More on the Density of Analytic Polynomials in Abstract Hardy Spaces

Alexei Karlovich and Eugene Shargorodsky

Abstract. Let \( \{F_n\} \) be the sequence of the Fejér kernels on the unit circle \( \mathbb{T} \). The first author recently proved that if \( X \) is a separable Banach function space on \( \mathbb{T} \) such that the Hardy-Littlewood maximal operator \( M \) is bounded on its associate space \( X' \), then \( \|f * F_n - f\|_X \to 0 \) for every \( f \in X \) as \( n \to \infty \). This implies that the set of analytic polynomials \( \mathcal{P}_A \) is dense in the abstract Hardy space \( H[X] \) built upon a separable Banach function space \( X \) such that \( M \) is bounded on \( X' \). In this note we show that there exists a separable weighted \( L^1 \) space \( X \) such that the sequence \( f * F_n \) does not always converge to \( f \in X \) in the norm of \( X \). On the other hand, we prove that the set \( \mathcal{P}_A \) is dense in \( H[X] \) under the assumption that \( X \) is merely separable.

Mathematics Subject Classification (2010). Primary 46E30, Secondary 42A10.

Keywords. Banach function space, abstract Hardy space, analytic polynomial, Fejér kernel.

1. Preliminaries and the main results

For \( 0 < p \leq \infty \), let \( L^p := L^p(\mathbb{T}) \) be the Lebesgue space on the unit circle \( \mathbb{T} := \{z \in \mathbb{C} : |z| = 1\} \) in the complex plane \( \mathbb{C} \). For \( f \in L^1 \), let

\[
\hat{f}(n) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta})e^{-in\theta} d\theta, \quad n \in \mathbb{Z},
\]

be the sequence of the Fourier coefficients of \( f \). Let \( X \) be a Banach space continuously embedded in \( L^1 \). Following [13, p. 877], we will consider the abstract Hardy space \( H[X] \) built upon the space \( X \), which is defined by

\[
H[X] := \{f \in X : \hat{f}(n) = 0 \quad \text{for all} \quad n < 0\}.
\]

This work was partially supported by the Fundação para a Ciência e a Tecnologia (Portuguese Foundation for Science and Technology) through the project UID/MAT/00297/2013 (Centro de Matemática e Aplicações).
It is clear that if $1 \leq p \leq \infty$, then $H[L^p]$ is the classical Hardy space $H^p$.

A function of the form

$$q(t) = \sum_{k=0}^{n} \alpha_k t^k, \quad t \in \mathbb{T}, \quad \alpha_0, \ldots, \alpha_n \in \mathbb{C},$$

is said to be an analytic polynomial on $\mathbb{T}$. The set of all analytic polynomials

is denoted by $\mathcal{P}_A$. It is well known that the set $\mathcal{P}_A$ is dense in $H^p$ whenever $1 \leq p < \infty$ (see, e.g., [3, Chap. III, Corollary 1.7(a)]). The density of the set $\mathcal{P}_A$ in the abstract Hardy spaces $H[X]$ was studied by the first author [8] for the case when $X$ is a so-called Banach function space.

Let us recall the definition of a Banach function space. We equip $\mathbb{T}$ with the normalized Lebesgue measure $dm(t) = |dt|/(2\pi)$. Let $L^0$ be the space of all measurable complex-valued functions on $\mathbb{T}$. As usual, we do not distinguish functions which are equal almost everywhere (for the latter we use the standard abbreviation a.e.). Let $L^0_+$ be the subset of functions in $L^0$ whose values lie in $[0, \infty]$. The characteristic function of a measurable set $E \subset \mathbb{T}$ is denoted by $\chi_E$.

Following [1, Chap. 1, Definition 1.1], a mapping $\rho : L^0_+ \to [0, \infty]$ is called a Banach function norm if, for all functions $f, g, f_n \in L^0_+$ with $n \in \mathbb{N}$, for all constants $a \geq 0$, and for all measurable subsets $E$ of $\mathbb{T}$, the following properties hold:

(A1) $\rho(f) = 0 \Leftrightarrow f = 0$ a.e., $\rho(af) = a\rho(f)$, $\rho(f + g) \leq \rho(f) + \rho(g)$,

(A2) $0 \leq g \leq f$ a.e. $\Rightarrow$ $\rho(g) \leq \rho(f)$ (the lattice property),

(A3) $0 \leq f_n \uparrow f$ a.e. $\Rightarrow$ $\rho(f_n) \uparrow \rho(f)$ (the Fatou property),

(A4) $m(E) < \infty \Rightarrow \rho(\chi_E) < \infty$,

(A5) $\int_E f(t) \, dm(t) \leq C_E \rho(f)$

with a constant $C_E \in (0, \infty)$ that may depend on $E$ and $\rho$, but is independent of $f$. When functions differing only on a set of measure zero are identified, the set $X$ of all functions $f \in L^0$ for which $\rho(|f|) < \infty$ is called a Banach function space. For each $f \in X$, the norm of $f$ is defined by $\|f\|_X := \rho(|f|)$. The set $X$ under the natural linear space operations and under this norm becomes a Banach space (see [1, Chap. 1, Theorems 1.4 and 1.6]). If $\rho$ is a Banach function norm, its associate norm $\rho'$ is defined on $L^0_+$ by

$$\rho'(g) := \sup \left\{ \int_{\mathbb{T}} f(t) g(t) \, dm(t) : f \in L^0_+, \, \rho(f) \leq 1 \right\}, \quad g \in L^0_+.$$

It is a Banach function norm itself [1, Chap. 1, Theorem 2.2]. The Banach function space $X'$ determined by the Banach function norm $\rho'$ is called the associate space (Köthe dual) of $X$. The associate space $X'$ can be viewed as a subspace of the (Banach) dual space $X^*$. 
Recall that $L^1$ is a commutative Banach algebra under the convolution multiplication defined for $f, g \in L^1$ by

$$(f * g)(e^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta - i\varphi})g(e^{i\varphi})\,d\varphi, \quad e^{i\theta} \in \mathbb{T}.$$ 

For $n \in \mathbb{N}$, let

$$F_n(e^{i\theta}) := \sum_{k=-n}^{n} \left( 1 - \frac{|k|}{n+1} \right) e^{i\theta k} = \frac{1}{n+1} \left( \frac{\sin \frac{n+1}{2}\theta}{\sin \frac{\theta}{2}} \right)^2, \quad e^{i\theta} \in \mathbb{T},$$

be the $n$-th Fejér kernel. For $f \in L^1$, the $n$-th Fejér mean of $f$ is defined as the convolution $f * F_n$.

Given $f \in L^1$, the Hardy-Littlewood maximal function is defined by

$$(Mf)(t) := \sup_{I \ni t} \frac{1}{m(I)} \int_I |f(\tau)|\,dm(\tau), \quad t \in \mathbb{T},$$

where the supremum is taken over all arcs $I \subset \mathbb{T}$ containing $t \in \mathbb{T}$. The operator $f \mapsto Mf$ is called the Hardy-Littlewood maximal operator.

**Theorem 1.1** ([8, Theorem 3.3]). Suppose $X$ is a separable Banach function space on $\mathbb{T}$. If the Hardy-Littlewood maximal operator is bounded on the associate space $X'$, then for every $f \in X$,

$$\lim_{n \to \infty} \|f * F_n - f\|_X = 0. \quad (1.1)$$

It is well known that for $f \in L^1$ one has

$$(f * F_n)(e^{i\theta}) = \sum_{k=-n}^{n} \hat{f}(k) \left( 1 - \frac{|k|}{n+1} \right) e^{i\theta k}, \quad e^{i\theta} \in \mathbb{T}$$

(see, e.g., [9, Chap. I]). This implies that if $f \in H[X] \subset H[L^1] = H^1$, then $f * F_n \in \mathcal{P}_A$. Combining this observation with Theorem 1.1 we arrive at the following.

**Corollary 1.2** ([8, Theorem 1.2]). Suppose $X$ is a separable Banach function space on $\mathbb{T}$. If the Hardy-Littlewood maximal operator $M$ is bounded on its associate space $X'$, then the set of analytic polynomials $\mathcal{P}_A$ is dense in the abstract Hardy space $H[X]$ built upon the space $X$.

Note that if a Banach function space $X$ is, in addition, rearrangement-invariant then the requirement of the boundedness of $M$ on the space $X'$ can be omitted in Corollary 1.2 (see [8, Theorem 1.1] or [11, Lemma 1.3(c)]). Leśnik [10] conjectured that the same fact should be true for arbitrary, not necessarily rearrangement-invariant, Banach function spaces.

In this note, we first observe that Theorem 1.1 does not hold for arbitrary separable Banach function spaces. For a function $K \in L^1$, consider the convolution operator $C_K$ with kernel $K$ defined by

$$C_Kf = f * K, \quad f \in L^1.$$
It follows from \cite[Theorem 2]{12} that there exists a continuous function 
\( p : \mathbb{T} \to [1, \infty) \) such that the sequence of the convolution operators \( C_{F_n} \) is 
not uniformly bounded in the variable Lebesgue space \( L^{p(\cdot)} \) defined as the set of all \( f \in L^0 \) such that 
\[
\int_{\mathbb{T}} |f(t)|^{p(t)} \, dm(t) < \infty.
\]
It is well known (see, e.g., \cite[Proposition 2.12, Theorem 2.78, Section 2.10.3]{4}) that if 
\( p : \mathbb{T} \to [1, \infty) \) is continuous, then \( L^{p(\cdot)} \) is a separable Banach function 
space equipped with the norm 
\[
\|f\|_{L^{p(\cdot)}} = \inf \left\{ \lambda > 0 : \int_{\mathbb{T}} \left| \frac{f(t)}{\lambda} \right|^ {p(t)} \, dm(t) \leq 1 \right\}.
\]
Since the norms of the convolution operators \( C_{F_n} \) may not be uniformly 
bounded on \( L^{p(\cdot)} \), the standard argument, based on the uniform boundedness 
principle, leads us to the following.

**Theorem 1.3.** There exist a separable Banach function space \( X \) on \( \mathbb{T} \) and a 
function \( f \in X \) such that (1.1) is not fulfilled.

We show that the separable Banach function space in Theorem 1.3 can be chosen as a weighted \( L^1 \) space, that is, the techniques of variable Lebesgue spaces can be omitted.

**Theorem 1.4 (Main result 1).** There exist a nonnegative function \( w \in L^1 \) 
such that \( w^{-1} \in L^\infty \) and a function \( f \) in the separable Banach function space 
\( X = L^1(w) = \{ f \in L^0 : fw \in L^1 \} \) such that (1.1) is not fulfilled.

In spite of the observation made in Theorems 1.3 and 1.4 we show that the requirement of the boundedness of the Hardy-Littlewood maximal operator \( M \) on the associate space \( X' \) of a separable Banach function space \( X \) in Corollary 1.2 can be omitted. Thus, Leśnik’s conjecture \cite{10} is, indeed, true.

**Theorem 1.5 (Main result 2).** If \( X \) is a separable Banach function space on \( \mathbb{T} \), then the set of analytic polynomials \( \mathcal{P}_A \) is dense in the abstract Hardy space \( H[X] \) built upon the space \( X \).

The paper is organized as follows. In Section 2 we prove that a convolution 
operator \( C_K \) with a nonnegative symmetric kernel \( K \in L^1 \) is bounded on 
a Banach function space \( X \) if and only if it is bounded in its associate space \( X' \). Further, we consider a special weight \( w \in L^1 \) such that \( w^{-1} \in L^\infty \). Then 
\( X = L^1(w) \) is a separable Banach function space with the associate space 
\( X' = L^\infty(w^{-1}) \). We show that the sequence of convolution operators \( \{ C_{K_n} \} \) 
with nonnegative bounded symmetric kernels \( K_n \), satisfying \( \| K_n \|_{L^1} = 1 \) and 
a natural localization property, is not uniformly bounded on \( X' = L^\infty(w^{-1}) \), 
and therefore, on its associate space \( X'' = X = L^1(w) \). Applying this result
to the sequence of the Fejér kernels \( \{F_n\} \), we prove Theorem 1.4 with the aid of the uniform boundedness principle.

In Section 3, we recall that the separability of a Banach function space \( X \) is equivalent to \( X^* = X' \). Further, we collect some facts on the identification of the Hardy spaces \( H^p \) on the unit circle and the Hardy spaces \( H^p(\mathbb{D}) \) of analytic functions in the unit disk \( \mathbb{D} \). Finally, we give the proof of Theorem 1.5 based on the application of the Hahn-Banach theorem, a corollary of the Smirnov theorem and properties of the identification of \( H^1 \) with \( H^1(\mathbb{D}) \).

2. Proof of the first main result

2.1. Norms of convolution operators on \( X \) and on its associate space \( X' \)

The Banach space of all bounded linear operator on a Banach space \( E \) is denoted by \( B(E) \).

Lemma 2.1. Let \( X \) be a Banach function space on \( \mathbb{T} \) and \( K \in L^1 \) be a nonnegative function such that \( K(e^{i\theta}) = K(e^{-i\theta}) \) for almost all \( \theta \in [-\pi, \pi] \). Then the convolution operator \( C_K \) is bounded on the Banach function \( X \) if and only if it is bounded on its associate space \( X' \). In that case

\[
\|C_K\|_{B(X')} = \|C_K\|_{B(X)}.
\] (2.1)

Proof. Suppose \( C_K \) is bounded on \( X' \). Fix \( f \in X \setminus \{0\} \). Since \( K \geq 0 \), we have \( |f \ast K| \leq |f| \ast K \). According to the Lorentz-Luxemburg theorem (see, e.g., [1, Chap. 1, Theorem 2.7]), \( X = X'' \) with equality of the norms. Hence

\[
\|f \ast K\|_X \leq \|f| \ast K\|_X = \|f| \ast K\|_{X''} = \sup \left\{ \int_{\mathbb{T}} (|f| \ast K)(t)|g(t)| \, dm(t) : g \in X', \|g\|_{X'} \leq 1 \right\}.
\]

Then for every \( \varepsilon > 0 \) there exists a function \( h \in X' \) such that \( h \geq 0 \), \( \|h\|_{X'} \leq 1 \), and

\[
\|f \ast K\|_X \leq (1 + \varepsilon) \int_{\mathbb{T}} (|f| \ast K)(t)h(t) \, dm(t).
\] (2.2)

Taking into account that \( K(e^{i\theta}) = K(e^{-i\theta}) \) for almost all \( \theta \in \mathbb{R} \), by Fubini’s theorem, we get

\[
\int_{\mathbb{T}} (|f| \ast K)(t)h(t) \, dm(t) = \int_{\mathbb{T}} (h \ast K)(t)|f(t)| \, dm(t).
\]

From this identity, Hölder’s inequality for \( X \) (see, e.g., [1, Chap. 1, Theorem 2.4]), and the boundedness of \( C_K \) on \( X' \), we obtain

\[
\int_{\mathbb{T}} (|f| \ast K)(t)h(t) \, dm(t) = \|f\|_X \|h \ast K\|_{X'} \leq \|f\|_X \|C_K\|_{B(X')}.
\] (2.3)

It follows from (2.2)–(2.3) that

\[
\|C_K\|_{B(X')} = \sup_{f \in X, f \neq 0} \frac{\|f \ast K\|_X}{\|f\|_X} \leq (1 + \varepsilon)\|C_K\|_{B(X')}.
\]
for every $\varepsilon > 0$, which implies the boundedness of $C_K$ on $X$ and the inequality
\[
\|C_K\|_{B(X)} \leq \|C_K\|_{B(X')}.
\] (2.4)

If $C_K$ is bounded on $X$, then using the Lorentz-Luxemburg theorem and (2.4) with $X'$ in place of $X$, we obtain that $C_K$ is bounded on $X'$ and
\[
\|C_K\|_{B(X')} \leq \|C_K\|_{B(X')} = \|C_K\|_{B(X)}.
\] (2.5)

Combining (2.4)–(2.5), we arrive at (2.1). □

2.2. Spaces $L^1(w)$ and $L^\infty(w^{-1})$ with a special weight $w$

Lemma 2.2. Let
\[
w(e^{i\theta}) := \begin{cases} \sqrt{m}, & \frac{\pi}{2m} \leq |\theta| \leq \frac{\pi}{2m-1}, \ m \in \mathbb{N}, \\ 1, & \frac{\pi}{2m+1} < |\theta| < \frac{\pi}{2m}, \ m \in \mathbb{N}. \end{cases}
\] (2.6)

Then the spaces
\[L^1(w) = \{f \in L^0 : fw \in L^1\}, \ L^\infty(w^{-1}) = \{f \in L^0 : fw^{-1} \in L^\infty\}\]
are Banach function spaces on $\mathbb{T}$ with respect to the norms
\[
\|f\|_{L^1(w)} = \|fw\|_{L^1}, \ \|f\|_{L^\infty(w^{-1})} = \|fw^{-1}\|_{L^\infty},
\]
and $(L^1(w))' = L^\infty(w^{-1})$. Moreover, the space $L^1(w)$ is separable.

Proof. It is clear that $w^{-1} \in L^\infty$ and, since
\[
\|w\|_{L^1} = \frac{1}{2\pi} \int_{-\pi}^{\pi} w(e^{i\theta}) \, d\theta = \sum_{m=1}^{\infty} \left( \frac{1}{2m} - \frac{1}{2m+1} \right) + \sum_{m=1}^{\infty} \sqrt{m} \left( \frac{1}{2m-1} - \frac{1}{2m} \right) < \infty,
\] (2.7)
we also have $w \in L^1$. Then it follows from [7, Lemma 2.5] that $L^1(w)$ and $L^\infty(w^{-1})$ are Banach function spaces and $(L^1(w))' = L^\infty(w^{-1})$. Finally, the separability of the space $L^1(w)$ follows from [4, Proposition 2.6] and [1, Chap. 1, Corollary 5.6]. □

2.3. Norms of convolution operators are not uniformly bounded on the spaces $L^1(w)$ and $L^\infty(w^{-1})$ with the special weight $w$

Theorem 2.3. Let $\{K_n\}$ be a sequence of bounded functions $K_n : \mathbb{T} \to \mathbb{C}$ such that
\[
K_n(e^{i\theta}) \geq 0, \quad K_n(e^{i\theta}) = K_n(e^{-i\theta}) \text{ a.e. on } [-\pi, \pi],
\] (2.8)
\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(e^{i\theta}) \, d\theta = 1,
\] (2.9)
and
\[
\lim_{n \to \infty} \sup_{\varepsilon \leq |\theta| \leq \pi} K_n(e^{i\theta}) = 0 \text{ for each } \varepsilon > 0.
\] (2.10)
If $w$ is the weight given by (2.6), then the convolution operators $C_{K_n}$ are bounded on $L^\infty(w^{-1})$ and on $L^1(w)$ for all $n \in \mathbb{N}$, however,

$$\sup_{n \in \mathbb{N}} \|C_{K_n}\|_{B(L^\infty(w^{-1}))} = \infty,$$

(2.11)

$$\sup_{n \in \mathbb{N}} \|C_{K_n}\|_{B(L^1(w))} = \infty.$$  

(2.12)

\textbf{Proof.} By (2.6)–(2.7), $w \in L^1$ and $w^{-1} \in L^\infty$. Therefore, for every $n \in \mathbb{N},$

$$\|C_{K_n}f\|_{L^1(w)} \leq \frac{1}{2\pi} \left\| \int_{-\pi}^{\pi} K_n(e^{i\theta}) |f(e^{i\theta})| d\theta \right\|_{L^1(w)}$$

$$\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \left\| K_n(e^{i\theta}) \right\|_{L^1(w)} |f(e^{i\theta})| d\theta$$

$$\leq \frac{1}{2\pi} \|K_n\|_{L^\infty} \|w\|_{L^1} \|f\|_{L^1}$$

$$= \frac{1}{2\pi} \|K_n\|_{L^\infty} \|w\|_{L^1} \|w^{-1}f\|_{L^1}$$

$$\leq \frac{1}{2\pi} \|K_n\|_{L^\infty} \|w\|_{L^1} \|w^{-1}\|_{L^\infty} \|f\|_{L^1(w)}.$$

Hence

$$\|C_{K_n}\|_{B(L^1(w))} \leq \frac{1}{2\pi} \|K_n\|_{L^\infty} \|w\|_{L^1} \|w^{-1}\|_{L^\infty}, \quad n \in \mathbb{N}.$$

It follows from (2.8) and Lemmas [2.1]–[2.2] that the operators $C_{K_n}$ are bounded on $L^\infty(w^{-1})$ for all $n \in \mathbb{N}$. Moreover, (2.11) implies (2.12).

Let us prove (2.11). Consider the sequence

$$v_m(e^{i\theta}) := \begin{cases} \sqrt{m}, & \frac{\pi}{2m} \leq \theta \leq \frac{\pi}{2m-1}, \\ 0, & \theta \in [-\pi, \pi] \setminus \left[ \frac{\pi}{2m}, \frac{\pi}{2m-1} \right], \end{cases} \quad m \in \mathbb{N}.$$

Then it follows from (2.6) that $\|v_m\|_{L^\infty(w^{-1})} = 1$ for all $m \in \mathbb{N}$.

Fix $m \in \mathbb{N}$. According to (2.9) and the localization property (2.10), there exists $n(m) \in \mathbb{N}$ such that

$$\int_{-\frac{\pi}{(2m)^2}}^{\frac{\pi}{(2m)^2}} K_n(e^{i\theta}) d\theta = \frac{1}{2} \int_{-\frac{\pi}{(2m)^2}}^{\frac{\pi}{(2m)^2}} K_n(e^{i\theta}) d\theta \geq \frac{1}{3} \quad \text{for all} \quad n \geq n(m).$$

Since $K_n \in L^1$, for every $n \geq n(m)$, there exists $\delta_n > 0$ such that

$$\int_{-\frac{\pi}{(2m)^2}}^{\frac{\pi}{(2m)^2}} K_n(e^{i\theta}) d\theta \geq \frac{1}{4},$$

$$n \in \mathbb{N}.$$
Therefore, for almost all $\vartheta \in \left[\frac{\pi}{2}m - \delta_n, \frac{\pi}{2}m\right]$, one gets

$$(C_{K_n} v_m) (e^{i\vartheta}) = \frac{\sqrt{m}}{2\pi} \int_{\frac{\pi}{2m}}^{\frac{\pi}{2m} + \frac{\pi}{(2m)^2}} K_n (e^{i\vartheta - i\theta}) \, d\theta \geq \frac{\sqrt{m}}{2\pi} \int_{\frac{\pi}{2m} - \frac{\pi}{(2m)^2}}^{\frac{\pi}{2m}} K_n (e^{i\eta}) \, d\eta \geq \frac{\sqrt{m}}{2\pi} \int_{\frac{-\pi}{(2m)^2}}^{-\frac{\pi}{2m}} K_n (e^{i\eta}) \, d\eta \geq \frac{\sqrt{m}}{2\pi} \int_{\frac{-\pi}{(2m)^2}}^{\frac{\pi}{(2m)^2}} K_n (e^{i\eta}) \, d\eta \geq \frac{\sqrt{m}}{8\pi}.$$  \hspace{1cm} (2.13)

In view of (2.6), $w(e^{i\vartheta}) = 1$ for all $\vartheta \in \left(\max\left\{\frac{\pi}{2m} - \delta_n, \frac{\pi}{2m+1}\right\}, \frac{\pi}{2m}\right)$. Hence, it follows from (2.13) that

$$\|C_{K_n} v_m\|_{L^\infty(w^{-1})} \geq \frac{\sqrt{m}}{8\pi} \quad \text{for all} \quad n \geq n(m),$$

while $\|v_m\|_{L^\infty(w^{-1})} = 1$. So

$$\|C_{K_n}\|_{B(L^\infty(w^{-1}))} \geq \frac{\sqrt{m}}{8\pi} \quad \text{for all} \quad n \geq n(m).$$

Since $m \in \mathbb{N}$ is arbitrary, the latter inequality immediately implies (2.11). \hspace{1cm} \Box

2.4. Proof of Theorem 1.4

Let $X = L^1(w)$, where $w$ is the weight given by (2.6). By Lemma 2.2, $X$ is a separable Banach function space. It is well known (and not difficult to check) that the sequence $\{F_n\}$ of the Fejér kernels is a sequence of bounded functions satisfying (2.8)–(2.10). By Theorem 2.3, the operators $C_{F_n}$ are bounded on $X$ for every $n \in \mathbb{N}$.

Assume that (1.1) is fulfilled for all $f \in X$. Then, for all $f \in X$, the sequence $\{C_{F_n} f\}$ is bounded in $X$. Therefore, by the uniform boundedness principle, the sequence $\{\|C_{F_n}\|_{B(X)}\}$ is bounded, but this contradicts (2.12). Thus, there exists a function $f \in X$ such that (1.1) does not hold. \hspace{1cm} \Box

3. Proof of the second main result

3.1. Separable Banach function spaces $X$ are spaces for which $X^*$ is isometrically isomorphic to $X'$

Combining [1, Chap. I, Corollaries 4.3 and 5.6] and observing that the measure $dm$ is separable (for the definition of a separable measure, see, e.g., [1, p. 27] or [3, Chap. I, Section 6.10]), we arrive at the following.

Theorem 3.1. Let $X$ be a Banach function space on $\mathbb{T}$. Then $X$ is separable if and only if its dual space $X^*$ is isometrically isomorphic to the associate space $X'$. 

3.2. Hardy spaces on the unit disk

Let $D$ denote the open unit disk in the complex plane $\mathbb{C}$. Recall that a function $F$ analytic in $D$ is said to belong to the Hardy space $H^p(D)$, $0 < p \leq \infty$, if the integral mean

$$M_p(r, F) = \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} |F(re^{i\theta})|^p \, d\theta \right)^{1/p}, \quad 0 < p < \infty,$$

$$M_\infty(r, F) = \max_{-\pi \leq \theta \leq \pi} |F(re^{i\theta})|,$$

remains bounded as $r \to 1$. If $F \in H^p(D)$, $0 < p \leq \infty$, then the nontangential limit

$$f(e^{i\theta}) = \lim_{r \to 1^-} F(re^{i\theta})$$

exists for almost all $\theta \in [-\pi, \pi]$ (see, e.g., [5, Theorem 2.2]) and the boundary function $f = f(e^{i\theta})$ belongs to $L^p$.

The following lemma is an immediate consequence of the Smirnov theorem (see, e.g., [5, Theorem 2.1]).

**Lemma 3.2.** If $F \in H^p(D)$ for some $p \in (0, 1)$ and its boundary function $f$ belongs to $L^1$, then $F \in H^1(D)$.

Recall that if $f \in H^1$ then its analytic extension $F$ into $D$, given by the Poisson integral

$$F(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P(r, \theta - \varphi) f(e^{i\varphi}) \, d\varphi, \quad 0 \leq r < 1, \quad -\pi \leq \theta \leq \pi,$$

where

$$P(r, \theta) = \frac{1 - r^2}{1 - 2r\cos \theta + r^2}, \quad 0 \leq r < 1, \quad -\pi \leq \theta \leq \pi,$$

is the Poisson kernel, belongs to $H^1(D)$ and the boundary function of $F$ coincides with $f$ a.e. on $T$ (see, e.g., [5, Theorem 3.1]).

It is important to note that the Taylor coefficients of $F \in H^p(D)$ coincide with the Fourier coefficients of its boundary function $f \in L^p$. More precisely, one has the following.

**Theorem 3.3 ([5 Theorem 3.4]).** Let $F(z) = \sum_{n=0}^{\infty} a_n z^n$ belong to $H^1(D)$ and let $\{\hat{f}(n)\}$ be the sequence of the Fourier coefficients of its boundary function $f \in L^1$. Then $\hat{f}(n) = a_n$ for all $n \geq 0$ and $\hat{f}(n) = 0$ for $n < 0$.

3.3. Proof of Theorem 1.5

Suppose $P_A$ is not dense in $H[X]$. Take any function $f \in H[X]$ that does not belong to the closure of $P_A$ with respect to the norm of $X$. Since $X$ is separable, it follows from Theorem 3.1 that $X^*$ is isometrically isomorphic to $X'$. Then, by a corollary of the Hahn-Banach theorem (see, e.g., [2 Chap. 7, Theorem 4.1]), there exists a function $g \in X' \subset L^1$ such that

$$\int_{-\pi}^{\pi} f(e^{i\theta}) g(e^{i\theta}) \, d\theta \neq 0 \quad (3.1)$$
and
\[ \int_{-\pi}^{\pi} p(e^{i\theta}) g(e^{i\theta}) \, d\theta = 0 \quad \text{for all} \quad p \in \mathcal{P}_A. \]
In particular, if \( p(e^{i\theta}) = e^{in\theta} \) with \( n = 0, 1, 2, \ldots \), then
\[ \hat{g}(-n) = 0 \quad \text{for all} \quad n = 0, 1, 2, \ldots \]  \hspace{1cm} (3.2)
Hence \( g \in H[X'] \subset H^1 \). For functions \( f \in H[X] \subset H^1 \) and \( g \in H[X'] \subset H^1 \), let \( F \) and \( G \) denote their analytic extensions to the unit disk \( \mathbb{D} \) by means of their Poisson integrals. Then \( F, G \in H^1(\mathbb{D}) \). It follows from (3.2) and Theorem 3.3 that \( G(0) = 0 \). Since \( F, G \in H^1(\mathbb{D}) \), by Hölder’s inequality, \( FG \in H^{1/2}(\mathbb{D}) \). On the other hand, since \( f \in X \) and \( g \in X' \), it follows from Hölder’s inequality for Banach function spaces (see [1, Chap. 1, Theorem 2.4]) that \( fg \in L^1 \). Then it follows from Lemma 3.2 that \( FG \in H^1(\mathbb{D}) \). Since \( (FG)(0) = F(0)G(0) = 0 \), applying Theorem 3.3 to \( FG \), we obtain \( \hat{f}g(0) = 0 \), that is,
\[ \int_{-\pi}^{\pi} f(e^{i\theta}) g(e^{i\theta}) \, d\theta = 0, \]
which contradicts (3.1). \( \square \)

Acknowledgment
We would like to thank the referee for the useful remarks.

References
[1] C. Bennett and R. Sharpley, Interpolation of Operators. Academic Press, Boston, 1988.
[2] Yu. M. Berezansky, Z. G. Sheftel, and G. F. Us, Functional Analysis, Vol. 1, Birkhäuser, Basel, 1996.
[3] J. B. Conway, The Theory of Subnormal Operators. American Mathematical Society, Providence, RI, 1991.
[4] D. Cruz-Uribe and A. Fiorenza, Variable Lebesgue Spaces. Birkhäuser, Basel, 2013.
[5] P. L. Duren, Theory of \( H^p \) Spaces. Academic Press, New York and London, 1970.
[6] L. V. Kantorovich and G. P. Akilov, Functional Analysis. Pergamon Press, Oxford, 2nd ed., 1982.
[7] A. Karlovich, Fredholmness of singular integral operators with piecewise continuous coefficients on weighted Banach function spaces. J. Integral Equations Appl. 15 (2003), 263–320.
[8] A. Karlovich, Density of analytic polynomials in abstract Hardy spaces. Comment. Math., to appear. Preprint is available at arXiv:1710.10078 [math.CA] (2017).
[9] Y. Katznelson, An Introduction to Harmonic Analysis, Dower Publications, Inc., New York, 1976.
[10] K. Lešnik, Personal communication to A. Karlovich. February 23, 2017.
[11] K. Leśnik, *Toeplitz and Hankel operators between distinct Hardy spaces*. arXiv:1708.00910 [math.FA] (2017).

[12] I. I. Sharapudinov, *Uniform boundedness in $L^p (p = p(x))$ of some families of convolution operators*. Math. Notes 59 (1996), 205–212.

[13] Q. Xu, *Notes on interpolation of Hardy spaces*. Ann. Inst. Fourier 42 (1992), 875–889.

Alexei Karlovich  
Centro de Matemática e Aplicações,  
Departamento de Matemática,  
Faculdade de Ciências e Tecnologia,  
Universidade Nova de Lisboa,  
Quinta da Torre,  
2829-516 Caparica, Portugal  
e-mail: oyk@fct.unl.pt

Eugene Shargorodsky  
Department of Mathematics  
King’s College London  
Strand, London WC2R 2LS  
United Kingdom  
e-mail: eugene.shargorodsky@kcl.ac.uk