Superfluid in a running lattice

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Quantum phase transitions are among the most fantastic phenomena in Nature. Here we show that observing them in different inertial frames may lead to new quantum phases through novel quantum phase transitions. We demonstrate this new effect by studying the Superfluid (SF)-Mott transitions of interacting bosons in a square lattice in a frame moving with a constant velocity $c$. We develop effective actions in the moving frame, then explore them by applying various methods such as field theory renormalization group, charge-vortex duality and scaling analysis. For the SF-Mott transition with the dynamic exponent $z = 1$, the charge conjugation $C$ symmetry, the parity $P$ and an emergent "Lorentz" invariance in the lab frame, the new phase in the moving frame is a boosted superfluid (BSF) carrying a finite momentum which breaks the $CP$ symmetry spontaneously. The transition splits into three new QPTs with $z = 1, (z_x, z_y) = (3/2, 3)$ and $z = 2$ in the moving frame. For the SF-Mott transition with the dynamic exponent $z = 2$ and an emergent Galileo invariance in the lab frame, The new phase in the moving frame is still a boosted superfluid (BSF). The transition splits into two new QPTs with $z = 2$ and $(z_x, z_y) = (3/2, 3)$ in the moving frame respectively. The intrinsic interplay between the Galileo transformation and symmetry breaking in many body quantum systems are analyzed. Contrast to the Doppler shifts in a relativistic quantum field theory and Unruh effects in an accelerating observer are made. Experimental detections in the moving frame are also discussed. The methods can be extended to all the quantum or topological phase transitions in any dimension. Doing measurements in a moving frame becomes an effective way to tune various quantum and topological phases through novel phase transitions.

I. INTRODUCTION

Quantum phase transitions (QPT) is one of the most fantastic phenomena in Nature\textsuperscript{1,3}. For example, Superfluid to Mott transitions\textsuperscript{4,6}, Anti-ferromagnetic state to Valence bond solid transition\textsuperscript{7,8}, magnetic states to quantum spin liquid transition\textsuperscript{9,15}, or quantum Hall (QH) to QH or insulator transitions\textsuperscript{16-20}, topological phase transitions of non-interacting fermions or interacting systems\textsuperscript{21-24} are among the most popular QPTs. In materials, one usually manipulate or control a system by applying a magnetic field, electric field, change doping, making a twist, adding a strain or pressure, to tune the system to go through various QPTs. Ultra-cold atoms loaded on optical lattices can provide unprecedented experimental systems for the quantum simulations and manipulations of some quantum phase and phase transitions. For example, the Superfluid to Mott (SF-Mott) transitions have been successfully realized by loading ultracold atoms in various optical lattice\textsuperscript{23-27}. However, due to the charge neutrality, it is difficult to tune the cold atom systems by applying a magnetic field or electric field\textsuperscript{25}. Here, we study the same QPTs observed in a moving inertial system and show that it leads to new quantum phase through novel quantum phase transition. It also demonstrates that doing measurements in a moving frame becomes a easy and effective way to realize various quantum and topological phases and to tune phase transitions.

We take the boson-Hubbard model of interacting bosons at integer fillings in a square lattice as the simplest example to show a QPT in a lab frame Fig:\ref{fig1}:

$$H_{BH} = -t \sum_{\langle ij \rangle} (b^\dagger_i b_j + h.c.) - \mu \sum_i n_i + \frac{U}{2} \sum_i n_i (n_i - 1)$$

(1)

which displays a SF-Mott transition around $t/U \sim 1$. Obviously, the very existence of the lattice breaks the Galileo and Lorentz invariance. It is also responsible for the existence of the Mott insulating state and the SF-Mott transition. This is in sharp contrast to the Helium4\textsuperscript{32,34} or Quantum Hall systems\textsuperscript{16,17,20} which respects the Galileo invariance. Its phase diagram\textsuperscript{4} is shown in Fig:\ref{fig8}.

Here we first focus on the 2d superfluid (SF) to Mott transitions\textsuperscript{4,6,23,27} in Fig\ref{fig8} with the dynamic exponent $z = 1$, then study the one with $z = 2$ in Fig\ref{fig8} in Sec.X. The effective action consistent with all the symmetries is:

$$S_L = \int d\tau d^2r [i|\partial_r \psi|^2 + v_r^2|\partial_x \psi|^2 + v_y^2|\partial_y \psi|^2 + r|\psi|^2 + u|\psi|^4 + \ldots ]$$

(2)

Where $\psi$ is the complex order parameter which has The effective mass $r$ tunes the SF-Mott transition with $z = 1$. 

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r > 0, ⟨ψ⟩ = 0 is in the Mott state which respects the U(1) symmetry, r < 0, ⟨ψ⟩ ̸= 0 is in the SF state which breaks the U(1) symmetry.

After the scaling away the v_{x}, v_{y}, it has an emergent Lorentz invariance whose characteristic velocity is the intrinsic velocity v_{x}, v_{y} instead of the speed of light c_{L} in the relativistic quantum field theory. The Lorentz transformation reduces to the Galileo transformation when v/c_{L} ≪ 1. It has a Time-reversal symmetry T. Even more importantly, it also has a particle-hole (PH) (because it is called charge conjugation (C) symmetry in the relativistic quantum field theory. So we adopt this notation in the following) under ψ(⃗x, t) ↔ ψ^{*}(⃗x, t) which dictates the particle spectrum is related to that of a hole by ω_{+}(k) ↔ ω_{-}(−k).

The lattice breaks the Galileo invariance and is static in the lab frame. Then one try to detect these SF-Mott transitions in a frame which is moving with the velocity ⃗v = cv along the y-axis with respect to the lab frame. This moving frame could be a fast-moving train or space-craft/satellite in the space. To address what will an observer in the moving frame see, one just perform a Galileo transformation (for notational simplicity, we drop the y-axis with respect to the lab frame see, one just perform a Galileo transformation instead of the speed of light c, because it is called charge conjugation (C) symmetry in the relativistic quantum field theory. If an object moves in a superfluid at a velocity below the critical velocity v < v_{c}, there is a first-order transition to a normal state [35]. More recent studies show that there could be a narrow window of gapless superconductors and superfluids in Fermi systems at low temperatures [36].

Before starting to analyze Eq.3, it is important to distinguish the 4 different cases. The case (1-3) have been discussed in some previous literatures. The case (4) is the main goal to be achieved in this manuscript.

(1) Controlling the moving object: An impurity moving in a superfluid. It was discussed in [32, 34] and more recently in [35]. If an object moves in a superfluid at T = 0 with a velocity below the critical velocity v < v_{c}, there is no viscosity. However, when v > v_{c}, a viscosity arises due to the emission of elementary excitations such as vortex rings [32, 36].

(2) Controlling the superfluid: The SF is flowing with a velocity v. It was discussed in [32, 34] and more recently in [37]. As shown in [37], the flow of a SF with v > v_{c} may not destroy SF, but the order parameter develops small additional components at the critical momentum, therefore reduce the superfluid density. However, when increasing v further, the fate of SF is still not known yet. This story was also reviewed in Appendix D by microscopic calculations such as Bogoliubov method and Galileo transformation.

(3) Controlling the supercurrent in a superconductor: There are similar phenomena in fermionic systems such as a gapped BCS superconductor: when the super-current is below a critical value j < j_{c1} ~ Δ/k_{F} where Δ is the gap and k_{F} is the Fermi momentum, there is no resistance, but when j > j_{c1}, there is a first-order transition to a normal state [38]. More recent studies [39] show that there could be a narrow window of gapless superconductors...
when \( j_{c1} < j < j_{c2} \sim e/2j_{c1} \) before it finally turns to normal at \( j > j_{c2} \). For gapless \( p^- \) wave such as He3-A phase \[40\] or \( d^- \) wave superconductors such as high temperature cuprate superconductors, it was believed any supercurrent drives it to a gapless superconductor, so \( j_{c1} = 0 \) \[41\], but there is still not a complete theory yet. This part is on fermions, so will be discussed in a separate work on moving topological superfluid \[42\].

(4) Controlling the ( optical ) lattice: The optical lattice is at rest in the lab frame, the ultra-cold atoms are loaded on top of it Fig.[4], but is observed in a moving frame ( Fig.[12] [7] ). This is different from the case (1), because the optical lattice is a macroscopic background instead of an moving object. This is different from the case (2) also, because we are controlling the optical lattice instead of the SF. The status of the Mott and the SF need to be determined by the Hamiltonian Eq.[1] This question is to be addressed in the present work. We are particularly interested in how the Mott and SF around \( U/t \sim 1 \) respond to the boost, especially when it is beyond a critical velocity. We develop symmetry based effective actions to achieve such a goal which is summarized in Fig.2 and 7.

We show that the quantum phase and phase transitions observed in the moving frame display quite different phenomena than those observed in the lab frame. For \( z = 1 \) case, when \( c \) is below a critical velocity, the Mott state and the SF state remain the same, but their excitation spectrum suffer a Doppler shift. By performing a field theory renormalization group (RG) developed for non-relativistic quantum field theory up to two loops, we show that the universality class from the Mott to the SF tuned by the ratio of the kinetic energy over the interaction at a fixed \( c \) below the critical velocity remains the same as the 3D XP class with \( z = 1 \). It is the CP symmetry of the charge conjugation (C ) ( also called the particle-hole ) symmetry and the parity (P) which dictates the result holds exactly to all loops. However, in the SF side, when \( c \) increases above the critical velocity, the SF becomes a boosted SF (BSF) carrying a finite momentum which spontaneously breaks the CP symmetry; the transition from the SF to BSF has an exotic, but exact dynamic exponent \( z_x = 3/2, 3 \). There are both Doppler shifted Goldstone and Riggs mode in the SF phase, but only Doppler shifted Goldstone mode inside the BSF phase. The Riggs mode disappears in the BSF phase due to the spontaneously broken CP symmetry. In the Mott side, when \( c \) increases above another critical velocity, the Mott phase also turns into the BSF phase with the dynamic exponent \( z = 2 \) and a Type-II dangerously irrelevant operator (DIOL). Charge-vortex duality transformation in the moving frame is also performed to study these new quantum phase transitions in the moving frame. Finite temperature properties of quantum or classical scaling functions of various physical quantities, especially their dependencies on the velocity \( c \) are explored. The intrinsic interplay between the symmetry breaking and the Galileo transformation are analyzed. Contrast to the Doppler shifts in a relativistic quantum field theory and Unruh effects in an accelerating observer is made. Crucial differences and connections between the Galileo invariance in single particle Schrodinger equation and many-body systems with emergent symmetry and symmetry breaking in the thermodynamic limit are discussed. For \( z = 2 \) case, the CP symmetry was explicitly broken, the QPTs from the Mott to BSF with the dynamic exponent \( z = 2 \) and the SF to BSF with \( z_x = 3/2, 3 \) still exist in the moving frame, but the Mott to the SF with \( z = 1 \) line is absent. Doing measurements in a moving frame becomes a easy and effective way to tune various quantum and topological phases and phase transitions. Experimental detections in the moving are also discussed. Our effective action methods can be extended to study all the quantum or topological phase transitions in any dimension. As expected, our results in the SF side is consistent with those achieved by microscopic perturbation calculations in (2) when the boost is below the critical velocity. But our methods can be applied to go beyond the critical velocity by finding the new phases and novel QPTs leading to these new phases.

II. THE GLOBAL PHASE DIAGRAM OF \( z = 1 \) IN A MOVING FRAME

Expanding Eq.[3] leads to

\[
S_M = \int d\tau d^3r [\partial_\tau \psi |^2 - i2c\partial_\tau \psi^* \partial_y \psi + v_y^2 |\partial_x \psi|^2 + (v_y^2 - c^2) |\partial_y \psi|^2 + a|\partial_y^2 \psi|^2 + b|\partial_y \partial_x \psi|^2 + r|\psi|^2 + u|\psi|^4 + \cdots ] \quad (4)
\]

where the higher derivative or higher order terms are not important in the lab frame \[43\], but may become important in the moving frame, especially near the new quantum phase transitions in Fig.2. When writing in terms of the metric \( g_{\mu\nu} \partial_\mu \psi^* \partial_\nu \psi \) in \( \tau, y \) space-time in Eq.2 it is \( g_{\tau,\tau} = 1, g_{\tau,y} = g_{y,\tau} = -ic, g_{y,y} = v_y^2 - c^2 \). The crossing ( or off-diagonal ) metric component \( g_{\tau,y} \) is the only new term in the moving frame generated by the Galileo transformation which does not appear in Eq.2 in the lab frame. It is this new term which plays important roles in the moving frame. Because the boost \( c \) adds a new tuning parameter, both \( \gamma = v_y^2 - c^2 \) and \( \mu \) can change sign and tune various QPTs in Fig.2. Taking the mean field ansatz \( \langle \psi \rangle = \sqrt{\rho} e^{i(\phi + k_0 y)} \) leads to the energy density

\[
E[\rho, k_0] = [(v_y^2 - c^2) k_0^2 + ak_0^4 + r] \rho + u \rho^2
\]  

(5)
Minimizing $E[\rho, \phi, k_0]$ with respect to $\rho$ and $k_0$ results in

$$k_0^2 = \begin{cases} 
0, & c^2 < v_y^2, \\
\frac{c^2 - v_y^2}{2u}, & c^2 > v_y^2.
\end{cases}$$

$$\rho = \begin{cases} 
0, & r > 0 \text{ and } c^2 < v_y^2, \\
-\frac{r}{2u}, & r < 0 \text{ and } c^2 < v_y^2, \\
-\frac{r}{2u} + \frac{(c^2 - v_y^2)^2}{8u}, & r > 0 \text{ and } c^2 > v_y^2, \\
0, & r < 0 \text{ and } c^2 > v_y^2.
\end{cases}$$

(6)

The phase diagram of the effective action Eq.6 with $z = 1$ as a function of $r$ and $c$ in the moving frame. $c = 0$ axis recovers that in the lab frame. The two tuning parameters are the effective mass $r$ and the boost $c$. The Mott, SF and Boosted SF (BSF) phase meet at the multi-critical point $M$. The BSF is the new phase which spontaneously breaks the CP symmetry. The Goldstone modes exist in both the SF and BSF, but the Riggs mode exists only in the SF protected by the CP symmetry, disappears in the BSF phase due to its spontaneously breaking of the CP symmetry. The three QPTs along the three paths I with $z = 1$, II with $z = (3/2, 3)$ and IIIa or IIIb with $z = 2$ are presented in the following sections. Exchanging the role of the lab and moving frame does not change the results, because both are related by Galileo transformation anyway.

The phase diagram is summarized in Fig.2. Due to the CP symmetry inside the Mott phase, $\omega_+ (\vec{k}) = \omega_- (-\vec{k})$, one can look at the instability from either particle or hole band $\omega_{\pm} = \sqrt{v_x^2 k_x^2 + v_y^2 k_y^2 \pm ck_y}$. In the sections III-V, we will discuss the three QPTs along the three path III-III in Fig.2 respectively. Then by employing the field theory renormalization group developed to study non-relativistic QFT, we will study the universality class from the Mott to SF transition along path I in Sec.VI. We will perform charge-vortex duality transformation in the moving frame along the path I in Sec.VII. We derive the scaling functions of various physical quantities along the three paths in Sec. VIII. Contrasts to relativistic Doppler effects, Unruh effects by an accelerating observer, Helium 4 phase diagram, also discussions on experimental detections in a moving frame are made in Sec.IX. In Sec.X, we present the $z = 2$ case in a moving frame. Conclusions and perspectives are presented in Sec.X. The Hamiltonian formalism is given in Appendix A which is complementary to the Lagrangian formalism. The breaking of Galileo invariance by a spin-orbit coupling is analyzed in Appendix B. The Galileo invariance in a single particle Schrodinger equation and its evolution into many body interacting systems in the thermodynamic limit are explored in Appendix C. The microscopic perturbation approach to Galileo transformation in a moving SF is presented in Appendix D.

III. THE MOTT TO SF TRANSITION ALONG THE PATH I WITH $z = 1$.

Along the path I in Fig.1b, at the mean-field level, we can substitute $\psi \rightarrow \sqrt{\rho_0} e^{i\phi_0}$ into the boosted effective action Eq.63

$$S = r\rho_0 + u\rho_0^2$$

(7)

When $r > 0$, it is in the Mott phase with $\langle \psi \rangle = 0$. When $r < 0$, it is in the SF phase with $\langle \psi \rangle = \sqrt{\rho_0} e^{i\phi_0}$ where $\rho_0 = -r/2u$ and $\phi_0$ is a arbitrary angle due to the $U(1)$ symmetry.

(a) The Mott state:
In the Mott phase, \( r > 0 \), one can write \( \psi = \psi_R + i\psi_I \) as its real part and imaginary part and expand the action up to second order

\[
S_{M1} = \int d\tau d^2r \sum_{\alpha=R,I} \left[ (\partial_\tau \psi_\alpha - ic\partial_y \psi_\alpha)^2 + v_x^2(\partial_x \psi_\alpha)^2 + v_y^2(\partial_y \psi_\alpha)^2 - \mu(\psi_\alpha)^2 \right]
\]  

which lead to 2 degenerate gapped modes with the effective mass \( r > 0 \):

\[
\omega_{R,I} = \sqrt{r + v_x^2 k_x^2 + v_y^2 k_y^2 - c k_y}
\]

which indicates the dynamic exponent \( z = 1 \).

(b) The SF phase:

In the SF phase, \( r < 0 \), we can write the fluctuations in the polar coordinates \( \psi = \sqrt{\rho_0 + \delta \rho e^{i(\phi_0 + \delta \phi)}} \) and expand the action up to the second order in the fluctuations:

\[
S = \frac{1}{2\rho_0} \int d\tau d^2r \left[ \left( \partial_\tau - ic\partial_y \right) \delta \rho \right] \] 

\[
+ \rho_0^2 \left[ \left( \partial_x + iv_x \partial_y \right) \delta \phi \right]^2 + \rho_0^2 \left[ \left( \partial_y \right) \delta \phi \right]^2
\]

which due to the CP symmetry dictating \( z = 1 \), leads to one gapless Goldstone \( \delta \phi \) mode and one gapped Riggs \( \delta \rho \) mode:

\[
\omega_G = \sqrt{v_x^2 k_x^2 + v_y^2 k_y^2 - c k_y}
\]

\[
\omega_H = \sqrt{4\rho_0 u + v_x^2 k_x^2 + v_y^2 k_y^2 - c k_y}
\]

Note that it is the CP symmetry which ensures the separation of the real part from the imaginary part when \( r > 0 \) in the Mott phase in Eq.9 and the separation of the Riggs mode from the Goldstone mode when \( r < 0 \) in the SF phase in Eq.12. Intuitively, one can say the two degenerate gapped modes in Eq.9 turn into the Goldstone mode and the Riggs mode in Eq.12 through the \( z = 1 \) QPT from the Mott phase to the SF phase.

(c) The QCP: a boosted SF-Mott transition with \( z = 1 \): RG analysis

If putting \( c = 0 \) in the effective action Eq.3 it is nothing but a 3D XP universality class with the critical exponents \( \nu = 0.67, \eta = 0.04 \) which is emergent Lorentz invariant. The \( z = 1 \) is protected by the Lorentz invariance at \( c = 0 \). The interaction \( u \) term is relevant at \( c = 0 \) and controlled by the 3D XP fixed point. Any \( c > 0 \) breaks Lorentz invariance. So the action is neither Lorentz invariant nor Galileo invariant. The \( c \) term is marginal suggesting a line of fixed points. The RG flow of \( u \) along the fixed line is determined by RG calculations in Sec.VI. It was shown that despite of the lack of Lorentz invariance at \( c \neq 0 \), the CP symmetry still detects the dynamic exponent \( z = 1 \). The Mott to the SF transition at \( c \neq 0 \) remains in the 3D XP universality class.

IV. THE SF TO BOOSTED SF TRANSITION ALONG PATH II WITH \((z_s, z_y) = (3/2, 3)\).

Inside the SF phase at a fixed \( r < 0 \), as \( c \) increases along the path II in Fig.1b, there is a quantum Lifshitz transition from the SF phase to the BSF driven by the boost of the Goldstone mode in Eq.11. Because the gapped Riggs mode remains un-critical across the transition, one can simply drop it. In fact, as shown below, the Riggs mode disappears in the BSF side due to the explicit CP symmetry breaking inside the BSF. Although the Goldstone mode to the quadratic order in Eq.11 is enough inside the SF phase, when studying the transition to the BSF, one must incorporate higher derivative terms and also higher order terms in Eq.3 to the Goldstone mode in Eq.11. A simple symmetry analysis leads to the following bosonic quantum Lifshitz transition from the SF to BSF in terms of the phase degree of freedom ( which can also be derived by substituting \( \psi = \sqrt{\rho_0 + \delta \rho e^{i(\phi_0 + \delta \phi)}} \) into Eq.3 then integrating out the Riggs mode \( \delta \rho \) )

\[
S_{SF-BSF} = \int d\tau d^2r \left[ (\partial_\tau \phi - ic\partial_y \phi)^2 + v_x^2(\partial_x \phi)^2 + v_y^2(\partial_y \phi)^2 + a(\partial_x^2 \phi)^2 + b(\partial_y \phi)^2 \right]
\]  

where \( a, b > 0 \) terms come from those in Eq.3, especially \( \gamma = v_y^2 - c^2 \) is the tuning parameter which drives the quantum Lifshitz transition from the SF phase to the BSF phase [46]. A simple scaling shows that when \( z = 1 \) inside the SF phase \( [a] = -2, [b] = -3 \), so they are irrelevant inside the SF phase, but become important near the SF-BSF transition as to be shown in the following.
The mean-field state can be written as \( \phi = \phi_0 + k_0 y \). Substituting it to the effective action Eq.12 leads to:
\[
S_0 \propto \left( v_y^2 - c^2 \right) k_0^2 + bk_0^4
\] (13)

At a low boost \( c^2 < v_y^2 \), \( k_0 = 0 \) is in the SF phase. At a high boost \( c^2 > v_y^2 \), \( k_0^2 = (c^2 - v_y^2) / 2b \) is in the BSF phase with the modulation \( k_0 \) along the \( y \)-axis. Note that the numerical value of \( k_0 \) is different than that listed in Eq.10. That is should not disturbing at all, because the former applies near the SF-BSF transition, the latter applies near the \( z = 2 \) Mott-SF transition to be discussed in the next section.

(a) The excitation spectrum in the SF phase

At a low boost \( c^2 < v_y^2 \) inside the SF phase, the quantum phase fluctuation can be written as \( \phi = \phi_0 + \delta \phi \). Expanding the action Eq.12 upto the second order leads to:
\[
S_{SF} = \int d\tau d^2 r \left[ \left( \partial_\tau \phi - ic \partial_y \phi \right)^2 + v_\tau^2 \left( \partial_\tau \phi \right)^2 + v_y^2 \left( \partial_y \phi \right)^2 \right]
\] (14)

which reproduces the gapless Goldstone mode in Eq.12 inside the SF phase:
\[
\omega_k = \sqrt{v_\tau^2 k_\tau^2 + v_y^2 k_y^2 - ck_y}
\] (15)

which is consistent with the Goldstone mode in Eq.12. The Riggs mode was dropped at very beginning, so can not be seen in Eq.14.

(b) The spontaneous CP symmetry breaking in the BSF phase

Due to the exact CP symmetry, \( \pm k_0 \) in Eq.13 are related by the CP symmetry, so it could take \( \pm \) sign or its any linear combination. To determine the ground state, we take the most general mean field ansatz:
\[
\langle \psi \rangle = \sqrt{p} (c_1 e^{ik_0 y} + c_2 e^{-ik_0 y})
\] (16)

with \( |c_1|^2 + |c_2|^2 = 1 \).

Substituting Eq.16 into Eq.1, integration over the space kill the oscillating parts and lead to its energy density:
\[
E[\rho, k_0, c_1, c_2] = \left( (v_y^2 - c^2) k_0^2 + ak_0^4 + r \right) \rho + (1 + 2|c_1|^2|c_2|^2) \rho^2
\] (17)

\( u > 0 \) and \( b > 0 \) dictates the minimization condition \( c_1 c_2 = 0 \). So the ground-state is either \( \langle \psi \rangle = \sqrt{p} e^{ik_0 y} \) or or \( \langle \psi \rangle = \sqrt{p} e^{-ik_0 y} \) which implies the spontaneously breaking of the CP symmetry. We call such a CP symmetry broken SF state the Boosted superfluid (BSF) [13].

(c) The excitation spectrum in the BSF phase

Inside the BSF phase, the quantum phase fluctuations can be written as \( \phi = \phi_0 + k_0 y + \delta \phi \). Expanding the action upto the second order in the phase fluctuations leads to
\[
S_{BSF} = \int d\tau d^2 r \left[ \left( \partial_\tau \phi - ic \partial_y \phi \right)^2 + v_\tau^2 \left( \partial_\tau \phi \right)^2 + \left( v_y^2 + 6bk_0^2 \right) \left( \partial_y \phi \right)^2 \right]
\] (18)

which leads to the gapless Goldstone mode inside the BSF phase:
\[
\omega_k = \sqrt{v_\tau^2 k_\tau^2 + (3c^2 - 2v_y^2) k_y^2 + ak_0^4 - ck_y}
\] (19)

where one can see \( 3c^2 - 2v_y^2 \geq 2(c^2 - v_y^2) + c^2 > c^2 \) when \( c^2 > v_y^2 \), thus the \( \omega_k \) is stable in BSF phase. Due to the spontaneous CP symmetry breaking, the Riggs mode may not even exist anymore in the BSF phase. This result will also be confirmed further in the next section from the Mott to BSF transition.

(d) The exotic QCP scaling with the dynamic exponents \( (z_x = 3/2, z_y = 3) \)

It is instructive to expand the first kinetic term in Eq.12 as:
\[
S = \int d\tau d^2 r \left[ Z \left( \partial_\tau \phi \right)^2 - 2ic \partial_\tau \phi \partial_y \phi + v_\tau^2 \left( \partial_\tau \phi \right)^2 + \gamma \left( \partial_y \phi \right)^2 + a \left( \partial_y^2 \phi \right)^2 + b \left( \partial_\phi \phi \right)^4 \right]
\] (20)

where \( Z \) is introduced to keep track of the renormalization of \( (\partial_\tau \phi)^2 \), \( \gamma = v_y^2 - c^2 \) is the tuning parameter.

The scaling \( \omega \sim k_\tau^3, k_x \sim k_y^2 \) leads to the exotic dynamic exponents \( (z_x = 3/2, z_y = 3) \). Then one can get the scaling dimension of \( [\gamma] = 2 \) which is relevant, as expected, to tune the transition, but \( [Z] = [b] = -2 < 0 \), so both are leading irrelevant operators [10] which determine the finite \( T \) behaviours and corrections to the leading scalings. Setting \( Z = b = 0 \) in Eq.20 leads to the Gaussian fixed action at the QCP where \( \gamma = 0 \). Exotically and interestingly, it is the crossing matrix \( y_{\tau, y} = g_{y, \tau} = -ic \) in Eq.21 which dictates the quantum dynamic scaling near the QCP.
V. THE MOTT TO BOOSTED SUPERFLUID TRANSITION ALONG THE PATH III WITH $z = 2$.

When $k_0 \neq 0$ along the path III in Fig.1b, it is convenient to introduce the new order parameter $\psi = \tilde{\psi} e^{ik_0 y}$, then the original action Eq.11 can be expressed in terms of $\psi$

$$ S = \int dx d^2r \left( |\partial_r \tilde{\psi}|^2 - 2ck_0 \tilde{\psi}^* \partial_r \tilde{\psi} + i2c\partial_r \tilde{\psi}^* \partial_y \tilde{\psi} + v^2_x |\partial_x \tilde{\psi}|^2 + (v^2_y - c^2 + 2ak_0^2)|\partial_y \tilde{\psi}|^2 \right)$$

where $\cdots$ denotes all the possible high-order derivative term.

Setting $i2k_0 (v^2_y - c^2 + 2ak_0^2) \tilde{\psi}^* \partial_y \tilde{\psi} = 0$ leads to $k^2_0 = -\frac{\nu^2 - c^2}{2a}$ which is the same as Eq.16 at $v^2 < c^2$. By using $2ak_0^2 = c^2 - v^2_y$, one can simplify the above action to

$$ S = \int dx d^2r \left[ |\partial_r \tilde{\psi}|^2 - i2c\partial_r \tilde{\psi}^* \partial_y \tilde{\psi} + 2ck_0 \tilde{\psi}^* \partial_r \tilde{\psi} + v^2_x |\partial_x \tilde{\psi}|^2 + 4ak_0^2 |\partial_y \tilde{\psi}|^2 + (r - ak_0^2)|\tilde{\psi}|^2 + u|\tilde{\psi}|^4 \right]$$

where one can observe $k_0 \neq 0$ leads to a linear derivative term $\tilde{\psi}^* \partial_r \tilde{\psi}$. It dictates the dynamic exponent $z = 2$.

(a) Scaling analysis near the $z = 2$ QCP

Simple scaling analysis shows that the first term $|\partial_r \tilde{\psi}|^2$ is irrelevant with scaling dimension $-2$, the second term is the metric crossing term $\partial_r \tilde{\psi}^* \partial_y \tilde{\psi}$ which is irrelevant with the scaling dimension $-1$, the third (linear derivative) term $\tilde{\psi}^* \partial_r \tilde{\psi}$ leads to $z = 2$.

After only keeping the leading irrelevant term which is the metric crossing term, we arrive at the effective action:

$$ S = \int dx d^2r (Z_1 \tilde{\psi}^* \partial_r \tilde{\psi} - iZ_2 \partial_r \tilde{\psi}^* \partial_y \tilde{\psi} + \tilde{\nu}_2 |\partial_x \tilde{\psi}|^2 + \tilde{\nu}_2 |\partial_y \tilde{\psi}|^2 - \tilde{\mu} |\tilde{\psi}|^2 + u|\tilde{\psi}|^4)$$

where $Z_1 = -2ck_0$, $Z_2 = 2c$, $\tilde{\nu}_2 = v^2_x$, $\tilde{\nu}_y = 4ak_0^2 = 2(c^2 - v^2_y)$. It is the effective chemical potential:

$$ \tilde{\mu} = -r + ak_0^2 = -(r - r_c), \quad r_c = ak_0^2 > 0 $$

which tunes the Mott to BSF transition. As shown in Path-IIIa and IIIb, there are two independent ways to tune $\tilde{\mu}$: Vertical path-IIIa, at a fixed $r > 0$, one increases $c$, therefore $k_0$ or Horizontal path-IIIb, at a fixed $\gamma < 0$, one increases $c$.

Now we focus on near the $z = 2$ quantum critical line. Due to $[Z_2] = -1$, $Z_2$ metric crossing term gets to zero quickly under the RG flows, so can be treated very small $Z_2 \ll Z_1$. In the following, we will use this fact to simplify the excitation spectrum in the Mott and BSF phase and also stress the roles of the leading irrelevant operator $Z_2$.

(b) Excitations in the Mott phase:

In the Mott phase, $\tilde{\mu} < 0$ and $\rho_0 = 0$, the excitation spectrum is

$$ \omega = \frac{-\tilde{\mu} + \tilde{\nu}_2 k_x^2 + \tilde{\nu}_y k_y^2}{Z_1 - Z_2 k_y} = \frac{\tilde{\nu}_2 k_x^2 - \tilde{\mu}}{Z_1 - Z_2 k_x} + \frac{(Z_1^2 \tilde{\nu}_2 - Z_2^2 \tilde{\mu})(k_y - k_0)^2}{(Z_1 - Z_2 k_x)^3} + \cdots $$

where $k_0 = \frac{Z_2 \tilde{\mu}}{Z_1 \tilde{\nu}_2}$ vanishes at the phase boundary $\tilde{\mu} = 0$. It indicates the condensation at $k_0$ with the dynamic exponent $z = 2$. When contrasting with Eq.9 near the $z = 1$, one can see it only contains the particle excitation spectrum, while the hole excitation is at much higher energy, so can be dropped. It is the leading irrelevant term $Z_2$ which leads to the shift of the minimum away from the origin.

(c) Excitations in the BSF phase:

In the BSF phase, $\tilde{\mu} > 0$ and $\rho_0 > 0$, the excitation spectrum are:

$$ \omega_\pm = Z_1^{-1} \left[ \pm \sqrt{4u \rho_0 (\tilde{\nu}_2 k_x^2 + (\tilde{\nu}_y + u \rho_0 Z_2^2) k_y^2) + 2u \rho_0 Z_2 k_y} \right] $$

which is always stable inside the BSF phase. $\pm$ corresponds to the P and H excitations respectively. Due to $z = 2$ QCP, the magnitude and phase are conjugate to each other, so the Rigg's mode inside the SF phase in Fig.2 does not exist anymore inside the BSF phase. As argued below Eq.11, this fact is also due to the spontaneous CP symmetry breaking inside the BSF phase.

Eq.20 and Eq.27 show that it is the type-II dangerously irrelevant operator $Z_2$ which leads to the Doppler shift term in the Mott and BSF phase respectively. They can be contrasted to Eq.6 and Eq.19 respectively. So we reach consistent results from the $z = 2$ line and the $z = (3/2, 3)$ line in Fig.2.
VI. FIELD THEORY RG ANALYSIS ON THE MOTT TO SF TRANSITION UNDER THE BOOST ALONG PATH I.

So far, we analyzed the effective action Eq.4 by mean field theory + Gaussian fluctuations. The results may be valid well inside the phases, but will surely break down near the QPTs. It becomes important to study the nature of the QPTs by performing renormalization group (RG) analysis. Unfortunately, the conventional Wilsonian momentum shell method seems inapplicable to study the RG in the moving frame. Very fortunately, the field theory methods developed for non-relativistic quantum field theory in [6, 18, 19] by one of the authors can be effectively applied in the moving frame. Following this method, we will perform the RG to investigate the nature of the QCP from Mott to SF in Fig.2 along the path I in Fig.2. We stress the important roles played by the CP symmetry.

1. RG of the self-energy at two-loops

\[ (4a) = \int \frac{d^4 q}{(2\pi)^4} \int \frac{d\nu}{2\pi} G(q, \nu) = \int \frac{d^4 q}{(2\pi)^4} \int \frac{d\nu}{2\pi} \frac{i}{2g} \left[ \frac{1}{\nu + i\epsilon_+(q)} - \frac{1}{\nu - i\epsilon_-(q)} \right] = \int \frac{d^4 q}{(2\pi)^4} \frac{1}{2g} = 0 \]  

where we first perform the integral over the frequency, then doing dimensional regularization in the momentum space. One can see that at one-loop order, \( c \) does not even appear, so it is identical to the relativistic case.

One loop is trivial. To find a non-vanishing anomalous dimension for the boson field, one must get to two loops.
\[ (4b) = \int \frac{d^d q}{(2\pi)^d} \int \frac{d\nu_0}{2\pi} \int \frac{d^{d} p}{(2\pi)^d} \int \frac{d^{d} \tilde{p}}{2\pi} G(\tilde{q}, q_0) G(\tilde{p}, p_0) G(\tilde{q} + \tilde{p}, k_0 + p_0 - q_0) = \]
\[ \times \left[ \frac{1}{p_0 + i\epsilon_+(\tilde{p})} - \frac{1}{p_0 - i\epsilon_-(\tilde{p})} \right] \frac{1}{k_0 + p_0 - q_0 + i\epsilon_+(\tilde{q} + \tilde{p} - q) - q_0 - i\epsilon_-(\tilde{q})} = \int \frac{d^d q}{(2\pi)^d} \int \frac{d^{d} p}{(2\pi)^d} \frac{1}{2q} \frac{1}{2|k + \tilde{p} - q|} \frac{1}{(k_0 + i\epsilon q_0)^2 + (q + i\epsilon |k + \tilde{p} - q|)^2} \]
\[ = \frac{u^2}{4\pi^2 \epsilon} [(k_0 + i\epsilon q_0)^2 + k^2] + \cdots \]  

where we only list the field renormalization UV divergence and \cdots means the UV finite parts. When one perform the frequency integral, one must pick two poles at the two opposite side of the frequency integral to get a non-vanishing answer. Putting \( c = 0 \) gives back to the relativistic case. Because \( c \) always appears in the combination of \( k_0 + i\epsilon q_0 \), so the UV divergency is identical to that of the \( c = 0 \) case. It gives the identical anomalous dimension to the \( c = 0 \) case. So the dynamic exponent \( z = 1 \) at least to two loops. In fact, we expect that the CP symmetry dictates \( z = 1 \) is exact in Fig.2.

2. RG of the interaction at one-loop

\[ (5a) = \int \frac{d^d q}{(2\pi)^d} \int \frac{d\nu}{2\pi} \int \frac{d^d p}{(2\pi)^d} G(\tilde{q}, \nu) G(\tilde{p} - \tilde{q}, \omega - \nu) = \]
\[ \int \frac{d^d q}{(2\pi)^d} \int \frac{d\nu}{2\pi} \frac{i}{2q} \frac{1}{|\tilde{q} - \nu + i\epsilon_+ (\tilde{q})| \omega - \nu - i\epsilon_-(\tilde{q})} - \frac{1}{|\nu - i\epsilon_-(\tilde{q})| \omega - \nu - i\epsilon_-(\tilde{q})} = \int \frac{d^d q}{(2\pi)^d} \frac{1}{2q} \frac{1}{|\tilde{q} - \nu|} (\omega + i\epsilon q_0)^2 + (q + i\epsilon |\tilde{q} - \nu|)^2 \]
\[ = \frac{u^2}{8\pi^2 \epsilon} + \cdots \]  

where \cdots means the UV finite parts. When one perform the frequency integral, one must pick two poles at the two opposite side of the frequency integral to get a non-vanishing answer. Putting \( c = 0 \) gives back to the relativistic case. Because \( c \) always appears in the combination of \( \omega + i\epsilon q_0 \), so the UV divergency is identical to that of the \( c = 0 \) case. One can get a similar expression in Fig.4(b), (c) cases. So the \( \beta \) function \( \beta(u) \) is identical to the \( c = 0 \) case.

Despite the critical exponent is the same as the \( c = 0 \) case, various physical quantities at a finite \( T \) still depend on \( c \) as calculated in the following.  

3. Finite temperature RG

Following the method developed in \[ \text{[R 18 19]} \], we can also study the RG at a finite temperature. The strategy is that even for a relativistic QFT at \( T = 0 \), any finite temperature breaks the Lorentz invariance, so the imaginary time direction has to be treated separately from the space, the summation over the imaginary frequency has to be performed first before doing the dimensional regularization in the momentum space only.
Now we look at the boson self-energy at one-loop and a finite temperature |\( T > 0 \)|, Eq\(^{29}\) becomes:

\[
(4a)(T > 0) = u \int \frac{d^d q}{(2\pi)^d} \frac{1}{\beta} \sum_{\nu} \frac{1}{2q} \left[ \frac{1}{i\nu - \epsilon_+(q)} - \frac{1}{i\nu + \epsilon_-(q)} \right]
\]

\[
= u \left[ \int \frac{d^d q}{(2\pi)^d} \frac{1}{2q} + 2 \int \frac{d^d q}{(2\pi)^d} \frac{1}{2q} e^{i\tau_\nu(q)} \right]
\]

\[
= 2u(k_B T)^2 \int \frac{d^d q}{(2\pi)^d} \frac{1}{2q} e^{i\tau_\nu(q) - 1}, \quad d = 3
\]

where we evaluate the last line at the upper critical dimension \( d_u = 3 \), drop the first term which is the \( T = 0 \) result listed in Eq\(^{29}\) So \( c \) does appear at any \( T > 0 \). Setting \( c = 0 \) recovers the result in the lab frame \( u(k_B T)^2/2\pi^2 \int_0^\infty \frac{d \nu}{\epsilon_\nu} = u(k_B T)^2/24 \). The interaction \( u \) is marginally irrelevant at \( d_u = 3 \) and will lead to logarithmic corrections to Eq\(^{32}\). For \( d = 2 < d_u = 3 \), the integral in Eq\(^{32}\) is IR divergent, this is expected because the Gaussian fixed point flows to Wilson-Fisher fixed point, then it goes to the \( d = 2 \) scaling analysis at Sec.VII.

Fig\(^{3b}\) can also be similarly evaluated at a finite \( T \), it is evaluated in Eq\(^{30}\) at \( T = 0 \) where the pole structure in the six terms considerably simplify the final UV divergent answer. But at a finite \( T \), all the six terms contribute due to the boson distribution factor. The cancellation of the \( T = 0 \) UV divergence in Eq\(^{30}\) by the counter-terms leads to a finite answer at \( T > 0 \).

### VII. THE CHARGE - VORTEX DUALITY ALONG THE PATH I

So far, we analyzed the effective action Eq\(^{41}\) by mean field theory + Gaussian fluctuations. Here we will study it by non-perturbative duality transformation. It was well known that there is a charge-vortex duality at \( z = 1 \) case in the lab frame \( \widehat{\mathbf{Q}}^{54 56} \). The Mott insulating phase is due to the condensation of vortices. The charge - vortex duality in the moving frame provides a non-perturbative proof of \( z = 1 \) from the Mott to SF transition along the path I when \( c < v_y \) tuned by \( r \).

1. The conserved Noether current in the moving frame

The global \( U(1) \) symmetry \( \psi \to \psi e^{i\chi} \) leads to the conserved Noether current \( \tilde{J}_\mu = (J_r, J_x, J_y) \) in Eq\(^{35}\)

\[
J_r = i(\psi^* \partial_\tau \psi - \psi \partial_\tau \psi^*) = i[(\psi^* \partial_\tau \psi - \psi \partial_\tau \psi^*) - ic(\psi^* \partial_y \psi - \psi \partial_y \psi^*)]
\]

\[
J_x = iv^2(\psi^* \partial_\tau \psi - \psi \partial_\tau \psi^*)
\]

\[
\tilde{J}_y = iv^2(\psi^* \partial_y \psi - \psi \partial_y \psi^*) - icJ_r = J_y - icJ_r
\]

which also show the current along the \( y \) direction \( \tilde{J}_y \) is the sum of the intrinsic one \( J_y \) and the one due to the boost \(-icJ_r \). They satisfy

\[
\partial_\tau J_r + \partial_x J_x + \partial_y (J_y - icJ_r) = 0
\]

which is equivalent to Eq\(^{33}\). This equivalence gives the physical meaning of the particle-hole 3-currents \( J_\mu = (J_r, J_x, J_y) \) introduced in the charge-vortex duality.

Now we evaluate the mean field 3-currents in all the three phases in Fig\(^2\). Plugging in \( \langle \psi \rangle = \sqrt{\rho_0} e^{i\phi_0 y} \) into Eq\(^{33}\) leads to:

\[
J_r = i2k_0 c \rho_0,
\]

\[
J_x = 0
\]

\[
\tilde{J}_y = 2k_0 (c^2 - v_y^2) \rho_0
\]

Near the \( z = 2 \) line in Fig\(^2\) \( k_0 = \pm \sqrt{\frac{c^2 - v_y^2}{2b}} \), so the BSF near the \( z = 2 \) line starts to carry both the density \( J_r \) and the current \( \tilde{J}_y \) which are opposite for particle and hole. Of course, as shown in Sec.V, due to the CP symmetry breaking inside the BSF phase, one can only pick of the \( \pm \). However, the Mott state with \( \rho_0 = 0 \) carries nothing.

Near the \( z = (3/2, 3) \) line in Fig\(^2\) \( k_0 = \pm \sqrt{\frac{c^2 - v_y^2}{2b}} \), so the BSF near the \( z = (3/2, 3) \) line starts to carry both the density \( J_r \) and the current \( \tilde{J}_y \) which are opposite for particle and hole. However, the SF state with \( k_0 = 0 \) carries nothing. We reach the same picture from the \( z = 2 \) line and the \( z = (3/2, 3) \) line in Fig\(^2\).

2. Duality transformation in the boson picture in the moving frame
We start from the hard-spin representation Eq(12)

\[ \mathcal{L}_b = (\partial_r \theta - ic \partial_y \theta)^2 + v_r^2 (\partial_x \theta)^2 + v_y^2 (\partial_y \theta)^2 \] (36)

where the angle \( \theta \) includes both the spin-wave and vortex excitation.

To simply the transformation, one can scale away \( v_x k_x \rightarrow k_x, v_y k_y \rightarrow k_y \), so \( c \rightarrow c/v_y \) near the Mott to the SF transition. To perform the duality transformation, one can decompose \( \theta \rightarrow \theta + \phi \) which stands for the the spin-wave and vortex respectively. Introducing the 3-currents \( J_{\mu} = (J_r, J_x, J_y) \) to decouple the three quadratic terms leads to:

\[ \mathcal{L}_b = \frac{1}{2} J_{\mu}^2 + i J_r (\partial_r \theta - ic \partial_y \theta + \partial_r \phi - ic \partial_y \phi) + i J_x (\partial_r \theta + \partial_x \phi) + i J_y (\partial_r \theta + \partial_y \phi) \] (37)

Then integrating out \( \theta \) leads to the conservation of the three current:

\[ (\partial_r - ic \partial_y) J_r + \partial_x J_x + \partial_y J_y = 0 \] (38)

which is equivalent to Eq(33). This equivalence shows that the 3-currents \( J_\mu = (J_r, J_x, J_y) \) is nothing but the conserved ones listed in Eq(33) directly derived from the Noether theorem.

According to Eq(38) and (33) there are two equivalent ways to proceed the duality transformation one is to introduce the three derivatives \( \partial_\mu = (\partial_r - ic \partial_y, \partial_x, \partial_y) \), so Eq(38) can be written as \( \partial_\mu J_\mu = 0 \) or introduce the three current \( \tilde{J}_\mu = (J_r, J_x, J_y - ic J_r) \), so Eq(33) can be written as \( \partial_\mu \tilde{J}_\mu = 0 \). It turns out the first way is physically more transparent, so we take it in the following. Note that the derivatives along the three directions still commute with each other in this boosted frame, so Eq(38) implies

\[ J_\mu = \epsilon_{\mu \nu \lambda} \tilde{\partial}_\nu a_\lambda \] (39)

where \( a_\lambda \) is a non-compact \( U(1) \) gauge field.

Then Eq(37) reduces to

\[ \mathcal{L}_v = \frac{1}{4} f_{\mu \nu}^2 + i 2 \pi a_\mu \tilde{J}_\mu \] (40)

where \( f_{\mu \nu} = \tilde{\partial}_\mu a_\nu - \tilde{\partial}_\nu a_\mu \) is the gauge invariant field strength (see below Eq(12)) and \( \tilde{J}_\mu = \frac{1}{2\pi} \epsilon_{\mu \nu \lambda} \tilde{\partial}_\nu a_\lambda \) is the vortex current.

Now we introducing the dual complex order parameter \( \tilde{\psi} \), and considering \( \tilde{\partial}_\mu = (\partial_r - ic \partial_y, \partial_x, \partial_y) \), Eq(11) can be written in terms of \( \tilde{\psi} \). It leads to Eq(12) where we explicitly wrote \( \tilde{\partial}_\mu = (\partial_r - ic \partial_y, \partial_x, \partial_y) \) out in the kinetic term, but only keep it implicitly in \( \tilde{f}_{\mu \nu} \).

3. Duality transformation in the vortex picture in the moving frame

In the lab frame, one can perform the well known charge-vortex duality on Eq(2)

\[ \mathcal{L}_v = |(\partial_r - i a_r) \psi_v |^2 + |(\partial_x - i a_x) \psi_v |^2 + |(\partial_y - i a_y) \psi_v |^2 \\
+ r_v |\psi_v |^2 + u_v |\psi_v |^4 + \frac{1}{4} f_{\mu \nu}^2 + \cdots \] (41)

When \( r_v < 0 \), it is in the Mott phase \( \langle \psi_v \rangle \neq 0, r_v > 0 \), it is in the SF phase \( \langle \psi_v \rangle = 0 \).

In addition to the emergent Lorentz invariance, the T, PH and P symmetry, the Global \( U(1) \) symmetry of the boson is promoted to the local (gauge) symmetry \( \tilde{\psi} \rightarrow \psi_v e^{i \chi}, a_\mu \rightarrow a_\mu + \tilde{\partial}_\mu \chi \). The gauge invariance is completely independent of the emergent Lorentz invariance, it is also much more robust than the emergent Lorentz invariance in materials or AMO systems.

Then going to the moving frame by substituting \( \partial_r \rightarrow \partial_r - ic \partial_y \) into Eq(11) leads to:

\[ \mathcal{L}_v = (\partial_r - ic \partial_y - i a_r) \tilde{\psi}_v (\partial_r - ic \partial_y + i a_r) \tilde{\psi}_v + |(\partial_x - i a_x) \tilde{\psi}_v |^2 + |(\partial_y - i a_y) \tilde{\psi}_v |^2 \\
+ r_v |\tilde{\psi}_v |^2 + u_v |\tilde{\psi}_v |^4 + \frac{1}{4} \tilde{f}_{\mu \nu}^2 + \cdots \] (42)

where \( \tilde{f}_{\mu \nu} = \tilde{\partial}_\mu a_\nu - \tilde{\partial}_\nu a_\mu \). As stressed in [30], one can see the sign difference between the boost \( ic \partial_y \) and the time component of the gauge field \( a_r \) in the first term. This different structure in the boost and gauge field could be important in the lattice version of Eq(12) to be discussed in the conclusion section. The boost also breaks the original gauge invariance \( \tilde{\psi} \rightarrow \psi_v e^{i \chi}, a_\mu \rightarrow a_\mu + \tilde{\partial}_\mu \chi \), but still has the generalized \( U(1) \) gauge invariance \( \psi_v \rightarrow \psi_v e^{i \chi}, a_\mu \rightarrow a_\mu + \tilde{\partial}_\mu \chi \).
To perform the duality transformation, it is convenient to get to the hard spin representation of Eq.12 (For the notational convenience in the following, we replace \( a_\mu \) in Eq.11 and 12 by \( A_\mu \) in Eq.13):

\[
\mathcal{L}_v = (\partial_\mu \theta - ic\partial_\mu \theta - A_\mu)^2 + (\partial_\phi \theta - A_\phi)^2 + (\partial_\gamma \theta - A_\gamma)^2 + \frac{1}{4} \tilde{F}_{\mu \nu}^2
\]  

(43)

where the angle \( \theta \) includes both the spin-wave and vortex excitation.

Following the similar procedures as done in the boson representation (1) decomposing \( \theta \to \theta + \phi \) which stands for the the "spin-wave" and "vortex" respectively. (2) Introducing the 3-currents \( J_\mu = (J_\tau, J_x, J_y) \) to decouple the three "quadratic" terms in Eq.13. (3) Integrating out \( \theta \) leads to the conservation of the three "boson" current: \( \partial_\mu J_\mu = 0 \) which implies \( J_\mu = \epsilon_{\mu \nu \lambda} \partial_\nu a_\lambda \) where \( a_\lambda \) is a non-compact \( U(1) \) gauge field.

Then we reach:

\[
\mathcal{L}_b = \frac{1}{4} \tilde{F}_{\mu \nu}^2 - iA_\mu \epsilon_{\mu \nu \lambda} \dot{\partial}_\lambda a_\lambda + i2\pi a_\mu \tilde{J}_\mu^v + \frac{1}{4} \tilde{j}_{\mu \nu}^2
\]  

(44)

where \( \tilde{J}_\mu^v = \tilde{\partial}_\mu a_\nu - \tilde{\partial}_\nu a_\mu \) is the gauge invariant field strength and \( \tilde{j}_{\mu \nu}^v = \frac{1}{2\pi} \epsilon_{\mu \nu \lambda} \tilde{\partial}_\lambda \phi \) is the "vortex" current.

Now integrating out \( A_\mu \) leads to a mass term for \( a_\nu \).

\[
\mathcal{L}_b = i2\pi a_\mu \tilde{J}_\mu^v + \frac{1}{4} \tilde{j}_{\mu \nu}^2 + \frac{1}{2} (a_\mu)^2
\]  

(45)

where the mass term makes the Maxwell term in-effective in the low energy limit. Note that this "vortex" current in the vortex representation is nothing but the original boson current in the boson representation. Now using \( \dot{\partial}_\mu = (\partial_\mu - ic\partial_\mu, \partial_x, \partial_y) \), it leads back to Eq.10. After introducing the dual complex order parameter \( \psi \) which is nothing but the original boson in Eq.3 it leads back to Eq.3.

It is easy to see Eq.11 has two sectors, the vortex degree of freedoms \( \psi_v \) and the gauge field \( a_\mu \). Then as shown in Eq.12 going to a moving frame adds a boost to both sectors. So far, our boson-vortex duality is limited to the path I with \( c < v_y \) it would be interesting to push it to path II and path III. So the boost could trigger instabilities in the two sectors respectively. It is worth to note that the boson-vortex duality transformation focus on only low energy sector, so the Riggs mode may not be seen in such a duality transformation. However, it was shown in Sec.IV, the Riggs mode is irrelevant anyway from the SF to BSF transition. It remains interesting to achieving Fig.2 from the vortex representation.

**VIII. FINITE TEMPERATURE PROPERTIES**

Here, we derive the finite temperature effects and quantum critical scaling functions of various physical quantizes along the three paths in Fig.1 and Fig.2 in the moving frame. Even we may not be able to get all the analytic expressions in some cases, we still stress the important effects of the boost \( c \) and compare to those in the lab frame.

**A. Near \( z = 1 \) QCP along the path I**

As shown in Sec.VI, the Mott to the SF transition at \( c = 0 \) is in the same universality class as that in the lab frame, namely in the 3D XP class with the critical exponent \( z = 1, \nu = 0.67, \eta = 0.04 \). One can write down the scaling functions for the Retarded single particle Green function, the Retarded density-density correlation function, compressibility and the specific heat in the Mott side near the QCP in Fig.2:

\[
G^R(\tilde{k}, \omega) = A(\frac{\sqrt{v_x v_y}}{k_B T})^2 (\frac{k_B T}{\Delta})^\nu \Psi(\frac{\hbar \tilde{k}}{k_B T}, \frac{\hbar k}{k_B T}, \frac{\Delta}{k_B T})
\]

\[
\kappa^R(\tilde{k}, \omega) = \frac{k_B T}{\hbar v_x v_y} \Phi(\frac{\hbar \tilde{k}}{k_B T}, \frac{\hbar k}{k_B T}, \frac{\Delta}{k_B T})
\]

\[
\kappa = \frac{k_B T}{\hbar^2 v_x v_y} A(\frac{\Delta}{k_B T}), \quad C_v = \frac{T^2}{\hbar^2 v_x v_y} B(\frac{\Delta}{k_B T})
\]  

(46)

where \( A \sim r^{\nu_{\nu'}} \) is the single particle residue, \( \Delta \sim r^{\nu} \) is the Mott gap inside the Mott phase, \( \tilde{\omega} = \omega - c \tilde{k}_y \) is the Doppler shifted frequency in the moving frame, \( \tilde{k} = \sqrt{k_x^2 + k_y^2 + v_y^2} k_y \) is the momentum and the compressibility is defined by \( \kappa = \lim_{\tilde{k} \to 0} \kappa^R(\tilde{k}, \omega = 0) \). From the two conserved quantities, one can also form the Wilson ratio \( W = \kappa T / C_v \).
FIG. 5. The finite temperature phase diagram along the path (I), (II) and (III) a and (III)b at $d = 2$ in the Fig. The SF density $\rho_s \sim (-r)^\nu$, $\nu = 0.67$ across the Mott-SF transition with $z = 1$ at $T = 0$ in (a), $\rho_s \sim |c - v_f|$ across the SF-BSF transition with $z = (3/2, 3)$ at $T = 0$ in (b), then $\rho_s \sim \tilde{\rho} \sim (r - r_c)$ or $\rho_s \sim (c - c_c)$ upto logarithmic correction with $z = 2$ at $T = 0$ in (c) and (d) respectively. The solid line in the SF or BSF side represents the finite temperature KT transition. The two dashed lines sandwiching the KT line stands for the narrow window of classical fluctuation regimes which are squeezed to zero at the QCP at $T = 0$. The dashed line in the Mott regime indicates the crossover from the Mott phase to the Quantum critical regime. (a) and (c) are tuned by the chemical potential $\mu$ in (c) and (d) respectively. The solid line in the SF or BSF side represents the finite temperature KT transition. The in the narrow window near the KT transition, the scaling function reduce to those of classical KT transition. In the SF side near the QCP, $\tilde{\mu}_{QCP}$ need to be replaced by $\mu_{QCP}$ where $\rho_s \sim |r|^{(d+z-2)\nu} \sim |r|^\nu$ is the SF density. As shown in Fig.5a and Sec.VI-B-3, all the scaling functions in the SF side will run into singularity at a finite $T = T_{KT}$ signaling the finite KT transition. In the narrow window near the KT transition, the scaling function reduce to those of classical KT transition.

In the following, from the effective action Eq. we can evaluate the three scaling functions explicitly.

From Eq.3 one can identify the single particle (boson) Green function in $(\vec{k}, \omega_n)$ space in the Mott side $r > 0$:

$$G_0(\vec{k}, \omega_n) = \langle \psi(\vec{k}, \omega_n)\psi^\dagger(\vec{k}, \omega_n) \rangle = \frac{1}{-(i\omega_n - ck)^2 + k^2 + r}$$

where $\epsilon_+(\vec{k}) = \sqrt{\vec{k}^2 + r \pm ck} > 0$ well inside the Mott phase $r > 0$.

From the three currents Eq.33 following the method developed in [48], one can also evaluate the retarded density-density correlation function $\kappa^R(\vec{k}, \omega_n)$.

From Eq.3 one can also get the free energy density well inside the Mott:

$$f = 2k_B T \int \frac{d^d\vec{k}}{(2\pi)^d} \log(1 - e^{-\beta \epsilon_+(\vec{k})}) + \int \frac{d^d\vec{k}}{(2\pi)^d} e^{\beta \epsilon_+(\vec{k})}$$

where we have used the CP symmetry $\epsilon_+(\vec{k}) = -\epsilon_-(\vec{k})$ to get rid of the hole excitation spectrum in favor or that of the particle. The last term is the ground state energy at $T = 0$.

From the free energy, one can immediately evaluate the specific heat: From Eq.3 one can also get the free energy well inside the Mott:

$$C_v = -T \frac{\partial^2 f}{\partial T^2} = \int \frac{d^d\vec{k}}{(2\pi)^d} \frac{e^{\beta \epsilon_+(\vec{k})}}{(e^{\beta \epsilon_+(\vec{k})} - 1)^2} \left(\frac{\epsilon_+(\vec{k})}{k_B T} \right)^2$$

Plugging in the $\epsilon_+(\vec{k})$ in the Mott phase leads to $C_v(T)$ in Eq.33 well inside the Mott side where one can just use the mean field theory $\Delta = r$. Eq.33 breaks down near the QCP. However, one can apply the simple scaling analysis $C_v \sim T^{d/z} \sim T^2$ for the $z = 1$ QCP.
B. Near \((z_x = 3/2, z_y = 3)\) QCP along the path II

Here the fundamental degree of freedoms is the phase \(\phi\) in Eq.12 of the boson \(\psi = \sqrt{\rho_0}e^{i\phi}\). If dropping the leading irrelevant operators \([Z] = [\beta] = -2\) and neglecting the vortex excitations, it becomes a Gaussian theory. So we first calculate the phase-phase correlation function, then the boson-boson correlation functions.

1. Correlation functions in the Quantum regimes

On both sides, the phase-phase correlation function:

\[
\langle \phi(-\vec{k}, -\omega_n)\phi(\vec{k}, \omega_n) \rangle = \frac{-1}{(i\omega_n - \epsilon_+(\vec{k}))(i\omega_n + i\epsilon_-(\vec{k}))}
\]  

(50)

where \(\epsilon_{\pm}(\vec{k}) = \sqrt{v_x^2k_x^2 + v_y^2k_y^2 + ak_y^4 \pm ck_y} > 0\), in the SF \(v_y > c\), at the QCP \(v_y = c\), \(v_y^2 \rightarrow 3c^2 - 2v_y^2\) in the BSF phase \(v_y < c\). \(\epsilon_+(\vec{k}) + \epsilon_-(\vec{k}) = 2c(\vec{k}) = 2\sqrt{v_x^2k_x^2 + v_y^2k_y^2 + ak_y^4}\) is an even function of \(\vec{k}\) where \(c\) drops out.

One can find the single particle (boson) Green function:

\[
G(x, \tau) = \langle \psi(x, \tau)|\psi^*(0, 0) \rangle = \rho_0 e^{-g(x, \tau)}
\]

\[
g(x, \tau) = \frac{1}{\beta} \sum_{\omega_n} \int \frac{d^dk}{(2\pi)^d} \frac{1 - e^{i(k \cdot x - \omega_n \tau)}}{2\epsilon(\vec{k})} \left\{ \frac{1}{i\omega_n - \epsilon_+(\vec{k})} - \frac{1}{i\omega_n + i\epsilon_-(\vec{k})} \right\}
\]

(51)

The frequency integral can be done first by paying the special attention to \(\tau = 0 = \beta\) at any finite \(T\):

\[
g(x, \tau) = \int \frac{d^dk}{(2\pi)^d} \frac{1}{2\epsilon(\vec{k})} e^{i\vec{k} \cdot \vec{x} - \epsilon_+(\vec{k})\tau} - e^{-\beta \epsilon_+(\vec{k})} - e^{\beta \epsilon_-(\vec{k})} - e^{-\epsilon_-(\vec{k})} \]

(52)

where one can use the CP symmetry \(\epsilon_+(\vec{k}) = \epsilon_-(\vec{k})\) to get an expression only in terms of \(\epsilon_+(\vec{k})\).

Now we look at several special cases:

(1) Putting \(T = 0\) leads to

\[
g(x, \tau) = \int \frac{d^dk}{(2\pi)^d} e^{i\vec{k} \cdot \vec{x} - \epsilon_+(\vec{k})\tau} - 1
\]

(53)

Its equal time at \(d = 2\) is \(g(x, 0) = \int \frac{d^2k}{(2\pi)^2} \frac{e^{i\vec{k} \cdot \vec{x}} - 1}{2\epsilon(\vec{k})} \sim 1/|x|\) is independent of \(c\), \(G(x, 0) \sim \rho_0 e^{-1/|x|}\). But its auto-correlation (equal-space) \(g(0, \tau) = \int \frac{d^2k}{(2\pi)^2} \frac{e^{i\vec{k} \cdot \vec{0}} - 1}{2\epsilon(\vec{k})} \) does depend on \(c\).

(2) Putting equal-time \(\tau = \beta\) in Eq.51 leads to the equal time : 

\[
g(x, 0) = \int \frac{d^dk}{(2\pi)^d} e^{i\vec{k} \cdot \vec{x}} - 1 \left\{ \frac{1}{e^{\beta \epsilon_+(\vec{k})} - 1} - \frac{1}{e^{-\beta \epsilon_-(\vec{k})} - 1} \right\}
\]

(54)

Putting \(\epsilon_+(\vec{k}) = \epsilon_-(\vec{k}) = \epsilon(\vec{k})\), one recovers the well-known \(c = 0\) result \(g(x, 0) = \int \frac{d^dk}{(2\pi)^d} \frac{e^{i\vec{k} \cdot \vec{x}} - 1}{2\epsilon(\vec{k})} \coth \frac{\beta \epsilon(\vec{k})}{2}\).

In the BSF phase, Eq.51 should be replaced by

\[
G(x, \tau) = \langle \psi(x, \tau)|\psi^*(0, 0) \rangle = \rho_0 e^{ik_{xy}y} e^{-g(x, \tau)}
\]

(55)

where in the \(g(x, \tau)\), one just replace \(v_y^2 \rightarrow 3c^2 - 2v_y^2\). It is the modulation \(e^{ik_{xy}y}\) in the correlation function which distinguishes the BSF from the SF.

2. The thermodynamic quantities in the quantum regimes

Applying Eq.49 to the SF or BSF side, one can find the specific heat:

\[
C_v = \frac{T^2}{v_x v_y} \int \frac{d^2k}{(2\pi)^2} \frac{e^{k + \alpha k_y (k + \alpha k_y)^2}}{(e^{k + \alpha k_y} - 1)^2} = f(\alpha)T^2
\]

(56)

where \(\alpha = c/v_y < 1\) in the SF side and \(\alpha = c/\sqrt{3c^2 - 2v_y^2}\) in the BSF side. This is consistent with the scaling \(C_v \sim T^{d/z} \sim f(\alpha)T^2\) with \(z = 1\).
At the QCP $f(\alpha = 1)$ diverges where we find
\[
C_v = \frac{T}{\nu x v_y} \int_{-\infty}^{\infty} \frac{dk_x}{2\pi} \int_0^{\infty} \frac{dk_y}{2\pi} e^{g(k_x, k_y, \alpha)} g^2(k_x, k_y, \alpha) + O(T^2) + \cdots, \quad g(k_x, k_y, \alpha) = \frac{k_x^2 + \alpha k_y^2}{2k_y}
\]
(57)
where the integral over $k_y$ is only half of the line, the other half contributes to the subleading $T^2$ term. It is consistent with the scaling $C_v \sim T^{1-z+1/z} \sim T$ with $(z_x = 3/2, z_y = 3)$ at the QCP.

The superfluid density $\rho_s \sim v_x \gamma \sim |c - v_y| \sim |\alpha - 1|$. So all the scaling functions can be written in terms of $\rho_s/k_BT$ on both sides.

3. In the classical regime near the KT transition

So far, we only look at the quantum effects of the boost $\partial_x \rightarrow \partial_x - ic\partial_y$. Now we look at its classical effects originating from such a substitution. At a finite temperature, setting the quantum fluctuations (the $\partial_x$ term) vanishing, in Eq(14) or Eq(18) then both equations reduce to
\[
S_{KT} = \frac{1}{k_BT} \int d^dr [\nu_x^2 (\partial_x \phi)^2 + \gamma^2 (\partial_y \phi)^2 + a(\partial_y^2 \phi)^2]
\]
(58)
where $\gamma^2 = v_y^2 - c^2$ inside the SF phase and $\gamma^2 = 2(c^2 - v_y^2)$ inside the BSF phase. By setting the $\partial_x$ term vanishing, the crucial crossing metric term also vanish. This facts suggest that the effects of boosts is mainly quantum effects, but still have important classical remanent effects. It indicates the finite temperature phase transition is still in Kosterlize-Thouless (KT) universality class with a reduced $T_{KT} \sim \rho_s \sim |c - v_y|$.

The classical boson correlation functions in the narrow window around the KT line show algebraic decay order, just like those in the classical KT transition.

\[
G(\vec{x}) = \langle \psi(\vec{x})\psi^*(0) \rangle = \rho_0 e^{-g(\vec{x})} = (|x|/a)^{-\frac{2}{z}}
\]
\[
g(\vec{x}) = k_BT \int \frac{d^d\vec{k}}{(2\pi)^d} \left\{ e^{i\vec{k} \cdot \vec{x}} - 1 \right\} \sim \frac{T}{\rho_s} \ln |x|/a
\]
(59)
which can be contrasted to the quantum regimes listed in Eq(51) and 52. As shown in Eq(55) inside the KT transition above the BSF phase has the modulation factor $e^{ik_0y}$.

Of course, the classical regime squeezes to zero at the QCP $\gamma = 0$ as shown in Fig(5).

C. Near the $z = 2$ QCP along the path III

The $z = 2$ QCP is in the same universality class as the zero density SF-Mott transition with the exact critical exponent $z = 1, \nu = 1/2, \eta = 0$ subject to logarithmic corrections at the upper critical dimension $d = 2$. The scaling functions for the Retarded single particle Green function, the Retarded density-density correlation function, compressibility and the specific heat near the QCP in Fig(5), d:
\[
G^R(\vec{k}, \omega_n) = e^{ik_0y} \frac{\hbar}{k_B T} \Psi\left( \frac{\hbar Z_1 \omega}{k_B T}, \frac{\hbar \tilde{k}}{\sqrt{k_B T}}, \frac{\tilde{\mu}}{k_B T} \right)
\]
\[
\kappa^R(\vec{k}, \omega_n) = \frac{1}{\hbar v_x v_y} \frac{A(\tilde{\mu})}{k_B T} \cdot \frac{\hbar Z_1 \omega}{k_B T} \cdot \frac{\hbar \tilde{k}}{\sqrt{k_B T}} \cdot \frac{\tilde{\mu}}{k_B T}
\]
\[
\kappa = \frac{1}{\hbar v_x v_y} A(\tilde{\mu}) \cdot \frac{\hbar}{k_B T} \cdot \frac{\hbar Z_1 \omega}{k_B T} \cdot \frac{\hbar \tilde{k}}{\sqrt{k_B T}} \cdot \frac{\tilde{\mu}}{k_B T}
\]
(60)
where $\tilde{\mu} = -(r - r_c)$ is listed in Eq(24) and $Z_1 \omega$ is the scaled frequency in the moving frame, $\tilde{k} = \sqrt{v_x^2 k_x^2 + v_y^2 k_y^2}$ is the momentum, $\tilde{\mu}$ is the effective chemical potential, especially the anisotropy between $k_x$ and $k_y$. Note the dramatic changes from the $z = 1$ scaling sets in Eq(16) to the $z = 2$ scaling sets in Eq(60).

In the BSF side near the QCP, $\frac{\hbar v_x v_y}{k_B T} A(\tilde{\mu})$ need to be replaced by $\frac{\hbar v_x}{k_B T}$ where $\rho_s \sim (d+z-2) \sim \tilde{\mu} = -(r - r_c)$ up to the logarithmic correction is the SF density in the BSF. As shown in Fig(5), d, all the scaling functions in the BSF side runs into a singularity at a finite $T = T_{KT}$ signaling the finite KT transition. In the narrow window near the KT transition, the scaling function reduce to those of classical KT transition.

The effects of the dangerously irrelevant $Z_2$ (the metric crossing term) has not been considered in the scaling function near the QCP, but become important inside the two phases as shown in Sec.IV. Note that it is the effective chemical potential $\tilde{\mu} = -(r - r_c), r_c = \alpha k_0^4 > 0$ listed in Eq(24) which tunes the Mott to the BSF transition. So one can
either tune the bare mass $r$ or the boost velocity to tune the transition (Fig.5 or Fig.51) respectively. Especially, at a fixed $r > 0$ inside the Mott state, one can tune it into the BSF just by increasing the boost velocity. The energy come from boosting the moving frame.

In summary, one can see the importance of the metric crossing term: it is marginal, dominant and irrelevant near the $z = 1$, $z = (3/2, 3)$ and $z = 2$ QCP respectively. The interaction $u$ is relevant and marginally irrelevant near the $z = 1$ and $z = 2$ respectively. Of course, despite the SF to BSF transition is a Gaussian one, it is the interaction which leads to the very existence of the SF, BSF and the QPT between the two.

**IX. COMMENTS ON LORENTZ INVARiance IN RELATIVISTIC QFT AND EXPERIMENTAL DETECTIONS**

In relativistic QFT, different inertial frames are just related by Lorentz transformations. relativistic Doppler shift is just a direct consequence of Lorentz transformations. For relativistic QFT, this is just the end of the story, Doppler shift will not lead to no new phases and new QPT. For non-relativistic quantum many body systems, Doppler shift is just a direct consequence of Galileo transformation. However, as shown in the previous sections, Doppler shift Eq.12,15,19,26,27 is far from being the end of the story. When the shift goes beyond the intrinsic velocity of the matter, it may trigger QPT and leads to new quantum phases. This is another powerful and explicit demonstration of P. W. Anderson’s view on quantum many body systems ” More is different ” which can be expanded to ”More is richer, more challenging, more interesting.....”.

**A. Contrast to Doppler shift and Unruh effects in relativistic QFT**

In relativistic quantum field theory, different inertial frames are related by Lorentz transformation, so they are completely equivalent. Even so, changing to a different inertial frame, the frequency changes as follows:

$$\omega' = \gamma(\omega - \vec{v} \cdot \vec{k}), \quad \gamma = \frac{1}{\sqrt{1-v^2/c^2}}. \quad (61)$$

which is the frequency in the boosted frame. Because $\omega^2 = c^2k^2 + m^2$ and $v < c$, so $\omega'$ always remains the same sign as $\omega$. For the light $m = 0$, then $\omega = ck$, then $\omega' = \omega\sqrt{\frac{1-\beta^2}{1+\beta^2}}$ where $\beta = v/c$ takes positive (negative) when the observer is moving away (towards) from the source. This is nothing but well known relativistic Doppler shift. Positive (negative) frequency stays invariant in different inertial frames, no QPT. Eq.(61) can be contrasted with the non-relativistic Doppler shift term in Eq.12,15,19,26,27 and also Eq.(D12) in the appendix F. When taking the non-relativistic limit $c \rightarrow \infty$ limit, as expected, they become identical. However, the crucial difference between the two groups is that: In the former, despite the frequency of a mode depends on the choice of the inertial frame, the decomposition into positive and negative frequencies is invariant. While, in the latter, the positive frequency can turn into a negative one driven by the boost, therefore trigger the QPTs shown in Fig.2 and Fig.7.

However, the in-equivalence may come from a non-inertial frame such as an uniformly accelerating frame. An uniformly accelerating observer would see quite different vacuum and excitations. Even an observer at rest in the Minkowski space-time see a true vacuum with no particles. This is the well known Unruh effects [71, 72]. The uniformly accelerating observer (with a constant 4-acceleration $a$) will see a thermal bath of particles with the temperature:

$$k_B T_U = \frac{\hbar a}{2\pi c} \quad (62)$$

Namely a pure state at $T = 0$ in Minkowski space-time becomes a mixed state at $k_B T_U = \frac{\hbar a}{2\pi c}$ thermo-field double? Unfortunately, the Unruh effect is so small that it is extremely difficult to detect. A proper acceleration of $a = 2.47 \times 10^{20} m/s^2$ corresponds to $T_U \sim 1K$. Or conversely, $a = 1m/s^2$ corresponds to $T_U \sim 4 \times 10^{-21} K$ which is beyond any current available cold atom experiment.

The Unruh effect is a quantum effect which vanishes as $\hbar \rightarrow 0$. It is also a relativistic effect which vanishes as $c \rightarrow \infty$ (namely when taking a non-relativistic limit by sending $c \rightarrow \infty$). The observer draws a worldline (trajectory) $x^2 = t^2 + (1/a)^2$ in the Minkowski space-time. The reference frame where an uniformly accelerating observer is at rest is called Rindler space-time which is related to the Minkowski space-time by $x = v \cosh \eta, t = r \sinh \eta$. It is confined to the wedge $x \geq |t|$ separated by the ”Rindler horizon” at $x = \pm \tau, t > 0$ from the rest of the space-time. The uniformly accelerating observer follows the trajectory with $r = 1/a, \eta = a \tau$ where $\tau$ is the proper time.

In relativistic QFT, the Unruh effect tells us that two different sets of observers such as inertial and Rindler will describe the same quantum state in very different ways. Here in a non-relativistic quantum field theory on a lattice,
we showed that even two different inertial observers will also see the same quantum state in very different ways. This shows that non-relativistic systems show much richer effects than their relativistic counter-parts, which can be much more easily detected experimentally in condensed matter or cold atom set-ups.

B. Contrast to $P - T$ phase diagram in Helium 4

As mentioned in the introduction, He4 is the oldest SF. So it would be instructive to contrast the phase diagram Fig.5 to that in a He4: the Mott, SF and BSF correspond to the normal, SF and solid phase respectively. While the boost $c$ and chemical potential $\mu$ correspond to the pressure $P$ and temperature $T$ respectively. Namely, here, we tune transitions by the boost $c$, while in Helium 4, one tune by pressure. Of course, the He4 system has the Galileo invariance which dictates the superfluid density $\rho_s = \rho$ at $T = 0$. Namely, all the quasi-particles are still part of the superfluid component at $T = 0$ instead of the normal component. Even so, the SF phase in He4 still breaks the Galileo invariance. The case (1) and case (2) mentioned in the introduction has been discussed previously in He4. Surprisingly, the case (3) has not been discussed yet. We expect our results achieved in Sec.II or appendix B also apply to the SF phase in He4 to some extents, but one may also need to consider the effects of rotons and topological vortex excitations which play important roles in class (1) and (2).

![Phase diagram comparison](image)

FIG. 6. Contrast Fig.2 at a 2d lattice and $T = 0$ against the liquid He4 diagram at 3d continuum and at a finite $T$. The former has no Galileo invariance, the latter does. To facilitate the comparison, we revert the horizontal axis in Fig.2. (a) The $z = 1$ line is in 3D XP for any $c < v_y$. (b) The liquid to SF transition is also in a classical 3d XP class. (a) The SF to BSF transition is a continuous quantum Lifshitz transition tuned by the boost $c$. (b) the SF to solid transition is a first order classical Lifshitz transition triggered by the lowering of roton surface tuned by the pressure. The roton surface is spherically symmetric. (a) the Mott to the BSF transition is a $z = 2$ class, (b) the liquid to solid transition is a first order Lifshitz transition driven by the peak in the density-density correlation tuned also by the pressure. This analogy suggests that the putative supersolid (SS) phase seems unlikely to exist in the He4.

C. Experimental detections in the moving frame

Only when the satellite velocity $c$ reaches the intrinsic velocity of the matter, the full phase diagram Fig.2 can be explored. For example, the sound velocity in He4 is about $v \sim 238 m/s$. In a conventional lab on the earth, taking a high way (magnetic levitated) train moving with a velocity $300km/h \sim 83 m/s$ is still below this characteristic velocity, a civil air-craft flight can reach even higher $800km/h \sim 240 m/s$ which just reaches the sound velocity in the Helium4. Travelling beyond any intrinsic velocity becomes easy when one can perform various kinds ultra-cold atom experiments and quantum simulations in the satellite orbiting around the earth.

Instead of going beyond the intrinsic velocity, one can also reduce the intrinsic velocity. The cold atom systems become good candidates. For example, in a weakly interacting BEC, it is much smaller $v \sim 1 cm/s$. So the full phase diagram Fig.2 can be easily explored just by putting the trap in a slowly moving trail.

We feel similar experimental techniques to detect the Doppler effects and the Unruh effects can also be applied here. Conventional 2-terminal or 4-terminal transport properties are hard to do, but other detection methods, especially Bragg spectroscopies and photo-emissions are effective in a moving frame. For example, the single-particle Green functions, the density and density correlation functions, the Goldstone and Riggs modes can be detected by all kinds of Bragg spectroscopies such as dynamic or elastic, energy or momentum resolved, longitudinal or transverse Bragg spectroscopies. The compressibility $\kappa$, the superfluid density $\rho_s$ and specific heat $C_v$ can be separately measured in Ref. [68], [70] and [67, 68] respectively, also In-Situ measurements.
X. Boosting an Emergent Galilean Invariant $z = 2$ SF-Mott Transition

In this section, we study the $z = 2$ SF-Mott transitions in a lab frame and observe it in a moving frame and contrast to the $z = 1$ ones addressed in the previous sections. We also analyze the intrinsic relations between the Galilean transformation and the symmetry breaking in the SF phase.

At integer fillings and in the absence of PH symmetry, the SF-Mott transition in Eq.1 in the lab frame (Fig.1a) and Fig.8) can be described by the $z = 2$ effective action

$$ S_L = \int d\tau d^2r [\bar{\psi} \gamma^0 \gamma^a \partial_a \psi + \frac{\mu}{4} |\partial_\tau \psi|^2 + v_y^2 |\partial_y \psi|^2 - \mu |\psi|^2 + u |\psi|^4 + \cdots] $$

(63)

Where the chemical potential $\mu$ tunes the SF-Mott transition with $z = 1$, $\mu < 0$, $\langle \psi \rangle = 0$ is in the Mott state which respects the $U(1)$ symmetry, $\mu > 0$, $\langle \psi \rangle \neq 0$ is in the SF state which breaks the $U(1)$ symmetry. It has an emergent Galileo invariance. So we expect performing a Galilean transformation to a moving frame should not change its form. This is indeed the case as demonstrated in the following.

Performing the Galileo transformation described in Sec.I leads to the following effective action in the moving frame (Fig.1b) (again, for the notational simplicity, we drop the $\tau$ in the moving frame):

$$ S_M = \int d\tau d^2r [\bar{\psi} (\partial_a - i e \partial_y) \psi + \frac{\mu}{4} |\partial_\tau \psi|^2 + v_y^2 |\partial_y \psi|^2 - \mu |\psi|^2 + u |\psi|^4 + \cdots] $$

(64)

The mean field ansatz $\psi = \sqrt{\rho} e^{i(\phi + k_0 y)}$ leads to the energy density

$$ E[\rho, k_0] = (c k_0 + v_y^2 k_0^2 - \mu) \rho + u \rho^2 $$

(65)

Minimizing $E[\rho, k_0]$ with respect to $\rho$ and $k_0$ results in

$$ k_0 = \frac{c}{2v_y^2}, \quad \rho = \begin{cases} 0, & \mu < \frac{c^2}{4v_y^2} \\ \mu + \frac{u}{8c^2v_y^2}, & \mu > \frac{c^2}{4v_y^2} \end{cases} $$

(66)

It is easy to see that due to the explicit PH symmetry breaking of $z = 2$ action Eq.64, the sign of $k_0$ is automatically given. This is in sharp contrast to the $z = 1$ case presented in the main text where the sign of $k_0$ is determined by the spontaneous CP-symmetry breaking.

It is convenient to introduce the new order parameter $\tilde{\psi} = \psi e^{i k_0 y}$, then $\partial_y \tilde{\psi} = e^{i k_0 y} \tilde{\psi}$ and

$$ S = \int d\tau d^2r (\tilde{\psi}^* \partial_a \tilde{\psi} - i (c + 2 k_0 v_y^2) \tilde{\psi}^* \partial_y \tilde{\psi} + v_y^2 |\partial_x \tilde{\psi}|^2 + v_y^2 |\partial_y \tilde{\psi}|^2 - (\mu - c k_0 - v_y^2 k_0^2) |\tilde{\psi}|^2 + u |\tilde{\psi}|^4 + \cdots) $$

(67)

Setting $i (c + 2 k_0 v_y^2) \tilde{\psi}^* \partial_y \tilde{\psi} = 0$ leads to $k_0 = -\frac{c}{2v_y^2}$ and

$$ S = \int d\tau d^2r (\tilde{\psi}^* \partial_a \tilde{\psi} + v_y^2 |\partial_x \tilde{\psi}|^2 + v_y^2 |\partial_y \tilde{\psi}|^2 - (\mu + v_y^2 k_0^2) |\tilde{\psi}|^2 + u |\tilde{\psi}|^4) $$

(68)

which as expected, remains the same as original $z = 2$ theory in the lab frame after identifying the chemical potential in the moving frame $\tilde{\mu} = \mu + v_y^2 k_0^2$. After identifying $v_y^2 = 1/2m$, then $\tilde{\mu} = \mu + v_y^2 k_0^2 = \mu + \frac{mc^2}{2}$ precisely match the Galilean transformation shown in Eq.110.

Setting $\tilde{\mu} = 0$ leads to the $z = 2$ phase boundary in the moving frame:

$$ \mu = -\frac{c^2}{4v_y^2} < 0 $$

(69)

which gives the $z = 2$ line in Fig.7.

In summary, Eq.63 is invariant under the Galilean transformation:

$$ \tilde{\psi} = \psi e^{-i k_0 y}, \quad \tilde{\mu} = \mu + \frac{mc^2}{2} $$

(70)

In the Mott phase $\mu < 0$ with a Mott gap $-\mu$, increasing the boost, the Mott gap decreases until to zero signifying the QPT to the BSF phase. However, in the SF phase, $\mu > 0$, then one need to get the effective action inside the SF phase first, then boosting the SF phase as done in the following subsection 4.4. The combination of sub-sections A-D lead to Fig.7.
FIG. 7. The phase diagram of action Eq.(64) with $z = 2$ observed in the moving frame Fig.1b as a function of $\mu$ and $c$. $c = 0$ recovers to that in the lab frame. Due to the explicit PH symmetry breaking, no Riggs mode inside the SF and BSF phase. The blue line with $z = 2$ and the green line with $(z_x, z_y) = (3/2, 3)$ are in the same universality class as the corresponding lines in Fig.2. However, the $z = 1$ red line in Fig.2 is missing here. The BSF phase in 2 spontaneously breaks the CP symmetry, while the CP symmetry was explicitly broken from very beginning.

A. Boosting a SF phase which breaks the emergent Galileo invariance spontaneously

It is important to observe that despite Eq.(63) has emergent Galileo-invariance. Its SF state breaks the Galileo invariance spontaneously. Then the procedures developed in Sec.III can also be applied here. We still start from the $z = 2$ action Eq.(63) in the lab frame.

Plugging the mean field ansatz $\psi = \sqrt{\rho} e^{i\phi}$ into the action Eq.(63) leads to the energy density

$$E[\rho, \phi] = -\mu \rho + \frac{u \rho^2}{2}$$

Minimizing $E[\rho, \phi]$ with respect to $\rho$ results in

$$\rho = \begin{cases} 0, & \mu < 0 \\ \frac{\mu}{2u}, & \mu > 0 \end{cases}$$

In the superfluid phase, $\mu > 0$ and $\rho > 0$, then $\psi = \sqrt{\rho_0 + \delta \rho} e^{i\delta \phi}$ and

$$S_{L:SF} = \int d\tau d^2 r \left\{ \frac{i}{4} \delta \rho \partial_\tau \delta \phi + \frac{1}{4\rho_0} \left[ v_x^2 (\partial_x \delta \rho)^2 + v_y^2 (\partial_y \delta \rho)^2 \right] + \rho_0 \left[ v_x^2 (\partial_x \delta \phi)^2 + v_y^2 (\partial_y \delta \phi)^2 \right] + U(\delta \rho)^2 \right\}$$

which shows $(\delta \rho, \delta \phi)$ becomes a conjugate variable, so no Riggs mode, in sharp contrast to the SF phase in the $z = 1$ case presented in Eq.(11). Of course, as stressed in Sec.IV, despite the existence of the Riggs mode inside the SF near the $z = 1$, it plays no roles in the SF to BSF transition in Fig.2. Due to the spontaneous CP symmetry breaking, it disappears in the BSF anyway.

Integrating out $\delta \rho$ lead to

$$S_{L:SF} = \int d\tau d^2 r \left\{ \frac{1}{4U} (\partial_\tau \delta \phi)^2 + \rho_0 \left[ v_x^2 (\partial_x \delta \phi)^2 + v_y^2 (\partial_y \delta \phi)^2 \right] \right\}$$

which gives the superfluid Goldstone mode:

$$\omega = \sqrt{4U \rho_0 (v_x^2 k_x^2 + v_y^2 k_y^2)}$$

Following the procedures developed in Sec.III leading to Eq.(12) the boost $\partial_\tau \rightarrow \partial_\tau - i c \partial_y$ leads to the action in the moving frame:

$$S_{M:SF} = \int d\tau d^2 r \left\{ \frac{1}{4U} \left[ (\partial_\tau - i c \partial_y) \delta \phi \right]^2 + \rho_0 \left[ v_x^2 (\partial_x \delta \phi)^2 + v_y^2 (\partial_y \delta \phi)^2 \right] \right\}$$
which takes the same form as Eq.\ref{eq:14} and leads to the exotic superfluid Goldstone mode in the moving frame:

$$
\omega = \sqrt{4U\rho_0(v_y^2k_x^2 + v_y^2k_y^2)} - ck_y
$$

(77)

As shown in Sec.III, the SF to BSF transition in Fig.2 is solely driven by the Goldstone mode, the Riggs mode is irrelevant. Here due to the explicit PH symmetry breaking, the Riggs mode does not exist in the first place. More importantly, some high derivative $a$ term or higher order $b$ term in Eq.20 which breaks the emergent Galileo invariance explicitly need also be considered. When applying the results achieved in Sec.III leads to the critical line between the SF and the boosted SF

$$
c^2 = 4U\rho_0v_y^2 = 2\mu v_y^2 \rightarrow \mu = \frac{c^2}{2v_y^2} > 0
$$

(78)

which gives the $(z_x, z_y) = (3/2, 3)$ line in Fig.7.

In fact, one can also achieve Eq.(77) directly by performing the Galileo transformation Eq.(D10) as shown in the Eq.(D8) in the appendix F.

Of course, both the $z = 2$ line and the $(z_x, z_y) = (3/2, 3)$ line may break down near the Multi-critical point $(\mu, c) = (0, 0)$ in Fig.7 but should hold when far away from it. The scaling analysis in Sec.VIII-B, VIII-C and Fig.5 (b),(c),(d) should also hold here. Compared to Fig.2, the $z = 1$ line is absent in Fig.7, the advantages for experimental detections is that the critical velocities to probe the $z = 2$ line and the $(z_x, z_y) = (3/2, 3)$ line is much lower than those in Fig.2

B. A leading irrelevant term breaking the emergent Galileo invariance explicitly

As said in the introduction, the microscopic system Eq.11 is not Galileo invariant. To see the effects of the boost which break the Galileo invariance, one must add some boosting terms to Eq.63 which are irrelevant near the $z = 2$ QCP, but break the Galileo invariance explicitly. This will be achieved in this section.

Let us consider the typical irrelevant second order derivative term which breaks the Galileo invariance explicitly:

$$
|\partial_\tau \psi|^2 \rightarrow |(\partial_\tau - ic\partial_y)\psi|^2 = |\partial_\tau \psi|^2 - i2c\partial_\tau \psi^* \partial_y \psi - c^2|\partial_y \psi|^2
$$

(79)

thus a more complete effective action than Eq.(64) is

$$
L_M = Z_1\psi^* (\partial_\tau - ic\partial_y)\psi + Z_2(\partial_\tau - ic\partial_y)\psi^* (\partial_\tau - ic\partial_y)\psi + v_y^2|\partial_y \psi|^2 + v_y^2|\partial_y \psi|^2 + a|\partial^2 \psi|^2 + b|\partial_y \psi|^4 - \mu|\psi|^2 + u|\psi|^4 + \cdots
$$

(80)

Plugging in the Mean field ansatz $\psi = \sqrt{\rho_0}e^{i(\phi+k_0y)}$ leads to the energy density

$$
E[\rho, k_0] = |Z_1ck_0 + (v_y^2 - Z_2c^2)k_0^2 + ak_0^2 - \mu|\rho + bk_0^2\rho^4 + u\rho^2
$$

(81)

In the following, we will drop the $b$ term which can be shown to be irrelevant in the following discussions.

Similar to Sec.IV, near the $z = 2$ QCP, due to $|Z_2| = -2$, one can study the $Z_2 \ll Z_1$ limit, then one can safely drop the $a$ term. The minimization with respect to $k_0$ and $\rho$ leads to the saddle point solution

$$
k_0 = -\frac{Z_1c}{2(v_y^2 - Z_2c^2)}, \quad \rho_0 = \max(0, \frac{\mu - Z_1ck_0 - (v_y^2 - Z_2c^2)k_0^2}{2U})
$$

(82)

which leads to a small correction to Eq.66

The critical line between the Mott ($\rho_0 = 0$) and the BSF ($\rho_0 > 0$) are given by

$$
\mu_c = -\frac{Z_2c^2}{4(v_y^2 - Z_2c^2)}
$$

(83)

which leads to a small correction to Eq.69

The mean-field phase diagram is qualitatively the same as $Z_2 = 0$ case with the slight change of $k_0$ and the phase boundary $\mu_c$ listed in Eq.22 and Eq.83.
When $k_0 \neq 0$, it is convenient to introduce a new order parameter $\psi = \psi e^{ik_0 y}$, Eq\[20\] becomes:

$$
S_M = \int d\tau d^2r \left[ (Z_1 - 2Z_2 c k_0) \dot{\psi}^* \partial_r \psi + Z_2 |\partial_r \dot{\psi}|^2 - i 2Z_2 c \partial_r \dot{\psi}^* \partial_y \psi + v_x^2 |\partial_x \dot{\psi}|^2 + (v_y^2 - Z_2 c^2) |\partial_y \dot{\psi}|^2 
- (\mu - Z_1 c k_0 - v_y^2 k_0^2 + Z_2 c^2 k_0^2) |\dot{\psi}|^2 + |U| |\psi|^4 \right].
$$

(84)

Setting the linear in $k_y$ term $Z_1 c + 2k_0 v_y^2 - 2Z_2 k_0 c^2 = 0$ recovers the $k_0$ in Eq\[22\]. One reach the final action in terms of $\dot{\psi}$:

$$
S = \int d\tau d^2r \left[ (Z_1 - 2Z_2 c k_0) \dot{\psi}^* \partial_r \dot{\psi} + Z_2 |\partial_r \dot{\psi}|^2 - i 2Z_2 c \partial_r \dot{\psi}^* \partial_y \dot{\psi} + v_x^2 |\partial_x \dot{\psi}|^2 + (v_y^2 - Z_2 c^2) |\partial_y \dot{\psi}|^2 - \mu |\dot{\psi}|^2 + U |\psi|^4 \right]
$$

(85)

where $\mu = \mu + \frac{Z_1^2 c^2}{4(v_y^2 - Z_2 c^2)}$ and $Z_1 - 2Z_2 c k_0 = \frac{Z_1 v_y^2}{v_y^2 - Z_2 c^2} > 0$. Setting $\mu = 0$ recovers Eq\[83\].

Now we arrive at the same form as the effective action Eq\[22\]. Similar scaling analysis below Eq\[22\] apply here also. After dropping the more irrelevant term $|\partial_r \dot{\psi}|^2$ and keep only the leading irrelevant metric crossing term, we arrive at the same action as Eq\[23\].

$$
S = \int d\tau d^2r \left[ (\tilde{Z}_1 \dot{\psi}^* \partial_r \tilde{\psi} - i \tilde{Z}_2 \partial_r \dot{\psi}^* \partial_y \dot{\psi} + v_x^2 |\partial_x \dot{\psi}|^2 + v_y^2 |\partial_y \dot{\psi}|^2 - \tilde{\mu} |\dot{\psi}|^2 + U |\psi|^4 \right]
$$

(86)

where $\tilde{Z}_1 = Z_1 - 2Z_2 c k_0$, $\tilde{Z}_2 = -2Z_2 c$, $\tilde{v}_2^2 = v_y^2$, $\tilde{v}_x^2 = v_y^2 - Z_2 c^2$. Then the discussions following Eq\[23\] apply here also. Of course, setting $\tilde{Z}_1 = 1$, $\tilde{Z}_2 = 0$ recovers Eq\[63\].

So we conclude that the leading irrelevant metric crossing term $\tilde{Z}_2$ just leads to the shift of the value $k_0$ and the critical chemical potential $\mu_c$. However, as shown in Sec. IV, it is this term which leads to the Doppler shift term in both the Mott and BSF phases. Its contribution to the conserved Noether currents will be addressed in the following sub-section. After considering the $\tilde{Z}_2$ term, we reach consistent physical picture on the BSF phase from the $z = 2$ QPT line and the $(z_x, z_y) = (3/2, 3)$ QPT line in Fig\[7\].

C. The conserved Noether current with $z = 2$ in the moving frame

1. The emergent Galileo invariant case

The global $U(1)$ symmetry $\psi \rightarrow \psi e^{ik_0 y}$ leads to the conserved Noether current $\tilde{J}_\mu = (J_\tau, J_x, J_y)$ in Eq\[64\]

\[
\begin{align*}
J_\tau &= -i \psi^* \dot{\psi}, \\
J_x &= iv_x^2 (\psi^* \dot{\psi} - \psi \partial_x \psi^*), \\
J_y &= iv_y^2 (\psi \dot{\psi}^* - \psi^* \partial_y \psi^*) - c \psi^* \psi = J_y - icJ_\tau
\end{align*}
\]

(87)

which also show the current along the $y$ direction $\tilde{J}_y$ is the sum of the intrinsic one $J_y$ and the one due to the boost $-icJ_\tau$, identical to the last equation in Eq\[33\]. They satisfy

$$
\partial_\tau J_\tau + \partial_x J_x + \partial_y (J_y - icJ_\tau) = 0
$$

(88)

which is identical to Eq\[34\].

Now we evaluate the 3-currents in all the three phase in Fig\[7\]. Plugging in $\langle \psi \rangle = \sqrt{\rho_0} e^{ik_0 y}$ into Eq\[87\] leads to:

\[
\begin{align*}
J_\tau &= -i \rho_0, \\
J_x &= 0, \\
J_y &= -2v_y^2 k_0 \rho_0 - c \rho_0
\end{align*}
\]

(89)

The Mott state with $\rho_0 = 0$ carries no current. The SF state with $k_0 = 0$ carries a current $J_\tau = -i \rho_0, J_x = 0$ and $J_y = -c \rho_0$. Near the $z = 2$ line in Fig\[7\] substituting $k_0 = -\frac{v_y^2}{Z_2 c} < 0$ into Eq\[89\] so the BSF near the $z = 2$ line carries $J_\tau = -i \rho_0, J_x = 0$ and $J_y = 0$ which means the intrinsic current just cancels that due to the boost. However, near the $z = (3/2, 3)$ line in Fig\[7\] then in the BSF state, adopting the results listed below Eq\[55\] to the current case, one can see the explicit PH symmetry breaking term picks up $k_0 = -\sqrt{\frac{v_x^2 - 44}{26} \rho_0 v_y^2} < 0$ which goes to zero as $k_0 \rightarrow 0^-$, so the BSF near the $z = (3/2, 3)$ line starts to reduce from the current $\tilde{J}_y = -c \rho_0$. Combining the picture reached
from the \( z = 2 \) line and the \( z = (3/2, 3) \) line, we expect that the current carrying by the BSF reduce from that value carried by the SF near the \( z = (3/2, 3) \) line, to zero near the \( z = 2 \) line in Fig 7.

2. The \( Z_2 \) term breaking the emergent Galilean invariance

It is important to stress that the current Eq 33 for the \( z = 1 \) case studied in Sec.V is particle-hole current, but here for \( z = 2 \), Eq 33 is either particle or hole current. By adding the leading irrelevant \( Z_2 \) term in Eq 33, one can also evaluate the Noether 3-currents in Eq 34.

\[
J_x = -iZ_1 \psi \partial_x \psi + iZ_2 (\psi \partial_x \psi - \psi \partial_x \psi^*) = -iZ_1 \psi \partial_x \psi + iZ_2 [(\psi^* \partial_x \psi - \psi \partial_y \psi^*) - ic(\psi^* \partial_y \psi - \psi \partial_y \psi^*)]
\]

\[
J_y = -ic_2 \psi \partial_y \psi - \psi \partial_y \psi^* - icJ_r = J_y - icJ_r
\]

Setting \( Z_1 = 0, Z_2 = 1 \) or \( Z_1 = 1, Z_2 = 0 \) recovers Eq 33 or Eq 37 respectively.

The first equation shows that \( J_r \) consists of the particle ( or hole ) current Eq 39 with the weight \( Z_1 \) and the particle-hole current Eq 33 with a small weight \( Z_2 \ll Z_1 \). The second shows that \( J_x \) remains the same. The third shows the current along the \( y \) direction \( \tilde{J}_y \) is still the sum of the intrinsic one \( J_y \) and the one due to the boost \( -icJ_r \).

Plugging in \( \langle \psi \rangle = \sqrt{\rho_0} e^{ik_0 y} \) into Eq 32 leads to:

\[
J_r = -iZ_1 \rho_0 + iZ_2 k_0 \rho_0 = -i \rho_0 [Z_1 - Z_2 k_0 c],
\]

\[
J_x = 0
\]

\[
\tilde{J}_y = -2ic_2 k_0 \rho_0 - c \rho_0 [Z_1 - Z_2 k_0 c]
\]

which need to be evaluated at the corrected \( k_0 \) in Eq 32.

The Mott phase with \( \rho_0 = 0 \) carries no current. The SF phase with \( k_0 = 0 \) carries \( J_r = -iZ_1 \rho_0, J_x = 0 \) and \( \tilde{J}_y = -c \rho_0 Z_1 \). In the BSF phase near the \( z = 2 \) line, plugging in the \( k_0 \) in Eq 32, one finds \( J_r = -i \rho_0 (\gamma \sqrt{Z_1 c/ (Z_1 c - Z_2 c)}), J_x = 0, \tilde{J}_y = 0 \). However near the \( z = (3/2, 3) \) line in the SF phase side, \( k_0 \to 0^- \), it starts from that value carried by the SF \( J_r = -iZ_1 \rho_0, J_x = 0 \) and \( \tilde{J}_y = -c \rho_0 Z_1 \). So considering all the terms which break the emergent Galilean invariance and become important away from the \( z = 2 \) line, we conclude that the physical picture listed below Eq 36 reached simply by setting \( Z_2 = 0 \) remains valid.

Similarly, as outlined in Sec.V, one can also achieve the phase diagram Fig 4 from the charge-vortex duality performed in the moving frame. Namely, the \( z = 2 \) line is driven by the instability in the vortex degree of freedoms, while the \( z = (3/2, 3) \) line is the instability in the dual gauge degree of freedoms.

XI. CONCLUSIONS AND PERSPECTIVES

In this work, we start from the microscopic model of boson-Hubbard model of interacting bosons at integer fillings in a square lattice which shows Mott to Superfluid (SF) transitions. In the continuum limit, the low energy effective actions have either the emergent Lorentz or the emergent Galilean invariance with the dynamic exponent \( z = 1 \) and \( z = 2 \) respectively. Then we construct effective actions in the moving frame to study the new quantum phases and QPTs in the moving frame. Fig 2 and 7 show that the quantum phases and QPTs depend sensitively on the inertial frames. Counter-intuitively, a Mott insulating phase may become a BSF phase or vice versa. An insulating state in a lab frame may become a BSF state in a frame moving with a sufficient large velocity. This not only holds for the Lorentz invariant \( z = 1 \) case Fig 2, but also for the Galilean invariant \( z = 2 \) case Fig 7. So if it is a Mott insulator or a dissipation-less SF depends on which observer you are asking. Along the \( z = 2 \) vertical line in Fig 8, one increase the chemical potential \( \mu \) to drive the Mott state to a SF state. Here, as shown in Eq 21 for \( z = 1 \) and in Eq 74 for \( z = 2 \), one increases the kinetic energy of the fast moving train which plays a similar role as increasing \( \mu \) to drive the Mott state to the BSF state.

On a lattice with the lattice potential \( V(x) = V(x + a) \), as seen from Eq 33, only a discrete subgroup \( t \to t + a/c, y' \to y + a \) of the Galilean transformation \( t' = t, y' = y + c t \) is kept invariant. So when performing the Galilean transformation in a lattice, one need to add a boost ( or current ) term to respect the subgroup. It remain important to investigate how the global phase diagram Fig 5 respond to such a boost term. Then derive the effective actions Eq 34 and Eq 36 from the lattice model with such a boost term. Fig 2 and 7 are valid when the BSF condensation momentum \( k_0 \) is small compared to the size of the BZ. However, it may break down otherwise, then one must resort the lattice model. The charge-vortex duality presented in Sec. VI is formulated in the continuum. It can be best formulated in a dual lattice where the vortices are hopping subject to a dual gauge field on the links. Then boosting such a dual lattice in the presence of the dual gauges field is interesting on its own right, it belongs to a new class.
of problems: boosting a lattice gauge theory with both matter on the lattice site and gauge fields on the link. So it will also bring important insights to quantum phases and transitions with matter and gauge fields in the dual lattice. However, putting a gauge field in a lattice is not guaranteed. For example, it is still not known how to put topological QFT such as Chern-Simon theory in a lattice. In this case, the continuum limit is the only way to proceed.

It is known that the Entanglement entropy (EE) dramatically increases near a QCP. The out of time correlation (OTOC) to describe the quantum information scramblings [73–75] at a finite $T$ also dramatically increases. As shown in Sec.VI, despite the Mott to SF transition along path I in Fig.2 is the same universality class of 3D XP with $z = 1$, its finite temperature properties does depend on $c$. So it would be interesting to study how the Lyapunov exponent $\lambda_L = f(c)T$ depends on $c$ at the QCP, especially near the critical velocity $v_p$ (or the M point) in Fig.2. The SF to BSF transition along the path II in Fig.2 provides a rare example of a Gaussian theory with highly non-trivial dynamic exponent $z = (3/2, 3)$, the $[Z] = -2$ term is still quadratic, so will not lead to any quantum chaos, but the $[b] = -2$ term is non-linear and will lead to quantum chaos, so it remains important to evaluate the OTOC due to this leading irrelevant operator.

The new quantum phases and novel quantum phase transitions discovered in the work still fall into Ginsburg-Landau symmetry breaking picture. It was known that topological transitions without accompanying symmetry breaking are beyond Ginsburg-Landau scheme. The methods developed in this work can be applied to study any other QPTs or topological phase transitions in interacting bosonic or non-interacting/interacting fermionic systems. For example, as reviewed in Appendix F, the Fractional Quantum Hall systems (FQH) [16, 17, 20] mentioned in the introduction have Galileo invariance. It dictates that the mass appearing in the Landau level spacing (or cyclotron frequency $\omega_c = eB/mc$) is simply the bare mass $m$ which is not renormalized by any inter-particle interaction $V(x_i - x_j)$ in Eq.C6 [16, 17]. This result may be considered as the counter-part in a strongly interacting bosonic system: the superfluid density $\rho_s$ of a Galileo invariant system at $T = 0$ such as Helium 4 is equal to its density, namely, it is 100% superfluid at $T = 0$ despite many atoms are kicked out of the condensate at zero momentum to high momentum states due to the atom-atom interaction $V(x_i - x_j)$. Indeed, despite the many-body wavefunctions of a QH system in a particular gauge such as the Laughlin wavefunction Eq.C4 may break the Galileo invariance, any gauge invariant quantities such as its topological orders and anyonic excitations should do. In fact, the general coordinate invariance [78] which combine both local gauge transformation and the local metric transformation perturbed around the flat Minkowski space-time has been developed to study the strongly interacting Fermi gas near the Feshbach resonance. Galileo invariance is just a subgroup of such a big General coordinate invariance. Then it has been employed to study the combined system of bulk + edge states of the FQH. In the presence of the edge, boosting along the edge keeps the Galileo invariance, but perpendicular to the edge breaks it [76, 77], it puts some constraints on the dynamic electrical conductive along the edge.

The topological order in any FQH phase seems compatible with the Galileo invariance. However it was known the QCP from one FQH phase to another FQH phase has the dynamic exponent $z = 1$ which seems in-compatible with the Galileo invariance. Usually, a QCP has enlarged emergent symmetries instead of less. It remains outstanding to resolve such a puzzle. The Bilayer Quantum Hall systems (BLQH) [20] have a charge neutral sector which hosts exciton SF, as shown in Sec.X-B, it still breaks the Galileo invariance. Then we expect our results achieved in Sec.IV and Sec.X also apply to the ESF phase in the charge neutral sector. How does it change the charge sector in a moving frame remains interesting to see.

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The important roles by gauge invariance has been stressed and explored in both high energy and condensed matter physics. The Lorentz invariance is essential ingredient in relativistic QFT. Somehow, the roles of Galileo transformation in the many body interacting non-relativistic system in a continuum has been much less explored. In fact, there exist also quite confusing and contradicting discussions in the previous literatures on Galileo invariance. In the several appendices, we discuss it in a systematic way and also make connections to the new results established in the main text.

Appendix A: Boosted Hamiltonians for $z = 1$ and $z = 2$.

In the main text, we only used the boosted Lagrangian approach, here we applied the corresponding Hamiltonians. Despite the two approaches are completely equivalent, the Hamiltonian approach may be more intuitive in the lattice approach on the boson Hubbard model Eq.1 mentioned in the conclusion section. Similar quantization method is useful to quantize fields in a curved space-time.

1. The boosted Hamiltonian in the $z = 1$ case
FIG. 8. The well known phase diagram [4] of the Boson Hubbard model Eq.1 in the lab frame. The Mott insulating phase resides inside the \( n = 1, 2, 3, \cdots \) lobes. The superfluid (SF) phase takes the other space. There is a particle-hole (PH) symmetry along the horizontal dashed lines going through the tip of the lobes, the Mott-SF transition has the dynamic exponent \( z = 1 \). There is no PH symmetry away from the horizontal dashed lines, the Mott-SF transition has the dynamic exponent \( z = 2 \).

The solid lines emerging from the joint point between \( n \)-th and \( n+1 \)-th Mott lobe delineates the contours of constant density with \( n - \epsilon \) and \( n + 1 - \epsilon \) respectively. The \( z = 1 \) and \( z = 2 \) SF-Mott transitions have emergent "Lorentz" invariance [29] and Galileo invariance respectively. As shown in Fig.2, 7, they response very differently to the boost of the moving frame.

From Eq.3 one can find the conjugate momentum,

\[
\Pi = \frac{\partial L_M}{\partial (\partial_\tau \psi)} = \partial_\tau \psi^* - ic\partial_y \psi^* \\
\Pi^* = \frac{\partial L_M}{\partial (\partial_\tau \psi^*)} = \partial_\tau \psi - ic\partial_y \psi
\]

(A1)

After imposing the equal-time commutation relations:

\[
[\psi(\vec{x}, t), \psi(\vec{x}', t)] = 0 \\
[\Pi(\vec{x}, t), \Pi(\vec{x}', t)] = 0 \\
[\psi(\vec{x}, t), \Pi(\vec{x}', t)] = i\hbar \delta(\vec{x} - \vec{x}')
\]

(A2)

where there is also an identical set for \( \psi^* \) and \( \Pi^* \). The two sets commute with each other.

One can find the corresponding Hamiltonian:

\[
H_{M, z=1} = -\Pi \partial_\tau \psi - \Pi^* \partial_\tau \psi^* + L_M = -\partial_\tau \psi^* \partial_\tau \psi - c^2 \partial_y \psi^* \partial_y \psi + v_x^2 |\partial_x \psi|^2 + v_y^2 |\partial_y \psi|^2 + r |\psi|^2 + u |\psi|^4 + \cdots
\]

(A3)

Now it is important to express \( \partial_\tau \psi^* = \Pi + ic\partial_y \psi^*, \partial_\tau \psi = \Pi^* + ic\partial_y \psi \) in terms of the conjugate momentum and substituting them into the Eq.A3 leads to:

\[
H_{M, z=1} = -\Pi \Pi^* - ic\Pi \partial_y \psi - ic\Pi^* \partial_y \psi^* + v_x^2 |\partial_x \psi|^2 + v_y^2 |\partial_y \psi|^2 + r |\psi|^2 + u |\psi|^4 + \cdots
\]

(A4)

which plus the commutation relations Eq.A2 gives the complete quantized boosted Hamiltonian for \( z = 1 \). Putting \( c = 0 \) recovers the \( H_M \) in the lab frame.

2. The boosted Hamiltonian in the \( z = 2 \) case

From Eq.64 one can find the conjugate momentum,

\[
\Pi = \frac{\partial L_M}{\partial (\partial_\tau \psi)} = \psi^t
\]

(A5)

After imposing the equal commutation relations:

\[
[\psi(\vec{x}, t), \psi(\vec{x}', t)] = 0 \\
[\psi^t(\vec{x}, t), \psi(\vec{x}', t)] = 0 \\
[\psi(\vec{x}, t), \psi^t(\vec{x}', t)] = i\hbar \delta(\vec{x} - \vec{x}')
\]

(A6)
One can find the corresponding Hamiltonian:

\[ H_{M,z=2} = -\Pi \partial_\tau \psi + L_M \]
\[ = -ic\psi^\dagger \partial_y \psi + v_y^2 |\partial_x \psi|^2 + v_x^2 |\partial_y \psi|^2 - \mu |\psi|^2 + u|\psi|^4 + \cdots \]
\[ = \psi^\dagger [-ic\partial_y \psi - v_y^2 \partial_x^2 - v_x^2 \partial_y^2 - \mu] \psi + u(\psi^\dagger \psi)^2 + \cdots \] (A7)

which is automatically in terms of the conjugate momentum \( \psi^\dagger \). It plus the commutation relations Eq.A6 gives the complete quantized boosted Hamiltonian for \( z = 2 \). Putting \( c = 0 \) recovers the \( H_M \) in the lab frame.

Following the procedures in Appendix D-2 and combining the \( z = 1 \) and \( z = 2 \) case, one can also derive the Hamiltonian corresponding Eq.80.

**Appendix B: A Spin-orbit coupling breaks Galileo invariance**

It was known that the Electro-Magnetic (EM) fields satisfy Maxwell equations which is intrinsically Lorentz invariant. Klein-Gordan (KG) equations for spin-0 bosons and Dirac equations for spin-1/2 fermions are also Lorentz invariant. So Klein-Gordan equations and Dirac equations couple to the EM fields in the minimal coupling scheme are gauge invariant and also Lorentz invariant. Taking the non-relativistic limit \( c \to \infty \) limit, KG equation just reduces to S-equation in a EM field, while the Dirac equation reduces to S-equation in an EM field plus several new terms such as the spin-orbit coupling term and Darwin term. The gauge invariance is kept in such a limit. However, despite Bosons and fermions have a well defined non-relativistic limit, but the EM fields does not. The free part in the resulting S-equation from KG equation and from the Dirac part is Galileo invariant. However, the EM terms in the resulting S-equation from KG equation and EM terms plus these extra terms from the Dirac equation may not. In this appendix, we first show that the Fermi Surface of non-interacting fermions breaks Lorentz invariance, but owns Galileo invariance. Then we show that as expected, any spin-orbit coupling breaks the Galileo invariance.

1. **A moving FS: Galileo invariance**

Using the prescription presented in the main text, we immediately find the FS observed in a moving frame with the velocity \( v \hat{y} \) in the first quantization form:

\[ H_M = \frac{\hbar^2}{2m} k^2 - \mu - ck_y, \quad \mu = \frac{\hbar^2 k_0^2 F}{2m} \] (B1)

where \( k_0 \) is the Fermi momentum in the lab frame.

Eq.B1 can be written as

\[ H_M = \frac{\hbar^2}{2m} [(k^2 - k_0^2)^2 - (k_{0,F}^2 + k_0^2)] \] (B2)

where \( k_0 = mc/\hbar^2 \hat{y} \) is the FS center shift due to the boost.

For 1d case, one can identify the two Fermi points in the moving frame:

\[ k^+_F = k_0 + \sqrt{k_0^2 + k_{0,F}^2} > 0 \]
\[ k^-_F = k_0 - \sqrt{k_0^2 + k_{0,F}^2} < 0 \] (B3)

which shows the Fermi momentum is shifted from \( \pm k_0F \).

In fact, Eq.B2 can also be viewed as the kinetic term in the interacting boson case in Eq.A7. In boson case, one focus on the BEC momentum, so the shift can be transformed away by introducing the new field and then the new effective chemical potential in Eq.70. So the boson case in Eq.A7 has the Galileo invariance. In the fermion case, one can perform the similar set of transformation as in Eq.A7, one can show Eq.B2 is also Galileo invariant. Both the boson and fermion case is also precisely due to the Galileo invariance of the non-relativistic Schrodinger equation at \( V = 0 \).

2. **Spin-orbit coupling: breaking Galileo invariance**

One can add a Weyl like SOC term \( \lambda \vec{k} \cdot \vec{S} \) to Eq.B1

\[ H_{M,SOC} = \frac{\hbar^2 k^2}{2m} - \mu - ck_y + \lambda \vec{k} \cdot \vec{S}, \quad \mu = \frac{\hbar^2 k_0^2 F}{2m} \] (B4)

In the helicity basis, the SOC term can also be written as a linear \( k \) term with \( \pm \) sign.
Eq. (B4) can be written as
\[
H_{M,SOC} = \frac{\hbar^2}{2m} [(\vec{k} - \vec{k}_0)^2 - (k_{0F}^2 + k_0^2)] + \lambda \vec{k} \cdot \vec{S}
\] (B5)
where \( \vec{k}_0 = mc/\hbar \vec{y} \) is the FS center shift due to the boost.

After performing the similar set of transformation as in Eq. (A7), Eq. (B5) becomes:
\[
H_{M,SOC} = \frac{\hbar^2}{2m} [q^2 - (k_{0F}^2 + k_0^2)] + \lambda q \cdot \vec{S} + \lambda \vec{k}_0 \cdot \vec{S}
\] (B6)
where the extra term \( \lambda \vec{k}_0 \cdot \vec{S} \) breaks the Galileo invariance. So we conclude the SOC breaks the Galileo invariance. Similarly, one can show that the more conventional \( \lambda \vec{S} \cdot \vec{L} \) also breaks the Galileo invariance. For the dramatic effects of SOC in interacting fermionic systems in a continuum, see [64–66].

Appendix C: Galileo transformation in single particle and many body Schrodinger equation, symmetry breaking in the thermodynamic limit

We first study how the Schrodinger equation transform under the Galileo transformation, then extend it to many body case. Then we show there could be intrigue interplay between symmetry breaking and Galileo invariance in the thermodynamic limit of SOC in interacting fermionic systems in a continuum, see [64–66].

1. Galileo transformation on the single particle Schrodinger equation

It was known that the non-relativistic Schrodinger equation (1st quantization form)
\[
i\hbar \frac{\partial \psi}{\partial t} = \left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x,t)\right] \psi
\] (C1)
transform under the Galileo transformation \( x' = x - vt, t' = t \) as:
\[
i\hbar \frac{\partial \psi'}{\partial t'} = \left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x'^2} + V'(x',t')\right] \psi'
\] (C2)
where the wavefunction and the potential in the lab frame are related to those in the moving frame by:
\[
\psi(x,t) = e^{i(k_0 x - E_0 t)} \psi'(x',t')
\]
\[
V(x + vt, t') = V'(x',t') \neq V(x',t')
\] (C3)
where \( k_0 = \frac{mv}{\hbar}, E_0 = \frac{\hbar^2 k_0^2}{2m} = \frac{1}{2}mv^2 \). In general, due to \( V'(x',t') \neq V(x',t') \) Schrodinger equation is not Galileo invariant. Of course, if \( V = 0 \), it is Galileo invariant. This is the case discussed above. If in a periodic lattice, \( V(x + a) = V(x) \), the momentum is defined only upto a reciprocal lattice, leading to the concept of Brillouin Zone (BZ), then S- equation is not Galileo invariant. For a random potential \( V(x) \) such as electrons moving in a disordered potential, then \( V(x) \) breaks the translational invariance completely. Only energy is conserved, the momentum is not defined. Then S- equation is not Galileo invariant. As shown in the last section, the SOC also breaks the Galileo invariance.

In a uniform magnetic field, the gauge potential \( \vec{A}(x) \) usually breaks the translational invariance, however, the magnetic field does not. For example, in the Landau gauge, the gauge potential breaks translational invariance along a given direction, in the symmetric gauge, the gauge potential breaks translational invariance completely, but keeps rotational symmetry, then the wavefunction such as the Laughlin wavefunction in the symmetric gauge indeed breaks translational invariance, but keeps rotational symmetry.

\[
\Psi(z_1, z_2, \cdots, z_N) = \Pi_{i<j}(z_i - z_j)^m e^{-\frac{\sum_i |z_i|^2}{\hbar \alpha}}
\] (C4)
where \( m \) is odd (even) for fermion (boson) respectively, \( \hbar \alpha \) is the magnetic length. The Jastraw factor keeps, but the Gaussian factor breaks the Galileo invariance. However, any gauge invariant quantity such as the energy gap, quasi-particle (fractional) charge, fractional statistics, topological order, Hall conductivity, etc must be Galileo invariant. We expect them to be also Galileo invariant.

2. Galileo transformation on many body Schrodinger equation and the symmetry breaking in the thermodynamic limit
Eq. [C1] can be easily generalized to many $N$ electronic or bosonic systems with short-range two-body $V(x_i - x_j)$ or long-range Coulomb interactions.

$$i\hbar \frac{\partial \Psi(x_1, x_2, \cdots x_N)}{\partial t} = \left[ \sum_i \frac{1}{2m} \left[ -i\hbar \frac{\partial}{\partial x_i} - e\vec{A}(x_i) \right]^2 + \sum_{i<j} V(x_i, x_j) \right] \Psi(x_1, x_2, \cdots x_N)$$

where $\vec{B} = \nabla \times \vec{A} = B\vec{z}$.

The many-body wavefunction and the gauge potential in the (symmetric gauge) in the lab frame are related to those in the moving frame by:

$$\Psi(x_i, t) = e^{i(k_0 \sum x_i - E_0 t)} \Psi'(x_i', t'),$$

$$A'_x(y') = -\frac{1}{2} By', \quad A'_y(x', t') = \frac{1}{2} Bx' + \frac{1}{2} Bvt'$$

where $k_0 = \frac{m_0}{2\hbar}, E_0 = \frac{\hbar^2 k_0^2}{2m_0} = \frac{1}{2} mv^2$. $1/2Bvt'$ can be removed by a gauge transformation $\Psi' \rightarrow \Psi' e^{iBvt' \sum \vec{q} \cdot \vec{y}}$. Obviously, $V(x_1' - x_2') = V(x_1 - x_2)$, so is translational and Galileo invariant. Of course, Eq. [C6] also applies to the interacting bosons in a rotating trap where the gauge field $\vec{A}$ comes from the rotation close to the trapping frequency. Setting $\vec{A} = 0$ recovers the Helium 4 case which is clearly Galileo invariance.

In the thermodynamic limit $N \rightarrow \infty$, various quantum or topological phases exist and connected by various quantum or topological phase transitions. For example, the Boson Hubbard model Eq. [D3] display the Mott, SF phase which are connected by the SF-Mott transition. Despite the effective action Eq. [C5] has emergent Galileo invariance, it is still spontaneously broken inside the SF phase. In QH system, QH phase does not break any symmetry, so it remains Galileo invariance. The topological order in any FQH phase seems compatible with the Galileo invariance.

### Appendix D: A moving superfluid, Doppler shifts and Galileo transformation

In this appendix, we perform some microscopic calculations on the superfluid to demonstrates the known phenomena of Doppler shifts due to the Galileo transformation. They are consistent with the mean field + Gaussian fluctuation analysis on the effective action approach used in the main text. However, the main limitation of the microscopic calculations used in this appendix is that it is not able to determine what is the quantum phase beyond the critical velocity, let alone the universality class of the quantum phase transitions driven by the boost. These questions can only be addressed by the effective action and RG approach used in the main text.

1. **Galileo invariance of a moving SF**

The Hamiltonian for weakly interacting bosons in a continuum system is

$$H_B = \sum_{\vec{k}} (\epsilon_{\vec{k}} - \mu) b_{\vec{k}}^\dagger b_{\vec{k}} + \frac{1}{2A} \sum_{\vec{k}, \vec{q}, \vec{p}} V(\vec{q}) b_{\vec{k} - \vec{p}}^\dagger b_{\vec{p}} b_{\vec{k} + \vec{q}}$$

where $\epsilon_{\vec{k}} = \hbar^2 k^2 / 2m$ is the free boson dispersion, $\mu$ is the chemical potential, $A$ is the 2d area, $V_d(\vec{q})$ is the boson-boson interaction.

Suppose the SF is moving with a momentum $\vec{q}$:

$$\psi_0 = \sqrt{N_0} e^{i\vec{q} \cdot \vec{x}}$$

which means a finite superflow $\vec{v} = \vec{q} / m$ where $m$ is the mass of an atom. Now one can write the boson operator as:

$$\psi_{\vec{q} + \vec{k}} = \sqrt{N_0} \delta_{\vec{k}, \vec{0}} + b_{\vec{q} + \vec{k}}$$

where $b_{\vec{q} + \vec{k}}$ stands for the quantum fluctuations with the momentum $\vec{k}$ measured relative to the BEC momentum $\vec{q}$.

Substituting Eq. [D3] into Eq. [D1] and expanding it to the quadratic order, one can determine the chemical potential $\mu$ by eliminating the linear term of $b_{\vec{q}}$ in the Hamiltonian $H_{SF}$ as

$$\mu = n_0 V_d(0) + \frac{1}{2} mv^2$$

where $n_0 = N_0 / A$ is the condensate density. Then one obtain the mean field Hamiltonian $H_{SF}$ to the quadratic order:

$$H_{SF} = \sum_{\vec{k}} \left[ \epsilon_{\vec{k}} + n_0 V_d(\vec{q} - \vec{k}) - \epsilon_{\vec{q}} b_{\vec{k}}^\dagger b_{\vec{k}} + \frac{n_0}{2} \sum_{\vec{k}} [V_d(\vec{k}) b_{\vec{q} + \vec{k}}^\dagger b_{\vec{q} - \vec{k}} + h.c.] \right]$$
which can be diagonalized by the Bogoliubov transformation
\[ \beta_{\vec{k}} = u_{\vec{k}} b_{\vec{q} + \vec{k}} + v_{\vec{k}} b_{\vec{q}}^\dagger \]  
(D6)

We obtain \( H_{SF} \) in terms of the quasi-particle creation and annihilation operators \( \beta_{\vec{k}} \) and \( \beta_{\vec{k}}^\dagger \):
\[ H_{SF} = E(0) + \sum_{\vec{k}} E_v(\vec{k}) \beta_{\vec{k}}^\dagger \beta_{\vec{k}} \]  
(D7)

where \( E(0) \) is the ground state energy and
\[ \begin{align*}
    u_{\vec{k}}^2 &= \frac{\epsilon_{\vec{k}} + n_0 V_\delta(\vec{k})}{2E(\vec{k})} + \frac{1}{2} \\
v_{\vec{k}}^2 &= \frac{\epsilon_{\vec{k}} + n_0 V_\delta(\vec{k})}{2E(\vec{k})} - \frac{1}{2} \\
    E_v(\vec{k}) &= E(\vec{k}) + \vec{k} \cdot \vec{v}
\end{align*} \]  
(D8)

where \( E(\vec{k}) = \sqrt{\epsilon_{\vec{k}}^2 + 2n_0 V_\delta(\vec{k})\epsilon_{\vec{k}}} \).

One can see that in the moving SF, \( u_{\vec{k}} \) and \( v_{\vec{k}} \) are the same as in the lab frame. However, the energy spectrum \( E_v(\vec{k}) \) contains a Doppler shift term \( \vec{k} \cdot \vec{v} \). In the low energy limit \( \vec{k} \to 0 \) limit, \( E_v(\vec{k}) \to u|\vec{k}| + \vec{k} \cdot \vec{v} \). If one picks up \( \vec{k} \parallel -\vec{v} \), one can identify the critical velocity \( v_c = u \). Unfortunately, one is not able to tell what will happen beyond the critical velocity from this mean field theory approach. This outstanding problem can only be addressed by the effective action and RG approach demonstrated in the main text.

One can also obtain normal and anomalous Green function:
\[ \begin{align*}
    G_n(\vec{q}; \vec{k}, \omega) &= i \frac{\omega - \vec{k} \cdot \vec{v} + \epsilon_{\vec{k}} + n_0 V_\delta(\vec{k})}{(\omega - \vec{k} \cdot \vec{v})^2 - E^2(\vec{k})} \\
    G_a(\vec{q}; \vec{k}, \omega) &= i \frac{n_0 V_\delta(\vec{k})}{(\omega - \vec{k} \cdot \vec{v})^2 - E^2(\vec{k})} 
\end{align*} \]  
(D9)

where one can identify the excitation spectrum \( \omega = \pm E(\vec{k}) + \vec{k} \cdot \vec{v} \) which is nothing but the last equation in Eq.\( \text{D8} \).

2. Galileo transformation
The Galilean transformation is:
\[ \begin{align*}
    \vec{k}' &= \vec{k} \\
    E' &= E - \vec{k} \cdot \vec{v} + \frac{1}{2} mv^2 \\
    \mu' &= \mu + \frac{1}{2} mv^2
\end{align*} \]  
(D10)

where \( \vec{k}', E', \mu' \) are the momentum, energy and chemical potential in the moving frame, where \( \vec{k}, E, \mu \) are those in the lab frame. Note that the momentum does not change, because as listed in Eq.\( \text{D3} \) it was measured from the BEC momentum \( \vec{q} = mv \) from very beginning.

In the moving frame, the Green functions take the same form as those in the lab frame:
\[ \begin{align*}
    G_n(\vec{k}', \omega') &= i \frac{\omega' + \epsilon_{\vec{k}} + n_0 V_\delta(\vec{k}')}{\omega'^2 - E^2(\vec{k}')} \\
    G_a(\vec{k}', \omega') &= i \frac{n_0 V_\delta(\vec{k}')}{\omega'^2 - E^2(\vec{k}')} 
\end{align*} \]  
(D11)

By substituting
\[ \begin{align*}
    \omega' &= E' - \mu' = E - \mu - \vec{k} \cdot \vec{v} = \omega - \vec{k} \cdot \vec{v}
\end{align*} \]  
(D12)

which is just the non-relativistic \( c \to \infty \) limit of Eq.\( \text{D11} \) one can see Eq.\( \text{D11} \) recovers Eq.\( \text{D9} \).

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[29] In fact, we should call such emergent "Lorentz" invariance as pseudo-Lorentz invariance. Because it is the intrinsic velocity v of the z = 1 system which plays the role of the speed of light cl in real Lorentz invariance in relativistic QFT. So when applying the real-Lorentz transformation to such a pseudo-Lorentz system, the system is not invariant. Instead the real-Lorentz transformation reduce to the Galileo transformation in the cl → ∞ limit. It is the main goal of this work to study the dramatic effects of such a Galileo transformation.

[30] If one had this term [(∂εψ - iε∂ψ)ψ], Eq.1 would be S_{eff} = \int dxdr[r(∂εψ)^2 + v_ε^2(∂εψ)^2 + (v_ε^2 + c^2)(∂ψ)^2 + a|∂ψ|^2 + r|ψ|^2 + u|ψ|^4] which is nothing but the original z = 1 theory Eq.1 with an increased v_ε^2. The most important crossing metric g_{L, y} term would be absent in this case. The only effect of the boost would just introduce an anisotropy to the velocity. This can not be right because the boost obviously break the PH ( C) symmetry and the parity ( P ), but keeps its combination CP. It is the crossing metric g_{L, y} term which does such a job.

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[44] Similar approach is applied in [22, 23] to study how the Mott phase (charge density wave (CDW) or valence bond solid (VBS)) at the half-filling 1/2 response to the filling $f$ slightly away from the half-filling $\delta f = f - 1/2$ in the dual vortex approach. One need to find the effective action inside the CDW or VBS phase which is a vortex condensed phase and breaks the translational symmetry, then apply a dual magnetic field $\delta f = f - 1/2$ to such an effective action. Then CDW supersolid or VB supersolid naturally results from such an approach.
[45] If $c_1c_2 \neq 0$, then the putative state breaks both $U(1)$ and the lattice translational symmetry, so possess both off-diagonal and diagonal symmetry, could be named a solid-state superfluid. See the discussions in Sec.III-B.
[46] Originated from Eq. [13], there are always higher derivative terms such as $v_{xx}(\partial^2 \phi)^2 + v_{xy}(\partial^2 \phi^2) + v_{xx}(\partial^2 \phi)(\partial^2 \phi^2)$, but including these extra terms will not change the results derived in the following, so we will not write them out explicitly.
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