Geometric Supergravity

R. D’Auria
Politecnico di Torino, DISAT Department,
Corso Duca degli Abruzzi 24, 10129 Torino, Italy.

1. Introduction.

In the year 1978 Y. Ne’eman and T. Regge proposed a new approach to the formulation of gauge theories, specifically gravity and supergravity, based on the formalism introduced by É. Cartan for the formulation of Riemannian geometry in a completely geometrical setting. Cartan’s approach implies a new, more geometrical and group-theoretical way of formulating General Relativity. Indeed, as the adopted formalism relies basically and consistently on the use of differential forms, the Cartan’s beautiful setting is independent of different coordinate frames, that is of the group of general coordinate transformations (GCTG). At the same time, it gives a prominent role to the gauge invariance of the theory under the Lorentz group which emerges quite naturally from the formalism. As a matter of fact, in Cartan’s view, Riemannian geometry must be seen as pertaining to finite Lie groups rather than to infinite group of general coordinate transformations. In the latter case it could be difficult to see how gravitation could be unified with gauge theories of other interactions, what instead seems quite natural in the geometrical formalism developed by him.

Following this line of approach, Y. Ne’eman and T. Regge further developed the Cartan’s formalism, proposing that in principle any diffeomorphic and gauge invariant theory should be constructed directly on the group manifold $G$ defining the Lie algebra valued gauge fields in the coadjoint representation of the group. This is consequence of the following considerations.

Referring to the pure gravity theory, the spin connection $\omega^{ab}$ and the vierbein $V^a$ 1-form gauge fields are just fragments of the adjoint multiplet $\mu^A$, $(A = 1, \ldots, 10)$, of the Poincaré Lie algebra with indices decomposed with respect to the Lorentz subgroup SO(1,3), and spanning a basis of the cotangent plane of the Poincare’ group, namely $\mu^A = (\omega^{ab}, V^a)$, $(a, b = 0, \ldots, 3)$. However in gravity theory, there is an essential difference between the two fields: While the vierbein $V^a = V^a_\mu \, dx^\mu$ propagate, this is not true for the spin connection.
This is a consequence of the well known fact that Riemannian geometry requires that the “curvature” of $V^a$, namely the torsion 2-form, must vanish. This makes the spin connection a functional of the vierbein and its derivatives. This disparity is essentially due to the factorization hypothesis which breaks from the very beginning the symmetry of the group, since it factorizes in a trivial way the dependence of the gauge fields from the Lorentz coordinates. In fact in the Cartan formulation of gravity the fields $(\omega^{ab}, V^a)$ live on the principal fiber bundle $[\mathcal{M}_4/\text{SO}(1,3)]$, where the base space $\mathcal{M}_4$ is the physical space-time which, in the vacuum configuration, reduces to the coset base space $[G/\text{SO}(1,3)]$ with $G = \text{ISO}(1,3)$.

It follows that if we now do not assume factorization, the dependence of the fields $\mu^A = (\omega^{ab}, V^a)$ on the group coordinates must be dictated by the field equations (and boundary conditions). It is then natural to try to construct gravity theory directly on the group manifold $G$, (ISO(1,3) in the gravity case), where the fields are represented by the Cartan-Maurer 1-forms in the coadjoint representation of the group. This implies that the gauge fields will depend not only on the coordinates $x^\mu$ ($\mu = 0, \ldots, 3$) related to the translation group, but also also on the coordinates of the Lorentz group $y^{\mu\nu}$. Out of the vacuum the left-invariant 1-forms, spanning a basis on the cotangent space of $G$, do not obey anymore the Cartan-Maurer equations, but become dynamical, that is they acquire curvatures, namely the field-strengths of the dynamical fields.

Quite generally, as can be shown in general and we will show in the simplest cases, when fields are defined directly on a (graded-)group manifold, this new approach makes it possible to give a more geometrical formulation of the theory. Actually, both for gravity theory as for supergravity in any dimensions $D$, the Lorentz invariance can be retrieved from the equations of motion as a result of the action principle, even if the $D$-form Lagrangian has to be integrated on a group manifold whose dimensionality, dim $G$, is greater than the form degree $D$ of the Lagrangian. In fact we will see that the integration of a $D$-form as a submanifold can be consistently performed as a result of its invariance under the GCTG, and the resulting equations of motion give horizontality of the curvatures in the Lorentz directions, leading to factorization of the Lorentz parameters. While this way to obtain Lorentz invariance starting from the whole group manifold $G$ seems to be of academic interest, its extension to graded groups (or supergroup, SG in the following) leads to a geometric interpretation of supersymmetry. Indeed, in this case, referring for simplicity to the $N = 1$, $D = 4$ case, the coadjoint supermultiplet now contains an extra fermionic vielbein, namely $\mu^A = (\omega^{ab}, V^a, \psi^\alpha)$, $a,b = 0, \ldots, 3$, $\alpha = 1, \ldots, 4$, where $\psi^\alpha$ is a Majorana spinor 1-form, so that the coadjoint multiplet will now depend, besides translation and Lorentz coordinates, also on the odd fermionic Grassman parameters $\theta^\alpha$. In this case the Lorentz factorization, obtained as a result of the field equations, is not sufficient alone to leave us on space-time, but rather on superspace defined in the vacuum configuration as $R^{4|4} = \text{OSp}(1,4)/\text{SO}(1,3)$, while out of vacuum superspace is a bundle whose base space is physical space-time. The superspace equations of motion, besides horizontality of the supercurvatures in the Lorentz directions, also give constraints on the super-curvatures in the “fermionic” directions, which allow to restrict the theory to space-time only. Indeed one finds that these components are linearly expressible in terms of the components of the super-curvatures on the (cotangent plane of the) space-time manifold. It is this property, dubbed rheonomy, which allows a
complete geometrical interpretation of supersymmetry\(^1\) and moreover allows the interpretation of the *superspace diffeomorphisms* as supersymmetry transformations on space-time.

In the following we shall try to give a short, albeit almost complete, account of these properties in the simplest case of \(\mathcal{N} = 1, D = 4\) pure supergravity. This is however sufficient, since, as we will explain, they work exactly in the same way for any other supergravity independently of the supergroup \(G\) the number of supersymmetry generators, space-time dimensionality and/or matter couplings, even if they often exhibit a much more intricate structure. Notwithstanding this the geometrical interpretation of supersymmetry can be shown to remain the same.

The development of this approach was proposed by T. Regge as soon as he came back to Torino from Princeton IAS. The developments of his ideas were initially a result of his collaboration with R. D’Auria and P. Fré and partly with A. D’Adda. The Torino group then developed the Regge initial ideas during many years\(^2\) and his geometric method has become one of the principal tools for investigation not only of supergravity, but also of any other topic where the geometrical and group-theoretical approach can be useful or even essential. Indeed a long series of achievements using the basic idea of Tullio Regge has been realized from the very beginning till our days and will probably continue also in the future.

Coming back to supergravity, let me just mention which other advantages of a purely geometric approach have emerged during the developments of the method.

- An important step has been the construction of the Lagrangian. Indeed, using the building principles of *geometricity* to be discussed below together with other obvious requirements like the presence of the Einstein term, the construction turns out to be essentially algorithmic and unique. Moreover, as outlined before, using the Ne’eman Regge’s geometrical action principle the superspace equations of motion give in one stroke, besides the space-time field equations, also linear relations between the components of the supercurvatures leading to the supersymmetry of the space-time Lagrangian.

- Besides, the steady use of the geometric approach also for theories having antisymmetric tensors in the gravitational multiplet has led the authors of reference\(^2\) to develop in a geometrical way a new structure for their treatment, by generalizing the Maurer-Cartan equations to integrable structures containing higher degree \(p\)-forms.\(^3\)

- As is well known, it is possible to develop supergravity theories in superspace using \(^1\)The same mechanisms of factorization and rheonomy also work in rigid theories. However, for lack of space we shall not consider theories where rigid supersymmetry is present.\(^2\)Torino group was essentially composed by D’Auria and Fré in the beginning, and later enlarged to L. Castellani and many other collaborators among whom an important role has been played by A. Ceresole. During the development of the approach many other collaborators joined our group in view of solving specific problems using, at least in part, our techniques, the most assiduous and important being S. Ferrara, L. Andrianopoli and lately M. Trigiante.\(^3\)The new structures generalizing Maurer-Cartan equations have led to a formulation of a mathematical structure which is nowadays recognized as a first example of the mathematical theory of \(L_{\infty}\) algebras (see e.g. references\(^3,4\)).
Bianchi identities. Since the geometric approach is essentially a superspace approach it is possible to derive directly (that is without writing an action principle) the equations of motion of any supergravity using only the Bianchi identities of the super-curvatures and using a priori rheonomy as a principle of the construction, as we shall explain in the following.

2. Pure Gravity in Cartan Formalism.

In this section we shall first remind some of the most important properties of the Cartan formulation of the Einstein gravity in order to establish the notations and thus setting the stage for the formulation of its extension to the Poincaré group manifold. This is a preparatory discussion in view of discussing the geometrical interpretation of supersymmetry (also called rheonomy) in supergravity theories.

In the Cartan geometrical framework the gauge fields are to be identified with the left-invariant differential forms \( \{ \sigma^A \} \) dual to the generators \( T_A \) of the Lie algebra of the Poincaré group \( G = ISO(1,3) \), namely we have \( \sigma^A(T_B) = \delta^A_B \). The indices \( A, B \) run on the (co-)adjoint representation of a Lie algebra. The left-invariant 1-forms satisfy the celebrated Maurer-Cartan equations which are a dual formulation of the Lie algebra:

\[
0 = d\sigma^A + \frac{1}{2} C^A_{BC} \sigma^B \sigma^C.
\]

Here \( C^A_{BC} \) are the structure constants of the Poincaré Lie algebra satisfying the Jacobi identities as a consequence the integrability condition \( d^2 = 0 \) of Eq. (2.1).

The left-invariant 1-forms \( \sigma^A \) describe the configuration of the physical vacuum, that is, vanishing field-strengths. In order to have non-vanishing field-strengths one needs a non-vanishing right hand side in equation (2.1), that is we must endow \( G \) with a set of non left-invariant forms \( \mu^A \), so that they can develop non-vanishing curvatures \( R^A \):

\[
R^A = d\mu^A + \frac{1}{2} C^A_{BC} \mu^B \mu^C,
\]

\( R^A \) being defined as the coadjoint multiplet of curvatures. In particular, the 1-forms \( \mu^A \) will be now dual to the non left-invariant vector fields \( \tilde{T}_A \) closing a Lie algebra of vector fields with structure functions instead of structure constants, namely \( C^A_{BC} \rightarrow C^A_{BC} + R^A_{BC} \), where \( R^A_{BC} \) are the components of the curvature 2-forms along \( \mu^B \mu^C \). Following Cartan’s geometrical setting, the 1-forms \( \mu^A \), indexed in the coadjoint representation of the Poincaré group, can be decomposed into their Lorentz content, that is the index \( A \) is decomposed with respect to indices of the Lorentz subgroup \( SO(1,3) \). In this way one defines the spin connection and the vierbein 1-forms: \( \mu^A = \{ \omega^a_b, V^a \} \), \( (a, b = 0, 1, 2, 3) \). Correspondingly, the curvatures or field-strengths of \( \omega^a_b, V^a \) take the following form:

\[
R^a_b = d\omega^a_b - \omega^a_c \wedge \omega^c_b,
\]

\[
T^a = dV^a - \omega^a_b \wedge V^b.
\]

The \( R^{ab} \) 2-form is named the Lorentz curvature and when expanded along a vierbein basis \( R^{ab} = \frac{1}{2} R_{cd}^{ab} V^c V^d \) its components coincide with minus the Riemann tensor in Lorentz
The $T^a$ 2-form is named the torsion. Moreover, the curvatures satisfy the Bianchi identities

$$ DR^{ab} = 0, \tag{2.5} $$

$$ DT^a - R^{ab} V_b = 0, \tag{2.6} $$

being $D = d - \omega$ the Lorentz covariant derivative. Using these definitions, the Cartan gravitational Lagrangian can be written as follows:

$$ L \approx R_{ab} \wedge V^c \wedge V^d \epsilon_{abcd}. \tag{2.7} $$

Note that the integrand of the action being a 4-form is automatically invariant under diffeomorphisms and, as it is shown below, it formally coincides with the usual Einstein-Hilbert Lagrangian written in terms of the metric and Levi-Civita connection.

In the original Cartan construction of the Lagrangian (2.7) the gauge fields $\{\omega^a, V^a\}$ depend only on the space-time coordinates while the dependence on the Lorentz parameters is factorized. In other words, the total space is taken to be a principal fiber bundle $[M_4, H]$, where $H = SO(1,3)$.

As the fiber bundle structure implies a factorization of the Lorentz parameters of the fiber, $M_4$ can be identified with a, generally non-flat, four manifold namely space-time. Therefore the gravitational action is obtained by integrating on $M_4$ the Lagrangian (2.7):

$$ A = \frac{1}{4\kappa^2} \int_{M_4} R_{ab} \wedge V^c \wedge V^d \epsilon_{abcd}, \tag{2.8} $$

where $\kappa = \sqrt{8\pi G}$, and $G$ is the gravitational constant.

Let us now observe that the formal equivalence between the Cartan and Einstein-Hilbert formulations just shown does not mean that they are completely equivalent. First of all Cartan vierbein formalism, showing explicitly the gauge invariance of the theory under Lorentz transformations, allows to introduce spinors in the General Relativity.

We note that our definition of curvature and torsion differ by the sign of the spin connection $\omega^{ab}$ compared to the more common definition in the literature. Since $\omega^{ab} \rightarrow -\omega^{ab}$ implies $R^{ab} \rightarrow -R^{ab}$, the relation between $R^{ab}$ and the Riemann tensor is $R^{ab}_{cd} = -V^a_{\mu} V^b_{\nu} V^c_{\rho} V^d_{\sigma} R_{\mu\nu\rho\sigma}$. Here and in the following we are using a mostly minus Minkowski metric.
framework, contrary to what happens in the usual formalism. Indeed in the world index setting, tensors transform under $GL(4, \mathbb{R})$ while spinors are in a $SO(1, 3)$ representation and therefore they can be naturally coupled in a formalism where Lorentz $SO(1, 3)$ covariance is present.

Furthermore, it is a first order Lagrangian, that is the gauge fields $\omega^{ab}, V^a$, being member of the same adjoint multiplet, are off-shell independent as it is natural in a geometric Lagrangian like (2.7). By geometric we mean that, besides containing the Einstein term, it is built only in terms of forms and wedge products.

Moreover it can be easily ascertained that requiring the Lagrangian to be geometric makes it uniquely determined. Indeed any other Lorentz invariant 4-forms to be added to Eq. (2.7) would be a wedge product of curvature 2-forms and would give rise to Chern characteristic classes (note that the term $T^a \wedge T_a$ is easily seen to reduce to the Cartan Lagrangian by partial integration).

Alternatively, uniqueness of the gravity Lagrangian is also obtained by requiring that no dimensional constant should enter the Lagrangian. In this case the (wedge) product of any two curvatures $R^A = (R^{ab}, T^a)$ would have different scaling with respect to the Lagrangian (2.7) and should therefore be omitted. Of course dimensional constants should appear in theories where coupling to matter, gauging and scalar potential appear.

Let us now write down the equations of motion derived from the action (2.8). Varying the action with respect to $\omega^{ab}$ and $V^d$ we find, respectively:

$$T^c \wedge V^d_{\epsilon abcd} = 0,$$

$$R^{ab} \wedge V^c_{\epsilon abcd} = 0,$$

where $T^a = \mathcal{D}V^a$ is the torsion 2-form, $\mathcal{D}V^a = dV^a - \omega^{ab}V_b$ denoting the Lorentz covariant derivative.

Before proceeding with the solutions to the above equations, it is important to stress that, besides the obvious invariance under GCTG, even if all the fields are valued in the Poincaré group, the action is invariant only with respect to the subgroup of the (local) Lorentz transformations.

This can be easily checked writing down the infinitesimal action of the Poincaré group on the gauge fields $\omega^{ab}, V^a$. Defining $\epsilon^A = \epsilon^{ab}, \epsilon^a$, being $\epsilon^{ab}$ and $\epsilon^a$ the parameters of the infinitesimal Lorentz and translation gauge transformations, respectively, we have:

$$\delta^{(gauge)}(A) = (\nabla\epsilon)^A,$$

where $\nabla$ denotes the Poincaré gauge covariant differential. Decomposing the (co)-adjoint index $A$ in indices of the Lorentz subgroup, from (2.14) follows

$$\delta^{(gauge)}\omega^{ab} = \mathcal{D}\epsilon^{ab},$$

$$\delta^{(gauge)}V^a = \mathcal{D}\epsilon^a + \epsilon^{ab}V_b,$$

7By curvature we mean both the $R^{ab}$ and $T^a$ 2-forms.

8For pure theories, however, like those described in terms of massless fields only, the pure gravity case being the simplest case, one dimensional constant of dimensions mass squared is allowed, adding the term $1/3\Lambda_{abcd}V^a V^b V^c V^d$ with $\Lambda$ having the dimension of a mass squared. This gives rise to a Einstein Lagrangian with a cosmological term. This kind of extensions, however, can be easily shown to be equivalent to starting with the group manifold of a (anti) de Sitter group instead of the Poincaré group and will not change anything in the mechanisms we are going to discuss both for gravity as for supergravity. Indeed we may note that the Poincaré group ISO(1, 3) is an Inönü-Wigner contraction of the SO(2, 3) group.
where $D = d - \omega$ denotes the Lorentz covariant differential. It is then easy to see that the Lagrangian (2.7) and the equations of motion are invariant under a local Lorentz transformation, but are not invariant under a local translation. Indeed performing an infinitesimal translation on (2.19), using the Bianchi identities (2.5) and (2.6) and integrating by parts, we have

$$\delta \int R^{ab} V^c V^d \epsilon_{abcd} = 2 \int R^{ab} D c V^d \epsilon_{abcd} = -2 \int c^e R^{cd} T^d \epsilon_{abcd} \neq 0. \quad (2.15)$$

The non-invariance under translations of the equations of motion (2.12) and (2.13) can be checked in an analogous way. We will see in the next section that the fact that a torsionless vierbein can acquire torsion under the action of a translation can be best understood using the notion of Lie derivative.

Let us now solve the equations of motion (2.12) and (2.13). From (2.12), expanding the torsion 2-form along the vierbein basis, $T^a = T^a_{bc} V^b V^c$, it is easy to find that the components $T^a_{bc}$ have a vanishing trace, $T^a_{ab} = 0$. This implies in turn $T^a_{bc} = 0$ as the unique solution. Therefore

$$T^a = 0 \longrightarrow dV^a - \omega^a_b V^b = 0. \quad (2.16)$$

Expanding along the differentials, we write

$$\partial_{[\mu} V^a_{\nu]} = \omega^a_{[\mu} V^b_{\nu]} b. \quad (2.17)$$

Equation (2.17) can be solved in an analogous way as the Levi-Civita connection is solved in terms of the metric and its derivatives, obtaining $\omega^a_{[\mu}$ in terms of the vierbein components and its derivatives. Therefore, implementing the purely algebraic equation of motion of the spin connection allows us to express $\omega^a_{b[\mu}$ as a functional of the vierbein and its first derivatives. This is strictly analogous to what happens in the first order Palatini formalism in the usual metric approach.

From the second equation, expanding the 2-form $R_{ab} = \frac{1}{2} R^{cd} V^c V^d$, we also find, after some algebraic manipulation, the Einstein equation

$$R_{ab} - \frac{1}{2} g_{ab} R = 0, \quad (2.18)$$

where now $R_{ab}$ is the Ricci tensor $R_{ab} = R^{cd}_{ab}$ and $R \equiv R^a_{a}$ is the scalar curvature.

2.1. Extending the Theory from $G/H$ to $G$.

We have seen in the previous section that the Einstein-Cartan Lagrangian is invariant under gauge Lorentz gauge transformations. It is therefore convenient to use the language of fiber bundles.

We will refer in the following to the Poincaré group, but most of the considerations are exactly the same for other gravity or supergravity theories, replacing the Poincare' group with a general (super-)group $G$ and the Lorentz group with a general gauge group $H$.

We know that in the vacuum configuration, the fiber bundle is $[\text{ISO}(1,3)/\text{SO}(1,3)]$, while for

\[\text{Note that we write Einstein equation } R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0 \text{ using rigid vierbein indices. Indeed, unless necessary, we are using rigid tangent indices throughout instead of coordinate indices. Of course the rigid indices can be traded with the world indices using the vierbein matrix in the usual way.}\]
a general configuration the gauge fields \( \mu^A = (\omega^{ab}, V^a) \) of the Cartan-Einstein Lagrangian (2.7) live on a fiber bundle \([M_4, H]\), where \( M_4 \) is the base space and \( H = \text{SO}(1,3) \) is the fiber. It is then convenient and natural to consider the base space \( M_4 \) as a deformation of the vacuum base space \( \text{ISO}(1,3) \). Indeed the base space out the vacuum can be obtained when the left-invariant 1-forms are deformed into a non-left-invariant dynamical forms \( \mu^A \) enjoying curvatures. In this case the base space can be thought of as a space \( G/H \) on which a dynamical metric has been defined, constructed out of the on left-invariant 1-forms \( \mu^A \), which no longer has \( G \) as isometry group. Equivalently we may also say that the group \( G \) has been deformed into a \( \tilde{G} \) so has to have dynamical fields. \( \tilde{G} \) is often referred to as a soft group manifold. With this nomenclature we can write the general structure of the fiber bundle as \( \tilde{G}/H \).

Let us now take the point of view of reference\(^1\) and assume that the set of 1-forms \( \mu^A = (\omega^{ab}, V^a) \) is defined, right from the beginning, on the whole Poincaré group \( G = \text{ISO}(1,3) \). In this case factorization of the Lorentz coordinates is absent since the gauge fields \( \mu^A = (\omega^{ab}, V^a) \) will now depend on the full set of ten-dimensional group coordinates, namely the \( x^\mu \) parameters of translations and the coordinates \( y^{\mu
u} \) associated with the Lorentz transformations of \( \text{SO}(1,3) \). The non left-invariant \( \mu^A \) 1-forms, generalizing the \( \sigma^A \), are naturally viewed as a set of vielbein spanning a local reference frame on the ten-dimensional cotangent plane at each point of \( \tilde{G}/H \).

We consider the Lagrangian \( L = R^{ab} \wedge V^c \wedge V^d_{\epsilonabcd} \) where now both the vielbein and curvatures are not factorized, but depend of the full set of the \( G \)-coordinates \((x^\mu, y^{\mu\nu})\).

Even if the Lagrangian (2.7) is formally the same when the fields 1-form and curvatures 2-forms are defined on the full group manifold of \( G \), there is, however, a problem to write down a suitable action, since we have to integrate the Lagrangian 4-form in the \( d \)-dimensional space of the group \( \tilde{G} \). \((d = 10 \text{ in the Poincaré case}).\)

A simple way out would be to embed the four-dimensional space-time as a four-dimensional hypersurface \( M_4 \) (with boundary value \( M_4 \)) in \( \tilde{G} \). However, the very presence of \( M_4 \) in the variational principle makes it a dynamical variable, thus subjected to variation. Indeed new fields enter in the action corresponding to the embedding functions of \( M_4 \) in \( \tilde{G} \). This implies the added complication that the equations describing the embedding of \( M_4 \) in \( \tilde{G} \) should include arbitrary functions which must be considered as fields, what would of course spoil the geometric nature of the Cartan geometric approach.

The crucial observation given in\(^1\) is that one can safely ignore the variation of \( M_4 \), since any variation of \( M_4 \) can be compensated by a change of coordinates \( x^M = (x^\mu, \eta^{\mu\nu}) \) in \( \tilde{G} \) under which the Lagrangian, built only in terms of differential forms (and wedge products among them), is invariant. When considering theories more general than pure gravity, this of course requires that the Lagrangian be geometrical in a larger sense as was previously explained. Indeed, besides the requirements of being built using only differential forms, wedge products and the differential operator \( d : d^2 = 0 \), we must also add the requirement of excluding the presence of the Hodge duality operator. Indeed as the hypersurface \( M_4 \) on which the integration is performed can be chosen arbitrarily, due to invariance under diffeomorphisms, the equations of motion will hold on the whole \( \tilde{G} \). The presence of the Hodge duality operator, instead, would give a dependence on the hypersurface \( M_4 \) and its metric.
In the case under study, namely simple gravity in the Cartan formalism, see equation (2.7), the requirement of geometricity in this larger sense is obviously satisfied and the action will be now written as follows:

$$A = \frac{1}{4\kappa^2} \int_{\mathcal{M}_4 \subset \tilde{G}} R^{ab} \wedge V^c \wedge V^d \epsilon_{abcd},$$  \hspace{1cm} (2.19)

where $\mathcal{M}_4$ is an arbitrary surface immersed in $\tilde{G} = \tilde{\text{ISO}}(1,3)$. The equations of motion derived from the action (2.19) are formally the same as equations (2.12) and (2.13), but are now valid on the whole group manifold, since, as already observed, the hypersurface $\mathcal{M}_4$ is arbitrary. Because of that, in analyzing their content, the Lorentz curvature and the torsion 2-forms must be expanded not only in terms of the vierbein $V^a \wedge V^b$, but also in terms of the 2-forms $\omega^{ab} \wedge V^c$ and $\omega^{ab} \wedge \omega^{cd}$, which are part of the $\mu^A \wedge \mu^B$ basis of the cotangent plane to $\tilde{G}$ at each point of $\mathcal{M}_4$. The projection along $V^a \wedge V^b$ leads of course to the same equations as in the Cartan original setting. On the other hand, expanding the curvature and the torsion equations also along the other two 2-forms of the basis containing at least one $\omega^{ab}$ 1-form, it is almost immediate to recognize that we obtain for their corresponding components the solution:

$$R^{ab} \mid (lm) = R^{ab} \mid (pq) = T^a \mid (lm) = T^a \mid (pq) = 0.$$  \hspace{1cm} (2.20)

These equations assert that the curvatures are horizontal along the Lorentz subgroup 1-forms containing at least one Lorentz gauge field $\omega^{ab}$, implying that the group manifold $\tilde{G}$ has acquired dynamically the structure of a fiber bundle, namely $[\tilde{G}/H, H]$, where in the present case $\tilde{G} = \tilde{\text{ISO}}(1,3)$, $H = \text{SO}(1,3)$. Indeed we recall that an equivalent way to say that a manifold $\tilde{G}$ has the structure of a fiber bundle $[\tilde{G}/H, H]$ is to require that the curvatures $R^A$ are horizontal, that is that their components along the $H$ directions spanned by the 1-forms dual to the generators of $H$ vanish. This is in fact the content of equation (3.21) since the curvature 2-forms $(R^{ab}, T^a)$ vanish along components $\omega^{ab} \wedge \omega^{cd}$ or $\omega^{ab} \wedge V^c$. It follows that only the components along two vielbein $R^A = R^A_{\mu\nu} V^a \wedge V^b$ will survive, that is, identifying $\mathcal{M}_4$ with the space-time and projecting them along the differentials, namely the $R^A_{\mu\nu}$ components.

Thus we have reobtained from the same Lagrangian, but an enlarged action principle, the same equations of motion as in the classical treatment of the Cartan Lagrangian given before. The new procedure of defining fields on the group manifold and of considering space-time as a hypersurface immersed in $G$, gives a conceptual advantage with respect to the usual formulation since the factorization of the Lorentz coordinates and the Lorentz invariance of the Lagrangian are a result of the field equations. When this is done the connection forms and the vierbein appear as a single g-bein on $\tilde{G}$ ($g$ being the dimensions of $G$) which also play the role of connections in computing covariant derivatives and curvatures.

On the other hand, implementing the equations of motion derived from the action principle of (2.19) one finds in a dynamical way that the theory lives effectively on a fiber bundle $[\tilde{G}/H, H]$. Note that even if in pure gravity the Hodge duality operator does not appear, in more general theories, like matter coupled gravity, it is precisely the absence of the Hodge duality operator which implies that the choice of $\mathcal{M}_4$ turns out to be irrelevant. Actually any other $\mathcal{M}_4$ could work equally well and the physics would be the same on any of them. Indeed a
diffeomorphism in the direction orthogonal to space-time, considered from a *passive* point of view, can be considered as a lifting of the hypersurfaces $\mathcal{M}_4$ to another hypersurfaces $\mathcal{M}_4'$ and corresponds to the same theory in a different Lorentz frame. In other words the lifting from $\mathcal{M}_4$ to $\mathcal{M}_4'$ corresponds to a Lorentz transformation. From an *active* point of view, however, restricting the Lorentz factorized theory to a fixed $\mathcal{M}_4$, identified as the physical space-time, the same diffeomorphism corresponds to consider the theory together with all possible Lorentz transformations. Indeed, as we will explain in the next subsection, a diffeomorphism on the group manifold reduces to a Lorentz gauge transformation when the curvatures along the Lorentz subgroup are horizontal.

Let us note that the requirements of geometricity given before for treating the simple case of pure Poincaré gravity work equally well in more complicated theories like matter coupled gravity theories in four or even higher dimensions.

In all these more realistic cases the requirement of the absence of the Hodge duality operator seems, at first sight, to be too strong. Indeed when any extended gravity theory is coupled to scalars or vector fields their kinetic terms require the presence of the Hodge duality operator. For example, a kinetic term for a vector field should be written as proportional to

$$-\int F_{\mu\nu} F^{\mu\nu} \sqrt{-g} d^4x = \frac{1}{2} \int F \wedge \ast F,$$

where $F_{\mu\nu} = \partial_{[\mu} A_{\nu]}$, $F = F_{ab} V^a V^b$ and $\ast F = \frac{1}{4} F_{ab} \epsilon_{abcd} V^c V^d$. This apparent obstacle can be easily overcome introducing a *first order formalism* for the vector field. Namely, we introduce a 0-form antisymmetric Lorentz tensor $\hat{F}_{ab}$ ($\hat{F}_{ab} = -\hat{F}_{ba}$) and write for the kinetic term of the 2-form $\tilde{F}$ the following action:

$$A_{\text{vec}} = -\int \hat{F}_{ab} \hat{F}_{ab} \Omega + \alpha \int \hat{F}_{ab} F^{cd} V^c V^d \epsilon_{abcd},$$

where $\Omega$ is the four-dimensional volume element ($-\frac{1}{4!} \epsilon_{pqrs} V^p V^q V^r V^s$). Varying the 0-form $\hat{F}_{ab}$ we find that choosing $\alpha = -\frac{1}{2}$ we obtain

$$\hat{F}_{ab} = F_{ab},$$

where $F_{ab}$ are the components along two vielbein of the 2-form $F$. Varying next the gauge field $A$, from the second term of the action (2.22) we find the usual equation of motion:

$$D_a F^{ab} = 0 \rightarrow D_\mu F^{\mu\nu} = 0.$$  
(2.24)

In this way we see that using a first order formalism for the vector fields Lagrangian, the kinetic term can be obtained from the equation of motion starting from a geometric Lagrangian where the Hodge operator is absent. The same trick can be obviously used for the kinetic term of scalar fields.

Quite generally the same requirement of geometricity should be made for any extension of the theory with additional fields (matter coupled gravity) and in particular in supergravity where, besides the mechanism of Lorentz coordinates factorization, the requirement of a geometric Lagrangian turns out to be necessary in order to implement in a geometrical way local supersymmetry. It is precisely this fact that makes the rather academic result of obtaining the factorization of Lorentz coordinates as a result of the field equations a simple standard example to understand in a similar way the supersymmetry transformations in supergravity.
2.2. **Gauge Transformations and Diffeomorphisms.**

It is interesting at this point to show in an explicit way how a diffeomorphism reduces to a gauge transformation when the curvatures are horizontal, while it differs by curvature terms in the general case. We perform the derivation in a general group-theoretical setting so that it may apply to any (softened) group or supergroup \( \tilde{G} \).

An infinitesimal element of the GCTG on \( \tilde{G} \) is given by a tangent vector on \( \tilde{G} \), \( \vec{t} = \epsilon^M T_M \), where the middle alphabet Latin capital indices are coordinate indices on \( \tilde{G} \). Using the vielbein \( \mu^A \) of the whole (soft) group \( \tilde{G} \) we can rewrite a tangent vector \( \vec{t} \) as follows:

\[
\epsilon^A = \epsilon^M \tilde{T}_A,
\]

(2.25)

where \( \epsilon^A = \epsilon^M \mu^A_M \), and \( \tilde{T}_A = T_M \mu^A_M \). Here \( \tilde{T}_A \) is the vector field generator dual to the non left-invariant 1-form \( \mu^A \), \( \mu^A(\tilde{T}_B) = \delta^A_B \), and \( \epsilon^A = \delta x^A \) is the infinitesimal parameter associated to the shift. An infinitesimal generator of diffeomorphisms generated by \( \epsilon^A \) is given by the Lie derivative

\[
\ell_{\epsilon} \mu^A = (\iota_\epsilon d + d\iota_\epsilon) \mu^A,
\]

(2.26)

where \( \iota_\epsilon \) is the contraction operator along \( \epsilon \).

On the other hand the Lie derivative (2.26) can be also rewritten as follows:

\[
\ell_{\epsilon} \mu^A = (\iota_\epsilon d + d\iota_\epsilon) \mu^A = \\
= \iota_\epsilon d\mu^A + d \left( \iota_{\epsilon (\iota_\epsilon T_A^M)} \mu^A \right) = \\
= \iota_\epsilon d\mu^A + d\epsilon A.
\]

(2.27)

Adding and subtracting \( C^A_{BC} \mu^B \wedge \mu^C \) to \( d\mu^A \) and using the definition of the covariant derivative

\[
\nabla \epsilon^A = d\epsilon A + C^A_{BC} \mu^B \epsilon^C,
\]

(2.28)

we find :

\[
\ell_{\epsilon} \mu^A = \iota_{\epsilon} \left( d\mu^A + \frac{1}{2} C^A_{BC} \mu^B \wedge \mu^C \right) - \epsilon B C^A_{BC} \mu^C + d\epsilon A.
\]

(2.29)

where we have used the antisymmetry of \( C^A_{BC} \) in the lower indices. The terms in brackets define the curvature \( R^A \) while the other two terms, using the antisymmetry of the structure constants in \( (B, C) \) define the gauge covariant differential of \( \epsilon^A \). Therefore, using the *anholonomized* parameter\(^{10} \) \( \epsilon^A \), the Lie derivative can be written as follows:

\[
\ell_{\epsilon} \mu^A = (\nabla \epsilon)^A + \iota_{\epsilon} R^A.
\]

(2.30)

Hence an infinitesimal diffeomorphism on the manifold \( \tilde{G} \) is a \( G \)-gauge transformation plus curvature correction terms.

In particular, if the curvature \( R^A \) has vanishing projection along a vector \( \epsilon^B T_B \), where \( B \) is an adjoint index of the subgroup \( H \subset \tilde{G} \) so that

\[
\iota_{\epsilon} R^A = \epsilon^B R^A_{BC} \epsilon^C = 0,
\]

(2.31)

\(^{10}\)By anholonomized parameter we mean that we are using the rigid group index of the vielbein \( \mu^A \).
then the action of the Lie derivative \(\ell_{e}\) coincides with a gauge transformation. In this case we recover the result that the curvatures are horizontal along the H directions and the group manifold acquires the structure of a principal fiber bundle whose base manifold is \(\tilde{G}/H\), and H the gauge group.

In conclusion, if the theory on \(\tilde{G}\) can predict horizontal curvatures, it lives on \([\tilde{G}/H, H]\) and it is equivalent to the original Cartan approach. This is in fact what we have found in the case of the Poincaré group.

We stress once again that the derivation of the formula AGCT, equation (2.30), makes no explicit reference to the specific group \(\tilde{G}\). It holds for any group, including supergroups, as we shall see in the supergravity case.

### 3. Geometric Supergravity.

The fact that the fiber bundle structure in H of a gravity theory can be obtained dynamically from a suitable action principle is certainly an interesting feature of these theories, since it sheds light on the geometrical origin of the theory and on the power of the action principle. However, from a purely physical point of view it does not seem to add anything important to our understanding of the theory. After all to write a theory possessing ab initio a fiber bundle structure does not change anything in the development of the theory and in its physical results.

The value of the previous detailed description of factorization of Lorentz parameters lies in the possibility to give a geometrical interpretation of supersymmetry analogous to the geometrical mechanism of factorization of the Lorentz coordinates shown in the pure gravity case.

Indeed we will show, using the simple example of \(N = 1, D = 4\) supergravity, that the invariance of the supergravity Lagrangian is due to the behavior of the supergroup curvatures along the fermionic components of the supervielbein in superspace. Indeed, while the curvatures in the direction of the “Lorentzian” vielbein \(\omega^{ab}\) of the supergroup have to be zero in order to obtain a fiber bundle structure for the H subgroup, exactly as it happens in gravity, this will not happen for the supercurvatures in the direction of the fermionic vielbein \(\psi^{\alpha}\).

What actually happens is the following: The dependence of the fields on the supergroup or superspace odd coordinates \(\theta^\alpha\), does not imply their complete factorization, rather one finds that such components of the curvature 2-forms can be expressed algebraically, actually linearly, in terms of the curvatures restricted to the bosonic cotangent plane of the embedded space-time hypersurface, namely in terms of \(V^a V^b\), the basis on the cotangent space of ordinary space-time. Inserting this result into the Lie derivative formula (2.30), one obtains a geometrical interpretation of the local supersymmetry transformations.

For the sake of brevity and simplicity we will show how this happens in the simple example of pure \(N = 1, D = 4\) supergravity. However, the relevant results hold exactly in the same way for any supergravity theory, pure or matter coupled, in any dimension \(4 \leq D \leq 11\) and for any number \(1 \leq N \leq 8\) of supersymmetry generators in the Lie superalgebra.

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11 Here and in the following \(\alpha\) is a spinor index in the relevant representation of SO(1,D-1).
3.1. $\mathcal{N} = 1, D = 4$ Supergravity in the Geometric Approach.

We will now give the explicit description of the group manifold formalism for the $\mathcal{N} = 1, D = 4$ pure supergravity theory.

The graded group of $\mathcal{N} = 1, D = 4$ supergravity is the super-Poincaré group $\tilde{G} = \text{OSp}(1|4)$, where the bar over OSp$(1|4)$ means Inönü-Wigner contraction of the super Anti-de Sitter group (recall that SO$(2,3) \simeq \text{Sp}(4)$). The coadjoint multiplet of left-invariant 1-forms is

\[ \sigma^A = (\dot{\omega}^{ab}, \dot{V}^a, \dot{\psi}^\alpha), \]

where $\dot{\psi}^\alpha$ is a Majorana spinor 1-form in the fermionic direction of the supergroup. The upper little ring means that, being left-invariant, they satisfy the Cartan-maurer equations (vanishing curvatures):

\[ d\sigma^A + \frac{1}{2} C^{A}_{BC} \sigma^B \wedge \sigma^C = 0. \]

These 1-form fields will now depend on the coordinates $(x^\mu, \eta^{\mu\nu}, \theta^\alpha)$ of the supergroup.

Deforming the $\sigma^A$ into the non left-invariant 1-form $\mu^A = (\omega^{ab}, V^a, \psi^\alpha)$ allows us to define the following multiplet of super-curvatures:

\[ R^a_{\ b} \equiv d\omega^a_{\ b} - \omega^a_{\ c} \wedge \omega^c_{\ b}, \]
\[ \hat{T}^a \equiv dV^a - \omega^a_{\ b} \wedge V^b - \frac{i}{2} \bar{\psi} \gamma^a \psi, \]
\[ \rho \equiv D\psi = d\psi - \frac{1}{4} \gamma_{ab} \psi \omega^{ab}, \]

where we adopt a matrix notation for the spinor current in equation (3.4) and we have denoted by $\rho$ the curvature 2-form of $\psi$; $\gamma_a$ and $\gamma_{ab}$ are Dirac gamma matrices in four dimensions. Moreover we have set a hat on the supertorsion, $\hat{T}^a$, to avoid confusion with the purely bosonic torsion $T^a$. The curvatures satisfy the following Bianchi identities:

\[ DR^{a}_{\ b} = 0, \]
\[ D\hat{T}^a + R^a_{\ b} V^b - i \bar{\psi} \gamma^a \rho = 0, \]
\[ D\rho - \frac{1}{4} \gamma_{ab} \psi R^{ab} = 0. \]

Note that all terms in the definition of the curvatures and in the Bianchi identities scale homogenously since $\omega^{ab}, V^a, \psi$ and their curvatures have length scaling $[L^0], [L^1]$ and $[L^{1/2}]$, respectively.

In order to write down the Lagrangian we require that it is geometric. For the sake of clarity let us repeat here what it amounts to, adding some more obvious requirements:

- It must be constructed using only differential forms, wedge products among them, and the $d$ exterior differential;
- It must not contain the Hodge duality operator;

\[ \rho \]

\[ \frac{1}{4} \gamma_{ab} \psi R^{ab} = 0. \]

Here and in the following we will mostly omit the spinor index $\alpha$ on the spinors. Moreover, if there is no risk of confusion, we shall often refer to the super-curvatures simply as curvatures.
As the Einstein term, which must be always present, scales as \([L^2]\), \([L^{D-2}]\) in \(D\) dimensions), all the terms must scale in the same way;

To these requirements one usually adds the following one:

- If all the curvatures \(R^A\) are zero (left-invariant \(\sigma^A\) or vacuum configuration), the Lagrangian and the equations of motion must vanish identically.

This last requirement is useful in constructing Lagrangians more complicated compared to the present \(N = 1, D = 4\) theory.

It is easily seen that the only term we can add satisfying the first three requirements, for the \(G = \text{OSp}(1|4)\) group, is the Rarita-Schwinger kinetic term (written in terms of differential forms). Thus we obtain

\[
\mathcal{A}^N=1_{D=4} = \frac{1}{4\kappa^2} \int_{M_4 \subset \text{OSp}(1|4)} \left[ R^{ab} V^c V^d \epsilon_{abcd} + \alpha \bar{\psi} \gamma^5 \gamma_a D \psi V^a \right]. \tag{3.9}
\]

where instead of a tilde we have used a boldface character to denote the soft supergroup manifold \(\text{OSp}(1|4)\). This supermanifold has ten bosonic and four fermionic coordinates, namely \((x^\mu, \eta^{\mu\nu}, \theta^a)\). The coefficient \(\alpha\) between the Einstein and Rarita-Schwinger is related to the normalization of the gravitino 1-form \(\psi\) and will be fixed in a moment. The equations of motion obtained by varying \(\omega^{ab}, V^a\), and \(\psi\) are respectively:

\[
\epsilon_{abcd} \left( D V^a + \frac{i\alpha}{8} \bar{\psi} \gamma^a \psi \right) \wedge V^d = 0, \tag{3.10}
\]

\[
2 R^{ab} \wedge V^c \epsilon_{abcd} - \alpha \bar{\psi} \gamma^5 \gamma_a D \psi = 0, \tag{3.11}
\]

\[
2 \gamma^5 \gamma_a D \psi \wedge V^a - \gamma^5 \gamma_a \psi \wedge \hat{T}^a = 0. \tag{3.12}
\]

As the equations of motion have to vanish identically when all the (super-)curvatures are zero (fourth requirement), we see that we must set in the left hand side of Eq.(3.10) \(\alpha = 4\) in order to have the super-torsion 2-form \(\hat{T}^a\) as defined in (3.4). With this value of \(\alpha\) equation (3.10) takes the form

\[
\hat{T}^c \wedge V^d \epsilon_{abcd} = 0 \tag{3.13}
\]

and we see that when all the supercurvatures are zero the equations of motion vanish identically.

To analyze the content of equations (3.11), (3.12) and (3.13) we expand the curvatures 2-forms along the basis \(\mu^A \wedge \mu^B\). Let us first consider their components along the basis 2-form containing at least on \(\omega^{ab}\), namely (omitting the wedge symbol):

\[
\omega^{ab} \omega^{cd}; \quad \omega^{ab} V^c; \quad \omega^{ab} \psi.
\]

It is then immediate to see that the equations of motion give horizontality of all curvatures \(R^{ab}, \hat{T}^a, \rho\) in the Lorentz directions, namely

\[
R^{A}_{(ab)(cd)} = R^{A}_{(ab)c} = R^{A}_{(ab)a} = 0, \tag{3.14}
\]

exactly as in the pure gravity case. In this case, however, the supermanifold where the theory lives has a base space the \(\text{OSp}(1|4)/\text{SO}(1,3)\) that is the softened super-coset of the vacuum \(\text{OSp}(1|4)/\text{SO}(1,3)\). We identify \(\text{OSp}(1|4)/\text{SO}(1,3)\) as the superspace ad will be
denoted $R^{(4|4)}$. Superspace has now as coordinates $x^\mu, \theta^\alpha$, since the Lorentz coordinates $\eta^{\mu\nu}$ have been factorized.

However as the physical space-time $\mathcal{M}_4$ is four-dimensional while superspace has four extra fermionic dimensions, we are left with the same problem as in the bosonic case of pure gravity, namely to construct a suitable action for a Lagrangian 4-form in the eight-dimensional superspace. It is then natural to resort to the same procedure used for the pure gravity theory. Namely we can identify the space-time with any four-dimensional bosonic hypersurface $\mathcal{M}_4$ embedded in superspace. Indeed, as the Lagrangian is completely geometrical and it does not contain the Hodge duality operator, the invariance under diffeomorphisms of the Lagrangian allows arbitrary deformations of $\mathcal{M}_4$ in superspace. Therefore the equations of motion, being independent of the particular hypersurface chosen, will hold on the full superspace. This is completely analogous to the pure gravity case, the only difference being that the diffeomorphisms on the bosonic group manifold have been replaced by diffeomorphisms in superspace.

Since we reduced ourselves to the study of a Lorentz invariant theory on superspace, $R^{(4|4)}$, let us shortly describe the geometric structure of the theory in superspace.

Since the Lorentz coordinates have already been factorized, on $R^{(4|4)}$ the general base of 1-forms is given by the set of super-vielbein $E^A$, namely $E^A = (V^a, \psi^\alpha)$, where $\psi^\alpha$, $\alpha = 1, \ldots, 4$, is the fermionic vielbein, that is a Majorana spinor 1-form named gravitino. The action (3.9) is now reduced to the following form:

$$A = \frac{1}{4\kappa^2} \int_{\mathcal{M}_4 \subset R^{(4|4)}} \left[ R^{ab} V^c V^d \epsilon_{abcd} + 4 \overline{\psi} \gamma^5 \gamma_a D \psi V^a \right].$$

(3.15)

Let us introduce for the sake of brevity the following notation. We denote by $R_A^{(p,q)}$, $A = (ab,a,\alpha)$, the components of the curvature along $p$ bosonic vielbein $V^a$ and $q$ fermionic vielbein $\psi$. Moreover we call outer all the components were $q \neq 0$, that is those components having at last one index along the $\psi$ direction, while when $q = 0$, that is when the only non vanishing components are along the bosonic vielbein, they will be called inner.

To analyze the equations (3.11), (3.12), and (3.13), now restricted to superspace, we must expand the curvatures along a complete basis of 2-forms in superspace. In this case we have

$$E^A \wedge E^B = \{(V^a V^b); \,(V^a \psi^\alpha); \,(\psi^\alpha, \psi^\beta)\}. \tag{3.16}$$

Let us first work out equation (3.13). Expanding $\hat{T}^a$ we have

$$\hat{T}^a = \hat{T}^a_{(2,0)} V^b V^c + \hat{H}^a_{\psi} \psi V^c + \psi K^a \psi, \tag{3.17}$$

where $\hat{T}^a_{(1,1)} = H^a_{\psi}$ is a spinor and $K^a$, defining the $\hat{T}^a_{(0,2)}$ component, is proportional to a gamma matrix in four dimensions. Considering equation (3.13), one easily conclude that the components of $\hat{T}^a_{(1,1)}$ must be zero, while $\hat{T}^a_{(0,2)}$ would only change the normalization of the gravitino 1-form in the definition (3.4), so we can put them also to zero. Then, equation (3.13) restricted to the $VV$ ($T^a_{(2,0)}$) components has exactly the same form as in the pure gravity case, that is (2.16), provided we replace $T^a$ with $\hat{T}^a$. Note, however, that in this case

13If the factorization of the Lorentz coordinates have not yet been implemented then we deform the hypersurface $\mathcal{M}_4$ on the full graded group manifold as it was done in the pure gravity case.
solving for the spin connection $\omega_{ab}^{\mu}$ with the usual procedure gives a spin connection which depends not only from the vielbein and their derivatives, but also from gravitino bilinears.\footnote{See e.g. reference.} With exactly the same computations as those made for pure gravity, one easily obtains the vanishing of the $\hat{T}_{ab}^{\mu}$ components and therefore the whole super-torsion 2-form is zero. In the same way, in order to solve the equations (3.11) and (3.12) we expand the curvatures $\rho$ and $R^{ab}$ along a complete basis of 2-forms in superspace. As $T^{a} = 0$ the equation (3.12) takes the form

$$\gamma_{a} D\psi V^{a} = 0$$ \hspace{1cm} (3.18)

and expanding $\rho \equiv D\psi$ as

$$\rho^{\alpha} = \tilde{\rho}_{ab}^{\alpha} V^{a} V^{b} + H_{a} \psi^{\alpha} V^{a} + \Omega_{\alpha\beta} \psi^{\alpha} \psi^{\beta},$$ \hspace{1cm} (3.19)

where, for the sake of clarity, we have made the spinor index explicit. From equation (3.18) one easily realizes that $H_{a} = \Omega_{\alpha\beta} = 0$, so that the 2-form $\rho$ has only components $\rho_{(2,0)}$ on the cotangent space of $\mathcal{M}_{4}$, namely

$$\rho = \tilde{\rho}_{ab} V^{a} V^{b}.$$ \hspace{1cm} (3.20)

We warn the reader that since we are now in superspace the rigid indices cannot be traded with coordinate indices using the bosonic vierbein $V_{\mu}$. Indeed, the full set of supervielbein is now given by $E^{A} = (V^{a}, \psi^{\alpha})$ and we should invert the matrix $E_{\mu}^{A}$ to find the space-time components. This is in fact the reason why we have denoted with a tilde the components of the supercurvatures along two bosonic vierbein. A simpler way to find the space components is to project the equations on the space-time basis $dx^{\mu} \wedge dx^{\nu}$. For example from Eq. (3.19), projecting on the space-time basis we obtain

$$\rho_{\mu\nu}^{\alpha} = \tilde{\rho}_{ab} V_{\mu}^{a} V_{\nu}^{b} + H_{a} \psi^{\alpha} V_{\mu}^{a} + \Omega_{\alpha\beta} \psi^{\alpha} \psi_{\nu}^{\beta},$$ \hspace{1cm} (3.21)

where the indices $\mu\nu$ are understood to be antisymmetric. We see that the tilded components of $\tilde{\rho}_{\mu\nu}$ differ from the the real space time components $\rho_{\mu\nu}$ by terms in the gravitino fields, namely outer terms. They are commonly named in the literature as supercovariant field strengths.\footnote{The name supercovariant means that their supersymmetry transformation law does not contain derivatives of the supersymmetry parameter $\epsilon^{\alpha}$.} However, in our case, as far the $\hat{T}_{ab}^{a}$ and $\rho$ curvatures are concerned, we can easily convert rigid Lorentz indices in world indices as usual, since in the present case they do not have outer components ($V^{c} \wedge V^{d}$) and ($\psi^{c} \wedge \psi^{d}$). Then, for the components of the aforementioned curvature 2-forms we can neglect the tilde symbol. Instead, as we will see in a while and further discuss in the sequel, the space-time components of the Lorentz curvature do not coincide with the components along $(V^{c} \wedge V^{d})$ expanded along the differentials of the coordinates. Indeed, expanding $R^{ab}$,

$$R^{ab} = \tilde{R}_{cd}^{ab} V^{c} V^{d} + \tilde{\Theta}_{c}^{ab} \psi V^{c} + \bar{\psi} K^{ab} \psi,$$ \hspace{1cm} (3.22)

from equation (3.11) we find

$$\tilde{\Theta}_{c}^{ab} = -\varepsilon^{abr}_{\gamma} \rho_{rs} \gamma^{5}_{rc} - \delta^{[a}_{c} \varepsilon^{b]}_{d} \rho_{st} \gamma^{5}_{tm}.$$ \hspace{1cm} (3.23)
and $K^{ab} = 0$.

Note that the value for the outer component of $R_{(1,1)}^{ab} = \overline{\Theta}_{c}^{ab} V^{c}$ given in (3.22) is written in terms of the components of the gravitino curvature along $V^{a} V^{b}$, namely $\rho_{ab}$. Thus an outer component of the Lorentz curvature 2-form is linearly expressed, on-shell, in terms of the inner components of the fermionic curvature $\rho$. This property is called rheonomy and will be discussed more generally in the following. Physically it just means that no new degree of freedom is introduced in the theory other than those already present on space-time. Actually, if rheonomy is assumed a priori, and we take advantage of the fact that all the coefficients in the expansion along the supervielbein must give rise to terms with same scale as the corresponding curvatures, then one easily recognizes that the previous results for the outer components of the curvature multiplet are easily recovered (as for the possibility to have dimensional constant, see footnote 8). In conclusion, the solution of the equations of motion (3.10), (3.11), (3.12) for the outer and inner projections of the curvature multiplet gives:

\begin{equation}
R^{ab} = \tilde{R}_{cd}^{ab} V^{c} V^{d} + \overline{\Theta}_{c}^{ab} \psi V^{c},
\end{equation}

\begin{equation}
\hat{T}^{a} = 0,
\end{equation}

\begin{equation}
\rho = \rho_{ab} V^{a} V^{b}.
\end{equation}

Finally, inserting the parameterizations (3.24), (3.25), (3.26) in the equations of motion (3.10), (3.11), (3.12), we find that the components of the equations of motion along $V^{a} V^{b} V^{c}$, that is $R^{A}_{(3,0)}$, give the space-time equations of motion:

\begin{equation}
\tilde{R}_{lm}^{mn} - \frac{1}{2} \delta_{lm}^{mn} \tilde{R}_{mn} = 0,
\end{equation}

\begin{equation}
\hat{T}^{m} = 0,
\end{equation}

\begin{equation}
\epsilon^{mnrs} \gamma_{5} \rho_{mn} = 0.
\end{equation}

To find the relation between the components on space-time of these equations, as observed before instead of inverting the supermatrix $E^{A}_{\alpha}$ we project the equations (3.24), (3.25), (3.26) on the space-time 2-form differentials $dx^{\mu} \wedge dx^{\nu}$. We obtain

\begin{equation}
R_{\mu\nu}^{ab} = \tilde{R}_{cd}^{ab} V^{c} V^{d} + \overline{\Theta}_{c}^{ab} \psi_{[\mu} V^{c]}^{\nu},
\end{equation}

\begin{equation}
\hat{T}_{\mu}^{a} = \hat{T}_{bc}^{a} V^{b} V^{c},
\end{equation}

\begin{equation}
\rho_{\mu\nu} = \rho_{ab} V_{a}^{\mu} V_{b}^{\nu}.
\end{equation}

We have already seen that the space-time components of $\hat{T}^{a}$ and $\rho$ curvatures are obtained as usual converting rigid indices in curve ones using the bosonic vierbein $V_{\mu}^{a}$ since their parametrization does not contain $\psi$ fields. Instead the space-time components of the Lorentz curvature expanded along the differentials of the coordinates, namely $R_{\mu\nu}^{ab}$ do not coincide with the components along $(V^{c} \wedge V^{d})$. In fact writing equation (3.30) as

\begin{equation}
\tilde{R}_{\mu\nu}^{ab} = R_{\mu\nu}^{ab} - \overline{\Theta}_{c}^{ab} \psi_{[\mu} V^{c]}_{\nu},
\end{equation}

where $\tilde{R}_{\mu\nu}^{ab} \equiv R_{\mu\nu}^{ab} V^{c} V^{d}$, we see that the Einstein equation of motion, written in terms of the $\tilde{R}_{\mu\nu}^{ab}$, contains extra terms linear in the inner components $\rho_{a} \equiv \rho_{\mu\nu} V_{a}^{\mu} V_{b}^{\nu}$. These terms give rise to the energy-momentum tensor of the gravitino field $\psi_{\mu}$.
3.2. Invariance of the Lagrangian.

Let us now check the supersymmetry invariance of the Lagrangian. In the geometric approach, the supersymmetry invariance of the Lagrangian is expressed by the vanishing of the Lie derivative of the Lagrangian for infinitesimal diffeomorphisms in the fermionic directions of superspace. Using the Lie derivative with a tangent vector $\vec{\epsilon} = \epsilon^\alpha \vec{D}_\alpha$, where $\vec{D}_\alpha$ is the tangent vector dual to $\psi^\alpha$, we must have

$$\delta_\epsilon L \equiv \epsilon_\epsilon L = \epsilon dL = 0,$$

where we have discarded the total derivative term $d(\epsilon_\epsilon L)$ and other possible exact 4-forms on the right hand side as we are assuming that the fields vanish at infinite so that any exact form does not contribute to the action.\(^{16}\) Taking into account the definition (3.4), a simple computation gives

$$dL = 2R^{ab}_c \hat{T}^c V^d \epsilon_{abcd} + i R^{ab}_c \bar{\psi}^\gamma \psi^\delta V^d \epsilon_{abcd} + 4\bar{\psi}^\gamma \gamma^c \gamma_\alpha \rho V^\alpha + \bar{\psi}^\gamma \gamma_\alpha \gamma_{ab} \psi^\delta R^{ab}_c \hat{T} - 2i \bar{\psi}^\gamma \gamma_\alpha \rho \psi^\delta \psi,$$

(3.35)

where we have used the Bianchi identities and $\epsilon_\epsilon \psi = \epsilon$, $\epsilon_\epsilon V^a = 0$. Using the Fierz identity for 1-form spinors $\gamma^a \psi \bar{\psi} \gamma^a \psi = 0$, (see reference\(^5\)) and performing the gamma matrix algebra, one finds:

$$dL = R^{ab}_c \hat{T}^c V^d \epsilon_{abcd} + \bar{\rho}^\gamma \gamma_\alpha \rho V^\alpha - 4\bar{\psi}^\gamma \gamma_\alpha \rho \hat{T}^a.$$

(3.36)

Finally, contracting with the tangent vector $\epsilon^\alpha \vec{D}_\alpha$, we obtain

$$\epsilon_\epsilon dL = 2(\epsilon_\epsilon R^{ab}) \hat{T}^c V^d \epsilon_{abcd} + 2R^{ab}_c (\epsilon_\epsilon \hat{T}^c) V^d \epsilon_{abcd} + 8(\epsilon_\epsilon \rho) \gamma^\gamma \gamma_\alpha \rho V^\alpha - 4\bar{\psi}^\gamma \gamma_\alpha \rho \hat{T}^a - 4\bar{\psi}^\gamma \gamma_\alpha \rho (\epsilon_\epsilon \hat{T}^a) = d(3 - \text{form}).$$

(3.37)

From (3.37) we see that we can have an invariant action if we require constraints on the components of the curvatures. Indeed, if we set

$$\epsilon_\epsilon T^a = 0; \quad \epsilon_\epsilon \rho = 0$$

(3.38)

and furthermore

$$2 \left( \epsilon_\epsilon R^{ab} \right) V^d \epsilon_{abcd} + 4\epsilon \gamma^\gamma \gamma_\gamma \rho = 0.$$

(3.39)

we find $\delta_\epsilon L = 0$, that is invariance of the Lagrangian under supersymmetry.

We note that the requirements (3.38) and (3.39) are the same of the on-shell constraints (3.25) and (3.26) found from the equations of motion. In particular (3.39) gives the solution

$$\epsilon_\epsilon R^{ab}_c = \epsilon \Theta^{ab}_c V^c,$$

(3.40)

where $\Theta^{ab}_c$ has been defined in equation (3.23).

In other words we retrieve exactly the same constraints on the curvatures as those found from the equations of motion.

We conclude that the supergravity Lagrangian is invariant under (local) supersymmetry transformations when the superspace curvatures are defined by the equations (3.24)-(3.26). However, this restricted form of the curvatures in superspace imply that the supersymmetry transformations given below leaving the Lagrangian invariant do not form a closed

\(^{16}\)Note that the left hand side of equation (3.34) is not zero since we are now in the (4+4)-dimensional superspace.
algebra, unless one uses the equations of motion. This is best understood looking at the supersymmetry transformation laws.

Indeed, since we have found the on-shell value of the curvatures, we may apply the Lie derivative formula (2.30) to write down the superspace diffeomorphisms of the gauge fields $\omega^{ab}, V^a, \psi$ using a generic tangent vector $\tilde{e} = e^{ab} D_{ab} + e^a D_a + e^a D_a$, where the tangent vectors $D_{ab} D_a, D_a$ are dual to the gauge field 1-forms $\omega^{ab}, V^a, \psi^a$. We find:

$$
\delta_\epsilon \omega^{ab} = (\nabla \epsilon)^{ab} + e^r V^r R^{ab}_{rs} + \Xi^{rb}_r \psi \epsilon^r + \Xi^{ab}_r \epsilon V^r,
$$

$$
\delta_\epsilon V^a = (\nabla \epsilon)^a,
$$

$$
\delta_\epsilon \psi^a = (\nabla \epsilon)^a + e^r \rho^{ab}_r V^s.
$$

Restricting ourselves to the Lie derivative along the fermionic supersymmetry parameter $\epsilon$ only, that is setting $e^{ab} = e^a = 0$, we have

$$
\delta_\epsilon \omega^{ab} = (\nabla \epsilon)^{ab} + \Xi^{rb}_r \epsilon V^r,
$$

$$
\delta_\epsilon V^a = (\nabla \epsilon)^a,
$$

$$
\delta_\epsilon \psi^a = (\nabla \epsilon)^a.
$$

Here the symbol $\nabla$ denotes the gauge covariant derivative of the coadjoint multiplet of OSp(1|4). The Lorentz content of the gauge covariant derivative when acting on the OSp(1|4) adjoint multiplet can be read off directly from the Bianchi identities (3.8). Indeed both the parameters $\epsilon^A$ and the curvatures are in the coadjoint multiplet of the supergroup. Therefore:

$$
\delta^{(\text{gauge})}_\epsilon \omega^{ab} = (\nabla \epsilon)^{ab} = D e^{ab},
$$

$$
\delta^{(\text{gauge})}_\epsilon V^a = (\nabla \epsilon)^a = D e^{ab} + \epsilon^{ab} V_b - i \psi \gamma^a \epsilon,
$$

$$
\delta^{(\text{gauge})}_\epsilon \psi = (\nabla \epsilon) = D \epsilon - \frac{1}{4} e^{ab} \gamma_{ab} \psi,
$$

where $D$ denotes the Lorentz covariant derivative. Setting again $e^{ab} = e^a = 0$ and substituting in (3.44), (3.45), (3.46) we find the final form of the supersymmetry transformations:

$$
\delta_\epsilon \omega^{ab} = \Xi^{rb}_r \epsilon V^r
$$

$$
\delta_\epsilon V^a = - i \psi \gamma^a \epsilon,
$$

$$
\delta_\epsilon \psi = D \epsilon.
$$

Now we recall that the Lie derivative along tangent vectors $\tilde{T}_A$ satisfy an algebra isomorphic to the Lie algebra of the vector fields $[\tilde{T}_A, \tilde{T}_B] = (C^A_{BC} + R^A_{BC}) \tilde{T}_C$, namely

$$
[\ell_{\tilde{T}_A}, \ell_{\tilde{T}_B}] = \ell_{[\tilde{T}_A, \tilde{T}_B]},
$$

if the supercurvatures $R^A_{BC}$ are completely general, that is if they do not satisfy any constraint. In our case they satisfy the constraints (3.24)-(3.26) and in general the Lie derivative

As it is well known there exist theories in $D = 4$ and $D = 5$ which admit auxiliary fields, that is fields that added to the coadjoint supermultiplet make the supersymmetry transformations, besides leaving the Lagrangian invariant, to close the supersymmetry algebra off-shell. This is related to the fact that their introduction pairs the number of off-shell degrees of freedom between boson and fermions. Moreover they are not dynamical as their equations of motion make them to vanish. However, it does not seem possible to extend their introduction to higher-dimensional theories nor to matter coupled supergravities. Therefore we do not treat them in this short review.
algebra, namely the algebra of supersymmetry transformations, cannot close off-shell. Actually, as will be discussed in subsection 3.3, requiring that the Bianchi identities on the constrained curvatures be satisfied, one finds that their components on the bosonic cotangent plane $R_{\alpha}^A$, satisfy the equations of motion of the theory. It follows that the supersymmetry algebra of the transformations leaving the Lagrangian invariant, associated to the two tangent vectors $\epsilon^\alpha D_\alpha$, will in general only close on-shell, that is, only if the equations of motion are satisfied.

3.3. Supersymmetry as Diffeomorphisms in Superspace and Rheonomy.

Let us now discuss the results obtained so far.

- Even if the supercurvatures $\hat{T}^a$ and $\rho$, equations (3.25) and (3.26), respectively, have no components along the fermionic vielbein $\psi$, a non-vanishing component along $\psi \wedge V^a$ does appear in the on-shell value of the Lorentz supercurvature, that is (3.23). This is sufficient to exclude factorization of the odd fermionic coordinates. Indeed its presence makes the supersymmetry transformation a diffeomorphism in superspace and not a gauge transformation. It must also be noted that the absence of such fermionic components in the (on-shell) gravitino curvature $\rho$ implies that the supersymmetry variation of $\psi$, given in equation (3.52), makes the transformation of the gravitino gauge field the same as if the Lagrangian were invariant under supersymmetry gauge transformations. However, as the supersymmetry transformations of the Lagrangian do not close an algebra, the gravitino transformation law is actually a diffeomorphism, and the Lagrangian cannot be a true gauge symmetry because of the absence of factorization, as we have previously shown.

The point is that such behavior of the gravitino transformation law is due to the very simple form of the minimal $\mathcal{N} = 1, D = 4$ pure supergravity. Any other supergravity with $\mathcal{N} > 1$ or $D > 4$ or even the same theory $\mathcal{N} = 1, D = 4$ coupled to matter multiplets exhibits a gravitino curvature with components $\rho_{(1,1)} \neq 0$ so that the $\delta \psi$ will have, besides the Lorentz covariant derivative of the supersymmetry parameter, also terms along $\psi \wedge V^a$.

As an example, let us consider $\mathcal{N} = 2, D = 4$ pure supergravity. Here the supergroup is $\text{OSp}(2|4)$. The coadjoint gauge supermultiplet is now given by $\mu^A = (\omega^{ab}, V^a, \psi_A, A)$, where $A$ is a $\text{U}(1)$ gauge field 1-form and the index $A$ enumerates the gravitinos in the two-dimensional representation of $\text{SO}(2)$. The definition of the associated supercurvatures are obtained as always starting from the Maurer-Cartan equations dual to the algebra of the (anti-)commutation generators and deforming the left-invariant 1-forms into non left-invariant ones. Without giving the derivation, we write, besides the definitions of the supercurvatures on the left hand side, also their on-shell parametrization as found from the analysis of the

18Unless the constraints coincide with horizontality of the full set of curvatures as it happens for the Lorentz gauge invariance.

19Note that if the Lagrangian were invariant under supersymmetry gauge transformations the superfields would only depend on the $x^a$ coordinates.
equations of motion:

\begin{align}
R^{ab} & \equiv d\omega^{ab} + \omega^{ac} \omega_{cb} \\
& = \tilde{R}^{ab}_{cd} V^c V^d + \overline{g}_{A|c} \psi^A V^c - \tilde{\psi}_A \left( F^{ab} + i \gamma^5 F^{ab} \right) \psi_B \epsilon^{AB}, \quad (3.54)
\end{align}

\begin{align}
\hat{T}^a & \equiv D V^a - i \frac{1}{2} \overline{\psi}_A \gamma^a \psi^A = 0, \\
F & \equiv F + \epsilon^{AB} \overline{\psi}_A \psi_B = F_{ab} V^a V^b, \\
\rho_A & \equiv D \psi_A = \tilde{\rho}_{A|ab} V^a V^b + \left( \gamma^a F_{ab} + i \gamma^5 \gamma^a F_{ab} \right) \epsilon_{AB} \overline{\psi}_B V^b, \quad (3.57)
\end{align}

where \( F = dA \) and \( F \) is the supercurvature, and \( \star F_{ab} \) is the Hodge dual of \( F_{ab} \).

The important thing to note is that the parametrization of the curvature 2-forms given by the equations of motion are all given in terms of their inner components, namely \( \tilde{R}^{ab}_{cd} \), \( \tilde{\rho}_{A|ab} \), and \( F_{ab} \) (\( \hat{T}^a \) is zero).\(^{20}\)

Since the on-shell values of the supercurvatures is known, the supersymmetry transformation laws of the coadjoint supermultiplet, now containing also \( A \), can be obtained at once from the general formula (2.29). Looking at the Lie derivative formula, we see that the transformation laws of the multiplet of fields can be simply obtained performing the contraction of the on-shell curvatures with respect to the tangent vector \( \tilde{\epsilon} D \) and adding to the gravitino transformation the Lorentz covariant derivative of the supersymmetry parameter as it happens in the \( \text{OSp}(1|4) \) case. We find:

\begin{align}
\delta \omega^{ab} & = \overline{g}^{ab}_{A|c} \epsilon^A V^c, \\
\delta \epsilon^a & = -i \overline{\psi}_A \gamma^a \epsilon^A, \\
\delta \psi_A & = D \epsilon_A + i \epsilon_{AB} F^{ab} V^b \gamma^a \epsilon_B + i \frac{1}{2} \epsilon_{AB} \epsilon_{abcd} F^{cd} V^b \gamma^a \epsilon_B, \\
\delta \epsilon_A & = 2 \epsilon^{AB} \overline{\psi}_A \epsilon_B. \quad (3.61)
\end{align}

From this example we see that in general not only the Lorentz curvature \( R^{ab} \), but also the other supercurvatures have non-vanishing components along the \( \psi \)-directions.

• We can now resume our analysis in the following way:

Supersymmetry can be interpreted geometrically as the requirement that the super-space equations of motion imply that the outer components of the super-curvatures are expressible algebraically (actually linearly) in terms of the components along two inner vielbein. As already mentioned this property has been called rheonomy. Note that rheonomy is just a geometrical interpretation of supersymmetry originally introduced on space-time. Explicitly, the occurrence of rheonomy can be written as follows:

\[ R^A_{\alpha C} = C^A_{\alpha |mn} R^B_{mn}, \quad (3.62) \]

where \( C^A_{\alpha |mn} \) are suitable invariant tensors of the supergroup \( \tilde{S} \tilde{G} \) defining the basic superalgebra on which the theory is constructed, \( \tilde{S} \tilde{G} = \text{OSp}(1|4) \) in our case.

The geometric meaning of this property can be better understood if we use the

\(^{20}\)Note that \( F_{ab} \) has no tilde since \( F \) has components only along \( V^a V^b \).
Lie derivative formula (2.30) in superspace. Inserting (3.62) in the Lie derivative formula (2.30) for a supergroup \( S_\widetilde{G} \) we obtain:

\[
\delta \mu^A = (\nabla \epsilon)^A + 2\epsilon C_{a,\alpha}^{A|mn} R_{mn}^B.
\]

(3.63)

On the other hand, the Lie derivative can be interpreted either from the passive or from the active point of view. From the passive point of view the supersymmetry transformation along the \( \epsilon^\alpha = \delta \theta^\alpha \) parameter is interpreted as the lift from a given \( \mathcal{M}_4 \) to an infinitesimally close \( \mathcal{M}_4' \) which does not change the physical content of the theory, since it is described by the same Lagrangian after a supersymmetry transformation (and a Lorentz gauge transformation) has been made.\(^{21}\)

From the active point of view, however, it transforms a given configuration on \( \mathcal{M}_4 \), which we can take as space-time, setting \( \theta^\alpha = \delta \theta^\alpha = 0 \), to another physically equivalent configuration on the same space-time hypersurface. This property allows us to restrict the theory, the Lagrangian, and the equations of motion to any such arbitrarily chosen hypersurface \( \mathcal{M}_4 (\theta^\alpha = d\theta^\alpha = 0) \) embedded in superspace and identified with space-time.

One can now appreciate why we have illustrated in detail the mechanism of the Lorentz coordinate factorization in the gravity case defined on the Poincaré manifold. Actually the interpretation of the rheonomy mechanism just illustrated is quite analogous to the interpretation of Lorentz transformations for gravity constructed directly on a group manifold. Indeed, in the case of pure gravity, we have seen that a transfer of information from any \( \mathcal{M}_4 \subset \tilde{G} \) to any other \( \mathcal{M}_4' \subset \tilde{G} \) implies a \( \text{SO}(1,3) \) transformation or, equivalently, a change of Lorentz configuration on the fixed space-time hypersurface.

On the other hand, in our example of \( \mathcal{N} = 1, \ D = 4 \) supergravity, besides deducing the factorization of the Lorentz coordinates exactly as in the pure gravity case, we have further illustrated that the equations of motion allow us to deduce that the transfer of information concerns not only Lorentz gauge transformations but, what is our main goal, also supersymmetry.

However the difference between \( \text{SO}(1,3) \) transformations and supersymmetry is that in the first case, due to the horizontality of the curvatures in the Lorentz directions, \( \omega_{\alpha \beta}^a \) the supergroup \( \tilde{G} \) acquires the structure of the fiber bundle \( \left[ \tilde{G}/\text{H}, \text{H} \right] \), and the Lie derivative reduces to a Lorentz gauge transformation. On the other hand, in the case of supersymmetry, curvatures are not horizontal along the \( \psi \) gauge fields and the Lie derivative gives the geometric interpretation of supersymmetry. Actually, it is the rheonomic mechanism one is interested in, and in fact, quite generally, in the construction of any supergravity theory the fiber bundle structure with a Lorentz fiber is assumed a priori as it can be considered of academic interest to obtain it from the variational principle. In a way, restricting a supergravity theory to a factorized superspace \( \tilde{G}/\text{SO}(1,3) \) includes on a \( \mathcal{M}_4 \) slice, identified as space-time, all possible supersymmetry related Lagrangians.

In conclusion, the entire physics is contained in any single \( \mathcal{M}_4 \) or, equivalently, the supersymmetry transformations relate the fields on \( \mathcal{M}_4 \) to the fields on any other submanifold.
ifold $\mathcal{M}'_D$. It must be kept in mind that, since supersymmetry is a Lie derivative (super-diffeomorphism) in superspace, it is not a gauge symmetry. Indeed, as we have seen, its algebra does not even close off-shell.

### 3.4. Building Rules of a General Lagrangian.

Our previous detailed examples give us simple rules for the construction of a general (geometric) Lagrangian in $D$ dimensions:

- Starting from the Cartan-Maurer equations of a supergroup one defines the super-curvatures of the supergroup on which the theory is based. More precisely one writes the dual form of its super-Lie algebra and deforms the left-invariant 1-forms so that we can define the super-curvatures.
- The Lagrangian must be a $D$-form built in terms of the 1-form gauge fields $\mu^A$ and their curvatures $R^A$. It must contain the Einstein term $R^{ab}V^c_1...V^{c_{D-2}}\epsilon_{abc_1...c_{D-2}}$ whose scale is $[L^{D-2}]$. All the other terms should also scale as the Einstein term. At this point the Lagrangian contains a set of undetemined coefficients which will be fixed from the superspace equations of motion.
- The Hodge operator must be absent; the kinetic terms of scalar and vector fields must be written in first order formalism, as shown in the example of Section 2.1.
- All the possible terms satisfying the previous requirements must be present.
- If gauge invariance under an $H$ subgroup of $\tilde{G}$ is imposed a priori, where $H$ is the gauge group of the theory containing the Lorentz group as a factor $H = SO(1, D-1) \otimes H'$, the action is obtained by integrating the Lagrangian on a (bosonic) hypersurface embedded in superspace, defined as the Deformed coset $\tilde{G}/H$. Alternatively, we could start integrating on the whole $\tilde{G}$ and obtaining the factorization of the coordinates of $H$ as field equations. Since factorization of the gauge group $H$ is actually always true if the Lagrangian is $H$-invariant, the customary way to proceed is to start with a $H$ invariant Lagrangian on superspace $\tilde{G}/H$.
- The field equations must reduce to identities if all the curvatures are zero, that is if we are in the vacuum configuration with left-invariant 1-forms $\mu^A = \sigma^A$.

The field equations derived from such Lagrangian give equations of two types:

1. Field equations relating outer components of the curvatures linearly in terms of the inner ones. These are the rheonomic conditions equivalent to saying that the Lagrangian is supersymmetric. Indeed as we have seen supersymmetry is an invariance of the Lagrangian under diffeomorphisms in superspace and the rheonomic relations simply express the fact that the theory is independent of the space-time identification of the hypersurface $\mathcal{M}_D$ embedded in superspace. Note that in solving these equations all undetermined coefficients of the various terms become fixed.

\[\text{22} \text{Indeed in most supergravity theories we may have a larger group of gauge invariance other than the Lorentz one. When this happens, and we want to start from the full supergroup manifold, the factorization of the extra coordinates belonging to $H'$ can be obtained from the action principle exactly as for the Lorentz group coordinates.}\]
24

Field equations which are differential equations in superspace. Restricted to the bosonic hypersurface $\mathcal{M}_D$ they are the space-time equations of motion.

3.5. The Role of the Bianchi Identities.

Till now we have extensively explained how to give a geometrical interpretation to the supersymmetry transformations of supergravity either constructed on superspace, when factorization of the Lorentz coordinates is assumed a priori, or directly starting from the group manifold locally identified by the underlying supersymmetry algebra. There is however an equivalent and powerful approach to the construction of the equations of motion and transformation laws which is based on a systematic use of the Bianchi identities assuming rheonomy from the very beginning.

To understand this point we note that the Bianchi identities are true identities only if no constraint is assumed among the supervielbein components of the curvatures. However what rheonomy does is exactly to give relations among the outer components $R^A_{\dot{B}\alpha}$ (those along at least one fermionic vielbein $\psi$) and inner components $R^a_{\dot{B}}$ (namely the components on the cotangent space to space-time). Moreover, in almost any case, one also assumes a further constraint called kinematical constraint, namely vanishing super-torsion $\hat{T}^a = 0$. The Bianchi identities then assume the form of differential constraints among the space-time components. These differential constraints, on the other hand, can be nothing else than the equations of motion, since Bianchi identities cannot conflict with the differential equations obtained from the Lagrangian. Once the field equations are obtained, the Lagrangian, if desired, can be easily reconstructed. In the actual computations one usually couples the two methods, namely the Lagrangian approach and the Bianchi identities equations, to arrive in the simplest way to the final determination of the parametrization of the curvatures in superspace (and thus to the supersymmetry transformation laws) and to the determination of all the coefficients in the Lagrangian.

4. Results of the Geometric Approach. Some Remarks.

Most of the previous considerations have been dedicated to the geometrical interpretation of supersymmetry and to the explicit geometric construction of a supergravity theory. A natural question is now what has been the impact of this kind of approach from the physical point of view. We cannot of course enter in a detailed exposition of the results obtained from the very beginning till nowadays. We limit ourselves to give some remarks and observations concerning its power in treating higher dimensional supergravities and matter coupled theories containing antisymmetric tensor fields, together with some unexpected properties of superspace. As far as the most relevant results obtained in the geometric approach is concerned, we limit ourself to give a short list of them in the Appendix. Here is a couple of short and hopefully interesting comments:

- Among the most interesting results there is certainly the reduction of theories containing antisymmetric tensors to equivalent theories formulated in terms of super-Lie
algebras. This result was obtained in reference\textsuperscript{2} from the analysis of the geometrical formulation of $D = 11$ supergravity formulated on space-time in reference\textsuperscript{6}. As $D = 11$ supergravity is thought to be the low energy limit of the so-called $M$-theory, this result can have a special relevance for a better understanding of the group theoretical structure of $M$-theory. Because of its importance, we shall devote the next section to a short review of this approach in the case of $D = 11$ supergravity.

- It is undeniable that the systematic use of the geometric and group-theoretical approach has been an essential tool to obtain many other interesting results. For example, we can say that, in general, the introduction of matter coupling to pure supergravities allows to put in light the global and local symmetries inherent non-linear interaction structure of the coupling to matter multiplets. Indeed very often the use of the geometrical approach has allowed to arrive to a complete answer to problems where other approaches often had given only limited answers.\textsuperscript{23} A typical example was the full construction of the $\mathcal{N} = 2$, $D = 4$ matter coupled supergravity\textsuperscript{5,8,9}, which was previously formulated using the superconformal approach in a coordinate dependent way.\textsuperscript{10} The geometrical approach provides a complete Lagrangian and transformation laws quite independently of the coordinate used for the scalar fields description of the $\sigma$ model and it makes the introduction of the notion of Special Geometry, the geometry of the Special and Quaternionic manifolds, the momentum maps and the related gaugings, together with a complete description of the scalar potential, very natural. These results also give insight into the related superconformal two-dimensional theories and Calabi-Yau compactifications in string theory. Further developments are also to be found in reference.\textsuperscript{7}

- Another interesting observation is the following: The geometric approach discussed in this pedagogical review is naturally formulated in superspace. One could ask whether this approach is exactly equivalent to the purely space-time approach. This seems not to be the case in some theories, like $\mathcal{N} = 1$, $D = 6$\textsuperscript{11}, and $D = 10$, IIB.\textsuperscript{12} In the $D = 6$ supergravity the field multiplet contains the sechsbein, a Weyl gravitino, and a 2-form (that is an antisymmetric two-index tensor), it was shown that a consistent theory on the group manifold might have no counterpart in the usual Noether approach. In our geometric $D = 6$ superspace model, the self-duality of the 2-form field-strength, necessary to match the Bose-Fermi on-shell degrees of freedom, follows from group manifold/superspace variational equations, but not from their $x$-space restriction. As a consequence, the theory is consistent, although the $x$-space Lagrangian is not supersymmetry invariant. Exactly in the same way can be treated the $D = 10$, IIB theory, so that also in this case the self-duality of the 5-form can be retrieved from the superspace equations of motion. These results hint to extra properties of superspace which can be traded on the embedded hypersurface $\mathcal{M}_4$ only after the superspace equations of motion have been implemented, that is they are not visible using a purely space-time approach.

\textsuperscript{23}Most of these developments and results can be found in the excellent review.\textsuperscript{7}
5. Higher p-Forms Supergravities and their Hidden Supergroups.

We have often stressed that the mechanism of rheonomy actually holds in all supergravities, independently of the number of supersymmetries, the dimensionality of space-time, and their matter couplings, if any. However, apart from few exceptions, most of the higher dimensionality theories have a gravitational multiplet containing antisymmetric tensors of higher rank, mostly of rank two. Similarly, matter supermultiplets also can have higher rank tensors. In these cases the group manifold interpretation presented before as a possible starting point for supergravities whose fields are defined on a group manifold cannot be maintained. Indeed the coadjoint multiplet of a (super-) group consists of 1-forms dual to the group generators, with no room for higher p-forms.

In the present section we will show, referring mainly to the case of $D = 11$ supergravity where this development was first presented,$^2$ that:

- The Maurer-Cartan equations can be generalized to more general structures, called Free Differential Algebras (FDAs), admitting in their multiplet also forms of degree higher than one. They represent a natural extension of Lie algebras in their dual formulation and can accommodate supermultiplets containing higher p-forms satisfying the integrability requirement $d^2 = 0$.
- Each higher p-form $A^{(p)}$ can be decomposed in terms of a set of trilinear (wedge) products of $p$ 1-forms, where besides the supervielbein basis $(V^a, \psi)$ there appear new 1-forms valued in tensor or spinor representations of the Lorentz group. The new 1-forms obey extra Maurer-Cartan equations, besides those of the super-Poincaré group. The decomposition can be done in such a way that the coefficients of the polynomial written as a sum of products of $p$ 1-forms assure the integrability of the original FDA equation for the p-form $A^{(p)}$.
- Together with the super-Poincaré dual generators, the new Maurer-Cartan equations describe the dual form of a new super-Lie algebra which, at least locally, describes a group manifold called the hidden supergroup of the FDA which in a sense can be considered as the group-theoretical starting point for a construction of the supergravity theories possessing higher p-forms in their gravitational multiplet.
- Among the new 1-forms needed to assure that the given decomposition reproduces the integrable equation of the FDA, there appear extra spinor 1-forms (one in the case of $D = 11$ supergravity) whose dual generators $Q'_\alpha$ in the Lie superalgebra are nilpotent. Their role, as recently clarified$^{13,14}$, is to assure that the new 1-form fields thus introduced are gauge fields living on the fiber bundle whose base space is ordinary superspace, so that their dependence on the new coordinates are completely factorized. This means that their curvatures are horizontal and do not add new degrees of freedom other than those already present in the original FDA. This property works exactly in the same way for all higher dimensional theories with $D < 11$ whose gravitational multiplet contain antisymmetric tensor fields, for example in $\mathcal{N} = 2$, $D = 7$ supergravity where two such nilpotent spinor generators are present.

These results were obtained in$^2$ by R. D’Auria and P. Fré in the case of the maximal
$D = 11$ supergravity trying to give a fully geometrical interpretation of the space-time formulation of the theory.\textsuperscript{6} Indeed, in this theory there appears an antisymmetric tensor of rank three in the gravitational multiplet. In their approach, R. D’Auria and P. Fré introduced for the first time the generalization of the Maurer-Cartan equations for integrable systems containing higher $p$-forms which they called Cartan Integrable Systems (CIS). Only later it was realized that structures of this kind were already introduced in mathematics and called Free Differential Algebras\textsuperscript{15}, which is the name now universally accepted.

Even if we shall not give any account of the underlying mathematics, it must be said that the relation between the FDA and groups or supergroups relies on the Chevalley-Eilenberg (super)-Lie algebra cohomology groups. The procedure we previously alluded to of decomposing a higher $p$-form in a polynomial of Lorentz valued 1-forms is the inverse of the construction of a FDA starting from a (super-)Lie algebra, which, as far as I know, was not treated in the Lie algebra cohomology theory.

In the following, we give a short account of the FDA of $D = 11$ supergravity and its resolution as a hidden ordinary Lie supergroup.

6. Free Differential Algebra and Hidden Supergroup of $D = 11$ Supergravity.

Eleven-dimensional supergravity can be founded on the following FDA:

\begin{align*}
R^{ab} &\equiv d\omega^{ab} - \omega^{ac} \wedge \omega^b_c = 0, \\
T^a &\equiv DV^a - \frac{i}{2} \Psi \wedge \Gamma^a \Psi = 0, \\
\rho &\equiv D\Psi = 0, \\
F^{(4)} &\equiv dA^{(3)} - \frac{1}{2} \Psi \wedge \Gamma_{ab} \Psi \wedge V^a \wedge V^b = 0, \\
F^{(7)} &\equiv dB^{(6)} - 15A^{(3)} \wedge dA^{(3)} - \frac{i}{2} \Psi \wedge \Gamma_{a_1 \ldots a_5} \Psi \wedge V^{a_1} \wedge \cdots \wedge V^{a_5} = 0.
\end{align*}

Note that this differential system is an extension of the usual Maurer-Cartan equations and as such it only describes the structure of the physical vacuum of the theory.

In our case the super-Poincaré Maurer-Cartan equations in $D = 11$ with the addition of the higher order differential of the 3-form $A^{(3)}$ and 6-form $B^{(6)}$.\textsuperscript{24} The consistency of the FDA requires the integrability of the last two equations, $d^2A^{(3)} = 0$, $d^2B^{(6)} = 0$. It can be shown that this is in fact satisfied as a consequence of 3-fermion Fierz identities obeyed by the gravitino 1-form field in eleven dimensions (see e.g. reference\textsuperscript{5}).\textsuperscript{25}

\textsuperscript{24}In the original paper the last equation (6.5) was not present. Actually, it was almost immediately realized (see e.g. reference\textsuperscript{5}, Vol. 2) that, besides the simplest FDA including as exterior form only $A^{(3)}$, one can extend the FDA to include also a (magnetic) 6-form potential $B^{(6)}$, related to $A^{(3)}$ by Hodge-duality of the corresponding field-strengths on space-time. Indeed there is a constructive procedure based on the Chevalley-Eilenberg Lie algebra cohomology to arrive to the maximal extension of the super-Poincaré algebra given by the FDA which implies as a maximal extension the presence of the $B^{(6)}$.\textsuperscript{5}

\textsuperscript{25}Note that here and in the following we do not elaborate on the theory out of vacuum, namely the interacting theory, since the topological (and cohomological) structure of the theory, which will be the object of the present investigation, is fully caught by the ground state of the FDA.
The authors of\(^2\) asked themselves whether one could trade the FDA structure on which the theory is based on an ordinary Lie superalgebra, written in its dual Cartan form, that is in terms of 1-form gauge fields which turn out to be valued in non-trivial tensor and spinor representations of Lorentz group SO(1, 10). This would allow to disclose the fully extended Lie superalgebra hidden in the supersymmetric FDA. This was proven to be true, and the hidden superalgebra underlying the FDA describing \( D = 11 \) supergravity was presented for the first time.

It was shown that this is indeed possible by associating, to the forms \( A^{(3)} \) and \( B^{(6)} \), the bosonic 1-forms \( B_{ab} \) and \( B_{a_1 \cdots a_5} \), in the antisymmetric representations of SO(1, 10), and furthermore an extra spinor 1-form \( \eta \). The Maurer-Cartan equations satisfied by these new 1-forms are:

\[
\mathcal{D} B_{a_1 a_2} = \frac{1}{2} \Psi \wedge \Gamma_{a_1 a_2} \Psi , \quad (6.6)
\]

\[
\mathcal{D} B_{a_1 \cdots a_5} = \frac{i}{2} \Psi \wedge \Gamma_{a_1 \cdots a_5} \Psi , \quad (6.7)
\]

\[
\mathcal{D} \eta = i E_1 \Gamma_a \Psi \wedge V^a + E_2 \Gamma^{ab} \Psi \wedge B_{ab} + i E_3 \Gamma_{a_1 \cdots a_5} \Psi \wedge B_{a_1 \cdots a_5} , \quad (6.8)
\]

\( \mathcal{D} \) being the Lorentz-covariant derivatives. Of course the whole consistence of this approach also requires the \( d^3 \) closure of the Maurer-Cartan newly introduced fields \( B_{ab} \), \( B_{a_1 \cdots a_5} \) and \( \eta \). For the two bosonic 1-form fields the \( d^3 \) closure is obvious in the ground state, because of the vanishing of the curvatures \( R^{ab} \) and \( \rho = \mathcal{D} \psi \), while \( \mathcal{D}^2 \eta = 0 \) requires the further condition:

\[
E_1 + 10 E_2 - 270 E_3 = 0 , \quad (6.9)
\]

which can be derived using the Fierz identities of the wedge product of three gravitino 1-forms in superspace.\(^2\)

In reference\(^2\) the most general decomposition of the 3-form \( A^{(3)} \) in terms of the 1-forms \( B_{ab} \), \( B_{a_1 \cdots a_5} \) and \( \eta \) was presented. It has the following form:

\[
A^{(3)} = T_0 B_{ab} \wedge V^a \wedge V^b + T_1 B_{ab} \wedge B^{(a} \wedge B^{b)} +
+ T_2 B_{a_1 \cdots a_4} \wedge B^{(a} \wedge B^{b_{a_1}} \cdots b_{a_4}) + T_3 \epsilon_{a_1 \cdots a_5 b_1 \cdots b_5} B_{a_1 \cdots a_5} \wedge B^{b_1 \cdots b_5} \wedge V^m +
+ T_4 \epsilon_{m_1 \cdots m_3 n_1 \cdots n_5} B_{m_1 m_2 m_3} \wedge B_{n_1 n_2 n_3 n_4 n_5} \wedge B_{a_1 \cdots a_5} +
+ i S_1 \tilde{\Psi} \Gamma_{a} \eta \wedge V^a + S_2 \tilde{\Psi} \Gamma^{ab} \eta \wedge B_{ab} + i S_3 \tilde{\Psi} \Gamma_{a_1 \cdots a_5} \eta \wedge B_{a_1 \cdots a_5} , \quad (6.10)
\]

where \( T_1 \) and \( S_1 \) are numerical coefficients. To show the equivalence of the FDA with a ordinary super-Lie algebra (in dual form) it is required that the integrability condition in superspace of the 3-form, \( dA^{(3)} \), computed in terms of differentials of the new 1-forms gives the same results as in the case of equation (6.4), namely \( dA^{(3)} + \frac{1}{2} \Psi \wedge \tilde{\gamma} \Gamma_{ab} \Psi \wedge V^a \wedge V^b = 0 \). To obtain such integrability the extra terms containing the currents involving the extra spinor 1-form \( \eta \) turn out to be necessary. The Ansatz (6.10) restricted to the bosonic 1-forms does not work. In other words the inclusion of the spinor 1-field \( \eta \) enters in the decomposition of the 3-form \( A^{(3)} \) in such a way to properly reproduce the vacuum FDA on ordinary superspace.

When the integrability is implemented all the coefficients in the decomposition become fixed in terms of the ratio \( E_3/E_2 \).\(^{26}\)

\( ^{26}\) In\(^2\) the first coefficient \( T_0 \) was arbitrarily fixed to \( T_0 = 1 \) giving only 2 possible solutions for the
In this way, one arrives at the following full set of Maurer-Cartan equations for the left-invariant 1-forms \((\omega^{ab}, V^a, \psi, B_{ab}, B_{a_1...a_5}, \eta)\):

\[
\begin{align*}
\omega^{ab} &= \omega^{ac} \wedge \omega^b_c, \\
DV^a &= \frac{1}{2} \bar{\Psi} \wedge \Gamma^a \Psi, \\
D\psi &= 0, \\
DB_{a_1a_2} &= \frac{1}{2} \bar{\Psi} \wedge \Gamma_{a_1a_2} \Psi, \\
DB_{a_1...a_5} &= \frac{1}{2} \bar{\Psi} \wedge \Gamma_{a_1...a_5} \Psi, \\
D\eta &= iE_1 \Gamma_a \Psi \wedge V^a + E_2 \Gamma^{ab} \Psi \wedge B_{ab} + iE_3 \Gamma^{a_1...a_5} \Psi \wedge B_{a_1...a_5}.
\end{align*}
\]

This set of Maurer-Cartan equations identifies the supergroup which is (locally) described, as anticipated, by the hidden super-Lie algebra underlying the theory (this same superalgebra written in terms of commutators is given below).

It must be noted that if we neglect the presence of the \(\eta\) spinor 1-form in the decomposition (6.10) we can obtain a closed algebra which, however, is not equivalent to the FDA we started from since it fails to give the closure \(d^2A^{(3)} = 0.\) Only when the extra currents containing \(\eta\) are present in (6.10) we obtain such integrability.

Let us finally write down the hidden superalgebra in terms of generators closing a set of (anti)commutation relations.

To recover the superalgebra in terms of (anti)commutators of the dual Lie superalgebra generators

\[T_A \equiv \{P_a, Q, J_{ab}, Z^{ab}, Z^{a_1...a_5}, Q'\},\]

which are dual to the 1-forms \(\{V^a, \psi, \omega^{ab}, B_{ab}, B_{a_1...a_5}, \eta\}\) respectively, one uses the duality between 1-forms and generators and finds that the \(D = 11\) FDA corresponds to the following hidden superalgebra (besides the Poincaré algebra):

\[
\begin{align*}
\{Q, Q\} &= -\left(i\Gamma^a P_a + \frac{1}{2} \Gamma^{ab} Z_{ab} + \frac{i}{5!} \Gamma^{a_1...a_5} Z_{a_1...a_5}\right), \\
\{Q', Q'\} &= 0, \\
\{Q, P_a\} &= -2iE_1 \Gamma_a Q', \\
\{Q, Z^{ab}\} &= -4E_2 \Gamma^{ab} Q', \\
\{Q, Z^{a_1...a_5}\} &= -2(5!)E_3 \Gamma^{a_1...a_5} Q', \\
\{J_{ab}, Z^{d}\} &= -8\delta_{[a} Z_{b]}^{d}, \\
\{J_{ab}, Z^{c_1...c_5}\} &= -20\delta_{[a}^{[c_1} Z_{b]}^{c_2...c_5]}, \\
\{J_{ab}, Q\} &= -\Gamma_{ab} Q, \\
\{J_{ab}, Q'\} &= -\Gamma_{ab} Q'.
\end{align*}
\]

set of parameters \(\{T_1, S_j, E_k\}\). It was pointed out later in\(^{16,17}\) that this restriction can be relaxed thus giving a more general solution in terms of one parameter. Indeed, as observed in the quoted reference, one of the \(E_i\) can be reabsorbed in the normalization of \(\eta\), so that, owing to the relation (6.8), we are left with one free parameter, say \(E_3/E_2\).
All the other commutators (beyond the Poincaré part) vanish. We shall identify the super-Lie algebra in either of the two dual forms given as Maurer-Cartan equations or (anti-commutators) as D’Auria-Fré algebra (DF-algebra in the following). In the Lie algebra version of the dual Maurer-Cartan equations the 1-forms $B_{a_1...a_5}$ and $B_{ab}$ are the 1-forms dual to the central generators $Z^{a_1...a_5}$ and $Z^{ab}$, respectively, of a central extension of the supersymmetry algebra given by the usual $D = 11$ super-Poincaré algebra and including the extra nilpotent generator $Q'$ (dual to the spinor 1-form $\eta$). Actually, as shown in reference\textsuperscript{14} from a cohomological point of view, to reproduce the integrability of $dA^{(3)}$, the presence of the 1-form $B_{a_1...a_5}$ in the decomposition (6.10) is not necessary, since all the terms where it appears sum up to give an exact 3-form. However, as we have seen, even if cohomologically trivial, its addition allows to extend the Lie superalgebra in a non-trivial way.\textsuperscript{27}

6.1. $D=11$ Supergravity and M-theory.

After several years, on the basis of different considerations, the same algebra, but without the inclusion of the extra nilpotent generator was rediscovered. This superalgebra, actually a subalgebra of the hidden superalgebra (6.18), was named $M$-algebra.\textsuperscript{18–22}

The presence in the relations (6.11)-(6.16) of the bosonic hidden 1-forms $B_{ab}, B_{a_1...a_5}$ can be considered as a generalization of the centrally extended supersymmetry algebra of\textsuperscript{23} (where the central generators were associated with electric and magnetic charges), and, as such, have in fact a topological meaning. This was recognized in\textsuperscript{18} and\textsuperscript{24}, where it was shown they are to be associated with extended objects (2-brane and 5-brane charges, respectively) in space-time. The $M$-algebra is commonly considered as the super-Lie algebra underlying $M$-theory\textsuperscript{25–27} in its low energy limit, corresponding to $D = 11$ supergravity in the presence of non-trivial $M$-brane sources.\textsuperscript{24, 28–32}

A field theory based on the $M$-algebra, however, is naturally described on the enlarged superspace whose cotangent space is spanned, besides the gravitino 1-form, also by bosonic fields $\{V^a, B_{ab}, B_{a_1...a_5}\}$. If we hold on the idea that the low energy limit of $M$-theory should be based on the same ordinary superspace, spanned by the supervielbein $(V^a, \psi)$, as in the original formulation of $D = 11$ supergravity,\textsuperscript{6} then the $M$-algebra cannot be the final answer since it does not contain the extra 1-form $\eta$ dual to the nilpotent generator fermionic generator $Q'$. Indeed we have shown that in order to reproduce the FDA on which $D = 11$ supergravity is based, the presence of $\eta$ among the 1-form generators is necessary. Actually

\textsuperscript{27}Here and in the following the term “central” for the charges $Z^{ab}, Z^{a_1...a_5}$, and for the spinorial charge $Q'$ refers to their commutators with all the generators apart from the Lorentz generator $J_{ab}$. The commutation relations with $J_{ab}$ are obviously dictated by their Lorentz index structure.

\textsuperscript{28}More precisely in reference\textsuperscript{14} it was shown that once formulated in terms of its hidden superalgebra of 1-forms, $A^{(3)}$ can be actually decomposed into the sum of two parts having different group-theoretical meaning: One of them does not depend on $B_{a_1...a_5}$ and allows to reproduce the FDA describing $D = 11$ supergravity, while the second one does not contribute to the 4-form cohomology, being a closed 3-form in the vacuum; however, the second part defines a one parameter family of trilinear forms invariant under a symmetry algebra that is related to $\mathfrak{osp}(1|32)$ by redefining the spin connection and adding a new Maurer-Cartan equation. Correspondingly, also the spinor 1-form $\eta$ can be analogously split into two different spinors appearing each one in just one of the two parts in which $A^{(3)}$ is decomposed.
the DF-algebra, (6.11)-(6.16) or (6.18), contains the $M$-algebra as a subalgebra since it also includes a nilpotent ($Q'^2 = 0$) fermionic generator $Q'$ dual to the spinor 1-form $\eta$ whose contribution to the Maurer-Cartan equations of the DF-algebra is given by equation (6.16). In other words the DF-algebra underlying the formulation of the eleven-dimensional FDA on superspace reproduces the eleven-dimensional theory on space-time introduced in reference\textsuperscript{6}, if and only if the decomposition of the 3-form $A^{(3)}$ also includes the 1-form $\eta$.

As it was shown in reference,\textsuperscript{13} this in turn implies that the group manifold generated by the DF-algebra has a fiber bundle structure whose base space is ordinary superspace, while the fiber is spanned, besides the Lorentz spin connection $\omega^{ab}$, also by the bosonic 1-form generators $B_{ab}, B_{a_1...a_5}$. It follows that if we would start from the group manifold generated by the DF-algebra, the coadjoint multiplet would now be $\mu^A = (\omega^{ab}, B_{ab}, B_{a_1...a_5}, V^a, \psi, \eta)$ and the action principle would factorize the coordinates associated to the first three 1-forms so that their degrees of freedom would not enter into the equations of motion. Indeed the presence of $Q'$, dual to the 1-form $\eta$, allows to consider the extra 1-forms $B_{ab}$ and $B_{a_1...a_5}$ as gauge fields in ordinary superspace instead of additional vielbeins of an enlarged superspace, that is, their curvatures on the fiber are horizontal. This is due to the dynamical cancellation of their unphysical contributions to the supersymmetry and gauge transformations with the supersymmetry and gauge transformations of $\eta$.\textsuperscript{29}

7. Conclusions.

We have shown how the original idea formulated by Y. Ne’eman and T. Regge of defining gravity and supergravity theories directly on a (super-)group manifold $\tilde{G}$ actually allows a completely geometrical and group-theoretical formulations of any gravity or supergravity extended theory.

We have explained how the generalized action principle of integrating the Lagrangian on a submanifold of $\tilde{G}$ is capable of obtaining the following results:

Factorization of coordinates belonging to gauge subgroups of $\tilde{G}$ can be obtained by using a generalized action principle where space-time is represented by a bosonic submanifold immersed in the (super-)group, so that $\tilde{G}$ actually becomes endowed by a fiber bundle structure.

For supergroups, where the notion of superspace as base space of the fiber bundle appears, the same procedure based on the extended action principle allows to add, to the notion of factorization, the notion of supersymmetry whose geometrical meaning is that the field-strengths (curvatures) in superspace are not horizontal, but can be expressed linearly in terms of the space time field-strengths. The corresponding geometrical interpretation has been named rheonomy.

It is possible to give very simple building principles that allow an almost algorithmic procedure for the construction of any supergravity Lagrangian, the most important principle being geometricity.

\textsuperscript{29}As observed in,\textsuperscript{13} all the above procedure of enlarging the field space to recover a well defined description of the physical degrees of freedom is strongly reminiscent of the BRST-procedure, and the behavior of $\eta$ is such that it can be actually thought of as a ghost for the 3-form gauge symmetry, when the 3-form is parametrized in terms of 1-forms.
Besides the original results obtained not only in supergravity, but also in related questions like duality, Calabi-Yau compactifications, monodromies etc, an important step has been the discovery of the Free Differential Algebras as a geometrical way to formulate supergravities containing higher p-forms fields by a suitable generalization of the Maurer-Cartan equations to higher p-forms. In particular using the notion of Lie algebra cohomology it has been possible to show that any such supergravity containing higher p-forms can be reduced to an ordinary supergravity based on an ordinary Lie superalgebra.

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9. Appendix

We give a short list of the most relevant and interesting results obtained in the geometric approach.

Besides giving the geometrical structure of the already existing supergravity theories, some supergravity theories were first obtained in the geometric approach, see e.g. \( \mathcal{N} = 3, D = 4 \) and \( \mathcal{N} = 2, D = 6, F_4 \) matter coupled supergravities.\(^{33,34}\) Construction of central and matter charges and symplectic structure of all \( D = 4, \mathcal{N} \) extended theories\(^{35}\).

- Symplectic invariant coupling of scalar-tensors\(^{7,36,37}\) and vector-tensor multiplets\(^{38}\) in \( \mathcal{N} = 2 \) supergravity and the role of magnetic charges.
- Derivation of the \( \mathcal{N} = 1 \) and \( \mathcal{N} = 2, D = 4 \) supergravity Lagrangians in the presence of a boundary.\(^{39}\)
- Unexpected interesting relation between \( \mathcal{N} = 2 \) supergravity in \( D = 4 \) and a three-dimensional theory describing the graphene electronic properties.\(^{40}\)

Other relevant results are the following: Anomaly Free supergravity in \( D = 10 \);\(^{41}\) Duality transformations in supersymmetric Yang-Mills theories coupled to supergravity;\(^{42}\) A detailed analysis of the role played by the Picard-Fuchs equations in supergravity;\(^{43-45}\) Symplectic structure of \( \mathcal{N} = 2 \) supergravity and its central extension;\(^{46}\) Symplectic structure and monodromy group for the Calabi-Yau two moduli space\(^{47,48}\) (as far as rigid supersymmetry is concerned, the necessity of introducing a Chern-Simon term in \( D = 10 \) super Yang-Mills \( \mathcal{N} = 1 \) Lagrangian was first realized using the geometric approach, see reference\(^{49}\)).

Finally, even if not concerning supersymmetry, an important result has been obtained in the theory of gravitation using a completely geometric Lagrangian coupled to a pseudo-scalar field in a non-canonical way. Using this Lagrangian it has been possible to show the existence of \textit{asymptotically flat gravitational instantons in gravity}.\(^{50}\) Further results in this approach were also obtained in reference.\(^{51}\)
Bibliography

1. Y. Ne’eman and T. Regge, “Gauge Theory of Gravity and Supergravity on a Group Manifold,” Riv. Nuovo Cim. 1N5 (1978) 1, and Phys.Lett. 74B (1978) 54-56;
2. R. D’Auria and P. Fré, “Geometric Supergravity in d = 11 and Its Hidden Supergroup,” Nucl. Phys. B 201 (1982) 101 Erratum: [Nucl. Phys. B 206 (1982) 496].
3. H. Sati and U. Schreiber, “Lie n-algebras of BPS charges,” JHEP 1703 (2017) 087 doi:10.1007/JHEP03(2017)087 [arXiv:1507.08692 [math-ph]].
4. https://ncatlab.org/nlab/show/L-infinity+algebras+in+physics
5. L. Castellani, R. D’Auria and P. Fré, “Supergravity and superstrings: A Geometric perspective,” Vol. 1 and 2. Singapore: World Scientific (1991)
6. E. Cremmer, B. Julia and J. Scherk, “Supergravity Theory in Eleven-Dimensions,” Phys. Lett. B 76 (1978) 409 [Phys. Lett. 76B (1978) 409].
7. M. Trigiante, “Gauged Supergravities,” Phys. Rept. 680 (2017) 1 doi:10.1016/j.physrep.2017.03.001 [arXiv:1609.09745 [hep-th]].
8. L. Andrianopoli, M. Bertolini, A. Ceresole, R. D’Auria, S. Ferrara, P. Fré and T. Magri, “N=2 supergravity and N=2 superYang-Mills theory on general scalar manifolds: Symplectic covariance, gaugings and the momentum map,” J. Geom. Phys. 23 (1997) 111 doi:10.1016/S0393-0440(97)00002-8 [hep-th/9605032].
9. R. D’Auria, S. Ferrara and P. Fré, “Special and quaternionic isometries: General couplings in N=2 supergravity and the scalar potential,” Nucl. Phys. B 359 (1991) 705. doi:10.1016/0550-3213(91)90077-B.
10. B. de Wit, P. G. Lauwers and A. Van Proeyen, “Lagrangians of N=2 Supergravity - Matter Systems,” Nucl. Phys. B 255 (1985) 569. doi:10.1016/0550-3213(85)90154-3
11. R. D’Auria, P. Fré and T. Regge, “Consistent Supergravity in Six-dimensions Without Action Invariance,” Phys. Lett. 128B (1983) 44. doi:10.1016/0370-2693(83)90070-9
12. L. Castellani and I. Pesando, “The Complete superspace action of chiral D = 10, N=2 supergravity,” Int. J. Mod. Phys. A 8 (1993) 1125.
13. L. Andrianopoli, R. D’Auria and L. Ravera, “Hidden Gauge Structure of Supersymmetric Free Differential Algebras,” JHEP 1608 (2016) 095. doi:10.1007/JHEP08(2016)095 [arXiv:1606.07328 [hep-th]].
14. L. Andrianopoli, R. D’Auria and L. Ravera, “More on the Hidden Symmetries of 11D Supergravity,” Phys. Lett. B 772 (2017) 578 doi:10.1016/j.physletb.2017.07.016 [arXiv:1705.06251 [hep-th]].
15. D. Sullivan, “Infinitesimal computations in topology”, Publications Mathématiques de l’IHES, 47 (1977), p. 269-331
16. I. A. Bandos, J. A. de Azcarraga, J. M. Izquierdo, M. Picon and O. Varela, “On the underlying gauge group structure of D=11 supergravity,” Phys. Lett. B 596 (2004) 145 doi:10.1016/j.physletb.2004.06.079 [hep-th/0406020].
17. I. A. Bandos, J. A. de Azcarraga, M. Picon and O. Varela, “On the formulation of D = 11 supergravity and the composite nature of its three-form gauge field,” Annals Phys. 317 (2005) 238 doi:10.1016/j.aop.2004.11.016 [hep-th/0409100].
18. J. A. de Azcarraga, J. P. Gauntlett, J. M. Izquierdo and P. K. Townsend, “Topological Extensions of the Supersymmetry Algebra for Extended Objects,” Phys. Rev. Lett. 63 (1989) 2443. doi:10.1103/PhysRevLett.63.2443.
19. E. Sezgin, “The M algebra,” Phys. Lett. B 392 (1997) 323 doi:10.1016/S0370-2693(96)01576-6 [hep-th/9609086].
20. P. K. Townsend, “M theory from its superalgebra,” In *Cargese 1997, Strings, branes and dualities* 141-177 [hep-th/9712004].
21. M. Hassaine, R. Troncoso and J. Zanelli, “Poincare invariant gravity with local supersymmetry as a gauge theory for the M-algebra,” Phys. Lett. B 596 (2004) 132 doi:10.1016/j.physletb.2004.06.067 [hep-th/0306258].
22. M. Hassaine, R. Troncoso and J. Zanelli, “11D supergravity as a gauge theory for the M-algebra,” PoS WC 2004 (2005) 006 [hep-th/0503220].
23. E. Witten and D. I. Olive, “Supersymmetry Algebras That Include Topological Charges,” Phys. Lett. B 78 (1978) 97. doi:10.1016/0370-2693(78)90357-X
24. A. Achucarro, J. M. Evans, P. K. Townsend and D. L. Wiltshire, “Super p-Branes,” Phys. Lett. B 198 (1987) 441. doi:10.1016/0370-2693(87)90896-3
25. J. H. Schwarz, “The power of M theory,” Phys. Lett. B 367 (1996) 97 doi:10.1016/0370-2693(95)01429-2 [hep-th/9510086].
26. M. J. Duff, “M theory (The Theory formerly known as strings),” Int. J. Mod. Phys. A 11 (1996) 5623 [Subnucl. Ser. 34 (1997) 324] [Nucl. Phys. Proc. Suppl. 52 (1997) no.1-2, 314] doi:10.1142/S0217751X96002583 [hep-th/9608117].
27. P. K. Townsend, “Four lectures on M theory,” In *Trieste 1996, High energy physics and cosmology* 385-438 [hep-th/9612121].
28. E. Bergshoeff, E. Sezgin and P. K. Townsend, “Supermembranes and Eleven-Dimensional Supergravity,” Phys. Lett. B 189 (1987) 75. doi:10.1016/0370-2693(87)91272-X
29. M. J. Duff, P. S. Howe, T. Inami and K. S. Stelle, “Superstrings in D=10 from Supermembranes in D=11,” Phys. Lett. B 191 (1987) 70. doi:10.1016/0370-2693(87)91323-2
30. E. Bergshoeff, E. Sezgin and P. K. Townsend, “Properties of the Eleven-Dimensional Super Membrane Theory,” Annals Phys. 185 (1988) 330. doi:10.1016/0003-4916(88)90050-4
31. P. K. Townsend, “The eleven-dimensional supermembrane revisited,” Phys. Lett. B 350 (1995) 184 doi:10.1016/0370-2693(95)00397-4 [hep-th/9501068].
32. P. K. Townsend, “P-brane democracy,” In *Duff, M.J. (ed.) : The world in eleven dimensions* 375-389 [hep-th/9507048].
44. A. Ceresole, R. D’Auria, S. Ferrara, W. Lerche, J. Louis and T. Regge, “Picard-Fuchs equations, special geometry and target space duality,” AMS/IP Stud. Adv. Math. 1 (1996) 281.
45. M. Billò, A. Ceresole, R. D’Auria, S. Ferrara, P. Fré, T. Regge, P. Soriani and A. Van Proeyen, “A Search for nonperturbative dualities of local N=2 Yang-Mills theories from Calabi-Yau threefolds,” Class. Quant. Grav. 13 (1996) 831 doi:10.1088/0264-9381/13/5/007 [hep-th/9506075].
46. A. Ceresole, R. D’Auria and S. Ferrara, “The Symplectic structure of N=2 supergravity and its central extension,” Nucl. Phys. Proc. Suppl. 46 (1996) 67 doi:10.1016/0920-5632(96)00008-4 [hep-th/9509160].
47. A. Ceresole, R. D’Auria and T. Regge, “Duality group for Calabi-Yau 2 moduli space,” Nucl. Phys. B 414 (1994) 517 doi:10.1016/0550-3213(94)90439-1 [hep-th/9307151].
48. A. C. Cadavid, A. Ceresole, R. D’Auria and S. Ferrara, “Eleven-dimensional supergravity compactified on Calabi-Yau threefolds,” Phys. Lett. B 357 (1995) 76 doi:10.1016/0370-2693(95)00691-N [hep-th/9506144].
49. R. D’Auria, P. Fré and A. J. da Silva, “Geometric Structure of N = 1D = 10 and N = 4D = 4 Superyang-mills Theory,” Nucl. Phys. B 196 (1982) 205. doi:10.1016/0550-3213(82)90036-0
50. R. D’Auria and T. Regge, “Gravity Theories With Asymptotically Flat Instantons,” Nucl. Phys. B 195 (1982) 308. doi:10.1016/0550-3213(82)90402-3
51. O. Chandia and J. Zanelli, “Topological invariants, instantons and chiral anomaly on spaces with torsion,” Phys. Rev. D 55, 7580 (1997) doi:10.1103/PhysRevD.55.7580 [hep-th/9702025].