The Effects of Finite Emission Height in Precision Pulsar Timing

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Precision timing is the key ingredient of ongoing pulsar-based gravitational wave searches and tests of general relativity using binary pulsars. The conventional approach to timing explicitly assumes that the radio emitting region is located at the center of the pulsar, while polarimetric observations suggest that radio emission is in fact produced at altitudes ranging from tens to thousands of kilometers above the neutron star surface. Here we present a calculation of the effects of finite emission height on the timing of binary pulsars using a simple model for the emitting region geometry. Finite height of emission changes the propagation path of radio photons through the binary and gives rise to a large spin velocity of the emission region co-rotating with the neutron star. Under favorable conditions these two effects introduce corrections to the conventional time delays at the microsecond level (for a millisecond pulsar in a double neutron star binary with a period of several hours and assuming $h = 100$ km). Exploiting the dependence of the emission height on frequency (radius-to-frequency mapping) and using multi-frequency observations one should be able to detect these timing corrections even though they are formally degenerate with conventional time delays. Although even in the most accurately timed systems the magnitude of the finite emission height effects is currently somewhat below timing precision, longer-term observations and future facilities like SKA will make measurement of these effects possible, providing an independent check of existing emission height estimates.

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I. INTRODUCTION

Since their discovery radio pulsars have been recognized as precise time-keepers. This unique property has been extensively exploited to test general relativity since 1975, when Hulse and Taylor discovered the first double neutron star (DNS) system PSR 1913+16 [1]. Thirty years later, pulsar surveys have found seven other DNS systems (among them a double pulsar system J0737-3039, in which both neutron stars are seen as radio pulsars, see [2]) and many white dwarf-neutron star (WD-NS) binaries. These systems have been used not only to test a number of general relativistic effects but also to set limits on the stochastic gravitational wave background [3].

Precise monitoring of the times of arrival (TOAs) of pulses on Earth is the key to success in this game. Although timing accuracy at the several $\mu$s level has been routinely achieved in a number of systems, only three WD-NS binary pulsars – J0737-4715, J1713+0747, J1909-3744 [4, 5] – have been timed to $0.1 - 0.3$ $\mu$s precision. At the same time, current instruments have not yet reached the level of intrinsic uncertainty present in pulsars (which will ultimately set a limit on their precision, see [6]). Ambitious new projects like the Parkes Pulsar Timing Array (PPTA), which aims to achieve daily timing residuals $\lesssim 0.1$ $\mu$s for more than ten millisecond pulsars [6], and proposed facilities like the Square Kilometer Array (SKA; [7], demand an understanding of the systematic timing uncertainties at the level of 1 ns [8]. With these new instruments, effects previously hidden in error will become observable and some of them may have not been accounted for by the current timing models.

The conventional physical model for interpreting timing data in terms of the binary system’s parameters, which was advanced by Damour and Deruelle [9, hereafter DD] and extended by Damour and Taylor [9, hereafter DT], treats the binary as two point masses in orbit around each other and assumes the radio emission region to be located at the very center of the radio pulsar. However, theoretical models of pulsar radio emission, detailed analysis of pulse structure and polarimetry of isolated pulsars suggest that their radio emission is in fact produced high in the magnetosphere. In slow pulsars it occurs at typical distances of order several hundred (or thousand in some cases) kilometers from the neutron star [10, 11, 12, 13]. For millisecond pulsars, which are more appropriate for precision timing, the typical height of the emission region $h$ is estimated to be several tens of kilometers [14, 15], although at low frequencies $h$ may be as high as $\approx 100$ km.

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A finite emission height has twofold effect on pulsar timing. First, the displacement of the emission region from the neutron star center affects photon propagation through the curved space-time within the binary. Second, a high-altitude emission region of a rapidly spinning millisecond pulsar must be moving at a rather large speed (in some cases reaching $0.2c$, where $c$ is the speed of light), leading to severe aberration of the pulsar radio beam. For a given spin period of a pulsar $P_p = 2\pi/\Omega_p$ the upper limit on the emission height $h$ is set by the size of the light cylinder $r_{LC} = c/\Omega_p \approx 500(P_p/10\,\text{ms})$ km, while the lower limit on $h$ is set by the neutron star radius $R_{NS} \approx 10$ km. The latter also sets a lower limit of $\Omega_p R_{NS} \sin \alpha \approx 0.02c(P_p/10\,\text{ms})^{-1}\sin \alpha$ on the rotational velocity of the emitting region, where $\alpha$ is the angle between the magnetic and spin axes of the pulsar.

Given the rapidly improving accuracy of pulsar timing and the increasing demand to understand systematic timing effects at the nanosecond level, which are driven by ongoing and future projects, the aforementioned effects of a non-zero emission height merit closer inspection. The physical change produces any measurable results. An outline of the paper runs as follows. In sec. II we describe the geometry of the emission in a dipolar magnetic field. In sec. III we consider different timing effects and compute corrections that result from considering a finite emission height. We discuss the measurability of these corrections in sec. IV and limitations and possible refinements of the model in sec. V.

**II. GEOMETRICAL MODEL OF RADIO EMISSION**

We consider a binary system with a semimajor axis $a$, eccentricity $e$, and inclination $i$ with respect to the observer’s line of sight. The instantaneous position of the pulsar is characterized by the true anomaly $\psi$ measured from the ascending node of the pulsar orbit, and $\omega$ is the longitude of periastron, see Figure 1. We use $R_p$ and $R_c$ for the position vectors of the pulsar and its companion with respect to the barycenter of the binary. In making numerical estimates we will adopt a fiducial model of the pulsar binary (motivated by the properties of the double pulsar J0737-3039, see [2]) which consists of two neutron stars ($M_p = M_c = 1.4\,M_\odot$ are the assumed pulsar and companion masses) in a compact orbit with a semimajor axis $a = 10^{11}$ cm (orbital period $P_b = 2.9$ hrs). We take pulsar spin period to be $P_p = 10$ ms and assume the height of the emission region to be $h = 10^7$ cm.

To illustrate our calculations we adopt a specific model for the geometry of pulsar radio emission closely related to the conventional rotating vector model [RVM, 17]. In the comoving coordinate system of the pulsar we assume neutron star magnetic field to have a dipolar geometry (which should be a good approximation at the distances where radio emission is produced), with the magnetic axis $\hat{m}$ ($|\hat{m}| = 1$) making an angle $\alpha$ with the pulsar spin axis $\hat{s}_p$ ($|\hat{s}_p| = 1$), see Figure 2. Vector $\hat{s}_p$ makes an angle $\zeta$ with the unit vector $\hat{n}_0$ from the binary system to observer at Earth ($\hat{s}_p \cdot \hat{n}_0 = \cos \zeta$), while the projection of $\hat{s}_p$ on the plane of the sky makes an angle $\eta$ with the ascending node of the binary orbit, see Figure 4. While the vector $\hat{s}_p$ is fixed, the orientation of $\hat{m}$ changes in time because the pulsar rotation always maintains the relation

$$\hat{m} \cdot \hat{s}_p = \cos \alpha.$$  

The position of the emitting region with respect to the pulsar center is $h\hat{r}$, where $|\hat{r}| = 1$ and $h$ is the emission height, see Figure 3. Emission is assumed to occur strictly along the local $B$ field since it is likely produced by the highly relativistic plasma streaming along the open field lines, so that the direction of emission $\hat{d}$ ($|\hat{d}| = 1$) is related to $\hat{r}$ and $\hat{m}$ as

$$\hat{d} = \frac{3 \cos \theta \hat{r} - \hat{m}}{(3 \cos^2 \theta + 1)^{1/2}},$$  

where $\theta$ is the angle between $\hat{r}$ and $\hat{m}$, i.e.

$$\hat{m} \cdot \hat{r} = \cos \theta.$$  

Vector $\hat{d}$ makes an angle $\rho$ with the magnetic axis $\hat{m}$ (see Figure 3) such that [see eq. 2]

$$\cos \rho = \frac{3 \cos^2 \theta - 1}{(3 \cos^2 \theta + 1)^{1/2}}.$$  

FIG. 1: Orbital configuration and spin orientation of the pulsar. Here $\hat{s}_p$ is the spin axis for the neutron star, $\hat{n}_0$ is the direction to the observer, $\hat{m}$ is the magnetic field axis, $R_p$ is the position vector of the pulsar. We use the orientation notation of DT so that the orbital angular momentum vector $\mathbf{L}$ constitutes angle $i$ with $-\hat{n}_0$. See text for details.
FIG. 2: Model of the emission cone geometry of the pulsar. Here we show the direction of emission \( \hat{d} \) when \( \hat{d} = \hat{n}_0 \) and the pulse is observed. Note that at the moment of emission \( \hat{m} \) has two values, which correspond to the two crossings of the emission cone by the observer’s line of sight. The coordinate system \((\hat{x}, \hat{y}, \hat{z})\) is also displayed.

We will be considering emission by all sources at the same height \( h \) belonging to a single flux surface for which the condition \( \Phi \) with a given value of \( \theta \) is fulfilled. Thus, \( h \) and \( \theta \) are fixed for this calculation (but see sec. V A for the consequences of relaxing this assumption). All such sources form a circular emission region around vector \( \hat{m} \), which gives rise to an emission cone with an opening angle \( 2\rho \) rigidly connected to (and co-axial with) the magnetic axis. When the emission cone crosses our line of sight as a result of pulsar rotation (that is, for some point \( h \hat{r} \) in the emission region, \( \hat{n}_0 \) becomes parallel to \( \hat{d} \)) a pulse of emission is sent to the observer on Earth. There are two emission episodes per spin period of pulsar corresponding to the two emission cone crossings by \( \hat{n}_0 \), separated by \( 2\Phi_0 \) in pulse phase (see Figure 2), where

\[
\cos \Phi_0 = \frac{\cos \rho - \cos \zeta \cos \alpha}{\sin \zeta \sin \alpha}.
\]  

By measuring the TOAs of these pulses one can obtain information about the motion of the pulsar and the propagation effects inside the binary.

The emission cone crosses our line of sight and makes the pulsar observable only if \( |\zeta - \alpha| < \rho \). In the absence of aberration caused by the motion of the emission region (i.e. when pulsar is not spinning and is at rest with respect to observer on Earth), favorable conditions for the detection of this conal emission,

\[
\hat{d} = \hat{n}_0,
\]  

can be achieved only for some specific orientations of vectors \( \hat{r} \) and \( \hat{m} \), which we call \( \hat{r}_0 \) and \( \hat{m}_0 \). To find these zeroth order (or static) values of \( \hat{r} \) and \( \hat{m} \) we form a stationary orthogonal basis using \( \hat{s}_p \) and \( \hat{n}_0 \):

\[
(\hat{x}, \hat{y}, \hat{z}) = \left( \frac{\hat{s}_p \times \hat{n}_0}{|\hat{s}_p \times \hat{n}_0|}, \frac{\hat{s}_p - \cos \zeta \hat{n}_0}{|\hat{s}_p \times \hat{n}_0|}, \hat{n}_0 \right).
\]
FIG. 3: An illustration of the dipole model used in our analysis. The field lines emanate from the surface of the star about the magnetic field axis, \( \hat{m} \), and the emission is produced at the point \( h \hat{r} \), in the direction \( \hat{d} \), which is tangent to the field line.

Then, using equations (1), (2), (3), and (6) one can easily find that

\[
\hat{m}_0 = \pm \sin \Phi_0 \sin \alpha \hat{x} + \frac{\cos \alpha - \cos \zeta \cos \rho}{\sin \zeta} \hat{y} \tag{8}
\]

\[
\hat{r}_0 = \pm \sin \Phi_0 \sin \alpha \frac{3 \cos \theta}{3 \cos \theta \sin \zeta} \hat{x} + \frac{\cos \rho + [3 \cos^2 \theta + 1]^\frac{1}{2}}{3 \cos \theta} \hat{z}, \tag{9}
\]

where \( \sin \Phi_0 \) can be found from equation (5). The \( \pm \) ambiguity in the \( x \)-components of \( \hat{m}_0 \) and \( \hat{r}_0 \) corresponds to the two crossings of the emission cone by \( \hat{n}_0 \) which results in two possible orientations of \( \hat{m}_0 \) and \( \hat{r}_0 \), see Figure 3. The upper and lower signs are for the leading and trailing pulses, respectively (cone crossings).

Slow pulsars typically have narrow beams meaning that \( \rho \ll 1, \Phi_0 \ll 1, \) and \( \theta \approx (2/3)^{1/2} \rho \ll 1. \) In this case it follows from equations (9) and (10) that \( \hat{m}_{0,x}, \hat{m}_{0,y}, \hat{r}_{0,x}, \hat{r}_{0,y} \sim \theta \ll 1 \) (remember that \( |\zeta - \alpha| < \rho \) so that both \( \hat{m}_0 \) and \( \hat{r}_0 \) are almost aligned with \( \hat{n}_0 \).

On the other hand, millisecond pulsars (the objects most suitable for precise timing), have wide beams with \( \rho, \Phi_0, \theta \sim 1, \) so that their \( \hat{m}_0 \) and \( \hat{r}_0 \) are generally quite mis-aligned with \( \hat{n}_0 \).

When the emission region is moving, the concomitant aberration changes the emission direction \( \hat{d} \) as seen by an observer at rest, so that the radio pulses are detected only at \( \hat{r} \) and \( \hat{m} \), which are somewhat different from \( \hat{r}_0 \) and \( \hat{m}_0 \). The conventional timing model (DD) assumes that the motion of the emission region comes only from the orbital motion of pulsar with speed \( v_o = \beta_o c \) (the proper motion of the binary as a whole is undetectable through timing). A finite height of emission introduces the spin velocity \( v_s = \beta_s c \), since the emission region co-rotates with the pulsar. In the case of millisecond pulsars this spin velocity is a considerable fraction of the speed of light even if emission takes place at the surface of the neutron star (\( v_s \approx 0.2 c \) for an orthogonal rotator with spin period 1 ms and \( h \sim 10 \) km, which is much larger than \( v_o \sim 10^{-3} c \)).

In the general case of nonzero orbital and spin velocities of the emitting region unit vectors \( \hat{r} \) and \( \hat{m} \) can be written as

\[
\hat{r} = \hat{r}_0 + \Delta_o \hat{r} + \Delta_s \hat{r} + \Delta_{so} \hat{r}, \tag{10}
\]

\[
\hat{m} = \hat{m}_0 + \Delta_o \hat{m} + \Delta_s \hat{m} + \Delta_{so} \hat{m}, \tag{11}
\]

where \( \Delta_o \) denotes corrections due to \( v_o \) (which have been previously calculated in the framework of DD timing model), \( \Delta_s \) corresponds to the corrections due to \( v_s \) only, while \( \Delta_{so} \) corresponds to the coupling between \( v_o \) and \( v_s \). The derivation of all these corrections is presented in Appendix and the expressions for \( \Delta_o \hat{r}, \Delta_s \hat{r}, \Delta_{so} \hat{r}, \Delta_o \hat{m}, \Delta_s \hat{m}, \Delta_{so} \hat{m} \) are given by equations (A4)-(A5) and (A12)-(A14). For the parameters of our fiducial pulsar binary model the typical relative mag-
nitudes of the different correction terms listed in equations (10) and (11) are
\begin{align}
\varepsilon_o & = \frac{v_o}{c} \approx 10^{-3} a_{11}^{-1/2}, \\
\varepsilon_s & = \frac{\Omega_p h}{c} = 0.21 h_7 \left( \frac{P_p}{10 \text{ ms}} \right)^{-1}, \\
\varepsilon_{so} & = \varepsilon_o \varepsilon_s \approx 2 \times 10^{-4} h_7 a_{11}^{-1/2} \left( \frac{P_p}{10 \text{ ms}} \right)^{-1}, \tag{14}
\end{align}
correspondingly, where \( h_7 \equiv h/(10^7 \text{ cm}) \) and \( a_{11} \equiv a/(10^{11} \text{ cm}) \).

### III. EFFECT ON TIMING

We are now in a position to calculate the corrections to the time delays that occur in binary pulsar systems. These delays appear when one relates the observer’s proper TOA of a given radio pulse \( \tau_o \) to the proper time of its emission in the comoving frame of the pulsar \( T_e \). As demonstrated by DD this transformation can be done in several steps, going first from \( \tau_o \) to the coordinate time of arrival \( t_o \) (measured in a frame comoving with the binary barycenter), which is linked to the coordinate time of emission \( t_e \) through the two propagation delays – the Römer (\( \Delta R \)) and Shapiro (\( \Delta S \)) delays (see eq. [4] of DD). In its turn, the coordinate time \( t_e \) is directly related to the proper time of emission in the pulsar frame \( T_e \) via the Einstein delay \( \Delta E \), so that \( t_e = T_e + \Delta E \). Finally, the deviation of the proper time of emission \( T_e \) from the purely periodic pattern in the pulsar frame is determined by the so-called aberration delay \( \Delta A \).

The introduction of the nonzero emission height modifies some of these delays. The Römer and Shapiro delays get affected because of the change in the path of the radio beam. The aberration delay gets modified because finite height introduces spin motion of the emission region (in addition to the orbital motion) which affects aberration of the radio signal. At the same time, the Einstein delay is not affected by the finite emission height as \( \Delta E \) is simply the difference between the proper time and the coordinate time of pulsar, which is independent of the geometry and physics of emission.

When calculating the timing contributions we will neglect the effect of gravitational lensing by the companion which mainly affects the Shapiro delay and may be relevant only for highly inclined binary systems [18, 19, 20, 21]. We will also neglect the retardation effect [22, 23] which can be easily accounted for afterwards.

#### A. Römer Delay

We begin with the Römer delay \( \Delta R \), the variation of the photon travel time as the pulsar moves through its orbit. This delay can be expressed as:
\[ \Delta R = -\frac{1}{c} \hat{n}_0 \cdot \mathbf{R}_e, \tag{15} \]
where \( \mathbf{R}_e \) is the position of the emission region relative to the binary barycenter. The conventional DD model assumes \( \mathbf{R}_e = \mathbf{R}_p \), while in our case \( \mathbf{R}_e = \mathbf{R}_p + \hat{h}_\mathbf{t} \), which gives rise to a correction to the Römer delay \( \Delta_{R,h} \). We use equation (10) to calculate \( \Delta_{R,h} \), in which \( \hat{h}_0 \) and \( \Delta_{r} \) are constant and would not contribute to the timing signal, while \( \Delta_{so} \mathbf{r} \) is small compared to \( \Delta_{r} \mathbf{r} \) by \( \sim \varepsilon_s \) (see below). Thus, we only retain \( \Delta_o \mathbf{r} \) and find
\[ \Delta_{R,h} = -\frac{(\beta_o \perp \hat{\mathbf{n}}_0) h}{3 \cos \theta} \tag{16} \]
where for any vector \( \mathbf{A} \), we denote its component perpendicular to \( \hat{\mathbf{n}}_0 \) as \( \mathbf{A}_\perp \equiv \mathbf{A} - (\mathbf{A} \cdot \hat{\mathbf{n}}_0) \hat{\mathbf{n}}_0 \). Note that the first term in the right-hand side of this expression has a sign ambiguity, which reflects the term’s opposite timing contribution to the leading and trailing components of the pulse profile (corresponding to the two crossings of the same emission cone by vector \( \hat{\mathbf{d}} \)). This ambiguity has the effect of periodically changing the separation between the two components of the pulse profile on an orbital timescale, which makes this timing contribution similar to the so-called latitudinal time delay caused by orbital aberration [4, 21]. The second term in (16) exhibits no such sign ambiguity, meaning that it is the same for both pulse components, and manifests itself as an overall homogeneous shift of the pulse profile, similar to other conventional time delays.

Assuming that all the angles specifying pulsar spin-magnetic orientation in (10) are of order unity (typical for rapidly spinning millisecond pulsars) one finds that the typical amplitude of \( \Delta_{R,h} \) is
\[ \tau_h = \frac{h}{c} \beta_o \approx 0.3 \mu s \ h_7 \left( \beta_o/10^{-3} \right), \tag{17} \]
which is within the timing accuracy of the most stable pulsars. For slower pulsars which typically have small \( \theta \sim \rho \) and \( \Phi_0 \), the expression in brackets in (16) would be of order \( \theta \ll 1 \), so that \( \Delta_{R,h} \sim \tau_t h \), making this contribution difficult to measure.

One may ask whether it is legitimate to discard \( \Delta_{so} \mathbf{r} \) contribution in (10) when \( \theta \ll 1 \). We believe it is, since the inclusion of \( \Delta_{so} \mathbf{r} \) would add terms of relative significance \( \sim \varepsilon_s \). At the same time, angles \( \rho \) and \( \theta \) are ultimately related to the position of the last open flux surface which touches the light cylinder at \( r_{LC} = c/\Omega_p \). At the distance \( h \) this flux surface is characterized by angle \( \theta \sim (h/r_{LC})^{1/2} = \varepsilon_s^{1/2} \). Thus, \( \Delta_{so} \mathbf{r} \) contributions are smaller than \( \Delta_{R,h} \sim \tau_t h \) by \( \sim \varepsilon_s^{1/2} \ll 1 \) and can be neglected.
B. Shapiro Delay

The Shapiro delay is caused by the curvature of spacetime near the companion, which is only substantial when the pulse passes near the companion. It is accurately measurable only in nearly edge-on systems. The conventional formula for the Shapiro delay is

\[ \Delta S_{0} = -\frac{R_{g}}{c} \log|\hat{n}_{0} \cdot (R_{p} - R_{c}) - (R_{p} - R_{c})| + \text{const}, \]  

(18)

where \( R_{g} \equiv 2GM/c^2 \) is the Schwarzschild radius of the companion.

A displacement of the emission region from the center of pulsar changes the propagation path of the radio beam and affects the Shapiro delay. The modified formula for the delay can be obtained by changing \( R_{p} \) in equation \( \text{(18)} \) to \( R_{p} = R_{p} + h\hat{r} \) and reads

\[
\Delta S = \Delta S_{0} + \Delta S_{h},
\]

(19)

\[
\Delta S_{h} = -\frac{R_{g}}{c} \frac{h\hat{r}_{0} \cdot (\hat{n}_{0} - \hat{n}_{R})}{(R_{p} - R_{c}) \cdot (\hat{n}_{0} - \hat{n}_{R})},
\]

(20)

Far from conjunction the height-related correction is very small: \( \Delta S_{h} \sim (h/a)(R_{g}/c) \lesssim 0.1 \text{ ns} \) since \( h/a \sim 10^{-4} \) and \( R_{g}/c \sim 10 \text{ µs} \) even in DNS binaries. However, around conjunction \( (R_{b} - R_{c}) \cdot (\hat{n}_{0} - \hat{n}_{R}) \sim b^2/(R_{b} - R_{c}) \ll a \), where \( b \approx a \cos \iota \) is the minimum projected separation between the pulsar and its companion in the plane of the sky.

At the same time,

\[
\hat{r}_{0} \cdot (\hat{n}_{0} - \hat{n}_{R}) = \pm \frac{\sin \Phi_{0} \sin \alpha}{3 \cos \theta} (\hat{n}_{0} - \hat{n}_{R})_{x} + \frac{\cos \alpha - \cos \zeta \cos \rho}{3 \cos \theta \sin \zeta} (\hat{n}_{0} - \hat{n}_{R})_{y} + \frac{\cos \rho + [3 \cos^{2} \theta + 1]^{1/2}}{3 \cos \theta} (\hat{n}_{0} - \hat{n}_{R})_{z},
\]

(22)

and at conjunction \( (\hat{n}_{0} - \hat{n}_{R})_{x} \sim b^2/(R_{b} - R_{c}) \), while the first two terms in the right-hand side are of order \( \theta b/(R_{b} - R_{c}) \). Thus, for millisecond pulsars with \( \rho, \theta \sim 1 \Delta S_{h} \) goes up to \( (h/b)(R_{g}/c) \) at conjunction. This can be substantial in a DNS binary: for \( a = 10^{11} \text{ cm}, h \sim 10^2 \text{ km}, \) and \( \iota = 89^\circ \) one has \( b \approx 17,000 \text{ km} \) so that \( h/b \sim 10^{-2} \) and at conjunction \( \Delta S_{h} \) can reach \( \sim 0.05 \text{ µs} \).

We also remark that the sign ambiguity of the first term in equation \( \text{(12)} \) gives rise to somewhat different height-related contributions to the Shapiro delay for the leading and trailing edges of the emission cone, if their pulse arrival times are measured separately.

C. Aberration

Another time delay arises when the aberration from the velocity of the emitting region changes the spin phase of the pulsar at which its radio emission reaches the observer on Earth. In a conventional approach, aberration is due to \( \beta_{0} \), while in our case it is due to \( \beta_{0} + \beta_{s} \).

As demonstrated in [24], DD, and [21] even in its conventional form (i.e. when \( \beta_{s} = 0 \) aberration leads not only to the overall homogeneous time delay of the profile (“longitudinal” aberration delay) but also to the distortion of the observed pulse shape. The latter can be interpreted as the so-called “latitudinal” timing delay which has opposite signs for the different components of the pulse profile. The same result occurs in our case as well and we now consider these two effects separately.

1. Longitudinal shift

The change \( \Delta \Phi \) of the pulsar spin phase that arises as a result of aberration can be expressed using simple geometric arguments as:

\[
\Delta \Phi = \frac{\Delta d \cdot (\hat{s}_{p} \times \hat{n}_{0})}{|\hat{s}_{p} \times \hat{n}_{0}|^2}.
\]

(23)

Then, substituting in (A12)-(A14), the longitudinal timing delay is found to be

\[
\Delta_{\text{long}} = \frac{\Delta \Phi}{\Omega_{p}} = \Delta_{\text{long},0} + \Delta_{\text{long},h},
\]

(24)

where

\[
\Delta_{\text{long},0} = -\frac{\beta_{0} \cdot (\hat{s}_{p} \times \hat{n}_{0})}{\Omega_{p}|\hat{s}_{p} \times \hat{n}_{0}|^2}
\]

(25)

is the conventional longitudinal aberration delay [24] and
The height-related correction is smaller than $\Delta A$, additional to $\Delta A$, longitudinal height-related contribution $\epsilon$ even longitudinal aberration leads to pulse distortion. Here we have retained only terms variable on the orbital time scale (proportional to $\beta_o$, i.e. orbital and spin-orbital contributions).

Orbital aberration alone gives $\Delta A_{0,0} \sim \beta_o \Omega_p^{-1} \approx 3.3\mu s$. The height-related correction is smaller than $\Delta A_{0,0}$ by $\epsilon$ so that $\Delta A_{0,h} \sim \beta_o h c^{-1} = t_h$, where $t_h$ is a fiducial timescale, see equation (17). We discuss the measurability of the longitudinal aberration delay in sec. IV.

Note that unlike $\Delta A_{0,0}$, which is the same for all pulse components, the longitudinal height-related contribution $\Delta A_{0,h}$ contains terms with sign ambiguity (terms proportional to $m_{0,x}$ or $r_{0,x}$), meaning that in the case of finite $h$ even longitudinal aberration leads to pulse distortion.

2. Latitudinal shift

Aberration also changes the latitude $\chi$ of vector $\hat{d}$ with respect to the pulsar spin axis. This latitudinal shift is

$$ \Delta \chi_A = \frac{\Delta d \cdot \hat{s}_p}{|\hat{s}_p \times \hat{n}_0|} = \Delta \chi_{A,0} + \Delta \chi_{A,h}, \quad (27) $$

where

$$ \Delta \chi_{A,0} = -\frac{\beta_o \hat{s}_p}{|\hat{s}_p \times \hat{n}_0|} \quad (28) $$

Here $\chi_0$ is the position angle of linear polarization given by $22$

$$ \tan \chi_0 = \frac{\sin \alpha \sin \Phi_0}{\cos \alpha \sin \zeta - \cos \Phi_0 \sin \alpha \cos \zeta}. \quad (31) $$

The magnitude of the latitudinal shift is in general (assuming $\tan \chi_0 \sim 1$) of the same order as the longitudinal one, i.e. one should expect $\Delta \chi_{A,h} \sim t_h$. However, in some cases, when the observer’s line of sight cuts the emission cone very close to its edge, $\tan \chi_0$ may become very small which increases $\Delta \chi_{A,h} \sim t_h / \tan \chi_0$, facilitating the detection of this delay $21$.

The height-related contribution $\Delta A_{0,h}$ contains both terms which are the same for all pulse profile components and terms which have sign ambiguity. This is different from the orbital latitudinal delay, which has identical magnitude but opposite signs for different crossings of the emission cone. Based on the results of sec. III C 1 and III C 2 we conclude that height-related corrections change the simple separation of the effects of longitudinal and latitudinal delays, i.e. a homogeneous shift of the pulse profile in the former case and pulse distortion...
in the latter. Both $\Delta_{A,0}^{\text{long}}$ and $\Delta_{A,0}^{\text{lat}}$ show a homogeneous shift as well as the distortion of the pulse profile.

IV. MEASURABILITY

Having calculated corrections to different time delays arising because of the finite height of emission region we can now ask whether these effects are measurable. The magnitude of these corrections depends on the type and orbital parameters of the system in question. In the double neutron star binary J0737-3039\( \text{[2]} \) one finds $t_h \approx 0.33h_7 \mu s$, while the correction to the Shapiro delay is about $0.06h_7$ ns, (neglecting factors which depend on angles $\alpha, \zeta, \eta$, etc.). At the same time, the current TOA uncertainty in this system is about 18 $\mu$s for a 30-s integration at 820 MHz\( \text{[2]} \), which makes height-dependent corrections unobservable on short timescales in this system, despite its large value of $\beta_0 = 10^{-3}$.

Much better timing accuracies have been achieved in some NS-WD binaries, however small companion masses make $\beta_0$ significantly lower in these systems than in the DNS binaries. In particular, using the data from\( \text{[4]} \) we find $t_h \approx 129h_7$ ns (for J0437-4715 which has RMS timing residual $\sigma_{\text{RMS}} = 200$ ns in 1-hour integration. For J1713+0747 we find $t_h \approx 129h_7$ ns ($\sigma_{\text{RMS}} = 125$ ns in 1-hour measurement), while for J1909-3744 $t_h \approx 30h_7$ ns ($\sigma_{\text{RMS}} = 150$ ns in 1-hour integration). Thus, finite emission height effects are not currently observable in these systems. However, even more precise timing of these WD-NS binaries should allow future measurement of these effects by using many independent TOA measurements taken on longer time intervals. The ongoing projects such as PPTA\( \text{[4]} \), and the advent of SKA\( \text{[3]} \), all of which dictate understanding the systematic timing errors at the level of 1 ns, will enable the measurement of the finite height effects both by lowering the timing errors in already known systems and by discovering new binary pulsars which may be better suited for this goal.

Another aspect of measurability has to do with the possible degeneracy between the finite height corrections and the already known delays, which could arise if the orbital dependence of these corrections mimics the orbital variation of other delays. In particular, DD have previously demonstrated that the longitudinal aberration delay\( \text{[4]} \)

\begin{equation}
\Delta_{A,0}^{\text{long}} = AS(\psi) + BC(\psi),
\end{equation}

\begin{equation}
A = \frac{\tan \eta}{\cos i} B = -\frac{\Omega_b a_p \sin \eta}{\Omega_p c \sqrt{1 - e^2 \sin \zeta}},
\end{equation}

\begin{equation}
S(\psi) \equiv \sin \psi + e \sin \omega,
\end{equation}

\begin{equation}
C(\psi) \equiv \cos \psi + e \cos \omega.
\end{equation}

(with $\Omega_b = 2\pi/P_b$) can be completely absorbed into the Römer delay by slightly rescaling the observed semimajor axis and eccentricity of the system from their true values, e.g.

\begin{equation}
\frac{e_{\text{obs}}}{e_{\text{true}}} = 1 + \epsilon_A, \quad \epsilon_A = \frac{c A}{a_p \sin i}.
\end{equation}

where $a_p = a M_c/(M_p + M_c)$. This makes $\Delta_{A,0}^{\text{long}}$ completely unobservable on an orbital timescale.

In fact, DD’s argument can be generalized for any timing contribution that depends on the pulsar orbital phase as $XC(\psi) + YS(\psi)$, where $X$ and $Y$ are constant coefficients. The part of such a delay that is the same for all pulse components is then degenerate with the Römer delay. By the same argument, the sign-ambiguous delay contributions (which are different for the leading and trailing pulse features and cause pulse profile distortions), are degenerate with the conventional latitudinal delay $\Delta_{A,0}^{\text{lat}}$, which has the orbital dependence analogous to $\Delta_{A,0}^{\text{long}}$, see Rafikov and Lai\( \text{[21]} \).

Using equations (10), (20), and (30) we see that $\Delta_{R,h}$, $\Delta_{A,h}^{\text{long}}$, and $\Delta_{A,h}^{\text{lat}}$ are linear functions of $\beta_0$, which in the coordinate system ($\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$) is given by

\begin{equation}
\beta_0 = \frac{\Omega_b a_p}{c \sqrt{1 - e^2 \sin \zeta}} [\cos i \cos \eta C(\psi) + \sin \eta S(\psi)] \hat{\mathbf{x}} + [\cos i \sin \eta C(\psi) - \cos \eta S(\psi)] \hat{\mathbf{y}} + \sin i C(\psi) \hat{\mathbf{z}}.
\end{equation}

Thus, these finite height contributions are of the form with their own constant coefficients of order $t_h$. Capitalizing on the result of DD we can immediately say that all these height-dependent contributions are degenerate with $\Delta_{R,0}$ and $\Delta_{A,0}^{\text{lat}}$ and are unobservable on an orbital timescale. The existence of $\Delta_{R,h}$ and $\Delta_{A,h}^{\text{long}}$ changes $A$ and $B$ in (33) to $A + A_h$ and $B + B_h$, where $A_h/A \sim B_h/B \sim \epsilon_\zeta$ and the dependence of $A_h$ and $B_h$ on $\zeta, \eta, i$ can be explicitly written out using equations (10), (20), and (30). Based on this argument we can also claim that the height dependent delays affect the observed values of orbital parameters by changing $\epsilon_A \rightarrow \epsilon_A + \epsilon_h$, where $\epsilon_h \sim c t_h/a_0 = (h/a_p) \beta_0 \sim 10^{-7}$.

The orbital dependence of the Shapiro delay is unique enough to preclude $\Delta_{S,h}$ from being absorbed into either Römer or latitudinal aberration delays. Nevertheless, we can look for a “new” orientation of the binary system characterized by the amended line of sight vector $\hat{\mathbf{n}}_{\text{new}} = \hat{\mathbf{n}}_0 + \delta \hat{\mathbf{n}}$, where $\delta \hat{\mathbf{n}}$ is small and $\hat{\mathbf{n}}_0 \cdot \delta \hat{\mathbf{n}} = 0$. Substituting $\hat{\mathbf{n}}_0 = \hat{\mathbf{n}}_{\text{new}} - \delta \hat{\mathbf{n}}$ into (19) and choosing

\begin{equation}
\delta \hat{\mathbf{n}} = \frac{h}{|\mathbf{R}_p - \mathbf{R}_c|} \hat{r}_{0,1}.
\end{equation}

we find that

\begin{equation}
\Delta_S = -\frac{R_g}{c} \log |\hat{\mathbf{n}}_{\text{new}} \cdot (\mathbf{R}_p - \mathbf{R}_c) - |\mathbf{R}_p - \mathbf{R}_c||,
\end{equation}

i.e. the expression for the Shapiro delay reduces to its standard form\( \text{[13]} \). In general the small correction vector $\delta \hat{\mathbf{n}}$ varies with the pulsar orbital phase. However, the
\[ \Delta_{S0} \] attains its highest value near conjunction where the relative contribution of the \( \Delta_{S,h} \) to \( \Delta_S \) is also largest and where \( |R_p - R_c| \) is roughly constant. Based on this we conclude that \( \Delta_{S,h} \) can be incorporated into \( \Delta_{S,0} \) by slight reorientation of the binary system from \( \mathbf{n}_0 \) to \( \mathbf{n}_{\text{new}} \). This makes \( \Delta_{S,h} \) unobservable on an orbital timescale due to its degeneracy with \( \Delta_{S,0} \).

Despite the fact that timing contributions due to finite height cannot be separately measured on an orbital timescale, one can hope to detect them on a longer geodetic precession timescale, analogous to the measurement of the conventional aberration delay \( \Delta_{A,0}^{\text{long}} \) \cite{DD, 21, 27}. Geodetic precession changes a pulsar’s spin orientation which, leads to periodic variations of angles \( \zeta \) and \( \eta \), affecting both the conventional aberration coefficients \( A \) and \( B \) and the height-dependent timing contributions. According to equation \cite{46}, this would lead to periodic variation of the inferred values of the binary orbital parameters (such as \( a \) and \( e \)), and would allow one to measure the combined effect of orbital aberration and finite height effects. However, separating these two contributors to the secular timing effects from each other may not be easy, as they vary in a similar fashion. In addition, the contribution to \( \varepsilon \) from the finite emission height is rather small compared to \( \varepsilon_A \). Additional information on the role of finite height effects compared to the orbital aberration may be provided by the pulse profile distortions on an orbital timescale (contributed both by \( \Delta_{A,0} \) and finite emission height effects), and by variation of the distortion pattern on the geodetic precession timescale.

Probably the easiest and most promising way of detecting the finite emission height effects is by using observations taken at different frequencies\(^2\). This method is based on the fact that the radio emission of different frequencies is produced at different heights in pulsar magnetospheres \cite{13, 17}, a manifestation of the so called radius-to-frequency mapping (RFM) that will be discussed in more detail in sec. \( \nabla \Delta \). In this case \( h = h(\nu) \) and by using timing measurements at different frequencies one would infer a different value of \( \varepsilon_{\text{obs}} \) and \( \varepsilon_{\text{obs}} \) at each frequency since the height-dependent \( \varepsilon_h \) is a function of \( \nu \) as well. As the orbital aberration is not frequency dependent, all variations of \( \varepsilon_{\text{obs}} \) and \( \varepsilon_{\text{obs}} \) that are frequency dependent must be ascribed to the finite emission height effects. By doing observations at sufficiently widely separated frequencies, one can achieve corresponding height difference \( \Delta h \sim h \), in which case the relative offset between the values of \( \varepsilon_{\text{obs}} \) and \( \varepsilon_{\text{obs}} \) at different frequencies would be of order \( \varepsilon_h \sim 10^{-7} \). This estimate suggests the level of accuracy to which \( \varepsilon_{\text{obs}} \) and \( \varepsilon_{\text{obs}} \) must be measured at a set of frequencies in order for the height-dependent effects to be detected.

\( ^2 \) This idea was suggested to us by Marten van Kerkwijk

V. DISCUSSION

Our use of a specific geometric model of the radio emission (similar to RVM) does not imply that our results would be vitiated if pulsar beam has a non-circular shape. The assumption of a specific beam shape was necessary in our case only to carry out an explicit calculation of various height related delays in terms of angles \( \eta, \zeta, \theta, \) etc. Clearly, even if pulsar has a non-circular beam, a finite emission height would still lead to aberration due to the pulsar spin and to the displacement of the emission region from the pulsar center. Of course, in this case we would not be able to predict exactly how the height-dependent timing signals scale with the pulsar orbital phase, which would complicate the interpretation of the secular evolution of these timing contributions on the geodetic precession timescale in terms of the pulsar spin orientation. However, even in this case one should still expect to see the difference in the observed orbital parameters of the binary measured at different frequencies, see the discussion in the end of sec. \( \nabla \).

We considered only the emission produced by a circular source (tied to a single flux surface) which has the form of two \( \delta \)-like spikes. This can be trivially generalized to the more realistic situation in which a set of different flux surfaces characterized by different values of \( \theta \) and \( \rho \) generates emission of different intensities which is observed on Earth as a complex pulse profile. This is done by a directly summing such signals with the proper delay for each flux surface, reflecting not only the conventional time delays but also the height-dependent contributions. Since the latter are explicit functions of \( \theta, \rho, \) etc. which are different for different flux surfaces, it is clear that the finite height effects would lead to the periodic distortions of the whole pulse profile (even within only the leading or trailing part of the pulse profile) on an orbital timescale, as delays of different flux surfaces would vary differently. The relative temporal distortion of a pulse feature with width \( \Delta \Phi \) due to the finite height effects should be at the level of \( \sim \Delta \Phi \Omega_{\text{orb}} \) which can reach \( \sim 10^{-4} \) for broad \( \Delta \Phi \sim 0.1 \) features in the profiles of the fastest \( (P_\rho \approx 2 \text{ ms}) \) millisecond pulsars.

In this respect we would like to emphasize the importance of timing of the different components of the pulse profile separately rather than just fitting single template to the whole profile as is usually done. This procedure has been previously proposed for detecting orbital aberration on an orbital timescale \cite{DD, 21, 27}, and it is clear from the previous discussion that it may also be very useful for interpreting the timing effects due to the finite emission height.

Our study was done in a monochromatic approximation while real radio receivers always have finite bandwidth. We do not expect this to affect our conclusions in any significant way, as the broadband pulse profile can be obtained by simply convolving the monochromatic pulse shape of the pulsar with the window function of the receiver. As long as one uses non-overlapping (even if
broad) bands one would still see the difference in the observed orbital parameters of the binary in different bands, which would in principle be a clear signature of the finite emission height effects.

A. Delays due to frequency dependent Emission Height

Our calculations have explicitly assumed the height of the emission region \( h \) to be constant. In reality, observations [11, 13, 15] find evidence for a radius-to-frequency mapping in which radio photons of different frequencies are emitted at different heights, so that \( h = h(\nu) \).

It is then easy to see that the orbital motion of pulsar should give rise to additional time delays proportional to \( dh/d\nu \). Indeed, because of the orbital Doppler shift, a monochromatic receiver detects radio photons which were emitted at slightly different frequency (\( \Delta \nu / \nu \sim \beta_o \sim 10^{-3} \)) than in the stationary case. The \( h(\nu) \) relation then implies that the height of emission also changes periodically to reflect this Doppler shift as \( \Delta h / h \sim \Delta \nu / \nu \sim \beta_o \) (observations suggest that \( dh/d\nu \)) is roughly constant). This clearly translates into another correction to the Römer delay which is about \( \Delta h / c \sim \beta_o h / c, i.e. \) has the same magnitude as \( \Delta R_{h,c} \) and all other height-dependent delays calculated in this paper.

Another effect arises because of the direct relationship between the distance from the pulsar center \( h \) and the magnetic latitude angle \( \theta \) for a given flux surface:

\[
h(\theta) = r_c \sin^2 \theta,
\]

where \( r_c \) is the equatorial radius of the flux surface, assuming that the dipolar geometry holds even beyond the light cylinder. The same Doppler effect that causes variations in \( h \) also causes the emission cone to widen at some orbital phases and narrow at others with the amplitude

\[
\Delta \hat{d} = \frac{\partial \hat{d}}{\partial \theta} \Delta \theta = \frac{3 \sin \theta (\hat{r} + \cos \theta \hat{m})}{(3 \cos^2 \theta + 1)^{3/2}} \Delta \theta,
\]

which causes pulse profile distortions. Because of that the direction of emission also changes by

\[
\Delta \hat{\sigma} = \frac{\partial \hat{\sigma}}{\partial \theta} \Delta \theta = \frac{\partial \hat{\sigma}}{\partial \theta} \Delta \theta = \frac{3 \sin \theta (\hat{r} + \cos \theta \hat{m})}{(3 \cos^2 \theta + 1)^{3/2}} \Delta \theta,
\]

see equation [2]. Similar to the latitudinal aberration delay [21] such a distortion can be interpreted in terms of the corresponding time delay \( \sim |\Delta \hat{\sigma} \hat{d}| \Omega_p^{-1} \) which, according to [10] and [11], is of order \( \Omega_p^{-1} \beta_o \sin^2 \theta = \beta_o h / (\Omega_p r_c) \). But since pulsar radio emission is produced in the open field line region, \( r_c \gtrsim r_{LC} = c / \Omega_p \), this additional delay is \( \lesssim \beta_o h / c \) – the same order of magnitude as all other height-dependent delays.

There are two reasons why we have not explicitly computed timing effects arising from RFM despite the fact that they are non-negligible compared to the contributions which we have already calculated. First, all of these effects are proportional to \( d \ln h / d \ln \nu \) while the height to frequency dependence is not very strong, \( d \ln h / d \ln \nu \approx 0.2 - 0.3 \) [17]. Second, the introduction of these terms does not change the qualitative picture of the finite emission height effects on timing and our conclusions about their measurability, but it does make calculations much more cumbersome. Because of this, we decided to leave the careful calculation of these effects for a future work.

VI. SUMMARY

We have extended the standard calculation of timing delays in binary millisecond pulsar systems by taking into account the fact that the emission region does not spatially coincide with the physical center of the pulsar but is located at some height (tens to thousands of kilometers) above the neutron star surface. This very natural extension has two consequences. First, such a displacement changes the propagation (Römer and Shapiro) delays. Second, the large spin velocity of the high-altitude emission region affects the aberration of the radio signal and contributes to the aberration delay. We have calculated the height-dependent corrections to these delays in the framework of a specific model of the pulsar emission geometry. The typical magnitude of corrections \( h / c \beta_o \) is of order several hundred nanoseconds in compact DNS systems (which is at the current precision level of a few most accurately timed millisecond pulsars in NS-WD binaries), and should be measurable in the future by projects like PPTA and facilities like SKA. Although these timing corrections are degenerate with other delays (Römer and aberration), they can be detected using multi-frequency observations and exploiting the RFM. Detection of these effects could yield a new measure of the emission height in pulsars, which would provide a consistency check on the estimates using other methods. Although our calculation assumes a specific model of the emission region geometry (that of a circular emission cone in a dipolar magnetic field), the main results of our analysis are qualitatively independent of that model.

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The linear character of the relation between \( \Delta h \) and \( \Delta r \) is not affected by the variation of the gravitational redshift in the field of pulsar and the change of spin velocity of the emission region. The coefficient in the relation changes because of these additional effects but only at the level \( R_g/h \) and \( \beta_s \) correspondingly, where \( R_g \) is the Schwarzschild radius of the pulsar.

**APPENDIX A: CORRECTIONS TO VECTORS \( \hat{\mathbf{m}} \) AND \( \hat{r} \)**

Here we compute changes in the vectors \( \hat{r} \) and \( \hat{m} \) that result from the orbital and spin motion of the emission region. The corrections we are looking for are small, typical orders are \( \varepsilon_{so} \ll \varepsilon_s \ll \varepsilon_o \ll 1 \) for the spin-orbital \( \Delta_{so} \), orbital \( \Delta_o \), spin \( \Delta_s \), and static terms correspondingly (the relative size of each of these terms is given in equations \( \text{(12)-(14)} \)). Thus, we can employ perturbation theory to compute these corrections by consecutively going to different orders in these small parameters.

In the static limit we neglect any motion of the emission region in which case equations \( \text{(11), (2), (3), and (6)} \) supplemented with the condition \( |\hat{m}| = 1 \) result in equations \( \text{(9) and (10)} \).

### 1. Contributions due to pulsar spin

In the next order we retain only terms proportional to the spin velocity \( \beta_s c \) of the emission region, where \( \beta_s = \varepsilon_s (\hat{\mathbf{s}}_p \times \hat{\mathbf{r}}_0) \). This motion aberrates the vector \( \hat{\mathbf{d}} \), producing an additional contribution \( \Delta_s \hat{\mathbf{d}} \) to \( \hat{\mathbf{d}}_0 = \hat{\mathbf{n}}_0 \) so that instead of \( \text{(10)} \) we now have

\[
\hat{d} = \hat{d}_0 + \Delta_s \hat{d} = \hat{n}_0 - \beta_s \perp \tag{A1}
\]

as the condition for the pulse to be emitted towards the observer. For any vector \( \mathbf{A} \), we denote its component perpendicular to \( \hat{n}_0 \) as \( \mathbf{A}_\perp \equiv \mathbf{A} - (\mathbf{A} \cdot \hat{n}_0)\hat{n}_0 \).

From the definition \( \text{(2)} \) we find that (assuming constant \( \theta \), i.e. no RFM, see sec. \( \text{V A} \))

\[
\Delta_s \hat{d} = \frac{3 \cos \theta \Delta_s \hat{r} - \Delta_s \hat{m}}{(3 \cos^2 \theta + 1)^{1/2}}, \tag{A2}
\]

while equations \( \text{(1), (2), and (6)} \) and condition \(|\hat{m}| = 1 \) lead to three more constraints:

\[
\Delta_s \hat{m} \cdot \hat{n}_0 = 0, \tag{A3}
\]

\[
\Delta_s \hat{m} \cdot \hat{s}_p = 0,
\]

\[
\Delta_s \hat{m} \cdot \hat{r}_0 = -\Delta_s \hat{r} \cdot \hat{n}_0.
\]

Combining equations \( \text{(A1), (A2), and (A3)} \) we obtain six equations for six unknowns (components of \( \Delta_s \hat{m} \) and
\( \Delta_s r \), which can be easily solved:

\[
\Delta_s m = \frac{(\beta s \cdot \hat{m}_0)(\hat{s}_p \times \hat{m}_0)}{\hat{n}_0 \cdot (\hat{s}_p \times \hat{m}_0)}, \quad (A4)
\]

\[
\Delta_s r = \frac{(\beta s \cdot \hat{m}_0)(\hat{s}_p \times \hat{m}_0)}{3 \cos \theta \hat{n}_0 \cdot (\hat{s}_p \times \hat{m}_0)} - \frac{(3 \cos^2 \theta + 1)^{1/2}}{3 \cos \theta} \beta s \perp, \quad (A5)
\]

providing us with the sought corrections due to the spin motion of the emission region. Note that although \( \varepsilon_s^2 \) can easily exceed \( \varepsilon_s \) we do not need to go to higher order in \( \varepsilon_s \) because the spin contributions \( \Delta_s \) do not vary with the orbital phase of the pulsar and thus are undetectable by timing. On the other hand, despite the fact that the constant \( \Delta_s m \) and \( \Delta_s r \) given by (A4) and (A5) do not affect timing directly, we need to compute them in order to calculate the spin-orbital corrections in the next section.

### 2. Corrections due to orbital motion and spin-orbit coupling

In general the aberration of vector \( \hat{d} \) is caused by the combination of both orbital and spin motions of the emission region. This gives rise to additional orbital contributions \( \Delta_o \) (which exist even if pulsar is not spinning or if \( h = 0 \), an assumption used by DD), and spin-orbital contributions \( \Delta_{so} \) to vectors \( \hat{m} \) and \( \hat{r} \), related to corresponding \( \Delta_s \hat{d} \) and \( \Delta_{so} \hat{d} \) via equations analogous to (A2).

Instead of equation (A1) we now have

\[
\hat{d} = \hat{d}_0 + \Delta_s \hat{d} + \Delta_o \hat{d} + \Delta_{so} \hat{d} = \hat{n}_0 - \beta_o + \hat{d}(\beta_o \cdot \hat{d}) - \beta_s + \hat{d}(\beta_s \cdot \hat{d}). \quad (A6)
\]

Substituting \( \hat{d} = \hat{d}_0 + \Delta_s \hat{d} + \Delta_o \hat{d} \) in the right-hand side of this equation (in doing that we ignore the spin-orbital term \( \Delta_{so} \hat{d} \) as it would lead to higher order corrections) one obtains

\[
\Delta_o \hat{d} = -\beta_o \perp, \quad (A7)
\]

\[
\Delta_{so} \hat{d} = -\varepsilon_s [\beta_o \perp (\hat{n}_0 \cdot (\hat{s}_p \times \hat{r}_0))] + 2\hat{n}_0(\beta_o \perp (\hat{s}_p \times \hat{r}_0)) + (\hat{s}_p \times \Delta_o \hat{r}) \perp, \quad (A8)
\]

Finally, instead of (A1) we have the following relations:

\[
(\Delta_o m + \Delta_{so} m) \cdot \hat{s}_p = 0, \quad (A9)
\]

\[
(\Delta_o m + \Delta_{so} m) \cdot (\hat{r}_0 + \Delta_o \hat{r}) = -(\hat{m}_0 + \Delta_o m) \cdot (\Delta_o \hat{r} + \Delta_{so} \hat{r}), \quad (A10)
\]

\[
(\Delta_o m + \Delta_{so} m) \cdot (\hat{m}_0 + \Delta_o m) = 0, \quad (A11)
\]

which provide us with enough information to solve for \( \Delta_o \hat{r}, \Delta_o m, \Delta_{so} \hat{r}, \Delta_{so} m \) by considering separately terms of order \( \varepsilon_o \) and \( \varepsilon_{so} \).

After tedious but straightforward calculations we find

\[
\Delta_o m = \frac{\hat{s}_p \times \hat{m}_0}{\hat{n}_0 \cdot (\hat{s}_p \times \hat{m}_0)} (\beta_o \perp \cdot \hat{m}_0), \quad (A12)
\]

\[
\Delta_o r = \frac{\hat{s}_p \times \hat{m}_0}{3 \cos \theta \hat{n}_0 \cdot (\hat{s}_p \times \hat{m}_0)} (\beta_o \perp \cdot \hat{m}_0) - \frac{(3 \cos^2 \theta + 1)^{1/2}}{3 \cos \theta} \beta_o \perp, \quad (A13)
\]

for the orbital terms and

\[
\Delta_{so} \hat{d} = -\varepsilon_s \left[ \beta_o \perp (\hat{n}_0 \cdot (\hat{s}_p \times \hat{r}_0)) + 2\hat{n}_0(\beta_o \perp \cdot (\hat{s}_p \times \hat{r}_0)) - \frac{(3 \cos^2 \theta + 1)^{1/2}}{3 \cos \theta} (\hat{s}_p \times \beta_o \perp) \perp + \frac{(\hat{s}_p \times (\hat{s}_p \times \hat{m}_0)) \perp}{3 \cos \theta} \beta_o \perp \cdot \hat{m}_0 \right], \quad (A14)
\]

for contributions to \( \hat{d} \) resulting from the spin-orbital coupling (we do not separately display expressions for \( \Delta_{so} \hat{r} \) and \( \Delta_{so} m \) as they are very cumbersome and are not used separately in the timing calculations anyway). These expressions are then used in sec. III to compute the height-related corrections to different time delays.