Lattice subalgebras of strongly regular vertex operator algebras

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Abstract
We prove a sharpened version of a conjecture of Dong-Mason about lattice subalgebras of a strongly regular vertex operator algebra $V$, and give some applications. These include the existence of a canonical conformal subVOA $W \otimes G \otimes Z \subseteq V$, and a generalization of the theory of minimal models.

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1. **Statement of main results**

This paper concerns the algebraic structure of strongly regular vertex operator algebras (VOAs). A VOA $V = (V,Y,1,\omega)$ is called regular if it is rational (admissible $V$-modules are semisimple) and $C_2$-cofinite (the span of $u(n)v$ ($u,v \in V, n \leq -2$) has finite codimension in $V$). It is strongly regular if, in addition, the $L(0)$-grading (or conformal grading) given by $L(0)$-weight has the form

$$V = \mathbb{C}1 \oplus V_1 \oplus \ldots$$

(1)

and all states in $V_1$ are quasiprimary (i.e. annihilated by $L(1)$). Apart from the still-undecided question of the relationship between rationality and $C_2$-cofiniteness, changing any of the assumptions in the definition of strong regularity will result in VOAs with quite different properties (cf. [DM5]). Such VOAs are of interest in their own right, but we will not deal with them here.

To describe the main results, we need some basic facts about strongly regular VOAs $V$ that will be assumed here and reviewed in more detail in later Sections. $V$ is equipped with an essentially unique nonzero, invariant, bilinear form $\langle , \rangle$, and $V$ is simple if, and only if, $\langle , \rangle$ is nondegenerate. We assume this is the case from now on. Then $V_1$ carries the structure of a reductive Lie algebra and all Cartan subalgebras of $V_1$ (maximal (abelian) toral Lie subalgebras) are conjugate in $\text{Aut}(V)$. We also refer them as Cartan subalgebras of $V$. We say that a subspace $U \subseteq V$ is nondegenerate if the restriction of $\langle , \rangle$ to $U \times U$ is nondegenerate. For example, the Cartan subalgebras of $V$ and the solvable radical of $V_1$ are nondegenerate. We refer to the dimension of $H$ as the Lie rank of $V$.

A subVOA of $V$ is a subalgebra $W = (W,Y,1,\omega')$ with a conformal vector $\omega'$ that may not coincide with the conformal vector $\omega$ of $V$. If $\omega = \omega'$ we say that $W$ is a conformal subVOA. $V$ contains a unique minimal conformal subVOA (with respect to inclusion), namely the Virasoro subalgebra generated by $\omega$. A basic example of a subVOA is the Heisenberg theory $(M_U, Y, 1, \omega_U)$ generated by a nondegenerate subspace $U$ of a Cartan subalgebra of $V$. $M_U$ has rank (or central charge) $\dim U$ and conformal vector

$$\omega_U := 1/2 \sum_i h^i(-1)h^i,$$

(2)

(for any orthonormal basis $\{h^i\}$ of $U$). A lattice theory is a VOA $V_L$ corresponding to a positive-definite, even lattice $L$. 

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We can now state the main result.

**Theorem 1**: Let $V$ be a strongly regular, simple VOA, and suppose that $U \subseteq H \subseteq V$ where $H$ is a Cartan subalgebra of $V$ and $U$ is a nondegenerate subspace. Let $\omega_U$ be as in (2). Then the following hold:

(a) There is a unique maximal subVOA $W \subseteq V$ with conformal vector $\omega_U$.

(b) $W \cong V_\Lambda$ is a lattice theory, where $\Lambda \subseteq U$ is a positive-definite even lattice with $\dim U = \text{rk}\Lambda$.  

\[ \text{(3)} \]

**Remark 2.** 1. Part (a) - the fact that there is a unique maximal subVOA $W$ with conformal vector $\omega_U$ - is elementary; it follows from the theory of commutants [FZ] (cf. Section 12). The main point of the Theorem is the identification of $W$ as a lattice theory.

2. $U$ is a Cartan subalgebra of $W$. Thus, every nondegenerate subspace of $H$ is a Cartan subalgebra of a lattice subVOA of $V$.

Theorem 1 has many consequences. We discuss some of them here, deferring a fuller discussion until later Sections. We can apply Theorem 1 with $U = \text{rad}(V_1)$, and this leads to the next result.

**Theorem 3**: Suppose that $V$ is a strongly regular, simple VOA. There is a canonical conformal subVOA

\[ T = W \otimes G \otimes Z, \]

the tensor product of subVOAs $W, G, Z$ of $V$ with the following properties:

(a) $W \cong V_\Lambda$ is a lattice theory and $\Lambda$ has minimal length at least 4;

(b) $G$ is the tensor product of affine Kac-Moody algebras of positive integral level;

(c) $Z$ has no nonzero states of weight 1: $Z = \mathbb{C}1 \oplus Z_2 \oplus \ldots$

**Remark 4.** The gradings on $W, G$ and $Z$ are compatible with that on $V$ in the sense that the $n^{th}$ graded piece of each of them is contained in $V_n$. $T$ has the tensor product grading, and in particular $T_1 = W_1 \oplus G_1 = V_1$. Indeed, $W_1 = \text{rad}(V_1)$ and $G_1$ is the Levi factor of $V_1$. Thus the weight 1 piece of $V$ is contained in a rational subVOA of standard type, namely a tensor product of a lattice theory and affine Kac-Moody algebras.
To a certain extent, Theorem 3 reduces the study of strongly regular VOAs to the following: (A) proof that \( Z \) is strongly regular; (B) study of strongly regular VOAs with no nonzero weight 1 states; (C) extension problem for strongly regular VOAs, i.e. characterization of the strongly regular VOAs that contain a \emph{given} strongly regular conformal subVOA \( T \). For example, we have the following immediate consequence of Theorem 3 and Remark 4.

**Theorem 5**: Suppose that \( V \) is a strongly regular, simple VOA such that the conformal vector \( \omega \) lies in the subVOA \( \langle V_1 \rangle \) generated by \( V_1 \). Then the canonical conformal subalgebra \([1]\) is a rational subVOA

\[
T = W \otimes G,
\]

where \( W \) and \( G \) are as in the statement of Theorem 2.

**Remark 6.** Let \( \mathcal{C} \) consist of the (isomorphism classes of) VOAs satisfying the assumptions of the Theorem. \( \mathcal{C} \) contains all lattice theories, all simple affine Kac-Moody VOAs of positive integral level (Siegel-Sugawara construction), and it is closed with respect to tensor products and extensions in the sense of (C) above. Theorem 5 says that every VOA in \( \mathcal{C} \) arises this way, i.e. an extension of a tensor product of a lattice theory and affine Kac-Moody theories.

There are applications of Theorem 1 to inequalities involving the Lie rank \( l \) and the effective central charge \( \tilde{c} \) of \( V \). These lead to characterizations of some classes of strongly rational VOAs \( V \) according to these invariants. For example, we have

**Theorem 7**: Let \( V \) be a strongly regular, simple VOA of effective central charge \( \tilde{c} \) and Lie rank \( l \). The following are equivalent:

(a) \( \tilde{c} < l + 1 \),

(b) \( V \) contains a conformal subalgebra isomorphic to a tensor product \( V_\Lambda \otimes L(c_{p,q},0) \) of a lattice theory of rank \( l \) and a simple Virasoro VOA in the discrete series.

**Remark 8.** 1. We \emph{always} have \( l \leq \tilde{c} \) ([DM1]).

2. Define a \emph{minimal model} as a strongly regular simple VOA whose Virasoro subalgebra lies in the \emph{discrete series}. The case \( l = 0 \) of Theorem 7 characterizes minimal models as those strongly regular simple VOAs which have
\[ \tilde{c} < 1 \]. This is, of course, very similar to the classification of minimal models in physics (cf. [FMS], Chapters 7 and 8), where attention is usually restricted to the unitary case, where \( c = \tilde{c} \), or equivalently \( q = p + 1 \), in the notation of Theorem 7. Minimal models with \( \tilde{c} = c \) were treated rigorously in [DW2]; our approach allows us to remove any assumptions about \( c \) and permits \( l \) to be nonzero.

Our results continue, and in some cases complete, lines of thought in [DM1] and [DM2] having to do with the weight 1 subspace \( V_1 \) of \( V \) and its embedding in \( V \). These include the invariant bilinear form of \( V \), the nature of the Lie algebra of \( V_1 \) and its action on \( V \)-modules, automorphisms of \( V \) induced by exponentiating weight one states, deformations of \( V \)-modules using weight one states, and (more recently [KM]) weak Jacobi form trace functions defined by weight one states. A fuller account might also have included simple currents arising from deformations by weight one states [DLM4], although we do not treat this subject here. These topics constitute a very satisfying Chapter in the theory of rational VOAs, and the Heidelberg Conference presented itself as a great opportunity to review this set of ideas. I am grateful to the organizers, Professors Winfried Kohnen and Rainer Weissauer, for giving me the chance to do so.

2. Background

A vertex operator algebra (VOA) is a quadruple \((V, Y, 1, \omega)\), often denoted simply by \( V \), satisfying the usual axioms. For these and other background results in VOA theory, we refer the reader to [LL]. We write vertex operators as

\[
Y(v, z) = \sum_{n \in \mathbb{Z}} v(n)z^{-n-1} \quad (v \in V),
\]

\[
Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n)z^{-n-2}.
\]

Useful identities that hold for all \( u, v \in V, p, q \in \mathbb{Z} \) include

\[
[u(p), v(q)] = \sum_{i=0}^{\infty} \binom{p}{i} (u(i)v)(p + q - i),
\]

(5)

\[
\{u(p)v\}(q) = \sum_{i=0}^{\infty} (-1)^i \binom{p}{i} (u(p - i)v(q + i) - (-1)^p v(q + p - i)u(i)),
\]
called the commutator formula and associativity formula respectively.

We assume throughout that $V$ is a simple VOA that is strongly regular as defined in Section 1. One of the main consequences of rationality is the fact that, up to isomorphism, there are only finitely many ordinary irreducible $V$-modules ([DLM2]). We let $\mathcal{M} := \{(M^1, Y^1), \ldots, (M^r, Y^r)\}$ denote this set, with $(M^1, Y^1) = (V, Y)$. It is conventional to use $u(n)$ to denote the $n^{th}$ mode of $u \in V$ acting on any $V$-module, the meaning usually being clear from the context, however it will sometimes be convenient to distinguish some of these modes. In particular, we often write $Y^j(u, z) := \sum_{n \in \mathbb{Z}} u_j(n) z^{-n-1}$, dropping the index $j$ from the notation when $j = 1$.

3. Invariant bilinear form

An invariant bilinear form on $V$ is a bilinear map $\langle \ , \rangle : V \times V \rightarrow \mathbb{C}$ satisfying

$$\langle Y(a, z)b, c \rangle = \langle b, Y(e^{zL(1)}(-z^{-2})L(0)a, z^{-1}) \rangle \quad (a, b, c \in V).$$

Such a form is necessarily symmetric ([FHL], Proposition 5.3.6).

A theorem of Li [L1] says that there is a linear isomorphism between $V_0/L(1)V_1$ and the space of invariant bilinear forms on $V$. Because $V$ is strongly regular then $V_0/L(1)V_1 = \mathbb{C}1$, so a nonzero invariant bilinear form exists and it is uniquely determined up to scalars.

If $a \in V_k$ is quasi-primary then (6) says that

$$\langle a(n)b, c \rangle = (-1)^k \langle b, a(2k - n - 2)c \rangle \quad (n \in \mathbb{Z}).$$

In particular, this applies if $a \in V_1$ (because $V$ is assumed to be strongly regular), or if $a = \omega$ is the conformal vector ($\omega$ is always quasiprimary). First apply (7) with $a = \omega$ and $n = 1$, noting that $\omega(1) = L(0)$. Then $k = 2$ and we obtain

$$\langle L(0)b, c \rangle = \langle b, L(0)c \rangle.$$ 

It follows that eigenvectors of $L(0)$ with distinct eigenvalues are necessarily perpendicular with respect to $\langle \ , \rangle$. Thus (1) is an orthogonal direct sum

$$V = \mathbb{C}1 \perp V_1 \perp \ldots$$
The radical $R$ of $\langle , \rangle$ is an ideal. Because we are assuming that $V$ is simple then $R = 0$ and $\langle , \rangle$ is nondegenerate. In particular, $\langle 1, 1 \rangle \neq 0$. In what follows, we fix the form so that
\[
\langle 1, 1 \rangle = -1. \tag{9}
\]
Note also that by (8), the restriction of $\langle , \rangle$ to each $V_n \times V_n$ is also nondegenerate.

4. The Lie algebra on $V_1$

The bilinear product $[uv] := u(0)v$ ($u, v \in V_1$) equips $V_1$ with the structure of a Lie algebra. Applying (7) with $u, v \in V_1$, we obtain $\langle u, v \rangle = \langle u(-1)1, v \rangle = -\langle 1, u(1)v \rangle$. With the convention (9), it follows that
\[
u(1)v = \langle u, v \rangle 1 \quad (u, v \in V_1).
\]

Because $V$ is strongly regular, a theorem of Dong-Mason [DM1] says that the Lie algebra on $V_1$ is reductive. (This result is discussed further in Section 7 below.) So there is a canonical decomposition $V_1 = A \perp S$ where $A = \text{Rad}(V_1)$ is an abelian ideal and $S$ is the (semisimple) Levi factor. The decomposition of $S$ into a direct sum of simple Lie algebras $\oplus_i g_i$ is also an orthogonal sum with respect to $\langle , \rangle$.

There is a refinement of this decomposition, established in [DM2], namely
\[
V_1 = A \perp g_{1,k_1} \perp \ldots \perp g_{s,k_s} \tag{10}
\]
where $g_i$ is a simple Lie algebra and $k_i$ is a positive integer (the level).

To explain what this means, for $U \subseteq V$ let $\langle U \rangle$ be the subalgebra of $V$ generated by $U$. $\langle U \rangle$ is spanned by states $u = u_1(n_1) \ldots u_t(n_t)1$ with $u_1, \ldots, u_t \in U$, $n_1, \ldots, n_t \in \mathbb{Z}$, and equipped with vertex operators defined as the restriction of $Y(u, z)$ to $\langle U \rangle$.

It is proved in [DM2] that there is an isomorphism of VOAs $\langle g_i \rangle \cong L_{g_i}(k_i, 0)$, where $L_{g_i}(k_i, 0)$ is the simple VOA (or WZW model) corresponding to the affine Lie algebra $\hat{g}_i$, determined by $g_i$, of positive integral level $k_i$. Orthogonal Lie algebras in [10] determine mutually commuting WZW
models. So the meaning of (10) is that the canonical subalgebra $G$ of $V$ generated by $S$ satisfies

$$G \cong L_{g_1}(k_1,0) \otimes \ldots \otimes L_{g_s}(k_s,0).$$

(11)

In particular, $G$ is a rational VOA equipped with the canonical conformal vector $\omega_G$ arising from the Sugawara construction associated to each tensor factor ([FZ], [LL]).

Because $V_1$ is reductive, it has a Cartan subalgebra, that is a maximal (abelian) toral subalgebra, and all Cartan subalgebras are conjugate in Aut($V_1$). (See the following Section for further discussion.) Let $H \subseteq V_1$ be a Cartan subalgebra of $V_1$, say of rank $l$. By Lie theory, the restriction of $\langle \ , \ \rangle$ to $H \times H$ is nondegenerate. We also call $H$ a Cartan subalgebra of $V$.

5. Automorphisms

An automorphism of $V$ is an invertible linear map $g : V \to V$ such that $g(\omega) = \omega$ and $g(a)(n)g^{-1} = g(a)(n)$ for all $a \in V, n \in \mathbb{Z}$, i.e.

$$gY(a,z)g^{-1} = Y(g(a),z).$$

(12)

The set of all automorphisms is a group Aut($V$). Because $g\omega(n)g^{-1} = g(\omega)(n) = \omega(n)$, it follows in particular that $g$ commutes with $L(0) = \omega(1)$. Therefore, Aut($V$) acts on each $V_n$. The uniqueness of $\langle \ , \ \rangle$ implies that

$$\text{Aut}(V) \text{ leaves } \langle \ , \ \rangle \text{ invariant}.$$  

(13)

So each $V_n$ affords an orthogonal representation of Aut($V$).

One checks (e.g. using induction and (5)) that for $n \geq 0$,

$$(u(0)^n v)(q) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} u(0)^{n-i} v(q) u(0)^i \quad (u,v \in V, \ q \in \mathbb{Z}).$$

Therefore,

$$\left( e^{u(0)} v \right) (q) = \sum_{n=0}^{\infty} \frac{1}{n!} (u(0)^n v)(q)$$

$$= \sum_{n=0}^{\infty} \sum_{i=0}^{n} \frac{(-1)^i}{i!(n-i)!} u(0)^{n-i} v(q) u(0)^i$$

$$= e^u(0) v(q) e^{-u(0)},$$
showing that (12) holds with $g = e^{u(0)}$. If we further assume that $u \in V_1$ then we obtain using (5) that

$$u(0)\omega = -[\omega(-1), u(0)]1 = -\sum_{i=0}^{\infty}(-1)^i(\omega(i)u)(-1-i)1 = -((L(-1)u)(-1) - (L(0)u)(-2))1 = 0.$$ 

It follows that $\{e^{u(0)} \mid u \in V_1\}$ is a set of automorphisms of $V$. Let $G = \langle e^{u(0)} \mid u \in V_1 \rangle$ be the group they generate. It is clear from the classical relation between Lie groups and Lie algebras that $G$ is the adjoint form of the complex Lie group associated with $V_1$. So there is a containment $G \subseteq \text{Aut}(V)$.

(Normality holds because if $g \in \text{Aut}(V)$ and $u \in V_1$ then $g(u) \in V_1$ and $ge^{u(0)}g^{-1} = e^{g(u)(0)}.$)

One consequence of this is the following. Because $G$ acts transitively on the set of Cartan subalgebras of $V_1$, it follows ipso facto that $\text{Aut}(V)$ also acts transitively on the set of Cartan subalgebras of $V_1$ (or of $V$). Thus the choice of a Cartan subalgebra in $V$ is unique up to automorphisms of $V$, in parallel with the usual theory of semisimple Lie algebras.

6. Projective action of $\text{Aut}(V)$ on $V$-modules

There is a natural action of $\text{Aut}(V)$ on the set $\mathcal{M}$ of (isomorphism classes of) irreducible $V$-modules $\{ (M^j, Y^j) \mid 1 \leq j \leq r \}$ [DM3]. Briefly, the argument is as follows. For $g \in \text{Aut}(V)$ and an index $j$, one checks that the pair $(M^j, Y^j_g)$ defined by $Y^j_g(v, z) := Y^j(gv, z)$ $(v \in V)$ is itself an irreducible $V$-module. Thus the action of $\text{Aut}(V)$ on $\mathcal{M}$ is defined by $g : (M^j, Y^j) \mapsto (M^j, Y^j_g)$. 

Because $\mathcal{M}$ is finite and $G$ is connected, the action of $G$ is necessarily trivial. Hence, if we fix the index $j$, then for $g \in \text{Aut}(V)$ there is an isomorphism of $V$-modules $\alpha_g : (M^j, Y^j) \mapsto (M^j, Y^j_g)$, i.e.

$$\alpha_g Y^j(u, z) = Y^j_g(u, z)\alpha_g = Y^j(gu, z)\alpha_g \quad (u \in V). \quad (14)$$
Because $M^j$ is irreducible, $\alpha_g$ is uniquely determined up to an overall nonzero scalar (Schur’s Lemma).

When $j = 1$, so that $M^j = V$, $\alpha_g$ coincides with $g$ itself, the scalar being implicitly determined by the additional condition $g(\omega) = \omega$. Generally, we find from (14) that

$$\alpha_{gh}Y^j(u, z)\alpha_{gh}^{-1} = \alpha_g\alpha_hY^j(u, z)\alpha_h^{-1}\alpha_g^{-1} \quad (g, h \in \text{Aut}(V)),$$

so that by Schur’s Lemma once more there are scalars $c_j(g, h)$ satisfying

$$\alpha_{gh} = c_j(g, h)\alpha_g\alpha_h.$$

The map $c_j : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}^*$, $(g, h) \mapsto c_j(g, h)$, is a 2-cocycle on $\mathfrak{g}$. It defines a projective action $g \mapsto \alpha_g$ of $\mathfrak{g}$ on $M^j$ that satisfies (14).

While the projective action of $\mathfrak{g}$ on $M^1 = V$ reduces to the linear action previously considered, the 2-cocycles $c_j$ are generally nontrivial, i.e. they are not 2-coboundaries. A well-known example is the VOA $V := \mathcal{L}_{\text{sl}_2}(1, 0)$, i.e. the level 1 WZW model of type $\text{sl}_2$, which is isomorphic to the lattice theory $V_{A_1}$ defined by the $A_1$ root lattice. In this case we have $V_1 = \mathfrak{sl}_2$, with adjoint group $\mathfrak{g} = \text{SO}_3(\mathbb{R})$. There are just two irreducible $V$-modules, corresponding to the two cosets of $A_1$ in its dual lattice $A_1^* := (1/\sqrt{2})A_1$, and their direct sum is the Fock space for the super VOA $V_{A_1^*}$. The automorphism group of this SVOA is $\text{SU}_2(\mathbb{C})$, in other words the projective action of $\mathfrak{g}$ on $M^2$ lifts to a linear action of its proper 2-fold (universal) covering group.

7. Complete reducibility of the $V_1$-action

We discuss the following result.

The Lie algebra $V_1$ is reductive, and its action on each simple $V$-module $(M^j, Y^j)$ is completely reducible. (15)

This follows from results in [DM1] and [DG]. We will need some of the details later, so we sketch the proof.

Each irreducible $V$-module $M^j$ has a direct sum decomposition into finite-dimensional $L_{\lambda}(0)$-eigenspaces

$$M^j = \bigoplus_{n=0}^{\infty} M^j_{n+\lambda_j},$$

where $\lambda_j$ is the highest weight of $M^j$.
where \( \lambda_j \) is a constant called the \textit{conformal weight} of \( M^j \). Each \( M^j_{i+\lambda_j} \) is a module for the Lie algebra \( V_1 \), acting by the zero mode \( u_j(0) \) (\( u \in V_1 \)), and \( L_5 \) amounts to the assertion that each of these actions is completely reducible. The simple summands \( g_i \) \( (1 \leq i \leq s) \) of \( V_1 \) act completely reducibly by Weyl’s theorem, so the main issue is to show that the abelian radical \( \mathfrak{A} \) of \( V_1 \) (cf. (10)) acts semisimply.

The first step uses a formula of Zhu \([Z]\). The case we need may be stated as follows (cf. \([DM1]\)):

\[
\text{Suppose that } u, v \in V_1. \text{ Then for } 1 \leq j \leq r,
\]

\[
\text{Tr}_{M^j} u_j(0) v_j(0) q^{L(0)} - c/24 = Z_{M^j} (u[-1] v, \tau) - \langle u, v \rangle E_2(\tau) Z_{M^j}(\tau).
\]

The notation, which is standard, is as follows (\([Z, DLM3]\)): for \( w \in V \),

\[
Z_{M^j}(w, \tau) := \text{Tr}_{M^j} \circ_j (w) q^{L(0)} - c/24
\]

is the graded trace of the zero mode \( \circ_j (w) \) for the action of \( w \) on \( M^j, u[-1] \) is the \(-1^{st}\) \textit{square bracket} mode for \( u \), and

\[
E_2(\tau) = -1/12 + 2 \sum_{n=1}^\infty \sum_{d|n} dq^n
\]

is the usual weight 2 Eisenstein series.

Next we show that if \( \langle u, v \rangle \neq 0 \) then for some index \( j \) we have

\[
Z_{M^j} (u[-1] v, \tau) \neq \langle u, v \rangle E_2(\tau) Z_{M^j}(\tau).
\]

Indeed, if this does not hold, we can obtain a contradiction using Zhu’s modular-invariance theorem \([Z]\) and the exceptional transformation law for \( E_2(\tau) \) (cf. \([DM1]\), Section 4 for details). From (17) we can conclude that if \( \langle u, v \rangle \neq 0 \) then there is an index \( j \) such that

\[
\text{Tr}_{M^j} u_j(0) v_j(0) \neq 0.
\]
By the same argument as above, the Lie subalgebra corresponding to the unipotent part necessarily vanishes, so that $\mathfrak{A}$ is a complex torus and $A$ consists of semisimple operators. In particular, (15) holds.

(15) was first stated in [DM1], although the proof there is incomplete. It would be interesting to find a proof that does not depend on the theory of algebraic groups.

8. The tower $L_0 \subseteq L \subseteq E$

Fix a Cartan subalgebra $H \subseteq V_1$ of rank $l$, say. We have seen in Section 7 that all of the operators $u_j(0)$ ($u \in H, 1 \leq j \leq r$) are semisimple. We set

\[
\begin{align*}
E &= \{ u \in H \mid u(0) \text{ has eigenvalues in } \mathbb{Q} \}, \\
L &= \{ u \in H \mid u_j(0) \text{ has eigenvalues in } \mathbb{Z}, 1 \leq j \leq r \}, \\
L_0 &= \{ u \in H \mid u(0) \text{ has eigenvalues in } \mathbb{Z} \}.
\end{align*}
\]

$E$ is a $\mathbb{Q}$-vector space in $H$ and $L \subseteq L_0 \subseteq E$ are additive subgroups.

Let $\mathfrak{H} \subseteq \mathfrak{G}$ be the group generated by exponentials $e^{2\pi i u(0)}$ ($u \in H$). From Section 7, $\mathfrak{H}$ is a complex torus $\mathfrak{H} \cong (\mathbb{C}^*)^l$. There is a short exact sequence

\[
0 \to L_0/L \to H/L \xrightarrow{\phi} \mathfrak{H} \to 1
\]

where $\phi$ arises from the morphism $u \mapsto e^{2\pi i u(0)}$ ($u \in H$). $H/L$ is the covering group of $H/L_0$ that acts linearly on each irreducible module $M^j$ as described in Section 6, and $H/L_0 \cong \mathfrak{H}$.

Because $V$ is f.g., there is an integer $n_0$ such that $V = \langle \bigoplus_{n=0}^{n_0} V_n \rangle$. Then $e^{2\pi i u(0)}$ ($u \in H$) is the identity if, and only if, its restriction to $\bigoplus_{n=0}^{n_0} V_n$ is the identity. It follows that the eigenvalues of $u(0)$ for $u \in E$ have bounded denominator, whence

\[
E/L_0 = \text{Torsion}(H/L_0) \cong (\mathbb{Q}/\mathbb{Z})^l.
\]

In particular, $E$ contains a $\mathbb{C}$-basis of $H$.

9. Deformation of $V$-modules

In [L2], Proposition 5.4, Li showed how to deform (twisted) $V$-modules using a certain operator $\Delta(z)$. We describe the special case that we need here. See [KM] for further details of the calculations below, and [DLM4] for further development of the theory.
Fix $u \in L_0$, and set
\[
\Delta_u(z) := z^{u(0)} \exp \left\{ - \sum_{k \geq 1} \frac{u(k)}{k} (-z)^{-k} \right\}.
\]
For an irreducible $V$-module $(M^{j'}, Y^{j'})$, set
\[
Y^{j'}_{\Delta_u(z)}(v, z) := Y^{j'}(\Delta_u(z)v, z) \quad (v \in V).
\]
Because $u(0)$ has eigenvalues in $\mathbb{Z}$ then $e^{2\pi i u(0)}$ is the identity automorphism of $V$. In this case, Li’s result says that there is an isomorphism of $V$-modules
\[
(M^{j'}, Y^{j'}_{\Delta_u(z)}) \cong (M^j, Y^j)
\]
for some $j$. (Technically, Li’s results deal with weak $V$-modules. In the case that we are dealing with, when $V$ is regular, the results apply to ordinary irreducible $V$-modules, as stated.) Thus there is a linear isomorphism $\psi : M^{j'} \xrightarrow{\cong} M^j$ satisfying
\[
\psi^{-1} Y^j(v, z) \psi = Y^{j'}(\Delta_u(z)v, z) \quad (v \in V).
\]
In (22) we choose $j' = 1$ (so $(M^{j'}, Y^{j'}) = (V, Y)$), $v = \omega$, and apply both sides to $1$. We obtain after some calculation that
\[
\psi^{-1} L^j(0) \psi(1) = 1/2 \langle u, u \rangle 1.
\]
The $L(0)$-grading on $M^j$ is described in (16). If $\psi(1) = \sum_n a_n$ with $a_n \in M^j_{n+\lambda_j}$, then $1/2 \langle u, u \rangle \sum_n a_n = \sum_n (n+\lambda_j) a_n$. This shows that $\psi(1) \in M^j_{n_0+\lambda_j}$ for some integer $n_0$, and moreover
\[
1/2 \langle u, u \rangle = n_0 + \lambda_j.
\]
We use (24) in conjunction with another Theorem ([DLM3], [AM]) that says that (for regular $V$) the conformal weight $\lambda_j$ of the irreducible $V$-module $M^j$ lies in $\mathbb{Q}$. Then it is immediate from (24) that $\langle u, u \rangle \in \mathbb{Q}$. The only condition on $u$ here is that $u \in L_0$. Because $E/L_0$ is a torsion group (20) we obtain
\[
\langle u, u \rangle \in \mathbb{Q} \quad (u \in E).
\]
Arguing along similar lines, we can also prove the following: (i) if \( u \in E \), all eigenvalues of the operators \( u_j(0) \) lie in \( \mathbb{Q} \) \((1 \leq j \leq r)\); (ii) if \( u \in L_0 \) then the denominators of the eigenvalues of \( u_j(0) \) divide the l.c.m. \( M \) of the denominators of the conformal weights \( \lambda_j \). In other words, \( L_0/L \) is a torsion abelian group of exponent dividing \( M \). (It is also f.g., as we shall see. So \( L_0/L \) is actually a finite abelian group.)

10. Weak Jacobi forms

The paper [KM] develops an extension of Zhu’s theory of partition functions [Z] to the context of weak Jacobi forms. We discuss background sufficient for our purposes. For the general theory of Jacobi forms, cf. [EZ].

We continue with a strongly regular VOA \( V \). Let \( h \in L \) (cf. (19)). For \( j \) in the range \( 1 \leq j \leq r \), define

\[
J_{j,h}(\tau, z) := \text{Tr}_{M_j} q^{L_j(0) - c/24} \zeta^{h_j(0)},
\]

where \( c \) is the central charge of \( V \). (The definition makes sense because we have seen that \( h_j(0) \) is a semisimple operator.) Notation is as follows: \( q := e^{2\pi i \tau} \), \( \zeta := e^{2\pi i z} \), \( \tau \in \mathbb{H} \) (complex upper half-plane), \( z \in \mathbb{C} \). The main result [KM] is that \( J_{j,h}(\tau, z) \) is holomorphic in \( \mathbb{H} \times \mathbb{C} \) and satisfies the following functional equations for all \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}), (u, v) \in \mathbb{Z}^2 \), \( 1 \leq i \leq r \):

(i) there are scalars \( a_{ij}(\gamma) \) depending only on \( \gamma \) such that

\[
J_{i,h}(\gamma \tau, \frac{z}{c\tau + d}) = e^{\pi i \varepsilon^2 (h, h)/(c\tau + d)} \sum_{j=1}^r a_{ij}(\gamma) J_{j,h}(\tau, z),
\]

(ii) there is a permutation \( j \mapsto j' \) of \( \{1, \ldots, r\} \) such that

\[
J_{j,h}(\tau, z + u\tau + v) = e^{-\pi i(h, h)(u^2\tau + 2uz)} J_{j',h}(\tau, z).
\]

This says that the \( r \)-tuple \( (J_{1,h}, \ldots, J_{r,h}) \) is a vector-valued weak Jacobi form of weight 0 and index \( 1/2(h, h) \). (By (25) we have \( (h, h) \in \mathbb{Q} \).) Part (i), which we do not need, is proved by making use of a theorem of Miyamoto [M], which itself extends some of the ideas in Zhu’s modular-invariance theorem [Z]. The proof of (ii) involves applications of the ideas of Section 9, and in particular the permutation in (27) is the same as the one we described earlier (loc. cit.)
11. The quadratic space \((E, \langle \cdot, \cdot \rangle)\)

We will prove the following result.

\[(E, \langle \cdot, \cdot \rangle)\] is a positive-definite rational quadratic space of rank \(l\), and \(L_0 \subseteq E\) is an additive subgroup of rank \(l\). \hspace{2cm} (28)

We have seen that both \(E/L_0\) and \(L_0/L\) are torsion groups. Hence \(E/L\) is also a torsion group, so in proving that \(\langle h, h \rangle > 0\) for \(0 \neq h \in E\), it suffices to prove this under the additional assumption that \(h \in L\). We assume this from now on, and set \(m = \langle h, h \rangle\). Note that the results of Section 10 apply in this situation.

We will show that \(m \leq 0\) leads to a contradiction. From (27) we know that \(J_{j,h}(\tau, z + u\tau + v) = J_{j',h}(\tau, z)\) for \(1 \leq j \leq r\). In terms of the Fourier series \(J_{j,h}(\tau, z) = \sum_{n,t} c(n, t)q^n \zeta^t, J_{j',h}(\tau, z) = \sum_{n,t} c'(n, t)q^n \zeta^t\), this reads

\[q^{\lambda_j - c/24} \sum_{n \geq 0, t} c(n, t)q^{n + mu^2/2 + tu} \zeta^{t + mu} = q^{\lambda_j' - c/24} \sum_{n \geq 0, t} c'(n, t)q^n \zeta^t\]

for all \(u \in \mathbb{Z}\), \(j'\) depending on \(u\). Suppose first that \(m = 0\). If for some \(t \neq 0\) there is \(c(n, t) \neq 0\) we let \(u \to -\infty\) and obtain a contradiction. Therefore, \(c(n, t) = 0\) whenever \(t \neq 0\). This says precisely that \(h_j(0)\) is the zero operator on \(M^j\). Furthermore, this argument holds for any index \(j\). But now (18) is contradicted. If \(m < 0\) the argument is even easier since we just have to let \(u \to -\infty\) to get a contradiction.

This proves that \(\langle \cdot, \cdot \rangle\) is positive-definite on \(E\), while rationality has already been established (25). Now we prove that \(E\) has rank \(l\), using an argument familiar from the theory of root systems (cf. [H], Section 8.5). We have already seen (cf. (20) and the line following) that \(E\) contains a basis of \(H\), say \(\{\alpha_1, \ldots, \alpha_l\}\). We assert that \(\{\alpha_1, \ldots, \alpha_l\}\) is a \(\mathbb{Q}\)-basis of \(E\).

Let \(u \in E\). There are scalars \(c_1, \ldots, c_l \in \mathbb{C}\) such that \(u = \sum_j c_j \alpha_j\). We have for \(1 \leq i \leq l\) that

\[\langle u, \alpha_i \rangle = \sum_j c_j \langle \alpha_i, \alpha_j \rangle.\hspace{2cm} (29)\]

Each \(\langle u, \alpha_i \rangle\) and \(\langle \alpha_i, \alpha_j \rangle\) are rational, and the nondegeneracy of \(\langle \cdot, \cdot \rangle\) implies that \(\langle \alpha_i, \alpha_j \rangle\) is nonsingular. Therefore, \(c_j = \langle u, \alpha_j \rangle/\det(\langle \alpha_i, \alpha_j \rangle) \in \mathbb{Q}\), as required.
We have proved that \( E \) is a \( \mathbb{Q} \)-form for \( H \), i.e. \( H = \mathbb{C} \otimes_{\mathbb{Q}} E \), so that \( E \) indeed has rank \( l \). That \( L_0 \subseteq E \) is a lattice of the same rank follows from (20). All parts of (28) are now established.

Now observe that the analysis that leads to the proof of (28) carries over verbatim to any nondegenerate subspace \( U \subseteq H \), say of rank \( l' \). For such a subspace we set \( E' := U \cap E, L' := U \cap L, L'_0 := U \cap L_0 \). The result can then be stated as follows:

\[
(U, \langle \ , \, \rangle) \text{ is a positive-definite rational quadratic space of rank } l', \text{ and } L'_0 \subseteq E \text{ is an additive subgroup of rank } l'. \tag{30}
\]

Another application of weak Jacobi forms allows us to usefully strengthen the statement (18) in some cases:

\[
\text{if } 0 \neq h \in E \text{ then } h_j(0) \neq 0 \text{ for each } 1 \leq j \leq r. \tag{31}
\]

Suppose false. Because \( E/L \) is a torsion group there is \( 0 \neq h \in L \) with \( h_j(0) = 0 \) for some index \( j \). Let \( m = \langle h, h \rangle \), so that \( m \neq 0 \). Then \( J_{j,h}(\tau, z) \) is a pure \( q \)-expansion, i.e. no nonzero powers of \( \zeta \) occur in the Fourier expansion. Indeed, it is just the partition function for \( M^j \), so it also does not vanish. By (27), \( e^{-\pi i (h, h)(u^2 \tau + 2uz)} J_{j', h}(\tau, z) = q^{-mu^2/2} \zeta^{-mu} \sum_{n \geq 0, t} c'(n, t) q^{n^2-\lambda j} \zeta^t \) is also a pure \( q \)-expansion. (As usual, \( j' \) depends on \( u \).) But because \( m > 0 \) we can let \( u \to \infty \) to see that in fact this power series is not a pure \( q \)-expansion. This contradiction proves (31).

12. Commutants

We retain previous notation. In particular, from now on we fix a Cartan subalgebra \( H \subseteq V_1 \) and a nondegenerate subspace \( U \subseteq H \) of rank \( l' \). Let \( M_U = (\langle U \rangle, Y, 1, \omega_U) \) be the Heisenberg subVOA of rank \( l' \) generated by \( U \) (cf. (2)). We set \( Y(\omega_U, z) := \sum_{n \in \mathbb{Z}} L_U(n) z^{-n-2} \).

Consider

\[
P_U := \{ (A, Y, 1, \omega_U) \mid A \subseteq V \}. \tag{32}\]

In words, \( P_U \) is the set of subVOAs \( A \subseteq V \) which have conformal vector \( \omega_U \). \( P_U \) is partially ordered by inclusion. It contains \( M_U \), for example.

One easily checks that \( L(1) \omega_U = 0 \). Therefore, the theory of commutants ([FZ], [LL], Section 3.11) shows that each \( A \in P_U \) has a compatible grading
with \((1)\). That is \(A_n := \{v \in A \mid L_U(0)v = n\} = A \cap V_n\). Moreover, \(\mathcal{P}_U\) has a unique maximal element. Indeed, the commutant \(C_V(A) = \ker V L_U(-1)\) for \(A \in \mathcal{P}_U\) is independent of \(A\), and the maximal element of \(\mathcal{P}_U\) is the double commutant \(C_V(C_V(A))\).

13. **U-weights**

Thanks to \((15)\) we can use the language of weights to describe the action of \(u(0) (u \in U)\). For \(\beta \in U\) set

\[V(\beta) := \{w \in V \mid u(0)w = \langle \beta, u \rangle w \mid u \in U\}\]

\(\beta\) is a \(U\)-weight, or simply weight (of \(V\)) if \(V(\beta) \neq 0\), \(V(\beta)\) is the \(\beta\)-weight space, and a nonzero \(w \in V(\beta)\) is a weight vector of weight \(\beta\).

Using the action of \(Y(u,z) (u \in U)\) on weight spaces, one shows that the set of \(U\)-weights

\[P := \{\beta \in U \mid V(\beta) \neq 0\}\]  

(33)

is a subgroup of \(U\). See [DM1], Section 4 for further details. By the complete reducibility of \(u(0) (u \in U)\) and the Stone von-Neumann theorem ([FLM], Section 1.7) applied to the Heisenberg subVOA \(M_U\), there is a weight space decomposition

\[V = M_U \otimes \Omega = \bigoplus_{\beta \in P} M_U \otimes \Omega(\beta)\]  

(34)

where \(\Omega := \{v \in V \mid u(n)v = 0 \mid u \in U, n \geq 1\}\), \(\Omega(\beta) := \Omega \cap V(\beta)\), and \(V(\beta) = M_U \otimes \Omega(\beta)\).

\(\Omega(0)\) is the commutant \(C_V(M_U)\), and \(M_U \otimes \Omega(0)\) the zero weight space. By arguments in [DM4] one sees that \(\Omega(0)\) is simple VOA (the simplicity of \(M_U\) is well-known), moreover each \(V(\beta)\) is an irreducible \(M_U \otimes \Omega(0)\)-module. So there is a tensor decomposition

\[V(\beta) = M_U(\beta) \otimes \Omega(\beta)\]

where \(M_U(\beta), \Omega(\beta)\) are irreducible modules for \(M_U, \Omega(0)\) respectively. Furthermore, \(V(\beta) \cong V(\beta')\) if, and only if, \(\beta = \beta'\). In particular, there is an identification

\[M_U(\beta) = M_U \otimes e^\beta\]  

(35)
where \( e^\beta \in \Omega(\beta) \).

14. **Lattice subalgebras of \( V \)**

We keep previous notation. In particular, \( P \) is the group of \( U \)-weights (33) and \( E' = U \cap E, L' = L \cap U, L'_0 = L_0 \cap U \) are as in Section 11 (cf. (30)).

Since \((E', \langle , \rangle)\) is a rational space (30) and contains a basis of \( U \), it follows that \( E' = \{ u \in U \mid \langle u, E' \rangle \subseteq \mathbb{Q} \} \). Because \( E'/L'_0 \) is a torsion group, we then see that \((L'_0)^0 \subseteq E'\). (Here, and below, we set \( F_0 := \{ u \in U \mid \langle u, F \rangle \subseteq \mathbb{Z} \} \) for \( F \subseteq E'\).) Now \( u \in P^0 \Leftrightarrow \langle P, u \rangle \subseteq \mathbb{Z} \Leftrightarrow \) all eigenvalues of \( u(0) \) are integral \( \Leftrightarrow u \in L'_0 \). We conclude that

\[
P = (L'_0)^0 \subseteq E' \tag{36}
\]

We will establish

\[
\text{there is a positive-definite even lattice } \Lambda \subseteq P \text{ such that } |P : \Lambda| \\
is finite and the maximal element } W \text{ of } \mathcal{P}_U \text{ satisfies } W \cong V_\Lambda. \tag{37}
\]

The argument utilizes ideas in [DM2]. Recall the isomorphism (21) which holds for all \( u \in L_0 \). Set

\[
\Gamma := \{ u \in L'_0 \mid (V, Y_{\Delta u(z)}) \cong (V, Y) \}. \tag{38}
\]

This is a subgroup of \( L'_0 \) of finite index. Although not necessary at this stage, we can show immediately that \( \Gamma \) is an even lattice. Indeed, if \( u \in \Gamma \) then the proof of (24) shows that we have \( \lambda_j = 0 \) in that display, whence \( \langle u, u \rangle = n_0 \) is a (nonnegative) integer. Now the assertion about \( \Gamma \) follows from (28).

There is another approach that gives more information. The isomorphism of \( V \)-modules defined for \( u \in \Gamma \) by (38) implies the following assertion concerning the weight spaces in (34):

\[
\Omega(\beta) \cong \Omega(\beta + u) \quad (u \in \Gamma, \beta \in P). \tag{39}
\]

In particular, taking \( \beta = 0 \) shows that \( \Omega(u) \neq 0 \) \( (u \in \Gamma) \), whence \( \Gamma \subseteq P \). Using (36) we deduce

\[
\Gamma \subseteq P, P^0 \subseteq \Gamma^0,
\]

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so $\Gamma$ is necessarily a positive-definite integral lattice of rank $l'$, and $|P : \Gamma| =: d$ is finite. (39) leads to a refinement of (34), namely a decomposition of $V$ into simple $M_U \otimes \Omega(0)$-modules

$$V = \bigoplus_{i=1}^{d} \bigoplus_{\beta \in \Gamma} M_U(\beta + \gamma_i) \otimes \Omega(\gamma_i),$$

where $\{\gamma_i \mid 1 \leq i \leq d\}$ are coset representatives for $P/\Gamma$.

Let

$$\Lambda := \{\beta \in P \mid \Omega(\beta) = \Omega(0)\},$$

(40)

$$W := \bigoplus_{\beta \in \Lambda} M_U(\beta).$$

(41)

Then $\Gamma \subseteq \Lambda$ and $W = C_V(\Omega(0)) = C_V(C_V(M_U))$. In particular, $\Lambda$ is an additive subgroup of $P$ of finite index and $W$ is a subVOA of $V$. Indeed, it is the maximal element of the poset $\mathcal{P}_U$ discussed in Section 12.

The $L(0)$-weight of $e^\beta \in W(\beta \in \Lambda)$ (cf. (35)) coincides with its $L_U(0)$-weight (cf. Section 12). Using the associativity formula, we have

$$L(0)e^\beta = L_0(0)e^\beta = 1/2 \sum_{t=1}^{l'} (h_t(-1)h_t)(1)e^\beta$$

$$= 1/2 \sum_{t=1}^{l'} \left\{ \sum_{k \geq 0} h_t(-1-k)h_t(1+k) + h_t(-k)h_t(k) \right\} e^\beta$$

$$= 1/2 \sum_{t=1}^{l'} h_t(0)h_t(0)e^\beta = 1/2 \sum_{t=1}^{l'} \langle \beta, h_t \rangle^2 e^\beta = 1/2 \langle \beta, \beta \rangle,$$

showing that $1/2\langle \beta, \beta \rangle \in \mathbb{Z}$ ($\beta \in \Lambda$).

This shows that $\Lambda$ is an even lattice of rank $l'$. The isomorphism $W \cong V_\Lambda$ then follows from the uniqueness of simple current extensions ([DM1], Section 5). This completes the proof of (37) and Theorem 1 is established.

15. The tripartite subVOA of $V$

We consider more closely the consequences of (37) in the case that $U := \text{rad}(V_1)$ is the radical of $V_1$. (37) is applicable here because $A$ is indeed nondegenerate (cf. Section 4). We keep the notation from previous Sections.
Observe that in this case, the lattice $\Lambda$ contains no roots, i.e. there is no $\beta \in \Lambda$ satisfying $\langle \beta, \beta \rangle = 2$. For if $\beta \in \Lambda$ is a root then $\beta$ is contained in an $\text{sl}_2$-subalgebra of $V_1$ and hence cannot lie in $A$. The commutant $\Omega(0)$ of $W$ contains the Levi factor $S \subseteq V_1$, hence also the subVOA $G$ that it generates (cf. ([1])) . We can then consider the commutant of $G$ in $\Omega(0)$, call it $Z$. In this way we obtain the canonical conformal subVOA of $V$ that we call the tripartite subalgebra

$$ T = W \otimes G \otimes Z. $$

By construction, $T$ is a conformal subalgebra of $V$, and the conformal gradings on $W, G, Z$ are compatible with the $L(0)$-grading on $V$. Because $(W \otimes G)_1 = W_1 \oplus G_1 = V_1$ then $Z_1 = 0$. This completes the proof of Theorem 3.

Conjecture: $Z$ is a strongly regular VOA.

This is just a special case of a more general conjecture, namely that the commutant of a rational subVOA (in a strongly regular VOA, say) is itself rational. If the Conjecture is true then the tripartite subalgebra $T$ is strongly regular, and $V$ reduces to a finite sum of irreducible $T$-modules. In this way, the classification of strongly regular VOAs reduces to the classification of strongly regular VOAs $Z$ with $Z_1 = 0$ and the extension problem as discussed in the Introduction.

16. The invariants $\tilde{c}$ and $l$

We give some applications of Theorem 1 exemplifying the philosophy of the previous paragraph. Let $V$ be a strongly regular VOA of central charge $c$ and $H \subseteq V$ a Cartan subalgebra of rank $l$. Recall ([DMI]) that the effective central charge of $V$ is the quantity

$$ \tilde{c}_V = \tilde{c} := c - 24\lambda_{\text{min}}. $$

Here, $\lambda_{\text{min}}$ is the minimum of the conformal weights $\lambda_j \ (1 \leq j \leq r)$ of the irreducible $V$-modules. It is known (loc. cit.) that $\tilde{c} \geq l$ and $\tilde{c} > 0$ if $\dim V > 1$. Because of these facts, $\tilde{c}$ is often a more useful invariant than $c$ itself. Note that $\tilde{c}$ is defined for any rational VOA.
We now give the proof of Theorem 7. The basic idea is to combine Zhu’s modular-invariance \([Z]\) together with growth conditions on the Fourier coefficients of components of vector-valued modular forms \([KnM]\). This method was first used in \([DM1]\). The availability of Theorem 1 brings added clarity.

It follows easily from the definitions that if \(W \subseteq V\) is a conformal subalgebra then \(\tilde{c}_V \leq \tilde{c}_W\). Moreover \(\tilde{c}\) is multiplicative over tensor products \(([FHL],\text{Section 4.6})\). So if (b) of Theorem 7 holds then \(\tilde{c}_V \leq \tilde{c}_V \Lambda + \tilde{c}_{L(c,p,q,0)}\).

Since \(\text{rk} \Lambda = l\) then \(\tilde{c}_V \Lambda = c = l\) because \(\lambda_{\text{min}} = 0\) for lattice theories \([D]\). Moreover, for the discrete series Virasoro VOA we have \(([DM1],\text{Section 4, Example (e)})\)

\[
\tilde{c} = 1 - \frac{6}{pq} \quad ((p, q) = 1, 2 \leq p < q),
\]

(42)

in particular we always have \(\tilde{c}_{L(c,p,q,0)} < 1\). Therefore \(\tilde{c}_V < l + 1\). This establishes the implication \((b) \Rightarrow (a)\) in Theorem 7.

Next, taking \(U = H\) in Theorem 1, we find that the maximal element of \(\mathcal{P}_H\) is a lattice subVOA \(W \cong V_\Lambda\) with \(\text{rk} \Lambda = \dim H = l\). Let \(C = C_V(W)\) be the commutant of \(W\). Then \(W \otimes C\) is a conformal subVOA of \(V\). Now suppose that part (b) of the Theorem does not hold. Thus the Virasoro subalgebra of \(C\), call it \(\text{Vir}_C\), has a central charge \(c'\), say, that is not in the discrete series. Then the known submodule structure of Verma modules over the Virasoro algebra shows that the partition function \(Z_{\text{Vir}_C}(\tau)\) of \(\text{Vir}_C\) satisfies

\[
Z_{\text{Vir}_C}(\tau) := \text{Tr}_{\text{Vir}_C} q^{L(0) - c'/24} = q^{-c'/24} \prod_{n=2}^{\infty} (1 - q^n)^{-1}.
\]

([DM1], Proposition 6.1 summarizes exactly what we need here.) Therefore,

\[
Z_{W \otimes \text{Vir}_C}(\tau) := Z_W(\tau)Z_{\text{Vir}_C}(\tau)
= \frac{\theta_\Lambda(\tau)}{\eta(\tau)^l} q^{-c'/24} \prod_{n=2}^{\infty} (1 - q^n)
= \frac{\theta_\Lambda(\tau)}{\eta(\tau)^{l+1}} q^{(1-c')/24} (1 - q).
\]

(\(\theta_\Lambda(\tau)\) and \(\eta(\tau)\) are the theta-function of \(\Lambda\) and the eta-function respectively.) It follows that for any \(\epsilon > 0\), the coefficients of the \(q\)-expansion of
\(\eta(\tau)^{l+1-\varepsilon}Z_{W \otimes \text{Vir}_C}(\tau)\) have exponential growth. Therefore, the same statement holds true ipso facto if we replace \(W \otimes \text{Vir}_C\) with \(V\). We state this as

the coefficients of \(\eta(\tau)^{l+1-\varepsilon}Z_V(\tau)\) have exponential growth \((\varepsilon > 0)\). \((43)\)

On the other hand, consider the column vector

\[
F(\tau) := (Z_{M_1}(\tau), \ldots, Z_{M_r}(\tau))^t
\]

whose components are the partition functions of the irreducible \(V\)-modules \(M^j\). By Zhu’s modular-invariance theorem, \(F(\tau)\) is a vector-valued modular form of weight 0 on the full modular group \(SL_2(\mathbb{Z})\) associated with some representation of \(SL_2(\mathbb{Z})\). (See [MT], Section 8 for a discussion of vector-valued modular forms in the context of VOAs.) Moreover, each \(Z_{M^j}(\tau)\) is holomorphic in the complex upper half-plane, so that their only poles are at the cusps. The very definition of \(\tilde{c}\), and the reason for its importance, is that the maximum order of a pole of any of the partition functions \(Z_{M^j}(\tau)\) is \(\tilde{c}/24\). It follows from this that

\[
\eta(\tau)^{\tilde{c}}F(\tau)
\]

is a holomorphic vector-valued modular form on \(SL_2(\mathbb{Z})\). As such, the Fourier coefficients of the component functions have polynomial growth \([KnM]\). In particular, this applies to \(\eta(\tau)^{\tilde{c}}Z_V(\tau)\), which is one of the components.

Comparing the last statement with \((43)\), it follows that \(\tilde{c} > l + 1 - \varepsilon\) for all \(\varepsilon > 0\), i.e. \(\tilde{c} \geq l + 1\). So we have shown that if part (b) of the Theorem does not hold, neither does part (a). Theorem 7 is thus proved.

The special case \(l = 0\) of the Theorem characterizes minimal models. We state it as

**Theorem 8:** Let \(V\) be a strongly regular VOA. Then \(\tilde{c} < 1\) if, and only if, the Virasoro subalgebra of \(V\) is in the discrete series.

**Corollary 9:** Let \(V\) be a strongly regular VOA with \(\dim V > 1\). Then \(\tilde{c} \geq 2/5\), and equality holds if, and only if, \(V \cong L(c_{2,5}, 0)\), the (Yang-Lee) discrete series Virasoro VOA with \(c = -22/5\).

Because \(\dim V > 1\) then \(\tilde{c} > 0\), and if \(\tilde{c} < 1\) then \(V\) is a minimal model by Theorem 8. Inspection of \((42)\) shows that the least positive value is \(2/5\), corresponding to the Yang-Lee model. This theory has only two irreducible
modules, of conformal weight 0 and $-1/5$. Therefore the second irreducible cannot be contained in $V$, so that $V \cong L(c_{2,5},0)$, as asserted in Corollary 9. Informally, the Corollary says that the Yang-Lee theory is the \textit{smallest} rational CFT.

We give a final numerical example. Suppose that $V$ is a strongly regular simple VOA such that $1 < \tilde{c} < 7/5$. Since $l \leq c$ we must have $l = 0$ or 1. In the latter case, by Theorem 7 we see that $V$ contains as a conformal subVOA a tensor product $V_\Lambda \otimes Vir$ where $Vir$ is a Virasoro algebra in the discrete series with $0 < \tilde{c}_{Vir} < 2/5$. This is impossible by Corollary 9. So in fact $l = 0$, i.e. $V$ has Lie rank 0, meaning that $V_1 = 0$. The smallest value of $\tilde{c}$ in the range $(1, 7/5)$ that I know of is a parafermion theory with $\tilde{c} = 8/7$.

References

[AM] G. Anderson and G. Moore, Rationality in conformal field theory, Comm. Math. Phys. \textbf{117} (1988), 441-450.

[B] G. Buhl, A spanning set for VOA modules, J. Alg. \textbf{254} No. 1 (2004), 125-151.

[D] C. Dong, Vertex algebras associated with even lattices, J. Alg \textbf{160} (1993), 245-265.

[DG] C. Dong and R. Griess, Automorphism groups and derivation algebras of finitely generated vertex operator algebras, Mich. Math. J. \textbf{50} (2002), 227-239.

[DLM1] C. Dong, H. Li and G. Mason, Regularity of rational vertex operator algebras, Adv. Math. \textbf{132} (1997), 148-166.

[DLM2] C. Dong, H. Li and G. Mason, Twisted representations of vertex operator algebras, Math. Ann. \textbf{310} (1998), 571-600.

[DLM3] C. Dong, H. Li and G. Mason, Modular-invariance of trace functions in orbifold theory and generalized moonshine, Comm. Math. Phys. \textbf{214} (2000), 1-56.

[DLM4] C. Dong, H. Li and G. Mason, Simple Currents and Extensions of Vertex Operator Algebras, Comm. Math. Phys. \textbf{180} (1996), 671-707.
[DLWY] C. Dong, C. Lam, Q. Wang and H. Yamada, The structure of parafermion vertex operator algebras, J. Algebra 323 (2010), 371-381.

[DM1] C. Dong and G. Mason, Rational vertex operator algebras and the effective central charge, Int. Math. Res. Not. 56 (2004), 2989-3008.

[DM2] C. Dong and G. Mason, Integrability of $C_2$-cofinite Vertex Operator Algebras, Int. Math. Res. Not. Vol. 2006, Art. ID 80468, 1-15.

[DM3] C. Dong and G. Mason, Nonabelian orbifolds and the boson-fermion correspondence, Comm. Math. Phys. 163 (1994), 523-559.

[DM4] C. Dong and G. Mason, On quantum Galois theory, Duke J. Math. 86 No. 2 (1997), 305-321.

[DM5] C. Dong and G. Mason, Shifted vertex operator algebras, Math. Proc. Camb. Phil. Soc. 141 (2006), 67-80.

[DW1] C. Dong and W. Zhang, Rational vertex operator algebras are finitely generated, J. Alg 320 No. 6 (2008), 2610-2614.

[DW2] C. Dong and W. Zhang, Toward classification of rational vertex operator algebras with central charges less than 1, arXiv:0711.4625v1.

[EZ] M. Eichler and D. Zagier, The Theory of Jacobi Forms, Progress in Mathematics 55, Birkhäuser, Boston, 1985.

[FHL] I. Frenkel, Y.-Z. Huang and J. Lepowsky, On Axiomatic Approaches to Vertex Operator Algebras, Memoirs AMS.

[FLM] I. Frenkel, J. Lepowsky and A. Meurman, Vertex Operator Algebras and the Monster, Academic Press, London, 1988.

[FMS] P. Di Francesco, P. Mathieu and D. Sénéchal, Conformal Field Theory, Springer, 1996.

[FZ] I. Frenkel and Y. Zhu, Vertex operator algebras associated to representations of affine and Virasoro algebras, Duke J. Math. 66 No. 1 (1992), 123-168.
[GN] M. Gaberdiel and A. Nietzke, Rationality, quasirationality and finite $W$-algebras, CMP 238 No. 1-2 (2003), 305-331.

[H] J. Humphreys, Introduction to Lie Algebras and Representation Theory, Graduate Texts in Mathematics 9, Springer, New York, 1972.

[KnM] M. Knopp and G. Mason, On vector-valued modular forms and their Fourier coefficients, Acta Arith. 110 (2003), 117-124.

[KM] M. Krauel and G. Mason, Vertex operator algebras and weak Jacobi forms, to appear in Intnl. J. Math.

[L1] H. Li, Symmetric invariant bilinear forms on vertex operator algebras, J. Pure Appl. Math 96 (1994), 279-297.

[L2] H. Li, Local systems of twisted vertex operators, vertex operator superalgebras and twisted modules, in Moonshine, the Monster, and related topics Contemp. Math. Vol. 193 (1996), 203-236.

[L3] H. Li, On abelian coset generalized vertex algebras, Comm. Contemp. Math. 3 No. 2 (2001), 287-340.

[LL] J. Lepowsky and H. Li, Introduction to vertex operator algebras, Birkhäuser, Boston,

[M] M. Miyamoto, A modular invariance on the theta functions defined on vertex operator algebras, Duke Math. J. 101 No. 2 (2000), 221-236.

[MT] G. Mason and M. Tuite, Vertex operators and modular forms, in A Window into Zeta and Modular Physics, MSRI Publ., CUP, 2010, 183-278.

[Z] Y. Zhu, Modular-invariance of characters of vertex operator algebras, JAMS. 9 No.1 (1996), 237–302.