Description of identical particles via gauged matrix models

: a generalization of the Calogero-Sutherland system

Jeong-Hyuck Park*

Korea Institute for Advanced Study
207-43 Cheongryangri-dong, Dongdaemun-gu
Seoul 130-012, Korea

Abstract

We elaborate the idea that the matrix models equipped with the gauge symmetry provide a natural framework to describe identical particles. After demonstrating the general prescription, we study an exactly solvable harmonic oscillator type gauged matrix model. The model gives a generalization of the Calogero-Sutherland system where the strength of the inverse square potential is not fixed but dynamical bounded by below.

*Electronic correspondence: jhp@kias.re.kr
1 Introduction

Recent advances in the noncommutative field theory have enabled us to realize that the quantum Hall system is closely related to the noncommutative Chern-Simons theories. After the pioneering work by Susskind [1], utilizing the fact that the noncommutative field theories can be formulated by matrices of infinite size, Polychronakos proposed a finite matrix Chern-Simons model for the description of the finite number of electrons in the quantum Hall system [2]. Soon after, a matrix version of the Laughlin’s wavefunction [3] was recovered by Hellerman and Raamsdonk [4].

In this paper, we wish to clarify the underlying principles of the above matrix model approach to the quantum Hall system in a more general setup. We elaborate the idea that the matrix models equipped with gauge symmetry provide a natural framework to describe identical particles (see e.g. [5, 6]).

After demonstrating the general prescription, we explicitly study an exactly solvable harmonic oscillator type gauged matrix model. We show the model gives a generalization of the Calogero-Sutherland system where the strength of the inverse square potential is not fixed but dynamical bounded by below.

2 Description of Identical Particles

One of the intrinsic properties of the fundamental particles in nature is the very fact that they are identical. Namely it is in principle impossible to identify each individual particle at different time slices. In the ordinary quantum mechanics, the conventional way of incorporating this idea is to anti-symmetrize the wavefunctions over the particle indices.

Matrix gauge theories provide a more natural framework to describe identical particles. To see this we first consider the classical Lagrangian for \( N \) electrons

\[
L_\text{cl} = \sum_{a=1}^{N} T(x_a, \dot{x}_a) - V(x_1, x_2, \cdots, x_N),
\]

where \( x \in \mathbb{R}^D \), the \( D \)-dimensional ‘space’, and the potential is totally symmetric over the particle indices. Formally introducing a diagonal \( N \times N \) matrix, \( X = \text{diag}(x_1, x_2, \cdots, x_N) \), one can always rewrite the Lagrangian in terms of the matrix

\[
L_\text{cl} = \text{tr}\left[T(X, \dot{X})\right] - V(\text{tr} f_l(X)),
\]

where we used the fact that the symmetric potential can be always written in terms of the traces of certain set of functions, \( \text{tr} f_l(X) = \sum_{a=1}^{N} f_l(x_a) \). We put the proof at the end of this section. In particular, when \( x_a \) carries no spatial index as in the \( D = 1 \) case, one can simply set \( f_l(x) = x^l, 1 \leq l \leq N \).

Allowing the off-diagonal elements with the restriction, \( X \) being hermitian to ensure the eigenvalues to be real, we encounter a new physical system. First, the action acquires a \( U(N) \)
symmetry for which the matrices transform in the adjoint representation

\[ X \to U X U^\dagger, \quad U \in U(N) . \]  

(3)

We note that the permutation group relevant to the relabeling of the electrons is a subgroup of \( U(N) \). As electrons are identical, the labeling is not physical so that the permutation symmetry is an auxiliary one. This suggests to gauge away the \( U(N) \) symmetry in the matrix formalism. Namely we introduce an auxiliary matrix, \( A_\alpha \) and replace the ordinary time derivative by a covariant time derivative

\[ D_t X = \dot{X} + i [A_\alpha, X] . \]  

(4)

Then what \( A_\alpha \) brings new is the Gauss’ constraint or the equation of the motion for \( A_\alpha \), which gives the quantum generator of the \( U(N) \) symmetry from the Noether theorem. Since we embed the discrete permutation group into the continuous group, the reparameterization of the electrons can be now realized by quantum operators. At the quantum level the constraint is to be imposed on the wavefunctions and this will mode out the auxiliary symmetry. We emphasize the point that the \( U(N) \) symmetry (3) can be now time dependent, and physically this amounts that we are requiring the physics to be invariant under not only time independent but also time dependent reparameterization of the particles. Once again, it is in principle impossible or meaningless to identify each individual particle at different time slices.

One interesting “generalization” is to add a term into the action which is linear in \( A_\alpha \)

\[ \kappa_{\text{bare}} \text{tr} A_\alpha . \]  

(5)

While writing the Gauss’ constraint at the quantum level there always occurs an ordering ambiguity, since the constraint contains the product of \( X \) and its conjugate momentum. The ambiguity amounts to adding an identity matrix to the Gauss’ constraint up to a factor. Thus, considering the above term (3) is not a mere “generalization” but rather a natural requirement. After all, the final form of the Gauss’ constraint which is to be imposed on the quantum wavefunctions should be written in the normal ordered way with the physical coefficient, \( \kappa \)

\[ \text{U(N) generator} = \kappa I_{N \times N} . \]  

(6)

normal ordered

After all, the actual integer value of the physical coefficient, \( \kappa \) is given by the superselction rule.

The consistency at the quantum level requires the physical coefficient, \( \kappa \) to be an integer \([6,7]\). At the quantum level, the Gauss’ constraint generates unitary transformations, \( U = e^{i\Lambda}, \Lambda^\dagger = \Lambda \) on all the arguments in the wavefunction

\[ |\Psi'\rangle = e^{i\kappa \text{tr} \Lambda} |\Psi\rangle . \]  

(7)

Taking the particular choice, \( \Lambda = \text{diag}(2\pi, 0, 0, \ldots, 0) \) gives the identity matrix, \( U = 1_{N \times N} \), and apparently the Gauss’ constraint on the wavefunctions works successfully only for integer, \( \kappa \). Essentially this quantization is identical to that of the coefficient in the noncommutative Chern-Simons theories \([8]\).

Nevertheless, non-zero \( \kappa \) is yet problematic for the finite matrix models. As the matrices are in the adjoint representation, the central \( U(1) \) transformation would leave the wavefunction...
invariant, and this is clearly inconsistent with Eq. (7) for non-zero $\kappa$. Curing the problem requires the presence of new variables in other representations. A natural candidate is a complex bosonic vector, $\phi$ in the fundamental representation so that $D_t\phi = \dot{\phi} + iA_\phi \phi$. As we will see later, this new variable governs the strength of the intrinsically existing repulsive potential in the matrix gauge theories.

Introducing the complex vector, the gauged matrix model can be self-consistent and is now of the general form

$$L = \text{tr}[T(X, D_tX) + \kappa_{\text{bare}} A_\phi] - \mathcal{V}(\text{tr}f_a(X)) + \text{vector parts}.$$  \hspace{1cm} (8)

As worked out in [4, 7] the general quantum wavefunction satisfying the Gauss’ constraint consists of $\text{U}(N)$ invariant part times $|\kappa|$ products of $\text{SU}(N)$ invariant parts (see Eq. (19)). Due to the latter, the wavefunction is an eigenstate of the particle exchange operator of the eigenvalue, $(-1)^\kappa$ capturing the identical nature of the particles.

The prescription to give the physical meaning to any expectation value in the matrix model is the map

$$\langle \Psi | \text{tr} f(X) | \Psi \rangle \longleftrightarrow \int dx^D \rho(x)f(x),$$

where $\rho(x)$ is the corresponding physical density function, while $f(x)$ is an arbitrary function. As we turn on the off-diagonal components in $X$ and introduce the vectors, the present non-matrix system is not simply equal to the initial one (1). Physically, the off-diagonal element, $X_{ab}^a$, $a \neq b$ corresponds to the interaction between the two particles, $a, b$, and integrating out the off-diagonal components generates an inverse square type repulsive potential. Thus the matrix gauge theories intrinsically contain a repulsive potential among the particles. However, this can be in principle eliminated in the matrix model by adding the counter term written in the matrix form, if necessary.

It is worth to note that due to the Gauss’ constraint on the wavefunction (7) any matrix valued expectation must be $\text{U}(N)$ invariant, and hence

$$\langle \Psi | f(X) | \Psi \rangle = \frac{1}{N} \langle \Psi | \text{tr} f(X) | \Psi \rangle I_{N \times N}.$$  \hspace{1cm} (10)

For the systems of kinetic terms quadratic in time derivatives in the $D$ dimensional space the phase space for the matrices has the dimension, $2DN^2$ so that subtracting the Gauss’ constraint and the gauge symmetry, the physical degrees of freedom for the matrices is $2(D - 1)N^2 + 1$. On the other hand the vector has the linear in $N$ degree of freedom. Hence when $D = 1$ the total degrees of freedom is linear in $N$, otherwise it is quadratic in $N$. Nevertheless if we include the BFSS type potential, $\text{tr}[X^I, X^J]^2$ [9], at the low energy limit the matrices tend to commute each other making the “off-diagonal components” negligible. This will result in linear degrees of freedom at low energy. In fact, in a special low energy limit the particles form a collinear motion [7, 10]. On the other hand, for the kinetic term linear in time derivatives like the $D = 2$ matrix Chern-Simons model, the dimension of the phase space is halved and the total degrees of freedom in linear in $N$. 

3
The proof of the fact
Consider a symmetric function depending on \( N \) particles’ coordinates, \( \bar{x}_a, a = 1, 2, \ldots, N \) of the form

\[
F(x_1, \ldots, x_N) = \sum_p f_1(x_{p_1}) f_2(x_{p_2}) \cdots f_N(x_{p_N}),
\]

where \( f_1, \ldots, f_N \) are functions of one particle coordinate and the sum is over the \( N! \) permutations so that the function is apparently symmetric over the particle indices. General symmetric functions can be written in terms of this kind of symmetric functions. For example, the Coulomb interaction can be written as a fraction of such two functions. In the below we show that \( F \) can be written in terms of \( \text{tr} f_1(X) \) with \( X = \text{diag}(x_1, x_2, \ldots, x_N) \). We prove this by the mathematical induction on the number of non-constant functions in \( \{ f_1, f_2, \ldots, f_N \} \) which we denote by \( \#_f \). If \( \#_f = 1 \), it is easy to see

\[
F(x_1, \ldots, x_N) = (N - 1)! \text{tr} f_\alpha(X) \prod_{\beta \neq \alpha} f_\beta,
\]

where \( f_\alpha \) is the only one non-constant function. Hence the statement holds for \( \#_f = 1 \). Now we assume that \( F \) can be written in terms of \( \text{tr} f_1(X) \) for \( \#_f < n \) cases, and consider \( F \) in the \( \#_f = n \) case. We let with out loss of generality \( f_1, \ldots, f_n \) be the \( n \) non-constant functions and set

\[
F' \equiv F - (N - n)! \left( \prod_{\alpha = 1}^n \text{tr} f_\alpha(X) \right) \left( \prod_{\beta > n} f_\beta \right).
\]

It is crucial to note that \( F' \) belongs to the classes, \( \#_f < n \) so that \( F' \) and hence \( F \) can be written in terms of \( \text{tr} f_1(X) \). This completes our proof.

3 Generalization of Calogero-Sutherland System

The \( D = 1 \) gauged matrix model we consider here is the harmonic oscillator type. With the column vector, \( \phi \), and the row vector, \( \bar{\phi} \equiv \phi^\dagger \), the Lagrangian is

\[
L = \text{tr} \left[ \frac{1}{2} m(D_\tau X)^2 - \frac{1}{2} g X^2 + mD_\tau \phi D_\tau \bar{\phi} - g' \phi \bar{\phi} + \kappa_{\text{bare}} A_\alpha \right].
\]

If \( g = g' \), writing a \( (N + 1) \times (N + 1) \) matrix \( (\begin{smallmatrix} X & \phi \\ \bar{\phi} & 0 \end{smallmatrix}) \), the model can be regarded as the truncation of the bigger matrix model with the broken gauge symmetry, \( (\begin{smallmatrix} A_\alpha \\ 0 \end{smallmatrix}) \), \( \text{U}(N + 1) \to \text{U}(N) \).

With the conjugate momenta, \( P = mD_\tau X, p_\phi = mD_\tau \phi \) we define the following dimensionless quantities

\[
C = \frac{1}{\sqrt{2}} \left( (mg)^{\dagger}X + i(mg)^{-\dagger}P \right), \quad \eta_\pm = \frac{1}{\sqrt{2}} \left( (mg')^{\dagger}\phi \pm i(mg')^{-\dagger}p_\phi \right).
\]

The standard quantization shows the non-vanishing commutators are

\[
[C^a_{bc}, \bar{C}^c_{bd}] = \delta^a_d \delta^c_b, \quad [\eta^a_+ + \bar{\eta}_+, b] = \delta^a_b, \quad [\eta^a_-, \bar{\eta}_-, b] = -\delta^a_b.
\]

Thus, \( \bar{C}, \bar{\eta}_+, \eta_- \) and \( C, \eta_+, \bar{\eta}_- \) are respectively creation and annihilation operators (our somewhat unconventional notation for \( \eta_- \) is to keep the consistent \( \text{U}(N) \) index notation.). The appearance
of the two sets of vector-valued harmonic oscillators is due to the fact that the kinetic term for the vector is written with the quadratic time derivatives. By introducing one more vector field, one could equivalently rewrite the vector parts in the action to contain two kinetic terms linear in time derivatives having the opposite signs (see [11] for further generalization of the kinetic terms linear in time derivatives).

In terms of the operators, the Gauss’ constraint reads in the normal ordered form,

$$\bar{C}^c_b C^a_c - \bar{C}^a_c C^c_b + \bar{\eta}_{+b} \eta^a_+ - \eta^a_- \bar{\eta}_{-b} = \kappa \delta^a_b,$$

(17)

where the left hand side generates the $U(N)$ transformations, while $\kappa$ on the right hand side is now of the physical value. The Hamiltonian is

$$H = \mathrm{tr} \left[ \frac{1}{2} \sqrt{\frac{2}{m}} (\bar{C} C + \bar{C} C) + \sqrt{\frac{2}{m}} (\eta_+ \bar{\eta}_+ + \eta_- \bar{\eta}_-) \right]$$

$$= \sqrt{\frac{2}{m}} \mathrm{tr}(\bar{C} C) + \sqrt{\frac{2}{m}} (\bar{\eta}_+ \eta_+ + \eta_- \bar{\eta}_-) + \frac{1}{2} \sqrt{\frac{2}{m}} N^2 + \sqrt{\frac{2}{m}} N.$$ (18)

Here the second expression is written in the normal ordered fashion so that $\mathrm{tr}(\bar{C} C)$, $\bar{\eta}_+ \eta_+$, $\eta_- \bar{\eta}_-$ are the number operators and $\frac{1}{2} \sqrt{\frac{2}{m}} N^2 + \sqrt{\frac{2}{m}} N$ is the zero-point fluctuation of the energy.

The exact wavefunctions satisfying the $U(N)$ covariance condition (7) due to the Gauss’ constraint are for $\kappa \geq 0$ and $\kappa < 0$ respectively [4, 7]

$$|\Psi\rangle = \begin{cases} 
G \times (e^{a_1 a_2 ... a_N} \bar{\eta}_{+a_1} (\bar{C} + \bar{C})_{a_2} ... (\bar{C}^{N-1})_{a_N})^{\kappa} |0\rangle \\
G \times (e_{a_1 a_2 ... a_N} \eta_+^{a_1} (\bar{C} \eta_-)_{a_2} ... (\bar{C}^{N-1} \eta_-)^{a_N})^{-\kappa} |0\rangle,
\end{cases}$$

(19)

where $G = G(\mathrm{tr}\bar{C}^l, \bar{\eta}_+ \bar{C}^m \eta_-)$ is an arbitrary function of the $U(N)$ invariant building blocks which are made of the creation operators only, $\mathrm{tr}\bar{C}^l$ and $\bar{\eta}_+ \bar{C}^m \eta_-$ with $1 \leq l \leq N$, $0 \leq m$. The trivial $G$ corresponds to the ground state whose energy is

$$E_0 = \sqrt{\frac{g'}{m}} (|\kappa| + 1) N + \frac{1}{2} \sqrt{\frac{g'}{m}} ((|\kappa| + 1) N^2 - |\kappa| N).$$ (20)

In particular when $\kappa = 0$, the vacuum, $|0\rangle$ is the ground state, and in this case we can calculate the exact density function, $\rho_0(x)$ in Eq.(11). First, using the large $N$ behaviour,

$$\langle 0 | \mathrm{tr}(C + \bar{C})^{2m} |0\rangle \sim \frac{(2m)!}{m!(m+1)!} N^{m+1},$$

(21)

one can check that the Fourier mode for $\rho_0(x)$ is given by the Bessel function, $J_1$. Realizing the Fourier transformation of a half circle is $J_1$ we get for large $N$

$$\rho_0(x) = \frac{1}{\pi} \sqrt{2N} (mg)^{\frac{1}{2}} \mathrm{Re} \left( \sqrt{1 - \frac{(mg)^{\frac{1}{2}}}{2N} x^2} \right).$$ (22)
In fact, this density function is identical to that in the Calogero-Sutherland system [12]. An intuitive way to see this result is to note the close relation to the matrix quantum Hall system with the confining harmonic potential [2], where the density function is constant on a disc. Essentially our density function is the one-dimensional projection of it. For general \( \kappa \) at the center, \( x = 0 \) we expect

\[
\lim_{N \to \infty} \left[ \rho_{\kappa}(0) \right]^2 = \frac{\sqrt{mg}}{\pi^2 (|\kappa| + 1)},
\]

and this is an analogue of the fractional filling factor, \( \nu = 1/(|\kappa| + 1) \) in the quantum Hall system. The “+1” in the denominator is again due to the zero-point fluctuation or the Vandermonde determinant.

Henceforth we discuss the classical dynamics of the system focusing on the \( \kappa \geq 0 \) case. The equations of motion are

\[
0 = mD_t D_t X + gX, \quad (24)
\]

\[
0 = mD_t D_t \phi + g' \phi \quad (\Leftrightarrow \quad 0 = D_t \eta_{\pm} \pm i \sqrt{\frac{g'}{m}} \eta_{\pm}). \quad (25)
\]

As in [7] we choose the gauge such that \( X \) is diagonal and \( \eta_{\pm} \) is real and non-negative. For the simplicity of notation we define a matrix, \( K_{ab} \equiv \eta_{+}^{a} \eta_{+}^{b} - \eta_{-}^{a} \eta_{-}^{b} \). Now the Gauss’ constraint determines \( \eta_{+}^{a} \) and the off-diagonal components of \( A_{\theta} \) completely

\[
\eta_{+}^{a} = \sqrt{\kappa + |\eta_{-}^{a}|^2}, \quad A_{\theta}^{a} = \frac{K_{ab}^{a}/m}{(x_{a} - x_{b})^2} \quad \text{for } a \neq b. \quad (26)
\]

Substituting these into the Hamiltonian we obtain a generalized Calogero-Sutherland model (cf. [14])

\[
H_{\text{eff}} = \sum_{a=1}^{N} \frac{1}{2m} \dot{x}_{a}^2 + \frac{1}{2} g x_{a}^2 + 2 \sqrt{\frac{g'}{m}} \eta_{-a} \eta_{-}^{a} + \sum_{a>b} \frac{|K_{ab}^{a}|^2/m}{(x_{a} - x_{b})^2}, \quad (27)
\]

where \( K_{ab}^{a} \) is now a function of \( \eta_{-} \) only

\[
K_{ab}^{a} = \sqrt{(\kappa + |\eta_{-}^{a}|^2)(\kappa + |\eta_{-}^{b}|^2)} - \eta_{-}^{a} \eta_{-}^{b}. \quad (28)
\]

Apparently the complex vector, \( \eta_{-} \) gives the novelty. Namely the strength of the repulsive potential, \( |K_{ab}^{a}| \) between two particles is not fixed but varies. The Schwarz inequality shows \( |K_{ab}^{a}| \geq \kappa \) and the saturation occurs when \( \eta_{-}^{a} \) is independent of the particle index, \( a \). In particular, for the finite energy configurations with the large \( g' \) limit, \( \eta_{-} \) vanishes classically and the saturation occurs.

The corresponding Lagrangian is with (28)

\[
L_{\text{eff}} = \sum_{a=1}^{N} \frac{1}{2} m \dot{x}_{a}^2 - \frac{1}{2} g x_{a}^2 - i \eta_{-a} \eta_{-}^{a} - 2 \sqrt{\frac{g'}{m}} \eta_{-a} \eta_{-}^{a} - \sum_{a>b} \frac{|K_{ab}^{a}|^2/m}{(x_{a} - x_{b})^2}. \quad (29)
\]
The minus sign for the kinetic term of $\eta_-$ is essentially from Eq.(16). Consistency requires that the dynamics of this generalized Calogero-Sutherland model must agree with the full equations of motion of the matrix model (24, 25). Some straightforward manipulation with Eq.(26) can show that only when the diagonal component of $A_\alpha$ is given by

$$A^a_{\alpha a} = -\sqrt{\frac{g}{m}} - \sum_{b \neq a} \frac{1}{2} (A^a_{\alpha b} + A^b_{\alpha a})(\eta^b_+ / \eta^a_+) ,$$ 

we get $D_t \eta_+ + i \sqrt{\frac{g}{m}} \eta_+ = 0$, and the matrix equations of motion reduce to those of the present generalized Calogero-Sutherland model

$$0 = m \ddot{x}_a + g x_a - \sum_{b \neq a} \frac{2|K^a_{\alpha b}|^2 / m}{(x_a - x_b)^2} ,$$

$$0 = i\dot{\eta}^a_+ + 2\sqrt{\frac{g}{m}} \eta^a_+ + \sum_{b \neq a} \frac{\partial}{\partial x^-_a} |K^a_{\alpha b}|^2 / m}{(x_a - x_b)^2} .$$

The former comes from the diagonal components in (24) while the off-diagonal components vanish due to (25), $(mD_t \ddot{X} + gX)^a_b = -i(D_t K)^a_b / (x_a - x_b) = 0, a \neq b$.

Especially when the effective charge saturates, $|K^a_{\alpha b}| = \kappa$, we get the solution for the two-particle system in a closed form

$$\eta^a_+ = -c e^{2i \sqrt{\frac{g}{m} (t+t_0)} ,}$$

and $x_1, x_2$ are

$$l_{CM} \sin(\sqrt{\frac{g}{m} t}) \pm l \sin(2\sqrt{\frac{g}{m} t} + \theta) + \sqrt{1 + k^2/(4gm^2)} .$$

It would be interesting to see any quantum correction to the inverse square potential in Eq.(29). When the vector freezes it should reduce to $\kappa^2 \to \kappa(\kappa + 1) [13]$. In any case, since $\kappa$ is an integer, the charge of the potential is quantized.

4 Conclusion

We have argued that the gauged matrix models provide a natural framework to describe identical particles since the particle indices which are not physical are gauged. The models intrinsically contain the interaction among the identical particles. Especially for $D = 1$ or the $[X^I, X^J] \to 0$ limit, it corresponds to the inverse square type repulsive potential. As an example we have considered an exactly solvable harmonic oscillator type gauged matrix model. We have obtained all the exact quantum wavefunctions and demonstrated that the model is a generalization of the Calogero-Sutherland system where the strength or the charge of the potential is not fixed but dynamical bounded by below. Furthermore, when the saturation occurs the charge is quantized.

The model we considered is non-relativistic as the kinetic term is quadratic in time derivatives, and this is essentially the reason why we get the inverse square potential. It is very much
desirable to find a matrix gauge model where the intrinsic interaction is given by the inverse potential or the Coulomb interaction. In this sense, it is worth to study the relativistic matrix model and analyze its intrinsic interaction such as

\[ S = \int d\tau \, \text{tr} \sqrt{(D_\tau T)^2 - (D_\tau X)^2}. \] (34)

Quantization of the system is analogue to that of the string theory. It would be interesting to see if the critical dimensions free of the Lorentz symmetry anomaly exists. A single particle system i.e. 1 × 1 matrix model is anomaly free in any dimension. On the other hand, the generic matrix model including the off diagonal elements may be anomalous just like the string theories where the anomaly comes from the oscillators not from the zero mode. Eq. (34) is a straightforward generalization of the single relativistic particle action, and an alternative approach to write the relativistic matrix model would be the dimensional reduction of the non-Abelian Born-Infeld action [15].

An electron in one place seems to be “distinguishable” from one at far distance. To explain this large scale phenomenon, the spontaneous symmetry breaking of the U(N) is to be considered.

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