The properties of strange quark matter under strong rotation

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We investigate the rotating quark matter in the three-flavor Nambu and Jona-Lasinio (NJL) model. The chiral condensation, spin polarization and number susceptibility of the light and strange quarks are carefully studied at finite temperature without or with finite chemical potential in this model. We find that the rotation suppresses the chiral condensation and enhances the first-order quark spin polarization, however for the second-order quark spin polarization and quark number susceptibility the effect is complicated and interesting. When extending to the situation with finite chemical potential, we find the angular velocity also plays a crucial role, at small angular velocity the chemical potential enhances the susceptibility, however in the middle region of angular velocity the effect of the chemical potential is suppressed by the angular velocity and susceptibility can be changed considerably, it can be observed that at very low temperature in the presence of quark chemical potential the quark number susceptibility has two maxima with increasing angular velocity. Furthermore, it is found that at sufficiently large angular velocity the contributions played by light quark and strange quark to these phenomena are almost equal. we also explored the phase diagram in the $T$-$\omega$ plane, we observe that there exist first order phase transitions for the rotating system and the first order phase transition lines move toward a higher temperature for decreasing angular velocity. It is also found that the different chemical potentials change the boundary of phase diagram, and that a larger chemical potential shifts down the critical temperature. We expect these studies to be used to understand the chiral symmetry breaking and restoration as well as probe the QCD phase transition.

I. INTRODUCTION

QCD thermodynamics has always been the subject of intense investigations for many years, which motivates various works to try to understand it better, and we know many about the equation of state of strongly interacting matter as a function of temperature $T$ and in a limited range of quark chemical potential $\mu$. Recently, QCD matter under rotation is of particular interest, there exist some interesting phenomena in rotating QCD matter, such as chiral vortical effect or chiral vortical wave [1–4], which is a key ingredient in theories that predict observable effects associated with chiral symmetry restoration and the production of false QCD vacuum states [5]. Many works can be investigated in various rotation-related phenomena, such as noncentral heavy-ion collisions in high energy nuclear physics [6–14], the mesonic condensation of isospin matter with rotation in hadron physics [15], the trapped non-relativistic bosonic cold atoms in condensed matter physics [16–20], the rapidly spinning neutron stars in astrophysics [21–29]. Quark matter under rotation has been studied in ultra-relativistic heavy-ion off-central collisions performed at Relativistic Heavy Ion Collider (RHIC) or Large Hadron Collider (LHC) as well as lattice simulation. It is known that for the region with very low temperature and very large chemical potential there exists uncertainty in lattice QCD due to the “sign problem", Ref. [30] has calculate the angular momenta of gluons and quarks in the rotating QCD vacuum, which would be very important for future theoretical research.

One interesting phenomenon is the quark spin polarization in noncentral collisions of heavy ions, where quark-gluon plasma (QGP) can be generated. And the global quark polarization could occur in the QGP due to the large angular momentum carried by two colliding nuclei, spin-orbit coupling can generate a spin alignment (polarization) along the direction of the system angular momentum. This polarization provides very valuable information about the QGP properties and can be measured experimentally with hyperons via parity-violating weak decays [31–42]. Experimental measurements of the $\Lambda$ hyperon polarization have been investigated at RHIC and LHC, which would be very helpful in the study of the hottest, least viscous and most vortical-fluid ever produced both for theoretical physics and experimental physics. Recently, the global spin polarizations of $\Lambda$ and $\bar{\Lambda}$ have been measured by the STAR collaboration in Au+Au collisions over a wide range of beam energies $\sqrt{s_{NN}} = 7.7 - 200$ GeV and by ALICE collaboration in Pb+Pb collisions at 2.76 TeV and 5.02 TeV [43–45]. On the other hand, theoretical research of spin polarization in the quark matter has been explored [46–53], which plays an important role to explain the origin of strong magnetic field in the magnetar as well as in changing the dynamical mass and some other phenomenon related to the chiral symmetric phase transition. It would be very interesting to take into account the influence of the rotating effect to the quark spin polarization, especially, in the case of $s$ quark matter under rotation. The study of quark spin polarization which is linked to the vorticity may help us understand the vortical nature of QGP and the chiral dynamics of the system.

Another interesting phenomenon in non-central collisions of heavy ions is the fluctuations and correlations of conserved charges as quantified by the corresponding susceptibilities, which are sensitive observable quantity in relativistic heavy-ion collisions and also considered as a useful probe for QGP.

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In this paper we mainly focus on the baryon number fluctuation which is also simply related the quark number susceptibility (QNS). QNS serves as a signature for the QGP formation in ultra relativistic heavy-ion collision and also plays an important factor to probe that the QCD phase transition as well as the equation of state (EOS) of strongly interacting matter [65–67]. Experimentally, various cumulants of net-kaon multiplicity distributions of Au+Au collisions at √sNN = 7.7 − 200 GeV have been measured by the STAR experiment [68–70], which are related to the thermodynamic susceptibilities. The study of QNS in lattice QCD has been interesting [71–75]. Although susceptibilities have been studied in the past [76–79], it has not been checked what is the influence of considering the contribution from the strange quark matter [80–91] in the rotation system.

In this paper, encouraged by the successful description of two-flavor QCD under rotation [92] in the Nambu-Jona-Lasinio (NJL) model, which embodies the spontaneous breaking of chiral symmetry via effective interactions between quarks, we further study the three-flavor NJL model in the framework of quark under rotation. Since there are several flavors and colors of quarks, several pairings are possible, which probably lead to a great variety of interesting phenomena. And the questions we are going to address are how the chiral condensate, quark spin polarization and quark number susceptibility are influenced by the rotation.

Our work is organized as follows. We first discuss the formalism of three-flavor NJL model in the presence of rotation in Section II, by using mean-field approach and the finite temperature field methods we obtain the grand potential of the fermions with rotation, and the corresponding analytical expressions for the gap equation, spin polarization and susceptibility of the quarks are given. Section III presents numerical results and discussions in detail. Section IV summarizes and concludes the paper. A brief description of fermions under rotation is given in Appendix A, which discuss the complete set of commuting operators in cylindrical coordinates and derive the eigenstates of these operators.

II. FORMALISM

We start from the NJL model, the simplest three-flavor NJL Lagrangian for fermions without rotation is given by [112]

\[ \mathcal{L} = \bar{\psi} (i \partial_{\mu} \gamma^{\mu} - m) \psi + G \sum_{a=0}^{8} (\bar{\psi} \lambda^a \psi)^2 + \mathcal{L}_{\text{det}}, \]

(1)

here, \( \psi \) is the quark field, \( m \) is the bare quark mass matrix, \( \lambda^a (a = 1, \ldots, 8) \) are the Gell-Mann matrices in flavor space with \( \lambda^0 = \sqrt{\frac{2}{3}} I \) where \( I \) is the unit matrix in the three-flavor space, and \( \mathcal{L}_{\text{det}} \) are given by

\[ \mathcal{L}_{\text{det}} = -K \{ \det[\bar{\psi}(1 + \gamma^5)\psi] + \det[\bar{\psi}(1 - \gamma^5)\psi] \}, \]

(2)

which is known as six-quark ’t Hooft term, here the det runs over the flavor degrees of freedom and indicates the favors become connected.

Using the mean field approximation which means that fluctuations are assumed to be small, the chiral condensates \( \langle \bar{\psi}_i \psi_j \rangle \) with \( i \neq j \) vanishes since it is assumed that flavor is conserved and after expanding the operators around their expectation values and neglecting the higher order fluctuations, we obtain

\[ \langle \bar{\psi}_i \psi_j \rangle^2 = -\langle \bar{\psi}_i \psi_j \rangle^2 + 2 \langle \bar{\psi}_i \psi_j \rangle \bar{\psi}_i \psi_j, \quad (i = j), \]

(3)

and the Lagrangian can reads

\[ \mathcal{L} = \bar{\psi} (i \partial_{\mu} \gamma^{\mu} - M) \psi - 2G \left( \langle \bar{u}u \rangle^2 + \langle \bar{d}d \rangle^2 + \langle \bar{s}s \rangle^2 \right) + 4K \langle \bar{u}u \rangle \langle \bar{d}d \rangle \langle \bar{s}s \rangle, \]

(4)

where we have defined the dynamical quark mass \( M \) as follows

\[ M_{fi} = m_{fi} - 4G \langle \bar{\psi}_{fi} \psi_{fi} \rangle + 2K \langle \bar{\psi}_{fi} \psi_{fi} \rangle \langle \bar{\psi}_{f_k} \psi_{f_k} \rangle \quad (i \neq j \neq k), \]

(5)

where \( i, j, k \) denote the flavor of the quarks and take \( u, d \) for two light flavors while \( s \) for strange quark.

The condensation of the QCD matter under the presence of rotation in the 2-flavor NJL model has been investigated in Ref. [92], which exhibit interesting behavior for the paring phenomena. In the present work we will extend to study the properties of the quark matter under rotation in 3-flavor NJL model with finite quark chemical potential. And the Lagrangian for spinor with rotation can be written in the following way:

\[ \mathcal{L} = \sum_{fi} \bar{\psi}_{fi} \left( i \gamma^{\mu} (\partial_{\mu} + \Gamma_{\mu}) - m + \gamma^0 \mu \right) \psi_{fi} + G \sum_{a=0}^{8} (\bar{\psi} \lambda^a \psi)^2 + \mathcal{L}_{\text{det}}, \]

(6)

here \( \gamma^{\mu} = e_{\alpha}^{\mu} \gamma^a \) with \( e_{\alpha}^{\mu} \) the tetrads for spinors, \( \Gamma_{\mu} = \frac{1}{4} \times \frac{1}{2}[\gamma^a, \gamma^b] \Gamma_{ab\mu} \) which is the spinor connection, where \( \Gamma_{ab\mu} = \gamma_{abc} (\epsilon^{\mu}_{\nu} G^{\nu}_{\alpha\beta} \epsilon_{\beta\nu} - \epsilon^{\nu}_{\mu} \partial_{\nu} \epsilon^{\nu}_{\alpha\beta}) \), and \( G^{\mu}_{\alpha\beta} \) is the affine connection determined by \( g^{\mu\nu} \). Considering the system with an angular velocity along the fixed z-axis, then \( \vec{\omega} = \vec{z} \times \vec{x} \) and choosing \( \epsilon^{\mu}_{\nu} = \delta^{\mu}_{\nu} + \delta^{\nu}_{\mu} \) \( v_i \) and \( \epsilon^{\mu}_{\nu} = \delta^{\mu}_{\nu} - \delta^{\nu}_{\mu} \) \( v_i \) (details can be found in Ref.), then we can expand the Lagrangian to the first order of angular velocity, finally, the Lagrangian is given by

\[ \mathcal{L} = \bar{\psi} \left[ i \gamma^{\mu} \partial_{\mu} - m + \gamma^0 \mu + (\gamma^0)^{-1} \left( (\vec{\omega} \times \vec{x}) \cdot (-i \bar{\partial}) + \bar{\omega} \cdot \vec{S} \right) \right] \psi + G \sum_{a=0}^{8} (\bar{\psi} \lambda^a \psi)^2 + \mathcal{L}_{\text{det}}, \]

(7)
where we can see as a result of rotation the Dirac operator includes the orbit-rotation coupling term and the spin-rotation coupling term.

The general definition of the partition function can be written as

$$Z = \int D[\psi]D[\bar{\psi}]e^{iS},$$  \hspace{1cm} (8)

here, $S$ denotes the quark action, which is the integration of the Lagrangian density $\mathcal{L}$. After by using the mean field approximation and carrying out the general approach of the path integral formulation for Grassmann variables, we are now able to exactly integrate out the fermionic fields and obtain

$$\log Z = -\frac{1}{T} \int d^3x \left( -2G \left( \langle uu \rangle^2 + \langle dd \rangle^2 + \langle ss \rangle^2 \right) + 4K \langle uu \rangle \langle dd \rangle \langle ss \rangle \right) + \sum_f \log \det \frac{D_f^{-1}}{T},$$  \hspace{1cm} (9)

here

$$D_f^{-1} = \gamma^0 \left( -i \omega_l + \left( n + \frac{1}{2} \right) \omega + \mu \right) - M - \gamma \cdot \vec{p},$$  \hspace{1cm} (10)

which is the inverse of propagator for quarks, and

$$\log \det \frac{D^{-1}}{T} = \text{tr} \log \frac{D^{-1}}{T} = \int d^3x \int \frac{d^3p}{(2\pi)^3} \left\langle \psi_p(x) \right| \log D^{-1} \left| \psi_p(x) \right\rangle.$$  \hspace{1cm} (11)

The Dirac fields can be defined in the terms of the wave functions $u(x), v(x)$

$$\psi_p(x) = \sum_{E,n,s,p} \left( u(x) + v(x) \right).$$  \hspace{1cm} (12)

In order to find solutions of the Dirac equation, we should firstly choose the complete set of commutating operators consists of $\hat{H}$ which can be obtained from three-flavor NJL Lagrangian, the momentum in the $z$-direction $\hat{p}_z$, the square of transverse moment $\vec{p}_t^2$, the $z$-component of the total angular momentum $\hat{J}_z$ and the transverse helicity $\hat{h}_t$. The positive and negative energy solutions of the Dirac field can be determined by calculating these eigenvalue equations of the complete set of commutating operators $\{\hat{H}, \hat{p}_z, \vec{p}_t^2, \hat{J}_z, \hat{h}_t\}$, here, for the derivation and detailed expression of the spinor $u, v$ can refer to the Appendix A, by substituting Eq. (12) to Eq. (11) we have

$$\log \det \frac{\hat{D}^{-1}}{T} = \sum_{E,n,s,p} \text{tr} \log \frac{D^{-1}_{u(x)}}{T} \int d^3x \int \frac{d^3p}{(2\pi)^3} \left\langle \langle u(x) | u(x) \rangle \right\rangle$$

$$+ \sum_{E,n,s,p} \text{tr} \log \frac{D^{-1}_{v(x)}}{T} \int d^3x \int \frac{d^3p}{(2\pi)^3} \left\langle \langle v(x) | v(x) \rangle \right\rangle,$$  \hspace{1cm} (13)

here, the concrete form of the $D^{-1}_{u(x)}$ that has been considered the rotation and nonzero chemical potential is

$$D^{-1}_{u(x)} = \begin{pmatrix}
-\iota \omega_l + \left( n + \frac{1}{2} \right) \omega + \mu & \hat{\sigma} \cdot \vec{p} \\
\hat{\sigma} \cdot \vec{p} & \left( \iota \omega_l + \left( n + \frac{1}{2} \right) \omega + \mu \right)
\end{pmatrix} - M$$

$$- \left( \iota \omega_l + \left( n + \frac{1}{2} \right) \omega + \mu \right) - M,$$  \hspace{1cm} (15)

which corresponding to the positive energy solution, and the concrete form for the $D^{-1}_{v(x)}$ is

$$D^{-1}_{v(x)} = \begin{pmatrix}
\iota \omega_l - \left( n + \frac{1}{2} \right) \omega + \mu & \hat{\sigma} \cdot \vec{p} \\
\hat{\sigma} \cdot \vec{p} & \left( \iota \omega_l - \left( n + \frac{1}{2} \right) \omega + \mu \right)
\end{pmatrix} - M$$

$$- \left( \iota \omega_l - \left( n + \frac{1}{2} \right) \omega + \mu \right) - M,$$  \hspace{1cm} (16)

which corresponding to the negative energy solution. Here, in order to study the rotating system at finite density, we have introduced quark chemical potential $\mu$ and note that the term $\left( n + \frac{1}{2} \right) \omega$ in above expressions denotes the rotational polarization energy, which are very useful when we study the polarization in the following sections. By using the general methods in the finite temperature fields [94], we obtain

$$\log \det \frac{\hat{D}^{-1}}{T} = \beta \left( \sqrt{M^2 + p_t^2 + p_z^2} + \left( n + \frac{1}{2} \right) \omega \right)$$

$$+ \log \left( e^{\beta \left( \sqrt{M^2 + p_t^2 + p_z^2} + \left( n + \frac{1}{2} \right) \omega \pm \mu \right)} + 1 \right)$$

$$+ \log \left( e^{\beta \left( \sqrt{M^2 + p_t^2 + p_z^2} + \left( n + \frac{1}{2} \right) \omega + \mu \right)} + 1 \right),$$  \hspace{1cm} (17)
and

\[
\log \det \frac{D^{-1}}{T} = \beta \left( -\sqrt{M^2 + p_t^2 + p_z^2} - (n + \frac{1}{2}) \omega \right) \\
+ \log \left( e^{\beta \left( -\sqrt{M^2 + p_t^2 + p_z^2} - (n + \frac{1}{2}) \omega - \mu \right)} + 1 \right) \\
+ \log \left( e^{\beta \left( -\sqrt{M^2 + p_t^2 + p_z^2} - (n + \frac{1}{2}) \omega + \mu \right)} + 1 \right),
\]

(18)

for each flavor, here \( \beta \) is the inverse temperature and the inner products of \( \langle u_{n,s} | u_{n,s} \rangle \), \( \langle v_{n,s} | v_{n,s} \rangle \) are derived with very simple expressions as follows,

\[
\int \frac{d^3p}{(2\pi)^3} \langle u_{n,s} | u_{n,s} \rangle = \frac{1}{2} \left( J_n (p_t r)^2 + J_{n+1} (p_t r)^2 \right),
\]

(19)

\[
\int \frac{d^3p}{(2\pi)^3} \langle v_{n,s} | v_{n,s} \rangle = \frac{1}{2} \left( J_n (p_t r)^2 + J_{n+1} (p_t r)^2 \right).
\]

(20)

Combining the Eqs. (9), (17), (18), (19) and (20) one could now derive the expression of the grand potential for strange quark when the momentum summation turns into the integration

\[
\sum_p \rightarrow V \int \frac{d^3p}{(2\pi)^3},
\]

(21)

and the energy summation performs over Matsubara frequency. Then the thermodynamic grand potential \( \Omega = -\frac{T}{V} \log Z \) has the following expression,

\[
\Omega = 2G (2\langle \bar{u}u \rangle^2 + \langle \bar{s}s \rangle^2) - 4K \langle \bar{u}u \rangle \langle \bar{s}s \rangle \\
- \frac{N_f N_c}{8\pi^2} \sum_{n=-\infty}^{\infty} \int_0^\Lambda p_t dp_t \int \frac{d^3p}{(2\pi)^3} \left( \left( J_{n+1} (p_t r)^2 + J_n (p_t r)^2 \right) \times \\
T \left\{ \log \left( e^{-\mu_u + \sqrt{M^2 + p_t^2 + p_z^2} - (n + \frac{1}{2}) \omega} + 1 \right) + \log \left( e^{\mu_u + \sqrt{M^2 + p_t^2 + p_z^2} - (n + \frac{1}{2}) \omega} + 1 \right) \\
+ \log \left( e^{-\mu_u + \sqrt{M^2 + p_t^2 + p_z^2} - (n + \frac{1}{2}) \omega} + 1 \right) + \log \left( e^{\mu_u + \sqrt{M^2 + p_t^2 + p_z^2} - (n + \frac{1}{2}) \omega} + 1 \right) \right\},
\]

(22)

\[
- \frac{N_f N_c}{8\pi^2} \sum_{n=-\infty}^{\infty} \int_0^\Lambda p_t dp_t \int \frac{d^3p}{(2\pi)^3} \left( \left( J_{n+1} (p_t r)^2 + J_n (p_t r)^2 \right) \times \\
T \left\{ \log \left( e^{-\mu_u + \sqrt{M^2 + p_t^2 + p_z^2} - (n + \frac{1}{2}) \omega} + 1 \right) + \log \left( e^{\mu_u + \sqrt{M^2 + p_t^2 + p_z^2} - (n + \frac{1}{2}) \omega} + 1 \right) \\
+ \log \left( e^{-\mu_u + \sqrt{M^2 + p_t^2 + p_z^2} - (n + \frac{1}{2}) \omega} + 1 \right) + \log \left( e^{\mu_u + \sqrt{M^2 + p_t^2 + p_z^2} - (n + \frac{1}{2}) \omega} + 1 \right) \right\}.
\]

Here, the isospin symmetry has been considered for the light quarks, so \( m_d = m_u, \mu_d = \mu_u, N_f = 2, N_c = 3 \) and \( \Lambda \) is the three-momentum. And \( \mu_u \) is the chemical potential for the \( u \) or \( d \) quark as well as \( \mu_s \) is that for \( s \) quark. When the isospin symmetry is satisfied, the dynamical quark masses are simplified to give:

\[
M_u = m_u + (2K \langle \bar{s}s \rangle - 4G \langle \bar{u}u \rangle),
\]

(23)

\[
M_s = m_s - 4G \langle \bar{s}s \rangle + 2K \langle \bar{u}u \rangle^2.
\]

(24)
We have discussed the evaluating of grand potential of quark under rotation in detail in the previous section. In this section, we list our final analytical expressions of the gap equation, quark spin polarization and quark number susceptibility in the situation with rotation. Firstly, we consider the gap equation which will be required to minimize the grand potential, the values are determined by solving the stationary condition, namely, \( \frac{\partial \Omega}{\partial (\bar{u}u)} = \frac{\partial \Omega}{\partial (ss)} = 0 \), and the detailed expressions for the stationary condition are listed as follows,

\[
8G \langle \bar{u}u \rangle - 8K \langle \bar{s}s \rangle - \frac{N_c}{8\pi^2} \sum_{n=-\infty}^{\infty} \int_0^1 p_t dp_t \int \frac{\sqrt{N^2-p_t^2}}{N^2-p_t^2} dp_z \left( J_{n+1}(p_t r)^2 + J_n(p_t r)^2 \right) \times
\]

\[
\begin{cases}
4Nf e \mu_p \left( e^{2\mu + \omega} - e^{2\sqrt{M_e^2 + p_t^2 + p_z^2}} \right) (2G - K(\bar{s}s)) M_u \\
\sqrt{M_e^2 + p_t^2 + p_z^2} \left( e^{\frac{\sqrt{M_e^2 + p_t^2 + p_z^2}}{2} + \frac{2\mu + \omega}{2}} + e^{\frac{\sqrt{M_e^2 + p_t^2 + p_z^2}}{2} + \frac{\mu - \omega}{2}} \right) \\
8Ke \mu_p \left( e^{2\mu + \omega} - e^{2\sqrt{M_e^2 + p_t^2 + p_z^2}} \right) (\bar{u}u) M_s \\
\sqrt{M_e^2 + p_t^2 + p_z^2} \left( e^{\frac{\sqrt{M_e^2 + p_t^2 + p_z^2}}{2} + \frac{2\mu + \omega}{2}} + e^{\frac{\sqrt{M_e^2 + p_t^2 + p_z^2}}{2} + \frac{\mu - \omega}{2}} \right)
\end{cases}
= 0,
\]

(25)

\[
4G \langle \bar{s}s \rangle - 4K\langle \bar{u}u \rangle^2 - \frac{N_c}{8\pi^2} \sum_{n=-\infty}^{\infty} \int_0^1 p_t dp_t \int \frac{\sqrt{N^2-p_t^2}}{N^2-p_t^2} dp_z \left( J_{n+1}(p_t r)^2 + J_n(p_t r)^2 \right) \times
\]

\[
\begin{cases}
-4KNf e \mu_p \left( e^{2\mu + \omega} - e^{2\sqrt{M_e^2 + p_t^2 + p_z^2}} \right) (\bar{u}u) M_u \\
\sqrt{M_e^2 + p_t^2 + p_z^2} \left( e^{\frac{\sqrt{M_e^2 + p_t^2 + p_z^2}}{2} + \frac{2\mu + \omega}{2}} + e^{\frac{\sqrt{M_e^2 + p_t^2 + p_z^2}}{2} + \frac{\mu - \omega}{2}} \right) \\
8Ke \mu_p \left( e^{2\mu + \omega} - e^{2\sqrt{M_e^2 + p_t^2 + p_z^2}} \right) M_s \\
\sqrt{M_e^2 + p_t^2 + p_z^2} \left( e^{\frac{\sqrt{M_e^2 + p_t^2 + p_z^2}}{2} + \frac{2\mu + \omega}{2}} + e^{\frac{\sqrt{M_e^2 + p_t^2 + p_z^2}}{2} + \frac{\mu - \omega}{2}} \right)
\end{cases}
= 0.
\]

(26)

Here we are going to study the quark spin polarization which can be defined as taking the partial derivative of minus grand potential with respect to angular velocity, and we introduce the following quark spin polarization as in Ref. [95]

\[
\mathcal{P}_1 = \frac{\partial (\bar{u}u)}{\partial (\frac{\omega}{\tau})} ,
\]

(27)

\[
\mathcal{P}_2 = \frac{\partial^2 (\bar{u}u)}{\partial (\frac{\omega}{\tau})^2} ,
\]

(28)

such definition ensures dimensionless polarization, then we list the detailed expression of the first-order polarization and second-order polarization as follows,
\[ P_1 = \frac{N_c}{8\pi^2 T^3} \sum_{n=-\infty}^{\infty} \int_0^\Lambda p^d p_t f^\Lambda \sqrt{\frac{A^2 - p_t^2}{A^2 - p^d_t}} \, dp_z \left( \left( J_{n+1}(p_tr)^2 + J_n(p_tr)^2 \right) (2n + 1) \times \right. \\
\left. \vphantom{\int} \begin{array}{c}
\frac{N_f \sinh \left( -\frac{2\sqrt{M_t^2 + p_t^2 + p_r^2 + 2n\omega + \omega}}{2T} \right)}{\cosh \left( -\frac{2\sqrt{M_t^2 + p_t^2 + p_r^2 + 2n\omega + \omega}}{2T} \right) + \cosh \left( \frac{4\pi}{T} \right)} + \\
\frac{\sinh \left( -\frac{2\sqrt{M_t^2 + p_t^2 + p_r^2 + 2n\omega + \omega}}{2T} \right)}{\cosh \left( -\frac{2\sqrt{M_t^2 + p_t^2 + p_r^2 + 2n\omega + \omega}}{2T} \right) + \cosh \left( \frac{4\pi}{T} \right)} \vphantom{\int} \end{array} \right), \tag{29} \]

\[ P_2 = \frac{N_c}{64\pi^2 T^3} \sum_{n=-\infty}^{\infty} \int_0^\Lambda p^d p_t f^\Lambda \sqrt{\frac{A^2 - p_t^2}{A^2 - p^d_t}} \, dp_z \left( \left( J_{n+1}(p_tr)^2 + J_n(p_tr)^2 \right) (2n + 1)^2 \times \right. \\
\left. \vphantom{\int} \begin{array}{c}
\begin{array}{c}
N_f \left( \text{sech}^2 \left( \frac{2\mu_r - 2\sqrt{M_t^2 + p_t^2 + p_r^2 + 2n\omega + \omega}}{4T} \right) + \text{sech}^2 \left( \frac{-2\mu_r - 2\sqrt{M_t^2 + p_t^2 + p_r^2 + 2n\omega + \omega}}{4T} \right) \right) \\
+ \left( \text{sech}^2 \left( \frac{2\mu_r - 2\sqrt{M_t^2 + p_t^2 + p_r^2 + 2n\omega + \omega}}{4T} \right) + \text{sech}^2 \left( \frac{-2\mu_r - 2\sqrt{M_t^2 + p_t^2 + p_r^2 + 2n\omega + \omega}}{4T} \right) \right) \vphantom{\int} \end{array} \right). \tag{30} \]

Next, we will investigate how much the rotation affects the baryon number fluctuations, these fluctuations can be quantified by the susceptibilities and the QNS is defined through the Taylor expansion coefficients of the pressure over the chemical potential [96–102]:

\[ \chi_n = \frac{\partial^n (P/T^4)}{\partial (\mu/T)^n}, \tag{31} \]

here we focus on the second order derivative of pressure with respect to \( \mu \), due to symmetry all the odd susceptibilities vanishes when \( \mu \to 0 \) (note, in the context of lattice calculations the susceptibilities are often defined as dimensionless quantities), using the relation of pressure \( P = -\Omega \), the actual calculation of the susceptibilities is straightforward and the detailed result is

\[ \chi_2^f = \frac{N_c}{16\pi^2 T^3} \sum_{n=-\infty}^{\infty} \int_0^\Lambda p^d p_t f^\Lambda \sqrt{\frac{A^2 - p_t^2}{A^2 - p^d_t}} \, dp_z \left( \left( J_{n+1}(p_tr)^2 + J_n(p_tr)^2 \right) \times \right. \\
\left. \vphantom{\int} \left( \text{sech}^2 \left( \frac{2\mu_r - 2\sqrt{M_t^2 + p_t^2 + p_r^2 + 2n\omega + \omega}}{4T} \right) + \text{sech}^2 \left( \frac{-2\mu_r - 2\sqrt{M_t^2 + p_t^2 + p_r^2 + 2n\omega + \omega}}{4T} \right) \right) \right), \tag{32} \]

here \( f = u, d, s \). With the analytical expressions given above, we show our numerical results in the next section.

III. NUMERICAL RESULTS AND DISCUSSIONS

In this section we will present our numerical results for the gap equation, quark spin polarization and quark number susceptibility. In our previous analytic expressions, the z-angular-momentum quantum number \( n = 0, \pm 1, \pm 2 ... \) in principle we should sum all the values of \( n \), fortunately, these expressions converge so fast that it is enough for us to sum \( n \) from \(-5 \) to \( 5 \), it should be noted that in order that the causality of a rigidly rotating system is maintained we should make sure that the local velocity is smaller than the light velocity, namely, the condition \( \omega r < 1 \) should be considered in all the calculations, and for simplicity, we take the same value of \( r \) as in Ref. [92]. Since any uniformly rotating system should be spatially bounded, it has been expected that the presence of boundaries can modify the properties of the rotating system [103–107], indeed, this is only true when the angular velocity \( \omega \) is much smaller than the inverse of the system’s size [108] as well as our discussion is mainly devoted to the bulk properties of rotating system, so in our analytic derivation we ignore the finite volume boundary effect and we leave it as our further study. In our calculations, for simplicity, we have let the chemical potential \( \mu_{u/d} = \mu_s = \mu \), and the input parameters in the NJL are the coupling constants \( G, K \), the quark masses \( m_{u/d}, m_s \) and the three-momentum cutoff \( \Lambda \). Then, in this context we use the second case in Table 1. of Ref. [109],
that \( m_u = m_d = 0.005 \text{ GeV} \), \( m_s = 0.1283 \text{ GeV} \), \( G = 3.672 \text{ GeV}^{-2} \), \( K = 59.628 \text{ GeV}^{-5} \), \( \Lambda = 0.6816 \text{ GeV} \), which have been estimated by the fitting in light of the following observations: \( m_\pi = 138 \text{ MeV} \), \( f_\pi = 92 \text{ MeV} \), \( m_K = 495 \text{ MeV} \) and \( m_{\rho} = 958 \text{ MeV} \).

Let us first discuss the results at chemical potential equals zero. We investigate the chiral condensation in rotating matter under the three-flavor NJL model, especially we consider the contribution from \( s \) quark. It is equivalent to study the gap equations since the gap equations correspond to the coupling of the quarks naked masses with the associated chiral condensates, firstly, in the plot of Fig. 1, we show the differences of squared gap masses between the case at nonzero angular velocity to those at zero angular velocity, \( M^2(\omega) - M^2(\omega = 0) \) with zero temperature and zero chemical potential for the up quark and strange quark, respectively, we found that the squared gap mass differences of all quarks decrease with increasing angular velocity, and at large angular velocity there is a sudden drop down for the squared gap mass differences, it obvious that the squared gap mass differences of lighter quark is more affected by the angular velocity, which decreases faster while that of strange quark decreases slower with angular velocity, this can be interpreted as that the quark with large current mass is less affected by the rotation. Then we plot the gap equation as a function of \( \omega \) with different values of \( T \) in Fig. 2 respectively for \( u \) and \( s \) quark. As one can see, both \( u \) quark and \( s \) quark gap equations show that the rotation has a suppression effect for the chiral condensation. It is clearly seen that at all temperatures the gap equations of both \( u \) and \( s \) decrease with increasing \( \omega \) and at very low temperature the chiral condensate experiences a first-order transition when \( \omega \) exceeds a certain value. It is very interesting when comparing the different flavor situation, in the left panel of Fig. 2 shows that \( M_u \) experiences a first-order transition around \( \omega = 0.6 \text{ GeV} \) at \( T = 0.01 \text{ GeV} \), in the right panel of Fig. 2 we observe that \( M_s \) also experiences a sharp change around \( \omega = 0.6 \text{ GeV} \) but changes not so much their mass compared to the light quark situation, due to the coupling between the different flavors. In addition, we also found that for the strange quark there is a fast drop from a high value for the quark effective mass to a small value around \( \omega = 1.0 \text{ GeV} \). From the figure we can see the role of the \( \omega \) as well as \( T \) are very important parameters for crossover or first-order transition. For the high temperatures the chiral condensate vanishes with the increasing \( \omega \) via a smooth crossover, and the temperature effect becomes weak with increasing the value of \( \omega \). As \( \omega \) further increases the gap equations for the both decrease more slowly, and both approach their naked masses. This can be interpreted as a hint that the chiral phase transitions for the \( s, u \) do not occur at the same angular velocity with the same temperature. It is found that the chiral condensation of \( u \) quark has produced results almost in agreement with those suggested in literature \([4]\), but with a slight difference due to they adopted the parameters in Ref. \([111]\).

![Figure 1](image1.png)

**FIG. 1.** (Color online) Differences of squared gap masses between the case at \( \omega \neq 0 \) and \( \omega = 0 \) with both \( \mu = 0 \) and \( T = 0 \) for up quark and strange quark as a function of \( \omega \). The first-order spin polarizations of the rotating system as a function of \( \omega \) for several temperatures with chemical potential equals zero are shown in Fig. 4, to be clear, the result of \( T = 0.01 \text{ GeV} \) is divided by 100. It is very clear to see that the angular velocity has a strong influence on the quark first-order spin polarization as well as the temperature is also important to the polarization of the quark. From the figure it is observed that the rotation system may induce a large polarization. At all temperatures the quark spin polarization increases with in-
creasing angular velocity for all quarks. At low temperature the quark first-order spin polarization increases very rapidly in a certain angular velocity window and then increases very slowly, while at high temperature the quark first-order spin polarization increases almost linearly. And at very low temperature an interesting phenomenon of the jump of the quark first-order spin polarization can be observed around $\omega \sim 0.6$ GeV for the rotating system in the figure. This jump of the first-order spin polarization is a hint for the first order phase transition to occur, and this distinguishing feature of the spin polarization may provide valuable insights for predicting the first order phase transition in experiments.

In order to have a better understanding for the spin polarization of quark with rotation, we plot the second-order spin...
polarizations of the rotating system as a function of $\omega$ for several temperatures with \( \mu = 0 \). In the case of the temperature is very low, the second-order spin polarization begins to occur at a certain value of the angular velocity and increasing with the angular velocity increases, until it reaches the highest value, when the angular velocity is increased further the second-order spin polarization becomes weaker and finally disappears, the reason why the second-order quark spin polarization disappears at large angular velocity is that when the angular velocity becomes larger, the dynamical mass of the quark becomes smaller and the chiral symmetry is gradually restored so that the second-order quark spin polarization does not occur.

On the other hand, in order to understand the properties of the quark matter under the rotation system better, it is helpful to study the behavior of baryon number susceptibility. We now turn to study the QNS and take into account the influence of angular velocity on QNS. Firstly, we consider the case of zero chemical potential, in Fig. 6 we plot the susceptibilities of $u$ and $s$ quark as a function of angular velocity for several fixed values of temperatures with zero chemical potential, to be clear, the result of $T = 0.01$ GeV is divided by 10. It is evident from the figure that the susceptibilities of $u$ and $s$ quark increase as the $\omega$ increases when $\omega$ is smaller than a certain value for all the temperatures, while the susceptibilities decrease as the $\omega$ increases when $\omega$ is larger than another certain value. And it also can be seen from the figure that the temperature and the angular velocity play an important role in the susceptibility, at low temperature the quark chiral symmetry is broken spontaneously, however with the increasing of the angular velocity the chiral symmetry is restored, so we can see at low temperature and low angular velocity the susceptibility is very small and at low temperature and large angular velocity the susceptibility disappears, however if the temperature is high the susceptibility always occurs and the angular velocity plays a slightly effect on the rotating matter.

It is fairly clear that the susceptibility is a finite quantity that is furthermore sensitive to the mass of the quark, so next we will discuss the differences of the $u$ and $s$ quark number susceptibilities under the rotation. When angular velocity is small and temperature is high we find that the susceptibility of $u$ is larger than the susceptibility of $s$ and when angular velocity and temperature are both small we find that it is easy for the susceptibility of $u$ quark to occur, for instance, at $T = 0.01$ GeV, the $u$ quark number susceptibility starts to occur at $\omega = 0.5$ GeV while for $s$ quark number susceptibility starts to occur at $\omega = 0.9$ GeV. As we can see from the figure that the curves have the highest values only for some certain regions of the angular velocity, and at low temperature we find that there exist a narrower region obvious changes the QNS when comparing that of $s$ quark with that of $u$ quark at $T = 0.01$ GeV, it is clearly can be seen from the curves that QNS changes little when the angular velocity is changed below 0.9 GeV or above 1.5 GeV for the $s$ quark, however for that of $u$ quark the angular velocity is below 0.5 GeV or above 1.5 GeV. It is very clear from the figure that the contribution from the angular velocity becomes dominant when $\omega \geq 1.1$, and the peaks of the susceptibilities appear at almost the same angular velocity. At all values of the chemical potential the behaviors of the $u$ quark and $s$ quark number susceptibility are very similar with increasing the angular velocity reveal that when the angular velocity is large the role played by the mass of different quarks becoming weaker and weaker and finally almost can be ignored.

Comparing with the Fig. 2, it is clear that a nonzero value of the chemical potential affects the first-order phase transition, at $T = 0.01$ GeV, there does not exist a sudden drop for the mean field mass gap when $\mu$ is large both for $u$ and
FIG. 6. (Color online) Susceptibilities of \( u \) and \( s \) quark as a function of \( \omega \) for several values of \( T \) with \( \mu = 0 \), here the result of \( T = 0.01 \) GeV is divided by 10.

FIG. 7. (Color online) The mean field mass gap of \( u \) and \( s \) quark as a function of \( \omega \) for several fixed values of \( \mu \) at \( T = 0.01 \) GeV. 

\( s \) quark, which indicating there exists a suppression effect for the chemical potential on the phase transition. At large chemical potential the chiral condensate vanishes with increasing \( \omega \) via a smooth crossover. From the figure we can also see that there is a different behavior between \( u \) and \( s \) quark, the \( u \) quark is more affected by the presence of the chemical potential and angular velocity than \( s \) quark because the \( s \) quark has a substantial mass even after the chiral phase transition.

In Fig. 8, the plots of the first-order spin polarization of the rotating system as a function of angular velocity at \( T = 0.01 \) GeV are presented for nonzero chemical potential \( \mu = 0.1, 0.2, 0.3, 0.4 \) GeV, respectively. As is clearly evident from the figure, the first-order spin polarizations of the rotating system increase with increasing angular velocity. And it can be seen that the first-order spin polarizations are affected obviously by the nonzero quark chemical potential. With increasing angular velocity, the first-order spin polarization of the system will first take place at larger quark chemical potential, for example, the first-order spin polarization of \( u \) start to occur around \( \omega = 0.2 \) GeV and \( \omega = 0.4 \) GeV for \( \mu = 0.2 \) GeV and \( \mu = 0.1 \) GeV, respectively. Fig. 8 also demonstrates that at very large quark chemical potential the first-order spin...
polarization of the system first quickly reach a certain value then is only relatively slowly varying with angular velocity, this can be understood by noting that at large value of quark chemical potential the chiral symmetry restored quickly.

Fig. 9 displays the results of the second-order polarizations of the system as a function of \( \omega \) for several fixed nonzero values of \( \mu \) at \( T = 0.01 \, \text{GeV} \). In particular, a non-monotonic is identified, with the second-order polarizations to first increase, reach a maximum, and then decrease sharply. This behaviour is the combined effect of the suppression in both the angular velocity itself and the chemical potential at very low temperature.

Next, we will analyze the patterns of second-order susceptibility of quark with rotation at nonzero chemical potential. Let us first consider the effect of angular velocity \( \omega \) on second-order susceptibility of the quarks at very low temperature. We plot the second-order susceptibilities of \( u \) and \( s \) quark as a function of \( \omega \) for several fixed nonzero values of chemical potential at \( T = 0.01 \, \text{GeV} \). From the figure we find the susceptibility have similar behavior with respect to angular velocity. We can see if one considers the case of nonzero chemical potential, the behavior of QNS changes considerably and is quite dependent on the angular velocity. As shown in the figure one can see the dependence of QNS on
angular velocity is complicated that the QNS increases with the increasing $\omega$ when $\omega$ is smaller than a certain value while the QNS decreases with the increasing $\omega$ when $\omega$ excesses another certain value, which indicates that the rotation matter may provide some new and helpful results to study the phase transition. And there are some interesting changes compared with the situation of zero chemical potential, at $T = 0.01$ GeV the curves of susceptibility of the quark have two peaks, which are very different compared with that in Fig. 6, the curves of QNS have such behavior because the gap mass with nonzero chemical potential are different from that the case with zero chemical potential and for any angular velocity which should satisfy the gap equation, whose constraint will have an effect to the susceptibility. In addition, a plausible explanation for this phenomenon is that the rotational velocity serves as an effective chemical potential and exhibits a non-trivial behavior such that the competition between the chemical potential and angular velocity renders the quark number susceptibility to reach its maximum in such manner.

![Graph](image)

**FIG. 11.** (Color online) The mean field gap $M_u$ and $M_s$ as a function of $\omega$ for several fixed values of $\mu$ at $T = 0.2$ GeV.

It may need a bit more explanation about the part played by the angular velocity $\omega$ with nonzero chemical potential at high temperature. Fig. 11 shows the mean field gap $M_u$ and $M_s$ versus $\omega$ with several fixed values of $\mu$ at $T = 0.2$ GeV, obviously from the figure we can see there is a generally rotational suppression effect on the chiral condensate for the system at high temperature with nonzero chemical potential. In order to have better understanding in Fig. 12 we plot $M_u(\omega)/M_u(\omega = 0)$ and $M_s(\omega)/M_s(\omega = 0)$ as a function of $\omega$ with several nonzero chemical potentials at $T = 0.2$ GeV, respectively. We can see the influence of the angular velocity to the light quark and strange quark is different. The figure shows that $M_u$ is much affected due to the current mass of the $u$ quark is very small and whose chiral symmetry can be easily restored compared to that of $s$ quark.

In Fig. 13, we plot the first-order spin polarization of the system as a function of $\omega$ for several fixed values of $\mu$ with $T = 0.2$ GeV. From Fig. 13 we find that at this temperature the first-order quark spin polarization of the system always occurs with increasing the angular velocity for all the nonzero chemical potential, at large chemical potential ($\mu = 0.4$ GeV), the first-order spin polarization of the system increase almost linearly with increasing the angular velocity. We now consider the effects of angular velocity $\omega$ on second-order spin polarization the system at $T = 0.2$ GeV with chemical potential $\mu = 0.1, 0.2, 0.3, 0.4$ GeV, respectively. From the Fig. 14 one immediately makes the following observations, in the case of high temperature the variation of second-order spin polarization of the system with angular velocity is complicated, we find that at small chemical potential the second-order spin polarization of the system have a peak structure with the increase of angular velocity, when $\omega < 1.0$ GeV at small chemical potential ($\mu = 0.1$ GeV) the second-order spin polarization of the system increases with increase in angular velocity, while at large chemical potential ($\mu = 0.4$ GeV) that decreases with increase in angular velocity.

Next, we would like to consider the effects of angular velocity $\omega$ on second-order susceptibility of the quarks at high temperature with several fixed nonzero values of chemical potential and the numerical results are shown in Fig. 15. It can
be found that at such high temperature the second-order susceptibility of the quarks can always occur, although the maximum value of susceptibility is very small compared to the situation in the Fig. 10, which means that the contribution of angular velocity to the susceptibility is suppressed by high temperature. From Fig. 15 one could infer the dependence of the second-order susceptibility of the quarks on the quark current mass when \( \omega < 0.5 \) GeV, the values of the susceptibility of the strange quark is smaller compared to that of light quarks for the same quark chemical potential. However, with the increase of \( \omega \), the behavior of the both quarks is very similar which means that at high temperature and large chemical potential the large angular velocity takes the predominant role compared to different current mass of the quarks. From Fig. 15 one could also infer the dependence of susceptibility on the chemical potential at high temperature, at small \( \omega \), the susceptibility increases with the increasing \( \mu \). at large \( \omega \), the susceptibility decreases with the increasing \( \mu \).

FIG. 12. (Color online)\( M_u(\omega)/M_u(\omega = 0) \) and \( M_s(\omega)/M_s(\omega = 0) \) as a function of \( \omega \) with several nonzero chemical potentials at \( T = 0.2 \) GeV.

FIG. 13. (Color online) First-order spin polarizations of the rotating system as a function of \( \omega \) for several fixed values of \( \mu \) at \( T = 0.2 \) GeV.

FIG. 14. (Color online) Second-order spin polarizations of the rotating system as a function of \( \omega \) for several fixed nonzero values of \( \mu \) at \( T = 0.2 \) GeV.

Let us move on to the topic of QCD phase diagram. The investigation of the phase diagram of QCD has been an active subject for many years. There has been much progress on the study of the QCD phase diagram with lattice QCD (LQCD) simulations, however, at large chemical potential the predictions made by the LQCD are not very reliable due to the sign problem of lattice gauge theory [110]. So, in order to investigate the QCD phase diagram many effective models have been proposed such as Nambu and Jona-Lasinio (NJL) models, quark-meson (QM) models, holographic QCD models, functional renormalization group (fRG), Dyson Schwinger equations (DSE) as well as some extending modes of these [111–132]. It is believed that the addition of external influences or new parameter ranges yield an increasing number of interesting results to the phase diagram [133].
And investigating the QCD matter under rotation is a fascinating topic, apart from the chiral condensation, spin polarization and number susceptibility of these matter, it is also of significant interest to explore the effects of rotation on the phase transitions. We now first explore the phase diagram in the $T$-$\omega$ plane, it is obvious that from the Fig. 2 there exist first order phase transitions for the rotating system. Here our main focus is on the first order phase transition and its associated critical end point with rotation in the presence of small chemical potential, in Fig. 16 we show the first order phase diagram of the system in the $T$-$\omega$ plane with different chemical potentials $\mu = 0.02, 0.03$ and $0.04$ GeV, respectively. We observe that the appearance of the first order phase transition line starting at $T \sim 0$ GeV and terminating at the critical end point identified with a full dot and these first order phase transition lines move toward a higher temperature for decreasing $\omega$. It is also found that the different chemical potentials change the boundary of phase diagram, and that a larger chemical potential shifts down the critical temperature. It is no doubt that there are many other discussions need to be done for the phase transition in the rotation system, here is just a beginning and this particular topic will be discussed in detail in future work.

IV. CONCLUSIONS AND OUTLOOK

Finally, we want to summarize our results and give a brief outlook. In this paper, we have presented detailed analytic formulae for the quark matter under rotation in three-flavor NJL model and related topics have been investigated. In order to have a better understanding of the rotating system with finite density we have also introduced the chemical potential. We studied the quark fields in cylindrical coordinates as well as investigated the effect of the rotating on the quark chiral condensate, quark spin polarization and quark number susceptibility at finite temperature with or without finite chemical potential in this model. We found that the angular velocity plays a very crucial role in these topics, at low temperature, small chemical potential and small angular velocity the chiral symmetry is broken spontaneously, while at large enough angular velocity the chiral symmetry is restored. Our numerical analysis shows that the rotation suppresses the chiral condensation and enhances the first-order quark spin polarization, however for the second-order quark spin polarization and quark number susceptibility the effect is very complicated, which can be found to have a peak structure.

We have also explicitly computed these quantities in the rotation system in the presence of chemical potential, we found that the nonzero chemical potential affects and makes the chiral...
eral condensate, quark spin polarization and quark number susceptibility have different behaviors. At very low temperature the chiral condensate experiences a first order transition when $\omega$ exceeds a certain value with zero chemical potential, while at larger chemical potential the first order transition is suppressed and changed to a crossover transition. It can be also observed that at very low temperature the quark number susceptibilities have two maxima, a plausible explanation for this phenomenon is that the rotational velocity serves as an effective chemical potential and exhibits a non-trivial behavior such that the competition between the chemical potential and angular velocity renders the quark number susceptibility to reach its maximum in such manner. In this paper we especially considered the contributions from $s$ quark and made some comparisons between $u$ quark and $s$ quark and found that at small angular velocity the part played by current mass to these phenomena is important, however, at sufficiently large angular velocity, the contributions played by different quarks to these phenomena are almost equal. In addition, we explored the phase diagram in the $T-\omega$ plane, we observe that there exist first order phase transitions for the rotating system and the first order phase transition lines move toward a higher temperature for decreasing angular velocity. It is also found that the different chemical potentials change the boundary of phase diagram, and that a larger chemical potential shifts down the critical temperature. Based on the interpretations made above, it would be possible to judge and forecast these phenomena of quark matter under rotation if we jointly take angular velocity, chemical potential and temperature factors into consideration in the three-flavor NJL model. We expect these studies to play an important role in help understanding the properties of strongly interacting rotating matter.

The theoretical interest in relativistic rotating systems is being revived in many different physical environments, which is undoubtedly calls for more investigation. For instance, the related effects of rotating fermions inside a cylindrical boundary [134], the investigation of a possible phase structure under rotation including the $s$ quark, especially the exploration to those regions of the phase diagram that cannot be reached on the lattice yet. The NJL model describes only quarks and antiquarks and neglects the gluons, so it is also very worth to extend the rotation system to the Polyakov extended Nambu and Jona-Lasinio (PNJL) model [135], which will consider the complex interactions between quarks and gluons and that the chiral symmetry restoration as well as the effect of quark confinement in PNJL under rotation may provide needed insight into the QCD. This model will have a clearer picture considering the constraint from the experimental related to rotation conducted at numerous research facilities worldwide as the Brookhaven National Laboratory (BNL), the European Organization for Nuclear Research (CERN) and the GSI Helmholtz Centre for Heavy Ion Research (GSI), due to one can see that some limits can be made for the parameter space under the given assumptions. It would be interesting to use the results obtained in this paper to investigate topics discussed here, and we leave these as our further study.

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Appendix A: The brief description of fermions under rotation

The properties of fermions under global rotation are relevant to a number of problems as discussed above, so it is important to choose an appropriate complete set of commuting operators in the cylindrical coordinates. In this section, we will start from the Dirac equation in the rotating frame, then we will derive the eigenvectors of the those complete set of commuting operators.

The general Lagrangian of the rotating fermions is written in the following way [16, 30, 92]

$$\mathcal{L} = \bar{\psi} \left[ i \gamma^\mu \partial_\mu - m + (\gamma^0)^{-1} \left( \left( \hat{\omega} \times \vec{x} \right) \cdot \left( -i \vec{\sigma} \right) + \hat{\omega} \cdot \hat{S} \right) \right] \psi, \tag{A1}$$

where $\psi$ is the quark field, $\omega$ is the angular velocity and $m$ is the bare quark mass matrix, as a result of rotation, we can see the Dirac operator includes the orbit-rotation coupling term and the spin-rotation coupling term, and we have defined

$$\hat{S} = \frac{1}{2} \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}, \tag{A2}$$

whose $z$-component related to the spin polarization of the quark. The corresponding Hamiltonian of Eq. (A1) in momentum space reads
\( \hat{H} = \gamma^0 \left( \vec{\gamma} \cdot \hat{p} + m \right) - \vec{\omega} \cdot \left( \vec{x} \times \hat{p} + \hat{S}_{4 \times 4} \right) = \hat{H}_0 + \vec{\omega} \cdot \hat{J}, \)

(A3)

here

\[ \hat{J} = \vec{x} \times \hat{p} + \hat{S}_{4 \times 4}, \]

(A4)

and the first term is the contribution of the angular momentum, the second term is the contribution of the spin angular momentum.

Now considering the energy eigenvalue equation

\[ \hat{H} \psi = E \psi, \]

(A5)

here \( \psi \) is the four-component spinors and can be written in terms of two-component spinors as

\[ \psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix}, \]

(A6)

substituting the Hamiltonian above to the energy eigenvalue equation

\[
\begin{pmatrix}
E - m + \omega_z \left( n + \frac{1}{2} \right) \\
E + m + \omega_z \left( n + \frac{1}{2} \right)
\end{pmatrix}
\begin{pmatrix}
\phi \\
\chi
\end{pmatrix} = \begin{pmatrix}
\vec{\sigma} \cdot \hat{p} \phi \\
\vec{\sigma} \cdot \hat{p} \chi
\end{pmatrix},
\]

(A7)

after some derivations, we can get the following equation

\[
\begin{pmatrix}
E - m + \omega_z \left( n + \frac{1}{2} \right) \\
E + m + \omega_z \left( n + \frac{1}{2} \right)
\end{pmatrix}
\begin{pmatrix}
\phi \\
\chi
\end{pmatrix} = \begin{pmatrix}
\vec{\sigma} \cdot \hat{p} \phi \\
\vec{\sigma} \cdot \hat{p} \chi
\end{pmatrix},
\]

(A10)

substituting Eq. (A12) into the Eq. (A10) after some tedious calculations, we find that \( g(r) \) satisfies the Bessel-type equation as follows,

\[
\begin{pmatrix}
E - m + \omega_z \left( n + \frac{1}{2} \right) \\
E + m + \omega_z \left( n + \frac{1}{2} \right)
\end{pmatrix}
\begin{pmatrix}
\phi \\
\chi
\end{pmatrix} = \begin{pmatrix}
\vec{\sigma} \cdot \hat{p} \phi \\
\vec{\sigma} \cdot \hat{p} \chi
\end{pmatrix},
\]

(A13)

the solutions of Eqs. (A13) and (A14) have the following form, respectively,

\[ g(r) = J_n(p_tr), J_{n+1}(p_tr), \]

(A15)

where \( J \) is the Bessel function. In order to commute with other operators we must define the helicity operator, the general helicity operator has the following form

\[ h_t = \gamma^5 \gamma^3 \sum \frac{\vec{p}_t}{|\vec{p}_t|} = \frac{1}{i|\vec{p}_t|} \begin{pmatrix}
0 & -P_- & 0 & 0 \\
P_+ & 0 & 0 & 0 \\
0 & 0 & 0 & P_- \\
0 & 0 & -P_+ & 0
\end{pmatrix}, \]

(A16)
here, $p_t$ is the transverse momentum, $P_+ = \hat{p}_x + ip_y$, $P_- = \hat{p}_x - ip_y$ and in Cylindrical coordinates they have such forms

$$ P_+ = -ie^{i\theta} \left( \frac{\partial}{\partial r} + \frac{1}{r} \frac{\partial}{\partial \theta} \right), \quad (A17) $$

$$ P_- = -ie^{-i\theta} \left( \frac{\partial}{\partial r} - \frac{1}{r} \frac{\partial}{\partial \theta} \right), \quad (A18) $$

which like shift operators when act on the terms including angular momentum quantum number $e^{i\theta} J_n(p_t r)$, $e^{i(n+1)\theta} J_{n+1}(p_t r)$, respectively, they satisfy the following relationship

$$ P_+ e^{i\theta} J_n(p_t r) = ip_t e^{i(n+1)\theta} J_{n+1}(p_t r), \quad (A19) $$

$$ P_- e^{i(n+1)\theta} J_{n+1}(p_t r) = -ip_t e^{i\theta} J_n(p_t r). \quad (A20) $$

Reconsidering the transverse helicity equation and the generalized orthogonality relation

$$ h_t \psi = s \psi, \quad (A21) $$

$$ \sum_{n=-\infty}^{\infty} \psi^\dagger \psi = 1, \quad (A22) $$

imposing the generalized completeness relation

$$ \sum_{n=-\infty}^{\infty} u^\dagger u = 1, \quad (A27) $$

these constant factors can be determined and finally we obtain, the positive energy particle solutions with positive and negative helicity in the Dirac representation, which take the following explicit form

$$ u = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{ip_z z} e^{i\eta \theta} J_n(p_t r) \\ se^{ip_z z} e^{i(n+1)\theta} J_{n+1}(p_t r) \\ e^{ip_z z} e^{i\eta \theta} J_n(p_t r) \\ se^{ip_z z} e^{i(n+1)\theta} J_{n+1}(p_t r) \end{pmatrix}, \quad (A23) $$

where, $u$ is a four-component spinor which must satisfy the Dirac equation

$$ (i\gamma^\mu \partial_\mu - m) u = 0, \quad (A24) $$

substituting $u$ into the Dirac equation gives

$$ \begin{pmatrix} E - m \\ -\sigma_p \\ E - m \end{pmatrix} \begin{pmatrix} c_A u_A \\ c_B u_B \end{pmatrix} = 0, \quad (A25) $$

here $u_A, u_B$ are two-component spinors and $c_A, c_B$ are normalization constants, after some calculations we get

$$ c_B u_B = c_A \begin{pmatrix} \frac{p_z - is p_t}{\sqrt{E + m}} e^{ip_z z} e^{i\eta \theta} J_n(p_t r) \\ \frac{-s p_z + ip_t}{\sqrt{E + m}} e^{ip_z z} e^{i(n+1)\theta} J_{n+1}(p_t r) \end{pmatrix}, \quad (A26) $$

in exactly the same way, the negative-energy antiparticle solutions are listed below

$$ v = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{p_z - is p_t}{\sqrt{E + m}} e^{-ip_z z} e^{i\eta \theta} J_n(p_t r) \\ \frac{-s p_z + ip_t}{\sqrt{E + m}} e^{-ip_z z} e^{i(n+1)\theta} J_{n+1}(p_t r) \\ \frac{-p_z - is p_t}{\sqrt{E + m}} e^{-ip_z z} e^{i\eta \theta} J_n(p_t r) \\ \frac{s p_z + ip_t}{\sqrt{E + m}} e^{-ip_z z} e^{i(n+1)\theta} J_{n+1}(p_t r) \end{pmatrix}. \quad (A29) $$

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