An $L^p$-$L^q$-Version of Morgan’s Theorem for the $n$-Dimensional Euclidean Motion Group

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We establish an $L^p$-$L^q$-version of Morgan’s theorem for the group Fourier transform on the $n$-dimensional Euclidean motion group $M(n)$.

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1. Introduction

An aspect of uncertainty principle in real classical analysis asserts that a function $f$ and its Fourier transform $\hat{f}$ cannot decrease simultaneously very rapidly at infinity. As illustrations of this, one has Hardy’s theorem [1], Morgan’s theorem [2], and Beurling-Hörmander’s theorem [3–5]. These theorems have been generalized to many other situations; see, for example, [6–10].

In 1983, Cowling and Price [11] have proved an $L^p$-$L^q$-version of Hardy’s theorem. An $L^p$-$L^q$-version of Morgan’s theorem has been also proved by Ben Farah and Mokni [7].

To state the $L^p$-$L^q$-versions of Hardy’s and Morgan’s theorems more precisely, we propose the following.

Let $a,b > 0$, $p, q \in [1, +\infty]$, $\alpha \geq 2$, and $\beta$ such that $1/\alpha + 1/\beta = 1$.
If we consider measurable functions $f$ on $\mathbb{R}$ such that

$$e^{a|x|^\alpha} f \in L^p(\mathbb{R}), \quad e^{b|y|^\beta} \hat{f} \in L^q(\mathbb{R}),$$

we obtain the following.

(i) If $(a\alpha)^{1/\alpha}(b\beta)^{1/\beta} > (\sin(\pi/2)(\beta - 1))^{1/\beta}$, then $f = 0$ a.e.
(ii) If $(a\alpha)^{1/\alpha}(b\beta)^{1/\beta} \leq (\sin(\pi/2)(\beta - 1))^{1/\beta}$, then one has infinitely many such $f$.

The case $\alpha = \beta = 2$, $p = q = +\infty$ corresponds to Hardy’s theorem.
The case $\alpha = \beta = 2$, $1 \leq p$, $q < +\infty$ corresponds to the Cowling-Price theorem.
The case $\alpha > 2$, $p = q = +\infty$ corresponds to Morgan’s theorem.
The case $\alpha > 2$, $1 \leq p$, $q < +\infty$ corresponds to the Ben Farah-Mokni theorem.
We remark that for each one of those cases there are further requirements for \( f \) if 
\[
(aa)^{1/\alpha}(b \beta)^{1/\beta} = (\sin(\pi/2)(\beta - 1))^{1/\beta},
\]
In this paper, we give an \( L^p - L^q \)-version of Morgan’s theorem for the \( n \)-dimensional Euclidean motion group \( M(n) \), \( n \geq 2 \).

We can note that for the motion group, theorems of Beurling and Hardy have been studied by Sarkar and Thangavelu [12]. For example, the condition in Theorem 1.1 below for \( f = 0 \) a.e. for the case \( \alpha = 2 \) follows from their work.

The motion group \( M(n) \) is the semidirect product of \( \mathbb{R}^n \) with \( K = SO(n) \). As a set \( M(n) = \mathbb{R}^n \times K \), and the group law is given by

\[
(x,k)(x',k') = (x + k \cdot x', kk'),
\]
here \( k \cdot x' \) is the nature action of \( K \) on \( \mathbb{R}^n \). The Haar measure of \( M(n) \) is \( dx \, dk \), where \( dx \) is the Lebesgue measure on \( \mathbb{R}^n \) and \( dk \) is the normalized Haar measure on \( K \).

Denote by \( \hat{M}(n) \) the unitary dual of the motion group. The abstract Plancherel theorem asserts that there is a unique measure \( \mu \) on \( \hat{M}(n) \) such that for all \( f \in L^1(M(n)) \cap L^2(M(n)) \),

\[
\int_{M(n)} | f(x,k) |^2 \, dx \, dk = \int_{\hat{M}(n)} \text{tr} (\pi(f)\pi(f)^*) \, d\mu(\pi),
\]
where \( \pi(f) = \int_{M(n)} f(x,k)\pi(x,k) \, dx \, dk \) is the group Fourier transform of \( f \) at \( \pi \in \hat{M}(n) \).

It is well known that \( \mu \) is supported by the set of infinite-dimensional elements of \( \hat{M}(n) \), which is parametrized by \( (r,\lambda) \in [0, \infty[ \times \hat{U} \), where \( U = SO(n-1) \) is the subgroup of \( SO(n) \) leaving fixed \( e_n = (0, \ldots, 0, 1) \) in \( \mathbb{R}^n \). As such an element \( \pi_{r,\lambda} \) is realized in a Hilbert space \( H_\lambda \), we note that for \( f \in L^1(M(n)) \cap L^2(M(n)) \), \( \pi_{r,\lambda}(f) \) is a Hilbert-Schmidt operator on \( H_\lambda \), moreover the restriction of the Plancherel measure on the part \([0, \infty[ \times \{\lambda\}\) is given up to a constant depending only on \( n \), by \( r^{n-1} \, dr \).

For the analogue of Morgan's theorem on \( M(n) \) we propose the following version, where we use the notation \( \hat{f}(r,\lambda) = \pi_{r,\lambda}(f) \).

**Theorem 1.1.** Let \( p, q \in [1, +\infty] \), \( a, b \in ]0, +\infty[ \), and \( \alpha, \beta \) positive real numbers satisfying \( \alpha > 2 \) and \( 1/\alpha + 1/\beta = 1 \).

Suppose that \( f \) is in \( L^2(M(n)) \) such that

(i) \( e^{a|\lambda|^\alpha} f(x,k) \in L^p(M(n)) \),
(ii) \( e^{b \lambda^\beta} \| \hat{f}(r,\lambda) \|_{HS} \in L^q([r, r^{n-1} \, dr]) \) for all fixed \( \lambda \) in \( \hat{U} \).

If \( (aa)^{1/\alpha}(b \beta)^{1/\beta} > (\sin(\pi/2)(\beta - 1))^{1/\beta} \), then \( f \) is null a.e.
If \( (aa)^{1/\alpha}(b \beta)^{1/\beta} \leq (\sin(\pi/2)(\beta - 1))^{1/\beta} \), then there are infinitely many such \( f \).

This paper is organized as follows.

In Section 2, we give a description of the unitary dual of the \( n \)-dimensional Euclidean motion group \( M(n) \). Section 3 is devoted to the above version of Morgan’s theorem for \( M(n) \).
2. Description of the unitary dual of $M(n)$

We are going to describe the infinite-dimensional elements of $\hat{M}(n)$, which are sufficient for the Plancherel formula. We start by some notations.

For any integer $m$, let $\langle \cdot, \cdot \rangle$ denote the Hermitian (resp., Euclidian) product on $\mathbb{C}^m$ (resp., on $\mathbb{R}^m$) and let $\| \cdot \|$ be the corresponding norm. For $y \neq 0$ in $\mathbb{R}^n$ let $U_y$ be the stabilizer of $y$ in $K$ under its natural action on $\mathbb{R}^n$. $U_y$ is conjugate to the subgroup $U = \text{SO}(n-1)$ of $\text{SO}(n)$ leaving fixed $\varepsilon_n = (0, \ldots, 0, 1)$ in $\mathbb{R}^n$.

We remark that $\hat{\mathbb{R}}^n$, the set of unitary characters of $\mathbb{R}^n$, is identified with $\mathbb{R}^n$. In fact any such character is of the form $\chi_y$, $y \in \mathbb{R}^n$, and is defined for all $x \in \mathbb{R}^n$ by $\chi_y(x) = e^{i(x,y)}$. The trivial character corresponds to $y = 0$.

To construct an infinite-dimensional irreducible unitary representation of the motion group $M(n)$, we use the following steps.

**Step 1.** Take a nontrivial element $\chi_y$ in $\hat{\mathbb{R}}^n$. It is stabilized under the action of $K$ by $U_y$.

**Step 2.** Take $\lambda \in \hat{U}_y$ and consider $\chi_y \otimes \lambda$ as a representation of the semidirect product of $\mathbb{R}^n$ by $U_y$ denoted by $\mathbb{R}^n \ltimes U_y$.

**Step 3.** Induce $\chi_y \otimes \lambda$ from $\mathbb{R}^n \ltimes U_y$ to $M(n)$ to obtain a representation $T_{y,\lambda}$ of $M(n)$.

We have then the following properties (see [13, 14] for details).

(a) For $y \neq 0$ and any $\lambda \in \hat{U}_y$, the representation $T_{y,\lambda}$ is unitary and irreducible.

(b) Every infinite-dimensional irreducible unitary representation of $M(n)$ is equivalent to $T_{y,\lambda}$ for some $y$ and $\lambda$ as above.

(c) The representations $T_{y_1,\lambda_1}$ and $T_{y_2,\lambda_2}$ are equivalent if and only if $\| y_1 \| = \| y_2 \|$ and $\lambda_1$ is equivalent to $\lambda_2$ under the obvious identification of $U_{y_1}$ with $U_{y_2}$.

In particular, when $\| y \| = r > 0$, $T_{y,\lambda}$ is equivalent to $T_{r\varepsilon_n,\lambda}$, so the different classes of infinite-dimensional representations of $M(n)$ can be parametrized by $(r, \lambda) \in [0, \infty) \times \hat{U}$. We use the notation $\pi_{r,\lambda}$ for $T_{r\varepsilon_n,\lambda}$ and for its equivalence class in $\hat{M}(n)$. Let us make this representation explicit.

$\lambda$ is an irreducible unitary representation of $U = \text{SO}(n-1)$, it is of finite dimension $d_\lambda$ and acts on $\mathbb{C}^{d_\lambda}$. Let $H_\lambda$ be the vector space of all measurable function $\psi : K \to \mathbb{C}^{d_\lambda}$ such that $\int_K \| \psi(k) \|^2 dk < \infty$ and $\psi(uk) = \lambda(u)(\psi(k))$ for all $u \in U$, $k \in K$. $H_\lambda$ is a Hilbert space with respect to the inner product defined by

$$
(\psi_1 | \psi_2) = d_\lambda \int_K \langle \psi_1(k), \psi_2(k) \rangle dk.
$$

(2.1)

$\pi_{r,\lambda}$ acts on $H_\lambda$ via

$$
[\pi_{r,\lambda}(a,k)\psi](k_0) = e^{i(k_0^{-1}r\varepsilon_n a)}\psi(k_0k), \quad \psi \in H_\lambda,
$$

(2.2)

for $a \in \mathbb{R}^n$, $k, k_0 \in K$.

The Plancherel measure $\mu$ is then supported by the subset of $\hat{M}(n)$ given by $\{ \pi_{r,\lambda} : \lambda \in \hat{U}, r \in \mathbb{R}^+ \}$, and on each “piece” $\{ \pi_{r,\lambda} : r \in \mathbb{R}^+ \}$ with $\lambda$ fixed in $\hat{U}$, it is given by $C_n r^{n-1} dr$, where $C_n$ is a constant depending only on $n$. 

The Fourier transform of a function $f$ in $L^1(M(n))$ is denoted as above by $\hat{f}$. It is defined for $(r, \lambda) \in [0, \infty) \times \hat{U}$ by

$$\hat{f}(r, \lambda) = \pi_{r, \lambda}(f) = \int_{\mathbb{R}^n} \int_K f(a, k) \pi_{r, \lambda}(a, k) dk da$$

(2.3)

(the integral being interpreted suitably, see [15]).

By the Plancherel theorem we know that for $f \in L^1(M(n)) \cap L^2(M(n))$, $\hat{f}(r, \lambda)$ is a Hilbert-Schmidt operator. Let $\|\hat{f}(r, \lambda)\|_{HS}$ be its Hilbert-Schmidt norm.

3. Morgan’s theorem for the motion group

Before giving Morgan’s theorem for the motion group $M(n)$, we state the following complex analysis lemma proved by Ben Farah and Mokni [7]. This lemma plays a crucial role in the proof of our main theorem.

**Lemma 3.1.** Suppose $\rho \in [1, 2[, q \in [1, +\infty]$, $\sigma > 0$, and $B > \sigma \sin(\pi/2)(\rho - 1)$.

If $g$ is an entire function on $\mathbb{C}$ satisfying the conditions

$$\|g(\rho + iy)\| \leq \text{const} e^{\sigma |y|^p} \text{ for any } x, y \in \mathbb{R},$$

$$e^{B|x|^p}g_{|\mathbb{R}} \in L^q(\mathbb{R}),$$

then $g = 0$.

We now give the $L^p-L^q$-version of Morgan’s theorem.

**Theorem 3.2.** Let $p, q \in [1, +\infty]$, $a, b \in [0, +\infty[$, and $\alpha, \beta$ positive real numbers satisfying $\alpha > 2$ and $1/\alpha + 1/\beta = 1$.

Suppose that $f$ is a measurable function on $M(n)$ such that

(i) $e^{a|x|^p} f(x, k) \in L^p(M(n))$,

(ii) $e^{b|\lambda|^q} \|\hat{f}(r, \lambda)\|_{HS} \in L^q(\mathbb{R}^+, r^{n-1} dr)$ for all fixed $\lambda$ in $\hat{U}$.

If $(aa^{1/\alpha}(b\beta)^{1/\beta} > (\sin(\pi/2)(\beta - 1))^{1/\beta}$, then $f$ is null a.e.

**Proof.** To prove that $f = 0$, we are going to prove that $\hat{f}(r, \lambda) = 0$. For this, it suffices to show that for fixed $\lambda \in \hat{U}$ and for any fixed $K$-finite vectors $\varphi$ and $\psi$ in $H_{\lambda}$, the condition $(aa^{1/\alpha}(b\beta)^{1/\beta} > (\sin(\pi/2)(\beta - 1))^{1/\beta}$ implies that $(\hat{f}(r, \lambda)\varphi | \psi) \equiv 0$ as a function of $r$ and $\lambda$.

Let $\lambda \in \hat{U}$ and let $\varphi, \psi$ be $K$-finite vectors in $H_{\lambda}$. We note that $\varphi$ and $\psi$ are continuous on $K$ and thus bounded. On the other hand, for $r \in \mathbb{R},$

$$(\hat{f}(r, \lambda)\varphi | \psi) = \int_K \int_{\mathbb{R}^n} f(x, k)(\pi_{r, \lambda}(x, k)\varphi | \psi) dx dk.$$
Let $\Phi_r(x, k) = (\pi_{r,\lambda}(x, k)\varphi \ | \ \psi)$ for $r \in \mathbb{R}$ and $(x, k) \in M(n)$. Then, by definition of $\pi_{r,\lambda}$, we have

$$
\Phi_r(x, k) = d_1 \int_K \langle (\pi_{r,\lambda}(x, k)\varphi)(0), \psi(0) \rangle dk_0
= d_1 \int_K e^{i(k_0^{-1} r_{xk})} \langle \varphi(k_0 k), \psi(k_0) \rangle dk_0
= d_1 \int_K e^{i(r_{xk}k_0)} \langle \varphi(k_0 k), \psi(k_0) \rangle dk_0. 
$$

(3.3)

Note that the integral on the right-hand side makes sense even if $r \in \mathbb{C}$. Hence, with $(x, k)$ fixed, the function $\Phi_r(x, k)$ of the variable $r$ extends to the whole complex plane. One can easily see that for fixed $(x, k)$, $z \mapsto \Phi_z(x, k)$ is an entire function on $\mathbb{C}$. Moreover, for $z \in \mathbb{C}$,

$$
| \Phi_z(x, k) | \leq d_1 \int_K | e^{i(z_{xk}k_0)} | \cdot | \varphi(k_0 k) | \cdot | \psi(k_0) | dk_0. 
$$

(3.4)

Then

$$
| \Phi_z(x, k) | \leq A \int_K e^{-\langle \text{Im} z_{xk}, k_0 x \rangle} dk_0, 
$$

(3.5)

where $A$ is a constant depending only on $\lambda$, $\varphi$, and $\psi$. (Note that $\varphi$ and $\psi$ are continuous functions on $K$ and hence are bounded.)

Using the fact that $dk_0$ is a normalized measure on $K$, we obtain

$$
| (\Phi_z(x, k)) | \leq A e^{\text{Im} z : \| x \|. 
$$

(3.6)

By definition of $\Phi_z(x, k)$, we have

$$
(\hat{f}(z, \lambda) \varphi \ | \ \psi) = \int_{\mathbb{R}^n} f(x, k) \Phi_z(x, k) dx dk. 
$$

(3.7)

Since $f$ satisfies hypothesis (i) of Theorem 3.2 and $| (\Phi_z(x, k)) | \leq A e^{\text{Im} z : \| x \|}$, we conclude that the function $r \mapsto (\hat{f}(r, \lambda) \varphi \ | \ \psi)$ can be extended to the whole of $\mathbb{C}$ and indeed it can be proved that the function

$$
z \mapsto (\hat{f}(z, \lambda) \varphi \ | \ \psi) \quad \text{is an entire function.} 
$$

(3.8)

Further, from (3.6) and (3.7), we deduce that

$$
| (\hat{f}(z, \lambda) \varphi \ | \ \psi) | \leq A \int_{\mathbb{R}^n} | f(x, k) | e^{\text{Im} z : \| x \|} dx dk. 
$$

(3.9)

Let $I = \{ (b\beta)^{-1/\beta} (\sin(\pi/2)(\beta - 1))^{1/\beta}, (aa\lambda)^{1/\alpha} \}$, and $C \in I$. Applying the convex inequality $|ty| \leq (1/\alpha) |t|^{\alpha} + (1/\beta) |y|^{\beta}$ to the positive numbers $C \| x \|$ and $|\text{Im} z|/C$, we obtain

$$
| \text{Im} z | \cdot \| x \| \leq \frac{C^\alpha}{\alpha} \| x \|^\alpha + \frac{1}{\beta C^\beta} | \text{Im} z |^\beta, 
$$

(3.10)
thus
\[ |(\hat{f}(z,\lambda)\varphi | \psi)| \leq Ae^{(1/\beta C^\beta)|\text{Im}z|^\beta} \int_\mathbb{R}^n |f(x,k)| e^{(C\alpha/\alpha)|x|^\alpha} \, dx \, dk. \] (3.11)

Then
\[ |(\hat{f}(z,\lambda)\varphi | \psi)| \leq Ae^{(1/\beta C^\beta)|\text{Im}z|^\beta} \int_\mathbb{R}^n e^{a|x|^\alpha} |f(x,k)| e^{(C\alpha/\alpha-a)|x|^\alpha} \, dx \, dk. \] (3.12)

Using this inequality, hypothesis (i), the fact that $dk$ is a normalized measure, and the inequality $a > c/\alpha$, we obtain
\[ |(\hat{f}(z,\lambda)\varphi | \psi)| \leq \text{const} e^{(1/\beta C^\beta)|\text{Im}z|^\beta}. \] (3.13)

On the other hand, since $\pi-r,\lambda$ and $\pi,r,\lambda$ are equivalent as representations of $M(n)$,
\[ \|\hat{f}(-r,\lambda)\|_{HS} = \|\hat{f}(r,\lambda)\|_{HS}. \] (3.14)

Hypothesis (ii) of Theorem 3.2 and the inequality (3.14) imply that the function
\[ r \mapsto e^{br}\|\hat{f}(r,\lambda)\|_{HS} \text{ belongs to } L^q(\mathbb{R}), \] (3.15)

thus
\[ r \mapsto e^{br}((\hat{f}(r,\lambda)\varphi | \psi)_{L^q(\text{H}^\lambda)}) \text{ belongs to } L^q(\mathbb{R}). \] (3.16)

It is clear from (3.8), (3.13), (3.16) that the function $z \mapsto (\hat{f}(z,\lambda)\varphi,\psi)$ satisfies the hypothesis of Lemma 3.1, and so
\[ (\hat{f}(z,\lambda)\varphi | \psi) \equiv 0 \] (3.17)
as a function of $z$.

Since $\varphi, \psi, \lambda$ are arbitrary, then $\hat{f}(r,\lambda) \equiv 0$ for all $r \in \mathbb{R}_+$ and $\lambda \in \hat{U}$. Hence, by the Plancherel formula, we get that $f = 0$ a.e. This completes the proof of the theorem. \(\square\)

In order to prove that our version respects the analogy with Morgan's theorem, let us now establish the sharpness of the condition
\[ (a\alpha)^{1/\alpha}(b\beta)^{1/\beta} > (\sin(\pi/2)(\beta-1))^{1/\beta} \] (3.18)
in Theorem 3.2.

**Proposition 3.3.** Let $p,q \in [1, +\infty]$, $a,b \in ]0, +\infty[$, and $a,\beta$ positive real numbers satisfying $\alpha > 2$ and $1/\alpha + 1/\beta = 1$.

If $(a\alpha)^{1/\alpha}(b\beta)^{1/\beta} \leq (\sin(\pi/2)(\beta-1))^{1/\beta}$, then there are infinitely many measurable functions on $M(n)$ satisfying
\begin{itemize}
  \item[(i)] $e^{a|x|^\alpha} f(x,k) \in L^p(M(n))$,
  \item[(ii)] $e^{br}\|\hat{f}(r,\lambda)\|_{HS} \in L^q(\mathbb{R}_+, r^{n-1} \, dr)$ for any $\lambda$ fixed in $\hat{U}$.
\end{itemize}
To prove this proposition, we use the following lemma for $a$, $b$, $\alpha$, $\beta$ as above.

**Lemma 3.4.** If $(a\alpha)^{1/\alpha}(b\beta)^{1/\beta} = (\sin(\pi/2)(\beta - 1))^{1/\beta}$, then for all $m \in \mathbb{R}$ and $m' = (2m + d(2 - \alpha))/(2\alpha - 2)$, there exists a nonzero measurable function on $M(n)$ satisfying

(i) $\left(1 + ||x||^{-m}e^{|a||x|^2}\right) f \in L^\infty(M(n))$,

(ii) $\left(1 + r\right)^{-m'}e^{br\beta} \left|\hat{f}(r, \lambda)\right|_{HS} \in L^\infty(\mathbb{R}^+, r^{n-1}dr)$ for any fixed $\lambda$ in $\hat{U}$.

**Proof.** We put for $(x, k) \in M(n)$

$$f(x, k) = -i \int_C z^\nu e^{z^q-qA||x||^2} dz,$$

where $q = a/(\alpha - 2)$, $A^\alpha = (1/4)((\alpha - 2)a)^2$, $\nu = (2m + 4 - \alpha)/(\alpha - 2)$, and $C$ is the path which lies in the half-plane $\text{Re} z > 0$, and goes to infinity, in the directions $\text{arg} z = \pm \theta_0$, $\pi/2q < \theta_0 < \pi/q$.

According to Morgan (see [2, page 190]), for $||x|| \to \infty$, we have

$$f(x, k) \sim (\alpha - 2)\left(\frac{(\alpha - 2)a}{2}\right)^{n/\alpha} \sqrt{\left(\frac{\pi}{\alpha}\right)} ||x||^{m} e^{-a||x||^2}.$$  

(3.20)

On the other hand, for $\lambda$ fixed in $\hat{U}$, $(\hat{f}(r, \lambda)\varphi \mid \psi)$ is equal to

$$-id_\lambda \int_K \int_{\mathbb{R}^n} \int_C \int_K z^\nu e^{z^q-qA||x||^2} e^{i(r\varphi, k0\alpha)} \langle \varphi(k0k), \psi(k0) \rangle dk0 dz da dk,$$

(3.21)

which by a change of variables $x = k_0^{-1}a$ is equal to

$$-id_\lambda \int_K \int_{\mathbb{R}^n} \int_C \int_K z^\nu e^{z^q-qA||x||^2} e^{i(r\varphi, x)} \langle \varphi(k0k), \psi(k0) \rangle dk0 dz dx dk.$$  

(3.22)

Using this equality and Fubini’s theorem, we obtain the following expression for $(\hat{f}(r, \lambda)\varphi \mid \psi)$:

$$-id_\lambda \left( \int_K \langle \varphi(k0k), \psi(k0) \rangle dk0 dk \right) \int_{\mathbb{R}^n} z^\nu e^{z^q-qA||x||^2} e^{i(r\varphi, x)} dx dz.$$  

(3.23)

Since

$$\int_{\mathbb{R}^n} e^{-qA||x||^2} e^{i(r\varphi, x)} dx = \left(\frac{\pi}{qAz}\right)^{n/2} e^{-r^2/4aqz},$$

we deduce that

$$(\hat{f}(r, \lambda)\varphi \mid \psi) = -id_\lambda \left( \frac{\pi}{qA} \right)^{n/2} \left( \int_K \langle \varphi(k0k), \psi(k0) \rangle dk0 \right) \int_C z^{y-n/2} e^{z^q-r^2/4aqz} dz.$$  

(3.25)

Now, we fix an orthonormal basis $\{e_j; j \in \mathbb{N}\}$ of $H_1$. Taking into account that $\hat{f}(r, \lambda)$ is a Hilbert–Schmidt operator, we then replace $\varphi$ by $e_i$, $\psi$ by $e_j$ and take the sum on $i, j \in \mathbb{N}$ to
Adapting the method of Morgan (see [2, page 191]), we obtain

$$
\| \hat{f}(r, \lambda) \|_{HS} = O(r^{m'} e^{-br\beta})
$$

(3.27)

with $m' = (2m + n(2 - \alpha))/(2\alpha - 2)$. We conclude by using the estimations (3.20) and (3.27).

Proof of Proposition 3.3. It suffices to prove the proposition for

$$(a\alpha)^{1/\alpha} (b\beta)^{1/\beta} = \left( \sin \frac{\pi}{2} (\beta - 1) \right)^{1/\beta},$$

(3.28)

and the rest is a deduction. Let $m$ be a real number verifying

$$
m < \min \left( -\frac{n}{p}, \frac{n(1 - \alpha)}{q} + \frac{n(\alpha - 2)}{2} \right)
$$

(3.29)

with the convention $1/r = 0$ when $r = \infty$. If $m' = (2m + n(2 - \alpha))/(2\alpha - 2)$, then $m' < -n/q$.

For fixed $\lambda$ in $\hat{U}$, Lemma 3.4 gives a nonzero measurable function $f$ on $M(n)$ satisfying the inequalities

$$e^{a\|x\|^\alpha} | f(x, k) | \leq \text{const.} (1 + \|x\|)^m,$$

$$e^{br\beta} \| \hat{f}(\lambda) \|_{HS} \leq \text{const.} (1 + r)^{m'}.$$

The conditions $m < -n/p$ and $m' < -n/q$ and the fact that $dk$ is a normalized measure imply that $e^{a\|x\|^\alpha} f$ belongs to $L^p(M(n))$ and $e^{br\beta} \| \hat{f}(\lambda) \|_{HS}$ belongs to $L^q(\mathbb{R}^+, C_n r^{n-1} dr)$ for fixed $\lambda$ in $\hat{U}$.

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