Classical Analysis of Degenerate Optomechanical Parametric Oscillators

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Abstract

Recent advances in the miniaturization of whispering gallery mode optical parametric oscillators have open the possibility of studying the interplay between parametric down-conversion and optomechanics, two of the most fundamental nonlinear optical processes. Even though a rigorous analysis of this scenario requires a fully quantum mechanical treatment where, e.g., the squeezing of the down-converted field or the cooling of the mechanical degree of freedom can be studied, having a clear idea of the classical properties of the system is useful or even necessary before starting such quantum mechanical analysis, and this is precisely what we offer in this work. We will restrict to the case of degenerate parametric down-conversion, obtaining then what we call a degenerate optomechanical parametric oscillator.

I. INTRODUCTION

Parametric down-conversion is a process happening in crystals with second-order optical nonlinearity, by which light at some frequency $2\omega_0$ is transformed into light at frequencies $\omega_s$ and $\omega_i$ such that $\omega_s + \omega_i \approx 2\omega_0$ [1–3]. When the crystal is introduced in an optical cavity, what has the effect of enhancing the nonlinear interaction and filter the fields all at once, we obtain a so-called optical parametric oscillator, in which the down-converted field starts oscillating in the cavity only once the power of the pumping laser exceeds some threshold value (such that the nonlinear gain can compensate for the cavity losses) [1, 3]. These devices have found many applications both in classical and quantum optics; in the classical case, they are among the most tunable sources of light, allowing to transform laser light into almost any (optical) frequency [1]. From a quantum point of view, the down-converted photons show strong quantum correlations; particularly relevant to this work is the degenerate optical parametric oscillator (DOPO), in which down-converted photons have the same frequency $\omega_s = \omega_i \approx \omega_0$, and show nearly-perfect quadrature squeezing when working close to threshold [2, 3]. Indeed, DOPOs are nowadays the sources of the highest-quality squeezed light [4–7], which can be used to increase the sensitivity of measurements beyond the standard quantum limit [8–11], or also to generate high quality entangled beams for quantum information purposes [12, 13].

On the other hand, we have optomechanical systems, where some mechanical degree of freedom is coupled to a light field via, e.g., radiation pressure [14–17]. When the interaction happens inside a cavity, the Lorentzian density of modes provided by the resonator, together with the injection of a coherent laser field with the proper detuning with respect to the cavity resonance, allow to cool down the mechanical degree of freedom to its quantum mechanical ground state through sideband cooling [18–29]. Since the mechanical degree of freedom is usually a mesoscopic system formed by many atoms, optomechanics seems a very promising platform where studying the transition from the microscopic quantum world to our natural macroscopic classical one, allowing, for example, to put bounds on collapse models [30–33]. From a practical point of view, apart from offering a new platform where performing traditional quantum optical tasks such as the generation of squeezed light [34–37], transparency windows [38–42], or photon blockade effects [43], optomechanical systems might be a perfect interface between optical and microwave technologies, since mechanical degrees of freedom couple to both electromagnetic scales [44–53].

As for actual optomechanical implementations, they come out in many different forms [17]: cavities with mirrors attached to cantilevers [18, 19] or suspended [21], flexible membranes placed inside optical cavities [22–26], drum-shaped capacitors coupled to superconducting circuits [27], or localized mechanical modes in photonic crystal cavities [28], are some examples. For our current purposes, the most relevant implementation consists in a whispery gallery mode resonator (a microtoroid or microdisk, for example), where light circulates around its edge via total internal reflection, pushing the whole structure, hence exciting some of its mechanical modes [20, 54].
FIG. 1: Sketch of the degenerate optomechanical parametric oscillator. It consists of a cavity containing a crystal with second-order optical nonlinearity, and formed by a fixed partially transmitting mirror and an oscillating perfectly reflecting mirror (sketched as a mirror coupled to a wall through a spring). When pumped with a laser beam with Gaussian transverse profile and frequency $\omega_L$ close to a cavity resonance $\omega_p$, the crystal is able to produce light at the subharmonic $\omega_L/2$, which is close to another cavity resonance $\omega_s$, through parametric down-conversion. In addition, the moving mirror feels the radiation pressure exerted from the light contained in the cavity.

From a fundamental point of view, the interplay between optomechanics and parametric down-conversion seems to be a natural and interesting problem to study within the nonlinear quantum optics community. In recent years the motivation has become also practical \[55–66\], since miniaturized whispering gallery mode resonators can be fabricated directly with the typical crystalline materials which possess second order optical nonlinearity, such that light can be parametric down-converted while circulating on the resonator. Even though current devices do not show optomechanical couplings able to compete with the down-conversion nonlinearity, it is to be expected that soon the limit in which these are comparable will be reached, and hence it seems worth offering predictions about what will be observed in such scenario. In particular, there are three obvious questions which one can ask: (i) how do the mechanics affect the squeezing properties of the down-converted field?, (ii) is it possible, or even more effective, to cool down the mechanics with the squeezed light?, and (iii) can we exploit the quantum-correlated light to perform more sensitive measurements of the mechanical motion?

While the rigorous answer to these questions can only be given through a fully quantum mechanical analysis, understanding the classical dynamics of the system is usually a prerequisite for such analysis, and this is what we offer in this work. For simplicity, we stick to the degenerate case, and assume that only the down-converted field is coupled to the mechanical mode, since it is the interplay between these two the one which seems most interesting and unexplored (the pump stays near-coherent for most parameters of the DOPO). Nevertheless, the analysis offered here can be trivially extended to the case in which only the pump mode is coupled to the mechanics. On the other hand, when having optomechanical coupling for both modes the analysis seems more difficult, especially from an analytic perspective, and it will require a generalization of the techniques presented here.

The work is organized as follows: first, we introduce the model of the system, which we have called degenerate optomechanical parametric oscillator (DOMPO), and then proceed to find its steady states and analyze their stability.

II. THE MODEL EQUATIONS

Even though the actual implementation can differ from the simple picture sketched in Fig. 1, a DOMPO can be schematically seen as an optical cavity formed by a fixed partially transmitting mirror and a perfectly reflecting oscillating mirror, containing a second-order nonlinear crystal, and pumped by an external laser at frequency $\omega_L$ close to resonance with a cavity mode at frequency $\omega_p$; the nonlinear crystal is capable of generating photons at another cavity resonance of frequency $\omega_s$ (signal mode) close to $\omega_L/2$, and the mirror can in principle feel the radiation pressure exerted by both the pump and signal modes. We write the electromagnetic vector potential inside the cavity as (we only write the relevant frequencies)

$$A(r, t) = \sum_{j=p,s} \sqrt{\frac{\hbar}{4\epsilon_0 n_j(z) \omega_j}} \varepsilon_j \alpha_j(t) \left\{ G(k_j; r_\perp, z) e^{i n_j k_j z - i \omega_j t} + G^*(k_j; r_\perp, z) e^{-i n_j k_j z - i \omega_j t} \right\} + c.c.; \quad (1)$$

in this expression, $\omega_p = \omega_L$, $\omega_s = \omega_L/2$, $z$ is the cavity axis, $n_j(z)$ is the refractive index felt by mode $j$ at position $z$, $k_j = \omega_j/c$ is its wave vector and $\varepsilon_j$ its polarization, and $r_\perp$ is the coordinate vector in the transverse plane,
$G(k_j; r_\perp, z)$ being a suitable Gaussian transverse mode \cite{3, 67, 68}. The prefactor $\sqrt{\hbar/4e_0 n_j(z)\omega_j}$ is chosen in order for the (free) electromagnetic energy to read \cite{3}$

E_{em} = \int_{cavity} d^3r \left[ \varepsilon_0 n^2(z)E^2(r, t) + \frac{1}{\mu_0} B^2(r, t) \right] = \sum_{j=p, s} \hbar \omega_j \alpha_j^*(t) \alpha_j(t), \quad (2)

so that $\alpha_j$ is interpreted as the complex amplitude of a unit mass harmonic oscillator with frequency $\omega_j$, position $Q_j = \hbar/2\omega_j (\alpha_j^* + \alpha_j)$, and momentum $P_j = i\hbar \omega_j/2(\alpha_j^* - \alpha_j)$. This is quite convenient to quantize the field, what can be done by imposing canonical commutation relations $[\hat{Q}_j, \hat{P}_j] = i\hbar \delta_{jl}$, hence replacing $\alpha_j$ ($\alpha_j^*$) by an annihilation $\hat{a}_j$ (creation $\hat{a}_j^\dagger$) operator for photons on mode $j$. However, in our case we will only study the classical nonlinear dynamics of the system, so that we won’t need to quantize, and will interpret $\alpha_j(t)$ as the normalized amplitude of the electromagnetic field, our basic optical variable.

As for the moving mirror of mass $M$, we model it as a harmonic oscillator of some frequency $\Omega_m$, denoting by $X$ its displacement (with respect to its equilibrium position) and $P$ the corresponding momentum. In order to get rid of the mass parameter, we will work with dimensionless versions of the position and momentum, $\xi = \sqrt{2\Omega_m M/\hbar X}$ and $\tilde{P} = \sqrt{2/\hbar \Omega_m M} P$, called quadratures.

With these definitions, the equations describing the classical dynamics of the system are just a combination of the usual DOPO \cite{69, 71} and Optomechanical \cite{34, 55, 72} equations, in particular:

\begin{align}
\frac{d\xi}{dt} &= \Omega_m \tilde{P}, \\
\frac{d\tilde{P}}{dt} &= -\gamma \xi - \Omega_m \xi + 2\tilde{g}_s |\alpha_s|^2 + 2\tilde{g}_p |\alpha_p|^2, \\
\frac{d\alpha_p}{dt} &= \xi_p - (\gamma_p - i\Delta_p - ig_p \xi) \alpha_p - \frac{\chi}{2} \beta_p^2, \\
\frac{d\alpha_s}{dt} &= -(\gamma_s - i\Delta_s - ig_s \xi) \alpha_s + \chi \alpha_p \alpha_s^*,
\end{align}

where $\gamma_j$ are damping rates for the corresponding degrees of freedom (associated with losses through the partially transmitting mirror in the case of light, and friction with the thermal environment in the case of the moving mirror), $\Delta_j = \tilde{\omega}_j - \omega_j$ are the detunings, $\xi_p$ is associated with the light fed into the cavity through the external laser (hence, it is proportional to the square root of the laser power), and $\tilde{g}_j$ and $\chi$ are parameters related to the strength of the radiation pressure and the parametric down-conversion, respectively.

Before studying the equations, it is recommendable to make some variable change which will allow us to see how many free parameters they actually have. To this aim, we define the following normalized parameters

\begin{align}
g_{PDC} &= \frac{\chi}{\sqrt{\gamma_p \gamma_s}}, \quad \sigma = \frac{\chi \xi_p}{\gamma_p \gamma_s}, \quad \kappa = \frac{\gamma_p}{\gamma_s}, \quad \delta_1 = \frac{\Delta_1}{\gamma_1}, \quad \gamma = \frac{\gamma_m}{\gamma_s}, \quad \Omega = \frac{\Omega_m}{\gamma_s}, \quad g_l = \frac{g_l}{g_{PDC} \sqrt{\Omega}}.
\end{align}

and variables

\begin{align}
\beta_s &= g_{PDC} \alpha_s, \quad \beta_p = \sqrt{k_{PDC} \alpha_p}, \quad p = \frac{g_{PDC}}{\sqrt{\Omega}} \tilde{P}, \quad x = g_{PDC} \sqrt{\Omega} \xi, \quad \tau = \gamma_0 t,
\end{align}

which lead to the following equations:

\begin{align}
\dot{\xi} &= \Omega^2 \tilde{P}, \\
\dot{\tilde{P}} &= -\gamma \xi - x + 2 g_p |\beta_p|^2 + 2 g_p \beta_p^2, \\
\kappa^{-1} \dot{\beta}_p &= \sigma - (1 - \delta_0 - ig_p x) \beta_p - \beta_p^2 / 2, \\
\dot{\beta}_s &= -(1 - \delta_0 - ig_s x) \beta_s + \beta_p \beta_s^*,
\end{align}

where the derivative is now made with respect the dimensionless time $\tau$. It is interesting to note that $g_l$ basically provides the ratio between the single-photon optomechanical and down-conversion couplings, and hence, assuming a $\sqrt{\Omega}$ of order 1, they inform us about which of the two nonlinear processes dominates.

As non-trivial nonlinear equations, it is not possible to find their time-dependent analytical solutions other than numerically. However, working with a dissipative system, we are mainly interested in its behavior for long times, and there is a lot that we can say about this without really solving the full nonlinear equations; in particular, we follow closely the procedure already applied to detuned DOPOs \cite{64, 71}. Moreover, as explained above, from a quantum point of view, the most interesting question is the interplay between the mechanics and the down-converted field, and hence, in the following we set $g_p = 0$. 

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III. STATIONARY SOLUTIONS

The simplest behavior that one can expect in the long-time term is that the system reaches some steady state. Hence, it is always convenient to start by finding the time-independent solutions to the nonlinear equations, which we denote by a bar, e.g., \( \bar{x} \); when needed, we will write the complex field amplitudes as \( \beta_j = \sqrt{T_j} \exp(i\varphi_j) \), with real variables \( \varphi_j \in ]-\pi, \pi] \) and \( \bar{T}_j \geq 0 \).

In the \( g_p = 0 \) case, all the stationary solutions of (6) have \( \bar{p} = 0 \) and \( \bar{x} = 2g_s I_s \), leaving us with

\[
\begin{align*}
\sigma &= (1 - i\delta_p)\bar{\beta}_p + \beta_s^2/2, \\
\bar{\beta}_p^*\bar{\beta}_s &= (1 - i\delta_s - 2(\bar{g}_s^2 I_s)(\beta_s).
\end{align*}
\] (7a) (7b)

We distinguish then two types of stationary solutions: trivial or below-threshold solutions, which have \( I_s = 0 \), and nontrivial or above-threshold solutions with \( I_s \neq 0 \).

In the trivial case, the solution is simply

\[
\bar{\beta}_s = 0 \quad \text{and} \quad \bar{\beta}_p = \sigma/(1 - i\delta_p).
\] (8)

As for the nontrivial solutions, we find their analytic expression as follows. First, note that (7b) implies

\[
\bar{\beta}_p = (1 - i\delta_s - 2g_s^2 I_s) e^{2i\varphi_s},
\] (9)

which plugged into (7a) leads to

\[
e^{-2i\varphi_s}\sigma = (1 - i\delta_p)(1 - i\delta_s - 2g_s^2 I_s) + \frac{I_s}{2},
\] (10)

whose absolute value squared gives us a second order polynomial for the signal intensity

\[
\sigma^2 = \left[ 1 + \frac{I_s}{2} - \delta_p(\delta_s + 2g_s^2 I_s) \right]^2 + (\delta_p + \delta_s + 2g_s^2 I_s)^2 \equiv q_0 + q_1 I_s + q_2 I_s^2,
\] (11)

with

\[
q_0 = (1 + \delta_p^2)(1 + \delta_s^2), \quad q_1 = -(1 - \delta_p \delta_s) + 4\delta_s(1 + \delta_p^2)g_s^2, \quad \text{and} \quad q_2 = 4g_s^4 + \frac{1}{2} - 2\delta_p \delta_s^2.
\] (12)

Depending on the value of the parameters, this equation can have a single real positive solution or two, as shown in Fig. 2. In order to find for which values of the parameters (in particular of the injection \( \sigma \) and the detunings \( \delta_j \)) this happens, we just need to obtain the expression for the turning point, marked as TP in Fig. 2 which is nothing but the extremum of \( \sigma^2(I_s) \), that is,

\[
\frac{\partial \sigma^2}{\partial I_s} \bigg|_{I_s=I_s^TP} = 0 \quad \implies \quad I_s^{TP} = -q_1/2q_2;
\] (13)

taking into account that \( q_2 > 0 \), the turning point will exist only if \( q_1 < 0 \), which gives us a condition on the detunings for a given optomechanical coupling:

\[
\delta_p \delta_s > 1 + 4\delta_s(1 + \delta_p^2)g_s^2.
\] (14)

Hence, when this condition is satisfied, we will have two possible steady-state signal intensities (three counting the trivial one) for injections \( \sigma^2 \in [q_0 - q_1^2/4q_2, q_0] \). Let us advance, however, that the branch connecting the trivial solution with the upper branch of the nontrivial one is unstable (see Fig. 2), so only two out of the three possible solutions can be observed in real experiments, leading to a bistability common in nonlinear optical systems. Finally, notice that condition (14) with \( g_s = 0 \) is in agreement with that found for detuned DOPOs [62, 71].

IV. LINEAR STABILITY ANALYSIS

The existence of a mathematical solution of the nonlinear equations is not enough to ensure its physical reality: it also needs to be stable against perturbations, since in the real world these are unavoidable, and therefore we would never be able to observe the system in the corresponding solution otherwise.
the eigenvalues of its corresponding linear stability matrix. Note that this fixes all the parameters but $g_s$ and $I_s$ (we are using this steady-state intensity $I_s$ as a parameter instead of $\sigma$, because the latter can be uniquely determined from the former, but in general not the other way around). In (b) we show in the space of these parameters the turning point (red thick line) and the single Hopf instability (blue thick line) found in this example, coloring the regions where they make the stationary solution unstable. Note that for this choice of parameters, in the absence of optomechanical coupling there are no instabilities apart from the trivial pitchfork bifurcation, as can be checked from the conditions [15] and [24]. Hence, we see that the effect of the optomechanical coupling in this case consists in introducing new instabilities which greatly reduce the domain of stability of the nontrivial stationary state. In (a) and (c) we show how the steady-state intensity $I_s$ depends on the injection $\sigma^2$ for two specific values of $g_s$, corresponding to the vertical grey dashed lines in the parameter space (b), denoting its unstable and stable regions by a dashed or solid line, respectively. In (a) we have chosen $g_s = 0.2$, for which no instabilities are present in the nontrivial solution as can be appreciated in (b), and hence only the pitchfork instability connecting the trivial and nontrivial solutions is present. In (c), on the other hand, we have chosen $g_s = 0.25$, for which we find both a turning point (and hence a domain of bistability between the trivial and nontrivial solutions) and a Hopf bifurcation leading to time-dependent long time term solutions.

Let us collect the variables of the system in a vector $r = \text{col}(x, p, \beta_p^*, \beta_p^0, \beta_s^*, \beta_s^0)$; the stability of a given stationary solution $F$ can be analyzed as follows [23]. We consider small fluctuations around it by writing $r(t) = F + \delta r(t)$, introduce this ansatz into the nonlinear system (8), and keep only terms which are linear in the fluctuations, obtaining a linear system $\delta \mathbf{r} = \mathbf{L} \delta \mathbf{r}$, where $\mathbf{L}$ is the so-called linear stability matrix. This matrix depends on the system parameters and the particular stationary solution whose stability we are considering, and in our case is given by

$$
\mathbf{L} = \begin{pmatrix}
0 & \Omega^2 & 0 & 0 & 0 & 0 \\
-1 & -\gamma & 0 & 0 & 2g_s \beta_p^s & 2g_s \beta_p^0 \\
0 & 0 & -\kappa(1 + i \beta_p^0) & 0 & -\kappa \beta_s^s & 0 \\
0 & 0 & -\kappa(1 - i \beta_p^0) & 0 & -\kappa \beta_s^0 & 0 \\
ig_s \beta_p^* & \beta_p^s & 0 & 0 & -(1 - i \beta_s^0 - ig_s \bar{x}) & \beta_p^0 \\
-ig_s \beta_s^* & 0 & 0 & \beta_s^s & -(1 + i \beta_s^0 + ig_s \bar{x}) & -ig_s \beta_p^0
\end{pmatrix}.
$$

(15)

Since the equation for the fluctuations is linear, it is then clear that their growth is controlled by the eigenvalues of this matrix; in particular, the fluctuations will be damped and disappear in the long-time term only if the real part of all the eigenvalues is negative. Hence, we say that a stationary solution $\mathbf{F}$ is stable (and therefore physical) when the eigenvalues of its corresponding linear stability matrix $\mathbf{L}(\mathbf{F})$ have all negative real part.

The points in the parameter space in which at least one of the eigenvalues has a zero real part are known as critical points, instabilities, or bifurcations, and they separate the regions in which the stationary solution changes from stable to unstable. We can distinguish two types of instabilities: pitchfork or static bifurcations, where the imaginary part of the relevant eigenvalue is also zero, which connect the stationary solution with another stationary solution; and Hopf or dynamic bifurcations, where the imaginary part of the relevant eigenvalue is non-zero, which connect the stationary solution with a time-dependent solution (usually some periodic solution, known in this context as a periodic orbit or limit cycle).

Before proceeding, let us comment on one subtle point concerning the system parameters. The linear stability matrix (15) does not depend explicitly on the injection $\sigma$, it does only implicitly through the intracavity stationary amplitudes $\beta_p^0$ and $\beta_s^0$. It is then convenient to use either $I_p$ or $I_s$ as a parameter instead of $\sigma$ when dealing with the trivial or nontrivial solutions, respectively, knowing that the latter can always be uniquely determined from the former by using (8) or (11).

Let’s now proceed to analyze the instabilities of the DOMPO, first for the trivial stationary solution, and then for the nontrivial ones.

Figure 2: We show the bifurcations and corresponding stable and unstable regions for one particular example for which we have chosen $\gamma = 0.005$, $\Omega = -\delta_s = 10$, $\delta_p = 5$, and $\kappa = 100$, the first three being typical parameters when aiming for sideband cooling in optomechanical systems. Note that this fixes all the parameters but $g_s$ and $I_s$.
A. Stability of the trivial solution

In the case of the trivial stationary solution ($\bar{\beta}_s = 0$), the linear stability matrix is highly simplified, acquiring in particular a box structure $L = L_m \oplus L_p \oplus L_s$, where the second block is already in diagonal form

$$L_{mp} = \begin{pmatrix} -\kappa(1-i\delta_p) & 0 \\ 0 & -\kappa(1+i\delta_p) \end{pmatrix},$$

and its two eigenvalues have negative real part, the first block is given by

$$L_{mp} = \begin{pmatrix} 0 & \Omega^2 \\ -1 & -\gamma \end{pmatrix},$$

whose eigenvalues $\lambda_m^{(\pm)} = -\left(\gamma \pm \sqrt{\gamma^2 - 4\Omega^2}\right)/2$ have also negative real part, and finally the last block reads

$$L_s = \begin{pmatrix} -1 + i\delta_s & \beta_p \\ \beta_p^* & -1 - i\delta_s \end{pmatrix},$$

with eigenvalues

$$\lambda_s^{(\pm)} = -1 \pm \sqrt{I_p - \delta_s^2}.$$  

Hence, we see that the only instability appears when $I_p = 1 + \delta_s^2 = I_p^0$, and in particular the trivial solution becomes unstable for $I_p > I_p^0$, or in terms of the injection, when $\alpha^2 > (1 + \delta_s^2)(1 + \delta_s^2)$. Note that this is precisely the point at which the trivial and nontrivial solutions coalesce, see the points marked as PB in Fig. 2 and hence this pitchfork bifurcation simply connects these two stationary solutions.

B. Stability of the nontrivial solution

In the case of the nontrivial solution the $6 \times 6$ linear stability matrix does not have a box structure, and hence their eigenvalues do not have a simple analytic expression. However, we are not as interested in the actual eigenvalues as we are in the points where the real part of some of them becomes zero, since those are the points marking the instabilities, and this points can be found by analyzing the characteristic polynomial of the stability matrix, which we write as $P(\lambda) = \sum_{m=0}^{6} c_m \lambda^m$. Most of the coefficients $c_m(I_s, \delta_s, \delta_p, g_s, \kappa, \gamma, \Omega)$ are too lengthy, and hence we don’t show them here, except for the independent one, which can be written as $c_0 = 4q_2 I_s + 2q_1$, where $q_1$ and $q_2$ are defined in (12).

Given the characteristic polynomial, the static instabilities can be found from $P(\lambda = 0) = 0$, that is, they are located in the region of the parameter space defined by the equation $c_0 = 0$, which in our case gives $I_s = I_s^{TP}$. Hence, we see that the turning point of the nontrivial solution is an instability, and it is simple to check that the lower branch of the nontrivial solution connecting the upper branch with the trivial solution is unstable (for example by evaluating the eigenvalues numerically for one set of parameters), as shown in Fig. 2. In other words, the turning point is a pitchfork bifurcation connecting the unstable lower branch with the upper branch, which is stable in all its domain of existence, except for possible Hopf bifurcations which we will describe in the next paragraphs.

We can then try to do the same with the Hopf bifurcations, but in that case the expressions are not as easy to handle. It is instructive to first consider the case without optomechanical coupling, $g_s = 0$. In this case the characteristic polynomial can be factorized as $P(\lambda) = P_{DOPO}(\lambda)P_m(\lambda)$, where $P_m(\lambda) = \lambda^2 + \gamma + \Omega^2$ is the characteristic polynomial associated to the free mechanical motion (hence showing no instabilities), while $P_{DOPO}(\lambda) = \sum_{n=0}^{4} d_n \lambda^n$, with

$$d_0 = \kappa^2 I_s(I_s + 2 - 2\Delta_p \Delta_s), \quad d_1 = 2\kappa[I_s + \kappa(I_s + \Delta_p)], \quad d_2 = \kappa[4 + 2I_s + \kappa(1 + \Delta_p)], \quad d_3 = 2(1 + \kappa), \quad \text{and} \quad d_4 = 1,$$

is the characteristic polynomial associated to the optical modes coupled through the parametric down-conversion process, that is, to the DOPO [65][71]. The Hopf instabilities are found by locating the points in the parameter space where the eigenvalues become purely imaginary, $\lambda = \omega_{HB}$, where the real parameter $\omega_{HB}$ is known as the Hopf frequency (providing the frequency of the periodic solution which is born right at the bifurcation). Applying this condition to the DOPO’s characteristic polynomial, we get

$$P_{DOPO}(\lambda = \omega_{HB}) = (d_0 - d_2\omega_{HB}^2 + d_4\omega_{HB}^4) + i\omega_{HB}(d_1 - d_3\omega_{HB}^2) = 0;$$

$$d_0 = \kappa^2 I_s(I_s + 2 - 2\Delta_p \Delta_s), \quad d_1 = 2\kappa[I_s + \kappa(I_s + \Delta_p)], \quad d_2 = \kappa[4 + 2I_s + \kappa(1 + \Delta_p)], \quad d_3 = 2(1 + \kappa), \quad \text{and} \quad d_4 = 1,$$
the imaginary part of this equation provides us with the Hopf frequency

\[ \omega_{\text{HB}}^2 = \frac{d_4}{d_3} = \frac{\kappa[I_s + \kappa(1 + I_s + \delta_p^2)]}{1 + \kappa}, \]  

(22)

which is well defined for every value of the parameters, while the real part of \([21]\) provides the condition \(d_0d_3^2 + d_4d_3^2 - d_2d_1d_3 = 0\), which can be solved for \(I_s\) analytically, leading to the simple expression

\[ I_{\text{HB}}^s = -\frac{(1 + \delta_p^2)(2 + \kappa)^2 + \kappa^2\delta_p^2}{(1 + \kappa)^2(2 + \kappa + \kappa\delta_p^2 + 2\delta_p\delta_s)}. \]  

(23)

This Hopf instability requires then

\[ \delta_p\delta_s < -1 - \kappa(1 + \delta_p^2)/2, \]  

(24)

to exist (otherwise \(I_{\text{HB}}^s < 0\)), which incidentally means that it does not exist when there is bistability in the system (what requires \(\delta_p\delta_s > 1\)). It is possible to show that the portion of the nontrivial solution with \(I_s > I_{\text{HB}}^s\) becomes unstable, and the limit cycles become chaotic for large enough injections [69–71].

Hence, we see that without optomechanical coupling, there is only one Hopf bifurcation. The main effect of optomechanics, that is, of increasing \(g_s\), is both to change the location of this instability already present for \(g_s = 0\), as well as create new ones. This is what we show in Fig. 2 for one example, where we plot the signal intensity of the Hopf instability that we have found as a function of \(g_s\), which is well defined for every value of the parameters, while the real part of (21) provides the condition

\[ \omega_{\text{HB}}^2 = \frac{\omega_{\text{HB}}^2 + \omega_{\text{HB}}^4 - \omega_{\text{HB}}^6}{3}, \]  

for \(I_s = 0\), we can proceed as follows. The second equation \([25b]\) can be solved for the Hopf frequency as

\[ \omega_{\text{HB}, \pm} = \frac{c_3 \pm \sqrt{c_3^2 - 4c_1c_5}}{2c_5}; \]  

(26)

these solutions can be introduced in \([25a]\), but unfortunately the resulting equation does not allow to find a simple analytic solution for \(I_s\). However, a symbolic program such as Mathematica allows us to find analytic solutions, provided that we write the equation as a more manageable polynomial. In particular, let us write \(\omega_{\text{HB}, \pm} = l \pm r\) with \(l = c_3/2c_5\) and \(r = \sqrt{c_3^2 - 4c_1c_5}/2c_5\), which allows us to rewrite \([25a]\) as

\[ c_0 - c_2l + c_4(l^2 + r^2) - c_6(l^3 + 3lr^2) = \pm r(c_2 - 2c_4l + c_6(3l^2 + r^2)). \]  

(27)

The square of this expression provides a sixth order polynomial equation for \(I_s\), whose solutions can be handled by a symbolic program. Note that by taking the square of the previous equation, we are indeed introducing extra fictitious solutions for \(I_s\), but we have checked that these extra solutions are always complex, and hence they do not provide anything which could be interpreted as instabilities. This procedure has allowed us to make an exhaustive numerical analysis of the Hopf instabilities for \(g_s \neq 0\), of which we show a characteristic example in Fig. 2, see the caption for a detailed explanation of the result.

V. CONCLUSIONS

In conclusion, in this work we have analyzed the classical properties of a DOMPO, when only the down-converted field is coupled to the mechanical degree of freedom. We believe that this analysis will be valuable when evaluating the interplay between down-conversion and optomechanics at the quantum level. Exactly the same techniques which we have applied in this case, can be trivially applied when only the pump mode is coupled to the mechanical degree
of freedom ($g_p \neq 0 = g_s$), while the case in which optomechanical coupling is allowed for both optical modes requires a nontrivial extension.

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[1] R. W. Boyd, Nonlinear Optics (Academic, 2003).
[2] P. Meystre and D. F. Walls (eds.), Nonclassical Effects in Quantum Optics (American Institute of Physics, 1991).
[3] C. Navarrete-Benlloch, Contributions to the Quantum Optics of Multi-mode Optical Parametric Oscillators (PhD thesis, Universitat de València, 2011).
[4] Y. Takeno, M. Yukawa, H. Yonezawa, and A. Furusawa, Opt. Express 15, 4321-4327 (2007).
[5] H. Vahlbruch, M. Mehmet, S. Chelkowski, B. Hage, A. Franzen, N. Lastzka, S. Gossler, K. Danzmann, and R. Schnabel, Phys. Rev. Lett. 100, 033602 (2008).
[6] T. Eberle, S. Steinlechner, J. Bauchrowitz, V. Handchen, H. Vahlbruch, M. Mehmet, H. Muller-Ebhardt, and R. Schnabel, Phys. Rev. Lett. 104, 251102 (2010).
[7] M. Mehmet, H. Vahlbruch, N. Lastzka, K. Danzmann, and R. Schnabel, Phys. Rev. A 81, 013814 (2010).
[8] K. Goda, O. Miyakawa, E. E. Mikhailov, S. Saraf, R. Adhikari, K. McKenzie, R. Ward, S. Vass, A. J. Weinstein, and N. Mavalvala, Nat. Phys. 4, 472-476 (2008).
[9] H. Vahlbruch, S. Chelkowski, B. Hage, A. Franzen, K. Danzmann, and R. Schnabel, Phys. Rev. Lett. 95, 211102 (2005).
[10] N. Treps, N. Grosse, W. P. Bowen, C. Fabre, H.-A. Bachor, and P. K. Lam, Science 301, 940-943 (2003).
[11] N. Treps, U. Andersen, B. Buchler, P. K. Lam, A. Maitre, H.-A. Bachor, and C. Fabre, Phys. Rev. Lett. 88, 203601 (2002).
[12] S. L. Braunstein and P. van Loock, Rev. Mod. Phys. 77, 513-577 (2005).
[13] C. Weedbrook, S. Pirandola, R. García-Patron, N. J. Cerf, T. C. Ralph, J. H. Shapiro, and S. Lloyd, Rev. Mod. Phys. 84, 621-669 (2012).
[14] T. J. Kippenberg and K. J. Vahala, Opt. Exp. 15, 17172 (2007).
[15] F. Marquardt and S. M. Girvin, Physics 2, 40 (2009).
[16] P. Meystre, Annalen der Physik 525, 215 (2013).
[17] M. Aspelmeyer, T. J. Kippenberg, and F. Marquardt, arXiv: 1303.0733 (2013).
[18] S. Gigan, H. R. Böhm, M. Paternostro, F. Blaser, G. Langer, J. B. Hertzberg, K. C. Schwab, D. Bäuerle, M. Aspelmeyer, and A. Zeilinger, Nature 444, 67 (2006).
[19] O. Arcizet, P.-F. Cohadon, T. Briant, M. Pinard, and A. Heidmann, Nature 444, 71 (2006).
[20] A. Schliesser, P. Del’Haye, N. Nooshi, K. J. Vahala, and T. J. Kippenberg, Phys. Rev. Lett. 97, 243905 (2006).
[21] T. Corbitt, Y. Chen, E. Innerhofer, H. Müller-Ebhardt, D. Ottaway, H. Rehbein, D. Sigg, S. Whitcomb, C. Wipf, and N. Mavalvala, Phys. Rev. Lett. 98, 150802 (2007).
[22] J. D. Thompson, B. M. Zwickl, A. M. Jayich, F. Marquardt, S. M. Girvin, and J. G. E. Harris, Nature 452, 72 (2008).
[23] D. J. Wilson, C. A. Regal, S. B. Papp, and H. J. Kimble, Phys. Rev. Lett. 103, 207204 (2009).
[24] J. D. Teufel, T. Donner, D. Li, J. W. Harlow, M. S. Allman, K. Cicak, A. J. Sirois, J. D. Whittaker, K. W. Lehnert, and R. W. Simmonds, Nature 475, 359 (2011).
[25] J. Chan, T. P. Mayer Alegre, A. H. Safavi-Naeini, J. T. Hill1, A. Krause1, S. Gröblacher, M. Aspelmeyer, and O. Painter, Nature 478, 89 (2011).
[26] M. Karuza, C. Molinelli, M. Galassi, C. Biancofiore, R. Natali, P. Tombesi, G. Di Giuseppe, and D. Vitali, New J. Phys. 14, 095015 (2012).
[27] I. Wilson-Rae, N. Nooshi, W. Zwerger, and T. J. Kippenberg, Phys. Rev. Lett. 99, 093901 (2007).
[28] F. Marquardt, J. P. Chen, A. A. Clerk, and S. M. Girvin, Phys. Rev. Lett. 99, 093902 (2007).
[29] C. Genes, D. Vitali, P. Tombesi, S. Gigan, and M. Aspelmeyer, Phys. Rev. A 77, 033804 (2008).
[30] W. Marshall, C. Simon, R. Penrose, and D. Bouwmeester, Phys. Rev. Lett. 91, 130401 (2003).
[31] D. Kleckner, I. Pikovskii, E. Jeffrey, L. Ament, E. Eliel, J. van den Brink, and D. Bouwmeester, New J. Phys. 10, 095020 (2008).
[32] O. Romero-Isart, A. C. Pflanzer, F. Blaser, R. Kaltenbaek, N. Krüger, M. Aspelmeyer, and O. Painter, Phys. Rev. Lett. 107, 020405 (2011).
[33] O. Romero-Isart, A. C. Pflanzer, F. Blaser, R. Kaltenbaek, N. Kiesel, M. Aspelmeyer, and O. Painter, Phys. Rev. Lett. 108, 053602 (2012).
[34] O. Romero-Isart, Phys. Rev. A 84, 052121 (2012).
[35] C. Fabre, Phys. Rev. A 49, 1337 (1994).
[36] S. Mancini and P. Tombesi, Phys. Rev. A 49, 4055 (1994).
[37] D. W. C. Brooks, T. Botter, S. Schreppler, T. P. Purdy, N. Bramhs, and D. M. Stamper-Kurn, Nature 488, 476 (2012).
[38] A. H. Safavi-Naeini, S. Gröblacher, J. T. Hill, J. Chan, M. Aspelmeyer, and O. Painter, Nature 500, 185 (2013).
[39] S. Weis, R. Rivière, S. Deléglise, E. Gavartin, O. Arcizet, A. Schliesser, and T. J. Kippenberg, Science 330, 1520 (2010).
[40] A. H. Safavi-Naeini, T. P. M. Alegre, J. Chan, M. Eichenfield, M. Winger, Q. Lin, J. T. Hill, D. E. Chang, and O. Painter, Nature 472, 69 (2011).
[41] J. D. Teufel, D. Li, M. S. Allman, K. Cicak, A. J. Sirois, J. D. Whittaker, and R. W. Simmonds, Nature 471, 204 (2011).
[41] F. Massel, S. U. Cho, J.-M. Pirkkalainen, P. J. Hakonen, T.T. Heikkilä, and M. A. Sillanpää, Nat. Commun. 3, 987 (2012).
[42] M. Karuza, C. Biancofiore, M. Bawaj, C. Molinelli, M. Galassi, R. Natali, P. Tombesi, G. Di Giuseppe, and D. Vitali, Phys. Rev. A 88, 013804 (2013).
[43] P. Rabl, Phys. Rev. Lett. 107, 063601 (2011).
[44] K. Stannigel, P. Rabl, A. S. Sørensen, P. Zoller, and M. D. Lukin, Phys. Rev. Lett. 105, 220501 (2010).
[45] A. H Safavi-Naeini and O. Painter, New J. Phys. 13, 013017 (2011).
[46] C. A. Regal and K. W. Lehnert, J. Phys.: Conference Series 264, 012025 (2011).
[47] J. M. Taylor, A. S. Sørensen, C. M. Marcus, and E. S. Polzik, Phys. Rev. Lett. 107, 273601 (2011).
[48] Y.-D. Wang and A. A. Clerk, Phys. Rev. Lett. 108, 153603 (2012).
[49] Y.-D. Wang and A. A. Clerk, New J. Phys. 14, 105010 (2012).
[50] Sh. Barzanjeh, M. Abdi, G. J. Milburn, P. Tombesi, and D. Vitali, Phys. Rev. Lett. 109, 130503 (2012).
[51] J. Bochmann, A. Vainsencher, D. D. Awschalom and A. N. Cleland, Nature Phys. 9, 712 (2013).
[52] T. Bagci, A. Simonson, S. Schmid, L. G. Villanueva, E. Zeuthen, J. Appel, J. M. Taylor, A. Sørensen, K. Usami, A. Schliesser, and E. S. Polzik, Nature 507, 81 (2014).
[53] R. W. Andrews, R. W. Peteron, T. P. Purdy, K. Cicak, R. W. Simmonds, C. A. Regal, and K. W. Lehnert, Nature Phys. 10, 321 (2014).
[54] A. Schliesser, Cavity Optomechanics and Optical Frequency Comb Generation with Silica Whispering-Gallery-Mode Microresonators (PhD thesis, Ludwig-Maximilians-Universität München, 2009).
[55] V. S. Ilchenko, A. A. Savchenkov, A. B. Matsko, and L. Maleki, J. Opt. Soc. Am. B 20, 333 (2003).
[56] V. S. Ilchenko, A. A. Savchenkov, A. B. Matsko, and L. Maleki, Phys. Rev. Lett. 92, 043903 (2004).
[57] A. A. Savchenkov, A. B. Matsko, M. Mohageg, D. V. Strekalov, and L. Maleki, Opt. Lett. 32, 157 (2007).
[58] J. U. Fürt, D. V. Strekalov, D. Elser, M. Lassen, U. L. Andersen, C. Marquardt, and G. Leuchs, Phys. Rev. Lett. 104, 153901 (2010).
[59] J. U. Fürt, D. V. Strekalov, D. Elser, A. Aiello, U. L. Andersen, Ch. Marquardt, and G. Leuchs, Phys. Rev. Lett. 105, 263904 (2010).
[60] J. U. Fürt, D. V. Strekalov, D. Elser, A. Aiello, U. L. Andersen, Ch. Marquardt, and G. Leuchs, Phys. Rev. Lett. 106, 113901 (2011).
[61] T. Beckmann, H. Linnenbank, H. Steigerwald, B. Sturman, D. Haertle, K. Buse, and I. Breunig, Phys. Rev. Lett. 106, 143903 (2011).
[62] C. S. Werner, T. Beckmann, K. Buse, and I. Breunig, Opt. Lett. 37, 4224 (2012).
[63] M. Förtsch, J. U. Fürt, C. Wittmann, D. Strekalov, A. Aiello, M. V. Chekhova, C. Silberhorn, G. Leuchs, and C. Marquardt, Nature Commun. 4, 1818 (2013).
[64] C. Marquardt, D. Strekalov, J. Fürt, M. Förtsch, and G. Leuchs, Opt. Phot. News 24, 38 (2013).
[65] M. Förtsch, G. Schunk, J. U. Fürt, D. Strekalov, T. Gerrits, M. J. Stevens, F. Sedlmieir, H. G. L. Schwefel, S. W. Nam, G. Leuchs, and C. Marquardt, [arXiv:1404.0593]
[66] M. Förtsch, T. Gerrits, M. J. Stevens, D. Strekalov, G. Schunk, J. U. Fürt, U. Vogl, F. Sedlmieir, H. G. L. Schwefel, G. Leuchs, S. W. Nam, and C. Marquardt, [arXiv:1410.6304]
[67] N. Hodgson and H. Weber, Laser resonators and beam propagation (Springer, New York, 2005).
[68] A. E. Siegman, Lasers (University Science Books, Sausalito, 1986).
[69] L. Lugliato, C. Oskjær, C. Fabre, E. Giacobino, and R. J. Horowicz, Il Nuovo Cimento 10, 959 (1988).
[70] N. P. Pettiaux, R.-D. Li, and P. Mandel, Optics Commun. 72, 256 (1989).
[71] C. Fabre, E. Giacobino, A. Heidmann, L. Lugliato, S. Reynaud, M. Vaidaczino, and W. Kaige, Quantum Opt. 2, 159 (1990).
[72] K. Jacobs, P. Tombesi, M. S. Collett, and D. F. Walls, Phys. Rev. A 49, 1961 (1994).
[73] L. M. Narducci and N. B. Abraham, Laser Physics and Laser Instabilities (World Scientific, Singapore, 1988).