On Fixed Points in the Setting of $C^*$-Algebra-Valued Controlled $F_\varepsilon$-Metric Type Spaces

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Abstract

In the present article, we first examine the conception of $C^*$-algebra-valued controlled $F_\varepsilon$-metric type spaces as a generalization of $F$-cone metric spaces over Banach algebra. Further, we prove some fixed point theorem with different contractive conditions in the framework of $C^*$-algebra-valued controlled $F_\varepsilon$-metric type spaces. Secondly, we furnish an example by means of the acquired result.

2010 AMS Classification: 47H10, 54H25.

Keywords and Phrases: $C^*$-algebra; $C^*$-algebra-valued controlled $F_\varepsilon$-metric type spaces; Contractive mapping; Fixed point theorem.

Article Type: Research Article.

1 Introduction

The conception of $b$-metric space was initiated by Bakhtin [10] as a generalization of metric spaces. In 1994, Matthews [12] proposed the concept of partial metric spaces where the self-distance of any point need not be zero. Tayyab Kamran et al. [11] introduced a new type of metric spaces, namely extended $b$-metric spaces by replacing the constant $s$ by a function $\theta(x,y)$ depending on the parameters of the left-hand side of the triangle inequality. Nabil Mlaiki et al. [23] proved Banach contraction principle in the setting of controlled metric type spaces which is a generalization of extended $b$-metric space. For more engrossing results in extended $b$-metric spaces, the reader may refer to [3–9]. In [2], Aiman Mukheimer have recently examined the hypothesis of extended partial $S_b$-metric spaces.

On the other hand, Fernandez et al. [1] established the notion of $F$-cone metric space over Banach algebra and investigated the existence and uniqueness of the fixed point under the same metric. In [24], Ma initiated the concept of $C^*$-algebra-valued metric spaces where the set of real numbers is replaced by the set of all positive elements of a unital $C^*$-algebra. For further probes on $C^*$-algebra, we refer to [13–22].

As noted above, a vigorous research on fixed point results in $C^*$-algebra-valued metric spaces, extended $b$-metric spaces and controlled metric type spaces has been developed in the
past few years, we focus our study on the concept of $C^*$-algebra-valued controlled $F_c$-metric type spaces in the present paper and prove fixed point theorem with disparate contractive condition.

2 Preliminaries

To start with, we recollect some necessary definitions which will be utilized in the main theorem.

Throughout this paper, $\mathbb{A}$ denotes an unital $C^*$-algebra. Set $\mathbb{A}_h = \{z \in \mathbb{A} : z = z^*\}$. We call an element $z \in \mathbb{A}$ a positive element, denote it by $\theta_{\mathbb{A}} \leq z$, if $z \in \mathbb{A}_h$ and $\sigma(z) \subseteq [0, \infty)$, where $\theta_{\mathbb{A}}$ is a zero element in $\mathbb{A}$ and $\sigma(z)$ is the spectrum of $z$. There is a natural partial ordering on $\mathbb{A}_h$ given by $z \leq w$ if and only if $\theta_{\mathbb{A}} \leq w - z$. We denote $\mathbb{A}_+$ and $\mathbb{A}_h'$ as $\{z \in \mathbb{A} : \theta_{\mathbb{A}} \leq z\}$ and the set $\{z \in \mathbb{A} : zw = wz, \forall w \in \mathbb{A}\}$ and $|z| = (z^*z)^{\frac{1}{2}}$ respectively.

Definition 2.1. [1] Let $X$ be a nonempty set. A function $F : X^3 \to A$ is called $F$-cone metric on $X$ if for any $\alpha, \beta, \gamma, \delta \in X$, the following conditions hold:
1. $\alpha = \beta = \gamma$ if and only if $F(\alpha, \alpha, \alpha) = F(\beta, \beta, \beta) = F(\gamma, \gamma, \gamma) = F(\alpha, \beta, \gamma)$;
2. $\theta \leq F(\alpha, \alpha, \alpha) \leq F(\alpha, \alpha, \beta) \leq F(\alpha, \beta, \gamma)$ for all $\alpha, \beta, \gamma \in X$ with $\alpha \neq \beta \neq \gamma$;
3. $F(\alpha, \beta, \gamma) \leq s[F(\alpha, \alpha, \beta) + F(\beta, \beta, \delta) + F(\gamma, \gamma, \delta)] - F(\delta, \delta, \delta)$.

Then the pair $(X, F)$ is called an $F$-cone metric space over Banach Algebra $\mathbb{A}$. The number $s \geq 1$ is called the coefficient of $(X, F)$.

Definition 2.2. [16] Let $X$ be a nonempty set and $A \in \mathbb{A}'$ such that $A \geq I_{\mathbb{A}}$. Suppose the mapping $S_b : X \times X \times X \to \mathbb{A}$ satisfies:
1. $\theta_{\mathbb{A}} \leq S_b(\alpha, \beta, \gamma)$ for all $\alpha, \beta, \gamma \in X$ with $\alpha \neq \beta \neq \gamma \neq \alpha$;
2. $S_b(\alpha, \beta, \gamma) = \theta_{\mathbb{A}}$ if and only if $\alpha = \beta = \gamma$;
3. $S_b(\alpha, \beta, \gamma) \leq A[S_b(\alpha, \alpha, \beta) + S_b(\beta, \beta, \delta) + S_b(\gamma, \gamma, \delta)]$ for all $\alpha, \beta, \gamma, \delta \in X$.

Then $S_b$ is said to be $C^*$-algebra-valued $S_b$-metric on $X$ and $(X, \mathbb{A}, S_b)$ is said to be a $C^*$-algebra-valued $S_b$-metric space.

Definition 2.3. [23] Given a non-empty set $X$ and $\delta : X \times X \to [1, \infty)$. A function $d : X \times X \to [0, \infty)$ is called a controlled metric type if:
1. $d(\alpha, \beta) = 0$ if and only if $\alpha = \beta$;
2. $d(\alpha, \beta) = d(\beta, \alpha)$;
3. $d(\alpha, \beta) \leq \delta(\alpha, \gamma)d(\alpha, \gamma) + \delta(\gamma, \beta)d(\gamma, \beta)$, for all $\alpha, \beta, \gamma \in X$.

The pair $(X, d)$ is called a controlled metric type space.

3 Main Results

In this main segment, as a generalization of $F$-cone metric space over Banach algebra, we introduce the notion of $C^*$-algebra valued controlled $F_c$-metric type spaces and furnish an example of the underlying spaces.

Hereinafter, $\mathbb{A}_I'$ will denote the set $\{z \in \mathbb{A} : zw = wz, \forall w \in \mathbb{A}$ and $z \geq I_{\mathbb{A}}\}$ respectively.
Definition 3.1. Let $X$ be a nonempty set and $C : X \times X \times X \to \mathbb{A}_l$. Suppose the mapping $F_c : X \times X \times X \to \mathbb{A}$ satisfies:
1. $\varpi = \bar{\nu} = \bar{\zeta}$ if and only if $F_c(\varpi, \varpi, \varpi) = F_c(\bar{\nu}, \bar{\nu}, \bar{\nu}) = F_c(\bar{\zeta}, \bar{\zeta}, \bar{\zeta}) = F_c(\varpi, \bar{\nu}, \bar{\zeta})$;
2. $\theta_{\mathbb{A}} \leq F_c(\varpi, \varpi, \varpi) \leq F_c(\varpi, \bar{\nu}, \bar{\nu}) \leq F_c(\varpi, \bar{\zeta}, \bar{\zeta})$;
3. $F_c(\varpi, \bar{\nu}, \bar{\zeta}) \leq C(\varpi, \varpi, \alpha)F_c(\varpi, \varpi, \alpha) + C(\bar{\nu}, \bar{\nu}, \alpha)F_c(\bar{\nu}, \bar{\nu}, \alpha) + C(\bar{\zeta}, \bar{\zeta}, \alpha)F_c(\bar{\zeta}, \bar{\zeta}, \alpha) - F_c(\bar{\alpha}, \bar{\alpha}, \bar{\alpha})$ for all $\varpi, \bar{\nu}, \bar{\zeta}, \bar{\alpha} \in X$.

Then $F_c$ is called a $C^*$-algebra-valued controlled $F_c$-metric type on $X$ and $(X, \mathbb{A}, F_c)$ is a $C^*$-algebra-valued controlled $F_c$-metric type spaces.

Remark 3.2. If $C(\varpi, \varpi, \alpha) = C(\bar{\nu}, \bar{\nu}, \alpha) = C(\bar{\zeta}, \bar{\zeta}, \alpha) = C(\varpi, \bar{\nu}, \bar{\zeta})$ for all $\varpi, \bar{\nu}, \bar{\zeta}, \bar{\alpha} \in X$, then we get

$$F_c(\varpi, \bar{\nu}, \bar{\zeta}) \leq C(\varpi, \bar{\nu}, \bar{\zeta})[F_c(\varpi, \varpi, \varpi) + F_c(\bar{\nu}, \bar{\nu}, \bar{\nu}) + F_c(\bar{\zeta}, \bar{\zeta}, \bar{\zeta}) - F_c(\bar{\alpha}, \bar{\alpha}, \bar{\alpha})].$$

In this case, $F_c$ is called a $C^*$-algebra-valued extended $F_c$-metric on $X$ and $(X, \mathbb{A}, c)$ is called a $C^*$-algebra-valued extended $F_c$-metric space.

Remark 3.3. In a $C^*$-algebra-valued controlled $F_c$-metric type space $(X, \mathbb{A}, F_c)$, if $\varpi, \bar{\nu}, \bar{\zeta} \in X$ and $F_c(\varpi, \bar{\nu}, \bar{\zeta}) = 0$, then $\varpi = \bar{\nu} = \bar{\zeta}$, but the converse need not be true.

Definition 3.4. A $C^*$-algebra-valued controlled $F_c$-metric type space $(X, \mathbb{A}, F_c)$ is said to be symmetric if it satisfies,

$$F_c(\varpi, \varpi, \bar{\nu}) = F_c(\varpi, \bar{\nu}, \varpi), \text{ for all } \varpi, \bar{\nu} \in X.$$ 

Example 3.5. Let $X = \{0, 1, 2, \ldots\}$ and $\mathbb{A} = \mathbb{R}^2$. If $\alpha, \beta \in \mathbb{A}$ with $\varpi = (\varpi_1, \varpi_2), \bar{\nu} = (\nu_1, \nu_2), k \varpi = (k \varpi_1, k \varpi_2), k \nu \bar{\nu} = (k \nu_1, \nu_2 \nu_2)$

Now define the metric $F_c : X \times X \times X \to \mathbb{A}$ and the control function $C : X \times X \times X \to \mathbb{A}_l$ as:

$$F_c(\varpi, \bar{\nu}, \bar{\zeta}) = \left( \frac{1}{2}(|\varpi + \bar{\zeta}|^2 + |\bar{\nu} + \bar{\zeta}|^2), \frac{1}{2}(|\varpi + \bar{\zeta}|^2 + |\bar{\nu} + \bar{\zeta}|^2) \right)$$

and

$$C(\varpi, \bar{\nu}, \bar{\zeta}) = \left( |\varpi + \bar{\nu} - \bar{\zeta} + 1|, |\varpi + \bar{\nu} - \bar{\zeta} + 1| \right).$$

It is easy to verify that $F_c$ is a $C^*$-algebra-valued controlled $F_c$-metric type space. Indeed for $\varpi = 1, \bar{\nu} = 2, \bar{\zeta} = 3$ and $\bar{\alpha} = 0$, we have

$$F_c(1, 2, 3) = (20.5, 20.5) \geq (1, 1)(1, 1) + (4, 4) + (9, 9) - (0, 0) = (14, 14) = C(1, 2, 3)[F_c(1, 1, 0) + F_c(2, 2, 0) + F_c(3, 3, 0)] - F_c(0, 0, 0).$$

Hence $F_c$ is not a $C^*$-algebra-valued extended $F_c$-metric space.
Definition 3.6. A sequence \( \{\varpi_n\} \) in a \( C^* \)-algebra-valued controlled \( F_c \)-metric type space is said to be:

(i) convergent sequence \( \iff \exists \varpi \in X \) such that \( F_c(\varpi_n, \varpi, \varpi) \to \theta_A \) as \( n \to \infty \) and we denote it by \( \lim_{n \to \infty} \varpi_n = \varpi \);

(ii) Cauchy sequence \( \iff F_c(\varpi_n, \varpi, \varpi_m) \to \theta_A \) as \( n, m \to \infty \).

Definition 3.7. A \( C^* \)-algebra-valued controlled \( F_c \)-metric type space \( (X, A, F_c) \) is said to be complete if every Cauchy sequence is convergent in \( X \) with respect to \( A \).

Theorem 3.8. Let \( (X, A, F_c) \) be a complete symmetric \( C^* \)-algebra-valued controlled \( F_c \)-metric type space and suppose \( T : X \to X \) is a mapping satisfying the following condition:

\[
F_c(T\varpi, T\varpi, T\varpi) \leq P^* F_c(\varpi, \varpi, \varpi) + Q^* F_c(\varpi, \varpi, T\varpi) + R^* F_c(\varpi, \varpi, T\varpi) R, \forall \varpi \in X, \tag{1}
\]

where \( P, Q, R \in A \) with \( \|P\|, \|Q\|, \|R\| \geq 0 \) satisfying \( \|P\|^2 + \|Q\|^2 + \|R\|^2 < 1 \) and for \( \varpi_0 \in X \), choose \( \varpi_n = T^n \varpi_0 \) assume that

\[
\sup_{m \geq 1} \lim_{n \to \infty} \|C(\varpi_{i+1}, \varpi_{i+1}, \varpi_{i+2})C(\varpi_{i+1}, \varpi_{i+1}, \varpi_m)\| < \frac{1 - \|R\|^2}{\|P\|^2 + \|Q\|^2}. \tag{2}
\]

In addition, for each \( \varpi \in X \), suppose that

\[
\lim_{n \to \infty} \|C(\varpi, \varpi, \varpi_n)\| \text{ and } \lim_{n \to \infty} \|C(\varpi_n, \varpi_n, \varpi)\| \tag{3}
\]

exist and are finite. Then \( T \) has a unique fixed point in \( X \).

Proof. Let \( \varpi_0 \in X \) be arbitrary and define the iterative sequence \( \{\varpi_n\} \) by:

\[
\varpi_{n+1} = T\varpi_n = \ldots = T^{n+1}\varpi_0, \quad n = 1, 2, \ldots. \tag{4}
\]

If follows from (1) and (4) that

\[
F_c(\varpi_n, \varpi_n, \varpi_{n+1}) = F_c(T\varpi_{n-1}, T\varpi_{n-1}, T\varpi_n) \\
\leq P^* F_c(\varpi_{n-1}, \varpi_{n-1}, \varpi_n) P + Q^* F_c(\varpi_{n-1}, \varpi_{n-1}, T\varpi_{n-1}) Q + \\
R^* F_c(\varpi_n, \varpi_n, T\varpi_n) R \\
\iff \|F_c(\varpi_n, \varpi_n, \varpi_{n+1})\| \leq \|P^* F_c(\varpi_{n-1}, \varpi_{n-1}, \varpi_n) P + Q^* F_c(\varpi_{n-1}, \varpi_{n-1}, T\varpi_{n-1}) Q + \\
R^* F_c(\varpi_n, \varpi_n, T\varpi_n) R\| \\
\leq \|P^* F_c(\varpi_{n-1}, \varpi_{n-1}, \varpi_n) P\| + \|Q^* F_c(\varpi_{n-1}, \varpi_{n-1}, T\varpi_{n-1}) Q\| + \\
\|R^* F_c(\varpi_n, \varpi_n, T\varpi_n) R\| \\
= (\|P\|^2 + \|Q\|^2) \|F_c(\varpi_{n-1}, \varpi_{n-1}, \varpi_n)\| + \|R\|^2 \|F_c(\varpi_n, \varpi_n, \varpi_{n+1})\| \\
\therefore \|F_c(\varpi_n, \varpi_n, \varpi_{n+1})\| \leq \frac{\|P\|^2 + \|Q\|^2}{1 - \|R\|^2} \|F_c(\varpi_{n-1}, \varpi_{n-1}, \varpi_n)\| \tag{5}
\]
Accordingly we get

\[ ||F_c(\omega_n, \omega_n, \omega_{n+1})|| \leq ||S||^2||F_c(\omega_{n-1}, \omega_{n-1}, \omega_n)|| \]
\[ = ||S^*S|| ||F_c(\omega_{n-1}, \omega_{n-1}, \omega_n)|| \]
\[ \leq ||S^*|| ||F_c(\omega_{n-1}, \omega_{n-1}, \omega_n)|| ||S|| \]

(6)

\[ \iff F_c(\omega_n, \omega_n, \omega_{n+1}) \leq S^* F_c(\omega_{n-1}, \omega_{n-1}, \omega_n) S, \]

where \( ||S||^2 = \frac{||P||^2+||Q||^2}{1-||R||^2} < 1 \). Recursively, we find that

\[ F_c(\omega_n, \omega_n, \omega_{n+1}) \leq (S^*)^n F_c(\omega_{n-1}, \omega_{n-1}, \omega_n) S^n \]  

(7)

For any \( n \geq 1 \) and \( q \geq 1 \), we have

\[ F_c(\omega_n, \omega_n, \omega_{n+q}) \leq C(\omega_n, \omega_n, \omega_{n+1}) F_c(\omega_n, \omega_n, \omega_{n+1}) + C(\omega_n, \omega_n, \omega_{n+1}) F_c(\omega_n, \omega_n, \omega_{n+1}) + \]
\[ C(\omega_n, \omega_n, \omega_{n+1}) F_c(\omega_n, \omega_n, \omega_{n+1}) - F_c(\omega_{n+1}, \omega_{n+1}, \omega_{n+1}) \]
\[ \leq 2C(\omega_n, \omega_n, \omega_{n+1}) F_c(\omega_n, \omega_n, \omega_{n+1}) + C(\omega_{n+q}, \omega_{n+q}, \omega_{n+1}) \]
\[ F_c(\omega_{n+1}, \omega_{n+1}, \omega_{n+q}) \]
\[ \leq 2C(\omega_n, \omega_n, \omega_{n+1}) F_c(\omega_n, \omega_n, \omega_{n+1}) + C(\omega_{n+q}, \omega_{n+q}, \omega_{n+1}) \]
\[ 2C(\omega_{n+1}, \omega_{n+1}, \omega_{n+2}) F_c(\omega_{n+1}, \omega_{n+1}, \omega_{n+2}) + C(\omega_{n+q}, \omega_{n+q}, \omega_{n+2}) F_c(\omega_{n+2}, \omega_{n+2}, \omega_{n+q}) - F_c(\omega_{n+2}, \omega_{n+2}, \omega_{n+2}) \]
\[ \vdots \]
\[ = 2C(\omega_n, \omega_n, \omega_{n+1}) F_c(\omega_n, \omega_n, \omega_{n+1}) + \]
\[ 2 \sum_{i=n+1}^{n+q-2} C(\omega_i, \omega_i, \omega_{i+1}) F_c(\omega_i, \omega_i, \omega_{i+1}) \prod_{j=n+1}^{i} C(\omega_{n+q}, \omega_{n+q}, \omega_j) + \]
\[ \prod_{i=n+1}^{n+q-1} C(\omega_{n+q}, \omega_{n+q}, \omega_i) F_c(\omega_{n+q-1}, \omega_{n+q-1}, \omega_{n+q}) \]
\[ \leq 2C(\omega_n, \omega_n, \omega_{n+1}) F_c(\omega_n, \omega_n, \omega_{n+1}) + \]
\[ 2 \sum_{i=n+1}^{n+q-1} C(\omega_i, \omega_i, \omega_{i+1}) F_c(\omega_i, \omega_i, \omega_{i+1}) \prod_{j=n+1}^{i} C(\omega_{n+q}, \omega_{n+q}, \omega_j) \]
\[ \leq 2C(\omega_n, \omega_n, \omega_{n+1})(S^*)^n S_0 S^n + \]
\[ 2 \sum_{i=n+1}^{n+q-1} C(\omega_i, \omega_i, \omega_{i+1})(S^*)^i S_0 S^i \prod_{j=1}^{i} C(\omega_{n+q}, \omega_{n+q}, \omega_j) \]
\[2 \left( S_0^2 C_0 \right) \left( S_0^2 C_0 \right)^\ast \left( S_0^2 C_0 \right)^\ast + \]
\[
2 \sum_{i=n+1}^{n+q-1} \left( S_0^2 \left( C, w_{i, w_{i+1}}, w_{i+1} \right) \prod_{j=1}^{i} C(w_{n+q, w_{n+q}, w_{j+1}}) \right)^\frac{1}{2} \leq \]
\[
2 \left| S_0^2 C(w_{n, w_{n+1}}) \right|^2 + \]
\[
2 \sum_{i=n+1}^{n+q-1} \left| S_0^2 \left( C, w_{i, w_{i+1}}, w_{i+1} \right) \prod_{j=1}^{i} C(w_{n+q, w_{n+q}, w_{j+1}}) \right|^\frac{1}{2} \]
\[
\leq 2 \left\| S_0 \right\| \left\| \left( C, w_{n, w_{n+1}} \right) \prod_{j=1}^{n+q} C(w_{n+q, w_{n+q}, w_{j+1}}) \right\| \left\| S \right\| ^\frac{1}{2} I_a + \]
\[
\left\| C(w_{i, w_{i+1}}) \prod_{j=1}^{n+q} C(w_{n+q, w_{n+q}, w_{j+1}}) \right\| \left\| S \right\| ^\frac{1}{2} I_a \]
\]

where \( I_a \) is the unit element in \( A \) and \( C(w_{i, w_{i+1}}, w_{0}) = S_0 \) for some \( S_0 \in A \). Let \( Y_m = \sum_{i=1}^{m} \left\| S \right\|^2 \left\| C(w_{i, w_{i+1}}, w_{i+1}) \right\| \prod_{j=1}^{i} C(w_{n+q, w_{n+q}, w_{j+1}}) \right\|. \) Consequently the above inequality implies,
\[
F_c(w_{n, w_{n+1}, w_{n+1}}) \leq 2 \left\| S_0 \right\| \left\| \left( C, w_{n, w_{n+1}} \right) \prod_{j=1}^{n+q} C(w_{n+q, w_{n+q}, w_{j+1}}) \right\| \left\| S \right\| ^\frac{1}{2} + \left( Y_{n+q-1} - Y_n \right) \right) I_a \]
\[
\text{(8)} \]
\]

The ratio test jointly with (2) implies that the limit of the sequence \( \{ Y_n \} \) exists and so \( \{ Y_n \} \) is Cauhy. Letting \( n \to \infty \) in the inequality above, we get
\[
\lim_{n \to \infty} F_c(w_{n, w_{n+1}, w_{n+1}}) = \theta_a. \]
\[
\text{(9)} \]
\]

Wherefore the sequence \( \{ w_n \} \) is Cauchy with respect to \( A \). Since \( (X, A, F_c) \) is a complete \( C^\ast \)-algebra-valued controlled \( F_c \)-metric type space, there exists a point \( w \in X \) such that
\[
\lim_{n \to \infty} F_c(w_{n, w_{n+1}, w_{n+1}}) = \theta_a. \]
\[
\text{(10)} \]
\]

Consider,
\[
F_c(w, w, w_{n+1}) \leq 2C(w, w, w_{n+1}) F_c(w, w, w_{n+1}) + C(w_{n+1}, w_{n+1}, w_{n+1}) F_c(w_{n+1}, w_{n+1}, w_{n+1}) \]
\[
\left. - F_c(w_{n+1}, w_{n+1}, w_{n+1}) \right) \]
\[
\iff \left\| F_c(w, w, w_{n+1}) \right\| \leq 2 \left\| C(w, w, w_{n+1}) \right\| \left\| F_c(w, w, w_{n+1}) \right\| + \left\| C(w_{n+1}, w_{n+1}, w_{n+1}) \right\| \left\| F_c(w_{n+1}, w_{n+1}, w_{n+1}) \right\| \]
\[
\left\| F_c(w_{n+1}, w_{n+1}, w_{n+1}) \right\| \]
\]
\]

It yields from (14) and (10) that
\[
\lim_{n \to \infty} \left\| F_c(w, w, w_{n+1}) \right\| = 0. \]
\[
\text{(11)} \]
Hence
\[ \|F_c(\varpi, \varpi, T\varpi)\| \leq 2\|C(\varpi, \varpi, \varpi_{n+1})\|\|F_c(\varpi, \varpi, \varpi_{n+1})\| + \|C(T\varpi, T\varpi, \varpi_{n+1})\| \]
\[ \|F_c(\varpi_{n+1}, \varpi_{n+1}, T\varpi)\| \]
\[ = 2\|C(\varpi, \varpi_{n+1})\|\|F_c(\varpi, \varpi, \varpi_{n+1})\| + \|C(T\varpi, T\varpi, \varpi_{n+1})\| \]
\[ \|F_c(T^{n+1}\varpi, T^{n+1}\varpi, T\varpi)\| \]

Regarding (11), we get \( \|F_c(\varpi, \varpi, \varpi_{n+1})\| \to 0 \) as \( n \to \infty \). Since \( T^n \to x \) and from continuity of \( T \), we acquire \( T^{n+1} \to Tx \) i.e., \( \|F_c(T^{n+1}\varpi, T^{n+1}\varpi, T\varpi)\| \to 0 \), as \( n \to \infty \). Thus
\[ \lim_{n \to \infty} \|F_c(\varpi, \varpi, T\varpi)\| = 0 \]
\[ \iff \lim_{n \to \infty} F_c(\varpi, \varpi, T\varpi) = \theta_k. \]

Hence \( T\varpi = \varpi \) i.e., \( \varpi \) is a fixed point of \( T \). Now to prove uniqueness, let \( \tilde{\nu} \neq \varpi \) be another fixed point of \( T \). Taking the expression (1) into account, we have
\[ F_c(\varpi, \varpi, \tilde{\nu}) = F_c(T\varpi, T\varpi, T\tilde{\nu}) \]
\[ \leq P^* F_c(\varpi, \varpi, \tilde{\nu}) P + Q^* F_c(\varpi, \varpi, T\varpi) Q + R^* F_c(\tilde{\nu}, \varpi, T\tilde{\nu}) R \]
\[ = P^* F_c(\varpi, \varpi, \tilde{\nu}) P + Q^* F_c(\varpi, \varpi, \varpi) Q + R^* F_c(\tilde{\nu}, \varpi, \varpi) R \]
\[ \|F_c(\varpi, \varpi, \tilde{\nu})\| \leq (\|P\|^2 + \|Q\|^2)F_c(\varpi, \varpi, \varpi) + \|R\|^2 \|F_c(\tilde{\nu}, \varpi, \varpi)\| \]
\[ \|F_c(\varpi, \varpi, \tilde{\nu})\| \leq \frac{\|R\|^2}{(1 - \|P\|^2 - \|Q\|^2)} \|F_c(\tilde{\nu}, \varpi, \varpi)\| \]
\[ < \|F_c(\tilde{\nu}, \varpi, \varpi)\| = \|F_c(\varpi, \varpi, \tilde{\nu})\| \]
which is a contradiction. Hence the fixed point is unique. \( \Box \)

In Theorem (3.8), if we take \( Q = R = \theta \), then the above theorem reduces to a Banach contraction principle, which can be stated as follows:

**Corollary 3.9.** Let \( (X, A, F_c) \) be a complete \( C^* \)-algebra-valued controlled \( F_c \)-metric type space and suppose \( T : X \to X \) is a mapping satisfying the following condition:
\[ F_c(T\varpi, T\varpi, T\tilde{\nu}) \leq P^* F_c(\varpi, \varpi, \tilde{\nu}) P, \ \forall \varpi, \tilde{\nu}, \in X, \] (12)
where \( P \in A \) with \( 0 \leq \|P\| < 1 \) and for \( \varpi_0 \in X \), choose \( \varpi_n = T^n\varpi_0 \) assume that
\[ \sup_{m \geq 1} \lim_{n \to \infty} \|C(\varpi_{i+1}, \varpi_{i+1}, \varpi_{i+2})C(\varpi_{i+1}, \varpi_{i+1}, \varpi_{m})\| < \frac{1}{\|P\|^2}. \] (13)

In addition, for each \( \varpi \in X \), suppose that
\[ \lim_{n \to \infty} \|C(\varpi, \varpi, \varpi_n)\| \text{ and } \lim_{n \to \infty} \|C(\varpi_n, \varpi_n, \varpi_n)\| \] (14)
either exist and are finite. Then \( T \) has a unique fixed point in \( X \).
Example 3.10. Let $X = [0, 4]$ and $\mathbb{A} = M_2(\mathbb{R})$ be the set of all $2 \times 2$ matrices under usual addition, multiplication and scalar multiplication. Define $F_c : X \times X \times X \to \mathbb{A}$ as follows:

$$F_c(\omega, \nu, \varsigma) = \begin{pmatrix} \max\{\omega, \varsigma\} + \max\{\nu, \varsigma\} & 0 \\ 0 & \max\{\omega, \varsigma\} + \max\{\nu, \varsigma\} \end{pmatrix}$$

Hence $(X, \mathbb{A}, F_c)$ is a $C^*$-algebra-valued controlled $F_c$-metric type space with $C(\omega, \nu, \varsigma) = 2 + \max\{\omega, \nu, \varsigma\}$. Now for any $A \in \mathbb{A}$, we define its norm as $\|A\| = \max\{\{a_i\}_1\leq 4\}$. Let $T : X \to X$ be defined as $T\omega = \frac{\omega}{8}$. Then

$$F_c(T\omega, T\omega, T\nu) = F_c\left(\frac{\omega}{8}, \frac{\omega}{8}, \frac{\nu}{8}\right) = \begin{pmatrix} 2\max\{\frac{\omega}{8}, \frac{\nu}{8}\} & 0 \\ 0 & 2\max\{\frac{\omega}{8}, \frac{\nu}{8}\} \end{pmatrix} = P^* F_c(\omega, \omega, \nu) P$$

where $P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ with $\|P\| = \frac{1}{2\sqrt{2}} < 1$. Now consider

$$C(\omega_{i+1}, \omega_{i+1}, \omega_{i+2}) = C(T^{i+1}\omega, T^{i+1}\omega, T^{i+2}\omega) = C\left(\frac{\omega}{8^{i+1}}, \frac{\omega}{8^{i+1}}, \frac{\omega}{8^{i+2}}\right) = \begin{pmatrix} 2 + \max\{\frac{\omega}{8^{i+1}}, \frac{\omega}{8^{i+1}}, \frac{\omega}{8^{i+2}}\} & 0 \\ 0 & 2 + \max\{\frac{\omega}{8^{i+1}}, \frac{\omega}{8^{i+1}}, \frac{\omega}{8^{i+2}}\} \end{pmatrix}$$

Similarly,

$$C(\omega_{i+1}, \omega_{i+1}, \omega_m) = \begin{pmatrix} 2 + \max\{\frac{\omega}{8^{i+1}}, \frac{\omega}{8^{i+1}}, \frac{\omega}{8^m}\} & 0 \\ 0 & 2 + \max\{\frac{\omega}{8^{i+1}}, \frac{\omega}{8^{i+1}}, \frac{\omega}{8^m}\} \end{pmatrix}$$

Thus

$$\lim_{i \to \infty} \|C(\omega_{i+1}, \omega_{i+1}, \omega_{i+2}) C(\omega_{i+1}, \omega_{i+1}, \omega_m)\| = \lim_{i \to \infty} \left\| \begin{pmatrix} (2 + \frac{\omega}{8^{i+1}})(2 + \max\{\frac{\omega}{8^{i+1}}, \frac{\omega}{8^m}\}) & 0 \\ 0 & (2 + \frac{\omega}{8^{i+1}})(2 + \max\{\frac{\omega}{8^{i+1}}, \frac{\omega}{8^m}\}) \end{pmatrix} \right\| = \lim_{i \to \infty} (2 + \frac{\omega}{8^{i+1}})(2 + \max\{\frac{\omega}{8^{i+1}}, \frac{\omega}{8^m}\}) = 4 + \frac{2\omega}{8^m}$$

and

$$\sup_{m \geq 1} \lim_{i \to \infty} \|C(\omega_{i+1}, \omega_{i+1}, \omega_{i+2}) C(\omega_{i+1}, \omega_{i+1}, \omega_m)\| = 4 + \frac{2\omega}{8} < 8 = \frac{1}{\|P\|^2}.$$
4 Conclusion

In this manuscript, we have analyzed the structure of $C^*$-algebra-valued controlled $F_c$-metric type spaces and acquired some fixed point theorem under different contractive conditions of the underlying spaces. Further, an example is conferred to show the effectiveness of the established result.

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