Ghost-spin chains, entanglement and $bc$-ghost CFTs

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Abstract

We study 1-dimensional chains of ghost-spins with nearest neighbour interactions amongst them, developing further the study of ghost-spins in previous work, defined as 2-state spin variables with indefinite norm. First we study finite ghost-spin chains with Ising-like nearest neighbour interactions: this helps organize and clarify the study of entanglement earlier and we develop this further. Then we study a family of infinite ghost-spin chains with a different Hamiltonian containing nearest neighbour hopping-type interactions. By defining fermionic ghost-spin variables through a Jordan-Wigner transformation, we argue that these ghost-spin chains lead in the continuum limit to the $bc$-ghost CFTs.
1 Introduction

Theories with gauge symmetry described in a covariant formulation are known to contain sectors with negative norm states in part described by ghost field excitations, although the physical content is often captured by a physical positive norm subspace alone. In 2-dim conformal field theories, ghost sectors have negative central charge, a reflection of the negative norm states. In [1], the entanglement entropy properties of certain 2-dim ghost conformal field theories were studied, with the finding that entanglement entropy was non-positive under certain conditions: we will discuss later the motivations that led to those investigations. Ghost-spins were constructed as a simple quantum mechanical toy model for theories with negative norm states and a study of their entanglement properties was also carried out. This was developed further in [2] where ensembles of ghost-spins entangled with ordinary spins was studied in more detail. While a single spin is defined as a 2-state spin variable with a positive definite inner product $\langle \uparrow | \uparrow \rangle = 1 = \langle \downarrow | \downarrow \rangle$ and $\langle \uparrow | \downarrow \rangle = 0 = \langle \downarrow | \uparrow \rangle$, a single ghost-spin is defined as a 2-state spin variable with the indefinite inner product $\langle \uparrow | \uparrow \rangle = 0 = \langle \downarrow | \downarrow \rangle$ and $\langle \uparrow | \downarrow \rangle = 1 = \langle \downarrow | \uparrow \rangle$, akin to the inner products in the $bc$-ghost system as is well-known (see e.g. [3]). The indefinite norm leads to negative norm states: tracing over a subset of these leads to a reduced density matrix for the remaining variables that is not positive definite, and thereby to non-positive entanglement entropy (EE) (reviewed in sec. 2). This study involves only information about the state of the system, with no recourse to dynamics.
In this work, we will study dynamical models of ensembles of ghost-spins. In sec. 3, we study 1-dimensional ghost-spin chains with a finite number of ghost-spins and discuss certain Ising-like nearest neighbour interactions. This helps organize the study of ghost-spin entanglement in [1,2]. In particular we describe the properties of the reduced density matrix and entanglement in such systems.

In sec. 4, we study a concrete example of a family of infinite ghost-spin chains motivated by the well-known family of $bc$-ghost conformal field theories. The $bc$-ghost system has been discussed extensively in e.g. [3, 4], as well as [5], and more recently [6–10]: they arise as Fadeev-Popov ghosts under gauge fixing, as is well-known from the $c = −26$ $bc$-CFTs in worldsheet string theory. The $bc$-ghost CFT with $c = −2$ can also be thought of as the nonlogarithmic subsector of $c = −2$ logarithmic CFTs consisting of 2-dim anticommuting (ghost) scalars, e.g. [7–13]. By constructing fermionic ghost-spin variables through a version of the Jordan-Wigner transformation, we show that the infinite ghost-spin chains here in fact lead to the $bc$-ghost CFTs in the continuum limit. Our investigation here is motivated by the well-known that an Ising spin chain in the continuum limit maps to a conformal field theory of free massless fermions (see e.g. [14], [15]). Sec. 5 contains a Discussion.

2 Reviewing ghost-spin ensembles and entanglement

Here we review “ghost-spins” constructed in [1], ensembles of ghost-spins and spins studied in [2], and their entanglement structures.

Firstly, for ordinary spin variables with a 2-state Hilbert space consisting of $\{\uparrow, \downarrow\}$, we take the usual positive definite norms in the Hilbert space

\begin{align}
\rangle \langle \uparrow | \uparrow \rangle &= \langle \downarrow | \downarrow \rangle = 1, \\
\rangle \langle \uparrow | \downarrow \rangle &= \langle \downarrow | \uparrow \rangle = 0. 
\end{align}

(2.1)

Then a generic state $| \psi \rangle = c_1 | \uparrow \rangle + c_2 | \downarrow \rangle$ has adjoint $\langle \psi | = c_1^* \langle \uparrow | + c_2^* \langle \downarrow |$ and a positive definite norm $\langle \psi | \psi \rangle = |c_1|^2 + |c_2|^2$. Thus states can be normalized as $\langle \psi | \psi \rangle = 1$. For 2-spin systems, entangled states $| \psi \rangle = \psi^{ij} | ij \rangle$ lead after tracing over say the second spin to a reduced density matrix with components $\rho_A^{ij} = \delta_{kl} \psi^{ik} (\psi^*)^{jl}$ which is automatically normalized as $tr \rho_A = \langle \psi | \psi \rangle = 1$. The positive definite norm structure here ensures that $\rho_A$ has von Neumann entropy $S_A = - tr \rho_A \log \rho_A = - \sum_i \rho_A(i) \log \rho_A(i)$ which is positive definite since each eigenvalue $0 < \rho_A(i) < 1$ makes the $- \log \rho_A(i) > 0$.

We define a single “ghost-spin” by a similar 2-state Hilbert space $\{\uparrow, \downarrow\}$, but with norms

\begin{align}
\rangle \langle \uparrow | \uparrow \rangle &= \langle \downarrow | \downarrow \rangle = 0, \\
\rangle \langle \uparrow | \downarrow \rangle &= \langle \downarrow | \uparrow \rangle = 1. 
\end{align}

(2.2)

This is akin to the normalizations in the $bc$-ghost system in [1] (see e.g. [3], Appendix, vol. 1
where this inner product appears). Now a generic state and its non-positive norm are
\[ |\psi\rangle = \psi^\dagger |\uparrow\rangle + \psi^\dagger |\downarrow\rangle \quad \Rightarrow \quad \langle \psi | \psi \rangle = \gamma_{\alpha\beta} \psi^\dagger (\psi^\dagger)^* = \psi^\dagger (\psi^\dagger)^* + (\psi^\dagger)^* \psi^\dagger, \]
(2.3)
where the adjoint is \( \langle \psi | = (\psi^\dagger)^* |\uparrow\rangle + (\psi^\dagger)^* |\downarrow\rangle \), and the ghost-spin inner product is given by the off-diagonal metric \( \gamma_{\uparrow\uparrow} = \gamma_{\downarrow\downarrow} = 1 \). An alternative convenient basis is
\[ |\pm\rangle \equiv \frac{1}{\sqrt{2}} (|\uparrow\rangle \pm |\downarrow\rangle), \quad \langle +|+\rangle = \gamma_{++} = 1, \]
(2.4)
\[ \langle -|-\rangle = \gamma_{--} = -1, \quad \langle +|-\rangle = \langle -|+\rangle = 0. \]
A generic state with nonzero norm can be normalized to norm \( +1 \) or \(-1 \). Then a negative norm state can be written as \( |\psi\rangle = \psi^+ |+\rangle + \psi^- |-\rangle \) with \( \langle \psi | \psi \rangle = |\psi^+|^2 - |\psi^-|^2 = -1 \). For a system of two ghost-spins, \( \{s_A, s_B\} \equiv \{ |\uparrow\rangle, |\uparrow\rangle, |\downarrow\rangle, |\downarrow\rangle \} \equiv \{ |++\rangle, |+-\rangle, |-+\rangle, |--\rangle \} \) are basis states. We define the states, adjoints and (indefinite) norms as
\[ |\psi\rangle = \sum \psi^{\alpha\beta} |\alpha\beta\rangle, \quad \langle \psi | = \sum \langle \alpha\beta | \psi^{\alpha\beta*}, \quad \langle \psi | \psi \rangle = \langle \kappa | \alpha \rangle \langle \beta | \psi^{\alpha\beta \kappa \lambda*} = \langle \gamma_{\alpha\beta} \psi^{\alpha\beta \kappa \lambda*} \]
(2.5)
where repeated indices, as usual, are summed over. A generic normalized positive/negative norm state \( |\psi\rangle = \psi^{++} |++\rangle + \psi^{+-} |+-\rangle + \psi^{+-} |+-\rangle + \psi^{--} |--\rangle \) has norm
\[ \langle \psi | \psi \rangle = |\psi^{++}|^2 + |\psi^{+-}|^2 - |\psi^{+-}|^2 - |\psi^{--}|^2 = \pm 1, \]
(2.6)
normalized with norm \( \pm 1 \), using the diagonal metric (2.4) in the \( |\pm\rangle \) basis.

The density matrix for the full system is \( \rho = |\psi\rangle \langle \psi | = \sum \psi^{\alpha\beta} \psi^{\alpha\beta \kappa \lambda*} |\alpha\beta\rangle \langle \kappa \lambda |. \) We split this system into two subsystems \( A \) and \( B \). The reduced density matrix for the subsystem \( A \), which consists of one ghost-spin is obtained by carrying out the trace over the subsystem \( B \) (environment) consisting of the other ghost-spin. This process can be defined on a multi-spin state using a partial contraction as
\[ \rho_A = tr_B \rho \equiv (\rho_A)^{\alpha\kappa} |\alpha\rangle \langle \kappa |, \quad (\rho_A)^{\alpha\kappa} = \gamma_{\alpha\beta} \psi^{\alpha\beta \kappa \lambda*} = \gamma_{\beta\alpha} \psi^{\alpha\beta \kappa \lambda*}, \]
(2.7)
\[ \Rightarrow \quad (\rho_A)^{++} = |\psi^{++}|^2 - |\psi^{+-}|^2, \quad (\rho_A)^{+-} = \psi^{++} \psi^{+-} - \psi^{+-} \psi^{+-*}, \]
(2.8)
\[ (\rho_A)^{+-} = \psi^{+-} \psi^{++*} - \psi^{+-} \psi^{+-*}, \quad (\rho_A)^{--} = |\psi^{+-}|^2 - |\psi^{--}|^2, \]
Then \( tr \rho_A = \gamma_{\alpha\kappa} (\rho_A)^{\alpha\kappa} = (\rho_A)^{++} - (\rho_A)^{--} \). Thus the reduced density matrix is normalized to have \( tr \rho_A = tr \rho = \pm 1 \) depending on whether the state (2.6) is positive or negative norm. The entanglement entropy calculated as the von Neumann entropy of \( \rho_A \) is
\[ S_A = -\gamma_{\alpha\beta} (\rho_A \log \rho_A)^{\alpha\beta} = -\gamma_{++} (\rho_A \log \rho_A)^{++} - \gamma_{--} (\rho_A \log \rho_A)^{--} \]
(2.9)
where the last expression pertains to the $|\pm\rangle$ basis with $\gamma_{\pm\pm} = \pm1$. This requires defining $\log \rho_A$ as usual as an operator expansion: here the contractions use the indefinite norm and are perhaps more transparent in terms of the mixed-index reduced density matrix $(\rho_A)^\alpha\kappa$.

As an illustration, consider a simple family of states studied in [1] with a diagonal reduced density matrix, so $\log \rho_A$ is also diagonal and easily calculated. For the states (2.6), this gives $\psi_{-+} = \frac{\psi^{++}}{\psi^{++}}$ and $\langle \psi | \psi \rangle = (|\psi^{++}|^2 - |\psi^{--}|^2) (1 + \frac{|\psi^{--}|^2}{|\psi^{++}|^2}) = \pm1$: so (2.8) gives

$$\langle \rho_A \rangle^{\alpha\beta} |\alpha\rangle \langle \beta| = \pm x |+\rangle \langle +| \mp (1-x) |-\rangle \langle -| , \quad x = \frac{|\psi^{++}|^2}{|\psi^{++}|^2 + |\psi^{--}|^2} \quad [0 < x < 1],$$

$$\langle \rho_A \rangle^\kappa = \gamma_{\alpha\beta} (\rho_A)^{\beta\kappa} : \quad (\rho_A)^+ = \pm x , \quad (\rho_A)^- = \pm (1-x) , \quad (2.10)$$

where the $\pm$ pertain to positive and negative norm states respectively. The location of the negative eigenvalue is different for positive and negative norm states, leading to different results for the von Neumann entropy. For negative norm states, $(\rho_A)^{++} < 0$, $(\rho_A)^-- > 0$. From the mixed-index RDM in the second line above, we see that $tr \rho_A = (\rho_A)^+ + (\rho_A)^- = \pm1$ manifestly. Now we obtain $(\log \rho_A)^+ = \log(|x\rangle)$ and $(\log \rho_A)^- = \log(|1-x\rangle)$, the $\pm$ referring again to positive/negative norm states respectively. The entanglement entropy (2.9) becomes $S_A = -(\rho_A)^+ (\log \rho_A)^+ - (\rho_A)^- (\log \rho_A)^-$ and so

$$\langle \rho_A \rangle > 0 : \quad S_A = -x \log x - (1-x) \log(1-x) > 0 , \quad (2.11)$$

For positive norm states, $S_A$ is manifestly positive since $x < 1$, just as in an ordinary 2-spin system. For negative norm states, we note that for the principal branch, $i.e. n = m$, the imaginary part is independent of $x$: in other words the imaginary part is the same for all such negative norm states provided we choose the same branch of the logarithm. In what follows whenever we get a logarithm with negative argument we will list all branches but in our analysis we will consider the principal branch only (with $n, m = 0$), $i.e.$ we will effectively set $\log(-1) = i\pi$. The real part of entanglement entropy is negative since $x < 1$ and the logarithms are negative.

We now review ensembles of ghost-spins and spins, possibly entangled, regarding them in general as toy models for quantum systems containing negative norm states. For multiple variables, the spin Hilbert space has a positive definite metric $g_{ij} = \delta_{ij}$, while the ghost-spin states have a non-positive metric $\gamma_{ij}$, with components $\gamma_{++} = 1$, $\gamma_{--} = -1$, as in (2.4) by a basis change $\{|\uparrow\rangle, |\downarrow\rangle\} \rightarrow \{|+\rangle, |-\rangle\}$ which makes negative norm states manifest. The entanglement entropy properties of the reduced density matrix after tracing over ghost-spins in such systems was studied in [2].
In general, the Hilbert space of spins and ghost-spins contains positive as well as negative norm states. One might ask if the entanglement entropy $S_A$ of $\rho_A^s$ is uniformly positive for all positive norm states, and uniformly negative for all negative norm states. This can be shown to be identically true when the spin sector is not entangled with the ghost-spin sector (both of which could be entangled within themselves). Firstly, considering observables of the spin variables alone, we expect that the correlation function satisfies $\langle \psi | O_s | \psi \rangle = tr_s (O_s \rho^s)$. Performing the trivial trace over all the ghost-spins shows that the reduced density matrix for the remaining spin sector alone is $\rho^s = tr_{gs} \rho$. Now disentangled ghost-spins and spins can be represented as product states with

$$\langle \psi | \psi \rangle = \langle s | \psi_s \rangle \langle \psi_s | \psi_s \rangle, \quad \langle s | \psi_s \rangle = \langle s | \psi_{gs} \rangle \langle \psi_{gs} | \psi_{gs} \rangle,$$

$$\langle \psi_s | \psi_s \rangle = g_{i_1 j_1} \ldots g_{n j_n} (\psi_i)^{i_1 i_2 \ldots} (\psi_j)^{j_1 j_2 \ldots} > 0, \quad \langle \psi_{gs} | \psi_{gs} \rangle = \gamma_{i_1 j_1} \ldots \gamma_{n j_n} (\psi_{gs})^{i_1 i_2 \ldots} (\psi_{gs})^{j_1 j_2 \ldots},$$

where the spin inner product is positive definite while $\langle \psi_{gs} | \psi_{gs} \rangle$ can be positive or negative. Normalizing positive/negative norm states to have norm $\pm 1$ respectively gives

$$\langle \psi_{gs} | \psi_{gs} \rangle \gtrless 0 \quad \Rightarrow \quad \langle \psi | \psi \rangle = \langle s | \psi_s \rangle \langle \psi_s | \psi_s \rangle = \pm 1 \quad [\langle \psi_s | \psi_s \rangle > 0].$$

The reduced density matrix after tracing over all ghost-spins is $\rho_A^s = tr_{gs} (\langle s | \psi_s \rangle \langle \psi_s | \psi_s \rangle \langle \psi_s | \psi_{gs} \rangle \langle \psi_{gs} | \psi_s \rangle)$ giving

$$(\rho_A^s)^{i_1 \ldots k_1 \ldots} = \langle \psi_{gs} | \psi_{gs} \rangle \langle \psi_s \rangle^{i_1 \ldots} \langle \psi_s \rangle^{k_1 \ldots} = \pm \frac{1}{\langle \psi_s | \psi_s \rangle} \langle \psi_s \rangle^{i_1 \ldots} \langle \psi_s \rangle^{k_1 \ldots}.$$

This implies the normalization $tr \rho_A^s = \pm 1$ for positive/negative norm states ($\langle \psi | \psi \rangle \gtrless 0$).

We see that the sign of the norm of the state enters as an overall sign in $\rho_A^s$. Thus for positive norm states, $\rho_A^s$ is positive definite with eigenvalues $0 < \lambda_i < 1$ satisfying $\sum_i \lambda_i = 1$ giving positive definite entanglement entropy $S_A = - tr_s \rho_A^s \log \rho_A^s = - \sum_i \lambda_i \log \lambda_i > 0$. For negative norm states however, we see that $\rho_A^s$ is negative definite with eigenvalues $\lambda_i$. Thus the von Neumann entropy is $S_A = - tr_s \rho_A^s \log \rho_A^s = - \sum_i (-\lambda_i) \log(-\lambda_i) = \sum_i \lambda_i \log \lambda_i + i\pi$ (taking $\log(-1) = i\pi$ as stated earlier). The entanglement entropy thus has a negative definite real part and a constant imaginary part, similar to the subfamily (2.10), (2.11), of two ghost-spin states.

When the spins are entangled with the ghost-spins, then this straightforward correlation between positive norm states and positivity of the entanglement entropy appears to not be true. With $\rho_A$ the reduced density matrix for the remaining spin variables after tracing over all the ghost-spins, the von Neumann entropy contains components of $\rho_A$ which in turn contains linear sub-combinations of the norm of the state. Thus even for positive norm states, some components of $\rho_A$ can be negative in general (while keeping positive the trace of $\rho_A$, which is the norm of the state): this leads to new entanglement patterns in general. Requiring that positive norm states give positive entanglement $S_A$ amounts to
requiring that the components \((\rho_A)^{IJ}\) are positive \((I, J\) being labels for the remaining spin variables): this is only true for specific subregions of the Hilbert space, \textit{i.e.} only certain families of states. When the number of ghost-spins is even, we can restrict to subfamilies of states which have correlated ghost-spins, \textit{i.e.} the ghost-spin values are the same in each basis state. This implies that all allowed states are positive norm, \textit{i.e.} negative norm states are excluded. This restricts to half the space of states which are now all positive norm, and the entanglement entropy is manifestly positive. The intuition here is in a sense akin to simulating \textit{e.g.} the \(X^\pm + bc\) subsector of the 2-dim sigma model representing the string worldsheet theory: in general negative norm states are cancelled between \(X^\pm\) and the \(bc\)-ghost subsectors in the eventual physical theory. The more general subsectors in the Hilbert space where \(\rho_A\) gives positive entanglement entropy for positive norm states can then be interpreted as the component of the state space that is connected to this correlated ghost-spin sector. As an example, consider a system of one spin entangled with two ghost-spins: the general state is \(\psi^{i,\alpha\beta}|i\rangle\alpha\beta\) and tracing over both ghost-spins leads to the reduced density matrix \((\rho_A)^{ik} = \gamma_{\alpha\sigma_\beta\rho}\psi^{i,\alpha\beta}(\psi^*)^{k,\sigma_\rho}\). The subfamily of states represented by \(|\psi\rangle = |\psi^{+++}|++\rangle + |\psi^{+-}|+-\rangle + |\psi^{-+}|-+\rangle + |\psi^{--}|--\rangle\) characterizes here the subspace of correlated ghost-spins: this is manifestly positive norm so that \(\rho_A\) is uniformly positive definite as is the entanglement entropy. This is also true for part of the component of the Hilbert space continuously connected to this subspace. For instance, the family of states \(|\psi\rangle = |\psi^{+++}|++\rangle + |\psi^{+-}|+-\rangle + |\psi^{-+}|-+\rangle + |\psi^{--}|--\rangle\) have norm \(\langle\psi|\psi\rangle = |\psi^{+++}|^2 - |\psi^{+-}|^2 - |\psi^{-+}|^2 + |\psi^{--}|^2\) and lead to a diagonal reduced density matrix \((\rho_A)^{++} = |\psi^{+++}|^2 - |\psi^{+-}|^2, (\rho_A)^{--} = -|\psi^{+-}|^2 + |\psi^{--}|^2\). This is positive definite as long as the states are “mostly” correlated ghost-spins, \textit{i.e.} the components \(\psi^{+-}, \psi^{--}\), are appropriately small. More generally, even ghost-spins allow sensible interpretations.

For systems with odd number of ghost-spins however, such a consistent subfamily of correlated ghost-spin states does not exist so it is not possible to uniformly pick a family of entangled states mentioned above such that positive norm states give positive entanglement entropy. For example, with one spin entangled with one ghost-spin, the general state is \(\psi^{i,\alpha}|i\rangle\alpha\) giving \((\rho_A)^{ik} = \gamma_{\alpha\beta}\psi^{i,\alpha}(\psi^*)^{k,\beta}\) as the reduced density matrix. A simple entangled state is \(|\psi\rangle = |\psi^{+++}|++\rangle + |\psi^{+-}|+-\rangle - |\psi^{-+}|-+\rangle - |\psi^{--}|--\rangle\) with \((\rho_A)^{++} = |\psi^{+++}|^2\), \((\rho_A)^{--} = -|\psi^{--}|^2\) and \(|\psi^{+-}|^2 = |\psi^{-+}|^2 = \pm 1\). Thus \((\log \rho_A)^+ = \log(|\psi^{+++}|^2), (\log \rho_A)^- = \log(-|\psi^{--}|^2),\) so the entanglement entropy is \(S_A = -|\psi^{+++}|^2 \log(|\psi^{+++}|^2) + |\psi^{--}|^2 \log(|\psi^{--}|^2) + |\psi^{-+}|^2 (i\pi)\). Thus a positive norm state does not give positive EE. Likewise, for a system of \(n\) ghost-spins with \(n\) odd (and no spins), the family of states \(|\psi\rangle = |\psi^{+++}|++\ldots + |\psi^{--}|--\ldots\) with norm \(\langle\psi|\psi\rangle = |\psi^{+++}|^2 + (\pm 1)^n|\psi^{--}|^2\) leads to a reduced density matrix \((\rho_A)^{++} = (\rho_A)^{+-} = |\psi^{+++}|^2, (\rho_A)^{--} = -(\rho_A)^{--} = (\pm 1)^n|\psi^{--}|^2\), structurally similar to the one spin and one ghost-spin case above. That is, there always exist positive norm states that lead to
entanglement entropy with negative real part (and nonzero imaginary part).

In the next section, we will discuss how introducing nearest neighbour interactions in the context of a finite ghost-spin chain organizes our understanding of this subspace of correlated ghost-spins and small deformations thereof.

3 Interactions and finite ghost-spin chains

We study 1-dimensional chains with a finite number of ghost-spins in this section: we imagine this to be a generalization to ghost-spins of ordinary 1-dimensional spin chains that are familiar in statistical physics and condensed matter systems (see e.g. [14], [15]). The simplest such configuration here consists of two ghost-spins: consider an Ising-like nearest neighbour interaction

\[ H = -J s s' , \quad s|\pm\rangle = \pm|\pm\rangle , \quad (3.1) \]

where \( s, s' \) are ghost-spin variables and we have written their action in the \{\pm\} basis. If \( J > 0 \), this has the same structure as for ordinary spin ferromagnetic interactions. For instance,

\[ H|\pm\pm\rangle = -J|\pm\pm\rangle, \quad H|\pm\mp\rangle = +J|\pm\mp\rangle . \quad (3.2) \]

The expectation values are the same as for spins,

\[ \langle H \rangle_{\pm\pm} = \frac{\langle \pm \pm | H | \pm \pm \rangle}{\langle \pm \pm | \pm \pm \rangle} = -J , \quad \langle H \rangle_{\pm\mp} = \frac{\langle \pm \mp | H | \pm \mp \rangle}{\langle \pm \mp | \pm \mp \rangle} = +J , \quad (3.3) \]

where e.g. \( \langle \pm \mp | \pm \mp \rangle = -1 \) using the norms in (2.5), and the minus sign cancels in the numerator and denominator in the expectation value (note that \( \langle \pm \mp | H | \pm \mp \rangle = +J(-1) \), i.e. these correlation functions acquire an additional minus sign).

The above nearest neighbour interaction implies that two positive/negative norm configurations attract while one positive and one negative norm repel. This suggests the mapping

\[ [\text{ghost spin}] \{+, -\} \equiv \{\uparrow, \downarrow\} [\text{spin}] \quad (3.4) \]

so that the ghost-spin ensemble with the interactions as defined here in the \{+, -\}-basis maps identically to an ordinary spin ensemble in the \{\uparrow, \downarrow\}-basis. Thus we have

ground states : \(|++\rangle, |--\rangle , \quad (\pm \pm | \pm \pm \rangle = +1 , \]

excited states : \(|+-\rangle, |-+\rangle , \quad (\pm \mp | \pm \mp \rangle = -1 , \quad (3.5) \]

i.e. the ground states are positive norm while the excited states are negative norm. The partition function is

\[ Z = \sum_n e^{-\beta E_n} = 2(e^{-\beta J} + e^{\beta J}) \equiv 2Z(2) , \quad (3.6) \]
identical to that for two Ising-like spins, as expected from the mapping \( \{+, -\} \equiv \{\uparrow, \downarrow\} \). This is despite the fact that the ghost-spin system has negative norm states. In this regard, we should note that this is akin to the partition function for the \( bc \)-ghost CFT with \( c = -2 \) which is positive definite although there is a plethora of negative norm states.

**Time evolution:** Let us now study the time evolution generated by this Hamiltonian (3.1) using the usual rules of quantum mechanics. It is clear that all eigenstates \( H|\Psi_E\rangle = E|\Psi_E\rangle \) evolve simply through phases so that \( |\Psi(t)\rangle = e^{-iHt}|\Psi(0)\rangle = e^{-iE\tau}|\Psi(0)\rangle \) in the Schrödinger picture. This implies that

\[
(\langle + + | \langle + + (t) \rangle = e^{-i(-J)t} \langle + + \rangle \langle + + (0) \rangle, \langle + - | \langle + + \rangle (t) \rangle = e^{-i(+J)t} \langle + - \rangle \langle + + (0) \rangle) .
\]

(3.7)

Then a generic state evolves as \( |\psi(t)\rangle = c_1 e^{+iJt} |++\rangle + c_2 e^{-iJt} |+-\rangle + c_3 e^{-iJt} |+-\rangle + c_4 e^{+iJt} |--\rangle \), and the norm \( \langle \psi(t) | \psi(t) \rangle = |c_1|^2 + |c_2|^2 - |c_2|^2 - |c_3|^2 = \langle \psi(0) | \psi(0) \rangle \) is time translation invariant. By comparison, for ordinary spins, we have \( |c_1|^2 + |c_2|^2 = 1 \) remains invariant under time evolution. The phases cancel out since the basis states are \( H \)-eigenstates and orthogonal to each other. This means that \( \pm \)ve norm states evolve to \( \pm \)ve norm states and do not mix.

To contrast the present case of negative norm states with ordinary spins, it is interesting to ask if a probabilistic interpretation exists. The amplitude for the state \( |\Psi(0)\rangle \) to evolve to itself is given by \( \langle \Psi(0) | e^{-iHt} |\Psi(0)\rangle \) which is

\[
\langle \Psi(0) | \Psi(t) \rangle = (|c_1|^2 + |c_4|^2) e^{iJt} - (|c_2|^2 + |c_3|^2) e^{-iJt} = (|c_1|^2 + |c_4|^2) (e^{iJt} - e^{-iJt}) \pm e^{-iJt}
\]

(3.8)

using the norm condition above. By comparison, for ordinary spins, we have \( \sum_i |c_i|^2 = 1 \) and \( \langle \Psi(0) | \Psi(t) \rangle = (|c_1|^2 + |c_4|^2) (e^{iJt} - e^{-iJt}) + e^{-iJt} \). So for positive norm states, the overlap amplitude for ghost-spins is of the same form as for ordinary spins: for negative norm states, the sign is different.

Consider now a state \( |\Psi(0)\rangle = c_1 |++\rangle + c_2 |+-\rangle \) normalized as \( |c_1|^2 - |c_2|^2 = \pm 1 \) then the probabilities to be measured in \( |++\rangle \) or \( |+-\rangle \) are

\[
P_{\psi}(++) = \langle ++ | \Psi(t) \rangle |^2 = |c_1|^2 > 0 , \quad P_{\psi}(+-) = \langle + - | \Psi(t) \rangle |^2 = |c_2|^2 = |c_1|^2 \mp 1 .
\]

(3.9)

Thus the total probability which is the sum of component probabilities \( P(++) + P(+-) \) is not unity, even when \( |\Psi(t)\rangle \) is positive norm: probability conservation does not hold, since the negative norm components give a minus sign as expected (by comparison, for ordinary spins, we have \( P(\uparrow \uparrow) + P(\uparrow \downarrow) = |c_1|^2 + |c_2|^2 = 1 \), with probability conserved as is familiar).

**3 ghost-spins:** The Hamiltonian for the ghost-spin chain is

\[
H = -J \sum_{nn} s s' = -J s_1 s_2 - J s_2 s_3 .
\]

(3.10)
There are $2^3 = 8$ states $|±±±⟩$ in all and their energies $E = ⟨H⟩$ are

$$E = -2J,$$

$$|++-⟩, |-+-⟩, |--+⟩, |---⟩ : E = +J - J = 0,$$

$$|+-+⟩, |++-⟩ : E = +2J.$$ (3.11)

It is clear that at each level, there are both positive and negative norm states: e.g. at the ground state level, $|+++⟩$ is positive norm while $|---⟩$ is negative norm. This structure also holds for $N$ ghost-spins with $N$ odd, i.e. the ground states contain $|--...--⟩$ which has negative norm. The partition function is

$$Z = 2(e^{2βJ} + 2 + e^{-2βJ}) = 2(e^{βJ} + e^{-βJ})^2 = 2Z_{(2)}^2,$$ (3.12)

where $Z_{(2)}$ is the partition function (3.6) for 2 ghost-spins.

**4 ghost-spins:** The Hamiltonian for the ghost-spin chain is

$$H = -J \sum_{nn} ss' = -Js_1s_2 - Js_2s_3 - Js_3s_4.$$ (3.13)

There are $2^4 = 16$ states $|±±±±⟩$ in all and the energies $E = ⟨H⟩$ are

$$E = -3J,$$

$$|++++⟩, |---−⟩ : E = -J,$$

$$|+++-⟩, |+-++⟩, |-+-+⟩, |--++⟩ : E = +J,$$

$$|+-++⟩, |++−−⟩, |−−−+⟩, |−++−⟩, |−−+−⟩, |−+−+⟩ : E = +3J.$$ (3.14)

In this case, the ground states are uniformly positive norm, as for two ghost-spins: these states fall in the category of “correlated ghost-spins” in [2]. Some (but not all) of the excited states are negative norm. The partition function is

$$Z = 2(e^{3βJ} + 3e^{βJ} + 3e^{-βJ} + e^{-3βJ}) = 2(e^{βJ} + 3e^{-βJ})^3 = 2Z_{(2)}^3,$$ (3.15)

and is identical to the case of 4 Ising-like ordinary spins. This sort of structure persists for an even number of ghost-spins.

**$N$ ghost-spins:** The Hamiltonian is

$$H = -J \sum_{nn} ss' = -J \sum_n s_ns_{n+1} = -Js_1s_2 - Js_2s_3 - ... - Js_{N-1}s_N.$$ (3.16)
There are $2^N$ states $|\pm^N\rangle$ in all. If $N$ is even, the form of the ground states, and the corresponding energy, are

$$\text{ground states : } |+^N\rangle, |-^N\rangle, \quad E = -(N - 1)J,$$

and are both positive norm. (If $N$ is odd, then $|-^N\rangle$ has negative norm.)

Some of the excited states have negative norm, somewhat similar in structure to the 4 ghost-spins case above. The highest energy states (and corresponding energy) are of the form

$$|+ + + + \cdots\rangle, \quad |+ + + - \cdots\rangle, \quad |+ - - - \cdots\rangle, \quad E = (N - 1)J,$$

i.e. maximally alternating $+,-$ ghost-spins (as in the case of 4 ghost-spins). These contain $\frac{N}{2}$ $\{-\}$ ghost-spins each (for $N$ even) and so are positive norm if $\frac{N}{2}$ is even.

The first excited level, with energy $\langle E \rangle = -(N - 3)J$, consists of states which have exactly one “kink” i.e. one $\{+-\}$ (or one $\{-+\}$) interface, as illustrated above for 4 ghost-spins (3.14)). In other words, (starting from the left) the first $-$ ghost-spin can be in one of $N - 1$ locations out of $N$ (as in the second line in (3.14)). Thus the first excited level comprises $2(N - 1)$ states, of the form

$$|+ + \cdots +\rangle, \quad |+ + \cdots + -\rangle, \quad \cdots \quad |+ - \cdots - -\rangle,$$

$$| - + \cdots + +\rangle, \quad |- + \cdots + +\rangle, \quad \cdots \quad |- - \cdots - +\rangle.$$

Higher excited states comprise multiple kinks and can be analysed similarly. Note that a kink here has a single $+-$ or $-+$ interface and is in general distinct from a “bulk” flipped spin, since (in 1-dim) that would have two interfaces. We are considering “open” chains here: if we consider “closed” chains instead, then the absence of endpoints means that each excitation of a flipped spin comes with two kinks.

Thus the partition function has the form

$$Z = 2(e^{(N-1)\beta J} + (N - 1)e^{\beta(N-3)J} + \cdots + e^{-(N-1)\beta J}) = 2(e^{\beta J} + e^{-\beta J})^{N-1} = 2Z_{(2)}^{N-1},$$

again a product over 2 ghost-spin partition functions (3.6).

Before discussing entanglement issues, we make a brief comment on the basis used here. Consider the action $s|\pm\rangle = \pm|\pm\rangle$ of the spin variable $s$: this means

$$s(|\uparrow\rangle \pm |\downarrow\rangle) = \pm(|\uparrow\rangle \pm |\downarrow\rangle) \quad \Rightarrow \quad s|\uparrow\rangle = |\downarrow\rangle, \quad s|\downarrow\rangle = |\uparrow\rangle.$$

Thus $s$ is like a spin-flip operator for these $\uparrow,\downarrow$ states, akin to the Pauli matrix $\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The Hamiltonian (3.1) itself, restricting to two ghost-spins for simplicity, can be written as

$$H = -J ss' = -J (|++\rangle\langle++| + |--\rangle\langle--|) + J (|--\rangle\langle+-| + |+-\rangle\langle-+|).$$
Explicitly changing basis using $|\pm\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle \pm |\downarrow\rangle)$ and expanding and simplifying $H$ gives

$$H = -J(|\uparrow\uparrow\rangle\langle\downarrow\downarrow| + |\downarrow\downarrow\rangle\langle\uparrow\uparrow| + |\uparrow\downarrow\rangle\langle\uparrow\downarrow|) \equiv -J\sigma_x\sigma'_x.$$ (3.23)

whereas $H \equiv -J\sigma_z\sigma'_z$ in the $\pm$-basis.

### 3.1 Entanglement: $N$ ghost-spins

For a chain of $N$ ghost-spins with Hamiltonian (3.16), the generic ground state and its norm are

$$|\psi_g\rangle = \psi_g^+|+\rangle^N + \psi_g^-|\rangle^N,$$

$$\langle\psi_g|\psi_g\rangle = |\psi_g^+|^2 + (-1)^N|\psi_g^-|^2,$$ (3.24)

which is positive norm for $N$ even: the norm is $\langle\psi_g|\psi_g\rangle = |\psi_g^+|^2 + |\psi_g^-|^2 = 1$, after normalizing. This is an entangled state: tracing over $N-1$ ghost-spins, the reduced density matrix for the remaining single ghost-spin (using the notation in sec. 2) has mixed index components $(\rho_A)^\pm = |\psi_g^\pm|^2$, $(\rho_A)^- = |\psi_g^-|^2$, with von Neumann entropy $S_A = -(\rho_A)^+ \log(\rho_A)^+ - (\rho_A)^- \log(\rho_A)^- = -x \log x - (1-x) \log(1-x) > 0$, where $x \equiv |\psi_g^+|^2$ satisfies $0 < x < 1$. Thus the ground state entanglement entropy of the $N$ ghost-spin chain is manifestly positive definite, for $N$ even.

The excited states include negative norm states and these can exhibit new entanglement patterns. For instance for the case of 4 ghost-spins above, consider from (3.14) the states

$$|\psi\rangle = \psi^{++++}|++++\rangle + \psi^{---}|---\rangle + \psi^{+++}|++-\rangle + \psi^{-++}|--\rangle,$$

$$\langle\psi|\psi\rangle = |\psi^{++++}|^2 + |\psi^{---}|^2 - |\psi^{+++}|^2 - |\psi^{-++}|^2 = \pm 1,$$ (3.25)

where we are normalizing positive/negative norm states to $\pm 1$ respectively. The negative norm states are necessarily “more excited”, i.e. have a larger contribution of the negative norm excited states to the total norm.

For odd number $N$ of ghost-spins, the ground state continues to be of the above form (3.24): however it is no longer uniformly positive norm, since $\langle\psi_g|\psi_g\rangle$ is negative for $|\psi_g^+\rangle = 0$. Thus no interpretation in terms of negative norm states being “more excited” is possible for $N$ odd since the ground states themselves are not uniformly positive norm. Instead, at each energy level, there are equal numbers of positive and negative norm states for $N$ odd.

For 4 ghost-spins, the reduced density matrix for the subsystem $A$ comprising a single ghost-spin (say the first index) after tracing over the 3 ghost-spins, generalizing (2.7), is

$$(\rho_A)^{\alpha\beta} = \gamma_{\sigma_1\rho_1}\gamma_{\sigma_2\rho_2}\gamma_{\sigma_3\rho_3}\psi^{\alpha\sigma_1\sigma_2\sigma_3}|\psi^\star\rangle^{\beta\rho_1\rho_2\rho_3} = \gamma_{\sigma_1\sigma_1}\gamma_{\sigma_2\sigma_2}\gamma_{\sigma_3\sigma_3}\psi^{\alpha\sigma_1\sigma_2\sigma_3}|\psi^\star\rangle^{\beta\sigma_1\sigma_2\sigma_3}.$$ (3.26)

Explicitly, for the states (3.25), this becomes

$$(\rho_A)^{++} = |\psi^{++++}|^2 - |\psi^{+++}|^2 = 0,$$

$$(\rho_A)^{+-} = 0,$$

$$(\rho_A)^{-+} = |\psi^{+++}|^2 - |\psi^{---}|^2.$$ (3.27)
We have chosen a family of states that lead to a diagonal RDM for convenience.) The mixed
index reduced density matrix then becomes

\[ (\rho_A)_{++} = |\psi^{++}|^2 - |\psi^{+-}|^2, \quad (\rho_A)_{--} = -|\psi^{-+}|^2 + |\psi^{--}|^2. \] (3.28)

The entanglement patterns here can be analysed using the norm and setting

\[ |\psi^{++}|^2 - |\psi^{+-}|^2 \equiv x, \quad \langle \psi|\psi \rangle = x + (\pm 1 - x) ; \]

\[ (\rho_A)_{++} = x, \quad (\rho_A)_{--} = \pm 1 - x, \]

\[ S_A = -x \log x - (\pm 1 - x) \log(\pm 1 - x) . \] (3.29)

This is very similar to the case of one spin and two ghost-spins analysed in [2], sec. 5. As
in that case, we see that for \( x > 0 \) we have \( 1 - x > 0 \) for positive norm states, implying
\( 0 < x < 1 \), and \( \rho_A \) is positive definite, with \( S_A > 0 \).
If \( x < 0 \), then \( (1 - x) > 0 \), giving \( S_A = |x| \log |x| - (1 + |x|) \log(1 + |x|) + i\pi |x| \), with
\( ReS_A < 0 \) and \( ImS_A \neq 0 \) for positive norm states. In this case, we have \( x < 0 \) implying that
\( |\psi^{++}|^2 < |\psi^{+-}|^2 \), i.e. there is a larger contribution from the negative norm excited state
component \( |++-\rangle \), than the positive norm ground state \( |++++\rangle \) component: it is these
components that arise since the reduced density matrix involves these particular segregations
of the full state. Note that there are several fully positive norm states comprising other states
at the first level, e.g. \( |++-\rangle, |---+\rangle \). These have a positive reduced density matrix
and positive entanglement.

This sort of structure is also true more generally: e.g. for two ghost-spins, the generic
state with norm is

\[ |\psi \rangle = \psi^{++}|++\rangle + \psi^{+-}|+-\rangle + \psi^{-+}|-+\rangle + \psi^{--}|--\rangle , \]

\[ \langle \psi|\psi \rangle = |\psi^{++}|^2 + |\psi^{+-}|^2 - |\psi^{-+}|^2 - |\psi^{--}|^2 = \pm 1 . \] (3.30)

From the entanglement patterns in [1], [2], reviewed in sec. 2, we have seen that the sub-
family (2.10), (2.11), with a diagonal RDM shows in fact that positive norm states lead to
a positive RDM and positive entanglement, while negative norm states have a negative de-
finite RDM (all eigenvalues negative) giving \( Re(EE) < 0 \) and \( Im(EE) \neq 0 \). To explore the
interpretation a little further, consider setting \( \psi^{-+} = 0 \), i.e. the state and reduced density
matrix from (2.7), (2.8), are

\[ |\psi \rangle = \psi^{++}|++\rangle + \psi^{--}|--\rangle + \psi^{+-}|+-\rangle + \psi^{-+}|-+\rangle , \quad (\rho_A)_{++} = |\psi^{++}|^2 - |\psi^{+-}|^2 , \quad (\rho_A)_{--} = |\psi^{--}|^2 > 0 , \]

(3.31)

where we suppress writing the off-diagonal components of \( \rho_A \) given the considerations be-
low. For small \( \psi^{+-} \), with \( |\psi^{+-}|^2 < |\psi^{++}|^2 \), this state is positive norm and thus gives a
positive RDM and entanglement entropy. However for larger $\psi^{+-}$, the effective probability $\rho^\pm_+$ decreases due to the negative norm nature of $\psi^{+-}$: for $|\psi^{+-}| = |\psi^{++}|$, this probability $\rho^+_+$ vanishes. Beyond this, the effective “probability” $\rho^+_+$ is negative and so the entanglement entropy is not positive definite. To give further perspective, consider an observable $O$ we have taken $O\rho O$ (obtaining a diagonal reduced density matrix)

$$\rho\psi\text{ positive RDM and entanglement entropy. However for larger } \psi^{+-}, \text{ the effective probability } \rho^+_+ \text{ decreases due to the negative norm nature of } \psi^{+-}: \text{ for } |\psi^{+-}| = |\psi^{++}|, \text{ this probability } \rho^+_+ \text{ vanishes. Beyond this, the effective “probability” } \rho^+_+ \text{ is negative and so the entanglement entropy is not positive definite. To give further perspective, consider an observable } O, \text{ we have taken } O\rho O \text{ (obtaining a diagonal reduced density matrix).}

The correlation function of $O_{s_1}$ in the state $|\psi\rangle$ is

$$\langle \psi | O_{s_1} | \psi \rangle = (\psi^\dagger)^{\alpha\beta} \langle \alpha | O_{s_1}^{\alpha\beta} | \alpha \rangle = (\psi^\dagger)^{\alpha\beta} O_{s_1}^{\alpha\beta} (\psi)^{\alpha\beta} \langle \alpha | \beta \rangle \langle \beta | \alpha \rangle \langle \rho | \beta \rangle$$

$$= \gamma \sigma \gamma \beta \alpha \rho_{\alpha}^{\alpha\beta} O_{s_1}^{\alpha\beta} = (\rho_{\alpha})^{\alpha\beta} (O_{s_1})^{\alpha\beta} = (\rho_{\alpha})^{\alpha\beta} (O_{s_1})^{\alpha\beta} + (\rho_{\alpha})^{\alpha\beta} (O_{s_1})^{\alpha\beta} = (O_{s_1})^{\alpha\beta} . \quad (3.32)$$

The expectation value becomes

$$\frac{\langle \psi | O_{s_1} | \psi \rangle}{\langle \psi | \psi \rangle} = \frac{(\rho_{\alpha})^{\alpha\beta} (O_{s_1})^{\alpha\beta} + (\rho_{\alpha})^{\alpha\beta} (O_{s_1})^{\alpha\beta}}{(\rho_{\alpha})^{\alpha\beta} + (\rho_{\alpha})^{\alpha\beta}} \quad (3.33)$$

For ordinary spins, $(\rho_{\alpha})^+^+$ and $(\rho_{\alpha})^-^-$ are always positive with $tr \rho_{\alpha} = (\rho_{\alpha})^+^+ + (\rho_{\alpha})^-^-$ = 1. For ghost-spins, $(\rho_{\alpha})^+_+$, $(\rho_{\alpha})^-^-$ can become negative, with $tr \rho_{\alpha} = (\rho_{\alpha})^+^+ + (\rho_{\alpha})^-^-$ = ±1 for positive/negative norm states. In particular, for (3.31) with positive norm states, as $(\rho_{\alpha})^+_+ \rightarrow 0$ through positive values, we have $(\rho_{\alpha})^-^-$ $\rightarrow 1$, and the expectation value becomes

$$(\rho_{\alpha})^+_+(O_{s_1})^+_+ + (\rho_{\alpha})^-^-(O_{s_1})^-^- \xrightarrow{(\rho_{\alpha})^+_+ \rightarrow 0} (O_{s_1})^-^- . \quad (3.34)$$

In other words, in the limit $|\psi^{+-}| \rightarrow |\psi^{++}|$ with $(\rho_{\alpha})^+_+ \rightarrow 0$, the state behaves as if the expectation value of an observable cares only about the $O^-^-$ component: the $O^-^-$ component, in general nonzero, does not contribute. Similar phenomena occur with $N$ ghost-spins as well.

For $N$ ghost-spins with $N$ even, the structure of ground states and excited states at the first level (3.17) (3.19) again suggests considering states of the form (for conveniently obtaining a diagonal reduced density matrix)

$$|\psi\rangle = \psi^{+N}|+^N\rangle + \psi^{-N}|-^N\rangle + \psi^{++\ldots+}|++\ldots+-\rangle + \psi^{-+\ldots-}|---\ldots-\rangle + \psi^{---\ldots+}|---\ldots+\rangle \quad (3.35)$$

$$|\psi^{+N}|^2 + |\psi^{-N}|^2 + |\psi^{++\ldots+}|^2 - |\psi^{-+\ldots-}|^2 = \pm 1 . \quad (3.36)$$

Again the reduced density matrix for the first index ghost-spin, after tracing over the remaining $N-1$ ghost-spins, can be seen to have the simple diagonal form

$$(\rho_{\alpha})^{++} = |\psi^{+N}|^2 - |\psi^{++\ldots+}|^2 , \quad (\rho_{\alpha})^{+-} = 0 , \quad (\rho_{\alpha})^{-+} = 0 , \quad (\rho_{\alpha})^{--} = |\psi^{-+\ldots+}|^2 - |\psi^{-N}|^2 , \quad (3.37)$$
and the mixed index reduced density matrix becomes
\[
(\rho_A)_+^+ = |\psi^+|^2 - |\psi^{+\ldots+}|^2, \quad (\rho_A)_-^- = -|\psi^{-\ldots+}|^2 + |\psi^{-N}|^2. \tag{3.38}
\]

With \(x \equiv |\psi^+|^2 - |\psi^{+\ldots+}|^2\), the entanglement patterns are similar to the 4 ghost-spin case (3.29). Thus \(x < 0\) leads to \(\text{Re} S_A < 0\) and \(\text{Im} S_A \neq 0\): \(x < 0\) means \(\text{Re} S_A < 0\) and \(\text{Im} S_A \neq 0\).

For ensembles of ghost-spins and spins, possible Hamiltonians might be of the form
\[
H = H_s + H_{gs} = -J_s \sum_{nn} s_s s'_s - J_{gs} \sum_{nn} s_{gs} s'_{gs},
\]
with the spin and ghost-spin sectors having nearest neighbour interactions within themselves but with the spins being decoupled from the ghost-spins. Then the ground states might be expected to be disentangled product states.

### 3.2 The reduced density matrix and its eigenvalues

Let us now study the reduced density matrix obtained after tracing over ghost-spins and its eigenvalues in some generality. First, it is useful to study explicit examples (e.g. 2, 4 ghost-spins etc). As a simple case, consider a 2-spin state
\[
|\psi\rangle = c_1 |++\rangle + c_2 |-\rangle + c_3 |++\rangle + c_4 |--\rangle. \tag{3.39}
\]

The reduced density matrix and eigenvalue equation are
\[
\begin{pmatrix}
  c_1 c_1^* & c_2 c_2^* & c_3 c_3^* & c_4 c_4^* \\
  c_3 c_1^* & c_4 c_2^* & c_3 c_3^* & c_4 c_4^*
\end{pmatrix}
\to
\begin{pmatrix}
  c_1 c_1^* & c_2 c_2^* - \lambda & c_3 c_3^* & c_4 c_4^* \\
  c_3 c_1^* & c_4 c_2^* & c_3 c_3^* & c_4 c_4^* + \lambda
\end{pmatrix} = 0
\]
\[
\begin{pmatrix}
  c_1 c_1^* & c_2 c_2^* & c_3 c_3^* & c_4 c_4^* \\
  c_3 c_1^* & c_4 c_2^* & c_3 c_3^* & c_4 c_4^*
\end{pmatrix}
\]
where the top signs correspond to ordinary spins, while the bottom signs pertain to ghost-spins. For ordinary spins, the basis states are all positive norm while for ghost-spins, a single minus sign gives negative norm as we have seen. Thus for an ensemble of ordinary spins, the eigenvalue equation is
\[
\det((\rho_A)^{jk} - \lambda \delta^{jk}) = 0 \to \lambda^2 - (\text{tr}\rho_A)\lambda + \det\rho_A = 0. \tag{3.41}
\]

Since there are only positive norm states here, \(\text{tr}\rho_A = 1\) giving
\[
\lambda^2 - \lambda - \det\rho_A = (\lambda - \frac{1}{2})^2 - \frac{1}{4} + \det\rho_A = 0 \implies \lambda = \frac{1}{2} = \pm \sqrt{\frac{1}{4} - \det\rho_A}. \tag{3.42}
\]
Simplifying from (3.40) above, it can be seen that
\[ \det \rho_A = |c_1c_4 - c_2c_3|^2. \]  
(3.43)

Since the norm condition on the state for ordinary spins is \( \sum_i |c_i|^2 = 1 \), we see that each \( c_i \) is bounded with \( 0 \leq |c_i| \leq 1 \) implying that \( \det \rho_A \) is bounded (with maximum value \( \frac{1}{4} \)). This in turn implies that the eigenvalues \( \lambda \) are always real.

Now consider ghost-spins: the eigenvalue equation for the mixed index reduced density matrix is
\[ (\rho_A)^k_i e_k = \lambda e_i = \lambda \delta_i^k e_k \quad \Rightarrow \quad (\rho_A)^j_i e_j = \gamma^{jk} (\rho_A)^j_k e_j = \gamma^{ij} \lambda e_j, \]  
(3.44)
giving
\[ \det \left( (\rho_A)^k - \lambda \gamma^{jk} \right) = 0 \quad \Rightarrow \quad \lambda^2 - (\text{tr} \rho_A) \lambda - \det \rho_A^{ik} = 0. \]  
(3.45)

Since \( \text{tr} \rho_A = (\rho_A)^{++} - (\rho_A)^{--} = \pm 1 \) for positive/negative norm states respectively, this becomes
\[ \lambda^2 - \lambda - \det \rho_A^{ik} = 0 \quad = \quad \left( \lambda + \frac{1}{2} \right)^2 - \frac{1}{4} - \det \rho_A^{ik} \quad \Rightarrow \]
\[ \lambda = \frac{1}{2} \pm \sqrt{\frac{1}{4} + \det \rho_A^{ik}} \quad [\text{+ve norm}]; \quad \lambda = -\frac{1}{2} \pm \sqrt{\frac{1}{4} + \det \rho_A^{ik}} \quad [\text{-ve norm}]. \]  
(3.46)

From (3.40) specializing to 2 ghost-spins, it can be seen that
\[ \det \rho_A^{ik} = -|c_1c_4 - c_2c_3|^2. \]  
(3.47)

However in this case, the norm condition gives
\[ |c_1|^2 + |c_4|^2 - |c_2|^2 - |c_3|^2 = \pm 1, \]  
(3.48)
so that \( |c_i| \) are not forced to be bounded, but in fact can be arbitrarily large while retaining the norm condition. For positive norm states, we have \( |c_1|, |c_4| > |c_2|, |c_3|, \) with \( 0 \leq |c_i| < \infty \), while for negative norm states, we have \( |c_2|, |c_3| > |c_1|, |c_4| \). This makes the determinant potentially unbounded, as we will see below. From the norm condition, we see that for positive norm states, the \( c_i \) can be parametrized as \( c_1 = \cosh \theta \cos \phi_1 e^{i\alpha_1}, \ c_4 = \cosh \theta \sin \phi_1 e^{i\alpha_4}, \ c_2 = \sinh \theta \cos \phi_2 e^{i\alpha_2}, \ c_3 = \sinh \theta \sin \phi_2 e^{i\alpha_3}, \) while for negative norm states, the parametrizations can be switched as \( c_1, c_4 \leftrightarrow c_2, c_3 \). On the real slice, we have all \( c_i \) real, \( i.e. \) the phases \( \alpha_i \) are all zero.

The minus signs in the norm make the determinant behaviour non-uniform: there are several branches. It is easiest to illustrate this on the real slice with all \( c_i \) real. First consider
the 1-parameter family of states (2.10) which give a diagonal reduced density matrix: from (3.40), these have

$$\rho_A^{\pm-} = c_1 c_3 - c_2 c_4 \Rightarrow (c_1^2 - c_2^2) \left(1 + \frac{c_3^2}{c_1^2}\right) = \pm 1 ,$$

and $$\rho^{++} = \pm x , \quad \rho^{--} = \mp (1 - x) , \quad x = \frac{c_1^2}{c_1^2 + c_4^2} ,$$

(3.49)

and

$$\det \rho_A^{ik} = - \left| c_1 c_4 - c_2 \left( \frac{c_2 c_4}{c_1} \right) \right|^2 = - \frac{c_1^2 c_4^2}{(c_1^2 + c_4^2)^2} = - x(1 - x) .$$

(3.50)

It can be seen now that

$$\frac{1}{4} + \det \rho_A^{ik} = \frac{1}{4} - x(1 - x) = \left(x - \frac{1}{2}\right)^2 > 0 ,$$

(3.51)

so that the eigenvalues \( \lambda \) on this diagonal 1-parameter branch are always real. At the point \( x = \frac{1}{2} \) the eigenvalues are both \( \frac{1}{2} \) for positive norm states.

On the other hand, a distinct branch arises taking \( c_3 = 0 \), i.e. the states (3.31): this gives

$$c_1^2 + c_4^2 - c_2^2 = \pm 1 , \quad \det \rho_A^{ik} = - |c_1 c_4|^2 .$$

(3.52)

For \( c_2 = 0 \), this state is positive norm: it continues to be positive norm for small nonzero \( c_2 \) so \( \det \rho_A^{ik} \) is small as well and the eigenvalues continue to be real. However for \( c_1, c_4, c_2 \) large satisfying the norm condition, the determinant is large and negative rendering the eigenvalues \( \lambda \) complex using (3.46), even for positive norm states. The real and complex \( \lambda \) branches intersect at a locus of coinciding eigenvalues \( \lambda = \frac{1}{2} \) when \( \det \rho_A^{ik} = - \frac{1}{4} \).

Thus we see that for 2 ghost-spins, the reduced density matrix exhibits several distinct branches with the eigenvalue spectrum varying from real to complex, much unlike the ordinary spin case.

It can also be checked that for an ensemble of one spin and two ghost-spins, tracing over all the ghost-spins leads to a reduced density matrix for the spin alone whose eigenvalue equation is again of the form above, for spins alone.

**Zero norm states**: There are zero norm states in ghost-spin systems of the sort we have been discussing: e.g. states \( |\uparrow\rangle \) and \( |\downarrow\rangle \) are zero norm themselves. To study entanglement in these cases, consider the 2 ghost-spin case: zero norm states (3.39) have \( \langle \psi | \psi \rangle = 0 \), i.e.

$$|c_1|^2 + |c_4|^2 = |c_2|^2 + |c_3|^2 \quad \text{and} \quad \text{tr} \rho_A = 0 .$$

(3.53)

We also have \( \det \rho_A^{ik} = - |c_1 c_4 - c_2 c_3|^2 \) as above. So the eigenvalue equation (3.45) is

$$\lambda^2 = \det \rho_A^{ik} < 0$$

(3.54)
since \( tr \rho_A = 0 \) and the eigenvalues are always pure imaginary. The entanglement entropy can of course have real and imaginary parts on evaluating this. Again \( det \rho^k_A \) can acquire large negative values: \textit{e.g.} on the branch \( c_3 = 0 \), we have \( det \rho^k_A = -|c_1 c_4|^2 \) which becomes large and negative when \( c_1, c_4 \) are large. Most basically however, zero norm states do not have any canonical normalization: an overall scaling changes the \( c_i \) and therefore \( \lambda \) as well.

### 3.3 RDM, eigenvalues and \( \uparrow\leftrightarrow\downarrow \) symmetry

In the \( \uparrow, \downarrow \)-basis, we have \( \gamma_{1\uparrow} = \gamma_{\downarrow\uparrow} = 1 \): then the 2 ghost-spin state (3.39) is

\[
|\psi\rangle = \psi^{\uparrow\uparrow}|\downarrow\downarrow\rangle + \psi^{\uparrow\downarrow}|\downarrow\uparrow\rangle + \psi^{\downarrow\uparrow}|\uparrow\downarrow\rangle + \psi^{\downarrow\downarrow}|\uparrow\uparrow\rangle
\]  

(3.55)

with norm

\[
\langle \psi | \psi \rangle = (\psi^{\uparrow\uparrow})^{\dagger} \psi^{\uparrow\downarrow} + (\psi^{\uparrow\downarrow})^{\dagger} \psi^{\uparrow\uparrow} + (\psi^{\downarrow\uparrow})^{\dagger} \psi^{\downarrow\downarrow} + (\psi^{\downarrow\downarrow})^{\dagger} \psi^{\downarrow\uparrow} = \pm 1 .
\]  

(3.56)

The reduced density matrix \( (\rho_A)^{\alpha\kappa} = \gamma_{\beta\lambda} \psi^{\alpha \beta} \psi^{\kappa \lambda} \) after tracing over the second ghost-spin is

\[
\det \left( \begin{array}{c}
(\psi^{\uparrow\uparrow})^{\dagger} \psi^{\uparrow\downarrow} + (\psi^{\uparrow\downarrow})^{\dagger} \psi^{\uparrow\uparrow} \\
(\psi^{\downarrow\uparrow})^{\dagger} \psi^{\downarrow\downarrow} + (\psi^{\downarrow\downarrow})^{\dagger} \psi^{\downarrow\uparrow}
\end{array} \right)
\rightarrow
\det \left( \begin{array}{c}
(\psi^{\uparrow\uparrow})^{\dagger} \psi^{\uparrow\downarrow} + (\psi^{\uparrow\downarrow})^{\dagger} \psi^{\uparrow\uparrow} \\
(\psi^{\downarrow\uparrow})^{\dagger} \psi^{\downarrow\downarrow} + (\psi^{\downarrow\downarrow})^{\dagger} \psi^{\downarrow\uparrow} - \lambda
\end{array} \right) = 0 .
\]  

(3.57)

The eigenvalue equation in the second line becomes

\[
\lambda^2 - (tr \rho^k_A) \lambda - \det \rho^k_A = 0
\]  

(3.58)

with

\[
tr \rho_A = \rho^{\uparrow\downarrow}_A + \rho^{\downarrow\uparrow}_A = \langle \psi | \psi \rangle = \pm 1 , \quad det \rho^k_A = -|\psi^{\uparrow\downarrow} \psi^{\uparrow\uparrow} - \psi^{\downarrow\uparrow} \psi^{\downarrow\downarrow}|^2 .
\]  

(3.59)

So for a state \( \frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle \pm |\downarrow\downarrow\rangle) \), we obtain \( \rho_A = (\begin{pmatrix} 0 & \pm 1/2 \\ \mp 1/2 & 0 \end{pmatrix}) \) giving \( (\lambda \mp \frac{1}{2})^2 = 0 \) which is \( \lambda = \frac{1}{2}, \frac{1}{2} \) for \( +ve \) norm and \( \lambda = -\frac{1}{2}, -\frac{1}{2} \) for \( -ve \) norm. Other previous cases can be recast in this basis as well.

Consider now a symmetry \( \uparrow\leftrightarrow\downarrow \) which exchanges up and down ghost-spins. Retaining only states invariant under this \( \uparrow\leftrightarrow\downarrow \) symmetry, the general state above collapses to

\[
|\psi\rangle = \psi^{\uparrow\downarrow}(|\downarrow\downarrow\rangle + |\uparrow\uparrow\rangle) + \psi^{\downarrow\uparrow}(|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle)
\]  

(3.60)

and the norm is

\[
\langle \psi | \psi \rangle = 2|\psi^{\uparrow\downarrow}|^2 + 2|\psi^{\downarrow\uparrow}|^2 > 0 , \quad tr \rho^k_A = \langle \psi | \psi \rangle = +1 .
\]  

(3.61)
This is always positive definite. The RDM above becomes
\[
\rho_{ik}^A = \begin{pmatrix}
(\psi^*)^\dagger \psi^\downarrow & + (\psi^*)^\dagger \psi^\uparrow \\
|\psi^\downarrow|^2 + |\psi^\uparrow|^2 & + |\psi^\downarrow|^2 + |\psi^\uparrow|^2 \\
(\psi^*)^\dagger \psi^\downarrow & + (\psi^*)^\dagger \psi^\uparrow
\end{pmatrix}
\] (3.62)

The determinant is
\[
\det \rho_{ik}^A = -\left| \frac{1}{2} - 2(\psi^\downarrow)^2 \right|^2
\] (3.63)
so
\[
\lambda^2 - \lambda + \left| \frac{1}{2} - 2(\psi^\downarrow)^2 \right|^2 = 0 \Rightarrow \left( \lambda - \frac{1}{2} \right)^2 = \frac{1}{4} - \left| \frac{1}{2} - 2(\psi^\downarrow)^2 \right|^2 > 0 .
\] (3.64)

This is quite like the case for ordinary spins. Since \(|\psi^\downarrow| < \frac{1}{2}\), we have the determinant bounded and so \(\lambda\) above is real, positive and bounded with \(\sum_i \lambda_i = 1\). So \(S_A > 0\).

In the \(\pm\)-basis, the \(\uparrow\leftrightarrow\downarrow\) symmetry is even more strikingly simple: we see that the \(|-\rangle\) state simply collapses as \(|-\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle - |\downarrow\rangle) \rightarrow 0\) leaving only the \(|+\rangle\) state which is positive definite. Thus truncating all states in any ensemble of ghost-spins to only those invariant under \(\uparrow\leftrightarrow\downarrow\) symmetry renders the ghost-spin Hilbert space manifestly positive definite.

### 3.4 Modified inner product and unitarity

It is known that various non-Hermitian Hamiltonians admit \(PT\)-symmetric extensions [16,17] which render the system unitary. In light of the fairly ordinary looking positive definite ghost-spin partition functions \textit{e.g.} (3.6), (3.20), it is interesting to ask if there is a modified inner product that leads to an effectively unitary structure for these systems (see \textit{e.g.} [18] for similar discussions in the context of 3-dim symplectic fermion theories).

Consider introducing an operator \(C\) such that nonzero expectation values are obtained only after a \(C\) insertion: \textit{i.e.}
\[
\langle \downarrow | \downarrow \rangle = 0 = \langle \uparrow | \uparrow \rangle, \quad \langle \downarrow | C | \downarrow \rangle = 1 = \langle \uparrow | C | \uparrow \rangle .
\] (3.65)

Then a generic ghost-spin state and its adjoint can be defined as
\[
|\psi\rangle = c_1 |\uparrow\rangle + c_2 |\downarrow\rangle, \quad (|\psi\rangle)^\dagger = c_1^* \langle \uparrow | + c_2^* \langle \downarrow | .
\] (3.66)

Using the above inner products with \(C\) insertions, we have the modified inner product for the state as
\[
\left( (|\psi\rangle)^\dagger, |\psi\rangle \right) \equiv \langle \psi | C | \psi \rangle = c_1 c_1^* \langle \uparrow | C | \uparrow \rangle + c_2 c_2^* \langle \downarrow | C | \downarrow \rangle = |c_1|^2 + |c_2|^2 ,
\] (3.67)
which is positive definite, thus defining a unitary structure on the Hilbert space. Thus all states are now positive norm: in particular the \(|\pm\rangle\) states have norm
\[
\langle \pm | C | \pm \rangle \propto \langle \uparrow | C | \uparrow \rangle + \langle \downarrow | C | \downarrow \rangle > 0 .
\] (3.68)
The fact that the partition functions previously discussed resemble those for an ordinary spin system can be taken to imply the existence of such an operator $C$ and the above unitary modification of the inner product to be positive definite. With this unitary inner product, the reduced density matrix for any subsystem of ghost-spin states is always positive definite and therefore so is the entanglement entropy.

Now consider coupling an ensemble of ghost-spins to an ensemble of ordinary spins. Define the inner product on states to be the usual one for the spin sector and to be the above unitary inner product on the ghost-spin sector. For instance in the one spin and two ghost-spins system, a family of states (which formerly contained negative norm states) and the associated inner product then are

$$|\psi\rangle = \psi_{\uparrow,++}|\uparrow,++\rangle + \psi_{\uparrow,+-}|\uparrow,+-\rangle + \psi_{\downarrow,-+}|\downarrow,-+\rangle + \psi_{\downarrow,--}|\downarrow,--\rangle,$$

$$\langle\psi|\psi\rangle = \ldots + |\psi_{\uparrow,+-}|^2 + |\psi_{\downarrow,-+}|^2 > 0,$$

(3.69)

using the above unitary inner products for $|\pm\rangle$. This is always positive definite so there are no negative norm states. In fact this now maps the spin and 2 ghost-spins system to a system of 3 ordinary spins. However the physical system originally was a single spin coupled with 2 ghost-spins: the ghost-spins are regarded as unphysical, reflecting the negative norm subsector arising from fixing a gauge symmetry. The physical subsector therefore is the single spin. From this point of view, the mapping to a system of 3 ordinary spins is a formal process since the physical subspace of the original system continues to be the single spin. The modified inner product in the “ghost spin” sector unitarises the system. This process in our case turns out to be a formal tool to “explain” why we get relatively ordinary looking partition functions for our choice of the Hamiltonian. So we will not pursue this $PT$-symmetric formulation further here.

Interesting generalizations of the finite ghost-spin chains we have been studying so far involve infinite ghost-spin chains and their possible continuum limits at criticality where a conformal field theory may emerge. We will study one concrete class of examples in the next section.

4 Ghost-spin chains and the $bc$-ghost CFT

In this section we will look at a family of infinite ghost-spin chains with a different interaction, although still based on the ghost-spins used so far treated as the underlying microscopic variables. Motivated by the well-known fact that the Ising spin chain at criticality is described by a CFT of free massless fermions (see e.g. [14], [15]), one might expect that infinite ghost-spin chains exhibit critical points at which a continuum description of the chain exists in terms of ghost-CFTs such as the $bc$-CFT (discussed extensively in e.g. [3, 4], as well
as [5], and more recently [6–10]). The off-diagonal inner products for states here reflect the off-diagonal oscillator algebra \( \{ b_n, c_m \} = \delta_{n+m,0} \) of the \( bc \)-ghost CFT.

In this light, consider an infinite 1-dimensional ghost-spin chain with a nearest neighbour interaction Hamiltonian

\[
H = J \sum_n \left( \sigma_b(n)\sigma_c(n+1) + \sigma_b(n)\sigma_c(n-1) \right),
\]

where \( \sigma_b \) and \( \sigma_c \) are two species of 2-state spin variables defined at each site and \( n \) labels the lattice site in the chain. The nearest neighbour interaction in this Hamiltonian is more akin to a hopping type interaction than the Ising type Hamiltonian in (3.1), (3.16): we will describe this in detail later. The spin variables \( \sigma_b, \sigma_c \) satisfy the (anti-)commutation relations

\[
\{ \sigma_b, \sigma_c \} = 1,
\]

\[
[\sigma_b, \sigma_b^\dagger] = [\sigma_c, \sigma_c^\dagger] = [\sigma_b, \sigma_c^\dagger] = 0,
\]

which are consistent with the off-diagonal inner product between ghost-spin states. These spin variables are self-adjoint and act on the two states \( |\uparrow\rangle, |\downarrow\rangle \), at each lattice site \( n \), as

\[
\sigma_b^\dagger = \sigma_b, \quad \sigma_c^\dagger = \sigma_c ;
\]

\[
\sigma_b |\downarrow\rangle = 0, \quad \sigma_b |\uparrow\rangle = |\downarrow\rangle, \quad \sigma_c |\uparrow\rangle = 0, \quad \sigma_c |\downarrow\rangle = |\uparrow\rangle.
\]

Thus the \( \sigma_b \) act as lowering operators while the \( \sigma_c \) act as raising operators. It is worth noting that the \( \sigma_b, \sigma_c \) cannot be \( 2 \times 2 \) Pauli matrices, since the latter satisfy \( \{ \sigma^-, \sigma^+ \} = 1 \) but with \( (\sigma^-)^\dagger = \sigma^+ \). The present algebra is off-diagonal, with hermitian operators.

As an example, for 2 ghost-spins, we have 4 states, \( |\downarrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\uparrow\uparrow\rangle \). These states can be expressed as

\[
|\downarrow\uparrow\rangle = \sigma_c^\dagger |\downarrow\downarrow\rangle, \quad |\uparrow\downarrow\rangle = \sigma_c |\downarrow\downarrow\rangle, \quad |\uparrow\uparrow\rangle = \sigma_c^\dagger \sigma_c |\downarrow\downarrow\rangle = \sigma_c^\dagger \sigma_c |\downarrow\downarrow\rangle,
\]

the last expression implying that the \( \uparrow \)-excitations in \( |\uparrow\uparrow\rangle \) have no particular order. As is natural for spin systems, the spin \( \sigma \)-variables at distinct lattice sites commute as in (4.2), e.g. \( \sigma_c\sigma_c' = \sigma_c',\sigma_c \). Now with the off-diagonal inner-products between states, we have

\[
\langle \uparrow\uparrow | \downarrow\downarrow \rangle = 1 = \langle \downarrow\downarrow | \uparrow\uparrow \rangle, \quad \langle \downarrow\uparrow | \downarrow\uparrow \rangle = 1 = \langle \uparrow\downarrow | \uparrow\downarrow \rangle.
\]

In the second set of inner products, note that the spins have been ordered right to left in the bra states: this is distinct from that used throughout the paper so far, e.g. (2.5). We have re-ordered in this manner anticipating our description of fermionic excitations in what follows.

The inner products above can be written explicitly in terms of the spin operators as e.g.

\[
\langle \downarrow\downarrow | \uparrow\uparrow \rangle = \langle \downarrow\downarrow | \sigma_c^\dagger \sigma_c^\dagger | \downarrow\downarrow \rangle = 1, \quad \langle \uparrow\downarrow | \uparrow\downarrow \rangle = \langle \uparrow\downarrow | \sigma_c^\dagger \sigma_c^\dagger | \downarrow\downarrow \rangle = 1.
\]

(4.6)
and so on, using $\sigma^\dagger_{ci} = \sigma_{ci}$: in this form, the ordering of spin operators is unimportant since they are commuting, however, it will be relevant once we have fermionic representations of these states. Now a basis of positive and negative norm states for 2 ghost-spins is

$$
|e^1_\pm\rangle = \frac{1}{\sqrt{2}} |\downarrow\downarrow\rangle \pm |\uparrow\uparrow\rangle \quad \Rightarrow \quad \langle e^1_\pm|e^1_\pm\rangle = \pm \frac{1}{2} (\langle \uparrow\uparrow | \downarrow\downarrow \rangle + \langle \downarrow\downarrow | \uparrow\uparrow \rangle) = \pm 1 ,
$$

$$
|e^2_\pm\rangle = \frac{1}{\sqrt{2}} |\downarrow\uparrow\rangle \pm |\uparrow\downarrow\rangle \quad \Rightarrow \quad \langle e^2_\pm|e^2_\pm\rangle = \pm \frac{1}{2} (\langle \downarrow\uparrow | \downarrow\uparrow \rangle + \langle \uparrow\downarrow | \uparrow\downarrow \rangle) = \pm 1 . \quad (4.7)
$$

### 4.1 Ghost-spin chains and fermionic excitations

We want to construct fermionic operators out of the commuting spin operators $\sigma_b, \sigma_c$. This can be achieved using a version of the Jordan-Wigner transformation [14, 15], which we will describe in the next subsection. These fermionic operators satisfy anti-commutation relations

$$
\{a_{bi}, a_{cj}\} = \delta_{ij} , \quad \{a_{bi}, a_{bj}\} = 0 , \quad \{a_{ci}, a_{cj}\} = 0 . \quad (4.8)
$$

So in particular unlike the $\sigma$ spin operators, these anticommute not just at the same site $i$ but also at distinct sites $i, j$. The ket and bra states exhibit the action

$$
a_{b|\downarrow} = 0 , \quad a_{b|\uparrow} = |\downarrow\rangle , \quad a_{c|\uparrow} = 0 , \quad a_{c|\downarrow} = |\uparrow\rangle , \quad \langle \downarrow | a_{b} = 0 , \quad \langle \uparrow | a_{b} = \langle \downarrow | , \quad \langle \downarrow | a_{c} = \langle \uparrow | , \quad \langle \uparrow | a_{c} = 0 . \quad (4.9)
$$

Now to construct ket states and their corresponding bra states, we have to be careful about the ordering of the operators and the spin excitations at the various sites, especially in constructing inner products of states. We adopt the convention that

$$
\langle \uparrow\uparrow | \downarrow\downarrow \rangle = 1 ; \quad |\uparrow\uparrow\rangle = a_{c1}a_{c2}|\downarrow\downarrow\rangle ; \quad \langle \downarrow\downarrow | = \langle \uparrow\uparrow | a_{b1}a_{b2} ,
$$

$$
\Rightarrow \quad \langle \downarrow\downarrow\uparrow\uparrow\rangle = \langle \uparrow\uparrow | a_{b2}a_{b1} a_{c1}a_{c2} |\downarrow\downarrow\rangle = \langle \uparrow\uparrow | \downarrow\downarrow \rangle = 1 , \quad (4.10)
$$

where we have illustrated two fermionic ghost-spins for simplicity. In other words, the underlining right arrow below the spins in the ket state displays the order of the operator excitations to be increasing to the right, and the underlining left arrow below the spins in the bra state shows the order to be increasing to the left. The states $|\downarrow\downarrow\rangle$ and $\langle \uparrow\uparrow |$ above are the empty and filled ket and bra states respectively so the ghost-spins in them are not underlined since they do not need ordering. Likewise for three fermionic ghost-spins, we have

$$
\langle \uparrow\uparrow\uparrow | \downarrow\downarrow\downarrow \rangle = 1 ; \quad |\uparrow\uparrow\uparrow\rangle = a_{c1}a_{c2}a_{c3}|\downarrow\downarrow\downarrow\rangle ; \quad \langle \downarrow\downarrow\downarrow | = \langle \uparrow\uparrow\uparrow | a_{b3}a_{b2}a_{b1} ,
$$

$$
\Rightarrow \quad \langle \downarrow\downarrow\downarrow\uparrow\uparrow\rangle = \langle \uparrow\uparrow\uparrow | a_{b3}a_{b2}a_{b1} a_{c1}a_{c2}a_{c3} |\downarrow\downarrow\downarrow\rangle = \langle \uparrow\uparrow\uparrow | \downarrow\downarrow\downarrow \rangle = 1 . \quad (4.11)
$$

The intuition here is that the ket state being $|\prod_i \downarrow_i\rangle$ corresponds to an empty state, and then an $a_{ci}$ operator acts on it to the right to fill it with a “particle”-like $\uparrow_i$-excitation.
These being fermionic have to be ordered towards the right. By contrast, the bra state corresponding to $\prod_i \psi_i^\dagger$ is a “filled” state and then an $a_{bi}$ operator acts on it to the left to remove a $\uparrow_i$-excitation or create a “hole”-like $\downarrow_i$-excitation. The $a_{bi}$ are ordered increasing towards the left.

Let us now focus on two fermionic ghost-spins and explore further. A state of the form below and its adjoint defined appropriately are

$$|\psi\rangle = \psi_1 |\downarrow\downarrow\rangle + \psi_2 |\uparrow\uparrow\rangle = \psi_1 a_{c1} a_{c2} |\downarrow\downarrow\rangle ,$$

$$\langle\psi| = \psi_1^* \langle\downarrow\downarrow| + \psi_2^* \langle\uparrow\uparrow| = \psi_1^* a_{b1} + \psi_2^* a_{b2} . \quad (4.12)$$

The first expression in each line is written purely in terms of the ordered fermionic ghost-spin basis states while the second expression expresses this in terms of the fermionic ghost-spin operators ordered appropriately, with the spins in the bra going right to left as the underlining arrow indicates. The inner product of these states then is

$$\langle\psi|\psi\rangle = \psi_1^* \psi_2 \langle\downarrow\downarrow|\uparrow\uparrow\rangle + \psi_2^* \psi_1 \langle\uparrow\uparrow|\downarrow\downarrow\rangle = \psi_1^* \psi_2 + \psi_2^* \psi_1 . \quad (4.13)$$

This is the expected indefinite norm so the system contains negative norm states: for instance $|\downarrow\downarrow\rangle - |\uparrow\uparrow\rangle$ has norm $-2$. This definition of the adjoints $(4.12)$ is consistent with the off-diagonal inner products of the commuting spin states $(4.5)$.

The rule for constructing the adjoint state is to write the bra state with the spins written as in the ket, but ordered right to left (along the underlining arrow in the bra states). The states $|\downarrow\downarrow\rangle$ and $\langle\uparrow\uparrow|$ as stated below $(4.10)$ do not need ordering, while for instance, the ket $|\downarrow\uparrow\rangle$ has adjoint $\langle\uparrow\downarrow|$. Thus using these basis states, we have states and their adjoints,

$$|\psi\rangle = \psi_1 |\downarrow\uparrow\rangle + \psi_2 |\uparrow\downarrow\rangle = \psi_1 a_{c1} a_{c2} |\downarrow\downarrow\rangle + \psi_2 a_{c1} a_{c2} |\downarrow\downarrow\rangle ,$$

$$\langle\psi| = \psi_1^* \langle\downarrow\uparrow| + \psi_2^* \langle\uparrow\downarrow| = \psi_1^* a_{b1} + \psi_2^* a_{b2} . \quad (4.14)$$

The inner product is

$$\langle\psi|\psi\rangle = \psi_1^* \psi_2 \langle\downarrow\uparrow|\uparrow\downarrow\rangle + \psi_2^* \psi_1 \langle\uparrow\downarrow|\downarrow\uparrow\rangle = \psi_1^* \psi_2 + \psi_2^* \psi_1 . \quad (4.15)$$

This is again the expected indefinite norm: e.g. $|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle$ is a negative norm state with norm $-2$. We see that a state $|\downarrow\uparrow\rangle$ (with $\psi_2 = 0$) has adjoint $\langle\uparrow\downarrow|$ and zero norm since $\langle\psi|\psi\rangle = \langle\uparrow\downarrow|\downarrow\uparrow\rangle = \langle\uparrow\downarrow|a_{b1} a_{c2} |\downarrow\downarrow\rangle = 0$.

Consider now 3 fermionic ghost-spins, and states/adjoints,

$$|\psi\rangle = \psi_1 |\uparrow\uparrow\uparrow\rangle + \psi_2 |\uparrow\downarrow\uparrow\rangle = \psi_1 a_{c1} a_{c2} a_{c3} |\downarrow\downarrow\downarrow\rangle ,$$

$$\langle\psi| = \psi_1^* \langle\uparrow\uparrow\uparrow| + \psi_2^* \langle\uparrow\downarrow\uparrow| = \psi_1^* a_{b1} a_{b2} |\downarrow\downarrow\downarrow\rangle + \psi_2^* a_{b2} |\downarrow\downarrow\downarrow\rangle . \quad (4.16)$$
The norm of this state $|\psi\rangle$ is given by the inner product

$$\langle \psi | \psi \rangle = \psi_1^* \psi_2 \langle \uparrow \uparrow \uparrow \uparrow | \uparrow \uparrow \uparrow \uparrow \rangle + \psi_2^* \psi_1 \langle \uparrow \uparrow \uparrow \uparrow | a_{b3} a_{b1} a_{c1} a_{c3} \downarrow \downarrow \downarrow \rangle + \psi_2^* \psi_1 \langle \uparrow \uparrow \uparrow \uparrow | a_{b2} a_{c2} \downarrow \downarrow \downarrow \rangle = \psi_1^* \psi_2 + \psi_2^* \psi_1 \ , \quad (4.17)$$

which is the expected indefinite norm. Along these lines, note that a state of the form $|\psi\rangle = |\downarrow \uparrow \rangle = a_{c2} a_{c3} \downarrow \downarrow \downarrow \rangle$ has its adjoint $\langle \psi | = \langle \uparrow \downarrow | = \langle \uparrow \downarrow | a_{b1}$; this has zero norm, since $\langle \psi | \psi \rangle = \langle \uparrow \downarrow \uparrow \downarrow \uparrow \rangle = \langle \uparrow \downarrow \uparrow \downarrow \uparrow | a_{b1} a_{c2} a_{c3} \downarrow \downarrow \downarrow \rangle = 0$.

Likewise for 4 fermionic ghost-spins, states/adjoints of the form

$$|\psi\rangle = \psi_1 \langle \downarrow \uparrow \downarrow \uparrow \rangle + \psi_2 \langle \uparrow \downarrow \downarrow \downarrow \rangle = \psi_1 a_{c2} a_{c3} \downarrow \downarrow \downarrow \rangle + \psi_2 a_{c1} a_{c3} \downarrow \downarrow \downarrow \rangle \ , \quad (4.18)$$

have norm given by the inner product

$$\langle \psi | \psi \rangle = \psi_1^* \psi_2 \langle \uparrow \uparrow \uparrow \uparrow | a_{b3} a_{b1} a_{c1} a_{c3} \downarrow \downarrow \downarrow \rangle + \psi_2^* \psi_1 \langle \uparrow \uparrow \uparrow \uparrow | a_{b4} a_{b2} a_{c2} a_{c4} \downarrow \downarrow \downarrow \rangle = \psi_1^* \psi_2 + \psi_2^* \psi_1 \ . \quad (4.19)$$

We see that a state $|\downarrow \uparrow \rangle$ (with $\psi_2 = 0$) is then zero norm, its adjoint being $\langle \uparrow \downarrow |$.

### 4.2 Ghost-spins and a Jordan-Wigner transformation

As stated earlier, we want to start with the ghost-spin chain described in terms of the commuting spin $\sigma_b, \sigma_c$-variables and go to the fermionic ghost-spin $a_b, a_c$-variables. Consider the following generalization of the usual Jordan-Wigner transformation $[14,15]$, written here for the commuting ghost-spin variables,

$$\sigma_{b1} = a_{b1} \ , \quad \sigma_{c1} = a_{c1} \ , \quad \sigma_{b2} = i(1 - 2a_{c1} a_{b1}) a_{b2} \ , \quad \sigma_{c2} = -i(1 - 2a_{c1} a_{b1}) a_{c2} \ , \quad \ldots \ ,$$
$$\sigma_{b3} = i(1 - 2a_{c1} a_{b1}) i(1 - 2a_{c2} a_{b2}) \ldots i(1 - 2a_{c(n-1)} a_{b(n-1)}) a_{b3} \ ,$$
$$\sigma_{c2} = -i(1 - 2a_{c1} a_{b1}) i(1 - 2a_{c2} a_{b2}) \ldots i(1 - 2a_{c(n-1)} a_{b(n-1)}) a_{c2} \ ,$$
$$\sigma_{cn} = (-i)(1 - 2a_{c1} a_{b1})(-i)(1 - 2a_{c2} a_{b2}) \ldots (-i)(1 - 2a_{c(n-1)} a_{b(n-1)}) a_{cn} \ ,$$

The inverse transformations for the fermionic ghost-spin variables are

$$a_{b1} = \sigma_{b1} \ , \quad a_{c1} = \sigma_{c1} \ , \quad a_{b2} = i(1 - 2\sigma_{c1} \sigma_{b1}) \sigma_{b2} \ , \quad a_{c2} = -i(1 - 2\sigma_{c1} \sigma_{b1}) \sigma_{c2} \ , \quad \ldots \ ,$$
$$a_{b3} = i(1 - 2\sigma_{c1} \sigma_{b1}) i(1 - 2\sigma_{c2} \sigma_{b2}) \ldots i(1 - 2\sigma_{c(n-1)} \sigma_{b(n-1)}) \sigma_{b3} \ ,$$
$$a_{c2} = -i(1 - 2\sigma_{c1} \sigma_{b1}) i(1 - 2\sigma_{c2} \sigma_{b2}) \ldots (-i)(1 - 2\sigma_{c(n-1)} \sigma_{b(n-1)}) \sigma_{c2} \ ,$$
$$a_{cn} = (-i)(1 - 2\sigma_{c1} \sigma_{b1}) (-i)(1 - 2\sigma_{c2} \sigma_{b2}) \ldots (-i)(1 - 2\sigma_{c(n-1)} \sigma_{b(n-1)}) \sigma_{cn} \ ,$$

The factor $(1 - 2\sigma_{c1} \sigma_{b1})$ is $-1$ or $+1$ depending on whether the $i$-th location is occupied ($\uparrow$) or not ($\downarrow$); this means $(1 - 2\sigma_{c1} \sigma_{b1})^2 = +1$ as can be checked explicitly as $(1 - 4\sigma_{c1} \sigma_{b1} + 4\sigma_{c1} \sigma_{b1} \sigma_{c1} \sigma_{b1}) = 1$. Furthermore we see that the term

$$\left[ \pm i(1 - 2\sigma_{c1} \sigma_{b1}) \right]^{\dagger} = \pm i(1 - 2\sigma_{c1} \sigma_{b1}) \ , \quad (4.22)$$
is hermitian as the \( \pm i \) factors ensure, thereby ensuring that the \( a_{bn}, a_{cn} \) operators are also hermitian: for instance

\[
a_{c2}^\dagger = \sigma_{c2}[i(1 - 2\sigma_{b1}\sigma_{c1})] = -i(1 - 2\sigma_{c1}\sigma_{b1})\sigma_{c2} = a_{c2} .
\]  

(4.23)

Now it can be seen that the \( a_b, a_c \)-variables are anticommuting: e.g.

\[
\{a_{b2}, a_{cn}\} = i(-i)^n \left[ \left( 1 - 2\sigma_{c1}\sigma_{b1} \right) \sigma_{b2} \left( 1 - 2\sigma_{c1}\sigma_{b1} \right) \cdots \left( 1 - 2\sigma_{c(n-1)}\sigma_{b(n-1)} \right) \sigma_{cn} \right. \\
+ \left. \left( 1 - 2\sigma_{c1}\sigma_{b1} \right) \left( 1 - 2\sigma_{c1}\sigma_{b2} \right) \cdots \left( 1 - 2\sigma_{c(n-1)}\sigma_{b(n-1)} \right) \sigma_{cn} \right] \\
= i(-i)^n \left[ \left( 1 - 2\sigma_{c1}\sigma_{b1} \right)^2 \sigma_{b2} \left( 1 - 2\sigma_{c2}\sigma_{b2} \right) \sigma_{cn} \cdots \left( 1 - 2\sigma_{c(n-1)}\sigma_{b(n-1)} \right) \right.
\\+ \left. \left( 1 - 2\sigma_{c1}\sigma_{b1} \right)^2 \left( 1 - 2\sigma_{c2}\sigma_{b2} \right) \sigma_{cn} \cdots \left( 1 - 2\sigma_{c(n-1)}\sigma_{b(n-1)} \right) \right] \\
= i(-i)^n \left( -\sigma_{b2}\sigma_{cn} + \sigma_{b2}\sigma_{cn} \right) \cdots \left( 1 - 2\sigma_{c(n-1)}\sigma_{b(n-1)} \right) = 0 
\]  

(4.24)

since the \( \sigma \)'s at distinct locations are commuting. Similarly other anticommutation relations can be checked. The fact that the \( a_{bn} \) contains a factor \( i \) whereas \( a_{cn} \) contains \( -i \) ensures that the anticommutator works out correctly: e.g.

\[
\{a_{bn}, a_{cn}\} = \left\{ \prod_{k=1}^{n-1} i(1 - 2\sigma_{ck}\sigma_{bk})\sigma_{bn}, \prod_{k=1}^{n-1} (-i)(1 - 2\sigma_{ck}\sigma_{bk})\sigma_{cn} \right\}
\\= i^{n-1}(-i)^{n-1}\{\sigma_{bn}, \sigma_{cn}\} = 1 ,
\]  

(4.25)

where we have used the fact that each \( 1 - 2\sigma_{ck}\sigma_{bk} \) factor commutes through the \( \sigma_{bn} \) and \( \sigma_{cn} \). Now note that

\[
-J \sum_n \sigma_{b(n)}\sigma_{c(n+1)} = -J \sum_n \left[ (1 - 2a_{c1}a_{b1})(1 - 2a_{c2}a_{b2}) \cdots (1 - 2a_{c(n-1)}a_{b(n-1)})a_{bn} \right] \times
\\\left[ (1 - 2a_{c1}a_{b1})(1 - 2a_{c2}a_{b2}) \cdots (1 - 2a_{c(n-1)}a_{b(n-1)})(1 - 2a_{cn}a_{bn})a_{c(n+1)} \right]
\\= -J \sum_n \left[ \prod_{k=1}^{n} (1 - 2a_{ci}a_{bi})^2 \right] a_{bn}(1 - 2a_{cn}a_{bn})a_{c(n+1)}
\\= +J \sum_n a_{bn}a_{c(n+1)} .
\]  

(4.26)

We have used the fact that \( a_{bn} \) commutes through each \( 1 - 2a_{ci}a_{bi} \) factor, leaving a nontrivial action with \( (1 - 2a_{cn}a_{bn}) \). It is now important to note that in the above calculation, we have assumed that the ghost-spin chain is infinite thereby allowing us to restrict to “bulk” terms: if the chain is finite, then there would be a boundary term of the form \( \sigma_{bN}\sigma_{c1} \) which would not simplify to the above form (with the exception of 2 ghost-spins). For instance, for a finite chain of 3 ghost-spins, this boundary term gives \( \sigma_{b3}\sigma_{c1} = (1 - 2a_{c1}a_{b1})(1 - 2a_{c2}a_{b2})a_{b3}a_{c1} \) which simplifies to give a term of the form \( -2a_{c2}a_{b2}a_{b3}a_{c1} \) which does not cancel with any other, and is not of the above quadratic form.
4.3 Ghost-spin chain for the bc-ghost CFT

Consider a 1-dimensional ghost-spin chain with a nearest neighbour interaction Hamiltonian

\[ H = J \sum_n (\sigma_{b(n)}\sigma_{c(n+1)} + \sigma_{b(n)}\sigma_{c(n-1)}) \]  \hspace{1cm} (4.27)

repeating (4.1), where \( n, n+1, n-1 \) label nearest neighbour lattice sites in the chain, which comprises 2-state spin variables at each site. This is not quite Ising-like: in fact it describes a “hopping” type Hamiltonian, which kills an \( \uparrow \)-spin at site \( n \) and creates it at site \( n+1 \), so that \( \uparrow_n \) hops to \( \uparrow_{n+1} \). It is useful to note that this Hamiltonian can also be written as

\[ H = J \sum_n (\sigma_{b(n)}\sigma_{c(n+1)} + \sigma_{b(n+1)}\sigma_{c(n)}) \], \hspace{1cm} (4.28)

and so on a nearest neighbour pair \((n, n+1)\) the action of \( H \) is seen quite generally to be

\[
\begin{align*}
H\left|\cdots \uparrow_n \downarrow_{n+1} \cdots \rightangle &\xrightarrow{b_{n+1}c_n} \left|\cdots \downarrow_n \uparrow_{n+1} \cdots \rightangle, \\
H\left|\cdots \downarrow_n \uparrow_{n+1} \cdots \rightangle &\xrightarrow{b_{n+1}c_n} \left|\cdots \uparrow_n \downarrow_{n+1} \cdots \rightangle. 
\end{align*}
\]  \hspace{1cm} (4.29)

While (4.27) represents an infinite ghost-spin chain, it is worth illustrating its action by considering finite chains: so consider a system of two ghost-spin lattice sites, with

\[ H = J(\sigma_{b1}\sigma_{c2} + \sigma_{b2}\sigma_{c1}) \]  \hspace{1cm} (4.30)

where we have imposed periodic boundary conditions (which thus gives the second term). We then see that \( H \) acts on the 4 states (in the commuting spin basis) as

\[
\begin{align*}
H|\downarrow\downarrow\rangle &= 0, \quad H|\uparrow\uparrow\rangle = 0, \\
H|\downarrow\uparrow\rangle &= J\sigma_{b2}\sigma_{c1}|\downarrow\uparrow\rangle = J|\uparrow\downarrow\rangle, \quad H|\uparrow\downarrow\rangle = J\sigma_{b1}\sigma_{c2}|\uparrow\downarrow\rangle = J|\downarrow\uparrow\rangle,
\end{align*}
\]  \hspace{1cm} (4.31)

since \( e.g. \sigma_{b1}\sigma_{c2} \) kills \( |\downarrow\uparrow\rangle \) and so on. The energy expectation values (for states with nonzero norm \( \langle\psi|\psi\rangle \)) are

\[
\langle E \rangle = \frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle}; \quad e_{\pm}^1 : \langle E \rangle = 0 \ [\text{ground states}]; \quad e_{\pm}^2 : \langle E \rangle = J \ [\text{excited states}],
\]  \hspace{1cm} (4.32)

using the basis in (4.7), \( i.e. e_{\pm}^1 = \frac{1}{\sqrt{2}}(|\downarrow\downarrow\rangle \pm |\uparrow\uparrow\rangle) \) etc. This gives the partition function

\[ Z = \sum_{s_i} e^{-\beta E[s_i]} = 2 \left( 1 + e^{-\beta J} \right) \]  \hspace{1cm} (4.33)

which is identical to that for 2 ordinary spins. Consider now 3 lattice sites (again with periodic boundary conditions): the Hamiltonian is

\[ H = J(\sigma_{b1}\sigma_{c2} + \sigma_{b2}\sigma_{c1} + \sigma_{b2}\sigma_{c3} + \sigma_{b3}\sigma_{c2} + \sigma_{b3}\sigma_{c1} + \sigma_{b1}\sigma_{c3}) . \]  \hspace{1cm} (4.33)
The action of $H$ on the 8 states is

\[
H | \downarrow\downarrow\rangle = 0 , \quad H | \uparrow\uparrow\rangle = 0 ,
\]

\[
| \downarrow\uparrow\rangle - b_1 c_1 + b_2 c_2 \rightarrow \quad | \uparrow\downarrow\rangle + | \downarrow\uparrow\rangle ,
\]

\[
| \uparrow\downarrow\rangle - b_1 c_3 + b_2 c_3 \rightarrow \quad | \down\uparrow\rangle + | \up\down\rangle ,
\]

\[
| \up\up\rangle - b_1 c_2 + b_3 c_2 \rightarrow \quad | \down\up\rangle + | \up\down\rangle .
\]

Thus some $H$ eigenstates with norms $\pm 1$ are

\[
| \down\down\rangle \pm | \up\up\rangle ; \quad (| \down\up\rangle + | \down\down\rangle + | \up\down\rangle) \pm (| \up\down\rangle + | \up\up\rangle + | \down\down\rangle) .
\] (4.34)

Norms for some generic states then are

\[
\alpha_1 (| \down\up\rangle + | \down\down\rangle + | \up\down\rangle) \pm \alpha_2 (| \up\down\rangle + | \up\up\rangle + | \down\down\rangle) \longrightarrow 3(\alpha_1^* \alpha_2 + \alpha_2^* \alpha_1)
\] (4.35)

\[
\alpha_1 | \down\down\rangle + \alpha_2 | \up\up\rangle \longrightarrow 2(\alpha_1^* \alpha_2 + \alpha_2^* \alpha_1)
\] (4.36)

while their energy eigenvalues are 0 and $2\alpha_1 \pm 2\alpha_2$ respectively.

gs $\rightarrow$ bc: Starting with the ghost-spin chain Hamiltonian in the commuting spin variables

\[
H = + J \sum_n \left( \sigma_{b(n)} \sigma_{c(n+1)} + \sigma_{b(n)} \sigma_{c(n-1)} \right) ,
\] (4.38)

we see using the Jordan-Wigner transformation (4.20), (4.21), and the simplification (4.26), that the Hamiltonian simplifies as

\[
H = J \sum_n \left( i^{n-1}[1][2] \ldots [n-1]a_{bn}(-i)^{n}[1][2] \ldots [n]a_{cn+1} + i^{n-1}[1][2] \ldots [n-1]a_{bn}(-i)^{n-2}[1][2] \ldots [n-2]a_{cn-1} \right) ,
\] (4.39)

where $[k] \equiv (1 - 2a_{ck}a_{bk})$. Commuting the various $[k]$ factors gives

\[
H = J \sum_n \left( (-i)a_{bn}(1 - 2a_{cn}a_{bn})a_{cn+1} + i(1 - 2a_{cn-1}a_{bn-1})a_{bn}a_{cn-1} \right)
\]

\[
= iJa_{bn} \left( a_{cn+1} - a_{cn-1} \right) .
\] (4.40)

In what follows, we will take a continuum limit of this system where $J$ is scaled as $J \sim \frac{1}{2\alpha}$: then we see that the difference becomes the derivative, i.e. $iJa_{bn}(a_{cn+1} - a_{cn-1}) \to -b\partial_c$, in the continuum limit. Note that the Hamiltonian (4.40) is hermitian: after anticommuting the $a_{bn}$ through, we have $H^\dagger = (-i)(a_{cn+1} - a_{cn-1})a_{bn} = H$. The ghost-spin Hamiltonian (4.27) that we began with was also hermitian of course.
**Momentum space variables:** So far we have been working with the lattice variables, which are real space representation of the spin degrees of freedom. To give the momentum space description of these operators let us consider the Fourier transform of the real space operators

\[
a_{bn} = \frac{1}{\sqrt{N}} \sum_{k} e^{-ikn} b_k , \quad a_{cn} = \frac{1}{\sqrt{N}} \sum_{k} e^{-ikn} c_k .
\]  

The hermiticity of \(a_{bn}, a_{cn}\) imposes a relation between negative Fourier modes and hermitian conjugate operators,

\[
b_{-k} = b_k^\dagger , \quad c_{-k} = c_k^\dagger .
\]  

The inverse transforms are

\[
b_k = \frac{1}{\sqrt{N}} \sum_{n} e^{ikn} a_{bn} , \quad c_k = \frac{1}{\sqrt{N}} \sum_{n} e^{ikn} a_{cn} , \quad \text{with} \quad \frac{1}{N} \sum_{n} e^{in(k+k')} = \delta_{k+k',0} .
\]  

The operators \(a_{bn}, a_{cn}\) are fermionic and satisfy anticommutation relations. The anticommutation relation between then translates into the following anticommutation relations between \(b\) and \(c\) Fourier modes,

\[
\{ b_k , c_k \} = \frac{1}{N} \sum_{n,m} e^{i(k+n+k')} \{ a_{bn} , a_{cn} \} = \delta_{k+k',0} , \quad \{ b_k , b_k' \} = 0 , \quad \{ c_k , c_k' \} = 0 .
\]  

In addition to these modes we see that there also exist “zero mode” operators, which are momentum space analogs of the centre of mass modes,

\[
k = 0 : \quad b_0 = \frac{1}{\sqrt{N}} \sum_{n} a_{bn} , \quad c_0 = \frac{1}{\sqrt{N}} \sum_{n} a_{cn} \quad \Rightarrow \quad \{ b_0 , c_0 \} = 1 .
\]  

Note that we are considering a chain of \(N\) fermionic ghost-spins with \(N\) odd and the momentum moding

\[
k = \pm m \frac{2\pi}{N} \equiv k_{m \pm} , \quad m = 1, 2, \ldots, \frac{N-1}{2} \quad [N \text{ odd}],
\]  

which in the large \(N\) continuum limit becomes \(k \to [-\pi/2, \pi/2]\). To illustrate this, consider 3 ghost-spins: the lattice sites are labelled by \(n = 0, 1, 2\), and \(k = -\frac{2\pi}{3}, 0, \frac{2\pi}{3}\), giving

\[
b_{\pm 2\pi/3} = \frac{1}{\sqrt{3}} \left( e^{\pm i(2\pi/3)(0)} a_{b0} + e^{\pm i(2\pi/3)(1)} a_{b1} + e^{\pm i(2\pi/3)(2)} a_{b2} \right) ,
\]

\[
b_0 = \frac{1}{\sqrt{3}} (a_{b0} + a_{b1} + a_{b2}) ,
\]

and likewise for the \(c_k\) operators. These Fourier modes allow a faithful mapping of the various spin states in terms of the momentum basis. The anticommutation relations are

\[
\{ b_{\pm 2\pi/3}, c_0 \} = \frac{1}{3} \left( \{ a_{b0}, a_{c0} \} + e^{\pm 2\pi i/3} \{ a_{b1}, a_{c1} \} + e^{\pm 4\pi i/3} \{ a_{b2}, a_{c2} \} \right) = \frac{1}{3} (1 + \omega_3 + \omega_3^2) = 0 ,
\]
with \( \omega_3 = e^{\pm i2\pi/3} \) a 3rd root of unity. Likewise for general odd \( N \), the anticommutation relation vanishes as \( \{ b_k, c_0 \} = \frac{1}{N} \sum_{n=0}^{N-1} e^{i(2\pi m/N)n} = \frac{1}{N}(1 + \omega_N + \ldots + \omega_N^{N-1}) = 0 \) with \( \omega_N = e^{i(2\pi m/N)} \), \( m = \pm 1, 2, \ldots, \frac{N-1}{2} \), a general \( N \)-th root of unity. It is worth pointing out at this stage that for \( N \) even, it turns out that the zero mode operators, if they exist, do not yield sensible anticommutation relations with the other modes. This is perhaps due to implicit anti-periodic boundary conditions. Our description of these momentum modes here with \( N \) odd is similar to the discussion in e.g. [14].

There is a pair of ground states for these momentum basis modes defined by the zero modes \( b_0, c_0 \), in (4.45),

\[
\begin{align*}
| b_0 \downarrow_{bc} \rangle = 0 & , & | b_0 \uparrow_{bc} \rangle = | \downarrow_{bc} \rangle ,
| c_0 \downarrow_{bc} \rangle = | \uparrow_{bc} \rangle , & & | c_0 \uparrow_{bc} \rangle = 0 ,
\end{align*}
\]

and all higher modes \( b_k, c_k \), with \( k > 0 \) annihilate \( | \downarrow_{bc} \rangle, | \uparrow_{bc} \rangle \). Note that these are distinct from the position basis states described earlier. Then states such as \( | \downarrow_{bc} \rangle - | \uparrow_{bc} \rangle \) clearly have negative norm. Excited states such as \( (b_{-k} - c_{-k})(| \downarrow_{bc} \rangle - | \uparrow_{bc} \rangle) \) also have negative norm \( (| \downarrow_{bc} \rangle - | \uparrow_{bc} \rangle)(b_{-k} - c_{-k})(| \downarrow_{bc} \rangle - | \uparrow_{bc} \rangle) = -2 \) using the \( b_k, c_k \) oscillator algebra.

In terms of the momentum basis modes, the ghost-spin chain Hamiltonian becomes

\[
H = \frac{i}{N} \sum_{n} \sum_{k,k'} e^{-ikn} b_k (e^{-i k'(n+1)} c_{k'} - e^{-i k'(n-1)} \bar{c}_{k'})
\]

\[
= iJ \sum_{k,k'} b_k c_{k'} \delta_{k+k',0} (e^{-i k'} - e^{+i k'}) = 2J \sum_k \sin k' b_k c_{k'} \delta_{k+k',0} .
\]

Reinstating the lattice spacing \( a \) by replacing \( k \) by \( ka \) in the sine function and then taking the continuum limit \( a \to 0 \) gives

\[
H = 2J \sum_k \sin(ka) b_{-k} c_k \quad \longrightarrow \quad 2Ja \sum_k k b_{-k} c_k .
\]

In order to obtain the critical theory we need to scale the coupling \( J \) as \( J \sim \frac{1}{2a} \) while taking the continuum limit to obtain a nonzero finite expression as \( a \to 0 \): this is simply a way of ensuring that the nearest neighbour lattice interaction leads to nontrivial continuum interactions as the lattice spacing goes to zero. This then gives

\[
H = \sum_{k>0} k (b_{-k} c_k + c_{-k} b_{k}) + \zeta .
\]

The constant \( \zeta \) is a normal ordering constant that arises as usual after rewriting creation operators to the left of annihilation operators.

The \( H \) above is of the same form as \( L_0 \) of the \( bc \)-ghost CFT with \( c = -2 \). We can construct other Virasoro generators by picking up appropriate Fourier modes of the ghost-spin chain Hamiltonian density \( i a_n (a_{c(n+1)} - a_{c(n-1)}) \). For example

\[
L_n = \sum_k (n-k) b_k c_{n-k} .
\]
Thus in the continuum limit we recover conformal invariance and we can express the Virasoro generators in terms of modes of $b$ and $c$ ghosts. In addition to the Virasoro symmetry we also have the ghost current symmetry $J_g(z) = :cb:(z)$.

It is worth asking what the symmetries of the original ghost-spin chain Hamiltonian (4.27) were. In this regard we note that $H$ term-by-term respects a phase rotation symmetry

$$
\sigma_{b(n)} \to e^{i\alpha} \sigma_{b(n)} , \quad \sigma_{c(n+1)} \to e^{-i\alpha} \sigma_{c(n+1)}. \tag{4.54}
$$

This is a microscopic reflection of the $U(1)$ symmetry in the continuum $bc$-CFT. In addition, note that there is a global scaling symmetry

$$
a \to \xi^{-1} a, \quad H \to \xi H, \quad \sigma_{b(n)} \to \xi^\lambda \sigma_{b(n)}, \quad \sigma_{c(n+1)} \to \xi^{1-\lambda} \sigma_{c(n+1)}. \tag{4.55}
$$

We see that the ghost-spin variables $(\sigma_b, \sigma_c)$ exhibit this symmetry for any constant $\lambda$ (although $\lambda = 1$ was implicit in most of our discussion above): this is the reflection of the fact that the $bc$-CFT is a conformal theory for any conformal weights $(h_b, h_c) = (\lambda, 1-\lambda)$. This arises from the fact that each term in $H$ involves two separate variables allowing a partial “cancellation” of the scaling factor $\xi$. This would not be possible for an Ising-like Hamiltonian, e.g. of the form (3.1), (3.16).

Further let us recall that for a general $bc$-CFT with weights $(h_b, h_c) = (\lambda, 1-\lambda)$, the energy-momentum tensor is $T(z) = (\partial b)c : -\lambda \partial (bc:) = -b\partial c : + (1-\lambda) \partial (bc:)$. This can be rewritten as $T(z)_{\lambda} = -\lambda : b\partial c : + (1-\lambda) : (\partial b)c :$. It is then useful to note that the lattice discretization of the last expression is

$$
\lambda J \sum_n i a_{bn} (a_{c(n+1)} - a_{c(n-1)}) - (1-\lambda)J \sum_n i (a_{b(n+1)} - a_{b(n-1)}) a_{cn} \nonumber
$$

$$
\longrightarrow \sum_n iJa_{bn} (a_{c(n+1)} - a_{c(n-1)}), \tag{4.56}
$$

where we have taken $iJa_{bn} (a_{c(n+1)} - a_{c(n-1)}) \to -b\partial c$ from (4.40), and the last simplification can be seen by appropriately recasting the $\sum_n$ in the second infinite lattice sum in the first line. In other words, the local expression $iJa_{bn} (a_{c(n+1)} - a_{c(n-1)})$ can be split into the two terms in $T(z)_{\lambda}$ for any $\lambda$. This is consistent with the fact that $T(z)_{\lambda}$ is $-b\partial c$ apart from a total derivative. Thus the lattice Hamiltonian (4.40) obtained from (4.27) captures the general $bc$-CFT$_{\lambda}$ equally well in the continuum limit $a \to 0$ with $J \sim \frac{1}{2a}$, along with the scaling (4.55).

We have thus argued that the ghost-spin chain with Hamiltonian (4.27) with weights $(\lambda, 1-\lambda)$ for the ghost-spin variables $\sigma_{bn}, \sigma_{cn}$, under the scaling symmetry (4.55) maps to the $bc$-ghost CFT with conformal weights $(h_b, h_c) = (\lambda, 1-\lambda)$ in the continuum limit. Note that while the scaling symmetry can be demonstrated in both the ghost-spin variables as
well as the fermionic ghost-spin variables $a_{bn}, a_{cn}$ representation, the Jordan-Wigner transformation which is a non-local relation between these two representations does not have this scaling symmetry. For ghost fields with conformal weights $(h_b, h_c) = (\lambda, 1 - \lambda)$, the Virasoro generators are given by $L_n = \sum_k (n\lambda - k)b_k c_{n-k}$, and the normal ordering constant $\zeta$ in (4.52) above is fixed by the Virasoro algebra of the $L_n$s of the $bc$-CFT as usual.

5 Discussion

We have studied 1-dimensional chains of ghost-spins with nearest neighbour interactions amongst them, developing the description of ghost-spins in [1, 2]. Ghost-spins, 2-state spin variables with indefinite norm, serve as simple quantum mechanical toy models for theories with negative norm states. In the finite ghost-spin chains, we have described how the Ising-like nearest neighbour interaction helps organize and clarify the study of entanglement earlier and we have further developed the properties of the reduced density matrix and its entanglement entropy. We have then studied a family of infinite ghost-spin chains with hopping type Hamiltonian, where defining fermionic ghost-spin variables through a Jordan-Wigner transformation maps these ghost-spin chains in the continuum limit to the $bc$-ghost CFTs. It may be interesting to explore other ghost-like field theories in this light and more generally the space of non-unitary CFTs that ghost-spin ensembles provide microscopic realizations for.

Along the lines of the Ising-like ghost-spin chains, a simple generalization of an infinite ghost-spin chain is the transverse Ising model for a ghost-spin chain with Hamiltonian e.g. $H = -J \sum_i (s^z_i s^z_{i+1} + g \sum_i s^x_i)$, where $s^z$ are the ghost-spin variables $s$ we have been describing so far, and $s^x$ are complementary variables (not commuting with $s^z$). For $g \sim 0$, the ground states are $s^z$ eigenstates $|+N\rangle, |-N\rangle$, as discussed previously, while for $g \gg 1$, the ground state is the $\sigma^x$ eigenstate $|\downarrow\rangle$: this is very similar to the ordinary transverse Ising spin chain, except that the variables here represent ghost-spins with indefinite norms and thus encode negative norm states. It would seem that $g = 1$ is a critical point where some scale invariant theory emerges. In light of our discussion here on the $bc$-ghost CFT which arises from a very different ghost-spin chain, it is unclear what this critical theory might be.

Another interesting system is a “ghost-spin glass”, with a Hamiltonian of the Ising spin glass form but with $N$ ghost-spins $H = -\sum J_{ij} s_i s_j$ with $i, j = 1, \ldots, N$. The couplings $J_{ij}$ are not restricted to nearest neighbour and so represent random nonlocal interaction couplings. In the $\{\pm\}$-basis, it would appear based on the discussions in sec. 3 that this system would have parallels with ordinary spin-glasses (see e.g. [19] for a relatively recent review), exhibiting many nearly degenerate ground states, but also containing negative norm states. It would be interesting to explore these.
The appearance of the bc-ghost system in the continuum limit of the infinite ghost-spin chain points towards a gauge symmetry which has been fixed using the Faddeev-Popov method. Such a symmetry would become manifest if this ghost system is coupled to ordinary matter. In familiar theories with gauge symmetry, the negative norm sector decouples from any physical process, a truncation which is technically implemented by the familiar BRST procedure. In the present case also, we expect that an appropriate BRST symmetry will enable a truncation of the full indefinite norm Hilbert space to the physical Hilbert space which comprises positive norm states alone, thereby leading in principle to positive entanglement entropy. We hope to report on this in the future.

The original motivation for constructing “ghost-spins” in [1] was to explore solvable toy models for ghost-CFTs and study their entanglement properties: this builds on earlier studies [20, 21] of generalizations of the Ryu-Takayanagi formulation [22, 23] to gauge/gravity duality for de Sitter space or dS/CFT [24–26]. In [20, 21], the areas of certain complex codim-2 extremal surfaces (involving an imaginary bulk time parametrization) were found to have structural resemblance with entanglement entropy of dual Euclidean CFTs, effectively equivalent to analytic continuation from the Ryu-Takayanagi expressions in AdS/CFT. In dS_4, the areas are real and negative. Certain attempts were made in [1] towards gaining some insight on this in CFT and quantum mechanical toy models: certain 2-dim ghost-CFTs under certain conditions were found to yield negative entanglement entropy using the replica formulation [27]. Likewise a toy model of two ghost-spins was found to yield the reduced density matrix and associated entanglement properties reviewed earlier in sec. 2. In the context of dS/CFT [24–26], de Sitter space is conjectured to be dual to a hypothetical Euclidean non-unitary CFT that lives on the future boundary $I^+$, with the dictionary $\Psi_{dS} = Z_{CFT}$ [26], where $\Psi_{dS}$ is the late-time wavefunction of the universe with appropriate boundary conditions and $Z_{CFT}$ the dual CFT partition function. This usefully organizes de Sitter perturbations, independent of the actual existence of the CFT. The dual CFT energy-momentum tensor correlators reveal central charge coefficients $C_d \sim i^{1-d} R^{d-1}_{dS} G^{d+1}_{dS}$ in $dS_{d+1}$ (effectively analytic continuations from AdS/CFT). This is real and negative in $dS_4$ so that $dS_4/CFT_3$ is reminiscent of ghost-like non-unitary theories. In [28], a higher spin $dS_4$ duality was conjectured involving a 3-dim CFT of anti-commuting $Sp(N)$ (ghost) scalars (studied previously in [18,29]). In this light, we are thinking of ensembles of ghost-spins as toy models for the latter $Sp(N)$ theories and thereby $dS_4$ possibly. In general such an ensemble of a large number $N$ of ghost-spins is non-unitary, containing large families of negative norm states. However as we have seen, there are subsectors of positive norm states as well, which in fact appear perfectly well-defined and sensible. It is interesting to speculate on possible parallels in the context of a possible dual cosmology.
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