Spaces with labelled partitions and isometric affine actions on Banach spaces

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Abstract

We define the structure of spaces with labelled partitions which generalizes the structure of spaces with measured walls and study the link between actions by automorphisms on spaces with labelled partitions and isometric affine actions on Banach spaces, and more particularly, on $L^p$ spaces. We construct natural spaces with labelled partitions on which direct sums of groups or semi-direct products of groups act and we apply this to prove that the wreath product of a group with property $PL^p$ by a group with Haagerup property has property $PL^p$.

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1 Introduction

A locally compact second countable group $G$ has Haagerup property (or is a-(T)-menable) if there exists a proper continuous isometric affine action of $G$ on a Hilbert space; this property can be seen as a strong negation of Kazhdan’s property (T) (an overview of the Haagerup property can be found in [CCJ+01]). Groups having Haagerup property are well known to satisfy the Baum-Connes conjecture by a result of Higson and Kasparov in [HK01] (see [Jul98] for further details). Haagerup property is closed by taking subgroups, direct products, amalgamated products over finite subsets but it is not stable by group extensions in general, even in the case of semi-direct products. However, Cornulier, Stalder and Valette recently proved in [CSV12] that it is stable by a particular kind of extension, namely the wreath product, using the connexion between Haagerup property and spaces with measured walls.

A space with walls is a pair $(X, W)$ where $X$ is a set and $W$ is a family of partitions of $X$ in two pieces called walls such that any pair of points of $X$ is separated by finitely many walls. This notion was introduced by Haglund and Paulin in [HP98] and generalized in a topological setting by Cherix, Martin and Valette in [CMV04] to space with measured walls (see Definition 3.7). It was gradually realised that the Haagerup property is equivalent to the existence of a proper action on a space with measured walls; more precisely, we have the following theorem: a locally compact second countable group has the Haagerup property if, and only if, it acts properly by automorphisms on a space with measured walls. Using results of Robertson and Steger (see [RS98]), Cherix, Martin and Valette in [CMV04], proved this theorem for discrete groups and Chatterji, Drutu and Haglund extended the equivalence to locally compact second countable groups in [CDH10] using the notion of median spaces. The stability of the Haagerup property by wreath product was established in [CSV12] by constructing a space with measured walls from the structures of measured walls on each factor, and moreover, in the same article, Cornulier, Stalder
and Valette generalized this result to the permutational wreath product (see Definition 5.1) when the index set $I$ is a quotient by a co-Haagerup subgroup of the shifting group $G$ (see [CI11] for a counter example when the pair $(G, I)$ has relative property (T)).

The notion of Haagerup property naturally extend to proper isometric affine action on Banach spaces. Recent works have been made about isometric actions on Banach spaces: in [HP06], Haagerup and Przybyszewska showed that every locally compact second countable group $G$ acts properly by affine isometries on the reflexive Banach space $\bigoplus_{n=1}^{\infty} L^{2n+1}(G, \mu)$ where $\mu$ is the Haar measure; Cornulier, Tessera, Valette introduced in [CTV08] property $(BP^0_V)$ for $V$ a Banach space; in [BFGM07], Bader, Furman Gelander and Monod, studied an analog of property (T) in terms of $L^p$ spaces and more generally, of superreflexive Banach spaces. See [Now13] for a recent overview of results and questions about isometric affine actions on Banach spaces.

Furthermore, we can focus on specific Banach spaces, namely, $L^p$ spaces. For $p \geq 1$, we say a locally compact second countable group $G$ has property $PL^p$ (or is $a$-$FL^p$-menable) if there exists a proper continuous isometric affine action on a $L^p$ space. See for instance [CDH10], for a characterisation of property $PL^p$ for $p \in [1, 2]$ in terms of Haagerup property. An important example is the following theorem due to Yu (see [Yu05]): let $\Gamma$ be a discrete Gromov hyperbolic group. Then there exists $p \geq 2$ such that $\Gamma$ has property $PL^p$. Yu proved this result by giving an explicit proper isometric affine action of $\Gamma$ on $\ell^p(\Gamma \times \Gamma | d(x, y) \leq R)$ and Nica (see [Nic12]) exhibited a proper isometric affine action of $\Gamma$ on $L^p(\partial \Gamma \times \partial \Gamma)$ with an explicit $\Gamma$-invariant measure on $\partial \Gamma \times \partial \Gamma$. A remarkable consequence of this result is that there exists infinite groups with property (T) (and hence, without Haagerup property) which have property $PL^p$ for some $p > 2$.

In this context, the goal of this paper is to define the notion of spaces with labelled partitions which generalizes the structure of space with measured walls in the general setting of continuous isometric affine actions on Banach spaces, and more particularly in the $L^p$ case (see Section 3). We establish in Paragraph 3.2 the following result which links isometric affine actions on Banach spaces and actions by automorphisms on spaces with labelled partitions:

**Theorem 1.** Let $G$ be topological group.

1) If $G$ acts properly continuously by affine isometries on a Banach space $B$ then there exists a structure $(G, \mathcal{P}, F(\mathcal{P}))$ of space with labelled partitions on $G$ such that $G$ acts properly continuously by automorphisms on $(G, \mathcal{P}, F(\mathcal{P}))$ via its left-action on
itself. Moreover, there exists a linear isometric embedding \( F(\mathcal{P}) \hookrightarrow B \).

2) If \( G \) acts properly continuously by automorphisms on a space with labelled partitions \((X, \mathcal{P}, F(\mathcal{P}))\) then there exists a proper continuous isometric affine action of \( G \) on a Banach space \( B \). Moreover, \( B \) is a closed subspace of \( F(\mathcal{P}) \).

This theorem can be rephrased in the particular case of \( L^p \) spaces as follows:

**Corollary 1.1.** Let \( p > 1 \) with \( p \notin 2\mathbb{Z} \setminus \{2\} \) and \( G \) be a topological group. \( G \) has property \( PL^p \) if, and only if, \( G \) acts properly continuously by automorphisms on a space with labelled partitions \((X, \mathcal{P}, F(\mathcal{P}))\) where \( F(\mathcal{P}) \) is isometrically isomorph to a closed subspace of a \( L^p \) space.

In Section 4, we give a natural construction of space with labelled partitions on direct sums and we exhibit an explicit proper action by automorphisms on this space given proper actions by automorphisms on each factor of the direct sum.

In Paragraph 4.3, given groups \( G_1, G_2 \) acting properly by automorphisms on spaces with labelled partitions, we observe that if a morphism \( \rho : G_1 \to \text{Aut}(G_2) \) “preserves” the structure of labelled partitions of \( G_2 \), then the semi-direct product \( G_1 \rtimes_\rho G_2 \) acts properly by automorphisms on the natural space with labelled partitions of the direct product; more precisely, we prove:

**Theorem 2.** Let \((X_1, \mathcal{P}_1, F_1(\mathcal{P}_1)), (X_2, \mathcal{P}_2, F_2(\mathcal{P}_2))\) be spaces with labelled partitions and \( G_1, G_2 \) be topological groups acting continuously by automorphisms on, respectively, \((X_1, \mathcal{P}_1, F_1(\mathcal{P}_1))\) and \((X_2, \mathcal{P}_2, F_2(\mathcal{P}_2))\) via \( \tau_1 \) and \( \tau_2 \).

Let \( \rho : G_2 \to \text{Aut}(G_1) \) be a morphism of groups such that \((g_1, g_2) \mapsto \rho(g_2)g_1\) is continuous for the product topology on \( G_1 \times G_2 \).

Assume there exists an action \( \bar{\rho} \) of \( G_2 \) on \( X_1 \) which is compatible with \( \tau_1 \) with respect to \( \rho \).

If \( \bar{\rho} \) induces a continuous action of \( G_2 \) by automorphisms on \((X_1, \mathcal{P}_1, F_1(\mathcal{P}_1)), \) then the semi-direct product \( G_1 \rtimes_\rho G_2 \) acts continuously by automorphisms on the natural structure of labelled partitions \((X_1 \times X_2, \mathcal{P}, F_q(\mathcal{P}))\) on the direct product of \( X_1 \times X_2 \).

Moreover, if, for \( i = 1, 2, \) \( G_i \) acts properly on \((X_i, \mathcal{P}_i, F_i(\mathcal{P}_i)))\), then \( G_1 \rtimes_\rho G_2 \) acts properly on \((X_1 \times X_2, \mathcal{P}, F(\mathcal{P}))\).

Finally, in Section 5, we apply these results to prove that the wreath product of a group with property \( PL^p \) by a group with the Haagerup property has property \( PL^p \) using the construction of space with measured walls provided in [CSV12]:
Theorem 3. Let $H, G$ be countable discrete groups, $L$ be a subgroup of $G$ and $p > 1$, with $p \notin 2\mathbb{Z} \setminus \{2\}$. Denote by $I$ the quotient $G/L$ and $W = \bigoplus_I H$. Suppose that $G$ is Haagerup, $L$ is co-Haagerup in $G$ and $H$ has property $PL_p$. Then the permutational wreath product $H \wr_I G = W \rtimes G$ has property $PL_p$.

Subsequently, all topological groups considered are assumed to be Hausdorff.

2 Preliminaries

2.1 Metrically proper actions

Definition 2.1. Let $G$ be a topological group acting continuously isometrically on a metric space $(X, d_X)$. The $G$-action on $X$ is said metrically proper if, for all (or equivalently, for some) $x_0 \in X$,

$$\lim_{g \to \infty} d_X(g.x_0, x_0) = +\infty.$$

Let $Y$ be a set endowed with a pseudo-metric $d$. We put on $Y$ the following equivalence relation: for $y, y' \in Y$, $y \sim y'$ if, and only if, $d(y, y') = 0$, and we denote by $X$ the quotient set $Y/\sim$. Then we can define a metric $\tilde{d}$ on $X$ by setting, for $y, y' \in Y$, $\tilde{d}([y], [y']) = d(y, y')$. Moreover, an isometric group action $(Y, d)$ preserves the classes of $\sim$ and then induces an isometric action on $(X, \tilde{d})$. This fact allows us to define metrically proper actions on pseudo-metric spaces:

Definition 2.2. Let $G$ be a topological group acting continuously isometrically on a pseudo-metric space $(Y, d)$. The $G$-action on $Y$ is said metrically proper if the induced $G$-action on the quotient metric space $(X, \tilde{d})$ is metrically proper.

In other words, a metrically proper action on a pseudo-metric space can be expressed this way:

Proposition 2.3. Let $G$ be a topological group acting continuously isometrically on a pseudo-metric space $(Y, d)$. The $G$-action on $Y$ is metrically proper if, for all (or equivalently, for some) $y_0 \in Y$,

$$\lim_{g \to \infty} d(g.y_0, y_0) = +\infty.$$

2.2 Isometric affine actions

Let $G$ be a topological group and $(B, \|\cdot\|)$ be a Banach space.
Definition 2.4. A continuous isometric affine action $\alpha$ of $G$ on $B$ is a strongly continuous morphism

$$\alpha : G \rightarrow \text{Isom}(B) \cap \text{Aff}(B).$$

Notice that if $B$ is a real Banach space, then, by Mazur-Ulam Theorem,

$$\text{Isom}(B) \cap \text{Aff}(B) = \text{Isom}(B).$$

Proposition 2.5. A continuous isometric affine action $\alpha$ of $G$ on $B$ is characterised by a pair $(\pi, b)$ where :

- $\pi$ is a strongly continuous isometric representation of $G$ on $B$,
- $b : G \rightarrow B$ is a continuous map satisfying the 1-cocycle relation : for $g, h \in G$,

$$b(gh) = \pi(g)b(h) + b(g).$$

And we have, for $g \in G$, $x \in X$ :

$$\alpha(g)x = \pi(g)x + b(g).$$

Definition 2.6. Let $\alpha$ be a continuous isometric affine action of $G$ on $B$. We say that $\alpha$ is proper if the action of $G$ on the metric space $(B, d_{\|\cdot\|})$ is metrically proper where $d_{\|\cdot\|}$ is the canonical metric on $B$ induced by the norm $\|\cdot\|$.

Proposition 2.7. A continuous isometric affine action $\alpha$ of $G$ on $B$ is proper if, and only if

$$\|b(g)\| \xrightarrow{g \rightarrow \infty} +\infty.$$ 

Definition 2.8. Let $p \geq 1$. We say that $G$ has property PL$^p$ (or is $a$-FL$^p$-menable) if there exists a proper continuous isometric affine action of $G$ on a $L^p$ space.

2.3 On isometries of $L^p$-spaces

In general, for $p \geq 1$, a closed subspace of a $L^p$-space is not a $L^p$-space (except the special case $p = 2$); but, in [HJ81], Hardin showed the following result about extension of linear isometries on closed subspace of a $L^p$ (here, we give a reformulation of this result coming from [BFGM07], Corollary 2.20) :

Theorem 2.9. Let $p > 1$ with $p \notin 2\mathbb{Z} \setminus \{2\}$ and $F$ be a closed subspace of $L^p(X, \mu)$. Let $\pi$ be a linear isometric representation of a group $G$ on $F$. Then there is a linear isometric
representation $\alpha'$ of $G$ on some other space $L^p(X', \mu')$ and a linear $G$-equivariant isometric embedding $F \hookrightarrow L^p(X', \mu')$.

An immediate consequence is the following:

**Corollary 2.10.** Let $p > 1$ with $p \notin 2\mathbb{Z} \setminus \{2\}$, $F$ be a closed subspace of a $L^p$-space and $G$ be a topological group. If $G$ acts properly by affine isometries on $F$, then $G$ has property $PL^p$.

In Section 4, we embed linearly isometrically into $L^p$ spaces some normed vector spaces isometrically isomorph to a direct sums of $L^p$ spaces thanks to the following basic result:

**Definition 2.11.** Let $I$ be a countable index set, $(B_i, \| \cdot \|_{B_i})_{i \in I}$ be a family of Banach spaces and $p \geq 1$. We call $\ell^p$-direct sum of the family $(B_i)$ the space:

$$B = \bigoplus_{i \in I} B_i := \left\{ (x_i)_{i \in I} \in \prod_{i \in I} B_i \mid \sum_{i \in I} \|x_i\|_{B_i}^p < +\infty \right\},$$

and we denote, for $x = (x_i) \in B$,

$$\|x\|_p := \left( \sum_{i \in I} \|x_i\|_{B_i}^p \right)^{\frac{1}{p}}.$$

The space $B = \bigoplus_{i \in I} B_i$ endowed with the norm $\| \cdot \|_p$ is a Banach space, and moreover, we have:

**Proposition 2.12.** Let $I$ be a countable index set, $p \geq 1$ and $(L^p(X_i, \mu_i))_{i \in I}$ be a family of $L^p$-spaces. Then $\left( \bigoplus_{i \in I} L^p(X_i, \mu_i), \| \cdot \|_p \right)$ is isometrically isomorph to a $L^p$-space.

### 3 Spaces with labelled partitions and actions on Banach Spaces

In this section we will introduce the structure of space with labelled partitions and record for further use a few basic properties.
3.1 Spaces with labelled partitions

3.1.1 Definitions

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Consider a set $X$ and a function $p : X \to \mathbb{K}$. There is a natural partition $P = P(p)$ of $X$ associated with $p$.

We have the following equivalence relation $\sim_p$ on $X$ : for $x, y \in X$,

$$x \sim_p y \text{ if, and only if, } p(x) = p(y).$$

We define the partition associated with $p$ by

$$P(p) = \{ \pi_p^{-1}(h) \mid h \in X/\sim_p \}$$

where $\pi_p$ is the canonical projection from $X$ to $X/\sim_p$.

Definition 3.1. Let $X$ be a set, and $\mathcal{P} = \{ p : X \to \mathbb{K} \}$ be a family of functions.

- We say that $p$ is a labelling function on $X$ and the pair $(P, p)$ is called a labelled partition of $X$.

- We say that $x, y \in X$ are separated by $p \in \mathcal{P}$ if $p(x) \neq p(y)$ and we denote by $\mathcal{P}(x|y)$ the set of all labelling functions separating $x$ and $y$.

Remark 3.2. The terminology "$x$ and $y$ are separated by $p$" comes from the fact that, if we denote by $P$ the partition of $X$ associated with $p$, $x$ and $y$ are separated by $p$ if, and only if, $x$ and $y$ belongs to two different sets of the partition $P$ i.e. $P$ separates $x$ and $y$.

Consider a set $\mathcal{P}$ of labelling functions on $X$, and the $\mathbb{K}$-vector space $\mathcal{F}(\mathcal{P}, \mathbb{K})$ of all functions from $\mathcal{P}$ to $\mathbb{K}$. Then we have a natural map $c : X \times X \to \mathcal{F}(\mathcal{P}, \mathbb{K})$ given by : for $x, y \in X$ and $p \in \mathcal{P}$,

$$c(x, y)(p) = p(x) - p(y).$$

Notice that $p$ belongs to $\mathcal{P}(x|y)$ if, and only if, $c(x, y)(p) \neq 0$.

Definition 3.3. Let $X$ be a set and $\mathcal{P}$ be a family of labelling functions. The map $c : X \times X \to \mathcal{F}(\mathcal{P}, \mathbb{K})$ such that, for $x, y \in X$ and for all $p \in \mathcal{P}$, $c(x, y)(p) = p(x) - p(y)$ is called the separation map of $X$ relative to $\mathcal{P}$.

We now define the notion of space with labelled partitions :

Definition 3.4 (Space with labelled partitions).

Let $X$ be a set, $\mathcal{P}$ be a family of labelling functions from $X$ to $\mathbb{K}$ and $(\mathcal{F}(\mathcal{P}), \| \|)$ be a
semi-normed space of $\mathbb{K}$-valued functions on $\mathcal{P}$ such that the quotient vector space $F(\mathcal{P})$ of $\mathcal{F}(\mathcal{P})$ by its subspace $\mathcal{F}(\mathcal{P})_0 = \{ \xi \in \mathcal{F}(\mathcal{P}) \mid \|\xi\| = 0 \}$ is a Banach space.

We say that $(X, \mathcal{P}, F(\mathcal{P}))$ is a space with labelled partitions if, for all $x, y \in X$:

$$c(x, y) : \mathcal{P} \to \mathbb{K}$$

belongs to $F(\mathcal{P})$.

**Definition 3.5.** If $(X, \mathcal{P}, F(\mathcal{P}))$ is a space with labelled partitions, we can endow $X$ with the following pseudo-metric: $d(x, y) = \|c(x, y)\|$ for $x, y \in X$.

We call $d$ the labelled partitions pseudo-metric on $X$.

**Remark 3.6.** If $(X, \mathcal{P}, F(\mathcal{P}))$ is a space with labelled partitions, then the separation map $c : X \times X \to F(\mathcal{P})$ is continuous where $X \times X$ is endowed with the product topology induced by the topology of $(X, d)$.

Our first example of spaces with labelled partitions is given by spaces with measured walls. Here we cite the definition of the structure of space with measured walls from [CSV12].

Let $X$ be a set. We endow $2^X$ with the product topology and we consider, for $x \in X$, the clopen subset of $2^X$, $\mathcal{A}_x := \{ A \subset X \mid x \in A \}$.

**Definition 3.7.** A measured walls structure is a pair $(X, \mu)$ where $X$ is a set and $\mu$ is a Borel measure on $2^X$ such that for all $x, y \in X$:

$$d_\mu(x, y) := \mu(\mathcal{A}_x \triangle \mathcal{A}_y) < +\infty$$

**Proposition 3.8.** Let $(X, \mu)$ be a measured space with walls. Then, for every real number $q \geq 1$, $(X, \mathcal{P}, L^q(\mathcal{P}, \mu))$ is a space with labelled partitions where $\mathcal{P} = \{ \mathbb{1}_h \mid h \in 2^X \}$.

Moreover, we have, for $x, y \in X$,

$$\|c(x, y)\|^q = d_\mu(x, y).$$

**Proof.** Denote $\mathcal{P} = \{ \mathbb{1}_h \mid h \in 2^X \}$. Then $\mathcal{P}$ is a family of labelling functions on $X$ and we denote by $c$ the separation map of $X$ associated with $\mathcal{P}$.

Let $x, y \in X$. For $h \in 2^X$, we have:

$$c(x, y)(\mathbb{1}_h) = \mathbb{1}_h(x) - \mathbb{1}_h(y) = \mathbb{1}_{\mathcal{A}_x}(h) - \mathbb{1}_{\mathcal{A}_y}(h).$$
The function \( f : 2^X \to \mathcal{P} \) such that, for \( h \in 2^X \), \( f(h) = \mathbb{1}_h \) is a bijection, and we endow \( \mathcal{P} \) with the direct image topology induced by \( f \). Then, \( \mu^* : \mathcal{P} \to \mathbb{R} \) such that, for any Borel subset \( A \) of \( \mathcal{P} \), \( \mu^*(A) = \mu(f^{-1}(A)) \) is a Borel measure on \( \mathcal{P} \).

We have \( \|c(x, y)\|_q^q = \int_{\mathcal{P}} |c(x, y)(p)|^q d\mu^*(p) = \int_{2^X} |\mathbb{1}_{A_x}(h) - \mathbb{1}_{A_y}(h)|^q d\mu(h) = \mu(A_x \triangle A_y) \), and then :

\[
\|c(x, y)\|_q^q = d_\mu(x, y) < +\infty.
\]

It follows that, for all \( x, y \in X \), \( c(x, y) \) belongs to \( L^q(\mathcal{P}, \mu) \) and hence, \( (X, \mathcal{P}, L^q(\mathcal{P}, \mu)) \) is a space with labelled partitions.

Using a proof of Yu (see [Yu05], Corollary 3.2) based on a construction of Mineyev in [Min01], we build a structure of labelled partitions on Gromov hyperbolic groups :

**Lemma 3.9** (Mineyev, Yu). Let \( \Gamma \) be a discrete \( \delta \)-hyperbolic group. Then there exists a \( \Gamma \)-equivariant function \( h : \Gamma \times \Gamma \to \mathcal{F}_c(\Gamma) \) where \( \mathcal{F}_c(\Gamma) = \{ f : \Gamma \to \mathbb{R} \text{ with finite support} \mid \| f \|_p = 1 \} \) such that :

\[
\text{Walls of } \mathbb{Z}^2 \text{ with their labels.}
\]
1. for all $a, x \in \Gamma$, $\text{supp } h(x, a) \subset B(a, 10\delta)$,

2. there exists constants $C \geq 0$ and $\varepsilon > 0$ such that, for all $x, x', a \in \Gamma$,

$$
\| h(x, a) - h(x', a) \|_p \leq Ce^{-\varepsilon(x|x')}a,
$$

3. there exists a constant $K \geq 0$ such that, for all $x, x' \in \Gamma$ with $d(x, x')$ large enough,

$$
\# \{ a \in \Gamma \mid \text{supp } h(x, a) \cap \text{supp } h(x', a) = \emptyset \} \geq d(x, x') - K.
$$

**Proposition 3.10** (Labelled partitions on a $\delta$-hyperbolic group). Let $\Gamma$ be a discrete $\delta$-hyperbolic group and denote $E_{10\delta} = \{(a, b) \in \Gamma \times \Gamma \mid d(a, b) \leq 10\delta\}$. There exists $p \geq 2$ such that $(\Gamma, E_{10\delta}, \ell^p(E_{10\delta}))$ is a space with labelled partitions.

**Proof.** We fix a finite generating set of $\Gamma$ and denote $d$ the word metric associated with it (and such that $\Gamma$ is Gromov hyperbolic of constant $\delta$ with respect to $d$). As $\Gamma$ is uniformly locally finite, there exists a constant $k > 0$ such that, for all $r > 0$ and $x \in \Gamma$, $\# B(x, r) \leq k^r$, and we consider $p \geq 2$ such that $ke^{-pe} < \frac{1}{2}$ whith $\varepsilon$ as in Lemma 3.9. We can see $E_{10\delta}$ as a set of labelling functions on $\Gamma$ using the function $h$ given by Lemma 3.9: we set, for $(a, b) \in E_{10\delta}$ and $x \in \Gamma$:

$$(a, b)(x) := h(x, a)(b).$$

Denote by $c$ the separation map associated with $E_{10\delta}$. We have, for $x, x' \in \Gamma$,

$$
\| c(x, x') \|_{\ell^p(E_{10\delta})}^p = \sum_{(a, b) \in E_{10\delta}} | h(x, a)(b) - h(x, a)(b) |^p,
$$

$$
= \sum_{a \in \Gamma} \| h(x, a) - h(x, a) \|_p^p \text{ by 1. Lemma 3.9},
$$

$$
\leq \sum_{a \in \Gamma} Cpe^{-pe(d(x, x'))}, \text{ by 2. Lemma 3.9},
$$

$$
\leq \sum_{a \in \Gamma} Cpe^{-pe(d(x, x'))},
$$

$$
\leq \sum_{n \in \mathbb{N}} Cpk^ne^{-pe(n - d(x, x'))}, \text{ and hence, by the choice of } p:
$$

$$
\| c(x, x') \|_{\ell^p(E_{10\delta})}^p \leq 2Cpe^{pe(d(x, x'))} < +\infty
$$

Thus $c(x, x')$ belongs to $\ell^p(E_{10\delta})$ for all $x, x' \in \Gamma$. It follows that $(\Gamma, E_{10\delta}, \ell^p(E_{10\delta}))$ is a space with labelled partitions. \qed
On countable sets, we can always define a structure of labelled partitions which gives
the discrete metric on this set:

**Proposition 3.11.** Let \( X \) be a countable set and \( \mathcal{P} = \{ \Delta_x \mid x \in X \} \) be the family of
labelling functions where, for \( x \in X \), \( \Delta_x = 2^{-\frac{1}{q}} \delta_x \).
Then, for every \( q \geq 1 \), \((X, \mathcal{P}, \ell^q(\mathcal{P}))\) is a space with labelled partitions.

**Proof.** We have, for \( x, y, z \in X \) with \( x \neq y \):

\[
c(x, y)(\Delta_z) = \Delta_z(x) - \Delta_z(y) = \begin{cases} 
0 & \text{if } z \notin \{x, y\} \\
\pm 2^{-\frac{1}{q}} & \text{otherwise.}
\end{cases}
\]

and then,

\[
\|c(x, y)\|_q^q = \sum_{z \in X} |c(x, y)(\Delta_z)|^q = |c(x, y)(\Delta_x)|^q + |c(x, y)(\Delta_y)|^q = 1.
\]

Notice that the labelled partitions pseudo-metric \( d \) on \( X \) in this case is precisely the
discrete metric on \( X \) i.e. \( d(x, y) = 1 \) for all \( x, y \in X, x \neq y \).

**Definition 3.12 (Naive \( \ell^q \) space with labelled partitions).** Let \( X \) be a countable set and
\( \mathcal{P} = \{ \Delta_x \mid x \in X \} \).
For \( q \geq 1 \), \((X, \mathcal{P}, \ell^q(\mathcal{P}))\) is called the naive \( \ell^q \) space with labelled partitions of \( X \).

### 3.1.2 Actions on spaces with labelled partitions

Here, we describe the maps which preserve the structure of space with labelled partitions.

**Definition 3.13 (homomorphism of spaces with labelled partitions).** Let \((X, \mathcal{P}, F(\mathcal{P}))\),
\((X', \mathcal{P}', F'(\mathcal{P}'))\) be spaces with labelled partitions and let \( f : X \to X' \) be a map from \( X \) to \( X' \).
We say \( f \) is a homomorphism of spaces with labelled partitions if:

1. for any \( p' \in \mathcal{P}' \), \( \Phi_f(p') := p' \circ f \) belongs to \( \mathcal{P} \),
2. for all $\xi \in F(P)$, $\xi \circ \Phi_f$ belongs to $F'(P')$ and,
\[
\|\xi \circ \Phi_f\|_{F'(P')} = \|\xi\|_{F(P)}.
\]

**Remark 3.14.**
- If $f$ is a homomorphism of spaces with labelled partitions, then $f$ is an isometry from $X$ to $X'$ endowed with their respective labelled partitions pseudo-metrics; indeed, for $x, y \in X$,
\[
d_X(x, y) = \|c(x, y)\|_{F(P)} = \|c(x, y) \circ \Phi_f\|_{F'(P')} = \|c'(f(x), f(y))\|_{F'(P')} = d_{X'}(f(x), f(y)),
\]

since we have $c(x, y) \circ \Phi_f = c'(f(x), f(y))$.
- An automorphism of the space with labelled partitions $(X, P, F(P))$ is a bijective map $f : X \to X$ such that $f$ and $f^{-1}$ are homomorphisms of spaces with labelled partitions from $(X, P, F(P))$ to $(X, P, F(P))$. Remark that, in this case, the map $\Phi_f$ is a bijection: $(\Phi_f)^{-1} = \Phi_{f^{-1}}$.

**Proposition 3.15.** Let $(X, P, F(P)), (X', P', F'(P')), (X'', P'', F''(P''))$ be spaces with labelled partitions and $f : X \to X'$, $f' : X' \to X''$ be homomorphisms of spaces with labelled partitions.

Denote $\Phi_f$ the map such that $\Phi_f(p') := p' \circ f$, for $p' \in P'$, and $\Phi_{f'}$ the map such that $\Phi_{f'}(p'') := p'' \circ f'$, for $p'' \in P''$.

Then $f' \circ f$ is a homomorphism of spaces with labelled partitions from $(X, P, F(P))$ to $(X'', P'', F''(P''))$ and we have, by denoting $\Phi_{f' \circ f}(p'') := p'' \circ (f' \circ f)$:
\[
\Phi_f \circ \Phi_{f'} = \Phi_{f' \circ f}.
\]

**Proof.** For all $p'' \in P''$, we have:
\[
\Phi_{f' \circ f}(p'') = p'' \circ (f' \circ f) \\
= (p'' \circ f') \circ f \\
= \Phi_{f'}(p'') \circ f \quad \text{with} \quad \Phi_{f'}(p'') \in P' \quad \text{by Definition 3.13} \\
= \Phi_f(\Phi_{f'}(p'')) \quad \text{and hence,} \\
\Phi_{f' \circ f}(p'') = \Phi_f \circ \Phi_{f'}(p'') \in P \quad \text{by Definition 3.13}.
\]

It follows that $\Phi_f \circ \Phi_{f'} = \Phi_{f' \circ f}$. 

Now, let $\xi \in F(\mathcal{P})$. Since $\xi \circ \Phi_f$ belongs to $F'(\mathcal{P}')$, 

$$\xi \circ \Phi_{f \circ f} = (\xi \circ \Phi_f) \circ \Phi_f' \in F''(\mathcal{P}''),$$

and we clearly have, using the previous equality,

$$\|\xi \circ \Phi_{f \circ f}\|_{F''(\mathcal{P}'')} = \|\xi \circ \Phi_f\|_{F'(\mathcal{P}')} = \|\xi\|_{F(\mathcal{P})}.$$

\[\square\]

**Remark 3.16.** Assume a group $G$ acts by automorphisms on $(X, \mathcal{P}, F(\mathcal{P}))$ and, for $g \in G$, denote by $\tau(g) : X \to X$, the map $x \mapsto \tau(g)x = gx$. Then, by Proposition 3.15, we have:

$$\Phi_{\tau(g_2)} \circ \Phi_{\tau(g_1)} = \Phi_{\tau(g_1g_2)}.$$  

**Example 3.17.** Let $X$ be a countable set, $q \geq 1$ and $G$ a group acting on $X$. Then $G$ acts by automorphisms on the naive $\ell^q$ space with labelled partitions of $X$.

In fact, if, for $g \in G$, we denote $\tau(g) : x \mapsto gx$, we have, for $z \in X$,

$$\Delta_z \circ \tau(g) = \Delta_{g^{-1}z} \in \mathcal{P},$$

and, for all $\xi \in \ell^q(\mathcal{P})$,

$$\|\xi \circ \Phi_{\tau(g)}\|_q^q = \sum_{x \in X} |\xi(\Delta_{gx})|^q = \sum_{g^{-1}x \in X} |\xi(\Delta_x)|^q = \sum_{x \in X} |\xi(\Delta_x)|^q = \|\xi\|_q^q.$$

**Definition 3.18.** Let $(X, \mathcal{P}, F(\mathcal{P}))$ be a space with labelled partitions and $G$ be a topological group acting by automorphisms on $(X, \mathcal{P}, F(\mathcal{P}))$.

- We say that $G$ acts continuously on $(X, \mathcal{P}, F(\mathcal{P}))$, if the $G$-action on $(X, d)$ is strongly continuous.

- We say that $G$ acts properly on $(X, \mathcal{P}, F(\mathcal{P}))$, if the $G$-action on $(X, d)$ is metrically proper where $d$ is the labelled partitions pseudo-metric on $X$.

**Remark 3.19.** Notice that if a topological Hausdorff group $G$ acts properly continuously by automorphisms on a space $(X, \mathcal{P}, F(\mathcal{P}))$ with labelled partitions, then it is locally compact and $\sigma$-compact: in fact, let $x_0 \in X$; for $r > 0$, $V_r = \{g \in G \mid d(gx_0, x_0) \leq r\}$ is
a compact neighbourhood of the identity element \( e \) in \( G \) since the action on \((X, d)\) is strongly continuous and proper, and we have \( G = \bigcup_{n \in \mathbb{N}} V_n \).

**Proposition 3.20.** Let \( G \) be a topological group. Assume \( G \) acts continuously by automorphisms on \((X, \mathcal{P}, F(\mathcal{P}))\).

The \( G \)-action on \((X, \mathcal{P}, F(\mathcal{P}))\) is proper if, and only if, for every (resp. for some) \( x_0 \in X \), \( \| c(gx_0, x_0) \| \to \infty \) when \( g \to \infty \).

**Proof.** It follows immediately from the definition of a metrically proper action. \( \square \)

An example of proper action by automorphisms is given by Gromov hyperbolic groups.

**Proposition 3.21.** Let \( \Gamma \) be a discrete \( \delta \)-hyperbolic group and \((\Gamma, E_{10\delta}, \ell^p(\Gamma))\) be the space with labelled partitions given by Proposition 3.10. Then the action of \( \Gamma \) by left-translation on itself induces a proper action of \( \Gamma \) by automorphisms on \((\Gamma, E_{10\delta}, \ell^p(\Gamma))\).

**Proof.** We keep notations used in the proof of Proposition 3.10. We first show that \( \Gamma \) acts by automorphisms on \((\Gamma, E_{10\delta}, \ell^p(\Gamma))\). Let \( \gamma, x \in \Gamma \) and \( (a, b) \in E_{10\delta} \). Since \( h \) is \( \Gamma \)-equivariant, we have:

\[
\Phi_\gamma((a, b))(x) = (a, b)(\gamma x) = h(\gamma x, a)(b) = h(x, \gamma^{-1}a)(\gamma^{-1}b) = (\gamma^{-1}a, \gamma^{-1}b)(x),
\]

And hence,

\[
\Phi_\gamma((a, b)) = (\gamma^{-1}a, \gamma^{-1}b) \in E_{10\delta}.
\]

Moreover, for \( \xi \in \ell^p(E_{10\delta}) \), we have:

\[
\|\xi \circ \Phi_\gamma\|_{\ell^p(E_{10\delta})} = \sum_{(a, b) \in E_{10\delta}} |\xi(\gamma^{-1}a, \gamma^{-1}b)|^p,
\]

\[
= \sum_{(\gamma a, \gamma b) \in E_{10\delta}} |\xi(a, b)|^p,
\]

\[
= \sum_{(a, b) \in E_{10\delta}} |\xi(a, b)|^p,
\]

\[
\|\xi \circ \Phi_\gamma\|_{\ell^p(E_{10\delta})} = \|\xi\|_{\ell^p(E_{10\delta})}.
\]

It follows that \( \Gamma \) acts by automorphisms on \((\Gamma, E_{10\delta}, \ell^p(E_{10\delta}))\).

Now, consider the identity element \( e \) of \( \Gamma \). Let \( \gamma \in \Gamma \) and denote \( A = \{a \in \Gamma \mid supp h(\gamma, a) \cap supp h(e, a) = \emptyset\} \). We have, by 3. Lemma 3.9, when \( d(\gamma, e) \) is large enough:
\[ \|c(\gamma, e)\|^p_{\ell^p(E_{10}\delta)} = \sum_{a \in \Gamma} \|h(\gamma, a) - h(e, a)\|^p, \]
\[ \geq \sum_{a \in A} \|h(\gamma, a) - h(e, a)\|^p = \sum_{a \in A} 2^p, \text{ since } \|h(x, a)\|^p = 1 \]
\[ \|c(\gamma, e)\|^p_{\ell^p(E_{10}\delta)} \geq 2^p(d(\gamma, e) - K). \]
And hence, when \( \gamma \to \infty \) in \( \Gamma \), we have : \( \|c(\gamma, e)\|^p_{\ell^p(E_{10}\delta)} \geq 2^p(d(\gamma, e) - K) \to +\infty. \]

### 3.2 Link with isometric affine actions on Banach spaces

In this section, we aim to prove the two statements of Theorem 1 which gives an analog of the equivalence between proper actions on spaces with measured walls and Haagerup property in terms of proper actions on spaces with labelled partitions and isometric affine actions on Banach spaces; and more particularly in the case of \( L^p \) spaces, using Hardin’s result about extension of isometries on closed subspaces of \( L^p \) spaces.

**Proof of Corollary 1.1.** The direct implication follows immediately from 1) Theorem 1.

Now, assume \( G \) acts properly continuously by automorphisms on a space \((X, \mathcal{P}, F(\mathcal{P}))\) and \( T : F(\mathcal{P}) \hookrightarrow L^p(X, \mu) \) is a linear isometric embedding.

By 2) Theorem 1, there is a proper continuous isometric affine action \( \alpha \) of \( G \) on a closed subspace \( B \) of \( F(\mathcal{P}) \) with \( \alpha(g) = \pi(g) + b(g) \). Thus, as \( T \) is a linear isometry, \( T(B) \) is a closed subspace of \( L^p(X, \mu) \) and \( \alpha' \) such that \( \alpha'(g) = T \circ \pi(g) \circ T^{-1} + T(b(g)) \) is a continuous isometric affine action of \( G \) on \( T(B) \). Then, by Corollary 2.10, \( G \) has property \( PL^p \).

**3.2.1 Labelled partitions associated with an isometric affine action**

In this part, we introduce the space with labelled partitions associated with a continuous isometric affine action of a topological group \( G \) and we give a proof of 1) Theorem 1 by defining an action of \( G \) by automorphisms on this structure.

Let us fix some notations and gather some facts about continuous isometric affine actions on Banach spaces :

Let \( E \) be a Banach space. We denote by \( E' \) its topological dual, by \( S_{E'} \) the unit ball of \( E' \) (with respect to the operator norm on \( E' \)) and by \( E'_{\text{aff}} \) the canonical structure of affine space on \( E' \) (elements of \( E'_{\text{aff}} \) are of the form \( f = f_0 + a \) where \( f_0 \in E' \) and \( a \in \mathbb{K} \)).
Lemma 3.22. Let $\alpha$ be a continuous isometric affine action of a topological group $G$ on a Banach space $E$. Then $\alpha$ induces an action of $G$ on $E'_\text{aff}$ where, for $g \in G$, $f \in E'_\text{aff}$ and $x \in E$,

$$\alpha(g)f(x) := f(\alpha(g^{-1})x).$$

Proof. Let $\alpha$ be a continuous isometric affine action of a topological group $G$ on $E$ with linear part $\pi$ and translation part $b$. Then $\pi$ induces an isometric representation of $G$ on $E'$ endowed with the operator norm where, for $g \in G$, $f \in E'$ and $x \in E$, $\pi(g)f(x) := f(\pi(g^{-1})x)$. In fact, since $\pi$ is an isometry of $E$, we have, for $g \in G$, $f_0 \in E'$:

$$\|\pi(g)f_0\|_{E'} = \|f_0\|_{E'}$$

and hence, $\pi(g)f_0 \in E'$ and $\pi$ is an isometry of $E'$. Then, for $g \in G$ and $f = f_0 + a \in E'_\text{aff}$, we have:

$$\alpha(g)f = \pi(g)f_0 + a + f(b(g^{-1})) \in E'_\text{aff}.$$ 

And since $\alpha$ is an action of $G$ on $E$, it follows that the formula : $\alpha(g)f(x) := f(\alpha(g^{-1})x)$ clearly defines an action of $G$ on $E'_\text{aff}$. \qed

Here is a description of the family of labelling functions used to build a structure of space with labelled partitions on $G$:

Definition 3.23. Let $G$ be a topological group and $\alpha$ be a continuous isometric affine action of $G$ on a Banach space $B$ with translation part $b$.
Consider the closure $E = \overline{\text{Vect}(b(G))}$ of the linear span of the image of $b$ in $B$.
For $f \in E'_\text{aff}$, we denote $p_f : G \to \mathbb{K}$ the labelling function such that, for $g \in G$:

$$p_f(g) = f(b(g)),$$

and we call the set

$$\mathcal{P}_\alpha = \{p_f \mid f \in \alpha(G)S_{E'}\}$$

the family of labelling functions on $G$ associated with $\alpha$ where $\alpha(G)S_{E'}$ is the $\alpha$-orbit in $E'_\text{aff}$ of the unit sphere $S_{E'}$ of $E'$.

Given a continuous isometric affine action of a topological group $G$ on a Banach space $B$, we build a structure of space labelled partitions on $G$ itself where the Banach space associated with this structure is isometrically isomorph to a Banach subspace of $B$. 

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Proposition 3.24. Let $G$ be a topological group, $\alpha$ be a continuous isometric affine action of $G$ on a Banach space $B$ with translation part $b$ and $P_\alpha$ be the family of labelling functions on $G$ associated with $\alpha$ with separation map denoted by $c_\alpha$.

There exists a Banach space $F_\alpha(P_\alpha)$ of $K$-valued functions on $P_\alpha$, isometrically isomorphic to a Banach subspace of $B$ such that $(G, P_\alpha, F_\alpha(P_\alpha))$ is a space with labelled partitions.

More precisely, there exists a linear isometry $T : F_\alpha(P_\alpha) \to E$ where $E = \overline{\text{Vect}(b(G))} < B$ such that, for all $g, h \in G$,

$$T(c_\alpha(g, h)) = b(g) - b(h).$$

Proof. Assume $\alpha$ is a continuous isometric affine action of $G$ on a Banach space $B$. Denote by $\pi$ the linear part of $\alpha$, and by $b$ its translation part.

Let $E = \overline{\text{Vect}(b(G))}$ be the closure in $B$ of the linear span of the image of $G$ by the map $b$. Since we have, for all $g, h \in G$, $\alpha(g)b(h) = b(gh)$, $E$ is an $\alpha$-invariant subspace of $B$ and hence, $\alpha$ is a continuous isometric affine action of $G$ on $E$. Consider the family of labelling functions on $G$ associated with $\alpha$,

$$P_\alpha = \{p_f | f \in \alpha(G)S_{E'}\},$$

and denote by $c_\alpha$ the separation map of $G$ associated with $P_\alpha$. We have, for $g, h \in G$ and $p_f \in P_\alpha$ where $f = f_0 + a \in E_{\text{aff}}$ :

$$(*) \quad c_\alpha(g, h)(p_f) = p_f(g) - p_f(h) = f(b(g)) - f(b(h)) = f_0(b(g) - b(h)).$$

Then $T : \text{Vect}(c_\alpha(g, h) | g, h \in G) \to E$ defined linearly by $T(c_\alpha(g, h)) = b(g) - b(h)$ is a linear embedding. In fact, $T$ is well defined and injective since if, for all $f_0 \in S_{E'}$, $f_0(\sum \lambda_i(b(g_i) - b(h_i))) = 0$ then, for all $f \in \alpha(G)S_{E'}$, $\sum \lambda_i c_\alpha(g_i, h_i)(p_f) = 0$.

For $\xi \in \text{Vect}(c_\alpha(g, h) | g, h \in G)$, we set

$$\|\xi\|_{P_\alpha} = \sup_{p_f \in P_\alpha} |\xi(p_f)|.$$ 

Then we have, for $\xi = \sum_{i=1}^n \lambda_i c_\alpha(g_i, h_i) \in \text{Vect}(c(g, h) | g, h \in G)$:
\[ \|\xi\|_{P_\alpha} = \sup_{p_f \in P_\alpha} \left| \sum_{i=1}^{n} \lambda_i c_\alpha(g_i, h_i)(p_f) \right|, \]

\[ = \sup_{f = f_0 + a \in c_\alpha(G) S \varphi'} \left| \sum_{i=1}^{n} \lambda_i f_0(b(g_i) - b(h_i)) \right|, \text{ by (\star)} \]

\[ = \sup_{f_0 \in S \varphi'} \left| f_0 \left( \sum_{i=1}^{n} \lambda_i (b(g_i) - b(h_i)) \right) \right|, \]

\[ \|\xi\|_{P_\alpha} = \left\| \sum_{i=1}^{n} \lambda_i (b(g_i) - b(h_i)) \right\|_B, \text{ by Hahn-Banach Theorem.} \]

It follows that \(\|\cdot\|_{P_\alpha}\) is a norm on \(\text{Vect}(c_\alpha(g, h) \mid g, h \in G)\) and \(T\) is an isometric embedding from \((\text{Vect}(c_\alpha(g, h) \mid g, h \in G), \|\cdot\|_{P_\alpha})\) to \((E, \|\cdot\|_B)\). Notice that, in particular, we have, for \(g, h \in G\),

\[ \|c_\alpha(g, h)\|_{P_\alpha} = \|b(g) - b(h)\|_B. \]

Then by setting:

\[ F_\alpha(P_\alpha) = \overline{\text{Vect}(c_\alpha(g, h) \mid g, h \in G)}^{\|\cdot\|_{P_\alpha}}, \]

the map \(T\) clearly extends to an isometric isomorphism from \(F_\alpha(P_\alpha)\) to \(E\) and since, for all \(g, h \in G\), \(c_\alpha(g, h)\) belongs to \(F_\alpha(P_\alpha)\), \((G, P_\alpha, F_\alpha(P_\alpha))\) is a space with labelled partitions.

**Definition 3.25.** Let \(G\) be a topological group, \(\alpha\) be a continuous isometric affine action of \(G\) on a Banach space \(B\) with translation part \(b\) and \(P_\alpha\) be the family of labelling functions on \(G\) associated with \(\alpha\) with separation map denoted by \(c_\alpha\).

The space with labelled partitions \((G, P_\alpha, F_\alpha(P_\alpha))\) is said to be the space with labelled partitions associated with \(\alpha\) where:

\[ F_\alpha(P_\alpha) = \overline{\text{Vect}(c_\alpha(g, h) \mid g, h \in G)}^{\|\cdot\|_{P_\alpha}}, \]

with, for \(\xi \in \text{Vect}(c_\alpha(g, h) \mid g, h \in G)\), \(\|\xi\|_{P_\alpha} = \sup_{p_f \in P_\alpha} |\xi(p_f)|\).

**Remark 3.26.** If the continuous isometric affine action \(\alpha\) is linear i.e. \(b(G) = \{0\}\), then the space \((G, P_\alpha, F_\alpha(P_\alpha))\) with labelled partitions associated with \(\alpha\) is degenerated in the sense that the quotient metric space associated with \((G, d)\) contains a single point, \(P_\alpha\) contains only the zero function from \(G\) to \(\mathbb{K}\) and \(F_\alpha(P_\alpha) = \{0\}\).

**Proposition 3.27.** Let \(G\) be a topological group, \(\alpha\) be a continuous isometric affine action of \(G\) on a Banach space \(B\) and \((G, P_\alpha, F_\alpha(P_\alpha))\) be the space with labelled partitions
associated with $\alpha$.

Then the action of $G$ on itself by left-translation induces a continuous action of $G$ by automorphisms on $(G, \mathcal{P}_\alpha, F_\alpha(\mathcal{P}_\alpha))$.

**Proof.** Denote by $\tau$ the action of $G$ on itself by left-translation and set $E = \text{Vect}(b(G)) < B$.

Let $p_f \in \mathcal{P}_\alpha$ and $g \in G$. We have, for all $\gamma \in G$,

$$\Phi_g(p_f) := p_f \circ \tau(g)(\gamma) = f(b(g\gamma)) = f(\alpha(g)b(\gamma)) = \alpha(g^{-1})f(b(\gamma)) = p_{\alpha(g^{-1})f}(\gamma).$$

Then $p_f \circ \tau(g) = p_{\alpha(g^{-1})f} \in \mathcal{P}_\alpha$ since $\alpha(g^{-1})f$ belongs to $\alpha(G)S_E$.

Now, for all $g \in G$ and for all $\xi \in \text{Vect}(c(g, h) \mid g, h \in G)$ with $\xi = \sum_{i=1}^n \lambda_i c_\alpha(g_i, h_i)$, we have:

$$\xi \circ \Phi_g = \sum_{i=1}^n \lambda_i c_\alpha(gg_i, gh_i) \in F_\alpha(\mathcal{P}_\alpha).$$

Consider the isometry $T : F_\alpha(\mathcal{P}_\alpha) \rightarrow E$ given by Proposition 3.24 such that for $\gamma_1, \gamma_2 \in G$, $T(c(\gamma_1, \gamma_2)) = b(\gamma_1) - b(\gamma_2)$. Hence, we have, for $g \in G$ and for $\xi \in \text{Vect}(c_\alpha(g, h) \mid g, h \in G)$:

$$\|\xi \circ \Phi_g\|_{\mathcal{P}_\alpha} = \|T(\xi \circ \Phi_g)\|_B,$$

$$= \|\sum_{i=1}^n \lambda_i (b(gg_i) - b(gh_i))\|_B,$$

$$= \|\sum_{i=1}^n \lambda_i \pi(g)(b(g_i) - b(h_i))\|_B,$$

$$= \|\pi(g) \sum_{i=1}^n \lambda_i (b(g_i) - b(h_i))\|_B,$$

$$= \|\sum_{i=1}^n \lambda_i (b(g_i) - b(h_i))\|_B,$$

$$= \|T(\xi)\|_B,$$

$$\|\xi \circ \Phi_g\|_{\mathcal{P}_\alpha} = \|\xi\|_{\mathcal{P}_\alpha};$$

and hence, by density of $\text{Vect}(c_\alpha(g, h) \mid g, h \in G)$ in $F_\alpha(\mathcal{P}_\alpha)$, we have, for all $g \in G$ and all $\xi \in F_\alpha(\mathcal{P}_\alpha)$:

$$\|\xi \circ \Phi_g\|_{\mathcal{P}_\alpha} = \|\xi\|_{\mathcal{P}_\alpha}.$$

It follows that the action of $G$ on itself by left-translation induces an action of $G$ by automorphisms on $(G, \mathcal{P}_\alpha, F_\alpha(\mathcal{P}_\alpha))$.

It remains to prove the $G$-action on $(G, d)$ is strongly continuous. To avoid any ambiguities, by $G$ we will understand the topological group $G$ and by $X = G$ we will understand
the set $G$ equipped with the topology induced by the pseudo-metric of labelled partitions $d(x, y) := \|c_\alpha(x, y)\|_{P_\alpha}$.

Let us show that, for all $x \in X$, the map $\tau_x : G \to X$ such that $g \mapsto \tau(g)x$ is continuous.

Since $b$ is continuous from $G$ to $E$ and $\pi$ is strongly continuous from $G$ to $E$, for all $x, y \in X$ the map $c'_{x, y} : G \times G \to E$ such that

$$(g, h) \mapsto b(gx) - b(hy) = \pi(g)b(x) + b(g) + \pi(h)b(y) + b(h)$$

is continuous.

Hence, $c_{x, y} : (g, h) \mapsto c_\alpha(gx, hy)$ is continuous from $G \times G \to E$ since $c_{x, y} = T^{-1} \circ c'_{x, y}$ and $T^{-1}$ is continuous from $E$ to $F(P)$ as an isometric isomorphism.

Let $x \in X$. Denote

$$\phi_y : g \mapsto \|c_\alpha(\tau_x(g), y)\|_{P_\alpha} = \|c_\alpha(gx, y)\|_{P_\alpha}.$$

We have $\phi_y(g) = \|c_{x, y}(g, e_G)\|_{P_\alpha}$ and then $\phi_y$ is continuous from $G$ to $\mathbb{R}_+$ for all $y \in X$.

Now, for every open ball $B(y, r)$ of $X$,

$$\tau_x^{-1}(B(y, r)) = \{g \in G \mid \|c_\alpha(\tau_x(g), y)\|_{P_\alpha} < r\} = \phi_y^{-1}([0, r[),$$

and $\phi_y^{-1}([0, r[)$ is an open subset of $G$ since $\phi_y$ is continuous.

It follows that, for all $x \in X$, $\tau_x : g \mapsto gx$ is continuous.

Proof of 1) Theorem 1. Assume $\alpha$ is a proper continuous isometric affine action of $G$ on a Banach space $B$ with translation part $b$ and denote $E = \overline{\text{Vect}(b(G))} < B$.

By Proposition 3.27, the $G$-action by left-translation on itself induces a continuous action by automorphisms on the space with labelled partitions associated with $\alpha$, $(G, P_\alpha, F_\alpha(P_\alpha))$. Let $T$ be the isometry from $F_\alpha(P_\alpha)$ to $E$ given by Proposition 3.24. Then we have, for all $g \in G$, $\|c_\alpha(g, e)\|_{P_\alpha} = \|T(c_\alpha(g, e))\| = \|b(g)\|$ and hence,

$$\lim_{g \to \infty} \|c_\alpha(g, e)\|_{P_\alpha} = \lim_{g \to \infty} \|b(g)\| = +\infty.$$

It follows that $G$ acts properly continuously by automorphisms on $(G, P_\alpha, F_\alpha(P_\alpha))$. 

\qed
3.2.2 From actions on a space with labelled partitions to isometric affine actions

We prove here statement 2) of Theorem 1 by giving a (non-canonical) way to build a proper continuous isometric affine action on a Banach space given a proper continuous action by automorphisms on space with labelled partitions.

**Lemma 3.28.** Let \( G \) be a topological group, \( (X, \mathcal{P}, F(\mathcal{P})) \) be a space with labelled partitions and denote \( E = \text{Vect}(c(x,y) \mid x, y \in X) \) where \( c \) is the separation map associated with \( \mathcal{P} \).

If \( G \) acts continuously by automorphisms on \( (X, \mathcal{P}, F(\mathcal{P})) \), then, for all \( x, y \in X \), \((g, h) \mapsto c(gx, hy)\) is continuous from \( G \times G \) to \( E \).

*Proof.* Consider on the subspace \( E \) of \( F(\mathcal{P}) \) the topology given by the norm \( \|\cdot\| \) of \( F(\mathcal{P}) \). If \( X \times X \) is endowed with the product topology of \( (X, d) \), as said in Remark 3.6, \( c : X \times X \to E \) is continuous and, since the \( G \)-action on \( X \) is strongly continuous, for all \( x, y \in X \), \((g, h) \mapsto (gx, hy)\) is continuous. Then, by composition, for all \( x, y \in X \), \((g, h) \mapsto c(gx, hy)\) is continuous. \(\Box\)

**Proposition 3.29.** Let \( G \) be a topological group acting continuously by automorphisms on a space with labelled partitions \( (X, \mathcal{P}, F(\mathcal{P})) \). Then there exists a continuous isometric affine action of \( G \) on a Banach subspace \( B \) of \( F(\mathcal{P}) \).

More precisely, \( B = \overline{\text{Vect}(c(x,y) \mid x, y \in X)} \) where \( c \) is the separation map associated with \( \mathcal{P} \) and \( \|\cdot\| \) is the norm of \( F(\mathcal{P}) \), and moreover, the linear part \( \pi \) and the translation part \( b \) of the affine action are given by, for a fixed \( x_0 \in X \):

\[
\pi(g)\xi = \xi \circ \Phi_\tau(g) \quad \text{for } g \in G \text{ and } \xi \in B;
\]

and

\[
b(g) = c(gx_0, x_0) \quad \text{for } g \in G.
\]

*Proof.* Let \( \tau \) be the \( G \)-action on \( X \).

By Definition 3.13 and Remark 3.16, the map \( \Phi_\tau(g) : \mathcal{P} \to \mathcal{P} \) such that \( \Phi_\tau(g)(p) = p \circ \tau(g) \) induces a linear representation \( \pi \) of \( G \) on \( F(\mathcal{P}) \) given by, for \( \xi \in F(\mathcal{P}) \) and \( g \in G \):

\[
\pi(g)\xi = \xi \circ \Phi_\tau(g).
\]

By the second requirement of Definition 3.13, we have \( \|\pi(g)\xi\| = \|\xi\| \). Thus, \( \pi \) is an isometric linear representation of \( G \) on \( F(\mathcal{P}) \).
Consider $E = \text{Vect}(c(x, y) \mid x, y \in X)$. Then the Banach subspace $B = \overline{E}^{\|\cdot\|}$ of $F(\mathcal{P})$ is stable under $\pi$ since $\pi(g)(c(x, y)) = c(gx, gy)$ for $x, y \in X$, $g \in G$. Let us show that the representation $\pi$ of $G$ on $B$ is strongly continuous. Let $\xi = \sum_{i=1}^{n} \lambda_i c(x_i, y_i) \in E$. We have, for $g \in G$,

$$\pi(g)\xi = \xi \circ \Phi_{\pi(g)} = \sum_{i=1}^{n} \lambda_i c(gx_i, gy_i) \in E,$$

and, by Lemma 3.28, for every $i$, $g \mapsto c(gx_i, gy_i)$ is continuous.

Hence, $g \mapsto \sum_{i=1}^{n} \lambda_i c(gx_i, gy_i) = \pi(g)\xi$ is continuous. Finally, by density, for all $\xi \in B$, $g \mapsto \pi(g)\xi$ is continuous from $G$ to $B$.

Now, let us define the translation part of the action. Fix $x_0 \in X$ and set, for all $g \in G$, $b(g) = c(gx_0, x_0) \in E$. We claim $b$ is a continuous 1-cocycle relative to $\pi$; indeed, we have, for $g \in G$, $x, y \in X$, $c(gx, gy) = c(x, y) \circ \Phi_{\pi(g)} = \pi(g)c(x, y)$ and then, for $g, h \in G$,

$$b(gh) = c(ghx_0, x_0) = c(ghx_0, gx_0) + c(gx_0, x_0) = \pi(g)b(h) + b(g).$$

The continuity of $b$ follows immediately from Lemma 3.28.

Hence, the morphism $\alpha : G \to \text{Isom}(B) \cap \text{Aff}(B)$ defined by, for all $g \in G$, $\xi \in B$, $\alpha(g)\xi = \pi(g)\xi + b(g)$ is a continuous isometric affine action of $G$ on $B$. \hfill \Box

**Remark 3.30.** In the case where $G$ is discrete, we do not have to find a subspace of $F(\mathcal{P})$ on which the representation is strongly continuous; then we have the following statement: If $G$ discrete acts by automorphisms on $(X, \mathcal{P}, F(\mathcal{P}))$, then there exists an isometric affine action of $G$ on $F(\mathcal{P})$.

**Proof of 2) Theorem 1.** Assume $G$ acts properly continuously on a space with labelled partitions $(X, \mathcal{P}, F(\mathcal{P}))$.

Consider the action $\alpha$ on the Banach subspace $B = \overline{E}^{\|\cdot\|}$ given by prop 3.29, where $E = \text{Vect}(c(x, y) \mid x, y \in X)$ and $\alpha(g)\xi = \pi(g)\xi + b(g)$, for $g \in G$, $\xi \in B$.

Then we have, if we denote by $d$ the pseudo-metric of labelled partitions on $X$ :

$$\|b(g)\| = \|c(gx_0, x_0)\|_\mathcal{P} = d(gx_0, x_0) \xrightarrow{g \to \infty} \infty$$

since the action of $G$ on $(X, \mathcal{P}, F(\mathcal{P}))$ is proper, and hence, $\alpha$ is a proper continuous isometric affine action of $G$ on $B$. \hfill \Box
4 Labelled partitions on a direct sum

In this section, we define a space with labelled partitions on the direct sum of a countable family of spaces with labelled partitions and we build on it a proper action given by proper actions on each factor.

4.1 Natural space with labelled partitions on a direct sum

Given a family of space with labelled partitions, we give a natural construction of a space with labelled partitions on the direct sum of this family. A similar construction in the case of spaces with walls can be found in [CMV04].

Definition 4.1. Let \( I \) be an index set, \( (X_i)_{i \in I} \) be a family of none empty sets and fix \( x_0 = (x_0^i)_{i \in I} \in \prod_{i \in I} X_i \).

The direct sum of the family \((X_i)_{i \in I}\) relative to \(x_0\) is defined by :

\[
\bigoplus_{i \in I} X_i := \left\{ (x_i)_{i \in I} \in \prod_{i \in I} X_i \mid x_i \neq x_0^i \text{ for finitely many } i \in I \right\}.
\]

For \( i \in I \), we denote by \( \pi^X_{X_i} : X \to X_i \) the canonical projection from the direct sum to the factor \( X_i \).

For \( x = (x_i)_{i \in I} \in \bigoplus_{i \in I} X_i \), the support of \( x \) is the finite subset of \( I \) :

\[
\text{supp}(x) = \{ i \in I \mid x_i \neq x_0^i \}.
\]

Definition 4.2. Let \( I \) be an index set, \((\langle (X_i, \mathcal{P}_i, F_i(\mathcal{P}_i)) \rangle)_{i \in I}\) be a family of spaces with labelled partitions and fix \( x_0 = (x_0^i)_{i \in I} \in \prod_{i \in I} X_i \). Denote \( X = \bigoplus_{i \in I} X_i \).

Let \( i \in I \). For \( p_i \in \mathcal{P}_i \), we define the labelling function \( p_i^{\oplus_i} : X \to \mathbb{K} \) by :

\[
p_i^{\oplus_i} = p_i \circ \pi^X_{X_i}.
\]

i.e., for \( x = (x_i)_{i \in I} \in X \), \( p_i^{\oplus_i}(x) = p_i(x_i) \).

We denote \( \mathcal{P}_i^{\oplus_i} = \{ p_i^{\oplus_i} \mid p_i \in \mathcal{P}_i \} \), and we call the set

\[
\mathcal{P}_X = \bigcup_{i \in I} \mathcal{P}_i^{\oplus_i}.
\]
the natural family of labelling functions on $X$ (associated with the family $(\mathcal{P}_i)_{i \in I}$).

Let $X_1, X_2$ be non empty sets and $\mathcal{P}_1, \mathcal{P}_2$ be families of labelling functions on, respectively, $X_1$ and $X_2$.

In terms of partitions, if $P_1$ is the partition of $X_1$ associated with $p_1 \in \mathcal{P}_1$, the partition $P_{1^1}$ of $X_1 \times X_2$ associated with $p_{1^1}$ is:

$$P_{1^1} = \{ h \times X_2 \mid h_1 \in P_1 \},$$

and similarly, for $p_2 \in \mathcal{P}_2$, we have:

$$P_{2^2} = \{ X_1 \times k \mid k_1 \in P_2 \}.$$

Partitions for the direct product

**Definition 4.3.** Let $I$ be a countable index set, $((X_i, \mathcal{P}_i, F_i(\mathcal{P}_i)))_{i \in I}$ be a family of spaces with labelled partitions and fix $x_0 = (x_i^0)_{i \in I} \in \prod_{i \in I} X_i$. Denote $X = x_0 \bigoplus_{i \in I} X_i$.

Let $i \in I$. For $\xi_i \in F_i(\mathcal{P}_i)$, we denote $\xi_i^\oplus : \mathcal{P}_X \to \mathbb{K}$ the function :

$$\xi_i^\oplus(p) = \begin{cases} 
\xi_i(p_i) & \text{if } p = p_i^\oplus \in \mathcal{P}_i^\oplus \\
0 & \text{if } p = p_j^\oplus \in \mathcal{P}_j^\oplus \text{ with } i \neq j
\end{cases}$$
Let $q \geq 1$. We denote $F_q(\mathcal{P}_X)$ the closure of 

$$E_q(\mathcal{P}_X) := \left\{ \sum_{i \in I} \xi_i \ | \ \xi_i \in F_i(\mathcal{P}_i) \text{ with } \xi_i \neq 0 \text{ for a finite number of } i \in I \right\},$$

endowed with the norm $\| \cdot \|_{N_q}$ defined by, for $\xi = \sum_{i \in I} \xi_i$ :

$$\|\xi\|_{N_q} := \left( \sum_{i \in I} \|\xi_i\|_{P_i(\mathcal{P}_i)}^q \right)^{\frac{1}{q}}.$$

The vector space $F_q(\mathcal{P}_X)$ is called the $q$-space of functions on $\mathcal{P}_X$ of $X$.

**Proposition 4.4.** Let $I$ be a countable index set and $((X_i, \mathcal{P}_i, F_i(\mathcal{P}_i)))_{i \in I}$ be a family of spaces with labelled partitions and fix $x_0 = (x_i^0)_{i \in I} \in \prod_{i \in I} X_i$. Denote $X = x_0 \bigoplus_{i \in I} X_i$. Then $(F_q(\mathcal{P}_X), \| \cdot \|_{N_q})$ is isometrically isomorph to $(\bigoplus_{i \in I}^q F_i(\mathcal{P}_i), \| \cdot \|_q)$. In particular, $F_q(\mathcal{P}_X)$ is a Banach space.

**Proposition 4.5** (Labelled partitions structure on a direct sum).

Let $I$ be a countable set, $((X_i, \mathcal{P}_i, F_i(\mathcal{P}_i)))_{i \in I}$ be a family of spaces with labelled partitions. Fix $x_0 = (x_i^0)_{i \in I} \in \prod_{i \in I} X_i$ and consider $X = x_0 \bigoplus_{i \in I} X_i$ together with its natural family of labelling functions $\mathcal{P}_X$.

Let $q \geq 1$ and $F_q(\mathcal{P}_X)$ be the $q$-space of functions on $\mathcal{P}_X$ of $X$. Then, the triple $(X, \mathcal{P}_X, F_q(\mathcal{P}_X))$ is a space with labelled partitions.

**Proof.** For $i \in I$, we denote by $c_i$ the separation map of $X_i$ associated with $\mathcal{P}_i$ and by $c_X$ the separation map associated with $\mathcal{P}_X$.

Let $x = (x_i)_{i \in I}, y = (y_i)_{i \in I} \in X$. For $p_i^{\oplus_i} \in \mathcal{P}_i^{\oplus_i}$, we have :

$$c_X(x, y)(p_i^{\oplus_i}) = p_i^{\oplus_i}(x) - p_i^{\oplus_i}(y) = p_i(x_i) - p_i(y_i) = c_i(x_i, y_i)(p_i).$$

It follows that $c_X(x, y) = \sum_{x_i \neq y_i} c_i(x_i, y_i)^{\oplus_i}$ which is a finite sum since $x, y$ belong to $X$. Thus, $c_X(x, y)$ belongs to $F_q(\mathcal{P}_X)$ and hence, $(X, \mathcal{P}_X, F_q(\mathcal{P}_X))$ is a space with labelled partitions.

**Definition 4.6.** Let $I$ be a countable set, $((X_i, \mathcal{P}_i, F_i(\mathcal{P}_i)))_{i \in I}$ be a family of spaces with labelled partitions and let $q \geq 1$. Consider $X = x_0 \bigoplus_{i \in I} X_i$ together with its natural family
of labelling functions $\mathcal{P}_X$ and let $F_q(\mathcal{P}_X)$ be the $q$-space of functions on $\mathcal{P}_X$ of $X$.

The triple $(X, \mathcal{P}_X, F_q(\mathcal{P}_X))$ is called the natural space with labelled partitions on the direct sum $X$.

### 4.2 Action on the natural space with labelled partitions of the direct sum

Let $I$ be an index set and $(H_i)_{i \in I}$ be a family of groups. Denote $e_W = (e_{H_i})_{i \in I}$ where, for $i \in I$, $e_{H_i}$ is the identity element of $H_i$.

We simply denote $\bigoplus_{i \in I} H_i$ the group $W = \bigoplus_{i \in I} H_i$ whose identity element is $e_W$.

**Proposition 4.7.**

Let $I$ be a countable set and $(H_i)_{i \in I}$ be a family of groups such that, for each $i \in I$, $H_i$ acts by automorphisms on a space with labelled partitions $(X_i, \mathcal{P}_i, F_i(\mathcal{P}_i))$. Denote $X = \bigoplus_{i \in I} X_i$ and $W = \bigoplus_{i \in I} H_i$.

Let $q \geq 1$. Then $W$ acts by automorphisms on the natural space with labelled partitions on the direct sum $(X, \mathcal{P}_X, F_q(\mathcal{P}_X))$ via the natural action of $W$ on $X$.

**Proof.** Denote by $\tau$ the $W$-action on $X$ and for $w \in W$, $p \in \mathcal{P}_X$, $\Phi_{\tau(w)}(p) := p \circ \tau(w)$ and, for $i \in I$, denote by $\tau_i$ the $H_i$-action on $X$ and for $h_i \in H_i$, $p_i \in \mathcal{P}_i$, $\Phi_{\tau_i(h_i)}(p_i) := p_i \circ \tau_i(h_i)$.

Let $p \in \mathcal{P}_X = \bigcup_{i \in I} \mathcal{P}_i^{\oplus i}$ and $w = (h_i)_{i \in I} \in W$. Then there exists $i \in I$ and $p_i \in \mathcal{P}_i$ such that $p = p_i^{\oplus i}$, and we have:

$$\Phi_{\tau(w)}(p^{\oplus i}) = (\Phi_{\tau_i(h_i)}(p_i))^{\oplus i} \in \mathcal{P}_i^{\oplus i} \subset \mathcal{P}_X,$$

since $\Phi_{\tau_i(h_i)}(p_i)$ belongs to $\mathcal{P}_i$.

For $\xi = \sum_{i \in I} s_i^{\oplus i} \in E_q(\mathcal{P}_X)$, we have:

$$\xi \circ \Phi_{\tau(w)}(p) = \xi((p_i^{\oplus i} \circ \tau(w))$$

$$= \xi((p_i \circ \tau_i(h_i))^{\oplus i})$$

$$= \xi_i^{\oplus i}((p_i \circ \tau_i(h_i))^{\oplus i})$$

$$= \xi_i(p_i \circ \tau_i(h_i))$$

$$= \xi_i \circ \Phi_{\tau_i(h_i)}(p_i)$$

$$\xi \circ \Phi_{\tau(w)}(p) = (\xi_i \circ \Phi_{\tau_i(h_i)})^{\oplus i}(p_i^{\oplus i}).$$
And hence,
\[ \xi \circ \Phi_{\tau(w)} = \sum_{i \in I} (\xi_i \circ \Phi_{\tau_i(h_i)})^{\oplus_i} \in F_q(P_X). \]

By completeness of \( F_q(P_X) \), for all \( \xi \in F_q(P_X), \xi \circ \Phi_{\tau(w)} \in F_q(P_X). \)
Moreover, for \( \xi = \sum_{i \in I} \xi_i^{\oplus_i} \in E_q(P_X) \), we have:
\[
\|\xi \circ \Phi_{\tau(w)}\|_{N_q}^q = \sum_{i \in I} \|\xi_i \circ \Phi_{\tau_i(h_i)}\|_{F_i(P_i)}^q = \sum_{i \in I} \|\xi_i\|_{F_i(P_i)}^q = \|\xi\|_{N_q}^q,
\]
since, for all \( i \in I \), \( \|\xi_i \circ \Phi_{\tau_i(h_i)}\|_{F_i(P_i)} = \|\xi_i\|_{F_i(P_i)} \).
Thus, by density of \( E_q(P_X) \) in \( F_q(P_X) \), for all \( \xi \in F_q(P_X), \|\xi \circ \Phi_{\tau(w)}\|_{N_q} = \|\xi\|_{N_q}. \)
It follows that \( W \) acts by automorphisms on \( (X, P_X, F_q(P_X)) \).

\[ \square \]

When \( I \) is finite, \( X = \bigoplus_{i \in I} X_i \) is simply the direct sum of the \( X_i \) and does not depend on \( x_0 \). In this case, proper continuous actions on each factor \( (X_i, P_i, F_i(P_i)) \) induce a proper continuous action on the natural space with labelled partitions of the direct sum \( (X, P_X, F_q(P_X)) \):

**Proposition 4.8.** Let \( n \in \mathbb{N}^* \). For \( i \in I = \{1, ..., n\} \), let \( H_i \) be a topological group acting properly continuously on a space with labelled partitions \( (X_i, P_i, F_i(P_i)) \); denote \( X = X_1 \times ... \times X_n \) and \( W = H_1 \times ... \times H_n \).

Let \( q \geq 1 \). Then \( W \) acts properly continuously by automorphisms on the natural space with labelled partitions of the direct product \( (X, P_X, F_q(P_X)) \) via the natural action of \( W \) on \( X \).

**Proof.** We denote by \( c \) the separation map associated with \( P_X \) and, for \( i \in I \), \( c_i \) the separation map associated with \( P_i \). By Proposition 4.7, \( W \) acts by automorphisms on \( (X, P_X, F_q(P_X)) \) and since \( I \) is finite, notice that \( w = (h_i) \to \infty \) in \( W \) if, and only if, there exists \( j \in I \) such that \( h_j \to \infty \) in \( H_j \).

Thus, for any \( x = (x_1, ..., x_n) \in X \), we have, for \( w = (h_i) \in W \) and any \( j \in I \):
\[
\|c(wx, x)\|_{N_q}^q = \sum_{i=1}^n \|c(h_ix_i, x_i)\|_{F_i(P_i)}^q \geq \|c(h_jx_j, x_j)\|_{F_j(P_j)}^q.
\]
It follows that, when \( w \to \infty \) in \( W \), \( \|c(wx, x)\|_{N_q} \to +\infty \) and then \( W \) acts properly on \( (X, P_X, F_q(P_X)) \). It remains to prove that the \( W \)-action on \( (X, d) \) is strongly continuous.
where $W$ is endowed with the product topology of the $H_i$’s. Remark that $d = (\sum_{i=0}^{n} d_i^q)^{\frac{1}{q}}$, then, the topology of $(X, d)$ is equivalent to the product topology of the $X_i$’s on $X$.

Let $x = (x_i)_{i \in I} \in X$. Denote by $\tau_x: W \to X$ the function $w \mapsto wx$. For all $i \in I$, $\pi_i^X \circ \tau_x: w \to h_i x_i$ is continuous since $h_i \to h_i x_i$ is continuous; hence it follows that $\tau_x$ is continuous.

If $I$ is countably infinite, even if each $H_i$-action on $(X_i, \mathcal{P}_i, F(\mathcal{P}_i))$ is proper, $W$ does not act properly on the natural space with labelled partitions on the direct sum $(X, \mathcal{P}_X, F_q(\mathcal{P}_X))$ in general. In fact, let $C$ be a positive real constant, and assume there exists, in each $H_i$, an element $h_i$ such that $\|c_i(h_i x_i^0, x_i^0)\|_{F_i(\mathcal{P}_i)} \leq C$. For $j \in I$, the element $\delta_j(h_j)$ of $W$ such that $\pi^W_i(\delta_j(h_j)) = e_{H_i}$ if $i \neq j$ and $\pi^W_i(\delta_j(h_j)) = h_j$ leaves every finite set of $W$ when $j$ leaves every finite set of $I$, but :

$$\|c(\delta_j(h_j)x_0, x_0)\|_{F_q(\mathcal{P}_X)} = \|c_i((h_j)x_i^0, x_i^0)\|_{F_i(\mathcal{P}_i)} \leq C.$$  

And then, $W$ does not act properly on $(X, \mathcal{P}_X, F_q(\mathcal{P}_X))$.

To make $W$ act properly on a space with labelled partitions in the case where $W$ is endowed with the discrete topology, we have to define a structure of labelled partitions on $W$ such that the labelled partitions metric between $e_W$ and $w$ goes to infinity when the support of $w$ leaves every finite set in $I$. To build this structure, we scale every labelling function of the naive $\ell^q$ space with labelled partitions on each factor $H_i$ by a weight depending on $i$ which grows as $i$ leaves every finite set in $I$.

**Notation 4.9.** Let $I$ be a countable index set and $X = \bigoplus_{i \in I} X_i$ be a direct sum of sets $X_i$’s.

We say, for $x \in X$, $\text{supp}(x)$ leaves every finite set in $I$ or $\text{supp}(x) \to \infty$ in $I$ if there exists $j \in \text{supp}(x)$ which leaves every finite set in $I$.

**Proposition 4.10.** Let $I$ be a countable index set and $X = \bigoplus_{i \in I} X_i$ be a direct sum of countable sets $X_i$’s. Then, an element $x = (x_i)_{i \in I} \in X$ leaves every finite set in $X$ if either there exists $j \in I$ such that $x_j$ leaves every finite set in $X_j$ or $\text{supp}(x)$ leaves every finite set in $I$.

Subsequently, for a countably infinite set $I$, we consider a function $\phi: I \to \mathbb{R}_+$ such that $\phi(i) \xrightarrow{i \to \infty} +\infty$ (such a function always exists when $I$ is countably infinite: for instance, take any bijective enumeration function $\phi$ from $I$ to $\mathbb{N}$).
**Definition 4.11.** Let $I$ be a countably infinite set and $X$ be a countable set. Let $i \in I$. We set, for $x \in X$:

$$(\phi(i))_{x, i} := 2^{-\frac{1}{q}} \phi(i) \delta_x : \mathbb{K},$$

where $\delta_x : X \to \{0, 1\}$ is the Dirac function at $x$, and we call the set

$$(\phi(i))_\Delta := \{(\phi(i))_{x, i} | x_i \in X_i\},$$

the $\phi(i)$-weighted naive family of labelling functions on $X$.

**Proposition 4.12.** Let $I$ be a countably infinite set, $X$ be a countable set and $i \in I$. Let $q \geq 1$. Then the triple $(X, (\phi(i))_\Delta, \ell_q((\phi(i))_\Delta))$ is a space with labelled partitions. Moreover, if a discrete group $H$ acts on $X$, then $H$ acts by automorphisms on $(X, (\phi(i))_\Delta, \ell_q((\phi(i))_\Delta))$.

**Proof.** It is a straightforward generalization of Proposition 3.11 and Proposition 3.17. □

**Lemma 4.13.** Let $I$ be a countably infinite set and $(H_i)_{i \in I}$ be a family of countable discrete groups and denote $W$ the group $\bigoplus_{i \in I} H_i$ endowed with the discrete topology. Consider, on each $H_i$, the $\phi(i)$-weighted naive family of labelling functions $(\phi(i))_\Delta$ and denote by $(\phi)_\Delta = \bigcup_{i \in I} (\phi(i))_{\Delta}$ the natural set of labelling functions associated with $(\phi(i))_\Delta_{i \in I}$.

Let $q \geq 1$. Then, $W$ acts by automorphisms on the natural space with labelled partitions on the direct sum $(W, (\phi)_\Delta, F_q((\phi)_\Delta))$. Moreover, we have:

$$\|c_\phi(w, e_W)\|_{F_q((\phi)_\Delta)} \to +\infty \text{ when supp}(w) \to \infty \text{ in } I,$$

where $c_\phi$ is the separation map associated with $(\phi)_\Delta$.

**Proof.** By Proposition 4.7, $W$ acts by automorphisms on $(W, (\phi)_\Delta, F_q((\phi)_\Delta))$ and we have, for $w = (h_i), w' = (h'_i) \in W$:

$$\|c_\phi(w, w')\|_{F_q((\phi)_\Delta)}^q = \sum_{i \in I} \|c_{\phi(i)}(h_i, h'_i)\|_q^q = \sum_{i \in \text{supp}(w^{-1}w')} \phi(i)^q.$$

Let $w \in W$ such that $\text{supp}(w) \to \infty$ in $I$. Then there exists $j \in \text{supp}(w)$ such that $j \to \infty$.
in $I$ and hence:

$$\|c_\phi(w, e_W)\|_{F_q((\phi^\Delta))}^q = \sum_{i \in \text{supp}(w)} \phi(i)^q \geq \phi(j)^q \to +\infty.$$ 

\[\square\]

**Proposition 4.14.** Let $I$ be a countably infinite set and $(H_i)_{i \in I}$ be a family of countable discrete groups such that, for each $i \in I$, $H_i$ acts properly by automorphisms on a space with labelled partitions $(X_i, \mathcal{P}_i, F_i(\mathcal{P}_i))$. Denote $X = X_0 \bigoplus_{i \in I} X_i$ and $W = \bigoplus_{i \in I} H_i$ endowed with the discrete topology.

Let $q \geq 1$. Then there exists a structure of space with labelled partitions $(Y, \mathcal{P}_Y, F_q(\mathcal{P}_Y))$ on which $W$ acts properly by automorphisms.

More precisely, $(Y, \mathcal{P}_Y, F_q(\mathcal{P}_Y))$ is the natural space with labelled partitions on the direct product $Y = X \times W$ where:

- on $X$, we consider the natural space with labelled partitions on the direct sum of the family $((X_i, \mathcal{P}_i, F_i(\mathcal{P}_i)))_{i \in I}$;
- on $W$, we consider the natural space with labelled partitions on the direct sum of the family $((H_i, (\phi(i))^\Delta, \ell^q((\phi(i))^\Delta)))_{i \in I}$ where for $i \in I$, $(\phi(i))^\Delta$ is the $\phi(i)$-weighted naive family of labelling functions on $H_i$.

**Proof.** By Proposition 4.7, $W$ acts by automorphisms on both $(X, \mathcal{P}_X, F_q(\mathcal{P}_X))$ and $(W, (\phi^\Delta), \ell^q((\phi^\Delta)))$. We set $Y = X \times W$ and consider the natural space with labelled partitions $(Y, \mathcal{P}_Y, F_q(\mathcal{P}_Y))$ on the direct product where:

$$\mathcal{P} = \mathcal{P}_X \oplus (\phi^\Delta)^\oplus 2,$$

and

$$F_q(\mathcal{P}) \simeq F_q(\mathcal{P}_X) \oplus \ell^q((\phi^\Delta)).$$

Then, by Proposition 4.7, $W \times W$ acts by automorphisms on $(Y, \mathcal{P}_Y, F_q(\mathcal{P}_Y))$ via the action $(w_1, w_2)(x, w) = (w_1x, w_2w)$. Hence, $W$ acts by automorphisms on $(Y, \mathcal{P}_Y, F_q(\mathcal{P}_Y))$, where $W$ is viewed as the diagonal subgroup $\{(w, w) \mid w \in W\} < W \times W$.

It remains to prove that the $W$-action on $(Y, \mathcal{P}_Y, F_q(\mathcal{P}_Y))$ is proper. We have, for $w = (h_i) \in W$:
\[ \|c_{\mathcal{Y}}(w.(x_0,e_W),(x_0,e_W))\|_{F_q(\mathcal{Y})}^q = \|c_{\mathcal{X}}(w.x_0,x_0)\|_{F_q(\mathcal{X})}^q + \|c_\phi(w,e_W)\|_q^q \]
\[ = \sum_{i \in \text{supp}(w)} \|c(h_i,x_i^0,x_i^0)\|_{F_i(\mathcal{X})}^q + \sum_{i \in \text{supp}(w)} \phi(i)q. \]

If there exists \( j \in I \), such that \( h_j \to \infty \) in \( H_j \), then, since the \( H_j \)-action is proper, we have :
\[ \|c_{\mathcal{Y}}(w.(x_0,e_W),(x_0,e_W))\|_{F_q(\mathcal{Y})} \geq \sum_{i \in \text{supp}(w)} \|c(h_i,x_i^0,x_i^0)\|_{F_i(\mathcal{X})}^q \to +\infty, \]
and if \( \text{supp}(w) \to \infty \) in \( I \), by Lemma 4.13, we have :
\[ \|c_{\mathcal{Y}}(w.(x_0,e_W),(x_0,e_W))\|_{F_q(\mathcal{Y})} \geq \|c_\phi(w,e_W)\|_q \to +\infty. \]

Hence, by Proposition 4.10, we conclude that \( W \) acts properly on \((\mathcal{Y},\mathcal{P}_Y,F_q(\mathcal{P}_Y))\). \( \square \)

4.3 Action of a semi-direct product on a space with labelled partitions

Definition 4.15 (compatible action). Let \( G_1, G_2 \) be groups and \( \rho : G_2 \to \text{Aut}(G_1) \) be a morphism of groups.
Consider a set \( X \) on which \( G_1 \) acts by \( \alpha_1 \) and \( G_2 \) acts by \( \alpha_2 \). We say \( \alpha_2 \) is compatible with \( \alpha_1 \) with respect to \( \rho \) if, for all \( g_1 \in G_1, \gamma_2 \in G_2 \), we have :
\[ \alpha_2(g_2) \circ \alpha_1(g_1) = \alpha_1(\rho(g_2)g_1) \circ \alpha_2(g_2). \]

Example 4.16. If \( \rho : G_2 \to \text{Aut}(G_1) \) is a morphism, then the action \( \rho \) of \( G_2 \) on \( G_1 \) is compatible with the action of \( G_1 \) on itself by translation with respect to \( \rho \).

Proposition 4.17. Let \( G_1, G_2 \) be groups, \( \rho : G_2 \to \text{Aut}(G_1) \) be a morphism of groups and \( \alpha_1, \alpha_2 \) be actions of \( G_1, G_2 \) respectively, on a space \( X \).
If \( \alpha_2 \) is compatible with \( \alpha_1 \) with respect to \( \rho \) then \( G_1 \rtimes_\rho G_2 \) acts on \( X \) by \( \alpha \) where \( \alpha(g_1,g_2) = \alpha_1(g_1) \circ \alpha_2(g_2). \)

Proof. Assume \( \alpha_2 \) is compatible with \( \alpha_1 \) with respect to \( \rho \). Then, for \((g_1,g_2),(g'_1,g'_2) \in G_1 \rtimes_\rho G_2\), we have :

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Example 4.18. Under the assumption of Example 4.16, the previous proposition gives the natural action of $G_1 \rtimes_\rho G_2$ on $G_1$ i.e. $(g_1, g_2)g'_1 = g_1 \rho(g_2)g'_1$.

Proof of Theorem 2. Denote by $\tau$ the action of $G = G_1 \rtimes_\rho G_2$ on $X = X_1 \times X_2$ defined by:

$$\tau(g_1, g_2)(x_1, x_2) = (\tau_1(g_1)(\tilde{\rho}(g_2)x_1), \tau_2(g_2)x_2).$$

We show that, via this action, $G$ acts by automorphisms on the direct product of spaces with labelled partitions $(X, \mathcal{P}, F_q(\mathcal{P}))$ where $\mathcal{P} = \mathcal{P}_1 \oplus \mathcal{P}_2$ and $F_q(\mathcal{P}) \simeq F_1(\mathcal{P}_1) \oplus F_2(\mathcal{P}_2)$ endowed with the $q$-norm of the direct sum for $q \geq 1$.

Let $p \in \mathcal{P}$ and $g = (g_1, g_2) \in G$. If $p = p_1^\oplus_1 \in \mathcal{P}_1^\oplus_1$, then, for all $x = (x_1, x_2) \in X$, we have:

$$\Phi_{\tau(g)}(p)(x) = p(\tau(g)x) = p_1^\oplus_1(\tau_1(g_1)(\tilde{\rho}(g_2)x_1), \tau_2(g_2)x_2) = p_1(\tau_1(g_1)(\tilde{\rho}(g_2)x_1)) = p_1 \circ \tau_1(g_1) \circ \tilde{\rho}(g_2)(x_1)$$

and since $G_1$ acts by automorphisms on $(X_1, \mathcal{P}_1, F_1(\mathcal{P}_1))$ via $\tau_1$, we have $p_1 \circ \tau_1(g_1) \in \mathcal{P}_1$, and $G_2$ acts by automorphisms on $(X_1, \mathcal{P}_1, F_1(\mathcal{P}_1))$ via $\tilde{\rho}$, then $p_1 \circ \tau_1(g_1) \circ \tilde{\rho}(g_2) \in \mathcal{P}_1$. Hence, $\Phi_{\tau(g)}(p) = (p_1 \circ \tau_1(g_1) \circ \tilde{\rho}(g_2))^\oplus_1$ belongs to $\mathcal{P}$.

For $p = p_2^\oplus_2 \in \mathcal{P}_2^\oplus_2$, we have $\Phi_{\tau(g)}(p) = (p_2 \circ \tau_2(g_2))^\oplus_2$ which belongs to $\mathcal{P}$ since $G_2$ acts by automorphisms on $(X_2, \mathcal{P}_2, F_2(\mathcal{P}_2))$ via $\tau_2$.

Then, for all $g \in G$ and all $p \in \mathcal{P}$,

$$\Phi_{\tau(g)}(p) = p \circ \tau(g) \in \mathcal{P}. $$

Let us fix some notations. We denote, for $g_1 \in G_1$, $g_2 \in G_2$:
Let $\xi$ be in $F(\mathcal{P})$ and $g = (g_1, g_2) \in G$. We have, for all $p_1 \in \mathcal{P}_1$ and all $p_2 \in \mathcal{P}_2$:

$$
\xi \circ \Phi_{\tau(g)}(p_1^{(1)}) = (\xi_1 \circ \Phi^{(p_1)}_{\tilde{\rho}(g_2)} \circ \Phi^{(1)}_{\tau_1(g_1)})^{(1)}(p_1^{(1)}),
$$

and

$$
\xi \circ \Phi_{\tau(g)}(p_2^{(2)}) = (\xi_1 \circ \Phi^{(2)}_{\tau_2(g_2)})^{(2)}(p_2^{(2)}).
$$

Hence, $\xi \circ \Phi_{\tau(g)} = (\xi_1 \circ \Phi^{(p_2)}_{\tilde{\rho}(g_2)} \circ \Phi^{(1)}_{\tau_1(g_1)})^{(1)} + (\xi_2 \circ \Phi^{(2)}_{\tau_2(g_2)})^{(2)}$ and we have:

$$
\|\xi \circ \Phi_{\tau(g)}\|_{N_q} = \|\xi_1 \circ \Phi^{(p_1)}_{\tilde{\rho}(g_2)} \circ \Phi^{(1)}_{\tau_1(g_1)}\|_{\mathcal{P}_1}^{q} + \|\xi_2 \circ \Phi^{(2)}_{\tau_2(g_2)}\|_{\mathcal{P}_2}^{q}
= \|\xi_1\|_{\mathcal{P}_1}^{q} + \|\xi_2\|_{\mathcal{P}_2}^{q}.
$$

It follows that $G_1 \rtimes_{\rho} G_2$ acts by automorphisms on the space with labelled partitions $(X_1 \times X_2, \mathcal{P}, F_q(\mathcal{P}))$.

It remains to check this action by automorphisms is continuous, i.e. for all $x \in X$, $g \mapsto \tau(g)x$ is continuous.

As a set $G_1 \rtimes_{\rho} G_2$ is simply $G_1 \times G_2$ and since $(g_1, g_2) \mapsto \rho(g_2)g_1$ is continuous, the product topology on $G_1 \times G_2$ is compatible with the group structure of $G_1 \rtimes_{\rho} G_2$ (see [Bou71], III.18 prop 20).

Moreover, $\tau_1$, $\tau_2$ and $\tilde{\rho}$ are strongly continuous, then, for all $(x_1, x_2) \in X$, the map $(g_1, g_2) \mapsto (\tau(g_1)(\tilde{\rho}(g_2)x_1), \tau_2(g_2)x_2)$ is continuous from $G_1 \times G_2$ endowed with the product topology to $(X, d)$ where $d$ is the labelled partitions pseudo-metric.

Hence, $G_1 \rtimes_{\rho} G_2$ acts continuously by automorphisms on $(X, \mathcal{P}, F_q(\mathcal{P}))$.

Assume, for $i = 1, 2$, $G_i$ acts properly on $(X_i, \mathcal{P}_i, F_i(\mathcal{P}_i))$ via $\tau_i$, and denote by $c_i$ the separation map associated with $\mathcal{P}_i$.

Fix $x_0 = (x_1, x_2) \in X_1 \times X_2$.

The following equality holds for every $g = (g_1, g_2) \in G_1 \rtimes_{\rho} G_2$:

$$
\|c(\tau(g)x_0, x_0)\|_{N_q}^{q} = \|c_1(\tau_1(g_1)(\tilde{\rho}(g_2)x_1), x_1)\|_{\mathcal{P}_1}^{q} + \|c_2(\tau_2(g_2)x_2, x_2)\|_{\mathcal{P}_2}^{q}.
$$

Since $G_1 \rtimes_{\rho} G_2$ is endowed with the product topology of $G_1$ and $G_2$, $g = (g_1, g_2) \to \infty$ in
$G_1 \rtimes_\rho G_2$ if, and only if, $g_1 \to \infty$ in $G_1$ or $g_2 \to \infty$ in $G_2$. Hence, we have two disjoint cases:

First case: $g_1 \to \infty$ in $G_1$ and $g_2$ belongs to a compact subset $K_2$ of $G_2$.

By continuity of $g_2' \mapsto \|c(\tilde{\rho}(g_2')x_1, x_1)\|_{F_1(P_1)}$, there exists $C(K_2) \geq 0$ such that, for every $g_2' \in K_2$, $\|c(\tilde{\rho}(g_2')x_1, x_1)\|_{F_1(P_1)} \leq C(K_2)$, and, hence,

$$\|c(\tau(g_1)\tilde{\rho}(g_2)x_1, x_1)\|_{F_1(P_1)} \leq \|c(\tau_1(g_1)\tilde{\rho}(g_2)x_1, x_1)\|_{F_1(P_1)} + \|c(\rho(g_2)x_1, x_1)\|_{F_1(P_1)}$$

But, since $G_1$ acts properly on $(X_1, P_1, F_1(P_1))$, $\|c(\tau(g_1)\tilde{\rho}(g_2)x_1, x_1)\|_{F_1(P_1)} \xrightarrow{g_1 \to \infty} +\infty$, and then,

$$\|c(\tau(g_1)\tilde{\rho}(g_2)x_1, x_1)\|_{F_1(P_1)} \xrightarrow{g_1 \to \infty} +\infty.$$

It follows that $\|c(\tau(g)x_0, x_0)\|_{N_q} \xrightarrow{g_1 \to \infty} +\infty$.

Second case: $g_2 \to \infty$ in $G_2$.

We have $\|c_2(\tau_2(g_2)x_2, x_2)\|_{F_2(P_2)} \xrightarrow{g_2 \to \infty} +\infty$ and then $\|c(\tau(g)x_0, x_0)\|_{N_q} \to +\infty$.

Finally, as required, we have

$$\|c(\tau(g)x_0, x_0)\|_{N_q} \xrightarrow{g \to \infty} +\infty,$$

and then, $G_1 \rtimes_\rho G_2$ acts properly by automorphisms on $(X, P, F_q(P))$.

5 Application to wreath products

Using Theorem 2, we simplify a part of the proof of Th 6.2 in [CSV12] where Cornulier, Stalder and Valette establish the stability of the Haagerup property by wreath product; and we generalize it in the following way: the wreath product of a group with property $PL^p$ by a Haagerup group has property $PL^p$.

We first introduce the notion of permutational wreath product:

**Definition 5.1.** Let $H, G$ be countable groups, $I$ be a $G$-set and $W = \bigoplus_{i \in I} H$. The permutational wreath product $H \wr_I G$ is the group:

$$H \wr_I G := W \rtimes_\rho G,$$
where $G$ acts by shift on $W$ via $\rho$ i.e. $\rho(g) : (h_i)_{i \in I} \mapsto (h_{g^{-1}i})_{i \in I}$, for $g \in G$.

To prove Theorem 3, we need the following structure of space with measured walls relative to the wreath product built in [CSV12], Theorem 4.2 (see [CSV12] § 6.1 for the definition and examples of co-Haagerup subgroups):

**Theorem 5.2** (Cornulier, Stalder, Valette). Let $H, G$ be countable discrete groups and let $L$ be a subgroup of $G$. Denote by $I$ the quotient $G/L$ and $W = \bigoplus_I H$.
Suppose that $G$ is Haagerup and that $L$ is co-Haagerup in $G$.
Then there exists a structure $(W \times I, \mu)$ of space with measured walls on $W \times I$, with wall pseudo-metric denoted by $d_\mu$, on which $W \rtimes G$ acts by automorphisms and which satisfies,
for any $x_0 = (w_0, i_0) \in W \times I$ and for all $g \in G$:

$$d_\mu((w,g)x_0, x_0) \to +\infty \text{ when } w \in W \text{ is such that } \text{supp}(w) \to \infty \text{ in } I.$$ 

**Lemma 5.3.** Let $H, G$ be countable discrete groups, $L$ be a subgroup of $G$ and $q \geq 1$. Denote by $I$ the quotient $G/L$ and $W = \bigoplus_I H$. Suppose that $G$ is Haagerup, $L$ is co-Haagerup in $G$ and $H$ has property $PL^q$.
Then $W$ and $G$ acts by automorphisms on a space $(X, \mathcal{P}, F(\mathcal{P}))$ with labelled partitions such that:
- the $W$-action is proper,
- the $G$-action is compatible with the $W$-action,
- the Banach space $F(\mathcal{P})$ is isometrically isomorphic to a Banach subspace of a $L^q$ space.

**Proof.** Consider the $W \times G$-action on the space with measured walls $(W \times I, \mu)$ given by Theorem 5.2. Then, by Proposition 3.8, $W \times G$ acts by automorphisms on the space with labelled partitions $(W \times I, \mathcal{P}_\mu, L^q(\mathcal{P}_\mu, \mu))$. Let $y_0 = (e_W, i_0) \in W \times I$. The separation map $c_\mu$ associated with $\mathcal{P}_\mu$ satisfies:

$$\|c_\mu((w,g)y_0, y_0)\|_q = d_\mu((w,g)y_0, y_0).$$

Now, consider the structure of space with labelled partitions on $H$ given by its proper isometric affine action on a space $L^q(E, \nu)$. By Proposition 4.7, $W$ acts by automorphisms on the natural structure of space with labelled partitions $(W, \mathcal{P}_W, F_q(\mathcal{P}_W))$ of the direct
sum of spaces with labelled partitions on \( H \). Moreover, \( G \) acts by automorphisms on 
\((W, \mathcal{P}_W, F_q(\mathcal{P}_W))\) by shift via its action on \( I \).

Denote \( X = (W \times I) \times W \) and consider the space with labelled partitions \((X, \mathcal{P}, F(\mathcal{P}))\) 
given by the direct product of spaces with labelled partitions \((W \times I, \mathcal{P}_\mu, L^2(\mathcal{P}_\mu, \mu))\) and 
\((W, \mathcal{P}_W, F_q(\mathcal{P}_W))\). Then we have actions by automorphisms \( \tau_W \) of \( W \) and \( \tau_G \) on \( X \) given by, for \( x = (w_1, i, w_2) \in X \), \( w \in W \) and \( g \in G \) :

\[
\tau_W(w)x = (ww_1, i, ww_2) \quad \text{and} \quad \tau_G(g)x = (\rho(g)w_1, gi, \rho(g)w_2).
\]

The action \( \tau_G \) is clearly compatible with \( \tau_W \) since \( W \rtimes \rho \) acts naturally on \( W \) and on 
\( W \times I \).

The Banach space \( F(\mathcal{P}) \) is isometrically isomorphic to the \( q \)-direct sum \( L^q(\mathcal{P}_\mu, \mu) \oplus F_q(\mathcal{P}_W) \), 
then \( F(\mathcal{P}) \) is isometrically isomorph to a Banach subspace of \( L^q(\mathcal{P}_\mu, \mu) \oplus (\bigoplus_j L^q(E, \nu)) \).

It follows that \( F(\mathcal{P}) \) is isometrically isomorph to a Banach subspace of a \( L^q \) space. Denote 
\( x_0 = (e_W, i_0, e_W) \in X \). We have, for \( w = (h_i)_{i \in I} \in W \) :

\[
\|c(\tau_W(w)x_0, x_0)\|_{F(\mathcal{P})}^q = \|c_{\mathcal{P}_\mu}((w, i_0, (e_W, i_0))\|_{F_\mu(\mathcal{P}_\mu)}^q + \|c_{\mathcal{P}_W}(w, e_W)\|_{F_q(\mathcal{P}_W)}^q = d_\mu((w, e_G)y_0, y_0) + \sum_{i \in \text{supp}(w)} \|c_{\mathcal{P}_H}(h_i, e_H)\|_{E_H(\mathcal{P}_H)}^q.
\]

Hence, \( W \) acts properly by automorphisms on \((X, \mathcal{P}, F(\mathcal{P}))\) : indeed, \( w = (h_i) \rightarrow \infty \) in 
\( W \) if, and only if, \( \text{supp}(w) \rightarrow \infty \) in \( I \) or there exists \( j \in I \) such that \( h_j \rightarrow \infty \) in \( H \); then, 
in the first case, by the previous theorem, \( d_\mu((w, e_G)y_0, y_0) \rightarrow +\infty \) and in the second case, 
\( \sum_{i \in \text{supp}(w)} \|c_{\mathcal{P}_H}(h_i, e_H)\|_{E_H(\mathcal{P}_H)}^q \geq \|c_{\mathcal{P}_H}(h_j, e_H)\|_{E_H(\mathcal{P}_H)}^q \rightarrow +\infty \).

\( \square \)

**Proof of Theorem 3.** By Lemma 5.3, \( W \) and \( G \) act by automorphisms on a space \((X, \mathcal{P}, F(\mathcal{P}))\) 
with labelled partitions such that the \( W \)-action is proper, and the \( G \)-action is compatible 
with the \( W \)-action with respect to \( \rho \). Moreover, since \( G \) is Haagerup, \( G \) acts properly 
by automorphisms on a space \((Y, \mathcal{P}', F'(\mathcal{P}'))\) with labelled partitions where \( F'(\mathcal{P}') \) is isometrically isomorph 
\( \text{to } L^q \) space.

Hence, by Theorem 2, \( H \rtimes \rho \) acts properly by automorphisms on a space 
\((Z, \mathcal{P}_Z, F_Z(\mathcal{P}_Z))\) where \( F_Z(\mathcal{P}_Z) \) is isometrically isomorph to \( F(\mathcal{P}) \oplus F'(\mathcal{P}') \) endowed with 
the \( q \)-norm of the direct sum. It follows that \( F_Z(\mathcal{P}_Z) \) is isometrically isomorph to a Banach 
subspace of a \( L^q \) space.

Thus, by Corollary 1.1, \( H \rtimes \rho \) has property \( PL^q \). \( \square \)
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