PROPERTIES OF GRADIENT MAPS ASSOCIATED WITH ACTION OF REAL REDUCTIVE GROUP

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Abstract. Let \((Z, \omega)\) be a Kähler manifold and let \(U\) be a compact connected Lie group with Lie algebra \(u\) acting on \(Z\) and preserving \(\omega\). We assume that the \(U\)-action extends holomorphically to an action of the complexified group \(U^C\) and the \(U\)-action on \(Z\) is Hamiltonian. Then there exists a \(U\)-equivariant momentum map \(\mu : Z \to u\). If \(G \subset U^C\) is a closed subgroup such that the Cartan decomposition \(U^C = U\exp(iu)\) induces a Cartan decomposition \(G = K\exp(p)\), where \(K = U \cap G\), \(p = g \cap iu\) and \(g = \mathfrak{k} \oplus p\) is the Lie algebra of \(G\), there is a corresponding gradient map \(\mu_p : Z \to p\). If \(X\) is a \(G\)-invariant compact and connected real submanifold of \(Z\), we may consider \(\mu_p\) as a mapping \(\mu_p : X \to p\). Given an \(\text{Ad}(K)\)-invariant scalar product on \(p\), we obtain a Morse-like function \(f = \frac{1}{2} \| \mu_p \|^2\) on \(X\). We point out that, without the assumption that \(X\) is real analytic manifold, the Lojasiewicz gradient inequality holds for \(f\). Therefore the limit of the negative gradient flow of \(f\) exists and it is unique. Moreover, we prove that any \(G\)-orbit collapses to a single \(K\)-orbit and two critical points of \(f\) which are in the same \(G\)-orbit belong to the same \(K\)-orbit. We also investigate convexity properties of the gradient map \(\mu_p\) in the Abelian cases. In particular, we study two orbits variety \(X\) and we investigate topological and cohomological properties of \(X\).

2010 Mathematics Subject Classification. 57S20; 32M05.

Key words and phrases. Cartan decomposition, Hamiltonian action, Momentum map, Norm square, Two orbit variety.

The first author was partially supported by the Project PRIN 2015, “Real and Complex Manifolds: Geometry, Topology and Harmonic Analysis”, Project PRIN 2017 “Real and Complex Manifolds: Topology, Geometry and holomorphic dynamics” and by GNSAGA INdAM.
1. Introduction

In this paper, we study the actions of real reductive Lie groups on real submanifolds of Kähler manifolds.

Let $U$ be a compact connected Lie group with Lie algebra $\mathfrak{u}$ and let $U^\mathbb{C}$ be its complexification. We say that a subgroup $G$ of $U^\mathbb{C}$ is compatible if $G$ is closed and the map $K \times \mathfrak{p} \to G$, $(k, \beta) \mapsto k \exp(\beta)$ is a diffeomorphism where $K := G \cap U$ and $\mathfrak{p} := \mathfrak{g} \cap i\mathfrak{u}$; $\mathfrak{g}$ is the Lie algebra of $G$. The Lie algebra $\mathfrak{u}^\mathbb{C}$ of $U^\mathbb{C}$ is the direct sum $\mathfrak{u} \oplus i\mathfrak{u}$. It follows that $G$ is compatible with the Cartan decomposition $U^\mathbb{C} = U \exp(i\mathfrak{u})$ (see Section 2.2), $K$ is a maximal compact subgroup of $G$ with Lie algebra $\mathfrak{k}$ and that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

Let $(Z, \omega)$ be a Kähler manifold with an holomorphic action of the complex reductive group $U^\mathbb{C}$. We also assume $\omega$ is $U$-invariant and that there is a $U$-equivariant moment map $\mu : Z \to \mathfrak{u}^\ast$. By definition, for any $\xi \in \mathfrak{u}$ and $z \in Z$, $d\mu^\xi = i_{\xi^Z} \omega$, where $\mu^\xi(z) := \mu(z)(\xi)$, and $\xi^Z$ denotes the
fundamental vector field induced on $Z$ by the action of $U$,

$$\xi_Z(z) := \frac{d}{dt} \bigg|_{t=0} \exp(t\xi) \cdot z.$$ 

The inclusion $i_p \hookrightarrow u$ induces by restriction, a $K$-equivariant map $\mu_p : Z \to (ip)^*$. Using an $\text{Ad}(U)$-invariant inner product on $u$, we can view $\mu_p$ as a map $\mu_p : Z \to p$. For $\beta \in p$ let $\mu^\beta_p$ denote $-i\beta$. i.e., $\mu^\beta_p(z) = -\langle \mu(z), i\beta \rangle$. Then $\text{grad}\mu^\beta_p = \beta_Z$, where $\text{grad}$ is computed with respect to the Riemannian metric induced by the Kähler structure. The map $\mu_p$ is called the gradient map associated with $\mu$ (see Section 2.3). For a $G$-stable locally closed real submanifold $X$ of $Z$, we consider $\mu_p$ as a mapping $\mu_p : X \to p$. Using the inner product on $p \subset iu$, we define the norm square of $\mu_p$ by

$$f(x) := \frac{1}{2} \| \mu_p(x) \|^2; \quad x \in X.$$ 

The set of semistable points associated with the critical points of the norm square of $\mu_p$ was studied in great details in [21]. The norm square of $\mu_p$ is in general far from being Morse-Bott and it’s critical sets may be very complicated. But can we have a particular case when the norm square will be Morse-Bott? We find this question to be true for two orbits variety. From now on, we always assume that $X$ is connected and compact.

Suppose that the action of $G$ on $X$ has two orbits. $X$ is called a two orbit variety. S. Cupit-Foutou obtained the classification of a complex algebraic varieties on which a reductive complex algebraic group acts with two orbits [13]. Applying standard Morse-theoretic results in [28] and [21], we prove that the norm square is Morse-Bott and obtain information on the cohomology and $K$-equivariant cohomology of $X$ (Theorem 6.1), generalizing [17].

A central ingredient to prove this result is the Ness Uniqueness Theorem which asserts that any two critical points of $f$ in the same $G$-orbit in fact belong to the same $K$-orbit (Theorem 4.6). Moreover, although we do not assume that $X$ is real analytic manifold, we point out that for any $G$-invariant compact and connected submanifold of $Z$, the Lojasiewicz gradient inequality holds for the norm square. Therefore, the limit of the negative gradient flow exists and it is unique and any $G$-orbit collapses to a single $K$-orbit (Theorem 4.9). We use the original ideas from [15] in a different context. By the stratification theorem, we have

$$\{ p \in X : G \cdot p \cap \mu_p^{-1}(0) \neq \emptyset \} = \{ p \in X : \lim_{t \to +\infty} \varphi_t(p) \in \mu_p^{-1}(0) \} = S_G(\mu_p^{-1}(0)).$$
where $\varphi_t(p)$ is the flow of the vector field $-\text{grad} f$. Then, there exist a $K$-equivariant strong deformation of $S_G(\mu_p^{-1}(0))$ onto the set $\mu_p^{-1}(0)$ (Theorem 3.10). Hence no analyticity assumption is necessary in the statement of the retraction Theorem answering Question 1 in [23, p.219].

Biliotti and Ghigi [8] proved a convexity theorem along orbits in a very general setting using only so-called Kempf-Ness function. The behaviour of the corresponding gradient map is encoded in the Kempf-Ness function. Recently, Biliotti [9] gives a new proof of the Hilbert-Mumford criterion for real reductive Lie groups stressing the properties of the Kempf-Ness functions. He shows that the Kempf-Ness function is Morse-Bott and it is convex along geodesics for the action of a linear group on $\mathbb{P}(V)$ where $V$ is a finite dimensional vector space. We prove this result in a general setting.

Results on convexity theorems are obtained. Let $\mathfrak{a} \subset \mathfrak{p}$ be an Abelian subalgebra. Let $\mu_\mathfrak{a} : X \to \mathfrak{a}$ denote the gradient map of $A = \exp(\mathfrak{a})$. If $\pi_\mathfrak{a} : \mathfrak{p} \to \mathfrak{a}$ is the orthogonal projection, then $\mu_\mathfrak{a} = \pi_\mathfrak{a} \circ \mu_\mathfrak{p}$. Although there is no counterexample, we do not know if the Abelian Convexity Theorem holds for any $G$-invariant connected submanifold (see for instance [1, 8, 22] for more details on the subject). If $G$ has a unique closed orbit $\mathcal{O}$, then we prove that $\mu_\mathfrak{a}(X) = \mu_\mathfrak{a}(\mathcal{O})$ and so a polytope. This result is new also if $G = U^C$ and $X = Z$. This means that $\mathcal{O}$ captures all the information of the $A$-gradient map. As an application, we prove that the Abelian convexity Theorem holds for a two orbits variety.

If $Z$ is connected and compact and $X \subset Z$ is a $A$-stable compact, connected coisotropic submanifold of $Z$, then we prove $\mu_\mathfrak{a}(X) = \mu_\mathfrak{a}(Z)$ (Theorem 5.10) and so it is a polytope as well. More precisely, there exists an open and dense subset $W$ of $X$ such that for any $p \in W$, we have $\mu_\mathfrak{a}(X) = \mu_\mathfrak{a}(A \cdot p)$.

2. Preliminaries

2.1. Convex geometry. In this section, some definitions and results in convex geometry are recalled. The reader can see e.g. [35] and [3] for further details on the topic.

Let $V$ be a real vector space with a scalar product $\langle \cdot, \cdot \rangle$ and let $E \subset V$ be a compact convex subset. The relative interior of $E$, denoted by $\text{relint} E$, is the interior of $E$ in its affine hull. For $a, b \in E$, denote the closed segment joining $a$ and $b$ by $[a, b]$. Then, a face of $E$ is a convex subset $F$ of $E$ such that if $a, b \in E$ and $\text{relint}[a, b] \cap F \neq \emptyset$, then $[a, b] \subset F$. The extreme points of $E$ denoted by $\text{ext} E$ are the points $a \in E$ such that $\{a\}$ is a face. By a Theorem of Minkowski, $E$
is the convex hull of its extreme points \[35\] p.19. The faces of \(E\) are closed \[35\] p. 62. The empty set and \(E\) are faces of \(E\); the other faces are called proper.

**Definition 2.1.** The support function of \(E\) is defined by the function \(h_E : V \to \mathbb{R}, h_E(u) = \max\{\langle x, u \rangle : x \in E\}\). If \(u \neq 0\), the hyperplane \(H(E, u) := \{x \in E : \langle x, u \rangle = h_E(u)\}\) is called the supporting hyperplane of \(E\) for \(u\).

The set

\[
F_u(E) := E \cap H(E, u)
\]

is a face and it is called the exposed face of \(E\) defined by \(u\).

Intuitively, the meaning of the support function is simple. For instance, consider a nonempty closed convex set \(E \subset \mathbb{R}^n\). Then for a unit vector \(u \in S^{n-1} \cap \text{dom} h_E\), the supporting function \(h_E(u)\) is the signed distance of the support plane to \(E\) with exterior normal vector \(u\) from the origin; the distance is negative if and only if \(u\) points into the open half-space containing the origin. In general not all faces of a convex subset are exposed. For instance, consider the convex hull of a closed disc and a point outside the disc: the resulting convex set is the union of the disc and a triangle. The two vertices of the triangle that lie on the boundary of the disc are non-exposed 0-faces.

A subset \(E \subset V\) is called a convex cone if \(E\) is convex, non empty and closed under multiplication by non negative real numbers. The following results about a compact convex set \(E\) and it’s faces are recalled from \[3\]

**Lemma 2.1.** If \(F \subset E\) is an exposed face, the set \(C_F := \{u \in V : F = F_u(E)\}\) is a convex cone. If \(G\) is a compact subgroup of \(O(V)\) that preserves both \(E\) and \(F\), then \(C_F\) contains a fixed point of \(G\).

**Theorem 2.2** \([35\] p. 62\]). If \(E\) is a compact convex set and \(F_1, F_2\) are distinct faces of \(E\), then relint \(F_1 \cap \text{relint} F_2 = \emptyset\). If \(G\) is a nonempty convex subset of \(E\) which is open in its affine hull, then \(G \subset \text{relint} F\) for some face \(F\) of \(E\). Therefore \(E\) is the disjoint union of the relative interiors of its faces.

The next result is a possibly well known but useful fact.

**Proposition 2.3.** Let \(C_1 \subseteq C_2\) be two compact convex set of \(V\). Assume that for any \(\beta \in V\) we have

\[
\max_{y \in C_1} \langle y, \beta \rangle = \max_{y \in C_2} \langle y, \beta \rangle.
\]
Then \( C_1 = C_2 \).

**Proof.** We may assume without loss of generality that the affine hull of \( C_2 \) is \( V \). Assume by contradiction that \( C_1 \subsetneq C_2 \). Since \( C_1 \) and \( C_2 \) are both compact, it follows that there exists \( p \in \partial C_1 \) such that \( p \in C_2 \). Since every face of a convex compact set is contained in an exposed face [35], there exists \( \beta \in V \) such that

\[
\max_{y \in C_1} \langle y, \beta \rangle = \langle p, \beta \rangle.
\]

This means the linear function \( x \mapsto \langle x, \beta \rangle \) restricted on \( C_2 \) achieves its maximum at an interior point which is a contradiction. \( \square \)

### 2.2 Compatible subgroups

In this section, the notion of compatible subgroup is discussed.

Let \( H \) be a Lie group with Lie algebra \( \mathfrak{h} \) and \( E, F \subset \mathfrak{h} \). Then, we set

\[
E^F := \{ \eta \in E : [\eta, \xi] = 0, \forall \xi \in F \}
\]

\[
H^F = \{ g \in H : \text{Ad}(g)(\xi) = \xi, \forall \xi \in F \}.
\]

If \( F = \{ \beta \} \) we write simply \( E^\beta \) and \( H^\beta \).

Let \( U \) be a compact Lie group and let \( U^C \) be its universal complexification in the sense of [26]. The group \( U^C \) is a reductive complex algebraic group with Lie algebra \( \mathfrak{u}^C \) [12]. \( \mathfrak{u}^C \) have the Cartan decomposition

\[
\mathfrak{u}^C = \mathfrak{u} + i\mathfrak{u}
\]

with a conjugation map \( \theta : \mathfrak{u}^C \to \mathfrak{u}^C \) and the corresponding group isomorphism \( \theta : U^C \to U^C \).

Let \( f : U \times i\mathfrak{u} \to U^C \) be the diffeomorphism \( f(g, \xi) = g \exp(\xi) \). The decomposition \( U^C = U \exp(i\mathfrak{u}) \) is referred to as the Cartan decomposition.

Let \( G \subset U^C \) be a closed real subgroup of \( U^C \). We say that \( G \) is compatible with the Cartan decomposition of \( U^C \) if \( f(K \times \mathfrak{p}) = G \) where \( K := G \cap U \) and \( \mathfrak{p} := \mathfrak{g} \cap i\mathfrak{u} \). The restriction of \( f \) to \( K \times \mathfrak{p} \) is then a diffeomorphism onto \( G \). It follows that \( K \) is a maximal compact subgroup of \( G \) and that \( \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \). Note that \( G \) has finitely many connected components. Since \( U \) can be embedded in \( \text{Gl}(N, \mathbb{C}) \) for some \( N \), and any such embedding induces a closed embedding of \( U^C \), any compatible subgroup is a closed linear group. By [30], Proposition 1.59 p.57, \( \mathfrak{g} \) is a real reductive Lie algebra, and so \( \mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{z}[\mathfrak{g}, \mathfrak{g}] \), where \( \mathfrak{z}(\mathfrak{g}) := \{ v \in \mathfrak{g} : [v, \mathfrak{g}] = 0 \} \) is the Lie algebra of the center of \( G \). Denote by \( G_{ss} \) the analytic subgroup tangent to \( [\mathfrak{g}, \mathfrak{g}] \) and by \( Z(G) \) the center of \( G \). Then \( G_{ss} \) is closed and \( G^o = Z(G)^o \cdot G_{ss} \) [30] p. 442, where \( G^o \), respectively \( Z(G)^o \), denotes the connected component of the identity of \( G \), respectively of \( Z(G) \).
Lemma 2.4 ([2, Lemma 7]).

a) If \( G \subset U^C \) is a compatible subgroup, and \( H \subset G \) is closed and \( \theta \)-invariant, then \( H \) is compatible if and only if \( H \) has only finitely many connected components.

b) If \( G \subset U^C \) is a connected compatible subgroup, then \( G_{ss} \) is compatible.

c) If \( G \subset U^C \) is a compatible subgroup, and \( E \subset p \) is any subset, then \( GE \) is compatible.

Indeed, \( GE = K^E \cap \exp(p^E) \), where \( K^E = K \cap G^E \) and \( p^E = \{ v \in p : [v, E] = 0 \} \).

2.3. Parabolic subgroups. Let \( G \subset U^C \) be a compatible, and let \( g = k \oplus p \) be the corresponding decomposition. A subalgebra \( q \subset g \) is parabolic if \( q^C \) is a parabolic subalgebra of \( g^C \). One way to describe the parabolic subalgebras of \( g \) is by means of restricted roots. If \( a \subset p \) is a maximal subalgebra, let \( \Delta(g, a) \) be the (restricted) roots of \( g \) with respect to \( a \), let \( g_\lambda \) denote the root space corresponding to \( \lambda \) and let \( g_0 = m \oplus a \), where \( m = z_k(a) \). Let \( \Pi \subset \Delta(g, a) \) be a base and let \( \Delta_+ \) be the set of positive roots. If \( I \subset \Pi \) set \( \Delta_I := \text{span}(I) \cap \Delta \). Then

\[
q_I := g_0 \oplus \bigoplus_{\lambda \in \Delta_I \cup \Delta_+} g_\lambda
\]

is a parabolic subalgebra. Conversely, if \( q \subset g \) is a parabolic subalgebra, then there are a maximal subalgebra \( a \subset p \) contained in \( q \), a base \( \Pi \subset \Delta(g, a) \) and a subset \( I \subset \Pi \) such that \( q = q_I \). We can further introduce

\[
a_I := \bigcap_{\lambda \in I} \ker \lambda \quad a^I := a_I^\perp
\]

\[
n_I = \bigoplus_{\lambda \in \Delta_+ - \Delta_I} g_\lambda \quad m_I := m \oplus a^I \oplus \bigoplus_{\lambda \in \Delta_I} g_\lambda.
\]

Then \( q_I = m_I \oplus a_I \oplus n_I \). Since \( \theta g_\lambda = g_{-\lambda} \), it follows that \( q_I \cap \theta q_I = a_I \oplus m_I \). This latter Lie algebra coincides with the centralizer of \( a_I \) in \( g \). It is a Levi factor of \( q_I \) and

\[
\alpha_I = z(q_I \cap \theta q_I) \cap p.
\]

Another way to describe parabolic subalgebras of \( g \) is the following. If \( \beta \in p \), the endomorphism \( ad(\beta) \in \text{End} g \) is diagonalizable over \( \mathbb{R} \). Denote by \( V_\lambda(\text{ad}(\beta)) \) the eigenspace of \( \text{ad}(\beta) \) corresponding to the eigenvalue \( \lambda \). Set

\[
g^{\beta +} := \bigoplus_{\lambda \geq 0} V_\lambda(\text{ad}(\beta)).
\]

Lemma 2.5. [2 Lemma 8] For any \( \beta \) in \( p \), \( g^{\beta +} \) is a parabolic subalgebra of \( g \). If \( q \subset g \) is a parabolic subalgebra, there is some vector \( \beta \in p \) such that \( q = g^{\beta +} \). The set of all such vectors is an open convex cone in \( z(q \cap \theta q) \cap p \).
A parabolic subgroup of $G$ is a subgroup of the form $Q = N_G(q)$ where $q$ is a parabolic subalgebra of $g$. Equivalently, a parabolic subgroup of $G$ is a subgroup of the form $P \cap G$ where $P$ is parabolic subgroup of $G^C$ and $p$ is the complexification of a subspace $q \subset g$. If $\beta \in p$ set

$$G^\beta := \{g \in G : \lim_{t \to -\infty} \exp(t\beta)g\exp(-t\beta) \text{ exists}\}$$

$$R^\beta := \{g \in G : \lim_{t \to -\infty} \exp(t\beta)g\exp(-t\beta) = e\} \quad r^\beta := \bigoplus_{\lambda > 0} V_\lambda(\text{ad}(\beta)).$$

$$G^\beta = \{g \in G : \operatorname{Ad}(g)(\beta) = \beta\}$$

Note that $g^\beta = g^\beta \oplus r^\beta$.

Lemma 2.6. [2, Lemma 9] If $G$ is connected, then $G^\beta$ is a parabolic subgroup of $G$ with Lie algebra $g^\beta$. Every parabolic subgroup of $G$ equals $G^\beta$ for some $\beta \in p$. $R^\beta$ is connected and it is the unipotent radical of $G^\beta$ and $G^\beta$ is a Levi factor.

2.4. Gradient map. Let $(Z, \omega)$ be a Kähler manifold. Let $U^C \times Z \to Z$ is holomorphic. Assume that $U$ preserves $\omega$ and that there is a $U$-equivariant momentum map $\mu : Z \to u$. If $\xi \in u$, we denote by $\xi_Z$ the induced vector field on $Z$ and we let $\mu^\xi \in C^\infty(Z)$ be the function $\mu^\xi(z) := \langle \mu(z), \xi \rangle$, where $\langle \cdot, \cdot \rangle$ is an $\operatorname{Ad}(U)$-invariant scalar product on $u^C$. We may also assume that the multiplication by $i$ is an isometry from $u$ onto $iu$ and $\langle u, iu \rangle = 0$ ([5, p.428]. By definition, we have

$$d\mu^\xi = i\xi_Z \omega.$$ 

Let $G \subset U^C$ be a compatible subgroup of $U^C$. For $x \in Z$, let $\mu_p^\beta(x)$ denote $-i$ times the component of $\mu(x)$ along $\beta$ in the direction of $ip$, i.e.,

$$(5) \quad \mu_p^\beta(x) := \langle \mu_p(x), \beta \rangle = \langle i\mu(x), \beta \rangle = \langle \mu(x), -i\beta \rangle = \mu^{-i\beta}(x)$$

for any $\beta \in p$. Then, the map defined by

$$\mu_p : Z \to p$$

is called the $G$ gradient map. Let $(\cdot, \cdot)$ be the Kähler metric associated to $\omega$, i.e. $(v, w) = \omega(v, Jw)$, for all $z \in Z$ and $v, w \in T_zZ$ where $J$ denotes the complex structure on $TZ$. Then $\beta_Z$ is the gradient of $\mu_p^\beta$, where $\beta_Z$ is the vector field on $Z$ corresponding to $\beta$ and the gradient is computed with respect to $(\cdot, \cdot)$. For the rest of this paper, fix a $G$-invariant locally closed submanifold $X$ of $Z$. We denote the restriction of $\mu_p$ to $X$ by $\mu_p$. Then

$$\operatorname{grad} \mu_p^\beta = \beta_X,$$
where grad is now computed with respect to the induced Riemannian metric on $X$. Since $X$ is $G$-stable, $\beta_X = \beta_Z$. Similarly, $\perp$ denotes perpendicularity relative to the Riemannian metric on $X$. We will now recall some of the properties of the gradient map.

**Lemma 2.7.** Let $x \in X$ and let $\beta \in p$. Then either $\beta_X(x) = 0$ or the function $t \mapsto \mu_p^\beta(\exp(t\beta) \cdot x)$ is strictly increasing.

**Proof.** Let $f(t) = \mu_p^\beta(\exp(t\beta) \cdot x) = \langle \mu_p(\exp(t\beta) \cdot x), \beta \rangle$. Then

$$f'(t) = \langle \beta_X(\exp(t\beta) \cdot x), \beta_X(\exp(t\beta) \cdot x) \rangle \geq 0.$$ 

Therefore $\beta_X(x) = 0$ or the function $f$ is strictly increasing. \hfill \Box

For any subspace $m$ of $g$ and $x \in X$, let

$$m \cdot x := \{ \xi_X(x) : \xi \in m \}.$$ 

**Lemma 2.8.** Let $x \in X$. Then

$$\ker d\mu_p(x) = (p \cdot x)\perp$$

**Proof.** From [5], $v \in \ker d\mu_p(x)$ if and only if for all $\beta \in p$

$$\langle d\mu_p(x)(v), \beta \rangle = 0 \iff d\mu_p^\beta(v) = 0 \iff \langle \beta_X(x), v \rangle = 0.$$ 

\hfill \Box

**Lemma 2.9.** Let $x \in X$. The following are equivalent:

a) $d\mu_p : T_xX \to p$ is onto;

b) $d\mu_p : g \cdot x \to p$ is onto;

c) the map $p \to T_xX$, $\beta \mapsto \beta_X$, is injective.

**Proof.**

$$\ker d\mu_p(x) = (p \cdot x)\perp,$$

it follows that $d\mu_p(x)$ is surjective if and only if $d\mu_p : p \cdot x \to p$ is surjective if and only if $\dim p \cdot x = \dim p$, concluding the proof. \hfill \Box
We recall the Slice Theorem, see [21]. For any Lie group $G$, a closed subgroup $H$ and any set $S$ with an $H$-action, the $G$-bundle over $G/H$ associated with the $H$-principal bundle $G 	o G/H$ is denoted by $G \times^H S$. This is the orbit space of the $H$-action on $G \times S$ given by $h \cdot (g, s) = (gh^{-1}, h \cdot s)$ where $g \in G$, $s \in S$ and $h \in H$. The $H$-orbit of $(g, s)$, considered as a point in $G \times^H S$, is denoted by $[g, s]$.

**Theorem 2.10** (Slice Theorem [21 Thm. 3.1]). If $x \in X$ and $\mu_p(x) = 0$, there are a $G_x$-invariant decomposition $T_x X = \mathfrak{g} : x \oplus W$, open $G_x$-invariant subsets $S \subset W$, $\Omega \subset X$ and a $G$-equivariant diffeomorphism $\Psi : G \times^{G_x} S \to \Omega$, such that $0 \in S$, $x \in \Omega$ and $\Psi([e, 0]) = x$.

Let $\beta \in \mathfrak{p}$. It is well-known that $G^\beta$ is compatible and

$$G^\beta = K^\beta \exp(\mathfrak{p}^\beta),$$

where $K^\beta = K \cap G^\beta = \{g \in K : \Ad(g)(\beta) = \beta\}$ and $\mathfrak{p}^\beta = \{v \in \mathfrak{p} : [v, \beta] = 0\}$ (see [30]).

**Corollary 2.10.1.** If $x \in X$ and $\mu_p(x) = \beta$, there are a $G^\beta$-invariant decomposition $T_x X = \mathfrak{g}^\beta : x \oplus W$, open $G^\beta$-invariant subsets $S \subset W$, $\Omega \subset X$ and a $G^\beta$-equivariant diffeomorphism $\Psi : G^\beta \times^{G_x} S \to \Omega$, such that $0 \in S$, $x \in \Omega$ and $\Psi([e, 0]) = x$.

This follows applying the previous theorem to the action of $G^\beta$ on $X$. Indeed, by Lemma 2.4 $G^\beta = K^\beta \exp(\mathfrak{p}^\beta)$ is compatible and the orthogonal projection of $i\mathfrak{m}$ onto $\mathfrak{p}^\beta$ is the $G^\beta$-gradient map $\mu_{\mathfrak{p}^\beta}$. The group $G^\beta$ is also compatible with the Cartan decomposition of $(U^\mathbb{C})^\beta = (U^\mathbb{C})^{i\beta} = (U^{1\beta})^\mathbb{C}$ and $i\beta$ is fixed by the $U^{1\beta}$-action on $\mathfrak{u}^{i\beta}$. This implies that $\mu_{\mathfrak{u}^{i\beta}} : Z \to \mathfrak{u}^{i\beta}$ given by $\mu_{\mathfrak{u}^{i\beta}}(z) = \pi_{\mathfrak{u}^{i\beta}} \circ \mu + i\beta$, where $\pi_{\mathfrak{u}^{i\beta}}$ is the orthogonal projection of $\mathfrak{u}$ onto $\mathfrak{u}^{i\beta}$, is the $U^{1\beta}$-shifted momentum map. The associated $G^\beta$-gradient map is given by $\mu_{\mathfrak{p}^\beta} := \mu_{\mathfrak{p}^\beta} - \beta$. Hence, if $G$ is commutative, then we have a Slice Theorem for $G$ at every point of $X$, see [21 p.169] for more details.

If $\beta \in \mathfrak{p}$, then $\beta_X$ is a vector field on $X$, i.e. a section of $TX$. For $x \in X$, the differential is a map $T_x X \to T_{\beta_X(x)}(TX)$. If $\beta_X(x) = 0$, there is a canonical splitting $T_{\beta_X(x)}(TX) = T_x X \oplus T_x X$. Accordingly $d\beta_X(x)$ splits into a horizontal and a vertical part. The horizontal part is the identity map. We denote the vertical part by $d\beta_X(x)$. It belongs to $\text{End}(T_x X)$. Let $\{\varphi_t = \exp(t\beta)\}$ be the flow of $\beta_X$. There is a corresponding flow on $TX$. Since $\varphi_t(x) = x$, the flow on $TX$ preserves $T_x X$ and there it is given by $d\varphi_t(x) \in \text{Gl}(T_x X)$. Thus we get a linear $\mathbb{R}$-action on $T_x X$ with infinitesimal generator $d\beta_X(x)$.
Corollary 2.10.2. If $\beta \in \mathfrak{p}$ and $x \in X$ is a critical point of $\mu^\beta_p$, then there are open invariant neighbourhoods $S \subset T_x X$ and $\Omega \subset X$ and an $\mathbb{R}$-equivariant diffeomorphism $\Psi : S \to \Omega$, such that $0 \in S, x \in \Omega, \Psi(0) = x$. (Here $t \in \mathbb{R}$ acts as $d\varphi_t(x)$ on $S$ and as $\varphi_t$ on $\Omega$.)

Proof. Since $\exp : \mathfrak{p} \to G$ is a diffeomorphism onto the image, the subgroup $H := \exp(\mathbb{R}\beta)$ is compatible. Hence, it is enough to apply the previous corollary to the $H$-action at $x$. □

Assume now that $\beta \in \mathfrak{p}$ and that $x \in \text{Crit}(\mu^\beta_p)$. Let $D^2 \mu^\beta_p(x)$ denote the Hessian, which is a symmetric operator on $T_x X$ such that

$$(D^2 \mu^\beta_p(x)v, v) = \frac{d^2}{dt^2}(\mu^\beta_p \circ \gamma)(0)$$

where $\gamma$ is a smooth curve, $\gamma(0) = x$ and $\dot{\gamma}(0) = v$. Denote by $V_-$ (respectively $V_+$) the sum of the eigenspaces of the Hessian of $\mu^\beta_p$ corresponding to negative (resp. positive) eigenvalues. Denote by $V_0$ the kernel. Since the Hessian is symmetric we get an orthogonal decomposition

$$T_x X = V_- \oplus V_0 \oplus V_+.$$  

(6)

Let $\alpha : G \to X$ be the orbit map: $\alpha(g) := gx$. The differential $d\alpha_e$ is the map $\xi \mapsto \xi X(x)$.

Proposition 2.11. If $\beta \in \mathfrak{p}$ and $x \in \text{Crit}(\mu^\beta_p)$ then

$$D^2 \mu^\beta_p(x) = d\beta_X(x).$$

Moreover $d\alpha_e(\nu^\beta_{\pm}) \subset V_{\pm}$ and $d\alpha_e(\nu^\beta_0) \subset V_0$. If $X$ is $G$-homogeneous these are equalities.

Proof. The first statement is proved in [21, Prop. 2.5]. Denote by $\rho : G_x \to T_x X$ the isotropy representation: $\rho(g) = dg_x$. Observe that $\alpha$ is $G_x$-equivariant where $G_x$ acts on $G$ by conjugation, hence $d\alpha_e$ is $G_x$-equivariant, where $G_x$ acts on $\mathfrak{g}$ by the adjoint representation and on $T_x X$ by the isotropy representation. Since $\beta_X(x) = 0$, $\exp(t\beta) \in G_x$ for any $t$ and $d\alpha_e$ is $\mathbb{R}$-equivariant. Therefore it interchanges the infinitesimal generators of the $\mathbb{R}$-actions, i.e. $d\alpha_e \circ \text{ad}\beta = d\beta_X = D^2 \mu^\beta_p(x)$. The required inclusions follow. If $G$ acts transitively on $X$ we must have $T_x X = d\alpha_e(\mathfrak{g})$. Hence the three inclusions must be equalities. □

Corollary 2.11.1. For every $\beta \in \mathfrak{p}$, $\mu^\beta_p$ is a Morse-Bott function.

Proof. Let $X^\beta := \{x \in X : \beta_X(x) = 0\}$. Corollary 2.10.2 implies that $X^\beta$ is a smooth submanifold. Since $T_x X^\beta = V_0$ for $x \in X^\beta$, the first statement of Proposition 2.11 shows that the Hessian is nondegenerate in the normal directions. □
From now on, we assume that \( X \) is compact and connected. Let \( \mu_p : X \to p \) the \( G \) gradient map. Let \( \beta \in p \). Let \( c_1 > \cdots > c_r \) be the critical values of \( \mu_p^\beta \). The corresponding level sets of \( \mu_p^\beta \), \( C_i := (\mu_p^\beta)^{-1}(c_i) \) are submanifolds which are union of components of \( \text{Crit}(\mu_p^\beta) \). The function \( \mu_p^\beta \) defines a gradient flow generated by its gradient which is given by \( \beta_X \). By Corollary 2.10.2, it follows that for any \( x \in X \) the limit:

\[
\varphi_\infty(x) := \lim_{t \to +\infty} \exp(t\beta)x,
\]

exists. Let us denote by \( W_i^\beta \) the unstable manifold of the critical component \( C_i \) for the gradient flow of \( \mu_p^\beta \):

\[
W_i^\beta := \{ x \in X : \varphi_\infty(x) \in C_i \}.
\]

(7)

Applying Corollary 2.11.1 we have the following well-known decomposition of \( X \) into unstable manifolds with respect to \( \mu_p^\beta \).

**Theorem 2.12.** In the above assumption, we have

\[
X = \bigcup_{i=1}^r W_i^\beta,
\]

(8)

and for any \( i \) the map:

\[
(\varphi_\infty)|_{W_i} : W_i^\beta \to C_i,
\]

is a smooth fibration with fibres diffeomorphic to \( \mathbb{R}^{l_i} \) where \( l_i \) is the index (of negativity) of the critical submanifold \( C_i \).

3. The Norm Square of the Gradient Map

We assume throughout that \( X \) is compact and connected \( G \) invariant submanifold of \( (Z, \omega) \) and \( G \) is connected. Let \( \mu_p : X \to p \) denote the \( G \) gradient map. Let \( \| \cdot \| \) denote the norm functions associated to the \( \text{Ad}(K) \)-invariant scalar product \( \langle \cdot, \cdot \rangle \) on \( p \). Define the function \( f : X \to \mathbb{R} \) by

\[
f(x) := \frac{1}{2} \| \mu_p(x) \|^2,
\]

for \( x \in X \).

In this section, the critical points of this function will be of central importance.

**Lemma 3.1.** The gradient of \( f \) is given by

\[
\nabla f(x) = \beta_X(x), \quad \beta := \mu_p(x) \in p \quad \text{and} \quad x \in X.
\]

(10)
Hence, $x \in X$ is a critical point of $f$ if and only if $\beta_X(x) = 0$.

**Proof.** Define a curve $\gamma(t)$ such that $\gamma(0) = x$ and $\gamma'(0) = v \in T_xX$. $f(x) = \frac{1}{2} \| \mu_p(x) \|^2 = \frac{1}{2} \langle \mu_p(x), \mu_p(x) \rangle$

$$
\frac{df(x)v}{dt} = \frac{df}{dt} \bigg|_{t=0} = \frac{1}{2} \frac{d}{dt} \bigg|_{t=0} \langle \mu_p(\gamma(t)), \mu_p(\gamma(t)) \rangle
$$

$$
= \langle d\mu_p(\gamma(t))\gamma'(t), \mu_p(\gamma(t)) \rangle \bigg|_{t=0}
$$

$$
= \langle d\mu_p(x)v, \mu_p(x) \rangle = \langle \beta_X(x), v \rangle, \quad \beta = \mu_p(x).
$$

Hence, $\nabla f(x) = \beta_X(x)$. □

**Corollary 3.1.1.** Let $x \in X$ and set $\beta := \mu_p(x)$. The following are equivalent.

a) $\beta_X(x) = 0$,

b) $d\mu_p^\xi(x) = 0$, $\xi \in p$,

c) $df(x) = 0$.

For the remaining part of this work, we fix $\beta = \mu_p(x)$. The negative gradient flow line of $f$ through $x \in X$ is the solution of the differential equation

\[
\begin{cases}
\dot{x}(t) = -\beta_X(x(t)), & t \in \mathbb{R} \\
x(0) = x.
\end{cases}
\]

The $G$-orbits are invariant under the gradient flow.

**Lemma 3.2.** Let $g : \mathbb{R} \to G$ be the unique solution of the differential equation

\[
\begin{cases}
g^{-1}g(t) = \beta_X(x(t)) \\
g(0) = e,
\end{cases}
\]

where $e$ is the identity of $G$.

Then,

$$
x(t) = g^{-1}(t)x
$$

for all $t \in \mathbb{R}$.

**Proof.** Define $y : \mathbb{R} \to X$ by

$$
y(t) = g^{-1}(t)x.
$$
Since $g^{-1} = -g^{-1}gg^{-1}$ and $g^{-1}g = \beta_X(x)$, it follows that
\[
\dot{y} = -g^{-1}\dot{g}g^{-1}x = -\beta_X(g^{-1}x) = -\beta_X(y(t))
\]
and
\[
y(0) = (g(0))^{-1}x = e^{-1}x = x.
\]
Hence $x(t) = y(t) = g^{-1}(t)x$ for all $t \in \mathbb{R}$.

The proof of the following Theorem is based on the Lojasiewicz gradient inequality, which holds in general for analytic gradient flows. A proof for the case of an action of a complex reductive group is given in [15].

**Theorem 3.3.** Let $x_0 \in X$ and $x : \mathbb{R} \to X$ be the negative gradient flow line of $f$ through $x_0$. There exist positive constants $\alpha$, $C$, $\psi$, and $\frac{1}{2} < \gamma < 1$ such that

\[
x_\infty := \lim_{t \to \infty} x(t)
\]
exists. Moreover, there exist a constant $T > 0$ such that for any $t > T$,
\[
d(x(t), x_\infty) \leq \int_t^\infty |\dot{x}(s)|ds
\]
\[
\leq \frac{\alpha}{1 - \gamma} (f(x(t)) - f(x_\infty))^{1 - \gamma}
\]
\[
\leq \frac{C}{(t - T)\psi}.
\]

**Proof.** Let $X = Z$. Using the Marle-Guiliemin-Sternberg local normal form, the moment map is locally real analytic. Since $\mu_p = \pi_p \circ i\mu$, where $\pi_p : iu \to p$ is the orthogonal projection, it follows that $\mu_p$ is locally real analytic. This implies that $f = \frac{1}{2} \mu_p \parallel X \to \mathbb{R}$ satisfies the Lojasiewicz gradient inequality. By Lemma 3.1, the gradient of $f : Z \to \mathbb{R}$ coincide with the gradient of $f : X \to \mathbb{R}$. Hence $f|_X$ also satisfies Lojasiewicz gradient inequality: there exists constants $\delta > 0$, $\alpha > 0$, and $\frac{1}{2} < \gamma < 1$ such that, for every critical value $a$ of $f$ and every $x \in X$,

\[
|f(x) - a| < \delta \quad \implies \quad |f(x) - a|^{\gamma} \leq \alpha |\nabla f(x)|.
\]

Let $x : \mathbb{R} \to X$ be a nonconstant negative gradient flow line of $f$.

\[
a = \lim_{t \to \infty} f(x(t))
\]
is a critical value of \( f \). Choose a constant \( T > 0 \) such that \( a < f(x(t)) < a + \delta \) for \( t \geq T \). Then, for \( t \geq T \),

\[
\frac{d}{dt} (f(x) - a)^{1-\gamma} = (1-\gamma) (f(x) - a)^{-\gamma} |\nabla f(x)| \geq \frac{1-\gamma}{\alpha} |\dot{x}|.
\]

Integrating the inequality over the interval \([t, \infty)\) gives

\[
(12) \int_t^\infty |\dot{x}(s)| ds \leq \frac{\alpha}{1-\gamma} (f(x(t)) - a)^{1-\gamma} \quad \text{for } t \geq T.
\]

This shows that

\[
x_\infty := \lim_{t \to \infty} x(t)
\]

exists and it is a critical point of \( f \) and hence satisfies \( \mu_p(x_\infty)_X = 0 \).

Set \( \xi(t) = (f(x(t)) - a)^{1-2\gamma} \).

\[
\xi(t) = (2\gamma - 1) (f(x(t)) - a)^{-2\gamma} |\nabla f(x(t))|^2 \geq \frac{2\gamma - 1}{\alpha^2} \quad \text{for } t \geq T.
\]

Which implies that

\[
\xi(t) \geq \frac{2\gamma - 1}{\alpha^2} (t - T) \quad \text{for } t \geq T.
\]

Hence

\[
(f(x(t)) - a)^{1-\gamma} = \xi(t)^{-\frac{1-\gamma}{2\gamma - 1}} \leq \left( \frac{2\gamma - 1}{\alpha^2} (t - T) \right)^{-\frac{1-\gamma}{2\gamma - 1}} \quad \text{for } t \geq T.
\]

Thus

\[
\frac{\alpha}{1-\gamma} (f(x(t)) - a)^{1-\gamma} \leq \frac{c}{(t - T)^\psi}, \quad \psi := \frac{1-\gamma}{2\gamma - 1}, \quad c := \frac{\alpha}{1-\gamma} \left( \frac{\alpha^2}{2\gamma - 1} \right)^\psi
\]

and by (12) the result follows.

\( \square \)

3.1. **Stratifications of the Norm Square of the Gradient map.** We recall the stratification theorem for actions of reductive group. First, we define a stratification of \( X \). For details see \[21\].

Given a maximal subalgebra \( a \subset p \), we pick \( a_+ \subset a \) a positive Weyl-chamber. Let \( f : X \to \mathbb{R} \) be the norm square of the gradient map \( \mu_p \), i.e.,

\[
f(x) := \frac{1}{2} \| \mu_p(x) \|^2,
\]

where \( \| \cdot \| \) denotes the norm functions associated to an \( \text{Ad}(K) \)-invariant scalar product \( \langle \cdot, \cdot \rangle \).

Let \( C \) denote the critical set of \( f \), \( \mathcal{B} := \mu_p(C) \) and \( \mathcal{B}_+ := \mathcal{B} \cap a_+ \).

Let \( X^{ss} := \{ x \in X : G \cdot x \cap \mu_p^{-1}(0) \neq \emptyset \} \). For \( \beta \in \mathcal{B}_+ \), following the notation introduced in \[21\], set
\[ X_{\|\beta\|^2} := \{ x \in X : \exp(\mathbb{R}\beta) \cdot x \cap (\mu_p^{-1}(\|\beta\|^2)) \neq \emptyset \} \]

\[ X^\beta := \{ x \in X : \beta X(x) = 0 \} \]

\[ X^\beta_{\|\beta\|^2} := X^\beta \cap X_{\|\beta\|^2} \]

\[ X^{\beta+}_{\|\beta\|^2} := \{ x \in X^{\beta+}_{\|\beta\|^2} : \lim_{t \to -\infty} \exp(t\beta) \cdot x \text{ exists and it lies in } X^{\beta+}_{\|\beta\|^2} \} \]

The set \( X^{\beta+}_{\|\beta\|^2} \) is \( G^{\beta+} \)-invariant. \( \mu_{p\beta} \) is a gradient map of the \( G^\beta \)-action on \( X^{\beta+}_{\|\beta\|^2} \). Set

\[ \tilde{\mu}_{p\beta} := \mu_{p\beta} - \beta. \]

Since \( \beta \) is in the center of \( g^\beta \) and \( G^\beta \) is a compatible subgroup of \( (U^\beta)^{\mathbb{C}} = (U^{\mathbb{C}})^{\beta} \), it is a gradient map too. Let

\[ S^{\beta+} := \{ x \in X^{\beta+}_{\|\beta\|^2} : \overline{G^\beta \cdot x} \cap \mu_p^{-1}(\beta) \neq \emptyset \}. \]

The set \( S^{\beta+} \) coincides with the set of semistable points of the group \( G^\beta \) in \( X^{\beta+}_{\|\beta\|^2} \) after shifting.

**Definition 3.1.** The \( \beta \)-stratum of \( X \) is given by \( S_\beta := G \cdot S^{\beta+} \).

**Theorem 3.4.** (Stratification Theorem)\([21, 7.3]\). Suppose \( X \) is a compact \( G \)-invariant submanifold of \( Z \). Then \( \mathfrak{B}_+ \) is finite and

\[ X = \bigsqcup_{\beta \in \mathfrak{B}_+} S_\beta. \]

Moreover

\[ \overline{S_\beta} \subset S_\beta \cup \bigcup_{\|\gamma\| > \|\beta\|} S_\gamma. \]

**Proposition 3.5.** \([21, 6.12]\) If \( z \in X \) satisfies

\[ f(z) = \max_{x \in X} f(x). \]

Then \( G \cdot z = K \cdot z \) and so it is closed orbit.

If \( v \in T_p X \), then \( |v| = \sqrt{(v, v)} \), where \( (\cdot, \cdot) \) is the scalar product induced by the Kähler form \( \omega \). The following proposition give the Hessian of \( f \).
where $\beta$.

In particular, if $x$ is a critical point, then
\begin{equation}
|f(x) - a| < \delta \quad \implies \quad |f(x) - a|^\gamma \leq \alpha|\nabla f(x)|.
\end{equation}

In particular, if $x$ is a critical point, then $f(x) = a$. Since $X$ is compact, by Theorem 3.3

$$X = \bigcup_{i=1}^{k} S_{\beta_i},$$

where $\beta_i \in \mathcal{B}_\pm$. We may assume that $\beta = \beta_1$ and so, $\| \beta_j \| > \| \beta \|$ for any $j = 2, \ldots, k$. Let

$$0 < \delta' < \min \left( \delta, \frac{\| \beta_2 \|^2 - \| \beta \|^2}{2}, \ldots, \frac{\| \beta_k \|^2 - \| \beta \|^2}{2} \right).$$
Let
\[ U = \{ x \in X : |f(x) - \| \beta \|^2 | < \delta \} \].

Let \( x_0 \in U \) and let \( x(t) \) the gradient flow of \(-\nabla f\) through \( x_0 \). Therefore,
\[
f(x(t)) \leq f(x_0) \leq |f(x_0) - \| \beta \|^2 | + \| \beta \|^2 < \delta' + \| \beta \|^2 \]
\[
< \frac{\| \beta_j \|^2 - \| \beta \|^2}{2} + \| \beta \|^2 \quad j = 2, \ldots, k
\]
\[
< \frac{\| \beta_j \|^2 + \| \beta \|^2}{2} \leq \| \beta_j \|^2, \quad j = 2, \ldots, k.
\]

Therefore, \( f(x_\infty) = \| \beta \|^2 \). This implies that \( U \subset S_\beta \) and so, using standard arguments of the gradient flow, \( S_\beta \) is open. \( \square \)

As in [33], we have the following deformation retraction.

**Theorem 3.10** (Retraction Theorem). Let \( \beta \in \mathfrak{a}_+ \) be a critical value of \( f \). Let \( S_\beta \) be the stratum associated to \( \beta \). Let \( \varphi_t(x) \) denote the gradient flow of \(-\nabla f\). Then there exist a \( K \)-equivariant strong deformation retraction of \( S_\beta \) onto \( S_\beta \cap \mu_p^{-1}(K \cdot \beta) \) given by
\[
[0, \infty] \times S_\beta \rightarrow S_\beta \cap \mu_p^{-1}(K \cdot \beta), \quad (t, p) \mapsto \varphi_t(p),
\]
and
\[
\varphi_\infty(p) = \lim_{t \rightarrow +\infty} \varphi_t(p).
\]

**4. Kempf-Ness Function**

Given \( G \) a real reductive group which acts smoothly on \( Z \); \( G = K \exp(\mathfrak{p}) \), where \( K \) is a maximal compact subgroup of \( G \). Let \( X \) be a \( G \)-invariant locally closed submanifold of \( Z \). As Mundet pointed out in [34], there exists a function \( \Phi : X \times G \rightarrow \mathbb{R} \), such that
\[
\langle \mu_p(x), \xi \rangle = \frac{d}{dt} \bigg|_{t=0} \Phi(x, \exp(t\xi)), \quad \xi \in \mathfrak{p},
\]
and satisfying the following conditions:

a) For any \( x \in X \), the function \( \Phi(x,. ) \) is smooth on \( G \).

b) The function \( \Phi(x,. ) \) is left-invariant with respect to \( K \), i.e., \( \Phi(x, kg) = \Phi(x, g) \).

c) For any \( x \in X, v \in \mathfrak{p} \) and \( t \in \mathbb{R} \);
\[
\frac{d^2}{dt^2} \Phi(x, \exp(tv)) \geq 0.
\]
Moreover:
\[ \frac{d^2}{dt^2} \Phi(x, \exp(tv)) = 0 \]
if and only if \( \exp(Rv) \subset G_x \).

d) For any \( x \in X \), and any \( g, h \in G \):
\[ \Phi(x, hg) = \Phi(x, g) + \Phi(gx, h). \]

This equation is called the cocycle condition. The proof is given in [7], see also [6].

The function \( \Phi : X \times G \to \mathbb{R} \) is called the Kempf-Ness function for \((X,G,K)\).

Let \( M = G/K \) and \( \pi : G \to M \). \( M \) is a symmetric space of non-compact type [11]. By (b), \( \Phi(x, kg) = \Phi(x, g) \), the function \( \Phi \) descend to \( M \) as
\[ \Phi : X \times M \to \mathbb{R}; \]

(15) \[ \Phi(x, \pi(g)) = \Phi(x, g^{-1}). \]

In this paper we have fixed an \( \text{Ad}(U) \)-invariant scalar product \( \langle \cdot, \cdot \rangle \) on \( u^\mathbb{C} \). We recall that \( \langle u, iu \rangle = 0 \) and the multiplication by \( i \) defines an isometry from \( u \) onto \( iu \). Hence \( g = \mathfrak{k} \oplus \mathfrak{p} \) is an orthogonal splitting with respect to \( \langle \cdot, \cdot \rangle \). Equip \( G \) with the unique left invariant Riemannian metric which agree with the scalar product \( \langle \cdot, \cdot \rangle \) on the tangent space \( g = \mathfrak{k} \oplus \mathfrak{p} \) of \( G \) at \( e \). This metric is \( \text{Ad}(K) \) invariant and so it induces a \( G \) invariant Riemannian metric of nonpositive curvature on \( M \) [25].

**Lemma 4.1.** For \( x \in X \), let \( \Phi_x(gK) = \Phi(x, g^{-1}) \). The differential of \( \Phi_x \) is given as
\[ d(\Phi_x)_{\pi(g)}(v_x) = -\langle \mu_p(g^{-1}x), \xi \rangle \]
where, \( v_x(g) = d(\pi \circ L_g)_e(\xi) \) and \( \xi \in \mathfrak{p} \). Therefore, \( \nabla \Phi_x(\pi(g)) = -d(\pi \circ L_g)_e(\mu_p(g^{-1}x)) \)

**Proof.** Let \( \pi(g) \in M \), \( \xi \in \mathfrak{p} \) and \( v_x \in T_{\pi(g)}G/K \). There exist \( \xi \in \mathfrak{p} \) such that
\[ v_x = \frac{d}{dt} \bigg|_{t=0} \exp(t\xi)K. \]

Take
\[ \gamma(t) = \pi(g \exp(t\xi)), \quad t \in [a, b], \ \xi \in \mathfrak{p}. \]

Then \( v_x = (d\pi)_g((dL_g)(\xi)) \),
\[ d(\Phi_x)_{\pi(g)}(v_x) = \frac{d}{dt} \bigg|_{t=0} \Phi(x, \gamma(t)) \]

\[ = \frac{d}{dt} \bigg|_{t=0} \Phi(x, \pi(g \exp(t\xi))) \]

\[ = \frac{d}{dt} \bigg|_{t=0} \Phi(x, \exp(-t\xi)g^{-1}) \quad \text{(by the definition of } \Phi) \]

\[ = \frac{d}{dt} \bigg|_{t=0} \big[ \Phi(x, g^{-1}) + \Phi(g^{-1}_x, \exp(-t\xi)) \big] \quad \text{(by condition (d))} \]

\[ = \frac{d}{dt} \bigg|_{t=0} \Phi(g^{-1}_x, \exp(-t\xi)) \]

\[ = -\langle \mu_p(g^{-1}_x), \xi \rangle. \]

This implies that

\[ d(\Phi_x)_{\pi(g)}(v_x) = -\langle \mu_p(g^{-1}_x), \xi \rangle. \]

□

We denote by the same symbol \( \Phi_x : G \to \mathbb{R} \), the function \( \Phi_x(g) = \Phi(x, g^{-1}) \). It is called the Kempf-Ness function at \( x \). Define \( \varphi_x : G \to G \cdot x \) as follows

\[ \varphi_x(g) = g^{-1}_x. \]

**Lemma 4.2.** The map \( \varphi_x \) intertwines the gradient of \( \Phi_x : G \to \mathbb{R} \) and the gradient of \( f \). i.e., \( \forall g \in G \)

\[ d(\varphi_x)_g \nabla \Phi_x = \nabla f((\varphi_x(g))). \]

**Proof.** Let \( \beta \in \mathfrak{p} \). Since

\[ \varphi_x(g \exp(t\beta)) = \exp(-t\beta)g^{-1}_x, \]

we have

\[ (d\Phi_x)_g(dL_g(\beta)) = -\beta_X(g^{-1}_x). \]

The result follow from taking \( \beta = \mu_p(x) \). □

**Remark 4.3.** A smooth curve \( \gamma(t) := \pi \circ g : \mathbb{R} \to M \) is a negative gradient flow line of \( \Phi_x \) if and only if the smooth curve \( g : \mathbb{R} \to M \) satisfies \( g^{-1} \dot{g} = \mu_p(g^{-1}_x) \). Indeed, from Lemma 4.1,

\[ \nabla \Phi_x(\pi(g)) = -d(\pi \circ dL_g)_e(\mu_p(g^{-1}_x)) \]
and by Lemma 4.2, $\varphi_x$ intertwines the gradient of $\Phi_x$ with $\nabla f$. Therefore, the gradient flow of $\Phi_x$ is such that $g^{-1}\dot{g} = \mu_p(g^{-1}x)$. On the other hand if $g : \mathbb{R} \to G$ satisfies $g^{-1}\dot{g} = \mu_p(g^{-1}x)$, the curve $\gamma = \pi \circ g(t)$ is a geodesic and
\[
\frac{d}{dt}(\Phi_x \circ \gamma) = -\langle \mu_p(g^{-1}x), g^{-1}\dot{g} \rangle.
\]

We recall some result from Riemannian geometry. We refer the reader to Appendix A in [15] for further details. Suppose $M$ is an Hadamard manifold, i.e., connected, complete, simply-connected with non-positive curvature. Let $\gamma_1, \gamma_2 : [a, b] \to M$ be smooth curves and for each $t \in [a, b]$, $\gamma(s, t) : [0, 1] \to M$ be the unique geodesic such that
\[
\gamma(0, t) = \gamma_1(t), \quad \gamma(1, t) = \gamma_2(t).
\]
Define the function $\rho : [a, b] \to \mathbb{R}$ by
\[
\rho(t) := d(\gamma_1(t), \gamma_2(t)).
\]
The following Lemma is proved in [15] Lemma A.2

**Lemma 4.4.** Suppose $f : M \to \mathbb{R}$ is a smooth function that is convex along geodesics. Let $\gamma_1, \gamma_2 : \mathbb{R} \to M$ be the negative gradient flow lines of $f$, and let $\gamma$ and $\rho$ be as defined above. Then $\rho$ is nonincreasing and, if $\rho(t) \neq 0$, then
\[
\dot{\rho}(t) = \frac{-1}{\rho(t)} \int_0^1 \frac{\partial^2}{\partial s^2} (f \circ \gamma)(s, t) ds.
\]

**Theorem 4.5.** Let $\Phi_x : M \to \mathbb{R}$. Then

a) $\Phi_x$ is a Morse-Bott function and it is convex along geodesics.
b) If $\gamma : \mathbb{R} \to M$ is a negative gradient flow of $\Phi_x$, then,
\[
\lim_{t \to \infty} \Phi_x(\gamma(t)) = \inf_{x \in M} \Phi_x.
\]

**Proof.** (a): By lemma 4.1, $g \in G$ is a critical of $\Phi_x$ if and only if
\[
\mu_p(g^{-1}x) = 0.
\]
\[
\text{Crit}(\Phi_x) = \{ \pi(g) \in M : \mu_p(g^{-1}x) = 0 \}.
\]
The next is to show that the Crit$(\Phi_x)$ is a submanifold of $M$. To do this, the Hessian of the function is computed along geodesics. The geodesic on $M$ passing through $\pi(g)$ in the direction
\( v = d\pi_g g\xi \) has the form \( \pi(g\exp(t\xi)) \). Hence, \( M \) is complete and by the Hadamard theorem,

\[
p \to M, \quad \xi \mapsto \pi(g\exp(\xi))
\]
is a diffeomorphism. This implies that \( \pi(g\exp(\xi)) \in \text{Crit}(\Phi_x) \) if and only if \( \mu_p(g^{-1}\exp(-\xi)x) = 0 \).

\[
\text{Hess}(\Phi_x) = d^2(\Phi_x)_{\pi(g)}(v), \quad \pi(g) \in \text{Crit}(\Phi_x)
\]

Moreover,

\[
d^2 \Big|_{t=0} \Phi^{-1}_x, \exp(t\xi) = 0
\]
if and only if \( \exp(t\xi) \subset G_{g^{-1}x} \). Hence,

\[
\text{Crit}(\Phi_x) = \{ \pi(g\exp t\xi) \in M : \exp t\xi \subset G_{g^{-1}x} \}
\]

which is a submanifold and the kernel of the Hessian. Therefore, \( \Phi \) is a Morse-Bott function and since the Hessian is non-negative along geodesics, it is convex along geodesics.

(b). Let \( \gamma_1, \gamma_2 \) be negative gradient flow of \( \Phi_x \). There exists \( g_1, g_2 : \mathbb{R} \to G \) such that \( \gamma_1 = \pi(g_1(t)) \) and \( \gamma_2 = \pi(g_2(t)) \). Let \( \beta : \mathbb{R} \to p \) and \( k : \mathbb{R} \to K \) be such that \( g_2(t) = g_1(t)\exp(\beta(t))k(t) \).

Define \( H : \mathbb{R} \times \mathbb{R} \to M \) by

\[
H(t, s) = \pi(g_1(t)\exp(s\beta(t))).
\]

The curve \( s \mapsto H(t, s) \) is the unique geodesic joining \( \gamma_1 \) and \( \gamma_2 \). By Lemma 4.4 the function \( \rho(t) = d_M(\gamma_1(t), \gamma_2(t)) = \| \beta(t) \| \) is nonincreasing.

Assume that \( \text{Crit}(\Phi_x) \) is not empty. Hence we may suppose that \( \mu_p(g_1(0)^{-1}x) = 0 \). This implies that the curve \( \gamma_1 \) is constant. Since \( \rho \) is nonincreasing, the image of \( \gamma_2 \) is contained in a compact subset of \( M \). Since \( \Phi_x \) is Morse-Bott, then \( \gamma_2 \) converges to a critical point of \( \Phi_x \). This
implies that if the critical manifold of $\Phi_x$ is nonempty, then $\Phi_x$ has a global minimal and every negative flow line of $\Phi_x$ converges to a critical point. Now suppose $\text{Crit}(\Phi_x)$ is empty. Assume by contradiction that

$$a := \lim_{t \to \infty} \Phi_x(\gamma_1(t)) \geq \inf_M \Phi_x.$$ 

Then, $\Phi_x(\gamma_1(t))$ is bounded from below. We can choose $\gamma_2$ such that $\Phi(\gamma_2(0)) < a$. Since the function $\rho = \| \beta(t) \|$ is nonincreasing, there exists a constant $C > 0$ such that $\rho(t) = \| \beta(t) \| \leq C$.

Hence,

$$\frac{d}{ds}\Phi(H(t,s)) = (d\Phi_x)_{\gamma_1(t)}(\dot{H}(t,0))$$

$$= -\langle \mu_p(g_1(t)^{-1}x), \beta(t) \rangle$$

$$\geq -\| \mu_p(g_1(t)^{-1}x) \| \| \beta(t) \|$$

$$\geq -C \| \mu_p(g_1(t)^{-1}x) \| .$$

Since for a fixed $t$, the function $s \to \Phi_x(H(t,s))$ is convex, this implies that the derivative $\frac{d}{ds}\Phi(H(t,s))$ increases. It follows that

$$\Phi_x(\gamma_2(t)) = \Phi_x(H(t,1))$$

$$= \Phi_x(H(t,0)) + \int_0^1 \frac{d}{ds}\Phi(H(t,s))ds$$

$$\geq \Phi_x(\gamma_1(t)) - C \| \mu_p(g_1(t)^{-1}x) \| .$$

Since the function $\Phi_x(\gamma_1(t))$ is bounded below and $\frac{d}{dt}\Phi_x(\gamma_1(t)) = -\| \mu_p(g_1(t)^{-1}x) \|^2$, there exists a sequence $t_i \to \infty$ such that $\lim_{i \to \infty} \| \mu_p(g_1(t_i)^{-1}x) \|^2 = 0$. This implies that

$$\lim_{i \to \infty} \Phi_x(\gamma_2(t_i)) \geq \lim_{i \to \infty} \Phi_x(\gamma_1(t_i)) = a.$$ 

This is a contradiction since by assumption $\Phi_x(\gamma_2(t)) < a$ and so $\lim_{t \to \infty} \Phi_x(\gamma_1(t)) < a$.

The following result asserts that any critical points of $f$ in the same $G$-orbit in fact belong to the same $K$-orbit. We use original ideas from [15] in a different context.

**Theorem 4.6.** Let $x_0, x_1 \in X$ be critical points of the norm square $f$. Then

$$x_1 \in G \cdot x_0 \implies x_1 \in K \cdot x_0.$$
Proof. Since $x_0, x_1 \in X$ are critical points of $f$, then by Lemma 3.1

\begin{equation}
\mu_p(x_0)_X = 0, \quad \mu_p(x_1)_X = 0
\end{equation}

(18)

Suppose $x_1 \in G \cdot x_0$. Let $g_0 \in G$ such that

$$x_1 = g_0^{-1}x_0$$

and $g, h : \mathbb{R} \to G$ be defined by

$$g(t) := \exp(t\mu_p(x_0)), \quad h(t) := g_0\exp(t\mu_p(x_1)).$$

Since $\mu_p(x_0)_X = 0$, then $g(t)^{-1}x_0 = \exp(-t\mu_p(x_0))x_0 = x_0$. Similarly

$$h(t)^{-1}x_0 = \exp(-t\mu_p(x_1))g_0^{-1}x_0 = g_0^{-1}x_0 = x_1$$

for all $t$. Thus $g(t)$ and $h(t)$ satisfy the differential equation $g^{-1}\dot{g} = \mu_p(g^{-1}x_0)$. These implies that the curves $\gamma_1 := \pi \circ g$ and $\gamma_2 := \pi \circ h$ are geodesics and are negative gradient flow lines of the Kempf-Ness function. Define $\xi(t) \in \mathfrak{p}$ and $k(t) \in K$ so that

$$h(t) := g(t)\exp(\xi(t))k(t).$$

$$x_1 = h(t)^{-1}g(t)x_0 = k(t)^{-1}\exp(-\xi(t))x_0.$$

Let

$$\rho(t) := d_M(\gamma_1(t), \gamma_2(t)) = \|\xi(t)\|, \quad \text{for all } t.$$

If $\rho \equiv 0$, then $\xi(t) = 0$ for all $t$ and

$$x_1 = k(t)^{-1}x_0$$

and this means that $x_1 \in K \cdot x_0$. Otherwise, for each $t$, let $\gamma(s, t) : [0, 1] \to M$ defined by

$$\gamma(s, t) := \pi(g(t)\exp(s\xi(t)))$$

be the unique geodesic. Note that $\gamma(0, t) = \gamma_1(t)$ and $\gamma(1, t) = \gamma_1(t)$.

By equation 16, we have

\begin{equation}
\dot{\rho}(t) = -\frac{1}{\rho(t)} \int_0^1 \frac{\partial^2}{\partial s^2} \Phi_{x_0}(g(t)\exp(s\xi(t)))ds
\end{equation}

(19)

\begin{equation}
= -\frac{1}{\rho(t)} \int_0^1 (\xi(t)_X (\exp(s\xi(t))x_0), \xi(t)_X (\exp(s\xi(t))x_0))ds.
\end{equation}

(20)
Choose a sequence $t_n \to \infty$ such that the limits

$$\lim_{n \to \infty} \dot{\rho}(t_n) = 0, \quad \xi_\infty := \lim_{n \to \infty} \xi(t_n), \quad k_\infty := \lim_{n \to \infty} k(t_n)$$

exist. By (20) and since $\lim_{n \to \infty} \dot{\rho}(t_n) = 0$, $\xi_\infty X(x_0) = 0$. Then,

$$x_1 = \lim_{n \to \infty} k(t_n)^{-1} \exp(-\xi(t_n)) x_0 = k_\infty^{-1} \exp(-\xi_\infty) x_0 = k_\infty^{-1} x_0.$$  

Hence $x_1 \in K \cdot x_0$ □

The following theorems also hold in the real case.

**Theorem 4.7.** Let $x_0 \in X$ and $x : \mathbb{R} \to X$ the negative gradient flow line of $f$ through $x_0$. Define $x_\infty := \lim_{t \to \infty} x(t)$. Then

$$\| \mu_p(x_\infty) \| = \inf_{g \in G} \| \mu_p(g x_0) \|.$$  

Moreover, the $K$-orbit of $x_\infty$ depends only on the $G$-orbit of $x_0$.

**Proof.** The limit $x_\infty$ exists by Theorem 3.3. The solution of the negative gradient flow line of $f$ through $x_0$ by Lemma 3.2 is given by

$$x(t) = g(t)^{-1} x_0,$$

where $g : \mathbb{R} \to G$ is the solution of

$$\begin{cases} g^{-1} \dot{g}(t) = \beta_X (x(t)) \\ g(0) = e, \text{ where } e \text{ is the identity of } G. \end{cases}$$

Fix an element $g_0 \in G$ and let $y : \mathbb{R} \to X$ and $h : \mathbb{R} \to G$ be the solutions of the differential equations

$$\dot{y} = -\beta_X (y(t)), \quad y(0) = g_0^{-1} x_0,$$

and

$$h^{-1} \dot{h} = \beta_X (y(t)), \quad h(0) = g_0.$$  

Define $\xi(t) \in p$ and $k(t) \in k$ by

$$h(t) = g(t) \exp(\xi(t)) k(t).$$

By Lemma 3.2

$$y(t) = h(t)^{-1} x_0 = k(t)^{-1} \exp(-\xi(t)) x(t), \quad \forall t \in \mathbb{R}.$$
Let $d_M : M \times M \to [0, \infty)$ be the distance function of the Riemannian metric on $M$. $\gamma_1 := \pi \circ g$ and $\gamma_2 := \pi \circ h$ are geodesics and are negative gradient flow lines of the so called Kempf-Ness function. Since $M$ is simply connected with nonpositive sectional curvature. Then

$$d_M(\gamma_1(t), \gamma_2(t)) = \| \xi(t) \|$$

is nonincreasing. Hence there exist a sequence $t_n \to \infty$ such that the limits

$$\xi_\infty := \lim_{n \to \infty} \xi(t_n), \quad k_\infty := \lim_{n \to \infty} k(t_n)$$

exist. Hence

$$y_\infty = \lim_{t \to \infty} y(t) = \lim_{t \to \infty} k(t)^{-1} \exp(-\xi(t)) x(t) = k_\infty^{-1} \exp(-\xi_\infty) x_\infty.$$ 
Which implies that $y_\infty$ and $x_\infty$ are critical points of the normed square of the gradient map belonging to the same $G$-orbit. Hence they belong to the same $K$-orbit by Theorem 4.6 and therefore,

$$\| \mu_p(x_\infty) \| = \| \mu_p(y_\infty) \| \leq \| \mu_p(g_0^{-1} x_0) \|.$$

\[\Box\]

**Theorem 4.8.** Let $x_0 \in X$ and

$$m := \inf_{g \in G} \| \mu_p(g x_0) \|.$$ 

Then

$$x, y \in \overline{G \cdot x_0}, \quad \| \mu_p(x) \| = \| \mu_p(y) \| = m \implies y \in K \cdot x.$$ 

**Proof.** The solution of the negative gradient flow line of $f$ through $x_0$ is given by

$$x(t) = g(t)^{-1} x_0,$$

Fix $g_0 \in G$ and we know that the limit $x_\infty$ of $x(t)$ exists. Then by Theorem 4.7

$$x_\infty \in \overline{G \cdot x}, \quad \| \mu_p(x_\infty) \| = m.$$ 

Let $x \in \overline{G \cdot x}$ such that $\| \mu_p(x) \| = m$.

Choose a sequence $g_n \in G$ such that

$$x = \lim_{n \to \infty} g_n^{-1} x_0.$$
and define $y_n : \mathbb{R} \to X$ and $x_i \in X$ by

$$y_n = -\beta x(y_n), \quad y_n(0) = g^{-1}_nx_0, \quad x_n := \lim_{t \to \infty} y_n(t).$$

Then from the estimate of Theorem 3.3 there exists a constant $c > 0$ such that, for $n$ sufficiently large,

$$d(x_n, g^{-1}_nx_0) \leq \int_0^\infty |\dot{y}_n(t)| dt \leq c(\|\mu_p(g^{-1}_nx_0)\|^2 - m^2)^{1-\alpha}.$$

Since

$$m = \|\mu_p(x)\| = \lim_{n \to \infty} \|\mu_p(g^{-1}_nx_0)\|,$$

which implies that $x = \lim_{n \to \infty} x_n$ and $x_n \in K \cdot x_\infty$ for all $n$ by Theorem 4.7. Therefore, $x \in K \cdot x_\infty$ because the group orbit $K \cdot x_\infty$ is compact. □

Let $x \in X$ be a critical point of $f$ and $y : \mathbb{R} \to M$ be the unique solution of the equation

$$\dot{y} = -\beta X(y(t)), \quad y(0) = y_0 \in X; \quad \beta = \mu_p(y).$$

We define the stable manifold of the critical set $K \cdot x$ by

$$(21) \quad S(K \cdot x) := \{y_0 \in X : \lim_{t \to \infty} y(t) = kx \text{ for some } k \in K\}.$$

By Theorem 3.3 $X$ is the union of these stable manifolds and each stable manifold is a union of $G$-orbits by Theorems 4.7 and 4.8. The stable manifolds of the gradient flow have a structure close to a stratification by stable manifolds corresponding to a Morse-Bott function.

By Theorems 3.3, 4.7 and 4.8 we have the following result. This result generalises Theorem 5.2 proved by Jablonski for the $G$-gradient map of a projective representation [27].

**Theorem 4.9.** Let $x \in X$ be a critical point of $f$ and $S(K \cdot x) \subset X$ be as defined above. The following holds:

a) $X = \bigcup_{x \in \text{Crit}(f)} S(K \cdot x)$.

b) Let $y_0 \in X$. Then $y_0 \in S(K \cdot x)$ if and only if

$$x \in G \cdot y_0, \quad \|\mu_p(x)\| = \inf_{g \in G} \|\mu_p(gy_0)\|$$

(22)

c) $S(K \cdot x)$ is a union of $G$-orbits.
Proof. (a) follows by Theorem 3.3.

To prove (b); let $y : \mathbb{R} \to M$ be the unique solution of the equation
\[ \dot{y} = -\beta X(y(t)), \quad y(0) = y_0 \in X; \quad \beta = \mu_p(y) \]
and $y_\infty := \lim_{t \to \infty} y(t)$. Then, by Lemma 3.2 and Theorem 4.7 we have
\[ y_\infty \in \overline{G \cdot y_0}, \quad \| \mu_p(y_\infty) \| = \inf_{g \in G} \| \mu_p(gy_0) \|. \]

From (21), $y_0 \in S(K \cdot x)$ if and only if $y_\infty \in K \cdot x$. Thus $y_0 \in S(K \cdot x)$ implies (22). Conversely, if $y_0$ satisfies (22), then it follows from (23) and Theorem 4.8 that $y_\infty \in K \cdot x$ and hence, $y_0 \in S(K \cdot x)$.

To prove (c); From (ii) and the uniqueness in Theorem 4.7 that $S(K \cdot x)$ is a union of $G$-orbits.

\[ \square \]

5. Convexity Properties of Gradient map

In this section, we prove convexity properties of the gradient map.

5.1. The Abelian Convexity Theorem. Suppose $X$ is compact and connected. Let $\beta \in \mathfrak{p}$ and let
\[ Y = \{ z \in X : \max_{x \in X} \mu^\beta_p = \mu^\beta_p(z) \}. \]

By Corollary 2.10.2, $Y$ is a smooth, possibly disconnected, submanifold of $X$.

Lemma 5.1. $Y$ is $G^\beta_+$ invariant.

Proof. Let $g \in G$ and let $\xi \in \mathfrak{p}$. It is easy to check that
\[ (dg)_p(\xi_x) = (\text{Ad}(g)(\xi))_X(gp), \]
and so $G^\beta$ preserves $X^\beta$. We claim that $Y$ is $G^\beta$-stable. In fact, by Lemma 2.4 it follows that $G^\beta = K^\beta \exp(\mathfrak{p}^\beta)$. $Y$ is $K^\beta$ invariant by $K$-equinvariant property of the gradient map. For each $y \in Y$, let $\xi \in \mathfrak{p}^\beta$ and let $\gamma(t) = \exp(t\xi)y$. Since $\beta_X(\gamma(t)) = 0$ it follows that $\mu^\beta_p(\gamma(t))$ is constant and so $\exp(t\xi) \cdot y \in Y$. Therefore $Y$ is $G^\beta$ invariant. By Lemma 2.6 $G^{\beta_+} = G^\beta R^{\beta_+}$ where $R^{\beta_+}$ is connected and then unipotent radical of $G^{\beta_+}$. By Proposition 2.11, $r^{\beta_+} \subset V_+$, where $V_+$ is the sum of the eigenspaces of the Hessian of $\mu^\beta_p$ corresponding to positive eigenvalues. Hence $r^{\beta_+} \cdot z \subset \mathfrak{g}_z$ for any $z \in Y$. This implies that $R^{\beta_+}$ acts trivially on $Y$ and the result follows.

\[ \square \]

Proposition 5.2. $Y$ contains a compact orbit of $G^{\beta_+}$ which coincides with a $K^\beta$ orbit.
Proof. By Lemma \[5.1\] \((G^\beta)^o\) preserves any connected component of \(Y\) and the restriction of \(\mu_p\) on any connected component defines a gradient map with respect to \((G^\beta)^o\) \[21\]. By Corollary 6.11 in \[21\] pag. 21, \((G^{\beta^+})^o\) has closed orbit which coincides with a \((K^\beta)^o\) orbit. Since \(G^\beta\) has a finite number of connected components and any connected component of \(G^\beta\) intersects \(K^\beta\), it follows that \(G^\beta\) has a closed orbit which coincides with a \(K^\beta\) orbit. This is also a closed orbit of \(G^{\beta^+}\) since \(R^{\beta^+}\) acts freely on \(Y\), concluding the proof. \[
\]

Let \(a \subset \mathfrak{p}\) be an Abelian subalgebra of \(\mathfrak{p}\) and let \(\pi_a : \mathfrak{p} \to a\) be the orthogonal projection onto \(a\). It is well known that \(\mu_a := \pi_a \circ \mu_p\) is the gradient map associated to \(A = \exp(\mathfrak{a})\). Let \(P = \text{Conv}(\mu_a(X))\). It is well-known, see for instance \[22\] Prop. 3] and \[5\] Prop. 3.1], that \(\mu(X^A)\) is finite and \(P\) is is the convex hull of \(\mu_a(X^A)\), where \(X^A = \{p \in X : A \cdot p = p\}\).

Suppose that the \(G\) action on \(X\) has a unique compact orbit, which is a \(K\) orbit \[21\] denoted by \(\mathcal{O}\). Let \(a'\) be a maximal Abelian subalgebra containing \(a\). Since \(\mu_a(\mathcal{O}) = \pi_a(\mu_a'(\mathcal{O}))\), By a Theorem of Kostant \[32\] it follows that \(\mu_a(\mathcal{O})\) is a polytope.

**Theorem 5.3.** Suppose the \(G\)-action on \(X\) has a unique closed orbit \(\mathcal{O}\). Then \(\mu_a(X) = \mu_a(\mathcal{O})\) and so it is a convex polytope.

**Proof.** Let \(\xi \in a\). Then

\[
\sigma = \{\alpha \in P : \max_{\gamma \in P} \langle \gamma, \xi \rangle = \langle \alpha, \xi \rangle\}
\]

is a face of \(P\). Since \(P\) is a polytope, any face of \(P\) is exposed \[35\]. We claim that

\[
\mu_a^{-1}(\sigma) = Y, \quad \text{where} \quad Y = \{z \in X : \max_{x \in X} \mu_{\mathfrak{p}}^\xi = \mu_{\mathfrak{p}}^\xi(z)\}.
\]

In fact, it is easy to see that \(Y \subset \mu_a^{-1}(\sigma)\). Suppose \(z \in \mu_a^{-1}(\sigma)\), then \(\mu_a(z) \in \sigma\) and \(\max_{\gamma \in P} \langle \gamma, \xi \rangle = \langle \mu_a(z), \xi \rangle = \mu_{\mathfrak{p}}^\xi(z)\). Hence, \(z \in Y\). By Lemma \[5.1\] \(Y\) is \(G^{\xi^+}\)-invariant. By Proposition \[5.2\], let \(z \in Y\) be such that \(G^{\xi^+} \cdot z\) is compact. But \(G = KG^{\xi^+}\), then \(G \cdot z\) is compact. Since the \(G\) action on \(X\) has a unique compact orbit, \(G \cdot z = \mathcal{O}\). Therefore,

\[
\max_{x \in P} \langle x, \xi \rangle = \max_{x \in \mu_a(\mathcal{O})} \langle x, \xi \rangle.
\]

By Proposition \[2.3\] \(\mu_a(\mathcal{O}) = P\). Hence, \(\mu_a(X) = \mu_a(\mathcal{O})\) is a polytope. \[
\]

**Remark 5.4.** In the above assumption, applying the main Theorem in \[5\], the convex hull of \(\mu_p(X)\) coincides with the convex hull of \(\mu_p(\mathcal{O})\). Hence the convex hull of \(\mu_p(X)\) is the convex hull of a \(K\)-orbit in \(\mathfrak{p}\) and so a polar orbitope \[2\].
5.2. **Abelian convexity from Non-Abelian convexity.** The Non-Abelian convexity theorem implies the Abelian convexity theorem. This is the purpose of this section.

Let \( a \subseteq p \) be a maximal Abelian, \( a_+ \) positive Weyl Chamber. If \( \lambda \in a_+ \), we denote by

\[
\Delta_\lambda = \text{Conv}\{w\lambda : w \in W\},
\]

where \( W = \frac{N_k(a)}{C_k(a)} \) is the Weyl-Group. If \( G = U^C \), then the following result is proved in [19]. The authors applied Kirwan’s Theorem [29] for the action \( U \times T \) on the cotangent bundle \( T^*U \), where \( T \) is a maximal torus of \( U \). Our proof uses a result of Gichev [16].

**Theorem 5.5.** If \( S \subseteq a_+ \) is a convex subset, then,

\[
S^# = \bigcup \{\Delta_\lambda : \lambda \in S\}
\]

is convex subset of \( a \).

*Proof.* Let

\[
\Delta_0 = \{ (\lambda, \mu) \in a_+ \times a : \mu \in \Delta_\lambda \}.
\]

We claim that \( \Delta_0 \) is convex. Let \( (\lambda_1, \mu_1), (\lambda_2, \mu_2) \in \Delta_0 \),

\[
t(\lambda_1, \mu_1) + (1-t)(\lambda_2, \mu_2) = (t\lambda_1 + (1-t)\lambda_2, t\mu_1 + (1-t)\mu_2).
\]

Now, from [16] we have

\[
\Delta_{(t(\lambda_1+(1-t)\lambda_2))} = \Delta_{t\lambda_1} + \Delta_{(1-t)\lambda_2} = t\Delta_{\lambda_1} + (1-t)\Delta_{\lambda_2},
\]

and so

\[
t\mu_1 + (1-t)\mu_2 \in \Delta_{(t(\lambda_1+(1-t)\lambda_2))}.
\]

This shows that \( \Delta_0 \) is convex. Let

\[
\pi_1 : a_+ \times a \to a_+
\]

and

\[
\pi_2 : a_+ \times a \to a.
\]

Then,

\[
S^# = \pi_2(\pi_1^{-1}(S) \cap \Delta_0)
\]

and so it is convex. \( \square \)
**Theorem 5.6.** Let $\mu_p : X \to \mathfrak{p}$ be the gradient map. Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal Abelian subalgebra and let $\mu_a = \pi_a \circ \mu_p$ be the corresponding gradient map. If $\mu_p(X) \cap \mathfrak{a}_+$ is convex, then

$$\mu_a(X) = (\mu_p(X) \cap \mathfrak{a}_+)^\#,$$

and so convex.

**Proof.** The action of $K$ on $\mathfrak{p}$ is polar and $\mathfrak{a}$ is a section [14]. Moreover, if $x \in \mathfrak{p}$, then

$$K \cdot x \cap \mathfrak{a}_+ = \{ \lambda \}.$$

By a beautiful Theorem of Kostant [32], $\pi_a(K \cdot x) = \Delta_{\lambda}$ and so a polytope. Therefore

$$\mu_a(X) = \{ \mu \in \mathfrak{a} : \mu \in \Delta_{\lambda}, \text{ where } \lambda \in \mu_p(X) \cap \mathfrak{a}_+ \} = (\mu_p(X) \cap \mathfrak{a}_+)^\#.$$

By the above Theorem, $\mu_a(X)$ is convex. \(\Box\)

5.3. **Convexity Results of the Gradient Map.** In this section, we continue to investigate the Abelian convexity property of the gradient map. We give a new proof of the convexity property of the gradient map for $X = Z$ avoiding the Linearization theorem.

**Theorem 5.7.** Suppose $(Z, \omega)$ is a connected and compact Kähler manifold. Then

$$\mu_a : Z \to \mathfrak{a}$$

is a convex polytope.

**Proof.** Let $\xi \in \mathfrak{a}$ and $\{t_n\}_{n \in \mathbb{N}}$ be a sequence such that $t_n \to \infty$. Denote by $g_n = \exp(t_n \xi)$. By Theorem 2 in [10], up to passing to a subsequence, there exist a proper analytic subset $U$ of $Z$ such that

$$g_n : Z - U \to Z, \quad g_n \to \varphi_\infty,$$

where $\varphi_\infty$ is Non-dominant meromorphic map. Note that $Z - U$ is connected and Zarinski open. Since $g_n = \exp(t_n \xi)$, it follows $\varphi_\infty(Z - U) \subset Z^\xi$ where

$$Z^\xi = \{ x \in Z : \xi_M(x) = 0 \}.$$

The vector $J\xi$ is a Killing vector field and so $Z^\xi$ is smooth, possibly disconnected, submanifold of $Z$ [31]. Since $Z - U$ is connected, $\varphi_\infty(Z - U)$ is contained in a connected component of $Z^\xi$ which we denoted by $Z_0$. 

Set \( \mu_\xi := \langle \mu, \xi \rangle \). \( \mu_\xi(Z_0) \) is constant. We claim that
\[
\mu_\xi(Z_0) = \max_{z \in Z} \mu_\xi.
\]
Indeed, let \( x_0 \in Z \):
\[
\mu_\xi(x_0) = c = \max_{z \in Z} \mu_\xi.
\]
Let \( \epsilon > 0 \). There exist neighbourhood \( N \) of \( x_0 \) such that
\[
\mu_\xi(N) \subset (c - \epsilon, c].
\]
\((Z - U) \cap N \neq \emptyset\). Pick \( p \in (Z - U) \cap N \). Then
\[
c - \epsilon \leq \mu_\xi(p) \leq \mu_\xi(\varphi_\infty(p)).
\]
Therefore
\[
\mu_\xi(Z_0) = c = \max_{z \in Z} \mu_\xi.
\]
Let \( P = \text{Conv}(\mu_\xi) \). By a Theorem of Atiyah \cite{1}, see also \cite{8, 23}, for any \( p \in Z \) we have
\[
\mu_\xi(A \cdot p) = \text{Conv}(\mu_\xi(Z^A \cap A \cdot p)) \subset \mu_\xi(p) + a_p,
\]
where \( Z^A = \{ z \in Z : A \cdot z = z \} \) and \( a_p \) is the Lie algebra of \( A_p \). Since \( Z \) is compact, the set \( Z^A \) has finitely many connected components. By \cite{26}, it follows that \( \mu_\xi(Z^A) \) is finite and \( P = \text{Conv}(\mu_\xi(Z^A)) \). This implies that \( P \) is a polytope.

Let \( x_0, x_1, \cdots, x_n \in P \) be vertices. Since \( P \) is a polytope any face is exposed. Then there exist \( \xi_0, \xi_1, \cdots, \xi_n \in a \) such that
\[
x_i = \{ \theta \in P : \langle \theta, \xi_i \rangle = \max_{y \in P} \langle y, \xi_i \rangle, i = 0, 1, \cdots, n \}.
\]
Denote \( c_i = \langle x_i, \xi_i \rangle \), for \( i = 1, \ldots, n \). There exists \( U_0, \cdots, U_n \) proper analytic subset and a sequence \( t_N \to +\infty \) such that \( \lim_{N \to \infty} \exp(t_N \xi_i) \) exists in \( Z - U_i \). Moreover, if \( z_i \in Z - U_i \), then
\[
\lim_{N \to \infty} \exp(t_N \xi_i) \cdot z_i \in (\mu_\xi)^{-1}(c_i) = \mu_\xi^{-1}(x_i).
\]
Now,
\[
(Z - U_0) \cap (Z - U_1) \cap \cdots \cap (Z - U_n) = Z - (U_0 \cup U_1 \cup \cdots \cup U_n) \neq \emptyset,
\]
then
\[
\overline{A \cdot p} \cap \mu_\xi^{-1}(x_i) \neq \emptyset.
\]
whenever $p \in Z - (U_0 \cup U_1 \cup \cdots \cup U_n)$ for any $i = 0, \ldots, n$. Applying, again a Theorem of Atiyah [1], we have

$$
\mu_a(A \cdot p) = \text{Conv}(\mu_a(Z^A) \cap \overline{A \cdot p}) = P.
$$

Therefore

$$
\mu_a(Z) = P.
$$

□

**Corollary 5.7.1.** In the above setting, the following hold true:

a) $\{p \in Z : \mu_a(A \cdot p) = \mu_a(Z)\}$ contains an open and dense subset of $Z$.

b) Any local maximal of $\mu^\xi$ is a global maximal. Indeed, we have proved that the unstable manifold of the critical component $C_0$ corresponding to the maximum is Zariski open.

We now prove the convexity property of the gradient map when $X$ is a connected, compact coisotropic submanifold of $(Z, \omega)$.

**Definition 5.1.** A submanifold $X \subset (Z, \omega)$ is coisotropic if for any $p \in X$, we have

$$(T_p X)^\perp \omega \subset T_p X.$$

Since $(Z, \omega)$ is Kähler,

$$(T_p X)^\perp \omega = J((T_p X)^\perp).$$

**Lemma 5.8.** If $X$ is coisotropic, then for any $p \in X$, we have

$$T_p X + J(T_p X) = T_p Z.$$

**Proof.**

$$J((T_p X)^\perp) \subset T_p X.$$

Applying $J$ we have

$$(T_p X)^\perp \subset J(T_p X).$$

And so

$$T_p X + J(T_p X) = T_p Z.$$
Lemma 5.9. Let $X$ be an $A$-invariant compact connected coisotropic submanifold of $(Z, \omega)$. Let $\xi \in \mathfrak{a}$. Then

$$\max_{p \in X} \mu^\xi_a = \max_{z \in Z} \mu^\xi_a.$$ 

Moreover, the unstable manifold associated to the maximum of $\mu^\xi_a$ is open and dense.

Proof. Let $W^\xi_0$ be the unstable manifold of the critical manifold $C_0$ satisfying $\mu^\xi_a(C_0) = c_0$. Assume that $C_0$ corresponds to a local maximum. Since

$$\nabla \mu^\xi_a|_X = \nabla \mu^\xi_a,$$

it follows that

$$W^\xi_0 = \tilde{W}^\xi_0 \cap X,$$

where $\tilde{W}^\xi_0$ is the unstable manifold in $Z$ of the critical components $\tilde{C}_0$ such that

$$\mu^\xi_a(\tilde{C}_0) = \mu^\xi_a(C_0) = c_0.$$ 

By a Linearization theorem in [20], $\tilde{W}^\xi_0$ is a complex manifold and $W^\xi_0$ is open in $X$. Let $p \in W^\xi_0$. Since

$$T_p W^\xi_0 = T_p X \subset T_p \tilde{W}^\xi_0,$$

it follows that

$$T_p X + J(T_p X) \subset T_p \tilde{W}^\xi_0.$$ 

By Lemma 5.8, $\tilde{W}^\xi_0$ is open. Since $\mu^\xi_a : Z \rightarrow \mathbb{R}$ is Morse-Bott of even index, it follows that $\mu^\xi_a : Z \rightarrow \mathbb{R}$ has a unique local maximum and so $\tilde{W}^\xi_0$ is open and dense. Therefore $\mu^\xi_a : X \rightarrow \mathbb{R}$ has also a unique local maximum. This proves

$$\max_{p \in X} \mu^\xi_a = \max_{z \in Z} \mu^\xi_a.$$ 

Since $\mu^\xi_a : X \rightarrow \mathbb{R}$ is Morse-Bott, applying Theorem 2.12 we get that, the unstable manifolds different from $W^\xi_0$ have codimension at least one. This implies that $W^\xi_0$ is also open and dense in $X$. □

Theorem 5.10. If $X$ is an $A$-invariant compact connected coisotropic submanifold of $(Z, \omega)$. Then

$$\mu_a(X) = \mu_a(Z),$$
and so a polytope. Moreover, there exists a subset an open and dense subset \( W \) of \( X \) such that for any \( p \in W \), we have

\[
\mu_a(X) = \mu_a(A \cdot p).
\]

**Proof.** Let \( \xi \in \mathfrak{a} \), by Lemma 5.9

\[
\max_{p \in X} \mu_a^\xi = \max_{z \in \mathbb{Z}} \mu_a^\xi,
\]

and the unstable manifold associated to the maximum of \( \mu_a^\xi \) is open and dense. By Proposition 3.1 in [8], \( \mu_a(X) \) is a polytope. Moreover, the set

\[
\{ p \in X : \mu_a(A \cdot p) = \mu_a(X) \}
\]

is open and dense. Finally, by Proposition 2.3 we have

\[
\mu_a(X) = \mu_a(Z)
\]

concluding the proof. \( \square \)

6. Two orbits variety

In this section we investigate two orbits variety.

**Definition 6.1.** Let \( X \) be a compact and connected \( G \)-stable submanifold of \( (\mathbb{Z}, \omega) \). We say that \( X \) is a two orbit variety if \( G \)-action on \( X \) has two orbits.

S. Cupit-Foutou obtained the classification of a complex algebraic varieties on which a reductive complex algebraic group acts with two orbits [13].

The norm square \( f \) has a maximum and a minimum. By the stratification theorem, keeping in mind that the strata are \( G \)-invariant, \( X \) is the union of a closed \( G \)-orbit \( S_{\beta_{\text{max}}} \), where the norm square achieves the maximum, and an open \( G \)-orbit \( S_{\beta_{\text{min}}} \), the stratum relative to the minimum of the norm square. We then show that \( f \) is a Morse-Bott function.

**Theorem 6.1.** If \( G \) acts on \( X \) with two orbits, then

a) the function \( f : X \to \mathbb{R} \) given by

\[
f(x) := \frac{1}{2} \| \mu_p(x) \|^2 \quad \text{for} \quad x \in X.
\]

is Morse-Bott; It has only two connected critical submanifolds given by the closed \( G \)-orbit \( S_{\beta_{\text{max}}} \), the stratum associated with the maximum of \( f \) and by a \( K \)-orbit \( S_{\beta_{\text{min}}} \), the stratum associated with the minimum of \( f \).
b) The Poincaré polynomial $P_X(t)$ of $X$ satisfies

$$P_X(t) = t^k \cdot P_{S_{\beta_{\text{max}}}}(t) + P_{S_{\beta_{\text{min}}}}(t) - (1 + t)R(t),$$

where $k$ is the real codimension of $S_{\beta_{\text{max}}}$ in $X$ and $R(t)$ is a polynomial with positive integer coefficients. In particular $\chi(X) = \chi(S_{\beta_{\text{max}}}) + \chi(S_{\beta_{\text{min}}});$

c) The $K$-equivariant Poincaré series of $X$ is given by

$$P^K_X(t) = t^k \cdot P^K_{S_{\beta_{\text{max}}}}(t) + P^K_{S_{\beta_{\text{min}}}}(t).$$

**Proof.** We first prove (a). Consider the function $f$ and its critical set $C$. $f$ is non constant on $X$; in fact if $f$ is constant, then every point of $X$ is a maximum point and in view of Proposition 3.5, all $G$-orbit would be closed.

By Theorem 4.6, we have that $S_{\beta_{\text{min}}}$ consist of a single $K$-orbit, and so it is connected.

Since $f$ realizes its maximum value at any critical point $x$ belonging to $S_{\beta_{\text{max}}}$, then by Proposition 3.7d

$$H_x(f) < 0 \quad \text{on} \quad T_x(S_{\beta_{\text{max}}})^\perp.$$  

Now we show that the Hessian of $f$ at a critical point $x$ belonging to $S_{\beta_{\text{min}}}$ is non degenerate in the normal direction. Set $\mu_p(x) = \beta_{\text{min}} = \beta$.

Suppose $\beta \neq 0$. By Remark 3.8,

$$T_xS_{\beta_{\text{min}}} = T_x(G \cdot x) = T_x(K \cdot x) \oplus p^\beta \cdot x \oplus r^{\beta^+} \cdot x,$$

where $r^{\beta^+}$ is the Lie algebra of $R^{\beta^+}$. By Proposition 3.7b,

$$H_x(f) > 0 \quad \text{on} \quad p^\beta \cdot x \oplus r^{\beta^+} \cdot x.$$  

Since $H_x(f) \geq 0$, it follows that

$$T_x(G \cdot x) = T_x(K \cdot x) \oplus (p^\beta \cdot x \oplus r^{\beta^+} \cdot x).$$

Suppose $\beta = 0$. Let $x = x_{\text{min}}$. By Theorem 4.6 $\mu_p^{-1}(0) = K \cdot x$.

$$\ker d\mu_p(x) = (p \cdot x)^\perp.$$  

By Proposition 3.6

$$H_x(f)|_{(p \cdot x)} > 0.$$  

$$T_xX = T_x(G \cdot x).$$
Since $K \cdot x_{\min} \subset \ker d\mu_p(x_{\min})$, it follows that

$$T_x(G \cdot x) = K \cdot x + \mathfrak{p} \cdot x = T_xX$$

$$H_x(f)|_{(K \cdot x)} = 0.$$

$$T_xX = K \cdot x \oplus \mathfrak{p} \cdot x.$$  

By dimensional reason, $H_x(f)$ is non degenerate. These show that $H_x(f)$ is non degenerate. Hence $f$ is Morse-Bott. The statements in (b) and (c) follow from the general theory in [28].

Finally, we point out that the Abelian convexity Theorem holds for a two orbit variety. Indeed, $X$ has a unique closed orbit. By Theorem 5.3 we derive the following result.

**Theorem 6.2.** Let $X$ be a two orbits variety. Let $\mathfrak{a} \subset \mathfrak{p}$ be an Abelian subalgebra. Then $\mu_{\mathfrak{a}}(X)$ is a polytope.

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