Exact separation phenomenon for the eigenvalues of large Information-Plus-Noise type matrices
Application to spiked models

M. Capitaine*

Abstract

We consider large Information-Plus-Noise type matrices of the form $M_N = (\sigma \frac{X_N}{\sqrt{N}} + A_N)(\sigma \frac{X_N}{\sqrt{N}} + A_N)^*$ where $X_N$ is an $n \times N$ ($n \leq N$) matrix consisting of independent standardized complex entries, $A_N$ is an $n \times N$ nonrandom matrix and $\sigma > 0$. As $N$ tends to infinity, if $n/N \rightarrow c \in [0, 1]$ and if the empirical spectral measure of $A_N A_N^*$ converges weakly to some compactly supported probability distribution $\nu \neq \delta_0$, Dozier and Silverstein established in [11] that almost surely the empirical spectral measure of $M_N$ converges weakly towards a nonrandom distribution $\mu_{\sigma, \nu, c}$. In [2], Bai and Silverstein proved, under certain assumptions on the model, that for some closed interval in $[0; +\infty[$ outside the support of $\mu_{\sigma, \nu, c}$ satisfying some conditions involving $A_N$, almost surely, no eigenvalues of $M_N$ will appear in this interval for all $N$ large. In this paper, we carry on with the study of the support of the limiting spectral measure previously investigated in [12] and later in [14, 15] and we show that, under the same assumptions as in [2], there is an exact separation phenomenon between the spectrum of $M_N$ and the spectrum of $A_N A_N^*$: to a gap in the spectrum of $M_N$ pointed out by Bai and Silverstein, it corresponds a gap in the spectrum of $A_N A_N^*$ which splits the spectrum of $A_N A_N^*$ exactly as that of $M_N$. We use the previous results to characterize the outliers of spiked Information-Plus-Noise type models.

1 Introduction

Let $\{X_{ij}, i \in \mathbb{N}, j \in \mathbb{N}\}$ be an infinite set of independent standardized complex entries $\mathbb{E}(X_{ij}) = 0, \mathbb{E}(|X_{ij}|^2) = 1$ in some probability space. Define for any nonnull integer numbers $n \leq N$ and $\sigma > 0$, the following matrix

$$M_N = (\sigma \frac{X_N}{\sqrt{N}} + A_N)(\sigma \frac{X_N}{\sqrt{N}} + A_N)^* \quad (1.1)$$

*CNRS, Institut de Mathématiques de Toulouse, Equipe de Statistique et Probabilités, F-31062 Toulouse Cedex 09. E-mail: mireille.capitaine@math.univ-toulouse.fr
where $X_N = (X_{ij})_{1 \leq i \leq n, 1 \leq j \leq N}$ and $A_N$ is an $n \times N$ nonrandom matrix. This model is referred to in the literature as the Information-Plus-Noise model. For any Hermitian $n \times n$ matrix $Y$, denote by

$$\lambda_1(Y) \geq \ldots \geq \lambda_n(Y)$$

the ordered eigenvalues of $Y$ and by $\mu_Y$ the empirical spectral measure of $Y$:

$$\mu_Y := \frac{1}{n} \sum_{i=1}^{n} \lambda_i(Y).$$

For a probability measure $\tau$ on $\mathbb{R}$, denote by $g_{\tau}$ its Stieltjes transform defined for $z \in \mathbb{C} \setminus \mathbb{R}$ by

$$g_{\tau}(z) = \int_{\mathbb{R}} \frac{d\tau(x)}{z - x}.$$ 

As $N$ tends to infinity, if $c_N = n/N \to c \in [0, 1]$, if the $X_{ij}$ are i.i.d and if the empirical spectral measure $\mu_{AN A_N^{*}}$ of $A_N A_N^{*}$ converges weakly to some probability distribution $\nu \neq \delta_0$, Dozier and Silverstein established in [11] that almost surely the empirical spectral measure $\mu_{N}$ of $M_N$ converges weakly towards a nonrandom distribution $\mu_{\sigma,\nu,c}$ which is characterized in terms of its Stieltjes transform which satisfies the following equation: for any $z \in \mathbb{C}^+$,

$$g_{\mu_{\sigma,\nu,c}}(z) = \int \frac{d\tau(x)}{(1 - \sigma^2 c g_{\mu_{\sigma,\nu,c}}(z)) z - 1 - \sigma^2 c g_{\mu_{\sigma,\nu,c}}(z) - \sigma^2 (1 - c) - \sigma^2 (1 - c)} d\nu(t). \quad (1.2)$$

Note that, since for any $z \in \mathbb{C}^+$, $\Im(z g_{\mu_{\sigma,\nu,c}}(z)) = -\int \frac{\Im z}{|1 - z|^2} d\mu_{\sigma,\nu,c}(z) \leq 0$ and $\Im(g_{\mu_{\sigma,\nu,c}}(z)) < 0$, one can easily see that the imaginary part of the denominator of the integrand in (1.2) is greater or equal to $\Im z$ so that the integral is well defined. Note also that in [11], the authors proved that the solution $m$ to the equation

$$m(z) = \int \frac{1}{(1 - \sigma^2 c m(z)) z - 1 - \sigma^2 c m(z) - \sigma^2 (1 - c)} d\nu(t), \text{ for any } z \in \mathbb{C}^+ \quad (1.3)$$

is unique if $\Im m(z) < 0$ and $\Im(z m(z)) \leq 0$ (specifically if $m$ is the Stieltjes transform of a probability measure with nonnegative support).

This result of convergence was extended to independent but non identically distributed random variables by Xie in [16].

In [12], Dozier and Silverstein investigated this limiting spectral measure $\mu_{\sigma,\nu,c}$ and proved in particular that its distribution function is continuous (it has a continuous derivative on $\mathbb{R} \setminus \{0\}$ and no mass at zero).

The support of the probability measure $\mu_{\sigma,\nu,c}$ (that is the measure whose Stieltjes transform satisfies (1.2) where $\nu$ is replaced by $\mu_{AN A_N^{*}}$ and $c$ by $c_N$) plays a fundamental role in the study of the spectrum of $M_N$ (see [2, 14, 15]).

Introducing, for any probability measure $\tau$ on $[0, +\infty]$ and any $0 < \gamma \leq 1$, $\sigma > 0$, the function $\omega_{\sigma,\tau,\gamma}$ defined in $\mathbb{R} \setminus \supp(\mu_{\sigma,\tau,\gamma})$ by

$$\omega_{\sigma,\tau,\gamma}(x) = x(1 - \sigma^2 \gamma g_{\mu_{\sigma,\tau,\gamma}}(x))^2 - \sigma^2 (1 - \gamma)(1 - \sigma^2 \gamma g_{\mu_{\sigma,\tau,\gamma}}(x))$$
Bai and Silverstein established the following result.

**Theorem 1.1.** [2] Assume that

1. \( X_{ij}, i, j = 1, 2, \ldots \) are independent standardized random variables.

2. There exists a \( K \) and a random variable \( X \) with finite fourth moment such that, for any \( x > 0 \)
   \[
   \frac{1}{n_1n_2} \sum_{i \leq n_1, j \leq n_2} P(|X_{ij}| > x) \leq KP(|X| > x)
   \]
   for any \( n_1, n_2 \).

3. There exists a positive function \( \Psi(x) \uparrow \infty \) as \( x \to \infty \) and \( M > 0 \) such that
   \[
   \max_{ij} E|X^2_{ij}|\Psi(|X^2_{ij}|) \leq M.
   \]

4. \( \|A_N\| \) is uniformly bounded.

5. As \( N \) tends to infinity, \( c_N = n/N \to c \in ]0, 1] \)

6. The empirical spectral measure \( \mu_{A_N A^*_N} \) of \( A_N A^*_N \) converges weakly to some probability distribution \( \nu \neq \delta_0 \).

7. Let \( [a, b] \) be such that the couple \( P(\sigma) \) of the following properties is satisfied:
   - (i) there exists \( 0 < \delta < a \) such that for all large \( N \), \( |a - \delta; b + \delta[\subset \mathbb{R} \setminus \text{supp}(\mu_{\sigma, \nu, c_{\text{SN}}}) \]
   - (ii) \( A_{Nj} \) denoting the matrix resulting from removing the \( j \)-th column from \( A_N \), there exists \( 0 < \tau < \delta \) and a positive \( d < 1 \) such that for all \( N \) large, the number of \( j \)'s with no eigenvalues of \( N/(N - 1)A_{Nj}A^*_N \) appearing in \( \omega_{\sigma, \nu, c_{\text{SN}}}([a - \tau, b + \tau]) \) is greater that \( N^{1-d} \).

Then
\[
\mathbb{P}[\text{for all large } N, \text{spect}(M_N) \subset \mathbb{R} \setminus [a, b]] = 1.
\]

**Remark 1.1.** Since \( \mu_{\sigma, \nu, c_{\text{SN}}} \) converges weakly towards \( \mu_{\sigma, \nu, c} \) (this can be deduced from [11] and Theorem 1 in [15]), Assumption 7. (i) implies that \( \forall 0 < \tau < \delta, [a - \tau; b + \tau[ \subset \text{supp} \mu_{\sigma, \nu, c} \).

Note that such a result was proved by a different approach in [14] when the \( X_{ij} \) are independent gaussian variables without assuming condition (ii) of 7. in Theorem 1.1. Note that when the \( X_{ij} \) are independent gaussian variables, it can be assumed that \( A_N \) is diagonal and, then, condition (ii) of 7. in Theorem 1.1 is not needed.

In [15], dealing with independent gaussian variables \( X_{ij} \), P. Loubaton and P. Vallet established an exact separation phenomenon between the spectrum of \( M_N \) and the spectrum of \( A_N A^*_N \): to a gap in the spectrum of \( M_N \), it corresponds a
gap in the spectrum of $A_N^*A_N$ which splits the spectrum of $A_N$ exactly as that of $M_N$. In this paper, we extend their result to the framework of non gaussian Information-Plus-Noise type matrices investigated in [2] since we establish the following

**Theorem 1.2.** Assume conditions [1-7] of Theorem 1.1 are satisfied and that moreover, if $c < 1$, $\omega_{\sigma,\nu,c}(a) > 0$. Then for $N$ large enough,

$$\omega_{\sigma,\nu,c}([a, b]) = [\omega_{\sigma,\nu,c}(a); \omega_{\sigma,\nu,c}(b)] \subset^c \text{supp}(\mu_{A_N^*A_N})$$

With the convention that $\lambda_0(M_N) = \lambda_0(A_N^*A_N) = +\infty$ and $\lambda_{N+1}(M_N) = \lambda_{N+1}(A_N^*) = -\infty$, for $N$ large enough, let $i_N \in \{0, \ldots, n\}$ be such that

$$\lambda_{i_N+1}(A_N^*A_N) < \omega_{\sigma,\nu,c}(a) \quad \text{and} \quad \lambda_{i_N}(A_N^*A_N) > \omega_{\sigma,\nu,c}(b). \quad (1.4)$$

Then

$$P[\text{for all large } N, \lambda_{i_N+1}(M_N) < a \text{ and } \lambda_{i_N}(M_N) > b] = 1.$$ 

**Remark 1.2.** We will prove in Proposition 2.2 that, if $c < 1$, the minimum of the support of $\mu_{\sigma,\nu,c}$ is strictly positive. Let us denote by $x_0$ this minimum. We will see that if $a \in [0; +\infty[ \setminus \text{supp}(\mu_{\sigma,\nu,c})$ is not included in $[0; x_0]$, then $\omega_{\sigma,\nu,c}(a) > 0$. Nevertheless this is not a necessary condition for $\omega_{\sigma,\nu,c}(a) > 0$ to hold as we will point out in Remark 3.1.

The technics of [15] completely differ from the approach used in the present paper which uses an argument similar to [9] which consists in introducing a continuum of matrices between $A_N^*A_N$ and $M_N$. Before using this approach, we carry on with the study of the support of the limiting spectral measure previously investigated in [12] and later in [14, 15]. The main results about the limiting spectral measure can be summarized in the following theorem.

**Theorem 1.3.** Let $c$ be in $[0; 1]$, $\sigma$ be in $[0; +\infty[$ and $\nu$ be a compactly supported probability measure on $[0; +\infty[$. Define differentiable functions $\omega_{\sigma,\nu,c}$ and $\Phi_{\sigma,\nu,c}$ on respectively $\mathbb{R} \setminus \text{supp}(\mu_{\sigma,\nu,c})$ and $\mathbb{R} \setminus \text{supp}(\nu)$ by setting

$$\omega_{\sigma,\nu,c} : \mathbb{R} \setminus \text{supp}(\mu_{\sigma,\nu,c}) \to \mathbb{R}$$

$$x \mapsto x(1 - \sigma^2c g_{\sigma,\nu,c}(x))^2 - \sigma^2(1 - c)(1 - \sigma^2c g_{\sigma,\nu,c}(x))$$

and

$$\Phi_{\sigma,\nu,c} : \mathbb{R} \setminus \text{supp}(\nu) \to \mathbb{R}$$

$$x \mapsto x(1 + c\sigma^2 g_{\nu}(x))^2 + \sigma^2(1 - c)(1 + c\sigma^2 g_{\nu}(x))$$

Set

$$E_{\sigma,\nu,c} := \left\{ u \in \mathbb{R} \setminus \text{supp}(\nu), \Phi_{\sigma,\nu,c}'(u) > 0, g_{\nu}(u) > -\frac{1}{\sigma^2c} \right\}.$$ 

We have the following results.

A) If $c < 1$ then $0 \notin \text{supp}(\mu_{\sigma,\nu,c})$.

If $c = 1$,
a) If $0 \in \text{supp}(\nu)$ then $0 \in \text{supp}(\mu_{\sigma,\nu,1})$.

b) If $0 \notin \text{supp}(\nu)$ and if $g_\nu(0) \leq -\frac{1}{\sigma}$ then $0 \notin \text{supp}(\mu_{\sigma,\nu,1})$.

c) If $0 \notin \text{supp}(\nu)$ and if $g_\nu(0) > -\frac{1}{\sigma}$ then $0 \notin \text{supp}(\mu_{\sigma,\nu,1})$.

B) 1. $\omega_{\sigma,\nu,c}$ is a homeomorphism from $\mathbb{R} \setminus \text{supp}(\mu_{\sigma,\nu,c})$ onto $\mathcal{E}_{\sigma,\nu,c}$ with inverse $\Phi_{\sigma,\nu,c}$.

2. $\omega'_{\sigma,\nu,c} > 0$ on $\mathbb{R} \setminus \text{supp}(\mu_{\sigma,\nu,c})$.

C) For any $y > x$ in $\mathcal{E}_{\sigma,\nu,c}$, $\Phi_{\sigma,\nu,c}(y) > \Phi_{\sigma,\nu,c}(x)$.

D) Assume that the support of $\nu$ has a finite number of connected components. Then there exists a nonnull integer number $p$ and $u_1 < v_1 < u_2 < \ldots < u_p < v_p$ (depending on $\sigma, \nu, c$) such that

$$\mathcal{E}_{\sigma,\nu,c} = ] - \infty; u_1 [ \cup \ldots \cup [ u_p ; v_1 ] ; \ldots \cup [ u_1 ; v_p ]; + \infty [.$$ 

We have

$$\text{supp}(\nu) \subset \bigcup_{l=1}^{p} [u_l; v_l]$$

and for each $l \in \{1, \ldots, p\}$, $[u_l; v_l] \cap \text{supp}(\nu) \neq \emptyset$.

Moreover,

$$\text{supp}(\mu_{\sigma,\nu,c}) = \bigcup_{l=1}^{p} \{ \Phi_{\sigma,\nu,c}(u^{-}_l); \Phi_{\sigma,\nu,c}(v^{+}_l) \},$$

with

$$\Phi_{\sigma,\nu,c}(u^{-}_1) < \Phi_{\sigma,\nu,c}(v^{+}_1) < \Phi_{\sigma,\nu,c}(u^{-}_2) < \Phi_{\sigma,\nu,c}(v^{+}_2) < \cdots < \Phi_{\sigma,\nu,c}(u^{-}_p) < \Phi_{\sigma,\nu,c}(v^{+}_p),$$

where $\Phi_{\sigma,\nu,c}(u^{-}_l) = \lim_{u \uparrow u_l} \Phi_{\sigma,\nu,c}(u)$ and $\Phi_{\sigma,\nu,c}(v^{+}_l) = \lim_{u \downarrow v_l} \Phi_{\sigma,\nu,c}(u)$.

Note that choosing a matricial model as introduced in Section 4 but without spikes, it is easy to deduce from Theorem 1, Proposition 1, and Theorem 1A and the weak convergence almost surely of the spectral measures the following

**Corollary 1.1.** Assume that the support of $\nu$ has a finite number of connected components. Let $[u_l, v_l]$ be a connected component of $\mathbb{R} \setminus \mathcal{E}_{\sigma,\nu,c}$, then

$$\mu_{\sigma,\nu,c}(\Phi_{\sigma,\nu,c}(u^{-}_l), \Phi_{\sigma,\nu,c}(v^{+}_l)) = \nu([u_l, v_l]).$$

The previous Theorem 1.2 and Theorem 1.3 allow us to investigate the spectrum of spiked models when the perturbation matrix is diagonal and to obtain in Theorem 1.4 a description of the convergence of the eigenvalues of $M_N$ depending on the location of the spikes of the perturbation with respect to $\mathcal{E}_{\sigma,\nu,c}$ and to the connected components of the support of $\nu$. This extends a previous result in [15] involving the Gaussian case and finite rank perturbations.

This paper is organized as follows. Section 2 is devoted to the proof of Theorem 1.3. Theorem 1.2 is proved in Section 3. Section 4 investigates the spectrum of spiked models.
2 The support of the limiting spectral measure

In this section, we split the results of Theorem 1.3 into different propositions that we prove successively. Firstly, we are going to take up the arguments and results of [12] that we will develop here for the reader’s convenience in order to state in Proposition 2.1 the characterization of the complement in $\mathbb{R} \setminus \{0\}$ of the support of $\mu_{\sigma,\nu,c}$. Secondly, in Proposition 2.2 we establish necessary and sufficient conditions for the inclusion of zero in the support of $\mu_{\sigma,\nu,c}$. These results allow us to put forward a complete characterization of the complement of the support of $\mu_{\sigma,\nu,c}$ in $\mathbb{R}$ in Proposition 2.3 and Proposition 2.4. Thirdly, in Proposition 2.5, we establish a relationship between the complement of the support of $\mu_{\sigma,\nu,c}$ and the complement of the support of $\mu_{\sigma,\sqrt{\nu},1}$ and in Proposition 2.6, we prove that $\Phi_{\sigma,\nu,c}$ is globally increasing on the set $\mathcal{E}_{\sigma,\nu,c}$. When the support of $\nu$ has a finite number of connected components, these results allow us to deduce a description of the support of $\mu_{\sigma,\nu,c}$ in Proposition 2.7 in terms of a finite union of closed disjoint intervals.

Actually it is possible to deduce from results of [12] the following characterization of the complement of $\text{supp}(\mu_{\sigma,\nu,c}) \cup \{0\}$ and the following relationships between $\Phi_{\sigma,\nu,c}$ and $\omega_{\sigma,\nu,c}$.

**Proposition 2.1.**

(i) \( \forall x \in \mathbb{R} \setminus \{\text{supp}(\mu_{\sigma,\nu,c}) \cup \{0\}\} \), we have $\omega_{\sigma,\nu,c}(x) \in \mathbb{R} \setminus \text{supp}(\nu)$, $\omega_{\sigma,\nu,c}'(x) > 0$, $g_{\nu}(\omega_{\sigma,\nu,c}(x)) > -\frac{1}{\sigma^2 c}$ and

\[ \Phi_{\sigma,\nu,c}(\omega_{\sigma,\nu,c}(x)) = x. \]  \hfill (2.1)

(ii) \( \forall x \in \mathbb{R} \setminus \text{supp}(\nu) \) such that $\Phi_{\sigma,\nu,c}'(x) > 0$ and $g_{\nu}(x) > -\frac{1}{\sigma^2 c}$, we have $\Phi_{\sigma,\nu,c}(x) \in \mathbb{R} \setminus \text{supp}(\mu_{\sigma,\nu,c})$ and

\[ \omega_{\sigma,\nu,c}(\Phi_{\sigma,\nu,c}(x)) = x. \]  \hfill (2.2)

(iii) Therefore

\[ \mathbb{R} \setminus \{\text{supp}(\mu_{\sigma,\nu,c}) \cup \{0\}\} = \Phi_{\sigma,\nu,c}\left\{ u \in \mathbb{R} \setminus \text{supp}(\nu), \Phi_{\sigma,\nu,c}(u) \neq 0, \Phi_{\sigma,\nu,c}'(u) > 0, g_{\nu}(u) > -\frac{1}{\sigma^2 c}\right\}. \]  \hfill (2.3)

**Proof:** [12] may be rewritten, for any $z \in \mathbb{C}^+$,

\[ \frac{g_{\mu_{\sigma,\nu,c}}(z)}{1 - \sigma^2 c g_{\mu_{\sigma,\nu,c}}(z)} = g_{\nu}\left[z(1 - \sigma^2 c g_{\mu_{\sigma,\nu,c}}(z))^2 - \sigma^2(1-c)(1-\sigma^2 c g_{\mu_{\sigma,\nu,c}}(z))\right] \]  \hfill (2.4)
According to Theorem 2.1 and Lemma 2.1 in [12], for any $x$ in $\mathbb{R} \setminus \{0\}$, one has
\[
\lim_{z \in \mathbb{C}^+ \to x} g_{\mu_{\sigma,\nu,c}}(z) := g_{\mu_{\sigma,\nu,c}}(x) \text{ exists and}
\]
\[
\forall z \in \mathbb{C}^+ \cup \mathbb{R} \setminus \{0\}, \quad \Re \left( \frac{1}{\sigma^2 c} - g_{\mu_{\sigma,\nu,c}}(z) \right) > 0 \tag{2.5}
\]
and according to Theorem 3.2 in [12],
\[
\text{if } x \in \mathbb{R} \setminus \{\text{supp}(\mu_{\sigma,\nu,c}) \cup \{0\}\} \text{ then } \omega_{\sigma,\nu,c}(x) \in \mathbb{R} \setminus \text{supp}(\nu). \tag{2.6}
\]
Therefore it makes sense to let $z \in \mathbb{C}^+$ tend to $x \in \mathbb{R} \setminus \{\text{supp}(\mu_{\sigma,\nu,c}) \cup \{0\}\}$ in (2.4) and get for any $x \in \mathbb{R} \setminus \{\text{supp}(\mu_{\sigma,\nu,c}) \cup \{0\}\}$,
\[
\frac{g_{\mu_{\sigma,\nu,c}}(x)}{1 - \sigma^2 c g_{\mu_{\sigma,\nu,c}}(x)} = g_{\nu}(\omega_{\sigma,\nu,c}(x)). \tag{2.7}
\]
Multiplying both sides of (2.7) by $\sigma^2 c$ and adding 1, it readily comes that
\[
1 - \sigma^2 c g_{\mu_{\sigma,\nu,c}}(x) = 1 + \sigma^2 c g_{\nu}(\omega_{\sigma,\nu,c}(x)) \tag{2.8}
\]
and then
\[
1 - \sigma^2 c g_{\mu_{\sigma,\nu,c}}(x) = \frac{1}{1 + \sigma^2 c g_{\nu}(\omega_{\sigma,\nu,c}(x))}. \tag{2.9}
\]
Replacing $1 - \sigma^2 c g_{\mu_{\sigma,\nu,c}}(x)$ in the expression of $\omega_{\sigma,\nu,c}(x)$ by the right hand side of (2.9), it follows that
\[
\omega_{\sigma,\nu,c}(x) = \frac{x}{(1 + \sigma^2 c g_{\nu}(\omega_{\sigma,\nu,c}(x)))^2} - \frac{\sigma^2 (1 - c)}{(1 + \sigma^2 c g_{\nu}(\omega_{\sigma,\nu,c}(x)))},
\]
and finally that
\[
\Phi_{\sigma,\nu,c}(\omega_{\sigma,\nu,c}(x)) = x \tag{2.10}
\]
Hence any $x \in \mathbb{R} \setminus \{\text{supp}(\mu_{\sigma,\nu,c}) \cup \{0\}\}$ can be written as $\Phi_{\sigma,\nu,c}(u)$ where $u = \omega_{\sigma,\nu,c}(x) \in \mathbb{R} \setminus \text{supp}(\nu)$. Let us prove that $\omega_{\sigma,\nu,c}(x) > 0$, $\Phi_{\sigma,\nu,c}(\omega_{\sigma,\nu,c}(x)) > 0$ and $g_{\nu}(\omega_{\sigma,\nu,c}(x)) > -\frac{1}{\sigma^2 c}$. By differentiating both sides of (2.8) we obtain that for any $x \in \mathbb{R} \setminus \{\text{supp}(\mu_{\sigma,\nu,c}) \cup \{0\}\}$
\[
\frac{g'_{\mu_{\sigma,\nu,c}}(x)}{(1 - \sigma^2 c g_{\mu_{\sigma,\nu,c}}(x))^2} = \omega_{\sigma,\nu,c}(x) g'_{\nu}(\omega_{\sigma,\nu,c}(x)). \tag{2.11}
\]
Therefore since $g_{\mu_{\sigma,\nu,c}}(x) < 0$ and $g'_{\nu}(\omega_{\sigma,\nu,c}(x)) < 0$ we can deduce that
\[
\text{for any } x \in \mathbb{R} \setminus \{\text{supp}(\mu_{\sigma,\nu,c}) \cup \{0\}\}, \omega_{\sigma,\nu,c}(x) > 0. \tag{2.12}
\]
Now by differentiating both sides of (2.10) we obtain that
\[
\text{for any } x \in \mathbb{R} \setminus \{\text{supp}(\mu_{\sigma,\nu,c}) \cup \{0\}\}, \Phi'_{\sigma,\nu,c}(\omega_{\sigma,\nu,c}(x)) > 0. \tag{2.13}
\]
Let us prove now that

$$\text{for } x \in \mathbb{R} \setminus \{\supp(\mu_{\sigma,\nu,c}) \cup \{0\}\}, g_\nu(\omega_{\sigma,\nu,c}(x)) > -\frac{1}{\sigma^2 c}, \quad (2.14)$$

According to Lemma 2.1 (c) in [12], $g_{\mu_{\sigma,\nu,c}}(x) < \frac{1}{\sigma^2 c}$. Moreover $y \mapsto \frac{y}{1-\sigma^2 cy}$ is a strictly increasing function from $]-\infty; -\frac{1}{\sigma^2 c}[\text{ onto } ]-\infty; +\infty[$. The result readily follows using (2.7). Hence, we have proved the inclusion

$$\mathbb{R} \setminus \{\supp(\mu_{\sigma,\nu,c}) \cup \{0\}\} \subset \Phi_{\sigma,\nu,c}(u \in \mathbb{R} \setminus \supp(\nu), \Phi_{\sigma,\nu,c}(u) \neq 0, \Phi'_{\sigma,\nu,c}(u) > 0, g_\nu(u) > -\frac{1}{\sigma^2 c}) \}.$$

Now let $u$ be in $\mathbb{R} \setminus \supp(\nu), \Phi'_{\sigma,\nu,c}(u) > 0, g_\nu(u) > -\frac{1}{\sigma^2 c}$. Following [12] let us prove that $\Phi_{\sigma,\nu,c}(u)$ belongs to $\mathbb{R} \setminus \{\supp(\mu_{\sigma,\nu,c}) \cup \{0\}\}$ and $\omega_{\sigma,\nu,c}(\Phi_{\sigma,\nu,c}(u)) = u$. Let $l_1; l_2 \subset [L_1; L_2] \subset \mathbb{R} \setminus \supp(\nu)$ such that $u \in [l_1; l_2]$ and for any $v \in [l_1; l_2], \Phi_{\sigma,\nu,c}(v) > 0, g_\nu(v) > -\frac{1}{\sigma^2 c}$. $g_\nu$ is strictly decreasing on $[l_1; l_2]$ and maps $[l_1; l_2]$ onto some interval $[d_1; d_2] \subset [-\frac{1}{\sigma^2 c}; +\infty[. h : b \mapsto \frac{b}{\sigma^2 c}(\frac{1}{b} - 1)$ is a strictly decreasing function from $]0; +\infty[\text{ onto } ]-\frac{1}{\sigma^2 c}; +\infty[$. Hence there is an interval $[k_1; k_2] \subset ]0; +\infty[\text{ such that } h \text{ is a one-to-one correspondence from } [k_1; k_2] \text{ to } [d_1; d_2].$ Therefore $g_\nu^{-1} \circ h$ is a one-to-one correspondence from $[k_1; k_2]$ to $[l_1; l_2].$ For any $v$ in $[l_1; l_2]$, there exists a unique $k$ in $[k_1; k_2]$ such that $v = g_\nu^{-1}(h(k));$ then $k = k(v) = h^{-1}(g_\nu(v)) = -\frac{1}{\sigma^2 c} \cdot \Phi_{\sigma,\nu,c}(v).$ Moreover

$$p(k) := \frac{1}{k^2} g_\nu^{-1}(h(k)) + \frac{1}{k} \sigma^2(1-c) = (1+\sigma^2 c g_\nu(v))^2 v + (1+\sigma^2 c g_\nu(v))^2(1-c) = \Phi_{\sigma,\nu,c}(v). \quad (2.15)$$

Differentiating both sides of (2.15) in $v$ we obtain that for any $v \in [l_1; l_2]$, $p'(k(v))k'(v) = \Phi'_{\sigma,\nu,c}(v) \text{ with } k'(v) = -\sigma^2 c \frac{g_\nu(v)}{(1+\sigma^2 c g_\nu(v))^2} > 0$. Therefore $\Phi'_{\sigma,\nu,c}(v) > 0$ implies $p'(k(v)) > 0$. According to Theorem 3.3 in [12], we can conclude that $p(k(u)) = \Phi_{\sigma,\nu,c}(u) \in \mathbb{R} \setminus \supp(\mu_{\sigma,\nu,c})$ and

$$k = 1 - \sigma^2 c \mu_{\sigma,\nu,c}(p(k)). \quad (2.16)$$

Moreover (2.16) implies

$$\frac{1}{1+\sigma^2 c g_\nu(u)} = 1 - \sigma^2 c g_\nu(u) \Phi'_{\sigma,\nu,c}(u).$$

It readily follows that

$$\Phi_{\sigma,\nu,c}(u)(1 - \sigma^2 c \mu_{\sigma,\nu,c}(\Phi_{\sigma,\nu,c}(u))^2 - \sigma^2(1-c)(1-\sigma^2 c g_\nu(u)) = u$$

that is $\omega_{\sigma,\nu,c}(\Phi_{\sigma,\nu,c}(u)) = u. \quad \square$
Interpreting $\mu_{\sigma,\nu,c}$ as the distribution of a Free Wishart process $FW(\frac{1}{\mu} ; \nu)$ introduced in [7] at time $t = \sigma^2 c$ will allow us to establish necessary and sufficient conditions for the inclusion of zero in the support of $\mu_{\sigma,\nu,c}$.

**Proposition 2.2.** 1. If $c < 1$ then $0 \notin \text{supp}(\mu_{\sigma,\nu,c})$.

2. If $c = 1$,

   a) If $0 \in \text{supp}(\nu)$ then $0 \in \text{supp}(\mu_{\sigma,\nu,1})$.
   
   b) If $0 \notin \text{supp}(\nu)$ and if $g_\nu(0) \leq -\frac{1}{\sigma}$ then $0 \in \text{supp}(\mu_{\sigma,\nu,1})$.
   
   c) If $0 \notin \text{supp}(\nu)$ and if $g_\nu(0) > -\frac{1}{\sigma}$ then $0 \notin \text{supp}(\mu_{\sigma,\nu,1})$.

**Proof:** Choosing gaussian entries for $X_N$, It is easy to see that $\mu_{\sigma,\nu,c}$ is the distribution of a Free Wishart process $FW(\frac{1}{\mu} ; \nu)$ introduced in [7] at time $t = \sigma^2 c$ (see Section 2.2 p 421 in [7]).

1. It is proved in Proposition 2.2 in [7] that if $c < 1$, the minimum of the support of $\mu_{\sigma,\nu,c}$ is strictly positive; therefore $0 \notin \text{supp}(\mu_{\sigma,\nu,c})$.

2. If $c = 1$, $\mu_{\sigma,\nu,1}$ is the distribution of $(Z(\sigma^2)^* + \sqrt{d_0})(Z(\sigma^2) + \sqrt{d_0})$ where $Z$ is a complex free Brownian motion in some $W^*$-non commutative probability space $(\mathcal{A}, \phi)$, $d_0$ is a a positive operator in $(\mathcal{A}, \phi)$ such that $\forall k \geq 0$, $\phi(d_0^k) = \int x^k d\nu(x)$, (see Definition 2.3 in [7]). From Lemma 5.1 of [4], $(Z(\sigma^2)^* + \sqrt{d_0})(Z(\sigma^2) + \sqrt{d_0})$ has the same distribution in $(\mathcal{A}, \phi)$ as $(S_{\sigma} + a)^2$ where $S_{\sigma}$ is a centered semi-circular variable with variance $\sigma^2$ and $a$ is a symmetrically variable which is free with $S_{\sigma}$ such that $\phi(a^{2k}) = \int x^k d\nu(x)$. Let us denote by $\tau$ the distribution of $a$; $\tau$ is the symmetrization\(^1\) of the pushforward of $\nu$ by the map $t \mapsto \sqrt{t}$. Then, denoting by $\mu_{sc}(\sigma)$ the centered semi-circular distribution with variance $\sigma^2$, proving 2. of Proposition 2.2 consists in studying if 0 is either in the support of the free convolution $\mu_{sc}(\sigma) \boxplus \tau$ or in its complement. The support of the free convolution of a probability measure on $\mathbb{R}$ and $\mu_{sc}(\sigma)$ has been deeply studied by P. Biane in [5]. Let $U_{\sigma,\tau} := \left\{ u \in \mathbb{R}, \int_R \frac{d\tau(x)}{(u-x)^2} > \frac{1}{\sigma^2} \right\}$.

According to (2.4) in [9],

$$U_{\sigma,\tau} = \text{supp}(\tau) \cup \left\{ u \in \mathbb{R} \setminus \text{supp}(\tau), \int_R \frac{d\tau(x)}{(u-x)^2} \geq \frac{1}{\sigma^2} \right\} \quad (2.17)$$

P. Biane obtained in [5] the following description of the support

$$\mathbb{R} \setminus \text{supp}(\mu_{sc}(\sigma) \boxplus \tau) = H_{\sigma,\tau}(\mathbb{R} \setminus U_{\sigma,\tau}). \quad (2.18)$$

where

$$H_{\sigma,\tau} : z \mapsto z + \sigma^2 g_\tau(z).$$

\(^1\)The symmetrization of a law $\mu$ on $[0; +\infty]$ is the law $\tau$ defined by $\tau(A) = \frac{\mu(A) + \mu(-A)}{2}$ for all Borel set $A$. 

9
Note that since $\tau$ is a symmetric distribution, $\overline{U_{\sigma,\tau}}$ is a symmetrical set and
\[ \forall z \in \mathbb{R} \setminus \overline{U_{\sigma,\tau}}, \quad H_{\sigma,\tau}(-z) = -H_{\sigma,\tau}(z) \] (2.19)

Therefore, if for some $z \in \mathbb{R} \setminus \overline{U_{\sigma,\tau}}$ we have $H_{\sigma,\tau}(z) = 0$ then $H_{\sigma,\tau}(-z) = H_{\sigma,\tau}(z)$. Since $H_{\sigma,\tau}$ is globally strictly increasing on $\mathbb{R} \setminus \overline{U_{\sigma,\tau}}$ (see Remark 2.1 in [9]), we can deduce that if $z \in \mathbb{R} \setminus \overline{U_{\sigma,\tau}}$ satisfies $H_{\sigma,\tau}(z) = 0$ it must be equal to zero. It follows that, according to (2.18), if $0 \in \overline{U_{\sigma,\tau}}$, then $0 \not\in \mathbb{R} \setminus \text{supp}(\mu_{\sigma,\nu,c}) = \sigma,\tau$. Since $0$ belongs to $\overline{U_{\sigma,\tau}}$ if and only if either $0$ belongs to sup$(\tau)$ (or equivalently $0 \in \text{supp}(\nu)$) or $0 \in \mathbb{R} \setminus \text{supp}(\nu)$ and $\int \frac{d\nu(x)}{x^2} = \int \frac{1}{x}d\nu(x) = -g_{\nu}(0) \geq \frac{1}{\sigma c}$, a) and b) follow.

If $0 \not\in \mathbb{R} \setminus \text{supp}(\mu_{\sigma,\nu,c})$ which means that $0 \not\in \text{supp}(\nu)$ and $g_{\nu}(0) > -\frac{1}{\sigma c}$, (2.19) yields $H_{\sigma,\tau}(0) = 0$ and then, (2.18) implies that $0 \in \mathbb{R} \setminus \text{supp}(\mu_{\sigma,\nu,c} \oplus \tau)$ and c) follows. \(\blacksquare\)

Proposition 2.1 and Proposition 2.2 easily allow to put foward the following complete characterization of the support of $\mu_{\sigma,\nu,1}$.

**Proposition 2.3.**

\[ \mathbb{R} \setminus \text{supp}(\mu_{\sigma,\nu,1}) = \Phi_{\sigma,\nu,1} \left\{ u \in \mathbb{R} \setminus \text{supp}(\nu), \Phi'_{\sigma,\nu,1}(u) > 0, g_{\nu}(u) > -\frac{1}{\sigma c} \right\} \] (2.20)

Now, assume that $c < 1$. According to Proposition 2.2, $0 \not\in \text{supp}(\mu_{\sigma,\nu,c})$. Since $z \mapsto \frac{1}{\sigma c} - g_{\mu_{\sigma,\nu,c}}(z)$ is strictly increasing on $]-\infty, \min \{ x, x \in \text{supp}(\mu_{\sigma,\nu,c}) \}$, and according to (2.3), \(\forall z \in \mathbb{R} \setminus \{ 0 \} \), \(\mathbb{R}(\frac{1}{\sigma c} - g_{\mu_{\sigma,\nu,c}}(z)) > 0\), we can readily deduce that \(\frac{1}{\sigma c} - g_{\mu_{\sigma,\nu,c}}(0) > 0\) and then

\[ \omega_{\sigma,\nu,c}(0) < 0. \]

Let $\epsilon > 0$ be such that $]-\epsilon; \epsilon[ \subset \mathbb{R} \setminus \text{supp}(\mu_{\sigma,\nu,c})$ and $\omega_{\sigma,\nu,c}(\epsilon) < 0$. According to Proposition 2.1 (i), we have for any $x \in [-\epsilon; \epsilon] \setminus \{ 0 \}$, $\omega_{\sigma,\nu,c}(x) \in \mathbb{R} \setminus \text{supp}(\nu), \omega_{\sigma,\nu,c}(x) > 0, \Phi'_{\sigma,\nu,c}(\omega_{\sigma,\nu,c}(x)) > 0, g_{\nu}(\omega_{\sigma,\nu,c}(x)) > -\frac{1}{\sigma c}$ and $\Phi'_{\sigma,\nu,c}(\omega_{\sigma,\nu,c}(x)) = x$. This readily implies that $\omega_{\sigma,\nu,c}(\epsilon) > \omega_{\sigma,\nu,c}(0)$ and therefore, since $g_{\nu}$ is strictly decreasing on $]-\infty; \omega_{\sigma,\nu,c}(\epsilon)],$ that

\[ g_{\nu}(\omega_{\sigma,\nu,c}(\epsilon)) > g_{\nu}(\omega_{\sigma,\nu,c}(0)) > -\frac{1}{\sigma c}. \]

Moreover, by continuity, we readily have

\[ \forall x \in ]-\epsilon; \epsilon[, \quad \Phi_{\sigma,\nu,c}(\omega_{\sigma,\nu,c}(x)) = x \] (2.21)

and $\omega_{\sigma,\nu,c}'(0) \geq 0$ and $\Phi'_{\sigma,\nu,c}(\omega_{\sigma,\nu,c}(0)) \geq 0$. Now, differentiating both sides of (2.21) at zero yields $\Phi'_{\sigma,\nu,c}(\omega_{\sigma,\nu,c}(0)) \neq 0$ and $\omega_{\sigma,\nu,c}'(0) \neq 0$ and then $\Phi'_{\sigma,\nu,c}(\omega_{\sigma,\nu,c}(0)) > 0$ and $\omega_{\sigma,\nu,c}'(0) > 0$. 

10
Lemma 2.1. Recall that $E_{\sigma,\nu,c} = \{ u \in \mathbb{R} \setminus \text{supp}(\nu), \Phi'_{\sigma,\nu,c}(u) > 0, g_{\nu}(u) > -\frac{1}{\sigma^c} \}$. We have for any $0 < c < 1$, $E_{\sigma,\nu,c} \subset E_{\sigma,\nu,c}'$. 

Proof: This readily follows from the fact that $\Phi'_{\sigma,\nu,c}(u) = \Phi'_{\sigma,\nu,c}'(u) = \Phi'_{\sigma,\nu,c}(u) + c \sigma^c (1 - c) g_{\nu}(u)$ and then $\Phi'_{\sigma,\nu,c}(u) < \Phi'_{\sigma,\nu,c}'(u)$. □

Lemma 2.2. The function $K$ defined by

$$K(z) = z + \frac{\sigma^2 (1 - c)}{1 - \sigma^2 g_{\mu,\nu,c}(z)}$$

is well defined on $\mathbb{R} \setminus \text{supp}(\mu_{\nu,c})$ and

$$\forall u \in E_{\sigma,\nu,c}', \Phi'_{\sigma,\nu,c}(u) = K(\Phi'_{\sigma,\nu,c}(u)). \quad (2.22)$$

Proof: According to (2.21), for any $z \neq 0$ in $\mathbb{R} \setminus \text{supp}(\mu_{\nu,c})$, $\frac{1}{\sigma^c} - g_{\mu,\nu,c}(z) > 0$. Now, if $0 \in \mathbb{R} \setminus \text{supp}(\mu_{\nu,c})$, since $\frac{1}{\sigma^c} - g_{\mu,\nu,c}(z)$ is strictly increasing on $]-\infty; \min\{x, x \in \text{supp}(\mu_{\nu,c})\}]$, we can deduce that $\frac{1}{\sigma^c} - g_{\mu,\nu,c}(z)(0) > 0$. Therefore for any $z$ in $\mathbb{R} \setminus \text{supp}(\mu_{\nu,c})$, $\frac{1}{\sigma^c} - g_{\mu,\nu,c}(z) > 0$ and $K$ is well defined. Now, (2.22) readily follows from (2.8), Proposition 2.3 and (2.21) for $u \neq 0$ and may be proved for $u = 0$ whenever $0 \in E_{\sigma,\nu,c}'$ by a continuity argument. □

Proposition 2.5.

$$\mathbb{R} \setminus \text{supp}(\mu_{\nu,c}) = K \left( \{ \mathbb{R} \setminus \text{supp}(\mu_{\nu,c}), K'(x) > 0 \} \right).$$

Proof: Let $x$ be in $\mathbb{R} \setminus \text{supp}(\mu_{\nu,c})$. According to Proposition 2.4, there exists $u \in E_{\sigma,\nu,c}$ such that $x = \Phi'_{\sigma,\nu,c}(u)$. According to Lemma 2.1 $u$ belongs to $E_{\sigma,\nu,c}'$ and according to (2.21), $x = K(\Phi'_{\sigma,\nu,c}(u))$. Proposition 2.3 implies
that $\Phi_{\sigma^{\sqrt{c},\nu,1}}(u)$ belongs to $\mathbb{R} \setminus \text{supp}(\mu_{\sigma^{\sqrt{c},\nu,1}})$. Differentiating both sides of (2.22) leads to
\[
\forall v \in \mathcal{E}_{\sigma^{\sqrt{c},\nu,1}}, \quad \Phi'_{\sigma^{\sqrt{c},\nu,1}}(v) = K'(\Phi_{\sigma^{\sqrt{c},\nu,1}}(v))\Phi'_{\sigma^{\sqrt{c},\nu,1}}(v). \tag{2.23}
\]
Then $K'(\Phi_{\sigma^{\sqrt{c},\nu,1}}(u)) > 0$ since $u \in \mathcal{E}_{\sigma^{\sqrt{c},\nu,1}} \subset \mathcal{E}_{\sigma^{\sqrt{c},\nu,1}}$. Therefore
\[
\mathbb{R} \setminus \text{supp}(\mu_{\sigma^{\sqrt{c},\nu,1}}) \subset K\left(\left\{ \mathbb{R} \setminus \text{supp}(\mu_{\sigma^{\sqrt{c},\nu,1}}), K'(x) > 0 \right\}\right).
\]
Now, let $x$ be in $\mathbb{R} \setminus \text{supp}(\mu_{\sigma^{\sqrt{c},\nu,1}})$ such that $K'(x) > 0$. According to Proposition 2.3 there exists $u$ in $\mathcal{E}_{\sigma^{\sqrt{c},\nu,1}}$ such that $x = \Phi_{\sigma^{\sqrt{c},\nu,1}}(u)$ and (2.22) implies $K(x) = K(\Phi_{\sigma^{\sqrt{c},\nu,1}}(u)) = \Phi_{\sigma^{\sqrt{c},\nu,1}}(u)$. Now, (2.23) implies that $\Phi'_{\sigma^{\sqrt{c},\nu,1}}(u) > 0$ and then $u \in \mathcal{E}_{\sigma^{\sqrt{c},\nu,1}}$ and, using Proposition 2.4 $\Phi_{\sigma^{\sqrt{c},\nu,1}}(u) \in \mathbb{R} \setminus \text{supp}(\mu_{\sigma^{\sqrt{c},\nu,1}})$. Therefore $K\left(\left\{ \mathbb{R} \setminus \text{supp}(\mu_{\sigma^{\sqrt{c},\nu,1}}), K'(x) > 0 \right\}\right) \subset \mathbb{R} \setminus \text{supp}(\mu_{\sigma^{\sqrt{c},\nu,1}})$ and the proof is complete. □

The following lemmas will allow us to establish in Proposition 2.6 that $\Phi_{\sigma^{\sqrt{c},\nu,1}}$ is globally increasing on the set $\left\{ u \in \mathbb{R} \setminus \text{supp}(\nu), \Phi'_{\sigma^{\sqrt{c},\nu,1}}(u) > 0, g_{\nu}(u) > -\frac{1}{\sigma^{\sqrt{c}}} \right\}$.

**Lemma 2.3.** Let $z_1 < z_2$ be in $\mathbb{R} \setminus \text{supp}(\mu_{\sigma^{\sqrt{c},\nu,1}})$ such that for $i = 1, 2$, $K'(z_i) > 0$. Then $K(z_1) < K(z_2)$.

**Proof:** We have for $i = 1, 2$,
\[
K'(z_i) = 1 - \frac{\sigma^4(1-c)c}{(1 - \sigma^2c g_{\mu_{\nu,1}}(z_i))^2} \int \frac{d\mu_{\sigma^{\sqrt{c},\nu,1}}(x)}{(z_i - x)^2} > 0.
\]
Then
\[
\frac{K(z_2) - K(z_1)}{(z_2 - z_1)} = 1 - \frac{\sigma^4(1-c)c}{(1 - \sigma^2c g_{\mu_{\nu,1}}(z_2))^2(1 - \sigma^2c g_{\mu_{\nu,1}}(z_1))} \int \frac{d\mu_{\sigma^{\sqrt{c},\nu,1}}(x)}{(z_2 - x)(z_1 - x)}
\]
Using Cauchy Schwartz inequality we have
\[
\frac{\sigma^4(1-c)c}{(1 - \sigma^2c g_{\mu_{\nu,1}}(z_2))^2(1 - \sigma^2c g_{\mu_{\nu,1}}(z_1))} \int \frac{d\mu_{\sigma^{\sqrt{c},\nu,1}}(x)}{(z_2 - x)(z_1 - x)} \leq \left\{ \frac{\sigma^4(1-c)c}{(1 - \sigma^2c g_{\mu_{\nu,1}}(z_2))^2(1 - \sigma^2c g_{\mu_{\nu,1}}(z_1))} \int \frac{d\mu_{\sigma^{\sqrt{c},\nu,1}}(x)}{(z_1 - x)^2} \right\}^{\frac{1}{2}} \leq \left\{ \frac{\sigma^4(1-c)c}{(1 - \sigma^2c g_{\mu_{\nu,1}}(z_2))^2(1 - \sigma^2c g_{\mu_{\nu,1}}(z_1))} \int \frac{d\mu_{\sigma^{\sqrt{c},\nu,1}}(x)}{(z_2 - x)^2} \right\}^{\frac{1}{2}} < 1.
\]
It follows that $K(z_2) - K(z_1) > 0$. □
Proposition 2.6. For any \( u_2 > u_1 \) in \( E_{\sigma,\nu,c} \),
\[
\Phi_{\sigma,\nu,c}(u_2) > \Phi_{\sigma,\nu,c}(u_1).
\] (2.24)

Proof: Let \( \tau \) be the symmetrization of the pushforward of \( \nu \) by the map \( t \mapsto \sqrt{t} \).
Let \( u \) be in \( E_{\sigma,\nu,c} \) such that \( u > 0 \). Define for any \( x \in \mathbb{R} \setminus \text{supp}(\tau) \),
\[
H_{\sigma,\nu,c}(x) = x + \sigma^2 cg_\tau(x)
\]
\[
= x + \sigma^2 cg_\nu(x^2).
\]
Note that \( \sqrt{u} \) is in \( \mathbb{R} \setminus \text{supp}(\tau) \) and
\[
\Phi_{\sigma,\nu,c}(u) = (H_{\sigma,\nu,c}(\sqrt{u}))^2
\] (2.25)
\[
\Phi'_{\sigma,\nu,c}(u) = \sigma^2(1 - c)\sigma^2 cg_\nu(u) + (1 + \sigma^2 cg_\nu(u))H'_{\sigma,\nu,c}(\sqrt{u}).
\]
It readily follows that
\[
1 - \sigma^2 c \int \frac{d\tau(x)}{(\sqrt{u} - x)^2} = H'_{\sigma,\nu,c}(\sqrt{u}) > 0.
\] (2.26)

Firstly, let \( u_2 > u_1 > 0 \) be in \( E_{\sigma,\nu,c} \). Then for \( i = 1, 2 \), \( \sqrt{u_i} \in \mathbb{R} \setminus \text{supp}(\tau) \) and by (2.25) and (2.26), \( \Phi_{\sigma,\nu,c}(u_i) = (H_{\sigma,\nu,c}(\sqrt{u_i}))^2 \), and
\[
\frac{H_{\sigma,\nu,c}(\sqrt{u_2}) - H_{\sigma,\nu,c}(\sqrt{u_1})}{\sqrt{u_2} - \sqrt{u_1}} = 1 - \sigma^2 c \int_\mathbb{R} \frac{d\tau(x)}{(\sqrt{u_1} - x)(\sqrt{u_2} - x)}
\]
\[
\geq 1 - \left( \sigma^2 c \int_\mathbb{R} \frac{d\tau(x)}{(\sqrt{u_1} - x)^2} \right)^{\frac{1}{2}} \left( \sigma^2 c \int_\mathbb{R} \frac{d\tau(x)}{(\sqrt{u_2} - x)^2} \right)^{\frac{1}{2}}
\]
\[
> 0.
\]
Since moreover for \( i = 1, 2 \), \( H_{\sigma,\nu,c}(\sqrt{u_i}) = \sqrt{u_i}(1 + \sigma^2 cg_\nu(u_i)) \) is positive, it follows that
\[
\Phi_{\sigma,\nu,c}(u_2) > \Phi_{\sigma,\nu,c}(u_1).
\]

Secondly, let \( u_1 < u_2 < 0 \) be in \( E_{\sigma,\nu,c} \). Since \( 1 + \sigma^2 cg_\nu \) is strictly decreasing on \( \mathbb{R} \) \(-\infty; \min\{x, x \in \text{supp}(\nu)\}\), we have \( \forall u \in \mathbb{R} \setminus \{0\} \), \( 1 + \sigma^2 cg_\nu(u) > 0 \) and \( \Phi_{\sigma,\nu,c}(u) \equiv (1 + \sigma^2 cg_\nu(u))(2uc\sigma^2 g_\nu(u) + (1 + \sigma^2 cg_\nu(u))) \) is obviously positive, it follows that
\[
\Phi_{\sigma,\nu,c}(u_2) > \Phi_{\sigma,\nu,c}(u_1).
\]

Finally, let \( u_1 < 0 < u_2 \) be in \( E_{\sigma,\nu,c} \). We have \( \Phi_{\sigma,\nu,c}(u_2) > 0 \) and \( \Phi_{\sigma,\nu,c}(u_1) < 0 \) so that
\[
\Phi_{\sigma,\nu,c}(u_2) > \Phi_{\sigma,\nu,c}(u_1).
\]

Therefore,
\[
\forall(u_1, u_2) \in (E_{\sigma,\nu,c} \setminus \{0\})^2, \text{ if } u_1 < u_2, \text{ then } \Phi_{\sigma,\nu,c}(u_2) > \Phi_{\sigma,\nu,c}(u_1). \] (2.27)
Now, (2.23) implies that for any \( u \in E_{\sigma,\nu,c} \subset E_{\sigma,\nu,1} \), \( K' \left( \Phi_{\sigma,\nu,1}(u) \right) > 0 \). It readily follows from (2.22), Lemma 2.3 and (2.27) that
\[
\forall (u_1, u_2) \in (E_{\sigma,\nu,c} \setminus \{0\})^2, u_1 < u_2, \Phi_{\sigma,\nu,c}(u_2) > \Phi_{\sigma,\nu,c}(u_1).
\] (2.28)

Assume now that 0 belongs to \( E_{\sigma,\nu,c} \); then there exists \( u_1 < 0 < u_2 \) such that \([u_1; u_2] \subset E_{\sigma,\nu,c} \). Since \( \Phi'_{\sigma,\nu,c} > 0 \) on \([u_1; u_2]\), we have for any \( u \) and \( v \) in \([u_1; u_2]\) such that \( u < 0 < v \), \( \Phi_{\sigma,\nu,c}(u) < \Phi_{\sigma,\nu,c}(0) < \Phi_{\sigma,\nu,c}(v) \). Then, it is easy to generalize (2.28) to (2.24).

Propositions 2.3 and 2.4, (2.2) and Proposition 2.6 readily yield Corollary 2.1.

For any \( y > x \) in \( \mathbb{R} \setminus \text{supp}(\mu_{\sigma,\nu,c}) \), \( \omega_{\sigma,\nu,c}(y) > \omega_{\sigma,\nu,c}(x) \).

When the support of \( \nu \) has a finite number of connected components, we have the following description of the support of \( \mu_{\sigma,\nu,c} \) in terms of a finite union of closed disjoint intervals.

**Proposition 2.7.** For any \( 0 < c \leq 1 \), recall that
\[
E_{\sigma,\nu,c} = \left\{ u \in \mathbb{R} \setminus \text{supp}(\nu), \Phi'_{\sigma,\nu,c}(u) > 0, g_{\nu}(u) > -\frac{1}{\sigma^2 c} \right\}.
\]

If the support of \( \nu \) has a finite number of connected components, there exists a nonnull integer number \( p \) and \( u_1 < v_1 < u_2 < \ldots < u_p < v_p \) depending on \( \sigma, \nu, c \), such that
\[
E_{\sigma,\nu,c} = (-\infty; u_1] \cup \bigcup_{l=1}^{p-1} [u_l; v_l] \cup [v_p; +\infty[.
\] (2.29)

We have
\[
\text{supp}(\nu) \subset \bigcup_{l=1}^{p} [u_l; v_l]
\] (2.30)

and for each \( l \in \{1, \ldots, p\} \), \([u_l; v_l] \cap \text{supp}(\nu) \neq \emptyset \). Moreover,
\[
\text{supp}(\mu_{\sigma,\nu,c}) = \bigcup_{l=1}^{p} [\Phi_{\sigma,\nu,c}(u_l^-); \Phi_{\sigma,\nu,c}(v_l^+)],
\] (2.31)

with for all \( l = 1, \ldots, p-1 \), \( \Phi_{\sigma,\nu,c}(v_l^+) < \Phi_{\sigma,\nu,c}(u_{l+1}^-) \), and for all \( l = 1, \ldots, p \),
\[
\Phi_{\sigma,\nu,c}(u_l^-) < \Phi_{\sigma,\nu,c}(v_l^+).
\] (2.32)

**Proof of Proposition 2.7**

- Assume \( c = 1 \). Denote by \( \mu_{sc}(\sigma) \) the centered semi-circular distribution with variance \( \sigma^2 \) and by \( \tau \) the symmetrization of the pushforward of \( \nu \) by the map \( t \mapsto \sqrt{t} \). We have seen in the proof of Proposition 2.4 that \( \mu_{\sigma,\nu,1} \) is the distribution of \( X^2 \) where \( X \) is a symmetrical absolutely continuous random variable with distribution \( \mu_{sc}(\sigma) \circ \tau \). If the support of \( \nu \) has a finite number of connected components, since \( \text{supp}(\tau) \subset U_{\sigma,\tau} \) (see (2.17)) and each connected component of \( U_{\sigma,\tau} \) contains at least an element of
supp(τ) (see Remark 2.2 in [9]), we can deduce that $\overline{U_{\sigma, \tau}}$ has a finite number of connected components:

$$\overline{U_{\sigma, \tau}} = \bigcup_{l=m}^{1} [s_l, t_l].$$

Then, by (2.18) and since $H_{\sigma, \tau}$ is globally strictly increasing on $\mathbb{R} \setminus \overline{U_{\sigma, \tau}}$ (see Remark 2.1 in [9]), we have

$$\text{supp}(\mu_{sc}(\sigma \boxplus \tau)) = \bigcup_{l=m}^{1} [H_{\sigma, \tau}(s_l^-), H_{\sigma, \tau}(t_l^+)].$$

Then, it readily follows from (2.33) that there exists an integer number $p$ and $a_1 \leq b_1 < a_2 \leq b_2 < \ldots < a_p \leq b_p$ such that

$$\text{supp}(\mu_{\sigma, \nu, 1}) = \bigcup_{i=1}^{p} [a_i; b_i].$$

- Assume $c < 1$. We need the following preliminary lemma.

**Lemma 2.4.** Assume that the support of $\nu$ has a finite number of connected components and let $I$ be a connected component of $\mathbb{R} \setminus \text{supp}(\mu_{\sigma, \nu, 1})$. Then $K(x \in I, K'(x) > 0)$ is connected.

**Proof:** Assume first that $I$ is bounded: $I = [a; b]$ for some $a$ and $b$ in $\mathbb{R}$.

**Lemma 2.5.** If $r$ and $t$ in $I$ satisfy $r < t, K'(r) > 0, K'(t) < 0$, then for all $z > t$, $z \in I$, we have $K'(z) \leq 0$.

**Proof:** Set $t_2 = \min\{z > r, z \in I, K'(z) < 0\}$. We have $K'(t_2) = 0$. Since $K'$ is holomorphic on $\{z \in \mathbb{C}, \Re(z) \in I\}$, its zeroes are isolated. Therefore, there exists $\epsilon > 0$ small enough such that $K' < 0$ on $[t_2, t_2 + \epsilon]$. Thus, $K(t_2 + \epsilon) < K(t_2)$. Moreover, we have $t_2 < r$ and $K' \geq 0$ on $[r; t_2]$. Using once more the isolated zeroes principle, we can find $\epsilon'$ small enough such that $K(t_2 + \epsilon) < K(t_2 - \epsilon') < K(t_2)$ and $K(t_2 - \epsilon') > 0$. Assume that there exists $z > t_2 + \epsilon$ in $I$ such that $K'(z) > 0$. Then, set $t_3 = \min\{z > t_2 + \epsilon, z \in I, K'(z) > 0\}$. We have $K(t_3) = 0$ and $K' \leq 0$ on $[t_2 + \epsilon, t_3]$. Thus $K(t_3) \leq K(t_2 + \epsilon) < K(t_2 - \epsilon')$. Using similar arguments as above we can find $\epsilon''$ small enough such that $K'(t_3 + \epsilon'') > 0$ and $K(t_3) < K(t_3 + \epsilon'') < K(t_2 - \epsilon')$ which leads to a contradiction with Lemma 2.3. Therefore there does not exist any $z > t_2 + \epsilon$ in $I$ such that $K'(z) > 0$. Moreover $K' < 0$ on $[t_2, t_2 + \epsilon]$. Hence we have $K'(z) \leq 0$ for any $z > t_2$ and Lemma 2.5 follows. □

Set $\mathcal{O} = \{z \in I, K'(z) > 0\}$. Assume that $\mathcal{O}$ is nonempty. Define

$$z_1 = \min\{z \in I, K'(z) > 0\},$$

15
Since 1 + \sigmaCG, the support of \(\sigmaCG\) cannot be a mass of \(l\), which is 0.

Assume that \(z_1 = z_2\) which implies \(z_2 \neq b\) and then \(\{z \in I, z > z_1, K'(z) < 0\} \neq \emptyset\). For any \(\epsilon > 0\) small enough, there exists \(t_3\) in \([z_1; z_1 + \epsilon]\) such that \(K'(t_3) > 0\). Now, there exists \(t_2\) in \([z_1, t_3]\), such that \(K'(t_2) < 0\) and there exists \(t_1\) in \([z_1, t_2]\) such that \(K'(t_1) > 0\), which leads to a contradiction with Lemma 2.6. Hence we must have \(z_1 < z_2\) and Lemma 2.6 readily yields that \(K(z) \leq 0\) for any \(z\) in \(I\) such that \(z > z_2\) if \(z_2 < b\).

Therefore \(O = \{z_1, z_2\} \setminus \{z \in z_1, z_2, \{z \in I, K'(z) > 0\}\}

Since \(\lim_{x \to \pm \infty} K(x) = 1\), there exists \(R > 0\) such that \(R > \max\{u \in \text{supp}(\muCG)\}\) and \(K > 0\) on \([-\infty; -R]\) and \([R; +\infty[.\) Since up to now, we assume that \(I\) is a bounded connected component of \(\mathbb{R} \setminus \text{supp}(\muCG)\), according to Lemma 2.3 we have for any \(z\) in \(O\), \(K(-R) < K(z) < K(R)\).

Then, if \(z_1 = a\) or \(z_2 = b\), we can define \(K(z_1) = \inf_{z \in O} K(z) = \inf_{z \in (z_1, z_2]} K(z)\) and \(K(z_2) = \sup_{z \in O} K(z) = \sup_{z \in z_1, z_2]} K(z)\). It comes easily that \(\mathcal{K}(O) = [K(z_1); K(z_2)]\).

Now, if \(I\) is a connected component of \(\mathbb{R} \setminus \text{supp}(\muCG)\) of the form \([-\infty; a[\) or \([a; +\infty[\), the previous study can be carried out for \([-R; a[\) or \([a; R[\) and it readily follows that \(\mathcal{K}\{x \in I, K'(x) > 0\}\) is connected.

According to Theorem 2.1 in \([12]\) and Theorem 1.3A), \(\muCG\) is absolutely continuous and its density is continuous. Therefore, the support of \(\muCG\) is the closure of its interior. It readily follows from \((2.34)\), Proposition 2.6 and Lemma 2.4 and the fact that \(\lim_{x \to -\infty} K(x) = -\infty\), \(\lim_{x \to +\infty} K(x) = +\infty\), that there exists an integer number \(p\) and \(a_1 \leq b_1 < a_2 \leq b_2 < \ldots a_{p} \leq b_{p}\) (depending on course on \(\sigmaCG\)) such that \(\text{supp}(\muCG) = \bigcup_{i=1}^{p}[a_i; b_i]\).

Now, using \((2.2)\), Propositions 2.3 and 2.4 and Corollary 2.1, we have

\[\mathcal{E}_{CG} = [-\infty; \omegaCG(a_1)][\omegaCG(b_1^+)\omegaCG(a_2^+)[\ldots \omegaCG(b_p)] + \infty[.\]

Therefore \((2.29)\) is satisfied with \(u_1 = \omegaCG(a_1\] and \(v_1 = \omegaCG(b_1^+)\). At this stage of the proof, we have \(u_1 \leq u_2 \leq v_2 \leq \ldots u_p \leq v_p\). Now, Propositions 2.3 and 2.4 and \((2.29)\) and Proposition 2.6 yield

\[\text{supp}(\muCG) = \bigcup_{i=1}^{p}[\PhiCG(u_i^+); \PhiCG(v_i^+)],\]

that is \((2.31)\), with moreover for all \(l = 1, \ldots, p-1, \PhiCG(v_i^+) \leq \PhiCG(u_i^-)\).

Since \(\muCG\) has no mass, we must have also for all \(l = 1, \ldots, p\),

\[\PhiCG(u_i^-) < \PhiCG(v_i^+).\]  \((2.35)\)

Assume that there exists \(l \in \{1, \ldots, p\}\) such that \(u_l = v_l\). We set \(v_0 := -\infty\) and \(u_{p+1} = +\infty\). We have \(v_{l-1}; v_l; v_{l+1} \subset \mathbb{R} \setminus \text{supp}(\nu)\). Since \(1 + \sigmaCG^2 \sigmaCG > 0\) on \([v_{l-1}; v_l[\), \(v_l\) cannot be a mass of \(\nu\); therefore, we must have \(v_l \in \mathbb{R} \setminus \text{supp}(\nu)\)

16
and then $\Phi_{\sigma,\nu,c}(v_i^-) = \Phi_{\sigma,\nu,c}(v_i^+)$ which leads to a contradiction with (2.35).

We can conclude that
\[ \forall l \in \{1, \ldots, p\}, \; u_l \neq v_i. \]

(2.30) is obvious since $E_{\sigma,\nu,c} \subset \mathbb{R} \setminus \text{supp}(\nu)$.

Actually, we have the following

**Lemma 2.6.** For each $l \in \{1, \ldots, p\}$, $[u_i; v_i] \cap \text{supp}(\nu) \neq \emptyset$.

Assume that there exists $l \in \{1, \ldots, p\}$ such that $[u_l; v_l] \subset \text{supp}(\nu)$. Let $\epsilon > 0$ be such that $\epsilon < \min\{(u_{k+1} - v_k); k = 1, \ldots, p - 1\}$. Since $1 + \sigma^2 c g_{\nu}$ is strictly decreasing on $[u_l - \epsilon; v_l + \epsilon]$ and $[v_l; v_l + \epsilon]$ is a subset of $E_{\sigma,\nu,c}$, we must have $\Phi_{\sigma,\nu,c}(u_l) = 0$ on $[u_l; v_l]$ and then, since $\Phi_{\sigma,\nu,c}$ is an holomorphic function on $\{z \in \mathbb{C}, \Re z \in [u_l; v_l]\}$, we have $\Phi_{\sigma,\nu,c}(v_l) < \Phi_{\sigma,\nu,c}(u_l)$ which leads to a contradiction with (2.35). \(\square\)

The proof of Proposition 2.7 is complete. \(\square\)

Proposition 2.7 corresponds to A) of Theorem 1.3. B) 2) of Theorem 1.3 can be readily obtained by differentiating the relation $\Phi_{\sigma,\nu,c}(\Phi_{\sigma,\nu,c}(u)) = u$, $\forall u \in E_{\sigma,\nu,c}$. Proposition 2.6 and Corollaire 2.1 yield C) of Theorem 1.3. Proposition 2.7 corresponds to D) of Theorem 1.3 \(\square\)

## 3 Exact separation phenomenon

### 3.1 Preliminary results

Our proof of Theorem 1.2 will need to establish the following preliminary proposition and lemmas.

**Proposition 3.1.** For all $0 < \sigma < \sigma^*$, $E_{\sigma,\nu,c} \subset E_{\sigma,\nu,c}$ so that it makes sense to consider the following composition of homeomorphisms

\[ \Phi_{\sigma,\nu,c} \circ \omega_{\sigma,\nu,c} : \mathbb{R} \setminus \text{supp}(\mu_{\sigma,\nu,c}) \to \Phi_{\sigma,\nu,c}(E_{\sigma,\nu,c}) \subset \mathbb{R} \setminus \text{supp}(\mu_{\sigma,\nu,c}), \]

which is strictly increasing on each connected component of $\mathbb{R} \setminus \text{supp}(\mu_{\sigma,\nu,c})$.

**Proof:** Let $0 < \sigma < \sigma^*$. Let $u$ be in $E_{\sigma,\nu,c}$. $g_{\nu}(u) > -\frac{1}{\sigma c}$ obviously implies $g_{\nu}(u) > -\frac{1}{\sigma c}$. We have

\[ \Phi_{\sigma,\nu,c}(u) = \left(1 + c \sigma^2 g_{\nu}(u)\right) \left(1 + c \sigma^2 g_{\nu}(u) + \left[2 u c \sigma^2 + \frac{\sigma^2 (1 - c) c \sigma^2}{1 + c \sigma^2 g_{\nu}(u)}\right] g_{\nu}(u)\right). \]

- If $2 u c \sigma^2 + \frac{\sigma^2 (1 - c) c \sigma^2}{1 + c \sigma^2 g_{\nu}(u)} \leq 0$, it is obvious that $\Phi_{\sigma,\nu,c}(u) > 0$.

- If $2 u c + \frac{(1 - c) c}{1 + c g_{\nu}(u)} > 0$, then $2 u c + \frac{(1 - c) c}{\sigma + c g_{\nu}(u)} > 0$. Moreover $\Phi_{\sigma,\nu,c}(u) > 0$ means that

\[ (1 + c \sigma^2 g_{\nu}(u))^2 + 2 u (1 + c \sigma^2 g_{\nu}(u)) c \sigma^2 g_{\nu}(u) + \sigma^2 (1 - c) c \sigma^2 g_{\nu}(u) > 0. \]

17
Since \((1 + c\sigma^2g_\nu(u)) > 0\) and \(2uc + \frac{(1 - c)c}{1 + c\sigma^2g_\nu(u)} > 0\), this implies
\[
g'_\nu(u) > - \frac{(1 + c\sigma^2g_\nu(u))}{2uc\sigma^2 + \frac{\sigma^2(1 - c)c\sigma^2}{1 + c\sigma^2g_\nu(u)}}.
\]

Equation (3.1) implies that
\[
(1 + c\sigma^2g_\nu(u)) + \left[2uc\sigma^2 + \frac{\sigma^2(1 - c)c\sigma^2}{1 + c\sigma^2g_\nu(u)}\right]g'_\nu(u)
> (1 + c\sigma^2g_\nu(u)) - \frac{\sigma^2}{\sigma^2} \left(1 + \frac{(1 - c)c\sigma^2}{1 + c\sigma^2g_\nu(u)}\right) \left(2uc + \frac{(1 - c)c\sigma^2}{1 + c\sigma^2g_\nu(u)}\right)
> (1 + c\sigma^2g_\nu(u)) - \frac{\sigma^2}{\sigma^2} \left(1 + \frac{(1 - c)c\sigma^2}{1 + c\sigma^2g_\nu(u)}\right) \left(2uc + \frac{(1 - c)c\sigma^2}{1 + c\sigma^2g_\nu(u)}\right)
= 1 - \frac{\sigma^2}{\sigma^2} + \Delta(u)
\]
where
\[
\Delta(u) = -(1 - c)c\frac{\sigma^2}{\sigma^2} \left(1 + \frac{(1 - c)c\sigma^2}{1 + c\sigma^2g_\nu(u)}\right) \left[\frac{\sigma^2}{1 + c\sigma^2g_\nu(u)} - \frac{\sigma^2}{1 + c\sigma^2g_\nu(u)}\right]
= (1 - c)c\frac{\sigma^2}{\sigma^2} \left(\sigma^2 - \hat{\sigma}^2\right) \left(1 + \frac{(1 - c)c\sigma^2}{1 + c\sigma^2g_\nu(u)}\right) \left(\sigma^2\right) \left(1 + c\sigma^2g_\nu(u)\right)
> 0
\]
It follows that \(\Phi'_{\sigma,\nu;c}(u) > 0\).

\(\Phi_{\sigma,\nu;c}\omega_{\sigma,\nu;c}\) is strictly increasing on each connected component of \(\mathbb{R}\setminus\text{supp(}\mu_{\sigma,\nu;c}\text{)}\) by B) 2. of Theorem 13 and the very definition of \(\hat{\varepsilon}_{\sigma,\nu;c}\) and the proof is complete. \(\square\)

**Lemma 3.1.** Let \(x\) be such that there exists \(\delta > 0\) such that for all large \(N\), \([x - \delta; x + \delta] \subset \sigma\mu_{\sigma,\nu;A_N}^{c,n}\). Then for all \(0 < \tau < \delta\),
\[
\omega_{\sigma,\nu;c}(\{x - \tau; x + \tau\}) = [\omega_{\sigma,\nu;c}(x - \tau); \omega_{\sigma,\nu;c}(x + \tau)] \subset \sigma\text{supp(}\mu_{A_N}^{c,n}\text{)}.
\]
Moreover, \(g_{\mu_{A_N}^{c,n}} \left(\omega_{\sigma,\nu;A_N}^{c,n}(x)\right)\) converges towards \(g_\nu(\omega_{\sigma,\nu;A_N}^{c,n}(x))\).

**Proof:** We will use several times the obvious fact that if a sequence \(\mu_n\) of probability measures weakly converges towards a probability measure \(\mu\), if for some \(\delta > 0\), \([u - \delta; u + \delta] \subset \text{supp(}\mu_n\) for all large \(n\), then \(g_{\mu_n}(u)\) converges towards \(g_{\mu}(u)\).

Let \(0 < \tau < \tau' < \delta\). Since \(\mu_{\sigma,\nu;A_N}^{c,n}\) weakly converges towards \(\mu_{\sigma,\nu;c}\), for any \(u \in [x - \tau; x + \tau']\), \(\omega_{\sigma,\nu;A_N}^{c,n}(u)\) converges towards \(\omega_{\sigma,\nu;c}(u)\) and \([x - \tau; x +\)
Theorem 1.1. Note that if $c^+$, $\omega'_{\sigma,\mu,c} > 0$ and $\omega'_{\sigma,\mu,c} > 0$ on $[x - \tau'; x + \tau']$. It readily follows that, for all large $N$,

$$
[\omega_{\sigma,\nu,c}(x - \tau); \omega_{\sigma,\nu,c}(x + \tau)] \subset \omega_{\sigma,\mu,c}^N(cN \cdot \omega_{\sigma,\nu,c}(x - \tau'); \omega_{\sigma,\mu,c}^N(cN \cdot \omega_{\sigma,\nu,c}(x + \tau')) \subset c^+ \sup(\mu_{AN^N})
$$

where we used B) 1. of Theorem 1.3 for the last inclusion. Therefore there exists $\alpha > 0$ such that for all large $N$,

$$
[\omega_{\sigma,\nu,c}(x) - \alpha; \omega_{\sigma,\nu,c}(x) + \alpha] \subset c^+ \sup(\mu_{AN^N}). \tag{3.2}
$$

It follows that $g_{\mu_{AN^N}}(\omega_{\sigma,\nu,c}(x))$ converges towards $g_{\nu}(\omega_{\sigma,\nu,c}(x))$.

Now, let $\epsilon$ be such that $0 < \epsilon < \alpha/2$ and $N_0(\epsilon)$ be such that for all $N \geq N_0(\epsilon)$, $|\omega_{\sigma,\mu,c}^N(cN \cdot \omega_{\sigma,\nu,c}(x) - \omega_{\sigma,\nu,c}(x))| < \epsilon$. We have

$$
|g_{\mu_{AN^N}}(\omega_{\sigma,\mu,c}^N(cN \cdot \omega_{\sigma,\nu,c}(x)) - g_{\mu_{AN^N}}(\omega_{\sigma,\nu,c}(x))|
\leq |\omega_{\sigma,\mu,c}^N(cN \cdot \omega_{\sigma,\nu,c}(x) - \omega_{\sigma,\nu,c}(x))| \int \frac{d\mu_{AN^N}(t)}{|\omega_{\sigma,\mu,c}^N(cN \cdot \omega_{\sigma,\nu,c}(x) - t)|}
\leq \frac{2\epsilon}{\alpha^2}
$$

where we used (3.2) in the last inequality. It follows that $g_{\mu_{AN^N}}(\omega_{\sigma,\mu,c}^N(cN \cdot \omega_{\sigma,\nu,c}(x)) - g_{\mu_{AN^N}}(\omega_{\sigma,\nu,c}(x))$ converges towards 0 when $N$ goes to infinity.

Writing

$$
g_{\mu_{AN^N}}(\omega_{\sigma,\mu,c}^N(cN \cdot \omega_{\sigma,\nu,c}(x)) - g_{\nu}(\omega_{\sigma,\nu,c}(x))
= g_{\mu_{AN^N}}(\omega_{\sigma,\mu,c}^N(cN \cdot \omega_{\sigma,\nu,c}(x)) - g_{\mu_{AN^N}}(\omega_{\sigma,\nu,c}(x))
+ g_{\mu_{AN^N}}(\omega_{\sigma,\nu,c}(x) - g_{\nu}(\omega_{\sigma,\nu,c}(x)),
$$

we can finally deduce that $g_{\mu_{AN^N}}(\omega_{\sigma,\mu,c}^N(cN \cdot \omega_{\sigma,\nu,c}(x))$ converges towards $g_{\nu}(\omega_{\sigma,\nu,c}(x))$. □

**Lemma 3.2.** Let $[a, b]$ ($a \neq b$) be a compact set satisfying the couple $P(\sigma)$ of properties defined in Theorem 1.1 7. and such that, if $c < 1$, $\omega_{\sigma,\nu,c}(a) > 0$. For all $0 < \hat{\sigma} < \sigma$, define $a(\hat{\sigma}) = \Phi_{\sigma,\nu,c}(\omega_{\sigma,\nu,c}(a))$ and $b(\hat{\sigma}) = \Phi_{\sigma,\nu,c}(\omega_{\sigma,\nu,c}(b))$. Then $[a(\hat{\sigma}), b(\hat{\sigma})]$ satisfies the couple $P(\hat{\sigma})$ of properties. Moreover, there exists $m_{a,b} > 0$ such that for all $0 < \hat{\sigma} < \sigma$,

$$
b(\hat{\sigma}) - a(\hat{\sigma}) \geq m_{a,b}. \tag{3.3}
$$

**Proof of Lemma 3.2.** Denote by $\delta$ and $\tau$ the parameters introduced in 7. of Theorem 1.1. Note that if $c = 1$, since $a > 0$ and using (2.3), we have

$$
\omega_{\sigma,\nu,c}(a) > 0 \tag{3.4}
$$

19
Moreover, according to B) 1. of Theorem 1.3 and Proposition 3.1 for any \(0 < c \leq 1\) and any \(0 < \sigma < \sigma\), we have \(1 + c\sigma_2^2g_\sigma(\omega_{\sigma,\nu,c}(a)) > 0\). Therefore it readily follows that if \(c = 1\), then \(a(\sigma) > 0\) and if \(c < 1\), if \(\omega_{\sigma,\nu,c}(a) > 0\), then \(a(\sigma) > 0\).

Moreover, since according to Proposition 3.1 \(x \mapsto \Phi_{\sigma,\nu,c} \circ \omega_{\sigma,\nu,c}(x)\) is strictly increasing on \([a; b]\), we have \(0 < a(\sigma) < b(\sigma)\).

According to Lemma 3.1 for all \(0 < \tau < \delta\) and for all \(x \in [a - \tau; b + \tau]\), \(g_{\sigma,\nu,c}(x)\) converges towards \(g_\nu(\omega_{\sigma,\nu,c}(x))\). It readily follows that for any \(\sigma > 0\), \(\Phi_{\sigma,\nu,c}(\omega_{\sigma,\nu,c}(x))\) converges towards \(\Phi_{\sigma,\nu,c}(\omega_{\sigma,\nu,c}(x))\). According to Proposition 3.1 \(x \mapsto \Phi_{\sigma,\nu,c}(\omega_{\sigma,\nu,c}(x))\) and \(x \mapsto \Phi_{\sigma,\nu,c}(\omega_{\sigma,\nu,c}(x))\) are strictly increasing on \([a - \tau; b + \tau]\).

It readily follows that, for all \(0 < \tau'' < \tau < \delta\), for all large \(N\),

\[
[\Phi_{\sigma,\nu,c} \circ \omega_{\sigma,\nu,c}(a - \tau''); \Phi_{\sigma,\nu,c} \circ \omega_{\sigma,\nu,c}(b + \tau'')] 
\subset \supp(\mu_{\sigma,\nu,c}(\omega_{\sigma,\nu,c}(x))).
\]

Therefore, there exists \(\alpha > 0\) such that \(\Phi_{\sigma,\nu,c}(\omega_{\sigma,\nu,c}(a)) - \alpha > 0\) and

\[
[\Phi_{\sigma,\nu,c}(\omega_{\sigma,\nu,c}(a)) - \alpha; \Phi_{\sigma,\nu,c}(\omega_{\sigma,\nu,c}(b)) + \alpha] \subset \supp(\mu_{\sigma,\nu,c}(\omega_{\sigma,\nu,c}(x)))
\]

which is the first property of \(\mathcal{P}(\sigma)\).

The second property of \(\mathcal{P}(\sigma)\) will be obviously satisfied if we prove that there exists \(\epsilon > 0\) such that for all large \(N\),

\[
\Phi_{\sigma,\nu,c}(\omega_{\sigma,\nu,c}(a)) - \epsilon; \Phi_{\sigma,\nu,c}(\omega_{\sigma,\nu,c}(b)) + \epsilon] \subset \supp(\mu_{\sigma,\nu,c}(\omega_{\sigma,\nu,c}(x))).
\]

We have proved that there exists \(\alpha > 0\) such that

\[
[\Phi_{\sigma,\nu,c}(\omega_{\sigma,\nu,c}(a)) - \alpha; \Phi_{\sigma,\nu,c}(\omega_{\sigma,\nu,c}(b)) + \alpha] 
\subset \supp(\mu_{\sigma,\nu,c}(\omega_{\sigma,\nu,c}(x))).
\]

(3.5) follows from the fact that, according to B) of Theorem 1.3 \(\omega_{\sigma,\nu,c}(x)\) is strictly increasing on the intervals involved in the inclusion and moreover \(\omega_{\sigma,\nu,c}(x) \in \mathcal{E}_{\sigma,\nu,c}(\omega_{\sigma,\nu,c}(x))\) for each \(x \in \mathcal{E}_{\sigma,\nu,c}(\omega_{\sigma,\nu,c}(x))\).

Let us prove (3.3). Let \(x\) be in \(\mathbb{R}\setminus \supp(\mu_{\sigma,\nu,c})\). Note that \(1 + c\sigma^2g_\nu(\omega_{\sigma,\nu,c}(x)) > 0\) since according to B) 1. of Theorem 1.3, \(\omega_{\sigma,\nu,c}(x)\) belongs to \(\mathcal{E}_{\sigma,\nu,c}\). We have

\[
\Phi_{\sigma,\nu,c}(\omega_{\sigma,\nu,c}(x)) = \omega_{\sigma,\nu,c}(x)(1 + c\sigma^2g_\nu(\omega_{\sigma,\nu,c}(x)))^{1/2} + \sigma^2(1 - c)(1 + c\sigma^2g_\nu(\omega_{\sigma,\nu,c}(x)))
\]

20
For any pair of integers $j, k$, we have

$$\hat{\Phi}_{\sigma, \nu, c}(\omega_{\sigma, \nu, c}(x)) = \sqrt{\lambda_{1}(XX^{*})} \leq \cdots \leq -\sqrt{\lambda_{n}(XX^{*})} \leq 0 = \cdots = 0 \leq \sqrt{\lambda_{n}(XX^{*})} \leq \cdots \leq \sqrt{\lambda_{1}(XX^{*})}.$$
3.2 Proof of Theorem 1.2

This proof is in the lineage of [1], [8], [9].

**Proof of Theorem 1.2** For any \( x \in \mathbb{R} \setminus \text{supp } \nu, \hat{\sigma} \mapsto \Phi_{\hat{\sigma},\nu,c}(x) \) is continuous and then bounded on \([0; \sigma]\). Therefore there exists \( M_{a,b} > 0 \) such that for all \( \hat{\sigma} \in [0; \sigma] \),

\[
0 < |\sqrt{\Phi_{\hat{\sigma},\nu,c}(\omega_{\sigma,\nu,c}(a))} + \sqrt{\Phi_{\hat{\sigma},\nu,c}(\omega_{\sigma,\nu,c}(b))}| \leq M_{a,b}.
\]

Then using (3.3) in Lemma 3.2, we can easily deduce that there exists \( \tilde{m}_{a,b} > 0 \) such that for all \( \hat{\sigma} \in [0; \sigma] \),

\[
\sqrt{\Phi_{\hat{\sigma},\nu,c}(\omega_{\sigma,\nu,c}(\omega_{\sigma,\nu,c}(a)))} - \sqrt{\Phi_{\hat{\sigma},\nu,c}(\omega_{\sigma,\nu,c}(\omega_{\sigma,\nu,c}(b)))} \geq \tilde{m}_{a,b}.
\]

Note that for \( \hat{\sigma} = 0 \), \( \Phi_{0,\nu,c}(\omega_{\sigma,\nu,c}(a)) = \omega_{\sigma,\nu,c}(a) > 0 \) (by assumption if \( c < 1 \) and according to (3.4) if \( c = 1 \)) and \( \Phi_{0,\nu,c}(\omega_{\sigma,\nu,c}(b)) = \omega_{\sigma,\nu,c}(b) > \omega_{\sigma,\nu,c}(a) \).

Therefore we can choose \( \tilde{m}_{a,b} > 0 \) such that (3.6) is true for any \( \hat{\sigma} \in [0; \sigma] \).

Since \( \hat{\sigma} \mapsto \Phi_{\hat{\sigma},\nu,c}(\omega_{\sigma,\nu,c}(\omega_{\sigma,\nu,c}(a))) \) and \( \hat{\sigma} \mapsto \Phi_{\hat{\sigma},\nu,c}(\omega_{\sigma,\nu,c}(\omega_{\sigma,\nu,c}(b))) \) are uniformly continuous on \([0; \sigma]\), there exists \( 0 < \delta_{a,b} < \frac{\tilde{m}_{a,b}}{4} \) (3.7) such that for any \( \hat{\sigma}, \hat{\sigma}' \in [0; \sigma] \) satisfying \( |\hat{\sigma} - \hat{\sigma}'| \leq \delta_{a,b} \), we have

\[
|\sqrt{\Phi_{\hat{\sigma},\nu,c}(\omega_{\sigma,\nu,c}(a))} - \sqrt{\Phi_{\hat{\sigma}',\nu,c}(\omega_{\sigma,\nu,c}(a))}| < \frac{\tilde{m}_{a,b}}{4},
\]

and

\[
|\sqrt{\Phi_{\hat{\sigma},\nu,c}(\omega_{\sigma,\nu,c}(b))} - \sqrt{\Phi_{\hat{\sigma}',\nu,c}(\omega_{\sigma,\nu,c}(b))}| < \frac{\tilde{m}_{a,b}}{4}.
\]

According to [10], we can choose \( C > 1 \) large enough such that almost surely, for all large \( N \),

\[
0 \leq \sqrt{\lambda_1(X_N X_N^*)} < C.
\]

Then, let us choose

\[
0 < C_{a,b} \leq \frac{\delta_{a,b}}{C\sigma}.
\]

Given \( k \geq 0 \) define

\[
\sigma_k = \sigma(1 - \frac{1}{1 + kC_{a,b}}),
\]

\[
s_k = \sqrt{\Phi_{\sigma_k,\nu,c}(\omega_{\sigma,\nu,c}(a))}
\]

and

\[
t_k = \sqrt{\Phi_{\sigma_k,\nu,c}(\omega_{\sigma,\nu,c}(b))}.
\]

Note that for all \( k \),

\[
|\sigma_{k+1} - \sigma_k| \leq C_{a,b} \sigma.
\]

According to (3.6), for any \( k \),

\[
t_k - s_k \geq \tilde{m}_{a,b}.
\]

22
and according to (3.8), (3.10) and (3.11),
\[|s_{k+1} - s_k| < \frac{\tilde{m}_{a,b}}{4} \quad \text{and} \quad |t_{k+1} - t_k| < \frac{\tilde{m}_{a,b}}{4}. \quad (3.13)\]

Now, let us introduce a continuum of matrices $M^{(k)}_N$ interpolating from $M_N$ to $A_N A_N^*$:
\[M^{(k)}_N := (\sigma_k X_N + A_N)(\sigma_k X_N + A_N)^*.\]

For all $k \geq 0$, set
\[E_k = \{ \text{no eigenvalues of } M^{(k)}_N \text{ in } [s_k^2, t_k^2], \text{ for all large } N \}.

By Lemma 3.2 and Theorem 1.1, we know that $P(E_k) = 1$ for all $k > 0$. Moreover, according to Lemma 3.1, for $N \geq N_0$, $[\omega_{\sigma,\nu,c}(a); \omega_{\sigma,\nu,c}(b)] \subset \text{supp}(\mu A_N A_N^*)$.

For $N \geq N_0$, let $i_N$ be such that $\lambda_{i_N+1}(A_N) < \omega_{\sigma,\nu,c}(a)$ and $\lambda_{i_N}(M^{(k)}_N) > \omega_{\sigma,\nu,c}(b)$. (3.14)

We shall show that by induction on $k$ that one has for all $k \geq 0$,
\[P[\lambda_{i_N+1}(M^{(k)}_N) < s_k^2 \quad \text{and} \quad \lambda_{i_N}(M^{(k)}_N) > t_k^2, \text{ for all large } N] = 1. \quad (3.15)\]

This is true for $k = 0$ since $\sigma_0 = 0$, $M^{(0)}_N = A_N A_N^*$ and $\Phi_{0,\nu,c}(x) = x$. Now, let us assume that (3.15) holds true. Defined
\[Y^{(k)}_N = \sigma_k \begin{pmatrix} 0 & X_N \\ X_N^* & 0 \end{pmatrix} + \begin{pmatrix} 0 & A_N \\ A_N^* & 0 \end{pmatrix}.\]

Using Theorem 3.1, (3.15) implies that
\[P[\lambda_{i_N+1}(Y^{(k)}_N) < s_k^2 \quad \text{and} \quad \lambda_{i_N}(Y^{(k)}_N) > t_k^2, \text{ for all large } N] = 1. \quad (3.16)\]

Since
\[Y^{(k+1)}_N = Y^{(k)}_N + \frac{\sigma C_{a,b}}{(1 + (k + 1)C_{a,b})(1 + kC_{a,b})} \begin{pmatrix} 0 & X_N \\ X_N^* & 0 \end{pmatrix},\]
we can deduce from Lemma 3.3 that
\[\lambda_{i_N+1}(Y^{(k+1)}_N) \leq \lambda_{i_N+1}(Y^{(k)}_N) + \sqrt{\lambda_1(X_N X_N^*) C_{a,b}^2}.\]

It follows using (3.9), (3.7) and (3.10) that
\[\lambda_{i_N+1}(Y^{(k+1)}_N) < s_k + \frac{\tilde{m}_{a,b}}{4} := \hat{s}_k \quad \text{a.s.}\]

Similarly, one can show that
\[\lambda_{i_N}(Y^{(k+1)}_N) > t_k - \frac{\tilde{m}_{a,b}}{4} := \hat{t}_k \quad \text{a.s.}\]
Inequalities (3.13) and (3.12) ensure that
\[ [\hat{s}_k, \hat{t}_k] \subset [s_{k+1}, t_{k+1}]. \]
As \( P(E_{k+1}) = 1 \), we deduce that, with probability 1,
\[ \lambda_{iN+1}(Y^{(k+1)}_N) < s_{k+1} \quad \text{and} \quad \lambda_{iN}(Y^{(k+1)}_N) > t_{k+1}, \]
for all large \( N \) and therefore
\[ \lambda_{iN+1}(M^{(k+1)}_N) < s_{k+1}^2 \quad \text{and} \quad \lambda_{iN}(M^{(k+1)}_N) > t_{k+1}^2, \]
for all large \( N \).
This completes the proof by induction of (3.15).

Now, we are going to show that there exists \( K \) large enough so that, for all \( k \geq K \), there is an exact separation of the eigenvalues of the matrices \( M_N \) and \( M^{(k)}_N \).

Let \( \alpha \) be such that \( 0 < \alpha < \sqrt{b} - \sqrt{a} \). There exists \( 0 < \delta < \frac{\alpha}{4} \) such that for any \( \hat{\sigma} \) in \( [0, \sigma] \) satisfying \( |\hat{\sigma} - \sigma| \leq \delta \), we have
\[ |\sqrt{\Phi_{\sigma,\nu,c}(\omega_{\sigma,\nu,c}(a))} - \sqrt{\Phi_{\sigma,\nu,c}(\omega_{\sigma,\nu,c}(a))}| \leq \frac{\alpha}{2}, \]
and
\[ |\sqrt{\Phi_{\sigma,\nu,c}(\omega_{\sigma,\nu,c}(b))} - \sqrt{\Phi_{\sigma,\nu,c}(\omega_{\sigma,\nu,c}(b))}| \leq \frac{\alpha}{2}. \]
Let \( K \) be a positive integer number such that
\[ \frac{C\sigma}{1 + KC_{a,b}} < \delta. \]
Using Lemma 3.3, (3.13) and (3.14), we get the following inequalities almost surely for all large \( N \).
If \( i_N < N \),
\[ \lambda_{iN+1}(Y_N) \leq \lambda_{iN+1}(Y^{(K)}_N) + (\sigma - \sigma_K)\sqrt{\lambda_1(X_N X_N^*)} \]
\[ < s_K + \frac{C\sigma}{1 + KC_{a,b}} \]
\[ = \sqrt{\Phi_{\sigma_K,\nu,c}(\omega_{\sigma_K,\nu,c}(a))} + \frac{C\sigma}{1 + KC_{a,b}} \]
\[ \leq \sqrt{a} + \alpha \]
If \( i_N > 0 \), for all large \( N \),
\[ \lambda_{iN}(Y_N) \geq \lambda_{iN}(Y^{(K)}_N) - (\sigma - \sigma_K)\sqrt{\lambda_1(X_N X_N^*)} \]
\[ > t_K - \frac{C\sigma}{1 + KC_{a,b}} \]
\[ = \sqrt{\Phi_{\sigma_K,\nu,c}(\omega_{\sigma_K,\nu,c}(b))} - \frac{C\sigma}{1 + KC_{a,b}} \]
\[ \geq \sqrt{b} - \alpha. \]
Since \( \sqrt{a} + \alpha; \sqrt{b} - \alpha \subset [\sqrt{a}; \sqrt{b}] \) and almost surely, for all \( N \) large enough \( [\sqrt{a}; \sqrt{b}] \) is a gap in the spectrum of \( Y_N \), we can deduce that, almost surely, for all \( N \) large enough
\[
\lambda_{i_N+1}(Y_N) < \sqrt{a} \quad \text{if } i_N < N, \quad (3.17)
\]
and
\[
\lambda_{i_N}(Y_N) > \sqrt{b} \quad \text{if } i_N > 0. \quad (3.18)
\]
It follows that, almost surely, for all \( N \) large enough
\[
\lambda_{i_N+1}(M_N) < a \quad \text{if } i_N < N, \quad (3.19)
\]
and
\[
\lambda_{i_N}(M_N) > b \quad \text{if } i_N > 0. \quad (3.20)
\]
Since \( \lambda_{N+1}(M_N) = -\lambda_0(M_N) = -\infty, \quad (3.19) \) (resp. \( (3.20) \)) is obviously satisfied if \( i_N = N \) (resp. \( i_N = 0 \)). This ends the proof of Theorem 1.2. \( \blacksquare \)

**Remark 3.1.** Assume that \( c < 1 \). It readily follows from Theorem 1.3 that \( a \in \mathbb{R} \setminus \text{supp}(\mu_{\sigma,\nu,c}) \) implies \( \omega_{\sigma,\nu,c}(a) > v_1 \geq 0 \). Nevertheless, the fact that \( a > 0 \) is not included in \( [0; \Phi_{\sigma,\nu,c}(u-1)] \) is not a necessary condition for \( \omega_{\sigma,\nu,c}(a) > 0 \) to hold as the following example shows. Let us choose \( \nu = \delta_2, \ c = \frac{1}{2}, \sigma^2 = 1 \). We have \( g_\nu(u) > -\frac{1}{\sigma^2} \iff u < \frac{1}{\sigma^2} \) and for any \( x \leq 0, \xi > 0 \). Let \( a_j(N), j = 1, \ldots, J \), and \( b_j(N) \geq 0, \ r + 1 \leq j \leq n \), be complex numbers such that
\[
\lim_{N \to +\infty} |a_j(N)|^2 = \theta_j,
\]

4 Application to spiked models

4.1 Matricial model and notations

We will consider the deformed model:
\[
M_N = (\sigma X_N + A_N)(\sigma X_N + A_N)^* \quad (4.1)
\]
- \( n \leq N, \ c_N = n/N \to c \in [0; 1] \).
- \( X_N \) satisfies conditions 1., 2. and 3. of Theorem 1.1.
- Let \( \nu \neq \delta_0 \) be a compactly supported probability measure whose support has a finite number of connected components. Let \( \theta_1 > \ldots > \theta_J > 0 \) be \( J \) fixed real numbers independent of \( N \) which are outside the support of \( \nu \). Let \( k_1, \ldots, k_J \) be fixed nonnull integer numbers independent of \( N \) and \( r = \sum_{j=1}^J k_j \). Let \( \beta_j(N) \geq 0, \ r + 1 \leq j \leq n \), be such that \( \frac{1}{n} \sum_{j=r+1}^n \delta_{\beta_j(N)} \) weakly converges to \( \nu \) and
\[
\max_{r+1 \leq j \leq n} \text{dist}(\beta_j(N), \text{supp}(\nu)) \to 0. \quad (4.2)
\]
Let \( a_j(N), j = 1, \ldots, J \), and \( b_j(N) \geq 0, \ r + 1 \leq j \leq n \), be complex numbers such that
\[ |b_j(N)|^2 = \beta_j(N). \]

Let us introduce the \( n \times N \) deterministic matrix \( A_N \) by setting for any \( j = 1, \ldots, J, \)

\[
(A_N)_{ii} = a_j(N) \quad \text{for } i = k_1 + \ldots + k_{j-1} + l, \ l = 1, \ldots, k_j,
\]

for any \( r + 1 \leq i \leq n, \)

\[
(A_N)_{ii} = b_i(N).
\]

and else \( (A_N)_{ij} = 0. \)

Thus, \( A_N A_N^* \) is a diagonal matrix, each \( |a_j(N)|^2 \) is an eigenvalue of \( A_N A_N^* \) with a fixed multiplicity \( k_j \) (with \( \sum_{j=1}^J k_j = r \)) and the other eigenvalues of \( A_N A_N^* \) are the \( \beta_j(N) \). Moreover the empirical spectral measure of \( A_N A_N^* \) weakly converges to \( \nu \). We will denote by \( \Theta \) the set

\[ \Theta := \{ \theta_1; \cdots; \theta_J \} \]

and by \( K_{\nu, \Theta} \) the set

\[ K_{\nu, \Theta} := \text{supp}(\nu) \cup \Theta. \quad (4.3) \]

### 4.2 Subordination property of rectangular free convolution of ratio \( c \)

For any \( c \in [0,1] \), the rectangular free convolution \( \boxplus_c \) is defined in [3] in the following way. Let \( M_{n,N}, N_{n,N} \) be \( n \) by \( N \) independent random matrices, one of them having a distribution which is invariant by multiplication by any unitary matrix on any side, the symmetrized empirical singular measures of which tend, as \( N \) tends to infinity in such a way that \( n/N \) tends to \( c \), to nonrandom probability measures \( \nu_1, \nu_2 \). Then the symmetrized empirical singular law of \( M_{n,N} + N_{n,N} \) tends to \( \nu_1 \boxplus_c \nu_2 \).

For any probability measure \( \tau \) on \([0;+\infty[\), let us denote by \( \sqrt{\tau} \) the symmetrization of the pushforward of \( \tau \) by the map \( t \mapsto \sqrt{t} \) and by \( \sigma^2 \tau \) the pushforward of \( \tau \) by the map \( t \mapsto \sigma^2 t \). Note that the limiting spectral measure \( \mu \) of \( M_N \) defined in \( (4.1) \) does not depend on the distribution of the entries of \( X_N \). Thus, choosing gaussian entries for \( X_N \), we can deduce that the limiting spectral measure \( \mu_{\sigma, \nu, c} \) of \( M_N \) satisfies

\[ \sqrt{\mu_{\sigma, \nu, c}}^* = \sqrt{\nu} \boxplus_c \sqrt{\sigma^2 \mu_c}^* , \]

where \( \mu_c \) is the well-known Marchenko-Pastur law defined by

\[ \mu_c(dx) = \max\{1 - \frac{1}{c}, 0\} \delta_0 + f(x)1_{[1-\sqrt{\sigma^2 (1+\sqrt{\sigma})}, (1+\sqrt{\sigma})]}(x)dx \quad (4.4) \]

2The empirical singular measure of a \( n \) by \( N \) \((n \leq N)\) matrix \( M \) is the uniform law on the eigenvalues of \( \sqrt{MM^*} \).
with
\[ f(x) = \frac{\sqrt{(x - (1 - \sqrt{c})^2)((1 + \sqrt{c})^2 - x)}}{2\pi cx}. \]

Let us now explain the intuition of our result. Using (2.7), one can easily see that for any \( x \in \mathbb{R} \setminus \{\text{supp}(\mu_{\sigma,\nu,c}) \cup \{0\}\}, \)
\[ c\omega_{\sigma,\nu,c}(x)g_\nu(\omega_{\sigma,\nu,c}(x))^2 + (1 - c)g_\nu(\omega_{\sigma,\nu,c}(x)) = cxg_{\mu_{\sigma,\nu,c}}(x)^2 + (1 - c)g_{\mu_{\sigma,\nu,c}}(x). \]

Note that this has to be related with the subordination result established in [4] involving the \( H \) transform related to rectangular free convolution with ratio \( c \).
Indeed, for any probability measure \( \tau \) on \([0; +\infty], \) the \( H \) transform of \( \sqrt{\tau} \) is such that
\[ H_{\sqrt{\tau}}(z) = czg_{\tau}(\frac{1}{z})^2 + (1 - c)g_{\tau}(\frac{1}{z}) \]
and the authors established in [4] that for any probability measures \( \mu \) and \( \tau \) on \([0; +\infty], \) there exists a meromorphic function \( F \) on \( \mathbb{C} \setminus [0; +\infty] \) such that
\[ H_{\sqrt{\nu} \boxplus c\sqrt{\mu}}(z) = H_{\sqrt{\nu}}(F(z)). \]
The intuition is that for large \( N, \)
\[ cxg_{\mu_{MN}}(x)^2 + (1 - c)g_{\mu_{MN}}(x) \]
and therefore that they will be eigenvalues of \( M_N \) that separate from the bulk whenever some of the equations
\[ \omega_{\sigma,\nu,c}(x) = \theta_j, \]
\( j = 1, \ldots, J, \) admits a solution outside the support of \( \mu_{\sigma,\nu,c}. \) According to Proposition [2.1], \( \omega_{\sigma,\nu,c}(x) = \theta_j \) admits a solution outside the support of \( \mu_{\sigma,\nu,c} \) if and only if \( \Phi'_{\sigma,\nu,c}(\theta_j) > 0, g_\nu(\theta_j) > -\frac{1}{\sigma^2c}. \) Moreover there is only one such a solution which is \( \Phi_{\sigma,\nu,c}(\theta_j). \)

This intuition lead us to introduce the following set
\[ \Theta_{\sigma,\nu,c} = \left\{ \theta \in \Theta, \Phi'_{\sigma,\nu,c}(\theta) > 0, g_\nu(\theta) > -\frac{1}{\sigma^2c} \right\}, \]
and to introduce for any \( \theta \) in \( \Theta_{\sigma,\nu,c}, \)
\[ \rho_{\theta} = \Phi_{\sigma,\nu,c}(\theta). \]

According to B) 1) of Theorem [1.3] \( \forall \theta \in \Theta_{\sigma,\nu,c}, \rho_{\theta} \notin \text{supp}(\mu_{\sigma,\nu,c}). \)

### 4.3 Inclusion of the spectrum

Let us define
\[ S = \text{supp}(\mu_{\sigma,\nu,c}) \cup \{\rho_{\theta}, \theta \in \Theta_{\sigma,\nu,c}\}. \] (4.5)
We have the following inclusion of the spectrum of \( M_N. \)
**Theorem 4.1.** A) For any $\epsilon > 0$,
\[
\mathbb{P}[\text{for all large } N, \text{spect}(M_N) \subset \{x \in \mathbb{R}, \text{dist}(x, \mathcal{S} \cup \{0\}) \leq \epsilon\}] = 1.
\]

B) Moreover, if $u_1 > 0$ ($u_1$ is defined in Theorem 1.3 D)), for any $\epsilon > 0$,
\[
\mathbb{P}[\text{for all large } N, \text{spect}(M_N) \subset \{x \in \mathbb{R}, \text{dist}(x, \mathcal{S}) \leq \epsilon\}] = 1.
\]

In order to prove Theorem 4.1, we first establish the following proposition.

**Proposition 4.1.** Let $[u, v] \subset \mathbb{R} \setminus \mathcal{S}$. For any $\delta > 0$ small enough, for all large $N$, $[u - \delta; v + \delta] \subset \mathbb{R} \setminus \text{supp}(\mu_{\sigma,\mu_A\hat{A}_N\gamma,c,N})$.

**Proof:** There exists $\delta_0 > 0$ such that $[u - \delta_0; v + \delta_0] \subset \mathbb{R} \setminus \mathcal{S}$. According to B) of Theorem 1.3, we have $[\omega_{\sigma,\mu}(u - \delta_0); \omega_{\sigma,\mu}(v + \delta_0)] \subset \mathbb{R} \setminus \text{supp}(\nu)$ and for any $x$ in $[\omega_{\sigma,\mu}(u - \delta_0); \omega_{\sigma,\mu}(v + \delta_0)]$, $\Phi'_{\sigma,\mu}(x) > 0$, $g_0(x) > -\frac{1}{\sigma^2}$. In particular $\Theta \setminus \Theta_{\sigma,\mu} \cap \omega_{\sigma,\mu}(u - \delta_0); \omega_{\sigma,\mu}(v + \delta_0) = \emptyset$. Moreover it is clear by B) of Theorem 1.3 that since for any $\theta$ in $\Theta_{\sigma,\mu}$, $\rho \notin [u - \delta_0; v + \delta_0]$, $\Theta_{\sigma,\mu} \cap \omega_{\sigma,\mu}(u - \delta_0); \omega_{\sigma,\mu}(v + \delta_0) = \emptyset$. Therefore,
\[
[\omega_{\sigma,\mu}(u - \delta_0); \omega_{\sigma,\mu}(v + \delta_0)] \subset \mathbb{R} \setminus K_{\nu,\theta},
\]
where $K_{\nu,\theta}$ is defined in 1.3. Note that using B) 2) of Theorem 1.3 we have $[\omega_{\sigma,\mu}(u - \frac{\delta_0}{2}); \omega_{\sigma,\mu}(v + \frac{\delta_0}{2})] \subset [\omega_{\sigma,\mu}(u - \delta_0); \omega_{\sigma,\mu}(v + \delta_0)]$. Then there exists $\alpha > 0$ such that
\[
d\left([\omega_{\sigma,\mu}(u - \frac{\delta_0}{2}); \omega_{\sigma,\mu}(v + \frac{\delta_0}{2})], K_{\nu,\theta}\right) > \alpha.
\]

Moreover according to the assumptions on the eigenvalues of $A_NA_N^*$, there exists some $N_0 \in \mathbb{N}$ such that for $N \geq N_0$, $\text{max}_{i+1 \leq i \leq n} d(\beta_i(N), \text{supp}(\nu)) < \alpha/2$ and $\text{max}_{i=1,\ldots,r} d(|a(N)|, \theta_i) < \alpha/2$ so that the spectrum of $A_NA_N^*$ is included in $\{x, d(x, K_{\nu,\theta}) < \frac{\alpha}{2}\}$. Therefore for $N \geq N_0$,
\[
d\left([\omega_{\sigma,\mu}(u - \frac{\delta_0}{2}); \omega_{\sigma,\mu}(v + \frac{\delta_0}{2})], \text{supp}(\mu_{A_NA_N^*})\right) > \frac{\alpha}{2} > 0.
\]

Moreover there exists $\epsilon > 0$ such that for any $x$ in $[\omega_{\sigma,\mu}(u - \frac{\delta_0}{2}); \omega_{\sigma,\mu}(v + \frac{\delta_0}{2})]$, $g_0(x) > -\frac{1}{\sigma^2} + \epsilon$ and $\Phi'_{\sigma,\mu}(x) > \epsilon$. (4.6)

Let us decompose $\mu_{A_NA_N^*}$ as
\[
\mu_{A_NA_N^*} = \mu_{\beta,N} + \mu_{\theta,N},
\]
where $\mu_{\beta,N} = \frac{1}{N} \sum_{j=r+1}^{n} \delta_{\beta_j(N)}$ and $\mu_{\theta,N} = \frac{1}{N} \sum_{j=r+1}^{J} k_j \delta_{|a_j(N)|^2}$.

We begin with a trivial technical lemma we will need in the following.
Lemma 4.1. \( g_{\mu A_N A_N^\star}; g'_{\mu A_N A_N^\star} \) and \( \Phi'_{\sigma,\mu A_N A_N^\star};c_N \) converge to \( g_\nu; g'_\nu \) and \( \Phi'_{\sigma,\nu};c \) respectively uniformly on \( [\omega_{\sigma,\nu};c(u - \frac{\delta_0}{2});\omega_{\sigma,\nu};c(v + \frac{\delta_0}{2})] \).

Proof of Lemma 4.1: First one can notice that, for any \( \epsilon > 0 \), for \( N \) large enough, for any \( x \) in \( [\omega_{\sigma,\nu};c(u - \frac{\delta_0}{2});\omega_{\sigma,\nu};c(v + \frac{\delta_0}{2})] \),

\[
- g'_{\bar{\mu};N}(x) = \frac{1}{N} \sum_{j=1}^{J} \frac{k_j}{(x - |a_j(N)|^2)^2} \leq \epsilon. \tag{4.7}
\]

We prove now that for all \( x \in [\omega_{\sigma,\nu};c(u - \frac{\delta_0}{2});\omega_{\sigma,\nu};c(v + \frac{\delta_0}{2})] \),

\[
- g'_{\bar{\mu};N}(x) = \frac{1}{N} \sum_{j=1}^{n} \frac{1}{(x - \beta_j)^2} \rightarrow \int \frac{d\nu(t)}{(x - t)^2} = -g'_\nu(x). \tag{4.8}
\]

For all \( x \in [\omega_{\sigma,\nu};c(u - \frac{\delta_0}{2});\omega_{\sigma,\nu};c(v + \frac{\delta_0}{2})] \), let \( h_x \) be a bounded continuous function defined on \( \mathbb{R} \) which coincides with \( f_x(t) = 1/(x-t)^2 \) on \( \text{supp}(\nu) + \left[-\frac{\alpha}{2}, \frac{\alpha}{2}\right] \). Since the sequence of measures \( \bar{\mu}_{\beta,N} \) weakly converges to \( \nu \), (4.8) follows, observing that 

\[
- g'_{\bar{\mu};N}(x) = \int h_x(t) d\bar{\mu}_{\beta,N}(t) - \int h_x(t) d\nu(t).
\]

The uniform convergence follows from Montel’s theorem, since \( g'_{\bar{\mu};N} \) and \( g'_\nu \) are analytic on \( D = \{z \in \mathbb{C}, \text{dist}(z, \text{supp}(\nu)) > \frac{\alpha}{2}\} \) and uniformly bounded on \( D \) by \( \frac{1}{\alpha} \) for all large \( N \).

The same study can be carried out for \( g_{\mu A_N A_N^\star} \) and the uniform convergence of \( \Phi'_{\sigma,\mu A_N A_N^\star};c_N \) towards \( \Phi'_{\sigma,\nu};c \) on \( [\omega_{\sigma,\nu};c(u - \frac{\delta_0}{2});\omega_{\sigma,\nu};c(v + \frac{\delta_0}{2})] \) readily follows. \( \square \)

Hence using (4.7) and Lemma 4.1, we can claim that for all large \( N \), for all \( x \) in \( [\omega_{\sigma,\nu};c(u - \frac{\delta_0}{2});\omega_{\sigma,\nu};c(v + \frac{\delta_0}{2})] \), \( g_{\mu A_N A_N^\star}(x) > -\frac{1}{\sigma_{c_E}} + \frac{\delta_0}{2} > -\frac{1}{\sigma_{c_E}} + \frac{\delta_0}{4} \) and \( \Phi'_{\sigma,\mu A_N A_N^\star};c_N(x) > \frac{\delta_0}{4} \). Therefore for all large \( N \),

\[
\left[\omega_{\sigma,\nu};c(u - \frac{\delta_0}{2});\omega_{\sigma,\nu};c(v + \frac{\delta_0}{2})]\right] \\
\cap \left\{ u \in \mathbb{R} \setminus \text{supp}(\mu_{A_N A_N^\star}) \middle| \Phi'_{\sigma,\mu A_N A_N^\star};c_N(u) > 0, g_{\mu A_N A_N^\star}(u) > -\frac{1}{\sigma_{c_E}} \right\}.
\]

According to B) 1) of Theorem 4.3, we can deduce that

\[
\Phi_{\sigma,\mu A_N A_N^\star};c_N \left([\omega_{\sigma,\nu};c(u - \frac{\delta_0}{2});\omega_{\sigma,\nu};c(v + \frac{\delta_0}{2})]\right)
\]

\[
= \left[\Phi_{\sigma,\mu A_N A_N^\star};c_N (\omega_{\sigma,\nu};c(u - \frac{\delta_0}{2}));\Phi_{\sigma,\mu A_N A_N^\star};c_N (\omega_{\sigma,\nu};c(v + \frac{\delta_0}{2}))\right] \subset \mathbb{R} \setminus \text{supp}(\mu_{\sigma,\mu A_N A_N^\star};c_N).
\]

Now since \( \Phi_{\sigma,\mu A_N A_N^\star};c_N (\omega_{\sigma,\nu};c(u - \frac{\delta_0}{2})) \) and \( \Phi_{\sigma,\mu A_N A_N^\star};c_N (\omega_{\sigma,\nu};c(v + \frac{\delta_0}{2})) \) converge respectively towards \( \Phi_{\sigma,\nu}(\omega_{\sigma,\nu};c(u - \frac{\delta_0}{2})) \) and \( \Phi_{\sigma,\nu}(\omega_{\sigma,\nu};c(v + \frac{\delta_0}{2})) \) and using B 1) of Theorem 4.3, we have for all large \( N \),

\[
\Phi_{\sigma,\mu A_N A_N^\star};c_N (\omega_{\sigma,\nu};c(u - \frac{\delta_0}{2})) \leq u - \frac{\delta_0}{4}.
\]

29
\begin{align*}
\Phi_{\sigma,\nu,c}(\omega_{\sigma,\nu,c}(v + \frac{\delta_0}{2})) & \geq v + \frac{\delta_0}{4} \\
and then \\
[u - \frac{\delta_0}{4}; v + \frac{\delta_0}{4}] & \subset \mathbb{R} \setminus \text{supp}(\mu_{\sigma,\nu,c}(A_N^*)).
\end{align*}

and the proof of Proposition 4.1 is complete. \(\square\)

**Proof of Theorem 4.1**
Using (3.9), assumption 4. of Theorem 1.1 and Theorem 11.8 of [2], one can easily see that there exists \(R > 0\) such that almost surely for all large \(N\), \(\|M_N\| < R\). Let us fix \(\epsilon > 0\) such that
\[
2\epsilon < \min \left\{ \frac{\Phi_{\sigma,\nu,c}(u_{l+1}) - \Phi_{\sigma,\nu,c}(v_l^+)}{d(\rho_{\theta_i}, \text{supp}(\mu_{\sigma,\nu,c}))}, \frac{d(\rho_{\theta_i}, \rho_{\theta_j})}{\theta_i \neq \theta_j} \in \Theta_{\sigma,\nu,c} \right\}.
\]
(4.9)

If \(\Phi_{\sigma,\nu,c}(u_{l+1}^-) > 0\), choose \(\epsilon\) such that (4.9) holds and moreover \(2\epsilon < \Phi_{\sigma,\nu,c}(u_{1}^-)\).

Theorem 4.1 A) follows using Theorem 1.1 for each \([\Phi_{\sigma,\nu,c}(v_l^+) - \epsilon; \Phi_{\sigma,\nu,c}(v_{l+1}^+)]\), \(|\rho_{\theta_i} - \rho_{\theta_j}| \neq \theta_i \in \Theta_{\sigma,\nu,c}\), and, if \(\Phi_{\sigma,\nu,c}(u_{1}^-) > 0\), \(\epsilon; \Phi_{\sigma,\nu,c}(u_{1}^-) - \epsilon\), since Proposition 4.1 implies 7. (i) of Theorem 1.1 and 7. (ii) of Theorem 1.1 is obviously satisfied since \(A_N\) is diagonal.

Now, if \(u_1 > 0\), we have \(\Phi_{\sigma,\nu,c}(u_1^-) > 0\). Let \(u = \min\{u_1, \theta_i \in \Theta_{\sigma,\nu,c}\} > 0\). Applying Proposition 4.1 and Theorem 1.2 to \([\Phi_{\sigma,\nu,c}(\frac{v_1}{2}); \Phi_{\sigma,\nu,c}(\frac{v_p}{2})]\), we obtain that almost surely for all large \(N\), there is no eigenvalue of \(M_N\) on the left hand side of \(\Phi_{\sigma,\nu,c}(\frac{v_1}{2})\) since \(\text{supp}(\nu) \subset \bigcup_{l=1}^p [u_l; v_l]\) and using the assumptions on the spectrum of \(A_N A_N^*\). Using also A), we deduce B). \(\square\)

### 4.4 Convergence of eigenvalues

In the non-spiked case i.e. \(r = 0\), the result of Theorem 4.1 reads as: \(\forall \epsilon > 0,\)
\[
\mathbb{P}(|\text{Spect}(M_N) \subset \text{supp}(\mu_{\sigma,\nu,c}) \cup \{0\} + (-\epsilon, \epsilon), \text{for all } N \text{ large}| = 1, \quad (4.10)
\]
and if \(u_1 > 0\), then
\[
\mathbb{P}(|\text{Spect}(M_N) \subset \text{supp}(\mu_{\sigma,\nu,c}) + (-\epsilon, \epsilon), \text{for all } N \text{ large}| = 1. \quad (4.11)
\]

This readily leads to the following asymptotic result for the extremal eigenvalues.

**Proposition 4.2.** Assume that the deformed model \(M_N\) is without spike i.e. \(r = 0\). Let \(k \geq 0\) be a fixed integer.

1. The first largest eigenvalues \(\lambda_{1+k}(M_N)\) converge almost surely to the right endpoint of the support of \(\mu_{\sigma,\nu,c}\).
2. If \(\Phi_{\sigma,\nu,c}(u_{1}^-) = 0\) that is when \(c = 1\) and either \(0 \in \text{supp}(\nu)\) or \(0 \notin \text{supp}(\nu)\) and \(g(0) \leq -\frac{1}{\sigma}\), then the last smallest eigenvalues \(\lambda_{n-k}(M_N)\) converge almost surely to zero.
3. If \( u_1 > 0 \) (which implies \( \Phi(u_1^+) > 0 \)) then the last smallest eigenvalues \( \lambda_{n-k}(M_N) \) converge almost surely to \( \Phi_{\sigma,\nu,c}(u_1^-) \).

**Proof of Proposition 4.2**

Recalling that \( \text{supp}(\mu_{\sigma,\nu,c}) = \bigcup_{i=1}^{p} [\Phi_{\sigma,\nu,c}(u_i^-), \Phi_{\sigma,\nu,c}(v_i^+)] \), from (4.10), one has that, for all \( \epsilon > 0 \),

\[
P[\lim_{N} \sup \lambda_1(M_N) \leq \Phi_{\sigma,\nu,c}(v_p^+) + \epsilon] = 1.
\]

But as \( \Phi_{\sigma,\nu,c}(v_p^+) \) is a boundary point of \( \text{supp}(\mu_{\sigma,\nu,c}) \), the number of eigenvalues of \( M_N \) falling into \( [\Phi_{\sigma,\nu,c}(v_p^+) - \epsilon, \Phi_{\sigma,\nu,c}(v_p^+) + \epsilon] \) tends almost surely to infinity as \( N \to \infty \). Thus, almost surely,

\[
\lim_{N} \inf \lambda_{1+k}(M_N) \geq \Phi_{\sigma,\nu,c}(v_p^+) - \epsilon.
\]

1) follows by letting \( \epsilon \to 0 \).

The proofs of 2) and 3) are similar to the proof of 1) using the fact that in these cases (4.11) holds.

\[\square\]

In the spiked case, one can deduce from Theorem 1.2 and Theorem 4.1 the following

**Theorem 4.2.** For any \( j = 1, \ldots, J \), we denote by \( n_{j-1} + 1, \ldots, n_{j-1} + k_j \) the descending ranks of \( |a_j(N)|^2 \) among the eigenvalues of \( A_N A_N^* \).

1) If \( \theta_j \in E_{\sigma,\nu,c} \) (i.e. \( \Theta_{\sigma,\nu,c} \)), the \( k_j \) eigenvalues \( (\lambda_{n_{j-1}+i}(M_N), 1 \leq i \leq k_j) \) converge almost surely outside the support of \( \mu_{\sigma,\nu,c} \) towards \( \rho_{\theta_j} = \Phi_{\sigma,\nu,c}(\theta_j) \).

2) If \( \theta_j \in \mathbb{R} \setminus E_{\sigma,\nu,c} \), then we let \( [u_{l_j}, v_{l_j}] \) (with \( 1 \leq l_j \leq p \)) be the connected component of \( \mathbb{R} \setminus E_{\sigma,\nu,c} \) which contains \( \theta_j \).

a) If \( \theta_j \) is on the right of any connected component of \( \text{supp}(\nu) \) which is included in \( [u_{l_j}, v_{l_j}] \) then the \( k_j \) eigenvalues \( (\lambda_{n_{j-1}+i}(M_N), 1 \leq i \leq k_j) \) converge almost surely to \( \Phi_{\sigma,\nu,c}(v_{l_j}^-) \) which is a boundary point of the support of \( \mu_{\sigma,\nu,c} \).

If \( u_{l_j} > 0 \) (which is always true if \( l_j \neq 1 \)) and if \( \theta_j \) is on the left of any connected component of \( \text{supp}(\nu) \) which is included in \( [u_{l_j}, v_{l_j}] \) then the \( k_j \) eigenvalues \( (\lambda_{n_{j-1}+i}(M_N), 1 \leq i \leq k_j) \) converge almost surely to \( \Phi_{\sigma,\nu,c}(u_{l_j}^-) \) which is a boundary point of the support of \( \mu_{\sigma,\nu,c} \).

b) If \( l_j = 1 \) and \( \Phi_{\sigma,\nu,c}(u_{l_j}^-) = 0 \) and if \( \theta_j \) is on the left of any connected component of \( \text{supp}(\nu) \) which is included in \( [u_{l_j}, v_{l_j}] \) then the \( k_j \) eigenvalues \( (\lambda_{n_{j-1}+i}(M_N), 1 \leq i \leq k_j) \) converge almost surely to 0.

c) If \( \theta_j \) is between two connected components of \( \text{supp}(\nu) \) which are included in \( [u_{l_j}, v_{l_j}] \) then the \( k_j \) eigenvalues \( (\lambda_{n_{j-1}+i}(M_N), 1 \leq i \leq k_j) \) converge almost surely to the \( \alpha_j \)-th quantile of \( \mu_{\sigma,\nu,c} \) (that is to \( q_{\alpha_j} \) defined by \( \alpha_j = \mu_{\sigma,\nu,c}([-\infty, q_{\alpha_j}]) \)) where \( \alpha_j \) is such that \( \alpha_j = 1 - \lim_{N} \frac{n_{j-1}}{N} = \nu([-\infty, \theta_j]) \).

31
The proof of Theorem 4.2 follows the lines of the proof of Theorem 8.1 [9]. We include it for the reader’s convenience. First, from Theorem 4.2 and Theorem 4.1, we readily deduce the following

**Corollary 4.1.** Let us fix $\epsilon > 0$ such that

$$2\epsilon < \min \left\{ \min_{l=1,\ldots,p} \left[ \Phi_{\sigma,\nu,c}(u_{l+1}^-) - \Phi_{\sigma,\nu,c}(u_l^+) \right], d(\rho_{\theta}, \text{supp}(\mu_{\sigma,\nu,c})) \right\},$$

$$d(\rho_{\theta}, \rho_{\theta_j}), \theta_i \neq \theta_j \text{ in } \Theta_{\sigma,\nu,c} \right\}$$

Let $u$ be in $\Theta_{\sigma,\nu,c} \cup \{v_l, l = 1, \ldots, m\}$ (resp. in $\Theta_{\sigma,\nu,c} \cup \{u_l, l = 1, \ldots, m\} \cap [0; +\infty]$). Let us choose $\delta > 0$ small enough so that for large $N$, $[u + \delta; u + 2\delta]$ (resp. $[u - 2\delta; u - \delta]$) is included in $E_{\sigma,\nu,c}[0; +\infty]$ and for any $0 \leq \delta' \leq 2\delta$, $\Phi_{\sigma,\nu,c}(u + \delta') - \Phi_{\sigma,\nu,c}(u^+) < \epsilon$ (resp. $\Phi_{\sigma,\nu,c}(u^-) - \Phi_{\sigma,\nu,c}(u - \delta') < \epsilon$). Let $i_N = i_N(u)$ be such that

$$\lambda_{i_N+1}(A_N^{-1}A_N^*) < u + \delta \quad \text{and} \quad \lambda_{i_N}(A_N^{-1}A_N^*) > u - \delta$$

(resp. $\lambda_{i_N+1}(A_N^{-1}A_N^*) < u - 2\delta \quad \text{and} \quad \lambda_{i_N}(A_N^{-1}A_N^*) < u - \delta$). Then

$$\mathbb{P}[\lambda_{i_N+1}(M_N) < \Phi_{\sigma,\nu,c}(u+\delta) \quad \text{and} \quad \lambda_{i_N}(M_N) > \Phi_{\sigma,\nu,c}(u^+)+\epsilon, \text{ for all large } N] = 1.$$  

(resp. $\mathbb{P}[\lambda_{i_N+1}(M_N) < \Phi_{\sigma,\nu,c}(u^-)-\epsilon \quad \text{and} \quad \lambda_{i_N}(M_N) > \Phi_{\sigma,\nu,c}(u-\delta) \text{ for all large } N] = 1$.)

**Proof of Theorem 4.2**

1) Choosing $u = \theta_j$ in Corollary 4.1 gives, for any $\epsilon > 0$,

$$\rho_{\theta} - \epsilon \leq \lambda_{n_{\theta}+k_j}(M_N) \leq \cdots \leq \lambda_{n_{\theta}+1}(M_N) \leq \rho_{\theta} + \epsilon, \text{ for large } N \ (4.12)$$

holds almost surely. Hence

$$\forall 1 \leq i \leq k_j, \quad \lambda_{n_{\theta}+i}(M_N) \xrightarrow{a.s.} \rho_{\theta}.$$  

2) a) We only focus on the case where $\theta_j$ is on the right of any connected component of $\text{supp}(\nu)$ which is included in $[u_l, v_l]$ since the other case may be considered with similar arguments. Let us consider the set $\{\theta_{j_1} > \ldots > \theta_{j_h}\}$ of all the $\theta_i$’s being in $[u_l, v_l]$ and on the right of any connected component of $\text{supp}(\nu)$ which is included in $[u_l, v_l]$. Note that we have for all large $N$, for any $0 \leq h \leq p$,

$$n_{j_h-1} + k_{j_h} = n_{j_h}$$

and $\theta_{j_h}$ is the largest eigenvalue of $A_N^{-1}A_N^*$ which is lower than $v_l$. Let $\epsilon > 0$. Applying Corollary 4.1 with $u = v_l$, we get that, almost surely,

$$\lambda_{n_{j_h}+1}(M_N) < \Phi_{\sigma,\nu,c}(v_l^+)+\epsilon \text{ and } \lambda_{n_{j_h}-1}(M_N) > \Phi_{\sigma,\nu,c}(v_l^+)+\epsilon$$

for all large $N$. Now, almost surely, the number of eigenvalues of $M_N$ being in $[\Phi_{\sigma,\nu,c}(v_l^+)-\epsilon, \Phi_{\sigma,\nu,c}(v_l^+)+\epsilon]$ should tend to infinity when $N$ goes to infinity. Since almost surely for all large $N$, $\lambda_{n_{j_h}-1}(M_N) > \Phi_{\sigma,\nu,c}(v_l^+)+\epsilon$ and $\lambda_{n_{j_h}+1}(M_N) < \Phi_{\sigma,\nu,c}(v_l^+)+\epsilon$, we should have

$$\Phi_{\sigma,\nu,c}(v_l^+) - \epsilon \leq \lambda_{n_{\theta}+k}\ (M_N) \leq \ldots \leq \lambda_{n_{\theta}+1}(M_N) < \Phi_{\sigma,\nu,c}(v_l^+) + \epsilon.$$
Hence, we deduce that: \( \forall 0 \leq l \leq p \) and \( \forall 1 \leq i \leq k_{jp}, \) \( \lambda_{n_{jp-1}+i}(MN) \xrightarrow{a.s.} \Phi_{\sigma,\nu,c}(u_l^+) \). The result then follows since \( j \in \{j_0, \ldots, j_p\} \).

b) In this case, \( \theta_j \) is one of the finite number of lowest eigenvalues of \( A_N \). Then \( \theta_j \) readily follows from the fact that the number of eigenvalues of \( \Phi_{\sigma,\nu,c} \) in \( [0, \epsilon] \) should tend to infinity when \( N \) goes to infinity.

c) Let \( \alpha_j = 1 - \lim_{N \to \infty} \frac{1}{2N} = \nu([-\infty, \theta_j]) \). Denote by \( Q \) (resp. \( Q_N \)) the distribution function of \( \mu_{\sigma,\nu,c} \) (resp. of the spectral measure of \( MN \)). \( Q \) is continuous on \( \mathbb{R} \) and strictly increasing on each interval \( [\Phi_{\sigma,\nu,c}(u_l^+), \Phi_{\sigma,\nu,c}(v_l^-)], 1 \leq l \leq m \).

From Corollary 1.1 and the hypothesis on \( \theta_j \), \( \alpha_j \in [Q(\Phi_{\sigma,\nu,c}(u_l^+)), Q(\Phi_{\sigma,\nu,c}(v_l^-))] \) and there exists a unique \( q_j \in [\Phi_{\sigma,\nu,c}(u_l^+), \Phi_{\sigma,\nu,c}(v_l^-)] \) such that \( Q(q_j) = \alpha_j \). Moreover, \( Q \) is strictly increasing in a neighborhood of \( q_j \).

Let \( \epsilon > 0 \). From the almost sure convergence of \( \mu_{MN} \) to \( \mu_{\sigma,\nu,c} \), we deduce
\[
Q_N(q_j + \epsilon) \xrightarrow{N \to \infty} Q(q_j + \epsilon) > \alpha_j, \quad \text{a.s.}
\]

From the definition of \( \alpha_j \), it follows that for large \( N, N-N-1, \ldots, n_{j-1}+k_j, \ldots, n_{j-1}+1 \) belong to the set \( \{k, \lambda_k(M_n) \leq q_j + \epsilon\} \) and thus,
\[
\limsup_{N \to \infty} \lambda_{n_{j-1}+1}(MN) \leq q_j + \epsilon.
\]

In the same way, since \( Q_N(q_j - \epsilon) \xrightarrow{N \to \infty} Q(q_j - \epsilon) < \alpha_j \),
\[
\liminf_{N \to \infty} \lambda_{n_{j-1}+k_j}(MN) \geq q_j - \epsilon.
\]

Thus, the \( k_j \) eigenvalues \( (\lambda_{n_{j-1}+i}(MN), 1 \leq i \leq k_j) \) converge almost surely to \( q_j \). \( \square \)

References

[1] Z. D. Bai and J. W. Silverstein. Exact separation of eigenvalues of large-dimensional sample covariance matrices. *Ann. Probab.*, 27(3):1536–1555, 1999.

[2] Z. Bai and J.W. Silverstein. No eigenvalues outside the support of the limiting spectral distribution of information-plus-noise type matrices. *Random Matrices: Theory and Applications* 1(1) 2012, pp. 1150004.

[3] F. Benaych-Georges. Rectangular random matrices, related convolution *Probab. Theory Related Fields* Vol. 144, no. 3 (2009) 471–515.

[4] F. Benaych-Georges, S. Belinschi and A. Guionnet. Regularization by free additive convolution, square and rectangular cases. *Complex Anal. Oper. Theory* Vol. 3, no. 3 (2009) 611–660.

[5] P. Biane. On the free convolution with a semi-circular distribution. *Indiana Univ. Math. J.*, 46(3):705–718, 1997.
[6] P. Biane, and F. Lehner. Computation of some examples of Brown’s spectral measure in free probability; Colloq. Math. 90 (2), 181-211, (2001).

[7] M. Capitaine, C. Donati-Martin. Free Wishart Processes Journal of Theoretical Probability, vol. 18, No 2, 2005) 413-438.

[8] M. Capitaine, C. Donati-Martin, and D. Féral. The largest eigenvalues of finite rank deformation of large Wigner matrices: convergence and nonuniversality of the fluctuations. Ann. Probab., 37(1):1–47, 2009.

[9] M. Capitaine, C. Donati-Martin, D. Féral and M. Février. Free convolution with a semi-circular distribution and eigenvalues of spiked deformations of Wigner matrices. Electronic Journal of Probability, 16: 1750–1792, 2011.

[10] R. Couillet, J. W. Silverstein, Z. Bai, and M. Debbah. Eigen-Inference for Energy Estimation of Multiple Sources. to appear in IEEE Trans. on Information Theory.

[11] R.B. Dozier and J.W. Silverstein. On the empirical distribution of eigenvalues of large dimensional information-plus-noise type matrices. J. Multivariate. Anal., vol. 98, no. 4: 678–694, 2007.

[12] R.B. Dozier and J.W. Silverstein. Analysis of the limiting spectral distribution of large dimensional information-plus-noise type matrices. J. Multivariate. Anal., vol. 98, no. 6: 1099–1122, 2007.

[13] R. A. Horn and C. R. Johnson. Matrix analysis. Cambridge University Press, Cambridge, 1990. Corrected reprint of the 1985 original.

[14] P. Vallet, P. Loubaton and X. Mestre. Improved Subspace Estimation for Multivariate Observations of High Dimension: The Deterministic Signal Case. 2010 Available at http://front.math.ucdavis.edu/1002.3234.

[15] P. Loubaton and P. Vallet. Almost sure localization of the eigenvalues in a Gaussian information-plus-noise model. Application to the spiked models 2010 Available at http://front.math.ucdavis.edu/1009.5807.

[16] J. Xie. The convergence on spectrum of sample covariance matrices for information-plus-noise type data. Appl. Math. J. Chinese Univ., vol. 27 (2): 181-191, 2012.