ON MULTI-VARIABLE ZASSENHAUS FORMULA

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ABSTRACT. In this paper, we give a recursive algorithm to compute the multi-variable Zassenhaus formula

\[ e^{X_1 + X_2 + \cdots + X_n} = e^{X_1}e^{X_2}\cdots e^{X_n} \prod_{k=2}^\infty e^{W_k} \]

and derive an effective recursion formula of \( W_k \).

1. Introduction

The celebrated Baker-Campbell-Hausdorff (BCH) is a fundamental identity in Lie theory [1, 2, 3] connecting Lie algebra with Lie group. The BCH says that for any linear operators \( X, Y \) in a bounded Hilbert space one has the formula

\[ e^X e^Y = e^{X+Y+\sum_{k=2}^\infty Z_k(X,Y)}, \tag{1.1} \]

where \( \exp \) is defined in the usual sense and \( Z_k(X,Y) \) is a degree \( k \) homogeneous Lie polynomial in the noncommutative variables \( X \) and \( Y \). The first few terms are

\[ Z_2 = \frac{1}{2} [X,Y], \quad Z_3 = \frac{1}{12} ([X,Y,X] - [Y,X,Y]), \]
\[ Z_4 = \frac{1}{24} [X,Y,Y,X]. \]

and the general expressions of \( Z_k(X,Y) \) can be explicitly computed by combinatorial formulas.

The dual form of the BCH is the famous Zassenhaus formula which establishes that the exponential \( e^{X+Y} \) can be uniquely decomposed as

\[ e^{X+Y} = e^X e^Y \prod_{m=2}^\infty e^{W_m(X,Y)} \tag{1.2} \]
\[ = e^X e^Y e^{W_2(X,Y)} e^{W_3(X,Y)} \cdots e^{W_k(X,Y)} \cdots, \]

where \( W_k(X,Y) \) is a homogeneous Lie polynomial in \( X \) and \( Y \) of degree \( k \) [4]. The first few terms are

\[ W_2 = -\frac{1}{2} [X,Y], \quad W_3 = \frac{1}{3} [Y,[X,Y]] + \frac{1}{6} [X,[X,Y]], \]
\[ W_4 = -\frac{1}{24} [X,[X,Y]] - \frac{1}{8} ([Y,[X,[X,Y]]] + [Y,[Y,[X,Y]]]). \]

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There are several methods to compute $W_k$ [5, 6, 7, 8]. In particular, a recursive algorithm has been proposed in [9] to express directly $W_k$ with the minimum number of independent commutators required at each degree $k$.

Similar to the BCH formula, the Zassenhaus formula is useful in many different fields: $q$-analysis in quantum groups [10], quantum nonlinear optics [13], the Schrödinger equation in the semiclassical regime [12], and splitting methods in numerical analysis [11], etc.

We now consider the multivariate BCH and Zassenhaus formulas. It is easy to obtain the multivariable BCH formula by repeatedly using the usual BCH:

$$e^{X_1}e^{X_2}\cdots e^{X_n} = e^{X_1+X_2+\cdots+X_n+\sum_{m=2}^{\infty} z_m(X_1,X_2,\ldots,X_n)},$$ (1.3)

where $z_m$ is a Lie polynomial in the $X_i$ of degree $m$. On the other hand, we also have the multivariable Zassenhaus formula

$$e^{X_1+X_2+\cdots+X_n} = e^{X_1}e^{X_2}\cdots e^{X_n} \prod_{k=2}^{\infty} e^{W_k}$$ (1.4)

where the product is ordered and $W_k$ is a homogeneous Lie polynomial in the $X_i$ of degree $k$. However, it is more complicated to express $W_k$ in terms of $X_i$’s.

The existence of the formula (1.4) is a consequence of Eq. (1.3). In fact, it is clear that $e^{-X_1}e^{X_1+X_2+\cdots+X_n} = e^{X_2+X_3+\cdots+X_n+D}$, where $D$ involves Lie polynomials of degree > 1. Then $e^{-X_2}e^{X_2+X_3+\cdots+X_n+D} = e^{X_3+X_4+\cdots+X_n+W_2+D_1}$, where $D_1$ is an infinite Lie power series in the $X_i$ with minimum degree > 2. Note that $W_2' \neq W_2$, we need to repeat the process $(n - 1)$ times to determine $W_2$, i.e.

$$e^{-X_{n-1}}e^{X_{n-1}+X_n+W_2^{(n-3)}+W_3^{(n-4)}+\cdots+W_{n-2}'+D_{n-3}}$$

$$= e^{X_n+W_2+W_3^{(n-3)}+\cdots+W_{n-2}'+D_{n-2}}$$

where $D_{n-2}$ involves Lie polynomials of degree > $(n - 1)$. Finally, we can get the formula (1.4) by repeating the process.

In this paper, we will give a new recursive algorithm to compute $W_k$ in (1.4). Our method is inspired by the recent algorithm in [9], and our formula is based on a new formula for $f_{1,k}$ using compositions of integers.

The paper is organized as follows. In Section 2, we give our recursive algorithm and a concrete procedure to compute $W_k, k = 1, 2, 3, 4, 5$. In Section 3, we establish a combinatorial formula of $f_{1,k}$ (see Theorem 3.1). We will show that our formula can give a slightly better recursion formula of $W_k$ when $k > 5$ in Theorem 3.2. Finally we use examples to show how the $f_{1,k}$ are used to derive Lie polynomial formulas of $W_k$ in terms of the operators $X_1, \ldots, X_n$. The latter set of formulas are expected be useful in the quantum control problem.

2. Multivariable Zassenhaus terms

2.1. A recurrence. For the operators $X_1, \ldots, X_n$ we consider the following function of $t$:

$$e^{t(X_1+X_2+\cdots+X_n)} = e^{tX_1}e^{tX_2}\cdots e^{tX_n}e^{2W_2}e^{3W_3}\cdots$$ (2.1)
where the $W_k$ can be determined by differential equations step by step, and it is easy to see that $W_k$ is a polynomial of degree $k$ in the $X_i$. Note that the multivariable
Zassenhaus formula (1.4) is the case when $t = 1$.

First we consider the iterated system of equations

$$R_1(t) = e^{-tX_n} \cdots e^{-tX_2} e^{-tX_1} e^{t(X_1 + X_2 + \cdots + X_n)}, \quad (2.2)$$
$$R_m(t) = e^{-tW_m} R_{m-1}(t), \quad m \geq 2. \quad (2.3)$$

It follows from (2.3) that

$$R_m(t) = e^{m+1W_{m+1}} e^{m+2W_{m+2}} \cdots, \quad m \geq 1. \quad (2.4)$$

We then take the logarithmic differentiation

$$F_m(t) = B_m(t) R_m(t)^{-1} \quad m \geq 1. \quad (2.5)$$

For $m = 1$, we have that

$$F_1(t) = -X_n - e^{-tad_{X_n}} X_{n-1} - e^{-tad_{X_n}} e^{-tad_{X_n-1}} X_{n-2} - \cdots - e^{-tad_{X_n}} \cdots e^{-tad_{X_2}} X_1$$
$$= e^{-tad_{X_n}} (e^{-tad_{X_n-1}} \cdots e^{-tad_{X_2}} e^{-tad_{X_1}} - I) X_n$$
$$+ e^{-tad_{X_n}} e^{-tad_{X_n-1}} (e^{-tad_{X_n-2}} \cdots e^{-tad_{X_2}} e^{-tad_{X_1}} - I) X_{n-1}$$
$$+ e^{-tad_{X_n}} e^{-tad_{X_n-1}} e^{-tad_{X_n-2}} (e^{-tad_{X_n-3}} \cdots e^{-tad_{X_2}} e^{-tad_{X_1}} - I) X_{n-2} + \cdots$$
$$+ e^{-tad_{X_n}} e^{-tad_{X_n-1}} \cdots e^{-tad_{X_3}} (e^{-tad_{X_2}} e^{-tad_{X_1}} - I) X_2$$
$$+ e^{-tad_{X_n}} e^{-tad_{X_n-1}} \cdots e^{-tad_{X_2}} (e^{-tad_{X_1}} - I) X_2$$
$$= \sum_{k=1}^{\infty} (-t)^k \sum_{i=2}^{n} \sum_{j_1 + \cdots + j_i \geq 1} \frac{ad_{X_n}^{j_n} \cdots ad_{X_2}^{j_2} ad_{X_1}^{j_1}}{j_1! j_2! \cdots j_i!} X_i,$$

where $ad_AB = [A, B]$ and we have used the well-known formula

$$e^{A}Be^{-A} = e^{ad_AB} = \sum_{n \geq 0} \frac{1}{n!} ad_A^n B,$$

as well as the fact that $e^{ad_X} X = X$. Write

$$F_1(t) = \sum_{k=1}^{\infty} f_{1,k} t^k, \quad (2.6)$$

then

$$f_{1,k} = (-1)^k \sum_{i=2}^{n} \sum_{j_1 + \cdots + j_i \geq 1} \frac{ad_{X_n}^{j_n} \cdots ad_{X_2}^{j_2} ad_{X_1}^{j_1}}{j_1! j_2! \cdots j_i!} X_i. \quad (2.7)$$
A similar expansion can be obtained for $F_m(t)$, $m \geq 2$, by using $R_m(t)$ in (2.3). More specifically,

$$F_m(t) = -m W_m t^{m-1} + e^{-t m W_m} R'_{m-1}(t) R_{m-1}^{-1}(t) e^{t m W_m}$$

$$= -m W_m t^{m-1} + e^{-t m W_m} F_{m-1}(t)$$

$$= e^{-t m W_m} (F_{m-1}(t) - m W_m t^{m-1}).$$

Writing $F_m(t) = \sum_{k=m}^{\infty} f_{m,k} t^k$, we immediately get that

$$f_{m,k} = \sum_{j=0}^{[\frac{k}{m}]-1} \frac{(-1)^j}{j!} a d W_m f_{m-1,k-mj}, \quad k \geq m \quad (2.8)$$

where $[\frac{k}{m}]$ denotes the integer part of $\frac{k}{m}$.

On the other hand, if we take the logarithmic derivative of $R_m(t)$ using the expression (2.4), we arrive at

$$F_m(t) = (m + 1) W_{m+1} t^m + \sum_{j=m+2}^{\infty} j t^{j-1} e^{t^{j-1} a d W_{m+1}} \cdots e^{t^{j-1} a d W_1} W_j. \quad (2.9)$$

Comparing the coefficients of the terms $t$, $t^2$, $t^3$ and $t^4$ in (2.6) and (2.9) for $F_1(t)$, we get that

$$f_{1,1} = 2 W_2, \quad f_{1,2} = 3 W_3, \quad f_{1,3} = 4 W_4, \quad f_{1,4} = 5 W_5 + 3[W_2, W_3],$$

so that

$$W_2 = \frac{1}{2} f_{1,1}, \quad W_3 = \frac{1}{3} f_{1,2}, \quad W_4 = \frac{1}{4} f_{1,3}, \quad W_5 = \frac{1}{5} f_{1,4} - \frac{1}{10} [f_{1,1}, f_{1,2}] \quad (2.10)$$

Similarly, comparing (2.8) and (2.9), we get

$$f_{m,m} = (m + 1) W_{m+1},$$

therefore

$$W_{m+1} = \frac{1}{m+1} f_{m,m} = \frac{1}{m+1} f_{\frac{m}{2},m}, \quad m \geq 4,$$

i.e.

$$W_m = \frac{1}{m} f_{\frac{m-1}{2},m-1}, \quad m \geq 5 \quad (2.12)$$

2.2. Examples of $W_k$. When $k = 1$ in the expression (2.7), the summation of the first $i - 1$ terms is already at least 1, so we have the formula

$$f_{1,1} = - \sum_{i=2}^{n} \sum_{j_1 + \cdots + j_{i-1} = 1} \frac{a d^{j_{i-1}} X_{i-1} \cdots a d^{j_2} X_2 a d^{j_1} X_1}{j_1! j_2! \cdots j_{i-1}!} X_i$$

$$= \sum_{1 \leq i < j \leq n} [X_j, X_i].$$

Thus

$$W_2 = \frac{1}{2} f_{1,1} = \frac{1}{2} \sum_{1 \leq i < j \leq n} [X_j, X_i].$$
Similarly for $k = 2$ in (2.7), we have

$$f_{1,2} = \sum_{i=2}^{n} \sum_{j_1, \ldots, j_{i-1} \geq 1, j_1 + \cdots + j_{i-1} = 2} \frac{ad_{X_i} \cdots ad_{X_{j_1}} ad_{X_{j_i}}}{j_1! j_2! \cdots j_{i-1}!} X_i$$

$$= \sum_{i=2}^{n} \left( \sum_{j_1, \ldots, j_{i-1} = 1, i \leq \leq n} \frac{ad_{X_i} ad_{X_{j_{i-1}}} \cdots ad_{X_{j_i}}}{j_1! j_2! \cdots j_{i-1}!} X_i \right) + \sum_{i=2}^{n} \frac{ad_{X_{j_{i-1}}} \cdots ad_{X_{j_i}}}{m_1! m_2!} X_i$$

$$= - \sum_{i \geq j} [X_i [X_i, X_j]] + \sum_{i > j} \frac{1}{m_j} ([X_i [X_{j_2}, X_{j_1}] - [X_i, [X_{j_2}, X_{j_1}]])$$

$$= - \sum_{i > j} [X_i [X_i, X_j]] - 2 \sum_{i > j} [X_i [X_i, X_j]] + \sum_{i > j} [[X_i X_j, X_k]] + \sum_{i > j} \frac{1}{2} [[X_i X_j, X_j]]$$

$$= \sum_{1 \leq i, j, k \leq n} [[X_j, X_i], X_k] + \frac{1}{2} \sum_{1 \leq i, j \leq n} [[X_j, X_i], X_i],$$

where $m_i$ is the multiplicity of $j_i$

Therefore

$$W_3 = \frac{1}{3} f_{1,2} = \frac{1}{3} \sum_{1 \leq i, j, k \leq n} [[X_j, X_i], X_k] + \frac{1}{6} \sum_{1 \leq i, j \leq n} [[X_j, X_i], X_i].$$

We list the first few other terms as follows:

$$f_{1,3} = -\left( \sum_{1 \leq i, j, k \leq n} [X_i, [X_k, [X_i, X_j]]] + \frac{1}{2} \sum_{1 \leq i, j, k \leq n} [X_k, [X_k, [X_i, X_j]]]$$

$$+ \frac{1}{2} \sum_{1 \leq i, j, k \leq n} [X_k, [X_i, [X_i, X_j]]] + \frac{1}{6} \sum_{1 \leq i, j \leq n} [X_i, [X_i, [X_i, X_j]]])$$

$$= \frac{1}{6} \sum_{1 \leq i, j \leq n} [[[X_j, X_i], X_i], X_i] + \frac{1}{2} \sum_{1 \leq i, j \leq n} [[[X_j, X_i], X_i], X_k]$$

$$+ [[[X_j, X_i], X_k], X_k] + \sum_{1 \leq i, j \leq n} [[[X_j, X_i], X_k], X_i].$$
so that

\[
W_4 = \frac{1}{24} \sum_{1 \leq i < j \leq n} [[[X_j, X_i], X_i], X_i] + \frac{1}{8} \sum_{1 \leq i < j, k \leq n} [[[X_j, X_i], X_i], X_k]
+ [[[X_j, X_i], X_k], X_k]) + \frac{1}{4} \sum_{1 \leq i < j, k \leq n} [[[X_j, X_i], X_i], X_k].
\]

\[
f_{1,4} = \frac{1}{24} \sum_{1 \leq i < j \leq n} [[[X_j, X_i], X_i], X_i], X_i] + \frac{1}{6} \sum_{1 \leq i < j, k \leq n} [[[X_j, X_i], X_i], X_i], X_k]
+ [[[X_j, X_i], X_k], X_k]) + \frac{1}{4} \sum_{1 \leq i < j, k \leq n} [[[X_j, X_i], X_i], X_k], X_k]
+ \frac{1}{2} \sum_{1 \leq i < j, k \leq n, k < l \leq n} ( [[[X_j, X_i], X_k], X_i], X_k) + [[[X_j, X_i], X_k], X_k])
+ [[[X_j, X_i], X_k], X_k]) + \sum_{1 \leq i < j, k \leq n, k < l \leq n} [[[X_j, X_i], X_k], X_k], X_k].
\]

\[
f_{1,5} = \frac{1}{120} \sum_{1 \leq i < j \leq n} [[[X_j, X_i], X_i], X_i], X_i] + \frac{1}{24} \sum_{1 \leq i < j, k \leq n} [[[X_j, X_i], X_i], X_i], X_k]
+ [[[X_j, X_i], X_k], X_k]) + \frac{1}{12} \sum_{1 \leq i < j, k \leq n} [[[X_j, X_i], X_i], X_k], X_k]
+ [[[X_j, X_i], X_k], X_k]) + \frac{1}{6} \sum_{1 \leq i < j, k \leq n, k < l \leq n} ( [[[X_j, X_i], X_k], X_i], X_k) + [[[X_j, X_i], X_k], X_k])
+ \frac{1}{4} \sum_{1 \leq i < j, k \leq n, k < l \leq n} ( [[[X_j, X_i], X_k], X_k], X_k) + [[[X_j, X_i], X_k], X_k])
+ \frac{1}{2} \sum_{1 \leq i < j, k \leq n, k < l \leq n} [[[X_j, X_i], X_k], X_k], X_k]
+ [[[X_j, X_i], X_k], X_k]) + \sum_{1 \leq i < j, k \leq n, k < l < m \leq n} [[[X_j, X_i], X_k], X_k], X_k], X_m].
\]

3. Iteration Formulas

To reveal the explicit rule for \( f_{1,k}(k \geq 1) \) based on the computations we gave in Section 2, we recall the definition of partitions and compositions [14].
3.1. **Formulation in terms of partitions.** A partition of a positive integer \( m \) is an integral unordered decomposition \( m = \lambda_1 + \cdots + \lambda_l \) such that \( \lambda_1 \geq \cdots \geq \lambda_l > 0 \), denoted by \( \lambda = (\lambda_1 \lambda_2 \ldots \lambda_l) \) and \( \lambda \vdash m \). Here \( \lambda_i \) are called the parts and \( l \) is the length of the partition. A composition is an ordered integral decomposition of \( m \): \( m = \lambda_1 + \cdots + \lambda_l \) such that \( \lambda_i > 0 \) and denoted by \( \lambda \models m \), in another words, compositions of \( m \) are obtained by permuting the unequal parts of the associated partition of \( m \). The set of partitions of \( m \) is denoted by \( \mathcal{P}(m) \) and the cardinality is denoted by \( p(m) \).

For example, the partitions of 4 are:

\[
\begin{align*}
(\lambda)^1 &= (4), \\
(\lambda)^2 &= (3, 1), \\
(\lambda)^3 &= (2, 2), \\
(\lambda)^4 &= (2, 1, 1), \\
(\lambda)^5 &= (1, 1, 1, 1).
\end{align*}
\]

Therefore, \( p(4) = 5 \). The associated compositions are distinct permutations of the partitions: \( (4), (3, 1), (1, 3), (2, 2), (2, 1, 1), (1, 2, 1), (1, 1, 2), (1, 1, 1, 1) \).

Accordingly the formulas of \( f_{1,k} \) go as follows. For \( f_{1,1} \), \( p(1) = 1 \),

\[
(\lambda)^1 = (1) : \sum_{1 \leq i < j \leq n} [X_j, X_i].
\]

For \( f_{1,2} \), \( p(2) = 2 \),

\[
\begin{align*}
(\lambda)^1 &= (2) : \frac{1}{2!} \sum_{1 \leq i < j \leq n} [[[X_j, X_i], X_i], \\
(\lambda)^2 &= (1, 1) : \sum_{1 \leq i < j, k \leq n} [[[X_j, X_i], X_k], X_k].
\end{align*}
\]

For \( f_{1,3} \), \( p(3) = 3 \),

\[
\begin{align*}
(\lambda)^1 &= (3) : \frac{1}{3!} \sum_{1 \leq i < j \leq n} [[[X_j, X_i], X_i], X_i], \\
(\lambda)^2 &= (2, 1) : \frac{1}{2!} \sum_{1 \leq i < j, k \leq n} ( [[[X_j, X_i], X_i], X_k] + [[[X_j, X_i], X_k], X_k]), \\
(\lambda)^3 &= (1, 1, 1) : \sum_{1 \leq i < j, k \leq n} [[[X_j, X_i], X_k], X_i].
\end{align*}
\]

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For $f_{1,4}$, $p(4) = 5$,

\[
(\lambda)^1 = (4) : \frac{1}{4!} \sum_{1 \leq i < j \leq n} [[[X_j, X_i], X_i], X_i], X_i],
\]

\[
(\lambda)^2 = (3, 1) : \frac{1}{3!} \sum_{1 \leq i < j, k \leq n} \left( [[[X_j, X_i], X_i], X_k] + [[[X_j, X_i], X_k], X_k]\right),
\]

\[
(\lambda)^3 = (2, 2) : \frac{1}{2!2!} \sum_{1 \leq i < j, k \leq n} [[[X_j, X_i], X_i], X_k], X_k],
\]

\[
(\lambda)^4 = (2, 1, 1) : \frac{1}{2!} \sum_{1 \leq i < j, k \leq n} \left( [[[X_j, X_i], X_i], X_k] + [[[X_j, X_i], X_k], X_k]\right)
+ [[[X_j, X_i], X_k], X_k], X_k],
\]

\[
(\lambda)^5 = (1, 1, 1, 1) : \sum_{1 \leq i < j, k \leq n} [[[X_j, X_i], X_k], X_k], X_h].
\]

We define the long commutator inductively as follows.

\[
[X_1, X_2] = X_1X_2 - X_2X_1,
\]

\[
[X_1, X_2, X_3] = [[X_1, X_2], X_3],
\]

\[
[X_1, X_2, X_3, \cdots, X_l] = [[X_1, X_2], X_3, \cdots, X_l].
\]

Fix a partition $\lambda = (\lambda_1\lambda_2 \cdots \lambda_l)$ of $k$, and for each composition out of $\lambda$: $(k_1k_2 \cdots k_l) \models k$ which is a rearrangement of $\lambda$ by permuting its parts, we associate the commutator

\[
[X_{j_1}, X_{i_1}, \cdots, X_{i_s}, X_{i_2}, \cdots, X_{i_l} \cdots, X_{i_l}] \tag{3.1}
\]

where the multiplicity of $i_s$ is $k_s$ for $1 \leq s \leq l$. For this reason, we will write (3.1) as $[X_jX_{i_1}^{k_1}X_{i_2}^{k_2} \cdots X_{i_l}^{k_l}]$. Then we have the following result.

**Theorem 3.1.** For each $k$, the following formula holds

\[
f_{1,k} = \sum_{(k_1 \cdots k_l) = k} \frac{1}{k_1!k_2! \cdots k_l!} \sum_{1 \leq i_1 < j_1 \leq n} \sum_{i_2 < i_3 < \cdots < i_l \leq n} [X_jX_{i_1}^{k_1}X_{i_2}^{k_2} \cdots X_{i_l}^{k_l}]
\]

3.2. **Determination of $W_m (m \geq 6$).** We have given the formulas of $W_k$ for $1 \leq k \leq 5$ in terms of $f_{1,k}$ (2.10). We now give the next a few terms as follows.
Theorem 3.2. For each $k \geq 2$ the exponents $W_m$ in the multi-variable Zassenhaus formula (1.4) for $m = 6k + i$, where $i = 0, 1, 2, 3, 4, 5$ are given by

\[
W_{6k} = \frac{1}{6k}(f_{2k-2,6k-1} - ad_{W_{2k-1}} f_{2k-2,4k} + \frac{1}{2!}ad_{W_{2k-1}}^2 f_{2k-2,2k+1} - ad_{W_{2k}} f_{2k-2,4k-1} + ad_{W_{2k}} ad_{W_{2k-1}} f_{2k-2,2k} - ad_{W_{2k+1}} f_{2k-2,4k-2} + ad_{W_{2k+1}} ad_{W_{2k-1}} f_{2k-2,2k-1} - ad_{W_{2k+2}} f_{2k-2,4k-3} - ad_{W_{2k+3}} f_{2k-2,4k-4} - \cdots - ad_{W_{3k}} f_{2k-2,3k}).
\]

(3.7)

\[
W_{6k+1} = \frac{1}{6k+1}(f_{2k-1,6k} - ad_{W_{2k}} f_{2k-1,4k} + \frac{1}{2!}ad_{W_{2k}}^2 f_{2k-1,2k} - ad_{W_{2k+1}} f_{2k-1,4k-1} + ad_{W_{2k+1}} ad_{W_{2k}} f_{2k-1,2k} - ad_{W_{2k+2}} f_{2k-1,4k-2} - ad_{W_{2k+3}} f_{2k-1,4k-3} - \cdots - ad_{W_{3k+1}} f_{2k-1,3k}).
\]

(3.8)

\[
W_{6k+2} = \frac{1}{6k+2}(f_{2k-1,6k+1} - ad_{W_{2k}} f_{2k-1,4k+1} + \frac{1}{2!}ad_{W_{2k}}^2 f_{2k-1,2k+1} - ad_{W_{2k+1}} f_{2k-1,4k} + ad_{W_{2k+1}} ad_{W_{2k}} f_{2k-1,2k} - ad_{W_{2k+2}} f_{2k-1,4k-1} - ad_{W_{2k+3}} f_{2k-1,4k-2} - ad_{W_{2k+4}} f_{2k-1,4k-3} - \cdots - ad_{W_{3k+1}} f_{2k-1,3k+1}).
\]

(3.9)

\[
W_{6k+3} = \frac{1}{6k+3}(f_{2k-1,6k+2} - ad_{W_{2k}} f_{2k-1,4k+2} + \frac{1}{2!}ad_{W_{2k}}^2 f_{2k-1,2k+2} - ad_{W_{2k+1}} f_{2k-1,4k+1} + ad_{W_{2k+1}} ad_{W_{2k}} f_{2k-1,2k+1} - ad_{W_{2k+2}} f_{2k-1,4k} + ad_{W_{2k+2}} ad_{W_{2k}} f_{2k-1,2k} - ad_{W_{2k+3}} f_{2k-1,4k-1} - ad_{W_{2k+4}} f_{2k-1,4k-2} - \cdots - ad_{W_{3k+1}} f_{2k-1,3k+1}.
\]

(3.10)

We postpone the verification of these formulas till the general result. The following result gives the general iterative formula for the multivariable Zassenhaus formula.

Theorem 3.2. For each $k \geq 2$ the exponents $W_m$ in the multi-variable Zassenhaus formula (1.4) for $m = 6k + i$, where $i = 0, 1, 2, 3, 4, 5$ are given by
\[
W_{6k+4} = \frac{1}{6k+4} (f_{2k,6k+3} - adW_{2k+1}f_{2k,4k+2} + \frac{1}{2!} ad^2_{W_{2k+1}} f_{2k,2k+1} - adW_{2k+2}f_{2k,4k+1} - adW_{2k+3}f_{2k,4k} - adW_{2k+4}f_{2k,4k-1} - \cdots - adW_{3k+1}f_{2k,3k+2}). \tag{3.11}
\]
\[
W_{6k+5} = \frac{1}{6k+5} (f_{2k,6k+4} - adW_{2k+1}f_{2k,4k+3} + \frac{1}{2!} ad^2_{W_{2k+1}} f_{2k,2k+2} - adW_{2k+2}f_{2k,4k+2} + adW_{2k+2}adW_{2k+1}f_{2k,2k+1} - adW_{2k+3}f_{2k,4k+1} - adW_{2k+4}f_{2k,4k} - adW_{2k+5}f_{2k,4k-1} - \cdots - adW_{3k+2}f_{2k,3k+2}). \tag{3.12}
\]

**Proof.** As we know that \(W_m = \frac{1}{m} f_{\left\lfloor \frac{m-1}{2} \right\rfloor, m-1}, \ m \geq 5\) in Section 2, we divide \(m\) into even and odd integers.

When \(m = 2a + 1, a \geq 2\),
\[
f_{\left\lfloor \frac{m-1}{2} \right\rfloor, m-1} = f_{a, m-1} \quad \left(\left\lfloor \frac{m-1}{a} \right\rfloor = 2\right)
= f_{a-1, m-1} - ad_{a} f_{a-1, a},
\]
if \(a = 2\), we stop the computation since we reach \(f_{1, k}\). Otherwise,
\[
\left\lfloor \frac{m-1}{a} \right\rfloor = \left\lceil \frac{2}{a-1} \right\rceil = \begin{cases} \begin{array}{ll} 3, & a = 3; \\ 2, & a \geq 4. \end{array} \end{cases}
\]
so that if \(a = 3\),
\[
f_{\left\lfloor \frac{m-1}{3} \right\rfloor, m-1} = f_{a-2, m-1} - ad_{a-1} f_{a-2, a+1} + \frac{1}{2!} ad^2_{a-1} f_{a-2, 2} - ad_{a} f_{a-2, a},
\]
we stop the computation. If \(a \geq 4\),
\[
f_{\left\lfloor \frac{m-1}{3} \right\rfloor, m-1} = f_{a-2, m-1} - ad_{a-1} f_{a-2, a+1} - ad_{a} f_{a-2, a}.
\]
Repeating the procedure, we obtain (3.3) and (3.5) as well as (3.8), (3.10), (3.12) in the theorem by using induction.

Similarly, when \(m = 2a, a \geq 3\),
\[
f_{\left\lfloor \frac{m-1}{2} \right\rfloor, m-1} = f_{a-1, m-1} \quad \left(\left\lfloor \frac{m-1}{a-2} \right\rfloor = 2, a \geq 3\right)
= f_{a-2, m-1} - ad_{a-1} f_{a-2, a},
\]
if \(a = 3\), we stop the computation. Otherwise,
\[
\left\lfloor \frac{m-1}{a-2} \right\rfloor = \left\lceil \frac{3}{a-2} \right\rceil = \begin{cases} \begin{array}{ll} 3, & a = 4; \\ 3, & a = 5; \\ 2, & a \geq 6. \end{array} \end{cases}
\]
\[
\left\lfloor \frac{a}{a-2} \right\rfloor = \left\lceil \frac{2}{a-2} \right\rceil = \begin{cases} \begin{array}{ll} 2, & a = 4; \\ 1, & a \geq 5. \end{array} \end{cases}
\]
so that if \( a = 4 \),

\[
\hat{f}_{\frac{m-1}{2},m-1} = f_{a-3,m-1} - ad_{W_{a-2}} f_{a-3,a+1} + \frac{1}{2!} ad_{W_{a-2}}^2 f_{a-3,3} - ad_{W_{a-1}} f_{a-3,a} + ad_{W_{a-1}} ad_{W_{a-2}} f_{a-3,2},
\]

we stop the computation. If \( a = 5 \),

\[
\hat{f}_{\frac{m-1}{2},m-1} = f_{a-3,m-1} - ad_{W_{a-2}} f_{a-3,a+1} + \frac{1}{2!} ad_{W_{a-2}}^2 f_{a-3,3} - ad_{W_{a-1}} f_{a-3,a}.
\]

If \( a \geq 6 \)

\[
\hat{f}_{\frac{m-1}{2},m-1} = f_{a-3,m-1} - ad_{W_{a-2}} f_{a-3,a+1} - ad_{W_{a-1}} f_{a-3,a}.
\]

Repeating the procedure, we obtain (3.2), (3.4) and (3.6) as well as (3.7), (3.9), (3.11) in Theorem 3.2 using induction. \(\square\)

According to Theorem 3.2, we know that \( W_m (m \geq 5) \) can be expressed as a linear combination of \( f_{1,k} (k \geq 1) \) in the end, then we use \( f_{1,k} (k \geq 1) \) given in Theorem 3.1 to obtain \( W_m (m \geq 5) \). To explain how this works, we give the explicit formulas of \( W_5, W_6 \) according to (2.10) and (3.2):

\[
W_5 = \frac{1}{5} (f_{1,4} - ad_{W_2} f_{1,2}) = \frac{1}{120} \sum_{1 \leq i<j \leq n} [[[X_j, X_i], X_i], X_i] + \frac{1}{30} \sum_{1 \leq i<j,k \leq n} ( [[[X_j, X_i], X_i], X_i], X_k] + \frac{1}{20} \sum_{1 \leq i<j,k \leq n} ( [[[X_j, X_i], X_k], X_k]) + \frac{1}{20} \sum_{1 \leq i<j,k \leq n} ( [[[X_j, X_k], X_k], X_k]) + \frac{1}{20} \sum_{1 \leq i<j,k \leq n} ( [[[X_k, X_i], X_k], X_k]) + \frac{1}{5} \sum_{1 \leq i<j,k \leq n} ( [[[X_j, X_i], X_i], X_i], X_h] + \frac{1}{10} \sum_{1 \leq i<j,k \leq n} ( [[[X_j, X_k], X_h], [X_{j1}, X_{i1}]] + \frac{1}{20} \sum_{1 \leq i<j,k \leq n} ( [[[X_{j1}, X_{i1}], X_{i1}], [X_{j1}, X_{i1}]]).
\]
$$W_0 = \frac{1}{6} (f_{1,5} - adW_{1,3})$$

$$= \frac{1}{720} \sum_{1 \leq i < j \leq n} \left( [[[X_j, X_i], X_i], X_i], X_i] + \frac{1}{144} \sum_{1 \leq i < j, k \leq n} [[[X_j, X_i], X_i], X_i], X_i], X_k]$$

$$+ [[[X_j, X_i], X_i], X_k], X_k], X_k] + \frac{1}{72} \sum_{1 \leq i < j, k \leq n} ((([[X_j, X_i], X_i], X_i], X_i], X_k]$$

$$+ [[[X_j, X_i], X_k], X_k], X_k], X_k] + \frac{1}{36} \sum_{1 \leq i < j, k \leq n} [[[X_j, X_i], X_i], X_i], X_k]$$

$$+ [[[X_j, X_i], X_i], X_k], X_k], X_k] + \frac{1}{24} \sum_{1 \leq i < j, k \leq n, k < l \leq n} ((([[X_j, X_i], X_i], X_i], X_k], X_l]$$

$$+ [[[X_j, X_i], X_k], X_k], X_k], X_k] + \frac{1}{12} \sum_{1 \leq i < j, k \leq n, k < l \leq n} ((([[X_j, X_i], X_k], X_k], X_k], X_k]$$

$$+ [[[X_j, X_k], X_k], X_k], X_k], X_k]$$

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