Percolation on infinite graphs and isoperimetric inequalities

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Abstract

We consider the Bernoulli bond percolation process (with parameter \(p\)) on infinite graphs and we give a general criterion for bounded degree graphs to exhibit a non-trivial percolation threshold based either on a single isoperimetric inequality if the graph has a bi-infinite geodesic, or two isoperimetric inequalities if the graph has not a bi-infinite geodesic. This new criterion extends previous criteria and brings together a large class of amenable graphs (such as regular lattices) and non-amenable graphs (such trees). We also study the finite connectivity in graphs satisfying the new general criterion and show that graphs in this class with a bi-infinite geodesic always have finite connectivity functions with exponential decay as \(p\) is sufficiently close to one. On the other hand, we show that there are graphs in the same class with no bi-infinite geodesic for which the finite connectivity decays sub-exponentially (down to polynomially) in the highly supercritical phase even for \(p\) arbitrarily close to one.

1 Introduction

Percolation is a subject which has been intensively studied during the last decades mainly on the \(d\)-dimensional unit cubic lattice \(\mathbb{Z}^d\). The study of percolation processes on infinite graphs other than \(\mathbb{Z}^d\), started basically in the early nineties, and has been focused essentially on non-amenable graphs (see e.g. [15] and reference therein) and transitive graphs. Some general results and conjectures about percolation on infinite graphs has been formulated in the seminal paper [5] where the authors prove, among other results, that non-amenable graphs do have a non-trivial percolation threshold (i.e. critical percolation probability \(p_c\) strictly less than 1). In [3] Babson and Benjamini have introduced a parameter depending on the graph’s structure which should be relevant in percolation. Basically, given an infinite graph \(G = (V, E)\), this parameter is the minimum \(t \in \mathbb{N}\) such that any minimal cut set in \(G\) is \(t\)-close, i.e. it is connected in the graph \(G^t\) which has the same vertex set \(V\) of \(G\) and edge set formed by those pairs \(\{x, y\} \subset V\) whose distance is less or equal than \(t\). A graph for which \(t\) is finite is said to have the quasi-connected minimal cut sets property (see e.g. [21]). Babson and Benjamini showed, via Peierls argument, that if \(t < \infty\) then \(p_c < 1\) for Cayley graphs of finitely presented infinite groups which are not a finite extension of \(\mathbb{Z}\) (most of the regular lattices fall in this class). Later, Procacci and Scoppola [18] pointed out that \(t < \infty\) was a sufficient condition for \(p_c < 1\) in a large class of bounded degree graphs, namely, graphs with a bi-infinite geodesic or satisfying a very mild isoperimetric
inequality. So, despite that most of the non-amenable graphs has \( t = \infty \) (but \( p_c < 1! \)), there was hope that the finiteness of \( t \) could still be a key information to establish whether \( p_c < 1 \), at least for a bounded degree amenable graph. In particular, Babson and Benjamini explicitly conjectured that the finiteness of \( t \) could be a sufficient condition for \( p_c < 1 \) for all amenable quasi-transitive graphs with one end. However recently Timár [23] provided a counterexample, i.e. a one-ended transitive amenable graph with \( p_c < 1 \) (the Diestel-Leader graph \( DL(2,2) \)) for which the Babson-Benjamini parameter \( t \) is infinite. This, together with the fact that most of the non-amenable graphs (e.g. trees) have \( t = \infty \) and \( p_c < 1 \), seems to suggest that \( t \) may not be the right quantity to look at in order to implement Peierls argument in general graphs.

In conclusion, even speculating that a necessary and sufficient criterion for a general graph to have \( p_c < 1 \) is probably too much to ask, it would be desirable to obtain at least a sufficient criterion able to include as much as possible graphs with known non-trivial percolation.

The results presented in this note are a little step in this direction. We present in fact a new sufficient criterion for a graph to have \( p_c < 1 \) which now brings together both amenable and non-amenable graphs (including graphs with Babson-Benjamini parameter \( t = \infty \)). The new criterion is based on a single isoperimetric inequality if the graph has a bi-infinite geodesic, while two isoperimetric inequalities are required if the graph has not a bi-infinite geodesic.

We also study the two-point connectivity function in graphs satisfying the general criterion above. In particular we investigate the possibility, originally discussed in [19], about the existence of percolation processes in infinite graphs for which the finite connectivity decays non-exponentially even in the highly supercritical phase. Indeed, given a graph \( G \) without bi-infinite geodesics which falls in the class satisfying our new criterion, we show that the decay of the finite connectivity may be not exponential, presenting an example. On the other hand, if \( G \) has a bi-infinite geodesic, then we show that the connectivity functions decay exponentially in the highly supercritical phase.

The paper is organized as follows. In Section 2 we rapidly review some definitions about graphs and remind the basic notions of the Bernoulli bond percolation process in graphs, defining in particular the finite connectivity functions and the critical percolation probability. In Section 3 we present our results in form of four theorems. Finally, in Section 4 we give the proofs of these theorems.

### 2 Notation and Definitions: graphs and percolation

#### 2.1 Infinite graphs

Throughout the paper, whenever \( X \) is a set, we will denote by \(|X|\) its cardinality. Let \( G = (V, E) \) be a graph with vertex set \( V \) and edge set \( E \). If \( x, y \in V \), we denote by \( d_G(x,y) \) the usual path distance in \( G \). If \( W \subset V \), let \( G[W] = (W, E[W]) \) denote the induced subgraph where \( E[W] = \{e \in E : e \cap W \neq \emptyset\} \). A set \( W \subset V \) is connected in \( G \) if \( G[W] \) is connected.

Given \( G = (V, E) \) connected and \( W \subset V \), we denote the edge boundary of \( W \) by \( \partial_eW = \{e \in E : |e \cap W| = 1\} \). We denote \( \partial^{\text{ext}}_vW = \{x \in V \setminus W : d_G(x, W) = 1\} \) the vertex external boundary of \( W \) and \( \partial^{\text{int}}_vW = \{x \in W : d_G(x, V \setminus W) = 1\} \) the vertex internal boundary of \( W \). We denote \( W^c = W \cup \partial^{\text{ext}}_vW \) the closure of \( W \). Note that \( W^c \) is connected and \( E[W^c] = E[W] \cup \partial_eW \). We finally denote \( \text{diam}(W) = \max_{x,y \in W} d_G(x,y) \).

Throughout the paper, the symbol \( G = (V,E) \) will always denote a graph which is infinite, connected and bounded degree, with \( \Delta(G) \equiv \max_{v \in V} |\partial_e v| < \infty \) being its maximum degree. We denote by \( \mathcal{A}_G = \{W \subset V : |W| < \infty \text{ and } G[W] \text{ is connected}\} \) the set of all finite and connected
subsets of vertices of an infinite graph $G$.

A geodesic ray $r = (V_r, E_r)$ in $G$ is an infinite sub-graph of $G$ such that $V_r = \{x_0, \ldots, x_n, \ldots\}$, $E_r = \{\{x_0, x_1\}, \ldots, \{x_{n-1}, x_n\}, \ldots\}$ and $d_G(x_0, x_n) = n$ for all $n \in \mathbb{N}$. Let $\rho$ and $\rho'$ be two geodesic rays in $G$, both starting at $x_0$, with vertex sets $V_\rho = \{x_0, x_1, \ldots, x_n, \ldots\}$ and $V_{\rho'} = \{x_0, y_1, \ldots, y_n, \ldots\}$ respectively. If $V_\rho$ and $V_{\rho'}$ are such that $d_G(x_n, y_m) = n + m$ for any $\{n, m\} \subset \mathbb{N}$, then the union $\rho \cup \rho' = (V_\rho \cup V_{\rho'}, E_\rho \cup E_{\rho'})$ is called a bi-infinite geodesic in $G$.

A finite connected graph $\tau = (V, E)$ is a tree if $|E| = |V| - 1$. If $G$ is a graph, we denote by $T(G)$ the set of all subgraphs of $G$ which are trees.

**Definition 2.1** Given a graph $G = (V, E)$, and given a connected set $W \in \mathcal{A}_G$, we define the tree distance $d_G^t(\partial_e W)$ of the edge-boundary $\partial_e W$ of $W$ as

$$d_G^t(\partial_e W) = \min_{\ell(\partial_e W) \in T(G)} |E|$$

**Remark.** Given $W \in \mathcal{A}_G$, let $\ell_{\partial_e W}$ be the number of edges of $E(W)$ necessary to connect $\partial_e W$, then $d_G^t(\partial_e W)$ can also be written as

$$d_G^t(\partial_e W) = \ell_{\partial_e W} + |\partial_e W|$$

Hence, since $\ell_{\partial_e W}$ is at most $|W| - 1$, we get immediately the following upper and lower bounds for $d_G^t(\partial_e W)$

$$|\partial_e W| \leq d_G^t(\partial_e W) \leq |\partial_e W| + |W| - 1$$

where the equation $|\partial_e W| = d_G^t(\partial_e W)$ holds if and only if $\partial_e W$ is connected and the equality $|\partial_e W| = |\partial_e W| + |W| - 1$ holds if and only if $G[W]$ is a tree.

A cut set of a graph $G = (V, E)$ is a set $\gamma \subset E$ such that the graph $G \setminus \gamma \equiv (V, E \setminus \gamma)$ is disconnected.

**Definition 2.2** Given $G = (V, E)$, a finite cut set $\gamma \subset E$ is called a contour if $G \setminus \gamma$ has exactly one finite connected component and is minimal with respect to this property, i.e., for all edges $e \in \gamma$ the graph $(V, E \setminus (\gamma \setminus e))$ has no finite connected component. We denote by $\mathcal{F}_G$ the set of all contours in $G$.

This definition generalizes in some sense the notion of Peierls contour in the Ising model. These objects were called “$(v, \infty)$- minimal cut set” in [3], “Peierls contours” in [18], and “fences” in [12]. It is worth to mention some recent extensions of the general notion of Pirogov-Sinai contours for trees in [20] and [12].

If $\gamma$ is a contour in $G$, we denote by $G_\gamma = (I_\gamma, E_\gamma)$ the unique finite connected component of $G \setminus \gamma$; the set $I_\gamma \subset V$ is called the vertex interior of the contour $\gamma$ and the set $E_\gamma \subset E$ is called the edge interior of the contour $\gamma$. We denote by $G_\gamma^e = (I_\gamma^e, E_\gamma^e)$ the graph with vertex set $I_\gamma^e = I_\gamma \cup \partial_e^{\text{int}} I_\gamma$ and edge set $E_\gamma^e = E_\gamma \cup \gamma$ and call it the closure of $G_\gamma$.

**Remark.** If $\gamma \in \mathcal{F}_G$ is a contour, then $I_\gamma \in \mathcal{A}_G$ and $\partial_e I_\gamma = \gamma$. So the tree distance $d_G^t(\gamma)$ of the contour $\gamma$ is

$$d_G^t(\gamma) = \min_{\ell(\partial_e I_\gamma) \in T(G_\gamma)} |E| = \ell_\gamma + |\gamma|$$

where $\ell_\gamma$ is the number of edges of $E_\gamma$ necessary to connect $\partial_e^{\text{int}} I_\gamma$.
Given a contour $\gamma$ in $G = (V, E)$ and a set of vertices $X \subset V$, we say that $\gamma$ surrounds $X$ and we write $\gamma \circ X$ if $X \subset I_\gamma$. We say that $\gamma$ separates $X$ and we write $\gamma \otimes X$ if $0 < |X \cap I_\gamma| < |X|$.

We denote by $F^G_n$ the set of all contours with cardinality $n$, by $F^G_X$ ($F^G_n(X)$) the set of $\gamma \in F^G_n$ ($\gamma \in F^G_n(X)$) such that $\gamma \circ X$ and finally, with a slight abuse of notation, for $e \in E$, we denote by $F^G_n(e)$ the set of contours $\gamma \in F^G_n$ such that $e \in \gamma$.

We now introduce two isoperimetric constants, $R_G$ and $P_G$, in a bounded degree graph $G$, which play a central role in all our results.

**Definition 2.3** Given $G = (V, E)$, let

$$R_G = \inf_{W \in \mathcal{A}_G} \frac{|\partial_e W|}{d^G_e(\partial_e W)} \tag{2.5}$$

and

$$P_G = \inf_{W \in \mathcal{A}_G} \frac{|\partial_e W|}{\log(\text{diam}(W))} \tag{2.6}$$

We call $R_G$ the “contour constant” of $G$ and $P_G$ the “wedge constant” of $G$.

The wedge constant $P_G$ of a graph $G$ was explicitly introduced in [18] and its role, as far as percolation on $G$ is concerned, has already been pointed out there. See also [14] and [8] where percolation in subsets of $\mathbb{Z}^d$ with wedge growing logarithmically with the diameter has originally been considered.

The contour constant $R_G$, as far as we know, is rather new in the literature. A related quantity has been recently introduced by Campari and Cassi [6] in the study of the Ising model on general graphs. In [6] the authors define a finite constant $l$ depending on the structure of the graph $G$, such that any contour in $G$ (according to definition 2.2) with cardinality $n$ is connectable with no more than $ln$ vertices. It is easy to see that $l$ is essentially the inverse of $R_G$. Indeed, if $R_G > 0$, then for any contour $\gamma$, $d^G_e(\gamma) \leq \frac{1}{R_G} |\gamma|$, or, in other word, recalling (2.4), $\gamma$ is connectable (by a tree which contains $\gamma$) using at most $\frac{1}{R_G} |\gamma|$ edges. So an easy computation yields $\frac{1}{R_G} - 1 \leq l \leq \frac{1}{R_G} + 1$.

The definition of the contour constant $R_G$ as formulated in (2.5) resembles that of the more usual and known Cheeger constant $C_G$, a.k.a. isoperimetric constant or expansion constant, defined (see e.g. [5] or [7]) as

$$C_G = \inf_{W \in \mathcal{A}_G} \frac{|\partial_e W|}{|W|} \tag{2.7}$$

However, we want to stress that the behavior of $R_G$ is quite different from that of $C_G$ as $G$ varies in the class of infinite graph. Indeed, by inequality (2.3), it is easy to see that

$$R_G \geq \frac{C_G}{C_G + 1} \tag{2.8}$$

so that $R_G$ is positive whenever the Cheeger constant of $G$ is positive. On the other hand, the converse is not true: the positivity of $R_G$ does not imply, in general, that of the Cheeger constant. In particular, $R_G$ is strictly positive in all amenable graphs (i.e. graphs with $C_G = 0$) for which the Babson-Benjamini parameter $t$ is finite, since, by definition (2.2) and by definition of $t$ given in the introduction, we have that $d^G_e(\partial_e W) \leq (1 + t)|\partial_e W|$ and hence

$$R_G \geq \frac{1}{t + 1} \tag{2.9}$$
The unit cubic lattice $\mathbb{Z}^d$ (for $d > 1$) is a topical example of a graph with $C_G = 0$, $t$ finite, and $R_G > 0$.

Finally, using the concept of contours, we introduce a new kind of distance between two vertices $x, y$ in a graph $G$.

**Definition 2.4** Given a graph $G = (\mathcal{V}, \mathcal{E})$, let $x, y \in \mathcal{V}$. We define the contour distance $f_G(x, y)$ between $x$ and $y$ by

$$f_G(x, y) = \min_{\gamma \subset \mathcal{E}(x, y)} |\gamma|$$

(2.10)

As we will show ahead, the contour distance can play an important role in the decay properties of the truncated connectivity functions of a graph in the supercritical phase.

### 2.2 Independent Percolation on infinite graphs

Given $G = (\mathcal{V}, \mathcal{E})$ and $p \in [0, 1]$, we associate to each edge $e \in \mathcal{E}$ i.i.d. Bernoulli variables $\omega(e)$, taking the value $\omega(e) = 1$ (meaning that the edge $e$ is open) with probability $p$, or else the value $\omega(e) = 0$ (meaning that the edge $e$ is closed) with probability $1 - p$. Let $P_p$ denote the standard product measure on the configurations of edges in $G$. A configuration $\omega$ of the process is a function $\omega : \mathcal{E} \to \{0, 1\} : e \mapsto \omega(e)$. We call $\Omega_G$ the set of configurations in $G$. Given $\omega \in \Omega_G$ we denote by $O(\omega)$ the subset of $\mathcal{E}$ given by $O(\omega) = \{e \in \mathcal{E} : \omega(e) = 1\}$ and by $C(\omega)$ the set $C(\omega) = \{e \in \mathcal{E} : \omega(e) = 0\}$.

If $G_N = (V_N, E_N)$ is a finite subgraph of $G$, let $\Omega_N$ be the of configurations in $G_N$, and let $\omega \in \Omega_N$, then the the probability $P_p(\omega)$ is given explicitly by

$$P_p(\omega) = p^{|O(\omega)|}(1 - p)^{|C(\omega)|}$$

(2.11)

Given a configuration $\omega \in \Omega_G$, an open cluster $g$ of $\omega$ is a connected subgraph $g = (V_g, E_g)$ of $G$ such that $\omega(e) = 1$ for all $e \in E_g$, and $\omega(e) = 0$ for all $e \in \partial g$ where $\partial g = \{e \in \mathcal{E} : |e \cap V_g| = 1\}$ is the external edge boundary of $g$. If $g = (V_g, E_g)$ is an open cluster of a given configuration $\omega$ and $X$ is a non-empty subset of $V_g$ we write shortly $X \subset g$, and write shortly $|g|$ in place of $|E_g|$.

**Definition 2.5** Let $G = (\mathcal{V}, \mathcal{E})$ be an infinite graph and let $X \subset \mathcal{V}$ such that $|X| = n$. The $n$-point finite connectivity function $\phi^n_p(X)$ is defined as

$$\phi^n_p(X) = P_p(\exists \text{ open cluster } g : X \subset g, |g| < \infty)$$

(2.12)

In this paper we will be interested only in two-points finite correlations and one-point finite correlation (i.e $|X| \leq 2$). In particular, the one-point finite correlation is directly related to the percolation probability. Given a graph $G = (\mathcal{V}, \mathcal{E})$ and a vertex $x \in \mathcal{V}$, the percolation probability, i.e. the probability that there is an infinite open cluster passing through $x$, is defined as

$$\theta_p(x) = P_p(\exists \text{ open cluster } g : \{x\} \subset g, |g| = \infty)$$

By (2.12), $\theta_p(x)$ may be written in term of connectivity functions as

$$\theta_p(x) = 1 - \phi^n_p(x).$$

(2.13)

A standard coupling argument shows that, in any graph $G$, $\theta_p(x)$ is an increasing function of $p$ (see e.g. [13], Theorem 2.1). And, as $G$ is connected, the critical percolation probability
$p_c$ does not depend on the choice of $x$, since if $\theta_p(x) > 0$, then, by FKG, $\theta_p(y) > 0$ for any two vertices $x, y$.

The critical percolation probability $p_c(G)$ for the graph $G$ is defined by

$$p_c = \sup_{p \in [0,1]} \{ p : \theta_p(x) = 0 \}$$

and we say that the system is in the subcritical phase if $p \in [0, p_c)$ and in the supercritical phase if $p \in (p_c, 1]$.

A way to show that $p_c < 1$ is to establish a non-trivial upper bound for the one-point correlation $\phi^f_p$, and this can be obtained via the so called Peierls argument. This very famous tool was originally stated for the Ising model on $\mathbb{Z}^2$ in the low-temperature phase, but, once the notion of contours (according to def. [2.2]) is introduced, the argument can be generalized for bond Bernoulli percolation in any graph $G$. With the notations and definitions previously introduced, the Peierls argument for a general graph $G = (V, E)$ can be stated as follows.

**Proposition 2.6 (Peierls argument)** Let $G = (V, E)$ be an infinite graph. If it is possible to find a finite positive constant $r$ and a vertex $x \in V$ such that, for all $n \in \mathbb{N}$,

$$|F^n_G(x)| \leq r^n,$$

then

$$p_c \leq 1 - \frac{1}{2r}$$

Such a proposition follows immediately by observing that if $g = (V_g, E_g)$ is finite open cluster such that $\{x\} \subset V_g$, then there is a contour $\gamma$ such that $\gamma \subset x$ and $\gamma \subset \partial g$, i.e. $\gamma$ is formed by closed edges. Therefore,

$$\phi^f_p(x) \leq \sum_{n \geq 1} \sum_{\gamma \in F^n_G(x)} (1 - p)^{|\gamma|} = \sum_{n \geq 1} [r(1 - p)]^n < 1$$

as soon as $p > 1 - \frac{1}{2r}$; whence $p_c \leq 1 - \frac{1}{2r}$.

In the next section we present our results in form of four theorems. The first two, Theorem [3.1] and Theorem [3.3] concern non triviality of percolation threshold $p_c$ in infinite graphs with finite maximum degree and provides new criteria to establish, via a sufficient condition, if a graph $G$ in this class has $p_c < 1$. Theorems [3.4] and [3.5] will concern the decay of the connectivity function for bounded degree graphs.

3 Results

3.1 New sufficient criteria for a graph $G$ to have $p_c < 1$

In order to implement the Peierls argument in a graph $G$ one must be able to bound exponentially the number of contours of cardinality $n$ that surrounds a fixed vertex. This is relatively easy if $G$ is the cubic lattice $\mathbb{Z}^d$ but it may be a desperate task for general graphs. In their seminal paper [5], Benjamini and Schramm wondered whether is possible to replace in a general graph the Peierls condition (2.15) by some more friendly isoperimetric inequality.

Our first result here below may be viewed, in our opinion, as a step in this direction.
Theorem 3.1 Let $G = (V, E)$ be an infinite graph with maximum degree $\Delta$ and with contour constant, defined in (2.5), equal to $R_G$. Suppose that

i) $R_G > 0$

ii) $G$ has a bi-infinite geodesic

Then $G$ has a non trivial percolation threshold and

$$p_c(G) \leq 1 - \frac{1}{2(2e\Delta^2)^{1/R_G}}$$  \hspace{1cm} (3.1)

Observing that any quasi-transitive graph has always a bi-infinite geodesic (see e.g. proposition 5.2 in [22] or lemma 5.7 of [11]), Theorem 3.1 immediately implies the following corollary.

Corollary 3.2 A quasi-transitive graph $G$ such that $R_G > 0$ has $p_c < 1$.

Remark 1. Theorem 3.1 is a genuine extension of Theorem 1 of [18]. In particular, by (2.8) and (2.9), one immediately sees that the class of graphs satisfying the hypothesis of Theorem 3.1 contains all graphs with positive Cheeger constant and all graphs with Babson-Benjamini parameter $t$ finite. So, for example, a regular tree, for which $t$ is infinite and $C_G > 0$, as well as a regular amenable lattice, for which $C_G = 0$, are both in the class of graphs satisfying Theorem 3.1. We would also draw the attention to the possibility that $R_G$ may play a role similar (or alternative) to that of the isoperimetric dimension $d_{iso}$ (see e.g. [5] or [6] for its definition). In particular, it seems interesting to inquire whether $R_G > 0$ for graphs with $d_{iso} > 1$, for in this case Theorem 3.1 could be regarded as a generalization, at least for graphs with a bi-infinite geodesic, of the conjecture stated in [5] (see there question 3.4).

Remark 2. Concerning Corollary 3.2 we recall that every quasi-transitive infinite graph $G$ has either 1 end, or two ends, or infinitely many ends (see e.g. [16] at the end of sec. 8 and references therein). If $G$ has infinitely many ends, then $G$ is non-amenable (see again [16] or [17], proposition 6.2), hence $C_G > 0$ and, by (2.8), $R_G > 0$. So $p_c < 1$. If $G$ has two ends, then $G$ is a finite extension of $\mathbb{Z}$ and hence $p_c = 1$. We are thus left with the quasi-transitive amenable graphs with one end, for which $R_G > 0$ is a sufficient condition for $G$ to have $p_c < 1$. As pointed out by Timár in [23], the Diestel-Leader graph $DL(2, 2)$ is an example of amenable (hence with Cheeger constant equal to zero) transitive graph with one end exhibiting contours which are not $t$-closed (in the Babson-Benjamini sense) for any $t \in \mathbb{N}$, such contours being the edge boundaries of the so-called tetrahedrons $T_n$. On the other hand, one can check immediately from the definition of $T_n$ (see Definition 2 in [4]) that $|\partial T_n| = 2^{n+2}$ and $d_{DL(2, 2)}(\partial T_n) = 2 \sum_{k=1}^{n} 2^k$, so that $d_{DL(2, 2)}(\partial T_n)/|\partial T_n| \leq 1$, i.e. tetrahedrons in $DL(2, 2)$ do satisfy (2.5). So the question raised by Babson and Benjamini ([3] question 3) and answered negatively by Timár can be replaced by the following.

Question. Are there a one-ended quasi-transitive amenable graph $G$ for which $R_G$ is zero?

Our second result refers to the class of graphs with no bi-infinite geodesic for which Theorem 3.1 cannot be applied.

Theorem 3.3 Let $G = (V, E)$ be an infinite graph with maximum degree $\Delta$ and with contour constant, defined in (2.5), equal to $R_G$ and wedge constant, defined in (2.6), equal to $P_G$. Suppose that

\hspace{1cm}
i) $R_G > 0$

ii) $P_G > 0$

Then $G$ has a non trivial percolation threshold and

$$p_c(G) < 1 - \frac{1}{2(2\Delta^2)^{1/R_G}e^{1/P_G}}$$

(3.2)

Remark 3. Theorem 5.3 can be directly compared with the results recently obtained by Campari and Cassi in [6], which can be resumed as follows (see there the theorem at pag. 021108-4).

For all graphs with isoperimetric dimension $d_{iso} > 1$ and contours (of cardinality $n$) which are connectable with no more than $ln$ vertices (where $l$ is a finite constant), the number of contours with cardinality $n$ surrounding a fixed vertex $x$ is bounded by $C^n$ with $C$ constant.

As previously discussed the condition $l$ finite is equivalent to require $R_G > 0$, however the first Campari-Cassi condition, i.e., $d_{iso} > 1$, is much stronger than requiring simply $P_G > 0$. Indeed, by definition, the isoperimetric dimension of $G$ is greater than one if and only if there exists an $\varepsilon > 0$ such that $\inf_W |\partial_W W|/|W|^\varepsilon > 0$. This clearly implies $P_G > 0$, since, for all $\varepsilon > 0$, $\log(\text{diam}(W)) \leq |W|^\varepsilon$ as soon as $|W|$ is sufficiently large. As an example, consider the graph $G = (V, E)$ whose vertex set $V$ is the subset of $\mathbb{Z}^2$ given by $V = \{(n_1, n_2) \in \mathbb{Z}^2 : n_1 \geq 0$ and $0 \leq n_2 \leq \ln(1 + n_1)\}$ and whose edge set is formed by the nearest neighbors in $V$. This graph has $d_{iso} = 1$ so it is outside the class of graphs considered in [6] but it has both $P_G > 0$ and $R_G > 0$.

3.2 Two-point finite connectivity in bounded degree graphs

In [19] the authors obtained (see there Theorem 4.1) an upper bound for the two-point connectivity function of the Random Cluster Model with parameters $p \in [0, 1]$ and $q > 0$ on a bounded degree graph $G$, showing that this function decays at least exponentially in $G$ as soon as, for fixed $q$, $p$ is sufficiently close to zero (i.e. in the highly subcritical phase). Since the independent percolation process on a graph $G$ coincides with Random Cluster Model on the same graph with parameter $q = 1$, Theorem 4.1. of [19] immediately implies the following well known result.

Claim. In any graph $G$ with maximum degree $\Delta$, as far as independent percolation with parameter $p$ on $G$ is considered, the two point connectivity function $\phi_p^G(x, y)$ always decays (at least) exponentially, as $d_G(x, y) \to \infty$, if $p$ is sufficiently small.

A proof of this statement specifically for percolation can be found e.g. in [14] (first part of the proof of Theorem 1.10). The argument there is performed for $\mathbb{Z}^d$ but it can be easily generalized for any bounded degree graph. We also mention that this claim has actually been proved to be true in the whole subcritical regime for special classes of graphs, namely, $\mathbb{Z}^d$ and regular lattices (see [1], section 5.3), quasi-transitive graphs (see [2], Theorem 3) and non-amenable graphs (see [21], Theorem 5.3).

In the same paper [19], authors also studied the decay properties of the two-point connectivity function of the Random Cluster Model with parameters $p \in [0, 1]$ and $q > 0$ on $G$ when $p$ is close to 1 (i.e. in the highly supercritical phase). In this case, however, they obtained (Theorem 5.9 of [19]) an upper bound for the connectivity functions of the form ($C_1, C_2$ are constants)

$$C_1 \exp\{-C_2 f_G(x, y)\}$$

(3.3)

where the function $f_G(x, y)$ is the contour distance $f_G(x, y)$ defined in (2.10). Now, it easy to provide examples of graphs for which $f_G(x, y)/d_G(x, y) \to 0$ as $d_G(x, y) \to \infty$. So, being able
get a lower bound of the same form of (3.3), one could raise the question (raised in fact in [19])
whether there are graphs for which the finite connectivity functions decay sub-exponentially, 
even for $p$ arbitrarily close to 1.

In this last section, motivated by the bounds obtained in Theorem 5.9 of [19], we present 
two theorems concerning the decay of the connectivity function for Bernoulli percolation in a 
bounded degree graph $G$ in the supercritical phase.

In the first one, Theorem 3.4 we show that the exponential decay of connectivities also holds 
in the supercritical phase, as $p$ is sufficiently close to one, if one restrict himself to the class of 
graphs satisfying the hypothesis of Theorem 3.7 i.e. graphs with a bi-infinite geodesic and with 
positive contour constant $R_G > 0$.

Our second result, Theorem 3.5 concerns the decay of connectivity functions in bounded 
degree graphs satisfying this time the hypothesis of Theorem 3.3 and essentially confirms the 
possibility raised in [19]. Namely, a sub-exponential decay of the connectivity functions may 
indeed occur in a graph $G$ with positive contour constant even for $p$ arbitrarily close to one (see 
the example below), but only if $G$ has no bi-infinite geodesic.

**Theorem 3.4** Let $G = (V, E)$ a bounded degree graph which satisfies the hypothesis of Theorem 
3.7 i.e. $G$ has a bi-infinite geodesic and $R_G > 0$. Then, as soon as $p \geq \frac{2r}{4p+1}$ and $x, y$ in a
bi-geodesic 

$$
\phi^f_p(x, y) \leq \frac{4}{3} \left[ \frac{r(1-p)}{p} \right]^{R_G d_G(x, y)}
$$

where $r = \left[ 2e\Delta^2 \right]^{1/R_G}$.

**Remark.** The exponential decay of the finite connectivity functions in the supercritical phase 
has been proved more than two decades ago, for the unit cubic lattice $\mathbb{Z}^d$, by Chayes, Chayes, 
Newman [10] (see also [9]). More recently Chen, Peres and Pete [7] have shown that also 
non-amenable graphs (i.e. graphs with $C_G > 0$) have finite connectivity functions decaying 
exponentially. We are not aware of any further generalization of such results in the literature.

**Theorem 3.5** Let $G = (V, E)$ a bounded degree graph which satisfies the hypothesis of Theorem 
3.3 i.e. $G$ is such that $R_G > 0$ and $P_G > 0$. Then, as soon as $p \geq \frac{r}{4p+1}$ and any $x, y \in V$

$$
\frac{1}{3} \left[ (1-p)p^{R_G^{-1}} \right]^{f_G(x, y)} \leq \phi^f_p(x, y) \leq \frac{4}{3} \left[ \frac{\tilde{r}(1-p)}{p} \right]^{f_G(x, y)}
$$

where $\tilde{r} = e^{1/P_G} \left[ 2\Delta^2 \right]^{1/R_G}$.

**Remark.** Theorem 3.5 above shows that in a graph $G$ with $R_G > 0$ and $P_G > 0$, the 
contour distance $f_G(x, y)$ defined in (2.10) controls the decay of the two-point finite connectivity 
function in the supercritical phase in the sense that $\phi^N_p(x, y) \propto (1-p)^{f_G(x, y)}$ as $d_G(x, y) \to \infty$.
Therefore if $G$ is such that $\lim_{d_G(x, y) \to \infty} f_G(x, y)/d_G(x, y) = 0$ the finite connectivity decays 
sub-exponentially. As an example, let us consider the graph $G = (V, E)$ whose vertex set $V$ is 
the subset of $\mathbb{Z}^2$ given by $V = \{(n_1, n_2) \in \mathbb{Z}^2 : n_1 \geq 0$ and $0 \leq n_2 \leq \ln(1+n_1)\}$ and whose 
edge set is formed by the nearest neighbors in $V$. It is easy to see that $R_G > 0$ and $P_G > 0$
and that $f_G((0,0), (n,0)) = \ln(1+n)$ and so, by Theorem 3.5 the connectivity function 
$\phi^N_p((0,0), (n,0))$ on $G$ is bounded below by $\frac{1}{3}(1+n)^{-\alpha_1(p)}$ and bounded above by $\frac{1}{3}(1+n)^{-\alpha_2(p)}$
with $\alpha_1(p) = |\ln[(1-p)p^{R_G^{-1}}]|$ and $\alpha_2(p) = |\ln[\tilde{r}(1-p)p^{-1}]|$. 


4 Proofs

We give here the proofs of the four theorems stated in the previous section. We preliminarily introduce some few notations.

We recall that, if $\gamma$ is a contour in $G$ and $x \in I_\gamma$, then for any ray $\rho = (V_\rho, E_\rho)$ in $G$ starting at $x$ we have that $E_\rho \cap \gamma \neq \emptyset$ (see e.g. Proposition 5.2 in [19]). So, for a fixed $x \in V$, a fixed $\gamma \in F_G(x)$, and a fixed geodesic ray $\rho$ starting at $x$, we define $e_x(\rho, \gamma)$ as the first edge of $E_\rho$, in the natural order of the ray $\rho$, which belongs to $\gamma$ and, for $n \in \mathbb{N}$, $x \in V$ and $\rho$ geodesic ray in $G$, let $r_n(x, \rho)$ be the set of edges of $\rho$ defined as follows.

$$r_n(x, \rho) = \{ e \in E_\rho : \exists \gamma \in F_G^n(x) \text{ such that } e = e_x(\rho, \gamma) \}. \quad (4.1)$$

4.1 Proof of Theorems 3.1 and 3.3

Suppose that $G$ has a bi-infinite geodesic $\delta$. Choose $x$ to be a vertex of $\delta$ and let $\rho$ and $\rho'$ be the two geodesic rays starting in $x$ such that $\delta = \rho \cup \rho'$. Then, since the ray $\rho$ is geodesic, we have $|r_n(x, \rho)| \leq \sup_{\gamma \in F_G^n(x)} d_G(x, e_x(\rho, \gamma))$ and since $\rho \cup \rho'$ is (bi-infinite) geodesic with $\rho$ and $\rho'$ starting at $x$, $d_G(x, e_x(\rho, \gamma)) \leq d_G(\rho, \gamma)$ for any $\gamma \in F_G(x)$. Moreover, if $e, e'$ are any two vertices of $\gamma \in F_G$, we have that $d_G(e, e') \leq d_G(\gamma)$, and, by (2.5), $d_G(\gamma) \leq |\gamma|/R_G$. So in the end we have

$$|r_n(x, \rho)| \leq \sup_{\gamma \in F_G^n(x)} d_G(x, e_x(\rho, \gamma)) \leq \sup_{\gamma \in F_G^n(x)} d_G(e_x(\rho', \gamma), e_x(\rho, \gamma)) \leq \sup_{\gamma \in F_G^n(x)} d_G(\gamma) \leq \frac{n}{R_G} \leq e^{n/R_G}. \quad (4.2)$$

Now observe that

$$|F_G^n(x)| \leq \sum_{e \in r_n(x, \rho)} |F_G^n(e)| \leq |r_n(x, \rho)| \sup_{e \in E} |F_G^n(e)|. \quad (4.3)$$

We estimate $|F_G^n(e)|$, i.e. the number of contours of fixed cardinality $n$ containing a fixed edge $e$. To do this we define a map $\tau$ which associates to each contour $\gamma$ of cardinality $n$ and such that $e \in \gamma$ a tree $\tau(\gamma) \subset G$ with edge set $E_{\tau(\gamma)}$ such that $|E_{\tau(\gamma)}| = d_G^n(\gamma)$. Now, by hypothesis contour $R_G > 0$. This implies that, for any contour $\gamma$ with cardinality $n$, $d_G^n(\gamma) \leq R_G^{-1}n$ and so we also have $|E_{\tau(\gamma)}| \leq R_G^{-1}n$. Moreover, by definition $\gamma \subset E_{\tau(\gamma)}$, and there are at most $(R_G^{-1}n)^2 \leq 2R_G^{-1}n$ ways to choose the set $\gamma$ in $E_{\tau(\gamma)}$. So we get

$$|F_G^n(e)| \leq \sum_{\gamma \text{ tree in } G \atop |E_{\tau(\gamma)}| = R_G^{-1}n, e \in \tau} 2R_G^{-1}n$$

and hence, using Euler’s Theorem and the fact that $G$ has maximum degree $\Delta$,

$$|F_G^n(e)| \leq \left[2\Delta^2 \right]^{n/R_G} \quad (4.4)$$

uniformly in $e \in E$. In conclusion, by (4.2)-(4.4) we get

$$|F_G^n(x)| \leq r^n \quad (4.5)$$
where
\[ r = \left[2e\Delta^2\right]^{1/R_G} \] (4.6)
and thus, by Proposition 2.6, Theorem 3.1 is proved.

Choose now a vertex \( x \in V \). Since \( G \) is connected infinite and bounded degree, there exists a geodesic ray \( \rho \) starting at \( x \). Let \( r_n(x, \rho) \) the subset of edges of \( \rho \) defined in (4.1). Since \( \rho \) is geodesic and since, by hypothesis, \( P_G > 0 \), we have, for any \( \gamma \in \mathcal{F}_G \), that
\[ |\gamma| \geq P_G \log[\text{diam } (I_\gamma)]. \]

So
\[ |r_n(x, \rho)| \leq \sup_{\gamma \in \mathcal{F}_G^n(x)} \text{diam } (I_\gamma) \leq e^{n/P_G}. \]

Since also \( R_G > 0 \) we can use the bound (4.4) together with bound (4.3) previously obtained to conclude that
\[ |\mathcal{F}_G^n(x)| \leq \bar{r}^n \] (4.7)
where now
\[ \bar{r} = e^{1/P_G} \left[2\Delta^2\right]^{1/R_G} \] (4.8)

Theorem 3.3 now follows once again from Proposition 2.6.

4.2 Proof of Theorem 3.4
Throughout this and the next subsections we will denote shortly \( \lambda \equiv \lambda(p) = 1 - p/p \)

We will also make use of the following definition: given an infinite graph \( G = (V, E) \), a sequence \( \{V_N\}_{N \in \mathbb{N}} \) of finite subsets of \( V \) is said to tend monotonically to \( V \), and we write \( V_N \nearrow V \), if, for all \( N \in \mathbb{N}, V_N \) is connected, \( V_N \subset V_{N+1} \), and \( \bigcup_{N \in \mathbb{N}} V_N = V \). We will denote shortly \( G_N = G[V_N] \) and \( E_N = E[V_N] \).

Let \( x, y \) be two vertices belonging to a bi-infinite geodesic of \( G \). We choose a sequence \( \{V_N\}_{N \in \mathbb{N}} \) tending monotonically to \( V \) and suppose \( N \) so large that \( \{x, y\} \in V_N \setminus \partial_{\text{int}}^V V_N \). Then, using the explicit representation (2.11) of the product measure \( P_p \) restricted to \( \Omega_N \), we may define the finite-volume finite connectivity functions
\[ \phi_{p,N}^f(x, y) = \sum_{\omega \in \Omega_N: \exists \text{open cluster } \{x,y\} \subset V_g, \partial g \subset E_N} p^{|O(\omega)|} (1 - p)^{|C(\omega)|} \] (4.9)

So that
\[ \phi_{p}^f(x, y) = \lim_{N \to \infty} \phi_{p,N}^f(x, y) \] (4.10)

hence, by continuity of the product measure \( P_p \), once we obtain an upper bound for \( \phi_{p,N}^f(x, y) \) uniformly in \( N \), the same bound also holds for the “infinite-volume” limit \( \phi_{p}^f(x, y) \).

It is now easy to see that l.h.s. of (4.9) can be rewritten as
\[ \phi_{p}^f, N(x, y) = \frac{1}{Z_{N}(p)} \sum_{\omega \in \Omega_N: \exists \text{open cluster } \{x,y\} \subset V_g, \partial g \subset E_N} \lambda^{|C(\omega)|} \] (4.11)

where

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$$Z_N(p) = \sum_{\omega \in \Omega_N} \lambda^{\lvert C(\omega) \rvert} = p^{-\lvert E_N \rvert}. \quad (4.12)$$

So, a configuration $\omega \in \Omega_N$ is given once we specify the set of closed edges $C(\omega)$ in $E_N$. If $C$ is a set of closed edges in $E_N$ we write $C \cap \{x, y\}$ if there is a contour $\gamma \subset C$ such that $\gamma \cap \{x, y\}$. Then we can write

$$Z_N(p) = \sum_{C \in E_N} \lambda^{\lvert C \rvert} = p^{-\lvert E_N \rvert} \quad (4.13)$$

and

$$\phi^f_N(p, x, y) = \frac{1}{Z_N(p)} \sum_{C \in E_N} \lambda^{\lvert C \rvert}. \quad (4.14)$$

Hence

$$\phi^f_N(x, y) \leq \frac{1}{Z_N(p)} \sum_{\gamma \in \mathcal{F}_d(x, y)} \sum_{C \in E_N \setminus \gamma \cap \{x, y\}} \lambda^{\lvert C \rvert} \leq \sum_{\gamma \in \mathcal{F}_d(x, y)} \lambda^{\lvert C \rvert}. \quad (4.15)$$

We now use the fact that $x, y$ belong to the bi-geodesic of $G$. Such a bi-infinite geodesic can be viewed as the union of two geodesic rays $\rho$ and $\rho'$ both starting at $x$ and such that $y \notin V_\rho$ and $y \in V_{\rho'}$. Fix now a contour $\gamma \cap \{x, y\}$. Let, $e(\rho, \gamma)$ be the first edge of the ray $\rho$ which belongs to $\gamma$ and let $e(\rho', \gamma)$ be the first edge of the ray $\rho'$ which belongs to $\gamma$.

Using the hypothesis that $R_G > 0$, by definition (2.5) we have that

$$\lvert \gamma \rvert \geq R_G \cdot d_G(\gamma) \geq R_G \cdot d_G(e(\rho, \gamma), e(\rho', \gamma)) \geq R_G \cdot d_G(x, y)$$

and so, using also the bound (4.5), we obtain

$$\phi^f_N(x, y) \leq \sum_{n \geq R_G d_G(x, y)} \sum_{\gamma \in \mathcal{F}_d^n(x, y)} \lambda^{n \lvert \gamma \rvert} \leq \sum_{n \geq R_G d_G(x, y)} \lambda^n \lvert \mathcal{F}_d^n(x, y) \rvert \leq \sum_{n \geq R_G d_G(x, y)} (\lambda R_G d_G(x, y))^n \leq \frac{(\lambda R_G d_G(x, y))^n}{1 - \lambda R_G d_G(x, y)} \quad (4.15)$$

So we get, uniformly in $N$

$$\phi^f_N(x, y) \leq \frac{4}{3} (r \lambda)^{R_G d_G(x, y)}, \quad \text{for } \lambda \leq \frac{1}{4r}$$

which completes the proof of Theorem 3.3

### 4.3 Proof of Theorem 3.5

To obtain the upper and lower bounds for the finite connectivity, we work again at finite volume.

We first obtain the upper bound which is easier. Proceeding analogously as we did in the proof of Theorem 3.4 we rewrite $\phi^f_N(x, y)$ as the ratio (4.14). Then as above

$$\phi^f_N(x, y) \leq \frac{1}{Z_N(p)} \sum_{\gamma \in \mathcal{F}_d(x, y)} \sum_{C \in E_N \setminus \gamma \cap \{x, y\}} \lambda^{\lvert C \rvert} \leq \sum_{\gamma \in \mathcal{F}_d(x, y)} \lambda^{\lvert \gamma \rvert}. \quad (4.15)$$
Then, summing only over these configurations, we get the lower bound
\[ \phi_p^{f,N}(x, y) \leq \sum_{n \geq f_0(x, y)} \sum_{\gamma \in \mathcal{F}^0_n(x, y)} \lambda^{\gamma} \leq \frac{4}{\bar{r}}(\bar{r}) \lambda f_0(x, y). \]

We now prove the lower bound of \( \phi_p^{f,N}(x, y) \). Let \( \gamma_0 \) be a minimum contour such that \( \gamma_0 \cup \{x, y\} \), that is, \( |\gamma_0| = f_0(x, y) \), and \( \gamma_0 \subset E_N \) (we can always suppose \( N \) sufficiently large to include that contour). We recall that by definition of contour, the set \( E_N \setminus \gamma_0 \) is partitioned in two disjoint sets \( E_{\gamma_0} \) (the edge interior of \( \gamma_0 \)) and \( E_N \setminus (\gamma_0 \cup E_{\gamma_0}) \) (the edge exterior of \( \gamma_{xy} \)) with \( G[I_{\gamma_0}] = (I_{\gamma_0}, E_{\gamma_0}) \) being a connected graph. Let \( \tau_0 \subset E_{\gamma_0} \) be a minimal tree in \( E_{\gamma_0} \) connecting the contour \( \gamma_0 \). By assumption (recall (2.5)) we have that \( |\tau_0| \leq R^{-1}\gamma_0 |\)

Now, among all configurations \( C \) of closed edges such that \( C \cup \{x, y\} \) there are those for which \( C \supset \gamma_0 \) and no subset of \( C \) can separate \( x, y \), and \( C \cap \tau_0 = \emptyset \) (i.e., all edges of \( \tau_0 \) are open). Then, summing only over these configurations, we get the lower bound
\[ \phi_p^{f,N}(x, y) \geq \frac{\lambda^{\gamma_0}}{Z_N(p)} \sum_{C \subset E_N \setminus \gamma_0 \subset \gamma_0 \cup \{x, y\} \subset \gamma_0} \lambda^{\gamma}. \]

Now, since \( E_N \setminus \gamma_0 \) is the disjoint union of \( E_N \setminus (\gamma_0 \cup E_{\gamma_0}) \) and \( E_{\gamma_0} \), and observing that there is no restriction over the sum of closed edges in \( E_N \setminus (\gamma_0 \cup E_{\gamma_0}) \) we have that
\[ \sum_{C \subset E_N \setminus \gamma_0 \subset \gamma_0 \cup \{x, y\}} \lambda^{\gamma} = Z_{E_N \setminus (\gamma_0 \cup E_{\gamma_0})} \sum_{C \subset E_{\gamma_0} \subset \gamma_0 \cup \{x, y\}} \lambda^{\gamma} = \]
\[ = p^{|\gamma_0|} \frac{Z_N}{Z_{E_{\gamma_0}}} \sum_{C \subset E_{\gamma_0} \subset \gamma_0 \cup \{x, y\}} \lambda^{\gamma} = p^{|\gamma_0|} \frac{Z_N}{Z_{E_{\gamma_0}}} Z_{E_{\gamma_0}} \sum_{C \subset E_{\gamma_0} \setminus \tau_0 \subset \gamma_0 \cup \{x, y\}} \lambda^{\gamma} = \]
\[ = p^{|\gamma_0|} \frac{Z_N}{Z_{E_{\gamma_0}}} \left[ Z_{E_{\gamma_0} \setminus \tau_0} - \sum_{C \subset E_{\gamma_0} \setminus \tau_0 \subset \gamma_0 \cup \{x, y\}} \lambda^{\gamma} \right] \]

where \( C \cup \{x, y\} \) means that \( C \) contains some contour \( \gamma \) such that \( \gamma \cup \{x, y\} \).

Hence we get
\[ \phi_p^{f,N}(x, y) \geq \frac{(\lambda p)^{|\gamma_0|}}{Z_{E_{\gamma_0}}} \left[ Z_{E_{\gamma_0} \setminus \tau_0} - \sum_{C \subset E_{\gamma_0} \setminus \tau_0 \subset \gamma_0 \cup \{x, y\}} \lambda^{\gamma} \right] = \]
\[ = (\lambda p)^{|\gamma_0|} p^{|\tau_0|} \left[ 1 - \frac{1}{Z_{E_{\gamma_0} \setminus \tau_0}} \sum_{C \subset E_{\gamma_0} \setminus \tau_0 \subset \gamma_0 \cup \{x, y\}} \lambda^{\gamma} \right] \geq \]
\[ \geq \left[ \lambda p^{1+R^{-1}} \right]^{|\gamma_0|} (1 - K_{\lambda}) \]

where
\[ K_{\lambda} = \frac{1}{Z_{E_{\gamma_0} \setminus \tau_0}} \sum_{C \subset E_{\gamma_0} \setminus \tau_0 \subset \gamma_0 \cup \{x, y\}} \lambda^{\gamma} \]
Now it is easy to get an upper bound for $K_\lambda$. Indeed, since $\tau_0$ is open in $E_{\gamma_0}$ and connects the boundary $\gamma_0$, a configuration $C$ of closed edges can separate $\{x, y\}$ only if there is at least a contour surrounding either $x$ or $y$. Hence

$$K_\lambda \leq \frac{1}{Z_{E_{\gamma_0}} \setminus \tau_0} \sum_{\gamma \in F_G \cap \gamma(x,y)} \lambda^{\vert \gamma \vert} \sum_{C \subset E_{\gamma_0} \setminus \tau_0} \lambda^{\vert C \vert} \leq \sum_{\gamma \in F_G \cap \gamma(x,y)} \lambda^{\vert \gamma \vert} \leq 2 \sup_{x \in V} \sum_{\gamma \in F_G(x)} \lambda^{\vert \gamma \vert} \leq 2 \sum_{n \geq 1} \lambda^n \sup_{x \in V} \vert F_G^n(x) \vert$$

Now, since $R_G > 0$ and $P_G > 0$ we can use the bound (4.7) to get

$$K_\lambda \leq 2 \sum_{n \geq 1} (\bar{r}\lambda)^n = 2 \frac{\bar{r}\lambda}{1 - \bar{r}\lambda}$$

and so $K_\lambda \leq \frac{2}{3}$ as soon as $\lambda < \frac{1}{4\bar{r}}$. In conclusion, we have obtained, again uniformly in $N$,

$$\phi_p^{t,N} (\{x, y\}) \geq \frac{1}{3} \left[ \lambda^{p^{1+R_G^{-1}}} f_G(x,y) \right]$$

as soon as $\lambda < \frac{1}{4\bar{r}}$.

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