The scalar-photon 3-point vertex in massless quenched scalar QED

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Abstract. Non perturbative studies of Schwinger-Dyson equations (SDEs) require their infinite, coupled tower to be truncated in order to reduce them to a practically solvable set. In this connection, a physically acceptable ansatz for the three point vertex is the most favorite choice. Scalar quantum electrodynamics (sQED) provides a simple and neat platform to address this problem. The most general form of the scalar-photon three point vertex can be expressed in terms of only two independent form factors, longitudinal and transverse. Ball and Chiu have demonstrated that the longitudinal vertex is fixed by requiring the Ward-Fradkin-Green-Takahashi identity (WFGTI), while the transverse vertex remains undetermined. In massless quenched sQED, we propose the transverse part of the non perturbative scalar-photon vertex.

1. Introduction
Gauge theories of fundamental interactions have been the cornerstone of describing the physical world at the most basic level. Their enormous success primarily lies in the region where the coupling strength is small enough and the tools of perturbation theory are reliable. However, not all interesting phenomena can be accessed in this approximation scheme. In the non perturbative sector of quantum chromodynamics (QCD), two major phenomena emerge: 1) color confinement, and 2) dynamical chiral symmetry breaking (DCSB). For studying strongly interacting bound states, a reliable understanding of these phenomena is essential. However, it can be achieved solely through non perturbative techniques such as lattice QCD, SDEs, [1, 2], chiral perturbation theory and effective quark models. Keeping this in mind, our interest is focussed on the study and physically acceptable truncations of SDEs beyond perturbation theory.

SDEs are the fundamental equations of motion of any quantum field theory (QFT). They form an infinite set of coupled integral equations that relate the n-point Green function to the (n + 1)-point function. As their derivation requires no assumption regarding the strength of the interaction, they are ideally suited for studying interactions like QCD, where one single theory has diametrically opposed perturbative and non perturbative facets in the ultraviolet and infrared regimes of momenta, respectively. Unfortunately, being an infinite set of coupled equations, they are intractable without some simplifying assumptions. Typically, in the non
perturbative region, SDEs are truncated at the level of two-point Green functions (propagators). We must then use an ansatz for the full three point vertex. This has to be done carefully.

- It must satisfy the Ward-Fradkin-Green-Takahashi identity (WFGTI), [3, 4, 5].
- It must satisfy the local gauge covariance properties of the theory.
- It must ensure the multiplicative renormalizability (MR) of the two point propagator.
- It should reduce to its perturbation theory Feynman expansion in the limit of weak coupling.
- It must have the same symmetry properties as the bare vertex under charge conjugation, parity and time reversal.
- One loop perturbation suggests that it should be free of any kinematic singularities.

The scalar-photon three point vertex $\Gamma^\mu(k, p)$ must be symmetric under the exchange of momenta $k$ and $p$.

2. The SDE for the Scalar Propagator

The SDE for the scalar propagator $S(k)$ in sQED, in the quenched approximation, is shown in Fig. 1:

**Figure 1.** The SDE for the scalar propagator. The color-filled solid blobs labelled with $S$ and $\Gamma_\nu$ stand for the full scalar propagator and the full scalar-photon vertex, respectively. The dots (\ldots) represent all the diagrams whose contribution begins at the two-loop level.

Mathematically, this is written as:

$$-iS^{-1}(k) = -iS_0^{-1}(k) + e^2 \int_M \frac{d^4\omega}{(2\pi)^4} (\omega + k)^\mu S(\omega)\Gamma_\nu(\omega, k)\Delta_{\mu\nu}(q)$$

$$-e^2 \int_M \frac{d^4\omega}{(2\pi)^4} \Gamma_0^{\mu\nu}(k, -\omega, k, \omega)\Delta_{\mu\nu}(\omega) - \int_M \frac{d^4\omega}{(2\pi)^4} S(\omega)\Gamma_0(\omega, k) + \ldots ,$$

(1)

where $e$ is the electromagnetic coupling, $q = \omega - k$, and the subscript $M$ indicates integration over the entire Minkowski space.

$\Delta_{\mu\nu}(\omega)$ and $S_0(k)$ are the bare photon and scalar propagators. $S(k)$ is the full scalar propagator. For massless scalars, $S(k)$ can be expressed in terms of the so-called wavefunction renormalization $F(k^2, \Lambda^2)$, so that

$$S(k) = \frac{F(k^2, \Lambda^2)}{k^2} ,$$

(2)
where $\Lambda$ is the ultraviolet cut-off used to regularize the divergent integrals involved. The bare scalar propagator is given by $S_0(k) = 1/k^2$. The bare photon propagator is

$$\Delta_{\mu\nu}(q) = \frac{1}{q^2} \left[ g_{\mu\nu} + (\xi - 1) \frac{g_{\mu\nu} q^2}{q^2} \right], \quad (3)$$

and it remains unrenormalized in the quenched approximation. $\Gamma_0^{\mu\nu}(k, -\omega, k, \omega) = 2ie^2g^{\mu\nu}$ and $\Gamma_0(k, \omega) = -i\lambda$ are the bare four point scalar-scallar-photon-photon and the four-scalar vertices, respectively. The last two diagrams of the gap equation, Eq. (1), will be referred to as the photon and the scalar bubble diagrams, in that order. $\Gamma^\nu(\omega, k)$ is the full three point scalar-photon vertex, for which we must make an ansatz in order to solve Eq. (1). The WFGTI for this vertex, i.e.,

$$q_\mu \Gamma^\mu(\omega, k) = S^{-1}(\omega) - S^{-1}(k), \quad (4)$$

allows us to decompose it as a sum of longitudinal and transverse components, as suggested by Ball and Chiu, [6]:

$$\Gamma^\mu(\omega, k) = \Gamma^\mu_L(\omega, k) + \Gamma^\mu_T(\omega, k). \quad (5)$$

The *longitudinal* part $\Gamma^\mu_L(\omega, k)$ satisfies the WFGTI, Eq. (4), by itself, and the *transverse* part $\Gamma^\mu_T(\omega, k)$, which remains completely undetermined, is naturally constrained by

$$q_\mu \Gamma^\mu_T(\omega, k) = 0. \quad (6)$$

Moreover,

$$\Gamma^\mu_L(k, k) = 0. \quad (7)$$

In order to satisfy the WFGTI in a manner free of kinematic singularities, we follow Ball and Chiu and write

$$\Gamma^\mu_L(\omega, k) = \frac{S^{-1}(\omega) - S^{-1}(k)}{\omega^2 - k^2} (\omega + k)^\mu. \quad (8)$$

This construction implies that the ultraviolet divergences solely reside in the longitudinal part. Moreover, recall the following relations between the renormalized and bare quantities:

$$S^R(p) = Z_2^{-1}S(p), \quad \Gamma^\mu_R(k, p) = Z_4 \Gamma^\mu(k, p). \quad (9)$$

Thus, the form of the longitudinal vertex in Eq. (8) guarantees the relation $Z_1 = Z_2$. Consequently, the running of the coupling is dictated by the corrections to the photon propagator alone. In the approximation of quenched sQED, the coupling does not run. If we unquench the theory, it is easy to calculate $Z_3$ and the running coupling constant with the well known expression:

$$\alpha(Q^2) = \frac{\alpha(Q_0^2)}{1 - (\alpha(Q_0^2)/12\pi) \ln(Q^2/Q_0^2)}. \quad (10)$$

The ultraviolet finite transverse vertex can be expanded out in terms of one unknown function $\tau(\omega^2, k^2, q^2)$, [6]:

$$\Gamma^\mu_T(\omega, k) = \tau(\omega^2, k^2, q^2) T^\mu(\omega, k), \quad (11)$$

where

$$T^\mu(\omega, k) = (\omega \cdot q) k^\mu - (k \cdot q) \omega^\mu \quad (12)$$

is the transverse basis vector in the Minkowski space and fulfils Eqs. (6,7). To begin with, the form factor $\tau(\omega^2, k^2, q^2)$ is an unconstrained scalar function (representing an 8-fold simplification of the spinor QED/QCD case).
3. The Transverse Vertex and Perturbation Theory Constraints

The three point vertex at one-loop level in perturbation theory has been calculated by Concha-Sánchez et al., [7], using dimensional regularization, in arbitrary gauge $\xi$ and dimensions $d$.

For the massless case, in dimension $d = 4$, they report

$$\tau_{BCD}(k^2, p^2, q^2) = \frac{\alpha}{8\pi^2} \left\{ (k^2 + p^2 - 4k \cdot p) \left( k \cdot p J_0 + \ln \left( \frac{q^4}{p^2 k^2} \right) \right) \\
+ \frac{(k^2 + p^2)q^2 - 8p^2 k^2}{p^2 - k^2} \ln \left( \frac{k^2}{p^2} \right) \\
+ (\xi - 1) \left[ k^2 p^2 J_0 + \frac{2[k^2 p^2 + k \cdot p(k^2 + p^2)]}{k^2 - p^2} \right] \ln \left( \frac{p^2}{k^2} \right) \\
+ \frac{2k \cdot p}{k^2 - p^2} \left[ k^2 \ln \left( \frac{q^2}{p^2} \right) - p^2 \ln \left( \frac{q^2}{k^2} \right) \right] \right\},$$  

(13)

where

$$J_0^{4,0} = \frac{2}{i\pi^2} \int_M d^4 \omega \frac{1}{\omega^2 (p - \omega)^2 (k - \omega)^2},$$  

(14)

with $q = k - p$. We propose the following vertex ansatz in Minkowski space [8]

$$\tau(k^2, p^2) = -\frac{3}{2} \frac{1}{(k^2 - p^2)} \left[ \frac{1}{F(k^2)} - \frac{1}{F(p^2)} \right] - \frac{\xi}{2} \frac{1}{(k^2 - p^2)} \frac{F(k^2) + F(p^2)}{s(k^2, p^2)} \left[ \frac{1}{F(k^2)} - \frac{1}{F(p^2)} \right] + \frac{1}{4} \frac{1}{(k^2 - p^2)} \frac{1}{s(k^2, p^2)} \left[ W \left( \frac{k^2}{p^2} \right) - W \left( \frac{p^2}{k^2} \right) \right],$$  

(15)

where we have introduced $F(k^2) \equiv F(k^2, \Lambda^2)$ as a simplifying notation. We also introduce the definition

$$s(k^2, p^2) = F(k^2) \frac{k^2}{p^2} + F(p^2) \frac{p^2}{k^2}$$  

(16)

and for the sake of simplicity we can choose the trivial solution $W(x) = 0$ for any dimensionless ratio $x$ of momenta.

$$\tau(k^2, p^2) = -\frac{3}{2} \frac{1}{(k^2 - p^2)} \left[ \frac{1}{F(k^2)} - \frac{1}{F(p^2)} \right] - \frac{\xi}{2} \frac{1}{(k^2 - p^2)} \frac{F(k^2) + F(p^2)}{s(k^2, p^2)} \left[ \frac{1}{F(k^2)} - \frac{1}{F(p^2)} \right].$$  

(17)

In order to compare the vertex ansatz, Eq. (17), based upon multiplicative renormalizability, against its one loop perturbative form, Eq. (13), it is convenient to take the asymptotic limit $k^2 \gg p^2$ of external momenta in the latter vertex. The resulting $\tau_{BCD}$ in the leading logarithms approximation (LLA) is

$$\tau_{BCD}^{\text{asym}}(k^2, p^2) \propto \frac{3\alpha}{4\pi k^2} \ln \left( \frac{k^2}{p^2} \right).$$  

(18)

Expectedly, it is independent of $q^2$ and hence we drop this dependence from its argument. Note that this expression is also independent of the covariant gauge parameter $\xi$. It is unlike spinor QED where the leading asymptotic vertex is proportional to $\xi$. For a numerical check, we define

$$\bar{\tau}(x) = -\frac{k^2}{\alpha \ln x} \tau(k^2, xk^2).$$  

(19)
where \( x = p^2/k^2 \) and we have suppressed the \( q^2 \) dependence for notational simplification. Thus:

\[
\tilde{\tau}_{\text{asym}}(x) = -\frac{3}{4\pi}.
\]  

In Fig. (2), we plot \( \tilde{\tau}_{\text{asym}}(x) \) and \( \tilde{\tau}_{\text{BCD}}(x) \) as a function of \( x \), the latter for different values of the gauge parameter \( \xi \) and for a fixed value of \( q^2 \), chosen arbitrarily. In the asymptotic limit, all curves converge to a single value, as expected.

Figure 2. The analytical result, long dashed lines representing a constant value given in Eq. (20) for the asymptotic transverse form factor \( \tilde{\tau}_{\text{asym}}(x) \), agrees with the numerical plot of \( \tilde{\tau}_{\text{BCD}}(x) \) obtained from Eq. (13) in the limit \( x \to 0 \) for different gauges and an arbitrarily chosen value of \( q^2 = -0.7\text{GeV}^2 \).

Using the perturbative expression for \( F(k^2) \) in Eq. (17), and taking the asymptotic limit \( k^2 \gg p^2 \), we have

\[
\tilde{\tau}_{\text{asym}}(k^2, p^2)^{k^2 \gg p^2} = \frac{3}{2} \frac{\alpha}{4\pi} \frac{(\xi - 3)}{k^2} \ln \left( \frac{k^2}{p^2} \right)
\]  

in the LLA. Note that the transverse form factors, Eqs. (18) and (21) have the functional form \((1/k^2) \ln(k^2/p^2)\). Furthermore, they are the same in the Feynman gauge \((\xi = 1)\). In order for them to be the same in an arbitrary gauge \( \xi \), we must seek a non-trivial \( W \)-function in Eq. (15). Perhaps the simplest such choice for \( W \) is

\[
W \left( \frac{k^2}{p^2} \right) = \lambda \frac{k^2}{p^2} \ln \left( \frac{k^2}{p^2} \right) + \frac{\lambda}{2} \frac{k^2}{p^2},
\]  

with \( \lambda = -3\alpha(\xi - 1)/2\pi \). In the Feynman gauge \((\xi = 1) W = 0 \), i.e., there is no necessity of a non-trivial \( W \)-function since both perturbative vertices, Eqs. (18) and (21) are already the same. Using the choice in Eq. (22) for \( W \) in the vertex, Eq. (15), we can define the transverse form factor as:

\[
\tau(k^2, p^2) = -\frac{3}{2} \frac{1}{(k^2 - p^2)} \left[ \frac{1}{F(k^2)} - \frac{1}{F(p^2)} \right] - \frac{\xi}{2} \frac{1}{(k^2 - p^2)} \frac{F(k^2) + F(p^2)}{s(k^2, p^2)} \times \left[ \frac{1}{F(k^2)} - \frac{1}{F(p^2)} \right] - \frac{(\xi - 1)}{3\alpha} \frac{1}{(k^2 - p^2) s(k^2, p^2)} \frac{[k^2 + p^2]}{k^2} \ln \left( \frac{k^2}{p^2} \right).
\]  

(23)
Note that the Eqs. (5,8,11,23) define our full vertex ansatz. It ensures the following key features of sQED:

- It satisfies the WFGTI by construction, [3, 4, 5].
- It guarantees the LKFT property of the scalar propagator and can be checked by employing it in its SDE. In other words, it ensures the multiplicative renormalizability (MR) of the two point scalar propagator.
- It reduces to its one loop perturbation theory Feynman expansion in the limit of small coupling and asymptotic values of momenta \( k^2 \gg p^2 \).
- It has the same symmetry properties as the bare vertex under charge conjugation, parity and time reversal, which imply symmetry under \( k \leftrightarrow p \).
- It is free of any kinematic singularities when \( k^2 \Rightarrow p^2 \), i.e.,

\[
\lim_{k^2 \Rightarrow p^2} (k^2 - p^2) \tau(k^2, p^2) = 0.
\]

Note that the kinematic dependence of the vertex on \( q^2 \) plays no role asymptotically and the standard analysis proceeds without reference to it. On the infrared domain, however, the kinematic dependence on \( q^2 \) may be important. Our vertex has this pitfall but its simplicity is reason enough for us to ignore this dependence.

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