Vaidya Solutions in General Covariant Hořava-Lifshitz Gravity without Projectability: Infrared Limit

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In this paper, we have studied nonstationary radiative spherically symmetric spacetime, in general covariant theory (U(1) extension) of Hořava-Lifshitz gravity without the projectability condition and in the infrared limit. The Newtonian prepotential \( \varphi \) was assumed null. We have shown that there is not the analogue of the Vaidya’s solution in the Hořava-Lifshitz Theory (HLT), as we know in the General Relativity Theory (GRT). Therefore, we conclude that the gauge field \( A \) should interact with the null radiation field of the Vaidya’s spacetime in the HLT.

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I. INTRODUCTION

One of the biggest problems of the GRT lies on the difficulty of its quantization, since it is a non-renormalizable theory. However, Hořava [1] has proposed a benchmark in renormalizable quantum gravity which has attracted a great interest. The theory was inspired by the Lifshitz scalar [2] and has often been called Hořava-Lifshitz theory gravity. He has formulated a theory of gravity, whose scaling at short distances exhibits a strong anisotropy between space and time [1],

\[
x \rightarrow b^{-1}x, \quad t \rightarrow b^{-z}t.
\]

In order for the theory to be power-counting renormalizable, in \((3 + 1)\)-dimensions the critical exponent \( z \) needs to be \( z \geq 3 \) [1, 3]. Since the literature about the theory is very extensive we suggest the reader the references [1–22].

In order to solve several problems in the HLT, Wang and collaborators have proposed a model without the projectability condition, but assuming that: (a) the detailed balance condition is softly broken; and (b) the symmetry of theory is enlarged to include a local \( U(1) \) symmetry [23, 24]. The enlarged symmetry was first introduced by Hořava and Melby-Thompson (HMT) in the case with the projectability condition and \( \lambda = 1 \) [25], and was soon generalized to the case with any \( \lambda \) [26], where \( \lambda \) is a coupling constant, which characterizes the deviation from GRT in the infrared limit [23, 30].

In this paper, we will analyze if the Vaidya’s spacetime can be described as a null radiation fluid in the general covariant HLT of gravity without the projectability condition [23, 24]. In Section II we present a brief introduction to the HLT. In Section III we show the Vaidya’s spacetime, expressed in ADM decomposition [1]. In Section IV we present the HLT equations for the infrared limit and their possible solutions. In Section V we analyze all the possible solutions for the HLT field equations. In Section VI we discuss the results. Finally, in Appendix A we present all the equations of HLT without projectability.

II. GENERAL COVARIANT HOŘAVA-LIFSHITZ GRAVITY WITHOUT PROJECTABILITY

In this section, we shall give a very brief introduction to the general covariant HLT gravity without the projectability condition. For detail, we refer readers to [23, 24]. The total action of the theory can be written as,

\[
S = \zeta^2 \int d^4x \sqrt{g} N \left( L_K - L_V + L_A + L_\varphi + \frac{1}{5} \zeta^2 L_M \right),
\]

where \( g = \det(g_{ij}), \) and

\[
L_K = K_{ij} K^{ij} - \lambda K^2,
\]

\[
L_V = -(\beta_0 \alpha_i a^i - \gamma_1 R),
\]

\[
L_A = \frac{A}{N} \left( 2\Lambda g - R \right),
\]

\[
L_\varphi = \varphi G^{ij} (2K_{ij} + \nabla_i \nabla_j \varphi + a_i \nabla_j \varphi) + (1 - \lambda) \left[ (\Delta \varphi + a_i \nabla^i \varphi)^2 + 2(\Delta \varphi + a_i \nabla^i \varphi) K \right] + \frac{1}{3} \hat{G}^{ijk} \left[ 4(\nabla_i \nabla_j \varphi) a_k \nabla_l \varphi + 5(a_i \nabla_j \varphi) a_k \nabla_l \varphi + 2(\nabla_i \varphi) a_0 \nabla_j \varphi \right].
\]

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\[ + 6K_{ij}a_l \nabla_l \varphi \], \quad (3) \]

where \(A\) and \(\varphi\) are the gauge field and the Newtonian prepotential, respectively [31]. Here \(\Delta \equiv g^{ij} \nabla_i \nabla_j\), \(\Lambda_g\) is a coupling constant, and all the coefficients, \(\beta_n\) and \(\gamma_n\), are dimensionless and arbitrary, except for the ones of the sixth-order derivative terms, \(\gamma_5\) and \(\beta_8\), which must satisfy the conditions,

\[ \gamma_5 > 0, \quad \beta_8 < 0, \quad (4) \]

in order to the theory to be unitary in the UV. The Ricci and Riemann tensors \(R_{ij}\) and \(R_{ijkl}\) all refer to the 3-metric \(g_{ij}\), with \(R_{ij} = R_k^{ikj}\) and

\[
K_{ij} = \frac{1}{2N} (-\dot{g}_{ij} + \nabla_i N_j + \nabla_j N_i),
\]

\[
G_{ij} = R_{ij} - \frac{1}{2} g_{ij} R + \Lambda_g g_{ij}.
\]

\[ \mathcal{L}_M \] is the Lagrangian of matter fields. To be consistent with observations in the infrared limit, we assume that

\[ \zeta^2 = \frac{1}{16\pi G}, \quad \gamma_1 = -1, \quad (6) \]

where \(G\) denotes the Newtonian constant, and

\[ \Lambda = \frac{1}{2} \zeta^2 \gamma_0, \quad (7) \]

is the cosmological constant. \(C_{ij}\) denotes the Cotton tensor, defined by

\[ C_{ij} = \epsilon^{ikl} \nabla_k \left( R_l^i - \frac{1}{4} R \delta_l^i \right), \quad (8) \]

with \(\epsilon^{123} = 1\). Using the Bianchi identities, one can show that \(C_{ij}C^{ij}\) can be written in terms of the five independent sixth-order derivative terms in the form

\[ C_{ij}C^{ij} = \frac{1}{2} R^3 - \frac{5}{2} RR_{ij}R^{ij} + 3R^2 R_{ij} R^{ij} + \frac{3}{8} R\Delta R \]

\[ + \left( \nabla_i R_{jk} \right) \left( \nabla^i R^{jk} \right) + \nabla_k G^k, \quad (9) \]

where

\[ G^k = \frac{1}{2} R^{jk} \nabla_j R - R_{ij} \nabla_j R^{ik} - \frac{3}{8} R \nabla^k R. \]

\[ (10) \]

Variations of the total action [2] with respect to \(N\), \(N^i\), \(A\), \(\varphi\) and \(g_{ij}\) yield, respectively, the Hamiltonian, momentum, \(A\)-, and \(\varphi\)-constraints, and dynamical equations, which are given explicitly in [24]. For the sake of reader's convenience, we include them in Appendix A.

In addition, assuming the translation symmetry of the action, one obtains the conservation laws of energy and momentum [24], which are also given in Appendix A.

### III. VAIDYA’S SPACETIME

The Arnowitt-Deser-Misner (ADM) form is given by [3],

\[
ds^2 = -N^2 dt^2 + g_{ij} \left( dx^i + N^i dt \right) \left( dx^j + N^j dt \right),
\]

\[ (i, j = 1, 2, 3). \quad (11) \]

Hereinafter, the Newtonian prepotential \(\varphi\) is assumed null and \(G = c = 1\).

The Vaidya’s spacetime with an ingoing null dust usually written in the form [32],

\[
ds^2 = - \left( 1 - \frac{2m(v)}{r} \right) dv^2 + 2dvdr + r^2 d\Omega^2, \quad (12)\]

where \(d\Omega^2 \equiv d\theta^2 + \sin^2 \theta d\phi^2\), and the corresponding energy-momentum tensor is given by

\[ T_{\mu \nu} = \rho(v, r) l_\mu l_\nu, \quad (13)\]

with

\[ \rho = \frac{2}{r^2} dm dv, \quad l_\mu = -\delta^v_\mu. \quad (14) \]

Introducing a time-like coordinate \(t\) via the relation, \(v = 2(t + r)\), the metric [12] can be cast in the form,

\[
ds^2 = - \frac{r}{M} dt^2 + \frac{4M}{r} \left[ dr + \left( 1 - \frac{r}{2M} \right) dt \right]^2 \]

\[ + r^2 d\Omega^2, \quad (15) \]

where

\[ M \equiv M(V) = 2m(v), \quad V \equiv t + r. \quad (16) \]

From equation [15], we immediately obtain

\[ N = \sqrt{\frac{r}{M}}, \quad N^i = \left( 1 - \frac{r}{2M} \right) \delta^i_v, \]

\[ g_{rr} = \frac{4M}{r}, \quad g_{\theta \theta} = r^2, \quad g_{\phi \phi} = r^2 \sin^2 \theta, \quad (17) \]

and

\[ N_i = g_{ij} N^j = -2 \left( 1 - \frac{2M}{r} \right) \delta^j_i, \]

\[ g^{rr} = \frac{r}{4M}, \quad g^{\theta \theta} = \frac{1}{r^2}, \quad g^{\phi \phi} = \frac{1}{r^2 \sin^2 \theta}, \quad (18) \]

\[ \rho = \frac{M^*}{2r^2}, \quad l_\mu = -2 \left( \delta^v_\mu + \delta_\mu^r \right), \quad (19) \]

where \(M^* \equiv dM/dV\).

Since \(M = M(V)\), introducing another independent variable, \(U = t - r\), we can find that

\[ M' = \frac{1}{2} M^*, \quad (20) \]

since \(dM(V)/dU = 0\).
Then, we find that the non null metric components are

\[ g_{tt} = - \left( N^2 - N_i N^i \right) = - \frac{4}{r} (M - r), \]

\[ g_{tt} = N_i = \frac{2}{r} (2M - r) \delta^i_t, \]

\[ g_{rr} = g_{rr} = \frac{4M}{r}, \]

\[ g_{\theta\theta} = \frac{g_{\theta\theta}}{r^2}, \]

\[ g_{\phi\phi} = \frac{g_{\phi\phi}}{r^2 \sin^2 \theta}, \]

\[ g_{rr} = - \frac{1}{N^2} = - \frac{M}{r}, \]

\[ g_{tt} = \frac{N_i^2}{N^2} = \frac{M}{r} \left( 1 - \frac{r}{2M} \right) \delta^i_t, \]

\[ g_{rr} = 1 - \frac{M}{r}, \]

\[ g_{\theta\theta} = \frac{1}{r^2}, \]

\[ g_{\phi\phi} = \frac{1}{r^2 \sin^2 \theta}. \]

(21)

For the projection tensor the non null components are

\[ h_{tt}^r = 1 - \frac{r}{2M}, \]

\[ h_{rr}^r = 1, \]

\[ h_{\theta\theta}^r = 1, \]

\[ h_{\phi\phi}^r = 1. \]

(22)

Then, it can be shown that

\[ n_\mu = N \delta^\mu_t = \sqrt{\frac{r}{M}}, \]

\[ n_\mu = (4) g^{\mu\nu} n_\nu = - \frac{1}{N} \left( \delta^\mu_t - N_i^i \delta^\mu_i \right), \]

\[ h_{\nu}^\mu = (4) g_{\nu\nu}^\mu n_\nu = \sqrt{\frac{M}{r}} \left[ - \delta^\mu_i + \left( 1 - \frac{r}{2M} \right) \delta^\mu_t \right], \]

\[ h_{\nu}^r = (4) g_{\nu\nu}^r n_\nu = 1 - \frac{r}{2M}, \]

\[ h_{\nu}^t = 1, \]

\[ h_{\nu}^\theta = 1, \]

\[ h_{\nu}^\phi = 1. \]

(24)

\[ J_i = T_{\mu\nu} n_\nu h_i^\mu = \frac{1}{r^2} \frac{\sqrt{M} dM}{r} dV \delta_i^t, \]

\[ = \frac{1}{r^2} \frac{\sqrt{M}}{r} \frac{dV}{dV} \delta_i^t \]

\[ \tau_{ij} = T_{\mu\nu} h_i^\mu h_j^\nu = 8M \frac{dM}{r^3} \frac{dV}{dV} \delta_i^t \delta_j^t, \]

(26)

(27)

where the prime and dot denotes the partial differentiation in relation to the coordinate \( r \) and \( t \), respectively.

**IV. INFRARED LIMIT**

In the infrared limit we must have

\[ J_i = -2\rho. \]

(29)

Besides, hereinafter, we have assumed that \( \lambda = 1 \). Thus we have

\[ K_{rr} = \frac{-2M'Mr - M'r^2 - 2M'Mr + 2M^2 + Mr}{\sqrt{r/MM^2}}, \]

\[ K_{\theta\theta} = \frac{r(-2M + r)}{2\sqrt{r/MM^2}}, \]

\[ K_{\phi\phi} = \frac{\sin^2 \theta (-2M + r)}{2\sqrt{r/MM^2}}, \]

\[ K = \frac{-2M'Mr - M'r^2 - 2M'Mr - 6M^2 + 5Mr}{\sqrt{r/MM^2}}, \]

\[ R_{rr} = \frac{M'r - M}{MM^2}, \]

\[ R_{\theta\theta} = \frac{M'r^2 + 8M^2 - 3Mr}{8M^2}, \]

\[ R_{\phi\phi} = \frac{\sin^2 \theta M'r^2 + 8M^2 - 3Mr}{8M^2}, \]

(30)

(31)

and

\[ \mathcal{L}_K = \frac{1}{2M^2r^2} \times \]

\[ [ -4M'^2M^2 + M'Mr^2 + 4MM^2 - 2MMr + 4M^2 - 2Mr ] \]

(32)

\[ \mathcal{L}_V = -\frac{1}{2M^2r^2} [ M'r^2 + 4M^2 - 2Mr ] \]

(33)
\[ F_V = \frac{\beta_0}{16M^3r}[-4M''Mr^2 + 7M'^2r^2 - 14M'Mr + 7M^2] \]  
\[ \text{(34)} \]

From equation (51) we have
\[ H = \mathcal{L}_K + \mathcal{L}_V + F_V = 8\pi J^2 = \frac{1}{16M^3r^2} \times \]
\[ [-4M''\beta_0Mr^3 + 7M'^2\beta_0r^2 - 14M'\beta_0Mr^2 - 32M'M^3 + 32M'M^3 - 16M'^2r + 7\beta_0M^2r^2] \]  
\[ \text{(35)} \]

\[ J_r = \frac{\dot{M}}{\sqrt{r/MMr}} \]  
\[ \text{(36)} \]

\[ J_\varphi = \frac{1}{16\sqrt{r/MM^4r^3}} \times \]
\[ [-4M''M^2r^3 + 2M''Mr^4 + 6M'^2Mr^3 - 5M'^2r^4 + 8M'\Lambda_gM^2r^4 - 24M^4r^2 - 8\dot{M}M^3r - 8M'^2 + 11M'Mr^3 - 8\dot{M}gM^3r^3 + 8\dot{M}^2Mr^2 + 24\Lambda_gM^4r^2 - 20\Lambda_gM^3r + 8M^4 + 8M^3r - 6M^2r^2] \]  
\[ \text{(38)} \]

From the dynamical equation (92) we have
\[ D^r = 8\pi \tau^{00} = \frac{d^r}{128\sqrt{r/MM^4r}}, \]  
\[ \text{(39)} \]

where
\[ d^r = \]
\[ -16A'M^2r^2 - \sqrt{r/MM^2\beta_0r^3} + 2\sqrt{r/MM'M^3} - 32\sqrt{r/MM'M^3} + 96\sqrt{r/MM'M^3} - 16r/\sqrt{r/MM'M^3} - \]
\[ \sqrt{r/MM'M^3} - 32\Lambda_gM^3r^2 + 32AM^3 - 8AM^2r \]  
\[ \text{(40)} \]

\[ D^{\theta\theta} = 8\pi \tau^{00} = \frac{d^{\theta\theta}}{32\sqrt{r/MM^3r^3}}, \]  
\[ \text{(41)} \]

where
\[ d^{\theta\theta} = \]
\[ -8A''M^2r^2 + 4A'M'M^2r - 12A'Mr^2 + 32\sqrt{r/MM'M^3} - 16r/\sqrt{r/MM'M^3} + \sqrt{r/MM'^2\beta_0r^2} - 2\sqrt{r/MM'M^3} + 4M'AMr - 16r/\sqrt{r/MM'M^3} + \]
\[ \sqrt{r/MM'M^3} - 32\Lambda_gM^3r - 4AM^2 \]  
\[ \text{(42)} \]

\[ D^{\phi\phi} = 8\pi \tau^{00} = \frac{D^{\theta\theta}}{\sin^2 \theta}. \]  
\[ \text{(43)} \]

V. POSSIBLE SOLUTIONS

We are looking for a HLT solution which is equivalent to the Vaidya’s solution in GRT. Then, initially we suppose that there is not any coupling between the matter field (the null radiation) and the gauge field A. It means that \( J_A = 0 \). From equations (60) and (51), we have
\[ J_A = M'r^2 - 4\Lambda_gM^2r^2 + 4M^2 - 2Mr = 0, \]  
\[ \text{(44)} \]

which give us the solution
\[ M = \frac{3r^2}{-4\Lambda_g r^3 + 12r + 3f(t)}, \]  
\[ \text{(45)} \]

where \( f(t) \) is an integration time function.

Using equations (27), (36) and (51) we have
\[ \frac{\dot{M}}{M/r \sqrt{r/MM}} (1 - \frac{M}{r}) = 8\pi \frac{1}{r^2} \sqrt{r/MM}. \]  
\[ \text{(46)} \]

Substituting equation (20) into (16) we get
\[ M(r, t) = \frac{r}{16\pi + 1}. \]  
\[ \text{(47)} \]

So, we can see that the mass depends only on the coordinate \( r \), i.e., it is a static solution. Therefore, we can conclude that there is not Vaidya’s solution in the theory of Hořava-Lifshitz, at least without coupling between the null radiation and the gauge field \( A \).

Moreover, since \( M = M(V) \) and \( U = t - r \), we can write the equation (44) in terms of \( V \) and \( U \), that is
\[ \frac{1}{8}(V - U)^2 \frac{dM}{dV} - \Lambda_gM^2(V - U)^2 + 4M^2 - M(V - U) = 0. \]  
\[ \text{(48)} \]

Deriving (48) twice, in terms of \( U \), and solving the differential equation, we find
\[ M(V) = -\frac{1}{M_0 + 8\Lambda_gV}. \]  
\[ \text{(49)} \]

Substituting the equation (49) into equation (14), we find that it does not satisfy for any \( M_0 \) and \( \Lambda_g \). Thus, again, we can conclude that, in fact, there is no Vaidya’s solution in the Hořava-Lifshitz theory if \( J_A = 0 \). In another words, the gauge field \( A \) must depend on the Vaidya’s mass, i.e., \( J_A = J_A(M) \).

Finally, let us suppose that \( J_A \neq 0 \), and considering the high complexity of the field equations, we use the follow ansatz: \( J_\varphi = 0 \) and a solution like \( M(V) = M_0V \), where \( M_0 \) is a constant. We can show using equation (38) that we have no consistent solution for it. This can imply in the need of the coupling between the null radiation and the pre-potential \( \varphi \) or, more probably, that the particular solution proposed is not consistent.
VI. CONCLUSION

In this paper, we have analyzed nonstationary radiative spherically symmetric spacetime, in general covariant theory of Hořava-Lifshitz gravity without the projection and condition in the infrared limit. The Newtonian prepotential \( \varphi \) was assumed null. We have shown that the gauge field \( A \) must interacts with the null radiation field of the Vaidya’s spacetime in the HLT, since we must have \( J_A \neq 0 \). Besides, we can conclude that there is not Vaidya’s solution, as we know in GRT, in the theory of Hořava-Lifshitz.

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VII. APPENDIX A: HLT FIELD EQUATIONS AND CONSERVATION LAWS

The variations of the action \( S \) with respect to \( N \) and \( N^i \) give rise to the Hamiltonian and momentum constraints,

\[
\mathcal{L}_K + \mathcal{L}_V^R + F_V - F_\varphi - F_\lambda = 8\pi G J^i, \tag{50}
\]

\[
\nabla_j \left\{ \pi^{ij} - \varphi \delta^{ij} - 2 \tilde{\mathcal{F}}^{jkl} a_i \nabla_k \varphi \right\} = 8\pi G J^i, \tag{51}
\]

where

\[
\mathcal{L}_V^R = \gamma_0 \frac{c^2}{\xi^2} - R + \gamma_2 \frac{\gamma_3 R_{ij} R^{ij}}{\xi^2} + \gamma_3 \frac{\gamma_4 C_{ij} C^{ij}}{\xi^2},
\]

\[
J^{j} = -N \frac{\delta \mathcal{L}_M}{\delta N_i}, \quad J^{i} = 2 \frac{\delta (N \mathcal{L}_M)}{\delta N}, \tag{52}
\]

and \( F_V \), \( F_\varphi \) and \( F_\lambda \) are given, respectively, by

\[
F_V = \beta_0 \left[ 2a_i^2 + 3(a_i a^i)^2 + 4 \nabla_i (a_k a^k a^i) \right],
\]

\[
F_\varphi = -G^{ij} \nabla_i \varphi \nabla_j \varphi - \frac{2}{N} \tilde{\mathcal{F}}^{jkl} \nabla_i (NK_{ij} \nabla_k \varphi),
\]

\[
F_\lambda = (1 - \lambda) \left\{ (\nabla^2 \varphi + a_i \nabla^i \varphi)^2 - \frac{2}{N} \nabla_i (NK \nabla^i \varphi) \right\}. \tag{55}
\]

\[
(F_n)_{ij}, \quad (F_\varphi)_{ij} \quad \text{and} \quad (F_\lambda)_{ij}, \quad \text{defined in equation (63)}
\]

are given, respectively, by

\[
(F_0)_{ij} = -\frac{1}{2} g_{ij},
\]

\[
(F_1)_{ij} = R_{ij} - \frac{1}{2} R g_{ij} + \frac{1}{N} (g_{ij} \nabla^2 N - \nabla_j \nabla_i N),
\]

\[
(F_2)_{ij} = -\frac{1}{2} g_{ij} R^2 + 2 R R_{ij} + \frac{1}{N} \left[ g_{ij} \nabla^2 (NR) - \nabla_j \nabla_i (NR) \right],
\]

\[
(F_3)_{ij} = -\frac{1}{2} g_{ij} R_{mn} R^{mn} + 2 R_{ik} R^k_i
\]
\[
\begin{align*}
(F_1)_{ij} &= -\frac{1}{2}g_{ij}R^3 + 3R^2R_{ij} \\
&+ \frac{3}{N}\left[g_{ij}\nabla^2 - \nabla_j\nabla_i\right](NR^2), \\
(F_3)_{ij} &= -\frac{1}{2}g_{ij}RR_{mn}R^{mn} \\
&+ R_{ij}RR_{mn}R^{mn} + 2R_{ij}R_{kj}R_k^i \\
&\quad + \frac{1}{N}\left[g_{ij}\nabla^2(NR_{mn}R^{mn}) \\
&\quad - \nabla_j\nabla_i(NR_{mn}R^{mn}) \\
&\quad + \nabla^2(NRR_{ij}) + g_{ij}\nabla_m\nabla_n(NR_{mn}R^{mn}) \\
&\quad - 2\nabla_m\nabla_i(R_{ij}^iNR)\right], \\
(F_6)_{ij} &= -\frac{1}{2}g_{ij}R^2 \nabla_i \nabla_j R \\
&+ \frac{1}{N}\left[g_{ij}\nabla^2(N\nabla^2R) - \nabla_j\nabla_i(N\nabla^2R) \\
&\quad + R_{ij}\nabla^2(NR) + g_{ij}\nabla^4(NR) - \nabla_i\nabla_j(\nabla^2(NR)) \\
&\quad - \nabla_j(NR\nabla_iR) + \frac{1}{2}g_{ij}\nabla_k(NR\nabla^2R)\right], \\
(F_8)_{ij} &= -\frac{1}{2}g_{ij}a^m_{a}a_{mn} + 2a^k_{a}a_{kj} \\
&\quad + \frac{1}{N}\left[g_{ij}\nabla^2(NR_{mn}R^{mn}) \\
&\quad - \nabla_j\nabla_i(NR_{mn}R^{mn}) \\
&\quad - \nabla^2R_{ij}(NR_{mn}R^{mn}) - g_{ij}\nabla_p\nabla_mR_{mn}(NR_{mn}R^{np}) \\
&\quad - 2\nabla_m(NR_{ij}(\nabla_{ij}R_{ij}^i)) - 2\nabla_n(NR_{ij}(\nabla_{ij}R_{ij}^n)) \\
&\quad + 2\nabla_k(NR_{ij}^k\nabla_{ij}R_{ij}^i)\right],
\end{align*}
\]

\[
\begin{align*}
(F_9)_{ij} &= -\frac{1}{2}g_{ij}akG_k + \frac{1}{2}\left[ \frac{1}{2}a^kR_{ki}^j\nabla_k R + a_{(i}R_{j)k}^k\nabla_k R \right] \\
&\quad - a_kR_{mij}\nabla^m_kR_{ij} + a_kR_{mij}\nabla_kR_{ij} \\
&\quad - \frac{1}{2}\left[a_kR_{mij}\nabla^m R_{kj} + a_jR^k_{mij}\nabla^m R_{kj}\right] \\
&\quad - \frac{3}{8}a_{(i}R^j\nabla_j R + \frac{3}{8}\left\{ R\nabla_k(Na^k)_{ij} \right\} \\
&\quad + g_{ij}\nabla^2\left[ R\nabla_k(Na^k) \right] - \nabla_i\nabla_j\left[ R\nabla_k(Na^k) \right] \right\} \\
&\quad + \frac{1}{4N}\left\{ - \frac{1}{2}\nabla^m\nabla_i(Na_{ij}) + \nabla_i\nabla_j[R\nabla_k(Na^k)] \right\}, \\
&\quad + \frac{1}{2\sqrt{gN}}\partial_{(\nu}R_{\mu)ij} - 2\varphi K_{(i}^iR_{j)\nu},
\end{align*}
\]

\[
\begin{align*}
(F_0)_{ij} &= -\frac{1}{2}g_{ij}a^k_{a}a_k + a_{a}ij, \\
(F_0')_{ij} &= -\frac{1}{2}g_{ij}(a^k_{a}a_k)^2 + 2(a^k_{a}a_k)a_{a}ij, \\
(F_2)_{ij} &= -\frac{1}{2}g_{ij}(a^k_{a})^2 + 2a^k_{a}a_{ij} \\
&\quad - \frac{1}{N}\left[2\nabla_i(Na_{a}a_{a}) - g_{ij}\nabla_n(a_\alpha Na_{a})\right], \\
(F_3)_{ij} &= -\frac{1}{2}g_{ij}(a^k_{a}a^k_{a})a^k_{a}a_{a} + a^k_{a}a_{a}a_{a} + a^k_{a}a_{a}a_{a}ij \\
&\quad - \frac{1}{N}\left[\nabla_i(Na_{a}a_{a}a_{a}) - \frac{1}{2}g_{ij}\nabla_n(a_\alpha Na_{a}a_{a})\right], \\
(F_4)_{ij} &= -\frac{1}{2}g_{ij}(a^k_{a}a_{a})R + a_{a}ij + a {a}k_{a}R_{ij} \\
&\quad + \frac{1}{N}\left[g_{ij}\nabla^2(Na^k_{a}a_{a}) - \nabla_i\nabla_j(Na^k_{a}a_{a}) \right], \\
(F_5)_{ij} &= -\frac{1}{2}g_{ij}a^m_{a}a_{mn} + 2a^k_{a}a_{kj} \\
&\quad - \frac{1}{N}\left[\nabla^k(Na^k_{a}a_{a}) - \nabla_i\nabla_j(Na^k_{a}a_{a}) \right], \\
(F_6)_{ij} &= -\frac{1}{2}g_{ij}a^m_{a}a_{mn}R^{mn} + 2a^m_{a}R_{m(i}a_{a}) \\
&\quad - \frac{1}{N}\left[\nabla^k(Na^k_{a}a_{a}) - \nabla_i\nabla_j(Na^k_{a}a_{a}) \right], \\
(F_7)_{ij} &= -\frac{1}{2}g_{ij}Ra^k_{a} + a^k_{a}R_{ij} + Ra_{ij} \\
&\quad + \frac{1}{N}\left[g_{ij}\nabla^2(Na^k_{a}a_{a}) - \nabla_i\nabla_j(Na^k_{a}a_{a}) \right], \\
&\quad - \nabla_i(Ra_{a}) + \frac{1}{2}g_{ij}\nabla^k(NRa_{a}) \right], \\
(F_8)_{ij} &= -\frac{1}{2}g_{ij}R_{ij}a^k_{a} + (\Delta a_{i}(\Delta a_{j}) + 2\Delta a_{k}\nabla_{(i}a_{j)k} \\
&\quad + \frac{1}{N}\left[\nabla_k(a_i\nabla^k(NRa_{a}) + a_iN\nabla a_{a}^k \right] - a^k_{a}\nabla_i(NRa_{a}) + g_{ij}Na^{a\beta}a_{a} - Na_{ij}\Delta a_{k} \right] \\
&\quad - 2\nabla_i(Na_{a}a_{a})\Delta a_{k} \right], \quad (57)
\end{align*}
\]
\[
\begin{align*}
(F_2^p)_{ij} &= -\frac{1}{2}g_{ij}\phi G^{mn}a_m\nabla_n\phi \\
&\quad -\frac{1}{2}g_{ij}\phi a_i a_k \nabla_k \phi + a_k R_{ijk}\nabla_j \phi \\
&\quad + \frac{1}{2}(R - 2\Lambda)\phi a_i \nabla_j \phi \\
&\quad - \frac{1}{N}\left\{ - \frac{1}{2}(R_{ij} + g_{ij})\nabla^2 - \nabla_i \nabla_j \right\}(N\phi^k \nabla_k \phi) \\
&\quad - \nabla^2(N\phi a_i \nabla_j \phi) + \frac{1}{2}g_{ij}\phi N \nabla_a \nabla_b \phi \\
&\quad + \frac{1}{2}\nabla^2(N\phi a_i \nabla_j \phi) \\
&\quad - \frac{1}{2}\nabla^2(N\phi a_i \nabla_j \phi) + \frac{1}{2}g_{ij}\phi (N\phi \nabla_a \nabla_b \phi) \\
&\quad + \frac{1}{2}g_{ij}\phi G^{mn}a_m\nabla_n\phi \\
&\quad - \frac{1}{2}(R - 2\Lambda)\phi a_k \nabla_j \phi \\
&\quad - \frac{1}{N}\left\{ - \frac{1}{2}(R_{ij} + g_{ij})\nabla^2 - \nabla_i \nabla_j \right\}(N\phi^k \nabla_k \phi) \\
&\quad - \nabla^2(N\phi a_i \nabla_j \phi) + \frac{1}{2}g_{ij}\phi N \nabla_a \nabla_b \phi \\
&\quad + \frac{1}{2}\nabla^2(N\phi a_i \nabla_j \phi) \\
&\quad - \frac{1}{2}\nabla^2(N\phi a_i \nabla_j \phi) + \frac{1}{2}g_{ij}\phi (N\phi \nabla_a \nabla_b \phi)
\end{align*}
\]

Variations of \( S \) with respect to \( \phi \) and \( A \) yield, respectively,
\[
\begin{align*}
\frac{1}{2}G^{ij}(2K_{ij} + \nabla_i \nabla_j \phi + a_i \nabla_j \phi) \\
+ \frac{1}{2N}\left\{ G^{ij}(2K_{ij} + \nabla_i \nabla_j \phi) - G^{ij}\nabla_j(N\phi a_i) \right\} \\
- \frac{1}{N}\tilde{G}^{ijkl}\left\{ \nabla_k(a_{ij}Nk_{ij}) + \frac{2}{3}\nabla_k(a_{ij}N\nabla_i \nabla_j \phi) \\
- 2\nabla_j(Na_{ij}\nabla_k \phi) + \frac{5}{3}\nabla_j(Na_{ij}\nabla_k \phi) \\
+ \frac{2}{3}\nabla_j(Na_{ik}\nabla_l \phi) \right\}
\end{align*}
\]
\[ + \frac{1 - \lambda}{N} \left\{ \nabla^2 [N(N^2 \varphi + a_k \nabla^k \varphi)] - \nabla^i [N(N^2 \varphi + a_k \nabla^k \varphi) a_i] + \nabla^2 (NK) - \nabla^i (NK a_i) \right\} = 8\pi G J_\varphi, \tag{59} \]

and

\[ R - 2\Lambda_g = 8\pi G J_A, \tag{60} \]

where

\[ J_\varphi = -\frac{\delta L_M}{\delta \varphi}, \quad J_A = 2 \frac{\delta (N L_M)}{\delta A}. \tag{61} \]

On the other hand, the variation of \( S \) with respect to \( g_{ij} \) yields the dynamical equations,

\[ \frac{1}{\sqrt{gN}} \frac{\partial}{\partial t} \left( \sqrt{gN} \pi^{ij} \right) + 2(\kappa_{ik} K_k^i - \lambda KK_{ij}) - \frac{1}{2} g^{ij} L_K + \frac{1}{N} \nabla_k (\pi^{ik} N^j + \pi^{kj} N^i - \pi^{ij} N_k) + F^{ij} + F^{ij}_A - \frac{1}{2} g^{ij} L_A + F^{ij}_\varphi \]

\[ - \frac{1}{N} (AR^{ij} + g^{ij} \nabla^2 A - \nabla^i \nabla^j A) = 8\pi G \tau^{ij}, \tag{62} \]

where

\[ \tau^{ij} = \frac{2}{\sqrt{gN}} \frac{\delta (\sqrt{gN} L_M)}{\delta g_{ij}}, \]

\[ F^{ij} = \frac{1}{\sqrt{gN}} \frac{\delta (-\sqrt{gN} L_A^V)}{\delta g_{ij}} = \sum_{s=0}^{n_s} \tilde{\gamma}_s \zeta^{n_s} (F_s)^{ij}, \tag{63} \]

with

\[ \tilde{\gamma}_s = \left( \gamma_0, \gamma_1, \gamma_2, \gamma_3, \frac{1}{2} \gamma_5, -\frac{5}{2} \gamma_5, 3 \gamma_5, \frac{3}{8} \gamma_5, 7 \gamma_5, \frac{1}{2} \gamma_5 \right), \]

\[ n_s = (2, 0, -2, -2, -4, -4, -4, -4, -4), \]

\[ m_s = (0, -2, -2, -2, -2, -2, -2, -2, -2), \]

\[ \mu_s = \left( 2, 1, 1, 2, 4, 5, 2, 3, 3, 1 - \lambda, 2 - 2\lambda \right). \tag{66} \]

In addition, the matter components \((J^i, J^j, J_\varphi, J_A, \tau^{ij})\) satisfy the conservation laws of energy and momentum,

\[ \int d^3 x \sqrt{gN} \left[ g_{ij} \tau^{ij} - \frac{1}{\sqrt{g}} \partial_k (\sqrt{g} J^k_j) + \frac{2N_s}{\sqrt{gN}} \partial_k (\sqrt{g} J^k_j) \right. \]

\[ \left. - \frac{1}{\sqrt{gN}} \partial_k (\sqrt{g} J_A) - 2 \dot{\varphi} J_\varphi \right] = 0, \tag{67} \]

\[ \frac{1}{N} \nabla^i (N \tau^{ik}) - \frac{1}{\sqrt{gN}} \partial_k (\sqrt{g} J^k_j) - \frac{J_A}{2N} \nabla_k A - \frac{j^f}{2N} \nabla_k N_k \]

\[ - \frac{N_k}{N} \nabla_i J^j - \frac{J_i}{N} (\nabla N_k - \nabla k N_i) + J_k \nabla k \varphi = 0. \tag{68} \]
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