MULTI-QUANDLES OF TOPOLOGICAL PAIRS

VLADIMIR TURAEV

ABSTRACT. In generalization of knot quandles we introduce similar algebraic structures associated with arbitrary pairs consisting of a path-connected topological space and its path-connected subspace.

1. INTRODUCTION

A quandle is a set with a binary operation whose axioms are inspired by the Reidemeister moves on knot diagrams. Every knot in the 3-sphere gives rise to a quandle which can be defined in terms of diagrams of the knot or in terms of homotopy classes of paths in the knot exterior. The study of quandles has been an active part of knot theory since the fundamental papers of Joyce [Jo] and Matveev [Ma], see a few sample papers on this subject in the list of references.

This paper was inspired by the following question: can one define analogs of knot quandles for arbitrary topological pairs? More precisely, consider a pair \((X, Y \subset X)\) of path-connected topological spaces. Pick a base point \(x \in X\). By a \(Y\)-path we shall mean a continuous path in \(X\) starting in \(x\) and terminating in \(Y\). By \(Y\)-homotopy (or just homotopy) of \(Y\)-paths we mean a continuous deformation in the class of \(Y\)-paths. During such a deformation, the starting point of the path does not move while the terminal point may slide along \(Y\) in an arbitrary way. Consider the set \(\pi(X, Y, x)\) of the homotopy classes of \(Y\)-paths. We ask whether this set carries a natural quandle-type structure.

The answer turns out to be positive with a few caveats. First, instead of quandles we involve more general algebraic structures which we call multi-quandles. Note that the multi-quandles of the pair \((X, Y)\) corresponding to different choices of the base point \(x \in X\) are isomorphic. Second, the isomorphism class of these multi-quandles is entirely determined by the fundamental groups of \(X, Y\) and the inclusion homomorphism \(\pi_1(Y) \to \pi_1(X)\). This, one can say, is a disappointment as we get no really new invariants of the pair \(X, Y\). Third, in this general setting we do not have knot diagrams which makes it impossible to apply diagrammatic techniques used in the theory of knot quandles. Nevertheless, the study of multi-quandles suggests a number of interesting questions. Presumably, various algebraic methods developed for quandles can be extended to multi-quandles.

The paper starts with the definition of so-called multi-racks. Then we define multi-quandles, discuss a few constructions of multi-quandles, and define multi-quandles of topological pairs.

Recently, algebraic objects analogous to multi-racks and multi-quandles were independently introduced by V. G. Bardakov and D. A. Fedoseev, see [BF].

2. MULTI-RACKS AND MULTI-QUANDLES

We define multi-racks, multi-quandles and give a few simple examples.
2.1. Basics. By a binary operation in a set $U$ we mean an arbitrary mapping $\triangleright : U \times U \to U$. The image of a pair $(u, v) \in U \times U$ under this mapping is denoted $u \triangleright v$. A binary operation $\triangleright$ in $U$ is non-degenerate if for any $v \in U$, the map $U \to U, u \mapsto u \triangleright v$ is bijective. A multi-rack is a pair consisting of a non-empty set $U$ and a collection $\{\triangleright_s\}_{s \in S}$ of non-degenerate binary operations in $U$ such that for any $s, t \in S, u, v, w \in U$ we have

\[(u \triangleright_s v) \triangleright_t w = (u \triangleright_t w) \triangleright_s (v \triangleright_t w)\]

and

\[(u \triangleright_t v) \triangleright_s w = (u \triangleright_s w) \triangleright_t (v \triangleright_s w)\].

In particular, for all $s \in S$ and $u, v, w \in U$ we must have

\[(u \triangleright_s v) \triangleright_s w = (u \triangleright_s w) \triangleright_s (v \triangleright_s w)\].

We call $U$ the underlying set of the multi-rack $(U, \{\triangleright_s\}_{s \in S})$ and call $S$ the set of binary operations in this multi-rack. Of course, if $T$ is a subset of $S$, then $(U, \{\triangleright_s\}_{s \in T})$ is also a multi-rack.

A morphism of multi-racks $(U, \{\triangleright_s\}_{s \in S}) \to (V, \{\triangleright_t\}_{t \in T})$ is a pair of maps $\Phi : U \to V, \varphi : S \to T$ such that for any $u, v \in U, s \in S$ we have

$\Phi(u) \triangleright_{\varphi(s)} \Phi(v) = \Phi(u \triangleright_s v)$.

Multi-racks and their morphisms form a category with respect to the obvious composition of morphisms and the obvious identity morphisms.

A multi-rack $(U, \{\triangleright_s\}_{s \in S})$ is a multi-quandle if $u \triangleright_s u = u$ for all $u \in U, s \in S$. A multi-rack (respectively, a multi-quandle) $(U, \{\triangleright_s\}_{s \in S})$ with $\text{card}(U) = 1$ is a rack (respectively, a quandle).

2.2. Examples. 1. Any non-empty set $U$ with the binary operation $u \triangleright v = u$ for all $u, v \in U$ is a quandle.

2. Consider a group $G$ and a family of commuting automorphisms $\{f_s : G \to G\}_{s \in S}$ of $G$. For $s \in S$, define a map $\triangleright_s : G \times G \to G$ by

$u \triangleright_s v = f_s(u^{-1})v$

for all $u, v \in G$. Since $f_s$ is an automorphism of $G$, the binary operation $\triangleright_s$ is non-degenerate. The identity $u \triangleright_s u = u$ is obvious. Direct computations show that for any $s, t \in S, u, v, w \in G$,

$(u \triangleright_s v) \triangleright_t w = f_s(u^{-1})v \triangleright_t w = f_t(f_s(u^{-1})v)w$

and

$(u \triangleright_t v) \triangleright_s (v \triangleright_t w) = f_t(u^{-1})w \triangleright_s f_t(v^{-1})w$

$= f_t(f_s(u^{-1})w)(f_t(v^{-1})w) = f_t(f_s(u^{-1})v)w$

$= f_s(f_t(u^{-1})w)w = f_t(f_s(u^{-1})v^{-1}w)$

where we use that $f_s, f_t$ are commuting automorphisms of $G$. Hence, the pair $(G, \{\triangleright_s\}_{s \in S})$ is a multi-quandle. To give a specific example, we can take as $G$ a module over a commutative ring and as $\{f_s\}_{s \in S}$ multiplications by invertible elements of this ring.

3. Consider a group $G$ and for each integer $n$ define a binary operations $\triangleright_n$ in $G$ as follows. For any $u, v \in G$, set

$\triangleright_n u, v = v^{-n}uw^n \in G$.

Direct computations show that the pair $(G, \{\triangleright_n\}_{n \in \mathbb{Z}})$ is a multi-quandle.
3. CONSTRUCTIONS OF MULTI-RACKS AND MULTI-QUANDES

The following constructions of multi-racks and multi-quandles play the key role in the definition of multi-quandles of topological pairs.

3.1. Multi-racks. For any group $G$ we define a multi-rack having $G$ as both the underlying set and the set of binary operations. Namely, for any $u, v, s \in G$, set

$$(3.1.1) \quad u \triangleright_s v = vsv^{-1}u \in G.$$  

**Lemma 3.1.** The pair $(G, \{\triangleright_s\}_{s \in G})$ is a multi-rack.

**Proof.** The non-degeneracy of $\triangleright_s$ is obvious. For any $u, v, w, s, t \in G$, we have

$$(u \triangleright_s v) \triangleright_t w = vsv^{-1}u \triangleright_t w = wtws^{-1}vsw^{-1}u = wtw^{-1}vs(wtw^{-1}v)^{-1}wtw^{-1}u = wtw^{-1}vsw^{-1}u.$$  

This proves (3.1.1). □

Note that if $G \neq \{1\}$ then the multi-rack in Lemma 3.1 is not a multi-quandle: for any $s \in G \setminus \{1\}$ we have $s \triangleright_s s = s^2 \neq s$.

3.2. Multi-quandles. Every pair (a group $G$, a subgroup $H \subset G$) gives rise to a multi-quandle as follows. Let $G/H$ be the set of left cosets of $H$ in $G$ and let $p = p_H : G \to G/H$ be the projection carrying any $u \in G$ to the left coset $uH$. Let $\{\triangleright_s\}_{s \in G}$ be the binary operations (3.1.1) in $G$.

**Lemma 3.2.** For any element $s$ of the center of $H$, there is a unique binary operation $\triangleright_s^s$ in $G/H$ such that the diagram

$$
\begin{array}{ccc}
G \times G & \overset{\triangleright_s}{\longrightarrow} & G \\
p \times p & \downarrow & p \\
G/H \times G/H & \overset{\triangleright_s^s}{\longrightarrow} & G/H
\end{array}
\tag{3.2.1}
$$

commutes.

**Proof.** Pick any $u, v \in G$ and $g, h \in H$. By definition,

$$ug \triangleright_s vh = vhs(wh)^{-1}ug = vhs(h^{-1}v^{-1}ug = vsv^{-1}ug = (u \triangleright_s v)g$$

where we use that $hs = sh$. Consequently,

$$p(ug \triangleright_s vh) = p(u \triangleright_s v).$$

Now it is clear that the formula

$$uH \triangleright_s^s vH = (u \triangleright_s v)H$$

defines the unique binary operation $\triangleright_s^s$ in $G/H$ satisfying the conditions of the lemma. □

We shall denote the center of a group $H$ by $Z(H)$.

**Theorem 3.3.** Under the assumptions of Lemma 3.2 the pair $(G/H, \{\triangleright_s^s\}_{s \in Z(H)})$

is a multi-quandle.
Proof. Pick \( s, t \in Z(H) \) and \( u, v, w \in G \). Lemma 3.2 implies that
\[
p((u \triangleright_s v) \triangleright_t w) = p(u \triangleright_s v) \triangleright_t^o p(w) = (p(u) \triangleright_s^o p(v)) \triangleright_t^o p(w).
\]
Similarly,
\[
p((u \triangleright_t v) \triangleright_s w) = (p(u) \triangleright_t^o p(w)) \triangleright_s^o (p(v) \triangleright_t^o p(w)).
\]
Now, the equality \([2.1.1] \) in \( G \) implies that
\[
(p(u) \triangleright_s^o p(v)) \triangleright_t^o p(w) = (p(u) \triangleright_t^o p(w)) \triangleright_s^o (p(v) \triangleright_t^o p(w)).
\]
Since the projection \( p : G \to G/H \) is surjective, we conclude that the equality \([2.1.1] \) holds for all elements of \( G/H \) and all \( s, t \in Z(H) \).

Next we observe that
\[
p(u) \triangleright_s^o p(u) = p(u \triangleright_s u) = p(usu^{-1}u) = p(us) = p(u)
\]
where the last equality holds because \( s \in H \). Thus, \( p(u) \triangleright_s^o p(u) = p(u) \). The surjectivity of \( p \) implies that \( x \triangleright_s^o x = x \) for all \( x \in G/H \).

It remains to prove the non-degeneracy of the operations \( \triangleright_s^o \) in \( G/H \). We must show that for any \( s \in Z(H), v \in G \), the map \( f : G/H \to G/H \) defined by \( f(x) = x \triangleright_s^o p(v) \) is bijective. First, we prove the surjectivity. Denote the bijection \( u \mapsto u \triangleright_s v : G \to G \) by \( F \). For any \( w \in G \), we have
\[
f(p(F^{-1}(w))) = p(F^{-1}(w)) \triangleright_s^o p(v) = p(F^{-1}(w) \triangleright_s v) = p(FF^{-1}(w)) = p(w).
\]
Therefore \( p(w) \in \text{Im} f \) and \( f \) is surjective. Suppose now that two elements \( x_1, x_2 \) of \( G/H \) satisfy \( f(x_1) = f(x_2) \). For \( i = 1, 2 \), pick any \( w_i \in p^{-1}(x_i) \subset G \). Then
\[
p(w_i \triangleright_s v) = p(w_i) \triangleright_s^o p(v) = x_i \triangleright_s^o p(v) = f(x_i).
\]
The equality \( f(x_1) = f(x_2) \) implies that the elements \( \{w_i \triangleright_s v\}_{i=1,2} \) of \( G \) lie in the same left coset of \( H \). By the definition of \( \triangleright_s^o \), this means that the elements \( \{usv^{-1}w_i\}_{i=1,2} \) of \( G \) lie in the same left coset of \( H \). Consequently, \( w_1, w_2 \in G \) lie in the same left coset of \( H \). Therefore \( x_1 = p(w_1) = p(w_2) = x_2 \). We conclude that the mapping \( f \) is injective. This fact and the surjectivity of \( f \) implies that \( f \) is a bijection. This completes the proof of the non-degeneracy of \( \triangleright_s^o \) and of the theorem. \( \square \)

4. Multi-quandles in topology

Consider a pair \( (X, Y \subset X) \) of non-empty path-connected topological spaces. For any base point \( x \in X \), we define a structure of a multi-quandle in the set \( \pi(X, Y, x) \) of \( Y \)-homotopy classes of \( Y \)-paths leading from \( x \) to \( Y \), as defined in the introduction. We also explain that our construction includes the standard knot quandles as a special case.

4.1. The case \( x \in Y \). The case where \( x \in Y \) is simpler and we start with it. Set \( G = \pi_1(X, x) \) and let \( H \subset G \) be the image of the inclusion homomorphism \( \pi_1(Y, x) \to \pi_1(X, x) = G \). We first construct a bijection \( G/H \to \pi(X, Y, x) \). Since \( x \in Y \), any path in \( X \) starting and ending in \( x \) is a \( Y \)-path. Considering the homotopy classes of such paths we obtain a mapping \( P : G \to \pi(X, Y, x) \). The path-connectedness of \( Y \) implies that the mapping \( P \) is surjective. Our definitions ensure that \( P \) is a composition of the projection \( p : G \to G/H \) and a mapping \( G/H \to \pi(X, Y, x) \). The latter mapping is bijective: its inverse is obtained by
assigning to any $Y$-path $\gamma : [0, 1] \to X$ the left coset $p(\gamma \beta) \in G/H$ where $\beta$ is any path in $Y$ from $\gamma(1) \in Y$ to $x$ and $[\gamma \beta] \in G$ is the homotopy class of the loop $\gamma \beta$.

Theorem 3.3 yields a structure of a multi-quandle in $G/H$. Transporting this structure along the bijection $G/H \to \pi(X, Y, x)$ we obtain a structure of a multi-quandle in the set $\pi(X, Y, x)$ with binary operations numerated by elements of the center of the group $H$.

4.2. The general case. Suppose now that $x$ is an arbitrary point of $X$. Fix a path $\alpha$ in $X$ leading from $x$ to a point $y \in Y$. This path determines a bijection $\pi(X, Y, y) \to \pi(X, Y, x)$ carrying the $Y$-homotopy class of any path $\rho$ from $y$ to $Y$ into the $Y$-homotopy class of the product path $\alpha \rho$ from $x$ to $Y$. Transporting the multi-quandle structure in $\pi(X, Y, y)$ constructed above to $\pi(X, Y, x)$ along this bijection we obtain a multi-quandle structure in $\pi(X, Y, x)$. Generally speaking, the latter structure depends on the choice of the path $\alpha$. However, the multi-quandles associated with different $x, y, \alpha$ are isomorphic (exercise). We denote the isomorphism class of these multi-quandles by $Q(X, Y)$.

4.3. The knot quandles. Consider a path-connected manifold $M$ of dimension $\geq 2$ and its codimension 2 path-connected submanifold $K \subset \text{Int}(M)$. We assume that the normal bundle of $K$ in $M$ is trivial and oriented. Let $E = E_K$ be the exterior of $K$ in $M$. The manifold $\partial E$ consists of $\partial M$ and the product $K \times S^1$ bounding a regular neighborhood of $K$ in $M$. Applying the constructions above to the pair $E \supset K \times S^1$, we obtain a multi-quandle $Q = Q(E, K \times S^1)$. The binary operations in $Q$ are numerated by the elements of the abelian group $Z(H)$ where $H$ is the image of the inclusion homomorphism $\pi_1(K \times S^1) \to \pi_1(E)$. The group $Z(H)$ has a distinguished element represented by the meridian circle of $K$.

Keeping only the corresponding binary operation in $Q$ (and forgetting the binary operations corresponding to all other elements of $Z(H)$), we obtain the Joyce-Matveev quandle of the pair $(M, K)$. This construction applies in particular to knots in $S^3$ and yields the Joyce-Matveev quandles of knots.

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Vladimir Turaev
Department of Mathematics
Indiana University
Bloomington IN47405, USA
and
IRMA, Strasbourg
7 rue Rene Descartes
67084 Strasbourg, France
vturaev@yahoo.com