Interference and correlated coherence in disordered and localized quantum walk

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Quantum walk has played an important role in development of quantum algorithms and protocols for quantum simulations. The speedup observed in quantum walk algorithms is attributed to quantum interference of spreading wave packet in position space. Similarly, localization in quantum walk due to disorder is also attributed to quantum interference effect. Therefore, it is intriguing to have a closer look and understand the way quantum interference manifests in different forms of quantum walk dynamics. Here we will use interference measure to quantify the interference in position space and coin space, together and independently. This interference measure helps us to differentiate localisation seen in quantum walks due to different forms of disorder and topological effects. We also shown that the interference measure in the coin space alone can serve as an indicator of the spatially localized quantum state. Comparing the result of interference and correlated coherence with the entanglement dynamics we show that all the intricate features of entanglement is captured by correlated coherence. This will give us a better understanding of role of quantum interference in quantum dynamics and its effect on entanglement. With the control over quantum walk dynamics and quantum interference, exploring the possibility of using interference as resource will be of immediate interest.

\section{I. INTRODUCTION}

Interference of quantum state traversing multiple paths in parallel has played an important role in quantum cryptography\cite{1,2}, quantum metrology\cite{3}, interferometry\cite{4,5}, and various other quantum information processing tasks\cite{6,7}. Though interference is widely studies and a universal theory of it is know\cite{8}, the intricacy involved in the dynamics during interference of quantum state and the way it can be quantified is still a topic of interest\cite{9,10}.

Quantum walks, developed using the aspects of quantum mechanics spreads quadratically faster in position when compared to its classical counterpart, classical random walk\cite{12-17}. Quantum walk has been used to model dynamics in many system such as photosynthesis\cite{18,19}, breakdown of electric field driven system\cite{20,21}, diffusion in quantum system\cite{22}, and localization\cite{23,24}. Universal computation\cite{25,26} and quantum simulations\cite{27-30} are some of the other important directions where quantum walk is considered to be one of the powerful algorithmic tool to establish quantum supremacy. Experimental implementation of quantum walk in various physical systems such as, ultracold atoms\cite{31,32}, ions\cite{33,34}, photons\cite{34-39} and NMR\cite{40} has also given it an edge over other dynamic processes used to demonstrate controlled quantum evolutions.

In quantum walks, superposition and interference play an important role in observing speedup in dynamics and in generation of entanglement\cite{41,42}. It is also one of the system where dynamics can be controlled to modify the way interference manifests leading to different interesting phenomena. That is, one can realize a ballistic spread of wave packet in position space and at the same time realize strong localization and weak localization by modifying the dynamics\cite{23,43}. Therefore, quantifying and understanding interference in quantum walks will play an important role in exploring the possibility of explicitly using quantum interference as resource in quantum information processing tasks. Interference between different computational paths which play an important role in quantum computation will also get a better understanding which could result in interest towards optimization of computational tasks.

From the theory of Anderson localization\cite{44} and weak localization\cite{45}, we know that broken symmetry in the dynamics of system due to disordered media leads to localisation of energy states and it has been experimentally verified\cite{46}. It is also established that the quantum interference is what results in Anderson localization and this effect is absent in classical systems.

In discrete-time quantum walks, interference leads to both, ballistic spread in the position space in homogeneous evolution and localization in presence of disorder evolution. Similarly, topological phases can also be engineered in discrete-time quantum walk and observe localization\cite{47}. Topological phases are very important in experimental realization of topological insulators. These phases do not break any symmetry, rather are described by the presence and absence of certain symmetries such as time-reversal, particle-hole and chiral symmetry\cite{48}. In one-dimension, due to $2\pi$ periodicity of quasi-energies, topological numbers of a 1D discrete-time quantum walk are defined for $0$ and $\pi$ quasi-energies which becomes $\mathbb{Z} \times \mathbb{Z}$ winding numbers\cite{49}. And at the interface where two domains of different winding numbers are connected we get topological phase. Since, winding number is a function of angle $\theta$, in 1D discrete-time quantum walk, allows us to identify different combinations of $\theta$ that leads to the localised state at the interface. The strongest localisation is obtained in the case of topological localisation\cite{50}.

In this work we will quantify the interference in different forms of discrete-time quantum walks (QW). We will compare the way interference manifest during homogeneous discrete-time quantum walk (HQW) and other configuration of quantum walks which will lead to strong and weak localizations. We will show that stronger localisation in the system implies minimum interference in the position space.

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Localization in the position space is seen due to alternative constructive and destructive pattern in particle coin (particle) space. Since particle state defines the probability amplitude with which particle moves in the position space with time. Therefore probability amplitude of the particle is defined by the interference in the particle space and spread in the distribution is defined by the interference in position space while correlated interference gives the effect of interference in particle space on the interference in the position space. This correlated interference, correlated coherence measure reproduces behaviour of entanglement between the particle and the position space.

In Sec. II, we have explained the dynamics of evolution in QW and simulation technique to achieve localisation in QW by changing the evolution operator and initial state. As coherence is the potential means to produce interference in the system as explained by Fresnel in context of wave \[51\], In Sec. III, a methodology to quantify the interference in terms of coherence in the system is explained. Sec. IV explains interference measure in localised state obtained in disorder induced QW and topological QW. A comparison between entanglement and interference is also explained in the same section before concluding in Sec. V.

II. EVOLUTION AND LOCALIZATION IN DISCRETE-TIME QUANTUM WALK

A. Discrete-time Quantum walk

Discrete-time quantum walk in one-dimension is defined on particle (or coin) Hilbert space, $H_c$ with internal states of the particle, $|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ as the basis states and position Hilbert space, $H_p$ defined by the basis states $|x\rangle$ where $x \in \mathbb{I}$. Each step of discrete-time quantum walk is evolved using a quantum coin operation,

$$B(\theta) = \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix}$$  \hspace{1cm} (1)

followed by a position shift operator

$$S_x = \sum_x [|\uparrow\rangle \langle x - 1| \otimes |x\rangle + |\downarrow\rangle \langle x + 1| \otimes |x\rangle],$$  \hspace{1cm} (2)

which evolves the particle into superposition of its basis states. Therefore, the unitary evolution operation at each step is given by,

$$W_x(\theta) = S_x [B(\theta) \otimes I]$$  \hspace{1cm} (3)

and after $t$-time steps, state of the system is $|\psi_t\rangle = W_x(\theta)^t |\psi_0\rangle$ for given initial state $|\psi_0\rangle = (\alpha|0\rangle + \beta|1\rangle) \otimes | x = 0\rangle$. The coin parameter $\theta$ controls the variance of the probability distribution in position space. Fig. 1-a shows the probability distribution for standard homogeneous discrete-time quantum walk (HQW) when $\theta = \pi/4$ after 100 steps of walk.

**FIG. 1:** (a) Probability distribution for one-dimensional homogeneous quantum walk (HQW) and for spatial-disordered QW (SQW) and temporal-disordered QW (TQW) after 200 time-steps, respectively. (b) Amount of interference($I(\rho) \equiv I$) in Hilbert space $H = H_c \otimes H_p$ with number of time-steps for HQW, SQW and TQW. The initial state of the particle is $\frac{1}{\sqrt{2}}(|\uparrow\rangle + i|\downarrow\rangle)$ and disordered system is averaged over 100-runs. A steep increase in interference is seem for HQW with time and for localized SQW, the amount of interference is saturated at a very small value. For weak localized TQW, continuous increase of interference in small quantity is seen with time.
B. Disorder induced localisation

Disorder is introduced in the discrete-time quantum walk evolution by using randomized quantum coin operation. Anderson localisation can be simulated using DTQW by using position dependent randomized quantum coin operation called spatial disordered Discrete-time quantum walk. A weak localisation can be simulated by using time dependent randomized quantum coin operation called temporal disorder in discrete-time quantum walk.

1. Spatial Disorder

The spatial disorder in discrete-time QW (SQW) evolution is introduced by a position dependent coin operation \( B(\theta_x) \), where \( \theta_x \) is randomly picked for each position from the range \( 0 \leq \theta_x \leq \pi \). The state after time \( t \) with spatial disorder using single parameter position dependent coin operation \( B(\theta_x) \) will be,

\[
|\psi_t\rangle_S = [W_x(\theta'_x)]^t |\psi_{in}\rangle = [S_x B(\theta'_x)]^t |\psi_{in}\rangle ,
\]

where \( B(\theta'_x) \equiv \sum_x [B(\theta_x) \otimes |x\rangle \langle x|] \). The state of the particle at each position \( x \) and at time \( t+1 \), in the form of the left-moving (\( \psi^L \)) and right-moving (\( \psi^R \)) components for evolution with spatial disorder are,

\[
\begin{align*}
\psi^L_{x,t+1} &= \begin{pmatrix} \cos \theta_{x+1} & \sin \theta_{x+1} \\ 0 & 0 \end{pmatrix} \psi^R_{x+1,t} \\
&\quad+ \begin{pmatrix} 0 & 0 \\ \sin \theta_{x-1} & -\cos \theta_{x-1} \end{pmatrix} \psi^R_{x-1,t}.
\end{align*}
\]

In general, spatial disorder induces a strong localization of the particle in position space with time in QW evolution [43]. This can also be seen seen in Fig. 1-a.

2. Temporal Disorder

The Temporal disorder in the DTQW evolution is introduced by a time dependent coin operation \( B(\theta_t) \), where \( \theta_t \) is randomly picked for each time-step from the range \( 0 \leq \theta_t \leq \pi \). The state in temporal disordered system after time \( t \), using a single parameter time dependent coin operation will be,

\[
|\psi_t\rangle_T = W_x(\theta_1)\ldots W_x(\theta_2) W_x(\theta_1) |\psi_{in}\rangle .
\]

The iterative form of the state of the particle at each position \( x \) and time \( (t+1) \) will be identical to Eq. (5) with a only a replacement of \( \theta_t \) in place of \( \theta_{x\pm1} \). Temporal disorder induces a weak localisation in position space in QW evolution [52, 53]. Fig. 1-a a shows localisation due to temporal disorder.

C. Topological localisation

Topological phases are associated with the presence of symmetries such as time-reversal symmetry, particle-hole symmetry and chiral symmetry. In one dimension all the three symmetries can be achieved in split-step QW therefore topological phases can be simulated [50]. Since the elements of the evolution operator are time independent and real, the time-reversal and particle-hole symmetries are attained. To ensure the chiral symmetry, different combination of \( \theta_1 \) and \( \theta_2 \) are chosen to satisfy the chirality relation,

\[
\Gamma W(\theta_1, \theta_2) \Gamma^{-1} = W(\theta_1, \theta_2)^{-1}
\]

where, \( W(\theta_1, \theta_2) \) is the evolution operator and chiral symmetry operator has the form,

\[
\Gamma \equiv \sigma_x \otimes \mathbb{I} ,
\]

\[
\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} .
\]

The strongest localization at interface where two different winding numbers are connected, is due to topological effects. Topological property can be introduced by considering a QW with each step split into two with different coin parameters \( \theta_i \) as

\[
W(\theta_1, \theta_2) = S_+ R_{\theta_2} S_- R_{\theta_1} ,
\]

where,

\[
R_{\theta} \equiv \begin{pmatrix} \cos(\theta/2) & \sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix} \otimes \mathbb{I}
\]

and for which the position split shift operators are,

\[
S_- = |\uparrow\rangle \langle \uparrow| \otimes |x-1\rangle \langle x+1| + |\downarrow\rangle \langle \downarrow| \langle x+1| \langle x+1|,
\]

\[
S_+ = |\uparrow\rangle \langle \uparrow| \otimes |x\rangle \langle x+1| + |\downarrow\rangle \langle \downarrow| \langle x+1| \langle x+1| .
\]

To create a real space boundary between topologically distinct phases and reveal non-trivial topological properties at the interface, one can choose different \( \theta_2 \) to the left \( (R_{\theta_2-}) \) and right side \( (R_{\theta_2+}) \) of a point in the position space, while defining the coin operation \( R_{\theta_1} \) uniformly on the entire position space. Different combinations of \( \theta_1 \) and \( \theta_2 \) gives different probability distribution across the interface but maximum localisation at the interface can be achieved for the combination \( (\theta_1, \theta_2, \theta_{2\pm}) = (\pi/2, -\pi, \pi) \) as seen in Fig. 2-(g). Probability distribution for different combinations of \( (\theta_1, \theta_2, \theta_{2\pm}) \) is shown in Fig. 2. For parameters when topological edge are created (different winding number) we see a localized state, Fig. 2-(c) and (g) [50].

Fig. 4, shows the standard deviation for homogeneous quantum walk, spatial and temporal disorder induced quantum walk obtained numerically. Homogeneous quantum walk shows increasing behaviour of standard deviation with time when compared to disordered induced quantum walk. Absence of increase in standard deviation in the system represent localisation in the system.

III. INTERFERENCE AND ENTANGLEMENT MEASURE

In quantum interference, probability amplitudes coherently superimpose while propagating and generate interference at
of light for double-slit experiment in context of quantum theory has already been derived \[ Y_{\text{oung}}'s \text{ double-slit experiment. Degree of spatial coherence of light for double-slit experiment in context of quantum theory has already been derived} \[ Y_{\text{oung}}'s \text{ double-slit experiment. Degree of spatial coherence of light for double-slit experiment in context of quantum theory has already been derived} \]

\[
|\psi_{in}(t)| = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle) \otimes |j = 0\rangle,
\]

respectively. For (a) and (b) \( \theta_1 = \pi/2 \) and \( (\theta_{2-}, \theta_{2+}) = (-\pi/4, \pi/4, \pi/4) \), (c) and (d) \( \theta_1 = \pi/2 \) and \( (\theta_{2-}, \theta_{2+}) = (-3\pi/4, 3\pi/4, \pi/4) \), (e) and (f) \( \theta_1 = -3\pi/2 \) and \( (\theta_{2-}, \theta_{2+}) = (5\pi/4, 3\pi/4) \) and for (g) and (h) \( \theta_1 = -3\pi/2 \) and \( (\theta_{2-}, \theta_{2+}) = (-\pi, \pi) \).

![Young's double-slit Scheme](image)

FIG. 3: A comparison of double-slit experiment with QW. In double-slit scheme, a photon impinges on the slits and renders into two possible paths which are detected on the screen. In QW, in the second step the probability of finding a particle at the point \( P \) is an interference of probability amplitude from \( Q_0 \) and \( Q_1 \). Therefore, QW as a whole can be seen as a multi-slit experiment with increase in number of slits with time.

different sites. Coherence of an optical field can be seen as the ability to produce interference as shown in schematic view of Young’s double-slit experiment. Degree of spatial coherence of light for double-slit experiment in context of quantum theory has already been derived\[54]. An isolated one step view of QW at position \( x \) and a nearest neighbouring site in one dimension has a similar set-up as the Young’s double-slit experiment as shown in Fig. 3. For a given Hilbert space \( H \) with basis \( i \in \mathbb{I} \), all the density matrices which are diagonal in this basis are incoherent\[10]. Therefore, coherence for a given density matrix can be defined by the off-diagonal elements of the matrix and coherence in system is a necessary condition for interference. The normalised coherence measure can be used to quantify the total amount of interference in a Hilbert space for a system at a given time \( t \) and the normalised coherence measure for a density matrix \( \rho \) with dimensionality \( N \) of the Hilbert space is given by,

\[
I(\rho) = \frac{1}{N - 1} \sum_{i \neq j} |\langle i | \rho | j \rangle| = \frac{1}{N - 1} \sum_{i \neq j} |\rho_{ij}|. \quad (14)
\]

If the dimensionality of the position Hilbert space of the QW is \( N \) then the dimensionality of the Hilbert space of QW is \( 2N \) due to the 2 dimensionality of the particle (or coin) Hilbert space in the Eq. (14). The normalised coherence measure above is same as the coherence measure based on the quantifier of quantum coherence\[9, 10].

Probability density to find a particle at \( i \)th position at time \( (t + 1) \) in the position space is due to the nearest neighbours i.e., due to probability densities of finding particle at \( (i + 1) \) and \( (i - 1) \) position at time \( t \). The probability amplitude at \( i \)th position at time \( (t + 1) \) is given by,

\[
\rho_{p_{t+1},t+1} = \cos^2 \theta \left( \rho_{t+1}^{i+1}(i + 1, i + 1) + \rho_{t+1}^{i-1}(i - 1, i - 1) \right) + \sin^2 \theta \left( \rho_{t+1}^{i+1}(i + 1, i - 1) + \rho_{t+1}^{i-1}(i + 1, i + 1) \right)
\]

\[
+ \sin \theta \cos \theta \left( \rho_{t+1}^{i+1}(i + 1, i + 1) - \rho_{t+1}^{i+1}(i - 1, i - 1) \right) + \sin \theta \cos \theta \left( \rho_{t+1}^{i-1}(i + 1, i + 1) - \rho_{t+1}^{i-1}(i - 1, i - 1) \right). \quad (15)
\]
shows coherence measure at each point in position space with respect to time-step for different topological phases as a function of coin parameter $\theta_1$ and $\theta_2$ and for $t = 100$. For (a) and (b) $\theta_1 = \pi/2$ and $(\theta_2, \theta_2) = (-\pi/4, \pi/4)$ and $(-3\pi/4, 3\pi/4)$, respectively. For (c) and (d) $\theta_1 = -3\pi/2$ and $(\theta_2, \theta_2) = (5\pi/4, 3\pi/4)$ and $(-\pi, \pi)$, respectively.

**Entanglement measure** between the particle and the position space using von Neumann entropy is,

$$E(\rho) = S(\rho_c) = -Tr[^{\rho_c} \log_2(\rho_c)]$$  

where $\rho_c$ is the density matrix of the particle space after tracing out the position space.

**Relative entropy of coherence** is an entropic measure of coherence as it leads to maximally coherent state and it satisfies the requirement of monotonicity under incoherent operation [10]. It is given by,

$$C_r(\rho) = S(\rho_{diag}) - S(\rho).$$  

where $\rho_{diag}$ denotes the diagonal part of $\rho$. A relation between $I(\rho) = C_l(\rho)/(N-1)$ and $C_r(\rho)$ has also been established [11] which is given by,

$$C_l \geq C_r(\rho) \geq (S(\rho_{diag}) - S(\rho))$$

for pure state $S(\rho) = 0$ and in Hilbert space $H = H_c \otimes H_p$ states are pure therefore in $H = H_p \otimes H_c$, $C_l \geq S(\rho_{diag})$. But in mixed state such as state in Hilbert space $H_c$ when $S(\rho) \neq 0$, if $C_l \leq S(\rho_{diag})$ we have a limit on entanglement in terms of correlated coherence $I_{cc}$ such as,

$$S(\rho) \geq S(\rho_{diag}) - C_l(\rho) \geq I_{cc}$$

where $S(\rho)$ is the von Neumann entropy given by $-Tr[^{\rho} \log_2(\rho)]$ and $N$ is the dimension of the Hilbert space.
FIG. 7: (a) Amount of interference in position Hilbert space ($I(\rho_p) \equiv I_p$) of the split-step QW and (b) Amount of interference in particle Hilbert space ($I(\rho_c) \equiv I_c$) with dimensionality 200 for position Hilbert space and 2 for particle Hilbert space respectively, with time for 1-D HQW, SQW and TQW. The initial state of the particle is $\frac{1}{\sqrt{2}}(\left|\uparrow\right> + i\left|\downarrow\right>)$ and disordered system is averaged over 100-runs. In particle space, $N = 2$ therefore $C_\text{tr}(\rho) = I_c(\rho)$ and $S(\rho_{\text{diag}}) > C_1$, therefore, Eq. (21) sets a limit to the entanglement measure between the particle and position space in quantum walk and also higher the value of $I_c$, implies $C_\text{tr}(\rho) \leq S(\rho_{\text{diag}})$ for any state and this bound gives the maximally coherent state [10].

FIG. 8: (a), (c), (e) and (g) represents the amount of interference in position Hilbert space ($I(\rho_p) \equiv I_p$) of the split-step QW and (b), (d), (f) and (h) represents the amount of interference in particle Hilbert space ($I(\rho_c) \equiv I_c$) with dimensionality 100 for position Hilbert space and 2 for particle Hilbert space respectively, after 100 time-steps with initial state $|\psi_{\text{in}}\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle) \otimes |x = 0\rangle$, respectively. For (a) and (b) $\theta_1 = \frac{\pi}{2}$ and $(\theta_2−, \theta_2+) = \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$, (c) and (d) $\theta_1 = \frac{\pi}{2}$ and $(\theta_2−, \theta_2+) = \left(-\frac{3\pi}{4}, \frac{3\pi}{4}\right)$, (e) and (f) $\theta_1 = -\frac{3\pi}{2}$ and $(\theta_2−, \theta_2+) = \left(\frac{5\pi}{4}, \frac{3\pi}{4}\right)$ and (g) and (h) $\theta_1 = -\frac{3\pi}{2}$ and $(\theta_2−, \theta_2+) = \left(-\pi, \pi\right)$.

FIG. 9: Entanglement between particle and position Hilbert space in 1-D HQW, SQW and TQW. The initial state of the particle is $\frac{1}{\sqrt{2}}(\left|\uparrow\right> + i\left|\downarrow\right>)$ and disordered system is averaged over 100-runs.
IV. INTERFERENCE MEASURE IN LOCALISED DISCRETE-TIME QUANTUM WALK

Discrete-time QW are known to interfere in both, coin and position Hilbert space during evolution and entangle the two Hilbert space. In this section we will try to quantify the role of interference as an effective resource to obtain the localised states in QW. For this we calculate the amount of interference generated by different localised QW and identify the contribution of interference that leads to strongly localised states.

Fig. 1-(b) shows the total amount of interference in complete Hilbert space \( H = H_c \otimes H_p \) for HQW and disorder induced localisations, SQW and TQW. When the interference measure in Hilbert space \( H = H_c \otimes H_p \) is compared to the interference measure in position and particle space separately as shown in Fig. 7-(a) and Fig. 7-(b), respectively, we see that total amount of interference is mostly dominated by the interference in the position space with some signature of interference in particle space. This measure of interference on a complete system does not give us any information of the interference in coin (particle) Hilbert space and position Hilbert space separately. That can be obtained by tracing out one of the Hilbert space from complete system and measuring the interference.

In position space, with increase in number of steps, amount of interference increases for HQW, a very small increase is seen for TQW and SQW with number of steps and very soon reaches a steady state value. From this observation we can say that as the localisation increases, amount of interference decreases. That is, amount of interference is minimum for spatial disorder induced localisation when compared to temporal disorder induced localisation or HQW as can be seen in Fig. 7-(a). This can be better understood from the Fig. 5 where amount of interference at each step and at each point in position space can be seen. It shows that in localised case interference happens near the point of localisation and hence in this case we have the minimum interference with respect to time. Similar explanation can be given for amount of interference in topological phases. Fig. 2-(a), 2-(c), 2-(e) and 2-(g) shows probability distribution for different topological phase and Fig. 2-(b), 2-(d), 2-(f) and 2-(h) shows corresponding amount of interference in Hilbert space \( H = H_c \otimes H_p \), respectively. The total amount of interference in Hilbert space \( H = H_c \otimes H_p \) is same as the amount of interference in position Hilbert space \( H_c \) as shown in Fig. 8-(a), 8-(c), 8-(e) and 8-(g) with the small signature of amount of interference in the particle Hilbert space \( H_p \) as shown in Fig. 8-(b), 8-(d), 8-(f) and 8-(h), respectively. Amount of interference in position space for topological localisation also shows that as the localisation in the system increases amount of interference in the position space decreases and for maximum localisation, amount of interference in position space is zero as shown in Fig. 8-(g) for topological phase \( (\theta_1, \theta_2, \theta_2, \theta_2) = (-3\pi/2, -3\pi/2, \pi) \).

Amount of interference at each point in position space with time is shown in Fig. 6. This behaviour of coherence measure in the position Hilbert space with number of steps is similar to the standard deviation with number of steps as shown in Fig. 4. Therefore, we can say that an absence of increase in coherence in an evolution process represents localisation in...
the system.

Fig. 7-(b) shows the amount of interference in coin space with number of steps for the HQW with SQW and TQW. Since coin operator defines the probability amplitude with which the particle moves in the position space of the QW, amount of interference in coin (or particle) Hilbert space $H_c$ gives information of the localisation in the position space. In case of HQW, when distribution is spread over the position space but the probability amplitude is mostly concentrated at the extreme points in that case, interference in coin space is almost same as the interference in SQW as shown in Fig. 7-(b). But as the localisation in the system increases (for SQW), the amount of interference shows oscillatory behaviour with respect to time (steps). It is due to the fact that when the wave packet is localized around the origin, every alternative step, the particle prominently interfere constructively and destructively. Interestingly, for TQW we see a decrease in the interference in coin space with number of steps.

Fig. 8-(b), 8-(d), 8-(f) and 8-(h) shows amount of interference in coin space for different topological phases. It can be seen that for $(\theta_1, \theta_2, \theta_3) = (\pi/2, -3\pi/4, 3\pi/4)$ and $(\theta_1, \theta_2, \theta_3) = (-3\pi/2, -\pi, \pi)$, we get strongest localisation as shown in Fig. 2-(c) and 2-(g), respectively and hence in this case we get the strongest and very clear alternate maxima and minima as shown in Fig. 8-(d) and 8-(h), respectively. An alternative constructive and destructive nature leads to the localisation and minimum interference in position space. This shows that the role of interference is coin space is small for TQW compared to the steady role it plays for SQW and HQW.

Fig. 9 and Fig. 11 shows amount of entanglement using von Neumann entropy for disorder induced quantum walk and topological phases, respectively. For SQW and localized topological QW we see lower value of entanglement compared to HQW and TQW. Entanglement measure is higher for TQW compared to HQW [43]. For strong localisation, the site in which particle is found in superposition is relatively less which results in decrease of entanglement therefore in case of maximum localisation in topological phase $(\theta_1, \theta_2, \theta_3) = (-3\pi/2, -\pi, \pi)$ in Fig. 2-(g), we get zero entanglement unlike the interference in particle Hilbert space where maximum amount of interference is obtained for maximum localisation as shown in Fig. 8-(h).

Fig. 10, shows correlation between coherence measure in coin (particle) space and the position space in HQW, SQW and TQW with time- step $t$ obtained using Eq. (20) after tracing out the position space. It gives the minimum possible entanglement in the system and the behaviour is identical to entanglement entropy measure. Since the value of $I_{cc}$ is higher for TQW then SQW and HQW which implies, maximally coherent state in TQW in particle space but minimum coherence (interference) measure in particle space as shown in Fig. 7-(b). This shows that the maximally coherent state does not imply maximum interference in the system.

V. CONCLUSION

DTQW which spreads quadratically faster in position space result in localizaition in presence of disorder in the evolution operator or due to topological effect. Interference during evolution plays a significant role both, for wide spread of wave packet in position space and for localization. Using the measure of interference we showed that for localised states, the interference is minimum when the localization is strong. In case of strong topological localisation we get maximum localization at the initial position and the amount of interference was observed to be zero. The probability distribution in position space is directly proportional to amount of interference at each position of HQW, SQW and TQW. For a topological QW, except for maximum localization (probability one at origin) probability will still serve as an indicator of measure of interference. We have also shown that the measure of the interference in coin Hilbert space alone can serve as an indicator of localization in position space and this will be very resourceful in localization studies.

Amount of interference in coin Hilbert space shows constructive and destructive pattern with respect to time for localized system which contributes to the decrease in amount of interference in the position Hilbert space for increases in the localisation in the system. From the correlated coherence measure (and entanglement) and coherence measure (interference) shows that maximally coherent state does not implies maximum interference in the system. Therefore, quantum dynamics with control over interference in dynamics and interference measure can be used as a strong resource in simulating the dynamics of a quantum system.

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