \( \eta \in M(\tilde{QC}) \).

Some aspects of the relationship between \( M(QC) \) and \( M(\tilde{QC}) \) were already mentioned by Power [7]. They were used by Silbermann [10] to establish a Fredholm theory for operators from the \( \mathcal{C}^* \)-algebra generated by Toeplitz and Hankel operators with \( PQC \)-symbols. Our motivation for presenting the results of this paper comes from the goal of establishing a Fredholm theory and a stability theory for the finite section method for operators taken from the \( \mathcal{C}^* \)-algebra generated by the singular integral operator on \( \mathbb{T} \), the flip operator, and multiplication operators by (operator-valued) \( PQC \)-functions [5]. This generalizes previous work [3, 4] and requires the results established here.
2 Approximate identities and VMO

In order to examine the relationship between $M(QC)$ and $M(\overline{QC})$, we need to recall some results and definitions concerning $QC$ and $M(QC)$. For $\tau = e^{i\theta} \in \mathbb{T}$ and $\lambda \in \Lambda := [1, \infty)$ let us define the moving average,

$$(m_\lambda a)(\tau) = \frac{\lambda}{2\pi} \int_{\theta-\pi/\lambda}^{\theta+\pi/\lambda} a(e^{ix}) dx. \quad (2.1)$$

Since each pair $(\lambda, \tau) \in \Lambda \times \mathbb{T}$ induces a bounded linear functional $\delta_{\lambda, \tau} \in QC^*$,

$$\delta_{\lambda, \tau} : QC \rightarrow \mathbb{C}, \quad a \mapsto (m_\lambda a)(\tau), \quad (2.2)$$

the set $\Lambda \times \mathbb{T}$ can be identified with a subset of $QC^*$. In fact, we have the following result, where we consider the dual space $QC^*$ with the weak-* topology (see [1, Prop. 3.29]).

**Proposition 2.1** $M(QC) = (\text{clos}_{QC^*}(\Lambda \times \mathbb{T})) \setminus (\Lambda \times \mathbb{T}).$

For $\tau \in \mathbb{T}$, let $M^\tau(QC)$ denote the points in $M(QC)$ that lie in the weak-* closure of $\Lambda \times \{\tau\}$ regarded as a subset of $QC^*$,

$$M^\tau(QC) = M(QC) \cap \text{clos}_{QC^*}(\Lambda \times \{\tau\}). \quad (2.3)$$

Obviously, $M^\tau(QC)$ is a compact subset of the fiber $M_\tau(QC)$. We remark that here and in the above proposition one can use arbitrary approximate identities (in the sense of Section 3.14 in [1]) instead of the moving average (see [1, Lemma 3.31]).

For $a \in L^1(\mathbb{T})$ and $\tau = e^{i\theta} \in \mathbb{T}$, the integral gap $\gamma_\tau(a)$ of $a$ at $\tau$ is defined by

$$\gamma_\tau(a) := \limsup_{\delta \to +0} \left| \frac{1}{\delta} \int_{\theta-\delta}^{\theta+\delta} a(e^{ix}) dx - \frac{1}{\delta} \int_{\theta-\delta}^{\theta} a(e^{ix}) dx \right|. \quad (2.4)$$

It is well-known [8] that $QC = VMO \cap L^\infty(\mathbb{T})$, where $VMO \subset L^1(\mathbb{T})$ refers to the class of all functions with vanishing mean oscillation on the unit circle $\mathbb{T}$. We will not recall its definition here, but refer to [8, 9, 1]. In the following lemma (see [9] or [1, Lemma 3.33]), $VMO(I)$ stands for the class of functions with vanishing mean oscillation on an open subarc $I$ of $\mathbb{T}$. Furthermore, we identify a function $q \in QC$ with its Gelfand transform, a continuous function on $M(QC)$.

**Lemma 2.2**

(a) If $q \in VMO$, then $\gamma_\tau(q) = 0$ for each $\tau \in \mathbb{T}$.

(b) If $q \in VMO(a, \tau) \cap VMO(\tau, b)$ and $\gamma_\tau(q) = 0$, then $q \in VMO(a, b)$.

(c) If $q \in QC$ such that $q|_{M^\tau(QC)} = 0$ and if $p \in PC$, then $\gamma_\tau(pq) = 0$.

Let $\chi_+$ (resp., $\chi_-$) be the characteristic function of the upper (resp., lower) semi-circle. The next lemma is based on the preceding lemma.
Lemma 2.3 Let $q \in QC$.

(a) If $q$ is an odd function, i.e., $q(t) = -q(1/t)$, then $q|_{M^0_+\{1\}(QC)} = 0$ and $q|_{M^0_-\{1\}(QC)} = 0$.

(b) If $q|_{M^0_+\{1\}(QC)} = 0$, then $pq \in QC$ whenever $p \in PC \cap C(\mathbb{T} \setminus \{\pm 1\})$.

(c) If $q|_{M^0\{1\}(QC)} = 0$ and $q|_{M^0_-\{1\}(QC)} = 0$, then $q\chi_+, q\chi_- \in QC$.

Proof. For part (a), since $q \in QC$ is an odd function, it follows from (2.1) that

$$\delta_{\lambda,\pm 1}(q) = (m\lambda q)(\pm 1) = 0 \quad \text{for all } \lambda \geq 1.\nonumber$$

Therefore, by (2.2) and (2.3), $q$ vanishes on $\Lambda \times \{\pm 1\} \subseteq QC^*$ and hence on its closure, in particular, also on $M^0_+\{1\}(QC)$.

For part (b) assume that $q|_{M^0_+\{1\}(QC)} = 0$. We use the fact that $QC = VMO \cap L^\infty$. It follows from the definition of $VMO$-functions that the product of a $VMO$-function with a uniformly continuous function is again $VMO$. Therefore, $pq$ is $VMO$ on the interval $\mathbb{T} \setminus \{\pm 1\}$. By Lemma 2.2(c), the integral gap $\gamma_{\pm 1}(pq)$ is zero. Hence $pq$ is $VMO$ on all of $\mathbb{T}$ by Lemma 2.2(b). This implies $pq \in QC$.

For case (c) decompose $q = q_{c_1} + q_{c_-}$ such that $c_{\pm 1} \in C(\mathbb{T})$ vanishes identically in a neighborhood of $\pm 1$. Then apply the result of (b). \hfill \Box

We will also need the following lemma.

Lemma 2.4 $\delta_{\lambda,\tau}$ is not multiplicative over $\widehat{QC}$ for each fixed $\lambda \in [1, \infty)$ and $\tau \in \mathbb{T}$.

Proof. Let $\tau = e^{i\theta}$ and consider $\phi(e^{ix}) = e^{ikx} + e^{-ikx}$ with $k \in \mathbb{N}$. Apparently, $\phi \in \widehat{QC}$.

Note that the moving average is generated by the function

$$K(x) = \frac{1}{2\pi} \chi(-\pi,\pi)(x), \quad \delta_{\lambda,\tau}(q) = (m\lambda q)(e^{i\theta}) = \int_{-\infty}^{\infty} \lambda K(\lambda x)q(e^{i(\theta-x)}) \, dx.\nonumber$$

Hence, by formula 3.14(3.5) in [1], or by direct computation,

$$\begin{align*}
\delta_{\lambda,\tau}(\phi^2) - \delta_{\lambda,\tau}(\phi)\delta_{\lambda,\tau}(\phi) \\
= (\hat{K}(2k/\lambda)e^{2ki\theta} + \hat{K}(-2k/\lambda)e^{-2ki\theta} + 2) - (\hat{K}(k/\lambda)e^{ki\theta} + \hat{K}(-k/\lambda)e^{-ki\theta})^2 \\
= 2 \cos(2k\theta) \left( \frac{\sin(2k\pi/\lambda)}{2k\pi/\lambda} - \left( \frac{\sin(k\pi/\lambda)}{k\pi/\lambda} \right)^2 \right) + 2 - 2 \left( \frac{\sin(k\pi/\lambda)}{k\pi/\lambda} \right)^2,
\end{align*}\nonumber$$

where $\hat{K}$ is the Fourier transform of the above $K$. Note that $\frac{\sin x}{x} \to 0$ as $x \to \infty$. Hence, for each fixed $\lambda$, one can choose a sufficiently large $k \in \mathbb{N}$, such that with the corresponding $\phi$,

$$\delta_{\lambda,\tau}(\phi^2) - \delta_{\lambda,\tau}(\phi)\delta_{\lambda,\tau}(\phi) > 1.\nonumber$$

Therefore $\delta_{\lambda,\tau}$ is not multiplicative for each $\lambda$ and $\tau$. \hfill \Box
3 Fibers of $M(QC)$ over $M(\widehat{QC})$

Now we are going to describe the fibers $M^\eta(QC)$. To prepare for it, we make the following definition. Given $\xi \in M(QC)$, we define its “conjugate” $\xi' \in M(QC)$ by

$$\xi'(q) := \xi(\tilde{q}), \quad q \in QC. \quad (3.1)$$

Recalling also definition (1.3), it is clear that $\hat{\xi} = \hat{\xi}' \in M(\widehat{QC})$. Furthermore, the following statements are obvious:

(i) If $\xi \in M_\tau(QC)$, then $\xi' \in M_{\bar{\tau}}(QC)$.

(ii) If $\xi \in M^\pm_\tau(QC)$, then $\xi' \in M^\mp_\tau(QC)$.

(iii) If $\xi \in M^0_\tau(QC)$, then $\xi' \in M^0_{\bar{\tau}}(QC)$.

For the characterization of the fibers $M^\eta(QC)$ we have to distinguish whether $\eta \in M_\tau(\widehat{QC})$ with $\tau \in \{+1, -1\}$ or with $\tau \in T_+$. In this connection recall the last formula in (1.6).

3.1 Fibers over $M_\tau(\widehat{QC})$, $\tau \in \{+1, -1\}$

For the description of $M^\eta(QC)$ with $\eta \in M_{\pm 1}(\widehat{QC})$ the following results is crucial.

Proposition 3.1 If $\xi_1, \xi_2 \in M^+_{\pm 1}(QC)$ and $\hat{\xi}_1 = \hat{\xi}_2$, then $\xi_1 = \xi_2$.

Proof. Each $q \in QC$ admits a unique decomposition

$$q = \frac{q + \tilde{q}}{2} + \frac{q - \tilde{q}}{2} =: q_e + q_o,$$

where $q_e$ is even and $q_o$ is odd. By Lemma 2.3(ac), we have $q_o\chi_- \in QC$, and

$$\xi_1(q) = \xi_1(q_e) + \xi_1(q_o) = \xi_1(q_e) + \xi_1(q_o - 2q_o\chi_-)$$

$$= \eta(q_e) + \eta(q_o - 2q_o\chi_-)$$

$$= \xi_2(q_e) + \xi_2(q_o - 2q_o\chi_-) = \xi_2(q_e) + \xi_2(q_o) = \xi_2(q).$$

Note that $q_o - 2q_o\chi_- = q_o(\chi_+ - \chi_-) \in \widehat{QC}$ and that $\lim_{t \to 1+0} q_o(t)\chi_-(t) = 0$, whence $\xi_i(q_o\chi_-) = 0$. It follows that $\xi_1 = \xi_2$. \qed

Theorem 3.2 Let $\eta \in M_{\pm 1}(\widehat{QC})$. Then either

(a) $M^\eta(QC) = \{\xi\}$ with $\xi = \xi' \in M^0_{\pm 1}(QC)$, or

(b) $M^\eta(QC) = \{\xi, \xi'\}$ with $\xi \in M^+_{\pm 1}(QC) \setminus M^0_{\pm 1}(QC)$ and $\xi' \in M^-_{\pm 1}(QC) \setminus M^0_{\pm 1}(QC)$. 

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Proof. From the statement (1.7) it follows that \( \hat{\xi} \in M_{\pm 1}(QC) \) implies \( \xi \in M_{\pm 1}(QC) \). Therefore \( \emptyset \neq M^0(QC) \subseteq M_{\pm 1}(QC) \) whenever \( \eta \in M_{\pm 1}(QC) \). Now the assertion follows from Proposition 1.1, Proposition 3.1 and the statements (i)-(iii) above.

Next we want to characterize of those \( \eta \in M_{\pm 1}(QC) \) which give rise to the first case. Consider the functionals \( \delta_{\lambda, \tau} \in QC^* \) associated with the moving average (2.2), and define, in analogy to (2.3),

\[
M^\tau(QC) := M(QC) \cap \text{clos}_{QC^*}(\Lambda \times \{\tau\}).
\]  

(3.2)

We will use this definition for \( \tau \in \mathbb{T}_+ = \mathbb{T} \cup \{+1, -1\} \).

**Theorem 3.3** The map \( \Psi : \xi \mapsto \hat{\xi} \) is a bijection from \( M^0_{\pm 1}(QC) \) onto \( M^0_{\pm 1}(QC) \).

Proof. Without loss of generality consider the case \( \tau = 1 \). First of all, \( \Psi \) maps \( M^0_1(QC) \) into \( M^1(QC) \). Indeed, it follows from (2.3) that for any \( \xi \in M^0_1(QC) \), any \( q_1, \ldots, q_k \in QC \subset QC \) and \( \varepsilon > 0 \), there exists \( \lambda \in \Lambda \) such that \( |\xi(q_i) - \delta_{\lambda, 1}(q_i)| < \varepsilon \) for all \( i \). But this is just \( |\hat{\xi}(q_i) - \delta_{\lambda, 1}(q_i)| < \varepsilon \). Therefore, \( \hat{\xi} \) lies in the weak-* closure of \( \{\delta_{\lambda, 1} : \lambda \in \Lambda\} \). Hence, by (3.2), \( \hat{\xi} \in M^0_1(QC) \). The injectiveness of the map \( \Psi|_{M^0_1(QC)} \) follows from Theorem 3.2 or Proposition 3.1.

It remains to show that \( \Psi|_{M^0_1(QC)} \) is surjective. Choose any \( \eta \in M^0_1(QC) \). By definition, there exists a net \( \{\lambda_\omega\}_{\omega \in \Omega} \), \( \lambda_\omega \in \Lambda \), such that the net \( \{\delta_{\lambda_\omega}\}_{\omega \in \Omega} := \{\delta_{\lambda_\omega, 1}\}_{\omega \in \Omega} \) converges to \( \eta \) (in the weak-* sense of functionals on \( QC \)). Note that \( \delta_{\lambda}(q) = 0 \) for any \( \lambda \in \Lambda \) whenever \( q \in QC \) is an odd function. Therefore the net \( \{\delta_{\lambda_\omega}\}_{\omega \in \Omega} \) (regarded as functionals on \( QC \)) converges to the functional \( \xi \in QC^* \) defined by

\[
\xi(q) := \eta(\frac{q + \bar{q}}{2}), \quad q \in QC.
\]

Indeed, \( \delta_{\lambda_\omega}(q) = \frac{1}{2}\delta_{\lambda_\omega}(q + \bar{q}) \rightarrow \frac{1}{2}\eta(q + \bar{q}) = \xi(q) \). It follows that \( \xi \in \text{clos}_{QC^*}(\Lambda \times \{1\}) \).

Next we show that \( \xi \) is multiplicative over \( QC \), i.e., \( \xi \in M(QC) \). Given arbitrary \( p, q \in QC \) we can decompose them into even and odd parts as \( p = p_e + p_o \), \( q = q_e + q_o \). The even part of \( pq \) equals \( p_eq_e + p_oq_o \). Therefore using the definition of \( \xi \) in terms of \( \eta \) we get

\[
\xi(p)\xi(q) = \eta(p_e)\eta(q_e) = \eta(p_e q_e), \quad \xi(pq) = \eta(p_o q_e + p_o q_o).
\]

Hence the multiplicativity of \( \xi \) follows if we can show that \( \eta(p_o q_o) = 0 \). To see this we argue as follows. By Lemma 2.3(ac), we have \( p_o q_o|_{M^0_{\pm 1}(QC)} = 0 \) and \( p_o q_o \chi_+ \in QC \), and hence by Lemma 2.2 the integral gap

\[
\gamma_1(p_o q_o \chi_+) = \lim_{\delta \to +0} \sup_{\delta} \frac{1}{\delta} \int_0^\delta (p_o q_o)(e^{ix}) \, dx = 0.
\]

In other word, as \( \lambda \to +\infty \),

\[
\delta_{\lambda}(p_o q_o) = \frac{\lambda}{2\pi} \int_{-\pi/\lambda}^{\pi/\lambda} (p_o q_o)(e^{ix}) \, dx = \frac{\lambda}{\pi} \int_0^{\pi/\lambda} (p_o q_o)(e^{ix}) \, dx \to 0.
\]
Since the net \( \{ \delta_{\lambda_\omega} \}_{\omega \in \Omega} \) (regarded as functionals on \( \widehat{QC} \)) converges to \( \eta \in M(\widehat{QC}) \), it follows from Lemma 2.3 that \( \lambda_\omega \to +\infty \). Therefore,
\[
\delta_{\lambda_\omega}(p_0 q_0) \to 0 \quad \text{and} \quad \delta_{\lambda_\omega}(p_0 q_0) \to \eta(p_0 q_0).
\]
We obtain \( \eta(p_0 q_0) = 0 \) and conclude that \( \xi \) is multiplicative. Combined with the above this yields \( \xi \in M^0_1(\widehat{QC}) \), while clearly \( \eta = \hat{\xi} \). Hence \( \Psi : M^0_1(\widehat{QC}) \to M^0_1(\widehat{QC}) \) is surjective. \( \square \)

The previous two theorems imply the following.

**Corollary 3.4** \( M^0_{\pm 1}(\widehat{QC}) \) is a closed subset of \( M_{\pm 1}(\widehat{QC}) \). Moreover,

(a) if \( \eta \in M^0_{\pm 1}(\widehat{QC}) \), then \( M^n(QC) = \{ \xi \} \) with \( \xi = \xi' \in M^0_{\pm 1}(QC) \);

(b) if \( \eta \in M^0_{\pm 1}(\widehat{QC}) \setminus M^0_{\pm 1}(\widehat{QC}) \), then \( M^n(QC) = \{ \xi, \xi' \} \) with \( \xi \in M^+_1(QC) \setminus M^-_1(QC) \) and \( \xi' \in M^-_1(QC) \setminus M^+_1(QC) \).

Note also that \( M^\pm_{\pm 1}(\widehat{QC}) \) decomposes into the disjoint union of
\[
M^\pm_{\pm 1}(\widehat{QC}) \setminus M^0_{\pm 1}(\widehat{QC}) \quad \text{and} \quad M^0_{\pm 1}(\widehat{QC}),
\]
and that \( \Psi \) is a two-to-one map from \( M^\pm_{\pm 1}(QC) \setminus M^0_{\pm 1}(QC) \) onto \( M^\pm_{\pm 1}(QC) \setminus M^0_{\pm 1}(QC) \).

### 3.2 Fibers over \( M_\tau(\widehat{QC}) \), \( \tau \in \mathbb{T}_+ \)

Now we consider the fibers of \( M^n(QC) \) over \( \eta \in M_\tau(\widehat{QC}) \) with \( \tau \in \mathbb{T}_+ \). This case is easier than the previous one.

**Proposition 3.5** If \( \hat{\xi}_1 = \hat{\xi}_2 \) for \( \xi_1, \xi_2 \in M_\tau(QC) \) with \( \tau \in \mathbb{T}_+ \), then \( \xi_1 = \xi_2 \).

Proof. Otherwise, there exists a \( q \in QC \), such that \( \xi_1(q) \neq 0, \xi_2(q) = 0 \). Since \( \tau \in \mathbb{T}_+ \), one can choose a smooth function \( c_\tau \) such that \( c_\tau = 1 \) in a neighborhood of \( \tau \) and such that it vanishes on the lower semi-circle. Now, construct \( \overline{q} = qc_\tau + \hat{q} c_\tau \in \widehat{QC} \). Note that \( \overline{q} - q \) is continuous at \( \tau \) and vanishes there, hence \( \xi_1(\overline{q} - q) = \xi_2(\overline{q} - q) = 0 \). But then, since \( \overline{q} \in QC \) and \( \hat{\xi}_1 = \hat{\xi}_2 \), we have
\[
0 \neq \xi_1(q) = \xi_1(\overline{q}) = \xi_2(\overline{q}) = \xi_2(q) = 0,
\]
which is a contradiction. \( \square \)

It has been stated in (1.7) that \( \Psi \) maps \( M_\tau(QC) \cup M_\tau(QC) \) onto \( M_\tau(\widehat{QC}) \). Taking the statements (i)-(iii) into account, the previous proposition implies the following.

**Corollary 3.6** Let \( \tau \in \mathbb{T}_+ \) and \( \eta \in M_\tau(\widehat{QC}) \). Then \( M^n(QC) = \{ \xi, \xi' \} \) with some (unique) \( \xi \in M_\tau(QC) \).

This corollary implies that \( \Psi \) is a bijection from \( M_\tau(QC) \) onto \( M_\tau(\widehat{QC}) \) for \( \tau \in \mathbb{T}_+ \). Clearly, \( \Psi \) is also a bijection from \( M_\tau(QC) \) onto \( M_\tau(\widehat{QC}) \). This suggests to define
\[
M^\pm_\tau(\widehat{QC}) := \{ \hat{\xi} : \xi \in M^\pm_\tau(QC) \}, \quad \tau \in \mathbb{T}_+.
\]
Recall that we defined \( M^0_\tau(QC) \) by equation (3.2).
Proposition 3.7 For $\tau \in \mathbb{T}_+$ we have

$$M_\tau(\widetilde{QC}) = M^+_\tau(\widetilde{QC}) \cup M^-_\tau(\widetilde{QC}), \quad M^0_\tau(\widetilde{QC}) = M^+_\tau(\widetilde{QC}) \cap M^-_\tau(\widetilde{QC}).$$

Proof. The first identity is obvious. Regarding the second one, note that by definition and by Proposition 3.5

$$M^+_\tau(\widetilde{QC}) \cap M^-_\tau(\widetilde{QC}) = \{ \xi : \xi \in M^0_\tau(\widetilde{QC}) \}.$$

It suffices to show that the map $\Psi : M^0_\tau(\widetilde{QC}) \to M^0_\tau(\widetilde{QC})$ is well-defined and bijective. Similar to the proof of Theorem 3.3 it can be shown that it is well-defined. Obviously it is injective. It remains to show that it is surjective.

Choose any $\eta \in M^0_\tau(\widetilde{QC})$. By definition, there exists a net $\{\lambda_\omega\}_{\omega \in \Omega}$, $\lambda_\omega \in \Lambda$, such that the net $\{\delta_{\lambda_\omega}\}_{\omega \in \Omega} := \{\delta_{\lambda_\omega,\tau}\}_{\lambda_\omega \in \Omega}$ converges to $\eta$ (in the weak-* sense of functionals on $\widetilde{QC}$). From Lemma 2.4 it follows that $\lambda_\omega \to +\infty$. Choose a continuous function $c_\tau$ such that $c_\tau = 1$ in a neighborhood of $\tau$ and such that it vanishes on the lower semi-circle. The net $\{\delta_{\lambda_\omega}\}_{\omega \in \Omega}$ (regarded as functionals on $\widetilde{QC}$) converges to the functional $\xi \in QC^*$ defined by

$$\xi(q) := \eta(\varphi), \quad q \in QC,$$

where $\varphi = q c_\tau + \bar{q} c_\tau \in \widetilde{QC}$. Indeed, $q - \varphi$ vanishes on a neighborhood of $\tau$, and hence $\delta_\lambda(q) = \delta_\lambda(\varphi)$ for $\lambda$ sufficiently large. Therefore, $\delta_{\lambda_\omega}(q) - \delta_{\lambda_\omega}(\varphi) \to 0$. This together with $\delta_{\lambda_\omega}(\varphi) \to \eta(\varphi) = \xi(q)$ implies that $\delta_{\lambda_\omega}(q) \to \xi(q)$. It follows that $\xi \in \text{clos}_{QC^*}(\Lambda \times \{\tau\})$.

In order to show that $\xi$ is multiplicative over $QC$, we write (noting $c_\tau c_\tau = 0$)

$$\bar{pq} - p \cdot \bar{q} = pq c_\tau + \bar{pq} c_\tau - (pc_\tau + \bar{p} c_\tau)(q c_\tau + \bar{q} c_\tau) = pq(c_\tau - c_\tau^2) + \bar{p}(c_\tau - c_\tau^2).$$

This is an even function vanishing in a neighborhood of $\tau$ and $\bar{\tau}$. Therefore $\eta(\bar{pq} - p \cdot \bar{q}) = 0$, which implies $\xi(pq) = \xi(p)\xi(q)$ by definition of $\xi$. It follows that $\xi \in M(QC)$. Therefore, $\xi \in M^0_\tau(QC)$ by definition (2.3). Since $\xi = \eta$ this implies surjectivity. \(\Box\)

A consequence of the previous proposition is that $M_\tau(\widetilde{QC})$ is the disjoint union of

$$M^+_\tau(\widetilde{QC}), \quad M^+_\tau(\widetilde{QC}) \setminus M^0_\tau(\widetilde{QC}), \quad \text{and} \quad M^-_\tau(\widetilde{QC}) \setminus M^0_\tau(\widetilde{QC}). \quad (3.5)$$

Comparing this with (1.8) we obtain that $\Psi$ is a two-to-one map from

(i) $M^+_\tau(QC) \setminus M^0_\tau(QC) \cup M^-_\tau(QC) \setminus M^0_\tau(QC)$ onto $M^+_\tau(\widetilde{QC}) \setminus M^0_\tau(\widetilde{QC})$,

(ii) $M^-_\tau(QC) \setminus M^0_\tau(QC) \cup M^+_\tau(QC) \setminus M^0_\tau(QC)$ onto $M^-_\tau(\widetilde{QC}) \setminus M^0_\tau(\widetilde{QC})$,

(iii) $M^0_\tau(QC) \cup M^0_\tau(QC)$ onto $M^0_\tau(\widetilde{QC})$.

4 Localization of $PQC$ over $\widetilde{QC}$

Now we are going to identify the fibers $M^0(PQC)$ over $\eta \in \widetilde{QC}$. This allows us to show that certain quotient $C^*$-algebras that arise from $PQC$ through localization are isomorphic to concrete $C^*$-algebras. What we precisely mean by the latter is the following.
Let $\mathfrak{A}$ be a commutative $C^*$-algebra and $\mathfrak{B}$ be a $C^*$-subalgebra, both having the same unit element. For $\beta \in M(\mathfrak{B})$ consider the smallest closed ideal of $\mathfrak{A}$ containing the ideal $\beta$,

$$J_\beta = \text{clos}\text{id}_{\mathfrak{A}}\{ b \in \mathfrak{B} : \beta(b) = 0 \}.$$

It is known (see, e.g., [1, Lemma 3.65]) that

$$J_\beta = \{ a \in \mathfrak{A} : a|_{M_\beta(\mathfrak{A})} = 0 \}.$$

Therein $a$ is identified with its Gelfand transform. Hence the map

$$a + J_\beta \in \mathfrak{A}/J_\beta \mapsto a|_{M_\beta(\mathfrak{A})} \in C(M_\beta(\mathfrak{A}))$$

is a well-defined *-isomorphism. In other words, the quotient algebra $\mathfrak{A}/J_\beta$ is isomorphic to $C(M_\beta(\mathfrak{A}))$. However, it is often more useful to identify this algebra with a more concrete $C^*$-algebra $\mathfrak{D}_\beta$. This motivates the following definition. A unital *-homomorphism $\Phi_\beta : \mathfrak{A} \to \mathfrak{D}_\beta$ is said to localize the algebra $\mathfrak{A}$ at $\beta \in M(\mathfrak{B})$ if it is surjective and if $\ker \Phi_\beta = J_\beta$. In other words, the induced *-homomorphism

$$a + J_\beta \in \mathfrak{A}/J_\beta \mapsto \Phi_\beta(a) \in \mathfrak{D}_\beta$$

is a *-isomorphism between $\mathfrak{A}/J_\beta$ and $\mathfrak{D}_\beta$.

Our goal is to localize $PQC$ at $\eta \in M(\widetilde{QC})$ in the above sense. The corresponding fibers are

$$M^\eta(PQC) = \{ z \in M(PQC) : z|_{\widetilde{QC}} = \eta \} = \{ z \in M_\xi(PQC) : \xi \in M^\eta(QC) \}.$$

Hence they can be obtained from the fibers $M^\eta(QC)$ and $M_\xi(PQC)$ (see Theorem 1.2). Recall the identification of $z \in M(PQC)$ with $(\xi, \sigma) \in M(QC) \times \{+1, -1\}$ given in (1.2). Furthermore, $\mathbb{C}^N$ is considered as a $C^*$-algebra with component-wise operations and maximum norm. (It is the $N$-fold direct product of the $C^*$-algebra $\mathbb{C}$.)

**Theorem 4.1**

(a) Let $\eta \in M^0_{\pm 1}(\widetilde{QC})$ and $M^\eta(QC) = \{\xi\}$. Then $M^\eta(PQC) = \{(\xi, +1), (\xi, -1)\}$ and $\Phi : PQC \to \mathbb{C}^2$ defined by

$$p \in PC \mapsto (p(\pm 1 + 0), p(\pm 1 - 0)), \quad q \in QC \mapsto (\xi(q), \xi(q))$$

extends to a localizing *-homomorphism.

(b) Let $\eta \in M_{\pm 1}(\widetilde{QC}) \setminus M^0_{\pm 1}(\widetilde{QC})$ and $M^\eta(QC) = \{\xi, \xi'\}$ with $\xi \in M^+_{\pm 1}(QC) \setminus M^0_{\pm 1}(QC)$. Then $M^\eta(PQC) = \{(\xi, +1), (\xi', -1)\}$ and $\Phi : PQC \to \mathbb{C}^2$ defined by

$$p \in PC \mapsto (p(\pm 1 + 0), p(\pm 1 - 0)), \quad q \in QC \mapsto (\xi(q), \xi'(q))$$

extends to a localizing *-homomorphism.
(c) Let \( \eta \in M^0(\tilde{QC}) \), \( \tau \in \mathbb{T}_+ \), and \( M^0\!(QC) = \{ \xi, \xi' \} \) with \( \xi \in M^0\!(QC) \). Then \( M^n\!(PQC) = \{ (\xi, +1), (\xi, -1), (\xi', +1), (\xi', -1) \} \) and \( \Phi : PQC \to \mathbb{C}^4 \) defined by
\[
p \in PC \mapsto (p(\tau + 0), p(\tau - 0), p(\tau + 1), p(\tau - 1)), \quad q \in QC \mapsto (\xi(q), \xi(q), \xi'(q), \xi'(q))
\]
extends to a localizing \(*\)-homomorphism.

(d) Let \( \eta \in M^\pm(\tilde{QC}) \setminus M^0(\tilde{QC}) \), \( \tau \in \mathbb{T}_+ \), and \( M^0(\tilde{QC}) = \{ \xi, \xi' \} \) with \( \xi \in M^\pm(\tilde{QC}) \setminus M^0(\tilde{QC}) \). Then \( M^n\!(PQC) = \{ (\xi, +1), (\xi', +1) \} \) and \( \Phi : PQC \to \mathbb{C}^2 \) defined by
\[
p \in PC \mapsto (p(\tau \pm 0), p(\tau \mp 1)), \quad q \in QC \mapsto (\xi(q), \xi'(q))
\]
extends to a localizing \(*\)-homomorphism.

Proof. Note that all cases of \( \eta \in M(\tilde{QC}) \) are considered (see (1.6), (3.3), and (3.5)). The description of \( M^n\!(QC) \) follows from Corollaries 3.4 and 3.6.

Let us consider only one case, say case (c). The other cases can be treated analogously.

We can write
\[
M^n\!(PQC) = \{ z \in M_\xi\!(PQC) : \xi \in M^n\!(QC) \}.
\]

Hence as \( M^n\!(QC) = \{ \xi, \xi' \} \) in this case, we get \( M^n\!(PQC) = M_\xi\!(PQC) \cup M_{\xi'}\!(PQC) \).

Now use Theorem 1.2 to get the correct description of \( M^n\!(PQC) \) as a set of four elements \( \{ z_1, z_2, z_3, z_4 \} \). Identifying \( C(M^n\!(PQC)) = C(\{ z_1, z_2, z_3, z_4 \}) \) with \( \mathbb{C}^4 \), the corresponding localizing homomorphism is given by
\[
\Phi : f \in PQC \mapsto (z_1(f), z_2(f), z_3(f), z_4(f)) \in \mathbb{C}^4.
\]

Using the identification of \( z \) with \( (\xi, \sigma) \in M(\tilde{QC}) \times \{ +1, -1 \} \) as given in (1.2), the above form of the \(*\)-homomorphism follows by considering \( f = p \in PC \) and \( f = q \in QC \). \( \square \)

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