Dephasing in disordered metals with superconductive grains

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Temperature dependence of electron dephasing time \( \tau_{\nu}(T) \) is calculated for a disordered metal with small concentration of superconductive grains. Above the macroscopic superconducting transition line, when electrons in the metal are normal, Andreev reflection from the grains leads to a nearly temperature-independent contribution to the dephasing rate. In a broad temperature range \( \tau_{\nu}^{-1}(T) \) strongly exceeds the prediction of the classical theory of dephasing in normal disordered conductors, whereas magnetoresistance is dominated (in two dimensions) by the Maki-Tompson correction and is positive.

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INTRODUCTION

During last few years, a number of experimental data on electron transport in disordered metal films and wires were shown to be in disagreement with the standard theory \( \Pi \) of electron dephasing in normal conductors. Namely, at sufficiently low temperatures \( T \leq T_1 \) the dephasing rate \( \tau_{\nu}^{-1}(T) \) was systematically found to deviate from the power-law dependence \( \Pi \):

\[
\frac{1}{\tau_{\nu}^{(0)}(T)} = \begin{cases} 
(T/h)^{3/2}\tau_1^{1/2}/(k_F l)^2, & \text{3D case,} \\
(T/2\pi hg)\ln(\pi g), & \text{2D case,}
\end{cases}
\]

with a tendency to apparent saturation in the \( T \to 0 \) limit (\( g = h/e^2R \gtrsim 1 \) is dimensionless conductance of the film). Since no dephasing rate may exist at strictly zero temperature \( \Pi \), such a behavior indicates a presence of some additional temperature scale(s) \( T_0 \) (which may occur to be extremely low), so that in the range \( T_0 \leq T \leq T_1 \) the main contribution to \( \tau_{\nu}^{-1}(T) \) comes from some new mechanism, different from the universal Nyquist noise considered in Ref. \( \Pi \). Among other suggestions (the presence of localized two-level systems \( \Pi \), nonequilibrium noise \( \Pi \), etc.) there were some speculations on a possible role of electron-electron interactions in \( \tau_{\nu}(T) \) “saturation”. Recent development \( \Pi \) of the theory \( \Pi \) have proved that perturbative account of electron-electron interactions does not lead to considerable corrections to Eq. \( \Pi \).

In this paper we show that electron-electron interaction considered nonperturbatively can indeed be responsible for strong deviation of dephasing rate from the standard predictions. Namely, we consider a system of small superconductive islands (of characteristic size \( a \)) situated in either bulk disordered metal matrix (3D case) or on the thin metal film (2D). The role of interaction here is to establish superconductivity in the islands, which is a nonperturbative effect. Such a system can exhibit \( \Pi \) a macroscopic superconducting transition mediated by the proximity Josephson coupling between the islands \( \Pi \), with the transition temperature \( T_c(n_i) \) depending on the concentration of the islands \( n_i \). Above this transition electrons in the metal are normal, but Andreev reflection of them from the superconducting islands leads to an additional contribution to the dephasing rate:

\[
1/\tau_{\nu}(T) = 1/\tau_{\nu}^{(0)}(T) + 1/\tau_{\nu}^{(c)}(T).
\]

Enhancement of dephasing rate due to superconductive fluctuations in homogeneous systems was considered previously both experimentally \( \Pi \) and theoretically \( \Pi \). Far above \( T_c \), the dephasing rate due to interaction in the Cooper channel is comparable to the dephasing rate \( h/\tau_{\nu}^{(0)}(T) \) due to the Coulomb interaction, being additionally suppressed as \( 1/\ln^2(T/T_c) \). Peculiarity of our result is that the superconductive contribution to the dephasing rate in inhomogeneous systems can be the dominant one in a broad range of temperatures above \( T_c(n_i) \).

To simplify calculations, we consider the model system \( \Pi \) where superconducting (SC) islands are connected to the metal matrix via tunnel barriers with normal-state tunnel conductances \( G_T \) (measured in units of \( e^2/h \)), which determine inter-islands resistance in the normal state. We are interested in the temperature range much below the critical temperature \( T_{c0} \) of islands, when charge transport between them and the metal occurs due to Andreev reflection processes. We assume large Andreev conductance, \( G_A \gg 1 \), thus Coulomb blocking of Andreev transport is suppressed. For small concentration of the islands, \( n_i < n_c \sim \exp(-\pi G_A/4) \), quantum fluctuations destroy macroscopic superconductive coherence through the whole system even at \( T = 0 \). In the opposite limit, \( n_i \gg n_c \), the thermally driven superconductor–metal transition takes place at \( T_c(n_i) \sim hDn_i^2/d \), where \( D \) is the diffusion coefficient and \( d \) is the dimensionality of space.

Here we focus on the temperature scale \( T \gg T_c(n_i) \), where macroscopic superconductivity is destroyed by thermal fluctuations, and the phases \( \varphi_j \) of superconductive order parameters on different islands fluctuate strongly and are uncorrelated with each other. Our main
result is the expression for the dephasing rate due to the processes of Andreev reflection from the SC islands:

\[
\frac{1}{\tau_{\varphi}(T)} = \begin{cases} 
\frac{n_i}{4\nu\hbar} G_A - \frac{4}{\pi} \ln \frac{G_A E_C}{2\pi^2 T}, & \text{3D case,} \\
\frac{n_i}{4\nu\hbar} G_A(T), & \text{2D case,}
\end{cases}
\]

where

\[
G_A(\omega) = \frac{G_T^2}{G_D(\omega)}
\]

is the (frequency-dependent) Andreev conductance of the island in the lowest tunneling approximation [12], with \(G_D^{-1} = (e^2/\hbar)(4\pi\sigma a)^{-1}\) for 3D spherical islands of radii \(a\), and \(G_D^{-1}(\omega) = (4\pi g)^{-1} \ln(D/a^2\omega)\), for 2D islands of radii \(a\). Here \(\sigma\) is the 3D conductivity of the metal matrix, \(g = 2\nu D \gg 1\) is the dimensionless film conductance per square, \(E_C = 2e^2/C\) is the bare charging energy of an island and \(\nu\) is the metal density of states per one spin projection.

Equation (3) is valid for \(T \gg \max(T_c(n_i), \tilde{E}_C)\), where \(\tilde{E}_C \propto E_C e^{-\pi G_A/4}\) is the renormalized charging energy (see below). In this temperature range the dephasing rate \(1/\tau_{\varphi}(T) \propto (T/\tilde{E}_C)\) [see Eq. (3)] vanishes at \(T \rightarrow 0\) in accordance with the general statement of Ref. [2].

In three dimensions we can also study the limit \(T \ll \tilde{E}_C\) available at \(n_i \ll n_c\), where macroscopic superconductivity never occurs due to quantum fluctuations. Here, the dephasing rate \(1/\tau_{\varphi}(T) \propto (T/\tilde{E}_C)\) [see Eq. (3)] vanishes at \(T \rightarrow 0\) in accordance with the general statement of Ref. [2].

Below we provide brief derivation of the result (3) and then discuss its physical origin and implications for observable \(\tau_{\varphi}(T)\) in 3D and 2D systems.

### DESCRIPTION OF THE FORMALISM

We start from the action functional \(S = S_D + S_T\) for the disordered metal \((S_D)\) and tunnel junctions with SC islands \((S_T)\), written in the replica form of the imaginary-time \(\sigma\)-model [13, 14, 15]:

\[
S_D = \frac{\pi\nu}{8} \text{Tr} \left[ D(\nabla Q)^2 - 4\gamma_3 E Q \right],
\]

\[
S_T = -\frac{\pi\nu}{8} \sum_j \int dA_j \text{Tr} Q(r_j) Q S_j.
\]

Integration in Eq. (5) goes over the contact areas \(A_j\), and \(\gamma = G_T / A_j\) is the tunnel conductance per unit area. The space- and time-dependent matrix \(Q(r, \tau, \tau')\) describing electron dynamics in the metal is the \(4 \times 4\) matrix in the direct product of the spin space (subscripts \(\alpha, \beta, \ldots\), and Pauli matrices \(\sigma\)) and the particle-hole (PH) space (Pauli matrices \(\tau\)). In general, \(Q\) should also act in the replica space with the number of relics \(N_r \rightarrow 0\). However, for the sake of perturbative calculations which do not involve closed loops of diffusive modes we can safely set \(N_r = 1\) thus omitting the redundant replica space. The \(Q\)-matrix obeys the constraint \(Q^2 = \mathbb{1}\) and the symmetry condition \(Q = \tau_2 Q^T \tau_2\). The usual Green functions of disordered metal correspond to the stationary uniform saddle-point \(\Lambda\) of the action \(S_D\) [written in the energy representation, with \(E_m = \pi T (2m + 1)\)]:

\[
\Lambda_{\alpha\beta}(m, n) = \delta_{\alpha\beta} \delta_{mn} \text{sign}(E_m) \tau_3.
\]

Equation (6) contains the superconductive matrix \(Q S_j\) of the \(j\)-th island:

\[
Q S_j(\tau) = \frac{1}{2} \begin{pmatrix} 0 & \sigma_2 e^{i\varphi_j(\tau)} \\ \sigma_2 e^{-i\varphi_j(\tau)} & 0 \end{pmatrix}.
\]

Diffusion modes of the disordered metal are accounted for by the \(Q\)-matrix fluctuations near the saddle-point \(\Lambda\). They can be parametrized as

\[
Q = \Lambda \left[ 1 + W + \frac{1}{2} W^2 + c_3 W^3 + c_4 W^4 + \ldots \right],
\]

in terms of the antithermien matrix \(W\) obeying the constraint \(\{A, W\} = 0\), and \(c_4 = c_3 - 1/8\). In the PH space the matrix \(W\) is given by

\[
W = \begin{pmatrix} d & c \\ -c & -d \end{pmatrix},
\]

with \(d = -d^T\) and \(c = c^T\) describing diffuson and cooperon modes, respectively. These matrices acting in the spin and Matsubara spaces are nonzero only if \(\varepsilon_m, \varepsilon_n < 0\) (diffusons) and \(\varepsilon_m, \varepsilon_n > 0\) (cooperons). Their bare propagators have the form:

\[
\langle d_{\alpha\beta}(m, n) r d^*_{\alpha\beta}(m, n, r') \rangle = \frac{2}{\pi \nu} D(r, r', \varepsilon_m, \varepsilon_n),
\]

\[
\langle c_{\alpha\beta}(m, n) r c^*_{\alpha\beta}(m, n, r') \rangle = \frac{2}{\pi \nu} C(r, r', \varepsilon_m, \varepsilon_n).
\]

**FIG. 1:** Schematic \((n_i, T)\) phase diagram of a metal with superconducting grains. The dephasing time \(\tau_{\varphi}^{(0)}\) due to Andreev reflection is shorter than \(\tau_{\varphi}^{(A)}\) in a broad range above \(T_c(n_i)\).
where
\[ D(r, r', \varepsilon_m, \varepsilon_n) = \theta(-\varepsilon_m \varepsilon_n) D_0(r, r', \varepsilon_m - \varepsilon_n), \quad (12a) \]
\[ C(r, r', \varepsilon_m, \varepsilon_n) = \theta(\varepsilon_m \varepsilon_n) D_0(r, r', \varepsilon_m + \varepsilon_n), \quad (12b) \]
and \( D_0(r, r', \omega) \) is the Green function of the diffusion operator:
\[ (-D \nabla^2 + |\omega|) D_0(r, r', \omega) = \delta(r - r') \quad (13) \]
with the boundary condition \( \nabla_n D_0(r, r', \omega) = 0 \) at the NS interface.

**DYNAMICS OF THE PHASE**

Integration over cooperon modes in the Gaussian approximation yields the action functional that describes phase dynamics of the array [13, 14]. For a single island, this action is of the form (\( \omega_k = 2\pi \nu k \)):
\[ S_A = T \sum_k \left[ \frac{\omega_k^2 |\omega_k|^2}{4E_C} + \frac{|\omega_k| G_A(\omega_k)}{8} (e^{i\varphi_k} - e^{i\varphi_k}) \right], \quad (14) \]
where \( E_C = 2e^2/C \) is the bare charging energy, with \( C \) being the total island capacitance, and the Andreev conductance \( G_A(\omega) \) is given by Eq. (3) (here we neglect the interaction-induced corrections to \( G_A \) studied in Refs. [3, 14]).

The action (14) had been studied extensively starting from the pioneering paper [12] (cf. Ref. [17] and references therein). At low enough frequencies, \( \omega \ll \Omega_0 \), where \( \Omega_0 = G_A(\Omega_0) E_C \), only the second term in Eq. (14) is relevant and the theory becomes logarithmic provided that \( G_A(\Omega_0) \gg 1 \). The latter condition which prohibits the Coulomb blocking of tunneling will be assumed hereafter.

Phase dynamics can be characterized by the imaginary-time phase autocorrelation function \( \Pi_M(\tau) = \langle e^{i\varphi(x, t)} e^{i\varphi(0,0)} \rangle \). This correlator decays at the time scale \( h/E_C \), where \( E_C \) is the renormalized effective charging energy, which is exponentially small in the considered regime of weak Coulomb blockade. For \( \omega \)-independent \( G_A(\omega) \) (corresponding to the 3D situation), the most detailed of existing estimates for \( E_C \) was found in Ref. [17] using the two-loop renormalization group (RG) together with the instanton analysis:
\[ E_C \approx \frac{E_C}{3\pi^2} \left( \frac{\pi G_A}{2} \right)^4 \exp \left( -\frac{\pi G_A}{4} \right). \quad (15) \]
At \( T \gg E_C \) the deviation of the autocorrelation function \( \Pi_M(\tau) \) from 1 can be determined by means of RG; in the one-loop approximation [valid at \( \Pi_M(\tau) \gg 1/G_A \)] the result is [14]:
\[ \Pi_M(\tau) = 1 - \frac{4}{\pi G_A} \ln \left( \frac{G_A E_C}{2\pi^2 h} \right). \quad (16) \]

In the 2D case, \( G_A(\omega) \propto \ln \omega \) which leads to an extremely slow \((\ln \ln \tau) \) correction to \( \Pi_M(\tau) \) and, hence, to negligibly small \( E_C \). To find \( E_C \) one then should take into account that the simple formula (3) is modified in the lowest-frequency limit due to i) Cooper-channel repulsion in the normal metal, and ii) breakdown of the lowest-order tunneling approximation, both these effects were considered in [11]. Below in this paper we assume (for the 2D case) that temperatures are not too low and approximation (3) is valid.

**PHASE TRANSITION**

**Thermal transition**

The temperature \( T_c(n_i) \) of the thermal superconducting transition is determined by the mean-field relation [7]
\[ T_c = \mathcal{J}(T_c)/2, \quad \mathcal{J}(T) = \sum_i E_J(r_i, T), \quad (17) \]
where \( E_J(r, T) \) is the \( (T \)-dependent) energy of proximity-induced Josephson coupling between two SC islands at the distance \( r \) in \( d \) dimensions:
\[ E_J(r, T) = \frac{G_T^{2}}{8\pi\nu E_C^{2}} \sum_{n=0}^{\infty} P_{d} \left( \frac{r}{\xi T} \sqrt{2n+1} \right), \quad (18) \]
where \( \xi_T = \sqrt{\hbar D / 2\pi T} \) is the thermal length, and we denoted \( P_3(x) = \exp(-x) \) and \( P_2(x) = K_0(x) \). Equation (17) is valid if the number of relevant terms in the sum for \( \mathcal{J}(T_c) \) is large, otherwise the transition is not of the mean-field type, but Eq. (17) can still serve as an estimate for \( T_c \).

The nature of the transition in \( d \) dimensions is determined by the parameter \( \delta_2 \):
\[ \delta_2 = \frac{G_T^2}{8\nu D} = \frac{3\pi^2 G_T^2}{4(k_Fl)(k_Fb)} = \frac{3T^2(k_Fa)^4}{4(k_Fl)(k_Fb)}, \quad (19a) \]
\[ \delta_2 = \frac{G_T^2}{8\nu D} = \frac{G_T^2}{4g^2}, \quad (19b) \]
which is an estimate for \( E_J(b, T)/T \) at \( T = hD/2\pi b^2 \), and \( b = n_i^{-1/d} \) is the typical distance between the islands. In Eq. (19a), we expressed \( G_T = \Gamma k_F^2 A_j/4\pi^2 \) through the characteristic transmission coefficient \( \Gamma \ll 1 \) of the S-I-N tunnel barrier.

In three dimensions the parameter \( \delta_3 \) can be arbitrary compared to 1. However, in two dimensions the parameter \( \delta_2 \) is bounded from below by the requirement of weak Coulomb blockade: \( G_A(\Omega_0) = (\delta_2/\pi) \ln(l/d) \gg 1 \), where we estimated the island’s capacity as \( C \sim a^2/d_1 \), with \( d_1 \) being the width of the insulating barrier. This condition requires \( \delta_2 \gg 1 \). Otherwise the transition is driven by quantum fluctuations and occurs at \( E_C \sim \mathcal{J} \).
If $\delta_d \ll 1$ then $T_c \ll D/2\pi b^2$, the Josephson coupling is long-range and the mean-field equation (17) gives for the transition temperature:

$$T_c = \frac{G_T^2 n_i}{16\nu} \ln \frac{1}{\delta_d} = \frac{hD}{2\pi b^2} \pi \delta_d \ln \frac{1}{\delta_d}, \quad \delta_d \ll 1. \quad (20)$$

If $\delta_d \gg 1$ then $T_c \gg hD/2\pi b^2$ and the Josephson coupling is short-range. The transition temperature can be estimated as

$$T_c = \frac{hDn_i^{2/d}}{2\pi} \ln^2 \delta_d = \frac{hD}{2\pi b^2} \ln^2 \delta_d, \quad \delta_d \gg 1. \quad (21)$$

### Quantum transition

Quantum transition can be described within the lowest tunneling approximation only in three dimensions (cf. [11] for discussion of quantum phase transition in a more complicated 2D case). Upon decreasing $n_i$, the transition temperature defined by Eq. (20) lowers eventually below $E_C$, then quantum fluctuations should be taken into account. At some critical concentration $n_c$ the temperature of the superconducting phase transition vanishes, marking the point of a quantum phase transition. The point of the quantum transition is determined by the equation similar to (17): $E_C = J(0)$ (cf. [7] for more details). However, the zero-temperature value of the integrated Josephson proximity coupling $J(0)$ cannot be determined by the simple formula (15) due to logarithmic divergency of the resulting expression. This divergency is cured by the account of the Cooper-channel repulsion constant in the metal $\lambda_n$ [8], leading to $J(0) = G_T^2 n_i/16\nu \lambda_n$. As a result, the critical concentration $n_c$ is found to be

$$n_c = \frac{16\pi \nu \lambda_n E_C}{G_T^2}, \quad (22)$$

where $E_C$ is defined in Eq. (18).

**COOPERON SELF-ENERGY**

In the presence of SC islands, cooperon modes are no longer gapless. To obtain the cooperon self-energy due to Andreev reflection we calculate the correction to the action in the lowest tunnel approximation:

$$\delta S = -\frac{(S_T^{(2)} S_T^{(2)})}{2} - \frac{(S_T^{(3)} S_T^{(4)})}{2} + \frac{(S_T^{(4)} S_T^{(1)} S_T^{(1)})}{2}, \quad (23)$$

where the vertices $S_T^{(i)}$ and $S_T^{(j)}$ come from expansion of the actions [9] and [10], respectively, to the order $W^i$. The second order in $G_T$ approximation (23) is valid provided that $G_T \ll G_D$ [12, 13]. The corresponding diagrams are shown in Fig. 2. Their sum is independent of the certain form of the parametrization [9]. Averaging in Eq. (23) goes over phase $\varphi_j(\tau)$ dynamics and bare diffusive modes [11]. It is important that at $T \gg T_c$ the phases on different islands are uncorrelated with each other. Upon averaging, one obtains the cooperon part of the induced action (23), which in the long-wavelength limit takes the form:

$$\delta S_C = -\frac{\pi \nu}{4} T^2 \sum_{m \neq n} \int dr \Sigma_{mn} |c_{\alpha\beta}(m, n, r)|^2, \quad (24)$$

where $c$ is the cooperon part of the matrix $W$, Eq. (10), and

$$\Sigma_{mn} \equiv \frac{n_i G_T^2}{16\nu^2} T \sum_k \frac{dA dA'}{A^2} [D(r, r'; m, n - k) - C(r, r'; m, n - k)] \Pi_M(k) + \{m \leftrightarrow -n\}, \quad (25)$$

and $\Pi_M(k)$ is an imaginary-frequency version of the autocorrelation function $\Pi_M(\tau)$. Equation (24) is valid provided that the cooperon wave vector $q$ is smaller than the inverse separation between the islands, $q \ll n_i^{1/4}$, which allows to pass from the discrete sum over the islands to the uniform integration over $r$. The self-energy $\Sigma_{mn}$ determines the low-$q$ behavior of the cooperon: $C(q, m, n) = (Dq^2 + |\epsilon_m + \epsilon_n| - \Sigma_{mn})^{-1}$.

Integrating diffusive modes over the area $A$ of the contacts yields the normal-metal resistance $G_D^{-1}$ which combines with $G_T^{-1}$ into the Andreev conductance. After simple algebra we obtain for $\epsilon_m, \epsilon_n > 0$:

$$\Sigma_{mn} = -\frac{n_i G_A}{8\nu} T \left[ \sum_{k=-m}^{m} \Pi_M(k) + \sum_{k=-n}^{n} \Pi_M(k) \right] \quad (26)$$

which is written for the case of $\omega$-independent $G_A$. Analytical continuation of Eq. (26) from $\epsilon_m, \epsilon_n > 0$ to real frequencies, $i\epsilon_m \to \epsilon + i0^+$, $i\epsilon_n \to \epsilon' + i0^+$, yields the cooperon self-energy

$$\Sigma(\epsilon, \epsilon') = -\frac{n_i G_A}{8\nu} \int_{-\infty}^{\infty} d\Omega \left\{ \Pi^K(\Omega) + \Pi^R(\Omega) F(\epsilon - \Omega) + \Pi^A(\Omega) F(\epsilon' + \Omega) \right\} \quad (27)$$

in terms of the Keldysh, retarded and advanced components $\Pi^{K,R,A}(\Omega)$ of the phase correlation function, and $F(\Omega) = \tanh(\Omega/2T)$.
To study the quantum corrections to conductivity at zero frequency we set $\varepsilon' = -\varepsilon$ leading to the cooperon decay rate $\gamma(\varepsilon) = -\hbar^{-1}\Sigma(\varepsilon, -\varepsilon)$:

$$\gamma(\varepsilon) = \frac{n_i G_A}{2\hbar
u} T \coth \frac{\varepsilon}{2T} \int_0^\infty \Pi(t) \frac{\sin \frac{\varepsilon t}{2T}}{\sinh \frac{\varepsilon t}{2T}} dt. \tag{28}$$

where we used the identity $\Pi^K(\Omega) = -2i\Pi(\Omega)$, where $\Pi(\Omega)$ is the Fourier-transform of the real-time symmetrized autocorrelation function of the island’s order parameter $\Pi(t) = \langle \cos[\varphi(t) - \varphi(0)] \rangle$. Another useful representation for $\gamma(\varepsilon)$ follows from Eq. (28) by means of the inverse Fourier transformation into the time domain:

$$\gamma(\varepsilon) = \frac{n_i G_A}{4\hbar
u} \int_0^\infty \frac{d\Omega}{2\pi} \Pi(\Omega) \{1 - F(\Omega) F(-\varepsilon - \Omega)\}. \tag{29}$$

It is interesting to note that the functional form of Eq. (29) coincides exactly with the expression for the tunneling density of states in the presence of the Coulomb zero-bias anomaly, cf. Eq. (58) of Ref. [19]. In the present case the island’s phase $\varphi(t)$ plays the role of the Coulomb-induced phase $K(t)$ introduced in [13], whose expression gives rise to the zero-bias anomaly: Then expression (29) can be rationalized with simple physical interpretation: “superconductive” contribution to the cooperon decay rate is just the average rate of Andreev processes which occur in the system. Indeed, quantum correction to conductivity comes from interference between different trajectories of the same electron; Andreev reflection transforms this electron into a hole, therefore destroying further interference.

**DEPHASING TIME**

Now we start to analyze the consequences of the result [20]. To evaluate the islands’ contribution into the dephasing rate, we need $\gamma(\varepsilon \approx T)$. Behavior of $\Pi(t \sim \hbar/T)$ is governed by the relation between temperature $T$ and the effective charging energy $\tilde{E}_C$ of SC islands.

**3D case**

**Moderately high temperatures, $T \geq \tilde{E}_C$**

At $T \geq \tilde{E}_C$, the integral in Eq. (29) converges at $t \sim \hbar/T$ where $\Pi(t)$ is given by Eq. (19). As a result, $\gamma(\varepsilon)$ is nearly energy-independent at $\varepsilon \sim T$ and can be identified with the dephasing rate leading to the 3D result [3]. The latter is valid as long as the expression in the brackets is large compared to unity.

Assuming that $\tau_\varphi^{(0)}(T)$ is given by Eq. (11), we can estimate the upper boundary $T^\text{3D}_\varphi(n_i)$ of the temperature range where $1/\tau_\varphi^A$ is the main contribution to the dephasing rate. Using the 3D expression for $G_A$ one finds

$$T^\text{3D}_\varphi(n_i) \sim \frac{\hbar}{2\pi} x_i^{2/3} (\Gamma k_F l)^{4/3}, \tag{30}$$

where $x_i = (4\pi/3)a^3 n_i$ is the volume fraction of the superconductive material in the matrix. From the low-temperature side applicability of the 3D result $[3]$ is limited by the thermal transition temperature $T^\text{3D}_c(n_i)$. Thus the relative width of the temperature window where Andreev reflection from the SC islands is the leading mechanism of dephasing is given by the ratio

$$\frac{T^\text{3D}_\varphi(n_i)}{T^\text{3D}_c(n_i)} \approx \begin{cases} \frac{500 G_A^{3/2}}{\ln^2(n_i/n_0)} & n_i \gg n_0, \\ \frac{50 G_A^{3/2}(n_0/n_i)^{1/3}}{\ln(n_0/n_i)} & n_i \ll n_0, \end{cases} \tag{31}$$

where we used Eqs. (13), (20) and (21), and defined $n_0 = (8\nu hD/G_\varphi^2)^3$ such that $\delta_3 = (n_i/n_0)^{1/3}$. Large factors in Eq. (31) result partly from the large factor in Eq. (30) hidden there in $x_i$ and $\Gamma$, and partly from writing $\ln \delta_3 = (1/3) \ln(n_i/n_0)$. We see that the condition $G_A \gg 1$ guarantees the existence of the broad temperature range where the dephasing time is nearly temperature independent and given by $\tau_\varphi^A$.

**Lowest temperatures, $T \ll \tilde{E}_C$**

The region of very low temperatures, $T \ll \tilde{E}_C$, can be traced only at very small concentration of the island, $n_i < n_c$ [cf. Eq. (22)], where superconductivity is absent even at $T = 0$ due to quantum fluctuations. Here the integral (29) converges at $t \sim \hbar/\tilde{E}_C$ and can be approximated as

$$\gamma(\varepsilon) = \frac{n_i G_A}{2\pi \nu} \coth \frac{\varepsilon}{2T} \int_0^\infty \Pi(t) dt. \tag{32}$$

The above integral is of the order of $\tilde{E}_C^{-1}$. Then the Andreev-reflection contribution to the dephasing rate can be estimated as

$$\frac{1}{\tau_\varphi^A(T)} \sim \frac{n_i}{2\pi \hbar \nu} \frac{T}{\tilde{E}_C}. \tag{33}$$

Since $1/\tau_\varphi^A$ scales $\propto T$ it always dominates the standard 3D result [11] at very low temperatures. However, the crossover temperature, where $\tau_\varphi^A = \tau_\varphi^{(0)}$, scales as $n_i^2$ and can be extremely low for small concentration of the islands.
2D case

As explained above, staying within the lowest tunneling approximation we can explore only the region of relatively high temperatures, \((G_T/4\pi g)\ln(hD/\alpha^2T) \ll 1\), where fluctuations are thermal. Substituting \(\Pi(t)\) by 1 in Eq. (21) we come to the result (4) for the 2D case. Here, contrary to the 3D case one can neglect the one-loop fluctuation correction \(\propto \ln \ln T\) compared to the bare \(\ln T\) dependence of \(G_A\).

Comparing with Eq. (1) one finds that the “superconductive” contribution to dephasing is dominant at \(T \leq T^{2D}_\star(n_i)\), where

\[ T^{2D}_\star(n_i) = \pi h D a G_A(T^{2D}_\star(n_i)) / (\ln(\pi g)). \tag{34} \]

The relative width of this window is then estimated by the ratio

\[ r^{2D}_\star(n_i) \approx 20 G_A(T_\star) / \ln(\pi g) \ln^2(G^2_c/4g) \tag{35} \]

and is large since \(G_A \gg 1\).

MAGNETORESISTANCE

Experimentally, \(\tau_\varphi\) is determined from the magnetoresistance data. For 2D systems, the low-field magnetoresistance is governed by the weak localization (WL) and Maki-Tompson (MT) corrections which have the same dependence on the magnetic field \[20]:

\[ \Delta R(H) / R^2 = -e^2 / 2\pi^2 h \left[ \alpha - \beta(T) \right] Y \left( 4D c H \tau_\varphi / hc \right), \tag{36} \]

with \(Y(x) = \ln(x) + \psi(1/2 + 1/x)\). Here \(\alpha = 1\) \((-1/2\) is the WL contribution in the limit of weak (strong) spin-orbit interaction, while the MT contribution is characterized by the function \(\beta(T)\) expressed through the Cooper channel interaction amplitude \(\Gamma(\omega_n)\) \[20]:

\[ \beta(T) = \frac{\pi^2}{4} \sum_m (-1)^m \Gamma(\omega_m) - \sum_{n \geq 0} \Gamma''(\omega_{2n+1}). \tag{37} \]

In a uniform system far above \(T_\varphi\), \(\beta(T) \sim 1 / \ln^2(T/T_\varphi)\) indicating that the MT contribution is smaller but in general comparable to the WL contribution.

For our system, effective attraction in the Cooper channel emerges as a result of Andreev reflection from the SC islands. Formally, integration over the phases \(\varphi_j(\tau)\) of the islands generates the standard Cooper interaction term in the action with \(\Gamma(\omega_k) = (n_c G^2_c / 16\nu) \Pi(\omega_k)\); where we made use of the fact that the phases of different islands are uncorrelated at \(T \gg T_\varphi\) and performed spacial average justified by the inequality \(\tau_\varphi \gg b\). In the temperature range considered, \(\Pi(\tau)\) is nearly constant on the time interval \(\tau \in [0, 1/T]\), so one can use the static approximation \(\Pi(\omega_k) = \delta_{k, 0}\). Substituting into Eq. (37) we obtain for \(T \gg T_\varphi\)

\[ \beta(T) = \frac{\pi^2}{64} \frac{n_c G^2_c}{\nu T}. \tag{38} \]

Comparing with the estimate (34) one finds that \(\beta(T) \gg 1\) at \(T \ll T_\varphi\), that is magnetoresistance is mainly due to the MT term and thus is positive irrespectively of the strength of the spin-orbit scattering.

Another relevant contribution to magnetoresistance is the Aslamazov-Larkin (AL) correction. In the range \(T \gg T_\varphi\), using the condition \(\delta_2 \gg 1\), one can estimate \(\Delta_{\text{AL}} \lesssim \ln n_i D / T\). Comparing with \(\Delta_{\text{MT}}\) following from Eqs. (36) and (38), one finds \(\Delta_{\text{AL}} / \Delta_{\text{MT}} \sim 1 / \delta_2 \ll 1\). Moreover, the relevant scale of magnetic field \(B_{\text{AL}}\) for the AL contribution to magnetoresistance is \(B_{\text{AL}} \sim cT / \epsilon D,\) i.e., it is by factor \(T \tau_\varphi / \hbar \gg 1\) larger than the corresponding WL scale \(B_{\text{WL}} \sim h c / \epsilon D \tau_\varphi\). Thus AL contribution to magnetoresistance is much smaller than quantum (WL and MT) corrections, and \(\tau_\varphi\) can be extracted from the standard low-field magnetoresistance measurements.

We believe the same conclusion to be valid in the 3D case. Here, however, the MT correction can be either large or small compared to the WL correction, depending on temperature and other parameters of the problem.

DISCUSSION

We demonstrate that small concentration of superconductive islands can enhance considerably the low-temperature dephasing rate in disordered bulk and thin-film metals as seen via the low-field magnetoresistance. In 2D the dominant quantum correction to magnetoresistance is the Maki-Tompson one, thus magnetoresistance is positive in the region of interest. Throughout the whole range where our results are valid, \(T \tau_\varphi / \hbar \gg 1\), which allows to neglect the energy dependence of the cooperon decay rate (20). This is why magnetoresistance follows the standard formula (20) derived for uniform metal films.

It was implicitly assumed while deriving Eq. (20) that \(L_\varphi = \sqrt{D \tau_\varphi}\) is much longer than inter-island separation \(b\). Using Eq. (3) one finds that in 2D case for this condition to be fulfilled the tunnel-limit inequality \(G_T / G_D \ll 1\) is required; for 3D case the condition \(L_\varphi \gg b\) is less restrictive. We expect that in the 2D case with SC islands strongly \((G_T \gg G_D)\) coupled to the film (11) the “Andreev” contribution to the dephasing rate at moderate temperatures can be estimated analogously to Eq. (3), with the proper expression \(G_A \approx G_D\) for the Andreev conductance, leading to \(1 / \tau_\varphi\sim n_i D / \ln(\xi T / a)\). Although we considered temperatures much below the intrinsic transition temperature of SC islands \(T_{\text{c0}}\), our approach can be adapted for \(T \sim T_{\text{c0}}\).
We note in passing that inhomogeneous in space superconducting gap function is known to affect the BSC peak in the density of states in a way very similar to that of magnetic impurities [21]. The present results show that analogy between inhomogeneous superconductivity and magnetic impurities extends to dephasing as well. The influence of the same dephasing mechanism upon other phase-coherent phenomena (e.g., mesoscopic fluctuations and persistent currents) remains to be studied.

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[1] B. L. Altshuler, A. G. Aronov and D. E. Khmelnitsky, J. Phys. C 15, 7367 (1982).
[2] Y. Imry, cond-mat/0202044
[3] Y. Imry, H. Fukuyama, and P. Schwab, Europhys. Lett. 47, 608 (1999).
[4] A. Zawadowski, Jan von Delft, and D. C. Ralph, Phys. Rev. Lett. 83, 2632 (1999).
[5] V. E. Kravtsov and B. L. Altshuler, Phys. Rev. Lett. 84, 3394 (2000).
[6] I. L. Aleiner, B. L. Altshuler and M. E. Gershenzon, Waves in Random Media, 9, 201 (1999).
[7] M. V. Feigel’man and A. I. Larkin, Chem. Phys. 235, 107 (1998).
[8] L. A. Aslamazov, A. I. Larkin and Yu. N. Ovchinnikov, ZhETF 55, 323 (1968) [Sov. Phys. JETP 28, 171 (1969)].
[9] J. M. Gordon, C. J. Lobb and M. Tinkham, Phys. Rev. B 29, 5232 (1984); J. M. Gordon and A. M. Goldman, Phys. Rev. B 34, 1500 (1986).
[10] B. R. Patton, Phys. Rev. Lett. 27, 1273 (1971); J. Keller and V. Korenman, Phys. Rev. B 5, 4367 (1972); W. Breining, M. C. Chang, E. Abrahams, and P. Wolfe, Phys. Rev. B 31, 7001 (1985).
[11] M. V. Feigel’man, A. I. Larkin, and M. A. Skvortsov, Phys. Rev. Lett. 86, 1869 (2001); Usp. Fiz. Nauk (Suppl.) 171, 99 (2001).
[12] Yu. V. Nazarov, Phys. Rev. Lett. 73, 1420 (1994).
[13] A. M. Finkelstein, Electron Liquid in Disordered Conductors, Vol. 14 of Soviet Scientific Reviews, edited by I. M. Khalatnikov (Harwood Academic, London, 1990).
[14] Y. Oreg, P. Brouwer, B. Simons, and A. Altland, Phys. Rev. Lett. 82, 1269 (1999).
[15] M. V. Feigelman, A. Kamenev, A. I. Larkin, and M. A. Skvortsov, Phys. Rev. B 66, 054502 (2002).
[16] J. M. Kosterlitz, Phys. Rev. Lett. 37, 1577 (1976).
[17] I. S. Beloborodov, A. V. Andreev and A. I. Larkin, Phys. Rev. B 68, 024202 (2003).
[18] M. A. Skvortsov, A. I. Larkin, and M. V. Feigel’man, Phys. Rev. B 63, 134507 (2001).
[19] A. Kamenev and A. Andreev, Phys. Rev. B 60, 2218 (1999).
[20] A. I. Larkin, Pis’ma v ZhETF 31, 239 (1980) [JETP Letters 31, 219 (1980)].
[21] A. I. Larkin and Yu. N. Ovchinnikov, ZhETF 61, 2147 (1971) [Sov. Phys. JETP 34, 1144 (1972)].