Parity Symmetry Breaking and Kink Hairy Black Hole

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Motivated by the Kibble-Zurek mechanism and the Gubser’s argument for the charged scalar hairs near the black hole horizon in a spacetime with negative cosmological constant, we realize the kink hairs in a planar Schwarzschild-AdS black hole. By increasing the chemical potential across the critical point of the dual boundary field theory, the former $Z_2$ symmetry of the real scalar fields spontaneously break and form kinks in the bulk. Correspondingly, in the AdS boundary it resembles the kinks in a one-dimensional chain. The relation between the kink numbers and the quench rate obeys a universal scaling law which satisfies the Kibble-Zurek’s prediction.

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I. INTRODUCTION

No hair conjecture of black hole states that the solution of black hole can be totally characterized by its mass, electric charge and angular momentum [1–3]. However, this conjecture was extended and challenged in some exceptional cases, such as the soft hairs near horizon [4], the black holes of non-Abelian Yang-Mills field [5], or in higher dimensional gravity [6], just to list some examples. One can refer to [7] and references therein. Among these, Gubser’s argument [8] for the hair of charged scalar fields in a spacetime with a negative cosmological constant was widely applied in the holographic superconducting phenomenon [9]. It brought in a surge of the applied holography especially in condensed matter [10].

In this paper, we dynamically realize the kink hairs of real scalar fields from the parity symmetry (Z$_2$ symmetry) breaking in a planar Schwarzschild-AdS black hole. We quench (by increasing) the chemical potential in the boundary field theory (effectively this is equivalent to decreasing the temperature), as crossing the critical point the scalar fields with former Z$_2$ symmetry will spontaneously break along a one-dimensional space. Due to the prominent Kibble-Zurek mechanism (KZM) [11–13], the symmetry breaking in a spatial region will induce topological defects as some symmetry breaking domains merge. In the one-dimensional case, kinks (a kind of topological defects) will finally form for the real scalar fields [14, 15].

By fixing the final temperature lower than the critical temperature, we will see the stable structures of the kinks in the final equilibrium states. This means these kink hairs are stable solutions.

Interestingly, its holographic dual can mimic the kink formations in one dimension from the mean field theory, such as the Ginzburg-Landau theory [14, 15]. From the KZM, the relation between the kink number density and the quench rate exhibits a universal power law. We find that from the holography of the kink hairy black hole, the number density of the kinks on the boundary field indeed obey a universal power law to the quench rate. Besides, the power is consistent with the KZM’s prediction in the mean field theory. However, we need to stress that the spatial dimension included in the power law should be the ‘effective spatial dimensions’, i.e., the dimension that really have impact to the formation of kinks. In our paper, we have implicitly assumed a homogeneous dependence on the y-direction, thus this spatial dimension has nothing to do with the kink numbers although the spatial dimension on the boundary is two. Therefore, the real effective spatial dimension for the kinks formation is one rather than two. As far as we know, this has not been reported previously.

II. PARITY SYMMETRY BREAKING

We start with the U(1) symmetric Abelian-Higgs model in the bulk (we work in the probe limit)

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - |D_\mu \tilde{\Psi}|^2 - m^2 |\tilde{\Psi}|^2$$

(1)

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $D_\mu = \nabla_\mu - i A_\mu$. The equations of motion (EoMs) read,

$$D_\mu D^\mu \tilde{\Psi} - m^2 \tilde{\Psi} = 0, \quad \nabla_\mu F^{\mu\nu} = i \left[ \tilde{\Psi}^* D^\nu \tilde{\Psi} - \tilde{\Psi} (D^\nu \tilde{\Psi})^* \right].$$

(2)

1 For complex scalar fields in a loop the topological defects are the winding numbers of the phase, see for example [16–20].
In order to have the $Z_2$ symmetry breaking, we need to have real scalar fields. We make the gauge transformations $\tilde{\Psi} = \Psi e^{i\lambda}$ and $A_\mu = M_\mu + \partial_\mu \lambda$, where $\Psi, M_\mu$ and $\lambda$ are real functions. Then we use the real functions to rewrite the EoMs. This procedure is equivalent to absorb the phase $\lambda$ into the real functions or to fix the gauge [21]. Therefore, the Eq.(2) become EoMs of real functions,

$$ (\nabla_\mu - i M_\mu \nabla^\mu) \Psi - m^2 \Psi = 0, \quad \nabla_\mu F^{\mu\nu} = 2 M^\nu \Psi^2. $$

Thus, if $+\Psi$ is a solution to the Eq.(3), $-\Psi$ is also a solution. This means the $Z_2$ symmetry $+\Psi \leftrightarrow -\Psi$ is a symmetry to the Eq.(3). So by gauge fixing, we have broken the former U(1) symmetry to $Z_2$ symmetry. Actually the equations in Eq.(3) satisfy a constraint. The imaginary part in the scalar equation can be derived from the EoMs of gauge fields, see the Appendix for details.

However, as a ground state the solution needs to choose a state with either $+\Psi$ or $-\Psi$, which spontaneously break the $Z_2$ symmetry in the equations. In addition, if the system has spatial directions, the condensate of real scalar field $\Psi$ will randomly choose $+\Psi$ or $-\Psi$ along the spatial direction. Thus, kinks form due to the $Z_2$ symmetry breaking. This is essentially the key point of KZM.

We adopt the idea of KZM to realize the kinks near the horizon, i.e. the kink hairs of a black hole. In this case we need to solve this system in a dynamical process due to KZM. A convenient choice is to use the Eddington-Finkelstein coordinates in the planar Schwarzschild-AdS black hole,

$$ ds^2 = \frac{1}{z^2} \left[-f(z)dt^2 - 2dtdz + dx^2 + dy^2\right]. $$

in which $f(z) = 1 - (z/z_h)^3$ with $z_h$ the horizon radius. Without loss of generality, we set the AdS radius be 1 and the horizon radius be $z_h = 1$ as well. Thus the AdS boundary is located at $z = 0$ and the temperature of the black hole is $T = 3/(4\pi)$.

### III. METHODS

We set the ansatz of fields as

$$ \Psi = \Psi(t, z, x), \quad M_t = M_t(t, z, x), \quad M_z = M_z(t, z, x), \quad M_x = M_x(t, z, x), $$

and we have turned off the fields $M_y$. In this ansatz we assume a homogeneous dependence on the $y$-direction, besides we set the periodic boundary condition along $x$-direction, thus the spatial directions look like a cylinder with $y$ as the axial direction. In the following we will see that the dependence of $x$-direction plays an important role in the kinks formation. Thus, the boundary field theory is effectively one-dimensional.

The ansatz of the fields (5) is self-consistent with the EoMs (3) since there are four independent equations to solve four real functions. We should stress that the inclusion of $M_z$ is important since we are working in the Eddington-Finkelstein coordinates. Even in the static case we cannot exclude $M_z$, instead $M_z = M_t/f(z)$ in static. See the Appendix for the details.

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2 Alternatively, one can rewrite the action (1) in a Stückelberg form, then gauge fix the phase to be zero [22]. These two procedures are equivalent.
The expansions of the fields near AdS boundary $z = 0$ is (for simplicity we choose $m^2 = -2$),
\[
\Psi \sim \Psi_1(t, x)z + \Psi_2(t, x)z^2 + \ldots, \quad M_t \sim \mu(t, x) - \rho(t, x)z + \ldots, \\
M_z \sim a_z(t, x) + b_z(t, x)z + \ldots, \quad M_x \sim a_x(t, x) + b_x(t, x)z + \ldots.
\]
We choose the standard quantization by setting $\Psi_1(t, x) \equiv 0$, thus $\Psi_2$ is related to the condensate of the superconducting order parameter $O(t, x)$ in the boundary field theory. $\mu$ and $\rho$ are interpreted as the chemical potential and charge density respectively in the boundary. From the expansions of the equation (3) we find that $a_z = \mu$. $a_x$ and $b_x$ are the velocity and current of the gauge fields on the boundary. We set $a_x = 0$ to have the holographic superfluid model. Near the horizon we set $M_t = 0$ as usual, and let other fields be finite.

In the KZM, one needs to quench the system across the critical point which is from the equilibrium states. In this paper we quench the chemical potential $\mu$. In the equilibrium or in the static case we find the critical potential is $\mu_c \approx 4.06$. We linearly quench the system from $T = 1.4T_c$ to $T = 0.8T_c$ and the quench profile is $T(t)/T_c = 1 - t/\tau_Q$, where $\tau_Q$ is the quench rate. From the dimensional analysis we know that $[T] = [\mu] = [\text{mass}]$, thus $T/\mu$ is massless. From the holographic superconductor [9] we see that decreasing $T$ is equivalent to increasing $\mu$, i.e., $T(t)/T_c = \mu_c/\mu(t)$. Therefore, we quench the chemical potential as $\mu(t) = \mu_c/(1 - t/\tau_Q)$.

At initial time, the system is in a state with vanishing scalar fields. In this case the simple solution to the gauge fields are $M_t = M_z f(z) = \mu(t_i) - \mu(t)z$ and $M_x = 0$. However, in order to quench the system and evolve it to a state with topological defects (kinks in our paper), a common procedure is to introduce white noise into the system at initial time. To this end, we introduce a very small Gaussian white noise $\zeta(x_i, t)$ for the scalar fields in the bulk with $\langle \zeta(x_i, t) \rangle = 0$ and $\langle \zeta(x_i, t)\zeta(x_j, t') \rangle = h\delta(t - t')\delta(x_i - x_j)$, in which $h = 0.001$. After the quench the system will evolve according to the equations (3). We adopt the 4th Runge-Kutta method in the time direction with time step $\Delta t = 0.01$. In the radial direction $z$, we use the Chebyshev pseudospectral methods with 21 grid points. In the periodic $x$-direction, we use the Fourier decomposition with 401 grid points. The length of the loop along $x$-direction is set to be $L = 200$.

IV. NUMERICAL EVIDENCE

A. Kink hairs in the bulk

We quench the system across the critical point $T_c$ (or $\mu_c$), then continue decrease the temperature to $0.8T_c$, then stop the quench. But we still need to hold the system at the final temperature until it equilibrates. During the whole course, the formation of kinks will turn out due to KZM. We exhibit the kink hairs in the bulk at the final equilibrium states in Fig.1. On the left panel, the 3D configuration of the real scalar field vividly shows the kink hairs in the bulk. They will tend to zero at the AdS boundary because of the boundary conditions.

\[3\] Although precisely we need to call these topological defects as domain walls since they have higher dimensions, we prefer calling them as “kinks” since the mechanism to generate these topological defects are the parity symmetry breaking and their effective spatial dimension is one.
FIG. 1: 3D configurations of the kink hairs of the real scalar fields in the bulk (left), the kinks at the horizon (middle) and the view of the kink hairs along $x$-direction (right).

On the middle panel we can see the one-dimensional kinks at the horizon. From this we clearly see that the absolute value of the maximum are equal to the absolute values of the minimum of the kinks. This phenomenon implies the $Z_2$ symmetry breaking and $|+\Psi| = |-\Psi|$. We can also find that some kinks are broader and others are narrower. This is because of the random properties of the KZM. In essence, the kinks can randomly choose their values, however, due to the superconducting condensate at the final temperature, averagely there is a coherence length $\xi$ to decide the width of the kinks. It also means we need to consider these kinks statistically to measure their average coherence length or the width.

The right panel shows the profile of the kink hairs from the view of $x$-direction. It is perfectly symmetric along $\Psi = 0$, which indicates the $Z_2$ symmetry breaking in the bulk along the radial direction. Therefore we realize the $Z_2$ symmetry breaking near the horizon and also in the bulk, which form the kink hairs. From the AdS/CFT correspondence, the kinks of the scalar field in the bulk may have some impact on the boundary field theory, which we will explore in the following.

**B. Holographic interpretations**

From the construction of holographic superconductor [9], the charged scalar hairs in the bulk will induce the condensate of the charged order parameters in the boundary field theory. Thus, we speculate that the kinks of the real scalar fields in the bulk will also induce some kinks of the order parameter in the boundary field theory.

In the Fig.2 we indeed see the kink formations of the order parameter in the boundary field theory. In this figure, we set the quench rate as $\tau_Q = 20$. In the left panels, we show the snapshots of the order parameter at four different time stages, from the initial to the final equilibrium state. At the initial time, the order parameter exhibits a random profile and then evolves and grows rapidly around $t \in [7, 13]$, later it goes to a profile of some stable kinks in the final equilibrium state. Their corresponding average absolute values $\langle |O| \rangle$ are shown in the right panel. This panel shows the time evolution of the $\langle |O(t)| \rangle$ from zero to a finite plateau. Therefore, clearly we get the kinks formation in the boundary field theory.

It is known that the boundary field theory in the AdS/CFT correspondence is a mean field theory with strong couplings [10]. Therefore, as we did, the kinks formation in the boundary field theory
from AdS/CFT looks like the kinks formation in a one-dimensional chain described by the Ginzburg-Landau theory [14, 15]. Since the Ginzburg-Landau theory is a mean field theory, it does not really involve the creation and annihilation operator. Instead, it has the real scalar fields in the action which resembles our setup in section II. The difference is that our setup is in the bulk. But from the AdS/CFT correspondence, we can really model the kinks in the chains from the bulk fields.

C. Scalings between kink numbers and quench rate

It is useful to see whether the kinks formation in our model satisfies the KZM’s scaling relation between the average kink numbers \( \langle n \rangle \) and the quench rate \( \tau_Q \),

\[
\langle n \rangle \propto \left( \frac{1}{\tau_Q} \right)^{\frac{d}{1+z\nu}},
\]

where \( d \) is the spatial dimension of the system in which the topological defects locate, \( \nu \) and \( z \) are the static critical exponent and dynamic critical exponent respectively. In particular, these exponents are from equilibrium physics with

\[
\xi \propto |\epsilon|^{-\nu}, \quad \tau \propto |\epsilon|^{-z\nu}.
\]

in which \( \epsilon = t/\tau_Q = 1 - T(t)/T_c \) is the reduced distance to the critical point, \( \xi \) and \( \tau \) are respectively the coherence length and relaxation time near the critical point. The symmetry breaking domain roughly has the size \( \hat{\xi} \propto \tau_Q^{\frac{1}{1+z\nu}} \). Therefore, the number density of topological defects are roughly

\[
\langle n \rangle \propto 1/\hat{\xi}^d \propto (1/\tau_Q)^{\frac{d}{1+z\nu}}.
\]

From the concise derivation of the KZ scaling law, we see that the spatial dimension \( d \) is indeed related to the dimension where the topological defects extend. In our case, the kinks extend in \( d = 1 \) spatial dimension, i.e., \( x \)-direction, although the AdS boundary has two spatial dimensions. The \( y \)-direction in fact does not have any impact on the kink formation.
FIG. 3: Double logarithmic plot of the average kink numbers $\langle n \rangle$ to the inverse of the quench rate $1/\tau_Q$. Smaller $1/\tau_Q$ implies slower quench. In the slow quench region, there is a universal power relation indicated by the red fitted line.

In the Fig.3, we show the relations between the average kink numbers $\langle n \rangle$ to the inverse of quench rate $1/\tau_Q$. We simulate independently 600 times to generate this figure. In the slow quench regime as $1/\tau_Q$ is smaller, we can see that $\langle n \rangle$ and $\tau_Q$ indeed satisfy a universal power relation with the red fitted line $\langle n \rangle \sim (30.96 \pm 1.2) \times (1/\tau_Q)^{0.256\pm0.012}$. The fitted power is consistent with the theoretical value $d\nu/(1 + z\nu) = 1/4$ in which $d = 1$. Since the boundary is a mean field theory we have $\nu = 1/2$ and $z = 2$ [23, 24]. Therefore, the kinks formation in our setup also match the KZ's scaling relation. But we need to stress that in this case the effective spatial dimension $d$ is equal to 1, rather than 2 although the boundary is two dimensions in space. The other y-dimension does not make any contribution in the kink's formation.

Our above arguments also extends to the formation of kinks in the bulk geometry. Since from the Fig.1 we see that the kinks in the bulk will have the same number as the kinks on the boundary (remember that $\frac{1}{2} \partial^2 \Psi|_{z \to 0} = O$). Therefore, although the bulk has three spatial dimensions ($z, x, y$), the effective dimension for the kink formation is still $d = 1$ since in other directions there are no kinks.

V. CONCLUSIONS AND DISCUSSIONS

In this paper we have realized the kink hairs in the AdS black hole, inspired from the parity symmetry breaking in KZM and the Gubser’s argument on the charged scalar hair in spacetime with negative cosmological constant. Holographically the kinks in the bulk geometry will induce the kinks in the boundary field theory. Moreover the kinks formed in the boundary or in the bulk all obey the KZ's scaling relation very well between the average kink numbers and the quench rate.

Our model resembles the applications of Ginzburg-Landau theory to produce the kinks in one-dimensional chains, which has wide applications in condensed matter physics, such as [14, 15] . Therefore, from this holographic model it will be interesting to study, such as the kink spatial distributions [25], the higher cumulants of the kink number beyond KZM [26], the entanglement entropy [27] and etc. It is expected that our initial work on the holographic kinks will bring new insights to the dualities between gravity and condensed matter.
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APPENDIX: EXPLICIT FORMS OF EQUATIONS OF MOTIONS

The EoMs (3) in the main context actually can be decomposed into

\[ \nabla_\mu \nabla^\mu \Psi - M_\mu M^\mu \Psi - m^2 \Psi = 0, \]  
\[ (\nabla_\mu M^\mu) \Psi + 2 M_\nu \nabla_\mu \Psi = 0, \]  
\[ \nabla_\mu F^{\mu\nu} = 2M^\nu \Psi^2. \]

The Eqs.(S1) and (S2) are respectively from the real part and imaginary part of the scalar equations in (3). As was emphasized in [28], these equations are not independent. In fact, from the EoMs of gauge fields (S3) one can derive the imaginary part EoMs (S2), such as

\[ 0 \equiv \nabla_\nu (\nabla_\mu F^{\mu\nu}) \Rightarrow \nabla_\nu (2M^\nu \Psi^2) = 0 \Rightarrow (\nabla_\nu M^\nu) \Psi + 2 M^\nu \nabla_\nu \Psi = 0. \]  

(S4)

The last equality is exactly the imaginary part of the scalar EoMS (S2).

In our ansatz of the fields (5) and in the frame of the line-element (4), the EoMs become

1). The gauge fields (S3) part:

\[ 0 = -\frac{2\Psi^2 M_t}{z^2} + \partial_x^2 M_t + f \partial_z^2 M_t - \partial_x \partial_z M_t - f \partial_{tt} M_z + \partial_t^2 M_z, \]  
\[ 0 = -\frac{2\Psi^2 M_z}{z^2} + \partial_x^2 M_z - \partial_x M_x + \partial_t^2 M_t - \partial_{tz} M_z, \]  
\[ 0 = -\frac{2\Psi^2 M_x}{z^2} - f' \partial_x M_z + f' \partial_z M_x + \partial_x M_t - f \partial_{xx} M_z + f \partial_z^2 M_x + \partial_{tx} M_z - 2 \partial_{tz} M_x, \]

(S5) (S6) (S7)

2). The real part of scalar fields (S1):

\[ 0 = -\frac{m^2 \Psi}{2z^2} - \frac{1}{2} \Psi M_x^2 + \Psi M_t M_z - \frac{1}{2} \Psi f M_z^2 + \frac{1}{2} \partial_z^2 \Psi - \frac{f \partial_z \Psi}{z} + \frac{1}{2} f' \partial_z \Psi + \frac{1}{2} \partial_z^2 \Psi \]  
\[ + \frac{\partial_t \Psi}{z} - \partial_{tz} \Psi \]  

(S8)

3). The imaginary part of scalar fields (S2):

\[ 0 = -\frac{2\Psi M_t}{z} + \frac{2\Psi f M_z}{z} - \Psi M_z f' - 2M_x \partial_x \Psi - \Psi \partial_x M_x + 2M_t \partial_z \Psi - 2 f M_z \partial_z \Psi + \Psi \partial_z M_t - \Psi f \partial_z M_z + 2 M_z \partial_t \Psi + \Psi \partial_t M_z. \]  

(S9)

in which \( f' = f'(z) \). There are five equations, but only four of them are independent due to the constraint (S4). Therefore, there are four independent equations for four real fields, i.e., \( \Psi, M_t, M_z \) and \( M_x \). Thus, our ansatz of the fields are self-consistent. We need to stress that including the \( (t, x) \)-dependence, our ansatz is the only possible choice. One cannot omit \( M_z \) or \( M_x \).
In order to get the initial condition for our quench, we need to solve the static as well as \( x \)-independent case of the EoMs. Therefore, the EoMs (S5),(S6) and (S7) become

\[
0 = -\frac{2\Psi^2 M_t}{z^2} + f \partial_z^2 M_t, \quad (S10)
\]
\[
0 = -\frac{2\Psi^2 M_z}{z^2} + \partial_z^2 M_t, \quad (S11)
\]
\[
0 = -\frac{2\Psi^2 M_x}{z^2} + f' \partial_z M_x + f \partial_z^2 M_x. \quad (S12)
\]

From Eq.(S12) we can safely set \( M_x = 0 \). From (S10) and (S11) we can readily get \( M_z = M_t/f \). At the initial time the system is in the normal state with vanishing scalar fields \( \Psi = 0 \). Thus we can solve \( M_t = \mu - \mu z \) and \( M_t = (\mu - \mu z)/f \) by imposing the boundary condition \( M_t(z \to 0) = \mu \) and \( M_t(z \to 1) = 0 \).

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