Preservation of Physical Properties of Stochastic Maxwell Equations with Additive Noise via Stochastic Multi-symplectic Methods

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Abstract

Stochastic Maxwell equations with additive noise are a system of stochastic Hamiltonian partial differential equations intrinsically, possessing the stochastic multi-symplectic conservation law. It is shown that the averaged energy increases linearly with respect to the evolution of time and the flow of stochastic Maxwell equations with additive noise preserves the divergence in the sense of expectation. Moreover, we propose three novel stochastic multi-symplectic methods to discretize stochastic Maxwell equations in order to investigate the preservation of these properties numerically. We made theoretical discussions and comparisons on all of the three methods to observe that all of them preserve the corresponding discrete version of the averaged divergence. Meanwhile, we obtain the corresponding dissipative property of the discrete averaged energy satisfied by each method. Especially, the evolution rates of the averaged energies for all of the three methods are derived which are in accordance with the continuous case. Numerical experiments are performed to verify our theoretical results.

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Key Words: Stochastic Maxwell equations; Stochastic Hamiltonian partial differential equations; Stochastic multi-symplectic conservation law; Dissipative property of averaged energy; Conservation law of averaged divergence.

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1 Introduction

The effects of stochastic perturbation play a no longer negligible role in modeling of physical phenomena. To quantify these effects, stochastic differential equations are required. Taking the context of electromagnetism as an example, [8] studied the propagation of ultra-short solitons with stochastic variations of media, which are modeled by stochastic Maxwell equations; the controllability of the Maxwell equations under stochastic forcing was considered by [1], via an abstract and a constructive approach using a generalization of the Hilbert uniqueness method.

Recently, the stochastic multi-symplectic structure for three dimensional (3-D) stochastic Maxwell equations with additive noise was proposed in [2], based on the stochastic version of variational principle, which means that stochastic Maxwell equations are a system of stochastic Hamiltonian partial differential equations (PDEs). It has been widely recognized that the structure-preserving numerical methods have the remarkable superiority to conventional numerical methods when applied to Hamiltonian ODEs and PDEs, such as long-term behavior, structure-preserving, physical properties-preserving (energy, divergence, charge) etc.; see [5] and references therein. Efforts have been devoted to the stochastic case. For example, authors in [4] established the theory for the stochastic multi-symplectic conservation law for the stochastic Hamiltonian PDEs and investigated a stochastic multi-symplectic method for stochastic nonlinear Schrödinger equation. And a stochastic multi-symplectic wavelet collocation method was proposed in [3], to approximate stochastic Maxwell equations with a class of multiplicative noise, while [2] proposed another stochastic multi-symplectic method, based on the stochastic variational principle.

Different from the approach of reference [2], we use a direct way to represent the stochastic Maxwell equations as another system of stochastic Hamiltonian PDEs, which avoids introducing extra variables and leads to cost efficiency. As a result, the stochastic Maxwell equations preserve the stochastic multi-symplectic conservation law almost surely. Meanwhile, we show that the averaged energy increases linearly as the growth of time, with the rate being \( K = 6\lambda^2 V(\Theta) Tr(Q) \). Here \( \lambda \) represents the level of noise, \( V(\Theta) \) is the volume of domain \( \Theta \) and \( Tr(Q) \) denotes the trace of operator \( Q \). This dissipative property of averaged energy may be due to the evolution of electromagnetic fields in bianisotropic medium takes place in an environment which is disturbed by some electromagnetic noise. For the divergence, it is proved that the flow of stochastic Maxwell equations preserves the divergence in the sense of expectation. It means that electric flux and magnetic flux are preserved in Gaussian random fields in the statistical sense. In this paper, we propose three numerical methods to discretize stochastic Maxwell equations with additive noise in order to investigate the preservation of these physical properties numerically. Method-I is based on the application of implicit midpoint method in both temporal and spatial directions to the equivalent stochastic Hamiltonian PDEs, while Method-II being a three-layer method is constructed by central difference in both temporal and spatial directions, which exhibits the grid staggering property of electromagnetism. We utilize central difference in spatial direction and implicit midpoint method in temporal direction to obtain Method-III. It is shown that all of the three numerical methods preserve the corresponding discrete versions of multi-symplectic conser-
vation law. Another aim of this paper is to investigate the numerical preservation of some important physical quantities including energy and divergence by numerical methods. For the energy, we obtain the corresponding dissipative property of the discrete averaged energy satisfied by each method. Furthermore, utilizing the adaptedness of solutions to stochastic Maxwell equations and properties of Wiener process, we estimate the dissipative rates with respect to time for three methods in our consideration, and we show that the discrete averaged energies evolve at most linearly with respect to time under certain assumptions. As for divergence, we show that all of the three methods preserve the discrete conservation law of averaged divergence well, as shown theoretically in Theorem 3.3, 3.6 and 3.10. Finally, numerical experiments are performed to validate the theoretical results.

The outline of this paper is as follows. In section 2 we present some preliminaries about stochastic Maxwell equations, including theorems about the evolution of energy and divergence, and the intrinsic stochastic multi-symplectic structure. Sections 3 is devoted to the comparison and analysis of three stochastic multi-symplectic numerical methods in the aspect of averaged energy and averaged divergence. Numerical experiments for stochastic Maxwell equations with additive noise are performed in section 4 to verify our theoretical results. Finally, concluding remarks are given in Section 5.

In the sequels, we let $e^T = (1, 1, 1)^T$ and denote by $< \cdot, \cdot >_{L^2}$ the $L^2(\Theta)$ inner product, by $< \cdot, \cdot >$ the Euclidean inner product, by $| \cdot |$ the Euclidean norm, and by $\mathcal{E}$ the expectation.

## 2 Stochastic Maxwell equations with additive noise

It is of interest to study phenomena where the densities of the electric and magnetic currents are assumed to be stochastic. These can be modeled by the following 3-D stochastic Maxwell equations with additive noise

$$
\begin{aligned}
\frac{\partial E}{\partial t} &= \nabla \times H - \lambda e^T \dot{\chi} \quad \text{in } (0, T) \times \Theta, \\
\frac{\partial H}{\partial t} &= -\nabla \times E + \lambda e^T \dot{\chi} \quad \text{in } (0, T) \times \Theta,
\end{aligned}
$$

(2.1)

with initial conditions

$$
E(0, x, y, z) = (E_{10}, E_{20}, E_{30}) \quad \text{in } \Theta,
$$

(2.2)

$$
H(0, x, y, z) = (H_{10}, H_{20}, H_{30}) \quad \text{in } \Theta,
$$

and perfectly electric conducting (PEC) boundary conditions

$$
E \times n = 0 \quad \text{or} \quad H \times n = 0 \quad \text{on } (0, T] \times \partial \Theta,
$$

(2.3)

where $T > 0$, $\Theta$ is a bounded and simply connected domain in $\mathbb{R}^3$ with smooth boundary $\partial \Theta$ and $n$ represents the unit outward normal of $\partial \Theta$. It is convenient at this point to give a precise mathematical definition of $\dot{\chi}$.

Hereafter, let $W$ be a $Q$-Wiener process defined on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with values in the Hilbert space $L^2(\mathbb{R}; \mathbb{R})$, which is a space of square integrable real-valued
functions. Let \( \{e_m\}_{m \in \mathbb{N}} \) be an orthonormal basis of \( L^2(\mathbb{R}; \mathbb{R}) \) consisting of eigenvectors of a symmetric, nonnegative and finite trace operator \( Q \), i.e., \( \text{Tr}(Q) < \infty \) and \( Qe_m = \eta_m e_m \). Then there exist a sequence of independent real-valued Brownian motions \( \{\beta_m\}_{m \in \mathbb{N}} \) such that

\[
W(t, x, \omega) = \sum_{m=0}^{\infty} \sqrt{\eta_m} e_m(x) \beta_m(t, \omega), \quad t \geq 0, x \in \mathbb{R}, \omega \in \Omega.
\]  

(2.4)

Set \( \dot{\chi} = \frac{dW}{dt} \) and note that the noise only depends on time \( t \) and one dimensional space variable \( x \).

We refer interested readers to [6] for the well-posedness of problem (2.1). The authors present some results on stochastic integrodifferential equations in Hilbert spaces, motivated from and applied to problems arising from the mathematical modeling of electromagnetics fields in complex random media. They examine the mild, strong and classical well posedness for Cauchy problem of the integrodifferential equation which describes Maxwell equations complemented with the general linear constitutive relations describing such media, with either additive or multiplicative noise.

### 2.1 Dissipative property of averaged energy

In this subsection, we consider the property of averaged energy for system (2.1). The following theorem shows that the averaged energy evolves linearly with respect to time \( t \) and with a growth rate \( K = 6\lambda^2 V(\Theta) \text{Tr}(Q) \), here \( V(\Theta) \) denotes the volume of space domain \( \Theta \).

**Theorem 2.1.** Let \( \mathbf{E} \) and \( \mathbf{H} \) be the solutions of the equations (2.1)-(2.3). Then for \( t \in [0, T] \), there exists \( K = 6\lambda^2 V(\Theta) \text{Tr}(Q) \) such that the averaged energy satisfies the following dissipative property

\[
\mathcal{E}(\Phi^{\text{exact}}(t)) = \mathcal{E}(\Phi^{\text{exact}}(0)) + Kt,
\]  

(2.5)

where \( \Phi^{\text{exact}}(t) = \int_\Theta(|\mathbf{E}(t)|^2 + |\mathbf{H}(t)|^2) dxdydz \).

**Proof.** We write (2.1) into

\[
\begin{align*}
\frac{dE}{dt} &= \nabla \times H dt - \lambda e^T dW, \\
\frac{dH}{dt} &= -\nabla \times E dt + \lambda e^T dW.
\end{align*}
\]

Let

\[
\begin{align*}
F_1(\mathbf{E}(t)) &= \int_\Theta |\mathbf{E}(t)|^2 dxdydz, \\
F_2(\mathbf{H}(t)) &= \int_\Theta |\mathbf{H}(t)|^2 dxdydz.
\end{align*}
\]
We obtain the first and second Fréchet derivatives of function $F_1$ as following

$$DF_1(E)(\varphi) = 2 \int_\Theta < E, \varphi > dxdydz, \quad (2.6)$$

$$D^2 F_1(E)(\varphi, \psi) = 2 \int_\Theta < \psi, \varphi > dxdydz.$$ 

Applying the infinite dimensional Itô formula to $F_1(E)$, we have

$$F_1(E(t)) = F_1(E(0)) + \int_0^t DF_1(E(s))(-\lambda e^T dW)$$

$$+ \int_0^t \left\{ DF_1(E(s))(\nabla \times H(s)) + \frac{1}{2} Tr \left[ D^2 F_1(E(s))(-\lambda e^T Q^\frac{1}{2})(-\lambda e^T Q^\frac{1}{2})^* \right] \right\} ds. \quad (2.7)$$

Substitute (2.6) into (2.7) leads to

$$F_1(E(t)) = F_1(E(0)) + 2 \int_0^t \int_\Theta < E(s), -\lambda e^T dW > dxdydz$$

$$+ \int_0^t \int_\Theta \left\{ 2 < E(s), \nabla \times H(s) > + Tr \left[ < -\lambda e^T, -\lambda e^T > (Q^\frac{1}{2})(Q^\frac{1}{2})^* \right] \right\} dxdydzds. \quad (2.8)$$

Similarly, we apply Itô formula to function $F_2(H(t))$ and obtain

$$F_2(H(t)) = F_2(H(0)) + 2 \int_0^t \int_\Theta < H(s), \lambda e^T dW > dxdydz$$

$$+ \int_0^t \int_\Theta \left\{ 2 < H(s), -\nabla \times E(s) > + Tr \left[ < \lambda e^T, \lambda e^T > (Q^\frac{1}{2})(Q^\frac{1}{2})^* \right] \right\} dxdydzds. \quad (2.9)$$

Summing (2.8) and (2.9) leads to

$$F_1(E(t)) + F_2(H(t)) = F_1(E(0)) + F_2(H(0))$$

$$+ 2 \int_0^t \int_\Theta < H(s) - E(s), \lambda e^T dW > dxdydz$$

$$+ \underbrace{2 \int_0^t \int_\Theta \left( < H(s), -\nabla \times E(s) > + < E(s), \nabla \times H(s) > \right)}_{(a)} dxdydzds$$

$$+ 2 \int_0^t \int_\Theta Tr \left[ < \lambda e, \lambda e^T > (Q^\frac{1}{2})(Q^\frac{1}{2})^* \right] dxdydzds. \quad (2.10)$$

Using the Green formula and PEC boundary conditions, we get

$$(a) = -2 \int_0^t \int_{\partial \Theta} (E \times H) \cdot n ds ds = 0.$$
Hence, there exists a constant $K = 6\lambda^2 V(\Theta) Tr(Q)$, such that

\begin{equation}
F_1(E(t)) + F_2(H(t)) = F_1(E(0)) + F_2(H(0)) \tag{2.11}
\end{equation}

\[ + 2 \int_0^t \int_{\Theta} < H(s) - E(s), \lambda e^T dW > dxdydz + Kt. \]

The assertion follows from applying expectation on equation (2.11). \qed

\section{2.2 Conservation law of averaged divergence}

As is well known that the electric field and magnetic field are divergence-free if the media is lossless in deterministic case. The following theorem shows that for stochastic Maxwell equations with additive noise (2.1) the electric field and magnetic field are still divergence-free, but in the sense of expectation.

\textbf{Theorem 2.2.} System (2.1) preserves the divergence in the sense of expectation, i.e.,

\begin{equation}
E(div(E(t))) = E(div(E(0))), \ E(div(H(t))) = E(div(H(0))). \tag{2.12}
\end{equation}

\textit{Proof.} Let

\begin{equation}
G(E) = divE = \frac{\partial E_1}{\partial x} + \frac{\partial E_2}{\partial y} + \frac{\partial E_3}{\partial z}. \tag{2.13}
\end{equation}

The corresponding first and second Fréchet derivatives of function $G$, respectively, are given by

\begin{equation}
DG(E)(\varphi) = div\varphi, \ D^2G(E)(\varphi, \psi) = 0. \tag{2.14}
\end{equation}

Applying the infinite dimensional Itô formula to $G(E)$, we have

\begin{equation}
G(E(t)) = G(E(0)) + \int_0^t DG(E(s))(-\lambda e^T dW) \tag{2.15}
\end{equation}

\[ + \int_0^t DG(E(s))(\nabla \times H(s))ds. \]

Substituting (2.14) into (2.15) and keeping in mind a fact $div(\nabla \times Y) = 0, \forall Y : \mathbb{R}^n \rightarrow \mathbb{R}^n$, we get

\begin{equation}
G(E(t)) = G(E(0)) + \int_0^t DG(E(s))(-\lambda e^T dW). \tag{2.16}
\end{equation}

The first assertion in (2.12) follows from taking the expectation on both sides of (2.16).

Analogously, we can get the second assertion in (2.12), by applying Itô formula to function $divH$. \qed

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2.3 Stochastic multi-symplectic conservation law

In this paper, we use a direct way to rewrite equation (2.1) into the form of stochastic Hamiltonian PDEs. Obviously, the direct approach may avoid introducing extra variables, see [2] for another approach based on the stochastic version of variational principle to rewrite Hamiltonian PDEs. By denoting \( Z \), see [2] for another approach based on the stochastic version of variational principle to rewrite equation (2.1). By denoting \( Z = (H_1, H_2, H_3, E_1, E_2, E_3)^T \), we have

\[
F d_t Z + K_1 Z_x dt + K_2 Z_y dt + K_3 Z_z dt = \nabla_Z S_1(Z) dt + \nabla_Z S_2(Z) \circ dW, \tag{2.17}
\]

where skew-symmetric matrices \( F, K_1, K_2, K_3 \) are given by

\[
F = \begin{pmatrix} 0 & -I_{3 \times 3} & 0 \\ I_{3 \times 3} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
K_p = \begin{pmatrix} \mathcal{D}_p & 0 \\ 0 & \mathcal{D}_p \end{pmatrix}, \forall p = 1, 2, 3, \tag{2.18}
\]

with \( I_{3 \times 3} \) being the identity matrix and

\[
\mathcal{D}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mathcal{D}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \mathcal{D}_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{2.19}
\]

We have the following theorem.

**Theorem 2.3.** The stochastic Hamiltonian PDEs (2.17) possess the stochastic multi-symplectic conservative law locally

\[
d_t \omega + \partial_x \kappa_1 dt + \partial_y \kappa_2 dt + \partial_z \kappa_3 dt = 0, \quad a.s.
\]

i.e.,

\[
\int_{y_0}^{y_1} \int_{x_0}^{x_1} \int_{z_0}^{z_1} \omega(t_1, x, y, z) dxdydz + \int_{y_0}^{y_1} \int_{t_0}^{t_1} \int_{z_0}^{z_1} \kappa_1(t_1, x_1, y_1, z) dt dxdydz \\
+ \int_{x_0}^{x_1} \int_{t_0}^{t_1} \int_{z_0}^{z_1} \kappa_2(t, x_1, y_1, z) dt dxdy + \int_{x_0}^{x_1} \int_{t_0}^{t_1} \int_{y_0}^{y_1} \kappa_3(t, x_1, y_1, z_1) dt dxdy \\
= \int_{y_0}^{y_1} \int_{x_0}^{x_1} \int_{z_0}^{z_1} \omega(t_0, x, y, z) dxdydz + \int_{y_0}^{y_1} \int_{t_0}^{t_1} \int_{z_0}^{z_1} \kappa_1(t_0, x_0, y_0, z) dt dxdydz \\
+ \int_{x_0}^{x_1} \int_{t_0}^{t_1} \int_{z_0}^{z_1} \kappa_2(t, x_0, y_0, z_1) dt dxdy + \int_{x_0}^{x_1} \int_{t_0}^{t_1} \int_{y_0}^{y_1} \kappa_3(t, x_0, y_0, z_0) dt dxdy,
\]

where \( \omega(t, x, y, z) = \frac{1}{2} dZ \wedge F dZ, \kappa_p(t, x, y, z) = \frac{1}{2} dZ \wedge K_p dZ \) \((p = 1, 2, 3)\) are the differential 2-forms associated with the skew-symmetric matrices \( F \) and \( K_p \), respectively, and \((t_0, t_1) \times (x_0, x_1) \times (y_0, y_1) \times (z_0, z_1)\) is the local definition domain of \( Z(t, x, y, z) \).
3 Stochastic multi-symplectic methods

In this section we mainly focus on the analysis of three stochastic multi-symplectic methods for the stochastic Maxwell equations (2.1), including the dissipative property of the discrete averaged energy and the conservative property of the discrete averaged divergence. The general difference operators are employed by:

\[
\begin{align*}
\partial_t Z^n_{i,j,k} &= \frac{Z_{i,j,k}^{n+1} - Z_{i,j,k}^n}{\Delta t}, \\
\partial_x Z^n_{i,j,k} &= \frac{Z^n_{i+1,j,k} - Z^n_{i,j,k}}{\Delta x}, \\
\partial_y Z^n_{i,j,k} &= \frac{Z^n_{i+1,j,k} - Z^n_{i-1,j,k}}{2\Delta y}, \\
\partial_z Z^n_{i,j,k} &= \frac{Z^n_{i,j+1,k} - Z^n_{i,j-1,k}}{2\Delta z}.
\end{align*}
\]  

(3.1)

The same definitions hold for operators \(\partial_y, \partial_y, \partial_z, \partial_\bar{z}\).

3.1 Method-I

Method-I is derived by applying the implicit midpoint method both in spatial and temporal directions to the equations (2.17), similarly as the approach in [2], but for the different for of stochastic Hamiltonian PDEs for equations (2.1). It is stated as follows

\[
\begin{align*}
F \partial_t Z^n_{i,\frac{1}{2}j+k+\frac{1}{2}} + K_1 \partial_x Z^n_{i,\frac{1}{2}j+k+\frac{1}{2}} + K_2 \partial_y Z^n_{i,\frac{1}{2}j+k+\frac{1}{2}} + K_3 \partial_z Z^n_{i,\frac{1}{2}j+k+\frac{1}{2}} &= (\tilde{\chi}^n_i)
+ K_3 \partial_z Z^n_{i,\frac{1}{2}j+k+\frac{1}{2}} = \nabla z S_2(Z^n_{i,\frac{1}{2}j+k+\frac{1}{2}})(\tilde{\chi}^n_i),
\end{align*}
\]

(3.2)

with \(F, K_p(p = 1, 2, 3), S_2\) are given by (2.18)-(2.20) and

\[
\begin{align*}
Z^n_{i,\frac{1}{2}j+k+\frac{1}{2}} &= \frac{1}{4}(Z^n_{i,\frac{1}{2}j+k+\frac{1}{2}} + Z^n_{i,\frac{1}{2}j+k+\frac{1}{2}} + Z^n_{i,\frac{1}{2}j+k+\frac{1}{2}} + Z^n_{i,\frac{1}{2}j+k+\frac{1}{2}}).
\end{align*}
\]

Terms \(Z^n_{i,\frac{1}{2}j+k+\frac{1}{2}}, Z^n_{i,\frac{1}{2}j+k+\frac{1}{2}}, Z^n_{i,\frac{1}{2}j+k+\frac{1}{2}}, Z^n_{i,\frac{1}{2}j+k+\frac{1}{2}}, Z^n_{i,\frac{1}{2}j+k+\frac{1}{2}}, Z^n_{i,\frac{1}{2}j+k+\frac{1}{2}}, Z^n_{i,\frac{1}{2}j+k+\frac{1}{2}}\) et al., are defined similarly. This method preserves the stochastic multi-symplectic conservation law, see for instance [2, Theorem 3].

Set the space domain to be \([x_1, x_r]\), and the uniform grid points are \(x_1 = x_0 < x_1 < x_2 < \cdots x_l = x_r\), \(I = (x_r - x_1) / \Delta x\). Thus \((\tilde{\chi}^n_i)\) can be calculated as follows

\[
(\tilde{\chi}^n_i) = \frac{(\Delta W)^n_i}{\Delta t} = \frac{W(t_{n+1}, x_i) - W(t_n, x_i)}{\Delta t}.
\]

We will give the discrete dissipative property of the averaged energy for Method-I in the following theorem.

Theorem 3.1. Assume that \(E^n_{i,j,k}\) and \(H^n_{i,j,k}\) are numerical solutions of (3.3), then under the periodic boundary condition the discrete averaged energy satisfies the following dissipative property

\[
\mathcal{E}(\Phi^{n+1}_i(t_{n+1})) = \mathcal{E}(\Phi^n_i(t_n)) + 2\lambda \Delta x \Delta y \Delta z \sum_{i,j,k} \mathcal{E}(\tilde{\chi}^n_{i,j,k} \Delta W^n_i),
\]

(3.3)
where

$$\Phi^{[n]}(t_n) = \Delta x \Delta y \Delta z \sum_{i,j,k} (|E_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}|^2 + |H_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}|^2),$$

(3.4)

and

$$\Theta_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}} = H_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}} + H_{2i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}} + H_{3i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}} - E_{2i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}} - E_{3i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}} - E_{3i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}},$$

(3.5)

Proof. We rewrite method (3.2) into the componentwise form of \( E \) and \( H \),

\[
\partial_t (E_1)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n} = \partial_x (H_3)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n} - \partial_y (H_2)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n} - \lambda (\chi)_{i}^{n},
\]

(3.6a)

\[
\partial_t (E_2)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n} = \partial_y (H_3)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n} - \partial_z (H_2)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n} - \lambda (\chi)_{i}^{n},
\]

(3.6b)

\[
\partial_t (E_3)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n} = \partial_z (H_1)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n} - \partial_x (H_2)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n} - \lambda (\chi)_{i}^{n},
\]

(3.6c)

\[
\partial_t (H_1)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n} = \partial_z (E_2)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n} - \partial_y (E_3)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n} + \lambda (\chi)_{i}^{n},
\]

(3.6d)

\[
\partial_t (H_2)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n} = \partial_y (E_3)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n} - \partial_x (E_1)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n} + \lambda (\chi)_{i}^{n},
\]

(3.6e)

\[
\partial_t (H_3)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n} = \partial_y (E_1)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n} - \partial_z (E_2)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n} + \lambda (\chi)_{i}^{n}.
\]

(3.6f)

Multiplying both sides of each equality from (3.6a) to (3.6f) with

\[
\Delta t \Delta x \Delta y \Delta z \left( (E_1)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n+1} + (E_1)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n} \right),
\]

\[
\Delta t \Delta x \Delta y \Delta z \left( (E_2)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n+1} + (E_2)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n} \right),
\]

\[
\Delta t \Delta x \Delta y \Delta z \left( (E_3)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n+1} + (E_3)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n} \right),
\]

\[
\Delta t \Delta x \Delta y \Delta z \left( (H_1)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n+1} + (H_1)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n} \right),
\]

\[
\Delta t \Delta x \Delta y \Delta z \left( (H_2)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n+1} + (H_2)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n} \right),
\]

\[
\Delta t \Delta x \Delta y \Delta z \left( (H_3)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n+1} + (H_3)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n} \right),
\]

respectively, summing all terms in the above equations over all spatial indices \( i, j, k \), it yields

\[
\Phi^{[n]}(t_{n+1}) = \Phi^{[n]}(t_n) + A_1 + A_2 + A_3 + A_4 + A_5 + A_6 + A_7,
\]

(3.7)
where

\[
A_1 = \Delta t \Delta x \Delta y \Delta z \sum_{i,j,k} \left[ \partial_y (H_3)^{n+\frac{1}{2}}_{i+\frac{1}{2},j,k} \left( (E_1)^{n+1}_{i+\frac{1}{2},j+\frac{1}{2},k} + (E_1)^{n}_{i+\frac{1}{2},j+\frac{1}{2},k} \right) \right. \\
- \left. \partial_x (H_2)^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2},k} \left( (E_1)^{n+1}_{i+\frac{1}{2},j+\frac{1}{2},k} + (E_1)^{n}_{i+\frac{1}{2},j+\frac{1}{2},k} \right) \right],
\]

\[
A_2 = \Delta t \Delta x \Delta y \Delta z \sum_{i,j,k} \left[ \partial_x (H_1)^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2},k} \left( (E_2)^{n+1}_{i+\frac{1}{2},j+\frac{1}{2},k} + (E_2)^{n}_{i+\frac{1}{2},j+\frac{1}{2},k} \right) \right. \\
- \left. \partial_y (H_3)^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2},k} \left( (E_2)^{n+1}_{i+\frac{1}{2},j+\frac{1}{2},k} + (E_2)^{n}_{i+\frac{1}{2},j+\frac{1}{2},k} \right) \right],
\]

\[
A_3 = \Delta t \Delta x \Delta y \Delta z \sum_{i,j,k} \left[ \partial_x (H_2)^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2},k} \left( (E_3)^{n+1}_{i+\frac{1}{2},j+\frac{1}{2},k} + (E_3)^{n}_{i+\frac{1}{2},j+\frac{1}{2},k} \right) \right. \\
- \left. \partial_y (H_1)^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2},k} \left( (E_3)^{n+1}_{i+\frac{1}{2},j+\frac{1}{2},k} + (E_3)^{n}_{i+\frac{1}{2},j+\frac{1}{2},k} \right) \right],
\]

\[
A_4 = \Delta t \Delta x \Delta y \Delta z \sum_{i,j,k} \left[ \partial_z (E_2)^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2},k} \left( (E_1)^{n+1}_{i+\frac{1}{2},j+\frac{1}{2},k} + (E_1)^{n}_{i+\frac{1}{2},j+\frac{1}{2},k} \right) \right. \\
- \left. \partial_y (E_3)^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2},k} \left( (E_1)^{n+1}_{i+\frac{1}{2},j+\frac{1}{2},k} + (E_1)^{n}_{i+\frac{1}{2},j+\frac{1}{2},k} \right) \right],
\]

\[
A_5 = \Delta t \Delta x \Delta y \Delta z \sum_{i,j,k} \left[ \partial_z (E_3)^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2},k} \left( (H_2)^{n+1}_{i+\frac{1}{2},j+\frac{1}{2},k} + (H_2)^{n}_{i+\frac{1}{2},j+\frac{1}{2},k} \right) \right. \\
- \left. \partial_z (E_1)^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2},k} \left( (H_2)^{n+1}_{i+\frac{1}{2},j+\frac{1}{2},k} + (H_2)^{n}_{i+\frac{1}{2},j+\frac{1}{2},k} \right) \right],
\]

\[
A_6 = \Delta t \Delta x \Delta y \Delta z \sum_{i,j,k} \left[ \partial_y (E_1)^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2},k} \left( (H_3)^{n+1}_{i+\frac{1}{2},j+\frac{1}{2},k} + (H_3)^{n}_{i+\frac{1}{2},j+\frac{1}{2},k} \right) \right. \\
- \left. \partial_x (E_2)^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2},k} \left( (H_3)^{n+1}_{i+\frac{1}{2},j+\frac{1}{2},k} + (H_3)^{n}_{i+\frac{1}{2},j+\frac{1}{2},k} \right) \right].
\]
and
\[ \mathcal{A}_7 = -\lambda \Delta x \Delta y \Delta z \sum_{i,j,k} ((E_1)^{n+1}_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}} + (E_1)^n_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}) (\Delta W)^n_i \]
- \lambda \Delta x \Delta y \Delta z \sum_{i,j,k} ((E_2)^{n+1}_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}} + (E_2)^n_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}) (\Delta W)^n_i \\
- \lambda \Delta x \Delta y \Delta z \sum_{i,j,k} ((E_3)^{n+1}_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}} + (E_3)^n_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}) (\Delta W)^n_i \\
+ \lambda \Delta x \Delta y \Delta z \sum_{i,j,k} ((H_1)^{n+1}_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}} + (H_1)^n_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}) (\Delta W)^n_i \\
+ \lambda \Delta x \Delta y \Delta z \sum_{i,j,k} ((H_2)^{n+1}_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}} + (H_2)^n_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}) (\Delta W)^n_i \\
+ \lambda \Delta x \Delta y \Delta z \sum_{i,j,k} ((H_3)^{n+1}_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}} + (H_3)^n_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}) (\Delta W)^n_i. \\

Using the periodic boundary condition, we can obtain that all terms from \( \mathcal{A}_1 \) to \( \mathcal{A}_6 \) are canceled out after tediously calculating. Taking expectation on both sides of (3.7) leads to the assertion of this theorem.

Specially, we can simplify the expression (3.3) in the case that \( W \) only depends on time. The evolution relationship for averaged energy coincides with the continuous case, see equations (2.5) and (3.8) for comparison.

**Theorem 3.2.** If \( W = W(t,\omega) \), then there exists a constant \( \bar{K} = 6\lambda^2 \bar{V}(\Theta) \) such that
\[ \mathcal{E}(\Phi^n(0)) = \mathcal{E}(\Phi^n(\omega)) + \bar{K} t. \] (3.8)

**Proof.** From the expressions (3.3) and (3.5), we present the following analysis as an example, because the other terms can be dealt similarly. Note that \( \Delta W_n = W(t_{n+1}) - W(t_n) \).
\[ 2\lambda \Delta x \Delta y \Delta z \sum_{i,j,k} \mathcal{E}((H_1)^{n+1}_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}) \Delta W_n \] (3.9)
\[ = 2\lambda \Delta x \Delta y \Delta z \sum_{i,j,k} \mathcal{E}((H_1)^n_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}} + \frac{1}{2} ((H_1)^{n+1}_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}} - (H_1)^n_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}) \Delta W_n. \]

Use the properties of the increment of Wiener process leads to
\[ \mathcal{E}((H_1)^n_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}) = 0. \]
And substituting the equation \([3.6d]\) into \((H_1)^{n+1}_{i+\frac{1}{2}, j+\frac{1}{2}, k+\frac{1}{2}} - (H_1)^{n}_{i+\frac{1}{2}, j+\frac{1}{2}, k+\frac{1}{2}}\) in \((3.9)\) and using the periodic boundary condition, we obtain

\[
2\lambda\Delta x\Delta y\Delta z \sum_{i,j,k} \mathcal{E}((H_1)^{n+\frac{1}{2}}_{i+\frac{1}{2}, j+\frac{1}{2}, k+\frac{1}{2}} \Delta W_n) \\
= \lambda\Delta x\Delta y\Delta z \sum_{i,j,k} \mathcal{E}\left[\frac{\Delta t}{\Delta z} \left((E_2)^{n+\frac{1}{2}}_{i+\frac{1}{2}, j+\frac{1}{2}, k+1} - (E_2)^{n+\frac{1}{2}}_{i+\frac{1}{2}, j+\frac{1}{2}, k}\right) \right] \Delta W_n \\
= \lambda^2\Delta x\Delta y\Delta z \sum_{i,j,k} \Delta t.
\]

Denote \(\bar{V}(\Theta) = \sum_{i,j,k} \Delta x\Delta y\Delta z\), then we have

\[
2\lambda\Delta x\Delta y\Delta z \sum_{i,j,k} \mathcal{E}((H_1)^{n+\frac{1}{2}}_{i+\frac{1}{2}, j+\frac{1}{2}, k+\frac{1}{2}} \Delta W_n) = \lambda^2\Delta t\bar{V}(\Theta).
\]

Similar results hold for others terms. Thus, we get

\[
\mathcal{E}(\Phi^{n+1}(t_{n+1})) = \mathcal{E}(\Phi^n(t_n)) + 6\lambda^2\bar{V}(\Theta)\Delta t,
\]

which proves the theorem.

In order to show that the Method-I preserves the discrete version of the averaged divergence, we may need the definition of discrete divergence operator at point \((x_i, y_j, z_k)\), which is given as follows \((9)\)

\[
\bar{\nabla}_{i,j,k}^{n+1} \cdot \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \partial_x\bar{\alpha}_{i-\frac{1}{2}, j, k} + \partial_y\bar{\beta}_{i, j-\frac{1}{2}, k} + \partial_z\bar{\gamma}_{i, j, k-\frac{1}{2}},
\]

\[(3.10)\]

where

\[
\bar{\alpha}_{i-\frac{1}{2}, j, k} = \alpha_{i-\frac{1}{2}, j+\frac{1}{2}, k+\frac{1}{2}} + \alpha_{i+\frac{1}{2}, j+\frac{1}{2}, k-\frac{1}{2}} + \alpha_{i-\frac{1}{2}, j-\frac{1}{2}, k+\frac{1}{2}} + \alpha_{i-\frac{1}{2}, j-\frac{1}{2}, k-\frac{1}{2}},
\]

\[
\bar{\beta}_{i, j-\frac{1}{2}, k} = \beta_{i+\frac{1}{2}, j-\frac{1}{2}, k+\frac{1}{2}} + \beta_{i-\frac{1}{2}, j-\frac{1}{2}, k+\frac{1}{2}} + \beta_{i+\frac{1}{2}, j-\frac{1}{2}, k-\frac{1}{2}} + \beta_{i-\frac{1}{2}, j-\frac{1}{2}, k-\frac{1}{2}},
\]

\[
\bar{\gamma}_{i, j, k-\frac{1}{2}} = \gamma_{i+\frac{1}{2}, j+\frac{1}{2}, k-\frac{1}{2}} + \gamma_{i+\frac{1}{2}, j-\frac{1}{2}, k-\frac{1}{2}} + \gamma_{i-\frac{1}{2}, j+\frac{1}{2}, k-\frac{1}{2}} + \gamma_{i-\frac{1}{2}, j-\frac{1}{2}, k-\frac{1}{2}}.
\]

The following theorem shows that the Method-I preserves the conservation law of discrete divergence in the sense of expectation.

**Theorem 3.3.** The numerical discretization \((3.6)\) to equations \((2.1)\) preserves the discrete divergence in the sense of expectation, i.e.,

\[
\mathcal{E}\left(\bar{\nabla}_{i,j,k}^{n+1} \cdot E^n\right) = \mathcal{E}\left(\bar{\nabla}_{i,j,k}^{n} \cdot E^n\right), \quad \mathcal{E}\left(\bar{\nabla}_{i,j,k}^{n+1} \cdot H^{n+1}\right) = \mathcal{E}\left(\bar{\nabla}_{i,j,k}^{n} \cdot H^n\right).
\]

\[(3.11)\]
Proof. The proof of the two assertions are similar, so here we only present that for electric field \( \mathbf{E} \).

\[
\mathcal{E}\left( \nabla_{i,j,k}^{[l]} \cdot \mathbf{E}^{n+1} - \nabla_{i,j,k}^{[l]} \cdot \mathbf{E}^n \right) = \mathcal{E} \left[ \partial_x \left( (E_1)^{n+1}_{i-\frac{1}{2},j,k} - (E_1)^n_{i-\frac{1}{2},j,k} \right) \right. \\
+ \partial_y \left( (E_2)^{n+1}_{i,j-\frac{1}{2},k} - (E_2)^n_{i,j-\frac{1}{2},k} \right) + \partial_z \left( (E_3)^{n+1}_{i,j,k-\frac{1}{2}} - (E_3)^n_{i,j,k-\frac{1}{2}} \right) \\
= \mathcal{E} \left[ \partial_x \left( (E_1)^{n+1}_{i-\frac{1}{2},j+\frac{1}{2},k} - (E_1)^n_{i-\frac{1}{2},j+\frac{1}{2},k} \right) + \partial_x \left( (E_1)^{n+1}_{i-\frac{1}{2},j+\frac{1}{2},k} - (E_1)^n_{i-\frac{1}{2},j+\frac{1}{2},k} \right) \right. \\
+ \partial_x \left( (E_2)^{n+1}_{i-\frac{1}{2},j+\frac{1}{2},k} - (E_2)^n_{i-\frac{1}{2},j+\frac{1}{2},k} \right) + \partial_y \left( (E_2)^{n+1}_{i-\frac{1}{2},j+\frac{1}{2},k} - (E_2)^n_{i-\frac{1}{2},j+\frac{1}{2},k} \right) \\
+ \partial_y \left( (E_3)^{n+1}_{i-\frac{1}{2},j+\frac{1}{2},k} - (E_3)^n_{i-\frac{1}{2},j+\frac{1}{2},k} \right) + \partial_z \left( (E_3)^{n+1}_{i-\frac{1}{2},j+\frac{1}{2},k} - (E_3)^n_{i-\frac{1}{2},j+\frac{1}{2},k} \right) \\
+ \partial_z \left( (E_3)^{n+1}_{i-\frac{1}{2},j+\frac{1}{2},k} - (E_3)^n_{i-\frac{1}{2},j+\frac{1}{2},k} \right) + \partial_z \left( (E_3)^{n+1}_{i-\frac{1}{2},j+\frac{1}{2},k} - (E_3)^n_{i-\frac{1}{2},j+\frac{1}{2},k} \right) \right].
\]
Replacing the difference of $E$ and $H$ by method (3.6) leads to

$$\mathcal{E}\left(\nabla_{i,j,k}^{[1]} \cdot \mathbf{E}^{n+1} - \nabla_{i,j,k}^{[1]} \cdot \mathbf{E}^n\right)$$

$$= \mathcal{E}\left\{ \partial_z \left[ \frac{\Delta t}{\Delta y} \left( (H_3)^{n+\frac{1}{2}}_{i-\frac{1}{2},j+1,k+\frac{1}{2}} - (H_3)^{n+\frac{1}{2}}_{i-\frac{1}{2},j,k+\frac{1}{2}} \right) - \frac{\Delta t}{\Delta z} \left( (H_2)^{n+\frac{1}{2}}_{i-\frac{1}{2},j+1,k+\frac{1}{2}} - (H_2)^{n+\frac{1}{2}}_{i-\frac{1}{2},j,k+\frac{1}{2}} \right) \right] \right\}$$

$$+ \partial_x \left[ \frac{\Delta t}{\Delta y} \left( (H_3)^{n+\frac{1}{2}}_{i-\frac{1}{2},j+1,k-\frac{1}{2}} - (H_3)^{n+\frac{1}{2}}_{i-\frac{1}{2},j,k-\frac{1}{2}} \right) - \frac{\Delta t}{\Delta z} \left( (H_2)^{n+\frac{1}{2}}_{i-\frac{1}{2},j+1,k-\frac{1}{2}} - (H_2)^{n+\frac{1}{2}}_{i-\frac{1}{2},j,k-\frac{1}{2}} \right) \right]$$

$$+ \partial_y \left[ \frac{\Delta t}{\Delta z} \left( (H_1)^{n+\frac{1}{2}}_{i+\frac{1}{2},j-\frac{1}{2},k+1} - (H_1)^{n+\frac{1}{2}}_{i+\frac{1}{2},j-\frac{1}{2},k} \right) - \frac{\Delta t}{\Delta x} \left( (H_3)^{n+\frac{1}{2}}_{i+1,j-\frac{1}{2},k+\frac{1}{2}} - (H_3)^{n+\frac{1}{2}}_{i+1,j-\frac{1}{2},k+\frac{1}{2}} \right) \right]$$

$$+ \partial_z \left[ \frac{\Delta t}{\Delta z} \left( (H_2)^{n+\frac{1}{2}}_{i+\frac{1}{2},j-\frac{1}{2},k-\frac{1}{2}} - (H_2)^{n+\frac{1}{2}}_{i+\frac{1}{2},j,k-\frac{1}{2}} \right) - \frac{\Delta t}{\Delta x} \left( (H_1)^{n+\frac{1}{2}}_{i+1,j+\frac{1}{2},k-\frac{1}{2}} - (H_1)^{n+\frac{1}{2}}_{i+1,j+\frac{1}{2},k-\frac{1}{2}} \right) \right]$$

$$-4\lambda \partial_x (\Delta W)^n_{i-1} - 2\lambda \partial_y (\Delta W)^n_{i-1} - 2\lambda \partial_y (\Delta W)^n_{i-1} - 2\lambda \partial_z (\Delta W)^n_{i-1}$$

$$- \mathcal{E}\left(\nabla_{i,j,k}^{[1]} \cdot \mathbf{E}^{n+1} - \nabla_{i,j,k}^{[1]} \cdot \mathbf{E}^{n}\right) = -4\lambda \mathcal{E}\left(\frac{(\Delta W)^n_{i-1} - (\Delta W)^n_{i-1}}{\Delta x}\right) = 0. \quad (3.12)$$

Thus the proof is completed.

3.2 Method-II

As is wellknown, for the numerical simulation of deterministic Maxwell equations Yee's method is the basis of the highly popular CEM numerical methods known as the finite-difference time-domain (FDTD) methods (see the original work [10]). It is constructed by
central difference in both spatial and temporal directions based on a half-step staggered grid. With the difference operators defined in (3.1), we generalize Yee’s method ([7]) to discretize the stochastic Maxwell equations (2.1) as follows:

\[
\begin{align*}
\bar{\partial}_t(H_1)_{i,j,k}^n &= -\bar{\partial}_y(E_3)_{i,j,k}^n + \bar{\partial}_z(E_2)_{i,j,k}^n + \lambda(\bar{\chi})_{i}^{n+1}, \\
\bar{\partial}_t(H_2)_{i,j,k}^n &= -\bar{\partial}_y(E_1)_{i,j,k}^n + \bar{\partial}_z(E_3)_{i,j,k}^n + \lambda(\bar{\chi})_{i}^{n+1}, \\
\bar{\partial}_t(H_3)_{i,j,k}^n &= -\bar{\partial}_y(E_2)_{i,j,k}^n + \bar{\partial}_z(E_1)_{i,j,k}^n + \lambda(\bar{\chi})_{i}^{n+1}, \\
\bar{\partial}_t(E_1)_{i,j,k}^n &= \bar{\partial}_y(H_3)_{i,j,k}^n - \bar{\partial}_z(H_2)_{i,j,k}^n - \lambda(\bar{\chi})_{i}^{n+1}, \\
\bar{\partial}_t(E_2)_{i,j,k}^n &= \bar{\partial}_y(H_1)_{i,j,k}^n - \bar{\partial}_z(H_3)_{i,j,k}^n - \lambda(\bar{\chi})_{i}^{n+1}, \\
\bar{\partial}_t(E_3)_{i,j,k}^n &= \bar{\partial}_y(H_2)_{i,j,k}^n - \bar{\partial}_z(H_1)_{i,j,k}^n - \lambda(\bar{\chi})_{i}^{n+1},
\end{align*}
\]

where

\[
(\bar{\chi})_{i}^{n+1} = \frac{W(t_{n+1}, x_i) - W(t_{n-1}, x_i)}{2\Delta t}.
\]

Clearly, the above method conserves the stochastic multi-symplectic conservation law. And in the following contents we will consider the properties of the discrete averaged energy and the discrete averaged divergence.

**Theorem 3.4.** Assume that \( E_{i,j,k}^n \) and \( H_{i,j,k}^n \) are numerical solutions of method (3.13), then under the periodic boundary condition the averaged energy satisfies

\[
\mathcal{E}(\Phi^{[II]}(t_{n+1})) = \mathcal{E}(\Phi^{[II]}(t_n)) + \lambda \Delta x \Delta y \Delta z \mathcal{E} \left( \sum_{i,j,k} \Upsilon_{i,j,k}^n (\Delta W)^{n-1}_i \right)
\]

where

\[
\Phi^{[II]}(t_{n+1}) = \Delta x \Delta y \Delta z \sum_{i,j,k} \left( (E_1)_{i,j,k}^{n+1} (E_1)_{i,j,k}^n + (E_2)_{i,j,k}^{n+1} (E_2)_{i,j,k}^n + (E_3)_{i,j,k}^{n+1} (E_3)_{i,j,k}^n + (H_1)_{i,j,k}^{n+1} (H_1)_{i,j,k}^n + (H_2)_{i,j,k}^{n+1} (H_2)_{i,j,k}^n + (H_3)_{i,j,k}^{n+1} (H_3)_{i,j,k}^n \right),
\]

and

\[
\Upsilon_{i,j,k}^n = (H_1)_{i,j,k}^n + (H_2)_{i,j,k}^n + (H_3)_{i,j,k}^n - (E_1)_{i,j,k}^n - (E_2)_{i,j,k}^n - (E_3)_{i,j,k}^n.
\]

**Proof.** Multiplying both sides of each equation from (3.13a) to (3.13f) with

\[
\begin{align*}
2\Delta t \Delta x \Delta y \Delta z (H_1)_{i,j,k}^n, & \quad 2\Delta t \Delta x \Delta y \Delta z (H_2)_{i,j,k}^n, & \quad 2\Delta t \Delta x \Delta y \Delta z (H_3)_{i,j,k}^n, \\
2\Delta t \Delta x \Delta y \Delta z (E_1)_{i,j,k}^n, & \quad 2\Delta t \Delta x \Delta y \Delta z (E_2)_{i,j,k}^n, & \quad 2\Delta t \Delta x \Delta y \Delta z (E_3)_{i,j,k}^n,
\end{align*}
\]
respectively. Summing all terms in over all spatial indices \(i, j, k\) together, and using the periodic boundary condition, we obtain

\[
\phi^{[\Pi]}(t_{n+1}) = \phi^{[\Pi]}(t_n) + \lambda \Delta x \Delta y \Delta z \sum_{i,j,k} (\mathcal{T}_{i,j,k}^n)(W_{i}^{n+1} - W_{i}^{n-1}).
\] (3.15)

We take expectation in both sides of (3.15) and use the independent properties of Wiener process to finish the proof.

Moreover, we have the following estimation for the averaged energy.

**Theorem 3.5.** There exists a constant \( \bar{K} = 6\lambda^2 \bar{V}^Q(\Theta) \) such that

\[
\mathcal{E}\left( \phi^{[\Pi]}(t_n) \right) = \mathcal{E}\left( \phi^{[\Pi]}(t_0) \right) + \bar{K} t_n.
\] (3.16)

**Proof.** We need to estimate each term in \( \lambda \Delta x \Delta y \Delta z \mathcal{E}\left( \sum_{i,j,k} (\mathcal{T}_{i,j,k}^n)(W_{i}^{n} - W_{i}^{n-1}) \right) \), with \( W_{i}^{n} = W(t_n, x_i) \). For the first term we have

\[
\lambda \Delta x \Delta y \Delta z \mathcal{E}\left[ \sum_{i,j,k} (H_1)_{i,j,k}^n(W_{i}^{n} - W_{i}^{n-1}) \right] \]

\[
= \lambda \Delta x \Delta y \Delta z \mathcal{E}\left[ \sum_{i,j,k} \left( (H_1)_{i,j,k}^n - (H_1)^{n-2}_{i,j,k} \right)(W_{i}^{n} - W_{i}^{n-1}) \right] \]

\[
= \lambda \Delta x \Delta y \Delta z \mathcal{E}\left\{ \sum_{i,j,k} \left[ \frac{-\Delta t}{\Delta y} (E_3)_{i,j+1,k}^{n-1} - (E_3)_{i,j-1,k}^{n-1} \right] + \frac{\Delta t}{\Delta z} (E_2)_{i,j,k+1}^{n-1} - (E_2)_{i,j,k}^{n-1} \right\} + \lambda(W_{i}^{n} - W_{i}^{n-2})(W_{i}^{n} - W_{i}^{n-1}) \]

\[
= \lambda^2 \Delta x \Delta y \Delta z \mathcal{E}\left\{ \sum_{i,j,k} (W_{i}^{n} - W_{i}^{n-2})(W_{i}^{n} - W_{i}^{n-1}) \right\} \]

\[
= \lambda^2 \Delta x \Delta y \Delta z \mathcal{E}\left\{ \sum_{i,j,k} (W_{i}^{n} - W_{i}^{n-1})^2 \right\} = \lambda^2 \bar{V}^Q(\Theta) \Delta t
\]

where \( \bar{V}^Q(\Theta) = \Delta x \Delta y \Delta z \sum_{i,j,k} \sum_m (\sqrt{\eta_m} e_m(x_i))^2 \). Here we mainly use the independent properties of Wiener increments. Because other terms could be estimated similarly, we finish the proof noting that \( \bar{K} = 6\lambda^2 \bar{V}^Q(\Theta) \).

Furthermore, the method (3.13) preserves the following discrete averaged divergence.

**Theorem 3.6.** The method (3.13) preserves the following discrete divergence in the sense of expectation

\[
\mathcal{E}\left( \bar{\nabla}^{[\Pi]} \cdot \mathbf{E}_{i,j,k}^{n+\frac{1}{2}} \right) = \mathcal{E}\left( \bar{\nabla}^{[\Pi]} \cdot \mathbf{E}_{i,j,k}^{n-\frac{1}{2}} \right),
\] (3.18)

\[
\mathcal{E}\left( \bar{\nabla}^{[\Pi]} \cdot \mathbf{H}_{i,j,k}^{n+\frac{1}{2}} \right) = \mathcal{E}\left( \bar{\nabla}^{[\Pi]} \cdot \mathbf{H}_{i,j,k}^{n-\frac{1}{2}} \right),
\]

where \( \bar{\nabla}^{[\Pi]} = (\bar{\partial}_x, \bar{\partial}_y, \bar{\partial}_z)^T \).

The proof of this theorem is similar to that of Theorem 3.3 so we omit it here.
3.3 Method-III

We use the central finite difference in spatial direction and implicit midpoint method in temporal direction, then we refer to this particular discretization as the Method-III (see [9] for deterministic case)

\[
F \partial_t Z_{i,j,k}^n + K_1 \partial_z Z_{i,j,k}^{n+\frac{1}{2}} + K_2 \partial_y Z_{i,j,k}^{n+\frac{1}{2}} + K_3 \partial_z Z_{i,j,k}^{n+\frac{1}{2}} = \nabla_z S_2(Z_{i,j,k}^{n+\frac{1}{2}})(\chi)_i^n. \tag{3.19}
\]

It is shown that method (3.19) preserves the stochastic multi-symplectic conservation law.

**Theorem 3.7.** The method [3.19] satisfies the discrete stochastic multi-symplectic conservation law a.s.,

\[
\frac{\omega_{i,j,k}^{n+1} - \omega_{i,j,k}^n}{\Delta t} + \frac{\kappa_{i,j,k}^{n+\frac{1}{2}} - \kappa_{i,j,k}^{n-\frac{1}{2}}}{\Delta x} + \frac{\kappa_{i,j,k}^{n+\frac{1}{2}} - \kappa_{i,j,k}^{n-\frac{1}{2}}}{\Delta y} + \frac{\kappa_{i,j,k}^{n+\frac{1}{2}} - \kappa_{i,j,k}^{n-\frac{1}{2}}}{\Delta z} = 0, \tag{3.20}
\]

where

\[
\omega_{i,j,k}^n = dZ_{i,j,k}^{n+\frac{1}{2}} \wedge FdZ_{i,j,k}^{n+\frac{1}{2}}, \quad \kappa_{i,j,k}^{n+\frac{1}{2}} = dZ_{i,j,k}^{n+\frac{1}{2}} \wedge K_1 dZ_{i+1,j,k}^{n+\frac{1}{2}},
\]

\[
\kappa_{i,j,k}^{n+\frac{1}{2}} = dZ_{i,j,k}^{n+\frac{1}{2}} \wedge K_2 dZ_{i,j+1,k}^{n+\frac{1}{2}}, \quad \kappa_{i,j,k}^{n+\frac{1}{2}} = dZ_{i,j,k}^{n+\frac{1}{2}} \wedge K_3 dZ_{i,j+1,k}^{n+\frac{1}{2}}.
\]

**Proof.** We take differential in the phase space on both sides of (3.19) to obtain

\[
\begin{align*}
\Delta x \Delta y \Delta z F(dZ_{i,j,k}^{n+\frac{1}{2}} - dZ_{i,j,k}^n) + \Delta t \Delta y \Delta z K_1(dZ_{i+1,j,k}^{n+\frac{1}{2}} - dZ_{i,j,k}^{n+\frac{1}{2}}) \\
+ \Delta t \Delta x \Delta z K_2(dZ_{i,j+1,k}^{n+\frac{1}{2}} - dZ_{i,j,k}^{n+\frac{1}{2}}) + \Delta t \Delta x \Delta y K_3(dZ_{i,j,k+1}^{n+\frac{1}{2}} - dZ_{i,j,k}^{n+\frac{1}{2}})
= 2\Delta x \Delta y \Delta z \nabla^2 S_2 \left( Z_{i,j,k}^{n+\frac{1}{2}} \right) (\Delta W)^{\frac{1}{2}}_i.
\end{align*}
\]

Then taking \( dZ_{i,j,k}^{n+\frac{1}{2}} = \frac{dZ_{i,j,k}^{n+1} + dZ_{i,j,k}^n}{2} \) and performing wedge product with the above equation yields

\[
\begin{align*}
\Delta x \Delta y \Delta z (dZ_{i,j,k}^{n+\frac{1}{2}} \wedge FdZ_{i,j,k}^{n+\frac{1}{2}} - dZ_{i,j,k}^n \wedge FdZ_{i,j,k}^n) \\
+ \Delta t \Delta y \Delta z (dZ_{i,j,k}^{n+\frac{1}{2}} \wedge K_1 dZ_{i+1,j,k}^{n+\frac{1}{2}} - dZ_{i,j,k}^{n+\frac{1}{2}} \wedge K_1 dZ_{i+1,j,k}^n) \\
+ \Delta t \Delta x \Delta z (dZ_{i,j,k}^{n+\frac{1}{2}} \wedge K_2 dZ_{i,j+1,k}^{n+\frac{1}{2}} - dZ_{i,j,k}^{n+\frac{1}{2}} \wedge K_2 dZ_{i,j+1,k}^n) \\
+ \Delta t \Delta x \Delta y (dZ_{i,j,k}^{n+\frac{1}{2}} \wedge K_3 dZ_{i,j,k}^{n+\frac{1}{2}} - dZ_{i,j,k}^{n+\frac{1}{2}} \wedge K_3 dZ_{i,j,k}^n)
= 0.
\end{align*}
\]

Thus we finish the proof by denoting

\[
\omega_{i,j,k}^{n+1} = dZ_{i,j,k}^{n+1} \wedge FdZ_{i,j,k}^{n+1}, \quad \kappa_{i,j,k}^{n+\frac{1}{2}} = dZ_{i,j,k}^{n+\frac{1}{2}} \wedge K_1 dZ_{i+1,j,k}^{n+\frac{1}{2}},
\]

\[
\kappa_{i,j,k}^{n+\frac{1}{2}} = dZ_{i,j,k}^{n+\frac{1}{2}} \wedge K_2 dZ_{i,j+1,k}^{n+\frac{1}{2}}, \quad \kappa_{i,j,k}^{n+\frac{1}{2}} = dZ_{i,j,k}^{n+\frac{1}{2}} \wedge K_3 dZ_{i,j+1,k}^{n+\frac{1}{2}}.
\]
We also rewrite (3.19) into the component form of \(E\) and \(H\) as follows:

\[
\partial_t E_n^{i,j,k} = \bar{\partial}_y (H_3)^{n+1/2}_{i,j,k} - \bar{\partial}_x (H_2)^{n+1/2}_{i,j,k} - \lambda (\dot{\chi})^n, \tag{3.21a}
\]

\[
\partial_t (E_2)^{n}_{i,j,k} = \bar{\partial}_y (H_1)^{n+1/2}_{i,j,k} - \bar{\partial}_x (H_3)^{n+1/2}_{i,j,k} - \lambda (\dot{\chi})^n, \tag{3.21b}
\]

\[
\partial_t (E_3)^{n}_{i,j,k} = \bar{\partial}_y (H_2)^{n+1/2}_{i,j,k} - \bar{\partial}_x (H_1)^{n+1/2}_{i,j,k} - \lambda (\dot{\chi})^n, \tag{3.21c}
\]

\[
\partial_t (H_1)^{n}_{i,j,k} = -\bar{\partial}_y (E_3)^{n+1/2}_{i,j,k} + \bar{\partial}_z (E_2)^{n+1/2}_{i,j,k} + \lambda (\dot{\chi})^n, \tag{3.21d}
\]

\[
\partial_t (H_2)^{n}_{i,j,k} = -\bar{\partial}_z (E_1)^{n+1/2}_{i,j,k} + \bar{\partial}_x (E_3)^{n+1/2}_{i,j,k} + \lambda (\dot{\chi})^n, \tag{3.21e}
\]

\[
\partial_t (H_3)^{n}_{i,j,k} = -\bar{\partial}_x (E_2)^{n+1/2}_{i,j,k} + \bar{\partial}_y (E_1)^{n+1/2}_{i,j,k} + \lambda (\dot{\chi})^n. \tag{3.21f}
\]

The following theorem states the dissipative property for the discrete averaged energy.

**Theorem 3.8.** Assume that \((E)^{n}_{i,j,k}\) and \((H)^{n}_{i,j,k}\) are numerical solutions of method (3.21), then under the periodic boundary condition, the averaged energy satisfies the following dissipative property

\[
\mathcal{E}(\Phi^{III}(t_{n+1})) = \mathcal{E}(\Phi^{III}(t_n)) + 2\lambda \Delta x \Delta y \Delta z \sum_{i,j,k} \mathcal{E}(\gamma_{i,j,k}^{n+1/2} (\Delta W)^{n}_{i,j,k}), \tag{3.22}
\]

where

\[
\Phi^{III}(t_n) = \Delta x \Delta y \Delta z \sum_{i,j,k} ( | E^{n}_{i,j,k} |^2 + | H^{n}_{i,j,k} |^2 ),
\]

and

\[
\gamma_{i,j,k}^{n+1/2} = (H_1)^{n+1/2}_{i,j,k} + (H_2)^{n+1/2}_{i,j,k} + (H_3)^{n+1/2}_{i,j,k} - (E_1)^{n+1/2}_{i,j,k} - (E_2)^{n+1/2}_{i,j,k} - (E_3)^{n+1/2}_{i,j,k}. \tag{3.23}
\]

The proof is similar to that of Theorem 3.1, so we omit it here.

In the following theorem we give an estimation about the evolution of the discrete averaged energy.

**Theorem 3.9.** There exists a constant \(\bar{K} > 0\) such that

\[
\mathcal{E}(\Phi^{III}(t_n)) - \mathcal{E}(\Phi^{III}(t_0)) \leq \bar{K} t_n. \tag{3.24}
\]
Proof. We rewrite (3.22) as following

\[
\mathcal{E}(\Phi_{n+1}^{[III]}) = \mathcal{E}(\Phi_n^{[III]}) + \mathcal{E}\left\{ \lambda \Delta x \Delta y \Delta z \sum_{i,j,k} \left( \left( H_1 \right)_{i,j,k}^{n+1} + \left( H_1 \right)_{i,j,k}^n \right) \right\} \\
+ \left( \left( H_2 \right)_{i,j,k}^{n+1} + \left( H_2 \right)_{i,j,k}^n \right) - \left( \left( H_2 \right)_{i,j,k}^{n+1} + \left( H_2 \right)_{i,j,k}^n \right) - \left( \left( E_1 \right)_{i,j,k}^{n+1} + \left( E_1 \right)_{i,j,k}^n \right) \\
\left( \left( H_3 \right)_{i,j,k}^{n+1} + \left( H_3 \right)_{i,j,k}^n \right) - \left( \left( H_3 \right)_{i,j,k}^{n+1} + \left( H_3 \right)_{i,j,k}^n \right) - \left( \left( E_1 \right)_{i,j,k}^{n+1} + \left( E_1 \right)_{i,j,k}^n \right) \\
- \left( \left( E_2 \right)_{i,j,k}^{n+1} + \left( E_2 \right)_{i,j,k}^n \right) - \left( \left( E_3 \right)_{i,j,k}^{n+1} + \left( E_3 \right)_{i,j,k}^n \right) - \left( \left( E_3 \right)_{i,j,k}^{n+1} + \left( E_3 \right)_{i,j,k}^n \right) - \left( \left( E_3 \right)_{i,j,k}^{n+1} + \left( E_3 \right)_{i,j,k}^n \right) \\
\left( \left( E_3 \right)_{i,j,k}^{n+1} + \left( E_3 \right)_{i,j,k}^n \right) - \left( \left( E_3 \right)_{i,j,k}^{n+1} + \left( E_3 \right)_{i,j,k}^n \right) - \left( \left( E_3 \right)_{i,j,k}^{n+1} + \left( E_3 \right)_{i,j,k}^n \right) - \left( \left( E_3 \right)_{i,j,k}^{n+1} + \left( E_3 \right)_{i,j,k}^n \right) \\
\left( \Delta W \right)_{i}^{n} \right\},
\] (3.25)

then estimate terms \((I) \sim (VI)\), respectively. For the first term \((I)\), using the independent property of Wiener increment and (3.21d), we get

\[
\mathcal{E}\left\{ \lambda \Delta x \Delta y \Delta z \sum_{i,j,k} \left( \left( H_1 \right)_{i,j,k}^{n+1} + \left( H_1 \right)_{i,j,k}^n \right) \left( \Delta W \right)_{i}^{n} \right\} \\
= \mathcal{E}\left\{ \lambda \Delta x \Delta y \Delta z \sum_{i,j,k} \left[ 2(\left( H_1 \right)_{i,j,k}^n) + \left( \left( H_1 \right)_{i,j,k}^{n+1} - \left( H_1 \right)_{i,j,k}^n \right) \right] \left( \Delta W \right)_{i}^{n} \right\} \\
= \mathcal{E}\left\{ \lambda \Delta x \Delta y \Delta z \sum_{i,j,k} \left[ \frac{\Delta t}{2\Delta z} \left( E_2 \right)_{i,j,k+1,1}^{n+\frac{1}{2}} - \left( E_2 \right)_{i,j,k-1}^{n+\frac{1}{2}} \right] \left( \Delta W \right)_{i}^{n} \right\} \\
- \frac{\Delta t}{2\Delta y} \left( E_3 \right)_{i,j+1,k}^{n+\frac{1}{2}} - \left( E_3 \right)_{i,j-1,k}^{n+\frac{1}{2}} + \lambda \Delta (W)_{i}^{n} \right\} \left( \Delta W \right)_{i}^{n} \right\} \\
= \mathcal{E}\left\{ \lambda^2 \Delta x \Delta y \Delta z \sum_{i,j,k} \left( \left( \Delta W \right)_{i}^{n} \right) \right\} \\
= \lambda^2 \tilde{V}_Q(\Theta) \Delta t,
\] (3.26)

where \( \tilde{V}_Q(\Theta) := \Delta x \Delta y \Delta z \sum_{i,j,k} m_{m}(\sqrt{\frac{1}{m} e_m(x_i)})^2 \), and we are benefit from that Wiener process is 1-D space and periodic boundary condition. Similarly, for term \((IV)\), we have

\[
\mathcal{E}\left[ - \lambda \Delta x \Delta y \Delta z \sum_{i,j,k} \left( \left( E_1 \right)_{i,j,k}^{n+1} + \left( E_1 \right)_{i,j,k}^n \right) \left( \Delta W \right)_{i}^{n} \right] = \lambda^2 \tilde{V}_Q(\Theta) \Delta t.
\] (3.27)

The estimations for terms \((II), (III), (V)\) and \((VI)\) are similar, so we present the estimation
of term (II) for an example.

\[
\mathcal{E}\left\{ \lambda \Delta x \Delta y \Delta z \sum_{i,j,k} \left( (H_2)_{i,j,k}^{n+1} + (H_2)_{i,j,k}^{n} \right) (\Delta W)_{i}^{n} \right\} = \mathcal{E}\left\{ \lambda \Delta x \Delta y \Delta z \sum_{i,j,k} \left[ 2(H_2)_{i,j,k}^{n} + \left( (H_2)_{i+1,j,k}^{n+1} - (H_2)_{i,j,k}^{n} \right) \right] (\Delta W)_{i}^{n} \right\} + \lambda^2 \hat{V}^Q(\Theta) \Delta t
\]

\[
= \mathcal{E}\left\{ \lambda \Delta x \Delta y \Delta z \sum_{i,j,k} \left[ \frac{\Delta t}{2\Delta x} \left( (E_3)_{i+1,j,k}^{n+1} - (E_3)_{i,j,k}^{n+1} \right) \right] (\Delta W)_{i}^{n} \right\} + \lambda^2 \hat{V}^Q(\Theta) \Delta t
\]

\[
\leq \Delta t \Delta x \Delta y \Delta z \sum_{i,j,k} \mathcal{E}\left\{ ((E_3)_{i,j,k}^{n+1})^2 \right\} + K_2(\Delta t)^2 \lambda^2 + \lambda^2 \hat{V}^Q(\Theta) \Delta t
\]

(3.29)

with \( \hat{V}^Q(\Theta) = \Delta x \Delta y \Delta z \sum_{i,j,k} \sum_{m} \eta_m \left( \frac{e_m(x_{i+1}) - e_m(x_{i-1})}{2\Delta x} \right)^2 \).

Hence

\[
\mathcal{E}\left( \Phi_{[III]}^{n+1}(t_{n+1}) \right) \leq \mathcal{E}\left( \Phi_{[III]}^{n+1}(t_{0}) \right) + 4\lambda^2 \hat{V}^Q(\Theta)(\Delta t)^2 + 6\lambda^2 \hat{V}^Q(\Theta) \Delta t.
\]

(3.30)

By Gronwall inequality, there exist constants \( \Delta t^* \) and \( \bar{K} = \bar{K}(\hat{V}^Q(\Theta), \hat{V}^Q(\Theta), \lambda) \) such that for \( \Delta t \leq \Delta t^* \)

\[
\mathcal{E}\left( \Phi_{[III]}^{n+1}(t_{n+1}) \right) - \mathcal{E}\left( \Phi_{[III]}^{n+1}(t_{0}) \right) \leq \bar{K} t_n.
\]

Thus the proof is finished. \( \square \)

**Remark 1.** If \( W = W(t, \omega) \), the same as Theorem 3.2, we have

\[
\mathcal{E}\left( \Phi_{[III]}^{n+1}(t_{n+1}) \right) = \mathcal{E}\left( \Phi_{[III]}^{n+1}(t_{0}) \right) + \bar{K} t_n,
\]

with \( \bar{K} = 6\lambda^2 \hat{V}(\Theta) \).

Define \( \hat{\nabla}^{[III]} = (\hat{\partial}_x, \hat{\partial}_y, \hat{\partial}_z)^T \), then the Method-III can preserve the following discrete averaged divergence. The proof is similar to that of Theorem 3.3.

**Theorem 3.10.** The numerical discretization (3.21) to stochastic Maxwell equations (2.1) preserves the following discrete averaged divergence

\[
\mathcal{E}\left( \hat{\nabla}^{[III]} \cdot E_{i,j,k}^{n+1} \right) = \mathcal{E}\left( \hat{\nabla}^{[III]} \cdot E_{i,j,k}^{n} \right),
\]

(3.31)

\[
\mathcal{E}\left( \hat{\nabla}^{[III]} \cdot H_{i,j,k}^{n+1} \right) = \mathcal{E}\left( \hat{\nabla}^{[III]} \cdot H_{i,j,k}^{n} \right)
\]
We may conclude that all of the three numerical methods are shown to be stochastic multi-symplectic and preserve the conservation law of the corresponding version of discrete averaged divergence. For the continuous problem, we prove that the averaged energy evolves linearly with respect to time, while each method in our consideration preserves this property to certain level. We show that this linear growth property is preserved well by Method-II, whereas Method-I and Method-III conserve this property in the case that the noise only depends on temporal variable. Moreover, we could prove that for space-time noise, the corresponding discrete averaged energy of Method-III grows at most linearly.

4 Numerical results

In this section, we mainly focus on the simulation of 2-D stochastic Maxwell equations with additive noise, for which the electric field and the magnetic field are

\[ E = (0, 0, E_3)^T, \quad H = (H_1, H_2, 0)^T, \]

respectively. I.e.,

\[ \begin{cases} 
\frac{\partial E_3}{\partial t} = \frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y} - \lambda \dot{\chi} & \text{in } (0, T) \times \Theta, \\
\frac{\partial H_1}{\partial t} = - \frac{\partial E_3}{\partial y} + \lambda \dot{\chi} & \text{in } (0, T) \times \Theta, \\
\frac{\partial H_2}{\partial t} = \frac{\partial E_3}{\partial x} + \lambda \dot{\chi} & \text{in } (0, T) \times \Theta, 
\end{cases} \tag{4.1} \]

with \( \Theta = [0, \frac{2}{3}] \times [0, \frac{1}{2}], T = 1 \) and initial data being

\[ E_3(x, y, 0) = \sin(3\pi x) \sin(4\pi y), \]

\[ H_1(x, y, 0) = - \frac{4}{5} \cos(3\pi x) \cos(4\pi y), \]

\[ H_2(x, y, 0) = - \frac{3}{5} \sin(3\pi x) \sin(4\pi y). \]

Hereafter, we choose the orthonormal basis \((e_m)_{m \in \mathbb{N}}\) of \( L^2([0, \frac{2}{3}]) \) and eigenvalue \( \{\eta_m\}_{m \in \mathbb{N}} \) of operator \( Q \) as

\[ e_m(x) = \sqrt{3} \sin\left(\frac{3}{2} m \pi x\right), \quad \eta_m = \frac{1}{m^2}. \tag{4.2} \]

By the definition of Wiener process \((2.4)\), we have

\[ (\Delta W)_i^n := W(t_{n+1}, x_i) - W(t_n, x_i) = \sum_{m=1}^{\infty} \frac{\sqrt{3}}{m} \sin\left(\frac{3}{2} m \pi x_i\right) \left(\beta_m(t_{n+1}) - \beta_m(t_n)\right). \tag{4.3} \]

In the performance of numerical methods, it is necessary to truncate this infinity sum. Figure 1 displays the value of \( a_m = \frac{\sqrt{3}}{m} \sin\left(\frac{3}{2} m \pi x_i\right) \left(\beta_m(t_{n+1}) - \beta_m(t_n)\right) \) with respect to \( m \). Observe that, after \( m = 200 \), the values of \( a_m \) fluctuate in a small range of zero. Thus we truncate the noise by the sum of the first 200 terms for the following experiments.

And we take the temporal step-size \( \Delta t = 0.001 \) and the spatial meshgrid-size \( \Delta x = \Delta y = \frac{1}{150} \). In order to show the influence of noise on the solution, we scale the value of \( \lambda \) by \( \lambda = 0, \)
\[ \lambda = 0.01, \lambda = 0.05 \text{ and } \lambda = 1, \text{ respectively.} \] Taking the electric field \( E_3 \) for an example, Figure 2 shows the contours until \( t = T \), by using the Method-I corresponding to different scales of the noise. We observe that the contour of electric wave \( E_3 \) has been destroyed due to the increase of the scale of the noise.

Next, we focus on numerically performing the dissipative properties of averaged energy. Based on Theorem 3.1, 3.4 and 3.8 for three numerical methods applied to 3-D stochastic Maxwell equations, we present the concrete form for 2-D case (4.1) respectively.

(1) Method-I

\[
\mathcal{E}(\Phi^{I}(t_{n+1})) = \mathcal{E}(\Phi^{I}(t_{n})) + 2\lambda \Delta x \Delta y \sum_{i,j} \mathcal{E}\left( \Upsilon^{n+\frac{1}{2}}_{i,j} (\Delta W)^{n}_{i} \right),
\]

where

\[
\Phi^{I}(t_{n+1}) = \Delta x \Delta y \sum_{i,j} \left[ \left( (E_3)^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} \right)^2 + \left( (H_1)^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} \right)^2 + \left( (H_2)^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} \right)^2 \right]
\]

and

\[
\Upsilon^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} = (H_1)^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} + (H_2)^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} - (E_3)^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}}.
\]

(2) Method-II

\[
\mathcal{E}(\Phi^{II}(t_{n+1})) = \mathcal{E}(\Phi^{II}(t_{n})) + \lambda \Delta t \Delta x \Delta y \mathcal{E}\left( \sum_{i,j} \Upsilon^{n}_{i,j} (W^{n}_{i} - W^{n-1}_{i}) \right)
\]
Figure 2: Contours of the $E_3$ for different sizes of noise $\lambda = 0$, $\lambda = 0.01$, $\lambda = 0.05$ and $\lambda = 1$.

Figure 3: The probability of density function of solution in the sense of $L^2$. 
where
\[
\Phi^{[II]}(t_{n+1}) = \Delta x \Delta y \sum_{i,j} \left( (E_3)_{i,j}^{n+1} (E_3)_{i,j}^n + (H_1)_{i,j}^{n+1} (H_1)_{i,j}^n + (H_2)_{i,j}^{n+1} (H_2)_{i,j}^n \right)
\]
and
\[
\Upsilon_{i,j}^n = (H_1)_{i,j}^n + (H_2)_{i,j}^n - (E_3)_{i,j}^n.
\]

(3) Method-III
\[
E\left( \Phi^{[III]}(t_{n+1}) \right) = E\left( \Phi^{[III]}(t_n) \right) + 2\lambda \Delta t \Delta x \Delta y \sum_{i,j} E\left( \Upsilon_{i,j}^{n+\frac{1}{2}} (\Delta W)_i^n \right),
\]
where
\[
\Phi^{[III]}(t_{n+1}) = \Delta x \Delta y \sum_{i,j} \left[ \left| (E_3)_{i,j}^{n+1} \right|^2 + \left| (H_1)_{i,j}^{n+1} \right|^2 + \left| (H_2)_{i,j}^{n+1} \right|^2 \right],
\]
and
\[
\Upsilon_{i,j}^n = (H_1)_{i,j}^{n+\frac{1}{2}} + (H_2)_{i,j}^{n+\frac{1}{2}} - (E_3)_{i,j}^{n+\frac{1}{2}}.
\]

Figure 4 presents the simulation of energies using the proposed methods in Section 3, where the blue lines denote energies along 100 trajectories respectively, and the red lines represent the averaged energy using Monte-Carlo method. It is shown that the averaged energy (red line) is linear growth with respect to the time, which coincides with the continuous case, see Theorem 2.1. It also extends the theoretical results for the estimation of the averaged energy in Section 3, since Theorem 3.2 says that for time-dependent noise, the averaged energy evolutes linearly and Theorem 3.9 states that for Method-III, the averaged energy evolutes at most linearly. Meanwhile, from Figure 3 which investigates the probability density function of random variable \( \max_n \sum_{i,j} \left( (E_3)_{i,j}^n + (H_1)_{i,j}^n + (H_2)_{i,j}^n \right) \), we may observe that the averaged energies for three methods are all bounded.

Finally, we consider the numerical simulation for the discrete conservation law of averaged divergence. Since the first two components of electric field \( \mathbf{E} \) are zero for 2-D system (4.1), which means that the averaged divergence-preserving property holds naturally. We consider that property of magnetic field \( \mathbf{H} \) in the following. The definitions of the corresponding discrete divergences are given as following:

(1) Method-I
\[
\nabla^{[I]}_{i,j} \cdot \mathbf{H}^n = \frac{1}{\Delta x} \left( 2(H_1)_{i+1,j}^n + (H_1)_{i+1,j+1}^n + (H_1)_{i+1,j-1}^n - 2(H_1)_{i,j}^n + (H_1)_{i-1,j+1}^n + (H_1)_{i-1,j-1}^n \right)
\]
\[
+ \frac{1}{\Delta y} \left( 2(H_2)_{i,j+1}^n + (H_2)_{i+1,j+1}^n + (H_2)_{i-1,j+1}^n - 2(H_2)_{i,j}^n - (H_2)_{i+1,j-1}^n - (H_2)_{i-1,j-1}^n \right),
\]
Figure 4: The averaged energy by the Method-I (left), Method-II (middle) and the Method-III (right) for $\lambda = 0.1$.

(2) Method-II
\[ \nabla_{i,j}^{[II]} \cdot H^{n+1}_i = \frac{1}{4\Delta x} ((H_1)_i^{n+1}_i - (H_1)^n_i - (H_1)^n_{i-1,j} + (H_1)^{n+1}_{i-1,j}) \]
\[ + \frac{1}{4\Delta y} ((H_2)_i^{n+1}_j - (H_2)^n_i - (H_2)^n_{i,j-1} + (H_2)^{n+1}_{i,j-1}), \]

(3) Method-III
\[ \nabla_{i,j}^{[III]} \cdot H^n_i = \frac{1}{2\Delta x} ((H_1)_i^{n+1}_i - (H_1)^n_i - (H_1)^n_{i-1,j}) + \frac{1}{2\Delta y} ((H_2)_i^{n+1}_j - (H_2)^n_i - (H_2)^n_{i,j-1}). \]

We numerically perform the error of divergence by Monte Carlo method, which is defined by

\[ \text{Error of divergence} = \max_{1 \leq n \leq N} \sum_i \sum_j \left| \frac{1}{P} \sum_{s=1}^{P} \left( \nabla_{i,j} \cdot H^{n+1} (\omega_s) - \nabla_{i,j} \cdot H^n (\omega_s) \right) \right| \Delta x \Delta y. \]

The error results for three methods are displayed in Figure 5. Observe that the scale of the error here is only of $10^{-3}$ for $P = 100$. This may be because the convergence rate of Monte Carlo method is $O(\frac{1}{\sqrt{P}})$, with $P$ being the total number of paths. Thanks to the special structure of the error of divergence, which means they can be rewritten as the difference of Wiener increments, i.e.

\[ \mathcal{E}(\nabla_{i,j}^{[I]} \cdot H^{n+1} - \nabla_{i,j}^{[I]} \cdot H^n) = \frac{2\lambda}{\Delta x} \mathcal{E}(\Delta W^n_i - \Delta W^{n+1}_i) = 0. \]  

(4.4)
Figure 5: The error of averaged divergence of the Method-I (left), Method-II (middle) and the Method-III (right) for $\lambda = 0.01$ $P = 100$.

\[
\mathcal{E}(\nabla_{i,j}^{[\text{II}]} \cdot H^{n+\frac{1}{2}} - \nabla_{i,j}^{[\text{II}]} \cdot H^{n-\frac{1}{2}}) = \frac{\lambda}{2\Delta x} \mathcal{E}(\Delta W_{i+1}^n + \Delta W_{i+1}^{n-1} - \Delta W_i^n - \Delta W_{i-1}^{n-1}) = 0. \tag{4.5}
\]

\[
\mathcal{E}(\nabla_{i,j}^{[\text{III}]} \cdot H^{n+1} - \nabla_{i,j}^{[\text{III}]} \cdot H^n) = \frac{\lambda}{2\Delta x} \mathcal{E}(\Delta W_{i+1}^n - \Delta W_i^{n-1}) = 0. \tag{4.6}
\]

We can utilize the right-hand sides of the above equalities to perform the influence of the number of paths $P$ without solving the equations themselves directly. Take $\lambda = 0.1$ in Method-I for an example. We take the number of trajectories $P = 10, 10^2, 10^3, 10^4, 10^5, 10^6, 10^7, 10^8$ respectively to obtain the corresponding values of $\frac{2\lambda}{\Delta x} \sum_{s=1}^{P} (\Delta W_i^n(\omega_s) - \Delta W_i^{n-1}(\omega_s))$, which represent the error of averaged divergence. From the numerical result, we know that the global residuals of the discrete averaged divergence become smaller and smaller with the increasing of the number of trajectories $P$.

5 Concluding Remarks

In this paper, we firstly studied some properties of continuous system of stochastic Maxwell equations driven by an additive noise. By using a direct approach, we rewrite stochastic Maxwell equations into the form of stochastic Hamiltonian PDEs, and we show that they preserve stochastic multi-symplectic structure almost surely. Furthermore, it is shown that the averaged energy increases linearly with respect to the evolution of time, and divergence is preserved in the sense of expectation.
Secondly, we proposed and analyzed three stochastic multi-symplectic numerical methods to discretize stochastic Maxwell equations with additive noise. Our start point is that in deterministic model, for lossless media, energy is a conserved quantity and divergence is free with no free charges or currents. They are important criteria to evaluate a numerical method is good or not. As is shown in continuous stochastic case, the averaged energy evolves linearly with the growth of time which is caused by random source, and the divergence is preserved in the sense of expectation which means electric flux and magnetic flux are preserved in Gaussian random fields in the statistical sense. It is meaningful to investigate the preservation of these physical properties by the three numerical methods. We showed that the three numerical methods conserve the corresponding versions of dissipative properties of the averaged energy, and the discrete averaged energies evolve at most linearly with respect to time. Furthermore, the three methods preserve the conservation law of the discrete divergence in the sense of expectation.

At last, some numerical experiments are performed to support our theoretical results. To truncate the infinite-dimensional Wiener process, which might be represented as an infinite summation of a sequence, we display the values of the sequence with respect to indices. We observe that for small noise, the electric and magnetic waves are not strongly perturbed, but when the noise level is higher and apparently the waves are destroyed. Furthermore, special attentions are needed to pay to the performance of Method-I, since the condition number of its iterates matrix is poorer than that of Method-II and Method-III, we utilize GMRES to deal with this problem. As for Method-II, it is a three-layer method, which needs another method to initialize, while the evolution of the discrete averaged energy is supported better in theoretical than Method-I and Method-II.
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