Quark mass function at finite density in real-time formalism

Hidekazu Tanaka* and Shuji Sasagawa

Department of Physics, Rikkyo University, Tokyo 171-8501, Japan

*E-mail: tanakah@rikkyo.ac.jp

Received December 7, 2019; Revised March 3, 2020; Accepted March 19, 2020; Published May 28, 2020

Chiral symmetry restoration of quarks is investigated at finite density in quantum chromodynamics. The critical chemical potentials at zero temperature, in which the chiral symmetry is restored, are calculated with the Schwinger-Dyson equation in the real-time formalism without the instantaneous exchange approximation. We present some properties of the quark mass functions and the quark propagators.

Subject Index B69

1. Introduction

Evaluation of phase transitions at finite density in quantum chromodynamics (QCD) is a difficult task even with lattice calculations, particularly due to an imaginary chemical potential [1].

In order to study the chiral phase transitions, one useful tool is the Schwinger–Dyson equation (SDE) [2,3], which can evaluate nonperturbative phenomena. Using the SDE, various works have been done for finite temperature and density.

For equilibrium systems, the imaginary-time formalism (ITF) is implemented in one-loop order with the SDE, which continues to Euclidean space at zero temperature limit. So far, the phase structures of the QCD transition has been studied [4–8].

On the other hand, the SDE in the real-time formalism (RTF), which can evaluate non-equilibrium systems, is formulated in Minkowski space. The SDE in the RTF has been studied with the instantaneous exchange approximation (IEA) [9,10], in which gauge boson energy is neglected. In the IEA, it has been shown that the mass function does not depend on the energy, and the critical coupling of the chiral symmetry breaking in quantum electrodynamics (QED) is about half that calculated with four-momentum integration in Euclidean space at zero temperature [10].

So far, the structure of the quark mass function in the strong coupling region in the entire range of energy and momentum space has not been fully studied in the RTF at finite temperature and density. In previous papers [11–13] we formulated the SDE for QED and QCD, in which the momentum integration is performed in Minkowski space without the IEA. In our method, real and imaginary parts of the mass functions are directly evaluated. Therefore, we can investigate the instability of systems due to imaginary parts of the quark energy with a real value of the chemical potential. On the other hand, in the ITF, the imaginary parts of the quark mass function due to an imaginary chemical potential is not regarded as a decay constant [5].

In our model, the critical temperature $T_C$, in which the chiral symmetry is restored at zero chemical potential, is $T_C \simeq 0.55\Lambda_{QCD}$. Furthermore, the effective quark mass evaluated at the resonance peak...
of an effective quark propagator is given as \( M_q \simeq \Lambda_{\text{QCD}} \) [13]. Here \( \Lambda_{\text{QCD}} \) denotes the QCD scale parameter.

In this paper, we study the chiral phase transition of the quark mass with the SDE in the RTF beyond the IEA for finite density at zero temperature, which corresponds to a high-density matter at low temperature.

In Sect. 2, we formulate the SDE in the RTF without the IEA. In Sect. 3, some numerical results for chiral phase transition in finite density are presented. In order to investigate the instability of the massive quark state, we evaluate time dependences of the effective quark propagator. Section 4 is devoted to the summary and some comments. The explicit expressions implemented in numerical calculations are given in Appendices A and B.

2. SDE for quark mass function

In the RTF, we implement two types of fields specified by “1” and “2” in the theory. The type-1 field is the usual field and the type-2 field corresponds to a ghost field in the heat bath.

In one-loop order, we calculate the 1–1 component of a self-energy of quark \( \Sigma^{11}(P) \) in QCD, which is given by

\[
- i \Sigma^{11}(P) = (i g_s)^2 C_F \int \frac{d^4 Q}{(2\pi)^4} \gamma^\mu i S^{11}(Q) \Gamma^\nu i D^{11}_{\mu \nu}(K),
\]

(2.1)

where \( S^{11}(Q) \) and \( D^{11}_{\mu \nu}(K) \) are the 1–1 components of thermal propagators for a quark with momentum \( Q = (q_0, \mathbf{q}) \) and a gluon with momentum \( K = P - Q = (k_0, \mathbf{k}) \), respectively. An external momentum of the quark is denoted by \( P = (p_0, \mathbf{p}) \). In our formulation, the time evolution of the system is generated with an operator \( \hat{H}' = \hat{H} - \mu \hat{N} \). Here \( \hat{H} \) and \( \hat{N} \) denote a Hamiltonian and a number operator of the quark, respectively. The energy eigenvalues for \( \hat{H}' \) are denoted by \( p_0 \) and \( q_0 \). In our calculation, the quark–gluon vertex is defined by \( \Gamma^\nu = \gamma^\nu \) with the gamma matrices \( \gamma^\nu \). The strong coupling constant and the color factor are denoted by \( g_s \) and \( C_F = 4/3 \), respectively.

The 1–1 component of the quark propagator in the RTF is given as

\[
i S^{11}(Q) = i \left[(S_F(Q))_R + i(S_F(Q))_I N_F(\mu, q_0)\right],
\]

(2.2)

where \( N_F(\mu, q_0) = \epsilon(q_0 + \mu) \epsilon(q_0) \) with a chemical potential \( \mu \) at zero temperature, in which we define \( \epsilon(z) = \theta(z) - \theta(-z) \) with the step function \( \theta(z) \).\(^1\) Here, \( (S_F(Q))_R \) and \( (S_F(Q))_I \) are the real

\(^1\) For a non-zero value of an imaginary part of the self-energy as \( \text{Im}(\Sigma^{11}) \neq 0 \), the quark propagator with a finite temperature \( T \) and a chemical potential \( \mu \) in the RTF may be written as [14]

\[
i S^{11}(Q) = i S_F(Q) - i S_F(Q) - S_F(Q)n_F(T, \mu, q_0),
\]

with

\[
n_F(T, \mu, q_0) = \frac{\theta(q_0 + \mu)}{e^{q_0/T} + 1} + \frac{\theta(-q_0 - \mu)}{e^{-q_0/T} + 1},
\]

in our formulation, which gives

\[
i S^{11}(Q) = i \left[(S_F(Q))_R + i(S_F(Q))_I \epsilon(q_0 + \mu) \tanh \left(\frac{q_0}{2T}\right)\right].
\]

Here, we approximate

\[
\tanh \left(\frac{q_0}{2T}\right) \rightarrow \epsilon(q_0)
\]

for \( T \rightarrow 0 \).
and imaginary parts of the quark propagator $S_F(Q)$, respectively.\(^2\)

The quark propagator $S_F(Q)$ is defined as

$$iS_F(Q) \equiv i(Q + \gamma^0 \mu + M(Q))I_F(Q). \quad (2.3)$$

with

$$I_F(Q) = \frac{1}{(q_0 + \mu)^2 - q^2 - M^2(Q) + i\epsilon}. \quad (2.4)$$

Here, $M(Q)$ denotes a quark mass function, which depends on the energy and momentum.

The 1–1 component of the gluon propagator is given as

$$iD^{11}_{\mu\nu}(K) = iD_{F\mu\nu}(K) = i\left[(D_{F\mu\nu}(K))_R + i(D_{F\mu\nu}(K))_I\right], \quad (2.5)$$

where

$$iD_{F\mu\nu}(K) = P^L_{\mu\nu}iD_L(K) + P^T_{\mu\nu}iD_T(K) \quad (2.6)$$

with

$$P^L_{\mu\nu} = -g_{\mu\nu} + \frac{K_{\mu}K_{\nu}}{K^2} - P^T_{\mu\nu} \quad (2.7)$$

and

$$P^T_{\mu\nu} = \left(-g_{\mu\nu} + \frac{K_{\mu}K_{\nu}}{K^2}\right)(1 - \delta_{0\mu})(1 - \delta_{0\nu}), \quad (2.8)$$

where the longitudinal and transverse components of the gluon propagator are given as

$$iD_L(K) = \frac{i}{K^2 - m_L^2 + i\epsilon} \quad (2.9)$$

and

$$iD_T(K) = \frac{i}{K^2 - m_T^2 + i\epsilon}, \quad (2.10)$$

respectively. Here, $m_L$ and $m_T$ denote the longitudinal and transverse gluon masses, respectively.

In this paper, we assume that the quark mass function only depends on $q_0$ and $q = |q|$ as $M(q_0, q)$. Integrating over the azimuthal angle of the quark momentum $q$, the trace of the self-energy $\Sigma^{11}$ is given by

$$M^{11}(p_0, p) \equiv \frac{1}{4} Tr[\Sigma^{11}(P)] = -\frac{iC_F}{2\pi^2} \int_{-\Lambda_0}^{\Lambda_0} dq_0 \int_{\delta}^{\Lambda} dq \frac{q}{p} a_s[M_{11}J_{11}] \quad (2.11)$$

\(^2\) The real and imaginary parts of a propagator $G$ are defined by

$$(G)_R = \frac{1}{2}(G + G^*)$$

and

$$(G)_I = \frac{1}{2}(G - G^*),$$

respectively. Here, $(G)_R$ and $(G)_I$ are separately evaluated by numerical calculations.
with $p = |p|$ and $\alpha_s = g^2/(4\pi)$, where

$$MI_{11} = (MI_F)_R + i(MI_F)_I(\mu, q_0)$$

(2.12)

and

$$J_{11} = (J_F)_R + i(J_F)_I.$$  

(2.13)

Here,

$$J_F = \int_{\eta_{++}}^{\eta_{--}} dkk [D_L(K) + 2D_T(K)]$$

(2.14)

with $\eta_{\pm} = |p \pm q|$ and $k = |k|$, respectively.

The real part $M_R$ and the imaginary part $M_I$ of the quark mass function $M$ are given by $M_R = (M_{11})_R$ and $M_I = (M_{11})_I/N_\eta(\mu, q_0)$, respectively. Furthermore, the real part $(M^2)_R$ and the imaginary part $(M^2)_I$ of $M^2$ are given by $(M^2)_R = (M_R)^2 - (M_I)^2$ and $(M^2)_I = 2M_RM_I$, respectively.

In numerical calculations, the real and imaginary parts of the quark energy $E_R$ and $E_I$ are defined by $E_R = |E| \cos(\Phi/2)$ and $E_I = |E| \sin(\Phi/2)$ with $\Phi = \text{arctan}((E^2)_1/(E^2)_R)$ and $|E| = \sqrt{|E^2|} = (\sqrt{(E^2)_R^2 + (E^2)_I^2})^{1/4}$. Here the real and imaginary parts of the squared quark energy $E^2$ are defined as $(E^2)_R = p^2 + (M^2)_R$ and $(E^2)_I = (M^2)_I - \varepsilon$, respectively, which are evaluated by numerically calculated $M_R$ and $M_I$ for each $q_0$ and $q$.

In Minkowski space, if the imaginary part of the squared mass function $(M^2)_I$ is small, the quark propagator $I_F$ in Eq. (2.4) varies rapidly near $q_0 + \mu^2 - q^2 \simeq (M^2)_R$. As implemented in the previous works [11–13], we divide the $q_0$ integration into small ranges and integrate the quark propagator over $q_0^{(i)} \leq q_0 \leq q_0^{(i+1)}$ ($q_0^{(1)} = -\Lambda_0, q_0^{(N)} = \Lambda_0$), in which the remaining contributions of the integrand are averaged over the range $q_0^{(i)} \leq q_0 \leq q_0^{(i+1)}$. The explicit expressions are summarized in Appendix A, in which we implement the running coupling constant for $\alpha_s$.

In order to investigate the instability of the massive quark state, we evaluate time dependences of the quark propagator

$$i\tilde{S}_F(x_0, p) = \int d^3x iS_F(x_0, x)e^{-ix\cdot p}$$

(2.15)

with

$$iS_F(x_0, x) = \int \frac{d^3Q}{(2\pi)^3} iS_F(Q)e^{-i(x_0q_0 - x\cdot q)} ,$$

(2.16)

where $x_0$ and $x$ denote time and space coordinates, respectively.

Integrating over $q$, the quark propagator $i\tilde{S}_F(x_0, p)$ is given as

$$i\tilde{S}_F(x_0, p) = \int \frac{dq_0}{2\pi} i\tilde{S}_F(\tilde{Q})e^{-i\tilde{Q}q_0}$$

(2.17)

with $\tilde{Q} = (q_0, p)$. We separate $i\tilde{S}_F(x_0, p)$ as

$$i\tilde{S}_F(x_0, p) = i\tilde{S}_F^{(+)}(x_0, p) + i\tilde{S}_F^{(-)}(x_0, p)$$

(2.18)

with

$$i\tilde{S}_F^{(\pm)}(x_0, p) = \pm \int \frac{dq_0}{2\pi} i\tilde{S}_F(\tilde{Q} \mp \gamma^0\mu + M(\tilde{Q})) \frac{1}{2E(\tilde{Q})} e^{-i\tilde{Q}q_0},$$

(2.19)
where $E(\tilde{Q})$ is the quark energy.
Here, $iS_F^{(\pm)}$ are further written by

$$
 iS_F^{(\pm)}(x_0, p) = \pm i\gamma_0 \int dq_0 \frac{1}{2\pi} \frac{1}{2E(\tilde{Q})} e^{-i\epsilon q_0} + iS_F^{(\pm)}(x_0, p) \tag{2.20}
$$

with

$$
 iS_F^{(\pm)}(x_0, p) = \pm \int dq_0 \frac{i(\tilde{Q}_\pm + M(\tilde{Q}))}{2\pi (q_0 + \mu) \mp E(\tilde{Q})} \frac{1}{2E(\tilde{Q})} e^{-i\epsilon q_0}, \tag{2.21}
$$

where $\tilde{Q}_\pm = (\pm E(\tilde{Q}), p)$. Here the first terms in Eq.(2.20) are canceled in $iS_F(x_0, p)$.
Using

$$
 \frac{1}{(q_0 + \mu) - E(\tilde{Q})} e^{-i\epsilon q_0} = -i \int_{-\infty}^{x_0} dy_0 e^{-iq_0(y_0 - (E(\tilde{Q}) - \mu)(x_0 - y_0))} \tag{2.22}
$$

for $E_1(\tilde{Q}) < 0$ and

$$
 \frac{1}{(q_0 + \mu) - E(\tilde{Q})} e^{-i\epsilon q_0} = i \int_{x_0}^{\infty} dy_0 e^{-iq_0(y_0 - (E(\tilde{Q}) - \mu)(x_0 - y_0))} \tag{2.23}
$$

for $E_1(\tilde{Q}) > 0$, the quark propagator $iS_F^{(+)}(x_0, p)$ is given as

$$
 iS_F^{(+)}(x_0, p) = \int dq_0 \frac{1}{2\pi} \int_{-\infty}^{x_0} dy_0 \frac{(\tilde{Q}_+ + M(\tilde{Q}))}{2E(\tilde{Q})} e^{-i\epsilon q_0(y_0 - (E(\tilde{Q}) - \mu)(x_0 - y_0))} \tag{2.24}
$$

for $E_1(\tilde{Q}) < 0$ and

$$
 iS_F^{(+)}(x_0, p) = -\int dq_0 \frac{1}{2\pi} \int_{x_0}^{\infty} dy_0 \frac{(\tilde{Q}_+ + M(\tilde{Q}))}{2E(\tilde{Q})} e^{-i\epsilon q_0(y_0 - (E(\tilde{Q}) - \mu)(x_0 - y_0))} \tag{2.25}
$$

for $E_1(\tilde{Q}) > 0$.

Similarly, using

$$
 \frac{1}{(q_0 + \mu) + E(\tilde{Q})} e^{-i\epsilon q_0} = i \int_{x_0}^{\infty} dy_0 e^{-iq_0(y_0 + (E(\tilde{Q}) + \mu)(x_0 - y_0))} \tag{2.26}
$$

for $E_1(\tilde{Q}) < 0$ and

$$
 \frac{1}{(q_0 + \mu) + E(\tilde{Q})} e^{-i\epsilon q_0} = -i \int_{-\infty}^{x_0} dy_0 e^{-iq_0(y_0 + (E(\tilde{Q}) + \mu)(x_0 - y_0))} \tag{2.27}
$$

for $E_1(\tilde{Q}) > 0$, the quark propagator $iS_F^{(-)}(x_0, p)$ is given as

$$
 iS_F^{(-)}(x_0, p) = \int dq_0 \frac{1}{2\pi} \int_{-\infty}^{x_0} dy_0 \frac{(\tilde{Q}_- + M(\tilde{Q}))}{2E(\tilde{Q})} e^{-i\epsilon q_0(y_0 + (E(\tilde{Q}) + \mu)(x_0 - y_0))} \tag{2.28}
$$

for $E_1(\tilde{Q}) < 0$ and

$$
 iS_F^{(-)}(x_0, p) = -\int dq_0 \frac{1}{2\pi} \int_{x_0}^{\infty} dy_0 \frac{(\tilde{Q}_- + M(\tilde{Q}))}{2E(\tilde{Q})} e^{-i\epsilon q_0(y_0 + (E(\tilde{Q}) + \mu)(x_0 - y_0))} \tag{2.29}
$$

for $E_1(\tilde{Q}) > 0$. 

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3. Numerical results

In this section, some numerical results are presented. We solve the SDE presented in the integrand of Eq. (2.11) by a recursion method. For each iteration, the quark mass function is normalized by a current quark mass at large $\zeta^2 = p_0^2 - p^2$, in which perturbative calculations are reliable. In the iteration, the quark mass function $M(p_0, p)$ in integrand of the SDE is replaced by the renormalized one obtained by the previous iteration.

First, we determine the QCD scale parameter $\Lambda_{QCD}$ which is implemented as an input parameter in our calculations.

In Fig. 1, the $\Lambda_{QCD}$ dependence of the pion decay constant $f_\pi$ at $T = \mu = 0$ is presented using the numerically calculated quark mass function $M(p_0, p)$. Detailed formulas for the calculation of $f_\pi$ are presented in Appendix B. As shown in Fig. 1, $0.30 \text{ GeV} \leq \Lambda_{QCD} \leq 0.34 \text{ GeV}$ gives $85.0 \text{ MeV} \leq f_\pi \leq 103 \text{ MeV}$.

In Ref. [13], we found that the critical temperature $T_C$, in which the chiral symmetry is restored at zero chemical potential, is $T_C \simeq 0.55 \Lambda_{QCD}$. Furthermore, the effective quark mass evaluated at the resonance peak of an effective quark propagator is given as $M_q \simeq \Lambda_{QCD}$. Therefore, our choice of the QCD scale parameter gives $0.30 \text{ GeV} \leq M_q \leq 0.34 \text{ GeV}$ and $0.17 \text{ GeV} \leq T_C \leq 0.19 \text{ GeV}$, respectively.

In order to investigate the chiral phase transition, we evaluate integrated mass functions $\langle |M| \rangle$, $\langle M_R \rangle$ and $\langle M_I \rangle$ as order parameters, in which the quark mass functions $|M(p_0, p)|$, $M_R(p_0, p)$ and $M_I(p_0, p)$ are integrated over $p_0$ and $p$, respectively.

In our model, the critical chemical potential $\mu_C$, in which the chiral symmetry is restored, depends on the energy scale entering the running coupling constant and the effective gluon mass. Though the running coupling constant $\alpha_s(t)$ is calculated by the renormalization group equation, the contributions from the chemical potential to the energy scale have some ambiguities in the one-loop approximation.

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3 The initial input parameters are $M_R = \Lambda_{QCD}$ and $M_I = 0$ at $\mu = 0$ with $\Lambda_0 = \Lambda = 10 \Lambda_{QCD}$ and $\delta = 0.1 \Lambda_{QCD}$ with $\epsilon = 10^{-6}$. In the evaluation of the quark mass function at $\mu + \Delta \mu$, we implement the solution of $M$ obtained at $\mu$ as the initial input.

4 We take the renormalized mass $m(\zeta^2) = 0$ at $\zeta = 10 \Lambda_{QCD}$. 

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Fig. 2. The $c_m$ dependences of the critical chemical potential $\mu_C$ for $c_\mu = 0$ and 1 with $\Lambda_{QCD} = 0.30$ GeV and 0.34 GeV, respectively.

Here we define $t = \log[(\bar{P}^2 + \bar{Q}^2 + c_\mu \mu^2)/\Lambda_{QCD}^2]$ with the parameter $c_\mu$, where $\bar{P}^2 = p_0^2 + p^2$ and $\bar{Q}^2 = q_0^2 + q^2$ (see Appendix A). Furthermore, we parametrize the effective gluon mass squared as $m_{T}^2 = 0$ and $m_{L}^2 = c_m \alpha_s(t) \mu^2 / \pi$ with a parameter $c_m$ [15,16].

In Fig. 2, the $c_m$ dependences of the critical chemical potential $\mu_C$, in which the order parameter $\langle |M| \rangle$ vanishes, are presented for $c_\mu = 0$ and 1 with $\Lambda_{QCD} = 0.30$ GeV and 0.34 GeV.

In Fig. 3, the $c_\mu$ dependences of the critical chemical potential $\mu_C$ with $\Lambda_{QCD} = 0.30$ GeV and 0.34 GeV are presented for $c_m = 0$ and 4.

For $c_m = c_\mu = 0$, the critical chemical potential is obtained in a range $0.53$ GeV $\leq \mu_C \leq 0.62$ GeV for $0.30$ GeV $\leq \Lambda_{QCD} \leq 0.34$ GeV. Our results may be compared with those calculated in the ITF [5,6,8], in which similar results are obtained. For example, the critical chemical potential is obtained as $\mu_C = 0.598$ GeV at $T = 0$ for $f_\pi = 93$MeV determined at $T = \mu = 0$ in Ref. [5].

As shown in Figs. 2 and 3, the critical chemical potential $\mu_C$ depends on the parameters $c_m$ and $c_\mu$ as well as $\Lambda_{QCD}$, particularly for small values of $c_m$ and $c_\mu$. On the other hand, the $c_m$ and $c_\mu$ dependences for the $\mu_C$ is rather stable for $c_m > 2$ and $c_\mu > 0.5$.

In Fig. 4, the $\mu$ dependences of the integrated quark mass functions $\langle |M| \rangle$, $\langle M_R \rangle$ and $\langle M_I \rangle$ with $c_m = 0$ and $c_\mu = 1$ are presented at $\Lambda_{QCD} = 0.32$ GeV. As shown in Fig. 4, the imaginary part of the quark mass function $\langle M_I \rangle$ is non-zero for the broken chiral symmetric phase below the critical chemical potential $\mu_C$, which means the massive quark state may be unstable if the energy scale rapidly changes. Furthermore, the real and imaginary parts vanish at the same critical chemical potential.

An imaginary part of the quark energy due to the imaginary part of the mass function may contribute time evolutions of the quark propagator. In order to investigate the instability of the massive quark state, we calculate time dependences of the quark propagator $i\tilde{S}_F^{(+)}(x_0,p)$ in Eq. (2.21), which is separated as

\[
i\tilde{S}_F^{(+)}(x_0,p) = \gamma_\mu i\tilde{S}_+^{(\mu)}(x_0,p) + i\tilde{S}_-^{(\mu)}(x_0,p) \quad (3.1)
\]
Fig. 3. The $c_\mu$ dependences of the critical chemical potential $\mu_C$ for $c_m = 0$ and 4 with $\Lambda_{QCD} = 0.30$ GeV and 0.34 GeV, respectively.

Fig. 4. The $\mu$ dependences of the integrated quark mass functions $\langle|M|\rangle$, $\langle M_R \rangle$ and $\langle M_I \rangle$ at $\Lambda_{QCD} = 0.32$ GeV with $c_m = 0$ and $c_\mu = 1$.

with

$$iS^\mu_+(x_0, p) = i \int dq_0 \frac{\bar{Q}^\mu_+}{2\pi} \frac{1}{(q_0 + \mu) - E(\tilde{Q})} e^{-ix_0q_0}$$

and

$$iS^m_+(x_0, p) = i \int dq_0 \frac{M(\tilde{Q})}{2\pi} \frac{1}{(q_0 + \mu) - E(\tilde{Q})} e^{-ix_0q_0}.$$  

From Eqs. (2.24) and (2.25), the real and imaginary parts of $iS^\mu_+(x_0, p)$ and $iS^m_+(x_0, p)$ are given as

$$\text{Re} \left[ iS^\mu_+(x_0, p) \right] = \int_{-\Lambda_0}^{\Lambda_0} \int_{-\Lambda_0}^{\Lambda_0} \frac{|\tilde{Q}^\mu_+|}{2|E(\tilde{Q})|} \cos(\Psi^\mu) e^{E_1(\tilde{Q})(x_0 - y_0)} \theta(-E_1(\tilde{Q}))$$
\[ - \int_{x_0}^{x_M} dy_0 \int_{-\Lambda_0}^{\Lambda_0} \frac{dq_0}{2\pi} \frac{|\hat{Q}_+^\mu|}{2|\hat{E}(\hat{Q})|} \cos(\Psi^\mu) e^{iE_1(\hat{Q})(\hat{x}_0 - \gamma_0)} \theta(E_1(\hat{Q})) \]  

and

\[ \text{Im} \left[ iS_+^{\mu}(x_0, p) \right] = - \int_{-x_M}^{x_0} dy_0 \int_{-\Lambda_0}^{\Lambda_0} \frac{dq_0}{2\pi} \frac{|\hat{Q}_+^\mu|}{2|\hat{E}(\hat{Q})|} \sin(\Psi^\mu) e^{E_1(\hat{Q})(\hat{x}_0 - \gamma_0)} \theta(-E_1(\hat{Q})) \]

\[ + \int_{x_0}^{x_M} dy_0 \int_{-\Lambda_0}^{\Lambda_0} \frac{dq_0}{2\pi} \frac{|\hat{Q}_+^\mu|}{2|\hat{E}(\hat{Q})|} \sin(\Psi^\mu) e^{E_1(\hat{Q})(\hat{x}_0 - \gamma_0)} \theta(E_1(\hat{Q})) \]  

with

\[ \Psi^\mu = q_0 \gamma_0 + (E_R(\hat{Q}) - \mu)(\hat{x}_0 - \gamma_0) + \Phi^\mu_\hat{Q} - \Phi/2 \]

for \( iS_+^{\mu}(x_0, p) \), and

\[ \text{Re} \left[ iS_+^{m}(x_0, p) \right] = \int_{-x_M}^{x_0} dy_0 \int_{-\Lambda_0}^{\Lambda_0} \frac{dq_0}{2\pi} \frac{|M(\hat{Q})|}{2|\hat{E}(\hat{Q})|} \cos(\Psi^m) e^{E_1(\hat{Q})(\hat{x}_0 - \gamma_0)} \theta(-E_1(\hat{Q})) \]

\[ - \int_{x_0}^{x_M} dy_0 \int_{-\Lambda_0}^{\Lambda_0} \frac{dq_0}{2\pi} \frac{|M(\hat{Q})|}{2|\hat{E}(\hat{Q})|} \cos(\Psi^m) e^{E_1(\hat{Q})(\hat{x}_0 - \gamma_0)} \theta(E_1(\hat{Q})) \]  

and

\[ \text{Im} \left[ iS_+^{m}(x_0, p) \right] = - \int_{-x_M}^{x_0} dy_0 \int_{-\Lambda_0}^{\Lambda_0} \frac{dq_0}{2\pi} \frac{|M(\hat{Q})|}{2|\hat{E}(\hat{Q})|} \sin(\Psi^m) e^{E_1(\hat{Q})(\hat{x}_0 - \gamma_0)} \theta(-E_1(\hat{Q})) \]

\[ + \int_{x_0}^{x_M} dy_0 \int_{-\Lambda_0}^{\Lambda_0} \frac{dq_0}{2\pi} \frac{|M(\hat{Q})|}{2|\hat{E}(\hat{Q})|} \sin(\Psi^m) e^{E_1(\hat{Q})(\hat{x}_0 - \gamma_0)} \theta(E_1(\hat{Q})) \]  

with

\[ \Psi^m = q_0 \gamma_0 + (E_R(\hat{Q}) - \mu)(\hat{x}_0 - \gamma_0) + \Phi_m - \Phi/2 \]

for \( iS_+^{m}(x_0, p) \). Here, \( \Phi^\mu_\hat{Q} \) and \( \Phi_m \) are defined as

\[ \Phi^\mu_\hat{Q} = \Phi/2 \]

and

\[ \Phi^i_m = \pi \theta(-p^i) \]

for \( i = 1, 2, 3 \), and

\[ \Phi_m = \arctan \frac{M_1(\hat{Q})}{M_R(\hat{Q})}, \]

respectively. In our calculations, we set the momentum \( p \) as \( p = (p^1, p^2, p^3) = (0, 0, p) \).

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\( x_M \) is taken as \( x_M = 10x_{\text{max}} \). Here, \( x_{\text{max}} \) is a maximum value of the plot for \( x_0 \).
Fig. 5. The $x_0$ dependences of the quark propagator $\text{Re} \left[ iS_+^k(x_0, \delta) \right]$ with $\mu = 0.0$ GeV and $p = \delta = 0.1 \Lambda_{\text{QCD}}$ for $k = 0, 3, m$, respectively, at $\Lambda_{\text{QCD}} = 0.32$ GeV. The dotted curve denotes the quark propagator $\text{Re} \left[ iS_+^0(x_0, \delta) \right]$ for $E_1(\tilde{Q}) < 0$.

In Fig. 5, the real part of $iS_+^k(x_0, p)$ with $p = \delta = 0.1 \Lambda_{\text{QCD}}$ for $k = 0, 3, m$, respectively, are presented at $\Lambda_{\text{QCD}} = 0.32$ GeV and $\mu = 0.0$ GeV. We can see that $iS_+^0(x_0, \delta) \approx iS_+^m(x_0, \delta) \gg iS_+^3(x_0, \delta)$, since $|E(\tilde{Q})| \simeq |M(\tilde{Q})| \gg \delta$.

As shown in Fig. 5, the amplitude of the quark propagator decreases as increasing $x_0$, which means the imaginary part of the quark energy $E_1(\tilde{Q})$ plays a role of a decay constant. The dotted curve denotes the quark propagator $\text{Re} \left[ iS_+^0(x_0, \delta) \right]$ for $E_1(\tilde{Q}) < 0$. Therefore, the contribution from $E_1(\tilde{Q}) > 0$ is not significant for the time evolution of the quark propagator.

In Fig. 6, the $x_0$ dependences of the quark propagator $\text{Re} \left[ iS_+^0(x_0, \delta) \right]$ for $\mu = 0.0, 0.1$ and 0.2 GeV are presented at $\Lambda_{\text{QCD}} = 0.32$ GeV with $c_m = 0$ and $c_{\mu} = 1$. As shown in Fig. 6, the wavelength is longer as $\mu$ increases due to the term $(E_\text{R}(\tilde{Q}) - \mu)x_0$ of the phase $\Psi^0$ in Eq. (3.6).
4. Summary and comments

In this paper, we have investigated quark mass functions solved by the Schwinger–Dyson equation (SDE) at finite density with zero temperature in the real-time formalism (RTF).

In our method, we can directly evaluate real and imaginary parts of the mass function. Therefore, we can investigate the instability of systems due to imaginary parts of the quark energy. Furthermore, there are no difficulties due to an imaginary chemical potential, which appears in the imaginary-time formalism (ITF).

In our model, the critical chemical potential $\mu_C$, in which the chiral symmetry is restored, depends on the QCD scale parameter $\Lambda_{\text{QCD}}$ as well as ambiguities of the energy scale in the running coupling constant $\alpha_s$ and the gluon masses ($m_L$ and $m_T$). Here, $m_L$ and $m_T$ denote the masses for a longitudinal component and a transverse component of the gluon propagator, respectively.

In order to choose the scale parameter $\Lambda_{\text{QCD}}$, we calculated the pion decay constant $f_\pi$ in Minkowski space. In our calculation, $0.30 \text{ GeV} \leq \Lambda_{\text{QCD}} \leq 0.34 \text{ GeV}$ gives $85.0 \text{ MeV} \leq f_\pi \leq 103 \text{ MeV}$ at zero temperature and chemical potential ($T = \mu = 0$).

Therefore, our choice of the QCD scale parameters give the effective quark mass evaluated at the resonance peak of an effective quark propagator $M_q$ as $0.30 \text{ GeV} \leq M_q \leq 0.34 \text{ GeV}$ and the critical temperature $T_C$, in which the chiral symmetry is restored at zero chemical potential, as $0.17 \text{ GeV} \leq T_C \leq 0.19 \text{ GeV}$, respectively, which may be qualitatively consistent with the results using lattice simulations [17–20].

In order to investigate ambiguities of our model, we introduced two parameters $c_m$ and $c_\mu$ for the gluon mass $m_L$ and the energy scale for the running coupling $\alpha_s$, respectively. These parameters are defined as coefficients for terms of the chemical potential squared. For $c_m = c_\mu = 0$, the critical chemical potential is obtained in a range $0.53 \text{ GeV} \leq \mu_C \leq 0.62 \text{ GeV}$ for $0.30 \text{ GeV} \leq \Lambda_{\text{QCD}} \leq 0.34 \text{ GeV}$. Our results may be compared with other calculations with the ITF at $T \simeq 0$. The critical chemical potential $\mu_C$ depends on the parameters $c_m$ and $c_\mu$ as well as $\Lambda_{\text{QCD}}$, particularly for small values of $c_m$ and $c_\mu$. On the other hand, $\mu_C$ is rather stable for large values of $c_m$ and $c_\mu$.

We found that the imaginary part of the integrated quark mass function $\langle M_I \rangle$ is non-zero for the broken chiral symmetric phase, which means that the massive quark state may be unstable for $\mu < \mu_C$ if the energy scale rapidly changes. Furthermore, the real and imaginary parts of the integrated mass functions vanish at the same critical point. The transition of the chiral symmetry restoration seems to be of the first order at $T = 0$.

In order to examine the effects of the imaginary part of the quark energy $E_I$, we calculated the time evolution of the quark propagator. The quark propagator decreases as the time increases, which suggests that main contribution of the imaginary part of the energy comes from $E_I < 0$. The contribution from $E_I > 0$ does not have a significant effect on the time evolution of the quark propagator.

As demonstrated in this paper, the SDE in the RTF may be useful for the investigation of the phase structures for large chemical potential and instability of systems in strong coupling physics.

Acknowledgements

This work was partially supported by MEXT-Supported Program for the Strategic Research Foundation at Private Universities, 2014-2017 (S1411024).

Funding

Open Access funding: SCOAP3.
Appendix A. Formulas for the quark mass function

The quark mass function is given by

\[ M^{11}(p_0, p) = -\frac{i C_F}{2\pi^2} \int_{-\Lambda_0}^{\Lambda} dq_0 \int_{\delta}^{\Lambda} dq \frac{q}{p} \frac{1}{\epsilon} \left[ \alpha_s M I_{11} J_{11} \right](p_0, p, q_0, q), \]

where the propagators \( M I_{11} \) and \( J_{11} \) are defined in Eq. (2.12) and Eq. (2.13), respectively.

Here, we approximate the integration over \( q_0 \) as

\[ M(p_0, p) \approx -\frac{i C_F}{2\pi^2} \int_{\delta}^{\Lambda} dq \frac{q}{p} \sum_{l=1}^{N-1} \langle \alpha_s J_{11}(q) \rangle_l [MI_{11}(q_0^{(l+1)}, q_0^{(l)})]. \]

with

\[ \langle MI_{11}(q_0^{(l+1)}, q_0^{(l)}) \rangle = \langle (MI_F)(q_0^{(l+1)}, q_0^{(l)}) \rangle_R + i \langle (MI_F)(q_0^{(l+1)}, q_0^{(l)}) \rangle_I N_F(T, \mu, (q_0)), \]

where \( \langle X \rangle_I \) denotes an average of \( X(q_0^{(l+1)}, q) \) and \( X(q_0^{(l)}, q) \) as \( \langle X \rangle_I = [X(q_0^{(l+1)}, q) + X(q_0^{(l)}, q)]/2 \), which depends on the momentum \( q \).

Here, \( ((MI_F)(q_0^{(l+1)}, q_0^{(l)}))_R \) and \( ((MI_F)(q_0^{(l+1)}, q_0^{(l)}))_I \) are given as

\[ ((MI_F)(q_0^{(l+1)}, q_0^{(l)}))_R \approx \langle M_{R_I}(I_{R_F}(q_0^{(l+1)}, q_0^{(l)})) \rangle_R - \langle M_{I_I}(I_{I_F}(q_0^{(l+1)}, q_0^{(l)})) \rangle_I, \]

and

\[ ((MI_F)(q_0^{(l+1)}, q_0^{(l)}))_I \approx \langle M_{R_I}(I_{R_F}(q_0^{(l+1)}, q_0^{(l)})) \rangle_I + \langle M_{I_I}(I_{I_F}(q_0^{(l+1)}, q_0^{(l)})) \rangle_R, \]

respectively, with

\[ I_{F}(q_0^{(l+1)}, q_0^{(l)}) = \int_{q_0^{(l)}}^{q_0^{(l+1)}} dq_0 I_F(q_0, q), \]

which depends on the momentum \( q \).

Here, the strong coupling constant \( \alpha_s \) is replaced by the running coupling constant \( \alpha_s(t) = g_s^2(t)/(4\pi)^2 \), which is defined as

\[ g_s^2(t) = \frac{1}{\beta_0} \times \begin{cases} \frac{1}{t} & \text{if } t_F < t < t_C \\ \frac{1}{t_F} + \frac{(t_F - t_C)^2 - (t - t_C)^2}{2t_F(t_C - t)} & \text{if } t_C < t < t_F \\ \frac{1}{t_C} + \frac{(t_C - t_F)^2 - (t - t_F)^2}{2t_C(t - t_F)} & \text{if } t < t_C \end{cases} \]

with \( \beta_0 = (33 - 2N_F)/(48\pi^2), t = \log[(\tilde{p}^2 + \tilde{Q}^2 + c_u \mu^2)/\Lambda^2_{QCD}], t_F = 0.5 \) and \( t_C = -2 \) for \( N_F \) flavors, where \( \tilde{p}^2 = p_0^2 + p^2 \) and \( \tilde{Q}^2 = q_0^2 + q^2. \)

For \( I_F, M \) and \( J_F \), we separate the real parts \( (I_F)_R, (M_R)_R \) and \( (J_F)_R \) and the imaginary parts \( (I_F)_I, (M_I)_I, (J_F)_I \), respectively.

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\(^6\) We implement the QCD coupling constant \( \alpha_s \) with Euclidean momenta. The difference between the argument of \( \alpha_s \) with the momenta in Minkowski space and that in Euclidean space is part of higher-order contributions to the one-loop approximation.
For $J_F$ in Eq. (2.14), we can integrate over $k$ as

$$(J_F)_R = (D_L)_R + 2(D_T)_R$$

with

$$(D_{T/L})_R = - \int_{\eta_-}^{\eta_+} \frac{dk}{(k^2 + m^2_{T/L} - k_0^2)^2 + \varepsilon^2} = - \frac{1}{4} \log \frac{(\eta_+^2 + m^2_{T/L} - k_0^2)^2 + \varepsilon^2}{(\eta_-^2 + m^2_{T/L} - k_0^2)^2 + \varepsilon^2}$$

and

$$(J_F)_I = (D_L)_I + 2(D_T)_I$$

with

$$(D_{T/L})_I = - \int_{\eta_-}^{\eta_+} \frac{dk}{(k^2 + m^2_{T/L} - k_0^2)^2 + \varepsilon^2}$$

$$= - \frac{1}{2} \left[ \arctan \frac{\eta_+^2 + m^2_{T/L} - k_0^2}{\varepsilon} - \arctan \frac{\eta_-^2 + m^2_{T/L} - k_0^2}{\varepsilon} \right],$$

respectively, with $\eta_{\pm} = |p \pm q|$ and $k = |k|$.

The real and imaginary parts of the quark propagator $I_F(q_0^{(l+1)}, q_0^{(l)})$ are given by

$$(I_F(q_0^{(l+1)}, q_0^{(l)}))_R = \frac{\epsilon((2(q_0 + \mu) - \frac{\text{d}E^2_{R}}{\text{d}q_0^{(l)}}))}{2|2(q_0 + \mu) - \frac{\text{d}E^2_{R}}{\text{d}q_0^{(l)}}|} \log \frac{t_M(q_0^{(l+1)}, q) (E^2_{R})_{l}^2}{[t_M(q_0^{(l)}, q) (E^2_{R})_{l}]^2}$$

and

$$(I_F(q_0^{(l+1)}, q_0^{(l)}))_I = \frac{\epsilon((2(q_0 + \mu) - \frac{\text{d}E^2_{R}}{\text{d}q_0^{(l)}}))\epsilon((E^2_{R})_{l})}{|2(q_0 + \mu) - \frac{\text{d}E^2_{R}}{\text{d}q_0^{(l)}}|}$$

$$\times \left[ \arctan \frac{t_M(q_0^{(l+1)}, q) (E^2_{R})_{l}^2}{|(E^2_{R})_{l}|} - \arctan \frac{t_M(q_0^{(l)}, q) (E^2_{R})_{l}}{|(E^2_{R})_{l}|} \right],$$

respectively, with

$$t_M(q_0, q) = (q_0 + \mu)^2 - (E^2_{R})$$

where we define $\epsilon(z) = \theta(z) - \theta(-z)$ with the step function $\theta(z)$. Here, the real and imaginary parts of the squared energy denoted by $(E^2_{R})$ and $(E^2)_I$, respectively, are given as

$$(E^2)_R = q^2 + (M^2)_R$$

and

$$(E^2)_I = (M^2)_I - \varepsilon,$$
Appendix B. Formulas for the pion decay constant

We calculate a squared pion decay constant in Minkowski metric as

\[(M^2)_R = \text{Re}(M^2) = (M_R)^2 - (M_I)^2\]

and

\[(M^2)_I = \text{Im}(M^2) = 2M_R M_I.\]

The real and imaginary parts of \(M, M^2\) and \(E^2\) depend on \(q_0\) and \(q\), respectively.

**Appendix B. Formulas for the pion decay constant**

We calculate a squared pion decay constant in Minkowski metric as

\[f_\pi^2 = 4N_C \int \frac{d^4p}{(2\pi)^4i} \frac{M(p_0, p) \left(M(p_0, p) - \frac{p}{3} \frac{\partial M(p_0, p)}{\partial p}\right)}{[p_0^2 - p^2 - M^2(p_0, p) + i\varepsilon]^2},\]

with a color factor \(N_C = 3\), which corresponds to an analytic continuation of the squared pion decay constant from Euclidian metric for \(T = \mu = 0\) limit \([5]\). Here, the quark mass function \(M\) depends on \(p_0\) and \(p = |p|\), which is calculated numerically by the SDE.

Integrating over the azimuthal angle of the quark momentum \(p\), the squared pion decay constant is given as

\[f_\pi^2 = -i\frac{N_C}{\pi^3} \int_{-\Lambda}^{\Lambda} dp_0 \int_{-\delta}^{\Lambda} dp \int_{-\delta}^{\Lambda} dp \int_{-\delta}^{\Lambda} dp L(p_0, p) I_F^2(p_0, p)\]

with

\[L(p_0, p) = M(p_0, p) \left(M(p_0, p) - \frac{p}{3} \frac{\partial M(p_0, p)}{\partial p}\right) = M^2(p_0, p) - \frac{p}{6} \frac{\partial M^2(p_0, p)}{\partial p},\]

\[= \left((M^2(p_0, p))_R - \frac{p}{6} \frac{\partial (M^2(p_0, p))_R}{\partial p}\right) + i \left((M^2(p_0, p))_I - \frac{p}{6} \frac{\partial (M^2(p_0, p))_I}{\partial p}\right) \equiv L_R + iL_I\]

and

\[I_F^2(p_0, p) = \frac{1}{[p_0^2 - p^2 - M^2(p_0, p) + i\varepsilon]^2} \equiv (I_F^2(p_0, p))_R + i(I_F^2(p_0, p))_I,\]

with

\[(I_F^2(p_0, p))_R = \frac{(p_0^2 - (E^2(p_0, p))_R)^2 - (E^2(p_0, p))_I^2}{[(p_0^2 - (E^2(p_0, p))_R)^2 + (E^2(p_0, p))_I^2]^2}\]

and

\[(I_F^2(p_0, p))_I = \frac{2(p_0^2 - (E^2(p_0, p))_R)(E^2(p_0, p))_I}{[(p_0^2 - (E^2(p_0, p))_R)^2 + (E^2(p_0, p))_I^2]^2}.\]
respectively. We define

\[(E^2(p_0,p))_R = p^2 + (M^2(p_0,p))_R\]

and

\[(E^2(p_0,p))_I = (M^2(p_0,p))_I - \varepsilon.\]

Here, we approximate the integration over \(p_0\) as

\[f^2_{\pi} \simeq -iN_C \sum_{l=1}^{N-1} \int_{-\Lambda}^\Lambda dp_0 \left( \langle (L^2)_R \rangle_I (I^2_{F}(p_0^{(l+1)}, p_0^{(l)}))_R - \langle L_I \rangle_I (I^2_{F}(p_0^{(l+1)}, p_0^{(l)}))_R \right) \]

\[+ i \langle (L^2)_R \rangle_I (I^2_{F}(p_0^{(l+1)}, p_0^{(l)}))_I + \langle L_I \rangle_I (I^2_{F}(p_0^{(l+1)}, p_0^{(l)}))_R \]

where we define

\[\langle X \rangle_I = (X(p_0^{(l+1)}, p) + X(p_0^{(l)}, p))/2,\]

which depends on the momentum \(p\), and

\[(I^2_{F}(p_0^{(l+1)}, p_0^{(l)}))_R = \int_{p_0^{(l)}}^{p_0^{(l+1)}} dp_0 (I^2_{F}(p_0))_R = \int_{p_0^{(l)}}^{p_0^{(l+1)}} dp_0 \left( \frac{p_0^2 - (E^2(p_0,p))_R}{(p_0^2 - (E^2(p_0,p))_R)^2 + (E^2)_I^2} \right)^2 \]

\[= - \epsilon \left( \frac{\langle (2p_0 - \frac{\partial (E^2)_R}{\partial p_0}) \rangle_I}{\langle (2p_0 - \frac{\partial (E^2)_R}{\partial p_0}) \rangle_I} \right) \left[ \frac{t_M(p_0^{(l+1)}, p)}{(t_M(p_0^{(l+1)}, p))^2 + (E^2)_I^2} - \frac{t_M(p_0^{(l)}, p)}{(t_M(p_0^{(l)}, p))^2 + (E^2)_I^2} \right] \]

and

\[(I^2_{F}(p_0^{(l+1)}, p_0^{(l)}))_I = \int_{p_0^{(l)}}^{p_0^{(l+1)}} dp_0 (I^2_{F}(p_0))_I = \int_{p_0^{(l)}}^{p_0^{(l+1)}} dp_0 \left( \frac{2(p_0^2 - (E^2(p_0,p))_R)(E^2)_I}{(p_0^2 - (E^2(p_0,p))_R)^2 + (E^2)_I^2} \right)^2 \]

\[= - \epsilon \left( \frac{\langle (2p_0 - \frac{\partial (E^2)_R}{\partial p_0}) \rangle_I}{\langle (2p_0 - \frac{\partial (E^2)_R}{\partial p_0}) \rangle_I} \right) \left[ \frac{(E^2)_I}{(t_M(p_0^{(l+1)}, p))^2 + (E^2)_I^2} - \frac{(E^2)_I}{(t_M(p_0^{(l)}, p))^2 + (E^2)_I^2} \right], \]

respectively, with \(t_M(p_0,p) = p_0^2 - (E^2(p_0,p))_R\). Finally, \(f_\pi\) is given as \(f_\pi = \sqrt{|f^2_{\pi}|}\).

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