Uniqueness of the 1D Compressible to Incompressible Limit

Rinaldo M. Colombo\textsuperscript{1} \hspace{1cm} Graziano Guerra\textsuperscript{2}

October 10, 2018

Abstract

Consider two compressible immiscible fluids in 1D in the isentropic approximation. The first fluid is surrounded and in contact with the second one. As the Mach number of the first fluid vanishes, the coupled dynamics of the two fluids results as the compressible to incompressible limit and is known to satisfy an ODE–PDE system. Below, a characterization of this limit is provided, ensuring its uniqueness.

Keywords: Compressible to Incompressible limit, Hyperbolic Conservation Laws, Uniqueness of the Zero Mach number Limit

2010 MSC: 35L65, 35Q35, 76N99

1 Introduction

The literature on the compressible to incompressible limit is vast. We refer for instance to the well known results \cite{12, 13, 15, 16}, the more recent \cite{3, 18}, the review \cite{17} and the references therein.

In this paper, following \cite{5}, we consider two compressible immiscible fluids and study the limit as one of the two becomes incompressible. A volume of a compressible inviscid fluid, say the liquid, is surrounded by another compressible fluid, say the gas. Using the Lagrangian formulation, in the isentropic case, we assume that the gas obeys a fixed pressure law $P_g(\tau)$, while for the liquid we assume a one parameter family of pressure laws $P_\kappa(\tau)$ such that $P'_\kappa(\tau) \to -\infty$ as $\kappa \to 0$. The total mass of the liquid is fixed so that in Lagrangian coordinates the liquid and gas phases fill the fixed sets (see Figure 1)

$\mathcal{L} = [0, m]$ \hspace{0.5cm} and \hspace{0.5cm} $\mathcal{G} = \mathbb{R} \setminus [0, m]$.

For an Eulerian description, see \cite{5}.

On $P_g(\tau)$ and $P_\kappa(\tau)$, we require the usual hypotheses and the incompressible limit assumption:

$P_g, P_\kappa \in C^4$, $P_g(\tau)$, $P_\kappa(\tau) > 0$; \hspace{0.5cm} $P'_g(\tau)$, $P'_\kappa(\tau) < 0$; \hspace{0.5cm} $P''_g(\tau)$, $P''_\kappa(\tau) > 0$; \hspace{0.5cm} $P'_\kappa(\tau) \xrightarrow{\kappa \to 0} -\infty$.

The standard choice $P_g(\tau) = k/\tau^\gamma$ satisfies (1.1) for all $k > 0$ and $\gamma > 0$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{In Lagrangian coordinates, the boundaries separating the two fluids are fixed.}
\end{figure}

\textsuperscript{1}INDAM Unit, University of Brescia, Italy. rinaldo.colombo@unibs.it
\textsuperscript{2}Department of Mathematics and Applications, University of Milano - Bicocca, Italy. graziano.guerra@unimib.it
The coupled dynamics of the two fluids is described by the p-system \([10]\) \((7.1.11)\)

\[
\begin{cases}
\partial_t \tau - \partial_z v = 0 \\
\partial_v + \partial_z P_s(z, \tau) = 0,
\end{cases}
\]

where \(P_s(z, \tau) = \begin{cases} P_s(\tau) & \text{for } z \in \mathcal{L}, \\
P_s(\tau) & \text{for } z \in \mathcal{G}, \end{cases}\) and 
\[v(t, z)\] being the fluid speed at time \(t\) and at the Lagrangian coordinate \(z\).

In Lagrangian coordinates, the conservation of mass and momentum are equivalent to the conservation of \(\tau\) and \(v\) which, in turn, are equivalent along the interfaces \(z = 0\) and \(z = m\) to the Rankine–Hugoniot conditions for \((1.2)\). Therefore, for a.e. \(t \geq 0\),

\[
\begin{align*}
v(t, 0-) &= v(t, 0+) \\
P_s(\tau(t, 0-)) &= P_s(\tau(t, 0+)), \\
P_s(\tau(t, 0-)) &= P_s(\tau(t, 0+)).
\end{align*}
\]

In other words, pressure and velocity have to be continuous across the interfaces. Hence, the pressure is a natural choice as unknown, rather than the specific volume. Following \([5, 7, 9, 11]\), we introduce the inverse functions of the pressure laws

\[
\mathcal{T}_p(p) = P_s^{-1}(p), \quad \mathcal{T}_s(p) = P_s^{-1}(p) \quad \text{where } \mathcal{T}_s(p) \xrightarrow{\kappa \to 0} 0,
\]

the last limit being a consequence of \((1.4)\). Rewrite system \((1.2)\) with \((p, v)\) as unknowns

\[
\begin{cases}
\partial_t \mathcal{T}_s(z, p) - \partial_z v = 0 \\
\partial_v + \partial_z p = 0,
\end{cases}
\]

where \(\mathcal{T}_s(z, p) = \begin{cases} \mathcal{T}_s(p) & \text{for } z \in \mathcal{L}, \\
\mathcal{T}_s(p) & \text{for } z \in \mathcal{G}. \end{cases}\)

The conditions at the interfaces become continuity requirements on the unknown functions:

\[
\begin{align*}
v(t, 0-) &= v(t, 0+) \\
p(t, 0-) &= p(t, 0+), \\
v(t, m-) &= v(t, m+) \\
p(t, m-) &= p(t, m+)
\end{align*}
\]

for a.e. \(t \geq 0\).

As in \([5]\), we fix a pressure law \(P\) and choose \(T = P^{-1}\), so that

\[
\mathcal{T}_s(p) = T\left(\bar{p} + \kappa^2(p - \bar{p})\right), \quad \lim_{\kappa \to 0}\mathcal{T}_s(p) = T(\bar{p}) = \bar{\tau},
\]

where \(\bar{\tau}\) is the constant specific volume at the incompressible limit and \(\bar{p} = P(\bar{\tau})\). For instance, the (modified) Tait equation of state \([14]\) Formula \((1)\) fits into \((1.4)\) with

\[
\mathcal{T}(p) = p^{-1/n} \quad \text{with} \quad \kappa^2 = \frac{n \beta_o}{\bar{\tau}^n}
\]

where \(\beta_o\) is the isothermal compressibility, \(n\) is a pressure independent parameter and \(\beta_o \to 0\) at the incompressible limit.

The main result in \([5]\) states the rigorous convergence (up to a subsequence) at the incompressible limit in the liquid phase of the solutions to \((1.4)\) to solutions to

\[
\begin{cases}
\partial_t \mathcal{T}_s(p) - \partial_z v = 0 \\
\partial_v + \partial_z p = 0
\end{cases}
\]

\[\mathcal{T}(p) = p^{-1/n} \quad \text{with} \quad \kappa^2 = \frac{n \beta_o}{\bar{\tau}^n}
\]

where \(\beta_o\) is the isothermal compressibility, \(n\) is a pressure independent parameter and \(\beta_o \to 0\) at the incompressible limit.

The existence of a Lipschitz continuous semigroup generated by \((1.7)\) is proved in \([1]\). On the other hand, a characterization yielding the uniqueness of solutions to \((1.7)\) is obtained in \([6]\).

In this paper we show that the incompressible limit obtained in \([5]\) satisfies the characterization in \([6]\). Hence, the solution \((p_o, v_o)\) to \((1.4)\) converges as \(\kappa \to 0\), the limit being the unique solution to \((1.7)\).

The next Section is devoted to the formal statements, while Section \([4]\) contains the technical proofs.
2 Main Result

Throughout, we denote by $\text{LC}$ the set of functions defined on $\mathbb{R} \setminus ]0, m[\,$ that are locally constant out of a compact set, i.e., they attain a constant value on $]-\infty, -M]$ and a, possibly different, constant value on $[M, +\infty]$, for a suitable positive $M$.

Below, solutions to (1.7) are understood in the sense of [5] Definition 3.2, see also [1] Definition 2.5], and are constructed in [5] as limits of solutions to (1.2). In solutions to (1.2), the propagation speed of waves in the gas region $\mathcal{G}$ is uniformly bounded, independently of $\kappa$. Therefore, to prove the uniqueness of solutions to (1.7) obtained as the compressible to incompressible semigroup $\mathcal{S}$, under the transformation

$$
U(x) = \begin{bmatrix} U_2 \\ -P_g(U_1 + \tau_\infty) \\ P_g(U_3 + \tau_\infty) \end{bmatrix},
$$

$$
F(U, w) = \begin{bmatrix} \frac{1}{m} \left[ P_g(U_1 + \tau_\infty) - P_g(U_2 + \tau_\infty) \right] \\ P_g(U_1 + \tau_\infty) - P_g(U_3 + \tau_\infty) \end{bmatrix},
$$

$$
b(U) = \begin{bmatrix} U_2 \\ U_4 \end{bmatrix}.
$$

the Cauchy Problem

$$
\begin{align*}
\partial_t \tau - \partial_x v &= 0 \\
\partial_t v + \partial_x P_g(\tau) &= 0 \\
v(t, 0- &= v(t) \\
v(t, m+) &= v(t) \\
(\tau, v)(0, x) &= (\tau_o, v_o)(x)
\end{align*}
$$

is formally equivalent to

$$
\begin{align*}
\partial_t U(t, x) + \partial_x f(U(t, x)) &= 0 & x \in \mathbb{R}^+ \\
b(U(t, 0)) &= g(w(t)) \\
w(t) &= F(U(t, 0), w(t)) \\
U(0, x) &= U_o(x) & x \in \mathbb{R}^+ \\
w(0) &= w_o
\end{align*}
$$

which fits in the well posedness theory developed in [6], as proved by the following Proposition.

**Proposition 2.1.** Let $P_g$ satisfy (1.4). Fix $\tau_\infty, \tau_\infty \in \mathbb{R}^+$. Then, system (2.4) generates a semigroup $\mathcal{S}: \mathbb{R}^+ \times \mathcal{D} \rightarrow \mathcal{D}$ uniquely characterized by the properties (i)-(iv) in [6] Theorem 4]. Moreover, for a suitable positive $\delta$,

$$
\mathcal{D} \supset \left\{ (U, w) \in (L^1 \cap \text{BV})(\mathbb{R}^+; \mathbb{R}^4) \times \mathbb{R}^2: TV(U) + \| b(U(0)) - g(w) \| < \delta \right\}.
$$
The above proposition leads to the main result of this paper.

**Theorem 2.2.** Let \( t \rightarrow ((\tau, v), v_I) (t) \) be a solution to (1.7) obtained as limit for \( \kappa \rightarrow 0 \) of solutions to (1.2), with an initial datum in \( L^C \) and satisfying for all \( t \in \mathbb{R}_+ \)

\[
TV ((\tau, v)(t); G) + \left\| \begin{bmatrix} v(t, 0^-) - v_I(t) \\ v(t, m^+) - v_I(t) \end{bmatrix} \right\| < \delta \tag{2.6}
\]

with \( \delta \) as in (2.5). Correspondingly, define \( t \rightarrow (U, w)(t) \) as in (2.1). Then,

1. for all \( t \in \mathbb{R}_+ \), the map \( t \rightarrow (U, w)(t) \) coincides with an orbit of the semigroup \( S \) defined in Proposition 2.1.

2. The semigroup \( S \) is defined globally in time for all initial data with sufficiently small total variation.

In the above statement, as well as below, we use the obvious notation

\[
TV ((\tau, v); G) = TV ((\tau, v); [-\infty, 0]) + TV ((\tau, v); [m, +\infty]) .
\]

### 3 Technical Proofs

**Proof of Proposition 2.1.** On the basis of (2.2) and with the help of (1.1), we verify that (2.4) satisfies the assumptions of [3, Theorem 4]. With reference to the notation therein, set \( \lambda \) as in (2.5). We obtain that for all \( t \in \mathbb{R}_+ \), \( (\tau, w) (t) \) coincides with an orbit of the semigroup \( S \) defined in Proposition 2.1.

Concerning (H2), \( b \) is clearly of class \( C^4 \) and \( b(0) = 0 \). Moreover,

\[
\det \left[ Db(U) \begin{bmatrix} r_3(U) \\ r_4(U) \end{bmatrix} \right] = \det \begin{bmatrix} \lambda_3(U) & 0 \\ 0 & -\lambda_4(U) \end{bmatrix} = -\lambda_3(U) \lambda_4(U)
\]

and the latter expression above is non zero by (1.1).

Assumptions (H3) and (H4) are immediate by (2.2) and (1.1).

An application of [3, Theorem 4] yields the existence of a Lipschitz continuous local semigroup \( S \) defined on a domain \( D \) enjoying [3, Properties (i)–(iv) in Theorem 4]. Note that (2.5) holds by [3, Formula (4) and Theorem 4].

**Proof of Theorem 2.2.** Given \( t \rightarrow ((\tau, v), v_I) (t) \), define \( t \rightarrow (U, w)(t) \) by means of (2.1). Since

\[
TV (U(t)) + \left\| b (U(t, 0^+)) - g (w(t)) \right\| = TV ((\tau, v)(t); G) + \left\| \begin{bmatrix} v(t, 0^-) - v_I(t) \\ v(t, m^+) - v_I(t) \end{bmatrix} \right\|
\]

thanks to (2.5) we obtain that for all \( t \in \mathbb{R}_+ \), \( (U, w)(t) \in D \), \( D \) being the domain defined in Proposition 2.1.
For $\varepsilon > 0$ and $\kappa > 0$, call $(p^{\kappa, \varepsilon}, v^{\kappa, \varepsilon})$ the wave front tracking approximate solutions to (1.4), see also [5] Formula (2.5) as defined in [5] Section 4, converging to $((P_{\varepsilon}(t), v_{\varepsilon})$ first as $\varepsilon \to 0$ and then as $\kappa \to 0$. To simplify the notation, here we omit the introduction of sequences and subsequences.

In the limit $\varepsilon \to 0$, by [5] Proof of Theorem 3.3 we have that

$$\lim_{\varepsilon \to 0} (p^{\kappa, \varepsilon}, v^{\kappa, \varepsilon})(t) = (p^{\kappa}, v^{\kappa})(t) \quad \text{for all} \quad t \geq 0 \quad \text{in} \quad L^1_{\text{loc}}(\mathbb{R}; \mathbb{R}^2),$$

where $(p^{\kappa}, v^{\kappa})$ solves [5] Formula (2.5) in the sense of [5] Definition 3.1.

In the limit $\kappa \to 0$, we have that

$$\lim_{\kappa \to 0} (p^{\kappa}, v^{\kappa})(t, \cdot) = (p, v)(t, \cdot) \quad \text{for all} \quad t \geq 0 \quad \text{in} \quad L^1_{\text{loc}}(G; \mathbb{R}^2)$$

and

$$\lim_{\kappa \to 0} v^\kappa(t, \cdot) = v(t) \quad \text{for all} \quad t \geq 0 \quad \text{in} \quad L^1(\mathcal{L}; \mathbb{R}^2)$$

$v_t$ being independent of $z$. Introduce

$$\overline{v}^{\kappa, \varepsilon}(t) = \frac{1}{m} \int_0^m v^{\kappa, \varepsilon}(t, z) \, dz$$

and

$$u^{\kappa, \varepsilon}(t) = \left( p^{\kappa, \varepsilon}, v^{\kappa, \varepsilon} \right)(s) \int_G \overline{v}^{\kappa, \varepsilon}(t) \, ds \quad \text{(3.1)}$$

and note that the above $L^1_{\text{loc}}$ convergence implies that

$$u^{\kappa, \varepsilon}(t) \to u^\kappa(t) = \left( (p^{\kappa}, v^{\kappa})(t) \right|_{G}, \frac{1}{m} \int_0^m v^{\kappa}(t, z) \, dz \right) \quad \text{as} \quad \varepsilon \to 0 \quad \text{for all} \quad t \geq 0, \quad \text{(3.2)}$$

$$u^\kappa(t) \to u(t) = \left( (p, v), v_t \right)(t) \quad \text{as} \quad \kappa \to 0 \quad \text{for all} \quad t \geq 0.$$

Following (2.1) and (3.1), introduce the variables

$$U^{\kappa, \varepsilon}(t, x) = \begin{bmatrix} T_g(p^{\kappa, \varepsilon}(t, x), \tau_{-\infty}) - v_{-\infty} \\ v^{\kappa, \varepsilon}(t, x + m) - \tau_{+\infty} \\ T_g(p^{\kappa, \varepsilon}(t, x), \tau_{+\infty}) - v_{+\infty} \end{bmatrix} w^{\kappa, \varepsilon}(t) = \begin{bmatrix} \overline{v}^{\kappa, \varepsilon}(t) - v_{-\infty} \\ \overline{v}^{\kappa, \varepsilon}(t) - v_{+\infty} \end{bmatrix} \quad \text{(3.3)}$$

and the distance

$$d \left( (U, w), (\tilde{U}, \tilde{w}) \right) = \left\| \tilde{U} - U \right\|_{L^1(\mathbb{R}^+; \mathbb{R}^4)} + \left\| \tilde{w} - w \right\|. \quad \text{(3.4)}$$

By the convergences (3.2), the definition (3.3) and the continuity of $S_t$

$$d \left( (U, w)(t), S_t ((U, w)(0)) \right) \leq \lim_{\kappa \to 0, \varepsilon \to 0} d \left( (U^{\kappa, \varepsilon}, w^{\kappa, \varepsilon})(t), S_t ((U^{\kappa, \varepsilon}, w^{\kappa, \varepsilon})(0)) \right). \quad \text{(3.4)}$$

By [2] Theorem 2.9, denoting by $L$ a Lipschitz constant of $S_t$,

$$d \left( (U^{\kappa, \varepsilon}, w^{\kappa, \varepsilon})(t), S_t ((U^{\kappa, \varepsilon}, w^{\kappa, \varepsilon})(0)) \right) \leq L \int_0^t \liminf_{h \to 0} \frac{1}{h} d \left( (U^{\kappa, \varepsilon}, w^{\kappa, \varepsilon})(s + h), S_h ((U^{\kappa, \varepsilon}, w^{\kappa, \varepsilon})(s)) \right) \, ds \leq L \int_0^t \liminf_{h \to 0} \frac{1}{h} d \left( (U^{\kappa, \varepsilon}, w^{\kappa, \varepsilon})(s + h), F(h) ((U^{\kappa, \varepsilon}, w^{\kappa, \varepsilon})(s)) \right) \, ds \quad \text{(3.5)}$$

where $F$ is the local flow defined in [6] Formula (5). By construction, the last term in the integrand above is

$$d \left( (U^{\kappa, \varepsilon}, w^{\kappa, \varepsilon})(s + h), F(h) ((U^{\kappa, \varepsilon}, w^{\kappa, \varepsilon})(s)) \right)$$

5
\[ \| U^{\kappa,\varepsilon}(s + h) - \mathcal{S}_h \left( U^{\kappa,\varepsilon}(s) \right) \|_{L^1(\mathbb{R}^2)} + \| w^{\kappa,\varepsilon}(s + h) - \left[ w^{\kappa,\varepsilon}(s) + h F \left( U^\sigma, w^{\kappa,\varepsilon}(s) \right) \right] \| \tag{3.6} \]

where \( F \) is as in [2.2], \( S \) is the Standard Riemann Semigroup [2 Chapter 9] generated by \( \partial_t U + \partial_x f(U) = 0 \), with \( f \) as in [2.2].

\[ \mathcal{U}^{\kappa,\varepsilon}(s, x) = \begin{cases} U^{\kappa,\varepsilon}(s, x) & x \geq 0 \\ U^\sigma & x < 0 \end{cases} \]

and \( U^\sigma \) is the unique state satisfying \( b(U^\sigma) = g \left( w^{\kappa,\varepsilon}(s) \right) \) that can be connected to \( U^{\kappa,\varepsilon}(t, 0+) \) by means of Lax waves with positive speed, with \( b \) and \( g \) as in [2.2].

Introduce

\[ (\bar{p}, \bar{v})(z) = \begin{cases} (p^{\kappa,\varepsilon}, v^{\kappa,\varepsilon})(s, z) & z < 0 \\ (p_0^\sigma, \bar{v}^{\kappa,\varepsilon})(s) & z \in [0, m/2] \\ (p_m^\sigma, \bar{v}^{\kappa,\varepsilon})(s) & z \in [m/2, m] \\ (p^{\kappa,\varepsilon}, v^{\kappa,\varepsilon})(s, z) & z > m \end{cases} \]

where \( \bar{v}^{\kappa,\varepsilon} \) is defined in [3.1] and the pressure \( p_0^\sigma \), respectively \( p_m^\sigma \), is such that the Riemann Problem

\[ \begin{align*}
\partial_t T(p) - \partial_x v &= 0 \\
\partial_t v + \partial_x p &= 0
\end{align*} \]

is solved by waves with negative, respectively positive, speed. Note that by [4 Lemma 4.1]

\[ |p^{\kappa,\varepsilon}(s, 0) - p_0^\sigma| = \mathcal{O}(1) \left| v^{\kappa,\varepsilon}(s, 0) - \bar{v}^{\kappa,\varepsilon}(s) \right|, \tag{3.7} \]

\[ |p^{\kappa,\varepsilon}(s, m) - p_m^\sigma| = \mathcal{O}(1) \left| v^{\kappa,\varepsilon}(s, m) - \bar{v}^{\kappa,\varepsilon}(s) \right|, \]

recall that \( z \to p^{\kappa,\varepsilon}(s, z) \) and \( z \to v^{\kappa,\varepsilon}(s, z) \) are locally constant in neighborhoods of \( z = 0 \) and \( z = m \), see [5 Formula (4.12)]. Call \( \Sigma \) the Standard Riemann Semigroup [2 Chapter 9] generated by the \( p \)-system

\[ \partial_t T(p) - \partial_x v = 0 \]

for \( z \) varying on all the real line. Observe that the first addend in [3.3] reads

\[ \| U^{\kappa,\varepsilon}(s + h) - \mathcal{S}_h \left( U^{\kappa,\varepsilon}(s) \right) \|_{L^1(\mathbb{R}^2)} \leq \int_G \left\| (p^{\kappa,\varepsilon}, v^{\kappa,\varepsilon})(s + h, z) - \left( \Sigma_h (\bar{p}, \bar{v}) \right)(z) \right\| \, dz. \tag{3.8} \]

Assume that at time \( s \) no interaction takes place and choose \( h \) sufficiently small so that in the time interval \( [s, s + h] \) no interaction takes place and no wave hits any of the lines \( z = \pm \varepsilon^2 \), \( z = 0 \), \( z = m \pm \varepsilon^2 \) and \( z = m \).

We now continue to estimate the right hand side in [3.3] limited to \( ]-\infty, 0] \). Let \( z_1, z_2, \ldots \) be the points of jump of the map \( z \to (p^{\kappa,\varepsilon}, v^{\kappa,\varepsilon})(s, z) \). Denote by \( \bar{\lambda} \) an upper bound for the characteristic speeds in the gas phase. Then, we have

\[ \int_{-\infty}^0 \left\| (p^{\kappa,\varepsilon}, v^{\kappa,\varepsilon})(s + h, z) - \left( \Sigma_h (\bar{p}, \bar{v}) \right)(z) \right\| \, dz \]

\[ = \sum_{z_i < -\varepsilon^2} \int_{z_i - \lambda h}^{z_i + \lambda h} \left\| (p^{\kappa,\varepsilon}, v^{\kappa,\varepsilon})(s + h, z) - \left( \Sigma_h (p^{\kappa,\varepsilon}, v^{\kappa,\varepsilon})(s) \right)(z) \right\| \, dz \tag{3.9} \]

\[ + \sum_{z_i \in [-\varepsilon^2, 0]} \int_{z_i - \lambda h}^{z_i + \lambda h} \left\| (p^{\kappa,\varepsilon}, v^{\kappa,\varepsilon})(s + h, z) - \left( \Sigma_h (p^{\kappa,\varepsilon}, v^{\kappa,\varepsilon})(s) \right)(z) \right\| \, dz \tag{3.10} \]
Consider now (3.11). We use [4, Point 2) in Theorem 2.2] to estimate the difference between (3.8) as follows:

\[ \left\| \left( p^{\epsilon, v}, v^{\epsilon, v} \right)(s + h, z) - \Sigma_h \left( \left( \tilde{p}, \tilde{v} \right) (s) \right) \right\| \, dz . \]  

(3.11)

A standard procedure yields the estimate of (3.9) by means of [2, (ii) in Lemma 9.1], so that

\[ [3.9] = \mathcal{O}(1) \epsilon \, h \, \text{TV} \left( p^{\epsilon, \sigma} (s) ; [ - \epsilon^2, \epsilon^2 ] \right) . \]

Similarly, since all waves in the strip \([- \epsilon^2, 0]\) have speed \(\pm 1\), by [2, (i) in Lemma 9.1] we have

\[ [3.10] = \mathcal{O}(1) h \, \text{TV} \left( p^{\epsilon, \sigma} (s) ; [ - \epsilon^2, 0 ] \right) . \]

Consider now (3.11). We use [3, Point 2) in Theorem 2.2] to estimate the difference between \(\left( p^{\epsilon, v}, v^{\epsilon, v} \right)\) and \(\Sigma_h (\tilde{p}, \tilde{v})\) that are solutions, respectively, to the two initial–boundary value problems

\[
\begin{align*}
\partial_t T(p) - \partial_z v &= 0 \\
\partial_t v + \partial_z p &= 0 \\
(p, v)(0, z) &= (p^{\epsilon, v}, v^{\epsilon, v})(s, 0)
\end{align*}
\]

and

\[
\begin{align*}
\partial_t T(p) - \partial_z v &= 0 \\
\partial_t v + \partial_z p &= 0 \\
(p, v)(0, z) &= (p^{\epsilon, \sigma}, v^{\epsilon, \sigma})(s, 0)
\end{align*}
\]

with the mean value \(\bar{p}^{\epsilon, \sigma}\) as defined in (3.1). Then, we apply [5, Proposition 4.9] to obtain

\[
\begin{align*}
[3.11] &\leq \mathcal{O}(1) \lambda h \left| v^{\epsilon, \sigma}(s, 0) - \bar{p}^{\epsilon, \sigma}(s) \right| \quad \text{by [3, Point 2) in Theorem 2.2]} \\
&\leq \mathcal{O}(1) \lambda h \, \text{TV} \left( v^{\epsilon, \sigma}(s) ; L \right) \quad \text{by (3.1)} \\
&\leq \mathcal{O}(1) h \, \kappa \quad \text{by [5, Proposition 4.9].}
\end{align*}
\]

Entirely analogous estimates can be applied to bound the similar terms on \([m, +\infty[\). We thus continue (3.8) as follows:

\[
\begin{align*}
\left\| U^{\epsilon, \sigma}(s + h) - \tilde{S}_h \left( U^{\epsilon, \sigma}(s) \right) \right\|_{L^1(\mathbb{R}_+; \mathbb{R}^4)} &\leq \mathcal{O}(1) h \epsilon \, \text{TV} \left( p^{\epsilon, \sigma}(s) ; [ - \epsilon^2, \epsilon^2 ] \cup [m, \infty[ \right) \\
&\quad + \mathcal{O}(1) h \, \text{TV} \left( p^{\epsilon, \sigma} (s) ; [ - \epsilon^2, 0 ] \cup [m, \epsilon^2 ] \right) \\
&\quad + \mathcal{O}(1) h \kappa .
\end{align*}
\]

We pass now to the second addend in (3.6), using (3.7) and [5, Proposition 4.9],

\[
\begin{align*}
\left\| w^{\epsilon, \sigma}(s + h) - \left[ w^{\epsilon, \sigma}(s) + hF \left( U^{\sigma}, w^{\epsilon, \sigma}(s) \right) \right] \right\| \\
&= \left\| \bar{p}^{\epsilon, \sigma}(s + h) - \bar{p}^{\epsilon, \sigma}(s) - hF \left( U^{\sigma}, w^{\epsilon, \sigma}(s) \right) \right\| \\
&= \sqrt{2} \left| \bar{p}^{\epsilon, \sigma}(s + h) - \bar{p}^{\epsilon, \sigma}(s) - \frac{1}{m} h \left( p_0^\sigma - p_m^\sigma \right) \right| \\
&= \sqrt{2} \frac{h}{m} \int_L v^{\epsilon, \sigma}(s + h, z) \, dz - \int_L v^{\epsilon, \sigma}(s, z) \, dz - \int_s^{s+h} \left( p^{\epsilon, \sigma}(\sigma, 0) - p^{\epsilon, \sigma}(\sigma, m) \right) \, d\sigma \\
&\quad + \sqrt{2} \frac{h}{m} \left| \left( p_0^\sigma - p_m^\sigma \right) - \left( p^{\epsilon, \sigma}(s, 0) - p^{\epsilon, \sigma}(s, m) \right) \right| \\
&\leq \sqrt{2} \frac{h}{m} \int_s^{s+h} \frac{d}{d\sigma} \int_L v^{\epsilon, \sigma}(\sigma, z) \, dz \, d\sigma - \int_s^{s+h} \left( p^{\epsilon, \sigma}(\sigma, 0) - p^{\epsilon, \sigma}(\sigma, m) \right) \, d\sigma + \mathcal{O}(1) h \kappa \\
&\leq \sqrt{2} \frac{h}{m} \int_s^{s+h} \left[ \sum_{z_i \in [0, m]} \left( v^{\epsilon, \sigma}(\sigma, z_i-) - v^{\epsilon, \sigma}(\sigma, z_i+) \right) \hat{z}_i \right]
\end{align*}
\]
while the cases of waves of the first family are entirely analogous.

\[ \sum_{z_i \in [0, m]} \left( p^{\kappa, \varepsilon}(\sigma, z_i^+) - p^{\kappa, \varepsilon}(\sigma, z_i^-) \right) \, d\sigma \leq O(1) \, h \, \kappa + \frac{\sqrt{2}}{m} \int_s^{s+h} \sum_{z_i \in [0, m]} \left| \left( v^{\kappa, \varepsilon}(\sigma, z_i^-) - v^{\kappa, \varepsilon}(\sigma, z_i^+) \right) \dot{z}_i + \left( p^{\kappa, \varepsilon}(\sigma, z_i^+) - p^{\kappa, \varepsilon}(\sigma, z_i^-) \right) \right| \, d\sigma \]

We estimate the integral term in the latter term above in different ways, depending on the location of \( z_i \):

\[ \int_s^{s+h} \sum_{z_i \in [0, m]} \left| \left( v^{\kappa, \varepsilon}(\sigma, z_i^-) - v^{\kappa, \varepsilon}(\sigma, z_i^+) \right) \dot{z}_i + \left( p^{\kappa, \varepsilon}(\sigma, z_i^+) - p^{\kappa, \varepsilon}(\sigma, z_i^-) \right) \right| \, d\sigma \]

since in \([0, \varepsilon^2] \cup m - \varepsilon^2, m]\) we have \( \dot{z}_i = 1 \). To bound the remaining terms in \([3.12]\), let \( z_i \in [\varepsilon^2, m - \varepsilon^2] \), use \([3.4] \) Lemma 4.1 and Formula (4.3) and assume that the jump at \( z_i \) is solved by a 2-rarefaction:

\[ \leq \varepsilon \left| v^{\kappa, \varepsilon}(\sigma, z_i^+) - v^{\kappa, \varepsilon}(\sigma, z_i^-) \right| + \left| p^{\kappa, \varepsilon}(\sigma, z_i^+) - p^{\kappa, \varepsilon}(\sigma, z_i^-) \right| \]

When dealing with a 2-shock we obtain the simpler estimate

\[ \left| - (v^{\kappa, \varepsilon}(\sigma, z_i^+) - v^{\kappa, \varepsilon}(\sigma, z_i^-)) \dot{z}_i + (p^{\kappa, \varepsilon}(\sigma, z_i^+) - p^{\kappa, \varepsilon}(\sigma, z_i^-)) \right| \leq \varepsilon \left| v^{\kappa, \varepsilon}(\sigma, z_i^+) - v^{\kappa, \varepsilon}(\sigma, z_i^-) \right| \]

while the cases of waves of the first family are entirely analogous.

Summarizing:

\[ [3.12] \leq O(1) \, h \, TV \left( p^{\kappa, \varepsilon}(s); [0, \varepsilon^2] \cup m - \varepsilon^2, m \right) \]
By formula (4.32) in Proposition 4.9, we finally obtain,

\[
\begin{align*}
&\quad d\left( (U^{\kappa,\varepsilon}, w^{\kappa,\varepsilon})(s + h), \mathcal{F}(h) \left( (U^{\kappa,\varepsilon}, w^{\kappa,\varepsilon})(s) \right) \right) \\
&\leq O(1) h \varepsilon + O(1) h \kappa + O(1) h \varepsilon \quad \text{TV}(p^{\kappa,\varepsilon}(s); [\varepsilon^2, m - \varepsilon^2]) \\
&\quad + O(1) h \varepsilon \quad \text{TV}(p^{\kappa,\varepsilon}(s); [-\varepsilon^2, \varepsilon^2]) \\
&\quad + O(1) h \varepsilon \quad \text{TV}(p^{\kappa,\varepsilon}(s); [m - \varepsilon^2, m + \varepsilon^2])
\end{align*}
\]

whence, by formula (4.33) in Proposition 4.9, we get

\[
\begin{align*}
&\quad d\left( (U^{\kappa,\varepsilon}, w^{\kappa,\varepsilon})(t), S_t ((U^{\kappa,\varepsilon}, w^{\kappa,\varepsilon})(0)) \right) \\
&\leq O(1) \int_0^t \left( \varepsilon + \kappa + \varepsilon \quad \text{TV}(p^{\kappa,\varepsilon}(s); [-\varepsilon^2, \varepsilon^2]) \right) \text{d}s \\
&\quad \int_0^t \text{TV}(p^{\kappa,\varepsilon}(s); [-\varepsilon^2, \varepsilon^2]) \text{d}z \\
&\quad \text{TV}(p^{\kappa,\varepsilon}(s); [m - \varepsilon^2, m + \varepsilon^2])
\end{align*}
\]

Changing the order of integration and using formula (4.33) in Proposition 4.9, we get

\[
\begin{align*}
&\quad \int_0^t \varepsilon + \kappa + \varepsilon \quad \text{TV}(p^{\kappa,\varepsilon}(s); [-\varepsilon^2, \varepsilon^2]) \text{d}s \\
&\quad \int_{[-\varepsilon^2, \varepsilon^2]} \text{TV}(p^{\kappa,\varepsilon}(s); [m - \varepsilon^2, m + \varepsilon^2]) \text{d}z \\
&\quad \text{TV}(p^{\kappa,\varepsilon}(s); [m - \varepsilon^2, m + \varepsilon^2])
\end{align*}
\]

so that

\[
\begin{align*}
&\quad d\left( (U^{\kappa,\varepsilon}, w^{\kappa,\varepsilon})(t), S_t ((U^{\kappa,\varepsilon}, w^{\kappa,\varepsilon})(0)) \right) = O(1) \left( \varepsilon + \kappa + t + \frac{\varepsilon^2}{\kappa} \right)
\end{align*}
\]

Using (3.4), the proof of 1. is completed.

Hence, the trajectory of the semigroup \( S \) with initial datum \( (U, w)(0) \) is defined for all \( t \in \mathbb{R}_+ \).

\[\square\]

**Acknowledgment:** The present work was supported by the PRIN 2012 project *Nonlinear Hyperbolic Partial Differential Equations, Dispersive and Transport Equations: Theoretical and Applicative Aspects* and by the GNAMPA 2014 project *Conservation Laws in the Modeling of Collective Phenomena.*

**References**

[1] R. Borsche, R. M. Colombo, and M. Garavello. Mixed systems: ODEs - balance laws. *J. Differential Equations*, 252(3):2311–2338, 2012.

[2] A. Bressan. *Hyperbolic systems of conservation laws*, volume 20 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, Oxford, 2000. The one-dimensional Cauchy problem.

[3] G.-Q. Chen, C. Christoforou, and Y. Zhang. Continuous dependence of entropy solutions to the Euler equations on the adiabatic exponent and Mach number. *Arch. Ration. Mech. Anal.*, 189(1):97–130, 2008.
[4] R. M. Colombo and G. Guerra. On general balance laws with boundary. *J. Differential Equations*, 248(5):1017–1043, 2010.

[5] R. M. Colombo and G. Guerra. BV solutions to 1D isentropic Euler equations in the zero Mach number limit. *J. Hyperbolic Differ. Equ.*, 2016. To appear. Preprint: [http://arxiv.org/abs/1509.01717](http://arxiv.org/abs/1509.01717).

[6] R. M. Colombo and G. Guerra. Characterization of the solutions to ODE–PDE systems. *Applied Mathematics Letters*, 62, 2016.

[7] R. M. Colombo and G. Guerra. A coupling between a non-linear 1D compressible-incompressible limit and the 1D p-system in the non smooth case. *Netw. Heterog. Media*, 11(2):313–330, 2016.

[8] R. M. Colombo, G. Guerra, M. Herty, and V. Schleper. Optimal control in networks of pipes and canals. *SIAM J. Control Optim.*, 48(3):2032–2050, 2009.

[9] R. M. Colombo, G. Guerra, and V. Schleper. The compressible to incompressible limit of 1D Euler equations: the non smooth case. *Arch. Rational Mech. Anal.*, 2015. To appear.

[10] C. M. Dafermos. *Hyperbolic conservation laws in continuum physics*, volume 325 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, third edition, 2010.

[11] G. Guerra and V. Schleper. A coupling between a 1D compressible-incompressible limit and the 1D p-system in the non smooth case. *Bull. Braz. Math. Soc. (N.S.)*, 47(1):381–396, 2016.

[12] S. Klainerman and A. Majda. Singular limits of quasilinear hyperbolic systems with large parameters and the incompressible limit of compressible fluids. *Comm. Pure Appl. Math.*, 34(4):481–524, 1981.

[13] S. Klainerman and A. Majda. Compressible and incompressible fluids. *Comm. Pure Appl. Math.*, 35(5):629–651, 1982.

[14] J. R. Macdonald. Some simple isothermal equations of state. *Rev. Mod. Phys.*, 38:669–679, Oct 1966.

[15] G. Métivier and S. Schochet. The incompressible limit of the non-isentropic Euler equations. *Arch. Ration. Mech. Anal.*, 158(1):61–90, 2001.

[16] S. Schochet. The compressible Euler equations in a bounded domain: existence of solutions and the incompressible limit. *Comm. Math. Phys.*, 104(1):49–75, 1986.

[17] S. Schochet. The mathematical theory of low Mach number flows. *M2AN Math. Model. Numer. Anal.*, 39(3):441–458, 2005.

[18] J. Xu and W.-A. Yong. A note on incompressible limit for compressible Euler equations. *Math. Methods Appl. Sci.*, 34(7):831–838, 2011.