CRITICAL STRONG FELLER REGULARITY FOR MARKOV SOLUTIONS TO THE NAVIER–STOKES EQUATIONS

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ABSTRACT. The main purpose of this paper is to show that Markov solutions to the 3D Navier–Stokes equations driven by Gaussian noise have the strong Feller property up to the critical topology given by the domain of the Stokes operator to the power one-fourth.

1. INTRODUCTION

It is not known whether the martingale problem for the Navier–Stokes equations driven by Gaussian noise is well–posed [7, 23]. In order to analyse the problem Da Prato and Debussche [6] (see also [9, 18]) showed the existence of Markov processes solutions to the equations and some regularity properties of the transitions semigroups.

A different approach to the existence and regularity of Markov solutions has been introduced in [13, 15] (see also [14, 22, 23, 21, 3, 25, 17]), based on an abstract selection principle for Markov families (see Theorem 2.3) and the short time coupling with a smooth process. A refined analysis of this coupling is one of the purposes of this paper (see Sections 3 and 5.1).

Here we consider the Navier–Stokes equations on the three dimensional torus $\mathbb{T}_3$ with periodic boundary conditions,

\begin{equation}
\begin{cases}
\dot{u} + (u \cdot \nabla)u + \nabla p = \nu \Delta u + \dot{\eta}, \\
\text{div } u = 0,
\end{cases}
\end{equation}

driven by a Gaussian noise. For simplicity we can represent the noise as

$$\dot{\eta} = \sum_{k \in \mathbb{Z}^3} \sigma_k \, d\beta_k(t) \, e^{ik \cdot x},$$

where $(\beta_k)_{k \in \mathbb{Z}^3}$ are (suitably) independent Brownian motions (precise definitions and assumptions will be given in the next section). The analysis originated in [15] used in a crucial way two main assumptions on the driving noise, namely regularity and non-degeneracy. The property of non-degeneracy can

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be translated, roughly speaking, in terms of the coefficients \((\sigma_k)_{k \in \mathbb{Z}}\) simply as \(\sigma_k > 0\). The possibility to relax this condition is analysed in Romito and Xu [25].

The main purpose of this paper is to complete the analysis developed in [13, 14, 15, 22, 23, 21] and relax the regularity assumption, namely to allow coefficients whose decay as \(|k| \to \infty\) is of order \(|\sigma_k| \approx |k|^{-3/2-2\alpha_0}\) for \(\alpha_0 > 0\). In [15] the restriction is \(\alpha_0 > \frac{1}{6}\), so the improvement seems tiny. On the other hand the following result achieved here is, in a way, the best possible.

**Theorem.** Assume non-degeneracy (as explained above) and let \(\alpha_0 > 0\). Then every Markov solution to the Navier-Stokes equations is strong Feller in the topology of \(D(A^{\alpha})\) for every \(\alpha > \frac{1}{2}\), where \(A\) is the Stokes operator.

This optimality has a twofold reason. On one hand, the value of the main parameter \(\alpha_0 \leq 0\) would correspond to non-trace class covariance and the analysis of the Navier-Stokes equations in this case is open. On the other hand the main theorem above states that under this assumption every solution has good regularity properties as long as the underlying equation admits local smooth solutions. In fact, the value \(\frac{1}{2}\) is the critical threshold for existence and uniqueness of smooth solution in the deterministic case, as proved by Fujita and Kato [16]. An explanation of the critical value, of the connection with the scaling properties of the equation and in general of the scaling heuristic for the Navier–Stokes equations can be found for instance in Cannone [4].

In conclusion in this paper we verify that every Markov diffusion generated by the Navier–Stokes equations has good properties of regularity as long as it lives in the largest possible space (at least in the hierarchy of hilbertian Sobolev spaces) dictated by the deterministic analysis.

The paper is organised as follows. Section 2 contains notations and a short summary of those definitions and result useful for this work. The strong Feller property in strong topologies is proved in Section 3. The main theorem (re-cast as Theorem 4.1) is proved in Section 4 and some additional properties of the Markov solutions which follow from it are given in Section 4.1. Finally in Section 5 we prove some technical results: the construction of the short time coupling with a smooth solutions and an inequality for the Navier–Stokes non-linearity.

### 2. Generalities and past results

Let \(T_3 = [0, 2\pi]^3\) and let \(\mathcal{D}^\infty\) be the space of infinitely differentiable divergence free periodic vector fields with mean zero on \(T_3\). Let \(H\) be the closure of \(\mathcal{D}^\infty\) in \(L^2(T_3, \mathbb{R}^3)\) and \(V\) be the closure in \(H^1(T_3, \mathbb{R}^3)\). Denote by \(A\), with domain \(D(A)\), the *Stokes operator* and for every \(\alpha \in \mathbb{R}\) set \(V_\alpha = D(A^{\alpha/2})\), with norm \(\|u\|_\alpha = \|A^{\alpha/2}u\|_H\) for \(u \in V_\alpha\). In particular we have \(V_0 = H\), \(V_1 = V\) and \(V_{-1} = V'\). Define the bi-linear operator \(B : V \times V \to V'\) as the projection onto
1.1

We recast problem (1.1) in the following abstract form,

\[ du + (\nu Au + B(u, u)) \, dt = \mathcal{Q}^{\frac{1}{2}} \, dW, \]

where \( W \) is a cylindrical Wiener process on \( H \) and \( \Omega \) is a linear bounded symmetric positive operator on \( H \) with finite trace.

The probabilistic framework for problem (2.1) is given as follows. Set \( \Omega_{\text{NS}} = \mathcal{C}([0, \infty); D(A)) \), let \( \mathcal{B} \) be the Borel \( \sigma \)-field on \( \Omega_{\text{NS}} \) and let \( \xi : \Omega_{\text{NS}} \rightarrow D(A)' \) be the canonical process on \( \Omega_{\text{NS}} \) (that is, \( \xi_t(\omega) = \omega(t) \)). Define the filtration \( \mathcal{B}_t = \sigma(\xi_s : 0 \leq s \leq t) \).

We give the definition of solutions following the approach presented in [21], which we briefly recall. For every \( \varphi \in \mathcal{D}^{\infty} \) consider the process \( (M^\varphi_t)_{t \geq 0} \) on \( \Omega_{\text{NS}} \) defined for \( t \geq 0 \) as

\[ M^\varphi_t = \langle \xi_t - \xi_0, \varphi \rangle_H + \nu \int_0^t \langle \xi_s, A\varphi \rangle_H \, ds - \int_0^t \langle B(\xi_s, \varphi), \xi_s \rangle_H \, ds. \]

**Definition 2.1.** Given \( \mu \in \text{Pr}(H) \), a probability \( P^\mu \) on \( (\Omega_{\text{NS}}, \mathcal{B}) \) with marginal \( \mu \) at time \( t = 0 \) is a **weak martingale solution** starting at \( \mu \) to problem (2.1) if

- \( P^\mu[L^2_{\text{loc}}([0, \infty); H)] = 1 \),
- for each \( \varphi \in \mathcal{D}^{\infty} \) the process \( (M^\varphi_t, \mathcal{B}_t, P^\mu) \) is a square integrable continuous martingale with quadratic variation \( [M^\varphi]_t = t\|\mathcal{Q}^{\frac{1}{2}}\varphi\|_H^2 \).

Let \( (\sigma^2_k)_{k \in \mathbb{N}} \) be the system of eigenvectors of the covariance \( \mathcal{Q} \) and let \( (e_k)_{k \in \mathbb{N}} \) be a corresponding complete orthonormal system of eigenfunctions. Define for every \( k \in \mathbb{N} \) the process \( \beta_k(t) = \sigma^{-1}_k M^e_k \). Under a weak martingale solution \( P \), \( (\beta_k)_{k \in \mathbb{N}} \) is a sequence of independent one dimensional Brownian motions, thus the process

\[ W(t) = \sum_{k=0}^{\infty} \sigma_k \beta_k(t) e_k \]

is a \( \Omega \)-Wiener process and \( z(t) = W(t) - \nu \int_0^t A e^{-\nu A(t-s)} W(s) \, ds \) is the associated Ornstein-Uhlenbeck process starting at 0, that is the solution to

\[ dz + \nu Az \, dt = \mathcal{Q}^{\frac{1}{2}} \, dW, \quad z(0) = 0. \]

Define the process \( v(t, \cdot) = \xi_t(\cdot) - z(t, \cdot) \). Since \( M^\varphi_t = \langle W(t), \varphi \rangle \) for every test function \( \varphi \), it follows that \( v \) is a weak solution of the equation

\[ \partial_t v + \nu Av + B(v + z, v + z) = 0, \quad P - \text{a. s.}, \]
with initial condition $v(0) = \xi_0$. The energy balance functional associated to $v$ is given as

$$E_t(v, z) = \frac{1}{2} \|v_t\|^2_H + v \int_0^t \|v_r\|^2_V dr - \int_0^t \langle z_r, B(v_r + z_r, v_r) \rangle dr.$$ 

**Definition 2.2.** Given $\mu \in \text{Pr}(H)$, a weak martingale solution $P_\mu$ starting at $\mu$ is an energy martingale solution if

- $P_\mu[v \in L^\infty_{\text{loc}}([0, \infty); H) \cap L^2_{\text{loc}}([0, \infty); V)] = 1$,
- there is a set $T_{P_\mu} \subset (0, \infty)$ of null Lebesgue measure such that for all $s \not\in T_{P_\mu}$ and all $t \geq s$, $P_\mu[E_t(v, z) \leq E_s(v, z)] = 1$.

The following theorem ensures existence of a Markov family of solutions to problem (1.1).

**Theorem 2.3 ([21]).** There exists a family $(P_x)_{x \in H}$ of energy martingale solutions such that $P_x[\xi_0 = x] = 1$ for every $x \in H$ and for almost every $s \geq 0$ (including $s = 0$), for all $t \geq s$ and all bounded measurable $\phi : H \to \mathbb{R}$,

$$E^{P_x} [\phi(\xi'_t) | \mathcal{F}_s] = E^{P_{\xi_0}} [\phi(\xi'_{t-s})].$$

In the rest of the paper, we shall consider the following assumption on the covariance operator.

**Assumption 2.4.** The covariance operator $Q$ of the driving noise satisfies

- [n1] there is $\alpha_0 > 0$ such that $A^{\frac{3}{2} + \alpha_0} Q^{\frac{1}{2}}$ is a linear bounded operator on $H$,
- [n2] $A^{\frac{3}{2} + \alpha_0} Q^{\frac{1}{2}}$ is a linear bounded invertible operator on $H$, with bounded inverse.

We shall emphasize when we need the stronger property [n2] or, vice versa, when the weaker property [n1] is sufficient for our purposes.

3. The Strong Feller Property

In this section we extend [15, Theorem 5.11] and [14, Theorem 3.1] to all the admissible values of $\alpha$ and $\alpha_0$ where a short time coupling with smooth solutions is possible (see Theorem 5.1).

**Definition 3.1.** A semigroup $(P_t)_{t \geq 0}$ is $V_\alpha$–strong Feller at time $t > 0$ if $P_t \phi \in C_b(V_\alpha)$ for every $\phi : H \to \mathbb{R}$ bounded measurable.

**Theorem 3.2.** Under Assumption 2.4, let $\alpha > \frac{1}{2}$ be such that

$$\max\{1 + \alpha_0, \frac{1}{2} + 2\alpha_0\} \leq \alpha < 1 + 2\alpha_0$$

(with $\alpha > \max\{1 + \alpha_0, \frac{1}{2} + 2\alpha_0\}$ if $\alpha_0 = \frac{1}{2}$). Then the transition semigroup $(P_t)_{t \geq 0}$ associated to any Markov solution $(P_x)_{x \in H}$ is $V_\alpha$–strong Feller for every $t > 0$. Moreover, there are $c > 0$ and $\gamma \geq 2$ (whose value is given in the proof) such that for all
\( \phi \in B_b(H), \, x \in V_\alpha \text{ and } h \in V_\alpha \text{ with } \|h\|_\alpha \leq 1, \)

\begin{align}
(3.1) \quad |\mathcal{P}_t \phi(x + h) - \mathcal{P}_t \phi(x)| &\leq \frac{c}{t \wedge 1} (1 + \|x\|_{\gamma}) \|h\|_\alpha \log(e \|h\|_{\alpha}^{-1}).
\end{align}

**Proof.** We follow the lines of the proof of \([15, \text{Theorem 5.11}]\). Let \(x \in V_\alpha \) and \(h \in V_\alpha \text{ with } \|h\|_\alpha \leq 1, \) and choose \(R \geq 3(1 + \|x\|_\alpha). \) Fix \(t > 0 \) and let \(\epsilon > 0 \) be such that \(\epsilon \leq c R^{-\gamma} \) (where \(c \gamma \) are so that Proposition 5.7 holds true) and \(\epsilon \not\in T_{P, x} \cup T_{P, x + h}, \) where \(T_P \) is the set of exceptional times where the energy inequality fails to hold for \(\mathbb{P} \) (see Definition 2.2). Then for every \(\phi \in B_b(H) \) with \(\|\phi\|_{\infty} \leq 1, \)

\[
|\mathcal{P}_t \phi(x + h) - \mathcal{P}_t \phi(x)| \leq |\mathcal{P}_t \phi(x + h) - \mathcal{P}_t^{(\alpha, R)} \psi(x + h)| + |\mathcal{P}_t^{(\alpha, R)} \psi(x + h) - \mathcal{P}_t \phi(x)|,
\]

where we have set \(\psi = \mathcal{P}_{t - \epsilon} \phi\) and we have used the Markov property (in the version of Theorem 2.3). Now, by Theorem 3.6 and Proposition 5.7,

\[
|\mathcal{P}_t^{(\alpha, R)} \psi(x) - \mathcal{P}_t \phi(x)| = E^{(\alpha, R)} [\psi(x, \xi_e) 1_{\{\tau_\alpha(x, R) < \epsilon\}}] - E^{(\alpha, R)} [\psi(s, \xi_e) 1_{\{\tau_\alpha(s, R) < \epsilon\}}] \leq c \|\phi\|_{\infty} e^{-c_2 \epsilon^2},
\]

and similarly for the term in \(x + h. \) The middle term can be estimated using either Propositions 3.3 or 3.4, depending on the value of \(\alpha. \) We consider first the case \(\alpha > \frac{3}{2}, \) so that

\[
|\mathcal{P}_t \phi(x + h) - \mathcal{P}_t \phi(x)| \leq c_1 e^{-c_2 \epsilon^2} + \frac{c_1}{\epsilon} \|h\|_\alpha e^{c_3 R^2 \epsilon}
\]

for constants \(c_1, \ldots, c_3 \) and \(R \geq 3(1 + \|x\|_\alpha), \epsilon \leq (c_4 R^{-2}) \) and \(\epsilon \leq \frac{1}{2}(t \wedge 1). \) As in the proof of \([14, \text{Theorem 3.1}]\), we choose the values \(R = 3(1 + \|x\|_\alpha) \) and \(\epsilon \approx (1 \wedge t \wedge c_4 R^{-2})/(\log(\|h\|_\alpha / \epsilon)) \) to get (3.1).

On the other hand, if \(\alpha \leq \frac{3}{2}, \) then

\[
|\mathcal{P}_t \phi(x + h) - \mathcal{P}_t \phi(x)| \leq c_1 e^{-c_2 \epsilon^2} + \frac{c_1}{\epsilon} \|h\|_\alpha e^{c_3 R^2 \epsilon}
\]

for \(R \geq 3(1 + \|x\|_\alpha), \epsilon \leq (c_4 R^{-\gamma}) \) and \(\epsilon \leq \frac{1}{2}(t \wedge 1), \) with \(\gamma = 4/(3 + 4 \alpha_0 - 2\alpha). \) A similar choice of \(\epsilon \) and \(R \) leads again to (3.1). \(\square\)

The rest of the section contains the arguments needed to complete the proof of the above theorem.

### 3.1. Differentiability of the approximated flow

Given \(\alpha \in (\frac{3}{4}, 1 + 2 \alpha_0), \) let \(\mathcal{P}_t^{(\alpha, R)} \varphi(x) = E[\varphi(u_{(\alpha, R)}^x(t))] \) be the transition semigroup associated to problem (5.1), with \(x \in V_\alpha \) and \(\varphi : H \to \mathbb{R} \) bounded measurable. In this section we analyse the regularity of this semigroup.
Proposition 3.3. Assume [n1] and [n2] of Assumption 2.4. Given \( R \geq 1 \) and \( \alpha \) such that
\[
\alpha > \frac{3}{2} \quad \text{and} \quad \frac{1}{2} + 2\alpha_0 \leq \alpha < 1 + 2\alpha_0,
\] the transition semigroup \( \{P^t_{\alpha, R}\}_{t \geq 0} \) associated to problem (5.1) is \( V_\alpha \)-strong Feller for all \( t > 0 \). Moreover, there are numbers \( c_1 > 0 \) and \( c_2 > 0 \) such that for every \( x_0 \in V_\alpha \), for every \( \varphi \in B_b(H) \) and for every \( h \in V_\alpha \),
\[
|P^t_{\alpha, R}(x_0 + h) - P^t_{\alpha, R}(x_0)| \leq \frac{c_1}{t^{\sqrt{v}}} \|h\|_\alpha e^{c_2 R^2 t} \|\varphi\|_\infty.
\]

Proof. Fix \( \alpha \) as in (3.2) and let \( t > 0 \) and \( \varphi \in B_b(H) \) with \( \|\varphi\|_\infty \leq 1 \). We proceed as in [15, Proposition 5.13]. By the Bismut, Elworthy and Li formula,
\[
|P^t_{\alpha, R}(x_0 + h) - P^t_{\alpha, R}(x_0)| \leq \frac{c}{t} \sup_{\eta \in [0,1]} \mathbb{E}^{P^0_{\alpha_0 + n h}} \left( \int_0^t \|Q^{-\frac{1}{2}} D_h \xi_s\|^2_{\|\varphi\|} \, ds \right)^{\frac{1}{2}}
\]
\[
\leq \frac{c}{t} \sup_{\eta \in [0,1]} \mathbb{E}^{P^0_{\alpha_0 + n h}} \left( \int_0^t \|D_h \xi_s\|^2_{\|\varphi\|} \, ds \right)^{\frac{1}{2}},
\]
since \( \|Q^{-\frac{1}{2}} D_h \xi_s\|_\|\varphi\| \leq C \|D_h \xi_s\|_{3/2+2\alpha_0} \), by [n2] on \( \Omega \), and so we only have to estimate the inner integral. For every \( x \in V_\alpha \) and \( h \in V_\alpha \), denote by \( u_\alpha^{(R)} \) the process solution to (5.1) starting at \( x \), and by \( \tilde{u} = D_h u_\alpha^{(R)} \) the derivative of the flow in the direction \( h \). Then \( \tilde{u} \) solves
\[
\partial_t \tilde{u} + vA \tilde{u} + \frac{X'_R(\|u_\alpha^{(R)}\|)}{\|u_\alpha^{(R)}\|} (u_\alpha^{(R)}, \tilde{u}) \nu_\alpha B(u_\alpha^{(R)}, u_\alpha^{(R)}) + \frac{X_R(\|u_\alpha^{(R)}\|)}{\|u_\alpha^{(R)}\|} (B(\tilde{u}, u_\alpha^{(R)}) + B(u_\alpha^{(R)}, \tilde{u})) = 0,
\]
with initial condition \( \tilde{u}(0) = h \), and so
\[
\frac{d}{dt} \|\tilde{u}\|_\alpha^2 + 2v\|\tilde{u}\|_{\alpha + 1}^2 \leq 2\frac{X'_R(\|u_\alpha^{(R)}\|)}{\|u_\alpha^{(R)}\|} (\tilde{u}, B(u_\alpha^{(R)}, u_\alpha^{(R)})) \nu_\alpha \|\tilde{u}\|_\alpha
\]
\[
+ 2X_R(\|u_\alpha^{(R)}\|) (\tilde{u}, B(\tilde{u}, u_\alpha^{(R)}) + B(u_\alpha^{(R)}, \tilde{u})) \nu_\alpha.
\]
In short, everything boils down to estimating the right-hand side (briefly denoted below by \( \textcircled{1} \)). By Lemma 5.11 (with \( a = b = \alpha \) and \( c = -\alpha \)) and Young’s inequality,
\[
\textcircled{1} \leq \frac{c}{R} R^2 \|\tilde{u}\|_\alpha \|\tilde{u}\|_{\alpha + 1} + cR \|\tilde{u}\|_\alpha \|\tilde{u}\|_{\alpha + 1} \leq v\|\tilde{u}\|_{\alpha + 1}^2 + \frac{c}{\sqrt{v}} R^2 \|\tilde{u}\|_\alpha^2
\]
and so, by Gronwall’s lemma,
\[
\mathbb{E} \left[ \int_0^t \|\tilde{u}\|_{\alpha + 1}^2 \, ds \right] \leq \frac{1}{\sqrt{v}} \|\tilde{h}\|_\alpha^2 e^{cR^2 t},
\]
which is enough to bound (3.3), as, by the choice of \( \alpha, 1 + \alpha \geq \frac{3}{2} + 2\alpha_0 \). \( \square \)
The strong Feller property as well as formula (3.5) (the transition semigroup \([P_t^{(\alpha, R)}])_{t \geq 0} associated to problem (5.1) is \(V_\alpha\)-strong Feller for all \(t > 0\). Moreover, there are numbers \(c_1 > 0\) and \(c_2 > 0\) such that for every \(x_0 \in V_\alpha\), for every \(\varphi \in B_b(H)\) and for every \(h \in V_\alpha\),

\[
|P_t^{(\alpha, R)} \varphi(x_0 + h) - P_t^{(\alpha, R)} \varphi(x_0)| \leq \frac{c_1}{t^{\frac{1}{2}}} \|h\|_{\frac{1}{2} + 2\alpha_0} \exp\left(\frac{c_2 t}{(2\alpha_1 - 1 + 4\alpha_0)} \right).
\]

The strong Feller property as well as formula (3.6) are also true if \(\alpha = \frac{3}{2}\) and \(\alpha_0 \in (\frac{1}{2}, \frac{3}{2})\).

**Proof.** Let \(\alpha\) be as in condition (3.5) and set \(\gamma = 2\alpha_0 + \frac{1}{2}\). Fix \(x \in V_\alpha\) and \(h \in V_\alpha\), and let \(\tilde{u} = D_h u_x^{(R)}\) be the derivative of the flow along \(h\), where \(u_x^{(R)}\) is the solution to problem (5.1) starting at \(x\). We proceed as in the proof of the previous proposition, so that we only need to estimate the right-hand side of (3.3). Again, \(\tilde{u}\) solves (3.4), but we estimate \(\tilde{u}\) in \(V_\gamma\). Since \(\alpha \geq 1 + \alpha_0\), we can use Lemma 5.11 with \(a = b = \alpha\) and \(c = -\gamma\), together with interpolation of \(V_\alpha\) between \(V_\gamma\) and \(V_{\gamma + 1}\) and Young’s inequality to get

\[
\frac{d}{dt} \|\tilde{u}\|_\gamma^2 + 2\nu \|\tilde{u}\|_{\gamma + 1}^2 \leq 2|\chi_\nu (\|u_x^{(R)}\|_{\alpha}) \langle \tilde{u}, B(u_x^{(R)}, u_x^{(R)}) \rangle_{V_\gamma}| \|\tilde{u}\|_{\alpha} \\
+ 2\chi_\nu (\|u_x^{(R)}\|_{\alpha}) \langle \tilde{u}, B(\tilde{u}, u_x^{(R)}) + B(u_x^{(R)}, \tilde{u}) \rangle_{V_\gamma} | \\
\leq c R \|\tilde{u}\|_{\alpha} \|\tilde{u}\|_{\gamma + 1} \\
\leq \nu \|\tilde{u}\|_{\gamma + 1}^2 + c (\nu^{-1 + \frac{1}{\gamma}} R^2)^{\frac{1}{1 - \gamma}} \|\tilde{u}\|_{\gamma}^2,
\]

**Figure 1.** The gray areas correspond to existence (Theorem 5.1), the slightly darker gray area corresponds to Proposition 3.3, the darkest area corresponds to Proposition 3.4.
and (3.6) follows as in the previous theorem.

In the case $\alpha = \frac{5}{2}$ we can choose $\epsilon \in (0, 1 - 2\alpha_0)$ and use Lemma 5.11 with $a = b = \frac{\epsilon}{\gamma}$ and $c = -\gamma$, with the same value $\gamma = 2\alpha_0 + \frac{1}{2}$. \hfill \square

Remark 3.5. The conclusions of the previous theorem imply that $(\mathbb{P}^{(\alpha, R)}_t)_{t \geq 0}$ extends to a semigroup on $V_\alpha$ (with a more careful estimate this can be seen to be true also in the range of values for the parameters $\alpha, \alpha_0$ given in Proposition 3.3). We shall obtain a stronger result in Section 4.

3.2. Short time coupling and weak–strong uniqueness. We show in this section that it is possible to couple for a short time any solution to the Navier–Stokes equations (1.1) to the unique solution to (5.1), for suitable values of $\alpha$ and $R$. The length of the short time is a stopping time whose size depends on the initial condition and the strength of the noise (see Proposition 5.7).

Given $\alpha \in (\frac{1}{2}, 1 + 2\alpha_0)$, $x \in V_\alpha$ and an energy martingale solution (see Definition 2.2) $\mathbb{P}_x$, consider the Wiener process (2.2) associated to $\mathbb{P}_x$ and the process $z$ solution to (2.3). Equation (5.4) has a unique solution $\mathbb{P}_x - a. s.$, hence $\mathbb{u}^{(\alpha, R)}_x = z + V^{(\alpha, R)}_x$ is well defined and the unique (path-wise and in law) solution to (5.1) on the probability space $(\Omega_{NS}, \mathcal{B}, \mathbb{P}_x)$ (in particular, it does not depend in an essential way from $\mathbb{P}_x$).

To summarise, we have realised the solutions $(\xi_t)_{t \geq 0}$ and $(\mathbb{u}^{(\alpha, R)}_x)_{t \geq 0}$ to (2.1) and (5.1) respectively (with the same noise) as stochastic processes on the probability space $(\Omega_{NS}, \mathcal{B}, \mathbb{P}_x)$. Define now

\[
\tau^{(\alpha, R)}_x(\omega) = \inf\{ t \geq 0 : \|u^{(\alpha, R)}_x(t)\|_{\alpha} \geq R \},
\]

if the above set is non-empty, and $\tau^{(\alpha, R)}_x = \infty$ otherwise.

Theorem 3.6 (Weak-strong uniqueness). Under [n1] in Assumption 2.4, let $\alpha \in (\frac{1}{2}, 1 + 2\alpha_0)$ and $R \geq 1$. Given $x \in V_\alpha$, let $\mathbb{P}_x$ be any energy martingale solution starting at $x$ and let $(\mathbb{u}^{(\alpha, R)}_x)_{t \geq 0}$ be the process solution to (5.1) defined above on $(\Omega_{NS}, \mathbb{P}_x)$. Then

\[
(\mathbb{u}^{(\alpha, R)}_x(t) - \xi_t) \mathbb{I}_{\{\tau^{(\alpha, R)}_x \geq t\}} = 0, \quad \mathbb{P}_x - a. s.
\]

for every $t \geq 0$. In particular,

\[
\mathbb{E}^{\mathbb{P}_x} [\varphi(\xi_t) \mathbb{I}_{\{\tau^{(\alpha, R)}_x \geq t\}}] = \mathbb{E}^{\mathbb{P}_x} [\varphi(\xi_t) \mathbb{I}_{\{\tau^{(\alpha, R)}_x \geq t\}}],
\]

for every $t \geq 0$ and every bounded continuous function $\varphi : H \to \mathbb{R}$, where $\mathbb{P}^{(\alpha, R)}_x$ is the distribution of $\mathbb{u}^{(\alpha, R)}_x$ on $\Omega_{NS}$.

Proof. If $\mathbb{P}[\tau^{(\alpha, R)}_x \geq t] = 0$, there is nothing to prove, so we assume that such probability is positive. For simplicity we shall write $u_R = u^{(\alpha, R)}_x$, $v_R$ the solution to (5.4) corresponding to $u_R$ and $\tau = \tau^{(\alpha, R)}_x$. 


We know that \( u_R(t) - \xi_s = v_R(t) - v(s) \), where \( v \) is the solution to (2.4), hence it is sufficient to show that \( v_R(t) = v(t) \) on \( \{ \tau_R \geq t \} \). By continuity (in \( H \) for the weak topology for instance), it is sufficient to show that \( v_R(s) = v(s) \) holds for \( s < \tau_R \). If \( s < \tau_R \), \( \|u_R\|_\alpha \leq R \) and \( \chi_R(\|u_R\|_\alpha) = 1 \), so we only need to prove that \( v_R \) is the unique weak solution to (2.4) for \( s < \tau_R \).

Set \( \delta = v_R - v \), then \( \delta \) satisfies

\[
\partial_t \delta + \nu \Delta \delta + B(\delta, u_R) + B(\xi, \delta) = 0,
\]

for \( s < \tau_R \). Moreover \( \delta \) satisfies the following energy inequality (with the same set of exceptional times corresponding to \( v \)),

\[
\frac{1}{2} \|\delta(s)\|^2_{H} + \nu \int_{0}^{s} \|\delta(r)\|^2_{V} \, dr + \int_{0}^{s} \langle \delta, B(\delta, u_R) \rangle_{H} \, dr \leq 0.
\]

Indeed by definition \( v \) satisfies an energy inequality (Definition 2.2), while by Theorem 5.1 \( v_R \) satisfies an energy equality, so we are left with the proof of an energy balance for \( \langle v_R, v \rangle_{H} \). We postpone this step to the end of the proof and we first show that \( \delta(s) = 0 \) for all \( s < \tau_R \). To this end, we estimate the nonlinear term in the energy balance for \( \delta \). If \( \alpha < \frac{3}{2} \), Lemma 5.11, (with \( a = \alpha, b = \frac{3}{2} - \alpha \) and \( c = 0 \)) and interpolation yield

\[
|\langle \delta, b(\delta, u_r) \rangle| \leq c \|\delta\|_{V}^{\alpha} \|\delta\|_{V}^{\frac{3}{2} - \alpha} \|u_R\|_{\alpha} \leq cR \|\delta\|_{V}^{\frac{3}{2} - \alpha} \|\delta\|_{H}^{\alpha - \frac{1}{2}} \leq \nu \|\delta\|_{V}^{2} + c(v, R) \|\delta\|_{H}^{2},
\]

and so \( \delta(s) = 0 \) for \( s < \tau_R \) by Gronwall’s lemma. If \( \alpha \geq \frac{3}{2} \) one can proceed similarly using an arbitrary value of \( a < \frac{3}{2} \).

To conclude the proof, we need to show that

\[
\langle v_R(t), v(t) \rangle_{H} + 2\nu \int_{s}^{t} \langle v_R, v \rangle_{V} \, dr = \\
= \langle v_R(s), v(s) \rangle_{H} - \int_{s}^{t} \langle v_R, B(u, u) \rangle \, dr - \int_{s}^{t} \chi_{R}(\|u_R\|_{\alpha}) \langle B(u_R, u_R) \rangle \, dr.
\]

We proceed as in Romito [24, Theorem 2.2]. As in the proof of the energy equality for \( v_R \) (see Lemmas 5.3 and 5.4), everything boils down in proving that \( \langle v_R(t), v(t) \rangle_{H} \) is differentiable in time with derivative \( \langle \dot{v}_R, v \rangle_{V} + \langle v_R, \dot{v} \rangle_{V} \). First we notice that both the equations for \( v \) and \( v_R \) are satisfied in \( V' \). Moreover we see by the proof of Lemmas 5.3 and 5.4 that \( v_R \in L_{loc}^{2}(0, \infty; V') \), hence \( \langle \dot{v}_R, v \rangle_{V', V} \) is well defined. On the other hand, since by Corollary 5.12 (with \( a = 1, b = 0 \)) \( B(v + z, v + z) \in L_{loc}^{2}(0, \infty; V_{\beta}) \) for all \( \beta > \frac{3}{2} \) and either \( v_R \in L_{loc}^{2}(0, \infty; V_{\alpha + 1}) \) (in the range of values of Lemma 5.3) or, by (5.8), \( v_R \in L_{loc}^{2}(0, \infty; V_{\beta}) \) for all \( \beta < \alpha + 1 \) (in the range of values of Lemma 5.4), it turns out that \( \langle \dot{v}_R, v \rangle_{V', V_{\beta}} \) is also well defined and in conclusion \( \langle v_R(t), v(t) \rangle_{H} \) is differentiable. The balance above then follows by the properties of the nonlinearity. \( \square \)
4. Critical regularity for the strong Feller property

In the previous section we have proved that the transition semigroup associated to any Markov solution has a regularising effect in strong topologies. Namely, the semigroup computed on bounded measurable functions gives back almost Lipschitz functions (see formula (3.1)). In this section we show that the space where the regularity of the semigroup holds can be relaxed, at the price of having continuity only. We remark that it may be possible to achieve strong Feller regularity including the value \( \alpha = \frac{1}{2} \), but this would require some more refined analytical method, which would make the paper much lengthier.

**Theorem 4.1.** Under Assumption 2.4, let \( (P_t)_{t \geq 0} \) be the transition semigroup associated to a Markov solution \( (P_x)_{x \in H} \). Then \( (P_t)_{t \geq 0} \) is \( V_\alpha \)-strong Feller for every \( \alpha > \frac{1}{2} \).

The theorem follows from Theorem 3.2 and Proposition 4.3 below, which contains the core idea. We first prove the following convergence lemma on the approximated problem examined in Appendix 5.1.

**Lemma 4.2.** Assume [n1] (from Assumption 2.4) and let \( \alpha \in \left( \frac{1}{2}, 1 + 2\alpha_0 \right) \) and \( \beta \in (\alpha, 1 + 2\alpha_0) \) such that \( \beta < \alpha + \left( \frac{1}{2} \wedge (\alpha - \frac{1}{2}) \right) \). If \( x_n \to x \) in \( V_\alpha \) and \( R \geq 1 \), then \( u_n^{(\alpha, R)}(t) \to u_x^{(\alpha, R)}(t) \) almost surely in \( V_\beta \) for all \( t > 0 \), where \( u_y^{(\alpha, R)} \) is the solution to (5.1) with initial condition \( y \).

**Proof.** Denote for simplicity \( u_n = u_n^{(\alpha, R)} \) and \( u = u_x^{(\alpha, R)} \). Let \( z \) be the solution to the Stokes problem (2.3) and set \( v_n = u_n - z, v = u - z \) and \( w_n = u_n - u \), which solves the following equation,

\[
\dot{w}_n + \nu A w_n + \chi_R(\|u_n\|_\alpha)B(u_n, w_n) + \chi_R(\|u\|_\alpha)B(w_n, u) + (\chi_R(\|u_n\|_\alpha) - \chi_R(\|u\|_\alpha))B(u_n, u) = 0.
\]

Assume first that \( \beta < \frac{3}{2} \), then

\[
\|w_n(t)\|_\beta \leq e^{-\nu \alpha t} w_n(0) + \int_0^t \left( \chi_R(\|u_n\|_\alpha) e^{-\nu \alpha |t-s|} B(u_n, w_n) \right) \beta ds + \left( \chi_R(\|u\|_\alpha) - \chi_R(\|u_n\|_\alpha) \right) e^{-\nu \alpha |t-s|} B(u_n, u) \beta ds + \left( \chi_R(\|u\|_\alpha) - \chi_R(\|u_n\|_\alpha) \right) e^{-\nu \alpha |t-s|} B(w_n, u) \beta ds.
\]

We use Corollary 5.12 (with \( a = \alpha, b = \beta \) for the first two terms in the integral and \( a = b = \alpha \) for the third term) and properties (5.10) and (5.12) to get

\[
\|w_n(t)\|_\beta \leq ct^{-\beta/2} \|x_n - x\|_\alpha + c_R \left( 1 + t^{\beta/2} \right) \int_0^t (t - s)^{-2\beta + 4\alpha} \|w_n(s)\|_\beta ds.
\]

Notice that the assumptions on \( \beta \) ensure that \( \frac{1}{2}(2\beta + 5 - 4\alpha) < 1 \). Fix \( T > 0 \) and let \( a_\beta \) be the weight function in Lemma 5.8 (with \( x = \frac{1}{2}(\beta - \alpha) \) and \( y = \)}
\[ \frac{1}{4}(2\beta + 5 - 4\alpha) \] so that
\[ c_R \left( 1 + T^{\frac{1}{2}}(\beta - \alpha) \right) a_\beta(t) \int_0^t (t - s)^{-\frac{1}{4}(2\beta + 5 - 4\alpha)} a_\beta(s)^{-1} \, ds \leq \frac{1}{2}. \]

With this choice, \( \sup_{s \in T} a_\beta(s) \| w_n(s) \|_\beta \leq c_{R, T} \| x_n - x \|_{\alpha} \) and so \( \| w_n(t) \|_\beta \to 0 \) for \( t > 0 \).

Consider now the case \( \beta > \frac{3}{2} \) (in particular this implies that \( \alpha \) is in the range of Lemma 5.3). The energy estimate, Lemma 5.11 (with \( \alpha = b = \beta, c = -\beta \)), formula (5.12) and Young’s inequality yield
\[ \frac{d}{dt} \| w_n \|_\beta^2 \leq c_R \left( 1 + \| z \|_\beta \right)^4 \left( 1 + \| v_n \|_{\alpha+1}^2 + \| v \|_{\alpha+1}^2 \right) \left( 2^2(\beta - \alpha) \right) \| w_n \|_\beta^2, \]

since \( \| u \|_\beta \leq \| z \|_\beta + (\| u \|_{\alpha} + \| z \|_\alpha)^{1+\alpha - \beta} \| v \|_{\alpha+1}^{\beta - \alpha} \) by interpolation of \( V_\beta \) between \( V_\alpha \) and \( V_{\alpha+1} \) (similarly for \( u_n \)). By assumption \( 2(\beta - \alpha) < 1 \), hence Gronwall’s lemma implies that for all \( s \leq t \),
\[ \| w_n(t) \|_\beta^2 \leq \| w_n(s) \|_\beta^2 \exp \left( c_R \int_s^t \left( 1 + \| z \|_\beta \right)^4 \left( 1 + \| v_n \|_{\alpha+1}^2 + \| v \|_{\alpha+1}^2 \right) \left( 2^2(\beta - \alpha) \right) \, dr \right). \]

By integrating for \( s \in [0, \frac{t}{2}] \), we get
\[ \| w_n(t) \|_\beta^2 \leq \frac{2}{t} \left( \int_0^t \| w_n(s) \|_\beta^2 \right) \exp \left( c_R \int_0^t \left( 1 + \| z \|_\beta \right)^4 \left( 1 + \| v_n \|_{\alpha+1}^2 + \| v \|_{\alpha+1}^2 \right) \left( 2^2(\beta - \alpha) \right) \, dr \right). \]

The exponential term is uniformly bounded in \( n \) (using inequality (5.5)), so we only need to show that the first integral on the right hand side converges to zero. If \( \beta \leq \alpha + \frac{1}{4} \) the result follows by applying inequality (5.13) to \( w_n = v_n - v \). On the other hand, if \( \beta > \alpha + \frac{1}{4} \), interpolation (between \( V_{\alpha+\frac{1}{4}} \) and \( V_{\alpha+1} \)) ensures convergence since, as above, \( \int \| w_n \|_{\alpha+1}^2 \to 0 \) and \( w_n \) is bounded uniformly in \( n \) in \( L^2(0, t; V_{\alpha+1}) \) (this can be proved using (5.5) on both \( v_n \) and \( v \)).

Finally, if \( \beta = \frac{3}{2} \), one can consider a slightly larger value \( \beta' > \beta \) which satisfies the same assumptions of \( \beta \) and apply the computations above.

\[ \square \]

**Proposition 4.3.** Assume \([\textbf{n1}]\) of Assumption 2.4 and let \( (P_t)_{t \geq 0} \) be the transition semigroup associated to a Markov solution \( (P_x)_{x \in H} \) to (1.1). If \( \alpha \in (\frac{1}{2}, 1 + 2\alpha_0) \) and there is a number \( \beta \in (\alpha, 1 + 2\alpha_0) \) such that \( (P_t)_{t \geq 0} \) is \( V_\beta \)-strong Feller, then \( (P_t)_{t \geq 0} \) is \( V_\alpha \)-strong Feller.

**Proof.** It is sufficient to show the theorem under the condition \( \beta < \alpha + (\frac{1}{2} \wedge (\alpha - \frac{1}{2})) \). The general case follows by iterating the argument.

Let \( x_n \to x \) in \( V_\alpha \). Choose \( R \geq 1 + 4 \sup_t \| x_n \|_{\alpha} \) and \( \epsilon_0 \leq c' R^{-\gamma} \), where \( c', \gamma \), \( \eta \) are the values given in Proposition 5.7. With such values, we know that, by Proposition 5.7,
\[ \left\{ \sup_{t \in [0, \epsilon_0]} \| z(t) \|_{\eta} \leq \frac{R}{3} \right\} \subset A_\epsilon = \left\{ \tau_{x_n}^{(\alpha, R)} \geq \epsilon \right\} \cap \bigcap_{n \in \mathbb{N}} \left\{ \tau_{x_n}^{(\alpha, R)} \geq \epsilon \right\} \]
for every $e \leq e_0$, where $\tau^{(\alpha, R)}$ is defined in (3.7). Notice that for any $\varphi \in B_b(H)$ and $e \leq e_0$ (so that it does not belong to any of the exceptional sets of $P_{x_n}, P_x$), by the Markov property and Theorem 3.6,
\[
P_t \varphi(y) = E^P_y[P_{t-e} \varphi(\xi_{t-e}) 1_{A_\varepsilon}] + E^P_y[P_{t-e} \varphi(\xi_{t-e}) 1_{A_\varepsilon^c}]
\]
\[
= P^{(\alpha, R)}_t \varphi(y) + E^P_y[P_{t-e} \varphi(\xi_{t-e}) 1_{A_\varepsilon}] - E^P_y[\varphi(\xi_{t-e}) 1_{A_\varepsilon^c}],
\]
with $y = x_n$ or $y = x$, where $(P^{(\alpha, R)}_t)_{t \geq 0}$ is the transition semigroup associated to problem (5.1). Since by Lemma 5.6 the term
\[
|o_{e,R}(y)| = |E^P_y[P_{t-e} \varphi(\xi_{t-e}) 1_{A_\varepsilon^c}] - E^P_y[\varphi(\xi_{t-e}) 1_{A_\varepsilon^c}]| \leq 2\|\varphi\|_{\infty} P^{(\alpha, R)}_t[A_\varepsilon^c] \leq 2\|\varphi\|_{\infty} P^{(\alpha, R)}_t[\sup_{t \leq e_0} \|z(t)\| \geq R] \leq c\|\varphi\|_{\infty} e^{-a_0 \varepsilon^2}
\]
converges to 0 as $e_0 \to 0$ uniformly in $n$, we have that
\[
P_t \varphi(x_n) - P_t \varphi(x) = P^{(\alpha, R)}_t P_{t-e} \varphi(x_n) - P^{(\alpha, R)}_t P_{t-e} \varphi(x) + o_{e,R}(x_n) - o_{e,R}(x).
\]
By assumptions, $P_{t-e} \varphi \in C_b(V_\beta)$, and by Lemma 4.2 $u^{(\alpha, R)}(\varepsilon) \to u^{(\alpha, R)}(\varepsilon)$ almost surely, where $u^{(\alpha, R)}(\varepsilon)$ is the solution to (5.1) with initial condition $y$. By Lebesgue theorem $P^{(\alpha, R)}_t P_{t-e} \varphi(x_n) \to P^{(\alpha, R)}_t P_{t-e} \varphi(x)$ as $n \to \infty$, and, in the limit as $e_0 \to 0$, we have that $P_t \varphi(x_n) \to P_t \varphi(x)$.

\[\square\]

4.1. A few consequences. As a preliminary result we show that under [n2] (see Assumption 2.4) each Markov solutions has Markov kernels supported on the whole state space. We follow the lines of [10]. For stronger results on the same lines we refer to [19, 20, 2, 26, 1, 25].

Lemma 4.4. Under [n2] consider a Markov solution $(P_x)_{x \in H}$. Then for every $\frac{1}{2} < \alpha < 1 + 2\alpha_0$, every $x \in V_\alpha$, every $t > 0$ and every open set $U \subset V_\alpha$, $P(t,x,U) > 0$, where $P(\cdot, \cdot, \cdot)$ is the Markov kernel associated to the given Markov solution.

Proof. Without loss of generality, we can assume $\alpha > 2\alpha_0$. We proceed as in [15, Proposition 6.1]: we need to show that $P_x[\|\xi_t - y\|_\alpha < e] > 0$ for all $t > 0$, $x, y \in V_\alpha$. This probability is bounded from below by $P^{(\alpha, R)}_x[\|\xi_t - y\|_\alpha < e, \tau^{(\alpha, R)} > t]$, hence it is sufficient to show that this last quantity is positive. This follows by solving a control problem as in Lemmas C.2, C.3 of [15].

\[\square\]

Corollary 4.5. Under Assumption 2.4, every Markov solution $(P_x)_{x \in H}$ to (1.1) admits a unique invariant measure, which is strongly mixing. Moreover, the convergence to the invariant measure is exponentially fast.

Finally, if $(P^1_x)_{x \in H}$ and $(P^2_x)_{x \in H}$ are different Markov solutions, then the corresponding Markov kernels $P^1_t(t, x, \cdot)$ and $P^2_t(t, x, \cdot)$ are equivalent measures for all $x \in V_\alpha$ and $\alpha > \frac{1}{2}$. Equivalence holds also for the corresponding invariant measures.
Proof. Given the above lemma, unique ergodicity is a consequence of strong Feller regularity and Doob’s theorem (see [8]). This extends [22, Corollary 3.2]. Exponential convergence is an extension of [22, Theorem 3.3] and follows with similar methods. Finally, equivalence of laws follows as in [14, Theorem 4.1]. □

We finally give a generalisation of Theorem 6.7 of [15].

**Proposition 4.6.** Under Assumption 2.4, let \((\mathbb{P}_x)_{x \in \mathbb{H}}\) be a Markov solution to 1.1. Then for any \(\alpha > \frac{1}{2}\), \((\mathbb{P}_x)_{x \in V_{\alpha}}\) is a Markov family.

**Proof.** We prove preliminarily the following claim: for every \(\alpha > \frac{1}{2}\), \(t_0 > 0\) and \(x \in V_{\alpha}\), \(\xi\) is continuous with values in \(V_{\alpha}\) in a neighbourhood of \(t_0\), \(\mathbb{P}_x\)-a. s.. Indeed, once this claim is proved, the proposition follows as in [15, Theorem 6.7], since the only necessary ingredient is that the transition semigroup is strong Feller.

Let \(\mu\) be the unique invariant measure of \((\mathbb{P}_x)_{x \in \mathbb{H}}\) and let \(\mathbb{P}^\star\) be the corresponding stationary solution (that is, the solution starting at \(\mu\)). We notice that, by [22, Corollary 3.2] (which depends only on Theorem A.2 in the same paper and whose assumption is [n1]), for every \(\beta < 1 + 2\alpha_0\) there is \(\eta = \eta(\beta) > 0\) such that \(\mathbb{E}^\mu||x||_\beta^\eta < \infty\).

Fix \(\alpha > \frac{1}{2}\), \(t_0 > 0\) and \(x \in V_{\alpha}\). For every \(0 < a < b\), set \(A(a, b) = C((a, b); V_{\alpha})\), we wish to show that \(\mathbb{P}_x[\xi \in \bigcup_{\epsilon} A(t_0 - \epsilon, t_0 + \epsilon)] = 1\). By the Markov property,

\[
\mathbb{P}^\star[A(t_0 - \epsilon, t_0 + \epsilon)] \geq \mathbb{E}^\mu[||\xi_{t_0 - 2\epsilon}||_\alpha \leq \frac{\epsilon}{2}] \inf_{||y||_a \leq \frac{\epsilon}{2}} \mathbb{P}_y[\xi \in A(\epsilon, 3\epsilon)] \\
\geq \left(1 - \frac{c}{R^\gamma} \mathbb{E}^\mu[||x||_\alpha]^\gamma \right) \inf_{||y||_a \leq \frac{\epsilon}{2}} \mathbb{P}_y[\xi \in A(\epsilon, 3\epsilon)]
\]

Using Theorem 3.6 and taking \(\epsilon \leq cR^{-\gamma}\) (where \(c, \gamma\) are from Proposition 5.7), we have

\[
\mathbb{P}_y[\xi \in A(\epsilon, 3\epsilon)] = \mathbb{P}_y^{(\alpha, R)}[\xi \in A(\epsilon, 3\epsilon)] + \\
+ \mathbb{P}_y[\xi \in A(\epsilon, 3\epsilon), \tau^{(\alpha, R)} < 3\epsilon] - \mathbb{P}_y^{(\alpha, R)}[\xi \in A(\epsilon, 3\epsilon), \tau^{(\alpha, R)} < 3\epsilon].
\]

Clearly, \(\mathbb{P}_y^{(\alpha, R)}[\xi \in A(\epsilon, 3\epsilon)] = 1\), while the last term on the right hand side converges to 0 for \(\epsilon \downarrow 0\) and \(R \uparrow \infty\). In conclusion \(\inf_{||y||_a \leq \frac{\epsilon}{2}} \mathbb{P}_y[\xi \in A(\epsilon, 3\epsilon)] \to 0\) and \(\mathbb{P}^\star[\xi \in \bigcup_{\epsilon} A(t_0 - \epsilon, t_0 + \epsilon)] = 1\). In particular \(\mathbb{P}_x[\xi \in \bigcup_{\epsilon} A(t_0 - \epsilon, t_0 + \epsilon)] = 1\) for \(\mu\)-a. e. \(x\), hence for all \(x\) by the strong Feller property and Lemma 4.4. □

5. **Technical tools**

5.1. **Short time coupling with a smooth problem.** We follow the approach of [12] (see also [15, 22]) to construct a regular process which coincides with any solution to (1.1) for a short time, using a cut-off of the nonlinearity. In this way
with large probability the two solutions have the same trajectories on a small time interval.

5.1.1. Existence for the regular problem. Let \( \chi : [0, \infty] \to [0, 1] \) be a non-increasing \( C^\infty \) function such that \( \chi \equiv 1 \) on \([0, 1]\) and \( \chi_R \equiv 0 \) on \([2, \infty)\) (see Figure 2). Given \( R \geq 1 \), set \( \chi_R(x) = \chi(\frac{x}{R}) \). Consider the following problem,

\[
\begin{aligned}
\chi_R U + \nu A u \, dt + \chi_R(\|u\|_\alpha) B(u, u) \, dt &= Q^\frac{1}{2} \, dW, \\
u(0) &= x.
\end{aligned}
\]

In the following we analyse for which values of \((\alpha, \alpha_0)\) the above problem is uniquely solvable.

**Theorem 5.1.** Assume [n1](Assumption 2.4). Given \( R \geq 1 \) and \( \frac{1}{2} < \alpha < 1 + 2\alpha_0 \), for every \( x \in V_\alpha \) problem (5.1) has a path-wise unique martingale solution \( P_{\chi}^{(\alpha, R)} \) on \( \Omega_{NS} \), with

\[
P_{\chi}^{(\alpha, R)}[C([0, \infty); V_\alpha)] = 1.
\]

Moreover, \((P_{\chi}^{(\alpha, R)})_{x \in V_\alpha}\) is a Markov family and its transition semigroup is Feller on \( V_\alpha \). Finally, for every \( 0 \leq s < t \),

\[
\frac{1}{2} \|v_t\|^2_H + \nu \int_s^t \|v_r\|^2_V \, dr - \int_s^t \chi_R(\|v_r + z_r\|_\alpha) \langle z_r, B(v_r + z_r, v_r) \rangle \, dr = \frac{1}{2} \|v_s\|^2_H,
\]

\( P_{\chi}^{(\alpha, R)} \)-a.s., where \( z \) is the solution to (2.3) and \( v \) solves (5.4) below.

**Remark 5.2.** The two bounds on \( \alpha \) required in the assumptions of the above theorem have a different justification. The requirement \( \alpha < 1 + 2\alpha_0 \) is due to the fact that the linearisation at 0 (that is, problem (2.3)) has that maximal regularity (see for instance [8]). On the other hand, \( \alpha > \frac{1}{2} \) because \( H^{1/2} \) is the largest space in the Sobolev–Hilbert hierarchy of spaces (see [16]).

We give a short sketch of the proof of the above theorem, which can be made rigorous by using suitable approximations (such as Galerkin approximations) as in the proof of existence for the Navier-Stokes equations themselves (see for instance [11]).

Let \( z \) denote the solution to the Stokes problem (2.3) starting at 0. By the assumption on \( \Omega \), trajectories of the noise belong to \( C^\gamma([0, \infty); V_{\alpha'}) \) for all \( \gamma \in \)
\( [0, \frac{1}{2}) \) and all \( \alpha' < 2\alpha_0 \). Hence, with probability one, \( z \in C([0, \infty); V_{1+2\alpha_0-\varepsilon}) \), for all \( \varepsilon > 0 \). In particular, \( z \in C([0, \infty); V_\alpha) \) with probability one.

Fix \( \alpha, R \geq 1 \) and \( x \in V_\alpha \) and write \( u = v + z \), where \( v \) is the solution to
\[
\partial_t v + v A v + \chi_R(||v + z||_\alpha)B(v + z, v + z) = 0.
\]
with initial condition \( v(0) = x \).

**Lemma 5.3.** Assume \([n1]\) from Assumption 2.4 and \( \frac{1}{2} < \alpha < \min(\frac{1}{2} + 4\alpha_0, 1 + 2\alpha_0) \). Then for every \( x \in V_\alpha \) there is a solution \( v \in C([0, \infty); V_\alpha) \cap L_2^\infty([0, \infty); V_{\alpha+1}) \) to problem (5.4). Moreover, \( v \) satisfies the balance (5.3).

**Proof.** For brevity, we only give details of the crucial estimates needed to prove that (5.4) can be solved pathwise and has a global weak solution in \( C([0, \infty); V_\alpha) \) and \( L_2^\infty([0, \infty); V_{\alpha+1}) \). The energy estimate in \( V_\alpha \) yields
\[
\frac{d}{dt}||v||_{\alpha}^2 + 2\nu||v||_{\alpha+1}^2 \leq 2\chi_R(\|u\|_\alpha)\langle v, B(u, u) \rangle_{V_\alpha}.
\]
If \( \alpha > \frac{2}{3} \), using Lemma 5.11 (with \( a = b = \alpha \) and \( c = -\alpha \)) and Young’s inequality (with exponent 2),
\[
2\chi_R(\|u\|_\alpha)\langle v, B(u, u) \rangle_{V_\alpha} \leq c\chi_R(\|u\|_\alpha)\|v\|_{1+\alpha}^2\|u\|_{\alpha} \leq \nu\|v\|_{\alpha+1}^2 + cR^4.
\]
which implies an a-priori estimate in \( L_2^\infty([0, \infty); V_\alpha) \) and \( L_2^\infty([0, \infty); V_{\alpha+1}) \).

If \( \alpha = \frac{2}{3} \), choose \( \varepsilon < 1 \) such that \( \alpha + \varepsilon < 1 + 2\alpha_0 \). Lemma 5.11 (\( a = \alpha, b = \alpha + \varepsilon, \) \( c = -\alpha \)), interpolation of \( V_{\alpha+\varepsilon} \) between \( V_\alpha \) and \( V_{\alpha+1} \), and Young’s inequality (with exponents 2 and \( \frac{2}{\alpha+\varepsilon} \)) yield
\[
2\chi_R(\|u\|_\alpha^2)\langle v, B(u, u) \rangle_{V_\alpha} \leq \nu\|v\|_{\alpha+1}^2 + cR^2\|z\|_{\alpha+\varepsilon}^2 + cR^{\frac{2}{\alpha+\varepsilon}}(R + \|z\|_\alpha)^2,
\]
and again an a-priori estimate for \( v \) in \( L_2^\infty([0, \infty); V_\alpha) \) and \( L_2^\infty([0, \infty); V_{\alpha+1}) \).

Finally, if \( \alpha < \frac{2}{3} \), we use Lemma 5.11 (\( a = b = \frac{2}{3}(2\alpha + 3), c = -\alpha \)), interpolation of \( V_{\frac{2}{3}(2\alpha+3)} \) and Young’s inequality,
\[
2\chi_R(\|u\|_\alpha)\langle v, B(u, u) \rangle_{V_\alpha} \leq c\chi_R(\|u\|_\alpha)\|v\|_{\alpha+1}^2\|u\|_{\frac{2}{3}(2\alpha+3)}^2
\]
\[
\leq \nu\|v\|_{\alpha+1}^2 + c\|z\|_{\frac{2}{3}(2\alpha+3)}^2 + c(R + \|z\|_\alpha)^{2/(2\alpha+1)}.
\]
Here we need \( \frac{2}{3}(2\alpha+3) < 1 + 2\alpha_0 \) (hence \( \alpha < \frac{1}{2} + 4\alpha_0 \), to have \( \|z\|_{\frac{2}{3}(2\alpha+3)} \) finite.

We also need an a-priori estimate for \( \partial_t v \) in \( L_2^0([0, T; V_{\alpha-1}) \), for all \( T > 0 \). This will imply continuity in time of \( v \) on \( V_\alpha \) (see for instance [27]). Together with continuity of \( z \), it implies (5.2). To do this, multiply the equations by \( A^{\alpha-1}v \) to get
\[
2\|v\|_{\alpha-1}^2 + \nu\frac{d}{dt}\|v\|_{\alpha}^2 = -2\chi_R(\|u\|_\alpha)\langle A^{\alpha-1}v, B(u, u) \rangle.
\]
The right hand side can be estimated in the three cases through Lemma 5.11 as in (5.5), (5.6) and (5.7) respectively (using the same values of $a$, $b$, $c$).

Finally, since $\partial_t v \in L^2(0, T; V_{\alpha-1})$, it follows that equation (5.4) is satisfied in $V'$ and $t \mapsto \|v(t)\|_{H^1}$ is differentiable with derivative $2(\partial_t v, v)_{V', V}$. Equality (5.3) follows easily from these two facts and the properties of the nonlinearity. □

**Lemma 5.4.** Assume [n1] from Assumption 2.4 and let $\alpha \in (\frac{1}{2} + 4\alpha_0, 1 + 2\alpha_0)$. Then for every $x \in V_{\alpha}$ there is a solution $v \in C([0, \infty); V_{\alpha})$ to problem (5.4). Moreover, $v$ satisfies the balance (5.3) and for every $\beta \in (\alpha, 1 + 2\alpha_0)\) and every $T > 0$ there is $c = c(\alpha, \beta, R, T) > 0$ such that

\[
\sup_{t \leq T}(t \wedge 1)^{\frac{1}{2}(\beta - \alpha)}\|v(t)\|_{\beta} \leq c(\|x\|_{\alpha} + \sup_{t \leq T}\|z(t)\|_{\beta}).
\]

**Proof.** The standard bounds in $L^\infty(0, T; H)$ and $L^2(0, T; V)$ ensure compactness of approximations (as in standard proofs for Navier–Stokes [27]). Convergence in $V_{\alpha}$ is needed in order to show that any limit point is a solution. This follows from Ascoli–Arzelà theorem. Indeed, Corollary 5.12 (with $a = b = \alpha$) implies that (we omit the subscript $n$ for simplicity),

\[
\|v(t)\|_{\alpha} \leq \|e^{-\nu At} x\|_{\alpha} + \int_0^t \chi_R(\|u\|_{\alpha}) e^{-\nu A(t-s)} B(u, u)_{\alpha} ds
\]

\[
\leq \|x\|_{\alpha} + c \int_0^t (t-s)^{-\frac{1}{2}(5-2\alpha)} \chi_R(\|u\|_{\alpha}) \|u\|_{\alpha}^2 ds
\]

\[
\leq \|x\|_{\alpha} + c R^2 t^{\frac{1}{2}(2\alpha - 1)},
\]

where we have used that

\[
\|A^\gamma e^{-\nu At} \|_{C(H)} \leq c t^{-\gamma}.
\]

Similarly, if $\beta > \alpha$, Corollary 5.12 ($a = \alpha, b = \beta$) yields

\[
\|v(t)\|_{\beta} \leq \|e^{-\nu At} x\|_{\beta} + \int_0^t \chi_R(\|u\|_{\alpha}) e^{-\nu A(t-s)} B(u, u)_{\beta} ds
\]

\[
\leq ct^{-\frac{1}{2}(\beta - \alpha)}\|x\|_{\alpha} + c R \int_0^t (t-s)^{-\frac{1}{2}(5-2\alpha)} (\|v(s)\|_{\beta} + \|z(s)\|_{\beta}) ds
\]

\[
\leq ct^{-\frac{1}{2}(\beta - \alpha)}\|x\|_{\alpha} + c R t^{\frac{2\alpha - 1}{2}} \sup_{s \leq T}\|z(t)\|_{\beta} + c R \int_0^t (t-s)^{-\frac{3-2\alpha}{2}} \|v(s)\|_{\beta} ds.
\]

Choose $a_\beta(t)$ as in Lemma 5.8 so that

\[
cRa_\beta(t) \int_0^t (t-s)^{-\frac{1}{2}(5-2\alpha)} a_\beta(s)^{-1} ds \leq \frac{1}{2},
\]

hence

\[
\sup_{t \leq T} a_\beta(t)\|v(t)\|_{\beta} \leq c\|x\|_{\alpha} + c \sup_{t \leq T}\|z(t)\|_{\beta}.
\]
Equicontinuity in time can be obtained by an estimate similar to (5.9), hence there is a subsequence of \((v_n)_{n \in \mathbb{N}}\) converging uniformly in \(V_\alpha\) on any interval \([\varepsilon, T]\). In particular, this implies that the limit point is a solution to (5.4) and it is continuous in \(V_\alpha\) on \((0, T]\). Continuity in 0 can be obtained with an estimate similar to (5.9). Finally, the bounds (5.8) can be obtained as in (5.11) and in turns they imply uniqueness, via Lemma 5.5 below.

Finally, we prove the energy balance (5.3). The estimate (5.8) implies that \(Av \in L^2_{\text{loc}}(0, \infty; V')\), while by Lemma 5.11 (with \(a = \alpha\), \(b = 1\) and \(c = 0\)) we know that \(\|\chi_R(\|u\|_\alpha)B(u, u)\|_{V'} \leq cR\|u\|_V\), hence \(\chi_R(\|u\|_\alpha)B(u, u)\) is in \(L^2_{\text{loc}}(0, \infty; V')\) and in conclusion \(\partial_t v \in L^2_{\text{loc}}(0, \infty; V')\) and equality (5.4) holds in \(V'\). Equality (5.4) again follows easily from these two facts and the properties of the nonlinearity. 

\[\alpha = 1 + 2\alpha_0\]

\[\alpha = \frac{1}{2} + 4\alpha_0\]

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yield
\[
\frac{d}{dt} \|w\|_{B}^{2} + 2 \|v\|_{B+1}^{2} \leq c|\chi_{R}(\|u_{2}\|_{\alpha}) - \chi_{R}(\|u_{1}\|_{\alpha})|\|u_{1}\|_{\alpha}\|u_{2}\|_{\alpha}\|w\|_{1+\beta}
\]
\[+ cR\|w\|_{\alpha}\|w\|_{1+\beta} \leq \nu\|w\|_{\alpha}^{2} + cR\|w\|_{2}. \tag{5.13}\]

If on the other hand \(\alpha < \frac{3}{4}\), we estimate \(w\) in \(H\). Lemma 5.11 (with \(a = \frac{3}{2} - \alpha\), \(b = \alpha\) and \(c = 0\)) and interpolation of \(V_\alpha\) and \(V_{3/2-\alpha}\) between \(H\) and \(V\) yield
\[
\frac{d}{dt} \|w\|_{H}^{2} + 2 \|v\|_{V}^{2} \leq c|\chi_{R}(\|u_{2}\|_{\alpha}) - \chi_{R}(\|u_{1}\|_{\alpha})|\|u_{1}\|_{\alpha}\|u_{2}\|_{\alpha}\|w\|_{V}
\]
\[+ cR\|w\|_{V}\|w\|_{\frac{3}{2}-\alpha} \leq \nu\|w\|_{V}^{2} + cR\|w\|_{H}^{2}(1 + \|u_{1}\|_{\frac{3}{2}-\alpha}), \]

where \(\|u_{1}\|_{\frac{3}{2}-\alpha}\) is integrable in time thanks to (5.8) and the fact that \(\alpha > \frac{1}{2}\). In both cases Gronwall’s lemma implies that \(w \equiv 0\), since \(w(0) = 0\). □

5.2. An estimate of the blow-up time. We next study the distribution of the random time \(\tau_{\alpha,R} : \Omega_{NS} \to [0, \infty)\), defined in (3.7). We start with an estimate of the tails of the solution \(z\) to (2.3), whose proof is standard (see [8] for instance, a proof in the case \(\beta = 2\) is given in [14]).

Lemma 5.6. Assume [n1] from Assumption 2.4 and let \(\beta < 1 + 2\alpha_{0}\). Then there are \(a_{0} > 0\) and \(c_{0} > 0\) (depending only on \(\alpha_{0}, \beta, \) and \(\nu\)) such that for all \(K \geq \frac{1}{2}\) and \(\epsilon > 0\),
\[
\mathbb{P}\left[\sup_{s \leq \epsilon} \|z(t)\|_{\beta} \geq K\right] \leq c_{0} e^{-a_{0}K^{2}/\epsilon}. \]

Proposition 5.7. Assume [n1] from Assumption 2.4 and let \(\alpha \in (\frac{1}{2}, 1 + 2\alpha_{0})\), with \(\alpha \neq \frac{3}{2}\). There exists \(c' = c'(\alpha) > 0\) such that if \(R \geq 1\), \(x \in V_{\alpha}\) with \(\|x\|_{\alpha} \leq \frac{R}{3}\) and if \(T \leq c'\frac{R}{(2\alpha - 1)^{\alpha/2}}\) then
\[
\left\{\sup_{[0, T]} |z(t)|_{\alpha} \leq \frac{R}{3}\right\} \subset \left\{\tau_{x}^{(\alpha, R)} \geq T\right\}, \]

where \(z\) is the solution to (2.3). In particular,
\[
\mathbb{P}_{x}^{(\alpha, R)}[\tau_{x}^{(\alpha, R)} \leq T] \leq c_{0} e^{-a_{0}R^{2}/3\epsilon}. \]

If \(\alpha = \frac{3}{2}\), then for every \(\epsilon < 1\) such that \(\alpha + \epsilon < 1 + 2\alpha_{0}\) there is \(c_{\epsilon} > 0\) such that the same holds true on the event \(\{\sup_{[0, T]} |z(t)|_{\alpha+\epsilon} \leq R/3\}\) for \(T \leq c_{\epsilon}R^{2/(1-\epsilon)}\).

Proof. Fix \(x \in V_{\alpha}\) with \(\|x\|_{\alpha} \leq \frac{R}{3}\), let \(z\) be the solution to (2.3) and set \(v^{(\alpha, R)} = u_{x}^{(\alpha, R)} - z\). Assume first \(\alpha > \frac{3}{2}\). If \(\sup_{[0, T]} |z(t)|_{\alpha} \leq \frac{R}{3}\), inequality (5.5) implies
Given Inequalities.

Lemma 5.9. The following two elementary estimates.

\[
\|u^{(\alpha, R)}_t\| \leq \|z(t)\| + \|v^{(\alpha, R)}(t)\| \leq \frac{R}{3} + R\sqrt{\frac{1}{9} + cR^2T} \leq R
\]

if \(T \leq c'R^{-2}\), for a suitable \(c'\). If on the other hand \(\alpha < \frac{3}{2}\), inequality (5.9) (which holds for the full range \(\alpha \in (\frac{1}{2}, \frac{3}{2})\)) yields \(\|v^{(\alpha, R)}\| \leq \frac{1}{3}R + cR^2T^{\frac{2\alpha - 1}{(2\alpha - 1)}}\), hence

\[
\|u^{(\alpha, R)}_t\| \leq R \text{ for } t \leq T, \text{ if } T \leq c'R^{-\frac{4}{(2\alpha - 1)}} \text{ and sup}_{[0, T]} |z(t)|\| \leq \frac{R}{3}.
\]

Finally, if \(\alpha = \frac{3}{2}\), we choose \(\epsilon > 0\) as we had done for (5.6) so that \(\|v(t)\| \leq c_\epsilon R^{2(\epsilon - 1)/(\epsilon - 1)} \sqrt{T}\) for \(t \leq T\) and hence \(\|u^{(\alpha, R)}_t\| \leq R\) for \(t \leq T\) if \(T \leq c_\epsilon R^{-\epsilon(1 - \epsilon)}\) and \(\sup_{[0, T]} |z(t)|\| \leq \frac{R}{3}\). \(\square\)

5.3. Inequalities.

Lemma 5.8. Given \(x, y \in [0, 1]\) and \(\delta > 0, \eta > 0,\) let

\[
a(t) = \begin{cases} t^x, & 0 \leq t \leq \delta, \\ \delta^x e^{-\eta(t-\delta)}, & t > \delta. \end{cases}
\]

Then \(a\) is continuous on \([0, \infty)\), \(|a(t)| \leq \delta^x\) and for all \(t \geq 0,\)

\[
a(t) \int_0^t (t-s)^{-y} a(s)^{-1} \, ds \leq B(1-x, 1-y)\delta^{1-y} + \eta^{\eta-1} \Gamma(1-y),
\]

where \(B\) and \(\Gamma\) are, respectively, the Beta and the Gamma functions.

Proof. Denote by \(A(t)\) the function in the statement of the lemma. If \(t \leq \delta,\) by a change of variables,

\[
A(t) = t^x \int_0^t (t-s)^{-y} s^{-x} \, ds = t^{1-y} B(1-x, 1-y) \leq \delta^{1-y} B(1-x, 1-y),
\]

while if \(t > \delta,\)

\[
A(t) = \delta^{x-\eta(t-\delta)} \int_0^\delta (t-s)^{-y} s^{-x} \, ds + \int_\delta^t (t-s)^{-y} e^{-\eta(t-s)} \, ds \leq \delta^{1-y} B(1-x, 1-y) + \eta^{\eta-1} \Gamma(1-y),
\]

where the first term is non-increasing in \(t \geq \delta\) and we have used a change of variables in the second term. \(\square\)

Finally, we prove a slight generalisation of [15, Lemma D.2] (a range of parameters is covered by [28, Lemma 2.1] or [5, Proposition 6.4]). First we need the following two elementary estimates.

Lemma 5.9. Let \(\alpha \in \mathbb{R},\) then there is a number \(c = c(\alpha)\) such that for all \(k_0 \geq 1,\)

\[
\sum_{k \in \mathbb{Z}^d: 0 \leq |k| \leq k_0} |k|^\alpha \leq \begin{cases} ck_0^{(\alpha+3)/\alpha}, & \alpha \neq -3, \\ c \log(1 + k_0), & \alpha = -3. \end{cases}
\]
Lemma 5.10. Let $\alpha, \beta, \gamma \in \mathbb{R}$ be such that $2(\alpha + \beta + \gamma) \geq 3$ if $\beta < \frac{3}{2}$, $\alpha + \gamma > 0$ if $\beta = \frac{3}{2}$ and $\alpha + \gamma > 0$ if $\beta > \frac{3}{2}$. Then there is a number $c = c(\alpha, \beta, \gamma)$ such that for every $1 \in \mathbb{Z}^3$, with $|l| > 1$,

$$\sum_{m : |l + m| > 2|m|} \frac{1}{|l|^{2\alpha} |m|^{2\beta} |l + m|^{2\gamma}} \leq c.$$  

Proof. First, notice that $\{m : |l + m| > 2|m|\} \subset \{m : |m| < |l|\}$ and so $|l + m| < 2|l|$. We prove that $\frac{2}{3}|l| \leq |l + m|$ holds as well. If $|m| < \frac{1}{3}|l|$, then $|l + m| \geq |l| - |m| > \frac{2}{3}|l|$. If on the other hand $|m| \geq \frac{2}{3}|l|$, then $|l + m| > 2|m| \geq \frac{2}{3}|l|$. The conclusion now follows using the previous lemma. □

Lemma 5.11. Let $a, b, c \in \mathbb{R}$ be such that $a \geq (-c) \lor 0$, $b \geq (-c) \lor 0$ and $2(a + b + c) \geq 3$ (with a strict inequality if at least one of the three numbers is equal to $3/2$). Then there is a number $c_B = c_B(a, b, c)$ such that

$$\langle B(u, v), w \rangle \leq c_B \|u\|_a \|v\|_b \|w\|_{c+1}.$$  

for all $u \in V_a$, $v \in V_b$ and $w \in V_{c+1}$.

Proof. We proceed as in the proof of [15, Lemma D.2]. In terms of Fourier series $u(x) = \sum u_k e^{ik \cdot x}$ and $v(x) = \sum v_k e^{ik \cdot x}$, hence

$$B(u, v) = i \sum_{k \neq 0} \left( \sum_{l + m = k} (k \cdot u_l) P_k v_m \right) e^{ik \cdot x},$$

where $P_k : \mathbb{R}^3 \to \mathbb{R}^3$ is the projection onto $\{y \in \mathbb{R}^3 : y \cdot k = 0\}$. Therefore,

$$\langle B(u, v), w \rangle = \mathbb{F} \left( \sum_{k \neq 0} \overline{w}_k \left( \sum_{l + m = k} (k \cdot u_l) P_k v_m \right) \right)$$

$$\leq \|w\|_{c+1} \left( \sum_{k \neq 0} |k|^{-2c} \sum_{l + m = k} |u_l| |v_m| \right)^{\frac{1}{2}}.$$

Divide the sum of the right-hand side of the above formula in the three terms $A, B$ and $C$, corresponding to the inner sum extended respectively to

$$A_k = \{l + m = k, \ |l| \geq \frac{|k|}{2}, |m| \geq \frac{|k|}{2}\},$$

$$B_k = \{l + m = k, |m| < \frac{|k|}{2}\}, \quad C_k = \{l + m = k, |l| < \frac{|k|}{2}\}.$$  

Set, for brevity, $U_k = |k|^a |u_k|$ and $V_k = |k|^a |v_k|$. We start with the estimate of $A$.

Since by Young’s and Cauchy–Schwartz’ inequalities,

$$A^2 \leq 2\|v\|_b^2 \sum_{k \neq 0} |k|^{-2c} \left( \sum_{l + m = k} |l|^{-2(a+b)} U_l^2 \right) + 2\|u\|_a^2 \sum_{k \neq 0} |k|^{-2c} \left( \sum_{l + m = k} |m|^{-2(a+b)} V_m^2 \right),$$
by exchanging the sums in $k$ and $l$ and using Lemma 5.9 (we only consider the first term, one can proceed similarly for the second),
\[
\sum_{k \neq 0} |k|^{-2c} \sum_{l \neq 0} |l|^{-2(a+b)} U_i^2 = \sum_{l \neq 0} |l|^{-2(a+b)} U_i^2 \sum_{|k| \leq |l|} |k|^{-2c} \leq c \|u\|^2_a,
\]
and so $A \leq c \|u\|_a \|v\|_b$. We estimate $B$ using Cauchy–Schwartz’ inequality, exchanging the sums and using Lemma 5.10,
\[
\begin{align*}
B^2 &\leq \|v\|^2_b \sum_{k \neq 0} |k|^{-2c} \sum_{B_k} |l|^{-2a} |m|^{-2b} U_i^2 \\
&= \|v\|^2_b \sum_{l \neq 0} |l|^{-2a} U_i^2 \sum_{m: |l+m| > 2|l|} |l+m|^{-2c} |m|^{-2b} \\
&\leq c \|u\|^2_a \|v\|^2_b.
\end{align*}
\]
Finally, the term $C$ can be obtained from $B$ by exchanging $u$ with $v$ and $l$ with $m$. \hfill \Box

**Corollary 5.12.** If $a, b \geq 0$, then there is $c_B > 0$ such that for all $u \in V_a$ and $v \in V_b$,
\[
\|A \delta^\perp B(u, v)\|_H \leq c_B \|u\|_a \|v\|_b,
\]
where $\delta = (a \wedge b - (3/2 - a \lor b)_+ - 1)$ if $a \lor b \neq 3/2$, and $\delta < (a \wedge b - 1)$ if $a \lor b = 3/2$ or $a \lor b = 0$.

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