MODULAR PROPERTIES OF NODAL CURVES ON \( K3 \) SURFACES

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Abstract. In this paper we partially address two questions which have been raised in [7]:
– The first is a rigidity property for pairs \((S, C)\) consisting of a general projective \( K3 \) surface \( S \), and a curve \( C \) obtained as the normalization of a nodal, hyperplane section of \( S \). We prove that a non-trivial deformation of such a pair \((S, C)\) induces a non-trivial deformation of \( C \);
– The second question concerns the Wahl map of curves \( C \) as above. We prove that the Wahl map of the normalization of a nodal curve contained in a general projective \( K3 \) surface is non-surjective.
In both cases, we impose upper bounds on the number of nodes of the hyperplane section.

Introduction

Curves on \( K3 \) surfaces have been investigated from various points of view, and there is an extensive literature concerning their properties. Most attention has been payed to the smooth curves. In a series of articles Mukai studied the properties of the morphism

\[
\left\{(S, C) \mid S \text{ is a general projective } K3 \text{ surface, and } C \text{ is a smooth hyperplane section of genus } g. \right\} \xrightarrow{\mu} \mathcal{M}_g,
\]

which associates to a pair \((S, C)\) consisting of a general, projective \( K3 \) surface the class of the curve \( C \) in the Deligne-Mumford space. He proved in [10] that the morphism \((\mu)\) is finite for \( g \geq 13 \), and then he went on proving that it is actually birational (see [11, theorem 1.2]). These topics are nicely surveyed in [12] and [1].

By contrast nodal curves on \( K3 \) surfaces have received somewhat less attention. The existence of nodal curves on \( K3 \) surfaces has been addressed in [9], and later on generalized in [6]. The deformation theory of nodal curves on \( K3 \) surfaces has been treated in [13], and recently in [7]. The goal of this paper is to (partially) address the following two questions raised in [7]:

(i) The first problem (see 5.7(ii) in loc. cit.) concerns the finiteness of the forgetful morphism \((\mu)\), where one considers now pairs \((S, C)\) such that \( C \) is the normalization of a nodal curve on \( S \).

(ii) The second problem is to find obstructions for embedding nodal curves into \( K3 \) surfaces. More precisely, a result due to Wahl says that for a smooth curve \( C \) lying on a projective \( K3 \) surface, the homomorphism

\[
w_C : \bigwedge^2 H^0(C, K_C) \to H^0(C, K_C^\otimes 3)
\]

is non-surjective (see [14, 2]). The question raised in [7, question 5.5] is the following: suppose that \( C \) is the normalization of a nodal curve on a projective \( K3 \) surface. Is it true that the homomorphism \( w_C \) is still non-surjective?

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For these questions we have the following two answers:

**Theorem** For two positive integers \( n \) and \( d \) (subject to the inequalities below), we define:

\[
\delta_{\text{max}}(n, d) = \begin{cases} 
\left\lfloor \frac{n}{2} \right\rfloor - 25 & \text{if } d = 1 \text{ and } n \geq 50; \\
2n - 27 & \text{if } d = 2 \text{ and } n \geq 14; \\
2(n - 1)(d - 1) - 25 & \text{if } d \geq 3 \text{ and } n = 11 \text{ or } n \geq 13.
\end{cases}
\]

(i) The forgetful morphism

\[
\left\{ (S, C, u) \mid (S, A) \in \mathcal{K}_n, C \in \mathcal{M}_{n-\delta}, \text{ and } u : C \rightarrow S \text{ is a morphism} \\
s.t. \ u_* C \hookrightarrow S \text{ is a reduced, nodal curve with } \delta \text{ nodes,} \\
which belongs to the linear system } |dA|.
\]

is generically finite onto its image.

(ii) Suppose \((S, A)\) is a polarized K3 surface, with \(\text{Pic}(S) = \mathbb{Z}A, A^2 = 2(n - 1)\), and consider a nodal, hyperplane section \(C\) of \(S\) of degree \(d\), with \(\delta\) nodes. Assume that

\[
\delta \leq \min \left\{ \delta_{\text{max}}(n, d), \frac{(n - 1)d^2 - 1}{3} \right\}.
\]

Let \(C\) be the normalization of \(\hat{C}\). Then the Wahl map of \(C\) is not surjective.

A remark concerning the upper bound appearing in (ii) above: there are few articles discussing the surjectivity properties of the normalization of nodal curves on surfaces. Actually, the author of this paper could find only the reference [5], which deals with the surjectivity of the Wahl map of plane nodal curves. In that reference, the authors impose an upper bound on the number of nodes too.

This article is structured as follows:

- In the first section we briefly recall basic facts concerning the deformation theory of curves on surfaces, and fix the notations used throughout the article.
- The second section contains our main technical tool used for answering the two above mentioned questions. It is well-known that the tangent bundle of any K3 surface \(S\) is stable. In proposition 2.1 we give an effective upper bound for the number of nodes of a nodal curve \(\hat{C} \hookrightarrow S\), such that the pull-back of the tangent bundle of \(S\) to the normalization of \(\hat{C}\) is still stable.
- The third and the fourth sections contain the proofs of the first, respectively the second main result.

1. Description of the problem

Throughout the article we will work over the field \(\mathbb{C}\) of complex numbers. Most of the material appearing in this section is contained in the articles [1] and [7]. Here we will introduce only those objects, and recall those properties, which are essential for our presentation.

**Definition 1.1.**

(i) We say that a polarized K3 surface \((S, A)\) is Picard general if

\[
\text{Pic}(S) = \mathbb{Z}A, \text{ with } A \rightarrow X \text{ ample.}
\]

In this case the self-intersection number \(A^2 = 2(n - 1)\), with \(n \geq 3\), and the linear system \(|A|\) induces an embedding \(S \hookrightarrow \mathbb{P}^n\).
(ii) We say that a morphism \( S \xrightarrow{\pi} \Delta \) between irreducible algebraic varieties is a family of Picard general, polarized K3 surfaces, if there is a relatively ample line bundle \( \mathcal{A} \to S \) such that fibres \((S_t, \mathcal{A}_t)\), \( t \in \Delta \), are Picard general K3 surfaces. In this case the function \( t \mapsto \mathcal{A}_t^2 \) is constant.

**Theorem 1.2.**

(i) Let \( \mathcal{X}_n \) be the set of Picard general, polarized K3 surfaces. Then \( \mathcal{X}_n \) can be endowed with the structure of a smooth stack, whose local charts are given by the local Kuranishi models of its points.

(ii) For any \( g \geq 1 \), let \( \mathcal{M}_g \) be the Deligne-Mumford stack of smooth and irreducible curves of genus \( g \). For \( d \geq 1 \) and \( 0 \leq \delta \leq (n - 1)d^2 \), we define

\[
\gamma_{n,\delta}^d := \left\{ (S, C, u) \mid (S, C) \in \mathcal{X}_n, C \in \mathcal{M}_{n-\delta}, \text{ and } u : C \to S \text{ is a morphism s.t. } u_* C \to S \text{ is a reduced, nodal curve with } \delta \text{ nodes, which belongs to the linear system } |dA| \right\}
\]

Then \( \gamma_{n,\delta}^d \) can be endowed with the structure of an analytic stack, which admits two forgetful morphisms

\[
\gamma_{n,\delta}^d \xrightarrow{\mu} \mathcal{M}_{g(d)-\delta} \xrightarrow{\kappa} \mathcal{X}_n
\]

(iii) There is a non-empty open subset \( \mathcal{X}_n^o \subset \mathcal{X}_n \) such that \( (\gamma_{n,\delta}^d)^o := \kappa^{-1}(\mathcal{X}_n^o) \) is smooth, the projection \( (\gamma_{n,\delta}^d)^o \to \mathcal{X}_n^o \) is submersive, and all the irreducible components of \( (\gamma_{n,\delta}^d)^o \) are \( 19 + g(d) - \delta \) dimensional.

Note that for \( d = 1 \) we recover the situation studied in [7] section 4: \( (\gamma_{1,\delta}^1)^o \) coincides with the stack \( \gamma_{n,\delta}^d \) introduced in loc. cit., definition 4.3.

**Proof.** (i) The detailed construction can be found for instance in [3] chap. VIII, sect. 12].

(ii) The analytic stack structure is obtained as follows: for a family \((S, \mathcal{A}) \xrightarrow{\pi} \Delta \) of Picard general K3 surfaces, \( \gamma_{n,\delta}^d(S) \) is naturally an open subscheme of the Kontsevich-Manin space of stable maps \( \mathcal{M}_{g(d)-\delta}(S; \beta) \), with suitable \( \beta \in H_2(S; \mathbb{Z}) \) such that \( \pi_* \beta = 0 \).

(iii) The proof is *ad litteram* the same as that of [7] proposition 4.8]. According to [6], there is a non-empty open subset \( \mathcal{X}_n^o \subset \mathcal{X}_n \) such that

\[ \forall (S, \mathcal{A}) \in \mathcal{X}_n^o, \text{ the linear system } |dA| \text{ contains irreducible, nodal curves with } \delta \text{ nodes.} \]

Consider a point \((S, C, u) \in \kappa^{-1}(\mathcal{X}_n^o)\), and denote \( \hat{C} := u_* C \). Then the short exact sequence

\[
0 \to T_S(\hat{C}) := \text{Ker}(\hat{\lambda}) \to T_S \xrightarrow{\hat{\lambda}} u_* \left( \frac{u^* T_S/T_C}{\cong \mathcal{K}_C} \right) \to 0
\]

induces the long exact sequence in cohomology:

\[
0 \to H^0(C, \mathcal{K}_C) \to H^1(S, T_S(\hat{C})) \xrightarrow{H^1(\iota)} H^1(S, T_S) \xrightarrow{H^1(\lambda)} H^1(C, \mathcal{K}_C) \to H^2(S, T_S(\hat{C})) \to 0
\]

The cohomology group \( H^1(S, T_S(\hat{C})) \) is naturally isomorphic to the Zariski tangent space \( T_{\gamma_{n,\delta}^d(S, C, u)} \) and the homomorphism \( H^1(\iota) \) can be identified with the differential of \( \kappa \) at
We deduce that $\text{Image}(d\kappa_{(S,C,u)}) = \text{Image}(H^1(\iota)) = \mathcal{T}_{\mathcal{X}_n,[S]}$, and $H^2(S,\mathcal{T}_S(\hat{C})) = 0$. □

The Zariski tangent space of $\mathcal{Y}_{n,\delta}^d$ is described in [7] section 4. Consider a triple $(S,C,u) \in \mathcal{Y}_{n,\delta}^d$, and let $\hat{S} \rightarrow S$ be the blow-up of $S$ at the $\delta$ double points of $\hat{C} = u(C)$. Then the morphism $u$ can be lifted to a morphism $\tilde{u}$ into $\hat{S}$, which is a closed embedding:

The infinitesimal deformations of $(S,C,u)$ are controlled by the (locally free) sheaf $(\sigma^*\mathcal{T}_S)(C)$ defined in the diagram below:

More precisely, the Zariski tangent space to $\mathcal{Y}_{n,\delta}^d$ at $(S,C,u)$ is isomorphic to

$$H^1(\hat{S},(\sigma^*\mathcal{T}_S)(C)) \cong H^1(S,\mathcal{T}_S(\hat{C})).$$

We are finally in position to precise the topic of this article. The first issue is:

**Question 1.3.** Is the morphism $\mathcal{Y}_{n,\delta}^d \xrightarrow{\mu} \mathcal{M}_{g(d)−\delta}$ (generically) finite?

The second issue is the following: one can easily see that $\dim(\mathcal{Y}_{n,\delta}^d) < \dim(\mathcal{M}_{g(d)−\delta})$ for $n$ sufficiently large.

**Question 1.4.** Is there any obstruction for a point $C \in \mathcal{M}_{g(d)−\delta}$ to lay in the image of $\mu$?

## 2. A vanishing result

In this section we prove the main technical ingredient needed for our approach to the question [1,3]. It is an application of Bogomolov’s effective restriction theorem for stable vector bundles over surfaces (see [3] section 7.3).
Proposition 2.1. Let $(S, \mathcal{A})$ be a Picard general $K3$ surface, with $\mathcal{A}^2 = 2(n-1)$, and let $\mathcal{E} \to S$ be a stable vector bundle of rank $r \geq 2$, and $c_1(\mathcal{E}) = 0$. Let $\hat{C} \to S$ be a reduced and irreducible, nodal curve having $\delta$ double points, with $\hat{C} \in |\mathcal{A}|$. We denote by $C \to \hat{C}$ its normalization, and by $C \xrightarrow{\nu} S$ the composed morphism.

Suppose that one of the conditions below are satisfied:

(i) $d = 1$ or $r = d = 2$, and $\delta \leq \left\lfloor \frac{(r-1)(n-1)d^2-1}{r} \right\rfloor - c_2(\mathcal{E})$, or 

(ii) $r \geq 2$ and $d \geq 2$, with $(r, d) \neq (2, 2)$, and $\delta \leq 2(n-1)d - c_2(\mathcal{E}) - \left\lceil \frac{r(n-1)+1}{r} \right\rceil$.

Then $H^0(C, u^*\mathcal{E}^\vee) = 0$.

Proof. Suppose that there is a non-zero section $\mathcal{O}_C \xrightarrow{s} u^*\mathcal{E}^\vee$. Then there is an effective divisor $\Delta$ on $C$, such that $s$ extends to a monomorphism of vector bundles $\mathcal{O}_C(\Delta) \to u^*\mathcal{E}^\vee$; equivalently, we obtain an epimorphism of vector bundles 

$$u^*\mathcal{E} \xrightarrow{\nu} \mathcal{O}_C(-\Delta),$$

with $\deg C \mathcal{O}_C(-\Delta) \leq 0$.

The direct image $\nu_*\mathcal{O}_C \to \hat{C}$ is a torsion free sheaf of rank one. We denote by $\hat{Q} := \Image(\nu_*q)$ appearing in the diagram 

$$\xymatrix{ \mathcal{E} \ar[r] & \mathcal{E} \otimes \mathcal{O}_C \ar[r] \ar[dl]_{\nu_*q} & \nu_*\mathcal{O}_C = \nu_*u^*\mathcal{E} \ar[r] & \nu_*\mathcal{O}_C. }$$

One can prove that $\hat{Q} \to \hat{C}$ is still a torsion free sheaf of rank one, and there is $0 \leq \delta' \leq \delta$ such that

$$\deg C \hat{Q} = \deg C \mathcal{O}_C + \delta';$$

equivalently, $\chi(\hat{Q}) = \delta' + \deg C \mathcal{O}_C - (n-1)d^2$.

One obtains a natural epimorphism of sheaves $\varepsilon : \mathcal{E} \to j_*\hat{Q}$ as follows:

$$\xymatrix{ \mathcal{E} \ar[r] & \mathcal{E} \otimes \mathcal{O}_{\hat{C}} \ar[r] \ar@/_/[d]_{\varepsilon} & j_*\hat{Q}. }$$

We denote by $\mathcal{G} := \Ker(\varepsilon)$; it is a locally free sheaf (a vector bundle) of rank $r$ over $S$. Using [8 reference] proposition 5.2.2, we compute its numerical invariants:

$$c_1(\mathcal{G}) = -[\hat{C}] = -dA, \quad c_2(\mathcal{G}) = c_2(\mathcal{E}) + (n-1)d^2 + \chi(\hat{Q}) = c_2(\mathcal{E}) + \delta' + \deg C \mathcal{O}_C \leq c_2(\mathcal{E}) + \delta,$$

$$\Delta(\mathcal{G}) := 2rc_2(\mathcal{G}) - (r-1)c_1^2(\mathcal{G}) = 2r \cdot \left[ c_2(\mathcal{G}) - \frac{r}{r-1} \cdot (n-1)d^2 \right] \leq 2r \cdot \left[ c_2(\mathcal{E}) + \delta - \frac{r}{r-1} \cdot (n-1)d^2 \right].$$

The hypothesis implies that $\Delta(\mathcal{G}) < 0$. Therefore [8 reference] theorem 7.3.4 implies the existence of a subsheaf $\mathcal{G}' \subset \mathcal{G}$ of rank $r'$, with torsion free quotient, such that:

$$0 < r'd - rm \Rightarrow 1 \leq (r-1)d - rm \Rightarrow m \leq \frac{(r-1)d - 1}{r}.$$

In particular $\frac{r+1}{r} \leq d$. For $d = 1$ and for $r = d = 2$, this gives already a contradiction, coming from the assumption that $u^*\mathcal{E}^\vee \to C$ has non-zero sections.
In higher degrees, we must go further a little bit, and use the second inequality in (2.1):
\[ \left( \frac{d}{r} - \frac{1}{r} \right)^2 \cdot 2(n - 1) \geq \left( \frac{d}{r} - \frac{m}{r'} \right)^2 \cdot 2(n - 1) \geq \frac{2(r-1)(n-1)d^2 - r(c_2(E) + \delta)}{r^2(r-1)} \]
\[ \Rightarrow \delta \geq 2(n-1)d - c_2(E) - \frac{r(n-1)}{r-1}. \]
This inequality contradicts again our hypothesis. \( \square \)

**Remark 2.2.** Let \( E \to S \) be as above, and suppose that \( r = 2 \). Then the proof of the proposition shows that actually \( u^*E \to C \) is a stable vector bundle, since we have used only that \( \deg_C Q \leq 0 \).

For \( r \geq 3 \), by applying the result to the exterior powers \( \bigwedge^\rho E \), \( \rho = 1, \ldots, r - 1 \) (which are still stable), it follows that \( u^*E^{\rho} \to C \) is a stable vector bundle itself, as soon as the number \( \delta \) of nodes is small enough. However, the formula for the upper bound of the number of nodes is lengthy, and we did not include it here.

The case when \( E \) is the tangent bundle \( T_S \) of \( S \) plays a privileged role for proving the rigidity of nodal curves on \( K3 \) surfaces. In this case, the previous theorem becomes:

**Corollary 2.3.** Suppose that \( S \) and \( u : C \to S \) are as in proposition [2.1]. If either
\begin{enumerate}[(i)]  
  \item \( d = 1 \) and \( \delta \leq \delta_{\text{max}}(n, 1) := \left\lfloor \frac{n}{2} \right\rfloor - 25 \) (hence \( n \geq 50 \)), or  
  \item \( d = 2 \) and \( \delta \leq \delta_{\text{max}}(n, 2) := 2n - 27 \) (hence \( n \geq 14 \)), or  
  \item \( d \geq 3 \) and \( \delta \leq \delta_{\text{max}}(n, d) := 2(n-1)(d-1) - 25 \) (hence \( (n-1)(d-1) \geq 13 \)),
\end{enumerate}
then \( u^*T_S \to C \) is a stable, rank two bundle, and therefore \( H^0(C, u^*T_S) = 0 \).

**Remark 2.4.** Notice that for degree one, nodal curves on Picard general \( K3 \) surfaces, the upper bound \( \delta_{\text{max}}(n, 1) \) appearing in the proposition above basically equals half of the arithmetic genus of the hyperplane section.

In this case \( \left\lfloor \frac{n}{2} \right\rfloor - 25 \) must be positive, and therefore the rigidity result holds for \( n \geq 50 \).

This bound is weaker than the (optimal) bound \( n \geq 13 \) obtained by Mukai in [10].

The upper bounds on the number of nodes obtained in corollary [2.3] are unlikely to be optimal. We are unable to address the following:

**Question 2.5.** Suppose that \( S \), and \( u : C \to S \) are as above, and that the genus of \( C \) is at least two. Is it true that \( u^*T_S \to C \) has no section?

A positive answer would allow to extend the rigidity results obtained in section [3].

3. THE RIGIDITY RESULT

In this section we are going to give a (partial) positive answer to the question [1.3].

**Theorem 3.1.** The morphism \( \mu : \mathcal{T}^d_{n,\delta} \to \mathcal{M}_{g(d)-\delta} \) is generically finite onto its image for all triples \((d, n, \delta)\) satisfying \( n \geq 13 \) and \( \delta \leq \delta_{\text{max}}(n, d) \), with \( \delta_{\text{max}}(n, d) \) as in corollary [2.3].

**Proof.** We must prove that for any smooth, quasi-projective curve \( \Delta \), and for any morphism \( \Delta \to \mathcal{T}^d_{n,\delta} \) such that \( \mu \circ U : \Delta \to \mathcal{M}_{g(d)-\delta} \) is constant, the morphism \( U \) is constant itself. Such a morphism \( U \) is equivalent to the following data:
- a smooth and irreducible curve \( C \) of genus \( g := g(d) - \delta \);
– a smooth family $(S, A) \xrightarrow{π} \triangle$ of Picard general $K3$ surfaces;
– a family of morphisms over the $1$-dimensional base $\triangle$

\[ C := \triangle \times C \xrightarrow{U=(u_t)_{t \in \triangle}} S = (S_t)_{t \in \triangle} \]

such that $\hat{C}_t := u_t(C) \hookrightarrow S_t$ are nodal curves, with $\delta$ ordinary double points, and $\hat{C}_t \in |dA_t|$ for all $t \in \triangle$. We must prove that, up to isomorphism, $S_t$ and $u_t$ are independent of $t \in \triangle$.

**Step 1** First of all, note that we may assume that $T_{\triangle} \to \triangle$ is trivializable. Otherwise we cover $\triangle$ with trivializable open subsets. We denote by $\partial/\partial t$ a trivializing section of $T_{\triangle}$.

The differentials of the various morphisms in (3.1) fit into the diagram:

\[ \begin{array}{cccccc}
0 & \longrightarrow & pr^*_C T_C & \longrightarrow & T_{\triangle \times C} = pr^*_C T_C \oplus pr^*_\triangle T_\triangle & \longrightarrow & pr^*_\triangle T_\triangle \cong O_{\triangle \times C} & \longrightarrow & 0 \\
0 & \longrightarrow & U^* T_S/\triangle & \longrightarrow & U^* T_S & \xrightarrow{\pi_*} & O_{\triangle \times C} & \longrightarrow & 0
\end{array} \]

We observe that $s := U_*(\partial/\partial t)$ is a section of $U^* T_S \to \triangle \times C$; let $s_t := s|_{\{t\} \times C} \in H^0(C, u_t^* T_S)$. The diagram (3.1) commutes, and therefore the tangential map $\pi_* : T S|_{S_t} \to T_\triangle$ sends $s$ into the trivializing section $\partial/\partial t \in H^0(\triangle, T_{\triangle})$. It follows that the second row in (3.2) is split, that is

\[ U^* T_S \cong O_{\triangle \times C} \oplus U^* T_S/\triangle. \]

With respect to this splitting $s = (s_0, \bar{s})$, where $\bar{s} \in H^0(\triangle \times C, U^* T_S/\triangle)$. By hypothesis $\hat{C}_t \hookrightarrow S_t$ are nodal curves for all $t \in \triangle$. Therefore corollary 2.3 implies that $u_t^* T_{S_t} \to C$ has no non-trivial sections, that is $s|_{\{t\} \times C} = 0$ for all $t$. We deduce that $\bar{s} = 0$, or intrinsically

\[ U_*(\partial/\partial t) = (s_0, 0). \]

**Step 2** We interpret the result in locally on $S$: there are local coordinates $(t, z, w)$ on $S$ such that the morphism $U$ is given by \[ U(t, x) = (t, z(t, x), w(t, x)), \quad \forall t \in \triangle, x \in C. \]

The equality (3.3) becomes:

\[ dz_{(t, x)}(\partial/\partial t) = 0 \quad \text{and} \quad dw_{(t, x)}(\partial/\partial t) = 0, \quad \forall (t, x) \in \triangle \times C. \]

It follows that $z(t, x) = z(x)$ and $w(t, x) = w(x)$, meaning that the ‘vertical’ component $u_t(x)$ of $U$ is independent of the parameter $t$.

Consider an arbitrary $t_0 \in \triangle$. Suppose that $\hat{x} \in \hat{C}_{t_0}$ is a double point, and let $x_1, x_2 \in C_{t_0}$ be the corresponding pair of points identified by the normalization map $C_{t_0} \xrightarrow{u_0} \hat{C}_{t_0}$. Then for all $t \in \triangle$ holds

\[ u_t(x_1) = (z(t, x_1), w(t, x_2)) = (z(t_0, x_1), w(t_0, x_2)) = u_0(x_1) = u_0(x_2) = \ldots = u_t(x_2), \]

that is the morphism $u_t$ will identify the same pairs of points of $C$. Since $t$ was arbitrary, we conclude that the curves $\hat{C}_t := u_t(C) \hookrightarrow S_t$ are all isomorphic to $\hat{C} := \hat{C}_{t_0}$. 
Step 3 The previous step reduces the initial problem to the study of deformations of pairs \((S, \hat{C})\), consisting of a K3 surface \(S\), and a nodal curve \(\hat{C} \hookrightarrow S\) (that is we forget about the normalization \(\nu : C \to \hat{C}\)). More precisely, we must prove that for any commutative diagram

\[
\begin{array}{ccc}
\triangle \times \hat{C} & \xrightarrow{j = (\nu)_t} & (S_t)_t \\
\downarrow \pi & & \downarrow \\
\triangle & & \\
\end{array}
\]

such that \(\hat{C} \cong j_t(\hat{C}) \hookrightarrow S_t, \forall t \in \triangle\) are nodal curves,

the family \((S, \triangle \times \hat{C}, j)\) is trivial.

The deformations of the pair \((S, \hat{C})\) are controlled by the locally free sheaf \(T_S(\hat{C})\), defined similarly as in [12] (see [7, section 2]). It fits into the exact sequence

\[
0 \to T_S(-\hat{C}) \to T_S(\hat{C}) \to T_C := \nu^*_C \to 0.
\]

The deformation \(j\) appearing in (3.4) keeps the nodal curve \(\hat{C}\) fixed, as an abstract curve. Therefore the infinitesimal deformation induced by \(j\) corresponds to an element

\[
\hat{e} \in \ker(H^1(S, T_S(\hat{C}))) \to H^1(\hat{C}, T_{\hat{C}}) = \text{Image}
\]

\[
H^1(S, T_S(-\hat{C})) \to H^1(S, T_S(\hat{C})).
\]

According to [10], \(H^1(S, T_S(-\hat{C})) = 0\) for a general \((S, A)\) with \(A^2 = 2(n - 1) \geq 24\). It follows that \(\hat{e} = 0\), which means that the deformation of the pair \((S, \hat{C})\) is trivial. □

Remark 3.2. The first step of the previous proof can be interpreted and proved at the level of the Zariski tangent space of \(\mathcal{V}^d_{n, \delta}\). Let \(e \in H^1(S, (\sigma^*T_S)(C))\) be the element corresponding to the deformation (3.1). By diagram chasing in [12] at the level of the long exact sequences in cohomology, we obtain the following (self-explanatory) diagram:

\[
\begin{array}{ccc}
0 & \xrightarrow{e} & H^1(S, (\sigma^*T_S)(C)) \\
& \searrow & \downarrow \pi \quad \text{It is injective by corollary} [24] \\
& & H^1(\hat{S}, (\sigma^*T_S)(\hat{C})) \\
\end{array}
\]

More precisely, we have the inclusion

\[
\ker(H^1(r)) \supseteq H^0(C, T_C) = 0 \quad H^1(\hat{S}, (\sigma^*T_S)(\hat{C})) \quad H^0(C, \nu^*T_S) = 0 \quad H^1(\hat{S}, \sigma^*T_S) \cong H^1(S, T_S)
\]

which shows that we can identify the deformation (3.1) with the induced infinitesimal deformation of \(S\), corresponding to \(e_0 = \pi_*(e)\). This is the differential theoretic counterpart of the claim that the section \(s\) appearing in the proof of (3.1) has the form \((s_0, 0)\).

The third step can be proved by differential methods too. However the second step is not proved at the tangential level: it uses effectively the fact that we are considering a 1-dimensional deformation.

As a byproduct we obtain:

Corollary 3.3. Suppose that \(S\) is a general K3 surface with \(\text{Pic}(S) = \mathbb{Z}A, A^2 = 2(n - 1) \geq 24\). Let \(\hat{C} \hookrightarrow S\) be a nodal curve of degree \(d\) with \(\delta\) nodes, such that \(\delta \leq \delta_{\text{max}}(n, d)\); we denote
by $N$ the set of nodes of $\hat{C}$, and let $\mathcal{I}_N \subset \mathcal{O}_S$ be their ideal sheaf. Then holds:

$$H^1(S, \Omega^1_S(\hat{C}) \otimes \mathcal{I}_N) = 0, \quad \text{or equivalently}$$

$$H^0(S, \Omega^1_S(\hat{C})) \to \bigoplus_{\hat{x} \in N} \Omega^1_S(\hat{x})$$

is surjective.

Proof. Simply consider the long exact sequence in cohomology corresponding to the first column in (1.2), and use the fact that $H^1(r)$ is injective. \hfill \Box

4. Applications to the Wahl map

The Wahl map for curves has been considered for the first time in [14]. The surjectivity of the Wahl map represents an obstruction for embedding a smooth curve into a $K3$ surface. For an overview of these results, and for further generalizations, we invite the reader to consult [15]. Here we recall only those notions which are necessary for this article.

Suppose that $L \to V$ is a line bundle over some variety $V$. The Wahl map is by definition

$$(4.1) \quad w_L : \bigwedge^2 H^0(V, L) \to H^0(V, \Omega^1_V \otimes L^2), \quad s \wedge t \mapsto sdt - tds.$$ 

Equivalently, the Wahl map is the restriction homomorphism $H^0(\text{res}_\Delta)$ at the level of sections, induced by the exact sequence

$$0 \to \mathcal{I}^2_{\Delta_V} \otimes (\mathcal{L} \boxtimes \mathcal{L}) \to \mathcal{I}_{\Delta_V} \otimes (\mathcal{L} \boxtimes \mathcal{L}) \xrightarrow{\text{res}_\Delta} \left(\mathcal{I}_{\Delta_V} / \mathcal{I}^2_{\Delta_V}\right) \otimes L^2 \to 0.$$ 

Much attention has been payed to the case when $V = C$ is a smooth projective curve, and $L = M = K_C$, where $K_C$ is the canonical line bundle of $C$. The importance of the map

$$w_C : \bigwedge^2 H^0(C, K_C) \to H^0(C, K^3_C)$$

relies in the fact that it gives an obstruction to realize the curve $C$ as a hyperplane section of a $K3$ surface. More precisely:

**Theorem 4.1.**

(i) (see [4]) The Wahl map $w_C$ is surjective for a general curve $C$ of genus at least 12.

(ii) (see [14, 2]) Suppose that $C \hookrightarrow S$ is a smooth hyperplane section of some $K3$ surface. Then the Wahl map is not surjective.

In other words, a generic smooth curve $C \in \mathcal{M}_g$ can not be realized as a hyperplane section of any $K3$ surface, as soon as $g \geq 12$. The surjectivity of the Wahl map is an obstruction for embedding a smooth curve into a $K3$ surface.

Remarkably enough, nodal curves escaped to the attention. Recently, in [7, Question 5.6], the authors asked whether there is an analogous obstruction for embedding nodal curves. We will apply the estimates obtained in section 2 to prove the following result:

**Theorem 4.2.** Let $S$ be a Picard general $K3$ surface, and let $\hat{C} \hookrightarrow S$ be a nodal curve of degree $d$ with $\delta$ nodes, with $\delta \leq \min \left\{ \delta_{\text{max}}(n, d), \frac{(n-1)d^2-1}{3} \right\}$. Then the Wahl map

$$w_C : \bigwedge^2 H^0(C, K_C) \to H^0(C, K^3_C)$$

is not surjective. (See corollary 2.3 for the definition of $\delta_{\text{max}}(n, d)$. )
We remark that for \( n \geq 146 \), the minimum between the two numbers above is
\[
\begin{align*}
\left\lfloor \frac{(n-1)d^2-1}{3} \right\rfloor & \quad \text{for } d = 1, \ldots, 4; \\
2(n-1)(d-1) - 25 & \quad \text{for } d \geq 5.
\end{align*}
\]

The proof of the theorem is inspired from \cite{2}, but contains several modifications needed to include the double points. Let us recall from loc. cit. that the proof of \ref{4.1}(ii) is based on the study of the diagram
\[
\begin{array}{c}
\xymatrix{
2 H^0(S, \mathcal{O}_S(C)) \ar[r]^{w_S} & H^0(S, \Omega^1_S \otimes \mathcal{O}_S(2C)) \ar[d]^{\rho} & H^0(S, \Omega^1_S \otimes \mathcal{K}^2_C) \ar[l]_{\rho_1} \\
2 H^0(C, \mathcal{K}_C) \ar[u]^{\rho} & H^0(C, \mathcal{K}^3_C) \ar[l]_{b}. 
}
\end{array}
\]

The surjectivity of \( w_C \) implies the surjectivity of \( b \), and one proves that this is impossible.

The main difficulty for extending this proof to the nodal case is that of finding appropriate substitutes for the cohomology groups appearing in \ref{4.2}. Since this task is rather computational, and is based on diagram chasing, it has been deferred to appendix A.

Now we introduce some notations which will be used in the proof of theorem \ref{4.2}. We denote by \( A \to S \) the ample generator of \( \text{Pic}(S) \). Let \( \tilde{C} \leftarrow S \) be a nodal curve of degree \( d \) with \( \delta \) nodes, and let \( N = \{ \hat{x}_1, \ldots, \hat{x}_\delta \} \subset S \) be its nodes. We denote by \( C \to \tilde{C} \) the normalization, and by \( x_1, x_2, \ldots, x_{\delta,1}, x_{\delta,2} \in C \) the pre-images by \( \nu \) of \( \hat{x}_1, \ldots, \hat{x}_\delta \) respectively. Let \( \tilde{S} := \text{Bl}_N(S) \to S \) be the blow-up of \( S \) at the nodes of \( \tilde{C} \), and \( E = E_1 + \ldots + E_\delta \) be the exceptional divisor in \( \tilde{S} \). Then the diagram
\[
\begin{array}{c}
\xymatrix{
C \ar[r]^{\nu} & \tilde{S} \\
\tilde{C} \ar[u]^{\tilde{u}} & S \ar[u]_{\sigma} \ar[l]_{\iota}
}
\end{array}
\]

is commutative, and \( \tilde{u} \) is an embedding. Note that the divisor \( \Delta := x_{1,1} + x_{1,2} + \ldots + x_{\delta,1} + x_{\delta,2} \) equals \( C \cdot E \), hence \( \mathcal{O}_C(\Delta) = \mathcal{O}_S(E) \mid_{C} = \mathcal{K}_S \mid_{C} \). We deduce the existence of the short exact sequence
\[
\begin{align*}
0 & \to T_C(\Delta) \to \Omega^1_S \otimes \mathcal{O}_C \to \mathcal{K}_C \to 0, \quad \text{and also that } \mathcal{K}_C \cong \sigma^* \mathcal{L}(-E).
\end{align*}
\]

**Lemma 4.3.** Let \( S, C, \) and \( u : C \to S \) be as above. If \( \delta \leq \min \left\{ \delta_{\max}(n, d), \frac{(n-1)d^2-1}{3} \right\} \), then the exact sequence \ref{4.3} is not split.

**Proof.** Let us assume that \( \tilde{u}^* \mathcal{T}_S \cong T_C \oplus \mathcal{K}_C(-\Delta) \). Then it follows from the diagram
\[
\begin{array}{c}
0 & \to T_C & \xrightarrow{\tilde{u}^*} \mathcal{T}_S & \to \mathcal{K}_C(-\Delta) & \to 0 \\
\xymatrix{ & 0 & \ar@{=}[u] & 0 & \\
0 & \to T_C & \xrightarrow{u^*} \mathcal{T}_S & \to \mathcal{K}_C & \to 0 }
\end{array}
\]

that \( H^0(C, \mathcal{K}_C(-\Delta)) \subset H^0(S, u^* \mathcal{T}_S) \). But the Riemann-Roch formula implies that
\[
\deg_C \mathcal{K}_C(-\Delta) - (g(C) - 1) = (g(C) - 1) - 2\delta = g(d) - 1 - 3\delta \geq 1.
\]
This contradicts corollary 2.3.

We denote by \( L := \mathcal{O}_S(C) \cong \mathcal{A}^d \). The Wahl maps of \( C \) and \( \tilde{S} \) fit into the following commutative diagram, which is important in the subsequent constructions:

\[
\begin{array}{ccc}
\bigwedge H^0(\tilde{S}, \sigma^* \mathcal{L}(-E)) & \xrightarrow{w_{\tilde{S}}} & H^0(\tilde{S}, \Omega^1_{\tilde{S}} \otimes \sigma^* \mathcal{L}(-2E)) \\
\downarrow \rho & & \downarrow \rho_1 \\
\bigwedge H^0(C, \mathcal{K}_C) & \xrightarrow{w_C} & H^0(C, \mathcal{K}_C^3). \\
\end{array}
\]

Proof. (of theorem 4.2) We assume that there is a curve \( \tilde{C} \hookrightarrow S \) such that the Wahl map of its normalization is surjective. We define

\[
R(C, \Delta) := w_C^{-1}(H^0(C, \mathcal{K}_C^3(-\Delta))) \subset \bigwedge H^0(C, \mathcal{K}_C),
\]

and denote by \( w_{C,\Delta} : R(C, \Delta) \to H^0(C, \mathcal{K}_C^3(-\Delta)) \) the restriction of the Wahl map to it. Since \( w_C \) is surjective, \( w_{C,\Delta} \) is surjective too. Furthermore, we define

\[
R := \rho^{-1}(R(C, \Delta)) \subset \bigwedge H^0(\tilde{S}, \sigma^* \mathcal{L}(-E))
\]

In the appendix we will construct the cube (A.3). Its rear face gives us the diagram

\[
\begin{array}{ccc}
R & \xrightarrow{w_{\tilde{C}}^{-1}} & H^1(\tilde{S}, \Omega^1_{\tilde{S}} \otimes \sigma^* \mathcal{L}^2(-3E)) \\
\downarrow \rho^\Delta & & \downarrow \rho_1 \\
R(C, \Delta) & \xrightarrow{w_{C,\Delta}} & H^0(C, \mathcal{K}_C^3(-\Delta)) \\
\end{array}
\]

which will be the substitute for the diagram (4.2) in the case of nodal curves.

Indeed, the surjectivity of \( \rho^\Delta \), and of \( w_{C,\Delta} \) implies the surjectivity of the homomorphism \( \delta \). But \( \delta \) is the restriction homomorphism at the level of sections in the exact sequence

\[
0 \to \mathcal{K}_C \to \Omega^1_{\tilde{S}|C} \otimes \mathcal{K}_C^2(-\Delta) \to \mathcal{K}_C^3(-\Delta) \to 0,
\]

generated by tensoring (4.3) with \( \mathcal{K}_C^2(-\Delta) \). Therefore the boundary map

\[
\partial : H^0(C, \mathcal{K}_C^3) \to H^1(C, \mathcal{K}_C)
\]

vanishes. By applying [2, Lemme 1], we deduce that the sequence (4.6) is split, hence (4.3) is split too. This contradicts the lemma 4.3.

Appendix A. Diagram chasing

In this section we continue to use the notations introduced in section 4.

Lemma A.1. The restriction homomorphisms

\[
H^0(\tilde{S}, \sigma^* \mathcal{L}(-E)) \to H^0(C, \mathcal{K}_C) \quad \text{and} \quad H^0(\tilde{S}, \sigma^* \mathcal{L}(-2E)) \to H^0(C, \mathcal{K}_C(-\Delta))
\]

are both surjective.
Proof. We have the exact sequence over \(\tilde{S}\): 
\[0 \to \sigma^* \mathcal{L}^{-1}(2E) \to \mathcal{O}_{\tilde{S}} \to \mathcal{O}_C \to 0.\]

(i) The first statement is obtained by tensoring it by \(\sigma^* \mathcal{L}(-E)\), and using that
\[H^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(E)) = H^1(\tilde{S}, \mathcal{K}_{\tilde{S}}) = 0.\]

(ii) The second statement is obtained by tensoring the exact sequence by \(\sigma^* \mathcal{L}(-2E)\), and using that
\[H^1(\tilde{S}, \mathcal{O}_{\tilde{S}}) = 0.\]

There is a natural restriction homomorphism \(\Omega^1_{\tilde{S}}^\text{res} \mathcal{K}_C\) which is surjective, and its kernel \(\mathcal{F} := \ker(\text{res}_C)\) is a locally free sheaf (a vector bundle) over \(\tilde{S}\) of rank two. The following commutative diagram is essential for the proof of theorem 4.2.

\[
\begin{array}{cccccccc}
0 & \to & \mathcal{F} \otimes \sigma^* \mathcal{L}^2(-3E) & \to & \Omega^1_{\tilde{S}} \otimes \sigma^* \mathcal{L}^2(-3E) & \to & \rho^1, \Delta & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{F} \otimes \sigma^* \mathcal{L}^2(-2E) & \to & \Omega^1_{\tilde{S}} \otimes \sigma^* \mathcal{L}^2(-2E) & \to & \rho^1 & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{F} \otimes \mathcal{O}_E & \to & \Omega^1_E \otimes \mathcal{O}_E & \to & \delta & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & & 0 & & 0 \\
\end{array}
\]

Actually the whole proof is based on the careful analysis of this diagram.

Every vector bundle on the projective line splits into the direct sum of line bundles. Hence the restriction of \(\Omega^1_{\tilde{S}}\) to each component \(E_j, j = 1, \ldots, \delta\), of the exceptional divisor \(E\) is the direct sum of line bundles. In fact
\[\Omega^1_{\tilde{S}} \otimes \mathcal{O}_{E_j} = \Omega^1_{E_j} \oplus \mathcal{N}^\vee_{E_j|\tilde{S}} = \mathcal{O}_{E_j}(-2) \oplus \mathcal{O}_{E_j}(1).\]

Therefore \(\Omega^1_{\tilde{S}} \otimes \mathcal{O}_E = \Omega^1_E \oplus \mathcal{N}^\vee_{E|\tilde{S}} = \mathcal{O}_E(-2) \oplus \mathcal{O}_E(1)\), where we use the shorthand notation
\[\mathcal{O}_E(-2) := \bigoplus_{j=1}^{\delta} \mathcal{O}_{E_j}(-2), \quad \text{and} \quad \mathcal{O}_E(1) := \bigoplus_{j=1}^{\delta} \mathcal{O}_{E_j}(1).\]

Since the homomorphism \(\text{res}_C\) is the restriction of 1-forms on \(\tilde{S}\) to 1-forms on \(C\), we deduce from the last line in (A.1) that
\[
\mathcal{F} \otimes \mathcal{O}_E = \mathcal{O}_E(-2) \oplus \mathcal{O}_E(-1).
\]

Lemma A.2. (i) \(H^0(\mathcal{F} \otimes \mathcal{O}_E) = 0\) and \(H^1(\mathcal{F} \otimes \mathcal{O}_E) = H^1(\mathcal{O}_E(-2)) \cong \mathbb{C} \oplus \ldots \oplus \mathbb{C} \) \(\delta\) times;
(ii) \(H^0(\tilde{S}, \mathcal{F} \otimes \sigma^* \mathcal{L}^2(-3E)) \cong H^0(\tilde{S}, \mathcal{F} \otimes \sigma^* \mathcal{L}^2(-2E))\) is an isomorphism;
(iii) \(H^1(\tilde{S}, \mathcal{F} \otimes \sigma^* \mathcal{L}^2(-3E)) \to H^1(\tilde{S}, \mathcal{F} \otimes \sigma^* \mathcal{L}^2(-2E))\) is injective.

Proof. (i) It follows from (A.2).
(ii) and (iii) Consider the long exact sequence in cohomology corresponding to the first column in (A.1). The claims follows from (i) above. \(\square\)
Standing hypothesis. From now on we will assume that the nodal curve $\hat{C} \hookrightarrow S$ has the property that the Wahl map of its normalization

$$w_C : \bigwedge^2 H^0(C, \mathcal{K}_C) \to H^0(C, \mathcal{K}_C^3)$$

is surjective.

Lemma A.3. (i) The homomorphisms

$$H^1(\tilde{S}, \mathcal{F} \otimes \sigma^* \mathcal{L}^2(-2E)) \to H^1(\tilde{S}, \Omega^1_S \otimes \sigma^* \mathcal{L}^2(-2E)),$$

and

$$H^1(\tilde{S}, \mathcal{F} \otimes \sigma^* \mathcal{L}^2(-3E)) \to H^1(\tilde{S}, \Omega^1_S \otimes \sigma^* \mathcal{L}^2(-3E))$$

are injective.

(ii) The restriction homomorphisms

$$\rho_1 : H^0(\tilde{S}, \Omega^1_S \otimes \sigma^* \mathcal{L}^2(-2E)) \to H^0(C, \mathcal{K}_C^3),$$

and

$$\rho_{1,\Delta} : H^0(\tilde{S}, \Omega^1_S \otimes \sigma^* \mathcal{L}^2(-3E)) \to H^0(C, \mathcal{K}_C^3(-\Delta))$$

are surjective.

(iii) $H^0(\tilde{S}, \Omega^1_S \otimes \sigma^* \mathcal{L}^2(-3E)) \cong \rho_{1,\Delta}^{-1}(H^0(C, \mathcal{K}_C^3(-\Delta)))$

$$= \{ s \in H^0(\tilde{S}, \Omega^1_S \otimes \sigma^* \mathcal{L}^2(-2E)) \mid \rho_1(s) \in H^0(C, \mathcal{K}_C^3(-\Delta)) \}.$$

Proof. (i) In the commutative diagram (A.3) the homomorphisms $w_C$ and $\rho$ are surjective, hence $\rho_1$ is also surjective. We deduce the injectivity of the first homomorphism from the second line in (A.1). On the other hand, it follows from (A.1) that we have the commutative square

$$\begin{array}{ccc}
H^1(\tilde{S}, \mathcal{F} \otimes \sigma^* \mathcal{L}^2(-3E)) & \xrightarrow{\text{injective by (A.3) i)}} & H^1(\tilde{S}, \Omega^1_S \otimes \sigma^* \mathcal{L}^2(-3E)) \\
\downarrow \text{as } \rho_1 \text{ surjective} & & \downarrow \\
H^1(\tilde{S}, \mathcal{F} \otimes \sigma^* \mathcal{L}^2(-2E)) & \xrightarrow{\text{injective}} & H^1(\tilde{S}, \Omega^1_S \otimes \sigma^* \mathcal{L}^2(-2E)).
\end{array}$$

It follows that the upper homomorphism is injective too, as claimed.

(ii) The surjectivity of $\rho_1$ has been proved already. For the second one, consider the long exact sequence in cohomology corresponding to the first line in (A.1), and use (i) above.

(iii) The first two rows of (A.1), together with (ii) above imply that we have the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^0(\tilde{S}, \mathcal{F} \otimes \sigma^* \mathcal{L}^2(-3E)) & \longrightarrow & H^0(\tilde{S}, \Omega^1_S \otimes \sigma^* \mathcal{L}^2(-3E)) & \longrightarrow & H^0(C, \mathcal{K}_C^3(-\Delta)) & \longrightarrow & 0 \\
\downarrow \cong & & \downarrow \cap & & \downarrow \cap & & \downarrow & & \\
0 & \longrightarrow & H^0(\tilde{S}, \mathcal{F} \otimes \sigma^* \mathcal{L}^2(-2E)) & \longrightarrow & H^0(\tilde{S}, \Omega^1_S \otimes \sigma^* \mathcal{L}^2(-2E)) & \longrightarrow & H^0(C, \mathcal{K}_C^3) & \longrightarrow & 0
\end{array}$$

The claim is a consequence of the fact that the first vertical arrow is an isomorphism. \qed

We define $R(C, \Delta) := w^{-1}_C(H^0(C, \mathcal{K}_C^3(-\Delta))) \subset \bigwedge^2 H^0(C, \mathcal{K}_C)$, and denote by $w_{C,\Delta}$ the restriction of the Wahl map to it. Then $w_{C,\Delta} : R(C, \Delta) \to H^0(C, \mathcal{K}_C^3(-\Delta))$ is surjective because $w_C$ is surjective.
The cohomology groups introduced so far fit into the following commutative cube:

\[
R := \rho^{-1}(R(\mathcal{C}, \Delta)) \xrightarrow{w_{\tilde{S}, E}} H^0(\tilde{S}, \Omega^1_{\tilde{S}} \otimes \sigma^* \mathcal{L}(-3E)) \]

\[
\xrightarrow{w_{\tilde{S}}} H^0(\tilde{S}, \Omega^1_{\tilde{S}} \otimes \sigma^* \mathcal{L}(-2E)) \]

\[
\xrightarrow{\rho_1 \Delta} H^0(\tilde{S}, \sigma^* \mathcal{L}(-E)) \]

\[
\xrightarrow{\rho_1} H^0(\mathcal{C}, K_{\mathcal{C}}^3(-\Delta)) \]

\[
\xrightarrow{w_{\mathcal{C}}} H^0(\mathcal{C}, K_{\mathcal{C}}^3) \]

\[
\xrightarrow{w_{\mathcal{C}, \Delta}} H^0(\mathcal{C}, K_{\mathcal{C}}^3) \]

\[
\xrightarrow{\rho} H^0(C, K_C^3) \]

\[
\xrightarrow{\rho_1} H^0(C, K_C^3) \]

\[
\xrightarrow{w_{\mathcal{C}}} H^0(C, K_C^3) \]

\[
\xrightarrow{w_{\mathcal{C} \Delta}} H^0(C, K_C^3(-\Delta)) \]

\[
\xrightarrow{\rho_1 \Delta} H^0(\mathcal{C}, K_{\mathcal{C}}^3(-\Delta)) \]

\[
\xrightarrow{w_{\mathcal{C}, \Delta}} H^0(\mathcal{C}, K_{\mathcal{C}}^3) \]

\[
\xrightarrow{\rho} H^0(C, K_C^3) \]

\[
\xrightarrow{w_{\mathcal{C}}} H^0(C, K_C^3) \]

\[
\xrightarrow{w_{\mathcal{C} \Delta}} H^0(C, K_C^3(-\Delta)) \]

\[
\xrightarrow{\rho_1 \Delta} H^0(\mathcal{C}, K_{\mathcal{C}}^3(-\Delta)) \]

\[
\xrightarrow{w_{\mathcal{C}, \Delta}} H^0(\mathcal{C}, K_{\mathcal{C}}^3) \]

The ′ ⊂ ′ signs on various arrows denote inclusions.

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