Recursive Prime Factorizations: Dyck Words as Representations of Numbers

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February 16, 2021

Abstract

I propose a class of numeral systems where numbers are represented by Dyck words, with the systems arising from a generalization of prime factorization. After describing two proper subsets of the Dyck language capable of uniquely representing all natural and rational numbers respectively, I consider “Dyck-complete” languages, in which every member of the Dyck language represents a number. I conclude by suggesting possible research directions.

1 Introduction

My fascination with patterns exhibited in the set of natural numbers \( \mathbb{N} = \{0, 1, 2, 3, \ldots \} \) led me to much experimentation and indeed quite a bit of frustration trying to discover and characterize such patterns. One of the most perplexing problems I encountered was the inherent arbitrariness of positional numeral systems. Consider the number 520:

\[
520 = 5 \times 10^2 + 2 \times 10^1 + 0 \times 10^0. \tag{1}
\]

The implicit selection of 10 as radix, though a convention tracing back to antiquity, reflects an arbitrary choice with consequences for patterns manifested in the representations. For instance, a well-known pattern is that if the sum of the digits in a decimal representation of a number is equal to a multiple of 3, then the number itself is divisible by 3; yet that is not the case with base-2 or base-5. This and many other such patterns may be generalized to apply to numeral systems of any base \( \geq 2 \), but for most of us the generalization detracts from the immediacy of the realization. It would be useful if the system of representation did not require any one number to
assume undue importance above the others, so that patterns would directly
reflect characteristics of those numbers under examination rather than being
obscured by the selection of some irrelevant number to serve as “the radix.”

There is another drawback inherent in positional numeral systems, at
least with regard to their use to identify and characterize patterns among
numbers. As is evident in Equation 1, evaluation of a number’s positional rep-
resentation requires three distinct operations, namely exponentiation, mul-
tiplication and addition. However, the number represented by decimal 520
can be more simply represented by a unique product of prime numbers

\[ 2^2 \times 5^1 \times 13^1, \]
called its prime factorization, the evaluation of which does not involve addi-
tion.

Remark. More precisely, the prime factorization of a number is unique up to
the order of the factors. Also, when I say “the evaluation of which does not
involve addition,” I am referring to addition as a distinct operation in the
evaluation; obviously the multiplication of natural numbers may be viewed
as iterated addition.

Thus I set out on a quest to discover systems for representing numbers
where the systems, being based upon prime factorization, neither involve the
concept of a radix nor require addition for evaluation. I furthermore sought
such systems with alphabets of the smallest size.

I succeeded in my quest, discovering a class of systems I call “Natural
Recursive Prime Factorizations” (“Natural RPFs”), where each of these sys-
tems can represent all members of \( \mathbb{N} \) using a language with an alphabet of
only two symbols. I subsequently realized natural RPFs can be modified to
yield another class of systems, “rational RPFs,” each of which is capable of
representing not merely all members of \( \mathbb{N} \) but all rational numbers. A re-
markable fact about rational RPF systems is that, unlike decimal and other
positional numeral systems, no enlargement of the alphabet is needed for
representation of the rationals; the same two-symbol alphabet is employed
as for natural RPF, without need for a negative sign or a radix point. There
is also no need for an overbar to designate repeating symbol sequences, since
every rational RPF system is able to represent all rational numbers by strings
of finite length.

I must warn the reader at the outset that these systems are impractical
for application to the mundane tasks of everyday life, such as balancing
checkbooks or enumerating street addresses. But they were never intended
for such purposes, instead being conceived to facilitate the study of patterns
among numbers, offering a convenient bridge between number theory and the
theory of formal languages. Natural RPF systems, for example, invite the analysis of their words using powerful techniques from computer science such as context-free grammars, parsers and finite state machines, providing direct connections between numbers and subsets of the well-studied Dyck language $\mathcal{D}$, including $\mathcal{D}$ itself. Words produced in these systems moreover do not involve an arbitrarily selected radix, eliminate the necessity for addition in their evaluation, and are closely related to the prime factorizations of almost all the numbers they represent.

*Remark.* I say *almost all* because 0 has no prime factorization, and because the question of whether 1 has a prime factorization is a matter of dispute [4]. Also, note that I regard prime numbers as having prime factorizations, the factorization of a prime number being the number itself, as given by the equation

$$p_k = \prod_{i=k}^{k} p_i.$$ 

Many interesting patterns arise in number sequences defined according to properties shared by their members’ representations as Dyck words (see Section 5.2 on page 43, for example).

## 2 The Standard Minimal RPF Natural Interpretation $\text{RPF}_{\mathbb{N}_{r,\min}}$

I begin by presenting a system capable of representing natural numbers by unique finite sequences of left and right parentheses. For now I will refer to this system as “minimal natural RPF,” abbreviated $\text{RPF}_{\min}$, in order to introduce the concept without first launching into a lengthy digression concerning languages and their interpretations. In Section 2.3, I will identify the system more precisely as “the standard minimal RPF natural interpretation $\text{RPF}_{\mathbb{N}_{r,\min}}$.”

### 2.1 Informal Treatment by Example

If challenged to describe minimal natural RPF in one sentence, I might say: “It is a numeral system in which 0 and 1 are represented by the empty string $\varepsilon$ and $( )$ respectively, with every other natural number $n$ being written as a product of powers of consecutive primes from 2 up to and including the greatest prime factor of $n$, each exponential term being surrounded by a single pair of parentheses and nonzero exponents being recursively treated in the same fashion as described for $n$, with the final resulting expression being stripped of all symbols except the parentheses, which are then rewritten on one line while preserving their order from left to right.”
I myself have difficulty digesting that long-winded sentence; let us therefore abandon it in favor of three examples, these being collectively sufficient to suggest how an arbitrary natural number may be represented in minimal natural RPF. To start with, the representations of zero and one are given by explicit definition:

- Zero is represented by the empty word $\epsilon$.
- One is represented by the word ().

The RPF$_{min}$ representation of every other natural number may be obtained by application of a recursive algorithm, as I will illustrate by finding the RPF$_{min}$ equivalent of decimal 520. But first I must introduce a function that will be used extensively in the algorithm.

### 2.1.1 The Minimal Parenthesized Padded Prime Factorization

We can express 520 as the exponential form of its prime factorization

$$p_1^3 p_3^1 p_6^1,$$

where $p_k$ is the $k$th prime number. The expression includes powers of $p_1$, $p_3$ and $p_6$, but not of $p_2$, $p_4$ or $p_5$, since these last three do not contribute to the prime factorization of 520. But let us rewrite Expression 2 as

$$p_1^3 p_2^0 p_3^1 p_4^0 p_5^0 p_6^1,$$

so that powers of all consecutive primes $p_1, \ldots, p_m$ are included, where $p_m$ is the greatest prime factor of 520. Now let us use single pairs of parentheses as grouping symbols around each exponential term, giving

$$(p_1^3)(p_2^0)(p_3^1)(p_4^0)(p_5^0)(p_6^1).$$

Expression 3 is the *minimal parenthesized padded prime factorization* (MPPPF, pronounced “MIP-fuh”) of 520. It is *minimal* because only powers of prime numbers up to and including the greatest prime factor of the number being represented are present, it is *parenthesized* for the obvious reason that all exponential terms are enclosed in parentheses, and it is *padded* because it includes exponential terms not appearing in the prime factorization.

Whether 1 has a prime factorization is a matter of dispute; we avoid the issue altogether by defining MPPPF(1) to be $(p_1^0)$. Certainly 0 has no prime factorization, but neither would one be useful for our purposes even if it were to exist. We accordingly define the domain of MPPPF to be the set of positive integers.
Observe that there cannot be more than one MPPPF corresponding to a given number, since MPPPF(1) is unique and MPPPFs for all other numbers n in the domain of MPPPF are the result of padding the unique prime factorization of n with the 0th powers of those noncontributing primes less than n’s greatest prime factor.

2.1.2 Finding the RPF\textsubscript{min} Equivalent of Decimal 520

We begin by expressing 520 as its MPPPF

\[(p_1^3)(p_2^0)(p_3^1)(p_4^0)(p_5^0)(p_6^1).\]

Our next step is to replace all nonzero exponents in the expression by their MPPPFs as well. We repeat this step until there no longer exist opportunities to replace exponents by MPPPFs:

\[
(p_1^3)(p_2^0)(p_3^1)(p_4^0)(p_5^0)(p_6^1) = \\
(p_1^3)(p_2^0)(p_3^1)(p_4^0)(p_5^0)(p_6^0),
\]

\[
(p_1^0)(p_2^0)(p_3^1)(p_4^0)(p_5^0)(p_6^0),
\]

\[
(p_1^0)(p_2^0)(p_3^1)(p_4^0)(p_5^0)(p_6^0).
\]

Now we proceed to the next and final step, which is to treat the expression as a string and delete all symbols except parentheses from it, writing the parentheses all on one line while preserving their order from left to right to yield the RPF\textsubscript{min} word

\[
(()(()))().()
\]

We may be certain \((()(()))().()())()()) is the only minimal natural RPF word corresponding to decimal 520. This is because the MPPPF of 520 is unique, and each nonzero exponent in Equation Set 4 has exactly one corresponding MPPPF.

Remark. Since RPF stands for recursive prime factorizations, the reader may wonder why I chose this name if words in the underlying languages do not appear to be prime factorizations at all, a prime factorization being by definition the product of only those prime numbers that are factors of the number being factorized. I offer this defense of my choice: we may indeed regard natural RPF words as prime factorizations, if we consider the empty pairs of matching parentheses arising from transcribing the zeroth powers of noncontributing primes to be markers collectively allowing us to deduce which prime numbers do contribute to the factorization.
2.1.3 Finding the Decimal Equivalent of

Having found the equivalent of decimal 520 in minimal natural RPF, let us go in the reverse direction, finding the decimal equivalent of the RPF\textsubscript{min} word

We begin by inserting 0 inside each empty matched pair of parentheses, yielding the expression

\[((0)((0)))(0)((0))(0)(0)((0)).\]

For the next step, we will treat expressions as containing zero or more “clusters,” by which I mean substrings beginning and ending with outermost matching parentheses; for example, the clusters from left to right in the expression above are \(((0)((0))), (0), ((0)), (0), (0) and ((0)). For each cluster \(w_k\), we replace \(w_k\) by the string

\[P_{k}^{\text{contents}_k},\]

where \(\text{contents}_k\) is the string obtained by deleting the outermost parentheses of \(w_k\). We do this repeatedly to the successive expressions until all the parentheses are gone:

\[
\begin{aligned}
((0)((0)))(0)((0))(0)(0)((0)) \\
\xrightarrow{\text{ }} \ 
\begin{array}{l}
p_{1}^{((0)((0)))}p_{2}^{0}p_{3}^{0}p_{4}^{0}p_{5}^{0}p_{6}^{0} \\
\xrightarrow{\text{ }} \ 
\begin{array}{l}
p_{1}^{p_{2}^{0}p_{3}^{0}}p_{2}^{0}p_{3}^{0}p_{4}^{0}p_{5}^{0}p_{6}^{0} \\
\xrightarrow{\text{ }} \ 
\begin{array}{l}
p_{1}^{p_{2}^{0}p_{3}^{0}p_{4}^{0}p_{5}^{0}p_{6}^{0}}.
\end{array}
\end{array}
\end{array}
\end{aligned}
\]

All that remains to be done is to evaluate the expression:

\[p_{1}^{p_{2}^{0}p_{3}^{0}p_{4}^{0}p_{5}^{0}p_{6}^{0}} = 2^{3} \cdot 5 \cdot 13 = 520.\]

We thus have a system capable of representing every natural number with an alphabet of only two symbols and not involving addition for evaluation. This fact may not seem particularly significant, given that the unary system of representing \(n\) by \(n\) contiguous marks uses an alphabet of only one symbol.
symbol. But unlike unary, minimal natural RPF directly and succinctly reflects the prime factorizations of the numbers it represents (for those numbers having prime factorizations, which includes all members of \( \mathbb{N} \) except 0 and 1). Indeed, for all natural numbers \( n \) greater than one, the minimal natural RPF word representing \( n \) contains not merely the prime factorization of \( n \), but also the prime factorizations of all factorizable numbers involved in the prime factorization of \( n \), the exponents in the prime factorization themselves being represented by their factorizations in recursive fashion.

**Remark.** See Table 1 on page 14 for the minimal natural RPF representations of the first 20 natural numbers.

Later (Section 3) I will show that minimal natural RPF can be modified to yield a system capable of representing every rational number, again using an alphabet of only two symbols, with no need for a negative sign or a radix point. Moreover, unlike decimal and other positional numeral systems, rational numbers can always be represented in this system by words of finite length, without the aid of such devices as continuation dots or overbars. What decimal represents as \(-0.333333\ldots\), for example, is represented in minimal rational RPF as \((())()()()\). But before we consider these matters in detail, let us move away from my hand-waving description of \( \text{RPF}_{\text{min}} \) and establish the concept upon a firmer foundation.

### 2.2 Interpretations and Representations

Recall that I introduced Equation 1 by writing “Consider the number 520.” My wording was intended as a device to illustrate an important point in the present section. So conflated in our minds are numbers with their representations, and with their decimal representations in particular, that I suspect few readers were bothered by the phrase “the number 520” as being meaningless, or at best an incomplete abbreviation of “the number represented by 520 in the decimal numeral system.” That is to say, most of us seldom stop to distinguish between *numbers* and *number words*. But there is in fact a distinction, and to ignore it can yield untoward consequences. For example, the set of natural numbers contains a unique multiplicative identity element 1 such that \( 1 \cdot n = n = n \cdot 1 \) for all \( n \in \mathbb{N} \). But all of the members of the following set are in the decimal system, and all evaluate to 1: \( \{1, 01, 001, 0001, \ldots\} \). There are then infinitely many “decimal numbers” (decimal number words) that can be considered the identity element for multiplication. Thus we might be tempted to conclude that the set of natural numbers contains infinitely many multiplicative identity elements, despite the existence of simple proofs to the contrary.
It is especially important that we maintain the distinction in this paper, which is intimately concerned with numbers and different ways of representing them. A sequence of symbols is one thing; what that sequence means is quite another. For example, 11 can be understood to mean $3_{10}$ in binary, $17_{10}$ in hexadecimal, or $11_{10}$ in decimal.

I find “meaning” a difficult concept to state with precision, so instead I offer a definition of the word “interpretation.”

**Definition 2.1.** Let $L$ be a formal language, and let $S$ be a set. If there exists some surjective function $f : L \to S$, then the triplet $(L, S, f)$ is an interpretation of $L$ as $S$, specifically the interpretation of $L$ as $S$ according to $f$, and we may say any of the following:

- $L$ interpreted according to $f$ is $S$ (equivalently: $L$ is the underlying language in the interpretation $(L, S, f)$).
- $S$ is $L$ interpreted according to $f$ (equivalently: $S$ is the target set in the interpretation $(L, S, f)$).
- $f$ interprets $L$ as $S$ (equivalently: $f$ is the evaluation function in the interpretation $(L, S, f)$).

As an example, let $B$ denote the set of nonempty strings over the alphabet $\{0, 1\}$, and let $f : B \to \mathbb{N}$, where $f(b)$ is the nonnegative integer corresponding to $b$ such that the latter is regarded as a word in unsigned binary. Then $(B, \mathbb{N}, f)$ is an interpretation of $B$ as the set of natural numbers, specifically the interpretation of $B$ as the set of natural numbers according to $f$. Now consider $g : B \to \mathbb{Z}$, where $g(b)$ is the integer corresponding to $b$ such that the latter is regarded as a word in 2s-complement binary, with the qualification that nonnegative integers always correspond to words containing the prefix 0. Then $(B, \mathbb{Z}, g)$ is an interpretation of $B$ as the set of integers, specifically the interpretation of $B$ as the set of integers according to $g$. Thus we see that the same language may underlie multiple interpretations.

The following definition allows us to speak of interpretations in terms of set members as well.

**Definition 2.2.** Let $(L, S, f)$ be an interpretation. For any $l \in L$, let $s \in S$ such that $s = f(l)$. Then we may say any of the following:

- $s$ is $l$ interpreted according to $f$.
- $l$ interpreted according to $f$ is $s$.
- $f$ interprets $l$ as $s$. 

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Sometimes the terminology of interpretations becomes awkward, resulting in a surfeit of passive participles (“l interpreted as...”; “l interpreted according to...”). We may remedy this to some extent by making use of the following definition.

**Definition 2.3.** Let \((L, S, f)\) be an interpretation, let \(l \in L\), and let \(s\) be a member of \(S\) satisfying the equation \(s = f(l)\). Then we say \(l\) represents \(s\) in \((L, S, f)\). If we do not wish to mention a particular interpretation (as for example when the interpretation would be clear from the context), we may simply say \(l\) represents \(s\), implying some interpretation exists such that \(l\) represents \(s\) in that interpretation.

In addition to speaking of members of languages representing members of sets, we can speak of languages representing sets.

**Definition 2.4.** If \((L, S, f)\) is an interpretation, then we say \(L\) represents \(S\) in \((L, S, f)\), or, equivalently, \(L\) is a representation of \(S\) in \((L, S, f)\). In cases where we do not wish to mention a specific interpretation, we may simply say \(L\) represents \(S\), or, equivalently, \(L\) is a representation of \(S\).

Definition 2.1 only requires that the function \(f\) be surjective. I introduce special terminology for the case where \(f\) is injective as well.

**Definition 2.5.** Let \((L, S, f)\) be an interpretation such that \(f\) is a bijection. Then we may say any of the following:

- \((L, S, f)\) is minimal.
- \(L\) is minimal with respect to \((L, S, f)\).
- The representation of \(S\) in \((L, S, f)\) is minimal.

If the interpretation is minimal, there is exactly one member of \(L\) representing any given \(s \in S\); otherwise we cannot exclude the possibility that \(s\) may have multiple representations. Regardless of whether the interpretation is minimal, the surjectivity of \(f\) ensures that every member of \(S\) has at least one representation in \(L\).

The following definition clarifies what I understand by the word *system* when I speak of RPF systems.

**Definition 2.6.** Let \(I\) be an interpretation such that the target set in \(I\) is numerical. Then we say \(I\) is a numeral system. We may simply say “\(I\) is a system,” if doing so does not incur ambiguity.
2.2.1 Standard and Other Prime-permuted RPF Interpretations

RPF systems are less arbitrary than positional numeral systems, not requiring the selection of a special number to serve as the radix. Yet some particular permutation of the sequence of prime numbers must be chosen for the interpretation of RPF words as numbers and the representation of numbers by RPF words. Returning momentarily to the concept of minimal parentheses padded prime factorizations (Section 2.1.1), MPPPFs involve powers of primes appearing in the same order as those primes occur in the sequence \((2, 3, 5, \ldots, p_k)\), where \(p_k\) is the greatest prime factor of the number of which the MPPPF is taken. But the descending sequence \((p_k, \ldots, 5, 3, 2)\) could be used to yield a “reverse MPPPF,” so that RPF words produced using reverse MPPPFs would be mirror images of their equivalents as produced using MPPPFs—I mean “mirror images” literally, in the sense that if we wrote down a reverse-MPPPF-derived word on a piece of paper and viewed it in a mirror, we would see in the reflection an image identical to its unreflected MPPPF-derived counterpart. We can therefore use either left-ascending or right-ascending sequences of primes upon which to base representations. But we need not stop there; we could use the shortest-length sequence \(S_{\text{swap}}\) in \((p_2, p_1, \ldots, p_{2j}, p_{2j-1}, \ldots)\) such that \(p_2\) was the first term in \(S_{\text{swap}}\) and all the primes from the least to the greatest prime factor of the argument of MPPPF were in \(S_{\text{swap}}\). In fact, any permutation of the sequence of prime numbers would suffice to determine the ordering of the prime powers in the definition of a proposed MPPPF. Nevertheless, in order to avoid a profusion of symbols designating the choice of the underlying prime permutation, and to have common ground for discussing RPF systems, it would be well to consider one sequence as the “standard,” with other permutations only being talked about when their existence was relevant to the discussion.

I select the identity permutation \(P = (2, 3, 5, 7, 11 \ldots)\) of prime numbers, where the terms appear in the same order as they occur in the sequence of natural numbers, to be the standard permutation. Indeed, we can regard \(P\) as not being a permutation at all, but the original sequence from which other prime sequences are derived by scrambling the terms in \(P\). Because values of successive terms in \(P\) increase as the terms are written in customary order from left to right, a standard RPF system can also be called a right-ascending or rightwise system. This is why I will often refer to a standard RPF system using the subscript \(r\), as in RPF\(_{N_{\text{min}}r}\) and RPF\(_{Q_{\text{min}}r}\).

**Definition 2.7.** The standard permutation, also called the rightwise or right-ascending permutation, is the sequence \(P\) of prime numbers \((2, 3, 5, 7, 11, \ldots)\).

**Remark.** Here we have an illustration of how the intimate relationship be-
tween prime factorizations and recursive prime factorizations results in parallels between the two. The fundamental theorem of arithmetic states that a number’s prime factorization is unique except for the order of the factors, so that the prime factorization of 520 could be written variously as $2^3 \times 5^1 \times 13^1$, $13^1 \times 5^1 \times 2^3$, etc. But in practice we usually write prime factorizations with the primes appearing in ascending order.

On occasion I will designate an RPF interpretation based upon a particular but arbitrary permutation of the prime number sequence (with the identity permutation being one of the possibilities); in such cases I will use lowercase Greek letters in the notation, as with $\text{RPF}_{N_r\text{min}}$.

In all of this, we must take care to remember that when we speak of “permutations,” we are not referring to the RPF systems themselves but rather to the sequences of prime numbers underlying them. I will therefore not refer to an RPF interpretation or its language as being a permutation, but rather as being prime-permuted, or, if I am referring to a specific prime permutation $\sigma$, as being $\sigma$-permuted.

### 2.3 $\gamma_{N_r}$ and $\text{RPF}_{N_{r\text{min}}}$

*Remark.* This section is confined to a discussion of mathematical objects relevant to those RPF systems arising from the standard permutation. See Section 2.7 for generalizations of the same objects for all prime permutations.

We can think of an RPF representation of a number as being a spelling of that number, with a spelling function mapping from the set of numbers onto the set of words in the language underlying the corresponding RPF interpretation. If the interpretation is minimal, we can then use the resulting possible spellings to define the language as the image of the spelling function.

We will employ a modification of this approach to define the standard minimal RPF natural interpretation $\text{RPF}_{N_{r\text{min}}}$, as follows. We will define a function $\gamma'_{N_r}$ (from $\gamma$ in ορθογραφία, orthographia, Greek for “word”) mapping a natural number $n$ to a unique string of parentheses, and then we will define the language underlying $\text{RPF}_{N_{r\text{min}}}$ as the set of all possible strings produced by the function. Since $\gamma'_{N_r}$ takes a natural number input and outputs an $\text{RPF}_{N_{r\text{min}}}$ spelling of the number, we could choose to regard $\gamma'_{N_r}$ as our spelling function; however, because we wish the spelling function to have an inverse, we will instead define a spelling function $\gamma_{N_r}$ identical to $\gamma'_{N_r}$ except with its codomain restricted to $\text{RPF}_{N_{r\text{min}}}$. Thus $\gamma_{N_r}$ will be a bijection, enabling us to speak of its inverse.

First I introduce two notations for convenience in concatenating strings; these will find extensive use throughout the rest of this paper.
Definition 2.8. The symbol $\triangleright$ is the string concatenation operator; $a \triangleright b$ denotes the concatenation of strings $a$ and $b$. The concatenation of $a$ and $b$ may also be written in customary fashion as $ab$, provided that doing so incurs no ambiguity.

Definition 2.9. Let $j, k \in \mathbb{N}_+$. Then we shall understand $\bigoplus_{i=j}^{k} s_i$ to mean the string concatenation $s_j \ldots s_k$ if $j \leq k$; otherwise the concatenation is the null string $\epsilon$.

Without further ado, let us define the nonsurjective precursor to our spelling function.

Definition 2.10. Let $\Sigma^*$ be the Kleene closure of the set $\{(, )\}$. Then the standard nonsurjective RPF natural transcription function, denoted by $\gamma'_{N_r}$, is given by $\gamma'_{N_r} : \mathbb{N} \to \Sigma^*$, where

- For $n = 0$, $\gamma'_{N_r}(n)$ is the empty string $\epsilon$.
- For $n = 1$, $\gamma'_{N_r}(n)$ is the string $()$.
- For $n > 1$, let $p_m$ be the greatest prime factor of $n$, and let $a = (a_1, \ldots, a_m)$ be the integer sequence satisfying the equation

$$n = \prod_{i=1}^{m} p_i^{a_i}.$$ 

Then

$$\gamma'_{N_r}(n) = \bigoplus_{i=1}^{m} (\cdot(\cdot \triangleright \gamma'_{N_r}(a_i) \triangleright \cdot)' ).$$

Now we can define our bijective spelling function by specifying its graph.

Definition 2.11. The standard RPF natural spelling function, denoted by $\gamma_{N_r}$, is given by

$$\gamma_{N_r} = \{(n, w) \in \mathbb{N} \times \gamma'_{N_r}(\mathbb{N}) \mid w = \gamma'_{N_r}(n)\},$$

where $\gamma'_{N_r}(\mathbb{N})$ is the image of $\gamma'_{N_r}$.

Remark. Definition 2.11 gives us our spelling function, but relies upon Definition 2.10 for computation of its values.
At times it will be convenient to use a notation for $\gamma_{N_r}(n)$ that does not include the parentheses inherent in function notation.

**Definition 2.12.** Let $k$ be a natural number. The expression $\gamma_{N_r}^k$ denotes the $k$th term in the sequence $(\gamma_{N_r}(n))_{n \in \mathbb{N}}$.

**Example 2.1.** Let us find the spelling corresponding to decimal 2646.

First we note that

$$2646 = 2^1 \cdot 3^3 \cdot 5^0 \cdot 7^2$$

$$= p_1^1 \cdot p_3^3 \cdot p_5^0 \cdot p_7^2.$$ 

Thus the standard RPF natural spelling of 2646 is

$$\gamma_{N_r}^{2646} = (\gamma_{N_r}^1)(\gamma_{N_r}^3)(\gamma_{N_r}^0)(\gamma_{N_r}^2)$$

$$= ((())((())))(())$$

(5)

Before we define the standard minimal RPF natural interpretation, let us give a name to its underlying language.

**Definition 2.13.** The *standard minimal RPF language*, denoted by $\mathcal{D}_{r_{\text{min}}}$, is the codomain of $\gamma_{N_r}$.

**Remark.** The presence of the symbol $\mathcal{D}$ in the notation is not intended to signify that $\mathcal{D}_{r_{\text{min}}}$ is a Dyck language; it is rather intended to suggest a relationship between the standard minimal RPF language and the Dyck language. This relationship is the subject of Section 2.4. Also, notice that I neither included $\mathbb{N}$ in $\mathcal{D}_{r_{\text{min}}}$ nor used the word *natural* in the appellation *standard minimal RPF language*. That is because the language has a nonnumerical alternative definition, as we will see in Section 2.8.

At long last we arrive at the definition of $\text{RPF}_{N_{r_{\text{min}}}}$.

**Definition 2.14.** The *standard minimal RPF natural interpretation*, denoted by $\text{RPF}_{N_{r_{\text{min}}}}$, is the interpretation $(\mathcal{D}_{r_{\text{min}}}, \mathbb{N}, \gamma_{N_r}^{-1})$.

### 2.4 The Language $\mathcal{D}$ and the Dyck Natural Numbers

Table 1 on the following page shows the standard minimal RPF spellings of the first twenty natural numbers, suggesting a resemblance between words in $\mathcal{D}_{r_{\text{min}}}$ and those in the Dyck language $\mathcal{D}$; indeed, $\mathcal{D}_{r_{\text{min}}}$ is a proper subset of $\mathcal{D}$. In light of this fact, and in light of the relevance of the Dyck language to all RPF systems, I provide here a brief description of $\mathcal{D}$. 

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Table 1: Standard RPF spellings of the first twenty natural numbers.

| Decimal | RPF_{N_{r_{\text{min}}}} | Decimal | RPF_{N_{r_{\text{min}}}} |
|---------|---------------------------|---------|---------------------------|
| 0       | $\varepsilon$            | 10      | (())(()(()))             |
| 1       | ()                        | 11      | (())(()())               |
| 2       | (())                      | 12      | (())(()())               |
| 3       | (())()                    | 13      | (())(()())()             |
| 4       | (())()()                  | 14      | (())()()()()             |
| 5       | (())()()                  | 15      | (())()()()()             |
| 6       | (())()()()               | 16      | (())()()()()             |
| 7       | (())()()()()             | 17      | (())()()()()()           |
| 8       | (())()()()()()           | 18      | (())()()()()()           |
| 9       | (())()()()()()()        | 19      | (())()()()()()()         |

Remark. We should avoid drawing too many conclusions from merely looking at the RPF spellings in Table 1. For example, we might hypothesize that the RPF spelling of every natural number greater than 0 is longer than its decimal counterpart. Yet that hypothesis is easily disproven by noting that

$$\gamma_{N_{r_{\text{min}}}}(443426488243037769948249630619149892803) = (())(()(()))(()(())).$$

Recall Section 2.1.2, where we found an RPF representation of the natural number represented by decimal 520. In doing so, we wrote 520 as the expression

$$p_1^{p_0}p_2^{p_0}p_3^{p_0}p_4^{p_0}p_5^{p_0}p_6^{p_0},$$

and then deleted everything except the parentheses to yield

$$(())(()(()))(()(())).$$

The result was a word in the Dyck language $D$, the set of all strings consisting of zero or more well-balanced parenthesis pairs. Informally, we can think of the property of being well-balanced as what distinguishes a syntactically correct use of grouping parentheses in an algebraic expression from a syntactically incorrect one. For example, the expression $(p_1^{p_0}p_2^{p_0})$ is nonsensical, the number of closing parentheses not equaling the number of opening parentheses; thus the string $(())$ is not well-balanced and is not a word in the Dyck language. The expression $(p_1^{p_0})(p_2^{p_0})$ is also nonsensical, even though the number of opening and closing parentheses is equal, because the closing parenthesis is not preceded by a matching opening parenthesis. Thus the string $(),$ not being well-balanced, is not a word in $D$. 

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Remark. The symbols in the language do not have to be parentheses; various authors use parentheses, square brackets, 1s and 0s, etc. Any binary symbol set will suffice.

**Definition 2.15.** Let $\Sigma^*$ be the Kleene closure of the alphabet $\{(,)\}$. The **Dyck language**, denoted by $\mathcal{D}$, is the set of all $w \in \Sigma^*$ such that the number of right parentheses in any prefix $w'$ of $w$ does not exceed the number of left parentheses in $w'$, and the number of left parentheses in $w$ is equal to the number of right parentheses in $w$.

The Dyck language is context-free, and can be generated by the following grammar:

$$S \to (S)S \mid \varepsilon.$$  

$\mathcal{D}$ has significance beyond that of merely being just another context-free language, rather having a deep relationship with all context-free languages. According to the Chomsky-Schützenberger representation theorem, every context-free language $L$ is a homomorphic image of the intersection of some regular language $R$ and $\mathcal{D}$. From a practical standpoint, this can be useful if we wish to prove $L$ is context-free but have not been able to isolate a set of rules defining a context-free grammar for $L$. (Indeed, I will soon employ the theorem to prove that $D_{r_{\text{min}}}$ is context-free.)

Now that I have provided a bare-bones introduction to $\mathcal{D}$, I return to my discussion concerning the relationship between that language and $D_{r_{\text{min}}}$.

The function $\gamma_{N_r}$ is recursive. In order that I might discuss sequences of recursive function evaluations, I now develop special notation and vocabulary. Recall Equation Set 5, which showed the steps in the evaluation of $\gamma_{N_r}(2646)$:

$$\begin{align*}
\gamma_{N_{r_{2646}}} &= (\gamma_{N_{r_1}})(\gamma_{N_{r_3}})(\gamma_{N_{r_0}})(\gamma_{N_{r_2}}) \\
&= ((())((\gamma_{N_{r_0}})(\gamma_{N_{r_1}})))((()\gamma_{N_{r_1}})) \\
&= ((())((())(())))).
\end{align*}$$

We can use these equations to construct a tree where every node represents an invocation of the spelling function to evaluate a number, with children of the node appearing in the same order as their invocations occur in the node’s evaluation:
And so we see that the invocation of $\gamma_{2646}$ leads to $\gamma_{2}$, being invoked to spell 1, followed by $\gamma_{3}$ being invoked to spell 3, followed by $\gamma_{0}$ being invoked to spell 0, followed by $\gamma_{1}$ being invoked to spell 2, with the spellings of 3 and 2 leading recursively to further invocations.

**Definition 2.16.** Let $S$ be a set, let $f$ be a unary function such that the domain of $f$ is $S$, and let $s \in S$. Then the *recursion tree for the evaluation of $f(s)$*, also called the *recursion tree of $f(s)$*, is a tree $T$ in which every node represents an invocation of $f$ to evaluate a member of $S$ such that the invocation arises from the evaluation of $f(s)$, with the root of $T$ representing the invocation $f(s)$, the children of a node having the same order as they occur in the invocation, and all invocations of $f$ arising from the evaluation of $f(s)$ appearing in $T$. A recursion tree consisting only of a root node is said to be *trivial*.

**Definition 2.17.** Let $f$ be a unary function such that the domain of $f$ is some set $S$, and let $T$ be the recursion tree for $f(s)$, where $s \in S$. Suppose there exist distinct nodes $A$ and $B$ in $T$ such that $A$ represents the invocation of $f$ to evaluate some $a \in S$ and $B$ represents the invocation of $f$ to evaluate some $b \in S$. If $B$ is a descendant of $A$, then we say $f(a)$ *entails* $f(b)$. If $B$ is a child of $A$, we may also say $f(a)$ *directly entails* $f(b)$.

**Example 2.2.** Referring to the recursion tree for the natural RPF spelling of 2646, we see that the spelling of 2646 entails the spelling of 0, since the spelling of 0 is a descendant of the spelling of 2646. Moreover, the spelling of 2646 directly entails the spelling of 0, since one of the children of the spelling of 2646 is the spelling of 0. However, the spelling of 3 does not entail the spelling of 2, or vice versa; although both are nodes in the tree, neither is a descendant of the other.

**Definition 2.18.** Let $f$ be a unary function such that the domain of $f$ is some set $S$, and let $T$ be a recursion tree such that $(f(s_1), \ldots, f(s_k))$ is the sequence of nodes in a path from the root of $T$ to one of its leaves. Then we call $(f(s_1), \ldots, f(s_k))$ a *recursion chain* in the evaluation of $f(s_1)$.

**Example 2.3.** Referring to the recursion tree shown earlier for the natural RPF spelling of 2646, we see that $\gamma_{2646}$ directly entails $\gamma_{3}$, which in turn directly entails $\gamma_{0}$. Furthermore, the tree is rooted at $\gamma_{2646}$, and $\gamma_{0}$ is a leaf. Therefore $(\gamma_{2646}, \gamma_{3}, \gamma_{0})$ is a recursion chain in the evaluation of $\gamma_{2646}$. Since $\gamma_{2646} = ((())((()))))((()))$, $\gamma_{3} = (())$, and $\gamma_{0} = \epsilon$, we can also write the recursion chain as $((())((()))))((()))', (())', \epsilon)$.

Now we have sufficient notation and vocabulary to prove that the standard minimal RPF language is a proper subset of the Dyck language.
Theorem 2.1. The language $D_{r_{min}}$ is a proper subset of the Dyck language $D$.

Proof. In order for $D_{r_{min}}$ to be a proper subset of $D$, every member of the former must also be a member of the latter, and at least one member of the latter must not be a member of the former. Let us address these criteria separately:

- Definition 2.13 identifies $D_{r_{min}}$ as the codomain of $\gamma'_{N_r}$, which in turn is the image of the standard nonsurjective RPF natural transcription function $\gamma'_{N'}$ according to Definition 2.10. That latter definition contains three cases for the spelling of a natural number $n$. The first case gives the spelling of 0 as $\varepsilon$, while the second case gives the spelling of 1 as $()$; both of these are members of $D$. For every other natural number $n$, the recursion tree of the spelling of $n$ must have as its leaves members of the set $\{\gamma_{N_{0}}, \gamma_{N_{1}}\}$, because only the spellings of 0 and 1 do not entail further spellings. The parent of a leaf in the recursion tree of the spelling of $n$ is a spelling where the leaf and its siblings are each surrounded by single pairs of matched parentheses, the parenthesized expressions then being concatenated in the same order as the siblings appear in the tree. Enclosing a Dyck word in matched single parentheses results in a Dyck word, and the concatenation of Dyck words also results in a Dyck word; thus the parents of leaves are spellings of Dyck words. The same process of surrounding children by parentheses and concatenating the resulting expressions to spell the parent applies at every level of the tree, so the spelling of $n$ is a Dyck word. Thus the spellings of all members of $N$ are Dyck words, allowing us to conclude that $D_{r_{min}} \subseteq D$.

- The string $s = '((())())'$ will suffice as a counterexample to demonstrate that at least one member of $D$ is not a member of $D_{r_{min}}$. The string satisfies the definition of a Dyck word: it contains no symbols other than left and right parentheses, with the number of left parentheses equal to the number of right parentheses and with every prefix of $s$ containing at least as many left parentheses as right parentheses. However, $s$ is not a member of $D_{r_{min}}$, as it is not in the codomain of $\gamma'_{N_r}$. To see why the spelling function is incapable of producing $s$, we observe that $s$ contains the suffix $'()'$ and then try to find a number $n$ such that its spelling contains that suffix. Let us treat the possibilities individually as follows. Certainly the spelling of natural number 0 does not contain $'()'$ as a suffix, since $\gamma_{N_r}(0) = \varepsilon$. While it is true that $\gamma_{N_r}(1) = '()'$ is a word containing suffix $'()'$, the word does not equal $s$. The final
possibility is $\gamma_{N_r}(n)$ for some $n \geq 2$. We note that regardless of which such $n$ we choose, its spelling only includes exponents $a_i$ from $a_1$ up to and including $a_m$, where $p_m$ is the greatest prime factor of $n$. However, spelling function $\gamma_{N_r}$ only outputs an empty parenthesis pair if it encounters the number 1, either as the original number being spelled or in the course of spelling a prime number raised to the 0th power. But $m$ cannot be 0, since any prime number $p$ raised to the 0th power is equal to one and is therefore not present in the prime factorization of $n$, implying $p$ cannot be the greatest prime factor of $n$. This in turn implies that the spelling of $n$ cannot contain the suffix '$(\cdot)'$, and we can thus be certain that $s \in \mathcal{D}$ but $s \notin \mathcal{D}_{r_{\text{min}}}$. We therefore conclude that $\mathcal{D}_{r_{\text{min}}}$ is a proper subset of $\mathcal{D}$.

We will see (Theorem 2.3) that every Dyck word represents a natural number. But the fact that $\mathcal{D}_{r_{\text{min}}}$ contains the spelling of every $n \in \mathbb{N}$, together with the fact that $\mathcal{D}_{r_{\text{min}}} \subset \mathcal{D}$, implies there must exist natural numbers possessing multiple representations in $\mathcal{D}$ (indeed, 0 is the only natural number having a unique representation in $\mathcal{D}$, every other natural number having infinitely many). Therefore we cannot view the number-words in $\mathcal{D}$ as being equivalent to the numbers they represent; to do so would result in absurdities such as the existence of more than one value of the product of two integers.

On the other hand, $\mathcal{D}_{r_{\text{min}}}$, enjoying a 1:1 correspondence with $\mathbb{N}$, may be treated as if its members are the natural numbers themselves. For instance, the equation

$$6 \cdot 2^3 + 2 = 50$$

is equivalent to

$$(()(()) \cdot ((())^{0(()())} + (()) = (())(()(())))$$

and the existence of a unique identity element for natural multiplication can be stated by asserting that for all $w \in \mathcal{D}_{r_{\text{min}}}$,

$$(()) \cdot w = w = w \cdot ()$$

In short, we may regard our $\mathcal{D}_{r_{\text{min}}}$ number-words as numbers. And so I give $\mathcal{D}_{r_{\text{min}}}$ a simpler name:

**Definition 2.19.** The set of Dyck natural numbers is the language $\mathcal{D}_{r_{\text{min}}}$.

**Remark.** There existing infinitely many $\sigma$-permuted minimal natural RPF languages, I could choose any one of them to be the Dyck naturals. But doing so would be no more advantageous than, say, reversing the order of writing digits in decimal numbers so that $102_{10}$ would instead be written as $201_{01}$. Thus I select the language underlying the standard interpretation.
2.5 The Standard RPF Natural Evaluation Functions $\alpha_{N_r}$ and $\alpha_{N_{r_{\text{min}}}}$

Definition 2.11 gave us the standard RPF natural spelling function $\gamma_{N_r}$, which is a bijection and therefore for which there exists an inverse function $\gamma_{N_r}^{-1}$. But even though we are aware that the inverse exists, we do not yet know how to compute its values; we would like to correct this deficiency, especially since $\gamma_{N_r}^{-1}$ is the evaluator in RPF$_{N_{r_{\text{min}}}}$. An evaluation function mapping $D_{r_{\text{min}}}$ onto $\mathbb{N}$ will thus be useful. First, though, I will define a function having a domain equal to $D$ rather than $D_{r_{\text{min}}}$, because I intend the function to eventually serve as the evaluator in the standard general RPF natural interpretation. I will then restrict the function to $D_{r_{\text{min}}}$ to yield the evaluation function we previously referred to as $\gamma_{N_r}^{-1}$.

The definition of the evaluation function involves regarding the input word as a concatenation of chunks and recursively evaluating these. Therefore before we go any further we must define exactly what we mean by a “chunk,” which in turn requires a definition of the “dimensionality” of a Dyck word.

Definition 2.20. The \textit{dimensionality} of a Dyck word $w$ is the number of outermost matching parenthesis pairs in $w$. More formally, the dimensionality of $w$ is given by the function $\text{dim} : D \to \mathbb{N}$ such that

- For $w = \varepsilon$, $\text{dim}(w) = 0$.
- For $w \neq \varepsilon$, $\text{dim}(w) = k$, where $k$ satisfies

$$w = \bigoplus_{i=1}^{k} (') \cdot d_i \cdot (')'$$

for some sequence of Dyck words $(d_1, \ldots, d_k)$.

\textit{Remark.} Dyck words $d_1, \ldots, d_k$ are certain to exist, because a criterion for the well-formedness of a Dyck word is that if the word can be expressed as a string $s$ enclosed within a matching pair of parentheses, then $s$ is also a Dyck word. Furthermore, the sequence $d$ is unique, since there is only one set of outermost matching parenthesis pairs in $w$, and $d_i$ is the Dyck word contained within the $i$th outermost matching parenthesis pair.

Definition 2.21. A \textit{chunk} is a Dyck word of dimensionality 1.

Lemma 2.2. If $w$ is a chunk, $w = ' (\cdot d \cdot ')'$ for some Dyck word $d$, and $|d| = |w| - 2$. 

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Proof. Being a chunk, \( w \) is a Dyck word of dimensionality 1. Therefore

\[
\begin{align*}
   w &= \bigoplus_{i=1}^{1} (\( ' ( ' \wedge d_i \wedge ' )' \)) \\
   &= ' ( ' \wedge d \wedge ' )' \\
   &= ' ( ' \wedge d \wedge ' )', \text{ where } d = d_1.
\end{align*}
\]

Since \( w = ' ( ' \wedge d \wedge ' )' \), \( d \) obviously has two fewer parentheses than \( w \); thus \( |d| = |w| - 2 \). \( \square \)

**Definition 2.22.** The content of a chunk \( w \) is the Dyck word \( d \), where \( d \) is given by:

- For \( w = () \), \( d \) is the Dyck word \( \varepsilon \).
- For \( w \neq () \), \( d \) is the Dyck word satisfying the equation \( ' ( ' \wedge d \wedge ' )' = w \).

**Definition 2.23.** The standard RPF natural evaluation of a Dyck word \( w \) is the function \( \alpha_{\text{N}} : D \rightarrow \mathbb{N} \) such that

- For \( w = \varepsilon \), \( \alpha_{\text{N}}(w) = 0 \).
- For \( w \neq \varepsilon \),

\[
\alpha_{\text{N}}(w) = \prod_{i=1}^{\dim(w)} p_i^{\alpha_{\text{N}}(d_i)}
\]

where \( d_i \) is the content of the \( i \)th chunk in \( w \).

**Remark.** The function is named \( \alpha \) for \( \acute{\alpha} \rho \iota \mu \acute{\omicron} \zeta \) (arithmos), Greek for "number."

**Remark.** A generalization of the standard RPF natural evaluation function to apply to all prime-permuted natural interpretations may be found in Section 2.7.

**Example 2.4.** Let us evaluate \( ()(()())() \):

\[
\begin{align*}
   \alpha_{\text{N}}'(')&(()())'(') = \prod_{i=1}^{2} p_i^{\alpha_{\text{N}}(d_i)} \\
   &= p_1^{\alpha_{\text{N}}(\varepsilon)} p_2^{\alpha_{\text{N}}'(')}(') \\
   &= p_1^{0} p_2^{0} p_1^{0} p_2^{0} \\
   &= p_2^{2} = 3^3 = 27.
\end{align*}
\]
Because it may not be obvious to the reader that the standard natural evaluation function is defined for all \( d \in D \), I offer the following:

**Theorem 2.3.** The domain of the standard RPF natural evaluation function \( \alpha_{N_r} \) is \( D \).

**Proof.** Let \( S \) be the domain of \( \alpha_{N_r} \). We must show that \( S \subseteq D \) and \( D \subseteq S \).

The first of the two requirements is satisfied by the language of Definition 2.23 itself, which explicitly restricts the domain of \( \alpha_{N_r} \) to \( D \). It therefore remains to be shown that \( D \subseteq S \). Toward that end, let \( d \) be a member of \( D \). Then either \( d = \varepsilon \) or \( d \neq \varepsilon \).

- For \( d = \varepsilon \), \( \alpha_{N_r}(\varepsilon) \) is explicitly and uniquely defined to be 0.

- For \( d \neq \varepsilon \), \( \alpha_{N_r}(d) \) is recursively defined as \( \prod_{i=1}^{\dim(d)} p_i^{\alpha_{N_r}(e_i)} \), where \( e_i \) is the content of the \( i \)th chunk in \( d \). If \( \alpha_{N_r}(e_i) \) exists for all \( i \in \{1, \ldots, \dim(d)\} \), then \( \prod_{i=1}^{\dim(d)} p_i^{\alpha_{N_r}(e_i)} \) exists as well, so that \( \alpha_{N_r}(d) \) evaluates to a number. Suppose that one of the terms \( p_i^{\alpha_{N_r}(e_i)} \) does not exist. There are only two possible reasons for its nonexistence: either \( e_i \) is not a Dyck word, or \( \alpha_{N_r}(e_i) \) involves an endless recursion of invocations and thus does not yield a value. Let us show that neither of these supposed possibilities can be the case:

  - Lemma 2.2 assures us that \( e_i \), being the content of a chunk, must itself be a Dyck word.

  - To see why the evaluation function cannot involve an endless recursion of invocations of itself, consider the evaluation of \( ()(()(())) \) from Example 2.4 on the previous page. The recursion tree for \( \alpha_{N_r}(()(()(()))) \) is:

    \[
    \begin{array}{c}
    \alpha_{N_r}(()(()(())))' \\
    \alpha_{N_r}(\varepsilon) & \alpha_{N_r}'(()(()))' \\
    \alpha_{N_r}(\varepsilon) & \alpha_{N_r}'(())' \\
    \alpha_{N_r}(\varepsilon) & \alpha_{N_r}'(') \\
    \alpha_{N_r}(\varepsilon)
    \end{array}
    \]

    The longest recursion chain arising from the evaluation of \( ()(()(())) \) is (\( \alpha_{N_r}(()(()(()))) \), \( \alpha_{N_r}'(()(()))' \), \( \alpha_{N_r}'(())' \), \( \alpha_{N_r}'(')' \), \( \alpha_{N_r}'(') \), \( \alpha_{N_r}(\varepsilon) \)). Now let \( k \) be the greatest integer such that there is a recursion chain of length \( k \) arising from the evaluation of an arbitrary Dyck word \( d \). Then \( k \) must be finite, because the first term in the chain is
the evaluation of a finite-length word and each subsequent term is the evaluation of the content of a chunk from the evaluation of its predecessor, with the length of the content being 2 less than the length of the chunk from which it came, according to Lemma 2.2. As the longest chain (or chains, if more than one chain is of length \( k \)) must be finite, the evaluation of \( d \) cannot involve an infinite sequence of recursive invocations of \( \alpha_{\mathcal{N}_{r}} \) upon itself.

Since we have demonstrated that \( \alpha_{\mathcal{N}_{r}}(d) \) exists for every \( d \in \mathcal{D} \), we have demonstrated that \( \mathcal{D} \subseteq \mathcal{S} \).

Having shown that \( \mathcal{S} \subseteq \mathcal{D} \) and \( \mathcal{D} \subseteq \mathcal{S} \), we conclude that \( \mathcal{S} = \mathcal{D} \). Therefore the domain of \( \alpha_{\mathcal{N}_{r}} \) is \( \mathcal{D} \).

Note that \( \alpha_{\mathcal{N}_{r}}(w) \) exists for every \( w \in \mathcal{D}_{r_{\text{min}}} \), since \( \mathcal{D}_{r_{\text{min}}} \subseteq \mathcal{D} \). However, the inverse of bijective function \( \gamma_{\mathcal{N}_{r}} \) is not \( \alpha_{\mathcal{N}_{r}} \), since the domain of the latter is a superset of \( \mathcal{D}_{r_{\text{min}}} \). We could opt to simply continue using \( \gamma_{\mathcal{N}_{r}}^{-1} \) to refer to the evaluation function in the standard minimal natural interpretation, using \( \alpha_{\mathcal{N}_{r}} \) to compute its values; nevertheless I prefer a more direct approach:

**Definition 2.24.** The standard minimal RPF natural evaluation function, denoted by \( \alpha_{\mathcal{N}_{r_{\text{min}}}} \), is the restriction of \( \alpha_{\mathcal{N}_{r}} \) to \( \mathcal{D}_{r_{\text{min}}} \).

I will make use of the following two lemmas to prove a theorem establishing that \( \alpha_{\mathcal{N}_{r_{\text{min}}}} = \gamma_{\mathcal{N}_{r}}^{-1} \).

**Lemma 2.4.** Let \( n \) be a natural number greater than 1. Then \( \dim(\gamma_{\mathcal{N}_{r}}(n)) = m \), where \( p_{m} \) is the greatest prime factor of \( n \).

**Proof.** Referring to Definition 2.10, which we use to calculate the value of \( \gamma_{\mathcal{N}_{r}}(n) \), we see that the spelling of \( n \) results in a word containing \( m \) outermost matching parenthesis pairs, where \( p_{m} \) is the greatest prime factor of \( n \).

**Lemma 2.5.** Let \( p_{k} \) be the \( k \)th prime number. Then \( \alpha_{\mathcal{N}_{r_{\text{min}}}}(\gamma_{\mathcal{N}_{r}}(p_{k})) = p_{k} \).

**Proof.** Observe that

\[
\gamma_{\mathcal{N}_{r}}(p_{k}) = \bigoplus_{i=1}^{k} (\textquotesingle {} \cap \gamma_{\mathcal{N}_{r}} (a_{i}) \cap \textquotesingle {}'),
\]

where every \( a_{i} \) is zero except for \( a_{k} = 1 \). We may thus rewrite \( \alpha_{\mathcal{N}_{r_{\text{min}}}}(\gamma_{\mathcal{N}_{r}}(p_{k})) \) as

\[
\alpha_{\mathcal{N}_{r_{\text{min}}}} \left( \bigoplus_{i=1}^{k} (\textquotesingle {} \cap \gamma_{\mathcal{N}_{r}} (a_{i}) \cap \textquotesingle {}') \right),
\]

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which in turn is equal to
\[ \prod_{i=1}^{k} p_i^{a_i}. \]
Because the only nonzero \( a_i \) is \( a_k = 1 \), we conclude that for every prime number \( p_k \),
\[ \alpha_{N_{\min}}(\gamma_{N_r}(p_k)) = p_k. \]
\[ \square \]

With these lemmas in hand, I am ready to present the theorem and its proof.

**Theorem 2.6.** The bijections \( \alpha_{N_{\min}} \) and \( \gamma_{N_r} \) are mutual inverses.

**Proof.** Our proof will be by mathematical induction. Let \( P_k \) be the proposition
\[ \alpha_{N_{\min}}(\gamma_{N_r}(k)) = k \]
for natural number \( k \).

Propositions \( P_0 \) and \( P_1 \) are easily verified by considering Definition 2.10 together with Definition 2.23:

- \( \alpha_{N_{\min}}(\gamma_{N_r}(0)) = \alpha_{N_{\min}}(e) = 0. \)
- \( \alpha_{N_{\min}}(\gamma_{N_r}(1)) = \alpha_{N_{\min}}(\langle' \rangle') = p_1^0 = 1. \)

Now let \( n \) be a natural number such that \( n \geq 2 \) and \( P_l \) is true for all \( l \) less than \( n \).

Choose nonnegative integers \( a_1, \ldots, a_m \) satisfying
\[ \prod_{i=1}^{m} p_i^{a_i} = n, \]
where \( p_m \) is the greatest prime factor of \( n \). Then
\[ \gamma_{N_r}(n) = \bigoplus_{i=1}^{m} \langle' \cap \gamma_{N_r}(a_i) \cap' \rangle'. \]

Applying \( \alpha_{N_{\min}} \) to both sides gives
\[ \alpha_{N_{\min}}(\gamma_{N_r}(n)) = \alpha_{N_{\min}}(\bigoplus_{i=1}^{m} \langle' \cap \gamma_{N_r}(a_i) \cap' \rangle'). \]

Observe that the dimensionality of \( \gamma_{N_r}(n) \) is \( m \), according to Lemma 2.4; thus
\[ \alpha_{N_{\min}}(\gamma_{N_r}(n)) = \prod_{i=1}^{m} p_i^{\alpha_{N_{\min}}(\gamma_{N_r}(a_i))}. \]
Note that either \( n \) is a product of powers of prime numbers all of which are less than or equal to \( n - 1 \), or \( n \) is itself prime. But since we are given that every natural number \( l \) less than \( n \) satisfies \( P_l \), and since we know from Lemma 2.5 that \( \alpha_{\text{min}}(\gamma_{\text{NR}}(p_k)) = p_k \) for any prime number \( p_k \), we can rewrite Equation 7 as
\[
\alpha_{\text{min}}(\gamma_{\text{NR}}(n)) = \prod_{i=1}^{m} p_i^{a_i}.
\] (8)

Considering Equation 6 together with Equation 8, we see that
\[
\alpha_{\text{min}}(\gamma_{\text{NR}}(n)) = n.
\]

Thus \( P_k \) is true for all \( k \in \mathbb{N} \), implying that \( \alpha_{\text{min}} \) is the inverse of \( \gamma_{\text{NR}} \). Furthermore, the two functions are mutual inverses, as the inverse of a bijection’s inverse is the bijection itself.

Theorem 2.6 gives us certainty that \( \alpha_{\text{min}} \) is the evaluation function in the standard minimal RPF natural interpretation.

### 2.6 Prime Factorization of Dyck Natural Numbers

Whether an algorithm exists capable of performing prime factorization in polynomial time on a classical (non-quantum) computer is one of the great unanswered questions of computer science, if the input and output are expressed using a positional numeral system such as binary or decimal. Using Dyck naturals, Algorithm 1 on the following page executes in linear time.

**Example 2.5.** Given \( ((())())()()((()))()((())) \) as input, the algorithm outputs the expression \([(()()()())(())(()())()((()))()((()))]([(()()()())(())(()())()((()))])\). Thus the algorithm tells us that
\[
(()()()()())(())()((())) = (()())(())()()()()()()((())).
\]

We can check the correctness of the equation by replacing Dyck naturals with their decimal equivalents, yielding
\[
83006 = 2^1 \cdot 7^3 \cdot 11^2.
\]

Listing 1 on page 26 is an implementation of the algorithm in Python. Looking at the listing, we can see that there is only one loop, with the number of iterations being equal to the length of the input string; thus the script runs in linear time. But while that is technically true, it may be misleading. For example, numbers represented in unary can be factorized in linear time, but still the running time is slower than with numbers represented in a
Algorithm 1: Finding the prime factorization of a Dyck natural number.

Input: An arbitrary Dyck natural \( w \).

Output: The prime factorization of \( w \) as an expression of Dyck naturals, with \( \land \) as the exponentiation operator, square brackets as grouping symbols, and multiplication of grouped expressions implied by their juxtaposition.

\[
\begin{align*}
\text{base} & \leftarrow \mathsf{(())}; \\
\text{foreach} \ c_k \ \text{in} \ \w & \ \text{do} \\
& \quad \text{if} \ c_k \neq \mathsf{()}' \ \text{then} \\
& \quad \quad \text{print} \ [''; \quad /* \text{print without newline} */ \\
& \quad \quad \quad \text{print} \ \text{base}; \\
& \quad \quad \quad \text{print} \ '\land'; \\
& \quad \quad \quad \text{print} \ '>'; \\
& \quad \quad \quad \text{print} \ \text{content of} \ c_k; \\
& \quad \quad \quad \text{print} \ ']'; \\
& \quad \text{end} \\
\text{end} \\
\text{base} & \leftarrow \mathsf{()}' \land \text{base};
\end{align*}
\]

positional numeral system, as the length of input for a unary representation of a number \( n \) is equal to the number itself, whereas the length of a base-\( k \) word representing \( n \) is proportional to \( \log_k(n) \). On the other hand, we might speed things up for the RPF-based algorithm by abbreviating words, e.g., by replacing sufficiently long strings of empty parenthesis pairs with hexadecimal words giving the lengths of unbroken sequences of such pairs; \( \mathsf{(())()()()()()()()()()()()} \) thus might be shortened to \( \mathsf{((())9}) \). Under such a scheme the following two factorizations would be equivalent:

\[
\begin{align*}
\mathsf{(())()()()()()()()()()()()()} & = (()^0 \cdot \mathsf{()()()()()()()()()()()()}^{0(0)}) \\
\mathsf{((())9}) & = (()^0 \cdot A((())^{0(0)}).
\end{align*}
\]

A prime factorization algorithm might use the compressed form for both input and output without ever having to reconstitute the corresponding Dyck numbers, since empty parenthesis pairs have no contents requiring processing.

Remark. The reader might object that finding the prime factorization of a Dyck natural number in linear time is a contrived problem, since the work involved with prime factorization has already been performed “up front” in order to get the RPF representation in the first place. But that is not necessarily true; in Theorem 2.7 on page 29 we will see that \( D_{r_{\text{min}}} \) has an
Listing 1: A Python implementation of Algorithm 1.

```python
w = input('Enter the Dyck word to be factorized: ')
base = '()'
parenthesisDifference = 0
leftIndex = 0
ignoreThisChunk = True
for i in range(0, len(w):
    if w[i] == '(': 
        parenthesisDifference = parenthesisDifference + 1
        if parenthesisDifference > 1:
            ignoreThisChunk = False
    elif w[i] == ')': 
        parenthesisDifference = parenthesisDifference - 1
        if parenthesisDifference == 0:
            if not ignoreThisChunk:
                print('[{}^{}]'.format(base, w[leftIndex+1 : i]), end='')
                ignoreThisChunk = True
            leftIndex = i + 1
            base = '()' + base
```

alternative nonnumerical definition, so that we can easily determine whether a given Dyck word is a Dyck natural, even though we may have no idea what number the Dyck natural represents.

### 2.7 Generalization to All Minimal Natural Interpretations

So far I have confined my treatment of minimal interpretations of \( \mathbb{N} \) to the “standard” or “rightwise” one; now I will provide a generalization to include all minimal prime-permuted natural interpretations. Using this generalization, we will be able to convert objects in one permuted interpretation into corresponding objects in another.

We will define a function \( \gamma'_{\mathbb{N}_0} \) mapping a natural number \( n \) to a unique string of parentheses, and then we will define the language underlying \( \text{RPF}_{\mathbb{N}^*_{\text{min}}} \) as the set of all possible strings produced by the function. Since \( \gamma'_{\mathbb{N}_0} \) takes a natural number input and outputs an \( \text{RPF}_{\mathbb{N}^*_{\text{min}}} \) spelling of the number, we could choose to regard \( \gamma'_{\mathbb{N}_0} \) as our spelling function; however, because
we wish the spelling function to have an inverse, we will instead define a spelling function \( \gamma'_{N_{\sigma}} \) identical to \( \gamma_{N_{\sigma}} \) except with its codomain restricted to \( \text{RPF}_{N_{\sigma \min}} \). Thus \( \gamma_{N_{\sigma}} \) will be a bijection, enabling us to speak of its inverse.

**Definition 2.25.** Let \( \Sigma^* \) be the Kleene closure of the set \( \{ (, ) \} \), and let \( \sigma \) be a permutation of \( P \), where \( P \) is the sequence of prime numbers \( (2, 3, 5, 7, \ldots) \). Then the \( \sigma \)-permuted nonsurjective RPF natural transcription function, denoted by \( \gamma'_{N_{\sigma}} \), is given by \( \gamma'_{N_{\sigma}} : \mathbb{N} \rightarrow \Sigma^* \), such that

- For \( n = 0 \), \( \gamma'_{N_{\sigma}}(n) \) is the empty string \( \epsilon \).
- For \( n = 1 \), \( \gamma'_{N_{\sigma}}(n) \) is the string \( () \).
- For \( n > 1 \), let \( (s_k)_{k=1}^{\infty} \) be the sequence \( (\sigma(p_1), \sigma(p_2), \sigma(p_3), \ldots) \), and let \( m \) be the smallest number for which an integer sequence \( (a_j)_{j=1}^{m} \) exists satisfying the equation \( n = \prod_{i=1}^{m} s_i^{a_i} \). Then
  \[
  \gamma'_{N_{\sigma}}(n) = \bigoplus_{i=1}^{m} (\text{'} \gamma'_{N_{\sigma}}(a_i) \text{'} \text{'}). 
  \]

At this point we can define our bijective spelling function by specifying its graph.

**Definition 2.26.** The \( \sigma \)-permuted RPF natural spelling function, denoted by \( \gamma_{N_{\sigma}} \), is given by

\[
\gamma_{N_{\sigma}} = \{(n, w) \in \mathbb{N} \times \gamma'_{N_{\sigma}}(\mathbb{N}) \mid w = \gamma'_{N_{\sigma}}(n)\},
\]

where \( \gamma'_{N_{\sigma}}(\mathbb{N}) \) is the image of \( \gamma'_{N_{\sigma}} \).

At times it will be convenient to use a notation for \( \gamma_{N_{\sigma}}(n) \) that does not include the parentheses inherent in function notation.

**Definition 2.27.** Let \( k \) be a natural number. The expression \( \gamma_{N_{\sigma k}} \) denotes the \( k \)th term in the sequence \( (\gamma_{N_{\sigma}}(n))_{n \in \mathbb{N}} \).

Before we define the \( \sigma \)-permuted minimal RPF natural interpretation \( \text{RPF}_{N_{\sigma \min}} \), we will assign a name to the language underlying the interpretation.

**Definition 2.28.** The \( \sigma \)-permuted minimal RPF language, denoted by \( D_{\sigma \min} \), is the codomain of \( \gamma_{N_{\sigma}} \).
Now we are ready to state the definition of RPF$_{N_{\min}}$.

**Definition 2.29.** The $\sigma$-permuted minimal RPF natural interpretation, denoted by RPF$_{N_{\sigma_{\min}}}$, is the interpretation $(D_{\sigma_{\min}}, N, \gamma_{N_{\sigma}}^{-1})$.

An evaluation function mapping $D_{\sigma_{\min}}$ into $\mathbb{N}$ will be useful. However, I will define the function so that its domain is $D$ rather than $D_{\sigma_{\min}}$, as I intend for the definition to apply equally well to the general natural RPF language as to its subset $D_{\sigma_{\min}}$.

**Definition 2.30.** Let $\sigma$ be a permutation of the sequence $(2, 3, 5, 7, \ldots)$ of prime numbers. Then the $\sigma$-permuted RPF natural evaluation of a Dyck word $w$ is the function $\alpha_{N_{\sigma}} : D \rightarrow \mathbb{N}$ such that:

- For $w = \epsilon$, $\alpha_{N_{\sigma}}(w) = 0$.
- For $w \neq \epsilon$, let $(s_k)_{k=1}^{\infty}$ be the sequence $(\sigma(p_1), \sigma(p_2), \sigma(p_3), \ldots)$. Then

$$\alpha_{N_{\sigma}}(w) = \prod_{i=1}^{\dim(w)} \sigma(p_i)^{\alpha_{N_{\sigma}}(d_i)},$$

where $d_i$ is the content of the $i$th chunk in $w$.

Note that $\alpha_{N_{\sigma}}(w)$ exists for every $w \in D_{\sigma_{\min}}$, since $D_{\sigma_{\min}} \subseteq D$. However, the inverse of bijective function $\gamma_{N_{\sigma}}$ is not $\alpha_{N_{\sigma}}$, the domain of the latter being a superset of $D_{\sigma_{\min}}$. The following definition gives us a $\sigma$-permuted evaluation function that is the inverse of the $\sigma$-permuted spelling function:

**Definition 2.31.** The $\sigma$-permuted minimal RPF natural evaluation function, denoted by $\alpha_{N_{\sigma_{\min}}}$, is the restriction of $\alpha_{N_{\sigma}}$ to $D_{\sigma_{\min}}$.

Now we have the ability to convert spellings in one minimal natural interpretation to equivalent spellings in another, according to the following straightforward procedure. Suppose we have a word $w_{\sigma}$ evaluating to some natural number $n$ in the $\sigma$-permuted minimal natural interpretation and wish to find the word $w_{\tau}$ evaluating to $n$ in the $\tau$-permuted minimal natural interpretation. We first find $n$ by applying $\alpha_{N_{\sigma_{\min}}}$ to $w_{\sigma}$; we then apply $\gamma_{N_{\tau}}$ to $n$, giving us $w_{\tau}$. In other words, we use the following equation:

$$w_{\tau} = \gamma_{N_{\tau}}(\alpha_{N_{\sigma_{\min}}}(w_{\sigma})). \quad (9)$$

We can also convert the evaluations in one minimal natural interpretation to equivalent evaluations in another, as follows. Suppose we have a number $n_{\sigma} \in \mathbb{N}$, the spelling of which is $w$ in the $\sigma$-permuted minimal natural interpretation, and we wish to find the number $n_{\tau} \in \mathbb{N}$ with the spelling $w$ in the
τ-permuted minimal natural interpretation. The following equation gives us $n_τ$:

$$n_τ = \alpha_{N_{τ_{\min}}} (\gamma_{N_{τ}}(n_τ)).$$  \hspace{1cm} (10)

For the rest of this paper, I will focus primarily upon interpretations arising from the standard permutation. Generalizations of the associated mathematical objects to all prime-permuted interpretations are somewhat tedious but straightforward.

### 2.8 The Standard Minimal RPF Language Reconsidered

We defined $D_{τ_{\min}}$ to be the codomain of $\gamma_{N_{τ}}$ (Definition 2.13). However, there is an interesting alternative definition—I say interesting, because it is a non-numerical definition of $D_{τ_{\min}}$, and because it can be used to prove that $D_{τ_{\min}}$ is a context-free language (see Theorem 2.8 on the next page).

**Theorem 2.7.** These definitions are equivalent:

1. The standard minimal RPF language $D_{τ_{\min}}$ is the codomain of the standard RPF natural spelling function $\gamma_{N_{τ}}$.

2. The standard minimal RPF language $D_{τ_{\min}}$ is the set

   $$\{ d \in D \mid (\,')()' \text{ is not a substring of } d \land (\,')()' \text{ is not a suffix of } d \}.$$ 

**Proof.** The spellings of 0 and 1 are $ε$ and () respectively, by explicit definition; neither of these contains the substring )())) or the suffix )().

Suppose that there exists some natural number $n$ greater than 1 such that its spelling contains the suffix )())), i.e., $\gamma_{N_{τ}}(n) = w = w' \sim (\,')'$, with $w'$ being a nonempty Dyck word. This would imply that the last chunk in $w$ corresponds to the zeroth power of the greatest prime factor of $n$, which is a contradiction, the zeroth power of a prime number not appearing in the (nonrecursive) prime factorization of $n$.

Now suppose that there exists some natural number $m$ greater than 1 such that its spelling contains the substring )(()). The presence of the closing parenthesis immediately following the empty matched pair of parentheses would imply that $\gamma_{N_{τ}}$ had been invoked with some number $n \geq 2$ as its argument as part of the recursion arising from the spelling of some other number, and that the spelling of $n$ contained the suffix )(), which we have already shown is a contradiction.
As for all other words \( w \) in \( D \) other than those we have already considered, these must also be members of \( D_{\text{min}} \). To see this, suppose that \( w \) is not in the codomain of \( \gamma_{\text{min}} \). Since the domain of \( \alpha_{\text{min}} \) is \( D \), \( \alpha_{\text{min}}(w) \) is a natural number. This would imply that a natural number exists (other than 0 or 1, these already having been considered) which cannot be represented by a product of prime powers

\[
n = \prod_{i=1}^{m} p_i^{a_i},
\]

where \( p_m \) is the greatest prime factor of \( n \) and \( a \) is the unique integer sequence satisfying the equation. Note that if we form a product of only the \( p^{a_i} \) where \( a_i \neq 0 \), we obtain the (nonrecursive) prime factorization of \( n \). Also note that all zero \( a_i \) designate the exponents of primes not contributing to the prime factorization. Thus \( n \) is an integer greater than or equal to 2 such that \( n \) has no prime factorization, which is a contradiction.

We now make use of the alternative definition of \( D_{\text{min}} \) to prove that \( D_{\text{min}} \) is context-free.

**Theorem 2.8.** Let \( \Sigma^* \) be the Kleene closure of the set \( \{(,\)\} \), and let \( R \) be the language accepted by the deterministic finite automaton (DFA) represented by the following state transition diagram:

![State Transition Diagram](image)

Then

\[
R = \{ w \in \Sigma^* \mid (')()' \text{ not a substring of } w \} \land (')()' \text{ not a suffix of } w \},
\]

and

\[
D_{\text{min}} = D \cap R,
\]

proving by the Chomsky-Schützenberger representation theorem that \( D_{\text{min}} \) is a context-free language.
Remark. I am indebted to Dr. Brian M. Scott for his assistance identifying the state transition table for the DFA used in Theorem 2.8.

Proof. We first verify that no word $w \in \Sigma^*$ containing the substring $())$ is accepted by the DFA. Suppose the automaton has read some arbitrary number of symbols and is currently at state $q_i$, with the remaining input starting with the string $())$. For each $i \in \{0, 1, 2, 3, 4\}$, the input sequence $())$ places the automaton in state $q_4$, from which no transition to another state is possible, so that $w$ ends with the automaton in state $q_4$. Since $q_4$ is not an acceptor state, $w$ is rejected. Thus the DFA rejects all words containing the substring $())$.

Next we verify that no word $w$ in $\Sigma^*$ containing the suffix $())$ is accepted by the DFA. Suppose the automaton has read some arbitrary number of symbols and is currently at state $q_i$, the remaining input being $())$. Now let us consider each of the possibilities.

- If $i \in \{0, 1, 2\}$, then the input ends with the automaton in state $q_3$. Since $q_3$ is not an acceptor state, $w$ is rejected.
- If $i \in \{3, 4\}$, then the input ends with the automaton in state $q_4$. Since $q_4$ is not an acceptor state, $w$ is rejected.

Thus the automaton rejects all words in $\Sigma^*$ containing the suffix $())$.

We must still verify that all words $w$ in $\Sigma^*$ other than those rejected above are recognized by the DFA. In our verification, we can ignore any words ending with the DFA in state $q_4$, since the only way to reach that state is for $w$ to contain the substring $())$, and all of these words have already been considered. We may likewise ignore words ending with the DFA in state $q_3$, these all containing the suffix $())$. The only other possibility is that $w$ ends in either $q_0$, $q_1$ or $q_2$, each of which is an acceptor state. Thus the automaton accepts all words in $\Sigma^*$ except for those containing the substring $())$ or the suffix $())$.

Because $D \subset \Sigma^*$, the DFA accepts every $d \in D$ such that $())$ is not a substring of $d$ and $())$ is not a suffix of $d$. Therefore, from Theorem 2.7,

$$D_{r_{\text{min}}} = D \cap R,$$

which we may rewrite as

$$D_{r_{\text{min}}} = h(D \cap R),$$
where $h : \mathcal{D} \to \mathcal{D}$ is the identity function, so that $h$ is a homomorphism as required by the Chomsky-Schützenberger representation theorem. And so we conclude that $\mathcal{D}_{r_{\text{min}}}$ is context-free.

I propose a context-free grammar for the generation of $\mathcal{D}_{r_{\text{min}}}$. The grammar is ambiguous, yielding two shift-reduce conflicts among the states generated by an LALR parser; nevertheless, with these conflicts resolved in favor of shifting I have used the parser to verify that the grammar generates minimal spellings of the first thousand natural numbers.

**Conjecture 2.1.** The language underlying $\text{RPF}_{\mathbb{N}_{r_{\text{min}}}}$ is generated by the following context-free grammar:

$$
\begin{align*}
N & \to \epsilon \\
N & \to ( ) \\
N & \to S \\
S & \to SS \\
S & \to ( )S \\
S & \to ( S ) \\
S & \to ( ( ))
\end{align*}
$$

3 The Standard Minimal RPF Rational Interpretation $\text{RPF}_{\mathbb{Q}_{r_{\text{min}}}}$

3.1 Generalization of Prime Factorization to Nonzero Rational Numbers

The prime factorization of each natural number $n \geq 2$ is a product of its prime factors such that the product evaluates to $n$ (if we understand prime numbers to be their own prime factorizations); we are guaranteed by the fundamental theorem of arithmetic that the product is unique up to the order of the factors. Thus we may write 40 as $2 \times 2 \times 2 \times 5$, or more briefly in exponential form as $2^3 \cdot 5^1$. Let us consider the general exponential form of the prime factorization of $n$

$$p_{a_1}^{b_1} \cdots p_{a_k}^{b_k},$$

where $p_{a_i}$ is the $i$th prime factor of $n$, $p_{a_k}$ is the greatest prime factor of $n$, and $(b_1, \ldots, b_k)$ is the unique sequence of positive integers such that the expression evaluates to $n$. 

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But let us now relax our requirement that the integers $b_i$ in $(b_i)_{i=1}^k$ be positive, allowing negative integers to be included as well. Then we could write $5.6$, for example, as

$$p_2^2 \cdot p_5^{-1} \cdot p_7^1.$$  

Observe that the above expression bears a striking resemblance to the exponential form of prime factorization. Indeed, just as with prime factorization, the expression is unique up to the order of its factors, since $5.6$ can be uniquely expressed as the reduced fraction $\frac{28}{5}$, the numerator and denominator each corresponding to unique prime factorizations:

$$5.6 = \frac{28}{5} = \frac{2 \times 2 \times 7}{5}.$$  

If we are willing to include $-1$ as a factor, we may represent negative rationals as well. Hence we may generalize prime factorization to nonzero rational numbers.

**Definition 3.1.** Let $q$ be a nonzero rational number, and let $(b_1, \ldots, b_k)$ be the integer sequence of shortest length such that $|q|$ is equal to $p_{a_1}^{b_1} \cdots p_{a_k}^{b_k}$, where $(p_{a_i})_{i=1}^k$ is a subsequence of the sequence $P$ of prime numbers. Then the rational prime factorization of $q$ is $p_{a_1}^{b_1} \cdots p_{a_k}^{b_k}$ if $q > 0$; otherwise, the rational prime factorization of $q$ is $-1 \cdot p_{a_1}^{b_1} \cdots p_{a_k}^{b_k}$.

**Remark.** I used the name “rational prime factorization” rather simply “prime factorization” in order to avoid controversy. For one thing, the fundamental theorem of arithmetic only addresses integers, specifically integers greater than 1. Also, note that while I was able to write the prime factorization of decimal 40 without using exponentiation, there is no integer $k$ satisfying

$$5.6 = 2 \times 2 \times 5 \times \cdots \times 5 \times \overbrace{7}^{k \text{ times}}.$$  

**3.2 Dyck Inflations and the Dyck Rationals**

We saw in Section 2.4 that $D_{r_{\min}}$ is not equal to $D$, being rather a proper subset of it. Recall that this is because $D_{r_{\min}}$ is the codomain of the spelling function $\gamma_{N_r}$, which is only capable of producing spellings where empty chunks correspond to the zeroeth powers of those prime numbers less than the greatest prime factor of the number being spelled (with the exception of the spelling of 1, that being '()'). We will relax this restriction to include additional words beyond those produced by $\gamma_{N_r}$, yielding nonminimal representations of natural numbers. But we will do so incrementally, because
making slight changes to $\alpha$ and $\gamma$ will give us the remarkable ability to regard every rational number as a unique Dyck word.

Definition 2.5 tells us what we are to understand by the term “minimal representation,” and of course a nonminimal representation is a representation that is not minimal. But there is another way to view the concept of nonminimality. In the decimal system, every member of the infinite sequence of strings $(102, 0102, 00102, \ldots)$ is understood to represent the number one hundred and two. We have no problem, for instance, determining the quantity represented by $000102$ on a car odometer. Nevertheless there is one particular string, $102$, that is the minimal word representing one hundred and two; all others may be regarded as a result of “inflating” the minimal word with zeroes without changing the quantity it represents. The situation is similar for words in nonminimal RPF languages, except that the inflations involve empty parenthesis pairs rather than zeros and there almost always exist multiple locations in the word where inflation may take place.

**Definition 3.2.** An **empty pair** is the string $(.)$.

Consider the three Dyck words $w = ()(()())$, $x = ()()(()())$ and $y = ()()()().$ Applying the standard natural evaluation function to each of these, we find that

$$\alpha_{N_r}(w) = \alpha_{N_r}(x) = \alpha_{N_r}(y) = 5.$$  

If we apply the standard natural spelling function to each of the above evaluations, we discover that

$$\gamma_{N_r}(\alpha_{N_r}(w)) = \gamma_{N_r}(\alpha_{N_r}(x)) = \gamma_{N_r}(\alpha_{N_r}(y)) = '()()()' = w,$$

so that $w$ is the only Dyck word of the three where the word equals the spelling of its evaluation. Also notice that of $w$, $x$ and $y$, $w$ is the word of shortest length; indeed, $()()()$, being a member of $D_{r_{\min}}$, is the shortest Dyck word evaluating to 5 in the standard natural interpretation.

**Definition 3.3.** Let $d$ and $d'$ be Dyck words. $d$ is said to be an **inflation** of $d'$ if $\alpha_{N_r}(d) = \alpha_{N_r}(d')$ and $|d| > |d'|$, where $|s|$ denotes the length of string $s$. If we do not wish to specify $d'$, we may simply say $d$ is an inflation, with the implication that some $w \in D$ exists such that $d$ is an inflation of $w$.

**Remark.** Dyck word $w$ is an inflation if and only if $w \neq \gamma_{N_r}(\alpha_{N_r}(w))$.

**Definition 3.4.** Let $d'$ and $d$ be Dyck words such that $d$ is an inflation of $d'$. Then we say $d'$ is a **deflation** of $d$.
Words in \( D_{r_{\text{min}}} \) cannot be inflations, as the only empty pairs they contain are those essential as separators to collectively designate indices of prime numbers contributing to the prime factorization (or, in the case of \((())\), to spell the number 1). Consider an arbitrary Dyck word \( w \). We can form a sequence with \( w \) as its first term, with every subsequent term being a deflation of its predecessor; the longest possible such sequence must have a member of \( D_{r_{\text{min}}} \) as its last term.

**Definition 3.5.** Let \( w \) and \( w' \) be distinct Dyck words. We say \( w \) collapses to \( w' \) if \( \alpha_{\mathbb{N}_r}(w) = \alpha_{\mathbb{N}_r}(w') \) and \( w' \in D_{r_{\text{min}}} \).

**Example 3.1.** Dyck word \( w = ((())((())())())()()() \) collapses to \( w' = ((())(())) \); the standard natural evaluation of both \( w \) and \( w' \) is 32, and \( w' \) is a member of \( D_{r_{\text{min}}} \).

The following gives us terminology to talk about strings consisting solely of empty pairs.

**Definition 3.6.** Let \( w = \bigoplus_{i=1}^{k} '()' \) for some \( k \in \mathbb{N} \). Then we say \( w \) is the string of \( k \) contiguous empty pairs. If \( k = 0 \), we say the string is trivial.

Thus the trivial string of zero contiguous empty pairs is \( \varepsilon \), the string of one contiguous empty pair is \( () \), the string of two contiguous empty pairs is \( ()() \), and so on.

The following definition allows us to quantify “how inflated” we consider a given Dyck word to be.

**Definition 3.7.** Let \( w \) be a Dyck word. The standard inflationary degree of \( w \), denoted by \( \text{dinf}_r(w) \), is the largest integer \( n \) such that a string of \( n \) contiguous empty pairs can be deleted from \( w \) to yield a Dyck word \( w' \) satisfying the equation \( \alpha_{\mathbb{N}_r}(w') = \alpha_{\mathbb{N}_r}(w) \).

**Example 3.2.** Let \( w = ((())((())())())() \). The longest substring that can be deleted from \( w \) to yield a Dyck word \( w' \) with the same standard natural evaluation as that of \( w \) is \( ()()() \), the string of 3 contiguous empty pairs. Thus \( \text{dinf}_r(w) = 3 \).

And now I introduce the language underlying the standard minimal RPF rational interpretation.

**Definition 3.8.** The standard quasiminimal RPF language, also called the set of Dyck rational numbers and denoted by \( D_{r_{\text{qmin}}} \), is the set of all Dyck words \( d \) such that \( \text{dinf}_r(d) \leq 1 \).

**Remark.** The designation quasiminimal is due to the fact that \( D_{r_{\text{qmin}}} \) is the language underlying both the standard minimal RPF rational interpretation and the standard nonminimal RPF natural interpretation \( (D_{r_{\text{qmin}}}, \mathbb{N}, \alpha_{\mathbb{N}_r}) \).
3.3 The Standard Rational Spelling Function $\gamma_{\mathbb{Q}_r}$

**Definition 3.9.** Let $q$ be a nonzero rational number. The greatest prime base of $q$ is given by the function $\text{gp}_b : \mathbb{Q} \setminus \{0\} \to \{p_1, p_2, p_3, \ldots\}$, where

- If $q = 1$, then $\text{gp}_b(q) = p_1$.
- Otherwise, let the integer sequence $(a_1, \ldots, a_m)$ satisfy the equation $q = \prod_{i=1}^{m} p_i^{a_i}$ such that $m$ is the greatest number for which $a_m \neq 0$. Then $\text{gp}_b(q) = p_m$.

**Remark.** Informally, the greatest prime base may be thought of as the largest prime number that must appear in the product in order for the product to evaluate to $q$.

**Definition 3.10.** Let $q$ be an arbitrary member of the set $\mathbb{Q}$ of rational numbers. The standard RPF rational spelling is the function $\gamma_{\mathbb{Q}_r} : \mathbb{Q} \to D_{\gamma_{\min}}$, where

- For $q = 0$, $\gamma_{\mathbb{Q}_r}(q) = \varepsilon$.
- For $q = 1$, $\gamma_{\mathbb{Q}_r}(q) = (')$.
- For $q < 0$, $\gamma_{\mathbb{Q}_r}(q) = \gamma_{\mathbb{Q}_r}(|q|) \sim (')$.
- Otherwise, let $p_m$ be the greatest prime base of $q$, and let integer sequence $(a_1, \ldots, a_m)$ satisfy the equation $q = \prod_{i=1}^{m} p_i^{a_i}$. Then

$$\gamma_{\mathbb{Q}_r}(q) = \bigoplus_{i=1}^{m} (')(\gamma_{\mathbb{Q}_r}(a_i) \sim (') ).$$

**Example 3.3.** Let us spell the Dyck rational corresponding to decimal $-0.\overline{2}$. First we note that $-0.\overline{2} = -\frac{2}{9} = -1 \cdot 2^1 \cdot 3^{-2} = -p_1^{1} p_2^{-2}$. Thus the standard RPF rational spelling corresponding to decimal $-0.\overline{2}$ is

$$\gamma_{\mathbb{Q}_r}(-0.\overline{2}) = \gamma_{\mathbb{Q}_r}(0.\overline{2}) \sim (')$$

$$= (')(\gamma_{\mathbb{Q}_r}(1) \sim ') \sim (')(\gamma_{\mathbb{Q}_r}(-2) \sim ') \sim ('$$

$$= (')(' \gamma_{\mathbb{Q}_r}(2) \sim (') \sim (') \gamma_{\mathbb{Q}_r}(1) \sim (') \sim ('))('$$

$$= (')(())'((',(())(())(())(())').$$

Observe how our modification of the spelling function allowed us to enlarge its domain from $\mathbb{N}$ to $\mathbb{Q}$ without requiring an intermediate modification to go from $\mathbb{N}$ to $\mathbb{Z}$. This is because the recursive nature of the spelling
function implies that if it can spell negative numbers, it can also spell negative exponents. Indeed, enlargement of the domain of the spelling function from $\mathbb{N}$ to merely $\mathbb{Z}$ requires that the set of permitted inflations be expressly limited to those where a string of inflationary empty pairs only occurs as a suffix of the original word being spelled; this constitutes an artificial restriction, intended to defeat the recursivity of the function. In other words, with recursive prime factorizations it is easier and more straightforward to go directly from representing natural numbers to representing rational numbers, than to go from representing natural numbers to representing integers and thence to representing rationals. When we modify the spelling function so we can spell negative numbers, we get the ability to spell rational numbers “for free.”

The standard quasiminimal RPF language $\mathcal{D}_{r_{qmin}}$ has alternative definitions equivalent to Definition 3.8. I leave the following as a proposition; its proof is tedious but not conceptually difficult, with my preceding treatment of $\mathcal{D}_{r_{min}}$ providing background for how to proceed (refer especially to Theorem 2.7 on page 29).

**Proposition 1.** The following definitions are equivalent:

- The *standard quasiminimal RPF language*, also called the *set of Dyck rational numbers* and denoted by $\mathcal{D}_{r_{qmin}}$, is the set of all Dyck words $d$ such that $\text{dinf}_r(d) \leq 1$.

- The *standard quasiminimal RPF language*, also called the *set of Dyck rational numbers* and denoted by $\mathcal{D}_{r_{qmin}}$, is the codomain of $\gamma_{\mathbb{Q}}$.

- The *standard quasiminimal RPF language*, also called the *set of Dyck rational numbers* and denoted by $\mathcal{D}_{r_{qmin}}$, is the set of all $d \in \mathcal{D}$ such that $d$ fulfills the following two criteria:
  - $d$ does not contain the substring $)()()$.
  - $d$ does not contain the suffix $)()$.

**Remark.** Using the nonnumerical alternative definition of $\mathcal{D}_{r_{qmin}}$, the definitions of “inflation” (Def. 3.3), “collapse” (Def. 3.5) and “inflationary degree” (Def. 3.7) can be restated so that they do not refer to evaluation function $\alpha_{\mathbb{N}_r}$, thus eliminating the necessity to perform prime factorization in the course of their application.
3.4 The Standard RPF Rational Evaluation Functions and the Standard Minimal RPF Rational Interpretation

I define a function capable of evaluating every member of the Dyck language as a rational number, and then I define a second function as a restriction of the first, yielding the bijection to serve as the evaluator in the standard minimal rational interpretation.

**Definition 3.11.** Let \( w \in \mathcal{D} \). Then the standard RPF rational evaluation of \( w \) is the function \( \alpha_{Q_r} : \mathcal{D} \rightarrow \mathbb{Q} \), where

- If \( w = \varepsilon \), \( \alpha_{Q_r}(w) = 0 \).
- Otherwise, let \( w' \) and \( z \) be Dyck words where \( w = w' \& z \), with \( z \) being the longest suffix of \( w \) such that \( w' \neq \varepsilon \) and \( z \) contains only empty parenthesis pairs. Then

\[
\alpha_{Q_r}(w) = (-1)^{\dim(z)} \prod_{i=1}^{\dim(w')} p_i^{\alpha_{Q_r}(d_i)},
\]

where \( d_i \) is the content of the \( i \)-th chunk in \( w' \).

**Example 3.4.** Let us evaluate the Dyck word \( w = (())((())())() \) as a standard RPF rational number. Note that \( w = w' \& z \), where \( w' = (())((())()) \) and \( z = () \). Thus we have

\[
\alpha_{Q_r}(w) = (-1)^{\dim(z)} \prod_{i=1}^{\dim(w')} p_i^{\alpha_{Q_r}(d_i)},
\]

with \( d = (')()', (())()' \). Therefore

\[
\alpha_{Q_r}(w) = (-1)^1 p_1^{\alpha_{Q_r}(')}, p_2^{\alpha_{Q_r}(()0')} = -p_1^{\alpha_{Q_r}(')}, p_2^{\alpha_{Q_r}((()0')} = -p_1^{\alpha_{Q_r}(')}, p_2^{\alpha_{Q_r}(()0')} = -p_1 p_2^{\alpha_{Q_r}((()0')} = -p_1 p_2^{\alpha_{Q_r}((()0')} = -p_1 p_2^{\alpha_{Q_r}((()0')} = -p_1 p_2^{\alpha_{Q_r}((()0')} = -2 \cdot 3^{-2} = -\frac{2}{9} = -0.2222\ldots.
\]

**Definition 3.12.** The standard minimal RPF rational evaluation function, denoted by \( \alpha_{Q_r^{\text{min}}} \), is the restriction of \( \alpha_{Q_r} \) to \( \mathcal{D}_{r_{\text{qmin}}} \).

Now we are ready to define the interpretation.
Definition 3.13. The \textit{standard minimal RPF rational interpretation}, denoted by $\text{RPF}_{\text{Q}_{r_{\text{min}}}}$, is the interpretation $(\mathcal{D}_{\text{r}_{\text{min}}}, \mathbb{Q}, \alpha_{\text{Q}_{r_{\text{min}}}})$.

\textbf{Remark.} The presence of both “min” and “qmin” in the definition above does not reflect a typographical error. The \textit{interpretation} is minimal, as evaluation function $\alpha_{\text{Q}_{r_{\text{min}}}}$ is a bijection. However, the underlying language is quasiminimal, as was noted in the remark following Definition 3.8 on page 35.

We found that $(()((())())))()$ evaluates to the number represented in decimal by $-0.2222\ldots$. Note that the decimal representation required three augmentations to the language: a negative sign to designate negative numbers, a radix point to mark the boundary between nonnegative and negative powers of the radix, and an ellipsis (or, alternatively, an overbar) to designate endlessly repeating digit sequences. Also note that the RPF representation of the same number did not require such augmentation; especially note that no ellipsis or overbar was required for the RPF representation to be of finite length. The following theorem tells us that \textit{all} rational numbers have finite-length RPF representations, with no addition to the symbol set required.

\textbf{Theorem 3.1.} Every rational number may be represented by a word of finite length in $\text{RPF}_{\text{Q}_{r_{\text{min}}}}$.

\textbf{Proof.} Let $q$ be an arbitrary rational number. To prove that $q$ has a finite-length representation in $\text{RPF}_{\text{Q}_{r_{\text{min}}}}$, we will find the representation.

- For $q = 0$, the finite-length representation of $q$ is $\epsilon$.
- For $|q| = 1$, $q$ can be either 1 or -1, the finite-length representations of which are () and () respectively.
- Otherwise choose $t \in \mathbb{N}$ and $u \in \mathbb{N}_+$ such that $\frac{t}{u}$ is a reduced fraction and $|q| = \frac{t}{u}$. By the definition of reduced fractions, $t$ and $u$ have no common factors. Let us express $t$ as $p^{|t|}_1 \cdot \cdots \cdot p^{|t|}_k$, where $p_k$ is the greatest prime factor of $t$. Similarly, let us express $u$ as $p^{|u|}_1 \cdot \cdots \cdot p^{|u|}_m$, where $p_m$ is the greatest prime factor of $u$. Thus

$$\frac{t}{u} = p^{c_1}_1 \cdots p^{c_j}_j,$$

where $j = \max(k, m)$ and

$$c_i = \begin{cases} a_i & \text{if } p_i \text{ is a factor of } t \\ -b_i & \text{if } p_i \text{ is a factor of } u \\ 0 & \text{otherwise} \end{cases}.$$
for $1 \leq i \leq j$. The expression $p_{i}^{c_{1}} \cdots p_{j}^{c_{j}}$ is then equal to $|q|$ and corresponds to the RPF$_{Q_{\min}}$ word $w = \gamma_{Q_{r}}(\frac{t}{u})$ of finite length, since $j$ is finite. If $q$ is positive, then $w$ is the representation of $q$ and we are done; otherwise the finite-length word $w \sim '(1)'$ is the representation of $q$.

Remark. In Section 4 we will see that the language underlying RPF$_{Q_{\min}}$ is a proper subset of the language $\mathcal{D}$ underlying the standard Dyck-complete interpretation of $Q$. Since $q$ as obtained above is a member of the former set, it is also a member of the latter, implying that every rational number has a representation of finite length in $\mathcal{D}$.

### 3.5 Why a Second Incremental Inflation Set Is Not Particularly Useful

Having modified RPF$_{N_{\min}}$ to yield RPF$_{Q_{\min}}$ by permitting first-degree inflations, we might imagine that further modifying it to permit second-degree inflations would buy us additional representational power, so that we could use terminal empty parenthesis pairs to encode additional two-state attributes—for example, an attribute called “spin,” which could either be clockwise or counterclockwise. Furthermore, we might imagine that the attributes of sign and of spin would be orthogonal, i.e., that the value of the sign could be determined without having to know that of the spin (and vice versa). But such a happy state of affairs is not the case, as the following theorem states for standard RPF interpretations.

**Theorem 3.2.** Let $R_0$ be the standard minimal RPF language of 0-degree inflations, let $R_1$ be the standard quasiminimal RPF language of inflations of degree $\leq 1$, and let $R_2$ be the standard RPF language of inflations of degree $\leq 2$. Then $R_2$ is no more powerful than $R_1$ for encoding orthogonal binary attributes.

**Proof.** $R_1$ includes minimal RPF words and their single terminal inflations, so it can be successfully used for encoding a single binary attribute modifying the evaluation of the word. For example, suppose the attribute is (north, south), such that $(())$ means “2 units due north,” whereas its terminal inflation $(())()$ evaluates to “2 units due south.” This is indeed possible using a subset of the Dyck language confined to inflations of degree $\leq 1$. But now suppose we try to extend the concept by using $R_2$ so we can encode an additional orthogonal binary attribute, say, that of (west, east), with the
next-to-last empty parenthesis pair signifying north versus south and the last 
empty parenthesis pair signifying west versus east. We might for instance 
naïvely claim that whereas \(((())\)) represents “2 units northwest”, \(((())(()))\) rep-
resents “2 units southeast.” Under such an interpretation, however, the word 
\(((())(()))\) is ambiguous; without further information aside from the word itself 
and its interpretation, we cannot tell whether it signifies 2 units northeast or 
2 units southwest. In other words, we can only be sure of the values of words 
that are minimal or have 
′(()′) as a proper suffix. We might be content to 
impose a convention for disambiguation, so that all occurrences of a minimal 
word concatenated with a single empty parenthesis pair would be resolved in 
favor of, say, south rather than east. But to impose such a convention would 
destroy the orthogonality of the two attributes.

Remark. The theorem states that the set of inflations of degree \(\leq 2\) is no more 
useful for encoding orthogonal binary attributes than the set of inflations of 
degree \(\leq 1\), but says nothing about sets including inflations of even higher 
degree. Obviously, though, permitting even longer strings of inflationary 
empty pairs does not eliminate the ambiguity.

## 4 Dyck-complete Interpretations

While a minimal RPF natural interpretation and its corresponding minimal 
RPF rational interpretation are sufficient to represent all natural numbers 
and all rational numbers respectively, and while each of these interpretations 
joins the property that a bijection exists between its underlying language 
and its target set, we can generalize our notion of RPF languages to permit 
all possible inflations. The result is the Dyck language, which underlies all 
generalized RPF interpretations, regardless of whether they are natural or 
rational, or which prime permutations determine the ordering of factors in 
their words.

Remark. The Dyck language has been a topic of much research, resulting in 
an extensive body of knowledge of potential use for the study of recursive 
prime factorizations.

**Definition 4.1.** An interpretation is *Dyck-complete* if its underlying lan-
guage is \(D\).

We already have our evaluation functions for the standard Dyck-complete 
natural and rational interpretations; these are \(\alpha_{N_r}\) and \(\alpha_{Q_r}\), respectively.

**Definition 4.2.** The *standard RPF Dyck-complete natural interpretation*, 
denoted by \(\text{RPF}_{N_r}\), is the interpretation \((D, \mathbb{N}, \alpha_{N_r})\).
**Definition 4.3.** The *standard RPF Dyck-complete rational interpretation*, denoted by $\text{RPF}_{\mathcal{Q}_r}$, is the interpretation $(\mathcal{D}, \mathcal{Q}, \alpha_{\mathcal{Q}_r})$.

These interpretations are nonminimal, their evaluation functions being noninjective. There is a spelling function associated with each interpretation—$\gamma_{\mathcal{N}_r}$ for $\text{RPF}_{\mathcal{N}_r}$, $\gamma_{\mathcal{Q}_r}$ for $\text{RPF}_{\mathcal{Q}_r}$—but words do not generally equal the spellings of their evaluations. Indeed, the evaluation functions define equivalence relations partitioning the underlying languages of their interpretations into equivalence classes, where each equivalence class contains Dyck words that evaluate to the same number. For example, let $R_{\mathcal{N}_r}$ be the equivalence relation

$$R_{\mathcal{N}_r} = \{ (w_1, w_2) \in \mathcal{D}^2 \mid \alpha_{\mathcal{N}_r}(w_1) = \alpha_{\mathcal{N}_r}(w_2) \}.$$

Then we can identify each equivalence class as $S_n$, where every member of the class evaluates to the natural number $n$. Thus $S_0 = \{ \epsilon \}$, $S_1 = \{ (\prime)(\prime), (\prime)(\prime)(\prime), \ldots \}$, and so on. In each equivalence class, there is exactly one word that is in the codomain of the spelling function; it is thus the only member of its class such that it is equal to the spelling of its evaluation.

Figure 1 shows the hierarchy of the languages $L$ underlying standard RPF interpretations, with the interpretations designated by subscripts. Also shown are the same sets as named according to their status as subsets of the Dyck language: the standard Dyck minimals, the standard Dyck quasiminimals, and the Dyck language itself.

![Euler diagram](image)

Figure 1: Euler diagram illustrating the hierarchy of standard RPF languages.
5 Conclusion: Research Directions and Possible Applications

I conclude with suggestions for further study of recursive prime factorizations, as well as possible applications in mathematics and computer science.

5.1 The Significance of OEIS Sequence A082582

Conjecture 5.1. Let $n \in \mathbb{N}$. The number of words in $D_{r_{\min}}$ of length $2n - 2$ (or, equivalently, of semilength $n - 1$) is given by the sequence

$$(a_n)_{n \in \mathbb{N}^+} = (1, 1, 1, 2, 5, 13, 35, 79, 192, 470, 1215, 3044, \ldots),$$

i.e., Sequence A082582 in the Online Encyclopedia of Integer Sequences (“Expansion of $(1 + x^2 - \sqrt{1 - 4x + 2x^2 + x^4})/(2x)$ in powers of $x^3$”) [3].

Remark. For example, the number of words of length $2(4 - 1) = 6$ in $D_{r_{\min}}$ is given by $a_4 = 2$. This may be easily verified for the case of $n = 4$ by enumerating all 6-character strings of left and right parentheses, and then eliminating those that are not Dyck naturals. The only two strings not eliminated are $()()$ and $((())$). One interpretation of sequence A082582 involves excluding Dyck paths containing the 4-character subpath DDUU. How that relates to $RPF_{N_{r_{\min}}}$ words is a question to be pondered.

5.2 Investigate the Properties of Stripes

Definition 5.1. Let $k$ be a natural number. The stripe of semilength $k$, denoted by $\theta(k)$, is the longest subsequence $s$ of $(0, 1, 2, 3, \ldots)$ such that every term $n$ in $s$ satisfies the equation $|\gamma_{N_r}(n)| = 2^k$, where $|w|$ is the length of string $w$. We call $\theta(0)$ and $\theta(1)$ trivial stripes.

Let $S$ be the sequence of the first four nontrivial stripes. Then $S$ is

$$((2), (3, 4), (5, 6, 8, 9, 16), (7, 10, 12, 15, 18, 25, 32, 64, 81, 256, 512, 65536)).$$

We see interesting patterns in $S$. For example, the first member of the $i$th stripe in $S$ is the prime number $p_i$, and no member $m$ exists in $S_i$ such that the spelling of $m$ entails the spelling of $p_i$. Also, the sequence of last terms of stripes in $S$ may be expressed as $(2, 2^2, 4, 2^2 2^2)$ . Do these patterns hold for all stripes of semilength $\geq 2$? What other patterns characterize stripes? Do these patterns suggest a nonnumerical algorithm for finding the successor of Dyck natural $w$, i.e., $\gamma_{N_r}(\alpha_{N_{r_{\min}}}(w) + 1)$? And so we are led to the next question:
5.3 Can We Develop an Algorithm to Compute the Successor of a Dyck Natural Number in Polynomial Time?

Whether numerical or nonnumerical, an algorithm to find the successor of a Dyck natural number such that the algorithm runs in polynomial time on a classical computer could be easily modified according to the Peano axioms to yield a second algorithm performing addition on Dyck naturals, again running on a classical computer in polynomial time; this algorithm in turn could be modified to yield a third algorithm capable of solving prime factorization of a number on a classical computer in polynomial time, with the input and output both being representations in a positional numeral system such as decimal. The last algorithm would constitute a solution to one of the currently unsolved problems of computer science. (As we saw in Section 2.6, prime factorization in linear time on a classical computer is already attainable, if we are willing to accept algorithms where the input and the output are expressed using Dyck natural numbers.)

5.4 Analyze the Time and Space Complexity of a Hybridized Modification of Algorithm 1

While Algorithm 1 accomplishes prime factorization in linear time, that does not mean the algorithm is faster than currently known prime factorization algorithms. In particular, the presence of long sequences of empty pairs in natural RPF words may result in very long input strings, with the result that execution time, though linear, is too large for the algorithm to be practical. I suggested a way to improve performance by using a hybrid scheme for input and output, where each sufficiently long substring of empty pairs would be replaced by a hexadecimal word giving the number of empty pairs in the substring. What would be the time and space complexities of such an algorithm?

5.5 The Vinculoid Conjectures

In decimal, we use a negative sign to extend the set of numbers we can exactly represent with a word of finite length from \( \mathbb{N} \) to \( \mathbb{Z} \); if we then use a radix point to imply the positions of coefficients of negative powers of the radix, we can exactly represent such members of \( \mathbb{Q} \) as \(-4.2\) and \(1.75\), but not those such as \(0.123123\ldots\). If we then use a vinculum (overbar or underbar) to designate an unending sequence of repeating digits, as in \(0.\overline{123}\), we can represent every member of \( \mathbb{Q} \) by a decimal word of finite length.
The Dyck rationals already represent \( \mathbb{Q} \) by words of finite length. Also, we know that irrational numbers can be expressed as infinite products of rationals, as is the case with the Wallis product:

\[
\frac{\pi}{2} = \prod_{n=1}^{\infty} \frac{4n^2}{4n^2 - 1}.
\]

Can we use a “vinculoid,” by which I mean a device serving a purpose analogous to that served by a vinculum in decimal to permit the exact representations of rational numbers, to extend the set of numbers we can represent by finite-length words in rational RPF from \( \mathbb{Q} \) to \( \mathbb{R} \)?

Perhaps a vinculoid would be a form of augmentation or “markup” such that marked-up Dyck rationals could represent irrational numbers.

**Conjecture 5.2.** It is possible to augment \( D_{\text{rmin}} \) to represent the set of real numbers by words of finite length.

Alternatively, a vinculoid might be a set of grammar productions from which an infinite sequence of Dyck rationals would be generated, such that the grammar would define a pattern manifested by successive Dyck rational approximations of an irrational number.

**Conjecture 5.3.** For every irrational number \( x \), there exists a context-free grammar \( G_x \) such that \( G_x \) generates an infinite sequence of Dyck rationals \( (s_i)_{i \in \mathbb{N}_+} \), with \( \alpha_{\mathbb{Q}}(s_i) \) converging to \( x \) as \( i \) approaches infinity.

### 5.6 Can We Use Grammar-based Compression to Detect and Identify Patterns in RPF Word Sequences?

Grammar-based compression algorithms such as Re-Pair [1] produce a context-free grammar for the string being compressed. Input with lower information entropy (i.e., input that is less random) will have a higher compression ratio than input with higher information entropy. Suppose we have two samples of input, these being of equal size but with one consisting of a concatenation of randomly-chosen Dyck naturals, the other consisting of a concatenation of Dyck naturals we hypothesize to be consecutive terms in a sequence manifesting some pattern. If the compression ratio obtained for the second input string is greater than that obtained for the first, we might take this as corroborating (but not proving, of course) our hypothesis. If we find the difference in compression ratios becomes more pronounced as the size of the input increases, we might take that to be an even stronger indication our hypothesis is correct.
Perhaps a conventional compression algorithm would serve as well as a grammar-based algorithm to accomplish what I described in the previous paragraph. But now suppose we have a sequence of Dyck rationals constituting successive approximations of an irrational number such as $\pi$, and we feed the algorithm longer and longer concatenations of terms in the sequence. Might we observe a pattern in the corresponding sequence of context-free grammars produced by the algorithm such that we can use the pattern to obtain a vinculoid (see Conjecture 5.3) enabling us to exactly represent the irrational number?

### 5.7 Describe a Minimal Language for Representing the Gaussian Rationals as Generalized Dyck Words

The Gaussian rationals are complex numbers of the form $q + ri$, such that $q$ and $r$ are rational. While we cannot represent such numbers as Dyck words (see Theorem 3.2 on page 40), the Dyck language can be generalized to include more than one type of delimiter, where the criteria for well-formedness apply to each type of delimiter in a word. Thus $w = (((())())())((())())$ is a word in that generalization of the Dyck language having the alphabet $\{(), [,,]\}$. We can regard $w$ as an RPF representation of the Gaussian rational $\frac{3}{2} + \frac{1}{2}i$, where the imaginary part is designated by being surrounded with square brackets. What would be the description of such a language $L$ with the additional property that $L$ is minimal? Does there exist a language $L'$ satisfying the criteria for $L$, with the additional property that $L'$ is context-free? What class of algebraic objects would $L'$ generate?

### 5.8 Techniques Making Use of RPF for Visualization of Patterns Among Numbers

My work on RPFs arose from a search for ways to make patterns in numbers more immediately accessible by avoiding certain drawbacks of positional numeral systems. While RPF spellings of numbers do show the prime factorizations involved, long strings of parentheses can become overwhelming, reminiscent of what programmers call “LISP hell.” I wonder whether graphical methods might be developed to aid the recognition of numerical patterns among RPF words, for example:

- Define an “RPF natural number spiral,” based on the Sacks natural number spiral but with the $k$th counterclockwise rotation of the spiral containing all naturals $n$ such that $|\gamma_{\text{S}}(n)| = 2k$, these appearing in numerical order. The sequence of numbers appearing on the zeroeth
rotation of the spiral, for instance, would be (0), while the sequence of numbers on the fifth rotation would be (5, 6, 8, 9, 16). If the patterns described in Section 5.2 hold for all nontrivial stripes, a feature of the spiral would be that the prime numbers all lie on the half-line separating quadrants I and IV, with only zero, one and every prime number appearing on the half-line, these occurring in ascending order from left to right.

One problem with such a spiral is that the lengths of the successive rotations grow so quickly. In the Sacks spiral, every rotation begins with a perfect square: rotation 0 begins with 0, rotation 1 with 1, rotation 2 with 4, rotation 3 with 9, and so on. There are thus 39 natural numbers falling within rotation 19: $19^2 \ldots 20^2 - 1$. But if Conjecture 5.1 is true, there are 54,857,506 numbers appearing in rotation 19 of the RPF natural spiral.

- Create a computer program that produces two-dimensional images according to an adaptation of escape-time algorithms such as those used to depict Mandlebrot and Julia sets, as follows. For every coordinate $(x, y)$ on the pixel display, let $z_0$ be a function of $(x, y)$, for example $z_0 = \lfloor x \rfloor \lfloor y \rfloor$, and let $z_{k+1}$ be defined as a function of $z_k$, for example $z_{k+1} = \alpha_{Q_0}(\gamma_{Q_0}(z_k))$. Now let $z_n$ be the smallest number greater than $L$ in the sequence, where $L$ is a large natural number of our choosing; we may regard $n$ as a measure of how quickly the values of $z_k$ are approaching infinity for the coordinate $(x, y)$. We then map values of $n$ to a range of colors, using the appropriate color to display the pixel corresponding to $(x, y)$. Of course, there may be instances where successive values of $z_k$ do not grow out of bounds, instead converging to some number; to handle this possibility, if $k$ reaches a certain large positive integer value $Q$ we have chosen, we quit the iterations and display the pixel for $(x, y)$ using the color black (for example).

5.9 The Union and Intersection of All $\sigma$-permuted Minimal Natural RPF Languages

Does $\bigcup_\sigma D_{\sigma_{\text{min}}}$ equal $D$? Does $\bigcap_\sigma D_{\sigma_{\text{min}}}$ equal $\{\epsilon, ()\}$?
5.10  Context-free Grammars for the Minimal and Quasi-
minimal Dyck Languages

Theorem 2.8 assures us that a context-free grammar (CFG) exists for $D_{\text{rmin}}$, and I have provided a candidate grammar as a conjecture. Is that grammar correct? If so, does an equivalent grammar with a smaller set of productions exist? Can we formulate a meta-grammar useful for deriving the productions of the language underlying an arbitrary prime-permuted minimal natural RPF interpretation? What would be an example of a CFG for $D_{\text{rqmin}}$?

5.11  Are Dyck Rationals Practical for Use in Compu-
tations Requiring Exact Representations of Ra-
tional Numbers?

On many occasions it is desirable to be able to specify, manipulate and test for the exact values of rational numbers; however, such numbers as $\frac{1}{7}$ can only be approximated in binary. This problem may be ameliorated to some extent by use of arbitrary-precision arithmetic, in which the amount of storage allocated for numbers is determined by the precision required for the application. I can imagine a modification of arbitrary-precision arithmetic to include an intermediate lookup to an array of RPF natural spellings, each of bit width $b$, the spellings stored as bit sequences encoding left parentheses (respectively, right parentheses) as 1s and right parentheses (respectively, left parentheses) as 0s, such that the spelling of natural number $k$ would reside at offset $bk$ from the start of the array. The purpose of this arrangement would be to allow “RPF shortcuts” in arbitrary-precision computations in those situations where taking such shortcuts would be advantageous in terms of time and/or space, or when an exact result was sufficiently important to warrant the implicit conversions. Of course, while it is true that every rational number has a finite-length representation in standard rational RPF (or in any other prime-permuted rational RPF interpretation, for that matter), there are several limitations that would impose constraints, and I do not know whether anything of value would arise from the idea. One limitation that can be overcome is the excessive size of words containing long substrings of empty pairs; in order for the scheme to be practical, it would have to be modified so that Dyck words are stored and accessed using a form of compression, such as replacing each sufficiently long substring of empty pairs by a hexadecimal word specifying the number of pairs replaced.
5.12 Can Recursive Prime Factorizations Be Useful for Cryptography?

I can imagine encryption algorithms involving recursive prime factorizations. As a very simple example, suppose we feed the ASCII values of plaintext into an algorithm that spells the values as Dyck naturals and then outputs as ciphertext the $\sigma$-permuted evaluations of the spellings for some prime permutation $\sigma$. Of course, this particular scheme is useless, being easily defeated by frequency analysis. Can recursive prime factorizations find practical use in cryptography?

5.13 Characterize Dyck Naturals Recognizable by Finite Automata

Although $D_{r_{\text{min}}}$ is not regular, many interesting subsets of it are. For example, the set \( \{ w \in D_{r_{\text{min}}} \mid \alpha_{N_{r_{\text{min}}}}(w) \text{ is prime} \} \) is recognized by the following DFA.

\[ \begin{array}{c}
q_0 \quad (,)
\quad q_5 \quad (,)
\quad q_4
\quad (,)
\quad q_3
\end{array} \]

Remark. We can also employ regular expressions to describe such subsets. In order to avoid confusion between grouping parentheses and symbols in the alphabet, let us use the binary alphabet \( \{1, 0\} \), where ’1' is equivalent to ’(’ and ’0' is equivalent to ’)’. Then \( (10)^*1100 \) denotes the set of prime Dyck naturals, and \( (10)^*1100(10)^*1100 \) denotes the set of squarefree semiprime Dyck naturals.

Let $k$ be a positive integer. What is the characterization of the set $N_k \subset \mathbb{N}$ such that each member $n$ of $N_k$ is represented according to $\alpha_{N_{r_{\text{min}}}}$ by a word $w$ in some regular subset $D \subset D_{r_{\text{min}}}$, where no spelling $\gamma_{N_{r_{\text{min}}}}(n)$ has a recursion chain of length greater than $k$?
References

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