SHORT PROOFS FOR \( q \)-RAABE FORMULA AND INTEGRALS FOR JACOBI THETA FUNCTIONS

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Abstract. We shall answer a question of Mező on the \( q \)-analogue of the Raabe’s integral formula for \( 0 < q < 1 \) and we shall evaluate an integral involving the first theta function. Moreover, we will reproduce short proofs for some identities of Mező.

1. Introduction

Recall that for a complex number \( q \) and a complex variable \( a \), the \( q \)-shifted factorials are given by

\[
(a; q)_0 = 1, \quad (a; q)_n = \prod_{i=0}^{n-1} (1-aq^i), \quad (a; q)_\infty = \lim_{n \to \infty} (a; q)_n = \prod_{i=0}^{\infty} (1-aq^i) \quad (|q| < 1)
\]

and recall the dilogarithm function

\[
\text{Li}(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2}
\]

which for \( z = 1 \) evaluates to \( \zeta(2) \). There are known two \( q \)-analogues of the gamma function which both were first introduced by Jackson in [5]. The first one is:

\[
\Gamma_q(x) = \frac{(q; q)_\infty}{(q^x; q)_\infty} (1-q)^{1-x} \quad (0 < q < 1)
\]

and the second one is:

\[
\Gamma_q(x) = \frac{(q^{-1}; q^{-1})_\infty}{(q^{-x}; q^{-1})_\infty} (q-1)^{1-x} q^{x(\frac{1}{q}-1)} \quad (q > 1).
\]

For more details on the version of \( \Gamma_q(x) \) for \( 0 < q < 1 \) we refer to [2, 3] and on the version of \( \Gamma_q(x) \) for \( q > 1 \) we refer to [8]. Raabe [9] gave the following integral

\[
\int_0^1 \log \Gamma(x + t) \, dx = \log \sqrt{2\pi} + t \log t - t \quad (t \geq 0)
\]

which as \( t \to 0^+ \) implies

\[
\int_0^1 \log \Gamma(x) \, dx = \log \sqrt{2\pi}.
\]

Recently, Mező found \( q \)-analogues for both [3] and [4] as in the following theorem.
Theorem 1. (Mezö [7, Theorem 2]) If \( q > 1 \), then for any \( t > 0 \),
\[
\int_0^1 \log \Gamma_q(x + t) \, dx = \log C_q - \frac{1}{2q^t \log q} \left( \frac{1}{1 - q^{-t}} \left( 2 \text{Li}_2(q^{-t}) + \log^2(1 - q^{-t}) \right) \right.
\]
\[
+ 2 \frac{1 - q^t}{1 - q^{-t}} \log \frac{1 - q}{1 - q^{-t}} \left( \log(1 - q^{-t}) - q^t \log^2 \frac{1 - q}{1 - q^{-t}} \right),
\]
where
\[
C_q = q^{-\frac{1}{24}} \frac{1}{(q - 1)^{\frac{1}{2}}} \log \left( \frac{1}{\log q} \right) \left( \frac{1}{q - 1} \right) \log \left( \frac{1}{q - 1} ; \frac{1}{q} \right) \infty.
\]
In particular, if \( t \to 0 \), then
\[
\int_0^1 \log \Gamma_q(x) \, dx = \frac{\zeta(2)}{\log q} + \log \left( \frac{q - 1}{\sqrt{q}} \right) + \log \left( \frac{1}{\log \left( \frac{1}{q} \right)} \right) \infty.
\]

To find these formulas, Mezö needed to evaluate the integral \( \int_0^1 \zeta_q(s, x + t) \, dx \) of the \( q \)-Hurwitz zeta function and made an appeal to a result by Kurokawa and Wakayama [6]. However, these ideas seem not to apply directly to the case \( 0 < q < 1 \) and therefore the author asked how identities (5) and (6) look like in the latter case. In this note we will answer Mezö’s question as in the following theorem.

Theorem 2. If \( 0 < q < 1 \) and \( t \geq 0 \), then
\[
\int_0^1 \log \Gamma_q(x + t) \, dx = \frac{1}{2} (1 - t) \log(1 - q) - \frac{1}{\log q} \text{Li}_2(q^t) + \log(q; q) \infty.
\]
In particular, if \( t = 0 \), then
\[
\int_0^1 \log \Gamma_q(x) \, dx = \frac{1}{2} \log(1 - q) - \frac{\zeta(2)}{\log q} + \log(q; q) \infty.
\]

Next, using the same approach we will reproduce a short, elementary proof for Mezö’s Theorem 1. In fact, Mezö’s main result in [7] is the following theorem involving the Jacobi’s fourth theta function
\[
\theta_4(x, q) = \sum_{n = -\infty}^{\infty} (-1)^n q^n e^{2nx}.\]

Theorem 3. (Mezö [7, Theorem 1]) If \( 0 < q < 1 \) is real, then
\[
\int_{-\frac{1}{2} \log q}^{\frac{1}{2} \log q} \log \theta_4(ix, q) \, dx = \zeta(2) + \log q \cdot \log(q^2; q^2) \infty.
\]
To prove the previous result, the author among other things made an appeal to Theorem 2. In this paper we will provide a short proof for Theorem 3. Furthermore, we will prove the following related theorem on the Jacobi’s first theta function
\[
\theta_1(x, q) = \sum_{n = -\infty}^{\infty} (-1)^{n-1/2} q^{(n+1)/2} e^{(2n+1)ix}.
\]

Theorem 4. If \( 0 < q < 1 \) is real, then
\[
\int_{-\log q}^{\log q} \log \theta_1(ix, q) \, dx = \zeta(2) + \log q \cdot \log(q^2; q^2) \infty.
\]
2. Proof of Theorem 2

We start by the second identity. It is clear that

\[(7) \log(q^x; q) = \log \prod_{k=0}^{\infty} (1 - q^{x+k}) = \sum_{k=0}^{\infty} \log(1 - q^{x+k})\]

and that for each \(k = 0, 1, \ldots\)

\[(8) \int \log(1 - q^{x+k}) \, dx = \int \sum_{n=1}^{\infty} \frac{(q^{x+k})^n}{n} = -\frac{1}{\log q} \log \log q \cdot \text{Li}_2(q^{x+k}) + \text{Constant}.\]

Then for each \(k = 0, 1, \ldots\)

\[
\int_0^1 \log(1 - q^{x+k}) \, dx = \frac{1}{\log q} \sum_{n=1}^{\infty} \frac{q^{kn}}{n^2} - \frac{1}{\log q} \sum_{n=1}^{\infty} \frac{q^{(k+1)n}}{n^2},
\]

which combined with (7) yields

\[(9) \int_0^1 \log(q^x; q) \, dx = \sum_{k=0}^{\infty} \int_0^1 \log(1 - q^{x+k}) \, dx = \frac{\zeta(2)}{\log q}.\]

Now using the previous integral and the definition (1) we find

\[
\int_0^1 \log \Gamma_q(x) \, dx = \int_0^1 \left( \log(q^x; q) + (1 - x) \log(1 - q) - \log(q^x; q)_{\infty} \right) \, dx
\]

\[= \log(q^x; q)_{\infty} + \frac{1}{2} \log(1 - q) - \frac{\zeta(2)}{\log q},\]

as desired. As to the first identity, the substitution rule applied to the indefinite integral \(k\) gives

\[
\int_0^1 \log(1 - q^{x+t+k}) \, dx = \int_t^{t+1} \log(1 - q^n) \, du
\]

\[= \frac{1}{\log q} \sum_{n=1}^{\infty} \frac{q^{(t+k)n}}{n^2} - \frac{1}{\log q} \sum_{n=1}^{\infty} \frac{q^{(t+k+1)n}}{n^2},\]

from which we get

\[
\int_t^{t+1} \log(q^n; q)_{\infty} \, du = \frac{1}{\log q} \log \log q \cdot \text{Li}_2(q^t).
\]

Now by definition \(k\) and the previous integral we conclude that

\[
\int_0^1 \log \Gamma_q(x + t) \, dx = \int_t^{t+1} \log \Gamma_q(u) \, du
\]

\[= \int_t^{t+1} \left( \log(q^x; q)_{\infty} + (1 - u) \log(1 - q) - \log(q^x; q)_{\infty} \right) \, du
\]

\[= \log(q^x; q)_{\infty} + \left( \frac{1}{2} - t \right) \log(1 - q) - \frac{1}{\log q} \log \log q \cdot \text{Li}_2(q^t).
\]

This completes the proof.
3. A short proof for Theorem 1

It is easy to check that if $q > 1$, then

$$
\Gamma_q(x) = \Gamma_{q^{-1}}(x)q^{(x-1)/2}.
$$

Thus with the help of Theorem 2 we have

$$
\int_0^1 \Gamma_q(x) \, dx = \int_0^1 \left( \log \Gamma_{q^{-1}}(x) + \frac{(x-1)(x-2)}{2} \log q \right) \, dx
= \frac{\zeta(2)}{\log q} - \frac{1}{12} \log q + \frac{1}{2} \log(q-1) + \log(q^{-1}; q^{-1})_\infty
= \frac{\zeta(2)}{\log q} + \log \sqrt{\frac{q-1}{q}} + \log(q^{-1}; q^{-1})_\infty.
$$

This proves identity (3). As to identity (5) we similarly get

$$
\int_0^1 \Gamma_q(x+t) \, dx = \int_0^1 \left( \log \Gamma_{q^{-1}}(x+t) + \frac{(x+t-1)(x+t-2)}{2} \log q \right) \, dx
= \log(q^{-1}; q^{-1})_\infty + \left( \frac{1}{2} - t \right) \log(1 - q^{-1}) - \frac{\text{Li}_2(q^{-t})}{\log q^{-1}}
+ \left( \frac{5}{12} + \frac{t(t-2)}{2} \right) \log q
= \log(q^{-1}; q^{-1})_\infty + \frac{\text{Li}_2(q^{-t})}{\log q} + \left( \frac{1}{2} - t \right) \log(q-1)
- \left( \frac{1}{2} - t \right) \log q + \left( \frac{5}{12} + \frac{t(t-2)}{2} \right) \log q
= \log(q^{-1}; q^{-1})_\infty + \frac{\text{Li}_2(q^{-t})}{\log q} + \left( \frac{1}{2} - t \right) \log(q-1)
+ \left( \frac{t^2}{2} - \frac{1}{12} \right) \log q,
$$

which by a straightforward but long calculation can be verified to agree with the right-hand-side of the formula (5).

4. A short proof for Theorem 3

By the triple product identity, \[4, 10\], we have

$$
\theta_4(ix, q) = (qe^{-2x}; q^2)_\infty (qe^{2x}; q^2)_\infty (q^2; q^2)_\infty.
$$

Then

$$
\int_{-\frac{1}{2} \log q}^{\frac{1}{2} \log q} \log q \theta_4(ix, q) \, dx = \int_{-\frac{1}{2} \log q}^{\frac{1}{2} \log q} \left( \log(qe^{-2x}; q^2)_\infty + \log(qe^{2x}; q^2)_\infty + \log(q^2; q^2)_\infty \right) \, dx.
$$

It is easy to check that

$$
\int_{-\frac{1}{2} \log q}^{\frac{1}{2} \log q} \log(1 - q^{2k+1}e^{-2x}) \, dx = -\sum_{n=1}^{\infty} \frac{(q^{2k+1})^n}{n} \int_{-\frac{1}{2} \log q}^{\frac{1}{2} \log q} e^{-2nx} \, dx
= \frac{1}{2} \sum_{n=1}^{\infty} \frac{(q^{2k})^n}{n^2} - \frac{1}{2} \sum_{n=1}^{\infty} \frac{(q^{2k+2})^n}{n^2}.
$$
implying that

\[(12) \int_{-\frac{1}{2} \log q}^{\frac{1}{2} \log q} \log(q e^{-2x}; q^2)_{\infty} \, dx = \frac{\zeta(2)}{2}.\]

A similar argument shows that

\[(13) \int_{-\frac{1}{2} \log q}^{\frac{1}{2} \log q} \log(q e^{2x}; q^2)_{\infty} \, dx = \frac{\zeta(2)}{2}.\]

Now putting (12) and (13) in (11) gives the desired integral.

5. Proof of Theorem 4

By the triple product identity (see \([4, 10]\))

\[\theta_1(ix, q) = (e^{-2x} q^2; q^2)_{\infty} (e^{2x} q^2; q^2)_{\infty} (q^2; q^2)_{\infty}\]

and therefore,

\[(14) \int_{0}^{\log q} \log \theta_1(ix, q) \, dx = \int_{0}^{\log q} \left( \log(e^{-2x} q^2; q^2)_{\infty} + \log(e^{2x} q^2; q^2)_{\infty} + \log(q^2; q^2)_{\infty} \right) \, dx.\]

Following the same ideas of our proof for Theorem 3 above, we get

\[\int_{0}^{\log q} (e^{-2x} q^2; q^2)_{\infty} \, dx = \int_{0}^{\log q} (e^{2x} q^2)_{\infty} \, dx = \frac{\zeta(2)}{2}.\]

Now putting together in (14) gives the desired identity.

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