Modified commutation relationships from the Berry-Keating program

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Abstract

Current approaches to quantum gravity suggest there should be a modification of the standard quantum mechanical commutator, $[\hat{x}, \hat{p}] = i\hbar$. Typical modifications are phenomenological and designed to result in a minimal length scale. As a motivating principle for the modification of the position and momentum commutator, we assume the validity of a version of the Bender-Brody-Müller variant of the Berry-Keating approach to the Riemann hypothesis. We arrive at a family of modified position and momentum operators, and their associated modified commutator, which lead to a minimal length scale. Additionally, this larger family generalizes the Bender-Brody-Müller approach to the Riemann hypothesis.

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I. INTRODUCTION

At present there is no agreed upon approach to quantizing gravity. However, there are general arguments that no matter what final form quantum gravity takes, it should have some non-zero minimal distance scale $\Delta x_0$. String theory based arguments lead to such a minimum absolute length scale (see [1] and the references therein for a survey). Many works have shown how a modification of the standard quantum commutator, $[\hat{x}, \hat{p}] = i\hbar \rightarrow [\hat{x}, \hat{p}] = i$ leads to a minimal length scale. There are also works which propose a minimal length scale by modifying the standard spatial and/or momentum commutators by allowing $[\hat{x}_i, \hat{x}_j]$ and/or $[\hat{p}_i, \hat{p}_j]$ to be non-zero. The two approaches of modifying either $[\hat{x}, \hat{p}]$ or $[\hat{x}_i, \hat{x}_j]$ and/or $[\hat{p}_i, \hat{p}_j]$ are related. In this work we will focus on the introduction of a minimal length scale via a modification of $[\hat{x}, \hat{p}] = i$.

In [5], a very simple modification of the quantum commutation relationship between the position operator ($\hat{x}$) and momentum operator ($\hat{p}$) was proposed of the form

$$[\hat{x}, \hat{p}] = i(1 + \beta \hat{p}^2),$$

where $\beta$ is an arbitrary parameter which is assumed to come from quantum gravity. Using this and the standard relationship between the quantum commutators and uncertainties gave

$$\Delta x \Delta p \geq \frac{1}{2} (1 + \beta \Delta p^2 + \beta \langle \hat{p} \rangle^2),$$

which in turn gave a minimal distance of $\Delta x_0 = \sqrt{\beta}$. One criticism of this approach is that it is purely phenomenological, bottom-up. The parameter $\beta$ is not determined, and even the specific form of the modified commutation relationship in (1) is simply an assumption. Having an undetermined parameter such as $\beta$ is similar to the introduction of the reduced Planck's constant, $\hbar$, which was originally introduced as a parameter to fit the observed blackbody spectrum.

In this work we propose a less phenomenological and more top-down approach to obtain a modified commutation relationship. Our method is motivated by the Bender-Brody-Müller approach to the Riemann hypothesis. The Riemann hypothesis deals with the non-trivial zeros of the Riemann zeta function and is connected with the distribution of prime numbers. The Riemann zeta function is given by

$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots$

In this paper we take $\hbar = 1$. 

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\[ \zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} = \frac{1}{\Gamma(z)} \int_0^{\infty} \frac{t^{z-1}}{e^t - 1} dt. \]  

(3)

\[ \Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt \]

is the usual gamma function. Using the integral expression in (3) one obtains a reflection formula for the Riemann zeta function

\[ \zeta(z) = 2^z \pi^{z-1} \sin(\pi z/2) \Gamma(1-z) \zeta(1-z). \]  

(4)

From (4) one can see that the Riemann zeta function has trivial zeros at the negative even integers, \( z = -2n \) due to the \( \sin(\pi z/2) \) term. Riemann noticed that there were also non-trivial zeros which occurred along the line \( \text{Re}(z) = \frac{1}{2} \). Specifically there were zeroes at the complex values \( z_n = \frac{1}{2} + it_n \) where \( n = 1, 2, 3, \ldots \) and \( t_1 = 14.135 \), \( t_2 = 21.022 \), \( t_3 = 25.011 \) etc. The Riemann hypothesis states that all of these nontrivial zeros lie on this line \( z = \frac{1}{2} + it \).

From the discrete nature of the imaginary part of the non-trivial zeros of the Riemann zeta function, it was conjectured that these non-trivial zeros were related to an eigenvalue problem. The general suggestion is there exists some operator, \( \hat{H} \), whose eigenvalues are the imaginary parts of the non-trivial zeros of the Riemann zeta function. This is called the Hilbert-Polya conjecture. The operator \( \hat{H} \) is called the “Hamiltonian”, although it is not connected with the energy of any system. Berry [12] and Keating [13] suggested a version of this proposal where the quantum version of the operator \( \hat{H} \) should reduce to the classical operator \( \hat{H} = xp \). One proposal is to take \( \hat{H} = \frac{1}{2}(\hat{x}\hat{p} + \hat{p}\hat{x}) \) thus taking into account that the order of the quantum operators \( \hat{x}\hat{p} \) matters. This form of \( \hat{H} \) is proportional to the one dimensional virial operator \(^3\) which is the generator for scaling/dilation transformations. Since our aim is to introduce a modification of the position and momentum operators that leads to a minimal length, the appearance of a modified dilation symmetry is natural.

Our proposal is to modify \( \hat{x} \) and \( \hat{p} \) so that they align with the recent attempt of Bender-Brody-Müller to address the Riemann hypothesis through the Berry-Keating program. The modified Hamiltonian proposed by Bender-Brody-Müller [10] is

\(^2\) The first hundred non-trivial zeros can be found at http://www.dtc.umn.edu/~odlyzko/zeta_tables/zeros2.

\(^3\) The virial operator is \( A = \frac{1}{2}(\hat{x}\hat{p} + \hat{p}\hat{x}) \), in terms of which the dilation transformation is \( D = e^{i\theta A} \) with \( \theta \) being some scaling parameter.
\[ \hat{H} = \hat{\Delta}^{-1}_{BBM}(\hat{x}\hat{p} + \hat{p}\hat{x})\hat{\Delta}_{BBM}, \]  

(5)

where \( \hat{\Delta}_{BBM} = 1 - e^{-i\hat{p}} \). When applied to an analytic function \( f(x) \), this is the unit difference operator, \( \hat{\Delta}_{BBM}f(x) = f(x) - f(x - 1) \). Taking into account that \( \hat{x} \) and \( \hat{p} \) satisfy the standard commutator relation \( [\hat{x}, \hat{p}] = i \), we “walk” the operator \( \hat{\Delta}_{BBM} \) through to the left and annihilate it with its inverse operator \( \hat{\Delta}^{-1}_{BBM} \). The Hamiltonian in (5) becomes

\[ \hat{H} = (\hat{x}\hat{p} + \hat{p}\hat{x}) - \frac{2\hat{p}e^{-i\hat{p}}}{(1 - e^{-i\hat{p}})}. \]  

(6)

We use this Hamiltonian to motivate a family of similar Hamiltonians, which lead to a family of modified position and momentum operators and their modified commutator. The modified operators, which we give below, are symmetric and lead to a minimum length scale similar to [5]. These operators are also symmetric in an inner-product space which requires the wave function to decay exponentially in the large momentum limit. These Hamiltonians satisfy a similar approach to the Riemann hypothesis as that suggested by Bender-Brody-Müller [10]. There are important open questions [14] and additional discussion [15] concerning the Bender-Brody-Müller approach to the Riemann hypothesis. We do not resolve the major criticism of “What is the Hilbert space used in the construction in reference [10]?” The family of Hamiltonians we present may provide alternative avenues to resolving this criticism. However, our main goal here is to use the Hamiltonians, such as given in equation (6), to give a top-down motivation for a modified commutation relationship between position and momentum.

II. MODIFIED POSITION AND MOMENTUM COMMUTATOR MOTIVATED BY THE BENDER-BRODY-MÜLLER HAMILTONIAN

We begin by writing down modified position and momentum operators of the form

\[ \hat{x}' = i(1 + g(p))\partial_p; \quad \hat{p}' = p(1 + f(p)). \]  

(7)

The functions, \( g(p) \) and \( f(p) \), will be fixed by the requirement that the new Hamiltonian, \( \hat{x}'\hat{p}' + \hat{p}'\hat{x}' \), with the new operators from (7), leads to the same Hamiltonian as in (6). We are working in momentum space since the extra term in \( \hat{H} \) from (6) involves only the momentum operator.
Using our modified position and momentum operators from (7) we find that the new Hamiltonian operator becomes

$$\hat{H} = (\hat{x}'\hat{p}' + \hat{p}'\hat{x}')$$

$$= [2ip(1 + f(p))(1 + g(p))\partial_p + i] + i(1 + g(p))(f(p) + pf'(p)) + ig(p).$$

The first term in (8) (i.e. $2ip(1 + g(p))(1 + f(p))\partial_p + i$) should correspond to the first term in (6) (i.e. $(\hat{x}\hat{p} + \hat{p}\hat{x}) = 2ip\partial_p + i$, using $\hat{x} = i\partial_p$ and $\hat{p} = p$). This correspondence is accomplished by requiring $(1 + f(p))(1 + g(p)) = 1$, i.e.

$$g(p) = \frac{-f(p)}{1 + f(p)}. \quad (9)$$

With this $g(p)$ the remaining terms in (8) become

$$i(1 + g(p))(f(p) + pf'(p)) + ig(p) = \frac{ipf'(p)}{1 + f(p)}.$$  

We determine $f(p)$ by requiring the above expression equal the last term in (6) yielding

$$\frac{ipf'(p)}{1 + f(p)} = -\frac{2pe^{-ip}}{1 - e^{-ip}} \rightarrow \frac{d}{dp}\ln(1 + f(p)) = \frac{2ie^{-ip}}{1 - e^{-ip}}. \quad (10)$$

Equation (10) is straightforward to solve and yields the solution

$$1 + f(p) = C(1 - e^{-ip})^2. \quad (11)$$

from which it follows

$$1 + g(p) = \frac{1}{C(1 - e^{-ip})^2}. \quad (12)$$

Using the modified position and momentum operators from equations (7), (11), and (12), we find that the associated modified commutator becomes

$$[\hat{x}', \hat{p}'] = i \left(1 + \frac{pf'(p)}{1 + f(p)}\right) = i + \frac{2p}{1 - e^{ip}}, \quad (13)$$

where we have used the expression for $g(p)$ from (9) to get to the intermediate form, and to obtain the final form we used (11). The first term, $i$, is the standard commutator, and the second term, $\frac{2p}{1 - e^{ip}}$, is the modification coming from the deformation of the position and momentum operators. It is this second term which represents the change that we associate with a modification of short distance/large momentum behavior coming from quantum gravity.
Equation (13) is the modification of the quantum commutator implied by the requirement that the modified position and momentum operators from equations (7), (11), and (12) give the Hamiltonian in (5) or (6).

We now impose the physical requirement that one should recover the standard operators in the low momentum limit, i.e. \( g(p), f(p) \to 0 \) as \( p \to 0 \). It is easy to see from equations (11) and (12) that \( f(p) \to -1 \) and \( g(p) \) diverges as \( p \to 0 \). Furthermore, as \( p \to 0 \) we see that \( \frac{2p}{1-e^{ip}} \to 2i \). In the limit \( p \to 0 \) the commutator in (13) becomes \( [\hat{x}', \hat{p}'] \to 3i \) which is not correct. Despite this initial failure we now ask if we can modify the Hamiltonian in (5) or (6) to get a modified commutator with the correct physical limit as \( p \to 0 \), while still preserving the potential approach to the Riemann hypothesis proposed in [10].

We begin by finding a new \( \hat{\Delta} \) which differs from the \( \hat{\Delta}_{BBM} \) from (5) and (6) and which will give more physical behavior in the \( p \to 0 \) limit. We will also need to check that this new \( \Delta \) still allows for the construction given in [10]. One problem with the original construction is that \( \hat{\Delta}_{BBM} = 1-e^{-ip} \to 0 \) as \( p \to 0 \). To avoid this, we could take a \( + \) sign so \( \hat{\Delta} = 1+e^{-ip} \). This new operator is a kind of averaging transformation of a function between points \( x \) and \( x-1 \) rather than a difference operator. Applying \( 1+e^{-ip} \) to a function, \( f(x) \), one finds \( \hat{\Delta} f(x) = f(x) + f(x-1) \). To make this a true averaging between the points \( x \) and \( x-1 \), we should divide by \( \frac{1}{2} \).

We will consider a \( \hat{\Delta} \) which is symmetric in \( p \): \( \hat{\Delta} = \frac{1}{2}(e^{kp} + e^{-kp}) = \cosh(kp) \), where \( k \) could be a real, imaginary, or complex constant. If we take \( k = i \) then this \( \hat{\Delta} \) when applied to \( f(x) \) would give the average of this function between \( x+1 \) and \( x-1 \). This choice of \( \hat{\Delta} \) takes a symmetric average of the wave function at values near \( x \). If there is a minimum length scale, then we expect the wave function would not vary much on an interval of that scale. This \( \hat{\Delta} \) would return a constant function to itself. This would also essentially bound the momentum scale above; this \( \hat{\Delta} \) sends functions which oscillate quickly approximately to zero.

With \( \hat{\Delta} = \cosh(kp) \), the Hamiltonian is

\[
\hat{H} = \hat{\Delta}^{-1}(\hat{x}\hat{p} + \hat{p}\hat{x})\hat{\Delta} = \hat{x}\hat{p} + \hat{p}\hat{x} + 2p(\Delta^{-1}[x, \hat{\Delta}])
= \hat{x}\hat{p} + \hat{p}\hat{x} + \frac{2ip}{\cosh(kp)}\partial_p(\cosh(kp)) = \hat{x}\hat{p} + \hat{p}\hat{x} + 2ipk \tanh(kp) . \quad (14)
\]

One obtains a differential equation, similar to (10), which is \( \frac{ip}{1+f(p)} = 2ipk \tanh(kp) \). This
gives the following solution for $f(p)$:

$$1 + f(p) = C \cosh^2(kp), \quad (15)$$

which can be used in (9) to obtain $g(p)$

$$1 + g(p) = C^{-1} \text{sech}^2(kp). \quad (16)$$

The functions $f(p), g(p)$ from (15) and (16) inserted in (7) give the modified position and momentum operators

$$\hat{x}' = \text{isech}^2(kp)\partial_p \quad ; \quad \hat{p}' = \cosh^2(kp)p \quad (17)$$

where $C = 1$ so $\hat{p}' \to p$ and $\hat{x}' \to i\partial_p$ as $p \to 0$. These operators are symmetric with respect to the inner product $\langle \psi(p)|\phi(p)\rangle = \int_{-\infty}^{\infty} \cosh^2(kp)\overline{\psi(p)}\phi(p)dp$. This inner product leads to the norm $||\psi||^2 = \int_{-\infty}^{\infty} \cosh^2(kp)|\psi(p)|^2 dp$. In order for this norm to be finite and give normalizable states, one needs exponential suppression of wave function at high momentum to counter the $\cosh^2(kp)$ factor.

The modified commutation relationship becomes

$$[\hat{x}', \hat{p}'] = \text{isech}^2(kp)\partial_p[cosh^2(kp)p] = i (1 + 2kp \tanh(kp)) \quad (18)$$

If $kp \ll 1$ the right hand side of (18) can be expanded using $\tanh(kp) \approx kp + \mathcal{O}(kp)^3$ with the result

$$[\hat{x}', \hat{p}'] \approx i + 2ik^2p^2. \quad (19)$$

This $kp \ll 1$ limit gives a commutator which is the same as the phenomenological commutator given by (11) with $\beta = 2k^2$. We recover a minimal length in a manner similar to that in reference [2].

In the calculations leading to (18) we have shown that by modifying the operator $\hat{\Delta}$ from $1 - e^{-ip}$ (the form taken in [11]) to $\cosh(kp)$ we get a physically reasonable modification of the position and momentum commutator. One of the aims of this work was to tie the modification of the position and momentum commutator to the Bender-Brody-Müller variant of the Berry-Keating program. We will show that choosing $\hat{\Delta} = \cosh(kp)$ does allow one to follow a similar construction to the one proposed in [10]. We will find that there are several variants of $\hat{\Delta}$ which work.
The basic idea of the Berry-Keating program is that there exists some Hamiltonian, \( \hat{H} \), which satisfies an eigenvalue equation \( \hat{H}\Psi = E\Psi \) whose eigenvalues, \( E \), give the imaginary part of the non-trivial zeros of the Riemann zeta function. The Hamiltonian we consider is of the Bender-Brody-Müller form \( \hat{H} = \Delta^{-1}(\hat{x}\hat{p} + \hat{p}\hat{x})\Delta \) where we take \( \Delta = \cosh(kp) \).

Following [10] we begin by re-writing the eigenvalue equation as

\[
\hat{H}\Psi = E\Psi \rightarrow (\hat{x}\hat{p} + \hat{p}\hat{x})(\Delta\Psi) = E(\Delta\Psi),
\]

which is an eigenvalue equation for \( \Delta\Psi \) with respect to the operator \( \hat{x}\hat{p} + \hat{p}\hat{x} \). Using the standard coordinate space representation \(^4\) of the position and momentum operators, \( \hat{x} = x \) and \( \hat{p} = -i\partial_x \), the eigenfunctions and eigenvalues to (20) are \( \Delta\Psi(z,x) = Ax - z \) and \( E_z = i(2z - 1) \) respectively, where \( A \) is a constant.

We want to find \( \Psi \) such that \( \Delta\Psi(z,x) = Ax^{-z} \). To this end we note that for analytic functions, \( f(x), e^{kp}f(x) = f(x-ik) \) i.e. \( e^{kp} \) is a generalized shift operator. When \( k = \pm i \) this is a shift of \( x \rightarrow x \pm 1 \). It follows \( \cosh(kp)f(x) = \frac{1}{2}(e^{kp} + e^{-kp})f(x) = \frac{1}{2}[f(x-ik) + f(x+ik)] \).

In reference [10] where \( \Delta = 1 - e^{-ip} \), the solution to \( \Delta\Psi(z,x) = Ax^{-z} \) was the Hurwitz zeta function, \( \zeta(z,x+1) \), defined as

\[
\Psi(z,x) \propto \zeta(z,x+1) = \sum_{n=0}^{\infty} \frac{1}{(n + x + 1)^z}.
\]

By imposing the boundary condition \( \Psi(z,0) = 0 \), this ‘forces’ the Riemann zeta function \( \zeta(z,1) \) to equal 0. If the spectrum of this operator can be shown to be real, then the \( z \)'s have the form \( \frac{1}{2} + it \) for real \( t \), i.e. the non-trivial zeros of the zeta function are on this critical line. We have no new argument on the reality of the spectrum.

For the case when \( \Delta = \cosh(kp) = \frac{1}{2}(e^{kp} + e^{-kp}) \), we use the Hurwitz-Euler eta function \(^{[16]}\)

\[
\eta(z,x+1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n + x + 1)^z}
\]

and show that it solves the equation \( \Delta\Psi(z,x) = Ax^{-z} \). This function is an alternating sign version of the Hurwitz zeta function of \(^{[21]}\). For \( x = 0 \), \( \eta(z,x+1) \) becomes the well known Dirichlet eta function \(^{[17]}\)

\[
\eta(z,1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n + 1)^z},
\]

\(^4\) Here we switch from momentum space operators to coordinate space operators to follow the construction of reference [10].
an alternating sign version of the Riemann zeta function.

The Hurwitz-Euler eta function satisfies $\hat{\Delta} \Psi(z, x) = Ax^{-z}$ by applying the shift $x \to \frac{x}{2ik} - \frac{1}{2}$ in [22] and arrive $^5$ at

$$\eta \left(z, \frac{x}{2ik} + \frac{1}{2}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n + \frac{x}{2ik} + \frac{1}{2})^z}. \tag{24}$$

Recalling that $\hat{\Delta} f(x) = \frac{1}{2}(f(x - ik) + f(x + ik))$ for $\hat{\Delta} = \frac{1}{2}(e^{kp} + e^{-kp})$ and applying this to $\eta \left(z, \frac{x}{2ik} + \frac{1}{2}\right)$ yields

$$\hat{\Delta} \eta \left(z, \frac{x}{2ik} + \frac{1}{2}\right) = \frac{1}{2} \left[ \eta \left(z, \frac{x}{2ik}\right) + \eta \left(z, \frac{x}{2ik} + 1\right) \right]$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(n + \frac{x}{2ik})^z} + \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(n + \frac{x}{2ik} + 1)^z} \tag{25}$$

$$= \frac{1}{2} \left(\frac{x}{2ik}\right)^{-z} \propto x^{-z},$$

where the alternating sign of the two series makes all the terms cancel between the series except for the $n = 0$ term of the first series. This shows that $\eta \left(z, \frac{x}{2ik} + \frac{1}{2}\right)$ satisfies the equation $\hat{\Delta} \Psi(z, x) = Ax^{-z}$.

To continue the Bender-Brody-Müller approach, we impose the boundary condition that the functions should be equal to zero at $x = ik$, i.e. $\eta \left(z, \frac{ik}{2ik} + \frac{1}{2}\right) = \eta(z, 1) = 0$, which makes the Dirichlet eta function equal to zero. The Dirichlet eta function has the same non-trivial zeros as the Riemann zeta function. This can be seen through the functional relationship between the Riemann zeta function and Dirichlet eta function $^[17]$:

$$\eta(z, 1) = (1 - 2^{1-z})\zeta(z, 1).$$

Thus both the trivial and non-trivial zeros of the Riemann zeta function are zeros of the Dirichlet eta function. The Dirichlet eta function has additional trivial zeros of the form $z = 1 + 2\pi ik/\ln(2)$ with $k \in \mathbb{Z}$ so the pre-factor $(1 - 2^{1-z}) = 0$. In $^[10]$, it is argued that the trivial zeros at $z = -2n$ of $\zeta(z, 1)$ correspond to the eigenfunctions which diverge as $x \to \infty$ and thus do not belong to the function space. The additional trivial zeros for the Dirichlet eta function at $z = 1 + 2\pi ik/\ln(2)$ could also be discarded for function space reasons. However, we note that without a well-defined function space, these arguments are

$^5$ so $x + 1 \to \frac{x}{2ik} + \frac{1}{2}$
suggestive at best. The boundary condition creates a correspondence between the non-trivial zeros of the Dirichlet eta function and the solutions of the eigenvalue equation. These zeros are exactly the non-trivial zeros of the Riemann zeta function and shows that the approach in [10] also works for $\hat{\Delta} = \cosh(kp)$.

One difference of the present construction, versus that the Bender-Brody-Müller (aside from the use of Hurwitz-Euler eta and Dirichlet eta functions versus Hurwitz zeta and Riemann zeta functions) is that now the non-trivial zeros are determined by setting the function equal to zero at $x = ik$ as opposed to $x = 0$ [10]. The reason for this shift of the location of the zero of the eigenfunction, from $x = 0$ to $x = ik$, can be seen by considering $\hat{\Delta} = \frac{1}{2}(1 + e^{2kp})$. One can follow through the steps in equations (20) - (25) and show that $\hat{\Delta} = \frac{1}{2}(1 + e^{2kp})$ also works for a construction similar to that given by Bender-Brody-Müller.

The $\Psi$ satisfying $\hat{\Delta}\Psi(z, x) = A x^{-z}$ are now of the form $\Psi(z, x) = \eta(z, \frac{x}{2ik})$ (note the lack of $+\frac{1}{2}$). Thus for $\hat{\Delta} = \frac{1}{2}(1 + e^{2kp})$, the boundary condition is set at $x = 0$, i.e. $\eta(z, x = 0) = 0$. Finally, we can get from $\hat{\Delta} = \frac{1}{2}(1 + e^{2kp})$ to $\hat{\Delta} = \cosh(kp) = \frac{1}{2}(e^{kp} + e^{-kp})$ by applying $e^{-kp}$ to $\frac{1}{2}(1 + e^{2kp})$. The operator $e^{-kp}$ shifts functions by $ik$ (i.e. $e^{-kp}f(x) \to f(x + ik)$) which would shift the boundary condition from $x = 0$, for the $\hat{\Delta} = \frac{1}{2}(1 + e^{2kp})$ case, to $x = ik$, for the $\hat{\Delta} = \cosh(kp) = \frac{1}{2}(e^{kp} + e^{-kp})$ case.

III. SUMMARY AND REMARKS

The main result of this work is that we arrive at a modification of the standard quantum position and momentum commutation relationship, using the Bender-Brody-Müller variant of the Berry-Keating program as a guide to give a specific form for the modified commutator. These modified operators and commutators are given in equations (17) and (18). This differs from earlier proposals for modified operators and commutators, such as (1), which are phenomenologically motivated. In addition to providing a theoretical, top-down approach to writing down the modified commutators, we also found that several different variants of the $\hat{\Delta}$ used in defining the Hamiltonian $\hat{H} = \hat{\Delta}^{-1}(\hat{x}\hat{p} + \hat{p}\hat{x})\hat{\Delta}$ allow one to tackle the Riemann hypothesis in the way proposed in reference [10]. In addition to $\hat{\Delta}_{BBM} = 1 - e^{-i\hat{p}}$, used by Bender-Brody-Müller, we have found that $\hat{\Delta} = \frac{1}{2}(e^{kp} + e^{-kp})$ and $\hat{\Delta} = \frac{1}{2}(1 + e^{kp})$ also lead to similar approaches to the Riemann hypothesis.

We also remark that the analysis here can be used to show that modifications of the
quantum commutation, such as given in (1), can be connected with different modifications of the position and momentum operators. In [5] the modified position and momentum operators connected with $[\hat{x}, \hat{p}] = i(1 + \beta \hat{p}^2)$ were given as

$$\hat{x} = i(1 + \beta p^2)\partial_p; \quad \hat{p} = p.$$  

(26)

The position operator is changed but the momentum operator is not. Using the analysis of position and momentum operators starting with (7), but having in mind the modified commutator given in (1), we find that the ansatz functions are $1 + f(p) = e^{\beta p^2/2}$ and $1 + g(p) = e^{-\beta p^2/2}$. These lead to modified position and momentum operators of the form

$$\hat{x}' = ie^{-\beta p^2/2}\partial_p; \quad \hat{p}' = e^{\beta p^2/2}p.$$  

(27)

Both sets of modified operators – those from equation (26) and equation (27) – lead to the same modified commutation relationship (1), however the form given in (27) is more symmetric, and recalls the duality between momentum and position, more than the form of the position and momentum operators given in (26).

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