Generalized selection problem with Lévy noise

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Abstract
Let $A_{\pm} > 0$, $\beta \in (0, 1)$, and let $Z^{(\alpha)}$ be a strictly $\alpha$-stable Lévy process with the jump measure $\nu(dz) = (C_{+} \mathbb{1}_{[0,\infty)}(z) + C_{-} \mathbb{1}_{[-\infty,0)}(z))|z|^{-1-\alpha} dz$, $\alpha \in (1, 2)$, $C_{\pm} \geq 0$, $C_{+} + C_{-} > 0$. The selection problem for the model stochastic differential equation $d\bar{X}^{\varepsilon} = (A_{+} \mathbb{1}_{[0,\infty)}(\bar{X}^{\varepsilon}) - A_{-} \mathbb{1}_{[-\infty,0)}(\bar{X}^{\varepsilon}))|\bar{X}^{\varepsilon}|^\beta dt + \varepsilon dZ^{(\alpha)}$ states that in the small noise limit $\varepsilon \to 0$, solutions $\bar{X}^{\varepsilon}$ converge weakly to the maximal or minimal solutions of the limiting non-Lipschitzian ordinary differential equation $d\bar{x} = (A_{+} \mathbb{1}_{[0,\infty)}(\bar{x}) - A_{-} \mathbb{1}_{[-\infty,0)}(\bar{x}))|\bar{x}|^\beta dt$ with probabilities $\bar{p}_{\pm} = \bar{p}_{\pm}(\alpha, C_{+}, C_{-}, \beta, A_{+}, A_{-})$, see [Pilipenko and Proske, Stat. Probab. Lett., 132:62–73, 2018]. In this paper we solve the generalized selection problem for the stochastic differential equation $dX^{\varepsilon} = a(X^{\varepsilon}) dt + \varepsilon b(X^{\varepsilon}) dZ$ whose dynamics in the vicinity of the origin in certain sense reminds of dynamics of the model equation. In particular we show that solutions $X^{\varepsilon}$ also converge to the maximal or minimal solutions of the limiting irregular ordinary differential equation $dx = a(x) dt$ with the same model selection probabilities $\bar{p}_{\pm}$. This means that for a large class of irregular stochastic differential equations, the selection dynamics is completely determined by four local parameters of the drift and the jump measure.

Keywords: Lévy process; stochastic differential equation; selection problem; zero noise limit; Peano theorem; non-uniqueness; irregular drift

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1 Introduction, setting, and the main result

The well known Peano existence theorem [Hartman, 1964, Theorem II.2.1] states that an ordinary differential equation (ODE) $dx = a(x) dt$ with a continuous function $a: \mathbb{R} \to \mathbb{R}$ has a local solution which however may

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be not unique. A classical example of such non-uniqueness is given by the non-Lipschitzian \( dx = \sqrt{|x|} \, dt \) which allows for a continuum of solutions starting at \( x = 0 \), namely \( x(t) = \frac{1}{2}(t - t_0)^2, \; t \geq 0 \).

On the contrary, the behaviour of stochastic differential equations (SDE) is often more regular. In particular and addition of a noise term allows to obtain unique solutions of SDEs with measurable or irregular coefficients. We refer the reader to, e.g., [Zvonkin, 1974; Strook and Varadhan, 1979; Veretennikov, 1981; Krylov and Röckner, 2005] for results on SDEs driven by a Brownian motion. General results on the existence and uniqueness of SDEs with measurable or irregular coefficients driven by Lévy processes can be found, e.g., in [Tanaka et al., 1974; Gikhman and Skorokhod, 1982; Situ, 2005; Priola, 2012; Chen and Wang, 2016; Priola, 2018; Kulik, 2019].

Consider now an SDE with a drift \( \beta x \) and assume that the underlying ODE \( dx = a(x) \, dt \) has multiple solutions. A natural question arises, what happens when the random perturbation vanishes. Heuristically, solutions of the small noise SDE should converge to one of the various deterministic solutions and the selection problem consists in description of this limit behaviour.

Originally, the selection problem was treated by Bafico and Baldi (1982), where the authors considered the SDE

\[
X^\varepsilon(t) = x + \int_0^t a(X^\varepsilon(s)) \, ds + \varepsilon \int_0^t b(X^\varepsilon(s)) \, dW(s), \tag{1.1}
\]

with a drift \( a \) being not Lipschitz continuous at \( x = 0 \), and a positive Lipschitz continuous diffusion coefficient \( b^2 \). They showed that under certain conditions, the limit law \( \text{Law}(X^\varepsilon|X^\varepsilon(0) = 0) \) is supported by the deterministic maximal and minimal solutions of the ODE \( dx = a(x) \, dt \) starting at zero with the weights \( p_\pm \) that can be explicitly determined, see [Bafico and Baldi, 1982, Theorem 4.1]. Veretennikov (1983) proved the uniqueness of the limit in the case of odd continuous concave drift and additive noise. Recently Delarue and Flandoli (2014) gave the new proof of the results by Bafico and Baldi (1982) for the piece-wise power drift

\[
\tilde{a}(x) = A_+ x^{\beta} 1_{[0,\infty)}(x) - A_- |x|^{\beta} 1_{(-\infty,0)}(x) \tag{1.2}
\]

with \( \beta \in (0,1) \) and \( A_+ > 0 \) in the case of additive Brownian perturbations. Trevisan (2013) studied the same equation with \( A_+ = 1 \) and \( \beta \in [0,1) \). Krykun and Makhno (2013) generalized the results by Bafico and Baldi (1982) to Itô SDEs with positive diffusion coefficient \( a \) of locally bounded variation. Gradinariu et al. (2001) analyzed large deviations of the laws of \( X^\varepsilon \) with additive noise and piece-wise power drift (1.2) with \( A_+ = 1, \beta \in [0,1) \).

Although the results obtained in Bafico and Baldi (1982) are very transparent and intuitively understandable, the intrinsic nature of the selection phenomena and especially the methods allowing one to derive the selection probabilities \( p_\pm \) in more general settings are far from being completely understood.

Thus, Pilipenko and Proske (2018a) considered a class of SDEs \( dX^\varepsilon = \tilde{a}(X^\varepsilon) \, dt + \varepsilon dB^{(\alpha)}(t) \) with \( \beta \in (-1,1) \) driven by \( \alpha \)-self-similar processes \( B^{(\alpha)} \), e.g. by a fractional Brownian motion or a strictly stable Lévy process. They showed that under some natural assumptions \( X^\varepsilon \) also selects the maximal and minimal solutions of the ODE \( dx = \tilde{a}(x) \, dt \) with some probabilities \( p_\pm, p_+ + p_- = 1 \). Unfortunately these probabilities cannot be always determined explicitly.

In this paper we address the selection problem for a Lévy driven SDE with multiplicative noise

\[
X^\varepsilon(t) = x + \int_0^t a(X^\varepsilon(s)) \, ds + \varepsilon \int_0^t b(X^\varepsilon(s)) \, dZ(s), \quad t \geq 0, \quad \varepsilon \to 0, \tag{1.3}
\]

whose drift \( a = a(x) \) has an irregular point at \( x = 0 \) but does not have the exact piece-wise power form (1.2).

The small jumps of the driving Lévy process \( Z \) remind of those of an \( \alpha \)-stable Lévy process. In other words we answer the question whether the selection dynamics are robust w.r.t. perturbations of the drift and the noise.

Let us formulate the precise assumptions.

**A\(_Z\):** Let \( Z = (Z_t)_{t \geq 0} \) be a Lévy process without a Gaussian component and the jump measure \( \nu \) such that for some \( \alpha \in (1,2) \) and some constants \( C_{\pm} \geq 0, \; C_- + C_+ > 0 \),

\[
\nu([-\infty, +\infty)) \sim C_+ z^{-\alpha} l_{\nu} \left( \frac{1}{z} \right), \quad \nu((-\infty, -\infty)) \sim C_- z^{-\alpha} l_{\nu} \left( \frac{1}{z} \right), \quad z \to +0, \tag{1.4}
\]

where \( l_{\nu} \) is the Lévy measure of \( Z \) and \( C_0 = \int_{\mathbb{R}} |x|^\alpha \rho(dx) \) for the probability density of \( Z \) given by the Lévy-Khintchine formula.
for a positive function \( l \) slowly varying at infinity.

**A\(_a\):** Let \( x \mapsto a(x) \) be a real valued continuous function of linear growth such that \( a(0) = 0 \) and that for \( \beta \in (0,1) \)
\[
a(x) = x^\beta L_+(x) \quad \text{for} \quad x > 0 \quad \text{and} \quad a(x) = -|x|^\beta L_-(|x|) \quad \text{for} \quad x < 0,
\]
with continuous functions \( L_\pm : (0, \infty) \to (0, \infty) \) that satisfy
\[
L_\pm(x) \sim A_\pm \left( \frac{1}{x} \right) \quad \text{as} \quad x \to +0,
\]
for a positive function \( l \) slowly varying at infinity, and \( A_\pm > 0 \).

**A\(_b\):** Let \( x \mapsto b(x) \) be a bounded continuous real valued function such that
\[
b_\pm(0) > 0.
\]
It follows from assumptions **A\(_a\), A\(_b\)** that equation (1.3) has a weak solution, see, e.g. Theorem 1 of §2 Chapter 5 in Gikhman and Skorokhod (1982).

**Remark 1.1.** We will see in the main result that the weak limit of the sequence \( \{X^\varepsilon\} \) as \( \varepsilon \to 0 \) is independent of the choice of weak solution \( X^\varepsilon \). So, further we assume that \( X^\varepsilon \) is any weak solution to (1.3). It should be also noticed that the presence of a noise often implies uniqueness of a solution and the strong Markov property, see references above.

Let us describe solutions of the limit ODE
\[
X^0_x(t) = x + \int_0^t a(X^0_x(s)) \, ds.
\]
Let \( A_\pm(\cdot) \) be continuous non-negative strictly increasing functions given by
\[
A_+(x) := \int_0^x \frac{dy}{a(y)}, \quad x > 0,
\]
\[
A_-(x) := \int_0^{-x} \frac{dy}{a(y)}, \quad x < 0,
\]
\[
A_\pm(0) = 0,
\]
and let \( A_\pm^{-1}(\cdot) : [0, \infty) \to [0, \infty) \) be their inverses. All these functions are well defined because of assumption **A\(_a\)**. Hence it is immediate to see that for \( x \neq 0 \)
\[
X^0_x(t) := A_+^{-1}(A_+(x) + t), \quad x > 0, \quad t \geq 0,
\]
and
\[
X^0_x(t) := -A_-^{-1}(A_-(x) + t), \quad x < 0, \quad t \geq 0,
\]
are unique solutions of the equation (1.8). For \( x = 0 \), there is a continuum of solutions and any solution either has the form
\[
X^0_\pm(t; t_0) = \begin{cases} 0, & t \in [0, t_0), \\ \pm A_\pm^{-1}(t - t_0), & t \in [t_0, \infty), \end{cases}
\]
where \( t_0 \in [0, +\infty) \) or is trivial \( X^0(0) = 0 \). Among the solutions (1.12) we single out the maximal and the minimal solutions
\[
x^\pm(t) := X^0_\pm(t; 0) = \pm A_\pm^{-1}(t), \quad t \geq 0.
\]
It is intuitively clear that any solution \( X^\varepsilon \) starting at zero should select one of the particular solutions \( x^\pm \) of (1.8) very quickly, so that one can expect that the selection is determined only by the small jumps of \( Z \).
Remark 1.3. If in $D$ are independent of and the probabilities $\bar{p}_\alpha$ to $\bar{\sigma}$ (although in Tanaka et al. (1974) the drift is supposed to be bounded, an extension of their results to $\alpha$ given by (1.2) follows easily from the sublinear growth of $\bar{a}$ at infinity).

The model equation (1.14) has a unique strongly Markovian solution due to Theorem 3.1 from Tanaka et al. (1974) (although in Tanaka et al. (1974) the drift is supposed to be bounded, an extension of their results to $\bar{a}$ given by (1.2) follows easily from the sublinear growth of $\bar{a}$ at infinity). The model ODE $dx = \bar{a}(x) dt$ has the following maximal and minimal solutions starting at $x = 0$:

$$x^\pm(t) = \pm(A_\pm(1-\beta)t)^{1/\beta}, \quad t \geq 0. \quad (1.17)$$

The selection problem for the model SDE (1.14) was solved by Pilipenko and Proske (2018a).

**Theorem 1.2** (Pilipenko and Proske (2018a)). Let $\bar{X}^\varepsilon$ be a solution to the model equation (1.14). Then

1) for any $\varepsilon > 0$

$$P\left( \lim_{t \to \infty} |\bar{X}^\varepsilon(t)| = +\infty \right) = 1 \quad (1.18)$$

and the probabilities

$$\bar{p}_\pm = P\left( \lim_{t \to \infty} \bar{X}^\varepsilon(t) = \pm \infty \right) \quad (1.19)$$

are independent of $\varepsilon$, and $\bar{p}_- + \bar{p}_+ = 1$;

2) the convergence

$$\text{Law } \bar{X}^\varepsilon \Rightarrow \bar{p}_- \delta_{x^-} + \bar{p}_+ \delta_{x^+}, \quad \varepsilon \to 0, \quad (1.20)$$

in $D([0, +\infty), \mathbb{R})$ holds true, where $\bar{x}^\pm$ are defined in (1.17).

**Remark 1.3.** If $\alpha = 2$, i.e. if $Z^{(\alpha)}$ is a Brownian motion then the probabilities $\bar{p}_\pm$ are known explicitly:

$$\bar{p}_- = \frac{A_+ \frac{1}{\alpha}}{A_- \frac{1}{\alpha - \beta} + A_+ \frac{1}{\alpha - \beta}}, \quad \text{and} \quad \bar{p}_+ = \frac{A_- \frac{1}{\alpha}}{A_- \frac{1}{\alpha - \beta} + A_+ \frac{1}{\alpha - \beta}}. \quad (1.21)$$

see Bafico and Baldi (1982); Delarue and Flandoli (2014).

**Remark 1.4.** It follows from the self-similarity of $Z^{(\alpha)}$ that for any $\varepsilon, \delta, \gamma > 0$ the rescaled process $\bar{X}^{\gamma, \delta, \varepsilon}(t) := \gamma \bar{X}^{\delta, \varepsilon}(\delta t)$, $t \geq 0$, satisfies the SDE

$$\bar{X}^{\gamma, \delta, \varepsilon}(t) = \int_0^t \gamma^{1-\beta} \delta \bar{a}(\bar{X}^{\gamma, \delta, \varepsilon}(s)) ds + \varepsilon \gamma \delta \frac{\bar{a}}{\bar{\sigma}} \cdot \bar{Z}^{(\alpha)}(t), \quad (1.22)$$

where $\bar{Z}^{(\alpha)} \overset{d}{=} Z^{(\alpha)}$. This implies that the selection probabilities $\bar{p}_\pm$ defined in (1.19) are the same for any model equation

$$\bar{X}^\varepsilon(t) = \int_0^t \bar{a}(\bar{X}^\varepsilon(s)) ds + \varepsilon \cdot \sigma \cdot Z^{(\alpha)}(t) \quad (1.23)$$

with any $\sigma > 0$. Moreover, they are completely determined by the four parameters $\alpha \in (1, 2)$, $C_+/C_- \in [0, +\infty]$, $\beta \in (0, 1)$, and $A_+/A_- \in (0, \infty)$. 


In the present paper we solve the generalized selection problem for the SDE (1.3). The main result of this paper is the following.

**Theorem 1.5.** Let assumptions $A_Z$, $A_a$, and $A_b$ hold true, and let $X^\varepsilon$ be a solution to (1.3) with the initial condition $x = 0$, namely

$$X^\varepsilon(t) = \int_0^t a(X^\varepsilon(s)) \, ds + \varepsilon \int_0^t b(X^\varepsilon(s-)) \, dZ(s), \quad t \geq 0.$$  

Then

$$\text{Law} \, X^\varepsilon \Rightarrow \bar{\mu}_- \delta_{x^-} + \bar{\mu}_+ \delta_{x^+}, \quad \varepsilon \to 0,$$

in $D([0, \infty), \mathbb{R})$ where functions $x^\pm$ are defined in (1.13) and the selection probabilities $\bar{\mu}_\pm$ are determined in Theorem 1.2 for the model equation (1.14).

Before proceeding with the proofs we give several clarifying remarks.

**Remark 1.6.** Theorem 1.5 states that the generalized selection probabilities of the equation (1.24) coincide with the selection probabilities $\bar{\mu}_\pm$ of the model equation (1.14). Hence the selection behaviour is robust with respect to appropriate perturbations of a) the drift, b) the Lévy measure in the vicinity of the origin, and c) with respect to incorporation of the multiplicative noise. Essentially, the selection probabilities for the whole class of SDEs (1.24) depend only on the four parameters of the model equation.

Our results agree with the results by Bafico and Baldi (1982) for Gaussian diffusions ($\alpha = 2$) where $\bar{\mu}_\pm$ were determined in terms of certain integrals of $a(x)/b(x)$, see Eq. (3.4) and Theorem 4.1 in Bafico and Baldi (1982).

**Remark 1.7.** If $x \neq 0$, then it is easy to verify that

$$\text{Law} \, X^\varepsilon_x \Rightarrow \delta_{X^0_x}, \quad \varepsilon \to 0,$$

with $X^0_x$ defined in (1.10) and (1.11).

**Remark 1.8.** We emphasize that although we do not assume uniqueness of (weak) solutions $X^\varepsilon$ of (1.24) for $\varepsilon \geq 0$, the weak limit (1.25) is unique.

**Remark 1.9.** The question how to determine the selection probabilities $\bar{\mu}_\pm$ is still open. Although the results by Pilipenko and Proske (2018a) establish the existence of $\bar{\mu}_\pm$ for the model equation for self-similar noises it is clear that quite different methods should be used for SDEs driven by Lévy processes or, say, a by a fractional Brownian motion.

**Remark 1.10.** Eventually we note that Theorem 1.5 gives us the existence and uniqueness of the weak limit. There is a number of works in which pathwise restoration of uniqueness for ODEs with an irregular or even distributional drift $a$ by adding a random perturbation is studied. For example the regularization by adding a sample Brownian path was studied by David (2007, 2011); Flandoli (2011a); Shaposhnikov (2016); Alabert and León (2017); Baños et al. (2018, 2019a). The same problem for the fractional Brownian motion was treated by Baños et al. (2019a); Catellier and Gubinelli (2019a); Barrini and Ouahhind (2016); Amine et al. (2017); Harang and Perkowski (2020); Galeati and Gubinelli (2020).

**Remark 1.11.** The selection problem in a multidimensional setting was also tackled recently by Pilipenko and Proske (2018a) and Delarue and Maurelli (2019). Small noise behaviour of multidimensional SDEs with discontinuous drift was also studied by Buckdahn et al. (2009) in the setting of differential inclusions.

The rest of the paper is devoted to the proof of the main result. To make the arguments more transparent we preface the proof with a heuristic description of the steps and explain the structure of the paper.

First we consider the process $X^\varepsilon$ and note that due to the boundedness of $b$ and the sublinear growth and the continuity of the drift $a$, the family of distributions $\{\text{Law}(X^\varepsilon)\}_{\varepsilon \in (0, 1]}$ is tight in $D([0, \infty), \mathbb{R})$ and any (weak) limit point is a solution of the ODE (1.18) with $x = 0$. All possible solutions to (1.18) have been described in (1.12).

To prove Theorem 1.5 it suffices to show two properties of the limit laws of $X^\varepsilon$ as $\varepsilon \to 0$. First, a process $X^\varepsilon$ can spend only infinitesimal time near zero and hence it chooses either the maximal or the minimal
solutions $x^\pm$ of the ODE (1.8); in other words, no solution $X^0(\cdot; t_0)$ with $t_0 > 0$ (see (1.12)) can support the limiting law of $X^\varepsilon$. Second, the deterministic solutions $x^\pm$ should be chosen with the probabilities $\tilde{p}_\pm$ determined in (1.19).

Hence we will show that the selection takes place with probabilities $\tilde{p}_\pm$ in an infinitesimal time-space box $t \in [0, T_0\varepsilon']$, $x \in [-R\varepsilon'', R\varepsilon'']$ with appropriately chosen bounds $\varepsilon' = \varepsilon'(\varepsilon) \to 0$ and $\varepsilon'' = \varepsilon''(\varepsilon) \to 0$ and $T_0 > 0$, $R > 0$ large enough. To achieve this, we introduce a rescaled process $Y^\varepsilon(t) := X^\varepsilon(\varepsilon't)/\varepsilon''$ and show that $Y^\varepsilon$ converges weakly to a solution of the model equation (1.23) with $\sigma = b(0)$. Hence the exit of $X^\varepsilon$ from the infinitesimal time-space box $[0, T_0\varepsilon'] \times [-R\varepsilon'', R\varepsilon'']$ is equivalent to the exit of $Y^\varepsilon$ from the $\varepsilon$-independent time-space box $[0, T_0] \times [0, R]$ which is controlled by Theorem 1.9.

The second step is to show that upon leaving the $\varepsilon$-dependent time-space box $[0, T_0\varepsilon'] \times [-R\varepsilon'', R\varepsilon'']$ with $R > 0$ sufficiently large, a solution $X^\varepsilon$ with high probability follows the maximal (minimal) solution $x^\pm$ as $\varepsilon \to 0$. Here it suffices to construct a deterministic increasing (decreasing) function that bounds $X^\varepsilon$ from below (above) with high probability.

The paper is organized as follows. In Section 2 we chose the appropriate scales $\varepsilon'$ and $\varepsilon''$ and show that the rescaled process $Z_\varepsilon(t) = Z(\varepsilon't)/\varepsilon''$ converges to the $\alpha$-stable process $Z^{(\alpha)}$ defined in (1.15) and the rescaled process $Y^\varepsilon$ converges to the solution of the $\varepsilon$-independent model equation. In Section 3 we obtain algebraic growth rates of the noise term $\int_0^1 b(X^\varepsilon(s-))\, dZ(s)$ that are uniform over $\varepsilon \in (0, 1]$ and the initial value $X^\varepsilon(0)$. In Section 4 we study the exit of $X^\varepsilon$ from the time-space box $[0, T_0\varepsilon'] \times [-R\varepsilon'', R\varepsilon'']$. In Section 5 we determine deterministic lower and upper bounds that push a solution $X^\varepsilon$ with an initial value $|X^\varepsilon(0)| \geq R\varepsilon''$ away from zero with high probability. This will finish the proof of Theorem 1.9.

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## 2 Preliminary considerations and time-space rescaling

Before starting the proof we make two technical assumptions that do not reduce the generality of the setting but simplify the arguments significantly.

**Remark 2.1.** To establish convergence (1.25) it suffices to show the weak convergence on the space $D([0, T], \mathbb{R})$ for each $T > 0$. We will use the truncation of large jumps procedure.

For $M > 0$, let

$$Z^M(t) = Z(t) - \sum_{s \leq t} \Delta Z(s) \cdot I(|\Delta Z(s)| > M) \tag{2.1}$$

be the Lévy process with bounded jumps. For each $T > 0$ and $\theta > 0$ we can find $M > 0$ large enough such that

$$P \left( Z(t) = Z(t)^M, t \in [0, T] \right) = 1 - \exp \left( -T \int_{|z| > M} \nu(dz) \right) \geq 1 - \theta. \tag{2.2}$$

Then for any solution $X^\varepsilon$ there exists a solution $X^{\varepsilon, M}$ of (1.3) driven by the process $Z^M$ such that

$$P \left( X(t)^\varepsilon = X(t)^{\varepsilon, M}, t \in [0, T] \right) \geq 1 - \theta \tag{2.3}$$

(we consider all processes on an appropriate probability space). Hence in order to prove weak convergence of the processes $X^\varepsilon$ it is sufficient to prove convergence for the processes $X^{\varepsilon, M}$ under the additional assumption that for some $M > 0$

$$\text{supp } \nu \subseteq [-M, M] \quad \text{and} \quad \nu(\{|z| = M\}) = 0. \tag{2.4}$$

From now on we assume (2.4) to hold for the process $Z$.

**Remark 2.2.** Similarly to the previous remark we also note that for any two drifts $a$ and $\tilde{a}$ both satisfying $A_a$ and such that $a(x) = \tilde{a}(x)$, $|x| \leq 1$, the corresponding solutions $X^\varepsilon$ and $\tilde{X}^\varepsilon$ coincide up to the exit from
for each $T$ criterion and boundedness.

**Lemma 2.3.** Assume that assumptions $A_a$ and $A_b$ are satisfied. Then the family of distributions $\{\text{Law}(X^\varepsilon)\}_{\varepsilon \in (0,1]}$ is tight in $D([0,\infty), \mathbb{R})$ and a limit $X$ of any weakly convergent subsequence $\{X^\varepsilon_n\}_{n \geq 1}$, $X^\varepsilon_n \Rightarrow X$, $n \to \infty$, satisfies the integral equation

$$X(t) = \int_0^t a(X(s)) \, ds.$$  

**Proof.** First we note that since $a$ is bounded,

$$\varepsilon \int_0^t b(X^\varepsilon(s-)) \, dZ(s) \Rightarrow 0, \quad \varepsilon \to 0,$$

weakly in the uniform topology. Tightness of $\{\text{Law}(X^\varepsilon)\}_{\varepsilon \in (0,1]}$ follows, e.g. from the continuity of $a$, Aldous’ criterion and boundedness

$$\sup_{\varepsilon \in (0,1]} \mathbb{E} \sup_{t \in [0,T]} |X^\varepsilon_t|^2 \leq C(T) < \infty$$

for each $T > 0$ and some $C(T) > 0$.

Finally, due to the continuity of $a$, for any weakly convergent subsequence $X^\varepsilon_n \Rightarrow X$ we get the weak convergence of the pairs

$$\left(X^\varepsilon_n(\cdot), \int_0^\cdot a(X^\varepsilon_n(s)) \, ds\right) \Rightarrow \left(X(\cdot), \int_0^\cdot a(X(s)) \, ds\right), \quad n \to \infty,$$

in $D([0,\infty), \mathbb{R}) \times C([0,\infty), \mathbb{R})$ which together with (2.7) implies the result. \hfill \Box

Let $X^\varepsilon$ be any solution of (1.24). For any $\varepsilon' = \varepsilon'(\varepsilon) > 0$ and $\varepsilon'' = \varepsilon''(\varepsilon) > 0$ consider a time-space rescaled process

$$Y^\varepsilon(t) = \frac{X^\varepsilon(\varepsilon't)}{\varepsilon''}, \quad t \geq 0,$$

which satisfies the SDE

$$Y^\varepsilon(t) = \frac{X^\varepsilon(\varepsilon't)}{\varepsilon''} = \frac{1}{\varepsilon''} \int_0^t a(X^\varepsilon(s)) \, ds + \frac{\varepsilon'}{\varepsilon''} \int_0^t b(X^\varepsilon(s-)) \, dZ(s)$$

$$= \int_0^t \frac{a(\varepsilon''Y^\varepsilon(s))}{\varepsilon''/\varepsilon'} \, ds + \int_0^t \frac{b(\varepsilon''Y^\varepsilon(s-))}{\varepsilon''/\varepsilon'} \, dZ(s)$$

$$= \int_0^t a_\varepsilon(Y^\varepsilon(s)) \, ds + \int_0^t b_\varepsilon(Y^\varepsilon(s-)) \, dZ(s),$$

where

$$a_\varepsilon(y) = \frac{a(\varepsilon''y)}{\varepsilon''/\varepsilon'}, \quad b_\varepsilon(y) = b(\varepsilon''y), \quad Z_\varepsilon(t) = \frac{Z(\varepsilon't)}{\varepsilon''/\varepsilon'}.$$  

**Lemma 2.4.** There exist positive null sequences $\varepsilon' = \varepsilon'(\varepsilon)$ and $\varepsilon'' = \varepsilon''(\varepsilon)$ such that

$$\lim_{\varepsilon \to 0} \frac{\varepsilon''}{\varepsilon'} = 0$$

and

$$\frac{\varepsilon''}{\varepsilon'} \sim (\varepsilon'')^\beta \left(\frac{1}{\varepsilon'}\right),$$

$$\varepsilon' \sim \left(\frac{\varepsilon''}{\varepsilon'}\right)^\alpha \left(\frac{1}{\varepsilon'}\right)^{-1} \quad \text{as} \quad \varepsilon \to 0.$$
Proof. Recall that a product, a sum, and a ratio of two positive slowly varying functions is again a slowly varying function (Proposition 1.3.6 in Bingham et al. (1987)). Furthermore due to Theorem 1.5.12 from Bingham et al. (1987), each regularly varying function \( f \) with index \( \gamma > 0 \) has an asymptotic inverse function \( g \) that is regularly varying with index \( 1/\gamma \), namely

\[
f(g(x)) \sim g(f(x)) \sim x, \quad x \to \infty.
\]  
(2.16)

Consider functions \( f_1(x) = x^{1-\beta}/l(x) \) and \( f_2(x) = x^\alpha l_\nu(x), \ x > 0 \), that are regularly varying at infinity and let

\[
g_1(x) = x^{-\frac{1}{\beta}}l_1(x) \quad \text{and} \quad g_2(x) = x^{\frac{1}{\alpha}}l_2(x)
\]  
(2.17)

be their asymptotic inverses, where \( l_1 \) and \( l_2 \) are slowly varying at infinity functions. Since \( \frac{1}{1-\beta} - \frac{1}{\alpha} > 0 \), the function

\[
f_3(x) = x^{-\frac{1}{\beta} + \frac{1}{\alpha}}l_1(x)
\]  
(2.18)

is also regularly varying with positive index. Let

\[
g_3(x) = x^{-\frac{1}{\beta} + \frac{1}{\alpha}}l_3(x)
\]  
(2.19)

be its asymptotic inverse.

We set

\[
\varepsilon'(\varepsilon) := \varepsilon^{-\frac{1}{\beta} + \frac{1}{\alpha}}l_1(\frac{1}{\varepsilon})^{-1} = \frac{1}{g_3(\frac{1}{\varepsilon})},
\]

\[
\varepsilon''(\varepsilon) := (\varepsilon'(\varepsilon))^{-\frac{1}{\gamma}}l_1(\frac{1}{\varepsilon'(\varepsilon)})^{-1} = \frac{1}{g_1(\frac{1}{\varepsilon'})}.
\]  
(2.20)

It is easy to see that \( \varepsilon \mapsto \varepsilon'(\varepsilon) \) and \( \varepsilon \mapsto \varepsilon''(\varepsilon) \) satisfy conditions of the Lemma. A straightforward verification yields the equivalence (2.14):

\[
f_1\left(\frac{1}{\varepsilon'}\right) = f_1\left(g_1\left(\frac{1}{\varepsilon'}\right)\right) \sim \frac{1}{\varepsilon'}.
\]  
(2.21)

Furthermore, since \( \frac{1}{\varepsilon} \sim f_3(\frac{1}{\varepsilon}) \) we get

\[
\frac{\varepsilon}{\varepsilon''} \sim \frac{g_1(\frac{1}{\varepsilon})}{f_3(\frac{1}{\varepsilon})} = g_2(\frac{1}{\varepsilon'}).
\]  
(2.22)

Since any regularly varying function preserves equivalence, see Buldygin et al. (2018, Theorem 3.42), we obtain (2.15) by application of \( f_2 \) to (2.22):

\[
f_2\left(\frac{\varepsilon}{\varepsilon''}\right) \sim f_2\left(g_2\left(\frac{1}{\varepsilon'}\right)\right) \sim \frac{1}{\varepsilon'}.
\]  
(2.23)

\( \square \)

Let \( \nu \) be the Lévy measure of the process \( Z \) satisfying \( A_Z \) and (2.24), and let \( \varepsilon', \varepsilon'' \) be the sequences chosen in Lemma 2.4. For \( \varepsilon \in (0, 1] \) let us define rescaled jump measures \( \nu_\varepsilon \) by setting

\[
\nu_\varepsilon([z, \infty)) = \varepsilon'\nu\left(\varepsilon''\frac{z}{\varepsilon}, \infty\right),
\]

\[
\nu_\varepsilon((-\infty, -z]) = \varepsilon'\nu\left(-\infty, -\varepsilon''\frac{z}{\varepsilon}\right), \quad z > 0.
\]  
(2.24)

**Lemma 2.5.** For the family of jump measures \( \{\nu_\varepsilon\}_{\varepsilon \in (0,1]} \) defined in (2.24) we have:

1. for each \( z > 0 \)

\[
\lim_{\varepsilon \to 0} \nu_\varepsilon([z, \infty)) = \nu^{(a)}([z, \infty)),
\]

\[
\lim_{\varepsilon \to 0} \nu_\varepsilon((-\infty, -z]) = \nu^{(a)}((-\infty, z]),
\]  
(2.25)

where \( \nu^{(a)} \) is defined in (1.16).

2. for each \( \delta > 0 \) there is \( C > 0 \) such that for all \( z > 0 \)

\[
\sup_{\varepsilon \in (0,1]} \left( \nu_\varepsilon((-\infty, -z]) + \nu_\varepsilon([z, \infty)) \right) \leq C \left( \frac{1}{z^{\alpha-\delta}} + \frac{1}{z^{\gamma+\delta}} \right).
\]  
(2.26)
Proof. Without loss of generality we consider only the right tail of $\nu_\varepsilon$. 
1. For any $z > 0$ we apply (2.14), (2.13) and (2.15) to get for $\varepsilon \to 0$ that

$$
\nu_\varepsilon([z, \infty)) = \varepsilon' \nu_\varepsilon\left(\left[\frac{\varepsilon}{\varepsilon}, \infty\right]\right) \sim C_+ \varepsilon' \left(\frac{\varepsilon^{\alpha}}{\varepsilon^{\alpha}}\right) - \alpha \nu_\varepsilon\left(\frac{\varepsilon^{\alpha}}{\varepsilon^{\alpha}}\right) - \frac{\varepsilon_+}{z^\alpha} \cdot \nu_\varepsilon\left(\frac{\varepsilon^{\alpha}}{\varepsilon^{\alpha}}\right) \sim C_+ \frac{\varepsilon^{\alpha}}{z^\alpha} = \nu_\varepsilon^{(\alpha)}([z, \infty)).
$$

(2.27)

2. Let $\delta > 0, \varepsilon \in (0,1], z > 0$. We consider two cases. First, let $0 < \frac{\varepsilon^{\alpha}}{\varepsilon} < M$. Then we take into account (2.15) and (1.4) and apply Potter’s theorem, see e.g. (Bingham et al., 1987, Theorem 1.5.6) to get

$$
\nu_\varepsilon([z, \infty)) = \varepsilon' \nu_\varepsilon\left(\left[\frac{\varepsilon^{\alpha}}{\varepsilon}, \infty\right]\right) = \varepsilon' \nu_\varepsilon\left(\left[\frac{\varepsilon^{\alpha}}{\varepsilon}, \infty\right]\right) \leq \sup_{\varepsilon \in (0,1]} \frac{\varepsilon'}{\nu_\varepsilon\left(\left[\frac{\varepsilon^{\alpha}}{\varepsilon}, \infty\right]\right)} \cdot \frac{\nu_\varepsilon([z, \infty))}{\varepsilon} \cdot \frac{C_+}{z^\alpha} \cdot (z^{-\delta} + \varepsilon^{\delta})
$$

(2.28)

Second, for $\frac{\varepsilon^{\alpha}}{\varepsilon} > M$ by (2.14) we have

$$
\nu_\varepsilon([z, \infty)) = 0.
$$

(2.29)

Theorem 2.6. Suppose that $\varepsilon'$ and $\varepsilon''$ satisfy (2.14) and (2.15), and assumptions of Theorem 1.5 hold true. Then

1.

$$
Z_\varepsilon \Rightarrow Z^{(\alpha)}, \ \varepsilon \to 0,
$$

(2.30)

where $Z^{(\alpha)}$ is defined in (1.15);  
2. there exists a weak limit

$$
Y_\varepsilon \Rightarrow Y, \ \varepsilon \to 0,
$$

(2.31)

which satisfies the SDE

$$
Y(t) = \int_0^t \bar{a}(Y(s)) \, ds + b(0)Z^{(\alpha)}(t), \ \ t \geq 0.
$$

(2.32)

The process $Y$ diverges to $\pm \infty$ with the selection probabilities $\bar{\nu}_\pm$ defined in Theorem 1.5.

Proof. 1. It is well known that in the case of Lévy processes convergence of marginal distributions implies the weak convergence in the Skorokhod space, see (Jacod and Shiryaev, 2003, Corollary VII.3.6).

For some $\mu \in \mathbb{R}$, the process $Z$ has the Lévy–Khintchine representation

$$
\ln \mathbb{E}e^{i\lambda Z^{(1)}} = i\mu \lambda + \int_{\mathbb{R}} (e^{i\lambda z} - 1 - i\lambda z) \nu(dz), \ \lambda \in \mathbb{R},
$$

(2.33)

whereas the rescaled process $Z_\varepsilon$ has the Lévy–Khintchine representation

$$
\ln \mathbb{E}e^{i\lambda Z_\varepsilon^{(1)}} = \ln \mathbb{E}e^{i\lambda \frac{Z^{(1)}}{\varepsilon^\alpha}} + \varepsilon' \int_{\mathbb{R}} (e^{i\lambda \frac{z}{\varepsilon^\alpha}} - 1 - i\lambda z) \frac{\nu(dz)}{\varepsilon^\alpha} = i\mu \lambda + \int_{\mathbb{R}} (e^{i\lambda z} - 1 - i\lambda z) \nu_\varepsilon(dz),
$$

(2.34)

with the jump measures $\nu_\varepsilon$ defined in (2.24).

Hence, the integration by parts formula, Lebesgue’s dominated convergence theorem, (2.24) and (2.25) yield that for each $\lambda \in \mathbb{R}$

$$
\int_{(0,\infty)} (e^{i\lambda z} - 1 - i\lambda z) \nu_\varepsilon(dz) = -i\lambda \int_{(0,\infty)} (e^{i\lambda z} - 1) \nu_\varepsilon([z, \infty)) dz
$$

(2.35)
The same convergence holds analogously for the negative tail.

Eventually it follows from the choice of $\varepsilon'$ and $\varepsilon''$ (see (2.17) and (2.22)) that

$$
\mu_\varepsilon = \mu \cdot \varepsilon' \cdot \varepsilon'' \sim \mu \cdot (\varepsilon')^{\frac{\gamma-1}{\gamma}} \cdot (\frac{1}{\varepsilon'})^2 \to 0, \quad \varepsilon \to 0.
$$

Therefore we obtain convergence of the characteristic functions

$$
E e^{i\lambda Z_\varepsilon(t)} \to E e^{i\lambda Z(\varepsilon) \varepsilon}, \quad \varepsilon \to 0.
$$

2. To show (2.31), first we note that for $\bar{a}$ defined in (1.2) and $b \in C_b(\mathbb{R}, \mathbb{R})$ the convergence

$$
\lim_{\varepsilon \to 0} a_{\varepsilon}(y) = \bar{a}(y) \quad \text{and} \quad \lim_{\varepsilon \to 0} b_{\varepsilon}(y) = b(0),
$$

holds point-wise and uniformly on compact intervals. To prove that solutions $Y^\varepsilon$ converge to $Y$ we follow the standard two-step scheme that consists in showing the tightness of the family $\{Y^\varepsilon\}$ and the identification of the limit.

To show tightness, one mimics the arguments of §2 of Chapter 5 of Gikhman and Skorokhod [1982]. Indeed, one shows that $Y^\varepsilon$ are bounded in probability on compact time intervals which together with the linear growth of $a$ implies the weak compactness of the integrals $\int_0^t a_{\varepsilon}(Y^\varepsilon(s)) \, ds$. The weak compactness of the noise term $\int_0^t b_{\varepsilon}(Y^\varepsilon(s)) \, dZ_\varepsilon(s)$ follows from the boundedness of $b_{\varepsilon}$ and the weak convergence (2.30).

Eventually the identification of the limit is obtained with the help of Theorem IX.4.8 from Jacob and Shiryaev [2003].

Due to Theorem 1.2 and Remark 1.4, the limiting process $Y$ diverges to $\pm \infty$ with the selection probabilities $\tilde{p}_\pm$.

## 3 Estimates for the noise

In this section we get estimates for a growth rate of the noise term $\int_0^t b_{\varepsilon}(Y^\varepsilon(s)) \, dZ_\varepsilon(s)$ as $t \to \infty$ that are uniform in $\varepsilon$. We start with the the following general result.

**Lemma 3.1.** Let $\bar{Z}$ be a zero mean Lévy process without a Gaussian component and with a jump measure $\nu$ such that for some $C > 0$ and $\gamma \in (1, 2)$ it satisfies

$$
\int_{|z| > x} \nu(dz) \leq \frac{C}{x^\gamma}, \quad x \geq 1.
$$

and

$$
\int_{|z| \leq 1} z^2 \nu(dz) \leq C.
$$

Then for any $\theta > 0$ and $\delta > 0$ there exists a generic constant $K = K(C, \gamma, \delta, \theta)$ such that for any predictable process $\{\sigma(t)\}_{t \geq 0}$, $|\sigma(t)| \leq 1$ a.s., we have

$$
P\left( \sup_{t \geq 0} \frac{\int_0^t \sigma(s) \, d\bar{Z}(s)}{1 + t^{\gamma+\delta}} \leq K \right) \geq 1 - \theta.
$$

**Proof.** Denote $T(x) := \nu((-x, x)^c)$, $x \geq 1$. With the help of the integration by parts and (3.1) we get for $x \geq 1$ that

$$
\int_{|z| > x} |z| \, \nu(dz) = - \int_x^\infty z \, dT(z) = -zT(z)|_x^\infty + \int_x^\infty T(z) \, dz \leq Cx^{1-\gamma} + \frac{C}{\gamma-1} x^{1-\gamma} = \frac{C\gamma}{\gamma-1} x^{1-\gamma}.
$$

Furthermore, for $x \geq 1$

$$
\int_{0 < |z| \leq x} z^2 \, \nu(dz) \leq \int_{0 < |z| \leq 1} z^2 \, \nu(dz) + \int_{1 < |z| \leq x} z^2 \, \nu(dz) \leq C - z^2T(z)|_1^x + 2 \int_1^x zT(z) \, dz
$$

$$
\leq 2C + \frac{2C}{2-\gamma} x^{2-\gamma} < 2C \frac{3-\gamma}{2-\gamma} x^{2-\gamma}.
$$

10
To show (3.3), we follow the reasoning by Pruitt [1981]. Let us use the Lévy–Itô representation of the process $\hat{Z}$, namely for a Poissonian random measure $N$ with the compensator $\nu(dz)dt$ we write

$$\hat{Z}(t) = \int_0^t z \, \hat{N}(dz, ds).$$

(3.6)

For arbitrary $A \geq 1$ and $T > 0$ we estimate

$$P\left( \sup_{t \in [0,T]} \left| \int_0^t \sigma(s) \, d\hat{Z}(s) \right| > A \right) \leq P\left( \sup_{t \in [0,T]} \left| \int_0^t \sigma(s) \, z \hat{N}(dz, ds) \right| > \frac{A}{3} \right)$$

$$+ P\left( \int_0^T \int_{|z| > A} N(dz, ds) > 0 \right) + P\left( \int_0^T \int_{|z| > A} |z| \nu(dz) ds > \frac{A}{3} \right) = I_1 + I_2 + I_3.$$  

(3.7)

By Doob’s inequality and (3.5) we obtain

$$I_1 \leq \frac{36 T^T}{A^2} E \sigma^2(s) \int_{|z| \leq A} z^2 \nu(dz) ds \leq \frac{36 T}{2 - \gamma} \frac{2 \gamma CT}{A^\gamma}.$$  

(3.8)

The inequality $1 - e^{-x} \leq x, x \geq 0$, and (3.1) imply that

$$I_2 = 1 - \exp \left( -T \int_{|z| > A} \nu(dz) \right) \leq T \int_{|z| > A} \nu(dz) \leq \frac{CT}{A^\gamma}.$$  

(3.9)

The item $I_3$ equals 0 if $T \int_{|z| > A} \nu(dz) ds < A/3$. By (3.3) this is true if $3CT \frac{T}{A^\gamma} = 1$. Hence for each $K > 0$ we have

$$P\left( \sup_{t \geq 0} \int_0^t \sigma(s) \, d\hat{Z}(s) > K \right) \leq \sum_{n=0}^\infty P\left( \sup_{t \in [2^n, 2^{n+1}]} \int_0^t \sigma(s) \, d\hat{Z}(s) > K \right)$$

$$\leq \sum_{n=0}^\infty \frac{C}{K^{\gamma}} \frac{2^n}{2^{\gamma n + \delta}} \leq \frac{C}{K^{\gamma}} \left( 1 + \frac{2^{2+\gamma \delta}}{2^{1+\gamma \delta} - 1} \right).$$

(3.10)

Let us apply (3.7), (3.8), (3.9) to the terms in the last line. Note that all the respective items $I_3$ are zero if $K > K_0 = (6C \frac{T}{A^\gamma})^{1/\gamma}$. Therefore for $C_1 = C(1 + 72 \frac{T}{A^\gamma})$ and $K > K_0$ we get

$$P\left( \sup_{t \geq 0} \int_0^t \sigma(s) \, d\hat{Z}(s) > K \right) \leq \frac{C_1}{K^{\gamma}} \sum_{n=0}^\infty \frac{C_1 2^{n+1}}{K^{2^{\gamma n + \delta}}} = \frac{C_1}{K^{\gamma}} \left( 1 + \frac{2^{2+\gamma \delta}}{2^{1+\gamma \delta} - 1} \right).$$

(3.11)

Choosing $K = K(C, \gamma, \delta, \theta)$ large enough we make the last probability less than $\theta$. 

\[\square\]

**Corollary 3.2.** Let $\theta > 0$. Let $H_\theta$, (2.3), (2.14) and (2.15) be satisfied. Let $X^\varepsilon$ be a solution to (1.3) with any starting point, and let $Y^\varepsilon(t) = X^\varepsilon(\varepsilon t)/\varepsilon^\alpha, t \geq 0$, be the rescaled process. Then for any $\theta > 0, T > 0$ and $\delta > 0$ there exists a generic constant $K = K(\alpha, \delta, \theta, T)$ such that for any $\varepsilon \in (0, 1]$ we have

$$P\left( \sup_{t \in [0,T]} \frac{2^n (Y^\varepsilon(s-)) dZ(s)}{1 + \varepsilon^{\frac{\alpha}{\gamma}} + \delta} \right) \leq K.$$  

(3.12)

**Proof.** The uniform estimate (2.20) from Lemma 2.5 implies that for any $\gamma \in (1, \alpha)$ there is a constant $C > 0$ such that the inequalities

$$\int_{|z| > x} \nu_z(dz) \leq \frac{C}{x^\gamma}, \ x \geq 1 \quad \text{and} \quad \int_{|z| \leq 1} z^2 \nu_z(dz) \leq C, \quad \text{for any } \gamma \in (1, \alpha).$$  

(3.13)
hold uniformly over \( \varepsilon \in (0,1] \).

The only difference between the statement of Lemma 3.1 and this corollary is that the processes \( \{Z_{\varepsilon}\} \) and the process \( Z \) respectively are not necessarily centered and that the supremum is taken over a finite \( \varepsilon \)-dependent interval. Hence we have to estimate the impact of the deterministic drift. It is more convenient to treat the deterministic linear mean value component \( \mu t \) of \( Z \), \( \mu \in \mathbb{R} \). Indeed, for \( \delta > 0 \) due to (2.22) there is a constant \( C_1 = C_1(\alpha, \delta) \) such that \( \varepsilon|\varepsilon''| \leq C_1 \cdot (\varepsilon')^{-\frac{1}{4}-\delta} \) for \( \varepsilon \in (0,1] \). Therefore for some constant \( C_2 > 0 \) we have

\[
\sup_{t \in [0,T]} \left| \frac{\varepsilon t}{\varepsilon \mu} \int_{0}^{T} b(X_{\varepsilon}^{s-}) \, ds \right| \leq \sup_{t \in [0,T]} \frac{t \cdot |\varepsilon| \cdot \sup_{y} |b(y)|}{1 + (\frac{1}{T})^{\frac{41}{4}+\delta}} \leq C_1 \cdot |\mu| \cdot \sup_{y} |b(y)| \cdot \sup_{t \in [0,T]} \frac{t(\varepsilon')^{-\frac{1}{4}-\delta}}{1 + (\frac{1}{T})^{\frac{4}{4}+\delta}} \leq C_1 \cdot |\mu| \cdot \sup_{y} |b(y)| \cdot s_{\varepsilon}^{-\frac{1}{4}+\delta} \cdot \sup_{s > 0} \frac{s_{\varepsilon}^{-\frac{1}{4}+\delta}}{1 + s_{\varepsilon}^{-\frac{1}{4}+\delta}} =: K_0(\alpha, \delta, T),
\]

that gives us the lower bound for \( K \) in (3.12).

4 Exit of \( X_{\varepsilon} \) from the time-space box \([0, T_0 \varepsilon'] \times [-R \varepsilon'', R \varepsilon'']\)

In the following Lemma we estimate the exit time of \( X_{\varepsilon} \) from a small neighborhood of 0. Here we essentially use the representation of \( X_{\varepsilon} \) in terms of \( Y_{\varepsilon} \) and establish the proper relations between its small time and small space behaviour. For \( R > 0 \) and a stochastic process \( X \) we denote the first exit times

\[
\tau_{X}^{R} = \inf\{t \geq 0 : X(t) > R\}, \quad \tau_{X}^{-R} = \inf\{t \geq 0 : X(t) < -R\}.
\]

Lemma 4.1. For any \( \theta > 0 \) and any \( R > 0 \) there is \( T_0 = T_0(R) > 0 \) such that

\[
\liminf_{\varepsilon \to 0} \mathbb{P}\left( \tau_{X_{\varepsilon}}^{R} \wedge \tau_{X_{\varepsilon}}^{-R} \leq T_0 \varepsilon' \right) \geq 1 - \theta.
\]

Proof. Recall that \( \varepsilon' = \varepsilon'(\varepsilon) \) and \( \varepsilon'' = \varepsilon''(\varepsilon) \) are chosen according to Lemma 2.4. Note that due to rescaling (2.10)

\[
\tau_{X_{\varepsilon}}^{R_{\varepsilon}} = \varepsilon' Y_{\varepsilon}^{R_{\varepsilon}}, \quad X_{\varepsilon}^{}(\tau_{X_{\varepsilon}}^{R_{\varepsilon}}) = \varepsilon'' Y_{\varepsilon}(\varepsilon' Y_{\varepsilon}^{R_{\varepsilon}}).
\]

Let \( R > 0 \) and choose \( \varepsilon_0 \in (0,1] \) be such that

\[
0 < \frac{b(0)}{2} \leq \inf_{|y| \leq R, \varepsilon(0, \varepsilon)} b(\varepsilon' y) = \inf_{|y| \leq R, \varepsilon(0, \varepsilon)} b(\varepsilon_0 y) \leq \sup_{|y| \leq R, \varepsilon(0, \varepsilon)} b(\varepsilon_0 y) \leq \sup_{|y| \leq R, \varepsilon(0, \varepsilon)} b(\varepsilon' y) \leq 2b(0).
\]

Also note that \( Z_{\varepsilon} \Rightarrow Z^{(\alpha)} \) by Theorem 2.4 so that \( Z_{\varepsilon} \) has unbounded jumps in the limit as \( \varepsilon \to 0 \). Let \( \sigma_\varepsilon \) be the first jump time such that \( |\Delta Z_{\varepsilon}(\sigma_\varepsilon)| > 6R/b(0) \). Then \( |\Delta Y^{(\varepsilon)}(\sigma_\varepsilon)| > 3R \) and hence \( \tau_{R}^{\varepsilon} < \tau_{-R}^{\varepsilon} \). Eventually (2.30) yields

\[
\lim_{\varepsilon \to 0} \mathbb{E} \sigma_\varepsilon = \left( \int_{|z| \geq 6R/b(0)} \nu^{(\alpha)}(dz) \right)^{-1}
\]

and the statement of the Lemma follows from (4.3) and Chebyshev’s inequality.

Corollary 4.2. For any \( \theta > 0 \) there exist \( R > 0 \) large enough and \( T_0 > 0 \) such that

\[
\limsup_{\varepsilon \to 0} \left| \mathbb{P}\left( \tau_{X_{\varepsilon}}^{R_{\varepsilon}} \wedge \tau_{X_{\varepsilon}}^{-R_{\varepsilon}} \leq T_0 \varepsilon' \right) - \tilde{p}_+ \right| \leq \theta,
\]

\[
\limsup_{\varepsilon \to 0} \left| \mathbb{P}\left( \tau_{X_{\varepsilon}}^{R_{\varepsilon}} \wedge \tau_{X_{\varepsilon}}^{-R_{\varepsilon}} \leq T_0 \varepsilon' \right) - \tilde{p}_- \right| \leq \theta.
\]

Proof. The result follows from (4.2), (4.3), and Theorems 1.2 and 2.6.
5 Behaviour of $X^\varepsilon$ upon exit from the time-space box $[0, T_0\varepsilon'] \times [-R\varepsilon'', R\varepsilon'']$. Proof of the main result

For definiteness, let us consider only dynamics on the positive half line.

**Lemma 5.1.** 1. For each $\gamma \in (0, \beta)$ there is $K_\gamma > 0$ such that for all $x \geq 1$ and $\varepsilon \in (0, 1]$

$$a_\varepsilon(x) \geq K_{\gamma} x^{\beta-\gamma}. \quad (5.1)$$

2. For any $\kappa \in (0, 1)$ there exists $\mu \in (0, 1)$ such that

$$\inf_{x \in [1-\mu, 1+\mu]} \frac{a_\varepsilon(x)}{a_\varepsilon(y)} > 1 - \kappa. \quad (5.2)$$

**Proof.** 1. Recall that according to Assumption $H_a$ and Remark 2.2, $a(x) = \varepsilon^\beta L_+(x \land 1)$, $x > 0$, and $a(0) = 0$. Hence

$$a_\varepsilon(x) = \frac{a(\varepsilon^\beta x)}{\varepsilon^\beta / \varepsilon'} = \varepsilon' \cdot (\varepsilon')^{\beta-1} \cdot x^\beta L_+((\varepsilon') x \land 1)$$

$$= \varepsilon' \cdot (\varepsilon')^{\beta-1} \cdot \left( \frac{1}{\varepsilon^\beta} \right) \cdot x^\beta \cdot L_+((\varepsilon') x \land 1) \cdot A_+ l\left( \frac{1}{\varepsilon'} \right) \frac{l(\frac{1}{\varepsilon'})}{l(\frac{1}{\varepsilon'})}. \quad (5.3)$$

The equivalence [2.14] guarantees that $\varepsilon' \cdot (\varepsilon')^{\beta-1} \cdot l\left( \frac{1}{\varepsilon'} \right) \geq C_1 > 0$ for some $C_1 > 0$ and $\varepsilon \in (0, 1]$. Let $\gamma \in (0, \beta)$, $x \geq 1$, and $\varepsilon \in (0, 1]$. We consider two cases.

a) For $x \varepsilon'' < 1$, with the help of Potter’s theorem [Bingham et al., 1987], Theorem 1.5.6 (ii) applied to the function $l$ we get

$$a_\varepsilon(x) \geq C_2 \cdot x^\beta \cdot \inf_{0 < y < 1} \frac{L_+(y)}{A_+ l\left( \frac{1}{y} \right)} \cdot A_+ \cdot \frac{l\left( \frac{1}{\varepsilon'} \right)}{l(\frac{1}{\varepsilon'})} \geq C_2 \cdot x^{\beta-\gamma} \quad (5.4)$$

for some $C_2 = C_2(\gamma) > 0$.

b) For $x \varepsilon'' \geq 1$ applying Potter’s theorem again we get

$$a_\varepsilon(x) \geq C_3 \cdot x^\beta \cdot \frac{L_+(1)}{l(1)} \cdot \frac{l(1)}{l(\frac{1}{\varepsilon'})} \geq C_3 \cdot x^\beta \cdot (\varepsilon')^\gamma \geq C_3 \cdot x^{\beta-\gamma} \quad (5.5)$$

for some $C_3 = C_3(\gamma)$, and (5.1) follows with $K_\gamma = C_2 \cdot C_3$.

2. To prove (5.2) we note that

$$\inf_{x \in [1-\mu, 1+\mu]} \frac{a_\varepsilon(x)}{a_\varepsilon(y)} = \inf_{x \in [1-\mu, 1+\mu]} \frac{a(x)}{a(y)} = (1 - \mu)^\beta \cdot \inf_{x \in [1-\mu, 1+\mu]} \frac{L_+(x \land 1)}{L_+(y \land 1)} = C(\mu) \cdot (1 - \mu)^\beta, \quad (5.6)$$

where $0 < C(\mu) \to 1$ as $\mu \to 0$ by continuity of $L_+$ and Potter’s bounds. Hence for any $\kappa \in (0, 1)$, the estimate (5.2) holds for $\mu$ small enough.

**Lemma 5.2.** Let $\gamma \in (0, \beta)$. Then for any $y \geq 1$, $\kappa \in (0, 1)$ and any $\varepsilon \in (0, 1]$ the solution of the ODE

$$\zeta_\varepsilon(t; y) = y + (1 - \kappa) \int_0^t a_\varepsilon(\zeta_\varepsilon(s; y)) \, ds \quad (5.7)$$

satisfies

$$\zeta_\varepsilon(t; y) \geq y + K t^{\frac{1}{1-\kappa}}, \quad t \geq 0, \quad (5.8)$$

with a constant $K = K(\beta, \gamma, \kappa) > 0$.

**Proof.** Let $\gamma \in (0, \beta)$ be fixed. For $y \geq 1$ we use (5.1) and compare $\zeta_\varepsilon(\cdot; y)$ with the solution of the auxiliary ODE

$$z_\kappa(t; y) = y + (1 - \kappa) \int_0^t (z_\kappa(s; y))^{\beta-\gamma} \, ds, \quad t \geq 0. \quad (5.9)$$
This solution has the explicit form
\[ z_\kappa(t; y) = \left( y^{1-\beta+\gamma} + (1 - \kappa)(1 - \beta + \gamma)K_\gamma t \right)^{\frac{1}{1-\beta+\gamma}}. \]  
(5.10)

Hence the application of an elementary inequality \((a + b)^p \geq a^p + b^p, a, b \geq 0, p \geq 1, \) yields \([5.8]\) with some \( K > 0 \).

We need the following comparison theorem for solutions of integral equations.

**Lemma 5.3.** Let for \( T > 0 \) and \( i = 1, 2 \), the functions \( u_i \) be solutions (not necessarily unique) to the equations

\[ u_i(t) = u_i(0) + \int_0^t U_i(s, u_i(s)) \, ds, \quad t \in [0, T]. \]  
(5.11)

Assume that \( u_1(0) \geq u_2(0) \), \( U_1(t, u_2(t)) > U_2(t, u_2(t)) \), \( t \in [0, T] \), and functions \( t \mapsto U_i(t, u_i(t)) \) are right-continuous. Then \( u_1(t) \geq u_2(t), \) \( t \in [0, T] \).

**Proof.** The proof of this Lemma is quite standard. Assume that there is

\[ \tau = \inf \{ t > 0 : u_1(t) < u_2(t) \} \in [0, T]. \]  
(5.12)

Then by continuity \( u_1(\tau) = u_2(\tau) \) and we necessarily have the inequality \( D^+ u_1(\tau) \leq D^+ u_2(\tau) \) for the right Dini derivatives of the solutions. However since \( t \mapsto U_i(t, u_i(t)) \) is right-continuous, by assumption

\[ D^+ u_1(\tau) = U_1(\tau, u_1(\tau)) = U_1(\tau, u_2(\tau)) > U_2(\tau, u_2(\tau)) = D^+ u_2(\tau), \]  
(5.13)

and we obtain a contradiction. \( \square \)

In the next Lemma we determine a lower bound for the process \( Y^\varepsilon \) starting sufficiently far from zero.

**Lemma 5.4.** For any \( \theta > 0, \kappa \in (0, 1) \) and \( T > 0 \) there are \( \mu = \mu(\kappa) \in (0, 1) \) and \( R = R(T, \kappa, \theta) \geq 1 \) such that for any \( \mathcal{F}_0 \)-measurable initial condition \( Y^\varepsilon(0) > R \) a.s. and all \( \varepsilon \in (0, 1] \)

\[ \mathbb{P}\left( Y^\varepsilon(t) \geq (1 - \mu)\bar{z}_\kappa^\varepsilon(t; Y^\varepsilon(0)), \quad t \in [0, T/\varepsilon'] \right) \geq 1 - \theta. \]  
(5.14)

A similar estimate from above also holds for \( Y^\varepsilon(0) < -R \) a.s.

**Proof.** For \( \varepsilon \in (0, 1] \) let

\[ g^\varepsilon(t) := \int_0^t b_\varepsilon(Y^\varepsilon(s-)) \, dZ_\varepsilon(s), \quad \tilde{Y}^\varepsilon(t) := Y^\varepsilon(t) - g^\varepsilon(t). \]  
(5.15)

Then \( \tilde{Y}^\varepsilon(t) \) satisfies the integral equation

\[ \tilde{Y}^\varepsilon(t) = Y^\varepsilon(0) + \int_0^t a_\varepsilon(\tilde{Y}^\varepsilon(s)) + g^\varepsilon(s) \, ds. \]  
(5.16)

Choose \( \gamma \in (0, \beta) \) small enough such that \( \frac{1}{1-\beta+\gamma} > \frac{1}{\alpha} \). For \( \theta > 0 \) fixed, we apply Corollary \([5.2]\) and find a constant \( K_1 = K_1(\beta, \gamma, \theta) > 0 \) such that for all \( \varepsilon \in (0, 1] \) and any initial starting point \( Y^\varepsilon(0) \in \mathbb{R} \)

\[ \mathbb{P}\left( \sup_{t \in [0, T/\varepsilon']} \left| \frac{g^\varepsilon(t)}{1 + t^{1-\beta+\gamma}} \right| \leq K_1 \right) \geq 1 - \theta. \]  
(5.17)

Consequently, for any \( \kappa \in (0, 1) \) and any \( y \geq 1 \) with the help of \([5.8]\) we get

\[ \mathbb{P}\left( \sup_{t \in [0, T/\varepsilon']} \left| \frac{g^\varepsilon(t)}{K_1(1 + t^{1-\beta+\gamma})} \right| \leq \frac{K_1(1 + t^{1-\beta+\gamma})}{y + Kt^{1-\beta+\gamma}} \right) \geq 1 - \theta. \]  
(5.18)
Let $\mu = \mu(\kappa) \in (0, 1)$ be such that (5.22) holds. For this $\mu$ choose $R \geq 1$ such that $\sup_{t \geq 0} \frac{K_1(1 + t^{1+\gamma})}{R + K(1 + t^{1+\gamma})} \leq \mu$. Then

$$\mathbb{P}\left( \sup_{t \in [0, T]} \left| \frac{g^\varepsilon(t)}{\zeta_n^\varepsilon(t; R \vee Y^\varepsilon(0))} \right| \leq \mu \right) \geq 1 - \theta. \quad (5.19)$$

In other words, for $Y^\varepsilon(0) \geq R$ a.s. we have

$$\mathbb{P}\left( a_\varepsilon(\zeta_n^\varepsilon(t; Y^\varepsilon(0)) + g^\varepsilon(t)) > (1 - \kappa)a_\varepsilon(\zeta_n^\varepsilon(t; Y^\varepsilon(0))), \ t \in [0, T / \varepsilon] \right) \geq 1 - \theta. \quad (5.20)$$

Therefore the comparison Lemma [5.3] applied to $u_1 = \hat{Y}^\varepsilon$ and $u_2 = \zeta_n^\varepsilon(\cdot; Y^\varepsilon(0))$ yields

$$\mathbb{P}\left( \hat{Y}^\varepsilon(t) \geq \zeta_n^\varepsilon(t; Y^\varepsilon(0)), \ t \in [0, T / \varepsilon] \right) \geq 1 - \theta \quad (5.21)$$

and hence

$$\mathbb{P}\left( Y^\varepsilon(t) \geq (1 - \mu)\zeta_n^\varepsilon(t; Y^\varepsilon(0)), \ t \in [0, T / \varepsilon] \right) \geq 1 - \theta. \quad (5.22)$$

Proof of Theorem [1.6]

Notice that for each $\kappa \in (0, 1), \ \varepsilon \in (0, 1)$ and $y > 0$ the function $\hat{\zeta}_n^\varepsilon(t; y) := \varepsilon^n \zeta_n^\varepsilon(t / \varepsilon; y), \ t \geq 0$, satisfies the equation

$$\hat{\zeta}_n^\varepsilon(t; y) = \varepsilon^n y + (1 - \kappa) \int_0^t a(\zeta_n^\varepsilon(s; y)) \, ds. \quad (5.23)$$

Hence according to (1.3), (1.12) and (1.13)

$$\hat{\zeta}_n^\varepsilon(t; y) = X^0_{\varepsilon y}((1 - \kappa)t) \geq x^+(1 - \kappa)t, \quad t \geq 0. \quad (5.24)$$

Let $\mu = \mu(\kappa) \in (0, 1)$ be chosen to satisfy (5.22).

Since the Lévy process $Z_\varepsilon$ is strong Markov, analogously to Lemma 5.4 and Corollary 5.2 we have the following. For any $T, \delta, \theta > 0$ there exists a generic constant $K = K(T, \alpha, \delta, \theta)$ such that for any $\varepsilon \in (0, 1]$ the estimate

$$\mathbb{P}\left( \sup_{t \in [0, T]} \left| \frac{\int_{\tau_{-R}^{\varepsilon}}^{\tau_{+R}^{\varepsilon}} b_\varepsilon(Y^\varepsilon(s^-)) \, dZ_\varepsilon(s)}{1 + t^{\frac{\alpha}{2} + \delta}} \right| \leq K \right) \geq 1 - \theta. \quad (5.25)$$

holds for any stopping time $\tau$. It follows from Corollary 5.2 Lemma 4.1 Corollary 4.2 Lemma 5.4 and (5.24) that for any $\theta > 0$ and $T > 0$ there are $R > 0$ and $T_0 > 0$ large enough such that

$$\liminf_{\varepsilon \to 0} \mathbb{P}\left( \tau_{-R}^{\varepsilon} < \tau_{-R}^{\varepsilon} \leq T_0 \varepsilon, \quad X^\varepsilon(\tau_{-R}^{\varepsilon} + t) \geq (1 - \mu)x^+(1 - \kappa)t, \ t \in [0, T] \right) \geq \bar{p}_+ - \theta, \quad (5.26)$$

$$\liminf_{\varepsilon \to 0} \mathbb{P}\left( \tau_{-R}^{\varepsilon} < \tau_{-R}^{\varepsilon} \leq T_0 \varepsilon, \quad X^\varepsilon(\tau_{-R}^{\varepsilon} + t) \leq (1 - \mu)x^-(1 - \kappa)t, \ t \in [0, T] \right) \geq \bar{p}_- - \theta. \quad (5.26)$$

In the last formula, Lemma 5.4 is applied to the process $Y^\varepsilon(t + \tau_{-R}^{\varepsilon} \wedge \tau_{-R}^{\varepsilon}), \ t \geq 0$, whose initial value belongs to the set $[-R, R]$, see 4.3. Corollary 4.2 holds true since $\tau_{-R}^{\varepsilon}$ are stopping times.

Since $\bar{p}_+ + \bar{p}_- = 1$ and any limit law of $\{X^\varepsilon\}$ is supported by the solutions $x^\pm$ (see Lemma 2.3 we get that for each $\delta > 0$

$$\limsup_{\varepsilon \to 0} \left| \mathbb{P}\left( \sup_{t \in [0, T_0 \varepsilon + T]} |X^\varepsilon(t) - x^+(t)| \leq \delta \right) - p_+ \right| \leq \theta, \quad (5.27)$$

and the proof is finished.
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