Diffusion in the Mean for an Ergodic Schrödinger Equation Perturbed by a Fluctuating Potential

Jeffrey Schenker

Department of Mathematics, Michigan State University, Wells Hall, 619 Red Cedar Road, East Lansing, MI 48823, USA. E-mail: jeffrey@math.msu.edu

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Abstract: Diffusive scaling of position moments and a central limit theorem are obtained for the mean position of a quantum particle hopping on a cubic lattice and subject to a random potential consisting of a large static part and a small part that fluctuates stochastically in time. If the static random potential is strong enough to induce complete localization in the absence of time dependent noise, then the diffusion constant is shown to go to zero, proportional to the square of the strength of the time dependent part.

1. Introduction

Proving diffusive propagation of the quantum wave function in a weakly disordered background over arbitrarily long time scales (in dimension $d \geq 3$) remains one of the outstanding open problems of mathematical physics. This is so despite the fact that there is a well developed physical theory of this phenomenon as a multiple scattering process—see [17] and also [18] and references therein. Heuristically, the multiple scattering picture of wave diffusion is as follows. Scattering by the disordered background leads to a build up of random phases over time, resulting in decoherence among different possible scattering paths. Thus we expect, to a high degree of accuracy, that propagation may be understood classically, as a superposition of reflections from random obstacles. Provided recurrence effects do not dominate, the central limit theorem suggests a diffusive evolution for the amplitude in the long run.

So far it has not been possible to turn the heuristic argument outlined above into mathematical analysis, at least without restricting to time scales that are not too long, as in [10, 11]. There are mathematical difficulties with each step of the heuristic argument. In particular, one substantial obstacle to making the analysis precise is recurrence. The wave
packet in a multiple scattering expansion may return often to regions visited previously. In a static medium, the environment seen at each return is identical to that seen before, denying us the stochastic independence needed to use a version of the central limit theorem.

In fact, recurrence is not simply a technical difficulty. The phenomenon of Anderson localization—which can be seen as a recurrence phenomenon [5] and is well understood mathematically, see [2,3,13,25]—shows that, under the right hypotheses (large disorder or low dimension), recurrence can dominate, resulting in complete localization of the wave function, up to exponentially small tails uniformly bounded for all time. It is worth noting that the above heuristic argument does not support diffusion in dimensions \( d = 1 \) or 2, because of the high recurrence of random walks in these dimensions. Not coincidentally, localization has been proved to dominate at any disorder strength in \( d = 1 \), e.g., [9,14]. The exact nature of the dynamics in \( d = 2 \) remains an open problem, although based on the scaling theory of localization [1] it is widely believed that localization occurs at any disorder strength in \( d = 2 \) as well.

It is reasonable to expect that diffusion occurs more readily for a model in which recurrence is eliminated or reduced. This was the idea behind prior work of the author and collaborators [15,19], in which diffusive propagation was shown to occur for solutions to a tight binding random Schrödinger equation with a random potential evolving stochastically in time. (The models treated in [15,19] had been considered previously by Tcheremchantsev [22,23], who obtained diffusive scaling for position moments up to logarithmic corrections.)

The aim of this paper is to consider the more general, and more subtle, situation in which the environment is a superposition of two parts: a large static part that, on its own, would lead to Anderson localization and a small dynamic part that evolves stochastically as in [15,19]. We will obtain diffusive propagation for the evolution, however diffusion will occur at a slow rate that can be controlled quantitatively in terms of the size of the dynamic part of the environment.

In some ways the problem considered here is a quantum analogue of the classical dynamics of disordered oscillator systems perturbed by noise in the form of a momentum jump process, considered in [6,7] and reviewed in [8]. In those works, heat transport is considered in the limit of weak noise in a regime for which transport is known to vanish for the disordered oscillator system without noise. A key feature of the noise in [6,7] is that energy is conserved in the system with noise; this is necessary so that one can speak about heat flux. By contrast, in the present work energy conservation is broken by the noise. Indeed the only conserved quantity for the evolution we consider is quantum probability; and it is this quantity which is subject to diffusive transport.

Specifically, we consider below solutions to a Schrödinger equation of the form
\[
i \partial_t \psi_t(x) = H_\omega \psi_t(x) + g V(x,t) \psi_t(x),
\]
(1.1)
on \ell^2(\mathbb{Z}^d)$, with $H_\omega$ an ergodic Schrödinger operator and $V(x,t)$ a random potential with time dependent stochastic fluctuations. The analysis below is applicable to a broad class of operators $H_\omega$ and $V(x,t)$—the specific assumptions are presented in Sect. 2.1. To avoid technicalities in this introduction, let us state the main results in terms of the following non-trivial, and somewhat typical, example of operators satisfying the general requirements:

1. Let $H_\omega$ be a discrete random Schrödinger operator of the form
\[
H_\omega \psi(x) = \sum_{|y-x|=1} \psi(y) + \lambda U_\omega(x) \psi(x),
\]
(1.2)