Non-Newtonian flow in macroscopic heterogeneous porous media: from power-law fluids to transitional rheology

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Many natural or industrial fluids exhibit non-linear behavior. Non-Newtonian fluids can therefore be found in different applications related to porous or fractured media such as mud flows, oil recovery, hydraulic fracturing or foam injection clean-up. This work focuses on the understanding of the flow of these fluids in large scale heterogeneous porous media, and particularly in the coupling between flow heterogeneity and rheology. This work considers fluids that exhibit a change in behavior, where the viscosity changes above a certain velocity threshold. This change of viscosity affects drastically the inhomogeneity of velocity field. In this paper, we analyse some general mathematical properties for non-linear rheologies at the microscopic and macroscopic scale. Most non-linear rheologies imply the existence of two asymptotic limits, at low and high flow rate. The velocity field of both limits are obtained by solving a power-law rheology in the medium, which is analysed using a perturbation expansion. As an example, we will consider fluids which are Newtonian for low shear-rate but becomes shear-thinning or shear-thickening at higher shear rate. Three macroscopic flowing regimes are then observed. It will be shown that the limits at low and high flow-rate are governed by the properties of power-law fluids. The transition between these two regimes is the analysed. The range depends on the heterogeneity permeability field but also on the rheological parameters. The statistical properties of the flow field displays interesting statistical and geometrical properties such as multi-scale (fractal), similarly to other critical systems (percolation, avalanches, etc.)

1. Introduction

Many natural or industrial fluids exhibit non-Newtonian behaviors, (Bird et al. 1987; Barnes et al. 1989; Coussot 2005), they are thus found in many applications related to porous or fractured media. An important industrial application is, for example, enhanced oil recovery (EOR) (see Sorbie et al. (1989); Sorbie (2013)). Indeed, oil is usually recovered by displacing it with another fluid (e.g. water). The main problem lies in the fact that the displacing fluid is often less viscous, resulting in viscous fingering. The fingering tends to create preferential flow paths, leaving many areas of oil inplace. The idea is then to inject a non-Newtonian fluid in order to prevent fingering. Another interesting application is the description of blood in the capillary network, which should be regarded as a suspension and thus with a non-Newtonian viscosity: shear-thinning or yield stress (Boyd et al. 2007; Bessonov et al. 2016). Non-Newtonian fluid is also present in the proppants used for hydraulic fracturing or the mud produced by drilling wells. Non-Newtonian fluids (cements, polymers, etc.) are also commonly used for fracture sealing (Tongwa et al. 2013).

A very recurring problem when dealing with porous media is that of upscaling. If the equations of motion are generally well known at the pore scale (e.g typically $\sim 10^{-3}$ m), a particular interest is in understanding the flow at much larger scales ($\sim 1 \times 10^{-3}$ m). This
is usually done by deriving constitutive equations for average quantities at an intermediate scale. This is illustrated by the famous Darcy’s law for Newtonian fluids, which relates linearly the mean gradient of pressure to the mean flow rate. At the microscopic level, fluids obey the Stokes equation (if inertia is neglected):

\[ \vec{0} = -\vec{\nabla}P + \mu \Delta \vec{u}, \quad (1.1) \]

where \( \vec{u} \) is the fluid velocity, \( P \) is the pressure and \( \mu \) is the viscosity. After averaging the velocity and pressure field over a large number of pores, it results Darcy’s law (Darcy 1856):

\[ \langle \vec{u} \rangle = -\frac{\kappa}{\mu} \langle \vec{\nabla}P \rangle, \quad (1.2) \]

where \( \kappa \) is the permeability of the porous medium. \( \kappa \) depends on its topology and is homogeneous to a squared length. In the following, for the sake of simplicity, the \( \langle . \rangle \) is usually omitted when the scale of interest is explicit. This scale, containing enough pores for Darcy’s law to be applicable, is called mesoscopic.

At the geological scale the type of material may however spatially vary leading to a macroscopic heterogeneous permeability field. To understand the flow at large scale then requires solving the heterogeneous Darcy’s law:

\[ \vec{u} = -\frac{\kappa(\vec{r})}{\mu} \vec{\nabla}P \quad \text{and} \quad \vec{\nabla} \cdot \vec{u} = 0. \quad (1.3) \]

It is also important to recall that a very similar equation is used for solving flow in rough fractures, usually referred as the Reynolds equation (Reynolds 1886; Zimmerman et al. 1991; Mourzenko et al. 1995). Indeed, in a fracture with varying opening and under the lubrication approximation (small thickness and small variation of the opening), flow obeys:

\[ \vec{q}(x, y) = -\frac{\kappa(x, y)}{\mu} \vec{\nabla}_{2D}P \quad \text{et} \quad \vec{\nabla}_{2D} \cdot \vec{q} = 0, \quad (1.4) \]

where the fracture is in the \((x, y)\)-plane, \( b(x, y) \) represents the local opening, \( \vec{q}(x, y) = \int_0^{b(x, y)} \vec{u} \, dz \) and \( \kappa(x, y) = \frac{b^2(x, y)}{12} \). The fracture can thus be treated as a 2D porous medium. There is, however, a small caveat because the dimensions are slightly different. In a porous medium, Darcy’s law implies an average velocity \( (m \cdot s^{-1}) \) and permeability has the dimension of \( m^2 \), whereas in a fracture, \( \vec{q} \) is a flow per unit length \( (m \cdot s^{-1}) \) and the "permeability" has the dimension of \( m^3 \). We will see later, this difference has some consequences in the model equations.

All the above mentioned equations apply to Newtonian fluids. A question that naturally arises is: how should this approach be modified when considering non-Newtonian fluids? While there is a very large variety of non-Newtonian fluids (Bird 1976; Bird et al. 1987; Coussot 2005), we will limit our study to fluids whose viscosity varies with shear rate, which are the most commonly non-Newtonian fluids. For these fluids, the rheology is characterized by the relationship between shear rate \( \dot{\gamma} \) and viscosity \( \mu(\dot{\gamma}) \), or equivalently between \( \dot{\gamma} \) and shear stress \( \tau(\dot{\gamma}) \), with \( \tau(\dot{\gamma}) = \mu(\dot{\gamma}) \dot{\gamma} \).

Due to the importance of the subject, a large number of experimental, numerical and theoretical studies have been carried out on the determination of a Darcy’s law for non-Newtonian fluids. The most classical approach (see for instance (Christopher & Middleman 1965; Sadowski & Bird 1965; Slattery 1967; Hirasaki et al. 1974; Chauvet and 1982; Sorbie et al. 1989; Pearson & Tardy 2002; Lopez et al. 2003; Sochi & Blunt 2008; Balhoff & Thompson 2004; Balhoff 2005)) consists in determining an effective shear rate \( \dot{\gamma}_{pm} \) in order to derive an effective viscosity. Another approach is to determine an effective stress (McKinley et al. 2005; Sadowski & Bird 1965; Sadowski & Bird 1965; Slattery 1967; Hirasaki et al. 1974; Chauvet 1982; Sorbie et al. 1989; Pearson & Tardy 2002; Lopez et al. 2003; Sochi & Blunt 2008; Balhoff & Thompson 2004; Balhoff 2005)).
1966), or an average viscosity (Eberhard et al. 2019). A common feature of these approaches is that they are based on the determination of effective quantities. They can be synthesized using scaling arguments. Indeed, by defining a typical length scale \( \lambda \) (pore size, grain diameter, \( \sqrt{k} \), etc.) and using the average flow rate \( u \), a typical shear rate \( \dot{\gamma}_{eff} = \alpha \dot{u}/\lambda \) can be defined. Similarly, using the mean pressure gradient, a typical shear stress \( \tau_{eff} = \beta \lambda \nabla P \) is also defined. The idea is then to use these quantities in the rheological function \( \dot{\gamma} = f(\tau) \) to derive a generalization of Darcy’s law in the form:

\[
 u \propto f(\beta \nabla P),
 \]

where the pre-factors must be determined (experimentally, numerically or theoretically). It is therefore expected that the flow/pressure curve will keep the overall shape of the rheological curve. We will refer to these approaches as the “mean field” approach.

Despite its apparent simplicity, it has successfully been used for different types of rheologies: see for example Christopher & Middleman (1965); Shah & Yortsos (1995, 1996); Auriault et al. (2002) for the power laws, Chauveteau (1982); Sorbie et al. (1989); Lopez et al. (2003); P. et al. (2006); Eberhard et al. (2019); Zami-Pierre et al. (2016) for the truncated power laws/file rheologies or Al-Fariss & Pinder (1987); Chevalier et al. (2014); Shahsavari & McKinley (2016) for yield stress. We note that most of the studies require a certain number of fitting parameters. Indeed, it seems quite natural that the prefactors of the effective quantities depends on the local topology and rheological parameters.

This approach has, however, some limitations which has been illustrated in the case of the yield stress fluid (Talon & Bauer 2013; Chevalier & Talon 2015; Liu et al. 2019). Indeed, these models assume that the whole fluid in the medium yields at the same mean gradient of pressure. The description is therefore particularly inaccurate for heterogeneous media as the shear rate (or stress) is heterogeneously distributed. Some regions are in fact easier to yield than others. This induces a transient flow regime where the material progressively yields and which cannot be taken into account by the mean field approach. A similar effect should be expected for any fluid with a change of rheological behavior: a transitional flow regime should occur as the material progressively and inhomogeneously changes its local rheology. However, as we will see later, the effective "mean field" approach can be considered approximately valid in the asymptotic limits (high or low flow rate).

The problem of non-Newtonian flow in porous media at the macroscopic scale has not been the subject of many studies. Entov (1967) studied homogeneous porous media with different geometries and we studied (Kostenko & Talon 2019) the case of the yield stress in heterogeneous porous media. This work has shown that the influence of the disorder is similar to that at the micro scale: there is a progressive creation of preferential flowing paths which induce a transient mean flow regime. The problem has been more studied for fractures considering different rheologies: Auradou et al. (2008); Boschan et al. (2008); Lavrov (2013b,a, 2015); Hewitt et al. (2016); Roustaei et al. (2016); Rodríguez de Castro & Radilla (2017). This studies showed a modification of the structure of the flow field as a function of the non-linear rheology (yield stress or shear-thinning) and of the mean aperture. There are also the works of Di Federico (1998); Talon et al. (2014); Felisa et al. (2018) which have analytically studied the particular cases where the opening is correlated in layers either perpendicular or transverse to the direction of the flow. In these cases, transitional flow regimes are observed depending on the distribution of the opening. There are also studies on foam flows in porous media or fracture Rossen (1990); Chen et al. (2005); Géraud et al. (2016) which can be, in some extend, be considered as non-Newtonian fluids.

It is important to note here that, although we are mainly dealing with non-Newtonian rheology, much of this work can also be used to understand the role of inertial effect in porous media or fracture. Indeed, since Forchheimer’s pioneering work, it is usually accepted that,
in order to include the effect of inertia, a modified Darcy’s law of the following form can be used: (Forchheimer 1901; Mei & Auriault 1991; Andrade Jr et al. 1999):

$$\vec{v} P \propto au^3 + bu^2 + cu.$$ (1.6)

The latter model can also be considered as a shear thickening rheology with a smooth transition from Newtonian at low flow rate to shear-thickening power-law at high flow rate.

The main objective of the article is to investigate transitional rheologies in macroscopic heterogeneous porous media. In particular, we are interested in understanding the interplay between the heterogeneities and the non-linearity of the viscosity. One motivation of this work is for instance to assess the use of non-Newtonian fluids to characterize the degree of heterogeneity of a field. Indeed, as each location modifies its behavior at a different applied pressure difference, recording the evolution of the average velocity can potentially give indications on the amplitude of field heterogeneities.

This paper is structured as follow. Section 2 is dedicated to the problem presentation. In section 3 will be presented some mathematical properties of the microscopic and macroscopic heterogeneous Darcy’s equation. First, a brief discussion will be made on the “mean field” approach at the microscale to derive the non-Newtonian Darcy’s law. Then will be discussed the problem of macroscopic heterogenous permeability field. It will be shown for instance that, if the rheological curve presents power law asymptotes at low and high flow rate, it is possible to deduce the asymptotes of the relationship between the mean flow rate and the mean pressure gradient, by considering a power-law rheology. A perturbation expansion method will be used to predict the asymptotic mean flow rate and the spatial fluctuations of the velocity field. Section 4 contains the numerical results using the so-called "truncated" rheology, which behaves as a Newtonian fluid at low shear rate and presents a power law rheology at high shear rate, with a sharp transition. In particular, the transition regime presents interesting statistical properties, similar to those of displaying criticality (e.g. percolation). Section 5 will be dedicated to a discussion and conclusion.

2. Problem description
We will first recall the basic equations governing the motion of non-Newtonian fluids both at the micro and meso (Darcy) scale.

2.1. Governing equations
2.1.1. Microscale: Stokes equation
At the micro scale, assuming a steady state and negligible inertia, the flow is governed by the non-Newtonian Stokes equation:

$$\nabla \cdot \Pi = \nabla P = 0$$ (2.1)

where $P$ is the pressure and $\Pi$ is the deviatoric stress tensor. The rheology of the fluid is then characterize by the relationship between the deviatoric stress tensor and the strain rate tensor defined as:

$$\Lambda_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right),$$ (2.2)

where $\vec{u}$ is the velocity field. For isotropic and incompressible fluids, it is sufficient to relate the second invariant of these tensors given by:

$$\dot{\gamma} = \sqrt{2\Lambda_{ij}\Lambda_{ij}} \quad \text{and} \quad \tau = \sqrt{\frac{1}{2} \Pi_{ij}\Pi_{ij}},$$ (2.3)
where \( \dot{\gamma} \) and \( \tau \) are the shear rate and shear stress respectively. The rheological response is thus defined by either the function \( \tau(\dot{\gamma}) \) or, equivalently, the effective viscosity function:

\[
\mu_{eff}(\dot{\gamma}) = \frac{\tau(\dot{\gamma})}{\dot{\gamma}}. \tag{2.4}
\]

And we have:

\[
\Pi = 2\mu(\dot{\gamma})\Delta = 2\frac{\tau(\dot{\gamma})}{\dot{\gamma}}. \tag{2.5}
\]

The simplest rheology is the Newtonian one, where viscosity is independent of shear rate. However, there is a wide variety of non-Newtonian rheologies (Bird 1976; Coussot 2005).

One common rheology is the power-law rheology which is defined as \( \mu_{eff}(\dot{\gamma}) = K\dot{\gamma}^{n-1} \), where \( K \) is the consistency and \( n \) the flow index. If \( n < 1 \), the viscosity decreases with the shear rate and the fluid is called shear-thinning. On contrary if \( n > 1 \), the fluid is shear-thickening.

Some fluids might display a change of behavior depending on the shear rate or stress. For instance, one can cite yield stress fluids often described by the Herschel-Bulkley rheology:

\[
\left\{ \begin{array}{ll}
\tau(\dot{\gamma}) < \tau_0 & \text{if } \dot{\gamma} = 0 \\
\tau(\dot{\gamma}) = \tau_0 + K\dot{\gamma}^n & \text{if } \dot{\gamma} > 0
\end{array} \right. \quad \text{or} \quad \left\{ \begin{array}{ll}
\dot{\gamma}(\tau) = 0 & \text{if } \tau < \tau_0 \\
\dot{\gamma}(\tau) = \left[ \frac{1}{K}(\tau - \tau_0) \right]^{1/n} & \text{if } \tau > \tau_0
\end{array} \right., \tag{2.6}
\]

where \( \tau_0 \) is the yield stress, \( n \) the flow index and \( K \) the consistency. If the applied stress is below the yielding threshold, the material responds like a solid. Above it, the fluid is able to be sheared and thus flowing.

Many fluids also display a change of rheological behavior, they have a constant viscosity at low shear rate, but as the shear rate increases they become shear-thinning of thickening with a power law viscosity. There are many different models to describe those fluids, such as the Carreau-Yasuda (Bird et al. 1987) model:

\[
\frac{\mu_{eff}(\dot{\gamma}) - \mu_\infty}{\mu - \mu_\infty} = \left[ 1 + \left( \frac{\dot{\gamma}}{\dot{\gamma}_0} \right)^{\alpha-1} \right]^\frac{\alpha}{\alpha-1}, \tag{2.7}
\]

where \( \mu \) and \( \mu_\infty \) are the viscosity at low and high shear rate respectively, and \( \dot{\gamma}_0 \) a shear rate characterizing the transition. If \( \mu_\infty = 0 \), the fluid is Newtonian at low shear rate and power-law at higher shear rate. Similarly, there is the Cross model:

\[
\mu_{eff}(\dot{\gamma}) = \frac{\mu}{1 + (\frac{\dot{\gamma}}{\dot{\gamma}_0})^n}. \tag{2.8}
\]

In this article, we will use a simplified model to describe such rheology which is the “truncated” rheology:

\[
\left\{ \begin{array}{ll}
\tau(\dot{\gamma}) = \mu\dot{\gamma} & \text{if } \dot{\gamma} < \dot{\gamma}_0 \\
\tau(\dot{\gamma}) = K\dot{\gamma}^n & \text{if } \dot{\gamma} > \dot{\gamma}_0
\end{array} \right. \quad \text{or} \quad \left\{ \begin{array}{ll}
\dot{\gamma}(\tau) = \frac{1}{\mu}\tau & \text{if } \tau < \tau_0 = \mu\dot{\gamma}_0 \\
\dot{\gamma}(\tau) = \left[ \frac{1}{K}\tau \right]^{1/n} & \text{if } \tau > \tau_0
\end{array} \right., \tag{2.9}
\]

where \( \mu \) is the viscosity at low shear rate and \( \dot{\gamma}_0 \) a threshold shear rate at which the fluid changes its behavior as sketched in Fig. 1 (left).

### 2.1.2. Mesoscale: Darcy’s law

These rheological laws are applicable to solve Stokes equation and thus to determine the flow at the microscopic level, inside each pore. As described in the introduction, in porous media, it is important to determine a mean constitutive equation, which relates the mean flow rate, \( u \)
to the mean pressure gradient, $\nabla P$ at the mesoscale, viz. a volume containing a large amount of pores. By applying the effective shear rate approach, it results the following generalization of Darcy’s law:

$$\nabla P = -g(u) \quad \text{or} \quad u = -f(\nabla P),$$

(2.10)

where $g$ and $f$ are assumed to be monotonically increasing (and thus invertible) functions.

To solve the flow field, a vector formulation of the non-Newtonian Darcy’s law is required. Assuming that the medium is locally isotropic, so that the mean flow is collinear and opposite to the mean pressure gradient, it follows:

$$\vec{\nabla} P = -g(\vec{r}; |\vec{u}|) \frac{|\vec{u}|}{|u|} \vec{u}$$

$$\text{or} \quad \vec{u} = -f(\vec{r}; \|\nabla P\|) \frac{\vec{\nabla} P}{\|\nabla P\|},$$

(2.11)

and

$$\vec{\nabla} \cdot \vec{u} = 0$$

(2.12)

This form is very generic and applies for instance to Carreau, Herschel-Bulkley or the truncated rheology. It is also very important to note that the functions $g$ and $f$ may vary spatially (e.g. heterogeneous permeability field). For better readability, the $\vec{r}$ will be omitted in the following.

For the truncated rheology, the equation then reads:

$$\begin{cases}
\vec{\nabla} P = -\frac{\mu}{\kappa(\vec{r})} \vec{u} & \text{if} \quad |\vec{u}| < u_c(\vec{r}) \\
\vec{\nabla} P = -\frac{\mu}{\kappa(\vec{r})} \left[ \frac{|\vec{u}|}{u_c(\vec{r})} \right]^{n-1} \vec{u} & \text{if} \quad |\vec{u}| > u_c(\vec{r})
\end{cases}$$

(2.13)

where $\kappa(\vec{r})$ is the local permeability and the prefactor in the second relation has been determined by continuity. Equivalently, by defining $u_c = \frac{\kappa g_c}{\mu}$, one may also write:

$$\begin{cases}
\vec{\nabla} P = -\frac{\kappa(\vec{r})}{\mu} \vec{u} & \text{if} \quad |\nabla P| < g_c(\vec{r}) \\
\vec{\nabla} P = -\frac{\kappa(\vec{r})}{\mu} \left[ \frac{|\nabla P|}{g_c(\vec{r})} \right]^{1/n-1} \vec{\nabla} P & \text{if} \quad |\nabla P| > g_c(\vec{r})
\end{cases}$$

(2.14)

As described in the introduction, this approach is based on the assumption that an effective
shear rate $\gamma_{eff}$ can be related to an effective shear stress $\tau_{eff}$. Despite its relative simplicity, this approach has been validated by my numerous experimental and numerical works (see introduction for references). In section 3.1, we will examine in more detail the validity of this approach. For example, we will show that these expressions actually represent the asymptotic limit at low and high flow rate.

In this article, where we intend to study the flow in a heterogeneous macroscopic field, eq. (2.13) is assume to be valid but with a hetergenous permeability field $\kappa(\vec{r})$ and threshold velocity field $u_c(\vec{r})$ (or $g_c(\vec{r})$) to be heterogenous.

### 2.2. Porous media

The permeability field was chosen to be distributed according to a log-normal distribution which is a common assumption (Gelhar & Axness 1983). This is obtained by generating a Gaussian field with zero mean and a given standard deviation. $\kappa$ is correlated with a correlation length (see (Kostenko & Talon 2019) for more details). The permeability field is then given by:

$$\kappa = \exp(f_0 + \delta f) = k_0 \exp(\delta f),$$

(2.15)

where $k_0 = \exp(f_0)$, is a parameter characterizing the average permeability of the medium.

As for the threshold parameters $u_c$ and $g_c$, intuitively they are expected to be somehow related to the permeability $\kappa$. A relationship can be established using phenomenological arguments. For porous media, defining the typical pore size $d$, a scaling analysis leads to $\kappa \sim d^2$ and $g_c \sim \tau_0/d$, thus: $g_c \sim \kappa^{-1/2}$ and $u_c \sim \kappa^{1/2}$.

For fracture, this reasoning is slightly different, defining $h$ as a typical opening leads to $\kappa \sim h^3$ and $g_c \sim \tau_0/h$. This leads to: $g_c \sim \kappa^{-1/3}$ and $u_c \sim \kappa^{2/3}$. As these arguments are purely phenomenological, we have studied a more general relation in the form

$$u_c = A \kappa^\gamma,$$

(2.16)

where $A$ is a prefactor and $\gamma$ is a parameter with $\gamma \in [0; 1]$. We also have:

$$g_c = \mu A \kappa^{\gamma-1}.$$  

(2.17)

### 2.3. Simulations

Macroscopic simulations were performed using a second order finite difference method combined with an augmented Lagragian approach described in the appendix. In practice, the flow rate is driven by imposing a pressure difference $\Delta P$ between inlet and outlet, but it is more convenient to use the mean pressure gradient $\langle \nabla P \rangle = \frac{\Delta P}{L}$ as control parameter, where $L$ is the length of the system. Lateral boundary conditions are periodic. Although there are many different parameters, the focus of this study will be on the amplitude of the disorder $\sigma_f$, the flow index $n$ and the exponent $\gamma$. Parameters $A$, $\mu$ and $k_0$ will thus be kept constant with $A = 1$, $\mu = 0.1$ and $k_0 = 1$. Based on these parameters a characteristic velocity

$$u_0 = A k_0^\gamma$$

and a characteristic pressure gradient

$$g_0 = \frac{\mu u_0}{k_0}$$

can be defined and be used for non-dimensionalization. $u_0$ and $g_0$ represent the velocity and the pressure gradient at which the system would change its behavior if the field were homogeneous (i.e $\sigma_f = 0$).
2.4. Flow examples

2.4.1. Homogenous permeability - heterogeneous critical velocities

In order to describe qualitatively the problem, a simplified version is presented, where the permeability is homogeneous but not the critical velocity $u_c$. The latter varies spatially according to a log-normal distribution.

In figure 2 (top) shows the flow field for different mean pressure gradients $\langle \nabla P \rangle$. For a small pressure difference, the flow field is homogeneous as for a Newtonian Darcy flow in a homogeneous field. As the pressure difference increases, flow field remains homogeneous until the first location reaches its local threshold (Fig. 2.a). Around this first point, since the viscosity has changed, the flow field is then perturbed in a way similar to a dipole flow. In this figure, the second regime is shear-thinning $n < 1$ so the viscosity has locally decreased. The flow has thus increased downstream and upstream, while it has decreased on the lateral sides. Consequently, the change of regime at one point should favour or defavour the change of behaviour in the vicinity. One can therefore expect some correlations of the regions that have changed their flow regime. For shear-thinning (resp. thickening) fluids the change of regime should thus be correlated in the streamwise (resp. perpendicular) direction. Figure 2 (bottom) represents the correspondings regions which have change their regime. It can be noted that this correlation effect seems in fact quite marginal. This might be explained by the fact that the distribution of $u_c$ is quite large $\sigma = 1$. The dipole velocity perturbation might thus be unable to compensate large fluctuation of the threshold value. As the pressure difference increases, more and more regions change their behavior until the entire domain is in the non-Newtonian regime.

Figure 3 presents the evolution of different interesting quantities. Figure 3.a displays the mean velocity as a function of the mean pressure gradient. As expected, for low pressure, the average flow rate varies linearly with the pressure as expected. At a certain point, the curve begins to deviate slowly from the linear trend to reach a power-law trend. This change of behavior can be magnified by plotting the amount $\langle u \rangle / \langle \nabla P \rangle$. We observe three flow regimes, the first regime is one where the flow is linear. At a high flow rate, the flow follows a power law regime of the exponent $\langle u \rangle \propto \langle \nabla P \rangle^{1/n}$. Between these two regimes, there is a transitional regime which is the main subject of this article. A convenient quantity to characterize this transition is the percentage of regions being in the non-linear regime $O(\langle \nabla P \rangle)$ ranging from 0 to 1 (Fig. 3.c). Another important quantity, shown in figure 3.d, is the relative standard deviation of the velocity field, $\frac{\sigma_u}{\langle u \rangle}$, which characterizes the flow heterogeneities and is of great importance in the problem of transport of species. Here, we observe that the standard deviation is zero at low pressure because the permeability is homogeneous. After the transition, it reaches a constant non-zero value. The reason behind this observation is that, even if the permeability is homogeneous, each location changes its behavior at a different mean pressure gradient. It thus introduces heterogeneities in the flow field.

2.4.2. Heterogeneous permeability - homogeneous critical velocities

Another interesting example is the opposite case, where the permeability $\kappa(\vec{r})$ is heterogeneous but not the critical velocity $u_c$, corresponding to $\gamma = 0$. Figure 4 represents the flow field distribution for different applied pressure difference. At low pressure drop, the velocity field corresponds to the heterogeneous Darcy’s equation and is thus heterogenous as well. As the mean flow rate increase, the first region to change its viscosity will be the one with the highest velocity in the Newtonian case. The velocity distribution is then drastically change above $u_c$ but the topology of the regions $u > u_c$ reflects the properties of the velocity field of the heterogeneous Newtonian Darcy’s equation. In particular, the correlations are in the streamwise direction as it will be discussed later.
Figure 2: For a shear-thinning truncated rheology $n = 1/2$ in a medium with a homogeneous permeability field but with a heterogeneous threshold, $\sigma = 1$. Top: Velocity field colormap for different imposed mean pressure gradients increasing from left to right. The average flow direction is from top to bottom. Middle row: in black are represented the regions above the threshold. Bottom: Distribution function of the velocity (blue) and threshold velocity $u_c$ (orange).

Figure 3: For the medium of Fig. 3 as function of the imposed mean gradient of pressure: a: Mean velocity $\langle u \rangle$. b: Mean velocity rescaled with the Darcy’s law. c: Percentage of regions which have change their behavior (i.e $u > u_c$). d: relative standart deviation of the velocity field $\sigma_{\langle u \rangle}$. Vertical dashed lines represent the transitions of the flow regimes.
3. Analytical aspects

In this section, we discuss some analytical aspects of the problem. It is important to note that many of the results presented here are very general, and can be applied to other non-linear rheologies than the truncated one.

3.1. Energy balance - Virtual work

The energy balance and the virtual work approaches lead to interesting properties. Virtual work consists in evaluating the work of the real forces on a virtual flow field, which is not necessarily the real one. Among all the possible velocity field, the solution of the problem is the minimum of a certain functional to be defined. This approach will be applied at two scales: the microscopic scale, where the fluid obeys Stokes non-Newtonian flow, and the macroscopic scale where the fluid obeys the non-Newtonian Darcy’s law.

At the microscopic level, this approach was for example introduced by Duvaut & Lions (1976) for yield stress fluids. Here, the description is summarized as it has already been discussed for instance for Hershel Bulkley in porous media in the appendix of Bauer et al. (2019). The method is however extended to any rheology. At the microscopic level, two limits, at high and low flow, can be formally predicted. This can partially justify the mean field approach in order to generalize Darcy’s law.

At the macroscopic level, to our knowledge, it has never been applied, so the method will be presented in more details. Here again, it allows to predict asymptotic behavior of the flow.

3.2. Microscopic description

A porous parallelepiped medium containing solid and porous regions is considered. The entry section is $S$, the length $L$ and a characteristic size (average pore, grain diameter, etc.) is named $\lambda$. A pressure $P_{in}$ and $P_{out}$ is applied to the inlet and outlet respectively. The total flow rate through $S$ is denoted $Q$. No flow on the lateral side of the porous media is assumed. In the porous regions, the rheology of the fluid is defined by the function $\tau(\dot{\gamma})$, a monotonically
increasing function. Following Chevalier et al. (2014), a standard energy balance gives:
\[
Q \Delta P = \int \tau(\dot{\gamma}) \dot{\gamma} \, d\tau^3.
\] (3.1)

This equation indicates that the relationship between the mean flow rate and the mean applied pressure can be deduced from the shear rate distribution \( \dot{\gamma}(\vec{r}) \). Its determination requires to solve the Stokes equation. Since pressure can be considered as an energy per unit volume, \( Q \Delta P \) represents the difference between the incoming and outgoing flux of this energy. It should therefore be equal to the total viscous dissipation within the medium given by the righthand term of eq. (3.1).

If the solution \( \dot{\gamma} \) is determined by Stokes equation, it can also be obtained from a minimum principle: among all admissible velocity fields \( \vec{v} \) (i.e. satisfying the free divergence and the boundary condition), the solution \( \vec{u} \) of the Stokes equation minimizes the functional:
\[
\vec{u} = \arg \min_{\vec{v}} \left\{ \int \phi(\dot{\gamma}[\vec{u}]) \, d\tau^3 - \Delta P Q[\vec{v}] \right\},
\] (3.2)

where \( \phi(\dot{\gamma}) = \int_0^{\dot{\gamma}} \tau(\dot{\gamma}') d\dot{\gamma}' \). \( Q[\vec{v}] = \int_{\text{inlet}} \vec{v} \cdot dS \) the total flow rate of the \( \vec{v} \) field. Physically, this relation can be considered as an extension of the minimum dissipation principle for the Newtonian Stokes equation (Keller et al. 1967; Happel & Brenner 1983). It is important to note that this minimum principle is applicable for an imposed pressure difference \( \Delta P \). A similar result can also be obtained for an imposed flow rate \( Q \) (Bauer et al. 2019).

3.2.1. Power-law fluids

Considering a simple power-law fluid, \( \tau(\dot{\gamma}) = K \dot{\gamma}^n \), it follows that \( \phi(\dot{\gamma}) = K \frac{1}{n+1} \dot{\gamma}^{n+1} \). Using eq. (3.2), the velocity field is the solution of:
\[
\vec{u} = \arg \min_{\vec{v}} \left\{ \int K \frac{1}{n+1} \dot{\gamma}^{n+1}[\vec{v}] \, d\tau^3 - \Delta P Q[\vec{v}] \right\}.
\]

It is then important to note that the result of the minimum should not change by multiplying the function by any real number. Multiplying by \( \epsilon^{n+1} \) leads to:
\[
\vec{u} = \frac{1}{\epsilon} \arg \min_{\vec{v}} \left\{ \int K \frac{1}{n+1} \dot{\gamma}^{n+1}[\epsilon \vec{v}] \, d\tau^3 - (\epsilon^n \Delta P) Q[\vec{v}] \right\},
\]
due to the linearity of the operator \( \dot{\gamma}[\vec{v}] \) and \( Q[\vec{v}] \). And thus:
\[
\vec{u} = \frac{1}{\epsilon} \arg \min_{\vec{v}} \left\{ \int K \frac{1}{n+1} \dot{\gamma}^{n+1}[\vec{v}] \, d\tau^3 - (\epsilon^n \Delta P) Q[\vec{v}] \right\},
\]
by changing the variable. This indicates that the velocity field, solution of the problem for the imposed pressure \( \epsilon^n \Delta P \) is the same solution at \( \Delta P \), but multiplied by \( \epsilon \). It follows the scaling relationships:
\[
\Delta P \rightarrow \epsilon^n \Delta P \\
\vec{u} \rightarrow \epsilon \vec{u} \\
Q \rightarrow \epsilon Q.
\]

It must be noted that this scaling relationship agrees with the first order expansion obtain by homogenization (Shah & Yortsos 1995; Auriault et al. 2002). For any power-law rheology of index \( n \), the field \( n \vec{\omega} = \frac{\vec{u}}{Q[\vec{u}]} \) is thus a constant independant of the applied pressure. And the non-dimensional shear rate field, defined as \( n \delta = \frac{\Delta \dot{\gamma}}{\dot{\gamma}} \) is also invariant with the applied pressure.
From the energy balance equation, eq. (3.1), it follows:

$$\Delta P = \frac{KQ^n}{\lambda^{n+1} S^{n+1}} \int n\delta^{n+1} \, dr^3$$  \hspace{1cm} (3.3)

$$= \frac{KQ^n L}{\lambda^{n+1} S^{n}} \langle (n\delta)^{n+1} \rangle. \hspace{1cm} (3.4)$$

And so

$$\lambda \frac{\Delta P}{L} = K \left( \frac{Q}{\lambda S} \right)^n \langle (n\delta)^{n+1} \rangle. \hspace{1cm} (3.5)$$

This equation shows that the mean field approach is thus able to predict correctly flow rate - pressure curve, providing the introduction of $\tau_{\text{eff}} = \lambda \frac{\Delta P}{L}$ and $\dot{\gamma}_{\text{eff}} = \frac{Q}{\lambda S} \frac{\sqrt{\langle n\delta^{n+1} \rangle}}{n}$, the quantity $\sqrt{\langle n\delta^{n+1} \rangle}$ is a constant field independant of the applied pressure (or flow rate). This field depends on the porous structure, so it requires to solve the Stokes equation once.

### 3.2.2. Rheology with change of behavior

For more complex rheologies, such as the truncated one, there is no guarantee that $\sqrt{\langle n\delta^{n+1} \rangle}$ is a constant field independant of the boundary conditions. However, it is interesting to consider the case where the rheological function $\tau(\dot{\gamma})$ admits power-law asymptotes:

$$\begin{align*}
\tau(\dot{\gamma}) &\sim K_0 \dot{\gamma}^a & \text{if } \dot{\gamma} \to 0 \\
\tau(\dot{\gamma}) &\sim K_\infty \dot{\gamma}^b & \text{if } \dot{\gamma} \to \infty.
\end{align*} \hspace{1cm} (3.6)$$

Since $\dot{\gamma} \sim Q$, it is expected that at high (resp. low) flow rates the highest (resp. lowest) exponent will dominate in $\phi(\dot{\gamma})$, so that

$$\phi(\dot{\gamma}) \xrightarrow{Q \to 0} \dot{\gamma}^{a+1} \quad \text{and} \quad \phi(\dot{\gamma}) \xrightarrow{Q \to \infty} \dot{\gamma}^{b+1}. \hspace{1cm} (3.7)$$

Using these limits in the minimization equation eq. (3.2), indicates that the solutions are equivalent to solving the flow for a power-law fluid with the dominant exponent. It follows that the fields $u/Q$ and the non-dimensional shear rate $\frac{\dot{\gamma}}{Q}$ become independent of the applied pressure at sufficiently high and low applied pressure (or flow rate). The knowledge of these two quantities allows then to derivate the effective Darcy’s law in the two limits, using the energy balance eq. (3.1)

As an example, we assume a polynomial rheology:

$$\tau(\dot{\gamma}) = A\dot{\gamma}^3 + B\dot{\gamma}^2 + C\dot{\gamma} + D. \hspace{1cm} (3.8)$$

It follows:

$$\phi(\dot{\gamma}) = \frac{1}{4} A\dot{\gamma}^4 + \frac{1}{3} B\dot{\gamma}^3 + \frac{1}{2} \dot{\gamma}^2 + D\dot{\gamma}. \hspace{1cm} (3.9)$$

At high (resp. low) flow rate, the non-dimensional shear rate will thus converge to the one obtained with a power-law fluid of $3\delta(\bar{r})$ (resp. $1\delta(\bar{r})$). Eq. (3.1) reads

$$Q\Delta P = A \int \dot{\gamma}^4 \, dr^3 + B \int \dot{\gamma}^3 \, dr^3 + C \int \dot{\gamma}^2 \, dr^3 + D \int \dot{\gamma} \, dr^3.$$ 

Using the non-dimensional shear rate field $\delta = \frac{\dot{\gamma}}{Q}$, it follows the two asymptotic behaviors:
\[
\frac{\Delta P}{L} = A \left( \frac{\langle u \rangle}{\lambda} \right)^3 + B \left( \frac{\langle u \rangle}{\lambda} \right)^2 \langle \delta^3 \rangle + C \left( \frac{\langle u \rangle}{\lambda} \right) \langle \delta^2 \rangle + D \langle \delta \rangle \quad \text{for} \quad \langle u \rangle \to \infty
\]
\[
\frac{\Delta P}{L} = A \left( \frac{\langle u \rangle}{\lambda} \right)^3 + B \left( \frac{\langle u \rangle}{\lambda} \right)^2 \langle \delta^3 \rangle + C \left( \frac{\langle u \rangle}{\lambda} \right) \langle \delta^2 \rangle + D \langle \delta \rangle \quad \text{for} \quad \langle u \rangle \to 0,
\]
with the notation \( \langle . \rangle = \frac{1}{S L} \int . d^3 \) and \( Q = S \langle u \rangle \). Here, both limits have a polynomial form, similar to the rheological function \( \tau (\dot{\gamma}) \). The coefficients depend on the \( k \)-th moments \( \langle n \delta^k \rangle \) which are constant but require solving a power law rheology with index 1 and 3.

This equation leads to a form similar to the mean field approach:
\[
\beta \frac{\Delta P}{L} = A \left( \frac{\alpha \langle u \rangle}{\lambda} \right)^3 + B \left( \frac{\alpha \langle u \rangle}{\lambda} \right)^2 + C \frac{\alpha \langle u \rangle}{\lambda} + D. \quad (3.9)
\]

However, there are two notable differences.

First of all, matching the main order term requires to define \( \dot{\gamma}_{eff} = \frac{\langle u \rangle}{\lambda} \langle 3 \delta^4 \rangle^{1/3} \) at high flow rate and \( \dot{\gamma}_{eff} = \frac{\langle u \rangle}{\lambda} \langle (1 \delta^4) \rangle^{1/3} \), at low flow rate. The definition of \( \dot{\gamma}_{eff} \) should thus change in the two limits. Secondly, the other coefficients then do not necessarily match. For example, there is a mismatch in the second term if \( \langle 3 \delta^4 \rangle^{1/3} \neq \langle 3 \delta^3 \rangle^{1/2} \). It is therefore generally not possible to determine \( \alpha \) and \( \beta \) so that all coefficients match, unless the polynomial has only two terms (as for Herschel-Bulkley).

This approach is applicable when the rheological function \( \tau (\dot{\gamma}) \) has a power-law limit \( \tau (\dot{\gamma}) \sim K_0 \dot{\gamma}^a \) (resp. \( \sim K_0 \dot{\gamma}^b \)) in the high (resp. low) flow rate limit. It is thus applicable with the truncated rheology (\( a = 1 \) and \( b = n \)) but also for yield stress fluids (for Herschel-Bulkley, \( a = 0 \) and \( b = n \)). It has been used by Chevalier et al. (2014) to propose a mesoscopic Darcy’s law for Herschel-Bulkley. The authors also used the Newtonian velocity distribution even for \( n \neq 1 \) to estimate the coefficients. The agreement with the experiments seems then to suggest that \( \langle n \delta^{n+1} \rangle^{1/n} \sim \langle 3 \delta^2 \rangle \), at least in the porous media used (bead packing).

### 3.3. Macroscopic description

In this section, a general nonlinear Darcy equation given by eqs. (2.10) is assumed. It will be demonstrated that the solution of the heterogenous non-linear Darcy’s equation is also the minimum of a functional \( \Phi [\vec{v}] \), among all the admissible field \( \vec{v} \). For this geometry, the pressure is prescribed at the inlet edge \( P_{in} \) and outlet edge \( P_{out} \). No flux flow through the lateral boundaries is assumed.

First, we define \( \Omega \) the set of admissible velocity fields satisfying the divergent free and the boundary conditions, Multiplying eq. (2.11) by any \( \vec{v} \in \Omega \), and integrating over the domain, yields to:
\[
\forall \vec{v} \in \Omega , \quad \int \nabla P \vec{v} \cdot d^3 = - \int g(\| \vec{u} \|) \frac{\vec{u} \cdot \vec{v}}{\| \vec{u} \|} d^3 \quad (3.10)
\]

Using the free divergence of \( \vec{v} \) and the divergent theorem, it follows
\[
\forall \vec{v} \in \Omega , \quad - \int g(\| \vec{u} \|) \frac{\vec{u} \cdot \vec{v}}{\| \vec{u} \|} d^3 = (P_{out} - P_{in}) \int_{\text{outlet}} \vec{v} \cdot dS = -Q[\vec{v}] \Delta P, \quad (3.11)
\]

where \( Q[\vec{v}] = \int_{\text{outlet}} \vec{v} \cdot dS = -\int_{\text{inlet}} \vec{v} \cdot dS \) is the total flow rate of the field \( \vec{v} \) and \( \Delta P = P_{in} - P_{out} \).

It can be demonstrated that the field \( \vec{u} \), solution of the Darcy’s equation eqs. (2.10), is also
It is equivalent to prove that:

\[
\Phi[\vec{v}; \Delta P] = \int G(||\vec{v}||) \, dr^3 - \Delta PQ[\vec{v}].
\] (3.12)

Where \(G(||\vec{v}||)\) is the primitive of \(g(||\vec{v}||)\):

\[
G(||\vec{v}||) = \int_0^{||\vec{v}||} g(y) \, dy.
\]

It is equivalent to prove that:

\[
\forall \vec{v} \in \Omega, \, \Phi[\vec{v} + \vec{u}; \Delta P] - \Phi[\vec{u}; \Delta P] \geq 0. \tag{3.13}
\]

Indeed, eq. (3.11) leads to:

\[
\forall \vec{v} \in \Omega, \, \Phi[\vec{v} + \vec{u}; \Delta P] - \Phi[\vec{u}; \Delta P] = \int \left\{ G(||\vec{v} + \vec{u}||) - G(||\vec{u}||) - \frac{g(||\vec{u}||)}{||\vec{u}||} \vec{u} \cdot \vec{v} \right\} \, dr^3. \tag{3.14}
\]

Since \(g\) is an increasing function, \(G\) is convex, and thus:

\[
G(||\vec{v} + \vec{u}||) - G(||\vec{u}||) \geq g(||\vec{u}||)(||\vec{v} + \vec{u}|| - ||\vec{u}||). \tag{3.15}
\]

It follows the required property:

\[
\forall \vec{v} \in \Omega, \, \Phi[\vec{v} + \vec{u}; \Delta P] - \Phi[\vec{u}; \Delta P] \geq \int \frac{g(||\vec{u}||)}{||\vec{u}||} \left[ ||\vec{u} + \vec{v}|| ||\vec{u}|| - ||\vec{u}||^2 - \vec{u} \cdot \vec{v} \right] \, dr^3 \geq 0, \tag{3.16}
\]

where Cauchy-Schwartz inequality has been used \((\vec{u} + \vec{v}).\vec{u} \leq ||\vec{u} + \vec{v}|| ||\vec{u}||\).

Several remarks can be made:

**Remark 1** As for the microscopic case, it is instructive to put the solution \(\vec{u}\) in eq. (3.11). Leading to the energy balance:

\[
Q[\vec{u}] \Delta P = \int g(||\vec{u}||) ||\vec{u}|| \, dr^3 > 0. \tag{3.17}
\]

This expression can be seen as a balance of energy and it also proves the intuitive result that the flow rate and the difference of pressure \(P_{in} - P_{out}\) have always the same sign.

**Remark 2:** Several examples can be given for different rheologies:

- **Newtonian (Darcy):**

\[
\nabla P = -\frac{\mu}{\kappa(r)} \vec{u}. \tag{3.18}
\]

In this case, \(g(||\vec{u}||) = \frac{\mu}{\kappa(r)} ||\vec{u}||\) yields to:

\[
\Phi[\vec{v}; \Delta P] = \int \frac{\mu}{2\kappa(r)} ||\vec{v}||^2 - \Delta PQ[\vec{v}].
\]

This expression is interesting because it shows the classical result that, in order to minimize dissipation, it is more favorable to have a higher velocity where permeability is high. However, it is important to note that the allowable field \(\vec{v}\) must satisfy the conservation of mass. This constraint can cause low permeability regions to have a high velocity (and vice versa). Conservation of mass tends to correlate the velocity field in the streamwise direction.

- **Herschel-Bulkley:** At Darcy’s scale, the velocity field can be described by (see
\( - \nabla P = c(\vec{r}) |\vec{u}|^{n-1} \vec{u} + g_c(\vec{r}) \frac{\vec{u}}{|\vec{u}|} \)  \hspace{1cm} (3.19) 

where \( g_c(\vec{r}) \) is the local critical pressure gradient below which there is no flow, \( c(\vec{r}) \) is a prefactor that depend on the consistency and the local geometry, and \( n \) the flow index. It then follows \( g(|\vec{u}|) = c(\vec{r}) |\vec{u}|^n + g_c(\vec{r}) \). Thus:

\[
\Phi[\vec{v}; \Delta P] = \int \left[ \frac{c(\vec{r})}{n+1} |\vec{v}|^{n+1} + g_c(\vec{r}) |\vec{v}| \right] d\tau^3 - \Delta P Q[\vec{v}].
\]

- **Truncated rheology**:

\[
\begin{cases}
\vec{v} = - \frac{\mu}{k} \vec{u} & \text{if} \quad |\vec{u}| < u_c(\vec{r}) \\
\vec{v} = - \frac{\mu}{k} \left[ \frac{|\vec{u}|}{u_c} \right]^{n-1} \vec{u} & \text{if} \quad |\vec{u}| > u_c(\vec{r})
\end{cases}
\]

leads to:

\[
\begin{cases}
\Phi[\vec{v}; \Delta P] = \int \left[ \frac{\mu}{2k(\vec{r})} |\vec{v}|^2 \right] d\tau^3 - \Delta P Q[\vec{v}] & \text{if} \quad |\vec{v}| < u_c(\vec{r}) \\
\Phi[\vec{v}; \Delta P] = \int \left[ \frac{\mu}{(n+1)k u_c |u_c|^{n-1}} \left[ |\vec{v}|^{n+1} - u_c n^{n+1} \right] + \frac{\mu}{2k(\vec{r})} u_c^2 \right] - \Delta P Q[\vec{v}] & \text{if} \quad |\vec{v}| > u_c(\vec{r})
\end{cases}
\]

- **Power-law rheology**: Darcy’s law now writes:

\[- \vec{v} P = c(\vec{r}) |\vec{u}|^{n-1} \vec{u}. \]  \hspace{1cm} (3.22) 

leading to:

\[
\vec{u} = \arg \min_{\vec{v}} \left\{ \int \frac{c(\vec{r})}{n+1} |\vec{v}|^{n+1} d\tau^3 - \Delta P Q[\vec{v}] \right\}. \]  \hspace{1cm} (3.23) 

Similarly to the microscopic case, the invariance of the velocity field can be proved. For any \( \epsilon \), multiplying by \( \epsilon^{n+1} \) the previous equation gives:

\[
\vec{u}(\Delta P) = \arg \min_{\vec{v}} \left\{ \int \frac{c(\vec{r})}{n+1} |\vec{v}|^{n+1} d\tau^3 - \epsilon^n \Delta P Q[\epsilon |\vec{v}|] \right\}
\]

\[
= \frac{1}{\epsilon} \arg \min_{\vec{v}} \left\{ \int \frac{c(\vec{r})}{n+1} |\vec{v}|^{n+1} d\tau^3 - \epsilon^n \Delta P Q[\vec{v}] \right\}
\]

\[
= \frac{1}{\epsilon} \vec{u}(\epsilon^n \Delta P).
\]

It follows the scaling relationships:

\[
\Delta P \rightarrow \epsilon^n \Delta P \\
\vec{u} \rightarrow \epsilon \vec{u} \\
Q \rightarrow \epsilon Q.
\]

Similarly, the field \( n \vec{u} = \frac{\vec{u}}{(\vec{u})} \) is thus a constant field, independant of the applied pressure or the mean velocity.

As for the microscopic case, it follows that for any rheology with power-law asymptotes, in the limit of a low or high flow rate, the \( \frac{\vec{u}(\vec{r})}{(\vec{u})} \) distribution should tend towards a field independent of the mean pressure or flow rate gradient. These constant fields are determined by solving a power-law fluid with the corresponding dominant exponent. This is true for
Herschel-Bulkley or truncated rheology but applies to any polynomial function (or series of power-laws).

One example could be the Forchheimer’s law, which corresponds to the generalization of Darcy’s law including the influence of inertia. It usually proposed a relation of the form (Forchheimer 1901):

$$\nabla P = Au^3 + Bu^2 + Cu,$$  \hspace{1cm} (3.24)

and thus $g(|\vec{u}|) = A|\vec{u}|^3 + B|\vec{u}|^2 + C|\vec{u}|$. In the high flow rate regime, the velocity field $\frac{\tilde{u}(\vec{r})}{u}$ should tend to the solution of a shear-thickening fluid eq. (3.22) with the exponent $n = 3$.

Using the energy balance equation eq. (3.17), it follows the two asymptotic regimes:

$$\frac{\Delta P}{L} = A\langle 1 \tilde{\sigma}^4 \rangle \langle u \rangle^3 + BA\langle 1 \tilde{\sigma}^3 \rangle \langle u \rangle^2 + C\langle 1 \tilde{\sigma}^2 \rangle \langle u \rangle^1 \quad \text{for} \quad \langle u \rangle \to 0$$

$$\frac{\Delta P}{L} = A\langle 3 \tilde{\sigma}^4 \rangle \langle u \rangle^3 + BA\langle 3 \tilde{\sigma}^3 \rangle \langle u \rangle^2 + C\langle 3 \tilde{\sigma}^2 \rangle \langle u \rangle^1 \quad \text{for} \quad \langle u \rangle \to \infty,$$

where the jth-moment $\langle n \tilde{\sigma}^j \rangle$ requires to determine the flow of a power-law fluid in the medium.

In conclusion, any nonlinear Darcy’s law with power law asymptotes converges to the solution of a power law fluid in heterogeneous media at low or high flow rate. The next section is devoted to analyzing the statistical properties of a power law fluid in heterogeneous fractured media or porous media. In particular, analytical solutions for the mean and standard deviation of the velocity field will be derived in the limit of small amplitudes of heterogeneities $\sigma_f \ll 1$.

### 3.4. Perturbation expansion for a power-law rheology

The objective of this section is to derive the mean properties of the flow field. The approach is to extend the work of Gelhar & Axness (1983) to power law fluids, which consists in expanding the solution around the mean value and assuming sufficiently small spatial fluctuations.

Assuming that the flow field is solution of a power-law rheology in heterogeneous porous media:

$$-\tilde{\nabla}P = c(\vec{r})|\vec{u}|^{n-1}\tilde{u},$$  \hspace{1cm} (3.25)

with the free divergence:

$$\nabla.\vec{u} = 0.$$  \hspace{1cm} (3.26)

In order to compare with the truncated rheology, $c(\vec{r})$ should be equal to $c(\vec{r}) = \frac{\mu}{k(\vec{r})\mu_c^{n}}$ with $k$ and $\mu_c$ distributed according to log-normal as described previously. However, in order to remain as general as possible, the generic function $c(\vec{r})$ will be kept as long as possible.

The principle is to decompose each field $\tilde{u}$, $P$, and $c(\vec{r})$ into a mean part and the spatially fluctuating part

$$\tilde{u}(\vec{r}) = U\vec{e}_x + \delta\tilde{u}(\vec{r}),$$  \hspace{1cm} (3.27)

$$\tilde{\nabla}P(\vec{r}) = G\vec{e}_x + \tilde{\nabla}\delta P(\vec{r}),$$  \hspace{1cm} (3.28)

$$c(\vec{r}) = C_0 + \delta c(\vec{r}),$$  \hspace{1cm} (3.29)

with $G = \langle \nabla P \rangle$ and $U = \langle u_x \rangle$. The mean flow is assumed to be along the $\vec{e}_x$ axis.

Because $c(\vec{r})$ is distributed according to a log-normal, it is more convenient, and without loss of generality, to introduce:

$$c(\vec{r}) = \exp (g_0 + \delta g) = C_0 \exp(\delta g).$$
Expanding the different terms in eq. (3.25) to the second order leads to,

$$c = C_0 (1 + \delta g + \frac{1}{2} \delta g^2)$$  \hspace{1cm} (3.30)

$$|\vec{u}|^{n-1} = |U \vec{e}_x + \delta \vec{u}|^{n-1}$$

$$= U^{n-1} + (n-1)U^{n-2}\delta u_x + \frac{n-1}{2}U^{n-3}(\delta \vec{u} \cdot \delta \vec{u}) + \frac{(n-1)(n-3)}{2}U^{n-3}\delta u_x^2$$  \hspace{1cm} (3.31)

Taking the spatial average \langle . \rangle of eq. (3.25) all the first order terms vanish by definition, yielding to:

$$-G \vec{e}_x = C_0 U^n \vec{e}_x + \frac{1}{2} C_0 \langle \delta g^2 \rangle U^n \vec{e}_x + C_0 \frac{n-1}{2} U^{n-2} \langle \delta \vec{u} \cdot \delta \vec{u} \rangle \vec{e}_x + \frac{(n-1)(n-3)}{2} U^{n-3} \langle \delta u^2 \rangle$$

$$+ C_0 U^{n-1} \langle \delta g \delta \vec{u} \rangle + C_0 (n-1) U^{n-1} \langle \delta g \delta u_x \rangle \vec{e}_x + C_0 (n-1) U^{n-2} \langle \delta u_x \delta \vec{u} \rangle.$$  \hspace{1cm} (3.32)

Along the \(e_x\) axis, it follows:

$$- \frac{G}{C_0 U^n} = 1 + \frac{1}{2} \langle \delta g^2 \rangle + \frac{n}{U} \langle \delta g \delta u_x \rangle + \frac{n(n-1)}{2U^2} \langle \delta u_x \delta u_x \rangle + \frac{n-1}{2U^2} \langle \delta u_x^2 \rangle.$$  \hspace{1cm} (3.33)

This expression relates the mean gradient to the mean velocity of the flow if we know the different cross-correlation terms of the spatially fluctuating fields, which are determined next.

The first order of eq. (3.25) gives:

$$- \nabla \delta p = C_0 U^n \delta g \vec{e}_x + C_0 (n-1) U^{n-1} \delta u_x \vec{e}_x + C_0 U^{n-1} \delta \vec{u}.$$  \hspace{1cm} (3.34)

Thus,

$$- \frac{1}{C_0 U^{n-1}} \nabla \delta p = U \delta g \vec{e}_x + (n-1) \delta u_x \vec{e}_x + \delta \vec{u}.$$  \hspace{1cm} (3.35)

Taking the curl leads to:

$$0 = \partial_x (\delta u_y) - \partial_y (U \delta g + n \delta u_x).$$  \hspace{1cm} (3.36)

It is now more convenient to write this equation in the Fourier space. Defining \(\hat{u}(k_x, k_y)\) and \(\hat{g}(k_x, k_y)\), respectively the Fourier transform of \(\delta \vec{u}\) and \(\delta g\), gives:

$$i k_x \hat{u}_y - i k_y U \hat{g} + n \hat{u}_x = 0.$$  \hspace{1cm} (3.37)

Using the free divergence in Fourier space:

$$k_x \hat{u}_x + k_y \hat{u}_y = 0,$$  \hspace{1cm} (3.38)

it follows the relationship between the fluctuation of \(\delta \vec{u}\) and \(\delta g\), in Fourier space:

$$\hat{u}_x = - \frac{k_y^2}{n k_y^2 + k_x^2} \hat{g}$$  \hspace{1cm} (3.39)

$$\hat{u}_y = \frac{k_y k_x}{n k_y^2 + k_x^2} \hat{g}.$$  \hspace{1cm}

From these expressions, it is possible to determine the different cross correlation terms using Parseval’s formula:

$$\iint F H \, dxdy = \iint \hat{F} \hat{H}^* \, dk_x dk_y,$$  \hspace{1cm}

for any field \(F(\vec{r})\) and \(H(\vec{r})\).
Thus,

\[ I_1 = \frac{1}{U^2} \langle \delta u_x \delta u_x \rangle = \iint \hat{u}_x \hat{u}_x \, dk_x dk_y = \iint \frac{k_y^4}{(nk_x^2 + k_y^2)^2} \hat{g}^* \hat{g}^* \, dk_x dk_y \tag{3.40} \]

\[ I_2 = \frac{1}{U} \langle \delta u_x \delta g \rangle = \iint \hat{u}_x \hat{g}^* \, dk_x dk_y = -\iint \frac{k_y^2}{nk_x^2 + k_y^2} \hat{g}^* \, dk_x dk_y \tag{3.41} \]

\[ I_3 = \frac{1}{U^2} \langle \delta u_y \delta u_y \rangle = \iint \hat{u}_y \hat{u}_y \, dk_x dk_y = \iint \frac{k_x^2}{(nk_x^2 + k_y^2)^2} \hat{g}^* \hat{g}^* \, dk_x dk_y. \tag{3.42} \]

These equations are very general and should apply to any distribution and correlation of permeability field, providing that the amplitude of the heterogeneities is small enough.

Using now the particular distribution field \( g(\vec{r}) \) with \( c(\vec{r}) = \frac{\mu}{\kappa(\vec{r})u_c n} \) and \( u_c = A\kappa^{\gamma} \) leads to:

\[ g_0 + \delta g = \log \mu + (1 - n) \log A - (1 + \gamma(n - 1))(f_0 + f). \tag{3.43} \]

\( \delta g \) is thus also a log-normal distribution with \( \sigma_g = \sqrt{(\delta g^2)} = (1 + \gamma(n - 1))\sigma_f \) and

\[ C_0 = \frac{\mu}{\kappa_0 u_0^{-1}} = \mu k_0^{-1-n} A^{1-n}. \tag{3.44} \]

It follows that \( g \) is also correlated with a Gaussian function with:

\[ \hat{g}^* \hat{g}^* = \beta^2 \exp^{-2\frac{k_x^2 + k_y^2}{k_0^2}}, \tag{3.45} \]

where \( \beta \) is a normalisation prefactor determined by \( \sigma_k = \sqrt{\langle \delta g^2 \rangle} \).

The different correlation functions can be derived after some manipulations,

\[ I_1(n) = \frac{\sigma_g^2}{2\pi} \int_0^{2\pi} \frac{\sin^4 \theta}{(n \cos^2 \theta + \sin^2 \theta)^2} \, d\theta = \frac{1}{2} \sigma_g^2 \frac{1}{\sqrt{n}} \left( \frac{n}{1 + \sqrt{n}} \right)^{3/2} \tag{3.46} \]

\[ I_2(n) = -\frac{\sigma_g^2}{2\pi} \int_0^{2\pi} \frac{-\sin^2 \theta}{n \cos^2 \theta + \sin^2 \theta} \, d\theta = -\sigma_g^2 \frac{1}{\sqrt{n + n}} \tag{3.47} \]

\[ I_3(n) = \frac{\sigma_g^2}{2\pi} \int_0^{2\pi} \frac{\sin^2 \theta}{(n \cos^2 \theta + \sin^2 \theta)^2} \, d\theta = \sigma_g^2 \Theta(n). \tag{3.48} \]

Unfortunately, no analytical form of the last integral could be found. It must therefore be computed numerically. Note that this integral has the property \( n^2 I_3(n) = I_3(1/n) \). Table 1 gives the value of \( \Theta(n) \) for the indexes \( n \) used in this manuscript. The most important correlation term is the standard deviation of the velocity field, Eq. (3.46), which will be used later:

\[ \frac{\sigma_u^2}{U^2} = \sigma_f^2 [1 + \gamma(n - 1)]^2 \frac{1 + 2\sqrt{n}}{2(1 + \sqrt{n})^{3/2}}. \tag{3.49} \]

These expressions can be used in eq. (3.33) to determine the mean pressure gradient \( G \) by imposing the mean flow rate \( U \):

\[ -\frac{G}{C_0 U^n} = 1 + \sigma_f^2 [1 + \gamma(n - 1)]^2 \left[ \frac{1}{2} - \frac{n}{n + \sqrt{n}} + \frac{n(n - 1)}{4} \frac{1 + 2\sqrt{n}}{(1 + \sqrt{n})^{3/2}} + \frac{n - 1}{2} \Theta(n) \right]. \tag{3.50} \]

A similar procedure could be made to express the mean flow rate \( U \) as function of the
imposed gradient of pressure $G$. The full calculation is left to the reader. The basic idea is to write the constitutive equation in the form $\vec{u} = -D|\nabla P|^{\alpha-1}\nabla P$, with $\alpha = 1/n$ and $D = C^{-\alpha}$. This results to:

$$-\frac{U}{D_0 G^\alpha} = 1 + \sigma_f^2 \left[ \alpha + \gamma(1-\alpha) \right]^2 \left[ \frac{1}{2} - \frac{\alpha}{\alpha + \sqrt{\alpha}} \right] + \frac{\alpha(\alpha - 1)}{4} \left( 1 + \sqrt{\alpha} \right)^2 \alpha^{3/2} + \frac{\alpha - 1}{2} \Theta(\alpha), \tag{3.51}$$

with $D_0 = C_0^{-\alpha} = \mu^{-\alpha} k_0^{\alpha+\gamma(1-\alpha)} A^{1-\alpha}$.

Surprisingly, this expression is very similar to eq. (3.50). This similarity originates from a symmetry property of 2D flow fields where the role of pressure and velocity can be switched. More detail will be provided in the next section. Eqs. (3.49) and (3.51) indicate that the mean flow rate is independent of the disorder amplitude for Newtonian fluid ($n = \alpha = 1$) as demonstrated by Gelhar & Axness (1983). For power fluids, the dependence is significantly more complicated.

Fig. 5 (left) represents the evolution of the average flow rate as a function of $n$ for different $\gamma$ as predicted by eq. (3.51) and using $D = G = \sigma_f = 1$. For any $\gamma$, the average flow rate is decreasing with $n$. For shear-thinning fluids, $n < 1$, the curve is greater than one, meaning that the average flow increases with the heterogeneity $\sigma_f$. In addition, this effect is more pronounced for lower values of $\gamma$. For shear thickening fluids, $n > 1$, the curve is below one, so the average flow rate decreases with $\sigma_f$. This effect is magnified by the higher value of $\gamma$.

Fig. 5 shows the width of the velocity distribution. For most of the $\gamma$ values, the heterogeneity of the velocity field decreases with $n$. Shear-thinning fluids have thus a much more heterogeneous velocity field than the shear thickening one. And this effect is more important for the lowest value of $\gamma$. Surprisingly, this trend seems to inverse above a certain value $\gamma \gtrsim 0.9$. A change in monotony is then observed, with shear thickening fluids being more heterogeneous than shear thinning ones.

### 3.5. Rotational symmetry

This argument originates from Matheron (1967) in a two-dimensional flow field for Newtonian Darcy’s law. It can however be generalized to non-Newtonian fluids. This symmetry is also applicable to the 2D pore network model (Straley 1977; Talon & Hansen 2020). As previously, a generic non-linear Darcy equation is assumed:

$$\nabla \cdot \vec{u} = 0, \tag{3.52}$$

and

$$\vec{u} = -f \left( |\nabla P| \right) \nabla P \tag{3.53}$$

$\dagger$ Another main difference is that in the first order expansion the divergent has to be taken instead of the curl, to eliminate the velocity and then relating the fluctuation in pressure as function of the fluctuation in $C$.  

| $n$  | 1/3 | 1/2 | 1 | 2 | 3  |
|------|-----|-----|---|---|----|
| $\Theta(n)$ | 0.348076 | 0.242641 | 0.125 | 0.0606602 | 0.0386751 |

Table 1: Numerical values of the function $\Theta(n)$ defined in eq. 3.48
or
\[
\vec{\nabla} P = -\frac{g(\|\vec{u}\|)}{\|\vec{u}\|} \vec{u}.
\]

The idea is to rotate the two fields \(\vec{u}\) and \(\vec{\nabla} P\) by 90°. In a coordinate system \((x, y, z)\), where the flow takes place in the plane \((x, y)\), this rotation is performed by making the cross product with the vector \(\vec{e}_z\):
\[
\vec{e}_z \times \vec{\nabla} P = -\frac{g(\|\vec{u}\|)}{\|\vec{u}\|} (\vec{e}_z \times \vec{u}).
\]

Defining the rotated fields \(\vec{q} = \vec{e}_z \times \vec{\nabla} P\) and \(\vec{Z} = \vec{e}_z \times \vec{u}\), it can be shown that \(\vec{V} \cdot \vec{q} = 0\) and \(\vec{V} \times \vec{Z} = \vec{0}\). This means that \(\vec{q}\) is a flux vector and \(\vec{Z}\) derives from a potential field \(\vec{Z} = \vec{V} \Psi\).

The two new fields satisfy:
\[
\vec{q} = -g(\|\vec{u}\|) \frac{\vec{V} \Psi}{\|\vec{V} \Psi\|}.
\]

Since \(g(\|\vec{u}\|) = \|\vec{\nabla} P\|\), it follows \(\|\vec{V} \Psi\| = g^{-1}(\|\vec{q}\|)\). Yielding to:
\[
\vec{V} \Psi = -g^{-1}(\|\vec{q}\|) \frac{\vec{q}}{\|\vec{q}\|}.
\]

As a result, the rotated fields \(\vec{q}\) and \(\vec{V} \Psi\) satisfy a non-Newtonian Darcy’s equation but with a rheology inverse to the original one. In particular, solving a shear-thinning fluid in one direction is then equivalent to solving a shear thickening in the other direction.

Considering the truncated model:
\[
\begin{cases}
\vec{V} P = -\frac{\mu}{k} \vec{u} & \text{ if } \|\vec{u}\| < u_c \\
\vec{V} P = -\frac{\mu}{k} \left(\frac{\|\vec{u}\|}{u_c}\right)^{n-1} \vec{u} & \text{ if } \|\vec{u}\| > u_c
\end{cases}
\]

is thus equivalent to:
\[
\begin{cases}
\vec{V} \Psi = -\frac{\kappa}{\rho} \vec{q} & \text{ if } \|\vec{q}\| < q_c \\
\vec{V} \Psi = -\frac{\kappa}{\rho} \left(\frac{\|\vec{q}\|}{q_c}\right)^{1/n-1} \vec{q} & \text{ if } \|\vec{q}\| > q_c
\end{cases}
\]
with $q_c = \mu u_c/\kappa$. Using the relation $u_c = A \kappa^\gamma$, the parameters have thus changed to:

\[
\begin{align*}
\bar{n} &\rightarrow 1/n \\
\bar{\mu} &\rightarrow 1/\mu \\
\bar{k} &\rightarrow 1/\kappa \\
\tilde{\eta} &\rightarrow -\gamma + 1 \\
\tilde{\lambda} &\rightarrow \mu A.
\end{align*}
\]

It is remarkable that for the most natural value $\gamma = 1/2$ in porous media, this coefficient is invariant with this transformation. It should also be noted that the inverse of a lognormal distribution remains lognormal with $\tilde{f}_0 \rightarrow -\tilde{f}_0$ and the same $\sigma_f$. It follow that the study can be limited to shear thinning fluids ($n < 1$) without loss of generality. Figure 6 illustrates this symmetry by comparing the average flow rate as a function of the average pressure gradient for the two exponents $n = 1/2$ and $n = 2$. A very good correspondance between the two curves can be observed.

4. Numerical results

4.1. Flow rate

The following section is dedicated to the numerical results obtain for the truncated theology. We will first analyse the two asymptotic limits and the study the intermediate flow regime.

4.1.1. Influence of the disorder

Fig. 7.a displays the mean flow-rate as a function of the mean pressure gradient $\nabla P$ for a sheared-thinning fluid, $n = 1/2$, and for the different amplitude of heterogeneities $\sigma_f$. As expected, and similarly to the homogeneous permeability medium, a transient behavior is observed around $u \sim u_0$ and $\nabla P \sim g_0$. At low pressure drop, the flow rate follows a linear relation $\langle u \rangle \propto \nabla P$, while at high pressure difference, it follows a power law $\langle u \rangle \propto \nabla P^{1/n}$. 

Figure 6: Blue curve: mean flow rate $\langle u \rangle$ as function of the mean gradient of pressure $\langle \nabla P \rangle$ for a truncated theology with $n = 1/2$, $\gamma = 1/2$ and $\sigma_f = 1$. Red circles: mean gradient of pressure as function of the mean velocity with $n = 2$, $\gamma = 1/2$ and $\sigma_f = 1$. 

with $q_c = \mu u_c/\kappa$. Using the relation $u_c = A \kappa^\gamma$, the parameters have thus changed to:

\[
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with $q_c = \mu u_c/\kappa$. Using the relation $u_c = A \kappa^\gamma$, the parameters have thus changed to:

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\bar{k} &\rightarrow 1/\kappa \\
\tilde{\eta} &\rightarrow -\gamma + 1 \\
\tilde{\lambda} &\rightarrow \mu A.
\end{align*}
\]
The influence of disorder is manifold. First, it smoothes out the transition and increases its range. For $\sigma_f = 0$, the transition is sharp, while for high $\sigma_f$, the transition is smoother, extending over half a decade. This effect is more pronounced after normalizing the average flow-rate by the Newtonian Darcy’s law $\frac{\mu(u)}{k_0 \nabla P}$ (Fig. 7.b). The second effect of heterogeneities is to increase the flow-rate in both asymptotic regimes. While it seems very marginal in the low pressure regime, the effect is quite significant in the high pressure regime, where the flow rate is almost doubled between $\sigma_f = 0$ and $\sigma_f = 2.5$.

Figures 7.c and 7.d represent the relative standard deviation of the flow field $\frac{\sigma_{\text{max}}}{\langle u \rangle}$ as a function of the applied pressure difference and for different $\sigma_f$. It displays a change of plateau when changing the flow regime. The presence of the two plateaus can be easily understood from the results of the section 3.1, where it has been shown that the $\frac{\langle u \rangle}{(\langle u \rangle)}$ field tends towards a constant in low and high flow regimes. Here again, the extension of the transition between these two asymptotes depends on the heterogeneities of the porous medium, sharp at low $\sigma_f$ while smoother at high $\sigma_f$. For shear thinning fluids, flow heterogeneity is increased in the high flow rate regime, while it is decreased for shear thickening fluids. In addition, flow heterogeneity increases with the disorder of the medium $\sigma_f$. For shear thinning fluids, this increase is more important in the power-law regime than in the linear one. Compared to the Newtonian regime the velocity field can either be more heterogenous for shear-thinning or less heterogenous for shear-thickening.
Figure 8: For a shear thinning fluid $n = 1/2$ and $\gamma = 1/2$. Left: mean velocity versus the amplitude of the heterogeneities $\sigma_f$ at low applied pressure drop $\nabla P/g_0 = 1e - 2$ (blue) and high pressure drop $\nabla P/g_0 = 100$ (red). Right: relative standard deviation as function of $\sigma_f$. Circles represent the simulations and the dashed line is the perturbation expansion prediction eqs. (3.49) and (3.51).

As described above, the flow field in both asymptotic regimes converges to the solution of a simple power law fluid whose characteristics were predicted at low $\sigma_f$ in the section 3.4. Fig. 8 compares the numerical mean and standard deviation of the flow field at sufficiently low (resp. high) applied pressure gradient $\nabla P/g_0 = 1e - 2$ (resp. $\nabla P/g_0 = 100$) against the prediction of eqs. (3.49) and (3.51). The two figures show a very good agreement between the analytical prediction and the simulations, even for an amplitude of heterogeneity as high as $\sigma_f \sim 2$, which is quite significant for a log-normal distribution. An expansion of order 2 seems therefore sufficient to predict both the mean flow and the standard deviation of the flow field as the variation seems to remain parabolic over a wide range of $\sigma_f$.

4.1.2. Range of the transitional flow regime

As described above, this range of the transitional flow regime depends significantly on $\sigma_f$. To understand this transition, it is useful to examine the evolution of the velocity distribution with the mean pressure gradient shown in Fig. 9.a. For a small pressure gradient, as long as the distribution is away from the mean critical velocity $u_0$, its shape remains constant while being shifted as $\nabla P$ increases.

As the distribution approaches the mean critical velocity $u_0$, it begins to deform. This reflects the fact that some regions begin to reach their critical velocity and thus changing their viscosity. At sufficiently high pressure, once this distribution is far enough from $u_0$, the distribution adopts another constant shape which is only shifted when increasing $\nabla P$. In Fig.9.b the two asymptotic forms $1\vec{w}$ and $1/2\vec{w}(\vec{r})$ are plotted. Notably, the shear thinning fluid $n = 1/2$ has a wider velocity distribution but is also slightly more asymmetrical towards the low velocity.

The invariance of the $n\vec{w}$ distribution can be exploited to estimate the range of the transition regime. Starting at a very low flow rate, all regions are in Newtonian regime and $|| \vec{u} || = \langle u \rangle |\vec{w}|$. Increasing the average flow rate, the first location to change its rheology occurs where $|| \vec{u} || = u_c$. Thus:

$$\max \frac{\langle u \rangle |1\vec{w}(\vec{r})|}{u_c(\vec{r})} = 1,$$

(4.1)
which means that the change of regime will start at the average velocity \( \langle u \rangle = u_1 \) satisfying:

\[
\frac{1}{u_1} = \max \left\{ \frac{1}{u_c} \text{vec}(\tilde{\omega}(\vec{r})) \right\}.
\]

(4.2)

Similarly, starting at high pressure drop, all the regions are in the power-law regime and \( \tilde{u}(\vec{r})/\langle u \rangle = n \tilde{\omega}(\vec{r}) \) is constant. While decreasing \( \langle u \rangle \), the first point to change its behavior occurs at:

\[
\frac{1}{u_2} = \min \left\{ \frac{n}{u_c} \text{vec}(\tilde{\omega}(\vec{r})) \right\}.
\]

(4.3)

Assuming that at order zero \( \nabla P_1 \approx \frac{\mu}{\kappa_0} u_1 \) and \( \nabla P_2 \approx \mu \kappa_0^{-1-\gamma(n-1)} A^{1-n} u_2^n \), it leads to an estimation for the bounds of the transient regime:

Lower bound:

\[
\nabla P_1 = \frac{\mu}{\kappa_0} \max \left\{ \frac{1}{u_c} \text{vec}(\tilde{\omega}(\vec{r})) \right\}.
\]

(4.4)

Upper bound:

\[
\nabla P_2 = \frac{\mu \kappa_0^{-1-\gamma(n-1)} A^{1-n}}{\left( \min \frac{n}{u_c} \text{vec}(\tilde{\omega}(\vec{r})) \right)^n}.
\]

(4.5)

The range of the transition hinges thus on the amplitude of the heterogeneities of the velocity field. If \( \tilde{\omega} \) is broad, it is expected that the transition will start at lower pressure gradient because the maximum is larger. And if \( \tilde{\omega} \) is broad, the transition should end at higher pressure gradient. It is important to emphasize, however, that the relevant quantity is in fact the width of \( \text{vec}(\tilde{\omega}(\vec{r})) \) which does not necessarily follow the width of \( \tilde{\omega} \), as will be discussed below.

It can also be noted that, since the extension of the transitional regime depends on the extrema of \( \tilde{\omega}/u_c \) and \( \tilde{\omega}/u_c \), this range is expected to depend on the width of these two distributions but also on the size of the system. Indeed, for a field of size \( L_x L_y \) with a correlation length \( \lambda \), it can approximately be considered as \( N = L_x L_y / \lambda^2 \) independent random numbers. On average, theory of extreme statistics suggests that:

\[
\langle \max \frac{n}{u_c} \text{vec}(\tilde{\omega}(\vec{r})) \rangle = \mathbb{P}_n^{-1}(1 - 1/N)
\]

(4.6)

and

\[
\langle \min \frac{n}{u_c} \text{vec}(\tilde{\omega}(\vec{r})) \rangle = \mathbb{P}_n^{-1}(1/N),
\]

(4.7)

where \( \mathbb{P}_n \) is the cumulative distribution function of \( n \tilde{\omega}/u_c \). The range of the transitional regime \( [\nabla P_1; \nabla P_2] \) is thus expected to increase with the size of the system, which seems quite intuitive. The larger the size of the system, the higher the maximum will be and the lower the minimum. The exact dependence of these extrema with the system size depends on the specific form of the distribution, particularly on the tails.

Fig. 10 represents different phase diagrams of the flow regime as a function of the mean pressure gradient and varying \( \sigma_f, \gamma \) or \( n \). A cross represents system which are in the intermediate regime. In addition, the bounds predicted by eqs. (4.4) and (4.5) have been plotted and they show good agreement.

Fig. 10.a represents the evolution of the transient regime as a function of the amplitude of the disorder \( \sigma_f \). As expected, the amplitude of the transition increases with the disorder. The upper bound \( \nabla P_2 \) increases however faster than the lower one \( \nabla P_1 \), this reflects the fact that the width of the distribution of \( \tilde{\omega}/u_c \) is larger than that of \( \tilde{\omega} \).
This inversion is in fact due to the presence of the exponent $\gamma$. Perturbation expansion results). This shows that the standard deviation of non-monotonic behavior. For low $\mu$, expected trend.

The correlation of the velocity field is thus a crucial characteristic to understand the evolution of the transition regime. This point will be discussed in more details in the next section.

**Fig. 9:** For a shear-thinning fluid, $n = 1/2$ and $\gamma = 1/2$. Left: Evolution of the probability distribution function (PDF) of the velocity as function of the applied mean pressure gradient. The vertical dashed line represents the mean threshold velocity $u_0$.

Right: normalized velocity distribution $\tilde{u}$ and $\tilde{u}$, obtained respectively at $\nabla P/g_0 = 1e-2$ and $\nabla P/g_0 = 1e2$.

**Fig. 10:** Flow regime diagram of the system as function of the amplitude of the heterogeneities $\sigma_f$ and the mean pressure gradient. Crosses represent systems in the intermediate flow regime defined as $O \in [10^{-5}, 1 - 10^{-5}]$ for different parameters: left $(n, \gamma) = (1/2, 0.5)$, center: $(n, \sigma) = (1/2, 1)$, and right $(\sigma, \gamma) = (1, 0.5)$. The red and blue dashed line represent respectively the bounds predicted by eqs. (4.4) and (4.5).

- Fig. 10.b shows the dependence of the extension with the rheological index $n$. Albeit the width of the $\tilde{u}$ distribution is decreasing with $n$ (see figure 11), the range increases with $n$. This inversion is in fact due to the presence of the exponent $n$ in eq. (4.5) which inverts the expected trend.

- The influence of $\gamma$ (Fig. 10.c) on the transition range is more surprising as it exhibits a non-monotonic behavior. For low $\gamma$ values, the range is decreasing with $\gamma$ but it becomes increasing beyond a certain $\gamma$ value. Here also, the transition doesn’t follow necessarily the amplitude of the velocity distribution which is decreasing with $\gamma$ (see Fig. 11 or the perturbation expansion results). This shows that the standard deviation of $\tilde{u}(\tilde{r}) = \frac{\tilde{u}(\tilde{r})}{Ak^\gamma(\tilde{r})}$ does not necessarily follow the one of $\tilde{u}(\tilde{r})$. This is due to the fact that the velocity and permeability field are not entirely correlated: higher velocity regions do not necessarily correspond to high permeability ($u_c$) ones.

- This can be understood by considering the correlations between the two fields. While the permeability field is anisotropic, the velocity field is much more correlated in streamwise direction due to the conservation of mass. A region of low permeability for instance can thus have a high velocity if it is located in front or behind a region of high permeability. The correlation of the velocity field is thus a crucial characteristic to understand the evolution of the transition regime. This point will be discussed in more details in the next section.
4.2. Statistical properties of the flow field in the transitional regime

In this section, the properties of the flow field within the transitional regime are analysed, which display some interesting statistical properties. One way to apprehend it is to notice a similarity with the percolation problem. Indeed, as shown in figure 12, when \( \langle u \rangle \) is increased, more and more regions meet the criteria \( |\vec{u}| > u_c \). This also allows to define cluster of connected regions. If the field \( |\vec{u}|/\langle u \rangle = \tilde{\omega}(\vec{r}) \) were constant, the transition would occur at

\[
\frac{\tilde{\omega}(\vec{r})}{u_c(\vec{r})} = \frac{1}{\langle u \rangle},
\]

which would correspond to a percolation problem. It is thus expected to observe similar behaviors such as fractal properties of these clusters and the presence of criticality. It is however important to remember that the problem is not strictly equivalent to percolation because the \( \frac{\vec{u}}{\langle u \rangle} \) field is not constant in the transient regime. In particular, the change in viscosity introduces correlations in the velocity field and thus modifies the shape of the clusters as described previously.

The shape of the clusters are characterized by their total size \( S \) and the two dimensions of the bounding rectangle containing it: \( L \) along the flow direction and \( W \) transversely to it.

4.2.1. Size distribution

Fig 13a displays the size distribution \( P(S) \) for different applied pressure drops and for the parameters \( n = 1/2 \) and \( \sigma = 1 \). For any applied pressure, the distribution follows a decaying power-law over a large range of sizes. There is, however, a large-scale cut-off, \( S_0 \), which varies with the pressure gradient. Reporting the variation of \( S_0 \) as a function of the pressure
Figure 13: Left: Probability distribution function of the cluster size for different applied pressure drops using the parameters \((n, \gamma, \sigma_\nu) = (1/2, 1/2, 1)\). Dashed black line represents the fitted power-law exponent. Right: Size of the largest cluster \(S_0\) versus the pressure gradient. The dotted vertical line represents \(\nabla P_\nu\). Inset: \(S_0\) as a function of \(\nabla P - \nabla P_\nu\) on a logarithmic scale, the line represents the fitted power law. Bottom: Size probability distribution function for different pressure gradients normalized according to eqs. (4.8) and (4.9).

(Fig. 13.b) shows that \(S_0\) diverges at a certain value of the mean pressure gradient \(\nabla P = \nabla P_\nu\) according to a power law:

\[
S_0 \propto |\nabla P - \nabla P_\nu|^{-\nu_S}.
\]

Combining the two observations, it follows the scaling law for the size distribution:

\[
p(S) \propto S^{-\tau_S} f\left(\frac{S}{S_0}\right),
\]

which is confirmed by the good collapse of the distributions rescaled according to eqs. (4.8) and (4.9) plotted in Fig. 13.c.

This scaling law is thus similar to the one found in other problems with a critical transition such as percolation (Stauffer & Aharony 1991) or avalanches (e.g. Amaral et al. (1995); Santucci et al. (2011); Chevalier et al. (2017)). The sizes are distributed on many scales up to a size limit. And this size limit diverges as the control parameter \(\nabla P\), approaches the critical value \(\nabla P = \nabla P_\nu\). The system is then scale invariant (e.g. fractal). These scaling laws are characterized by the two exponents \(\tau_S\) and \(\nu_S\). It is important to note that a similar scaling law can be observed for the length \(L\) of the clusters:

\[
p(S) \propto S^{-\tau_L} f\left(\frac{L}{L_0}\right) \text{ with } L_0 \propto |\nabla P - \nabla P_\nu|^{-\nu_L},
\]

(4.10)
allowing to define the exponents $\tau_L$ and $\nu_L$.

The measured exponents $\tau_S$, $\tau_L$, $\nu_S$ and $\nu_L$ for the different sets of parameters are reported in tables 2-4. The most remarkable result is the fact that $\tau_S$ seems to be indeed independent of the parameters.

It is important to recall that in many critical systems, some exponents are independent of the details of the disorder distribution. This behavior is often referred as “universal”. This is the case for instance for the exponents $\tau$ and $\nu$ in percolation or avalanches of an elastic line in a random media (Barabasi & Stanley 1995). The results here seem then to suggest an universal behavior for the exponent $\tau_S$. However, the value $\tau_S = 1.65 \pm 0.05$ is quite different from the standard percolation problem ($\tau_{perc} = 2.05$), which indicates that it would be of a different universality class.

The trend is less clear for the exponent $\nu_S$, which seems to vary with the rheological index $n$ and also the heterogeneities $\sigma_f$. However, it is worth noting that the determination of the exponent $\nu_S$ is generally more prone to errors as it requires the determination of $\nabla P_c$, which is also subject to uncertainties. The error can be estimated to be about 10%. It is then difficult to conclude on the universality of this exponent.

4.2.2. Cluster’s shape

The shape of the clusters can also be characterized by the aspect ratio $L/W$. Fig. 14.a represents the width of each cluster according to their length $L$ for two different parameter sets. A power-law relationship is observed:

$$\langle W \rangle_L \propto L^\zeta,$$  \hspace{1cm} (4.11)

which is a characteristic of the self-affine structure generally found in anisotropic critical systems (e.g. avalanches, directed percolation, front propagations, etc.). It characterizes the fact that, if many different cluster sizes are present, the aspect ratio is not the same at each scale. If $\zeta < 1$, bigger clusters are more elongated than the smaller ones as represented in Fig. 14.a (with $\gamma = 0.25$). For $\zeta > 1$, as for $\gamma = 0.75$ in this figure, bigger clusters are elongated in the direction transverse to the flow.

Because clusters are not compact, viz. they may contain holes, another interesting quantity to analyze is the area (mass) as a function of the enclosing box size $WL$, as shown in figure 14.b. Here, again the relationship observed is a power-law:

$$S \propto (LW)^\beta,$$

which is a characteristic of fractal structure.

The measured exponents for different parameter sets, $\sigma$, $n$ and $\gamma$, are displayed in table 2-4. The size exponent $\beta$ appears to be almost constant, within the error bar, for any parameter value: $\beta \approx 0.77 \pm 0.05$. This exponent seems to be universal. More surprising is the evolution of the aspect ratio exponent $\zeta$ which takes only two values: either $\zeta \approx 0.85 \pm 0.05$ or $\zeta \approx 1.15 \pm 0.05$ depending on the parameters. The two cases display in Figure 15 are in fact the only two observable exponents. In this figure, the two cases appear very similar if they were rotated by $90^\circ$. This is reminiscent to the rotational symmetry described above. Indeed, concerning the influence of the rheological index $n$, this change of correlation direction can be apprehended from the rotational symmetry argument described in section 3.5.

Indeed, inversing the rheological index is equivalent to a rotation by $90^\circ$. One can thus expect that, by inversing $n \rightarrow 1/n$, the shape of the clusters are remaining identical providing that we inverse the role of $W$ and $L$. This argument leads to the relation:

$$\zeta(n) = \frac{1}{\zeta(1/n)}.$$
Table 2: Measured exponents for different amplitude of heterogeneity $\sigma_f$ with $(n, \gamma) = (0.5, 0.5)$.

| $\sigma_f$ | 0.5 | 1.0 | 2.0 |
|------------|-----|-----|-----|
| $\tau_S$   | 1.65| 1.65| 1.67|
| $\nu_S$    | 3.0 | 3.25| 3.2 |
| $\nu_L$    | 1.8 | 1.8 | 1.75|
| $\zeta$    | 0.85| 0.85| 0.85|
| $\beta$    | 0.77| 0.77| 0.77|

which seems to be satisfied by the two observed exponents, so the symmetry is respected. The value $\zeta(n = 1) \approx 1$ appears thus to be a marginal value. It remains quite remarkable that this exponent is constant for any shear-thinning (or shear-thickening) $n$ exponent.

More unexpected is the similar change when $\gamma$ is varied (table 4). $\zeta$ is constant up to a certain value $\gamma \sim 0.6$, where it switches to its inverse value. Some conjecture can be made to interpret this switching.

First of all, this change of correlation confirms the fact that the spatial correlation of the velocity field, correlated in the direction of the current, doesn’t necessarily follow the correlation of the $\bar{u}(\vec{r})/u_c(\vec{r})$ field. The value of $\gamma$ at which the change occurs, seems to correspond to the change in trend observed in Fig. 11 for the evolution of the standard deviation $\sigma_{\omega}/u_c$.

To understand this change, it is useful to consider the two extreme cases $\gamma = 0$ and $\gamma = 1$. It should be noted that the criterion $|\bar{u}| > u_c(\vec{r})$ also corresponds to a criterion for local pressure gradient $|\bar{\nabla} p(\vec{r})| > g_c(\vec{r})$, with $u_c = A\kappa^\gamma$ and $g_c = \mu A\kappa^\gamma$. If $\gamma = 0$, $u_c$ is a constant: it is then expected that the cluster defined by $|\bar{u}| > u_c$, follows the correlation of the velocity field. The cluster should then be more elongated in the streamwise direction.

But for $\gamma = 1$, the situation is different because $u_c(\vec{r})$ is now a random variable but $g_c$ is a constant. The clusters, also defined by $|\bar{\nabla} p(\vec{r})| > g_c$, should have a correlation direction similar to that of $\bar{\nabla} p$. If the velocity field is naturally correlated in the flow direction, the pressure gradient tends to be correlated in the transverse direction. If the clusters, $|\bar{\nabla} p(\vec{r})| > g_c$, have a correlation direction similar to that of $\bar{\nabla} p$, transverse to the flow. For intermediate $\gamma$, there is thus a balance between these two effects. The argument remains qualitative because, as discussed earlier, shear-thinning fluids should promote slightly the correlation in the streamwise direction.

If the cluster elongation changes with $\gamma$, it is still quite surprising that the correlation exponent $\zeta$ switches from one value to another. This seems to suggest that this exponent is universal but a preferential direction of correlation is determined by the value of $\gamma$.

5. Conclusion

In this paper, we have investigated the flow in macroscopic heterogeneous porous media with a non-linear rheology displaying a change of behavior such as the truncated rheology. First, we have discussed the method to derive an extension of the Darcy’s law for such rheology. The classical approach, which consists in introducing an effective viscosity or effective stress as function of an effective shear rate, has been discussed. In particular, it has been shown that, if for power-law fluid the approach can be correct, it becomes less accurate for more

† This can be seen from the rotational symmetry or using the expansion method for power-law fluids.
Table 3: Measured exponents for different amplitude of rheological index $n$ with $(\sigma_f, \gamma) = (1, 0.5)$.

| $n$ | 1/3 | 0.5 | 2/3 | 1 | 2 | 3 |
|-----|-----|-----|-----|---|---|---|
| $\tau_S$ | 1.65 | 1.65 | 1.65 | 1.65 | 1.65 | 1.65 |
| $\nu_S$ | 3.3 | 3.25 | 3.5 | 3.6 | 3.7 | 3.9 |
| $\nu_L$ | 1.95 | 1.8 | 1.8 | 1.9 | 1.9 | 1.9 |
| $\zeta$ | 0.83 | 0.85 | 0.85 | 0.95 | 1.15 | 1.17 |
| $\beta$ | 0.77 | 0.77 | 0.8 | 0.8 | 0.78 | 0.78 |

Table 4: Measured exponents for different $\gamma$ with $(n, \sigma_f) = (0.5, 1)$.

| $\gamma$ | 0 | 1/4 | 1/2 | 2/3 | 3/4 | 1 |
|----------|---|-----|-----|-----|-----|---|
| $\tau_S$ | 1.65 | 1.65 | 1.65 | 1.65 | 1.65 | 1.65 |
| $\nu_S$ | 3.35 | 3.3 | 3.25 | 3.15 | 3.3 | 3.4 |
| $\nu_L$ | 2.1 | 2.1 | 1.8 | 1.9 | 1.9 | 1.9 |
| $\zeta$ | 0.85 | 0.85 | 0.85 | 1.15 | 1.15 | 1.13 |
| $\beta$ | 0.77 | 0.77 | 0.77 | 0.77 | 0.77 | 0.7 |

Figure 14: Left: Width ($W$) of clusters according to their length ($L$). Right: surface of the cluster $S$ according to the bounding size $WL$. Blue: $(n, \gamma, \sigma_f) = (0.5, 0.75, 1)$ and green: $(n, \gamma, \sigma_f) = (0.5, 0.25, 1)$. The red lines correspond to the power-law fit.

complicated rheological laws. Using the energy balance/virtual work approach, it has been shown that the method predicts qualitatively the asymptotic behaviors. However, one should expect a different definition of the effective shear rate at low and high flow rate.

The influence of a macroscopic heterogeneous permeability and velocity threshold fields has then been considered. In this case, an energy balance/virtual work approach was able to predict some mathematical properties of the flow field. It shows also the existence of two asymptotic behaviors, at high and low flow rate. In both limits, the normalized velocity field becomes constant and corresponds to the solution of a power-law fluid with the dominant exponent.

The flow field properties of such power-law fluid in a heterogeneous porous media was studied analytically using a perturbation approach. This allows us to predict the mean and the
Figure 15: Regions above the threshold $u > u_c$ (in black) for two different sets of parameters and close to the critical point. The mean flow direction is from top to bottom. Left: $(n, \gamma, \sigma_f) = (0.5, 0.25, 1)$ and right: $(n, \gamma, \sigma_f) = (0.5, 0.75, 1)$. The $\gamma$ parameter drastically affects the orientation of the correlation from the direction of the flow to perpendicular to it.

standard deviation of the velocity field in both asymptotic regimes. These predictions were found to be in good agreement with the simulations. Another useful theoretical feature is the rotational symmetry of a 2D porous media, which shows that solving a shear thinning fluid in one direction is similar to solving a shear thickening in the other direction. This symmetry property is very interesting in this problem because it allowed to understand the change of correlation observed in the simulations.

Then the numerical simulations were presented. First, it shows that two asymptotic behaviors can be well understood from the power-low flow field with the dominant exponent. The range of the transitional regime was investigated. Qualitatively, it is expected that the transition range is related to the width of the velocity distribution of the power-law fluids in the same permeability field. For wider velocity distribution should begin the transition at a lower mean velocity and finish the transition at a higher velocity. If this trend has been observed, the results have shown that it is more complicated to make quantitative predictions. Indeed, one very important aspect of the problem is the relationship between the velocity, the permeability and the local threshold.

In this paper, a power-law dependency between the permeability and the threshold have been assumed. The study revealed that the exponent $\gamma$ modifies drastically the correlation of the clusters. The reason behind this observation is that there are different mechanisms at work. First, high permeability regions have a higher velocity field but also a higher velocity threshold. If the velocity threshold vary weakly with the permeability, the correlation should follow the one of the velocity field, in the streamwise direction. Conversely, if the velocity threshold varies greatly with the permeability (i.e. $\gamma$ high), the transition might not correspond to the highest velocities. At the same time, a similar reasoning considering the gradient of pressure field can be made: low permeability regions are more likely to have a higher pressure gradient but also a lower pressure threshold. So the correlations might be dominated by the pressure gradient field, transversely to the flow direction. As a result, there is thus a competition between this two effects which is balanced by the exponent $\gamma$. The relationship between the permeability and the two thresholds seems then to be very important. It was assumed a power-law but it is important to stress that the dependence is probably more complicated than this.

A remark can be made. In the literature, a very common approach is to model a porous media or a fracture by a bundle of parallel tubes or layers (Di Federico 1998; Chen et al.
It is then important to note that, in such models, the flow field is, by construction infinitely correlated in the streamwise direction, while the pressure gradient is not correlated in the transverse direction. These models are thus expected to be unable to capture behaviors related to the change of correlation. They must consequently be taken with caution while applied to 2D or 3D medium.

The statistical properties of the flow field in the transitional regime appeared to be quite rich also. Indeed, in this regime, the regions above their threshold define clusters which display properties found in critical systems. The cluster sizes are distributed according to a power-law with a cut-off. And the latter diverges at a particular critical mean gradient of pressure. The shape of the cluster also displays self-affine fractality. An important result is that some of the exponents (size distribution, shape exponents) do not vary with the parameters and the amplitude of the heterogeneity, which tends to suggest the presence of a universal behaviors. One very interesting feature is that the shape exponent $\zeta$ is constant but only switches to its inverse depending on the value of $\gamma$.

This statistical feature is thus very reminiscent to related problems such as percolation (Stauffer & Aharony 1991) and yield stress fluid in porous media (Kostenko & Talon 2019). The flow structure of yield stress fluid is, however, quite different because below the threshold there is no flow. Regions above the threshold are then necessarily channel paths connecting the inlet to the outlet. The exponents are thus expected to be different. It was found $\tau_S \approx 1.15$ and $\zeta \approx 0.75$. The size distribution exponent is different while the roughness exponent is similar to the present case. More closely would be the percolation problem (directed or not), but as discussed in the paper, the problem is expected to be a little bit different because the correlation of the velocity field evolve with the applied gradient of pressure. The observed exponent are expected to be different: $\tau_S \approx 2.1$ for percolation and $\tau_S \approx 1.26$ for directed percolation. The present case seems to be intermediate, the exponent are “universal” they fall in a different universality class.

There are many interesting directions to pursue this work. One important question is how this generalize to 3D permeability field. While the rotational symmetry is no longer applicable in 3D, it remains the important fact that the velocity field is correlated along the streamwise direction while the pressure gradient correlates in the two transverse directions. A change of correlation should thus be expected depending on $\gamma$. Critical behaviors should also be probably observed but with different exponents. Another interesting study would be to investigate the problem of transport of species in such system. Indeed, the dispersion of a tracer hinges on the heterogeneities of the velocity field and its correlation. It is thus expected to observe a change of behavior due to the change of rheology. In addition, since the molecular diffusion coefficient is usually linked to the viscosity of the fluid, the coefficient is expected to be different when the rheology is below and above the threshold. Finally, another direction of investigation would be the similarities with others in porous media displaying similar behaviors. For instance, two phase flow problem (Tallakstad et al. 2009; Yiotis et al. 2013; Sinha et al. 2017; Yiotis et al. 2019), emulsion driven in porous (Le Blay et al. 2020) or erosion of granular bed (Aussillous et al. 2016), display similar critical behavior with the occurrence of preferential flow paths depending on the flow rate. These problems have in common the property that at a certain critical velocity, the local flow condition is drastically modified, because bubbles are mobilized or because the grains rearrange with each other. It would thus be very interesting to investigate the similarities and differences between these problems.
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Appendix A. Numerical method

A.1. Augmented Lagrangian method

This augmented Lagrangian method has been introduced to solve non-Newtonian Stokes equation and has been used by many authors (see for instance Glowinski & Le Tallec (1989); Roquet & Saramito (2003)). In this paper the method was adapted to solve the non-linear Darcy’s equation:

\[
\nabla P = -g(\|\vec{u}\|) \frac{\vec{u}}{\|\vec{u}\|},
\]

(A 1)

\[
\nabla \cdot \vec{u} = 0.
\]

As for boundary conditions, pressure is imposed at the inlet and outlet, \(P_{in}\) and \(P_{out}\) respectively. Periodic conditions are assumed at the lateral sides. The mean flow is taken in the \(x\)–direction. As described in the paper, the solution of such system of equation is equivalent to finding, among all admissible velocity fields the functional:

\[
\Phi[\vec{u}] = \int G(\|\vec{u}\|) dr^2 - P_{in} \int_{in} \vec{u}_x dy + P_{out} \int_{out} u_x dy,
\]

(A 2)

with

\[
G(\|\vec{u}\|) = \int_0^{\|\vec{u}\|} g(y) dy.
\]

The main idea of the augmented Lagrangian method is to introduce a secondary field \(\vec{v}\) in order to decouple the non-linear problem \(G(\|\vec{u}\|)\) to the flow equation. The equality \(\vec{u} = \vec{v}\) is then guaranteed by the introduction of a Lagrangian vector field \(\vec{\mu}\). An extra term \(\frac{\alpha}{2}(\vec{u} - \vec{v})^2\) is also added to enhance the convergence, where \(\alpha\) is a small parameter. Another Lagrangian field \(\lambda\) is introduced to impose the free divergence, the problem can thus be recast into a saddle point determination:

\[
\min_{\vec{u}, \vec{v}} \max_{\vec{\mu}, \lambda} \Psi[\vec{u}, \vec{v}, \vec{\mu}, \lambda],
\]

(A 3)

with

\[
\Psi[\vec{u}, \vec{v}, \vec{\mu}, \lambda] = \int \left[ G(\|\vec{u}\|) + \vec{\mu} \cdot (\vec{v} - \vec{u}) + \frac{\alpha}{2}(\vec{u} - \vec{v})^2 - \lambda \nabla \cdot \vec{v} \right] dr^2 - P_{in} \int_{in} v_x dy + P_{out} \int_{out} v_x dy.
\]

(A 4)

If we now differentiate this functional, we obtain:

\[
\forall \delta\vec{v}, \quad \frac{\delta \Psi}{\delta \vec{v}} \cdot \delta\vec{v} = \int [\vec{\mu} + \alpha \vec{v} - \alpha \vec{u}] \cdot \delta\vec{v} - \int \lambda \nabla \cdot \delta\vec{v} dr^2 - P_{in} \int_{in} \delta v_x dy + P_{out} \int_{out} \delta v_x dy,
\]

(A 5)

\[
\forall \delta\lambda, \quad \frac{\delta \Psi}{\delta \lambda} \cdot \delta\lambda = - \int \delta\lambda \nabla \cdot \vec{v} dr^2
\]

(A 6)

\[
\forall \delta\vec{u}, \quad \frac{\delta \Psi}{\delta \vec{u}} \cdot \delta\vec{u} = \int \left[ g(\|\vec{u}\|) \frac{\vec{u}}{\|\vec{u}\|} - \vec{\mu} + \alpha \vec{u} - \alpha \vec{v} \right] \cdot \delta\vec{u} dr^2
\]

(A 7)
\[ \forall \delta \mu, \quad \frac{\delta \Psi}{\delta \mu} \cdot \delta \mu = \int (\vec{v} - \bar{u}) \cdot \delta \mu dr^2. \] (A 8)

This set of equations is quite cumbersome. However, the main advantage of this approach lies in the fact that, for a given \( \bar{u} \) and \( \bar{\mu} \), finding the saddle point for \( \bar{u} \) and \( \lambda \) eqs (A 5)-(A 6), is equivalent to solving:

\[ \vec{v} = -\frac{1}{\alpha} (\vec{\nabla} \lambda + \bar{\mu}) + \bar{u} \] (A 9)
\[ \vec{\nabla} \cdot \vec{v} = 0, \] (A 10)

with the boundary conditions \( \lambda = P_{in} \) and \( \lambda = P_{out} \). This is a standard linear Darcy’s equation with a source term, which can thus be solved using standard methods. Here, a second order finite difference method was used. It is important note that in these equation the permeability is homogeneous and constant, which allows a very fast solution at each step.

For the given field \( \vec{v}, \lambda \) and \( \bar{\mu} \), the minimization of \( \Psi \) with respect to \( \bar{u} \), is equivalent to solve:

\[ (g(||\bar{u}||) + \alpha) \bar{u} = \bar{\mu} + \alpha \bar{v}, \] (A 11)

which represents an implicit problem. This can be solve numerically or analytically. With our particular function \( g(||u||) \), an analytical solution can be found for some exponents \( n = 1/4, 1/3, 1/2, 3/4 \) and 1. Solutions are given in the next section.

After defining initial fields \( \bar{u}_0, \vec{v}_0, \bar{\mu}_0 \) and \( \lambda_0 \), the algorithm is thus decomposed in the following step:

1. Solve \( \vec{v}_{n+1} \) and \( \lambda_{n+1} \) with the Darcy’s equation:
\[ \vec{v}_{n+1} = -\frac{1}{\alpha} (\vec{\nabla} \lambda_{n+1} + \bar{\mu}_n) + \bar{u}_n \] (A 12)
\[ \vec{\nabla} \cdot \vec{v}_{n+1} = 0, \] (A 13)

2. Determine \( \bar{u}_{n+1} \), by solving
\[ (g(||\bar{u}_{n+1}||) + \alpha) \bar{u}_{n+1} = \bar{\mu}_n + \alpha \vec{v}_{n+1}, \] (A 14)

3. Advancing \( \bar{\mu} \) toward the gradient eq. (A 8):
\[ \bar{\mu}_{n+1} = \bar{\mu}_n + (\vec{v}_{n+1} - \bar{u}_{n+1}) d\mu, \] (A 15)

where \( d\mu \) is a small parameter, taken equal to \( \alpha \) for simplicity.

A.2. Solutions of eq. (A 11)

For the given fields \( \vec{v} \) and \( \bar{\mu} \), the algorithm requires to find \( \bar{u} \) satisfying:

\[ (g(||\bar{u}||) + \alpha) \bar{u} = \bar{\mu} + \alpha \vec{v} \equiv \bar{B}, \] (A 16)

with

\[ \begin{align*}
g(||\bar{u}||) & = \frac{\mu}{\kappa} ||\bar{u}|| \quad \text{if} \quad ||\bar{u}|| < u_c \\
g(||\bar{u}||) & = \frac{\mu}{\kappa} \left[ \frac{||\bar{u}||}{u_c^{n-1}} \right] \quad \text{if} \quad ||\bar{u}|| > u_c
\end{align*} \] (A 17)

It must be noted that \( \bar{u} \) and \( \bar{B} \) are colinear and with the same orientation because the left term in eq. (A 16) is positive. It is thus sufficient to determine the norm of \( ||\bar{u}|| = u \). The
equations then become:

\[
\begin{cases}
\left( \frac{\mu}{K} + \alpha \right)u = B & \text{if } B < \left( \frac{\mu}{K} + \alpha \right)u_c \\
\frac{A u^n + \alpha u}{B} = B & \text{if } B > \left( \frac{\mu}{K} + \alpha \right)u_c
\end{cases}
\]  

(A 18)

where \( A = \frac{\mu}{k u_c n-1} \).

If the first equation is trivial, the second one has an analytical solution only for specific value of \( n \). We give here the ones used in this work.

- \( n = \frac{1}{3} \):

\[
u = -\sqrt[3]{\frac{3}{\Delta}} + \frac{\Delta}{18^{1/3} \alpha^3} + \frac{B}{\alpha},
\]

with

\[
\Delta = \left( \sqrt{3} \sqrt[3]{4 A^9 \alpha^9 + 27 A^6 \alpha^{10} B^2 - 9 A^3 \alpha^5 B} \right)^{1/3}
\]

- \( n = \frac{1}{2} \):

\[ u = \left( -A + \sqrt{A^2 + 4 \alpha B} \right)^2 \]

- \( n = \frac{2}{3} \):

if \( B < \frac{4 A^3}{27 \alpha^2} \),

\[
u = \frac{1}{3 \alpha^3} \left[ -(A^3 - 3 \alpha^2 B) - 2 A^{3/2} \sqrt{A^3 - 6 \alpha^2 B \cos \left( \phi/3 \right)} \right], \quad \text{with}
\]

\[
\phi = \arg \left( -2 A^6 + 18 A^3 \alpha^2 B - 27 \alpha^4 B^2 + i 3 \sqrt{3} \alpha^3 \sqrt{4 A^3 B^3 - 27 \alpha^2 B^4} \right)
\]

if \( B > \frac{4 A^3}{27 \alpha^2} \),

\[
u = \frac{1}{3 \alpha^3} \left[ -(A^3 - 3 \alpha^2 B) - 2^{1/3} \left( -A^6 + 6 A^3 \alpha^2 B \right) \frac{1}{\Delta} + \frac{\Delta}{2^{1/3}} \right], \quad \text{with}
\]

\[
\Delta = -2 A^6 + 18 A^3 \alpha^2 B - 27 \alpha^4 B^2 + 3 \sqrt{3} \alpha^3 \sqrt{-4 A^3 B^3 + 27 \alpha^2 B^4}
\]

- \( n = 2 \):

\[
u = -\alpha + \sqrt{\alpha^2 + 4 A B}
\]

- \( n = 3 \):

\[
u = \frac{\Delta}{18^{1/3} A} - \frac{3 \sqrt{3} \alpha}{\Delta},
\]

with

\[
\Delta = \left( 9 A^2 B + \sqrt{3} \sqrt{27 A^4 B^2 + 4 A^3 \alpha^3} \right)^{1/3}.
\]

REFERENCES

Al-Fariss, T. & Pinder, K. L. 1987 Flow through porous media of a shear-thinning liquid with yield stress. *Can. J. Chem. Eng.* 65 (3), 391–405.
Amaral, L. A. N., Barabási, A. L., Buldyrev, S. V., Harrington, S. T., Havlin, S., Sadraei, R., & Stanley, H. E. 1995 Avalanches and the directed percolation depinning model - experiments, simulations, and theory. *Phys. Rev. E* **51** (5), 4655–4673.

Andrade Jr, J. S., Costa, UMS, Almeida, MP, Makse, HA & Stanley, HE 1999 Inertial effects on fluid flow through disordered porous media. *Physical Review Letters* **82** (26), 5249.

Auradou, H., Boschan, A., Chertcoff, R., Gabbanelli, S., Hulin, J.-P. & Ippolito, I. 2008 Enhancement of velocity contrasts by shear-thinning solutions flowing in a rough fracture. *Journal of non-Newtonian fluid mechanics* **153** (1), 53–61.

Auriault, J.-L., Royer, P. & Geindrau, C. 2002 Filtration law for power-law fluids in anisotropic porous media. *International Journal of Engineering Science* **40** (10), 1151 – 1163.

Aussillous, P., Zou, Z., Guazzelli, É., Yan, L. & Wyatt, M. 2016 Scale-free channeling patterns near the onset of erosion of sheared granular beds. *Proceedings of the National Academy of Sciences*, arXiv: http://www.pnas.org/content/early/2016/10/04/1609023113.full.pdf.

Balhoff, M. 2005 Modeling the flow of non-Newtonian fluids in packed beds at the pore scale. *PhD Thesis*.

Balhoff, M. T. & Thompson, K. E. 2004 Modeling the steady flow of yield-stress fluids in packed beds. *AIChE J.* **50** (12), 3034–3048.

Barabási, A-L & Stanley, H. E. 1995 *Fractal concepts in surface growth*. Cambridge university press.

Barnes, H.A., Hutton, J.F. & Walters, K. 1989 *An introduction to rheology*, vol. 3. Elsevier Science Limited.

Bauer, D., Talon, L., Peysson, Y., Ly, H. B., Batôt, G., Chevalier, T. & Fleury, M. 2019 Experimental and numerical determination of darcy’s law for yield stress fluids in porous media. *Phys. Rev. Fluids* **4**, 063301.

Bessonov, N., Sequeira, A., Simakov, S., Vassilevski, Y. & Volpert, V. 2016 Methods of blood flow modelling. *Mathematical modelling of natural phenomena* **11** (1), 1–25.

Bird, R B 1976 Useful non-newtonian models. *Annual Review of Fluid Mechanics* **8** (1), 13–34, arXiv: https://doi.org/10.1146/annurev.fl.08.010176.000305.

Bird, R B., Armstrong, R C & Hassager, O. 1987 Dynamics of polymeric liquids. vol. 1: Fluid mechanics.

Boschan, A., Ippolito, I., Chertcoff, R., Auradou, H., Talon, L. & Hulin, J. P. 2008 Geometrical and taylor dispersion in a fracture with random obstacles: An experimental study with fluids of different rheologies. *Water Resour. Res.* **44** (6), W06420.

Boyd, J., Buick, J. M. & Green, S. 2007 Analysis of the casson and carreau-yasuda non-newtonian blood models in steady and oscillatory flows using the lattice boltzmann method. *Physics of Fluids* **19** (9), 093103, arXiv: https://doi.org/10.1063/1.2772250.

Chauveteau, G. 1982 Rodlike polymer solution flow through fine pores: Influence of pore size on rheological behavior. *Journal of Rheology* **26** (2), 111–142, arXiv: https://doi.org/10.1122/1.549660.

Chen, M., Rossen, W. & Yortsos, Y. C. 2005 The flow and displacement in porous media of fluids with yield stress. *Chem. Eng. Sci.* **60** (15), 4183 – 4202.

Chevalier, T., Dubey, A. K., Atis, S., Rosso, A., Salin, D. & Talon, L. 2017 Avalanches dynamics in reaction fronts in disordered flows. *Phys. Rev. E* **95**, 042210.

Chevalier, T., Rodts, S., Chatex, X., Chevalier, C. & Coussot, P. 2014 Breaking of non-newtonian character in flows through a porous medium. *Phys. Rev. E* **89**, 023002.

Chevalier, T. & Talon, L. 2015 Generalization of Darcy’s law for Bingham fluids in porous media: From flow-field statistics to the flow-rate regimes. *Phys. Rev. E* **91**, 023011.

Christopher, R. H & Middleman, S. 1965 Power-law flow through a packed tube. *Industrial & Engineering Chemistry Fundamentals* **4** (4), 422–426.

Coussot, P. 2005 *Rheometry of pastes, suspensions, and granular materials: applications in industry and environment*. John Wiley and Sons.

Darcy, H. 1856 *Les fontaines publiques de la ville de Dijon: exposition et application*... Victor Dalmont.

Dideferico, V. 1998 Non-newtonian flow in a variable aperture fracture. *Transport in Porous Media* **30**, 75–86.

DuVaut, Georges & Lions, Jacques Louis 1976 *Inequalities in mechanics and physics*. Springer.

Eberhard, U., Seybold, H. J., Flörioncic, M., Bertsch, P., Jiménez-Martínez, J., Andrade, J. S. & Holzner, M. 2019 Determination of the effective viscosity of non-newtonian fluids flowing through porous media. *Frontiers in Physics* **7**, 71.
Entov, V.M. 1967 On some two-dimensional problems of the theory of filtration with a limiting gradient. *Prikl. Mat. Mekh.* **31**, 820–833.

Felisa, G., Lenci, A., Lauriola, I., Longo, S. & Federico, V. Dr 2018 Flow of truncated power-law fluid in fracture channels of variable aperture. *Advances in Water Resources* **122**, 317 – 327.

Forchheimer, P. 1901 Wasserbewegung durch boden. *Z. Ver. Deutsch. Ing.* **45**, 1782–1788.

Gelhar, L. W. & Axness, C. L. 1983 Three-Dimensional Stochastic Analysis of Macrodispersion in Aquifers. *Water Resour. Res.* **19**, 161–180.

Glowinski, R. & Le Tallec, P. 1989 Augmented Lagrangian and operator-splitting methods in nonlinear mechanics., vol. 9. SIAM.

Géraud, B., Jones, S. A., Cantat, I., Dollet, B. & Méheust, Y. 2016 The flow of a foam in a two-dimensional porous medium. *Water Resources Research* **52** (2), 773–790, arXiv: https://agupubs.onlinelibrary.wiley.com/doi/pdf/10.1002/2015WR017936.

Happel, J. & Brenner, H. 1983 *Low Reynolds number hydrodynamics*. D. Reidel Publishing Co., Hingham, MA.

Hewitt, D. R., Daneshi, M., Balmbforth, N. J. & Martinez, D. M. 2016 Obstructed and channelized viscoplastic flow in a hele-shaw cell. *Journal of Fluid Mechanics* **790**, 173–204.

Hirasaki, GJ, Pope, GA & others 1974 Analysis of factors influencing mobility and adsorption in the flow of polymer solution through porous media. *Society of Petroleum Engineers Journal* **14** (04), 337–346.

Keller, J. B., Rubenfeld, L. A. & Molyneux, J. E. 1967 Extremum principles for slow viscous flows with applications to suspensions. *Journal of Fluid Mechanics* **30** (1), 97–125.

Kostenko, R. & Talon, L. 2019 Numerical study of bingham flow in macroscopic two dimensional heterogeneous porous media. *Physica A: Statistical Mechanics and its Applications* **528**, 121501.

Lavrov, A. 2013a Numerical modeling of steady-state flow of a non-newtonian power-law fluid in a rough-walled fracture. *Computers and Geotechnics* **50**, 101 – 109.

Lavrov, A. 2013b Redirection and channelization of power-law fluid flow in a rough-walled fracture. *Chemical Engineering Science* **99**, 81 – 88.

Lavrov, A. 2015 Flow of truncated power-law fluid between parallel walls for hydraulic fracturing applications. *Journal of Non-Newtonian Fluid Mechanics* **223**, 141 – 146.

Le Blay, M., Adda-Bedia, M. & Bartolo, D. 2020 Emergence of scale-free smectic rivers and critical depinning in emulsions driven through disorder. *Proceedings of the National Academy of Sciences* **117** (25), 13914–13920, arXiv: https://www.pnas.org/content/117/25/13914.full.pdf.

Liu, C., De Luca, A., Rosso, A. & Talon, L. 2019 Darcy’s law for yield stress fluids. *Phys. Rev. Lett.* **122**, 245502.

Lopez, X., Valvatne, P. H. & Blunt, M. J. 2003 Predictive network modeling of single-phase non-newtonian flow in porous media. *Journal of Colloid and Interface Science* **264** (1), 256 – 265.

Matheron, G. 1967 *Eléments pour une théorie des milieux poreux*. Masson Paris.

McKinley, R. M., Jahn, H. O., Harris, W. W. & Greenkorn, R. A. 1966 Non-newtonian flow in porous media. *AIChE Journal* **12** (1), 17–20, arXiv: https://aiche.onlinelibrary.wiley.com/doi/pdf/10.1002/aic.690120106.

Mei, C. C. & Auriault, J.-L. 1991 The effect of weak inertia on flow through a porous medium. *Journal of Fluid Mechanics* **222**, 647–663.

Mourzenko, V.V., Thovert, J.-F. & Adler, P.M. 1995 Permeability of a single fracture; validity of the reynolds equation. *J. Phys. II France* **5** (3), 465–482.

Nash, Sarah & Rees, D Andrew S 2016 The effect of microstructure on models for the flow of a bingham fluid in porous media. *Transp. Porous Media* .

P., Christian L., Tardy, P. M.J., Sorbie, K. S. & Crawshaw, J. C. 2006 Experimental and modeling study of newtonian and non-newtonian fluid flow in pore network micromodels. *Journal of Colloid and Interface Science* **295** (2), 542 – 550.

Pearson, JRA & Tardy, PMJ 2002 Models for flow of non-newtonian and complex fluids through porous media. *J. Non-Newtonian Fluid Mech.* **102** (2), 447–473.

Reynolds, Osborne 1886 On the theory of lubrication and its application to mr. beauchamp tower’s experiments, including an experimental determination of the viscosity of olive oil. *Phil. Trans. R. Soc. Lond.* **177**, 157–234.

Rodríguez de Castro, A. & Radilla, G. 2017 Non-darcian flow of shear-thinning fluids through packed beads: Experiments and predictions using forchheimer’s law and ergun’s equation. *Advances in Water Resources* **100**, 35 – 47.
Roquet, N. & Saramito, P. 2003 An adaptive finite element method for Bingham fluid flows around a cylinder. *Computer Methods in Applied Mechanics and Engineering* **192** (192), 3317–3341.

Rossen, William R 1990 Theory of mobilization pressure gradient of flowing foams in porous media: I. incompressible foam. *J. Colloid Interface Sci.* **136** (1), 1 – 16.

Roustaei, A., Chevalier, T., Talon, L. & Frigaard, I. A. 2016 Non-Darcy effects in fracture flows of a yield stress fluid. *Journal of Fluid Mechanics* **805**, 222–261.

Sadowski, T. J. & Bird, R. B. 1965 Non-newtonian flow through porous media. i. theoretical. *Transactions of the Society of Rheology* **9** (2), 243–250, arXiv: https://doi.org/10.1122/1.549000.

Santucci, S., Planet, R., Maloyy, K. Jørgen & Ortin, J. 2011 Avalanches of imbibition fronts: Towards critical pinning. *Europhys. Lett.* **94** (4), 46005.

Shah, C.B. & Yortsos 1996 The permeability of strongly disordered systems. *Phys. Fluids* **8**, 280–282.

Shah, C. B. & Yortsos, Y. C. 1995 Aspects of flow of power-law fluids in porous media. *AIChE J.* **41** (5), 1099–1112.

Shahsavari, Setareh & McKinley, Gareth H. 2016 Mobility and pore-scale fluid dynamics of rate-dependent yield-stress fluids flowing through fibrous porous media. *Journal of Non-Newtonian Fluid Mechanics* **235**, 76 – 82.

Sinha, Santanu, Bender, Andrew T., Danczyk, Matthew, Keepseagle, Kayla, Prather, Cody A., Bray, Joshua M., Thrane, Linn W., Seymour, Joseph D., Codd, Sarah L. & Hansen, Alex 2017 Effective rheology of two-phase flow in three-dimensional porous media: Experiment and simulation. *Transport in Porous Media* **119** (1), 77–94.

Slattery, John C. 1967 Flow of viscoelastic fluids through porous media. *AIChE Journal* **13** (6), 1066–1071, arXiv: https://aiche.onlinelibrary.wiley.com/doi/pdf/10.1002/aic.690130606.

Sochi, Taha & Blunt, Martin J. 2008 Pore-scale network modeling of ellis and herschel-bulkley fluids. *J. Pet. Sci. Eng.* **60** (2), 105 – 124.

Sorbie, K.S, Clifford, P.J & Jones, E.R.W 1989 The rheology of pseudoplastic fluids in porous media using network modeling. *Journal of Colloid and Interface Science* **130** (2), 508 – 534.

Sorbie, Kenneth S 2013 Polymer-improved oil recovery. Springer Science & Business Media.

Stauffer, Dietrich & Aharony, Amnon 1991 *Introduction to percolation theory*. Taylor and Francis.

Straley, Joseph P 1977 Critical exponents for the conductivity of random resistor lattices. *Physical Review B* **15** (12), 5733.

Tallakstad, Ken Torre, Knudsen, Henning Arendt, Ramstad, Thomas, Løvoll, Grunde, Måløy, Knut Jørgen, Toussaint, Renaud & Flekkøy, Eirik Grude 2009 Steady-state two-phase flow in porous media: Statistics and transport properties. *Phys. Rev. Lett.* **102** (7), 074502–.

Talon, L., Auradou, H. & Hansen, A. 2014 Effective rheology of Bingham fluids in a rough channel. *Front. Physics* **2** (24), 24.

Talon, L. & Bauer, D. 2013 On the determination of a generalized Darcy equation for yield-stress fluid in porous media using a lattice-Boltzmann trt scheme. *Eur. Phys. J. E* **36** (12), 139.

Talon, Laurent & Hansen, Alex 2020 Effective rheology of bi-viscous non-newtonian fluids in porous media. *Front Phys* **7**, 225.

Tongwa, Paul, Nygaard, Runar, Blue, Aaron & Bai, Baojun 2013 Evaluation of potential fracture-sealing materials for remediating co2 leakage pathways during co2 sequestration. *International Journal of Greenhouse Gas Control* **18**, 128–138.

Yiotis, A. G., Dollari, A., Kainourgakis, M. E., Salin, D. & Talon, L. 2019 Nonlinear darcy flow dynamics during ganglia stranding and mobilization in heterogeneous porous domains. *Phys. Rev. Fluids* **4**, 114302.

Yiotis, A. G., Talon, L. & Salin, D. 2013 blob population dynamics during immiscible two-phase flows in reconstructed porous media. *Phys. Rev. E* **87**, 033001.

Zami-Pierre, F., de Loubens, R., Quintard, M. & Davit, Y. 2016 Transition in the flow of power-law fluids through isotropic porous media. *Phys. Rev. Lett.* **117**, 074502.

Zimmerman, R.W., Kumar, S. & Bodvarsson, G.S. 1991 Lubrication theory analysis of the permeability of rough-walled fractures. *International Journal of Rock Mechanics and Mining Sciences & Geomechanics Abstracts* **28** (4), 325–331.