On Keller’s conjecture in dimension seven

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Abstract
A cube tiling of \( \mathbb{R}^d \) is a family of pairwise disjoint cubes \([0,1)^d + T = \{[0,1)^d + t : t \in T \} \) such that \( \bigcup_{t \in T} ([0,1)^d + t) = \mathbb{R}^d \). Two cubes \([0,1)^d + t, [0,1)^d + s \) are called a twin pair if \( |t_j - s_j| = 1 \) for some \( j \in [d] = \{1, \ldots, d\} \) and \( t_i = s_i \) for every \( i \in [d] \setminus \{j\} \). In 1930, Keller conjectured that in every cube tiling of \( \mathbb{R}^d \) there is a twin pair. Keller’s conjecture is true for dimensions \( d \leq 6 \) and false for all dimensions \( d \geq 8 \). For \( d = 7 \) the conjecture is still open. Let \( x \in \mathbb{R}^d \), \( i \in [d] \), and let \( L(T, x, i) \) be the set of all \( i \)th coordinates \( t_i \) of vectors \( t \in T \) such that \( ([0,1)^d + t) \cap ([0,1)^d + x) \neq \emptyset \) and \( t_i \leq x_i \). It is known that if \( |L(T, x, i)| \leq 2 \) for some \( x \in \mathbb{R}^7 \) and every \( i \in [7] \) or \( |L(T, x, i)| \geq 6 \) for some \( x \in \mathbb{R}^7 \) and \( i \in [7] \), then Keller’s conjecture is true for \( d = 7 \). In the present paper we show that it is also true for \( d = 7 \) if \( |L(T, x, i)| = 5 \) for some \( x \in \mathbb{R}^7 \) and \( i \in [7] \). Thus, if there is a counterexample to Keller’s conjecture in dimension seven, then \( |L(T, x, i)| \in \{3,4\} \) for some \( x \in \mathbb{R}^7 \) and \( i \in [7] \).

Key words: box, cube tiling, Keller’s conjecture, rigidity.

1 Introduction

This paper is a continuation of the paper [10] and therefore the following introduction is limited to essential information. For a more detailed historical sketch on Keller’s cube tiling conjecture and related problems on cube tilings we refer the reader to [10].

A cube tiling of \( \mathbb{R}^d \) is a family of pairwise disjoint cubes \([0,1)^d + T = \{[0,1)^d + t : t \in T \} \) such that \( \bigcup_{t \in T} ([0,1)^d + t) = \mathbb{R}^d \). Two cubes \([0,1)^d + t, [0,1)^d + s \)
$[0, 1]^d + s$ are called a twin pair if $|t_j - s_j| = 1$ for some $j \in [d] = \{1, \ldots, d\}$ and $t_i = s_i$ for every $i \in [d] \setminus \{j\}$. In 1907, Minkowski \cite{Minkowski} conjectured that in every lattice cube tiling of $\mathbb{R}^d$, i.e. when $T$ is a lattice in $\mathbb{R}^d$, there is a twin pair, and in 1930, Keller \cite{Keller} generalized this conjecture to arbitrary cube tiling of $\mathbb{R}^d$. Minkowski’s conjecture was confirmed by Hajós \cite{Hajo} in 1941. In 1940, Perron \cite{Perron} proved that Keller’s conjecture is true for all dimensions $d \leq 6$.

In 1992, Lagarias and Shor \cite{Lagarias}, using ideas from Corrádi’s and Szabó’s papers \cite{Corradi, Szabo}, constructed a cube tiling of $\mathbb{R}^{10}$ which does not contain a twin pair and thereby refuted Keller’s cube tiling conjecture. In 2002, Mackey \cite{Mackey} gave a counterexample to Keller’s conjecture in dimension eight, which also shows that this conjecture is false in dimension nine. For $d = 7$ Keller’s conjecture is still open.

Let $[0, 1)^d + T$ be a cube tiling, $x \in \mathbb{R}^d$ and $i \in [d]$, and let $L(T, x, i)$ be the set of all $i$th coordinates $t_i$ of vectors $t \in T$ such that $([0, 1)^d + t) \cap ([0, 1)^d + x) \neq \emptyset$ and $t_i \leq x_i$ (Figure 1). For every cube tiling $[0, 1)^d + T$, $x \in \mathbb{R}^d$ and $i \in [d]$ the set $L(T, x, i)$ contains at most $2^{d-1}$ elements.

![Figure 1](image)

**Fig. 1.** A portion of a cube tiling $[0, 1)^2 + T$ of $\mathbb{R}^2$. The number of elements in $L(T, x, i)$ depends on the position of $x \in \mathbb{R}^2$. For $x = (2, 3)$, we have $L(T, x, 1) = \{3/2\} (= \{t_1\})$ and $L(T, x, 2) = \{5/2, 11/4\} (= \{t_2, t'_2\})$, while for $x' = (4, 15/4)$, we have $L(T, x', 1) = \{7/2\}$ and $L(T, x', 2) = \{13/4\}$.

In 2010, Debroni et al. \cite{Debroni} computed that Keller’s conjecture is true for all cube tilings $[0, 1)^7 + T$ of $\mathbb{R}^7$ with $T \subset (1/2)\mathbb{Z}^7$ or equivalently, $T \subset a + \mathbb{Z}^7 \cup b + \mathbb{Z}^7$, where fixed $a, b \in [0, 1)^7$ are such that $a_i \neq b_i$ for every $i \in [7]$. Observe now that the condition $|L(T, x, i)| \leq 2$ for some $x \in \mathbb{R}^7$ and every $i \in [7]$ means that $T_1 \subset a + \mathbb{Z}^7 \cup b + \mathbb{Z}^7$, where $T_1 \subset T$ consists of all $t$ for which $([0, 1)^7 + t) \cap ([0, 1)^7 + x) \neq \emptyset$. Thus, it is easy to show that the result of Debroni et al. proves also that the conjecture is true for cube tilings
of $\mathbb{R}^7$ with $|L(T, x, i)| \leq 2$ for some $x \in \mathbb{R}^7$ and every $i \in [7]$. Indeed, if there is no twin pairs in the set $\{(0, 1)^7 + t : t \in T_1\}$, then extending this family to the two-periodic tiling $[0, 1)^7 + T$ of $\mathbb{R}^7$, where $T = T_1 + 2\mathbb{Z}^7$, we obtain a cube tiling with $T \subset a + \mathbb{Z}^7 \cup b + \mathbb{Z}^7$ without twin pairs, which contradicts the result of Debroni et al.

In [10] we showed that Keller’s conjecture is true for cube tilings $[0, 1)^7 + T$ of $\mathbb{R}^7$ for which $|L(T(x, i))| \geq 6$ for some $x \in \mathbb{R}^7$ and $i \in [7]$. In this paper we prove that

\textit{Keller’s conjecture is true for cube tilings $[0, 1)^7 + T$ of $\mathbb{R}^7$ for which $|L(T(x, i))| = 5$ for some $x \in \mathbb{R}^7$ and $i \in [7]$.}

It follows from the above results that if there is a counterexample to Keller’s conjecture in dimension seven, then $|L(T, x, i)| \in \{3, 4\}$ for some $x \in \mathbb{R}^7$ and $i \in [7]$.

The proof of the crucial result (Theorem 5.2) that allows us to prove the above assertion on Keller’s conjecture is based on computations, and these need reductions. The two longest and most arduous sections of the paper, Section 3 and 4, contain the preparation results for adequate reductions. So, the reader who wants to have a first overview of how discussed case of Keller’s conjecture is proven may leave these sections and continue reading from Theorem 5.2.

The present paper is organized as follows. In Section 2 we give basic notions concerning the systems of boxes and abstract words (details can be found in [6, 11]). These issues were developed in [9, 11]. Since they are not widely known, we present them in detail. In Section 3 we give results on the structure of systems of boxes. In the next section we examine some special system of 12 boxes (written down as a system of abstract words). In Section 5, based on the result from the previous two sections, we first establish initial configurations for computations, and next we prove the theorem on the structure of the above mentioned system of 12 boxes (Theorem 5.2). In the final Section 6 using Theorem 5.2 we prove that Keller’s cube tiling conjecture is true for tilings $[0, 1)^7 + T$ of $\mathbb{R}^7$ with $|L(T(x, i))| = 5$ for some $x \in \mathbb{R}^7$ and $i \in [7]$.

2 Basic notions

In this section we present the basic notions on dichotomous boxes and words (details can be found in [6, 11]). We start with systems of boxes.

A non-empty set $K \subseteq X = X_1 \times \cdots \times X_d$ is called a box if $K = K_1 \times \cdots \times K_d$ and $K_i \subseteq X_i$ for each $i \in [d]$. By Box($X$) we denote the set
of all boxes in $X$. The set $X$ will be called a $d$-box. The box $K$ is said to be proper if $K_i \neq X_i$ for each $i \in [d]$. Two boxes $K$ and $G$ in $X$ are called dichotomous if there is $i \in [d]$ such that $K_i = X_i \setminus G_i$. A suit is any collection of pairwise dichotomous boxes. A suit is proper if it consists of proper boxes. A non-empty set $F \subseteq X$ is said to be a polybox if there is a suit $\mathcal{F}$ for $F$, i.e. if $\bigcup \mathcal{F} = F$. A polybox $F$ is rigid if it has exactly one suit. (Figure 2 presents the suit for a rigid polybox. The polyboxes $\bigcup \mathcal{F}^3,A$ and $\bigcup \mathcal{F}^3,A'$ in Figure 3 are not rigid).

Fig. 2. The suit for a rigid polybox in the 3-box $[0,2]^3$. The two brightest boxes are $[0,1]^3$ and $[1,2]^3$. The remaining three boxes are $[3/4,7/4) \times [0,1) \times [1,2]$ (the light box), $[0,1) \times [1,2] \times [3/4,7/4)$ (the darker box) and $[1,2] \times [3/4,7/4] \times [0,1)$ (the darkest box).

The important property of proper suits is that, for every proper suits $\mathcal{F}$ and $\mathcal{G}$ for a polybox $F$, we have $|\mathcal{F}| = |\mathcal{G}|$. Thus, we can define a box number $|F|_0$ = the number of boxes in any proper suit for $F$ (In Figure 3 we have $|\bigcup \mathcal{F}^3,A|_0 = 3$). A proper suit for a $d$-box $X$ is called a minimal partition of $X$ (Figure 3). Every minimal partition of a $d$-box has $2^d$ boxes.

Two boxes $K, G \subset X$ are said to be a twin pair if $K_j = X_j \setminus G_j$ for some $j \in [d]$ and $K_i = G_i$ for every $i \in [d] \setminus \{j\}$. Observe that, the suit for a rigid polybox can not contain a twin pair.

Every two cubes $[0,1)^d + t$ and $[0,1)^d + p$ in an arbitrary cube tiling $[0,1)^d + T$ of $\mathbb{R}^d$ satisfy Keller’s condition ([8]): there is $i \in [d]$ such that $t_i - p_i \in \mathbb{Z} \setminus \{0\}$, where $t_i$ and $p_i$ are $i$th coordinates of the vectors $t$ and $p$. For any cube $[0,1)^d + x$, where $x = (x_1, ..., x_d) \in \mathbb{R}^d$, the family $\mathcal{F}_x = \{([0,1)^d + t) \cap ([0,1)^d + x) \neq \emptyset : t \in T\}$ is a partition of the cube $[0,1)^d + x$, in which, because of Keller’s condition, every two boxes $K, G \in \mathcal{F}_x$ are dichotomous, i.e. there is $i \in [d]$ such that $K_i$ and $G_i$ are disjoint and $K_i \cup G_i = [0,1] + x_i$. Moreover, since cubes in cube tilings are half-open, every box $K \in \mathcal{F}_x$ is proper, and consequently the family $\mathcal{F}_x$ is a minimal partition of $[0,1)^d + x$. The structure of the partition $\mathcal{F}_x$ reflects the local structure of the cube tiling $[0,1)^d + T$. Obviously, a cube tiling $[0,1)^d + T$ contains a twin pair if and only if the partition $\mathcal{F}_x$ contains a twin pair for some $x \in \mathbb{R}^d$ (see Figure 1).
Below we sketch our approach to the problem of the existence of twin pairs in a cube tiling \([0,1]^7 + T\) of \(\mathbb{R}^7\) with \(|L(T,x,i)| = 5\). To do this we describe the structure of a minimal partition. A graph-theoretic description of this structure can be found in [2] (see also [14]).

Let \(X\) be a \(d\)-box. A set \(F \subseteq X\) is called an \(i\)-cylinder if

\[ l_i \cap F = l_i \quad \text{or} \quad l_i \cap F = \emptyset, \]

where \(l_i = \{x_1\} \times \ldots \times \{x_{i-1}\} \times X_i \times \{x_{i+1}\} \times \ldots \times \{x_d\}\) and \(x_j \in X_j\) for \(j \in [d] \setminus \{i\}\) (Figure 3).

Let \(\mathcal{F}\) be a minimal partition and let \(A \subseteq X_i\) be a set such that there is a box \(K \in \mathcal{F}\) with \(K_i \in \{A, X_i \setminus A\}\). Let

\[ \mathcal{F}^{i,A} = \{K \in \mathcal{F} : K_i = A\} \quad \text{and} \quad \mathcal{F}^{i,A'} = \{K \in \mathcal{F} : K_i = X_i \setminus A\}. \]

Since the boxes in \(\mathcal{F}\) are pairwise dichotomous, the set \(\bigcup(\mathcal{F}^{i,A} \cup \mathcal{F}^{i,A'})\) is an \(i\)-cylinder, and the set of boxes \(\mathcal{F}^{i,A} \cup \mathcal{F}^{i,A'}\) is a suit for it. As \(|\mathcal{F}| = 2^d\), it follows that the boxes in \(\mathcal{F}\) can form at most \(2^{d-1}\) pairwise disjoint \(i\)-cylinders. More precisely, for every \(i \in [d]\) there are sets \(A^1, \ldots, A^k \subseteq X_i\) such that \(A^n \notin \{A^m, X_i \setminus A^m\}\) for every \(n, m \in [k], n \neq m\), and

\[ \mathcal{F} = \mathcal{F}^{i,A^1} \cup \mathcal{F}^{i,(A^1)'} \cup \ldots \cup \mathcal{F}^{i,A^k} \cup \mathcal{F}^{i,(A^k)'} = \bigcup_{n=1}^{k} \mathcal{F}^{i,A^n}. \]

The boxes in \(\mathcal{F}\) are proper, and hence \(|\mathcal{F}^{i,A^n} \cup \mathcal{F}^{i,(A^n)'}| \geq 2\) for every \(i \in [d]\) and \(n \in [k]\). Thus \(k \leq 2^{d-1}\) and consequently \(|L(T,x,i)| \leq 2^{d-1}\) for every cube tiling \([0,1]^d + T, x \in \mathbb{R}^d\) and \(i \in [d]\), as \(|L(T,x,i)|\) is the number of all \(i\)-cylinders in the partition \(\mathcal{F}_x\).

If \(K\) is a box in \(X\), \(\mathcal{F}\) is a family of boxes, \(x \in X\) and \(i \in [d]\), then

\[ (K)_i = K_1 \times \ldots \times K_{i-1} \times K_{i+1} \times \ldots \times K_d, \]

\[ (\mathcal{F})_i = \{(K)_i : K \in \mathcal{F}\} \]

and

\[ (x)_i = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_d). \]

Since \(\bigcup(\mathcal{F}^{i,A} \cup \mathcal{F}^{i,A'})\) is an \(i\)-cylinder, the sets of boxes \((\mathcal{F}^{i,A})_i\) and \((\mathcal{F}^{i,A'})_i\) are two suits for the polybox \(\bigcup(\mathcal{F}^{i,A})_i = \bigcup(\mathcal{F}^{i,A'})_i\), which is a polybox in the \((d-1)\)-box \(X\), (Figure 3). Note that, as \((\mathcal{F}^{i,A})_i\) and \((\mathcal{F}^{i,A'})_i\) are proper suits for the polybox \(\bigcup(\mathcal{F}^{i,A})_i\), we have \(|(\mathcal{F}^{i,A})_i| = |(\mathcal{F}^{i,A'})_i|\).
Fig. 3. The minimal partition $\mathcal{F} = \mathcal{F}^{3,A} \cup \mathcal{F}^{3,A'} \cup \mathcal{F}^{3,B} \cup \mathcal{F}^{3,B'}$ of the 3-box $X = [0,1]^3$ ($A = [0,1/2)$, $B = [0,3/4]$), two 3-cylinders and its suits.

Let now $[0,1)^7 + T$ be a cube tiling of $\mathbb{R}^7$ and let $\mathcal{F}_x$ be as defined above. If $|L(T,x,i)| = 5$ for some $i \in [7]$, then

$$\mathcal{F}_x = \mathcal{F}^{i,A^1} \cup \mathcal{F}^{i,(A^1)'} \cup \ldots \cup \mathcal{F}^{i,A^5} \cup \mathcal{F}^{i,(A^5)'}.$$ 

Assume that there are no twin pairs in the tiling $[0,1)^7 + T$. Then $\mathcal{F}_x$ does not contain a twin pair. It follows from [10, Theorem 3.1] (see Theorem 3.1 in Section 3) that $|\mathcal{F}^{i,A^k}| \geq 12$ for every $k \in [5]$. Thus, there is at least one $k \in [5]$ such that $|\mathcal{F}^{i,A^k}| = 12$ because $|\mathcal{F}_x| = 128$ and $|\mathcal{F}^{i,A^k}| = |\mathcal{F}^{i,(A^k)'}|$ for every $k \in [5]$. The main effort in the paper will be rely on describing precisely the structure of all twin pairs free suits $\mathcal{F}^{i,A^k} \cup \mathcal{F}^{i,(A^k)'}$ for $i$-cylinders $\bigcup (\mathcal{F}^{i,A^k} \cup \mathcal{F}^{i,(A^k)'})$ such that $|\mathcal{F}^{i,A^k}| = 12$. Knowing this structure, we will be able to prove that Keller’s conjecture is true for a cube tiling $[0,1)^7 + T$ with $|L(T,x,i)| = 5$.

Similarly like in [10] instead of suits we will consider systems of abstract words. We collect below basic notions concerning words (details can be found in [11]).

A set $S$ of any objects will be called an alphabet, and the elements of $S$ will be called letters. A permutation $s \mapsto s'$ of the alphabet $S$ such that $s'' = (s')' = s$ and $s' \neq s$ is said to be a complementation. We add an extra element $\ast$ to the set $S$ and the set $S \cup \{\ast\}$ is denoted by $\ast S$. We set $\ast' = \ast$. Each sequence of letters $s_1 \ldots s_d$ from the set $\ast S$ is called a word. The set of all words of length $d$ is denoted by $(\ast S)^d$, and by $S^d$ we denote the set of
all words $s_1 \cdots s_d$ such that $s_i \neq *$ for every $i \in [d]$. Two words $u = u_1 \cdots u_d$ and $v = v_1 \cdots v_d$ are dichotomous if there is $j \in [d]$ such that $u_j \neq *$ and $u'_j = v_j$. If $V \subseteq \ast S^d$ consists of pairwise dichotomous words, then we call it a polybox code (or polybox genome). Two words $u, v \in \ast S^d$ are a twin pair if there is $j \in [d]$ such that $u'_j = v_j$, where $u_j \neq *$ and $u_i = v_i$ for every $i \in [d] \setminus \{j\}$.

If $A \subseteq [d]$ and $A^c = [d] \setminus A = \{i_1 < \cdots < i_n \}$, then $(u)_A = u_{i_1} \cdots u_{i_n}$ and $(V)_A = \{(v)_A : v \in V\}$ for $V \subseteq \ast S^d$. If $A = \{i\}$, then we write $(u)_i$ and $(V)_i$ instead of $(u)_{\{i\}}$ and $(V)_{\{i\}}$, respectively. If $V \subseteq \ast S^d$, $s \in \ast S$ and $i \in [d]$, then let $V^i\{s\} = \{v \in V : v_i = s\}$. If $V$ is a polybox code, then the representation

$$V = V^{i_1}_{l_1} \cup V^{i_2}_{l_2} \cup \cdots \cup V^{i_k}_{l_k} \cup V^{i'k}_{l'k},$$

where $l_j, l'_j \in \ast S$ for $j \in [k]$, will be called a distribution of words in $V$.

Suppose now that for each $i \in [d]$ a mapping $f_i : \ast S \to \text{Box}(X_i)$ is such that $f_i(s') = X_i \setminus f_i(s)$ for $s \neq *$ and $f_i(*) = X_i$. We define the mapping $f : \ast S^d \to \text{Box}(X)$ by

$$f(s_1 \cdots s_d) = f_1(s_1) \times \cdots \times f_d(s_d).$$

About such defined $f$ we will say that it preserves dichotomies. If $V \subseteq \ast S^d$, then the set of boxes $f(V) = \{f(v) : v \in V\}$ is said to be a realization of the set of words $V$. Clearly, if $V$ is a polybox code, then $f(V)$ is a suit for $\bigcup f(V)$. The realization is said to be exact if for each pair of words $v, w \in V$, if $v_i \not\in \{w_i, w'_i\}$, then $f_i(v_i) \not\in \{f_i(w_i), X_i \setminus f_i(w_i)\}$.

A polybox code $V \subseteq \ast S^d$ is called a partition code if any realization $f(V)$ of $V$ is a suit for a $d$-box $X$. Observe that, if $V \subseteq S^d$ is a partition code, then $f(V)$ is a minimal partition.

We will exploit some abstract but very useful realization of polybox codes. This sort of realization was invented in [1], where it was the crucial tool in proving the main theorem of that paper.

Let $S$ be an alphabet with a complementation, and let

$$ES = \{B \subseteq S : |\{s, s'\} \cap B| = 1, \text{whenever } s \in S\},$$

$$Es = \{B \in ES : s \in B\} \text{ and } E* = ES.$$ 

Let $V \subseteq \ast S^d$ be a polybox code, and let $v \in V$. The equicomplementary realization of the word $v$ is the box

$$\bar{v} = Ev_1 \times \cdots \times Ev_d$$

in the $d$-box $(ES)^d = ES \times \cdots \times ES$. The equicomplementary realization of the code $V$ is the family

$$E(V) = \{\bar{v} : v \in V\}.$$
If $S$ is finite, $s_1, \ldots, s_n \in S$ and $s_i \not\in \{s_j, s_j'\}$ for every $i \neq j$, then

$$|E_{s_1} \cap \cdots \cap E_{s_n}| = (1/2^n)|ES|.$$  \hfill (2.1)

In the paper we will assume that $S$ is finite. The value of the realization $E(V)$, where $V \subset S^d$, lies in the above equality. In particular, boxes in $E(V)$ are of the same size; for $w \in E(V)$ we have $|\bar{w}| = (1/2^d)|ES|^d$. Thus, two boxes $\bar{v}, \bar{w} \subset (ES)^d$ are dichotomous if and only if $\bar{v} \cap \bar{w} = \emptyset$.

Moreover, from (2.1) we obtain the following important lemma which was proven in \[10\]:

**Lemma 2.1** Let $w, u, v \in S^d$, and let $D$ be a simple partition of the $d$-box $\bar{w}$. If boxes $\bar{w} \cap \bar{u}$ and $\bar{w} \cap \bar{v}$ belong to $D$, then there is a simple partition code $C \subset S^d$ such that $u, v \in C$. In particular, if $\bar{w} \cap \bar{u}$ and $\bar{w} \cap \bar{v}$ form a twin pair, then $u$ and $v$ are a twin pair. \hfill $\square$

Let $V, W \subset (*S)^d$ be polybox codes, and let $v \in (*S)^d$. We say that $v$ is covered by $W$, and write $v \succeq W$, if $f(v) \subseteq \bigcup f(W)$ for every mapping $f$ that preserves dichotomies. If $v \subseteq W$ for every $v \in V$, then we write $V \subseteq W$.

Polybox codes $V, W \subset (*S)^d$ are said to be equivalent if $V \subseteq W$ and $W \subseteq V$. A polybox code $V \subset S^d$ is called rigid if there is no code $W \subset S^d$ which is equivalent to $V$ and $V \neq W$. Observe that, rigid polybox codes can not contain a twin pair.

Let $g: S^d \times S^d \rightarrow \mathbb{Z}$ be defined by the formula

$$g(v, w) = \prod_{i=1}^{d}(2[v_i = *] + [v_i \not\in \{v_i, v_i'\}]),$$  \hfill (2.2)

where $[p] = 1$ if the sentence $p$ is true and $[p] = 0$ if it is false.

Let $w \in S^d$, and let $V \subset S^d$ be a polybox code. Then

$$\bar{w} \subseteq \bigcup E(V) \iff w \subseteq V \iff \sum_{v \in V} g(v, w) = 2^d.$$  \hfill (2.3)

It follows from the definition of equivalent polybox codes $V, W \subset S^d$ and \[10\] that $V$ and $W$ are equivalent if and only if $\bigcup E(V) = \bigcup E(W)$.

Let $s_* = * \cdots * \in (*S)^d$ and let $\bar{g}(\cdot, s_*) : (*S)^d \rightarrow \mathbb{Z}$ be defined as follows:

$$\bar{g}(v, s_*) = \prod_{i=1}^{d}(2[v_i = *] + [v_i \neq *]).$$

The proofs of the last two results can be found in \[10\].
Lemma 2.2 Let $V \subset (\ast S)^d$. The set $V$ is a partition code if and only if 
$\sum_{v \in V} \bar{g}(v, s_v) = 2^d$.

Corollary 2.3 Let $V \subset S^d$ be a polybox code and let $u \in S^d$. For every $v \in V$ let $\bar{v} \in (\ast S)^d$ be defined in the following way: if $v_i \neq u_i$, then $\bar{v}_i = v_i$, and if $v_i = u_i$, then $\bar{v}_i = \ast$. Let $\bar{u} \cap \bar{v} \neq \emptyset$ for every $v \in V$. If $u \subseteq V$, then $\bar{V} = \{\bar{v} : v \in V\}$ is a partition code.

3 Polybox codes with a few words

To show that Keller’s conjecture is true in dimension seven for a cube tiling $[0, 1]^7 + T$ for which $|L(T, x, i)| \geq 6$, it was sufficient to prove the following theorem (10):

Theorem 3.1 If $V, W \subset S^d, d \geq 4$, are disjoint and equivalent polybox codes without twin pairs, then $|V| \geq 12$.

To show that the conjecture is true in dimension seven for a cube tiling $[0, 1]^7 + T$ with $|L(T, x, i)| = 5$, we have to know precisely the structure of all twin pairs free disjoint and equivalent polybox codes $V$ and $W$, with 12 words each, in dimensions four, five and six. To find this structure we need to know the structure of some polybox codes having a few words. We start with partition codes.

If $v \in (\ast S)^d$, and $\sigma$ is a permutation of the set $[d]$, then $v_\sigma = v_{\sigma(1)} \cdots v_{\sigma(d)}$.

For every $i \in [d]$ let $h_i : \ast S \to \ast S$ be a bijection such that $h_i(\ast) = \ast$ and $(h_i(l'))' = h_i(l)$ for every $l \in S$. We say that polybox codes $V, W \subset (\ast S)^d$ are isomorphic if there are $\sigma$ and $h_1, \ldots, h_d$ such that $W = \{h_1(v_{\sigma(1)}) \cdots h_d(v_{\sigma(d)}) : v \in V\}$.

Lemma 3.2 Let $V \subset (\ast S)^d$ be a partition code.

If $|V| = 3$, then

$$ (V)_A^c = \{l_1, l'_1, l_2, l'_2\}, $$

where $l_1, l_2 \in S$, $A = \{i_1 < i_2\} \subseteq [d]$ and $(V)_A = \{\ast \cdots \ast\} \subset (\ast S)^{d-2}$.

If $|V| = 4$ and $V$ contains only one twin pair, then

$$ (V)_A^c = \{l_1l_2l_3, l'_1l'_2l'_3, * * l_3\} $$

where $l_1, l_2, l_3 \in S$, $A = \{i_1 < i_2 < i_3\} \subseteq [d]$ and $(V)_A = \{\ast \cdots \ast\} \subset (\ast S)^{d-3}$.

If $|V| = 5$ and $V$ does not contain a twin pair, then

$$ (V)_A^c = \{l_1l_2l_3, l'_1l'_2l'_3, l_1l_3, l'_1l'_3, l'_1l_2\} $$

where $l_1, l_2, l_3 \in S$, $A = \{i_1 < i_2 < i_3\} \subseteq [d]$ and $(V)_A = \{\ast \cdots \ast\} \subset (\ast S)^{d-4}$. 

2 Theorem 2.3 Let $V \subset S^d$ be a polybox code and let $u \in S^d$. For every $v \in V$ let $\bar{v} \in (\ast S)^d$ be defined in the following way: if $v_i \neq u_i$, then $\bar{v}_i = v_i$, and if $v_i = u_i$, then $\bar{v}_i = \ast$. Let $\bar{u} \cap \bar{v} \neq \emptyset$ for every $v \in V$. If $u \subseteq V$, then $\bar{V} = \{\bar{v} : v \in V\}$ is a partition code.
where \( l_1, l_2, l_3 \in S \), \( A = \{ i_1 < i_2 < i_3 \} \subseteq [d] \) and \((V)_A = \{* \cdots *\} \subset (S)^{d-3}\).

If \(|V| = 6\) and \(V\) does not contain a twin pair, then
\[
(V)_A^v = \{* * * * l_4, l_1 l_2 l_3 l'_4, l'_1 l'_2 l'_3 l'_4, l_1 l'_3 l'_4, l'_1 l_2 * l'_4\},
\]
where \( l_1, l_2, l_3, l_4 \in S \), \( A = \{ i_1 < i_2 < i_3 < i_4 \} \subseteq [d] \) and \((V)_A = \{* \cdots *\} \subset (S)^{d-4}\).

The above partition codes are given up to an isomorphism.

Proof. Let \( V = \{v^1, v^2, v^3\} \). By Lemma 2.2, \( \sum_{v \in V} \tilde{g}(v, s_v) = 2^d \), and thus \( \tilde{g}(v^1, s_v) = 2^{d-1} \), \( \tilde{g}(v^2, s_v) = \tilde{g}(v^3, s_v) = 2^{d-2} \). Let \( i_1 \in [d] \) be such that \( v^1_{i_1} \neq * \). Then \( v^2_{i_2} = v^3_{i_3} = (v^1_{i_1})' \). The words \( v^2, v^3 \) are dichotomous, and therefore \( v^2_{i_2} = (v^1_{i_1})', v^2_{i_2} \neq * \), for some \( i_2 \in [d] \setminus \{i_1\} \) (Figure 4a). (Clearly, we can assume that \( i_1 < i_2 \).) Obviously, \((V)_A = \{* \cdots *\} \subset (S)^{d-2}\), where \( A = \{i_1, i_2\} \).

Let now \( V = \{v^1, v^2, v^3, v^4\} \). There are two solutions of the equation
\[
\sum_{v \in V} \tilde{g}(v, s_v) = 2^d: \tilde{g}(v^1, s_v) = 2^{d-2} \text{ for every } i \in [4] \text{ and } \tilde{g}(v^1, s_v) = 2^{d-1},
\tilde{g}(v^2, s_v) = 2^{d-2}, \tilde{g}(v^3, s_v) = \tilde{g}(v^4, s_v) = 2^{d-3}. \]
Since the words are pairwise dichotomous, it can be easily checked that in both cases there are \( i_1, i_2, i_3 \in [d], i_1 < i_2 < i_3 \), such that \( v_i = * \) for every \( v \in V \) and \( i \in [d] \setminus \{i_1, i_2, i_3\} \).

Thus, we have to determine all partitions of a 3-dimensional box into four pairwise dichotomous boxes with only one twin pair. It is easy to see that the first solution corresponds to partitions with more than one twin pair (examples of such partitions are presented in Figure 4b and 4c). The second solution corresponds to partition codes with one twin pair (Figure 4d).

The proofs of (3.3) and (3.4) (Figure 4e and 4f) can be found in [10]. □
Fig. 4. Figure a: the realization $f((V)_A)$ of the partition code $(V)_A = \{l_1, l_1l_2, l_1l_2l_3\}$ in the 2-box $X = [0, 1]^2$, where $f_i(l_i) = [0, 1/2), f_i(*) = [0, 1]$ for $i = 1, 2$. Figure b and c: examples of partitions of a 3-box $X = [0, 1]^3$ into four pairwise dichotomous boxes with more than one twin pair. Figure d: the realization $f((V)_A)$ in $X = [0, 1]^3$, where $f_3(l_3) = [0, 1/2)$ and $f_3(*) = [0, 1)$, of the partition code $(V)_A = \{l_1l_2l_3, l_1l_2l_3, l_1l_2l_3, l_1l_2l_3\}$ with one twin pair. Figure e: the realization $f((V)_A)$ in $X = [0, 1]^4$ of the partition code $(V)_A = \{l_1l_2l_3, l_1l_2l_3, l_1l_2l_3, l_1l_2l_3\}$. Figure f: the realization $f((V)_A)$ in the 4-box $X = [0, 1]^4$, where $f_4(l_4) = [0, 1/2)$, $f_4(*) = [0, 1)$ of the code $(V)_A = \{**l_4, l_1l_2l_3l_4, l_1l_2l_3l_4, l_1l_2l_3l_4\}$. We consider the two halves of the 4-box $X = [0, 1]^4$, $[0, 1]^3 \times [0, 1/2)$ (on the left) and $[0, 1]^3 \times [1/2, 1)$ (on the right), the fourth axis is omitted, in which we see the realizations of the codes $(\{l_1l_2l_3l_4, l_1l_2l_3l_4, l_1l_2l_3l_4\})_i$ (on the left) and $(\{**l_4\})_i$ (on the right).

**Lemma 3.3** Let $V, W \subset (S)^d$ be disjoint and equivalent polybox codes without twin pairs such that $|V| \in \{2, 3\}$ and $|W| \in \{2, 3, 4\}$, and let $l_1, l_2, l_3, l_4 \in S$.

| If $|V| = |W| = 2$ and $d \geq 2$, then | If $|V| = |W| = 2$ and $d \geq 3$, then |
| --- | --- |
| $(V)_A = \{l_1, l_1l_2, l_1l_2l_3\}$ | $(V)_A = \{l_1l_2, l_1l_2l_3\}$ |
| $(W)_A = \{l_1l_2, l_1l_2l_3\}$ | $(W)_A = \{l_1l_2, l_1l_2l_3\}$ |
| where $A = \{i_1, i_2\}$, $(V)_A = (W)_A = \{(p)_A\}$ for some $p \in (S)^d$ | |

| If $|V| = |W| = 3$ and $d \geq 3$, then | If $|V| = |W| = 4$ and $d \geq 3$, then |
| --- | --- |
| $(V)_A = \{**l_3, l_1l_2l_3\}$ | $(V)_A = \{**l_3, l_1l_2l_3\}$ |
| $(W)_A = \{l_1l_2, l_1l_2l_3, l_1l_2l_3\}$ | $(W)_A = \{l_1l_2, l_1l_2l_3, l_1l_2l_3\}$ |
| where $A = \{i_1, i_2, i_3\}$, $(V)_A = (W)_A = \{(p)_A\}$ for some $p \in (S)^d$ | |

| If $|V| = |W| = 3$ and $d \geq 3$, then | If $|V| = |W| = 3$ and $d \geq 3$, then |
| --- | --- |
| $(V)_A = \{**l_3, l_1l_2l_3\}$ | $(V)_A = \{**l_3, l_1l_2l_3\}$ |
| $(W)_A = \{l_1l_2, l_1l_2l_3, l_1l_2l_3\}$ | $(W)_A = \{l_1l_2, l_1l_2l_3, l_1l_2l_3\}$ |
| where $A = \{i_1, i_2, i_3\}$, $(V)_A = (W)_A = \{(p)_A\}$ for some $p \in (S)^d$ | |

The above polybox codes are given up to an isomorphism.
Proof of the case $|V| = 2, |W| = 4$. Let $V = \{u, v\}$ and $W = \{w, p, q, r\}$.

We can assume that $\bar{u} \cap \bar{w} \neq \emptyset$ and $\bar{v} \cap \bar{w} \neq \emptyset$. Then there is $i \in [d]$ such that $(\bar{v})_i \cap (\bar{u})_i \neq \emptyset$ and $(\bar{w})_i \subseteq (\bar{v})_i \cap (\bar{u})_i$. We will show that $w_i = *$. Suppose this is not true. Since $(\bar{v})_i \cap (\bar{u})_i \neq \emptyset$ and $u, v$ are dichotomous, we have $v_i = u'_i$, $u_i \neq *$. Then $w_i \notin \{u_i, u'_i\}$, and, by (2.1), we can choose $x \in \bar{u} \setminus \bar{w}$ and $y \in \bar{v} \setminus \bar{w}$ such that $(x)_i = (y)_i$ and $(x)_i \in (\bar{w})_i$. The words in $W$ are pairwise dichotomous, and thus there is a word in $W$, say $p$, such that $x, y \in \bar{p}$. Note that $p_i = w'_i$ and consequently, $(\bar{p})_i \subseteq (\bar{v})_i \cap (\bar{u})_i$.

Moreover, $(\bar{u})_i \setminus (\bar{v})_i \cup (\bar{v})_i \setminus (\bar{u})_i \neq \emptyset$, for otherwise $u$ and $v$ would be a twin pair. Let $(\bar{v})_i \setminus (\bar{u})_i \neq \emptyset$ and take $z \in \bar{v}$ such that $(z)_i \in (\bar{v})_i \setminus (\bar{u})_i$. Clearly, $z \notin \bar{w} \cup \bar{p}$, and thus, $z \in \bar{q}$. Then, by (2.1), $q_i = v_i$. Since $w$ and $p$ are not a twin pair, $(\bar{w})_i \setminus (\bar{p})_i \cup (\bar{p})_i \setminus (\bar{w})_i \neq \emptyset$. Assume without loss of generality that $(\bar{p})_i \setminus (\bar{w})_i \neq \emptyset$ and choose $z^1 \in \bar{u} \setminus \bar{p}$ such that $(z^1)_i \in (\bar{p})_i \setminus (\bar{w})_i$. Then $z^1 \in \bar{r}$, and since $p$ and $r$ are dichotomous, we have $r_i = p'_i = w_i$. Now it can be easily seen that $(\bar{w})_i \cup (\bar{r})_i = (\bar{u})_i$, which implies that $w$ and $r$ form a twin pair, a contradiction. This completes the proof that $w_i = *$.

We now show that exactly one box from the set $E(W)$ has nonempty intersection with both boxes $\bar{u}$ and $\bar{v}$. Assume on the contrary that there are exactly two boxes in $E(W)$, say $\bar{w}$ and $\bar{p}$, having nonempty intersections with $\bar{u}$ and $\bar{v}$ simultaneously. Then, as we have just shown, $w_i = p_i = *$ and $q_i, r_i \in \{u_i, u'_i\}$. If $(\bar{u})_i = (\bar{w})_i \cup (\bar{p})_i$ or $(\bar{v})_i = (\bar{w})_i \cup (\bar{p})_i$, then $(\bar{w})_i$ and $(\bar{p})_i$ are a twin pair, and consequently, $w$ and $p$ are a twin pair, which is a contradiction. Therefore, $(\bar{u})_i = (\bar{w})_i \cup (\bar{p})_i \cup (\bar{q})_i$ and $(\bar{v})_i = (\bar{w})_i \cup (\bar{p})_i \cup (\bar{r})_i$ and $q_i = r'_i$. By (3.1), there are twin pairs in the sets of boxes $(\bar{w}_i, (\bar{p})_i, (\bar{q})_i)$ and $(\bar{w}_i, (\bar{p})_i, (\bar{r})_i)$. Since $w_i = p_i$, the boxes $(\bar{w}_i)$ and $(\bar{p})_i)$ cannot form a twin pair which means that the set $(\bar{w}_i) \cup (\bar{p})_i$ is not a box. But $(\bar{w})_i \cup (\bar{p})_i \cup (\bar{q})_i$ and $(\bar{w})_i \cup (\bar{p})_i \cup (\bar{r})_i$ are boxes. Therefore, $(\bar{q})_i = (\bar{r})_i$ and consequently, $q$ and $r$ are a twin pair, a contradiction.

If $w_i = p_i = q_i = *$ and $r_i \neq *$, then $(\bar{w})_i \cup (\bar{p})_i \cup (\bar{q})_i = (u_i)$ or $(\bar{w})_i \cup (\bar{p})_i \cup (\bar{q})_i = (v_i)$. By (3.1), two of the three boxes $(\bar{w})_i, (\bar{p})_i, (\bar{q})_i$ are a twin pair, and therefore there is a twin pair among the words $w, p, q, r$, which is impossible.

Similarly, if $w_i = p_i = q_i = r_i = *$, then $(\bar{w})_i \cup (\bar{p})_i \cup (\bar{q})_i \cup (\bar{r})_i = (u_i)$ and therefore, by (3.2) and the proof of Lemma 3.2 (the case $|V| = 4$), there is a twin pair in the set $\{w, p, q, r\}$, which is a contradiction.

We have shown that $w_i = *$, $u_i = v'_i$ and $p_i, q_i, r_i \in \{u_i, u'_i\}$.

Since $(\bar{w})_i \subseteq (\bar{u})_i$, for every $j \in [d] \setminus \{i\}$ we have $Ew_j \subseteq Eu_j$, and by (2.1), if $Ew_j \neq Eu_j$, then $w_j \notin S$ and $u_j = *$.

Now two cases may occur:

\[ \bar{u} = \bar{u} \cap \bar{w} \cup \bar{p} \cup \bar{q} \cup \bar{r} \quad \text{and} \quad \bar{v} = \bar{v} \cap \bar{w} \quad (\text{Figure 5b, c}) \]
or

\[ \tilde{u} = \tilde{u} \cap \tilde{w} \cup \tilde{p} \cup \tilde{q} \quad \text{and} \quad \tilde{v} = \tilde{v} \cap \tilde{w} \cup \tilde{r} \quad (\text{Figure 5a, d, e}). \]

In the first case the \(d\)-box \(\tilde{u}\) is divided into four pairwise dichotomous boxes, and thus the structure of this partition is given by \(3.2\) which contains exactly one twin pair. Hence, the partition \(\{\tilde{u} \cap \tilde{w}, \tilde{p}, \tilde{q}, \tilde{r}\}\) contains one twin pair, and \(W\) does not contain a twin pair. Therefore, the box \(\tilde{u} \cap \tilde{w}\) must be one of the twins. We may assume that \(\tilde{p}\) is the second one. Thus, there are \(i_1, i_2, i_3 \in [d]\), \(i_1 < i_2 < i_3 \) and the letters \(l_1, l_2, l_3 \in S\) such that (we assume without loss of generality that \(i_3 < i\)): \((u)_{AC} = * * * l_4, (w)_{AC} = l_1l_2l_3*, \) and therefore \((\tilde{u} \cap \tilde{w})_{AC} = E l_1 \times E l_2 \times E l_3 \times E l_4\), where \(l_4 = u_i, A = \{i_1 < i_2 < i_3 < i\} \) and \((p)_{AC}\) has one of the forms: \(l_1'l_2l_3l_4, l_1l_2'l_3l_4\) or \(l_1l_2l_3l_4\). We consider the first case as in the rest of them we obtain isomorphic forms. Let \((p)_{AC} = l_1l_2l_3l_4.\) By \(3.2\), \((q)_{AC} = * l_2l_3l_4\) and \((r)_{AC} = * * l_3l_4\) or \((q)_{AC} = * l_2l_3l_4\) and \((r)_{AC} = * l_3^4 * l_4\). Since \((w)_i = (v)_i\), we have \((v)_{AC} = l_1l_2l_3l_4.\) By \(3.2\), \((u)_A = (p)_A = (q)_A = (r)\) and, since \((\tilde{w})_i \subset (\tilde{u})_i \cap (\tilde{v})_i, (u)_A = (w)_A = (v)_A.\)

In the second case there are two possibilities: \((\tilde{u})_i \cap (\tilde{r})_i \neq \emptyset\) or \((\tilde{u})_i \cap (\tilde{r})_i = \emptyset.\) Since now the \(d\)-box \(\tilde{u}\) is divided into three pairwise dichotomous boxes, the structure of the partition \(\{\tilde{u} \cap \tilde{w}, \tilde{p}, \tilde{q}\}\) is given by \(3.1.\) Clearly, as above, we may assume that the boxes \(\tilde{u} \cap \tilde{w}\) and \(\tilde{p}\) are the only twin pair in this partition. Thus, there are \(i_1, i_2 \in [d] \setminus \{i\}, i_1 < i_2,\) and letters \(l_1, l_2 \in S\) such that such that \((u)_{BE} = * * l_3, (w)_{BE} = l_1l_2l_3*\) and \((u)_B = (w)_B,\) where \(l_3 = u_i\) and \(B = \{i_1 < i_2 < i_3\}, i = i_3.\) Furthermore, \((p)_{BE} = l_1l_2l_3\) and \((q)_{BE} = * l_2l_3\) or \((p)_{BE} = l_1l_2l_3\) and \((q)_{BE} = l_1^4 * l_3\) (in this second case we obtain an isomorphic form). In both cases, \((p)_B = (q)_B = (u)_B.\)

Let \((\tilde{u})_i \cap (\tilde{r})_i \neq \emptyset.\) Since \((\tilde{w})_i \cap (\tilde{r})_i\) are a twin pair and \((u)_B = (w)_B,\) there is \(k \in \{i_1, i_2\}\) such that \(w_k = r_k'\) and \((w)_{i_1,k} = (r)_{i_1,k}.\) Then \((w)_B = (r)_B.\) Taking \(k = i_1,\) we obtain \((r)_B = l_1' l_2' l_3'.\) Thus, we have to exclude the case \((p)_B = l_1'l_2'l_3\) and \((q)_B = * l_2l_3,\) for otherwise \(p\) and \(r\) are a twin pair (if we take \(k = i_2\), then \((r)_B = l_1l_2l_3\) and the case \((p)_B = l_1l_2l_3\) and \((q)_B = l_1l_2l_3\) has to be excluded). Since \(\tilde{v} = \tilde{v} \cap \tilde{w} \cup \tilde{r},\) we have \((v)_B = * l_2l_3\) and \((v)_{i_1,i} = (w)_{i_1,i},\) and then \((v)_B = (w)_B \) if \(k = i_2,\) then \((v)_B = l_1l_3').\)

Summing up, in the case \((\tilde{u})_i \cap (\tilde{r})_i \neq \emptyset\) we have: \((u)_B = * * l_3, (v)_B = * l_2l_3\) and \((w)_B = l_1l_2l_3*\) or \((u)_B = (w)_B = (v)_B = (r)_B.\)

Let now \((\tilde{u})_i \cap (\tilde{r})_i = \emptyset.\) Since the boxes \((\tilde{w})_i \cap (\tilde{v})_i\) and \((\tilde{r})_i\) are a twin pair, if \(w_{i_1} = r_{i_1}'\) or \(w_{i_2} = r_{i_2}'\) then \((\tilde{u})_i \cap (\tilde{r})_i = \emptyset.\) Therefore there is exactly one \(j \in B^c\) such that \(w_j = r_j'\) and \(w_j \neq *\). Assume without loss of generality that \(j = i_4 > i_3 = i.\) Then \((r)_A = l_1l_2l_3l_4',\) where \(l_4 = u_{i_4} \neq *\) because \(w_j = u_j\) and \(A = \{i_1, i_2, i_3, i_4\}.\) Thus, \((u)_A = * * l_4, (v)_A = l_1l_2l_3'\) and \((w)_A = l_1l_2l_4.\) Clearly, \((p)_A = l_1l_2l_3l_4\) and \((q)_A = l_1l_2l_3l_4\) if \((p)_A = l_1l_2l_3l_4\)
and \((q)_{A^c} = l_1^*l_2l_4\) we get an isomorphic form of \(W\). Since \(\bar{u} = \bar{p} \cup \bar{q} \cup \bar{w} \cap \bar{u}\), we have \((p)_{A} = (q)_{A} = (u)_{A} = (w)_{A}\). Similarly, since \(\bar{v} = \bar{r} \cup \bar{w} \cap \bar{v}\), we have \((w)_{A} = (r)_{A} = (v)_{A}\).

Permuting the letters at the third and the fourth position in every word in \(V\) and \(W\) we get the form as it is in the lemma.

Fig. 5. Figure a: the realizations \(f((V)_{A^c})\) and \(f((W)_{A^c})\) of the codes \((V)_{A^c} = \{l_1^*l_2^*\}\) (on the left) and \((W)_{A^c} = \{l_1l_2l_3, l_1^*l_2^*l_3, l_1^*l_2l_3^*\}\) (on the right) in the 3-box \(X = [0,1]^3\), where \(f_i(l_i) = [0,1/2)\) and \(f_i(*) = [0,1]\) for \(i = 1,2,3\). In a four dimensional case we consider the two halves of the 4-box \(X = [0,1]^4\): \([0,1]^3 \times [0,1/2]\) (always on the left) and \([0,1]^3 \times [1/2,1]\) (always on the right). Clearly, the fourth axis is omitted. Figure b and c: the realizations \((f((V)_{A^c}))_{i_4}\) and \((f((W)_{A^c}))_{i_4}\) of the codes \((V)_{A^c} = \{l_1^*l_2^*l_3^*\}\) and \((W)_{A^c} = \{l_1l_2l_3^*, l_1^*l_2^*l_3, l_1^*l_2l_3^*, l_1^*l_2^*l_3^*\}\) in the 4-box \(X = [0,1]^4\), where \(f_i(l_i) = [0,1/2)\) and \(f_i(*) = [0,1]\). The two darkest boxes in Figure c are in fact one box which is a realization of the word \(l_1^*l_2^*l_3^*\). Figure d and e: the realizations \((f((V)_{A^c}))_{i_4}\) and \((f((W)_{A^c}))_{i_4}\) of the codes \((V)_{A^c} = \{l_1^*l_2l_3^*, l_1^*l_2^*l_3, l_1^*l_2l_3^*, l_1^*l_2^*l_3^*\}\) and \((W)_{A^c} = \{l_1^*l_2^*l_3^*, l_1l_2l_3^*, l_1^*l_2^*l_3^*\}\) in \(X = [0,1]^4\). Similarly like above, the two darkest boxes in Figure e are one box which is a realization of the word \(l_1^*l_2^*l_3^*\).

The sketches of the proofs of the rest of the cases of Lemma 3.3.

Let \(|V| = 2, |W| = 3\), and let \(V = \{v, u\}\) and \(W = \{w, p, q\}\). In the same way as above we show that there is exactly one word in \(W\), say \(w\), and there is \(i \in [d]\) such that \(w_i = \star\), \(u_i = v_i\), \(\langle w \rangle_i \subseteq \langle u \rangle_i \cap \langle v \rangle_i\) and \(p_i, q_i \in \{u_i, u'_i\}\).

If \(\bar{u} = \bar{u} \cap \bar{w} \cup \bar{p} \cup \bar{q}\) and \(\bar{v} = \bar{v} \cap \bar{w} \cup \bar{q}\), then the structure the partition \(\{\bar{u} \cap \bar{w}, \bar{p}, \bar{q}\}\) of the \(d\)-box \(\bar{u}\) is given by \((3.1)\), and \((\bar{v})_i = (\bar{w})_i\). This case is illustrated in Figure 6b.

If \(\bar{u} = \bar{u} \cap \bar{w} \cup \bar{p}\) and \(\bar{v} = \bar{v} \cap \bar{w} \cup \bar{q}\), then the boxes \(\bar{u} \cap \bar{w}, \bar{p}\) are a twin pair and \(\bar{v} \cap \bar{w}, \bar{q}\) are a twin pair. Note that \((\bar{u})_i \cap (\bar{q})_i = \emptyset\), for otherwise \(p\) and \(q\)
are a twin pair, which is impossible. This case is illustrated in Figure 6c.

The proof of the case $|V| = 2, |W| = 2$ (Figure 6a) can be found in [10].

Let $|V| = \{v^1, v^2, v^3\}$ and $|W| = \{w^1, w^2, w^3\}$. Since in this case realizations are 3-dimensional boxes, we will establish only the values $\bar{g}(v^i, s_*)$ and $\bar{g}(w^i, s_*)$ for $i = 1, 2, 3$. Recall that $\bar{g}(v^i, s_*) = 2k$ if and only if the words $v^i$ contains $k$ stars and, by Lemma 2.2, $\sum_{i=1}^{3} \bar{g}(v^i, s_*) \leq 8$.

If $\bar{g}(v^1, s_*) = 4$ and $\bar{g}(v^2, s_*) = 2$, then $\bar{g}(v^3, s_*) = 1$ because if $\bar{g}(v^3, s_*) = 2$, then $v^2$ and $v^3$ form a twin pair. This is easy to see that $\bar{g}(v^1, s_*) = 4$, $\bar{g}(w^2, s_*) = 2$, $\bar{g}(w^3, s_*) = 1$. This case is illustrated in Figure 6e.

It can be easily verified that the case $\bar{g}(v^1, s_*) = 4$, $\bar{g}(v^2, s_*) = 1$ and $\bar{g}(v^3, s_*) = 1$ is impossible.

Let now $\bar{g}(v^i, s_*) = 2$ for $i = 1, 2, 3$. This is obvious that $\bar{g}(v^i, s_*) = 2$ for $i = 1, 2, 3$ (Figure 6d).

Similarly, this is not hard to find the forms of $V$ and $W$ in the case when $\bar{g}(v^1, s_*) = \bar{g}(v^2, s_*) = 2$ and $\bar{g}(v^3, s_*) = 1$ (Figure 6f).

Finally, this is easy to check that the cases $\bar{g}(v^1, s_*) = 2$, $\bar{g}(v^2, s_*) = \bar{g}(v^3, s_*) = 1$ and $\bar{g}(v^i, s_*) = 1$ for $i = 1, 2, 3$ are impossible.

\[\square\]

---

**Fig. 6.** Figure a: the realizations $f((V)_{A^c})$ and $f((W)_{A^c})$ of the codes $(V)_{A^c} = \{l_1*, l'_1l_2\}$ (on the left) and $(W)_{A^c} = \{*l_2, l_1l'_2\}$ (on the right) in the 2-box $X = [0,1]^2$, where $f_i(l_i) = [0,1/2]$ and $f_i(*) = [0,1]$ for $i = 1, 2$. Figure b: the realizations $f((V)_{A^c})$ and $f((W)_{A^c})$ of the codes $(V)_{A^c} = \{l_1**l'_2l_3\}$ and $(W)_{A^c} = \{l_1*l_3, l_1l_2l'_3, *l'_2l_3\}$ in the 3-box $X = [0,1]^3$, where $f_i(l_i) = [0,1/2]$ and $f_i(*) = [0,1]$. Figure c: the realizations $f((V)_{A^c})$ and $f((W)_{A^c})$ of the codes $(V)_{A^c} = \{l_1l_2*, *l'_2l_3\}$ and $(W)_{A^c} = \{l_1l_2l_3, l_1* l'_2, *l'_4l_2l_3\}$.

Figure d: the realizations $f(V)$ and $f(W)$ of the codes $V = \{l_1l_2*, l'_1* l_3, *l'_2l_3\}$ (on the left) and $W = \{*l_2l_3, l_1* l'_2, *l'_2l_3\}$ (on the right). Figure e: the realizations $f(V)$ and $f(W)$ of the codes $V = \{l_1**, l'_1l_2l_3, l'_2l_3\}$ and $W = \{*l_3, l_1l_2l'_3, *l'_2l_3\}$. Figure f: the realizations $f(V)$ and $f(W)$ of the codes $V = \{l_1l_2*, l'_1l_2l_3, *l'_2l'_3\}$ and $W = \{*l_2l_3, l_1* l'_3, *l'_1l_2l'_3\}$. 
We now collect the above results in the forms in which they will be used later in the paper.

**Statement 3.4** Let $V \subset S^d$ be a polybox code and let $w \subset V$, where $w \in S^d$ and $w \notin V$.

(a) If the polybox code $V$ does not contain a twin pair, then it contains at least five words. The code $V$ has exactly five words and does not contain a twin pair if and only if it is of the form $(3.3)$ of Lemma 3.2, where instead of a twin pair if and only if it is of the form $(3.4)$ of Lemma 3.2, where instead of $\star$ at a position $j \in [d]$ we take $w_j$, and $l_k \notin \{w_{i_k}, w'_{i_k}\}$ for $k \in \{1, 2, 3\}$. Moreover, $V$ is rigid. Similarly, the code $V$ has exactly six words and does not contain a twin pair if and only if it is of the form $(3.4)$ of Lemma 3.2, where instead of $\star$ at a position $j \in [d]$ we take $w_j$, and $l_k \notin \{w_{i_k}, w'_{i_k}\}$ for $k \in \{1, 2, 3, 4\}$.

(b) If $u \subset V$, where $u \in S^d$ and $u \notin V$, the code $V$ does not contain twin pairs and the words $w, u$ are dichotomous but they do not form a twin pair, then $|V| \geq 7$.

(c) Let $\{\tilde{w} \cap \tilde{v} \neq \emptyset : v \in V^{i,j}\} \neq \emptyset$, where $w_i \notin \{l, l'\}$. Then the set $\tilde{w} \cap \bigcup E(V^{i,j}) \cup \tilde{w} \cap \bigcup E(V^{i,j'})$ is an $i$-cylinder in the d-box $\tilde{w}$. Consequently, $\bigcup \{\tilde{w} \cap \tilde{v} : v \in Q\} = \bigcup \{\tilde{w} \cap \tilde{v} : v \in P\}$, where $P = \{v \in V^{i,j} : \tilde{w} \cap \tilde{v} \neq \emptyset\}$ and $Q = \{v \in V^{i,j'} : \tilde{w} \cap \tilde{v} \neq \emptyset\}$.

If $|P| = 1$, $|Q| = 5$ and $V$ does not contain a twin pair, then there is a set $A = \{i_1 < i_2 < i_3\} \subset [d]$ such that

$$(P)_{A^c} = \{w_{i_1}w_{i_2}w_{i_3}\} \text{ and } (Q)_{A^c} = \{l_1l_2l_3, l'_1l'_2l'_3, w_{i_1}l'_2l'_3, l_1w_{i_2}l'_3, l'_1l_2w_{i_3}\},$$

where $l_k \notin \{w_{i_k}, w'_{i_k}\}$ for $k = 1, 2, 3$ and $(P)_{A(U(i))} = (Q)_{A(U(i))}$.

If $(|P|, |Q|) \in \{(2, 2), (2, 3), (2, 4), (3, 3)\}$ and $V$ does not contain a twin pair, then the structure of $(P)$, and $(Q)$ is such as in Lemma 3.3, but in all those polybox codes we put $w_j$ instead of $\star$, if the star appears at the $j$-th position, and $l_k \notin \{w_{i_k}, w'_{i_k}\}$ for $k \in \{1, 2, 3, 4\}$.

(d) Let $P$ and $Q$ be such as in (c). If $|P| = 1$ and $1 \leq |Q| \leq 4$, then there is a twin pair in $V$.

**Proof of (a).** For $|V| = 5$ it can be found in [10], and the case $|V| = 6$ is proven in the same manner.

**Proof of (b).** Let $W = \{v \in V : \tilde{v} \cap \tilde{w} \neq \emptyset\}$ and $U = \{v \in V : \tilde{v} \cap \tilde{u} \neq \emptyset\}$. By (a), $|W| \geq 5$ and $|U| \geq 5$.

Suppose that $|W| = 5$ and $|U| = 5$. Again by (a), there is a set $A = \{i_1 < i_2 < i_3\} \subset [d]$ and letters $l_1, l_2, l_3 \in S$, $l_j \notin \{w_{i_j}, w'_{i_j}\}$ for $i = 1, 2, 3$, such that

$$(W)_{A^c} = \{l_1l_2l_3, l'_1l'_2l'_3, w_{i_1}l'_2l'_3, l_1w_{i_2}l'_3, l'_1l_2w_{i_3}\}.$$
and \((w)_A = (v)_A\) for every \(v \in W\).

Clearly, if \(|W \cap U| \leq 3\), then \(|V| \geq 7\).

Let \(|W \cap U| = 4\). Since \(w\) and \(u\) are dichotomous, there is \(i \in [d]\) such that \(w_i = u_i\). If \(i \in A^c\), then \(W \cap U = \emptyset\) because \((w)_A = (v)_A\) for every \(v \in W\). Therefore, \(i \in A\). Observe that, for the rest two \(j \in A \setminus \{i\}\) we have \(u_j \neq w_j\) and \(u_j \not\in \{l_j, l'_j\}\), for otherwise \(|W \cap U| < 4\). Assume without loss of generality that \(i = i_1\). Since the structure of \(U\) is such as predicted in (a) and \(l_1l_2l_3, l'_1l'_2l'_3 \in (U)_A^c\), it follows that

\[
(U)_A^c = \{l_1l_2l_3, l'_1l'_2l'_3, w_{i_1}l_2l_3, l_1w_{i_2}l'_3, l'_1l_2w_{i_3}\},
\]

and \((w)_A = (u)_A\). Thus, \(w_{i_1}' = u_{i_1}, w_{i_2}' = u_{i_2}\) and \(w_{i_3}' = u_{i_3}\), and hence \(w\) and \(u\) are a twin pair, a contradiction.

If \(W = U\), then \(u = w\), which contradicts the assumption.

Suppose now that \(|W| = 6\) and assume on the contrary that \(|V| = 6\). Then \(V = W\). By (3.3) in Lemma 3.2 and (a),

\[
(W)_A^c = \{w_{i_1}w_{i_2}w_{i_3}l_4, l_1l_2l_3l_4, l'_1l'_2l'_3, w_{i_1}l'_2l'_3l'_4, l_1w_{i_2}l'_3l'_4, l'_1l_2w_{i_3}l'_4\},
\]

and \((w)_A = (v)_A\) for \(v \in W\), where \(A = \{i_1 < i_2 < i_3 < i_4\} \subseteq [d]\) and \(l_j \not\in \{w_{i_j}, w_{i_j}'\}\) for \(j = 1, 2, 3, 4\).

Let \(v^1 \in W\) be such that \((v^1)_A^c = w_{i_1}w_{i_2}w_{i_3}l_4\). We have \((w)_i = (v^1)_i\). Note that the structure of \((W \setminus \{v^1\})_i\) is such as in (a). In particular, \((v^1)_i\) is one and only word which is covered by \((W \setminus \{v^1\})_i\). Hence, \((w)_i = (v^1)_i\). Consequently, \(w\) and \(u\) are a twin pair, a contradiction.

**Proof of (c).** The set \(\bar{w} \cap \bigcup E(V^1) \cup \bar{w} \cap \bigcup E(V^{i'})\) is an \(i\)-cylinder in the \(d\)-box \(\bar{w}\) because \(\bar{w} \cap \bar{v} : v \in V\) is a suit for \(\bar{w}\) (Figure 7).

Since \(V\) does not contain twin pairs, the set of boxes \(\{\bar{w} \cap \bar{v} : v \in V\}\) is a partition of the \(d\)-box \(\bar{w}\) into pairwise dichotomous boxes which, by Lemma \(2.1\), does not contain twin pairs (Figure 7).

Since the set \(\bigcup \{\bar{w} \cap \bar{v} : v \in V^1\} \cup \bigcup \{\bar{w} \cap \bar{v} : v \in V^{i'}\}\) is an \(i\)-cylinder in the box \(\bar{w}\), we have \(\bigcup \{\bar{v} \cap \bar{w} : v \in V^{i}\}\) = \(\bigcup \{\bar{v} \cap \bar{w} : v \in V^{i'}\}\).

We prove only the case \(|P| = 1, |Q| = 5\). The rest of the cases is proven in the very similar way (compare Example \(5.5\)).

Let \(P = \{u\}\). The \((d - 1)\)-box \((\bar{w} \cap \bar{u})\) is divided into five pairwise dichotomous boxes \(\{\bar{w} \cap \bar{v}_i : v \in Q\}\). Thus, \(\bar{w} \cap \bar{v}_i \subseteq (\bar{w} \cap \bar{u})\) for every \(v \in Q\), and then \(E_{w_j} \cap E_{v_j} \subseteq E_{w_j} \cap E_{u_j}\) for every \(j \in [d] \setminus \{i\}\). It follows that, by \((2.1)\), if \(w_j \neq u_j\), then \(v_j = u_j\). Moreover, by Lemma \(2.1\), the boxes of the partition \(\{\bar{w} \cap \bar{v}_i : v \in Q\}\) do not form twin pairs. Therefore, a code of the partition \(\{\bar{w} \cap \bar{v}_i : v \in Q\}\) of the box \((\bar{w} \cap \bar{u})\) is given by \(\{88\}\). Since for every \(j \in A = \{i_1 < i_2 < i_3\}\) (\(A\) is such as in (a)) there is \(l \in S\) such that
v_j = l$ and $q_j = l'$ for some $v, q \in Q$ and $Ew_j \cap Ew_j = Ew_j \cap Ew_j \cup Ew_j \cap Eq_j$, it must be, by (2.1), $w_j = u_j$ for every $j \in A$. Thus, $(P)_{A^c} = \{w_{i_1}w_{i_2}w_{i_3}\}$ and $(Q)_{A^c} = \{l_1l_2l_3, l'_1l'_2l'_3, w_{i_1}l_2l'_3, l_1w_{i_2}l'_3, l'_1l_2w_{i_3}\}$. Since, by (3.3), $Ew_j \cap Ev_j = Ew_j \cap Ev_j$ for every $v \in Q$ and $j \in [d] \setminus (A \cup \{i\})$, we have, by (2.1), $u_j = v_j$, and thus $(P)_{A \cup \{i\}} = (Q)_{A \cup \{i\}}$.

Proof of (d). Let $P = \{u\}$. The $(d - 1)$-box $(\bar{w} \cap \bar{v})$, is divided into pairwise dichotomous boxes $\{(\bar{u} \cap \bar{v})_j : v \in Q\}$. By (3.1), (3.2) and the proof of Lemma 3.2 (the case $|V| = 4$), this partition contains a twin pair. In the same manner as in (c) we show that there is a twin pair in $Q$. □

Example 3.5 In Figure 7 the five boxes on the left are a realization of the polybox code $V = \{aaa, a'aa', baa', a'ba, aa'b\}$, and the box in the middle is a realization of the word $w = bbb$. Since $w \subset V$, we have $\bar{w} \subset \bigcup E(V)$. Thus, the 3-box $\bar{w}$ is divided into pairwise dichotomous boxes $\bar{w} \cap \bar{v}$ for $v \in V$, and the set $\bigcup \{(\bar{w} \cap \bar{v}) : v \in P\} = \{\bar{w} \cap \bar{v} : v \in P\}$, where $P = \{v \in V^{3,a} : \bar{w} \cap \bar{v} \neq \emptyset\} = \{aaa, a'ba\}$ and $Q = \{v \in V^{3,a'} : \bar{w} \cap \bar{v} \neq \emptyset\} = \{a'aa', baa\}$, is an 3-cylinder in the box $\bar{w}$. Therefore, $\bigcup \{(\bar{w} \cap \bar{v})_3 : v \in Q\} = \bigcup \{(\bar{w} \cap \bar{v})_3 : v \in P\}$. Now, the polybox $\bigcup \{(\bar{w} \cap \bar{v})_3 : v \in Q\}$ is divided twice into pairwise dichotomous boxes without twin pairs. Since $|Q| = |P| = 2$, we apply Lemma 3.3 for the case $|Q| = |W| = 2$ to get the structure of $(Q)_3$ and $(P)_3$. Recall that in that case we have $(V)_{A^c} = \{*l_2, l_1l'_2\}$ and $(W)_{A^c} = \{l'_1l_2, l_1*\}$, where $A = \{i_1, i_2\}$ and $(V)_A = (W)_A = \{(p)_A\}$. In our case we have $A = \{1, 2\}$.

Making in $(V)_{A^c}$ and $(W)_{A^c}$ the substitutions $l_1 = a'$, $l_2 = a$ and $* = b$ we obtain $(Q)_3 = \{ba, a'aa'\}$ and $(P)_3 = \{aa, a'b\}$.

![Fig. 7](image_url)

Fig. 7. On the top: the light box (in the middle) is contained in a sum of five pairwise dichotomous boxes (the boxes on the left). These boxes determine a partition of the light box into pairwise dichotomous boxes (the partition on the right). On the bottom: the boxes in this partition are arranged into 3-cylinders.
4 The structure of equivalent polyboxes codes with 12 words: necessary conditions

In this section we determine necessary conditions which have to be fulfilled by disjoint and equivalent twin pairs free polyboxes codes $V$ and $W$ having 12 words each. This conditions will serve us to establish the initial configurations of words for the computations.

Similarly like in [10] we define a graph on a polybox code $V$. A pair of words $v, u \in S^d$ such that $v_i \notin \{u_i, u'_i\}$ for some $i \in [d]$ and $(u)_i$ and $(v)_i$ are a twin pair is called an $i$-siblings (in Figure 7 the two upper boxes on the left upper picture are 1-siblings).

Let $V \subset S^d$ be a polybox code. A graph of siblings in $V$ is a graph $G = (V, \mathcal{E})$ in which two vertices $u, v \in V$ are adjacent if they are $i$-siblings for some $i \in [d]$. We colour each edge in $\mathcal{E}$ with the colours from the set $[d]$:

an edge $e \in \mathcal{E}$ has a colour $i \in [d]$ if its ends are $i$-siblings. The graph $G$ is simple and, if $V$ does not contain a twin pair, $d(v) \leq d$ for every $v \in V$, where $d(v)$ denotes the number of neighbors of $v$. Observe that the graph $G$ does not contain triangles.

In [10], we proved the following two lemmas.

**Lemma 4.1** Let $G = (V, \mathcal{E})$ be a graph of siblings in a polybox code $V \subset S^d$, $u$ and $v$ be adjacent vertices and let $d(u) = n$ and $d(v) = m$. If $n + m = 2d$, then there are $i \in [d]$ and $l \in S$ such that

$$|V^{i,l} \cup V^{i,l'}| \geq 2d - 2,$$

and if $n + m \leq 2d - 1$, then

$$|V^{i,l} \cup V^{i,l'}| \geq n + m - 1$$

for some $i \in [d]$ and $l \in S$.

By $d(G)$ we denote the average degree of a graph $G$.

**Lemma 4.2** Let $G = (V, \mathcal{E})$ be a simple graph, and let $m = \max\{d(v) + d(u) : v, u \in V \text{ and } v, u \text{ are adjacent}\}$. Then $d(G) \leq m/2$.

For fixed $x \in ES$ and $i \in [d]$ let

$$\pi^i_x = ES \times \cdots \times ES \times \{x\} \times ES \times \cdots \times ES,$$

where $\{x\}$ stands at the $i$th position. If $V \subset (ES)^d$ is a polybox code, then the slice $\pi^i_x \cap \bigcup E(V)$ is a "flat" polybox in $(ES)^d$ (boxes which are contained
in this polybox have the factor \( \{x\} \) at the \( i \)th position). Therefore we define a polybox \( (\pi_x^i \cap \bigcup E(V))_i \) in the \((d-1)\)-box \((ES)^{d-1}\):

\[
(\pi_x^i \cap \bigcup E(V))_i = \bigcup \{ (\tilde{v})_i : v \in V \text{ and } \pi_x^i \cap \tilde{v} \neq \emptyset \}.
\]

The polybox \( (\pi_x^i \cap \bigcup E(V))_i \) does not depend on a particular choice of a polybox code, because if \( W \) is an equivalent polybox code to \( V \), then \( \bigcup E(V) = \bigcup E(W) \), and hence \( (\pi_x^i \cap \bigcup E(V))_i = (\pi_x^i \cap \bigcup E(W))_i \).

We will slice a polybox \( \bigcup E(V) \) by the set \( \pi_x^i \) for various \( x \in ES \). In particular, we will pay attention whether the polybox code \( \{(v)_i : v \in V, \pi_x^i \cap \tilde{v} \neq \emptyset\} \) is rigid or it contains a twin pair.

In [10] we showed that any polybox code without twin pairs having at most seven words is rigid. Now we need a slightly better rigidity result:

**Lemma 4.3** If a polybox code \( V \subset S^d, S = \{a, a', b, b'\} \), does not contain a twin pair and \(|V| \leq 9\), then it is rigid.

**Proof.** We will show that if \( W \) is an equivalent polybox code to \( V \) and \( V \cap W = \emptyset \), then \(|V| > 9\). We proceed by induction on \( d \). In [10] Corollary 3.5 we showed that for \( d \leq 3 \) every polybox code \( V \subset S^d \) without twin pairs is rigid. Thus, the lemma is true for \( d \leq 3 \). Let \( d \geq 4 \).

We may assume that for every \( i \in [d] \) there is at least one letter \( l \in S \) such that \( V^{i,l} \neq \emptyset \) and \( V^{i,l'} \neq \emptyset \), for otherwise if \( V^{i,l} \neq \emptyset \), \( V^{i,s} \neq \emptyset \) and \( V^{i,l'} = \emptyset \), \( V^{i,s'} = \emptyset \), where \( l \not\in \{s, s'\} \), then \( \bigcup E(V^{i,l}) \cap \bigcup E(W \setminus V^{i,l}) = \emptyset \) and \( \bigcup E(V^{i,s}) \cap \bigcup E(W \setminus V^{i,s}) = \emptyset \). Thus, \( \bigcup E(V^{i,l}) = \bigcup E(W^{i,l}) \) and \( \bigcup E(V^{i,s}) = \bigcup E(W^{i,s}) \). By the inductive hypothesis, \( V^{i,l} = W^{i,l} \) and \( V^{i,s} = W^{i,s} \), and hence \( V \cap W = \emptyset \), a contradiction.

Assume that \( V^{i,a} \neq \emptyset \), \( V^{i,a'} \neq \emptyset \) and \( V^{i,b} \cup V^{i,b'} = \emptyset \) for some \( i \in [d] \) and let \( x \in Ea \) and \( y \in Ea' \). It follows from the inductive hypothesis that the polybox code \( \{(v)_i : v \in V^{i,l}\} \) is rigid for \( l \in \{a, a'\} \), and therefore

\[
\{(\tilde{v})_i : v \in V^{i,a}\} = \{(\tilde{w})_i : w \in W, \tilde{w} \cap \pi_x^i \neq \emptyset\}
\]

and

\[
\{(\tilde{v})_i : v \in V^{i,a'}\} = \{(\tilde{w})_i : w \in W, \tilde{w} \cap \pi_x^i \neq \emptyset\}
\]

because \( \bigcup E(V) \cap \pi_z^i = \bigcup E(W) \cap \pi_z^i \) for every \( z \in ES \). Thus, there are \( u \in V^{i,a}, v \in V^{i,a'} \) and \( w \in W \setminus W^{i,a}, q \in W \setminus W^{i,a'} \) such that \( (u)_i = (w)_i \) and \( (v)_i = (q)_i \). Then, \( \tilde{w} \setminus \tilde{u} \subset \bigcup E(V^{i,a'}) \) and \( \tilde{q} \setminus \tilde{v} \subset \bigcup E(V^{i,a}) \). By (2.3), \( (u)_i \subseteq (V^{i,a})_i \) and \( (v)_i \subseteq (V^{i,a})_i \), and by Statement 3.3 (a), \(|(V^{i,a})_i| \geq 5 \) and \(|(V^{i,a'})_i| \geq 5 \). Thus, \(|V| = |(V^{i,a})_i| + |(V^{i,a'})_i| \geq 10 \).

Let now \( V^{i,a} \neq \emptyset, V^{i,a'} \neq \emptyset \) and \( V^{i,b} \neq \emptyset \) for some \( i \in [d] \). Taking \( x \in Ea \cap Eb \) and \( y \in Ea' \cap Eb' \), in the same manner as above, we show that
\{(\bar{v})_i : v \in V^{i,a}\} = \{(\bar{w})_i : w \in W, \bar{w} \cap \pi_b^i \neq \emptyset\} \text{ and } \{(\bar{v})_i : v \in V^{i,a'}\} = \\
{(\bar{w})_i : w \in W, \bar{w} \cap \pi_b^i \neq \emptyset}\), and consequently \(|V| > |(V^{i,a})_i| + |(V^{i,a'})_i| \geq 10\).

Thus, in the rest of the proof we assume that \(V^{i,l} \neq \emptyset\) for every \(i \in [d]\) and \(l \in S\).

Let us suppose that there are \(i \in [d]\) and two letters in \(S\), say \(a\) and \(b\), such that the polybox code \((V^{i,a} \cup V^{i,b})_i\) does not contain a twin pair, i.e. there are no \(i\)-siblings in \(V^{i,a} \cup V^{i,b}\). This means, by the inductive hypothesis, that the polybox code \(\{(v)_i : v \in V^{i,a} \cup V^{i,b}\}\) is rigid, and hence \(\{(v)_i : v \in V^{i,a} \cup V^{i,b}\} = \{(w)_i : w \in W, \bar{w} \cap \pi_b^i \neq \emptyset\}\), where \(x \in Ea \cap Eb\).

Therefore, there are \(u \in V^{i,a}\), \(v \in V^{i,b}\), \(w \in W^{i,b}\) and \(q \in W^{i,a}\) such that \((u)_i = (w)_i\) and \((v)_i = (q)_i\).

Thus, \((u)_i \subseteq (V^{i,a'}), (v)_i \subseteq (V^{i,b'}),\) and hence \(|V| \geq |(V^{i,a'})_i| + |(V^{i,b'})_i| \geq 10\).

Therefore, we assume that for every \(l, s \in S\), \(l \not\in \{s, s'\}\), the set \((V^{i,l} \cup V^{i,s})_i\) contains a twin pair, i.e. there is an \(i\)-siblings in \(V^{i,l} \cup V^{i,s}\).

We now consider the graph \(G = (V, \mathcal{E})\) of siblings in \(V\). It follows from the above assumption that for every \(\{l, s\} \in \{(b, a), (b, a'), (b', a), (b', a')\}\) and every \(i \in [d]\) there is an edge \((v, u) \in \mathcal{E}\) such that \(\{v_i, u_i\} = \{l, s\}\). In particular, there are at least 4 edges with the colour \(i \in [d]\), and therefore \(|\mathcal{E}| \geq 4d\).

Let \(u, v \in V\) be such that

\[d(v^0) + d(u^0) = \max\{d(v) + d(u) : v, u \in V\} \text{ and } v, u \text{ are adjacent}\}.

Let \(d \geq 5\) and suppose that \(d(u^0) + d(v^0) = 9\). By Lemma 4.1, there are \(i \in [d]\) and \(l \in S\) such that \(|V^{i,l} \cup V^{i,l'}| \geq 8\). Since \(V^{i,s} \neq \emptyset\) and \(V^{i,s'} \neq \emptyset\), where \(s \not\in \{l, l'\}\), it follows that \(|V| \geq 10\).

Let \(d(u^0) + d(v^0) = 8\). Similarly like above, by Lemma 4.1, there are \(i \in [d]\) and \(l \in S\) such that \(|V^{i,l} \cup V^{i,l'}| \geq 7\). If \(|V^{i,l} \cup V^{i,l'}| = 8\), then \(|V| \geq 10\). Let \(|V^{i,l} \cup V^{i,l'}| = 7\). This means that \(|V^{i,s} \cup V^{i,s'}| = 2\), where \(s \not\in \{l, l'\}\).

It can be easily seen that there are \(i, j \in [d], i \neq j,\) and \(l, s \in \{a, b\}\) such that \(|V^{i,l} \cup V^{i,l'}| = 7\) and \(|V^{j,s} \cup V^{j,s'}| = 7\). We can assume without loss of generality that \(l = s = a\), as \(i \neq j\). Let \(\{u\} = (N(u^0) \cup N(v^0))) \setminus (V^{i,a} \cup V^{i,a'})\) and \(\{v\} = (N(u^0) \cup N(v^0))) \setminus (V^{j,a} \cup V^{j,a'})\), where \(N(u^0)\) and \(N(v^0)\) denote the set of all neighbors of vertices \(u^0\) and \(v^0\), respectively. We have \(u_i \in \{b, b'\}\) and \(v_j \in \{b, b'\}\). Moreover, \(w_i, w_j \in \{b, b'\}\), where \(\{w\} = V \setminus (N(u^0) \cup N(v^0))\). Note that \(u \neq v\), for otherwise \(u \not\in N(u^0) \cup N(v^0)\). Assume without loss of generality that \(u_i = b\) and \(v_j = a\). Since \(p_i, p_j \in \{a, a'\}\) for every \(p \in (N(u^0) \cup N(v^0)) \setminus \{u, v\}\), the only vertices from the set \(N(u^0) \cup N(v^0)\) which can be adjacent to the vertex \(w\) are \(u\) and \(v\). This means that, there is no \(i\)-siblings \(q, t \in V\) such that \(q_i = b'\) and \(t_i = a'\), which is a contradiction.
Let now \( d = 4 \) and assume without loss of generality that \( u^0 = aaaaa \) and \( v^0 = ba'aa \). By just considered case, we assume that for every \( i, j \in [d] \), \( i \neq j \) and \( l, s \in \{a, b\} \) we have \(|V^{i,l} \cup V^{i,l'}| \leq 6 \) or \(|V^{j,s} \cup V^{j,s'}| \leq 6 \). Thus, it suffices to consider three cases: \( n_2 = n_3 = n_4 = 2 \); \( n_2 = 3, n_3 = 2, n_4 = 1 \), and \( n_2 = n_3 = 2, n_4 = 1 \), where

\[
  n_i = |(V^{i,b} \cup V^{i,b'}) \cap (N(u^0) \cup N(v^0))|
\]

for \( i \in \{2, 3, 4\} \).

Let \( \{v^1, \ldots, v^6\} = (N(u^0) \cup N(v^0)) \setminus \{u^0, v^0\} \). Observe that in the first two cases, if two vertices \( u, v \in (N(u^0) \cup N(v^0)) \setminus \{u^0, v^0\} \) are adjacent, then \( u, v \in (V^{i,b} \cup V^{i,b'}) \cap (N(u^0) \cup N(v^0)) \) for some \( i \in \{2, 3, 4\} \). (Recall that, since \( u^0 = aaaaa \) and \( v^0 = ba'aa \), if \( v_1 \in \{b, b'\} \) for some \( v \in N(v^0) \cup N(u^0) \) and \( i \in \{2, 3, 4\} \), then \( v_1 \in \{a, a'\} \) for every \( j \in \{2, 3, 4\} \setminus \{i\} \).) Therefore, in these two cases, the maximal number of edges with ends in \( N(u^0) \cup N(v^0) \) is achieved if the vertices \( v^1, \ldots, v^6 \) are arranged as presented in Figure 8a and 8b for the first and the second case, respectively (recall that the graph \( G \) does not contain triangles). Since \( d(w) \leq 4 \), where \( \{w\} = V \setminus (N(u^0) \cup N(v^0)) \), we have \(|\mathcal{E}| < 16 \), which contradicts the assumption on the number of edges in \( G \).

![Fig. 8. The maximal numbers of edges with ends in the set \( N(u^0) \cup N(v^0) \) in the first two cases. In the first case we have \( v_2, v_3 \in \{b, b'\} \) and \( v_4, v_5 \in \{b, b'\} \) and \( v_6 \in \{b, b'\} \). In the second case we have \( v_2, v_3 \in \{b, b'\} \) and \( v_4, v_5 \in \{b, b'\} \) and \( v_6 \in \{b, b'\} \).](image-url)
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To obtain 12 edges with ends in the set \( N(u^0) \cup N(v^0) \), which is the maximal number of edges with ends in this set, the vertices \( v^1, \ldots, v^6 \) have to be arranged as pictured in Figure 9. Since \( d(v^1) = 4 \) and the graph \( G \) does not contain triangles, it must be \( d(w) < 4 \), and then \( |E| < 16 \), a contradiction.

![Figure 9](image)

Fig. 9. The maximal number of edges with ends in the set \( N(u^0) \cup N(v^0) \) in the third cases.

Finally, let \( d(v^0) + d(u^0) \leq 7 \). It follows from Lemma 4.2 that

\[
d(G) \leq \frac{7}{2},
\]

and since \( d(G)|V| = 2|E| \) and \( 2|E| \geq 32 \), we have \( |V| > 9 \). \( \square \)

In the next two lemmas we give forbidden distributions of words in the considered codes \( V \) and \( W \).

**Lemma 4.4** If \( V, W \subset S^d \), \( S = \{a, a', b, b'\} \), are disjoint and equivalent polybox codes without twin pairs and \( |V| = 12 \), then for every \( i \in [d] \) the distribution

\[
|V^i.a| = 5, \quad |V^i.a'| = 1, \quad |V^i.b| = 5, \quad |V^i.b'| = 1
\]

is impossible.

**Proof.** Assume on the contrary that there is \( i \in [d] \) such that \( |V^i.a| = 5, |V^i.a'| = 1 \) and \( |V^i.b| = 5, |V^i.b'| = 1 \). Let \( \{v\} = V^i.a' \) and \( \{u\} = V^i.b' \). If \( u \) and \( v \) is not an \( i \)-siblings, then the polybox code \( \{(u), (v), i\} \) is rigid. Since \( \pi_x \cap \bigcup E(V) = \pi_x \cap \bigcup E(W) \) for \( x \in Ea' \cap Eb' \), it follows that \( (v)_i = (w)_i \) and \( (u)_i = (q)_i \), where \( w \in W^i.b' \) and \( q \in W^i.a' \). In the similar way as in the first part of the proof of Lemma 4.3 we show that \( (w)_i \subseteq (W^i,b)_i \) and \( (q)_i \subseteq (W^i,a)_i \). Thus, by Statement 3.4(a), \( |W^i.b| \geq 5 \) and \( |W^i.a| \geq 5 \). But \( |W^i.a'| \geq 1 \) and \( |W^i.b'| \geq 1 \), and therefore \( |W^i.b| = 5 \). Since \( (w)_i \subseteq (W^i,b)_i \) and \( (w)_i \subseteq (V^i.a)_i \), the structure of \( (W^i,b)_i \) and \( (V^i,a)_i \) are...
such as given in Statement 3.3(a). Therefore, assuming without loss of generality that \((w)_i = b \cdots b\), we have \((V^{i,a})_i = \{l_1i_2l_3, l_1l_2l_3, l_1l_3l_2, l_1l_2l_3\}\) and \((W^{i,b})_i = \{s_1s_2s_3, s_1s_2s_3, s_1s_2s_3, s_1s_2s_3\}\), where \(l_i, s_i \in \{a, a'\}\) for \(i = 1, 2, 3\). It can be easily checked, using (2.1), that for every \(l_i, s_i \in \{a, a'\}\), \(i = 1, 2, 3\), there is \(y \in (ES)^d\) such that \((y)_i \in \bigcup (E(V^{i,a}))_i \cap \bigcup (E(W^{i,b}))_i\), and \((y)_i \notin (w)_i\). Observe now that, again by (2.1), the point \(y\) can be chosen such that \(yi \in Eb \setminus Ea\), and then \(y \in \bigcup E(W^{i,b}) \setminus \bigcup E(V^{i,a})\). Thus, \(y \notin \bigcup E(W)\) and \(y \notin \bigcup E(V)\), a contradiction.

Before we consider the case when \(u\) and \(v\) are an \(i\)-siblings note that there is \(w \in W\) such that \(\bar{w} \cap \bigcup E(V^{i,a}) \neq \emptyset\) and \(\bar{w} \cap \bigcup E(V^{i,a'}) \neq \emptyset\), for otherwise \(V^{i,a} \subseteq W^{i,a}\) and \(V^{i,a'} \subseteq W^{i,a'}\). Then, by Statement 3.3(b) and (a), respectively, \(|W^{i,a}| \geq 7\) and \(|W^{i,a'}| \geq 5\), and thus \(|W^{i,a}| = 7\) and \(|W^{i,a'}| = 5\), as \(|V| = 12\). Then \(W^{i,b} \cup W^{i,b'} = \emptyset\), which is impossible, because any \(z \in \bigcup E(W^{i,b})\) such that \((z)_i \in \bigcup E((W^{i,b})_i) \setminus \bigcup E((W^{i,b'})_i)\) is covered by a box from \(E(W^{i,b})\). Thus, the structures of \(V^{i,a}\) and \(V^{i,a'}\) are such as predicted in Statement 3.3(d). In particular, \((v)_i \subseteq (V^{i,a})_i\).

Let now \(u\) and \(v\) be an \(i\)-siblings. Then \((v)_i\) and \((u)_i\) are a twin pair. We can assume without loss of generality that \((v)_i = b \cdots b\). Since \((v)_i \subseteq (V^{i,a})_i\), it follows that, by Statement 3.3(a), \((V^{i,a})_A = \{l_1i_2l_3, l_1l_2l_3, l_1l_3l_2, l_1l_2l_3\}\), where \(l_i \in \{a, a'\}\) for \(i = 1, 2, 3\), \(A = \{i, i_1, i_2, i_3\}\) and \(i_1, i_2, i_3\) are such as in (3.3) of Lemma 3.2. The words \((v)_i, (u)_i\) are a twin pair, and therefore the polybox code \((V^{i,a})_i \cup (V^{i,b})_i\) does not contain a twin pair, and hence, by Lemma 4.3, it is rigid. Then \((V^{i,a})_i \cup (V^{i,b})_i = (W^{i,a})_i \cup (W^{i,b})_i\), and consequently \((V^{i,a})_i = (W^{i,b})_i\), because \(V\) and \(W\) are disjoint. Thus, for \(w \in W^{i,b}\) we have \((w)_i \subseteq (V^{i,a})_i\), and then \(|W^{i,a}| \geq 5\), a contradiction. □

**Lemma 4.5** Let \(V, W \subseteq S^d\) be disjoint and equivalent polybox codes without twin pairs. If there are \(i \in [d]\) and \(l, s \in S, s \not\in \{l, l'\}\), such that \(|V^{i,l}| = |V^{i,l'}| = 1\) and \(|V^{i,s}| \neq |V^{i,s'}|\) or \(|V^{i,l}| = 1\) and \(2 \leq |V^{i,l'}| \leq 4\), then \(|V| > 12\).

**Proof.** By Theorem 3.3 |\(V| \geq 12\). Suppose on the contrary that \(|V| = 12\). Let \(|V^{i,l}| = |V^{i,l'}| = 1\). By Statement 3.3(d), \(V^{i,l} \subseteq W^{i,l}\) and \(V^{i,l'} \subseteq W^{i,l'}\), and thus, by Statement 3.3(a), \(|W^{i,l}| \geq 5\) and \(|W^{i,l'}| \geq 5\).

Since \(|V^{i,s}| \neq |V^{i,s'}|\), we may assume that \(\bigcup E((V^{i,s})_i) \subseteq E((V^{i,s'})_i) \neq \emptyset\). Note that each point \(z \in \bigcup E((V^{i,s})_i)\) with \((z)_i \in \bigcup E((V^{i,s})_i) \setminus \bigcup E((V^{i,s'})_i)\) must be covered by a box from the set \(E(W^{i,s})\), and thus \(W^{i,s} \neq \emptyset\). Consequently, we may assume that \(|W^{i,l}| = 5\).

Suppose now that for every \(r \in S, r \not\in \{l, l'\}\) we have \(W^{i,r} = \emptyset\) or \(W^{i,r'} = \emptyset\). Then, by (2.1), there is \(x \in E\) such that \(\bigcup E(W) \cap \pi_x = \bigcup E(W^{i,l}) \cap \pi_x\). By Lemma 4.3 the polybox code \((w)_i : w \in W^{i,l}\) is rigid, and therefore \((v)_i = (w)_i\) for some \(w \in W^{i,l}\) and \(v \in V^{i,l}\) because
\[ \bigcup E(W) \cap \pi_x^i = \bigcup E(V) \cap \pi_x^i. \] Thus, \( v = w \), a contradiction. Hence, there is \( r \in S, r \notin \{l, l'\} \) such that the sets \( W^{i,r} \) and \( W^{i,r'} \) are non-empty. Clearly, \( r = s \) and then \( |W^{i,s}| = |W^{i,s'}| = 1 \) because \( |W^{i,l}| \geq 5 \), \( |W^{i,l'}| \geq 5 \) and \( |V| = 12 \). It follows from Statement 3.4(d) that \( W^{i,s} \subseteq V^{i,s} \) and \( W^{i,s'} \subseteq V^{i,s'} \), and thus, by Statement 3.3(a), \( |V^{i,s}| \geq 5 \) and \( |V^{i,s'}| \geq 5 \). Since \( |V^{i,s}| \neq |V^{i,s'}| \), we have \( |V^{i,s}| + |V^{i,s'}| \geq 11 \), and consequently \( |V| > 12 \), a contradiction.

Let now \( |V^{i,l}| = 1 \) and \( 2 \leq |V^{i,l'}| \leq 4 \). By Statement 3.4(d), \( V^{i,l} \subseteq W^{i,l} \) and \( V^{i,l'} \subseteq W^{i,l'} \), and by Statement 3.4(a) and (b), respectively, \( |W^{i,l}| \geq 5 \) and \( |W^{i,l'}| \geq 7 \). Thus, \( |W^{i,l}| = 5 \) and \( |W^{i,l'}| = 7 \) because \( |V| = 12 \). This means that \( W^{i,s} \cup W^{i,s'} = \emptyset \) for every \( s \in S \setminus \{l, l'\} \), and therefore the set \( \bigcup E(V^{i,s} \cup V^{i,s'}) \) is an \( i \)-cylinder, as \( V^{i,s} \cup V^{i,s'} \neq \emptyset \) for some \( s \notin \{l, l'\} \). Indeed, if \( \bigcup E(V^{i,s} \cup V^{i,s'}) \) is not an \( i \)-cylinder, then we may assume that \( \bigcup E((V^{i,s})_i) \cup \bigcup E((V^{i,s'})_i) \neq \emptyset \). A point \( z \in \bigcup E(V^{i,s}) \) with \( (z)_i \in \bigcup E((V^{i,s})_i) \cup \bigcup E((V^{i,s'})_i) \) is covered by a box from \( E(W^{i,s}) \), which is impossible. Thus, the polybox codes \( (V^{i,s}) \), and \( (V^{i,s'})_i \) are equivalent. By Theorem 3.1 \( |V| > |V^{i,s}| + |V^{i,s'}| \geq 24 \), a contradiction. □

An important role in determining the structure of the polybox codes \( V \) and \( W \) will play the following lemma.

**Lemma 4.6** Let \( V, W \subset S^d \), where \( S = \{a, a', b, b'\} \), be disjoint and equivalent polybox codes without twin pairs. Assume that there are \( i \in [d] \) and \( l, p \in S, l \neq p \), such that the sets \( V^{i,l} \) and \( V^{i,p} \) are non-empty. If there is \( x \in ES \) such that the polybox code \( \{(v)_i : v \in V, \pi_x^i \cap v \neq \emptyset \} \) does not contain a twin pair, then \( |V| > 12 \).

**Proof.** By Theorem 3.1 \( |V| \geq 12 \). For future applications, in the first part of the proof we do not assume that \( S = \{a, a', b, b'\} \).

If \( V^{i,s} = \emptyset \) or \( V^{i,s'} = \emptyset \) for every \( s \in S \), then in the similar way as at the beginning of the proof of Lemma 4.3 we show that \( V^{i,l} \) and \( V^{i,l'} \) are equivalent and similarly, \( V^{i,p} \) and \( W^{i,p} \) are equivalent. By Theorem 3.1 \( |V| \geq |V^{i,l}| + |V^{i,l'}| \geq 24 \).

Let \( p = l' \), and let \( V = V^{i,l} \cup V^{i,l'} \). If \( W = W^{i,l} \cup W^{i,l'} \), then the polybox codes \( V^{i,l} \) and \( W^{i,l} \) are equivalent and similarly, \( V^{i,l'} \) and \( W^{i,l} \) are equivalent. Then, by Theorem 3.1 \( |V| = |V^{i,l}| + |V^{i,l'}| \geq 24 \).

If \( W^{i,s} \neq \emptyset \), where \( s \notin \{l, l'\} \), then in the same manner as in the second part of the proof of the previous lemma we show that the set \( \bigcup E(W^{i,s} \cup W^{i,s'}) \) is an \( i \)-cylinder, which gives \( |W| \geq |W^{i,s}| + |W^{i,s'}| \geq 24 \), and thus \( |V| > 12 \).

Suppose now on the contrary that \( |V| = 12 \). Furthermore, assume that \( V^{i,s} = \emptyset \) or \( V^{i,s'} = \emptyset \) for every \( s \in S \), where \( s \notin \{l, l'\} \).

Suppose that \( V^{i,s} \neq \emptyset \) for at least one \( s \notin \{l, l'\} \). By (2.1), we can choose
If \( V^{i,l} \) is not rigid, then, by Lemma 4.3, \(| V^{i,l} | \geq 10\). Since \(| V | = 12\), we have \(| V^{i,l} | = 10\), \(| V^{i,b} | = 1\) and \(| V^{i,s} | = 1\). Then \( V^{i,s} \subseteq W^{i,s} \), and consequently, by Statement 3.4 (a), \(| W^{i,s} | \geq 5\). Therefore, for every \( x \in E_l \cap E_s \) and \( y \in E_l' \cap E_s \) we have

\[
|((\pi^{i}_x \cap \bigcup E(V) )_{i})|_0 \geq 10 \text{ and } |((\pi^{i}_y \cap \bigcup E(V) )_{i})|_0 \geq 5.
\]

Now it is easy to see (compare Lemma 3.6) that \(| V | \geq 15\), a contradiction.

If the codes \( V^{i,l} \) and \( V^{i,l'} \) are rigid, then \( W^{i,l} \cup W^{i,l'} = \emptyset \), for otherwise taking \( w \in W^{i,l} \) we get, by Lemma 4.3 and the rigidity of \( V^{i,l} \), \((v)_i = (w)_i \) for some \( v \in V^{i,l} \), and thus \( v = w \), a contradiction. Since \( W^{i,l} \cup W^{i,l'} = \emptyset \), the set \( \bigcup E(V^{i,l} \cup V^{i,l'}) \) is an \( i \)-cylinder (compare the proof of the previous lemma), and since \( V^{i,l} \) and \( V^{i,l'} \) are rigid, the set \( V^{i,l} \cup V^{i,l'} \) consists of twin pairs, which is a contradiction.

Thus we may assume that the sets \( V^{i,l}, V^{i,l'}, V^{i,s} \) and \( V^{i,s'} \) are non-empty.

Let now \( S = \{a, a', b, b'\} \). (We still assume that \(| V | = 12\).) It follows from the above that \( V^{i,l} \neq \emptyset \) for \( l \in \{a, a', b, b'\} \).

Let \( x \in E_a \cap E_b \) be such that the set \( \{(v)_i : v \in V, \pi^i_x \cap \bar{v} \neq \emptyset \} \) does not contain a twin pair.

If \(|(\pi^i_x \cap \bigcup E(V)_i)|_0 \geq 10\), then \(|(\pi^i_x \cap \bigcup E(V)_i)|_0 = 10 \text{ and } |(\pi^i_y \cap \bigcup E(V)_i)|_0 = 2\) for \( y \in E_a' \cap E_b' \), and consequently \(| V^{i,a} | = | V^{i,b} | = 1\).

Since \( V = V^{i,a} \cup V^{i,a'} \cup V^{i,b} \cup V^{i,b'} \), by Lemma 4.5, \(| V^{i,a} | = 5\), \(| V^{i,a'} | = 1\) and \(| V^{i,b} | = 5\), \(| V^{i,b'} | = 1\), which is, by Lemma 4.4, impossible.

If \(|(\pi^i_x \cap \bigcup E(V)_i)|_0 \leq 9\), then, by Lemma 4.3, the polybox code \( \{(v)_i : v \in V, \pi^i_x \cap \bar{v} \neq \emptyset \} \) is rigid. Thus, there are \( v \in V^{i,s}, u \in V^{i,b} \) and \( w, p \in W \) with \( w_i \in \{b, b'\} \) and \( p_i \in \{a, a'\} \) such that \((v)_i = (w)_i \) and \((u)_i = (p)_i \). Then \(| V^{i,a} | \geq 5\), \(| V^{i,b} | \geq 5\). Since \(| V | = 12\), we have \(| V^{i,a} | = 5\), \(| V^{i,a'} | = 1\) and \(| V^{i,b} | = 5\), \(| V^{i,b'} | = 1\), which is, by Lemma 4.4, impossible. Thus \(| V | > 12\). \( \square \)

We now once again indicate a forbidden distribution of words in \( V \) and \( W \).

Lemma 4.7 If \( V, W \subseteq S^d \), where \( d \geq 5 \) and \( S = \{a, a', b, b'\} \), are disjoint and equivalent polybox codes without twin pairs and \(| V | = 12\), then the distribution of words in \( V \)

\[
| V^{i,a} | = | V^{i,a'} | = | V^{i,b} | = | V^{i,b'} | = 3 \text{ for every } i \in [d]
\]

is impossible.
Proof. By Lemma 4.6, we may assume that for every \( i \in [d] \) and \( l, s \in \{a, a', b, b'\}, l \not\in \{s, s'\} \), there are \( i \)-siblings in the set \( V^{i,l} \cup V^{i,s} \).

Suppose on the contrary that \( V \) has the distribution \( |V^{i,l}| = |V^{i,l'}| = 3 \) for every \( i \in [d] \) and \( l \in \{a, b\} \). Let \( G = (V, \mathcal{E}) \) be a graph of siblings in \( V \).

Note that, it follows from the assumption on \( i \)-siblings in \( V \) that \( |\mathcal{E}| \geq 4d \).

Let \( u^0, v^0 \in V \) be such that

\[
  d(v^0) + d(u^0) = \max \{d(v) + d(u) : v, u \in V \text{ and } v, u \text{ are adjacent}\}
\]

If \( d(v^0) + d(u^0) \geq 8 \), then, by Lemma 4.1 there are \( i \in [d] \) and \( l \in \{a, b\} \) such that \( |V^{i,l} \cup V^{i,l'}| \geq 7 \), which contradicts the assumption on the distribution of words in \( V \). On the other hand, if \( d(v^0) + d(u^0) \leq 6 \), then, by Lemma 4.2, \( d(G) \leq 3 \). Since \( |\mathcal{E}| \geq 20 \), we have \( |V| > 12 \), which is a contradiction.

Let \( d(v^0) + d(u^0) = 7 \). It can be easily shown that there are \( i, j \in [d], i \neq j, \) \( l, s \in \{a, b\} \) such that \( |V^{i,l} \cup V^{i,l'}| = 6 \) and \( |V^{j,s} \cup V^{j,s'}| = 6 \), where \( V^{i,l}, V^{i,l'}, V^{j,s}, V^{j,s'} \subset N(u^0) \cup N(v^0) \). We can assume without loss generality, as \( i \neq j \), that \( l = s = a \). Let \( \{u\} = (N(u^0) \cup N(v^0)) \setminus (V^{i,a} \cup V^{i,a'}) \) and \( \{v\} = (N(u^0) \cup N(v^0)) \setminus (V^{j,a} \cup V^{j,a'}) \). Clearly, \( u_i \in \{b, b'\} \) and \( v_j \in \{b, b'\} \).

Thus, \( u \neq v \), for otherwise \( u \not\in N(u^0) \cup N(v^0) \). Assume without loss of generality that \( u_i = b \) and \( v_i = a \). Since \( w_i, w_j \in \{b, b'\} \) for every \( w \in V \setminus (N(u^0) \cup N(v^0)) \) and \( p_i, p_j \in \{a, a'\} \) for every \( p \in (N(u^0) \cup N(v^0)) \setminus \{u, v\} \), a vertex \( w \in V \setminus (N(u^0) \cup N(v^0)) \) can be joined only with \( u \) or \( v \). This means that there is no \( i \)-siblings \( q, t \in V \) such that \( q_i = b' \) and \( t_i = a' \), a contradiction.

In the next lemma we show that to find the structure of the polyboxes \( V \) and \( W \) we may assume that they are written down in the alphabet \( S = \{a, a', b, b'\} \).

**Lemma 4.8** Let \( V, W \subset S^d \) be disjoint and equivalent polybox codes without twin pairs, and let \( V \) be extensible to a partition code. If there is \( i \in [d] \) such that \( V^{i,l} \cup V^{i,l'} \neq \emptyset \) for at least three \( l \in S \), then \( |V| > 12 \). Thus, if \( |V| = |W| = 12 \), then \( V, W \subset \{a, a', b, b'\}^d \).

**Proof.** It follows from Theorem 3.1 that \( |V| \geq 12 \). Suppose on the contrary that \( |V| = 12 \).

In the same way as in the first part of the proof of Lemma 4.6 we show that for every \( i \in [d] \) there are at least two letters \( l, s \in S, l \not\in \{s, s'\} \), such that

\[
V^{i,l}, V^{i,l'}, V^{i,s}, V^{i,s'} \neq \emptyset
\]

(4.4)

Fix \( i \in [d] \). If \( W^{i,l} \cup W^{i,l'} = \emptyset \), then, in the same way as in the proof of Lemma 4.5 we show that the set \( \bigcup E(V^{i,l} \cup V^{i,l'}) \) is an \( i \)-cylinder, and
consequently \(|V| > |V_{i,l}^{j} \cup V_{i,r}^{j'}| \geq 24\), a contradiction. Thus, \(W_{i,l}^{j} \cup W_{i,r}^{j'} \neq \emptyset\) and \(W_{i,s}^{j} \cup W_{i,s'}^{j'} \neq \emptyset\).

Suppose that \(V_{i,r}^{j} \neq \emptyset\) and \(V_{i,r'}^{j'} = \emptyset\) for some \(r \in S \setminus \{l, l', s, s'\}\). For every \(w \in W\) such that \(w \cap \bigcup E(V_{i,r}^{j}) \neq \emptyset\) we have \(w_i = r\), and hence \(V_{i,l}^{j} \subseteq W_{i,r}^{j'}\) from where, by Statement 3.4 (a), we obtain \(|W_{i,r}^{j'}| \geq 5\). Then \(|W_{i,l}^{j} \cup W_{i,r}^{j'}| \leq 3\) or \(|W_{i,s}^{j} \cup W_{i,s'}^{j'}| \leq 3\) because \(|W| = 12\).

Let \(|W_{i,l}^{j} \cup W_{i,r}^{j'}| \leq 3\) and \(W_{i,l}^{j} \neq \emptyset\), \(W_{i,r}^{j'} \neq \emptyset\). Since \(|W_{i,r}^{j'}| \geq 5\) and \(W_{i,s}^{j} \cup W_{i,s'}^{j'} \neq \emptyset\), by Lemma 4.5, \(|W| > 12\), a contradiction.

If \(|W_{i,l}^{j} \cup W_{i,r}^{j'}| \leq 3\) and \(W_{i,l}^{j} \neq \emptyset\), \(W_{i,r}^{j'} = \emptyset\), then \(W_{i,l}^{j} \subseteq V_{i,l}^{j}\), and by Statement 3.4 (a), \(|V_{i,l}^{j}| \geq 5\). Moreover, since \(W_{i,r}^{j'} = \emptyset\), we have

\[
(V_{i,l}^{j'})_{i} \subseteq (V_{i,l}^{j})_{i}.
\]

Let \(|V_{i,l}^{j'}| = 1\) and \(|V_{i,l}^{j}| \geq 5\). Then \(|W_{i,l}^{j}| \geq 4\) because a point \(z \in \bigcup E(V_{i,l}^{j})\) such that \((z)_{i} \in \bigcup E((V_{i,l}^{j})); \bigcup E((V_{i,r}^{j'})_{i})\) is covered by a box from \(E(V_{i,l}^{j'})\) and \(|\hat{v}| = |\hat{w}|\) for every \(v \in V\) and \(w \in W\). Similarly, if \(|V_{i,r}^{j'}| \in \{2, 3\}\), then, by Statement 3.4 (b), \(|V_{i,l}^{j} \geq 7\), and consequently \(|W_{i,l}^{j}| \geq 4\). In both cases we get a contradiction to \(|W_{i,l}^{j} \cup W_{i,r}^{j'}| \leq 3\).

If \(|V_{i,l}^{j'}| \geq 4\) and \(|V_{i,l}^{j}| \geq 7\), then, by (4.4), \(|V| > 12\), a contradiction.

Therefore, \(V_{i,r}^{j}, V_{i,r'}^{j'} \neq \emptyset\). Moreover, we showed that for every \(j \in [d]\) and \(p \in S\)

\[
V_{i,p} \neq \emptyset \Leftrightarrow V_{i,p} \neq \emptyset. \tag{4.5}
\]

Suppose that \(|V_{i,l}^{j}| = |V_{i,l}^{j'}| = 1\). It follows from Statement 3.4 (c) that \(V_{i,l}^{j} \subseteq W_{i,l}^{j}\) and \(V_{i,l}^{j'} \subseteq W_{i,l}^{j'}\). By Statement 3.4 (a), we have \(|W_{i,l}^{j}| \geq 5\) and \(|W_{i,l}^{j'}| \geq 5\). Clearly, similarly like in the distribution (4.4), there are \(s_1, r_1 \in S \setminus \{l, l'\}, s_1 \notin \{r_1, r'_1\}\), such that the sets \(W_{i,s_1}^{j}, W_{i,s_1'}^{j'}, W_{i,r_1}^{j}\) and \(W_{i,r_1}^{j'}\) are non-empty. Thus, \(|W| > 12\), a contradiction.

If \(|V_{i,l}^{j}| = 1\) and \(|V_{i,l}^{j}| = 2\), then, by Lemma 4.5, \(|V| > 12\), a contradiction.

Thus \(|V_{i,p} \cup V_{i,p'}| = 4\), and by Lemma 4.5, \(|V_{i,p}| = |V_{i,p'}| = 2\) for \(p \in \{l, s, r\}\).

Let \(d = 4\). Since \(V\) can be extended to a partition code \(U \subseteq S^4\), i.e. \(|U| = 16\), we have \(V_{i,p} \cup V_{i,p'} \subseteq U_{i,p} \cup U_{i,p'}\) for \(p \in \{l, s, r\}\). As \(|U_{i,p}| = |U_{i,p'}|\) and \(\bigcup E(V_{i,p}) \neq \bigcup (E(V_{i,p}'))_{i}\) for \(p \in \{l, s, r\}\), at least two words are needed to complete the set \(V_{i,p} \cup V_{i,p'}\) to the set \(U_{i,p} \cup U_{i,p'}\) for \(p \in \{l, s, r\}\). But then \(|U_{i,p} \cup U_{i,p'}| \geq 6\) for \(p \in \{l, s, r\}\), and thus \(|U| > 16\) which is a contradiction.

Let now \(d \geq 5\). For every \(p \in \{l, s, r\}\) there is \(w \in W\) such that \(w \cap \bigcup E(V_{i,p}) \neq \emptyset\) and \(\hat{w} \cap \bigcup E(V_{i,p'}) \neq \emptyset\), for otherwise \(V_{i,p} \cap W_{i,p} \cup V_{i,p'} \subseteq W_{i,p'}\) for some \(p\), and thus, by Statement 3.4 (b), \(|W_{i,p}| \geq 7\) and \(|W_{i,p'}| \geq 7\). Then \(|W| > 12\), a contradiction.
By Statement 3.3 (d), for every \( p \in \{l, s, r\} \) there are \( i_1(p), i_2(p) \in [d] \setminus \{i\}, i_1(p) < i_2(p) \), such that

\[
(V^{i,p})_{A^c} = \{(l_1(p), l_2(p), l_1(p)s_2(p))\}, \quad (V^{i,p})_{A^c} = \{(l_1(p), l_2(p)'\}, s_1(p)l_2(p)\}
\]

and

\[
(V^{i,p})_{A \cup \{i\}} = (V^{i,p})_{A \cup \{i\}} = \{o(p)\}
\]

where \( o(p) \in S^{d-3} \) for \( p \in \{l, s, r\} \), \( A = \{i_1(p), i_2(p)\} \) and \( l_i(p), s_i(p) \in S \), \( l_i(p) \not\in \{s_i(p), s_i(p)'\} \) for \( i = 1, 2 \).

Let \( j \in [d] \setminus \{i\} \) be such that \( v_j = o_j(p) \) for every \( v \in V^{i,p} \cup V^{i,p}' \) and \( p \in \{l, s, r\} \). Then \( V^{i,o_j(p)} \neq \emptyset \) and \( V^{i,o_j(p)'} = \emptyset \) for some \( p \in \{l, s, r\} \), which contradicts (4.5) or \( V = V^{i,o_j(p)} \cup V^{i,o_j(p)'} \), which contradicts (4.4).

Since \( d \geq 5 \), there is \( j \in [d] \setminus \{i\} \) and there are two letters in \( S \), say \( l \) and \( s \), such that

\[
v_j = o_j(l) \quad \text{for} \quad v \in V^{i,l} \cup V^{i,l}' \quad \text{and} \quad v_j = o_j(s) \quad \text{for} \quad v \in V^{i,s} \cup V^{i,s}',
\]

where \( o_j(l), o_j(s) \in S \) and

\[
v_j \in \{l_1(r), l_1(r)', s_1(r)\} \quad \text{for} \quad v \in V^{i,r} \cup V^{i,r}'.
\]

or

\[
v_j \in \{l_2(r), l_2(r)', s_2(r)\} \quad \text{for} \quad v \in V^{i,r} \cup V^{i,r}',
\]

where \( l_k(r) \not\in \{s_k(r), s_k(r)\} \) for \( k = 1, 2 \). We consider the first case (the second case is considered in the same manner).

Let \( o_j(l) = o_j(s) \). Then \( |V^{j,o_j(l)}| = 8 \). By (4.5), we have \( V^{j,o_j(l)'} \neq \emptyset \), and by (4.4) we have \( V^{j,s} \cap V^{j,s'} \neq \emptyset \) for at least one \( s \not\in \{o(j), o(j)'\} \). Thus, by Lemma 4.5, \( |V| > 12 \), a contradiction.

Let \( o_j(l) = o_j(s)' \). If \( l_1(r) \not\in \{o_j(l), o_j(l)\} \), then \( V^{j,s_1(r)'} = \emptyset \), which contradicts (4.5), as \( V^{j,s_1(r)'} \neq \emptyset \). If \( l_1(r) \not\in \{o_j(l), o_j(l)\} \), then \( |V^{j,l_1(r)}| = 2 \), \( |V^{j,l_1(r)'}| = 1 \) and, by Lemma 4.5, \( |V| > 12 \), a contradiction.

Finally, let \( o_j(l) \not\in \{o_j(s), o_j(s)'\} \). If \( s_1(r) \not\in \{o_j(l), o_j(l)', o_j(s), o_j(s)'\} \), then \( V^{j,s_1(r)'} = \emptyset \), which contradicts (4.5). Let \( s_1(r) \in \{o_j(l), o_j(l)\} \). Then

\[
l_1(r) \not\in \{o_j(l), o_j(l)\}, \quad \text{and} \quad V^{j,s_1(r)} = \emptyset \quad \text{or} \quad |V^{j,s_1(r)}| = 1 \quad \text{and} \quad |V^{j,s_1(r)'}| = 4.
\]

In the first case we get a contradiction to (4.5). In the second case, by Lemma 4.5, \( |V| > 12 \), which is also a contradiction.

To show that \( V \cap W \subseteq \{a, a', b, b'\}^d \) assume on the contrary that \( V \subseteq \{a, a', b, b'\}^d \) and \( W \subseteq \{c, c', d, d'\}^d \), where \( \{a, a', b, b'\} \neq \{c, c', d, d'\} \). Let \( V^{i,c} \cup V^{i,c'} = \emptyset \). Then in the same way as in the second part of Lemma 4.5, we show that the set \( \bigcup E(W^{i,c} \cup W^{i,c'}) \) is an \( i \)-cylinder and consequently

\[
|W^{i,c}| + |W^{i,c'}| \geq 24,
\]

a contradiction. \( \square \)

At the end of this section we show that the computations will be made mainly for \( d = 4, 5 \) and only in one case for \( d = 6 \).
Lemma 4.9. Let \( V, W \subset S^6 \), where \( S = \{ a, a', b, b' \} \), be disjoint and equivalent polybox codes without twin pairs such that the distribution of words in \( V \) is different from \( |V_{i,a}| = |V_{i,a'}| = 5 \) and \( |V_{i,b}| = |V_{i,b'}| = 1 \) for every \( i \in [6] \). Then \( |V| > 12 \).

Proof. By Theorem 3.1, \( |V| \geq 12 \). Let \( G = (V, \mathcal{E}) \) be a graph of siblings in \( V \). By Lemma 4.6, we assume that for every \( i \in [d] \) and \( l, s \in \{ a, a', b, b' \}, l \notin \{ s, s' \} \), there is an \( i \)-siblings in the set \( V^{i,l} \cup V^{i,s} \). Thus, for every \( i \in [6] \) and every \( \{ l, s \} \in \{ \{ b, a \}, \{ b, a' \}, \{ b', a \}, \{ b', a' \} \} \) there is an edge \((v, u) \in \mathcal{E}\) such that \( \{ v, u \} = \{ l, s \} \). In particular, for every \( i \in [6] \) there are at least 4 edges with the colour \( i \), and therefore \( |\mathcal{E}| \geq 24 \).

Let \( u^0, v^0 \in V \) be such that

\[
d(u^0) + d(v^0) = \max\{d(v) + d(u) : v, u \in V \text{ and } v, u \text{ are adjacent}\}.
\]

We may assume without loss of generality that \( u^0 = aaaaaaaaa \) and \( v^0 = bdaaaaa \).

By the assumption on the distribution of words in \( V \) and Lemma 4.5,

\[
|V^{i,l} \cup V^{i,s}| \geq 4 \tag{4.6}
\]

for every \( i \in [6] \) and \( l \in \{ a, b \} \).

If \( d(u^0) + d(v^0) \geq 10 \), then, by Lemma 4.1, there are \( i \in [6] \) and \( l \in \{ a, b \} \) such that \( |V^{i,l} \cup V^{i,s}| \geq 9 \). By (4.6), \( |V| > 12 \).

If \( d(u^0) + d(v^0) = 9 \), then it can be easily seen that there are \( i, j \in [6] \), \( l, s \in \{ a, b \} \) such that \( |V^{i,l} \cup V^{i,s}| \geq 8 \) and \( |V^{j,l} \cup V^{j,s}| \geq 8 \). Hence, by (4.4), \( |V^{i,l} \cup V^{i,s}| = 8 \) and \( |V^{j,l} \cup V^{j,s}| = 8 \). If \( |V| = 12 \), then along the same lines as in the proof of the second part of Lemma 4.4, we show that there is \( i \in [6] \) such that the set \( \mathcal{E} \) contains less than 4 edges with the colour \( i \), which contradicts the assumption on \( \mathcal{E} \).

Let \( d(u^0) + d(v^0) = 8 \), \( \{ w_1, w_2, w_3, w_4 \} = V \setminus (N(u^0) \cup N(v^0)) \) and

\[
|(V^{i,l} \cup V^{i,s}) \cap (N(u^0) \cup N(v^0))| = n_i.
\]

Suppose on the contrary that \( |V| = 12 \).

Assume first that there is \( i \in \{ 2, \ldots, 6 \} \), say \( i = 6 \), such that \( n_6 = 0 \). Then, by (4.6), \( \{ w_1, w_2, w_3, w_4 \} = V^{6,b} \cup V^{6,b'} \). Note that, there is \( w \in W \) such that \( \hat{w} \cap \bigcup E(V^{6,b}) \neq \emptyset \) and \( \hat{w} \cap \bigcup E(V^{6,b'}) \neq \emptyset \), for otherwise \( V^{6,b} \subset W^{6,b} \) and \( V^{6,b'} \subset W^{6,b'} \), and consequently, by Statement 3.4, \( a \) and \( b \), \( |W^{6,b}| + |W^{6,b'}| \geq 12 \). Since \( W^{6,a} \cup W^{6,a'} \neq \emptyset \), we have \( |W| > 12 \), which is a contradiction. By Lemma 4.5, \( |V^{6,b}| = |V^{6,b'}| = 2 \), and the structure of \( V^{6,b} \) and \( V^{6,b'} \) is such as described in Statement 3.4. In particular, there are \( i, j \in \{ 2, 3, 4, 5 \} \), say \( i = 4 \), \( j = 5 \) and letters \( l_4, l_5 \in S \) such that \( w_i \in \{ l_4, l_4' \} \) and \( w_5 \in \{ l_5, l_5' \} \) for every \( i \in \{ 1, 2, 3, 4 \} \).
Let $l_4 = l_5 = b$. Then $w^i, w^j \in \{b, b'\}$ for $i \in \{1, 2, 3, 4\}$. Since $n_6 = 0$, an edge $(w, v)$ has the colour 6 if and only of $w \in \{w^1, w^2, w^3, w^4\}$ and $v \in N(v^0) \cup N(v^0)$. Then $v_4, v_5 \in \{b, b'\}$, and consequently $v \notin N(v^0) \cup N(v^0)$. Thus, there are no edges of the colour 6 in $\mathcal{E}$, a contradiction.

Let now $l_5 = a$. Then, by (4.6), $n_5 \geq 4$. Note that, if $v_i, v_j \in \{b, b'\}$, $i, j \in \{2, 3, 4, 5\}$, $i \neq j$, then $v \notin N(v^0) \cup N(v^0)$. Thus, $n_i = 0$ for some $i \in \{2, 3, 4\}$, and consequently an edge $(w, v)$, where $w \in \{w^1, w^2, w^3, w^4\}$ and $v \in N(v^0) \cup N(v^0)$, has the colour $i$. On the other hand $(w, v)$ must be of the colour 6 as $n_6 = 0$, a contradiction.

Thus, $n_i \geq 1$ for every $i \in \{6\}$, and therefore it suffices to consider two cases: $n_2 = 2, n_3 = \cdots = n_6 = 1$ and $n_1 = \cdots = n_6 = 1$.

It follows from (4.6) that in the first case there are at least two words in the set $\{w^1, w^2, w^3, w^4\}$, say these are $w^1$ and $w^2$, which have the letters $b$ or $b'$ at at least three positions $i, j, k \in \{3, 4, 5, 6\}$, which means that they cannot be adjacent to vertices from the set $N(v^0) \cup N(v^0)$. In the second case there are at least three such words; assume that these are $w^1, w^2$ and $w^3$. It is easy to verify that in the first case there are at most 8 edges with ends in the set $N(v^0) \cup N(v^0)$, and in the second case there are at most 12 such edges (compare the second part of the proof of Lemma 4.3). The maximal number of edges with ends in $\{w^1, w^2, w^3, w^4\}$ is four as the graph $G$ does not contain triangles. Thus, in the second case in order to obtain $|\mathcal{E}| \geq 24$ it must be $d(w^4) \geq 8$, which is impossible since $d(v) \leq 6$ for every $v \in V$. For the same reason in the first case it must be $d(w^3) = d(w^4) = 6$. Then for every $v \in (N(v^0) \cup N(v^0)) \setminus \{v^0, v^0\}$ the vertices $w^3$ and $w^4$ are adjacent to $v$. Since $G$ does not contain triangles, $w^3$ and $w^4$ cannot be adjacent, and hence $|\mathcal{E}| < 24$, which contradicts the assumption on $\mathcal{E}$.

Let now $d(v^0) + d(v^0) \leq 7$. It follows from Lemma 4.2 that $d(G) \leq 7/2$, and since $d(G)|V| = 2|\mathcal{E}|$ and $2|\mathcal{E}| \geq 48$, we have $|V| > 12$.

\section{Computations}

In this section we describe the computations which lead to the determination of all possible twin pairs free equivalent and disjoint polybox codes $V, W \subset S^d$ having 12 words each, where $S = \{a, a', b, b'\}$ and $d \in \{4, 5, 6\}$. The structure of such polybox codes $V, W$ is given in Theorem 5.2.

The longest part of the paper was devoted to the preparations of the computations, since it seems hopeless to do this without any initial configurations of words, where by an initial configuration of words we mean a some number of words or their fragments in the constructing code $V$ (see tables in this section).
An immediate consequence of lemmas 4.4–4.8 is the following result on the distribution of words in $V$:

**Corollary 5.1** Let $V, W \subset S^d$, $S = \{a, a', b, b'\}$, be disjoint and equivalent polybox codes without twin pairs, and let $|V| = 12$. Then for every $i \in [d]$ the distribution of words in $V$ takes one of the forms:

1. $|V^{i,a}| = 7$, $|V^{i,a'}| = 1$, $|V^{i,b}| = 2$, $|V^{i,b'}| = 2$
2. $|V^{i,a}| = 6$, $|V^{i,a'}| = 2$, $|V^{i,b}| = 2$, $|V^{i,b'}| = 2$
3. $|V^{i,a}| = 6$, $|V^{i,a'}| = 1$, $|V^{i,b}| = 2$, $|V^{i,b'}| = 3$
4. $|V^{i,a}| = 5$, $|V^{i,a'}| = 3$, $|V^{i,b}| = 2$, $|V^{i,b'}| = 2$
5. $|V^{i,a}| = 5$, $|V^{i,a'}| = 2$, $|V^{i,b}| = 2$, $|V^{i,b'}| = 3$
6. $|V^{i,a}| = 5$, $|V^{i,a'}| = 1$, $|V^{i,b}| = 3$, $|V^{i,b'}| = 3$
7. $|V^{i,a}| = 5$, $|V^{i,a'}| = 1$, $|V^{i,b}| = 2$, $|V^{i,b'}| = 4$
8. $|V^{i,a}| = 5$, $|V^{i,a'}| = 5$, $|V^{i,b}| = 1$, $|V^{i,b'}| = 1$
9. $|V^{i,a}| = 4$, $|V^{i,a'}| = 4$, $|V^{i,b}| = 2$, $|V^{i,b'}| = 2$
10. $|V^{i,a}| = 4$, $|V^{i,a'}| = 3$, $|V^{i,b}| = 2$, $|V^{i,b'}| = 3$
11. $|V^{i,a}| = 4$, $|V^{i,a'}| = 2$, $|V^{i,b}| = 2$, $|V^{i,b'}| = 4$
12. $|V^{i,a}| = 3$, $|V^{i,a'}| = 3$, $|V^{i,b}| = 2$, $|V^{i,b'}| = 4$

If $d = 4$, then

13. $|V^{i,a}| = 3$, $|V^{i,a'}| = 3$, $|V^{i,b}| = 3$, $|V^{i,b'}| = 3$.

Moreover, in every case $1 – 13$, except the case 8, for every $l \in \{a, b\}$ there is $w \in W$ such that $\bar{w} \cap \bigcup E(V^{i,l}) \neq \emptyset$ and $\bar{w} \cap \bigcup E(V^{i,l'}) \neq \emptyset$.

**Proof.** We prove the second part of the corollary. For every $w \in W$ we have $w \subseteq V$. It follows from Lemma 2.1 that the family of pairwise dichotomous boxes $\{\bar{w} \cap \bar{v} \neq \emptyset : v \in V\}$ is a suit for the $d$-box $\bar{w}$ without twin pairs. Suppose on the contrary that for every $w \in W$ at most one of the sets $\bar{w} \cap \bigcup E(V^{i,l})$ and $\bar{w} \cap \bigcup E(V^{i,l'})$ is non-empty for some $l \in \{a, b\}$. Then $V^{i,l} \subseteq W^{i,l}$ and $V^{i,l'} \subseteq W^{i,l'}$. Since $|V^{i,l}| \geq 2$ or $|V^{i,l'}| \geq 2$ (recall that the case 8 has been excluded), we have, by Statement 3.1 (b), $|W^{i,l}| \geq 7$ or $|W^{i,l'}| \geq 7$. But $|W^{i,l}| \geq 5$ and $|W^{i,l'}| \geq 5$, by Statement 3.1 (a). Thus, $|W^{i,l} \cup W^{i,l'}| \geq 12$, and as $|W^{i,s} \cup W^{i,s'}| > 0, s \not\in \{l, l'\}$, we have $|W| > 12$, a contradiction. \(\square\)
5.1 Initial configurations of the words.

It follows from Corollary 5.1 that there is \( w \in W \) such that \( \bar{w} \cap \bigcup E(V_{i,b}) \neq \emptyset \) and \( \bar{w} \cap \bigcup E(V_{i,b'}) \neq \emptyset \) when the polybox code \( V \) has the distributions of words 1–5 and 9–13 and there is \( w \in W \) such that \( \bar{w} \cap \bigcup E(V_{i,a}) \neq \emptyset \) and \( \bar{w} \cap \bigcup E(V_{i,a'}) \neq \emptyset \) when \( V \) has the distributions 6 and 7.

Now we show how the initial configurations of words are established. We do it in detail for the distributions 1, 2, 4 and 9. The rest configurations are determined in the similar manner.

Let \( V \) has the distribution of words of the form 1, 2, 4 or 9 of Corollary 5.1. Then, by Statement 3.4 (d), there is \( w \in W \) such that \(|\{ \bar{w} \cap \bar{v} \neq \emptyset : v \in V_{i,b} \}| = 2\) and \(|\{ \bar{w} \cap \bar{v} \neq \emptyset : v \in V_{i,b'} \}| = 2\). By Statement 3.4 (c), the set \( \bar{w} \cap \bigcup E(V_{i,b}) \cup \bar{w} \cap \bigcup E(V_{i,b'}) \) is an \( i \)-cylinder in the \( d \)-box \( \bar{w} \). Again by Statement 3.4 (c), there are \( i_1, i_2 \in [d] \setminus \{ i \} \), \( i_1 < i_2 \), such that

\[
(V_{i,b})_{A} = \{ w_{i_1}l_2, l_1l_2' \}, \quad (V_{i,b'})_{A} = \{ l_1l_2, l_1'l_2w_{i_2} \}
\]

and

\[
(V_{i,b})_{\{i_1,i_2\}} = (V_{i,b'})_{\{i_1,i_2\}} = \{ p \},
\]

where \( A = \{ i_1, i_2 \} \), \( l_j \not\in \{ w_{i_j}, w_{i_j}' \} \) for \( j = 1, 2 \), and \( p \in S^{d-3} \) (compare Figure 7). Clearly, without loss of generality we may assume that \( w_{i_1} = w_{i_2} = b \) and \( p = b \cdots b \) (recall that \( V, W \subset \{ a, a', b, b' \}^d \)). Then, by Statement 3.4 (c), \( l_1, l_2 \in \{ a, a' \} \), so we may assume that \( l_1 = l_2 = a \).

Let \( V \) be such as in the case 1 of Corollary 5.1, and let \( i = 1, i_1 = 2, i_2 = 3 \) and \( d = 5 \). For the computations we take \( a = +1, a' = -1, b = +2 \) and \( b' = -2 \). We arrange the words from the set \( V_{1,2} \cup V_{1,-2} \) as the first four rows of the matrix \( M \) of the size \( 12 \times 5 \). The rest eight words from \( V_{1,1} \cup V_{1,-1} \) are unknown; we know only their first letters, i.e. +1 and −1 and that \( |V_{1,1}| = 7 \) and \( |V_{1,-1}| = 1 \). The matrix \( M \) has a form

\[
M = \begin{bmatrix}
+2 & +2 & +1 & +2 & +2 \\
+2 & +1 & -1 & +2 & +2 \\
-2 & +1 & +1 & +2 & +2 \\
-2 & -1 & +2 & +2 & +2 \\
+1 & : & +1 & & -1 \\
\end{bmatrix}
\]

These four full-length words and the eight first letters of the rest of words from \( V \), seven letters +1 and one −1, are the initial configurations of words for the computations corresponding to the case 1 of Corollary 5.1 for \( d = 5 \).
5. Our task is to compute the missing cells of the matrix $M$ such that the rows in the resulting matrix form the polybox code $V$ having 12 words without twin pairs and, by Lemma 4.6, for every $i \in [5]$ and every $l, s \in \{+1, -1, +2, -2\}, l \notin \{s, s'\}$, the set $V^{i,l} \cup V^{i,s}$ contains an $i$-siblings.

In the case 1, 2, 4, and 9 the initial configurations of words are the following (we write $M$ in the form of a table):

| Case 1, $d = 4, 5$ | Case 2, $d = 4, 5$ | Case 4, $d = 4, 5$ | Case 9, $d = 4, 5$ |
|-------------------|-------------------|-------------------|-------------------|
| $+2$              | $v^1$             | $+2$              | $v^1$             |
| $+2$              | $v^2$             | $+2$              | $v^2$             |
| $-2$              | $v^3$             | $-2$              | $v^3$             |
| $-2$              | $v^4$             | $-2$              | $v^4$             |
| $7 \times +1$    | $6 \times +1$    | $5 \times +1$    | $4 \times +1$    |
| $1 \times -1$    | $2 \times -1$    | $3 \times -1$    | $4 \times -1$    |

where

$v^1 = +2 + 1 + 2(+2), v^2 = +1 - 1 + 2(+2), v^3 = -1 + 1 + 2(+2), v^4 = +1 + 2 + 2(+2),$

where $(+2)$ at the end of $v^i$ means that the letter $+2$ has to be placed at the fifth position in the words $v^i, i = 1, 2, 3, 4$, for the case $d = 5$. This notation will be used also below.

Observe that in the cases 3, 5 and 10 of Corollary 5.1 by Statement 3.4 (d), it can be $|\{\hat{\alpha} \cap \hat{\beta} \neq \emptyset : v \in V^{i,b}\}| = 2$ and $|\{\hat{\alpha} \cap \hat{\beta} \neq \emptyset : v \in V^{i,b'}\}| = 3$ or $|\{\hat{\alpha} \cap \hat{\beta} \neq \emptyset : v \in V^{i,b}\}| = 2$ and $|\{\hat{\alpha} \cap \hat{\beta} \neq \emptyset : v \in V^{i,b'}\}| = 2$. Therefore, by Statement 3.4 (c),

| Case 3, $d = 4, 5$ | Case 5, $d = 4, 5$ | Case 10, $d = 4, 5$ |
|-------------------|-------------------|-------------------|
| $+2$              | $w^1$             | $+2$              | $w^1$             |
| $+2$              | $w^2$             | $+2$              | $w^2$             |
| $-2$              | $w^3$             | $-2$              | $w^3$             |
| $-2$              | $w^4$             | $-2$              | $w^4$             |
| $-2$              | $w^5$             | $-2$              | $w^5$             |
| $6 \times +1$    | $5 \times +1$    | $4 \times +1$    |
| $1 \times -1$    | $2 \times -1$    | $3 \times -1$    |

| Case 3, $d = 4, 5$ | Case 5, $d = 4, 5$ | Case 10, $d = 4, 5$ |
|-------------------|-------------------|-------------------|
| $+2$              | $v^1$             | $+2$              | $v^1$             |
| $+2$              | $v^2$             | $+2$              | $v^2$             |
| $-2$              | $v^3$             | $-2$              | $v^3$             |
| $-2$              | $v^4$             | $-2$              | $v^4$             |
| $1 \times -2$    | $1 \times -2$    | $1 \times -2$    |
| $6 \times +1$    | $5 \times +1$    | $4 \times +1$    |
| $1 \times -1$    | $2 \times -1$    | $3 \times -1$    |
where

\[ w_1 = +2 + 2 + 1(+2), \quad w_2 = +1 + 1 - 1(+2), \]
\[ w_3 = +1 + 1 + 2(+2), \quad w_4 = -1 + 1 + 1(+2), \quad w_5 = +2 - 1 + 1(+2) \]

or

\[ w_1 = +2 + 1 + 1(+2), \quad w_2 = +1 + 2 - 1(+2), \]
\[ w_3 = +1 + 1 + 2(+2), \quad w_4 = -1 + 1 + 1(+2), \quad w_5 = +1 - 1 - 1(+2) \]

**Remark 5.1** In the codes above the sets of words \( \{ w_1, w_2 \} \) and \( \{ w_3, w_4, w_5 \} \) correspond to the codes \( V \) and \( W \) in Lemma 3.3 for the case \( |V| = 2, |W| = 3 \). Observe that the words \( w_3, w_4, w_5 \) can also take the form: \( w_3 = +1 + 1 + 2(+2), \quad w_4 = +1 - 1 + 1(+2), \quad w_5 = -1 + 2 + 1(+2) \), but permuting the letters at the first two positions we see that the polybox codes \( w_1 = +2 + 2 + 1(+2), \quad w_2 = +1 + 1 - 1(+2), \quad w_3 = +1 + 1 + 2(+2), \quad w_4 = -1 + 1 + 1(+2), \quad w_5 = +2 - 1 + 1(+2) \) and \( w_1 = +2 + 2 + 1(+2), \quad w_2 = +1 + 1 - 1(+2), \quad w_3 = +1 + 1 + 2(+2), \quad w_4 = -1 + 1 + 1(+2), \quad w_5 = +1 - 1 + 1(+2) \) are isomorphic. Clearly, we have to make the computations only for non-isomorphic codes.

Moreover, the forms of the words \( \{ w_1, \ldots, w_6 \} \) are such as in Statement 3.4 (c) (see also Example 3.5).

Similarly, in the case of the polybox codes \( U = \{ u_1, \ldots, u_6 \} \) and \( P = \{ p_1, \ldots, p_6 \} \) which are given below, the codes \( \{ u_1, u_2 \}, \{ u_3, u_4, u_5, u_6 \} \) and \( \{ p_1, p_2, p_3 \}, \{ p_4, p_5, p_6 \} \) correspond to the codes \( V \) and \( W \) in Lemma 3.3 for the cases \( |V| = 2, |W| = 4 \) and \( |V| = 3, |W| = 3 \), respectively. Also in these cases we make the computations for non-isomorphic codes.

By Statement 3.4 (c), (d) and the above remark, the cases 11 and 12 give the initial configurations of words:
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| Case 12, $d = 4, 5$ | Case 12, $d = 4, 5$ | Case 12, $d = 4, 5$ |
|---------------------|---------------------|---------------------|
| $u^1$               | $w^1$               | $v^1$               |
| $u^2$               | $w^2$               | $v^2$               |
| $u^3$               | $w^3$               | $v^3$               |
| $u^4$               | $w^4$               | $v^4$               |
| $u^5$               | $w^5$               |                     |
| $u^6$               |                     |                     |

where

$u^1 = +2 + 2 + 1$, $u^2 = +1 + 1 + 1$, $u^3 = +1 + 1 + 1 + 2$, $u^4 = +1 + 1 + 1 + 2$, $u^5 = +2 - 1 + 1 + 1$, $u^6 = +2 + 2 - 1 + 1$

or

$u^1 = +2 + 2 + 1 + 1$, $u^2 = +1 + 1 + 2 - 1$, $u^3 = +1 + 1 + 1 + 2$, $u^4 = +1 + 1 + 1 + 1 + 1 + 1 + 1 - 1 - 1$

or

$u^1 = +2 + 2 + 1(2)$, $u^2 = +1 + 1 + 2(2)$, $u^3 = +1 + 1 + 2(2)$, $u^4 = +1 + 1 + 1(2)$, $u^5 = +2 - 1 + 1(2)$, $u^6 = +1 - 1 - 1(2)$

By Lemma 4.7, Statement 3.4 (c), (d) and Remark 5.1.

| Case 13, $d = 4$ | Case 13, $d = 4$ | Case 13, $d = 4$ |
|------------------|------------------|------------------|
| $p^1$            | $w^1$            | $v^1$            |
| $p^2$            | $w^2$            | $v^2$            |
| $p^3$            | $w^3$            | $v^3$            |
| $p^4$            | $w^4$            | $v^4$            |
| $p^5$            | $w^5$            |                  |
| $p^6$            |                  |                  |

where

$p^1 = +2 + 1 + 1$, $p^2 = +1 + 2 - 1$, $p^3 = -1 - 1 + 2$, $p^4 = +2 - 1 + 2$, $p^5 = -1 + 2 + 1$, $p^6 = +1 + 1 + 2$

or

$p^1 = +1 + 2 + 2$, $p^2 = -1 - 1 + 2$, $p^3 = -1 + 1 + 1$, $p^4 = +2 - 1 + 2$, $p^5 = +1 + 2 + 1$, $p^6 = +1 + 1 + 1$

or

$p^1 = +1 + 2 + 2$, $p^2 = -1 - 1 + 2$, $p^3 = -1 + 1 + 1$, $p^4 = +2 + 2 + 1$, $p^5 = +1 + 2 - 1$, $p^6 = -1 - 1 - 1$

or

$p^1 = +1 + 2 + 2$, $p^2 = -1 - 1 + 2$, $p^3 = -1 + 1 + 1$, $p^4 = +2 + 2 + 1$, $p^5 = +2 - 1 - 1$, $p^6 = +1 + 1 - 1$, $p^7 = +1 + 1 - 1$.
\[ p^1 = +1 + 1 + 2, \quad p^2 = +2 - 1 - 1, \quad p^3 = -1 - 1 + 1, \]
\[ p^4 = -1 - 1 + 2, \quad p^5 = +1 + 2 - 1, \quad p^6 = +1 + 1 + 1. \]

As we noted above, in the cases 6 and 7 we have \( \hat{w} \cap \bigcup E(V^{i,a}) \neq \emptyset \) and \( \hat{w} \cap \bigcup E(V^{i,a'}) \neq \emptyset \). In this two cases we have \( w_i = b \). Thus, by Lemma 4.5, \( |\{\hat{w} \cap \hat{v} \neq \emptyset : v \in V^{i,a}\}| = 5 \) and \( |\{\hat{w} \cap \hat{v} \neq \emptyset : v \in V^{i,a'}\}| = 1 \). Therefore, by Statement 3.4(c),

| Case 6, \( d = 4, 5 \) | Case 7, \( d = 4, 5 \) |
|----------------|----------------|
| +1 | +1 |
| +1 | +1 |
| +1 | +1 |
| +1 | +1 |
| −1 | −1 |
| \( 3 \times +2 \) | \( 2 \times +2 \) |
| \( 3 \times -2 \) | \( 4 \times -2 \) |

where
\[ q^1 = +1 + 1 + 1( +2), \quad q^2 = -1 - 1 - 1( +2), \quad q^3 = +2 + 1 - 1( +2), \]
\[ q^4 = -1 + 2 + 1( +2), \quad q^5 = +1 - 1 + 2( +2), \quad q^6 = +2 + 2 + 2( +2). \]

Finally, we consider the initial configurations of words corresponding to the case 8 of Corollary 5.1. In this case, by Lemma 4.9, besides computations for \( d = 4, 5 \) we have to make computations for \( d = 6 \).

Observe first that, by Statement 3.4(d), \( V^{i,b} \subseteq W^{i,b} \) and \( V^{i,b'} \subseteq W^{i,b'} \), and thus, by Statement 3.4(a), \( |W^{i,b}| \geq 5 \) and \( |W^{i,b'}| \geq 5 \), and since, by Lemma 4.6, \( |W^{i,a}| \geq 1 \) and \( |W^{i,a'}| \geq 1 \), we have \( |W^{i,b}| = 5 \), \( |W^{i,b'}| = 5 \) and \( |W^{i,a}| = 1 \), \( |W^{i,a'}| = 1 \). Clearly, \( V^{i,a} \subseteq V^{i,a'} \) and \( W^{i,a} \subseteq W^{i,a'} \), and thus the structure of \( V^{i,a} \) and \( V^{i,a'} \) are such as in Statement 3.4(a).

We do not know what is a relation between the polybox codes \( V^{i,a} \) and \( V^{i,a'} \). Therefore, we fix the structure of \( V^{i,a} \), say that \( (V^{i,a})_i = \{q^1, \ldots, q^5\} \) for \( d = 4, 5 \) and \( (V^{i,a'})_i = \{q^1 + 2, \ldots, q^5 + 2\} \) for \( d = 6 \), and \( V^{i,a'} \) will take all possible forms, i.e., \( (V^{i,a'})_i = \{x^1, \ldots, x^5\} \), where
\[
x^1 = l_{1}l_{2}l_{3}(u_{4})(u_{5}), \quad x^2 = l'_{1}l'_{2}l'_{3}(u_{4})(u_{5}), \quad x^3 = s_{1}l_{2}l'_{3}(u_{4})(u_{5})
\]
\[
x^4 = l'_{1}s_{2}l_{3}(u_{4})(u_{5}), \quad x^5 = l_{1}l'_{2}s_{3}(u_{4})(u_{5}),
\]
\( s_{1}, s_{2}, s_{3}, l_{1}, l_{2}, l_{3}, u_{4}, u_{5} \) range over the set \( \{+1, -1, +2, -2\} \), \( s_{i} \notin \{l_{i}, l'_{i}\} \) for \( i = 1, 2, 3 \), \( \sigma \) is a permutation of the set \( [d] \) and \( v_{\sigma} = v_{\sigma(1)} \cdots v_{\sigma(d)} \), where \( v = v_{1} \cdots v_{d} \). Now, for every selection of letters \( s_{1}, s_{2}, s_{3}, l_{1}, l_{2}, l_{3}, u_{4}, u_{5} \in \{+1, -1, +2, -2\} \) and a permutation \( \sigma \) we have the initial configuration
The case 8 generates a quite big package of the initial configurations of words, but each of them contains much as ten words, which makes every single computation very quick.

Below we present an algorithm using in the computations.

Let $S = \{+1, -1, +2, -2\}$, and let $S^d_+, S^d_-$ and $S^d_{-2}$ denote the sets of all words of length $d$, for $d = 4, 5, 6$, having the letter +1, −1 or −2 at the first position, respectively. By $M^d_r(n_{+1}, n_{-1}, n_{+2})$ we denote the set of $r$ full-length words in the initial configuration having $n_{+1}$, $n_{-1}$ and $n_{+2}$ letters +1, −1 and +2 at the first position, respectively.

**Algorithm**

**Input:** The set $M^d_r(n_{+1}, n_{-1}, n_{+2})$

**Output:** A polybox code $V$ having 12 words without twin pairs such that for every $i \in [d]$ and every $l, s \in \{+1, -1, +2, -2\}$, $l \notin \{s, s'\}$, the set $V^{i,l} \cup V^{i,s}$ contains an $i$-siblings

| Case 8, $d = 4, 5, 6$ |
|-----------------------|
| +1                    | $q^1$ |
| +1                    | $q^2$ |
| +1                    | $q^3$ |
| +1                    | $q^4$ |
| +1                    | $q^5$ |
| −1                    | $x^1_\sigma$ |
| −1                    | $x^2_\sigma$ |
| −1                    | $x^3_\sigma$ |
| −1                    | $x^4_\sigma$ |
| −1                    | $x^5_\sigma$ |
| 1 × +2                |      |
| 1 × −2                |      |
Let
Theorem 5.2
For every two words v and w by vw we denote a concatenation of v and w. If A and B are sets of words, then

\[ AB = \{vw : v \in A, w \in B\}. \]

The results of the computations are given in the following theorem:

**Theorem 5.2** Let \( l, s \in \{a, a', b, b'\}, l \notin \{s, s'\} \), and let

\[
\begin{align*}
A_1 &= \{ss', l's', s's\}, \quad B_1 = \{ll', sl, s'l\}, \\
A_2 &= \{s'l'\}, \quad B_2 = \{ss\}, \quad C_1 = \{ll, l'l\}, \\
A_1' &= \{ss, l's', s's\}, \quad B_1' = \{ll, sl, s'l\}, \\
A_2' &= \{s'l'\}, \quad B_2' = \{ss\}, \quad C_1' = \{ll, l'l\}.
\end{align*}
\]

If \( V, W \subseteq S^d \), where \( d \in \{4, 5, 6\} \), are disjoint and equivalent polybox codes without twin pairs having twelve words and \( V \) is extensible to a partition code, then there is a set \( A = \{i_1 < i_2 < i_3 < i_4\} \subseteq [d] \) such that

\[
(V)_{A'} = W_1 \setminus W_1 \cap W_2, \quad (W)_{A'} = W_2 \setminus W_1 \cap W_2,
\]

and \( (V)_A = (W)_A = \{(p)_A\} \) for some \( p \in S^d \). The representation of \( V \) and \( W \) is given up to an isomorphism.
Proof. To show that $V$ and $W$ do not contain a twin pair observe that the partition code $W_1$ contains four twin pairs

$$ls's's,  l's's's; \ l'l's'l, \ l'l's'l; \ ssl'l', \ s's'l'; \ sll's', \ s'l's',$$

and the partition code $W_2$ also contains four twins

$$ls's's, \ l's's's; \ l'l's'l, \ l'l's'l; \ ssl'l', \ s's'l'; \ sll's', \ s'l's',$$

where the marked words form the set $W_1 \cap W_2$. Therefore, the polybox codes $V = W_1 \setminus W_1 \cap W_2$ and $W = W_2 \setminus W_1 \cap W_2$ are disjoint and equivalent, they do not contain twin pairs and $|V| = |W| = 12$.

By Corollary 5.1, the distributions of the letters in $V$ cannot be other than the distributions $1-13$ of this corollary. Thus, the code $V$ has to contain at least one initial configuration from the list of the initial configurations of words given in the tables in this section.

The computations showed that the initial configurations corresponding to the cases $1-7$ and $9-12$ cannot be completed to a twin pair free polybox code $V$ with twelve words such that for every $i \in [d]$ and $l, s \in \{a, a', b, b'\}$, $l \notin \{s, s'\}$, the set $V^{i,l} \cup V^{i,s}$ contains an $i$-siblings (compare Lemma 4.6). Similarly, this cannot be done in the case 8 for $d = 5$ and 6. The only configurations that can be completed to a polybox code $V$ with the above properties are the case 8 for $d = 4$ and the configurations given in the last two tables for the case 13. The structures of all these complemented codes $V$ (for $d = 4$ we have $(V)_A = V$ as $A^c = \emptyset$) with 12 words are, up to an isomorphism, as stated in the theorem. The polybox code $W$ is a complementation of the words $W_1 \cap W_2 = \{ls's's, l'l's'l, ssl'l', sll's\}$ to a partition codes. It can be easily check that there is exactly one such complementation which is disjoint with $V$, and its structure is such as given in the theorem. (The words $ls's's, l'l's'l, ssl'l', sll's'$ can be complemented to a partition code exactly in two ways: $V$ and $W$).

Since for $d = 5$ and 6 all initial configurations cannot be completed to a twin pair free polybox code with twelve words such that for every $i \in [d]$ and $l, s \in \{a, a', b, b'\}$, $l \notin \{s, s'\}$, the set $V^{i,l} \cup V^{i,s}$ contains an $i$-siblings, it follows that for the above two dimensions we have, by Lemma 4.6 $(V)_A = (W)_A = \{(p)_A\}$ for some $p \in S^d$, and $(V)_{A^c}$, $(W)_{A^c}$ are such as in dimension four.

□

Remark 5.2 The representation of $V$ and $W$ given in Theorem 5.2 can be found in [12]. The codes $V$ and $W$ were used by Lagarias and Shor[13, 14] and later on by Mackey[15] to construct the counterexamples to Keller's cube tiling conjecture. In the context of this conjecture one of these codes was
given first by Corrádi and Szabó in [3], as an example of the maximal clique in a 4-dimensional Keller graph.

6 Twin pairs in cube tilings of $\mathbb{R}^7$

From Theorem 3.1 and 5.2 we obtain the following

**Theorem 6.1** Let $U \subset S^7$ be a partition code. If there are $i \in \{7\}$ and $l \in S$ such that $|U_i^l| \leq 12$, then there is a twin pair in $U$.

**Proof.** Since $\bigcup E(U_i^l \cup U_i^{l'})$ is an $i$-cylinder in $(ES)^7$, the codes $U_i^l$ and $U_i^{l'}$ are equivalent. If $U_i^l$ or $U_i^{l'}$ contains a twin pair, then clearly $U$ does. Thus, we assume that these two codes do not contain a twin pair. If $(U_i^l)_i \neq \emptyset$ and $(v)_i \in (U_i^l)_i \cap (U_i^{l'})_i$, then the words $w \in U_i^l$ and $p \in U_i^{l'}$ such that $(v)_i = (w)_i = (p)_i$ are a twin pair. Therefore we may assume that $(U_i^l)_i$ and $(U_i^{l'})_i$ are disjoint and do not contain a twin pair. It follows from Theorem 3.1 that $|U_i^l| = 12$, and Theorem 5.2 precisely describes the structure of the codes $(U_i^l)_i$ and $(U_i^{l'})_i$ (clearly, $(U_i^l)_i$ and $(U_i^{l'})_i$ are extensible to partition codes). In Theorem 5.2 we take $A = \{1, 2, 3, 4\}$, $(p)_A = ll$ and $(U_i^l)_i = V$. Without loss of generality we can take $i = 7$. We consider only the code $U_i^l$ which has the form:

\[
\begin{align*}
&l \quad s \quad l \quad s \quad l \quad l \quad l \\
&l' \quad s \quad l' \quad l \quad l \quad l \\
&l' \quad l \quad l' \quad s \quad l \quad l \\
&l' \quad s' \quad l' \quad l \quad l \\
&l' \quad s' \quad s' \quad s \quad l \quad l \\
&l' \quad l' \quad l' \quad l \quad l \\
&s' \quad l' \quad l' \quad l \quad l \\
&s' \quad l' \quad l' \quad l \quad l \\
&s \quad l' \quad s' \quad l \quad l \\
&s \quad l \quad s' \quad l \quad l \\
\end{align*}
\]

We choose four words from $U_i^l$:

\[
\begin{align*}
v &= l \quad s \quad l \quad s \quad l \quad l \\
u &= l \quad s \quad l' \quad l \quad l \quad l \\
p &= l \quad s' \quad l' \quad s' \quad l \quad l \\
q &= s' \quad s \quad l' \quad l' \quad l \quad l \\
\end{align*}
\]
Let $W \subset U^4$ be the set of all $w \in U^4$ such that $(u)_4 \subseteq (W)_4$ and $(\tilde{w})_4 \cap (\tilde{w})_4 \neq \emptyset$. Since $U$ is a partition code, $W \neq \emptyset$. Clearly, for every $i \in [7], i \neq 4$, and $w \in W$ we have $w_i \neq u'_i$, for otherwise $(\tilde{w})_4 \cap (\tilde{w})_4 = \emptyset$ which contradicts the definition of $W$. Every $w \in W$ is dichotomous to the words $v, q$ and $p$, and therefore $w_1 = w_2 = s$ and $w_3 = w_4 = l'$ for every $w \in W$. Then

$$l \ll (W)_{\{1,2,3,4\}}.$$

If $(W)_{\{1,2,3,4\}}$ does not contain a twin pair, then its structure is such as in Statement 3.4 (a). Without loss of generality we may assume that

$$(W)_{\{1,2,3,4\}} = \{s_1s_2s_3, s'_1s'_2s'_3, l's'_2s_3, s_1l's_3, s'_1s_2l\},$$

where $s_i \notin \{l, l'\}$ for $i = 1, 2, 3$. But then $ss'l's'_2s_1l \in U^7$ which is not true. Therefore, the code $(W)_{\{1,2,3,4\}}$ contains a twin pair, and hence $W$ contains a twin pair. \hfill \Box

**Corollary 6.2** Let $[0,1]^7 + T$ be a cube tiling of $\mathbb{R}^7$. If $|L(T, x, i)| = 5$ for some $x \in \mathbb{R}^7$ and $i \in [7]$, then there is a twin pair in $[0,1]^7 + T$.

**Proof.** As it was noted at the beginning of Section 2, the set of boxes $\mathcal{F}_x = \{([0,1]^7 + t) \cap ([0,1]^7 + x) \neq \emptyset : t \in T\}$ is a minimal partition of $[0,1]^7 + x$. Let $U$ be a partition code such that $\mathcal{F}_x$ is an exact realization of $U$ (compare [10, Theorem 4.2]). Since $|L(T, x, i)| = 5$, we have $U = U^{i_1} \cup U^{i_2} \cup \ldots \cup U^{i_5} \cup U^{i_5}$ and $|U^{i_j}| \leq 12$ for some $j \in [5]$. By Theorem 6.1 there is a twin pair in $U$, and consequently there is a twin pair in $\mathcal{F}_x$. Then the tiling $[0,1]^7 + T$ contains a twin pair. \hfill \Box

From the result of Debroni et al., [10] Corollary 4.2] and the above corollary we obtain the following

**Corollary 6.3** If there is a counterexamples to Keller’s conjecture in dimension seven, then $|L(T, x, i)| \in \{3, 4\}$ for some $x \in \mathbb{R}^7$ and $i \in [7]$.

In [10] we extended the notion of a $d$-dimensional Keller graph: if $S$ is an alphabet with a complementation, then a $d$-dimensional Keller graph on the set $S^d$ is the graph in which two vertices $u, v \in S^d$ are adjacent if they are dichotomous but do not form a twin pair.

From Theorem 6.1 we obtain the following

**Corollary 6.4** Every clique in a 7-dimensional Keller graph on $S^7$ which contains at least five vertices $u^i, \ldots, u^5$ such that $u^i \notin \{u^m, (u^m)'\}$ for some $i \in [7]$ and every $n, m \in \{1, \ldots, 5\}, n \neq m$, has less than $2^7$ elements.
Proof. Assume on the contrary that there is a clique $U$ containing vertices $u^1, \ldots, u^5$ and $|U| = 2^7$. Thus, $U$ is a partition code without twin pairs. Since $u^n_i \not\in \{u^n_m, (u^m)\}'$ for every $n, m \in \{1, \ldots, 5\}, n \neq m$, it follows that $|U^{i,n}| \leq 12$ for some $m \in [5]$. By Theorem 6.1 there is a twin pair in $U$, a contradiction. \hfill \square

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