The Critical Value of the Contact Process with Added and Removed Edges

Paul Jung

Department of Mathematics, Cornell University

Abstract

We show that the critical value for the contact process on a vertex-transitive graph \( G \) with finitely many edges added and/or removed is the same as the critical value for the contact process on \( G \). This gives a partial answer to a conjecture of Pemantle and Stacey.

Keywords: Interacting particle system; Contact process; Phase transition; Infinitesimal coupling

1 Introduction

The contact process with infection rate \( \lambda \) is an interacting particle system whose state space is \( X = \{0, 1\}^G \) for some connected graph \( G \) with countably many vertices (we use \( G \) to describe both the vertex set and the graph, depending on the context). This process was first introduced by Harris (1974) as a stochastic model for the spread of an infection (1’s mark infected sites and 0’s mark healthy sites). A simple description of the process is as follows. Each healthy site \( x \in G \) becomes infected at an exponential rate which is proportional to the number of infected neighbors of \( x \) and each infected site becomes healthy at exponential rate 1. The contact process has been extensively studied since the 1970’s; for a detailed account of what is known about this process we refer the reader to Liggett(1999).

To define the process more precisely, let \( \eta \in X \). We denote

\[
\eta_x(u) = \begin{cases} 
1 - \eta(u) & \text{if } u = x \\
\eta(u) & \text{if } u \neq x 
\end{cases}
\]

and

\[
c_G(x, \eta) = \begin{cases} 
1 & \text{if } \eta(x) = 1 \\
\lambda \sum_{|y-x|=1} \eta(y) & \text{if } \eta(x) = 0
\end{cases}.
\]

The generator of the contact process \( \eta_t \) is now given by the closure of the operator \( L_G \) on \( D(X) \), the set of all functions on \( X \) depending on finitely many coordinates:

\[
L_G f(\eta) = \sum_x c_G(x, \eta) [f(\eta_x) - f(\eta)], \quad f \in D(X).
\]
The subscript \(\mathcal{G}\) reminds us that the generator and the transition rates depend on the graph \(\mathcal{G}\). We write \(S_\mathcal{G}(t)\) for the semigroup of this process.

If the initial state of the process is such that \(\eta_0(x) = 1\) for only finitely many \(x \in \mathcal{G}\), then the process is just a Markov chain, and its state at time \(t \geq 0\) is given by \(A_t\) where \(A\) is the finite subset of \(\mathcal{G}\) which is exactly the set of all infected sites at time \(t\). In the sequel, \(A\) will always be a finite subset of \(\mathcal{G}\). One of the basic tools used in the study of the contact process is its self-duality. In other words:

\[
P^n[\eta_t(x) = 0 \text{ for all } x \in A] = P^A[\eta_t(x) = 0 \text{ for all } x \in A_t].
\]

The above result is based on a graphical construction of the contact process, and it can be found in any reference on the contact process (for example Liggett(1999)).

One of the goals in the treatment of any interacting particle system is to study the invariant measures. For the contact process, it is easily seen that any reference on the contact process (for example Liggett(1999)). We note here that \(\delta_0\) is the point mass on all 0’s, is an invariant measure. Also, if \(\delta_1\) is the point mass on all 1’s then simple coupling and monotonicity arguments show that the upper invariant measure

\[
\lim_{t \to \infty} \delta_1S_\mathcal{G}(t) = \mu_\mathcal{G}
\]

exists and \(\mu_\mathcal{G}\) stochastically dominates every other invariant measure. If \(\mu_\mathcal{G} = \delta_0\) then \(\delta_0\) is the only invariant measure and the process is ergodic. We have the following definition for \(\lambda_c\), the critical value of the contact process:

\[
\lambda_c = \sup\{\lambda \geq 0 : \mu_\mathcal{G} = \delta_0\}.
\]

We note here that \(\lambda_c\) is often called the global survival critical value or the lower critical value. We mention this because there is another natural critical value for the contact process which is known as the local survival critical value or the upper critical value, however, in this paper we have no need to define this other critical value.

Suppose \(\mathcal{G}\) is a connected graph with countably many vertices. Let \(\mathcal{G}'\) be a graph formed by adding \(n\) edges to the graph \(\mathcal{G}\). We will say that the \(i^{th}\) new edge is placed between the vertices \(u_i\) and \(v_i\) where \(u_i \neq v_i\) since loops are meaningless in the contact process. The new edges can be placed between two vertices that already have an edge in \(\mathcal{G}\), however, to simplify things we assume that each element in the set \(\{u_1, v_1, \ldots, u_n, v_n\}\) is distinct. We use the phrase “to simplify things” here because this requirement is not necessary, but it makes the proofs easier to follow.

Let \(\lambda'_c\) be the critical value of the contact process on \(\mathcal{G}'\). If \(\lambda_c\) is the critical value for the contact process on \(\mathcal{G}\), then Pemantle and Stacey(2000) have conjectured that \(\lambda'_c = \lambda_c\). When \(\mathcal{G} = \mathbb{Z}^d\), the following argument which uses duality together with a result of Bezuidenhout and Grimmett(1991) shows that this is true. For simplicity we consider the case where \(\mathcal{G}'\) differs from \(\mathcal{G}\) only by the addition of one edge between the vertices \(u\) and \(v\).

Let \(A_t\) be the process on \(\mathcal{G}\) and let \(A'_t\) be the process on \(\mathcal{G}'\). Now suppose there exists a nontrivial upper invariant measure \(\mu_{\mathcal{G}'}\) for some \(\lambda < \lambda_c\). If \(||\cdot||\) denotes graph distance from some distinguished vertex labelled the origin, then

\[
\mu_{\mathcal{G}'} \{ \eta : \eta(x) = 1 \} = P^{\lambda}_x(\{A'_t \neq \emptyset \text{ for all } t\})
\]

\[
= P^{\lambda}_x(\{A'_t \neq \emptyset \text{ for all } t\} \cap \{u \in A'_t \text{ or } v \in A'_t \text{ for some } t\})
\]

\[
\leq e^{-c||x||} \text{ for some } c > 0
\]
1 INTRODUCTION

where the inequality comes from Theorem 1.7 of Bezuidenhout and Grimmett(1991). But now the above inequality implies that \( \sum_{x} \mu_{G'} \{ \eta : \eta(x) = 1 \} < \infty \) which means that \( \mu_{G'} \) concentrates on configurations with finitely many ones contradicting its invariance. Therefore \( \lambda'_c = \lambda_c \).

The above argument can easily be extended to adding any finite number of edges to \( G = \mathbb{Z}^d \). The issue of removing edges is a bit trickier; it requires us to start from a graph \( G \) which is exactly \( \mathbb{Z}^d \) with finitely many edges removed. This is not much of a problem since it seems that the argument used to show exponential decay in Theorem 1.7 in Bezuidenhout and Grimmett(1991) can be extended to such graphs. However, we would also like to know that the conjecture of Pemantle and Stacey holds for graphs such as \( \mathbb{T}^d \), and here we run into a problem since the above argument depends on the amenability of \( \mathbb{Z}^d \). The goal of this paper is to introduce an alternate argument which shows that \( \lambda'_c = \lambda_c \) whenever we can satisfy a certain integrability condition known to hold for the subcritical contact process even on nonamenable graphs.

Since \( G \) and \( G' \) have the same vertex sets, we will always use the notation \( G \) when referring to the vertex set of either graph. As noted above, it should be clear from the context when \( G \) refers to the vertex set rather than the graph.

**Theorem 1.1.** If \( A_{t}^{(o)} \) is the contact process on \( G \) starting from one infection at the origin, \( o \), and for \( \lambda < \lambda_c \) we have

\[
\int_{0}^{\infty} E|A_{t}^{(o)}|dt < \infty,
\]

then \( \lambda'_c = \lambda_c \).

Note that \( \int_{0}^{\infty} E|A_{t}^{(o)}|dt < \infty \) if and only if \( \int_{0}^{\infty} E|A_{t}^{A}|dt < \infty \) for all finite \( A \subset G \).

**Corollary 1.2.** If \( G \) is a vertex-transitive graph and \( G' \) is formed by adding and/or removing finitely many edges from \( G \), then the critical values for the contact process on \( G \) and \( G' \) are the same.

The corollary follows from the arguments of Aizenman and Barsky(1987) where it is shown that \( \check{\Box} \) holds for all \( \lambda < \lambda_c \) whenever \( G \) is transitive. Aizenman and Barsky(1987) actually concentrate on a discrete-time percolation model, but we have learned through personal communication that Aizenman has an unpublished extension to the contact process. The current author also has proved such an extension in a forthcoming paper. It can be seen in the arguments of Aizenman and Barsky(1987), that the proof is also valid for transitive graphs with finitely many edges removed.

Some comments are in order concerning the techniques used in the proofs below. Theorem \( \Box \) is a result that gives information about the contact process when the infection rates are perturbed at a finite number of sites. The techniques used in the proof work equally well when the healing rates are perturbed (these processes are known as inhomogeneous contact processes). In particular, it can be seen from the proofs below that if the healing rates are lowered at a finite number of sites for the contact process on a vertex-transitive graph, then the critical value is left unchanged. This is a special case of a result proved by Madras, Schinazi, and Schonmann(1994) for inhomogeneous contact processes on \( \mathbb{Z}^d \). We also note that the techniques in the proofs can also be applied to other spin systems. For example, one can extend to all dimensions, Theorem 2 of Handjani(1999) which concerns a perturbed biased-voter model on \( \mathbb{Z} \).
2 A generator computation

In this section we prove a crucial lemma which uses certain coupled processes to gain information about the evolution of the upper invariant measure for the contact process on \( G' \) (which we denote as \( \mu_{g'} \)) under the semigroup \( S_g \). As we will see, the lemma basically boils down to a generator computation, giving us the title of this section. The methods of this section are motivated by the infinitesimal coupling of the exclusion process used in Andjel, Bramson, and Liggett(1988) and again in Jung(2004).

Before stating the lemma, we describe the couplings to be used. All couplings to be used will follow the motion of the basic coupling for two processes \( \eta_t \) and \( \xi_t \). The basic coupling is the coupling of \( \eta_t \) and \( \xi_t \) which allows the two processes to move together as much as possible (see Liggett(1985) Chapter III for more details). One of the most useful properties of the basic coupling is the fact that it preserves stochastic domination in time. In particular, if \( \eta_t \) and \( \xi_t \) are coupled using a basic coupling then \( \eta_0(x) \leq \xi_0(x) \) for all \( x \in G \) implies that \( \eta_t(x) \leq \xi_t(x) \) for all \( x \in G \).

We now describe the various initial measures we use for the couplings below. Let \( \{u_i\} \) and \( \{v_i\} \) be as before. If \( \eta \) is given the measure \( \mu_u \), define

\[
D_u^i = \{ \eta : \eta(u_i) = 0, \eta(v_i) = 1 \} \quad \text{and} \quad D_v^i = \{ \eta : \eta(u_i) = 1, \eta(v_i) = 0 \}.
\]

The measures \( \mu_u^i \), and \( \mu_v^i \) are defined by conditioning \( \mu_{g'} \) on the events \( D_u^i \) and \( D_v^i \) respectively. Also, define the measures \( \hat{\mu}_u^i \) and \( \hat{\mu}_v^i \) to be exactly equal to \( \mu_u^i \) and \( \mu_v^i \) except that we change the values at \( u_i \) and \( v_i \) so that \( \eta(u_i) = \eta(v_i) = 1 \).

We can now define the initial measures \( \nu_u^i \) and \( \nu_v^i \) for the coupled processes \( (\eta_u^{i,\cdot}, \xi_u^{i,\cdot}) \) and \( (\eta_v^{i,\cdot}, \xi_v^{i,\cdot}) \). The measure \( \nu_z^i \) for \( z = u, v \) has marginal measures \( \mu_z^i \) and \( \hat{\mu}_z^i \) corresponding to \( \eta_0^{i,\cdot} \) and \( \xi_0^{i,\cdot} \) respectively and the marginals are coupled so that \( \eta_0^{i,\cdot}(x) \leq \xi_0^{i,\cdot}(x) \) for all \( x \).

**Lemma 2.1.**

\[
\frac{d}{dt} \mu_{g'} S_g(t) \{ \eta : \eta(x) = 1 \} = \sum_{i=1}^{n} \sum_{z = u, v} \lambda_{\mu_{g'}} \{ D_z^i \} E[\eta_z^{i,\cdot}(x) - \xi_z^{i,\cdot}(x)]
\]

**Proof.** We write \( 1_x(\eta) = \eta(x) \) and \( 1_x^+(\eta) = S_g(t) 1_x(\eta) \). Letting \( \mu_g^t_{g'} = \mu_{g'} S_g(t) \) we have

\[
\frac{d}{dt} \mu_{g'} S_g(t) \{ \eta : \eta(x) = 1 \} = \lim_{s \to 0} \frac{1}{s} \int 1_x \ d\mu_{g'}^{t+s} - \int 1_x \ d\mu_{g'}^t \]

\[
= \lim_{s \to 0} \frac{1}{s} \int 1_x^+ \ d\mu_{g'} - \int 1_x^- \ d\mu_{g'}.
\]
By the definition of the generator, the above is equal to
\[
= \int L_{\mathcal{G}} 1_x^i \, d\mu_{\mathcal{G}}^i
= \int L_{\mathcal{G}} 1_x^i \, d\mu_{\mathcal{G}}^i + \sum_{i=1}^{n} \sum_{z=u,v} \int (c_{\mathcal{G}}(z_i, \eta) - c_{\mathcal{G}^i}(z_i, \eta)) [1_x^i(\eta_{z_i}) - 1_x^i(\eta)] \, d\mu_{\mathcal{G}}^i
= \sum_{i=1}^{n} \left[ \int \eta(\nu_i)(1 - \eta(\nu_i))(-\lambda)[1_x^i(\eta_{u_i}) - 1_x^i(\eta)] \, d\mu_{\mathcal{G}}^i + \int \eta(\nu_i)(1 - \eta(\nu_i))(-\lambda)[1_x^i(\eta_{v_i}) - 1_x^i(\eta)] \, d\mu_{\mathcal{G}}^i \right]
= \sum_{i=1}^{n} \lambda \left[ \int \eta(\nu_i)(1 - \eta(\nu_i)) [1_x^i(\eta) - 1_x^i(\eta)] \, d\mu_{\mathcal{G}}^i + \int \eta(\nu_i)(1 - \eta(\nu_i)) [1_x^i(\eta) - 1_x^i(\eta)] \, d\mu_{\mathcal{G}}^i \right]
= \sum_{i=1}^{n} \sum_{z=u,v} \lambda \mu_{\mathcal{G}}^i \{ D_z^i \} \int [1_x^i(\eta) - 1_x^i(\eta_{z_i})] \, d\mu_z^i
= \sum_{i=1}^{n} \sum_{z=u,v} \lambda \mu_{\mathcal{G}}^i \{ D_z^i \} E[\eta^z_i(x) - \eta^z_i(x)].
\]

The third equality above follows since \( \int L_{\mathcal{G}} 1_x^i \, d\mu_{\mathcal{G}}^i = 0 \) by the invariance of \( \mu_{\mathcal{G}}^i \) under \( L_{\mathcal{G}}^i \). □

3 Proof of Theorem 1.1

Proof of Theorem 1.1 Let \( \zeta^z_i = \xi^z_i - \eta^z_i \). It is clear from the way that \( \eta^z_i \) and \( \xi^z_i \) are coupled that \( \zeta^z_i(x) \leq A_t^{\{z_i\}}(x) \) for all \( x \in \mathcal{G} \). Therefore
\[
\sum_{x \in \mathcal{G}} E[\zeta^z_i(x) - \eta^z_i(x)] \leq E[A_t^{\{z_i\}}].
\]

By Lemma 2.1 we get that
\[
(-1) \sum_{x \in \mathcal{G}} \frac{d}{dt} \mu_{\mathcal{G}}^i S_{\mathcal{G}}(t) \{ \eta(x) = 1 \} \leq \sum_{i=1}^{n} \sum_{z=u,v} \lambda \mu_{\mathcal{G}}^i \{ D_z^i \} E[A_t^{\{z_i\}}].
\]

Using monotonicity arguments it is easy to show that \( \lim_{t \to \infty} \mu_{\mathcal{G}}^i S_{\mathcal{G}} = \mu_{\mathcal{G}} \) so integrating both sides of (3) with respect to \( t \) from 0 to \( \infty \) gives
\[
\sum_{x \in \mathcal{G}} [\mu_{\mathcal{G}}^i \{ \eta(x) = 1 \} \mu_{\mathcal{G}} \{ \eta(x) = 1 \}] = (-1) \int_0^\infty \sum_{x \in \mathcal{G}} \frac{d}{dt} \mu_{\mathcal{G}}^i S_{\mathcal{G}}(t) \{ \eta(x) = 1 \} \, dt
\leq \int_0^\infty \sum_{i=1}^{n} \sum_{z=u,v} \lambda \mu_{\mathcal{G}} \{ D_z^i \} E[A_t^{\{z_i\}}] \, dt
\]

When \( \lambda < \lambda_c \), the right-hand side is finite and \( \mu_{\mathcal{G}} = \delta_0 \), therefore
\[
\sum_{x \in \mathcal{G}} \mu_{\mathcal{G}}^i \{ \eta(x) = 1 \} < \infty.
\]
But this implies that when $\lambda < \lambda_c$, $\mu_{G'}$ concentrates on configurations with finitely many infected sites. Since $\mu_{G'}$ is a stationary distribution for a Markov chain which has $\delta_0$ as its only absorbing state, it must be that $\mu_{G'} = \delta_0$ when $\lambda < \lambda_c$ which implies $\lambda'_c = \lambda_c$.

Acknowledgement. We thank Rick Durrett for many useful discussions concerning the contact process and for his mentorship this past year.

References

[1] Aizenman, M. and Barsky, D. J. (1987) Sharpness of the phase transition in percolation models. *Comm. Math. Phys.*, 108, 489-526.

[2] Andjel, E. D., Bramson, M. D. and Liggett T. M. (1988) Shocks in the asymmetric exclusion process. *Probab. Th. Rel. Fields*, 78, 231-247.

[3] Bezuidenhout, C. and Grimmett, G. (1991) Exponential decay for subcritical contact and percolation processes. *Ann. Probab.*, 19, 984-1009.

[4] Harris, T. E. (1974) Contact interactions on a lattice. *Ann. Probab.*, 2, 969-988.

[5] Handjani, S. (1999) Spatial perturbations of one-dimensional spin systems. *Stoch. Proc. Appl.*, 81, 73-79.

[6] Jung, P. H. (2004) Perturbations of the symmetric exclusion process, to appear in *Markov Proc. Rel. Fields*.

[7] Liggett, T. M. (1985) *Interacting Particle Systems*. Springer-Verlag, New York.

[8] Liggett, T. M. (1999) *Stochastic Interacting Systems: Contact, Voter, and Exclusion Processes*. Springer-Verlag, Berlin Heidelberg.

[9] Madras, N., Schinazi, R. and Schonmann, R. H. (1994) On the critical behavior of the contact process in deterministic inhomogeneous environments. *Ann. Probab.*, 22, 1140-1159.

[10] Menshikov, M. V. (1986) Coincidence of the critical points in percolation problems. *Soviet Math. Dokl.*, 33, 856-859.

[11] Pemantle, R. and Stacey, A. M. (2001) The branching random walk and contact process on non-homogeneous and Galton-Watson trees. *Ann. Probab.*, 29, 1563-1590.