Grammic monoids with three generators

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Abstract
Young tableaux are combinatorial objects whose construction can be achieved from words over a finite alphabet by row or column insertion as shown by Schensted sixty years ago. Recently Abram and Reutenauer studied the action of the free monoid on the set of columns defined by the famous insertion algorithm. Since the number of columns is finite, this action yields a finite transformation monoid. Here we consider the action on the set of rows. We investigate the corresponding infinite monoid in the case of a 3 letter alphabet. In particular we show that it is the quotient of the free monoid relative to a congruence generated by the classical Knuth rules plus a unique extra rule.

Keywords Young tableaux · Monoid congruences

1 Introduction

Let $A$ be a finite ordered alphabet. A Young tableau is a finite labeling of the right top quarter discrete plane justified to the left and to the bottom in such a way that the rows are nondecreasing from left to right and the columns are increasing from bottom to top, see Example 1 below. Using Schensted’s method of inserting a letter in a row (or in a column), a Young tableau can be associated to a word so that all letter occurrences appear once and only once in the tableau. Two words are associated to the same tableau if and only if they are equivalent in a congruence $\equiv$ whose rules were given by Knuth [3, Theorem 6.6]. The quotient thus defined is the plactic monoid.

Schensted’s left insertion of a letter in a column results in a column plus a possible letter. Ignoring this extra letter yields an action of the free monoid on the (finite) set of columns. The quotient of the free monoid by the nuclear congruence is a finite monoid Styl$(A)$ (the stylistic monoid), many interesting properties of which are studied...
in [1], such as its $J$-classes, the notion of $N$-tableaux as representatives of congruence classes in the same way that Young tableaux represent congruence classes of the plactic monoid, and a minimum set of rules defining the stylic congruence. Here we consider the “dual” problem of acting on the set of rows which are finite nondecreasing sequences. The main departure is that this monoid, called Grammic monoid as suggested to us by Antoine Abram and which comes from the Greek word for “row”, is clearly no longer finite. In this note we consider a specific aspect, namely the presentation of the monoid when $A$ has three generators. Based on the characterization of the pairs of words defining the same mapping, our main result states that these pairs are precisely those that are equivalent in the congruence generated by the Knuth rules and the single new rule $cbab = bcab$ where $a < b < c$. This result is optimal since on 3 generators the plactic congruence is strictly finer than the grammic congruence. An open issue is to generalize it to an arbitrary number of generators. We conjecture that for 4 letters the congruence is generated by the Knuth rules and the single new rule $dbac = bdac$ where $a < b < c < d$.

I would like to relate the present result to the paper [4] investigating the algebra of the plactic monoid of rank 3 over a field $K$ and which is a continuation of [2]. In Proposition 2.5 it is shown that this plactic monoid is isomorphic to a subdirect product of its quotient by the congruence generated by the relation $ac = ca$ and its quotient by the congruence generated by the relation $cbab = bcab$. In view of our main theorem this latter quotient is isomorphic to the Grammic monoid over 3 generators.

2 Preliminaries

2.1 Young tableaux

The reader is referred to the chapter [6] for an introduction of the plactic monoid but the basics is recalled now.

Let $A$ be a totally ordered alphabet with $k$ elements $a_1 < \cdots < a_k$. Given an element $u$ of the free monoid $A^*$ and a letter $a \in A$, we let $|u|_a$ denote the number of occurrences of $a$ in $u$ and $|u|$ denote its length. We let the plactic congruence on $A^*$ be denoted by $\equiv$ and we recall that it is generated by the Knuth relations [3, Expression 6.7]

$$\begin{align*}
    bac & \equiv bca \text{ where } a < b \leq c, \\
    acb & \equiv cab \text{ where } a \leq b < c
\end{align*}$$

(1)

A Young tableau is a labelling by occurrences of letters in $A$ of a finite lower order ideal of $\mathbb{N}^2$ where the ordering is the product of the natural ordering on $\mathbb{N}$. Every row is nondecreasing from left to right and every column is strictly decreasing from top to bottom.

**Example 1** We assume $A = \{a, b, c\}$ with $a < b < c$. The following is a Young tableau

```
c
bbc
aabb
```
The height of a Young tableau is the number of its rows. The notions of bottom, top rows, and the like have their natural meaning. The bottom row is at height 1, the next above is at height 2 etc... In Example 1 the tableau has height 3, the bottom row is \( aabb \), and the top row is \( c \).

Schensted gave in [7] an algorithm associating to a word \( w \) a Young tableau \( P(w) \) in which every letter occurrence appears once and only once in such a way that two words \( u \) and \( v \) are \( \equiv \)-congruent if they are associated to the same tableau, see e.g., [6, Proposition 6.2.3]. We recall it and reserve henceforth the term “row” for a nondecreasing sequence of letters and “column” for a decreasing sequence of letters. The algorithm proceeds by insertion of a letter \( b \) in a row \( b_1 \cdots b_p \). If \( b \geq b_p \) then the insertion results in the row \( b_1 \cdots b_pb \). Otherwise, if \( b_i \) is the leftmost letter greater than \( b \) then the insertion results in the row obtained by substituting \( b \) for \( b_i \). More generally, the insertion of \( b \) in a Young tableau consists of inserting \( b \) in the bottom row. If \( b \) is greater than or equal to the greatest element in the row the procedure stops. Otherwise \( b \) bumps the letter \( b_i \) which is inserted in the next upper row and the process is iterated as long as necessary. The Young tableau associated with a word \( w \) is obtained incrementally by starting from the empty tableau and successively inserting the letters of \( w \).

Dually the tableau can be constructed by inserting a letter \( c \) in a column \( c_1 \cdots c_p \). If \( c > c_1 \cdots c_p \) the resulting column is \( cc_1 \cdots c_p \). Otherwise if \( c_i \) is the least letter greater than or equal to \( c \) (note the difference between row and column insertion) then \( c \) is substituted for \( c_i \) in the column. The construction of the tableau follows the same pattern as in the case of row insertion except that instead of proceeding from bottom to top, it proceeds from left to right.

As a consequence of Schensted construction, there are two specific representatives of an \( \equiv \)-equivalence class, namely the row normal form which is the sequence of the rows of the Young tableau from top to bottom and the column normal form which is the sequence of the columns of the tableau from left to right.

In Example 1 the row and column normal forms are \( c/bbc/aabb \) and \( cba/ba/cb/b \) where we used the backslash symbol for visual convenience. In particular \( cbbcaabb \equiv cbabacb \). The height of the tableau is 3.

### 2.2 Grammicit congruence

We consider the action of a letter \( b \) on a row (we recall that it is a nondecreasing sequence of letters) \( \alpha_1 \cdots \alpha_n \) which is the insertion of \( b \) as explained above in Schensted procedure except that the possible letter which is expelled is definitely lost. More precisely, we set

\[
\alpha_1 \cdots \alpha_n \cdot b = \begin{cases} 
\alpha_1 \cdots \alpha_n \cdot b & \text{if } \alpha_n \leq b \\
\alpha_1 \cdots \alpha_{i-1}b\alpha_{i+1} \cdots \alpha_n & \text{if } \alpha_i \text{ is the leftmost letter greater than } b,
\end{cases}
\]
The mapping extends to $A^*$ and factors through the plactic monoid because of the following equalities which can be readily verified.

$$\alpha_1 \cdots \alpha_n \cdot bac = \alpha_1 \cdots \alpha_n \cdot bca \text{ where } a < b \leq c,$$

$$\alpha_1 \cdots \alpha_n \cdot acb = \alpha_1 \cdots \alpha_n \cdot cab \text{ where } a \leq b < c$$

We let $u \equiv_{\text{gram}} v$ denote the grammic congruence between two words $u$ and $v$ defining the same mappings on the rows and call grammic monoid the quotient of $A^*$ by $\equiv_{\text{gram}}$. The congruence $\equiv_{\text{gram}}$ is clearly coarser than the congruence $\equiv$. Alternatively, by identifying $a_1^{n_1} \cdots a_k^{n_k}$ with the $k$-tuple $(n_1, \ldots, n_k)$ we may consider that the free monoid acts on $\mathbb{N}^k$ in an obvious way.

**Lemma 1** If two words $u, v \in A^*$ define the same action on the set of rows, they have the same commutative image.

**Proof** Consider a word $u = \alpha_1 \cdots \alpha_n \in A^*$ of length $n$ and the vector $\lambda = (n, n, \ldots, n) \in \mathbb{N}^k$. By the definition of the action of letter on vectors, for all proper prefixes $w$ of $u$, no component of $\lambda \cdot w$ vanishes. Thus

$$(\lambda \cdot u)_i = \begin{cases} n + |u_{a_1}| & \text{if } i = 1 \\ n + |u_{a_i}| - |u_{a_{i-1}}| & \text{if } 1 < i \leq k \end{cases}$$

Therefore $(\lambda \cdot u)_i = (\lambda \cdot v)_i$ implies $|u|_{a_i} = |v|_{a_i}$.

**Lemma 2** The bottom row of the Young tableau $u$ is $(0, 0, \ldots, 0) \cdot u$.

**Proof** Indeed, we make two preliminary remarks. From the very definition of the action of the plactic monoids on the rows, it is clear that the image of $(0, 0, \ldots, 0)$ by the product of $p$ columns with the same rightmost letter equal to $a_i$ is the vector with all components equal to 0 except component $i$ which is equal to $p$. Furthermore, the action of $a_i$ on a vector in $\mathbb{N}^k$ affects no components smaller than $i$. In other words for $u \in (A \setminus \{a_1, \ldots, a_{i-1}\})^*$ and $(n_1, \ldots, n_k) \in \mathbb{N}^k$ there exists $(m_i, \ldots, m_k)$ such that $(n_1, \ldots, n_k) \cdot u = (n_1, \ldots, n_{i-1}, m_i, \ldots, m_k)$. The claim of the lemma results from the previous observations and the fact that an element of the plactic monoid has a representative as a product of nondecreasing columns in the following partial ordering over decreasing sequences of letters

$$a_1 \cdots a_p \geq b_1 \cdots b_q \text{ if } p \geq q \text{ and } a_{p-q+1} \leq b_1, \ldots, a_p \leq b_q$$

(e.g., with Example 1 we have $cba \geq ba \geq cb \geq b$).

Lemmas 1 and 2 have simple consequences for the case of two- and three-letter alphabets.

---

1 Christophe Reutenauer told me that Lemmas 1 and 2 were known to his coauthor Antoine Abram.
Corollary 1  Over a two letter alphabet we have

\[ u \equiv v \iff u \equiv_{\text{gram}} v \]

Corollary 2  If \( u \equiv_{\text{gram}} v \) and \( u \not\equiv v \) then the row normal form of at least one of the two words has height 3.

**Proof**  If \( u \equiv_{\text{gram}} v \) the bottom rows of their row normal forms are equal. If they have only two rows, then the top rows are also equal by Lemma 1.

Lemma 3  Let \( 0 < i \leq k \) and \( u \in A^* \) in row normal form, \( u_1/u_2/\ldots/u_p \). Set \( u_j = b_jw_j \) where \( b_1 > b_2 > \ldots > b_p \) are the initial letters of the rows. Consider

\[ m_i = \sum_{i<j} \sum_{a<b_i} |u_j|_a \]

Then for all vectors \( x \in \mathbb{N}^k \) with \( x_i > m_i \) and for all proper prefixes \( w \) of \( u \) it holds \((x \cdot w)_i > 0\).

**Proof**  Let \( r \) be the greatest integer less than \( i \) or 0 if such an integer does not exist. Since it is clear that \((x \cdot u_1u_2\cdots u_p)_i = (x \cdot u_{r+1}u_2\cdots u_p)_i\) we may assume \( r = 0 \) and compute \((x \cdot u_1\cdots u_p)_i\).

Rewrite \( m_i = \sum_{i<j} \mu_j \) and assume \( x_i > \sum_{i<j} \mu_j \). Then by induction on \( j = 1, \ldots, p \) for all prefixes \( v \) of \( u_j \) we have

\[ (x \cdot u_1\cdots u_{j-1}v)_i - \sum_{\ell>j} \mu_\ell > 0 \]

With Example 1 the previous computation yields \( x_1 > 0, x_2 > 2 \) (actually it can be checked that \( x_3 > 6 \) and thus \((x_1, x_2 + 2, x_3 + 6).w = (x_1 + |w|_a, x_2 + 2 + |w|_b - |w|_a, x_3 + 6) + |w|_c - |w|_b\).

In the next lemma given a formal vector \( X = (\omega_1, \ldots, \omega_{i-1}, x_i, \omega_{i+1}, \ldots, \omega_k) \) where \( \omega_j, j \neq i \) is an integer constant and \( x_i \) a variable and a word \( u \in A^* \), we view the expression \( X \cdot u \) as a function of \( \mathbb{N} \) into \( \mathbb{N}^k \).

Lemma 4  Let \( u, v \in A^* \) be two words of length \( n \). If for all \( |x_i| \leq n + 1 \) the functions \( X \cdot u \) and \( X \cdot v \) are equal, then these functions are equal for all values of \( x_i \).

**Proof**  We set \( \Delta_u(x_i) = X \cdot u - X \) and we show that, viewed as a function of \( x_i \), it is constant for all \( x_i > |u| \). We set \( u = b_1 \cdots b_n \) and observe that by Lemma 3 for all prefixes \( w = b_1 \cdots b_r, r = 0, \ldots, n \), we have \((X \cdot w)_i \geq n - r \). We show by induction that the function \( \Delta_{b_1 \cdots b_r}(X) \) is a constant for all \( x_i \geq n \). If \( r = 0 \) then \( \Delta_1(x_i) \) is the zero vector thus we assume \( r > 0 \). We set \( w = vb_r \) and \( b_r = a_j \) and we compute the action of \( v \) on \( X \).

\[ \text{Springer} \]
Case 1 \( j < i \) and \((X \cdot v)_h = 0\) for all \( j < h < i \) then

\[
\begin{align*}
(X \cdot w)_j &= (X \cdot v)_j + 1, \\
(X \cdot w)_i &= (X \cdot v)_i - 1, \text{ (Lemma 3)} \\
(X \cdot w)_\ell &= (X \cdot v)_\ell, \ell \neq i, j
\end{align*}
\]

Case 2 \( j < i \) and \((X \cdot v)_h \neq 0\) for some least \( j < h < i \) then

\[
\begin{align*}
(X \cdot w)_j &= (X \cdot v)_j + 1, \\
(X \cdot w)_h &= (X \cdot v)_h - 1, \\
(X \cdot w)_\ell &= (X \cdot v)_\ell, \ell \neq j, h
\end{align*}
\]

Case 3 \( j = i \) and \((X \cdot v)_h \neq 0\) for some least \( i < h \leq n \) then

\[
\begin{align*}
(X \cdot w)_j &= (X \cdot v)_j + 1, \\
(X \cdot w)_h &= (X \cdot v)_h - 1, \\
(X \cdot w)_\ell &= (X \cdot v)_\ell, \ell \neq j, h
\end{align*}
\]

Case 4 \( j = i \) and \((X \cdot v)_h = 0\) for all \( h \leq n \) then

\[
\begin{align*}
(X \cdot w)_j &= (X \cdot v)_j + 1, \\
(X \cdot w)_\ell &= (X \cdot v)_\ell, \ell \neq i
\end{align*}
\]

As a result for \( x_i > n \) we have

\[ X(x_i) \cdot u = X(n) \cdot u + (x_i - n)[(X(n + 1) \cdot u - X(n) \cdot u) \text{ } \]}

In particular if \( u \) and \( v \) satisfy the hypothesis of the lemma, the two functions \( X(x_i) \cdot u \) and \( X(x_i) \cdot v \) are equal for all \( x_i \in \mathbb{N} \).

The next claim is a refinement of Lemma 1 and shows that the equivalence \( \equiv_{\text{gram}} \) can be computed.

**Proposition 1** Two words \( v \) and \( v \) define the same mappings on the rows if and only if they have the same length and their restrictions coincide over the subset \( \{x_1, \ldots, x_k\} \in \mathbb{N}^k \setminus \{0 \leq x_i \leq \max\{|u|, |v|\} + 1, 1 \leq i \leq k\} \).

**Proof** Observe that if they have the same restriction over the set of vectors of maximum coordinate \( \max\{|u|, |v|\} + 1 \) then by Lemma 1 they have the same length \( n \). Let \( F \) be the set of partial functions \( f : \{1, \ldots, n\} \mapsto \mathbb{N} \) and let \( \text{supp}(f) \) be their domain of definition. With \( f \in F \) let \( X_f = (\alpha_1, \ldots, \alpha_k) \) be a formal vector where \( \alpha_j \) is a constant \( f(j) \) when \( j \in \text{supp}(f) \) and \( \alpha_j \) is a variable otherwise. The set of variables is \( \{x_{i_1}, \ldots, x_{i_p}\} \) where \( i_1, \ldots, i_p \) is the ordered subset \( \{1, \ldots, n\} \setminus \text{supp}(f) \). Given a word \( u \in A^* \) we interpret \( X_f \cdot u \) as a function of \( \mathbb{N}^p \) into \( \mathbb{N}^k \) in the natural way. For
(\xi_1, \ldots, \xi_p) \in \mathbb{N}^p$ the value of the function is denoted $X_f(\xi_1, \ldots, \xi_p) \cdot u$. We claim that for all $f \in F$, for all formal vectors $X_f$ and all words $u, v$, it holds

$$\forall \xi_1, \ldots, \xi_p : X_f(\xi_1, \ldots, \xi_p) \cdot u = X_f(\xi_1, \ldots, \xi_p) \cdot v \quad \text{if } \forall \xi_1, \ldots, \xi_p \leq n + 1 : X_f(\xi_1, \ldots, \xi_p) \cdot u = X_f(\xi_1, \ldots, \xi_p) \cdot v \quad (3)$$

We show that the bottom statement in (3) implies the top statement. We proceed by induction on $p$ by skipping the trivial case $p = 0$ and the case $p = 1$ which is covered by Lemma 4. Define $f' \in F$ by the condition

$$f'(i) = \begin{cases} f(i) & \text{if } i \in \text{supp}(f) \\ \xi_i & \text{if } i = i_p \\ \text{undefined otherwise} \end{cases}$$

If $\xi_{i_p} \leq n + 1$ then we have

$$X_f(\xi_1, \ldots, \xi_p) \cdot u = X_{f'}(\xi_1, \ldots, \xi_{p-1}) \cdot u \overset{\text{induction}}{=} X_{f'}(\xi_1, \ldots, \xi_{p-1}) \cdot v = X_f(\xi_1, \ldots, \xi_p) \cdot v$$

Now define $g \in F$ by the condition

$$g(i) = \begin{cases} f(i) & \text{if } i \in \text{supp}(f) \\ \xi_i & \text{if } 1 \leq s < r \\ \text{undefined otherwise} \end{cases}$$

Observe that $\text{supp}(g) = \{1, \ldots, n\} \setminus [i_r]$. Then we have

$$X_f(\xi_1, \ldots, \xi_p) \cdot u = X_g(\xi_i_p) \cdot u \overset{\text{Lemma 4}}{=} X_{f'}(\xi_{i_p}) \cdot v = X_f(\xi_1, \ldots, \xi_p) \cdot v$$

\[\square\]

The bound $n + 1$ is not sharp and could probably be improved by using Lemma 4 at the price of a more confuse statement for Proposition 1.

### 3 Congruence in the case of three generators

Henceforth we work with the grammic monoid of rank 3. We specialize Lemma 1 to three generators. The set of rows is identified with the set of triples $(x_1, x_2, x_3) \in \mathbb{N}^3$
and the alphabet is renamed as \( \{1, 2, 3\} \). Then

\[
(x_1, x_2, x_3) \cdot 1 = \begin{cases} 
(x_1 + 1, x_2 - 1, x_3) & \text{if } x_2 > 0 \\
(x_1 + 1, x_2, x_3 - 1) & \text{if } x_2 = 0 \text{ and } x_3 > 0 \\
(x_1 + 1, x_2, x_3) & \text{otherwise}
\end{cases}
\]  

(4)

\[
(x_1, x_2, x_3) \cdot 2 = (x_1, x_2 + 1, \max(0, x_3 - 1))
\]

(5)

\[
(x_1, x_2, x_3) \cdot 3 = (x_1, x_2, x_3 + 1)
\]

Because of the very definition of Young tableaux the sequence of rows has the form

\[3^a 2^b 3^c 1^d 2^e 3^f\]

with

\[0 \leq a \leq b \leq d, \text{ and } b + c \leq d + e.\]  

(5)

When speaking of row representatives the conditions (5) are always implicitly assumed and will not be recalled.

### 3.1 Characterization of pairs of congruent words

We start with necessary conditions for two \( \equiv \) - representatives to be \( \equiv_{\text{gram}} \)-equivalent.

**Lemma 5** If two words \( u, v \) are \( \equiv_{\text{gram}} \)-equivalent then the row representatives of \( u \) and \( v \) in the plactic monoid are of the form

\[3^a 2^b 3^c 1^d 2^e 3^f \quad \text{and} \quad 3^{a'} 2^{b'} 3^{c'} 1^d 2^e 3^f\]  

with \( a, a' \leq b \) and \( a + c = a' + c' \).

If furthermore \( b = b' = 0 \) or \( d = d' = 0 \) then the two words are \( \equiv \)-equivalent.

**Proof** Consider two arbitrary \( \equiv_{\text{gram}} \)-equivalent row normal forms

\[3^a 2^b 3^c 1^d 2^e 3^f \quad \text{and} \quad 3^{a'} 2^{b'} 3^{c'} 1^d 2^e 3^{f'}\]

By Lemma 2 they have the same bottom rows, hence \( d = d' \), \( e = e' \) and \( f = f' \). Because of Lemma 1 they have the same commutative image which implies \( b = b' \).

The condition \( d = d' = 0 \) implies that the two representatives have no occurrence of 1 thus by Corollary 1 that they are \( \equiv \) equivalent. By Corollary 2 condition \( b = b' = 0 \) implies that they are \( \equiv \) equivalent. Equality \( a + c = a' + c' \) is yet another consequence of the commutative equivalence. \( \square \)

**Proposition 2** Let \( u, v \) be two words which are not \( \equiv \)-equivalent. They are \( \equiv_{\text{gram}} \)-equivalent if and only if their representatives in row normal form are of the form

\[u = 3^a 2^b 3^c 1^d 2^e 3^f \quad \text{and} \quad v = 3^{a'} 2^{b'} 3^{c'} 1^d 2^e 3^{f'}\]  

(7)
with \( a \neq a', a + c = a' + c' \) and \( c, c' \leq e \).

**Proof** We compute the action of \( 3^a 2^b 3^c 1^d 2^e 3^f \) on \((x_1, x_2, x_3) \in \mathbb{N}^3\) with \( a \leq b \leq d \) and \( b + c \leq d + e + f \). Observe that

\[
(x_1, x_2, x_3) \cdot 3^a 2^b 3^c 1^d 2^e 3^f = (x_1, x_2, x_3) \cdot 3^a 2^b 3^c 1^d 2^e 3^f + (0, 0, f)
\]

Thus we may suppose \( f = 0 \) and it suffices to compute \((x_1, x_2, x_3) \cdot 3^a 2^b 3^c 1^d 2^e\). Because of Lemma 5 it all amounts to determine under which conditions, for fixed \( b, d, e \), there exist different pairs \((a, c)\) which define the same mappings on the rows. We compute via the rules (4)

\[
(x_1, x_2, x_3) \quad \xrightarrow{3^a} \quad (x_1, x_2, x_3 + a) \\
\xrightarrow{2^b} \quad (x_1, x_2 + b, \max(0, x_3 + a - b)) \\
\xrightarrow{3^c} \quad (x_1, x_2 + b, \max(c, x_3 + a - b + c))
\]

**Case 1** If \( x_2 + b \geq d \) then

\[
(x_1, x_2 + b, \max(c, x_3 + a - b + c)) \\
\xrightarrow{1^d} \quad (x_1 + d, x_2 + b - d, \max(c, x_3 + a - b + c)) \\
\xrightarrow{2^e} \quad (x_1 + d, x_2 + b - d + e, \max(0, \max(c, x_3 + a - b + c)) - e) \\
= (x_1 + d, x_2 + b - d + e, \max(0, c - e, x_3 + a - b + c - e))
\]

**Case 2** If \( x_2 + b < d \) then

\[
(x_1, x_2 + b, \max(c, x_3 + a - b + c)) \\
\xrightarrow{1^d} \quad (x_1 + d, 0, \max(0, \max(c, x_3 + a - b + c)) - (d - (x_2 + b))) \\
= (x_1 + d, 0, \max(0, c - (d - (x_2 + b)), x_3 + a - b + c - (d - (x_2 + b))) \\
= (x_1 + d, 0, \max(0, x_2 + c - d + b, x_2 + c - d + x_3 + a)) \\
\xrightarrow{2^e} \quad (x_1 + d, e, \max(0, \max(0, x_2 + c - d + b, x_2 + c - d + x_3 + a)) - e) \\
= (x_1 + d, e, \max(0, x_2 + c - d + b - e, x_2 + c - d + x_3 + a - e))
\]

The same computation holds with \( a' \) and \( c' \) substituted for \( a \) and \( c \). Observe that in both cases 1 and 2 the first two components do not depend on \( a \) and \( c \).

Consider the third component. In expression \( \max(0, c - e, x_3 + a - b + c - e) \) of case 1, for \( x_3 = 0 \), because of \( a - b \leq 0 \) the condition \( c - e > 0 \) yields \( \max(0, c - e, x_3 + a - b + c - e) = c - e \). Thus if \( c \geq e \), the two words are equivalent if and only if \( c' = c' \) and \( a = a' \). If \( c - e < 0 \) then for whatever values for \( x_3 \) we have \( \max(0, c - e, x_3 + a - b + c - e) = \max(0, x_3 + a - b + c - e). \) Then the two words are \( \equiv_{\text{gram}} \)-equivalent if and only if \( e' < c \) because \( \max(0, x_3 + a - b + c - e) = \max(0, x_3 + a' - b + c' - e) \). \( \square \)
The previous proposition allows us to compute the number of \( \equiv \)-classes included in a given \( \equiv_{\text{gram}} \)-class. If this number is different from 1 then a row normal form of a \( \equiv \)-class looks like \( 3^a 2^b 3^c 1^d 2^e 3^f \) with \( 0 < b, d \) and \( c \leq e \). There are as many \( \equiv \)-classes as there are pairs \((a', c')\) such that \( a + c = a' + c' \) and \( a', c' \geq 0 \), i.e., as many \( c' \) that satisfy the inequalities \( \max\{a + c - b, 0\} \leq c' \leq \min\{e, a + c\}\).

**Example 2** The following four row normal forms define the same \( \equiv_{\text{gram}} \)-class.

\[
\begin{align*}
ccc/bbb/aaabbbbb & \equiv_{\text{gram}} cc/bbbc/aaabbbbb \\
c/bbcc/aaabbbbb & \equiv_{\text{gram}} bbbc/aaabbbbb
\end{align*}
\]

### 3.2 Projections on subalphabets

Given \( B \subseteq A \) we let \( \pi_B \) denote the projection of \( A^* \) over \( B^* \), i.e., the morphism defined by \( \pi_B(a) = a \) if \( a \in B \) and \( 1 \) if \( a \not\in B \). Routine computations show that over \( A = \{1, 2, 3\} \), if two words are \( \equiv_{\text{gram}} \)-equivalent, then so are their projections over all subalphabets \( B \subseteq A \). However, the converse does not hold. Indeed, we have \( 2331122 \not\equiv_{\text{gram}} 23331122 \) but the row normal forms of their projections over the subalphabets \( \{1, 2\}, \{1, 3\} \) and \( \{2, 3\} \) are equal, respectively 21122, 33113 and 33222.

With \( |A| = 4 \) the projections of \( \equiv_{\text{gram}} \)-equivalent words are no longer necessarily \( \equiv_{\text{gram}} \)-equivalent. Indeed, we have \( 4213 \equiv_{\text{gram}} 2413 \) since for both words the image of \((x_1, x_2, x_3, x_4)\) is \((x_1 + 1, x_2, x_3 + 1, \max\{x_4 - 1, 0\})\) if \( x_3 = 0 \) and \((x_1 + 1, x_2, x_3, x_4)\) otherwise. However, \( 421 \not\equiv_{\text{gram}} 241 \) because the bottom rows of the tableaux are respectively 1 and 14.

### 4 The Grammic monoid over \( \{1, 2, 3\} \)

**Theorem 1** Let \( \sim \) be the relation consisting of the pairs \((xuy, xvy)\) \( \in \{0, 1, 2\}^* \) where \((u, v)\) is one of the rules in (1) or the new rule

\[
(3212, 2132).
\]  

Then \( \equiv_{\text{gram}} \) is the transitive closure \( \ast \) of the relation \( \sim \). Furthermore, if the words are in the form of Lemma 5, the number of applications of the rule (8) to prove that they are \( \equiv_{\text{gram}} \)-equivalent is equal to \( |c - c'| \).

**Proof** We have \( 3212 \equiv_{\text{gram}} 2132 \) because for both words the image of \((x_1, x_2, x_3)\) is \((x_1 + 1, x_2 + 1, \max\{x_3 - 1, 0\})\) thus \( u \sim v \) implies \( u \equiv_{\text{gram}} v \). We prove the converse. By Corollary 1 we may assume that \( u \) and \( v \) have an occurrence of each of the three letters. We start with two different \( \equiv_{\text{gram}} \)-congruent row normal forms

\[
w(c) = 3^a 2^b 3^c 1^d 2^e 3^f \quad \text{and} \quad w(c') = 3^a' 2^b 3^c' 1^d 2^e 3^f
\]
Observe that it suffices to consider the case \( c' = c + 1 \) since if \( c' = c + t \leq e' \) then we have \( w(c) \sim w(c + 1) \sim \cdots \sim w(c + t) \). We want to show that by successive substitutions of one side of Eq. (8) for the other side we can rewrite \( w(c) \) into \( w(c') \).

We proceed by case study where Case 1 assumes \( d \leq b + c + 1 \) and Case 2 assumes \( d > b + c + 1 \). We implicitly and repeatedly use the fact that two columns commute if and only if one is a subset of the other.

Case 1. See Fig. 1 for a visual support. The two words are of the following column normal form.

\[
(321)^a (21)^b (31)^y (32)^\delta 2^e 3^f \quad \text{and} \quad (321)^{a-1} (21)^{\beta+1} (31)^y (32)^{\delta+1} 2^e 1^f \quad \epsilon > \gamma
\]

We may factorize the words respectively as

\[
\begin{align*}
(321)^a \cdot (321)^{\beta} (31)^y (32)^\delta 2^e 3^f \\
(321)^a \cdot (21)^{\beta+1} (31)^y (32)^{\delta+1} 2^e 1^f
\end{align*}
\]

with \( \epsilon \geq \gamma + 1, \delta \geq 0 \)

Therefore it suffices to show how to pass from

\[
\begin{align*}
w_1 &= (321)(21)^{\beta}(31)^y(32)^{\delta}2^{\gamma+1} \\
w_2 &= (21)^{\beta+1}(31)^y(32)^{\delta+1}2^{\gamma}
\end{align*}
\]

This is obtained by applying the elementary rules concerning the commutation of two columns and equalities of the form \( 1^y (32)^y = (31)^{y} 2^y \)

\[
\begin{align*}
(321)(21)^{\beta}(31)^y(32)^{\delta}2^{\gamma} &
\equiv (321)(21)^{\beta}1^y (32)^{\delta+\gamma} 2 \\
\equiv (21)^{\beta}1^y (321)^y (32)^{\delta+\gamma} 2 \\
\equiv (21)^{\beta}1^y (321)^y (32)^{\delta+\gamma} \\
\sim (21)^{\beta+1}1^y (21)(32)(32)^{\delta+\gamma} \\
\equiv (21)^{\beta+1}1^y (32)^{\delta+\gamma+1} \\
\equiv (21)^{\beta+1}(31)^y 2^y (32)^{\delta+1} \\
\equiv (21)^{\beta+1}(31)^y (32)^{\delta+1}2^y
\end{align*}
\]
thus $w_1 \sim w_2$. Furthermore the computation used the substitution $3212 \sim (21)(32)$ on a unique occurrence which means that there exist two word $u, v$ such that

$$w_1 \equiv u3212v \sim u2312v \equiv w_2.$$ 

Case 2 The two words are of the following form, cf. Fig. 2

$$((321)^\varepsilon (21)^\beta (31)^\gamma 1^\delta 2^\epsilon 3^f)_{\varepsilon \geq \gamma + 1, \delta > 0}$$

and as above is suffices to show how to pass from $w_1 = (321)(21)^\beta (31)^\gamma 1^\delta 2^\gamma + 1$ to $w_2 = (21)^\beta + 1(31)^\gamma + 11^\delta - 12^\gamma + 1$. This results from the following boring computation

$$
\begin{align*}
(321)(21)^\beta (31)^\gamma 1^\delta 2^\gamma + 1
\equiv (321)(21)^\beta 1^\delta (31)^\gamma 2^\gamma 2 \\
\equiv (321)(21)^\beta 1^\gamma + \delta (32)^\gamma 2 \\
\equiv (21)^\beta 1^\gamma + \delta (32)^\gamma \\
\sim (21)^\beta 1^\gamma + \delta (21)(32)^\gamma \\
\equiv (21)^\beta + 11^\delta (32)^\gamma + 1 \\
\equiv (21)^\beta + 11^\delta (31)^{\gamma + 1}2^\gamma 2 \\
\equiv (21)^\beta + 11^\delta (31)^{\gamma + 1}2^\gamma + 1 
\end{align*}
$$

As in the previous case we have $w_1 \sim w_2$ and the computation used the substitution $3212 \sim 2132$ on a unique occurrence. \qed

5 Further works

Our characterization of $\equiv_{\text{gram}}$-congruent words over a three-letter alphabet allows one to answer some questions concerning arbitrary alphabets. E.g., in [1] the authors show the following property for all alphabets $B \subseteq A$: if two words in $B^*$ have the same action of all columns labelled by $B$ they have the same action on all columns labeled by $A$. It is routine to check that the result holds when $|B| \leq 3$ and when “row” is substituted for “column”.

A modest amount of computation led us to conjecture that in the case of four letters, the congruence is generated by the Knuth rules plus the rule $dbac = badc$ with $a < b \leq c < d$ which could probably be proved or disproved by case study.
performed by a theorem prover. However, it is not clear how far a brute force approach could be drawn and therefore a more conceptual approach is desirable.

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