Abstract. We consider improvements of Dirichlet’s Theorem on space of matrices $M_{m,n}(\mathbb{R})$. It is shown that for a certain class of fractals $K \subset [0,1]^{mn} \subset M_{m,n}(\mathbb{R})$ of local maximal dimension Dirichlet’s Theorem cannot be improved almost everywhere. This is shown using entropy and dynamics on homogeneous spaces of Lie groups.

1. Introduction

1.1. Dirichlet’s theorem. Let $m,n$ be positive integers and denote by $M_{m,n} = \mathbb{R}^{mn}$ the space $m \times n$ matrices with real entries. Dirichlet’s Theorem (hereafter abbreviated by DT) on simultaneous diophantine approximations says the following:

\[ \text{DT}(m,n): \text{Given } Y \in M_{m,n} \text{ and } N \geq 1, \text{ there exist } q = (q_1, \ldots, q_m) \in \mathbb{Z}^m \setminus \{0\} \subset M_{1,m} \text{ and } p = (p_1, \ldots, p_n) \in \mathbb{Z}^n \subset M_{1,n} \text{ with } \]

\[ \| qY + p \| \leq \frac{1}{N^m} \quad \text{and} \quad \| q \| \leq N^n. \]

Here and hereafter, unless otherwise specified, $\| \cdot \|$ stands for the sup norm on $\mathbb{R}^k$, i.e. $\| (x_1, \ldots, x_k) \| = \max_{1 \leq i \leq k} |x_i|$. We use $B_s(x)$ (or $B_s$ if $x = 0$) to denote the ball of radius $s$ centered at $x$ in this norm.

Given $Y$ as above and a positive number $\sigma < 1$, we say DT can be $\sigma$-improved for $Y$, and write $Y \in DI_\sigma(m,n)$ or $Y \in DI_\sigma$ when the dimensions are clear from the context, if for every $N$ large enough one can find $q = (q_1, \ldots, q_m) \in \mathbb{Z}^m \setminus \{0\}$ and $p = (p_1, \ldots, p_n) \in \mathbb{Z}^n$ with

\[ \| qY + p \| \leq \frac{\sigma}{N^m} \quad \text{and} \quad \| q \| \leq \sigma N^n. \]

We say that DT can be improved for $Y$ if $Y \in DI_\sigma$ for some $0 < \sigma < 1$. The following theorem of Davenport and Schmidt says that for most $Y$ DT cannot be improved.

**Theorem 1.1 (DS).** For any $m,n \in \mathbb{N}$ and positive number $\sigma < 1$, the set $DI_\sigma(m,n)$ has Lebesgue measure zero.
In fact only the cases with $m = 1$ or $n = 1$ are proved in [DS]. But the method there can be generalized to the settings above. After [DS], there are different strengthens and generalizations of Theorem 1.1. There are detailed reviews of the history of these developments in [KW] and [Sh]. In these two papers, they successfully strengthen Theorem 1.1 for the cases of $m = 1$ or $n = 1$. In the case $m = 1$, [KW] showed that for a large class of measures (e.g. friendly measures in [KLW]) DT can not be $\sigma$-improved for almost every element if $\sigma < \sigma_0$ for some positive number $\sigma_0$ depending on the measure. After that, Shah improved the result by removing the upper bound $\sigma_0$ for a special kind of measures concentrated on analytic curves. More precisely,

**Theorem 1.2 (Sh).** Let $\varphi : [a, b] \to \mathbb{R}^k$ be an analytic curve such that $\varphi([a, b])$ is not contained in a proper affine subspace. Then Dirichlet’s theorem $DT(1, k)$ and $DT(k, 1)$ can not be improved for $\varphi(s)$ for almost all $s \in [a, b]$.

1.2. Nonimprovability of DT for fractal measures. Our aim is to generalize Theorem 1.1 in a direction in some sense opposite to Theorem 1.2. Instead of a smooth one-dimensional submanifold, we are going to consider measures supported on a full Hausdorff dimension subset of $M_{m,n}$ and show that for $\mu$ almost every point DT can not be improved. Without loss of generality, we are going to work with measures on $J = [0, 1] \subset \mathbb{R}^k$. Let $f$ and $g$ be real valued functions depending on $\epsilon$, then $f \ll \epsilon g$ means $f \leq Cg$ for some constant $C > 0$ depending only on $\epsilon$.

**Definition 1.3.** Let $\mu$ be a probability measure on $J$. We say $\mu$ has local maximal dimension if there exists $s_0 > 0$ such that for any $\epsilon > 0$, $0 < \delta < 1$, $0 < s \leq s_0$, and $x \in J$ one has

\[(1.2) \quad \mu(B_{\delta s}(x)) \ll_{\epsilon} \delta^{k-\epsilon} \mu(B_s(x)).\]

We also say $\mu$ has $s_0$-local maximal dimension if $s_0$ is known.

**Remark 1.4.** (1.2) implies supp$(\mu)$ has Hausdorff dimension $k$.

In Theorem 6.3 we prove that DT$(m, n)$ can not be improved almost everywhere if $\mu$ implies some non-escape of mass property. In particular, we have:

**Theorem 1.5.** Let $\mu$ be a Borel probability measure on $[0, 1]^n \subset M_{1,n}$ with local maximal dimension. If $\mu$ is Federer (see Section 3), then DT$(1, n)$ can not be improved for $\mu$ almost every element.

1.3. Example of fractal measures. It is easy to see that the Lebesgue measure on $[0, 1]^k$ has local maximal dimension and is Federer. Next we give an example (suggested by Einsiedler) of a fractal measure on $[0, 1]$ with the same property but singular to the Lebesgue measure. First we divide $[0, 1]$ into 3 subintervals of the same length $\frac{1}{3}$ and cut the middle open interval out. We denote the remaining two closed subintervals by $[1]$ and $[2]$ with the natural ordering from left to right. Next we divide these two intervals
into 5 subintervals of the same length \( \frac{1}{3^5} \) and cut the middle interval out. We denote the remaining closed intervals inside \([1]\) by \([1], [1, 2], [1, 3], [1, 4]\) with the left to right ordering. We denote the remaining closed intervals inside \([2]\) in a similar way. In this construction, we allow some overlappings of end points so that all the remaining intervals are closed.

This process is continued for all natural numbers \(n\). That is after \(n\)-th step we have

\[
\begin{align*}
\text{(1.3)} & \quad 2 \cdot 4 \cdots (2n) \\
\text{(1.4)} & \quad \frac{1}{3} \cdots \frac{1}{2n+1}.
\end{align*}
\]

Each of them is denoted by \([y_1, \ldots, y_n]\) where \(1 \leq y_i \leq 2i\). Such a closed interval is said to be of stage \(n\). Then we cut all of them into \(2n+3\) subintervals of the same length and take the middle open interval out. For the stage \(n\) interval \([y_1, \ldots, y_n]\), we denote the remaining \(2n+2\) subintervals by \([y_1, \ldots, y_n, y_{n+1}]\) with the left to right ordering where \(0 \leq y_{n+1} \leq 2n+2\). See figure 1 for the process of dividing a stage \(n\) subinterval.

We use \(C_n\) to denote the union of all stage \(n\) subintervals. Let \(C = \bigcap_n C_n\), then in view of (1.3) and (1.4) we have

\[
\text{(1.5)} \quad m(C) = \lim_{n \to \infty} m(C_n) = \lim_{n \to \infty} \frac{2}{3} \cdots \frac{2n}{2n+1} = 0
\]

where \(m\) is the Lebesgue measure. The last equality of (1.5) follows from

\[
\left( \frac{2}{3} \cdots \frac{2n}{2n+1} \right)^2 \leq \left( \frac{2}{3} \cdots \frac{2n}{2n+1} \right) \left( \frac{3}{4} \cdots \frac{2n+1}{2n+2} \right) = \frac{2}{2n+2} \to 0.
\]

One can define a measure \(\mu\) on \(C \subset [0,1]\) by assigning

\[
\mu([y_1, \ldots, y_n]) = \frac{1}{2} \cdots \frac{1}{2n}.
\]

**Proposition 1.6.** Let \(\mu\) on \([0,1]\) be the probability measure above, then \(\mu\) has local maximal dimension and is Federer.

We omit the proof here, the reader can consult Section 4.1 of the author’s thesis [S] for a proof. Many other examples can be constructed in a similar way. It is easy to see that local maximal dimension is invariant under products. It is mentioned in [KLM] that Federer is invariant under products,
too. So we may see many examples of measures on \([0, 1]^k\) with local maximal dimension, or in addition Federer and singular to the Lebesgue measure.

1.4. \textbf{Method of proof}. We are going to translate the diophantine properties to properties of trajectories for the action of a diagonal matrix on the homogeneous space \(X = \text{SL}(m + n, \mathbb{Z})/\text{SL}(m + n, \mathbb{R})\) in Section 6. This method is developed in \cite{Da1} and \cite{KM} and then was used also in \cite{KLM}, \cite{KW} and \cite{Sh} for various kinds of problems.

Our diophantine approximation result follows from an equidistribution result in Section 5. We put a measure of local maximal dimension on \([0, 1]^{mn}\) in the unstable submanifold of \(X\). We denote the new measure by \(\nu\) and translate the property of \(\mu\) into the homogeneous setting where we say \(\nu\) has local maximal dimension in the unstable horospherical direction. We prove that the average of \(\nu\) along the orbit is equidistributed with respect to the Haar measure \(m_X\) if there is no loss of mass.

We will use the entropy theory developed by Margulis and Tomanov in \cite{MT} to prove the equidistribution result. They proved that the measure on \(X\) of maximal entropy under diagonal actions is precisely the Haar measure \(m_X\) and the maximal entropy can be computed according to the entries of the diagonal matrix. This method will be reviewed in Section 4.

To use the entropy theory, we need to show that the average of \(\nu\) along the orbit has no loss of mass. In general we do not know whether this is true since \(X\) is noncompact. Einsiedler and Kadyrov are working on this question under weaker assumptions and have obtained some positive results on special cases. If \(m = 1\), we can also use Theorem 3.3 of \cite{KLM} to establish the non-escape of mass property. In Section 3 we show that local maximal dimension and Federer imply absolutely decaying, hence friendly. Therefore with an additional Federer assumption, we get non-escape of mass property and the corresponding diophantine approximation result.

\textbf{Acknowledgements:} The author would like to thank his advisor Manfred Einsiedler for his help in preparing this paper and his advice on how to write articles.

2. \textbf{Preliminaries}

We fix a locally compact topological space \(X\) and a continuous map \(T : X \to X\). Let \(\mathcal{B}\) stand for the Borel \(\sigma\)-algebra of \(X\). We assume all measures on \(X\) are Radon and the convergence of measures is under the weak* topology.

2.1. \textbf{Equidistribution and non-escape of mass}. A sequence of probability measures \(\mu_n\) on \(X\) is said to be equidistributed with respect to a probability measure \(\lambda\), if

\[
\lim_n \mu_n = \lambda.
\]
Definition 2.1. Let $\mu$ and $\lambda$ are probability measures on $X$. We say that $\mu$ is equidistributed on average with respect to $\lambda$ if the sequence

$$\mu_k = \frac{1}{k} \sum_{l=0}^{k-1} T^l \mu$$

is equidistributed in the sense of (2.1).

It is well known that any limit measure of the sequence (2.2) is $T$-invariant.

The following lemma tells us how to compute the value of the limit measure on a good Borel set.

Lemma 2.2. Suppose $\mu_n$ ($n \geq 1$) and $\mu$ are probability measures on $X$ and $B \in \mathcal{B}$ is relatively compact. If $\mu(\partial B) = 0$ and $\mu_n \to \mu$, then $\mu_n(B) \to \mu(B)$.

Definition 2.3. For a probability measure $\mu$ on $X$, we say there is no loss of mass (or non-escape of mass) on average if for any limit point $\nu$ of the sequence

$$\frac{1}{k} \sum_{l=0}^{k-1} T^l \mu,$$

one has $\nu(X) = 1$.

Lemma 2.4. Let $\mu_i$ ($i = 1, 2$) be probability measures on $X$ and $\mu = c \mu_1 + (1 - c) \mu_2$ for some $0 < c < 1$. If $\mu$ has no loss of mass on average then $\mu_i$ ($1 \leq i \leq 2$) has no loss of mass on average.

2.2. Entropy. Next we review the definition of entropy. More details can be found in [EW] and [Wa]. Let $\mathcal{P} \subset \mathcal{B}$ be a finite or countable partition of $X$ by Borel measurable subsets, then the entropy of $\mathcal{P}$ is

$$H_\mu(\mathcal{P}) = \sum_{P \in \mathcal{P}} \mu(P)(-\log \mu(P)).$$

Let $\mathcal{Q}$ be another partition. Then the common refinement of $\mathcal{P}$ and $\mathcal{Q}$ is denoted by

$$\mathcal{P} \vee \mathcal{Q} = \{P \cap Q \neq \emptyset : P \in \mathcal{P}, Q \in \mathcal{Q}\}.$$

The common refinement of finite collection of partitions is defined similarly. We use $T^{-1}(\mathcal{P})$ to denote the partition of $X$ consisting subsets of the form $T^{-1}(P)$ for $P \in \mathcal{P}$.

Definition 2.5. Let $(X, \mathcal{B}, \mu, T)$ be a measure preserving system and let $\mathcal{P}$ be a partition of $X$ with finite entropy, then the entropy of $T$ with respect to $\mathcal{P}$ is

$$h_\mu(T, \mathcal{P}) = \lim_{n \to \infty} \frac{1}{n} H_\mu \left( \bigvee_{i=0}^{n-1} T^{-i} \mathcal{P} \right).$$

The entropy of $T$ is

$$h_\mu(T) = \sup_{\mathcal{P} : H_\mu(\mathcal{P}) < \infty} h_\mu(T, \mathcal{P}).$$
3. Friendly measure and non-escape of mass

3.1. Non-escape of mass. Friendly measure is defined in [KLW], so let us review some concepts in that paper. In this section the norm on $\mathbb{R}^n$ is $\|\cdot\|_E$ which is induced from the standard inner product of $\mathbb{R}^n$. For $x \in \mathbb{R}^n$ and $r > 0$, $B(x, r)$ stands for the open ball of radius $r$ centered at $x$ under $\|\cdot\|_E$. For an affine hyperplane $\mathcal{L} \subset \mathbb{R}^n$, we denote by $d_{\mathcal{L}}(x)$ the distance from $x$ to $\mathcal{L}$. By $\mathcal{L}(\epsilon)$ we denote the $\epsilon$-neighborhood of $\mathcal{L}$, that is the set

$$(3.1) \quad \mathcal{L}(\epsilon) \overset{\text{def}}{=} \{ x \in \mathbb{R}^n : d_{\mathcal{L}}(x) < \epsilon \}.$$ 

Let $\mu$ be a Radon measure on $\mathbb{R}^n$ and $U$ be an open subset. We say $\mu$ is Federer on $U$ if there exists $c, \beta > 0$ such that for all $x \in \operatorname{supp}(\mu) \cap U$ and every $0 < \delta \leq s$ with $B(x, s) \subset U$ one has

$$(3.2) \quad \mu(B(x, \delta)) \geq c \left( \frac{\delta}{s} \right)^\beta \mu(B(x, s)).$$

We will say that $\mu$ is Federer if for $\mu$-a.e. $x \in X$, there exist a neighborhood $U$ of $x$ such that $\mu$ is Federer on $U$.

Let $C, \alpha > 0$ and $U$ be an open subset of $\mathbb{R}^n$. We say $\mu$ is absolutely $(C, \alpha)$-decaying on $U$ if for any non-empty open ball $B = B(z, r) \subset U$ with $z \in \operatorname{supp}(\mu)$, any affine hyperplane $\mathcal{L} \subset \mathbb{R}^n$ and any $\epsilon > 0$ one has

$$(3.3) \quad \mu(B \cap \mathcal{L}(\epsilon)) \leq C \left( \frac{\epsilon}{r} \right)^\alpha \mu(B).$$

We will say $\mu$ is absolutely decaying if for $\mu$-a.e. $y_0 \in \mathbb{R}^n$, there exist a neighborhood $U$ of $y_0$ and $C, \alpha > 0$ such that $\mu$ is absolutely $(C, \alpha)$-decaying on $U$.

Friendly measure in [KLW] is defined as Federer, nonplanar and decaying. The measures interested to us are absolutely decaying which implies nonplanar and decaying.

The non-escape of mass is related to Theorem 3.3 of [KLW]. The homogeneous space is a special case of Section 6. Here $n > 0$, $G = SL_{n+1}(\mathbb{R})$, $\Gamma = SL_{n+1}(\mathbb{Z})$ and $X = \Gamma \backslash G$. Let $t > 0$ and

$$(3.4) \quad a = \text{diag}(e^t, \ldots, e^t, e^{-nt}) \in G.$$ 

The dynamical system is $T = T_a : X \to X$ which sends $x \in X$ to $xa^{-1}$. We define the following maps from $\mathbb{R}^n$ to $G$ and $X$:

$$\phi(y) \overset{\text{def}}{=} \begin{pmatrix} I_n & 0 \\ y & 1 \end{pmatrix}, \quad \tau(y) \overset{\text{def}}{=} \Gamma \phi(y).$$

Recall that $X$ can be identified with the space $\Omega$ of unimodular lattices of $\mathbb{R}^{n+1}$. For $\epsilon > 0$, we define

$$(3.5) \quad F_\epsilon \overset{\text{def}}{=} \{ \Delta \in \Omega : \|v\|_E \geq \epsilon \quad \forall \ v \in \Delta \backslash \{0\} \},$$

i.e., $F_\epsilon$ is the collection of all unimodular lattices in $\mathbb{R}^{n+1}$ which contain no nonzero vector smaller than $\epsilon$. It is easy to see that $\{F_\epsilon\}_{\epsilon > 0}$ is an exhaustion.
of $X$. With these preparations, we can state Theorem 3.3 of [KLW] as follows:

**Theorem 3.1.** Suppose $\mu$ is a friendly measure on $\mathbb{R}^n$ and $a$ as in (3.4). Then for $\mu$-almost every $y_0 \in \mathbb{R}^n$, there is a ball $B$ centered at $y_0$ and $\tilde{C}, \alpha > 0$ such that for any $l \in \mathbb{Z}_{\geq 0}$ and $\epsilon > 0$,

$$\mu(\{y \in B : \tau(y)a^{-l} \notin F_\epsilon\}) \leq \tilde{C}\epsilon^\alpha. \tag{3.6}$$

Now let us fix a probability measure $\mu$ on $\mathbb{R}^n$ and assume it is friendly. We can cover $\text{supp}(\mu)$ by countably many open balls such that Theorem 3.1 holds. Therefore given a positive number $\delta$ (close to 0), there exist balls $B_1, \ldots, B_m$ such that Theorem 3.1 holds for all of them with the same $\tilde{C}, \alpha$ and $\mu(\cup B_i) \geq 1 - \delta$. So for any integer $l \geq 0$ and any $\epsilon > 0$,

$$\mu(\{y \in \mathbb{R}^n : \tau(y)a^{-l} \notin F_\epsilon\}) \leq \delta + \sum_{i=1}^{m} \mu(\{y \in B_i : \tau(y)a^{-l} \notin F_\epsilon\}) \leq \delta + m\tilde{C}\epsilon^\alpha \tag{3.7}$$

This allows us to prove the following non-escape of mass result:

**Corollary 3.2.** Let $\mu$ be a probability measure on $\mathbb{R}^n$ and $\tau, a$ as above. If $\mu$ is friendly, then $\nu = \tau_*\mu$ has no loss of mass on average with respect to $T = T_a$.

**Proof.** Let $\eta$ be a limit point of the sequence $\frac{1}{k} \sum_{l=0}^{k-1} T_*^l \nu$. Without loss of generality we may assume $\eta = \lim_{k \to \infty} \frac{1}{k} \sum_{l=0}^{k-1} T_*^l \nu$.

Given $\epsilon > 0$, we want to compute $\eta(F_\epsilon)$. It is easy to see that if $\epsilon_1 < \epsilon$, then $F_\epsilon$ is contained in the interior of $F_{\epsilon_1}$. Therefore we may assume $\eta(\partial F_\epsilon) = 0$. $F_\epsilon$ is relatively compact by Mahler’s criterion ([Ra] Chapter 10). According to Lemma 2.2,

$$\eta(F_\epsilon) = \lim_{k \to \infty} \frac{1}{k} \sum_{l=0}^{k-1} T_*^l \nu(F_\epsilon) = \lim_{k \to \infty} \frac{1}{k} \sum_{l=0}^{k-1} T_*^l \tau_*\mu(F_\epsilon)$$

$$= \lim_{k \to \infty} \frac{1}{k} \sum_{l=0}^{k-1} \mu(\{y \in \mathbb{R}^n : \tau(y)a^{-l} \in F_\epsilon\})$$

$$= 1 - \lim_{k \to \infty} \frac{1}{k} \sum_{l=0}^{k-1} \mu(\{y \in \mathbb{R}^n : \tau(y)a^{-l} \notin F_\epsilon\}). \tag{3.8}$$

Apply estimate (3.7) for (3.8), we have

$$\eta(F_\epsilon) \geq 1 - \delta + m\tilde{C}\epsilon^\alpha$$

for some constants $m, \alpha, \tilde{C} > 0$ which do not depend on $\epsilon$. By taking $\epsilon \to 0$ (for those with $\eta(\partial F_\epsilon) = 0$), we have

$$\eta(X) \geq 1 - \delta.$$ 

Since $\delta$ is arbitrary, $\eta(X) = 1$. \qed
3.2. Local maximal dimension and friendly. Let $\mu$ be a Radon measure on $[0,1]^n$, then we say $\mu$ is Federer, absolutely decaying or friendly if as a measure on $\mathbb{R}^n$, it is Federer, absolutely decaying or friendly. We will show that if $\mu$ has local maximal dimension and is Federer, then it is absolutely decaying and therefore friendly. To avoid confusion we review some notations. We use $\| \cdot \|$ to denote the sup norm on $\mathbb{R}^n$ and $B_s(x)$ for the ball of radius $s$ center $x$ under this norm. $\| \cdot \|_E$ stands for the Euclidean norm on $\mathbb{R}^n$ and $B(x,s)$ stands for the ball under this norm.

If $\mu$ has local maximal dimension, then as a measure on $\mathbb{R}^n$ it has the following property: There exists $s_0 > 0$ such that for any $\epsilon > 0$, $0 < \delta < 1$, $0 < s \leq s_0$, and $x \in \mathbb{R}^n$, one has
\begin{equation}
\mu(B_{\delta s}(x)) \leq \delta^{n-\epsilon} \mu(B_s(x)).
\end{equation}
Since $\mu$ is Federer, for $\mu$-a.e. $y \in \mathbb{R}^n$, there is a neighborhood $U$ of $y$ such that $\mu$ is Federer on $U$, that is (3.2) holds.

Let us fix $y$ and $U$ as above. Suppose $r_0 > 0$ such that $B_{9nr_0}(y) \subset U$ and $9nr_0 < s_0$ where $s_0$ is the upper bound of $s$ in (3.9). Here the radius $9nr_0$ is used so that the balls we are considering below are inside $U$. In the following three lemmas, we use (3.9) and (3.2) to show $\mu$ is absolutely $(C,\alpha)$-decaying on $V = B_{r_0}(y)$ for some $C,\alpha > 0$.

**Lemma 3.3.** Let $B = B(z,r) \subset V$ where $z \in \text{supp}(\mu)$ and $\mathcal{L}$ be an affine hyperplane of $\mathbb{R}^n$. Suppose $0 < \epsilon < r$, then $B \cap \mathcal{L}(\epsilon)$ can be covered (measure theoretically) by as few as $2\left(\frac{r}{\epsilon}\right)^{n-1}$ sets of the form $B_{3\epsilon n}(x)$ where $x \in B$.

**Proof.** Let us fix some notations first. In a Euclidean space with a fixed orthonormal basis ball and box mean the usual figure in Euclidean geometry. We will say $n$-ball or $n$-box if we want to emphasize the dimension. Without loss of generality, we assume $B \cap \mathcal{L}(\epsilon)$ is nonempty.

The closure of $\mathcal{L}(\epsilon)$ in $\mathbb{R}^n$ is a family of affine hyperplanes parallel to $\mathcal{L}$. Each of them is an Euclidean space under the induced inner product if we fix an origin. We can fix an orthonormal basis for all of them so that we can talk about box and ball as above. Under these frames a hyperplane intersects $B$ in a ball of radius $\leq r$. Let $L$ be a hyperplane such that $L \cap B$ has the largest area. Since $L \cap B$ is a $(n-1)$-ball of radius $\leq r$, it is contained in a $(n-1)$-box of length $2r$. Such a box can be covered by
\begin{align*}
(n-1)\text{-boxes of length } 2\epsilon. \text{ From Euclidean geometry, we know each } (n-1)\text{-box of length } 2\epsilon \text{ is contained in an } (n-1)\text{-ball of radius } \\
\epsilon\sqrt{n-1} \leq cn.
\end{align*}
So we can find a covering of $B \cap L$ by $(n-1)$-balls $B_1, \ldots, B_m$ in $L$ centered at $B \cap L$ with radius $cn$ for some integer $m \leq 2\left(\frac{r}{\epsilon}\right)^{n-1}$.

Assume $B_i$ has center $x_i$, then the ball $B(x_i,3\epsilon n)$ in $\mathbb{R}^n$ contains $B_i$. We claim that $B(x_i,3\epsilon n)$ for $1 \leq i \leq m$ cover $B \cap \mathcal{L}(\epsilon)$. To see this, let
\( x \in B \cap \mathcal{L}^{(\epsilon)} \). Since \( B \cap L \) has the largest area, there exists \( b \in B \cap L \) such that \( \|x - b\|_E < 2\epsilon \). Note \( b \in B_i \) for some \( i \), so \( \|b - x_i\|_E < \epsilon n \). Therefore

\[
\|x - x_i\|_E \leq \|x - b\|_E + \|b - x_i\|_E < 2\epsilon + \epsilon n \leq 3\epsilon n.
\]

The lemma follows from the fact that \( B(x_i, 3\epsilon n) \subset B_{3\epsilon n}(x_i) \).

\[\square\]

**Lemma 3.4.** Let \( B = B(z, r) \subset V \) where \( z \in \text{supp}(\mu) \). If \( 0 < \epsilon < r \) and \( x \in B \), then

\[
\mu(B_{3\epsilon n}(x)) \leq C \left( \frac{\epsilon}{r} \right)^{n-1} \mu(B(z, r))
\]

where the constant \( C \) does not depend on \( B \), \( x \) and \( \epsilon \).

**Proof.** By (3.9),

\[
\mu(B_{3\epsilon n}(x)) \leq C_1 \left( \frac{3\epsilon n}{r} \right)^{n-1} \mu(B_r(x)) = C \left( \frac{\epsilon}{r} \right)^{n-1} \mu(B_r(x))
\]

for some constant \( C_1 \) and hence \( C \) depending on the exponent \( 0.1 \). Since \( x \in B = B(z, r) \subset B_r(z) \), we have \( B_r(x) \subset B_{2r}(z) \). Apply this for (3.11),

\[
\mu(B_{3\epsilon n}(x)) \leq C \left( \frac{\epsilon}{r} \right)^{n-1} \mu(B(2r, z)) \leq C \left( \frac{\epsilon}{r} \right)^{n-1} \mu(B(z, 2r\sqrt{n}))
\]

since \( n \)-box \( B_{2r}(z) \) is contained in \( n \)-ball \( B(z, 2r\sqrt{n}) \). Recall that \( z \in \text{supp}(\mu) \) and \( B(z, 2r\sqrt{n}) \subset U \) by the technical choice of \( V \). If we take \( 2r\sqrt{n} \) and \( r \) as radius in (3.2), we have

\[
\mu(B(z, r)) \geq c \left( \frac{r}{2r\sqrt{n}} \right)^\beta \mu(B(z, 2r\sqrt{n}))
\]

for some \( c, \beta > 0 \) which depend on \( U \). (3.13) implies that

\[
\mu(B(z, 2r\sqrt{n})) \leq C_2 \mu(B(z, r))
\]

where \( C_2 \) depends on \( U \). Combine (3.12) and (3.14), we have

\[
\mu(B_{3\epsilon n}(x)) \leq C C_2 \left( \frac{\epsilon}{r} \right)^{n-1} \mu(B(z, r)).
\]

The dependence of \( C \) and \( C_2 \) implies \( CC_2 \) is independent of \( B \), \( x \) and \( \epsilon \).

\[\square\]

**Lemma 3.5.** \( \mu \) is absolutely decaying on \( V \).

**Proof.** Let \( B = B(z, r) \subset V \) where \( z \in \text{supp}(\mu) \) and \( \mathcal{L} \) be an affine hyperplane of \( \mathbb{R}^n \). Suppose \( 0 < \epsilon < r \), then by Lemma 3.3 we can cover \( B \cap \mathcal{L}^{(\epsilon)} \) by balls \( B_{3\epsilon n}(x_i) \) for \( x_i \in B \) and \( 1 \leq i \leq m \leq 2 \left( \frac{\epsilon}{r} \right)^{n-1} \). So

\[
\mu(B \cap \mathcal{L}^{(\epsilon)}) \leq \sum_{i=1}^{m} \mu(B_{3\epsilon n}(x_i)).
\]

By the estimate for \( \mu(B_{3\epsilon n}(x_i)) \) in Lemma 3.4, we have

\[
\mu(B \cap \mathcal{L}^{(\epsilon)}) \leq m C \left( \frac{\epsilon}{r} \right)^{n-1} \mu(B(z, r)).
\]
where $C$ is independent of $B$, $L$ and $\epsilon$. By the upper bound of $m$ above,

$$\mu(B \cap L^{(\epsilon)}) \leq 2C \left( \frac{\epsilon}{r} \right)^{0.9} \mu(B(z, r)).$$

If $\epsilon \geq r$, (3.15) holds for $C = 1$. 

Therefore, we have proved that for $\mu$-a.e. $y$ there is a neighborhood $V$ of $y$ such that $\mu$ is absolutely decaying on $V$. We summarize the result as follows:

**Theorem 3.6.** Let $\mu$ be a probability measure on $[0, 1]^n$. If $\mu$ has local maximal dimension and is Federer, then $\mu$ is absolutely decaying, hence friendly.

4. **Diagonal actions on homogeneous spaces**

4.1. **General setup for homogeneous spaces.** In this section we setup the general concepts and notations for Lie groups and their homogeneous spaces that are used in Section 5.

Let $G \subset SL(N, \mathbb{R})$ be a closed and connected subgroup with identity element $e$. Let $\Gamma \subset G$ be a discrete subgroup and define $X = \Gamma \backslash G$. Any $g \in G$ acts on $X$ by right translation $g.x = xg^{-1} = \Gamma(hg^{-1})$ for $x = \Gamma h \in X$.

Recall that $\Gamma$ is a lattice if $X$ carries a $G$-invariant probability measure $m_X$, which is called the Haar measure on $X$. From now on we assume that the discrete subgroup $\Gamma$ is a lattice.

We fix a left invariant metric $d^G$ on $G$ and use $B^G_g(r)$ (or $B^G_e(r)$ if $x = e$) to denote the ball of radius $r$ centered at $x \in G$. We define a metric $d$ on $X$ by

$$d(\Gamma g, \Gamma h) = \inf_{\gamma \in \Gamma} d^G(\gamma g, h).$$

For any compact subset $K$ of $X$, there exists $r > 0$, such that the map $B^G_e(r) \to X$ defined by sending $g \in G$ to $xg$ where $x \in K$ is an isometry. We call $r$ an injectivity radius on $K$.

Let $a \in G$ and consider the map $T = T_a : X \to X$ defined by $T(x) = a.x = xa^{-1}$. We define the stable horospherical subgroup for $a$ by

$$G^- = \{ g : a^l ga^{-l} \to e \text{ as } l \to \infty \}$$

which is a closed subgroup of $G$. Similarly one can define the unstable horospherical subgroup by

$$G^+ = \{ g : a^l ga^{-l} \to e \text{ as } l \to -\infty \}$$

which is also a closed subgroup of $G$. The centralizer of $a$ is the closed subgroup

$$G^0 = C_G(a) = \{ h : ah = ha \}$$

Next we define a special kind of diagonalizable elements which are first defined by Margulis and Tomanov in [MT] in the setting of real and p-adic algebraic groups. Here we use the more general concept in [EL], Section 7. We say that $a$ is $\mathbb{R}$-semisimple if as an element of $SL(N, \mathbb{R})$ $a$ is conjugate to a diagonal element of $SL(N, \mathbb{R})$. In particular, this implies that the adjoint
action $\text{Ad}_a(a^g a^{-1})$ of $a$ on the Lie algebra $g$ of $G$ has eigenvalues in $\mathbb{R}$ so is diagonalizable over $\mathbb{R}$. We say furthermore that $a$ is class $A$ if the following properties hold:

- $a$ is $\mathbb{R}$-semisimple.
- 1 is the only eigenvalue of absolute value 1 for $\text{Ad}_a$.
- No two different eigenvalues of $\text{Ad}_a$ have the same absolute value.

For a class $A$ element $a$ we have a decomposition of the Lie algebra $g$ into subspaces

$$g = g_{-} \oplus g_{0} \oplus g_{+}$$

where $g_{0}$ is the eigenspace for eigenvalue 1, $g_{-}$ is the direct sum of the eigenspaces with eigenvalues less than 1 in absolute value, and $g_{+}$ is the direct sum of the eigenspaces with eigenvalues greater than 1 in absolute value. These are precisely the Lie algebras of $G_{0}, G^{-}, G^{+}$, respectively.

Here and hereafter, we assume $g_{+}$ is an eigenspace of $\text{Ad}_a$ and $G^{+}$ is abelian. Let $t > 0$ be the logarithm of the absolute value of the eigenvalue on $g_{+}$. We fix a basis $e_{1}, \ldots, e_{n}$ of $g_{+}$ and use $\| \cdot \|_{+}$ to denote the sup norm under this basis, i.e.

$$\|b_{1} e_{1} + \cdots + b_{n} e_{n}\|_{+} = \sup_{1 \leq i \leq n} |b_{i}|.$$  \hfill (4.2)

Let $B_{s}^{+} (\text{or } B^{+}(s))$ be the ball of radius $s$ centered at zero of $g_{+}$ under this norm. Similarly we fix a basis consisting of eigenvectors for $g_{0}$ and $g_{-}$. We use $\| \cdot \|_{0}$ and $\| \cdot \|_{-}$ to denote the sup norm under these basis. There are corresponding concepts $B_{s}^{0}$ and $B_{s}^{-}$.  \hfill (4.3)

There exists $\alpha > 0$ and an open subset $\tilde{G}$ of $e$ in $G$ such that the map

$$\varphi : B_{\alpha}^{-} + B_{\alpha}^{0} + B_{\alpha}^{+} \to \tilde{G}$$

which sends $(x, y, z)$ to $\exp x \exp y \exp z$ is a diffeomorphism. $\alpha$ and $\varphi$ are fixed for Section 4 and 5. Each element of $\tilde{G}$ naturally corresponds to an element

$$x + y + z \in B_{\alpha}^{-} + B_{\alpha}^{0} + B_{\alpha}^{+} \subset g$$

via the above diffeomorphism $\varphi$.

We define the projection map $\pi : \tilde{G} \to g_{+}$ by

$$\pi(\exp u^{-} \exp u^{0} \exp u^{+}) = u^{+}$$

for $u^{-} \in B_{\alpha}^{-}$, $u^{0} \in B_{\alpha}^{0}$ and $u^{+} \in B_{\alpha}^{+}$. With these definitions we can say that the multiplication in $G$ is local Lipschitz in the sense of the following lemma:

**Lemma 4.1.** Given $\epsilon, t > 0$, there exist $r, s > 0$ such that

$$\exp(B_{s}^{-}) \exp(B_{s}^{0}) \exp(B_{t}^{+}) B^{G}_{r} \subset \tilde{G}$$

and

$$\|\pi(h_{1}) - \pi(h_{2})\|_{+} \leq \epsilon \|\pi(h_{1} g) - \pi(h_{2} g)\|_{+}$$

for any $h_{1}, h_{2} \in \exp(B_{s}^{-}) \exp(B_{s}^{0}) \exp(B_{t}^{+})$ and $g \in B^{G}_{r}$.
The above lemma follows from the fact that \( \pi \) is smooth and we can give each space a proper Riemannian metric according to the norm. We omit the proof, and the reader may see Section 5.1 of the author’s thesis [S] for a detailed proof.

**Definition 4.2.** We say that \( (r, s) \) is \( (t, \epsilon) \)-regular, if they satisfy (4.5) and (4.6) above.

In the following lemma we are going to consider more precisely how \( \text{Ad}_a \) changes elements of \( g^- \) and \( g^+ \).

**Lemma 4.3.** Suppose \( u^- \in B^-_s \) and \( u^+ \in B^+_s \), then \( \text{Ad}_a(u^-) \in B^-_s \) and \( \text{Ad}_a(u^+) \in B^+_e \).

**Proof.** We prove the part concerning \( g^+ \) and the other part can be proved similarly. Recall that \( t \) is the logarithm of the absolute value of the eigenvalue on \( g_+ \). So by the definition of \( \| \cdot \|_+ \) in (4.2),

\[
\| \text{Ad}_a(u^+) \|_+ = e^t \| u^+ \|_+.
\]

\[\Box\]

4.2. **Entropy and measure.** Let \( a \in G \) be a class \( \mathcal{A} \) element and \( T = T_a : X \to X \) be the map which sends \( x \in X \) to \( a.x = xa^{-1} \). In this section we review the results about using entropy to classify \( T \)-invariant measures on the homogeneous space \( X \). The method dates back to Ledrappier and Young [LY] who used entropy to classify invariant probability measures on compact Riemannian manifolds under a smooth map which answered a question by Pesin. Later their method was adapted by Margulis and Tomanov in [MT] to the settings of products of real and \( p \)-adic algebraic groups. In [MT] measures invariant under unipotent flows are classified. Along the way measures of maximal entropy for diagonal flows are also characterized. A convenient modern reference of these results is [EL].

**Theorem 4.4** ([MT]). Let \( \mu \) be a \( T \)-invariant probability measure on \( X \), then

\[
(4.7) \quad h_\mu(T) \leq -\log |\det \text{Ad}_a|_{g^-}|
\]

and equality holds \textit{iff} \( \mu \) is \( G^- \) invariant.

A Lie group \( G \) which has a lattice as a discrete subgroup is unimodular. This implies \( \det(\text{Ad}_g) = 1 \) for all \( g \in G \). Thus

\[
(4.8) \quad -\log |\det \text{Ad}_a|_{g^-}| = \log |\det \text{Ad}_a|_{g^+}| = nt.
\]

Since \( h_\mu(T) = h_\mu(T^{-1}) \), if equality holds in (4.7), we will have a similar equality for \( T^{-1} \) which is defined by \( a^{-1} \) action. Thus \( \mu \) is invariant under the closed subgroup generated by \( G^+ \) and \( G^- \). It is not hard to see from the definition of \( G^+ \) and \( G^- \) that they are \( a \)-normalized subgroups of \( G \). Furthermore the closed subgroup generated by them is normal since \( G \) is connected. In the literature, this subgroup is called the \textit{Auslander normal}
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subgroup for the element $a$. In many cases this theorem shows that the Haar measure on $X$ is the unique measure of maximal entropy, e.g.

**Corollary 4.5.** Let $\Gamma$ be a lattice of $G$ and $X = \Gamma \backslash G$. If the action of the Auslander normal subgroup of $a$ is uniquely ergodic on $X$, then $X$ has a unique measure $m_X$ of maximal entropy under map $T$.

**Remark 4.6.** If the Auslander normal subgroup of $a$ is the whole group $G$, then its action is automatically uniquely ergodic. Hence Corollary 4.5 is true.

5. Equidistribution of measures on homogeneous spaces

In this section notations are the same as in Section 4. So $G$ is a closed connected linear group with identity $e$, $\Gamma$ is a lattice of $G$, $X$ is the homogeneous space $\Gamma \backslash G$, and $m_X$ is the probability Haar measure on $X$. Also $a \in G$ is an element of class $A$ and $T = T_a : X \to X$ is the map that sends $x$ to $a.x = xa^{-1}$. Recall that we assume $G^+$ is abelian and its Lie algebra $g_+$ is an eigenspace of $\Ad_a$ with dimension $n$.

5.1. Properties of measures.

**Definition 5.1.** Suppose $\kappa > 0$ and $\mu$ is a Borel probability measure with compact support on $X$. We say $\mu$ has local dimension $\kappa$ in the unstable horospherical direction if there exist $s_0 > 0$ and a finite measure $\lambda$ on $X$ such that for any $0 < \tilde{s} \leq s < s_0$, $u \in B_s^+ + u \subset B_{\tilde{s}}^+$, $0 < \delta < 1$ and $x \in \text{supp}(\mu)$ one has

\[
\mu(\exp(B^-_s) \exp(B^0_s) \exp(B^+_{\delta s} + u).x) \ll_{\kappa, \delta} \lambda(\exp(B^-_{\tilde{s}}) \exp(B^0_{\tilde{s}}) \exp(B^+_{\tilde{s}} + u).x).
\]

We will say $\mu$ has local maximal dimension in the unstable horospherical direction if there exist $s_0$ and $\lambda$ as above such that $\kappa < n$ holds for any $\kappa < n$.

We say $\mu$ has $s_0$-local dimension $\kappa$ in the unstable horospherical direction if $s_0$ is known. If $s_0 < \alpha$ for the $\alpha$ in (4.3), then $\varphi$ can be used and (5.1) is the same as

\[
\mu(\varphi(B^-_s + B^0_s + B^+_{\delta s} + u).x) \ll_{\kappa, \delta} \lambda(\varphi(B^-_s + B^0_s + B^+_{\tilde{s}} + u).x).
\]

For the fixed basis $e_1, \ldots, e_n$ of $g_+$, we define a map $\psi : \mathbb{R}^n \to g_+$ which sends $(x_1, \ldots, x_n)$ to $x_1 e_1 + \cdots + x_n e_n$. $\psi$ is an isometric isomorphism with respect to the sup norm of $g_+$ under the chosen basis. The composite $\varphi \circ \psi : \mathbb{R}^n \to G^+ \subset G$ is a homomorphism of Lie groups as $G^+$ is assumed to be abelian. Let us fix some $x \in X$ and define $\tau : \mathbb{R}^n \to X$ that sends $b \in J$ to $x \varphi \circ \psi(b)$. See Figure 2 for the relationship of these maps. Recall that $B_s(x)$ (or $B_s$ if $x = 0$) stands for the ball of radius $s$ centered at $x$ in $\mathbb{R}^n$. 
Proposition 5.2. Let \( \mu \) be a Borel probability measure on \( J \) with local maximal dimension. There exists \( s_0 > 0 \) such that if \( B_\sigma(x) \subset J \) and \( \mu(B_\sigma(x)) \neq 0 \) for some \( 0 < \sigma < s_0 \), then \( \tau_\ast \nu \) where \( \nu = \frac{1}{\mu(B_\sigma(x))} \mu|_{B_\sigma(x)} \) has local maximal dimension in the unstable horospherical direction.

Proof. Suppose \( r \) is an injectivity radius on \( \tau(J) \). We choose some \( 0 < s_0 < \alpha/2 \) for the \( \alpha \) in [1,3] such that \( \mu \) has \( s_0 \)-local maximal dimension and

\[
\varphi(B_{s_0}^- + B_{s_0}^0 + B_{s_0}^+) \subset B_r^G.
\]

For \( \sigma < s_0 \) and \( \epsilon > 0 \), we prove that \( \nu \) has \( s_0 \)-local dimension \( n - \epsilon \) in the unstable horospherical direction. So it suffices to prove

\[
\tau_\ast(\mu|_{B_\sigma(x)})(\varphi(B_{s_0}^- + B_{s_0}^0 + B_{s_0}^+) + u).y
\]

where \( \delta, \tilde{s}, s, u \) are as in the setting of Definition 5.1 and \( y = \tau(b) \) for some \( b \in B_\sigma(x) \).

To analyze the \( \tau_\ast(\mu|_{B_\sigma(x)}) \) part in (5.3), it is convenient to write

\[
\tau(B_\sigma(x)) = x \varphi \circ \psi(B_\sigma(x) + b) = y \varphi \circ \psi(B_\sigma(x) + b).
\]

Since \( b \in B_\sigma(x), \sigma < s_0 < \alpha/2 \) and \( \psi \) is an isometry,

\[
\tau(B_\sigma(x)) \cap \varphi(B_{s_0}^- + B_{s_0}^0 + B_{s_0}^+) + u).y
\]

\[
= \varphi(B_{s_0}^+ + \psi(b - x)).y \cap \varphi(B_{s_0}^- + B_{s_0}^0 + B_{s_0}^+) + u).y
\]

\[
= \varphi(B_{s_0}^+ + \psi(b - x)).y \cap \varphi(B_{s_0}^+ + u).y
\]

\[
= y \varphi \circ \psi(B_\sigma + x - b) \cap y \varphi \circ \psi(B_{s_0} - c)
\]

where \( c = \psi^{-1}(u) \in \mathbb{R}^n \). In view of (5.5),

\[
A_{\delta \tilde{s}} \overset{\text{def}}{=} \tau^{-1}(\varphi(B_{s_0}^- + B_{s_0}^0 + B_{s_0}^+) + u).y \cap B_\sigma(x)
\]

consists exactly \( z \in B_\sigma(x) \) such that \( z - b \in B_{\delta \tilde{s}} - c \). So

\[
A_{\delta \tilde{s}} = B_{\delta \tilde{s}}(b - c) \cap B_\sigma(x) \subset B_{\delta \tilde{s}}(b - c).
\]

To compute the \( \tau_\ast \mu \) part of (5.3), let

\[
A_{\delta} \overset{\text{def}}{=} \tau^{-1}(\varphi(B_{s_0}^- + B_{s_0}^0 + B_{s_0}^+) + u).y \cap \tau^{-1}(\varphi \circ \psi(B_{s_0} + c).y)
\]

\[
= \tau^{-1}(\varphi \circ \psi(B_{s_0} + b - c)) \cap B_\delta(b - c).
\]
Since \( \mu \) has local maximal dimension, (3.9) holds, i.e.
\[
\mu(B_{\delta \tilde{\alpha}}(b - c)) \ll_t \delta^{-t} \mu(B_{\tilde{\alpha}}(b - c)).
\]
By (5.6), (5.7) and (5.8)
\[
\mu(A_{\delta \tilde{\alpha}}) \leq \mu(B_{\delta \tilde{\alpha}}(b - c)) \ll_t \delta^{-t} \mu(B_{\tilde{\alpha}}(b - c)) \leq \delta^{-t} \mu(A_{\tilde{\alpha}}).
\]
This completes the proof.

5.2. Equidistribution of measures.

**Theorem 5.3.** Let \( \mu \) be a Borel probability measure on \( X \). Suppose \( G^+ \) is abelian and \( g_+ \) is an eigenspace of \( \text{Ad}_a \). If \( \mu \) has local dimension \( \kappa \) in the unstable horospherical direction for some \( \kappa > 0 \) and \( \rho \) is a limit point of the sequence \( \frac{1}{k} \sum_{i=0}^{k-1} T_i \mu \) such that \( \rho(X) > 0 \), then \( h_\nu(T) \geq \kappa t \) where \( \nu = \frac{\rho}{\rho(X)} \).

**Proof.** Without loss of generality, we may assume
\[
\nu = \lim_{k \to \infty} \frac{1}{k} \sum_{i=0}^{k-1} T_i \mu.
\]
We fix some \( 0 < \epsilon < \min\{\frac{\kappa}{2}, 1\} \) and will construct a finite partition \( P \) of \( X \) such that
\[
h_\nu(T, P) \geq \kappa t + f(\epsilon) \quad \text{with} \quad \lim_{\epsilon \to 0} f(\epsilon) = 0.
\]
In view of the definition of \( h_\nu(T) \) in (2.3) and (5.9),
\[
h_\nu(T) \geq \kappa t + f(\epsilon).
\]
Let \( \epsilon \to 0 \) and we see \( h_\nu(T) \geq \kappa t \) which completes the proof. The proof of (5.9) is divided into four steps.

**Step one:** Construction of the partition \( P \). Fix a compact set \( K \supset \text{supp}(\mu) \) with \( \nu(K) > 1 - \epsilon^2 \). Choose some positive numbers \( r \) and \( s_0 \) such that \( 2r \) is an injectivity radius on \( K \) and \( \mu \) has \( s_0 \)-local dimension \( \kappa \) in the unstable horospherical direction. By shrinking \( r \) and \( s_0 \) we may require that \( (r, s_0) \) is \((t, c)\)-regular as in Definition 4.2 and \( e^c s_0 < \alpha \) for the \( \alpha \) in (4.3), so that (4.5), (4.6) hold and \( \varphi \) in (4.3) can be used. Fix some \( 0 < s < s_0 \) such that
\[
\varphi(\tilde{B})^{-1} \varphi(\tilde{B}) \subset B_r^G
\]
where
\[
\tilde{B} = B_s^- + B_s^0 + B_s^+ \subset \mathfrak{g}.
\]
Consider the covering of \( K \) by sets of the form \( \varphi(B).x \) where \( x \in K \) and
\[
B = B_s^- + B_s^0 + B_s^+ \subset \mathfrak{g}.
\]
Since \( K \) is compact there exists a finite covering with centers \( x_i \) for \( 1 \leq i \leq q \).
We may assume that \( x_1, \ldots, x_p \in \text{supp}(\mu) \) and \( \varphi(B).x_i \) for \( 1 \leq i \leq p \) cover \( \text{supp}(\mu) \). Furthermore by enlarging \( s \) a little bit but still requiring \( s < s_0 \) and (5.10), we may assume that \( \nu(\partial(\varphi(B).x_i)) = 0 \) for each \( i \). Let
\[
P_i = \varphi(B).x_i \quad \text{for} \quad 1 \leq i \leq q, \quad \tilde{P}_0 = X - K
\]
and
\[ (5.14) \quad \tilde{P} = \{ \tilde{P}_i : 1 \leq i \leq p \}. \]
Note that elements of $\tilde{P}$ cover $\text{supp}(\mu)$. The construction of $\mathcal{P}$ is as follows:
\[ (5.15) \quad P_1 = \tilde{P}_1, P_2 = \tilde{P}_2 \setminus P_1, P_3 = \tilde{P}_3 \setminus \bigcup_{i=1}^{q-1} P_i, \ldots, P_q = \tilde{P}_q \setminus \bigcup_{i=1}^{q-1} P_i. \]
and
\[ P_0 = X \setminus \bigcup_{i=1}^{q} P_i. \]
It follows that
\[ (5.16) \quad \nu(P_0) \leq \nu(X \setminus K) < \epsilon^2. \]
Note that $P_0$ may be an empty set if $X$ is compact but we may assume $P_i \neq \emptyset$ for each $1 \leq i \leq q$. We set
\[ \mathcal{P} = \{ P_0, P_1, \ldots, P_q \}. \]

**Step two:** General estimate. Let
\[ \mathcal{P}_m = \bigvee_{i=0}^{m-1} T^{-i} \mathcal{P}. \]
Then from Definition 2.5
\[ (5.17) \quad h_\nu(T, \mathcal{P}) = \lim_{m \to \infty} \frac{1}{m} H_\nu(\mathcal{P}_m) = \lim_{m \to \infty} \frac{1}{m} \sum_{Q \in \mathcal{P}_m} \nu(Q)(-\log \nu(Q)). \]
In the above equation the sum runs over all the nonempty sets of the form
\[ (5.18) \quad Q = Q_0 \cap T^{-1} Q_1 \cap \cdots \cap T^{-(m-1)} Q_{m-1} \quad \text{where} \quad Q_i \in \mathcal{P}. \]
Let
\[ (5.19) \quad \alpha(Q) = \sup_{1 \leq l \leq m} \frac{|\{0 \leq i < l : Q_i = P_0\}|}{l}, \]
\[ (5.20) \quad B_\epsilon = \{ x \in X : \sup_{l \geq 1} \frac{1}{l} \sum_{i=0}^{l-1} \chi_{P_0}(T^i x) > \epsilon \} \]
where $\chi_{P_0}$ is the characteristic function of $P_0$. Then $\alpha(Q) > \epsilon$ implies $Q \subset B_\epsilon$. By the maximal ergodic theorem,
\[ (5.21) \quad \epsilon \nu(B_\epsilon) \leq \nu(P_0) < \epsilon^2, \]
which implies $\nu(B_\epsilon) < \epsilon$. Let
\[ (5.22) \quad \mathcal{Q}_m = \{ Q \in \mathcal{P}_m : \alpha(Q) \leq \epsilon \}, \]
then
\[ \sum_{Q \notin \mathcal{Q}_m} \nu(Q) \leq \nu(B_\epsilon) < \epsilon. \]
Therefore
\begin{equation}
\sum_{Q \in Q_m} \nu(Q) > 1 - \epsilon
\end{equation}

In view of (5.17) and (5.23), an estimate of $\nu(Q)$ for $Q \in Q_m$ will be enough to prove (5.9) and hence the theorem.

**Step three:** Estimate of $\nu(Q)$ for $Q \in Q_m$ where $Q$ is in the form of (5.18).

Recall that $Q = Q_0 \cap \cdots \cap Q_{m-1} = P_1 \cap \cdots \cap P_{m-1} \subset \tilde{P}_1 \cap \cdots \cap \tilde{P}_{m-1}$

where $\tilde{P}_i$ is the open subset defined in (5.13). For simplicity of notations we set $Q_i = \tilde{P}_i$, so $Q_i = \varphi(B).y_i$ for some $y_i \in K$ if $Q_i \neq \tilde{P}_0$. Under these notations

$Q \subset \tilde{Q}_0 \cap \cdots \cap \tilde{T}^{-m}(m-1)\tilde{Q}_{m-1} \overset{\text{def}}{=} \tilde{Q}$

Since $\tilde{Q}_i$ is open and $\tilde{Q}_0 \neq \tilde{P}_0$, we may assume $\tilde{Q} = \varphi(U).y_0$ for some open subset $U \subset B$.

Let $N = N(Q) \overset{\text{def}}{=} |\{i \in \mathbb{Z} : 0 \leq i < m, Q_i = P_0\}|$. Since $Q \in Q_m$, we have

\begin{equation}
N \leq m \epsilon.
\end{equation}

Then by Lemma 5.9, $U$ can be covered by as few as

\begin{equation}
2^{Nn}e^{ntN+\epsilon(m-1)-N} < 2^{Nn}e^{ntN+m\epsilon}
\end{equation}

tube-like sets (see (5.43) for the precise definition) of the form

\begin{equation}
B_s^- + B_0^0 + B^+(e^{-(l-\epsilon)(m-1)s}) + u \subset B
\end{equation}

where $u \in B_s^+$. Let us fix such a covering $R$ of $U$, then (5.24) and (5.25) imply

\begin{equation}
|R| \leq 2^{Nn}e^{ntN+m\epsilon} \leq e^{ntm\epsilon+m\epsilon}2 = e^{m\epsilon A}
\end{equation}

where $A = nt + n + n \log 2$ is a constant since the system $T : X \to X$ is fixed. Recall that $\nu(\partial \tilde{Q}) = 0$, by Lemma 2.2 we have

\begin{equation}
\nu(Q) \leq \nu(\tilde{Q}) \ll \rho(X) \lim_{k \to \infty} \frac{1}{k} \sum_{i=0}^{k-1} T_i \mu(\tilde{Q}) = \lim_{k \to \infty} \frac{1}{k} \sum_{i=0}^{k-1} T_i \mu(\varphi(U).y_0)
\end{equation}

\begin{equation}
\leq \limsup_{k \to \infty} \frac{1}{k} \sum_{i=0}^{k-1} \sum_{R \in R} T_i \mu(\varphi(R).y_0) = \limsup_{k \to \infty} \frac{1}{k} \sum_{i=0}^{k-1} \sum_{R \in R} \mu(T^{-i}(\varphi(R).y_0))
\end{equation}

Recall that the elements of $\tilde{P}$ defined in (5.14) cover the support of $\mu$, so for each $R \in R$ we have

\begin{equation}
\mu(T^{-i}(\varphi(R).y_0)) = \sum_{P \in \tilde{P}} \mu(T^{-i}(\varphi(R).y_0) \cap P).
\end{equation}
Therefore

\[ \nu(Q) \ll_{\rho(X)} \limsup_{k \to \infty} \frac{1}{k} \sum_{l=0}^{k-1} \sum_{R \in \mathcal{R}} \sum_{P \in \mathcal{P}} \mu(T^{-l}(\varphi(R), y_0) \cap P). \]

Since \(|\mathcal{P}|\) is fixed and \(|\mathcal{R}|\) is bounded above efficiently in (5.27), it suffices to estimate

\[ \mu(T^{-l}(\varphi(R), y_0) \cap P) \]

for each \(R\) and \(P\). So let us fix some \(R \in \mathcal{R}\) in the form of (5.26) and \(P = \varphi(B). y \in \mathcal{P}\) for some \(y \in \text{supp}(\mu)\). We first cover \(B\) by tube-like sets of the form

\[ V = B^+_s + B^0_s + B^+(e^{-tl} s) + v \subset B \quad \text{where} \quad v \in B^+_s \]

in a way that there are not many (bounded absolutely) overlaps. For an interval \(B^\mathbb{R}_s \subset \mathbb{R}\), it can be covered by \(\leq e^tl + 1\) intervals of the form

\[ B^\mathbb{R}_{e^{-tl} s} + u \subset B^\mathbb{R}_s \quad \text{where} \quad u \in B^\mathbb{R}_s \]

such that each of them intersects at most 2 of others. Since \(B^+_s\) is the same as a direct product of \(n\) copies of \(B^\mathbb{R}_s\), we see that \(B^+_s\) can be covered by as few as

\[ (e^tl + 1)^n \]

sets of the form (5.30) and each of them intersects at most 3\(^n\) elements including itself. This not many overlapping property later will give (5.36). Now let us fix a covering \(\mathcal{E}\) of \(B\) as above, then

\[ T^{-l}(\varphi(R), y_0) \cap P \subset \bigcup_{E \in \mathcal{E}} T^{-l}(\varphi(R), y_0) \cap \varphi(E). y. \]

Let us fix some \(E \in \mathcal{E}\) in the form of (5.30). Then by Lemma 5.6 and Remark 5.7

\[ T^{-l}(\varphi(R), y_0) \cap \varphi(E). y \subset \varphi(W). y \]

where \(W\) is a tube-like set of the form

\[ B^-_s + B^0_s + B^+(e^{-l(m-1+l)+m \epsilon} s) + w \subset E \subset B \]

with \(w \in B^+_s \cap E\).

By assumption \(\mu\) has local dimension \(\kappa\) in the unstable horospherical dimension, so according to Definition 5.1 there exists a finite measure \(\lambda\) on \(X\) such that

\[ \mu(\varphi(W), y) = \mu(\varphi(B^-_s + B^0_s + B^+(e^{-l(m-1+l)+m \epsilon} s) + w), y) \ll_{\kappa} (e^{-l(m-1)+m \epsilon})^{\kappa} \lambda(\varphi(B^-_s + B^0_s + B^+(e^{-tl} s) + w), y). \]

Strictly speaking \(w\) in (5.31) depends on \(R\), \(E\) and \(P\), but we will index it by \(E\) for simplicity since we are trying to estimate (5.29) where \(R\) and \(P\) are...
fixed. As the multiplicity of the intersections of the sets in $E$ are bounded by $3^n$, we have

$$\sum_{E \in E} \lambda(\varphi(B_+ + B_0 + B^+(e^{-t}s) + w_E), y) \ll 1.$$  \hspace{1cm} (5.36)

Now combining (5.32), (5.33), (5.35) and (5.36) we have

$$\mu(T^{-1}(\varphi(R), y_0) \cap P) \leq \sum_{E \in E} \mu(\varphi(R \cap E), y_0 \cap \varphi(E), y) \ll \kappa e^{(m \kappa + m \epsilon A + m \epsilon \kappa)}.$$  \hspace{1cm} (5.37)

By (5.27), (5.28) and (5.37),

$$\nu(Q) \ll_{\kappa, \rho(X), \bar{P}} |\bar{P}| |R| \exp((-t(m - 1) + m \epsilon \kappa)$$

$$= |\bar{P}| \exp(-mkt + me \kappa + meA + m \epsilon A).$$  \hspace{1cm} (5.38)

Since $t$ and $\kappa$ are fixed, $e^{\kappa t}$ is a constant. We are trying to estimate $h_{\nu}(T, \mathcal{P})$, so the number $|\bar{P}|$ determined by $\mathcal{P}$ is fixed. Therefore,

$$\nu(Q) \ll_{\kappa, \rho(X), \mathcal{P}} \exp(m(-\kappa t + \epsilon \kappa + \epsilon A))$$

for any $Q \in \mathcal{Q}_m$.

**Step four:** Conclusion. With the results of step three we can complete the estimate of $h_{\nu}(T, \mathcal{P})$. By (5.40) and (5.22)

$$h_{\nu}(T, \mathcal{P}) \geq \lim \inf_{m \to \infty} \frac{1}{m} \sum_{Q \in \mathcal{Q}_m} \nu(Q)(-\log \nu(Q)).$$  \hspace{1cm} (5.40)

By (5.39),

$$-\log \nu(Q) \geq m(\kappa t - \epsilon \kappa - \epsilon A) + M$$

for some constant $M$ depending on $\kappa$, $\rho(X)$ and $\mathcal{P}$. In view of (5.40) and (5.41),

$$h_{\nu}(T, \mathcal{P}) \geq (\kappa t - \epsilon \kappa - \epsilon A) \lim \inf_{m \to \infty} \sum_{Q \in \mathcal{Q}_m} \nu(Q)$$

$$\geq (\kappa t - \epsilon \kappa - \epsilon A)(1 - \epsilon)$$

where the last inequality follows from (5.23). Note that

$$\lim_{\epsilon \to 0} (\epsilon \kappa - \epsilon A)(1 - \epsilon) - \kappa t \epsilon = 0.$$ 

This establishes (5.9) hence the theorem. \hfill \square

**Theorem 5.4.** Let $\mu$ be a Borel probability measure on $X$. Suppose $G^+$ is abelian, $g_+$ is an eigenspace of $Ad_a$, and the action of the Auslander normal subgroup is uniquely ergodic. If $\mu$ has local maximal dimension in the unstable horospherical direction and there is no loss of mass on average with respect to $T$, then

$$\lim_{k \to \infty} \frac{1}{k} \sum_{l=0}^{k-1} T^l_* \mu = m_X.$$
Proof. Let \( \nu \) be a limit point of the sequence \( \frac{1}{k} \sum_{l=0}^{k-1} T^l \mu \) under the weak* topology. The assumption about no loss of mass implies that \( \nu \) is a probability measure on \( X \). According to Definition 5.1 and the assumption about the measure, \( \mu \) has local dimension \( \kappa \) in the unstable horospherical direction for any \( \kappa < n \). Therefore Theorem 5.3 implies \( h_\nu(T) \geq \kappa t \) for any \( \kappa < n \). So \( h_\nu(T) \geq nt \). On the other hand, Theorem 4.4 and (4.8) imply \( h_\nu(T) \leq nt \). Thus \( h_\nu(T) = nt \). By assumption, the action of the Auslander normal subgroup is uniquely ergodic, so \( \nu = m_X \) by Corollary 4.5.\( \square \)

Remark 5.5. The no loss of mass assumption is superfluous in many cases, see Corollary 3.2.

5.3. Proof of lemmas. In this section we are going to prove the lemmas that are used in the proof of Theorem 5.3. Before doing this, let us fix some notations according to the construction of \( \mathcal{P} \) in step one of the proof. Suppose \( 0 < \epsilon < \min\{\frac{t}{2}, 1\} \), \( K \) is a compact subset of \( X \) and \( 2r \) be an injectivity radius on \( K \). Let \( 0 < s < e^{-t} \alpha \) for the \( \alpha \) in (4.3) and set \( \tilde{B} = B_s^- + B_s^0 + B_s^+ \), \( B = B_s^- + B_s^0 + B_s^+ \) so that \( \varphi \) can be used for elements of \( \tilde{B} \). An element of \( B \) is usually represented by \( u^- + u^0 + u^+ \) where \( u^- \in g_-, u^0 \in g_0, \) and \( u^+ \in g_+ \). This will be referred to as the standard representation of elements in \( g \). We also assume

\[
\varphi(\tilde{B})^{-1}\varphi(\tilde{B}) \subset B_s^G
\]

and \( (r, s) \) is \( (t, \epsilon) \)-regular so that (4.5) and (4.6) hold. An open subset of \( B \) is called tube-like if it is of the form \( B_s^- + B_s^0 + B_s^+ + u \) where

\[
\begin{align*}
&u \in g_+ \quad \text{and} \quad B_s^+ + u \subset B_s^+.
\end{align*}
\]

For \( g, h \in G \), we use \( \eta_g(h) \) to denote \( ghg^{-1} \). In the proof of the following lemmas the assumption \( G^+ \) is abelian is used.

Lemma 5.6. Suppose \( m \geq 0, l \geq 1 \) and we have tube-like sets

\[
\begin{align*}
V &= B_s^- + B_s^0 + B^+(e^{-(l-1)m}s) + v \\
W &= B_s^- + B_s^0 + B^+(e^{-l(l-1)s}) + w.
\end{align*}
\]

Then for any \( x, y \in K \),

\[
T^{-1}(\varphi(V), x) \cap \varphi(W), y = \varphi(U), y
\]

where \( U \) (possibly empty) is contained in a tube-like set of the form

\[
B_s^- + B_s^0 + B^+(e^{-(m+l)+(m+l)s}) + u \subset W.
\]

Remark 5.7. It is obvious that the conclusion is still valid if we replace \( W \) by

\[
W_1 = B_s^- + B_s^0 + B^+(e^{-tl} s) + w \subset W.
\]
Proof. We may assume $T^{-1}(\varphi(V).x) \cap \varphi(W).y \neq \emptyset$, otherwise the conclusion is trivial. Let $g \in \varphi(U) \subset \varphi(W)$. Suppose $c = \exp(w)$, then from the shape of $W$ in (5.45), we have

$$\pi(gc^{-1}) \in B^+(e^{-t(l-1)}s).$$

So $gc^{-1} = \varphi(w^- + w^0 + w^+)$ where $w^- + w^0 + w^+$ is the standard representation for $\varphi^{-1}(gc^{-1})$ and $w^+ \in B^+(e^{-t(l-1)}s)$. Thus

$$a^l gc^{-1} a^{-l} = \eta_{a^l} (\varphi(w^- + w^0 + w^+)) = \eta_{a^l} (\exp w^-) \eta_{a^l} (\exp w^0) \eta_{a^l} (\exp w^-)$$

$$= \varphi(\Ad_{a^l}(w^-) + w^0 + \Ad_{a^l}(w^+))$$

Therefore,

$$a^l g.y = (a^l gc^{-1} a^{-l})a^l c.y = \varphi(\Ad_{a^l}(w^-) + w^0 + \Ad_{a^l}(w^+)) a^l c.y.$$  \hspace{1cm} (5.50)

According to Lemma 4.3, $\Ad_{a^l}(w^-) \in B_{\varepsilon}^-$ and $\Ad_{a^l}(w^+) \in B_{\varepsilon}^+$. So

$$\Ad_{a^l}(w^-) + w^0 + \Ad_{a^l}(w^+) \in \tilde{B}.$$  \hspace{1cm} (5.51)

Since $g \in \varphi(U)$, (5.46) implies $g.y \in T^{-1}(\varphi(V).x)$. So

$$a^l g.y = \varphi(\tilde{v}).x \quad \text{for some } \tilde{v} \in V \subset B \subset \tilde{B}.$$  \hspace{1cm} (5.52)

From the two expressions of $a^l g.y$ in (5.50) and (5.52), we have

$$a^l c.y = \varphi(\Ad_{a^l}(w^-) + w^0 + \Ad_{a^l}(w^+))^{-1} \varphi(\tilde{v}).x.$$  \hspace{1cm} (5.53)

By (5.42), (5.51) and (5.52),

$$\varphi(\Ad_{a^l}(w^-) + w^0 + \Ad_{a^l}(w^+))^{-1} \varphi(\tilde{v}) \in B_r^G.$$  \hspace{1cm} (5.54)

Therefore,

$$a^l c.y = k.x \quad \text{for some } k \in B_r^G$$

and

$$a^l g.y = a^l gc^{-1} a^{-l} (a^l c.x) = a^l gc^{-1} a^{-l} k.x.$$  \hspace{1cm} (5.55)

Let $h \in \varphi(U)$ be another element, then

$$a^l h.y = a^l hc^{-1} a^{-l} (a^l c.x) = a^l hc^{-1} a^{-l} k.x.$$  \hspace{1cm} (5.56)

By (5.42), (5.49), (5.51) and similar results for $h$,

$$a^l gc^{-1} a^{-l}, a^l hc^{-1} a^{-l} \in B_r^G.$$  \hspace{1cm} (5.57)

Since $(r, s)$ is $(t, \varepsilon)$-regular (see Definition 4.2), we have

$$\|\pi(a^l gc^{-1} a^{-l}) - \pi(a^l hc^{-1} a^{-l})\|_+ \leq \varepsilon \|\pi(a^l gc^{-1} a^{-l} k) - \pi(a^l hc^{-1} a^{-l} k)\|_+.$$  \hspace{1cm} (5.58)

In view of (5.57) and the fact $k \in B_r^G$,

$$a^l gc^{-1} a^{-l} k, a^l hc^{-1} a^{-l} k \in B_{2r}^G.$$
Recall that \( a^ig.y, a^ih.y \in \varphi(V).x \) and \( 2r \) is an injectivity radius of \( x \), so (3.50) and (5.56) imply \( a^i ge^{-t}a^{-l}k, a^ihc^{-1}a^{-l}k \in \varphi(V) \). According to the shape of \( V \) in (5.44), we have

\[
\|\pi(a^i ge^{-t}a^{-l}k) - \pi(a^ihc^{-1}a^{-l}k)\|_+ \leq 2e^{-(t-\epsilon)m}s. \tag{5.59}
\]

By (5.58), (5.59) and Lemma 4.3, (5.58) and (5.59) imply

\[
\|[\pi(\varphi_a - \pi)(\varphi_{a^i} a^{-l} - \pi(\varphi_{a^i} h^{-1}a^{-l}k) - \pi(a^i hc^{-1}a^{-l}k)]_+ \leq e^{-t}l \|u(\pi(a^i ge^{-t}a^{-l}k) - \pi(a^ihc^{-1}a^{-l}k))_+ \leq e^{-t}l e^s (2e^{-(t-\epsilon)m}s, \\
\leq e^{-t}(m+1)\epsilon s. \tag{5.60}
\]

Lemma 5.8. Suppose \( Q_i = \varphi(B)_x \) for \( x_i \in K \) and \( 0 \leq i \leq m \), which are open subsets of \( X \). If \( Q = Q_0 \cap T^{-1}Q_1 \cap \cdots \cap T^{-m}Q_m \), then \( Q \subset \varphi(U).x_0 \) for some tube-like set \( U \subset B \) of the form

\[
B_s^- + B_s^0 + B^+(e^{-(t-\epsilon)m}s) + u. 
\]

Proof. The lemma is proved by induction on \( m \). If \( m = 0 \), then \( Q_0 = \varphi(B)_x \) and the lemma is true in this case.

Now assume the lemma is true for \( m-1 \), then we may assume

\[
Q_1 \cap T^{-1}Q_2 \cap \cdots \cap T^{-(m-1)}Q_m \subset \varphi(V).x \tag{5.61}
\]

where

\[
V = B_s^- + B_s^0 + B^+(e^{-(t-\epsilon)(m-1)s} + v 
\]

is a tube-like set of \( B \). It follows from Lemma 5.6 (\( m \) and \( l \) there equal \( m-1 \) and \( 1 \)) that

\[
Q \subset Q_0 \cap T^{-1}(\varphi(V).x) \subset \varphi(U).x_0 
\]

for some tube-like set

\[
U = B_s^- + B_s^0 + B^+(e^{-tm+me}s) + u. 
\]

Lemma 5.9. Let \( \mathcal{N} \) be a subset of \( \{ 1, \ldots, m \} \) with \( N \) elements, \( Q_i \) be an open subset of \( X \) for \( 0 \leq i \leq m \) such that \( Q_i = \varphi(B)_x \) for some \( x_i \in K \) if \( i \notin \mathcal{N} \). Let

\[
Q = Q_0 \cap T^{-1}Q_1 \cap \cdots \cap T^{-m}Q_m = \varphi(U).x_0
\]

for some open subset \( U \) of \( B \). Then \( U \) can be covered by as few as

\[
2^{Nn}e^{ntN + (m-N)n}
\]

tube-like sets of the form

\[
B_s^- + B_s^0 + B^+(e^{-(t-\epsilon)m}s) + u
\]

where \( u \in B^+_s \).
Proof. Let $D$ be the total number of blocks of numbers in $\mathcal{N}$, i.e. $D = 1$ if $\mathcal{N} = \{i, i+1, \ldots, i+j\} \subset \mathbb{N}$. We are going to prove the lemma by induction on $D$. If $D = 0$, then $N = 0$ and it is proved in Lemma 5.8.

Now suppose $D > 0$ and the lemma is true for $D - 1$. Let $i + 1, \ldots, i+j$ be the first block such that $\{i+1, \ldots, i+j\} \subset \mathcal{N}$. So

$$k \not\in \mathcal{N} \text{ if } k \leq i \quad \text{and} \quad i+j+1 \not\in \mathcal{N}.$$  

Lemma 5.8 applies to $Q_0 \cap T^{-1} Q_1 \cap \cdots \cap T^{-i} Q_i = \varphi(W)_x x_0$ and tells us that $W$ is contained in a tube-like set of the form

$$B_{s}^{-} + B_{s}^{0} + B^{+}(e^{-(t-\epsilon)i}s) + u.$$  

For the interval $B_{e^{-\epsilon i}i}^{R}$, it can be covered by as few as

$$2 e^{-\epsilon i} = 2^{e^{t}+i}$$  

open intervals of the form $B_{e^{-\epsilon (i+j)}}^{R} + b \subset B_{e^{-\epsilon i}}^{R}$ where $b \in \mathbb{R}$. Since $B^{+}(e^{-(t-\epsilon)i}t)$ is isomorphic to the product of $n$ copies of $B_{e^{-\epsilon i}}$, it can be covered by as few as

$$2^{e^{tj+e}}n = 2^{n\epsilon ntj+ein}$$  

tube-like sets of the form

$$B_{s}^{-} + B_{s}^{0} + B^{+}(e^{-t(i+j)}s) + w.$$  

Let $W_{1}$ be one of them. Now let us consider

$$Q_{i+1} \cap T^{-1} Q_{i+2} \cap \cdots \cap T^{-(m-i-j-1)} Q_{m} = \varphi(V).x_{i+j+1}$$  

for some open subset $V$ of $B$. The induction hypothesis implies that $V$ can be covered by as few as

$$2^{(N-j)\epsilon N + \epsilon (m-i-j-1)(N-j)n} = 2^{(N-j)\epsilon N + \epsilon (m-i-N-1)n}$$  

tube-like sets of the form

$$B_{s}^{-} + B_{s}^{0} + B^{+}(e^{-(t-\epsilon)(m-i-j-1)}s) + v.$$

Let $V_{1}$ be one of them. As the product of (5.63) and (5.65) is bounded by the number in (5.61), it remains to see

$$\varphi(W_{1}).x_{0} \cap T^{-i-j-1}(\varphi(V_{1}).x_{i+j+1}) = \varphi(U).x_{0}$$  

for some open subset $U$ which is contained in a tube-like set of the form

$$B_{s}^{-} + B_{s}^{0} + B^{+}(e^{-(t-\epsilon)m}s) + u.$$  

Lemma 5.6 ($m$ and $l$ there equal $m-i-j-1$ and $i+j+1$) implies that $U$ is a contained in a tube-like set

$$B_{s}^{-} + B_{s}^{0} + B^{+}(e^{-(t-\epsilon)m+(m-i-j)\epsilon}s) + u$$  

which is a subset of (5.67).
6. Applications

In this section we are going to interpret the improvements of DT in the setting of homogeneous space. Let $G = SL(m + n, \mathbb{R})$, $\Gamma = SL(m + n, \mathbb{Z})$, $X = \Gamma \backslash G$ and $m_X$ be the usual probability Haar measure on $X$. $G$ acts on $\mathbb{R}^{m+n}$ (considered as $M_{1,m+n}$) by $g(\xi) = \xi g$ as matrix multiplication. Let $\Omega$ be the set of unimodular lattices in $\mathbb{R}^{m+n}$. $G$ acts on $\Omega$ by $g(\Delta) = \Delta g = \{ vg : v \in \Delta \}$. $G$ acts transitively on $\Omega$ and the stabilizer of $\mathbb{Z}^{m+n}$ is $\Gamma$. Thus $\Gamma \backslash G \cong \Omega$ as a set. We endow $\Omega$ with the natural locally compact topology of $\Gamma \backslash G$. In this topology, a sequence $\{ \Delta_i \}_{i}$ converges to a lattice $\Delta$ iff $\Delta_i$ has a basis $\{ b_1^{(i)}, \ldots, b_{m+n}^{(i)} \}$ and $\Delta$ has a basis $\{ b_1, \ldots, b_{m+n} \}$ such that

\begin{equation}
\lim_i b_1^{(i)} = b_1, \ldots, \lim_i b_{m+n}^{(i)} = b_{m+n}.
\end{equation}

$M_{m,n} = \mathbb{R}^{mn}$ stands for the space of $m \times n$ matrices with real entries. Recall that $\| \cdot \|$ stands for the sup norm of $\mathbb{R}^{k}$ and $B_s(x)$ (or $B_s$ if $x = 0$) stands for the ball of radius $s$ centered at $x$ under the sup norm. There is a map

\begin{equation}
\phi : M_{m,n} \rightarrow SL(m + n, \mathbb{R})
\end{equation}

which sends $Y \in M_{m,n}$ to the block matrix $\begin{pmatrix} I_n & 0 \\ Y & I_m \end{pmatrix}$. Let $N$ be a positive integer and $t = \log N$. We set

$$a_N = \begin{pmatrix} e^{-tm}I_n & 0 \\ 0 & e^{tn}I_m \end{pmatrix} = \begin{pmatrix} N^{-m}I_n & 0 \\ 0 & N^nI_m \end{pmatrix}.$$

Recall $Y \in D I_{\sigma}$ iff there exist $q \in \mathbb{Z}^m \setminus \{0\}$ and $p \in \mathbb{Z}^n$ such that

\begin{equation}
N^m\|qY + p\| \leq \sigma \quad \text{and} \quad N^{-n}\|q\| \leq \sigma
\end{equation}

for $N$ large enough. \eqref{6.3} is equivalent to

\begin{equation}
\| (p, q)\phi(Y)a_N^{-1} \| = \| (N^m(p + qY), N^{-n}q) \| \leq \sigma.
\end{equation}

So $Y \in D I_{\sigma}$ iff

\begin{equation}
\min\{\|\xi\| : \xi \in \mathbb{Z}^{m+n} \phi(Y)a_N^{-1}, \xi \neq 0 \} \leq \sigma
\end{equation}

for all large enough $N$ depending on $Y$ and $\sigma$.

Let

$$K_{\sigma} = \{ \Delta \in \Omega : \min_{\xi \in \Delta \setminus 0} \|\xi\| > \sigma \} \quad \text{and} \quad x = \mathbb{Z}^{m+n} = \Gamma \in X.$$

It is well-known that if $0 < \sigma < 1$, then $K_{\sigma}$ is an open neighborhood of $x$. Also, the larger the $\sigma$ is, the smaller the set $K_{\sigma}$ would be. In these notations \eqref{6.5} is the same as

\begin{equation}
x\phi(Y)a_N^{-1} \notin K_{\sigma}.
\end{equation}

So DT can be $\sigma$-improved for $Y$ if \eqref{6.6} holds for all $a_N$ with large enough $N$. Let $\tau : M_{m,n} \rightarrow X$ be the map which sends $Y$ to $x\phi(Y)$. For $a \in G$, we use $T_a : X \rightarrow X$ to denote the map that sends $x \in X$ to $xa^{-1}$.
**Theorem 6.1.** Let \( \mu \) be a locally finite measure on \( M_{m,n} \) and \( T = T_a \) where \( a = a_M \) for some integer \( M > 1 \). If there exists \( s_0 > 0 \) such that for any \( s < s_0 \) and any ball \( B_s(x) \) one has

\[
\lim_{k \to \infty} \frac{1}{k} \sum_{l=0}^{k-1} T^l_s(\tau_s(\mu|_{B_s(x)})) = \mu(B_s(x))m_X,
\]

then \( DT \) can not be improved for \( \mu \) almost every element.

**Proof.** We need to show for any \( 0 < \sigma < 1 \), \( DT \) can not be \( \sigma \)-improved for \( \mu \) almost every \( Y \). So let us fix some \( 0 < \sigma < 1 \) and prove \( \mu(\text{DI}_\sigma) = 0 \).

Since \( \mu \) is locally finite, there are sufficiently large real numbers \( R \) such that \( \mu(\partial B_R) = 0 \). So it suffices to prove \( \mu(\text{DI}_\sigma \cap B_R) = 0 \) if \( \mu(\partial B_R) = 0 \). Let us fix such a positive number \( R \).

Claim: there exists \( 0 < \tau < 1 \) depending on \( \sigma \) such that for any \( s < s_0 \) one has

\[
\mu(\text{DI}_\sigma \cap B_s(x)) \leq \tau \mu(B_s(x))
\]

for any \( x \in M_{m,n} \). Let us assume the claim for the moment and prove \( \mu(\text{DI}_\sigma \cap B_R) = 0 \). Suppose otherwise, then we may choose an open subset \( U \) of \( B_R \) containing \( \text{DI}_\sigma \cap B_R \) such that

\[
\mu(U) \leq \frac{1}{\tau} \mu(\text{DI}_\sigma \cap B_R).
\]

Since \( \mu \) is locally finite, \( U \) can be covered (measure theoretically) by countably many disjoint balls \( B_{s_i}(x_i) \subset U \) for \( s_i < s_0 \) and \( x_i \in X \). By (6.8),

\[
(6.10) \quad \mu(U) = \sum_i \mu(B_{s_i}(x_i)) \geq \frac{1}{\tau} \sum_i \mu(\text{DI}_\sigma \cap B_{s_i}(x_i)) = \frac{1}{\tau} \mu(\text{DI}_\sigma \cap B_R).
\]

By (6.9) and (6.10), we have

\[
\mu(U) > \mu(U).
\]

This contradiction shows that \( \mu(\text{DI}_\sigma \cap B_R) = 0 \).

Let us prove the claim. We fix \( 0 < s < s_0 \) and some \( x \in M_{m,n} \). Since \( \sigma < 1 \), \( K_\sigma \) is a open neighborhood of \( x \). Hence there exists \( \epsilon > 0 \) such that

\[
m_X(K_\sigma) > \epsilon.
\]

So there exists a continuous function \( 0 \leq f \leq 1 \) such that

\[
supp(f) \subset K_\epsilon \quad \text{and} \quad \int_X f \, dm_X > \frac{\epsilon}{2}.
\]

We apply this \( f \) for (6.7), then

\[
(6.11) \quad \lim_{k \to \infty} \frac{1}{K} \sum_{l=0}^{k-1} \int_{B_s(x)} f(x, \phi(b)a_{-l}^{-1}) \, d\mu(b) = \mu(B_s(x)) \int_X f \, dm_X > \frac{\epsilon}{2} \mu(B_s(x)).
\]

Let

\[
E = \{ b \in B_s(x) : x, \phi(b)a_{-l}^{-1} \notin K_\sigma \quad \text{for } l \text{ large enough} \}.
\]
As in (6.6), if \( b \in \mathbf{D}_\sigma \), then \( x\phi(b)a_N^{-1} \notin K_\sigma \) for all large \( N \). In particular
\[
x\phi(b)a_M^{-l} = x\phi(b)a_M^{-1} \notin K_\sigma
\]
for \( l \) large enough. So we have
\[
(6.12) \quad \mathbf{D}_\sigma \cap B_s(x) \subset E
\]
From the definition of \( E \) and \( f \) we see that
\[
(6.13) \quad \lim_{k \to \infty} \frac{1}{k} \sum_{l=0}^{k-1} f(x\phi(b)a_M^{-l}) = 0
\]
if \( b \in E \). Note as a function of \( b \), \( \frac{1}{k} \sum_{l=0}^{k-1} f(x\phi(b)a_M^{-l}) \) is bounded above by the constant function 1, so the dominated convergence theorem implies
\[
(6.14) \quad \lim_{k \to \infty} \frac{1}{k} \sum_{l=0}^{k-1} \int_{B_s(x) \setminus E} f(x\phi(b)a_M^{-l}) \, d\mu(b) = 0.
\]
By (6.11) and (6.14),
\[
\mu(B_s(x) \setminus E) \geq \lim_{k \to \infty} \frac{1}{k} \sum_{l=0}^{k-1} \int_{B_s(x) \setminus E} f(x\phi(b)a_M^{-l}) \, d\mu(b) > \epsilon \mu(B_s(x)).
\]
Combine this with (6.12), we have
\[
(6.15) \quad \mu(\mathbf{D}_\sigma \cap B_s(x)) \leq \mu(E) \leq \left(1 - \frac{\epsilon}{2}\right) \mu(B_s(x)).
\]
Since \( \epsilon \) only depends on \( \sigma \), we may set \( \tau = 1 - \frac{\epsilon}{2} \). This completes the proof of the claim.

Remark 6.2. Suppose \( \text{supp}(\mu) \) is contained in a compact set \( A = \text{clo}(B_R(y)) \) for some \( R > 0 \) and \( y \in M_{m,n} \) such that \( \mu(\partial A) = 0 \). It is easy to see from the proof of Theorem 6.1 that it suffices to assume (6.7) holds for \( B_s(x) \subset A \).

**Theorem 6.3.** Let \( \mu \) be a Borel probability measure on \([0, 1]^{mn} \subset M_{m,n} \) with local maximal dimension. If \( \tau_* \mu \) has no loss of mass on average with respect to \( T = T_a \) where \( a = a_M \) for some integer \( M > 0 \), then \( DT \) cannot be improved for \( \mu \) almost every element.

**Proof.** By Proposition 5.2 there exists \( s_0 > 0 \) such that if \( B_s(x) \subset J \) and \( \mu(B_s(x)) \neq 0 \) for some \( 0 < s < s_0 \), then \( \tau_* \nu \) where \( \nu = \frac{1}{\mu(B_s(x))} \mu|_{B_s(x)} \) has local maximal dimension in the unstable horospherical direction.

Since \( \tau_* \mu \) has no loss of mass on average, Lemma 2.4 implies \( \tau_* \nu \) has no loss of mass on average. Therefore Theorem 5.4 implies
\[
\lim_{k \to \infty} \frac{1}{k} \sum_{l=0}^{k-1} T_*^l(\tau_* \nu) = m_X.
\]
That is
\[
\lim_{k \to \infty} \frac{1}{k} \sum_{l=0}^{k-1} T_*^l(\tau_* (\mu|_{B_s(x)})) = \mu(B_s(x))m_X.
\]
It is easy to see from the local maximal dimension property of \( \mu \) that \( \mu(\partial J) = 0 \), so the assumptions of Theorem 6.1 and Remark 6.2 are satisfied. Therefore the conclusion follows.

In author’s opinion, the assumption of non-escape of mass is superfluous in Theorem 6.3. The following are some facts about non-escape of mass property of a measure with local maximal dimension:

- \( m = n = 1 \) and \( G = SL(2, \mathbb{R}) \).
  - This is proved in an unpublished paper of Einsiedler, Lindenstrauss, Michel and Venkatesh using hyperbolic geometry of the upper half plane.
- \( m = 1, n = 2 \) and \( G = SL(3, \mathbb{R}) \).
  - This is proved by Einsiedler and Kadyrov. In fact they are working on the non-escape of mass problem under weaker assumptions and trying to generalize their method to the cases with \( m = 1 \) or \( n = 1 \).
- \( \mu \) is in addition Federer.
  - Since Federer and local maximal dimension imply friendly (Theorem 3.6), Corollary 3.2 gives the conclusion. This proves Theorem 1.5.

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