HODGE-DELIENGE EQUIVARIANT POLYNOMIALS AND MONODROMY OF HYPERPLANE ARRANGEMENTS

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Abstract. We investigate the interplay between the monodromy and the Deligne mixed Hodge structure on the Milnor fiber of a homogeneous polynomial. In the case of hyperplane arrangement Milnor fibers, we obtain a new result on the possible weights. For line arrangements, we prove in a new way the fact due to Budur and Saito that the spectrum is determined by the weak combinatorial data, and show that such a result fails for the Hodge-Deligne polynomials. In an appendix, we also establish a connection between the Hodge-Deligne polynomials and rational points over finite fields.

1. INTRODUCTION

Let $E = \mathbb{C}^{n+1}$, with $\mathcal{A}$ a finite set of hyperplanes through 0 in $E$, $Z = Z_\mathcal{A} = \bigcup_{H \in \mathcal{A}} H$ and $N = N_\mathcal{A} = E \setminus Z$ the corresponding complement. To keep notation simple, we will also denote by $\mathcal{A}$ the associated hyperplane arrangement in the complex projective space $\mathbb{P}^n$ and use $H$ for an affine hyperplane in $E$ and also for the associated projective hyperplane in $\mathbb{P}^n$.

Let $Q = 0$ be a reduced equation for the union $V = \bigcup_{H \in \mathcal{A}} H \subset \mathbb{P}^n$ of (projective) hyperplanes in $\mathcal{A}$ and $F = \{x \in \mathbb{C}^{n+1} \mid Q(x) = 1\}$ the associated Milnor fiber. If $d = |\mathcal{A}|$ is the number of hyperplanes in $\mathcal{A}$, then $d = \deg Q$ and there is a monodromy isomorphism

\begin{equation}
    h : F \to F, \ h(x) = \lambda \cdot x,
\end{equation}

with $\lambda = \exp(2\pi i/d)$. This may be regarded as an action on $F$ of the cyclic group $\mu_d$ generated by $\lambda$. Any irreducible representation of $\mu_d$ is one-dimensional, and has the form $\mathbb{C}_\alpha$ ($\alpha \in \mu_d$), where for $1 \in \mathbb{C}_\alpha$, $\lambda \cdot 1 = \alpha$. If $V$ is any $\mu_d$-module, we shall denote by $V_\beta$ ($\beta \in \mu_d$) its $\mathbb{C}_\beta$-isotypic component.

It is an open question whether the Betti numbers $b_j(F)$ of $F$ or, more generally, the dimension of the isotypic components $H^j(F, \mathbb{C})_\beta$ for $0 < j \leq n$ and $\beta \in \mu_d$, are determined by the combinatorics of $\mathcal{A}$. This is a natural question, since the cohomology algebra $H^*(M, \mathbb{Z})$ of the complement $M = \mathbb{P}^n \setminus V$ is known to be determined by the combinatorics of $\mathcal{A}$, see [16]. The same applies to $N = M \times \mathbb{C}^\times$. In particular,
using the degree $d$ covering projection $p : F \to M$, we see that $\chi(F) = d \cdot \chi(M)$ is determined by the combinatorics.

A recent result of Budur and Saito in [3] asserts that a related invariant, the spectrum of a hyperplane arrangement $\mathcal{A}$ in $\mathbb{P}^n$ defined by

$$Sp(\mathcal{A}) = \sum_{\alpha \in \mathbb{Q}} m_\alpha t^\alpha,$$

with $m_\alpha = \sum_j (-1)^j n \dim Gr^j_p \tilde{H}^j(F, \mathbb{C})_\beta$ where $p = [n+1-\alpha]$ and $\beta = \exp(-2\pi i \alpha)$, is also determined by the combinatorics.

On the other hand, it was shown in [10] and in [4], that for $n = 2$ (i.e. for a line arrangement) the eigenspace decomposition

$$(1.3) \quad H^1(F, \mathbb{Q}) = H^1(F, \mathbb{Q})_1 \oplus H^1(F, \mathbb{Q})_{\neq 1},$$

is actually a decomposition of mixed Hodge structures (denoted henceforth by MHS) such that $H^1(F, \mathbb{Q})_1 = p^*(H^1(M, \mathbb{Q}))$ is pure of type $(1, 1)$, and $H^1(F, \mathbb{Q})_{\neq 1}$ is pure of weight 1.

One may consider a more general setting where the union of hyperplanes $V$ is replaced by a degree $d$ hypersurface $V$ in $\mathbb{P}^n$ given by a reduced equation $Q_V(x) = 0$, the associated Milnor fiber $F_V$ being defined by $Q_V(x) = 1$, and ask which of the above results remain true. It is natural to replace the combinatorics of $\mathcal{A}$ by the local type of singularities of $V$ and the topology of the dual complex associated to a normal crossing exceptional divisor arising in the resolution of singularities for $V_{\text{sing}}$ as in [2]. Some questions have easy negative answers in this more general setting, for instance the classical example of Zariski of cuspidal sextics implies that $b_1(F)$ depends on the position of singularities in general.

In this paper we investigate the relationship between the monodromy action and the MHS on the Milnor fiber cohomology in this more general setting. To do this, we first refine the equivariant weight polynomials introduced in [9] to get the equivariant Hodge-Deligne polynomials $P^\Gamma(X)$ associated to a finite group $\Gamma$ acting (algebraically) on a complex algebraic variety $X$.

More precisely, let $X$ be a quasi-projective variety over $\mathbb{C}$ and consider the Deligne MHS on the rational cohomology groups $H^*(X, \mathbb{Q})$ of $X$. Since this MHS is functorial with respect to algebraic mappings, if $\Gamma$ acts algebraically on $X$, each of the graded pieces

$$(1.4) \quad H^{p,q}(H^j(X, \mathbb{C})) := \text{Gr}^p_H \text{Gr}^W_{p+q} H^j(X, \mathbb{C})$$

becomes a $\Gamma$-module, and these modules are the building blocks of the polynomial $P^\Gamma(X)$. In this situation we refer to $H^*(X, \mathbb{Q})$ as being a graded $\Gamma$-MHS.

This viewpoint is applied to the hypersurface $X_V$, the projective closure of $F_V$, given by the equation

$$(1.5) \quad Q_V(x) - t^d = 0$$

in $\mathbb{P}^{n+1}$. The main results can be stated as follows.
Theorem 1.1. Let $H^0_0(X_V, \mathbb{Q}) = \text{coker}\{H^*(\mathbb{P}^{n+1}, \mathbb{Q}) \to H^*(X_V, \mathbb{Q})\}$ be the primitive cohomology of $X_V$, where the morphism is induced by the inclusion $i : X_V \to \mathbb{P}^{n+1}$. Then, for any hypersurface $V \subset \mathbb{P}^n$, there are natural isomorphisms of $\mu_d$-MHS

$$H^j(F_V, \mathbb{Q})_{\#1} = H^0_0(X_V, \mathbb{Q})^\vee(-n)$$

for any $0 < j \leq n$.

Here, for a MHS $H$, we denote by $H^\vee$ the dual MHS, and $H(m)$ denotes the Tate twist, see [17] for details. We can restate this result more explicitly using the equivariant Hodge-Deligne numbers $h^{p,q}(H, \alpha)$, see Example 2.1 for the definition and the basic properties of these numbers.

Corollary 1.2. For any hypersurface $V \subset \mathbb{P}^n$ of degree $d$, any $\alpha \in \mu_d$, $\alpha \neq 1$ and any $0 < j \leq n$,

$$h^{p,q}(H^j(F_V, \mathbb{C}), \alpha) = h^{n-p,n-q}(H^0_0(X_V, \mathbb{C}), \alpha) = h^{n-q,n-p}(H^0_0(X_V, \mathbb{Q}), \alpha).$$

Theorem 1.3. Let $F$ be the Milnor fiber of a hyperplane arrangement $\mathcal{A}$ in $\mathbb{P}^n$. Then

$$\text{Gr}^W_{2j} H^j(F, \mathbb{Q})_{\#1} = 0$$

for any $0 < j \leq n$.

Remark 1.4. (i) For any hyperplane arrangement $\mathcal{A}$ in $\mathbb{P}^n$ and any $j \geq 0$, it is known that $H^j(F, \mathbb{Q})_1 = p^*(H^j(M, \mathbb{Q}))$ is pure of type $(j, j)$, see [13]. That is why we consider here only the summand $H^*(F, \mathbb{Q})_{\#1}$.

(ii) The result in Theorem 1.3 is optimal, since even for a line arrangement $H^2(F, \mathbb{Q})$ may have weights 2 and 3, see Example 5.3 below. Moreover, this result does not hold for a general hypersurface $V$, in fact not even for $V$ an irreducible curve, see Example 4.6 below.

We have the following more precise result for some hypersurfaces $V$.

Theorem 1.5. Let $F_V$ be the Milnor fiber of a hypersurface $V$ in $\mathbb{P}^n$ having only isolated singularities. Then the following hold.

(i) $H^j(F_V, \mathbb{Q}) = 0$ for $0 \leq j \leq n-2$;

(ii) $H^{n-1}(F_V, \mathbb{Q})_{\#1}$ is a pure Hodge structure of weight $n-1$;

(iii) If in addition the singularities of $V$ are weighted homogeneous, then $H^n(F_V, \mathbb{Q})_{\#1}$ is a MHS with weights $n$ and $n+1$, i.e. $\text{Gr}^W_j H^n(F, \mathbb{Q})_{\#1} = 0$ for $j > n + 1$.

The first part of the next Theorem gives a new proof of a result already present in [2]. For a line arrangement $\mathcal{A}$, let us refer to the following as the weak combinatorial data: $d$, the number of lines in $\mathcal{A}$ and $m_k$, the numbers of points of multiplicity $k$ in $V$, for all $k \geq 2$.

Theorem 1.6. For a line arrangement $\mathcal{A}$, one has the following.

(i) The spectrum $\text{Sp}(\mathcal{A})$ is determined by the weak combinatorial data. More generally, the spectrum $\text{Sp}(V)$ of a hypersurface $V$ having only isolated singularities is determined by $d = \deg(V)$ and by the local type of its singularities.

(ii) The Hodge-Deligne polynomial $P^{\mu_d}(F)$ corresponding to the monodromy action on $F$ is not determined by the weak combinatorial data.
In fact, a weaker invariant, the weight polynomial $W(F)$, as recalled in Remark 2.3 (take $\Gamma = 1$), is itself not determined by the weak combinatorial data (same example as in the proof of Theorem 1.6).

Explicit numerical formulas for the coefficients $m_{\alpha}$ in the spectrum of $A$ were obtained in [3], Theorem 3. On the other hand, our proof gives geometric description of these coefficients in terms of specific $\mu_d$-actions on Milnor fibers and on a degree $d$ smooth surface, see for instance (5.8).

We mention also Corollary 3.2, computing the Hodge-Deligne polynomial of the link (or deleted neighbourhood) of the singular locus $\Sigma$ of a projective variety $X$ in terms of the Hodge-Deligne polynomial of the exceptional divisor $D$ of an embedded resolution of the pair $(X, \Sigma)$.

In the Appendix, we use $p$-adic Hodge theory to prove that quite generally, whenever a $\Gamma$-variety $X$ is defined over a number field, the number of rational points of its reductions modulo prime ideals can be used in certain cases to compute the Hodge-Deligne polynomial $P^\Gamma_c(X)(u, v)$. We thank Mark Kisin for discussions about this subject.

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2. EQUIVARIANT HODGE-DELINE POLYNOMIALS

Let $\Gamma$ be a finite group and denote by $R(\Gamma)$ the complex representation ring of $\Gamma$. If $V$ is a (finite dimensional) $\Gamma$-module, we denote by the same symbol $V$ the class of $V$ in $R(\Gamma)$.

When $E$ is a $\Gamma$-module, the dual module $E^\vee$ is defined in the usual way, that is

$$(g \cdot h)(v) = h(g^{-1} \cdot v),$$

for any $h \in E^\vee$, $v \in E$ and $g \in \Gamma$. This gives rise to an involution

$$\iota : R(\Gamma) \to R(\Gamma), \ E \mapsto E^\vee$$

of the representation ring $R(\Gamma)$.

Example 2.1. If $\Gamma = \mu_d$, then as explained in the Introduction, any irreducible $\Gamma$-module is of the form $\mathbb{C}_\alpha$ for some $\alpha \in \mu_d$. One then has $\iota(\mathbb{C}_\alpha) = \mathbb{C}_{\bar{\alpha}} = \mathbb{C}_{\alpha^{-1}}$. Moreover, if a $\Gamma$-module $E$ is defined over $\mathbb{Q}$, then $\dim E_\alpha = \dim E_{\bar{\alpha}}$ for any $\alpha \in \mu_d$. It follows that in this case $\iota(E) = E$.

If $H$ is a $\mu_d$-MHS and $\alpha \in \mu_d$, we write $h^{p,q}(H, \alpha)$ for the multiplicity of the irreducible representation $\mathbb{C}_\alpha$ in the representation $H^{p,q}(H)$. That is, $h^{p,q}(H, \alpha) = \dim H^{p,q}(H)_\alpha$. With this notation, we have

$$h^{p,q}(H^\vee, \alpha) = h^{-p,-q}(H, \bar{\alpha}).$$

To see this, note that $H^{p,q}(H^\vee)$ is the dual of $H^{-p,-q}(H)$. One has also

$$h^{p,q}(H(m), \alpha) = h^{p+m,q+m}(H, \alpha) \text{ and } h^{p,q}(H, \alpha) = h^{q,p}(H, \bar{\alpha}).$$

For the second equality, recall that complex conjugation establishes an $\mathbb{R}$-linear isomorphism $H^{p,q} \to H^{q,p}$.
Definition 2.2. The equivariant Hodge-Deligne polynomial of the $\Gamma$-variety $X$ is the polynomial $P^\Gamma(X) \in R(\Gamma)[u,v]$ defined as the sum
\[ P^\Gamma(X)(u,v) = \sum_{p,q} E^\Gamma_{p,q}(X) u^p v^q \]
where $E^\Gamma_{p,q}(X) = \sum_j (-1)^j H^{p,q}(H^j(X,\mathbb{C}))$. Similarly, the equivariant Hodge-Deligne polynomial with compact supports of the $\Gamma$-variety $X$ is the polynomial $P^\Gamma_c(X) \in R(\Gamma)[u,v]$ defined by the sum
\[ P^\Gamma_c(X)(u,v) = \sum_{p,q} E^\Gamma_{c,p,q}(X) u^p v^q \]
where $E^\Gamma_{c,p,q}(X) = \sum_j (-1)^j H^{p,q}(H^j_{\text{c}}(X,\mathbb{C}))$.

A similar notation $P^\Gamma(H^*)$ will be used when $H^*(X,\mathbb{Q})$ is replaced by any graded $\Gamma$-MHS $H^*$.

Remark 2.3. If we set $u=v$ in the above formulas, we get exactly the equivariant weight polynomials of the $\Gamma$-variety $X$ introduced in [9], namely $P^\Gamma(X)(u,u) = W^\Gamma(X,u)$ and $P^\Gamma_c(X)(u,u) = W^\Gamma_c(X,u)$. So the equivariant Hodge-Deligne polynomials are refinements of the equivariant weight polynomials.

It follows from the Poincaré Duality, see Theorem 6.23, p. 155 in [17], that when $X$ is a smooth connected $n$-dimensional variety, one has the following relation
\[ \iota(P^\Gamma(X)(u,v)) = u^n v^n P^\Gamma_c(X)(u^{-1},v^{-1}) \]
where the involution $\iota$ acts on the coefficients of these polynomials. In fact, the formula (1.6) in [9] should be modified to read
\[ \iota(W^\Gamma(X,u)) = u^{2n} W^\Gamma_c(X,u^{-1}). \]
However, since the weight filtration is defined over $\mathbb{Q}$, in many cases, e.g. when $\Gamma = \mu_d$, the involution $\iota$ acts trivially on the coefficients of $W^\Gamma(X,u)$, recall Example 2.1 above.

Remark 2.4. Larger, hence more interesting, symmetry groups acting on the Milnor fiber of a hyperplane arrangement may occur as follows. Let $G \subseteq \text{GL}(E)$ be a finite group which preserves $\mathcal{A}$, and leaves invariant a polynomial (not necessarily reduced) $Q$ such that $\bigcup_{H \in \mathcal{A}} H = Q^{-1}(0)$; for example, $G$ might be a unitary reflection group, which leaves invariant a suitable product of linear forms corresponding to $\mathcal{A}$ [14].

Write $d$ for the degree of $Q$. Then $\Gamma := G \times \mu_d$ acts on $E$: $(g, \xi) \cdot v = \xi^{-1} g v$ for $v \in E$, and $\Gamma F \subseteq F$.

Define $\tilde{N} := \{(v, \xi) \in N \times \mathbb{C}^\times \mid Q(v) = \xi^d \} = \{(v, \xi) \mid \xi^{-1} v \in F\}$. Then $\mathbb{C}^\times$ acts on $\tilde{N}$ diagonally: $\alpha : (v, \xi) \mapsto (\alpha v, \alpha \xi)$, and we have the following commutative diagram
\[ \begin{array}{ccc}
\tilde{N} & \xrightarrow{p_1} & N \\
\pi_1 \downarrow & & \downarrow \pi \\
F & \xrightarrow{p} & M 
\end{array} \]
Remark 2.5. If the algebraic variety \( X \) is defined over an algebraic number field, it may be reduced modulo various primes, and properties of \( P^\Gamma(X) \) may be deduced by considering rational points of twisted Frobenius maps in such reductions. Although we do not explore this theme extensively in this work, we provide some basic background results in the Appendix. In particular, we give a proof of the following result for a \( \Gamma \)-variety defined over a number field.

Theorem 2.6. (See Theorem A.8 below) Suppose there are polynomials \( P_X(t; w) = \sum_{i=0}^{2 \dim X} a_{2i}(w)t^i \in \mathbb{C}[t] \) such that for almost all \( q \), and all \( w \in \Gamma \), we have \( |X(\overline{F}_q)^\Gamma \text{Frob}_q| = P_X(q; w) \). Then \( E^\Gamma_{c,d,e} = 0 \) if \( d \neq e \), and \( P^\Gamma_c(X)(x,y) = W^\Gamma_c(X)(\sqrt[3]{ty}) \). Moreover \( P_X(t^2; w) = W^\Gamma_c(X)(t; w) \).

Here we write, for any polynomial \( Q(x,y) = \sum_{i,j} Q_{i,j} x^i y^j \in \mathbb{R}(\Gamma)[x,y] \) and \( w \in \Gamma \), \( Q(x,y; w) := \sum_{i,j} \text{Trace}(w, Q_{i,j}) x^i y^j \).

3. Localization at the singular locus

Let \( X \) be an \( n \)-dimensional projective algebraic variety, \( n \geq 2 \), with singular locus \( \Sigma \). Let \( X^* = X \setminus \Sigma \) be the regular part of \( X \).

Assume that a finite group \( \Gamma \) acts algebraically on \( X \); then \( \Sigma \) is \( \Gamma \)-invariant, i.e. for \( g \in \Gamma \) and \( a \in \Sigma \) one has \( g \cdot a \in \Sigma \).

As in [7], we consider the following exact sequence of \( \Gamma \)-MHS, see for instance [17], p. 136.

\[
\ldots \to H^k_{\Sigma}(X) \to H^k(X) \to H^k(X^*) \to H^{k+1}_{\Sigma}(X) \to \ldots
\]

For the equivariant Hodge-Deligne polynomials, this yields (with obvious notation)

\[
P^\Gamma(X) = P^\Gamma(X^*) + P^\Gamma(H^\Sigma_{c}(X)).
\]

On the other hand, let \( T \) be an \( \Gamma \)-stable algebraic neighbourhood of \( \Sigma \) in \( X \), and \( T^* = T \setminus \Sigma \) the corresponding deleted neighbourhood of \( \Sigma \) in \( X \), which is homotopically equivalent to the link \( K(\Sigma) \) of \( \Sigma \) in \( X \). Then Remark 6.17 in [17], p.151, yields an exact sequence of \( \Gamma \)-MHS

\[
\ldots \to H^k_{\Sigma}(X) \to H^k(X) \to H^k(T^*) \to H^{k+1}_{\Sigma}(X) \to \ldots
\]

which, at the level of equivariant Hodge-Deligne polynomials, gives

\[
P^\Gamma(H^\Sigma_{c}(X)) = P^\Gamma(\Sigma) - P^\Gamma(T^*).
\]

Further, by the additivity of the equivariant Hodge-Deligne polynomials with compact supports, see for instance [9], one has

\[
P^\Gamma_c(X^*) = P^\Gamma(X) - P^\Gamma(\Sigma)
\]

Finally, the Poincaré Duality formula (2.4) for smooth varieties implies that

\[
P^\Gamma(X^*)(u,v) = u^nv^w(P^\Gamma_c(X^*))^{-1}(u^{-1}, v^{-1}).
\]
Assembling the above relations yields a proof of the following result, which is an equivariant form of Proposition 1.7 in \[7\].

**Proposition 3.1.**

\[
P^\Gamma(X)(u, v) - u^n v^n i(P^\Gamma(X))(u^{-1}, v^{-1}) = P^\Gamma(\Sigma)(u, v) - u^n v^n i(P^\Gamma(\Sigma))(u^{-1}, v^{-1}) - P^\Gamma(T^*)(u, v).
\]

Now suppose we have a $\Gamma$-equivariant log resolution $\pi: (\tilde{X}, D) \to (X, \Sigma)$, see \[1\]. Then the deleted neighbourhoods $T^*$ of $\Sigma$ in $X$ and $T^*_D$ of $D$ in $\tilde{X}$ clearly coincide. Applying Proposition 3.1 to $(\tilde{X}, D)$ and using (2.4), we obtain the following generalization of the known relation between the resolution and MHS on the link for an isolated surface singularity, see \[6\].

**Corollary 3.2.** With the above notation, one has

\[
P^\Gamma(T^*)(u, v) = P^\Gamma(D)(u, v) - u^n v^n i(P^\Gamma(D))(u^{-1}, v^{-1}).
\]

For the remainder of this section we confine attention to the following particular situation. Let $X$ be a hypersurface in $\mathbb{P}^{n+1}$, with $n \geq 2$, having only isolated singularities

\[\Sigma = \{a_1, ..., a_m\}.\]

Assume that the finite group $\Gamma$ acts algebraically on $\mathbb{P}^{n+1}$ in such a way that the hypersurface $X$ is $\Gamma$-stable, i.e. for $g \in \Gamma$ one has $g \cdot X \subset X$, and $\Sigma$ is point-wise invariant, i.e. for $g \in \Gamma$ and $a_j \in \Sigma$ one has $g \cdot a_j = a_j$.

Now the link $K_s$ of each singular point $a_s$ may be chosen to be $\Gamma$-invariant, and hence the cohomology groups $H^s(K_s)$ acquire a $\Gamma$-mixed Hodge structure. Moreover, for any $k$, one has the following isomorphism of $\Gamma$-MHS:

\[
H^k(T^*, \mathbb{Q}) = \bigoplus_s H^k(K_s, \mathbb{Q}).
\]

(3.5)

Let us look at the polynomial $P^\Gamma(X)$ in more detail. To do this, let $i: X \to \mathbb{P}^{n+1}$ be the inclusion and define the primitive cohomology of $X$ to be

\[H^0_s(X, \mathbb{Q}) = \text{coker} \{i^*: H^s(\mathbb{P}^{n+1}, \mathbb{Q}) \to H^s(X, \mathbb{Q})\}.
\]

This is clearly a graded $\Gamma$-MHS and one has

\[
P^\Gamma(X) = P^\Gamma(H^0_s(X, \mathbb{Q})) + P_n
\]

(3.6)

where $P_n(u, v) = 1 + uv + ... + u^n v^n$. To see this, notice that $\Gamma$ acts trivially on $H^s(\mathbb{P}^{n+1})$ and recall the known facts on the cohomology of hypersurfaces with isolated singularities, see \[6\].

In particular, it is known that $H^j_0(X, \mathbb{Q}) = 0$ except for $j = n$ (here the weights are $\leq n$ since $X$ is proper) and for $j = n + 1$, when $H^{n+1}_0(X, \mathbb{Q})$ is pure of weight $n + 1$ by Steenbrink’s results, see \[21\].

Henceforth we assume in addition that each of the isolated singularities $(X, a_s)$ is weighted homogeneous. Then the possible weights on $H^n_0(X, \mathbb{Q})$ are just $n - 1$ and $n$, see Example (B), formula (i), p. 381 in \[7\].
Moreover, the only nonzero cohomology groups of a link $K_s$ are the following: $H^0(K_s)$ (1-dimensional, of type (0,0)), $H^{n-1}(K_s)$ (of pure weight $n-1$), $H^n(K_s)$ (of pure weight $n+1$), and $H^{2n-1}(K_s)$ (1-dimensional, of type $(n,n)$).

As a consequence of this discussion, we see that knowledge of the Hodge-Deligne polynomials of the links $K_s$ and of the primitive cohomology group $H^0_{n+1}(X, \mathbb{Q})$ determine the terms of weight $n-1$ in the Hodge-Deligne polynomial of $X$. More precisely, we have the following result extending Corollary 1.8 in [7].

**Corollary 3.3.** For $p+q = n-1$, one has the following equality in $R(\Gamma)$

$$H^{p,q}(H^n_0(X)) = \sum_s H^{p,q+1}(H^n(F_s)) - H^{n-p,n-q}(\iota(H^{n+1}_0(X))).$$

Here $F_s$ is the Milnor fibre corresponding to $K_s$, and to obtain the formula, we use the isomorphism of $\mu_d$-representations

$$H^{p,q}(H^{n-1}(K_s)) = H^{p+1,q+1}(H^n(F_s))$$

valid for $p+q = n-1$ and any weighted homogeneous hypersurface singularity $(X, a_s)$ of dimension $n$.

The Hodge-Deligne polynomials of the Milnor fibers $F_s$ are local invariants easy to compute in general (since one has explicit bases for these cohomology groups in terms of algebraic differential forms), while the the Hodge-Deligne polynomial of $H^{n+1}_0(X, \mathbb{Q})$ may be computed in many cases using the results in (6.3.15), p. 202 in [6], see for instance Theorem 6.4.15 in [6].

4. **Monodromy of Milnor fibers of line arrangements**

Recall the following notation from the Introduction: we consider a degree $d$ reduced hypersurface $V$ in $\mathbb{P}^n$, $n \geq 2$, given by $Q_V = 0$ and the associated Milnor fiber $F_V$ defined in $\mathbb{C}^{n+1}$ by $Q_V(x) = 1$. We further consider $X_V$, the projective closure of $F_V$ in $\mathbb{P}^{n+1}$, given by $Q_V(x) - t^d = 0$. When $V$ is the union of the hyperplanes in $\mathcal{A}$, we may drop the subscript $V$ from $F_V$ and $X_V$.

We consider the (monodromy) $\mu_d$-action on $F_V$ given by

$$\alpha \cdot (x_0, \ldots, x_n) = (\alpha x_0, \ldots, \alpha x_n)$$

for any $\alpha \in \mu_d$ and $(x_0, \ldots, x_n) \in F_V$, and the related $\mu_d$-action on $\mathbb{P}^{n+1}$ given by

$$\alpha \cdot (x_0 : \ldots : x_n : t) = (x_0 : \ldots : x_n : \alpha^{-1}t)$$

for any $\alpha \in \mu_d$ and $(x_0 : \ldots : x_n : t) \in \mathbb{P}^{n+1}$. Then the obvious isomorphism $F_V \to X_V \setminus V$ given by $(x_0, \ldots, x_n) \mapsto (x_0 : \ldots : x_n : 1)$ is $\mu_d$-equivariant, in particular

$$P^\mu_{\mu_d}(F) = P_{\mu_d}(X) - P^\mu_{\mu_d}(V).$$

The last term $P^\mu_{\mu_d}(V)$ is easy to compute (and depends only on the combinatorics in the case of hyperplane arrangements), since the $\mu_d$-action on $V$ is trivial. In this way we arrive at the situation studied in the previous section.
4.1. Proof of Theorem 1.1.

Proof. This proof is very simple and quite general; it does not require the results obtained above. We have the following exact sequence of $\Gamma$-MHS.

(4.4) \[ \cdots \to H_0^{i-1}(V) \to H_0^i(F_{V'}) \to H_0^i(X_V) \to H_0^i(V) \to \cdots \]

Recall that $H_c^i(F_V)$ is dual to $H^i(F_V)$, and hence $\dim H_c^i(F_V)_1 = \dim H^{2n-i}(F_V)_1 = b_{2n-i}(M)$. On the other hand, Alexander Duality implies that

\[ \dim H^{2n-i}(M) = \dim H^i(\mathbb{P}^n, V) = \dim H_0^{i-1}(V). \]

Moreover, since $X_V/\mu_d = \mathbb{P}^n$, it follows that $H_0^i(X)_1$, the fixed part under $\Gamma = \mu_d$, is trivial. As a result we get the following identification of $\mu_d$-MHS

(4.5) \[ H_c^i(F)_{\neq 1} = H_0^i(X). \]

This identification yields Theorem 1.1 by Poincaré Duality. \qed

4.2. Proof of Theorem 1.3.

Proof. This proof requires a number of results due to Budur and Saito in [3], [19], [20]. First, notice that by taking a generic hyperplane section and applying the affine version of the Lefschetz Theorem, see for instance [6], p. 25, it is enough to prove Theorem 1.3 for $j = n$. To proceed, we need the following result, see Lemma (3.5) in [19].

Proposition 4.3. If $Gr^n_2 H^n(F, \mathbb{C})_\alpha \neq 0$, then $N^n_\alpha \neq 0$ on $\psi_{Q,\alpha} \mathbb{C}Z$, where $Z = \mathbb{C}^{n+1}$ and $N$ is the logarithm of the unipotent part of the monodromy.

For the general properties of the nearby cycles $\psi_{Q,\alpha} \mathbb{C}Z$ we refer to [8], and for the weight filtration on them to [18]. Let $D \subset Z$ be the affine cone over $V$, i.e. $D = Q^{-1}(0)$. There is a canonical way to construct an embedded resolution of $D$ in $Z$, see section (2.1) in [3] for a projective version and section (3.5) in [20] for a special affine case.

Let $Z_0 = Z$ and denote by $p_0 : Z_1 \to Z_0$ the blow-up of the origin in $Z_0 = \mathbb{C}^{n+1}$. Then for $1 \leq i \leq n-1$, let $p_i : Z_{i+1} \to Z_i$ be the blow-up with center $C_i$, the disjoint union of the proper transforms in $Z_i$ of the linear spaces (edges) $V \in L(A)$ (regarded as subspaces in $Z$) with $\dim V = i$. Let $\tilde{Z} = Z_n$, $\tilde{p} : \tilde{Z} \to Z$ the composition of the $p_i$’s and $\tilde{D} = \tilde{p}^{-1}(D)$. Then $\tilde{D}$ is a normal crossing divisor in $\tilde{Z}$, with irreducible components parametrized by all the edges $V \in L(A)$ with $\dim V \leq n-1$. Let $\tilde{D}_V$ denote the irreducible component of $\tilde{D}$ corresponding to the edge $V$. We need the following result, see Proposition (2.3) in [3] (our situation is slightly different, but the same proof applies).

Proposition 4.4. The intersection of a family of irreducible components $(\tilde{D}_V)_k=1,r$ is empty, unless $V_1 \subset V_2 \subset \ldots \subset V_r$ up to a permutation. In particular, the multiplicity of $\tilde{D}$ at any point $y \in \tilde{D}$ is bounded by $n$. 
Let $\tilde{Q} = Q \circ \tilde{p}$. Then one has an isomorphism
\begin{equation}
R\tilde{p}_*\psi_{\tilde{Q},\alpha}C_{\tilde{Z}} = \psi_{Q,\alpha}C_{Z}
\end{equation}
compatible with the $N$-actions. The order of $N$, acting on the right hand side is bounded by $n$, by the results in the section (3.2) of [20] and Proposition 4.4. Hence $N^n = 0$ on both sides of (4.6). One may also use Theorem 2.14 in [18]. In view of Proposition 4.3 this completes the proof of Theorem 1.3. □

4.5. Proof of Theorem 1.5

Proof. The first claim is rather obvious in view of Kato-Matsumoto Theorem, see [6], Theorem (3.2.2).

The second claim follows from Theorem 1.1: indeed, it follows from [21] that in this case $H^{n+1}_0(X_V)$ is a pure HS of weight $n + 1$. This fact was also proved in [10].

For the last claim, using again Theorem 1.1, we have to show that $H^n_0(X_V, \mathbb{Q})$ has only weights $n - 1$ and $n$. But this was already noticed in the final part of section 3. □

The following example shows that Theorems 1.3 and 1.5 are quite sharp.

Example 4.6. We show that $Gr^p_H^2(F_V, \mathbb{Q}) \neq 0$ for some curves $V$ in $\mathbb{P}^2$. As above, again using Theorem 1.1, we have to show that one may have $W_0H^2_0(X_V, \mathbb{Q}) \neq 0$. Let $V$ be an irreducible plane curve having a singularity $(V, a)$ whose local monodromy operator is not of finite order, e.g. suppose that a local equation for $(V, a)$ is $(x^2 + y^3)(y^2 + x^3) = 0$.

Then the resolution graph of the corresponding singularity for the surface $X_V$ has at least one cycle, see [15]. This implies that the cohomology $H^1(K)$ of the corresponding link has elements of weight 0 (dual to the elements of weight 4 in $H^2(K)$ described in [6], Example (C29), p. 245). Then an application of Corollary 3.3 with $p = q = 0$ yields the claimed result.

5. Computation of Hodge-Deligne polynomials for line arrangements

Now we start the proof of Theorem 1.6. For this we use an idea already present in [7], p. 380. Let $X$ be a hypersurface in $\mathbb{P}^{n+1}$, with $n \geq 2$, having only isolated singularities $\Sigma = \{a_1, \ldots, a_m\}$. Let $F_s$ be the Milnor fiber of the singularity $(X, a_s)$. Steenbrink [21] has constructed a MHS on $H^*(F_s)$ such that the following is a MHS exact sequence.
\begin{equation}
0 \to H^n(X) \to H^n(X_\infty) \to \oplus s H^n(F_s) \to H^{n+1}(X) \to 0.
\end{equation}

Here $X_\infty$ is a smooth surface in $\mathbb{P}^{n+1}$, nearby $X$, regarded as a generic fiber in a 1-parameter smoothing $X_w$ of $X$. Moreover, $H^n(X_\infty)$ is endowed with the Schmid-Steenbrink limit MHS, whose Hodge filtration will be denoted by $F_{SS}$. The Hodge filtration $F_{SS}$ on $H^n(X_\infty)$, being the limit of the Deligne Hodge filtration $F$ on $H^n(X_w)$, yields for any $p$ isomorphisms
\begin{equation}
Gr^p_{F_{SS}} H^n(X_\infty) = Gr^p_F H^n(X_w)
\end{equation}
of $\mathbb{C}$-vector spaces (i.e. equality of dimensions). Note that our smoothing $X_w$ can be constructed in a $\mu_d$-equivariant way, e.g. just take $X_w$ to be the zero set in $\mathbb{P}^{n+1}$ of a polynomial of the form $Q_1(x) + wR_1(x)$ with $R_1$ a generic homogeneous polynomial of degree $d$ in $\mathbb{C}[x]$. With such a choice, the isomorphisms (5.2) become equalities in the representation ring $R(\mu_d)$. Moreover, these representations can be explicitly determined, since they coincide with the representations computed as in the Example below (by a standard deformation argument).

Example 5.1. Let $Y$ be the smooth surface in $\mathbb{P}^3$ defined by $x^d + y^d + z^d + t^d = 0$ with the $\mu_d$-action induced by that on $\mathbb{P}^3$ described above. Using the description of the vector spaces $Gr^p_FH^2_0(Y)$ in terms of rational differential forms à la Griffiths, see for instance [6], it follows that $H^{p,2-p}(d) = Gr^p_H^2(0)$ is the following $\mu_d$-representation:
(i) if $p = 2$, then the multiplicity of the representation $C_{\lambda^k}$ is 0 for $k = 1, 2$ and $(k-1)$ for $k = 3, \ldots, d - 1$.
(ii) if $p = 0$, since $H^{2,0}(0)(d) = H^{0,2}(2)(d)$, it follows by conjugating (i) that the multiplicity of the representation $C_{\lambda^k}$ is 0 for $k = d-1, d-2$ and $(d-k-1)$ for $k = 1, \ldots, d-3$.
(iii) If we denote by $h^{p,q}(\alpha)$ the multiplicity of the representation $C_{\alpha}$ for $\alpha \in \mu_d$, $\alpha \neq 1$ in the representation $H^{p,q}(d)$ above, the multiplicities $h^{1,1}(\alpha)$ are determined using (i), (ii) and the formula
\[ h^{2,0}(\alpha) + h^{1,1}(\alpha) + h^{0,2}(\alpha) = d^2 - 3d + 3. \]

For each $p = 0, 1, 2$, the exact sequence (5.1) yields an exact sequence of $\mu_d$-modules
\[ 0 \to Gr^p_FH^2_0(X) \to Gr^p_FH^2_0(X) \to \bigoplus_s Gr^p FH^2(F_s) \to Gr^p FH^2(X) \to 0. \]

If $H$ is $\mu_d$-MHS and $\alpha \in \mu_d$, we use the notation $h^{p,q}(H, \alpha)$ for the multiplicity of the representation $C_{\alpha}$ in the representation $H^{p,q}(H)$. With this notation, the exact sequence (5.3) and Corollary 3.3 yield the following.

Proposition 5.2. For $p + q = n$, one has
\[ h^{p,q}(H^0_0(X), \alpha) = h^{p,q}(\alpha) + h^{p,q+1}(H^{n+1}(X), \alpha) + h^{p+1,q}(H^{n+1}(X), \alpha) - \sum_s (h^{p,q}(H^n(F_s), \alpha) + h^{p,q+1}(H^n(F_s), \alpha) + h^{p+1,q}(H^n(F_s), \alpha)). \]

Example 5.3. The Ceva (or Fermat) arrangement $A(3, 3, 3)$ is defined by
\[ Q = (x^3 - y^3)(x^3 - z^3)(y^3 - z^3) = 0. \]

The monodromy action on $H^1(F, \mathbb{C})$ has only three eigenvalues: 1 (with multiplicity 8), $\alpha_1 = \exp(2\pi i/3) = \lambda^3$ (with multiplicity 2), and $\alpha_2 = \exp(4\pi i/3) = \lambda^6$ (with multiplicity 2), see [4], 3.2. (i) and also [23] for more on this line arrangement.

This arrangement has nine lines, no ordinary double points, and twelve triple points. It follows that the surface $X$ has in this case 12 singularities with local equation
\[ a^3 + b^3 + c^9 = 0. \]
A local computation using [6], see particularly p. 194, implies that each $H^2(F_s)$ has weights 2 and 3. The part of weight 3 has dimension 2 and the corresponding representations are
\begin{equation}
H^{2,1}(H^2(F_s)) = \mathbb{C}_{\lambda^6} \quad \text{and} \quad H^{1,2}(H^2(F_s)) = \mathbb{C}_{\lambda^3}
\end{equation}
with $\lambda = \exp(2\pi i/9)$. The part of weight 2 has dimension 30 and one has
\begin{equation}
H^{2,0}(H^2(F_s)) = \mathbb{C}_{\lambda^7} \oplus \mathbb{C}_{\lambda^8} \quad \text{and} \quad H^{0,2}(H^2(F_s)) = \mathbb{C}_{\lambda} \oplus \mathbb{C}_{\lambda^2}.
\end{equation}
The remaining representation $H^{1,1}(H^2(F_s))$ has dimension 26 and is determined by the equalities $h^{1,1}(H^2(F_s), \lambda^k) = 3$ for $0 < k < 4$ or $5 < k < 9$ and $h^{1,1}(H^2(F_s), \lambda^k) = 4$ for $k = 4, 5$. It follows that
\begin{equation}
H^{2,1}(H^3(X)) = 2\mathbb{C}_{\lambda^6} \quad \text{and} \quad H^{1,2}(H^3(X)) = 2\mathbb{C}_{\lambda^3}.
\end{equation}
Finally, using Corollary 3.3, we get
\[ h^{1,2}(H^2(F), \alpha) = h^{1,0}(H^2_0(X), \overline{\alpha}) = 7 \]
for $\alpha = \lambda^3$; in particular there are elements of weight 3 in $H^2(F)_{\neq 1}$. And using Proposition 5.2 we get also that
\[ h^{0,2}(H^2(F), \alpha) = h^{2,0}(H^2_0(X), \overline{\alpha}) \neq 0 \]
for $\alpha = \lambda^5$; in particular there are also elements of weight 2 in $H^2(F)_{\neq 1}$.

Finally, we prove Theorem 1.6. For the first claim we have to show that the multiplicities $m_a$ can be expressed in terms of the listed invariants and $\beta = \exp(-2\pi ia)$. The case when $a$ is an integer is very easy, using the identification $H^*(F, \mathbb{Q})_1 = H^*(M, \mathbb{Q})$ and the well known fact that the Betti numbers of $M$ can be expressed in terms of the listed invariants.

We treat the case $1 < a < 2$ and leave the other cases, which are entirely similar and easier, to the reader. In the case $1 < a < 2$, one has, with notation from the Introduction
\[ m_a = -h^{1,0}(H^1(F), \beta) + h^{1,1}(H^2(F), \beta) + h^{1,2}(H^2(F), \beta). \]
Using Corollary 1.2 we have
\[ m_a = -h^{1,2}(H^3(X), \overline{\beta}) + h^{1,1}(H^2_0(X), \overline{\beta}) + h^{1,0}(H^2_0(X), \overline{\beta}). \]
By Proposition 5.2 we get
\begin{equation}
(5.7) \quad h^{1,1}(H^2_0(X), \overline{\beta}) = h^{1,1}(\overline{\beta}) + h^{1,2}(H^3(X), \overline{\beta}) + h^{2,1}(H^3(X), \overline{\beta}) - LC1
\end{equation}
where $LC1$ is a number depending only on local invariants at the singularities, i.e. computable from the invariants $d$ and $m_k$ for $k \geq 2$ and $\beta$. Using Corollary 3.3 we also get
\[ h^{1,0}(H^2_0(X), \overline{\beta}) = LC2 - h^{2,1}(H^3(X), \overline{\beta}) \]
where $LC2$ is another local constant as above. It follows that in the last formula for $m_a$ the subtle invariants related to $X$ cancel out and the result involves only the local constants $LC1$ and $LC2$ in addition to the number $h^{1,1}(\overline{\beta})$ which depends only on $d$ and $\beta$. 
In fact, this computation yields the following formula

\[(5.8) \quad m_a = h^{1,1}(\gamma) - \sum_s (h^{1,1}(H^2(F_s), \gamma) + h^{1,2}(H^2(F_s), \gamma))\]

with \(1 < a < 2\) and \(\gamma = \exp(2\pi i a)\).

A similar argument applies to any any hypersurface having only isolated singularities, in view of our Theorem 1.5.

To show that the Hodge-Deligne polynomial \(P^\mu_d(F)\) corresponding to the monodromy action is not determined by \(d\) and by the numbers \(m_k\) of points of multiplicity \(k \geq 2\) in \(V\), it is enough to find one coefficient which involves invariants associated to \(X\). Indeed, it is known that there are line arrangements \(A\) and \(A'\), having the same list of invariants and with different values for some \(h^{1,2}(H^3(X), \beta)\), see for instance Theorem 6.4.15 and its proof, pp. 212-213 in [6]. If \(\beta \neq 1\), then the multiplicity of \(C\beta\) in the virtual representation \(E^\mu_d H^1_0(F)\), which is the coefficient of \(uv\) in \(P^\mu_d(F)\), is \(h^{1,1}(H^3_0(X), \beta)\). Now the formula (5.7) with \(\beta\) replacing \(\beta\) completes the proof.

**Appendix A. Rational points over finite fields and equivariant Hodge-Deligne polynomials.**

**A.1. The setting.** Let \(X\) be a variety over \(O_{[\overline{\nu}]}\), where \(O\) is the ring of integers of an algebraic number field \(F\), and suppose that the finite group \(\Gamma\) acts as a group of scheme automorphisms on \(X\). Then \(X\) has (compact support) equivariant Hodge-Deligne modules \(H^d_e (H^j_\gamma (X(\mathbb{C}), \mathbb{C}) \in R(\Gamma)\) defined as in 2.2 above, and correspondingly, the equivariant Hodge-Deligne polynomial \(P^\mu_c (X)(x, y) \in R(\Gamma)[x, y]\), also defined in Definition 2.2. If \(\Gamma = 1\), we have the usual Hodge-Deligne numbers [12] given by

\[(A.1) \quad h^{d,e}(j) := \dim_\mathbb{C} \text{Gr}^d_F \text{Gr}^e_F H^j_\gamma (X(\mathbb{C}), \mathbb{C}).\]

The Euler-Hodge numbers of \(X\) are given by

\[(A.2) \quad h^{d,e} := \sum_j (-1)^j h^{d,e}(j),\]

and the (non-equivariant, compact supports) Hodge-Deligne polynomial by

\[(A.3) \quad P^c_c (X)(x, y) := \sum_{d,e} h^{d,e} x^d y^e.\]

In this appendix, we amplify some of the results of [12] to make more explicit connections between the Hodge-Deligne polynomials of \(X\) and the eigenvalues of (possibly twisted) Frobenius endomorphisms on the \(p\)-adic étale cohomology of the reduction modulo various primes of \(X\).

In particular, we show how to deduce a result of Katz [11] by this means, and give an equivariant analogue (Theorem A.8) of that result. Our argument uses the \(K\) group of representations of the Galois group, rather than the \(K\) group of schemes, as Katz did.
A.2. Background in $p$-adic Hodge theory. We recall the basic setup, and amplify some results of [12].

Notation We shall use the notation of [12]. In particular, $F$ is a number field, $S$ a finite set of primes in $F$, $\bar{F}$ an algebraic closure of $F$, and $F_S \subset \bar{F}$ the maximal extension of $F$ which is unramified outside $S$. Write $G_{F,S} = \text{Gal}(F_S/F)$, and for a prime of $F$ $v \notin S$, write $\text{Frob}_v$ for the corresponding geometric Frobenius automorphism in $G_{F,S}$. Write $q_v$ for the cardinality of the residue field of $v$. We shall often write $\text{Frob}_q$ for $\text{Frob}_v$ if $q_v = q$.

For a rational prime $p$ such that $S$ contains all the prime divisors of $p$ in $F$, denote by $\mathbb{Q}_p(i)$ the $i^{th}$ tensor power of the one dimensional cyclotomic representation of $G_{F,S}$ over $\mathbb{Q}_p$. For each prime $p$ dividing $p$, fix an algebraic closure $\bar{F}_p$ of the $p$-adic completion $F_p$ of $F$ at $p$. Fix an embedding $\bar{F} \to \bar{F}_p$ and denote the corresponding decomposition group by $G_{F_p}$. There is then a canonical homomorphism $G_{F_p} \to G_{F,S}$, and representations of $G_{F,S}$ may therefore be restricted to $G_{F_p}$.

We refer to [12] §2 for properties of Fontaine’s filtered field $B_{dR}$. The relevant notation we require is as follows. The field $B_{dR}$ is discretely valued and contains $F_p$. Its residue field is denoted $\mathbb{C}_p$, and its decreasing filtration is denoted $\text{Fil}^\bullet B_{dR}$. If $V$ is a finite dimensional continuous $\mathbb{Q}_p G_{F_p}$-module, recall that $V$ is said to be de Rham if $\dim_{\mathbb{Q}_p} B_{dR} \otimes_{\mathbb{Q}_p} V)^{G_{F_p}} = \dim_{\mathbb{Q}_p}(V)$. We note that it is pointed out in [12] that it follows from arguments of Faltings, Tsuji and Kisin that any subquotient of $H^j_c(X,\mathbb{Q}_p)$ is de Rham.

A.3. Cohomology and eigenvalues of Frobenius. Recall that the de Rham cohomology $H^j_{c,dR}(X)$ is an $F$-vector space with a decreasing (Hodge) filtration $F^*$, whose complexification $H^j_c(X(\mathbb{C}),\mathbb{C}) := H^j_{c,dR}(X) \otimes_F \mathbb{C}$ correspondingly has two decreasing filtrations $F^*, F^0*$; as well as the increasing weight filtration $W_*$. The associated graded components of these filtrations are related by

\begin{equation}
\text{Gr}_{m}^W H^j_c(X(\mathbb{C}),\mathbb{C}) = \oplus_{d+e=m} \text{Gr}_{m}^d F_{m} \otimes_{F} H^j_c(X(\mathbb{C}),\mathbb{C}).
\end{equation}

The $p$-adic cohomology $H^j_c(X,\mathbb{Q}_p)$ also has a weight filtration (cf. [12] (2.1.5)) $W_*^0 H^j_c(X,\mathbb{Q}_p)$, whose associated graded parts are denoted $\text{Gr}_{m}^W H^j_c(X,\mathbb{Q}_p)$. The eigenvalues of $\text{Frob}_v$ (see above) on $\text{Gr}_{m}^W H^j_c(X,\mathbb{Q}_p)$ are known to be of the form $\zeta q_v^k$, where $\zeta$ is an algebraic number which has absolute value 1 in any embedding $\mathbb{Q}_p \to \mathbb{C}$. We fix such an embedding, and denote the eigenvalues of $\text{Frob}_v$ on $\text{Gr}_{m}^W H^j_c(X,\mathbb{Q}_p)$ by $\zeta_{m,k}^j q_v^k$, $k = 1, 2, \ldots, d_m^j$, where $d_m^j = \dim_{\mathbb{Q}_p} \text{Gr}_{m}^W H^j_c(X,\mathbb{Q}_p)$.

A.4. Filtrations and comparison theorems. Recall the following facts from [12]. We have (cf. [12] (2.1.3)) the following isomorphism of filtered $F_p G_{F_p}$-modules for each $j$:

\begin{equation}
H^j_{c,dR}(X) \otimes_{F} B_{dR} \sim \text{H}^j_{p}(X,\mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{dR},
\end{equation}

where on the left, the filtration is the tensor product of $\text{Fil}^*$ on $B_{dR}$ and $F^*$ on $H^j_{c,dR}(X)$, while on the right side the filtration comes from just the filtration on
$B_{dR}$. Moreover the isomorphism (A.5) respects the weight filtrations on the two cohomology theories (see \[ Lemma (2.1.4), Cor. (2.1.5)]).

Since $\Gr_{F^\c}(B_{dR}) \simeq \mathbb{C}_p(k)$ as $F_pG_{F_p}$-module, where $\mathbb{C}_p(k)$ denotes the $k$th Tate twist of the cyclotomic character, we obtain the following isomorphism of $F_pG_{F_p}$-modules by taking the weight $m$ component of the degree $d$ associated graded of the filtered spaces in (A.5).

\[
\bigoplus_{i=0}^{d} \Gr^i_F \Gr^W_{m} H^{j}_{c,dR}(X) \otimes_{F} \mathbb{C}_p(d) \sim \Gr^W_{m} H^{j}_{c}(X, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_p(d).
\]

Now take $G_{F^\c}$-fixed points of both sides of (A.6). Since $G_{F_p}$ has trivial action on $H^{j}_{c,dR}(X)$, and $\mathbb{C}_p(d)^{G_{F_p}} = 0$ if $d \neq 0$, while $\mathbb{C}_p(0)^{G_{F_p}} = F_p$ (see [22]) the left side becomes $\Gr^d_F \Gr^W_{m} H^{j}_{c,dR}(X) \otimes_{F} F_p$, which has $F_p$-dimension $h^{d,m-d}(j)$. We therefore have, for each $j, m$ and $d$,

\[
\Gr^d_F \Gr^W_{m} H^{j}_{c,dR}(X) \otimes_{F} F_p \sim (\Gr^W_{m} H^{j}_{c}(X, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_p(d))^{G_{F_p}},
\]

and taking dimensions over $F_p$ we obtain

\[
h^{d,m-d}(j) = \dim_{F_p} (\Gr^W_{m} H^{j}_{c}(X, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_p(d))^{G_{F_p}}.
\]

We shall make use of the Grothendieck ring $R(\mathbb{Q}_pG_{F,S})$ of finite dimensional continuous $\mathbb{Q}_pG_{F,S}$ representations. If $R$ is such a representation, we write $[R]$ for its class in $R(\mathbb{Q}_pG_{F,S})$. Every such element is equal to $\sum_i [S_i]$ where $S_i$ is a simple $\mathbb{Q}_pG_{F,S}$-module, and if $[R] = \sum_i [S_i]$, we say that $\oplus_i S_i$ is the semisimplification of $R$, and write $R_{ss} = \oplus_i S_i$.

A.5. Katz’s theorem. We shall show how the above considerations may be used to prove the following result of Katz.

**Theorem A.6.** (Katz, Appendix) Suppose there is a polynomial $P_{X}(t) \in \mathbb{C}[t]$ such that for almost all $q$, $|X(\mathbb{F}_q)| = P_{X}(q)$. Then $h^{d,e} = 0$ if $d \neq e$, and $P_{c}(X)(x,y) = P_{X}(xy)$.

The term “almost all” here means that for all but finitely many rational primes $\ell$, there is a power $q_\ell$ of $\ell$ such that the assertion holds for $q = q_\ell$, for any $r$.

**Proof of Katz’s theorem.** We are given a polynomial $P_{X}(t) = \sum_{i=0}^{n} a_{2n} t^n$, such that for almost all $q$, $|X(\mathbb{F}_q)| = P_{X}(q)$. Define the constants $c_{i}$, $i = 0, 1, \ldots, n$ by

\[
c_{i} = \begin{cases} a_{i} & \text{if } i \text{ is even} \\ 0 & \text{otherwise} \end{cases}
\]

By the Grothendieck fixed point theorem, we have

\[
|X(\mathbb{F}_q)| = \sum_{j=0}^{2\dim(X)} (-1)^j \text{Trace}(\text{Frob}_q, H^{j}_{c}(X, \mathbb{Q}_p))
\]

\[
= \sum_{m} \sum_{j=0}^{2\dim(X)} (-1)^j \text{Trace}(\text{Frob}_q, \Gr^W_{m} H^{j}_{c}(X, \mathbb{Q}_p))
\]
Now all modules $\text{Gr}_m^W H^j_c(X, \mathbb{Q}_p)$ are represented in the Grothendieck ring $R(\mathbb{Q}_p G_{F,S})$, and modules with equal trace functions on almost all Frobenius are equal in $R(\mathbb{Q}_p G_{F,S})$.

Using the fact that the concept of weight as defined by the eigenvalues of Frobenius coincides with that arising from Hodge theory \[5\], it follows by taking the pieces of weight $m$ in (A.10) that the following equation holds in $R(\mathbb{Q}_p G_{F,S})$.

\[(A.11) \quad \sum_j (-1)^j [\text{Gr}_m^W H^j_c(X, \mathbb{Q}_p)] = c_m \mathbb{Q}_p(-\frac{m}{2}).\]

Note that the right side of (A.11) is zero if $c_m = 0$, in particular if $m$ is odd. Write

\[V^e_m := \bigoplus_{j \text{ even}} \text{Gr}_m^W H^j_c(X, \mathbb{Q}_p) \quad \text{and} \quad V^o_m := \bigoplus_{j \text{ odd}} \text{Gr}_m^W H^j_c(X, \mathbb{Q}_p).\]

It follows from (A.7) that (cf. (A.2))

\[(A.12) \quad h^{d,m-d} = \dim_{F_p}(V^e_m \otimes \mathbb{C}_p(d))^{G_{F_p}} - \dim_{F_p}(V^o_m \otimes \mathbb{C}_p(d))^{G_{F_p}}.\]

We observe next that in (A.12), we may replace $V^e_m$ etc. by their semisimplifications. To see this, let $V = V^e_m$; then clearly $(V \otimes_{\mathbb{Q}_p} \mathbb{C}_p(d))^{ss} = V^{ss} \otimes_{\mathbb{Q}_p} \mathbb{C}_p(d)$ as $G_{F_p}$-modules. But it follows from (A.6) that $V \otimes_{\mathbb{Q}_p} \mathbb{C}_p$, and hence also $V^{ss} \otimes_{\mathbb{Q}_p} \mathbb{C}_p(d)$, is semisimple, and therefore equal to its semisimplification. Thus $V^{ss} \otimes_{\mathbb{Q}_p} \mathbb{C}_p(d) \simeq V \otimes_{\mathbb{Q}_p} \mathbb{C}_p(d)$.

It then follows from (A.12) and (A.11) that

\[(A.13) \quad h^{d,m-d} = c_m \dim_{F_p}(\mathbb{Q}_p(-\frac{m}{2}) \otimes \mathbb{C}_p(d))^{G_{F_p}}.\]

Hence $h^{d,m-d} = 0$ unless $m$ is even and $m = 2d$, and if this condition is satisfied, then $h^{d,d} = c_{2d}$. The result is now clear. \(\square\)

### A.7. Equivariant theory

Let $\Gamma$ be a finite group of automorphisms of $X$, where $X$ is as in $\text{§A.1}$. Then $\Gamma$ preserves all the filtrations discussed above, and we define the equivariant Hodge numbers by

\[(A.14) \quad h^{d,e}(j, w) := \text{Trace}(w, \text{Gr}_F^d \text{Gr}_F^e H^j_c(X(\mathbb{C}), \mathbb{C})),\]

for $w \in \Gamma$.

Similarly we define

\[(A.15) \quad h^{d,e}(w) := \sum_j (-1)^j h^{d,e}(j, w),\]

and the equivariant Hodge polynomials by

\[(A.16) \quad P^\Gamma_c(X)(x, y; w) := \sum_{d,e} h^{d,e}(w)x^dy^e.\]

We shall prove the following equivariant generalization of Katz’s theorem.
Theorem A.8. Suppose there are polynomials \( P_X(t; w) = \sum_{i=0}^{2 \dim X} a_{2i}(w)t^i \in \mathbb{C}[t] \) such that for almost all \( q \) and all \( w \in \Gamma \), we have \( |X(F_q)_{\text{Frob}_q}| = P_X(q; w) \). Then \( h^{d,e}(w) = 0 \) if \( d \neq e \), and \( P^\Gamma(X)(x, y; w) = P_X(xy, w) \) for each \( w \in \Gamma \). Moreover, the function \( w \mapsto a_{2j}(w) \) is a virtual character of \( \Gamma \) for each \( j \).

Proof. This is similar to the proof of Katz’s theorem above, and we maintain the above notation. Write \( \Theta := \mathcal{R}(\Gamma \times G_{F,S}) \) be the Grothendieck group of finite dimensional continuous representations of \( \Gamma \times G_{F,S} \) over \( \mathbb{Q}_p \), with \( \Gamma \) having the discrete topology. Note that the set of elements \( (w, \text{Frob}_q) \) is dense in \( \Gamma \times G_{F,S} \), so that two elements of \( \Theta \) are equal if and only if each element \( (w, \text{Frob}_q) \) has equal traces on the two modules.

Now any element \( \theta \) of \( \Theta \) may be written uniquely in the form \( \theta = \sum_\phi \chi_\phi \otimes \phi \), where the (finite) sum is over the simple representations \( \phi \) of \( G_{F,S} \), and for each \( \phi \), \( \chi_\phi \in R(\Gamma) \) is a virtual representation of \( \Gamma \). This applies in particular to the \( \Gamma \times G_{F,S} \) modules \( \gr_m^W H_c^j(X, \mathbb{Q}_p) \).

By the Grothendieck fixed point theorem, we have, for any \( w \in \Gamma \),

\[
|X(F_q)_{\text{Frob}_q}| = \sum_{j=0}^{2 \dim(X)} (-1)^j \text{Trace}(w \text{Frob}_q, H_c^j(X, \mathbb{Q}_p))
\]

(A.17)

\[
= \sum_{m} \sum_{j=0}^{2 \dim(X)} (-1)^j \text{Trace}(w \text{Frob}_q, \gr_m^W H_c^j(X, \mathbb{Q}_p))
\]

\[
= P_X(q, w),
\]

and taking the weight \( m \) piece of \( (A.17) \), it follows that we have the following equation in \( \Theta = R(\Gamma \times G_{F,S}) \).

\[
\sum_j (-1)^j [\gr_m^W H_c^j(X, \mathbb{Q}_p)] = \chi_m \otimes \mathbb{Q}_p(-\frac{m}{2}),
\]

(A.18)

where \( \chi_m \) is a virtual representation of \( \Gamma \) whose character at \( w \in \Gamma \) is \( a_{2m}(w) \).

Next, for any element \( \theta = \sum_\phi \chi_\phi \otimes \phi \) of \( \Theta \), define the \( G_{F,p} \)-invariant part \( \theta^{G_{F,p}} \) by \( \theta^{G_{F,p}} = \chi_1 \in R(\Gamma) \), the coefficient of the trivial representation of \( G_{F,S} \). This coincides with the \( 1_{G_{F,p}} \)-isotypic part of \( \theta \) in the case of proper representations. It follows from \( (A.7) \) that as \( \Gamma \)-module,

\[
\gr_d^F \gr_m^W H_{c,dR}(X) \otimes F_p = (\gr_m^W H_c^j(X, \mathbb{Q}_p) \otimes \mathbb{Q}_p (1_\Gamma \otimes \mathbb{C}_p(d)))^{G_{F,p}}.
\]

(A.19)

Hence by \( (A.18) \) we have the following equation in \( R(\Gamma) \). Write \( H^{d,m-d} \) for the element of \( R(\Gamma) \) represented by \( \sum_j (-1)^j \gr_d^F \gr_m^W H_{c,dR}(X) \otimes F_p \). Then

\[
H^{d,m-d} = (\chi_m \otimes (\mathbb{Q}_p(-\frac{m}{2}) \otimes \mathbb{Q}_p(d)))^{G_{F,p}}.
\]

(A.20)

The right side of \( (A.20) \) is 0 unless \( m = 2d \), and is equal to \( \chi_m \) when \( m = 2d \); the result is now clear. □
Remark A.9. In Remark 2.3 it was pointed out that for the equivariant weight polynomials $W^\Gamma_c(X)(x)$ of [9], we have the relation $P^\Gamma_c(X)(x,x) = W^\Gamma_c(X)(x)$. Hence given the conditions of Theorem A.8 the conclusion may be stated as $P^\Gamma_c(X)(x,y) = W^\Gamma_c(X)(\sqrt{xy})$.

Further remarks. We note that the following result is an easy consequence of [12].

Proposition A.11. Let $V$ be a continuous $\mathbb{Q}_p G_{F,S}$ module. Then

(i) For fixed integer $i$, let $V_i$ be the subset of vectors $x \in V$ such that for almost all $q$, $\text{Frob}_q x = \zeta^i q^i x$, for some root of unity $\zeta$. Then $V_i$ is a subspace of $V$.

(ii) $\text{Frob}_q$ acts semisimply on the subspace $V_T := \sum_i V_i$ for almost all $q$.

Proof. (i) If $x$ and $y$ are in $V_i$, then for almost all $q$, $(\text{Frob}_q)^{n(x)} = q^{n(x)}i x$ for some integer $n(x)$, and similarly for $y$; so $\text{Frob}_q^{n(x)n(y)}(x+y) = q^{n(x)n(y)}i(x+y)$, whence $x+y \in V_i$.

(ii) The proof of [12, Prop (1.2)] shows that $\text{Frob}_q$ acts semisimply on $V_i$, and hence on $V_T := \sum_i V_i$. □

It follows from Proposition A.11 and (A.8) that $\dim \left( H^j_c(X, \mathbb{Q}_p) \right) \leq h^{d,d}(j)$.

When $X$ is smooth and projective the space $V_T$ (which in this case consists just of a single $V_i$) is the subject of the Tate conjecture, which asserts that it should be the subspace spanned by cycle classes. This equality above would mean that the cohomology is spanned by cycle classes. This is satisfied only in certain special cases - for example if $X$ has a stratification by affine spaces.

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