Lifelong Learning of Compositional Structures

Jorge A. Mendez and Eric Eaton
Department of Computer and Information Science
University of Pennsylvania
{mendezme,eeaton}@seas.upenn.edu

Abstract

A hallmark of human intelligence is the ability to construct self-contained chunks of knowledge and adequately reuse them in novel combinations for solving different yet structurally related problems. Learning such compositional structures has been a significant challenge for artificial systems, due to the combinatorial nature of the underlying search problem. To date, research into compositional learning has largely proceeded separately from work on lifelong or continual learning. We integrate these two lines of work to present a general-purpose framework for lifelong learning of compositional structures that can be used for solving a stream of related tasks. Our framework separates the learning process into two broad stages: learning how to best combine existing components in order to assimilate a novel problem, and learning how to adapt the set of existing components to accommodate the new problem. This separation explicitly handles the trade-off between the stability required to remember how to solve earlier tasks and the flexibility required to solve new tasks, as we show empirically in an extensive evaluation.

1 Introduction

A major goal of artificial intelligence is to create an agent capable of acquiring a general understanding of the world. Such an agent would require the ability to continually accumulate and build upon its knowledge as it encounters new experiences. Lifelong machine learning addresses this setting, whereby an agent faces a continual stream of diverse problems and must strive to capture the knowledge necessary for solving each new task it encounters. If the agent is capable of accumulating knowledge in some form of compositional representation, it could then selectively reuse and combine relevant pieces of knowledge to construct novel solutions.

Many different compositional representations for handling multiple tasks have been proposed over the last few years [32, 9, 13, 19]. In this work, we address the novel question of how to learn these compositional structures in a lifelong learning setting. We design a general-purpose framework that is agnostic to the specific algorithms used for learning and the form of the structures being learned. Evoking Piaget’s [23] assimilation and accommodation stages of intellectual development, this framework embodies the benefits of dividing the lifelong learning process into two distinct stages. In the first stage, the learner strives to solve a new task by combining existing components it has already acquired. The second stage uses discoveries from the new task to improve existing components and to construct fresh components if necessary.

Our proposed framework is capable of incorporating various forms of compositional structures, as well as different mechanisms for avoiding catastrophic forgetting [18]. As examples of the flexibility of our framework, it can incorporate naïve fine-tuning, experience replay, and elastic weight consolidation [12] as knowledge retention mechanisms, and linear combinations of linear models [14, 27], soft layer ordering [19], and a soft version of gating networks [13, 26] as the compositional structures. We instantiate our framework with the nine combinations of these examples, and evaluate it on eight different data sets, consistently showing that separating the lifelong learning process into...
two stages increases the capabilities of the learning system, reducing catastrophic forgetting and achieving higher overall performance. Qualitatively, we show that the components learned by an algorithm that subscribes our framework correspond to reusable, self-contained functions.

2 Related work

2.1 Lifelong learning

Recent years have seen a growing interest in the study of continual or lifelong learning. In this setting, agents must handle a variety of tasks over their lifetimes, and should accumulate knowledge in a way that enables them to more efficiently learn to solve new problems. Recent efforts have mainly focused on avoiding catastrophic forgetting. At a high level, different algorithms define parts of parametric models (e.g., deep neural networks) to be shared across tasks. As the agent encounters tasks sequentially, it strives to retain the knowledge that enabled it to solve earlier tasks. One common approach is to impose some data-driven regularization to prevent parameters from deviating too much in directions that are harmful for performance on the early tasks \cite{12, 33, 16, 25}. Another approach retains a small buffer of data from all tasks, and continually updates the model parameters utilizing data from all tasks, thereby maintaining the knowledge required to solve them \cite{17, 24, 10}. A related technique is to learn a generative model to “hallucinate” data, reducing the memory footprint of the algorithm at the cost of using lower-quality data and increasing the cost of accessing data \cite{1}.

These approaches, although effective in avoiding the problem of catastrophic forgetting, make no substantial effort toward the discovery of reusable knowledge. One could argue that the model parameters are learned in such a way that they are reusable across all tasks. However, it is unclear what the reusability of these parameters means, and moreover the way in which parameters are reused is hard-coded into the architecture design. This latter issue is a major drawback when attempting to learn tasks with high degree of variability, as the exact form in which tasks are connected is often unknown. We would hope that the algorithm could determine these connections autonomously.

To tackle this problem, some approaches focus on lifelong learning of factored or clustered models \cite{27, 20}. Here, the algorithms learn a set of models that are reusable across many tasks and automatically select how to reuse them. However, such methods have been limited to selectively reusing entire models, enabling agents to reuse knowledge, but not explicitly in a compositional manner.

2.2 Compositional knowledge

A mostly distinct line of parallel work has explored the learning of compositional knowledge. The majority of such methods either learn the structure for piecing together a given set of components \cite{30, 4, 5} or learn the set of components given a known structure for how to compose them \cite{3}.

A more interesting case is when neither the structure nor the set of components are given, and the agent must autonomously discover the compositional structure underlying a set of tasks. Some approaches for solving this problem assume access to a solution descriptor (e.g., in the form of natural language), which can be mapped by the agent to a solution structure \cite{9, 11, 22}. Unfortunately, it is not always possible to receive this kind of supervision, as many agents (e.g., service robots) are expected to learn in more autonomous settings. Other approaches do not make such an assumption, and instead learn the structure directly from optimization of a cost function \cite{26, 13, 19, 2, 6}.

However, note that the approaches above rely on the assumption that the agent will have access to a large batch of tasks, from which it is possible to evaluate numerous combinations of components and structures on all tasks simultaneously. A more realistic scenario is one in which the agent faces a sequence of tasks in a lifelong learning fashion. Most work in this line assumes that each component can be fully learned by training on a single task, and then can be reused for other tasks \cite{24, 7, 29}. Unfortunately, this is infeasible in many real-world scenarios in which the agent has access to little data for each of the tasks. One notable exception was proposed by Gaunt et al. \cite{8}, which improves early components with experience in new tasks, but is limited to very simplistic settings.

In contrast to prior work, our approach explicitly learns compositional structures in a lifelong learning setting. We do not assume access to a large batch of tasks or the ability to learn definitive components after training on a single task. Instead, we train on a small initial batch of tasks (four tasks, in all our experiments), and then autonomously update the existing components to accommodate new tasks.
3 The lifelong learning problem

We frame lifelong learning as online multitask learning. The agent, illustrated in Figure 1, will face a sequence of prediction tasks \( \mathcal{T}(1), \ldots, \mathcal{T}(T_{\text{max}}) \) over its lifetime. Each task will be a learning problem defined by a cost function \( \mathcal{L}(t)(f(t)) \), where the agent must learn a prediction function \( f(t) \in \mathcal{F} : \mathcal{X}(t) \rightarrow \mathcal{Y}(t) \) to minimize the cost. Each task’s solution is assumed to be parameterized by \( \theta(t) \), such that \( f(t) = \theta(t) \). The goal of the lifelong learner is to find the set of parameters \( \{\theta(1), \ldots, \theta(T_{\text{max}})\} \) that minimizes the overall cost across all tasks:

\[
\frac{1}{T_{\text{max}}} \sum_{t=1}^{T_{\text{max}}} \mathcal{L}(t)(f(t)).
\]

The total number of tasks, the order in which these tasks will arrive, and the task relationships are all unknown.

The agent will be given access to limited data for each task, and will strive to discover any relevant information to 1) relate it to previously stored knowledge in order to permit transfer and 2) store any new knowledge for future reuse. At any time, the agent may be evaluated on any previous task. Crucially, this formulation requires the agent to perform well on all tasks, so it must strive to retain knowledge from even the earliest tasks.

4 The lifelong compositional learning framework

We now present our framework for lifelong learning of compositional structures. Knowledge will be stored in a set of shared components \( M = \{m_1, \ldots, m_k\} \). Each component \( m_i \in M \) is a self-contained, reusable function parameterized by \( \phi_i \) that can be combined with other components. Given these components, the agent will reconstruct each task’s predictive function via a task-specific structure \( s(t) : \mathcal{X}(t) \times M^k \rightarrow \mathcal{F} \), such that \( f(t)(x) = s(t)(x, M)(x) \), where \( s(t) \) is parameterized by a vector \( \psi(t) \). Note that \( s(t) \) yields a function that maps inputs \( \mathcal{X}(t) \) to outputs \( \mathcal{Y}(t) \). The structure functions are in charge of selecting the components from \( M \) to execute for the current task and input, and the order in which to execute these components. Examples of specific forms of components and structures under this definition are described in Section 4.1.

The intuition behind our framework is that, at any point in the lifetime of an agent, it will have acquired a strong set of components. If these components, with minor adaptations, can be combined to solve the current task, then the agent should first learn how to reuse them before making any modifications to the components. The rationale for keeping components fixed initially is that modifications in the early stages of training, before the agent has acquired sufficient knowledge about the current task, could be catastrophically damaging to the set of existing components. Once the structure has been learned, we consider that the agent has captured sufficient knowledge about the current task, and it would be sensible to update the components to better accommodate it. If, instead, it is not possible to capture the current task with the existing components, then new components should be added.

Algorithms under our framework take the form of Algorithm 1 split into the following steps.

**Initialization** The components should be initialized in a way that encourages them to be reusable, both across tasks and across “positions” within a task. The former signifies that the components should achieve a particular sub-problem regardless of the objective of the task at hand. The latter means that components may be reused multiple times for a single task, or at different points across different tasks. For example, in deep nets, this means that the components could be used at different depths. One way to achieve this is to enforce a random structure for an initial set of tasks that reuses components both at different positions and across different tasks, and train these initial tasks jointly.

**Assimilation** Different algorithms for finding compositional knowledge vary in how they optimize the compositional structure for each task. In the context of learning modular deep networks, different approaches include treating the component selection as a reinforcement learning...
We now present three different instantiations of our framework with varying compositional structures. As proposed by Meyerson and Miikkulainen [19], we assume that each module is one layer, which holds for all the above examples.

**Accommodation (1)** The first step of accommodation is incorporating newly discovered knowledge into existing components. An effective approach should avoid modifying the components in a way that harms performance on earlier tasks, but should also be flexible enough to incorporate new knowledge required to solve the current task. Different approaches in the setting of non-compositional structures have been to naively fine-tune models with data from the current task, to impose some data-driven regularization scheme to selectively freeze weights [12, 25], or to store a portion of data from previous tasks and use experience replay to avoid forgetting [10, 17]. We will instantiate our framework by using any of these methods to accommodate new knowledge into existing components once the current task has been assimilated. For this to be possible, we require that the method can be selectively applied to a subset of the parameters being learned (in this case, the component parameters \( \phi \)).

**Accommodation (2)** Often, the existing components are insufficient to construct good solutions to the current problem. At this point, in the second step of accommodation, the learner must incorporate new knowledge required to solve the current task. This is related to work on dynamically expandable networks [31].

### 4.1 Compositional structures

We now present three different instantiations of our framework with varying compositional structures.

**Linear combinations of models** In the simplest setting, each component is a linear model, and the composition of them is obtained via linear combinations. Specifically, we assume that \( X(t) \subseteq \mathbb{R}^d \), and each task-specific function is given by \( f_{\theta(t)}(x) = \theta(t)^{\top} x \), with \( \theta(t) \in \mathbb{R}^d \). The predictive functions are constructed from a set of linear component functions \( m_{\phi_i}(x) = \phi_i^{\top} x \), with \( \phi_i \in \mathbb{R}^d \), by linearly combining them via a task-specific weight vector \( f(t)(x) = s(t)(x, M)(x) = \psi(t)^{\top} (\Phi^t x) \), where \( \psi(t) \in \mathbb{R}^k \), and we have constructed the matrix \( \Phi = [\phi_1, \ldots, \phi_k] \) to collect all \( k \) components.

**Soft layer ordering** In order to handle more complex models, we construct compositional deep nets that compute each layer’s output as a linear combination of the outputs of multiple modules. As proposed by Meyerson and Miikkulainen [19], we assume that each module is one layer, the number of components matches the network’s depth, and all components share the input and output dimensions. Concretely, each component is a deep layer net \( m_{\phi_i}(x) = \sigma(\phi_i^{\top} x) \), where \( \sigma \) is any nonlinear activation and \( \phi_i \in \mathbb{R}^{d \times d} \). A set of parameters \( \psi(t) \in \mathbb{R}^{k \times k} \) weights the output of the components at each depth: \( s(t) = \mathcal{D}(t) \circ \sum_{i=1}^{k} \psi_{i,1} m_i \circ \cdots \circ \sum_{i=1}^{k} \psi_{i,k} m_i \circ \mathcal{E}(t) \), where \( \mathcal{E}(t) \) and \( \mathcal{D}(t) \) are task-specific input and output transformations such that \( \mathcal{E}(t) : X(t) \rightarrow \mathbb{R}^d \) and \( \mathcal{D}(t) : \mathbb{R}^d \rightarrow Y(t) \), and the weights are restricted to sum to one at each depth \( j: \sum_{i=1}^{k} \psi_{i,j} = 1 \).
Soft gating  Noted in both approaches above, the learner uses a fixed structure \( s^{(t)} \) for any given task. However, it has been shown that, in the presence of large data, it is beneficial to modify the network architecture for each input \( x \) \footnote{\cite{salimans2016evolution}}. In consequence, we devise an input-dependent compositional architecture that maintains the advantages of fully differentiable models. As above, we assume that each component is a single layer \( m_{\phi_i}(x) = \sigma(\phi_i \circ x) \). In this case, the weights for each component at depth \( j \) are computed by an input-dependent soft gating net \( s^{(t)}_j : \mathcal{X}^{(t)} \rightarrow \mathbb{R}^k \), where we again restrict the weights to sum to one at each depth: \( \sum_{i=1}^k [s^{(t)}_j(x)]_i = 1 \). With this, the complete predictive function is given by \( s^{(t)} = \mathcal{D}^{(t)} \circ \sum_{i=1}^k [s^{(t)}_i(x)]_i \circ \cdots \circ \sum_{i=1}^k [s^{(t)}_k(x)]_i \circ \mathcal{E}^{(t)} \).

4.2 Reduction in computational cost

Typical approaches to lifelong learning, which do not consider separate assimilation and accommodation processes, tend to be computationally intensive. For example, Kronecker-factored elastic weight consolidation requires pre-multiplying all the gradients by a large matrix at every step. Similarly, experience replay requires sampling from all previous tasks at every training iteration. In contrast, our framework only carries out these expensive operations for every accommodation (1) step, which is done far less frequently than every iteration. Moreover, the gradient steps taken by algorithms under our framework during the assimilation phase are carried out only on the structure parameters for the current task \( \psi^{(t)} \), instead of all network parameters, making each step more efficient.

5 Experimental evaluation

5.1 Framework instantiations and baselines

Instantiations  We evaluated our framework with the three architectures of Section 4.1. For each, we trained three algorithms within our framework, varying the way they execute the accommodation (1) step. Naïve fine-tuning (VAN) updates the components via backpropagation with data for the current task, ignoring all past tasks. Elastic weight consolidation (EWC) places a quadratic penalty for deviating from the previous tasks’ parameters: \( \lambda \sum_{t=1}^{T-1} ||\theta - \theta^{(t)}||^2_F \), where \( F^{(t)} \) is the Fisher information matrix around the optimum of task \( t \). Backpropagation is carried out on the penalized objective. We approximated \( F^{(t)} \) as a Kronecker-factored matrix \footnote{\cite{salimans2016evolution}}. Experience replay (ER) stores a small amount of data for each task into a replay buffer, and during accommodation (1) takes backpropagation steps with data from the replay buffer along with the current task’s data. All algorithms assimilate the current task \( t \) via backpropagation on the structure’s parameters \( \psi^{(t)} \). To enable our deep learners to discover new components, we created an accommodation (2) step where the agent considers adding a single new component for each task. After assimilation and accommodation (1), the learner compares its validation performance with and without the newly created component, and keeps it only if it yields a substantial improvement in accuracy (5% relative, in our experiments). Assimilation and accommodation (1) alternate backpropagation steps with and without the new component. Intermittently bypassing the new component ensures that, if the agent chooses to discard it, the structure over only existing components is optimal. We denote algorithms with and without accommodation (2) as dynamic + compositional and compositional, respectively.

Baselines  For every accommodation (1) method, we constructed two baselines. Joint baselines use compositional structures, but do not separate assimilation and accommodation, and instead update components and structures jointly. On the contrary, no-components baselines optimize a single architecture to be used for all tasks. We also trained an ablated version of our framework that keeps all components fixed after initialization (FM), only taking assimilation steps for each new task.

5.2 Data sets

Linear models  We used three data sets for experimenting with the linear models that have previously been used for evaluating linear lifelong learning models \footnote{\cite{finn2017model}}. The Landmine data set consists of \( T_{\text{max}} = 29 \) tasks, which require detecting land mines in radar readings from different regions. The Facial Recognition (FERA) data set tasks are recognizing one of three facial expressions for one of seven individuals, for a total of \( T_{\text{max}} = 21 \) tasks. Finally, the London Schools (Schools) data set contains \( T_{\text{max}} = 139 \) tasks, each corresponding to exam score prediction in a different school.
Table 1: Average final performance across tasks using factored linear models—accuracy for FERA and Landmine (higher is better) and RMSE for Schools (lower is better). Standard errors after ±.

| Base | Algorithm  | FERA | Landmine | Schools |
|------|------------|------|----------|---------|
| ER   | Compositional | 79.0 ± 0.4% | 93.6 ± 0.1% | 10.66 ± 0.04 |
|      | Joint       | 78.2 ± 0.4% | 90.5 ± 0.3% | 11.56 ± 0.09 |
|      | No Comp.    | 66.4 ± 0.3% | 93.5 ± 0.1% | 10.34 ± 0.03 |
| EWC  | Compositional | 79.0 ± 0.4% | 93.7 ± 0.1% | 10.55 ± 0.03 |
|      | Joint       | 68.0 ± 0.6% | 73.0 ± 2.4% | 23.32 ± 1.99 |
|      | No Comp.    | 57.0 ± 0.9% | 92.7 ± 0.4% | 18.01 ± 1.04 |
| VAN  | Compositional | 79.0 ± 0.4% | 93.7 ± 0.1% | 10.87 ± 0.07 |
|      | Joint       | 68.0 ± 0.6% | 72.8 ± 2.5% | 25.80 ± 2.35 |
|      | No Comp.    | 57.0 ± 0.9% | 92.7 ± 0.4% | 18.00 ± 1.04 |

Table 2: Average final accuracy across tasks using soft layer ordering. Standard errors after ±.

| Base | Algorithm  | MNIST | Fashion | CUB | CIFAR | Omniglot |
|------|------------|-------|---------|-----|-------|----------|
| ER   | Dyn. + Comp. | 97.7 ± 0.2% | 96.3 ± 0.4% | 77.6 ± 0.8% | 75.9 ± 0.5% | 69.0 ± 0.7% |
|      | Compositional | 96.5 ± 0.2% | 95.3 ± 0.7% | 79.2 ± 0.7% | 56.0 ± 0.8% | 67.8 ± 1.0% |
|      | Joint       | 94.2 ± 0.3% | 94.7 ± 0.7% | 76.8 ± 0.5% | 63.8 ± 0.6% | 67.9 ± 0.5% |
|      | No Comp.    | 91.2 ± 0.3% | 93.1 ± 0.6% | 43.1 ± 1.0% | 49.5 ± 0.8% | 40.0 ± 3.9% |
| EWC  | Dyn. + Comp. | 97.3 ± 0.2% | 96.1 ± 0.4% | 72.7 ± 0.9% | 71.7 ± 0.9% | 67.7 ± 0.6% |
|      | Compositional | 96.7 ± 0.2% | 95.3 ± 0.6% | 72.4 ± 1.2% | 43.4 ± 1.1% | 52.2 ± 7.3% |
|      | Joint       | 66.3 ± 1.4% | 69.1 ± 1.4% | 65.3 ± 0.7% | 41.3 ± 0.8% | 61.2 ± 0.7% |
|      | No Comp.    | 64.3 ± 0.8% | 58.5 ± 2.9% | 47.7 ± 1.4% | 34.1 ± 0.9% | 66.2 ± 1.2% |
| VAN  | Dyn. + Comp. | 97.4 ± 0.3% | 96.0 ± 0.4% | 72.5 ± 0.8% | 69.6 ± 1.2% | 67.1 ± 0.6% |
|      | Compositional | 96.4 ± 0.2% | 95.3 ± 0.6% | 73.7 ± 1.1% | 52.5 ± 1.3% | 65.3 ± 1.2% |
|      | Joint       | 67.4 ± 1.4% | 66.1 ± 2.4% | 64.4 ± 0.8% | 41.4 ± 0.8% | 60.2 ± 1.1% |
|      | No Comp.    | 64.4 ± 1.1% | 59.4 ± 2.7% | 48.3 ± 1.9% | 34.1 ± 0.8% | 64.7 ± 1.0% |
| FM   | Dyn. + Comp. | 99.1 ± 0.0% | 97.0 ± 0.3% | 78.2 ± 0.4% | 74.3 ± 0.9% | 67.7 ± 0.7% |
|      | Compositional | 84.1 ± 0.8% | 85.9 ± 1.3% | 79.2 ± 0.6% | 46.0 ± 1.6% | 58.3 ± 3.0% |

**Deep models** We used five data sets for evaluating our deep models. Binary MNIST (MNIST) is a common benchmark for lifelong learning algorithms, where each task is a binary classification problem between a pair of digits. We constructed $T_{\text{max}} = 10$ tasks by randomly sampling the digits with replacement across tasks. The Binary Fashion MNIST (Fashion) data set is equivalent in format to MNIST, but labels correspond to items of clothing. For these two data sets, all models used a task-specific input transformation layer $E^{(i)}$ initialized at random and kept fixed throughout training, to ensure that the input spaces were sufficiently different [19]. A more complex lifelong learning problem commonly used in the literature is Split CUB-200 (CUB), where the agent must classify bird species. We created $T_{\text{max}} = 20$ tasks by randomly sampling ten species for each, without replacement across tasks. All agents used a frozen ResNet-18 pre-trained on ImageNet as a feature extractor $E^{(i)}$ shared across all tasks. For these first three data sets, all architectures were simple fully-connected networks. To show that our framework supports more complex convolutional architectures, we used two additional data sets. We constructed a lifelong learning version of CIFAR-100 (CIFAR) with $T_{\text{max}} = 20$ tasks by randomly sampling five classes per task, without replacement across tasks. Finally, we used the Omniglot data set, which consists of $T_{\text{max}} = 50$ multi-class classification problems, each corresponding to detecting handwritten symbols in a given alphabet. The inputs to all architectures for these two data sets were the images directly, without any transformation $E^{(i)}$.

We conducted ten trials of each experiment with varying random seeds. Additional details on data preprocessing, train/test split, and hyper-parameters are provided in Appendix A. The code and data sets used for our experiments are available at [github.com/GRASP-ML/Mendez2020Compositional](https://github.com/GRASP-ML/Mendez2020Compositional).

5.3 Results

We now present our empirical results. More detailed results, evaluations with different amounts of data, and a visualization of learned components are included in Appendices B and C.
We then evaluated how different algorithms performed when learning deep nets with soft layer ordering. Algorithms with all steps in our framework (Dyn + Comp.) considerably outperformed the others. With EWC and VAN, the version of our framework without accommodation (2) (Compositional) yielded better performance than baselines. Moreover, the gaps between forward and final performance indicate that algorithms within our framework are less prone to catastrophic forgetting.

Table 1 summarizes the results of obtained with linear models. The compositional versions of ER, EWC, and VAN clearly outperformed all the joint versions, which learn the same form of models but by jointly optimizing structures and components. This suggests that the separation of the learning process into assimilation and accommodation stages enables the agent to better capture the structure of the problem. Interestingly, the no-components variations, which learn a single linear model for all tasks, performed better than the jointly trained versions in two out of the three data sets, and even outperformed our compositional ER algorithm in one case. This indicates that the tasks in those two data sets (Landmine and Schools) are so closely related that a single model can capture them.

5.3.1 Linear combinations of models

We then evaluated how different algorithms performed when learning deep nets with soft layer ordering. Results in Table 2 show that all the algorithms conforming to our framework outperformed the joint and no-components learners. In four out of the five data sets, the dynamic addition of new components yielded either no or marginal improvements. However, in the CIFAR data set, it was crucial for the agent to be capable of detecting when new components were needed. This added flexibility enables our learners to handle more varied tasks, where new problems may not be solved without substantially new knowledge. Algorithms with accommodation (1) outperformed the ablated compositional FM agent, showing that it is necessary to accommodate new knowledge into the set of components in order to handle a diversity of tasks. When FM was allowed to dynamically add new components (keeping old ones fixed), it yielded the best performance on MNIST and Fashion by adding far more components than methods with accommodation (1), as we show in Appendix B.

To study how flexibly our agents learn new tasks and how stable they are in retaining information about earlier tasks, Figure 2 shows the average gain in accuracy w.r.t. no-components VAN immediately after each task was learned (forward) and after all tasks had been learned (final). Compositional learners with no dynamically added components struggled to match the performance of joint baselines in the forward stage, indicating that learning the ordering over existing layers during much of training
Table 3: Average final accuracy across all tasks using soft gating. Standard errors after the ±.

| Base | Algorithm       | MNIST    | Fashion  | CIFAR    | Omniglot |
|------|----------------|----------|----------|----------|----------|
|      | Dyn. + Comp.   | 98.2 ± 0.1% | 96.7 ± 0.4% | 72.2 ± 0.7% | 71.5 ± 0.7% |
|      | Compositional  | 98.0 ± 0.2% | 96.3 ± 0.4% | 71.5 ± 0.8% | 70.8 ± 0.6% |
|      | Joint          | 93.7 ± 0.4% | 93.8 ± 1.5% | 68.9 ± 1.0% | 70.3 ± 0.3% |
|      | No Comp.       | 91.2 ± 0.3% | 93.1 ± 0.6% | 49.5 ± 0.8% | 40.0 ± 3.9% |
|      | Dyn. + Comp.   | 98.2 ± 0.1% | 96.5 ± 0.4% | 65.9 ± 0.8% | 67.3 ± 1.3% |
|      | Compositional  | 98.0 ± 0.3% | 96.6 ± 0.4% | 73.4 ± 0.7% | 70.4 ± 0.7% |
|      | Joint          | 66.1 ± 1.0% | 65.3 ± 1.7% | 49.8 ± 1.2% | 62.4 ± 0.5% |
|      | No Comp.       | 64.3 ± 0.8% | 58.5 ± 2.9% | 34.1 ± 0.9% | 66.2 ± 1.2% |
|      | Dyn. + Comp.   | 98.2 ± 0.1% | 96.8 ± 0.4% | 64.4 ± 0.8% | 65.5 ± 1.3% |
|      | Compositional  | 98.0 ± 0.2% | 96.5 ± 0.4% | 65.9 ± 0.8% | 69.4 ± 0.7% |
|      | Joint          | 67.3 ± 1.7% | 62.6 ± 3.4% | 49.2 ± 0.8% | 62.3 ± 1.6% |
|      | No Comp.       | 64.4 ± 1.1% | 59.4 ± 2.7% | 34.1 ± 0.8% | 64.7 ± 1.0% |
|      | Dyn. + Comp.   | 98.5 ± 0.1% | 96.7 ± 0.4% | 72.9 ± 0.7% | 71.7 ± 0.6% |
|      | Compositional  | 94.8 ± 0.4% | 95.8 ± 0.4% | 75.2 ± 0.7% | 72.0 ± 0.5% |

Figure 4: Average gain w.r.t. no-components VAN across tasks and data sets immediately after training on each task (forward) and after all tasks had been trained (final), using soft gating. The separation of the learning process into two phases substantially increased performance over baselines.

is less flexible than modifying the layers themselves, as expected. However, the added stability enabled our compositional agents to dramatically decrease forgetting w.r.t. non-compositional methods. The dynamic addition of new layers yielded substantial improvements in the forward stage, while still reducing of catastrophic forgetting w.r.t. baselines. Figure 3 shows the learning curves of all tasks on the MNIST and Fashion data sets using ER, the highest-performing accommodation (2) method. The curves for compositional and dynamic + compositional ER display almost no decrease after training on a task, whereas the joint and no-components versions both show diminishing accuracy as the agent learns subsequent tasks. Most notably, as more tasks were seen by dynamic ER, the existing components became better able to assimilate new tasks, shown by the trend of increasing performance as the number of tasks increases. This suggests that the later tasks’ accommodation stage can successfully determine which new knowledge should be incorporated into existing components (enabling them to better generalize across tasks), and which must be incorporated into a new component.

5.3.3 Soft gating

Finally, we tested our algorithms when the compositional structures were given by a soft gating net. Table 5.6 shows the results on the considered data sets, where once again there was a substantial improvement of compositional algorithms w.r.t. baselines. We hypothesized that the gating network granted our assimilation step substantially more flexibility, which is confirmed in Figure 5.9, where we see that the forward accuracy of compositional methods was nearly identical to that of jointly trained and no-components versions. In fact, this added flexibility enabled our simplest version of a compositional algorithm, FM, to perform better than the full versions of our algorithm in the two most complex data sets with convolutional gating nets, showing that even the components initialized with only a few tasks (in this case, four) are sufficient for top lifelong learning performance. We attempted to run experiments with this method on the CUB data set, but found that all algorithms were incapable of generalizing to test data. This is consistent with findings in prior work, which showed that gating nets require vast amounts of data, unavailable for the CUB data set. [26] [13].
Conclusion

We presented a general framework for learning compositional structures in a lifelong learning setting. The key piece of our framework is the separation of the learning process into two stages: assimilation of new problems with existing components, and accommodation of newly discovered knowledge into the set of components. These stages have connections to Piagetian theory of development, opening the door for future investigations that bridge between lifelong machine learning and developmental psychology. We showed the flexibility of our framework by capturing nine different concrete algorithms within our framework, and demonstrated empirically in an extensive evaluation that algorithms within our framework are stronger lifelong learners than existing approaches.

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References

[1] Achille, A., Eccles, T., Matthey, L., Burgess, C., Watters, N., Lerchner, A., and Higgins, I. (2018). Life-long disentangled representation learning with cross-domain latent homologies. In Advances in Neural Information Processing Systems 31 (NeurIPS-18), pages 9873–9883.

[2] Alet, F., Lozano-Perez, T., and Kaelbling, L. P. (2018). Modular meta-learning. In Proceedings of the 2nd Conference on Robot Learning (CoRL-19), pages 856–868.

[3] Bošnjak, M., Rocktäschel, T., Naradowsky, J., and Riedel, S. (2017). Programming with a differentiable forth interpreter. In Proceedings of the 34th International Conference on Machine Learning (ICML-17), pages 547–556.

[4] Bunel, R., Hausknecht, M., Devlin, J., Singh, R., and Kohli, P. (2018). Leveraging grammar and reinforcement learning for neural program synthesis. In 6th International Conference on Learning Representations (ICLR-18).

[5] Cai, J., Shin, R., and Song, D. (2017). Making neural programming architectures generalize via recursion. In 5th International Conference on Learning Representations (ICLR-17).

[6] Chang, M., Gupta, A., Levine, S., and Griffiths, T. L. (2019). Automatically composing representation transformations as a means for generalization. In 7th International Conference on Learning Representations (ICLR-19).

[7] Fernando, C., Banarse, D., Blundell, C., Zwols, Y., Ha, D., Rusu, A. A., Pritzel, A., and Wierstra, D. (2017). PathNet: Evolution channels gradient descent in super neural networks. arXiv preprint arXiv:1701.08734.

[8] Gaunt, A. L., Brockschmidt, M., Kushman, N., and Tarlow, D. (2017). Differentiable programs with neural libraries. In Proceedings of the 34th International Conference on Machine Learning (ICML-17), pages 1213–1222.

[9] Hu, R., Andreas, J., Rohrbach, M., Darrell, T., and Saenko, K. (2017). Learning to reason: End-to-end module networks for visual question answering. In Proceedings of the 2017 IEEE International Conference on Computer Vision (ICCV-17), pages 804–813.

[10] Isele, D. and Cosgun, A. (2018). Selective experience replay for lifelong learning. In Proceedings of the Thirty-Second AAAI Conference on Artificial Intelligence (AAAI-18), pages 3302–3309.

[11] Johnson, J., Hariharan, B., van der Maaten, L., Hoffman, J., Fei-Fei, L., Lawrence Zitnick, C., and Girshick, R. (2017). Inferring and executing programs for visual reasoning. In Proceedings of the 2017 IEEE International Conference on Computer Vision (ICCV-17), pages 2989–2998.
[12] Kirkpatrick, J., Pascanu, R., Rabinowitz, N., Veness, J., Desjardins, G., Rusu, A. A., Milan, K., Quan, J., Ramalho, T., Grabska-Barwinska, A., et al. (2017). Overcoming catastrophic forgetting in neural networks. Proceedings of the National Academy of Sciences (PNAS), 114(13):3521–3526.

[13] Kirsch, L., Kunze, J., and Barber, D. (2018). Modular networks: Learning to decompose neural computation. In Advances in Neural Information Processing Systems 31 (NeurIPS-18), pages 2408–2418.

[14] Kumar, A. and Daumé III, H. (2012). Learning task grouping and overlap in multi-task learning. In Proceedings of the 29th International Conference on International Conference on Machine Learning (ICML-12), pages 1723–1730.

[15] Lee, S., Stokes, J., and Eaton, E. (2019). Learning shared knowledge for deep lifelong learning using convolutional networks. In Proceedings of the Twenty-Eighth International Joint Conference on Artificial Intelligence, IJCAI-19, pages 2837–2844.

[16] Li, Z. and Hoiem, D. (2017). Learning without forgetting. IEEE Transactions on Pattern Analysis and Machine Intelligence (TPAMI), 40(12):2935–2947.

[17] Lopez-Paz, D. and Ranzato, M. (2017). Gradient episodic memory for continual learning. In Advances in Neural Information Processing Systems 30 (NeurIPS-17), pages 6467–6476.

[18] McCloskey, M. and Cohen, N. J. (1989). Catastrophic interference in connectionist networks: The sequential learning problem. In Psychology of Learning and Motivation, volume 24, pages 109–165. Elsevier.

[19] Meyerson, E. and Miikkulainen, R. (2018). Beyond shared hierarchies: Deep multitask learning through soft layer ordering. In 6th International Conference on Learning Representations (ICLR-18).

[20] Nagabandi, A., Finn, C., and Levine, S. (2019). Deep online learning via meta-learning: Continual adaptation for model-based RL. In 7th International Conference on Learning Representations (ICLR-19).

[21] Nguyen, C. V., Li, Y., Bui, T. D., and Turner, R. E. (2018). Variational continual learning. In 6th International Conference on Learning Representations (ICLR-18).

[22] Pahuja, V., Fu, J., Chandar, S., and Pal, C. (2019). Structure learning for neural module networks. In Proceedings of the Beyond Vision and LANGUAGE: Integrating Real-world Knowledge (LANTERN), pages 1–10. Association for Computational Linguistics.

[23] Piaget, J. (1976). Piaget’s Theory, pages 11–23. Springer Berlin Heidelberg, Berlin, Heidelberg.

[24] Reed, S. and de Freitas, N. (2016). Neural programmers-interpreters. In 4th International Conference on Learning Representations (ICLR-16).

[25] Ritter, H., Botev, A., and Barber, D. (2018). Online structured Laplace approximations for overcoming catastrophic forgetting. In Advances in Neural Information Processing Systems 31 (NeurIPS-18), pages 3738–3748.

[26] Rosenbaum, C., Klinger, T., and Riemer, M. (2018). Routing networks: Adaptive selection of non-linear functions for multi-task learning. In 6th International Conference on Learning Representations (ICLR-18).

[27] Ruvolo, P. and Eaton, E. (2013). ELLA: An efficient lifelong learning algorithm. In Proceedings of the 30th International Conference on Machine Learning (ICML-13), pages 507–515.

[28] Shazeer, N., Mirhoseini, A., Maziark, K., Davis, A., Le, Q., Hinton, G., and Dean, J. (2017). Outrageously large neural networks: The sparsely-gated mixture-of-experts layer. In 5th International Conference on Learning Representations (ICLR-17).

[29] Valkov, L., Chaudhari, D., Srivastava, A., Sutton, C., and Chaudhuri, S. (2018). Houdini: Lifelong learning as program synthesis. In Advances in Neural Information Processing Systems 31 (NeurIPS-18), pages 8687–8698.
[30] Xu, D., Nair, S., Zhu, Y., Gao, J., Garg, A., Fei-Fei, L., and Savarese, S. (2018). Neural task programming: Learning to generalize across hierarchical tasks. In Proceedings of the 2018 IEEE International Conference on Robotics and Automation (ICRA-18), pages 3795–3802.

[31] Yoon, J., Lee, J., Yang, E., and Hwang, S. J. (2018). Lifelong learning with dynamically expandable network. In 6th International Conference on Learning Representations (ICLR-18).

[32] Zaremba, W., Mikolov, T., Joulin, A., and Fergus, R. (2016). Learning simple algorithms from examples. In Proceedings of the 33rd International Conference on Machine Learning (ICML-16), pages 421–429.

[33] Zenke, F., Poole, B., and Ganguli, S. (2017). Continual learning through synaptic intelligence. In Proceedings of the 34th International Conference on Machine Learning (ICML-17), pages 3987–3995.
Appendices to
“Lifelong Learning of Compositional Structures”
by Jorge A. Mendez and Eric Eaton

A Experimental setting

Below, we give additional details describing the experimental setting used in the main paper.

A.1 Data sets

The data sets used for linear experiments underwent the same processing and train/test split of Ruvolo and Eaton [27]. For MNIST and Fashion, we randomly sampled pairs of digits to act as the positive and negative classes in each task, and allowed digits to be reused across tasks. For CUB and CIFAR, ten and five classes were used per task, respectively, without reusing of classes across different tasks. CUB images were cropped by the bounding boxes available with the data set, and resized to 224 × 224. For these four data sets, we used the standard train/test split, and further divided the training set into 80% for training and 20% for validation. Finally, for Omniglot, we used each alphabet as one task, and split the data into 80% for training, 10% for validation, and 10% for test, for each task. For each of the ten trials, we varied the random seed which controlled the tasks (whenever the tasks were not fixed by definition), the random splits for training/validation/test, and the order in which the tasks were presented to the agent. Validation sets were only used by dynamic + compositional learners for selecting whether to keep a new components. Details are summarized in Table A.1.

Table A.1: Data set details summary.

|       | FERA | Landmine | Schools | MNIST | Fashion | CUB | CIFAR | Omniglot |
|-------|------|----------|---------|-------|---------|-----|-------|----------|
| tasks | 21   | 29       | 139     | 10    | 10      | 20  | 20    | 30       |
| classes | 2   | 2        | —       | 2     | 2       | 10  | 5     | 14–55    |
| features | 100 | 9        | 27      | 784   | 784     | 512 | 32×32×3 | 105×105  |
| feat. extract. | PCA | —        | —       | —     | —       | —   | ResNet-18 | —        |
| train | 225–499 | 222–345 | 11–125 | ~9500 | ~9500   | ~120 | ~2000 | 224–880  |
| val   | —    | —        | —       | ~2500 | ~2500   | ~30  | ~500  | 28–110   |
| test  | 225–500 | 223–345 | 11–126 | ~2000 | 2000    | ~150 | 500   | 28–110   |

A.2 Network architectures

We used $k = 4$ components for all compositional algorithms with fixed $k$. This is the only architecture parameter for linear models. Below, we describe the architectures used for the remaining experiments.

**Soft layer ordering**  We based our soft layer ordering architectures on those used by Meyerson and Miikkulainen [19], whenever possible. For MNIST and Fashion, we used a random and fixed linear input transformation $E(t)$ for each task, and each component was a fully-connected layer of 64 units. For CUB, all tasks shared a fixed ResNet-18 pre-trained on ImageNet$^1$ as an input transformation, followed by a task-specific input transformation $E(t)$ given by a linear trained layer, and each component was a fully-connected layer of 256 units. For CIFAR, there was no input transformation, and each component was a convolutional layer of 50 channels with $3 \times 3$ kernels and padding 1, followed by a max-pooling layer of size $2 \times 2$. Finally, for Omniglot, there was also no input transformation, and each component was a convolutional layer of 53 channels with $3 \times 3$ kernels and no padding, followed by max-pooling of $2 \times 2$ patches. The input images to the convolutional nets in CIFAR and Omniglot were padded with all-zero channels in order to match the number of channels required by all component layers (50 and 53, respectively). All component layers were followed by ReLU activation and a dropout layer with dropout probability $p = 0.5$. The output of each network was a linear task-specific output transformation $D(t)$ trained individually on each task. The architectures for jointly trained baselines were identical to these, and those for no-components baselines had the same layers but no mechanism to select the order of the layers.

$^1$The pre-trained ResNet-18 is provided by Pytorch, and we followed the pre-processing recommended at https://pytorch.org/docs/stable/torchvision/models.html
Soft gating} The soft gating architectures mimicked those of the soft layer ordering architectures closely, all having the same input and output transformations, as well as the same components. The only difference was in the structure architectures. For fully-connected networks, at each depth, the structure function $s^{(t)}$ was a linear layer that took as input the previous depth’s output and whose output was a soft selection over the component layers for the current depth. For convolutional networks, there was one gating net per task with the same architecture as the main network. The structure $s^{(t)}$ was computed by passing the previous depth’s output in the main network through the remaining depths in the gating network (e.g., the output of depth 2 in the original network was passed through depths 3 and 4 in the gating network to compute the structure over modules at depth 3).
We now present detailed results that expand upon those presented in Section 5.3 in the main paper. All agents trained for 100 epochs on each task, with a mini-batch of size 32. Compositional agents used the first 99 epochs solely for assimilation and the last epoch for accommodation (1). dynamic + compositional agents followed this same process, but every assimilation step was concurrently done with and without addition of a new component; after the accommodation (1) step, the agent kept the new component if its validation performance with the added component represented at least a 5% relative improvement over the performance without the additional component. Joint agents trained all components and the structure for the current task jointly during all 100 epochs, keeping the structure for the previous tasks fixed, while no-components agents trained the whole model at every epoch.

ER-based algorithms used a replay buffer of a single mini-batch per task. Similarly, EWC-based algorithms used a single mini-batch to compute the approximate Fisher information matrix required for regularization, and used a fixed regularization parameter $\lambda = 10^{-1}$. To ensure a fair comparison, all algorithms, including our baselines, used the same initialization procedure by training the first $n_{\text{init}} = 4$ tasks jointly, in order to encourage the network to generalize across tasks. For soft ordering networks, the order of modules for the initial tasks was initialized as a random one-hot vector for each task at each depth, ensuring that each component was selected at least once, and for soft gating networks, the gating networks were randomly initialized. The structures over initial tasks were kept fixed during training, modifying only the weights of the components.

B Additional results for quantitative experiments

We now present detailed results that expand upon those presented in Section 5.3 in the main paper. For completeness, we include expanded results from Figures 2 and 3 in the main paper, corresponding to soft layer ordering. Figure B.1 is a more detailed version of Figure 2 and shows the test accuracy immediately after each task was trained and after all tasks had been trained, separately for each data set. Compositional algorithms conforming to our proposed framework achieve a better trade-off than others in flexibility and stability, leading to good adaptability to each task with little forgetting of previous tasks. Similarly, Figure B.2 shows learning curves similar to those in Figure 3 in the main paper, for each data set. Baselines that train components and structures jointly all exhibit a decay in...
the performance of earlier tasks as learning of future tasks progresses, whereas methods conforming to our framework do not. Results for soft gating nets display a similar behavior.

The gap between the first and second bar for each algorithm in Figure B.1 is an indicator of the amount of catastrophic forgetting. However, it hides details of how forgetting affects each individual task. On the other hand, the decay rate of each task in Figure B.2 shows how each task is forgotten over time, but does not measure quantitatively how much forgetting occurred. Based on prior work [15], we evaluated the ratio of performance after each task was trained to after all tasks had been trained as a metric of knowledge retention. Results in Figure B.3 show that compositional methods exhibit substantially less catastrophic forgetting, particularly for the earlier tasks seen during training.

In our experiments, it was in many cases necessary to incorporate an accommodation (2) step in order for our algorithm to be sufficiently flexible to handle the stream of incoming tasks. This accommodation (2) step enables our methods to dynamically add new components if the existing ones are insufficient to achieve good performance in the new task. Table B.1 shows the number of components learned by each dynamic algorithm using both soft ordering and soft gating, averaged across all ten trials. Notably, in the soft ordering case, in order for our methods to work on the CIFAR data set, they required learning almost one component per task. This explains why compositional algorithms without dynamic component additions were incapable of performing well on CIFAR. It is also worth noting that soft gating nets typically required adding fewer new components, which is to be expected, since the gating structure gives the learner substantially more flexibility. Recall that, as mentioned in Section 5.3, soft gating networks were unable to perform well on the CUB data set because of the small sample size, so it is omitted from the table.

One of the key aspects of lifelong learning is the ability to learn in the presence of little data for each task, using knowledge acquired from previous tasks to acquire better generalization for new tasks. To evaluate the sample efficiency of our algorithm, we varied the number of data points used for training for MNIST, Fashion, and CUB using the soft ordering structure and ER. We repeated the evaluation for fifty trials, controlling the selection of classes and samples for each task, and the order over tasks. Learners were trained for 1,000 epochs, with our compositional methods alternating nine epochs of assimilation and one epoch of accommodation (1). We used a batch size of $b = 32$, and limited the replay buffer size to $\min (\max (\lfloor 0.1n \rfloor, 1), b)$ for each data size $n$. Figure B.4 shows the learning accuracy for ER-based algorithms as a function of the number of training points, revealing that compositional algorithms work better than baselines even in the presence of very little data.

### Table B.1: Number of learned components. Standard errors reported after the ±.

| Structure | Base | MNIST | Fashion | CUB | CIFAR | Omniglot |
|-----------|------|-------|---------|-----|-------|----------|
| Soft ordering | ER | 5.2 ± 0.3 | 4.9 ± 0.3 | 5.9 ± 0.3 | 19.1 ± 0.3 | 9.3 ± 0.3 |
|           | EWC | 5.0 ± 0.3 | 4.7 ± 0.2 | 5.7 ± 0.3 | 19.5 ± 0.2 | 10.1 ± 0.4 |
|           | VAN | 5.0 ± 0.2 | 4.8 ± 0.3 | 6.1 ± 0.3 | 17.7 ± 0.3 | 10.0 ± 0.7 |
|           | FM  | 10.0 ± 0.0 | 8.8 ± 0.2 | 6.5 ± 0.4 | 19.1 ± 0.4 | 10.2 ± 0.6 |
| Soft gating | ER | 4.0 ± 0.0 | 4.2 ± 0.1 | — | 4.1 ± 0.1 | 7.1 ± 0.4 |
|           | EWC | 4.1 ± 0.1 | 4.0 ± 0.0 | — | 4.2 ± 0.1 | 7.1 ± 0.4 |
|           | VAN | 4.1 ± 0.1 | 4.2 ± 0.1 | — | 4.1 ± 0.1 | 7.2 ± 0.3 |
|           | FM  | 5.4 ± 0.2 | 4.7 ± 0.2 | — | 4.4 ± 0.2 | 7.3 ± 0.4 |

Figure B.4: Accuracy of ER-based methods with varying data sizes. Compositional methods achieved better performance even with extremely little data per task. Shaded area represents std. errors.
C Visualization of the learned components

The primary motivation for our framework was the creation of lifelong learning algorithms capable of discovering self-contained, reusable components, useful for solving a variety of tasks. In this section, we qualitatively evaluate the behavior of components learned under our framework as compared to components learned with traditional lifelong learning methods. To do so, we ran a visualization experiment on MNIST, where the goal of each task was to predict a specific pixel’s intensity.

More concretely, we followed the visualization experiment of Meyerson and Miikkulainen [19], where each task corresponded to a single image of the digit “4”, and each pixel in the image constituted one data-point. The x, y coordinates of the pixel were used as features, and the pixel’s intensity was the associated label. Pixel coordinates and intensities were normalized to [0, 1]. All pixels in the image were treated as training data, since we were interested in understanding the learned representations, as opposed to generalizing to unseen data. Our network had \( k = 4 \) components shared across all tasks, and used soft layer ordering to learn the structure \( s^{(t)} \) for each task. We used a linear input transformation layer \( E^{(t)} \) shared across all tasks, and a shared sigmoid output transformation layer \( D^{(t)} \). Sharing the input and output transformations across tasks ensures that the only differences across the models of the different tasks are due to the structure of each task over the components. We trained the network to minimize the binary cross-entropy loss on \( T_{\text{max}} = 10 \) tasks for 1,000 epochs via the compositional and jointly trained versions of ER with a replay buffer and batch size of 32 pixel instances, updating the components of the compositional version every 100 epochs.

To evaluate the ability of compositional ER to capture reusable functional primitives, we observed the reconstructed images output by our network as we varied the intensity \( \psi_{i,j}^{(t)} \) with which one specific component \( i \) is chosen at different depths \( j \) in the network. We focused our evaluation on the last two tasks seen by the learner, in order to disregard the effects of catastrophic forgetting, which rendered the visualizations of the outputs of the joint ER baseline incomprehensible for earlier tasks. Figure C.1 shows these reconstructions as the intensity of component \( i = 0 \) varies at different depths. The component learned with compositional ER learned to vary the thickness of the digit regardless of the task at hand, with a more extreme effect at the initial layers. In contrast, joint ER learned a component whose effect is different for different tasks and at different depths.

![Figure C.1](image-url)

Figure C.1: Visualization of reconstructed MNIST “4” digits on the last two tasks seen by the compositional and joint variants of ER with soft layer ordering, varying the intensity with which a single specific component is selected. Compositional ER learned a component that performs a functional primitive: the more intensely the component is selected (moving from left to right on each row), the thinner the lines of the digit become. The magnitude of this effect decreases with depth (moving from top to bottom), with the digit completely disappearing as the component is more intensely selected at the earliest layers, but only becoming slightly sharper with intensity at the deepest layers. This effect is consistent across both tasks. Joint ER did not exhibit this consistent behavior, with different effects observed at different depths and for the different tasks.