Large amplitude electromagnetic solitons in a fully relativistic magnetized electron-positron-pair plasma

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Nonlinear propagation of purely stationary large amplitude electromagnetic (EM) solitary waves in a magnetized electron-positron (EP) plasma is studied using a fully relativistic two-fluid hydrodynamic model which accounts for physical regimes of both weakly relativistic ($P \ll nmc^2$) and ultrarelativistic ($P \gg nmc^2$) random thermal energies. Here, $P$ is the thermal pressure, $n$ the number density and $m$ the mass of a particle, and $c$ is the speed of light in vacuum. Previous theory in the literature [Phys. Plasmas 11, 3078 (2004)] is advanced and generalized by the relativistic thermal motion of both electrons and positrons. While both the sub-Alfvénic and super-Alfvénic solitons coexist in the weakly relativistic regime, the ultrarelativistic EP plasmas in contrast support only the sub-Alfvénic solitons. Different limits of the Mach numbers and soliton amplitudes are also examined in these two physical regimes.

I. INTRODUCTION

Electron-positron (EP) plasmas have been known to play important roles in many physical situations, such as active galactic nuclei 1, 2, pulsars 3, quasars 4, black holes 5, accretion disks 6, the early universe 7, 8, near the polar cap of fast rotating neutron stars 9–12, as well as in laboratories 13. In the latter, it has been shown that the production of the ion-free high-density neutral EP-pair plasmas and their identification as collective modes can be possible in a controlled laboratory environment.

Linear and nonlinear waves in EP-pair plasmas differ fundamentally from those in ordinary electron-ion plasmas or from a purely electronic beam due to their intrinsic and complete symmetry with equal charge (but opposite in sign) and mass. The Sagdeev or pseudopotential approach has been the most suitable technique for the description of nonlinear large amplitude waves 14–23 which also works well in pair plasmas 24. However, when relativistic dynamics is included together with thermal pressure of plasma particles for the description of large amplitude EM waves, the Sagdeev’s approach may not be suitable. In this context, an alternative procedure has also been developed by McKenzie et al. 25 to study the properties of nonlinear waves in its own frame of reference. Although both approaches are analogous to each other especially for electrostatic waves, the McKenzie approach provides a better perception and usefulness than the Sagdeev’s approach especially when one is concerned with the propagation of electromagnetic (EM) solitary waves in plasmas 26, 27. In the latter, Verheest and Cattaert have studied the propagation of large amplitude EM waves in nonrelativistic and relativistic EP-pair plasmas without any thermal flow of electrons and positrons using the McKenzie approach.

In this work, our aim is to advance and generalize the theory of Verheest and Cattaert 26 by considering the fully relativistic fluid models for electrons and positrons which account for physical regimes of both weakly relativistic and ultrarelativistic random thermal energies. We show that in contrast to the weakly relativistic plasmas which support both sub-Alfvénic and super-Alfvénic solitons, only the sub-Alfvénic solitons can be formed in EP-pair plasmas with ultrarelativistic energies.

II. RELATIVISTIC FLUID MODEL AND MULTISPECIES INTEGRALS

We consider the nonlinear propagation of EM solitary waves along the constant magnetic field $B_0\hat{z}$ in an EP-pair plasma with relativistic flow of thermal electrons and positrons. We assume that the effective collision frequency in an EP-pair plasma, which includes the recombination and photon annihilation effects, is assumed to be much smaller than the plasma oscillation frequency of electrons and positrons. From the energy momentum tensor, the basic equations for the relativistic dynamics of a $j$-th species particle can be written as 28, 29

$$\frac{\partial}{\partial t}(\gamma_j n_j) + \nabla \cdot (\gamma_j n_j \mathbf{v}_j) = 0,$$

(1)
\[
\frac{H_j}{c^2} \left( \frac{\partial}{\partial t} + v_j \cdot \nabla \right) (\gamma_j v_j) = n_j q_j \left( E + \frac{1}{c} v_j \times B \right) - \frac{1}{\gamma_j} \nabla P_j - \frac{v_j v_j}{\gamma_j} \frac{dP_j}{dt},
\]

where \( d/dt \equiv \partial_t + v_j \cdot \nabla \), \( n_j, q_j, m_j, v_j, \gamma_j, P_j \) and \( H_j \) are, respectively, the number density, charge, mass, fluid velocity, relativistic factor, thermal pressure and enthalpy per unit volume of \( j \)-species fluid. Also, \( E \) and \( B \) are the electric and magnetic (total) fields respectively. Introducing \( \mathcal{E}_j \) as the total energy density and \( \epsilon_j \) the internal energy density of the \( j \)-species fluid, we have \( H_j = \mathcal{E}_j + P_j \) and \( \mathcal{E}_j = n_j m_j c^2 + \epsilon_j \). We consider the polytropic pressure law as \( P_j = (\Gamma - 1) \epsilon_j = n_j k_B T_j \), where \( k_B \) is the Boltzmann constant, so that \( \epsilon_j = n_j k_B T_j / (\Gamma - 1) \) and \( \mathcal{E}_j = n_j m_j c^2 + \Gamma P_j / (\Gamma - 1) = n_j m_j c^2 [1 + \Gamma \beta_j / (\Gamma - 1)] \) with the energy ratio \( \beta_j = k_B T_j / m_j c^2 \) and the polytropic index \( 4/3 \leq \Gamma \leq 5/3 \). In particular, \( \Gamma = 5/3 \) and \( 4/3 \), respectively, correspond to the weakly relativistic (classical) and ultrarelativistic regimes. So, in the weakly relativistic limit \( P_j \ll n_j m_j c^2 \) (applicable for low-energy plasma), we have for \( \Gamma = 5/3 \), \( H_j = n_j m_j c^2 + (5/2) n_j k_B T_j \approx n_j m_j c^2 \), and in the regime of ultrarelativistic energies where \( P_j \gg n_j m_j c^2 \), we have instead \( H_j = n_j m_j c^2 + 4 n_j k_B T_j \approx 4 n_j k_B T_j \).

The system is then closed by the following Maxwells equations.

\[
\nabla \cdot E = 4\pi \sum_j q_j n_j \gamma_j, \quad \nabla \cdot B = 0, \quad \nabla \times E = -\frac{1}{c} \frac{\partial B}{\partial t}, \quad \nabla \times B = \frac{4\pi}{c} \sum_j q_j n_j \gamma_j v_j + \frac{1}{c} \frac{\partial E}{\partial t},
\]

In order to derive an evolution equation for purely stationary nonlinear solitary EM waves and their properties from Eqs. (1) to (7) we follow the McKenzie approach as used in, e.g., Ref. [26, 27]. First, we derive various conserved quantities for a general species \( j \) before we apply it for an EP plasma. We look for the excitation of solitary waves that propagate along the constant magnetic field \( B_0 \), i.e., the \( x \)-axis. In a frame moving with the constant speed \( V \) along the \( x \)-axis, all plasma species have the same constant velocity \( V \) along the direction. Since in the wave frame there is no time derivative, Eqs. (1) and (2) reduce to

\[
\frac{d}{dx} (\gamma_j n_j v_{jx}) = 0,
\]

\[
\frac{\alpha_j}{c^2} \gamma_j n_j v_{jx} \frac{d}{dx} (\gamma_j v_j) = \gamma_j n_j q_j \left( E + \frac{1}{c} v_j \times B \right) - \frac{dP_j}{dx} \hat{x}.
\]

Also, from Eqs. (10) to (13) we successively obtain the following equations.

\[
\frac{dE_j}{dx} = 4\pi \sum_j q_j n_j \gamma_j, \quad \frac{dB_j}{dx} = 0, \quad \hat{x} \times \frac{dE_j}{dx} = 0, \quad \hat{x} \times \frac{dB_j}{dx} = \frac{4\pi}{c} \sum_j q_j n_j \gamma_j v_j
\]

Now, Eq. (11) gives on integration \( B_x = B_0 \), a constant. Also, from Eq. (12) it follows that \( E_\perp = 0 \) under the boundary condition \( E \to 0 \) as \( x \to \pm \infty \), and so only \( E_x = \partial \phi / \partial x \) (\( \phi \) is the scalar potential) and \( B_\perp \) are the variables, which also tend to zero as \( x \to \pm \infty \), i.e., in the undisturbed plasma far away from the region of the nonlinear structure. Next, from the equation of continuity (8), we obtain the following conservation of mass (parallel flux).

\[
\gamma_j n_j v_{jx} = \gamma_j n_j o_j V
\]

From Eq. (9), after summing over all the species and using Eqs. (10), (13) and (14), we obtain

\[
\frac{V}{c^2} \sum_j \gamma_j n_j o_j \alpha_j \frac{d}{dx} (\gamma_j v_j) = \frac{1}{4\pi} \left[ E_x \frac{dE_x}{dx} + \left( \hat{x} \times \frac{dB_\perp}{dx} \right) \right. \\
\left. \times B_\perp \right] - \sum_j \frac{dP_j}{dx} \hat{x}
\]

Integrating Eq. (15) with respect to \( x \) we obtain the following two distinct integrals of motion.

\[
\frac{V}{c^2} \sum_j \gamma_j n_j o_j \alpha_j (\gamma_j v_{jx} - \gamma_j o_j V) = \frac{1}{8\pi} \left( E_x^2 - B_\perp^2 \right) \\
- \sum_j (P_j - P_{j0})
\]
Furthermore, the projection of Eq. (9) on \( \mathbf{v}_{j \perp} \) gives
\[
\mathbf{v}_{j \perp} \cdot \left[ \frac{\gamma_j n_j \alpha_j}{c^2} \frac{d}{dx} (\gamma_j \mathbf{v}_{j \perp}) \right] = \frac{q_j \gamma_j n_j}{c} \mathbf{v}_{j \perp} \cdot (\mathbf{e}_\perp \times \mathbf{B}_\perp).
\]
Multiplying Eq. (13) by \( n_j \alpha_j / q_j \), summing over all the species and integrating we obtain
\[
\sum_j \frac{n_j}{q_j} v_{j \perp}^2 = 0,
\]
where we have used Eq. (17). We can also project Eq. (9) on \( \mathbf{v}_j \) to yield
\[
\frac{\alpha_j}{2c^2} \frac{d}{dx} \left( \gamma_j^2 v_j^2 \right) = q_j \gamma_j E_j - K_B T_j \frac{d}{dx} \log (n_j).
\]

III. RELATIVISTIC EP PLASMAS: ENERGY INTEGRAL

We focus our attention to an EP-pair plasma. The results obtained in Sec. II will be modified with \( q_e = -e \), \( q_p = e \), \( m_e = m_p = m = n_{e0} = n_{p0} = n_0 \), \( T_e = T_p = T \), \( \gamma_e \alpha_e = \gamma_p \beta_p = \gamma_0 \) and \( \beta_e = \beta_p = \beta \), where the subscripts \( j = e \) and \( p \), respectively, stand for electrons and positrons. Thus, for EP plasmas the invariants \( (14), (16), (17) \) and \( (19) \), respectively, reduce to
\[
\gamma_e n_e v_{px} = \gamma_p n_p v_{px} = V \gamma_0 n_0,
\]
\[
\frac{V}{c^2} \gamma_0 n_0 \alpha (\gamma_e v_{ex} + \gamma_p v_{px} - 2 \gamma_0 V) = \frac{1}{8 \pi} (E_x^2 - B_x^2) - (P_e + P_p - 2 P_0),
\]
\[
\frac{V}{c^2} \gamma_0 n_0 \alpha (\gamma_e v_{e \perp} + \gamma_p v_{p \perp}) = \frac{B_0}{4 \pi} \mathbf{B}_\perp,
\]
\[
v_{e \perp}^2 = v_{p \perp}.
\]
Using Eq. (24), we obtain from Eq. (23) the following two results.
\[
(v_{i \perp} - v_{e \perp}) \cdot \mathbf{B}_\perp = 0
\]
\[
(v_{i \perp} + v_{e \perp}) \times \mathbf{B}_\perp = 0
\]
Thus, it follows from Eqs. (25) and (26) that while the component of \( (v_{i \perp} - v_{e \perp}) \) is orthogonal to \( \mathbf{B}_\perp \), the other component of \( (v_{i \perp} + v_{e \perp}) \) is parallel to \( \mathbf{B}_\perp \).

In the weakly nonlinear theory, the truly stationary solutions are only possible at linear polarization of EM fields. So, we can assume without loss of generality that \( \mathbf{B}_\perp \) is along the \( y \)-axis. Then Eqs. (25) and (26) give \( v_{ey} = v_{py} = v_y \) and \( v_{pz} = -v_{ez} = v_z \), and so the \( y \)-component of Eq. (13) gives
\[
v_y \sum q_j \gamma_j n_j = 0.
\]
Since from Eq. (23), \( v_y \propto B_y \neq 0 \), Eq. (27) yields the following charge neutrality condition.
\[
\sum_{j=e,p} q_j \gamma_j n_j = 0,
\]
and so we have from Eq. (21), \( v_{ex} = v_{px} = v_x \) and from the Gauss law \( \int \mathbf{E} \cdot d\mathbf{A} = 0 \). Conversely, using the charge neutrality condition \( (28) \), the Ampére law \( (13) \) reduces to
\[
\mathbf{e}_x \times \frac{d \mathbf{B}_\perp}{dx} = \frac{4 \pi n \gamma}{c} (\mathbf{v}_{i \perp} - \mathbf{v}_{e \perp}),
\]
where \( \gamma = 1 / \sqrt{1 - v^2/c^2} \). Furthermore, a scalar multiplication of \( d \mathbf{B}_\perp / dx \) with \( \mathbf{B}_\perp \) gives
\[
\mathbf{B}_\perp \times \frac{d \mathbf{B}_\perp}{dx} = 0,
\]
meaning that the wave magnetic field \( \mathbf{B}_\perp \) is linearly polarized. Thus, our assumption of linear polarization of EM fields and quasineutrality condition are valid. Next, from Eqs. (22) and (23), we obtain
\[
\gamma v_x = \frac{1}{2} \left[ \gamma_0 V - \frac{mc^2 V_A^2 b^2}{2V_0 \alpha} \right] + \frac{c^2 K_B T}{2V_0 \alpha} - \frac{4 \pi n \gamma}{c} \frac{d}{dx} \frac{v_y}{V_0 \alpha},
\]
\[
\gamma v_y = \frac{mc^2 V_A^2}{2V_0 \alpha} b
\]
where \( b = B_y / B_0 \) is the dimensionless wave magnetic field, \( \gamma_0 = 1 / \sqrt{1 - V^2/c^2} \) and \( V_A \) is the Alfvén velocity in an EP plasma, defined by, \( V_A^2 = B_0^2 / (8 \pi n_0 m) \). Next, rearranging the \( y \)-component of Eq. (6), we obtain another velocity component
\[
v_z = \frac{\alpha v_x}{ce B_0} \frac{d}{dx} (\gamma v_y)
\]
Note that Eq. (21) results into \( n = (V_0 \gamma_0 n_0)/(\gamma v_x) \) which when applied to Eq. (20) gives, after integration and summation over electron and positron species, the following conservation of kinetic energy.
\[
\gamma^2 (v_x^2 + v_y^2 + v_z^2) = \gamma_0^2 V^2 - \frac{2c^2 K_B T}{\alpha} \log \left( \frac{\gamma_0 V}{\gamma v_x} \right).
\]
We define the Mach number as \( M = V / V_A \) and a dimensionless coordinate \( \zeta = x \omega p / c \), where \( \omega p^2 \equiv \omega_{pe}^2 + \omega_{pp}^2 \)
\(8\pi n_0e^2/m\) is the squared total plasma oscillation frequency of electrons and positrons. Finally, using Eqs. (31) to (33), we obtain from Eq. (34) the following equation.

\[
\frac{1}{2} \left( \frac{db}{d\zeta} \right)^2 + \psi(b) = 0, \quad (35)
\]

where \(\psi\) is the Sagdeev potential or pseudopotential, given by,

\[
\psi(b) = \frac{M^2}{2} \left( 1 - \frac{f}{g^2} \right) \quad (36)
\]

and

\[
f = 1 - \frac{m^2 e^4}{\gamma_0^4 M^2 \alpha^2} b^2 + \frac{2e^2 K_B T}{\alpha \gamma_0^2 V^2} \log(g), \quad (37)
\]

\[
g = \frac{1}{2} \left[ 1 - \frac{m^2 c^4}{2 \gamma_0^2 M^2 \alpha^2} b^2 + \frac{c^2 K_B T}{\alpha \gamma_0^2 V^2} \right] + \sqrt{\left( 1 - \frac{m^2 c^4}{2 \gamma_0^2 M^2 \alpha^2} b^2 + \frac{c^2 K_B T}{\alpha \gamma_0^2 V^2} \right)^2 - \frac{4e^2 K_B T}{\alpha \gamma_0^2 V^2}}. \quad (38)
\]

Equation (35) represents an energy integral for a pseudo particle of unit mass at pseudo time \(\zeta\) moving with the pseudo velocity \(db/d\zeta\) with a pseudopotential energy \(\psi(b)\). In particular, in absence of the effects of relativistic flow (\(\gamma_0 \sim 1\)) and thermal pressures of electrons and positrons (\(\beta \sim 0\)), the pseudopotential [Eq. (36)] reduces to

\[
\psi(b) = \frac{M^2}{2} \left[ 1 + \frac{4 \left( b^2 - M^2 \right)}{(b^2 - 2M^2)^2} \right] \quad (39)
\]

which is exactly the same as in Ref. [26]. Introducing the parameter \(v_0 = V/c\) and noting that \(\beta \equiv K_B T/mc^2 \ll 1\) defines the regimes of weakly relativistic (classical) plasmas and \(\beta \gg 1\) that of ultra-relativistic plasmas, we recast \(f\) and \(g\) as

\[
f = 1 - \frac{b^2(1 - v_0^2)^2}{M^4 [1 + \Gamma \beta/(\Gamma - 1)]^2} + 2S \log g, \quad (40)
\]

\[
g = \frac{1}{2} \left[ 1 - \frac{b^2(1 - v_0^2)}{2M^2 [1 + \Gamma \beta/(\Gamma - 1)]} + S + \sqrt{\left( 1 - \frac{b^2(1 - v_0^2)}{2M^2 [1 + \Gamma \beta/(\Gamma - 1)]} + S \right)^2 - 4S} \right], \quad (41)
\]

where \(S = (1 - v_0^2)\beta/v_0^2 [1 + \Gamma \beta/(\Gamma - 1)]\).

A general discussion of Eq. (35) is almost similar to the Sagdeev’s approach for large amplitude nonlinear waves. The necessary conditions for the existence of solitary waves are (i) \(\psi(b) = 0\) and \(d\psi/db = 0\) at \(b = 0\),

(ii) \(d^2\psi/db^2 < 0\) at \(b = 0\) (iii) \(\psi(b_m \neq 0) = 0\), \(\psi(b) < 0\) for \(0 < |b| < |b_m|\) and \((d\psi/db)|_{b=b_m} \geq 0\) according to when the solitary waves are compressive (with \(b > 0\)) or rareactive (with \(b < 0\)). Here, \(b_m\) corresponds to the amplitude of the solitary waves. It is straightforward to verify that the condition (i) is satisfied. However, the condition (ii) is satisfied for \(M > M_c\), where \(M_c\) is the critical value of \(M\), given by,

\[
M_c = \sqrt{\frac{1 - v_0^2}{1 + \Gamma \beta/(\Gamma - 1)}}, \quad (42)
\]

i.e., for \(M \leq M_c\) solitary wave propagation may not be possible in EP plasmas. Later, we will verify the condition (iii) numerically in two different regimes, i.e., weakly relativistic and ultrarelativistic regimes. Furthermore, since the pseudopotential \(\psi(b)\) is to be a real valued function, the expression under the square root in \(g\) must be either zero or positive, yielding \(|b| < |b_m| \leq b_c\) where

\[
b_c = \sqrt{2M \left( 1 - \sqrt{S} \right)} - \sqrt{\frac{1 + \Gamma \beta/(\Gamma - 1)}{1 - v_0^2}}. \quad (43)
\]

It follows that for some given values of \(M, \beta\) and \(v_0\), the wave amplitude will not exceed the critical value \(b_c\). The upper limit of the Mach number \(M_u\) can be obtained in terms of \(\beta\) and \(v_0\) from the condition \(\psi(b_c) \geq 0\) as

\[
M_u = \frac{\sqrt{2(1 - v_0^2) \left( 1 - \sqrt{S} \right)}}{\sqrt{1 + \Gamma \beta/(\Gamma - 1)} \left[ 1 + S(\log S - 1) \right]} \quad (44)
\]

Thus, in order that the EP plasmas support large amplitude solitary waves, we must have \(M_c < M < M_u\). In particular, for \(\beta \to 0\) (cold plasmas) and \(\gamma_0 \sim 1\), i.e., \(v_0 \ll 1\) (nonrelativistic plasmas) we have \(M_c \approx 1\) and \(M_u \sim \sqrt{2}\), i.e., super-Alfvénic solitons may exist with the Mach number satisfying \(1 < M < \sqrt{2}\). This is in agreement with the results of Verheest and Cattart [25], who reported in nonrelativistic cold electron-positron plasmas. Next, in order that \(M_u < M_u\) holds, the function \(A(v_0, \beta)\) must be positive, where

\[
A(v_0, \beta) = 2(1 - \sqrt{S})^2 - S(\log S - 1), \quad (45)
\]

together with \(0 < S < 1\). In what follows, we examine numerically the conditions and different limits of the wave amplitude and the Mach number stated above for the existence of large amplitude EM solitons. We focus our discussion on two particular physical regimes of weakly relativistic (\(\beta \ll 1\)) and ultrarelativistic (\(\beta \gg 1\)) plasmas. These are demonstrated in the two subsections [11A] and [11B]. Note that one can, in principle, consider some other finite values of \(\beta\), which may be neither much smaller nor much larger than unity, however, a corresponding choice of the polytropic index in between 4/3 \(\leq \Gamma \leq 5/3\) may not be appropriate, and can lead to some incorrect results.
A. Weakly relativistic regime ($\beta \ll 1$)

We consider $\Gamma = 5/3$. Since $0 < S < 1$ and $0 < \beta \ll 1$, we have two cases of interest (i) $0 < v_0 < \sqrt{2}/9$, $0 < \beta < v_0^2/(1 - 7v_0^2/2)$, i.e., when the upper limits of $\beta$ depend on $v_0$ (ii) $\sqrt{2}/9 \leq v_0 < 1$, $0 < \beta < 1$, i.e., when the upper limit of $\beta$ is independent of $v_0$. Figure [1] is the contour plot of $A(v_0, \beta) = 0$ showing the possible existence region of solitary waves in the $(v_0, \beta)$-plane. Within the domain $0 < v_0 < \sqrt{2}/9$, the ranges of values of $\beta$ change according to case (i). For example, the admissible range of $\beta$ at $v_0 = 0.3$ is $0 < \beta < 0.13$ and at $v_0 = 0.4$ it is $0 < \beta < 0.36$. So, smaller the values of $v_0$, lower is the upper limit of $\beta$. On the other hand, when $\sqrt{2}/9 \leq v_0 < 1$ and $\beta$ is independent of $v_0$, there is a wide range of values of $\beta$: $0 < \beta < \beta_1$ for which the solitary waves exist. However, in all the domains the solitary waves must have a maximum amplitude $b_c$, provided the admissible Mach number lies in $M_c < M < M_u$.

![Figure 1](image1.png)

**FIG. 1.** $A(v_0, \beta) = 0$ [Eq. (45)] is contour plotted to show the existence and non-existence regions of EM solitary waves in weakly relativistic ($\beta \ll 1$) plasmas.

Figure 2 displays the plots of the lower (solid line) and upper (dashed line) limits of the Mach number within the domain $0 < \beta < 1$ for different values of $v_0$ in two cases discussed before [cf. Fig. 1]. The subplots (a) and (b) correspond to the case (i) where $\beta$ depends on $v_0$, while (c) and (d) that for the case (ii) where $\beta$ does not depend on $v_0$. We note that the values of $M_c$ are always less than unity, while those of $M_u$ can be less than or greater than unity depending on the values of $\beta$ and $v_0$ within the regimes. Here, the values of $\beta$ at which both $M_c$ and $M_u$ coincide are not admissible, because otherwise $M = M_c = M_u$ would violate the condition for the existence of solitary waves. If we scale $\beta \leq 0.05$ to interpret its smallness in the weakly relativistic regime, then from the subplots (a) and (b) of Fig. 2 we find that there are, in fact, two subregimes of $\beta$, namely $0 < \beta < \beta_1$ and $\beta_1 < \beta \leq 0.05$. In the former regime, we have $1 < M_u < 1.4$, while in the other one has $0 < M_u < 1$. The threshold value $\beta_1$ shifts towards lower values as the value of $v_0$ is increased within the admissible domain. In fact, for values of $v_0 \geq 0.7$, the threshold value disappears and only we have $0 < M_u < 1$ in $0 < \beta \leq 0.05$. Thus, it follows that the EP plasmas with weakly relativistic ($0 < \beta \leq 0.05$) energies can support both the sub-Alfvénic ($0 < M < 1$) and super-Alfvénic ($1 < M < 1.4$) solitons in the regime $0 < v_0 < 0.7$, while only the sub-Alfvénic solitons may exist for $0.7 < v_0 < 1$. From Fig. 2 it is also noticed that the values of both $M_c$ and $M_u$ decrease with increasing values of $v_0$, and they tend to become smaller than unity as $v_0$ approaches 1, implying that as the phase velocity of EM solitary waves approaches the speed of light in vacuum, it is more likely that the sub-Alfvénic solitons can exist in relativistic EP-pair plasmas.

![Figure 2](image2.png)

**FIG. 2.** Plots of the lower ($M_c$) and upper ($M_u$) limits of the Mach number, given by Eqs. (42) and (44), are shown for different values of $v_0$ in weakly relativistic ($0 < \beta \ll 1$) plasmas. The subplots (a) and (b) correspond to the regimes $0 < v_0 < \sqrt{2}/9$, $0 < \beta < v_0^2/(1 - 7v_0^2/2)$, while the subplots (c) and (d) for $\sqrt{2}/9 \leq v_0 < 1$, $0 < \beta < 1$. Note that $M_c$, $M_u$ ≤ 1 for $0.7 < v_0 < 1$.

In what follows, we numerically examine the variations of the wave amplitude $b_m$ [at which $\psi(b) = 0$] against the parameter $\beta$ ($0 \leq \beta \ll 1$) for different values of the Mach number, $M_c < M < M_u$ and with two different values of $v_0$, taking one from each of the regimes $0 < v_0 < \sqrt{2}/9$ and $\sqrt{2}/9 < v_0 < 1$. In these regimes of $M$ and $\beta$, the values of $b_m$ are always found to be $\lesssim b_c$. We consider (a) $v_0 = 0.3$ when the upper limit of $\beta$ depends on $v_0$, i.e., $0 < \beta < v_0^2/(1 - 7v_0^2/2)$ and (b) $v_0 = 0.6$ when $\beta$ does not depend on $v_0$. The results are shown in Fig. 3. It mainly displays the contour plots of $\psi(b_m) = 0$ in the $(\beta, b)$-plane. It is interesting to note from subplot (a) that within the domain $0 \leq \beta \leq 0.06$ and for a fixed value of $v_0 = 0.3$ in $0 < v_0 < \sqrt{2}/9$, the amplitude $b_m$ increases with increasing values of $M$ in $M_c < M < M_u$. However, the same decrease with increasing values of $\beta$ until $1.06 \lesssim M < M_u$. However, as $M$ decreases...
from $M = 1.06$ to lower values within the domain $M_c < M < 1.06$, the values of $b_m$ increase in a subinterval $0 < \beta \lesssim \beta_2$, while those decrease in another subinterval $\beta_2 < \beta < 0.06$. Here, $\beta_2$ is some threshold value of $\beta$ which shifts to higher values as $M$ decreases from 1.06 to $M_c$. On the other hand, for a fixed value $v_0 = 0.6$ in $\sqrt{2/9} < v_0 < 1$ [subplot (b)], the wave amplitude always increases with increasing values of both $\beta$ ($0 \lesssim \beta < 0.05$) and $M$ ($M_c < M < M_u$). From the subplots (a) and (b) it is also seen that the ranges of values of $\beta$ where $b_m$ is defined differ and increase with decreasing values of $M$.

Having obtained various parameter regimes for the existence of EM solitary waves as discussed before, we now plot the profiles of the pseudopotential $\psi(b)$ and the corresponding solitary structures as in Fig. 4 for different values of $v_0$, $\beta$ and the Mach number $M$ in two different regimes (i) $0 < v_0 < \sqrt{2/9}$, $0 < \beta < v_0^2/(1 - 7v_0^2/2)$, (ii) $\sqrt{2/9} < v_0 < 1$, $0 < \beta < 1$, $M_c < M < M_u$ [subplots (a) and (b)] and (ii) $\sqrt{2/9} < v_0 < 1$, $0 < \beta < 1$, $M_c < M < M_u$ [subplots (c) and (d)]. As expected, the amplitudes of the solitons exactly correspond to the cut-off values of $\psi$ at $b = b_m \neq 0$ (i.e., the points where $\psi$ crosses the $b$-axis). From the profiles of $\psi$ and $b$, the soliton widths can also be verified by the formula: width $W = |b_m/\psi_{\text{min}}|$. An enhancement of the amplitude and broadening of the soliton profile (width) are seen to occur with an increase of the Mach number, however, the amplitude increases but the width decreases with increasing value of $v_0$ and $\beta$ within the admissible regimes [subplots (a) and (b)]. On the other hand, subplots (c) and (d) show the same qualitative behaviors, i.e., with an increase of any one of $v_0$, $\beta$ and $M$, the amplitude increases and the width decreases.

**B. Ultra-relativistic regime ($\beta \gg 1$)**

We consider the polytropic index $\Gamma = 4/3$. In this case, since $0 < S < 1$ and $\beta \gg 1$, we can have also two possible regimes similar to the weakly relativistic case, namely (i) $\sqrt{1/6} < v_0 < \sqrt{1/5}$, $1 \ll \beta < v_0^2/(1 - 5v_0^2)$, i.e., when the upper limits of $\beta$ depend on the values of $v_0$ and (ii) $\sqrt{1/5} < v_0 < 1$, $\beta \gg 1$, i.e., when the upper limits of $\beta$ do not depend on $v_0$. However, looking at the expressions of $M_c$ and $M_u$, we find that within the range $\sqrt{1/6} < v_0 < \sqrt{1/5}$, the ratio $M_u/M_c = \sqrt{2(1 - \sqrt{5})}/\sqrt{1 + S(\log S - 1)}$ varies from 0.9814 to 0.9996, i.e., $M_u/M_c \sim 1$ for $\beta \gg 1$. A numerical estimation also reveals that in this regime of $v_0$, $|\psi(b)| \lesssim 10^{-9}$ and the soliton amplitude $|b_m| \lesssim 0.01$. So, we are not interested in this short regime of $v_0$, and only the regime to be considered for analysis is $\sqrt{1/5} < v_0 < 1$, $\beta \gg 1$.

Figure 5 shows the plots of $M_c$ (the lower limit of the Mach number, solid line) and $M_u$ (the upper limit of the Mach number, dashed line) within the domain $\sqrt{1/5} < v_0 < 1$ for different values of $v_0$. We find that both $M_c$ and $M_u$ decrease with increasing values of $\beta$ and they remain less than unity even for $\beta \gg 1$. Furthermore, it is noticed that the values of both $M_c$ and $M_u$ decrease with increasing values of $v_0$. Thus, it follows that in contrast to the weakly relativistic regime, the EP plasmas with ultrarelativistic energies may support only sub-Alfvénic solitons. Such a feature in relativistic EP plasmas has not been reported before.

Similar to the case of weakly relativistic plasmas, we also show the variation of the soliton amplitude $b_m$ for different values of the Mach number $M$ within $M_c < M < M_u$ and with a fixed value of $v_0$ in $\sqrt{1/5} < v_0 < 1$ as shown in Fig. 6. It is found that the values of $b_m$ increase with increasing values of $\beta$, however, the threshold values of $\beta$ shift to lower ones as the values of $M$ are increased. Since $\beta \gg 1$, relatively lower values of $M$ would favor the existence of EM solitary waves in ultrarelativistic regimes.

The pseudopotential $\psi(b)$ and the corresponding soliton profiles of the magnetic field $b$ are also shown in Fig.
FIG. 4. Plots of the pseudopotential $\psi(b)$ [subplots (a) and (c)] and the corresponding soliton profile [(b) and (d)] for different values of $v_0$, $\beta$ and $M$ as in the legends in two different regimes: (i) $0 < v_0 < \sqrt{2/9}$, $0 < \beta < v_0^2/(1 - 7v_0^2/2)$, $M_c < M < M_u$ [subplots (a) and (b)] and (ii) $\sqrt{2/9} \leq v_0 < 1$, $0 < \beta \ll 1$, $M_c < M < M_u$ [subplots (c) and (d)].

FIG. 5. Plots of the lower ($M_c$) and upper ($M_u$) limits of the Mach number, given by Eqs. (42) and (44), are shown for different values of $v_0$ ($\sqrt{1/5} < v_0 < 1$) in ultrarelativistic plasmas ($\beta \gg 1$): (a) $v_0 = 0.6$ and (b) $v_0 = 0.7$. 
The soliton amplitude $b_m$ is shown against $\beta$ in the ultrarelativistic regime ($\beta \gg 1$) for different values of the Mach number $M$ and for a fixed value of $v_0 = 0.7$ in $\sqrt{1/5} \leq v_0 < 1.$

[7] for different values of $v_0$, $\beta$ and the Mach number $M$. It is seen that with increasing values of these parameters, the soliton amplitude increases and the width decreases.

IV. CONCLUSION

We have studied the nonlinear propagation of purely stationary large amplitude electromagnetic solitary waves in a magnetized relativistic electron-positron-pair plasma. A fully relativistic two-fluid model is considered which accounts for both the weakly relativistic ($\beta \ll 1$) and ultrarelativistic ($\beta \gg 1$) thermal motions of electrons and positrons where $\beta \equiv k_B T/mc^2$. Thus, previous theory in the literature [26] is advanced and generalized. Using the McKenzie approach, the system of fluid equations is reduced to an energy-like equation which describes the evolution of EM solitary waves in its own reference frame. Different parameter regimes of the wave phase velocity $v_0 \equiv V/c$ and the energy ratio $\beta$ for the existence of solitary waves, as well as different limits of the soliton amplitude ($b_m$) and the Mach number $M \equiv V/V_A$ are demonstrated both in the limits of weakly relativistic and ultrarelativistic energies. It is found that

- In the weakly relativistic limit, EM solitary waves may exist in two different regimes (i) $0 < v_0 < \sqrt{2/9}$, $0 < \beta < v_0^2/(1 - 7v_0^2/2)$ and (ii) $\sqrt{2/9} \leq v_0 < 1$, $0 < \beta \ll 1$. The solitary waves can appear as the sub-Alfvénic ($0 < M < 1$) or super-Alfvénic ($1 < M < \sqrt{2}$) solitons with amplitude $0 < b_m < 2$.
- In the ultrarelativistic limit, EM solitary waves exist in the regime $\sqrt{1/5} < v_0 < 1$, $\beta \gg 1$. In this case, only sub-Alfvénic ($0 < M < 0.4$) solitons may exist with amplitude $0 < b_m < 1$.

It is to be noted that both the sub-Alfvénic and super-Alfvénic solitons exist symmetrically for the wave magnetic field $b \equiv B_e/B_0 > 0$ or $< 0$ owing to the obvious symmetry of EP-pair plasmas with equal mass and opposite charges. This means that the EM solitary waves can propagate as compressive or rarefactive type solitons. The energy integral is expressed in terms of the magnetic field instead of the electrostatic potential as the latter may be relevant for electrostatic solitary waves not for EM waves. Furthermore, we have considered the isothermal pressure law for mathematical simplicity. Instead, one can use the adiabatic pressure law, i.e., $P/P_0 = (n/n_0)^\Gamma$ with polytropic index $\Gamma$, however, in this case, the relativistic fluid equations may not be reducible to the energy integral form [35] either by the McKenzie approach or Sagdeev approach.

To conclude, the nonlinear excitation of EM waves and the formation of solitary structures in pair plasmas are known to have significant relevance not only in space and astrophysical environments but also in laboratory experiments [14]. Furthermore, in pulsars and active galactic nuclei with violent surroundings, these nonlinear phenomena would not occur with small amplitude only. In this context, the present theory in magnetized electron-positron plasmas can help understand certain aspects of these stronger nonlinear phenomena with large wave amplitude.

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FIG. 7. Profiles of the pseudopotential $\psi(b)$ [subplot (a)] and the corresponding soliton [subplot (b)] are shown in ultrarelativistic ($\beta \gg 1$) regime for different values of $v_0$, $\beta$ and $M$ as in the legends with $\sqrt{1/5} \leq v_0 < 1$ and $M_c < M < M_u$.

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