Vacuum GR in Chang–Soo variables: Hilbert space structure in anisotropic minisuperspace

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Abstract

In this paper we address the major criticism of the pure Kodama state, namely its normalizability and its existence within a genuine Hilbert space of states, by recasting Ashtekar’s general relativity into set of new variables attributed to Chang–Soo/CDJ. While our results have been shown for anisotropic minisuperspace, we reserve a treatment of the full theory for a following paper which it is hoped should finally bring this issue to a close. We have performed a canonical treatment of these new variables from the level of the classical/quantum algebra of constraints, all the way to the construction of the Hilbert space of states, and have demonstrated their relevance to the principle of the semiclassical-quantum correspondence. It is hoped that these new variables and their physical interpretation should provide a new starting point for investigations in classical and quantum GR and in the construction of a consistent quantum theory.
1 Introduction

The main focus of the present paper to establish a new Hilbert space structure for quantum gravity by recasting general relativity in terms of a set of new variables. The traditional approach in loop variables utilizes the spin network states, which have been rigorously shown to meet the requirements of a Hilbert space [1]. While the spin network states have been constructed to solve the kinematic constraints of general relativity and to diagonalize geometric operators, the implementation of the Hamiltonian constraint on these states and its resulting dynamics remains an open issue. In the present paper and series of papers we propose a new approach to resolve the issue of dynamics corresponding to the set of physical states. The primary motivation for our work is to address once and for all the existing criticisms of the Kodama state and its relatives as a genuine physical and rigorous state of quantum gravity.

Two of the common objections to the pure Kodama state $\Psi_{Kod}$ relate to its normalizability and its existence within a well-defined Hilbert space. The issue of normalizability has been raised in various works including [2],[3]. One attempt to address this has been made in [4], which found the linearization of $\Psi_{Kod}$ to be delta function normalizable for spaces of Euclidean signature. It is hoped that the present paper will address these objections, starting from within the realm of anisotropic minisuperspace with the full theory to follow in a subsequent paper. The means toward this end necessitate an interpretation of general relativity most suited to the construction of a Hilbert space consistently implementing its dynamics. This brings in the Chang–Soo/CDJ variables. We will show that when cast in terms of these new variables, the route to resolution of the aformentioned issues is clear.

In line with the development of any new model we will perform the standard procedures for the Chang–Soo/CDJ variables, starting with the verification of consistency of the Dirac algebra of constraints and culminating with the construction of a Hilbert space of states. In section 2 we transform from the Ashtekar into Chang–Soo variables and perform a reduction into anisotropic minisuperspace. In section 3 we compute the classical algebra of constraints, moving on to the quantum algebra in section 4. In section 5 we rigorously define and construct the Hilbert space using a kind of Bargmann representation and apply it to the Chang–Soo variables in section 6, devoted to the solution to the constraints. Section 6 culminates with a demonstration of role of the pure Kodama state $\Psi_{Kod}$ within this Hilbert space structure, which it is hoped addresses the main issues. In section 7, we provide some additional arguments in support of the Hilbert structure by arguments from geometric and from path integral quantization. Section
2 Transformation into Chang–Soo variables

Our starting point for the transformation into the new variables will be the Ashtekar variables \((A^a_i, \tilde{\sigma}^a_i)\) where \(A^a_i\) is the left-handed \(SU(2)_-\) Ashtekar connection with densitized triad \(\tilde{\sigma}^a_i\).\(^1\) The 3 + 1 decomposition of the action for vacuum general relativity in the Ashtekar variables is given by [5],[6],[7]

\[
I_{Ash} = \int_0^T dt \int_\Sigma d^3x \tilde{\sigma}^a_i \dot{A}^a_i - i \sum N H - N^i H_i + A^a_0 G_a, \tag{1}
\]

where \(N\) is the lapse function with lapse density \(\bar{N} = N/\sqrt{\text{det} \tilde{\sigma}}\), and \(N^i\) and \(A^a_0\) are respectively the shift vector and \(SU(2)_-\) rotation angles. Here \(B^i_a\) is the Ashtekar curvature given by \(B^i_a = \epsilon^{ijk} (\partial_j A^a_k + \frac{1}{2} f_{abc} A^c_j A^k_c)\), with structure constants \(f_{abc}\). The action (1) represents a totally constrained first class constrained system. The Gauss’ law and diffeomorphism constraints, the kinematic constraints, are given by

\[
H_i = \epsilon^{ijk} \tilde{\sigma}^j_a B^k_a, \quad G_a = D_i \tilde{\sigma}^i_a \tag{2}
\]

with \(SU(2)_-\) covariant derivative \(D_i = (D_i)^{ab} = \delta^{ab} \partial_i + f^{abc} A^c_i\) and the Hamiltonian constraint is given by

\[
H = \epsilon^{ijk} \epsilon^{abc} \left( \frac{\Lambda}{6} \tilde{\sigma}^i_a \tilde{\sigma}^j_b \tilde{\sigma}^k_c + \tilde{\sigma}^i_a \tilde{\sigma}^j_b B^k_c \right) \tag{3}
\]

where \(\Lambda\) is the cosmological constant.

We would like to make a change of variables from (1) into a new set of variables, whose ultimate purpose will be to illuminate the Hilbert space structure of quantum gravity as it applies to the pure and the generalized Kodama states. To this end, we lay the groundwork for the quantization of gravity in the new variables, applying the concepts of [8],[9],[10]. In the present paper we will start with anisotropic minisuperspace, reserving the full theory for following works.

To transform into the new variables we introduce the CDJ Ansatz,

\[
\tilde{\sigma}^i_a = \Psi_{ae} B^i_e. \tag{4}
\]

\(^1\)By convention, lowercase Latin indices from the beginning of the alphabet \(a, b, c, \ldots\), denote internal \(SU(2)_-\) indices, while those from the middle \(i, j, k, \ldots\) denote spatial indices in three space \(\Sigma\).
Here $\Psi_{ae}$ is known as the CDJ matrix, named after Capovilla, Dell and Jacobson. The CDJ matrix $\Psi_{ae}$ is normally used as a Lagrange multiplier designed to enforce the metricity, which allows the spacetime metric $g_{\mu\nu}$ to be derivable from tetrads. However, we will utilize the CDJ matrix in a different way whose purpose in the present paper and toward our programme will soon become abundantly clear. First, we substitute (4) into (1). The kinematic constraints (2) then transform into

$$H_i = \epsilon_{ijk}\tilde{\sigma}_a^j B_a^k = \epsilon_{ijk}B_a^j B_e^k \Psi_{ae}; \quad G_a = D_i\tilde{\sigma}_a^i = B_a^i D_i\Psi_{ae},$$

(5)

where we have used the Bianchi identity $D_i B_a^i = 0$. The Hamiltonian constraint (3) under the substitution (4) transforms into

$$H = (\det B)(\text{Adet}\Psi + \text{Var}\Psi)$$

(6)

where we have defined $\text{Var}\Psi = (\text{tr}\Psi)^2 - \text{tr}\Psi^2$. Substitution of (4) into the canonical structure of (1) yields

$$\tilde{\sigma}_a^i \dot{A}_i^a = \Psi_{ae} B_e^i \dot{A}_i^a.$$  

(7)

The canonical structure of (7) suggests that the CDJ matrix $\Psi_{ae}$ can be viewed as a dynamical variable canonically conjugate to a variable whose time derivative is given by $B_e^i \dot{A}_i^a$. It is at this point that we introduce a new conjecture or principle.

Let us regard the CDJ matrix $\Psi_{ae}$ as a fully dynamical variable, and no longer part of an Ansatz. Equation (4) can then rather be seen, under the assumption of a nondegenerate Ashtekar curvature, as

$$\Psi_{ae} = \tilde{\sigma}_a^i (B^{-1})_l^e.$$  

(8)

From this perspective we regard the Ashtekar curvature $B_a^i$ as freely specifiable with the densitized triad being a derived quantity from the CDJ matrix $\Psi_{ae}$, seen as more fundamental.

Next, we define a new set of variables $X^{ae}$, such that

$$\Psi_{ae} B_e^i \dot{A}_i^a = \dot{X}^{ae}.$$  

(9)

Hence, as implied by the canonical structure $\Psi_{ae}\dot{X}^{ae}$, the variable $X^{ae}$ then takes on the interpretation of the set of ‘coordinates’ canonically conjugate to the ‘momenta’ $\Psi_{ae}$.

We can now rewrite the Lagrangian (1), regarding $X^{ae}$ and $\Psi_{ae}$ as the fundamental dynamical variables, obtaining
\[ L_{S^{\text{soo}}} = \int_0^T dt \int_S d^3x \left[ \Psi_a \dot{X}^{ae} - i N \sqrt{\det B} \sqrt{\det \Psi} (\Lambda + \text{tr} \Psi^{-1}) \right. \]
\[ + N^i \epsilon_{ijk} B^j_a B^k_e \Psi_a + A^a_0 \Psi_e (\Psi_{ae}), \]  \tag{10} \]

where \( w_e \equiv B^i_e D_i \) is the Gauss' law operator which acts on the CDJ matrix. The Hamiltonian constraint in the Chang–Soo variables arises from the manipulations

\[ NH = \frac{N}{\sqrt{\det \sigma}} \epsilon_{ijk} \epsilon^{abc} \left( \frac{\Lambda}{6} \bar{\sigma}_a \bar{\sigma}_b \bar{\sigma}_c + \bar{\sigma}_a \bar{\sigma}_b B^k_c \right) \]
\[ = \frac{N}{\sqrt{\det B} \sqrt{\det \Psi}} (\det B) (\Lambda \text{det} \Psi + Var \Psi) \]
\[ = \frac{N \sqrt{\det B} \det \Psi}{\sqrt{\det \Psi}} (\Lambda + \frac{Var \Psi}{\det \Psi}) = N \sqrt{\det B} \sqrt{\det \Psi} (\Lambda + \text{tr}^{-1} \Psi). \tag{11} \]

There are some differences in the action (10) as compared to (1). One main difference is that the lapse function \( N \) is no longer densitized. Hence, using the right hand side of (10) as a starting point, one sees that the Hamiltonian constraint in the new variables is nonpolynomial unlike in the Ashtekar variables.²

The variables \( X^{ae} \) are known as the Chang–Soo variables, attributed to Chopin Soo [11],[12], and arise on connection superspace from a set of connection one forms \( \delta X^{ae} = B^i_e \delta A^a_i \). We will focus in the present paper upon the spatially homogeneous and anisotropic sector of the full theory for pure gravity in the Chang–Soo variables, using (10) as a starting point. In this regard we will lay down the basic framework for the Hilbert space in the Chang–Soo variables.³ Toward this end we will first reduce the equations of motion and the constraints into the spatially homogeneous sector. After this we will compute the Dirac algebra of classical and quantum constraints and then construct a Hilbert space. We will then illuminate the relation of the Hilbert space to the canonical procedure by showing that they satisfy the quantum constraints, and how the pure Kodama state \( \Psi_{Kod} \) fits into this overall scheme.

²This difference among others, as we will ultimately show, will have important implications for the quantum theory and for the resulting Hilbert space of states.

³Hence we will take (10) through the standard analysis of in anisotropic minisuperspace, with a view toward establishing a Hilbert space of quantum states for the full theory. These states will ultimately generalise the pure Kodama state \( \Psi_{Kod} \) for vacuum GR.
2.1 A few useful identities

Let us now collect a few useful identities and relations regarding the Chang–Soo/CDJ variables in anisotropic minisuperspace. By the terminology ‘anisotropic minisuperspace’ is meant the full theory reduced to the spatially homogeneous sector, with no symmetry reductions of the variables. Hence, in the case of the Ashtekar variables, all nine components of the connection $A_i^a$ are to be regarded as independent dynamical variables. The set of Chang–Soo one forms and corresponding vector fields then are given, in anisotropic minisuperspace, by

$$dX^{ae} = (\det A) (A^{-1})^i_e dA^a_i; \quad \frac{\partial}{\partial X^{ae}} = (\det A)^{-1} A^e_i \frac{\partial}{\partial A^a_i}. \quad (12)$$

where we have used $B^i_a = (\det A) (A^{-1})^i_a$ in minisuperspace, for which the Ashtekar curvature retains all nine components as independent variables. Some other recurring quantities are

$$B^i_a = (\det A) (A^{-1})^i_a, \quad C_{ae} = B^i_e A^a_i = (\det A) \delta_{ae}. \quad (13)$$

We have defined the $X^{ae}$ coordinates by integrating along a direction in the space of connections $\Gamma$, by

$$X^{ae} = \sum_i \int_0^{A^a_i} (\det \alpha) (\alpha^{-1})^i_e d\alpha^a_i. \quad (14)$$

Lastly, note that the vector fields and corresponding one forms (12) are invariant under $SO(3)$ transformations on the spatial indices.

Next, we must rewrite the constraints in terms of the Chang–Soo/CDJ variables in anisotropic minisuperspace. For the kinematic constraints (2) we have for the Gauss’ law constraint that

$$w_a = f_{abc} A^b_i \Psi_{ce} B^e_c = f_{abc} C_{be} \Psi_{be} = f_{abc} \delta_{be} (\det A) \Psi_{ce} = - (\det A) f_{ace} \Psi_{ce} \quad (15)$$

and for the diffeomorphism constraint generator we have

$$v_i = \epsilon_{ijk} B^j_b \Psi_{ac} = (\det A)^2 \epsilon_{ijk} (A^{-1})^j_b (A^{-1})^c_a \Psi_{ae} = (\det A) \epsilon^{i}_{ade} A^d_j \Psi_{ae} = (\det A) A^d_i \epsilon_{dae} \Psi_{ae}. \quad (16)$$

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Note the position of the indices in (12) and (13) for $X^{ae}$ such as to implement the required operation relative to the $SU(2)_-$ index of the connection $A_i^a$.

Another convention is to arrange indices such that the contractions occur in the same sequence as the indices appear, hence the negative sign on the Gauss’ law constraint in (15).
It will also be useful to have expressions for the vector fields in the original Ashtekar connection variables. Hence for the Gauss’ law generators we have

\[ w_a = -(\operatorname{det} A) f_{ace} \frac{\partial}{\partial X^{ce}} = -(\operatorname{det} A)(\operatorname{det} A)^{-1} f_{ace} A^a_i \frac{\partial}{\partial A^c_i}. \]  

(17)

Likewise, for the diffeomorphism generators we obtain

\[ v_i = (\operatorname{det} A) A^d_i \epsilon_{dae} \frac{\partial}{\partial X^{ae}} = (\operatorname{det} A) A^d_i \epsilon_{dae}(\operatorname{det} A)^{-1} A^e_j \frac{\partial}{\partial A^j_d} = -\epsilon_{dea} A^d_i A^e_j \frac{\partial}{\partial A^j_d}. \]  

(18)

Observe the shift in sign in accordance with the convention in (18) and well as (17), when switching between the Ashtekar and the Chang–Soo variables.

A few other useful identities include

\[ \frac{\partial}{\partial X^{bf}} A^e_i = (\operatorname{det} A)^{-1} A^f_j \frac{\partial}{\partial A^j_f} A^e_i = (\operatorname{det} A)^{-1} A^f_j \delta^e_i = (\operatorname{det} A)^{-1} \delta_{bc} A^e_i; \]

\[ \frac{\partial (\operatorname{det} A)}{\partial X^{bf}} = (\operatorname{det} A)^{-1} A^f_j \frac{\partial}{\partial A^j_f}(\operatorname{det} A) = (\operatorname{det} A)^{-1} A^f_j (\operatorname{det} A)(A^{-1})^j_b = \delta_{bf}. \]  

(19)

The Hamiltonian constraint generator is given in the Chang–Soo variables in the homogeneous sector by (11)

\[ H = (\operatorname{det} A) \sqrt{\det \Psi} \left( \Lambda + \operatorname{tr} \Psi^{-1} \right), \]  

(20)

where we have used the minisuperspace relation \( \det B = (\operatorname{det} A)^2 \).

To recapitulate, the constraint generators are given by

\[ w_a = -(\operatorname{det} A) f_{ace} \Psi_{ce} = f_{ace} A^a_i \frac{\partial}{\partial A^c_i}; \]

\[ v_i = (\operatorname{det} A) A^d_i \epsilon_{dae} \Psi_{ae} = -A^d_i A^e_j \epsilon_{dea} \frac{\partial}{\partial A^j_d}; \]

\[ H = (\operatorname{det} A) \sqrt{\det \Psi} \left( \Lambda + \operatorname{tr} \Psi^{-1} \right). \]  

(21)

It will sometimes be convenient to represent the constraints as generating transformations by contracting them on the corresponding gauge parameters. Hence we have that

\[ ^6 \text{A handy mnemonic device in understanding the algebra of constraints is to balance mass dimensions } [X^{ae}] = [\operatorname{det} A] = 3 \text{ when evaluating variational derivatives of the various terms.} \]
\[ \vec{G}[\vec{\theta}] = \vec{w}_a(\theta^a) = -(\det A) \theta^a \Psi_{ae} \equiv \vec{w}^{ae}(\vec{\theta}) \Psi_{ae}; \]
\[ \vec{H}[N] = \vec{v}_i(N^i) = (\det A) N^i A^d_i \Psi_{ae} \equiv \vec{v}^{ae}(N) \Psi_{ae}; \]
\[ H[N] = N (\det A) \sqrt{\det \Psi} \left( \Lambda + \text{tr} \Psi^{-1} \right). \quad (22) \]

Lastly, let us redisplay the useful identities (19) for completeness.

\[ \frac{\partial}{\partial X^{bf}} A^e_i = (\det A)^{-1} \delta_{be} A^f_i; \quad \frac{\partial (\det A)}{\partial X^{bf}} = \delta_{bf}. \quad (23) \]

### 2.2 Hamilton’s equations of motion

We now derive the equations of motion for the homogeneous sector of the starting Lagrangian (10) for Lorentzian signature. The starting action is, given by

\[ I_{Soo} = \int_0^T dt \left( -\frac{i}{G} \Psi_{ae} \dot{X}^{ae} \right) \]
\[ + (\det A) \left[ -i N \sqrt{\det \Psi} (\Lambda + \text{tr} \Psi^{-1}) + (N^i A^d_i - \theta^d) f_{dae} \Psi_{ae} \right] \quad (24) \]

The symplectic structure from (24) provides the following Poisson brackets

\[ \{ X^{ae}, \Psi_{bf} \} = i G \delta^a_e \delta^c_f. \quad (25) \]

Let us now put into place the machinery thus far developed in the computation of the necessary Poisson brackets. Starting with the diffeomorphism constraint, the variational derivative with respect to the conjugate momentum is given by

\[ \frac{\delta \vec{H}[N]}{\delta \Psi_{bf}} = \frac{\partial}{\partial \Psi_{bf}} (\det A) (A^d_i N^i) \Psi_{ae} \]
\[ = (\det A) (N^i A^d_i) \epsilon_{dae} \frac{\partial \Psi_{ae}}{\partial \Psi_{bf}} = (\det A) (N^i A^d_i) \epsilon_{dae} \delta^b_c \delta^f_e = (\det A) N^i A^d_i f_{bf} \quad (26) \]

Note in (26) the antisymmetry in the indices \( bf \). Next we compute the variational derivative with respect to the configuration variables

\[ \frac{\delta \vec{H}[N]}{\delta X^{bf}} = \frac{\partial}{\partial X^{bf}} (\det A) (A^d_i N^i) \epsilon_{dae} \Psi_{ae} = \left[ A^d_i \frac{\partial (\det A)}{\partial X^{bf}} + (\det A) \frac{\partial}{\partial X^{bf}} A^d_i \right] N^i f_{dae} \Psi_{ae} \]
\[ = N^i (A^d_i \delta_{bf} + (\det A) (\det A)^{-1} \delta_{bd} A^f_i) f_{dae} \Psi_{ae} = N^i (A^d_i \delta_{bf} + A^f_i \delta_{bd}) f_{dae} \Psi_{ae}. \quad (27) \]
Moving on to the variational derivatives of the Gauss’ law constraint with respect to the momentum variables,

\[
\frac{\delta G_a[\theta^a]}{\delta \Psi_{bf}} = -\frac{\delta}{\delta \Psi_{bf}}(\det A)^a f_{acg \Psi_{cg}}
\]

\[
= -(\det A)^a f_{acg} \frac{\partial \Psi_{cg}}{\partial \Psi_{bf}} = -(\det A)^a f_{acg} \delta^b_{cg} \delta^f_{fg} = -(\det A)^a f_{abf}
\]

(28)

Moving on to the configuration variables, we have

\[
\frac{\delta G_a[\theta^a]}{\delta X^{bf}(x)} = \frac{\partial}{\partial X^{bf}(x)}(\det A)^a f_{agh \Psi_{gh}} = -\delta_{bf}^a f_{agh \Psi_{gh}}.
\]

(29)

For the Hamiltonian constraint we have

\[
\frac{\delta H[N]}{\delta \Psi_{bf}} = \frac{\partial}{\partial \Psi_{bf}} \left[ N(\det A)^{1/2} \sqrt{\det \Psi} \left( \Lambda + \text{tr} \Psi^{-1} \right) \right]
\]

\[
= N(\det A)^{1/2} \frac{\partial (\det \Psi)^{1/2}}{\partial \Psi_{bf}} + \sqrt{\det \Psi} \frac{\partial}{\partial \Psi_{bf}} \left( \Lambda + \text{tr} \Psi^{-1} \right)
\]

\[
= N(\det A)^{1/2} \left[ \frac{1}{2} (\Psi^{-1})^{bf} \left( \Lambda + \text{tr} \Psi^{-1} \right) - (\Psi^{-1})^{bf} \right].
\]

(30)

Another way to depict this is by

\[
\frac{\delta H[N]}{\delta \Psi_{bf}} = \frac{1}{2} (\Psi^{-1})^{bf} H[N] - N(\det A)^{1/2} \sqrt{\det \Psi} (\Psi^{-1})^{bf},
\]

(31)

which contains a part proportional to the constraint and a correction term. Likewise, computing the variational derivative with respect to the configuration variable, we have

\[
\frac{\delta H[N]}{\delta X^{bf}} = \frac{\partial}{\partial X^{bf}} \left[ N(\det A)^{1/2} \sqrt{\det \Psi} \left( \Lambda + \text{tr} \Psi^{-1} \right) \right]
\]

\[
= \delta_{bf} N \sqrt{\det \Psi} \left( \Lambda + \text{tr} \Psi^{-1} \right) = \delta_{bf} (\det A)^{-1} H[N],
\]

(32)

The Hamilton’s equations of motion read, using (34), (27) and (29),

\[
X^{ae} = \frac{\delta H}{\delta \Psi^{ae}} = i \left[ \frac{1}{2} (\Psi^{-1})^{ae} H[N] - N(\det A) \sqrt{\det \Psi} (\Psi^{-1})^{ae} \right]
\]

+ (\det A)^i N^i f_{bf} - (\det A)^a f_{abf}

(33)

\[
\dot{\Psi}_{ae} = \frac{\delta H}{\delta X^{ae}} = -i \delta_{ae} (\det A)^{-1} H[N] - N^i (A_i^d \delta_{bf} + A_j^d \delta_{bd}) f_{dae} \Psi_{ae} - \delta_{ae} \theta^a f_{dgf} \Psi_{gh}
\]

A useful mnemonic device is that a diffeomorphism with parameter \(N^i\) can be seen as a gauge transformation with a field-dependent parameter \(A_i^a N^a\).
We will relegate a more detailed study of the equations (33) to the next work, which will cover inflation in anisotropic minisuperspace.

3 Classical algebra of constraints

One indication of the degree of similarity or dissimilarity between the theories defined by (10) and (1) is the structure of the constraint algebra, which we shall examine in the present section. A diffeomorphism in anisotropic minisuperspace in Chang–Soo variables can be represented in a form resembling an observable with a field-dependent parameter, as in

\[ \vec{H}[\vec{N}] = \Psi_{[ae]}[\nu_{ae}(\vec{N})] = (\det A) N^j A^d_i \epsilon_{dae} \Psi_{[ae]} \].

(34)

Hence the spatial diffeomorphisms are generated by the antisymmetric part of the conjugate momentum \( \Psi_{ae} \). The gauge transformations are given by

\[ \vec{G}[\vec{\theta}] = w_{ae}(\vec{\theta}) \Psi_{ae} = -(\det A) \theta^f f_{fae} \Psi_{[ae]}, \]

(35)

which likewise are generated by the antisymmetric part of \( \Psi_{ae} \). We now compute the Poisson brackets, given by

\[ \{A, B\} = \sum_{bf} \left( \frac{\delta A}{\delta \Psi_{bf}} \frac{\delta B}{\delta X^{bf}} - \frac{\delta B}{\delta \Psi_{bf}} \frac{\delta A}{\delta X^{bf}} \right). \]

(36)

Starting with the commutator of two diffeomorphisms, we have

\[ \{ \vec{H}[\vec{M}], \vec{H}[\vec{N}] \} = \sum_{bf} \left[ \frac{\delta \vec{H}[\vec{M}]}{\delta \Psi_{bf}} \frac{\delta \vec{H}[\vec{N}]}{\delta X^{bf}} - \frac{\delta \vec{H}[\vec{N}]}{\delta \Psi_{bf}} \frac{\delta \vec{H}[\vec{M}]}{\delta X^{bf}} \right] \]

\[ = \left( (\det A) A^b_j M^j f_{gbf} \right) \left( N^i (A^d_i \delta_{bf} + A^d_i \delta_{bd}) f_{dae} \Psi_{ae} \right) 
- \left( (\det A) A^b_j N^j f_{gbf} \right) \left( M^i A^d_i (A^d_i \delta_{bf} + A^d_i \delta_{bd}) f_{dae} \Psi_{ae} \right) \]

\[ = (\det A) (M^j N^i - N^j M^i) A^d_i (A^d_i \delta_{bf} + A^d_i \delta_{bd}) f_{gbf} f_{dae} \Psi_{ae} \]

(37)

where the first term of (37) vanishes due to contraction of \( \delta_{bf} \) on \( f_{gbf} \) due to antisymmetry of the structure constants. Continuing along, we have
\{\bar{H}[\bar{M}], \bar{H}[\bar{N}]\} = (\det A)(M^j N^i - N^j M^i)A^q_i \delta_{pb} f_{gbf} f_{dae} \Psi_{ae}
= -(\det A)(M^j N^i - N^j M^i)A^q_i f_{gbf} f_{dae} \Psi_{ae}
= (\det A)^2(M^j N^i - N^j M^i)\epsilon_{ijk}(A^{-1})^{q}_i f_{dae} \Psi_{ae}
= (\det A)^2(\bar{N} \times \bar{M})_k(A^{-1})^{q}_i f_{dae} \Psi_{ae} \quad (38)

on account of the antisymmetry in the structure constants, where we have defined \((M^j N^i - N^j M^i)\epsilon_{ijk} = (\bar{N} \times \bar{M})_k\). Continuing on, we have

\[(\det A)^2(\bar{N} \times \bar{M})_k(A^{-1})^{q}_i f_{dae} \Psi_{ae} = (\det A)^2(\bar{N} \times \bar{M})_k(A^{-1})^{q}_i \delta_{pb} f_{gbf} f_{dae} \Psi_{ae}
= (\det A)^2(\bar{N} \times \bar{M})_k(A^{-1})^{q}_i \delta_{pb} f_{gfb} f_{dae} \Psi_{ae}
= (\det A)(A^{-1} A^{-1})^{ki}(\bar{N} \times \bar{M})_k(\det A) f_{dae} \Psi_{ae} = \Psi_{ae}[V_{ae}(\bar{N} \times \bar{M})]. \quad (39)\]

The Poisson bracket of two diffeomorphisms can be seen as another diffeomorphism with field-dependent parameter \((\det A)(A^{-1} A^{-1})^{ki}(\bar{N} \times \bar{M})_k\), where we have used the nondegeneracy requirement \(\det A \neq 0\).

Moving on to the Poisson bracket of two gauge transformations,

\[\{\bar{G}[\bar{\theta}], \bar{G}[\bar{\lambda}]\} = \sum_{bf} \left[ \frac{\delta \bar{G}[\bar{\theta}]}{\delta \Psi_{bf}} \frac{\delta \bar{G}[\bar{\lambda}]}{\delta X^{bf}} - \frac{\delta \bar{G}[\bar{\lambda}]}{\delta \Psi_{bf}} \frac{\delta \bar{G}[\bar{\theta}]}{\delta X^{bf}} \right] = (-(\det A)\theta^a f_{abf})(-\delta_{bf} \lambda^e f_{egb} \Psi_{gb}) - (-(\det A)\lambda^a f_{abf})(-\delta_{bf} \theta^e f_{egb} \Psi_{gb}) = 0 - 0 = 0. \quad (40)\]

Two gauge transformations strongly commute, due to annihilation of symmetric indices by the antisymmetric indices on the structure constants.

Let us now examine the effect of a gauge transformation and a diffeomorphism.

\[\{\bar{H}[\bar{N}], \bar{G}[\bar{\theta}]\} = \sum_{bf} \left[ \frac{\delta \bar{H}[\bar{N}]}{\delta \Psi_{bf}} \frac{\delta \bar{G}[\bar{\theta}]}{\delta X^{bf}} - \frac{\delta \bar{G}[\bar{\theta}]}{\delta \Psi_{bf}} \frac{\delta \bar{H}[\bar{N}]}{\delta X^{bf}} \right] = \left( (\det A) N^i A^q_i f_{dbf} \right) \left( -\delta_{bf} \theta^a f_{agb} \Psi_{gb} \right)
- \left( - (\det A) \theta^a f_{gbf} \right) \left( N^i (A^q_i \delta_{bf} + A^f_i \delta_{bd}) f_{dae} \Psi_{ae} \right) \quad (41)\]

The first and second terms on the right hand side of (41) vanish due to antisymmetry of the structure constants, leaving just the last third term. Proceeding along,

\[\{\bar{H}[\bar{N}], \bar{G}[\bar{\theta}]\} = (\det A)\theta^a N^i A^q_i f_{gbf} \delta_{bd} f_{dae} \Psi_{ae}
= - (\det A)\theta^a f_{gbf} (N^i A^q_i) f_{dae} \Psi_{ae} = \Psi_{ae}[W^{ae}(\bar{\theta} \wedge (\bar{A}. \bar{N})]. \quad (42)\]
Hence, the commutator of a gauge transformation with a diffeomorphism can be seen as a gauge transformation with field-dependent parameter \((\hat{\theta} \wedge (\vec{N} \cdot \vec{A}))_b = f_{g\ell b} \theta^{g\ell}(N^i A^\ell_i)\), where we use the wedge notation to denote cross products in \(SU(2)\).

We now compute the Poisson bracket of a spatial diffeomorphism and a deformation normal to the spatial hypersurface \(\Sigma\)

\[
\{H[N], \vec{H}[\vec{N}]\} = \left[\frac{\delta H[N]}{\delta \Psi_{bf}} \frac{\delta \vec{H}[\vec{N}]}{\delta X^{bf}} - \frac{\delta \vec{H}[\vec{N}]}{\delta \Psi_{bf}} \frac{\delta H[N]}{\delta X^{bf}}\right]
\]

\[
= \left[\frac{1}{2}(\Psi^{-1})^{bf} H[N] - N(detA)\sqrt{\det \Psi}(\Psi^{-1})^{bf}\right] (N^i A^\ell_i (\delta_{bf} f_{dae} + \delta_{bd} f_{fae}) \Psi_{ae})
\]

\[
- ((detA)A^\ell_i N^i f_{dbf})(\delta_{bf}(detA)^{-1} H[N]).
\]

The third term on the right hand side of (43) vanishes due to antisymmetry of the structure constants. Hence, proceeding along, we have

\[
\left(\frac{1}{2}(\Psi^{-1})^{bf} H[N] - N(detA)\sqrt{\det \Psi}(\Psi^{-1})^{bf}\right) N^i (A^\ell_i \delta_{bf} + A^\ell_j \delta_{bd}) f_{dae} \Psi_{ae}
\]

\[
- ((detA)A^\ell_i N^i f_{dbf}) (\delta_{bf}(detA)^{-1} H[N]).
\]

The second term of (44) drops out due to contraction of \(\delta_{bf}\) on \(f_{dbf}\), for \(detA \neq 0\). Hence we have

\[
\{H[N], \vec{H}[\vec{N}]\} = \left[\frac{1}{2}(\Psi^{-1})^{bf} (N^i A^\ell_i) H[N] - N(detA)\sqrt{\det \Psi} tr(\Psi^{-1})^{bf}(N^i A^\ell_i)\right]
\]

\[
+ \frac{1}{2}(\Psi^{-1})^{df} H[N](N^i A^\ell_i) - N(detA)\sqrt{\det \Psi}(\Psi^{-1})^{df}(N^i A^\ell_i) f_{dae} \Psi_{ae}
\]

Making the definition

\[
\eta^{ae} = \sqrt{\det \Psi}(\Psi^{-1} \Psi^{-1})^{ae}
\]

along with the definition \(M^{ae} = M^{ae} + \delta^{ae} trM\) for an arbitrary matrix \(M\), we have

\[
\{H[N], \vec{H}[\vec{N}]\} = (N^i A^\ell_i) \left[\frac{1}{2}(\Psi^{-1})^{df} + \delta^{df} tr\Psi^{-1}\right] H[N] - N(detA)(\eta^{df} + \delta^{df}\eta) f_{dae} \Psi_{ae}
\]

\[
= (N^i A^\ell_i) \left[\frac{1}{2}(\Psi^{-1})^{df} H[N] - N(detA)\eta^{df}\right] f_{dae} \Psi_{ae}
\]

where we have defined \(\eta = tr\eta^{ae}\). We recognize the result as a diffeomorphism with momentum dependent structure functions.
\{H[N], \tilde{H}[\tilde{N}]\} = N^i A_i^a N(\det A) \mathcal{M}_\psi f_{ae} \Psi_{ae} = \Psi_{[ae]}[V_{ae}(NN^i)] \sim \tilde{H}[N\tilde{H}; \mathcal{M}], (48)

where the boldsymbol \( V_{ae} \) signifies a diffeomorphism with momentum-dependent structure functions. Here we have defined

\begin{equation}
\mathcal{M}_\psi = \frac{1}{2} (\Psi^{-1}) \sqrt{\det \Psi} (\Lambda + \text{tr} \Psi^{-1}) - \pi^\psi
\end{equation}

(49)

to signify a field-dependent parameter, containing momentum-dependent structure functions.\(^8\)

Moving on to the Poisson bracket between a normal deformation and a gauge transformation, we have

\begin{equation}
\{H[N], \tilde{G}[\tilde{\theta}]\} = \sum \left( \frac{\delta H[N]}{\delta \Psi_{bf}} \frac{\delta \Psi_{bf}(x)}{\delta \Psi_{bf}(x)} - \frac{\delta \Psi_{bf}(x)}{\delta \Psi_{bf}(x)} \frac{\delta H[N]}{\delta \Psi_{bf}} \right)
\end{equation}

\begin{equation}
= \left[ \frac{1}{2} (\Psi^{-1}) H[N] - N(\det A) \sqrt{\det \Psi} (\Psi^{-1}) \right] \theta^a f_{agh} \Psi_{gh}
\end{equation}

\begin{equation}
- \left( - (\det A) \theta^a f_{abf} (- \delta_{bf} (\det A)^{-1} H[N]) \right)
\end{equation}

(50)

The last term of (50) vanishes on account of the antisymmetry in the structure constants. Continuing on, we have that

\begin{equation}
\{H[N], \tilde{G}[\tilde{\theta}]\} = \left( \frac{1}{2} (\Psi^{-1}) H[N] - N(\det A) \right) \theta^a f_{agh} \Psi_{gh}
\end{equation}

\begin{equation}
= \mathcal{M} N \theta^a f_{agh} \Psi_{gh} = \Psi_{[agh]}[W_{gh}(N\tilde{\theta})] \sim \tilde{G}[N\tilde{\theta}; \mathcal{M}]
\end{equation}

(51)

where we have defined \( \mathcal{M} = \text{tr} \mathcal{M}_\psi \). Hence the kinematic constraints transform covariantly under the action of the Hamiltonian constraint, with momentum dependent structure functions.

Finally, we can move on to the Poisson bracket between two normal evolutions. This is given by

\begin{equation}
\{H[N], H[M]\} = \sum \left( \frac{\delta H[N]}{\delta \Psi_{bf}(x)} \frac{\delta H[M]}{\delta \Psi_{bf}(x)} - \frac{\delta H[M]}{\delta \Psi_{bf}(x)} \frac{\delta H[N]}{\delta \Psi_{bf}(x)} \right)
\end{equation}

\begin{equation}
= \left( \frac{1}{2} (\Psi^{-1}) H[N] - N(\det A) \sqrt{\det \Psi} (\Psi^{-1}) \right) (- \delta_{bf} (\det A)^{-1} H[M])
\end{equation}

\begin{equation}
- \left( \frac{1}{2} (\Psi^{-1}) H[M] - M(\det A) \sqrt{\det \Psi} (\Psi^{-1}) \right) (- \delta_{bf} (\det A)^{-1} H[N]) = 0
\end{equation}

(52)

---

\(^8\)One interpretation is that an ‘open’ algebra prohibits the group-theoretical implementation of the constraints, since one must determine in general a nonlinear action of momenta on states in the Schrödinger representation when one quantizes the theory. However, restricted to a special class of states, it is possible to obtain a group action. Note also that the presence of coordinates \( \det A \) and \( A_i^a \) do not count as structure functions, since they are part of the definition of the diffeomorphism and gauge generators.
Hence, two Hamiltonian constraints in anisotropic minisuperspace strongly commute.

3.1 Recapitulation

To summarize the results of this section, the classical constraint algebra in the homogeneous sector of the Chang–Soo variables is given by

\[
\{ \hat{\mathcal{H}}[\vec{N}], \hat{\mathcal{H}}[\vec{M}] \} = \{ \Psi_{\[ae]}[\mathbf{v}_{ae}(\vec{N})], \Psi_{\[bf]}[\mathbf{v}_{bf}(\vec{M})] \} = \Psi_{ae}[\mathbf{V}_{ae}(\vec{N} \times \vec{M})] = 0;
\]

\[
\{ \hat{\mathcal{H}}[\vec{N}], \hat{\mathcal{H}}[\vec{M}] \} = \{ \Psi_{\[ae]}[\mathbf{w}_{ae}(\vec{N})], \Psi_{\[bf]}[\mathbf{w}_{bf}(\vec{M})] \} = 0;
\]

\[
\{ \hat{\mathcal{H}}[\vec{N}], \hat{\mathcal{G}}[\vec{\theta}] \} = \{ \Psi_{\[ae]}[\mathbf{V}_{ae}(\vec{N})], \Psi_{\[bf]}[\mathbf{W}_{bf}(\vec{\theta})] \} = \Psi_{ae}[\mathbf{W}_{ae}(\vec{\theta}) \wedge (\vec{A} \cdot \vec{N})] = 0;
\]

\[
\{ \hat{\mathcal{H}}[\vec{N}], \hat{\mathcal{H}}[\vec{M}] \} = \{ \Psi_{\[ae]}[\mathbf{M}_{ae}(\vec{N})], \Psi_{\[bf]}[\mathbf{M}_{bf}(\vec{M})] \} = \Psi_{ae}[\mathbf{M}_{ae}(\vec{N} \times \vec{M})] = 0;
\]

\[
\{ \hat{\mathcal{H}}[\vec{N}], \hat{\mathcal{G}}[\vec{\theta}] \} = \{ \Psi_{\[ae]}[\mathbf{M}_{ae}(\vec{N})], \Psi_{\[bf]}[\mathbf{M}_{bf}(\vec{\theta})] \} = \Psi_{ae}[\mathbf{M}_{ae}(\vec{\theta}) \wedge (\vec{A} \cdot \vec{N})] = 0.
\]

There are two independent Abelian subalgebras within the algebraic structure (53), corresponding to gauge transformations and normal deformations. The gauge transformations and spatial diffeomorphisms transform covariantly under the action of the Hamiltonian constraint, albeit with momentum dependent structure functions. Let us display the constraint algebra in Ashtekar variables in minisuperspace for comparison.

\[
\{ \hat{\mathcal{H}}[\vec{N}], \hat{\mathcal{H}}[\vec{M}] \} = H_k [\mathbf{N}^i \partial^k M_i - M^i \partial^k N_i] \sim 0
\]

\[
\{ \hat{\mathcal{H}}[\vec{N}], G_\alpha[N_\alpha] \} = G_\alpha [\mathbf{N}^i \partial_i \theta^\alpha] \sim 0
\]

\[
\{ G_\alpha[\theta^\alpha], G_\beta[\lambda^\beta] \} = G_\alpha [f_{bc} \theta^b \lambda^c].
\]

The algebra of constraints in the Ashtekar variables, a semi-direct product of SU(2) with spatial diffeomorphisms, is a Lie algebra. The inclusion of the Hamiltonian constraint enlarges the kinematic algebra into an open algebra due to the structure functions[5],[6],[7]

\[
\{ \hat{\mathcal{H}}[\vec{N}], \hat{\mathcal{H}}[\vec{M}] \} = H_i [\mathbf{N}^i \partial_i \vec{M} - \mathbf{M}^i \partial_i \vec{N}] \sim 0
\]

\[
\{ \hat{\mathcal{H}}[\vec{N}], \hat{\mathcal{G}}[\vec{\theta}] \} = H_i [\mathbf{N}^i \partial_i \vec{M} - \mathbf{M}^i \partial_i \vec{N}] \sim 0.
\]

\[9\] The notation is that bold text lowercase \(v\) and \(w\) respectively denote diffeomorphisms and gauge transformations with single parameters while uppercase \(V\) and \(W\) denote composite diffeomorphisms and gauge transformations. Boldsymbol \(V\) and \(W\) respectively denote diffeomorphisms and gauge transformations with momentum dependent structure functions.

\[10\] This is unlike in the Ashtekar variables where the structure functions appear from commuting two Hamiltonian constraints.
We have set all terms containing spatial gradients to zero to reduce the algebra to the homogeneous sector. There is a big difference between the structure of the algebra (53) and that of (54), (55). The diffeomorphisms do not act trivially in the Chang–Soo variables whereas they do in the Ashtekar variables, and the kinematic constraints transform covariantly under normal deformations, whereas in the Ashtekar variables they are trivialized in minisuperspace.

These structural differences in the algebra implies an inherent difference between the manifestation of general relativity in the Chang–Soo as compared with the Ashtekar variables, even though the former was motivated by a seemingly innocuous Ansatz $\tilde{\sigma}_a^i = \Psi_{ae} \tilde{B}_c^i$. It will be convenient, for visualization, to display the multiplication table for the minisuperspace algebra. Any objects in bold denote the existence of momentum-dependent structure functions. For the Ashtekar variables we have

$$Poisson_{Ash} \sim \begin{pmatrix} \times & \vec{G} & \vec{H} & H \\ \vec{G} & \vec{G} & \vec{G} & 0 \\ \vec{H} & -\vec{G} & \vec{H} & H \\ H & 0 & -H & \vec{H} \end{pmatrix} \rightarrow \begin{pmatrix} \times & \vec{G} & \vec{H} & H \\ \vec{G} & \vec{G} & \vec{G} & 0 \\ \vec{H} & 0 & 0 & 0 \\ H & 0 & 0 & 0 \end{pmatrix}$$

For the Chang–Soo and CDJ variables it is given by

$$Poisson_{Soo} \sim \begin{pmatrix} \times & \vec{G} & \vec{H} & H \\ \vec{G} & 0 & \vec{G} & \vec{G} \\ \vec{H} & -\vec{G} & 0 & \vec{H} \\ H & -\vec{G} & -\vec{H} & 0 \end{pmatrix}$$

The kinematic constraints form an ideal, with the gauge transformations forming an ideal within that ideal. This should have some interesting effects when one considers the representation theory of the algebra of constraints. We relegate this investigation to future work.

### 3.2 Transformations on the configuration space

Let us now compute the action of the transformations generated by the constraints on the phase space variables. We will start first with the configuration variables $X^{ae}$.

$$X^{\tau^{ae}}[\vec{\theta}] = e^{\vec{\Phi}[\vec{\theta}]} X^{ae} e^{-\vec{\Phi}[\vec{\theta}]} = \sum_n \frac{1}{n!} Ad^n (\vec{H}[N]; X^{ae})$$  \hspace{1cm} (56)

we have, for the adjoint action of a general constraint on $X^{ae}$,
\[
Ad(\Phi_I[\theta^I]; X^{ae}(x)) = \int_\Sigma d^3y \theta^I(y) [\Phi_I(y), X^{ae}(x)]
\]  
(57)

with \(Ad^n(A; B) = [A, Ad^{n-1}(A; B)]\). For constraints linear in momenta such as the kinematic constraints, the general result for an infinitesimal transformation

\[
Ad(\Phi_I[\theta^I]; X^{ae}(x)) = \int_\Sigma d^3y \theta^I(y) \frac{\delta}{\delta X^{ae}(y)} X^{ae}(x)
\]

\[
= \int_\Sigma d^3y \theta^I(y) \theta^{ae}(x - y) \delta_{\alpha}^a \delta_{\beta}^e = \theta^{ae}(x)
\]

(58)
is a translation of \(X^{ae}\) in field space at the spatial point \(x\). Hence a finite transformation generated by the kinematic constraints should result in a translation.\(^{11}\)

We would like to obtain the interpretation for these transformations in the homogeneous sector. Starting with diffeomorphisms, we have

\[
X'^{ae} = e^{\vec{H}[\vec{N}]} X^{ae} e^{-\vec{H}[\vec{N}]} = \sum N^{i_1} \epsilon_{i_1 j_1 k_1} B_{a_1}^{j_1} B_{e_1}^{k_1} \frac{\partial}{\partial X^{ae}_{a_1 e_1}} N^{i_2} \epsilon_{i_2 j_2 k_2} B_{a_2}^{j_2} B_{e_2}^{k_2} \frac{\partial}{\partial X^{ae}_{a_2 e_2}}
\]

\[
\ldots N^{i_{n-1}} \epsilon_{i_{n-1} j_{n-1} k_{n-1}} B_{a_{n-1}}^{j_{n-1}} B_{e_{n-1}}^{k_{n-1}} \frac{\partial}{\partial X^{ae}_{a_{n-1} e_{n-1}}} N^{i_n} \epsilon_{i_n j_n k_n} B_{a_n}^{j_n} B_{e_n}^{k_n}
\]

\[
= \sum_n \frac{1}{n!} N^{i_1} \epsilon_{i_1 j_1 k_1} B_{a_1}^{j_1} B_{e_1}^{k_1} \frac{\partial}{\partial X^{ae}_{a_1 e_1}} \ldots N^{i_n} \epsilon_{i_n j_n k_n} B_{a_n}^{j_n} B_{e_n}^{k_n}
\]

\[
X^{ae} = e^{N^i \frac{\partial}{\partial \eta^i}} X^{ae}(\vec{\eta}) = X^{ae}(\vec{\eta} + \vec{N})(59)
\]

The end result is a translation of \(X^{ae}\), seen as a function of the characteristic directions \(\eta^i\), by the amount \(N^i\) in the functional space of fields. One might wonder the manner in which diffeomorphisms would get implemented in the homogeneous sector. Reducing the diffeomorphism vector fields into this sector via the identifications

\[
\vec{H}[\vec{N}] = N^i \epsilon_{ijk} B_{a}^{j} B_{e}^{k} \frac{\delta}{\delta X^{ae}} \rightarrow (\det A) N^i A^d_i f_{d ae} \frac{\partial}{\partial X^{ae}},
\]

we have the following transformation

\[
X^{ae}(\vec{N}) = X^{ae} + (\det A) N^i A^d_i f_{d ae}
\]

(61)

Hence, the antisymmetric part of \(X^{ae}\) gets transformed, leaving the symmetric part untouched.

\(^{11}\)The net effect is that the observables remain invariant under these transformations.
Moving on to gauge transformations, we have

\[ X'^{ae} = e^{\tilde{G}[\tilde{\theta}]} X^{ae} e^{-\tilde{G}[\tilde{\theta}]} = \sum_n \frac{1}{n!} (d_{\alpha_1}^{f_1} \theta^{\alpha_1}) \frac{\partial}{\partial X^{f_1 \alpha_1}} (d_{\alpha_2}^{f_2} \theta^{\alpha_2}) \frac{\partial}{\partial X^{f_2 \alpha_2}} \]

\[ \ldots \]

\[ = \sum_n \left( d_{\alpha_n}^{f_n} \theta^{\alpha_n} \right) \frac{\partial}{\partial X^{f_n \alpha_n}} X^{ae} = e^{\theta^a \delta_a^e} X^{ae} (\zeta) = X^{ae} (\zeta + \tilde{\theta}) \]

(62)

We see that the result of a gauge transformation is another translation in field space on \( X^{ae} \). Making the identification in the homogeneous sector of

\[ \tilde{G}[\tilde{\theta}] = \theta^a \delta_a^h \frac{\delta}{\delta X^{gh}} \rightarrow (\text{det} A) \theta^a f_{ab} \frac{\partial}{\partial X^{bf}}, \]

(63)

then we have the effect of a gauge transformation upon the Chang–Soo variables of

\[ X^{fg}(\tilde{\theta}) = X^{fg} - X \theta^a f_{af} \]

(64)

where we have defined \( X = \text{tr} X^{ae} \). As for the diffeomorphisms, there is a nonlinear translation in the antisymmetric part of the Chang–Soo matrix \( X^{ae} \). Hence, a diffeomorphism can be seen as a gauge transformation in the opposite direction with a field-dependent parameter.

4 Quantum algebra of constraints

We now compute the quantum algebra of constraints to check for anomalies in concert with [13]. First, we promote the classical Chang–Soo variables to quantum operators \((X^{ae}, \Psi^{ae}) \rightarrow (\hat{X}^{ae}, \hat{\Psi}^{ae})\) which act on a Hilbert space of states \(|\psi\rangle\). Since we are interested in the time evolution of the quantum states, we must obtain the equal time commutators reduced to within the homogeneous anisotropic sector. Hence, Poisson brackets (25) get promoted directly to to equal-time commutators as in

\[ [\hat{X}^{ae}(T), \hat{\Psi}_{bf}(T)] = \hbar G^a_b \delta^e_f, \]

(65)

along with the trivial commutators

\[ [\hat{X}^{ae}(T), \hat{X}^{bf}(T)] = [\hat{\Psi}_{ae}(T), \hat{\Psi}_{bf}(T)] = 0 \]

(66)
for each time \( T \). The classical constraints \( \Phi_I \) get promoted to quantum operators \( \hat{\Phi}_I \) which act on the state \( |\psi\rangle = |\psi(T)\rangle \) at the given time \( T \). Without committing yet to a specific form of the quantum state \( |\psi\rangle \), let us first compute the quantum algebra of observables corresponding to the constraints.\(^\text{12}\)

There are two types of observables that will concern us in this section. First, there are observables corresponding to constraints linear in conjugate momenta, which take on the form of linear functionals \( \hat{\Psi}_{ae} F(X^{ae}) \), where \( F \) is a function whose argument is the operator \( \hat{X}^{ae} \). Then there are also observables corresponding to constraints nonlinear in momenta, such as

\[
O[X, \Psi] = \sum_{k=0}^{N} O_{a_1 e_1 \ldots a_k e_k}^{b_k+1 f_k+1 \ldots b_N f_N} X^{a_1 e_1} \ldots X^{a_k e_k} \hat{\Psi}_{b_k+1 f_k+1} \ldots \hat{\Psi}_{b_N f_N}.
\]  

Upon quantization we must decide on an operator-ordering for these functionals. The preferred operator ordering is one for which the quantum and the classical algebra of constraints are isomorphic to one another. Hence (67) gets promoted to

\[
O[\hat{X}, \hat{\Psi}] = \sum_{k=0}^{N} O_{a_1 e_1 \ldots a_k e_k}^{b_k+1 f_k+1 \ldots b_N f_N} \hat{X}^{a_1 e_1} \ldots \hat{X}^{a_k e_k} \hat{\Psi}_{b_k+1 f_k+1} \ldots \hat{\Psi}_{b_N f_N}.
\]  

Hence, we choose an ordering with all momenta to the right of the configuration variables, and also demonstrate that the quantum constraint algebra of constraints closes for this ordering.

We will make repeated use of the following operator identity involving bosonic operators, strictly maintaining the ordering of the operators

\[
[\hat{A}\hat{B}, \hat{C}\hat{D}] = \hat{A}[\hat{B}, \hat{C}]\hat{D} + [\hat{A}, \hat{C}]\hat{B}\hat{D} + \hat{C}\hat{A}[\hat{B}, \hat{D}] + \hat{C}[\hat{A}, \hat{D}]\hat{B}.
\]  

The kinematic constraints fall into the general form of functionals linear in momenta, so we start with a pair of kinematic constraints with \( F_{ae} = F_{ae}(X) \) and \( G_{ae} = G_{ae}(X) \) being functions solely of the configuration variables \( X^{ae} \)

\[
\left[ \hat{\Psi}_{ae}(F_{ae}), \hat{\Psi}_{bf}(G_{bf}) \right] = \left[ F_{ae} \hat{\Psi}_{ae}, \hat{G}_{bf} \hat{\Psi}_{bf} \right] = \left[ F_{ae} \hat{\Psi}_{ae}, \hat{G}_{bf} \hat{\Psi}_{bf} \right] + \hat{G}_{bf} F_{ae} \left[ \hat{\Psi}_{ae}, \hat{\Psi}_{bf} \right] + \hat{G}_{bf} \hat{F}_{ae} \left[ \hat{\Psi}_{ae}, \hat{\Psi}_{bf} \right] + \hat{G}_{bf} \hat{F}_{ae} \left[ \hat{\Psi}_{ae}, \hat{\Psi}_{bf} \right] \hat{\Psi}_{ae}.
\]  

\(^{12}\)There is value in being cognizant of the transformations of all states, physical or not, by the constraints implemented at the quantum level.

\(^{13}\)In the most general case, one needn’t be restricted to polynomial functions of the basic variables.
The middle two terms of (70) vanish due to trivial commutators between like phase space variables, yielding

\[
\hat{\Psi}_{ae} \hat{\Psi}_{bf} \left[ \hat{F}_{ae}, \hat{G}_{bf} \right] = \hat{F}_{ae} \left( \frac{\partial \hat{G}_{bf}}{\partial X_{ae}} \right) \hat{\Psi}_{bf} - \hat{G}_{ae} \left( \frac{\partial \hat{F}_{bf}}{\partial X_{ae}} \right) \hat{\Psi}_{bf} = \hat{F}_{ae} \left( \frac{\partial \hat{G}_{bf}}{\partial X_{bf}} \right) \hat{\Psi}_{bf} - \hat{G}_{bf} \left( \frac{\partial \hat{F}_{ae}}{\partial X_{bf}} \right) \hat{\Psi}_{bf} = \hat{F}_{ae} \left( \hat{\Psi}_{bf} \right) \hat{G}_{bf} \left( \hat{\Psi}_{ae} \right) = V_{ae} \left( \{ \hat{F}, \hat{G} \} \right) \hat{\Psi}_{ae}. \tag{71}
\]

Observe that the operator ordering is still maintained with the momenta to the right, hence the quantum algebra of observables linear in momenta closes. Additionally, the quantum algebra for the chosen ordering is directly isomorphic to its Poisson-bracketed counterpart for the kinematic constraints (53). This can be seen when one makes the identifications

\[
\hat{F}_{ae} = \frac{\delta(\Psi_{cg}[F_{cg}])}{\delta \Psi_{ae}}, \quad \hat{G}_{bf} = \frac{\delta(\Psi_{cg}[G_{cg}])}{\delta \Psi_{bf}}, \tag{72}
\]

whereupon the structure functions operators in (71) take on the interpretation of quantized Poisson brackets.

Moving on to the commutator of normal deformations with constraints linear in momenta, we have

\[
\left[ \hat{H}, \hat{\Psi}_{ae} \hat{F}_{ae}(X) \right] = \left[ N(\det \hat{A}) \sqrt{\det \hat{\Psi}} \left( \Lambda + \text{tr} \hat{\Psi}^{-1} \right), \hat{F}_{ae} \hat{\Psi}_{ae} \right] = N(\det \hat{A}) \left[ \sqrt{\det \hat{\Psi}} \left( \Lambda + \text{tr} \hat{\Psi}^{-1} \right), \hat{F}_{ae} \hat{\Psi}_{ae} \right] + N(\det \hat{A}) \hat{F}_{ae}(X) \left[ \sqrt{\det \hat{\Psi}} \left( \Lambda + \text{tr} \hat{\Psi}^{-1} \right), \hat{\Psi}_{ae} \right] + N(\det \hat{A}) \hat{F}_{ae}(X) \left[ \det \hat{A}, \hat{\Psi}_{ae} \right] \sqrt{\det \hat{\Psi}} \left( \Lambda + \text{tr} \hat{\Psi}^{-1} \right) \hat{\Psi}_{ae} \tag{73}
\]

The second and third terms on the right hand side of (73) vanish, due to commutation between like phase space variables, leaving the first and the last term. Continuing on with the derivation using the Liebniz rule, we get, starting with the first term,

\[
(\det \hat{A}) \left[ \sqrt{\det \hat{\Psi}} \left( \Lambda + \text{tr} \hat{\Psi}^{-1} \right), \hat{F}_{ae} \right] \hat{\Psi}_{ae} = (\det \hat{A}) \left( \sqrt{\det \hat{\Psi}} \left( \Lambda + \text{tr} \hat{\Psi}^{-1} \right), \hat{F}_{ae} \right) + (\det \hat{A}) \left[ \sqrt{\det \hat{\Psi}} \left( \Lambda + \text{tr} \hat{\Psi}^{-1} \right), \hat{F}_{ae} \right] \left( \Lambda + \text{tr} \hat{\Psi}^{-1} \right) \hat{\Psi}_{ae} \tag{74}
\]

Continuing on with the expansion, we get
\[
\begin{align*}
&= (\det \hat{A})\left(-\sqrt{\det \hat{\Psi}} \delta_{bf} ((\hat{\Psi}^{-1})^{bc} [\hat{\Psi}_{cd}, \hat{F}_{ae}](\hat{\Psi}^{-1})^{df})
\right. \\
&\quad + \frac{1}{2} \sqrt{\det \hat{\Psi}} (\hat{\Psi}^{-1})^{cd} [\hat{\Psi}_{cd}, \hat{F}_{ae}](\Lambda + \text{tr} \hat{\Psi}^{-1}) \hat{\Psi}_{ae}
\left.\right) \\
&= (\det \hat{A})\sqrt{\det \hat{\Psi}} \left(-((\hat{\Psi}^{-1})^{bc} [\hat{\Psi}_{cd}, \hat{F}_{ae}](\hat{\Psi}^{-1})^{df})
\right. \\
&\quad + \frac{1}{2} ((\hat{\Psi}^{-1})^{cd} (\partial_{X} \hat{F}_{ae}) \Lambda + \text{tr} \hat{\Psi}^{-1}) \right) \hat{\Psi}_{ae}
\end{align*}
\]

The momentum appears to the right in (75), therefore one could ascribe the interpretation of a kinematic transformation with structure functions in the quantum theory.

The fourth term of (73) yields

\[
\hat{F}_{ae} (\det \hat{A}, \hat{\Psi}_{ae}) \sqrt{\det \hat{\Psi}} (\Lambda + \text{tr} \hat{\Psi}^{-1}) = -\hat{F}_{ae} \frac{\partial (\det \hat{A})}{\partial X^{ae}} \sqrt{\det \hat{\Psi}} (\Lambda + \text{tr} \hat{\Psi}^{-1})
\]

\[
= -\delta_{ae} \hat{F}_{ae} \sqrt{\det \hat{\Psi}} (\Lambda + \text{tr} \hat{\Psi}^{-1}) = -\text{tr} \hat{F} \sqrt{\det \hat{\Psi}} (\Lambda + \text{tr} \hat{\Psi}^{-1}).
\]

This would be a Hamiltonian constraint operator if not for a missing factor of \(\det \hat{A}\). Hence let us perform an insertion of the identity operator \(\hat{I} = (\det \hat{A})^{-1}(\det \hat{A})\) between the first and the second term of the right hand side of (76). Then we obtain

\[
-\text{tr} \hat{F} \sqrt{\det \hat{\Psi}} (\Lambda + \text{tr} \hat{\Psi}^{-1})
\]

\[
= -(\text{tr} \hat{F})(\det \hat{A})^{-1}(\det \hat{A}) \sqrt{\det \hat{\Psi}} (\Lambda + \text{tr} \hat{\Psi}^{-1}) = -(\text{tr} \hat{F})(\det \hat{A})^{-1} \hat{H}.
\]

Hence the operator (77) as long as the zero is not within the spectrum of \((\det \hat{A})^{-1}\).\(^{14}\) Again, the momenta appear to the right with any structure functions appearing to the left. Hence we obtain, using the results of (74) and (77), that

\[
[\hat{H} [N], \hat{\Psi}_{ae} [\hat{F}_{ae}(X)]] = N (\hat{V}_{ae}(\partial \hat{F}; \hat{\Psi}^{-1})) \hat{\Psi}_{ae} - (\text{tr} \hat{F})(\det \hat{A})^{-1} \hat{H}.
\]

where we have defined

\[
\hat{V}_{ae}(\partial \hat{F}; \hat{\Psi}^{-1}) = (\det \hat{A})\sqrt{\det \hat{\Psi}} \left(-((\hat{\Psi}^{-1})^{bc} [\hat{\Psi}_{cd}, \hat{F}_{ae}](\hat{\Psi}^{-1})^{df})
\right. \\
&\quad + \frac{1}{2} ((\hat{\Psi}^{-1})^{cd} (\partial_{X} \hat{F}_{ae}) \Lambda + \text{tr} \hat{\Psi}^{-1}) \right) \hat{\Psi}_{ae}
\]

\(^{14}\)This condition can be implemented at the level of the Hilbert space, by restricting one’s self to configurations corresponding to a nondegenerate Ashtekar magnetic field.
It remains to compute the commutator of two normal deformations. Hence

\[
\left[ \hat{H}[N], \hat{H}[M] \right] = \left[ N(\det \hat{A}) \sqrt{\det \Psi (\Lambda + \text{tr} \Psi^{-1})}, M(\det \hat{A}) \sqrt{\det \Psi (\Lambda + \text{tr} \Psi^{-1})} \right] = 0. \tag{80}
\]

This commutator vanishes due to the antisymmetric property of the commutator.

So the quantum algebra of Dirac observables on an arbitrary state \( |\psi\rangle \) is given by

\[
\left[ \hat{\Psi}_{ae}[\hat{F}_{ae}], \hat{\Psi}_{bf}[\hat{G}_{bf}] \right] = V_{ae}(\{\hat{F}, \hat{G}\}) \hat{\Psi}_{ae} |\psi\rangle \\
\left[ \hat{H}[N], \hat{\Psi}_{ae}[\hat{F}_{ae}] \right] = \left( (\hat{V}_{ae}(\partial \hat{F}; \hat{\Psi}^{-1})) \hat{\Psi}_{ae} + (\text{tr} \hat{F})(\det \hat{A})^{-1} \hat{H} \right) |\psi\rangle \\
\left[ \hat{H}[N], \hat{H}[M] \right] |\psi\rangle = 0 \tag{81}
\]

The algebra of quantum constraints closes in analogy to its classical counterpart, therefore the quantum theory is Dirac consistent and free of anomalies.

The Heisenberg equations of motion for the field operators are given by

\[
\dot{\hat{X}}_{ae} = -\frac{i}{\hbar} \left[ \hat{X}_{ae}, \hat{H} \right]; \quad \dot{\hat{\Psi}}_{ae} = -\frac{i}{\hbar} \left[ \hat{\Psi}_{ae}, \hat{H} \right]. \tag{82}
\]

with general solution

\[
\hat{X}_{ae}(T) = \hat{U}(T, 0) \hat{X}_{ae}(0) \hat{U}^{-1}(T, 0); \quad \hat{\Psi}_{ae}(T) = \hat{U}(T, 0) \hat{\Psi}_{ae}(0) \hat{U}^{-1}(T, 0) \tag{83}
\]

where \( \hat{U}(T, 0) = e^{i \int_0^T d\tau \hat{H}(\tau)} \) is the time evolution operator from an initial spatial hypersurface \( \Sigma_0 \) to a final spatial hypersurface \( \Sigma_T \), and where \( \hat{H} = \hat{H}[-\vec{N}] + G_a[\theta^a] + H[\vec{N}] \).

### 4.1 Examination for anomalies

We have shown that the algebra of constraints closes at the quantum level. Therefore, the quantum algebra of constraints is consistent in the sense of Dirac. However, it is possible that the quantum algebra may close, yet still be inconsistent with the classical algebra of the Poisson brackets. If this were to be the case, then one should hope the quantum algebra to be the more general. Let us examine the consistency of the algebra. First, we know already that it is consistent with respect to the commutator of two Hamiltonian constraints for the operator ordering chosen. Hence
\{H[N], H[M]\} = [\tilde{H}[N], \tilde{H}[M]] = 0. Hence it remains to check the situation for the Gauss’ law and the diffeomorphism constraints. For the classical theory we have that \{\tilde{G}[\hat{\theta}], \tilde{G}[\hat{\lambda}]\} = 0, or that two gauge transformations commute. To check this in the quantum theory let us make use of the representation of the constraints as differential operators in the Schrödinger representation. Hence

\[
[\hat{\tilde{G}}[\hat{\theta}], \hat{\tilde{G}}[\hat{\lambda}]] = \left(- (\text{det}A)^d \theta^d f_{dae} \frac{\partial}{\partial X^{ae}} \right) \left( (\text{det}A) \lambda^g f_{gbf} \frac{\partial}{\partial X^{bf}} \right) \\
- \left( (\text{det}A) \lambda^d f_{dae} \frac{\partial}{\partial X^{ae}} \right) \left( - (\text{det}A) \theta^g f_{gbf} \frac{\partial}{\partial X^{bf}} \right) \\
= (\text{det}A) \left( (\text{det}A)^2 \theta^d \lambda^g f_{dae} f_{gbf} \delta_{ae} \frac{\partial}{\partial X^{bf}} \right) \\
- 2 (\text{det}A) \theta^d \lambda^g f_{dae} f_{gbf} \delta_{ae} \frac{\partial}{\partial X^{bf}} = 0 \quad (84)
\]

on account of the antisymmetry of the structure constants. So the Poisson brackets are consistent with the commutators with respect to the Gauss’ law constraint.

Next, we examine the consistency of the commutator between a diffeomorphism and a gauge transformation. This is given by

\[
[\hat{\tilde{G}}[\hat{\theta}], \hat{\tilde{H}}[\hat{N}]] = \left(- (\text{det}A)^d \theta^d f_{dae} \frac{\partial}{\partial X^{ae}} \right) \left( (\text{det}A) N^i A^g_i f_{gbf} \frac{\partial}{\partial X^{bf}} \right) \\
- \left( (\text{det}A) N^i A^g_i f_{dae} \frac{\partial}{\partial X^{ae}} \right) \left( - (\text{det}A) \theta^g f_{gbf} \frac{\partial}{\partial X^{bf}} \right). \quad (85)
\]

The contribution due to the derivatives acting on \text{det}A vanish since they produce the isotropic matrix \delta_{ae} which is annihilated by the structure constants. Hence there remain the terms involving the functional derivatives acting on the Ashtekar connection. This is given by the first term of (85), as in

\[
-(\text{det}A)^2 \theta^d N^i f_{dae} f_{gbf} \frac{\partial}{\partial X^{ae}} A^i_g \frac{\partial}{\partial X^{bf}} = -(\text{det}A)^2 \theta^d N^i f_{dae} f_{gbf} (\text{det}A)^{-1} \delta_{ag} A^i_e \frac{\partial}{\partial X^{bf}} \\
= (\text{det}A) \theta^d (N^i A^g_i) f_{dae} f_{gbf} \frac{\partial}{\partial X^{bf}} = (\text{det}A) \theta^d (N^i A^g_i) f_{dae} f_{gbf} \frac{\partial}{\partial X^{bf}} = (\text{det}A) \hat{\theta} \times (\hat{N} \cdot A)_{bf} \hat{\Psi}_{bf} \quad (86)
\]

which is a diffeomorphism, as obtained via Poisson brackets. The now arises a problem when we evaluate the commutator of two diffeomorphisms. This is given by

\[
[\hat{\tilde{H}}[\hat{N}], \hat{\tilde{H}}[\hat{M}]] = \left( (\text{det}A) N^i A^g_i \epsilon_{dae} \frac{\partial}{\partial X^{ae}} \right) \left( (\text{det}A) M^k A^g_k \epsilon_{gbf} \frac{\partial}{\partial X^{bf}} \right) \\
- \left( (\text{det}A) M^i A^g_i \epsilon_{dae} \frac{\partial}{\partial X^{ae}} \right) \left( (\text{det}A) N^k A^g_k \epsilon_{gbf} \frac{\partial}{\partial X^{bf}} \right) \quad (87)
\]
Likewise, all terms in (87) with derivatives acting on $\text{det} A$ become annihilated, leaving behind the following expression

$$(\text{det} A)^2 (N^i M^k - M^i N^k) \epsilon_{dae} \epsilon_{gbf} A^d_i (\text{det} A)^{-1} \left( \frac{\partial}{\partial X^{ae}} A^a_i \frac{\partial}{\partial X^{bf}} \right)$$

$$= (\text{det} A) (N^i M^k - M^i N^k) \epsilon_{dae} \epsilon_{gbf} A^d_i \frac{\partial}{\partial X^{bf}}$$

$$= (\text{det} A) \left[ (N^i A^d_i) (M^k A^k) - (M^i A^d_i) (N^k A^k) \right] \epsilon_{dae} \epsilon_{gbf} \frac{\partial}{\partial X^{bf}}$$

$$= V^{bf} ((\vec{N} \cdot A) \times (\vec{M} \cdot A)) \tilde{\Psi}^{bf}.$$  \hspace{1cm} (88)

The conclusion of this section is that the quantum and classical algebra of constraints are mutually consistent.

5 Hilbert space of states

We now arrive at the main deliverable of this paper, namely to establish the requisite structure to perform quantum mechanics of gravity in the Chang–Soo variables starting with anisotropic minisuperspace. It is hoped that the results of this section should address most objections to the existence of a Hilbert space for quantum gravity for the pure Kodama state $\Psi^{Kod}$ and its relatives. We perform this arrangement for anisotropic minisuperspace in the present work, reserving the full theory for a subsequent paper.

5.1 Structure of the kinematic Hilbert space

Define a linear vector space $\gamma$ over the field of complex numbers $C$ so that $\forall \lambda \equiv \lambda_{ae} \in (3, C) \in \gamma$ we have that $\lambda_{ae} = \alpha_{ae} + i \beta_{ae}$, for some $\alpha_{ae}, \beta_{ae} \in (3, R)$. Hence $\lambda_{ae}$ forms a $3 \times 3$ matrix of complex numerical constants. The complex conjugate of $\lambda$ is given by $\lambda^* \in \gamma$ with $\lambda^*_{ae} = \alpha_{ae} - i \beta_{ae}$. We will make use of two prototype vectors $\lambda, \zeta \in \gamma$ in what follows, such that

$$\lambda_{ae} = \alpha_{ae} + i \beta_{ae}; \quad \zeta_{ae} = \alpha'_{ae} + i \beta'_{ae} \quad \forall a, e$$  \hspace{1cm} (89)

Next, endow $\gamma$ with the structure of an inner product space with inner product $\pi(\gamma \otimes \gamma) \rightarrow C$, given by

$$\langle \lambda | \zeta \rangle = \sum_{a, e} \lambda^*_{ae} \zeta_{ae} = \lambda^* \cdot \zeta.$$  \hspace{1cm} (90)

Hence in components we have that
\[ \langle \lambda | \zeta \rangle = (\alpha - i\beta) \cdot (\alpha' + i\beta') = \alpha \cdot \alpha' + \beta \cdot \beta' + i(\alpha \cdot \beta' - \beta \cdot \alpha'). \]  

\[ (91) \]

Next, define a vector space \( \Gamma_C \) over the field of functions\(^{15} \) such that for any \( Z \equiv Z^{ae} \in \Gamma_C \), we have that \( Z^{ae} = X^{ae} + iY^{ae} \) for all \( a,e \). Hence \( Z^{ae} \in GL(3, C[f]) \).

Define a linear functional or a map \( \Psi \) from \( \gamma \otimes \Gamma_C \to \psi_{Kin} \), where \( \psi_{Kin} \) will ultimately play the role of the kinematic Hilbert space \( H_{Kin} \), by

\[ \Psi_{\gamma}(Z) = \langle Z | \Psi_{\gamma} \rangle = e^{\lambda(Z)}, \]

where we have defined

\[ \lambda(Z) = \sum_{a,e} \lambda_{ae} Z^{ae} = \lambda \cdot Z. \]

\[ (93) \]

In terms of the constituents, (93) is given by

\[ \lambda(Z) = (\alpha + i\beta) \cdot (X + iY) = \alpha(X) - \beta(Y) + i(\beta(X) + \alpha(Y) \sim \alpha \cdot X - \beta \cdot Y + i(\beta \cdot X + \alpha \cdot Y)). \]

\[ (94) \]

The second line of (94) is an abuse of notation made pure for convenience, but should be understood from the context. Hence we can now make the following identification of

\[ \Psi_{\lambda}(Z) = \langle X, Y | \Psi_{\lambda} \rangle = \Psi_{\lambda}(X, Y). \]

\[ (95) \]

### 5.2 Formulation of a norm on the space \( \gamma \)

**Lemma(1):** The first claim is that the inner product \( \pi \) induces a norm on \( \gamma \), given by

\[ \pi_{\gamma}(\lambda) = \| \lambda \| = \sqrt{\langle \lambda | \lambda \rangle} = \sqrt{\lambda^* \cdot \lambda} = \sqrt{\alpha \cdot \alpha + \beta \cdot \beta} \]

\[ (96) \]

It is clear that the norm as defined in (96) is positive definite, since \( \alpha \) and \( \beta \) are real. We must now prove that (96) satisfies the requirements for the definition of a norm.

\(^{15}\)As distinguished from the field of numerical constants. We will denote the field of holomorphic functions by \( C[f] \).
(i) Prove $\pi_\gamma(c\lambda) = |c|\pi_\gamma(\lambda)$ for all $\lambda \in \gamma$ and $c \in C$. This automatically follows from

$$
\pi_\gamma(c\lambda) = \sqrt{(c\lambda)^* \cdot (c\lambda)} = (c^* c)^{1/2} \sqrt{\lambda^* \cdot \lambda} = |c|\pi_\gamma(\lambda).
$$

(97)

(ii) Next, we must prove that the norm $\pi$ satisfies the Minkowski inequality. Hence, prove that $\pi_\gamma(\lambda + \zeta) \leq \pi_\gamma(\lambda) + \pi_\gamma(\zeta)$, for all $\lambda, \zeta \in \gamma$. Let us proceed by comparing the squares of the inequality. Define the square of left hand side of the inequality by $l$, given by

$$
l = |\pi_\gamma(\lambda + \zeta)|^2 = \langle \lambda + \zeta | \lambda + \zeta \rangle = (\lambda^* + \zeta^*) \cdot (\lambda + \zeta)
$$

$$
= \lambda^* \cdot \lambda + \lambda^* \cdot \zeta + \zeta^* \cdot \lambda + \zeta^* \cdot \zeta
$$

$$
|\lambda|^2 + |\zeta|^2 + \lambda^* \cdot \zeta + \zeta^* \cdot \lambda.
$$

(98)

The last two terms in (98) are given by

$$
\lambda^* \cdot \zeta + \zeta^* \cdot \lambda = (\alpha - i\beta) \cdot (\alpha' + i\beta') + (\alpha' - i\beta') \cdot (\alpha + i\beta) = 2(\alpha \cdot \alpha' + \beta \cdot \beta').
$$

(99)

Hence we have that

$$
l = |\lambda|^2 + |\zeta|^2 + 2(\alpha \cdot \alpha' + \beta \cdot \beta').
$$

(100)

Define the square of the right hand side of the Minkowski inequality by $r$, given by $r = (|\pi_\gamma(\lambda)| + |\pi_\gamma(\zeta)|)^2$. Then we have that

$$
r = \sqrt{\langle \lambda | \lambda \rangle + \langle \zeta | \zeta \rangle}^2 = \langle \lambda | \lambda \rangle + \langle \zeta | \zeta \rangle + 2\sqrt{\langle \lambda | \lambda \rangle \langle \zeta | \zeta \rangle}
$$

$$
= |\lambda|^2 + |\zeta|^2 + 2\alpha \cdot \alpha' + \beta \cdot \beta'.
$$

(101)

Next, we construct the inequality through the quantity $(r - l)/2$. Using the results of (100) and (101), we have

$$
\frac{r - l}{2} = \sqrt{\alpha \cdot \alpha' + \beta \cdot \beta'} \sqrt{\alpha' \cdot \alpha' + \beta' \cdot \beta'} - (\alpha \cdot \alpha' + \beta \cdot \beta') = ||\lambda||\|\zeta\| - \text{Re}\{\lambda^* \cdot \zeta\},
$$

(102)

But $\text{Re}\{\lambda^* \cdot \zeta\} \leq ||\lambda \cdot \zeta||$, since the real part of any complex number cannot exceed the the norm, and by the Caucy–Scwharz inequality we have that $||\lambda \cdot \zeta|| \leq ||\lambda||\|\zeta\|$. Hence it follows that $||\lambda||\|\zeta\| - \text{Re}\{\lambda^* \cdot \zeta\} \geq 0$. Therefore $\sqrt{r - l} \geq 0$, which leads to

$$
\pi_\gamma(\lambda) + \pi_\gamma(\zeta) \geq \pi_\gamma(\lambda + \zeta)
$$

(103)
as desired.

(iii) The last item to prove is positivity, namely that \( \pi_\gamma(\lambda) = 0 \) iff \( \lambda = 0 \). This automatically follows, since

\[
|\pi_\gamma(\lambda)|^2 = \alpha \cdot \alpha + \beta \cdot \beta \geq 0.
\]

(104)

Since \( \lambda, \beta \in \mathbb{R}^9 \) are real, the inner product \( \pi \) is positive definite. Therefore, \( \pi_\gamma(\lambda) = 0 \) iff \( \alpha_{ae} = \beta_a = 0 \), \( \forall a, e \), or that \( \lambda = 0 \).

We have successfully defined a norm on the complex vector space of complex-valued three by three matrices. Hence we are part of the way toward our goal of constructing a kinematic Hilbert space for quantum gravity in Chang–Soo variables. Next, we must formulate the resulting Hilbert space structure induced on the space of states \( \psi \).

### 5.3 Formulation of a norm on the space \( \psi \)

We will now show how the inner product \( \pi(\gamma \otimes \gamma) \rightarrow \mathbb{C} \) given by \( \langle \lambda | \zeta \rangle \) induces an inner product \( \Pi(\psi_{\text{K in}} \otimes \psi_{\text{K in}}) \rightarrow \mathbb{C} \), given by \( \langle \psi_\lambda | \psi_\zeta \rangle \) with respect to the Gaussian measure \( D\mu(Z) = D\mu(X,Y) \), given by

\[
D\mu(Z) = \nu^{-9} \prod_{a,e} \delta Z \ e^{-\nu^{-1}|Z|^2} = \nu^{-9} \prod_{a,e} \delta X \delta Y \ e^{-\nu^{-1}(X \cdot X + Y \cdot Y)}. \tag{105}
\]

We will show three things. First, that the inner product \( \Pi \) is finite iff the inner product \( \pi \) is finite.\(^{16}\) Secondly, the inner product \( \Pi(\psi_{\text{K in}} \otimes \psi_{\text{K in}}) \) induces a norm \( \Pi_\Gamma(\Psi_\lambda) \) given by

\[
\Pi_\Gamma(\Psi_\lambda) = \langle \Psi_\lambda | \Psi_\lambda \rangle = e^{\nu \langle \lambda | \lambda \rangle}. \tag{106}
\]

The norm (106) is clearly positive definite.

First, for the inner product we have that

\[
\langle \Psi_\lambda | \Psi_\zeta \rangle = \nu^{-9} \int D\mu(Z) \langle \Psi_\lambda | Z \rangle \langle Z | \Psi_\zeta \rangle = \int \nu^{-9} D\mu(X,Y) \langle \Psi_\lambda | X,Y \rangle \langle X,Y | \Psi_\zeta \rangle. \tag{107}
\]

Proceeding from (107), we have

\(^{16}\) Another way to state this is that a necessary and sufficient condition for \( \Pi \) to be square integrable is for \( \pi \) to be square summable.
\[ \langle \Psi_\lambda | \Psi_\zeta \rangle = \nu^{-9} \int DXDY \ e^{-\nu^{-1}(X \cdot X + Y \cdot Y)} \Psi^*_\lambda(X, Y) \Psi_\zeta(X, Y), \quad (108) \]

where we have made the identifications

\[ \Psi^*_\lambda(X, Y) = e^{\alpha X - \beta Y - i(\beta \cdot X + \alpha Y)}; \quad \Psi_\zeta(X, Y) = e^{\alpha' X - \beta' Y + i(\beta' \cdot X + \alpha' Y)} \quad (109) \]

and we have made use of the definitions

\[ X \cdot X + Y \cdot Y = \sum_{a,e} X^{ae} X^{ae} + Y^{ae} Y^{ae}. \quad (110) \]

Hence, the inner product on \( \psi_{\text{Kin}} \), continuing from (108) is given by

\[ \langle \Psi_\lambda | \Psi_\zeta \rangle = \left( \nu^{-9/2} \int DX e^{-\nu^{-1}(X \cdot X + r \cdot X)} \right) \left( \nu^{-9/2} \int DY e^{-\nu^{-1}(Y \cdot Y + r \cdot Y)} \right) \]
\[ = e^{\frac{\nu}{4}(r \cdot r + s \cdot s)} = \exp \left[ \frac{\nu}{4} \left( |\pi_\gamma(r)|^2 + |\pi_\gamma(s)|^2 \right) \right]. \quad (111) \]

where we have made the definitions \( r \equiv r_{ae} = \alpha_{ae} + \alpha'_{ae} + i(\beta_{ae} + \beta'_{ae}) \) and \( s \equiv s_{ae} = -(\beta_{ae} + \beta'_{ae}) + i(-\alpha_{ae} + \alpha'_{ae}) \). The following holds

\[ \frac{1}{4}(r \cdot r + s \cdot s) = \alpha \cdot \alpha' + \beta \cdot \beta' + i(\alpha \cdot \beta' - \alpha' \cdot \beta) = \lambda^* \cdot \zeta = \langle \lambda | \zeta \rangle. \quad (112) \]

The end result is that

\[ \langle \Psi_\lambda | \Psi_\zeta \rangle = e^{\nu <\lambda | \zeta>} = e^{\nu \lambda^* \cdot \zeta}. \quad (113) \]

Hence the inner product \( \Pi(\psi_{\text{Kin}} \otimes \psi_{\text{Kin}}) \to C \) is induced by the inner product \( \pi(\overline{\gamma} \otimes \gamma) \to C \).

It now remains to prove that \( \psi_{\text{Kin}} \) is a normed linear space with norm derived form the inner product \( \Pi \). If so, then the norm would be given by

\[ \Pi(\Psi_\lambda) = \sqrt{\langle \Psi_\lambda | \Psi_\lambda \rangle} = \sqrt{|\Psi_\lambda|^2} = e^{\frac{\nu}{2} |\lambda|^2}. \quad (114) \]

We must now prove that (114) satisfies the requirements of a norm.

(i) Prove \( \Pi(c\Psi_\lambda) = |c|\Pi(\Psi_\lambda) \) for all \( c \in C \) and all \( \Psi_\lambda \in \psi_{\text{Kin}} \). Hence we have that
\[ \|c\Psi\| = (c^*c)^{1/2}\|\Psi\|, \quad (115) \]

which follows directly from the linearity of the integral defining the inner product. Next, we must verify that the Minkowski inequality is satisfied.

(ii) Prove \( \Pi(\Psi_\lambda + \Psi_\zeta) \leq \Pi(\Psi_\lambda) + \Pi(\Psi_\zeta) \). First we define the right and left hand sides of the inequality by \( r = \Pi(\Psi_\lambda) + \Pi(\Psi_\zeta) \) and \( l = \Pi(\Psi_\lambda + \Psi_\zeta) \), where

\[ \Pi(\Psi_\lambda) = e^{\frac{\nu}{2}|\lambda|^2}; \quad \Pi(\Psi_\zeta) = e^{\frac{\nu}{2}|\zeta|^2}. \quad (116) \]

Hence we have that

\[ r = e^{\frac{\nu}{2}|\lambda|^2} + e^{\frac{\nu}{2}|\zeta|^2} \quad (117) \]

and \( l \) is given by

\[
l = \sqrt{\langle \Psi_\lambda + \Psi_\zeta | \Psi_\lambda + \Psi_\zeta \rangle} \\
= \sqrt{\langle \Psi_\lambda | \Psi_\lambda \rangle + \langle \Psi_\zeta | \Psi_\zeta \rangle + \langle \Psi_\lambda | \Psi_\zeta \rangle + \langle \Psi_\zeta | \Psi_\lambda \rangle} \\
= \sqrt{e^{\nu|\lambda|^2} + e^{\nu|\zeta|^2} + e^{\nu\lambda^*\zeta} + e^{\nu\zeta^*\lambda}}. \quad (118) \]

To examine the Minkowski inequality, let us form the difference \( r^2 - l^2 \). This is given by

\[
r^2 - l^2 = e^{\nu|\lambda|^2} + e^{\nu|\zeta|^2} + 2e^{\frac{\nu}{2}(|\lambda|^2+|\zeta|^2)} - \left( e^{\nu|\lambda|^2} + e^{\nu|\zeta|^2} + e^{\nu\lambda^*\zeta} + e^{\nu\zeta^*\lambda} \right) \\
= 2e^{\frac{\nu}{2}(|\lambda|^2+|\zeta|^2)} - \left( e^{\nu\lambda^*\zeta} + e^{\nu\zeta^*\lambda} \right). \quad (119) \]

To proceed, we recall the definitions \( \lambda = \alpha + i\beta, \lambda^* = \alpha - i\beta \) and \( \zeta = \alpha' + i\beta' \) and \( \zeta^* = \alpha' - i\beta' \). The following identities are in place

\[
\zeta^* \cdot \lambda = \alpha' \cdot \alpha + \beta' \cdot \beta + i(\alpha' \cdot \beta - \beta' \cdot \alpha); \quad \lambda^* \cdot \zeta = \alpha \cdot \alpha' + \beta \cdot \beta' - i(\alpha \cdot \beta - \beta' \cdot \alpha) \quad (120) \]

as well as the identities

\[
|\lambda|^2 = \alpha \cdot \alpha + \beta \cdot \beta; \quad |\zeta|^2 = \alpha' \cdot \alpha' + \beta' \cdot \beta'. \quad (121) \]

Hence we have for first term on the right hand side of (119) that
\[ e^{\frac{\nu}{2}(|\lambda|^2 + |\zeta|^2)} = e^{\frac{\nu}{2}(\alpha \cdot \alpha + \alpha' \cdot \alpha' + \beta \cdot \beta' + \beta' \cdot \beta')} \]

and for the second term on the right hand side of (119) that

\[ e^{\nu \lambda^* \cdot \zeta} + e^{\nu \zeta^* \cdot \lambda} = e^{i\nu(\alpha' \cdot \beta - \beta' \cdot \alpha)} + e^{\nu(\alpha \cdot \alpha' + \beta \cdot \beta')} e^{-i(\alpha' \cdot \beta' \cdot \alpha)} \]

\[ = 2 e^{\nu(\alpha' + \beta')} \cos[\nu(\alpha' \cdot \beta' - \beta' \cdot \alpha)] \leq 2 e^{\nu(\alpha' + \beta')}, \quad (123) \]

where we have used that \( \cos \theta \leq 1 \) for \( \theta \in \mathbb{R} \), and it is the case that the argument of the cosine function in the last line of (123) is real. Since we have put both sides of the Minkowski inequality on the same footing, in terms of exponentials, we can now prove the inequality. Define two 18-vectors with components \( a = (\alpha, \beta) \) and \( b = (\alpha', \beta') \). Then the Minkowski inequality reduces to

\[ \frac{r^2 - l^2}{2} \geq e^{\frac{\nu}{2}(|a|^2 + |b|^2)} - e^{\nu a \cdot b}. \quad (124) \]

Since \( a = (\alpha_{ae}, \beta_{ae}) \) and \( b = (\alpha'_{ae}, \beta'_{ae}) \) are real, then \( |a| + |b| \geq 0 \) implies that \( |a| + |b| \geq 2a \cdot b \). Hence comparing the arguments of the exponentials in (124) the result follows. Thus

\[ \Pi(\Psi_\lambda + \Psi_\zeta) \leq \Pi(\Psi_\lambda) + \Pi(\Psi_\zeta) \quad (125) \]

There is one last obstacle to the claim of a Hilbert space, which is the completion of the inner product space in the given norm, namely positivity.

(iii) Prove that \( \Pi(\Psi_\lambda) = 0 \) iff \( \Psi_\lambda = 0 \). This is problematic in that \( \Pi(\Psi_\lambda) = e^{\frac{\nu}{2} |\lambda|^2} \geq 1 \), on account of \( |\lambda|^2 \geq 0 \). Hence there currently does not exist a state \( \psi_{kin} \) which can play the role of the zero vector. The zero vector is necessary in order for our Hilbert space to contain physical states annihilated by the constraints. The physical Hilbert space must be invariant under the constraints in order for it to be possible to map states in the kernel of the constraints into the zero vector.\[ \text{The zero vector is necessary in order for our Hilbert space to contain physical states annihilated by the constraints. The physical Hilbert space must be invariant under the constraints in order for it to be possible to map states in the kernel of the constraints into the zero vector.} \]

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5.4 Kinematical Hilbert space and a resolution of the problem of time

The usual method of quantization in the loop representation is to first define a kinematical Hilbert space $H_{Kin}$, an appropriate space of quantum states forming an irreducible representation of the algebra of quantum observables at the level prior to implementation of the constraints, with corresponding Haar measure. Our view in this work is that $H_{Kin}$ might be unnecessarily big, and therefore it is not necessary to have a Hilbert space structure on this whole space in order to have a complete and consistent quantum theory. Though $H_{Kin}$ may provide a stepping stone which provides useful insights, one should be ultimately interested in the space of physical states $|\psi_{phys}\rangle \in \text{Ker}\{\hat{\Phi}\}$, those states which are strongly annihilated by all of the quantum constraints as in

$$\hat{\Phi}_I |\psi_{phys}\rangle = 0.$$  \hspace{1cm} (126)

Additionally, one should be able to construct a sufficiently large set of states $|\psi\rangle \in H_{phys}$ to enable verification of the semiclassical limit of the theory. As implied by (81), any arbitrary state $|\psi\rangle$ can serve as a representation of the algebra in Chang–Soo variables due to the manifest closure of the algebra. Hence we can define the kinematic Hilbert space by $|\psi\rangle \in H_{Kin}$ subject to normalizability. We are now free to deduce the physical states $|\psi_{phys}\rangle$ as elements of $H_{phys} \in H_{Kin}$.

In determining the Hilbert space structure it will be convenient to work in the functional Schrödinger representation. By choosing a complete set of states $|X\rangle$ of the configuration variables at a given time $T$, one may define a projection operator on to configuration basis eigenstates for each time $T$, with corresponding orthogonality and completeness relations

$$\prod_{a,e} \int_{\Gamma} \delta\mu(X) |X(T)\rangle\langle X(T)| = I(T); \quad \langle X(T) | X'(T) \rangle = \prod_{a,t} \delta(X(T) - X'(T))$$  \hspace{1cm} (127)

which allows any arbitrary state $|\psi\rangle \in H_{Kin}$ to be expanded in this basis at a given time $T$\(^{18}\)

$$|\psi\rangle = \prod_{a,e} \int_{\Gamma} \delta X(T) |X(T)\rangle\langle X(T)|\psi\rangle.$$  \hspace{1cm} (128)

\(^{18}\)The states are decomposed with respect to a fixed time $T$, due to the fact that the relationship amongst the operators has been defined only for equal times via the equal-time commutation relations. Additionally, the projection operator onto a complete set of states $I = I(T)$ should be explicitly independent of time.
One claim of the present paper is that the decomposition (128) solves the problem of time in quantum gravity, at least in minisuperspace. This is because the state $|\psi\rangle$ is an abstract entity until projected onto a complete basis. By defining $\Psi(T) = \langle X(T) |\psi\rangle$ we can associate a wavefunction to any desired time.

The elementary quantum operators in the Schrödinger representation forming a representation of the equal-time commutation relations (65) can be realized by multiplication and functional differentiation of the wavefunction $\Psi(T)$ at a given time $T$, as in

$$\hat{X}^{ae}(T)\psi[X] = X^{ae}(T)\psi[X]; \quad \hat{\Psi}_{ae}(T)\psi = hG\delta_{\delta X^{ae}(T)}\psi[X] \quad \text{(129)}$$

where we have taken $\psi[X] = \langle X |\psi\rangle$.

5.5 Normalizability of the wavefunctions

As regards the measure on the physical Hilbert space $H_{phys}$, we will see that for pure gravity for Lorentzian signature it will be convenient to use a Gaussian measure in order to obtain square-integrable wavefunctions, whereas for Euclidean signature one additionally has the option of the Lebesgue measure for delta-function normalizable wavefunctions. This implies the Hilbert space structure for each point $x \in \Sigma$ in the Lorentzian case of $H = L^2(C,d\mu)$, with a resolution of the identity as in the Bargmann representation

$$\prod_{a,e} \int \delta X(T) \exp \left[ \frac{1}{2\nu} \sum_{a,e} X^{ae}(T)X_{ae}(T) \right] X(T)\langle X(T) | \rangle |X(T)| = I. \quad \text{(130)}$$

To illustrate, let us construct an auxiliary Hilbert space made up of linear functionals of the Chang–Soo variables. Hence define the wavefunction

$$\Psi_{\lambda}(X) = \exp \left[ \sum_{a,e} \lambda_{ae} X^{ae} \right] \quad \text{(131)}$$

where the $\lambda_{ae} \in \gamma$ are not dynamical variables, but simply a three by three matrix of labels. For Euclidean signature, an orthogonal basis of states can

\footnote{We will sometimes omit the time label on the wavefunction $\psi(X)$. It is then implied that the Chang–Soo variables $X^{ae}$ play the role of a clock with respect to which the wavefunction evolves.}

\footnote{Here $\nu$ is a numerical constant of mass dimension $[\nu] = -6$, necessary to make the argument of the exponential dimensionless.}
be chosen by restricting the variables $X^{ae}$ to be real. One can see this by inserting a complete set of states at the chosen time into the inner product of two wavefunctions labeled by matrices $\lambda \equiv \lambda^{ae}$ and $\zeta \equiv \zeta^{ae}$. Hence, choosing the Lebesgue measure with the replacements $\lambda \rightarrow i\lambda$ and $\zeta \rightarrow i\zeta$, we have that

$$\langle \zeta | \lambda \rangle_{\text{Eucl.}} = \prod_{a,e} \int_{\Gamma} \delta X^{ae} \langle \zeta | X(T) \rangle \langle X(T) | \lambda \rangle = \prod_{a,e} \int_{\Gamma} \delta X^{ae} \Psi^{*}_\zeta(X) \Psi^{*}_\lambda(X)$$

$$= \int DX e^{iX \cdot (\lambda - \zeta)} = \prod_{a,e} \int_{\Gamma} \delta(\lambda^{ae}(T) - \zeta^{ae}(T)) = \delta_{\lambda \zeta} \forall T$$

Hence the states labelled by two matrices in $GL(3, R)$ are orthogonal if any of the corresponding matrix elements are not the same.

For both Euclidean and Lorentzian signatures, a Hilbert space of coherent states can be built. This is necessary in order to obtain square integrable wavefunctions in the Lorentzian case. To illustrate take the Hilbert space consisting of holomorphic functions of $Z \in \Gamma'$ with $\psi_\lambda[Z] = e^{Z(\lambda)}$, where now both $Z^{ae}$ and $\lambda^{ae}$ are complex-valued. $Z \in \Gamma'$ are elements of the complex vector space $\Gamma'$ and $\lambda \in \Gamma'$ is in its dual. Then we can define the inner product

$$\lambda \cdot Z = \sum_{ae} \lambda^{ae} Z^{ae}. \quad (133)$$

We will see how this complex inner product on $\Gamma'$ induces an inner product on the space of states $|\psi\rangle$. First, we must decompose $\lambda$ and $Z$ into their real and their imaginary parts as in $\lambda^{ae} = \alpha^{ae} + i\beta^{ae}$ and $Z^{ae}(T) = X^{ae}(T) + iY^{ae}(T)$. This can be interpreted in the representation as the decompositon

$$\langle Z | \lambda \rangle = \exp \left[ \sum_{a,e} \alpha^{ae} X^{ae} - \beta^{ae} Y^{ae} + i(\beta^{ae} X^{ae} + \alpha^{ae} Y^{ae}) \right] = e^{\lambda \cdot Z} \quad (134)$$

Now we can evaluate the inner product between two states labelled by $\lambda = \alpha_i + i\beta_i$ and $\zeta = \alpha_j + i\beta_j$ in the measure $D\mu(Z, \overline{Z})$. \(^{21}\) Hence

$$\langle \lambda | \zeta \rangle = \nu^{-9} \int dZ e^{-\frac{1}{2} |Z|^2} \langle \lambda | Z \rangle \langle Z | \zeta \rangle$$

$$= \nu^{-9} \int dX dY e^{-\frac{1}{2} (X^2 + Y^2)} e^{(\alpha_i + \alpha_j + i(\beta_i - \beta_j)) \cdot X - (\beta_i - \beta_j + i(\alpha_i - \alpha_j)) \cdot Y}$$

$$= \exp \left[ \nu (\alpha_i \cdot \alpha_j + \beta_i \cdot \beta_j - \frac{i}{2} (\alpha_i \cdot \beta_j - \alpha_j \cdot \beta_i)) \right] = e^{\nu X \cdot \zeta}. \quad (135)$$

\(^{21}\)Here $i \sim a, e, j \sim a, e$ is a shorthand notation to label both the components of the vector and also to identify them as distinct vectors.
We see that in order for the inner product of two wavefunctions to be finite, then the real and the imaginary parts must form a square-summable series, which is the case since we are dealing with a finite dimensional space $\Gamma'$.\(^{22}\) Hence, the inner product on the space $\Gamma'$ induces a corresponding inner product on the space of states $|\psi\rangle$. Written out fully, this is given by

$$\langle \lambda | \zeta \rangle_{\text{Lor.}} = \exp \left[ \nu \sum_{a,e} \lambda_{ae}^* \zeta_{ae} \right].$$ \(^{(136)}\)

No further reference to the Chang–Soo variables exists and we have obtained a well-defined Hilbert space with normalizable wavefunctions.

### 5.6 Expectation values and observables

Now that we have defined a Hilbert space, we must next give a prescription for calculating matrix elements of observables. Hence, we have for an observable $O = O[Z, \overline{Z}]$

$$\langle \lambda | O | \zeta \rangle = \int D\mu(Z, \overline{Z}) \langle \lambda | Z(T) \rangle \langle Z(T) | \hat{O} | \zeta \rangle$$

$$= \int D\mu(Z, \overline{Z}) \Psi_\lambda^*(Z) \left[ \hat{O}[Z, \overline{Z}] \Psi_\zeta(Z) \right]$$ \(^{(137)}\)

The crucial observation is to exploit the techniques of generating functionals. Hence multiplication by $Z$ under the integral (137) corresponds to differentiation with respect to $\lambda$, and likewise multiplication by $\overline{Z}$ corresponds to differentiation with respect to $\overline{\zeta}$. Hence we have that

$$\langle \lambda | Z^{ae} | \zeta \rangle = \frac{\partial}{\partial \zeta_{ae}} e^{\nu \lambda^* \cdot \zeta} = \nu \lambda_{ae}^* \langle \lambda | \zeta \rangle;$$

$$\langle \lambda | Z^{ae} | \zeta \rangle = \frac{\partial}{\partial \lambda_{ae}} e^{\nu \lambda^* \cdot \zeta} = \nu \zeta_{ae} \langle \lambda | \zeta \rangle;$$ \(^{(138)}\)

The procedure (138) then generalizes to functions of the Chang–Soo variables. Hence

$$\langle \lambda | O[Z, \overline{Z}] | \zeta \rangle = O[\nu \zeta, \nu \overline{\lambda}] e^{\nu \lambda^* \cdot \zeta}$$ \(^{(139)}\)

\(^{22}\)This automatically excludes configurations with infinite values for $(\lambda, \zeta)$.\)
The observables acquire the dimensionful constant \( \nu \).\(^{23}\)

Note also, when taking expectation values of derivatives,

\[
\langle \lambda \mid \frac{\partial}{\partial Z_{ae}} \mid \zeta \rangle = \int dZ dZ e^{-\nu^{-1}|Z|^2} e^{\lambda(Z)} \frac{\partial}{\partial Z_{ae}} e^{\zeta(Z)} \]

\[
= \zeta_{ae} \int dZ dZ e^{-\nu^{-1}|Z|^2} e^{\lambda(Z)} e^{\zeta(Z)} = \zeta_{ae} \langle \lambda \mid \zeta \rangle \forall a, e.
\]

Hence, as one would expect in the Bargmann representation \([ \ldots \] \), we have

\[
Z_{ae} \sim \nu \frac{\partial}{\partial Z_{ae}}
\]

Another feature which one can exploit is to obtain the expectation value of the reciprocal of operators.\(^{24}\) Using the result for holomorphic functions that

\[
\frac{1}{z} = \int_{-\infty}^{0} ds e^{sz} \equiv (\partial/\partial s)^{-1} s^z,
\]

we have the relations

\[
\langle \lambda \mid \frac{1}{Z_{ae}} \mid \zeta \rangle = \int_{-\infty}^{0} d\zeta_{ae} e^{\nu \lambda \zeta} = \frac{1}{\nu \zeta_{ae}} \forall a, e.
\]

It is hoped that the present section has addressed \([12]\), which attempts to define an inner product of the Kodama state in the loop representation, at least in anisotropic minisuperspace. Defining the state on a loop \( \gamma \) by

\[
\Psi_{\gamma}[A] = \Psi_{Kod}[A] T_{\gamma}[A],
\]

where \( A \sim A_{i}^{a} \) is the Ashtekar connection defined by \( A_{i}^{a} = \Gamma_{i}^{a} - iK_{i}^{a} \), where \( \Gamma \) and \( K \) are respectively the three dimensional spin connection and the extrinsic curvature of a spatial slice \( \Sigma \), and \( T_{\gamma}[A] = e^{\int_{\gamma} A} \) is the holonomy of the Ashtekar connection, one attempts to compute the inner product of two states via the prescription

---

\(^{23}\)The mass dimension of the constant \( \nu \) is \([\nu] = 6\) in order to make the argument of the exponential defining the measure dimensionless. Its value can further be fixed based upon parameters from quantum gravity.

\(^{24}\)This procedure applies only to scalars and is well defined only when the scalar is nonzero.
In the prescription that we have outlined in the present work we have provided a possible resolution, at least in the homogeneous sector, by using in essence a kind of coherent basis for the states. The analogy is that the states in the Chang–Soo variables can be labelled by the matrix $\lambda \equiv \lambda_{ae}$, which as we will see consists of five independent elements in minisuperspace upon solution of the constraints. The matrix $\lambda_{ae}$ is the analogue of the loop label $\gamma$, except it has a well-defined semiclassical interpretation which we will amplify in a separate work. Since the states are holomorphic functions of the Chang–Soo variables, then the inner product between two states is well defined in the Bargmann representation, whether for Lorentzian or for Euclidean signature. Hence in the language of [12] taking $Z = X + iY$,

$$\langle \Psi | O \hat{X}, \hat{Y} | \Psi \rangle = \left[ O(\partial/\partial a,a/\partial b) e^{a-b} \right]_{a=\eta+\gamma, \ b=\eta-\gamma} = O(\eta - \gamma, \eta + \gamma) e^{\gamma^* \Psi}(147)$$

This as well applies for reciprocals of operators.

One last question concerns the existence of self-adjoint bounded operators on the Hilbert space. It is clear, due to the properties of the Bargmann representation and as shown, that $Z^{\alpha e}$ and $\overline{Z}^{eic}$ constitute such operators in the Gaussian measure. Since the constraints annihilate the physical states $\Psi_{Physics}$ by definition and as we will show in the next section, then the quantum constraints are by default strongly self-adjoint. Now that we have put in place the requisite Hilbert space structure, we will now demonstrate the relevance of the Hilbert space we have introduced to quantum gravity in the Chang–Soo variables.

25 Suffice it to say that the inverse of the matrix $\lambda_{ae}$ directly correlates to the anti self-dual part of the Weyl curvature tensor, which is related to the Petrov classification of the spacetime which can also be labelled by various invariants of the CDJ matrix.
6 Solution to the quantum constraints and association with the physical Hilbert space

We will now correlate a set of normalizable wavefunctions solving the quantum constraints in Chang–Soo variables to the Hilbert space just established, starting from the most general form of a trial wavefunction $\Psi \in \psi_{Kin}$ at a given time $T$

$$\Psi[X] = \exp \left[ \int_T \Psi_{ae}[X]\delta X^{ae} \right], \quad (148)$$

where $\Psi_{ae}$, at the level prior to implementation of the constraints, can have the most general functional dependence on the Chang–Soo variables $X^{ae}$.

We will often suppress the time label $T$ in what follows in order to avoid cluttering up the notation. Through explicitly satisfying the constraints, we must then narrow down the specific functional dependence of the matrix elements $\lambda_{ae}$ upon the configuration variables $X^{ae}$.

Starting with the quantized diffeomorphism constraint we have

$$\hat{H}_i \Psi = \epsilon_{ijk} B^j_a B^k_e \hat{\Psi}_{ae} \Psi = 0. \quad (149)$$

Expanding (149) on the wavefunction (148), we obtain

$$\hbar G \epsilon_{ijk} B^j_a B^k_e \frac{\partial}{\partial X^{[ae]}} \Psi = ((\det A) A^f_i \epsilon_{fae} \Psi_{[ae]}) \Psi = 0 \quad (150)$$

Since the $\det A \neq 0$, then the antisymmetric part of the semiclassical eigenvalue must vanish. Hence $\Psi_{[ae]} = 0$, or the matrix $\Psi_{ae} = \Psi_{(ae)}$ must be symmetric. This reduces the solution space from $GL(3,C)$ to $C^6$, with six independent components.$\textsuperscript{26}$

$$\hat{\Psi}_{[ae]} \Psi = \hbar G \frac{\delta}{\delta X^{[ae]}} \Psi. \quad (151)$$

From (151) one concludes that the diffeomorphism invariant state $|\psi_{Diff}\rangle$ does not contain any dependence upon the antisymmetric part of the Chang–Soo variables $X^{[ae]}$, which can now be considered as unphysical.

The Gauss’ law constraint is given by

$$\hat{G}_a \Psi = (\det A) f_{ade} \hat{\Psi}_{de} \Psi = (\det A) f_{ade} \Psi_{de} \Psi = 0. \quad (152)$$

$\textsuperscript{26}$Note that this is the same condition that would have arisen from solving the constraints at the classical level. Hence, there is a semiclassical-quantum correspondence.
Again, since \( \det A \neq 0 \), the Gauss’ law constraint also implies that the anti-symmetric part of the CDJ matrix vanishes in minisuperspace. We already know this from the diffeomorphism constraint, therefore the Gauss’ law constraint in anisotropic minisuperspace is redundant and does not result in a reduction of the degrees of freedom of \( \Psi_{ae} \).\(^{27}\)

A complex symmetric matrix can generally be diagonalized by a complex orthogonal transformation \(^{15}\) parametrized by three complex angles \((\theta_1, \theta_2, \theta_3)\) such that \( O_{ab}(\vec{\theta}) = e^{i\vec{\theta} \cdot T} \), where \( T \) is the set of generators for the group. Hence one can under fairly general circumstances write down the following relation\(^{28}\)

\[
\Psi_{ae} = O_{ae}(\vec{\theta})e^{i\int_{\bar{\theta}} O_{bc}^{-1}(\vec{\theta})\lambda_f}
\]

which parametrizes the matrix \( \Psi_{ae} \) by the diagonal matrix of its eigenvalues \( \lambda_f = \text{Diag}(\lambda_1, \lambda_2, \lambda_3) \) and a complex orthogonal rotation matrix \( O_{ae} \). Let us now solve the quantum Hamiltonian constraint, exploiting this observation. Classically, the Hamiltonian constraint involves just the invariants of the matrix \( \Psi_{ae} \) and is given by

\[
H = (\det A)\sqrt{\det \Psi} (\Lambda + \text{tr}\Psi^{-1}) = 0, \quad (154)
\]

where we have made use of the invariance of the trace under the complex orthogonal transformations.\(^{29}\) This produces a solution for \( \lambda_3 \) as a function of \( \lambda_1 \) and \( \lambda_2 \).

It requires some care to interpret the quantization of the reciprocal of the eigenvalues \( \bar{\lambda} \) in (154). At the classical level for a nondegenerate \( B_i^n \), a nontrivial solution exists by dividing through by \( \det A \) to obtain

\[
\sqrt{\frac{1}{\lambda_1} \frac{1}{\lambda_2} \lambda_3} (\Lambda + \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3}) = 0. \quad (155)
\]

To obtain the wavefunctional it helps to put the Hamiltonian constraint into a form for which the quantum version can be solved directly. Since we have eliminated any dependence upon the configuration variables, we can rescale the constraint by an arbitrary function of momenta without incurring any operator ordering ambiguities upon quantization.\(^{30}\) Rescaling by \( \sqrt{\det \Psi} \), we have classically that

\(^{27}\)In the full theory, the Gauss’ law constraint does result in a genuine reduction due to the spatial gradients, a nonlocal effect which we show in a separate work in progress.

\(^{28}\)We have excluded cases for which three independent eigenvectors do not exist in this work. Such ‘degenerate’ cases must be treated separately.

\(^{29}\)This excludes the degenerate configurations, for which the CDJ matrix is in general not diagonalizable. Such cases must be treated separately.

\(^{30}\)This is because we have chosen an operator ordering in the algebra of quantum constraints with the momenta to the right of the coordinates.
\[ \det \lambda + \text{Var} \lambda = \Lambda \lambda_1 \lambda_2 \lambda_3 + \lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_1 = 0 \]  

Equation (156) suggests a more efficient quantization from the level of the phase space using just the eigenvalues \( \lambda \) of the symmetric part of the CDJ matrix \( \Psi_{(ae)} \) along with its conjugate configuration variables \( X^{ae} \) as dynamical variables. The commutation relations (65) can then be replaced, by an appropriate rotation of variables, by

\[ \left[ \hat{X}^a(T), \hat{\Psi}_e(T) \right] = \hbar \beta G \delta_e^a; \quad \left[ \hat{X}^a(T), \hat{X}^e(T) \right] = \left[ \hat{\Psi}_a(T), \hat{\Psi}_e(T) \right] = 0. \quad (157) \]

Going through the analogous procedure in the Schrödinger representation of the physical variables \( \lambda \), the quantum Hamiltonian constraint would read

\[ \hat{H} \Psi = \hbar^2 G^2 \left[ h \Lambda \frac{\delta^3}{\delta X^1 \delta X^2 \delta X^3} + \frac{\delta^2}{\delta X^1 \delta X^2} + \frac{\delta^2}{\delta X^2 \delta X^3} + \frac{\delta^2}{\delta X^3 \delta X^1} \right] \Psi = 0. \quad (158) \]

There exists a subspace of (148) solving (158) free of ordering ambiguities, consistent with numerically constant \( \Psi_{ae} \). Hence, the matrix \( \lambda_{ae} \) serves as the label for a linear functional of \( X^{ae} \), and is free of functional dependence upon the dynamical variables \( X^{ae} \). We can now form a solution at time \( T \) by evaluating the starting action (24) on the solution to the constraints \( \Phi_a \sim \hat{H}_i \sim G_a \sim \hat{H} \sim 0. \)

\[ \psi_{\text{phys}} = \beta e^{i(hG)^{-1} \int_S} = \beta \exp \left[ (hG)^{-1} \int_0^T dt \Psi_{ae} \hat{X}^{ae} \right] \bigg|_{\Phi_a=0} \quad (159) \]

where \( \beta \) is a normalization constant for the state. But the initial value constraints must be satisfied consistently with the equations of motion. The classical equations of motion for the CDJ matrix \( \Psi_{ae} \) are given by (33)

\[ \Psi_{ae} = -\frac{\delta H}{\delta X^{ae}} = - \left[ i \delta_{ae} (\det A)^{-1} H[N] - N^i \left( A_i^d \delta_{bf} + A_i^f \delta_{bd} \right) f_{dae} \Psi_{ae} - \delta_{ae} \theta^d f_{dge} \Psi_{ge} \right] \quad (160) \]

which under the condition of a nondegenerate magnetic field \( \det A \neq 0 \) is completely a linear combination of constraints. Therefore each term

\(31\) Incidentally, the general solution of the classical equations of motion consistent with the constraints is that \( \Psi_{ae} = \lambda_{(ae)} = \text{const.} \forall a, e \) as we will show below. Due to the semiclassical-quantum correspondence, the classical solution evaluated on the starting action should produce a semiclassical wavefunction that also equals the quantum state \([8]\).
on the right hand side of (160) vanishes on account of the Hamiltonian, 
diffeomorphism and Gauss’ law constraints respectively, which implies for 
vacuum GR in anisotropic minisuperspace that \( \dot{\Psi}_{ae} = 0 \ \forall a, e \). Hence 
\( \Psi_{ae} = \lambda_{(ae)} = \text{const. for all} \ a, e \). Consistency with the Hamiltonian con-
straint reduces \( \lambda_{ae} \) to a five dimensional complex vector space.

The wavefunction (162) then reduces to

\[
\psi_{\text{Phys}} = \beta \exp \left[ \left( \frac{\hbar G}{\lambda_{ae}} \right) \int_0^T dt \lambda_{(ae)} X^{ae} \right] \mid_{\lambda_{ae} = \text{const.} \ \forall a, e,} \tag{161}
\]

Since \( \lambda_{ae} \) are a set of numerical constants, we can factor them out of the 
time integral in (162) thus leaving the integral of \( X^{ae} \) which is a total time 
derivative. Hence,

\[
\psi_{\text{Phys}}(T) = \beta \exp \left[ \int_0^T dt \lambda_{(ae)} X^{ae} \right] \mid_{\lambda_{ae} = \text{const.} \ \forall a, e} = \beta \exp \left[ (\hbar G)^{-1} \lambda_{ae} \int_0^T dt X^{ae} \right] = \beta \exp \left[ (\hbar G)^{-1} \lambda_{ae} (X^{ae}(T) - X^{ae}(T_0)) \right] = \psi_{\text{Phys}}(T_0) e^{(\hbar G)^{-1} \lambda \cdot X(T)} \tag{162}
\]

where we have defined \( \Psi_{\text{Phys}}(T_0) = \beta e^{- (\hbar G)^{-1} \lambda \cdot X(T_0)} \) as the wavefunction 
at some initial time \( T_0 \), for some normalization constant \( \beta \) which depends 
on the matrix elements \( \lambda_{ae} \). Hence the wavefunction is defined at the final 
time irrespectively of how the field \( X^{ae} \) evolved to that time.\(^{32}\) Equation 
(162) represents a Hilbert space labelled by five arbitrary constants. Two 
constants correspond to two of the eigenvalues of the CDJ matrix in \( \lambda_1, \lambda_2 \), 
with \( \lambda_3 \) determined by (156). The remaining three constants reside in the 
\( SO(3, C) \) angles used to diagonalize the CDJ matrix. If one regards these 
angles as being unphysical, then one can write the state as

\[
\psi_{\lambda_1, \lambda_2}[X] = \langle X | \lambda_1, \lambda_2 \rangle \sim e^{(\hbar G)^{-1} X \cdot \lambda} \tag{163}
\]

which is a normalizable set of states labelled by two arbitrary constants 
corresponding to two freely specifiable eigenvalues of \( \Psi_{ae} \). Hence we have 
constructed a Hilbert space. As for the pure Kodama state \( \Psi_{Kod} \), this 
forms an element of (163) where the three eigenvalues are equal as in \( \lambda_1 = \lambda_2 = \lambda_3 = -\frac{6}{\chi} \). Since (163) is normalizable in the measure introduced 
in the previous section, then the pure Kodama state \( \Psi_{Kod} \) in anisotropic 
minisuperspace is also normalizable with respect to the same measure.

\(^{32}\)This is our interpretation of the analogue of the noboundary proposal [16] for the 
Chang–Soo variables in anisotropic minisuperspace, and the addressal of the problem of 
time in quantum gravity, a result we hope to extend to the full theory in a subsequent 
paper.
7 Final Hilbert space arguments

The normalization factor $\beta$ can now be found to within an arbitrary phase factor by requiring that the wavefunctions be normalized in Lorentzian signature with respect to the measure defined in the previous section. Hence

\[ |\Psi_\lambda|^2 = |\beta|^2 \exp \left[ -2(hG)^{-1} \Re \{ \lambda \cdot X_0 \} \right] e^{i\lambda^* \cdot \lambda} = 1 \]  

(164)

where we have defined $X_0 = X(T_0)$ and where we have used the shorthand notation $\lambda \equiv (\lambda_1, \lambda_2)$ to denote the labels for the state. Equation (164) leads to the condition

\[ \beta = \beta(\lambda) = e^{i\alpha} \exp \left[ (hG)^{-1} \Re \{ \lambda \cdot X_0 \} \right] e^{-\frac{\pi}{4} \lambda^* \cdot \lambda}, \]  

(165)

where $\alpha$ is an arbitrary real number depicting the arbitrariness of the phase, whereupon $\beta$ has acquired the label of the state. We can finally write the wavefunction solving the constraints in the form

\[ \psi_\lambda[X] = e^{\frac{\pi}{4} \lambda^* \cdot \lambda} e^{(hG)^{-1} \lambda \cdot X}. \]  

(166)

Using these considerations, the Hilbert space of states is then given by

\[ \psi_\lambda[X] = e^{\frac{\pi}{4} \lambda^* \cdot \lambda} e^{(hG)^{-1} \lambda \cdot X}. \]  

(167)

We have suppressed the time label $T$, which is understood to be implicit in (167).

7.1 Arguments from geometric quantization and the semiclassical-quantum correspondence

The form of the wavefunctions forming the physical states can also be seen from arguments on geometric quantization. At the classical level prior to implementation of the quantum constraints the phase space for the Chang–Soo/CDJ variables is 18 dimensional. The symplectic two form \( \Omega \) on the phase space at this level is given by

33 One could alternatively use this phase to label the state by its initial value at \( t = 0 \) in the projective representation of the state, and then place all wavefunctions at \( t = 0 \) into the same equivalence class of states. In this way one circumvents the need to define the initial state of the universe.
Ω = (ℏG)^{-1} \sum_{a,e} \delta \Psi_{ae} \wedge \delta X^{ae} = (ℏG)^{-1} \sum_{a,e} \delta(\Psi_{ae} \delta X^{ae}). \quad (168)

For the class of states within the Hilbert space \( \psi_\lambda \in \psi_{phys} \), the symplectic two form \( \Omega \) vanishes. This is due to the fact that for these states, \( \Psi_{ae} = \lambda_{ae} \) which are numerical constants. Since the exterior derivative of a numerical constant is zero, then the symplectic two form evaluated on the solution to the constraints consistent with the initial value problem is given by

\[
\Omega \bigg|_{\dot{\phi}_\alpha = 0} = (ℏG)^{-1} \sum_{a,e} \delta \lambda_{ae} \wedge \delta X^{ae} = 0 \ \forall \lambda_{ae} \quad (169)
\]

where we have used \( \delta \lambda_{ae} = 0 \). By the Poincare Lemma, a closed two form is locally exact. Hence \( \Omega = 0 \), implies that \( \Omega = \delta \theta \) for some one form \( \theta \). The one form can be found from (168) using

\[
\Omega = 0 = (ℏG)^{-1} \sum_{a,e} \delta(\lambda_{ae} \delta X^{ae}) = \delta \theta_\lambda. \quad (170)
\]

The one form \( \theta = \theta_\lambda \) has then acquired the label of the states, as in

\[
\theta_\lambda = (ℏG)^{-1} \sum_{a,e} \lambda_{ae} \delta X^{ae}. \quad (171)
\]

The wavefunction of the universe for the states within our Hilbert space is then given by the integral of this one form over the space \( X^{ae} \), as in

\[
\Psi_\lambda [X] \propto e^{(ℏG)^{-1} \int \theta_\lambda} \propto e^{\lambda \cdot X} \quad (172)
\]
to within the normalization factor, for each \( \lambda \).

We have just shown that the canonical quantization procedure is consistent with geometric quantization for the class of states within our Hilbert space \( \psi_\lambda \in \psi_{phys} \), which implies an equivalence between these procedures. While restriction of the states to within the physical Hilbert space \( H_{phys} \) is a sufficient condition for this equivalence, the question arises as to whether it is necessary. An examination of (168) indicates that for any \( \Psi_{ae} = F_{ae} [X^{bf}] \), where the CDJ matrix can contain arbitrary functional dependence on the Chang–Soo potentials, that \( \Omega \) can be made to vanish. This can be seen from the manipulations

\[
\Omega = \sum_{a,e} \delta \Psi_{ae} \wedge \delta X^{ae} = \sum_{a,e} \sum_{b,f} \left( \frac{\partial F_{ae} [X]}{\partial X^{bf}} \right) \delta X^{bf} \wedge \delta X^{ae} \quad (173)
\]
For the choice $F_{ae}[X] = \partial I/\partial X^{ae}$, for some arbitrary scalar function $I = I[X]$ of $X^{ae}$, we have that

$$
\Omega = \sum_{a,e} \delta \Psi_{ae} \wedge \delta X^{ae} = \sum_{a,e} \sum_{b,f} \left( \frac{\partial^2 I[X]}{\partial X^{ae} \partial X^{bf}} \right) \delta X^{bf} \wedge \delta X^{ae} = 0, \quad (174)
$$

Due to contraction of symmetric indices from the second partial derivative with antisymmetric indices from the two forms. Furthermore, by restriction $X^{ae} = X^{(ae)}$ to symmetric indices, the functions $F_{ae} = F_{(ae)}[X]$ can be chosen to satisfy the classical Hamiltonian constraint via the restriction

$$
\text{Var} F + \Lambda \text{det} F = 0.
$$

The functions $F$ would lead to wavefunctions of the form $\Phi_F = e^{(hG)^{-1}}$, where $I$ is an arbitrary function of $X^{ae}$.

One possibility for exclusion might be the criterion of normalizability. However, the wavefunctions $\Phi$ would be normalizable in the measure $D\mu(X)$ since for any two functions $F \equiv F_{ae}[X]$ and $G \equiv G_{ae}[X]$, the inner product would be given by

$$
\langle \Phi_F | \Phi_G \rangle = \int DX e^{-\frac{1}{2}|X|^2} \Phi_F[X] \Phi_G[X]
$$

$$
= \Phi_F[\partial/\partial \lambda] \Phi_G[\partial/\partial \lambda^*] e^{\nu \lambda^* \lambda} \bigg|_{\lambda = \lambda^* = 0} = \Phi_F[\nu \lambda^*] \Phi_G[\nu \lambda], \quad (176)
$$

where we have made use of the generating functional techniques of the previous sections. Hence the requirement of square integrability would still be met as long as the wavefunctions $\Phi$ are themselves finite, since

$$
|\Phi_F|^2 = \Phi_F[\nu \lambda^*] \Phi_F[\nu \lambda] < \infty. \quad (177)
$$

Certainly, this requirement can be met by a fairly general class of appropriate choices of the function $I[X]$. So then the question resurfaces, since there exists a more general set of normalizable wavefunctions $\Phi_F[X]$ satisfying the classical version of the constraints, as to why one should restrict oneself to wavefunctions for which $F_{ae} = \lambda_{ae} = \text{const}$.. Hence, why label the states by five arbitrary constants as opposed to five arbitrary functions of $X$?

The answer resides in the observation that while the states $\psi_\lambda$ satisfy the semiclassical-quantum correspondence (SQC), the states $\Phi_F$ do not. This can be seen by consideration of the quantum version of the constraints. The action of the momentum operator on the states $\Phi_F$ would be given by
\[ \hat{\Psi}_{ae} \Phi_F = hG \frac{\partial}{\partial X_{ae}} e^{(hG)^{-1} I} = F_{ae}[X] \Phi_F. \] (178)

For \( F_{ae} = F_{(ae)} \), the kinematic constraints are already satisfied due to being linear in momenta. Hence for the diffeomorphism constraint,

\[ \hat{H}_i \Phi_F[X] = (hG)(\det A)^d \epsilon_{d_{ae}} \frac{\partial}{\partial X_{ae}} \Phi_F = (\det A) A^d_{ae} F_{ae} \Phi_F = 0 \] (179)

which for nondegenerate curvatures \( \det A \neq 0 \), requires that \( F_{ae} \) be symmetric in the indices \( a,e \). Likewise, the Gauss' law constraint would produce the same condition, being linear in momenta.

However, for the quantum Hamiltonian constraint we have, upon rescaling

\[ \hat{H} \Phi_F = (hG)^2(\det A)^3 \epsilon_{abc} \epsilon_{def} \left[ hG \Lambda \frac{\partial^3}{\partial X_{ae} \partial X_{bf} \partial X_{cg}} + \delta_{cg} \frac{\partial^2}{\partial X_{ae} \partial X_{bf}} \right] \Phi_F \]

\[ = (\det A)^\sqrt{\det \hat{\Psi}} (q_0(X) + (hG)q_1(X) + (hG)^2q_2(X)) \Phi_F = 0 \] (180)

for some coefficients \( q_0, q_1 \) and \( q_2 \) which we do not display here. The term \( q_0 \) is the semiclassical term, which is the same as would be for the classical implementation of the Hamiltonian constraint for all wavefunctions. The terms \( q_1 \) and \( q_2 \) are quantum terms which would result from the action of the derivatives on \( F_{ae}[X] \). For the states \( \psi_\lambda \) these terms would be zero since \( F_{ae} = \lambda_{ae} \) would be numerical constants. Hence, without any further restrictions on \( q_1 \) and \( q_2 \), the wavefunctions \( \Phi_F \) would satisfy the quantum constraints not of general relativity in Chang–Soo variables, but rather of a different theory.\(^{34}\)

To provide a further argument for the necessity of states residing in the Hilbert space that we have constructed, note that the states \( \psi_\lambda \) which satisfy the SQC are also consistent with the solution to the classical equations of motion, \( \hat{\Psi}_{ae} = 0 \ \forall a,e \) as we have shown. This requires that \( \lambda_{ae} \) be numerical constants in anisotropic minisuperspace. This is not the case for the functions \( F_{ae} = F_{ae}[X(T)] \), since these functions contain time dependence through the time dependence of the Chang–Soo variables \( X^{ae} = X^{ae}(T) \). Therefore, it is the conclusion of this subsection that a consistent quantum theory must be consistent with the quantum constraints. Since this condition, as we have just argued for \( \psi_\lambda \), requires a consistency of the state with

\(^{34}\)Since we are interested in quantizing general relativity, we must therefore reject the states \( \Phi_F \), restricting to the states \( \psi_\lambda \). Note that \( \Phi_F[X] \) can still satisfy the classical version of the constraints \( q_0 = 0 \) without satisfying the quantum version. However, \( \psi_\lambda \) satisfy both the classical and the quantum versions, hence are consistent with the SQC.
the classical equations of motion, it then follows that the SQC is a strong condition limiting the allowable class of physical states to those forming a consistent quantum theory.\textsuperscript{35}

\section*{7.2 Arguments from path integral quantization}

We now illustrate the manner by which the wavefunction solving the quantum constraints can arise from the path integration procedure. This should establish its equivalence to the canonical and geometric approaches under a special restriction.\textsuperscript{36} We will show that in order for this equivalence to exist, the path integral must implement the solution to the constraints, which would in turn holographically project the wavefunction to the final time $T$.

The transition amplitude from an initial time $T_0$ to a final time $T$ is given by

$$\langle X, T; X_0, T_0 \rangle = \int DX e^{S[X]} \quad (181)$$

where $S[X]$ is the starting action on configuration space. To illustrate the holographic effect, let us take the path integral (181) ‘off-shell’ into its phase space representation.\textsuperscript{37}

$$\langle X, T; X_0, 0 \rangle = \int DX \int D\Psi \int D\theta e^{\int_{T_0}^{T} dt (\Psi \dot{X} - i\Phi(\theta))}. \quad (182)$$

Here, $X$ refer to the 9-complex dimensional space of Chang–Soo variables $X^{ae}$ and $\Psi$ the 9-complex dimensional space of CDJ matrix elements $\Psi_{ae}$ at the level prior to implementation of the constraints. We have incorporated the constraints into a row vector $\Phi \sim (G_a, H_i, H)$ contracted into the corresponding column vector formed by the Lagrange multipliers $\theta \sim (\theta^a, N^i, N).$\textsuperscript{38} Performing the path integral over the Lagrange multipliers we obtain, assuming a Euclidean signature,

\textsuperscript{35}While we have shown this for anisotropic minisuperspace, we will extend it to the full theory in a subsequent paper. The implication is that the solution to the constraints should as well be tantamount to the Cauchy problem for general relativity. We will then test the hypothesis, in subsequent work, that this latter implication is a natural consequence of the SQC in the full theory.

\textsuperscript{36}This is the restriction to Euclidean signature spacetimes, which allow the constraints to be implemented by delta functions upon path integration over the auxiliary variables.

\textsuperscript{37}Hence, the configuration space path integral can be seen as a continuation of the steps of the path integral originating from the full phase space.

\textsuperscript{38}The factor of $i$ in (182) is put in so that the constraints can be implemented by delta functions. Note that this fixes the signature of the spacetime in order to place the Hamiltonian constraint on the same footing as the kinematic constraints.
\( \langle X, T; X_0, T_0 \rangle = \int DX \left( \int D\Psi e^{\int_0^T dt \Psi \cdot X} \right) \left( \int D\theta e^{-i \int_0^T dt \Phi(\theta)} \right) \)
\[ = \int DX \left( \int D\Psi e^{\int_0^T dt \Psi \cdot X} \right) \delta(\Phi) \quad (183) \]

The delta functions arise due to the implementation of the constraints in for Euclidean signatures. For the class of states \( \Psi \in \psi_{phys} \), the path integral implements both the classical and the quantum form of the constraints identically.\(^{39}\)

Next, the path integration over the momenta must be carried out in order to substitute the solution to the constraints into the remainder of the path integral. Let us start with the constraints linear in momenta. For these constraints the following contribution must be performed due to the antisymmetric elements.

\[ \int d\Psi_{[12]} d\Psi_{[23]} d\Psi_{[31]} \delta(\Psi_{[12]}) \delta(\Psi_{[23]}) \delta(\Psi_{[31]}) e^{\int_0^T dt \Psi_{[ae]} \hat{X}^{[ae]}} = 1. \quad (184) \]

Upon implementation of the diffeomorphism constraint we are reduced from an eighteen to a twelve dimensional phase space containing symmetric CDJ matrix elements. Since the Gauss’ law constraint is redundant, there remains just the Hamiltonian constraint, which is a single condition relating six CDJ matrix elements. Let us pick one of them \( \Psi_{33} \) without loss of generality. The Hamiltonian constraint should then implement

\[ \Psi_{33} = \Psi_{33} \left( \Psi_{(12)}, \Psi_{(23)}, \Psi_{(31)}, \Psi_{(11)}, \Psi_{(22)} \right) \sim \Psi_{33} (\Psi'_{(ae)}), \quad (185) \]

where \( \Psi' \) signifies the symmetric elements not including \( \Psi_{33} \). So five CDJ matrix elements should be freely specifiable.\(^{40}\)

\[ \delta(\Lambda + \text{tr}\Psi^{-1}) \quad (186) \]

We would like for the path integral to capture the fact that there is a five-fold infinity of solutions to the Hamiltonian constraint. The corresponding delta function at the level prior to integrating over \( \Psi_{33} \) appears in the form

\[ \prod_{A=1}^5 \int d\Psi_A \delta(\Psi_A - \lambda_A) \delta(\prod_{\zeta_A}(\lambda_A - \zeta_A)) e^{\int_0^T dt \Psi_A \hat{X}^A(t)}. \quad (187) \]

\(^{39}\)This signifies a semiclassical-quantum correspondence for these states.

\(^{40}\)According to the classical equations of motion for minisuperspace, these elements should all be numerical constants.
Equation (187) requires some explanation. We are performing a path integral over the five dimensional space of symmetric CDJ matrix elements not including $\Psi_{33}$. The rightmost delta function states that there are an infinite number of possible values that these elements $\lambda_A$ can take on for each $A$, and each possibility contributes to the path integral by making the argument of the delta function vanish. The leftmost delta function then equates the particular value being considered to the CDJ matrix element. If one views $\lambda_A$ as a vector in 5 dimensional complex space, then each possible vector constitutes a solution to the Hamiltonian constraint. We will be using the infinite dimensional analogue of the identity

$$\delta(f(x)) = \sum_r \frac{\delta(x - r)}{f'(r)},$$  \hspace{1cm} (188)

where $r$ are the roots of the function $f(x)$. Hence we have the relation

$$\delta \left( \prod_{\zeta_A} (\lambda_A - \zeta_A) \right) = \sum_{\zeta_A} \frac{\delta(\lambda_A - \zeta_A)}{Z(\zeta_A)}$$  \hspace{1cm} (189)

for each $A$, where we have defined

$$Z(\zeta_A) = \prod_{m \neq \zeta_A} (\Psi_A - m)$$  \hspace{1cm} (190)

which is formally infinite. The procedure is then to perform the above steps (189) and (190) for a given $A$ and then take the product over all $A$. Performing the path integral, we then obtain a wavefunction $\Psi(\vec{\lambda})$ labelled by the five arbitrary CDJ matrix elements as in

$$\Psi(\vec{\lambda}) = \prod_{A=1}^{5} \left( \sum_{\zeta_a} \frac{\delta(\lambda_A - \zeta_A)}{Z(\zeta_A)} \right) \exp \left[ \int_0^T dt \lambda_A (X^A(T) - X^A(0)) \right]$$  \hspace{1cm} (191)

where we have performed the following steps

$$\int_0^T dt \lambda_A \dot{X}^A(t) = \lambda_A \int_0^T dt \dot{X}^A = \lambda_A X^A \bigg|_0^T.$$  \hspace{1cm} (192)

A given $\lambda$ is selected for each term in the sum. Since the $\lambda_A$ are numerical constants, they can be factored out of the integral upon implementation of

\footnote{Another way to view this is as a polynomial on infinite degree whose roots are the real numbers.}
the delta function by integrating $d\Psi_A$. The result is the integral of a total time derivative which leads to a boundary term independent of the history between those times.

Next remains the integral over $\Psi_{33}$. This must be performed for each $\bar{\lambda}$. Hence we have

$$\int \beta d\Psi_{33} \delta(\Psi_{33} - H(\bar{\lambda})) \Psi(\bar{\lambda}) e^{\int_0^T dt \Psi_{33} X^{33}} = \Psi(\bar{\lambda}) e^{f(\bar{\lambda})(X^{33}(T) - X^{33}(0))}. \quad (193)$$

Hence, the path integral representation has thus far led to

$$\psi_\lambda = \sum_\zeta \beta(\bar{\zeta}) \delta[\zeta - \lambda] e^{\zeta \cdot X} \quad (194)$$

where we have defined $\lambda_{33} = f(\bar{\lambda})$. All normalization factors are contained in $\beta$, and $\zeta$ ranges over the entire range of values that $\lambda$ can take on.

We still have one more integration to complete, namely over the configuration variables $X(t)$ for $0 < t < T$. Since the wavefunction (195) is defined at the final time $T$, which is a boundary term, then (195) is immune to this integration. Hence we have

$$\int DX \Psi(\bar{\lambda}) \psi_\lambda = \psi_\lambda \int DX = (Vol_X) \psi_\lambda. \quad (195)$$

Here, $Vol_X$ is the volume of the nine complex-dimensional space of Chang–Soo variable configurations with is an infinite numerical constant.\footnote{This infinity is not an issue since it is common to all states and will cancel out in relative probabilities and in the computation of observables.} Hence the end result is that

$$\langle X, T; X_0, T_0 \rangle = \int DX e^{S[X]} = (Vol_X) \sum_\zeta \beta(\bar{\zeta}) \delta[\zeta - \lambda] e^{\zeta \cdot X}. \quad (196)$$

Equation (196) encapsulates the collection of all possible values of $\lambda_{ae}$, which are picked out by the path integral for Euclidean signatures. Hence this establishes the equivalence of the path integration procedure, for Euclidean signature, to the geometric and the canonical procedures.
8 Conclusion

The main results of the present paper are as follows. We have introduced a new approach to general relativity in terms of a new set of dynamical variables first introduced by CDJ and Chang–Soo. Our implementation of the variables as full dynamical variables signifies a departure from the conventional implementation, which uses the CDJ matrix as an auxiliary variable. Regarded as fundamental dynamical variables, the Chang–Soo/CDJ variables exhibit a different algebraic structure with respect to the classical and the quantum constraints relative to the Ashtekar variables. These differences have pointed to a difference in the fundamental structure of the two theories already at the level of minisuperspace, which we expect will carry over into the full theory.

We have shown that the classical and the quantum algebra of constraints in Chang–Soo variables in anisotropic minisuperspace is consistent in accordance with the Dirac procedure. Additionally, we have constructed a Hilbert space of states for our model by utilizing a Gaussian measure for Lorentzian signature, with the option of a Lebesgue measure in Euclidean signature spacetime. The result is a set of normalizable coherent-like states labelled by two free parameters which solve the quantum constraints. These particular states satisfy the semiclassical-quantum correspondence (SQC) introduced in [8] in that they simultaneously solve the classical and the quantum initial value constraints in congruity with the equations of motion. Additionally, our states address the problem of time in quantum gravity, at least in the anisotropic minisuperspace sector, in that they automatically ‘evolve’ to the desired time as a result of the SQC.

The pure Kodama $\Psi_{Kod}$ state clearly belongs to this aforementioned class of states. Therefore we claim as another result of this paper that any issues or objections raised about the state regarding normalizability and existence within a well-defined Hilbert space, at least for anisotropic minisuperspace, have been resolved: a result we hope to extend to the full theory in additional works in this series. Additionally, we have extended the arguments of [8] to incorporate a wider class of states, at least for anisotropic minisuperspace, and have provided arguments of the equivalence of the canonical and geometric approaches to quantization for these states. We additionally have provided an argument for equivalence to the path integration approach for Euclidean signature spacetime.

Future directions along this line of research using the Chang–Soo variables include and are not limited to the following: (i) The addressal of any outstanding issues regarding the Kodama states and their generalizations (ii) An in-depth analysis of general relativity in these variables both including the classical and the quantum theories, (iii) Solution to the initial value

\footnote{Five free parameters if one includes the $SO(3, C)$ angles diagonalizing the CDJ matrix.}
constraints problem of general relativity and the search for and construction of new classes of solutions. (iv) Extension of the algorithm for the construction of the generalized Kodama states to incorporate more general solutions to GR, and ultimates (v) the construction of a consistent and finite quantum theory. The next immediate work will examine the classical equations of motion in anisotropic minisuperspace and their effect on inflation, as well as some phenomenological investigations [17].

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