Néron models and base change

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Abstract. We study various aspects of the behaviour of Néron models of semi-abelian varieties under finite extensions of the base field, with a special emphasis on wildly ramified Jacobians. In Part 1, we analyze the behaviour of the component groups of the Néron models, and we prove rationality results for a certain generating series encoding their orders. In Part 2, we discuss Chai’s base change conductor and Edixhoven’s filtration, and their relation to the Artin conductor. All of these results are applied in Part 3 to the study of motivic zeta functions of semi-abelian varieties. Part 4 contains some intriguing open problems and directions for further research. The main tools in this work are non-archimedean uniformization and a detailed analysis of the behaviour of regular models of curves under base change.
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CHAPTER 1

Introduction

1. Motivation and background

1.1. Néron models of abelian varieties. Let $R$ be a complete discrete valuation ring with quotient field $K$ and algebraically closed residue field $k$. We fix a separable closure $K^s$ of $K$. Let $A$ be an abelian $K$-variety. There exists a canonical “best” extension of $A$ to a smooth commutative group scheme $\mathcal{A}$ over $R$, known as the Néron model of $A$. Its distinguishing feature is the so-called Néron mapping property, which says that if $Z$ is a smooth $R$-scheme, then any morphism of $K$-schemes $Z \times_R K \to A$ extends uniquely to an $R$-morphism $Z \to \mathcal{A}$. In particular, $\mathcal{A}$ is the minimal smooth $R$-model of $A$.

The existence of these models was proved by A. Néron [Né64], and they have since become an invaluable tool for studying arithmetic properties of abelian varieties. There are also numerous geometrical applications, such as the study of compactifications of moduli spaces of principally polarized abelian varieties. For a detailed scheme theoretical account of the construction and basic properties of Néron models, we refer to the excellent textbook [BLR90].

Taking the special fiber of the Néron model allows one to define a canonical reduction

$$\mathcal{A}_k := \mathcal{A} \times_R k$$

of $A$. This is a smooth commutative group scheme over $k$, but the geometric structure of $\mathcal{A}_k$ can be substantially more complicated than that of $A$. In general one cannot expect that the reduction of $A$ is again an abelian variety; the reduction $\mathcal{A}_k$ may not be proper, or even connected.

Let $\mathcal{A}_k^\circ$ be the identity component of $\mathcal{A}_k$. The group

$$\Phi(A) = \mathcal{A}_k / \mathcal{A}_k^\circ$$

of connected components of $\mathcal{A}_k$ is a finite abelian group, known as the Néron component group of $A$. The identity component $\mathcal{A}_k^\circ$ is canonically an extension of an abelian $k$-variety $B$ by a commutative smooth linear algebraic $k$-group $G$; this is the so-called Chevalley decomposition of $\mathcal{A}_k^\circ$. One can show that $G$ is the product of a torus $T$ and a unipotent group $U$, whose dimensions are called the toric and unipotent rank of $A$, respectively. In the special case where $U = \{0\}$, one says that $A$ has semi-abelian reduction. In the literature, one often uses the term semistable reduction instead, but we prefer to avoid it because it might lead to confusion with semistable models of $K$-varieties.

The most important structural result concerning Néron models is Grothendieck’s Semistable Reduction theorem [SGA7-1, IX.3.6]. It asserts the existence of a unique minimal finite extension $L$ of $K$ in $K^s$ such that $A \times_K L$ has
semi-abelian reduction. We say that $A$ is tamely ramified if $L$ is a tame extension of $K$, and wildly ramified else.

The Néron model is functorial in $A$, but otherwise it does not have good functorial properties. A first problem is that it behaves poorly in exact sequences. Another important complication is that its formation does not commute with base change. Let $K'$ be a finite separable extension of $K$ and denote by $R'$ the integral closure of $R$ in $K'$. We set $A' = A \times_K K'$, and we denote by $\mathcal{A}'$ the Néron model of $A'$. Since $\mathcal{A} \times_R R'$ is smooth, the Néron mapping property of $\mathcal{A}'$ implies that there exists a unique morphism of $R'$-group schemes

$$h : \mathcal{A} \times_R R' \to \mathcal{A}'$$

that extends the canonical isomorphism on generic fibers. If $A$ has semi-abelian reduction, then $h$ is an open immersion; in particular, it is an isomorphism on identity components. This property underlies the importance of Grothendieck’s Semistable Reduction Theorem. In the general case, it is quite hard to describe the properties of the base change morphism $h$.

### 1.2. Motivic zeta functions

The problem of describing how Néron models behave under base extensions lies at the heart of our work on motivic zeta functions of abelian varieties. In order to explain this notion, we first need to introduce some notation. Let $\mathbb{N}'$ be the set of positive integers not divisible by the characteristic of $k$. For each $d \in \mathbb{N}'$, we denote by $K(d)$ the unique degree $d$ extension of $K$ in $K'$. We put $A(d) := A \times_K K(d)$ and we denote by $\mathcal{A}(d)$ the Néron model of $A(d)$. In [HN11a], we defined the motivic zeta function $Z_A(T)$ as

$$Z_A(T) = \sum_{d \in \mathbb{N}'} \mathcal{A}(d) |_{\mathbb{L}}^{\text{ord}_\Lambda(T)} T^d \in K_0(\text{Var}_k)[[T]].$$

Here $K_0(\text{Var}_k)$ denotes the Grothendieck ring of $k$-varieties, $\mathbb{L}$ denotes the class $[\mathbb{A}_k^1]$ of the affine line in $K_0(\text{Var}_k)$, and $\text{ord}_\Lambda$ is a function from $\mathbb{N}'$ to $\mathbb{N}$ whose definition will be recalled in Chapter 4.

One can roughly say that $Z_A(T)$ is the generating series for the reductions $\mathcal{A}(d)_{|_{\mathbb{L}}}$ (up to a certain scaling by $\mathbb{L}$), and thus encodes in a very precise way how the Néron model of $A$ changes under tamely ramified extensions of $K$. Moreover, one can view this object as an analog of Denef and Loeser’s motivic zeta function for complex hypersurface singularities. This link will not be pursued further in this monograph; we refer to [HN12] for a detailed survey, and an overview of the literature.

Since each of the connected components of $\mathcal{A}(d)_{|_{\mathbb{L}}}$ is isomorphic to the identity component $\mathcal{A}(d)_{|_{\mathbb{L}}}^0$, we have the relation

$$[\mathcal{A}(d)_{|_{\mathbb{L}}}] = [\Phi(A(d))] \cdot [\mathcal{A}(d)_{|_{\mathbb{L}}}^0]$$

in $K_0(\text{Var}_k)$ for every $d \in \mathbb{N}'$. Because of this fact, many properties of $Z_A(T)$, such as rationality and the nature of its poles, are closely linked to analogous properties of the Néron component series

$$S_A(T) = \sum_{d \in \mathbb{N}'} \Phi(A(d)) |T|^d \in \mathbb{Z}[[T]]$$

that we introduced in [HN10] (there it was denoted $S_o(A; T)$). This series measures how the number of Néron components varies under tame extensions of $K$. We were able to prove in [HN10, 6.5] that it is a rational function when $A$ is tamely ramified.
or $A$ has potential multiplicative reduction. This was a key ingredient of our proof that the motivic zeta function $Z_A(T)$ is rational if $A$ is tamely ramified [HN11a]. Moreover, setting $T = L^{-s}$ and viewing $s$ as a formal variable, we showed that $Z_A(L^{-s})$ has a unique pole. Interestingly, this pole coincides with an important arithmetic invariant of the abelian variety $A$: the base change conductor $c(A)$, which was introduced for tori by Chai and Yu [CY01] and for semi-abelian varieties by Chai [Ch00]. It is a nonnegative rational number that measures the defect of semi-abelian reduction of $A$.

1.3. Aim. One of the main purposes of this monograph is to extend the above mentioned results beyond the case of tamely ramified abelian varieties. For one thing, it is natural to ask what can be said without the tameness assumption. In general, it is not even clear if $Z_A(T)$ is rational if $A$ is wildly ramified. We establish this fact for Jacobians in Chapter 7. It remains a considerable challenge to understand the motivic zeta function of a wildly ramified abelian variety that is not a Jacobian, and likewise, to understand the Néron component series of a wildly ramified abelian variety that is not a Jacobian and does not have potential multiplicative reduction.

Going in a different direction, one can ask about the situation for more general group schemes than abelian varieties. In [HN11a] we developed the general theory of motivic zeta functions to also include, in particular, the class of semi-abelian varieties. The existence of Néron models of semi-abelian $K$-varieties was proven in [BLR90]. An important difference with the case of abelian varieties is that the Néron model of a semi-abelian variety will, in general, only be locally of finite type. In order to get a meaningful definition of the motivic zeta function, one has to consider the maximal quasi-compact open subgroup scheme of the Néron model. On the level of component groups, this means that we consider the torsion subgroup of the component group.

While the methods we developed to study the behaviour of component groups of abelian varieties under base change can easily be extended to semi-abelian varieties, the torsion subgroup is a much more subtle invariant. An important complication is the lack of a geometric characterization of this object. A natural candidate would be the following: the Néron component group of the maximal split subtorus of a semi-abelian $K$-variety $G$ is a lattice of maximal rank inside the component group $\Phi(G)$, and one might hope to capture the torsion part by showing that this injection is split. We will show that this is usually not the case. For algebraic tori one can encompass this problem by passing to the dual torus, but no similar technique seems to exist for general semi-abelian varieties. Our approach consists in defining a suitable notion of non-archimedean uniformization for semi-abelian varieties. However, the uniformization space will no longer be an algebraic object, so that the existence of a (formal) Néron model is no longer guaranteed; instead, we will make a careful study of the properties of the sheaf-theoretic Néron model defined by Bosch and Xarles [BX96].

1.4. A guiding principle. Before we move on to present an overview of the contents of this monograph, it may be instructional to point out some of the main themes and strategies. One of the basic ideas in our work on component series and motivic zeta functions of abelian varieties is the expectation that the Néron model of an abelian $K$-variety $A$ changes “as little as possible” under a tame
extension $K'/K$ that is “sufficiently orthogonal” to the minimal extension $L/K$ where $A$ acquires semi-abelian reduction. This principle was a crucial ingredient in establishing rationality and determining the poles of the component series $S_A^r(T)$ and the motivic zeta function $Z_A(T)$ in [HN10] and [HN11a].

What do we mean by “as little as possible”? It is unreasonable to require that the base change morphism

$$h : \mathcal{A} \times_R R' \to \mathcal{A}'$$

is an isomorphism: even when $A$ has semi-abelian reduction, the number of connected components of the Néron model might still change (as we’ll see, the rate of growth is determined by the toric rank $t(A)$ of $A$). It can be shown by elementary examples that if $A$ does not have semi-abelian reduction, we cannot even require $h$ to be an open immersion. The best we can ask for is that the following two properties are satisfied.

1. The number of components grows as if $A$ had semi-abelian reduction, i.e., if $K'/K$ is a tame extension of degree $d$, then the equality

$$|\Phi(A \times_K K')| = d^{t(A)} \cdot |\Phi(A)|$$

holds;

2. The $k$-varieties $\mathcal{A}^0_k$ and $(\mathcal{A}')^0_k$ define the same class in $K_0(\text{Var}_k)$. By [Ni11b] 3.1, this is equivalent to the property that $\mathcal{A}^0_k$ and $(\mathcal{A}')^0_k$ have the same unipotent and reductive ranks, and isomorphic abelian quotients in the Chevalley decomposition.

The meaning of “sufficiently orthogonal” is less clear. The most natural guess is that the extensions $K'$ and $L$ should be linearly disjoint over $K$, which is equivalent to asking that $[K':K]$ and $[L:K]$ are coprime because the extension $K'/K$ is tame. We’ve shown in [HN10] and [HN11a] that this condition is indeed sufficient when $A$ is tamely ramified or $A$ has potential multiplicative reduction. However, we will see that, for the Jacobian $\text{Jac}(C)$ of a $K$-curve $C$, the condition has to be modified: one needs to replace the degree of $L$ over $K$ by another invariant $e(C)$ that we call the stabilization index of $C$. It is defined in terms of the geometry of the $R$-models of the curve $C$. For general wildly ramified abelian $K$-varieties, it is not even clear if a suitable notion of orthogonality exists; we will come back to this problem in Part 3.

2. Content of this monograph

2.1. Overview of the chapters. We’ll now give an overview of the chapters of this text. A brief summary can also be found at the beginning of each chapter.

Chapter 2 contains preliminary material on group schemes, models of curves and related topics. We recall key results from the literature and prove some basic new properties that will be needed in the remainder of the text.

Part II is the longest part of this monograph; it is devoted to the study of Néron component groups of semi-abelian $K$-varieties. One of the main objectives is to prove the rationality of the Néron component series and to determine the order of its pole at $T = 1$; this is the pole that influences the behaviour of the motivic zeta function. In Chapter 3 we investigate wildly ramified Jacobian varieties. Even in this situation, many of the methods we used for tamely ramified abelian varieties are no longer sufficient, or applicable. To point out just one problem, let us mention
that one can find examples already for elliptic curves where properties (1) and (2) in Section 1.4 above do not hold for tame extensions of degree coprime to \([L : K]\).

The definition of the stabilization index \(e(C)\) and the study of its basic properties will occupy an important part of the chapter.

The approach we take is to make use of the close relationship between Néron models of the Jacobian \(\text{Jac}(C)\) of a \(K\)-curve \(C\) and regular models of \(C\). More precisely, if we fix a regular, proper and flat \(R\)-model \(\mathcal{C}\) of a smooth and proper \(K\)-curve \(C\) of index one, then the relative Picard scheme \(\text{Pic}^{0}_{C/R}\) is canonically isomorphic to the identity component \(\mathcal{A}_{0}\) of the Néron model \(\mathcal{A}\) of the Jacobian \(A = \text{Jac}(C)\); this is a fundamental theorem of M. Raynaud [Ra70]. Because of this link, many invariants associated to \(\mathcal{A}\) can also be computed on \(\mathcal{C}\). In particular, this is true for the component group. A key technical step in Chapter 3 is therefore to provide a detailed description of the behaviour of regular models of \(K\)-curves under tame extensions of \(K\).

Assume that the special fiber \(\mathcal{C}_k\) is a strict normal crossings divisor. Then we’ll call \(\mathcal{C}\) an \(sncd\)-model of \(C\). For any \(d \in \mathbb{N}'\), we denote by \(\mathcal{C}_d\) the normalization of \(\mathcal{C} \times_R R(d)\) and by \(\mathcal{C}(d)\) the minimal desingularization of \(\mathcal{C}_d\). We will show that \(\mathcal{C}(d)\) is an \(sncd\)-model of \(C \times_K K(d)\) and we explain how its structure can be determined from the structure of \(\mathcal{C}\). In order to obtain these results, we show that \(\mathcal{C}_d\) has at most locally toric singularities, which can be explicitly resolved using the results in Kiraly’s PhD thesis [Ki10]. We provide an appendix with some basic results on the resolution of locally toric singularities, because this part of [Ki10] has not been published. Alternatively, one could use the language of logarithmic geometry [Ka94].

Next, we define the stabilization index \(e(C)\). If \(C\) is tamely ramified, it coincides with the degree of the minimal extension over which \(C\) acquires semi-stable reduction, but this is not true in general. The main property of \(e(C)\) is that one can make a very precise comparison of the special fibers of \(\mathcal{C}\) and \(\mathcal{C}(d)\) for any \(d \in \mathbb{N}'\) prime to \(e(C)\); see Proposition 2.3.3. The key point is that the combinatorial structure of \(\mathcal{C}(d)_k\) only depends on the combinatorial structure of \(\mathcal{C}_k\), and not on the characteristic of \(k\). This results allows us to extend certain facts from [HN10] from tamely ramified abelian varieties to wildly ramified Jacobians, which is crucial for proving the rationality of the component series. Our strategy is to use Winters’ theorem on the existence of curves with fixed reduction type to reduce to the case where \(K\) has equal characteristic zero; then every abelian \(K\)-variety is tamely ramified. It is surprisingly hard to give a direct combinatorial proof of our results on the component series, even for tamely ramified Jacobians.

In Chapter 3, we switch our attention to Néron component groups of semi-abelian varieties. As we’ve explained before, the main problem here is to understand the behaviour of the torsion part of the component group under finite extensions of the base field \(K\).

Our approach is based on two main tools. The first one is non-archimedean uniformization in the sense of [BX96]. This theory shows that, in the rigid analytic category, one can write every abelian \(K\)-variety as a quotient of a semi-abelian \(K\)-variety \(E\) by an étale lattice such that the abelian part of \(E\) has potential good reduction; in this way, one can try to reduce the study of Néron component groups to the case of tori and abelian varieties with potential good reduction. We extend this construction to semi-abelian varieties. The main complication is that, in this
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In the case, one needs to replace $E$ by an analytic extension of a semi-abelian $K$-variety by a $K$-torus. This extension will usually not be algebraic, so that one cannot use the theory of Néron models for algebraic groups. Neither can one apply the theory of formal Néron models in [BS95], because the set of $K$-points on the extension can be unbounded.

This brings us to the second tool in our approach: the sheaf-theoretic Néron model of Bosch and Xarles [BX96]. They interpret the formation of the Néron model as a pushforward operation on abelian sheaves from the smooth rigid site of $\text{Sp}K$ to the smooth formal site on $\text{Spf}R$, and show how one can recover the component group from this sheaf-theoretic interpretation. An advantage of this approach, which we already exploited in [HN10], is that one can quantify the lack of exactness of the Néron functor by means of the derived functors of the pushforward. A second advantage is that one can associate a sheaf-theoretic Néron model and a component group to any smooth abelian sheaf on $\text{Sp}K$, and in particular to any commutative rigid $K$-group, even when a geometric (formal) Néron model does not exist. This is particularly useful when considering non-archimedean uniformizations of semi-abelian varieties as above.

With these tools at hand, we can control the behaviour of the torsion part of the component group of a semi-abelian variety under finite extensions of the base field $K$. Our main result is Theorem 3.4.2, which says, in particular, that the component series of a semi-abelian $K$-variety $G$ is rational if the abelian part $G_{\text{ab}}$ of $G$ is tamely ramified or a Jacobian, and also if $G_{\text{ab}}$ has potential multiplicative reduction.

In Part 2, we study Chai’s base change conductor of a semi-abelian variety, and some related invariants that were introduced by Edixhoven. These invariants play a key role in the determination of the poles of motivic zeta functions of semi-abelian varieties. In Chapter 5 we define a new invariant for wildly ramified abelian $K$-varieties, the tame base change conductor $c_{\text{tame}}(A)$. This value is defined as the sum of the jumps (counting multiplicities) in Edixhoven’s filtration on the special fiber of the Néron model of $A$. Equivalently, one can also define $c_{\text{tame}}(A)$ as a limit of base change conductors with respect to all finite tame extensions of $K$ in $K^s$. If $A$ is the Jacobian of a curve $C$, then we show that these jumps, and hence also $c_{\text{tame}}(A)$, only depend on the combinatorial data of the special fiber of the minimal sncd-model of $C$.

By construction, it is clear that $c_{\text{tame}}(A) = c(A)$ if $A$ is tamely ramified. In the wild case, this equality may no longer hold. For instance, we’ll show that for an elliptic $K$-curve $E$, the equality $c(E) = c_{\text{tame}}(E)$ holds if and only if $E$ is tamely ramified, and we interpret the error term in the wildly ramified case.

It is natural to ask how $c(A)$ is related to other arithmetic invariants that one can associate to an abelian $K$-variety $A$. Recall that, for algebraic tori, the main result of [CY01] states that the base change conductor of the torus equals one half of the Artin conductor of its cocharacter module. The situation is more delicate for abelian varieties: the base change conductor is not invariant under isogeny, so that it certainly cannot be computed from the Tate module of $A$ with $\mathbb{Q}_\ell$-coefficients. In Chapter 6 we study the case where $A$ is the Jacobian of a curve $C$. A promising candidate to consider is the so called Artin conductor $\text{Art}(C)$ of the curve $C$. This important invariant was introduced by T. Saito in [Sa88] and measures the difference between the Euler characteristics of the geometric generic and special
fibers of the minimal regular model of \( C \), with a correction term coming from the Swan conductor of the \( \ell \)-adic cohomology of \( C \). It is not reasonable to expect that \( \text{Art}(C) \) and \( c(A) \) contain equivalent information, since \( \text{Art}(C) \) vanishes if and only if \( C \) has good reduction while \( c(A) \) vanishes if and only if \( C \) has semistable reduction. However, one can hope to express \( c(A) \) in terms of \( \text{Art}(C) \) and a suitable measure for the defect of potential good reduction of \( C \).

We will establish such an expression for curves of genus 1 or 2. For elliptic curves, we get a clear picture: we will show that \(-12 \cdot c(E)\) is equal to \( \text{Art}(E) \) if \( E \) has potential good reduction, and to \( \text{Art}(E) - v_K(j(E)) \) else. In the latter case, one can view \(-v_K(j(E))\) as a measure for the defect of potential good reduction by noticing that it is equal to \( |\Phi(E \times_K L)|/[L : K] \), for any finite extension \( L \) of \( K \) such that \( E \times_K L \) has semi-abelian reduction.

Our method for genus 2 curves is somewhat indirect; in this case, the curve is hyperelliptic, which allows one to define a minimal discriminant \( \Delta_{\text{min}} \in K \) and its valuation \( v_K(\Delta_{\text{min}}) \). We will compare \( v_K(\Delta_{\text{min}}) \) with the base change conductor \( c(A) \). The relationship to \( \text{Art}(C) \) then becomes clear due to work of Liu \[ Li94 \] and Saito \[ Sa88 \], where \( \text{Art}(C) \) and \( v(\Delta_{\text{min}}) \) are compared. One should point out that already for genus 2, the invariants \( c(A) \) and \( \text{Art}(C) \) seem to “diverge”; the base change conductor \( c(A) \) rather seems to behave like \( v(\Delta_{\text{min}}) \). However, it is not clear how to generalize the definition of \( v(\Delta_{\text{min}}) \) to curves of higher genus.

We finally turn our focus to motivic zeta functions in Part 8. We recall the definition of the motivic zeta function \( Z_G(T) \) of a semi-abelian \( K \)-variety \( G \) in Chapter 7. Our principal result, Theorem 8.1.2 extends the main theorem in \[ HN11a \] to Jacobians and to tamely ramified semi-abelian varieties. More precisely, let \( G \) be a semi-abelian \( K \)-variety with abelian part \( G_{ab} \), and assume either that \( G \) is tamely ramified or that \( G \) is a Jacobian. Then we prove that \( Z_G(T) \) is a rational function and that \( Z_G(L^{-s}) \) has a unique pole at \( s = c_{\text{tame}}(G) \) of order \( t_{\text{tame}}(G_{ab}) + 1 \). Here \( t_{\text{tame}}(G_{ab}) \) denotes the tame potential toric rank of \( G_{ab} \), that is, the maximum of the toric ranks of \( G_{ab} \times_K K' \) as \( K' \) runs over the finite tame extensions of \( K \). We will also discuss similar results for Prym varieties.

In Chapter 8, we establish a cohomological interpretation of the motivic zeta function \( Z_G(T) \) by means of a trace formula; this is similar to the Grothendieck-Lefschetz trace formula for varieties over finite fields and the resulting cohomological interpretation of the Hasse-Weil zeta function. Let \( p \) be the characteristic exponent of \( k \) and let \( \ell \) be a prime different from \( p \). Denote by \( K' \) the tame closure of \( K \) in \( K^s \) and choose a topological generator \( \sigma \) of the tame inertia group \( \text{Gal}(K'/K) \). We denote by \( P_G(T) \) the characteristic polynomial of \( \sigma \) on the tame \( \ell \)-adic cohomology of \( G \). One of our main results says that, if \( G \) has maximal unipotent rank, the prime-to-\( p \) part of the order of the Néron component group of \( G \) is equal to \( P_G(1) \). We deduce from this result that, for every tamely ramified semi-abelian \( K \)-variety \( G \), the \( \ell \)-adic Euler characteristic of the special fiber of the quasi-compact part of the Néron model of \( G \) is equal to the trace of \( \sigma \) on the \( \ell \)-adic cohomology of \( G \). This yields a trace formula for the specialization of the motivic zeta function \( Z_G(T) \) with respect to the \( \ell \)-adic Euler characteristic

\[ \chi : K_0(\text{Var}_k) \to \mathbb{Z} : [X] \mapsto \chi(X). \]

We also give an alternative proof of this result for a Jacobian \( A = \text{Jac}(C) \) using a computation on the tame nearby cycles on an sncd-model \( \mathcal{C} \) for the curve \( C \). On
the way, we recover a formula of Lorenzini for the characteristic polynomial $P_A(T)$ in terms of the geometry of $\mathcal{C}_k$.

To conclude, we formulate some interesting open problems and directions for further research in Part 4.

### 2.2. Notation.

**2.2.1** We denote by $K$ a complete discretely valued field with ring of integers $R$ and residue field $k$. We denote by $\mathfrak{m}$ the maximal ideal of $R$, and by $v_K : K \to \mathbb{Z} \cup \{\infty\}$ the normalized discrete valuation on $K$. We assume that $k$ is separably closed and denote by $p$ the characteristic exponent of $k$. The letter $\ell$ will always denote a prime different from $p$. For most of the results in this memoir, the conditions that $R$ is complete and $k$ separably closed are not serious restrictions: since the formation of Néron models commutes with extensions of ramification index one [BLR90 10.1.3], one can simply pass to the completion of a strict henselization of $R$. For some of the results we present, we need to assume that $k$ is perfect (and thus algebraically closed); this will be clearly indicated at the beginning of the chapter or section.

**2.2.2** We fix a uniformizing parameter $\pi$ in $R$ and a separable closure $K_s$ of $K$. We denote by $K_t$ the tame closure of $K$ inside $K_s$, and by $\mathcal{P} = \text{Gal}(K_s/K_t)$ the wild inertia subgroup of the inertia group $I = \text{Gal}(K_s/K)$ of $K$.

**2.2.3** We denote by $\mathbb{N}'$ the set of positive integers prime to $p$. For every $d \in \mathbb{N}'$, there exists a unique degree $d$ extension of $K$ inside $K_t$, which we denote by $K(d)$. It is obtained by joining a $d$-th root of $\pi$ to $K$. The integral closure $R(d)$ of $R$ in $K(d)$ is again a complete discrete valuation ring. For every $K$-scheme $Y$, we put $Y(d) = Y \times_K K(d)$.

**2.2.4** We’ll consider the special fiber functor

$$(\cdot)_k : (\text{Schemes}/R) \to (\text{Schemes}/k) : \mathcal{X} \mapsto \mathcal{X}_k = \mathcal{X} \times_R k$$

as well as the generic fiber functor

$$(\cdot)_K : (\text{Schemes}/R) \to (\text{Schemes}/K) : \mathcal{X} \mapsto \mathcal{X}_K = \mathcal{X} \times_R K.$$

We will use the same notations for the special fiber functor from the category of formal $R$-schemes locally topologically of finite type to the category of $k$-schemes locally of finite type, resp. Raynaud’s generic fiber functor from the category of formal $R$-schemes locally topologically of finite type to the category of quasi-separated rigid $K$-varieties.

**2.2.5** We denote by $(\cdot)_{\text{an}}$ the rigid analytic GAGA functor from the category of $K$-schemes of finite type to the category of quasi-separated rigid $K$-varieties. From now on, all rigid $K$-varieties will tacitly be assumed to be quasi-separated.

**2.2.6** For every separated $k$-scheme of finite type $X$, we denote by $\chi(X)$ its $\ell$-adic Euler characteristic

$$\chi(X) = \sum_{i \geq 0} (-1)^i \dim H^i_\ell(X, \mathbb{Q}_\ell).$$

The value $\chi(X)$ does not depend on $\ell$. 

(2.2.7) For every field $F$, an algebraic $F$-group is a group scheme of finite type over $F$. A semi-abelian $F$-variety is a smooth commutative algebraic $F$-group that is an extension of an abelian $F$-variety by an algebraic $F$-torus.

(2.2.8) For every finitely generated abelian group $\Phi$, we denote by $\Phi_{\text{tors}}$ the subgroup of torsion elements and by $\Phi_{\text{free}}$ the free abelian group $\Phi/\Phi_{\text{tors}}$. These operations define functors from the category of finitely generated abelian groups to the category of finite abelian groups and the category of free abelian groups of finite rank, respectively.

3. Acknowledgements

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1. Galois theory of $K$

In this section, we assume that $k$ is algebraically closed.

1.1. The Artin conductor.

(1.1.1) Let $K'$ be a tame finite extension of $K$ of degree $d$. Denote by $R'$ the integral closure of $R$ in $K'$, and by $m'$ the maximal ideal of $R'$. Then the quotient $m'/(m')^2$ is a rank one vector space over $k$, and the left action of $\Gamma = \text{Gal}(K'/K)$ on $K'$ induces an injective group morphism

$$\Gamma \rightarrow \text{Aut}_k(m'/(m')^2) = k^*$$

whose image is the group $\mu_d(k)$ of $d$-th roots of unity in $k$. Thus we can identify $\text{Gal}(K'/K)$ and $\mu_d(k)$ in a canonical way.

(1.1.2) If $L$ is a finite Galois extension of $K$ with Galois group $\Gamma = \text{Gal}(L/K)$, then the lower numbering ramification filtration $(\Gamma_i)_{i \geq -1}$ on $\Gamma$ is defined in [Se68, IV §1]. We quickly recall its definition. Denote by $R_L$ the valuation ring of $L$, by $m_L$ its maximal ideal and by $\pi_L$ a uniformizer of $L$. For every $i \geq -1$, the subgroup $\Gamma_i$ consists of all elements $\sigma$ in $\Gamma$ such that $v_L(\pi_L - \sigma(\pi_L)) \geq i + 1$.

Note that $\sigma$ belongs to $\Gamma_i$ if and only if it acts trivially on $R_L/m_L^{i+1}$, with $i$ the smallest integer greater than or equal to $i$. Since $k$ is algebraically closed, we have $\Gamma_i = \Gamma$ for all $i \leq 0$.

(1.1.3) If $K'$ is a finite extension of $K$ in $L$, and if we denote by $H$ the Galois group $\text{Gal}(L/K')$, then $H_i = \Gamma_i \cap H$ for all $i$ by Proposition 2 in [Se68, IV §1]. If $K'$ is Galois over $K$ and if we set $\Gamma' = \text{Gal}(K'/K)$, then the image of $\Gamma_i$ under the projection morphism $\Gamma \rightarrow \Gamma'$ equals $\Gamma_{\varphi_{L/K'}(i)}$ for every $i \geq -1$, where $\varphi_{L/K'}$ is the Herbrand function of the extension $L/K'$ [Se68, IV §3]. This result is known as Herbrand’s theorem.

(1.1.4) Let $V$ be a finite dimensional vector space over a field $F$, and consider an action of the Galois group $\text{Gal}(K^*/K)$ on $V$. We assume that there exists a finite extension $L$ of $K$ in $K^*$ such that the Galois action on $V$ factors through the finite quotient $\Gamma = \text{Gal}(L/K)$ of $\text{Gal}(K^*/K)$. The (exponent of the) Artin conductor $\text{Art}(V)$ of $V$ is defined by

$$\text{Art}(V) = \sum_{i \in \mathbb{N}} 1_{[\Gamma : \Gamma_i]} \dim(V/V^\Gamma_i) \in \mathbb{Q}.$$
It can be written as the sum of the so-called tame part
\[ \dim(V) - \dim(V^\Gamma) \]
and the wild part
\[
\text{Sw}(V) = \sum_{i \in \mathbb{Z}_{>0}} \frac{1}{[\Gamma : \Gamma_i]} \dim(V/V^\Gamma_i)
\]
which is also called the Swan conductor of \( V \). The Artin and Swan conductor do not depend on the choice of \( L \). They measure the ramification, resp. wild ramification, of the Galois representation \( V \), and vanish if and only if \( \text{Gal}(K^s/K) \), resp. \( P \), act trivially on \( V \).

\[ \text{(1.1.5)} \] If \( V \) is a finite dimensional \( \mathbb{Q}_\ell \)-vector space endowed with a continuous and quasi-unipotent action of \( \text{Gal}(K^s/K) \), then we define the Artin and Swan conductor of \( V \) as the corresponding conductor of the semi-simplification \( V^{ss} \) of the Galois representation \( V \). Recall that the Galois action on the \( \ell \)-adic cohomology spaces with compact supports of an algebraic \( K \)-variety is always quasi-unipotent, by the Monodromy Theorem \[ \text{SGA7-I, Exp. I, 1.3} \].

1.2. Isolating the wild part of the conductor. The following easy corollary of Herbrand’s theorem will be quite useful for our purposes.

**Proposition 1.2.1.** Let \( L \) be a finite Galois extension of \( K \) in \( K^s \), and let \( K' \) be a finite tame extension of \( K \) of degree \( d \) in \( K^s \). We set \( L' = K'L \subset K^s \), \( \Gamma = \text{Gal}(L/K) \) and \( \Gamma' = \text{Gal}(L'/K') \), and we denote by \( e \) the greatest common divisor of \( d \) and \( [L : K] \). Then for every \( i > 0 \), the image of \( \Gamma'_i \) under the natural morphism \( \Gamma' \to \Gamma' \) equals \( \Gamma^i_e/d \).

**Proof.** This follows immediately from Herbrand’s theorem, since \( L'/L \) is a tame extension of degree \( d/e \) so that \( \varphi_{L'/L}(i) = ie/d \) for all \( i \geq 0 \). \( \square \)

\[ \text{(1.2.2)} \] Proposition \[ \text{1.2.1} \] has the following interesting consequence: even though base change to a finite tame extension \( K' \) of \( K \) of degree prime to \( [L : K] \) does not change the Galois group \( \Gamma = \text{Gal}(L/K) \), it does alter the ramification filtration by pushing the higher ramification groups towards infinity. This allows us to isolate the wild part of the Artin conductor of a Galois representation of \( K \) in the following way.

**Proposition 1.2.3.** Let \( V \) be a finite dimensional vector space over \( \mathbb{Q} \) or \( \mathbb{Q}_\ell \), endowed with a continuous action of \( \text{Gal}(K^s/K) \). We denote by \( \text{Art}(V) \) and \( \text{Sw}(V) \) the Artin conductor, resp. Swan conductor, of the Galois representation \( V \). For every \( d \in \mathbb{N}' \), we denote by \( V(d) \) the restriction of \( V \) to \( \text{Gal}(K^s/K(d)) \). If we order the set \( \mathbb{N}' \) by the divisibility relation, then
\[
\text{Sw}(V) = \lim_{d \in \mathbb{N}'} \frac{\text{Art}(V(d))}{d}.
\]

**Proof.** Proposition \[ \text{1.2.1} \] implies that \( \text{Sw}(V(d)) = d \cdot \text{Sw}(V) \) for all \( d \in \mathbb{N}' \), while the tame part of \( \text{Art}(V(d)) \) is bounded by the dimension of \( V \). \( \square \)
2. Subtori of algebraic groups

2.1. Maximal subtori.

(2.1.1) Let $F$ be a field, and let $G$ be a smooth commutative algebraic group over $F$. It follows from [HN10 3.2] that the abelian presheaf
$$ T \mapsto \text{Hom}_{F}(T, G) $$
onumber
on the category of $F$-tori is representable by an $F$-torus $G_{\text{tor}}$, and that the tautological morphism $G_{\text{tor}} \to G$ is a closed immersion. We call $G_{\text{tor}}$ the maximal subtorus of $G$. It follows from [HN10 3.1] and the subsequent remark that, for every field extension $F'$ of $F$, the torus $G_{\text{tor}} \times_{F} F'$ is the maximal subtorus of $G \times_{F} F'$. The dimension of $G_{\text{tor}}$ is called the reductive rank of $G$ and denoted by $\rho(G)$. By [HN10 3.4], the reductive rank of $G/G_{\text{tor}}$ is zero. The algebraic group $G$ is a semi-abelian $F$-variety if and only if $G/G_{\text{tor}}$ is an abelian variety; in that case, we denote this quotient by $G_{\text{ab}}$.

(2.1.2) Every $F$-torus $T$ has a unique maximal split subtorus $T_{\text{spl}}$ [HN10 3.5]. The cocharacter module of $T_{\text{spl}}$ is the submodule of Galois-invariant elements of the cocharacter module of $T$. It follows that every smooth commutative algebraic $F$-group $G$ has a unique maximal split subtorus, which we denote by $G_{\text{spl}}$. If $T'$ is a split $F$-torus, then every morphism of algebraic groups $T' \to G$ factors through $G_{\text{spl}}$. We call the dimension of $G_{\text{spl}}$ the split reductive rank of $G$, and denote it by $\rho_{\text{spl}}(G)$. By the remark after [HN10 3.6], the split reductive rank of $G/G_{\text{spl}}$ is zero.

(2.1.3) An $F$-torus $T$ is called anisotropic if $T_{\text{spl}}$ is trivial, i.e., if its only Galois-invariant character is the trivial character. By the duality between tori and their character modules, all morphisms from an anisotropic torus to a split torus or from a split torus to an anisotropic torus are trivial.

2.2. Basic properties of the reductive rank.

Lemma 2.2.1. The reductive rank of a connected smooth commutative algebraic $F$-group is invariant under isogeny.

Proof. Let $f : G \to H$ be an isogeny of connected smooth commutative algebraic $F$-groups, and denote by $d$ the degree of $f$. Since the kernel of $f$ is killed by $d$, there exists a morphism of algebraic $F$-groups $g : H \to G$ such that $g \circ f = d_{G}$, where we denote by $d_{G}$ the multiplication by $d$ on $G$. Then
$$ f \circ g \circ f = f \circ d_{G} = d_{H} \circ f $$
so that $f \circ g = d_{H}$ because $f$ is faithfully flat.

The morphisms $f$ and $g$ induce morphisms $f_{\text{tor}} : G_{\text{tor}} \to H_{\text{tor}}$ and $g_{\text{tor}} : H_{\text{tor}} \to G_{\text{tor}}$ such that $f_{\text{tor}} \circ g_{\text{tor}}$ and $g_{\text{tor}} \circ f_{\text{tor}}$ are given by multiplication by $d$. In particular, $f_{\text{tor}}$ and $g_{\text{tor}}$ are isogenies, so that $\dim G_{\text{tor}} = \dim H_{\text{tor}}$. \hfill \square

Lemma 2.2.2. Let $\ell$ be a prime invertible in $F$, and let
$$ f : G \to H $$
be a morphism of semi-abelian $F$-varieties such that the induced morphism of $\ell$-adic Tate modules
$$ T_{\ell} G \to T_{\ell} H $$
is an isomorphism. Then $f$ is an isogeny.
PROOF. Let \( F^a \) be an algebraic closure of \( F \), and denote by \( F^s \) the separable closure of \( F \) in \( F^a \). Let \( J \) be a connected smooth commutative algebraic \( F \)-group. By definition,

\[
T_\ell J = \lim_{\longleftarrow n} (\ell^n J(F^a))
\]

where \( \ell^n J \) is the kernel of multiplication by \( \ell^n \) on \( J \). Since \( \ell^n J \) is an étale scheme \([\text{SGA3-II}]\), XV.1.3], the map

\[
\ell^n J(F^a) \to \ell^n J(F^s)
\]

is a bijection. Thus, as a \( \mathbb{Z}_\ell \)-module without Galois structure, \( T_\ell J \) is invariant under base change to \( F^a \), so that we may assume that \( F \) is algebraically closed.

The reduction of the identity component of \( \ker(f) \) is a semi-abelian \( F \)-variety, by \([\text{HN11a}]\), 5.2]. It has trivial \( \ell \)-adic Tate module, so that it must be trivial. It follows that \( \ker(f) \) is finite. Likewise, the image of \( f \) is a semi-abelian subvariety of \( H \) with the same \( \ell \)-adic Tate module as \( H \). Since the Tate module of a semi-abelian \( F \)-variety \( J \) is a free \( \mathbb{Z}_\ell \)-module of rank

\[
\dim J_{\text{tor}} + 2 \dim J_{\text{ab}},
\]

it follows that \( f \) is surjective. Thus \( f \) is an isogeny. \( \square \)

**Lemma 2.2.3.** For every exact sequence

\[
0 \to G_1 \to G_2 \to G_3 \to 0
\]

of connected smooth commutative algebraic \( F \)-groups, we have

\[
\rho_{\text{spl}}(G_2) = \rho_{\text{spl}}(G_1) + \rho_{\text{spl}}(G_3).
\]

**Proof.** Base change to the perfect closure of \( F \) does not affect the split reductive rank of an algebraic \( F \)-group. Thus we may assume that \( F \) is perfect. For \( i = 1, 2, 3 \), we denote by \( T_i \) the maximal split subtorus \((G_i)_{\text{spl}} \) of \( G_i \).

**Case 1:** \( G_1 \) is a split torus. The morphism \( T_2 \to G_3 \) factors through \( T_3 \). Replacing \( G_2 \) by \( T_3 \) and \( G_3 \) by the semi-abelian \( F \)-variety \( G_2 \times_{G_1} T_3 \), we may assume that \( G_3 \) is a split \( F \)-torus. Then \( G_2 \) is an extension of two split \( F \)-tori, and thus it is again a split \( F \)-torus (it is diagonalizable \([\text{SGA3-II}]\), IX.8.2], smooth \([\text{SGA3-II}]\), VI.B.9.2] and connected). Therefore, \( \rho_{\text{spl}}(G_i) = \dim(G_i) \) for \( i = 1, 2, 3 \) and the result is clear.

**Case 2:** General case. Dividing \( G_1 \) and \( G_2 \) by \( T_1 \) and applying Case 1 to the exact sequences \( 0 \to T_1 \to G_1 \to G_1/T_1 \to 0 \) and \( 0 \to T_1 \to G_2 \to G_2/T_1 \to 0 \), we can reduce to the case where \( T_1 \) is trivial. Arguing as in Case 1, we may assume that \( G_3 \) is a split torus, so that \( T_3 = G_3 \).

The kernel of the morphism \( T_2 \to G_3 \) is diagonalizable \([\text{SGA3-II}]\), IX.8.1]. It is also a closed subgroup of \( G_1 \). Since \( T_1 \) is trivial, the kernel of \( T_2 \to G_3 \) must be finite. Thus it suffices to show that \( T_2 \to G_3 \) is surjective. Denote by \( H \) the schematic image of \( T_2 \to G_3 \). This is a closed subgroup of the split torus \( G_3 \). The quotient \( G_3/H \) is again a split \( F \)-torus (it is connected, smooth \([\text{SGA3-II}]\), VI.B.9.2(xii]) and diagonalizable \([\text{SGA3-II}]\), IX.8.1]).

One deduces from the Chevalley decomposition of \( G_2 \) that the quotient \( G_2/T_2 \) is an extension of an abelian \( F \)-variety by the product of a unipotent \( F \)-group and an anisotropic \( F \)-torus. Thus the morphism of \( F \)-groups \( G_2/T_2 \to G_3/H \) must be trivial, because all morphisms from an anisotropic torus, a unipotent group or an abelian variety to a split torus are trivial; this follows from \([2.1.3]\) and \([\text{SGA3-II}]\).
XVII.2.4], and the fact that every regular function on an abelian variety is constant. On the other hand, the morphism $G_2/T_2 \to G_3/H$ is surjective by surjectivity of $G_2 \to G_3$, so that $G_3/H$ must be trivial, and $H = G_3$. Since the image of $T_2 \to G_3$ is closed [SGA3-I, VI 1.2], it follows that $T_2 \to G_3$ is surjective. \(\square\)

**Corollary 2.2.4.** For every exact sequence
\[
0 \to G_1 \to G_2 \to G_3 \to 0
\]
of connected smooth commutative algebraic $F$-groups, we have
\[
\rho(G_2) = \rho(G_1) + \rho(G_3).
\]

**Proof.** This follows at once from Lemma 2.2.3, since $\rho(G) = \rho_{\text{spl}}(G \times F F^*)$ for every commutative algebraic $F$-group $G$. \(\square\)

We will also need the following variant of Corollary 2.2.4.

**Lemma 2.2.5.** Let $\ell$ be a prime invertible in $F$. Let
\[
G_1 \to G_2 \to G_3
\]
be a complex of connected smooth commutative algebraic $F$-groups such that the sequence of Tate modules
\[
0 \to T_\ell G_1 \to T_\ell G_2 \to T_\ell G_3 \to 0
\]
is exact. Then
\[
\rho(G_2) = \rho(G_1) + \rho(G_3).
\]

**Proof.** As in the proof of Lemma 2.2.2, we may suppose that $F$ is algebraically closed. If we denote by $U_i$ the unipotent radical of $G_i$, then the morphism
\[
T_\ell G_i \to T_\ell (G_i/U_i)
\]
is an isomorphism, since multiplication by $\ell$ defines an automorphism of $U_i$. Therefore, dividing $G_i$ by its unipotent radical, we may assume that $G_i$ is a semi-abelian $F$-variety for $i = 1, 2, 3$.

Then $G_1$ is $\ell$-divisible, so that the sequence
\[
0 \to T_\ell G_1 \to T_\ell G_2 \to T_\ell (G_1/G_2) \to 0
\]
is exact. This means that
\[
T_\ell (G_1/G_2) \to T_\ell G_3
\]
is an isomorphism. But $G_1/G_2$ is a semi-abelian $F$-variety, so that the morphism
\[
G_1/G_2 \to G_3
\]
is an isogeny by Lemma 2.2.2, and $\rho(G_2/G_1) = \rho(G_3)$ by Lemma 2.2.1. Now the result follows from Corollary 2.2.4. \(\square\)
3. Néron models

3.1. The Néron model and the component group.

(3.1.1) A Néron lft-model of a smooth commutative algebraic $K$-group $G$ is a separated smooth $R$-scheme $\mathcal{G}$, endowed with an isomorphism $\mathcal{G} \times R K \to G$, such that for every smooth $R$-scheme $Z$, the natural map $\text{Hom}_R(Z, \mathcal{G}) \to \text{Hom}_K(Z \times R K, G)$ is a bijection. This universal property implies that a Néron lft-model is unique up to unique isomorphism if it exists, so that we can safely speak of the Néron model of $G$. It also entails that the group law on $G$ extends uniquely to a commutative group law on $G$ that makes $G$ into a separated smooth group scheme over $R$. Moreover, the formation of Néron models is functorial: if $f : G \to H$ is a morphism of smooth commutative algebraic $K$-groups such that $G$ and $H$ have Néron lft-models $G$ and $H$, respectively, then the universal property implies that $f$ extends uniquely to a morphism of group schemes $G \to H$ over $R$.

(3.1.2) Every semi-abelian $K$-variety $G$ admits a Néron lft-model, by [BLR90, 10.2.2]. The constant $k$-group scheme $\Phi(G) = G_k / G_o k$ of connected components of $G$ is called the Néron component group of $G$, or component group for short. We will identify it with the abelian group $\Phi(G)(k)$. The specialization morphism $G(K) = \mathcal{G}(R) \to \mathcal{G}(k)$ induces an isomorphism $G(K) / G^o(R) \to \Phi(G)$.

(3.1.3) The group $\Phi(G)$ is finitely generated [HN11a, 5.4], and its rank is equal to the dimension of the maximal split subtorus of $G$ [BX96, 4.11]. In particular, the Néron lft-model of $G$ is quasi-compact if and only if $G$ does not contain a subgroup isomorphic to $G_m,K$. This happens, for instance, if $G$ is an abelian $K$-variety.

(3.1.4) By [BLR90, 7.2.1] and [BS95, 6.2], the formal $m$-adic completion $\hat{\mathcal{G}}$ of $\mathcal{G}$ is a formal Néron model of the rigid $K$-group $G^\text{an}$ in the sense of [BS95, 1.1]. The special fibers of $\mathcal{G}$ and $\hat{\mathcal{G}}$ are canonically isomorphic. This means that we can use formal and rigid geometry to study component groups of semi-abelian $K$-varieties. In particular, we will make extensive use of rigid uniformization [BX96, 1.1].

(3.1.5) From the fact that the torsion part of $\Phi(G)$ is finite, one can easily deduce that $\mathcal{G}$ has a unique maximal quasi-compact open subgroup scheme $\mathcal{G}^\text{rc}$ over $R$ [HN11a, 3.6], which we call the Néron model of $G$. It is characterized by a universal property in [HN11a, 3.5]. The component group $\mathcal{G}_k^\text{rc} / (\mathcal{G}_k^\text{rc})^o$ is the torsion part of the component group $\Phi(G)$.

(3.1.6) The notation that we use in this article is slightly different from the one in [HN11a]: there we denoted the Néron lft-model by $\mathcal{G}^{lft}$ and the Néron model by $\mathcal{G}$. In the present article, the notation introduced above will be more convenient.

3.2. The toric rank.

(3.2.1) Besides the component group, we can define the following fundamental invariants of a semi-abelian $K$-variety $G$. Let $\mathcal{G}$ be the Néron lft-model of $G$. The toric rank $t(G)$ of $G$ is the reductive rank of $\mathcal{G}_k^\text{rc}$, i.e., the dimension of the maximal subtorus $T$ of $\mathcal{G}_k^\text{rc}$. If $G$ is a torus, then $t(G)$ is equal to the dimension of the maximal split subtorus of $G$ [HN10, 3.13].
(3.2.2) If $k$ is perfect, then we define the abelian rank $a(G)$ and the unipotent rank $u(G)$ of $G$ as the dimension of the abelian quotient $B$, resp. the unipotent part $U$, in the Chevalley decomposition

$$0 \to T \times_k U \to \mathcal{G}_k^0 \to B \to 0$$

of $\mathcal{G}_k^0$. Note that the sum $t(G) + a(G) + u(G)$ equals the dimension of $G$.

**Proposition 3.2.3.** For every exact sequence of semi-abelian $K$-varieties

$$0 \to G_1 \to G_2 \to G_3 \to 0$$

with $G_1$ a torus, we have

$$t(G_2) = t(G_1) + t(G_3).$$

**Proof.** We denote by $G_i$ the Néron lft-model of $G_i$, for $i = 1, 2, 3$. For every prime $\ell$ invertible in $k$, the sequence of Tate modules

$$0 \to T_\ell G_1 \to T_\ell G_2 \to T_\ell G_3 \to 0$$

is exact because the group $G_1(K^s)$ is $\ell$-divisible. Moreover, if we denote by $X(G_1)$ the character module of $G_1$, then there exists an $I$-equivariant isomorphism of $\mathbb{Z}_\ell$-modules

$$T_\ell G_1 \cong X(G_1)^{\vee} \otimes_{\mathbb{Z}_\ell} \mathbb{Z}_\ell(1)$$

and thus an isomorphism

$$H^1(I, T_\ell G_1) \cong H^1(I, X(G_1)^{\vee}) \otimes_{\mathbb{Z}_\ell} \mathbb{Z}_\ell.$$

Thus, choosing the prime $\ell$ such that it does not divide the degree of the splitting field of $G_1$ over $K$, we may assume that

$$H^1(I, T_\ell G_1) = 0.$$

Then, taking $I$-invariants in the sequence (3.2.4), we get an exact sequence

$$0 \to T_\ell G_1(K) \to T_\ell G_2(K) \to T_\ell G_3(K) \to 0$$

that we can identify with the sequence

$$0 \to T_\ell (G_1)^{\vee}_k \to T_\ell (G_2)^{\vee}_k \to T_\ell (G_3)^{\vee}_k \to 0$$

by the arguments in [SGA7-I IX.2.2.5]. Now the result follows from Lemma 2.2.5. $\square$

**Corollary 3.2.5.** For every semi-abelian $K$-variety $G$, we have

$$t(G) = t(G_{tor}) + t(G_{ab}).$$

### 3.3. Néron models and base change.

(3.3.1) Let $G$ be a semi-abelian $K$-variety, with Néron lft-model $\mathcal{G}$. Let $K'$ be a finite extension of $K$, with valuation ring $R'$, and denote by $\mathcal{G}'$ the Néron lft-model of $G' = G \times_k K'$. By the universal property of the Néron lft-model, there exists a unique morphism of $R'$-group schemes

$$h : \mathcal{G} \times_R R' \to \mathcal{G}'$$

that extends the natural isomorphism between the generic fibers. It is not an isomorphism, in general. The morphism $h$ induces a morphism of component groups

$$\alpha_G : \Phi(G) \to \Phi(G').$$

One of the principal aims of this monograph is to study the properties of $\alpha_G$. Here we give an elementary example, which we will need in some of the proofs in Part [I]
3.4. Example: the Néron lifft-model of a split algebraic torus.

(3.4.1) For every integer $n > 0$, the Néron lifft-model of $G_{m,K}$ is constructed by gluing copies of $G_{m,R}^n$ along their generic fibers. For $n = 1$, this procedure is described in [BLR90 10.1.5]. The general case follows from the fact that the formation of Néron lifft-models commutes with products.

(3.4.2) Let $T$ be a split $K$-torus of dimension $n$, and let $\mathcal{S}$ be its Néron lifft-model. It follows from the construction of the Néron lifft-model of $G_{m,K}$ that there exists a canonical isomorphism

$$\Phi(T) \cong X(T)^{\vee} \otimes_{\mathbb{Z}} (K^*/R^*) = \text{Hom}_{\mathbb{Z}}(X(T), K^*/R^*)$$

where $X(T)$ denotes the character group of $T$. In particular, $\Phi(T)$ is a free $\mathbb{Z}$-module of rank $n = \dim(T)$.

(3.4.4) Under the isomorphism (3.4.3), a group morphism $f : X(T) \to K^*/R^*$ corresponds to the unique connected component of $\mathcal{S}$ that contains the specializations of all the $K$-points $x$ in $T$ such that $f(\chi)$ is the class of $\chi(x)$ in $K^*/R^*$ for every character $\chi$ of $T$.

(3.4.5) Let $K'$ be a finite extension of $K$, with valuation ring $R'$. Then under the isomorphism (3.4.3) applied to $T$ and $T' = T \times_K K'$, the morphism

$$\alpha_T : \Phi(T) \to \Phi(T')$$

corresponds to the group morphism

$$X(T)^{\vee} \otimes_{\mathbb{Z}} (K^*/R^*) \to X(T')^{\vee} \otimes_{\mathbb{Z}} ((K')^*/(R')^*)$$

induced by the inclusion of $K^*$ in $(K')^*$ and the isomorphism $X(T) \to X(T')$. It follows that $\alpha$ is an isomorphism from $\Phi(T)$ onto the sublattice $e \cdot \Phi(T')$ of $\Phi(T')$, where $e$ denotes the ramification index of the extension $K'/K$. Thus $\alpha$ is injective, and its cokernel is isomorphic to $(\mathbb{Z}/e\mathbb{Z})^n$.

3.5. The Néron component series.

(3.5.1) Let $G$ be an abelian $K$-variety. In [HN10], we introduced a generating series that encodes the orders of the component groups of $G$ after base change to finite tame extensions of $K$. The Néron component series $S^\Phi_G(T)$ of $G$ is defined as

$$S^\Phi_G(T) = \sum_{d \in \mathbb{N}} |\Phi(G(d))| \cdot T^d \in \mathbb{Z}[[T]].$$

Recall that we denote by $G(d)$ the abelian variety obtained from $G$ by base change to the unique degree $d$ extension $K(d)$ of $K$ in $K^*$.

(3.5.2) We can extend this definition to semi-abelian $K$-varieties. Since in this case, the component group $\Phi(G)$ might be infinite, we will consider the order of the torsion part $\Phi(G)_{\text{tors}}$. As we’ve seen, this torsion part is precisely the group of connected components of the special fiber of the Néron model $\mathcal{S}^e_{\text{qc}}$ of $G$. We define the Néron component series $S^\Phi_G(T)$ of $G$ by

$$S^\Phi_G(T) = \sum_{d \in \mathbb{N}} |\Phi(G(d))_{\text{tors}}| \cdot T^d \in \mathbb{Z}[[T]].$$

3.6. Semi-abelian reduction.
It can be quite difficult to describe the behaviour of the Néron model of a semi-abelian $K$-variety under finite extensions of the base field $K$. The most important tool is Grothendieck’s Semi-Stable Reduction Theorem, which we will now recall.

Let $G$ be a semi-abelian $K$-variety with Néron left-model $\mathcal{G}$. We say that $G$ has multiplicative, resp. semi-abelian, reduction if $\mathcal{G}_p^\circ$ is a torus, resp. a semi-abelian $k$-variety. We say that $G$ has good reduction if it has semi-abelian reduction and, in addition, $G_{ab}$ has good reduction, i.e., if the Néron model of $G_{ab}$ is an abelian $R$-scheme.

If $G$ is a $K$-torus, then $G$ has semi-abelian reduction if and only if it has multiplicative reduction; by (1.1.2), this happens if and only if $G$ is split. It follows from \([HN10], 4.1\) that a semi-abelian $K$-variety $G$ has semi-abelian reduction if and only if $G_{ab}$ has semi-abelian reduction and $G_{tor}$ is split.

**Proposition 3.6.4.** Let $G$ be a semi-abelian $K$-variety with semi-abelian reduction, and let $H$ be a subtorus of $G$. Then $H$ and $G/H$ have semi-abelian reduction.

**Proof.** The torus $H$ is a subtorus of $G_{tor}$, so that $H$ and $G_{tor}/H$ must be split \([SGA3-II], IX.8.1\). The quotient $G/H$ is an extension of $G_{ab}$ by the split torus $G_{tor}/H$. Since $G_{ab}$ has semi-abelian reduction, we obtain that $G/H$ has semi-abelian reduction.

**Theorem 3.6.5 (Semi-Stable Reduction Theorem).**

1. The semi-abelian variety $G$ has semi-abelian reduction if and only if the action of $\text{Gal}(K^s/K)$ on $T_IG$ is unipotent.

2. There exists a unique minimal finite extension $L$ of $K$ in $K^s$ such that $G_{ab} \times_K L$ has semi-abelian reduction. The field $L$ is a finite Galois extension of $K$.

3. If $G$ has semi-abelian reduction, then $G \times_K K'$ has semi-abelian reduction for every finite separable extension $K'$ of $K$.

**Proof.** Point (1) follows from \([HN10], 4.1\), and (3) follows from \([3.6.3] \text{ and } SGA7-I, 3.3\). Thus it is enough to prove (2). It follows from Grothendieck’s Semi-Stable Reduction Theorem for abelian varieties \([SGA7-I], IX.3.6\) that there exists a finite separable extension $K_0$ of $K$ such that $G_{ab} \times_K K_0$ has semi-abelian reduction. By (3), we may assume that the torus $G_{tor} \times_K K_0$ is split; then $G \times_K K_0$ has semi-abelian reduction by \([3.6.3]\). Denote by $I'$ the subset of $\text{Gal}(K^s/K)$ consisting of elements that act unipotently on $T_IG$. This is a normal subgroup of $\text{Gal}(K^s/K)$. It is open, because it contains the open subgroup $\text{Gal}(K^s/K_0)$ of $\text{Gal}(K^s/K)$, by (1). The fixed field $L$ of $I'$ satisfies the properties in the statement.

The importance of the Semi-Stable Reduction Theorem lies in the following fact. A semi-abelian $K$-variety $G$ has semi-abelian reduction if and only if the base change morphism $f$ in \([3.3.1]\) is an open immersion for every finite extension $K'$ of $K$. This is an immediate consequence of the Semi-Stable Reduction Theorem and \([SGA7-I], 3.1(e)\). Thus if $G$ has semi-abelian reduction, then the formation of the identity component $\mathcal{G}_0^\circ$ of the Néron left-model of $G$ commutes with base change to finite extensions of $K$. However, we shall see later on that the component group $\Phi(G)$ will still change, unless $G$ is an abelian $K$-variety with good reduction.
The potential toric rank of $G$ is defined as the toric rank of $G \times_K L$ and denoted by $t_{\text{pot}}(G)$. The potential abelian rank is defined analogously. We say that $G$ has potential good reduction if $G \times_K L$ has good reduction. Likewise, we say that $G$ has potential multiplicative reduction if $G \times_K L$ has multiplicative reduction.

3.7. Non-archimedean uniformization.

(3.7.1) Let $A$ be an abelian $K$-variety. We denote by $L$ the minimal extension of $K$ in $K^s$ such that $A \times_K L$ has semialbelian reduction.

(3.7.2) The non-archimedean uniformization of $A$ consists of the following data [BX96 1.1]:

- a semi-abelian $K$-variety $E$ that is the extension of an abelian $K$-variety $B$ with potential good reduction by a $K$-torus $T$:
  
  $$0 \to T \to E \to B \to 0.$$ 

- an étale lattice $M$ in $E$, of rank dim $T$, and an étale covering of $K$-analytic groups
  
  $$E^\text{an} \to A^\text{an}$$

with kernel $M$.

The lattice $M$ and the torus $T$ split over $L$, and $B \times_K L$ has good reduction. In particular, we can view $M$ as a Gal($L/K$)-module.

(3.7.3) The non-archimedean uniformization behaves well under base change, in the following sense. Let $K'$ be a finite extension of $K$. If we denote by $(\cdot)'$ the base change functor from $K$ to $K'$, then

$$0 \to (M')^\text{an} \to (E')^\text{an} \to (A')^\text{an} \to 0$$

is the non-archimedean uniformization of $A' = A \times_K K'$.

4. Models of curves

In this section, we assume that $k$ is algebraically closed.

4.1. Sned-models and combinatorial data.

(4.1.1) Let $C$ denote a smooth, proper and geometrically connected $K$-curve of genus $g$. An $R$-model of $C$ is a proper and flat $R$-scheme $\mathscr{C}$ endowed with an isomorphism of $K$-schemes $\mathscr{C}_K \to C$. We say that a model $\mathscr{C}$ of $C$ is an $\text{sncd}$-model if $\mathscr{C}$ is regular and the special fiber

$$\mathscr{C}_k = \sum_{i \in I} N_i E_i$$

is a divisor with strict normal crossings on $\mathscr{C}$. It is well known that $C$ always admits an $\text{sncd}$-model. Moreover, if $g > 0$, there exists a minimal $\text{sncd}$-model of $C$, which is unique up to unique isomorphism (see [Li02 9.3]). We will often impose the condition that $C$ has index one, i.e., that $C$ admits a divisor of degree one. By [Ra70 7.1.6], this condition is equivalent to the property that

$$\gcd\{N_i \mid i \in I\} = 1.$$
4. MODELS OF CURVES

(4.1.2) Let \( C \) be a regular flat \( R \)-scheme whose special fiber \( C_k = \sum_{i \in I} N_i E_i \) is a divisor with strict normal crossings. For each irreducible component \( E_i \) of \( C_k \), we put \( E_i^0 = E_i \setminus \cup_{j \neq i} E_j \). We associate a graph \( \Gamma(C_k) \) to \( C_k \) as follows. We let the vertex set \( \{v_i\}_{i \in I} \) correspond bijectively to the set of irreducible components \( \{E_i\}_{i \in I} \) of \( C_k \). Whenever \( i \neq j \), the vertices \( v_i \) and \( v_j \) are connected by \(|E_i \cap E_j|\) distinct edges. If \( C \) is proper over \( R \), then \( \Gamma(C_k) \) is simply the dual graph associated to the semi-stable curve \( (C_k)^{\rm{red}} \). By the combinatorial data of \( C_k \) we mean the graph \( \Gamma(C_k) \) with each vertex \( v_i \) labelled by a couple \((N_i, g_i)\), where \( N_i \) denotes the multiplicity of \( E_i \) in \( C_k \) and \( g_i \) denotes the genus of \( E_i \) if \( E_i \) is proper over \( k \), and \( g_i = -1 \) else. We use this definition for \( g_i \) to treat in a uniform way the case where \( C \) is proper over \( R \) and the case where \( C \) is obtained by resolving the singularities of an excellent local \( R \)-scheme.

4.2. A theorem of Winters. The following result by Winters will play a crucial role in this paper. It will allow us to transfer certain results from residue characteristic 0 to positive residue characteristic.

**Theorem 4.2.1 (Winters).** Let \( C \) be a smooth, proper, geometrically connected curve over \( K \), and let \( C_k \) be an sncd-model for \( C \). Assume that \( C \) admits a divisor of degree one. Then there exist a smooth, proper, geometrically connected curve \( D \) over \( \mathbb{C}((t)) \) and an sncd-model \( D \) for \( D \) over \( \mathbb{C}[[t]] \) such that \( C_k \) and \( D_k \) have the same combinatorial data.

**Proof.** This follows from [Wi74, 3.7 and 4.3].

(4.2.2) Note that the curve \( D \) automatically has the same genus as \( C \), since the genus can be computed from the combinatorial data of \( C_k \) (see for instance [Ni12, 3.1.1]). If \( C \) is relatively minimal, then so is \( D \), since the existence of \((-1)\)-curves can also be read off from the combinatorial data.

4.3. Néron models of Jacobians.

(4.3.1) Let \( C \) be a smooth, proper, geometrically connected curve over \( K \) of index one. One can use the geometry of \( R \)-models of \( C \) to study the Néron model of the Jacobian variety \( A = \text{Jac}(C) \) of \( C \), because of the following fundamental theorem of Raynaud [BLR90, 9.5.4]: if \( C \) is a regular \( R \)-model of \( C \), then the relative Picard scheme \( \text{Pic}^0_{C/R} \) is canonically isomorphic to the identity component of the Néron model \( \mathcal{A} \) of \( A \).

(4.3.2) This result has several interesting consequences. Let \( C/R \) be an sncd-model of \( C \), with special fiber \( C_k = \sum_{i \in I} N_i E_i \). The abelian quotient in the Chevalley decomposition of \( \mathcal{A}^0_k \) is isomorphic to

\[
\prod_{i \in I} \text{Pic}^0_{E_i/k},
\]

by [BLR90, 9.2.5 and 9.2.8], and the toric rank \( t(A) \) of \( A \) is equal to the first Betti number of the graph \( \Gamma(C_k) \) [BLR90, 9.2.5 and 9.2.8].

(4.3.3) It is also possible to compute the component group \( \Phi(A) \) from the combinatorial data of \( C_k \), as follows. Consider the complex of abelian groups

\[
\mathbb{Z}^I \xrightarrow{\alpha} \mathbb{Z}^I \xrightarrow{\beta} \mathbb{Z},
\]
where \( \alpha = (E_i \cdot E_j)_{i,j \in I} \) is the intersection matrix of \( \mathcal{C}_k \) and \( \beta \) sends the \( i \)-th standard basis vector of \( \mathbb{Z}^I \) to \( N_i \in \mathbb{Z} \), for every \( i \in I \). Then, by [BLR90, 9.6.1], there is a canonical isomorphism \( \Phi(A) \cong \ker(\beta)/\text{im}(\alpha) \).

**4.3.4** In particular, both the component group \( \Phi(A) \) and the toric rank \( t(A) \) only depend on the combinatorial data of \( \mathcal{C}_k \), and not on the characteristic exponent \( p \) of \( k \).

**4.4. Semi-stable reduction.**

**4.4.1** Let \( C \) be a smooth, proper, geometrically connected curve over \( K \). An sncd-model \( \mathcal{C} \) is called semi-stable if its special fiber \( \mathcal{C}_k \) is reduced. We say that \( C \) has semi-stable reduction if \( C \) has a semi-stable sncd-model. T. Saito has proven in [Sa87, 3.8] that, unless \( C \) is a genus one curve without rational point, the curve \( C \) has semi-stable reduction if and only if the action of \( \text{Gal}(K^s/K) \) on

\[
H^1(C \times_K K^s, \mathbb{Q}_\ell)
\]

is unipotent; see also [Ni12, 3.4.2].

**4.4.2** Assume that \( C \) has genus \( g \neq 1 \) or that \( C \) is an elliptic curve, and set \( A = \text{Jac}(C) \). Then there exist canonical Galois-equivariant isomorphisms

\[
H^1(A \times_K K^s, \mathbb{Q}_\ell) \cong H^1(C \times_K K^s, \mathbb{Q}_\ell),
\]

\[
H^1(A \times_K K^s, \mathbb{Q}_\ell) \cong \text{Hom}_{\mathbb{Z}_\ell}(T_{\ell}A, \mathbb{Q}_\ell).
\]

Thus Theorem 3.6.5(1) implies that \( A \) has semi-abelian reduction if and only if \( C \) has semi-stable reduction. If \( C \) has index one, this equivalence can also be deduced (with some additional work) from Raynaud’s isomorphism

\[
\text{Pic}^0_{C/R} \cong \mathcal{A}^0.
\]
Part 1

Néron component groups of semi-abelian varieties
CHAPTER 3

Models of curves and the Néron component series of a Jacobian

In this chapter, we assume that $k$ is algebraically closed. Let $C$ be a smooth, proper, geometrically connected curve over $K$. We will study the behaviour of $sncd$-models of $C$ under finite tame extensions of the base field $K$. Our main technical result is that these models can be compared in a very explicit way if the degree of the base change is prime to the stabilization index $e(C)$ of $C$, a new invariant that we introduce in Definition 2.2.3. Using this result, we prove the rationality of the Néron component series of a Jacobian variety over $K$ (Theorem 3.1.5).

1. Sncd-models and tame base change

1.1. Base change and normalization.

(1.1.1) Let $C$ be a smooth, proper, geometrically connected curve over $K$. Let $\mathcal{C}/R$ be an $sncd$-model of $C$ and let $d$ be an element of $\mathbb{N}'$. We denote by $\mathcal{C}_d$ the normalization of $\mathcal{C} \times_R R(d)$ and by

$$f : \mathcal{C}_d \to \mathcal{C}$$

the canonical morphism. We denote by

$$\rho : \mathcal{C}(d) \to \mathcal{C}_d$$

the minimal desingularization of $\mathcal{C}_d$. For the applications we have in mind, it is important to describe $\mathcal{C}_d$ and $\mathcal{C}(d)$ in a precise way. Such a description will be given in Proposition 1.3.2. In particular, we will show that $\mathcal{C}(d)$ is an $sncd$-model. These results are well known in more restrictive settings (cf. [Ha10a]). However, to our best knowledge, they have not appeared in the literature in the generality that we need (although some of them are claimed without proof in Section 3 of [Lo93]). In particular, we need to deal with the situation where $\mathcal{C}_k$ contains irreducible components with multiplicities divisible by $p$ that intersect each other, and this case is not covered in [Ha10a].

(1.1.2) The main tool we use is the description of the minimal desingularization of a locally toric singularity given in Kiraly’s PhD thesis [Ki10]. This provides a convenient combinatorial description of the minimal desingularization of a tame cyclic quotient singularity that also applies to the case where $R$ has mixed characteristic (the equal characteristic case was worked out in [CES03]). Since this part of [Ki10] has not been published, we have gathered the results that we need as an appendix, in Section 4. Alternatively, one could use tools from logarithmic geometry (desingularization of log-regular schemes) since Kiraly’s locally toric singularities correspond precisely to Kato’s toric singularities [Ka94 3.1].
(1.1.3) Let $x$ be a closed point on $\mathcal{C}_k$, and let $y_1, \ldots, y_r$ denote the finitely many points in the special fiber of $\mathcal{C}_k$ mapping to $x$. Since normalization of an excellent scheme commutes with localization and completion [Ha10a] §2.2, the homomorphism $\hat{\varphi}_{x} \rightarrow \hat{\varphi}_{d,y_i}$ induced by $f$ can be identified with the composed sequence of homomorphisms

$$\hat{\varphi}_{x} \rightarrow \hat{\varphi}_{x} \otimes_R R(d) \rightarrow (\hat{\varphi}_{x} \otimes_R R(d))^{\text{nor}} \cong \prod_{i=1}^{r} \hat{\varphi}_{d,y_i} \rightarrow \hat{\varphi}_{d,y_i},$$

where, for any reduced ring $A$, we write $A^{\text{nor}}$ for the integral closure of $A$ in its total ring of fractions.

(1.1.4) The situation where only one irreducible component of $\mathcal{C}_k$ passes through $x$ is completely described in [Ha10a] §2.4, so we assume in the following that there are two distinct irreducible components of $\mathcal{C}_k$ passing through $x$. In that case, we can find an isomorphism

$$\hat{\varphi}_{x} \cong R[[x_1, x_2]]/(\pi - u \cdot x_1^{m_1} x_2^{m_2}),$$

where $m_1$ and $m_2$ are the multiplicities of the components of $\mathcal{C}_k$ intersecting at $x$, and where $u \in R[[x,y]]$ is a unit. As we do not want to assume that either $m_1$ or $m_2$ is prime to $p$, we cannot get rid of the unit $u$ by a coordinate change. This prevents us from simply transferring the results in [Ha10a].

1.2. Local computations.
(1.2.1) Let $d \in \mathbb{N}$ and put $R' = R(d)$. We choose a uniformizer $\pi'$ in $R'$ such that $\pi'^{d} = \pi$. We will now explain how to normalize $A \otimes_R R'$, where

$$A = R[[x_1, x_2]]/(\pi - u \cdot x_1^{m_1} x_2^{m_2}).$$

To do this, write $c = \gcd(d, m_1, m_2)$. Since $c$ is prime to $p$, we can choose a unit $v \in R[[x_1, x_2]]$ such that $v' = u$. Then

$$(A \otimes_R R')^{\text{nor}} \cong \prod_{\xi \in \mu_c(k)} \left( R'[x_1, x_2]/(\pi'^{d'} - \xi v \cdot x_1^{m'_1} x_2^{m'_2}) \right)^{\text{nor}},$$

where $d' = d/c, m'_1 = m_1/c$ and $m'_2 = m_2/c$. Let us put $e_i = \gcd(d', m'_i)$ so that $d'' = e_1 e_2 d''$ and $m''_i = e_i m''_i$ for $i = 1, 2$, with $d''$ and $m''_i$ in $\mathbb{N}$. We moreover fix a unit $w \in R[[x_1, x_2]]$ such that $w^{e_1 e_2} = \xi v$. Then one can argue as in the proof of [Ha10a] 2.4 to see that the $R'$-homomorphism

$$\alpha : R'[x_1, x_2]/(\pi'^{d'} - \xi v \cdot x_1^{m'_1} x_2^{m'_2}) \rightarrow R'[y_1, y_2]/(\pi'^{d''} - w \cdot y_1^{m''_1} y_2^{m''_2})$$

defined by $x_1 \mapsto y_1^{e_1^{r}}$ and $x_2 \mapsto y_2^{e_2^{r}}$ is finite and injective, and that it induces an isomorphism of fraction fields. Thus the source and target of $\alpha$ have the same normalization.

We next fix a unit $w' \in R[[x_1, x_2]]$ such that $w'^{d''} = w$, and define an $R'$-homomorphism

$$\beta : R'[y_1, y_2]/(\pi'^{d''} - w \cdot y_1^{m''_1} y_2^{m''_2}) \rightarrow R'[z_1, z_2]/(\pi' - w' \cdot z_1^{m''_1} z_2^{m''_2})$$

by $y_i \mapsto z_i^{m''_i}$ for $i = 1, 2$. We let $\mu_{d'}(k)$ act on the latter ring by $\zeta \cdot z_1 = \zeta z_1$ and $\zeta \cdot z_2 = \zeta'^{r} z_2$, where $0 < r < d''$ is the unique integer such that $rm''_2 + m''_1 \equiv 0$ modulo $d''$. Note that $w' \in R[[x_1, x_2]]$ is invariant under this action.
Lemma 1.2.2. The image of $\beta$ is contained in
\[(R'[\{z_1, z_2\}]/(\pi' - w' \cdot z_1^{m_1'} z_2^{m_2'}))^{\mu_{d'}(k)},\]
and the homomorphism
\[R'[[y_1, y_2]]/(\pi' - w' \cdot z_1^{m_1'} y_1^i y_2^j) \to (R'[\{z_1, z_2\}]/(\pi' - w' \cdot z_1^{m_1'} z_2^{m_2'}))^{\mu_{d'}(k)}\]
is a normalization morphism. Thus we obtain an isomorphism
\[(R'[[x_1, x_2]]/(\pi' - w' x_1^{m_1'} x_2^{m_2'}))^{\mu_{d'}(k)} \cong (R'[\{z_1, z_2\}]/(\pi' - w' \cdot z_1^{m_1'} z_2^{m_2'}))^{\mu_{d'}(k)}.

Proof. This can be proved in a similar way as [Ha10a] Prop. 2.7. □

1.3. Minimal desingularization.

(1.3.1) Let $\mathcal{C}/R$ be an sncd-model of $C$ with special fiber $\mathcal{C}_k = \sum_{i \in I} N_i E_i$. For any $d \in \mathbb{N}'$, let $f: \mathcal{C}_d \to \mathcal{C}$ be the composition of base change to $R(d)$ and normalization and let $\rho: \mathcal{C}(d) \to \mathcal{C}_d$ be the minimal desingularization as in Section 1.2.2. Proposition 1.3.2 below lists some properties of these morphisms, and of the schemes $\mathcal{C}_d$ and $\mathcal{C}(d)$, that are relevant for the applications further on.

Proposition 1.3.2. For every $d \in \mathbb{N}'$, the following properties hold:

(1) For each irreducible component $E_i$ of $\mathcal{C}_k$, the scheme $F_i = \mathcal{C}_d \times_\mathcal{C} E_i$

is a disjoint union of smooth irreducible curves $F_{ij}$. The multiplicity $N_i'$ of $(\mathcal{C}_d)_k$ along each component $F_{ij}$ is given by $N_i' = N_i / \gcd(d, N_i)$, and the morphism

$\mathcal{C}_d \times_\mathcal{C} E_i^a \to E_i^a$

is a Galois cover of degree $\gcd(N_i, d)$.

(2) If $E_i$ is a rational curve that intersects the other components of $\mathcal{C}_k$ in precisely one (resp. two) points, then each $F_{ij}$ is a rational curve that intersects the other components of $(\mathcal{C}_d)_k$ in precisely one (resp. two) points. In both cases, the number of connected components of $F_i$ is equal to $n_i = \gcd(N_i, N_a, d)$ where $a$ is any element of $I \setminus \{i\}$ such that $E_a$ intersects $E_i$. In particular, $n_i$ does not depend on the choice of $a$.

(3) Each singular point of $\mathcal{C}_d$ is an intersection point of two distinct irreducible components of the special fiber. Moreover, let $x$ be a point that belongs to the intersection of two distinct irreducible components $F$ and $F'$ of $(\mathcal{C}_d)_k$ which dominate irreducible components $E$ and $E'$ of $\mathcal{C}_k$, respectively. Let $N$ and $N'$ be the multiplicities of $E$ and $E'$ in $\mathcal{C}_k$. Then the special fiber of the minimal desingularization $\mathcal{D}$ of the local germ $\text{Spec} \mathcal{O}_{\mathcal{C}_d, x}$ is a divisor with strict normal crossings whose combinatorial data only depend on $N$, $N'$ and $d$. Moreover, each exceptional component of $\mathcal{D}_k$ is a rational curve that meets the other irreducible components of $\mathcal{D}_k$ in precisely two points.

In particular, the $R(d)$-scheme $\mathcal{C}(d)$ is an sncd-model of $C(d)$.

Proof. (1) For any $i \in I$, an easy local computation shows that

$\mathcal{C}_d \times_\mathcal{C} E_i^a \to E_i^a$.
is a Galois cover of degree \( \gcd(N_i, d) \) and that \( N_i' = N_i / \gcd(N_i, d) \) (cf. [Ha10a §2.4]). Let \( x \) be a closed point of \( C_d \) such that \( f(x) \) belongs to the intersection of \( E_i \) with another component \( E_j \). Then, using the explicit computation of \( \tilde{O}_{C_d,x} \) in Lemma 1.2.2, the proof of [Ha10a 2.9] shows that \( F_i \) is smooth at \( x \) also.

2) This is shown in the proofs of Propositions 3.1 and 3.2 in [Ha10a]. To be precise, these proofs are written in [Ha10a] under an additional assumption on the multiplicities of the components of \( C_k \), but this assumption can be removed by using the computations in Section 1.2 instead of the ones in [Ha10a §2].

3) Since \( C_d \) is normal, the singular locus consists of finitely many closed points in the special fiber. If \( y \in C_d \) is a closed point that belongs to a unique irreducible component of \( (C_d)_k \), then \( C_d \) is regular at \( y \) by [Ha10a Cor. 2.2]. Let now \( x \) be a point as in the statement of the lemma. Then, by Lemma 1.2.2 the germ Spec \( \tilde{O}_{C_d,x} \) is a tame cyclic quotient singularity, and its formal structure only depends on the triple \( (N, N', d) \) by the computations in Section 1.2. The structure of its minimal resolution is described in Proposition 12.3. In particular, it follows from that description that the special fiber of the minimal resolution is a divisor with strict normal crossings. □

2. The characteristic polynomial and the stabilization index

In this section, we introduce two invariants of a smooth, proper and geometrically connected \( K \)-curve \( C \): the characteristic polynomial \( P_C(t) \) and the stabilization index \( e(C) \). They have a natural cohomological interpretation when \( C \) is tamely ramified, but their meaning is somewhat mysterious in the wildly ramified case. A crucial feature of the stabilization index \( e(C) \) is that the behaviour of a relatively minimal sncd-model of \( C \) under tamely ramified base change can be controlled completely if the degree of the base change is prime to \( e(C) \). This property will be essential for our results on Néron component groups of Jacobians.

2.1. The characteristic polynomial.

(2.1.1) Let \( C \) be a smooth, proper, geometrically connected \( K \)-curve of genus \( g \). Let \( C/R \) be an sncd-model of \( C \) with special fiber \( C_k = \sum_{i \in I} N_i E_i \).

**Definition 2.1.2.** The characteristic polynomial of \( C \) is the monic polynomial

\[
P_C(t) = (t - 1)^2 \prod_{i \in I} (t^{N_i} - 1)^{-\chi(E_i^o)}
\]

in \( \mathbb{Z}[t] \) of degree \( 2g \).

The fact that \( P_C(t) \) is indeed a polynomial of degree \( 2g \) was proven by the second author in [Ni12 3.1.6], and previously by Lorenzini in [Lo93] under the assumption that \( \gcd\{N_i | i \in I\} = 1 \). Although \( P_C(t) \) is defined in terms of the sncd-model \( C \), it is easy to see that it does not depend on the choice of such a model, since the expression in Definition 2.1.2 does not change if we blow up \( C \) at a closed point of \( C_k \). We do not know how to define \( P_C(t) \) intrinsically on \( C \), without reference to an sncd-model. However, in [Ni12], the second author proved the following result.

**Proposition 2.1.3.** Let \( \sigma \) be a topological generator of the tame inertia group \( \Gal(K^t/K) \), and denote by \( P_C^\sigma(t) \) the characteristic polynomial of \( \sigma \) on

\[H^1(C \times_K K^t, \mathbb{Q}_\ell).\]
For every $i \in I$, we denote by $N'_i$ the prime-to-$p$ part of $N_i$. Then the following properties hold:

1. $P_C'(t) = (t - 1)^2 \prod_{i \in I} (t^{N'_i} - 1)^{-\chi(E^*_i)}$,
2. $P_C'(t)$ divides $P_C(t)$, and they are equal if and only if $C$ is tamely ramified.

**Proof.** This follows from formula (3.4) and Corollary 3.1.7 in [Ni12]. □

We will now explain the behaviour of $P_C(t)$ under tamely ramified base change.

**Definition 2.1.4.** For every monic polynomial

$$Q(t) = \prod_{j=1}^{r} (t - \alpha_j) \in \mathbb{C}[t]$$

and every integer $d > 0$, we set

$$Q^{(d)}(t) = \prod_{j=1}^{r} (t - \alpha_j^d) \in \mathbb{C}[t].$$

**Lemma 2.1.5.** Consider integers $a, b, d > 0$, and set $e = \gcd(a, d)$. If $Q(t) = (t^a - 1)^b$, then

$$Q^{(d)}(t) = (t^a/e - 1)^{eb}.$$

**Proof.** The morphism

$$\mu_a(\mathbb{C}) \to \mu_{a/e}(\mathbb{C}) : \zeta \mapsto \zeta^d$$

is a surjection, and every fiber contains precisely $e$ elements. □

If $C$ is tamely ramified, then Proposition 2.1.2 implies at once that $P_{C^{(d)}}(t) = P_C^{(d)}(t)$ for every element $d$ in $\mathbb{N}$. The following proposition states that this remains true in the wildly ramified case (see also [Lo93 2.6]).

**Proposition 2.1.6.** Let $d$ be an element in $\mathbb{N}$. For every $i \in I$, we set $d_i = \gcd(d, N_i)$. Then we have

$$P_{C^{(d)}}(t) = P_C^{(d)}(t) = (t - 1)^2 \prod_{i \in I} (t^{N_i/d_i} - 1)^{-d_i \cdot \chi(E^*_i)}.$$

**Proof.** We will compute $P_{C^{(d)}}(t)$ on the sncd-model $\mathcal{C}(d)$ for $C(d)$ from Section 1.3. By Proposition 1.3.2(3), every exceptional component $F$ of the minimal desingularization $\rho : \mathcal{C}(d) \to \mathcal{C}_d$ satisfies $\chi(F^o) = 0$, so that these exceptional components do not contribute to $P_{C^{(d)}}(t)$. For every $i \in I$, we have that

$$\mathcal{C}(d) \times_{\mathcal{C}_d} E^*_i$$

is a disjoint union of strata $F^o_{ij}$, with $F_{ij}$ the irreducible components of $\mathcal{C}(d)_k$ dominating $E_i$. By Proposition 1.3.2(1), each of these components $F_{ij}$ has multiplicity $N_i/d_i$, and

$$\mathcal{C}(d) \times_{\mathcal{C}_d} E^*_i = E^*_i$$

is a Galois cover of degree $d_i$. Since $d$ is prime to $p$, we know that

$$\sum_j \chi(F^o_{ij}) = \chi(\mathcal{C}(d) \times_{\mathcal{C}_d} E^*_i) = d_i \cdot \chi(E^*_i)$$
where the first equality follows from the additivity of the Euler characteristic and the second from Hurwitz’ theorem. Thus
\[ P_{C(d)}(t) = (t - 1)^2 \prod_{i \in I} (t^{N_i/d_i} - 1)^{-d_i \chi(E^o_i)}. \]

It remains to show that this expression is equal to \( P_C^{(d)}(t) \). Since the operator \( Q(t) \mapsto Q^{(d)}(t) \) is clearly multiplicative, this follows immediately from Lemma [2.1.5].

### 2.2. The stabilization index.

(2.2.1) We keep the notations introduced in (2.1.1).

**Definition 2.2.2.** We say that \( E_i \) is a principal component of \( \mathcal{C}_k \) if the genus of \( E_i \) is non-zero or \( E_i \setminus E^o_i \) contains at least three points. We denote by \( I_{\text{prin}} \subseteq I \) the subset corresponding to principal components of \( \mathcal{C}_k \).

**Definition 2.2.3.** We set \( e(\mathcal{C}) = \text{lcm}_{i \in I_{\text{prin}}} \{ N_i \} \).

If \( \mathcal{C}_{\text{min}} \) is a relatively minimal sncd-model of \( C \), then we set \( e(C) = e(\mathcal{C}_{\text{min}}) \).

We call \( e(C) \) the stabilization index of \( C \).

If \( g = 0 \), then \( C = \mathbb{P}^1_K \) and the special fiber of every relatively minimal sncd-model of \( C \) is isomorphic to \( \mathbb{P}^1_k \), so that \( e(C) = 1 \). If \( g > 0 \), then \( C \) has a unique minimal sncd-model, so that it is clear that \( e(C) \) is well-defined. It should be noted that the value \( e(\mathcal{C}) \) depends on the model \( \mathcal{C} \) and not only on \( C \). Every irreducible component \( E \) of \( \mathcal{C}_k \) can be turned into a principal component by blowing up \( 3 - |E \setminus E^o| \) distinct points on \( E^o \). On the other hand, \( e(C) \) always divides \( e(\mathcal{C}) \), since no additional principal components can be created in the contraction of \( \mathcal{C} \) to a relatively minimal sncd-model.

In the tamely ramified case, the stabilization index can be interpreted as follows.

**Proposition 2.2.4.** Assume that \( g \neq 1 \) or that \( C \) is an elliptic curve. Let \( L \) be the minimal extension of \( K \) in \( K^s \) such that \( C \times_K L \) has semi-stable reduction, and let \( \sigma \) be a topological generator of the tame inertia group \( \text{Gal}(K^t/K) \).

The curve \( C \) is tamely ramified if and only if \( e(C) \) is prime to \( p \). In that case, \( e(C) \) is equal to \([L : K]\), and this value is the smallest element \( d \) in \( \mathbb{N} \) such that \( \sigma^d \) acts unipotently on \( H^1(C \times_K K^t, \mathbb{Q}_l) \).

**Proof.** First, assume that \( C \) is tamely ramified. Then \( e(C) = [L : K] \) by [Ha10a 7.5] or [Ni12 3.4.4]. Since \([L : K]\) is prime to \( p \), we see that \( e(C) \) is prime to \( p \). Suppose, conversely, that \( e(C) \) is prime to \( p \), and that \( \mathcal{C} \) is relatively minimal. Then we have \( \chi(E^o_i) \geq 0 \) for every component \( E_i \) of \( \mathcal{C}_k \) such that \( N_i \) is not prime to \( p \), because such a component can not be principal. Now [Ni12 3.1.5] implies that \( \chi(E^o_i) = 0 \) for every such component \( E_i \), and [Ni12 3.1.7] tells us that \( C \) is tamely ramified. The remainder of the statement is a consequence of [Sa87 3.11]; see also [Ni12 3.4.2].
The equality \( e(C) = [L : K] \) can fail for wildly ramified curves: even the prime-to-\( p \) parts of \( e(C) \) and \( [L : K] \) can be different, as is illustrated by Examples 2.2.5 and 2.2.6. It would be quite interesting to find a cohomological interpretation of \( e(C) \) in the wildly ramified case.

**Example 2.2.5.** Assume that \( k \) has characteristic 2 and that \( R = W(k) \). Let \( C \) be the elliptic \( K \)-curve with Weierstrass equation

\[ y^2 = x^3 + 2. \]

It is easily computed, using Tate’s algorithm, that this equation is minimal, and that \( C \) has reduction type II over \( R \) and acquires good reduction over the wild Kummer extension \( L = K(\sqrt{2}) \) of \( K \). Thus \( e(C) = 6 \) whereas \( [L : K] = 2 \).

**Example 2.2.6.** Assume that \( k \) has characteristic 2 and that \( R = k[[\pi]] \). Let \( C \) be the elliptic \( K \)-curve with Weierstrass equation

\[ y^2 + \pi^2 y = x^3 + \pi^3. \]

Using Tate’s algorithm, we find that \( C \) has reduction type \( I_0^* \) over \( R \), so that \( e(C) = 2 \).

On the other hand, let \( \alpha \) be an element of \( K^* \) satisfying

\[ \alpha^2 + \pi^2 \alpha = \pi^3, \]

and set \( K' = K(\alpha) \). This is a quadratic Artin-Schreier extension of \( K \). Let \( L = K'(3) \) be the unique tame extension of \( K' \) in \( K^* \) of degree 3. Then it is easy to check that \( C \times_K L \) has good reduction and that \( L \) is the minimal extension of \( K \) with this property, so that \( [L : K] = 6 \).

(2.2.7) Thanks to Propositions 2.1.3 and 2.2.4, one can compute \( e(C) \) from the characteristic polynomial \( P_C(t) \) if \( C \) is tamely ramified. We’ll now show that this recipe is also valid in the wild case. This will then allow us to control the behaviour of \( e(C) \) under tame base change, using Proposition 2.1.6.

**Proposition 2.2.8.** Assume that \( g \neq 1 \) or that \( C \) is an elliptic curve. The stabilization index \( e(C) \) is the smallest integer \( e > 0 \) such that \( \xi^e = 1 \) for every complex root \( \zeta \) of \( P_C(t) \).

**Proof.** Assume that \( \mathcal{C} \) is a relatively minimal \( sncd \)-model of \( C \). Let \( e \) be the smallest strictly positive integer such that \( \xi^e = 1 \) for every complex root \( \zeta \) of \( P_C(t) \). It is clear that \( e|e(C) \), because every irreducible component \( E \) of \( \mathcal{C}_k \) with \( \chi(E) < 0 \) is principal. Thus it suffices to show that \( N \) divides \( e \) if \( N > 1 \) is the multiplicity of a principal component in \( \mathcal{C}_k \). This follows at once from [Ni2] 3.2.3, because [Ni2] 3.2.2 implies that \( \mathcal{C} \) is not \( N \)-tame.

**Proposition 2.2.9.** For every integer \( d \in \mathbb{N} \), we have that

\[ e(\mathcal{C}(d)) = e(\mathcal{C})/gcd(e(\mathcal{C}), d). \]

**Proof.** Let \( E \) be an irreducible component of \( \mathcal{C}_k \), and let \( E' \) be an irreducible component of \( \mathcal{C}(d)_k \) that dominates \( E \). Then \( g(E') \geq g(E) \) by [Ha77] IV.2.5.4 and it is clear that, if \( E \) meets the other components of \( \mathcal{C}_k \) in \( n \) distinct points, then \( E' \) meets the other components of \( \mathcal{C}(d)_k \) in at least \( n \) points. Thus \( E' \) is principal if \( E \) is principal. By Proposition 1.3.2(1), the multiplicity \( N' \) of \( E' \) in \( \mathcal{C}(d)_k \) equals \( N/gcd(N, d) \), where \( N \) is the multiplicity of \( E \) in \( \mathcal{C}_k \).
On the other hand, it follows from Proposition 1.3.2(2) that $E'$ is non-principal if $E$ is non-principal. By Proposition 1.3.2(3) any exceptional component of $\rho : C(d) \to C_d$ is non-principal. If we write $\mathcal{C}(d)_k = \sum_{j \in I(d)} N_j E'_j$, it follows that

$$e(\mathcal{C}(d)) = \text{lcm}_{j \in I(d)_\text{prin}} \{N'_j\} = \text{lcm}_{i \in I_{\text{prin}}} \{N_i / \gcd(N_i, d)\} = e(\mathcal{C}) / \gcd(e(\mathcal{C}), d).$$

\[ \square \]

**Corollary 2.2.10.** Assume that $C$ is not a genus one curve without rational point whose Jacobian has additive reduction and potential multiplicative reduction. Then for every $d \in \mathbb{N}'$, we have that

$$e(C(d)) = e(C) / \gcd(e(C), d).$$

**Proof.** If $g \neq 1$ or $C$ is an elliptic curve, this follows from Propositions 2.1.6 and 2.2.8. Thus we may assume that $C$ is a genus one curve without rational point. Its Jacobian is an elliptic curve $E$, and $C$ is an $E$-torsor over $K$. If we denote by $m$ the order of the class of $C$ in the Weil-Châtelet group $H^1(K, E)$, then the reduction type of $C$ is $m$ times the reduction type of $E$, by [LLR04] 6.6. Assume that $\mathcal{C}$ is the minimal sncd-model of $C$. Looking at the Kodaira-Néron reduction table, we see that all principal components of the special fiber $C_k$ have the same multiplicity $N$. We’ve shown in the proof of Proposition 2.2.9 that every component of $\mathcal{C}(d)_k$ that dominates a principal component of $C_k$ is itself principal, and that these are the only principal components of $\mathcal{C}(d)_k$. Their multiplicities are all equal to

$$N_i / \gcd(N, d) = e(C) / \gcd(e(C), d).$$

No new principal components are created by the contraction of $\mathcal{C}(d)$ to the minimal sncd-model $\mathcal{C}(d)_{\text{min}}$ of $C(d)$, and the special fiber of $\mathcal{C}(d)_{\text{min}}$ contains at least one principal component, unless $E(d)$ has multiplicative reduction. In the latter case, it follows from our assumptions that $E$ already had multiplicative reduction, so that $e(C) = e(C(d)) = 1$. \[ \square \]

**2.2.11** Let us briefly comment on the case where $C$ is a genus one curve without rational point; this case is irrelevant for the applications in the following sections, since we will only be interested in the Jacobian of $C$, which is an elliptic curve.

Let $E$ be an elliptic curve over $K$. The genus one $K$-curves with Jacobian $E$ are classified by the Weil-Châtelet group $H^1(K, E)$. It is known that the group $H^1(K, E)$ is non-trivial when $E$ has semi-stable reduction. If $E$ has additive reduction, then $H^1(K, E)$ is a $p$-group, and it is non-trivial when $p > 1$ [LLR04] 6.7. Now let $C$ be a genus one curve with Jacobian $E$. We denote by $m$ the order of the class of $C$ in $H^1(K, E)$. As we already recalled in the proof of Corollary 2.2.10 the reduction type of $C$ is $m$ times the reduction type of $E$ [LLR04] 6.6. This implies that $m$ is equal to the index of $C$ (the greatest common divisor of the degrees of the closed points of $C$), and also to the minimum of the degrees of the closed points on $C$ [BLR90] 9.1.9.

1. Proposition 2.2.4 fails for $C$ if $p > 1$, $\mathcal{C}_k$ has a principal component, $m$ is divisible by $p$ and $E$ is tamely ramified. This can happen, for instance, if $p \geq 5$ and $E$ has additive reduction.

2. Even if $k$ has characteristic zero, Proposition 2.2.4 can fail for $C$. If $E$ has multiplicative reduction, then $e(C) = 1$ but $[L : K] = m$. Likewise, Proposition 2.2.8 might fail: if $E$ has good reduction, then $P_C(t) = (t-1)^2$ while $e(C) = m$. 

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(3) The case excluded in the statement of Corollary 2.2.10 never occurs if $p = 1$, because then $H^1(K, E) = 0$ if $E$ has additive reduction. However, it does occur if $p > 1$. Assume that $E$ has additive reduction and acquires multiplicative reduction over $K(d)$, for some $d \in \mathbb{N}'$. Then $m$ is a power of $p$ and $e(C) = m \cdot e(E)$ while $e(C(d)) = 1$, so that the property in Corollary 2.2.10 does not hold if $m > 1$.

2.3. Applications to $\text{sn}_{d\mathbb{N}}$-models and base change.

(2.3.1) We keep the notations from (2.1.1). An important feature of the invariant $e(\mathcal{C})$ is that we can give a rather precise description of the special fiber of $\mathcal{C}(d)$ in terms of the special fiber of $\mathcal{C}$, as long as $d \in \mathbb{N}'$ is prime to $e(\mathcal{C})$.

Lemma 2.3.2. Assume that $C$ is not a genus one curve without rational point whose Jacobian has multiplicative reduction. Let $d$ be an element of $\mathbb{N}'$ that is prime to $e(\mathcal{C})$. For every irreducible component $E_i$ of $\mathcal{C}_k$, the $k$-scheme

$$F_i = \mathcal{C}_d \times_{\mathcal{C}} E_i$$

is smooth and irreducible. We denote by $N_i'$ the multiplicity of $(\mathcal{C}_d)_k$ along $F_i$. Then $N_i' = N_i / \gcd(N_i, d)$, and $F_i \rightarrow E_i$ is a ramified tame Galois cover of degree $\gcd(N_i, d)$.

If $E_i$ is principal, or $E_i$ is a rational curve such that $E_i \cdot \sum_{j \neq i} E_j = 1$, then $F_i \rightarrow E_i$ is an isomorphism, and $N_i' = N_i$. If $E_i$ is a rational curve such that $E_i \cdot \sum_{j \neq i} E_j = 2$, then $F_i \cong \mathbb{P}^1_k$ and $F_i \rightarrow E_i$ is either an isomorphism, or ramified over the two points of $E_i \setminus E_i^o$. Moreover, if $i$ and $j$ are distinct elements of $I$, then over any point of $E_i \cap E_j$ lies exactly one point of $F_i \cap F_j$.

Proof. We’ve already seen in Proposition 1.3.2 that $F_i$ is smooth, that $N_i' = N_i / \gcd(N_i, d)$ and that $\mathcal{C}_d \times_{\mathcal{C}} E_i^o \rightarrow E_i^o$ is a tame Galois cover of degree $\gcd(N_i, d)$.

If $E_i$ is principal, then $N_i$ divides $e(\mathcal{C})$, so that $N_i$ is prime to $d$, $N_i' = N_i$ and $F_i \rightarrow E_i$ is an isomorphism. Now assume that $E_i$ is a non-principal component; in particular, it is rational. If $E_i$ is the only component of $\mathcal{C}_k$, then it follows from [Ni12 3.1.1] that $g = 0$ and $\mathcal{C}_k = E_i$, so that the result is clear. Thus we may suppose that $E_i$ meets another component $E_j$ of $\mathcal{C}_k$. We choose a point $x$ in $E_i \cap E_j$.

We claim that $\gcd(N_i, N_j, d) = 1$. Point (2) in Proposition 1.3.2 then implies that $F_i$ is a connected smooth rational curve. If $F_i \rightarrow E_i$ is not an isomorphism, one deduces from Hurwitz’ theorem that $F_i \rightarrow E_i$ must be ramified over $E_i \setminus E_i^o$ and that $|E_i \setminus E_i^o| = 2$.

It remains to prove our claim that $\gcd(N_i, N_j, d) = 1$. We are in one of the following three cases:

(1) $E_i$ is part of a chain of rational components $\mathcal{C}_k$ that meets a principal component $E$ of $\mathcal{C}_k$,

(2) $\mathcal{C}_k$ is a chain of rational curves,

(3) $\mathcal{C}_k$ is a loop of rational curves.

In case (1), the multiplicity $N$ of $E$ in $\mathcal{C}_k$ is prime to $d$ by our assumption that $d$ is prime to $e(\mathcal{C})$. In case (2), one deduces from [Ni12 3.1.1] that $g = 0$ and that the ends of the chain have multiplicity one. In case (3), we must have $g = 1$ by [Ni12 3.1.1] and $\text{Jac}(C)$ has multiplicative reduction by [LLR04 6.6]. Then by our assumptions, $C$ has a rational point so that at least one of the components of
has multiplicity one. In all cases, the argument in [Ha10a, 6.3] now shows that gcd($N_i, N_j, d) = 1$.

**Proposition 2.3.3.** For each element $d$ of $\mathbb{N}'$ prime to $e(C)$, the combinatorial data of $C_k(d)$ only depend on the combinatorial data of $C_k$. In particular, they do not depend on the characteristic exponent $p$ of $k$.

**Proof.** This follows immediately from Proposition 1.3.2 and Lemma 2.3.2, unless $C$ is a genus one curve without rational point whose Jacobian has multiplicative reduction. In fact, even that case is covered by the proof of Lemma 2.3.2, unless $C_k$ is a loop of rational curves. In that situation, the result follows easily from Proposition 1.3.2.

3. The Néron component series of a Jacobian

In this Section, we study the Néron component series of a Jacobian variety. This series was defined in Section 3.5 of Chapter 2. We keep the notations from (2.1.1). We assume that $g$ is positive, and we denote by $A$ the Jacobian $\text{Jac}(C)$ of the curve $C$.

### 3.1. Rationality of the component series.

**Proposition 3.1.1.** Let $K'/K$ be a finite tame extension of $K$ whose degree $d$ is prime to $e(C)$. Then

$$|\Phi(A \times_K K')| = d^t(A) \cdot |\Phi(A)|.$$

**Proof.** Embedding $K'$ in $K^s$, we may assume that $K = K(d)$. Let $C$ be the minimal sncd-model of $C$. By Theorem 1.2.1 we can find a smooth, proper and geometrically connected $\mathbb{C}((t))$-curve $D$ with minimal sncd-model $\mathcal{D}$ over $\mathbb{C}[[t]]$ such that the special fiber $\mathcal{D}_C$ has the same combinatorial data as $C_k$. This implies that $e(C) = e(D)$. By Lemma 2.3.2 the combinatorial data of $\mathcal{D}(d)_{K}$ and $C_k(d)$ coincide. Thus if we set $B = \text{Jac}(D)$, then $t(A) = t(B)$, $\Phi(A) \cong \Phi(B)$ and $\Phi(A(d)) \cong \Phi(B(d))$, so that we may assume that $K = \mathbb{C}((t))$. Then $C$ is tamely ramified. By Proposition 2.2.4 the integer $e(C)$ equals the degree of the minimal extension of $K$ over which $C$ acquires semi-stable reduction, so that the result follows from [HN10, 5.7].

The following example, which was included already in [HN10, 5.9], shows that in the statement of Proposition 3.1.1 we cannot replace $e(C)$ by the degree of the minimal extension of $K$ where $C$ acquires semi-stable reduction.

**Example 3.1.2.** We consider again the elliptic curve $C$ in Example 2.2.5. Then $\Phi(C) = 0$ because $C$ has reduction type II. On the other hand, using Tate’s algorithm one checks that $C(3)$ has reduction type $I_2^*$ so that $|\Phi(C(3))| = 4$.

**Lemma 3.1.3.** For every $d \in \mathbb{N}'$ prime to $e(C)$, we have that $t(A(d)) = t(A)$.

**Proof.** Let $C/R$ be the minimal sncd-model of $C$. We consider the graphs $\Gamma = \Gamma(C_k)$ and $\Gamma(d) = \Gamma(C_k(d))$, but we forget the weights. From Proposition 1.3.2 and Lemma 2.3.2 we deduce that $\Gamma(d)$ is obtained from $\Gamma$ by subdividing every edge $\varepsilon$ into a chain of $n_\varepsilon$ edges, for some $n_\varepsilon \geq 1$. This clearly implies that $\text{Gamma}$ and $\Gamma(d)$ have the same first Betti number, which means that $t(A(d)) = t(A)$.
Before we can state the main result in this section, we need to recall the definition of the tame potential toric rank of $A$ which was introduced in [HN10 6.3]. This value is defined by

$$t_{tame}(A) = \max \{ t(A(d)) \mid d \in \mathbb{N}' \}.$$ 

If we denote by $L$ the minimal extension of $K$ in $K^s$ such that $A \times_K L$ has semi-abelian reduction, and by $d$ the prime-to-$p$ part of $[L : K]$, then $t_{tame}(A) = t(A(d))$ by [HN10 6.4]. In particular, if $A$ is tamely ramified, $t_{tame}(A)$ equals the potential toric rank $t_{pot}(A)$ of $A$. In general, we have $t_{tame}(A) \leq t_{pot}(A)$ by [HN10 3.9].

**Theorem 3.1.5.** Let $C/K$ be a smooth, projective and geometrically connected curve of genus $g > 0$. Assume that $C$ admits a zero divisor of degree one, and set $A = \text{Jac}(C)$. The component series

$$S^\Phi_A(T) = \sum_{d \in \mathbb{N}'} |\Phi(A(d))| T^d$$

is rational. More precisely, it belongs to the subring

$$\mathcal{R} = \mathbb{Z} \left[ T, \frac{1}{T_j - 1} \right]_{j \in \mathbb{Z}_{>0}}$$

of $\mathbb{Z}[[T]]$. It has degree zero if $p = 1$ and $A$ has potential good reduction, and negative degree in all other cases. Moreover, $S^\Phi_A(T)$ has a pole at $T = 1$ of order $t_{tame}(A) + 1$.

**Proof.** To ease notation, we put $e = e(C)$ in this proof. Since the tame case is covered in [HN10 6.5], we can and will assume here that $A$ is wildly ramified. This means in particular that $p > 1$ and $p | e$, by Proposition 2.2.4.

Let us denote by $\mathcal{S}$ the set of divisors of $e$ that are prime to $p$. We define the auxiliary series

$$S'_A(T) = \sum_{d \in \mathbb{N}', \gcd(d,e) = 1} |\Phi(A(d))| T^d.$$ 

Then we can write

$$S^\Phi_A(T) = \sum_{a \in \mathcal{S}} S'_{A(a)}(T^a),$$

since $e(C(a)) = e(C)/a$ by Corollary 2.2.10.

By Lemma 3.1.3 we have $t_{tame}(A) = \max \{ t(A(a)) \mid a \in \mathcal{S} \}$. Using [HN10 6.1], we see that it is enough to prove the following claims, for every $a$ in $\mathcal{S}$:

1. the series $S'_{A(a)}(T^a)$ belongs to $\mathcal{R}$ and has a pole at $T = 1$ of order $t(A(a)) + 1$.
2. the degree of $S'_{A(a)}(T^a)$ is negative.

To prove these claims, it suffices to consider the case $a = 1$. We denote by $\mathcal{S}_e$ the set of elements in $\{1, \ldots, e\}$ that are prime to $e$. Since $p$ divides $e$, we can write

$$S'_A(T) = |\Phi(A)| \cdot \sum_{d \in \mathbb{N}', \gcd(d,e) = 1} d^{(A)} T^d$$

$$= |\Phi(A)| \cdot \sum_{b \in \mathcal{S}_e} \left( \sum_{q \in \mathbb{N}} (qe + b)^{(A)} T^{qe+b} \right).$$

where the first equality follows from Proposition 3.1.1. An easy computation shows that for each \( b \in \mathcal{P} \), the series \( \sum_{q \in \mathbb{N}} (qe + b) t_A^{q + b} \) belongs to \( \mathcal{Z} \), has a pole at \( T = 1 \) of order \( t_A + 1 \), and has negative degree. This clearly implies claim (2), and claim (1) now follows from [HN10, 6.1].

(3.1.6) We expect that Theorem 3.1.5 is valid for all abelian \( K \)-varieties \( A \). If \( A \) is tamely ramified or \( A \) has potential multiplicative reduction, we proved this in [HN10, 6.5]. The crucial open case is the one where \( A \) is a wildly ramified abelian variety with potential good reduction; in that case, it is not clear how to control the behaviour of the \( p \)-part of \( \Phi(A) \) under finite tame extensions of \( K \). If \( A \) is tamely ramified and \( A \) has potential good reduction, then the \( p \)-part of \( \Phi(A) \) is trivial by Theorem 1 in [ELL96].

4. Appendix: Locally toric rings

4.1. Resolution of locally toric singularities.

Definition 4.1.1. Let \( A \) be a complete Noetherian local \( R \)-algebra. We say that \( A \) is a locally toric ring if it is of the form

\[
A \cong R[[S]]/\mathcal{I} = R[[\chi^s : s \in S]]/\mathcal{I},
\]

where \( S \) is a finitely generated integral saturated sharp monoid and the ideal \( \mathcal{I} \) is generated by an element of the form \( \pi - \varphi \) with

\[
\varphi \equiv 0 \mod (\chi^s : s \in S, s \neq 0).
\]

We have made a slight modification of Definition 2.6.2 in [Ki10] in order to treat the cases where \( R \) is of equal, resp. mixed characteristic in a uniform way. The condition that \( S \) is sharp seems to be missing in [Ki10] (without this condition, the ring \( R[[S]] \) is not well-defined). The condition that \( S \) is saturated is equivalent to the assumption in [Ki10, 2.6.2] that \( A \) is normal. The condition that \( S \) is finitely generated, integral and sharp implies that the abelian group \( S_{\text{gp}} \) is a free \( \mathbb{Z} \)-module of finite rank and that the natural morphism \( S \to S_{\text{gp}} \) is injective. Thus \( S \) is a submonoid of a free \( \mathbb{Z} \)-module of finite rank, as required in [Ki10, 2.6.2]. Definition 4.1.1 is equivalent to Kato’s definition of a toric singularity in [Ka94, 2.1], by [Ka94, 3.1].

(4.1.2) We shall only be concerned here with the case where \( A \) has Krull dimension 2; then \( S_{\text{gp}} \) has rank two. By [Li69, 27.3] there exists a minimal desingularization \( \rho : X \to \text{Spec} A \) which, if \( A \) is locally toric, turns out to be entirely determined by the monoid \( S \). We explain briefly the main ingredients in this procedure, following the treatment in [Ki10, Ch. 2].

(4.1.3) Set \( M = S_{\text{gp}} \) and denote by \( N \) the dual lattice \( M^\vee = \text{Hom}(M, \mathbb{Z}) \). We put \( M_R = M \otimes \mathbb{Z} R \) and \( N_R = N \otimes \mathbb{Z} R \). The monoid \( S \) yields a cone \( \sigma^\vee = S \otimes N \mathbb{Z} \supseteq M_R \) and a dual cone \( \sigma = (\sigma^\vee)^\vee \subset N_R \). There exists a fan \( F = \{ \sigma_t : t \in T \} \) of cones in \( N_R \) that is a regular refinement of \( \sigma \). Moreover, since \( M \) has rank two, there exists a minimal regular refinement \( F_{\text{min}} \), in the sense that any other regular refinement of \( \sigma \) is also a refinement of \( F_{\text{min}} \).
(4.1.4) For each \( \sigma_i \in F_{\text{min}} \), put \( S_{\sigma_i} = (\sigma_i)^{\vee} \cap M \) and define \( A_{\sigma_i} = A[S_{\sigma_i}] \) (the notation is ambiguous; this is not the monoid \( A \)-algebra associated to \( S_{\sigma_i} \), but the subring of the quotient field of \( A \) obtained by joining to \( A \) the elements \( \chi^s \) with \( s \) in \( S_{\sigma_i} \)). Note that \( A_{\sigma_i} \) will in general not be local.

**Lemma 4.1.5.** Let \( \sigma_i \subset N_G \) be part of a regular refinement of \( \sigma \). Then \( A_{\sigma_i} \) is regular.

**Proof.** This follows from [Ki10] 2.6.8, but the proof that is given there is not complete (one cannot directly carry over the methods from the classical theory of toric varieties over an algebraically closed field, since the equation \( \pi - \varphi \) is not homogeneous with respect to the torus action). Instead, one can invoke the logarithmic approach in [Ka94] 10.4. We will give below a proof by explicit computation for tame cyclic quotient singularities (Proposition 4.2.5); this is the only case we used in this chapter. \( \Box \)

(4.1.6) The natural maps \( \text{Spec} A_{\sigma_i} \rightarrow \text{Spec} A \) are birational, and glue to give a desingularization

\[ \rho_{F_{\text{min}}} : X_{F_{\text{min}}} = \bigcup_i \text{Spec} A_{\sigma_i} \rightarrow \text{Spec} A. \]

The minimality of \( F_{\text{min}} \) implies that \( \rho_{F_{\text{min}}} \) is the minimal toric desingularization of \( \text{Spec} A \). In fact, it is also the minimal resolution: this can be seen, for instance, by computing the self-intersection numbers of the exceptional curves in the resolution. In the case we’re interested in, these self-intersections will be at most \(-2\), so that none of the exceptional curves in the resolution satisfies Castelnuovo’s contractibility criterion and the resolution is indeed minimal (see Proposition 4.2.5).

### 4.2. Tame cyclic quotient singularities.

(4.2.1) We consider the regular local ring

\[ A = R[[t_1, t_2]]/(\pi - w \cdot t_1^{m_1} t_2^{m_2}), \]

where \( w \) is a unit in \( R[[t_1, t_2]] \). Let \( n \) be an integer prime to \( m_1 \) and \( m_2 \), and let \( r \) be the unique integer \( 0 < r < n \) such that \( m_1 + rm_2 \equiv 0 \) modulo \( n \).

We let \( \mu_n(k) \) act on the \( R \)-algebra \( R[[t_1, t_2]] \) by \( (\xi, t_1) \mapsto \xi t_1 \) and \( (\xi, t_2) \mapsto \xi^r t_2 \) for any \( \xi \in \mu_n(k) \). We also assume that the unit \( w \) is \( \mu_n(k) \)-invariant, so that there is an induced \( \mu_n(k) \)-action on \( A \).

**Lemma 4.2.2.** We set

\[ M = \{(s_1, s_2) \in \mathbb{Z}^2 \mid n \text{ divides } s_1 + s_2 \} = (n, 0)\mathbb{Z} + (-r, 1)\mathbb{Z} \subset \mathbb{Z}^2 \]

and \( S = M \cap (\mathbb{Z}_{\geq 0})^2 \). Then \( B = A^{\mu_n(k)} \) is a locally toric ring with respect to the monoid \( S \). More precisely,

\[ B \cong R[[t_1^{s_1} t_2^{s_2} \mid (s_1, s_2) \in S]]/(\pi - w \cdot t_1^{m_1} t_2^{m_2}). \]

**Proof.** Since \( n \) is prime to \( p \), we have an isomorphism of \( R \)-algebras

\[ B \cong R[[t_1, t_2]]^{\mu_n(k)}/(\pi - w \cdot t_1^{m_1} t_2^{m_2}). \]

The lemma then follows from the fact that the \( R \)-algebra \( R[[t_1, t_2]]^{\mu_n(k)} \) is topologically generated by invariant monomials, which are exactly the elements \( t_1^{s_1} t_2^{s_2} \) with \( (s_1, s_2) \in S \). \( \Box \)
The R-scheme \( \text{Spec } B \) is called a tame cyclic quotient singularity. Thanks to Lemma 4.2.2, we can use the theory of locally toric singularities to construct the minimal resolution of \( \text{Spec } B \). First, we need to introduce some notation.

**Definition 4.2.4.** For relatively prime integers \( a, b \in \mathbb{Z}_{>0} \) with \( a > b \) we use the compact notation \( a/b = [z_1, \ldots, z_L]_{\text{HJ}} \) for the Hirzebruch-Jung continued fraction expansion

\[
a/b = z_1 - \frac{1}{z_2 - \frac{1}{z_3 - \ldots}}
\]

with \( z_i \in \mathbb{Z}_{\geq 2} \).

**Proposition 4.2.5.** Let \( \rho : X = X_{\text{Fmin}} \rightarrow \text{Spec } B \) be the minimal toric desingularization of \( \text{Spec } B \). The special fiber \( X_k = \sum_{i=0}^{L+1} \mu_i E_i \) is a strict normal crossings divisor and, renumbering the irreducible components of \( X_k \) in a suitable way, we can arrange that the following properties hold:

1. \( E_0 \) and \( E_{L+1} \) are the strict transforms of the irreducible components of the special fiber of \( \text{Spec } B \).
2. \( E_i \cong \mathbb{P}^1_k \) for each \( 1 \leq i \leq L \).
3. For each \( 1 \leq i \leq L \), \( E_i \) intersects \( E_{i-1} \) and \( E_{i+1} \) in a unique point, and no other components of \( X_k \). Moreover, \( E_0 \) (resp. \( E_{L+1} \)) intersects \( E_1 \) (resp. \( E_L \)) in a unique point, and no other components of \( X_k \).
4. \( E_i^2 \leq -2 \) for each \( 1 \leq i \leq L \).

In particular, \( \rho \) is the minimal resolution of \( \text{Spec } B \).

**Proof.** Let \( n \) and \( r \) be as above, and write \( n/r = [b_1, \ldots, b_L]_{\text{HJ}} \). The monoid \( S \) from Lemma 4.2.2 defines the lattice

\[
N = (0, 1)\mathbb{Z} + \frac{1}{n}(1, r)\mathbb{Z},
\]

which is dual to \( M = S_{\text{sep}} \) and the cone \( \sigma = \mathbb{R}_{\geq 0}^2 \) in \( N \otimes_\mathbb{Z} \mathbb{R} \). The minimal regular refinement of \( \sigma \) is the union of subcones \( \sigma_0, \ldots, \sigma_L \) that can be computed as in \([\text{Fu93}, \text{§2.6}]\). If we put \( e_0 = (0, 1) \), \( e_1 = \frac{1}{n}(1, r) \), and inductively define \( e_{i+1} = b_i e_i - e_{i-1} \) (in particular \( e_{L+1} = (1, 0) \)), then the cone \( \sigma_i \) is generated by \( e_i \) and \( e_{i+1} \).

We put \( u_0 = t_1^n \) and \( v_0 = t_1^{-1}t_2 \), and define inductively elements \( u_i = (v_{i-1})^{-1} \) and \( v_i = u_{i-1}(v_{i-1})^{b_i} \) in the fraction field of \( B \), for \( 1 \leq i \leq L \). We moreover put \( \mu_0 = m_2 \) and \( \mu_1 = (m_1 + rm_2)/n \) and define inductively integers \( \mu_{i+1} = b_i \mu_i - \mu_{i-1} \). In particular \( \mu_i \in \mathbb{N} \) for all \( i \) and \( \mu_{L+1} = m_1 \). Moreover, an easy induction argument shows that \( t_1^{m_i}t_2^{m_i} = u_i^{\mu_i}v_i^{\mu_i} \) for all \( i \).

We identify \( \mathbb{Z}^2 \) with the group of monomials in the variables \((t_1, t_2)\) via the map \((a_1, a_2) \mapsto t_1^{a_1} t_2^{a_2}\). Then for each \( i \in \{0, \ldots, L\} \), the elements \( u_i \) resp. \( v_i \) are dual to \( e_i \) resp. \( e_{i+1} \) and generate the monoid \( S_{\sigma_i} \). Moreover, we get

\[
B_{\sigma_i} = R[[S]][S_{\sigma_i}]/(x - w \cdot u_i^{\mu_i} v_i^{\mu_i}).
\]

First, we show that \( \text{Spec } B_{\sigma_i} \) is regular. The scheme \( \text{Spec } B \) has an isolated singularity at its unique closed point \( x \), and \( \rho \) is an isomorphism over the complement of this closed point. Thus we only need to prove regularity of \( \text{Spec } B_{\sigma_i} \) at the points lying above \( x \). In such points we either have \( u_i = 0 \) or \( v_i = 0 \), so that the result follows from the fact that \( \text{Spec } B_{\sigma_i} \) is two-dimensional and the zero loci of \( u_i \) and \( v_i \) are regular one-dimensional schemes. More precisely, the zero locus
of $v_i$ is isomorphic to $\text{Spec } k[[u_0]]$ if $i = 0$ and to $\text{Spec } k[u_i]$ else; likewise, the zero locus of $u_i$ is isomorphic to $\text{Spec } k[[v_L]]$ if $i = L$ and to $\text{Spec } k[v_i]$ else.

The ideal $(u_i, v_i)$ is maximal in $B_{\sigma_i}$, so that $u_i, v_i$ form a regular system of parameters and the special fiber of $\text{Spec } B_{\sigma_i}$ is a divisor with strict normal crossings. Thus we can conclude that $X$ is regular and that $X_k$ is a strict normal crossings divisor of the form described in (1) and (3). The relation $u_i = (v_{i-1})^{-1}$ implies that the components $E_1, \ldots, E_L$ are all isomorphic to $\mathbb{P}^1_k$, so that (2) holds as well. Moreover, the component $E_j$ has multiplicity $\mu_j$ in $X_k$, for each $j$ in $\{0, \ldots, L+1\}$, so that

$$0 = X_k \cdot E_i = \left( \sum_{j=0}^{L+1} \mu_j E_j \right) \cdot E_i = \mu_{i-1} + \mu_i E_i^2 + \mu_{i+1}$$

for all $1 \leq i \leq L$. This implies that the self-intersection number of $E_i$ is equal to $-b_i$. In particular, each of these exceptional components has self-intersection number at most $-2$, such that (4) holds and $\rho$ is minimal. \qed
In this chapter, we will study the behaviour of the torsion part of the Néron component group of a semi-abelian $K$-variety under ramified extension of the base field $K$. Our main goal is to prove the rationality of the component series (Theorem 3.4.2). We discussed the case of an abelian $K$-variety in [HN10]; in that case, the component group is finite. The main complication that arises in the semi-abelian case is the fact that it is difficult in general to identify the torsion part of the component group in a geometric way. This problem is related to the index of the semi-abelian $K$-variety, an invariant that we introduce in Section 2. For tori, the torsion part of the component group has a geometric interpretation in terms of the dual torus, and we can explicitly compute the index from the character group. The case of a semi-abelian variety is substantially more difficult; there we need to construct a suitable uniformization, which is no longer an algebraic group but a rigid analytic group. In order to deal with Néron component groups of rigid analytic groups, we will use the cohomological theory of Bosch and Xarles [BX96], that we recall and extend in Section 1. We correct an error in their paper, which was pointed out by Chai, and we show that all of the principal results in [BX96] remain valid.

1. Component groups of smooth sheaves

1.1. The work of Bosch and Xarles.

(1.1.1) In [BX96], Bosch and Xarles developed a powerful cohomological approach to the study of component groups of abelian $K$-varieties. They interpret the Néron model in terms of a push-forward functor from the rigid smooth site on Sp $K$ to the formal smooth site on Spf $R$, and they show that the Néron component group of an abelian variety can be recovered from the smooth sheaf associated to the Néron model. Combining this interpretation with non-archimedean uniformization of abelian varieties, they deduce several deep results on the structure of the component group.

(1.1.2) Unfortunately, it is known that the perfect residue field case of Lemma 4.2 in [BX96] is not correct (see [Ch00, 4.8(b)]), and the proofs of some of the main results in [BX96] rely on this lemma. We will now show that one can replace the erroneous lemma by another statement that suffices to prove the validity of all the other results in [BX96].

**Proposition 1.1.3.** Assume that $k$ is algebraically closed. Let

\[ 0 \rightarrow T \rightarrow G \rightarrow H \rightarrow 0 \]
be a short exact sequence of semi-abelian $K$-varieties such that $T$ is a torus. We denote by $f : \mathcal{G} \to \mathcal{H}$ the unique morphism of Néron lift-models that extends the morphism $G \to H$. Then the following properties hold.

1. The map 
   
   \[
   f^o(R) : \mathcal{G}^o(R) \to \mathcal{H}^o(R)
   \]

   is surjective.

2. The sequence of component groups

   \[
   (1.1.4) \quad \Phi(T) \to \Phi(G) \to \Phi(H) \to 0
   \]

   is exact.

Proof. (1) By [Ch00, 4.3], the Galois cohomology group $H^1(K,T)$ vanishes, so that the sequence

\[
(1.1.5) \quad 0 \to T(K) \to G(K) \to H(K) \to 0
\]

is exact. In particular, the map

\[
G(K) = \mathcal{G}(R) \to \mathcal{H}(R) = H(K)
\]

is surjective. Since $\mathcal{G}(R)/\mathcal{G}^o(R)$ is finitely generated by [HN11a, 3.5], it follows from [BLR90, 9.6.2] that

\[
f^o(R) : \mathcal{G}^o(R) \to \mathcal{H}^o(R)
\]

is surjective.

(2) It follows at once from the exactness of (1.1.3) that $\Phi(G) \to \Phi(H)$ is surjective. Then by an elementary diagram chase one deduces from point (1) that the sequence

\[
\Phi(T) = T(K)/T^o(R) \to \Phi(G) = G(K)/\mathcal{G}^o(R) \to \Phi(H) = H(K)/\mathcal{H}^o(R) \to 0
\]

is exact.

(1.1.6) Let us check that replacing [BX96, 4.2] (perfect residue field case) by Proposition [1.1.3] suffices to prove all the subsequent results in [BX96] (note that one can immediately reduce to our setting where $R$ is complete and $k$ is separably closed, since the formation of Néron models commutes with base change to the completion of a strict henselization). The result [BX96, 4.2] was applied at the following places.

- In the proof of [BX96, 4.11(i)] and [BX96, 5.8(ii)], the result in [BX96, 4.2] was used to prove the exactness of the sequence

\[
\Phi(T_{spl}) \longrightarrow \Phi(T) \longrightarrow \Phi(T') \longrightarrow 0
\]

where $T$ is a $K$-torus and $T' = T/T_{spl}$. This also follows from the proof of [BLR90, 10.1.7], since $T'$ is anisotropic. If $k$ is perfect, it follows from Proposition [1.1.3].

- The proof of [BX96, 4.11(ii)] (perfect residue field case); this result is a special case of Proposition [1.1.3].

1.2. Identity component and component group of a smooth sheaf.
(1.2.1) We denote by \((\text{Spf} R)_{\text{sm}}\) and \((\text{Sp} K)_{\text{sm}}\) the small smooth sites over \(\text{Spf} R\) and \(\text{Sp} K\), respectively [BX96 §3]. For every smooth affinoid \(K\)-algebra \(A\) and every abelian sheaf \(F\) on \((\text{Sp} K)_{\text{sm}}\), we will write \(F(A)\) instead of \(F(\text{Sp} A)\), and we will use the analogous notation for smooth topological \(R\)-algebras of finite type and abelian sheaves on \((\text{Spf} R)_{\text{sm}}\).

(1.2.2) An important property of the site \((\text{Spf} R)_{\text{sm}}\) is the following: if \(\mathfrak{X}\) is a non-empty smooth formal \(R\)-scheme, then the structural morphism \(\mathfrak{X} \to \text{Spf} R\) has a section. This follows from the infinitesimal lifting criterion for smoothness and the fact that the \(k\)-rational points on \(\mathfrak{X} \times_R k\) are dense because \(k\) is separably closed [BLR90 2.2.13].

(1.2.3) The generic fiber functor that associates to a (smooth) formal \(R\)-scheme its (smooth) rigid generic fiber over \(K\) induces a morphism of sites

\[ j : (\text{Sp} K)_{\text{sm}} \to (\text{Spf} R)_{\text{sm}}. \]

For every abelian sheaf \(F\) on \((\text{Sp} K)_{\text{sm}}\), we define the Néron model \(\mathcal{F}\) of \(F\) by

\[ \mathcal{F} = j_*F \]

as in [BX96 §3]. This is an abelian sheaf on \((\text{Spf} R)_{\text{sm}}\).

(1.2.4) Bosch and Xarles define in [BX96 §4] the identity component \(\mathcal{F}^o\) of an abelian sheaf \(\mathcal{F}\) on \((\text{Spf} R)_{\text{sm}}\), which is a subsheaf of \(\mathcal{F}\), and the component sheaf

\[ \Phi(\mathcal{F}) = \mathcal{F}/\mathcal{F}^o. \]

The identity component and component group are functorial in \(\mathcal{F}\). The sheaf property of \(\mathcal{F}\) is never used in the construction of the identity component, so that we can immediately extend this definition to abelian presheaves on \((\text{Spf} R)_{\text{sm}}\), as follows.

(1.2.5) Let \(\mathcal{F}\) be an abelian presheaf on \((\text{Spf} R)_{\text{sm}}\). We define \(\mathcal{F}^o(R)\) as the subgroup of \(\mathcal{F}(R)\) consisting of elements \(\sigma\) such that there exists a smooth connected formal \(R\)-scheme \(\mathfrak{X}\), an element \(\tau\) in \(\mathcal{F}(\mathfrak{X})\) and points \(x_0\) and \(x_1\) in \(\mathfrak{X}(R)\) such that \(x_0^*\tau = 0\) and \(x_1^*\tau = \sigma\) in \(\mathcal{F}(R)\). If \(Y\) is any smooth formal \(R\)-scheme, then an element of \(\mathcal{F}(Y)\) belongs to \(\mathcal{F}^o(Y)\) if and only if its image in \(\mathcal{F}(R)\) lies in \(\mathcal{F}^o(R)\) for every \(R\)-morphism \(\text{Spf} R \to Y\). We will say that \(\mathcal{F}\) is connected if \(\mathcal{F}^o = \mathcal{F}\). If \(\mathcal{F}\) is a sheaf, then so is \(\mathcal{F}^o\).

Proposition 1.2.6. If \(\mathcal{F}\) is an abelian presheaf on \((\text{Spf} R)_{\text{sm}}\), \(\mathfrak{X}\) is a connected smooth formal \(R\)-scheme and \(x\) is an element of \(\mathfrak{X}(R)\), then an element \(\sigma\) of \(\mathcal{F}(\mathfrak{X})\) belongs to \(\mathcal{F}^o(\mathfrak{X})\) if and only if \(x^*\sigma\) belongs to \(\mathcal{F}^o(R)\).

Proof. The “only if” part is a direct consequence of the definition of the identity component \(\mathcal{F}^o\), so that it suffices to prove the converse implication. Let \(y\) be a point of \(\mathfrak{X}(R)\). We must show that \(y^*\sigma\) lies in \(\mathcal{F}^o(R)\). Denote by

\[ f : \mathfrak{X} \to \text{Spf} R \]

the structural morphism, and set

\[ \sigma_0 = \sigma - f^*x^*\sigma \quad \in \mathcal{F}(\mathfrak{X}). \]

Then \(x^*\sigma_0 = 0\), so that \(y^*\sigma_0\) lies in \(\mathcal{F}^o(R)\). But

\[ y^*\sigma_0 = y^*\sigma - x^*\sigma \]

and since \(x^*\sigma\) lies in \(\mathcal{F}^o(R)\), we find that \(y^*\sigma\) lies in \(\mathcal{F}^o(R)\), as well. \(\square\)
Proposition 1.2.7. For every abelian sheaf \( \mathcal{F} \) on \((\text{Spf } R)_{\text{sm}}\), the component sheaf \( \Phi(\mathcal{F}) \) is the constant sheaf on \((\text{Spf } R)_{\text{sm}}\) associated to the abelian group 
\[ \mathcal{F}(R)/\mathcal{F}^o(R). \]

Proof. Let \( \mathfrak{X} \) be a non-empty connected smooth formal \( R \)-scheme. We will show that the morphism 
\[ i : \mathcal{F}(R)/\mathcal{F}^o(R) \to \mathcal{F}(\mathfrak{X})/\mathcal{F}^o(\mathfrak{X}) \]
induced by the structural morphism \( f : \mathfrak{X} \to \text{Spf } R \) is an isomorphism. Since \( \mathfrak{X} \) is smooth and \( k \) is separably closed, the morphism \( f \) has a section \( x : \text{Spf } R \to \mathfrak{X} \), which induces a section 
\[ s : \mathcal{F}(\mathfrak{X})/\mathcal{F}^o(\mathfrak{X}) \to \mathcal{F}(R)/\mathcal{F}^o(R) \]
of \( i \). If \( \tau \) is an element of \( \mathcal{F}(\mathfrak{X}) \), then 
\[ x^*(\tau - f^*x^*\tau) = 0 \]
in \( \mathcal{F}(R) \), so that \( \tau - f^*x^*\tau \) must lie in \( \mathcal{F}^o(\mathfrak{X}) \). This implies that 
\[ \tau \mod \mathcal{F}^o(\mathfrak{X}) = (i \circ s)(\tau \mod \mathcal{F}^o(\mathfrak{X})). \]
Thus \( s \) is inverse to \( i \). \( \square \)

(1.2.8) With a slight abuse of notation, we will usually write \( \Phi(\mathcal{F}) \) for the abelian group \( \mathcal{F}(R)/\mathcal{F}^o(R) \). When \( F \) is an abelian sheaf on \((\text{Sp } K)_{\text{sm}}\), we write \( \Phi(F) \) for \( \Phi(j_*F) \).

(1.2.9) If \( G \) is a smooth commutative rigid \( K \)-group, then the associated presheaf on \((\text{Sp } K)_{\text{sm}}\) is a sheaf \( [\text{BX96}, \text{3.3}] \), which we’ll denote again by \( G \). If \( G \) admits a formal Néron model \( \mathcal{G} \) in the sense of \( [\text{BS95}] \), then \( \mathcal{G} \) represents the Néron model \( j_*G \) on \((\text{Spf } R)_{\text{sm}}\), and \( \mathcal{G}^o \) represents the identity component \( (j_*G)^o \). It follows that the component group \( \Phi(G) \) of the abelian sheaf \( G \) is canonically isomorphic to the group \( \mathcal{G}^o/\mathcal{G}^o_k \) of connected components of \( \mathcal{G}_k \).

1.3. Some basic properties of the component group.

Lemma 1.3.1.

1. If \( \mathcal{F} \) is an abelian presheaf on \((\text{Spf } R)_{\text{sm}}\) and \( \mathcal{F} \to \mathcal{F}' \) is a sheafification, then \( \mathcal{F}(R) \to \mathcal{F}'(R) \) is an isomorphism.

2. The functor \( \mathcal{F} \to \mathcal{F}(R) \) from the category of abelian sheaves on \((\text{Spf } R)_{\text{sm}}\) to the category of abelian groups is exact.

Proof. (1) Injectivity of \( \mathcal{F}(R) \to \mathcal{F}'(R) \) follows immediately from the fact that every surjective smooth morphism of formal schemes \( \mathfrak{X} \to \text{Spf } R \) has a section, so that an element of \( \mathcal{F}(R) \) vanishes as soon as it vanishes on some smooth cover of \( \text{Spf } R \). It remains to prove surjectivity. Any element \( \sigma' \) of \( \mathcal{F}'(R) \) can be represented by an element \( \sigma \) of \( \mathcal{F}(\mathfrak{X}) \) where \( \mathfrak{X} \) is a non-empty smooth formal \( R \)-scheme and \( \sigma \) satisfies the gluing condition with respect to the smooth cover \( \mathfrak{X} \to \text{Spf } R \). If \( x \) is any point in \( \mathfrak{X}(R) \), then the image of \( x^*(\sigma) \in \mathcal{F}(R) \) by the morphism \( \mathcal{F}(R) \to \mathcal{F}'(R) \) equals \( \sigma' \).

(2) Taking sections of an abelian sheaf always defines a left exact functor; right exactness is proven by applying (1) to the sheafification of the image presheaf of a surjective morphism of sheaves. \( \square \)
PROPOSITION 1.3.2. Let $\mathcal{F}$ be an abelian presheaf on $(\text{Spf } R)_{\text{sm}}$, and denote by $\mathcal{F} \to \mathcal{F}'$ its sheafification.

1. If $\mathcal{F}$ is connected, then so is $\mathcal{F}'$.
2. The identity component $(\mathcal{F}')^0$ is the sheafification of $\mathcal{F}^0$.
3. The component sheaf $\Phi(\mathcal{F}')$ is the sheafification of the quotient presheaf $\mathcal{F}/\mathcal{F}^0$.

PROOF. (1) We know by Lemma 1.3.1 that the morphism $\mathcal{F}(R) \to \mathcal{F}'(R)$ is surjective, so that this is a direct consequence of Proposition 1.2.7 and the functoriality of the identity component.

(2) The sheafification $\mathcal{G}$ of $\mathcal{F}^0$ is a subsheaf of $\mathcal{F}'$, and by (1), it is contained in $(\mathcal{F}')^0$. We will show that $\mathcal{G} = (\mathcal{F}')^0$.

Step 1. Let $\mathcal{X}$ be a connected smooth formal $R$-scheme, let $\sigma$ be an element in $\mathcal{F}(\mathcal{X})$ and assume that the image $\tau$ of $\sigma$ in $(\mathcal{F}')^0(\mathcal{X})$. Assume, moreover, that there exists a point $x$ in $\mathcal{X}(R)$ such that $x^*\tau$ lies in $\mathcal{G}(R)$. We will prove that $\tau$ lies in $\mathcal{G}(\mathcal{X})$.

By Lemma 1.3.1, we can find an element $\rho$ in $\mathcal{F}^0(R)$ such that the image of $\rho$ in $(\mathcal{F}')^0(R)$ is equal to $x^*\tau$. We denote by $h$ the structural morphism $\mathcal{X} \to \text{Spf } R$, and we set

$$\sigma_0 = h^*(x^*\sigma - \rho)$$

in $\mathcal{F}(\mathcal{X})$. Then $\sigma_0$ lies in the kernel of $\mathcal{F} \to \mathcal{F}'$, so that we may assume that $x^*\sigma \in \mathcal{F}^0(R)$ by replacing $\sigma$ by $\sigma - \sigma_0$. This implies that $\sigma$ lies in $\mathcal{F}^0(\mathcal{X})$, by Proposition 1.2.4, so that $\tau$ lies in $\mathcal{G}(\mathcal{X})$.

Step 2. Let $\mathcal{X}$ be a connected smooth formal $R$-scheme, and let $\tau$ be an element of $(\mathcal{F}')^0(\mathcal{X})$. Assume that there exists a point $x_1$ in $\mathcal{X}(R)$ such that $x_1^*\tau$ lies in $\mathcal{G}(R)$.

We will prove that $\tau$ lies in $\mathcal{G}(\mathcal{X})$.

Let $x_2$ be a point of $\mathcal{X}(R)$. For each $i$ in $\{1, 2\}$, we can find a smooth morphism of connected formal $R$-schemes $\mathcal{Y}_i \to \mathcal{X}$ whose image contains $x_i$, and such that the restriction $\tau_i$ of $\tau$ to $\mathcal{Y}_i$ lifts to an element $\sigma_i$ of $\mathcal{F}(\mathcal{Y}_i)$. Since $\mathcal{X}$ is connected, the intersection of the images of $\mathcal{Y}_1 \to \mathcal{X}$ and $\mathcal{Y}_2 \to \mathcal{X}$ is non-empty, and thus contains an $R$-point $x_3$.

By Step 1, we know that $\tau_1$ and $\tau_2$ lies in $\mathcal{G}(\mathcal{Y}_1)$ and $\mathcal{G}(\mathcal{Y}_2)$, respectively. We can lift $x_1$ to a point $y_1$ in $\mathcal{Y}_1(R)$ and $y_2^*\tau_2 = x_1^*\tau$ in $\mathcal{G}(R)$. Then $x_3^*\tau$ lies in $\mathcal{G}(R)$, because $x_3$ lies in the image of $\mathcal{Y}_1 \to \mathcal{X}$. Again applying Step 1, we find that $\tau_3$ and $\tau_2$ lies in $\mathcal{G}(\mathcal{Y}_2)$. This means that $\tau$ is a section of $\mathcal{G}$ locally at every $R$-point of $\mathcal{X}$ with respect to the smooth topology, so that $\tau$ must lie in $\mathcal{G}(\mathcal{X})$.

Step 3. Now we prove that $\mathcal{G} = (\mathcal{F}')^0$. Step 2 implies at once that $\mathcal{G}(R) = (\mathcal{F}')^0(R)$: for every element $\rho$ of $((\mathcal{F}')^0(R))$, we can find a connected smooth formal $R$-scheme $\mathcal{X}$, an element $\tau$ in $((\mathcal{F}')^0(\mathcal{X}))$ and points $x_0, x_1$ in $\mathcal{X}(R)$ such that $x_0^*\tau = 0$ and $x_1^*\tau = \rho$. By Step 2, we know that $\tau$ lies in $\mathcal{G}(\mathcal{X})$, so that $\rho$ must lie in $\mathcal{G}(R)$. Now, again by Step 2, we see that $\mathcal{G}(\mathcal{Y}) = (\mathcal{F}')^0(\mathcal{Y})$ for every connected smooth formal $R$-scheme, which implies that $\mathcal{G} = (\mathcal{F}')^0$.

(3) This follows from (2) and exactness of the sheafification functor [Mi80 II.2.15].

LEMMA 1.3.3. Let $K'$ be a finite extension of $K$ with valuation ring $R'$, and let $\mathcal{X}$ be a smooth formal $R'$-scheme. Then we can cover $\mathcal{X}$ by open formal subschemes $\mathcal{U}$ with the following property: there exist a connected smooth formal $R$-scheme $\mathcal{Y}$
and a smooth surjective morphism of $R'$-schemes $h : \mathcal{Y} \times_R R' \to \mathfrak{U}$ such that, for every point $x$ of $\mathcal{U}(R')$, there exists a point $y$ in $\mathcal{Y}(R)$ whose image in $(\mathcal{Y} \times_R R')(R')$ is mapped to $x$ by the morphism $h$.

**Proof.** We denote by $k'$ the residue field of $R'$. This is a finite purely inseparable extension of the separably closed field $k$. We choose a basis $e_1, \ldots, e_d$ for the $R$-module $R'$.

Shrinking $X$, we may assume that $X$ is connected and admits an étale $R'$-morphism to

$$\mathcal{B} = \text{Spf } R'[X_1, \ldots, X_m],$$

for some integer $m \geq 0$. We consider the morphism of formal $R'$-schemes

$$g : \mathcal{B} \to \mathcal{Y} \times_R R',$$

defined by

$$X_i \mapsto \sum_{j=1}^d e_j X_{i,j}.$$

This morphism is clearly smooth and surjective. Moreover, over every $R'$-point of $\mathcal{B}$, we can find a point of $\mathcal{B}(R')$ whose $X_{i,j}$-coordinates lie in $R$.

We set

$$\mathcal{Y}' = \mathcal{Y} \times_R \mathcal{B}.$$

The second projection morphism

$h : \mathcal{Y}' \to X$

is smooth and surjective, and the first projection morphism

$$\mathcal{Y}' \to \mathcal{B}$$

is étale. Since the morphism

$$\text{Spec } R'/m^n R' \to \text{Spec } R/m^n$$

is finite, radical and surjective for every integer $n > 0$, the invariance of the étale site under such morphisms [SGA1, IX.4.10] implies that there exists an étale morphism of formal $R$-schemes

$$\mathcal{Y} \to \mathcal{B} = \text{Spf } R[X_1, \ldots, X_m]$$

together with an isomorphism of formal $\mathcal{B}$-schemes

$$\mathcal{Y}' \to \mathcal{Y} \times_R R'.$$

Now let $x$ be any point of $X(R')$. We will construct a point $y$ in $\mathcal{Y}(R) \subset \mathcal{Y}'(R')$ whose image in $X(R')$ is $x$. Let $b$ be a point of $\mathcal{B}(R)$ with the same image as $x$ in $\mathcal{B}(R')$. The couple $(x, b)$ defines a point $y$ in

$$\mathcal{Y}(R') = \mathcal{B}(R') \times_{\mathcal{B}(R')} X(R').$$

Since $\mathcal{Y} \to \mathcal{B}$ is étale, the reduction $y_0$ of $y$ in $\mathcal{Y}'(k')$ lies in $\mathcal{Y}(k)$, because the reduction of $b$ is $k$-rational and $k'$ is purely inseparable over $k$. Moreover, the point $b$ can be lifted in a unique way to a point $z$ of $\mathcal{Y}(R)$ whose reduction in $\mathcal{Y}(k)$ coincides with $y_0$. Repeating this uniqueness argument after base change to $R'$, we see that $z$ must coincide with $y$. In particular, $y$ lies in $\mathcal{Y}(R) \subset \mathcal{Y}'(R')$. $\square$
Lemma 1.3.4. Let $K'$ be a finite extension of $K$ with valuation ring $R'$. We denote the morphism $\text{Spf } R' \to \text{Spf } R$ by $h$. Then the functor $h_*$ from the category of smooth abelian sheaves on $\text{Spf } R'$ to the category of smooth abelian sheaves on $\text{Spf } R$ is exact.

Proof. A direct image functor such as $h_*$ is always left exact, because it has a right adjoint $h^*$. It follows from Lemma 1.3.3 that every smooth formal $R'$-scheme is Zariski-locally of the form $X \times_R R'$, for some smooth formal $R$-scheme $X$. This easily implies that $h_*$ is right exact. □

Proposition 1.3.5. Let $K'$ be a finite extension of $K$ with valuation ring $R'$. We denote the morphism $\text{Spf } R' \to \text{Spf } R$ by $h$. For every abelian sheaf $\mathcal{F}$ on $(\text{Spf } R')_{\text{sm}}$, the following properties hold.

1. If $\mathcal{F}$ is connected, then $h_* \mathcal{F}$ is connected.
2. The natural morphism $h_*(\mathcal{F}^o) \to h_* \mathcal{F}$ induces an isomorphism $h_*(\mathcal{F}^o) \to (h_* \mathcal{F})^o$.
3. The natural morphism $h_* \mathcal{F} \to h_* \Phi(\mathcal{F})$ induces an isomorphism $\Phi(h_* \mathcal{F}) \to h_* \Phi(\mathcal{F})$.

Proof. (1) Let $\sigma$ be an element of $(h_* \mathcal{F})(R) = \mathcal{F}(R')$. Since $\mathcal{F}$ is connected, we can find a connected smooth formal $R'$-scheme $\mathcal{X}$, a section $\tau$ in $\Phi(\mathcal{X})$ and points $x_0$, $x_1$ in $\mathcal{X}(R')$ such that $x^*_\tau = 0$ and $x^*_\tau(\sigma) = 0$. By Lemma 1.3.3 we can find for each $\mathcal{Y}_i$ an open neighbourhood $\mathcal{U}_i$, a connected smooth formal $R$-scheme $\mathcal{Y}_i$ and a smooth surjective morphism of formal $R'$-schemes

$\mathcal{Y}_i \times_R R' \to \mathcal{U}_i$

such that every $R'$-point on $\mathcal{U}_i$ lifts to an $R$-point of $\mathcal{Y}_i$. We choose for each $i$ a point $y_i$ on $\mathcal{Y}_i(R)$ whose image in $\mathcal{U}_i(R')$ is $x_i$. We write $\tau_i$ for the restriction of $\tau$ to $\mathcal{Y}_i \times_R R'$, and for the corresponding element of $(h_* \mathcal{F})(\mathcal{Y}_i)$.

Since $\mathcal{X}$ is smooth and connected, we can find a point $x_2$ of $\mathcal{X}(R')$ that lies in the intersection of $\mathcal{Y}_0$ and $\mathcal{Y}_1$. We can lift this point to a point $y_{2,i}$ in $\mathcal{Y}_i(R)$, for $i = 0, 1$. Then $y^*_{2,0} \tau_0$ lies in $(h_* \mathcal{F})^o(R)$ by Proposition 1.2.6 because $\mathcal{Y}_0$ is smooth and connected and $y^*_{0,0} \tau_0 = x^*_\tau = 0$. But

$y^*_{2,0} \tau_0 = x^*_\tau \tau = y^*_{2,1} \tau_1$

so that $y^*_{2,1} \tau_1$ and $y^*_{0,1} \tau_1 = \sigma$ must also lie in $(h_* \mathcal{F})^o(R)$.

(2) By (1) and left exactness of $h_*$, it is enough to show the following property: if $\mathcal{X}$ is a connected smooth formal $R$-scheme and $\sigma$ is an element of $(h_* \mathcal{F})^o(\mathcal{X})$, then the corresponding element of $\mathcal{F}(\mathcal{X} \times_R R')$ belongs to $\mathcal{F}^o(\mathcal{X} \times_R R')$. To prove this property, it suffices to consider the case $\mathcal{X} = \text{Spf } R$, which follows immediately from the definition of the identity component and the fact that $\mathcal{Y} \times_R R'$ is connected for every connected formal $R$-scheme $\mathcal{Y}$.

(3) This follows from (2) and right exactness of $h_*$. □

1.4. The trace map.
Let $K'$ be a finite separable extension of $K$, with valuation ring $R'$. Then we have a commutative diagram of morphisms of sites
\[
\begin{array}{ccc}
(\text{Sp } K')_{\text{sm}} & \xrightarrow{j'} & (\text{Sp } R')_{\text{sm}} \\
\downarrow h_K & & \downarrow h \\
(\text{Sp } K)_{\text{sm}} & \xrightarrow{j} & (\text{Sp } R)_{\text{sm}}.
\end{array}
\]

Let $F$ be an abelian sheaf on $(\text{Sp } K)_{\text{sm}}$. Since $(h_K)_*$ is left adjoint to $h_K^*$, we have a tautological morphism
\[
\tau : F \to (h_K)_*(h_K)^* F.
\]
Applying the functor $j_*$, this yields a morphism
\[
j_* F \to h_* j'_* h_K^* F
\]
of smooth abelian sheaves on $\text{Spf } R$. In [HN10, 2.3], we defined a trace map
\[
\text{tr} : (h_K)_* h_K^* F \to F
\]
such that the composition $\text{tr} \circ \tau$ is multiplication by $d = [K' : K]$. Applying the functor $j_*$ to $\text{tr}$, we obtain a morphism of smooth abelian sheaves
\[
h_* j'_* h_K^* F \cong j_*(h_K)_* h_K^* F \to j_* F
\]
on $\text{Spf } R$.

Now we apply the functor $\Phi(\cdot)$ to the morphisms \((1.4.3)\) and \((1.4.4)\). This yields morphisms of component groups
\[
\alpha : \Phi(F) \to \Phi(h_K^* F) \\
\text{tr} : \Phi(h_K^* F) \to \Phi(F),
\]
where we used Proposition \((1.3.3)\) to identify the component group of $h_* j'_* h_K^* F$ with $\Phi(h_K^* F)$. The composition $\text{tr} \circ \alpha$ is multiplication by $d$. In particular, for every smooth commutative rigid $K$-group, we obtain a trace map
\[
\Phi(G \times_K K') \to \Phi(G)
\]
such that the precomposition with the base change morphism
\[
\Phi(G) \to \Phi(G \times_K K')
\]
is multiplication by $d$ on $\Phi(G)$.

**Proposition 1.4.6.** Let $K'$ be a finite separable extension of $K$, and let $G$ be a smooth commutative rigid $K$-group. Then the kernel of the base change morphism
\[
\Phi(G) \to \Phi(G \times_K K')
\]
is killed by $[K' : K]$.

**Proof.** This in an immediate consequence of the existence of the trace map. \[\square\]
2. The index of a semi-abelian $K$-variety

2.1. Definition of the index.

**Proposition 2.1.1.** Let $G$ be a semi-abelian $K$-variety, denote by $G_{\text{spl}}$ its maximal split subtorus, and set $H = G/G_{\text{spl}}$. Then $\Phi(G_{\text{spl}})$ is a free $\mathbb{Z}$-module of rank $\rho_{\text{spl}}(G)$, and the sequence

$$0 \to \Phi(G_{\text{spl}}) \to \Phi(G) \to \Phi(H) \to 0$$

is exact. Moreover, $\Phi(H)$ is finite, and the rank of $\Phi(G)$ equals $\rho_{\text{spl}}(G)$.

**Proof.** It follows from the example in Section 3.4 of Chapter 2 that $\Phi(G_{\text{spl}})$ is free of rank $\rho_{\text{spl}}(G)$. Since $H$ is an extension of an abelian variety by an anisotropic torus, the Néron $\text{ft}$-model of $H$ is quasi-compact, so that $\Phi(H)$ is finite. There are no non-trivial morphisms of algebraic $K$-groups from $H$ to $\mathbb{G}_{m,K}$, so that the conditions of [BLR90, 10.1.7] are fulfilled. The exactness of (2.1.2) is shown in the proof of [BLR90, 10.1.7]. It follows that $\Phi(G)$ has rank $\rho_{\text{spl}}(G)$. \qed

(2.1.3) We keep the notations of Proposition 2.1.1. Since $\Phi(H)$ is finite, the injective morphism of free $\mathbb{Z}$-modules

$$\Phi(G_{\text{spl}}) \to \Phi(G)_{\text{free}}$$

has finite cokernel.

**Definition 2.1.4.** We define the index of $G$, denoted by $i(G)$, as

$$i(G) = |\text{coker}(\Phi(G_{\text{spl}}) \to \Phi(G)_{\text{free}})|.$$ 

(2.1.5) The torsion part of $\Phi(G)$ is isomorphic to the component group $\Phi(H)$ if and only if the index $i(G)$ is one. This happens, for instance, if $G$ is the product of $G_{\text{spl}}$ and $H$, but $i(G)$ can be different from one in general, as shown by the example below. In the next section, we will study the behaviour of the index under finite extensions of $K$.

2.2. Example: The index of a $K$-torus.

(2.2.1) Let $T$ be a $K$-torus. We will compute the index of $T$ in terms of the character module $X(T)$ of $T$. Let $K'$ be a splitting field of $T$, and put $\Gamma = \text{Gal}(K'/K)$. We consider the trace map

$$\text{tr} : X(T) \to X(T)^{\Gamma} : x \mapsto \sum_{g \in \Gamma} g \ast x.$$ 

By the proof of [HN10, 3.5], the maximal split subtorus $T_{\text{spl}}$ of $T$ has character module $X(T)/\ker(\text{tr})$, so that we have a canonical isomorphism

$$\Phi(T_{\text{spl}}) \cong (X(T)/\ker(\text{tr}))^\vee.$$ 

On the other hand, we can look at the maximal anisotropic subtorus $T_a$ of $T$. It has character module $X(T)/X(T)^{\Gamma}$, and the quotient $T/T_a$ is a split $K$-torus with character module $X(T)^{\Gamma}$. It is the dual of the maximal split subtorus of the dual torus of $T$. 

4. COMPONENT GROUPS AND NON-ARCHIMEDEAN UNIFORMIZATION

(2.2.2) We will see in Proposition 3.3.1 that the sequence
\[ \Phi(T_{a}) \longrightarrow \Phi(T) \longrightarrow \Phi(T/T_{a}) = (X(T)^{\Gamma})^{\vee} \longrightarrow 0 \]
is exact. Since \( \Phi(T_{a}) \) is finite, the induced morphism
\[ \Phi(T)_{\text{free}} \longrightarrow (X(T)^{\Gamma})^{\vee} \]
is an isomorphism. Thus the index \( i(T) \) of \( T \) is equal to the cardinality of the cokernel of the injective morphism of abelian groups
\[ X(T)^{\Gamma} \longrightarrow X(T)/\ker(\text{tr}) \]
that is induced by the inclusion \( X(T)^{\Gamma} \subset X(T) \).

(2.2.3) This index can be different from one. For instance, let \( K \) be the field \( \mathbb{C}((t)) \) of complex Laurent series and denote by \( K' \) the degree two Galois extension \( \mathbb{C}((\sqrt{t})) \) of \( K \). Let \( T \) be the torus corresponding to the character module \( X(T) = \mathbb{Z}^{2} \) with an action of \( \text{Gal}(K'/K) \cong \mathbb{Z}[2](\mathbb{C}) \) given by
\[ (-1)^{*}v = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot v. \]
Then \( X(T)^{\Gamma} \) is the submodule of \( \mathbb{Z}^{2} \) generated by \((1, 1)\) and \( \ker(\text{tr}) \) is the submodule of \( \mathbb{Z}^{2} \) generated by \((1, -1)\), so that \( i(T) = 2 \).

3. Component groups and base change

3.1. Uniformization of semi-abelian varieties.

(3.1.1) Our aim is to study the behaviour of the torsion part of the component group of a semi-abelian \( K \)-variety \( G \) under finite extensions of the field \( K \). To extend the results for abelian varieties from [HN10], we need a suitable notion of uniformization for \( G \).

(3.1.2) Let
\[ 0 \rightarrow M \rightarrow E^{\text{an}} \rightarrow (G_{ab})^{\text{an}} \rightarrow 0 \]
be the uniformization of the abelian \( K \)-variety \( G_{ab} \), with \( E \) a semi-abelian \( K \)-variety with potential good reduction and \( M \) an étale lattice in \( E \) of rank \( \rho(E) \); see Chapter 2 (3.7.2). We put
\[ H = E^{\text{an}} \times (G_{ab})^{\text{an}} G_{G}^{\text{an}}. \]
This is a smooth rigid \( K \)-group that fits into the short exact sequences of rigid \( K \)-groups
\[ 0 \rightarrow M \rightarrow H \rightarrow G^{\text{an}} \rightarrow 0, \]
and
\[ 0 \rightarrow G_{\text{tor}}^{\text{an}} \rightarrow H \rightarrow E^{\text{an}} \rightarrow 0. \]
We will call the sequence (3.1.3) the non-archimedean uniformization of \( G \).
It is important to keep in mind that $H$ is usually not algebraic, so that the theory of Néron lft-models in [BLR90] cannot be applied. One can deduce from [BS95, 1.2] and Proposition 3.2.4 below that the rigid $K$-group $H$ admits a quasi-compact formal Néron model in the sense of [BS95, 1.1] if and only if the split reductive ranks of $G_{tor}$ and $E$ are zero. It seems plausible that $H$ always has a formal Néron model, but we will not prove this. Instead, we will work with the smooth sheaf associated to $H$ and use the theory of component groups of smooth abelian sheaves developed in Section 1.

### 3.2. Bounded rigid varieties and torsors under analytic tori.

**Proposition 3.2.1.** Let $T$ be a split algebraic $K$-torus with Néron lft-model $\mathcal{T}$. Let $\mathfrak{H}$ be a smooth connected formal $R$-scheme, and let $G$ be a $T_{an}$-torsor on $H = \mathfrak{H}_K$. Let $g$ be a point of $G(K)$, and let $c$ be an element of the component group $\Phi(T)$. Then there exists a unique map

$$
\psi: G(K) \to \Phi(T)
$$

with the following properties.

1. We have $\psi(g) = c$.
2. The map $\psi$ is equivariant with respect to the action of $T(K)$ on $G(K)$ and $\Phi(T) = T(K)/\mathcal{T}^\circ(R)$.
3. For every smooth connected formal $R$-scheme $\mathfrak{X}$, every $R$-morphism $\mathfrak{X} \to \mathfrak{H}$ and every morphism of rigid $H$-varieties $f: \mathfrak{X}_K \to G$,

$$
\psi \circ f: \mathfrak{X}_K(K) \to \Phi(T)
$$

is constant.

Moreover, for every finite subset $\Phi_0$ of $\Phi(T)$, the set $\psi^{-1}(\Phi_0)$ is contained in a quasi-compact open subvariety of $G$.

**Proof.** We may assume that $c = 0$, since we can always compose $\psi$ with a translation. By [BX96, 4.2], we can cover $\mathfrak{H}$ by open formal subschemes $\mathfrak{U}$ such that the torsor $G$ is trivial over $\mathfrak{U}_K$. Any intersection of such opens $\mathfrak{U}$ will contain a $K$-valued point in its generic fiber, because $\mathfrak{H}$ is smooth and connected. Thus we may assume that $G = H \times_K T_{an}$. Let $\mathcal{T}$ be the Néron lft-model of $T$ and denote for each $t$ in $T(K)$ by $\overline{t}$ the residue class of $t$ in $\Phi(T) = T(K)/\mathcal{T}^\circ(R)$. Then the map

$$
\psi: G(K) = H(K) \times T(K) \to \Phi(T): (h, t) \mapsto \overline{t}
$$

is clearly the unique map satisfying properties (1)-(3) in the statement. If we denote by $\mathcal{T}_0$ the union of the connected components of $\mathcal{T}$ that belong to $\Phi_0$, and by $T_0$ the generic fiber of the $m$-adic completion of $\mathcal{T}_0$, then $T_0$ is a quasi-compact open subvariety of $T_{an}$ and the set $\psi^{-1}(\Phi_0)$ is contained in the quasi-compact open subvariety $H \times_K T_0$ of $G$. 

**Definition 3.2.2.** We say that a rigid $K$-variety $X$ is bounded if $X$ admits a quasi-compact open rigid subvariety that contains all the $K$-rational points of $X$.
(3.2.3) It follows from [BS95 1.2] that a smooth rigid $K$-group is bounded if and only if it admits a quasi-compact formal Néron model in the sense of [BS95 1.1]. If $G$ is a semi-abelian $K$-variety, then $G^{an}$ is bounded if and only if it is the extension of an abelian $K$-variety by an anisotropic $K$-torus.

**Proposition 3.2.4.** Let $H$ be a bounded smooth rigid $K$-variety and let $T$ be an anisotropic $K$-torus. Then every $T^{an}$-torsor $G$ on $H$ is bounded.

**Proof.** Let $\mathcal{H}$ be a formal weak Néron model of $H$. Recall that this means that $\mathcal{H}$ is a quasi-compact smooth formal $R$-scheme, endowed with an open immersion of rigid $K$-varieties $\mathcal{H}_K \to H$ whose image contains all the $K$-rational points of $H$. Every bounded smooth rigid $K$-variety admits a formal weak Néron model, by [BS95 3.3] (recall that we assume all rigid $K$-varieties to be quasi-separated). If we let $\mathcal{U}$ run through the connected components of $\mathcal{H}$, then the set $H(K)$ is covered by the quasi-compact open subvarieties $\mathcal{U}_K$ of $H$. Thus it suffices to prove the result after base change from $H$ to any of these subvarieties $\mathcal{U}_K$. Therefore, we may assume that $H$ has a connected smooth formal $R$-model.

We may also assume that there exists a $K$-rational point $x$ on $G$, since $G$ is obviously bounded if $G(K)$ is empty. Let $K'$ be a finite Galois extension of $K$ such that $T' = T \times_K K'$ is split, let $R'$ be the integral closure of $R$ in $K'$, and set $G' = G \times_K K'$ and $H' = H \times_K K'$. Then $H'$ has a connected smooth formal $R'$-model, and $G'$ is a $(T')^{an}$-torsor over $H'$. Let

$$\psi : G'(K') \to \Phi(T')$$

be the function from Proposition 3.2.1 such that $\psi(x) = 0$. We will prove that $\psi$ sends each element of $G(K) \subset G'(K')$ to 0. Then $G(K)$ is contained in a quasi-compact open subvariety of $G'$, by Proposition 3.2.1, and this implies that $G$ is bounded.

The canonical isomorphism

$$\Phi(T') \cong X(T)^\vee \otimes_\mathbb{Z} (K')^*/(R')^*$$

from (3.4.3) in Chapter 2 is equivariant with respect to the Galois action of $\text{Gal}(K'/K)$. The Galois action on the value group $(K')^*/(R')^*$ is trivial, and $X(T)^{\text{Gal}(K'/K)} = 0$ because $T$ is anisotropic. Thus $\Phi(T')^{\text{Gal}(K'/K)} = 0$. On the other hand, the uniqueness of the map

$$\psi : G'(K') \to \Phi(T')$$

easily implies that it is $\text{Gal}(K'/K)$-equivariant: if $\sigma$ is an element of $\text{Gal}(K'/K)$, then $\psi^\sigma := \sigma \circ \psi \circ \sigma^{-1}$ maps $x$ to 0 and still satisfies properties (2) and (3) in Proposition 3.2.1. Thus $\psi$ maps each element of $G(K)$ to an element of $\Phi(T')^{\text{Gal}(K'/K)} = 0$. □

**3.3. Behaviour of the component group under base change.**

**Proposition 3.3.1.** Assume that $k$ is algebraically closed. Let

$$0 \to T \to G \to H \to 0$$

be a short exact sequence of semi-abelian $K$-varieties, such that $T$ is a torus. Then the sequence of component groups

$$(3.3.2) \quad \Phi(T) \to \Phi(G) \to \Phi(H) \to 0$$

is exact, and the kernel of $\Phi(T) \to \Phi(G)$ is torsion.
Proof. We already proved exactness of the sequence (3.3.2) in Proposition 1.1.3, so it remains to prove that the kernel of $\Phi(T) \to \Phi(G)$ is torsion. It is enough to show that

$$\text{rank}(\Phi(G)) = \text{rank}(\Phi(T)) + \text{rank}(\Phi(H)).$$

This follows from Proposition 2.1.1 and Lemma 2.2.3 in Chapter 2. □

Corollary 3.3.3. Let $G$ be a semi-abelian $K$-variety, and let $L$ be a finite separable extension of $K$ such that $G \times_K L$ has good reduction. Then $\Phi(G)_{\text{tors}}$ is killed by $[L : K]$.

Proof. The torus $G_{\text{tor}} \times_K L$ is split, and the abelian variety $G_{\text{ab}} \times_K L$ has good reduction. Thus it follows from Proposition 3.3.1 that $\Phi(G \times_K L)$ is free, so that $\Phi(G)_{\text{tors}}$ must be contained in the kernel of the base change morphism

$$\Phi(G) \to \Phi(G \times_K L).$$

This kernel is killed by $[L : K]$, by Proposition 1.4.6. □

(3.3.4) We will need the following generalization of Proposition 1.1.3. The main example to keep in mind is the following: $H$ is the analytification of a semi-abelian $K$-variety, and $G$ is an analytic extension of $H$ by an algebraic $K$-torus $T$. Then for every finite separable extension $L$ of $K$, the rigid $K$-group $H \times_K L$ has a formal Néron model in the sense of \cite[1.1]{BS95}, by \cite[6.2]{BS95}.

Proposition 3.3.5. Assume that $k$ is algebraically closed. Let

$$0 \to T^\text{an} \to G \to H \to 0$$

be an exact sequence of smooth commutative rigid $K$-groups, where $T^\text{an}$ is the rigid analytification of an algebraic $K$-torus $T$. Assume that $H$ has a formal Néron model and that the abelian sheaf $G/T^\text{an}_{\text{sp}}$ on $(\text{Sp} K)_{\text{sm}}$ is representable by a smooth commutative rigid $K$-group $G'$. Then the following properties hold.

1. The sequence of component groups

$$0 \to \Phi(T) \to \Phi(G) \to \Phi(H) \to 0$$

is exact.

2. If $H \times_K L$ admits a formal Néron model for every finite separable extension $L$ of $K$, then the kernel of $\Phi(T) \to \Phi(G)$ is finite.

Proof. The main problem is that we no longer know if $G$ admits a formal Néron model, so that we cannot apply the arguments in the proof of Proposition 3.3.1 in a direct way. Instead of trying to prove the existence of a formal Néron model for $G$, we will adapt the arguments of Proposition 3.3.1 to the Néron model $j_*G$ of the smooth sheaf $G$ — see (1.2.3).

11 The sequence (3.3.6) defines an exact sequence of étale sheaves on $\text{Sp} K$, and

$$H^1_{\text{et}}(\text{Sp} K, T^\text{an}) = H^1(K, T) = 0$$

by \cite[4.3]{Ch00}. It follows that the sequence

$$0 \to T(K) \to G(K) \to H(K) \to 0$$

is exact. We denote by \( \mathcal{G} = j_*G \) and \( \mathcal{H} = j_*H \) the Néron models of \( G \) and \( H \) on \((\text{Spf } R)_{\text{sm}}\). Looking at the proof of Proposition 1.1.3 and applying Proposition 1.2.7, we see that it is enough to show that the map

\[
\mathcal{G}^o(R) \to \mathcal{H}^o(R)
\]

is surjective.

First, we reduce to the case where \( T_{\text{an}} \) is bounded. It is shown in \cite{BX96} (split case) that \( R^i j_* T_{\text{an}}^{\text{spl}} = 0 \). Thus, applying the functor \( j_* \) to the exact sequence

\[
0 \to T_{\text{an}}^{\text{spl}} \to G \to G' \to 0
\]

of abelian sheaves on \((\text{Sp } K)_{\text{sm}}\), we get an exact sequence

\[
0 \to j_* T_{\text{an}}^{\text{spl}} \to \mathcal{G} \to \mathcal{G}' \to 0
\]

of abelian sheaves on \((\text{Spf } R)_{\text{sm}}\). In particular, \( \mathcal{G} \to \mathcal{G}' \) is an epimorphism. Then \cite{BX96} implies that \( \mathcal{G}^o \to (\mathcal{G}')^o \) is an epimorphism, as well. It follows from Lemma 1.3.1 that \( \mathcal{G}^o(R) \to (\mathcal{G}')^o(R) \) is surjective. Therefore, it is enough to prove that the map \((\mathcal{G}')^o(R) \to \mathcal{H}^o(R)\) is surjective, so that we may replace the sequence \((3.3.6)\) by the exact sequence

\[
0 \to (T/T_{\text{an}})_{\text{an}}^{\text{an}} \to G' \to H \to 0
\]

of abelian sheaves on \((\text{Sp } K)_{\text{sm}}\). This means that we may assume that \( T_{\text{spl}} \) is trivial, and thus that \( T_{\text{an}} \) is bounded. Then \( G' = G \).

Now, we reduce to the case where \( H \) is quasi-compact. By our assumptions, the Néron model \( \mathcal{H} \) of \( H \) is representable by a quasi-compact smooth formal \( R \)-scheme, that we denote again by \( \mathcal{H} \). We denote by \( \mathcal{H}^o \) its identity component. The generic fiber of \( \mathcal{H}^o \) is a quasi-compact open rigid subgroup of \( H \), which we denote by \( H^o \). It is easily seen that \( \mathcal{H}^o \) is a formal Néron model for \( H^o \).

The inverse image \( \tilde{G} \) of \( H^o \) in \( G \) is an open rigid subgroup of \( G \) that fits into a short exact sequence of smooth rigid \( K \)-groups

\[
0 \to T_{\text{an}} \to \tilde{G} \to H^o \to 0.
\]

Since the functor \( j_* \) commutes with fibred products, the Néron model \( \tilde{\mathcal{G}} = j_* \tilde{G} \) is isomorphic to \( \mathcal{G} \times \mathcal{H}^o \). Clearly, the morphism \( \tilde{\mathcal{G}} \to \mathcal{G} \) induces an isomorphism between the respective identity components. Thus we may assume that \( H = H^o \) and \( G = \tilde{G} \). In particular, we may assume that \( H \) is quasi-compact.

We’ve reduced to the case where both \( T_{\text{an}} \) and \( H \) are bounded. Then \( G \) is bounded by Proposition 3.2.4 so that its Néron model \( \tilde{\mathcal{G}} \) is represented by a quasi-compact smooth formal \( R \)-scheme \cite{BS95} 1.2. Exactness of \((3.3.8)\) implies that the natural map \( \mathcal{G}(R) \to \mathcal{H}(R) \) is surjective. Then it follows from \cite{BLR90} 9.6.2 that \( \mathcal{G}^o(R) \to \mathcal{H}^o(R) \) is surjective, which is what we wanted to prove. To be precise, \cite{BLR90} 9.6.2 is formulated for algebraic schemes, but the proof of \cite{BLR90} 9.6.2 immediately carries over to the formal scheme case, since it only involves the Greenberg schemes of the algebraic schemes \( \mathcal{G} \times_R (R/m^n) \) and \( \mathcal{H} \times_R (R/m^n) \) for \( n > 0 \).
Let $L$ be a finite separable extension of $K$ such that the torus $T \times_K L$ is split. Then the square

\[
\begin{array}{ccc}
\Phi(T) & \longrightarrow & \Phi(G) \\
\downarrow & & \downarrow \\
\Phi(T \times_K L) & \longrightarrow & \Phi(G \times_K L)
\end{array}
\]

commutes, and the kernel of $\Phi(T) \to \Phi(T \times_K L)$ is torsion by [HN10, 5.3]. Thus it is enough to show that $\Phi(T \times_K L) \to \Phi(G \times_K L)$ is injective. Therefore, we may assume that $T$ is split.

We set $\tilde{G} = G \times_H H^0$ and $\tilde{\mathcal{G}} = j_* \mathcal{G}$ as in (1). Then $\tilde{\mathcal{G}}$ is a subsheaf of $\mathcal{G}$ and these sheaves have the same identity component. It follows that $\Phi(\tilde{G})$ is a subgroup of $\Phi(G)$ that contains the image of $\Phi(T)$ in $\Phi(G)$. Thus replacing $H$ by $H^0$ and $G$ by $\tilde{G}$ does not affect the kernel of $\Phi(T) \to \Phi(G)$. Therefore, we may assume that $H = H^0$. In particular, $H$ admits a smooth connected formal $R$-model.

We denote by $\psi$ the function associated to the $T^\text{an}$-torsor $G \to H$ as in Proposition 3.2.1, normalized by $\psi(e_G) = 0$ (here $e_G$ denotes the identity point of $G$). Let $t$ be an element of $T(K)$ whose class $t$ in $\Phi(T) = T(K)/T^\text{o}(R)$ belongs to the kernel of $\Phi(T) \to \Phi(G)$. We denote again by $t$ the image of $t$ in $G(K)$. Then it follows from property (2) in Proposition 3.2.1 that $\psi(t) = \tilde{7}$. Thus it is enough to show that $\psi(t) = 0$.

By definition of $(j_* G)^\circ$, there exist a smooth connected formal $R$-scheme $\mathfrak{X}$, a morphism of rigid $K$-varieties

\[f : \mathfrak{X}_K \to G\]

and elements $x_0$ and $x_1$ of $\mathfrak{X}_K(K)$ such that $f$ maps $x_0$ to $e_G$ and $x_1$ to $t$. By the universal property of the formal Néron model, the induced morphism $\mathfrak{X}_K \to H$ extends uniquely to an $R$-morphism $\mathfrak{X} \to \mathcal{H}$. Thus we can apply property (3) of Proposition 3.2.1 and we see that $\psi(t) = \psi(e_G) = 0$, as required. □

(3.3.9) Let $e$ be a positive integer, and let $G$ be an abelian $K$-variety. We denote by $B$ the abelian $K$-variety with potential good reduction that appears in the non-archimedean uniformization of $G$ (see (3.7.2) in Chapter 2). Then we consider the following condition on the couple $(G, e)$:

For every finite separable extension $K'$ of $K$ of degree prime to $e$, the base change morphism $\Phi(B) \to \Phi(B \times_K K')$ is an isomorphism.

(3.3.10) This condition is satisfied, for instance, in each of the following cases:

- $G$ has potential multiplicative reduction and $e$ is any positive integer (in this case, $B$ is trivial);
- $G$ is tamely ramified and $e$ is a multiple of the degree of the minimal extension $L$ of $K$ such that $G \times_K L$ has semi-abelian reduction [HN10, 5.5];
- $B$ is isomorphic to the Jacobian of a smooth projective $K$-curve $C$ of index one, and $e$ is a multiple of the stabilization index $e(C)$ (Proposition 3.1.1 in Chapter 3).
Moreover, if \((G, e)\) and \((G', e')\) are two couples satisfying \([3.3.9]\), then the couple \\
\((G \times_K G', \operatorname{lcm}\{e, e'\})\) also satisfies \([3.3.9]\), since the formation of Néron models commutes with finite products.

**Proposition 3.3.11.** Assume that \(k\) is algebraically closed. Let \(G\) be a semi-abelian \(K\)-variety, with toric part \(G_{\text{tor}}\) and abelian part \(G_{\text{ab}}\). Suppose that \(G_{\text{ab}}\) has potential good reduction, and denote by \(L\) the minimal extension of \(K\) in \(K^s\) such that \(G \times_K L\) has semi-abelian reduction. Then the group \(\Phi(G)_{\text{tors}}\) is killed by \([L : K]\). For every finite separable extension \(K'\) of \(K\) of degree \(d\) prime to \([L : K]\), the base change morphism \\
\(\alpha : \Phi(G) \rightarrow \Phi(G \times_K K')\)

is injective.

Let \(e > 0\) be a multiple of \([L : K]\) such that \((G_{\text{ab}}, e)\) satisfies condition \([3.3.9]\). If \(d = [K' : K]\) is prime to \(e\), then image of the morphism \\
\(\alpha_{\text{free}} : \Phi(G)_{\text{free}} \rightarrow \Phi(G \times_K K')_{\text{free}}\)

equals \(d \cdot \Phi(G \times_K K')_{\text{free}}\), and the morphism \\
\(\alpha_{\text{tors}} : \Phi(G)_{\text{tors}} \rightarrow \Phi(G \times_K K')_{\text{tors}}\)

is an isomorphism. In particular, the cokernel of \(\alpha\) is isomorphic to \((\mathbb{Z}/d\mathbb{Z})^{\rho_{\text{spl}}(G)}\).

**Proof.** For notational convenience, we’ll denote by \((\cdot)'\) the base change functor from \(K\) to \(K'\). In particular, \(G' = G \times_K K'\). We proved the injectivity of \(\alpha\) in \([HN10\text{, 5.5}]\), and we showed in Corollary \([3.3.8]\) that \(\Phi(G)_{\text{tors}}\) is killed by \([L : K]\). Since, by Proposition \([3.3.5]\), \(G'\) acquires semi-abelian reduction on the degree \([L : K]\) extension \(K' \otimes_K L\) of \(K'\), Corollary \([3.3.9]\) also implies that \(\Phi(G')_{\text{tors}}\) is killed by \([L : K]\). We prove the remainder of the theorem by considering the following cases.

**Case 1:** \(G\) is a split torus. This case was discussed in Example \([3.4]\) in Chapter \([2]\). Recall that \(\Phi(G)\) and \(\Phi(G')\) are free \(\mathbb{Z}\)-modules of rank \(\rho_{\text{spl}}(G) = \dim G\).

**Case 2:** \(\rho_{\text{spl}}(G) = 0\). In this case, we need to show that \(\alpha\) is an isomorphism. We have a commutative diagram \\
\[
\begin{array}{cccccc}
\Phi(G_{\text{tor}}) & \longrightarrow & \Phi(G) & \longrightarrow & \Phi(G_{\text{ab}}) & \longrightarrow & 0 \\
\downarrow{\beta} & & \downarrow{\alpha} & & \downarrow{\gamma} & & \\
\Phi(G'_{\text{tor}}) & \longrightarrow & \Phi(G') & \longrightarrow & \Phi(G'_{\text{ab}}) & \longrightarrow & 0
\end{array}
\]
whose rows are exact by Proposition \([1.1.3]\) and such that all vertical morphisms are injective. Moreover, \(\beta\) and \(\gamma\) are isomorphisms, by \([HN10\text{, 5.5}]\) and condition \([3.3.9]\) for \((G_{\text{ab}}, e)\). A straightforward diagram chase shows that \(\alpha\) is an isomorphism.

**Case 3:** General case. Set \(H = G/G_{\text{spl}}\) and consider the commutative diagram \\
\[
\begin{array}{cccccc}
0 & \longrightarrow & \Phi(G_{\text{spl}}) & \longrightarrow & \Phi(G) & \longrightarrow & \Phi(H) & \longrightarrow & 0 \\
\downarrow{\beta} & & \downarrow{\alpha} & & \downarrow{\gamma} & & \\
0 & \longrightarrow & \Phi(G'_{\text{spl}}) & \longrightarrow & \Phi(G') & \longrightarrow & \Phi(H') & \longrightarrow & 0
\end{array}
\]

The rows are exact by Proposition 3.3.1 and the fact that \( \Phi(G_{\text{spl}}) \) and \( \Phi(G'_{\text{spl}}) \) are free \( \mathbb{Z} \)-modules. By Lemma 3.3.3, we know that \( H \times_K L \) has semi-abelian reduction. Thus Case 2 implies that \( \gamma \) is an isomorphism. By the Snake Lemma, \( \delta \) induces an isomorphism between the cokernels of \( \beta \) and \( \alpha \). It follows from Case 1 that the cokernel of \( \beta \), and thus the cokernel of \( \alpha \), is isomorphic to \( (\mathbb{Z}/d\mathbb{Z})_{\text{tors}}(G) \).

By Proposition 2.1.1, the \( \mathbb{Z} \)-module \( \Phi(G)_{\text{free}} \) has rank \( \rho_{\text{spl}}(G) \). Since \( d \) is prime to \( e \) and \( \Phi(G)_{\text{tors}} \) and \( \Phi(G')_{\text{tors}} \) are killed by \( [L : K] \), the morphism \( \alpha_{\text{tors}} \) must be an isomorphism (it is injective because \( \alpha \) is injective, and its cokernel is killed by both \( d \) and \( e \)). It follows that the natural morphism \( \text{coker}(\alpha) \to \text{coker}(\alpha_{\text{free}}) \) is an isomorphism, so that the image of \( \alpha_{\text{free}} \) equals \( d \cdot \Phi(G')_{\text{free}} \).

**Proposition 3.3.12.** Assume that \( k \) is algebraically closed. Let \( C \) be a smooth projective \( K \)-curve of index one with Jacobian \( G \). We denote by \( L \) the minimal extension of \( K \) in \( K^* \) such that \( G \times_K L \) has semi-abelian reduction. If \( e \) is a multiple of both \( [L : K] \) and the stabilization index \( e(C) \), then \( (G, e) \) satisfies condition 3.3.9.

**Proof.** Let \( K' \) be a finite separable extension of \( K \) of degree \( d \) prime to both \( [L : K] \) and to \( e(C) \). Let \( E^\text{an} \to G^\text{an} \) be the non-archimedean uniformization of \( G \), and denote by \( (\cdot)' \) the base change functor from \( K \) to \( K' \). The proof of [HN10 5.7] shows that the base change morphisms \( \Phi(G) \to \Phi(G') \) and \( \Phi(E) \to \Phi(E') \) have isomorphic cokernels (in that proof, the assumption that \( A \) is tamely ramified or has potential multiplicative reduction is only used at the very end, to apply [HN10 5.6]). By Proposition 3.3.1 this implies that the cokernel of \( \Phi(E) \to \Phi(E') \) has cardinality \( d^{e(G)} \).

We denote by \( T \) and \( B \) the toric, resp. abelian part of \( E \). Consider the commutative diagram

\[
\begin{array}{ccc}
\Phi(T) & \xrightarrow{\delta} & \Phi(E) \\
\alpha \downarrow & & \beta \downarrow \\
\Phi(T') & \xrightarrow{\varepsilon} & \Phi(E')
\end{array}
\]

The rows are exact by Proposition 1.1.3, the vertical morphisms are injective by [HN10 5.7], and the cokernel of \( \alpha \) also has cardinality \( d^{e(G)} = d^{e(T)} \) by [HN10 5.6]. The kernels of \( \delta \) and \( \varepsilon \) are torsion by Proposition 3.3.1 and thus killed by \( [L : K] \), by Corollary 3.3.3. But \( d \) is prime to \( [L : K] \), and, moreover, \( \alpha_{\text{tors}} \) is an isomorphism by Proposition 3.3.11. Thus \( \alpha \) maps \( \text{ker}(\delta) \) surjectively onto \( \text{ker}(\varepsilon) \), and the Snake Lemma implies that \( \gamma \) is an isomorphism.

**Theorem 3.3.13.** Assume that \( k \) is algebraically closed. Let \( G \) be a semi-abelian \( K \)-variety, with toric part \( G_{\text{tor}} \) and abelian part \( G_{\text{ab}} \). Let \( L \) be the minimal extension of \( K \) in \( K^* \) such that \( G \times_K L \) has semi-abelian reduction. Let \( K' \) be a finite separable extension of \( K \) of degree \( d \).

1. If \( d \) is prime to \( [L : K] \), then the base change morphism

\[
\alpha : \Phi(G) \to \Phi(G \times_K K')
\]

is injective.
Let $e$ be a multiple of $[L : K]$ in $\mathbb{Z}_{>0}$ such that $(G_{ab}, e)$ satisfies condition (3.3.19). If $d$ is prime to $e$, then the cokernels of the morphisms

$$\begin{align*}
\alpha & : \Phi(G) \rightarrow \Phi(G \times_K K') \\
\alpha_{\text{tors}} & : \Phi(G)_{\text{tors}} \rightarrow \Phi(G \times_K K')_{\text{tors}} \\
\alpha_{\text{free}} & : \Phi(G)_{\text{free}} \rightarrow \Phi(G \times_K K')_{\text{free}}
\end{align*}$$

are isomorphic to $(\mathbb{Z}/d\mathbb{Z})^{\ell(G)}$, resp. $(\mathbb{Z}/d\mathbb{Z})^{\ell(G_{ab})}$, resp. $(\mathbb{Z}/d\mathbb{Z})^{\rho_{\text{sp}}(G)}$.

**Proof.** To simplify notation, we'll denote by $(\cdot)'$ the base change functor from $K$ to $K'$. First, we prove (1). It follows from Proposition 1.4.6 that the commutative diagram

$$
\begin{CD}
\Phi(G_{\text{tor}}) @>{\delta}>> \Phi(G) @>{\epsilon}>> \Phi(G_{ab}) @>>> 0 \\
@V{\beta}VV @V{\alpha}VV @V{\gamma}VV \\
\Phi(G'_{\text{tor}}) @>{\delta'}>> \Phi(G') @>{\epsilon'}>> \Phi(G'_{ab}) @>>> 0
\end{CD}
$$

has exact rows. Let $x$ be an element of $\Phi(G)$ such that $\alpha(x) = 0$. Then $dx = 0$ by Proposition 1.4.6. Moreover, $\epsilon(x) = 0$ because $\gamma$ is injective [HN10 5.7], so that $x$ lifts to an element $\tilde{x}$ in $\Phi(G_{\text{tor}})$. This element $\tilde{x}$ must be torsion, because the kernel of $\delta$ is torsion by Proposition 3.3.1. But the torsion part of $\Phi(G_{\text{tor}})$ is killed by $[L : K]$ (Corollary 3.3.3), so that $x = 0$ since $d$ is prime to $[L : K]$. Hence, $\alpha$ is injective.

Now, we prove (2). The restriction of $\beta$ to $\ker(\delta)$ is a surjection onto $\ker(\delta')$, since $\alpha$ is injective, the kernel of $\delta'$ is torsion and $\beta$ is an isomorphism on torsion parts (Proposition 3.3.11). Thus it follows from the Snake Lemma that the sequence

$$0 \rightarrow \text{coker}(\beta) \rightarrow \text{coker}(\alpha) \rightarrow \text{coker}(\gamma) \rightarrow 0$$

is exact. Applying Proposition 3.3.11 to $G_{\text{tor}}$, we see that the cokernel of $\beta$ is isomorphic to $(\mathbb{Z}/d\mathbb{Z})^{\rho_{\text{sp}}(G)}$ (note that $\rho_{\text{sp}}(G_{\text{tor}}) = \rho_{\text{sp}}(G)$).

Assume for a moment that we can prove that the cokernel of $\alpha_{\text{tors}}$ is isomorphic to $(\mathbb{Z}/d\mathbb{Z})^{\ell(G_{ab})}$. Replacing $G$ by $G_{ab}$, this also implies that the cokernel of $\gamma$ is isomorphic to $(\mathbb{Z}/d\mathbb{Z})^{\ell(G_{ab})}$, because the component group of an abelian variety is finite. Moreover, the cokernel of $\alpha$ is killed by $d$, by the existence of the trace map (1.4.2), and this implies that the short exact sequence (3.3.14) is split. Thus $\text{coker}(\alpha)$ is isomorphic to

$$\text{coker}(\beta) \oplus \text{coker}(\gamma) \cong (\mathbb{Z}/d\mathbb{Z})^{\rho_{\text{sp}}(G)} \oplus (\mathbb{Z}/d\mathbb{Z})^{\ell(G_{ab})} \cong (\mathbb{Z}/d\mathbb{Z})^{\ell(G)}$$

where the equality $\rho_{\text{sp}}(G) + t(G_{ab}) = t(G)$ follows from Corollary 3.2.5. Another application of the Snake Lemma shows that the natural morphism $\text{coker}(\alpha_{\text{tors}}) \rightarrow \text{coker}(\alpha)$ is injective, and that its cokernel is isomorphic to $\text{coker}(\alpha_{\text{free}})$. This implies that $\text{coker}(\alpha_{\text{free}})$ is isomorphic to $(\mathbb{Z}/d\mathbb{Z})^{\rho_{\text{sp}}(G)}$.

Thus it suffices to determine the cokernel of the morphism

$$\alpha_{\text{tors}} : (\Phi(G))_{\text{tors}} \rightarrow (\Phi(G'))_{\text{tors}}.$$ 

We consider the non-archimedean uniformization

$$0 \rightarrow M \rightarrow H \rightarrow G_{\text{an}} \rightarrow 0$$
of $G$ as in [3.12]. We set $I = \text{Gal}(K^*/K)$ and $I' = \text{Gal}(K^*/K')$. Like in the proof of [HN10 5.7], we get a commutative diagram with exact rows

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & M^I & \longrightarrow & \Phi(H) & \longrightarrow & \Phi(G) & \longrightarrow & H^1(I, M) \\
\alpha_1 \downarrow \iota & & \alpha_2 \downarrow & & \alpha_3 \downarrow & & \iota \downarrow \alpha_4 & & \\
0 & \longrightarrow & M^{I'} & \longrightarrow & \Phi(H') & \longrightarrow & \Phi(G') & \longrightarrow & H^1(I', M) \\
\beta_3 \downarrow & & \iota \downarrow \beta_4 & & & & & & \\
\Phi(G) & \longrightarrow & \tilde{\gamma}_1 & \longrightarrow & H^1(I, M) \\
\end{array}
\]

where $\tilde{\beta}_3 \circ \tilde{\alpha}_3$ and $\beta_4 \circ \alpha_4$ are multiplication by $d$, and where $\alpha_1$, $\alpha_4$ and $\beta_4$ are isomorphisms. Here we view $M$ as a discrete $I$-module. Since $H^1(I, M)$ is killed by $[L : K]$, and $d$ is prime to $[L : K]$, the isomorphism $\alpha_4$ identifies $\tilde{\gamma}_1(\Phi(G)_{\text{tors}})$ and $\tilde{\gamma}_2(\Phi(G')_{\text{tors}})$ (see the argument for Claim 3 in the proof of [HN10 5.7]). Hence, looking at the commutative diagram with exact rows

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & (\Phi(H)/M^I)_{\text{tors}} & \longrightarrow & \Phi(G)_{\text{tors}} & \longrightarrow & \tilde{\gamma}_1(\Phi(G)_{\text{tors}}) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (\Phi(H')/M^{I'})_{\text{tors}} & \longrightarrow & \Phi(G')_{\text{tors}} & \longrightarrow & \tilde{\gamma}_2(\Phi(G')_{\text{tors}}) & \longrightarrow & 0 \\
\end{array}
\]

we see that it is enough to show that the cokernel of the injective morphism

$\phi : (\Phi(H)/M^I)_{\text{tors}} \rightarrow (\Phi(H')/M^{I'})_{\text{tors}}$

is isomorphic to $(\mathbb{Z}/d\mathbb{Z})^{t(G_{\text{ab}})}$. Since $M^I$ is a free $\mathbb{Z}$-module of rank $t(G_{\text{ab}})$ [HN10 3.13], it suffices to prove the following claims.

Claim 1: The morphism $\Phi(H) \rightarrow \Phi(H')$ is injective.

Claim 2: The morphism $\Phi(H)_{\text{tors}} \rightarrow \Phi(H')_{\text{tors}}$ is an isomorphism.

Claim 3: The image of the morphism $\Phi(H)_{\text{free}} \rightarrow \Phi(H')_{\text{free}}$ equals $d \cdot (\Phi(H')_{\text{free}})_{\text{free}}$.

From the exact sequence [3.14], we deduce the commutative diagram

\[
\begin{array}{ccccccccc}
\Phi(G_{\text{tor}}) & \xrightarrow{\phi_1} & \Phi(H) & \xrightarrow{\phi_2} & \Phi(E) & \longrightarrow & 0 \\
\delta_1 \downarrow & & \delta_2 \downarrow & \downarrow \delta_3 & & & & \\
\Phi(G'_{\text{tor}}) & \xrightarrow{\phi'_1} & \Phi(H') & \xrightarrow{\phi'_2} & \Phi(E') & \longrightarrow & 0 \\
\epsilon_2 \downarrow & & \epsilon_3 \downarrow & & & & & & \\
\Phi(H) & \xrightarrow{\phi_2} & \Phi(E) & \longrightarrow & 0 \\
\end{array}
\]

where $\epsilon_2$ and $\epsilon_3$ are the trace maps; thus $\epsilon_i \circ \delta_i$ is multiplication by $d$, for $i = 2, 3$. The rows of this diagram are exact and the kernels of $\phi_1$ and $\phi'_1$ are torsion, by Proposition [3.3.5]. Moreover, $\delta_1$ and $\delta_3$ are injective, by point (1). An easy diagram chase shows that the kernel of $\delta_2$ must be contained in the torsion part of $\Phi(G_{\text{tor}})$. But $\Phi(G_{\text{tor}})_{\text{tors}}$ is killed by $[L : K]$ (Corollary [3.3.3] and ker($\delta_2$) is killed by $d$, so that $\delta_2$ must be injective. This settles Claim 1.
We know that $\Phi(G_{\text{tor}})_{\text{tors}}$ and $\Phi(E)_{\text{tors}}$ are killed by $[L : K]$, and that the kernel of $\phi_1$ is torsion. This easily implies that $\Phi(H)_{\text{tors}}$ is killed by $[L : K]^2$. Moreover, $(\delta_1)_{\text{tors}}$ and $(\delta_3)_{\text{tors}}$ are isomorphisms, by Proposition 3.3.11. Since $d$ is prime to $[L : K]$, it is invertible in $\Phi(E)_{\text{tors}}$, and $(1/d)\epsilon_3_{\text{tors}}$ is inverse to $(\delta_3)_{\text{tors}}$. Likewise, $(1/d)\epsilon_2_{\text{tors}}$ is a right inverse of $(\delta_2)_{\text{tors}}$. Thus to prove that $(\delta_2)_{\text{tors}}$ is an isomorphism, it is enough to show that $(\epsilon_2)_{\text{tors}}$ is injective. Let $x$ be an element of $\Phi(H')_{\text{tors}}$, and assume that $\epsilon_2(x) = 0$. Then $\phi_2'(x) = 0$, so that we can lift $x$ to an element $y$ of $\Phi(G'_{\text{tor}})$. This element must be torsion, because the kernel of $\phi'_1$ is torsion. If we set

$$z = (\phi_1 \circ (\delta_1)^{-1}_{\text{tors}})(y)$$

then $\delta_2(z) = x$ and

$$z = (1/d)\epsilon_2 \circ \delta_2(z) = (1/d)\epsilon_2(x) = 0.$$

Thus $x = 0$ and $\delta_2$ is injective. This proves Claim 2.

Now we prove Claim 3. We have a commutative diagram with exact rows

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & \Phi(G_{\text{tor}})_{\text{free}} & \overset{(\phi_1)_{\text{free}}}{\longrightarrow} & \Phi(H)_{\text{free}} & \overset{\tilde{\phi}_2}{\longrightarrow} & \Phi(E)/\text{Im}((\phi_2)_{\text{tors}}) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Phi(G'_{\text{tor}})_{\text{free}} & \overset{(\phi'_1)_{\text{free}}}{\longrightarrow} & \Phi(H')_{\text{free}} & \overset{\tilde{\phi}'_2}{\longrightarrow} & \Phi(E')/\text{Im}((\phi'_2)_{\text{tors}}) & \longrightarrow & 0
\end{array}
\]

where all the vertical maps are injective. Since $\epsilon_2 \circ \delta_2$ is multiplication by $d$, and $\Phi(H)$ and $\Phi(H')$ have the same rank, we know that

$$d \cdot \Phi(H')_{\text{free}} \subset \text{Im}((\delta_2)_{\text{free}})$$

so that it remains to prove the converse implication.

Since $E'$ acquires semi-abelian reduction over the degree $[L : K]$ extension $L \otimes_K K'$ of $K'$, and $d$ is prime to $[L : K]$, Corollary 3.3.13 implies that $d$ is invertible in $\Phi(E')_{\text{tors}}$. Likewise, $d$ is invertible in $\Phi(G'_{\text{tor}})_{\text{tors}}$. Applying Proposition 3.3.11 to $E$ and $G_{\text{tor}}$, we see that every element in the image of $\delta_1$ or $\tilde{\delta}_3$ is divisible by $d$. Thus every element in the image of $(\delta_1)_{\text{free}}$ or $\tilde{\delta}_3$ is divisible by $d$. We will deduce that every element in the image of $(\delta_2)_{\text{free}}$ is divisible by $d$.

Let $x$ be an element of $\Phi(H)_{\text{free}}$ and set $y = (\delta_2)_{\text{free}}(x)$. Then there exists an element $z$ in $\Phi(E')/\text{Im}((\phi'_2)_{\text{tors}})$ such that

$$dz = (\delta_1 \circ \tilde{\phi}_2)(x) = \tilde{\phi}_2(y).$$

We can lift $z$ to an element $y'$ in $\Phi(H')_{\text{free}}$ such that $\tilde{\phi}_2(y - dy') = 0$. By (3.3.15), we know that $dy'$ lies in the image of $(\delta_2)_{\text{free}}$, so that the element $y - dy'$ can be written as $(\delta_2)_{\text{free}}(x')$, with $x' \in \Phi(H)_{\text{free}}$. By injectivity of $\tilde{\delta}_3$, we know that $x'$ belongs to $\Phi(G_{\text{tor}})_{\text{free}}$. Thus $y - dy'$ lies in the image of $(\delta_1)_{\text{free}}$, and we can find an element $w$ in $\Phi(G'_{\text{tor}})_{\text{free}}$ such that $dw = y - dy'$. It follows that $y$ is divisible by $d$ in $\Phi(H')_{\text{free}}$. 

\textbf{Corollary 3.3.16.} Assume that $k$ is algebraically closed. Let $G$ be a semi-abelian $K$-variety, and let $L$ be the minimal extension of $K$ in $K^s$ such that $G \times_K L$ has semi-abelian reduction. Let $e$ be a multiple of $[L : K]$ in $\mathbb{Z}_{>0}$ such that $(G_{\text{ab}}, e)$ satisfies condition (3.3.9). Then for every finite separable extension $K'$ of $K$ of degree prime to $e$, we have

$$i(G) = i(G \times_K K').$$
3. COMPONENT GROUPS AND BASE CHANGE

Proof. We set \( G' = G \times_K K' \) and \( d = [K' : K] \). The index of \( G \) is given by
\[
i(G) = |\text{coker}(\Phi(G_{\text{spl}}) \to \Phi(G)_{\text{free}})|.
\]
We have a commutative square of free \( \mathbb{Z} \)-modules
\[
\begin{array}{ccc}
\Phi(G_{\text{spl}}) & \to & \Phi(G)_{\text{free}} \\
\alpha_{\text{spl}} \downarrow & & \downarrow \alpha_{\text{free}} \\
\Phi((G')_{\text{spl}}) & \to & \Phi(G')_{\text{free}}
\end{array}
\]
with injective morphisms. By the proof of [HN10, 4.2], we know that the natural morphism \( G_{\text{spl}} \times_K K' \to (G')_{\text{spl}} \) is an isomorphism. Thus all the \( \mathbb{Z} \)-modules in the diagram have rank \( t(G) = t(G') \). The cokernels of \( \alpha_{\text{spl}} \) and \( \alpha_{\text{free}} \) have the same cardinality, namely, \( d^{t(G_{\text{tor}})} = d^{t_{\text{spl}}(G)} \). Thus the cokernels of the horizontal morphisms also have the same cardinality, which means that \( i(G) = i(G') \). □

(3.3.17) If \( G \) is an abelian variety with potential multiplicative reduction or a tamely ramified abelian variety, then we proved Theorem 3.3.13 in [HN10, 5.7] without the assumption that \( k \) is algebraically closed. To be precise, [HN10, 5.7] only gives a formula for the cardinality of the cokernel of \( \alpha \), but using the trace map as in the proof of Theorem 3.3.13 one can determine the full structure of the cokernel. Unfortunately, there is a slight error in the statement of [HN10, 5.7]: the cokernel of the base change morphism \( \alpha \) is isomorphic to \( (\mathbb{Z}/r\mathbb{Z})^{t_{\text{spl}}(G)} \), with \( r \) the ramification index of \( K'/K \) instead of the degree. Of course, when \( k \) is algebraically closed, these values coincide.

3.4. The component series of a semi-abelian variety.

(3.4.1) Theorem 3.3.13 makes it possible to extend our previous results on the component series for abelian varieties ([HN10, 5.7] and Theorem 3.1.5 in Chapter 3) to semi-abelian varieties.

Theorem 3.4.2. Assume that \( k \) is algebraically closed. Let \( G \) be a semi-abelian \( K \)-variety with toric part \( G_{\text{tor}} \) and abelian part \( G_{\text{ab}} \). Let \( L \) be the minimal extension of \( K \) in \( K' \) such that \( G \times_K L \) has semi-abelian reduction, and let \( e \) be a multiple of \( [L : K] \) in \( \mathbb{Z}_{>0} \) such that \( (G(d), e/gcd(d, e)) \) satisfies condition 3.3.9 for all \( d \) in \( \mathbb{N}' \).

Then the component series
\[
S^G_{\Phi}(T) = \sum_{d \in \mathbb{N}'} |\Phi(G(d))_{\text{tors}}| T^d
\]
is rational. More precisely, it belongs to the subring
\[
\mathcal{Y} = \mathbb{Z}[T, \frac{1}{T}, \frac{1}{T_j - 1}]_{j \in \mathbb{Z}_{>0}}
\]
of \( \mathbb{Z}[T] \). It has degree zero if \( p = 1 \) and \( G \) has potential good reduction, and degree \(< 0 \) in all other cases. Moreover, \( S^G_{\Phi}(T) \) has a pole at \( T = 1 \) of order \( t_{\text{tame}}(G_{\text{ab}}) + 1 \).

Proof. One can simply copy the arguments in [HN10, 6.5], invoking Theorem 3.3.13 instead of [HN11a, 5.7]. □
(3.4.3) The conditions on $G$ and $e$ in the statement of Theorem 3.4.2 are satisfied, for instance, if $G_{ab}$ is tamely ramified or has potential multiplicative reduction and $e = [L : K]$, and also if $G_{ab}$ is isomorphic to the Jacobian of a smooth projective curve $C$ of index one and $e$ is the least common multiple of $[L : K]$ and the stabilization index $e(C)$ of $C$ (Corollary 2.2.10 in Chapter 3 and Proposition 3.3.12). We expect that Theorem 3.1.5 is valid for all semi-abelian $K$-varieties $G$. 
Part 2

Chai and Yu’s base change conductor and Edixhoven’s filtration
CHAPTER 5

The base change conductor and Edixhoven’s filtration

In this chapter, we recall the definition of the base change conductor of a semi-abelian $K$-variety $G$ and of Edixhoven’s filtration on the special fiber of the Néron model of $G$. We use Edixhoven’s construction to define a new invariant, the tame base change conductor, which is important for the applications to motivic zeta functions in Part 3. We compare the base change conductor and its tame counterpart on some explicit examples. The main result of this section states that the jumps of the Jacobian variety of a $K$-curve $C$ only depend on the combinatorial reduction data of $C$ (Theorem 3.1.3). This generalizes a previous result of the first author, where an additional condition on the reduction data was imposed.

1. Basic definitions

1.1. The conductor of a morphism of modules.

Definition 1.1.1. Let $f : M \to N$ be an injective morphism of free $R$-modules of finite rank $r$. The tuple of elementary divisors of $f$ is the unique monotonically increasing tuple of non-negative integers $(c_1(f), \ldots, c_r(f))$ such that

$$\text{coker}(f) \cong \bigoplus_{i=1}^r R/m^{c_i(f)}.$$ 

The conductor of $f$ is the non-negative integer

$$c(f) = \sum_{i=1}^r c_i(f) = \text{length}_R \text{coker}(f).$$

Note that the elementary divisors of $f$ are the valuations of the diagonal elements in a Smith normal form of $f$, so that $c(f)$ is equal to the exponent of the determinant ideal of $f$ in $R$.

1.2. The base change conductor of a semi-abelian variety.

(1.2.1) Let $G$ be a semi-abelian $K$-variety of dimension $g$ with Néron model $\mathcal{G}/R$. Let $K'/K$ be a finite separable field extension of ramification index $e(K'/K)$, and denote by $R'$ the integral closure of $R$ in $K'$. We denote by $\mathcal{G}'/R'$ the Néron model of $G' = G \times_K K'$.

The canonical base change morphism

$$h : \mathcal{G} \times_R R' \to \mathcal{G}'$$

induces an injective homomorphism

(1.2.2) \[ \text{Lie}(h) : \text{Lie}(\mathcal{G}) \otimes_R R' \to \text{Lie}(\mathcal{G}') \]

of free $R'$-modules of rank $g$. 

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5. THE BASE CHANGE CONDUCTOR AND EDIXHOVEN'S FILTRATION

**Definition 1.2.3.** We call
\[
\left( \frac{c_1(\text{Lie}(h))}{e(K'/K)}, \ldots, \frac{c_g(\text{Lie}(h))}{e(K'/K)} \right)
\]
the tuple of \(K'\)-elementary divisors of \(G\), and we denote it by
\[
(c_1(G, K'), \ldots, c_g(G, K')).
\]
If \(G'\) has semi-abelian reduction, we set \(c_i(G) = c_i(G, K')\) for every \(i\) in \(\{1, \ldots, g\}\) and we call \((c_1(G), \ldots, c_g(G))\) the tuple of elementary divisors of \(G\).

We call the rational number
\[
c(G, K') := \sum_{i=1}^{g} c_i(G, K') = \frac{1}{e(K'/K)} \cdot \text{length}_{K'}(\text{coker}(	ext{Lie}(h)))
\]
the \(K'\)-base change conductor of \(G\). If \(G'\) has semi-abelian reduction, we set \(c_i(G) = c_i(G, K')\) and we call this invariant the base change conductor of \(G\).

(1.2.4) The values \(c_i(G)\) and \(c(G)\) are independent of the choice of a finite separable extension \(K'\) of \(K\) such that \(G \times_K K'\) has semi-abelian reduction. This follows from the fact that \(\text{Lie}(h)\) is an isomorphism for every \(K'\) if \(G\) has semi-abelian reduction.

(1.2.5) The base change conductor and the elementary divisors were introduced for tori by Chai and Yu in [CY01] and for semi-abelian varieties by Chai in [Ch00]. The base change conductor \(c(G)\) vanishes if and only if \(G\) has semi-abelian reduction [HN11a, 4.16], and one can consider \(c(G)\) as a measure for the defect of semi-abelian reduction of \(G\). More generally, \(c(G, K')\) measures the difference between the identity components \((\mathcal{G} \times_R R')^o\) and \((\mathcal{G})^o\), and vanishes if and only if the morphism
\[
h : \mathcal{G} \times_R R' \to \mathcal{G}'
\]
is an open immersion (same proof as [HN11a, 4.16]).

(1.2.6) It is straightforward to check that \(c(G, K')\) behaves additively in towers, in the following sense: if \(K \subset K' \subset K''\) is a tower of finite separable extensions, then
\[
c(G, K'') = c(G, K') + \frac{c(G', K'')}{e(K''/K)}.
\]
Moreover, if \(c(G, K')\) is zero, then
\[
c_i(G, K'') = \frac{c_i(G', K'')}{e(K'/K)}
\]
for all \(i\) in \(\{1, \ldots, g\}\), and if \(c(G', K'')\) is zero, then \(c_i(G, K') = c_i(G, K'')\) for all \(i\) in \(\{1, \ldots, g\}\). Choosing \(K''\) in such a way that \(G \times_K K''\) has semi-abelian reduction, we see in particular that
\[
c(G) = c(G, K') + \frac{c(G')}{e(K'/K)}
\]
for all finite separable extensions \(K'\) of \(K\), and that
\[
c_i(G) = \frac{c_i(G')}{e(K'/K)}
\]
for all \(i\) in \(\{1, \ldots, g\}\) if \(c(G, K') = 0\).

1.3. Jumps and Edixhoven’s filtration.
(1.3.1) One can define a filtration on $G_k$ by closed subgroups that measures the behaviour of $G$ under tame base change. This construction is due to Edixhoven if $G$ is an abelian variety [Ed92] and extends to semi-abelian varieties in a straightforward way; see [HN11a] §4.1. We’ll briefly recall its construction.

(1.3.2) Let $d$ be an element of $N'$. Recall that we denote by $K(d)$ the unique degree $d$ extension of $K$ in $K^s$, and by $R(d)$ its valuation ring. We denote by $\mu$ the Galois group $\text{Gal}(K(d)/K)$, and by $G(d)$ the Néron model of $G(d) = G \times_K K(d)$. The Weil restriction

$$W = \prod_{R(d)/R} G(d)$$

carries a natural $\mu$-action, and its fixed locus $W^\mu$ is canonically isomorphic to $G$ [HN11a] 4.1. Denote by $\mathfrak{m}(d)$ the maximal ideal in $R(d)$. For every $i$ in $\{0, \ldots, d\}$, the reduction modulo $\mathfrak{m}(d)^i$ defines a morphism of $k$-group schemes

$$W^\mu_k \to \prod_{(R(d)/\mathfrak{m}(d)^i)/k} (G(d) \times_{R(d)} (R(d)/\mathfrak{m}(d)^i))$$

whose kernel we denote by $F_i^d G_k$. In this way, we obtain a descending filtration

$$G_k = F_d^0 G_k \supset F_d^1 G_k \supset \ldots \supset F_d^d G_k = 0$$

on $G_k$ by closed subgroups, and $F_d^i G_k$ is a smooth connected unipotent $k$-group for all $i > 0$ [HN11a] 4.8. The graded quotients of this filtration are denoted by

$$\text{Gr}_d^i G_k = F_d^i G_k / F_d^{i+1} G_k.$$

**Definition 1.3.3.** We say that $j \in \{0, \ldots, d - 1\}$ is a $K(d)$-jump of $G$ if $\text{Gr}_d^j G_k \neq 0$. We denote the set of $K(d)$-jumps of $G$ by $J_{G,K(d)}$. The dimension of $\text{Gr}_d^j G_k$ is called the multiplicity of the $K(d)$-jump $j$ and is denoted by $m_{G,K(d)}(j)$. We extend $m_{G,K(d)}$ to a function

$$m_{G,K(d)} : [0, d[ \to \mathbb{N} : j \mapsto m_{G,K(d)}(j)$$

by sending $j$ to 0 if $j$ is not a $K(d)$-jump of $G$, and we call this function the $K(d)$-multiplicity function of $G$.

(1.3.4) It is explained in [Ed92] 5.3 and [HN11a] 4.8 how the function $m_{G,K(d)}$ can be computed from the action of $\text{Gal}(K(d)/K) \cong \mu_d(k)$ on $\text{Lie}(G(d)_k)$: for every $i$ in $\{0, \ldots, d - 1\}$, the value of $m_{G,K(d)}$ at $i/d$ equals the dimension of the maximal subspace of $\text{Lie}(G(d)_k)$ where each $\zeta \in \mu_d(k)$ acts by multiplication with $\zeta^i$.

(1.3.5) In [Ed92], Edixhoven also introduced a filtration of $G_k$ with rational indices that captures the filtrations $F_d^i G_k$ introduced above simultaneously for all $d$ in $N'$. It is defined as follows. If $p = 1$ then we set $Q' = \mathbb{Q}$, and if $p > 1$ we set $Q' = \mathbb{Z}_{(p)}$. For each rational number $\alpha = a/b$ in $\mathbb{Q'} \cap [0, 1]$ with $a \in \mathbb{N}$ and $b \in \mathbb{N'}$, we put

$$F^\alpha G_k = F_b^a G_k.$$

By [HN11a] 4.11] this definition does not depend on the choice of $a$ and $b$, and we obtain in this way a decreasing filtration $F^\bullet G_k$ of $G_k$ by closed subgroups. Note that only finitely many subgroups occur in the filtration $F^\bullet G_k$ since $G_k$ is Noetherian. One can define jumps also for this filtration, in the following way. Let $\rho$ be an element of $\mathbb{R} \cap [0, 1]$. We put $F^{\rho} G_k = F^\beta G_k$, where $\beta$ is any value in $\mathbb{Q'} \cap [\rho, 1]$ such that $F^\beta G_k = F^\beta G_k$ for all $\beta' \in \mathbb{Q'} \cap [\rho, \beta]$. If $\rho \neq 0$, we put $F^{<\rho} G_k = F^{\bullet} G_k$,
where $\gamma$ is any value in $\mathbb{Q}' \cap [0, \rho]$ such that $\mathcal{F}^{\gamma} \mathcal{G}_k = \mathcal{F}^{\gamma} \mathcal{G}_k$ for all $\gamma' \in \mathbb{Q}' \cap [\gamma, \rho]$. We set $\mathcal{F}^{<0} \mathcal{G}_k = \mathcal{G}_k$. Then we define

$$\text{Gr}^\rho \mathcal{G}_k = \mathcal{F}^{<\rho} \mathcal{G}_k / \mathcal{F}^{>\rho} \mathcal{G}_k$$

for every $\rho$ in $\mathbb{R} \cap [0, 1]$. 

**Definition 1.3.6.** Let $j$ be an element of $\mathbb{R} \cap [0, 1]$. We say that $j$ is a jump of $\mathcal{G}_k$ if $\text{Gr}^j \mathcal{G}_k \neq 0$. We denote the set of jumps of $G$ by $\mathcal{J}_G$. The multiplicity of $j$ is the dimension of $\text{Gr}^j \mathcal{G}_k$ and is denoted by $m_G(j)$. If $j$ is not a jump of $G$, we set $m_G(j) = 0$. In this way, we obtain a function

$$m_G : \mathbb{R} \cap [0, 1] \to \mathbb{N}$$

which we call the multiplicity function of $G$. We define the tame base change conductor of $G$ to be the sum

$$c_{\text{tame}}(G) = \sum_{j \in \mathcal{J}_G} m_G(j) \cdot j.$$

(1.3.7) The jumps of $G$ can also be described as follows. Let $K_0 \subset K_1 \subset \ldots$ be a tower of finite tame extensions of $K$ in $K^s$ that is cofinal in the set of all finite tame extensions of $K$ in $K^s$, ordered by inclusion. Set $d_n = [K_n : K]$ for every $n \in \mathbb{N}$. Then $\mathcal{J}_G$ is the limit of the sequence of subsets

$$\frac{1}{d_n} \cdot \mathcal{J}_{G,K_n}$$

of $[0, 1]$ for $n \to \infty$. More precisely, if we count the $K_n$-jumps of $G$ with their multiplicities and put them in ascending order, we get a tuple

$$0 \leq j_{n,1} \leq \ldots \leq j_{n,g}$$

in $\{0, \ldots, d_n - 1\}$. For every $i \in \{1, \ldots, g\}$, the sequence $(j_{n,i}/d_n)_{n \in \mathbb{N}}$ is a monotonically increasing sequence in $[0, 1]$, whose limit we denote by $j_i$. Then it is easy to see that $j_i < 1$ for every $i$ and that

$$0 \leq j_1 \leq \ldots \leq j_g < 1$$

are the jumps of $G$, counted with multiplicities.

(1.3.8) For tamely ramified abelian $K$-varieties, there exists an interesting relation between the jumps and the Galois action on the $\ell$-adic Tate module of the abelian variety. We refer to [HNT11] for details.

(1.3.9) We established in [HNT11a] the following relationship between the base change conductor and the jumps.

**Proposition 1.3.10.** For every finite tame extension $K'$ of $K$ in $K^s$ of degree $d$, the tuple

$$(c_1(G, K') \cdot d, \ldots, c_g(G, K') \cdot d)$$

is equal to the tuple of $K'$-jumps of $G$, if we count every $K'$-jump with its multiplicity and put them in ascending order. In particular,

$$c(G, K') \cdot d = \sum_{j \in \mathcal{J}_{G,K'}} m_{G,K'}(j) \cdot j.$$

If $G$ is tamely ramified, then the tuple

$$(c_1(G), \ldots, c_g(G))$$
is equal to the tuple of jumps of \( G \), if we count every jump with its multiplicity and put them in ascending order. In particular,
\[
c(G) = c_{\text{tame}}(G).
\]

**Proof.** See [HN11a, 4.18 and 4.18]. □

**Corollary 1.3.11.**
(1) If \( d \) and \( d' \) are elements of \( \mathbb{N}' \) such that \( d \) divides \( d' \), then
\[
c_i(G, K(d)) \leq c_i(G, K(d'))
\]
for every \( i \) in \( \{1, \ldots, g\} \). In particular,
\[
c(G, K(d)) \leq c(G, K(d')).
\]

(2) We have
\[
c_{\text{tame}}(G) = \sup_{d \in \mathbb{N}'} c(G, K(d)) = \lim_{d \to \mathbb{N}'} c(G, K(d))
\]
where we order \( \mathbb{N}' \) by the divisibility relation.

**Proof.** This follows from Proposition [1.3.10] and the description of the jumps in [1.3.7]. □

If \( G \) is wildly ramified, it can happen that \( c_{\text{tame}}(G) \neq c(G) \), as we will illustrate in Examples [1.3.13, 2.2.3 and 2.2.4].

(1.3.12) As noted by Edixhoven in [Ed92, 5.4(5)], it is not at all clear whether the jumps of a semi-abelian \( K \)-variety \( G \) are always rational numbers. We will come back to this issue in Part 4. Let \( L \) be the minimal extension of \( K \) in \( K^s \) such that \( G \times K L \) has semi-abelian reduction, and set \( e = [L : K] \). If \( L \) is a tame extension of \( K \), then it follows from Proposition [1.3.10] that \( m_G(j/e) = m_{G,L}(j) \) for every \( j \) in \([1, e]\). In particular, the jumps belong to the set \( (1/e) \mathbb{Z} \). Moreover, in that case, one can easily compute the \( K' \)-jumps of \( G \) and their multiplicities from the jumps of \( G \), for all finite tame extensions \( K' \) of \( K \) [HN11a, 4.13]. The crucial point is that the identity component of the Néron model of \( A \times_K L \) is stable under finite separable extensions of \( L \), by [SGA7-I, IX.3.3]. Combining the formula in [HN11a, 4.13] with [HN11a, 4.20], we see that \( e \) equals the smallest positive integer \( m \) such that \( mj \) is integer for every jump \( j \) of \( G \). If \( G \) is the (possibly wildly ramified) Jacobian of a smooth proper geometrically connected \( K \)-curve \( C \) of index one, then we’ll show in Corollary [3.1.5] that the jumps of \( G \) are rational, and that the stabilization index \( e(C) \) defined in Chapter 3 is the smallest positive integer \( m \) such that \( mj \) is integer for every jump \( j \) of \( G \).

**Example 1.3.13.** Assume that \( K \) has equal characteristic two. Let \( \alpha \) be an element of \( K \), and assume that the valuation \( v_K(\alpha) \) of \( \alpha \) is odd and strictly negative. We denote by \( L \) the Artin-Schreier extension
\[
L = K[t]/(t^2 + t + \alpha)
\]
of \( K \), and by \( R_L \) its valuation ring. We denote by \( T \) the unique non-split one-dimensional \( K \)-torus with splitting field \( L \). It has character module \( X(T) \cong \mathbb{Z} \) where the action of the generator \( \sigma \) of \( \text{Gal}(L/K) \cong \mathbb{Z}_2 \) on \( \mathbb{Z} \) is given by \( m \mapsto -m \).
The torus \( T \) is the norm torus of the quadratic extension \( L/K \).
First, we compute the tame base change conductor \( c_{\text{tame}}(T) \). For notational convenience, we set \( s = (1 - v_K(\alpha))/2 \). This is an element of \( \mathbb{Z}_{>0} \). The torus \( T \) is isomorphic to

\[
\text{Spec } K[u, v]/(u^2 + \alpha v^2 + uv + 1)
\]

with multiplication given by

\[ (u_1, v_1) \cdot (u_2, v_2) = (u_1u_2 + \alpha v_1v_2, u_1v_2 + u_2v_1 + v_1v_2) \]

and inversion given by

\[ (u, v)^{-1} = (u + v, v). \]

Let \( \pi \) be a uniformizer in \( K \). For every \( d \in \mathbb{N}' \), we choose a uniformizer \( \pi(d) \) in \( K(d) \) such that \( \pi(d)^d = \pi \). We set \( \alpha' = \pi^{2s-1} \alpha, x = \pi(d)^{(d-1)/2}d^s(u + 1) \) and \( y = \pi^{1-2s}v \). Then \( \alpha' \) is a unit in \( R \) and we can rewrite the equation of \( T \) as

\[
\pi(d)x^2 + \alpha' y^2 + \pi(d)^{d_2-2}(d_1/2)xy + y = 0.
\]

The coefficients of this equation belong to \( R(d) \). We set

\[
\mathcal{T}(d) = \text{Spec } R(d)[x, y]/(\pi(d)x^2 + \alpha' y^2 + \pi(d)^{d_2-2}(d_1/2)xy + y).
\]

It is easily checked that the group structure on \( T(d) \) extends to \( \mathcal{T}(d) \). Moreover, for every point \( (x_0, y_0) \) in \( T(K(d)) \), we have the following properties.

1. If \( v_{K(d)}(x_0) < v_{K(d)}(y_0) \), then the terms \( \pi(d)x_0^2 \) and \( y_0 \) must have the same valuation;

2. If \( v_{K(d)}(x_0) \geq v_{K(d)}(y_0) \), then \( y_0^2 \) and \( y_0 \) must have the same valuation.

Thus the coordinates \( (x_0, y_0) \) lie in \( R(d) \). Now \( \mathcal{T}(d) \) is a Néron left-model of \( T(d) \). We’ll write \( \mathcal{T} \) instead of \( \mathcal{T}(1) \).

The base change morphism

\[
\mathcal{T} \times_R R(d) \to \mathcal{T}(d)
\]

is defined by the morphism of \( R(d) \)-algebras

\[
R(d)[x, y]/(\pi(d)x^2 + \alpha' y^2 + \pi(d)^{d_2-2}(d_1/2)xy + y) \to R(d)[x, y]/(\pi x^2 + \alpha' y^2 + \pi x y + y)
\]

that sends \( x \) to \( \pi(d)^{(d-1)/2}x \) and \( y \) to \( y \). The \( R(d) \)-module \( \text{Lie}(\mathcal{T}(d)) \) is generated by \( \partial/\partial x \), so that \( c(T, K(d)) = (d - 1)/2 \) for every \( d \) in \( \mathbb{N}' \). Using Corollary \[3.11\]

we see that

\[
c_{\text{tame}}(T) = \sup_{d \in \mathbb{N}'} c(T, K(d)) = 1/2.
\]

Now, we compute the base change conductor \( c(T) \). If we set \( u' = u + tv + 1 \) and \( v' = u + (t + 1)v + 1 \) then

\[
\mathcal{T}_L^o = \text{Spec } R_L[u', v']/(u'v' + u' + v')
\]

is the identity component of the Néron model of the split torus \( T \times_K L \). The \( R_L \)-module \( \text{Lie}(\mathcal{T}_L^o) \) is generated by \( \partial/\partial u' \), and the base change morphism

\[
\mathcal{T}^o \times_R R_L \to \mathcal{T}_L^o
\]

is given by

\[
u' \mapsto \pi^s x + t \pi^{2s-1}y \text{ and } v' \mapsto \pi^s x + (t + 1) \pi^{2s-1}y.
\]

Thus \( c(T) = s \). In particular, \( c_{\text{tame}}(T) \neq c(T) \).

2. Computing the base change conductor

2.1. Invariant differential forms.
(2.1.1) Let $G$ be a semi-abelian $K$-variety of dimension $g$ with Néron $lft$-model $\mathcal{G}$ and let $K'$ be a finite separable extension of $K$ with valuation ring $R'$. We will explain a possible strategy to compute the $K'$-base change conductor $c(G, K')$ in concrete examples.

(2.1.2) For every smooth group scheme $\mathcal{H}$ of relative dimension $g$ over a local scheme $S = \text{Spec } A$, we denote by $e_{\mathcal{H}}$ the unit section $S \rightarrow \mathcal{H}$ and we set $\omega_{\mathcal{H}/S} = e_{\mathcal{H}}^* \Omega_{\mathcal{H}/S}$. This is a free $A$-module of rank 1, which can be identified with the free $A$-module of translation-invariant elements of $\Omega^2_{\mathcal{H}/S}(\mathcal{H})$ by [BLR90, 4.2.1]. The module $\omega_{\mathcal{H}/S}$ is the dual of the determinant of the free $A$-module $\text{Lie}(\mathcal{H})$ of rank $g$.

(2.1.3) We set $G' = G \times_K K'$ and we denote by $\mathcal{G}'$ the Néron $lft$-model of $G'$. Let

$$h : \mathcal{G} \times_R R' \rightarrow \mathcal{G}'$$

be the canonical base change morphism. Pulling back the natural morphism

$$h^* \Omega^1_{\mathcal{G}'/R'} \rightarrow \Omega^1_{\mathcal{G} \times_R R'/R'}$$

through the unit section $e_{\mathcal{G} \times_R R'}$, we obtain a morphism of $R'$-modules

$$\phi : e_{\mathcal{G} \times_R R'}^* \Omega^1_{\mathcal{G}'/R'} \rightarrow e_{\mathcal{G} \times_R R'}^* \Omega^1_{\mathcal{G} \times_R R'/R'}$$

which is the dual of the morphism $\text{Lie}(h)$. Putting $\text{Lie}(h)$ in Smith normal form, it is easy to see that the cokernels of $\phi$ and $\text{Lie}(h)$ have the same length, and that this length is equal to the valuation of the determinant of $\phi$ in $R'$. Thus we can also compute the $K'$-base change conductor of $G$ as

$$c(G, K') = v_{K'}(\det(\phi))/e(K'/K)$$

$$= \frac{1}{e(K'/K)} \cdot \text{length}_{R'}(\text{coker}(\det(\phi) : \omega_{\mathcal{G}'/R'} \rightarrow \omega_{\mathcal{G} \times_R R'} \otimes_R R'))$$

where $v_{K'}$ denotes the normalized discrete valuation on $K'$.

(2.1.4) In practice, one computes $v_{K'}(\det(\phi))$ as follows: choose a translation-invariant volume form $\omega$ on $G$ that extends to a relative volume form on the $R$-scheme $\mathcal{G}$, and denote by $\omega'$ its pullback to $G'$. Then $\omega'$ will, in general, not extend to a relative volume form on $\mathcal{G}'$, but it will have poles along the components of the special fiber $\mathcal{G}'_K$. The order of the pole does not depend on the choice of the component, by translation invariance of $\omega'$, and it is equal to $-v_{K'}(\det(\phi))$.

2.2. Elliptic curves.

PROPOSITION 2.2.1 (see also Proposition 3.7.2 in [Lu10]). We denote by $v_K$ the normalized discrete valuation on $K$. If $E$ is an elliptic curve over $K$ with $j$-invariant $j(E)$ and minimal discriminant $\Delta$, then

$$c(E) = \begin{cases} v_K(\Delta)/12 & \text{if } E \text{ has potential good reduction,} \\ (v_K(\Delta) + v_K(j(E)))/12 & \text{if } E \text{ has potential multiplicative reduction.} \end{cases}$$

Moreover, for every finite separable extension $K'$ of $K$, we have

$$c(E, K') = \frac{1}{12} (v_K(\Delta) - v_{K'}(\Delta')/e(K'/K))$$

where $\Delta'$ denotes a minimal discriminant of $E \times_K K'$ and $v_{K'}$ denotes the normalized discrete valuation on $K'$. 
Proof. Let $K'$ be a finite separable extension of $K$, and set $E' = E \times_K K'$. Let $\mathcal{E}$ be the Néron model of $E$. We consider a Weierstrass equation

$$y^2 + (a_1x + a_3)y = x^3 + a_2x^2 + a_4x + a_6$$

for $E$, with $a_i \in R$ for all $i$. To Equation (E) one associates a translation invariant differential form

$$\omega = \frac{dx}{2y + a_1x + a_3} \in \Omega^1_{E/K}(E).$$

If we also assume that (E) is a minimal equation, then $\omega$ extends to a generator of the free rank one $\mathcal{O}_E$-module $\Omega^1_{E/K}(E)$ (see e.g. [Li02], 9.4.35). Moreover, pulling back $\omega$ via the unit section $e$ yields a generator of the free rank one $R$-module $\omega_{E/R}$.

Let $(E')$ be a minimal Weierstrass equation for the elliptic curve $E'$. It yields an invariant differential form $\omega' \in \Omega^1_{E'/K'}(E')$ in the same way as above. If we denote by $\pi$ the projection $E' \to E$, then $\omega' = r \cdot \pi^* \omega$ for some element $r \in R' = \mathcal{O}_{E'}(E')$, and

$$c(E, K') = \frac{v_{K'}(r)}{e(K'/K)}.$$ 

Moreover, if we denote by $\Delta$ and $\Delta'$ the discriminants of the Weierstrass equations (E) and (E'), then

$$v_{K'}(\Delta') = e(K'/K) \cdot v_K(\Delta) - 12v_{K'}(r)$$

by [Si86], III.1.1.2. Thus we find that

$$c(E, K') = \frac{1}{12} \left( v_K(\Delta) - \frac{v_{K'}(\Delta')}{e(K'/K)} \right).$$

Now assume that $E'$ has semi-abelian reduction. Then $v_{K'}(\Delta') = 0$ if $E'$ has good reduction, and $v_{K'}(\Delta') = -v_K(\omega(E)) \cdot e(K'/K)$ if $E'$ has multiplicative reduction. This yields the required formula for $c(E') = c(E, K')$.

(2.2.2) Using Proposition 2.2.1 we can easily compute the base change conductor of a tamely ramified elliptic curve from the data in the Kodaira-Néron reduction table (Table 4.1 in [Si94], IV.9). We will see in Theorem 3.1.3 that the tame base change conductor of an elliptic curve only depends on the reduction type, even for wildly ramified curves. If $p = 1$, then $c_{\text{tame}}(E) = c(E)$ for every elliptic $K$-curve $E$ (Proposition 1.3.10), so that we obtain the following table of values for the tame base change conductor (see also [Ed92], 5.4.5 and [Ha10b], Table 8.1).

| Type | $I_{\geq 0}$ | II | III | IV | $I^*_{\geq 0}$ | IV* | $III^*$ | $II^*$ |
|------|-------------|----|-----|----|---------------|-----|----------|--------|
| $c_{\text{tame}}(E)$ | 0 | 1/6 | 1/4 | 1/3 | 1/2 | 2/3 | 3/4 | 5/6 |

Table 2.2.2. The tame base change conductor for elliptic curves

Example 2.2.3. Assume that $k$ is an algebraically closed field of characteristic 2 and that $R = W(k)$. Let $E$ be the elliptic curve given by the minimal Weierstrass equation

$$y^2 = x^3 + 2.$$
As we’ve already noticed in Example 2.2.3, $E$ has reduction type $II$ and potential good reduction. Therefore, $c_{\text{tame}}(E) = 1/6$ by Table 2.2.2, and 
\[ c(E) = v_K(\Delta)/12 = 6/12 = 1/2 \]
by Proposition 2.2.1. In particular, $c(E) \neq c_{\text{tame}}(E)$.

**Example 2.2.4.** In this example we assume that $K$ has equal characteristic 2. We’ll construct an elliptic curve $E/K$ with potential multiplicative reduction such that $c(E) \neq c_{\text{tame}}(E)$.

Lorenzini has proven in [Lo10, 2.8] that every elliptic curve $E/K$ with additive reduction and potential multiplicative reduction has reduction of type $I_{\nu+4s}^{\nu}$ for certain integers $\nu > 0$ and $s > 0$ and acquires multiplicative reduction over a degree two extension of $K$. In particular, such a curve $E$ is wildly ramified and $c_{\text{tame}}(E) = 1/2$ by Table 2.2.2. We will now compute $c(E)$.

By the proof of [Lo10, Thm. 2.8], we can find a Weierstrass equation
\[ y^2 + xy = x^3 + Dx^2 + a_6 \]
over $K$, where $D = u\pi^{-2s+1}$ for a suitable unit $u \in R$ and $a_6 \in R$ with $v_K(a_6) > 0$.

We can use Tate’s algorithm to verify that
\[ (\xi) \quad y^2 + \pi^s xy = x^3 + \pi^{2s}Dx^2 + \pi^{5s}a_6, \]
is a minimal Weierstrass equation for $E$ over $R$. Using Proposition 2.2.1, we compute that $c(E) = s$. Thus $c(E) \neq c_{\text{tame}}(E)$. Also, note that this example shows that $c(E)$ can be arbitrarily large, whereas the tame base change conductor of a semi-abelian variety is always strictly bounded by the dimension.

**2.3. Behaviour under non-archimedean uniformization.**

**(2.3.1)** Another technique that is quite useful to compute the jumps and elementary divisors of an abelian variety is the use of non-archimedean uniformization. Let $A$ be an abelian $K$-variety, and let
\[ 0 \to M \to E^{\text{an}} \to A^{\text{an}} \to 0 \]
be its non-archimedean uniformization as in (3.7.2) of Chapter 3. Thus $E$ is the extension of an abelian $K$-variety with potential good reduction by a $K$-torus, and $M$ is an étale $K$-lattice in $E^{\text{an}}$.

**Proposition 2.3.2.** For every finite separable extension $K'$ of $K$, the $K'$-elementary divisors of $E$ and $A$ coincide, so that $c(E, K') = c(A, K')$. In particular, the elementary divisors of $E$ and $A$ coincide and $c(E) = c(A)$. Moreover, if $K'$ is tame over $K$, then the multiplicity functions $m_{E, K'}$ and $m_{A, K'}$ are equal. Thus $m_E = m_A$ and $c_{\text{tame}}(E) = c_{\text{tame}}(A)$.

**Proof.** By Proposition 1.3.10, we only need to prove the claim about the $K'$-elementary divisors. This can be done by using the same arguments as in [Ch00, 5.1]: if we denote by $\mathcal{E}$ and $\mathcal{A}$ the Néron models of $E$ and $A$, and by $\tilde{\mathcal{E}}$ and $\tilde{\mathcal{A}}$ their formal $m$-adic completions, then the morphism $\tilde{E}^{\text{an}} \to \tilde{A}^{\text{an}}$ extends uniquely to a morphism of formal $R$-group schemes $\tilde{\mathcal{E}} \to \tilde{\mathcal{A}}$. This morphism is a local isomorphism by [BX96, 2.3], and thus induces an isomorphism of $R$-modules
\[ \text{Lie}(\mathcal{E}) \to \text{Lie}(\mathcal{A}). \]
Now the result follows from the fact that $E^{\text{an}} \times_K K' \to A^{\text{an}} \times_K K'$ is the non-archimedean uniformization of $A \times_K K'$.

Example 2.3.3. Let $E$, $K$ and $D$ be as in Example 2.2.4 and consider the Artin-Schreier extension

$$L = K[w]/(w^2 + w + D)$$

of $K$. We claim that $E \times_K L$ has multiplicative reduction. To see this, it suffices to make a change of variables $y = y' + wx$, which yields an integral Weierstrass equation

$$(E') \quad y'^2 + xy' = x^3 + a_6$$

over the valuation ring of $L$. Tate’s algorithm shows that $(E')$ is minimal and that $E \times_K L$ has multiplicative reduction. Thus the uniformization morphism of $E$ is of the form $u : Tan \to Ean$, where $T$ is the norm torus of $L/K$, because this is the unique one-dimensional $K$-torus with minimal splitting field $L$. The norm torus of $L/K$ is precisely the torus from Example 1.3.13, with $\alpha = D$. We’ve seen that, both for $E$ and for $T$, the tame base change conductor equals $1/2$ and the base change conductor equals $(1 - v_K(D))/2$.

(2.3.4) In view of Proposition 2.3.2 it would be quite useful to have a formula that expresses the base change conductor of a semi-abelian $K$-variety in terms of the conductors of its toric and abelian part, even if one is only interested in the base change conductors of abelian varieties. The natural guess is the following.

Conjecture 2.3.5 (Chai). Assume that $k$ is algebraically closed. If $G$ is a semi-abelian $K$-variety with toric part $G_{tor}$ and abelian part $G_{ab}$, then

$$c(G) = c(G_{tor}) + c(G_{ab}).$$

If $G$ is tamely ramified, then one can prove this statement in an elementary way; see [HN11a 4.23]. In [Ch00], Chai proves his conjecture when $K$ has characteristic zero and also when $k$ is finite. In [CLN11], the characteristic zero case is reproven by reducing it to a Fubini property for motivic integrals.

3. Jumps of Jacobians

In this section, we assume that $k$ is algebraically closed.

3.1. Dependence on reduction data.

(3.1.1) Let $C$ be a smooth, projective, geometrically connected $K$-curve of genus $g > 0$ and index one. We denote by $A$ the Jacobian of $C$. It is possible to obtain detailed information concerning the jumps of $A$ and their multiplicities in terms of the geometry of $sncd$-models of $C$. Let $\mathcal{C}/R$ be an $sncd$-model of $C$, and consider the following condition on $\mathcal{C}$:

\[ (*) \text{ If } E \text{ and } E' \text{ are irreducible components of } \mathcal{C}_k \text{ such that } E \cap E' \text{ is non-empty, then the multiplicity of } E \text{ or } E' \text{ in } \mathcal{C}_k \text{ is prime to } p. \]

It is easy to see that if condition $(*)$ holds for some $sncd$-model of $C$, then it also holds for the minimal $sncd$-model. In particular, this is the case if $C$ is tamely ramified, by Saito’s criterion [Sa87 3.11]. In [Ha10b], the first author proved the following result.
3. Jumps of Jacobians

Theorem 3.1.2. Let $\mathcal{C}/R$ be an sncd-model of $C$ and assume that $\mathcal{C}$ satisfies condition $(\ast)$. Then the jumps of $A$ and their multiplicities only depend on the combinatorial data of $\mathcal{C}_k$.

Proof. See [Ha10b, 8.2].

We will now show that the condition $(\ast)$ can be omitted, which yields the following stronger theorem.

Theorem 3.1.3. Let $C$ be a smooth, projective, geometrically connected $K$-curve of genus $g > 0$ and index one. We denote by $\mathcal{A}$ the Jacobian of $C$. Let $\mathcal{C}/R$ be an sncd-model of $C$. Then the jumps of $A$ and their multiplicities only depend on the combinatorial data of $\mathcal{C}_k$. In particular, they do not depend on the characteristic exponent $p$ of $k$.

Proof. We will indicate how the proof of [Ha10b, 8.2] can be generalized. We write $\mathcal{C}_k = \sum_{i \in I} N_i E_i$. For every $d$ in $\mathbb{N}$, we denote by $\mathcal{C}(d)$ the minimal desingularization of the normalization $\mathcal{C}_d$ of $\mathcal{C} \times_R R(d)$ as in Chapter 3, (1.1.1). Then $\mathcal{C}(d)$ is an sncd-model of $C \times_K K(d)$, by Proposition 1.3.2 in Chapter 3. The jumps in $\mathcal{F} \cdot \mathcal{A}_k$ can be computed if one has a sufficiently good description of the Galois action on the sncd-models $\mathcal{C}(d)$ as $d$ varies in $\mathbb{N}$. Indeed, there is a natural $\text{Gal}(K(d)/K)$-action on $C \times_K K(d)$ which extends uniquely to $\mathcal{C}(d)$, in such a way that the natural isomorphism

$$\text{Pic}^0_{\mathcal{C}_d/R(d)} \cong \mathcal{A}(d)^\circ$$

is equivariant. This yields an equivariant isomorphism

$$H^1(\mathcal{C}(d)_k, \mathcal{O}_{\mathcal{C}(d)_k}) \cong \text{Lie}(\mathcal{A}(d)_k)$$

(cf. [Ha10b, 2.4]).

Moreover, it is not necessary to treat all $d \in \mathbb{N}$. Let us put

$$\lambda(\mathcal{C}) = \text{lcm}\{N_i \mid i \in I\}.$$ 

In order to prove Theorem 3.1.3 it suffices to show, for every $d \in \mathbb{N}$ prime to $\lambda(\mathcal{C})$, that the $K(d)$-jumps and their multiplicities only depend on the combinatorial structure of $\mathcal{C}_k$. This follows from the fact that the set $\mathbb{Z}_{(\lambda(\mathcal{C}))} \cap \mathbb{Q}' \cap [0, 1]$ is dense in $\mathbb{Q}' \cap [0, 1]$. The reason for restricting to these values $d$ is that both the geometry of $\mathcal{C}(d)$ and the $\text{Gal}(K(d)/K)$-action can be described sufficiently well. This is not clear for general $d \in \mathbb{N}$.

Thus let $d$ be an element of $\mathbb{N}$ that is prime to $\lambda(\mathcal{C})$. Using Lemma 2.3.2 in Chapter 8 and the results in Section 4 of that chapter, it is easy to check that both the statement and proof of [Ha10b, 3.4] extend directly to our situation. This means that the following properties hold.

1. Every irreducible component $F$ of $\mathcal{C}(d)_k$ is stable under the $\text{Gal}(K(d)/K)$-action.

2. $\text{Gal}(K(d)/K)$ acts trivially on $F$ unless $F$ belongs to the exceptional locus of the minimal desingularization $\rho : \mathcal{C}_d \to \mathcal{C}_d$.

3. Every intersection point $x \in F \cap F'$ of distinct irreducible components of $\mathcal{C}(d)_k$ is a fixed point for $\text{Gal}(K(d)/K)$.

Then in order to compute the irreducible components of the action of $\text{Gal}(K(d)/K) \cong \mu_d(k)$ on

$$H^1(\mathcal{C}(d)_k, \mathcal{O}_{\mathcal{C}(d)_k}),$$
it suffices to determine the irreducible components of the action of $\text{Gal}(K(d)/K)$ on the cotangent space $\Omega_x$ of $\mathcal{C}(d)$ at $x$ for each intersection point $x \in F \cap F'$ in the special fiber $\mathcal{C}(d)_k$. In fact, the proof in [Ha10b] extends directly to our situation.

The Galois action on the cotangent space $\Omega_x$ can be computed on the completed local ring $\hat{\mathcal{O}}_{\mathcal{C}(d),x}$. The point $x$ maps to an intersection point $y$ in the special fiber of $\mathcal{C}_d$. We gave an explicit description of $\hat{\mathcal{O}}_{\mathcal{C}_d,y}$ in Lemma 1.2.2 and with this presentation it is straightforward to describe the Galois action on the cotangent space $\Omega_y$ of the special fiber $\mathcal{C}_d$. In fact, the proof in [Ha10b] §4.1. Moreover, using the explicit description of the minimal desingularization

$$\rho_y : \mathcal{Z}_y \to \text{Spec} \hat{\mathcal{O}}_{\mathcal{C}_d,y}$$

in Section 4 of Chapter 3, it is easy to describe the unique lifting of the $\text{Gal}(K(d)/K)$-action to $\mathcal{Z}_y$, and we can use the same arguments as in [Ha10b] 8.2.

**Corollary 3.1.4.** We keep the notations of Theorem 3.1.3. For every $d \in \mathbb{N}'$, the $K(d)$-jumps of $A$ and their multiplicities only depend on the combinatorial data of the special fiber $\mathcal{C}_d$.

**Proof.** An integer $i \in \{0, \ldots, d-1\}$ is a $K(d)$-jump of multiplicity $m > 0$ of $A$ if and only if

$$\dim(F_d^i \mathfrak{A}_k) - \dim(F_d^{i+1} \mathfrak{A}_k) = m.$$ 

But by construction, $F_d^i \mathfrak{A}_k = \mathfrak{A}_k^{k/d} \mathfrak{A}_k$ for every $k \in \{0, \ldots, d-1\}$, so the result follows from Theorem 3.1.3. 

**Corollary 3.1.5.** We keep the notations of Theorem 3.1.3. The jumps of $A$ are rational numbers, and the stabilization index $e(C)$ is the smallest integer $n > 0$ such that $n \cdot j$ is integer for every jump $j$ of $A$. In particular, $e(C)$ only depends on the abelian $K$-variety $A$, and not on $C$.

**Proof.** Let $\mathcal{C}$ be the minimal $\text{sncd}$-model of $C$. By Theorem 4.2.4 we can find a smooth, projective and geometrically connected $\mathbb{C}(t)$-curve $D$ of genus $g$ and an $\text{sncd}$-model $\mathcal{D}/\mathbb{C}[[t]]$ of $D$ such that the special fibers $\mathcal{C}_d$ and $\mathcal{D}_C$ have the same combinatorial data. Then $\mathcal{D}$ is the minimal $\text{sncd}$-model of $D$ and $e(C) = e(D)$. Set $B = \text{Jac}(D)$. By Theorem 3.1.3 the jumps of $B$ and their multiplicities are the same as those of $A$. Thus we may assume that $K = \mathbb{C}((t))$. Then $A$ is tamely ramified, and $e(A)$ equals the degree of the minimal extension of $\mathbb{C}((t))$ where $A$ acquires stable reduction, by Proposition 2.2.4. The result now follows from [Ha10b] 2.4.

The following corollary shows how the jumps of $A = \text{Jac}(C)$ are related to the characteristic polynomial $P_C(t)$ that we introduced in Chapter 3 Definition 2.1.2.

**Corollary 3.1.6.** We keep the notations of Theorem 3.1.3. For every jump $j$ of $A$, the value $\exp(2\pi ij)$ is a root of the characteristic polynomial $P_C(t)$ of multiplicity at least $m_A(j)$.

**Proof.** Let $\mathcal{C}$ be the minimal $\text{sncd}$-model of $C$. It is clear that $P_C(t)$ can be computed from the combinatorial data of $\mathcal{C}_k$, and by Theorem 3.1.3 the same is true for the jumps of $A$ and their multiplicities. Thus we can reduce to the case $K = \mathbb{C}((t))$ as in the proof of Corollary 3.1.5.
Let $\sigma$ be a topological generator of the absolute Galois group $\text{Gal}(K^*/K)$. We know by Proposition 2.1.3 that $P_C(t)$ is the characteristic polynomial of the action of $\sigma$ on 

$$H^1(C \times_K K^*, \mathbb{Q}_l) \cong (T_lA)^\vee \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$$

so that the result follows from [HN11a, 5.14]. \hfill $\square$
CHAPTER 6

The base change conductor and the Artin conductor

In this chapter, we assume that $k$ is algebraically closed. We will compare the base change conductor of the Jacobian variety of a $K$-curve $C$ to Saito’s Artin conductor of $C$ and other invariants of the curve, assuming that the genus of $C$ is 1 or 2.

1. Some comparison results

1.1. Algebraic tori.

(1.1.1) Let $T$ be an algebraic torus over $K$. We denote its cocharacter module by $X_*(T)$. The Artin conductor $\text{Art}(T)$ of $T$ is defined as the Artin conductor of the $\text{Gal}(K^s/K)$-module $X_*(T) \otimes_\mathbb{Z} \mathbb{Q}$. Likewise, we define the Swan conductor $\text{Sw}(T)$ as the wild part of the Artin conductor of $X_*(T) \otimes_\mathbb{Z} \mathbb{Q}$.

(1.1.2) A deep result of Chai and Yu [CY01, §11 and §12] states that the base change conductor $c(T)$ of $T$ is related to the Artin conductor by the formula

\[
c(T) = \frac{\text{Art}(T)}{2},
\]

In particular, $c(T)$ is invariant under isogeny. This is not the case for the elementary divisors of $T$, which depend on the integral structure of the Galois module $X_*(T)$ [Ch00, 8.5]. We will now give a similar interpretation of the tame base change conductor of $T$.

**Proposition 1.1.4.** Let $T$ be an algebraic $K$-torus of dimension $g$ with character module $X(T)$. Denote by $\text{Sw}(T)$ the wild part of the Artin conductor $\text{Art}(T)$. The tame part

\[
\text{Art}(T) - \text{Sw}(T) = g - \text{rank}_\mathbb{Z} X(T)^l
\]

of the Artin conductor is equal to the unipotent rank $u(T)$ of $T$, i.e., the unipotent rank of the identity component of the special fiber of the Néron model of $T$. Moreover, ordering $\mathbb{N}'$ by the divisibility relation, we have the following equalities:

\[
\frac{\text{Sw}(T)}{2} = \lim_{d \in \mathbb{N}'} \frac{c(T(d))}{d},
\]

\[
\frac{u(T)}{2} = c_{\text{tame}}(T).
\]

**Proof.** It is well-known that

\[
u(T) = g - \text{rank}_\mathbb{Z} X(T)^l.
\]
A proof can be found, for instance, in [HN10 3.14]. The equality (1.1.3) is a direct consequence of (1.1.5) and Proposition 1.2.3 in Chapter 2. The equality (1.1.6) follows immediately from (1.1.5), using (1.2.6) in Chapter 5 and Corollary 1.3.11.

1.2. Saito’s discriminant-conductor formula.

(1.2.1) For abelian varieties, the base change conductor can change under isogenies [Ch00 6.10], so that we cannot hope to express the base change conductor in terms of the $\mathbb{Q}_\ell$-adic Tate module – except in some special cases; see for instance [Ch00 5.2] for the case of potential multiplicative reduction.

(1.2.2) Nevertheless, it is possible to rewrite the formula for the base change conductor of an elliptic curve in Proposition 2.2.1 in such a way that the Artin conductor of the curve appears in the expression. The proper notion of Artin conductor to use in this setting is the one from Saito’s influential paper [Sa88]. If $X$ is a regular proper flat $R$-scheme and $\ell$ is a prime different from $p$, then the $\ell$-Artin conductor of $X$ is defined as

$$\text{Art}_\ell(X) = \chi(X_K) - \chi(X_k) + \sum_{i \geq 1} (-1)^i \text{Sw}_RH^1(X_K \times_K K^\times, \mathbb{Q}_\ell)$$

where Sw$_R$ denotes the Swan conductor. If $X$ has relative dimension at most one over $R$, then it is known that the Artin conductor is independent of $\ell$; it will be simply denoted by Art$(X)$. When $C$ is a geometrically connected smooth projective $K$-curve of genus $g \geq 1$, then we set

$$\text{Art}(C) = \text{Art}(\mathcal{C})$$

where $\mathcal{C}$ is the minimal regular $R$-model of $C$. The invariant Art$(C)$ vanishes if and only if $C$ has good reduction, except if $C$ is a genus one curve without rational point whose Jacobian has good reduction. We define the tame part of the Artin conductor of $C$ by

$$\text{Art}_{\text{tame}}(C) = \chi(C) - \chi(C_k).$$

(1.2.3) If $K'$ is a finite separable extension of $K$ with valuation ring $R'$, and if $S = \text{Spec } R'$, then

$$\text{Art}(S) = [K':K] - 1 + \text{Sw}_R \mathbb{Q}_\ell[K'/K]$$

which is the usual Artin conductor of the extension $K'/K$. Thus Art$_\ell(X)$ is a natural generalization of the Artin conductor for finite separable extensions of $K$. The classical discriminant-conductor formula states that

$$\text{Art}(S) = v_K(\Delta_{K'/K})$$

where $\Delta_{K'/K}$ denotes the discriminant of the extension $K'/K$. Saito generalized this result to curves $C$ over $K$, where the valuation of the discriminant is defined by measuring the degeneration of a certain morphism of rank one free $R$-modules that is an isomorphism over $K$ [Sa88 Thm.1]. Here we will only consider elliptic and hyperelliptic curves, in which case the valuation of the discriminant can be defined in a more explicit way.

2. Elliptic curves

2.1. The potential degree of degeneration.
For elliptic curves, the valuation of Saito’s discriminant coincides with the valuation of the classical minimal discriminant of a Weierstrass equation [Sa88, §1, Cor.2]. Let $E$ be an elliptic $K$-curve with minimal discriminant $\Delta$ and $j$-invariant $j(E)$, and let $\mathcal{X}$ be the minimal regular $R$-model of $E$. Then

$$\text{Art}(E) = -\chi(\mathcal{X}_k) - \delta(E)$$

where $\delta(E)$ denotes the wild part of the conductor of $E$ (i.e., the Swan conductor of $T_1E \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$). Since $\text{Art}(E)$ vanishes if and only if $E$ has good reduction, it is not reasonable to expect that we can compute $c(E)$ directly from $\text{Art}(E)$ (recall that $c(E)$ vanishes as soon as $E$ has semi-abelian reduction). There should be a second term involved that measures the potential degeneration. A natural candidate is the invariant $\delta_{\text{pot}}(E)$ that we define as follows.

**Definition 2.1.2.** Let $E$ be an elliptic curve over $K$, and let $L$ be a finite separable extension of $K$ such that $E \times_K L$ has semi-abelian reduction. We define the potential degree of degeneration $\delta_{\text{pot}}(E)$ as

$$\delta_{\text{pot}}(E) = \left\{ \begin{array}{cl} 0 & \text{if } E \times_K L \text{ has good reduction,} \\ \frac{1}{[L:K]} |\Phi(E \times_K L)| & \text{if } E \times_K L \text{ has multiplicative reduction.} \end{array} \right.$$ 

This definition does not depend on the choice of $L$, by the following lemma.

**Lemma 2.1.3.** For every elliptic $K$-curve $E$, we have

$$\delta_{\text{pot}}(E) = \left\{ \begin{array}{cl} 0 & \text{if } E \text{ has potential good reduction,} \\ -v_K(j(E)) & \text{if } E \text{ has potential multiplicative reduction.} \end{array} \right.$$ 

**Proof.** This follows immediately from the equality $|\Phi(E)| = -v_K(j(E))$ for elliptic $K$-curves $E$ with multiplicative reduction [Si94, 9.2(d)]. $\square$

### 2.2. A formula for the base change conductor.

We will now explain how to compute $c(E)$ from $\text{Art}(E)$ and $\delta_{\text{pot}}(E)$.

**2.2.1** Saito’s discriminant-conductor formula for curves [Sa88, Thm.1] implies that

$$v_K(\Delta) = -\text{Art}(E) = \chi(\mathcal{X}_k) + \delta(E),$$

which can be rewritten as Ogg’s formula

$$v_K(\Delta) = f(E/K) + m - 1$$

where $f(E/K)$ is the conductor of $E$ and $m$ denotes the number of irreducible components of $\mathcal{X}_k$. The equality

$$-\text{Art}(C) = f(C/K) + m - 1$$

holds for curves of arbitrary genus $g \geq 1$, by [Li94, Prop.1]; for elliptic curves, it can also easily be checked by a case-by-case computation on the Kodaira-Néron reduction table.
Using Saito’s discriminant-conductor formula, we can rewrite the formula for \( c(E) \) in the following way.

**Proposition 2.2.3.** For every elliptic \( K \)-curve \( E \), we have

\[
    c(E) = -\frac{1}{12}(\text{Art}(E) + \delta_{\text{pot}}(E)).
\]

**Proof.** Immediate from Proposition 2.2.1 and Lemma 2.1.3. \( \square \)

Using this result, we can also interpret the wild part of the base change conductor, i.e., the difference \( c(E) - c_{\text{tame}}(E) \), when \( E \) is wildly ramified.

**Proposition 2.2.4.** Let \( E \) be an elliptic curve over \( K \), and denote by \( \delta(E) \) the wild part of the conductor of \( E \).

1. We have

\[
    c_{\text{tame}}(E) = -\frac{1}{12}(\text{Art}_{\text{tame}}(E) + \delta_{\text{pot}}(E)),
\]

\[
    c(E) - c_{\text{tame}}(E) = \frac{\delta(E)}{12},
\]

unless \( p = 2 \) and \( E \) has reduction type \( I_n^* \) for some \( n > 0 \).

2. Assume that \( p = 2 \) and that \( E \) has reduction type \( I_n^* \) for some \( n > 0 \). Then \( E \) is wildly ramified. If \( E \) has potential good reduction, then

\[
    c(E) - c_{\text{tame}}(E) = \frac{1}{12}(\delta(E) + n).
\]

If \( E \) has potential multiplicative reduction, then

\[
    c(E) - c_{\text{tame}}(E) = \frac{\delta(E)}{4}.
\]

Moreover, in the latter case,

\[
    v_K(j(E)) = 2\delta(E) - n \quad \text{and} \quad c(E) = \frac{1}{4}(\delta(E) + 2).
\]

3. We have \( c(E) = c_{\text{tame}}(E) \) if and only if \( E \) is tamely ramified.

**Proof.** Let \( E' \) be an elliptic curve over \( \mathbb{C}(t) \) with the same reduction type as \( E \). Theorem 3.1.3 tells us that \( c_{\text{tame}}(E) = c_{\text{tame}}(E') \), and we know that \( c_{\text{tame}}(E') = c(E') \) because \( E' \) is tamely ramified. Thus

\[
    c(E) - c_{\text{tame}}(E) = c(E) - c(E').
\]

1. First, assume that \( p \neq 2 \). Then \( E \) and \( E' \) have the same potential reduction type, since \( E \) has potential multiplicative reduction if and only if it is of type \( I_n \) or \( I_n^* \), with \( n > 0 \), in which case \( \delta_{\text{pot}}(E) = n \) (this can be deduced from Table 4.1 in [Si86, IV]). Proposition 2.2.3 now yields

\[
    c_{\text{tame}}(E) = c(E') = -\frac{1}{12}(\text{Art}_{\text{tame}}(E) + \delta_{\text{pot}}(E)),
\]

\[
    c(E) - c(E') = \frac{1}{12} \cdot \text{Sw}_{R}H^1(E \times_K K^*, \mathbb{Q}_l) = \frac{\delta(E)}{12}.
\]

The case \( p = 2 \) is more delicate, because there exist elliptic curves with reduction type \( I_n^* \), \( n > 0 \), and potential good reduction. The elliptic curves with potential multiplicative reduction have been classified by Lorenzini in [Lo10, 2.8].

[2.2.2]
His result implies that each such curve \( E \) has reduction type \( I_n^* \) for some \( n > 0 \) and that \( \varpi_{\text{pot}}(E) = n \). Thus if \( E \) does not have reduction type \( I_n^* \), our argument from the \( p \neq 2 \) case again yields
\[
c_{\text{tame}}(E) = -\frac{1}{12}(\text{Art}_{\text{tame}}(E) + \varpi_{\text{pot}}(E)),
\]
\[
c(E) - c(E') = \frac{\delta(E)}{12}.
\]
This concludes the proof of (1).

(2) Assume that \( p = 2 \) and that \( E \) has reduction type \( I_n^* \) for some \( n > 0 \). Then \( E \) is wildly ramified, by Saito’s criterion [Sa87 3.11]. If \( E \) has potential good reduction, Proposition 2.2.3 yields
\[
c(E) - c(E') = \frac{1}{12}(\delta(E) + \varpi_{\text{pot}}(E')) = \frac{1}{12}(\delta(E) + n).
\]
If \( E \) has potential multiplicative reduction, then we have \( \varpi_{\text{pot}}(E) = -v_K(j(E)) \) so that
\[
c(E) - c(E') = \frac{1}{12}(\delta(E) + n + v_K(j(E))).
\]
Moreover, \( c(E') = 1/2 \) by Table 2.2.2 and it follows from [Ch00 5.2] that
\[
c(E) = \frac{2 + \delta(E)}{4}.
\]
Thus
\[
c(E) - c(E') = \frac{\delta(E)}{4} \text{ and } v_K(j(E)) = 2\delta(E) - n.
\]
(3) This is obvious, since \( \delta(E) > 0 \) when \( E \) is wildly ramified. \( \square \)

3. Genus two curves

Looking at the formula in Proposition 2.2.3, it is natural to wonder if there exists a similar formula for higher genus curves, expressing the base change conductor of the Jacobian in terms of the Artin conductor of the curve and a suitable measure of potential degeneration. To conclude this chapter we will now briefly discuss the case of genus 2 curves, and we will see that already there the situation gets more delicate. Let \( C \) be a smooth geometrically connected projective \( K \)-curve of genus 2. We will describe the relationship between the base change conductor of the Jacobian of \( C \) and the minimal discriminant and Artin conductor of \( C \). Detailed proofs, as well as generalizations to curves of higher genus, will be given in future work.

3.1. Hyperelliptic equations.

(3.1.1) We start by recalling a few facts that can be found in [Li94]. By a \textit{hyperelliptic equation} for \( C \), we mean an affine equation
\[
(E) \quad y^2 + Q(x)y = P(x)
\]
where \( P, Q \in K[x] \) satisfy \( \deg(Q) \leq 3 \) and \( \deg(P) \leq 6 \), and with the property that
\[
\text{Spec } K[x, y]/(y^2 + Q(x)y - P(x)) \text{ is isomorphic to an open dense subset of } C.
\]
Such an equation can always be found.
The discriminant of \((E)\) is defined as
\[
\Delta(E) = 2^{-12} \text{disc}_6(4P(x) + Q(x)^2)
\]
where \(\text{disc}_6(4P(x) + Q(x)^2)\) denotes the discriminant of \(4P(x) + Q(x)^2\) considered as a polynomial of degree 6 in the variable \(x\).

Also associated to \((E)\) are the differential forms
\[
\omega_i(E) = \frac{x^{i-1}dx}{2y + Q(x)}
\]
for \(i \in \{1, 2\}\), which form a \(K\)-basis of \(H^0(C, \Omega^1_C)\).

If \((E')\)
\[
z'^2 + Q'(v)z = P'(v)
\]
is another hyperelliptic equation for \(C\), there exists an element \(u \in K\) such that
\[
\Delta(E') = u^{10} \cdot \Delta(E)
\]
and
\[
\omega_1(E') \land \omega_2(E') = u^{-1} \cdot \omega_1(E) \land \omega_2(E).
\]
Thus the element
\[
\Delta(E)(\omega_1(E) \land \omega_2(E)) \otimes 10 \in (\wedge^2 H^0(C, \Omega^1_C/K)) \otimes 10
\]
is independent of the choice of hyperelliptic equation.

### 3.2. Minimal equations.

#### (3.2.1) Let \(\mathcal{X}'\) be a regular model of \(C\) over \(R\), and denote by \(\omega_{\mathcal{X}'/R}\) its relative canonical sheaf. Then \(H^0(\mathcal{X}', \omega_{\mathcal{X}'/R})\) is a free \(R\)-module of rank 2, and we have a canonical morphism of \(K\)-vector spaces
\[
H^0(\mathcal{X}', \omega_{\mathcal{X}'/R}) \otimes_R K \to H^0(C, \Omega^1_C/K)
\]
so that we can view \(H^0(\mathcal{X}', \omega_{\mathcal{X}'/R})\) as an \(R\)-lattice in \(H^0(C, \Omega^1_C/K)\).

**Definition 3.2.2 (Definition 1 of \([Li94]\)).** Let \(\mathcal{C}\) be the minimal regular model of \(C\). We say that \((E)\) is a minimal equation of \(\mathcal{C}\) if \(\{\omega_1(E), \omega_2(E)\}\) is an \(R\)-basis of the lattice \(H^0(\mathcal{C}, \omega_{\mathcal{C}/R})\). In this case, we call the integer \(v_K(\Delta(E))\) the minimal discriminant of \(C\), and we denote it by \(v(\Delta(C))\).

#### (3.2.3) By \([Li94]\) Prop. 2] one can always find a minimal equation \((E)\). The definition of \(v(\Delta(C))\) is independent of choice of such \((E)\).

#### (3.2.4) Let \(K'/K\) be a finite separable field extension and let \((E)\) be a minimal equation for \(C\). It can also be viewed as a hyperelliptic equation for \(C' = C \times_K K'\), but over \(K'\), the equation \((E)\) may no longer be minimal. If \((E')\) is a minimal equation for \(C'\), we have
\[
\Delta(E') = r^{10} \cdot \Delta(E)
\]
and
\[
\omega_1(E) \land \omega_2(E) = r \cdot \omega_1(E') \land \omega_2(E')
\]
for a certain element \(r \in K'\).
3.3. Comparison of the base change conductor and the minimal discriminant.

(3.3.1) Assume that the curve $C$ has index one. We put $A = \text{Jac}(C)$, and we denote by $\mathcal{A}$ the Néron model of $A$. Let moreover $f : \mathcal{C} \to \text{Spec } R$ be the minimal regular model of $C$. One can show that there exists a canonical isomorphism

$$\omega_{\mathcal{A}/R} \cong f_*(\omega_{\mathcal{C}/R}),$$

where $\omega_{\mathcal{A}/R}$ is the module of invariant differential forms associated to $\mathcal{A}$. In particular, the differentials $\omega_1(\mathcal{E})$ and $\omega_2(\mathcal{E})$ form an $R$-basis of $\omega_{\mathcal{A}/R}$, and using this fact it is straightforward to see that, for every finite separable extension $K'/K$, we can compute the $K'$-base change conductor as

$$c(A, K') = \frac{1}{[K': K]} \cdot v_{K'}(r)$$

where the element $r$ of $K'$ is defined as in (3.2.4).

(3.3.2) We assume now in addition that $A$ acquires semi-abelian reduction over $K'$, which is equivalent to the property that $C' = C \times_K K'$ has a semi-stable model over $R'$. Equation (3.2.5) yields the following formula:

$$v(\Delta(C')_{\text{min}}) = 10 \cdot c(A) + v(\Delta(C')_{\text{min}})/[K': K].$$

We next compute $v(\Delta(C')_{\text{min}})$ purely in terms of a semi-stable reduction of $C'$. In order to do this, recall that Saito introduced in [Sa89] three “discriminants” $\Delta(C)$, $\Delta'(C)$ and $\Delta_1(C)$ to which he associated three values $\text{ord } \Delta(C)$, $\text{ord } \Delta'(C)$ and $\text{ord } \Delta_1(C)$ in $\mathbb{Z}$. Moreover, he showed that

$$\text{ord } \Delta'(C) = -\text{Art}_{\mathcal{C}/S}$$

and that

$$\text{ord } \Delta_1(C) = \text{Ar}_{\mathcal{C}/S} + \text{ord } \Delta_1(C).$$

On the other hand, Liu showed in [Li94] that

$$\text{ord } \Delta'(C) = v(\Delta(C)_{\text{min}}).$$

Let $\mathcal{C}'$ denote the minimal regular model of $C'$. By our assumption, the special fiber $\mathcal{C}'_k$ is a semi-stable curve. We denote by $\sigma$ the cardinality of the set $\text{Sing}(\mathcal{C}'_k)$ of singular points on $\mathcal{C}'_k$. It is easy to verify that

$$-\text{Ar}_{\mathcal{C}'/R} = \sigma$$

(see for instance [Li94 Prop. 1]). Let moreover $\tau$ denote the cardinality of the set of points $p \in \text{Sing}(\mathcal{C}'_k)$ such that $\mathcal{C}'_k - \{p\}$ is disconnected. Then the proposition on page 234 of [Sa89] tells us that

$$\text{ord } \Delta_1(C') = \tau.$$

Combining Equations (3.3.3) - (3.3.7) now yields an interesting relationship between the minimal discriminant and the base change conductor. Note the similarity to the formula for elliptic curves in Proposition 2.2.3: here the value $\sigma + \tau$ plays the role of the degree of potential degeneration. In particular, it vanishes if and only if $C$ has potential good reduction.

**Proposition 3.3.8.** With the notation introduced above, we have

$$v(\Delta_{\text{min}}) = 10 \cdot c(A) + (\sigma + \tau)/[K': K].$$
(3.3.9) The relationship between the base change conductor $c(A)$ and the Artin conductor $\text{Art}_{\mathcal{E}/S}$ is more subtle. By (3.3.4) and (3.3.5), the difference between $v(\Delta(C)_{\text{min}})$ and $-\text{Art}_{\mathcal{E}/S}$ is precisely measured by $\text{ord} \Delta_1$. In [Li94], Liu shows that $\text{ord} \Delta_1$ can be computed entirely from the numerical type of the minimal regular model $\mathcal{C}$. In particular, $\text{ord} \Delta_1 \neq 0$ for certain numerical types (see [Li94 Prop. 7+8]). In view of Proposition (3.3.8), this means that $-\text{Art}_{\mathcal{E}/S}$ differs “more” from $c(A)$ than $v(\Delta_{\text{min}})$. 
Part 3

Applications to motivic zeta functions
Motivic zeta functions of semi-abelian varieties

In this chapter, we assume that \( k \) is algebraically closed. We will prove in Theorem 3.1.2 the rationality of the motivic zeta function of a Jacobian variety, and we show that it has a unique pole, which coincides with the tame base change conductor from Chapter 5. We will also investigate the case of Prym varieties.

1. The motivic zeta function

1.1. Definition.

(1.1.1) We'll recall the definition of the motivic zeta function \( Z_G(T) \) of a semi-abelian \( K \)-variety \( G \), which was introduced in [HN11a 8.1]. It measures how the Néron model of \( G \) varies under tamely ramified extensions of \( K \).

(1.1.2) First we need to introduce some notation. Let \( \mathcal{K}_0(\text{Var}_k) \) be the Grothendieck ring of \( k \)-varieties. For each \( k \)-variety \( V \), we denote by \([V]\) its class in \( \mathcal{K}_0(\text{Var}_k) \). We set \( \mathcal{L} = [\mathbb{A}^1_k] \). We refer to [NS11] for the definition of the Grothendieck ring and its basic properties.

(1.1.3) For every \( d \in \mathbb{N}' \), we denote by \( \mathcal{G}^\mathrm{qc}(d) \) the Néron model of \( G \times_k K(d) \) as defined in Chapter 2 (3.1.5). Recall that, by definition, \( \mathcal{G}^\mathrm{qc}(d) \) is the largest quasi-compact open subgroup scheme of the Néron lft-model \( \mathcal{G}(d) \) of \( G(d) \). We’ll write \( \mathcal{G} \) and \( \mathcal{G}^\mathrm{qc} \) instead of \( \mathcal{G}(1) \) and \( \mathcal{G}^\mathrm{qc}(1) \) for the Néron lft-model, resp. Néron model, of \( G \). We denote by \( h_d \) the canonical base change morphism

\[
h_d : \mathcal{G}^\mathrm{qc} \times_R R(d) \to \mathcal{G}^\mathrm{qc}(d).
\]

**Definition 1.1.4.** The order function

\[
\text{ord}_G : \mathbb{N}' \to \mathbb{N}
\]

is defined by

\[
\text{ord}_G(d) = c(G, K(d)) \cdot d = \text{length}_{\text{cok}(\text{Lie}(h_d))}.
\]

(1.1.5) The values \( \text{ord}_G(d) \) were defined in a different way in [HN11a 7.2], but we proved the formula \( \text{ord}_G(d) = c(G, K(d)) \cdot d \) in [HN11a 7.5]. For the purpose of this paper, this formula can serve as a definition. Note that we would have obtained the same function \( \text{ord}_G \) by working with Néron lft-models instead of Néron models. The reason for working with Néron models is that in the following definition, we need the property that \( \mathcal{G}^\mathrm{qc}(d)_k \) is of finite type over \( k \), in order to take its class in the Grothendieck ring \( \mathcal{K}_0(\text{Var}_k) \) (note that the Grothendieck ring of \( k \)-schemes locally of finite type is trivial by the existence of infinite coproducts).
We define the motivic zeta function $Z_G(T)$ of $G$ as

$$Z_G(T) = \sum_{d \in \mathbb{N}'} [\mathcal{G}^\text{qc}(d)_k]^{\text{ord}_G(d)} T^d \in K_0(\text{Var}_k)[[T]].$$

1.2. Decomposing the identity component.

(1.2.1) We see from the definition that the motivic zeta function $Z_G(T)$ depends on two factors: the behaviour of the class $[\mathcal{G}^\text{qc}(d)_k]$, and the order function $\text{ord}_G$. We will now analyze the class $[\mathcal{G}^\text{qc}(d)_k]$ in more detail. For each $d \in \mathbb{N}'$, we consider the Chevalley decomposition

$$0 \to T(d) \times U(d) \to \mathcal{G}(d)_k^\sigma \to B(d) \to 0$$

of $\mathcal{G}(d)_k^\sigma$, with $T(d)$ a $k$-torus, $U(d)$ a unipotent $k$-group and $B(d)$ an abelian $k$-variety. The dimensions of $T(d)$ and $U(d)$ are called the toric and unipotent rank of $G(d)$ and denoted by $t(G(d))$ and $u(G(d))$; see Section 3.2 in Chapter 2.

Proposition 1.2.2. For each $d \in \mathbb{N}'$, we have

$$[\mathcal{G}^\text{qc}(d)_k] = [\Phi(G(d))_{\text{tors}}] \cdot [\mathcal{G}(d)_k^\sigma] = [\Phi(G(d))_{\text{tors}}] \cdot (L - 1)(L)^{(G(d))} [B(d)]$$

in $K_0(\text{Var}_k)$.

Proof. Since $k$ is algebraically closed, each connected component of $\mathcal{G}^\text{qc}(d)_k$ is isomorphic to the identity component $\mathcal{G}(d)_k^\sigma$. The statement now follows from the computation of the class in $K_0(\text{Var}_k)$ of a connected smooth commutative $k$-group in [Ni11b] 3.1.

(1.2.3) We’ve made an extensive study of the groups $\Phi(G(d))_{\text{tors}}$ in Part 1. If $G$ is tamely ramified, then we have analyzed the behaviour of the order function $\text{ord}_G$ and the invariants $t(G(d))$, $u(G(d))$ and $B(d)$ in [HN11a] §6-7. In the next section, we will extend these results to (possibly wildly ramified) Jacobians, using our results in Chapters 3 and 5.

2. Motivic zeta functions of Jacobians

2.1. Behaviour of the identity component.

(2.1.1) Let $C$ be a smooth, projective, geometrically connected $K$-curve of genus $g > 0$ and index one. We denote by $\mathcal{C}$ the minimal $\text{snec}$-model of $C$ over $R$. Recall that for every $d \in \mathbb{N}'$ we construct from $\mathcal{C}$ an $\text{snec}$-model $\mathcal{C}(d)$ of $C(d) = C \times_k K(d)$ by the procedure explained in Section 1 of Chapter 3. We write its special fiber as

$$\mathcal{C}(d)_k = \sum_{i \in I(d)} N(d)_i E(d)_i.$$
3. Rationality and Poles

3.1. Rationality of the zeta function.

Theorem 3.1.2. Let $G$ be either a tamely ramified semi-abelian $K$-variety or the Jacobian of a smooth, projective, geometrically connected $K$-curve of index one. We denote by $G_{ab}$ the abelian part of $G$; in particular, $G = G_{ab}$ when $G$ is a Jacobian.

The motivic zeta function $Z_G(T)$ is rational, and belongs to the subring

$$\mathcal{R}_{c_{\text{tame}}(G)} = K_0(\text{Var}_k) \left[ T^\frac{1}{1 - \sum_{a/b \in Z_{>0}, a/b = c_{\text{tame}}(G)} T^a} \right]$$

of $K_0(\text{Var}_k)[[T]]$. The zeta function $Z_G(\mathbb{L}^{-s})$ has a unique pole at $s = c_{\text{tame}}(G)$, whose order is equal to $t_{\text{tame}}(G_{ab}) + 1$. The degree of $Z_G(T)$ is zero if $p = 1$ and $G$ has potential good reduction, and negative in all other cases.
Then, for every topological generator $\sigma$ the action of $\sigma$ every prime $p$, by Proposition 2.2.3 in Chapter 3. The proof of this case follows the same lines as the proof of Theorem 3.1.5 in Chapter 3.

We set $e = e(C)$ to ease notation. For every element $\alpha$ in $\{1, \ldots, e\} \cap \mathbb{N}$ we define the auxiliary series

$$Z_{G}^{(\alpha)}(T) = \sum_{d \in (\alpha + \mathbb{N})} \left( |\Phi(G(d))| [G(d)_{k}]^{0} \right)^{e_{\text{ord}}(d) T^{e}} \in K_{0}(\text{Var}_{k})[[T]].$$

Then we have

$$Z_{G}(T) = \sum_{\alpha \in \{1, \ldots, e\} \cap \mathbb{N}} Z_{G}^{(\alpha)}(T).$$

We fix an element $\alpha$ in $\{1, \ldots, e\} \cap \mathbb{N}$ and we set $\alpha' = \gcd(\alpha, e)$. We know that $e(C(\alpha')) = e/\alpha'$ by Proposition 2.2.3 in Chapter 3. Then for every integer $d = \alpha + qe$ with $q \in \mathbb{N}$, we have $[\alpha'(d)_{k}]^{0} = [\alpha'(\alpha')_{k}]^{0}$ in $K_{0}(\text{Var}_{k})$ by Proposition 2.1.3

$$|\Phi(G(d))| = (d/\alpha')^{t(G(\alpha'))} |\Phi(G(\alpha'))|$$

by Proposition 3.1.1 in Chapter 3 and

$$\text{ord}_{G}(d) = \text{ord}_{G}(\alpha) + qe \text{e_{tame}(G)}$$

by Proposition 2.2.2. We can therefore write

$$Z_{G}^{(\alpha)}(T) = |\Phi(G(\alpha))| [\alpha'(\alpha')_{k}]^{0} T^{e_{\text{ord}}(\alpha)} S_{G}^{(\alpha)}(T)$$

with

$$S_{G}^{(\alpha)}(T) = \sum_{q \in \mathbb{N}} \left( \frac{qe + \alpha}{\alpha'} \right)^{t(G(\alpha'))} T^{q \text{e_{tame}(G)}}.$$ 

The rest of the proof is completely analogous to [HN11a, 8.6].

3.2. Poles and monodromy.

(3.2.1) If $G$ is a tamely ramified abelian $K$-variety, we can relate the unique pole $s = c_{\text{tame}}(G)$ of $Z_{G}(T)$ to the monodromy action on the cohomology of $G$, as follows. As we’ve explained in [HN11a, §2.5], this result can be viewed as a global version of the Motivic Monodromy Conjecture for semi-abelian $K$-varieties.

**Theorem 3.2.2.** Let $G$ be a tamely ramified semi-abelian $K$-variety of dimension $g$. Let $m$ be the order of $c(G) = c_{\text{tame}}(G)$ in $\mathbb{Q}/\mathbb{Z}$, and let $\Phi_{m}(T)$ be the cyclotomic polynomial whose roots are the primitive $m$-th roots of unity. Then, for every topological generator $\sigma$ of the tame inertia group $\text{Gal}(K^{1}/K)$ and every prime $\ell \neq p$, the polynomial $\Phi_{m}(T)$ divides the characteristic polynomial of the action of $\sigma$ on $H^{g}(G \times_{K} K^{1}, \mathbb{Q}_{\ell})$. Thus for every embedding of $\mathbb{Q}_{\ell}$ in $\mathbb{C}$, the value $\exp(2\pi i c(G) i)$ is an eigenvalue of $\sigma$ on $H^{g}(G \times_{K} K^{1}, \mathbb{Q}_{\ell})$.

**Proof.** See Corollary 5.15 in [HN11a].
It is not at all clear how to find a similar cohomological interpretation of $c_{\text{tame}}(G)$ when $G$ is wildly ramified. The tame cohomology spaces $H^i(G \times_K K^t, \mathbb{Q}_\ell)$ certainly contain too little information; for instance, $H^1(G \times_K K^t, \mathbb{Q}_\ell)$ is trivial for every wildly ramified elliptic $K$-curve. On the other hand, the inertia group $I$ is not procyclic if $p > 1$, so that we cannot associate eigenvalues to its action on $H^0(G \times_K K^t, \mathbb{Q}_\ell)$ in any straightforward way.

### 3.3. Prym varieties.

(3.3.1) The results that we’ve obtained for Jacobians can be extended to a larger class of abelian $K$-varieties, the so-called Prym varieties. In particular, we will explain how one can prove the rationality of Edixhoven’s jumps and of the motivic zeta function for this class. A suitable reference for the theory of Prym varieties is [ABH02], where it is not assumed that the ground field is algebraically closed. However, whenever we work with Prym varieties, we assume that $\text{char}(k) \neq 2$.

(3.3.2) Let $C$ be a smooth, projective and geometrically connected $K$-curve of genus $g > 0$. Assume that $C$ carries an involution $\iota : C \to C$ with either 0 or 2 fixed points. We’ll refer to the first situation as case (0) and to the latter as case (2). We denote by

$$\pi : C \to C_1$$

the quotient map. Put $A = \text{Jac}(C)$ and $A_1 = \text{Jac}(C_1)$. There is induced a norm map

$$Nm : A \to A_1$$

which can be described (on points) as

$$\sum m_Q [Q] \to \sum m_Q [\pi(Q)].$$

The following result is well-known (cf. [ABH02] Ch. 1).

**Proposition 3.3.3.**

1. The norm map is surjective with smooth kernel. The identity component $P := \ker(Nm)^0$ is an abelian $K$-variety.
2. In case (0), the kernel $\ker(Nm)$ has 2 connected components, and in case (2) it is connected.
3. Let $\mathcal{L} = \mathcal{L}(\Theta)$ be the theta divisor on $A$. The restriction $\mathcal{M} = \mathcal{L}|_P$ is twice a principal polarization.

**Definition 3.3.4.** The abelian $K$-variety $P$ is called the Prym variety associated to $(C, \iota)$.

(3.3.5) We denote by $u_K : P \to A$ the inclusion morphism. Let $\phi_{\mathcal{L}} : A \to A^\vee$ be the principal polarization associated to $\mathcal{L}$. We denote by $\overline{P}$ the complement of $P$ in $A$; it is given by

$$\overline{P} := (\ker(u_K^\vee \circ \phi_{\mathcal{L}})_{\text{red}})^0.$$ 

We consider the difference map

$$u_K : P \times_K \overline{P} \to A : (x, y) \mapsto x - y.$$
and the morphism $w_K : \overline{P} \to A_1$ induced by the norm map.

**Lemma 3.3.6.** The morphisms $v_K$ and $w_K$ are isogenies whose degree is a power of 2.

**Proof.** The kernel of $v_K$ can be identified with $P \times_A \overline{P}$. By construction, it is a subgroup of the kernel of $u_K^* \circ \phi_L \circ u_K : P \to P^\vee$, but this morphism is precisely the polarization $\phi_M$ associated to $M = u_K^* \mathcal{L}$. Since $M$ is twice a principal polarization by Proposition 3.3.3, the degree of $\phi_M$ is a power of 2. Thus the degree of $v_K$ is also a power of 2.

Now we turn our attention to $w_K : \overline{P} \to A_1$. We can factor $w_K$ as $\alpha : P \to A/P$ followed by $\beta : A/P \to A_1$. We've already seen that the degree of $\ker(\alpha) = P \times_A \overline{P}$ is a power of 2. The morphism $\pi : C \to C_1$ induces a morphism of abelian $K$-varieties $\gamma : A_1 \to A/P$ such that the composition $\beta \circ \gamma$ is multiplication by $\deg(\pi) = 2$. Thus the degree of $\beta$ is a power of 2, as well, and the same conclusion holds for the composition $W_K = \alpha \circ \beta$. $\Box$

**Proposition 3.3.7.** For every $d \in \mathbb{N}'$, we have

$$m_{A,K}(d) = m_{P,K}(d) + m_{A_1,K}(d).$$

In particular,

$$m_A = m_P + m_{A_1}.$$

**Proof.** For each $d \in \mathbb{N}'$ we have isogenies

$$v_{K(d)} : P(d) \times_R \overline{P}(d) \to A(d)$$

and

$$w_{K(d)} : \overline{P}(d) \to A_1(d)$$

whose degree is prime to $p$, by Lemma 3.3.6 and our assumption that $p \neq 2$. By [BLR90] 7.3.6, these morphisms extend uniquely to Galois equivariant isogenies

$$v(d) : \mathcal{P}(d) \times_R \overline{\mathcal{P}}(d) \to \mathcal{A}(d)$$

and

$$w(d) : \overline{\mathcal{P}}(d) \to \mathcal{A}_1(d)$$

on the level of Néron models, and the degrees of these isogenies are still prime to $p$.

Passing to the special fibers and considering the tangent spaces at the origin, we get isomorphisms of $k[\mu_d(k)]$-modules

$$\text{Lie}(v(d)_k) = \text{Lie}(\mathcal{P}(d)_k) \oplus \text{Lie}(\overline{\mathcal{P}}(d)_k) \to \text{Lie}(\mathcal{A}(d)_k)$$

and

$$\text{Lie}(w(d)_k) : \text{Lie}(\overline{\mathcal{P}}(d)_k) \to \text{Lie}(\mathcal{A}_1(d)_k).$$

By (1.3.4) in Chapter 5, this implies that $m_{P,K(d)} + m_{\overline{\mathcal{P}},K(d)} = m_A$ and $m_{\overline{\mathcal{P}},K(d)} = m_{A_1,K(d)}$, so that $m_{A,K}(d) = m_{P,K(d)} + m_{A_1,K}(d)$. $\square$

**Corollary 3.3.8.** The jumps of $P$ are rational numbers.

**Proof.** This is an immediate consequence of Proposition 3.3.7 and Corollary 3.1.5 in Chapter 5. $\square$
Theorem 3.3.9. Assume that $P$ has potential multiplicative reduction. Then the motivic zeta function $Z_P(T)$ is rational, and belongs to the subring

$$\mathcal{R}_k^{\text{tame}}(P) = \mathcal{M}_k \left[ T, \frac{1}{1 - L a T^b} \right]_{(a,b) \in \mathbb{Z} \times \mathbb{N}, a/b = \text{c}_{\text{tame}}(P)}$$

of $\mathcal{M}_k[[T]]$. The zeta function $Z_P(\mathbb{L}^{-s})$ has a unique pole at $s = \text{c}_{\text{tame}}(P)$, whose order is equal to $t_{\text{tame}}(P) + 1$. The degree of $Z_P(T)$ is zero if $p = 1$ and $P$ has potential good reduction, and strictly negative in all other cases.

Proof. We can assume that $P$ is wildly ramified, since the tame case was settled in [HN11a, 8.6]. Let $L$ be the minimal extension of $K$ in $K^s$ such that $P \times_K L$ has semi-abelian reduction, and set $e_P = [L : K]$. Then we define

$$e(P) = \text{lcm}\{e_P, e(C), e(C_1)\}.$$  

For any $d \in \mathbb{N}'$ we have that $e(P(d)) = e(P)/\text{gcd}(d, e(P))$, since the same property holds for $e_P, e(C)$ and $e(C_1)$ individually.

We first observe that by Proposition 3.3.3, the equalities

$$\text{ord}_P(d) = c(P, K(d)) \cdot d = c(A, K(d)) \cdot d - c(A_1, K(d)) \cdot d = \text{ord}_A(d) - \text{ord}_{A_1}(d)$$

hold for all $d \in \mathbb{N}'$. Hence, for all integers $\alpha \in \mathbb{N}'$ and $q$ such that $\alpha + q \cdot e(P) \in \mathbb{N}'$, it follows from Proposition 2.2.2 that

$$\text{ord}_P(\alpha + q \cdot e(P)) = \text{ord}_A(\alpha) + q \cdot e(P) \cdot c_{\text{tame}}(A)$$

$$- (\text{ord}_{A_1}(\alpha) + q \cdot e(P) \cdot c_{\text{tame}}(A_1))$$

$$= (\text{ord}_A(\alpha) - \text{ord}_{A_1}(\alpha)) + q \cdot e(P) \cdot (c_{\text{tame}}(A) - c_{\text{tame}}(A_1))$$

$$= \text{ord}_P(\alpha) + q \cdot e(P) \cdot c_{\text{tame}}(P).$$

Secondly, we claim that $[\mathcal{R}_k^0] = [\mathcal{R}(d)^0_k]$ for each $d \in \mathbb{N}'$ that is prime to $e(P).$ To see this, note that

$$[\mathcal{R}(n)^0_k] = \mathbb{L}^{u(P(n))} \cdot (\mathbb{L} - 1)^{t(P(n))}$$

for all $n \in \mathbb{N}'$ because we assume that $P$ has potential multiplicative reduction. Thus it suffices to show that $u(P(d)) = u(P)$ and $t(P(d)) = t(P)$ for all prime to $e(P)$.

If $G$ denotes either $A$ or $A_1$, we know from Proposition 2.1.3 that $a(G(d)) = a(G), t(G(d)) = t(G)$ and $u(G(d)) = u(G)$ for each prime to $e(P)$. It follows from [BLR90, 7.3.6] that the invariants $a, t$ and $u$ are preserved under isogenies of degree prime to $p$, and it is obvious that they behave additively with respect to products of abelian varieties. We therefore conclude that

$$t(P(d)) = t(A_1(d)) = t(A) - t(A_1) = t(P)$$

and likewise that

$$u(P(d)) = u(A_1(d)) = u(A) - u(A_1) = u(P).$$

This proves the claim.

Since we assume that $P$ has potential multiplicative reduction, [HN10, 5.7] asserts that

$$\phi_P(d) = d^t(P) \cdot \phi_P$$

for every $d \in \mathbb{N}'$ prime to $e(P)$. This gives us all the necessary ingredients to repeat the proof of Theorem 3.1.2. $\square$
In this chapter, we assume that $k$ is algebraically closed. We will show how the motivic zeta function of a tamely ramified semi-abelian $K$-variety admits a cohomological interpretation by means of a trace formula, which is quite similar to the Grothendieck-Lefschetz trace formula for varieties over finite fields. The material in this chapter supersedes the unpublished preprint [Ni09] of the second author on the trace formula for abelian varieties. The case of algebraic tori was discussed in [Ni11b].

1. The trace formula for semi-abelian varieties

1.1. The rational volume.

1.1.1 We say that a $K$-variety $X$ is bounded if $X(K)$ is bounded in $X$, in the sense of [BLR90 1.1.2]. This boundedness property is equivalent to the existence of a quasi-compact open subvariety of the rigid analytification $X^\text{an}$ of $X$ that contains all $K$-rational points of $X$ [Ni11a 4.3].

1.1.2 Let $X$ be a smooth $K$-variety. A weak Néron model for $X$ is a smooth $R$-variety $\mathcal{X}$, endowed with an isomorphism of $K$-varieties

$$\mathcal{X} \times_R K \to X,$$

such that the natural map

$$\mathcal{X}(R) \to X(K)$$

is a bijection [BLR90 3.5.1]. It follows from [BLR90 3.1.3 and 3.5.7] that $X$ has a weak Néron model if and only if $X$ is bounded.

1.1.3 A weak Néron model $\mathcal{X}$ is not unique, but, using the change of variables formula for motivic integrals, one can show that the Euler characteristic $\chi(\mathcal{X}_k)$ of the special fiber of $\mathcal{X}$ does not depend on the choice of the weak Néron model; see [LS03 4.5.3], [Ni11a 5.2] and [NS11 §2.4]. Therefore, the following definition only depends on $X$, and not on $\mathcal{X}$.

**Definition 1.1.4.** Let $X$ be a smooth and bounded $K$-variety, and let $\mathcal{X}$ be a weak Néron model of $X$. Then the rational volume of $X$ is the integer

$$s(X) := \chi(\mathcal{X}_k) \in \mathbb{Z}.$$
(1.1.5) One can consider the rational volume \(s(X)\) as a measure for the set of \(K\)-rational points on \(X\); loosely speaking, via the reduction map

\[ X(K) = \mathcal{X}(R) \rightarrow \mathcal{X}_k(k), \]

we can view \(X(K)\) as a family of balls in \(K^d\) parameterized by the \(k\)-variety \(\mathcal{X}_k\). In particular, \(s(X)\) vanishes if \(X(K)\) is empty, but the converse implication does not hold.

(1.1.6) As explained in [Ni11b §1], it is possible to extend the definition of the rational volume to varieties that are not smooth and bounded. For a semi-abelian \(K\)-variety \(G\), the rational volume can be computed directly on the Néron model \(\mathcal{G}^{qc}\) of \(G\): it is simply given by

\[ s(G) = \chi(\mathcal{G}^{qc}_k). \]

Note that \(\mathcal{G}^{qc}_k\) is a weak Néron model of \(G\) if and only if \(G\) is bounded, in which case \(\mathcal{G}^{qc}\) coincides with the lft-Néron model \(\mathcal{G}\) of \(G\). We can further refine this expression for \(s(G)\) in the following way. We say that \(G\) has additive reduction if \(G^\circ\) is unipotent.

Proposition 1.1.7. Let \(G\) be a semi-abelian \(K\)-variety. Then \(s(G) = 0\) unless \(G\) has additive reduction. In that case, \(G\) is bounded and

\[ s(G) = |\Phi(G)|. \]

Proof. It follows from [Ni11b 3.2] that the toric rank of \(G\) is non-zero if \(G\) contains a non-trivial split subtorus. Thus we deduce from (3.1.3) in Chapter 2 that \(G\) is bounded when \(G\) has additive reduction. The remainder of the statement follows from Proposition 1.2.2 in Chapter 7 by applying the Euler characteristic \(\chi(\cdot)\), since the Euler characteristics of a non-trivial \(k\)-torus and a non-trivial abelian \(k\)-variety are zero. \(\square\)

1.2. The trace formula and the number of Néron components.

(1.2.1) Let \(X\) be a smooth, proper, geometrically connected \(K\)-variety. Assume that \(X(K^t)\) is non-empty, and that the wild inertia \(P\) of \(K\) acts trivially on the \(\ell\)-adic cohomology spaces

\[ H^i(X \times_K K^t, \mathbb{Q}_\ell) \]

for all \(i \geq 0\). Let \(\sigma\) be a topological generator of the tame inertia \(\text{Gal}(K^t/K)\). In [Ni12 4.1.4], the second author asked whether the following cohomological interpretation of the rational volume \(s(X)\) always holds:

\[ s(X) = \sum_{i \geq 0} (-1)^i \text{Trace}(\sigma | H^i(X \times_K K^t, \mathbb{Q}_\ell)). \]

He proved this result when \(k\) has characteristic zero [Ni11a 6.5], and when \(X\) is a curve [Ni11a §7]. Moreover, in [Ni12 4.2.1], he gave an explicit formula for the error term in this formula in terms of an sncd-model of \(X\). We will now prove that the trace formula (1.2.2) is valid if we replace \(X\) by a tamely ramified semi-abelian \(K\)-variety.
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(1.2.3) Let $G$ be a semi-abelian $K$-variety. We denote by $P_G(T)$ the characteristic polynomial

$$P_G(T) = \det(T \cdot \Id - \sigma \mid H^1(G \times_K K^t, \mathbb{Q}_\ell))$$

of $\sigma$ on $H^1(G \times_K K^t, \mathbb{Q}_\ell)$.

**Proposition 1.2.4.**

(1) If we denote by $G_{\text{tor}}$ and $G_{\text{ab}}$ the toric and abelian part of $G$, then

$$P_G(T) = P_{G_{\text{tor}}}(T) \cdot P_{G_{\text{ab}}}(T).$$

(2) The polynomial $P_G(T)$ is independent of the prime $\ell \neq p$. It is a product of cyclotomic polynomials in $\mathbb{Z}[T]$, and its roots are roots of unity of order prime to $p$.

(3) We have $P_G(1) \neq 0$ if and only if $G$ has additive reduction.

(4) If $G$ is tamely ramified, then

$$P_G(1) = \sum_{i \geq 0} (-1)^i \text{Trace}(\sigma \mid H^i(G \times_K K^t, \mathbb{Q}_\ell)).$$

**Proof.** (1) The sequence of $\ell$-adic Galois representations

$$0 \to H^1(G_{\text{ab}} \times_K K^t, \mathbb{Q}_\ell) \to H^1(G \times_K K^t, \mathbb{Q}_\ell) \to H^1(G_{\text{tor}} \times_K K^t, \mathbb{Q}_\ell) \to 0$$

is exact, as it can be obtained by dualizing the exact sequence of $\ell$-adic Tate modules

$$0 \to T_1G_{\text{tor}} \to T_1G \to T_1G_{\text{ab}} \to 0$$

and inverting $\ell$. Since the wild inertia $P$ is a pro-$p$-group and $p$ is different from $\ell$, taking $P$-invariants yields an exact sequence of $\ell$-adic $\text{Gal}(K^t/K)$-representations

$$0 \to H^1(G_{\text{ab}} \times_K K^t, \mathbb{Q}_\ell) \to H^1(G \times_K K^t, \mathbb{Q}_\ell) \to H^1(G_{\text{tor}} \times_K K^t, \mathbb{Q}_\ell) \to 0.$$

The result now follows from multiplicativity of the characteristic polynomial in short exact sequences.

(2) By (1), we may assume that $G$ is a $K$-torus or an abelian $K$-variety. If $G$ is a torus with character module $X(G)$, then the statement follows from the canonical isomorphism of $\ell$-adic $\text{Gal}(K^t/K)$-representations

$$H^1(G \times_K K^t, \mathbb{Q}_\ell) \cong (X(G) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell(1))^P$$

and the fact that the order of $\sigma$ on $X(G)^P$ is finite and prime to $p$. If $G$ is an abelian $K$-variety, then it was proven in [Lo93, 2.10] that $P_G(t)$ belongs to $\mathbb{Z}[T]$ and is independent of $\ell$. Then $P_G(t)$ must be a product of cyclotomic polynomials by quasi-unipotency of the Galois action on $T_1G$ [SGA7-I, IX.4.3], and the orders of its roots are prime to $p$ by triviality of the pro-$p$ part of $\text{Gal}(K^t/K)$.

(3) The inequality $P_G(1) \neq 0$ is equivalent to the property that $(T_1G)^I = 0$. Denoting by $\mathcal{G}$ the Néron $I$-model of $G$, we can identify $(T_1G)^I$ with $T_1\mathcal{G}_k^\ell$ by [SGA7-I, IX.2.2.3.3 and IX.2.2.5] (the results in [SGA7-I, IX] are formulated for abelian $K$-varieties, but the proofs remain valid for semi-abelian varieties). Thus it follows from [SGA7-I, IX.2.1.11] that the rank of the free $\mathbb{Z}_\ell$-module $(T_1G)^I$ equals the toric rank of $G$ plus twice the abelian rank of $G$. Therefore, $G$ has additive reduction if and only if $(T_1G)^I = 0$.

(4) By [HN11a, 5.7] and the fact that $P$ acts trivially on the cohomology spaces

$$H^i(G \times_K K^t, \mathbb{Q}_\ell),$$
there exists for every \( i \geq 0 \) a Galois-equivariant isomorphism
\[
H^i(G \times_K K^t, \mathbb{Q}_\ell) \cong \bigwedge H^1(G \times_K K^t, \mathbb{Q}_\ell).
\]

A straightforward computation now shows that
\[
P_G(1) = \sum_{i \geq 0} (-1)^i \text{Trace}(\sigma | H^i(G \times_K K^t, \mathbb{Q}_\ell)).
\]

\[\square\]

**Corollary 1.2.5.** If \( G \) is a tamely ramified abelian \( K \)-variety that does not have additive reduction, then
\[
s(G) = \sum_{i \geq 0} (-1)^i \text{Trace}(\sigma | H^i(G \times_K K^t, \mathbb{Q}_\ell)) = 0.
\]

**Proof.** The equality \( s(G) = 0 \) follows from Proposition 1.1.7. By Proposition 1.2.4, we have
\[
\sum_{i \geq 0} (-1)^i \text{Trace}(\sigma | H^i(G \times_K K^t, \mathbb{Q}_\ell)) = P_G(1) = 0
\]
because \( G \) does not have additive reduction. \( \square \)

In order to investigate the case where \( G \) has additive reduction, we’ll need some elementary lemmas.

**Lemma 1.2.6.** Fix an integer \( d > 1 \) and let \( \Phi_d(T) \in \mathbb{Z}[T] \) be the cyclotomic polynomial whose roots are the primitive \( d \)-th roots of unity. Then \( \Phi_d(1) \) is a positive divisor of \( d \).

**Proof.** We proceed by induction on \( d \). For \( d = 2 \) the result is clear, so assume that it holds for each value \( d' \) with \( 1 < d' < d \). We can write
\[
\frac{T^d - 1}{T - 1} = \prod_{e \mid d, e > 1} \Phi_e(T)
\]
and evaluating at \( T = 1 \) we get
\[
d = \prod_{e \mid d, e > 1} \Phi_e(1)
\]
so \( \Phi_d(1) \mid d \). By the induction hypothesis, \( \Phi_e(1) > 0 \) for \( 1 < e < d \), so \( \Phi_d(1) > 0 \) as well. \( \square \)

**Lemma 1.2.7.** Let \( q \) be a prime, \( M \) a free \( \mathbb{Z}_q \)-module of finite type, and \( \alpha \) an endomorphism of \( M \). Then \( M/\alpha M \) is torsion if and only if \( \alpha \) induces an automorphism on \( M \otimes_{\mathbb{Z}_q} \mathbb{Q}_q \). In this case, the order \( |M/\alpha M| \) of \( M/\alpha M \) satisfies
\[
|M/\alpha M| = |\text{det}(\alpha | M \otimes_{\mathbb{Z}_q} \mathbb{Q}_q)|_q^{-1}
\]
where \( | \cdot |_q \) denotes the \( q \)-adic absolute value.

**Proof.** The module \( M/\alpha M \) is torsion if and only if \( (M/\alpha M) \otimes_{\mathbb{Z}_q} \mathbb{Q}_q = 0 \), i.e., if and only if \( \alpha \) induces a surjective and, hence, bijective endomorphism on \( M \otimes_{\mathbb{Z}_q} \mathbb{Q}_q \). In this case, we have
\[
M/\alpha M \cong \mathbb{Z}_q/q^\epsilon_1 \mathbb{Z}_q \oplus \ldots \oplus \mathbb{Z}_q/q^\epsilon_r \mathbb{Z}_q
\]
where $q^{c_1}, \ldots, q^{c_r}$ are the invariant factors of $\alpha$ on $M$. Since $\det(\alpha | M \otimes_{\mathbb{Z}_q} \mathbb{Q}_q)$ equals $q^{c_1 + \cdots + c_r}$ times a unit in $\mathbb{Z}_q$, we find

$$|M/\alpha M| = q^{c_1 + \cdots + c_r} = |\det(\alpha | M \otimes_{\mathbb{Z}_q} \mathbb{Q}_q)|_q^{-1}. \tag{1.2.9}$$

\begin{proof}

Theorem 1.2.8. Let $G$ be a semi-abelian $K$-variety with additive reduction. If we denote by $\Phi(G)'$ the prime-to-$p$ part of the component group $\Phi(G)$, then

$$P_G(1) = |\Phi(G)'|. \tag{4.1}$$

If, moreover, $G$ is tamely ramified, then

$$\sum_{i \geq 0} (-1)^i \operatorname{Trace}(\sigma | H^i(G \times_K K^t, \mathbb{Q}_l)) = |\Phi(G)'| = |\Phi(G)|. \tag{1.2.8}$$

Let $G$ be a semi-abelian $K$-variety. If, moreover, $G$ is tamely ramified, then

$$\sum_{i \geq 0} (-1)^i \operatorname{Trace}(\sigma | H^i(G \times_K K^t, \mathbb{Q}_l)) = |\Phi(G)'| = |\Phi(G)|. \tag{1.2.8}$$

Proof. We first show that the $p$-primary part of $\Phi(G)$ is trivial when $G$ is tamely ramified. By Proposition 1.1.3, the sequence

$$\Phi(G_{\text{tor}}) \rightarrow \Phi(G) \rightarrow \Phi(G_{\text{ab}}) \rightarrow 0$$

is exact. Let $L$ be the minimal extension of $K$ in $K^t$ such that $G \times_K L$ has semi-abelian reduction. It follows from [HN10 4.1] that the torus $G_{\text{tor}} \times_K L$ is split, and that the abelian variety $G_{\text{ab}} \times_K L$ has semi-abelian reduction. Thus we can deduce from [LL01 1.8] that $\Phi(G_{\text{ab}})$ is killed by $[L : K]^2$, and from [Ni11b 3.4] that $\Phi(G_{\text{tor}})$ is killed by $[L : K]$. (In fact, $G_{\text{tor}}$ is anisotropic because $G$ has additive reduction, so that $\Phi(G_{\text{tor}})$ is torsion, but we don’t need this fact here.) Hence, $\Phi(G)$ is killed by $[L : K]^3$. In particular, its $p$-primary part is trivial, since $L$ is a tame extension of $K$.

Thus, in view of Proposition 1.2.4(4), it suffices to prove the first assertion in the statement. Let $q$ be a prime different from $p$, and denote by

$$T_q^iG = (T_qG)^P$$

the tame $q$-adic Tate module of $G$. By [HN10 4.4], the order of the $q$-primary part $\Phi(G)_q$ of $\Phi(G)$ equals the cardinality of

$$H^1(G(K^t/K), T_qG)_{\text{tors}} \cong H^1(G(K^t/K), T_q^iG) \cong T_q^iG/(\text{Id} - \sigma)T_q^iG$$

where the first isomorphism follows from the fact that the functor $(\cdot)^P$ is exact on pro-$q$-groups, and the second from the fact that $\sigma$ is a topological generator of $\text{Gal}(K^t/K)$.

Since $G$ has additive reduction, Proposition 1.2.4 tells us that 1 is not an eigenvalue of $\sigma$ on

$$T_q^iG \otimes_{\mathbb{Z}_q} \mathbb{Q}_q \cong H^1(G \times_K K^t, \mathbb{Q}_q)^\vee.$$

Thus by Lemma 1.2.7, the $\mathbb{Z}_q$-module $T_q^iG/(\text{Id} - \sigma)T_q^iG$ is torsion, and its order is given by

$$|\det(\text{Id} - \sigma | H^1(G \times_K K^t, \mathbb{Q}_q))|_q^{-1} \tag{1.2.9}$$

where $| \cdot |_q$ denotes the $q$-adic absolute value. Since $P_G(T)$ is independent of $\ell$, we find

$$|\Phi(G)_q| = |P_G(1)|_q^{-1}$$

for every prime $q \neq p$. 

We know by Proposition 1.2.4 that each root of $P_G(T)$ is a root of unity of order prime to $p$. Thus Lemma 1.2.6 implies that $|P_G(1)|_p = 1$ if $p > 1$. Hence, taking the product of (1.2.9) over all primes $q \neq p$, we get

$$|\Phi(G)'| = \prod_{q \neq p} |P_G(1)|_q^{-1} = \prod_{r \text{ prime}} |P_G(1)|_r^{-1} = |P_G(1)| = P_G(1)$$

where the last equality follows from Lemma 1.2.6. This concludes the proof.

**Corollary 1.2.10** (Trace formula for semi-abelian varieties). If $G$ is a tamely ramified semi-abelian $K$-variety, then

$$s(G) = \sum_{i \geq 0} (-1)^i \text{Trace}(\sigma | H^i(G \times_K K^t, \mathbb{Q}_\ell)).$$

**Proof.** Combine Proposition 1.1.7, Corollary 1.2.5 and Theorem 1.2.8. □

**Corollary 1.2.11.** If $G$ is a semi-abelian $K$-variety with additive reduction, then $|\Phi(G)'|$ is invariant under isogeny.

**Proof.** This is an immediate consequence of Theorem 1.2.8 since an isogeny induces an isomorphism on $\mathbb{Q}_\ell$-adic cohomology spaces. Note that the property of having additive reduction is preserved under isogeny, by the same arguments as in [SGA7-I, IX.2.2.7]. □

**Corollary 1.2.12.** If $G$ is a tamely ramified semi-abelian $K$-variety with additive reduction, then

$$0 \to \Phi(G_{tor}) \to \Phi(G) \to \Phi(G_{ab}) \to 0$$

is exact.

**Proof.** We already know from Proposition 1.1.3 that this sequence is right exact. Thus it is enough to show that $\Phi(G_{tor})$ is finite and

(1.2.13) $$|\Phi(G)| = |\Phi(G_{tor})| \cdot |\Phi(G_{ab})|.$$ 

The torus $G_{tor}$ and the abelian variety $G_{ab}$ must have additive reduction, since $P_G(1) \neq 0$ so that $P_{G_{tor}}(1) \neq 0$ and $P_{G_{ab}}(1) \neq 0$ (see Proposition 1.2.4). Thus $G_{tor}$ is anisotropic and $\Phi(G_{tor})$ is finite. Proposition 1.2.4 also tells us that

$$P_G(1) = P_{G_{tor}}(1) \cdot P_{G_{ab}}(1)$$

so that the equality (1.2.13) follows from Theorem 1.2.8. □

**Remark 1.2.14.** In the proof of Theorem 1.2.8 we invoked [LL01, 1.8]. The proof of [LL01, 1.8] is based on [BX96, 5.6 and 5.9]; we refer to (1.1.6) in Chapter 4 for a comment on these results.

### 1.3. Cohomological interpretation of the motivic zeta function.
Let $G$ be a semi-abelian $K$-variety. The trace formula in Corollary 1.2.10 yields a cohomological interpretation of the motivic zeta function $Z_G(T)$ from Chapter 7, in the following way. We denote by $\chi(Z_G(T))$ the element of $\mathbb{Z}[[T]]$ that we obtain by taking the images of the coefficients of the series $Z_G(T) \in K_0(\text{Var}_k)[[T]]$ under the ring morphism

$$\chi : K_0(\text{Var}_k) \to \mathbb{Z}$$

that sends the class of a $k$-variety $X$ to the Euler characteristic $\chi(X)$. Explicitly, we have

$$\chi(Z_G(T)) = \sum_{d > 0} \chi(\mathcal{G}(d)^{\text{qc}})T^d \in \mathbb{Z}[[T]]$$

where $\mathcal{G}(d)^{\text{qc}}$ denotes the Néron model of $G(d) = G \times_K K(d)$.

**Theorem 1.3.2 (Cohomological interpretation of the motivic zeta function).**

Let $G$ be a semi-abelian $K$-variety, and denote by $\text{Add}_G$ the set of elements $d$ in $\mathbb{N}'$ such that $G(d)$ has additive reduction. Then

$$\chi(Z_G(T)) = \sum_{d \in \mathbb{N}'} s(G \times_K K(d))T^d$$

$$= \sum_{d \in \text{Add}_G} |\Phi(G(d))|T^d$$

$$= \sum_{d \in \mathbb{N}'} \sum_{i \geq 0} (-1)^i \text{Trace}(\sigma^d | H^i(G \times_K K, \mathbb{Q}_l))T^d$$

in $\mathbb{Z}[[T]]$.

**Proof.** The first equality follows from the definition of the rational volume, and the second from Proposition 1.1.7 (these two equalities do not require $G$ to be tamely ramified). The last equality is a consequence of the trace formula in Corollary 1.2.10.

**(1.3.3)** Theorem 1.3.2 was stated for abelian varieties in [HN11a 8.4], with a reference to the second author’s unpublished preprint [Ni09] for the proof. Theorem 1.3.2 supersedes that statement, and extends it to semi-abelian varieties.

2. The trace formula for Jacobians

To conclude this chapter, we give an alternative proof of Theorem 1.2.8 if $G$ is the Jacobian $\text{Jac}(C)$ of a $K$-curve $C$. The proof is based on an explicit expression for the characteristic polynomial $P_G(T)$ in terms of an sncd-model of the curve $C$.

2.1. The monodromy zeta function.

**(2.1.1)** Let $C$ be a smooth, projective, geometrically connected $K$-curve of genus $g(C) > 0$. Let $\mathcal{C}$ be a minimal sncd-model of $C$, with special fiber

$$\mathcal{C}_k = \sum_{i \in I} N_i E_i.$$ 

We set $\delta(C) = \gcd\{N_i | i \in I\}$. For each $i \in I$, we denote by $N'_i$ the prime-to-$p$ part of $N_i$. Moreover, we set $E_i^\circ = E_i \setminus \cup_{j \neq i} E_j$ and we denote by $d_i$ the cardinality of $E_i \setminus E_i^\circ$. 

(2.1.2) We denote by $A$ the Jacobian of $C$. Then there exists an isomorphism of $G(K^*/K)$-representations

$$H^1(C \times_K K^t, Q_\ell) \cong H^1(A \times_K K^t, Q_\ell).$$

Thus $C$ is cohomologically tame if and only if $A$ is tamely ramified, and the polynomial $P_A(T)$ is equal to the characteristic polynomial of $\sigma$ on

$$H^1(C \times_K K^t, Q_\ell).$$

We denote by $\zeta_C(T)$ the reciprocal of the monodromy zeta function of $C$, i.e.,

$$\zeta_C(T) = \prod_{i=0}^2 \det(T \cdot \text{Id} - \sigma | H^i(X \times_K K^t, Q_\ell))^{(-1)^{i+1}} \in \Omega(T).$$

**Theorem 2.1.3.** We have

$$\zeta_C(T) = \prod_{i \in I} (T^{N'_i} - 1)^{-\chi(E'_i)},$$

$$P_A(T) = (T - 1)^2 \prod_{i \in I} (T^{N'_i} - 1)^{-\chi(E'_i)}.$$  \hspace{1cm} (2.1.4)

If $C$ is cohomologically tame, and either $\delta(C)$ is prime to $p$, or $g(C) \neq 1$, then

$$\zeta_C(T) = \prod_{i \in I} (T^{N_i} - 1)^{-\chi(E_i)},$$

$$P_A(T) = (T - 1)^2 \prod_{i \in I} (T^{N_i} - 1)^{-\chi(E_i)}.$$ \hspace{1cm} (2.1.5)

**Proof.** The expressions for $P_A(T)$ follow immediately from the expressions for $\zeta_C(T)$, since $\sigma$ acts trivially on the degree 0 and degree 2 cohomology of $C$. Formula (2.1.4) is a special case of the arithmetic A’Campo formula in [Ni12, 2.6.2].

Now assume that $C$ is cohomologically tame and that $g(C) \neq 1$ or that $\delta(C)$ is prime to $p$. Note that the property that $\delta(C)$ is prime to $p$ implies that $C(K^t)$ is non-empty, by [Ni12, 3.1.4]. Then Saito’s geometric criterion for cohomological tameness [Sa87, 3.11], phrased in the form of [Ni12, 3.3.2], implies that $E_i \cong G_{m,k}$ if $N_i$ is not prime to $p$. Thus $\chi(E'_i) = 0$ for all $i \in I$ such that $N_i \neq N'_i$. Therefore, (2.1.6) follows from (2.1.3). \hfill \Box

(2.1.8) As an immediate corollary, we obtain an alternative proof of Theorem 2.1(i) in [Lo93]. Denote by $g(E_i)$ the genus of $E_i$, for each $i \in I$. We put $a = \sum_{i \in I} g(E_i)$. We denote by $\Gamma$ the dual graph of $G_k$, and by $t$ its first Betti number.

**Corollary 2.1.9 (Lorenzini).** We have

$$P_A(T) = (T - 1)^{2a + 2t} \prod_{i \in I} \left(\frac{T^{N'_i} - 1}{T - 1}\right)^{2g(E_i) + d_i - 2}.$$  \hspace{1cm} (2.1.7)

**Proof.** This follows immediately from Theorem 2.1.3 the formula

$$\chi(E'_i) = 2 - 2g(E_i) - d_i,$$

and the fact that

$$2 - 2t = \sum_{i \in I} (2 - d_i)$$

(both sides equal twice the Euler characteristic of $\Gamma$). \hfill \Box
(2.1.10) In \[\text{Lo93}\], it was assumed that \(\delta(C) = 1\), but our arguments show that this is not necessary. If \(\delta(C) = 1\), then \(a\) equals the abelian rank of \(A = \text{Jac}(C)\) and \(t\) its toric rank (see \[\text{Lo90}\ p. 148\]).

2.2. The trace formula for Jacobians. We can use the above results to give an alternative proof of Theorem \[\text{1.2.8}\] and Corollary \[\text{1.2.10}\] for the Jacobian \(A = \text{Jac}(C)\) of the curve \(C\).

**Proposition 2.2.1.** We keep the notations of Section 2.1, and we assume that \(\delta(C) = 1\).

1. If \(A\) does not have additive reduction, then \(P_A(1) = 0\).
2. If \(A\) has additive reduction, then \(P_A(1) = |\Phi(A)'|\), where \(\Phi(A)'\) denotes the prime-to-\(p\) part of \(\Phi(A)\).
3. If \(A\) has additive reduction and is tamely ramified, then \(\Phi(A)' = \Phi(A)\) and

\[
s(A) = \sum_{i \geq 0} (-1)^i \text{Trace}(\sigma|H^i(A \times_k K^t, \mathbb{Q}_\ell))\]

**Proof.** (1) It follows from Corollary \[\text{2.1.9}\] that the order of 1 as a root of \(P_A(T)\) equals \(2a + 2t\). Hence, if \(A\) does not have additive reduction, then \(P_A(1) = 0\).

(2) By \[\text{Lo90}\ 1.5\], we have

\[
|\Phi(A)| = \prod_{i \in I} N_i^{d_i - 2}
\]

because the toric rank of \(A\) is zero. By Corollary \[\text{2.1.9}\] we know that

\[
P_A(T) = \prod_{i \in I} \left(\frac{T^{N_i'} - 1}{T - 1}\right)^{d_i - 2}
\]

because \(a = t = 0\). This yields

\[
P_A(1) = \prod_{i \in I} (N_i')^{d_i - 2} = |\Phi(A)'|.
\]

(3) If \(A\) is tamely ramified, then Saito’s criterion for cohomological tameness \[\text{Sa87}\ 3.11\] implies that \(d_i = 2\) if \(N_i \neq N_i'\), so that

\[
|\Phi(A)'| = \prod_{i \in I} (N_i')^{d_i - 2} = \prod_{i \in I} N_i^{d_i - 2} = |\Phi(A)|.
\]

Combining this with (1), (2) and Propositions \[\text{1.1.7}\] and \[\text{1.2.4}\], we find

\[
s(A) = \sum_{i \geq 0} (-1)^i \text{Trace}(\sigma|H^i(A \times_k K^t, \mathbb{Q}_\ell)).
\]

\[\square\]
Part 4

Some open problems
To conclude, we will formulate some open problems and directions for future research stemming from the results in the preceding chapters. We assume that $k$ is algebraically closed.

1. The stabilization index

Let $A$ be an abelian $K$-variety, and let $L/K$ be the minimal extension of $K$ in $K^s$ such that $A \times_K L$ has semi-abelian reduction. Let $K'$ be a finite tame extension of $K$ and denote by $R'$ the integral closure of $R$ in $K'$. We set $A' = A \times_K K'$. We denote by $\mathcal{A}^\prime$ the Néron model of $A'$, and by

$$h : \mathcal{A} \times_R R' \to \mathcal{A}^\prime$$

the canonical base change morphism.

A central theme in this monograph was the study of the properties of this base change morphism $h$. As we've explained in Section 1.4 of the introduction, the basic idea is that the Néron models $\mathcal{A}$ and $\mathcal{A}^\prime$ should differ as little as possible if the extension $K'/K$ is sufficiently orthogonal to the extension $L/K$. This principle was a crucial ingredient in establishing rationality and determining the poles of the component series $S^\Phi_A(T)$ and the motivic zeta function $Z_A(T)$. The qualification “as little as possible” includes, in particular, the following properties.

- The number of components grows as if $\mathcal{A}$ had semi-abelian reduction, i.e., the equality in Proposition 3.1.1 of the introduction holds:

$$|\Phi(A(d))| = d^{\ell(A)} \cdot |\Phi(A)|.$$  

- The $k$-varieties $\mathcal{A}_k^\circ$ and $(\mathcal{A}^\prime)_k^\circ$ define the same class in $K_0(\text{Var}_k)$.

As we've seen, the meaning of “sufficiently orthogonal” is less clear. We'd like to express this property by saying that the degree of the extension $K'/K$ is coprime to a certain invariant $e(A)$ of the abelian $K$-variety $A$, that we will call the stabilization index. We can define the stabilization index in the following cases.

1. If $A$ is tamely ramified or $A$ has potential multiplicative reduction, then we can set $e(A) = [L : K]$. As we've explained in Section 1.4 of the introduction, the two above properties are satisfied for every finite tame extension $K'/K$ of degree prime to $e(A)$.

2. If $A$ is the Jacobian of a smooth proper $K$-curve $C$ of index one, then we can set $e(A) = e(C)$, where $e(C)$ is the stabilization index of $C$ that we introduced in Chapter 3 Definition 2.2.3. We've shown in Corollary 3.1.5 of Chapter 3 that $e(A)$ only depends on $A$, and not on $C$. The two above properties are satisfied for every finite tame extension $K'/K$ of degree prime to $e(A)$, by Chapter 3 Proposition 3.1.1 and Chapter 4 Proposition 2.1.3.

This immediately raises several questions. If $A$ is a tamely ramified Jacobian, then the two definitions of $e(A)$ are equivalent, by Proposition 2.2.4 in Chapter 3. We expect that the two definitions are also equivalent for wildly ramified Jacobians with potential multiplicative reduction; this is true if and only if the following question has a positive answer. We say that $C$ has multiplicative reduction if $\text{Jac}(C)$ has multiplicative reduction; this is equivalent to the property that the special fiber of the minimal $\text{sncd}$-model of $C$ consists of rational curves.
2. The characteristic polynomial

QUESTION 1.1. Let \( C \) be a smooth proper \( K \)-curve of index one. Let \( L \) be the minimal extension of \( K \) in \( K^s \) such that \( C \times_K L \) has semistable reduction, and assume that \( C \times_K L \) has multiplicative reduction. Is it true that \( [L : K] = e(C) \)?

If \( L/K \) is tame this follows from Proposition 2.2.4 in Chapter 3 but if \( L/K \) is wild, it is a very difficult problem to determine the degree of \( L/K \) in terms of the geometry of an sncd-model of \( C \). The equality \( e(C) = [L : K] \) can fail if we don’t assume that \( C \times_K L \) has multiplicative reduction: see Examples 2.2.5 and 2.2.6 in Chapter 3.

A second question is whether we can find a suitable definition of the stabilization index \( e(A) \) for arbitrary abelian \( K \)-varieties \( A \). A possible candidate is the following. Assume that all the jumps of \( A \) are rational numbers. This is currently still an open problem, except in the tame case and for Jacobians. Then we define the stabilization index \( e(A) \) of \( A \) as the smallest integer \( e > 0 \) such that \( e \cdot j \) belongs to \( \mathbb{Z} \) for every jump \( j \) of \( A \). Corollary 3.1.5 in Chapter 3 guarantees that \( e(A) = e(C) \) if \( A \) is the Jacobian of a \( K \)-curve \( C \). Moreover, it follows easily from [HN1a, 4.20 and 5.1] that \( e(A) \) is equal to \( [L : K] \) when \( A \) is tamely ramified. Thus our new definition of the stabilization index \( e(A) \) is equivalent to Definition (1) above for tamely ramified abelian varieties and to Definition (2) for Jacobians. We believe that it is also equivalent to Definition (1) if \( A \) is a wildly ramified abelian \( K \)-variety with potential multiplicative reduction. In this case, the uniformization space of \( A \) is an algebraic \( K \)-torus. Thus, in view of Proposition 2.3.2 it would be enough to show that the following question has a positive answer.

QUESTION 1.2. Let \( T \) be an algebraic \( K \)-torus, and let \( L \) be the minimal splitting field of \( K \) in \( K^s \). Is it true that the jumps of \( T \) are rational, and that \( [L : K] \) is the smallest integer \( e > 0 \) such that \( e \cdot j \) belongs to \( \mathbb{Z} \) for every jump \( j \) of \( T \)?

Note that a positive answer to Question 1.2 implies that Definitions (1) and (2) are equivalent for Jacobians with potential multiplicative reduction, so that it would also provide an affirmative answer to Question 1.1. This seems to be a promising approach to solve Question 1.1. A key step would be to find a suitable interpretation for the jumps of a torus in terms of its character module.

Finally, for arbitrary abelian \( K \)-varieties \( A \), the question remains whether the jumps of \( A \) are indeed rational numbers and whether our candidate for \( e(A) \) has the required properties. All of these problems will be investigated in future research.
3. The motivic zeta function and the monodromy conjecture

Let $A$ be a tamely ramified abelian $K$-variety of dimension $g$. We proved in [HN11a, 8.6] that the base change conductor $c(A)$ is the only pole of the motivic zeta function $Z_A(L^{-s})$. Moreover, it follows from our result in [HN11a, 5.13] that for every embedding of $Q_\ell$ in $C$ and for every topological generator $\sigma$ of $\text{Gal}(K^t/K)$, the value $\exp(2\pi ic(A))$ is an eigenvalue of the action of $\sigma$ on the tame $\ell$-adic cohomology group

$$H^g(A \times_K K^t, \mathbb{Q}_\ell).$$

As we've explained in [HN11a, 2.5], this result can be viewed as a version of Denef and Loeser's motivic monodromy conjecture for abelian varieties.

We've seen in Chapter 7, Theorem 3.1.2, that the tame base change conductor $c_tame(A)$ is still the unique pole of the motivic zeta function $Z_A(L^{-s})$ when $A = \text{Jac}(C)$ is a (possibly wildly ramified) Jacobian. But it is not clear at all how a suitable version of the monodromy conjecture could be formulated in this case, because the tame cohomology spaces do not contain enough information. For instance, for every wildly ramified elliptic $K$-curve $E$, the tame cohomology space $H^1(E \times_K K^t, \mathbb{Q}_\ell)$ is trivial.

This question is closely related to the problem of finding a cohomological interpretation of the characteristic polynomial $P_C(t)$ (Section 2). We know by Proposition 3.1.6 in Chapter 5 that every jump $j$ of $A$ is a root of $P_C(t)$ of multiplicity at least $m_A(j)$. If $A$ is a tamely ramified Jacobian, then by Proposition 2.1.3 in Chapter 3 this implies that $\exp(2\pi ij)$ is an eigenvalue of order at least $m_A(j)$ of the action of $\sigma$ on $H^1(A \times_K K^t, \mathbb{Q}_\ell)$

so that

$$\exp(2\pi ic_{tame}(A)) = \prod_{j \in J_A} \exp(2\pi im_A(j)j)$$

is an eigenvalue of the action of $\sigma$ on

$$H^g(A \times_K K^t, \mathbb{Q}_\ell) \cong \bigwedge^g H^1(A \times_K K^t, \mathbb{Q}_\ell).$$

Thus it seems plausible that a suitable cohomological interpretation of $P_C(t)$ in the wildly ramified case would also give rise to a cohomological interpretation of $c_{tame}(A)$.

We expect that, for arbitrary wildly ramified abelian $K$-varieties $A$, the motivic zeta function $Z_A(L^{-s})$ is still rational, with a unique pole at $c_{tame}(A)$. The key step in the proof of such a result would be a suitable characterization of the stabilization index (Section 1). One may also ask for a cohomological interpretation of $c_{tame}(A)$ in this case.

4. Base change conductor for Jacobians

Let $A$ be the Jacobian of a $K$-curve $C$ of index one. In Chapter 6 we compared the base change conductor $c(A)$ to the Artin conductor $\text{Art}(C)$ when $C$ had genus 1 or 2. Based on these results, one is led to speculate whether it is still possible to compute $c(A)$ in terms of other arithmetic invariants of the curve $C$ if the genus of
5. Component groups of Jacobians

A key element in the proof of our results on the component series and the motivic zeta function of a Jacobian \(A = \text{Jac}(C)\) is Proposition 3.1.1 in Chapter 3, which asserts that

\[
|\Phi(A(d))| = d^t \cdot |\Phi(A)|
\]

for every \(d \in \mathbb{N}'\) prime to \(e(C)\). We deduced this formula in a somewhat indirect fashion, using Winters’ theorem to transfer it from the equal characteristic zero case. It would be quite interesting to have a more direct proof of Proposition 3.1.1 that avoids using Winters’ theorem.

In the literature one can find numerous results concerning the computation of the order of \(\Phi(A)\) in terms of a regular \(R\)-model \(C\) of \(C\) (see [BLR90, 9.6] for an overview). All of these results essentially involve computing certain minors of the intersection matrix associated to the special fiber \(C_k\). Even though, starting with the minimal \(sncd\)-model \(C\) of \(C\), our results in Chapter 3 provide us with good control over the special fiber of \(C(d)\) for every \(d \in \mathbb{N}'\) prime to \(e(C)\), one is faced with the problem that the rank of the intersection matrix of \(C(d)_k\) grows together with \(d\). If \(t(A) = 0\), an easy application of [BLR90, 9.6/6] enables one to derive the formula mentioned above, but whenever \(t(A) > 0\) it remains a combinatorial challenge to use this approach for computing the order of the component group of \(A(d)\) as a function of \(d\).
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