Inner and Outer Approximations of Star-Convex Semialgebraic Sets

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Abstract—We consider the problem of approximating a semialgebraic set with a sublevel-set of a polynomial function. In this setting, it is standard to seek a minimum volume outer approximation and/or maximum volume inner approximation. As there is no known relationship between the coefficients of an arbitrary polynomial and the volume of its sublevel sets, previous works have proposed heuristics based on the determinant and trace objectives commonly used in ellipsoidal fitting. For the case of star-convex semialgebraic sets, we propose a novel objective which yields both an outer and an inner approximation while minimizing the ratio of their respective volumes. This objective is scale-invariant and easily interpreted. Numerical examples demonstrate that the approximations obtained are often tighter than those returned by existing heuristics. We also provide methods for establishing the star-convexity of a semialgebraic set by finding inner and outer approximations of its kernel.

I. INTRODUCTION

Consider a compact, semialgebraic set $\mathcal{X} \subset \mathbb{R}^n$ given by the intersection of the 1-sublevel sets of $m$ polynomial functions $g_i(x) \in \mathbb{R}[x]$:

$$\mathcal{X} = \{x \mid g_i(x) \leq 1, i \in [m]\}. \quad (1)$$

Semialgebraic sets arise naturally in many control applications. The set of coefficients for which a polynomial is Schur or Hurwitz stable is given by a semialgebraic set. For Hurwitz stability, the polynomial inequalities can be derived from the Routh array. These sets are often complicated and cumbersome to analyze. As such, it is common to seek simpler representations which closely approximate the set but are more amenable to further analysis [1]. Examples of “simple” representations include hyperrectangles and ellipsoids.

A number of publications have explored the use of sum-of-squares (SOS) optimization for approximating a semialgebraic set with a simpler representation [1]–[8]. The most common parameterization is to seek a SOS polynomial whose 1-sublevel set $\mathcal{F} = \{x \mid f(x) \leq 1\}$ provides either an inner ($\mathcal{F} \subseteq \mathcal{X}$) or outer ($\mathcal{F} \supseteq \mathcal{X}$) approximation of the set $\mathcal{X}$. In this formulation, an open question is the choice of the objective function. For outer (resp. inner) approximations, a natural objective is to minimize (resp. maximize) the volume of the 1-sublevel set. For an ellipsoid $\mathcal{E} = \{x \mid x^T Ax + b^T x + c \leq 1\}$ where $A \succeq 0$, the volume is proportional to $\det A^{-1}$. Using the logarithmic transform, ellipsoidal volume minimization can be posed as the convex objective $-\log \det A$ [2]. More generally, in the case of homogeneous polynomials it is possible to find the minimum volume outer approximation by solving a hierarchy of semidefinite programs [9].

Ellipsoids and homogeneous polynomials are not ideal candidates for approximating asymmetric shapes due to their inherent symmetry. General polynomials offer a more flexible basis for approximating sets. The caveat is that we lack expressions for computing the volume of the 1-sublevel set as a function of the polynomial coefficients. The most common approach is to mimic the determinant ([2], [4]) or trace [1] objectives used in ellipsoidal fitting. These objectives often yield qualitatively good approximations. However, they have no explicit relationship to the volume beyond upper bounding it in some cases [1]. Thus it is difficult to infer the quality of an approximation from the objective value attained.

A. Contributions

This paper makes the following contributions:

- We propose and justify an algorithm based on SOS optimization for jointly finding an inner and outer approximation of a semialgebraic set. The algorithm minimizes the volume of the outer approximation relative to the volume of the inner approximation. This objective is easily interpreted and scale-invariant.
- We provide numerical examples showing that our algorithm tends to yield better approximations than existing methods when applied to star-convex sets.
- We provide algorithms for finding inner and outer approximations of the kernel of a star-convex set as shown in Figure 1.

The paper is organized as follows. Section II defines the problem we address and reviews the notion of star-convexity. Section III surveys existing volume heuristics for SOS-based set approximation. Section IV proposes a new volume heuristic for finding outer and inner approximations. Section V provides methods for approximating the kernel of a star-convex set. Section VI provides numerical examples. Section VII concludes the paper.
B. Notation

Let \( i \in [k] := \{1, \ldots, k\} \). Let \( \mathbb{Z}^+ \) denote the set of positive integers. Let \( S^{n-1} := \{ x \in \mathbb{R}^n \mid \|x\| = 1 \} \). The notation \( P \succeq 0 \) indicates that the symmetric matrix \( P \) is positive semidefinite (PSD). Given a compact set \( \mathcal{X} \subseteq \mathbb{R}^n \), its volume (formally, Lebesgue measure) is denoted \( \text{vol} \mathcal{X} := \int_{\mathcal{X}} dx \).

Let \( \sigma_{\mathcal{X}}(c) := \max c^T x \) denote the support function of \( \mathcal{X} \) where \( c \in S^{n-1} \). Given sets \( \mathcal{A}, \mathcal{B} \subseteq \mathbb{R}^n \) the (bi-directional) Hausdorff distance is \( d_H(\mathcal{A}, \mathcal{B}) := \max \{ h(\mathcal{A}, \mathcal{B}), h(\mathcal{B}, \mathcal{A}) \} \) where \( h(\mathcal{A}, \mathcal{B}) := \max_{\alpha \in \mathcal{A}} \min_{b \in \mathcal{B}} \| \alpha - b \|_2 \).

The \( \alpha \)-sublevel set of a function \( f(x) : \mathbb{R}^n \to \mathbb{R} \) is \( \{ x \in \mathbb{R}^n \mid f(x) \leq \alpha \} \). For \( x \in \mathbb{R}^n \), let \( \mathbb{R}[x] \) denote the set of polynomials in \( x \) with real coefficients. Let \( \mathbb{R}_d[x] \) denote the set of all polynomials in \( \mathbb{R}[x] \) of degree less than or equal to \( d \). A polynomial \( p(x) \in \mathbb{R}[x] \) is a SOS polynomial if there exists polynomials \( q_i(x) \in \mathbb{R}[x], i \in [j] \) such that \( p(x) = q_1^2(x) + \ldots + q_j^2(x) \). We use \( \Sigma[x] \) to denote the set of SOS polynomials in \( x \). A polynomial of degree 2\( d \) is a SOS polynomial if and only if there exists \( P \succeq 0 \) (the Gram matrix) such that \( p(x) = z(x)^T P z(x) \) where \( z(x) \) is the vector of all monomials of \( x \) up to degree \( d \) [10]. Letting \( m := \binom{n+d}{d} \) denote the length of \( z(x) \), we have that \( P \in \mathbb{R}^{m \times m} \). To minimize notational clutter, we will sometimes list a polynomial \( f(x) \) as a decision variable. It is implied that \( d \) is specified and matrix \( P \) is introduced as a decision variable such that \( f(x) = z(x)^T P z(x) \).

II. PROBLEM STATEMENT

Definition 1 (Star-Convex Set [11]). A set \( S \subseteq \mathbb{R}^n \) is star-convex if it has a non-empty kernel. The kernel is

\[
\ker S := \{ x \mid tx + (1-t)y \in S \forall t \in [0,1], y \in S \}. \tag{2}
\]

The kernel is the set of points in \( S \) from which one can “see” all of \( S \) as shown in Figure [1]. It is easily shown that the kernel is convex. If \( S \) is convex then \( \ker S = S \).

We will be interested in approximating the set \( \{1\} \) for the case in which it is star-convex with respect to the origin.

Problem 1 (Star-Convex Set Approximation). Given a compact, semialgebraic set \( \mathcal{X} \) with \( 0 \in \text{int} \mathcal{X} \cap \ker \mathcal{X} \) and \( d \in \mathbb{Z}^+ \) find a polynomial \( f_o(x) \in \mathbb{R}_{2d}[x] \) \( (f_i(x) \in \mathbb{R}_{2d}[x]) \) whose 1-sublevel set \( \mathcal{F}_o \) \( (\mathcal{F}_i) \) is of minimum (maximum) volume and is an outer (inner) approximation of \( \mathcal{X} \):

\[
\min_{f_o(x) \in \mathbb{R}_{2d}[x]} \text{vol} \mathcal{F}_o \text{ s.t. } \mathcal{X} \subseteq \mathcal{F}_o
\]

\[
\left( \max_{f_i(x) \in \mathbb{R}_{2d}[x]} \text{vol} \mathcal{F}_i \text{ s.t. } \mathcal{F}_i \subseteq \mathcal{X} \right).
\]

To establish star-convexity of \( \mathcal{X} \), we seek polytopic approximations of its kernel.

Problem 2 (Kernel Approximation). Given a semialgebraic set \( \mathcal{X} \subseteq \mathbb{R}^n \) find a polytope \( \mathcal{K}_o \) \( (\mathcal{K}_i) \) of minimum (maximum) volume that is an outer (inner) approximation of \( \ker \mathcal{X} \):

\[
\min_{\mathcal{K}_o} \text{vol} \mathcal{K}_o \text{ s.t. } \ker \mathcal{X} \subseteq \mathcal{K}_o
\]

\[
(\max_{\mathcal{K}_i} \text{vol} \mathcal{K}_i \text{ s.t. } \mathcal{K}_i \subseteq \ker \mathcal{X}).
\]

III. EXISTING VOLUME HEURISTICS FOR SET APPROXIMATION

We review existing heuristics for approximating semialgebraic set \( \mathcal{X} \) using SOS optimization. Each of these methods finds an even-degree polynomial \( f(x) = z(x)^T P z(x) \). The variations between the methods largely relate to the objective applied to Gram matrix \( P \). For general polynomials, there is no known relationship between \( P \) and the volume of the sublevel sets. Thus the following objectives are all heuristics in some sense.

A. Determinant Maximization \((−\det P)\)

In [2], the authors propose maximizing the determinant of the Hessian \( \nabla^2 f(x) \) of SOS polynomials. If \( f \) is a polynomial of degree \( 2 \), this reduces to the ellipsoidal objective \( −\det A \) for \( \mathcal{E} = \{ x \mid x^T A x + b^T x + c \leq 1 \} \). As the Hessian must be PSD, the outer approximation is convex. This makes it ill-suited to approximating non-convex shapes.

In [4], the authors propose performing determinant maximization directly on the Gram matrix \( P \). The Hessian is no longer required to be PSD. This allows non-convex outer approximations to be found.

B. Inverse Trace Minimization \((\text{tr} P^{-1})\)

The determinant maximization objective minimizes the product of the eigenvalues of \( P^{-1} \). In [4], the authors propose an alternative heuristic of minimizing the sum of the eigenvalues of \( P^{-1} \). This requires an additional matrix variable \( V \) and constraint \( V \succeq P^{-1} \). Using the Schur complement this can be written as a block matrix constraint involving \( V \) and \( P \) (vice \( P^{-1} \)). The objective \( \min \text{tr} V \) then indirectly minimizes the sum of the eigenvalues of \( P^{-1} \).

C. \( l_1 \) Minimization

In [1] the authors propose minimizing the \( l_1 \) norm of a polynomial evaluated over a bounding box \( B \supseteq \mathcal{X} \). This approach was first introduced in [12] for approximating the volume of semialgebraic sets. Using hyperrectangles as bounding boxes, one can integrate the polynomial over \( B \). The resulting objective \( l_1(f(x)) := \int_B f(x) dx \) is linear in terms of \( P \). The outer approximation consists of the intersection of the 1-superlevel set of \( f(x) \) and \( B \):

\[
\mathcal{X} \subseteq (B \cap \{ x \mid f(x) \geq 1 \}). \tag{3}
\]

This differs from other objectives which do not rely on bounding boxes as part of the set approximation. In this setting, \( f(x) \) is approximating the indicator function of \( \mathcal{X} \) over a compact set \( B \). Convergence of \( f(x) \) to the true indicator function in the limit (as degree \( d \to \infty \)) can be shown by leveraging the Stone-Weierstrass theorem. The asymptotic rate of convergence is at least \( O(1/\log \log d) \) [13]. Inner approximations can be found by outer approximating the complement of \( \mathcal{X} \).

*One application of approximating semialgebraic sets is to yield a single sufficient condition for ensuring \( x \not\in \mathcal{X} \), which can be incorporated into a nonlinear optimization problem (e.g. obstacle avoidance in motion planning [7]). The presence of the bounding box in the resulting set description would require logical constraints to represent \( f(x) < 1 \forall x \not\in B \) \( \implies x \not\in \mathcal{X} \) which are generally unsupported in nonlinear optimization solvers.*
IV. INNER AND OUTER APPROXIMATIONS OF STAR-CONVEX SETS

We propose a new volume heuristic for solving Problem 1. Our heuristic is inspired by the following two lemmas.

Lemma 1. Let \( X, \mathcal{F} \) be compact sets in \( \mathbb{R}^n \) such that \( \mathcal{F} \subseteq X \). Let \( 0 \in \text{int} \, \mathcal{F} \). Then there exists a scaling \( s \geq 1 \) such that \( X \subseteq s \mathcal{F} \).

Lemma 2. Let \( X \subseteq \mathbb{R}^n \). Let \( sX = \{sx \mid x \in X\} \) denote the scaled set where \( s \geq 0 \). Then \( \text{vol} \, sX = s^n \cdot \text{vol} \, X \).

Thus given an inner approximation \( \mathcal{F} \), we can obtain an outer approximation \( s \mathcal{F} \) for some \( s \geq 1 \) with relation
\[
\frac{\text{vol} \, s\mathcal{F}}{\text{vol} \, \mathcal{F}} = s^n. \tag{4}
\]

By minimizing \( s \) we minimize the ratio of the outer approximation volume to the inner approximation volume. Figure 2 visualizes this intuitive heuristic for approximating a set.

We seek a polynomial \( f : \mathbb{R}^n \to \mathbb{R} \) whose 1-sublevel set \( \mathcal{F} = \{x \mid f(x) \leq 1\} \) is an inner approximation of \( X \). We turn this into a condition involving the complement of \( X \):
\[
\mathcal{F} \subseteq X \iff f(x) > 1 \forall x \in X^c. \tag{5}
\]

Optimization methods require non-strict inequalities. We approximate the strict inequality by introducing a small constant \( \epsilon > 0 \) and working with the closure of the complement of \( X \). Define the following:
\[
\bar{X} = \bigcup_{i \in [m]} \{x \mid g_i(x) \geq 1\}. \tag{6}
\]

We then use the following approximation of (5):
\[
\mathcal{F} \subset \text{int} \, \bar{X} \iff f(x) \geq 1 + \epsilon \forall x \in \bar{X}. \tag{7}
\]

Next, we scale the set \( \mathcal{F} \) by a scaling variable \( s > 1 \) to obtain an outer approximation:
\[
s \mathcal{F} \supseteq X \iff f(\frac{x}{s}) \leq 1 \forall x \in X. \tag{8}
\]

Combining the above we arrive at the following:
\[
\begin{align*}
\min \ s \\
\text{s.t. } & f(x) \geq 1 + \epsilon \forall x \in \bar{X}, \\
& f(\frac{x}{s}) \leq 1 \forall x \in X.
\end{align*} \tag{9}
\]

Remark 1. Our scaling heuristic is applicable to approximating any compact set containing the origin in its interior. However, it is best suited to approximating star-convex sets in which \( 0 \in \text{int} \, X \cap \ker \, X \) as visualized in Figure 2. Otherwise there exists a lower bound \( s_{lb} \) such that \( 1 < s_{lb} \leq s \) in (9).

Lemma 3. Let \( X \) and \( \mathcal{F} \) be compact sets in \( \mathbb{R}^n \). Let \( \mathcal{F} \subseteq X \subseteq s^* \mathcal{F} \) for some \( s^* > 1 \). Let \( 0 \in \text{int} \, \mathcal{F} \). Let \( x, sx \in X \) and \( tx \notin X \forall t \in (1, s) \) for some \( s > 1, x \neq 0 \). Then \( s^* \geq s \).

Proof. See appendix.

We let \( s_{lb} \) denote the lowest lower bound given by Lemma 3. This imposes a minimum volume ratio between the inner and outer approximation. Figure 2(right) visualizes this result. The set is not star-convex and therefore \( 0 \notin \ker \, X \).

The black line segment connecting the origin to point \( s_{lb} \) is not contained in \( X \). This point imposes a lower bound on \( s \), preventing the inner and outer approximations from coming closer together.

We introduce SOS polynomials \( \lambda_i(x), \mu_i(x), i \in [m] \) and replace the set-containment conditions in (9) with SOS conditions. If \( s \) is left as a decision variable, we would have bilinear terms involving the coefficients of \( f(x) \) and \( s \).

Instead we perform a bisection over \( s \), solving a feasibility problem at each iteration as given by (10). Algorithm 1 details the bisection method.

Optimization Problem: FindApprox \((s, g_i)\)
\[
\begin{align*}
\min \ f(x), & \lambda_i(x), \mu_i(x) \quad 0 \\
\text{s.t. } & f(x) - (1 + \epsilon) - \lambda_i(x)(g_i(x) - 1) \in \Sigma[x], i \in [m], \\
& 1 - f(\frac{x}{s}) - \sum_{i=1}^{m} \mu_i(x)(1 - g_i(x)) \in \Sigma[x], \\
& \lambda_i(x), \mu_i(x) \in \Sigma[x], \quad i \in [m].
\end{align*} \tag{10}
\]

Remark 2. The objective is scale-invariant. Let solution \((f^*(x), s^*)\) define an outer and inner approximation of \( X \). Scale \( X \) by \( \alpha > 0 \), replacing constraints \( g_i(x) \) with \( g_i(\frac{x}{\alpha}) \).

Then the solution pair \((f^*(\frac{x}{\alpha}), s^*)\) defines the new approximation, where the objective value remains unchanged. The objective is not translation-invariant however. For example, assume we approximate a star-convex set exactly with \((f^*(x), s^* = 1)\). Translate \( X \) by \( t \in X \setminus \ker \, X \), replacing \( g_i(x) \) with \( g_i(x - t) \). Then \( 0 \notin \ker \, X \) and \( s^* > 1 \) for any approximation by Lemma 3.

Remark 3. If \( \mathcal{F} \) is convex we can relate the scaling \( s \) to the Hausdorff distance between the approximations.
given by the following semialgebraic set:

\[ \nabla g_i(x) = 0 \quad \text{for all } g_i(x) = 1. \]

Lemma 5. Let \( \mathcal{X} \) be a semialgebraic set as defined in (1). Let \( \nabla g_i(x) \neq 0 \) \( \forall x \in \partial \mathcal{X}_i, i \in [m] \). The kernel of \( \mathcal{X} \) is given by the following semialgebraic set:

\[ \ker \mathcal{X} = \{ x_k | \nabla g_i(x_k)^T (x_k - x_b) \leq 0 \forall x_b \in \partial \mathcal{X}_i, i \in [m] \}. \]

Remark. From Lemma 5 we see that the kernel of \( \mathcal{X} \) is defined by cutting-planes tangent to the active constraint \( g_i(x_k) = 1, x_b \in \partial \mathcal{X} \) as shown in Figure 1.

**Algorithm 1 Inner and Outer Approximation of \( \mathcal{X} \)**

**Input:** \( \mathcal{X} = \{ x \in \mathbb{R}^n | g_i(x) \leq 1, i \in [m] \}, s_{tot} > 0 \)

**Output:** \( \mathcal{F}, s \mathcal{F} \) s.t. \( \mathcal{F} \subseteq \mathcal{X} \subseteq s \mathcal{F} \)

\[
\begin{align*}
& s_{ub} \leftarrow 1 + s_{tot}, s_{tb} \leftarrow 1 \\
& \textbf{while } \text{FindApprox}(s_{ub}, g_i) = \text{Infeasible} \textbf{ do} \\
& \quad s_{tb} \leftarrow s_{ub} \\
& \quad s_{ub} \leftarrow 2s_{tb} \\
& \textbf{while } s_{ub} - s_{tb} > s_{tot} \textbf{ do} \\
& \quad s_{try} \leftarrow 0.5(s_{ub} + s_{tb}) \\
& \quad \textbf{if } \text{FindApprox}(s_{try}, g_i) = \text{Infeasible} \textbf{ then} \\
& \quad \quad s_{tb} \leftarrow s_{try} \\
& \quad \textbf{else} \\
& \quad \quad s_{ub} \leftarrow s_{try} \\
& \textbf{return } \text{FindApprox}(s_{ub}, g_i)
\end{align*}
\]

**Proof.** See appendix.

**V. SAMPLING-BASED APPROXIMATIONS OF THE KERNEL**

Algorithm 1 assumed the set \( \mathcal{X} \) contained the origin in its kernel. If this does not hold, but there exists a point \( x^* \in \ker \mathcal{X} \cap \text{int} \mathcal{X} \), we can apply Algorithm 1 to the translated set \( \{ x - x^* | x \in \mathcal{X} \} \). As our objective is not invariant with respect to translation, it is useful to approximate the kernel to establish possible choices for \( x^* \) in this section we provide algorithms for finding polytopic approximations of \( \ker \mathcal{X} \).

It will be convenient to represent the boundary of \( \mathcal{X} \) in terms of the inequality that is active. Define the following:

\[ \partial \mathcal{X}_i = \{ x | g_i(x) = 1, g_j(x) \leq 1, j \in [m] \setminus i \}. \]

The boundary of \( \mathcal{X} \) is given by the union

\[ \partial \mathcal{X} = \bigcup_{i \in [m]} \partial \mathcal{X}_i. \]

**Algorithm 2 Outer Approximation of \( \ker \mathcal{X} \)**

**Input:** \( \mathcal{X} = \{ x \in \mathbb{R}^n | g_i(x) \leq 1, i \in [m] \}, n_s \geq 1 \)

**Output:** Outer Approximation \( \mathcal{K}_o \supseteq \ker \mathcal{X} \)

\[
\begin{align*}
& \mathcal{K}_o \leftarrow \mathbb{R}^n \\
& \textbf{for } j = 1 \text{ to } n_s \textbf{ do} \\
& \quad x_{o_b}, \mathcal{I} \leftarrow \text{Sample}(\partial \mathcal{X}) \\
& \quad \mathcal{K}_o \leftarrow \mathcal{K}_o \cap \{ x | \nabla g_i^T(x_{o_b})(x - x_b) \leq 0, i \in \mathcal{I} \} \\
& \quad \textbf{if } (\mathcal{K}_o = \emptyset) \textbf{ then} \\
& \quad \quad \textbf{return } \mathcal{K}_o
\end{align*}
\]

**Proof.** See appendix.

**Remark.** From Lemma 5 we see that the inner approximation is empty, this is sufficient to conclude that the set \( \mathcal{X} \) is not star-convex. Conversely, if the inner approximation is not empty this is sufficient to establish that \( \mathcal{X} \) is star-convex. In the case that the outer approximation is not empty and the inner approximation is empty we cannot conclude anything about the star-convexity of the set.

A. \textbf{Outer Approximation}

We assume the existence of an oracle \( \text{Sample}(\partial \mathcal{X}) \) which allows us to randomly sample points \( x_{o_b} \in \partial \mathcal{X} \) and identify possible choices for \( \ker \mathcal{X} \).

B. \textbf{Inner Approximation}

Consider finding a point \( x_k \in \ker \mathcal{X} \) that maximizes a linear cost \( c^T x_k \) where \( c \in \mathbb{R}^{n-1} \) (i.e. the support function of \( \ker \mathcal{X} \)). From Lemma 5 the resulting convex optimization problem requires set containment constraints:

\[
\begin{align*}
& \max_{x_k} c^T x_k \\
& \text{s.t.} \\
& -\nabla g_i(x)^T (x_k - x_b) \geq 0 \forall x_b \in \partial \mathcal{X}_i, i \in [m].
\end{align*}
\]

We replace the set containment conditions with SOS conditions using Putinar’s Positivstellensatz [14].

\[
\begin{align*}
& \max_{x_k, \lambda_i^{(j)}(x)} c^T x_k \\
& \text{s.t.} \\
& -\nabla g_i(x)^T (x_k - x) - \sum_{j=1}^m \lambda_j^{(i)}(x)(1-g_j(x)) \in \Sigma[x], i \in [m] \\
& \lambda_j^{(i)}(x) \in \Sigma[x], \quad i \in [m], j \in [m] \setminus i.
\end{align*}
\]

For a given direction \( c \in \mathbb{R}^{n-1} \) this program lower bounds the support function of \( \ker \mathcal{X} \). The lower bound monotonically increases with \( \text{deg}(\lambda_j^{(i)}) \). If the problem is feasible, the maximizing argument \( x_k \) belongs to \( \ker \mathcal{X} \) and therefore \( \mathcal{X} \) is star-convex. If infeasible we cannot make any

A practical heuristic is to let \( x^* \) be the Chebyshev center of \( \ker \mathcal{X} \).
conclusions about the star-convexity of $\mathcal{X}$. By solving for random directions $c_i \in S^{n-1}$, $i \in [n_x]$ the convex hull of points $x_k$ provides an inner approximation of the kernel as given by Algorithm 3.

**Algorithm 3** Inner Approximation of $\ker \mathcal{X}$

**Input:** $\mathcal{X} = \{x \in \mathbb{R}^n | g_i(x) \leq 1, i \in [m], \{c_i\} \subset S^{n-1}, i \in [n_x] \}$

**Output:** Inner Approximation $\mathcal{K}_i \subseteq \ker \mathcal{X}$

$\mathcal{K}_i \leftarrow \emptyset$

for $j = 1$ to $n_x$

$x_k \leftarrow \text{FindSupport}(c_j, g_i)$

if $\text{FindSupport}(c_j, g_i) = \text{Infeasible}$ then

return $\mathcal{K}_i = \emptyset$

$\mathcal{K}_i \leftarrow \text{conv} \{ \mathcal{K}_i, x_k \}$

**C. Kernel of Unions and Intersections**

Given sets $A, B \subseteq \mathbb{R}^n$ and their kernels, we can find inner approximations of the kernel of their intersection and union using the following lemma.

**Lemma 6.** Let $A, B \subseteq \mathbb{R}^n$. Then the following holds:

$$\ker(A \cap B) \supseteq \ker A \cap \ker B$$

$$\ker(A \cup B) \supseteq \ker A \cap \ker B.$$  \hspace{1cm} (16, 17)

**Proof.** See appendix.

Thus if $A, B$ are star-convex and have kernels that intersect, their union and intersection is also star-convex. This is useful for establishing star-convexity without resorting to numerical algorithms.

**VI. Examples**

We evaluate Algorithm 1 on various examples and compare the results to the existing heuristics reviewed in Section III. We focus our comparison on outer approximations as more heuristics apply to this case. We use percent error as our metric, calculated as $100 \times \frac{\text{vol}_{\text{conv}} \mathcal{K}_{\text{outer}} - \text{vol}_{\text{conv}} \mathcal{K}_{\text{inner}}}{\text{vol}_{\text{conv}} \mathcal{K}_{\text{outer}}}$ where $\mathcal{K}_{\text{outer}}$ is the outer approximation of $\mathcal{X}$. We first consider approximating two examples from the literature with polynomials of increasing degree. In all instances, our algorithm yielded the tightest outer approximation as shown in Figure 1. Next we consider 100 randomly generated convex polytopes in $\mathbb{R}^2$. In the majority of cases, our heuristic yielded the tightest outer approximation as shown in Table I. Lastly, we approximate a set that is not star-convex. Our heuristic degrades with increasing lower bound $s_{lb}$ as suggested by Lemma 3.

**A. Polynomial matrix inequality** [3]

$$\mathcal{X} = \{ x \in \mathbb{R}^2 \mid \begin{bmatrix} 1 - 16 x_1 x_2 & x_1 \\ x_1 & 1 - x_1^2 - x_2^2 \end{bmatrix} \succeq 0 \}.$$  \hspace{1cm} (18)

Using Algorithms 2 and 3 we find the kernel ($\mathcal{K}_{\text{outer}} = \mathcal{K}_i = \text{conv} \{ \pm (-0.1752, 0.3335), \pm (0.1268, 0.2213) \}$) as shown in Figure 2 (left) shows the 4th-order approximation obtained with each objective. For the $l_1$ approximation we also show the bounding box from [1].

**B. Discrete-time stabilizability region** [3],[1]

$$\mathcal{X} = \{ x \in \mathbb{R}^2 \mid 1 + 2 x_2 \geq 0, 2 - 4 x_1 - 3 x_2 \geq 0, 10 - 28 x_1 - 5 x_2 - 24 x_1 x_2 - 18 x_1^2 \geq 0, 1 - x_2 - 8 x_1 x_2 - 2 x_1^2 - 8 x_1^2 x_2 - 6 x_1 x_2^2 \geq 0 \}.$$  \hspace{1cm} (19)

The set contains the origin in its kernel. Figure 3 shows the percent error for increasing degree. Figure 4 shows the 6th-order approximations obtained with each objective. For the $l_1$ approximation we also show the bounding box from [1].

**C. Convex Polytopes**

We generate 100 random convex polytopes in $\mathbb{R}^2$ with their Chebyshev center at the origin. We find outer approximations using the different objectives. Table I lists the number of times each objective obtained the smallest percent error relative to the other objectives for a given polytope.

**D. Non-Star-Convex Set**

$$\mathcal{X} = \{ x \in \mathbb{R}^2 \mid r^2 \leq (x_1 - c)^2 + x_2^2 \leq 1, x_1 \leq c \}.$$  \hspace{1cm} (20)

Let $0 < r < c < 1$ so the origin is in the interior of the set. Figure 2 shows the set for the case in which $c = 0.9$ and $r = 0.4$. Points $(c, \pm r) \in \partial \mathcal{X}$ yield cutting planes $x_2 \geq r$ and $x_2 \leq -r$ such that $\ker \mathcal{X} = \emptyset$. Table II gives the outer approximation error for $c = 0.9$ and varying $r$. For the scaling objective, we also report the objective value $s^*$ and its scaling objective.

Figure 1 Figure 2 (left) shows the 4th-order approximation obtained with Algorithm 1. Figure 3 shows the percent error as we increase the degree. Although each objective value (not shown) decreases monotonically with increasing degree, the percent error occasionally increases. This demonstrates the heuristic nature of the objectives for minimizing volume.

**Fig. 3.** Approximation percent error and solve times for examples A and B.

**TABLE I**

| Deg. | # Trials | $s$ | $-\text{det} P$ | $\text{tr} P^{-1}$ | $l_1$ |
|------|----------|----|-----------------|-----------------|------|
| 4    | 100      | 73 | 0               | 0               | 2    |
| 6    | 100      | 98 | 0               | 0               | 2    |

Figure I Figure II (left) shows the 4th-order approximation obtained with Algorithm I. Figure III shows the percent error as we increase the degree. Although each objective value (not shown) decreases monotonically with increasing degree, the percent error occasionally increases. This demonstrates the heuristic nature of the objectives for minimizing volume.

The set contains the origin in its kernel. Figure IV shows the 6th-order approximations obtained with each objective. For the $l_1$ approximation we also show the bounding box from [1].

**TABLE II**

| Deg. | # Trials | $s$ | $-\text{det} P$ | $\text{tr} P^{-1}$ | $l_1$ |
|------|----------|----|-----------------|-----------------|------|
| 4    | 100      | 73 | 0               | 0               | 2    |
| 6    | 100      | 98 | 0               | 0               | 2    |

**The $l_1$ objective failed to improve upon the bounding box $B$ supplied.**
lower bound $s_{lb}$ as $s_{lb}$ increases the percent error increases, confirming our heuristic is best suited to star-convex sets.

E. Solver Performance

Figure 3 shows the solve times for the various objectives on a logarithmic scale. Applied to a matrix $P \in \mathbb{R}^{m \times m}$, the $-\det P$ and $tr P^{-1}$ objectives introduce a PSD matrix $H \in \mathbb{R}^{2m \times 2m}$ due to reformulations involving the exponential cone [15] and Schur complement [4] respectively. In contrast, the scaling $(s)$ and $l_1$ objectives work directly with $P$, yielding smaller semidefinite programs. The $l_1$ objective has the best computational performance. Due to the use of bisection, the total solve time for the scaling objective is an integer multiple of the time shown in Figure 3. Accounting for this, the scaling objective still remains competitive with the $-\det P$ and $tr P^{-1}$ objectives.

F. Implementation Details

YALMIP [16] and MOSEK [15] were used to solve the SOS programs. Volumes of non-star-convex sets were approximated by evaluating the indicator function over a discrete grid. Volumes of star-convex sets were approximated using numerical integration in polar coordinates.

8The line segments connecting $(0,0)$ to $(c,\pm r)$ define the maximum lower bound on $s$ in Lemma 5. It can be shown that $s_{lb} = \frac{r}{2|p_2|}$ where $p_2 = (c,r), p_1 = (c + r \cos \phi, r \sin \phi)$ and $\phi = \frac{\pi}{2} + 2 \arctan \frac{r}{c}$.

9Supporting code will be released upon publication.

VII. CONCLUSIONS

An algorithm for finding approximations of semialgebraic sets using sum-of-squares optimization was proposed. The algorithm relies on a novel objective which minimizes the scaling necessary to transform an inner approximation into an outer approximation of the set. Numerical examples demonstrated this objective often finds tighter approximations compared to existing heuristics when applied to star-convex sets. Applied to non-star-convex sets, our proposed heuristic performs poorly. A promising direction to address this is through star-convex decompositions [17]. We leave this exploring this option for future work.

ACKNOWLEDGEMENTS

The author thanks Enrique Mallada and the anonymous reviewers for their valuable feedback.

APPENDIX

A. Proof of Lemma 3

Proof. Assume $1 < s^* < s$ satisfies $F \subseteq X \subseteq s^*F$. Let $x, sx \in X, x \neq 0$ such that $tx \notin X \forall t \in (1, s)$. Given $sx \in X \implies sx \in s^*F \implies \frac{s}{s^*}x \in F$. However, $1 < \frac{s}{s^*} < s \implies \frac{s}{s^*}x \notin F$, a contradiction. Thus $s^* \geq s$. \qed

B. Proof of Lemma 4

Proof. Recall the Hausdorff distance between two compact, convex sets can be written in terms of their support functions.

$$d_H(sF, F) = \max_{c \in \mathcal{S}^{n-1}} |\sigma_F(c) - \sigma_F(c)|$$

(18)

$$= \max_{c \in \mathcal{S}^{n-1}} |\sigma_F(c) - \sigma_F(c)|$$

(19)

$$= (s - 1) \cdot \max_{c \in \mathcal{S}^{n-1}} \sigma_F(c)$$

(20)

$$= (s - 1) \cdot \max_{x \in F} \|x\|_2.$$  

(21)

\qed

C. Proof of Lemma 6

1) ker$(A \cap B) \supseteq$ ker$(A \cap \ker B)$: Let $l(x, y) = \{\lambda x + (1 - \lambda) y \mid \lambda \in [0, 1]\}$ for some $x \in \ker(A \cap \ker B)$ and $y \in A \cap B$. As $x \in \ker(A, y \in A \implies l(x, y) \subseteq A$ and similarly, $x \in \ker(B, y \in B \implies l(x, y) \subseteq B$, we see that $x \in \ker(A \cap B)$.

2) ker$(A \cup B) \supseteq$ ker$(A \cap \ker B)$: Let $l(x, y) = \{\lambda x + (1 - \lambda) y \mid \lambda \in [0, 1]\}$ for some $x \in \ker(A \cap \ker B)$ and $y \in A \cup B$. For the case when $y \in A$, then $x \in \ker A \implies l(x, y) \subseteq A \implies l(x, y) \subseteq A \cup B$. Similarly, for the case when $y \in B$, then $x \in \ker(B \implies l(x, y) \subseteq B \implies l(x, y) \subseteq A \cup B$. Therefore $x \in \ker(A \cup B)$.

\qed

Remark. Note that there is no relation between ker$(A \cap B)$ and ker$(A \cup B)$ in general. We gives examples in which one set is a subset of the other.

ker$(A \cup B) \supseteq$ ker$(A \cap B)$: Let $A \setminus B \neq \emptyset$ and $B \setminus A \neq \emptyset$. Let $A \cup B$ be a convex set. Then ker$(A \cup B) = A \cup B \supseteq (A \cap B) \supseteq$ ker$(A \cap B)$.

ker$(A \cup B) \supseteq$ ker$(A \cap B)$: Let $A$ be a compact set that is not star-convex with non-empty interior. Let $B$ be a non-empty convex set satisfying $B \subseteq A$. Then ker$(A \cap B) = B \supseteq \emptyset =$ ker$(A \cup B)$. 

TABLE II

| r   | Degree | $s(s^*/s_{lb})$ | $-\det P$ | $tr P^{-1}$ |
|-----|--------|----------------|-----------|-------------|
| 0.1 | 4      | 12.0 (1.096 / 1.025) | 13.0 | 11.8 |
| 0.2 | 4      | 13.6 (1.104 / 1.104) | 16.1 | 14.0 |
| 0.3 | 4      | 35.1 (1.250 / 1.250) | 18.5 | 17.8 |
| 0.4 | 4      | 81.7 (1.492 / 1.492) | 17.3 | 22.9 |

Fig. 4. 6th-order outer approximations of example B

TABLE II

| s   | Error |
|-----|-------|
| 1.04 | 4.9%  |
| 6.53 \times 10^9 | 9.7%  |
D. Proof of Lemma 5

Proof. For convenience, define the following:

$$\mathcal{H} := \{ p \mid \nabla g_i(q)^T(p-q) \leq 0, \forall q \in \partial \mathcal{X}, i \in [m] \}.$$  

We show that $\ker \mathcal{X} \subseteq \mathcal{H}$ and $\ker \mathcal{X} \supseteq \mathcal{H}$ and therefore $\ker \mathcal{X} = \mathcal{H}$. Assume $p \not\in \ker \mathcal{X}$ but there exists a point $q \not\in \partial \mathcal{X}$ for some $i \in [m]$ such that $\nabla g_i(q)^T(p-q) > 0$.

Recall the definition of the directional derivative: 

$$\lim_{t \to 0} \frac{g_i(tp + (1-t)q) - g_i(q)}{t} = \nabla g_i(q)^T(p-q).$$

Given $g_i(q) = 1$ and $\nabla g_i(q)^T(p-q) > 0$ implies there exists an open set $U$ such that $\nabla g_i(q)^T(p-q) > 0$. The line segment over this open interval does not belong to $\mathcal{X}$. Thus $p \not\in \ker \mathcal{X}$, a contradiction.

$$\nabla \phi = \frac{\partial g_i}{\partial x_n}(q) > 0 \implies \frac{\partial g_i}{\partial x_n}(q) > 0 \text{ for some open ball around } q \text{ as } g_i \text{ is smooth. Assuming } \phi_n(t) > q_n \implies g_i(\phi(t)) > g_i(\lim l(t)) > 1 \text{ for points sufficiently close to } q, \text{ a contradiction. Thus } \phi_n(t) < q_n \text{ for some interval } t \in (0,e), e > 0.$$  

From this we have

$$\frac{\partial g_i}{\partial x_n}(\phi(t))(q_n - \phi_n(t)) > 0, \forall t \in (0,e). \quad (26)$$

Given $q_n = \phi_n(0) > \phi_n(t)$ for some interval $t \in (0,e)$, by the mean value theorem there exists $t_* \in (0,e)$ such that $\frac{\partial g_i}{\partial x_n}(t_*) < 0$. This yields the following relation:

$$\frac{\partial g_i}{\partial x_n}(\phi(t_*)) \frac{d\phi_n(t_*)}{dt} < 0. \quad (27)$$

Given $g_i(\phi(t)) = 1 \forall t \in (-c,d) \implies \frac{d\phi}{dt}(\phi(t)) = 0$. We expand this at the point $t_*$ obtaining

$$0 = \frac{\partial g_i}{\partial x_n}(\phi(t_*)) \frac{d\phi_n-t_1(t_*)}{dt} + \frac{\partial g_i}{\partial x_n}(\phi(t_*)) \frac{d\phi_n(t_*)}{dt} = \frac{\partial g_i}{\partial x_n}(\phi(t_*))^T(p_{[n-1]} - q_{[n-1]}) + \frac{\partial g_i}{\partial x_n}(\phi(t_*)) \frac{d\phi_n(t_*)}{dt}. \quad (28)$$

From equations (27) and (28) we obtain

$$\frac{\partial g_i}{\partial x_n}(\phi(t_*))^T(p_{[n-1]} - q_{[n-1]}) > 0. \quad (29)$$

Finally, we evaluate the stated constraint on $p \in \mathcal{H}$ at the boundary point $\phi(t_*)$ giving

$$\nabla g_i(\phi(t_*))^T(p - \phi(t_*)) = \frac{\partial g_i}{\partial x_n}(\phi(t_*)) (p_n - \phi_n(t_*)) + \frac{\partial g_i}{\partial x_n}(\phi(t_*))^T (p_{[n-1]} - q_{[n-1]}) (1 - t_*). \quad (30)$$

From (26) and (28) and noting that $(1 - t_*) > 0$ and $q_n = p_n$ gives

$$\nabla g_i(\phi(t_*))^T(p - \phi(t_*)) > 0. \quad (31)$$

Thus $p \not\in \mathcal{H}$, a contradiction. \qed

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