Multitape automata and finite state transducers with lexicographic weights

Aleksander Mendoza-Drosik

Abstract—Finite state transducers, multitape automata and weighted automata have a lot in common. By studying their universal foundations, one can discover some new insights into all of them. The main result presented here is the introduction of lexicographic finite state transducers, that could be seen as intermediate model between multitape automata and weighted transducers. Their most significant advantage is being equivalent, but often exponentially smaller than even smallest nondeterministic automata without weights. Lexicographic transducers were discovered by taking inspiration from Eilenberg’s algebraic approach to automata and Solomonoff’s graphic transducers were discovered by taking inspiration from A treatment of a priori probability. Therefore, a quick and concise survey of those topics is presented, prior to introducing lexicographic transducers.

Index Terms—Mealy machines, transducers, sequential machines, computability, complexity

I. INTRODUCTION

A. Preliminaries

Product of sets $B$ and $C$ is the set $B \times C$ of all ordered pairs $(b,c)$ such that $b \in B$ and $c \in C$. A (partial) function $B \rightarrow C$ is a subset of $B \times C$ such that $(b,c), (b,c)' \in B \rightarrow C$ implies $c = c'$. Given some function $A \subset B \rightarrow C$, we say that $A$ is total if for every $b$ there exists some $c$, such that $(b,c) \in A$. We shall not differentiate between $(B \times C) \times D$ and $B \times (C \times D)$. We also assume that $\rightarrow$ binds weaker than $\times$, hence $B \times C \rightarrow D$ stands for $(B \times C) \rightarrow D$. One can easily check that $B \rightarrow (C \rightarrow D)$ is the same as $B \times C \rightarrow D$, but it’s different from $(B \rightarrow C) \rightarrow D$. Given some set $A$ that is a subset of $B \times C$, we use $[A]_B$ to denote left projection $[b \in B : \exists c \in C(b,c) \in A]$. Analogically for right projection $[A]_C$.

Suppose $\circ$ is some total function $\circ \subset A \times A \rightarrow A$, then set $A$ together with $\circ$ is called a monoid if two criteria are met. First there must exist some element $1_A \in A$ (called identity element) such that $\circ(1_A,a) = \circ(a,1_A) = a$ for all $a \in A$. Second, it must always hold that $\circ(\circ(a_1,a_2),a_3) = \circ(\circ(a_1,a_2),a_3))$. Instead of writing $\circ(a_1,a_2)$, one can also use infix notation $a_1 \circ a_2$. Thanks to the second criterion, the order of brackets doesn’t matter and we can omit them, as in $a_1 \circ a_2 = a_3$.

If $A$ contains two elements $a_1$ and $a_2$ such that $a_1 \circ a_2 = 1_A$, then we call them invertible, $a_1$ can be denoted as $a_1^{-1}$ and called the inverse of $a_1$. Monoid, in which every element has some inverse, is called group.

If $B \times C$ is a monoid, then $B$ and $C$ must be monoids themselves, with $1_{B \times C} = (1_B,1_C)$. This is called direct product of monoids.

Basic understanding of measure theory is assumed.

B. Algebraic foundations

Suppose $A$ is some set of labels, $Q$ is set of vertices and $\delta \subset Q \times A \times Q$ a set of edges. Then $(Q,A,\delta)$ is a labelled directed graph. Define path to be a finite sequence of edges $(q_{k_1}, x_1, q_{k_2}), (q_{k_2}, x_2, q_{k_3}), \ldots(q_{k_r}, x_r, q_{k_{r+1}})$ where $q_{k_1}, q_{k_{r+1}} \in Q$, $x_1 \in A$ and $(q_{k_i}, x_i, q_{k_{i+1}}) \in \delta$ for every index $i$.

If $A$ together with operation $\cdot$ (which we call "multiplication") is a monoid, then define signature $\Pi$ of a path as the result of multiplying consecutive labels $x_1 \cdot x_2 \cdot \ldots \cdot x_m$.

Automaton $[A]$ is defined as tuple $(Q,I,A,\delta,F)$ where $Q$ and $\delta$ are finite, $A$ is finitely generated and both $I$ and $F$ are subsets of $Q$. It’s common to refer to elements of $Q$ as states, instead of vertices. Similarly $A$ is called set of strings or words instead of labels. All elements belonging to some (usually fixed and known from context) generator of $A$ are called symbols or letters. Elements of $\delta$ are called transitions instead of edges. States that belong to $I$ are called initial and those belonging to $F$ are final. We also use $e$, called empty string, instead writing of $1_A$, when we put emphasis that $1_A$ is a string.

Path is accepting if it starts in some initial $q_k$, and ends in final $q_{k,m+1}$. An automaton accepts string $x \in A$ if it is a signature of some accepting path.

The elements of $A$ need not be “actual strings”. For instance they might be pairs or triples of elements from other sets. In such cases when $A = B \times C$, we call the automaton to be multitape. Note that $B$ or $C$ itself might be nested product of other sets. When $A = (B_1 \times B_2) \times B_3$, then automaton has 3 tapes. The reader might notice that the distinction between single-tape and multitape automata is blurry. Indeed, a pair of letters $(b,c)$ could always be encoded as a single letter $a_{bc}$ (if $B$ has $n$ letters and $C$ has $m$, then $B \times C$ has $n \cdot m$), hence one tape can be used to encode multiple other tapes within.

There is not much distinction between $A$ and $A \times \{\epsilon\}$. Indeed, a tape that can only read an empty string, isn’t read at all and doesn’t make any difference. We call $\{\epsilon\}$ a trivial tape. Usually there is also not much difference between $A = B \times C$ and $A = C \times B$. The order of tapes can be switched and a nearly identical automaton can always be built.

If $A = B \times C$, then we can say that automaton is sequential up to $B$ if $b \neq b'$ implies $(q,(b',c),q') \notin \delta$. This ensures that as we read consecutive symbols from input tape, the transition that wasn’t taken before doesn’t suddenly become valid. Automaton is sequential if it is sequential up to entire $A$. (Note that $A$ is same as $A \times \{\epsilon\}$). For instance, automata that allow entire strings on their edges, are not sequential, white automata that only allow individual symbols, are sequential.

Automaton with $A = B \times C$ is deterministic up to $B$ if it is sequential up to $B$ and $|I| = 1$ and $\delta \subset Q \times (B_1 \setminus \{1_B\}) \rightarrow C \times Q$ (when the function is partial, then automaton is called partial, otherwise it’s called complete). Automaton is deterministic when it is deterministic up to entire $A$. 
Automaton with \( A = B \times C \) is \( \varepsilon \)-free up to \( B \) if \( \delta \) is subset of \( Q \times (B \setminus \{1_B\}) \times C \times Q \). Automaton is \( \varepsilon \)-free if it is \( \varepsilon \)-free up to entire \( A \).

**Language** is any subset of \( A \). Language \( L \) is rational if and only if there exists some automaton accepting all strings in \( L \) and rejecting all those not in \( L \). If \( A \) is a direct product of several monoids, then we call \( L \) a rational relation. If rational relation is a function, then automaton recognizing it is called functional. If \( M = (Q, I, A, \delta, F) \) is some automaton, then \( \mathcal{L}(M) \) is used to denote language recognized by \( M \). If \( \mathcal{L}(M) \) is a relation \( B \times C \), then we use \( M(b) \) to denote \( \mathcal{L}(M)^C \). If automaton has 3 tapes, say \( A = B \times C \times D \), then \( M(b) \) treats it like \( B \times (C \times D) \). Similarly \( M(b, c) \) treats \( A \) as if it was \( (B \times C) \times D \). Also, because order of tapes makes little difference, we can always switch it implicitly and write \( M(b, d) \) to mean \( (B \times D) \times C \).

Automaton with \( A = B \times C \) is \( k \)-valued up to \( B \) if the number of outputs \( M(b) \) for any \( b \) is bounded by constant \( k \) (precisely \( |M(b)| \leq k \)). Functional automata are exactly those that are 1-valued. Automaton is \( k \)-ambigious up to \( B \) if for every accepted \( b \) there are at most \( k \) distinct accepting paths with signature \( b \). Ambigious automaton may still be functional if all the accepting paths generate the same outputs. It’s possible to decide functionality of automaton in polynomial time[3][4].

Subset \( L \) of strings \( A = B \times C \) (language) is prefix free up to \( B \) if there is no element \( b_1 \) that would be a prefix of another \( b_2 \) (the notion of string prefix makes the most sense when \( B \) is a free monoid). More formally if \( (b_1, c_1) \) and \( (b_2, c_2) \) are both in \( L \) and \( b_1 \) is a prefix of \( b_2 \), then \( b_1 = b_2 \). Automaton is subsequential up to \( B \) if it is sequential up to \( B \) and \( \mathcal{L}(M) \) is prefix free up to \( B \). This definition is very different from those found in other papers[5][6][7]. Usually most authors extend their automata with additional output function for accepting states. If automaton ends in that state, then some additional final output is appended before accepting. Such functionality can be emulated by adding special symbol \# as end marker [3]. Those familiar with syntactic transformation monoid[9], might notice that symbols are in a sense functions, therefore end marker is the same as “state output function”. Reader should also notice that any language with end marker is indeed prefix free. The resemblance is analogical to that between plain kolmogorov complexity and prefix free complexity[10].

In essence, sequential automata continue working as long as there is input to read, whereas subsequential automata can “decide on their own” when input should end and can take some additional action. It’s easy to prove that any subsequential machine on minimal number of states can have at most one accepting state. One could say that automata with “state output function” also have such unique accepting state but it’s “secretly hidden” in the definition of automaton, rather than explicitly specified in \( Q \).

There is no formal distinction between input and output tapes. For instance, given \( \mathcal{L}(M) \subset B \times C \times D \), we may consider \( B \) to be input and \( C \times D \) to be output, when we use \( M(b) \). If instead we use \( M(b, c) \), then \( B \times C \) become input tapes and \( D \) becomes output. If we use \( M(b, c, d) \) then all tapes are input. Automata having 2 tapes, with first one designated as input, are often called transducers.

Norm \( \|\cdot\| \) is a function that assigns real number to every element of some set. If \( A \) is a free monoid, then we can define the norm \( |a| \) to be length of string \( a \). If \( A \) is not free, then it’s much less obvious what the length should be. (For instance, if \( a_1a_2 = a_2, then a_1(a_2a_2) \) might have length 4 or it might have length 2 because \( a_1a_2a_2 = a_1a_2 = a_1a_2 \).

When \( A \) is a direct product of several free monoids, then one can study the relationship between their lengths. The most notable property is that if \( A = \Sigma^* \times \Gamma^* \) and \( \delta \subset Q \times \Sigma \times \Gamma \times Q \), then for every accepted \((\sigma, \gamma) \in A \) the lengths \( |\sigma| \) and \( |\gamma| \) are equal. This can be further generalized to \( A = \Sigma_1^* \times \Sigma_2^* \times ... \Sigma_n^* \). If \( \delta \) is of the form such that at least one \( \Sigma_i^* \) is required to be of length exactly 1 on each transition (that is \( \delta \subset Q \times \Sigma_1^* \times ... \Sigma_i^* \times ... \Sigma_n^* \times Q \)) then we can induce norm on the accepted subset of \( A \), that is, \(|\sigma_1, ..., \sigma_i, ..., \sigma_n| = |\sigma_i| \). Such norm also coincides with length of accepting path, therefore it allows us to generalise and apply pumping lemma to multitape automata.

If \( A \) contains some elements with inverses then the automaton cannot be sequential. In particular suppose \( aa^{-1} = 1_A \) and \((q, a', q') \in \delta \) then \( a' = a'1_A = a'(aa^{-1}) = (a'a)^{-1} \) but \( a'a \neq 1_A \) and \( a^{-1} \neq 1_A \), hence sequentiality is violated.

Every input tape can be seen as a read-only tape and every output tape can be thought of as write-only. Just as there is no formal distinction between input and output, there is no distinction between read-only and write-only. The difference becomes significant only when we allow read-write tapes, also known as stacks. In particular, if \( B \) is a group, then pushing \( b \) onto stack is the same a reading \( b \) from tape. Popping \( b \) off of the stack can be seen as reading \( b^{-1} \).

A may or may not contain commuting elements. If \( A = B \times C \) then all the elements of the form \((1_B, c)\) and \((b, 1_C)\) commute. This phenomenon characterizes nonsequential machines which are used to encode concurrent systems (see theory of traces[11]). For this reason, all multitape automata with \( \epsilon \)-transitions are in a sense “concurrent” machines.

**Configuration** is defined to be a subset of \( Q \). Given configuration \( K \) and \( x \in A \), define \( \hat{\delta} \) to be transitive closure of \( \delta \), that is, \( \delta(K, x) \) is the set of all states \( q \), for which there exists a path starting in \( K \) and ending in \( q \) with signature \( x \).

In cases when \( A = B \times C \) we can extend the concept of configuration to include \( C \), that is, define superposition as a subset of \( Q \times C \). Given some superposition \( S \) we define \( \delta_C \) such that \( (q', y') \in \delta_C(S, x) \) whenever there exists \((q, y) \in S \) and path starting in \( q \) and ending in \( q' \) with signature \((x, y') \).

Configuration is a way of capturing what states are “active” at a particular moment of computation. Superposition keeps track outputs associated with a each “active” state. Every element of superposition represents one possible branch of nondeterministic computation and the output accumulated along the way.

If some automaton \( M \) is sequential up to \( B \) and \( \varepsilon \)-free up to \( B \), then we can define image of configuration

\[
\delta_C(K, b) = \{ q' \in Q : \exists q \in K(q, (b, c'), q') \in \delta \}
\]

and image of superposition

\[
\delta_C(S, b) = \{ (q', c') \in Q \times C : \exists (q, c) \in S(q, (b, c'), q') \in \delta \}
\]
Therefore the number of elements in superposition cannot exceed one because there is at most one transition that can be taken at each step. This gives us an effective way of computing the output of \( M \), that is \( M(x) = (\hat{\delta}_C(I \times \{1\}, x))^G \) for all \( x \) in \( B \).

**Theorem 1 (Deterministic superposition).** If automaton over \( A = B \times C \) is deterministic up to \( B \) then \( |\hat{\delta}_C(S, x)| \leq 1 \) for all \( x \in B \) and all initial superpositions \( |S| = 1 \).

*Proof:* Determinism states that \( \delta \subset Q \times B \to C \times Q \), so there is at most one transition that can be taken at each step. Therefore the number of elements in superposition cannot increase.

As a consequence one may easily show that in deterministic automata initial state and signature uniquely determine path. This leads us to introduce the following theorem.

**Theorem 2 (Preservation of prefixes).** Let \( M \) be some automaton over \( A = B \times C \) deterministic up to \( B \). For all strings \( x, x' \in B \) if \( M(x \alpha') = y' \neq \emptyset \) and \( M(x) = y \neq \emptyset \), then \( y \) is \( \gamma \), a prefix of \( y' \).

*Proof:* It follows directly from uniqueness of path that corresponds to signature \( xx' \).

Note that theorem 1 also applies to single-tape automata, because \( A \) can be treated like \( A \times \{\ell\} \). Superposition belonging to \( Q \times \{\ell\} \) is the same as configuration.

**Theorem 3 (Infinite superposition).** Let \( M \) be an automaton over \( A = B \times C \) sequential up to \( B \). \( |M(x)| = \infty \) for some \( x \in B \) only if \( M \) contains \( \epsilon \)-cycle \((y_1, (1_B, y_1), y_2), ..., (y_m, (1_B, y_m), y_1)\) where \( y_i \in C \) and \((1_B, y_1, ..., y_m) \neq A\).

*Proof:* Every time a non-\( \epsilon \)-transition from \( \delta \subset Q \times B \) to \( (1_B, y) \) occurs, \( Q \times 1_B \) contains \( \epsilon \)-cycles \((y_1, (1_B, y_1), y_2), ..., (y_m, (1_B, y_m), y_1)\) where \( y_i \in C \) and \((1_B, y_1, ..., y_m) \neq A\).

**Theorem 4 (Functional superposition).** Let \( M \) be a functional automaton over \( A = B \times C \), sequential up to \( B \) and whose recognized language is of the form \( L \subset B \to C \). Then there exists an equivalent automaton such that \( \hat{\delta}_C(S, x) \subset Q \to C \) for all \( S \subset Q \to C \) and \( x \in B \).

*Proof:* Suppose to the contrary that there is \( x \) and \( q \) for which \( \hat{\delta}_C(S, x) \) returns relation \( Q \times C \) that is not a function \( Q \to C \). Then there are two possibilities: either there is a path that starts in \( q \) and ends in \( F \) or \( q \) is not. If the first case is true, then \( M \) is not functional, because we might follow that path and accept with multiple \( C \) outputs. If the second case applies, then the state \( q \) is redundant and we are free to delete it.

**C. Stochastic languages and weighted automata**

Suppose that \( B \) is a tape of some automaton. If \( B \) is a complete semiring then it is called the tape of weights and the automaton itself is weighted. We require completeness because infinite sum may arise, though this requirement can be relaxed for \( \epsilon \)-free automata (theorem 3).

**Probabilistic automaton** is any automaton, whose \( A \) is a measure space with total measure \( \mu(A) \) equal 1. Every measurable subset of \( A \) is called a stochastic language and can be treated like a random event.

Now we can present a way of constructing automata with probabilistic weights. Let \( N \geq 1 \) be some natural number. Take the segment \((0, 1)\) of real number line and split it into \( N \) equally sized intervals. Let \( \Omega = \{\omega_1, \omega_2, ..., \omega_N\} \) be the set of all those intervals \((\frac{1}{N}, \frac{2}{N}), \) including \( \omega_0 \) representing \((0, 1)\). Set \( \Omega \) generates a monoid with multiplication \( \omega_x \cdot \omega_y \) defined as

\[
\omega_x \cdot \omega_y = (x_0, x_1) \cdot \omega_y = x_0 + (x_1 - x_0) \cdot \omega_y
\]

In other words, \( \omega_x \) determines linear transformation that treats \( \omega_x \) as the new unit interval \((0, 1)\) and \( \omega_y \) is made relative to it (for instance, if \( \omega_1 = (0, 0.5) \) and \( \omega_2 = (0.5, 1) \) then \( \omega_1 \omega_2 = (0.25, 0.5) \)). Norm \( |\omega| \) is equal to the length of interval. It holds that \( |\omega_x \cdot \omega_y| = |\omega_x| \cdot |\omega_y| \). Define complete semiring \( B \) generated by \( \Omega \) with union of intervals as additive operation (hence \( \omega_1 + \omega_2 = \omega_1 \)). Norm of \( b \) is equal to summing and multiplying norms of individual elements of \( \Omega \) (note \( |b_1 + b_2| \neq |b_1| + |b_2| \)). As \( N \) approaches \( \infty \), the accuracy of \( \Omega \) increases and their sums can approximate any real number. Consider \( \Omega_b \) to be the set of all infinite strings starting with \( b \) and \( \Omega_b \) is the set of all possible infinite strings. We can turn \( B \) into measure space by mapping every \( b \) into the corresponding measurable set \( \Omega_b \). Such definition of measure space corresponds to Solomonoff’s a priori prefix complexity [10]. Norm \( |b| \) coincides with measure \( \mu(\Omega_b) \).

For any subset \( B' \) of \( B \) the measure of \( B' \) is equal to sum \( \mu(\Omega_{B'}) = |\sum_{b \in B'} b| \). Note that the subset \( \{\omega_1, ..., \omega_N\} \) itself has uniform distribution but if we partitioned \((0, 1)\) in some irregular way, we might obtain different distributions. Moreover this subset can be seen as a random variable and every sequence of random variables "falls into" some \( b \) in \( B \) with probability \( |b| \).

Consider automaton \( M \) with single initial state and transitions of the form \( Q \times (\Omega \backslash \{\omega_0\}) \to C \times Q \). For any input \( c \) take the set \( M(c) \subset B \) and turn it into prefix-free set \( B' \) (that is, if \( b_1, b_2 \in M(c) \) and \( b_1 \) is prefix of \( b_2 \), then don’t include \( b_2 \) in \( B' \)). Such set is a random event with probability \( P(c) = \mu(\Omega_{B'}) \). To prove that \( P(c) \) never exceeds 1, notice that in every prefix-free subset of \( B \), no segments of \((0, 1)\) overlap, so we can sum them without double-counting. This also implies that \( |\sum_{b \in B'} b| = |\sum_{b \in B'} |b| \). Note that every string \( b \) uniquely determines some path, so if both \((c, b_1) \) and \((c, b_2) \) belong to \( L(M) \) but \( b_1 \) is prefix of \( b_2 \), means there is \( \epsilon \)-cycle starting and ending in some final state (so it’s only natural and intuitive to discard \( b_2 \) when counting \( P(c) \)).

If automaton has transitions of the form \( Q \times (\Omega \backslash \{\omega_0\}) \to C \times D \times Q \), then we can calculate probability of any...
output $D$ given some input $C$. We could say that $P(c, d)$ equals $\mu(\Omega_{M(c,d)})$ and $P(c)$ equals $\sum_{d \in D} P(c, d).$ Then conditional probability $P(d|c)$ is obtained from $P(\cdot|c) = \sum_{d \in D} P(d|c)$. 

The construction described above is called the probabilistic semiring. Those familiar with the theory of weighted automata[12][13] might notice that this definition is completely different from the "standard" one. We haven’t used formal power series[14] or weight function for transitions[13]. Apart from assuming that $B$ is a measure space, we didn’t need to extend our definition of automata in any way. Perhaps, the most significant difference is that we didn’t resort to summation of all possible paths but instead defined everything in terms of formal languages and strings. This presents an alternative approach to weighted automata, that lies much closer to theory formal languages. In fact, we can also introduce tropical semiring (and all others) in similar fashion.

Suppose that $A = B \times C$ (the order doesn’t matter much) and $C$ is a complete semiring. Given some language $L$ we introduce quotient of $L$ denoted with $L \cap B$ and defined as

$$(c, b) \in L \cap B \iff b = \sum_{(k, c) \in L} b'$$

$L$ can be any subset of $A$ but $L \cap B$ is specifically a function $C \to B$. Note that in case of probabilistic semiring, the probability $P(c) = \mu(\Omega_{M(c)})$ is in fact the same as $P(c) = |(L(M) \setminus B)(c)|$. This will be our starting point for defining tropical semiring in terms of strings and languages.

Consider automaton $M$ over $A = B \times C$, where $\leq_B$ is some relation of total order on $B$. Then we can turn $B$ into semiring with $\max$ (or $\min$) as additive operation. Hence $B$ can be treated as tape of weights. We will call this max semiring (or min semiring). If additionally $B$ commutes under multiplication, then we call it arctic semiring (or tropical semiring). Note that in other papers[6][15][12], $B$ is required to represent real numbers, but we don’t assume it here, because that would require infinitary alphabets[15]. If $B$ is a free monoid with $\leq_B$ representing lexicographic order, then $B$ is a special case of max semiring (min semiring), called lexicographic arctic semiring (or lexicographic tropical semiring). The lexicographic order itself might be defined by comparing strings from left to right or right to left. Because each time the automaton takes the transition, the weight is appended, rather than prepended, it makes more sense to consider right-to-left order (otherwise only the first transition would matter and the remaining steps of computation would be of little relevance). Therefore in this paper we go with

$$b_1w_1 > b_2w_2 \iff w_1 > w_2 \text{ or } (w_1 = w_2 \text{ and } b_1 > b_2)$$

where $w_1, w_2$ belong to generator of $B$. This semiring is a new discovery, which we will investigate in depth in the next part of this paper.

Let $M = (Q, I, C \times D, \delta, F)$ be some automaton that may or may not be deterministic. We define $\delta' \subset Q \times B \times C \to D \times Q$ to be a disambiguation up to $C$ for $M$ if $(Q, I, B \times C \times D, \delta', F)$ is deterministic and $\delta$ coincides with $\delta'$ in the following sense:

$$(q, (c, d), q') \in \delta \iff \exists b \in B(q, (b, c, d), q') \in \delta'$$

Notice that weighted automata can often be seen as disambiguations of some nondeterministic automata.

Given any $L \subset C \times D \to B$ we can maximize $B$ with respect to $D$ written as $\max_{D \to B}$

$$(c, d) \in \max_{D \to B} L \land (c, d', b) \in L \implies b \leq L(c, d)$$

Analogically we can also define $\min_{D \to B}$ Once we obtain a quotient of some weighted automaton, we can erase the weights completely by either minimising or maximizing them.

Consider automaton over $A = B \times C$ with $C = C_1 \times C_2 \times \ldots C_n$ where every $C_i$ is a max (min) semiring. Then we can turn $C$ into max (min) semiring by treating $C_i$ to the left as "more important" than those to the right. More formally $(c_1, (c_2, \ldots, c_n)) > (c'_1, (c'_2, \ldots, c'_n))$ if and only if either $c_1 > c'_1$ or $c_1 = c'_1$ and recursively $(c_2, \ldots, c_n) > (c'_2, \ldots, c'_n)$. Such construction of $C$ is known as lexicographic semiring[15].

II. LEXICOGRAPHIC TROPICAL SEMIRING

Consider automaton $M$ over $A = W^* \times \Sigma^* \times D$ with transitions $\delta \subset Q \times W \times \Sigma \times D \times Q$ and total order $\leq_W$, which induces lexicographic order on $W^*$, making it a lexicographic tropical semiring. We can be sure that for any $b \in W^*$, $c \in \Sigma^*$ and $d \in D$ if $(b, c, d)$ is in $L(M)$, then lengths $|b|$ and $|c|$ are equal. For any input $c$ we can compute $M(c)$ and after dividing it by $W^*$, the quotient $M(c)W^*$ is a function $D \to W^*$ assigning (lexicographically) lowest possible path to every obtainable output $D$. Because $\min$ is used as semiring addition, the quotient $M(c)W^*$ becomes

$$(d, b) \in M(c)W^* \iff b = \min_{(b, d) \in M(c)} b'$$

and all $b'$ are of equal lengths (same as $|c|$). Because there are only finitely many strings of any fixed length, the we don’t require $W^*$ to be a complete semiring. Moreover, we don’t need to worry about defining lexicographic comparison for strings of different lengths. We will refer to such $M$ as lexicographic transducer.

An interesting property emerges, when we study superpositions $Q \times W^* \times D$. Suppose that $S$ is some superposition obtained on lexicographic transducer by reading string $\sigma_1, \sigma_2, \ldots, \sigma_n \in \Sigma^*$. Let $(q, b, d) \in S$ and imagine that the automaton reads next symbol $\sigma_{k+1}$ and enters new superposition $S'$. It takes some transition $(q, \sigma_{k+1}, w, d', q')$, it causes the element $(q', b, d')$ to be included in $S'$. If there were two elements $(q_1, b_1, d_1)$ and $(q_2, b_2, d_2)$ in $S$ and $b_1 < b_2$, then the inequality would still be preserved for $b_1w < b_2w$ in $S'$. In that sense, the superpositions are monotonous and we can safely remove $(q_1, b_1, d_1)$ from $S$ without making any difference to $M(c)W^*$.

On the other hand, suppose that $(q_1, b_1, d_1)$ and $(q_2, b_2, d_2)$ are in $S$ and then automaton takes transitions $(q_1, \sigma_{k+1}, w_1, d'_1, q')$ and $(q_2, \sigma_{k+1}, w_1, d'_2, q')$ both leading to the same $q'$ over the same $\sigma_{k+1}$. We say that the states are conflicting. In order to determine whether $b_1w_1 > b_2w_2$, all we need to know is $b_1 > b_2$ and $w_1 > w_2$ but we don’t have to know the actual strings, because by definition

$$b_1w_1 > b_2w_2 \iff w_1 > w_2 \text{ or } (w_1 = w_2 \text{ and } b_1 > b_2)$$

This gives us everything that’s necessary for the following theorem.
Theorem 5 (Weights can be erased). Let $M$ be some lexicographic transducer over $A = W^* \times \Sigma^* \times D$ then there exists automaton $N$ over $\Sigma^* \times D$ equivalent to $\min_{D \rightarrow W^*} \mathcal{L}(M) \setminus W^*$.

Proof: Suppose that $M = (Q, I, \alpha, \delta, F)$ and $N = (Q', I', \Sigma^* \times D, \delta', F')$. Conversion can be carried out using "extended" powererset construction. Instead of using $Q' = 2^Q$, which can keep track of current configuration in $Q$, we need to keep track of superposition $Q \times W^*$. However, because there are infinitely many strings $W^*$, such powerset would result in infinite $Q'$. To make $Q'$ bounded, we abstract the exact strings $W^*$ away and only focus on the order relationship between them. More precisely, let $S \subseteq Q \rightarrow W^*$ be some superposition and let $\phi_S$ be a formula of the following form:

$$\epsilon < S(q_1) < S(q_2) = S(q_3) < ... = ... < = S(q_n)$$

where $q_1, ..., q_n$ are all the states included in given $S$. Let $\Phi$ be the set of formulas for all possible superpositions. We can treat $\Phi$ as equivalence classes for $Q \rightarrow W^*$. Note that it's enough to only consider $Q \rightarrow W^*$ instead of $Q \times W^*$, because (as shown a few paragraphs before) the weights are monotonous and we can remove all but the smallest one.

Having said all this, we can put $Q' = Q \times \Phi$. The extra $Q$ is needed, because we want to pick one representative state $q$ from every formula $\phi$. We can immediately add all those elements $(q, \phi)$ of $Q'$ for which $q$ cannot be found in $\phi$. The state $q$ we be used to keep track of $D$. Hence superposition $S' \in N$ that corresponds to $Q' \times D$ translates to $Q \times (\phi) \times D$, which can be seen as entire class of superpositions $Q \times W^* \times D$.

The set of states used in any given $\phi$ determines some configuration $K_\phi$. For every two states $(q_1, \phi_1), (q_2, \phi_2) \in Q'$ we put transition from $(q_1, \phi_1)$ to $(q_2, \phi_2)$ over symbol $\sigma$ with output $d$, whenever configuration $K_{\phi_1}$ transitions to $K_{\phi_2}$ over $\sigma$ (formally $K_{\phi_1} \times D(K_{\phi_2}, \sigma) = K_{\phi_2}$) and the state $q_1$ itself also transitions to $q_2$ formally $(q_1, w, \sigma, d, q_2) \in \delta$ and the formula $\phi_2$ indeed holds true (after transitioning from $\phi_1$ over $\sigma$). In a moment some of those transitions will need to be removed in order to simulate the effect of erased weights. Before that, we should first add one more extra state $f$ to $Q'$, which will be the only accepting state of $N$. Every time we put transition from $(q_1, \phi_1)$ to $(q_2, \phi_2)$ and $q_2$ is an accepting state, we need to put the exact same transition from $(q_1, \phi_1)$ to $f$. (This way we can simulate $\epsilon$-transition from $(q_2, \phi_2)$ to $f$.) For every initial state $q_i$ of $M$, we designate $(q_i, \phi)$ as initial state of $N$, where $\phi$ is the formula

$$\epsilon = S(q_1) = ... = S(q_i) = ... = S(q_n)$$

and $\{q_1, ..., q_n\}$ is the set of initial states $I$. If any initial state is also an accepting state, then we additionally set $f$ as initial state of $N$.

Finally, the last step of conversion is to find all conflicting states and remove the transitions with lower weights. Recall that if there are two states $(q_1, \phi), (q_2, \phi) \in Q'$ transitioning to the same third state $q'_3 \in Q'$ over the same symbol $\sigma$, then we call $(q_1, \phi)$ and $(q_2, \phi)$ conflicting. Remember that every transition in $\delta'$ is a "copy" of some weighted transition in $\delta$ (including those leading to $f$). Let’s say that $w_{1}$ is the weight that "would be" put between $(q_1, \phi)$ and $q'_3$, if we hadn’t erased it. Similarly fro $w_2$ and $(q_2, \phi)$. Next we need to lookup if according to $\phi$ the state $q_1$ carries lower or higher weight than $q_2$. This, together with the $w_{1}$ and $w_2$, gives us enough information to decide which of the transitions should be erased (if any).

This concludes the construction of $N$. Note that $N$ is non-deterministic and it’s not possible to reach such configuration of $Q'$ in which two states would have different $\phi$ (This does not include $f$ which doesn’t have any $\phi$ associated with it. There is no problem, because $f$ has no outgoing transitions).

We shall say that a lexicographic transducer $M$ is functional when the relation $\min_{D \rightarrow W^*} \mathcal{L}(M) \setminus W^*$ is functional. Theorem 4 together with theorem 5 tell us that every functional $M$ (after removing dead-end states) has all reachable superpositions of the form $Q \times W^* \rightarrow D$. After removing all but the lowest weight (due to monotonicity), that leaves us with only $Q \rightarrow W^* \rightarrow D$. This implies that any time there are two conflicting states, either the weights on transitions are different, or they are the same but also the associated $D$ outputs are the same. If there was only conflicting pair of states with equal weights but different $D$, that would break functional nature of automaton and lead to ambiguous output.

Suppose that the automaton $M$ has no reachable conflicting states with equal weights and only one single accepting state. Then $M$ is guaranteed to be functional. We will call such automata strongly functional. Note that, the lexicographic tropical semiring becomes "unnecessary" because in the formula

$$b_{1}w_{1} > b_{2}w_{2} \iff w_{1} > w_{2} \text{ or } (w_{1} = w_{2} \text{ and } b_{1} > b_{2})$$

always falls into the left side of "or" and the recursion on the right never happens ($w_{1} > w_{2}$ always holds). Therefore it’s not necessary to keep the history of weights $W^*$ in the superposition $Q \rightarrow W^* \times D$. We can drop them altogether and only compute with $Q \rightarrow D$. Another special property of such $M$ is that it’s "deterministic in reverse", that is, given sequence of configurations $K_0, K_1, ..., K_n$ for each step $\sigma_1, \sigma_2, ..., \sigma_n$ of computation, we can backtrack from accepting state back to initial state in a deterministic way, because when we know one particular state $q_i$ in $K_i$, $q_i$ is the unique final state) and $\sigma_i$ then there is always only one smallest weight that could have brought us to $q_i$ from some state of $K_{i-1}$. Also, the unique final state is not a limitation, because we can use nondeterminism to simulate $\epsilon$-transitions (similarly to the way we did it in theorem 5) or we might introduce special symbol $\#$ that we always put at the end of string. We can also show that weights of such automata can be erased in a simpler way than in theorem 5.

Theorem 6 (Weights can be erased (strongly functional case)). Let $M$ be some lexicographic transducer over $A = W^* \times \Sigma^* \times D$ that has only one final state and no conflicting states with equal weights. Then there exists automaton $N$ over $\Sigma^* \times D$ equivalent to $\min_{D \rightarrow W^*} \mathcal{L}(M) \setminus W^*$.

Proof: Similar to the previous case but this time we put $Q' = 2^Q \times Q \cup \{f\}$. If $S$ is configuration in $M$, we know that we only need to keep track of $Q \rightarrow D$, instead of $Q \times W^* \times D$. Therefore we can convert $S$ to superposition $S'$ of $N$, by setting $((K_S, q), d) \in S'$ for every $(q, d) \in S$, where $K_S$ is the configuration corresponding to $S$. 
We put transition from \((K_1, q_1)\) to \((K_2, q_2)\) in \(Q'\) over \(\sigma\) with output \(d\) whenever \(K_1\) transitions to \(K_2\) over \(\sigma\) (formally \(\delta_{W*} (K_1, x) = K_2\)) and \(q_1\) transitions to \(q_2\) over \(\sigma\) and \(d\) (formally \((q_1, w, \sigma, d, q_2) \in \delta\)).

We make state \((K, q)\) of \(N\) final whenever \(q\) is final. The initial states of \(N\) are all of the form \((I, q)\) for each \(q\) in \(I\).

The last step is to find all conflicting states, that is, two states \((K, q_1), (K, q_2) \in Q'\) having the same configuration \(K\) and transition to some \(q' \in Q'\) over the same \(\sigma\). The transitions \(((K, q_1), \sigma, d, q')\) and \(((K, q_2), \sigma, d, q')\) are called conflicting transitions. Every time we encounter them, we delete the one with higher weight. We will find out their weights by looking put what transition from \(\delta\) lead to their creation. It will never happen that two conflicting transitions have equal weights.

Lexicographic transducers (even the strongly functional ones) can be exponentially smaller than even the smallest nondeterministic equivalent 2-tape automata. To show this we will need help of Myhill-Nerode theorem.

Let \(L \subset B \rightarrow C\) be some (partial) function and let \((b_1, b_2) \in B\). Element \((b, c) \in A\) is a distinguishing extension up to \(B\) of \(b_1\) and \(b_2\) if exactly one of \((b_0, b, L(b_0)c)\) or \((b_0, b, L(b_0)c)\) belongs to \(L\). Define an equivalence relation \(\equiv\) on \(A\) such that \(a_0 \equiv a_1\) if and only if there is no distinguishing extension up to \(B\) for \(a_0\) and \(a_1\).

**Theorem 7** (Generalized Myhill-Nerode theorem). Let \(L \subset B \rightarrow C\). Assume that \(L(bb') = c \neq \emptyset\) and \(M(b) = c \neq \emptyset\) implies \(c\) is a prefix of \(c'\) (preservation of prefixes holds). \(L\) can be recognized by automaton deterministic up to \(B\) if and only if there are only finitely many equivalence classes induced by \(\equiv\).

**Proof:** ( \(\Leftarrow\) ) First assume there are finitely many equivalence classes. Let \(G\) be the smallest generator of \(B\). Then build an automaton by treating every class as a state \(Q\). Put a transition from class \(q\) to \(q'\) over \(b' \in G\setminus 1_B\) whenever there exists \((b, c) \in q\) and \((b', c') \in q'\). Preservation of prefixes guarantees existence of suffix \(s\) such that \(cs = c'\). This suffix shall be used as transition output. By \(b'\) only from \(G\setminus 1_B\) we ensure that automaton is sequential up to \(B\) and has no \(\epsilon\)-transitions. The class that contains \(1_A\) is designated as the unique initial state (hence the automaton is deterministic up to \(B\)). All the classes intersecting \(L\) are accepting states (note that if \(q \cap L \neq \emptyset\) then \(q \in L\)).

( \(\Rightarrow\) ) Conversely, if there is an automaton deterministic up to \(B\) and recognizing \(L\), then there could be found a homomorphism from states of machine to equivalence classes. (The exact proof is well covered in most introductory courses to automata theory and this generalized version is largely analogous, so we won’t elaborate on this proof much further.)

Note that this theorem no longer works when automaton is not deterministic at least up to \(B\).

**Theorem 8.** There exists a family of strongly functional lexicographic transducers such that their equivalent minimal 2-tape nondeterministic automata (after erasing weights) are exponentially larger.

**Proof:** We define family of strongly functional lexicographic transducers in such a way that for every \(i \geq 3\) there is one defined on \(i\) states. The number of states of minimal equivalent 2-tape automaton is \(O(2^n)\). Figure 1 presents a way to build such automata. State \(q_0\) is initial. Using strings from \(\{0,1\}^*\) one can obtain any configuration of states \(q_1\) to \(q_n\). Let’s associate each configuration with a string \(z \in \{0,1\}^n\) (for instance \(z = 011\) would be a configuration \(q_2, q_3\)). That gives \(2^n\) possible states. State \(q_{n+1}\) is accepting and all the states \(q_1...q_n\) are connected to it. Essentially the relation described by this automaton is a subset of \(\{0,1\}^2 \times \{y_1,...,y_n\}\). Also suppose that weights \(w_1...w_n\) are in strictly ascending order. Then the automaton maps every \(z\) determined by \(x \in \{0,1\}^+\) to some \((x, y_k)\) such that \(k\) indicates the least significant bit in \(z\).

For instance suppose \(n = 4\) and \(x = 0011\) \(\sim z = 1101\) then \((x2, y_1),(x02, y_3),(x002, y_4),(x0002, y_4),(x00002, \emptyset)\). Notice that one can reconstruct \(z\) from such sequence of \(y\)’s.

Each 2-tape automaton that has disjoint alphabets in each tape can be simulated by a single-tape automaton reading union of those alphabets. In this case such union is \(D = \{0,1,2,y_1,...,y_n\}\). This way the language becomes subset of \(\{0,1\}^\pm 2\{y_1,...,y_n\}\) and every pair \((x, y_k)\) becomes a string \(xy_k\).

Using Myhill-Nerode theorem, it can be seen that no two \(x\) strings that map to two different \(y\) are equivalent, hence the smallest deterministic FSA must have at least \(2^n\) states. Call this minimal automaton \(A\). The most difficult problem is to show that no nondeterministic automaton polynomially smaller than \(A\) can be build. The rigorous proof can be obtained with help of Theorem 7 and Lemma 7 presented in Kameda Weiner [17]. We can build RAM using \(D(A)\) and \(D(\overline{A})\) (all defined in (17)). Notice that every configuration ("configuration" in (17)) of states \(q_1...q_n\) has different succeeding event[17] and none of them is subset of the other (because there exists bijection between \(z\) and sequence of \(y\)’s produced by \((x2, y_{k0}),(x02, y_{k1}),...\)). Hence the minimal legitimate grid cannot be extended for any of them and the nondeterministic FSA cannot be much smaller than \(2^n\).

One can easily notice that every 2-tape automaton can be treated like a lexicographic transducer with all weights equal, therefore the opposite of theorem 7 doesn’t hold (there is no family of 2-tape automata such that lexicographic transducers would be larger).
III. Conclusions

Weighted automata don’t have to be an alien concept that requires any special extensions. All weights can be viewed as tapes over alphabets with some particular properties. This more general approach give us necessary foundations for defining lexicographic transducers. They were invented by trying to generalize and simplify weighted automata. There is yet a lot to discover. Theorem 8 gives certain clues, that perhaps they could be inferred more efficiently, or at least generalize better. Solomonoff’s theory of inductive inference says that simpler and shorter automata, should be the preferred solution to inference problems. Lexicographic transducers can express complex "replace-all" functions in simpler and more reliable ways than other weighted automata, specifically thanks to lack of commutativity in lexicographic tropical semiring. They seem perfectly suited for tasks that require the automaton to "forget history" of their weights.

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References

[1] J.-E. Pin, Mathematical Foundations of Automata Theory. American Mathematical Society, 2017.
[2] S. Eilenberg, Automata, Languages and Machines Vol. A. Academic Press, 1974.
[3] M.-P. Béal, O. Carton, C. Prieur, and J. Sakarovitch, “Squaring transducers: An efficient procedure for deciding functionality and sequentiality of transducers,” in LATIN 2000: Theoretical Informatics, G. H. Gonnet and A. Viola, Eds. Berlin, Heidelberg: Springer Berlin Heidelberg, 2000, pp. 397–406.
[4] I. O. Gurari, E.M., "A note on finite-valued and finitely ambiguous transducers," Math. Systems Theory, 1983.
[5] F. P. Mehryar Mohri and M. Riley, “Weighted finite-state transducers in speech recognition,” AT&T Labs Research, 2004.
[6] C. de la Higuera, Grammatical Inference: Learning Automata and Grammars. Cambridge University Press, 2010.
[7] C. E. Hasan Ibne Akram, Colin de la Higuera, “Actively learning probabilistic subsequential transducers,” JMLR: Workshop and Conference Proceedings, 2012.
[8] S. Eilenberg, Automata, Languages and Machines Vol. B. Academic Press, 1976.
[9] N. V. A. Shen, V. A. Uspensky, Kolmogorov Complexity and Algorithmic Randomness. IRIF, 2019.
[10] V. Diekert, The Book Of Traces. Wspc, 1995.
[11] M. Droste, W. Kuich, and H. Vogler, Handbook of Weighted Automata, 01 2009.
[12] M. Droste and D. Kuske, “Weighted automata,” Institut fur Informatik, Universitat Leipzig, 2010.
[13] M. S. Arto Salomaa, Automata-Theoretic Aspects of Formal Power Series. Springer-Verlag New York.
[14] K. Meer and A. Naif, “Generalized finite automata over real and complex numbers,” vol. 591, 04 2014.
[15] B. Roark, R. Sproat, and I. Shafran, “Lexicographic semirings for exact automata encoding of sequence models,” in Proceedings of the 49th Annual Meeting of the Association for Computational Linguistics: Human Language Technologies. Portland, Oregon, USA: Association for Computational Linguistics, Jun. 2011, pp. 1–5. [Online]. Available: https://www.aclweb.org/anthology/P11-2001
[16] P. W. Tsunehiko Kameda, “On the state minimization of nondeterministic finite automata,” IEEE Transactions on Computers, 1970.
[17] R.J. Solomonoff, “A formal theory of inductive inference. part i,” Information and Control, 1964.
[18] ———, “A formal theory of inductive inference. part ii,” Information and Control, 1964.