MOTION PLANNING IN POLYHEDRAL PRODUCTS OF GROUPS AND A FADELL-HUSSEINI APPROACH TO TOPOLOGICAL COMPLEXITY

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Abstract. We compute the topological complexity of a polyhedral product $Z$ defined in terms of an LS-logarithmic family of locally compact connected CW topological groups. The answer is given by a combinatorial formula that involves the LS category of the polyhedral-product factors. As a by-product, we show that the Iwase-Sakai conjecture holds true for $Z$. The proof methodology uses a Fadell-Husseini viewpoint for the monoidal topological complexity (MTC) of a space, which, under mild conditions, recovers Iwase-Sakai’s original definition. In the Fadell-Husseini context, the stasis condition —MTC’s raison d’être— can be encoded at the covering level. Our Fadell-Husseini-inspired definition provides an alternative to the MTC variant given by Dranishnikov, as well as to the ones provided by García-Calcines, Carrasquel-Vera and Vandembroucq in terms of relative category.

1. Introduction

As originally introduced in [12], the monoidal topological complexity of a path-connected space $X$, denoted by $\text{TC}^M(X)$, is apparently more restrictive than Farber’s topological complexity of $X$, $\text{TC}(X)$. For a locally finite simplicial complex $X$, Iwase and Sakai showed in [13, Theorem 1] that $\text{TC}^M(X)$ differs from $\text{TC}(X)$ by at most one unit, and conjectured that the equality $\text{TC}^M(X) = \text{TC}(X)$ holds true.

In [5, Lemma 2.7] Dranishnikov proved Iwase-Sakai’s conjecture for a connected Lie group $G$,

\begin{equation}
\text{TC}(G) = \text{cat}(G) = \text{TC}^M(G),
\end{equation}

where the equality between $\text{TC}(G)$ and $\text{cat}(G)$, the LS category of $G$, is well known. The first main result in this paper generalizes (1) to the realm of polyhedral products defined by LS-logarithmic families of locally compact connected CW topological groups. In particular, we show that Iwase-Sakai’s conjecture remains valid in such a context. Explicitly:

Theorem 1.1. Let $G^K$ denote the polyhedral product associated to an abstract simplicial complex $K$ with vertex set $\{1, 2, \ldots, m\}$ and a based family $\mathcal{G} = \{(G_i, e_i)\}_{i=1}^m$ of locally compact connected CW topological groups, where $e_i$ stands for the neutral element of $G_i$. If $\mathcal{G}$ is an LS-logarithmic family, i.e., if the equality

\[\text{cat}(G_{i_1} \times \cdots \times G_{i_k}) = \text{cat}(G_{i_1}) + \cdots + \text{cat}(G_{i_k})\]

holds true for any strictly increasing sequence $1 \leq i_1 < \cdots < i_k \leq m$, then

\begin{equation}
\text{TC}(G^K) = \text{TC}^M(G^K) = \max \left\{ \sum_{i \in \sigma_1 \cup \sigma_2} \text{cat}(G_i) : \sigma_1, \sigma_2 \in K \right\}.
\end{equation}

The expression in (2) should be compared to the fact that the LS category of $G^K$ is also determined by the LS category of the polyhedral-product factors. Namely, under the hypotheses

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of Theorem 1.1, [1, Theorem 1.4] gives
\[ \text{cat}(G^K) = \max \left\{ \sum_{i \in \sigma} \text{cat}(G_i) : \sigma \in K \right\}. \]

The proof of Theorem 1.1 involves a Fadell-Husseini flavored definition of monoidal topological complexity of a space. Such a concept arises naturally by inputting Dranishnikov’s and García-Calcines’ views —reviewed below— into Iwase-Sakai’s original definition.

Dranishnikov noticed in [5] that, if \( X \) is an ENR space (or more generally an ANR space as we will see in Proposition 2.3 below), Iwase-Sakai’s definition of \( \text{TC}^M(X) \) can be relaxed in the sense that the diagonal of \( X \) does not have to be contained in each open domain covering \( X \times X \) (yet, the continuous local sections of the end-points evaluation map \( e_{01} : X^{[0,1]} \to X \times X \) are still required to yield constant paths on points of the diagonal). Furthermore, in [9, Remark 2.19], García-Calcines proved that, when \( X \) is an ANR space, \( \text{TC}^M(X) \) can be defined in terms of general (not necessarily open) covers of \( X \times X \) by following the lines in Iwase-Sakai’s work, that is, by requiring that the diagonal lies in each subset covering \( X \times X \), and that the corresponding local sections yield constant paths when restricted to the diagonal of \( X \). On the other hand, Fadell and Husseini defined in [7] the relative category of a cofibered pair \( (X, A) \), denoted by \( \text{cat}^{FH}(X, A) \), in terms of open sets covering \( X \). In contrast to the definition of \( \text{cat}(X) \), in the relative-category context only one of the covering open subsets is required to contain \( A \) and deform to \( A \) (rel \( A \)) within \( X \), while the rest of the open sets are actually required to deform within \( X \) to a point.

We introduce the Fadell-Husseini monoidal topological complexity of a space \( X \) by blending the definition of \( \text{cat}^{FH}(X, A) \) into Dranishnikov’s and García-Calcines’ viewpoints for \( \text{TC}^M(X) \). Explicitly:

**Definition 1.2.**

(a) The Fadell-Husseini monoidal topological complexity of a path connected space \( X \), denoted by \( \text{TC}^{FH}(X) \), is the smallest nonnegative integer \( n \) for which there is an open cover \( \{ U_0, \ldots, U_n \} \) of \( X \times X \) by \( n+1 \) subsets, on each of which there exists a continuous section \( s_i : U_i \to X^{[0,1]} \) of the end-points evaluation map \( e_{01} \) so that:

1. \( U_0 \) contains the diagonal \( \Delta X = \{(x, x) : x \in X\} \);
2. \( s_0(x, x) = e_x \), the constant path at \( x \), for all \( x \in X \);
3. \( \Delta X \cap U_i = \emptyset \) for all \( i \geq 1 \).

(b) The Fadell-Husseini generalized monoidal topological complexity \( \text{TC}^{gFH}(X) \) is defined as above, except that the elements of the covers \( \{ U_0, \ldots, U_n \} \) are not required to be open.

We agree to set \( \text{TC}^{gFH}(X) = \infty \) or \( \text{TC}^{FH}(X) = \infty \), if the required coverings fail to exist.

The second main result in this paper asserts that Definition 1.2 recovers the ones given by Iwase-Sakai (\( \text{TC}^M \)), Dranishnikov (\( \text{TC}^{DM} \)) and García-Calcines (\( \text{TC}^g \)) when working with ANR spaces.

**Theorem 1.3.** If \( X \) is an ANR space, then
\[ \text{TC}^{FH}(X) = \text{TC}^{gFH}(X) = \text{TC}^{DM}(X) = \text{TC}^M(X) = \text{TC}^g(X). \]

As noted above, the last equality in Theorem 1.3 is due to García-Calcines, whereas the next-to-last equality is due to Dranishnikov (see Proposition 2.3 below).

In the generalized setting, condition (3) in Definition 1.2 can be omitted without altering the value of \( \text{TC}^{FH}(X) \), as the diagonal \( \Delta X \) can be removed, if needed, from the sets \( U_1, \ldots, U_n \). A similar observation applies in the non-generalized setting as long as \( X \) is a Hausdorff space. Far more striking is the role of the stasis condition (2) in Definition 1.2. The next result was pointed out to the authors by J. M. García-Calcines:
Theorem 1.4. If \( X \) is a locally equiconnected Hausdorff space such that \( X \times X \) is normal, then the stasis condition (2) in Definition 1.2 can be ignored without altering the resulting numerical value of \( TC^{FH}(X) \). The same conclusion holds in the generalized setting if \( X \) is an ANR space.

Consequently, the equality \( TC(X) = TC^M(X) \) in the Iwase-Sakai conjecture holds true for any ANR space \( X \) for which there is a (not necessarily monoidal) motion planner with \( TC(X) + 1 \) (not necessarily open) local domains one of which contains the diagonal \( \Delta X \) (cf. Corollary 3 in [13]).

The proof of Theorem 1.3 follows the guideline established by García-Calcines in [9] to show that \( TC^M(X) = TC^M_g(X) \) when dealing with an ANR space \( X \). In fact, paralleling many of his techniques, we introduce in Definitions 3.1 and 3.4 below the notions of \( \text{relcat}_{op}^{FH}(i_X) \) and \( \text{relcat}_g^{FH}(i_X) \), where \( i_X : A \hookrightarrow X \) stands for a cofibration. Both concepts represent a Fadell-Husseini version of the concepts \( \text{relcat}_{op}(i_X) \) and \( \text{relcat}_g(i_X) \), which were widely studied in [9] in order to provide a characterization by covers of relative category in the sense of Doeraene-El Haouari. We show that if \( X \) is an ANR space and \( i_X : \Delta X \hookrightarrow X \times X \) denotes the canonical cofibration, then

\[
\text{relcat}_g^{FH}(i_X) = \text{relcat}_{op}^{FH}(i_X) = \text{relcat}_g(i_X) = \text{relcat}_{op}(i_X),
\]

where the last two equalities were proved in [9, Theorems 1.6 and 2.16]. Finally, the crucial ingredient to prove Theorem 1.3 comes from [2, Theorem 12], where the central term in (4) is shown to agree with \( TC^M(X) \).

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2. Preliminaries

2.1. Topological complexity and its monoidal version.

Definition 2.1 (Farber). For a path-connected space \( X \), let \( e_{01} : X^{[0,1]} \to X \times X \) be the end-points evaluation map that sends each path \( \gamma \) in \( X \) into the ordered pair \( (\gamma(0), \gamma(1)) \). The topological complexity of \( X \), denoted by \( TC(X) \), is defined as the smallest \( n \) for which there is an open cover \( \{U_0, \ldots, U_n\} \) of \( X \times X \) by \( n + 1 \) open sets, on each of which there is a continuous local section \( s_i : U_i \to X^{[0,1]} \) of \( e_{01} \). If the coverings fail to exist, we agree to set \( TC(X) = \infty \).

The open sets \( U_i \) covering \( X \times X \) and the corresponding local sections \( s_i \) of \( e_{01} \) are called local domains and local rules, respectively, whereas the family \( \{(U_i, s_i)\} \) is known as a motion planner.

As we pointed out at the beginning of the introduction, monoidal topological complexity is an apparently stronger version of \( TC \) satisfying an additional condition: if \( X \) is thought of as the configuration space of a mechanical system and \( A, B \in X \) is a pair of initial-final configurations of the system with \( A = B \), then the continuous motion of the system from \( A \) to \( B \) is required to be the constant path at \( A \). Such a requirement appears to be quite natural in actual applications. Explicitly:

Definition 2.2 (Iwase-Sakai). The monoidal topological complexity of a path-connected space \( X \), denoted by \( TC^M(X) \), is the smallest \( n \) for which there is an open cover \( \{U_0, \ldots, U_n\} \) of \( X \times X \) by \( n + 1 \) open sets, each one containing the diagonal \( \Delta X = \{(x, x) : x \in X\} \), and on each of which there is a continuous local section \( s_i : U_i \to X^{[0,1]} \) of the end-points evaluation map \( e_{01} : X^{[0,1]} \to X \times X \) such that, for each \( x \in X \), \( s_i(x, x) = c_x \), the constant path at \( x \). Such a section is called reserved. If the coverings fail to exist, we agree to set \( TC^M(X) = \infty \).
The example exhibited in [9, p.13] shows that, unlike the usual topological complexity, the monoidal topological complexity fails to be a homotopy invariant. It is known that $TC^M(X)$ is a homotopy invariant if $X$ is locally equiconnected, i.e., provided the canonical diagonal embedding $\Delta X \hookrightarrow X \times X$ is a cofibration (see [2, Theorem 12] and [9, Proposition 2.17]). An important instance of locally equiconnected spaces is given by an absolute neighborhood retract (ANR) which, throughout this paper, means a metrizable space $X$ satisfying the following property: every continuous map $f: A \to X$ defined on a closed subset $A$ of a metrizable space $Y$ can be continuously extended over an open neighborhood $U$ of $A$ in $Y$.

The next proposition claims that the condition $\Delta X \subseteq U_i$, imposed on each set of the open cover $\{U_i\}_{i=0}^n$ of $X \times X$ in Definition 2.2, can be omitted in the case when $X$ is an ANR space.

**Proposition 2.3.** If $X$ is an ANR space, then $TC^M(X) = TC^{DM}(X)$, where the latter expression is defined to be the smallest nonnegative integer $n$ for which there is an open cover $\{U_0, \ldots, U_n\}$ of $X \times X$, on each of which there is a continuous local section $s_i: U_i \to X^{[0,1]}$ of the fibration $e_{01}: X^{[0,1]} \to X \times X$ such that $s_i(x,x) = c_x$ for all $x \in X$ with $(x,x) \in U_i$.

**Proof.** Clearly $TC^{DM}(X) \leq TC^M(X)$. We show the opposite inequality by following the indications at the bottom of page 4365 of [5].

Let $\{U_0, \ldots, U_n\}$ be an open cover of $X \times X$ by sets that admit continuous local sections $s_i: U_i \to X^{[0,1]}$ of the fibration $e_{01}$ such that $s_i(x,x) = c_x$ for all $x \in X$ with $(x,x) \in U_i$. Since $X \times X$ is a normal space, we can assure the existence of a closed cover $\{V_0, \ldots, V_n\}$ of $X \times X$ with $V_i \subseteq U_i$ for all $i \in \{0,1,\ldots,n\}$. Then we have a continuous extension $\overline{s}_i: V_i \cup \Delta X \to X^{[0,1]}$

of $s_i|_{V_i}$ defined by

$$
\overline{s}_i(x,x') = \begin{cases} 
    s_i(x,x'), & \text{if } (x,x') \in V_i; \\
    c_x, & \text{if } (x,x') \in \Delta X.
\end{cases}
$$

For $i \in \{0,1,\ldots,n\}$, let $\Gamma_i$ stand for the closed subset $((V_i \cup \Delta X) \times [0,1]) \cup (X \times X \times \{0,1\})$ of $X \times X \times [0,1]$ and define a continuous function $u_i: \Gamma_i \to X$ by

$$
u_i(x,x',t) = \begin{cases} 
    \overline{s}_i(x,x')(t), & \text{if } (x,x',t) \in (V_i \cup \Delta X) \times [0,1]; \\
    x, & \text{if } (x,x',t) \in X \times X \times \{0\}; \\
    x', & \text{if } (x,x',t) \in X \times X \times \{1\}.
\end{cases}
$$

Note that $u_i$ is well-defined because $\overline{s}_i$ is a continuous local section of $e_{01}$. Since $X$ is an ANR space, there are open neighborhoods $W_i$ of $\Gamma_i$ in $X \times X \times [0,1]$ and continuous maps $\overline{s}_i|_{W_i} : W_i \to X$ with $\overline{s}_i|_{\Gamma_i} = u_i$. By the compactness of $[0,1]$, we can take an open set $N_i$ in $X \times X$ containing $V_i \cup \Delta X$ such that $N_i \times [0,1] \subseteq W_i$. Finally, the required reserved section $s'_i: N_i \to X^{[0,1]}$ of the fibration $e_{01}$ is defined by $s'_i(x,x')(t) = \overline{s}_i(x,x',t)$ for all $(x,x') \in N_i$ and $t \in [0,1]$. Indeed, by construction, $s'_i$ is a continuous extension of $\overline{s}_i$. Therefore, the new open cover $\{N_0, \ldots, N_n\}$ of $X \times X$ fulfills the requirements of Definition 2.2. \hfill $\Box$

**Remark 2.4.** For a Hausdorff space $X$ (so that $\Delta X$ is closed), the inequalities $TC^{DM}(X) \leq TC^{FH}(X) \leq TC^M(X)$ follow directly from the definitions. Furthermore, these inequalities are in fact equalities if $X$ is an ANR, in view of Proposition 2.3.

The proof of the equalities $TC(G) = cat(G) = TC^M(G)$ given in [5, Lemma 2.7], where $G$ is a connected Lie group, can be adapted to show that the Iwase-Sakai conjecture holds true for a locally compact connected CW topological group. We spell out the details (in Proposition 2.5 below) as they will be useful in Section 5 for constructing an explicit motion planner leading to
the inequality
\[ \text{TC}^M(G^K) \leq \max \left\{ \sum_{i \in \sigma_1 \cup \sigma_2} \text{cat}(G_i) : \sigma_1, \sigma_2 \in K \right\} \]
in Theorem 1.1.

**Proposition 2.5.** If $G$ is a locally compact connected CW topological group, then $\text{TC}(G) = \text{cat}(G) = \text{TC}^M(G)$.

**Proof.** Since $\text{TC}(G) \leq \text{TC}^M(G)$, with the former agreeing with $\text{cat}(G)$ (see [8, Lemma 8.2], where the same proof works for topological groups), it suffices to show that $\text{TC}^M(G) \leq \text{cat}(G)$. Furthermore, since a locally compact CW complex is an ANR space (see Appendix II of [14]), we only need to show that $\text{TC}^{EH}(G) \leq \text{cat}(G)$, in view of Remark 2.4.

Let $n := \text{cat}(G)$ and choose an open cover $\{N_0, \ldots, N_n\}$ of $G$ together with homotopies $H_i : N_i \times [0, 1] \to G$ satisfying $H_i(a, 0) = a$ and $H_i(a, 1) = e$, $a \in N_i$, for all $i \in \{0, 1, \ldots, n\}$ (here $e$ denotes the neutral element of $G$). We can assume that $e \notin N_i$ for all $i > 0$ and $H_0(e, t) = e$ for all $t \in [0, 1]$, where the latter requirement follows from [3, Lemma 1.25] and the fact that $\{e\}$ is a cofibration (recall that CW complexes have non-degenerate base points). For each $i \in \{0, 1, \ldots, n\}$, set $V_i := \{(a, b) \in G \times G : b^{-1}a \in N_i\}$. On each $V_i$ of the open cover $\{V_0, \ldots, V_n\}$ of $G \times G$ we have the continuous section $s_i : V_i \to G^{[0, 1]}$ of $e_{01}$ defined by $s_i(a, b)(t) = bH_i(b^{-1}a, t)$, $t \in [0, 1]$. Note that $\Delta G \cap V_i = \emptyset$ for all $i \in \{1, \ldots, n\}$ and $s_0(a, a)(t) = aH_0(a^{-1}a, t) = aH_0(e, t) = ae = a$ with $(a, a) \in \Delta G$ and $t \in [0, 1]$. Therefore $\text{TC}^{EH}(G) \leq \text{cat}(G)$ and the result follows. \qed

### 2.2. Relative category.
We start by recalling the definition of the join of two maps having the same target. Let $f : X \to Z$ and $g : Y \to Z$ be maps, the join of $f$ and $g$, denoted by $X \ast_Z Y$, is defined as the homotopy pushout of the homotopy pullback of $f$ and $g$.

![Diagram of the join of two maps](image)

where the dashed arrow, called join map or whisker map, is given by the weak universal property of the homotopy pushout.

The previous definition enables us to set forth the main notion of this section, the relative category of a map as introduced in [4]. The $n$-th Ganea map of a given map $i_X : A \to X$, denoted by $g_n : G_n(i_X) \to X$ ($n \geq 0$), is the join map inductively defined by the join construction

![Diagram of Ganea maps](image)

where $g_0 := i_X$ and, if $g_{n-1} : G_{n-1}(i_X) \to X$ is already given, $G_n(i_X)$ is the join of $i_X$ and $g_{n-1}$. Then, the **relative category** of $i_X$, denoted by $\text{relcat}(i_X)$, is defined as the least nonnegative
integer \( n \) such that \( g_n: G_n(i_X) \to X \) admits a homotopy section \( \sigma: X \to G_n(i_X) \) satisfying \( \sigma \circ i_X \simeq \alpha_n \).

Doeraene and El Haouari proved in \cite{DoeraeneElHaouari} that the relative category of a map possesses a Whitehead-type characterization in terms of the \( n \)-th sectional fat-wedge \( t_n: T^n(i_X) \to X^{n+1} \) of \( i_X \) (see Theorem \ref{thm:Whitehead} below), which is inductively defined as follows: For \( n = 0 \), set \( T^0(i_X) := A \) and \( t_0 = i_X: A \to X \). If \( t_{n−1}: T^{n−1}(i_X) \to X^n \) is already defined, then \( t_n \) is the join map rendering a homotopy commutative diagram

\[
\begin{array}{ccc}
T^{n−1}(i_X) \times X & \xrightarrow{t_{n−1} \times 1_X} & X^{n+1} \\
1_X \times i_X & \searrow & \\
T^n(i_X) & \swarrow & \\
& X^n \times A &
\end{array}
\]

where \( T^n(i_X) \) is the join of \( t_{n−1} \times 1_X \) and \( 1_X \times i_X \).

The next result pieces together \cite{DoeraeneElHaouari}*{Proposition 26} and \cite{Gonzalez}*{Corollary 11}. We have chosen the statement in Theorem \ref{thm:Whitehead} below for multiple reasons. To start with, the \( n \)-th sectional fat-wedge and the formulas for all maps appearing in diagram \( (5) \) below have a simple description in the case when \( i_X: A \to X \) is a cofibration. On the other hand, we are primarily interested in the case of the diagonal inclusion in \( X \times X \). Nonetheless, we remark that the characterization of relative category given by Doeraene and El Haouari in \cite{DoeraeneElHaouari}*{Proposition 26} applies for any map \( i_X: A \to X \).

\begin{thm}
Let \( i_X: A \to X \) be a cofibration. We have \( \text{relcat}(i_X) \leq n \) if and only if there exists a map \( f: X \to T^n(i_X) \) filling in a homotopy commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\tau_n} & T^n(i_X) \\
i_X & \downarrow f & \downarrow t_n \\
X & \xrightarrow{\Delta_{n+1}} & X^{n+1},
\end{array}
\]

where \( \Delta_{n+1} \) is the diagonal map, \( \tau_n = \Delta_{n+1}|_A \), and \( t_n: T^n(i_X) \to X^{n+1} \) is the inclusion of the subspace \( T^n(i_X) = \{(x_0, \ldots, x_n) \in X^{n+1} : x_i \in A \text{ for some } i \} \).
\end{thm}

3. The Fadell-Husseini monoidal topological complexity

The main goal of this section is to complete the proof of Theorem \ref{thm:main}. By Remark \ref{rem:oprelcat} and the work in \cite{FadellHusseini}, the equality \( \mathbf{T} \text{C}^M(X) = \mathbf{T} \text{C}^F_H(X) \) is the only one requiring argumentation. The crux of the proof relies on a Fadell-Husseini type definition of the notions \( \text{relcat}^{\text{op}}(i_X) \) and \( \text{relcat}^{\text{op}}(i_X) \) for a cofibration \( i_X: A \to X \) (see Remarks \ref{rem:oprelcat} and \ref{rem:oprelcat2}).

\begin{defn}
Let \( i_X: A \to X \) be a cofibration. A subset \( U \) of \( X \) is \( A \)-relatively sectional if \( A \subseteq U \) and there exists a homotopy of pairs \( H: (U, A) \times [0, 1] \to (X, A) \) such that \( H(x, 0) = x \) and \( H(x, 1) \in A \) for all \( x \in U \). The Fadell-Husseini relative category of \( i_X \), denoted by \( \text{relcat}^{\text{F_H}}(i_X) \), is the least nonnegative integer \( n \) such that \( X \) admits an open cover \( \{U_i\}_{i=0}^n \) satisfying:

(1) \( U_0 \) is \( A \)-relatively sectional;
(2) for \( i \geq 1 \), \( U_i \cap A = \emptyset \) and there are homotopies \( H_i: U_i \times [0, 1] \to X \) with \( H_i(x, 0) = x \) and \( H_i(x, 1) \in A \) for all \( x \in U_i \).

If such an integer does not exist, we set \( \text{relcat}^{\text{op}}(i_X) = \infty \).
Remark 3.2. If each $U_i$ is required to be $A$-relatively sectional in Definition 3.1, we obtain the notion of $\text{relcat}_{op}(i_X)$. The latter concept agrees with the relative category of $i_X$ because of [9, Theorem 1.6]. In the next proposition we show that the equality $\text{relcat}(i_X) = \text{relcat}_{op}^{FH}(i_X)$ holds as well by following techniques similar to those exposed in [9].

**Proposition 3.3.** Let $X$ be a normal space. If $i_X: A \hookrightarrow X$ is a cofibration, then $\text{relcat}(i_X) = \text{relcat}_{op}^{FH}(i_X)$.

**Proof.** Let $n := \text{relcat}(i_X)$, which agrees with $\text{relcat}_{op}(i_X)$ in view of [9, Theorem 1.6]. Choose an open cover $\{U_0, \ldots, U_n\}$ of $X$ such that, for any $i \geq 0$, $U_i$ is $A$-relatively sectional. Then there are homotopies of pairs $H_i: (U_i, A) \times [0, 1] \to (X, A)$ such that $H_i(x, 0) = x$ and $H_i(x, 1) \in A$ for all $x \in U_i$. It is clear that, by setting $U_0^* := U_0$, $U_i^* := U_i \setminus A$ and by restricting the homotopies $H_i$ to $U_i^*$ for $i \geq 1$, the two items of Definition 3.1 are fulfilled. Thus

$$\text{relcat}_{op}^{FH}(i_X) \leq \text{relcat}(i_X).$$

Next, in order to prove the opposite inequality, let $m := \text{relcat}_{op}^{FH}(i_X)$ and consider an open cover $\{U_i\}_{i=0}^m$ of $X$ such that:

1. $U_0$ is $A$-relatively sectional, i.e., $A \subseteq U_0$ and there exists a homotopy of pairs $H_0: (U_0, A) \times [0, 1] \to (X, A)$ with $H_0(x, 0) = x$ and $H_0(x, 1) \in A$ for all $x \in U_0$;
2. for $i \geq 1$, $U_i \cap A = \emptyset$ and there are homotopies $H_i: U_i \times [0, 1] \to X$ with $H_i(x, 0) = x$ and $H_i(x, 1) \in A$ for all $x \in U_i$.

Since $X$ is a normal space, there are closed sets $A_i, B_i$ and open sets $\Theta_i (i = 0, \ldots, m)$ fulfilling $A \subseteq A_0$ and $A_i \subseteq \Theta_i \subseteq B_i \subseteq U_i$ for all $i$, with $\{A_i\}_{i=0}^m$ covering $X$. Furthermore, there exist Urysohn maps $h_i: X \to [0, 1]$ such that $h_i(A_i) = \{1\}$ and $h_i(X \setminus \Theta_i) = \{0\}$. For $i \geq 0$, let $L_i: (X, A) \times [0, 1] \to (X, A)$ be the continuous map defined by

$$L_i(x, t) = \begin{cases} x, & \text{if } x \in X \setminus B_i; \\ H_i(x, h_i(x) \cdot t), & \text{if } x \in U_i. \end{cases}$$

Observe that $L_i$ is well-defined because $x = H_i(x, h_i(x) \cdot t)$ for all $x \in U_i \setminus B_i$. Likewise, the facts $L_0(a, t) = H_0(a, t) \in A$ and $L_i(a, t) = a$ for $t \in [0, 1], a \in A$ and $i \geq 1$ (recall, $U_i \cap A = \emptyset$ for $i \geq 1$) imply that $L_i$ restricts to a map $A \times [0, 1] \to A$. Then define $L: (X, A) \times [0, 1] \to (X^{m+1}, T^m(i_X))$ by $L := (L_0, \ldots, L_m)$. Since $\{A_i\}_{i=0}^m$ covers $X$, we get a well-defined map $f: X \to T^m(i_X)$ by setting $f(x) := L(x, 1)$. Such a map satisfies $L: \Delta_{m+1} \simeq t_m \circ f$ and $L_{|A \times [0, 1]} : \tau_m \simeq f \circ i_X$. Therefore, by Theorem 2.6, $\text{relcat}(i_X) \leq \text{relcat}_{op}^{FH}(i_X)$. \hfill \square

We now discuss the Fadell-Husseini generalized relative category $\text{relcat}_g^{FH}(i_X)$, which is determined just as in Definition 3.1, but without requiring that the covers should consist of open sets. We show that, under mild hypotheses, dropping such a condition is immaterial (Proposition 3.7 below).

**Definition 3.4.** Let $i_X: A \hookrightarrow X$ be a cofibration. We define the Fadell-Husseini generalized relative category of $i_X$, denoted by $\text{relcat}_g^{FH}(i_X)$, as the least nonnegative integer $n$ such that $X$ admits a (not necessarily open) cover $\{U_0, \ldots, U_n\}$ satisfying:

1. $U_0$ is $A$-relatively sectional;
2. for $i \geq 1$, $U_i \cap A = \emptyset$ and there are homotopies $H_i: U_i \times [0, 1] \to X$ with $H_i(x, 0) = x$ and $H_i(x, 1) \in A$ for all $x \in U_i$.

We set $\text{relcat}_g^{FH}(i_X) = \infty$ if the required coverings do not exist.

**Remark 3.5.** If each subset $U_i$ in Definition 3.4 is required to be $A$-relative sectional, we obtain the notion of $\text{relcat}_g(i_X)$. In view of [9, Theorem 2.16], the latter concept coincides with
relcat($i_X$) provided $i_X: A \hookrightarrow X$ is a cofibration between ANR spaces. Paralleling the proof of such a fact, we will prove an analogous conclusion in the Fadell-Husseini context.

Before delving into the equality relcat($i_X$) = relcat$_g^{FH}(i_X)$, we prove the following technical fact (cf. [9, Lemma 2.14]):

**Lemma 3.6.** Let $i_X: A \hookrightarrow X$ be a cofibration with $X$ a normal space. Assume that

1. $\{U_0, \ldots, U_n\}$ is an open cover of $X$;
2. $A \subseteq U_0$ and there is a homotopy $H_0: U_0 \times [0, 1] \to X$ so that $H_0(x, 0) = x$, $H_0(x, 1) \in A$, and $H_0(-1, 1) \mid A \simeq 1_A$;
3. for $i \geq 1$, $U_i \cap A = \emptyset$ and there are homotopies $H_i: U_i \times [0, 1] \to X$ with $H_i(x, 0) = x$ and $H_i(x, 1) \in A$ for any $x \in U_i$.

Then relcat($i_X$) $\leq n$.

**Proof.** The first half of the argument follows the constructions in the proof of Proposition 3.3: Let $A_i, B_i$ be closed sets and $\Theta_i$ be open sets ($i = 0, \ldots, n$), with $\{A_i\}_{i=0}^n$ covering $X$, such that $A \subseteq A_0$ and $A_i \subseteq \Theta_i \subseteq B_i \subseteq U_i$ for all $i$. Choose Urysohn maps $h_i: X \to [0, 1]$ such that $h_i(A_i) = \{1\}$ and $h_i(X \setminus \Theta_i) = \{0\}$. For $i \geq 0$, let $L_i: X \times [0, 1] \to X$ be defined by

$$L_i(x, t) = \begin{cases} x, & \text{if } x \in X \setminus B_i; \\ H_i(x, h_i(x) \cdot t), & \text{if } x \in U_i. \end{cases}$$

Set $L := (L_0, \ldots, L_n): X \times [0, 1] \to X^{n+1}$ and note that we still have $L_i\mid_{A \times [0, 1]}: A \times [0, 1] \to A$, as well as $L: \Delta_{n+1} \simeq t_n \circ f$, where

$$f := L(-, 1): X \to T^n(i_X).$$

On the other hand, let $G_0: A \times [0, 1] \to A$ be the homotopy between $1_A$ and $H_0$ $(-, 1)\mid_A$, this is, $G_0(a, 0) = a$ and $G_0(a, 1) = H_0(a, 1)$ for all $a \in A$. Define $G: A \times [0, 1] \to A^{n+1} \subseteq T^n(i_X)$ to be $G' = (G_0, L_1\mid_{A \times [0, 1]}, \ldots, L_n\mid_{A \times [0, 1]})$. Observe that $L_0(a, 1) = H_0(a, 1) = G_0(a, 1)$, so $G: \tau_n \simeq f \circ i_X$. Therefore, the desired inequality relcat($i_X$) $\leq n$ comes from Theorem 2.6. □

**Proposition 3.7.** Let $i_X: A \hookrightarrow X$ be a cofibration between ANR spaces. We have relcat($i_X$) $= \operatorname{relcat}_g^{FH}(i_X)$.

**Proof.** Clearly relcat$_g^{FH}(i_X) \leq \operatorname{relcat}_g^{FH}(i_X) = \operatorname{relcat}(i_X)$, where the latter relation holds in view of Proposition 3.3 (recall that $X$ is a normal space since it is metrizable). We show the inequality relcat($i_X$) $\leq \operatorname{relcat}_g^{FH}(i_X)$.

Let $n := \operatorname{relcat}_g^{FH}(i_X)$ and consider a (not necessarily open) cover $\{U_i\}_{i=0}^n$ of $X$ such that:

1. $U_0$ is $A$-relatively sectional, i.e., $A \subseteq U_0$ and there exists a homotopy of pairs $H_0: (U_0, A) \times [0, 1] \to (X, A)$ with $H_0(x, 0) = x$ and $H_0(x, 1) \in A$ for all $x \in U_0$;
2. for $i \geq 1$, $U_i \cap A = \emptyset$ and there are homotopies $H_i: U_i \times [0, 1] \to X$ with $H_i(x, 0) = x$ and $H_i(x, 1) \in A$ for all $x \in U_i$.

The argument below for $i = 0$ is the one in the proof of [9, Theorem 2.16]. We review the details since we will then describe a slight variant in order to deal with the case of $i > 0$. Consider the following factorization of $i_X$ through its mapping cocylinder:

$$A \xleftarrow{i_X} X \xrightarrow{p} \hat{A},$$
where \( \tilde{A} = \{(a, \beta) \in A \times X^{[0,1]} : i_X(a) = \beta(0)\} \), \( p \) is a fibration and \( j \) a homotopy equivalence. As observed in \cite[Lemma 2.13]{9}, \( \tilde{A} \) is also an ANR. Define \( s_0 : U_0 \to \tilde{A} \) to be \( s_0 = j \circ H_0(-,1) \), then \( p \circ s_0 \simeq U_0 \hookrightarrow X \) and \( s_0|_{A} \simeq j \). Actually, since \( p \) is a fibration, there is no problem in assuming that \( p \circ s_0 = U_0 \hookrightarrow X \). Following the proof of \cite[Theorem 2.7]{9}, there exist an open neighborhood \( V_0 \) of \( U_0 \) in \( X \) and a map \( \sigma_0 : V_0 \to \tilde{A} \) such that \( p \circ \sigma_0 = V_0 \hookrightarrow X \) and \( \sigma_0|_{U_0} \simeq s_0 \). In particular,

\[
\sigma_0|_{A} = (\sigma_0|_{U_0})|_{A} \simeq s_0|_{A} \simeq j.
\]

If \( j' : \tilde{A} \to A \) denotes a homotopy inverse of \( j \), then

\[
i_X \circ j' \circ \sigma_0 = p \circ j \circ j' \circ \sigma_0 \simeq p \circ \sigma_0 = V_0 \hookrightarrow X.
\]

Hence, there exists a homotopy \( G_0 : V_0 \times [0,1] \to X \) such that \( G_0(0,0) = x \), \( G_0(0,1) = i_X \circ j' \circ \sigma_0(x) \in A \) and \( G_0(-,1)|_{A} = j' \circ \sigma_0|_{A} \simeq j' \circ j \simeq 1_{A} \).

On the other hand, for \( i \geq 1 \), set \( s_i := j \circ H_i(-,1) : U_i \to \tilde{A} \). An examination of the proof above (omitting the steps that involve \( A \subseteq U_0 \)) reveals that there are open neighborhoods \( V_i \) of \( U_i \) in \( X \) together with maps \( \sigma_i : V_i \to \tilde{A} \) so that \( p \circ \sigma_i = V_i \hookrightarrow X \) and \( \sigma_i|_{U_i} \simeq s_i \). Without losing generality we may assume that \( V_i \cap A = \emptyset \) for, otherwise, we simply set \( V'_i := V_i \setminus A \) and \( \sigma'_i = \sigma_i|_{V'_i} \). Furthermore, we have homotopies \( G_i : V_i \times [0,1] \to X \) such that \( G_i(x,0) = x \) and \( G_i(x,1) = i_X \circ j' \circ \sigma_i(x) \in A \) for all \( x \in V_i \).

The required inequality \( \operatorname{relcat}(i_X) \leq n \) then follows from Lemma 3.6.

We complete the proof of Theorem 1.3 by bringing together the main results of this section.

**Proof of Theorem 1.3.** Since \( X \) is an ANR space, the canonical embedding \( i_X : \Delta X \hookrightarrow X \times X \) is a cofibration between ANR spaces, so that \( \operatorname{relcat}(i_X) = \operatorname{relcat}_{op}(i_X) = \operatorname{relcat}_{g}(i_X) \), by \cite[Theorems 1.6 and 2.16]{9}. Furthermore, the equalities \( \operatorname{relcat}(i_X) = \operatorname{relcat}_{op}(i_X) = \operatorname{relcat}_{g}(i_X) \) come from Propositions 3.3 and 3.7.

In order to show the equality \( \TC^M(X) = \TC_{g}^{EH}(X) \), we first note that \( \operatorname{relcat}_{g}^{EH}(i_X) \leq \TC_{g}^{EH}(X) \). The latter fact comes by noticing that, if \( \{U_i\}_{i=0}^{n} \) is a (not necessarily open) cover of \( X \times X \) and \( s_i : U_i \to X^{[0,1]} \) are the local sections of the fibration \( e_{01} \) coming from Definition 1.2, then one can define homotopies \( H_i : (U_0, \Delta X) \times [0,1] \to (X \times X, \Delta X) \) and \( H_i : U_i \times [0,1] \to X \times X \) \((i \geq 1)\) as \( H_i(x,y,t) = (s_i(x,y)(t),y) \) \((i \geq 0)\) satisfying the two items of Definition 3.4. Likewise, it is clear that \( \operatorname{relcat}_{g}^{EH}(X) \leq \operatorname{TC}^{EH}(X) \); nevertheless, the latter expression agrees with \( \operatorname{TC}^{M}(X) \) due to Remark 2.4. Finally, the equality \( \operatorname{TC}^{M}(X) = \operatorname{relcat}(i_X) \) follows from \cite[Theorem 12]{2}, while \( \operatorname{relcat}(i_X) = \operatorname{relcat}_{g}^{EH}(i_X) \) by our initial discussion. In summary, we have

\[
\operatorname{relcat}_{g}^{EH}(i_X) \leq \TC_{g}^{EH}(X) \leq \operatorname{TC}^{EH}(X) = \operatorname{TC}^{M}(X) = \operatorname{relcat}_{g}^{EH}(i_X),
\]

which completes the proof.

4. **Proof of Theorem 1.4**

We start in the non-generalized setting, i.e., by proving that the stasis condition (2) in Definition 1.2 can be omitted without altering the numerical value of \( \operatorname{TC}^{EH}(X) \). Let \( \{(U_i, s_i)\}_{i=0}^{n} \) be a motion planner with \( \Delta X \subseteq U_0 \) and \( \Delta X \cap U_i = \emptyset \) for all \( i \geq 1 \). We do not assume that the section \( s_0 \) of \( e_{01} \) yields constant paths when restricted to \( \Delta X \), but we do assume that all subsets \( U_i \) are open. The task is to construct a motion planner \( \{(V_i, \sigma_i)\}_{i=0}^{n} \) of a Fadell-Husseini type, that is, one that consists of open sets \( V_i \) satisfying \( \Delta X \cap V_i = \emptyset \) for all \( i \geq 1 \), as well as \( \Delta X \subseteq V_0 \) with \( \sigma_0(x,x) = c_x \), the constant path at \( x \), for all \( x \in X \).

If \( n = 0 \), then \( X \) is in fact contractible, so that the homotopy invariance of the monoidal topological complexity for locally equiconnected spaces \( \left[ \cite[Proposition 2.17]{9} \right] \) gives the required
motion planner of a Fadell-Husseini type. We can therefore assume \( n \geq 1 \). By [6, Theorem II.1], there is an open neighborhood \( W \) of \( \Delta X \) in \( X \times X \) and a local section \( \lambda : W \to X^{[0,1]} \) of the end-points evaluation map \( e_{01} \) satisfying \( \lambda(x,x) = c_x \) for all \( x \in X \). Furthermore, by the normality assumption, there is an open cover \( \{W_i\}_{i=0} \) of \( X \times X \) such that \( W_i \subseteq \overline{W_i} \subseteq U_i \) for all \( i \geq 0 \). Consider the open neighborhood \( N \) of \( \Delta X \) given by

\[
N = W \cap U_0 \cap \left((X \times X) \setminus \overline{W_i}\right) \cap \cdots \cap \left((X \times X) \setminus \overline{W_n}\right).
\]

Using once more the normality of \( X \times X \), take an open set \( M \) in \( X \times X \) with \( \Delta X \subseteq M \subseteq \overline{M} \subseteq N \). Let \( V_0 = (U_0 \setminus \overline{M}) \cup M \) and define the reserved section \( \sigma_0 : V_0 \to X^{[0,1]} \) of \( e_{01} \) by

\[
\sigma_0(x,x') = \begin{cases} 
s_0(x,x'), & \text{if } (x,x') \in U_0 \setminus \overline{M}; \\
\lambda(x,x'), & \text{if } (x,x') \in M.
\end{cases}
\]

Lastly, for \( 1 \leq i \leq n \), set \( V_i = W_i \cup (N \setminus \Delta X) \) and define the local sections \( \sigma_i : V_i \to X^{[0,1]} \) of \( e_{01} \) by

\[
\sigma_i(x,x') = \begin{cases} 
s_i(x,x'), & \text{if } (x,x') \in W_i; \\
\lambda(x,x'), & \text{if } (x,x') \in N \setminus \Delta X.
\end{cases}
\]

Then \( \{(V_i, \sigma_i)\}_{i=0}^n \) is the required motion planner of Fadell-Husseini type, for \( \{V_0, V_1, \ldots, V_n\} \) covers \( X \times X \). Indeed, since \( W_i \subseteq V_i \) for \( i \geq 1 \), the covering assertion follows by observing that \( W_0 \setminus (V_1 \cup \cdots \cup V_n) \subseteq U_0 \setminus (V_1 \cup \cdots \cup V_n) \subseteq (U_0 \setminus \overline{M}) \cup (M \setminus M) \setminus (V_1 \cup \cdots \cup V_n) \subseteq (V_0 \cup (\overline{M} \setminus M)) \cup (V_1 \cup \cdots \cup V_n) \subseteq (V_0 \cup (V_1 \cup \cdots \cup V_n)) \cup (\overline{M} \setminus (M \cup V_1 \cup \cdots \cup V_n)) \subseteq N \setminus \Delta X \setminus V_1 \cup \cdots \cup V_n = \emptyset 
\]

as \( n \geq 1 \).

We end up by sketching the argument for the generalized case. Let \( \{(U_i, s_i)\}_{i=0}^n \) be a generalized motion planner consisting of a cover \( \{U_i\}_{i=0}^n \) of \( X \times X \) by not necessarily open subsets \( U_i \) such that \( \Delta X \subseteq U_0 \) and \( \Delta X \cap U_i = \emptyset \) for all \( i \geq 1 \), and of sections \( s_i : U_i \to X^{[0,1]} \) of \( e_{01} \). Again, without assuming that \( s_0 \) is a reserved section of \( e_{01} \), the task is to assure the existence of a corresponding generalized motion planner, one of whose rules is defined on the whole diagonal via constant paths. Following the proof of [9, Theorem 2.7], we can construct a new motion planner \( \{(V_i, \sigma_i)\}_{i=0}^n \) so that, for all \( i \geq 0 \), \( V_i \) is an open subset of \( X \times X \), \( U_i \subseteq V_i \) (so \( \Delta X \subseteq V_0 \)), and \( \sigma_i|_{U_i} \simeq s_i \). Furthermore, without loss of generality we can assume \( \Delta X \cap V_i = \emptyset \) for all \( i \geq 1 \). Then, by the argument in the non-generalized case, we can fix the situation so to have in addition \( \sigma_0(x,x) = c_x \), the constant path at \( x \), for all \( x \in X \), thus completing the argument.

5. Polyhedral products of groups

This section is devoted to proving Theorem 1.1 via Theorem 1.3. We start by giving a quick overview on polyhedral products and discussing a direct consequence (Corollary 5.4) of Theorem 1.1.

Definition 5.1. Let \( \langle X, A \rangle = \{(X_i, A_i)\}_{i=1}^m \) be a family of pairs of spaces and \( K \) be an abstract simplicial complex with \( m \) vertices labeled by the set \( \{1,2,\ldots,m\} \). The polyhedral product determined by \( \langle X, A \rangle \) and \( K \) is

\[
\langle X, A \rangle^K = \cup_{\sigma \in K} \langle X, A \rangle^{\sigma},
\]

\footnote{Note that \( \langle X, A \rangle^{\sigma_1} \) is contained in \( \langle X, A \rangle^{\sigma_2} \) provided \( \sigma_1 \subseteq \sigma_2 \). Therefore, it suffices to take the union over all the maximal simplices of \( K \) in (7), this is, simplices that are not contained in any other simplex of \( K \).}
where \((X, A)^\sigma = \prod_{i=1}^m Y_i\) and
\[
Y_i = \begin{cases} 
A_i, & \text{if } i \in \{1, 2, \ldots, m\} \setminus \sigma; \\
X_i, & \text{if } i \in \sigma.
\end{cases}
\]

We are interested in the case where all \(A_i = \ast\), in which case \((X, \ast)^K\) and \((X, \ast)^\sigma\) are simply denoted by \(X^K\) and \(X^\sigma\), respectively. Moreover, it is clear that, for any \(\sigma \in K\), \(X^\sigma\) is a retract of \(X^K\) and \(X^\sigma \approx \prod_{i \in \sigma} X_i\).

**Definition 5.2.** A family of based spaces \(X = \{(X_i, \ast)\}_{i=1}^m\) is said to be LS-logarithmic if
\[
\text{cat}(X_{i_1} \times \cdots \times X_{i_k}) = \text{cat}(X_{i_1}) + \cdots + \text{cat}(X_{i_k})
\]
holds for all strictly increasing sequences \(1 \leq i_1 < \cdots < i_k \leq m\).

**Example 5.3.** The family \(\mathcal{G} = \{(U(n), e_i)\}_{i=1}^m\), where \(U(n)\) denotes the \(n\)-th unitary group, fulfills the requirements of Theorem 1.1. The LS-logarithmicity hypothesis comes from [15, Example 3.3], while \(\text{cat}(U(n)) = n\) is guaranteed by [3, Theorem 9.47]. Theorem 1.1 thus gives
\[
\text{TC}(\mathcal{G}^K) = \text{TC}^M(\mathcal{G}^K) = \max \left\{ \sum_{i \in \sigma_1 \cup \sigma_2} \text{cat}(U(n)) : \sigma_1, \sigma_2 \in K \right\}
\]
\[
= \max \left\{ \sum_{i \in \sigma_1 \cup \sigma_2} n : \sigma_1, \sigma_2 \in K \right\} = n \cdot \max\{|\sigma_1 \cup \sigma_2| : \sigma_1, \sigma_2 \in K\}.
\]
Setting \(n = 1\), we recover the result obtained in [11, Theorem 2.7] (for \(r = 2\) and all spheres being 1-dimensional). In fact, Theorem 1.1 determines the topological complexity and the monoidal topological complexity of a polyhedral product whose factors are unitary groups or special unitary groups of possibly different dimensions. In such a case, the LS-logarithmicity hypothesis is guaranteed by [15, Example 3.3].

**Corollary 5.4.** Let \(\mathcal{G}\) be an LS-logarithmic based family as the one in Theorem 1.1. If no \(G_i\) is contractible, then \(\mathcal{G}^K\) admits an \(H\)-space structure if and only if \(K\) is the standard \((m - 1)\)-simplex.

**Proof.** If \(K\) is the standard \((m - 1)\)-simplex, then \(\mathcal{G}^K = G_1 \times \cdots \times G_m\) is a topological group, and hence it is an \(H\)-space. On the other hand, suppose that \(\mathcal{G}^K\) admits an \(H\)-space structure. Being connected and cellular, \(\mathcal{G}^K\) satisfies
\[
\max \left\{ \sum_{i \in \sigma_1 \cup \sigma_2} \text{cat}(G_i) : \sigma_1, \sigma_2 \in K \right\} = \text{TC}(\mathcal{G}^K) = \text{cat}(\mathcal{G}^K)
\]
\[
= \max \left\{ \sum_{i \in \sigma} \text{cat}(G_i) : \sigma \in K \right\},
\]
where the first equality comes from Theorem 1.1, the second one follows from [15, Theorem 1], and the third one is guaranteed by (3). Finally, bearing in mind that both maximums above agree, and that \(\text{cat}(G_i) \geq 1\) for all \(i \in \{1, \ldots, m\}\), we conclude that \(K\) is the standard \((m - 1)\)-simplex. \(\square\)

We now delve into the proof of Theorem 1.1, starting with the following auxiliary result:

**Lemma 5.5.** Let \(\mathcal{G}^K\) be as in Theorem 1.1. Then \(\text{TC}(\mathcal{G}^K)\) is no less than
\[
C(G_1, \ldots, G_m; K) := \max \left\{ \sum_{i \in \sigma_1 \cup \sigma_2} \text{cat}(G_i) : \sigma_1, \sigma_2 \in K \right\}.
\]

Note that the maximum in (8) is realized by maximal simplices of \(K\).
Proof. From [1, Corollary 6.15] we get \( \text{TC}(G^K) \geq \text{cat}(G^\sigma_1 \times G^\sigma_2) \) for any disjoint simplices \( \sigma_1, \sigma_2 \in K \). The result follows in view of the LS-logarithmicity hypothesis. \( \square \)

Since \( \text{TC}(G^K) \leq \text{TC}^M(G^K) \), the proof of Theorem 1.1 will be complete once we prove:

**Proposition 5.6.** Let \( G^K \) be as in Theorem 1.1. Then \( \text{TC}^M(G^K) \leq C(G_1, \ldots, G_m; K) \).

5.1. **Proof of Proposition 5.6.** As we remarked in the proof of Proposition 2.5, locally compact CW complexes are ANR spaces. Consequently, \( G^K \) is an ANR for each \( \sigma \in K \), and therefore \( G^K = \bigcup_{\sigma \in K} G^\sigma \) is an ANR as well. Furthermore, in view of Theorem 1.3, it suffices to show that \( \text{TC}^F(G^K) \leq C(G_1, \ldots, G_m; K) \).

In order to attain the latter task, we will construct a general cover (not necessarily open) of \( G^K \) fulfilling the conditions of Definition 1.2 (Proposition 5.7 below).

For each \( i \in \{1, \ldots, m\} \), let \( \{V_{i0}, \ldots, V_{ic_i}\} \) be an open cover of \( G_i \times G_i \) together with reserved sections \( \lambda_{ik} : V_{ik} \to G_i^{[0,1]} \) of the end-points evaluation map \( e_{01} : G_i^{[0,1]} \to G_i \times G_i \). Here, \( c_i \) stands for the LS category of the corresponding polyhedral product factor \( G_i \). As shown in the proof of Proposition 2.5, we can assume that the diagonal of \( G_i \) is contained in \( V_{i0} \), as well as \( \Delta G_i \cap V_{ik} = \emptyset \) for all \( k \in \{1, \ldots, c_i\} \).

The open sets \( V_{ik} \) might not be disjoint; however, this requirement can be achieved by redefining

\[
U_{ik} = V_{ik} \setminus (V_{i0} \cup \cdots \cup V_{i(k-1)})
\]

for each \( k \in \{0, 1, \ldots, c_i\} \) and \( i \in \{1, \ldots, m\} \) (so \( U_{i0} = V_{i0} \)). We say that a pair \((a, b)\) in \( G_i \times G_i \) produces \( k \) closed conditions if \((a, b) \in U_{ik}, \) with \( k \in \{0, 1, \ldots, c_i\}\). The number of closed conditions produced by \((a, b)\) is denoted by \( C(a, b) \).

We regard an element \((a_1, a_2)\) of \( G^K \times G^K \) as a matrix of size \( m \times 2 \), i.e.,

\[
(a_1, a_2) = \begin{pmatrix} a_{11} & a_{12} \\ \vdots & \vdots \\ a_{m1} & a_{m2} \end{pmatrix},
\]

where each column belongs to \( G^K \), say

\[
(a_{11}, \ldots, a_{m1}) \in G^{\sigma_1}, \text{ and } (a_{12}, \ldots, a_{m2}) \in G^{\sigma_2},
\]

with \( \sigma_1, \sigma_2 \in K \). We know that each row \((a_1, a_2)\) of the matrix \((a_1, a_2)\) lies in a unique set \( U_{ik} \) for \( k = C(a_{i1}, a_{i2}) \in \{0, 1, \ldots, c_i\} \), so the number of closed conditions determined by \((a_1, a_2)\), denoted by \( C(a_1, a_2) \), is defined to be the sum of closed conditions produced by the rows \((a_{i1}, a_{i2})\) of \((a_1, a_2)\), this is,

\[
C(a_1, a_2) := \sum_{i=1}^{m} C(a_{i1}, a_{i2}).
\]

It is clear that \( C(a_1, a_2) \leq \sum_{i \in \sigma_1 \cup \sigma_2} c_i \leq C(G_1, \ldots, G_m; K) \), and hence we have proved:

**Proposition 5.7.** The sets \( W_j = \{(a_1, a_2) \in G^K \times G^K : C(a_1, a_2) = j\} \), with \( j \) belonging to \( \{0, 1, \ldots, C(G_1, \ldots, G_m; K)\} \), form a pairwise disjoint cover of \( G^K \times G^K \).

The proof of Proposition 5.6 will be complete once a (continuous) local rule is constructed on each \( W_j \). The task is attained by splitting \( W_j \) into topological disjoint subsets (see Proposition 5.8 below), and then defining a (continuous) local section of the fibration \( e_{01} \) on each one of them.

A partition of \( j \), with \( j \in \{0, 1, \ldots, C(G_1, \ldots, G_m; K)\} \), is an ordered tuple \((j_1, \ldots, j_m)\) of nonnegative integers such that \( j = j_1 + \cdots + j_m \) and \( 0 \leq j_i \leq c_i \) for each \( i \in \{1, \ldots, m\} \). For
such a partition of \( j \), set

\[
W_{(j_1, \ldots, j_m)} = \{(a_1, a_2) \in G^K \times G^K : C(a_{i_1}, a_{i_2}) = j_i, \text{ for } i \in \{1, \ldots, m\}\}.
\]

It is straightforward to see that

\[
W_j = \bigcup_{(j_1, \ldots, j_m)} W_{(j_1, \ldots, j_m)},
\]

where the disjoint union runs over all partitions of \( j \). We next show that (9) is in fact a topological union, this is, \( W_j \) has the weak topology determined by the several \( W_{(j_1, \ldots, j_m)} \).

**Proposition 5.8.** Let \( j \in \{0, 1, \ldots, |C(G_1, \ldots, G_m; K)|\} \). If \( (j_1, \ldots, j_m) \) and \( (r_1, \ldots, r_m) \) are two different partitions of \( j \), then

\[
W_{(j_1, \ldots, j_m)} \cap W_{(r_1, \ldots, r_m)} = \emptyset = W_{(j_1, \ldots, j_m)} \cap W_{(r_1, \ldots, r_m)}.
\]

**Proof.** Since \( (j_1, \ldots, j_m) \neq (r_1, \ldots, r_m) \), there is a natural number \( \ell \in \{1, \ldots, m\} \) with \( j_\ell \neq r_\ell \), say \( j_\ell < r_\ell \), while the equality \( j_1 + \cdots + j_m = j = r_1 + \cdots + r_m \) forces the existence of another natural number \( q \in \{1, \ldots, m\} \) such that \( r_q < j_q \).

For elements \( (a_1, a_2) \in W_{(j_1, \ldots, j_m)} \) and \( (b_1, b_2) \in W_{(r_1, \ldots, r_m)} \) we have

\[
(a_{t_1}, a_{t_2}) \in U_{t_\ell} \quad \text{and} \quad (b_{t_1}, b_{t_2}) \in U_{t_\ell},
\]

then \( (a_{t_1}, a_{t_2}) \in V_{t_\ell} \) and \( (b_{t_1}, b_{t_2}) \notin V_{t_\ell} \) since \( j_\ell < r_\ell \). It is clear that the latter condition is inherited by elements of \( W_{(r_1, \ldots, r_m)} \), so the second equality of the proposition follows.

The statement \( W_{(j_1, \ldots, j_m)} \cap W_{(r_1, \ldots, r_m)} = \emptyset \) follows by using the condition \( r_q < j_q \). \( \square \)

In the rest of the section we construct a continuous local section of the end-points evaluation map \( e_{01} \) on each \( W_{(j_1, \ldots, j_m)} \). Such a task is performed in the following way: For \( i \in \{1, \ldots, m\} \), let \( d_i \) denote a metric on \( G_i \). Since \( d_i \) is always equivalent to a bounded metric, we can assume that the diameter of \( G_i \), defined by \( \delta(G_i) = \text{sup}\{d_i(a, b) : a, b \in G_i\} \), is finite. Likewise, there is no problem in assuming that each diameter \( \delta(G_i) \) is positive.

We reparametrize the initial navigational instructions \( \lambda_{ik} \) as follows: For \( k \in \{0, 1, \ldots, c_i\} \), consider the section \( \tau_{ik} : U_{ik} \rightarrow G_i^{[0,1]} \) of the end-points evaluation map \( e_{01} : G_i^{[0,1]} \rightarrow G_i \times G_i \) where, for \( (a_1, a_2) \in U_{ik} \),

\[
\tau_{ik}(a_1, a_2)(t) = \begin{cases} 
  a_1, & \text{if } d_{i_1} + d_{i_2} = 0; \\
  \lambda_{ik}(a_1, a_2)(\frac{2\delta(G_i)}{d_{i_1} + d_{i_2}}), & \text{if } 0 \leq t \leq \frac{d_{i_1} + d_{i_2}}{2\delta(G_i)} \neq 0; \\
  a_2, & \text{if } 0 \neq \frac{d_{i_1} + d_{i_2}}{2\delta(G_i)} \leq t \leq 1;
\end{cases}
\]

and \( d_{ij} = d_i(a_j, e_i), \ j = 1, 2 \). Recall that \( e_i \) denotes the neutral element of \( G_i \).

The path \( \tau_{ik} \) is clearly continuous on the open subset of \( U_{ik} \) determined by the condition \( d_{i_1} + d_{i_2} \neq 0 \). The latter open subset of \( U_{ik} \) equals in fact \( U_{ik} \) unless \( k = 0 \), so that \( \tau_{ik} \) is continuous on the whole \( U_{ik} \) for \( k \in \{1, \ldots, c_i\} \). The continuity of \( \tau_{i0} \) on \( U_{i0} \) follows from the continuity of the reserved section \( \lambda_{i0} \).

Define the (not necessarily continuous) section

\[
\varphi : G^K \times G^K \rightarrow \prod_{i=1}^m G_i^{[0,1]}
\]

of the fibration \( e_{01} \) to be \( \varphi(a_1, a_2) = (\varphi_1(a_{i_1}, a_{i_2}), \ldots, \varphi_m(a_{m_1}, a_{m_2})) \), whose \( i \)th coordinate \( \varphi_i(a_{i_1}, a_{i_2}) \) is the path in \( G_i \) from \( a_{i_1} \) to \( a_{i_2} \), given by

\[
\varphi_i(a_{i_1}, a_{i_2})(t) = \begin{cases} 
  a_{i_1}, & \text{if } 0 \leq t \leq t_{a_{i_1}}; \\
  \mu(a_{i_1}, a_{i_2})(t - t_{a_{i_1}}), & \text{if } t_{a_{i_1}} \leq t \leq 1.
\end{cases}
\]
Here, $t_{a_{i1}} = \frac{1}{2} - \frac{d((a_{i1}, e_i))}{2\delta(G_i)}$, and

$$
\mu(a_{i1}, a_{i2}) = \begin{cases} 
\tau_{i0}(a_{i1}, a_{i2}), & \text{if } (a_{i1}, a_{i2}) \in U_{i0}; \\
\vdots & \\
\tau_{ic_i}(a_{i1}, a_{i2}), & \text{if } (a_{i1}, a_{i2}) \in U_{ic_i}.
\end{cases}
$$ (11)

By construction, the map $\varphi$ is a section of the fibration

$$e_{01} : \left( \prod_{i=1}^{m} G_i \right)^{[0,1]} \rightarrow \prod_{i=1}^{m} G_i \times \prod_{i=1}^{m} G_i.$$

Although $\varphi$ fails to be a continuous global section of $e_{01}$, its restriction to each $W(j_1, \ldots, j_m)$, where $(j_1, \ldots, j_m)$ is a partition of $0 \leq j \leq C(G_1, \ldots, G_m; K)$, is continuous since formulas (11) can be rewritten as

$$\mu = \begin{cases} 
\tau_{i0}, & \text{if } j_i = 0; \\
\vdots & \\
\tau_{ic_i}, & \text{if } j_i = c_i.
\end{cases}$$

**Remark 5.9.** In preparation for the proof of our final result, we spell out formulas (10) in order to unravel the motion provided by $\varphi$ at the level of each polyhedral product factor $G_i$. Concretely, if $(a_{i1}, a_{i2}) \in U_{ik}$ for some $k \in \{0, 1, \ldots, c_i\}$, the path $\varphi_i(a_{i1}, a_{i2})$ is described as follows:

- if $0 \leq t \leq \frac{1}{2} - \frac{d((a_{i1}, e_i))}{2\delta(G_i)}$, then stay at $a_{i1}$;
- if $\frac{1}{2} - \frac{d((a_{i1}, e_i))}{2\delta(G_i)} \leq t \leq \frac{1}{2} + \frac{d((a_{i2}, e_i))}{2\delta(G_i)}$, then move from $a_{i1}$ to $a_{i2}$ at constant speed via $\tau_{ik}$;
- if $\frac{1}{2} + \frac{d((a_{i2}, e_i))}{2\delta(G_i)} \leq t \leq 1$, then stay at $a_{i2}$.

Recall that $\varphi$ was defined as a map from $G^K \times G^K$ to $(\prod_{i=1}^{m} G_i)^{[0,1]}$. We next show that the image of $\varphi$ is contained in $(G^K)^{[0,1]}$, thus completing the proof of Proposition 5.6.

**Proposition 5.10.** The image of $\varphi$ is contained in $(G^K)^{[0,1]}$.

**Proof.** Let $(a_1, a_2) \in G^K \times G^K$, we need to prove that $\varphi(a_1, a_2)((0,1]) \subseteq G^K$. So, assume $(a_{11}, \ldots, a_{m1}) \in G^{a_1}$ and $(a_{12}, \ldots, a_{m2}) \in G^{a_2}$ with $\sigma_1, \sigma_2 \in K$. By Remark 5.9, for all $i \notin \sigma_1$, $a_{i1} = e_i$ keeps its position through time $t \leq 1/2$, so that $\varphi(a_1, a_2)((0,1/2]) \subseteq G^{a_1} \subseteq G^K$. Again, Remark 5.9 shows that, for $i \notin \sigma_2$, the path $\varphi_i(a_{i1}, a_{i2})$ has reached its final position $a_{i2} = e_i$ at time 1/2, so that $\varphi(a_1, a_2)((1/2, 1]) \subseteq G^{a_2} \subseteq G^K$, and the proof is complete. \(\square\)

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