Fusion of $SO(N)$ reflection matrices

N. J. MacKay*

Dept of Applied Maths and Theoretical Physics,
Cambridge University,
Cambridge, CB3 9EW,
England

ABSTRACT

We examine the reflection matrix acting on the $SO(N)$ vector multiplet and fuse it to obtain that acting on the rank two particle multiplet, and give its decomposition.

1 Introduction

There has been much interest lately in the integrability properties of massive 1+1-D theories on a half-line. However, relatively little of this has been directed at the construction of explicit solutions of the reflection equation, the analogue for reflection ($K$-)matrices of the Yang-Baxter equation (YBE) for scattering ($S$-)matrices. In this paper we apply the fusion procedure to the reflection matrix in the vector representation of $SO(N)$ to find that in the second particle multiplet, an $SO(N)$-reducible representation whose components are the second rank antisymmetric tensor and the singlet. Such $K$-matrices would be expected to apply to the $SO(N)$ principal chiral model on the half-line and thus to differ in pole structure from the $SO(N)$ $\sigma$-model investigated by Ghoshal, which is not

*n.j.mackay@damtp.cam.ac.uk
expected to have bulk bound states. In carrying this out we start from the $K$-matrices provided by Cherednik and investigate their properties, comparing the matrix structure with that of Ghoshal, before fusing them. A proper interpretation of our results in terms of the integrability (or otherwise) of the physical boundary condition is, however, lacking.

2 Reflection matrix in the vector representation

First let us recall that the $S$-matrix solution of the YBE for two particles in vector representations of $SO(N)$ is

$$S_{11}(\theta) = X_{11}(\theta)\sigma(\theta) \mathbf{P} (P_S + [2]P_A + [2][h]P_0) ,$$

where

$$[a] \equiv \frac{\theta + a\pi}{\theta - a\pi} ,$$

$h = N - 2$ is the dual Coxeter number of $SO(N)$, $\mathbf{P}$ indicates transposition of states in a tensor product, and $P_S$, $P_A$, and $P_0$ are projectors onto the second rank symmetric traceless and antisymmetric tensors and the singlet respectively. $\sigma(\theta)$ is a scalar factor chosen such that with $X_{11} = 1$ there are no poles in the physical strip $0 \leq \text{Im} \theta \leq \pi$,

$$\sigma(\theta) = \frac{\Gamma(\frac{1}{2} + \theta/2i\pi)\Gamma(\frac{1}{2} + \frac{\theta}{2i\pi})\Gamma(\frac{1}{2} + \frac{\theta}{2i\pi} - \theta/2i\pi)\Gamma(-\theta/2i\pi)\Gamma(\frac{1}{2} - \frac{\theta}{2i\pi})\Gamma(\frac{1}{2} + \frac{\theta}{2i\pi})\Gamma(\frac{1}{2} + \theta/2i\pi)\Gamma(+\theta/2i\pi)}{\Gamma(\frac{1}{2} - \theta/2i\pi)\Gamma(\frac{1}{2} - \theta/2i\pi)\Gamma(\frac{1}{2} + \frac{\theta}{2i\pi})\Gamma(-\theta/2i\pi)\Gamma(\frac{1}{2} + \frac{\theta}{2i\pi})\Gamma(\frac{1}{2} - \frac{\theta}{2i\pi})\Gamma(1/2 + \theta/2i\pi)\Gamma(1/2 - \theta/2i\pi)},$$

and $X_{11}(\theta)$ is a so-called CDD factor which is used here to give the physical pole structure.

The $SO(N)$ $\sigma$-model is expected to have no bound states and so has $X_{11} = 1$. Here we wish to assume the existence of a second, fused particle state proportional to $P_A + P_0$ (so that the second particle multiplet is the adjoint $\oplus$ singlet representation of $SO(N)$, which is an irreducible representation of the underlying Yangian charge algebra), and so take

$$X_{11}(\theta) = -(2)(N - 4) , \quad \text{where} \quad (x) \equiv \frac{\sinh \left( \frac{\theta}{2} + \frac{x\pi}{h} \right)}{\sinh \left( \frac{\theta}{2} - \frac{x\pi}{h} \right)} .$$

We next recall some ideas from Cherednik’s paper. First, this $S$-matrix, $S \equiv \pm S$ (dropping for the moment the ‘11’ suffices), in principle only applies when both particles
are right-moving. If the reflection matrix is $K_1(\theta)$ (where, as with $S$, for the moment we drop the suffix '1') then we also need to define scattering matrices for one left- and one right-moving particle, $0S$, and for two left-moving particles, $-S$. We then have

\begin{align*}
+_S(\theta) &= 1 \otimes K(0) . 0S(\theta) . 1 \otimes K(0) \\
-_S(\theta) &= K(0) \otimes 1 . 0S(\theta) . K(0) \otimes 1 ,
\end{align*}

in terms of which the reflection equation is

\begin{equation}
-_S(\phi - \theta) . 1 \otimes K(\phi) . 0S(\phi + \theta) . 1 \otimes K(\theta) = 1 \otimes K(\theta) . 0S(\phi + \theta) . 1 \otimes K(\phi) . -S(\phi - \theta) . (3)
\end{equation}

This $K$ must satisfy both unitarity

\begin{equation}
K(\theta) K(-\theta) = I (4)
\end{equation}

and a combined crossing and unitarity relation\footnote{Note that, in contrast to the diagonal case\footnote{This relation cannot be rewritten to give the $S$-matrix as a product of reflection matrices, so that the interpretation of scattering as being equivalent to the placing of a two-sided mirror at the point of scattering fails.}}: if we write the $S$-matrix acting on two vectors with indices $k, l$ as $S_{kl}^{ij}$ and the $K$-matrix acting on a vector with index $k$ as $K_k^i$, then this is

\begin{equation}
K_j^i (i\pi/2 - \theta) = 0_S^{ij} (2\theta) K_k^i (i\pi/2 + \theta) . (5)
\end{equation}

Cherednik gave a solution to (3):

\begin{equation}
K(\theta) \propto E + c\theta I ,
\end{equation}

where $I$ is the $N \times N$ identity matrix, $c$ is an (undetermined) constant and $E$ some matrix such that $E^2 = I$. For $SU(N)$ (which we can recover by setting the trace operator to zero in the above $S$-matrix) this is a solution for all $c, E$, whilst in the $SO(N)$ case it is only a solution if

\begin{equation}
c = -\frac{2h}{i\pi \text{Trace}E} . (6)
\end{equation}

(At this stage we should also point out how this and all subsequent calculations are most easily performed. We use Brauer’s diagrammatic representation\footnote{We use Brauer’s diagrammatic representation\footnote{It is then simple to augment the algebra with $E$, the}} for the identity, transposition and trace operators on two vectors in which multiplication in the algebra corresponds to concatenation of the symbols, and which has been used to calculate the fused $S$-matrices\footnote{acting in $P_A + P_0$. It is then simple to augment the algebra with $E$, the}} acting in $P_A + P_0$. It is then simple to augment the algebra with $E$, the
corresponding symbol being a bead on one of the strands, with a bead on a loop equalling
Trace\(E\) and two beads on the same strand disappearing.)

The given matrix structure of \(K\) satisfies (4), and satisfies (5) provided (6) holds. In the \(su(N)\) case (5) is more subtle since representations are not self-conjugate, and we need to use \(S_{11}\). Once again, however, we find that (3) must hold for (5) to be satisfied, this time with \(h = N\). This situation for \(K\)-matrices is analogous to that for the crossing parameter \(h\) for the \(S\)-matrices: in the \(SO(N)\) case this parameter was explicit in the \(S\)-matrix (1) (because the vector representation is self-conjugate) but in the \(SU(N)\) case the YBE was not enough to fix it, and it only appeared after the separate implementation of crossing symmetry.

Cherednik required the distinction between \(+S, -S\) and \(0S\) because his \(K(0) \neq I\). We shall choose instead to work with \(K(\theta) \propto I + c\theta E\), so that we can require the more physical condition \(K(0) = I\), with \(+S = -S = 0S\). All of our results apply also to Cherednik’s \(K\) under the change \([\ ] \mapsto -[\ ]\). We write the solution, with an overall factor to be determined, as

\[
K(\theta) = \tau(\theta) \left( P^- - \left[ \frac{h}{ci\pi} \right] P^+ \right),
\]

where

\[
P^\pm = \frac{1}{2}(I \pm E)
\]

are projectors. This \(K\) solves (3), and satisfies (4,5) provided

\[
\tau(\theta) \tau(-\theta) = 1
\]

\[
\frac{\tau(i\frac{\theta}{2} - \theta)}{\tau(i\frac{\theta}{2} + \theta)} = X_{11}(2\theta)\sigma(2\theta) \left[ \frac{h}{2} \right] \left[ \frac{h}{ci\pi} - \frac{h}{2} \right],
\]

which is solved by

\[
\tau(\theta) = -(1 - N/2)(-N/2)(3 - N) \frac{\Gamma(\frac{1}{4} + \theta/2i\pi)\Gamma(\frac{1}{4} + \frac{1}{2i\pi} - \theta/2i\pi)\Gamma(\frac{1}{4} + \frac{1}{2i\pi} + \theta/2i\pi)}{\Gamma(\frac{1}{4} - \theta/2i\pi)\Gamma(\frac{1}{4} + \frac{1}{2i\pi} + \theta/2i\pi)\Gamma(\frac{1}{4} + \frac{1}{2i\pi} - \theta/2i\pi)}
\]

\[
\times \frac{\Gamma(1 - \theta/2i\pi)\Gamma(\frac{1}{2} - \frac{1}{2i\pi} + \theta/2i\pi)\Gamma(1 - \frac{1}{2i\pi} - \theta/2i\pi)}{\Gamma(1 + \theta/2i\pi)\Gamma(\frac{1}{2} - \frac{1}{2i\pi} - \theta/2i\pi)\Gamma(1 - \frac{1}{2i\pi} + \theta/2i\pi)}.
\]
The product of gamma functions alone solves (8) when $X_{11} = 1$, and has no poles on the physical strip $0 \leq \text{Im} \theta \leq \pi$; with the given $X_{11}$ we need also the ( ) prefactors, which are the reflection analogue of the CDD factors. Possible such factors have been given by various authors$^8$ investigating the (scalar) reflection factors in affine Toda field theory. We choose those given by Fring and Köberle, which are trivial at $\theta = 0$ and are ‘minimal’ for our problem, in the sense that they introduce no new physical poles. Thus the only pole in $K_1(\theta)$ is that at $1/c$, which corresponds to a boundary bound state proportional to $P^+$. It still remains to relate $E$ to physical boundary conditions. It is not difficult to show that any real $N \times N$ matrix $E$ such that $E^2 = I_N$ is similar to

$$\begin{pmatrix} I_M & * \\ 0 & -I_{N-M} \end{pmatrix}.$$  

$P^-$ then leaves an $SO(N - M)$-symmetric subspace invariant, whilst $P^+$ projects onto an $SO(M)$ subspace corresponding to the boundary bound state at $1/c = \frac{N-2M}{2\hbar} i\pi$. The boundary condition must therefore correspond to the free condition on the $SO(N - M)$ subgroup, and a tentative suggestion is that for the principal chiral field the Neumann boundary condition $g'(x = 0) = 0$ and $g(x = 0) \in SO(M)$ may be integrable. The physics of sigma models on the half-line remains an interesting but largely unexplored problem. In particular, it may be interesting to investigate the Yangian charges$^9$ on the half-line.

For $M = 1$ we recover Ghoshal’s condition that the boundary scattering preserve $SO(N - 1)$ symmetry. We then have $c = 2/i\pi$, one of the [ ] factors disappears from $\tau$ and our expressions are those of Ghoshal, who has also, implicitly, made the choice $K(0) = 1$. Note that his $\sigma$ is rescaled by $[2][\hbar]$ from ours because of a difference in how we write the $S$-matrix$^1$. The boundary bound state pole is then at the edge of the physical strip, at $i\pi/2$. 

5
3 The fused reflection matrix

We are now in a position to fuse the $K$-matrix (7). The way to do this is well-known: as with $S$-matrix fusion, we make use of the fact that when $\phi - \theta = \frac{2i\pi}{h}$, the $S$-matrix $-S$ in (3) projects onto the second rank particle and so we can replace it by $P_A + P_0$, onto which the action of (3) can now consistently be restricted. We therefore define the reflection matrix for the second particle to be

$$K_2(\theta) = (P_A + P_0) \cdot 1 \otimes K(\theta + i\pi/h) \cdot 0 S(2\theta) \cdot 1 \otimes K(\theta - i\pi/h).$$

The rather complicated ensuing expression for $K_2$ can eventually be cast into quite a neat form:

$$K_2(\theta) = \tau_2(\theta) \left( P_A^- - [h/ci\pi - 1]P_2 + [h/ci\pi - 1][h/ci\pi + 1]P_A^+ \right),$$

where

$$P_A^\pm = P_A \otimes P_\pm \cdot P_A$$

$$P_2 = (P_+ \otimes P_+ - P_- \otimes P_+) P_A + P_0$$

are projectors and

$$\tau_2(\theta) = [1] \tau(\theta + i\pi/h) \tau(\theta - i\pi/h) \sigma(2\theta) X_{11}(2\theta).$$

Unitarity, cross-unitarity and solution of (3) are preserved by fusion and automatically follow, as does $K_2(0) = P_A + P_0$. This time the poles are of two types: those introduced by the bulk $S$-matrix CDD factor (2), and the boundary bound states at $1/c \pm i\pi/h$. The former are a particle pole at $\theta = \frac{i\pi}{h}$ and its crossed-channel partner. Of the latter, certainly that at $1/c + i\pi/h$ will be a boundary bound state, projecting onto the antisymmetric component (only) of the $P^+ \otimes P^+$ space (the $SO(M)$-symmetric component for the $E$ discussed above): note that at $c = \frac{2}{i\pi}$ and $M = 1$ this projector vanishes, and the pole is off the physical strip. The interpretation of the pole at $1/c - i\pi/h$ eludes us.

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