A Proof of Alon’s Second Eigenvalue Conjecture

Joel Friedman*

March 27, 2002

Abstract

A $d$-regular graph has largest or first (adjacency matrix) eigenvalue $\lambda_1 = d$. In this paper we show the following conjecture of Alon. Fix an integer $d > 2$ and a real $\epsilon > 0$. Then for sufficiently large $n$ we have that “most” $d$-regular graphs on $n$ vertices have all their eigenvalues except $\lambda_1 = d$ bounded by $2\sqrt{d-1} + \epsilon$ in absolute value. (Alon conjectured this only for $\lambda_2$, but our methods, being trace methods, also bound negative eigenvalues.)

1 Introduction

The eigenvalues of the adjacency matrix of an undirected graph, $G$, are real and hence can be ordered

$$\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G),$$

where $n$ is the number of vertices in $G$. If $G$ is $d$-regular, i.e. each vertex is of degree $d$, then $\lambda_1 = d$. In [Alo86], Noga Alon conjectured that for any $d \geq 3$ and $\epsilon > 0$, $\lambda_2(G) \leq 2\sqrt{d-1} + \epsilon$ for “most” $d$-regular graphs on a sufficiently large number of vertices. The Alon-Boppana bound shows that the $2\sqrt{d-1}$ cannot be improved upon (see [Alo86, Nil91, Fri93]). The main goal of this

---

*A Departments of Computer Science and Mathematics, University of British Columbia, Vancouver, BC V6T 1Z4 (V6T 1Z2 for Mathematics), CANADA. jf@cs.ubc.ca. Research supported in part by an NSERC grant.
paper is to prove this conjecture for various models of a “random $d$-regular graph.”

Our methods actually show that for “most” $d$-regular graphs, $|\lambda_i(G)| \leq 2\sqrt{d-1} + \epsilon$ for all $i \geq 2$, since our methods are variants of the standard “trace method.”

Our primary interest in Alon’s conjecture, which was Alon’s motivation, is that fact that graphs with $|\lambda_i|$ small for $i \geq 2$ have various nice properties, including being expanders or magnifiers (see [Alo86]).

For a fixed $n$ we can generate a random $d$-regular graph on $n$ vertices by taking $d/2$ permutations on $V = \{1, \ldots, n\}$, $\pi_1, \ldots, \pi_{d/2}$, each $\pi_i$ chosen uniformly among all $n!$ permutations with all the $\pi_i$ independent. We then form

$$E = \left\{(i, \pi_j(i)), (i, \pi_j^{-1}(i)) \mid j = 1, \ldots, d/2, \quad i = 1, \ldots, n\right\},$$

yielding a directed graph $G = (V, E)$, which we may view as undirected. We call this probability space of random graphs $\mathcal{G}_{n,d}$. $G$ can have multiple edges and self-loops, and each self-loop contributes 2 to the appropriate diagonal entry of $G$’s adjacency matrix$^1$.

The main goal of this paper is to prove theorems like the following, which prove Alon’s conjecture, for various models of a random $d$-regular graph; we start with the model $\mathcal{G}_{n,d}$.

**Theorem 1.1** Fix a real $\epsilon > 0$ and an even positive integer $d$. Then there is a constant, $c$, such that for a random graph, $G$, in $\mathcal{G}_{n,d}$ we have that with probability $1 - c/n^\tau$ we have for all $i > 1$

$$|\lambda_i(G)| \leq 2\sqrt{d-1} + \epsilon,$$

where $\tau = \tau_{\text{fund}} = \lceil(\sqrt{d-1} + 1)/2\rceil - 1$. Furthermore, for some constant $c' > 0$ we have that $\lambda_2(G) > 2\sqrt{d-1}$ with probability $\geq c'/n^s$, where $s = \lfloor (\sqrt{d-1} + 1)/2 \rfloor$. (So $s = \tau_{\text{fund}}$ unless $(\sqrt{d-1} + 1)/2$ is an integer, in which case $\tau_{\text{fund}} = s - 1$.)

Left open is the question of whether or not this theorem holds with $\epsilon = 0$ or even some function $\epsilon = \epsilon(n) < 0$. Calculations such as those in [Fri93] suggest that it does, even for some negative function $\epsilon(n)$. Examples of “Ramanujan graphs,” i.e. graphs where $|\lambda_i(G)| \leq 2\sqrt{d-1}$ except $i = 1$

$^1$Such a self-loop is a whole-loop in the sense of [Fri93]; see also Section 2 of this paper.
(and, at times, $i = n$ when $\lambda_n = -d$) have been given in [LPS88, Mar88, Mor94] where $d$ is one more than an odd prime or prime power. Theorem 1.1 demonstrates the existence of “nearly Ramanujan” graphs of any even degree. We shall soon address odd $d$, as well.

Another interesting question arises in the gap between $\tau_{\text{fund}}$ and $s$ in Theorem 1.1 in the case where $(\sqrt{d-1} + 1)/2$ is an integer; it is almost certain that one of them can be improved upon. In the language of Section 4 of this paper, $\tau_{\text{fund}}$ is the smallest order of a supercritical tangle, and $s$ that of a hypercritical tangle; a gap between $\tau_{\text{fund}}$ and $s$ can only occur when there is a critical tangle of order smaller than that of any hypercritical tangle.

Previous bounds of the form $\lambda_2 \leq f(d) + \epsilon$ include $f(d) = (2d)^{1/2}(d-1)^{1/4}$ of the author (see [Fri91]), which is slight improvement over the Broder-Shamir bound of $f(d) = 2^{1/2}d^{3/4}$ (see [BS87]). Asymptotically in $d$, the bounds $f(d) = C\sqrt{d}$ of Kahn and Szemerédi (see [FKS89]) and $f(d) = 2\sqrt{d-1} + 2\log d + C$ of the author (see [FKS89, Fri91]) are improvements over the first two bounds, however both involve an undetermined absolute constant $C$.

The value of $\tau_{\text{fund}}$ in Theorem 1.1 depends on the particular model of a random graph. Indeed, consider the model $H_{n,d}$ of a random graph, which is like $G_{n,d}$ except that we insist that each $\pi_i$ be one of the $(n-1)!$ permutations whose cyclic decomposition consists of one cycle of length $n$. The same methods used to prove Theorem 1.1 will show the following variant.

**Theorem 1.2** Theorem 1.1 holds with $G_{n,d}$ replaced by $H_{n,d}$ and $\tau_{\text{fund}} = \lceil \sqrt{d-1} \rceil - 1$ and $s = \lfloor \sqrt{d-1} \rfloor$, except that when $d = 4$ we take $s = 2$.

Once again, $\tau_{\text{fund}} = s$, unless a certain expression, in this case $\sqrt{d-1}$ (excepting $d = 4$), is an integer. Note that for $H_{n,d}$, the value of $\tau_{\text{fund}}$ is roughly twice as large as that for $G_{n,d}$ for $d$ large.

Next consider two more models of random graphs. Let $I_{n,d}$, for positive integers $n, d$ with $n$ even, be the model of a random $d$-regular graph formed from $d$ random perfect matchings on $\{1, \ldots, n\}$.

For an odd positive integer $n$, let a *near perfect matching* be a matching of $n-1$ elements of $\{1, \ldots, n\}$; such a matching becomes a 1-regular graph if it is complemented by a single half-loop\(^2\) at the unmatched vertex. Taking $d$ independent such 1-regular graphs gives a model, $J_{n,d}$, of a $d$-regular graph on $n$ vertices for $n$ odd.

\(^2\)Readers unfamiliar with half-loops (i.e., self-loops contributing only 1 to a diagonal entry of the adjacency matrix) can see Section 2 of this paper or [Fri93].
Theorem 1.3 Theorem 1.2 holds with $\mathcal{H}_{n,d}$ replaced by $\mathcal{I}_{n,d}$ and with no $d = 4$ exception (i.e., $s = 1$ for $d = 4$). Theorem 1.1 holds with $\mathcal{G}_{n,d}$ replaced by $\mathcal{J}_{n,d}$.

The model $\mathcal{I}_{n,d}$ is known to be contiguous\(^3\) with the model, $\mathcal{K}_{n,d}$, of choosing a $d$-regular graph uniformly from those $d$-regular graphs with vertex set $\{1, \ldots, n\}$ (see [Wor99], Corollary 4.17).

Corollary 1.4 Fix an $\epsilon > 0$ and an integer $d \geq 2$. Let $\mathcal{L}_n$ be any family of probability spaces of $d$-regular graphs on $n$ vertices (possibly defined for only certain $n$) contiguous with $\mathcal{G}_{n,d}$, $\mathcal{H}_{n,d}$, $\mathcal{I}_{n,d}$, or $\mathcal{J}_{n,d}$ (such as $\mathcal{K}_{n,d}$). Then for $G$ in $\mathcal{L}_n$ we have that with probability $1 - o(1)$ (as $n \to \infty$) for all $i$ with $2 \leq i \leq n$ we have

$$|\lambda_i(G)| \leq 2\sqrt{d-1} + \epsilon.$$ 

Our method for proving Theorems 1.1, 1.2, and 1.3 is a variant of the well-known “trace method” (see, for example [Wig55, Gem80, FK81, BS87, Fri91]) originated by Wigner, especially the author’s refinement in [Fri91] of the beautiful Broder-Shamir style of analysis in [BS87]. The standard trace method involves taking the expected value of the trace of a reasonably high power\(^4\) of the adjacency matrix. In our situation we are unable to analyze this trace accurately enough to prove Theorem 1.1, as certain infinite sums involved in our analysis diverge (for example, the infinite sum involving $W(T; \vec{m})$ and $P_{i,T,\vec{m}}$ just above the middle of page 351 in [Fri91], for types of order $> d$). This divergence is due to certain “tangles” that can occur in a random graph and can adversely affect the eigenvalues (see Sections 2 and 4). To get around these “tangles” we introduce a selective trace. The selective trace is the trace of a power of the adjacency matrix where we disregard any contribution to this trace except those that are “irreducible” and have no small path that “traces out” a “tangle.” Since these “tangles” occur with probability at most proportional to $n^{-\tau}$, with $\tau = \tau_{\text{fund}}$ as in Theorem 1.1, the selective trace usually agrees with the standard “irreducible” trace.

Analyzing the selective traces involves a new concept of the “new type,” which is a refinement of the “type” of [Fri91].

\(^3\)Given a family of measurable spaces, $(\Omega_n, \mathcal{F}_n)$, we say that two corresponding families of measures, $\{\mu_n\}$ and $\{\nu_n\}$ are contiguous if for any family of measurable events, $\{E_n\}$ (i.e., $E_n \in \mathcal{F}_n$), we have $\mu_n(E_n) \to 0$ as $n \to \infty$ iff $\nu_n(E_n) \to 0$ as $n \to \infty$.

\(^4\)In [BS87, Fri91] this power is roughly $c \log n$, where $c$ depends on $d$ and on aspects of the method.
We caution the reader about the notation used here. In this paper we work with only $d$-regular graphs. In [BS87] $2d$-regular graphs were studied; in [Fri91] the graphs are usually $2d$-regular, although for a part of Section 3 the graphs are $d$-regular. We also caution the reader that here we use the term “irreducible” (as used in [BS87, Fri91] and, for example, in the text [God93]) as opposed to “reduced” (which is quite common) or “non-backtracking” (sometimes used in [Fri91]) in describing walks and related concepts.

We hope to generalize or “relativize” the theorems here to theorems about new eigenvalues of random covers (see [Fri] for a relativized Broder-Shamir theorem). In this paper we occasionally go out of our way to use a technique that will easily generalize to this setting.

The rest of this paper is organized as follows. In Section 2 we review the trace method used in [Fri91] and explain why it requires modification to prove Alon’s conjecture; as a byproduct we establish the part of Theorem 1.1 involving $s$. In Section 3 we give some background needed for some technical details in later sections. In Section 4 we formalize the notion of a tangle, and discuss their properties; we prove the part of Theorem 1.2 and 1.3 involving $s$. In Section 5 we describe “types” and “new types,” explaining how they help to estimate “walk sums;” walk sums are generalizations of all notions of “trace” used here. In Section 6 we describe the “selective trace” used in this paper; we give a crucial lemma that counts certain types of selective cycles in a graph. In Section 7 we explain a little about “$d$-Ramanujan” functions, giving a theorem to be used in Section 14 that also illustrates one of the main technical points in Section 8. In Section 8 we prove that certain selective traces have an asymptotic expansion (in $1/n$) whose coefficients are “$d$-Ramanujan.” In Section 9 we show that the expansion in Section 8 still exists when we count selective traces of graphs not containing any finite set of tangles of order $\geq 1$. In Section 10 we introduce strongly irreducible traces, that simplify the proofs of the main theorems in this paper. In Section 11 we prove a crucial lemma that allows us to use the asymptotic expansion to make conclusions about certain eigenvalues; this lemma sidesteps the unresolved problem of (even roughly) determining the coefficients of the asymptotic expansion (in [Fri91] we actually roughly determine the coefficients for the shorter expansion developed there). In Section 12 we prove the magnification (or “expansion”) properties needed to apply the sidestepping lemma of Section 11. In Section 13 we complete the proof of Theorem 1.1. In Section 14 we complete the proof of Theorems 1.2 and 1.3, giving general conditions on a model of random graph that are sufficient to
imply the Alon conjecture. In Section 15 we make some closing remarks.

Throughout the rest of this paper we will work with $G_{n,d}$ unless we explicitly say otherwise, and we understand $d$ to be a fixed integer $> 2$. At times we insist that $d$ be even (for example, in dealing with $G_{n,d}$ and $H_{n,d}$).

2 Problems with the Standard Trace Method

In this section we summarize the trace method used in [Fri91], and why this method cannot prove Alon’s conjecture. During this section we will review some of the ideas of [Fri91], involving asymptotic expansions of various types of traces, which we modify in later sections to complete our proof of Alon’s conjecture.

2.1 The Trace Method

We begin by recalling the trace method as used in [Fri91], and why it did not yield the Alon conjecture.

The trace method (see [Wig55, Gem80, FK81, McK81, BS87, Fri91], for example) determines information on the eigenvalues of a random graph in a certain probability space by computing the expected value of a sufficiently high power of the adjacency matrix, $A$; this expected value equals the expected value of the sum of that power of the eigenvalues, since

$$\text{Trace} (A^k) = \lambda_1^k + \cdots + \lambda_n^k.$$

Now $\text{Trace} (A^k)$ may also be interpreted as the number of cycles (walks in the graph that start and end at the same vertex) of length $k$. Now restrict our discussion to $G_{n,d}$. A word, $w = \sigma_1 \cdots \sigma_k$, of length $k$ over the alphabet

$$\Pi = \{\pi_1, \pi_1^{-1}, \ldots, \pi_{d/2}, \pi_{d/2}^{-1}\}$$

(i.e. each $\sigma_i \in \Pi$), determines a random permutation, and the $i, j$-th entry of $A^k$, is just the number of words, $w$, of length $k$, taking $i$ to $j$. But given a word, $w$, the probability, $P(w)$, that $w$ takes $i$ to $i$ is clearly independent of $i$. Hence we have

$$E \left[ \text{Trace} (A^k) \right] = n \sum_{w \in \Pi^k} P(w)$$
In [BS87], Broder and Shamir estimated the right-hand-side of the above equality to obtain an estimate on $\lambda_2$. This analysis was refined in [Fri91]. We review the ideas there.

First, $w$ is said to be irreducible if $w$ contains no consecutive occurrence of $\sigma, \sigma^{-1}$. It is well-known that any word, $w$, has a unique reduction to an irreducible word\footnote{In fact, the irreducible word has length which is its distance to the identity in the Cayley graph over the free group on $d/2$ elements (see [FTP83], Sections 1 and 7 of chapter 1). Alternatively, see Proposition 2.5 of [DD89] or Theorem 1 of [Joh90] (this theorem says that a free group on a set, $X$, is in one-to-one correspondence with the set of reduced words, $X \cup X^{-1}$, which means that every word over $X \cup X^{-1}$ has a unique corresponding reduced word; here “reduced” is our “irreducible”).} (or reduced word), $w'$, obtained from $w$ by repeatedly discarding any consecutive occurrences of $\sigma$ and $\sigma^{-1}$ in $w$, and $P(w) = P(w')$. If we let $\text{Irred}_k$ be the set of irreducible words of length $k$, then to evaluate $\text{Trace}(A^m)$ it suffices to evaluate

$$\text{IrredTr}(A,k) = n \sum_{w \in \text{Irred}_k} P(w),$$

for $k = m, m-2, \ldots$ (see below). It is easy to see that for any fixed word, $w$, we have a power series expansion

$$P(w) = a_0(w) + \frac{a_1(w)}{n} + \frac{a_2(w)}{n^2} + \cdots$$

(see, for example, Theorem 5.4); for example, $a_0(\pi_1 \pi_1^{-1}) = 1$, $a_0(\pi_1 \pi_1) = 0$, $a_1(\pi_1 \pi_1) = 2$, $a_i(\pi_1)$ is 1 or 0 according to whether or not $i = 1$, etc. So set

$$g_i(k) = \sum_{w \in \text{Irred}_k} a_{i+1}(w)$$

(1)

(we easily see $a_0(w) = 0$ for $w \in \text{Irred}_k$ and $k \geq 1$ and so $g_{-1}(k) = 0$ for $k \geq 1$).

**Definition 2.1** A function, $f(k)$, on positive integers, $k$, is said to be $d$-Ramanujan if there is a polynomial $p = p(k)$ and a constant $c > 0$ such that

$$|f(k) - (d - 1)^kp(k)| \leq ck^{\epsilon}(d - 1)^{k/2}$$

for all $k$. We call $(d - 1)^kp(k)$ the principal term of $f$, and $f(k) - (d - 1)^kp(k)$ the error term (both terms are uniquely determined if $d > 2$).
In [Fri91] it was shown (among other things) that for all $i \leq \sqrt{d - 1}/2$ we have that $g_i$ as above is $d$-Ramanujan. This, it turns out, gives a second eigenvalue bound of roughly $2\sqrt{d - 1} + \log d + C + O(\log \log n/\log n)$ for a universal constant, $C$. We now explain why.

A standard counting and expansion argument is given in [Fri91] (specifically Theorem 3.1 there) to establish the following lemma.

**Lemma 2.2** For fixed even $d \geq 4$ there is an $\eta > 0$ such that with probability $1 - n^{1-d} + O(n^{2-2d})$ we have that a $G$ in $\mathcal{G}_{n,d}$ has $\max(\lambda_2, -\lambda_n) \leq d - \eta$ (also with probability $n^{1-d} + O(n^{2-2d})$ we have that $\lambda_2 = d$).

These eigenvalues control the trace of $A^k$. Next we establish the precise relationship between the traces of the $A^k$ and the IrredTr $(A,k)$. Let $A_k$ be the matrix whose $i,j$-th entry is the number of irreducible walks from $i$ to $j$, i.e., walks that never take a step that is the reverse of the step just taken (along the same edge, if multiple edges exist).

**Lemma 2.3** The $A_k$ are given by

$$A_k = q_k(A),$$

where $q_k$ is the degree $k$ polynomial given via

$$q_k(2\sqrt{d - 1}\cos \theta) = (\sqrt{d - 1})^k \left( \frac{2}{d - 1} \cos k\theta + \frac{d - 2}{d - 1} \frac{\sin(k + 1)\theta}{\sin \theta} \right) \tag{2}$$

(which is a type of Chebyshev polynomial); alternatively we have $q_1(x) = x$, $q_2(x) = x^2 - d$, and for $k \geq 3$ we have

$$q_k(x) = x q_{k-1}(x) - (d - 1) q_{k-2}(x).$$

Also

$$\text{IrredTr}(A,k) = \text{Trace}(A_k) = \sum_{i=1}^{n} q_k(\lambda_i).$$

The proof is given in [LPS86] and [Fri91]. In Section 10 we shall use the fact that for fixed $\lambda$, $q_k = q_k(\lambda)$ satisfy the recurrence

$$(\sigma_k^2 - \lambda \sigma_k + (d - 1))q_k = 0,$$

where $\sigma_k$ is the “shift in $k$” operator (i.e., $\sigma_k q_k = q_{k+1}$)
To go the other way we note:

\[ A^k = \sum_{i=k,k-2,k-4,...} N_{k,i} A_i, \]

where \( N_{k,i} \) is the number of words of length \( k \) that reduce to a given irreducible word of length \( i \). Thus

\[ \text{Trace} (A^k) = \sum_{i=k,k-2,k-4,...} N_{k,i} \text{IrredTr} (A,i). \]

**Lemma 2.4** For \( k, i \) even we have

\[ N_{k,i} \leq \left(2\sqrt{d-1}\right)^k (d-1)^{-i/2} \sqrt{(d-1)/d} \]

if \( i > 0 \) and

\[ N_{k,0} \leq \left(2\sqrt{d-1}\right)^k. \]

An exact formula for \( N_{k,i} \) is given in [McK81]. A weaker estimate than the above lemma was used in [Fri91]. The proof of this estimate is a simple spectral argument used by Buck (see [Buc86, Fria]).

**Proof** Consider the adjacency matrix, \( A_T \), of the infinite \( d \)-regular tree, \( T \).

**Lemma 2.5** \( A_T \) has norm \( \leq 2\sqrt{d-1} \).

Actually, it is well-known that the norm of \( A_T \) is exactly \( 2\sqrt{d-1} \) (see, for example, page 9 of [Woe00], and the theorems on \( \lambda_1 \) in Section 3 here). However, the proof below is simple and generalizes to many other situations (and can be used in many cases to determine the exact norm of \( A_T \)).

**Proof** Let \( f \) be a function on \( L^2(T) \). Fix a vertex, \( v_0 \), of \( T \), to be viewed as the root of \( T \); the children of a vertex, \( v \), are defined to be those vertices connected to \( v \) of distance one greater to \( v_0 \). We have

\[ (A_T f, f) = \sum_v \sum_{w \in \text{children}(v)} 2f(v)f(w) \]

which, by Cauchy-Schwarz, is

\[ \leq \sum_v \sum_{w \in \text{children}(v)} \left(f^2(v)\sqrt{d-1} + \frac{f^2(w)}{\sqrt{d-1}}\right) \]
\[ f^2(u_0)d / \sqrt{d-1} + \sum_{v \neq u_0} f^2(v)2\sqrt{d-1}. \]
\[ \leq \sum_v f^2(v)2\sqrt{d-1} = 2\sqrt{d-1}\|f\|^2. \]

Thus the norm of \( A_T \) is \( \leq 2\sqrt{d-1} \).

(To see that the norm of \( A_T \) is exactly \( 2\sqrt{d-1} \) we find functions, \( f \), (of finite support) for which the applications of Cauchy-Schwarz in the above proof are “usually” tight. Namely, we can take \( f(v) = (d-1)^{-\text{dist}(v,v_0)} \) for \( \text{dist}(v,v_0) \leq s \) and \( f(v) = 0 \) otherwise, where \( s \) is a parameter which tends to \( \infty \). This technique works for some other graphs.)

We resume the proof of Lemma 2.4. Let \( v \) be a vertex, and let \( S \) be the vertices of distance \( i \) to \( s \). Then \( |S| N_{k,i} \) is the dot product of \( A_T^k \chi_{\{v\}} \) with \( \chi_S \), where \( \chi_U \) denotes the characteristic function of \( U \), i.e. the function that is 1 on \( U \) and 0 elsewhere. So by Cauchy-Schwarz

\[ |S| N_{k,i} = (A_T^k \chi_{\{v\}}, \chi_S) \leq \|A_T\|^k \|\chi_{\{v\}}\| |\chi_S| = \left(2\sqrt{d-1}\right)^k \sqrt{|S|}. \]

We finish with the fact that \( |S| = 1 \) if \( i = 0 \), and otherwise \( |S| = d(d-1)^{i-1} \).

Notice that clearly \( N_{k,k} = 1 \), and so for \( i = k \) Lemma 2.4 is off by a multiplicative factor of roughly \( 2^k \); according to [McK81, FTP83], the Lemma 2.4 estimate of \( N_{k,0} \) is off by roughly a factor of \( k^{3/2} \). The roughness of Lemma 2.4 is unimportant for our purposes.

Now notice that by Lemmas 2.2 and 2.3 we have

\[ \text{E} [\text{IrredTr} (A, k)] = q_k(d) \left(1 + n^{d-1} + O(n^{2d-2})\right) + \text{error}, \]

where

\[ |\text{error}| \leq (n-1) \max_{|\lambda| \leq d-\eta} |q_k(\lambda)|. \]

It is easy to see (see [Fri91]) that \( q_k(d) = (d-1)^k \), and for some \( \alpha > 0 \) we have

\[ \max_{|\lambda| \leq d-\eta} |q_k(\lambda)| \leq (d-1-\alpha)^k \eta^k, \]

for an absolute constant \( c \) (with any \( \eta \) as in Lemma 2.2). We wish to draw some conclusions about the principal term of the \( g_i 's \). We need the following lemma:
Lemma 2.6  For fixed \(d, r\) there is a constant, \(c\), such that for \(k \geq 1\) we have that in \(G_{n,d}\) we have

\[
E[\text{IrredTr}(A, k)] = g_0(k) + \frac{g_1(k)}{n} + \frac{g_2(k)}{n^2} + \cdots + \frac{g_{r-1}(k)}{n^{r-1}} + \text{error},
\]

where

\[
|\text{error}| \leq c(d - 1)^{k-1}k^{4r+2}/n^r.
\]

Proof  This follows from the calculations on page 352 of [Fri91]; for each \(i\), the \(f_i\) in [Fri91] is the polynomial in the principal term of \(g_i\) (and we mean \(f_i\) corresponds precisely to \(g_i\), not \(g_{i-1}\) or \(g_{i+1}\)).  (Actually we shall later\(^6\) see that the \(4r + 2\) in the error term estimate can be replaced by \(4r\).)

\[\square\]

We now take \(k\) of order \(\log^2 n\) and use standard facts about expansion and expansion’s control on eigenvalues (namely our Lemma 2.2) to conclude, as in [Fri91], the following theorem.

Theorem 2.7  Let \(g_0, g_1, \ldots, g_r\) be \(d\)-Ramanujan for some \(r \leq d\). Then the principal term of \(g_i\) vanishes for \(1 \leq i \leq r\), and the principal term of \(g_0\) is \(d(d - 1)^{k-1}\).

Proof  See Theorem 3.5 of [Fri91].

\[\square\]

We next apply Lemma 2.4 to estimate the expected value of a trace of \(A^k\) where \(k\) is roughly

\[
h(n, r, d) = \frac{(r + 1)\log n}{\log(\frac{d}{(2\sqrt{d - 1})})}\; (3)
\]

as in [Fri91] and conclude the following theorem.

Theorem 2.8  With the same hypotheses as Theorem 2.7, we have (in \(G_{n,d}\))

\[
E\left[\sum_{i=2}^{n} \lambda_i^k\right] \leq \rho^k
\]

\(^6\)This stems from the fact that in [Fri91], the \(e^{(r+1)k/n}k^{2r+2}\) just above equation (21) (page 352) could have been replaced with \(e^{rk/n}k^{2r}\).
for all \( k \leq h(n, r, d) \), with \( h \) as above, where

\[
\rho = 2\sqrt{d-1} \left( \frac{d}{2\sqrt{d-1}} \right)^{1/(r+1)} \left( 1 + \frac{c \log \log n}{\log n} \right)
\]  

and \( c \) depends only on \( r, d \).

**Proof** First we take \( k = \lfloor h(n, r, d) \rfloor \) and find that \( \rho \) can be taken as above. For smaller \( k \) we appeal to Jensen’s inequality. See [Fri91] for details.

\( \Box \)

We now see that for \( r \) roughly \( \sqrt{d} \), we have \( \rho = 2\sqrt{d-1} + \log d + C + o(1) \) for an absolute constant, \( C \), where \( o(1) \) is a quantity that for fixed \( d \) tends to 0 (proportional to \( \log \log n / \log n \)) as \( n \to \infty \).

### 2.2 Limitations of the Trace Expansion

In this subsection we will show that for some \( i \leq O(\sqrt{d} \log d) \), \( g_i \) is not \( d \)-Ramanujan. We similarly show the part of Theorem 1.1 involving \( s \). Both these facts are due to the possible occurrence of what we call *tangles*. Tangles and avoiding them are the main themes in this paper.

We begin by describing an example of a *tangle*, and its effect on eigenvalues and traces. Consider \( G_{n,d} \) for a fixed, even \( d \) and a variable \( n \) which we view as large. Fix an integer \( m \) with \( 1 \leq m \leq d/2 \) (assume \( d \geq 4 \)). Consider the event, \( T \), that

\[ \pi_i(1) = 1 \text{ for } i = 1, \ldots, m. \]

Clearly \( T \) occurs with probability \( 1/n^m \).

Assume \( T \) occurs in a fixed \( d \)-regular graph, \( G = (V, E) \). Consider the characteristic functions \( \chi_{\{1\}}, \chi_{V \setminus \{1\}} \), where \( \chi_U \) is the function that is 1 on the vertices in \( U \), and 0 elsewhere. If

\[
\mathcal{R}_A(v) = \frac{(Av, v)}{(v, v)}
\]

is the Rayleigh quotient associated to the adjacency matrix, \( A \), of \( G \), then

\[ \mathcal{R}_A(\chi_{\{1\}}) \geq 2m \]
(since \((A\chi_U,\chi_U)\) counts twice the number of edges with both endpoints in \(U\)). Also
\[
(A\chi_{V\setminus\{1\}},\chi_{V\setminus\{1\}}) = (A\chi_V,\chi_V) - 2(A\chi_V,\chi_{\{1\}}) + (A\chi_{\{1\}},\chi_{\{1\}}) 
\geq dn - 2(d - m) + 2m,
\]
so
\[
\mathcal{R}_A(\chi_{V\setminus\{1\}}) = \frac{dn - 2d + 4m}{n - 1} = d - O(1/n)
\]
viewing \(m, d\) as fixed. Since \(\chi_{\{1\}}, \chi_{V\setminus\{1\}}\) are orthogonal, the min-max principle implies that
\[
\lambda_2 \geq \min(2m, d - O(1/n)).
\]
Next consider the probability that
\[
\pi_i(r) = r \text{ for } i = 1, \ldots, m
\]
for at least one value of \(r\). Inclusion/exclusion shows that the probability of this is at least
\[
\sum_r \text{Prob}\{\pi_i(r) = r \text{ for } i = 1, \ldots, m\} - 
\sum_{r,s} \text{Prob}\{\pi_i(r) = r \text{ and } \pi_i(s) = s \text{ for } i = 1, \ldots, m\}
\geq n^{1-m} - \binom{n}{2} n^{-2m}.
\]
We summarize the above observations.

**Theorem 2.9** For fixed \(m \geq 1\) we have that \(\lambda_2 \geq 2m\) for sufficiently large \(n\) with probability \(n^{1-m} - (1/2)n^{2-2m}\).

The proof of the above theorem did not exploit the fact that aside from having \(m\) self-loops, the vertex 1 is still adjacent to \(d - 2m\) other vertices of a \(d\)-regular graph. We seek a stronger theorem that exploits this fact.

**Theorem 2.10** For fixed integers \(m \geq 1\) and \(d \geq 4\), with \(2m - 1 > \sqrt{d - 1}\), we have that \(\lambda_2 > 2\sqrt{d - 1}\) for sufficiently large \(n\) with probability at least \(n^{1-m} - (1/2)n^{2-2m}\).
(Notice that Theorem 2.9 would require \( m > \sqrt{d-1} \) for the same conclusion.) We are very interested to know if one can prove Theorem 2.10 when \( 2m - 1 = \sqrt{d-1} \) for integer \( m \) and even integer \( d \). We expect not. (The situation where \( 2m - 1 = \sqrt{d-1} \) gives rise to what we will call a “critical tangle,” and \( 2m - 1 \geq \sqrt{d-1} \) to a “supercritical tangle,” in Section 4.)

**Proof** Note: in Theorems 3.11 and 4.2 we give a proof of a generalization of this theorem requiring far less calculation (but requiring more machinery).

Again, assume that \( \pi_i(v_0) = v_0 \) for \( i = 1, \ldots, m \) and some \( v_0 \). It suffices to show \( \lambda_2 > 2\sqrt{d-1} \) for sufficiently large \( n \).

Let

\[
\alpha(m) = (2m - 1) + \frac{d - 1}{2m - 1}.
\]

By Cauchy-Schwarz we have \( \alpha > 2\sqrt{d - 1} \) (equality does not hold, since \( 2m - 1 \neq (d - 1)/(2m - 1) \) since \( 2m - 1 > \sqrt{d-1} \)).

Let \( \rho(v) \) denote \( v \)’s distance to \( v_0 \). For a fixed \( r \), let

\[
f(v) = \begin{cases} 
(2m - 1)^{-\rho} & \text{if } \rho \leq r, \\
0 & \text{otherwise},
\end{cases}
\]

where \( \rho = \rho(v) \). It is easy to check that \( (Af)(v) \geq \alpha f(v) \) provided that \( \rho(v) < r \). It follows that

\[
\frac{(Af, f)}{(f, f)} \geq \frac{\alpha(f, f)_{r-1}}{(f, f)_r},
\]

where

\[
(f, f)_t = \sum_{\rho(v) \leq t} f^2(v).
\]

But \( 1 = f^2(v_0) \leq (f, f)_{r-1} \) if \( r \geq 1 \), and also

\[
(f, f)_r \leq (f, f)_{r-1} + (d - 2m)(d - 1)r^{-1}(2m - 1)^{-2r}
\]

(since, by induction, the number of vertices at distance \( r \) from \( v \) is at most \( (d - 2m)(d - 1)^{r-1} \)). So

\[
\frac{(f, f)_r}{(f, f)_{r-1}}
\]

can be made arbitrarily close to 1 by taking \( r \) sufficiently large (since \( (d - 1)(2m - 1)^{-2} < 1 \)).
Let \( R \) be the Rayleigh quotient of \( A \)'s adjacency matrix. The last paragraph, especially equations (6) and (7) implies that for \( \alpha' \) there is an \( r = r(\alpha') \) such that \( R(f) \geq \alpha' \).

So let \( N \) be the support of \( f \), which is bounded as a function of \( d \) and \( r \). \( f \) is orthogonal to \( g = \chi_{V \setminus N} \), and counting edges as before we see

\[
R(g) \geq \frac{d|V| - 2d|N|}{|V| - |N|} = d - O(|N|d/|V|).
\]

It follows that by taking \( n \) sufficiently large, we can make \( \lambda_2 \geq \alpha' \); since \( \alpha > 2\sqrt{d-1} \), we can choose \( \alpha' > 2\sqrt{d-1} \), making \( \lambda_2 > 2\sqrt{d-1} \).

\[ \square \]

Theorem 2.10 proves the part of Theorem 1.1 involving \( s \), by taking \( s = m - 1 \) with \( m \) as small as possible (namely \( m = \lfloor (\sqrt{d-1} + 1)/2 \rfloor + 1 \)). The analogous parts of Theorems 1.2 and 1.3 are slightly trickier, since the “tangle” involved has automorphisms; we shall delay their proof (see Theorem 4.12) until we give a more involved discussion of tangles in Section 4.

Notice that our proof is really computing the norm of \( A_H \) where \( H \) is the \( d \)-regular graph with the vertex 1 having \( m \) self-loops, and which is a (an infinite) tree when these loops are removed. The \( f \) as above shows that \( A_H \)'s norm is at least \( \alpha \). The statement and proof of Lemma 2.5 for the \( d \)-regular tree applies to the above tree (with \( 2m - 1 \) replacing \( \sqrt{d-1} \), and with \( \alpha \) replacing \( 2\sqrt{d-1} \)). In this way our proof of Theorem 2.10 is very much like one proof of the Alon-Boppana theorem (see [Nil91, Fri93]).

The discussion in this section leads to the following theorem.

**Theorem 2.11** There is an absolute constant (independent of \( d \)), \( C \), such that the \( g_i \) of equation (1) cannot be \( d \)-Ramanujan for all \( i \leq C\sqrt{d}\log d \).

**Proof** We fix an integer \( s \) to be chosen later with \( 0 < s < d/2 \), and set \( s + 1 = m \) and apply Theorem 2.10. Since \( \alpha \geq 2s \) with \( \alpha \) as in equation (5), we have for \( k \) even,

\[
E \left[ \lambda_2^k \right]^{1/k} \geq \left( n^{-s} + O(n^{-s-1}) \right)^{1/k} 2s.
\]

According to Theorem 2.8, if \( g_0, \ldots, g_r \) are \( d \)-Ramanujan for some \( r \leq d \), then we have

\[
E \left[ \lambda_2^k \right]^{1/k} \leq \rho,
\]

15
with $\rho$ as in equation (4), provided that $k$ is even and bounded by $h(n, r, d)$ as in Theorem 2.8. For some constant $c$ we have that for any $C$ and for $r = C\sqrt{d}\log d$, equation (4) gives

$$
\rho = 2\sqrt{d-1} \left( 1 + cC^{-1}d^{-1/2} + c(\log n)^{-1}\log\log n \right).
$$

In other words,

$$
\left( n^{-s} + O(n^{-s-1}) \right)^{1/k} 2s \leq 2\sqrt{d-1} \left( 1 + cC^{-1}d^{-1/2} + c(\log n)^{-1}\log\log n \right). \quad (8)
$$

Take $k$ even and as close to $h(n, r, d)$ as possible; note that by equation (3),

$$
\log n = \log\left( \frac{d/(2\sqrt{d-1})}{r+1} \right) \leq \frac{\log d}{(C\sqrt{d}\log d) + 1} \leq 1/(C\sqrt{d});
$$

hence, taking $n \to \infty$ in equation (8) implies that for a universal constant, $c'$, we have

$$
e^{-sd^{-1/2}/C} 2s \leq 2\sqrt{d-1} \left( 1 + cC^{-1}d^{-1/2} \right) \leq 2\sqrt{d-1} \left( 1 + cC^{-1} \right).
$$

Choosing $s = C\sqrt{d}$ and dividing by 2 yields

$$
C\sqrt{d}/e \leq \sqrt{d-1} \left( 1 + cC^{-1} \right).
$$

Choosing $C$ large enough so that $C/e > 1 + (c/C)$ makes this impossible.

\[\square\]

We have proven that not all $g_i$ are $d$-Ramanujan for $i \leq r$ where $r = C\sqrt{d}\log d$. Notice that in our terminology, Theorem 2.18 of [Fri91] says that $g_i$ is $d$-Ramanujan for $i = \lfloor \sqrt{d-1}/2 \rfloor - 1$; again, for each $i$ the $f_i$ in [Fri91] is the polynomial in the principal part of our $g_i$. This leaves the question of whether Theorem 2.11 can be improved to an $r$ value closer to $\lfloor \sqrt{d-1}/2 \rfloor - 1$; we conjecture that it can be improved to $r = \lfloor (\sqrt{d-1} + 1)/2 \rfloor$, and that the tangle with $m = \lfloor (\sqrt{d-1} + 1)/2 \rfloor + 1$ already “causes” this $g_r$ (or a lower one) to fail to be $d$-Ramanujan.

This also leaves open the question of what can be said about the $g_i$ that are not $d$-Ramanujan. Perhaps such $g_i = g_i(k)$ are a sum of $\nu^k p_\nu(k)$ over various $\nu$ with some added error term. In this paper we avoid this issue, working with a modified trace (i.e., “selective” traces) for which the corresponding $g_i$ are $d$-Ramanujan.
3 Background and Terminology

In this section we review some ideas and techniques from the literature needed here. We also give some convenient terminology that is not completely standard.

3.1 Graph Terminology

We use some nonstandard notions in graph theory, and we carefully explain all our terminology and notions here.

A directed graph, $G$, consists of a set of vertices, $V = V_G$, a set of edges, $E = E_G$, and an incidence map, $i = i_G: E \to V \times V$; if $i(e) = (u, v)$ we will write $e \sim (u, v)$, say that $e$ is of type $(u, v)$, and say that $e$ originates in $u$ and terminates in $v$. (If $i$ is injective then it is usually safe to view $E$ as a subset of $V \times V$, and we say that $G$ has no multiple edges.) The adjacency matrix, $A = A_G$, is a square matrix indexed on $V$, where $A(u, v)$ counts the number of edges of type $(u, v)$. The outdegree at $v \in V$ is the row sum of $A$ at $v$, i.e., the number of edges originating in $v$; the indegree is the column sum or number of edges terminating in $v$.

A graph, $G$, is a directed graph, $\tilde{G}$, such that each edge of type $(u, v)$ is “paired” with an “opposite edge” of type $(v, u)$; in other words, we have a map $\text{opp} = \text{opp}_G: E_{\tilde{G}} \to E_{\tilde{G}}$, such that $\text{opp}(\text{opp})$ is the identity, and if $e \in E_{\tilde{G}}$ has $e \sim (u, v)$, then $\text{opp}(e) \sim (v, u)$; in other words, the edges $E_{\tilde{G}}$ come in “pairs,” except that a self-loop, i.e., an $e \in E_{\tilde{G}}$ with $e \sim (v, v)$, can be paired with itself (which is a “half-loop” in the terminology of [Fri93]) or paired with another self-loop at $v$ (which is a “whole-loop”). Half-loops about $v$ contribute 1 to the adjacency matrix entry at $v, v$, and whole-loops contribute 2. In this paper we primarily work with whole-loops, needing half-loops only in the model $J_{n,d}$. We refer to the (undirected) edges, $E_G$, of a graph, $G$, as the set of “pairs” of edges, $\{e, \text{opp}(e)\}$. $G$’s vertex set and adjacency matrix are just those of the directed graph, $\tilde{G}$, i.e., $V_G = V_{\tilde{G}}$ and $A_G = A_{\tilde{G}}$.

A numbering of a set, $S$, is a bijection $\iota: S \to \{1, 2, \ldots, s\}$, where $s = |S|$. A partial numbering of a set, $S$, is a numbering of some subset, $S'$, of $S$ (we allow $S'$ to be empty, in which case none of $S$ is numbered). We can speak of a graph, directed or not, as having numbered or partially numbered vertices and/or edges. A numbering can be viewed as a total ordering.

Each letter $\pi \in \Pi = \{\pi_1, \pi_1^{-1}, \ldots, \pi_{d/2}^{-1}\}$ has its associated inverse,
\[ \pi^{-1} \in \Pi, \text{ and every word } w = \sigma_1 \ldots \sigma_k \text{ over } \Pi \text{ has its associated inverse, } \]
\[ w^{-1} = \sigma_k^{-1} \ldots \sigma_1^{-1}. \]

If \( \mathcal{W} \) is any set of words over \( \Pi \), then a \( \mathcal{W} \)-labelling of an undirected graph, \( G \), is a map or “labelling” \( \mathcal{L} : E_G \rightarrow \mathcal{W} \) such that \( \mathcal{L}(\text{opp}(e)) = (\mathcal{L}(e))^{-1} \) for each \( e \in E_G \). For example, any graph \( G \in G_{n,d} \) automatically comes with a \( \Pi \)-labelling, namely \((i, \pi_j(i))\) is labelled \( \pi_j \), and \((i, \pi_j^{-1}(i))\) is labelled \( \pi_j^{-1} \).

An orientation of an undirected graph, \( G \), is the distinguishing for each \( e \in E_G \) of one of the two directed edges corresponding to \( e \).

The following definition is special to this paper.

**Definition 3.1** Fix sets \( V, E \) and a set of words, \( \mathcal{W} \), over \( \Pi \), with \( \mathcal{W}^{-1} = \mathcal{W} \). A structural map is a map \( s : E \rightarrow \mathcal{W} \times V \times V \). A structural map defines a unique \( \mathcal{W} \)-labelled, oriented graph, \( G \), with \( V_G = V \) and \( E_G = E \), as follows: for each \( e \in E \) with \( s(e) = (\sigma, u, v) \), we form a directed edge of type \((u, v)\) labelled \( \sigma \) and declare it distinguished, and pair it with a directed edge of type \((v, u)\) labelled \( \sigma^{-1} \).

### 3.2 Variable Length Graphs and Subdivisions

Two techniques in this paper use Shannon’s idea of capacity (of a variable length graph\(^7\)) and his algorithm (in \([SW49]\)) for computing it. First, we state and prove the lemma below as a stepping stone to the more difficult modification of this lemma that we shall need later (in Theorem 6.6). Second, we prove the very easy Theorem 3.4 that is crucial to Lemma 9.2.

**Lemma 3.2** For any \( \rho_1, \rho_2 \geq 1 \) and \( \epsilon > 0 \) there is a \( C \) such that the following holds. Let \( G \) be a directed graph with \( E_G \) partitioned into two sets, \( E_1 \) and \( E_2 \). Let the maximum number of \( E_i \) edges leaving any vertex be \( \leq \rho_i \) for \( i = 1, 2 \). For positive integers \( m_1, m_2 \), let \( W(m_1, m_2) \) be the number of cycles in \( G \) that pass through \( m_i \) edges in \( E_i \), for \( i = 1, 2 \). If \( G \) has \( n \) vertices then

\[ W(m_1, m_2) \leq n(\rho_2 + \epsilon)^{(m_1+1)C+m_2}. \]

Notice that there is a more tedious proof of this lemma that does not use VLG’s and Shannon’s algorithm\(^8\). However, VLG’s and Shannon’s algorithm simplify our arguments, so we use them here.

---

\(^7\)In Shannon’s terminology of \([SW49]\), Chapter 1, Section 1, the edges have various “times,” such as a dot versus a dash in Morse code.

\(^8\)Indeed, any such cycle first visits \( a_1 \) edges in \( E_1 \), then \( b_1 \) in \( E_2 \), then \( a_2 \) in \( E_1 \), etc., where \( a_1 \geq 0, b_1 \geq 1, a_2 \geq 1 \), etc. We then proceed to count in how many ways \( m_1 \) can be
Proof First we give some background on VLG’s.

Consider a directed VLG (variable length graph) (see [SW49, AFKM86, Fri93]), $G$, which is a directed graph with a positive integer “edge length” assigned to each edge. $\lambda_1(G)$ is defined to be the limit as $k \to \infty$ of the $k$-th root of the number of cycles of length $\leq k$ (where the “length” of a cycle or path is the sum of all its edge lengths). If all edge lengths are 1, then $\lambda_1(G)$ is the usual largest (or Perron-Frobenius) eigenvalue of $G$.

A bead in a directed graph is a vertex with indegree and outdegree 1 and without a self-loop. A beaded path is a path where every vertex except possibly the endpoints are beads.

**Definition 3.3** If $G$ is a directed VLG, then $G$’s subdivided form, $G_{sbd}$, is a directed graph where each directed edge of length $\ell$ from $u$ to $v$ is replaced by a beaded path of length $\ell$ from $u$ to $v$ (introducing $\ell - 1$ new vertices for each edge).

Let $G = (V, E)$ be a directed VLG whose maximal edge length is $s$. It is easy to see that

1. the number of cycles about a vertex $v \in V$ of a given length in $G$ is the same number as in $G_{sbd}$,
2. hence $\lambda_1(G) = \lambda_1(G_{sbd})$,
3. $|V_{G_{sbd}}| \leq |V_G| + (s - 1)|E_G|$, 
4. hence the total number of cycles of length $k$ in $G$, which is at most the same number as in $G_{sbd}$, is at most

\[
\text{Trace} \left( A_{G_{sbd}}^k \right) \leq |V_{G_{sbd}}| \lambda_1^k \leq (|V_G| + (s - 1)|E_G|) \lambda_1^k,
\]

with $\lambda_1 = \lambda_1(G) = \lambda_1(G_{sbd})$.

Shannon gives the following algorithm (see [SW49], Chaper 1, Section 1) for computing $\lambda_1(G)$ (or the “valence” or “capacity”): let $Z_G = Z_G(z)$ be the matrix whose $i,j$ entry is the sum of $z^\ell$ over all edge lengths, $\ell$, of edges from $i$ to $j$, with $z$ a formal parameter. Then $\lambda_1(G)$ is the reciprocal of the smallest real root in $z$ of

\[
\det (I - Z_G(z)) = 0.
\]

written as $a_1 + a_2 + \cdots$ as above, and similarly for $m_2$. Then we consider, for appropriate and small $\nu > 0$, the cases $m_1 \geq \nu m_2$ and $m_1 < \nu m_2$ separately.
Proof of Lemma 3.2: Fix an $\epsilon, \rho_1, \rho_2 > 0$ as in the statement of the lemma; we wish to find the appropriate $C$. For every positive integer, $s$, let $G_s$ be the directed VLG which is $G$ with all $E_1$ edges of length $s$ and all $E_2$ edges of length 1. We claim

$$\lambda_1(G_s) \leq \rho_2 + \epsilon$$

for some integer $s$ depending only on $\rho_1, \rho_2, \epsilon$ (and not on $G$). Indeed, it suffices to show that $Z_{G_s}(z)$ has all its eigenvalues $< 1$ for $z < (\rho_2 + \epsilon)^{-1}$ for all $s$ sufficiently large. But

$$Z_{G_s}(z) = z^s A_{E_1} + z A_{E_2},$$

where $A_{E_i}$ is the adjacency matrix of the graph on $(V, E_i)$. The row sums of $Z_{G_s}(z)$ are at most

$$z^s \rho_1 + z \rho_2,$$

and hence all eigenvalues of $Z_{G_s}(z)$ (which has non-negative entries) are at most

$$z^s \rho_1 + z \rho_2 < (\rho_2 + \epsilon)^{-s} \rho_1 + (\rho_2 + \epsilon)^{-1} \rho_2.$$

Since the right-hand-side of the above inequality is $< 1$ for $s$ sufficiently large (depending only on $\rho_1, \rho_2, \epsilon$), the claim that $\lambda_1(G_s) \leq \rho_2 + \epsilon$ for sufficiently large $s$ holds.

Now fix an $s$ that is sufficiently large to have $\lambda_1(G_s) \leq \rho_2 + \epsilon$. A cycle through $m_i$ edges of $E_i$ in $G$ determines a unique cycle of length $sm_1 + m_2$ in $G_s$. Given the above observations about subdivisions (just after Definition 3.3), and applying them to $G_s$, we see that

$$W(m_1, m_2) \leq (\text{number of } G_s \text{ cycles of length } sm_1 + m_2)$$

$$\leq \left( |V_G| + (s - 1)|E_G| \right) \lambda_1^{sm_1+m_2}(G_s) \leq \left( |V_G| + |E_G| \right) s(\rho_2 + \epsilon)^{sm_1+m_2}.$$

So the lemma follows.

Our second use of VLG’s is the following crucial fact (see Lemma 9.2).

Theorem 3.4 Let $G$ be a graph or VLG with $\lambda_1(G) > 1$, and fix some $u, v \in V_G$. For each integer $i \geq 1$, let $G_i$ be $G$ plus one additional edge of type $(u, v)$ and of length $i$. Then

$$\lim_{i \to \infty} \lambda_1(G_i) = \lambda_1(G).$$
Furthermore, the same is true if $G_i$ is $G$ plus a bounded number of edges of a specified type, each of length $\geq i$.

**Proof** Let $z$ be the smallest real root of equation (9), and let $z_i$ be the smallest real root of equation (9) with $G_i$ replacing $G$. Clearly the number of cycles of length $\leq k$ does not increase as $i$ increases, and so $z_i$ is nondecreasing in $i$ and $z_i \leq z < 1$ for all $i$. So $z_i$ has a limit, $\bar{z}$, as $i \to \infty$, and clearly $\bar{z} \leq z$ so $\bar{z}^i \to 0$ as $i \to \infty$; hence $Z_{G_i}(z_i) \to Z_G(\bar{z})$ as $i \to \infty$. But then $\bar{z}$ is a real root of equation (9), and by definition we must have $\bar{z} \geq z$. Hence $\bar{z} = z$ and therefore $z_i \to z$ as $i \to \infty$; taking reciprocals gives $\lambda_1(G_i) \to \lambda_1(G)$.

\[ \Box \]

### 3.3 Supression and More on Subdivision

In this paper we will have occasion to perform the opposite of a subdivision, which is a special case of “supression.”

**Definition 3.5** Let $G$ be a strongly connected directed graph, with $W \subset V_G$ a subset of beads in $G$ such that $W$ contains no cycle. The supression of $W$ (in $G$), denoted $G[W]_{sup}$, is the directed VLG on vertices $V_G \setminus W$ obtained by replacing each beaded directed path of length $\ell$ in $G$ between $V_G \setminus W$ vertices by a single directed edge of length $\ell$ between the path’s endpoints.

The supression (by the supressed vertices) of a supression returns the original directed graph.

(There is a more general notion in symbolic dynamics, called “restriction” in [Fri91], that allows the supression of an arbitrary proper subset of $V$ (perserving various cycle counts and $\lambda_1$); if the supressed set contains a cycle, then the resulting VLG will have infinitely many edges.)

We will also have occasion to use subdivision and supression on undirected graphs. A *bead* in an undirected graph is a vertex of degree 2 with no self-loops. Undirected VLG’s, subdivisions, and suppressions can be defined; however, Shannon’s algorithm must be modified, since a walk entering a beaded path has much more freedom when the graph is undirected; in this paper we only use Shannon’s algorithm applied to directed graphs. We remark that if $G$ is a $\Pi$-labelled graph, then any supression in $G$ has a natural $\Pi^+$-labelling, where $\Pi^+$ is the set of words on $\Pi$ of length $\geq 1$. 

21
3.4 Irreducible Eigenvalues

Let $G$ be an undirected graph with corresponding directed graph $\hat{G}$. Let $G_{\text{Irred}}$ be the graph with vertices $E_{\hat{G}}$ and an edge from $e_1$ to $e_2$ iff $e_1e_2$ forms an irreducible path in $G$; i.e., $e_1$ and $e_2$ are not opposites (i.e., paired) in $G$, and $e_1$ terminates in the vertex where $e_2$ originates. Then walks in $G_{\text{Irred}}$ give “irreducible” (or “reduced” or “non-backtracking”) walks in $G$. A cycle of length $k$ in $G_{\text{Irred}}$ gives a cycle in $G$ (with specified starting vertex) that is strongly irreducible, meaning that the cycle is irreducible and the last step in the cycle is not the inverse of the first step. We define the irreducible eigenvalues of $G$ to be those of $G_{\text{Irred}}$, and we define the largest or Perron-Frobenius eigenvalue of $G_{\text{Irred}}$ to be $\lambda_{\text{Irred}} = \lambda_{\text{Irred}}(G)$, the largest irreducible eigenvalue of $G$.

We now state a theorem for use later in this paper; the theorem requires a definition.

**Definition 3.6** A connected graph, $G$, is loopy if $|E_G| \geq |V_G|$, or equivalently if $G$ contains an irreducible cycle, or equivalently if $G$ is not a tree. $G$ is 1-loopy if $G$ is connected and the removal of any edge from $G$ leaves a graph each of whose connected components are loopy.

**Theorem 3.7** A connected graph, $G$, has $G_{\text{Irred}}$ strongly connected iff $G$ is 1-loopy. In particular, if $G$ is connected and $d$-regular for $d \geq 3$, then $G_{\text{Irred}}$ is strongly connected.

**Proof** Consider a directed edge, $e \sim (u, v)$, of $G$, and let $e'$ be $e$’s opposite (we permit $e = e'$, i.e., the case of a half-loop). If the removal of $e$ leaves the connected component of $v$ being a tree, then there is no irreducible walk from $e$ to $e'$. On the other hand, if this connected component is not a tree, then a cycle in this component about $v$ of minimum length is irreducible, which then extends to an irreducible walk from $e$ to $e'$. To summarize, if $G$ is not 1-loopy, then $G_{\text{Irred}}$ is not strongly connected; otherwise, each edge has an irreducible path to its opposite, i.e., each edge is connected to its opposite in $G_{\text{Irred}}$.

Next if $e_1, e_2$ are two distinct, unpaired directed edges originating in the same vertex, $v$, then an irreducible path from $e_1$ to its opposite followed by $e_2$ shows that $e_1$ and $e_2$ are connected by an irreducible path. Thus any two edges that share a vertex are strongly connected in $G_{\text{Irred}}$.
Finally if $e_1, e_2$ are two undirected edges without a common vertex, then a shortest path connecting them gives an irreducible path from some orientation of $e_1$ to some of $e_2$. By the above we can follow them with irreducible paths to the edge of opposite orientation. Thus any two edges that do not share a vertex are strongly connected in $G_{\text{Irred}}$. So $G_{\text{Irred}}$ is strongly connected.

For the later part of the theorem, if $G$ is connected and $d$-regular with $d \geq 3$, then removing an edge leaves components with $e$ edges and $v$ vertices where $e \geq (3v/2) - 1$. So $e - v \geq (v/2) - 1$ and so $e - v \geq -1/2$ (since $v > 0$) so $e - v \geq 0$. Hence $G$ is 1-loopy, and we are done.

\[ \Box \]

### 3.5 $\lambda_1$ and cycles

In this subsection we recall some facts about $\lambda_1$ and cycles of graphs, either indicating the proofs or giving references. This section is geared to infinite graphs, since most of the facts are very easy when the graph is finite.

Let $G$ be a graph of bounded degree, i.e. there is an $r$ such that the degree of each vertex is at most $r$. Then $A_G$, $G$’s adjacency matrix, is a bounded linear operator (bounded by $r$) on $L^2(G)$, the square summable functions on $G$’s vertices. We let $\lambda_1 = \|A\|$.

**Theorem 3.8** For every $\epsilon > 0$ there is an $f \neq 0$ such that $\|Af\| \geq (\lambda_1 - \epsilon)\|f\|$, where $f$ has finite support.

**Proof** By definition of norm, there is a $g \neq 0$ with $\|Ag\| \geq (\lambda_1 - (\epsilon/2))\|g\|$. For any $\nu > 0$ we may write $g = g_1 + g_2$ where $g_1$ is of finite support and $\|g_2\| \leq \nu$. It is easy to see (using the fact that $A$ is bounded) that if $\nu$ is sufficiently small we can take $f = g_1$ to satisfy the above theorem.

\[ \Box \]

**Theorem 3.9** Assume $G$ is connected. For any vertex, $v$, of $G$ and positive integer, $k$, let $c(v, k)$ be the number of cycles of length $k$ in $G$ about $v$. Then $c(v, k) \leq \lambda_1^k$, and

$$\lim_{r \to \infty} [c(v, 2r)]^{1/(2r)}$$

exists and equals $\lambda_1$. 

23
The VLG algorithm for $\lambda_1$ and/or generating functions can be used, in certain situations, to count cycles. We shall do so in the important relationship described below (Theorem 3.11).

**Definition 3.10** Let $\psi$ be a finite graph with each vertex of degree $\leq d$ for an integer $d > 2$. By $\text{Tree}_d(\psi)$ we mean the unique (up to isomorphism) undirected graph, $G$, that has an inclusion $\iota: \psi \to G$ such that $G$ is $d$-regular and such that $G$ becomes a forest when we remove (the image under $\iota$ of) $\psi$'s edges.

We have seen an example of this construction in Section 2, where $\psi$ is one vertex with $m$ self-loops, in the proof of Theorem 2.10. See Figure 1 for another example.

In the category of $d$-regular graphs with a $\psi$ inclusion, $\text{Tree}_d(\psi)$ is none other than the universal cover.

The methods used to prove Theorems 1.1, 1.2, and 1.3 suggest the following curious theorem.
Theorem 3.11  Let \( d \geq 3 \), and let \( \psi \) be a finite graph with each vertex of degree \( \leq d \). Then

\[
\lambda_1(\text{Tree}_d(\psi)) = 2\sqrt{d-1} \quad \iff \quad \lambda_{\text{tred}}(\psi) \leq \sqrt{d-1},
\]

\[
\lambda_1(\text{Tree}_d(\psi)) > 2\sqrt{d-1} \quad \iff \quad \lambda_{\text{tred}}(\psi) > \sqrt{d-1}.
\]

The same is true for any real \( d > 2 \), provided that \( \lambda_1(\text{Tree}_d(\psi)) \) is interpreted with an appropriate analytic continuation in \( d \) (described below).

Before proving the theorem, we describe the analytic continuation to which we refer.

Let \( a_n \) for \( n = 2, 4, \ldots \) be the number of walks from the root to itself (never passing through the root except at the beginning and end) on an undirected rooted tree, \( \tilde{T} \), where every vertex has \( d-1 \) children, with \( d > 1 \) an integer (for now). (\( \tilde{T} \) has degree \( d-1 \) at the root and degree \( d \) elsewhere, so \( \tilde{T} \) is not regular.) It is easy to see that

\[
S = S_d(z) = \sum_{n=2}^{\infty} a_n z^n,
\]

satisfies the recurrence \( S = z^2(d-1)(1 + S + S^2 + \cdots) \) (since the walks counted by \( S \) take one step, in \( d-1 \) possible ways, to a child of the root, followed by some number (possibly zero) of \( S \) walks, followed by the step back to the root); so \( S(1-S) = z^2(d-1) \), so that near \( z = 0 \)

\[
S = S_d(z) = \frac{1 - \sqrt{1 - 4(d-1)z^2}}{2}. \tag{10}
\]

This above expression makes sense for any real \( d > 1 \). By \( \lambda_1(\text{Tree}_d(\psi)) \) for \( d > 1 \) real (assuming each degree in \( \psi \) is \( \leq d \)) we mean \( \lambda_1 \) of the VLG formed from \( \psi \) with an additional self-loop about each vertex of degree \( r \) that has “formal weight” \( (d-1)S/(d-1) \) (this can be viewed as adding an infinite set of self-loops of given weights and lengths corresponding to this power series\(^9\), or can simply be viewed as a factor to add to the diagonal of \( Z_G(z) \) in Shannon’s algorithm). We add this self-loop since in \( \tilde{T} \) each root-child contributes \( S/(d-1) \) to the generating function of self-loops about the root.

\(^9\)In other words, we view a power series \( \sum a_n z^n \) as the sum of \( a_n \) edges of length \( n \) (even when \( a_n \) is not an integer).
So in \( \text{Tree}_d(\psi) \) we are adding \( d - r \) root-children, in a formal sense, to each degree \( r \) vertex.

**Proof** We give two proofs of the above theorem. The first is to consider 
\[
z \leq z_0 = \frac{1}{2\sqrt{d-1}}
\]
in the series for \( \lambda_1 \left( \text{Tree}_d(\psi) \right) \). We have \( S_d(z_0) = 1/2 \), so a self-loop of \( (d - r)S/(d - 1) \) on a vertex of degree \( r \) adds a \( (d - r)/(2d - 2) \) to the diagonal of \( Z_G(z_0) \). We get

\[
Z_G(z_0) = z_0A + \left( \frac{dI - D}{2d - 2} \right),
\]
where \( D \) is the diagonal matrix whose entry at vertex \( v \) is the degree of \( v \), and \( I \) is the identity matrix. We wish to know when \( I - Z_G(z_0) \) has all positive eigenvalues (and \( I - Z_G(z) \) for all \( z < z_0 \)); but

\[
I - Z_G(z_0) = I - \frac{A}{2\sqrt{d-1}} - \left( \frac{dI - D}{2d - 2} \right) = \left( I - y_0A + y_0^2(D - I) \right)/2,
\]
where \( y_0 = 1/\sqrt{d-1} \). Our theorem now follows from the fact that \( \lambda_{\text{Irred}}(A) \) is given by \( 1/y \) of the smallest root, \( y \), of

\[
\det \left( I - yA + y^2(D - I) \right) = 0
\]
(see [God93], exercise 13 page 72).

We give another proof of the theorem; we first assume that \( d \) is integral. Consider the universal cover, \( T \), of \( \text{Tree}_d(\psi) \). \( T \) is a \( d \)-regular tree. Let \( v \) be a vertex of \( \psi \), which we may also view as a vertex of \( \text{Tree}_d(\psi) \), and let \( \bar{v} \) be a vertex of \( T \) that maps to \( v \). Let \( b_k \) be the number of cycles of length \( k \) about \( v \) in \( \text{Tree}_d(\psi) \). Let \( c_k \) be the number of cycles of length \( k \) in \( T \) about \( \bar{v} \) (or about any vertex, since \( T \) is a \( d \)-regular tree). Let \( a_k \) be the number vertices in \( T \) of distance \( k \) to \( \bar{v} \) that map to \( v \), or equivalently the number of irreducible cycles of length \( k \) about \( v \) in \( \psi \) (or in \( \text{Tree}_d(\psi) \)). Set \( f = f(z) \) to be the generating function for \( a_k \), i.e.,

\[
f(z) = \sum_{k=0}^{\infty} a_k z^k,
\]
and set \( g, h \) to be the generating functions for, respectively \( b_k, c_k \).

First of all, we claim that

\[
h(z) = \left( 1 - \frac{dz^2}{1 - S_d} \right)^{-1} = \left( 1 - \frac{dS_d}{d - 1} \right)^{-1};
\]
indeed, the generating function for the cycles about a \( \bar{v} \) that do not pass through \( \bar{v} \) anywhere in the middle is

\[
R_d = dz^2/(1 - S_d) = dS_d/(d - 1),
\]

by the same argument used to derive the equation for \( S_d \); allowing walks that pass through \( \bar{v} \) any number of times is

\[
1 + R_d + R_d^2 + \cdots = 1/(1 - R_d) = h(z).
\]

We see that \( h(z) \) has radius of convergence \( 1/(2\sqrt{d - 1}) \), given equation (10). Next, we claim that the generating function for the walks in \( T \) from one vertex, \( v_0 \), to a neighbouring vertex, \( v_1 \), that don’t pass through \( v_1 \) in the middle, is \( z/(1 - S_d) \); indeed, such a walk from \( v_0 \) to \( v_1 \) takes a walk from \( v_0 \) to itself, avoiding \( v_1 \) (which corresponds to a walk from the root of \( \tilde{T} \) to itself), followed by one step from \( v_0 \) to \( v_1 \). Now a cycle about \( v \) in \( \text{Tree}_d(\psi) \) has a unique lifting to a walk in \( T \) beginning at \( \bar{v} \) and ending at a vertex, \( \bar{v}' \), that maps to \( v \); if the distance of \( \bar{v} \) to \( \bar{v}' \) is \( m \), then the generating function walks from \( \bar{v} \) to \( \bar{v}' \) is

\[
U_{m,d}(z) = (z/(1 - S_d))^m h(z),
\]

since such a walk first hits the vertex of distance \( m - 1 \) to \( \bar{v}' \) at some first time, then the vertex of distance \( m - 2 \), etc., and then makes some cycle about \( \bar{v}' \). Since the number of \( \bar{v}' \) mapping to \( v \) is counted by the \( a_k \) and the generating function, \( f \), and since each \( \bar{v}' \) contributes a \( U_{m,d} \) to \( g(z) \), we have

\[
g(z) = \sum_{k=0}^{\infty} a_k U_{k,d}(z) = h(z) \sum_{k=0}^{\infty} a_k (z/(1 - S_d))^k = h(z)f(z/(1 - S_d)). \quad (11)
\]

It follows that the radius of convergence of \( g \) is less than \( 1/(2\sqrt{d - 1}) \) if and only if the same is true for \( f(z/(1 - S_d)) \). But at \( z = 1/(2\sqrt{d - 1}) \) we have \( S_d = 1/2 \), so \( z/(1 - S_d) = (d - 1)^{-1/2} \). So the question of \( g \)'s radius of convergence boils down to whether or not \( f \)'s radius of convergence is less than \( (d - 1)^{-1/2} \), and this proves the theorem (for \( d \) integer); here our argument is simplest if we make use of the fact that a power series about \( z = 0 \) with radius of convergence \( \rho \) and with non-negative coefficients always has a singularity at \( \rho \) (see the proof of Theorem 3.12).
If \( d \) is real, the same proof carries over, although we treat self-loops of formal weight, representing root-children additions, as being collections of trees. So we take \( T \) to be the universal cover of \( H \), and then to each vertex of degree \( r \) we add a self-loop of formal weight \((d - r)S/(d - 1)\); in other words, the self-loops of formal weight in \( \text{Tree}_d(\psi) \) are lifted to \( T \) but left intact in \( T \) (since these self-loops are collections of trees). The rest of the proof goes through with the above minor adjustment.

\[2\]

The following consequence of the above proof is worth stating, but will not be used in the rest of this paper.

**Theorem 3.12** Let \( z_0 = 1/\lambda_1(\text{Tree}_d(\psi)) \) and \( z_1 = 1/\lambda_{\text{Irred}}(\psi) \), with notation as in Theorem 3.11 (\( d > 2 \) is real, and each vertex of \( \psi \) has degree \( \leq d \)). Then if \( \lambda_1(\text{Tree}_d(\psi)) > 2\sqrt{d - 1} \), we have

\[
z_1 = \frac{z_0}{1 - S(z_0)}.
\]

\[2\]

**Proof** This follows from equation (11) and the well-known fact that a power series \( \sum \alpha_n z^n \) with \( \alpha_n \geq 0 \) has a first (smallest) singularity on the positive real axis\(^{10}\); indeed, if \( g(z) \) has its first positive singularity at \( z_0 \), then \( f(z) \) must have one at \( z_1 = z_0/(1 - S_d(z_0)) \), since \( h \) is analytic at \( z = z_0 \). Since \( z/(1 - S) = (S/z)/(d - 1) \), \( z/(1 - S) \) can be viewed as a power series with non-negative coefficients; hence \( z/(1 - S) \) is increasing in real, positive \( z \). If \( f \) has a singularity before \( z_1 \), then \( g \) would have a positive real singularity before \( z_0 \), which is impossible. Thus \( f \)'s first positive singularity is \( z_1 \).

\[2\]

\(^{10}\)If not, there is a series \( s(z) = \sum \alpha_n z^n \) with \( \alpha_n \geq 0 \) with radius of convergence 1 such that \( \sum \beta_m (z - 1 + \epsilon)^m \) agrees with the former power series near \( z = 1 - \epsilon \), and \( \sum \beta_m u^m \) has radius of convergence \( 3\epsilon \), for some \( \epsilon > 0 \). But the \( \beta_m \), arising from derivatives of \( s \) at \( z = 1 - \epsilon \), are also non-negative. Thus \( \sum \beta_m u^m = \sum \alpha_n (u + 1 - \epsilon)^n \) can be expanded in a series of non-negative terms \( \alpha_n u^m (1 - \epsilon)^n \). Since \( \sum \beta_m (2\epsilon)^m \) converges, so does \( \sum \alpha_n (1 + \epsilon)^n \) (by rearranging terms in a non-negative infinite sum); but then \( s(z) \)'s radius of convergence is \( > 1 \).
4 Tangles

In Section 2 we saw that a vertex with \( m \) self-loops in a \( G_{n,d} \) graph, with \( m \) “large,” gives rise to a “large” second eigenvalue (i.e., larger than \( 2\sqrt{d-1} \) for sufficiently large \( m \), as \( n \to \infty \)). Here we generalize this observation to what we call a “tangle.” Our proof of the Alon conjecture via a trace method must somehow overcome all “hypercritical” tangles.

\textbf{Definition 4.1} Given two \( \Pi \)-labelled graphs, \( G \) and \( H \), we say \( G \) contains \( H \) (or \( H \) occurs in \( G \)) if there is an inclusion\(^{11} \) \( \iota : H \to G \) that preserves the labelling; the number of times a graph, \( H \), occurs in \( G \) is the number of distinct\(^{12} \) such \( \iota \). A tangle (or \( G_{n,d} \)-tangle) is a \( \Pi \)-labelled connected graph, \( \psi \), that is contained in some element of \( G_{n,d} \).

For example, in Section 2 we studied the tangle with one vertex and \( m \) self-loops labelled \( \pi_1, \ldots, \pi_m \). We define \( H_{n,d} \)- and \( I_{n,d} \)- and \( J_{n,d} \)-tangles similarly.

\textbf{Theorem 4.2} Fix a positive integer, \( d \geq 3 \), and a graph \( \psi \). Any graph, \( G \), on \( n \) vertices, that contains \( \psi \) has second eigenvalue at least \( \rho - o(1) \), where \( o(1) \) is a function of \( n \) tending to 0 as \( n \to \infty \), and where \( \rho \) is the norm of the adjacency matrix of Tree\(_d(\psi)\).

This generalizes Theorem 2.10.

\textbf{Proof} Fix \( \epsilon > 0 \); we will show that for \( n \) sufficiently large any such \( G \) has \( \lambda_2(G) \geq \rho - \epsilon \). First, there is a finitely supported \( f \neq 0 \) on Tree\(_d(\psi)\) with \( \|Af\| \geq \|f\|(\rho - \epsilon) \), where \( A \) is the adjacency matrix of Tree\(_d(\psi)\). If \( V_\psi \) is the set of vertices on which \( f \) is non-zero, then replacing \( f \) with the non-negative first Dirichlet eigenfunction on \( V_\psi \) (see [Fri93]) we may assume \( f \) is non-negative, nonzero, and that \( Af \geq f(\rho - \epsilon) \) (with equality everywhere except at the boundary of \( V_\psi \)). There is a covering map (see [Fri93] and [Fria]) \( \pi : \text{Tree}_d(\psi) \to G \); set \( \pi_*f \) to be the function on \( G \) defined by

\[
(\pi_*f)(v) = \sum_{\pi(w)=v} f(w).
\]

\(^{11}\)By an inclusion we mean a graph homomorphism that is a injection on the vertices and on the edges.

\(^{12}\)For example, if \( H = G \) consists of one edge joining two distinct vertices, \( u, v \), then the identity is considered distinct from the morphism interchanging the vertices. By the same principle, if \( H \) has exactly \( k \) automorphisms, then the number of times \( H \) occurs in a graph is always a multiple of \( k \).
If $A_G$ is the adjacency matrix of $G$, then clearly
\[ A_G(\pi_* f) \geq (\rho - \epsilon)(\pi_* f). \]

So, on the one hand, $\mathcal{R}(\pi_* f) \geq \rho - \epsilon$, where $\mathcal{R}$ is the Rayleigh quotient for $A_G$. On the other hand, the support, $N$, of $\pi_* f$ is of size no greater than that of $f$, and this size is bounded (independent of $G$ and $n$). So the same reasoning as in the proof of Theorem 2.10 shows that
\[ \mathcal{R}(g) \geq d - o(1), \quad \text{where} \quad g = \chi_{V \setminus N}. \]

Since $\pi_* f$ and $g$ are orthogonal, we are done (by the min-max principle).

\[ \square \]

**Definition 4.3** A tangle, $\psi$, is critical (respectively, supercritical, hypercritical) if $\lambda_{\text{Irred}}(\psi)$ equals (respectively, $\geq$ and $>$) $\sqrt{d - 1}$.

According Theorems 3.11 and 4.2, a fixed hypercritical tangle can only occur in a graph with sufficiently many vertices if the graph has $\lambda_2 > 2\sqrt{d - 1}$.

Now that we know how tangles affect eigenvalues, we want to know how often the tangles occur. This discussion, and the particular application to $H_{n,d}$, $I_{n,d}$, and $J_{n,d}$, will take the rest of this section.

**Definition 4.4** A leaf on a graph is a vertex of total degree 1 (whether the graph is directed or not). We say that a graph is pruned if it has no leaves. A simple pruning is the act of removing one leaf and its incident edge from a graph; pruning is the repeated performance of some sequence of simple prunings; complete pruning is the act of pruning until no more pruning can be done.

For example, completely pruning a tree results in a single vertex with no edges; completely pruning a cycle leaves the cycle unchanged.

**Proposition 4.5** Given a graph, $G$, there is a unique pruned graph $H$ obtainable from completely pruning $G$. Furthermore, $H$ is completely pruned iff each edge of $H$ lies on an irreducible cycle.
Proof Let \( e_1, \ldots, e_t \) denote the edges pruned in one pruning of \( G \), in the order in which they are pruned. Let \( G' \) be a different complete pruning of \( G \), which we assume does not contain all the \( e_i \). Let \( j \) be the smallest integer such that \( e_j \) lies in \( G' \). On the one hand, the removal of \( e_1, \ldots, e_{j-1} \) from \( G \), or any subgraph of \( G \), allows \( e_j \) to be pruned from \( G' \), or any subgraph of \( G' \), including \( G' \). On the other hand, the prunability of \( e_j \) from \( G' \) contradicts the completeness of the pruning that formed \( G' \). It follows that any complete pruning contains all the edges of any other, and so any two are the same.

We now address the last statement of the theorem. If \( H \) has a leaf, then the edge incident upon this leaf does not lie in an irreducible cycle. Conversely, if \( H \) has no leaves, and if \( e \) is an edge with endpoints \( u, v \), consider the graph, \( H' \), obtained by removing \( e \) from \( H \). If \( u \) and \( v \) are connected in \( H' \), then a minimal length path that joins them, along with \( e \), gives an irreducible cycle containing \( e \). Otherwise, since \( u \)'s connected component is not a tree (or else \( H \) would have leaves), this component has an irreducible cycle, and a shortest walk from \( u \) to this cycle, once around the cycle, and back to \( u \) gives an irreducible cycle beginning and ending at \( u \). Similarly there is such a cycle about \( v \). The cycle about \( u \), followed by \( e \), followed by the \( w \) cycle, and back through \( e \) (in the other direction), gives an irreducible cycle containing \( e \).

\[ \square \]

A morphism of a tangles is a morphism of \( \Pi \)-labelled graphs, i.e., a graph morphism that preserves the edge labelling.

**Definition 4.6** The order of a tangle, \( \psi \), is \( \text{ord}(\psi) = |E_\psi| - |V_\psi| \) (for the model \( J_{n,d} \), to be considered soon, a half-loop is counted as one edge). More generally, for a graph, \( G \), or any structure with an underlying graph, \( G \) (such as a “form” or “type” to be defined in Section 5), its order is \( \text{ord}(G) = |E_G| - |V_G| \).

**Theorem 4.7** For any tangle, \( \psi \), the expected number of occurrences of \( \psi \) in an element, \( G \), of \( G_{n,d} \) is \( n^{-r} + O(n^{r-1}) \), where \( r \) is the order of \( \psi \). The probability that at least one occurrence occurs is at least \( n^{-r} / (2c^2) + O(n^{r-1}) \), with \( r \) as before and where \( c \) is the number of automorphisms of the complete pruning of \( \psi \).
Notice that the number of automorphisms of the complete pruning of a graph, \( \psi \), is at least as many as that of \( \psi \), and it may be strictly greater\(^{13}\). The proof following will imply that the probability of \( \psi \)'s occurrence is also \( \leq n^{-r/c} + O(n^{-r-1}) \), which matches the lower bound to first order, provided that \( r \geq 1 \).

**Proof** Let \( V_\psi = \{u_1, \ldots, u_s\} \), and for a tuple \( \bar{m} = (m_1, \ldots, m_s) \) of distinct integers between 1 and \( n \), let \( \iota_{\bar{m}} \) denote the event that the map, \( \iota \), mapping \( u_i \) to \( m_i \), is an occurrence of \( \psi \) in \( G \). If \( a_i \) is the number of \( \psi \)'s edges labelled \( \pi_i \), then each event \( \iota_{\bar{m}} \) involves setting \( a_i \) values of \( \pi_i \), all of which occur with probability

\[
\frac{(n-a_1)! \cdots (n-a_{d/2})!}{n!}.
\]

Since the sum of the \( a_i \) is \( |E_\psi| \), this probability is \( n^{-|E_\psi|} \). Since there are \( n!/(n-|V_\psi|)! = n^{|V_\psi|} + O(n^{|V_\psi|-1}) \) different \( \iota_{\bar{m}} \)'s, the expected number of occurrences is \( n^{-r} + O(n^{-r-1}) \), where \( r = \text{ord}(\psi) \).

Next notice that if \( \psi' \) is a pruning of a tangle, \( \psi \), then the probability that \( \psi \) occurs is \( 1 + O(n^{-1}) \) times the probability that \( \psi' \) occurs (adding each pruned edge adds a condition that occurs with probability between 1 and \( (n-c_1)/(n-c_2) \) with \( c_1, c_2 \) constants). Hence we may assume \( \psi \) is pruned.

An automorphism of \( \psi \) can be viewed as a permutation on \( V_\psi \), which is the same as a permutation, \( \sigma \), on \( \{1, \ldots, s\} \) (identifying a \( u_i \in V_\psi \) with \( i \)). Such a permutation, \( \sigma \), acts by permuting the components of the \( \bar{m} \)'s. Say that \( \iota_{\bar{m}} \) is equivalent to \( \iota_{\bar{k}} \) if \( \bar{m} \) and \( \bar{k} \) differ by a permutation, \( \sigma \), associated to an automorphism of \( \psi \); i.e., if \( \iota_{\bar{m}} \) and \( \iota_{\bar{k}} \) correspond to the same subgraph of \( G \). Let \( R \) be a set of representatives in the equivalence classes of all \( \bar{m} \)'s.

By inclusion/exclusion, the probability that \( \psi \) occurs at least once is at least

\[
\sum_{\bar{m} \in R} \text{Prob} \{ \iota_{\bar{m}} \} - \frac{1}{2} \sum_{\bar{k} \neq \bar{m}} \sum_{\bar{k}, \bar{m} \in R} \text{Prob} \{ \iota_{\bar{m}} \cap \iota_{\bar{k}} \}.
\]

The first summand is \( (1/c)n^{-r} + O(n^{-r-1}) \), by the argument given for the expected number. For the second summand, we may write \( \iota_{\bar{m}} \cap \iota_{\bar{k}} \) as \( \iota_{\bar{q}}(\psi') \),

\(^{13}\)Indeed, a structural induction argument (i.e., by pruning one leaf) shows that any automorphism of the complete pruning of \( \psi \) has at most one extension to \( \psi \). On the other hand, if \( \psi \) is a cycle of length \( q \) with all edges in one “direction” labelled \( \pi_1 \), then \( \psi \) has \( q \) automorphisms; yet if we add one edge labelled \( \pi_2 \) to \( \psi \) at any vertex, the new graph has only the trivial automorphism.
where \( \vec{q} \) is a vector comprised of the distinct components of \( \vec{m} \) and \( \vec{k} \), and where \( \psi' \) is the tangle obtained by gluing two copies of \( \psi \) along certain vertices (corresponding to where the components of \( \vec{m} \) and \( \vec{k} \) coincide). If \( \vec{m} \) is disjoint from (i.e., nowhere coincides with) \( \vec{k} \), then \( \psi' \) is two disjoint copies of \( \psi \); for fixed \( \vec{k} \in R \) we have

\[
\sum_{\vec{m} \in R} \text{Prob} \left\{ \iota_{\vec{m}} \mid \iota_{\vec{k}} \right\} = n^{-r} / c + O(n^{-r-1}),
\]

the summation being over conditional probabilities, since the conditioning of \( \iota_{\vec{k}} \) and summing over \( \vec{m} \) disjoint only affects equation (12) by changing \( n!/(n-a_i)! \) terms into \( (n-c_i)/(n-c_i-a_i)! \) terms for constants \( c_i \), which is a second order change. Hence

\[
\sum_{\vec{k}, \vec{m} \in R} \text{Prob} \{ \iota_{\vec{m}} \cap \iota_{\vec{k}} \} = \sum_{\vec{k} \in R} \text{Prob} \{ \iota_{\vec{k}} \} \sum_{\vec{m} \in R} \text{Prob} \{ \iota_{\vec{m}} \mid \iota_{\vec{k}} \} \]

\[
= \sum_{\vec{k} \in R} \text{Prob} \{ \iota_{\vec{k}} \} \left( n^{-r} / c + O(n^{-r-1}) \right) = n^{-2r} / c^2 + O(n^{-2r-1}).
\]

To understand the situation where \( \vec{m} \) and \( \vec{k} \) overlap somewhere, we pause for some lemmas.

**Lemma 4.8** Let \( \iota: \psi \to G \) be an inclusion of graphs, with \( G \) connected. Then the order of \( G \) is at least that of \( \psi \).

**Proof** Let \( G_\psi \) be \( G \) with the vertices of \( \iota(\psi) \) identified, and all \( \iota(\psi) \) edges discarded. Then \( G_\psi \) is connected, and so has order \( \geq -1 \); on the other hand, clearly the order of \( G_\psi \) is the order of \( G \) minus that of \( \psi \) minus 1 (for the vertex that is the identification of all \( \iota(\psi) \) vertices). Hence

\[
\text{ord}(G) = \text{ord}(G_\psi) + \text{ord}(\psi) + 1 \geq -1 + \text{ord}(\psi) + 1 = \text{ord}(\psi).
\]

\[\square\]

**Lemma 4.9** Let \( \iota: \psi \to G \) be an inclusion of pruned graphs. Then the order of \( G \) is at least that of \( \psi \), and \( G \)’s order is strictly greater than \( \psi \)’s if \( \iota(\psi) \) is strictly contained in some connected component of \( G \).
Proof A connected component of a graph that is pruned (and non-empty) has non-negative order. So we may assume \( \iota(\psi) \) meets every connected component of \( G \). It suffices to prove the case where \( \iota(\psi) \) meets one connected component of \( G \), i.e., the case where \( G \) is connected. But the lemma then follows by applying Lemma 4.8 to \( \psi \) and \( G \) or to \( \psi \) and \( G \setminus \{e\} \) (i.e., \( G \) with \( e \) removed), where \( e \) is an edge missed by \( \iota(\psi) \).

\[
\square
\]

Lemma 4.10 Let \( \iota_1, \iota_2 : \psi \to H \) be two inclusions of a tangle, \( \psi \), in a connected (labelled) graph, \( H \), such that \( \iota_1(\psi) \cup \iota_2(\psi) = H \). Assume that \( \iota_1(\psi) \neq H \). Then the order of \( H \) is greater than that of \( \psi \).

Proof Ignoring \( \iota_2 \), the preceding lemma applies to \( \iota_1 \) to immediately yield the lemma.

\[
\square
\]

Lemma 4.10 shows that

\[
\sum_{\vec{k}, \vec{m} \text{ not disjoint} \atop \vec{k}, \vec{m} \in R, \vec{k} \neq \vec{m}} \Pr \{ \iota_{\vec{m}} \cap \iota_{\vec{k}} \} = O(n^{-r-1}),
\]

since the summation can be broken down into a finite number of sums over tangles of order at least \( r + 1 \). Thus

\[
\sum_{\vec{m} \in R} \Pr \{ \iota_{\vec{m}} \} - \frac{1}{2} \sum_{\vec{k} \neq \vec{m} \atop \vec{k}, \vec{m} \in R} \Pr \{ \iota_{\vec{m}} \cap \iota_{\vec{k}} \}.
\]

\[
= n^{-r}/c + O(n^{-r-1}) - n^{-2r}/(2c^2) + O(n^{-2r-1}) + O(n^{-r-1}),
\]

which completes the proof of Theorem 4.7.

\[
\square
\]

It is easy to see that the above proof of Theorem 4.7 uses very little about the model of random graph, and therefore generalizes as follows.
Theorem 4.11 Let $K_n$ be a model of $d$-regular random graphs on $n$ vertices labelled $\{1, \ldots, n\}$, defined for some values of $n$. Further assume that (1) $K_n$ is invariant under renumbering $\{1, \ldots, n\}$, and (2) any tangle, $\psi$, has expected number of occurrences $n^{-r} + O(n^{-r-1})$ where $r$ is the order of $\psi$. Then Theorem 4.7 holds for $K_n$.

Theorem 4.12 For $G$ drawn from $H_{n,d}$ or $I_{n,d}$, we have that $\lambda_2(G) > 2\sqrt{d-1}$ with probability at least $n^{-s}/2 + O(n^{-s-1})$ where $s = \lfloor \sqrt{d-1} \rfloor$, except for when $d = 4$ in $H_{n,d}$, where we may take $s = 2$. The same holds for $J_{n,d}$ with probability $n^{-s} + O(n^{-s-1})$, where $s = \lfloor (\sqrt{d-1}+1)/2 \rfloor$ (and no exceptional values of $d$).

Proof For $J_{n,d}$, the tangle consisting of 1 vertex with a number of self-loops (in this case half-loops), proves the theorem for $J_{n,d}$ just as it did for $G_{n,d}$ in Theorem 2.10.

Consider the tangle, $\psi$, with two vertices and $m$ edges joining the two vertices labelled $\pi_1, \pi_2, \ldots$; $\psi$ is an $H_{n,d}$-tangle provided that $m \leq d/2$, and an $I_{n,d}$-tangle if $m \leq d$. $\lambda_{\text{Irred}}(\psi)$ is clearly $m-1$, $s = \text{ord}(\psi) = m-2$, and the automorphism group of $\psi$ is of order 2. So if $m-1 > \sqrt{d-1}$ and if $G$ contains this tangle, then we have $\lambda_2(G) > 2\sqrt{d-1}$ for $n$ sufficiently large. If we take $s = \lfloor \sqrt{d-1} \rfloor$, then $m-1 > \sqrt{d-1}$; we require $m \leq d$ for $I_{n,d}$, amounting to

$$\lfloor \sqrt{d-1} \rfloor + 2 \leq d,$$

which is satisfied for all $d \geq 3$. For $H_{n,d}$ we require

$$\lfloor \sqrt{d-1} \rfloor + 2 \leq d/2,$$

which is satisfied for all $d \geq 7$. Since $d$ is even and $\geq 4$ in $H_{n,d}$, we finish by examining the cases $d = 4$ and $d = 6$.

For $d = 4$ consider the tangle, $\psi$, with vertices $v_1, v_2, v_3, v_4$, edges labelled $\pi_1, \pi_2$ from $v_1$ to $v_2$, from $v_2$ to $v_3$, and from $v_3$ to $v_4$. Then $\text{ord}(\psi) = 2$ and $\lambda_{\text{Irred}}(\psi) > \lambda_{\text{Irred}}(\psi')$ where $\psi'$ is the subgraph of $\psi$ induced on $v_1, v_2, v_3$. But in the proof of Theorem 6.10 we compute $\lambda_{\text{Irred}}(\psi') = \sqrt{3}$. Hence $\psi$ is hypercritical of order 2, so we may take $s = 2$ in the theorem in the case of $d = 4$ and $H_{n,d}$.

For $d = 6$, the proof of Theorem 6.10 gives a tangle of order 2 with $\lambda_{\text{Irred}} > \sqrt{6}$ (see equation (23) and the discussion around it). So we may take $s = 2$ in our theorem when $d = 6$ in $H_{n,d}$.

$\square$
Walk Sums and New Types

In this section we give some general techniques that are used in estimating the expected values of all the various traces that are used in this paper. The main idea, originated in [BS87] and strengthened in [Fri91], is to group contributions to the trace in the following way. Consider the word, \( w \), and vector, \( \vec{t} \)

\[
\begin{align*}
  w &= (\sigma_1, \ldots, \sigma_{10}) = (\pi_2^{-1}, \pi_1, \pi_3, \pi_1^{-1}, \pi_3, \pi_1^{-1}, \pi_3, \pi_1^{-1}, \pi_2), \\
  \vec{t} &= (t_0, \ldots, t_{10}) = (5, 2, 4, 3, 7, 4, 3, 7, 4, 2, 5)
\end{align*}
\]

(see Figure 2). This represents a possible or potential irreducible cycle of length 10, from 5 to 2 along the edge labelled \( \pi_2^{-1} \), from 2 to 4 along \( \pi_1 \), etc. Graphically we depict this potential walk by the subgraph it traces out, called its “form” (see Figure 2). \((w; \vec{t})\) represents a possible contribution to \( \text{IrredTr}(A, 10) \). For any word, \( w \), of length \( k \) and vector, \( \vec{t} \), of length \( k + 1 \), let \( P(w; \vec{t}) \) be the probability that \( \sigma_i(t_{i-1}) = t_i \) for all \( i \), i.e., that the cycle does occur. Then the expected value of \( \text{IrredTr}(A, 10) \) is just a sum of appropriate \( P(w; \vec{t}) \)’s, what we will call a “walk sum.”

In the form of Figure 2, we see that the numbers of the vertices 5, 2, 4, 3, 7, are irrelevant, since our random graph model is “symmetric,” or invariant under renumbering the vertices. So we may replace the vertex 5 by an abstract symbol \( v_1 \), 2 by \( v_2 \), etc. (for reasons to made clear later, we do want to
remember the order in which the vertices were traversed); this replacement is not necessary, but makes clear that there is no particular preference for any numbering of the vertices.

In our example of Figure 2, \((w; \vec{t})\), a loop about the vertex 4 is traversed twice before we return to the starting vertex 2. Broder and Shamir realized that the other vertices, those of degree 2 that are not the starting vertex (here the vertices 2, 3, 7) are less interesting features of the “form.” By supressing these “less interesting” vertices we get the “type” of the form (see Figure 2); in [Fri91], walk sums were grouped by their form, and the sums for each form were grouped by the type of the form. A type is a graph with certain features, but when a form gives rise to a type then the edges of the type inherit \(\Pi^+\) labels from the form, and inherit “lengths,” which are the lengths of the \(\Pi^+\) labels (recall that \(\Pi^+\) is the set of nonempty words over \(\Pi\)). For example, the edge \((5, 4)\) of the type in Figure 2 inherits a length of 2 and a label of \(\pi^{-1}\pi_1\) from the form next to it; edge \((4, 4)\) inherits a length of 3 and a label of \(\pi_3\pi^{-1}_{\pi_3}\).

This paper introduces a “new type,” used in analyzing walk sums corresponding to selective traces. A new type is a type with some additional information, primarily fixing the lengths of certain type edges and requiring all other lengths to be sufficiently large.

5.1 Walk sums

By a potential \((k, n)\)-walk, \((w; \vec{t})\), we mean a pair of a word \(w = \sigma_1 \ldots \sigma_k\) of length \(k\) over \(\Pi\), and a vector, \(\vec{t} = (t_0, \ldots, t_k)\), with each \(t_i \in \{1, \ldots, n\}\); we sometimes refer to \(k\) as the length and \(n\) as the size of the potential walk. Given such a \((w; \vec{t})\), let \(\mathcal{E}(w; \vec{t})\) denote the event that the \(\pi_i\) are chosen so that \(\sigma_i(t_{i-1}) = t_i\) for all \(i = 1, \ldots, k\). Let \(P(w; \vec{t})\) denote the probability that \(\mathcal{E}(w; \vec{t})\) occurs.

A potential \((k, n)\)-cycle is a potential walk, \((w, \vec{t})\) as above with \(t_k = t_0\).

All our variants of traces, irreducible and not irreducible, selective and not, can be viewed as sums of \(P(w; \vec{t})\) over appropriate \(w\)’s and \(\vec{t}\)’s. In this subsection we formalize this notion and make preliminary remarks about such sums and asymptotic expansions.

**Definition 5.1** A walk collection, \(\mathcal{W} = \mathcal{W}(k, n)\), is a collection, for any two positive integers \(k\) and \(n\), of pairs \((w; \vec{t})\) as above, i.e. \(w\) is a word over \(\Pi\) of length \(k\), and \(\vec{t} = (t_0, \ldots, t_k)\) is a \(k + 1\) dimension vector over \(\{1, \ldots, n\}\).
The walk sum associated to \( \mathcal{W} \) is

\[
\text{WalkSum}(\mathcal{W}, k, n) = \sum_{(w, \vec{t}) \in \mathcal{W}(k, n)} P(w; \vec{t}).
\]

The main goal of this paper is to organize the various \((w, \vec{t})\) pairs into groups over which we can easily sum \(P(w; \vec{t})\). One simple organizational remark is that symmetries in the \((w, \vec{t})\) pairs often simplify matters. Specifically, given a permutation, \(\tau\), of \(\{1, \ldots, n\}\), and a vector \(\vec{t}\) as above, let

\[
\tau(\vec{t}) = (\tau(t_0), \ldots, \tau(t_k)).
\]

Say that \(\vec{s}\) and \(\vec{t}\) differ by a symmetry if \(\vec{s} = \tau(\vec{t})\) for some \(\tau\); in this case clearly \(P(w, \vec{s}) = P(w, \vec{t})\) for any word \(w\) of length \(k\). We use \(\vec{s} \sim \vec{t}\) to denote that \(\vec{s}\) and \(\vec{t}\) differ by a symmetry.

**Definition 5.2** A walk collection, \(\mathcal{W}\), is SSIIC if it is

1. symmetric, i.e. \((w, \vec{t}) \in \mathcal{W}(k, n)\) implies that \((w, \vec{s}) \in \mathcal{W}(k, n)\) for all \(\vec{s} \sim \vec{t}\),

2. size invariant, i.e. if \((w, \vec{t})\) is a potential \((k, n)\)-walk, then for any \(n' > n\), \((w, \vec{t}) \in \mathcal{W}(k, n)\) iff \((w, \vec{t}) \in \mathcal{W}(k, n')\),

3. irreducible, meaning that \((w, \vec{t}) \in \mathcal{W}\) implies that \(w\) is irreducible, and

4. cyclic, meaning that \((w, \vec{t}) \in \mathcal{W}\) implies that \(t_0 = t_k\).

The walk sums of interest here, namely traces that are irreducible or strongly irreducible and possibly selective, will all be SSIIC. We now make a series of remarks about walk sums that apply to all SSIIC walk sums; some of the remarks apply more generally.

**Definition 5.3** Given \(\vec{t}\) as above, i.e., \(\vec{t}\) is a positive integer valued vector, define the equivalence class of \(\vec{t}\) to be

\[
[\vec{t}] = \{\vec{s} \mid \vec{s} \sim \vec{t}\},
\]

i.e., the set of all positive integer valued vectors differing from \(\vec{t}\) by a symmetry, i.e., differing by a renumbering of the integer values. Define the \(n\)-th equivalence class of \(\vec{t}\)

\[
[\vec{t}]_n = \{\vec{s} \mid \vec{s} \sim \vec{t} \text{ and all } s_i \leq n\}
\]

(we may omit the \(n\) subscript if \(n\) is understood).
Let \( n \) be fixed, and set
\[
E_{\text{symm}}(w; \vec{t}) = E_{\text{symm}}(w; \vec{t})_n = \sum_{\vec{s} \in [\vec{t}]_n} P(w; \vec{s})
\]
\[
= n(n - 1) \cdots (n - v + 1) P(w; \vec{t}),
\]
where \( v \) is the number of distinct elements of \( \vec{t} \). A symmetric walk sum is just the sum of certain \( E_{\text{symm}}(w, \vec{t}) \)'s, and we can write
\[
\text{WalkSum}(W, k, n) = \sum_{(w; \vec{t}) \in W} E_{\text{symm}}(w; \vec{t}),
\]
where we understand that in the above sum we sum over one \( \vec{t} \) in each equivalence class.

Each \( \vec{t} \) of length \( k + 1 \) has an \( \vec{s} \sim \vec{t} \) where the size of \( \vec{s} \)'s entries are at most \( k + 1 \). So for each \( k \) there are a finite number of equivalence classes, \( \mathcal{W}(k) \), of \( (w; \vec{t}) \) such that \( w \) is of length \( k \) and \( \vec{t} \) is of some finite size. We refer to \( \mathcal{W}(k) \) as the set of potential walk classes of length \( k \) (or potential cycles classes when we restrict to those \( \vec{t} \)'s with \( t_k = t_0 \)). So if \( \mathcal{W} \) is size invariant we may write
\[
\text{WalkSum}(W, k, n) = \sum_{(w; \vec{t}) \in \mathcal{W}(k)} E_{\text{symm}}(w; \vec{t}),
\]
where the right-hand-side represents a finite sum (for fixed \( k \)).

Our next step is to comment about \( E_{\text{symm}}(w, \vec{t})_n \). Notice that if the conditions \( \sigma_i(t_{i-1}) = t_i \) involve determining \( a_j \) values of \( \pi_j \), then
\[
P(w; \vec{t}) = \prod_{i=1}^{d/2} \frac{1}{n(n - 1) \cdots (n - a_j + 1)} = \prod_{i=1}^{d/2} \frac{(n - a_j)!}{n!}.
\]
(14)

Let \( e = a_1 + \cdots + a_{d/2} \).

**Theorem 5.4** For any \( w, \vec{t} \) and any integer \( q \geq 0 \) we have
\[
E_{\text{symm}}(w, \vec{t})_n = n^{e - e} \left( p_0 + \frac{p_1}{n} + \cdots + \frac{p_q}{n^q} + \frac{\text{error}}{n^{q+1}} \right)
\]
(15)

where
\[
|\text{error}| \leq e^{(q+1)k/n} k^{2q+2},
\]
and the \( p_i = p_i(w; \vec{t}) \) are polynomials of degree \( \leq i \) in \( v, a_1, \ldots, a_{d/2} \).
The proof is contained between Lemma 2.7 and Corollary 2.10 of [Fri91]. We will need variants of this theorem, so we review its proof here. If
\[
g(x) = (1 - b_1x) \cdots (1 - b_rx)(1 - c_1x)^{-1} \cdots (1 - c_sx)^{-1},
\] (16)
where the \(b_i\) and \(c_j\) are positive constants, then \(g\)'s \(i\)-th derivative satisfies the bound
\[
|g^{(i)}(x)|/i! \leq (1 - xc_{\text{max}})^{-i} \left( \sum b_j + \sum c_j \right)^i
\]
where \(c_{\text{max}}\) is the maximum of the \(c_j\) (by equation (6) in [Fri91] on page 339). This estimate, using Taylor’s theorem, expanding in \(x = 1/n\) about \(x = 0\), gives the error term for Theorem 5.4 of
\[
e^{(q+1)k/n} \left( \sum b_j + \sum c_j \right)^{q+1},
\] (17)
In the case of Theorem 5.4, the \(\sum b_j\) represents the sum of 0, 1, \ldots, \(v - 1\), which is at most \(\binom{k}{2}\), and the \(\sum c_j\) represents the sum over \(j\) of all sums of 0, 1, \ldots, \(a_j - 1\), which is at most \(\binom{k}{2}\). Since \(2\binom{k}{2} \leq k^2\), we get an error term at most \(e^{(q+1)k/n}k^{2q+2}\). (See [Fri91] for details on anything in this paragraph.)

**Definition 5.5** Given a pair, \((w, \vec{t})\), as above, its order is \(e - v\), with \(e, v\) as above.

**Lemma 5.6** Given a word, \(w\), of length \(k\) over \(\Pi\), we have
\[
\sum_{\vec{t} \text{ such that } (w, \vec{t}) \text{ is of order } \geq r} P(w, \vec{t}) \leq n \left( \frac{k}{r+1} \right) \left( \frac{k}{n-k} \right)^{r+1},
\]
which for \(k \leq n/2\) is at most \(ck^{2r+2}n^{-r}\) for some constant \(c\) depending only on \(r\).

**Proof** See [Fri91], second displayed equation and discussion before on page 352 (this is the same idea used in the \(r = 1\) case proven in [BS87]). (The extra factor of \(n\) appears here but not in [Fri91], since we do not fix the initial vertex of the walk. Also note that the “order,” used here, is one less than the “number of coincidences,” used in [Fri91].)
Lemma 5.7 For any irreducible word, $w$, over $\Pi$, of length $k$, there are at most
\[ \sum_{j=0}^{r} \binom{k}{j} k^j \leq ck^{2r} \]
equivalence classes $[\vec{t}]$ whose order with $w$ is $\leq r - 1$.

Proof See the third displayed equation of page 352 of [Fri91] and the discussion preceding.

(The proofs of the two preceding lemmas, both based on page 352 of [Fri91], are easy consequences of the idea of “forced/fixed choices” and “coincidences” of a walk of [BS87] and [Fri91].)

Theorem 5.8 Let $W$ be SSIIC and let $r \geq 1$. Then for all $k \leq n/2$ we have
\[ \text{WalkSum}(W, k, n) = f_0(k) + \frac{f_1(k)}{n} + \cdots + \frac{f_{r-1}(k)}{n^{r-1}} + \frac{\text{error}}{n^r}, \]
where
\[ f_i(k) = \sum_{j=0}^{r-1} \left( \sum_{(w;[\vec{t}]) \text{ order } j \in W(k)} p_{i-j}(w; [\vec{t}]) \right) \]
(with $p_i$ as in equation (15)) and where for some $c$ depending only on $r$,
\[ |\text{error}| \leq ck^{4r}(d - 1)^k. \]

Proof By Lemma 5.6, we introduce an error of at most $ck^{2r+2}n^{-r}$ per word by ignoring potential walks of order $\geq r$. Each word, $w$, has at most $ck^{2r}$ associated potential walk classes of order $\leq r - 1$ (by Lemma 5.7), and truncating the associated asymptotic expansion, as in equation (15), of each associated potential walk class results in an error of at most $ck^{2r}$ (by Theorem 5.4). So each word, $w$, of length $k$ involved in $W$ contributes an error of at most $ck^{4r}$, and there are at most $d(d-1)^{k-1}$ such words (for $k \geq 1$) since $W$ consists of only irreducible words.

\[ \square \]
5.2 The Loop

Here we analyze walk sums associated with simple loops. This gives some ideas and a lemma to be used in Section 8.

5.2.1 The Singly Traversed Simple Loop

Let \( w \) be an irreducible word over \( \Pi \) of length \( k \) and \( \vec{t} \) a \( k + 1 \) tuple over \( \{1, \ldots, n\} \) as in the previous subsection. If \( t_0 = t_k \) and the \( t_i \)'s are otherwise distinct, we say \( E(w; \vec{t}) \) (i.e., the event that \( \sigma_i(t_{i-1}) = t_i \) for all \( i \)) is a \textit{singly traversed simple loop} or STSL for short; we define \( W_{\text{STSL}} \) to be the collection of all STSL’s. When \( E(w; \vec{t}) \) occurs and \( (w; \vec{t}) \in W_{\text{STSL}} \), the walk (or, more precisely, cycle) from \( t_0 \) following \( w \) traces out a “simple loop” once, that begins and ends at \( t_0 \), moving through distinct edges and vertices throughout the cycle.

Clearly \( E_{\text{STSL}} \) is SSIIC, so according to Theorem 5.8 we have an asymptotic expansion in \( 1/n \) with coefficients \( f_i(k) \) for the associated walk sum. We now briefly indicate why the \( f_i \) are \( d \)-Ramanujan. This is a mildly tedious exercise, covered (in much greater generality) in [Fri91]. We quote the main points here.

First, note that there is exactly one equivalence class \([\vec{t}]\) of \( \vec{t} \)'s that appear in \( W_{\text{STSL}} \). So we may write \( E_{\text{symm}}(w) \) for \( E_{\text{symm}}(w; \vec{t}) \) for any \( \vec{t} \) of the equivalence class, and we may write \( p_i(w) \) for the \( p_i(w; \vec{t}) \) in Theorem 5.8 or 5.4 (note also that \( v = k \) in the notation of Theorem 5.4).

Let for \( \sigma, \tau \in \Pi \), let \( \text{Irred}_{k,\sigma,\tau} \) denote the irreducible words of length \( k \) beginning with \( \sigma \) and ending with \( \tau \).

**Lemma 5.9** Let \( p = p(a_1, \ldots, a_{d/2}, k) \) be a polynomial. For every \( \sigma, \tau \) there are polynomials \( Q_1, Q_2, Q_3 \) of \( k \) of degree at most the degree of \( p \) such that

\[
\sum_{w \in \text{Irred}_{k,\sigma,\tau}} p(a_1(w), \ldots, a_{d/2}(w), k) = (d - 1)^k Q_1(k) + (-1)^k Q_2(k) + Q_3(k).
\]

**Proof** This is immediate from Lemma 2.11 of [Fri91] or Corollary 2.12 (note that our \( d \) is \( 2d \) in [Fri91]).

\[\Box\]

Note that the formula for the above lemma comes about from the fact that \( d - 1, -1, 1 \) are the eigenvalues of \( G_{\text{Irred}} \) where \( G \) is the graph with one vertex and \( d/2 \) whole-loops.
Corollary 5.10 We have each $f_i(k)$ as in Theorem 5.8 for $W = W_{STSL}$ is $d$-Ramanujan.

Proof We have

$$f_i(k) = \sum_{\sigma \neq \tau^{-1}} \sum_{\text{Irred}_{k,\sigma,r}} p_i(a_1(w), \ldots, a_{d/2}(w), k),$$

where the $p_i$ are the polynomials given in Theorem 5.8 or 5.4. The result now follows from Lemma 5.9.

5.2.2 Simple Loops

Consider any irreducible $(w, \vec{t})$ that traces out a simple loop, i.e., the vertices and edges visited form one cycle, but now we don’t require that the loop is traversed only once. Corresponding to this geometric picture of a simple loop we can form the associated walk collection of simple loop cycles; such cycles, being irreducible, must traverse the loop traced out some number of times.

So let $W_{SL}$ be the set of $(w, \vec{t})$ pairs with

1. $w = \sigma_1 \ldots \sigma_k$ irreducible,
2. $\sigma_1 \neq \sigma_k^{-1}$,
3. $t_0, \ldots, t_{r-1}$ distinct for some $r$ dividing $s$,
4. $t_{i+r} = t_i$ for $0 \leq i \leq k - r$, and
5. $w = u^s$ for some word $u$ with $rs = k$.

$W_{SL}$ is the walk collection of simple loop walks.

Clearly we have

$$\text{WalkSum}(W_{SL}, k, n) = \sum_{s|k} \text{WalkSum}(W_{STSL}, s, n). \quad (18)$$

We easily conclude the following theorem.

Theorem 5.11 The $f_i(k)$ corresponding to $W_{SL}$ are $d$-Ramanujan, and have the same principle term as the $f_i(k)$ corresponding to $W_{STSL}$. 

43
Proof  By equation (18),
\[ |\text{WalkSum}(W_{SL}, k, n) - \text{WalkSum}(W_{STSL}, k, n)| \]
\[ \leq \sum_{s \leq (k/2)} \text{WalkSum}(W_{STSL}, s, n) \leq ck^c(d - 1)^{k/2}. \]
But \( \text{WalkSum}(W_{STSL}, k, n) \) is \( d \)-Ramanujan.

\[ \square \]

5.3 Forms, Types, and New Types

In this subsection we will classify potential walks, \((w; \vec{t})\), or more generally potential walk classes, according to some characteristics of the subgraph that the walk traces out.

Definition 5.12 A form, \( \Gamma \), is an oriented, \( \Pi \)-labelled graph, \( G_\Gamma = (V_\Gamma, E_\Gamma) \), with edges and vertices numbered. A specialization of a form, \( \Gamma \), is an injection \( \iota: V_\Gamma \rightarrow \{1, \ldots, n\} \).

With each potential walk, \((w; \vec{t})\), we associate a form, \( \Gamma = \Gamma(w; \vec{t}) \) with a specialization \( \iota \), as follows:

1. Set \( V_\Gamma = \{v_1, \ldots, v_r\} \) to be any numbered \((v_i \text{ numbered } i)\) set of size \( r \), where \( r = |V_\Gamma| \) is the number of distinct elements among the \( t_i \) (where \( \vec{t} = (t_0, \ldots, t_k) \))

2. \( \iota(v_i) \) is the \( i \)-th distinct element of the sequence \( t_0, t_1, \ldots, t_k \),

3. Set \( E_\Gamma = \{e_1, \ldots, e_m\} \) to be any numbered set of size \( m \) \((e_i \text{ numbered } i)\), where \( m = |E_\Gamma| \) is the number of distinct triples \( \{(\sigma, t_{i-1}, t_i)\}_{i=1,2,\ldots,k} \), where we identify a triple \( (\sigma, s, t, s) \) with \( (\sigma^{-1}, t, s) \),

4. if \((\sigma_j, t_{j-1}, t_j)\) is the \( r \)-th distinct tuple in \( \{(\sigma, t_{i-1}, t_i)\}_{i=1,2,\ldots,k} \) (with the previous identification), then \( s(e_r) = (\sigma_j, t^{-1}(t_{j-1}), t^{-1}(t_j)) \) defines the structural map (see Definition 3.1) of \( G_\Gamma \).

(see the example in Figure 2 explained at the beginning of this section).

In other words, the form is the subgraph traced out by \((w, \vec{t})\), with some additional information (we remember the order in which the vertices and
edges are visited, and the direction each edge is first traversed). We say that forms $\Gamma_1$ and $\Gamma_2$ are isomorphic if they are isomorphic as oriented, numbered, $\Pi$-labelled graphs. Because of the numbering, there is at most one isomorphism between any two forms (or a form and itself). We say that $(w, \vec{t})$ is of form $\Gamma$ or associated to $\Gamma$, written $(w, \vec{t}) \in \Gamma$, if one (or any) of the forms associated to $(w, \vec{t})$ is isomorphic to $\Gamma$. Given $(w, \vec{t})$, there is always an associated form, $\Gamma$, with $V_\Gamma = \{t_0, \ldots, t_k\}$ and associated specialization, $\iota$, being the identity; however, we usually view $V_\Gamma$ as any numbered set of the right size, since all of our random graph models are symmetric.

If $(w, \vec{t}) \in \Gamma$, then define

$$E[\Gamma]_n = E_{\text{symm}}(w, \vec{t})_n$$

depends only on $\Gamma$, and not on the particular $(w, \vec{t})$ to which $\Gamma$ is associated; indeed,

$$E[\Gamma]_n = \frac{n!}{(n-v)!} \prod_{i=1}^{d/2} \frac{(n-a_i)!}{n!},$$

where $v = |V_\Gamma|$, and $a_i$ is the number of edges in $\Gamma$ labelled with $\pi_i$ and $\pi_i^{-1}$.

Hence, if $\mathcal{W}$ is symmetric and size invariant, we may write

$$\text{WalkSum}(\mathcal{W}, k, n) = \sum_{\Gamma} W_{\Gamma}(\mathcal{W}, k) E[\Gamma]_n,$$

where $W_{\Gamma}(\mathcal{W}, k)$ is the number of potential walk classes in $\mathcal{W}(k)$ associated to $\Gamma$.

**Definition 5.13** A legal walk in a form, $\Gamma$, is a walk starting in $v_1$ that visits all the vertices of $G_\Gamma$ in order (of their numbering), all the edges in order, and any edge is first traversed in the direction of its orientation. Each legal walk of length $k$ generates a walk class in the natural way.

The following easy proposition is worth stating formally.

**Proposition 5.14** $W_{\Gamma}(\mathcal{W}, k)$ is the number of legal walks on $\Gamma$ of length $k$.

**Proof** A potential walk $(w; \vec{t})$, to which is associated a form, $\Gamma$, and a specialization, $\iota$, determines a legal walk, $W = W(w; \vec{t})$, on $G_\Gamma$ by applying $\iota^{-1}$ to $\vec{t}$. Clearly $W = W(w; \vec{t})$ depends only on the equivalence class of $\vec{t}$. Conversely, any legal walk, $W$, traces out a word, $w = w(W)$, via the
edge labelling, and any injection \( \iota: V_\Gamma \to \{1, 2, \ldots\} \) determines a vector, \( \vec{t} = \vec{t}(W, \iota) \); as \( \iota \) varies, \( \vec{t}(W, \iota) \) varies over an equivalence class. So we have an association, \((w, [\vec{t}]) \mapsto W(w, [\vec{t}])\) and an association, \(W \mapsto (w(W), [\vec{t}](W))\), from a potential class to a legal walk and vice versa, that are clearly inverses of each other.

\[ \square \]

**Definition 5.15** The order of a form, \( \Gamma \), is \( \text{ord}(\Gamma) = |E_\Gamma| - |V_\Gamma| \). (It equals the order of any potential walk to which it is associated.)

Next we group the forms together by their “type.” Before doing so, we recall that for a form \( \Gamma = \Gamma(w; \vec{t}) \) with specialization, \( \iota \), the potential walk \( (w; \vec{t}) \) pulls back under \( \iota^{-1} \) to a walk on \( G_\Gamma \); we remark that the vertex and edge numberings of \( \Gamma \) serve to remember in which order the vertices and edges were traversed in the walk.

**Definition 5.16** A type, \( T \), is a connected, oriented graph \( G_T = (V_T, E_T) \), with vertex and edge numberings such that all vertices except possibly the first one are of degree \( \geq 3 \). A labelling of a type means a \( \Pi^+ \)-labelling (recall \( \Pi^+ \) is the set of words over \( \Pi \) of length at least 1).

To each form, \( \Gamma \), we associate a type, \( T = T(\Gamma) \), as follows. Let \( W \) be the set of beads of \( G_\Gamma \) numbered greater than 1. We claim \( W \) cannot contain a cycle, for then this cycle would be disconnected from the vertex numbered 1. So we may form the supression of \( W \) in \( G_\Gamma \). This supression inherits a vertex and edge numbering from \( G_\Gamma \) (ordering a vertex before another in the supression if it is numbered less in \( G_\Gamma \), and ordering edges in the supression by any associated edge in \( E_\Gamma \)). Of course, the \( \Pi \)-labelling of \( \Gamma \) gives rise to a \( \Pi^+ \)-labelling of \( G_T(\Gamma) \).

In other words, the beads of a form are “less important” features, and the type is just the form with these “less important” features supressed.

A \( \Pi^+ \)-labelled type uniquely determines a form, and vice versa. We wish to group together forms corresponding to one type (inducing different \( \Pi^+ \)-labellings), the prototypical example being STSL’s or SL’s discussed earlier in this section. To do this it will be helpful to remember a small part of the labelling, namely the starting and ending letter of each \( \Pi^+ \)-label.
Definition 5.17 A lettering of a type, $T$, is the assignment to each directed edge a starting letter in $\Pi$ and an ending letter in $\Pi$ (such that opposite directed edges are lettered with the starting letter of one being the inverse of the ending letter of the other). Given a form, $\Gamma$, with $T = T(\Gamma)$, or equivalently a $\Pi^+$-labelling of $T$, the associated lettering assigns to each edge the starting and ending letter of the $\Pi^+$-label assigned to it in its orientation.

Definition 5.18 A $B$-new type is a collection, $\tilde{T} = (T; E_{\text{long}}, E_{\text{fixed}}; k_{\text{fixed}})$, of (1) a lettered type, $T$, (2) a partition of $E_T$ into two sets, $E_{\text{long}}, E_{\text{fixed}}$, (3) for each $e_i \in E_{\text{fixed}}$ an edge length, $k_i^{\text{fixed}}$, with $0 < k_i^{\text{fixed}} < B$, and (4) a $\Pi^+$-labelling of $E_{\text{fixed}}$ with each $e_i \in E_{\text{fixed}}$ labelled with a word of length $k_i^{\text{fixed}}$. A $\Pi^+$-labelling of $T$ (or, equivalently, a form, $\Gamma$), is said to be of $B$-new type $\tilde{T}$ if each $E_{\text{long}}$ label is of length $\geq B$, and each label corresponding to $e_i \in E_{\text{fixed}}$ is of length $k_i^{\text{fixed}}$ and agrees with the label specified by $\tilde{T}$. $\tilde{T}$ is said to be based on $T$.

Theorem 5.19 For each $r > 0$ there are finitely many types of order $\leq r$. For each type, $T$, and each $B > 0$ there are finitely many $B$-new types based on $T$.

Proof The first statement is just Lemma 2.3 of [Fri91], except that “coincidence” is used instead of “order” (and the coincidence is the order plus one). The second statement is clear since there are finitely many (1) letterings, (2) partitions of $E_T$, (3) choices of $k_i^{\text{fixed}}$ with $e_i \in E_{\text{fixed}}$, and (4) labellings of each $E_{\text{fixed}}$ edge, $e_i$, with a length $k_i^{\text{fixed}} < B$.

6 The Selective Trace

In this section we define a selective trace, and discuss some of its properties.

6.1 The General Selective Trace

Fix a graph, $G = (V, E)$, coming from $G_{n,d}$, so that $V = \{1, \ldots, n\}$ and $G$ is $\Pi$-labelled.
By a path\textsuperscript{14} of length $k$ in $G$ we shall mean a vertex, $v \in V$, and a word of length $k$, $w = \sigma_1 \cdots \sigma_k$, over $\Pi$ (i.e., with each $\sigma_i \in \Pi$). Such a path determines a subgraph in $G$ of those vertices and edges traversed. We say a path traverses a tangle, $\psi$, if the subgraph traversed by the path contains the tangle, $\psi$.

**Definition 6.1** Let $\Psi = \{\psi_1, \ldots\}$ be a (finite or infinite) collection of tangles. For positive integer, $S$, the set of $(S, \Psi)$-selective cycles (respectively, walks) are those irreducible cycles (respectively, walks) that have no subpath of length $\leq S$ that traverses a tangle in $\Psi$. The $k$-th irreducible $(S, \Psi)$-selective trace of $G$, $\text{IrSelTr}_{S,\Psi}(G; k)$, is the number of $(S, \Psi)$-selective cycles of length $k$.

Intuitively, the selective trace modifies the standard irreducible trace on those graphs that have a tangle in $\Psi$, and avoids those cycles that in some short part trace out such a tangle.

### 6.2 A Lemma on Selective Walks

What is the point of the selective trace? We can answer this question in two ways. First, since hypercritical tangles give large eigenvalues, any trace with an arbitrarily long asymptotic expansion in $1/n$ with $d$-Ramanujan coefficients must avoid hypercritical tangles (according to Theorems 3.11 and 4.2); a trace must be selective or its asymptotic expansion coefficients will not all be $d$-Ramanujan. Second, there is a crucial technical theorem, Theorem 6.6, about counting irreducible contributions to a selective trace. This lemma makes certain infinite sums converge for the selective trace that would have to diverge for the standard trace— for example, the infinite sum involving $W(T; \vec{m})$ and $P_{i,T,\vec{m}}$ just above the middle of page 351 in [Fri91], for types of order $> d$; for the same reason, this crucial theorem makes the $1/n$ expansion for a selective trace have $d$-Ramanujan coefficients when they don’t for a trace that is not selective— indeed, the $(2d-1)^{k/2}$ bound in equation (24) of [Fri91] depends critically on $2i+2 \leq \sqrt{2d-1}$, and this equation corresponds

\textsuperscript{14}By a path one often means a sequence of vertices. In case there are multiple edges in the graph, one needs to note also which edge is traversed. Finally, in the case of whole-loops in an undirected graph, one needs to remember in which “direction” each whole-loop is being traversed. In the present situation, all the above information is contained simply in the initial vertex and the permutations, $\pi_i$ or $\pi_i^{-1}$, being taken on each step of the path.
to the error in the $n^{-i}$ term in the expansion of the expected value of the irreducible trace (recall that $2d$ in [Fri91] corresponds to our $d$). We shall finish this subsection with the crucial technical theorem, Theorem 6.6, after setting up the necessary terminology.

A relabelling of a tangle, $\psi$, is a tangle, $\psi'$, that differs from $\psi$ only in its edge labels.

**Definition 6.2** A set, $\Psi$, of tangles is called closed under pruning (respectively, relabelling) if $\psi \in \Psi$ implies $\psi' \in \Psi$ for any pruning (respectively, relabelling), $\psi'$, of $\psi$.

Note: In the definition above, $\psi$ and $\psi'$ must be $G_{n,d}$-tangle (or tangles in whatever model is discussed)— a vertex with two self-loops labelled both labelled $\pi_1$ is not a $G_{n,d}$-tangles and is therefore not considered a relabelling of the tangle where the self-loops are labelled $\pi_1$ and $\pi_2$.

**Definition 6.3** For a positive integer, $\tau$, let $\Psi_{\text{ord}}(\tau)$ be the set of tangles whose order is $\geq \tau$. For positive integers $\tau_1 \leq \tau_2$, let $\Psi_{\text{ord}}(\tau_1, \tau_2)$ be the set of all tangles whose order is $\geq \tau_1$ and $\leq \tau_2$. We also write $\Psi_{\text{ord}}(\tau, \infty)$ for $\Psi_{\text{ord}}(\tau)$.

Since pruning a tangle does not affect its order, $\Psi_{\text{ord}}(\tau_1, \tau_2)$ is closed under pruning; clearly $\Psi_{\text{ord}}(\tau_1, \tau_2)$ is closed under relabelling.

Consider a form, $\Gamma$, of type $T$, in which $T$’s edges, $e_i$, have length $k_i$ (as beaded paths arising from $\Gamma$). For each $e_i \in E_T$ fix an integer $m_i \geq 1$. Let $\Psi$ be a set of tangles closed under relabelling. Let $W_\Gamma(\bar{m}; S, \Psi)$ be the number of legal cycles (in particular, beginning at the first vertex) in $\Gamma$ that traverse each $e_i$ exactly $m_i$ times (in either direction) and that are $(S, \Psi)$-selective. Since $\Psi$ is closed under relabelling, $W_\Gamma$ depends only on the length, $k_i$, of $e_i$ in $\Gamma$, not on the particular $k_i$ length $\Pi^+$ labels. So we may write

$$W_\Gamma(\bar{m}; S, \Psi) = W_T(\bar{m}, \bar{k}; S, \Psi).$$

In other words, for any $\Pi^+$-labelling of $T$, the number of such walk classes with a given $k_i$ and $m_i$ for $T$, $W_T(\bar{m}, \bar{k}; S, \Psi)$, that are irreducible cylces that are $(S, \Psi)$ selective, is independent of the labelling.

Now given the above setting, call an edge, $e_i$, of $T$ long if $k_i > S$, and short otherwise. If a walk contains some $\psi \in \Psi$ in any consecutive $S$ steps, then by possibly pruning these consecutive steps along long edges at the
beginning and end, we get a consecutive walk over short edges that contains a pruning of $\psi$. In particular, if $B > S$ and $\tilde{T} = (T; E_{\text{long}}, E_{\text{fixed}}; \vec{k}_{\text{fixed}})$ is a $B$-new type based on $T$, then $W_T$ depends on only $\tilde{T}$, $\vec{m}$, $S$, and $\Psi$ provided that $k_i = k_{i,\text{fixed}}$; hence we may write

$$W_T(\vec{m}, k; S, \Psi) = W_{\tilde{T}}(\vec{m}; S, \Psi).$$

**Definition 6.4** We say that a collection of tangles, $\Psi$, is $r$-supercritical if it contains all supercritical tangles of order $\leq r$.

Next we give two examples of very natural $r$-supercritical tangle sets.

**Definition 6.5** Let $\tau_{\text{fund}}$ be the smallest order of a supercritical tangle, and let $\Psi_{\text{fund}} = \Psi(\tau_{\text{fund}})$. Let $\Psi_{\text{eig}}$ be the set of all supercritical tangles.

$\Psi_{\text{fund}}$ and $\Psi_{\text{eig}}$ are clearly $r$-supercritical for any $r$; $\Psi_{\text{ord}}(\tau_{\text{fund}}, r)$ and $\Psi_{\text{eig}}[r] = \Psi_{\text{eig}} \cap \Psi_{\text{ord}}(\tau_{\text{fund}}, r)$ are also clearly $r$-supercritical. We arrive at our crucial technical theorem, that is the key to the selective trace.

**Theorem 6.6** Let $T$ by any type, with specified edge set partition $E_{\text{long}}, E_{\text{fixed}}$, and a $\Pi$-lettering specified. Let the edges be indexed so that

$$E_{\text{long}} = \{e_1, \ldots, e_t\}, \quad E_{\text{fixed}} = \{e_{t+1}, \ldots, e_b\}.$$

Then there is a $c$, $\epsilon > 0$, and an $S_0$ such that the following is true for all $S \geq S_0$. Let $\Psi$ be a set of tangles containing all supercritical tangles included in a form of type $T$; e.g., by Lemma 4.8 we may take $\Psi$ to be any $r$-supercritical set for $r = \text{ord}(T)$. Let

$$W_{\tilde{T}, S}(M_1, M_2) = \sum_{\substack{m_1 + \ldots + m_t = M_1 \\ m_{t+1} + \ldots + m_b = M_2 \\ m_i \geq 1}} W_{\tilde{T}}(\vec{m}; S, \Psi),$$

for a $B$-new type, $\tilde{T}$, with $B > S$ and with $\tilde{T}$ having edge set partition $E_{\text{long}}, E_{\text{fixed}}$. Then

$$W_{\tilde{T}, S}(M_1, M_2) \leq cB(\sqrt{d - 1} - \epsilon)^{CM_1 + M_2}.$$
Proof First consider, \( \rho_1 = \rho_1(k) \), the maximum number of \((S, \Psi)\)-selective irreducible walks of length \( k \) in \( T \) there are from any given vertex through only \( E_{\text{fixed}} \) edges, for any \( S \geq k \), where each \( E_{\text{fixed}} \) edge is, for now, taken to be of unit length. Such a walk traces out a tangle, \( \psi \), and this tangle must have \( \lambda_{\text{Irred}} < \sqrt{d - 1} \). This tangle, \( \psi \), is also a subgraph of \( T \) (since each edge is of length one, for now). So such a walk is contained in one of a finite number of subgraphs, \( T' \), of \( T \), each with \( \lambda_{\text{Irred}}(T') < \sqrt{d - 1} \) (since the walk is entirely in \( E_{\text{fixed}} \), and all \( E_{\text{fixed}} \)'s edges are taken to be of length 1). For each \( T' \) there is a \( c \) such that the number of \( E_{\text{fixed}} \) walks of length \( k \) there is

\[
\leq c k^c (\lambda_{\text{Irred}}(T'))^k,
\]

by considering the Jordan form of \( A_{T'} \); for any \( \eta > 0 \) this number is

\[
\leq c' (\lambda_{\text{Irred}}(T') + \eta)^k,
\]

for all \( k \), for sufficiently large \( c' \). Hence taking \( \epsilon > 0 \) to be any number such that \( \lambda_{\text{Irred}} < \sqrt{d - 1} - \epsilon \) for these finitely many graphs, we conclude that there is a \( c \) such that

\[
\rho_1(k) \leq c(\sqrt{d - 1} - \epsilon)^k.
\]

Next let \( T' \) be \( T \) with each \( e_i E_{\text{fixed}} \) is subdivided into \( k_{i_{\text{fixed}}} \) edges, and consider \( \rho_2(k) \), the maximum number of \((S, \Psi)\)-selective irreducible walks, \( \omega \), in \( T' \), through only \( E_{\text{fixed}} \) edges. Each such walk, \( \omega \), in \( T' \), gives rise to a path, \( \omega' \), in \( T \), by possibly extending \( \omega \) at the beginning and end to an irreducible \( T' \) walk until the walk hits a \( V_T \) vertex. The length of \( \omega' \) (in \( T \)) is at most \( k \). Each \( \omega' \) walk can come from at most four \( \omega \)'s (the four arises from an ambiguity at the beginning and end of \( \omega' \) with subdivided self-loops in \( E_{\text{fixed}} \)). Hence

\[
\rho_2(k) \leq 4c \sum_{j=1}^{k} (\sqrt{d - 1} - \epsilon)^j \leq c' (\sqrt{d - 1} - \epsilon)^k.
\]

Finally consider a \( B \)-new type, \( \tilde{T} \), under the assumption \( B > S \geq k \). Let \( G \) be a VLG with each \( E_{\text{long}} \) edge of length \( \geq R + 1 \) (for some parameter \( R \geq S \) to be specified shortly), and \( E_{\text{fixed}} \) edges subdivided according to their \( \tilde{T} \) lengths. (Any form, \( \Gamma \), corresponding to \( \tilde{T} \) has each \( E_{\text{long}} \) \( \Pi^+ \)-label of length \( \geq R + 1 \) provided that \( R \leq S \), since \( S < B \)). The row sum of \( Z^G_k(z) \) represents how many walks there are originating from a given vertex.
of various lengths. Such a row sum consists of a sum over \( j \) of representations of walks with \( k - j \) steps in \( E_{\text{fixed}} \) and \( j \) steps in \( E_{\text{long}} \); there are \( \binom{k}{j} \) ways of choosing the \( j \) steps in \( E_{\text{long}} \), and the contribution from these \( j \) steps is \( \leq z^{Rj} \); the \( k - j \) steps in \( E_{\text{fixed}} \) occur in at most \( j + 1 \) contiguous \( E_{\text{fixed}} \) walks for length \( q_1, \ldots, q_r \) with \( r \leq j + 1 \), and the \( i \)-th contiguous walk contributes at most

\[
c(\sqrt{d-1} - \epsilon)^{q_i} z^{q_i},
\]

for a total contribution, assuming \( c \geq 1 \), of at most

\[
c^{j+1}(\sqrt{d-1} - \epsilon)^{k-j} z^{k-j}.
\]

The total contribution is therefore at most

\[
c \sum_{j=0}^{k} \binom{k}{j} c^{j}(\sqrt{d-1} - \epsilon)^{k-j} z^{k+Rj} = c(\sqrt{d-1} - \epsilon + cz^R)^{k} z^{k}.
\]

(19)

Now choose \( R \) large enough so that

\[
\sqrt{d-1} - \epsilon + c(\sqrt{d-1} - \epsilon)^{-R} \leq \sqrt{d-1} - (\epsilon/2).
\]

Then for \( z \leq (\sqrt{d-1} - (\epsilon/4))^{-1} \) we have

\[
(\sqrt{d-1} - \epsilon + cz^R)^{k} z^{k} \leq (1 - \epsilon')^{k}
\]

for some \( \epsilon' > 0 \). So choose \( k \) large enough so that \( c(1 - \epsilon')^{k} < 1 \). From equation (19), each row sum of \( Z_k^G(z) \) is then \( < 1 \) for all \( z \leq (\sqrt{d-1} - (\epsilon/4))^{-1} \). Hence \( \lambda_1(A_G) \leq \sqrt{d-1} - (\epsilon/4) \).

It follows that for \( S_0 = \max(k, R) \) we have that for any \( S \geq S_0 \) we have

\[
W_{T,S}(M_1, M_2) \leq (\sqrt{d-1} - (\epsilon/4))^{(R+1)M_1+M_2} |V_G|;
\]

\(|V_G|\), the number of vertices in the subdivided form of \( G \), is at most the number of vertices in the type plus \( B - 2 \) times the number of \( E_{\text{fixed}} \) edges plus \( R \) times the number of \( E_{\text{long}} \) edges. In total, \(|V_G|\) is at most a constant times \( B \).

\( \square \)
6.3 Determining $\tau_{fund}$

In order to use the selective trace, we must determine $\tau_{fund}$. We begin by doing so for the model $G_{n,d}$, and then we use similar techniques for the models $H_{n,d}$, $I_{n,d}$, and $J_{n,d}$.

More generally, for a given $\tau$, consider the task of finding the tangle, $\psi$, in $G_{n,d}$, of order $\leq \tau$ with $\lambda_{irred}(\psi)$ as large as possible. To simplify this task, notice that pruning leaves the order and $\lambda_{irred}$ invariant; hence we may restrict our search to those $\psi$'s that are completely pruned.

**Lemma 6.7** Let $G$ be a graph with edge $e = \{u,v\}$ with $u \neq v$. Let $G_e$ be the contraction of $G$ along $e$, i.e. the graph obtained by discarding $e$ and identifying $u$ with $v$. Then $\lambda_{irred}(G) \leq \lambda_{irred}(G_e)$.

**Proof** Consider an irreducible cycle, $c$, about $u$ in $G$. Then we can associate to this cycle one in $G_e$, $\iota(c)$, by discarding all occurrences of $e$. This association, $\iota$, is an injection, since given a $G_e$ irreducible cycle about $u$ of the form $\iota(c)$, we can infer when $e$ was taken (since $e = \{u,v\}$ with $u \neq v$) in the $G$ cycle, giving rise to (at most) a single $G$ cycle. Since this injection does not increase the length of the cycles, we conclude that the number of irreducible cycles about $u$ in $G$ of length $\leq k$ is no more than the number in $G_e$. Hence the conclusion of the lemma.

Since edge contraction reduces the number of vertices and of edges by one each, edge contraction leaves the order invariant. So in looking for a $\lambda_{irred}$ tangle of a given order, we may always assume the tangle is completely pruned and cannot be edge contracted (and remain a tangle).

We now claim (by Lemma 6.7) that for $G_{n,d}$ and $\tau \leq (d/2) - 1$, a loop with $\tau + 1$ whole-loops has the largest $\lambda_{irred}$ of all tangles of order $\tau$. For this graph we clearly have $\lambda_{irred} = 2\tau + 1$; hence $\tau_{fund}$ is the smallest integer, $\tau$ with $2\tau + 1 \geq \sqrt{d - 1}$, provided that this $\tau$ is $\leq (d/2) - 1$. But we easily verify that this $\tau$,

$$\tau_{fund} = \lceil (\sqrt{d-1} + 1)/2 \rceil - 1 = \lceil (\sqrt{d-1} - 1)/2 \rceil$$

is indeed $\leq (d/2) - 1$ for all $d \geq 4$. We have just established the following theorem.
Theorem 6.8 For the model $G_{n,d}$, we have $	au_{\text{fund}} = \lceil (\sqrt{d-1} + 1)/2 \rceil - 1$.

For $H_{n,d}$ we have to remember that tangles can’t have self-loops. Thus contractions can only be done along non-multiple edges, and $\tau_{\text{fund}}$ will not generally be the same for $H_{n,d}$ and $G_{n,d}$.

Lemma 6.9 Let $u,v$ be vertices of distance two in a graph, $G$, i.e., there are no edges joining $u$ and $v$, but there is a $w$ with edges to each of $u,v$. Let $G'$ be the graph obtained by identifying $u$ and $v$ and deleting one of the edges from $w$ to $u$ (or to $v$) (so that the order of $G'$ is the same as that of $G$). Then $\lambda_{\text{Irred}}(G) \leq \lambda_{\text{Irred}}(G')$.

Proof Let $U$ be the vertex in $G'$ which is the identification of $v$ and $u$. Let the edges from $u$ to $w$ be enumerated $e_1, \ldots, e_s$, and those from $v$ to $w$ enumerated $f_1, \ldots, f_t$. The edges from $U$ to $w$ are $g_1, \ldots, g_r$, where $r = s + t - 1$.

First consider the case when $V_G = \{u,v,w\}$, and consider the irreducible cycles about $w$ (which are necessary of even length). Such a cycle begins in $w$ and takes two steps, visiting either $u$ or $v$, in, respectively, $s(s-1)$ or $t(t-1)$ ways. After coming back from a $u$ vertex, another step of length 2 can either (1) visit a $u$ vertex, in $(s-1)^2$ ways, or (2) visit a $v$ vertex, in $t(t-1)$ ways; similarly for coming back from a $v$ vertex. Thus “coming back from a $u$ vertex” and “coming back from a $v$ vertex” forms a Markov chain, and the total number of irreducible cycles of length $k$ about $w$ is

$$I_1(k) = \left[ \begin{array}{cc} s(s-1) & t(t-1) \\ (s-1)^2 & (t-1)^2 \end{array} \right]^{(k-2)/2} \left[ \begin{array}{c} 1 \\ 1 \end{array} \right]$$

(20)

We wish to compare this to the number of irreducible $G'$ cycles about $w$, of which there are clearly

$$I_2(k) = r(r-1)^{k-1} = (s + t - 1)(s + t - 2)^{k-1}.$$  

For starters, we see

$$I_2(2) - I_1(2) = 2(s - 1)(t - 1)$$

which is non-negative, since both $s, t \geq 1$. Now since the maximum row sum in the $2 \times 2$ matrix of equation (20) is

$$s^2 + t^2 - 2(s + t) + 1 + \max(s,t),$$

54
we have

\[ I_1(k + 2) \leq I_1(k)m_1, \quad \text{where} \quad m_1 = s^2 + t^2 - 2(s + t) + 1 + \max(s, t) \]

for all \( k \). But

\[ I_2(k + 2) = I_2(k)m_2, \quad \text{where} \quad m_2 = (s + t - 2)^2, \]

and

\[ m_2 - m_1 = 2st - 2(s + t) + 3 - \max(s, t) = 1 + 2(s - 1)(t - 1) - \max(s, t), \]

which is positive unless \( s \) or \( t \) is 1. Thus, provided that \( s \geq 2 \) and \( t \geq 2 \), we have

\[ \lambda_{\text{Irred}}(G) \leq \sqrt{m_1} < \sqrt{m_2} = \lambda_{\text{Irred}}(G'), \]

and

\[ I_1(k) \leq I_1(2)m_1^{(k-2)/2} < I_2(2)m_2^{(k-2)/2}I_2(k) = I_2(k) \quad (21) \]

for all even \( k \). If \( t = 1 \) we calculate

\[ I_1(k) = s(s - 1)^{k-1} = I_2(k), \quad (22) \]

and similarly when \( s = 1 \).

We shall use the above calculation below. We can now assume that \( V_G \) has a vertex, \( x \), different from \( u, v, w \).

There is a natural bijection of edges, \( \iota \) from \( E_G \setminus \{e_i \cup \{f_i\}\} \) to \( E_{G'} \setminus \{g_i\} \). Extend \( \iota \) to a map on all of \( E_G \) by defining \( \iota(e_i) \) and \( \iota(f_i) \) to be a formal symbol \( S \). For any irreducible \( G \) cycle about \( x \) specified by its edges, \( c = (c_1, \ldots, c_k) \) with \( c_i \in E_G \), we associate a sequence

\[ \iota(c) = (\iota(c_1), \ldots, \iota(c_k)). \]

We claim that the number of \( c \) with a given image \( \iota(c) \) is no more than the number of \( E_{G'} \) cycles corresponding to \( \iota(c) \) by changing all \( g_i \) edges into \( S \)'s. Indeed, consider a block of consecutive \( S \)'s in \( \iota(c) \), i.e. \( \iota(c_a) = \iota(c_{a+1}) = \cdots = \iota(c_b) = S \), and \( \iota(c_{a-1}) \neq S \) and \( \iota(c_{b+1}) \neq S \); \( \iota(c) \) cannot begin or end with an \( S \), since the cycle begins at \( x \), and so we can assume \( a \geq 2 \) and \( b \leq \ell - 1 \). By looking at \( \iota(c_{a-1}) \) and \( \iota(c_{b+1}) \) we can determine whether or not the \( S \)-block begins in \( u, v, \) or \( w \), and ends in \( u, v, \) or \( w \). If the \( S \)-block begins in \( w \) and ends in \( w \), then equations (21) and (22) show that there are
no fewer $G'$ sequences for the corresponding $S$-block than $G$ sequences. Next compare those $S$-blocks that begin in a $u$ and end in a $w$. The number of such sequences in $G$ is

$$\begin{bmatrix} s & 0 \\ s(s-1) & (t-1)^2 \end{bmatrix} \begin{bmatrix} (s-1)^2 & t(t-1) \end{bmatrix}^{(b-a)/2} \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

whereas the number in $G'$ is $(s + t - 1)(s + t - 2)^{b-a}$ (since the non-$S$ edge $\ell(c_{a-1})$ can be followed by any $U$ to $w$ edge in $G'$); so the $G'$ number is no less than the $G$ number for $b - a = 0$ (since $t \geq 1$), and each time $b - a$ is increased by 2, the former number gets multiplied by an $m_2$, the latter gets multiplied by no more than $m_1$, where $m_1 < m_2$, provided that $s \geq 2$ and $t \geq 2$; the $s = 1$ or $t = 1$ case is easily checked to result in equality. The same argument holds for $v$ to $w$ $S$-blocks. For an $S$-block starting and ending in $u$, we wish to compare

$$\begin{bmatrix} s & 0 \\ s(s-1) & (t-1)^2 \end{bmatrix} \begin{bmatrix} (s-1)^2 & t(t-1) \end{bmatrix}^{(b-a-1)/2} \begin{bmatrix} s-1 \\ 0 \end{bmatrix},$$

with $(s + t - 1)(s + t - 2)^{b-a}$. Again, it suffices to compare when $b - a = 1$, which is immediate, and to check $s = 1$ or $t = 1$ separately. We argue for $S$-blocks starting in either $u$ or $v$ and ending in either $u$ or $v$ similarly.

\[ \square \]

**Theorem 6.10** For the model $\mathcal{H}_{n,d}$, we have $\tau_{\text{fund}} = \lceil \sqrt{d - 1} \rceil - 1$.

**Proof** As before, we consider a $\tau$ and search for those $\psi$ of order $\leq \tau$ with $\lambda_{\text{Irred}}(\psi)$ as large as possible. By Lemma 6.9, and by contractions (in Lemma 6.7), we may restrict our search to those $\psi$ with two or more edges between every pair of nodes.

First assume that $\tau + 2 \leq d/2$. If $\psi$ has two vertices, then $\psi$ has $\tau + 2$ edges joining the two vertices (since there are no self-loops in $\mathcal{H}_{n,d}$). In this case $\lambda_{\text{Irred}}(\psi) = \tau + 1$. We claim that this is as large a $\lambda_{\text{Irred}}$ as possible (again, assuming $\tau + 2 \leq d/2$). Indeed, if $\psi$ has $r > 2$ vertices, then the maximum degree of a vertex is $|E|$ minus the edges not involved with that particular vertex, which is at least 2 for each pair of the $r - 1$ other vertices. So the maximum degree is at most

$$|E| - \binom{r-1}{2} 2 \leq (|V| + \tau) - \binom{r-1}{2} 2 = \tau + r - (r-1)(r-2).$$

56
Since $\lambda_{\text{Irred}}$ is no greater than the maximum degree minus 1, we have

$$\lambda_{\text{Irred}} \leq \tau + r - (r - 1)(r - 2) - 1 = \tau + 1 - (r - 2)^2.$$  

It follows that if $r > 2$, $\lambda_{\text{Irred}}$ is strictly less than $\tau + 1$.

To achieve $\lambda_{\text{Irred}}(\psi) = \tau + 1$ with our $\psi$ having two vertices, we required $\tau + 2 \leq d/2$. To get $\lambda_{\text{Irred}}(\psi) = \tau + 1$ to equal or exceed $\sqrt{d - 1}$, we require $\tau + 1 = \left\lfloor \sqrt{d - 1} \right\rfloor$, for which we must have

$$\left\lfloor \sqrt{d - 1} \right\rfloor + 1 \leq d/2.$$  

Since $d/2$ is an integer, this is equivalent to

$$\sqrt{d - 1} + 1 \leq d/2,$$

which we easily see holds for all even $d > 2$ except $d = 4, 6$.

We conclude that $\tau_{\text{fund}} = \left\lfloor \sqrt{d - 1} \right\rfloor - 1$ for even $d \geq 8$. It suffices to analyze the cases $d = 4, 6$.

For each order, $\tau$, and $d = 4, 6$, we must examine those tangles of order $\tau$ and determine the largest possible $\lambda_{\text{Irred}}(\psi)$. Let us note that if $\psi$ is a tangle of order $-1$, then it is a tree and has $\lambda_{\text{Irred}}(\psi) = 0$. If $\psi$ is a completely pruned tangle of order 0, then $\psi$ is a cycle and has $\lambda_{\text{Irred}}(\psi) = 1$.

If $d = 4$, then consider the tangle of order 1 with three vertices, consisting of one “middle” vertex joined by two edges to each of two vertices. (This is a tangle by labelling the left to middle edges and the middle to right edges $\pi_1, \pi_2$.) We easily compute $\lambda_{\text{Irred}} = \sqrt{3}$, as this graph is bipartite and the number of irreducible walks of length $2m$ from the middle vertex, all such walks being cycles, is clearly $4 \cdot 3^{m-1}$. So for $d = 4$, $\tau_{\text{fund}} = 1$.

For $d = 6$, consider the tangle, $\psi = (V, E)$ with $V = \{v_1, v_2, v_3\}$ with three edges connecting $v_1$ to $v_2$ (labelled $\pi_1, \pi_2, \pi_3$) and two edges connecting $v_2$ to $v_3$ (labelled $\pi_1, \pi_2$). We claim that $\lambda_{\text{Irred}}(\psi) > \sqrt{5}$. Say that a cycle about $v_2$ ends in “state A” if the last vertex before $v_2$ was $v_1$, and otherwise in “state B” (i.e. the second to last vertex is $v_3$). From state A, taking two additional irreducible steps, there are 4 ways to reach another state A, and two ways to reach another state B. From state B, taking two irreducible steps, there is one way to reach another state B and six ways to reach another state A. It easy follows that $\lambda_{\text{Irred}}(\psi)$ is the square root of the largest eigenvalue of

$$\begin{bmatrix} 4 & 2 \\ 6 & 1 \end{bmatrix}.$$  

(23)
But $\lambda_1$ of this matrix is $(5 + \sqrt{57})/2$, and this $\lambda_1$ is just $\lambda_{\text{Irred}}(\psi)$. It follows that $\lambda_{\text{Irred}}(\psi) > \sqrt{6} > \sqrt{5}$, and hence $\tau_{\text{fund}} \leq 2$.

We wish to rule out $\tau_{\text{fund}} = 1$ when $d = 6$. Since we are considering only completely pruned graphs, $\psi$, each vertex has degree $\geq 2$. Such a graph, $\psi$, of order 1 has all vertices of degree 2 except for one of degree 4 or two of degree 3. In the case where there are vertices, $u, v$, of degree 3, therefore joined by three disjoint beaded paths, then $\lambda_{\text{Irred}}(\psi)$ is greatest when the beaded paths are each of length 1 (by setting up an obvious map from irreducible cycles about $u$ from the general graph to the one with beaded paths of length 1); hence $\lambda_{\text{Irred}}(\psi) \leq 2$ in this case, since the graph of two vertices joined by three edges has $\lambda_{\text{Irred}} = 2$. Similarly, in the case with $u$ of degree 4, therefore having two beaded cycles from $u$, $\lambda_{\text{Irred}}(\psi)$ is greatest when the lengths of the two cycles are two (they cannot be one since $H_{n,d}$ does not permit self-loops); hence $\lambda_{\text{Irred}}(\psi) \leq \sqrt{3}$ in this case. Hence $\tau_{\text{fund}} > 1$ and therefore $\tau_{\text{fund}} = 2$.

We conclude that $\tau_{\text{fund}} = \lceil \sqrt{d - 1} \rceil - 1$ also when $d = 4, 6$. 

\[ \square \]

**Theorem 6.11** For the model $I_{n,d}$, we have $\tau_{\text{fund}} = \lceil \sqrt{d - 1} \rceil - 1$ for all $d \geq 3$.

**Proof** We argue as with $H_{n,d}$. The only difference is that in $I_{n,d}$, two vertices can have as many as $d$ edges between them in a tangle (as opposed to $d/2$ edges in an $H_{n,d}$ tangle). So the argument in the previous theorem shows that the two-vertex tangles give that $\tau = \lceil \sqrt{d - 1} \rceil - 1$ equals $\tau_{\text{fund}}$ provided that $\tau + 2 \leq d$ (as opposed to $\tau + 2 \leq d/2$ for $H_{n,d}$). But we easily verify that

\[ \lceil \sqrt{d - 1} \rceil + 1 \leq d \]

for all $d \geq 3$ (indeed, we have equality for $d = 3$, and each time we increase $d \geq 3$ by one, $\sqrt{d - 1}$ increases by less than one).

\[ \square \]

**Theorem 6.12** For the model $J_{n,d}$, we have $\tau_{\text{fund}} = \lceil \sqrt{d - 1} \rceil - 1$ for all $d \geq 3$.

**Proof** As in $I_{n,d}$, for any $\tau \leq d - 2$ there is a tangle $G_\tau$ that is a pair of vertices with $\tau + 2$ edges joining them. $G_\tau$ has order $\tau$ and $\lambda_{\text{Irred}} = \tau + 1$; since when $\tau + 2 = d$ we have $\lambda_{\text{Irred}}(G_\tau) = d - 1 \geq \sqrt{d - 1}$ giving a supercritical
tangle, we need worry only about whether or not there is a tangle of order \( \tau \leq d - 2 \) that can beat the \( \lambda_{\text{Irred}} \) of \( G_{\tau} \). Again, as with \( I_{n,d} \), Lemma 6.9 can be applied to graphs with half-loops, and so by the same argument as for \( I_{n,d} \) we have that only graphs on one or two vertices can possibly beat \( G_{\tau} \).

So consider a graph on vertices \( u, v \) with \( a \) half-loops about \( u \), \( c \) half-loops about \( v \), and \( b \) edges from \( u \) to \( v \). An irreducible path traverses edges of four different states: (1) half-loops about \( u \), (2) edges from \( u \) to \( v \), (3) edges from \( v \) to \( u \), and (4) half-loops about \( v \). Now we write a transition matrix about the states: for example in state (1) we may either continue on one of \( a - 1 \) half-loops in state (1) or continue on one of \( b \) edges in state (2). We find the transition matrix

\[
\begin{bmatrix}
  a - 1 & b & 0 & 0 \\
  0 & 0 & b - 1 & c \\
  a & b - 1 & 0 & 0 \\
  0 & 0 & b & c - 1
\end{bmatrix},
\]

and \( \lambda_{\text{Irred}} \) of our graph is this matrix's largest eigenvalue. The order of the graph is \( a + b + c - 2 \) (recall, each half-loop contributes one to the order of a graph). But the row sum is never greater than \( a + b + c - 1 \) (and always less unless \( a \) or \( c \) vanishes), and so if this graph has order \( \tau \) its \( \lambda_{\text{Irred}} \) is no more than \( \tau + 1 \). Hence no \( J_{n,d} \) tangle of order \( \tau \) beats \( G_{\tau} \), provided that \( \tau \leq d - 2 \). Thus \( \tau_{\text{fund}} \) is the smallest number with \( \tau_{\text{fund}} + 1 \geq \sqrt{d - 1} \).

\( \square \)

## 7 Ramanujan Functions

In this section we discuss Ramanujan functions in order to (1) explain their significance, and (2) give some intuition on some very technical issues surrounding the asymptotic expansion for irreducible traces (as in Section 8).

**Definition 7.1** A function, \( f(k) \), on positive integers, \( k \), is said to be \( d \)-Ramanujan of order \( \alpha > 0 \) if there is a polynomial \( p = p(k) \) and a constant \( c > 0 \) such that

\[
|f(k) - (d - 1)^k p(k)| \leq ck^\alpha
\]

for all \( k \). We call \( (d - 1)^k p(k) \) the principal term of \( f \), and \( f(k) - (d - 1)^k p(k) \) the error term (both terms are uniquely determined if \( \alpha < d - 1 \)). A function is super-\( d \)-Ramanujan if it is \( d \)-Ramanujan of order 1.
A \( d \)-Ramanujan function as defined before, in Definition 2.1, is just a \( d \)-Ramanujan of order \( \sqrt{d-1} \).

Let \( N(k) \) be the number of irreducible cycles of length \( k \) in a \( d \)-regular graph. Then in [LPS86] it is shown that if \( N(k) \) is \( d \)-Ramanujan, then any eigenvalue, \( \lambda \neq \pm d \), of the graph satisfies \( |\lambda| \leq 2\sqrt{d-1} \). The discussion there also shows that in any case, if \( \lambda \) is the eigenvalue of largest absolute value < \( d \), then \( N(k) \) is \( d \)-Ramanujan of order \( \alpha \) with

\[
\alpha = \frac{|\lambda| + \sqrt{\lambda^2 - 4(d-1)}}{2}
\]

(and not for any smaller an \( \alpha \)). Any discussion of irreducible traces and eigenvalues is bound to be tied to \( d \)-Ramanujan functions.

One important property of \( d \)-Ramanujan functions of order \( \alpha \) is that they are closed under addition. Another very important property is that they are closed under convolution, which we now formally explain. This property will be used in Section 14, and refined versions of it will be used in Section 8.

**Theorem 7.2** Let \( f_1, f_2 \) be \( d \)-Ramanujan of order \( \alpha \) with \( \alpha < d - 1 \). Then their convolution,

\[
g(k) = (f_1 * f_2)(k) = \sum_{j=1}^{k-1} f_1(j) f_2(k-j)
\]

is also \( d \)-Ramanujan of order \( \alpha \).

The techniques in Section 8 prove a more precise version of this theorem (keeping track of the sizes of the the error term and the coefficients of the principle part); for this reason we keep the argument below concise.

**Proof** For \( i = 1, 2 \) let

\[
f_i(k) = (d-1)^k p_i(k) + e_i(k)
\]

where \( p_i \) are polynomials and the \( |e_i(k)| \) are bounded by \( ck^c \alpha^k \) for some \( k \).

We may also write

\[
f_i(k) = (d-1)^k (p_i(k) + \tilde{e}_i(k)), \quad \text{where } \tilde{e}_i(k) = (d-1)^{-k} e_i(k).
\]

Since convolution is bilinear, we easily see

\[
f * g = e_1 * e_2 + (d-1)^k (p_1 * p_2 + p_1 * \tilde{e}_2 + p_2 * \tilde{e}_1).
\]
It suffices to show that
\[ e_1 \ast e_2, \quad (d - 1)^k (p_1 \ast p_2)(k), \quad (d - 1)^k (p_1 \ast \tilde{e}_2)(k), \quad (d - 1)^k (p_2 \ast \tilde{e}_1)(k) \]
are \(d\)-Ramanujan of order \(\alpha\).

According to Sublemma 2.15 of [Fri91], \(p_1 \ast p_2\) is a polynomial. Next
\[
(p_1 \ast \tilde{e}_2)(k) = \sum_{j=1}^{k-1} (d - 1)^{-j} p_1(k - j)e_2(j) = \Sigma_1 - \Sigma_2
\]
where
\[
\Sigma_1 = \sum_{j=1}^{\infty} p_1(k - j)(d - 1)^{-j}e_2(j),
\]
\[
\Sigma_2 = \sum_{j=k}^{\infty} p_1(k - j)(d - 1)^{-j}e_2(j).
\]
Writing
\[
p_1(k - j) = \sum a_{r,s} k^r j^s,
\]
we see that \(\Sigma_1\) is a polynomial, and \(\Sigma_2\) is bounded by \(ck^\alpha \alpha^k (d - 1)^{-k}\) (see Section 8, especially Lemma 8.9, for details). This shows \((d - 1)^k (p_1 \ast \tilde{e}_2)\) is \(d\)-Ramanujan of order \(\alpha\). Similarly, so is \((d - 1)^k (p_2 \ast \tilde{e}_1)\); \(e_1 \ast e_2\) is easily also seen to be so (with zero principal term).

\[ \square \]

8 An Expansion for Some Selective Traces

In this section we prove the first crucial expansion theorem. Our second such theorem, Theorem 9.3, will extend these ideas.

**Theorem 8.1** Let \(r\) be a positive integer, and let \(\Psi\) be a set of tangles containing all supercritical tangles of order < \(r\). Then there is an \(S_0 = S_0(r)\) such that for all \(S > S_0\) the following holds. We have
\[
E[\text{IrSelTr}_{S,\Psi}(G; k)] = f_0(k) + \frac{f_1(k)}{n} + \cdots + \frac{f_{r-1}(k)}{n^{r-1}} + \frac{\text{error}}{n^r},
\]
where the \(f_i\) are \(d\)-Ramanujan and the error term satisfies the bound given in Theorem 5.8.
This theorem is an immediate consequence of Theorems 5.4 and 5.8 and the following theorem.

**Theorem 8.2** Fix a lettering, \(\mathcal{L}\), of type \(T\), and fixed non-negative integers \(\ell_1, \ldots, \ell_{d/2}\)

\[
R_{T,\mathcal{L}}(k_1, \ldots, k_b) = \sum_{(w_1, \ldots, w_b)} \prod_{j=1}^{d/2} (a_j(w_1) + \ldots + a_j(w_b))^{\ell_j}
\]

where the sum is over all tuples of words \((w_1, \ldots, w_b)\) such that each \(w_i\) is irreducible and of length \(k_i\) and is compatible with \(\mathcal{L}\). Then if \(\tilde{T}\) is a \(B\)-new type, then

\[
f(k) = \sum_{m_i \geq 1} \sum_{k_1 m_1 + \ldots + k_b m_b = k \atop k_i \geq B \text{ if } e_i \in E_{\text{long}}} \sum_{k_i = k_i^{\text{fixed}} \text{ if } e_i \in E_{\text{fixed}}} W_{\tilde{T}}(\tilde{m}; S, \Psi) R_{T,\mathcal{L}}(k_1, \ldots, k_b) \tag{24}
\]

(with \(W\) as in Theorem 6.6), then \(f\) is \(d\)-Ramanujan for all \(B \geq B_0 = B_0(T)\).

**Proof** Clearly it suffices to prove the following theorem.

**Theorem 8.3** Theorem 8.2 holds with \(R_{T,\mathcal{L}}\) replaced by

\[
S_{T,\mathcal{L}}(k_1, \ldots, k_b) = \sum_{(w_1, \ldots, w_b)} \prod_{i=1}^{b} \prod_{j=1}^{d/2} a_j^{\ell_{ij}}(w_i),
\]

with \(\ell_{ij}\) any set of non-negative integers.

Our Lemma 5.9 reduces the above theorem to the following.

**Theorem 8.4** With notation as in Theorem 8.2, let \(K_1, K_2, K_3\) be a partition of \(k_1, \ldots, k_b\), and let \(|K_i|\) for \(i = 1, 2, 3\) denote the sum of the \(k_j\) in \(K_i\). Then for fixed non-negative integers \(\ell_1, \ldots, \ell_b\), Theorem 8.2 holds with \(R_{T,\mathcal{L}}\) replaced by

\[
R_{T,\mathcal{L}}(k_1, \ldots, k_b) = (d - 1)^{|K_1|} (-1)^{|K_2|} k_1^{\ell_1} \ldots k_b^{\ell_b}. \tag{25}
\]

More generally, Theorem 8.2 holds with \(R_{T,\mathcal{L}}\) replaced by

\[
R_{T,\mathcal{L}}(k_1, \ldots, k_b) = (d - 1)^{|K_1|} k_1^{\ell_1} \ldots k_u^{\ell_u} \beta(k_{u+1}, \ldots, k_b), \tag{26}
\]

62
where the edges are ordered so that
\[ \{ i | e_i \in E_{\text{long}} \text{ and } k_i \in K_1 \} = \{ 1, \ldots, u \}, \]
and where \( \beta \) is function such that
\[ |\beta(k_{u+1}, \ldots, k_b)| \leq c(|k_{u+1}| + \cdots + |k_b|)^c \]
for some constant \( c \).

The \( R \) of equation (25) is all that is needed for \( G_{n,d} \); it will be convenient (if not necessary) to use the \( R \) of equation (26) for \( J_{n,d} \) (see section 14).

**Proof** It suffices to deal with the \( R \) of equation (26). Let
\[ \tilde{K}_i = \{ k_j \in K_i \mid e_j \in E_1 \}. \]

We may assume \( E_1 = \{ e_1, \ldots, e_{t} \} \) and \( \tilde{K}_1 = \{ k_1, \ldots, k_u \} \); set
\[ M_1 = m_1 + \ldots + m_t, \quad M_2 = m_{t+1} + \ldots + m_b, \]
\[ j_1 = k_1 m_1 + \ldots + k_t m_t, \quad \text{and} \quad j_2 = k_{t+1} m_{t+1} + \ldots + k_b m_b. \]

Clearly it suffices to prove the theorem for
\[ f(k) = \sum_{\tilde{m}} W_T(\tilde{m}; S, \Psi) \sum_{\begin{subarray}{c} k_1 m_1 + \ldots + k_b m_b = k \medskip \kappa \geq \beta \end{subarray}} (d - 1)^{|\tilde{K}_1|k_1^{\ell_1}} \cdots k_u^{\ell_u} \beta(k_{u+1}, \ldots, k_t), \]
understanding that \( k_{t+1}, \ldots, k_t \) are fixed by \( \tilde{k}^{\text{fixed}} \).

**Definition 8.5** The (coefficient) norm, \( |p| \), of a polynomial, \( p \) (which is possibly multivariate), is the largest absolute value among its coefficients.

Working with this notion of a norm is a bit “weak,” (i.e., sometimes much stronger statements would hold with other norms), but this notion is sufficient for our purposes.

Let
\[ f_{\tilde{m}}(k) = \sum_{k \cdot \tilde{m} = k} (d - 1)^{|\tilde{K}_1|k_1^{\ell_1}} \cdots k_u^{\ell_u} \beta(k_{u+1}, \ldots, k_t). \]
Theorem 8.6 For any vector of positive integers, $\vec{m}$, we have $f_{\vec{m}}$ is $d$-Ramanujan with principal term $(d-1)^k p_{\vec{m}}(k)$ and error term $e_{\vec{m}}(k)$ satisfying
\[
|p_{\vec{m}}| \leq c(d-1)^{(-BM_1-M_2+c)/2},
\]
and
\[
|e_{\vec{m}}(k)| \leq ck^c(d-1)^{k-1BM_1-M_2+Be^c/2},
\]
where $c$ depends only on the $\ell_i$ and $\beta$.

Proof Fix a value of $\vec{m}$. Without loss of generality we may assume $m_1 = \cdots = m_s = 1$ and $m_{s+1}, \ldots, m_u \geq 2$. For now assume that $s \geq 1$; we will later indicate the minor changes needed for the situation $s = 0$ (i.e. when there are no $m_i$ belonging to $\tilde{K}_1$ that equal one). Let
\[
g_{\vec{m}}(r) = \sum_{\substack{k_{s+1}, \ldots, k_b \text{ s.t.} \\ k_{s+1}m_{s+1} + \cdots + k_bm_b = r \\ k_i \geq B \text{ for } i \leq t}} (d-1)^{k_{s+1}+\cdots+k_u} \beta(k_{u+1}, \ldots, k_t).
\]

Lemma 8.7 If $\tilde{T}$ is a $B$-new type, then
\[
|g_{\vec{m}}(r)| \leq cr^c(d-1)^{r-1BM_1-M_2+Be^c/2}
\]
for some constant $c$ depending only on $\tilde{T}$.

Proof Since each $k_i$ is at most $r$,
\[
\beta(k_{u+1}, \ldots, k_t)
\]
is bounded by $cr^c$, and it suffices to prove the estimate for $g_{\vec{m}}$ replaced with
\[
\sum_{\substack{k_{s+1}m_{s+1} + \cdots + k_bm_b = r \\ k_i \geq B \text{ for } i \leq t}} (d-1)^{k_{s+1}+\cdots+k_u}.
\]
But there are only $\binom{r+b-s-1}{b-s-1}$ ways of writing $r$ as the sum of $b-s$ positive integers. So it suffices to show
\[
(d-1)^{k_{s+1}+\cdots+k_u} \leq (d-1)^{r-1BM_1-M_2+Be^c/2}.
\]
Now we have
\[
r = k_{s+1}m_{s+1} + \cdots + k_bm_b,
\]
64
$$2k_{s+1} + \cdots + 2k_u = r - (k_{s+1}m_{s+1} + \cdots + k_{b}m_{b}) + (2k_{s+1} + \cdots + 2k_u)$$

$$= r - (m_{s+1} - 2)k_{s+1} - \cdots - (m_{u} - 2)k_{u} - m_{u+1}k_{u+1} - \cdots - m_{b}k_{b}.$$ 

Since $m_i \geq 2$ for $i$ between $s + 1$ and $u$, and since $k_i \geq B$ for $i \leq t$ (and $k_i \geq 1$ for all $i$), we conclude

$$2k_{s+1} + \cdots + 2k_u \leq r - (m_{s+1} - 2)B - \cdots - (m_{u} - 2)B$$

$$- m_{u+1}B - \cdots - m_{t}B - m_{t+1} - \cdots - m_{b}$$

$$= r - (m_{s+1} + \cdots + m_{t})B + 2(u - s)B - (m_{t+1} + \cdots + m_{b})$$

$$= r - (M_1 - s - 2(u - s))B - M_2.$$ 

Hence

$$k_{s+1} + \cdots + k_u \leq (2k_{s+1} + \cdots + 2k_u)/2 \leq \left( r - (M_1 - s - 2(u - s))B - M_2 \right)/2.$$ 

But $u, s$ are bounded by the number of edges in $\tilde{T}$. 

We will need another lemma.

**Lemma 8.8** For any non-negative integers $\ell_1, \ldots, \ell_s$ there is a polynomial $Q$ such that for all $k \geq s$ we have

$$\sum_{k_1 + \cdots + k_s = k \atop \text{integers } k_i \geq 1} k_{1}^{\ell_1} \cdots k_{s}^{\ell_s} = Q(k).$$ 

**Proof** This is a special case of Sublemma 2.15 of [Fri91] (proven in a straightforward induction on $s$).

Now let

$$j_{11} = k_{1}m_{1} + \cdots + k_{s}m_{s} = k_{1} + \cdots + k_{s}, \quad j_{12} = k_{s+1}m_{s+1} + \cdots + k_{t}m_{t}$$

...
(so that \(j_{11} + j_{12} = j_1\)). In the notation of the above two lemmas we have

\[
\bar{f}(k) = \sum_{j_{11} + j_{12} + j_2 = k} (d-1)^{j_{11}} Q(j_{11}) g_m(j_{12} + j_2)
\]

\[
= \sum_{r=1}^{k-s} (d-1)^{k-r} Q(k-r) g_m(r);
\]

(27)

here we sum until \(r = k - s\) since Lemma 8.8 requires \(k \geq s\) (and \(Q(k-r)\) vanishes for \(k < s\)), and we sum from \(r = 1\) to simplify the expression, despite the fact that \(g_m(r)\) clearly vanishes for \(r < B(m_{s+1} + \cdots + m_t) + m_{t+1}k_{t+1} + \cdots + m_bk_b\).

The sum in equation (27) is clearly \(\Sigma_1(k) - \Sigma_2(k)\), where

\[
\Sigma_1(x) = \sum_{r=1}^{\infty} (d-1)^{k-r} Q(x-r) g_m(r),
\]

\[
\Sigma_2(x) = \sum_{r=k-s+1}^{\infty} (d-1)^{k-r} Q(x-r) g_m(r),
\]

assuming these sums converge.

We claim \(\Sigma_1(k)\) will be the principal part of \(f_m(k)\), and \(\Sigma_2(k)\) will be the error term. First we need the following lemma.

**Lemma 8.9** For any positive integer, \(D\), there is a \(C_2\) such that the following holds. Let \(g(r)\) be a function defined on non-negative integers, \(r\), such that \(|g(r)| \leq C_1 r^D \rho^r\), with \(\rho < 1\). Let \(Q = Q(x)\) be any polynomial of degree at most \(D\). Then (1) the infinite sum

\[
h(x) = \sum_{r=1}^{\infty} Q(x-r) g(r)
\]

is convergent (in coefficient norm), (2) the degree of \(h\) is that of \(Q\), and (3) we have

\[
|h| \leq C_1 C_2 (1 - \rho)^{-2D} |Q|.
\]

The same is true for the sum

\[
h_u(x) = \sum_{r=u+1}^{\infty} Q(x-r) g(r),
\]

(66)
for any positive integer $u$, except that we replace the last claim with the estimate

$$|h_u| \leq C_1 C_2 u^{2D} (1 - \rho)^{-2D} \rho^u |Q|$$

(however, the $C_2$ in this equation might need to be larger than that in the estimate for $h$).

**Proof** First we observe that since

$$\sum_{r=0}^{\infty} \binom{r}{j} \rho^r = \frac{\rho^{j+1}}{(1-\rho)^j},$$

we have

$$\sum_{r=0}^{\infty} r^j \rho^r = \frac{q_j(\rho)}{(1-\rho)^j},$$

where $q_j$ is some polynomial of degree $\leq j + 1$.

We first prove the claims on $h$. By the binomial theorem, and the fact that $Q$’s degree is bounded, it suffices to examine only the cases where $Q(x-r)$ is replaced by $x^i r^j$ for $i + j \leq D$. In this case $h(x)$ becomes

$$\sum_{r=1}^{\infty} x^i r^j g(r) = x^i \sum_{r=1}^{\infty} r^j g(r),$$

and we have

$$\sum_{r=1}^{\infty} r^2 |g(r)| \leq \sum_{r=0}^{\infty} r^j C_1 r^{D} \rho^r = \frac{C_1 \rho^{j+D+1}}{(1-\rho)^{j+D}}.$$

This establishes the claim on $h$. $h_u$ is reduced to $h$ via

$$h_u(x) = \sum_{r=1}^{\infty} \tilde{Q}(x-r) \tilde{g}(r),$$

where $\tilde{Q}(x) = Q(x-u)$ and $\tilde{g}(r) = g(r+u)$. So $\tilde{g}$ satisfies the same estimate as does $g$, except with an extra factor of $(r + u)^D r^{-D} \rho^u \leq C u^D \rho^u$; the binomial theorem implies that $|\tilde{Q}|$ is at most $|Q| u^D$ times a constant depending on $D$. 

$\square$
We continue with the proof of Theorem 8.6. We have
\[(d - 1)^{-k}\Sigma_1(k) = \sum_{r=1}^{\infty} Q(k - r)[(d - 1)^{-r}g_{\vec{m}}(r)] = \sum_{r=1}^{\infty} Q(k - r)\tilde{g}(r),\]
where \(\tilde{g}(r) = (d - 1)^{-r}g_{\vec{m}}(r)\). Now \(Q\) is fixed in the theorem, so \(|Q|\) can be regarded as a constant. Also, since 
\[|g_{\vec{m}}(r)| \leq cr^c(d - 1)^{(r - BM_1 - M_2 + Bc)/2},\]
according to Lemma 8.7, we have 
\[|\tilde{g}(r)| \leq cr^c(d - 1)^{(-BM_1 - M_2 + Bc)/2}.\]
It follows that 
\[(d - 1)^{-k}\Sigma_1(k) = h(k),\]
where \(h\) is a polynomial with 
\[|h| \leq c(d - 1)^{(-BM_1 - M_2 + Bc)/2},\]
amuming that \(d > 2\) (so that \(1 - \rho\) with \(\rho = (d - 1)^{-1/2}\) is strictly positive).
Furthermore, Lemma 8.9 also implies that 
\[|\Sigma_2| \leq c(d - 1)^k(k - s + 1)^{2D}(d - 1)^{-(k-s)/2}(d - 1)^{(-BM_1 - M_2 + Bc)/2} \leq c'k^{2D}(d - 1)^{(-BM_1 - M_2 + Bc + k)/2}.\]
Now we see that \(\Sigma_1(k) + \Sigma_2(k)\) is the decomposition of \(f_{\vec{m}}(k)\) into principal and error terms, as claimed before, and that these terms satisfy the bounds stated in Theorem 8.6.
Finally we indicate the minor changes when \(s = 0\). In this case we take \(Q\) to be the function \(Q(0) = 1\) and \(Q\) vanishing elsewhere. Then \(f_{\vec{m}}(k) = g_{\vec{m}}(k)\), so Lemma 8.7 shows that \(f_{\vec{m}}\) is \(d\)-Ramanujan with zero principal part.

\[\square\]

We continue with the proof of Theorem 8.4. We are studying 
\[f(k) = \sum_{\vec{m}} W_{\hat{T}}(\vec{m}; S, \Psi)f_{\vec{m}}(k).\]
Set 
\[F(M_1, M_2; k) = \sum_{m_1 + \ldots + m_d = M_1 \atop m_{t+1} + \ldots + m_d = M_2} W_{\hat{T}}(\vec{m}; S, \Psi)f_{\vec{m}}(k),\]

68
so that

\[ f(k) = \sum_{M_1, M_2 > 0} F(M_1, M_2; k). \]

Theorem 6.6 combined with Theorem 8.6 gives that for fixed \( M_1, M_2 \) we have that \( F(M_1, M_2; k) \) is \( d \)-Ramanujan with principle term \((d−1)^k P_{M_1, M_2}(k)\) and error term \( E_{M_1, M_2}(k) \) where

\[
|P_{M_1, M_2}| \leq (d - 1)^{(-BM_1 - M_2 + c)/2} cB(\sqrt{d - 1} - \epsilon)^c M_1 + M_2,
\]

\[
|E_{M_1, M_2}(k)| \leq c k^c (d - 1)^{(k - BM_1 - M_2 + Bc)/2} cB(\sqrt{d - 1} - \epsilon)^c M_1 + M_2,
\]

and the degree of \( P_{M_1, M_2} \) is bounded independent of \( M_1, M_2 \). So we take \( B > c \) and sum over all \( M_1, M_2 \) to conclude that \( f \) is \( d \)-Ramanujan.

\[ \square \]

### 9 Traces Without Tangled Graphs

In this section we take any of the selective traces discussed in this paper and to remove the contribution from graphs with tangles. What is left comes from graphs without tangles, where the selective trace (based on those tangles) will always agree with the trace without selectivity. This eliminates the need to analyze the effect of selectivity.

The expansions of Section 8 carry over to traces where we exclude the contribution from tangled graphs. This is a tedious but fairly straightforward consequence of the methods already developed, along with the following very important lemma.

**Definition 9.1** Recall that \( \Psi_{\text{eig}} \) is the set of supercritical tangles, i.e., whose \( \lambda_{\text{irred}} \) is \( \geq \sqrt{d - 1} \). Recall that \( \Psi_{\text{eig}}[r] \) is the subset of elements of \( \Psi_{\text{eig}} \) of order at most \( r \). Let \( \Psi_{\text{min}}[r] \) be the set of tangles of \( \Psi_{\text{eig}}[r] \) that are minimal with respect to inclusions, i.e., that don't have another element of \( \Psi_{\text{eig}}[r] \) properly included in it.

**Lemma 9.2** \( \Psi_{\text{min}}[r] \) is finite.

This means that containing a supercritical tangle of order \( \leq r \) is equivalent to containing one of a finite set of tangles.
Proof Assume that $\Psi_{\text{min}}[r]$ is not finite. With each tangle we associate a type which is the labelled graph obtained by suppressing the degree two vertices. Since there are finitely many types of order $\leq r$, there must be an infinite number of $\Psi_{\text{min}}[r]$ tangles of some type, $T$. By passing to a subsequence we may assume there is an infinite sequence of $\Psi_{\text{min}}[r]$ elements, $\{\psi_i\}$, such that for each edge of $T$ the associated labelling is either constant or has length tending to infinity; furthermore, the length must tend to infinity along at least one $T$ edge. Let $\psi_\infty$ be the limiting tangle, where we discard all edges with length tending to infinity.

We claim $\lambda_{\text{Irred}}(\psi_i) = \lambda_1(T_{\text{Irred}}^i)$, where $T_{\text{Irred}}^i$ is the VLG with underlying directed graph $T_{\text{Irred}}$ and where $e = (v_1, v_2) \in E_{T_{\text{Irred}}}$ has length equal to the length, $\ell(v_1)$, of $v_1$ in $\psi_i$ (recall $v_1$ can be viewed as a directed edge of $T$); indeed, with a vertex path $v_1, \ldots, v_r$ in $T_{\text{Irred}}$ with $v_j = (u_j, u_{j+1})$ for $u_j \in V_T$, we associate the walk $u_1, \ldots, u_{r+1}$, which has $\psi_i$ length equal to the sum of the $\ell(v_i)$. For a cycle, where $u_{r+1} = u_1$, its length (in $T_{\text{Irred}}^i$) $\ell(v_1) + \cdots + \ell(v_r)$, which corresponds to a unique $(\psi_i)_{\text{Irred}}$ cycle of the same length, arising from the subdivided $v_j$. This correspondence is clearly a length preserving bijection between $T_{\text{Irred}}^i$ cycles and $(\psi_i)_{\text{Irred}}$ cycles. Hence $\lambda_{\text{Irred}}(\psi_i) = \lambda_1(T_{\text{Irred}}^i)$.

Similarly $\lambda_{\text{Irred}}(\psi_\infty) = \lambda_1(T_{\text{Irred}}^\infty)$ with $T_{\text{Irred}}^\infty$ defined similarly. Now by Theorem 3.4, $\lambda_1(T_{\text{Irred}}^i) \rightarrow \lambda_1(T_{\text{Irred}}^\infty)$, and so $\lambda_{\text{Irred}}(\psi_\infty) \geq \sqrt{d-1}$. Also $\psi_\infty$’s order is less than that of the $\psi_i$ (because of the edge removal(s)). Hence $\psi_\infty$ is again a $\Psi_{\text{elg}}[r]$ tangle. But $\psi_\infty$ properly contains (all) $\psi_i$, which contradicts the supposed minimality of the $\psi_i$. Hence $\Psi_{\text{min}}[r]$ is finite.

We illustrate the above lemma with an example. Let $\psi_i$ be a sequence of tangles whose underlying graph is the same except for one beaded cycle of length $i$ about some vertex. This infinite collection of tangles would prove troublesome to the methods of this section. However either (1) $\lambda_{\text{Irred}}(\psi_i) < \sqrt{d-1}$ for some $i$, at which point only finitely many of the $\psi_i$ are relevant, or (2) the limiting tangle, $\psi_\infty$, has $\lambda_{\text{Irred}}(\psi_\infty) \geq \sqrt{d-1}$, in which case a $\psi_i$ inclusion implies a $\psi_\infty$ inclusion.

**Theorem 9.3** Let $\Psi$ be a finite set of pruned (nonempty) tangles of order $\geq 1$. Let $\chi_\Psi$ be the indicator function of the event that $G \in G_{n,d}$ contains a
(i.e., at least one) tangle from $\Psi$, i.e.,
\[
\chi_\Psi(G) = \begin{cases} 
1 & \text{if } G \text{ contains a tangle from } \Psi, \\
0 & \text{if not.}
\end{cases}
\]

Let $\Psi'$ be a set of tangles including all supercritical tangles of order $< r$. Then for any $r$ there is an $S_0 = S_0(r)$ such that for all $S \geq S_0$ we have an expansion
\[
E[\chi_\Psi \text{IrSelTr}_{S,\Psi'}(G; k)] = f_0(k) + \frac{f_1(k)}{n} + \cdots + \frac{f_{r-1}(k)}{n^{r-1}} + \frac{\text{error}}{n^r},
\]
where the $f_i$ are $d$-Ramanujan and the error term satisfies
\[
|\text{error}| \leq ck^2(d-1)^k
\]
with $c$ and $\tilde{r}$ depending only on $r$, $\Psi$, and $\Psi'$. Therefore
\[
E[(1 - \chi_\Psi) \text{IrSelTr}_{S,\Psi'}(G; k)]
\]
also has such an expansion.

We believe this theorem is true even if $\Psi$ contains cycles, i.e., tangles of order 0. But to prove this would be more difficult, since Lemma 9.12 would not be true.

**Proof** Let $\chi_{(w; \vec{t})}$ be the indicator function of the event, $\mathcal{E}(w; \vec{t})$, of the potential walk $(w; \vec{t})$. Then for fixed, irreducible $w$ we have
\[
\sum_{\vec{t} \text{ such that } (w, \vec{t}) \text{ is of order } \geq r} E[\chi_\Psi \chi_{(w; \vec{t})}] \leq \sum_{\vec{t} \text{ such that } (w, \vec{t}) \text{ is of order } \geq r} E[\chi_{(w; \vec{t})}]
\]
\[
= \sum_{\vec{t} \text{ such that } (w, \vec{t}) \text{ is of order } \geq r} P(w, \vec{t}) \leq ck^{2r+2}n^{-r},
\]
by Lemma 5.6. The sum of the above over all irreducible words of length $k$ therefore satisfies a bound of $c(d-1)^k k^{2r+2}n^{-r}$. Therefore (assuming $r \geq 1$) it suffices to give an expansion for the sum over all irreducible $w$ and $t$ of
\[
\sum_{(w, \vec{t}) \text{ of order } \leq r-1} E[\chi_\Psi \chi_{(w; \vec{t})}].
\]
For a set, $\Psi$, of tangles, let $\Lambda = \Lambda_\Psi(G)$ be the indicator function for the event that all tangles in $\Psi$ occur in $G$. By inclusion-exclusion we have that

$$
\chi = \sum_{k=1}^{\binom{|\Psi|}{k}} (-1)^{k+1} \sum_{\Omega \subset \Psi \atop |\Omega|=k} \Lambda_\Omega.
$$

Therefore it suffices to give an expansion for the quantity in equation (29) with $\chi$ replaced with $\Lambda$

Let $\Psi = \{\psi_1, \ldots, \psi_r\}$ be disjoint tangles, i.e., $V_{\psi_i}$ are disjoint, such that no two $\psi_i$ are isomorphic. A $\Psi[\vec{\mu}]$ inclusion in $G$ is a collection of distinct inclusions, $\{\iota_{i,j}\}$ with $j = 1, \ldots, \mu_i$ for each $i = 1, \ldots, r$, such that $\iota_{i,j}$ is an inclusion of $\psi_i$ into $\{1, \ldots, n\}$. We say that two $\Psi[\vec{\mu}]$ inclusions are the same if they differ by a permutation of the $\iota_{i,j}$’s. Let

$$N_{\Psi,\vec{\mu}} = N_{\Psi,\vec{\mu}}(G)$$

be the number of such $\Psi[\vec{\mu}]$ inclusions in $G$. Again, by inclusion-exclusion we have

$$\left| \Lambda - \sum_{\max(\mu_i)<R} (-1)^{\mu_1+\cdots+\mu_r} N_{\Psi,\vec{\mu}} \right| \leq \sum_{\max(\mu_i)=R} N_{\Psi,\vec{\mu}}. \quad (30)$$

It therefore suffices to show that for some $R$ we have (1) $N_{\Psi,\vec{\mu}}(d-1)^k$ satisfies the error bound of $ck^r(d-1)^k/n^r$ for all $\vec{\mu}$ with $\max(\mu_i) = R$, and (2) the quantity in equation (29) with $\chi$ replaced with $N_{\Psi,\vec{\mu}}$ for any particular $\mu$ has an expansion like the right-hand-side of equation (28). First we proceed to define “formoids” and “typoids,” which are variants of forms and types, which will let us describe a suitable value for $R$.

Recall that a potential walk gives rise to a form, which in turn gives rise to a type. Define a potential tangle specialization as a tangle, $\psi$, along with an inclusion of $\psi$’s vertices, $V_\psi$, into $\{1, \ldots, n\}$; a potential tangle set specialization, $(\Omega, \sigma)$, is an ordered collection of distinct tangles, $\Omega$, with an inclusion, $\sigma$, of $\Omega$’s vertices,

$$V_\Omega = \bigcup_{\psi \in \Omega} V_\psi,$$

into $\{1, \ldots, n\}$. (Notice that the different tangles of $\Omega$ may share vertices or edges, so that in the specialization the images of the tangles may intersect.)
Say that two potential tangle set specializations, \((\Omega_1, \sigma_1)\) and \((\Omega_2, \sigma_2)\), are isomorphic, written \((\Omega_1, \sigma_1) \sim (\Omega_2, \sigma_2)\), if \(\sigma_1(V_{\Omega_1}) = \sigma_2(V_{\Omega_2})\) and \(\sigma_2^{-1}\sigma_1: V_{\Omega_1} \rightarrow V_{\Omega_2}\) extends to isomorphisms \(\omega_{1,j} \rightarrow \omega_{2,j}\), where \(\omega_{i,1}, ..., \omega_{i,m}\) are the elements of \(\Omega_i\), in order. We say that \((\Omega_1, \sigma_1)\) and \((\Omega_2, \sigma_2)\) are rearrangeably isomorphic, written \((\Omega_1, \sigma_1) \sim_{RA} (\Omega_2, \sigma_2)\) if the same is true, except that \(\sigma_2^{-1}\sigma_1\) gives isomorphism \(\omega_{1,j} \rightarrow \omega_{2,\tau(j)}\) for a permutation, \(\tau\); we write \(\tau = \text{perm}((\Omega_1, \sigma_1, \Omega_2, \tilde{\sigma}))\) (\(\tau\) is uniquely determined, since the \(\{\omega_{2,i}\}\) are distinct); we also say that \(\Omega_1\) and \(\Omega_2\) are rearrangeably isomorphic. Given \((\Omega, \sigma)\), let the rearrangeable automorphism group of \((\Omega, \sigma)\) be
\[
\text{ReAut}(\Omega, \sigma) = \{\text{perm}((\Omega, \sigma, \Omega, \tilde{\sigma})) | (\Omega, \sigma) \sim_{RA} (\Omega, \tilde{\sigma})\}.
\]
ReAut\((\Omega, \sigma)\) is clearly independent of \(\sigma\), and can therefore be written simply as ReAut\((\Omega)\), the rearrangeable isomorphism group of \(\Omega\).

We associate to each \(\Psi[\mu]\) inclusion in \(G\), \(\{t_{i,j}\}\), an isomorphism class of potential tangle set specializations, \((\Omega, \sigma)\), as follows. Set \(\Omega = \{\omega_{i,j}\}\), where \(\omega_{i,j} = t_{i,j}(\psi_i)\). We therefore have
\[
V_\Omega = \bigcup t_{i,j}(V_{\psi_i}) \quad \text{and} \quad E_\Omega = \bigcup t_{i,j}(E_{\psi_i}),
\]
and set \(\sigma\) to be the identity; \(E_\Omega\) is labelled (provided that \(\{t_{i,j}\}\) gives a consistent labelling).

In Figure 3 we give an example. There \(\Psi\) consists of a single tangle, \(\psi\), the complete graph on four vertices. \(\{t_{i,j}\}\) consists of four inclusions, and
these inclusions are indicated in the bottom row of four graphs; for example, the first inclusion is $a \mapsto 3$, $b \mapsto 5$, $c \mapsto 9$, and $d \mapsto 6$. So $\Omega = \{ \omega_1, \ldots, \omega_4 \}$ has four tangles, and these tangles overlap on some of their vertices; indeed, this overlapping indicates how $\{ i_{i,j} \}$ places four copies of $\psi$ into the vertex set $\{1, \ldots, n\}$. As we can see, $\psi$ has automorphisms, and we view $i_{1,1}$ and $i_{1,2}$, the first and second inclusions, as distinct. (In the figure we have omitted all edge labels.)

N.B.: The tangle set, $\Omega$, has a corresponding graph, $G_{\Omega} = (V_{\Omega}, E_{\Omega})$, and we often only need deal with $G_{\Omega}$ (or, say, $V_{\Omega}$). However, it is very important to remember that $\Omega$ really is an ordered set of tangles, $\{ \omega_1, \omega_2, \ldots \}$, with each $\omega_i$ a graph in its own right.

For a potential tangle set specialization, $(\Omega, \sigma)$, let $\chi_{\Omega, \sigma}$ denote the indicator function of the event that $(\Omega, \sigma)$ occurs.

**Theorem 9.4** For any $\Psi$ (finite, disjoint, and non-isomorphic, as before) and $\vec{\mu}$ there is a finite set, $\mathcal{F}$, of tangle sets such that

$$N_{\Psi, \vec{\mu}} = \sum_{\Omega \in \mathcal{F}} |\text{ReAut}(\Omega)|^{-1} \sum_\sigma \chi_{\Omega, \sigma},$$  \hspace{1cm} (31)

where our sum over $\sigma$ is over all $\sigma: V_{\Omega} \to \{1, \ldots, n\}$. Hence for any potential walk, $(w, \vec{t})$, we have

$$\chi_{(w, \vec{t})} N_{\Psi, \vec{\mu}} = \sum_{\Omega \in \mathcal{F}} |\text{ReAut}(\Omega)|^{-1} \sum_\sigma \chi_{(w, \vec{t})} \chi_{\Omega, \sigma}.$$

**Proof** Let $(\Omega_i, \sigma_i)$ for $i = 1, 2$ come from the same $\Psi[\vec{\mu}]$ inclusion. Then $\Omega_1$ and $\Omega_2$ are rearrangeably isomorphic via $\sigma_2^{-1} \sigma_1$. Thus the function $f$ defined by

$$f(\Omega) = |\text{ReAut}(\Omega)|^{-1} \sum_\sigma \chi_{\Omega, \sigma}$$

is (1) invariant in rearrangeable isomorphism classes, (2) represents a sum over indicator functions of distinct $\Psi[\vec{\mu}]$ inclusions, and (3) if $\Omega'$ is not rearrangeably isomorphic to $\Omega$ then $f(\Omega)$ and $f(\Omega')$ can be viewed as sums over indicator functions of disjoint collections of $\Psi[\vec{\mu}]$ inclusions. Recall that the $\Psi[\vec{\mu}]$ inclusions are regarded as the same if they differ by rearranging the $\{i_{i,j}\}$; so given $(\Omega, \sigma)$, there are exactly $|\text{ReAut}(\Omega)| \tilde{\sigma}$'s such that $(\Omega, \sigma)$ and $(\tilde{\Omega}, \tilde{\sigma})$ give the same $\Psi[\vec{\mu}]$ inclusion. Furthermore, every $\Psi[\vec{\mu}]$ inclusion occurs in some $f(\Omega)$, by taking one of the inclusion’s associated potential tangle set
specializations, \((\Omega, \sigma)\). Hence equation (31) holds with \(F\) containing one \(\Omega\) in each rearrangeable isomorphism class corresponding to a \(\Psi[\vec{\mu}]\) inclusion.

It remains to see that \(F\) is finite, i.e., that there are only finitely many rearrangeable isomorphism classes of \(\Omega\) that correspond to \(\Psi[\vec{\mu}]\) inclusions for a given \(\Psi\) and \(\vec{\mu}\). But it suffices to let \(\Omega\) range over a subset of the \(\Pi\)-labelled graphs on a bounded number of vertices (no more than the sum of the \(\mu_i|V_{\psi_i}|\)) to get representatives of \(F\). Hence \(F\) is finite.

\[\square\]

**Definition 9.5** A formoid, \(\Gamma\), is a collection \((\Gamma_{\text{total}}, \Gamma_{\text{walk}}, \Gamma_{\text{tset}})\) of a form, \(\Gamma_{\text{walk}}\), \(\Pi\)-labelled graph, \(\Gamma_{\text{tset}}\), and their union, \(\Gamma_{\text{total}}\); here, by union we mean that we set
\[
V_{\Gamma_{\text{total}}} = V_{\Gamma_{\text{walk}}} \cup V_{\Gamma_{\text{tset}}}
\]
\[
E_{\Gamma_{\text{total}}} = E_{\Gamma_{\text{walk}}} \cup E_{\Gamma_{\text{tset}}}
\]
(so \(\Gamma_{\text{walk}}\) and \(\Gamma_{\text{tset}}\) may share vertices and edges). As such we may view \(\Gamma_{\text{total}}\) as a \(\Pi\)-labelled graph that is partially oriented and numbered (in its edges and vertices). A specialization of a formoid, \(\Gamma\), is a map, \(\iota: V_{\Gamma_{\text{total}}} \to \{1, \ldots, n\}\).

By a potential pair we mean tuple \((w; \vec{t}; \Omega; \sigma)\) consisting of a potential walk, \((w; \vec{t})\), coupled with a potential tangle set specialization, \((\Omega; \sigma)\); a potential pair determines a formoid, \(\Gamma\), with a specialization, \(\iota\), which we now describe. Given a potential pair, the potential walk determines a form, \(\Gamma_{\text{walk}}\) with a specialization \(\iota_{\text{walk}}: V_{\Gamma_{\text{walk}}} \to \{1, \ldots, n\}\). Next we form a graph, \(\Gamma_{\text{tset}}\), to represent the tangle set, \(\Omega\), and its specialization, \(\sigma\). Namely, we take a set, \(V_{\Gamma_{\text{tset}}}\), disjoint from \(V_{\Gamma_{\text{walk}}}\), of size so that there is a bijection \(\iota_{\text{tset}}: V_{\Gamma_{\text{tset}}} \to \sigma(V_{\Omega})\) (and we fix such an \(\iota_{\text{tset}}\)). We form an edge \((v_1, v_2)\) labelled \(\nu \in \Pi\) in \(E_{\Gamma_{\text{tset}}}\) for each directed edge \(e \in E_{\Omega}\) (\(E_{\Omega}\) being the directed edges of \(E_{\Omega}\)) of type \((u_1, u_2)\) labelled \(\nu\) such that \(\sigma(u_i) = \iota_{\text{tset}}(v_i)\) for \(i = 1, 2\). Finally we identify a vertex \(u \in V_{\Gamma_{\text{walk}}}\) with one \(v \in V_{\Gamma_{\text{tset}}}\) if \(\iota_{\text{walk}}(u) = \iota_{\text{tset}}(v)\). This identification gives rise to an identification of edges in \(E_{\Gamma_{\text{walk}}}\) and \(E_{\Gamma_{\text{tset}}}\) of compatible labelling. \(\iota_{\text{walk}}\) and \(\iota_{\text{tset}}\) give rise to a specialization \(\iota: V_{\Gamma_{\text{total}}} \to \{1, \ldots, n\}\).

**Definition 9.6** Given a potential pair \((w; \vec{t}; \Omega; \sigma)\), we define the \(n\)-th equivalence class of \(\vec{t}\) and \(\sigma\), denoted \([\vec{t}; \sigma]_n\), to be the pairs, \((\vec{s}, \tau)\), such that there is a permutation of the integers taking \(\vec{t}\) to \(\vec{s}\) and \(\sigma\) to \(\tau\) with the property that the components of \(\vec{s}\) and the image of \(\tau\) lie in \(\{1, \ldots, n\}\). Similarly we define
the $n$-th equivalence class of $\sigma$, denoted $[\sigma]_n$, to be those $\tau$ equivalent to $\sigma$ (i.e., differing by a permutation of the integers) and mapping $\Omega$ to $\{1, \ldots, n\}$. We also denote by $[w;\vec{t};\Omega;\sigma]$ the collection of all potential pairs $(w,\vec{s};\Omega,\tau)$ such that $(\vec{s},\tau) \in [\vec{t};\sigma]$.

We set

$$E_{\text{symm}}(w;\vec{t};\Omega;\sigma)_n = \sum_{(\vec{s},\tau) \in [\vec{t};\sigma]} E\left[\chi(w;\vec{s})\chi_\Omega;\tau\right].$$

As before, we see that $E_{\text{symm}}(w;\vec{t};\Omega;\sigma)_n$ depends only on the underlying formoid, $\Gamma$, and $n$, and furthermore

$$E_{\text{symm}}(w;\vec{t};\Omega;\sigma)_n = E[\Gamma]_n = \frac{n!}{(n-v)!} \prod_{i=1}^{d/2} \frac{(n-a_i)!}{n!},$$

(33)

where $v = |V_\Gamma|$ and $a_i$ is the number of $\pi_i$ labels occurring in $\Gamma$.

**Lemma 9.7** Given $w, \Omega$ with $|w| = k$, we have that the number of equivalence classes $[\vec{t};\sigma]$ with $t$ of length $k + 1$ and $\sigma : \Omega \to \{1, \ldots, n\}$ (i.e. so that $(w;\vec{t};\Omega;\sigma)$ is a potential pair) such that $(w;\vec{t})$ is of order $\leq r - 1$ is at most $ck^{2r+|V_\Omega|}$, where $c$ depends only on $\Omega$ and $r$.

**Proof** There are at most $ck^{2r}$ equivalence classes $[\vec{t}]$ for $w$ of order $\leq r - 1$ (by Lemma 5.7). Given $w,\vec{t}$, a $\sigma : V_\Omega \to \{1, \ldots, n\}$ will have the image of $\sigma$ intersect the components of $\vec{t}$ in, say, $s$ positions, in $\leq \binom{k}{s}$ possible ways, and the equivalence class of $(w;\vec{t};\Omega;\sigma)$ depends only on this intersection. Hence the total number of equivalence classes is

$$\leq ck^{2r} \sum_{s=0}^{k} \binom{k}{s} \leq c k^{2r+|V_\Omega|}. $$

Formoids will be grouped and summed according to their associated “typoid,” which we now define.

**Definition 9.8** A typoid, $T$, is a collection $(T_{\text{total}}, T_{\text{walk}}, T_{\text{tset}}, V_{\text{diff}})$ such that (1) $T_{\text{walk}}$ is a type, (2) $T_{\text{tset}}$ is a $\Pi$-labelled graph, (3) $T_{\text{total}}$ is the union $T_{\text{walk}} \cup T_{\text{tset}}$, and is partially $\Pi$-labelled, partially numbered, partially oriented,
Figure 4: $T^+_{\text{walk}}$ arising from $T_{\text{walk}}$ and $T_{\text{tset}}$. $V_{\text{diff}}$ consists of the two $T_{\text{tset}}$ vertices in $T^+_{\text{walk}}$; $e$ is $\Pi$-labelled and of unit length.

(4) $V_{\text{diff}}$ is a subset of vertices of $T_{\text{total}}$, and (5) $T_{\text{walk}}$ is a subgraph of the suppression $T_{\text{total}}[V_{\text{diff}}]_{\text{sup}}$. We also set $T^+_{\text{walk}}$ to be $T_{\text{walk}}$ subdivided by $V_{\text{diff}}$, and partially labelled at those edges which are $T_{\text{tset}}$ edges. The order of a typoid is the order of $T_{\text{total}}$ (i.e., $|E_{T_{\text{total}}}| - |V_{T_{\text{total}}}|$).

So consider a formoid, $\Gamma$. To $\Gamma_{\text{walk}}$ we associate its type, $T_{\text{walk}}$ (an oriented, numbered, unlabelled graph). To $\Gamma_{\text{total}} = \Gamma_{\text{walk}} \cup \Gamma_{\text{tset}}$ we associate a variant of the notion of “type,” as follows. $V_{T_{\text{walk}}}$ are those vertices of $\Gamma_{\text{walk}}$ that are either (1) the first vertex (according to the numbering), or (2) a vertex of degree $\geq 3$. Set $V_{\text{diff}}$ to be $(\Gamma_{\text{walk}} \setminus V_{T_{\text{walk}}}) \cap V_{\Gamma_{\text{tset}}}$, i.e., those vertices that are suppressed in forming $T_{\text{walk}}$ that are also $\Gamma_{\text{tset}}$ vertices. Let $T_{\text{total}}$ be $\Gamma_{\text{total}}$ with $\Gamma_{\text{walk}} \setminus (V_{\text{diff}} \cup V_{T_{\text{walk}}})$ supressed. So $T_{\text{total}}$ is $\Gamma_{\text{total}}$ with the supression of all vertices that are not in $\Gamma_{\text{tset}}$ nor “distinguished” in $\Gamma_{\text{walk}}$. Then $T_{\text{walk}}$ is a subgraph of $T_{\text{total}}[V_{\text{diff}}]_{\text{sup}}$. The numbering and orientation of $\Gamma_{\text{walk}}$ gives rise to one on $T_{\text{walk}}$ and $T^+_{\text{walk}}$, and to a partial numbering and orientation of $T_{\text{total}}$ (basically we remember which vertices and edges are traversed first, and in which orientation first).

Figure 4 gives an example. The numberings and orientation have been omitted. The bends in the edges with no vertices represent “insignificant” form vertices of degree two, although the type forgets such vertices and how many there are. There we see that $T^+_{\text{walk}}$ has two additional vertices and one
edge, $e$, fixed of length 1 (since $e \in T_{tset}$). Note that in this example each (of
the two) $T_{tset}$ components “meet” $T_{walk}$, and so $T_{walk}^+$ has only one connected
component; but, in general, $T_{walk}^+$ can have many connected components.

**Definition 9.9** By a $B$-new typoid, $\tilde{T}$ we mean a collection

$$\tilde{T} = (T; E_{long}, E_{fixed}; \vec{k}^{\text{fixed}})$$

where (1) $T$ is a typoid, (2) $T_{tset}$ is lettered, (3) $E_{total}$ is partitioned into two
sets, $E_{long}$ and $E_{fixed}$, (4) for each $e_i \in E_{fixed}$ we have an edge length $k_i^{\text{fixed}}$,
where $0 < k_i^{\text{fixed}} < B$, (5) we have a $\Pi^+$-labelling of each $e_i \in E_{fixed}$ with a
label of size $k_i^{\text{fixed}}$, and (6) for each $e_i$ labelled in $T_{total}$ (i.e., originally coming
from $T_{tset}$), $e_i \in E_{fixed}$ and has the same label and length (namely one) as in
$T_{total}$. We say that $\tilde{T}$ is based on $T$. The subdivision of $T_{walk}$ by adding $V_{diff}$,
namely $T_{walk}^+$, becomes a $B$-new type except that it may have some vertices of
degree two (namely those of $V_{diff}$).

**Theorem 9.10** Let $T', \Omega$ be fixed. Then there are only finitely many typoids,
$T$, such that (1) $T_{walk} = T'$, and (2) $T$ arises from a formoid arising from
a potential pair $(w; \vec{t}; \Omega; \sigma)$ for some $w, \vec{t}, \sigma$. Also, for a fixed typoid, $T$, and
any $B$, there are only finitely many $B$-new typoids based on $T$.

**Proof** $T_{walk}$ is fixed. Also $T_{tset}$, viewed without regard for $T_{walk}$, is deter-
mined by $\Omega$. $T_{total}$ is determined by these two fixed abstract graphs ($T_{walk}$
and $T_{tset}$) and the way they intersect, meaning (1) which $T_{tset}$ vertices are
$T_{walk}$ vertices, (2) which $T_{tset}$ vertices lie in a $T_{walk}$ edge (i.e., belong to $V_{diff}$),
and (3) which $T_{tset}$ edges coincide with or lie in a $T_{walk}$ edge. Since $T_{walk}$ and
$T_{tset}$ are finite, this can happen in only finitely many ways.

The second part of the theorem (concerning finitely many new typoids
based on $T$) is clear (there are finitely many letterings, finitely many $E_{total}$
partitions, finitely many $k_i^{\text{fixed}}$ values for $e_i \in E_{fixed}$, and finitely may labels).

\[ \square \]

We are almost ready to describe the $R$ to be used in equation (30).

**Lemma 9.11** Let $\psi$ be a nonempty, pruned tangle of order $\geq 1$. Let $c_r(\psi)$
denote the maximum number of distinct inclusions of $\psi$ into a $T_{tset}$ of a
typoid, $T$, of order $< r$. Then $c_r(\psi)$ is finite.
Proof There can be at most \( r - 1 \) connected components of \( T_{\text{set}} \), since each component has order \( \geq 1 \). It suffices to show that there are a finite number of inclusions \( \iota: \psi \to G \) of a connected labelled graph of order \( \leq r - 1 \) and \( \geq 1 \). Let \( G_{\text{type}} \) be \( G \) with its degree two vertices suppressed (no cycle of degree two can exist in \( G \)).

Each \( \iota \), gives rise to a map \( \bar{\iota}: V_\psi \to V_{G_{\text{type}}} \cup E_{G_{\text{type}}} \), where \( \bar{\iota}(v) \) is either the \( V_{G_{\text{type}}} \) vertex it is sent to under \( \iota \), or the edge \( e \in E_{G_{\text{type}}} \) where \( \iota(v) \) is sent to (after subdividing the edge). If \( \bar{\iota} \) does not have any \( V_{G_{\text{type}}} \) element in its image, then since \( \psi \) is connected and consists of edges of length 1, \( \iota \) includes \( \psi \) as a union of subpaths of a single, subdivided edge. But this contradicts the fact that \( \psi \) is pruned, nonempty, and of order \( \geq 1 \).

So we may assume that \( \bar{\iota}(v) \in V_{G_{\text{type}}} \) for some \( v \). Since \( \psi \) is connected we see by induction on the distance to \( v \) that every \( V_\psi \) vertex has its image in \( G_{\text{type}} \) determined either by the labelling of \( \psi \) and the lettering of \( G_{\text{type}} \) or by the fact that the \( G_{\text{type}} \) labellings must be irreducible words.

Hence the knowledge of \( \bar{\iota} \), of which there are only finitely many possibilities, determines \( \iota \), and we are done.

\[ \square \]

Now let \( \Psi \) be a set of distinct, pruned, nonempty tangles, and set

\[ c_r(\Psi) = \max_{\psi \in \Psi} c_r(\psi). \]

Let \( R = c_r(\Psi) + 1 \). A \( \Psi[\vec{\mu}] \) inclusion with \( \max \mu_i = R \) implies that the typoid corresponding has order \( \geq r \).

\[ \square \]

We have proven the following.

**Lemma 9.12** Given \( \Psi \) and \( r \) there is an \( R \) such that the following holds. Let \( (w; \vec{t}; \Omega; \sigma) \) be a potential pair corresponding to a \( \Psi[\vec{\mu}] \) inclusion with \( \max \mu_i = R \). Then the typoid corresponding to \( (w; \vec{t}; \Omega; \sigma) \) is of order \( \geq r \).

Now if \( (\Omega, \sigma) \) corresponds to a \( \Psi[\vec{\mu}] \) inclusion where \( \max(\mu_i) = R \), and \( \Gamma \) is the formoid associated to a potential pair \( (w; \vec{t}; \Omega; \sigma) \), then

\[ E[\Gamma](n) \leq \frac{n(n-1) \cdots (n-|V_\Gamma|+1)}{n(n-1) \cdots (n-|E_\Gamma|+1)} \leq (n-|E_\Gamma|+1)^{-r}, \]
since $\Gamma$ must be of order $\geq r$. If $|w| = k$, we have

$$|E_\Gamma| \leq k + R|E_\Psi|.$$  

Hence

$$E[\Gamma] \leq (n/2)^{-r}$$  

provided that $k \leq n/2 - c$, where $c$ is a constant depending only on $\Psi$ and $R$.

**Lemma 9.13** For any finite $\Psi$, take $R$ as above. For any SSSI $\mathcal{W}$ we have

$$E \left[ \sum_{(w;\vec{t}) \in \mathcal{W}(k,n)} \sum_{\max(\mu_i) = R} N_{\Psi,\vec{t}} \chi_{(w,\vec{t})} \right] \leq ck^{\tilde{r}}(d-1)^k n^{-r}$$  

where $c$ and $\tilde{r}$ depend on only $\Psi$ and $R$.

**Proof** There are $\leq c(d-1)^k$ irreducible words, $w$, of length $k$ (where $c = d/(d-1)$). By Theorem 9.4 the $N_{\Psi,\vec{t}}$ sum can be replaced by a sum over

$$f(\Omega) = |\text{ReAut}(\Omega)|^{-1} \sum_\sigma \chi_{\Omega,\sigma}$$

summed over a finite collection, $\mathcal{F}$, of $\Omega$’s. For each fixed $w$ and $\Omega$ there are $\leq ck^{2r+|V_{\Omega}|}$ equivalence classes $[\vec{t},\sigma]_{(n)}$ with $(w;\vec{t};\Omega;\sigma)$ a potential pair. So by equation (34) the symmetric sum corresponding is of size at most $(n/2)^{-r}$. So the expected value in equation (35) is at most $c(d-1)^k k^{\tilde{r}}(n/2)^{-r}$, where

$$\tilde{r} = 2r + \max_{\mathcal{F} \text{ for } \Psi[\vec{t}], \Omega \in \mathcal{F}} \max_{\max(\mu_i) = R} |V_{\Omega}|,$$

with $\mathcal{F}$ as in Theorem 9.4.

By the above lemma we wish to get an asymptotic expansion for

$$E \left[ \sum_{(w;\vec{t}) \in \mathcal{W}(k,n)} \sum_{\tau \in [\sigma]_n, \text{ of order } \leq r-1} \chi_{\Omega,\tau} \chi_{(w,\vec{t})} \right]$$
for fixed $\Omega, \sigma$ in order to prove Theorem 9.3. We know this expression equals
\[
\sum_{\Gamma \text{ order } \leq r-1} \mathbb{E}[\Gamma]_n \text{ (number } [w; \vec{t}; \Omega; \sigma] \text{ classes of formoid } \Gamma). \tag{36}
\]
It suffices to choose a $B$ and then find an asymptotic expansion for the sum over only those $\Gamma$ of a fixed $B$-new typoid. Before doing so, let us remark that we may write
\[
\mathbb{E}[\Gamma]_n = p_0 + \frac{p_1}{n} + \cdots + \frac{p_{r-1}}{n^{r-1}} + \frac{\text{error}}{n^r}, \tag{37}
\]
where $p_i$ is a polynomial in the $a_i$ and $v$ of equation (33), and
\[
|\text{error}| \leq e^{rk/n} \left( v(v - 1)/2 + a_1(a_1 - 1)/2 + \cdots + a_d/2(a_d/2 - 1)/2 \right)^r
\]
(by equation (17)). Since our potential walk is of length $k$, we have
\[
v \leq k + |V_\Omega|,
\]
and
\[
\sum a_i \leq k + |E_\Omega|.
\]
Also if $\Gamma$ is of a fixed typoid, since $\Omega$ is fixed we have that $v$ is a function of the $a_i$. The upshot is that in equation (37) we have that the $p_i$ are polynomials in the $a_i$ alone (if $\Gamma$ is of a fixed typoid) and the error term satisfies
\[
|\text{error}| \leq ck^{2r}
\]
for all $k \leq n/2$.

We now apply equation (37) to equation (36) to study
\[
\sum_{\Gamma \text{ order } \leq r-1} \left( p_0(\Gamma) + \frac{p_1(\Gamma)}{n} + \cdots + \frac{p_{r-1}(\Gamma)}{n^{r-1}} + \frac{\text{error}(\Gamma)}{n^r} \right) \text{Cl}(\Gamma), \tag{38}
\]
where Cl($\Gamma$) is the number of $[w; \vec{t}; \Omega; \sigma]$ equivalence classes of $\Gamma$.

Since for a fixed $w$, the number of $[w; \vec{t}; \Omega; \sigma]$ equivalence classes is bounded by $ck^{2r + |V_\Omega|}$, we have
\[
\sum_{\Gamma} \text{Cl}(\Gamma) \leq ck^{2r + |V_\Omega|}(d - 1)^k,
\]
81
and so
\[
\sum_{\Gamma} \Cl(\Gamma) \error(\Gamma) \leq c k^{d+|V_\Gamma|} (d-1)^k.
\]

It remains to show that for suitable \( B \), each \( B \)-new typoid, \( \tilde{T} = (T; E_{\text{long}}, E_{\text{fixed}}; \tilde{k}_{\text{fixed}}) \), has
\[
\sum_{\Gamma \in \tilde{T}} \Cl(\Gamma) p_i(\Gamma),
\]
(summed over \( \Gamma \) of new typoid \( \tilde{T} \)) is \( d \)-Ramanujan. For positive integers \( \{m_i\} \) indexed over \( e_i \) that are edges of \( T_{\text{walk}}^+ \), set
\[
W_{\tilde{T}}(\tilde{m}; S, \Psi')
\]

which is the number of walk classes of new type \( \tilde{T} \), traversing the edge \( e_i m_i \) times, that are irreducible (\( S, \Psi' \)) selective cycles and that respect the \( T_{\text{walk}}^+ \) numbering and orientation; assuming \( \tilde{T} \) is a \( B \)-new type with \( B > S \), this number does not depend on the “long” (i.e., \( E_{\text{long}} \)) edge lengths. Index the edges so that \( E_{\text{long}} = \{e_1, \ldots, e_t\} \) and \( E_{\text{fixed}} = \{e_{t+1}, \ldots, e_b\} \). We claim that the proof of Theorem 6.6 shows that
\[
W_{\tilde{T}, S}(M_1, M_2) = \sum_{\substack{m_1 + \ldots + m_1 = M_1 \\ m_{t+1} + \ldots + m_k = M_2}} W_{\tilde{T}}(\tilde{m}; S, \Psi'),
\]
satisfies the bound
\[
W_{\tilde{T}, S}(M_1, M_2) \leq c B (\sqrt{d} - 1 - \epsilon)^{C M_1 + M_2}.
\]
for \( S \geq S_0 \) for some \( c, S_0, \) and \( \epsilon \) depending only on \( \tilde{T} \). This is because the argument of Theorem 6.6 is unaffected by the two essential new features that \( T_{\text{walk}}^+ \) has over types, which are the possible presence of (1) some degree 2 vertices, and (2) some edges whose lengths are fixed at 1 (namely “tangle” or \( T_{\text{tset}} \) edges). But then Theorem 8.2 holds for \( \tilde{T} \) being a \( B \)-new typoid. But each \( a_i = a_i(\Gamma) \), which is the number of edges labelled \( \pi_i \) in \( \Gamma \) (or \( \Gamma_{\text{total}} \)), has
\[
a_i(\Gamma) = a_i(\Gamma_{\text{walk}}^+) + a_i(\Gamma_{\text{tset}} \setminus \Gamma_{\text{walk}}^+),
\]
and \( a_i(\Gamma_{\text{tset}} \setminus \Gamma_{\text{walk}}^+) \) is fixed in a new typoid, \( \tilde{T} \). Thus for any polynomial \( P \) and new typoid, \( \tilde{T} \), there is a polynomial \( \tilde{P} \) such that
\[
P(a_1(\Gamma), \ldots, a_{d/2}(\Gamma)) = \tilde{P}(a_1(\Gamma_{\text{walk}}^+), \ldots, a_{d/2}(\Gamma_{\text{walk}}^+)).
\]
Thus for the polynomials $p_i$ (of equations (35) and (38)) and a fixed $B$-new typoid $\tilde{T}$ there are $\tilde{p}_i$ with

$$\sum_{\Gamma \in \tilde{T}} \text{Cl}(\Gamma)p_i(\Gamma) = \sum_{\Gamma \in T^+_\text{walk}} \tilde{p}_i(a_1(\Gamma^+_\text{walk}), \ldots, a_{d/2}(\Gamma^+_\text{walk})).$$

By the analogue of Theorem 8.2 with $\tilde{T}$ being a $B$-new typoid, for all $i$ the right-hand-side of this equation is $d$-Ramanujan. Summing over the finitely many $B$-new typoids gives Theorem 9.3.

### 10 Strongly Irreducible Traces

We wish to use Theorem 9.3 to estimate eigenvalues. However, it is easier to use strongly irreducible traces rather than irreducible traces. We explain this and develop the properties of the strongly irreducible trace in this section.

**Definition 10.1** A word $w \in \Pi^*$ is strongly irreducible if $w$ is irreducible and $w = \sigma_1 \cdots \sigma_k$ with $\sigma_1 \neq \sigma_k^{-1}$.

For any of our irreducibe traces, selective irreducible traces, irreducible walk sums, etc., we can form its “strongly irreducible” version where we discard contributions from words that are not strongly irreducible. In any graph, labelled or not, one can speak of strongly irreducible cycles as those cycles that are irreducible and whose last step is not the opposite of its first step.

**Definition 10.2** The $k$-th strongly irreducible trace of a graph, $G$, is the number of strongly irreducible cycles of length $k$ for a positive integer $k$; we denote it $\text{SIT}(G, k)$ or $\text{SIT}(A, k)$ if $A$ is the adjacency matrix of $G$. If $G$ has half-loops, we consider each half-loop to be a strongly irreducible cycle (of length 1) and include it in our count for $\text{SIT}(G, 1)$ or $\text{SIT}(A, 1)$.

Half-loops are only a concern for us in the model $\mathcal{J}_{n,d}$; the reason that we count half-loops as strongly irreducible is to make Lemma 10.4 hold.

With a cycle of length $k$ in $G_{\text{Irred}}$ about a directed edge, $e$, of $G$, we may associate the strongly irreducible cycle in $G$ about the vertex in which $e$ originates. It follows that if $\mu_i$ are the eigenvalues of $G_{\text{Irred}}$, we have

$$\text{SIT}(G, k) = \sum_{i=1}^{nd} \mu_i^k$$

83
for all $k \geq 2$; for $k = 1$ we must add the number of half-loops to the right-hand-side of the above equation, since there is no edge in $G_{\text{Irred}}$ from a vertex to itself when the vertex corresponds to a half-loop.

We will study the relationship between $\text{IrredTr}(G, k)$ and $\text{SIT}(G, k)$ and its consequences. The most important consequence is the following theorem.

**Theorem 10.3** For $|\lambda| \leq d$, let

$$
\mu_{1,2}(\lambda) = \frac{\lambda \pm \sqrt{\lambda^2 - 4(d-1)}}{2},
$$

and set

$$
\tilde{q}_k(\lambda) = \mu_1^k(\lambda) + \mu_2^k(\lambda) + (1 + (-1)^k)(d-2)/2.
$$

(40)

If $G$ is a $d$-regular graph with no half-loops and adjacency matrix eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$, then

$$
\text{SIT}(G, k) = \sum_{i=1}^{n} \tilde{q}_k(\lambda_i).
$$

(In a sense, to each eigenvalue, $\lambda$, of $G$, there correspond eigenvalues $\mu_{1,2}(\lambda)$ of multiplicity one each and eigenvalues $1$ and $-1$ of multiplicity $(d-2)/2$ each in $G_{\text{Irred}}$.) Furthermore, if instead $G$ has no whole-loops, then the same is true with $\tilde{q}_k$ replaced by $\hat{q}_k$ where

$$
\hat{q}_k(x) = \begin{cases} 
\tilde{q}_k(x) & \text{if } k \text{ is even or } k = 1, \\
\tilde{q}_k(x) - x & \text{if } k \geq 3 \text{ is odd.}
\end{cases}
$$

Furthermore, we shall see that $\tilde{q}_k$, like the $q_k$ of Lemma 2.3, are polynomials of degree $k$ that may alternatively be expressed as a simple linear combination of Chebyshev polynomials (plus the $\pm 1$ eigenvalue contribution for the $\tilde{q}_k$).

**Lemma 10.4** Let $G$ be a $d$-regular graph on $n$ vertices with $h$ half-loops. Then for all integers $k \geq 2$ we have\(^{15}\)

$$
\text{IrredTr}(G, k) = \text{SIT}(G, k) + (d-2) \sum_{i=1}^{[(k-1)/2]} (d-1)^{i-1}\text{SIT}(G, k-2i)
$$

+ \begin{cases} 
0 & \text{if } k \text{ is even,} \\
(d-1)^{(k-3)/2}h & \text{if } k \text{ is odd.}
\end{cases}
$$

\(^{15}\)For $k = 2$ the summation in the formula to follow is ignored, since it ranges from $i = 1$ to $i = [(k-1)/2] = 0.$
Therefore for all \( k \geq 4 \) (and for \( k = 3 \) when \( h = 0 \)) we have

\[
\text{IrredTr} (G, k) - (d - 1)\text{IrredTr} (G, k - 2) = \text{SIT} (G, k) - \text{SIT} (G, k - 2) \quad (41)
\]

**Proof** The last equation follows from the previous one, with a little care when \( k = 3 \); indeed, for \( k = 3 \) we have \( \text{SIT} (G, k - 2) = \text{SIT} (G, 1) = 2w + h \), where \( w, h \) are the number of whole- and half-loops. So

\[
\text{IrredTr} (G, 3) = \text{SIT} (G, 3) + (d - 2)\text{SIT} (G, 1) + h,
\]

and

\[
\text{IrredTr} (G, 1) = \text{SIT} (G, 1) = \text{Trace} (A) = 2w + h.
\]

Hence

\[
\text{IrredTr} (G, 3) - (d - 1)\text{IrredTr} (G, 1) - \text{SIT} (G, 3) + \text{SIT} (G, 1) = h.
\]

Thus equation (41) holds with \( k = 3 \) if \( h = 0 \). We similarly show that this equation holds regardless of \( h \) for \( k \geq 4 \).

So it suffices to prove the first equation of the lemma. Each irreducible cycle about a vertex, \( v \), begins by traversing a path, \( p \), to a vertex, \( w \), then follows a strongly irreducible (nonempty) cycle about \( w \), and then backtracks over \( p \) (and this statement is only true if count half-loops as strongly irreducible). So we may count irreducible cycles of length \( k \) in \( G \) by counting how many paths of length \( i \) there are from \( w \) that when combined with a strongly irreducible cycle about \( w \) of length \( k - 2i \) yield an irreducible cycle, \( C \), of length \( k \). The strongly irreducible cycle’s length, \( k - 2i \), must be positive, or else \( C \) isn’t irreducible. If \( C \) is of length \( \geq 2 \) or is a whole-loop, then there are \( d - 2 \) possibilities for the first edge of the path (since two edges are ruled out by \( C \) in either case), and \( d - 1 \) possibilities for all edge choices thereafter. For a half-loop there are \( d - 1 \) possibilities for the first edge (since only the single half-loop edge is ruled out). So the contribution per strongly irreducible cycle of length \( k - 2i \) is 1 for \( i = 0 \), \( (d - 2)(d - 1)^{i - 1} \) for \( i \geq 1 \), except in a half-loop, where the contribution per half-loop is \( (d - 1)^i \), i.e., an additional \( (d - 1)^{i - 1} \) beyond the standard contribution. Since \( k \) is odd and \( i = (k - 1)/2 \) in the case of a half-loop, the additional amount beyond the standard contribution is \( (d - 1)^{(k-3)/2} \) per half-loop.

\( \Box \)
We return to the proof of the theorem. First assume that $G$ has no half-loops. We will prove by induction on $k \geq 1$ that

$$\text{SIT}(G, k) = \sum_{i=1}^{n} \tilde{q}_k(\lambda_i),$$

where $\tilde{q}_k$ are polynomials of degree $k$. Clearly

$$\text{SIT}(G, 1) = \text{Trace}(A),$$

and so $\tilde{q}_1$ exists as desired with $\tilde{q}_1(\lambda) = \lambda$. Of the cycles of length 2, all irreducible cycles are strongly irreducible, so $\tilde{q}_2(\lambda) = \lambda^2 - d$. Lemmas 10.4 and 2.3 now imply (by induction on $k$) that polynomials $\tilde{q}_k(\lambda)$ exist of degree $k$ satisfying equation (42), and that the $\tilde{q}_k$, for $k \geq 2$, are annihilated by

$$(\sigma_k^2 - 1)(\sigma_k^2 - \lambda \sigma_k + (d - 1)), \quad (43)$$

where $\sigma_k$ is the “shift in $k$” operator, i.e., $\sigma_k(f(k)) = f(k + 1)$ (here we use the fact that the $q_k$ are annihilated by $\sigma_k^2 - \lambda \sigma_k + (d - 1)$, mentioned below Lemma 2.3). Since $\mu_{1,2}(\lambda), \pm 1$ are the four roots in $\sigma_k$ of equation (43), we have

$$\tilde{q}_k(\lambda) = c_1 \mu_1^k + c_2 \mu_2^k + c_3 + c_4 (-1)^k,$$

where the $c_i = c_i(\lambda)$ assuming that the four roots are distinct. There are now two ways to finish the theorem.

The first method to finish the theorem is to calculate $\tilde{q}_3, \tilde{q}_4$ and verify equation (40) holds for $k = 2, 3, 4, 5$, i.e., that $c_1 = c_2 = 1$ and $c_3 = c_4 = (d - 2)/2$ work in those cases. Then by uniqueness (i.e., the nonvanishing of a van der Monde determinant), those $c_i$’s must be the unique $c_i$’s that work for all $k$.

Another way to finish the theorem is to use the fact that $c_1(d) = 1$ (see the remark after Lemma 5.9), and then argue that $c_1(\lambda) = c_2(\lambda) = 1$ by analytic continuation. First, the $c_i$’s are the unique solutions to a $4 \times 4$ system of equations with coefficient analytic in $\lambda$; hence the $c_i$ are, indeed, analytic in $\lambda$. Next, notice that $\mu_1(\lambda)$ at $\lambda = d$ analytically continues to $\mu_2(\lambda)$ at $\lambda = d$ by one loop about $2\sqrt{d - 1}$; thus is suffices to prove that $c_1 = 1$ near $d$. Next notice that different $\lambda$’s give different $\mu$’s (indeed, $r^2 - \lambda_1 r + (d - 1) = 0$ and $r^2 - \lambda_2 r + (d - 1) = 0$ for the same $r$ implies $r(\lambda_2 - \lambda_1) = 0$, so if $\lambda \neq d$ is an eigenvalue of multiplicity $k$ in a graph, then $c_1(\lambda)$ times $k$ must be an
integer. But there is a sequence, \( z_n \to 1 \), with \( c_1(z_n) \) an integer or half-integer (namely a cycle of length \( n \) where each edge has multiplicity \( d/2 \) has \( z_n = (d/2) \cos(2\pi/n) \) as an eigenvalue of multiplicity two). So by continuity \( c_1(z_n) = 1 \) for sufficiently large \( n \), and thus \( c_1 \) is identically 1.

It suffices to determine \( c_3, c_4 \), which from \( \tilde{q}_1, \tilde{q}_2 \) we find are (the constant functions) \( c_3 = c_4 = (d - 2)/2 \).

Finally, if \( G \) has only half-loops (no whole-loops), then the \( k \) even formula and \( k = 1 \) formula are unchanged. We easily see by induction on odd \( k \geq 3 \) that the polynomial \( \tilde{q}_k = \tilde{q}_k(x) - x \) works (we use the fact that \( h \), the number of half-loops, is the trace of \( A_G \), i.e., \( h = \sum \lambda_i \)).

\[ \text{Theorem 10.5} \]

Let an integer \( d > 2 \) and a real \( \epsilon > 0 \) be fixed. There is an \( \eta > 0 \) such that if \( G \) is a \( d \)-regular graph with \( |\lambda_i| \leq d - \epsilon \) for all \( i > 1 \), then the \( \mu_i \) of equation (39) satisfy \( \mu_1 = d - 1 \) and \( |\mu_i| \leq d - \eta \) for all \( i > 1 \).

\[ \text{Proof} \]

The \( \mu_i \) must be \( \pm 1 \) or roots of the equation in \( \mu \)

\[
\mu^2 - \mu \lambda_i + (d - 1) = 0,
\]

or

\[
\mu = \frac{\lambda_i \pm \sqrt{\lambda_i^2 - 4(d - 1)}}{2}.
\]

For \( i = 1 \) we have \( \lambda_1 = d \) and the corresponding \( \mu \)’s are \( \mu = d - 1, 1 \). The other \( \mu \)’s are either 1 or come from \( \lambda_i \) with \( i > 1 \). But for \( |\lambda_i| \leq d - \epsilon \) it is easy to see that the corresponding \( \mu \)’s are bounded away from \( d - 1 \).

\[ \square \]

We can form a selective, strongly irreducible trace by taking

\[ \text{SSIT}_{S,\Psi'}(G; k) \]

to be the number of strongly irreducible cycles of length \( k \) that are \( (S, \Psi') \)-selective. Notice that there are no more strongly irreducible cycles than irreducible cycles in any lettered type, and the strong irreducibility of a potential walk can be determined from its image in the corresponding lettered type. Hence the expansion theorems of Sections 6–9, especially Theorem 9.3, carry over to SSIT replacing IrSelTr, by simply replacing

\[ W_{\tilde{T},S}(M_1, M_2) \]

by the same number of walk classes of the new typoid, \( \tilde{T} \), except requiring that the walks are strongly irreducible.
11 A Sidestepping Lemma

Lemma 11.1 Fix integers $r, \tilde{r}, d$ with $d > 2$, polynomials $p_0, \ldots, p_r$, a constant, $c$, and integer $D$. Assume that for each $n$ we have random variables $\theta_1, \ldots, \theta_m$ such that $m = Dn$, and $1 - \theta_i$ is either a real number of absolute value between 1 and $(d - 1)^{-1/2}$ or a complex number of absolute value $(d - 1)^{-1/2}$ (or both). Furthermore assume that for all integer $k \geq 1$ we have

\[
E \left[ \sum_i (1 - \theta_i)^k \right] = \sum_{j=0}^{r-1} p_j(k)n^{-j} + O(k^{r-1}n^{r-1} + k^c(d - 1)^{-k/2}).
\]

(44)

Then for sufficiently large $n$ we have

\[
E \left[ \sum_i \chi_{\{|\theta_i| > \log^{-1/2} n\}} (1 - \theta_i)^k \right] = O(Dn^{1-(r/3)} + k^c(d - 1)^{-k/2})
\]

for all $k$ with $1 \leq k \leq n^\gamma$ for some constant $\gamma > 0$, where the constant $\gamma$ and the constant in the $O(\cdot)$ notation depends only on $r, \tilde{r}, d$ and the maximum degree of the $p_i$.

After this section we will apply this lemma with the $(d - 1)(1 - \theta_i)$ being the eigenvalues of $G_{\text{Irred}}$, with $k$ proportional (for fixed $r$) to log $n$.

If $\sigma_k$ denotes the “shift with respect to $k$,” i.e., $\sigma_k(f(k)) = f(k+1)$, then some fixed power of $\sigma_k - 1$ annihilates the $p_j(k)$, and also

\[(\sigma_k - 1)^i(1 - \theta)^k = (-\theta)^i(1 - \theta)^k.\]

This allows us to say a lot about the $\theta_i$ by applying some power of $\sigma_k - 1$ to equation (44). For the irreducible trace, the $(1 - \theta_i)^k$ are replaced by certain Chebyshev polynomials of $\theta_i$, and applying powers of $\sigma_k - 1$ to them seems more awkward; this is why we have introduced strongly irreducible traces.

In [Fri91], we worked with irreducible traces, not strongly irreducible traces. There we had the $(1 - \theta_i)^k$ replaced by Chebyshev polynomials of $1 - \theta_i$; however (1) we knew that $\theta_1 = 0$ and $\theta_i$ was bounded away from 0 with probability $1 - O(n^{1-d})$, and (2) we could only prove the asymptotic expansion up to $r$ which was roughly proportional to $d^{1/2}$. So we could directly apply the analogue of equation (44) with $k$ roughly $\log^2 n$ to determine that $p_0(k) = 1$ and the higher $p_j$ vanish (up to $j$ roughly proportional to $d^{1/2}$). In this paper the arbitrary length of the asymptotic expansion for a type of
trace comes at the cost of having far less control over the $\theta_i$, and we have no ability to determine the $p_j$ exactly. Fortunately Lemma 11.1 allows us to control the $\theta_i$ bounded away from 0, and fortunately we will see that the $\theta_i$ for $i > 1$ are bounded away from 0 with probability $1 - O(n^{-\tau_{\text{fund}}})$.

We wish to comment that one expects polynomials $p_j = p_j(k)$ to arise from the binomial expansion. Namely (by Taylor’s theorem),

$$
(1 - \theta)^k = \sum_{i=0}^{s-1} \binom{k}{i} (-\theta)^i + \binom{k}{s} (-\xi)^s (1 - \xi)^{k-s},
$$

for some $\xi \in [0, \theta]$. So for those $\theta \leq n^{-\beta}$ for some constant $\beta > 0$, we may take $s$ roughly $r/\beta$ and get an error term in $\xi$ bounded by roughly $O(n^{-r})$; we get a similarly bounded error term when taking expected values of $(1 - \theta)^k \chi_E$ where $E$ is the event that $\theta \leq n^{-\beta}$. In this way, equation (45) could give rise to the terms of an asymptotic expansion.

**Proof** (of Lemma 11.1). Let $s$ be a fixed even integer such that the maximum degree of the $p_j$ is $\leq s - 1$. We apply $(\sigma_k - 1)^s$ to equation (44) to conclude that

$$
E \left[ \sum_i (-\theta_i)^s (1 - \theta_i)^k \right] = O(k^7 n^{-r} + k^c (d - 1)^{-k/2})
$$

for any $k$, where the constant in the $O(\cdot)$ notation depends on $d$ and $s$. We conclude that for $\log^2 n \leq k \leq n^{r/(2\tau)}$ we have

$$
E \left[ \sum_i \theta_i^s (1 - \theta_i)^k \right] = O(n^{-r/2}).
$$

Applying this for $k = \lfloor n^\gamma \rfloor$ and $k = \lfloor \log^2 n \rfloor$, where $\gamma = r/(2\tau)$, and subtracting we conclude

$$
E \left[ \sum_i \theta_i^s ((1 - \theta_i)^{\lfloor \log^2 n \rfloor} - (1 - \theta_i)^{\lfloor n^\gamma \rfloor}) \right] = O(n^{-r/2}).
$$

Since $\theta_i$ is real unless $1 - \theta_i = (d - 1)^{-1/2}$, we conclude that

$$
E \left[ \sum_i |\theta_i|^s ((1 - \theta_i)^{\lfloor \log^2 n \rfloor} - |1 - \theta_i|^{\lfloor n^\gamma \rfloor}) \right]
$$

89
\[= O(n^{-r/2}) + O\left(n(d - 1)^{-\left(\log^2 n\right)/2}\right) = O(n^{-r/2}).\]

Now for any \(\theta_i \leq \log^{-2} n\) we have \((1 - \theta_i)^{\log^2 n}\) is at least roughly \(1/e\) for large \(n\); also for \(\theta_i \geq n^{-\alpha}\) for a constant \(\alpha > 0\) we have \((1 - \theta_i)^{n^\gamma}\) is near 0 for large \(n\) provided \(\alpha < \gamma\), and also \(\theta_i \geq n^{-\alpha}\). We conclude that

\[E\left[\sum_i (\chi_{n^{-\alpha} \leq \theta_i \leq \log^{-2} n}) n^{-s\alpha}\right] = O(n^{-r/2}),\]

and hence, since \(\theta_i\) is real for \(|\theta_i| < \log^{-2} n\) for \(n\) large,

\[E\left[\chi_{n^{-\alpha} \leq \theta_i \leq \log^{-2} n}\right] = O(n^{s\alpha-(r/2)}). \tag{46}\]

Let \(\alpha > 0\) be fixed with \(s\alpha < r/6\), and set

\[
\begin{align*}
A_1[i, n] &= \text{The event that } \theta_i < n^{-\alpha}, \\
A_2[i, n] &= \text{The event that } n^{-\alpha} \leq \theta_i \leq \log^{-2} n, \\
A_3[i, n] &= \text{The event that } \log^{-2} n < |\theta_i|.
\end{align*}
\]

Equation (46) implies that

\[\text{Prob}\{A_2[i, n]\} = O(n^{-r/3}).\]

Since the number of \(\theta_i\) is linear in \(n\), we conclude that

\[E\left[\sum_i \chi_{A_2[i, n]}(1 - \theta_i)^k\right] = O(n^{1-(r/3)}). \tag{47}\]

By the comment just before the proof, we have (using Taylor’s theorem)

\[E\left[\chi_{A_1[i, n]}(1 - \theta_i)^k\right] = \sum_{j=0}^{j \leq 2r/\alpha} \binom{k}{j} E\left[\chi_{A_1[i, n]}(-\theta_i)^j\right] + O(k^{1+(2r/\alpha)}n^{-2r}).\]

Summing over \(i\) in the above involves summing over \(i\) in the expected values and multiplying the error term by a number linear in \(n\). So let

\[q(k, n) = \sum_i \sum_{j=0}^{j \leq r/\alpha} \binom{k}{j} E\left[\chi_{A_1[i, n]}(-\theta_i)^j\right],\]

90
which is a polynomial of fixed degree in $k$ whose coefficients depend on $n$ (and the $\theta_i$ which are given for each value of $n$). Fix a $\gamma$ for which

$$O(k^{1+(2r/\alpha)n^{1-2r}}) = O(n^{-r/3})$$

for all $k \leq n^\gamma$, i.e., fix a $\gamma$ with

$$\gamma(1 + (r/\alpha)) \leq 5r/3 - 1.$$  

Then for all $k \leq n^\gamma$ we have

$$E\left[\sum_i \chi_{A_1[i,n]}(1 - \theta_i)^k\right] = q(k, n) + O(n^{-r/3}). \quad (48)$$

Now combine

$$E\left[\sum_i (1 - \theta_i)^k\right] = \sum_{j=1}^3 E\left[\sum_i \chi_{A_j[i,n]}(1 - \theta_i)^k\right]$$

with equations (48) and (47) to conclude that for $k \leq n^\gamma$ we have

$$E\left[\sum_i (1 - \theta_i)^k\right] = q(k, n) + E\left[\sum_i \chi_{A_3[i,n]}(1 - \theta_i)^k\right] + O(n^{1-(r/3)}).$$

On the other hand, equation (44) just says

$$E\left[\sum_i (1 - \theta_i)^k\right] = p(k, n) + O(k^\tilde{r} n^{-r} + k^c(d - 1)^{-k/2}),$$

where $p(k, n)$ is the polynomial in $k$ given as the sum of the $p_j(k)/n^j$. Therefore

$$E\left[\sum_i \chi_{A_3[i,n]}(1 - \theta_i)^k\right] = p(k, n) - q(k, n) + O(n^{1-(r/3)}) + O(k^\tilde{r} n^{-r} + k^c(d - 1)^{-k/2}) \quad (49)$$

for all $k \leq n^\gamma$. Bounding the right-hand-side of this equality will finish the lemma by adding this equality to equation (47). Since $A_3[i,n]$ implies $|1 - \theta_i| \leq (1 - \log^{-2} n)$ and thus $|1 - \theta_i|^k \leq e^{O(\log^{-2} n)}$ for $k \geq \log^4 n$, we have

$$E\left[\sum_i \chi_{A_3[i,n]}|1 - \theta_i|^k\right] = O(n^{-r/3})$$
for \( k \geq \log^4 n \) for \( n \) sufficiently large. We conclude that

\[
p(k; n) - q(k, n) = O(n^{1-r/3}) \tag{50}
\]

for all \( k \) with \( \log^4 n \leq k \leq n^\gamma \).

**Sublemma 11.2** Let \( g(k) \) be a polynomial in \( k \) of degree \( \leq s - 1 \) such that \( |g(i)| \leq 1 \) for integers \( i = a, a + 1, \ldots, b \) for some integers \( a, b \) with \( a \leq b \). Then \( |g(i)| \leq 2^s - 1 \) for integers \( i \) with

\[
a - \frac{b - a}{s - 1} \leq i \leq a.
\]

**Proof** We have \((\sigma_k - 1)^s g = 0\), and therefore

\[
g(x) = \sum_{i=1}^{d} \binom{s}{i} (-1)^{i-1} g(x + hi)
\]

for any \( x \) and \( h \). Given \( i < a \), let \( h = a - i \) and \( x = i \) in the above; \( x + h, x + 2h, \ldots, x + sh \) are integers between \( a \) and \( b \) provided that

\[
i + (a - i)s \leq b,
\]

so \( i \geq (as - b)/(s - 1) \) or \( i \geq a - (b - a)/(s - 1) \). If so, then

\[
|g(i)| \leq \sum_{i=1}^{d} \binom{s}{i} |g(x + hi)| \leq \sum_{i=1}^{d} \binom{s}{i} = 2^s - 1.
\]

Recall equation (50) and the fact that \( p \) and \( q \) are polynomials in \( k \) (for fixed \( n \)) of bounded degree. So applying the above sublemma for \( a = 2[(\log^4 n)/2] \) and \( b = 2[n^\gamma/2] \) implies that \( p(k, n) - q(k, n) = O(n^{1-(r/3)}) \) for \( 1 \leq k \leq \log^4 n \). Equation (50) now holds for all \( 1 \leq k \leq n^\gamma \); adding this equation to equation (49) completes our proof of Lemma 11.1.
12 Magnification Theorems

In this section we use standard counting arguments to prove theorems implying “magnification” or “expansion” for “most” on random graphs; here “most” means all graphs excepting a set of probability $O(n^{-\gamma_{\text{fund}}})$. These theorems will then be used with Lemma 11.1 to prove Theorems 1.1, 1.2, and 1.3.

A graph, $G$, with $n$ vertices is said to be a $\gamma$-magnifier if for all subsets of vertices, $A$, of size $\leq n/2$ we have

$$|\Gamma(A) - A| \geq \gamma |A|,$$

where $\Gamma(A)$ denotes those vertices connected to some member of $A$ by an edge. Alon has shown that any $d$ regular $\gamma$-magnifier has

$$\lambda_2(G) \leq d - \frac{\gamma^2}{4 + 2\gamma^2}$$

(see [Alo86]; see [Dod84, SJ89, JS89] for related “edge magnification” results).

**Definition 12.1** Say that a $d$-regular graph on $n$ vertices is a $\gamma$-spreader if for every subset, $A$, of at most $n/2$ vertices we have

$$|\Gamma(A)| \geq (1 + \gamma)|A|.$$  

**Theorem 12.2** Let $G$ be a $d$-regular $\gamma$-spreader. Then for all $i > 1$ we have

$$\lambda_i^2(G) \leq d^2 - \frac{\gamma^2}{4 + 2\gamma^2}.$$  

**Proof** Since the graph is $d$-regular, we have $|\Gamma(B)| \geq |B|$ for all subsets of vertices, $B$. Taking $B = \Gamma(A)$ yields

$$|\Gamma^2(A)| \geq |\Gamma(A)| \geq (1 + \gamma)|A|.$$  

Hence $G^2$, the graph on $V_G$ whose edges are paths in $G$ of length 2 (and whose adjacency matrix is $A_G^2$), is a $d^2$-regular $\gamma$-magnifier. Now apply Alon’s result on magnification and eigenvalues to $G^2$, whose eigenvalues are $\lambda_i^2(G)$. $$\square$$
We now establish that for all our models, a graph will be a $\gamma$-spreader for some fixed $\gamma = \gamma(d) > 0$ with probability $1 - O(n^{-\tau_{fund}})$.

**Theorem 12.3** For any $\epsilon > 0$ and even $d \geq 4$ there is a $\gamma > 0$ such that $G \in \mathcal{G}_{n,d}$ is a $\gamma$-spreader with probability $1 - O(n^{-\tau_{fund}})$.

Later we shall prove this theorem for other models of random graphs, by very similar calculations. This theorem is easy for $d$ sufficiently large; but when $d = 4$ (or later possibly $d = 3$) one has to calculate fairly carefully.

**Proof** Fix $A, B \subset \{1, \ldots, n\}$, and consider the event that $\Gamma(A) \subset B$. We will impose the condition that $a = |A| \leq n/2$ and $|B| = a + \lceil \gamma a \rceil$.

First notice that for any constant, $C$, if $\gamma < 1/C$, then $|B| = |A| = a$. But since $G$ is $d$-regular and we have $d|A|$ edges leaving $A$, these edges comprise all edges incident upon $B$ (since $|B| = |A|$). Thus $A \cup B$ is a union of connected components of $G$. But this cannot occur if $G$ has no supercritical tangles of size $\leq 2C$ (since each connected component of $G$ has $\lambda_{Irred} = d - 1$). For a constant $C$ there are only a constant number of tangles of size $\leq 2C$. Thus, by forsaking a probability of $O(n^{-\tau_{fund}})$, we may assume that $a = |A| \geq C$ for any fixed constant, $C$ (provided that we then take $\gamma < 1/C$) for sufficiently large $n$.

So consider a random permutation, $\pi = \pi_i$, and consider the event that $\pi$ and $\pi^{-1}$ map $A$ to $B$. Let $C_1 = A \cap B$, $C_2 = A \setminus B$, $C_3 = B \setminus A$, and let $c_i = |C_i|$. We view $\pi$ as determined by a perfect matching of a bipartite graph on inputs, $I$, and outputs, $O$, with $I, O$ being copies of $\{1, \ldots, n\}$ (and $i \in I$ mapped to $\pi(i) \in O$). Viewing $\pi$ as a bipartite matching, it consists of

1. $r$ edges from $C_1$ to $C_1$ (i.e., the $I$ vertices corresponding to $C_1$ to those $O$ vertices corresponding to $C_1$),
2. $c_1 - r$ edges from $C_1$ to $C_3$,
3. $c_1 - r$ edges from $C_3$ to $C_1$,
4. $c_2$ edges from $C_2$ to $C_3$, and
5. $c_2$ edges from $C_3$ to $C_2$.

(This is true since a $C_2$ vertex, either input or output, must be paired with a $C_3$ vertex, and a $C_1$ vertex must be paired with either a $C_1$ or $C_3$ vertex.)

So the event that $\pi$ and $\pi^{-1}$ map $A$ to $B$ with $c_1, c_2, c_3, r, A, B$ all held fixed has probability

$$p(c_1, c_2, c_3, r) = \left[\binom{c_1}{r} \frac{r!}{c_1!}\right] \left[\binom{c_3}{c_1 - r} \frac{(c_1 - r)!}{c_1!}\right] \times \left[\binom{c_3 - c_1 + r}{c_2} \frac{c_2!}{c_3!}\right]^2 \frac{[n(n-1) \cdots (n-2c_1-2c_2+r+1)]^{-1}}{n!}$$
(The first expression in square brackets corresponds to choosing $r C_1$ to $C_1$ edges; the second expression corresponds to choosing $c_1 - r C_1$ to $C_3$ edges, and is squared to include choosing the $C_3$ to $C_1$ edges; etc.) The probability taken over all $A, B$ of a given $c_1, c_2, c_3$ (and with $r$ fixed) is therefore at most

$$\left( \begin{array}{c} n \\ c_1, c_2, c_3, n - c_1 - c_2 - c_3 \end{array} \right) p^{d/2}(c_1, c_2, c_3, r).$$

(51)

It suffices to show that this expression is $O(n^{-s})$ with $s = \tau_{\text{fund}} + 4$, provided that $a$ is sufficiently large (and $\leq n/2$), since then we can sum equation (51) over the $\leq n^4$ relevant values of $c_1, c_2, c_3, r$. We should remind ourselves that $c_1, c_2, c_3, r$ range over integers with $c_1 + c_2 = a$, $c_1 + c_3 = a + \lfloor \gamma a \rfloor$, $r \leq c_1$.

Furthermore, considering the expression defining $p$, we have $c_3 - c_1 + r \geq c_2$.

We now write

$$b = b(c_1, c_2, c_3, r, n) = \left( \begin{array}{c} n \\ c_1, c_2, c_3, n - c_1 - c_2 - c_3 \end{array} \right)$$

(52)

$$= \frac{n!}{c_1! c_2! c_3! (n - c_1 - c_2 - c_3)!}$$

and

$$p = p(c_1, c_2, c_3, r, n) = \frac{(c_1! c_3!)^2 (n - 2c_1 - 2c_2 + r)!}{((c_1 - r)! (c_3 - c_1 - c_2 + r)!)^2 r! n!}.$$ 

(53)

We make some general remarks about analyzing the factorials in the above two equations:

1. All factorials in the above equations are of the form $(\mu n)!$ for some $\mu \in [0, 1]$. Stirling’s formula $m! \sim (m/e)^m \sqrt{2\pi m}$ implies that

$$\frac{1}{n} \log[(\mu n)!] = \mu \log(n/e) + \mu \log \mu + O\left( \frac{\log n}{n} \right),$$

(54)

where the constant in the $O(\cdot)$ is independent of $n$ and $\mu \in [0, 1]$.

2. In analyzing $b$ and $p$ above, we may ignore the $\mu \log(n/e)$ term in equation (54). This is because $b, p$ are balanced in that the sum of the numbers to which factorials are applied is the same in the numerator and denominator; in other words, the $\mu \log(n/e)$ terms in the numerator will exactly cancel those in the denominator.
3. Let \( f(\theta) = -\theta \log \theta \). We claim that for \( \theta_1, \theta_2 \in [0, 1] \) we have
\[
|f(\theta_1) - f(\theta_2)| \leq \max\{f(|\Delta \theta|), f(1 - |\Delta \theta|)\}, \quad \text{with} \quad \Delta \theta = \theta_2 - \theta_1.
\]

Indeed, since \( f''(\theta) = -1/\theta < 0 \) for \( \theta > 0 \), \( f \) is concave in \([0, 1]\), and so \( g(\theta) = f(\theta + \Delta \theta) - f(\theta) \) is decreasing in \( \theta \) for \( \Delta \theta \) fixed; so \( |g| \)'s maximum over an interval is taken at its endpoints, and since \( f(0) = f(1) = 0 \), the above claim is established.

Next, a Taylor expansion shows that \(-\epsilon \log \epsilon \geq -(1 - \epsilon) \log(1 - \epsilon)\) for sufficiently small \( \epsilon > 0 \). Hence there is an \( \epsilon_0 \) such that
\[
|f(\theta_1) - f(\theta_2)| \leq f(|\theta_1 - \theta_2|) \tag{55}
\]
for all \( \theta_1, \theta_2 \in [0, 1] \) with \( |\theta_1 - \theta_2| \leq \epsilon_0 \).

Let \( \nu_i, \rho, \alpha, \delta \ (i = 1, 2, 3) \) be the non-negative reals given by
\[
c_i = \nu_i n, \quad r = \rho n, \quad a = \alpha n, \quad \lfloor \gamma a \rfloor = \delta n.
\]
We have that
\[
\nu_1 + \nu_2 = \alpha, \quad \nu_1 + \nu_3 = \alpha + \delta, \quad \rho \leq \nu_1, \quad \rho \geq \nu_1 + \nu_2 - \nu_3.
\]
We conclude that
\[
|\nu_2 - \nu_3| \leq \delta, \quad |\nu_1 - \rho| \leq \delta.
\]
It follows from equation (54), remark (2) below it, and equation (55), that we may replace \( \nu_3 \) with \( \mu_2 \) and \( \rho \) with \( \nu_1 \) in calculating \( (\log b)/n \) and incur an additive error term of at most \( O(\delta \log \delta) \). Thus we get
\[
\frac{\log b}{n} = h(\nu_1, \nu_2) + O\left(\frac{\delta \log \delta}{n} + \frac{\log n}{n}\right),
\]
where
\[
h(\nu_1, \nu_2) = -\nu_1 \log \nu_1 - 2\nu_2 \log \nu_2 - (1 - \nu_1 - 2\nu_2) \log(1 - \nu_1 - 2\nu_2). \tag{56}
\]
Similarly we calculate
\[
\frac{-\log p}{n} = h(\nu_1, \nu_2) + O\left(\frac{\delta \log \delta}{n} + \frac{\log n}{n}\right),
\]
96
i.e., we have the exact same equation (!) for \( \log b \) replaced by \(- \log p\) (this “coincidence” happens for the other models as well). Hence

\[
\frac{\log(bp^2)}{n} = -h(\nu_1, \nu_2) + O\left( |\delta \log \delta| + \frac{\log n}{n} \right).
\]

Since \( \nu_1 + \nu_2 = \alpha \), we have either (or both) \( \nu_i \) are \( \geq \alpha/2 \). Hence

\[
h(\nu_1, \nu_2) \geq - (\alpha/2) \log(\alpha/2).
\]

Now we claim that for any constant \( C > 0 \) there is a constant \( \gamma > 0 \) such that for all \( \alpha \in [0, 1/2] \) we have

\[
-\alpha \log \alpha \geq -C(\gamma \alpha) \log(\gamma \alpha).
\]

Indeed, for \( \gamma < 1 \) fixed we have

\[
g(\alpha) = \frac{(\gamma \alpha) \log(\gamma \alpha)}{\alpha \log \alpha} = \gamma + \frac{\gamma \log \gamma}{\log \alpha}
\]

is increasing for \( \alpha \in [0, 1/2] \). Hence it suffices to choose a \( \gamma > 0 \) sufficiently small so that

\[
g(1/2) = \frac{(1/2) \log(1/2)}{(\gamma/2) \log(\gamma/2)} \geq C,
\]

so that

\[
g(\alpha) \geq g(1/2) \geq C,
\]

which along with equation (58) yields equation (57).

It follows that for sufficiently small \( \gamma > 0 \) we have

\[
\frac{\log(bp^2)}{n} \geq - (\alpha/2) \log(\alpha/2) + O \left( |\delta \log \delta| + \frac{\log n}{n} \right),
\]

and, since \( \delta \leq \gamma a/n = \gamma \alpha \), this expression is

\[
\geq - (\alpha/4) \log(\alpha/2) + O \left( \frac{\log n}{n} \right).
\]

Hence for any constant, \( C_1 \), there is a \( C_2 \) such that if \( a \geq C_2 \) (i.e., \( \alpha \geq C_2/n \)) then

\[
\frac{\log(bp^2)}{n} \leq -C_1 \log \frac{n}{n}
\]

for all \( n \) sufficiently large. In other words \( bp^{\delta/2} \), i.e., the expression in equation (51), is \( \leq n^{-C_1} \); this, by the discussion after equation (51), completes the proof.

97
Theorem 12.4. Theorem 12.3 holds in the models $\mathcal{H}_{n,d}$, $\mathcal{I}_{n,d}$, and $\mathcal{J}_{n,d}$.

**Proof**. In $\mathcal{H}_{n,d}$ each permutation occurs with probability at most $n$ times its probability in $G_{n,d}$. Therefore the same analysis goes through, except that $p$ is multiplied by at most a factor of $n$. This changes the expression for $n^{-1} \log(bp^2)$ by an $O(n^{-1} \log n)$ factor, so the same proof carries over.

For $\mathcal{I}_{n,d}$ we again set $C_i$ and $c_i$ as before. A perfect matching in $\{1, \ldots, n\}$ will have (1) $r$ vertices of $C_1$ paired amongst themselves, (2) $c_1 - r$ vertices of $C_1$ paired with $C_3$ vertices, and (3) $c_2$ vertices of $C_2$ paired with $C_3$ vertices. This data determines the pairing for $r + 2(c_1 - r) + 2c_2$ vertices. The expression for $b$, representing the number of ways the $C_i$ can be chosen, is the same as before. We now derive an expression for $p$, the probability that a single perfect matching matches all $A$ vertices to those in $B$.

For an even integer, $m$, let $m$ odd factorial be

$$m!_{\text{odd}} = (m - 1)(m - 3) \cdots 3 = \frac{m!}{2^{m/2}(m/2)!},$$

which is just the number of perfect matchings of $m$ elements. Stirling’s formula yields

$$m!_{\text{odd}} \sim \sqrt{2} (m/e)^{m/2}$$

(so that for our purposes $m!_{\text{odd}}$ can be regarded as replacable by the square root of $m!$).

We have

$$p = p(c_1, r, n) = \left[\binom{c_1}{r} r!_{\text{odd}}\right] \left[\binom{c_3}{c_1 - r} (c_1 - r)!\right]$$

$$\times \left[\binom{c_3 - c_1 + r}{c_2} c_2!\right] \frac{(n - 2c_1 - 2c_2 - r)!_{\text{odd}}}{n!_{\text{odd}}}.$$

We get that $- \log p$ is

$$\frac{h(\nu_1, \nu_2)}{2} + O\left(|\delta \log \delta| + \frac{\log n}{n}\right),$$

with $h$ as in equation (56). Since $b$ is unchanged, by analyzing as before we see that there is a fixed $\gamma > 0$ such that for any constant $C_1$ there is a $C_2$ such that $bp^3 = O(n^{-C_1})$ provided that $a \geq C_2$.  

\[ \Box \]
Next we consider $J_{n,d}$. Let $G$ be a random graph in $J_{n,d}$, so $V_G = \{1, \ldots, n\}$. Consider the graph $G'$ formed by adding one new vertex, $w = n + 1$, to $G$ and replacing each half-loop about a vertex, $v$, in $G$ by an edge from $v$ to $w$. Then $G'$ is precisely distributed as an element of $I_{n+1,d}$; indeed, a perfect matching on $V_G$ matches $w$ to some element of $V_G = \{1, \ldots, n\}$ and then randomly matches the remaining $n-1$ elements of $V_G$.

Now we know that $G'$ is a $\gamma$-spreader with probability $1 - O(n^{-\tau_{\text{fund}}})$. But for any $A \subset V_G$, $\Gamma_G'(A)$ consists of at most one more vertex than $\Gamma_G(A)$. Hence for $|A| \leq |V_G|/2$ and $G'$ being a $\gamma$-spreader, we have

$$|\Gamma_G(A) \setminus A| \geq \gamma |A| - 1 \geq \gamma' |A|,$$

where $\gamma' = \gamma - (1/c')$, provided that $|A| \geq c'$. Hence $G$ is a $\gamma$-spreader on sets, $A$, of size $\max(c, c') \leq |A| \leq n/2$. On smaller sets, $A$, we have $G$ is a $\gamma'$-spreader with probability $1 - O(n^{-\tau_{\text{fund}}})$, assuming $\gamma' \leq 1/\max(c, c')$, by the argument given before for $G_{n,d}$. (Notice that the $\tau_{\text{fund}}$ for $I_{n,d}$ and $J_{n,d}$ are the same.) Hence a random graph in $J_{n,d}$ is a $\gamma'$-spreader with probability $1 - O(n^{-\tau_{\text{fund}}})$.

13 Finishing the $G_{n,d}$ Proof

Here we quickly finish the proof of Theorem 1.1, which proves Alon’s conjecture for $G_{n,d}$.

Fix a value of $r$ to be specified later. Let $\Psi' = \Psi_{\text{eig}}[r-1]$ be the set of supercritical tangles of order $< r$, and let $\Psi = \Psi_{\text{min}}[r-1]$ (which we recall is the set of minimal $\Psi' = \Psi_{\text{eig}}[r-1]$ elements with respect to inclusion); we know that $\Psi$ is finite by Lemma 9.2, and we recall that if $G$ contains a $\Psi'$ tangle then it contains an element of $\Psi$. The probability that $\chi_{\Psi}(G) = 1$, i.e., that $G$ contains at least one element of $\Psi$, is at most $O(n^{-\tau_{\text{fund}}})$, since every one of the finitely many tangles in $\Psi$ occurs with probability proportional to $1/n$ to the order of tangle (Theorem 4.7), and each tangle order is at least $\tau_{\text{fund}}$. Given that $\chi_{\Psi}(G) = 0$, we have that $G$ contains no supercritical tangle of order $< r$, and hence no irreducible cycle can fail to be $(S, \Psi')$-selective for any $S$. Hence for all $S$ and $k$ we have

$$\chi_{\Psi}(G) = 0 \quad \text{implies} \quad \text{SSIT}_{S,\Psi'}(G; k) = \text{SIT}(G; k).$$
Thus
\[ E \left[ (1 - \chi_{\Psi})SSIT_{S,\Psi'}(G; k) \right] = E \left[ (1 - \chi_{\Psi})SIT(G; k) \right]. \]

Now according to Theorem 10.3 we have
\[ SIT(G; k) = \sum_{i=1}^{n} \mu_i^k(\lambda_i) + \mu_2^k(\lambda_i) + (1 + (-1)^k)(d - 2)/2, \]
for even \( k \geq 2 \), where
\[ \mu_{1,2}(\lambda) = \frac{\lambda \pm \sqrt{\lambda^2 - 4(d-1)}}{2}. \]

In other words, there are \( nd \) numbers \( \nu_i \), such that \( SIT(G; k) \) is the sum of the \( k \)-th powers of these numbers. Also for each \( i \) we have \( \nu_i = \pm 1 \) or else \( \nu_i \) is real unless it is of absolute value \( \sqrt{d-1} \). Combining this and Theorem 9.3 we see that
\[ \theta_i = 1 - (1 - \chi_{\Psi})\nu_i/(d-1) \]
are random variables that satisfy the conditions of Lemma 11.1 for each \( G \).

It follows that
\[
E \left[ (1 - \chi_{\Psi}) \sum_{i=1}^{n} \sum_{j \text{ such that } |\mu_j(\lambda_i)| \leq (d-1)(1-\log^{-2} n)} \mu_j(\lambda_i)^k \right] = O(Dn^{1-(r/3)} + k^c(d-1)^{-k/2})
\]
for all \( k \) with \( 1 \leq k \leq n^\gamma \) for some constant \( \gamma > 0 \) depending only on \( r \).

According to Theorems 12.2 and 12.3 there is an \( \epsilon > 0 \) such that with probability \( 1 - O(n^{-r_{\text{fund}}}) \) we have \( |\lambda_i| \leq d - \epsilon \) for all \( i \neq 1 \); in this case there is an \( \epsilon' = \epsilon'(\epsilon) > 0 \) such that \( |\mu_j(\lambda_i)| \leq (d-1) - \epsilon' \) for all \( j = 1, 2 \) and \( i \neq 1 \).

Now let \( A \) be the event that \( \chi_{\Psi} = 0 \) and that \( |\mu_j(\lambda_i)| \leq (d-1) - \epsilon' \) for all \( j = 1, 2 \) and \( i \neq 1 \). Let \( B = B(\eta) \) be the event that \( |\mu_j(\lambda_i)| \geq e^{\eta}\sqrt{d-1} \) for an arbitrary fixed \( \eta > 0 \). Then \( A \cap B \) implies that for even integer \( k \) we have
\[
\sum_{i=2}^{n} \sum_{j=1}^{2} \mu_j(\lambda_i)^k \geq \left(e^{\eta}\sqrt{d-1}\right)^k - 2(n-2)(d-1)^{k/2}
\]
(since each \( \mu_j(\lambda_i) \) is either of absolute value \( \sqrt{d-1} \), or is real so that \( \mu_j(\lambda_i) \) raised to an even power is non-negative). By the same argument, for any \( G \)
we have
\[ \sum_{i \text{ s.t. } \mu_1, \mu_2 \text{ not real}} 2 \sum_{j=1}^2 \mu_j(\lambda_i)^k \geq -2(n-1)(d-1)^{k/2}. \]

It follows, using equation (59), that for even \( k \),
\[
\begin{align*}
\text{Prob}\{A \cap B\} \left( e^{\eta \sqrt{d-1}} \right)^k & \leq \mathbb{E} \left[ \sum_{i \text{ s.t. } \mu_1, \mu_2 \text{ real}} 2 \sum_{j=1}^2 \mu_j(\lambda_i)^k \right] \\
& = \mathbb{E} \left[ \sum_{i=2}^n \sum_{j=1}^2 \mu_j(\lambda_i)^k \right] - \mathbb{E} \left[ \sum_{i \text{ s.t. } \mu_1, \mu_2 \text{ not real}} 2 \sum_{j=1}^2 \mu_j(\lambda_i)^k \right] \\
& \leq O(Dn^{1-(r/3)}(d-1)^k + k^c(d-1)^{k/2}) + 2(n-1)(d-1)^{k/2}.
\end{align*}
\]

We now take
\[ k = 2 \left[ \frac{r \log n}{3 \log (d-1)} \right]. \]

We have
\[ (k/2) - 1 \leq \frac{r \log n}{3 \log (d-1)} \leq k/2. \]

Hence
\[ n^{-r/3} \leq (d-1)^{-(k/2)+1}, \]

and so
\[
\begin{align*}
\text{Prob}\{A \cap B\} & \leq c \max(k^c, n)e^{-k\eta} \\
& \leq cne^{-k\eta} = cnn^{-\alpha r},
\end{align*}
\]

where \( \alpha = (2/3)\eta/\log(d-1) \), i.e. \( \alpha \) is a positive constant (depending only on \( \eta \) and \( d \)). Choosing \( r \) so that \( \alpha r - 1 > \tau_{\text{fund}} \), we have
\[
\text{Prob}\{A \cap B\} = O(n^{-\tau_{\text{fund}}}).
\]

But we have already seen (Theorems 12.2 and 12.3) that
\[
\text{Prob}\{A^c\} = O(n^{-\tau_{\text{fund}}}),
\]

where \( A^c \) is the complement of \( A \). Hence
\[
\text{Prob}\{B\} = \text{Prob}\{B \cap A\} + \text{Prob}\{B \cap A^c\} = O(n^{-\tau_{\text{fund}}}).
\]
For any $\epsilon > 0$ there is an $\eta > 0$ such that $|\lambda| \geq 2\sqrt{d-1} + \epsilon$ implies $|\mu_i(\lambda)| \geq e^n \sqrt{d-1}$ for at least one $i$, which is the event $B = B(\eta)$ above. It follows that for any $\epsilon > 0$ we have

\[
\text{Prob} \{ |\lambda_i| \geq 2\sqrt{d-1} + \epsilon \text{ for some } i > 1 \} = O(n^{-\tau_{\text{rand}}}).
\]

This (and Theorem 2.10) proves Theorem 1.1.

14 Finishing the Proofs of the Main Theorems

We now complete the proofs of Theorems 1.2 and 1.3, i.e., we establish the Alon conjecture for $H_{n,d}$, $I_{n,d}$, and $J_{n,d}$.

The proofs of the theorems are as the proof for $G_{n,d}$. We only need to establish the following results for the different models of random graph:

1. Labelling: The model comes with edges labelled from a set $\Pi$ such that to each $\pi \in \Pi$ we associate a $\pi^{-1} \in \Pi$ such that $(\pi^{-1})^{-1} = \pi$ (in other words, the elements of $\Pi$ are paired, with the possibility that an element is paired with itself).

2. Coincidence: If $k$ of the random edges have been determined, and if we fix any two vertices, $v, w$, in the graph, then the probability that an edge of a given label takes $v$ to $w$ is at most $c/(n - ck)$ for some constant $c$. We have only briefly mentioned coincidences in this paper, but our Lemmas 5.6 and 5.7, proven in [Fri91], require a property like this.

3. Expansion with Error: Consider a $\Pi$-labelled graph, $H$, with vertices a subset of $\{1, \ldots, n\}$, that can occur as a subgraph of a graph in our model. The probability that $H$ occurs must depend only on the number of edges, $a_{\pi}$, of each label, $\pi$ (of course, $a_{\pi} = a_{\pi^{-1}}$). Furthermore, this probability times the number subsets of $\{1, \ldots, n\}$ of size $V_H$ is, for every positive integer $r$,

\[
E_{\text{symm}}(H)_n = \left( \sum_{i=0}^{r-1} \frac{p_i(\bar{a})}{n^i} \right) + \frac{\text{error}}{n^r},
\]
where $p_i$ are polynomials in $\vec{a}$ (where $\vec{a}$ is the collection of all $a_\pi$) and where
\[ |\text{error}| \leq ck'
\] for all $k \leq n/c$, where $c_1, r'$ depend only on $r$. Furthermore, $p_i = 0$ if $i$ is less than the order of $H$.

4. Simple Word Sum: Let $\text{Irred}_{k,\sigma,\tau}$ be those words that begin with $\sigma$, end in $\tau$, and are irreducible (meaning no consecutive occurrence of $\pi$ and $\pi^{-1}$). Then for any polynomial, $p = p(\vec{a})$ (with $\vec{a}$ as above), we require
\[
\sum_{w \in \text{Irred}_{k,\sigma,\tau}} p(a_1(w),\ldots,a_{d/2}(w),k) = (d-1)^k Q_1(k) + E(k) \tag{60}
\]
for a polynomial, $Q_1$, and a function $E$ with $|E(k)| \leq ck$ for some constant $c$ (i.e., the above sum is super-$d$-Ramanujan).

5. $\tau_{\text{fund}}$ determination: We must determine $\tau_{\text{fund}}$ for the model.

6. Spreading: There is a constant $\gamma > 0$ such that the probability that a random graph has $|\lambda_i| \geq d - \gamma$ for some $i \neq 1$ is of order at most $n^{-\tau_{\text{fund}}}$.

We have already shown spreading and determined $\tau_{\text{fund}}$ for all three models. The labelling of the models is quite simple: $H_{n,d}$ is labelled like $G_{n,d}$; $I_{n,d}$ is labelled with $\Sigma = \{\sigma_1,\ldots,\sigma_d\}$ where $\sigma_i^{-1} = \sigma_i$ (each $\sigma_i$ represents a perfect matching); $J_{n,d}$ is labelled with $\Sigma \cap T$ with $\Sigma$ as before and $T = \{\tau_1,\ldots,\tau_d\}$ with $\tau_i^{-1} = \tau_i$, and where the $\sigma_i$ represent the near perfect matching and the $\tau_i$ represents the single completing half-loop for $\sigma_i$.

Coincidence is easily checked for all three models.

We address the issue of Simple Word Sum. The word sum for $H_{n,d}$ is the same as for $G_{n,d}$. For $I_{n,d}$, the technique of Lemma 2.11 of [Fri91] reduces the matter to the irreducible eigenvalues of a vertex with $d$ half-loops; since these eigenvalues are the eigenvalues of a $d \times d$ matrix which is 0 on the diagonal and 1’s elsewhere, the eigenvalues are $d - 1$ with multiplicity 1 and $-1$ with multiplicity $d - 1$. Hence the simple word sum of equation (60) is given by
\[
(d-1)^k Q_1(k) + (-1)^k Q_2(k) \tag{61}
\]
where $Q_i$ are polynomials. For $J_{n,d}$ we can break the sum by how many half-loops are involved. For a fixed set of half-loops involved in the irreducible
word, the sum is a convolution of functions of the form in equation (61), which by Theorem 7.2 is again super-$d$-Ramanujan.

We now establish Expansion with Error for the three models. Equation (14) has the $\mathcal{H}_{n,d}$ analogue

$$P(w; \vec{t}) = \prod_{i=1}^{d/2} \frac{(n - a_i - 1)!}{(n - 1)!}.$$  

Now recall the proof of Theorem 5.4, especially equations (16) and (17). For $\mathcal{H}_{n,d}$, we have

$$E_{\text{symm}}(H)_n = n(n - 1) \cdots (n - v + 1) \prod_{i=1}^{d/2} \frac{(n - a_i - 1)!}{(n - 1)!} = n^{v - e} g(1/n),$$

with $g$ as in equation (16) with $b_1, \ldots, b_v$ being 0, 1, \ldots, $v - 1$ and $c_1, \ldots, c_e$ being the collection of the sequences $1, 2, \ldots, a_j$. Hence for a walk of length at most $k$ we have

$$\sum b_j + \sum c_j \leq \binom{k}{2} + \binom{k + 1}{2} = k^2.$$  

Accordingly Expansion with Error holds for $\mathcal{H}_{n,d}$ with expansion polynomials determined by equation (62), and with error term bounded by

$$e^{rk/n} k^{2r};$$

this bound is $\leq c k^{r'}$ for all $k \leq n$ with $r' = 2r$ and $c = e^r$.

Similarly for $\mathcal{I}_{n,d}$ we have the analogue

$$P(w; \vec{t}) = \prod_{i=1}^{d} \frac{(n - a_i)!_{\text{odd}}}{n!_{\text{odd}}}.$$  

The analysis goes through essentially as before; in the error bound we have $\sum b_j$ is again $\binom{k}{2}$, but this time the $\sum c_j$ is as large as

$$1 + 3 + 5 + \cdots + (2k - 1) = k^2.$$  

104
(taking one \(a_i = 2k\) and the rest 0). So Expansion with Error holds for \(I_{n,d}\) with error term bounded by

\[
e^{rk/n} \left( k^2 + \binom{k}{2} \right)^r \leq e^{rk/n}(2k)^{2r}.
\]

For \(J_{n,d}\), consider a random 1-regular graph, \(G'\), consisting of a near perfect matching plus one complementing half-loop on the vertex set \(\{1, \ldots, n\}\). Notice that the number of such graphs is \(n(n-1)!_{\text{odd}}\). Hence the probability of occurrence of a specified half-loop and \(a\) other matchings in \(G'\) is

\[
\frac{(n - 1 - 2a)!_{\text{odd}}}{n(n-1)!_{\text{odd}}} = \frac{1}{n(n-2) \cdots (n-2a)},
\]

and the probability of a specified matchings (with no specified half-loop) is

\[
\frac{(n - 2a)(n - 1 - 2a)!_{\text{odd}}}{n(n-1)!_{\text{odd}}} = \frac{1}{n(n-2) \cdots (n-2a-2)}.
\]

So for any specification of half-loops in \(H\), i.e., any fixing of each \(a_{\tau_i}\) to 0 or 1, \(E_{\text{symm}}(H)_n\) is a polynomial in the \(a_{\sigma_i}\)'s; this makes \(E_{\text{symm}}(H)_n\) a polynomial in the \(\vec{a}\), namely

\[
\sum_{I \subseteq \{1, \ldots, d\}} \left( p_I(a_{\sigma_1}, \ldots, a_{\sigma_d}) \prod_{i \in I} a_{\tau_i} \prod_{i \notin I} (1 - a_{\tau_i}) \right).
\]

We also see that, in the terminology above, \(\sum b_j = \binom{k}{2}\) and \(\sum c_j \leq 2^{(k-1)}\). Hence Expansion with Error holds for \(J_{n,d}\) as well.

This establishes the six required results mentioned at the beginning of this section for the models \(H_{n,d}\), \(I_{n,d}\), and \(J_{n,d}\). Theorems 1.2 and 1.3 follow.

## 15 Closing Remarks

We make a number of final remarks.

**Stronger conjectures:** As mentioned before, numerical experiments indicate that the average (and median) \(\lambda_2\) for a random graph is \(2\sqrt{d - 1 + \epsilon(n)}\), where \(\epsilon(n)\) is a negative function (tending to 0 as \(n \to \infty\)). By the results of Friedman and Kahale (extending the Alon-Boppana result), \(-\epsilon(n) \leq O(\log^{-2} n)\) (see [Fri93]). However, the trace method, even with selective traces, seems to require some fundamental new idea in order to have any hope of achieving \(\epsilon(n)\) that is zero or negative.
Critical $d$: As mentioned before, when there is a critical tangle of order strictly less than that of any hypercritical tangle, then our techniques leave a gap in that we can only prove $\lambda_2 > 2\sqrt{d-1}$ with probability $\geq c/n^s$ where $s > \tau_{\text{fund}}$. This case is extremely interesting, since it seems that there should be a theorem that closes this gap, and such a theorem would either get around a poorly bounded $W_{\bar{T},\bar{S}}(M_1, M_2)$ or improve the very interesting Theorem 3.11 (or do something else).

Relative Alon Conjecture: Following [Fria], it seems quite possible to relativize the main theorems in this paper. Namely, fix a “base” graph, $B$, (or, more generally, a “base” pregraph, in the sense of [Fri93]). Fix an $\epsilon > 0$. Then we believe that most random coverings of $B$ of degree $n$ have all “new” eigenvalue $\leq \epsilon + \rho$, where $\rho$ is the spectral radius of the universal cover of $B$. Similarly, we can ask for $\epsilon$ to be zero or even a negative function of $n$. See [Fria, FT] for further discussion.

Alternate Proof with Trace (see the end of Section 2): It may be possible to analyze the expected irreducible trace over all of $G_{n,d}$. As remarked in Theorem 2.11 and the discussion thereafter, the coefficients of $g_{i}(k)$ there could no longer be $d$-Ramanujan. It may be possible to analyze selective traces without discarding contributions from tangled graphs. In other words, if we better understood how selectivity affected irreducible traces, we might not need Section 9 (and certain parts of our understanding of these traces might improve). Selectivity in $G$ can clearly be expressed in terms of walks in an induced subgraph of a “higher block presentation” of $G$ (see [LM95, Kit98]). However, it is not clear what can be said about the eigenvalues of induced subgraphs of a higher block presentation; the author has only some weak results in this directions (see [Frib]).

References

[AFKM86] R. Adler, J. Friedman, B. Kitchens, and B. Marcus. State splitting for variable length graphs. IEEE Transactions on Information Theory, 32(1):108–115, January 1986.

[Alo86] N. Alon. Eigenvalues and expanders. Combinatorica, 6(2):83–96, 1986. Theory of computing (Singer Island, Fla., 1984).
Andrei Broder and Eli Shamir. On the second eigenvalue of random regular graphs. In *28th Annual Symposium on Foundations of Computer Science*, pages 286–294, 1987.

Marshall W. Buck. Expanders and diffusers. *SIAM J. Algebraic Discrete Methods*, 7(2):282–304, 1986.

Warren Dicks and M. J. Dunwoody. *Groups acting on graphs*. Cambridge University Press, Cambridge, 1989.

Jozef Dodziuk. Difference equations, isoperimetric inequality and transience of certain random walks. *Trans. Amer. Math. Soc.*, 284(2):787–794, 1984.

Z. Füredi and J. Komlós. The eigenvalues of random symmetric matrices. *Combinatorica*, 1(3):233–241, 1981.

J. Friedman, J. Kahn, and E. Szemerédi. On the second eigenvalue of random regular graphs. In *21st Annual ACM Symposium on Theory of Computing*, pages 587–598, 1989.

J. Friedman. Relative expanders or weakly relatively Ramanujan graphs. Preprint.

J. Friedman. SVD misalignment and eigenvalue separation. In preparation.

J. Friedman. On the second eigenvalue and random walks in random $d$-regular graphs. *Combinatorica*, 11(4):331–362, 1991.

J. Friedman. Some geometric aspects of graphs and their eigenfunctions. *Duke Math. J.*, 69(3):487–525, 1993.

J. Friedman and Jean-Pierre Tillich. Generalized Alon-Boppana theorems and error-correcting codes. In preparation.

Alessandro Figà-Talamanca and Massimo A. Picardello. *Harmonic analysis on free groups*. Marcel Dekker Inc., New York, 1983.

S. Geman. A limit theorem for the norm of random matrices. *Ann. of Prob.*, 8(2):252–261, 1980.
[God93] C. D. Godsil. *Algebraic combinatorics*. Chapman & Hall, New York, 1993.

[Joh90] D. L. Johnson. *Presentations of groups*. Cambridge University Press, Cambridge, 1990.

[JS89] Mark Jerrum and Alistair Sinclair. Approximating the permanent. *SIAM J. Comput.*, 18(6):1149–1178, 1989.

[Kit98] Bruce P. Kitchens. *Symbolic dynamics*. Springer-Verlag, Berlin, 1998.

[LM95] Douglas Lind and Brian Marcus. *An introduction to symbolic dynamics and coding*. Cambridge University Press, Cambridge, 1995.

[LPS86] A. Lubotzky, R. Phillips, and P. Sarnak. Explicit expanders and the ramanujan conjectures. In *18th Annual ACM Symposium on Theory of Computing*, pages 240–246, 1986.

[LPS88] A. Lubotzky, R. Phillips, and P. Sarnak. Ramanujan graphs. *Combinatorica*, 8(3):261–277, 1988.

[Mar88] G. A. Margulis. Explicit group-theoretic constructions of combinatorial schemes and their applications in the construction of expanders and concentrators. *Problemy Peredachi Informatsii*, 24(1):51–60, 1988.

[McK81] B. McKay. The expected eigenvalue distribution of a large regular graph. *Lin. Alg. Appl.*, 40:203–216, 1981.

[Mor94] Moshe Morgenstern. Existence and explicit constructions of $q+1$ regular Ramanujan graphs for every prime power $q$. *J. Combin. Theory Ser. B*, 62(1):44–62, 1994.

[Nil91] A. Nilli. On the second eigenvalue of a graph. *Discrete Math.*, 91(2):207–210, 1991.

[SJ89] Alistair Sinclair and Mark Jerrum. Approximate counting, uniform generation and rapidly mixing Markov chains. *Inform. and Comput.*, 82(1):93–133, 1989.
[SW49] Claude E. Shannon and Warren Weaver. *The Mathematical Theory of Communication*. University of Illinois Press, 1971/1949.

[Wig55] E. Wigner. Characteristic vectors of bordered matrices with infinite dimensions. *Annals of Math.*, 63(3):548–564, 1955.

[Woe00] Wolfgang Woess. *Random walks on infinite graphs and groups*. Cambridge University Press, Cambridge, 2000.

[Wor99] N. C. Wormald. Models of random regular graphs. In *Surveys in Combinatorics, 1999 (Canterbury)*, pages 239–298. Cambridge Univ. Press, Cambridge, 1999.