Collapse and Revival of Entanglement between Qubits Interacting via a Quantum Bus

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We study the dynamics of the Jaynes-Cummings Model for two level systems (or qubits) interacting with a quantized single mode electromagnetic cavity (or ‘quantum bus’). We show that there is a time in between the collapse and revival of Rabi oscillations when the state of the qubit sub-system, $|\psi\rangle_{\text{attractor}}$, is largely independent of its initial state. This generalizes to many qubits the discovery by Gea-Banacloche for the one qubit case. The qubits in such ‘attractor’ states are not entangled either with the field or among themselves, even if they were in the initial state. Subsequently the entanglement between the qubits revives. Finally, it is argued that the collapse and revival of entanglement and the persistence of ‘non-classicality’ is a generic feature of multiple qubits interacting via a ‘quantum bus’.

At the heart of Quantum Information Science there is the essential quantum mechanical notion of entanglement. Measures and dynamics of entanglement are at the centre of much current research as there are many fundamental questions yet unanswered. Under such circumstances the study of simple models which feature interesting time evolution of entanglement, respectively. The simplest model which captures the salient features of the relevant physics in these fields is the Jaynes-Cummings model (JCM) for the one qubit case and its generalization for multi qubit systems by Tavis and Cummings. In this letter we wish to focus on the interesting dynamics of entanglement between the qubits as described by these models.

One of the most interesting and surprising predictions of the JCM is the ‘collapse and revival’ of Rabi oscillations of the occupation probabilities for various qubit states as the system evolves, from an initial state which is a product of a coherent state $|\alpha\rangle$, for the radiation field, and a generic qubit state $|\psi_{Ni}\rangle$. It is central to our present concern that such remarkable dynamics occurs only because both the matter and the cavity field are treated fully quantum mechanically. Thus, in the language of quantum computation, we may regard the above system as a collection of qubits interacting via a quantum bus. Indeed our aim here is to study the ‘collapse and revival’ of entanglement between non-interacting qubits induced by a quantum bus.

For clarity let us recall the multi qubit JCM Hamiltonian, for which each qubit labelled $i$ can be either in its ground state $|g_i\rangle$, with energy $\epsilon_{g,i}$, or its excited state $|e_i\rangle$, with energy $\epsilon_{e,i}$. Up to a constant the Hamiltonian may be written in the following conventional form

$$\hat{H} = \frac{\hbar\omega}{2} \hat{a} + \sum_{i=1}^{N_q} \Omega_i \hat{\sigma}_i^z + \hbar \sum_{i=1}^{N_q} \lambda_i (\hat{a} \hat{\sigma}_i^+ + \hat{a}^{\dagger} \hat{\sigma}_i^-)$$

$$\hat{\sigma}_i^z = |e_i\rangle\langle e_i| - |g_i\rangle\langle g_i|, \quad \hat{\sigma}_i^+ = |e_i\rangle\langle g_i|, \quad \hat{\sigma}_i^- = |g_i\rangle\langle e_i|$$

where $\hat{a}$ and $\hat{a}^{\dagger}$ are the creation and annihilation operators of photons with frequency $\omega$, $\hbar \Omega_i = \epsilon_{e,i} - \epsilon_{g,i}$ and $\lambda_i$ is the cavity-qubit coupling constant. Here we consider only the case of resonance between the qubits and the cavity e.g. $\omega = \Omega_i$ for all $i$ and uniform coupling $\lambda_i = \lambda$.

The celebrated ‘collapse and revival’ can be observed in the one qubit case. We define the initial state:

$$|\Psi(0)\rangle = |\psi_1\rangle |\alpha\rangle$$

where $|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$, $\alpha = \sqrt{\pi} e^{-i\theta}$ and $|\psi_1\rangle = (C_g |g\rangle + C_e |e\rangle)$. $\pi$ is the average number of photons in the field. The Rabi oscillations of the probability that the qubit is in the initial state at first collapse, on a time scale of $t_c \approx \sqrt{\frac{\hbar}{2}}$, and then revive at $t_r \approx \frac{\pi}{4} \frac{\lambda}{\Omega}$. This is illustrated in Fig. 1 for $C_g = 1, C_e = 0$ by plotting $\sum_{n=0}^{\infty} |\langle g,n|\Psi(t)\rangle|^{2}$ where $|\langle g,n|\rangle$ is the state for the qubit in the ground state with $n$ photons in the cavity.

A second notable feature of this time evolution, discovered by Gea-Banacloche, is that at $t = t_r$, $|\Psi(t)\rangle$ again factorises into a qubit part $|\psi_{\text{attractor}}\rangle$ and a cavity part $|\Phi(t)\rangle$. Moreover, remarkably, the former is given by

$$|\psi_{\text{attractor}}\rangle = \frac{1}{\sqrt{2}} (e^{-i\theta} |e\rangle \pm i |g\rangle)$$

where $\theta$ is the phase of the initial coherent state, for all initial conditions such that $|C_g|^2 + |C_e|^2 = 1$. Note also $|\psi_{\text{attractor}}\rangle$ is attained at $t = 3t_r/2$. Because of this strikingly non-linear behavior, following Phoenix and Knight, we shall refer to these states, Eq. (3), as...
‘attractors’. The probability that the qubit is in the state $|\psi_{attractor}^+\rangle$, as depicted by $\sum_{n=0}^{\infty} |\langle \psi_{attractor}^+, n | \Psi_1(t) \rangle|^2$ is also shown in Fig. 1 together with the von Neumann entropy $S^Q(t) = -\text{Tr} (\rho^Q(t) \ln \rho^Q(t))$ associated with the reduced density matrix $\rho^Q(t) = \text{Tr}_F (|\Psi_1(t)\rangle \langle \Psi_1(t)|)$ of the qubit by tracing over the field. Clearly, at $t = \frac{\tau}{2}$ the entropy $S^Q(t)$ tends to zero and attains it as $\bar{\tau}_r \to \infty$, indicating that the radiation field and the qubit are not entangled $[12]$. 

Prompted by these results we have investigated the $N_q > 1$ qubit evolution, starting in the state $|\Psi_{N_q}(0)\rangle = |\psi_{N_q}\rangle |\alpha\rangle$, and found that the spin coherent states $[14]$ 

$$|\psi_{N_q}\rangle \pm = \frac{1}{\sqrt{2^{N_q}}} (e^{-i\theta}|e\rangle \pm i|g\rangle)^\otimes N_q \tag{4}$$

can also be regarded as ‘attractors’ in a similar, dynamical, sense as outlined above. The only difference is that in the $N_q > 1$ case $|\psi_{N_q}\rangle $ will occur only for a restricted range of initial conditions which we shall call basin of attraction. At the attractor time there is no entanglement between the qubits and the radiation field, furthermore because $|\psi_{N_q}\rangle $ is a product of one qubit states, the qubits are not entangled with each other. Below we explore the implications of this observation for the dynamics of entanglement between the qubits.

From the above point of view the simplest case of interest is that of two qubits e.g. $N_q = 2$. In this case the time evolution described by $|\Psi_2(t)\rangle$ is readily found $[13]$. For the most general, normalized, initial state 

$$|\psi_2\rangle = C_{ee} |ee\rangle + C_{eg} |eg\rangle + C_{ge} |ge\rangle + C_{gg} |gg\rangle \tag{5}$$

of the qubit sector the exact analytical solution will be given elsewhere $[16]$. Here we consider only the sector determined by the restrictions: $a = e^{i\theta} C_{ee} = e^{-i\theta} C_{gg}$ and $\frac{1}{2} - |a|^2 = C_{eg} = C_{ge}$. As will be illustrated presently these define the ‘basin of attraction’ for the ‘attractor’ $|\psi_2\rangle^+_{attractor}$. Namely, for any values of $a$ satisfying $0 \leq |a| \leq 1/\sqrt{2}$ in 

$$|\psi_2\rangle = a (e^{-i\theta} |ee\rangle + e^{-i\theta} |gg\rangle) + \frac{1}{\sqrt{2}} (|eg\rangle + |ge\rangle) \tag{6}$$

the probability that the two qubits are in the state $|\psi_2\rangle^+_{attractor}$ (given by $P_{2\text{attractor}}(t) = |\langle \psi_{2\text{attractor}}^+ | p^Q(t) | \psi_{2\text{attractor}}^+ \rangle|^2$, will reach 1 at some time $t^*$. An example of such behavior (for $\theta = 0$) is shown in Fig. 2. To highlight the similarity with the analogous phenomena in the one qubit case (Fig. 1) we also show the entropy $S^Q(t)$. This is calculated from $\rho^Q$, the two qubit density matrix reduced with respect to the cavity field coordinate, which describes a mixed state for most times $t$. Notably, at $t^* = \frac{\tau}{4}$, where $P_{2\text{attractor}}(t) = 1$, the entropy $S^Q(t)$ tends to zero in the large $\bar{\tau}_r$ limit, indicating that the system of two qubits is not entangled with the field.

The interesting new feature of the two qubit case as opposed to the one qubit case is that the former is in general host to entanglement between qubits and this provides an opportunity to study the dynamics of such entanglement. For example, whilst almost all of the initial states in the ‘basin of attraction’ given in Eq. (6) describe entangled qubits they all evolve into $|\psi_2\rangle^+_{attractor}$ at $t = \frac{\tau}{4}$, where they are not entangled. We plot the pure state tangle of the initial condition defined as $\tau = 4 |C_{ee} C_{gg} - C_{eg} C_{ge}|^2$ $[17]$ as a function of $a$ in Fig. 3. Note that although there are only two points where $\tau = 0$, all values of entanglement, including $\tau = 1$ meaning maximal entanglement, are present in the ‘basin of attraction’. Thus we
are observing the time evolution of a generic amount of entanglement. To throw further light on the matter we show, in Fig. 4, the time evolution of the mixed state tangle calculated from $\rho^Q$ for the maximally entangled initial state $|\psi_2\rangle = \frac{1}{\sqrt{2}}(|ee\rangle + |gg\rangle)$ [18]. Evidently, just as the occupation of the initial qubit states collapses and revives, so does the entanglement. This phenomenon was first noted by Rodrigues et al in a similar context [11].

Surprisingly, $\tau$ remains near zero for long periods between revivals. Thus, we are dealing with a phenomenon which was dubbed the ‘death of entanglement’ by Yu and Eberly [19] and is the center of much current interest [2]. Qing et al [20] have found a similar collapse and revival for the same model we have studied here but for very different initial conditions. What makes our results even more surprising is that the phenomenon occurs for a well defined range of initial conditions, namely the ‘basin of attraction’ for all the $N_q$ qubit ‘attractor’ states, and the defining features of these can be generalized to an arbitrary number of qubits interacting with the same quantum bus. In fact, using the large $N$ expansion of Meinier et al [21] we have found a ‘basis of attraction’ for all the ‘attractor’ states in Eq. (3) given by

$$|\psi_{N_q}\rangle = \sum_{k=0}^{N_q} A(N_q, a) e^{-i(\frac{N_q}{2} - k)\theta} \sqrt{\frac{(N_q)!}{k!(N_q-k)!}} |N_q, \frac{N_q}{2} - k\rangle$$

$$A(N_q, a) = \begin{cases} a & \text{if } k \text{ is even} \\ \frac{1}{\sqrt{2}a^2} - |a|^2 & \text{if } k \text{ is odd} \end{cases} \tag{7}$$

where $0 \leq |a| \leq \frac{1}{\sqrt{2}}$, and the states $|N_q, m\rangle$ are the fully symmetrized $N_q$ qubit states. $m$ is the difference between the number of qubits in the excited state $N_e$ and those in the ground states $N_g$.

As noted before, the ‘attractor’ states are manifestly not entangled, but the states in the basin of attraction are. Although there is no unique measure of entanglement for $N_q > 2$ qubits it is reasonable to assume that an arbitrary $N_q$ qubit state in its ‘basin of the attraction’ is generically entangled. Thus the collapse and revival of entanglement should be expected to be a generic feature of the $N_q$ qubit JCM [27].

Interestingly, unlike in the ‘two qubit, two cavity’ model studied by Yünaç, Yu and Eberly [22], in the above calculations $\tau$ decays smoothly to zero with no discontinuities in the gradient, but does not actually go to zero before it revives. That is to say there is no ‘sudden death of entanglement’ [2, 20] and hence there is no need for ‘rebirth’ [23]. In fact at $t = \frac{1}{4}t_r$, when the qubit subsystem is in the ‘attractor’ state, the entanglement is encoded in the radiation field. At this time both the qubit-resonator entanglement and qubit-qubit entanglement vanish. To investigate the form this encoding takes we present in Fig. 5 the $Q$ function $Q(\alpha, t) = \langle \alpha | \rho^F(t) | \alpha \rangle$, where $\rho^F(t)$ is the reduced density matrix for the radiation field at various times. Note that whilst at $t = 0$ and $t_r$ there is only one circle which represents a coherent state $|\alpha\rangle$, at other times there are two circles. At the interesting time $t = \frac{1}{4}t_r$, when the radiation field is disentangled from the qubits, there are two macroscopically different circles on opposite sides of phase space so the state of the cavity is a superposition of the two coherent states, $|\alpha\rangle$ and $|-\alpha\rangle$.

Such ‘Schrödinger cat’ states have been studied by several authors with various perspectives [24, 25, 26]. Both ‘Schrödinger cat’ states and entangled states may be regarded as particularly ‘non-classical’ [10], while coherent states of the field and product states of the qubits are regarded as more classical states. As a consequence, this fact prompts the following observation: the entanglement present in the qubit part of the system at $t = 0$ is encoded in the state of the radiation field, $|\Phi(t)\rangle$, at $t = \frac{1}{4}t_r$, which becomes highly ‘non-classical’. This is demonstrated by

![FIG. 3: (color online) The value of the tangle for the states in the basin of attraction for different values of $a$. We notice that there are only two points where the tangle is zero, $a = \pm \frac{1}{2}$.

![FIG. 4: (color online) The qubit system started in the maximally entangled state $|\psi_2\rangle = (|ee\rangle + |gg\rangle)/\sqrt{2}$ and $\tau = 50$. (a) the entropy of the qubit system. (b) the probability of being in the state $|gg\rangle$. (c) the mixed state tangle of the qubit system.](image-url)
FIG. 5: (color online) Phase space sketches of the $Q$ function at six different times when the qubits start in the ‘basin of attraction’. (a) the time $t = 0$, where the cavity is in a coherent state which is shown by a circle of uncertainty in phase space. (b) a time a little after $t = 0$. (c) the time $t = t_1/4$. (d) just before the time $t = t_2$. (e) the time $3t_1/4$. (f) the time $t = t_2$, when both the circles have returned to their original position. The qubit dipole states are represented as arrows. The single arrow at $t = t_2/4$ corresponds to the spin coherent attractor state.

For example, if the initial qubit state is not entangled, $(\tau = 0)$, namely $a = \pm \frac{1}{2} e^{i\phi}$, then the field state is a more ‘classical’ coherent state. However for $a = \frac{1}{2} e^{i\phi}$, where $\phi$ is an arbitrary phase, or $a = i r$, where $r$ is a real number, the qubits are maximally entangled $(\tau = 1)$ and the field is in the ‘non-classical’ ‘Schrödinger cat’ state $|\Phi(1/4t_r)\rangle \propto (|\alpha\rangle + |-\alpha\rangle)$ characterized by a Wigner function which takes negative values near the origin. In short the ‘non-classicality’ which was in the qubit subsystem at $t = 0$ is conserved at $t = 4t_r/\tau$ when it is in the field subsystem. Remarkably, this also implies a new strategy for producing ‘Schrödinger cat’ states. We shall elaborate on this interesting possibility in a future publication [10].

Evidently, whilst we have described the time evolution of entanglement in detail only in the 2-qubit limit the existence of the ‘attractor’ states, with a finite basin of attraction, for an arbitrary number of qubits implies that an oscillatory flow of ‘non-classicality’ between the qubits and the quantum bus, the cavity, is a generic feature of dynamics described by the multi-qubit JCM. As the properties of this model are relatively readily accessible, either analytically or numerically, further study of the ‘collapse and revival’ of multipartite entanglement dynamics outlined above is clearly called for. In particular the effect of decoherence on the persistence of non-classicality discovered above remains an open question.

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