THE ALGEBRA OF HARMONIC FUNCTIONS FOR A MATRIX-VALUED TRANSFER OPERATOR

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Abstract. We analyze matrix-valued transfer operators. We prove that the fixed points of transfer operators form a finite dimensional $C^*$-algebra. For matrix weights satisfying a low-pass condition we identify the minimal projections in this algebra as correlations of scaling functions, i.e., limits of cascade algorithms.

Contents

1. Introduction 1
2. Transfer operators 3
   2.1. The setting 4
3. The peripheral spectrum 6
4. The $C^*$-algebra of continuous harmonic maps 9
5. Refinement operators 10
6. An example: low-pass filters 13
   6.1. Strong convergence of the cascade algorithm 17
References 19

1. Introduction

In this paper we study a class of harmonic functions which come from the theory of dynamical systems. While there is an analogy to the classical theory of harmonic functions for Laplace operators, this analogy has two steps: First the continuous variable Laplace, or elliptic operators in PDE admit a variety of discrete approximations leading to random walk models, with variable coefficients PDEs corresponding to variable weight functions in the discrete version.

As a result of work by numerous authors, and motivation from applications, there is now a rich harmonic analysis which is based on a certain transfer operator. It is based on path models, paths originating in a compact space, and with finite branching. The simplest instances come from endomorphisms in compact spaces $X$, onto, but with finitely branched inverses. A weight function on $X$ is prescribed, and the corresponding transfer operator (alias Ruelle operator) is denoted $R_W$.

Our motivation comes mainly from problems in wavelet analysis, but transfer operators are ubiquitous in applied mathematics: As it turns out, a study of transfer operators has now emerged as a subject of independent interest, with numerous applications. Examples: operator theory (locating the essential spectrum) [CI91, Hel96]; $C^*$-algebras [MS06]; groupoids [MSS06]; infinite determinants [BR96, Rue92]; anisotropic Sobolev spaces [Bal95]; tiling spaces [MM05]; number theory (zeta functions) [Hen02]; optimization [Hag05]; and quantum statistical mechanics [Ara80, Rue92].

Our present paper uses tools from at least three areas, $C^*$-algebras, dynamics, and wavelets:

1. We will be using both $C^*$-algebras, and their representations, especially an important family of matrix algebras.

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2. Dynamical systems. The connection between the algebras are relevant to the kind of dynamical systems which are built on branching-laws. The reason for this is that the spectral properties of the transfer operator and of the associated eigenspace algebras are connected to the ergodic properties of the dynamical system.

3. Wavelet analysis. The connection to basis constructions using wavelets is this: The context for wavelets is a Hilbert space $H$, where $H$ may be $L^2(\mathbb{R}^d)$ where $d$ is a dimension, $d = 1$ for the line (signals), $d = 2$ for the plane (images), etc. The more successful bases in Hilbert space are the orthonormal bases ONBs, but until the mid 1980s, there were no ONBs in $L^2(\mathbb{R}^d)$ which were entirely algorithmic and effective for computations.

Originating with \cite{Law90}, a popular tool for deciding whether or not a candidate for a wavelet basis is in fact an ONB uses a wavelet version of the transfer operator, still based on $n$-fold branching laws, but now with the branching corresponding to frequency bands. The wavelet Ruelle operator weights input over $n$ branching possibilities, and the weighting is assigned by a prescribed scalar function $W$, the modulus squared of a low-pass filter function, often called $m_0$. We are interested in the top part of the spectrum of $R_W$, an infinite-dimensional version of the so called Perron-Frobenius problem from finite-dimensional matrix theory.

This is especially useful for wavelets that are initialized by a single function, called the scaling function. These are called the multiresolution analysis (MRA) wavelets, or for short the MRA-wavelets. But there are multiwavelets (i.e., more than one scaling function) for example for localization in frequency domain, where then asking for an ONB is not feasible, but instead frame wavelets are natural.

We attack this problem by introducing a matrix version of the weight function $W$: so our $W$ is no longer scalar valued, but rather matrix-valued; the size of the matrices depending on an optimal number of scaling functions. We show that the space of harmonic functions for the $W$-Ruelle operator with matrix valued weights acquires the structure of a $C^*$-algebra. It serves to decide ONB vs frame properties, and the stability properties needed in applications, for example in the analogue to digital signal problem.

A transfer operator, also called Ruelle operator, is associated to a finite-to-one endomorphism $r : X \to X$ and a weight function $W : X \to [0, \infty)$, and it is defined by

$$R_W f(x) = \sum_{r(y) = x} W(y) f(y)$$

for functions $f$ on $X$. Here $X$ is a compact Hausdorff space, and $W$ is a non-negative continuous function on $X$. The function $W$ is said to be normalized if

$$\sum_{y \in r^{-1}(x)} W(y) = 1, \quad (x \in X).$$

Transfer operators have been extensively used in the analysis of discrete dynamical systems \cite{Bal00} and in wavelet theory \cite{BJ02}.

In multivariate wavelet theory (see for example \cite{JS99} for details) one has an expansive $n \times n$ integer matrix $A$, i.e., all eigenvalues $\lambda$ have $|\lambda| > 1$, and a multiresolution structure on $L^2(\mathbb{R}^n)$, i.e., a sequence of closed subspaces $\{V_j\}_{j \in \mathbb{Z}}$ of $L^2(\mathbb{R}^n)$ such that

(i) $V_j \subset V_{j+1}$, for all $j$;
(ii) $\bigcup_j V_j$ is dense in $L^2(\mathbb{R}^n)$;
(iii) $\cap_j V_j = \{0\}$;
(iv) $f \in V_j$ if and only if $f((A^T)^{-1}) \in V_{j-1}$;
(v) There exist $\varphi_1, ..., \varphi_d \in V_0$ such that $\{\varphi_k (\cdot - j) \mid k \in \{1, ..., d\}, j \in \mathbb{Z}^n\}$ forms an orthonormal basis for $V_0$.

The functions $\varphi_1, ..., \varphi_d$ are called scaling functions, and their Fourier transforms satisfy the following scaling equation:

$$\hat{\varphi}_i(x) = \sum_{j=1}^d m_{ji} (A^{-1} x) \hat{\varphi}_j (A^{-1} x), \quad (x \in \mathbb{R}^n, i \in \{1, ..., d\},)$$

where $m_{ji}$ are some $\mathbb{Z}^n$-periodic functions on $\mathbb{R}^n$. 

The orthogonality of the translates of $\varphi_i$ imply the following QMF equation:
$$\sum_{Ay=x \mod \mathbb{Z}^d} m^*(y)m(y) = 1, \quad (x \in \mathbb{R}^n/\mathbb{Z}^n),$$
where $m$ is the $d \times d$ matrix $(m_{ij})_{i,j=1}^d$. When the translates of the scaling functions are not necessarily orthogonal one still obtains the following relation: if we let
$$h_{ij}(x) := \sum_{k \in \mathbb{Z}^n} \hat{\varphi}_i(x + k)\hat{\varphi}_j(x + k), \quad (x \in \mathbb{R}^n/\mathbb{Z}^n),$$
then the matrix $h = (h_{ij})_{i,j=1}^d$ satisfies the following property:
$$R_h(x) := \sum_{Ay=x \mod \mathbb{Z}^n} m^*(y)h(y)m(y) = h(x), \quad (x \in \mathbb{R}^n/\mathbb{Z}^n),$$
i.e., $h$ is a fixed point for the matrix-valued transfer operator $R$. The fixed points of a transfer operator are also called harmonic functions for this operator. Thus the orthogonality properties of the scaling functions are directly related to the spectral properties of the transfer operator $R$.

This motivates our study of the harmonic functions for a matrix-valued transfer operator. The one-dimensional case (numbers instead of matrices) was studied in [Bjo02, Dut04b, Dut04a]. These results were then extended in [DJ06a, DJ06b], by replacing the map $x \mapsto Ax \mod \mathbb{Z}^n$ on the torus $\mathbb{T}^n := \mathbb{R}^n/\mathbb{Z}^n$, by some expansive endomorphism $r$ on a metric space.

Here we are interested in the case when the weights defining the transfer operator are matrices, just as in equation (1.1). We keep a higher level of generality because of possible applications outside wavelet theory, in areas such as dynamical systems or fractals (see [DJ06a, DJ06b]). However, for clarity, the reader should always have the main example in mind, where $r : x \mapsto Ax \mod \mathbb{Z}^n$ on the $n$-torus. The quotient map $\mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$ defines a simply connected covering space, and $r$ lifts to the dilation $\tilde{r}(x) = Ax$ on $\mathbb{R}^n$.

In Section 2 we introduce the main notions. Since we are especially interested in continuous harmonic maps we used the language of vector bundles (see also [Pck04]). Packer and Rieffel introduced projective multiresolution analyses (PMRA’s) in [Pck01]. In their formalism, the scaling spaces correspond to sections in vector bundles over $\mathbb{T}^n$. Motivated by their work, we introduce transfer operators that act on bundlemaps on vectorbundles. This gives a way to construct new PMRA’s.

In Section 5 we perform a spectral analysis of the transfer operator, prove that it is quasi-compact and give an estimate of the essential spectral radius. Section 4 contains one of the main results of the paper: the continuous harmonic functions form a $C^*$-algebra, with the usual addition, multiplication by scalars and adjoint, and with the multiplication defined by a certain spectral projection of $R$.

In Section 5 we define the refinement operator. This operates at the level of the covering space $\tilde{X}$, and extends the usual refinement operator from wavelet theory (see [Bjo02]). The correlations of the scaling functions correspond to fixed points of the refinement operator. We give the intertwining relation between the transfer operator and the refinement operator in Theorem 5.3. With this relation we will see that the fixed points of the transfer operator correspond to the fixed points of the refinement operator, i.e., to scaling functions.

In Section 6 we consider the case of low-pass filters. In the one-dimensional case the low-pass condition amounts to $m(0) = 1$, i.e., $m(0)$ is maximal. This is needed in order to obtain solutions of the scaling equation in $L^2(\mathbb{R}^n)$. In the matrix case, the appropriate low-pass condition was introduced in [IS99] under the name of the $E(l)$-condition (see Definition 6.1). If this condition is satisfied we show that the associated scaling functions exist in our case as pointwise limits of iterates of the refinement operator (Theorem 6.2). Their correlation functions give minimal projections in the algebra of continuous harmonic functions (Theorem 6.5).

In Theorem 6.7 we show that, if the peripheral spectrum of the transfer operator is “simple”, and we make an appropriate choice of the starting point, then the iterates of the refinement operator converge strongly.

### 2. Transfer operators

We describe now our setting, starting with the main example, the one used in multivariate wavelet theory.

**Example 2.1.** Let $\tilde{X} = \mathbb{R}^n$ and let $G \subset \tilde{X}$ be a discrete subgroup such that $\tilde{X}/G$ is compact, i.e. $G$ is a full-rank lattice, $G \cong \mathbb{Z}^n$ and $X := \tilde{X}/G \cong \mathbb{T}^n$ and let $A \in GL(\mathbb{R}^n)$ such that $AG \subset G$ and such that if $\lambda$
is an eigenvalue of $A$ then $|\lambda| > 1$. Then there exists a norm $\| \cdot \|_A$ on $\mathbb{R}^n$ and a number $\rho > 1$ such that 
\[
\|Ax\|_A \geq \rho\|x\|_A \quad \text{for all } x \in \mathbb{R}^n.
\]

Indeed, let $1 < \rho < \min |\lambda|$ and define $\| \cdot \|_A : \mathbb{R}^n \to [0, \infty)$ as
\[
\|x\|_A = \sum_{j=0}^{\infty} \rho^j \|A^{-j}x\|
\]

By the spectral radius formula and the root test, this converges and defines a norm on $\mathbb{R}^n$ such that 
\[
\|Ax\|_A \geq \rho\|x\|_A \quad \text{for all } x \in \mathbb{R}^n.
\]

Let $p : \hat{X} \to X$ denote the quotient map and define a map $r : X \to X$ as $r(p(x)) = p(Ax)$. The map $r$ is a regular covering map with deck transformations $G/AG$. We have that $\#G/AG = |\det(A)|$ and denote this number by $q := \#G/AG$. If $\mu$ is the normalized Haar measure on $X$, then
\[
(2.1) \quad \int_X f d\mu = \int_X q^{-1} \sum_{r = t} f(s)d\mu(t),
\]

i.e., $\mu$ is strongly $r$-invariant. Finally, there exists a normalization of the Haar measure on $\hat{X}$, say $\hat{\mu}$ such that $
\int_{\hat{X}} f d\hat{\mu} = \int_X \sum_{g \in G} f \circ gd\mu$ for every $f \in C_c(\hat{X})$.

We are mainly concerned with the above situation, but our results apply to a more general setting:

2.1. The setting. Let $\hat{X}$ be a locally compact metric space with an isometric and properly discontinuous action of a discrete group $G \times \hat{X} \ni g, \hat{x} \mapsto g\hat{x}$ such that $X := \hat{X}/G$ is compact.

Let $p : \hat{X} \to X$ be the quotient covering map.

Suppose $\hat{r}$ is a strictly expansive homeomorphism on $\hat{X}$, i.e., for some $0 < \theta < 1$
\[
d(\hat{r}x, \hat{r}y) \theta \geq d(x, y), \quad (x, y \in \hat{X}).
\]

Let $\hat{x}_0$ be the fixed point of $\hat{r}$ and $x_0 := p(\hat{x}_0)$.

Assume that there exists $A \in \text{End}(G)$ such that $AG \subset G$ is a normal subgroup, and $\hat{r}g\hat{x} = (Ag)\hat{r}\hat{x}$ for every $g \in G$, $\hat{x} \in \hat{X}$.

Let $r : X \to X$, $r(p(x)) = p(r(x))$, for all $x \in \hat{X}$.

Moreover, let $\hat{\mu}$ and $\mu$ be regular measures on $\hat{X}$ and $X$ such that $\mu$ is strongly $r$-invariant as in (2.1), and
\[
(2.2) \quad \int_{\hat{X}} f d\hat{\mu} = \int_X \sum_{g \in G} f \circ gd\mu, \quad (f \in C_c(\hat{X})).
\]

Let $\rho : \xi \to X$ be a Lipschitz continuous $d$-dimensional complex vectorbundle over $X$ with a hermitian metric, i.e. a map $\langle \cdot, \cdot \rangle : \xi \times \xi \to \mathbb{C}$ that restricts to positive definite sesquilinear forms on each fiber. We say that $\xi$ is Lipschitz continuous if there exists a trivializing system of bundlemaps $\phi_U : \xi|_U \to U_i \times \mathbb{C}^d$ such that every $\phi_U, \phi_{U'}^{-1}$ is Lipschitz continuous on $(U_i \cap U_j) \times \mathbb{C}^d$.

Let $S$ denote the continuous sections in $\xi$. By the Serre-Swan theorem [Atiyah80], this is a projective $C(X)$-module and the endomorphisms $F$ on $S$ are exactly the bundlemaps on $\xi$ acting as $F_S(x) = F(x)s(x)$. In fact, $\text{End}(\xi)$ equipped with the norm $\|F\|_\infty = \sup_{x \in X} \|F(x)\|$ and the pointwise involution with respect to the form $\langle \cdot, \cdot \rangle_x : \xi_x \times \xi_x \to \mathbb{C}$ is a $C^*$-algebra.

Let $p^*\xi$ be the pull-back of the vector bundle $\xi$ by the map $p$, i.e.,
\[
p^*\xi := \{(x, v) \in \hat{X} \times \xi \mid p(x) = p(v)\}.
\]

Assumption: We assume that $p^*\xi$ is a trivial $d$-bundle.

This is always the case if $X$ is contractible. We claim that we can take $p^*\xi$ to be the trivial bundle $\hat{X} \times \mathbb{C}^d$ with the canonical Hermitian inner product.

Given $d$-linearly independent sections $s_1, \ldots, s_d$ in $p^*\xi$ such that $\phi_U, s_i|_U$ is Lipschitz continuous for every $i, j$ we get a Lipschitz continuous bundle isomorphism $\sigma : \hat{X} \times \mathbb{C}^d \to p^*\xi$ such that $\sigma e_i = s_i$, where $e_i$, $1 \leq i \leq d$ are the canonical sections in the trivial bundle $\hat{X} \times \mathbb{C}^d$. We equip the product bundle with the standard inner product on $\mathbb{C}^d$ and $p^*\xi$ with pullback of the inner product on $\xi$. Let $u_x^*|_\xi = \sigma_x$ denote the pointwise polar decomposition.

$\sigma^*\sigma$ is positive and invertible, i.e. the holomorphic functional calculus on Banach algebras tells us that we can apply the square root and still get a bounded operator. This means that $|\sigma|$ is a bounded operator.
on the Lipschitz continuous $\tilde{X} \to \mathbb{C}^d$. Now $x,v \mapsto u_xv$ defines a Lipschitz continuous bundle isomorphism $\tilde{X} \times \mathbb{C}^d \to p^*\xi$ that is isometric over each $x$. This means that we can identify $p^*\xi$ with the trivial product bundle equipped with the ordinary inner product.

Let $S_1 \subset S$ denote the set of $s \in S$ such that $p^*s \in C_b(\tilde{X}, \mathbb{C}^d)$ is Lipschitz continuous. $S_1$ is a projective Lip$_1(\tilde{X})$ module of finite rank and a Banach space with the norm

$$\|s\|_1 = \sup_{x \in \tilde{X}} \|s(x)\| + \sup_{x,y \in \tilde{X}} \|p^*s(x) - p^*s(y)\|_{C^*} d(x,y).$$

Let $L$ denote the endomorphisms on $S_1$. $L$ is dense in $\text{End}(\xi)$ and a Banach algebra with the usual operator norm

$$\|g\|_1 = \sup_{\{s \in S_1 \mid \|s\|_1 \leq 1\}} \|gs\|_1.$$

Let $r^*\xi$ denote the pullback of $\xi$ along $r$ (see [AL89, 1.1]) and let

$$m : r^*\xi \to \xi$$

be a bundle map. The pullback of the inner product on $\xi$ gives an inner product on $r^*\xi$ and we get a unique $m^* : \xi \to r^*\xi$ such that $(v, mw) = \langle m^*v, w \rangle$ for every pair $v \in \xi, w \in r^*\xi|x$.

Let $g_1, \ldots, g_q$ be a complete system of representatives for the right cosets of $AG$ in $G$, with $g_1 = 1$, and define $\psi_1, \ldots, \psi_q : \tilde{X} \to \tilde{X}$ as

$$(2.3) \quad \psi_i = \tilde{r}^{-1} \circ g_i,$$

we see that $d(\psi_i(x), \psi_i(y)) \leq \theta d(x,y)$, and $\{p\psi_1x, \ldots, p\psi_qx\} = r^{-1}(px)$ for all $x \in \tilde{X}$.

**Definition 2.2.** We define the transfer operator associated to $m$ as the operator $R$ acting on $h \in \text{End}(\xi)$ such that

$$(Rh)(px) = \sum_{i=1}^q m^*(p\psi_i x)h(p\psi_i x)m(p\psi_i x), \quad (x \in \tilde{X}).$$

An element $h \in \text{End}(\xi)$ is called harmonic for the transfer operator $R$, if $Rh = h$. Define

$$m^{(k)}(x) := m(px)m(rx)\ldots m(r^{k-1}x).$$

Moreover, let $\Omega_k := \times_{j=1}^{q} \{1, \ldots, q\}$, $\omega \in \Omega_k$ and let $\omega_j$ denote the $j$’th coordinate of $\omega$.

A computation yields the following identity:

$$(R^kh)(px) = \sum_{\omega \in \Omega_k} m^{(k)}(\psi_{\omega_k} \ldots \psi_{\omega_1} x)h(p\psi_{\omega_k} \ldots \psi_{\omega_1} x)m^{(k)}(\psi_{\omega_k} \ldots \psi_{\omega_1} x).$$

We will use repeatedly the following Cauchy-Schwarz type inequality:

**Lemma 2.3.** If $m_1, \ldots, m_k, \tilde{m}_1, \ldots, \tilde{m}_k, h_1, \ldots, h_d \in M_d(\mathbb{C})$ then

$$\left\| \sum_{i=1}^k \tilde{m}_i^* h_i m_i \right\| \leq \left\| \sum_{i=1}^k \tilde{m}_i^* \tilde{m}_i \right\|^{1/2} \left\| \sum_{i=1}^k m_i^* m_i \right\|^{1/2} \max_{1 \leq j \leq k} \|h_j\|.$$

**Proof.** The space $E := \oplus_{j=1}^k M_d(\mathbb{C})$ with the sesquilinear map

$$\langle \cdot, \cdot \rangle : E \times E \to M_d(\mathbb{C}), \quad \langle a_1 \oplus \cdots \oplus a_k, b_1 \oplus \cdots \oplus b_k \rangle = \sum_{i=1}^k a_i^* b_i$$

form an $M_d(\mathbb{C})$-Hilbert module.

If $H \in \text{End}_{M_d(\mathbb{C})}(E)$ then by the Cauchy-Schwarz inequality for Hilbert Modules

$$\|\langle \eta, H\zeta \rangle\| \leq \|H\zeta\| \|\langle \eta, \eta \rangle\|^{1/2} \leq \|H\| \|\langle \zeta, \zeta \rangle\|^{1/2} \|\langle \eta, \eta \rangle\|^{1/2}.$$

The claim follows with $H(a_1 \oplus \cdots \oplus a_k) = (h_1a_1 \oplus \cdots \oplus h_k a_k)$, $\zeta = m_1 \oplus \cdots \oplus m_k$ and $\eta = \tilde{m}_1 \oplus \cdots \oplus \tilde{m}_k$. \qed
If there exists \( h \in \text{End}(\xi) \) that is positive, invertible and harmonic with respect to \( R \), then there exists a \( c > 0 \) such that \( c1 \leq h \); this implies that for all \( k \geq 0 \), \( h = R^k h \geq cR^k 1 \geq 0 \), and with Lemma 2.3 for all \( h_0 \in \text{End}(\xi) \):
\[
\| R^k h_0 \|_\infty \leq \| R^k 1 \|_\infty \| h_0 \|_\infty \leq c^{-1} \| h \|_\infty \| h_0 \|_\infty ,
\]
so the existence of such an element implies that \( \sup_k \| R^k \|_\infty < \infty \).

We assume from now on that \( m \) satifies the following conditions.

(i) \( m : r^* \xi \to \xi \) is Lipschitz continuous.

(ii) \( \sup_k \| R^k \|_\infty < \infty \)

Remark 2.4. Since \( p^* \xi \) is the trivial bundle, we have \( p^* r^* \xi = \tilde{r}^* p^* \xi = \text{graph}(\tilde{r}) \times \mathbb{C}^d \). This means that \( p^* m \) defines a bundle map \( \text{graph}(\tilde{r}) \times \mathbb{C}^d \to \tilde{X} \times \mathbb{C}^d \). Now \( p^* m(x, y, v) = (x, m_0(x)v) \) for some map \( m_0 : \tilde{X} \to M_d(\mathbb{C}) \). \( p^* m^* \) defines a map \( \tilde{X} \times \mathbb{C}^d \to \text{graph}(\tilde{r}) \times \mathbb{C}^d \) such that \( p^* m^*(x, v) = (x, \tilde{r} x, m_0(x)v) \) where \( m_0^* (x) \) is the unique element map \( \tilde{X} \to M_d(\mathbb{C}) \) such that \( \langle m_0^* (x)v, w \rangle = \langle v, m_0(x)v \rangle \) for every \( x \in \tilde{X} \) and \( v, w \in \mathbb{C}^d \). Therefore we can identify \( p^* m \) with \( m_0 \).

3. The peripheral spectrum

The essential spectral radius of a bounded linear operator \( T \) on a Banach space is the infimum of positive numbers \( \rho \geq 0 \) such that \( \lambda \in \sigma_p(T) \) and \( |\lambda| > \rho \) implies \( \lambda \) is an isolated eigenvalue of finite multiplicity. If the essential spectral radius of \( T \) is strictly less than the spectral radius then \( T \) is said to be quasicompact.

Whenever \( Y \) is a complete metric space and \( Z \subset Y \) we define the Ball measure of noncompactness of \( Z \), say \( \gamma_Y(Z) \) to be
\[
\gamma_Y(Z) = \inf \{ \delta > 0 \mid \text{There exist } z_1, \ldots, z_k \in Z \text{ s.t. } Z \subset \cup_i B(z_i, \delta) \}.
\]

If \( B \) is a Banach space with unit ball \( B_1 \) and \( R \) is a bounded linear operator on \( B \), define
\[
\gamma(R) = \gamma(R(B_1)).
\]
The following theorem is due to Nussbaum [Nus70]

Theorem 3.1. \( \lim_k \gamma(R^k)^{1/k} \) exists and equals the essential spectral radius of \( R \).

The corollary is due to Hennion [Hen93]

Corollary 3.2. If \( (L, \| \cdot \|) \) is a Banach space with a second norm \( | \cdot | \) and an operator \( R \) such that

(i) \( R : (L, \| \cdot \|) \to (L, | \cdot |) \) is a compact operator.

(ii) For every \( n \in \mathbb{N} \) there exist positive numbers \( r_n \) and \( R_n \) such that \( \liminf_n r_n^{1/n} \leq r \) and
\[
\| R^n f \| \leq R_n | f | + r_n \| f \|,
\]
then the essential spectral radius of \( R \) is less than \( r \).

Proof. Let \( B_1 = \{ b \in L \mid \| b \| = 1 \} \), \( D(z, \delta) = \{ y \in L \mid \| y - z \| < \delta \} \) and \( B(z, \delta) = \{ y \in L \mid \| y - z \| < \delta \} \). Since \( R(B_1) \) is relatively compact with respect to \( (L, | \cdot |) \) there exists a sequence \( b_1, \ldots, b_m \in B_1 \) for every \( \delta > 0 \) such that
\[
R(B_1) \subset \bigcup_{i=1}^m D(R b_i, \delta) \cap B(0, \| R \|).
\]
If \( a \in D(R b_i, \delta) \cap B(0, \| R \|) \) then
\[
\| R^n R b_i - R^n a \| \leq R_n | R b_i - a | + r_n \| R b_i - a \| \leq R_n \delta + 2 \| R \| r_n.
\]
Thus \( R^{n+1}(B_1) \) can be covered by \( \{ B(R^{n+1} b_i, R_n \delta + 2 \| R \| r_n) \} \). Such a sequence can be found for arbitrary \( \delta > 0 \), i.e. \( \gamma(R^{k+1}) \leq 2 \| R \| r_n \) and
\[
\lim_n \gamma(R^n)^{1/n} \leq \liminf_n (2 \| R \| r_n)^{1/n} \leq r.
\]

We intend to give an estimate of the essential spectral radius of \( R|_L \). First we need some lemmas.
Lemma 3.3. If $m \circ p$ is Lipschitz continuous, there exists a $D > 0$ such that

$$
(\sum_{\omega \in \Omega_k} \| m^{(k)}(\psi_{\omega_k} \ldots \psi_{\omega_1}, x) - m^{(k)}(\psi_{\omega_k} \ldots \psi_{\omega_1}, y) \|^2)^{1/2} \leq d(x, y)D.
$$

For every $x, y \in \tilde{X}$ and $k \in \mathbb{N}$.

Proof. Since $\tilde{r}$ is expansive, the maps $\psi_j$ are contractive, with contraction constant $\theta$.

$$
\sum_{\omega \in \Omega_k} \| m^{(k)}(\psi_{\omega_k} \ldots \psi_{\omega_1}, x) - m^{(k)}(\psi_{\omega_k} \ldots \psi_{\omega_1}, y) \|^2
$$

$$
= \sum_{\omega \in \Omega_k} \sum_{j=1}^{k} \| m^{(k-j)}(\psi_{\omega_k} \ldots \psi_{\omega_1}, x)(m(p\psi_j \ldots \psi_{\omega_1}, x) - m(p\psi_j \ldots \psi_{\omega_1}, y))m^{(j-1)}(\psi_{\omega_{j-1}} \ldots \psi_{\omega_1}, y) \|^2
$$

$$
\leq \text{const} \sum_{\omega \in \Omega_k} \sum_{j=1}^{k} \| m^{(k-j)}(\psi_{\omega_k} \ldots \psi_{\omega_1}, x) \| \theta^j d(x, y) \| m^{(j-1)}(\psi_{\omega_{j-1}} \ldots \psi_{\omega_1}, y) \|^2.
$$

By the Cauchy-Schwarz inequality, this is dominated by

$$
\text{const} \ d(x, y)^2 \sum_{\omega \in \Omega_k} ((\sum_{i=1}^{k} \theta^{2i})^{1/2})(\sum_{j=1}^{k} \| m^{(k-j)}(\psi_{\omega_k} \ldots \psi_{\omega_1}, x) \|^2 \| m^{(j-1)}(\psi_{\omega_{j-1}} \ldots \psi_{\omega_1}, y) \|^2)^{1/2}.
$$

Since $\| H \| \leq \text{tr}(H) \leq d\| H \|$ for any positive $d \times d$ matrix, we obtain

$$
\sum_{i} \| H_i \|^2 = \sum_{i} \| H_i^* H_i \| \leq \sum_{i} \text{tr}(H_i^* H_i) = \text{tr}(\sum_{i} H_i^* H_i) \leq d \sum_{i} H_i^* H_i.
$$

This, with $\theta < 1$, implies that our expression is bounded by

$$
\text{const} \ d(x, y)^2 \sum_{\omega_1, \ldots, \omega_j} \| R^{k-j}(1)(\psi_{\omega_j} \ldots \psi_{\omega_1}, x) \| \| m^{(j-1)}(\psi_{\omega_{j-1}} \ldots \psi_{\omega_1}, y) \|^2
$$

$$
\leq \text{const} \ d(x, y)^2 \sup_n \| R^n(1) \| \| R^j(1)(y) \| \leq \text{const} \ d(x, y)^2 (\sup_n \| R^n(1) \|)^2.
$$

□

Lemma 3.4. There exist $D_1, D_2 > 0$ such that

$$
\| p^* R^k h(x) - p^* R^k h(y) \| \leq D_1 \| h \|_1 \| d(\psi_{\omega_k} \ldots \psi_{\omega_1}, x, \psi_{\omega_k} \ldots \psi_{\omega_1}, y) + \| h \|_\infty D_2 d(x, y).
$$

for every $k \in \mathbb{N}$ and $\omega \in \Omega_k$.

Proof.

$$
\| p^* R^k h(x) - p^* R^k h(y) \|
$$

$$
\leq \sum_{\omega \in \Omega_k} m^{(k)}(\psi_{\omega_k} \ldots \psi_{\omega_1}, x)(h(p\psi_{\omega_k} \ldots \psi_{\omega_1}, x) - h(p\psi_{\omega_k} \ldots \psi_{\omega_1}, y))m^{(k)}(\psi_{\omega_k} \ldots \psi_{\omega_1}, x)
$$

$$
+ \| \sum_{\omega \in \Omega_k} m^{(k)}(\psi_{\omega_k} \ldots \psi_{\omega_1}, x)h(p\psi_{\omega_k} \ldots \psi_{\omega_1}, y)(m^{(k)}(\psi_{\omega_k} \ldots \psi_{\omega_1}, x) - m^{(k)}(\psi_{\omega_k} \ldots \psi_{\omega_1}, y))
$$

$$
+ \| \sum_{\omega \in \Omega_k} (m^{(k)}(\psi_{\omega_k} \ldots \psi_{\omega_1}, x) - m^{(k)}(\psi_{\omega_k} \ldots \psi_{\omega_1}, y))h(p\psi_{\omega_k} \ldots \psi_{\omega_1}, y)m^{(k)}(\psi_{\omega_k} \ldots \psi_{\omega_1}, y).\]
Moreover, for every $x,y \in X$ and $\varepsilon > 0$, we can find a finite cover of $X$ such that

$$\left| \sum_{\omega \in \Omega_k} (m(k)^* \psi_{\omega_k} \cdots \psi_{\omega_1})(x,m(k)^* \psi_{\omega_k} \cdots \psi_{\omega_1}) \right| \leq \sup_{\omega \in \Omega_k} \| h(p\psi_{\omega_k} \cdots \psi_{\omega_1}) - h(p\psi_{\omega_k} \cdots \psi_{\omega_1}) \| .$$

This is bounded by

$$\| R^k (p \psi_{\omega_k} \cdots \psi_{\omega_1}) \| \sup_{\omega} \| (h(p\psi_{\omega_k} \cdots \psi_{\omega_1}) - h(p\psi_{\omega_k} \cdots \psi_{\omega_1})) \| + \| h \| \left( \| R^k (p \psi_{\omega_k} \cdots \psi_{\omega_1}) \| + \| R^k (p \psi_{\omega_k} \cdots \psi_{\omega_1}) \| \right).$$

The claim follows from this and Lemma 3.3.

Lemma 3.5. The map

$$h \mapsto \| h \|_\infty + \sup_{x \neq y} \frac{|p^* h(x) - p^* h(y)|}{d(x, y)}$$

defines an equivalent norm on $L^1$.

Proof. A straightforward computation gives the following estimate

$$\sup_{\| p^* s \| \leq 1} \| h \| \infty + \sup_{x \neq y} \frac{|p^* h s(x) - p^* h s(y)|}{d(x, y)} \leq 2 (\| h \| \infty + \sup_{x \neq y} \frac{|p^* h(x) - p^* h(y)|}{d(x, y)}).$$

On the other hand, we can find a finite cover of $X$ by open sets $V$ such that the restriction of $\xi$ to these sets is trivial. By the Lebesgue covering lemma there exists an $\varepsilon > 0$ such that whenever $x,y \in X$ have $d(x,y) < \varepsilon/2$, there exists a $\nu$ in this cover such that $\| \psi_{\omega_k} \cdots \psi_{\omega_1} \|_\infty$.

Moreover, for every $x,y \in X$ with $p x, p y \in V$, we can construct a section $s^{px,py} \in S_1$ such that $p^* s^{px,py}(z) = u_{px,py}$ for every $y \in V$ and $p \in \mathbb{R}$ and, by constructing a Lipschitz partition of unity subordinated to our cover, we can assume that $\| s_{px,py} \|_1$ is uniformly bounded by some constant $D$ that does not depend on $V, x, y$.

Now

$$\frac{|p^* h(x) - p^* h(y)|}{d(x, y)} = \frac{|p^* h(x) - p^* h(y)||u_{px,py}|}{d(x, y)} \leq \sup_{\| p^* s \| \leq D} \frac{|p^* h s(x) - p^* h s(y)|}{d(x, y)} \leq \sup_{\| p^* s \| \leq D} \sup_{d(x,y) < \varepsilon/2} \frac{|p^* h s(x) - p^* h s(y)|}{d(x, y)}.$$

If $d(px, py) \geq \varepsilon/2$ then

$$\frac{|p^* h(x) - p^* h(y)|}{d(x, y)} \leq 4 \| h \| \infty / \varepsilon,$$

so we get a constant such that

$$\| h \| \infty + \sup_{x \neq y} \frac{|p^* h(x) - p^* h(y)|}{d(x, y)} \leq D \| h \|.$$
Theorem 3.6. The essential spectral radius of \( R : L \rightarrow L \) is less than \( 0 \). Since \( \sup_k \| R^k h \|_\infty < \infty \), the spectral radius is at most 1.

Moreover, in the main example with \( r(px) = p(Ax) \) where \( A \) is a linear map on \( \mathbb{R}^n \), the essential spectral radius is less than the spectral radius of \( A^{-1} \).

Proof. By the Lemmas 3.3 and 3.5 we get \( D_1, D_2 > 0 \) such that
\[
\| R^k h \|_1 \leq D_1 \theta^k \| h \|_1 + D_2 \| h \|_\infty,
\]
in the general case, and
\[
\| R^k h \|_1 \leq D_1 \| A^{-k} \| \| h \|_1 + D_2 \| h \|_\infty,
\]
in the case with the linear map \( A \).

By the Arzela-Ascoli theorem, bounded sets in \( S_1 \) are precompact in \( S \). This implies that bounded sets in \( L \) are precompact in \( \text{End}(\xi) \). The claim now follows from Corollary 3.2 and the spectral radius formula.

Theorem 3.7. The Cesaro means \( k^{-1} \sum_{j=1}^k R^j g \) converges with respect to the \( \| \cdot \|_\infty \)-norm for every \( g \in \text{End}(\xi) \). The map \( T_1 : g \mapsto \lim_k k^{-1} \sum_{j=1}^k R^j g \) defines a completely positive idempotent acting on \( \text{End}(\xi) \) such that \( T_1|_{\mathcal{L}} \in B(L) \) and \( T_1 \text{End}(\xi) = \{ g \in \text{End}(\xi) | Rg = g \} = \{ g \in L | Rg = g \} \).

Proof. Theorem [DSSS VIII.5.1], states that if \( R \) is a bounded operator on a Banach space \( B \) with uniformly bounded iterates, then the set of \( x \in B \) such that the Cesaro means \( n^{-1} \sum_{j=1}^n R^j x \) converge is a closed subspace consisting of all \( y \in B \) such that \( n^{-1} \sum_{j=1}^n R^j y \) is weakly sequentially compact and \( \lim_n n^{-1} R^n y = 0 \). To apply this to our setting, note that the sequence \( n^{-1} \sum_{j=1}^n R^j g, n \in \mathbb{N} \), is uniformly bounded with respect to the \( \| \cdot \|_1 \)-norm whenever \( g \in L \), so, using the Arzela-Ascoli theorem, it has a convergent subsequence with respect to the \( \| \cdot \|_\infty \)-norm. Moreover, \( \lim_n n^{-1} R^n g = 0 \) for every \( g \in L \). Since \( L \subset \text{End}(\xi) \) is dense with respect to the \( \| \cdot \|_\infty \)-norm, this theorem implies that \( n^{-1} \sum_{j=1}^n R^j g \) converges, \( RT_1 g = T_1 g \), and \( Rg = g \) implies that \( T_1 g = g \) for every \( g \in \text{End}(\xi) \).

Since the uniform limit of a sequence of uniformly bounded Lipschitz continuous sections is also Lipschitz continuous with Lipschitz constant less than the bound, the limit of the convergent subsequence is in \( L \) and the restriction of \( T_1 \) is bounded with respect to the \( \| \cdot \|_1 \)-norm.

Let \( h \in \text{End}(\xi) \) such that \( Rh = h \) and pick a sequence \( \{ h_n \} \subset L \) such that \( \lim_n \| h_n - h \|_\infty = 0 \). Now \( T_1 h_n \in T_1 L \) converges to \( h \) with respect to the \( \| \cdot \|_\infty \)-norm. Moreover, we have that \( T_1|_{\mathcal{L}} = \{ g \in L | Rg = g \} \) is finite dimensional so it is closed in both \( L \) and \( \text{End}(\xi) \). This implies that \( h \in T_1 L \) and \( \{ g \in \text{End}(\xi) | Rg = g \} = \{ g \in L | Rg = g \} \).

A map \( P \in B(\text{End}(\xi)) \) is completely positive if \( a \otimes h \mapsto a \otimes P(h) \) defines a positive map \( P^{(k)} \) on \( M_m(\mathbb{C}) \otimes \text{End}(\xi) \) for every \( k \in \mathbb{N} \). We see that \( h \mapsto m^2 (p \cdot h)(p \cdot m(p \cdot h)) \) defines a completely positive map. This implies that \( R \) is also completely positive. Since \( T_1 \) is the limit of Cesaro means of the completely positive operators \( R^k \), that converges in norm for every \( h \in \text{End}(\xi) \), we see that \( T_1 \) is also completely positive. Indeed, if \( (h_{ij})_{i,j=1}^k \) is a positive matrix with coefficients in \( \text{End}(\xi) \), then \( (R^k h_{ij})_{i,j} \) is a positive matrix, and since each Cesaro mean \( \frac{1}{k} \sum_{i=1}^k R^i h_{ij} \) converges to \( T_1 h_{ij} \) we get that \( (T_1 h_{ij})_{i,j} \) is a positive matrix, so \( T_1 \) is completely positive.

The essential spectral radius of similar transfer operators is analyzed in several other places in the literature. In [Bj02] this is computed with the help of a theorem by Ionescu and Tulcea [ITM50], while in [Rue89] this is computed with techniques from symbolic dynamics and dynamical zeta functions.

4. The \( C^* \)-Algebra of Continuous Harmonic Maps

Definition 4.1. An element \( h \in \text{End}(\xi) \) is called harmonic with respect to the transfer operator \( R \) if \( Rh = h \). We denote by \( \mathcal{H} \) the set of all continuous harmonic functions \( h \in \text{End}(\xi) \).

Suppose \( h \) is an invertible and strictly positive \( R \)-harmonic bundlemap. We introduce the normalized transition operator \( \tilde{R} \) on the bundlemaps on \( \xi \). Let \( \tilde{m} = h^{1/2} m (h^{-1/2} \circ r) \). Note that \( \tilde{m} \) is Lipschitz because \( h \) is Lipschitz (by Theorem 3.7) and \( \det h \) is bounded away from 0.

Now \( \tilde{R} \) is a completely positive map that preserves the identity with a corresponding completely positive idempotent \( \tilde{T}_1 \). A theorem due to Choi and Effros [CE77] states that the image of a completely positive
map $T$ that preserves the identity, equipped with the usual norm, $\ast$-operation and the product $a,b \mapsto T(ab)$ form a $C^*$-algebra, i.e. the set of $\hat{R}$-harmonic maps is a $C^*$-algebra when equipped with usual $\ast$-operation, the usual norm and the product $a,b \mapsto T(ab)$.

The map $g \mapsto h^{-1/2}gh^{-1/2}$ is a linear bijection from the $R$-harmonic bundle maps to the $\hat{R}$-harmonic bundlemaps. This means that the $R$-harmonic bundlemaps with the usual $\ast$-operation, the norm $\|a\|_h = \|h^{-1/2}ah^{-1/2}\|$ and the product $a,b \mapsto h^{-1/2}T_1(h^{-1/2}ah^{-1/2}b)h^{1/2} = T_1(ah^{-1}b)$ form a $C^*$-algebra. Let $\mathcal{H}$ denote this algebra and define $a \ast b = T_1(ah^{-1}b)$.

Since the essential spectral radius of $R$ is strictly less that 1, $\mathcal{H}$ is finite dimensional, so we get the following:

**Theorem 4.2.** The set $\mathcal{H}$ of continuous harmonic functions for the transfer operator $R$ forms a finite dimensional $C^*$-algebra with the usual addition, adjoint and multiplication by scalars and with the product defined by $h_1 \ast h_2 := T_1(h_1h^{-1/2}h_2)$ and norm $\|f\|_h := \|h^{-1/2}fhh^{-1/2}\|$.

Every finite dimensional $C^*$-algebra is isomorphic to $M_{k_1}(C) \oplus \cdots \oplus M_{k_r}(C)$ for $k_1,\ldots,k_r \in \mathbb{N}$. See for instance Theorem [Dav96, III.1.1] or [RLL00, 7.1]. In what follows, we will give a partial description of this algebra in some situations.

5. **Refinement operators**

We define here the refinement operator $M$. This operates at the level of $\hat{X}$, the simply connected covering space of $X$. On $\hat{X}$ we have a unitary “dilation operator” $U$. It is defined by the following:

**Proposition 5.1.** The measure $\hat{\mu}$ satisfies the following quasi-invariance equation:

$$\frac{d\hat{\mu}}{d\mu} \circ \hat{r}^{-1} = q.$$ 

The operator $U$ defined on $\mathcal{H} := L^2(\hat{X},\hat{\mu})$ by $Uf = q^{1/2}f \circ \hat{r}$ is unitary, with $U^*f = q^{-1/2}f \circ \hat{r}^{-1}$.

Define $\pi(a) f(x) = a \circ p(x) f(x)$ for $a \in C(X)$ and $f \in \mathcal{H}$. Then $\pi$ is a representation of $C(X)$ in $B(\mathcal{H})$, and $(\mathcal{H}, \pi, U)$ form a covariant representation of the dynamical system $r : X \to X$ in the sense that $U \pi(f) U^* = \pi(f \circ r)$ for every $f \in C(X)$.

**Proof.** Using the stong invariance (2.1) of the measure $\mu$, equation (2.2) and (2.3), we have for all $f \in C_c(\hat{X})$:

$$\int_X f \, d\hat{\mu} = \int_X f \circ \hat{r} \, d\hat{\mu} \circ \hat{r}^{-1} \, d\mu = \int_X \sum_{g \in G} (f \circ \hat{r} \circ \hat{r}^{-1}(Ag)) g \, d\mu = \int_X q^{-1} \sum_{i=1}^q \sum_{g \in G} (f \circ \hat{r} \circ \hat{r}^{-1}(Ag)) g \, d\mu$$

$$= \int_X q^{-1} \sum_{i=1}^q \sum_{g \in G} (f \circ \hat{r} \circ \hat{r}^{-1}(Ag)) \circ \hat{r}^{-1} \circ \hat{r} d\mu$$

$$= \int_X q^{-1} \sum_{i=1}^q \sum_{g \in G} (f \circ \hat{r} \circ \hat{r}^{-1}(Ag)) \circ \hat{r}^{-1} d\mu.$$ 

The other statements follow from some easy computations. \hfill \square

**Definition 5.2.** As in [PR04], we define $\Xi$ to be $f \in C_b(\hat{X})$ (bounded continuous functions on $\hat{X}$) such that $\sum_{g \in G} |f(gx)|^2$ is bounded and continuous for every $x \in \hat{X}$. Following their proof, we see that when $\Xi \subset C_b(\hat{X})$ is equipped with the inner product

$$\langle f_1, f_2 \rangle = \sum_{g \in G} T_1 \circ gf_2 \circ g,$$

we have a $C(X)$-Hilbert module.

Let $\text{Hom}_{C(X)}(S,\Xi)$ be the set of adjointable $C(X)$-linear maps between the Hilbert modules $S$ and $\Xi$.

The **refinement operator** associated to the map $m$,

$$M : \text{Hom}_{C(X)}(S,\Xi) \to \text{Hom}_{C(X)}(S,\Xi)$$

is defined by

$$MWs = U^{-1}W(q^{-1/2}ms \circ r) = (Wms \circ r) \circ \hat{r}^{-1}, \quad (W \in \text{Hom}_{C(X)}(S,\Xi), s \in S).$$
Proposition 5.3. If \( W \in \text{Hom}_{C(X)}(S, \Xi) \) then \( UWS \subset WS \) if and only if there exists a bundlemap \( \tilde{m} : r^*\xi \to \xi \) such that \( W \) is a fixed point for the refinement operator \( \tilde{M} \) associated to \( \tilde{m} \).

Proof. If \( W \in \text{Hom}_{C(X)}(S, \Xi) \) such that \( MW = W \), we see immediately from the definition of \( M \) that \( UWS \subset WS \). Conversely, suppose \( UWS \subset WS \). Let \( \zeta \) be another \( C(X) \)-module such that \( \xi \oplus \zeta \simeq X \times \mathbb{C}^k \) (see [Ati89, Corollary 1.4.14]). Let \( s_1, \ldots, s_k \) be linearly independent sections in \( \xi \oplus \zeta \). The sections \( Pr_1s_j \), \( 1 \leq j \leq k \) generate \( S \) as a \( C(X) \)-module. There exist \( c_{i,j} \in C(X) \), \( 1 \leq i, j \leq k \) such that \( UWPr_1s_j = \sum_i c_{i,j}Pr_1Ws_i \) for every \( j \). Moreover \( r^*\xi \oplus r^*\zeta \simeq r^*(\xi \oplus \zeta) \) and

\[
x \mapsto (x, Pr_1s_i(rx)) \oplus (x, Pr_2s_i(rx)), \quad (1 \leq i \leq k),
\]

form \( k \) linearly independent sections \( s'_1, \ldots, s'_k \) in the bundle in \( r^*\xi \oplus r^*\zeta \). Now define \( \mathcal{C} : r^*\xi \oplus r^*\zeta \to \xi \oplus \zeta \) such that \( \mathcal{C}s'_i = \sum_j c_{i,j}s_j \), and \( \tilde{m} = q^{1/2}Pr_1CPr_1 \). This gives us that \( U^{-1}W(q^{-1/2}\tilde{m}s \circ r) = Ws \) for every \( s \in S \).

Proposition 5.4. If \( W \in \text{Hom}(S, \Xi) \) such that

\begin{enumerate}
  \item \( MW = W \);
  \item \( W^*W \) is an idempotent (Note that here \( W^* \) is the adjoint of \( W \) as a Hilbert module map between \( S \) and \( \Xi \! \));
  \item There exists an \( f \in WS \) such that \( f(\bar{x}_0) \neq 0 \);
\end{enumerate}

then \( \{U^kWS\}_{k \in \mathbb{Z}} \) form a projective multiresolution analysis, i.e. the following conditions are satisfied

\begin{enumerate}
  \item \( WS \) is a projective submodule in \( \Xi \);
  \item \( WS \subset U^{-1}WS \);
  \item \( \cup_{k \in \mathbb{Z}} U^kWS \) is dense in \( \Xi \);
  \item \( \cap_{k \in \mathbb{Z}} U^kWS = \{0\} \);
\end{enumerate}

Proof. If \( W^*W \) is a projection, then \( WW^* \) is also a projection, \( WS = W^*W\Xi \) and \( W|_{W^*WS} \) is an isometry onto its image, [Lan95]. This implies that \( WS \) is isomorphic to the image of an idempotent on a projective module and therefore it must be projective also. By proposition 13 and 14 in [PR04], we get conditions 2, 3 and 4.

We have the following important intertwining relation between the refinement operator and the transfer operator:

Theorem 5.5. If \( W_1, W_2 \in \text{Hom}_{C(X)}(S, \Xi) \), then

\[
R(W_1^*W_2) = (MW_1)^*MW_2.
\]

In particular, if \( MW_1 = W_1 \) and \( MW_2 = W_2 \) then \( R(W_1^*W_2) = W_1^*W_2 \), i.e., \( W_1^*W_2 \) is harmonic.

Proof. Let \( s_1, s_2 \in S \)

\[
\langle MW_1s_1, MW_2s_2 \rangle' \circ p = \sum_{g \in G} (W_1ms_1 \circ r) \circ \tilde{p}^{-1} \circ g(W_2ms_2 \circ r) \circ \tilde{p}^{-1} \circ g
\]

\[
= \sum_{i=1}^q \sum_{g \in G} (W_1ms_1 \circ r) \circ \tilde{p}^{-1} \circ Ag \circ g_i(W_2ms_2 \circ r) \circ \tilde{p}^{-1} \circ Ag \circ g_i
\]

\[
= \sum_{i=1}^q \sum_{g \in G} (W_1ms_1 \circ r) \circ g \circ \psi_i(W_2ms_2 \circ r) \circ g \circ \psi_i
\]

\[
= \sum_{i=1}^q \langle W_1ms_1 \circ r, W_2ms_2 \circ r \rangle' \circ p \circ \psi_i
\]

\[
= \sum_{i=1}^q \langle s_1 \circ r, (m^*W_1W_2m)s_2 \circ r \rangle \circ p \circ \psi_i
\]

\[
= \langle s_1, R(W_1^*W_2)s_2 \rangle \circ p.
\]
This suggests a method to construct embeddings $W : S \to \Xi$, such that $U W S \subset WS$. Pick an $m : r^* \xi \to \xi$ and let $R$ be the corresponding transfer operator with a basis of $R$-invariant functionals on $\text{End}(\xi)$, $\tau_1, \ldots, \tau_r$, and a positive and invertible harmonic map $h$. Then look for a fixed point $W$ of the refinement operator $M$, such that $\tau_j(h) = \tau_j(W^* W)$ for every $1 \leq j \leq r$. Then $W^* W = h$ and $W$ is injective. Moreover, if $h = 1$ we also know that $W$ is an isometry.

This method is essentially the same as a technique often encountered in the wavelet literature. With some assumptions on the low-pass filter, the infinite product expansion yields a scaling function that generates a multiresolution analysis in $L^2(\mathbb{R})$. This scaling function has orthonormal $\mathbb{Z}$-translates if and only if the constants are the only fixed points for the corresponding transfer operator, see for instance Law90.

Suppose $W : S \to \Xi$ is of the form $W s = \langle s_0, s \rangle f$, with $s_0 \in S$ and $f \in \Xi$, then $W$ is adjointable, and a direct computation gives the following two identities:

\begin{equation}
M^k W s(x) = \langle s_0 \circ p(\tilde{r}^{-k} x), m^{(k)}(\tilde{r}^{-k} x) s \circ p(x) \rangle f(\tilde{r}^{-k} x).
\end{equation}

\begin{equation}
(M^k W)^* f \circ p(x) = \sum_{\omega \in \Omega_h} m^{(k)}(\psi_{\omega_1} \ldots \psi_{\omega_1} x)(W^* f \circ \tilde{r}^k) \circ p(\psi_{\omega_1} \ldots \psi_{\omega_1} x).
\end{equation}

Moreover, there exist $M_1, M_2 > 0$ such that $\|M^k W\| \leq M_1 \|W\|$ and $\|(M^k W)^*\| \leq M_2 \|W^*\|$ for every $k \geq 0$.

**Definition 5.6.** Let $\Xi_1$ be the set of $f \in \Xi$ such that there exists a $D > 0$ such that

\[ (\sum_{g \in G} |f(gx) - f(gy)|^2)^{1/2} \leq Dd(x, y), \]

for every $x, y \in \hat{X}$.

**Lemma 5.7.** $\Xi_1$ is a Lip$_1(X)$-module and $f_1, f_2 \in \Xi_1$ implies that $\langle f_1, f_2 \rangle' \in \text{Lip}_1(X)$.

**Proof.**

\[
\|\langle f_1, f_2 \rangle'(x) - \langle f_1, f_2 \rangle'(y)\| \leq \left| \sum_{g \in G} \mathcal{F}_1(g)(f_1(gx) - f_2(gy)) \right| + \left| \sum_{g \in G} (f_1(gx) - f_2(gy))f_2(gy) \right|
\]

\[
\leq (\langle f_1, f_1 \rangle'(x))^1/2(\sum_{g \in G} |f_2(gx) - f_2(gy)|^2)^{1/2} + (\langle f_2, f_2 \rangle'(y))^1/2(\sum_{g \in G} |f_1(gx) - f_1(gy)|^2)^{1/2}.
\]

\[\square\]

**Lemma 5.8.** Let $W \in \text{Hom}_{C(X)}(S, \Xi)$ such that $W s(x) = \langle s'(px), s(px) \rangle f'(x)$ with $s' \in S_1$ and $f' \in \Xi_1$. There exist $D_1, D_2 > 0$ such that

\[
\left( \sum_{g \in G} |(M^k W s)(gx) - (M^k W s)(gy)|^2 \right)^{1/2} \leq D_1 \|s\|_{\infty} d(x, y) + D_2 \|s(px) - s(py)\|,
\]

for every $x, y \in \hat{X}$.

**Proof.** We have

\begin{equation}
\left( \sum_{g \in G} |(M^k W s)(gx) - (M^k W s)(gy)|^2 \right)^{1/2}
\end{equation}

\begin{equation}
\leq \left( \sum_{g \in G} |\langle s' \circ p(\tilde{r}^{-k} gx) - s' \circ p(\tilde{r}^{-k} gy), m^{(k)}(\tilde{r}^{-k} gy) s \circ p(y) \rangle f'(\tilde{r}^{-k} gy) \rangle^2 \right)^{1/2}
\end{equation}

\begin{equation}
+ \left( \sum_{g \in G} |\langle s' \circ p(\tilde{r}^{-k} gx), m^{(k)}(\tilde{r}^{-k} gx) - m^{(k)}(\tilde{r}^{-k} gy) s \circ p(y) \rangle f'(\tilde{r}^{-k} gy) \rangle^2 \right)^{1/2}
\end{equation}

\begin{equation}
+ \left( \sum_{g \in G} |\langle s' \circ p(\tilde{r}^{-k} gx), m^{(k)}(\tilde{r}^{-k} gx) (s \circ p(x) - s \circ p(y)) f'(\tilde{r}^{-k} gy) \rangle^2 \right)^{1/2}
\end{equation}

\begin{equation}
+ \left( \sum_{g \in G} |\langle s' \circ p(\tilde{r}^{-k} gx), m^{(k)}(\tilde{r}^{-k} gx) (s \circ p(x) - f'(\tilde{r}^{-k} gy)) f'(\tilde{r}^{-k} gy) \rangle^2 \right)^{1/2}.
\end{equation}
We estimate the last four summands separately. (5.4) is bounded by
\begin{equation}
\left(\sum_{g \in G} \sum_{\omega \in \Omega_k} \|s' \circ p(\psi_{\omega k} \ldots \psi_{\omega 1}) - s' \circ p(\psi_{\omega k} \ldots \psi_{\omega 1} x)\|^2 \cdot \|m^{(k)}(\psi_{\omega k} \ldots \psi_{\omega 1})s \circ p(y)\|^2 \cdot |f'(g\psi_{\omega k} \ldots \psi_{\omega 1})|^2 \right)^{1/2} \leq (\sum_{\omega \in \Omega_k} \|s' \circ p(\psi_{\omega k} \ldots \psi_{\omega 1}) - s' \circ p(\psi_{\omega k} \ldots \psi_{\omega 1} y)\|^2 \cdot \|m^{(k)}(\psi_{\omega k} \ldots \psi_{\omega 1} y)s \circ p(y)\|^2 \cdot |f'(g\psi_{\omega k} \ldots \psi_{\omega 1} y)|^2)^{1/2} \leq \text{const} \theta^k d(x, y) \|R^k 1\|_{\infty} \|s\|_{\infty} \|f', f''\|_{\infty}^{-1/2}.
\end{equation}

(We used (3.1) for the last estimate.)

(5.5) is bounded by
\begin{equation}
\left(\sum_{g \in G} \sum_{\omega \in \Omega_k} \|s' \circ p(\psi_{\omega k} \ldots \psi_{\omega 1})\|^2 \cdot \|m^{(k)}(\psi_{\omega k} \ldots \psi_{\omega 1}) - m^{(k)}(\psi_{\omega k} \ldots \psi_{\omega 1} x)\|^2 \cdot \|m^{(k)}(\psi_{\omega k} \ldots \psi_{\omega 1} y)s \circ p(y)\|^2 \cdot |f'(g\psi_{\omega k} \ldots \psi_{\omega 1} y)|^2 \right)^{1/2} \leq \text{const} \theta^k d(x, y) \|R^k 1\|_{\infty} \|s\|_{\infty} \|f', f''\|_{\infty}^{-1/2}.
\end{equation}

and we used Lemma 3.3 in the last inequality.

Moreover, (5.6) is bounded by
\begin{equation}
\left(\sum_{g \in G} \sum_{\omega \in \Omega_k} \|s' \circ p(\psi_{\omega k} \ldots \psi_{\omega 1})\|^2 \cdot \|m^{(k)}(\psi_{\omega k} \ldots \psi_{\omega 1}) - m^{(k)}(\psi_{\omega k} \ldots \psi_{\omega 1} x)\|^2 \cdot \|m^{(k)}(\psi_{\omega k} \ldots \psi_{\omega 1} y)s \circ p(y)\|^2 \cdot |f'(g\psi_{\omega k} \ldots \psi_{\omega 1} y)|^2 \right)^{1/2} \leq \text{const} \theta^k d(x, y) \|R^k 1\|_{\infty} \|s\|_{\infty} \|f', f''\|_{\infty}^{-1/2}.
\end{equation}

Finally, (5.7) is bounded by the following expression
\begin{equation}
\left(\sum_{g \in G} \sum_{\omega \in \Omega_k} \|s' \circ p(\psi_{\omega k} \ldots \psi_{\omega 1})\|^2 \cdot \|m^{(k)}(\psi_{\omega k} \ldots \psi_{\omega 1}) - m^{(k)}(\psi_{\omega k} \ldots \psi_{\omega 1} x)\|^2 \cdot \|m^{(k)}(\psi_{\omega k} \ldots \psi_{\omega 1} y)s \circ p(y)\|^2 \cdot |f'(g\psi_{\omega k} \ldots \psi_{\omega 1} y)|^2 \right)^{1/2} \leq \text{const} \theta^k d(x, y) \|R^k 1\|_{\infty} \|s\|_{\infty} \|f', f''\|_{\infty}^{-1/2}.
\end{equation}

\hfill \Box

6. AN EXAMPLE: LOW-PASS FILTERS

In this section we will assume that $m$ satisfies a certain low-pass condition. This will imply that the iterates of the refinement operator will converge to some fixed points, which correspond to scaling functions. These in turn will generate harmonic functions which are minimal projections in the algebra $\mathfrak{A}$.

To simplify the notation we will identify $m \circ p$ and $p^* m$ with $m$.

Definition 6.1. Let $1 \leq l \leq d$. We say that a $d \times d$ matrix $a$ satisfies the $E(l)$ condition if $\|a\| = 1$, 1 is the only eigenvalue of $a$ on the unit circle and both the geometric and algebraic multiplicities of 1 are equal to $l$.

In the next theorem, we show how certain “scaling functions” $W_v$ can be constructed from each eigenvector $v$ of $m(x_0)$ corresponding to the eigenvalue 1.

Theorem 6.2. Assume $R1 = 1$ and $m(x_0)$ satisfies the $E(l)$ condition.

(i) The limit
\[ \mathcal{P}(\tilde{x}) := \lim_{k \to \infty} m^{(k)}(r^{-k}(\tilde{x})), \quad (\tilde{x} \in \tilde{X}) \]
exists for all $\tilde{x} \in \tilde{X}$, and is uniform on compact sets. The following refinement equation is satisfied:
\begin{equation}
\mathcal{P}(r^{-1} \tilde{x}) m(\tilde{r}^{-1} \tilde{x}) = \mathcal{P}(\tilde{x}), \quad (\tilde{x} \in \tilde{X}).
\end{equation}
\( \mathcal{P}(\tilde{x}_0) \) is the orthogonal projection onto the eigenspace \( E_1 \) of \( m(x_0) \) that corresponds to the eigenvalue 1. The range of \( \mathcal{P}(\tilde{x}) \) is contained in \( E_1 \), i.e.
\[
m(x_0)\mathcal{P}(\tilde{x}) = \mathcal{P}(\tilde{x}), \quad (\tilde{x} \in \tilde{X}).
\]
(6.2)

For all \( g \in G, g \neq 1 \), and all \( v \in E_1 \),
\[
\mathcal{P}(g\tilde{x}_0) v = 0.
\]
(6.3)

(ii) Fix a function \( f \in \Xi_1 \) with \( f, f' = 1 \) and \( f(g\tilde{x}_0) = 0 \) for all \( g \in G, g \neq 1 \). Let \( s_0 \in S_1 \) with \( s_0(x_0) =: v \). Define \( W \in \text{Hom}(S, \Xi) \). \( W_{s_0}(s) = (s_0, s) f \). Then, for all \( s \in S \), \( M^k W_{s_0} s \) converges uniformly on compact sets to a continuous \( \mathcal{W}_s, s \in \Xi \), \( \mathcal{W}_s s(\tilde{x}) = (v, \mathcal{P}(\tilde{x}) s(\tilde{x})) \), and this defines a \( \mathcal{W}_v \in \text{Hom}(S, \Xi) \). Moreover, \( \mathcal{W}_v \) depends only on the value \( v \) of \( s_0 \) at \( x_0 \), not on the entire section \( s_0 \). In addition \( \mathcal{W}_v \) satisfies the refinement equation
\[
M \mathcal{W}_v = \mathcal{W}_v.
\]
(6.4)

Also, \( \mathcal{W}_v s(\tilde{x}_0) = (\mathcal{P}(\tilde{x}_0) v, s(x_0)) f(\tilde{x}_0) \).

Proof. (i) First note that, since the vector bundle \( p^* \xi \) is trivial, the terms of the product defining \( m^{(k)} \) act on the same vector space so it makes sense to talk about the convergence of this product (see also Remark 2.4).

Since \( m(x_0) \) satisfies the \( E(l) \) condition, we can find an invertible matrix \( u \) such that \( J := u^{-1} m(x_0) u \) is in Jordan canonical form, with the \( l \times l \) leading principal submatrix of \( u^{-1} m(x_0) u \) being the identity \( l \times l \) matrix. The other Jordan blocks correspond to eigenvalues \( \lambda \) with \( |\lambda| < 1 \).

Then one can see that \( J^p \) converges to the matrix \( \begin{bmatrix} I_l & 0 \\ 0 & 0 \end{bmatrix} \). This shows in particular that \( \|m(x_0)^p - m(x_0)^{p+p'}\| \) can be made arbitrarily small.

Take now a compact subset \( \tilde{K} \) of \( \tilde{X} \). Since \( \tilde{r}^{-1} \) is contractive towards the fixed point \( \tilde{x}_0 \), there are some constants \( C > 0, 0 < \theta < 1 \) depending only on the set \( \tilde{K} \) such that \( d(\tilde{r}^{-k} \tilde{x}, \tilde{x}_0) \leq CG^k \) for all \( \tilde{x} \in \tilde{K} \) and \( k \geq 0 \).

We have for \( k \geq 0, p, p' \geq 0 \)
\[
\|m^{(k+p)}(\tilde{r}^{-k} \tilde{x}) - m^{(k+p+p')} (\tilde{r}^{-k} \tilde{x})\| = \|m^{(p)} (\tilde{r}^{-k} \tilde{x}) m^{(k)} (\tilde{r}^{-k} \tilde{x}) - m^{(p+p')} (\tilde{r}^{-k} \tilde{x}) m^{(k)} (\tilde{r}^{-k} \tilde{x})\| \leq \|m^{(k)} (\tilde{r}^{-k} \tilde{x})\| \|m^{(p)} (\tilde{r}^{-k} \tilde{x}) - m^{(p+p')} (\tilde{r}^{-k} \tilde{x})\|.
\]
Since \( R^k 1 = 1 \) we have that \( \|m^{(k)} (\tilde{r}^{-k} \tilde{x})\| \leq 1 \). On the other hand, we have with Lemma 3.3
\[
\|m^{(p)} (\tilde{r}^{-k} \tilde{x}) - m^{(p+p')} (\tilde{r}^{-k} \tilde{x})\| \leq \|m^{(p)} (\tilde{r}^{-k} \tilde{x}) - m^{(p)} (\tilde{r}^{-k} \tilde{x}_0)\| + \|m(x_0)^p - m(x_0)^{p+p'}\| + \|m(x_0)^{p+p'} - m(x_0)^{p+p'}\|.
\]
This proves that the sequence is uniformly Cauchy, therefore it is uniformly convergent.

The scaling equation (6.1) follows from the convergence of the product. Now let us compute \( \mathcal{P}(\tilde{x}_0) \). We have \( \mathcal{P}(\tilde{x}_0) = \lim_k m(x_0)^k \). Then \( \mathcal{P}(\tilde{x}_0) = \lim_k m(x_0)^{2k} = \lim_k m(x_0)^k m(x_0)^k = \mathcal{P}(\tilde{x}_0) \).

If \( v \in C^*, m(x_0) \mathcal{P}(\tilde{x}_0) v = \lim_k m(x_0)^{k+1} v = \mathcal{P}(\tilde{x}_0) v \), so the range of \( \mathcal{P}(\tilde{x}_0) \) is contained in the eigenspace \( E_1 \) of \( m(x_0) \) corresponding to the eigenvalue 1. We have the following lemma:

**Lemma 6.3.** The eigenspaces corresponding to the eigenvalue 1 for \( m(x_0) \) and \( m^*(x_0) \) are the same.

Proof. Let \( v \in C^n \) such that \( m(x_0) v = v \). We may assume \( \|v\| = 1 \). Since \( R1 = 1 \), we have \( m^*(x_0) m(x_0) \leq 1 \) so \( \|m^*(x_0)\| \leq 1 \). Using the Schwarz inequality we obtain
\[
1 = (m(x_0) v, v) = (v, m^*(x_0) v) \leq \|v\| \|m^*(x_0) v\| \leq 1,
\]
therefore we have equalities in all these inequalities. This implies that \( m^*(x_0) v = \lambda v \) for some \( \lambda \in C^n \), and \( \lambda \) has to be 1. Hence \( m^*(x_0) v = v \). The reverse implication can be proved analogously. \(\Box\)
With Lemma 6.3 we have that $P(x_0)^* = \lim_k m^*(x_0)^k$ leaves $E_1$ fixed, so $P(x_0)^*P(\tilde{x}_0) = P(\tilde{x}_0)$. But this implies that $P(\tilde{x}_0)$ is a positive operator, and since it is also an idempotent, it must be equal to the orthogonal projection onto its range $E_1$.

To prove equation (6.2) we estimate, for $\tilde{x} \in \tilde{X}$:

\[
\|m(x_0)P(\tilde{x}) - P(\tilde{x})\| \leq \|m(x_0)P(\tilde{x}) - m(x_0)m(p\tilde{x}^{-k})\| + \|m(x_0)m(p\tilde{x}^{-k})\| + \|m(x_0)m(p\tilde{x}^{-k})\|.
\]

But

\[
\|m(x_0)m(p\tilde{x}^{-k})\| \leq \|m(x_0) - m(p\tilde{x}^{-k})\| \leq \|m(x_0) - m(p\tilde{x}^{-k})\|,
\]

since $R_1 = 1$ and $\|m(\tilde{x})\| \leq 1$ for all $\tilde{x}$.

Therefore all the terms can be made as small as we want, provided $k$ is big, and this proves (6.2).

**Lemma 6.4.** For all $v$ in the eigenspace $E_1$,

\[
m(\psi_\tilde{x}_0)v = 0, \quad (2 \leq k \leq q).
\]

**Proof.** Since $R_1 = 1$ one has

\[
\langle v, v \rangle = \sum_{k=1}^q \langle v, m^*(p\psi_\tilde{x}_0)m(p\psi_\tilde{x}_0)v \rangle = (m(x_0)v, m(x_0)v) + \sum_{k=2}^q \langle m(p\psi_\tilde{x}_0)v, m(p\psi_\tilde{x}_0)v \rangle =
\]

\[
\langle v, v \rangle + \sum_{k=2}^q \|m(p\psi_\tilde{x}_0)v\|^2.
\]

This implies the lemma. □

Take now $g \in G$, $g \neq 1$. Let $k$ be the first positive integer such that $r^{-k}g\tilde{x}_0 \notin G\tilde{x}_0$. Then $p(r^{-l}g\tilde{x}_0) = x_0$ for $1 \leq l \leq k - 1$ and $r(p(r^{-k}g\tilde{x}_0)) = x_0$, but $p(r^{-k}g\tilde{x}_0) \neq x_0$. Using Lemma 6.4 we obtain for $j \geq k$

\[
m^{(j)}(r^{-k}g\tilde{x}_0)v = m(p\tilde{x}^{-j}g\tilde{x}_0)m(x_0)\ldots m(x_0)v = 0.
\]

So $P(g\tilde{x}_0)v = 0$.

(ii) With (5.1) we have

\[
M^kW_{s_0}s(p\tilde{x}) = \left\langle s_0 \circ p(r^{-k}\tilde{x}), m^{(k)}(r^{-k}\tilde{x})s \circ p(\tilde{x}) \right\rangle f(r^{-k}x)
\]

Each term in this formula is uniformly convergent on compact sets so $M^kW_{s_0}s$ is uniformly convergent on compact sets to $\langle s_0(x_0), P(\tilde{x})s \circ p(\tilde{x}) \rangle$. Note that $f(\tilde{x}_0) = 1$ because $\langle f, f, f \rangle = 1$ and $f(g\tilde{x}_0) = 0$ for $g \neq 1$. Using Fatou’s lemma, and the intertwining relation in Theorem 6.5, we have

\[
\langle W_{\psi_\tilde{x}}, W_{\psi_\tilde{x}} \rangle'(p\tilde{x}) = \sum_{g \in G} \langle W_{\psi_\tilde{x}}, W_{\psi_\tilde{x}} \rangle(g\tilde{x}) \leq \liminf_k \sum_{g \in G} \langle M^kW_{s_0}s, M^kW_{s_0}s \rangle(g\tilde{x}) =
\]

\[
\liminf_k \langle M^kW_{s_0}s, M^kW_{s_0}s \rangle'(p\tilde{x}) = \liminf_k \langle s, R^k(W^*_{s_0}W_{s_0}s) \rangle'(p\tilde{x}) \leq \langle s, s \rangle \sup_{x \in X} \|W^*_{s_0}W_{s_0}(x)\|,
\]

and we used $R^k(W^*_{s_0}W_{s_0}) \leq R^k(\sup_x \|W^*_{s_0}W_{s_0}\|) = \sup_x \|W^*_{s_0}W_{s_0}\|$ in the last inequality ($R_1 = 1$). Moreover, we have, with Lemma 6.8

\[
|\langle W_{\psi_\tilde{x}}, W_{\psi_\tilde{x}} \rangle'(p\tilde{x}) - \langle W_{\psi_\tilde{x}}, W_{\psi_\tilde{x}} \rangle'(p\tilde{y})| = |\left\langle \sum_{g \in G} W_{\psi_\tilde{x}}(g\tilde{x}), W_{\psi_\tilde{x}}(g\tilde{y}) \right\rangle^2 - \left\langle \sum_{g \in G} W_{\psi_\tilde{x}}(g\tilde{x}), W_{\psi_\tilde{x}}(g\tilde{y}) \right\rangle^2 |,
\]

\[
\leq \left( \sum_{g \in G} \|W_{\psi_\tilde{x}}(g\tilde{x}) - W_{\psi_\tilde{x}}(g\tilde{y})\|^2 \right)^{\frac{1}{2}} \leq \liminf_k \left\langle \sum_{g \in G} M^kW_{s_0}s(g\tilde{x}) - M^kW_{s_0}s(g\tilde{y}) \right\rangle^{\frac{1}{2}}
\]

\[
\leq \text{const} \left( \|s\|_{\infty}d(\tilde{x}, \tilde{y}) + \|s(\tilde{x}) - s(\tilde{y})\| \right).
\]

This shows that $p\tilde{x} \mapsto \langle W_{\psi_\tilde{x}}, W_{\psi_\tilde{x}} \rangle'(p\tilde{x})$ is continuous, so $\langle W_{\psi_\tilde{x}}, W_{\psi_\tilde{x}} \rangle'$ is, and this implies that $W_{\psi_\tilde{x}} \in \mathcal{E}$.

The refinement equation is clearly satisfied because $\lim_k M^kW_{s_0}s(\tilde{x}) = (M\lim_k W_{s_0}s)(\tilde{x})$.

The last statement follows from the fact that $P(x_0)$ is the orthogonal projection onto the eigenspace $E_1$. □
From the “scaling functions” $\mathcal{W}_v$ defined in Theorem 6.2, we can construct the harmonic functions $h_{v_1,v_2}$ as the correlations $\mathcal{W}_{v_2}^* \mathcal{W}_{v_1}$ between these scaling functions.

**Theorem 6.5.** For $v_1, v_2 \in \mathbb{C}^n$ define $h_{v_1,v_2} := \mathcal{W}_{v_2}^* \mathcal{W}_{v_1}$. Denote by $h_{v_1} := h_{v_1,v_1} = \mathcal{W}_{v_1}^* \mathcal{W}_{v_1}$.

(i) The function $h_{v_1,v_2}$ is continuous and harmonic (i.e., $R h_{v_1,v_2} = h_{v_1,v_2}$).

(ii) For all $v \in E_1$, with $\|v\| = 1$, and all continuous harmonic functions $h \geq 0$, with $h(x_0)v = v$, one has $h \leq h_v$.

(iii) For all continuous harmonic functions $h$, $h(x_0)$ commutes with the projection $\mathcal{P}(x_0)$ onto the eigenspace $E_1$ (see also Theorem 6.2).

(iv) The map $\Psi_{x_0} : h \mapsto h(x_0) \mathcal{P}(x_0)$ is a surjective $*$-algebra morphism between the algebra of continuous harmonic functions and $\mathcal{M}(\mathbb{C})$.

Proof. (i) The continuity of $h_{v_1,v_2}$ follows from the fact that $\mathcal{W}_{v_1}$ and $\mathcal{W}_{v_2}$ are in $\text{Hom}(S, \Xi)$. Using the intertwining relation in Theorem 5.5, we have

$$R(h_{v_1,v_2}) = R(\mathcal{W}_{v_2}^* \mathcal{W}_{v_1}) = (\mathcal{W}_{v_2}^*)^* \mathcal{W}_{v_1} = \mathcal{W}_{v_2}^* \mathcal{W}_{v_1} = h_{v_1,v_2}.$$

(ii) Since $h \geq 0$, we have $h^{1/2}(x_0)v = v$ and $h^{1/2} \in \mathcal{E}(\xi)$. Then, as in the proof of Theorem 6.2, taking $s_0 \in S_1$ with $s_0(x_0) = v$, and $\|s_0\|_{\infty} \leq 1$, one has that for all $s \in S$, $\{M^k(W_{s_0}h^{1/2})s\}_k$ converges uniformly on compact sets to $\mathcal{W}_v$. Then, using Fatou’s lemma:

$$\langle s, W_{s_0}^* W_v s \rangle = \langle W_v s, W_v s \rangle' \leq \liminf_k \left\langle M^k(W_{s_0}h^{1/2})s, M^k(W_{s_0}h^{1/2})s \right\rangle' = \liminf_k \left\langle s, R^k(h^{1/2}W_{s_0}^* W_{s_0}h^{1/2})s \right\rangle \leq \liminf_k \left\langle s, R^k(h^{1/2}W_{s_0}^* W_{s_0}h^{1/2})s \right\rangle \leq \langle s, h s \rangle,$$

because

$$\langle W_{s_0}^* W_v s, W_{s_0}^* W_v s \rangle' = \langle f, f' \rangle \langle s_0, s \rangle^2 \leq \langle f, f' \rangle \|s_0\|_{\infty}^2 \|s\|_{\infty}^2 \leq \|s\|_{\infty}^2.$$

so $\|W_{s_0}^* W_v\| \leq 1$.

This implies that $h \leq h_v$.

(iii) Let $h$ be a continuous harmonic function. Using Lemma 6.3, we have for all $v \in E_1$, $k \geq 1$:

$$h(x_0)v = R^k h(x_0)v = m^{(k)}(\psi_0^k x_0) h(p_x^k \tilde{x}_0) m^{(k)}(\psi_0^k x_0) v = \langle m(x_0)^{k'} h(x_0)m(x_0)^k \rangle v = \langle h(x_0)^{k'} h(x_0) \rangle v.$$

Letting $k \to \infty$, and using Theorem 6.2, we obtain

$$h(x_0)v = \mathcal{P}(x_0)^* h(x_0)v,$$

and this implies $h(x_0) \mathcal{P}(x_0) = \mathcal{P}(x_0) h(x_0) \mathcal{P}(x_0)$. Apply this equality to the harmonic function $h^*$, and take the adjoint to obtain $\mathcal{P}(x_0) h(x_0) = \mathcal{P}(x_0) h(x_0) \mathcal{P}(x_0)$. Thus $h(x_0)$ commutes with $\mathcal{P}(x_0)$.

(iv) The map is clearly linear and preserves the adjoint. We prove that it preserves multiplication: take two continuous functions $a, b$ and $v \in E_1$. Then we have $a(x_0)b(x_0)v \in E_1$. Using a similar argument as above, we obtain

$$a \ast b(x_0)v = \lim_{k \to \infty} \frac{1}{k} \sum_{j=1}^{k} R^j(ab)(x_0)v = \lim_{k \to \infty} \frac{1}{k} \sum_{j=1}^{k} (m(x_0)^{j'}) a(x_0)b(x_0)v = a(x_0)b(x_0)v,$$

and we used Lemma 6.3 for the last equality. This shows that $a \ast b(x_0) \mathcal{P}(x_0) = a(x_0)b(x_0) \mathcal{P}(x_0) = a(x_0) \mathcal{P}(x_0)b(x_0) \mathcal{P}(x_0)$. Therefore $\Psi_{x_0}$ is $*$-algebra morphism.

To see that $\Psi_{x_0}$ is surjective, let $b \in \mathcal{M}(\mathbb{C})$. Pick a continuous $a \in \text{End}(\xi)$ such that $a(x_0)v = bv$ for all $v \in E_1$. Then for $v \in E_1$:

$$T_1(a)(x_0)v = \lim_{k \to \infty} \frac{1}{k} \sum_{j=1}^{k} R^j(a)(x_0)v = \lim_{k \to \infty} \frac{1}{k} \sum_{j=1}^{k} (m(x_0)^{j'}) a(x_0)v = bv.$$
Therefore $\Psi(x_0)(T_1(\alpha)) = b$, and $\Psi_{x_0}$ is surjective.

(v) Let $s_1, s_2 \in S$ with $s_1(x_0) = w_1 \in E_1$, $s_2(x_0) = w_2 \in E_1$. Then

$$\langle w_1, h_{v_1, v_2}(x_0)w_2 \rangle = \langle W_{v_1}s_1, W_{v_2}s_2 \rangle(x_0) = \sum_{g \in G} \langle P(g\bar{x}_0)w_1, v_1 \rangle \langle P(g\bar{x}_0)w_2, f(\bar{x}_0), f(\bar{x}_0) \rangle$$

and using (6.3) and Theorem 6.2.

$$\langle w_1, v_2 \rangle \langle v_1, w_2 \rangle = \langle w_1, \langle v_1, w_2 \rangle v_2 \rangle.

This proves (v).

(vi) Since $\mathcal{H}$ is a finite dimensional $C^*$-algebra, there exist $n_1, \ldots, n_p$ and an isomorphism $\beta : \mathcal{H} \to \bigoplus_{i=1}^p M_{n_i}(\mathbb{C})$. Consider the kernel $K := \text{Ker}(\Psi_{x_0} \circ \beta^{-1})$. This is an ideal in $\bigoplus_{i=1}^p M_{n_i}(\mathbb{C})$. It is easy to see that $K = \bigoplus K_i$, where $K_i$ is an ideal in $M_{n_i}(\mathbb{C})$. Then $K_1 = 0$ or $K_i = M_{n_i}(\mathbb{C})$. Denote the complements of these ideals $K_i^c := M_{n_i}(\mathbb{C}) \ominus K_i$. Then $\Psi_{x_0} \circ \beta^{-1}$ restricts to an isomorphism on $\bigoplus_{i=1}^p K_i^c$. Since $M_{i}(\mathbb{C})$ has trivial center, only one of the summands $K_i^c$ is non-trivial. After some relabeling we may assume $K_i^c$ is the non-trivial one, so $n_1 = l$. Therefore $\gamma_1 : x_1 \mapsto \Psi_{x_0} \circ \beta^{-1}(x_1, 0, \cdots, 0)$ is an isomorphism of $M_{i}(\mathbb{C})$ onto itself. Then, we have that

$$\Psi_{x_0} \circ \beta^{-1}(\gamma_1^{-1} x_1, x_2, \cdots, x_p) = \Psi_{x_0} \circ \beta^{-1}(\gamma_1^{-1} x_1, 0, \cdots, 0) = \gamma_1(\gamma_1^{-1} x_1) = x_1$$

for all $(x_1, \ldots, x_p) \in \bigoplus_{i=1}^p M_{n_i}(\mathbb{C})$. Then, define $\alpha(h) = (\gamma_1 \beta_1(h), \beta_2(h), \cdots, \beta_p(h))$ for $h \in \mathcal{H}$. Then $\alpha$ is an isomorphism, and if $h = (x_1, \ldots, x_p)$ then $\beta_1(h) = \gamma_1^{-1} x_1$, and $\beta_2(h) = x_2, \cdots, \beta_p(h) = x_p$, so $\Psi_x h = \Psi_{x_0} \beta^{-1}(\gamma_1^{-1} x_1, x_2, \cdots, x_p) = x_1 = \alpha_1(h)$. This gives the desired isomorphism $\alpha$.

(vii) Consider $\alpha(h_v)$. If $b = (b_1, \ldots, b_p) \in M_l(\mathbb{C}) \ominus M_{n_i}(\mathbb{C}) \cdots \ominus M_{n_p}(\mathbb{C})$ is positive, and $b_1 v = v$, then $\alpha^{-1}(b)$ is positive, and $\alpha^{-1}(b)(x_0)v = b_1 v = v$. Using (ii), we obtain that $\alpha^{-1}(b) \geq h_v$, so $b \geq \alpha(h_v)$. Since we also have $\text{proj}_i \alpha(h_v) v = v$, we obtain that $\alpha(h_v)$ is a minimal projection. Therefore $h_v$ is.

Since $\alpha(h_{v_1})$ and $\alpha(h_{v_2})$ are minimal projections onto $v_1$ and $v_2$ respectively, the affirmation follows.

(viii) We have, with Lemma 6.3, for all $g \in \text{End}(\xi)$,

$$\tau_{v_1, v_2}(Rg) = \sum_{j=1}^k \langle v_2, m^*(\psi_j x_0)g(\psi_j x_0)m(\psi_j x_0)v_1 \rangle = \langle m(x_0)v_2, g(x_0)m(x_0)v_1 \rangle = \langle v_2, g(x_0)v_1 \rangle.$$ 

The other statements follow from (v) and (vi).

6.1. **Strong convergence of the cascade algorithm.** We show here that, when the peripheral spectrum of $R$ is completely described by Theorem 6.3, then the iterates of the refinement operator converge strongly to the fixed points $W_v$. We begin with a result that applies to a more general situation, and gives a sufficient condition for strong convergence.

**Theorem 6.6.** Assume that 1 is the only eigenvalue of $R$ on the unit circle. Let $W \in \text{Hom}_{C^*(\xi)}(S, \Xi)$, $W_s = \sum_{j=1}^k (s_j, s) f_j$ with $s_j \in S_1$, $f_j \in \Xi_1$ for all $j \in \{1, \ldots, k\}$, and let $\tau_1, \ldots, \tau_l$ be a family of continuous linear functionals on $\text{End}(\xi)$ invariant under $R$ and that separates points in $\mathcal{H}$. If $\tau_l((M^k W - W)^*(M^k W - W)) = 0$ for every $k \geq 0$ and every $i$ then $M^k W$ converges with respect to the operator norm on $\text{Hom}_{C^*(\xi)}(S, \Xi)$ and

$$(\lim_k M^k W)^*(\lim_k M^k W) = T_1(W^*W).$$

**Proof.** The intertwining relation gives us

$$\|\langle (M^{k+1} W S - M^k W S, M^{k+1} W S - M^k W S) \rangle \|_\infty = \|\langle (M^k - 1) M^k W S, (M^k - 1) M^k W S \rangle \|_\infty$$

$$= \|\langle s, R_l((M^k W - W)^*(M^k W - W)) s \rangle \|_\infty \leq \| s, s \|_\infty \| R_l((M^k W - W)^*(M^k W - W)) \|_\infty.$$ 

Moreover, since $\tau_l((M^k W - W)^*(M^k W - W)) = 0$ for every $k \geq 0$ and $\tau_l$ is invariant for $R$, we get that $\tau_l(T_1((M^k W - W)^*(M^k W - W)) = 0$, and since $\tau_l$ is separating, we obtain $T_1((M^k W - W)^*(M^k W - W)) = 0$. Since $R$ is quasicompact with no other eigenvalues on the unit circle than 1, we obtain a decomposition $R = T_1 + S$ as operators on $L_2^\infty$, so $S T_1 = T_1 S = 0$ and $R^n = T_1 + S^n$ for all $n$. Since $T_1$ is compact, the essential spectral radius of $S$ is the same as the one of $R$ (see [Nus70]), but as $S$ does not have eigenvectors for the eigenvalue 1, the spectrum of $S$ is contained in a disc of radius $1 - \epsilon$ with $\epsilon > 0$. By the spectral...
radius formula, we see that \( \lim_n \| S^n \|_L = 0 \). Since \( \sup_k \| (M^k W - W)^*(M^k W - W) \|_1 < \infty \) (see Lemma 5.8), we see that
\[
\| R^n((M^k W - W)^*(M^k W - W)) \| \leq \| R^n((M^k W - W)^*(M^k W - W)) \|_L
\]
for every \( k \), i.e. \( M^k W \) form a Cauchy sequence in \( \text{Hom}_{C(X)}(S, \Xi) \).

The last claim is true since
\[
\langle \lim_k M^k W s_1, \lim_k M^k W s_2 \rangle = \langle \lim_k M^k W s_1, M^k W s_2 \rangle = \langle \lim_k s_1, R^k(W^* W)s_2 \rangle = \langle \lim_k(s_1, T_1(W^* W))s_2 \rangle = \langle s_1, T_1(W^* W)s_2 \rangle.
\]

**Theorem 6.7.** Assume \( R1 = 1, m(x_0) \) satisfies the \( E(1) \) condition, \( \dim S = l^2 \) and 1 is the only eigenvalue for \( R \) on the unit circle. Let \( \{v_1, \ldots, v_l\} \) be an orthogonal basis for \( E_1 = \{ v \in \mathbb{C}^d \mid m(x_0)v = v \} \) and let \( s_1, \ldots, s_l \in S_1 \) with \( s_j(x_0) = v_j \) for all \( 1 \leq j \leq l \). Define \( W_{s_1}, \ldots, W_{s_l} \) as in Theorem 6.2.

(i) For all \( 1 \leq j \leq l \), \( M^k W_s \) converges to \( W_{s_j} \) with respect to the norm in \( \text{Hom}(S, \Xi) \);

(ii) For all \( 1 \leq j_1, j_2 \leq l \),
\[
h_{s_{j_1}, s_{j_2}} := W_{s_{j_2}}^* W_{s_{j_1}} = T_1(W_{s_{j_2}}^* W_{s_{j_1}}).
\]

(iii) The algebra \( S \) is isomorphic to \( M_l(\mathbb{C}) \), and
\[
h_{v_1} + \cdots + h_{v_l} = 1.
\]

**Proof.** We can use Theorem 6.2 and we have that \( M^k W_{s_j} \) converges pointwise to \( W_{s_j} \). Also, \( h_{v_i, v_j} \) form a basis for \( S \) and \( \tau_{i,j} \) form a family of functionals invariant under \( R \) that separate the points in \( S \).

We use Theorem 6.6 and we perform the following computation: since \( s_i(x_0) = v_i \), we have:
\[
\tau_{i,1}(M^k W_{s_j} - W_{s_j})^*(M^k W_{s_j} - W_{s_j}) = \langle (M^k W_{s_j} - W_{s_j})s_i, (M^k W_{s_j} - W_{s_j})s_i \rangle^*(x_0)
\]
\[
= \sum_{g \in G} |\langle s_j \circ p(\tilde{r}^{-k}g \tilde{x}_0), m^{(k)}(\tilde{r}^{-k}g \tilde{x}_0)s_i(x_0) \rangle f(\tilde{r}^{-k}g \tilde{x}_0) - \langle s_j(x_0), s_i(x_0) \rangle f(g \tilde{x}_0)|^2
\]
(see also 5.1).

For \( g \in G \), we have a unique factorization
\[
\tilde{r}^{-k} g = g' \psi_{w_1} \cdots \psi_{w_l},
\]
where \( g' \in G \). The elements \( g \in A^G \) are exactly those for which the above factorization is \( g' \psi_1 \cdots \psi_1 \). Moreover, with Lemma 6.3, we have that \( m^{(k)}(\psi_{w_1} \cdots \psi_{w_l} \tilde{x}_0)v_i = \delta_{w_1 \cdots w_l}v_i \). Also \( \| f(\tilde{x}_0) \| = 1 \) and \( f(g \tilde{x}_0) = 0 \) for \( g \neq 1 \).

This implies that the above expression equals
\[
\sum_{g' \in G} \sum_{\omega \in \Omega_k} |\langle s_j \circ p(\psi_{w_1} \cdots \psi_{w_l} \tilde{x}_0), m^{(k)}(\psi_{w_1} \cdots \psi_{w_l} \tilde{x}_0)s_i(x_0) \rangle f(g' \psi_{w_1} \cdots \psi_{w_l} \tilde{x}_0) - \langle s_j(x_0), s_i(x_0) \rangle f(g \tilde{x}_0)|^2
\]
\[
= \sum_{g' \in G} \sum_{\omega \in \Omega_k} |\langle s_j(x_0), s_i(x_0) \rangle|^{2}\delta_{1 \cdots 1, \omega} f(g \psi_{w_1} \cdots \psi_{w_l} \tilde{x}_0) - f(\tilde{r}^{-k} g' \psi_{w_1} \cdots \psi_{w_l} \tilde{x}_0)|^2
\]

Then for \( i \neq i' \), by an application of Cauchy-Schwarz, we have:
\[
\| \tau_{i,i'}((M^k W_{s_j} - W_{s_j})^*(M^k W_{s_j} - W_{s_j})) \| = 0.
\]
Thus \( M^k W_{s_j} \) converges strongly, by Theorem 6.6 and (i) follows.

(ii) follows directly from (i) as in Theorem 6.6.

(iii) follows directly from Theorem 6.5.

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