Quadratic relations of the deformed $W$-superalgebra $\mathcal{W}_{qt}(\mathfrak{sl}(2|1))$

December 10, 2019

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Dedicated to Professor Boris Feigin on the occasion of his 65th birthday

Abstract

This paper is a continuation of study by J. Ding and B. Feigin [1]. We find a bosonization of the deformed $W$-superalgebras $\mathcal{W}_{qt}(\mathfrak{sl}(2|1))$ that commutes up-to total difference with deformed screening currents. Using our bosonization, we derive a set of quadratic relations of generators of the deformed $W$-superalgebra $\mathcal{W}_{qt}(\mathfrak{sl}(2|1))$. The deformed $W$-superalgebra is independent of the choice of a Dynkin-diagram for the superalgebra $\mathfrak{sl}(2|1)$, though the deformed screening currents depend on it.

1 Introduction

Drinfeld and Sokolov [2] introduced the classical $W$-algebra $W(g)$ as Poisson structure of a hierarchy of integrable Hamiltonian equations associated with simple Lie algebra $g$. There are two kinds of one-parameter deformations of the classical $W$-algebra. One is the $q$-Poisson $W$-algebra [3] and the other is the $W$-algebra [4]. The deformed $W$-algebra $W_{qt}(g)$ is a two-parameter deformation of the classical $W$-algebra $W(g)$, including the $q$-Poisson $W$-algebra and the $W$-algebra as special cases. Various applications are known for each algebra. For instance, the $W$-algebra $W_{\beta}(g)$ plays an important role in conformal field theory. In particular the $W$-algebra $W_{\beta}(\mathfrak{sl}(2))$ is nothing but the Virasoro algebra. Shiraishi, Kubo, Awata, and Odake [5] introduced a bosonization of the deformed Virasoro algebra $\mathcal{W}_{qt}(\mathfrak{sl}(2))$ that is a one-parameter deformation of the Virasoro algebra, to construct a deformation of the correspondence between conformal field theory and the Calogero-Sutherland model. They also constructed the deformed screening currents, that commute with the deformed Virasoro algebra up-to total difference, and found that a set of quadratic relations of generators of the deformed Virasoro algebra. In [6] [7] [8] [9] [10], the results of Shiraishi et.al. were extended to the deformed $W$-algebra $\mathcal{W}_{qt}(\mathfrak{sl}(N))$ and the deformed twisted $W$-algebra $\mathcal{W}_{qt}(A_{2}^{(1)})$. In [11] Frenkel and Reshetikhin proposed the deformed $W$-algebra $\mathcal{W}_{qt}(g)$ associated with a simple Lie algebra $g$ and twisted algebra $A_{2N}^{(2)}$ by starting from a two-parameter deformation of
the Cartan matrix. In [12], Sevostyanov introduced the deformed W-algebra \( \mathcal{W}_{qt}(\mathfrak{g}) \) associated with a simple Lie algebra \( \mathfrak{g} \) by means of deformed Drinfeld-Sokolov reduction for affine quantum group, and checked that Sevostyanov’s deformed Virasoro algebra coincides with those introduced by Shiraiishi et al. [5]. It is still an open problem to find quadratic relations of the deformed W-algebra \( \mathcal{W}_{qt}(\mathfrak{g}) \) except for \( \mathfrak{g} = \mathfrak{sl}(N) \) [5, 8] and twisted algebra \( A_2^{(2)} \) [9].

In the works above [5, 6, 7, 8, 9, 10], the deformed screening currents appear as a by-product of the deformed W-algebras. In [1] Ding and Feigin tried to construct a bosonization of the deformed W-algebra \( \mathcal{W}_{qt}(\mathfrak{g}) \) and the deformed screening currents associated with \( \mathfrak{g} = \mathfrak{sl}(3) \) and \( \mathfrak{sl}(2|1) \), starting from the opposite direction of such an approach. First, they tried to construct a bosonization of the deformed screening currents associated with two kinds of Dynkin-diagrams of \( \mathfrak{sl}(N) \). Representation theory of the superalgebras is much more complicated than non-superalgebra and has rich structures [13, 14, 15, 17, 18, 19, 20]. For instance, there are two kinds of Dynkin-diagrams associated with the superalgebra \( \mathfrak{sl}(2|1) \), i.e., \( \circlearrowright \circlearrowright \circlearrowleft \) and \( \circlearrowleft \circlearrowright \circlearrowright \). We check that quadratic relations of the deformed W-superalgebra \( \mathcal{W}_{qt}(\mathfrak{sl}(2|1)) \) are independent of the choice of a Dynkin-diagram, even though the deformed screening currents depend on it.

This paper is a continuation of the work of J.Ding and B.Feigin [1]. We revisit bosonization of the deformed W-superalgebra \( \mathcal{W}_{qt}(\mathfrak{sl}(2|1)) \), because Ding and Feigin stopped calculation in the middle and there are small mistakes in their paper. In this paper we start from a review of Ding and Feigin’s construction and report a bosonization of the deformed W-superalgebra \( \mathcal{W}_{qt}(\mathfrak{sl}(2|1)) \) by \( a_i(m) \) \( (i = 1, 2, m \in \mathbb{Z}_{\geq 0}) \) satisfying \([a_i(m), a_j(n)] = \frac{1}{m} A_{i,j}(m) \delta_{m+n,0}\) with \( A_{1,2}(m) = A_{2,1}(-m) \) and \( A_{1,1}(m) = A_{2,2}(m) = 1\). Using our bosonization, we derive a set of quadratic relations of generating functions \( T_i(z) \) \( (i = 1, 2, 3, \cdots) \) of the deformed W-superalgebra \( \mathcal{W}_{qt}(\mathfrak{sl}(2|1)) : f_{i,j}(z_2/z_1)T_i(z_1)T_j(z_2) - f_{j,i}(z_1/z_2)T_j(z_2)T_i(z_1) = (\text{terms containing delta-function}) \). Representation theory of the superalgebras is much more complicated than non-superalgebra and has rich structures [13, 14, 15, 17, 18, 19, 20]. For instance, there are two kinds of Dynkin-diagrams associated with the superalgebra \( \mathfrak{sl}(2|1) \), i.e., \( \circlearrowright \circlearrowright \circlearrowleft \) and \( \circlearrowleft \circlearrowright \circlearrowleft \). We check that quadratic relations of the deformed W-superalgebra \( \mathcal{W}_{qt}(\mathfrak{sl}(2|1)) \) are independent of the choice of a Dynkin-diagram, even though the deformed screening currents depend on it.

The text is organized as follows. In Section 2 we present the notation and the settings. In Section 3 we summarize the main theorem, which will be proven in Sections 4, 5, and 6. In Section 4, starting from a review of Ding and Feigin’s construction [1], we derive a bosonization of the deformed screening currents associated with two kinds of Dynkin-diagrams of \( \mathfrak{sl}(2|1) \). In Section 5 we construct a bosonization of the first generating function \( T_i(z) \) of the deformed W-superalgebra \( \mathcal{W}_{qt}(\mathfrak{sl}(2|1)) \) associated with two kinds of Dynkin-diagrams. In Section 6 we derive a set of quadratic relations of the generating functions \( T_i(z) \) for the deformed W-superalgebra \( \mathcal{W}_{qt}(\mathfrak{sl}(2|1)) \). In Section 7 we study two special limits. One is the classical limit and the other is the conformal limit. We discuss some open problems. In Appendix A we reproduce the same deformed W-algebra \( \mathcal{W}_{qt}(\mathfrak{gl}(N)) \) given in [5, 7, 8] by Ding and Feigin approach. In Appendix B we summarize some normal ordering rules. In Appendix C we study \( \mathcal{W}_{qt}(\mathfrak{sl}(2|1)) \) in another regime.
2 Preliminaries

In this Section we prepare the notation and the settings.

2.1 Notation

For any integer $n$, define

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}. \tag{2.1}$$

We use symbol for infinite product

$$(z; q)_\infty = \prod_{n=0}^{\infty} (1 - q^n z). \tag{2.2}$$

We use the symbols $\Theta_q(\varepsilon)$ and $[\varepsilon]$ for the elliptic theta function. The theta function $\Theta_q(\varepsilon)$ satisfies $\Theta_q(q\varepsilon) = -\varepsilon^{-1}\Theta_q(\varepsilon)$ and $\Theta_q(e^{2\pi i} \varepsilon) = \Theta_q(\varepsilon)$. The theta function $[\varepsilon]$ satisfies $[\varepsilon + \tau] = -e^{2\pi i (\varepsilon + \frac{r}{2})}/r[\varepsilon]$ where $\tau = \frac{\imath}{\log \varepsilon}$. Explicitly they are given by

$$[\varepsilon] = x^{\frac{\varepsilon^2}{2} - u \Theta_{x^{2u}}(x^{2u})}, \quad \Theta_q(z) = (q; q)_\infty(z; q)_\infty(q/z; q)_\infty. \tag{2.3}$$

Define the formal delta-function $\delta(z)$ by

$$\delta(z) = \sum_{m \in \mathbb{Z}} z^m. \tag{2.4}$$

2.2 Lie superalgebra $\mathfrak{sl}(2|1)$

Let $\varepsilon_1, \varepsilon_2, \delta_1$ be a basis with inner product $(\mid \ )$ such that

$$(\varepsilon_i | \varepsilon_j) = \delta_{i,j}, \quad (\varepsilon_i | \delta_1) = (\delta_1 | \varepsilon_i) = 0, \quad (\delta_1 | \delta_1) = -1 \quad (i, j = 1, 2). \tag{2.5}$$

The different root systems of $\mathfrak{sl}(2|1)$ with corresponding Dynkin-diagrams and Cartan matrices are given as follows.

(i) One-fermion case : Simple root system $= \{ \alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = \varepsilon_2 - \delta_1 \}$

$$\begin{array}{c}
\alpha_1 \\
\longrightarrow \\
\alpha_2
\end{array}
\quad \text{Cartan matrix} = \begin{pmatrix}
2 & -1 \\
-1 & 0
\end{pmatrix}$$

(ii) Two-fermion case : Simple root system $= \{ \alpha_1 = \varepsilon_1 - \delta_1, \alpha_2 = \delta_1 - \varepsilon_2 \}$

$$\begin{array}{c}
\alpha_1 \\
\longrightarrow \\
\alpha_2
\end{array}
\quad \text{Cartan matrix} = \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}$$

We denote by $\circ$ an even simple root and by $\otimes$ an odd isotropic simple root. In what follows we call $\circ \dashv \otimes$ one-fermion diagram, and $\otimes \dashv \otimes$ two-fermion diagram.
2.3 Deformed screening current

We introduce the Heisenberg algebra \( \mathcal{H}_{qt}(2|1) \) generated by \( a_i(m), Q_i \) \((m \in \mathbb{Z}, i = 1, 2)\) satisfying

\[
[a_i(m), a_j(n)] = \frac{1}{m} A_{i,j}(m) \delta_{m+n,0} \quad (m, n \neq 0, i, j = 1, 2),
\]
\[
[a_i(0), Q_j] = A_{i,j}(0) \quad (i, j = 2).
\] (2.6)

The remaining commutators vanish. Here we assume

\[
A_{1,2}(m) = A_{2,1}(-m), \quad A_{1,1}(m) = A_{2,2}(m) = 1 \quad (m \neq 0).
\] (2.7)

We define \( A_{i,j}(0) \) in (2.12) and (2.13). In what follows we work on Fock space of the Heisenberg algebra. We introduce the deformed screening currents \( S^+_i(z) \) \((i = 1, 2)\) as follows.

\[
S^+_i(z) = \exp(Q_i + a_i(0) \log z) : \exp \left( \sum_{m \neq 0} s^+_i(m) a_i(m) z^{-m} \right) : \quad (i = 1, 2).
\] (2.8)

Later we will determine parameters \( s^+_i(m) \) \((i = 1, 2)\) from the ansatz (2.18). We note

\[
\exp(Q_i + a_i(0) \log z) = z^{\frac{1}{2} A_{i,i}(0)} e^{Q_i z a_i(0)}.
\]

We note that our bosonization is slightly different from those in [1] by a scalar factor \( z^{\frac{1}{2} A_{i,i}(0)} \). The normal ordering formulae are given by

\[
S^+_i(z_1)S^+_j(z_2) = z^{A_{i,j}(0)} h_{i,j}(z_2/z_1) : S^+_i(z_1)S^+_j(z_2) : \quad (i, j = 1, 2),
\] (2.9)

where we set

\[
h_{i,j}(z) = \exp \left( \sum_{m=1}^{\infty} \frac{1}{m} s^+_i(m) A_{i,j}(m) s^+_j(-m) z^m \right) \quad (i, j = 1, 2).
\] (2.10)

We assume the symmetry of \( h_{i,j}(z) \) as follows.

\[
h_{1,2}(z) = h_{2,1}(z).
\] (2.11)

The deformed screening currents \( S^+_i(z) \) is a one-parameter deformation of those of the \( W \)-superalgebra. In the conformal limit \( q \to 1 \), our deformed screening currents reproduce those of the \( W \)-superalgebra. Hence, we impose the following conditions.

(i) In the case of one-fermion diagram \( \bigcirc \rightarrow \bigotimes \), we set

\[
A_{1,1}(0) = 2\alpha, \quad A_{2,2}(0) = 1, \quad A_{1,2}(0) = A_{2,1}(0) = -\beta.
\] (2.12)

(ii) In the case of two-fermion diagram \( \bigotimes \rightarrow \bigotimes \), we set

\[
A_{1,1}(0) = 1, \quad A_{2,2}(0) = 1, \quad A_{1,2}(0) = A_{2,1}(0) = \beta - 1.
\] (2.13)

Here \( \alpha, \beta \) \((\alpha \neq \frac{1}{2})\) are a generic parameter.
2.4 Deformed W-superalgebra \( \mathcal{W}_{q,t}(\mathfrak{sl}(2|1)) \)

We introduce the first generating function \( T_1(z) \) of the deformed W-superalgebra \( \mathcal{W}_{q,t}(\mathfrak{sl}(2|1)) \) by

\[
T_1(z) = g_1 \Lambda_1(z) + g_2 \Lambda_2(z) + g_3 \Lambda_3(z),
\]

where we set

\[
\Lambda_i(z) = \exp\left(\sum_{j=1,2} \lambda_{i,j}(0) a_j(0)\right) : \exp\left(\sum_{j=1,2} \sum_{m \neq 0} \lambda_{i,j}(m)s^+_j(m)a_j(m)z^{-m}\right) : (i = 1, 2, 3).
\]

Later we will determine parameters \( g_i \) \((i = 1, 2, 3)\) and parameters \( \lambda_{i,j}(m) \) \((i = 1, 2, 3, j = 1, 2, m \in \mathbb{Z})\) from the ansatz (2.18).

We assume that normal ordering formulae of \( \Lambda_i(z) \) and \( S^+_j(w) \) have at most one pole and one zero.

\[
\begin{align*}
\Lambda_1(z_1)S^+_1(z_2) &= \frac{q_1 (z_1 - p_1 z_2)}{p_1 (z_1 - q_1 z_2)} : \Lambda_1(z_1)S^+_1(z_2) : (|z_1| > |z_2|), \\
S^+_1(z_2)\Lambda_1(z_1) &= \frac{q_1 (z_1 - p_1 z_2)}{p_1 (z_1 - q_1 z_2)} : \Lambda_1(z_1)S^+_1(z_2) : (|z_2| > |z_1|), \\
\Lambda_1(z_1)S^+_2(z_2) &= \Lambda_1(z_1)S^+_2(z_2) := S^+_2(z_2)\Lambda_1(z_1), \\
\Lambda_2(z_1)S^+_1(z_2) &= \frac{q_2 (z_1 - p_2 z_2)}{p_2 (z_1 - q_2 z_2)} : \Lambda_2(z_1)S^+_1(z_2) : (|z_1| > |z_2|), \\
S^+_1(z_2)\Lambda_2(z_1) &= \frac{q_2 (z_1 - p_2 z_2)}{p_2 (z_1 - q_2 z_2)} : \Lambda_2(z_1)S^+_1(z_2) : (|z_2| > |z_1|), \\
\Lambda_2(z_1)S^+_2(z_2) &= \Lambda_2(z_1)S^+_2(z_2) := S^+_2(z_2)\Lambda_2(z_1), \\
\Lambda_3(z_1)S^+_1(z_2) &= \Lambda_3(z_1)S^+_1(z_2) := S^+_1(z_2)\Lambda_3(z_1), \\
\Lambda_3(z_1)S^+_2(z_2) &= \Lambda_3(z_1)S^+_2(z_2) := S^+_2(z_2)\Lambda_3(z_1), \\
S^+_2(z_2)\Lambda_3(z_1) &= \frac{q_3 (z_1 - p_3 z_2)}{p_3 (z_1 - q_3 z_2)} : \Lambda_3(z_1)S^+_2(z_2) : (|z_2| > |z_1|).
\end{align*}
\]

Here parameters \( p_i, p'_i, q_i, q'_i \) \((i = 1, 2)\) satisfy

\[
p_i, p'_i, q_i, q'_i > 0, \quad q_i \neq q'_i, \quad p_i \neq q_i, \quad p'_i \neq q'_i \quad (i = 1, 2).
\]

Later we determine parameters \( p_i, p'_i, q_i, q'_i \) from the ansatz (2.18). From (2.16) we have

\[
[T_1(z), S^+_i(w)] = g_i \left(\frac{q_i}{p_i} - 1\right) : \Lambda_i(z)S^+_i(w) : \delta\left(\frac{q_i w}{z}\right) + g_{i+1} \left(\frac{q'_i}{p'_i} - 1\right) : \Lambda_{i+1}(z)S^+_i(w) : \delta\left(\frac{q'_i w}{z}\right) \quad (i = 1, 2).
\]

Here we used the formal delta-function \( \delta(z) \) defined in (2.4). We assume the following ansatz.

**Ansatz** We assume that the first generating function \( T_1(z) \) of the deformed W-superalgebra \( \mathcal{W}_{q,t}(\mathfrak{sl}(2|1)) \) commutes with the deformed screening currents \( S^+_i(w) \) up-to total difference.

\[
[T_1(z), S^+_i(w)] = g_i \left(\frac{q_i}{p_i} - 1\right) : \Lambda_i(z)S^+_i(w) : \left(\delta\left(\frac{q_i w}{z}\right) - \delta\left(\frac{q'_i w}{z}\right)\right) \quad (i = 1, 2).
\]
From the above ansatz (2.18), we will determine bosonization of the screening currents $S_i^+(z)$ and first generating function $T_1(z) = g_1\Lambda_1(z) + g_2\Lambda_2(z) + g_3\Lambda_3(z)$.

## 3 Main theorem

In this Section we summarize the main theorem, which will be proven in Sections 4, 5, and 6. We give a set of quadratic relations of generators of the $W$-superalgebra $\mathcal{W}_{q,t}(\mathfrak{sl}(2|1))$.

We introduce the parameters $x, r, s > 0$ as follows.

\[
\frac{q_1^i}{q_1} = x^{2r}, \quad \beta = \frac{r-1}{r}, \quad q_1 = s.
\]

We will write all parameters $p_i, p_i^j, q_i, q_i^j, A_{i,j}(m), s_i^j(m), \lambda_{i,j}(m), g_i$ by only three parameters $x, r, s$, later.

The parameters $x, r$ are similar as those of the deformed Virasoro algebra in [21] and vertex operators for the SOS model in [22]. The parameters $q, t$ in the symbol $\mathcal{W}_{q,t}(\mathfrak{sl}(2|1))$ are given by $q = x^{2r}, t = x^{2(r-1)}$.

### 3.1 One-fermion diagram

In this Section we focus our attention to the case of one-fermion diagram $\circ \rightarrow \otimes$ and $0 < x < 1$. Define the functions $\Delta_i(z)$ ($i = 0, 1, 2, 3, \cdots$) by

\[
\Delta_i(z) = \frac{(1-x^{2r-i}z)(1-x^{-2r+i}z)}{(1-x^2z)(1-x^{-2}z)}.
\]

Define the structure function $f_{i,j}(z)$ ($i, j = 1, 2, 3, \cdots$) by

\[
f_{i,j}(z) = \exp \left( -\sum_{m=1}^{\infty} \frac{1}{m} [\text{Min}(i,j)m]_x |(r+1-\text{Max}(i,j))m]_x \frac{[(r-1)m]_x[rm]_x}{[m]_x[(r+1)m]_x} (x-x^{-1})^2 z^m \right).
\]

We have

\[
\Delta_i(z) - \Delta_i(1/z) = \frac{[r]_x[r-i]_x}{[i]_x} (x-x^{-1})(\delta(x^{-i}z) - \delta(x^i z)) \quad (i = 1, 2, 3, \cdots),
\]

\[
\Delta_i(z) = \frac{1}{f_{1,i+1}(z)} \prod_{k=1}^{i+1} f_{1,1}(x^{i+2-2k}z) \quad (i = 1, 2, 3, \cdots).
\]

For $0 < x < 1$ we have infinite product formulae as follows.

\[
f_{1,1}(z) = \frac{1}{(1-z)} \left( \frac{x^{2(r-1)}; x^{2(r+1)}}{(x^{4r}; x^{2(r+1)})} \right) \left( \frac{x^{2r}; x^{2(r+1)}}{(x^2; x^{2(r+1)})} \right)^2,
\]

\[
f_{1,j}(z) = \frac{x^{2(r-j+1)}; x^{2(r+1)}}{(x^{j+1}; x^{2(r+1)})} \left( \frac{x^{2(r-j+1)}; x^{2(r+1)}}{(x^{-j}; x^{2(r+1)})} \right) \left( \frac{x^{2r-j+1}z; x^{2(r+1)}}{(x^{-j+3}; x^{2(r+1)})} \right) \left( \frac{x^{2r-j+1}z; x^{2(r+1)}}{(x^{j+3}; x^{2(r+1)})} \right).
\]

The structure functions $f_{1,j}(z)$ ($j = 1, 2, 3, \cdots$) are the very-well-poised basic hypergeometric series [23].

\[
f_{1,j}(x^{-i}z) = 6 \phi_5 \left[ \begin{array}{c} z, x^{2(r+1)} z^{\frac{1}{2}}, -x^{2(r+1)} z^{\frac{1}{2}}, x^{2r}, x^{2(1-r)}, x^{2(r+1-j)}, x^2 z, x^{4r} z, x^{2j} z \end{array} \right].
\]
Ratio of the structure functions $f_{1,j}(z)/f_{1,j}(z^{-1})$ ($j = 1, 2, 3, \cdots$) is ratio of the theta functions.

$$\frac{f_{1,1}(z)}{f_{1,1}(z^{-1})} = -\frac{1}{z} \left( \frac{\Theta_{x^2(r+1)}(x^{4r}z) \left( \frac{\Theta_{x^2(r+1)}(x^{2r}z)}{\Theta_{x^2(r+1)}(x^{2r}z)} \right)^2}{\Theta_{x^2(r+1)}(x^{4r}z)} \right),$$

$$\frac{f_{1,j}(z)}{f_{1,j}(z^{-1})} = \frac{\Theta_{x^2(r+1)}(x^{4j+1}z) \Theta_{x^2(r+1)}(x^{j-1}z) \Theta_{x^2(r+1)}(x^{-j+3}z) \Theta_{x^2(r+1)}(x^{4r-j+1}z)}{\Theta_{x^2(r+1)}(x^{4j+1}z)}.$$  \hspace{1cm} (3.10, 3.11)

Here we used the theta function $\Theta_q(z)$ defined in (2.3).

**Proposition 3.1**  In the case of one-fermion diagram and $0 < x < 1$, the bosonization of $\Lambda_i(z)$ ($i = 1, 2, 3$) given in Theorem [5.1] satisfies the following normal ordering rules.

$$f_{1,1}(z_2/z_1)\Lambda_i(z_1)\Lambda_j(z_2) = \Delta_1(x^{-1}z_2/z_1) : \Lambda_i(z_1)\Lambda_j(z_2) : \quad (1 \leq i < j \leq 3),$$

$$f_{1,1}(z_2/z_1)\Lambda_i(z_1)\Lambda_j(z_2) = \Delta_1(x z_2/z_1) : \Lambda_i(z_1)\Lambda_j(z_2) : \quad (1 \leq i < j \leq 3),$$

$$f_{1,1}(z_2/z_1)\Lambda_i(z_1)\Lambda_j(z_2) = : \Lambda_i(z_1)\Lambda_j(z_2) : \quad (i = 1, 2),$$

$$f_{1,1}(z_2/z_1)\Lambda_3(z_1)\Lambda_3(z_2) = \Delta_2(z_2/z_1) : \Lambda_3(z_1)\Lambda_3(z_2) :.$$ \hspace{1cm} (3.12)

Here we set $\Delta_i(z)$ ($i = 1, 2$) and $f_{1,1}(z)$ in (3.2 and 3.3), respectively.

We introduce the generating functions $T_i(z)$ ($i = 1, 2, 3, \cdots$) associated with one-fermion diagram as follows.

$$T_i(z) = \Lambda_1(z) + \Lambda_2(z) + \frac{[r-1]}{[1]} \Lambda_3(z),$$

$$T_2(z) = : \Lambda_1(x^{-1}z)\Lambda_2(xz) : + \frac{[r-1]}{[1]} : \Lambda_1(x^{-1}z)\Lambda_3(xz) : + \frac{[r-1]}{[1]} : \Lambda_2(x^{-1}z)\Lambda_3(xz) : + \frac{[r-1]}{[1]} : \Lambda_3(x^{-1}z)\Lambda_3(xz) :,$$

$$T_i(z) = \prod_{j=1}^{i-2} \frac{[r-j]}{[j]} \Lambda_1(x^{-i+1}z)\Lambda_2(x^{-i+3}z) \prod_{j=1}^{i-2} \Lambda_3(x^{-i+2j+3}z) : + \frac{[r-1]}{[1]} \Lambda_1(x^{-i+1}z) \prod_{j=1}^{i-2} \Lambda_3(x^{-i+2j+3}z) : + \frac{[r-1]}{[1]} \Lambda_1(x^{-i+1}z) \prod_{j=1}^{i-2} \Lambda_3(x^{-i+2j+3}z) : + \frac{[r-1]}{[1]} \Lambda_1(x^{-i+1}z) \prod_{j=1}^{i-2} \Lambda_3(x^{-i+2j+3}z) : = \Lambda_3(x^{-i+2j+3}z) : \quad (i = 3, 4, 5, \cdots).$$ \hspace{1cm} (3.13)

The following is the main theorem of this paper.

**Theorem 3.2**  In the case of one-fermion diagram and $0 < x < 1$, the generating functions $T_i(z)$ ($i = 1, 2, 3, \cdots$) introduced in (3.13) satisfy a set of the following quadratic relations.

$$f_{i,j}(z_2/z_1)T_i(z_1)T_j(z_2) - f_{j,i}(z_1/z_2)T_j(z_2)T_i(z_1) = \sum_{k=1}^{k-1} \Delta_1(x^{2l+1}) \left( \delta \left( \frac{x^{j+i+2k}z_2}{z_1} \right) f_{i-k,j+k}(x^{j-i})T_{i-k}(x^kz_1)T_{j+k}(x^{-k}z_2) - \delta \left( \frac{x^{j-1+i+2k}z_2}{z_1} \right) f_{i-k,j+k}(x^{-j+i})T_{i-k}(x^{-k}z_1)T_{j+k}(x^kz_2) \right) \quad (j \geq i \geq 1).$$ \hspace{1cm} (3.14)

Here we set $\Delta_1(z)$ and $f_{i,j}(z)$ ($i, j \geq 1$) in (3.3 and 3.3), respectively.
**Definition 3.3** The deformed $W$-superalgebra $W_{sl}(\mathfrak{sl}(2|1))$ associated with one-fermion diagram is an associative algebra over $\mathbb{C}$ with the generators $T_i[m]$ ($m \in \mathbb{Z}, i = 1, 2, 3, \cdots$). The defining relations of the generators are given by the relations (3.14). Here we use the generating functions $T_i(z) = \sum_{m \in \mathbb{Z}} T_i[m] z^{-m}$ ($i = 1, 2, 3, \cdots$).

We note that $W_{sl}(\mathfrak{sl}(2|1))$ is completely different from $W_{sl}(\mathfrak{sl}(N))$. The generating functions $T_i^\text{slN}(z)$ for $W_{sl}(\mathfrak{sl}(N))$ defined in (A.33) satisfy $T_i^\text{slN}(z) = 1$. On the other hand, in the case of generic parameter $r > 0$, the generating functions $T_i(z)$ of $W_{sl}(\mathfrak{sl}(2|1))$ satisfy $T_i(z) \neq 1$ ($i = 1, 2, 3, \cdots$). Degeneration of the generating functions $T_i(z) = 0$ ($i \geq r + 2$) holds, where $r$ is an integer $r \geq 1$. In language of solvable lattice models, it corresponds to the $\mathfrak{sl}(2|1)$ RSOS model [24]. The structure functions $f_{i,j}(z)$ for $W_{sl}(\mathfrak{sl}(2|1))$ are connected with the structure functions $f_{i,j}^\text{slN}(z)$ for $W_{sl}(\mathfrak{sl}(N))$ given in (A.33) as specialization.

$$f_{i,j}(z) = f_{i,j}^\text{slN}(z)\big|_{N \rightarrow r+1}.$$  \hspace{1cm} (3.15)

**3.2 Two-fermion diagram**

In this Section we focus our attention to the case of the two-fermion diagram $\circlearrowright \circlearrowleft$ and $0 < x < 1$.

**Proposition 3.4** In the case of two-fermion diagram and $0 < x < 1$, the bosonization of $\Lambda_i(z)$ ($i = 1, 2, 3$) given in Theorem 3.4 satisfies the following normal ordering rules.

$$f_{1,1}(z_2/z_1)\Lambda_i(z_1)\Lambda_j(z_2) = \Delta_1(x^{-1}z_2/z_1) : \Lambda_i(z_1)\Lambda_j(z_2) : \ (1 \leq i < j \leq 3),$$

$$f_{1,1}(z_2/z_1)\Lambda_j(z_1)\Lambda_i(z_2) = \Delta_1(xz_2/z_1) : \Lambda_j(z_1)\Lambda_i(z_2) : \ (1 \leq i < j \leq 3),$$

$$f_{1,1}(z_2/z_1)\Lambda_i(z_1)\Lambda_i(z_2) = : \Lambda_i(z_1)\Lambda_i(z_2) : \ (i = 1, 3),$$

$$f_{1,1}(z_2/z_1)\Lambda_i(z_1)\Lambda_2(z_2) = \Delta_2(z_2/z_1) : \Lambda_2(z_1)\Lambda_2(z_2) : . \hspace{1cm} (3.16)$$

Here we set $\Delta_i(z)$ ($i = 1, 2$) and $f_{1,1}(z)$ in (3.3) and (3.3), respectively.

We introduce the generating functions $T_i(z)$ ($i = 1, 2, 3, \cdots$) associated with two-fermion diagram as follows.

$$T_1(z) = \Lambda_1(z) + \frac{[r-1]_x}{[1]_x} \Lambda_2(z) + \Lambda_3(z),$$

$$T_2(z) = : \Lambda_1(x^{-1}z)\Lambda_3(xz) : + \frac{[r-1]_x}{[1]_x} : \Lambda_1(x^{-1}z)\Lambda_2(xz) : + \frac{[r-1]_x}{[1]_x} : \Lambda_2(x^{-1}z)\Lambda_3(xz) : +$$

$$+ \frac{[r-1]_x[r-2]_x}{[1]_x[r]_x} : \Lambda_2(x^{-1}z)\Lambda_2(xz) :,$$

$$T_i(z) = \prod_{j=1}^{i-2} \frac{[r-j]_x}{[j]_x} : \Lambda_1(x^{-i+1}z) \prod_{j=1}^{i-2} \Lambda_2(x^{-i+j+1}z) \Lambda_3(x^{i-1}z) :$$

$$+ \prod_{j=1}^{i-1} \frac{[r-j]_x}{[j]_x} : \Lambda_1(x^{-i+1}z) \prod_{j=1}^{i-1} \Lambda_2(x^{-i+j+1}z) : + \prod_{j=1}^{i-1} \frac{[r-j]_x}{[j]_x} \prod_{j=1}^{i-1} \Lambda_2(x^{-i+j-1}z) \Lambda_3(x^{i-1}z) :$$

$$+ \prod_{j=1}^{i} \frac{[r-j]_x}{[j]_x} : \prod_{j=1}^{i} \Lambda_2(x^{-i+j-1}z) : \ (i = 3, 4, 5, \cdots). \hspace{1cm} (3.17)$$
Theorem 3.5  In the case of two-fermion diagram and 0 < x < 1, the generating functions $T_i(z)$ ($i = 1, 2, 3, \ldots$) defined in (3.14) satisfy the same set of the quadratic relations (3.14) with the same structure functions $f_{i,j}(z)$ given in (3.14).

The deformed $W$-superalgebra $W_{q^2}(\mathfrak{sl}(2|1))$ associated with two-fermion diagram is the same as those associated with one-fermion diagram. We checked that the quadratic relations of the deformed $W$-superalgebra $W_{q^2}(\mathfrak{sl}(2|1))$ are independent of the choice of a Dynkin-diagram, even though the deformed screening currents depend on it.

4 Deformed screening current

In this Section we revisit bosonization of the deformed screening currents, following Ding and Feigin’s approach [1].

4.1 Quantum difference equation

In this Section we study $q$-difference equations of the structure functions $h_{i,j}(z)$ ($i, j = 1, 2$) defined in (2.10). In this Section we determine parameters $q_i, p_i, q_i^{'}, p_i^{'}$, $A_{i,j}(m)$, $s_i^{+}(m)$, $g_i$. We start from the ansatz (2.18), which is equivalent to the following two relations.

\[ q_i^{A_{i,i}(0)} : \Lambda_i(z)S_i^{+}(z/q_i) := (q_i^{'})^{A_{i,i}(0)} : \Lambda_{i+1}(z)S_i^{+}(z/q_i^{'}) : \quad (i = 1, 2), \quad \text{(4.1)} \]

\[ \frac{g_{i+1}}{g_i} = -\left(\frac{q_i^{'}}{q_i}\right)^{A_{i,i}(0)} \left(\frac{p_i}{p_i^{'}} - 1\right) \left(\frac{q_i^{'}}{q_i} - 1\right) \quad (i = 1, 2). \quad \text{(4.2)} \]

First, we derive a set of $q$-difference equations of $h_{i,i}(z)$ ($i = 1, 2$). Multiplying $S_i^{+}(w)$ on (4.1) from left, we obtain

\[ \frac{1 - z/p_i w}{1 - z/q_i w} h_{i,i}(w) = \frac{1 - z/p_i^{'} w}{1 - z/q_i^{'} w} h_{i,i}(w) \quad (i = 1, 2). \quad \text{(4.3)} \]

Multiplying $S_i^{+}(w)$ on (4.1) from right, we obtain

\[ \left(\frac{q_i}{q_i^{'}}\right)^{A_{i,i}(0)-1} \frac{p_i}{p_i^{'}} \frac{1 - p_i w/z}{1 - q_i w/z} h_{i,i}(w) = \frac{1 - p_i^{'} w/z}{1 - q_i^{'} w/z} h_{i,i}(w) \quad (i = 1, 2). \quad \text{(4.4)} \]

A set of $q$-difference equations (4.3) and (4.4) are equivalent to the following equations.

\[ \frac{p_i}{p_i^{'}} \left(\frac{q_i^{'}}{q_i}\right)^{A_{i,i}(0)-1} = 1, \quad (p_i - p_i^{'})(p_i p_i^{'} - q_i q_i^{'}) = 0 \quad (i = 1, 2), \quad \text{(4.5)} \]

\[ \frac{1 - p_i z}{1 - q_i z} h_{i,i}(w) = \frac{1 - p_i^{'} z}{1 - q_i^{'} z} h_{i,i}(w) \quad (i = 1, 2). \quad \text{(4.6)} \]

Solving equations (4.5) and (4.6), we have the following Propositions.

Proposition 4.1  For $i$ such that $A_{i,i}(0) \neq 1$, we have

\[ p_i = q_i \left(\frac{q_i^{'}}{q_i^{'}}\right)^{A_{i,i}(0)}, \quad p_i^{'} = q_i^{'} \left(\frac{q_i}{q_i^{'}}\right)^{A_{i,i}(0)}, \quad \text{(4.7)} \]

\[ h_{i,i}(z) = \exp \left( - \sum_{m=1}^{\infty} \frac{1}{m} \frac{(1 - (q_i^{'} / q_i))(1 + (q_i^{'} / q_i)(1 - q_i^{'} / q_i)^m)}{1 - (q_i^{'} / q_i)^m} z^m \right). \]
Proposition 4.2 For $i$ such that $A_{i,i}(0) = 1$, we have

$$p_i = p_i', \quad h_{i,i}(z) = (1 - z).$$

(4.8)

Next, we study $q$-difference equations of $h_{1,2}(z)$ and $h_{2,1}(z)$. For $1 \leq i \neq j \leq 2$ we multiply $S_j^+(w)$ on (4.1) from left or right. Then we have the following $q$-difference equations.

$$\frac{h_{1,2}(q_1 z)}{h_{1,2}(q_1' z)} = \frac{q_2}{p_2} \left( \frac{q_1}{q_1'} \right)^{A_{1,2}(0)} \frac{(1 - p_2 z)}{(1 - q_2 z)},$$

(4.9)

and

$$\frac{h_{1,2}(z/q_2)}{h_{1,2}(z/q_2')} = \frac{1 - z/p_2}{(1 - z/q_2')}.$$

(4.10)

Upon the specialization:

$$\frac{q_2}{p_2} \left( \frac{q_1}{q_1'} \right)^{A_{1,2}(0)} = 1,$$

(4.13)

we obtain solutions of the $q$-difference equations (4.9) and (4.10):

$$h_{1,2}(z) = \exp \left( - \sum_{m=1}^{\infty} \frac{1}{m} \frac{(p_2/q_1)^m - (q_2/q_1)^m}{1 - (q_1'/q_1)^m} z^m \right),$$

(4.14)

$$h_{1,2}(z) = \exp \left( - \sum_{m=1}^{\infty} \frac{1}{m} \frac{(q_2'/q_2)^m - (q_2'/q_1)^m}{1 - (q_2'/q_2)^m} z^m \right).$$

(4.15)

Upon the specialization:

$$\frac{q_1'}{p_1'} \left( \frac{q_2'}{q_2} \right)^{A_{2,1}(0)} = 1,$$

(4.16)

we obtain solutions of the $q$-difference equations (4.11) and (4.12):

$$h_{2,1}(z) = \exp \left( - \sum_{m=1}^{\infty} \frac{1}{m} \frac{(q_1'/q_2)^m - (q_1'/q_1)^m}{1 - (q_1'/q_1)^m} z^m \right),$$

(4.17)

$$h_{2,1}(z) = \exp \left( - \sum_{m=1}^{\infty} \frac{1}{m} \frac{(q_2'/q_1)^m - (p_1'/q_2)^m}{1 - (q_2'/q_2)^m} z^m \right).$$

(4.18)

From compatibility of two formulæ of $h_{1,2}(z)$ in (4.14) and (4.15) (or $h_{2,1}(z)$ in (4.17) and (4.18)), there are two possibilities of choice of $q_1, q_2, q_1', q_2'$:

(i) $\frac{q_1'}{q_1} = \frac{q_2'}{q_2}$ or (ii) $\frac{q_1'}{q_1} = \frac{q_2}{q_2'}$.

(4.19)

First, we study the case of (i) $\frac{q_1'}{q_1} = \frac{q_2'}{q_2}$. From compatibility of two formulæ of $h_{1,2}(z)$ in (4.14) and (4.16), we have

$$p_2 = q_2 \left( \frac{q_1'}{q_1} \right)^{-A_{1,2}(0)} \quad (i = 1, 2),$$

$$h_{1,2}(z) = \exp \left( - \sum_{m=1}^{\infty} \frac{1}{m} \frac{(q_1'/q_1)^{A_{1,2}(0)m} - (q_2'/q_1)^m z^m}{1 - (q_1'/q_1)^m} \right) \quad (i = 1, 2).$$

(4.20)
From compatibility of two formulae of \( h_{2,1}(z) \) in (4.17) and (4.18), we have
\[
p_i' = q_i \left( \frac{q_i'}{q_i} \right)^{1+A_{2,1}(0)} (i = 1, 2),
\]
\[
h_{2,1}(z) = \exp \left( -\sum_{m=1}^{\infty} \frac{1}{m} \frac{(q_i'/q_i)^m - (q_i'/q_i)^{(1+A_{2,1}(0))m}}{1 - (q_i'/q_i)^m} (q_1/q_2)^m z^m \right) (i = 1, 2). \tag{4.21}
\]

We have necessary and sufficient conditions of \( h_{1,2}(z) = h_{2,1}(z) \) as follows.
\[
h_{1,2}(z) = h_{2,1}(z) \iff q_2 = \left( \frac{q_1}{q_i} \right)^{\frac{1}{2}(1+A_{1,2}(0))} (i = 1, 2). \tag{4.22}
\]

Hence, we have the following Propositions.

**Proposition 4.3** In the case of \( \frac{q_1}{q_i} = \frac{q_2}{q_i} \), we have
\[
p_i' = q_i \left( \frac{q_i'}{q_i} \right)^{1+A_{2,1}(0)}, \quad p_2 = q_2 \left( \frac{q_i'}{q_i} \right)^{-A_{1,2}(0)}, \quad q_2 = \left( \frac{q_i'}{q_i} \right)^{\frac{1}{2}(1+A_{1,2}(0))} (i = 1, 2),
\]
\[
h_{1,2}(z) = h_{2,1}(z) \exp \left( -\sum_{m=1}^{\infty} \frac{1}{m} \frac{(q_i'/q_i)^m - (q_i'/q_i)^{(1+A_{1,2}(0))m}}{1 - (q_i'/q_i)^m} (q_1/q_2)^m z^m \right) (i = 1, 2). \tag{4.23}
\]

Next, we study the case of (ii) \( \frac{q_1}{q_i} = \frac{q_2}{q_i} \). From compatibility of two formulae of \( h_{1,2}(z) \) in (4.14) and (4.15), we have
\[
q_1 = q_2 = \frac{q_2}{q_i}, \quad p_1 = q_1, \quad p_2 = q_2, \quad A_{1,2}(0) = A_{2,1}(0) = 1,
\]
\[
h_{1,2}(z) = (1 - q_2 z/q_1). \tag{4.24}
\]

From compatibility of two formulae of \( h_{2,1}(z) \) in (4.17) and (4.18), we have
\[
q_1 = q_2 = \frac{q_2}{q_i}, \quad p_1 = q_1, \quad p_2 = q_2, \quad A_{1,2}(0) = A_{2,1}(0) = 1,
\]
\[
h_{2,1}(z) = (1 - q_2 z/q_2). \tag{4.25}
\]

Imposing the condition \( h_{1,2}(z) = h_{2,1}(z) \), we have the following Proposition.

**Proposition 4.4** In the case of \( \frac{q_1}{q_i} = \frac{q_2}{q_i} \), we have
\[
q_1 = q_2 = p_1 = p_2, \quad q_1' = q_2, \quad A_{1,2}(0) = A_{2,1}(0) = 1,
\]
\[
h_{1,2}(z) = h_{2,1}(z) = (1 - z). \tag{4.26}
\]

We assume \( A_{2,2}(0) = 1 \) in the case of both one-fermion diagram and two-fermion diagram. Hence, we have \( q_2 = p_2' \) from Proposition 4.2 and 4.4. However it contradicts with the first assumption of parameters \( q_2 \neq p_2' \) in (2.17). Hence, in what follows, we study only the case (i) \( \frac{q_1}{q_i} = \frac{q_2}{q_i} \).

### 4.2 One-fermion diagram

In this Section we study the case of one-fermion diagram \( \bigcirc \rightarrow \bullet \). In what follows we study the case (i) \( \frac{q_1}{q_i} = \frac{q_2}{q_i} = x^{2r} \). From Proposition 4.1, 4.2, and 4.3 we have the following Propositions.
Proposition 4.5 In the case of one-fermion diagram, parameters \( q, q', p, p', A_{i,j}(0) \) \((i,j = 1,2)\) and \( g_i \) \((i = 1,2,3)\) are parametrized by \( x, r, s \) as follows.

\[
\begin{align*}
A_{1,1}(0) &= \frac{2(r-1)}{r}, \quad A_{2,2}(0) = 1, \quad A_{1,2}(0) = A_{2,1}(0) = \frac{1-r}{r}, \\
q_1 &= s, \quad q_1' = sx^{2r}, \quad p_1 = sx^{2(r-1)}, \quad p_1' = sx^2, \\
q_2 &= sx, \quad q_2' = sx^{2r+1}, \quad p_2 = sx^{2r-1}, \quad p_2' = sx^{2r-1}, \\
g_2 &= g_1, \quad g_3 = [r-1]x^g_2.
\end{align*}
\]

(4.27)

Proposition 4.6 In the case of one-fermion diagram, the structure functions \( h_{i,j}(z) \) \((i,j = 1,2)\) introduced in (4.10) are given by

\[
\begin{align*}
h_{1,1}(z) &= \exp \left( -\sum_{m=1}^{\infty} \frac{1}{m} \frac{[(r-1)m]_x[2m]_x}{[rm]_x[m]_x} z^m \right), \\
h_{2,2}(z) &= (1-z), \\
h_{1,2}(z) &= h_{2,1}(z) = \exp \left( \sum_{m=1}^{\infty} \frac{1}{m} \frac{[(r-1)m]_x}{[rm]_x} z^m \right). 
\end{align*}
\]

(4.28)

We study commutation relation of the Heisenberg algebra. To calculate normal orderings of \( S_j^\pm(z) \) and \( A_i(z) \), we use the following dressed commutation relations.

\[
[s_i^+(m)a_i(m), s_j^+(n)a_j(n)] = \frac{1}{m} B_{i,j}(m) \delta_{m+n,0}, \quad B_{i,j}(m) = s_i^+(m)A_{i,j}(m)s_j^+(m).
\]

(4.29)

By definitions of \( A_{i,j}(m) \) and \( B_{i,j}(m) \), we have

\[
B_{1,2}(m) = B_{2,1}(m) \iff A_{1,2}(m) = A_{2,1}(m) \frac{s_1^+(m)s_2^+(m)}{s_1^+(m)s_2^+(m)}, \\
B_{1,2}(m) = B_{2,1}(-m) \iff A_{1,2}(m) = A_{2,1}(-m).
\]

(4.30)

Proposition 4.7 In the case of one-fermion diagram, \( B_{i,j}(m) \) \((i,j = 1,2, m \neq 0)\) in (4.28) are given by

\[
\begin{align*}
B_{1,1}(m) &= -\frac{[(r-1)m]_x[2m]_x}{[rm]_x[m]_x}, \quad B_{2,2}(m) = -1, \\
B_{1,2}(m) &= B_{2,1}(m) = \frac{[(r-1)m]_x}{[rm]_x}.
\end{align*}
\]

(4.31)

From the explicit formulae above we have \( B_{1,2}(m) = B_{1,2}(-m) = B_{2,1}(m) = B_{2,1}(-m) \). Hence, we have \( A_{1,2}(m) = A_{2,1}(-m) \). \( B_{i,j}(m) \) is uniquely determined, though \( A_{i,j}(m) \) and \( s_i^+(m) \) are not determined uniquely. There is ambiguity for choice of \( A_{i,j}(m) \) and \( s_i^+(m) \). For instance, we have a solution as follows.

For \( m \neq 0 \), we have

\[
\begin{align*}
A_{1,1}(m) &= A_{2,2}(m) = 1, \\
A_{1,2}(m) &= -x^{-m} \sqrt{\frac{[(r-1)m]_x[2m]_x}{[rm]_x[m]_x}}, \quad A_{2,1}(m) = -x^{-m} \sqrt{\frac{[(r-1)m]_x[2m]_x}{[rm]_x[m]_x}}.
\end{align*}
\]

(4.32)

For \( m > 0 \), we have

\[
\begin{align*}
s_1^+(m) &= \sqrt{\frac{[(r-1)m]_x[2m]_x}{[rm]_x[m]_x}}, \quad s_1^+(m) = - \sqrt{\frac{[(r-1)m]_x[2m]_x}{[rm]_x[m]_x}}, \\
s_2^+(m) &= x^{-m}, \quad s_2^+(m) = -x^{-m}.
\end{align*}
\]

(4.33)
Proposition 4.8 In the case of one-fermion diagram, a bosonization of the deformed screening currents $S_i^+(z)$ ($i = 1, 2$) introduced in (2.5) is given by $A_{i,j}(m)$ ($i, j = 1, 2, m \in \mathbb{Z}$) in (4.27), (4.32), and $s_i^+(m)$ ($i = 1, 2, m \in \mathbb{Z}_{\neq 0}$) in (4.33).

Here we focus our attention to the regime $0 < x < 1$.

Proposition 4.9 In the case of one-fermion diagram and $0 < x < 1$, the deformed screening currents $S_i^+(z)$ introduced in (2.5) satisfy the following commutation relations.

\[
S_i^+(z_1)S_i^+(z_2) = \frac{[u_1 - u_2 - 1]}{[u_1 - u_2 + 1]}S_i^+(z_2)S_i^+(z_1),
\]

\[
S_1^+(z_1)S_2^+(z_2) = -S_2^+(z_2)S_1^+(z_1),
\]

\[
S_i^+(z_1)S_j^+(z_2) = -\frac{[u_1 - u_2 + 1]}{[u_1 - u_2 - 2]}S_j^+(z_2)S_i^+(z_1),
\]

where we set $z_i = x^{2u_i}$ ($i = 1, 2$). Here we used the theta function $[u]$ defined in (2.3).

4.3 Two-fermion diagram

In this Section we study the case of two-fermion diagram $\bigotimes \bigotimes$. In what follows we study the case (i) $\frac{q'_1}{q_1} = \frac{q'_2}{q_2} = x^{2r}$. From Proposition 4.1, 4.2, and 4.3 we have the following Propositions.

Proposition 4.10 In the case of two-fermion diagram, parameters $q_i, q'_i, p_i, p'_i, A_{i,j}(0)$ ($i, j = 1, 2$) and, $g_i$ ($i = 1, 2, 3$) are parametrized by $x, r, s$ as follows.

\[
A_{1,1}(0) = A_{2,2}(0) = 1, \quad A_{1,2}(0) = A_{2,1}(0) = -\frac{1}{r},
\]

\[
q_1 = s, \quad q'_1 = sx^{2r}, \quad p_1 = p'_1 = sx^{2(r-1)},
\]

\[
q_2 = sx^{r-1}, \quad q'_2 = sx^{3r-1}, \quad p_2 = p'_2 = sx^{r+1},
\]

\[
g_2 = [r - 1]_x g_1, \quad g_3 = \frac{1}{[r - 1]_x} g_2.
\]

Proposition 4.11 In the case of two-fermion diagram, the structure functions $h_{i,j}(z)$ ($i, j = 1, 2$) introduced in (2.10) are given by

\[
h_{1,1}(z) = h_{2,2}(z) = (1 - z),
\]

\[
h_{1,2}(z) = h_{2,1}(z) = \exp \left( \sum_{m=1}^{\infty} \frac{1}{m} \frac{|m|_x}{|r_m|_x} z^m \right).
\]

Proposition 4.12 In the case of two-fermion diagram, $B_{i,j}(m)$ ($i, j = 1, 2, m \neq 0$) defined in (4.29) are given by

\[
B_{1,1}(m) = B_{2,2}(m) = -1,
\]

\[
B_{1,2}(m) = B_{2,1}(m) = \frac{|m|_x}{|r_m|_x}.
\]

From the explicit formulae above we have $B_{1,2}(m) = B_{1,2}(-m) = B_{2,1}(m) = B_{2,1}(-m)$. Hence, we have $A_{1,2}(m) = A_{2,1}(-m)$. $B_{i,j}(m)$ is uniquely determined, though $A_{i,j}(m)$ and $s_i^+(m)$ are not determined
uniquely. There is ambiguity for choice of $A_{i,j}(m)$ and $s_i^+(m)$. For instance, we have a solution as follows. For $m \neq 0$ we have

\begin{align*}
A_{1,1}(m) &= A_{2,2}(m) = 1, \\
A_{1,2}(m) &= -x^{-2m} \frac{[m]_x}{[rm]_x}, \quad A_{2,1}(m) = -x^{2m} \frac{[m]_x}{[rm]_x}. \tag{4.38}
\end{align*}

For $m > 0$ we have

\begin{align*}
s_1^+(m) &= x^m, \quad s_1^+(-m) = -x^{-m}, \quad s_2^+(m) = x^{-m}, \quad s_2^+(-m) = -x^m. \tag{4.39}
\end{align*}

**Proposition 4.13** In the case of two-fermion diagram, a bosonization of the deformed screening currents $S_i^+(z) \ (i = 1, 2)$ introduced in (2.8) is given by $A_{i,j}(m) \ (i, j = 1, 2, m \in \mathbb{Z})$ in (4.39), (4.38), and $s_i^+(m) \ (i = 1, 2, m \in \mathbb{Z}_{\neq 0})$ in (4.39).

Here we focus our attention to the regime $0 < x < 1$.

**Proposition 4.14** In the case of two-fermion diagram and $0 < x < 1$, the deformed screening currents $S_i^+(z)$ introduced in (2.8) satisfy the following commutation relations.

\begin{align*}
S_i^+(z_1)S_i^+(z_2) &= -S_i^+(z_2)S_i^+(z_1) \quad (i = 1, 2), \\
S_i^+(z_1)S_i^+(z_2) &= \frac{[u_l - u_2]}{[u_l - u_2 + r + 1]} S_i^+(z_2)S_i^+(z_1), \tag{4.40}
\end{align*}

where we set $z_i = x^{u_i} \ (i = 1, 2)$. Here we used the theta function $[u]$ defined in (2.3).

The deformed screening currents associated with two-fermion diagram are completely different from those associated with one-fermion diagram.

## 5 Deformed W-superalgebra $\mathcal{W}_{ql}(\mathfrak{sl}(2|1))$

In this Section we construct a bosonization of the first generating function $T_i(z)$ of the deformed W-superalgebra $\mathcal{W}_{ql}(\mathfrak{sl}(2|1))$.

### 5.1 One-fermion diagram

In this Section we study the case of one-fermion diagram $\overline{\mathbb{O}} \rightarrow \overline{\mathbb{O}}$. We consider the assumption (2.16) upon the parametrization given in Section 4.2. In the case of one-fermion diagram, the assumptions (2.16) are equivalent to the following set of equations. Parameters $\lambda_{i,j}(m) \ (m \neq 0)$ introduced in (2.15) are determined by

\begin{align*}
\begin{pmatrix}
\frac{[u_l - u_2]}{[u_l - u_2 + r + 1]} S_i^+(z_2)S_i^+(z_1) \\
-1
\end{pmatrix}
= \frac{1}{s^m(x - x^{-1})}
\begin{pmatrix}
-x^{(r-1)m}[(r-1)m]_x & x^{(r+1)m}[(r-1)m]_x & 0 \\
0 & -x^{rm}[(r-1)m]_x & x^{2rm}[m]_x
\end{pmatrix}
\begin{pmatrix}
\lambda_{1,1}(m) & \lambda_{2,1}(m) & \lambda_{3,1}(m) \\
\lambda_{1,2}(m) & \lambda_{2,2}(m) & \lambda_{3,2}(m)
\end{pmatrix}
(m \neq 0). \tag{5.1}
\end{align*}
Because of $A_{1,2}(m) = A_{2,1}(-m)$, we have unified equations for both $m > 0$ and $m < 0$.

Parameters $\lambda_{i,j}(0)$ are determined by

$$
\begin{pmatrix}
\frac{2(r-1)}{r} & \frac{1-r}{r} & 1 \\
\frac{1-r}{r} & \frac{1}{r} & 1 \\
1 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
\lambda_{1,1}(0) \\
\lambda_{2,1}(0) \\
\lambda_{1,2}(0)
\end{pmatrix}
= 2 \log x
\begin{pmatrix}
(1-r) & (r-1) & 0 \\
0 & (1-r) & 1
\end{pmatrix}.
$$

(5.2)

Solving the above equations we have the following Theorem.

**Theorem 5.1** In the case of one-fermion diagram, a bosonization of $\Lambda_i(z)$ ($i = 1, 2, 3$) introduced in [2, 15] is given as follows. Parameters $\lambda_{i,j}(m)$ ($i = 1, 2, 3, j = 1, 2, m \in \mathbb{Z}_{>0}$) are given by

$$
\begin{align*}
\lambda_{1,1}(m) &= s^{m_x} [(r+1)m]_x (x - x^{-1}), \\
\lambda_{1,2}(m) &= s^{m_x} [(r+1)m]_x (x - x^{-1}), \\
\lambda_{2,1}(m) &= -s^{m_x} [2r] [(r+1)m]_x (x - x^{-1}), \\
\lambda_{2,2}(m) &= s^{m_x} [(r+1)m]_x (x - x^{-1}), \\
\lambda_{3,1}(m) &= -s^{m_x} [2r] [(r+1)m]_x (x - x^{-1}), \\
\lambda_{3,2}(m) &= -s^{m_x} [2r] [(r+1)m]_x (x - x^{-1}).
\end{align*}
$$

(5.3)

Parameters $\lambda_{i,j}(0)$ ($i = 1, 2, 3, j = 1, 2$) are given by

$$
\begin{align*}
\lambda_{1,1}(0) &= \frac{2r^2}{r+1} \log x, \\
\lambda_{1,2}(0) &= \frac{-2r(r-1)}{r+1} \log x, \\
\lambda_{2,1}(0) &= \frac{2r}{r+1} \log x, \\
\lambda_{2,2}(0) &= \frac{2r(r-1)}{r+1} \log x, \\
\lambda_{3,1}(0) &= \frac{2r}{r+1} \log x, \\
\lambda_{3,2}(0) &= \frac{-2r}{r+1} \log x.
\end{align*}
$$

(5.4)

A set of solutions of $A_{i,j}(m)$ ($i, j = 1, 2, m \in \mathbb{Z}$) and $s^+_i(m)$ ($i = 1, 2, m \in \mathbb{Z}_{>0}$) is given in (4.27), (4.32), and (4.33). From the bosonizations of $\Lambda_i(z)$ and $S^+_i(w)$, we have the following Propositions.

**Proposition 5.2** In the case of one-fermion diagram, the first generating function $T_i(z)$ commutes with the deformed screening currents $S^+_i(w)$ up-to total difference.

$$
[T_i(z), S^+_i(w)] = g_i \left( x^{2(1-r)} - 1 \right) : \Lambda_i(z) S^+_i(w) : 
\left( \delta \left( \frac{sx^{i-1}}{z} \right) - \delta \left( \frac{sx^{2r+i-1}}{z} \right) \right) (i = 1, 2).
$$

(5.5)

**Proposition 5.3** In the case of one-fermion diagram, the normal ordering rules of $\Lambda_i(z)$ ($i = 1, 2, 3$) are given as follows.

$$
\begin{align*}
\Lambda_i(z_1) \Lambda_i(z_2) &= \Lambda_i(z_1) \Lambda_i(z_2) : \\
\times \ exp \left( \sum_{m=1}^{\infty} \frac{1}{m} \left[ (r+1)m \right]_x [r]_x (z_2/z_1)^m \right) (i = 1, 2),
\end{align*}
$$

$$
\Lambda_3(z_1) \Lambda_3(z_2) &= \Lambda_3(z_1) \Lambda_3(z_2) : \\
\times \ exp \left( \sum_{m=1}^{\infty} \frac{1}{m} \left[ 2m \right]_x [r]_x (z_2/z_1)^m \right),
$$

(5.6)

$$
\Lambda_i(z_1) \Lambda_j(z_2) &= \Lambda_i(z_1) \Lambda_j(z_2) : \\
\times \ exp \left( -\sum_{m=1}^{\infty} \frac{1}{m} \left[ (r-1)m \right]_x (z_2/z_1)^m (x - x^{-1})^m \right) (1 \leq i < j \leq 3),
$$

$$
\Lambda_j(z_1) \Lambda_i(z_2) &= \Lambda_j(z_1) \Lambda_i(z_2) : \\
\times \ exp \left( -\sum_{m=1}^{\infty} \frac{1}{m} \left[ (r-1)m \right]_x (z_2/z_1)^m (x - x^{-1})^m \right) (1 \leq i < j \leq 3).
$$
5.2 Two-fermion diagram

In this Section we study the case of two-fermion diagram $\otimes \otimes$. We consider the assumption $\ref{2.16}$ upon the parametrization given in Section 4. In the case of two-fermion diagram, the assumptions $\ref{2.16}$ are equivalent to the following set of equations. Parameters $\lambda_{i,j}(m) \ (m \neq 0)$ introduced in $\ref{2.15}$ are determined by

$$
\begin{pmatrix}
-1 & \frac{m_x}{[rm]_x} \\
\frac{m_x}{[rm]_x} & -1
\end{pmatrix}
\begin{pmatrix}
\lambda_{1,1}(m) & \lambda_{1,2}(m) & \lambda_{3,1}(m) \\
\lambda_{1,2}(m) & \lambda_{2,2}(m) & \lambda_{3,2}(m)
\end{pmatrix}
\begin{pmatrix}
(1 - r) & 1 & 0 \\
0 & -1 & (r - 1)
\end{pmatrix}
$$

Because of $A_{1,2}(m) = A_{2,1}(-m)$, we have unified equations for both $m > 0$ and $m < 0$.

Parameters $\lambda_{i,j}(0)$ are determined by

$$
\begin{pmatrix}
1 & -\frac{1}{r} \\
-\frac{1}{r} & 1
\end{pmatrix}
\begin{pmatrix}
\lambda_{1,1}(0) & \lambda_{1,2}(0) & \lambda_{3,1}(0) \\
\lambda_{1,2}(0) & \lambda_{2,2}(0) & \lambda_{3,2}(0)
\end{pmatrix}
= 2 \log x
$$

Solving the above equations, we have the following Theorem.

**Theorem 5.4** In the case of two-fermion diagram, a bosonization of $\Lambda_i(z) \ (i = 1, 2, 3)$ introduced in $\ref{2.15}$ is given as follows. Parameters $\lambda_{i,j}(m) \ (i = 1, 2, 3, j = 1, 2, m \in \mathbb{Z}_{\neq 0})$ are given by

$$
\lambda_{1,1}(m) = s m x^{(r-1)m} \frac{[rm]_x^2}{[r(r+1)]_x} (x - x^{-1}), \quad \lambda_{1,2}(m) = s m x^{(r-1)m} \frac{[rm]_x^2}{[r(r+1)]_x} (x - x^{-1}),
$$

$$
\lambda_{2,1}(m) = -s m x^{2rm} \frac{[rm]_x^2}{[r(r+1)]_x} (x - x^{-1}), \quad \lambda_{2,2}(m) = s m x^{(r-1)m} \frac{[rm]_x^2}{[r(r+1)]_x} (x - x^{-1}),
$$

$$
\lambda_{3,1}(m) = -s m x^{2rm} \frac{[rm]_x^2}{[r(r+1)]_x} (x - x^{-1}), \quad \lambda_{3,2}(m) = -s m x^{2rm} \frac{[rm]_x^2}{[r(r+1)]_x} (x - x^{-1}).
$$

Parameters $\lambda_{i,j}(0) \ (i = 1, 2, 3, j = 1, 2)$ are given by

$$
\lambda_{1,1}(0) = -\frac{2r^2}{r+1} \log x, \quad \lambda_{1,2}(0) = -\frac{2r^2}{r+1} \log x,
$$

$$
\lambda_{2,1}(0) = \frac{2r}{r+1} \log x, \quad \lambda_{2,2}(0) = -\frac{2r}{r+1} \log x,
$$

$$
\lambda_{3,1}(0) = \frac{2r}{r+1} \log x, \quad \lambda_{3,2}(0) = \frac{2r^2}{r+1} \log x.
$$

A set of solutions of $A_{i,j}(m) \ (i, j = 1, 2, m \in \mathbb{Z})$ and $s_{i}^{\uparrow}(m) \ (i = 1, 2, m \in \mathbb{Z}_{\neq 0})$ is given in $\ref{4.35}$, $\ref{4.38}$, and $\ref{1.30}$. From the bosonization of $\Lambda_i(z)$ and $S_{i}^{\uparrow}(w)$, we have the following Propositions.

**Proposition 5.5** In the case of two-fermion diagram, the first generating function $T_1(z)$ commutes with the deformed screening currents $S_{i}^{\uparrow}(w)$ up to total difference.

$$
[T_1(z), S_{i}^{\uparrow}(w)] = g_i \left( x^{2(r-2)(i-2)-2} - 1 \right) : \Lambda_i(z) S_{i}^{\uparrow}(w) :
$$

$$
\times \left( \delta \left( \frac{s_{2}^{r-1}(i-1)w}{z} \right) - \delta \left( \frac{s_{2}^{2r+1}(i-1)w}{z} \right) \right) \ (i = 1, 2).
$$
Proposition 5.6 In the case of two-fermion diagram, the normal ordering rules of $\Lambda_i(z)$ ($i = 1, 2, 3$) are given as follows.

$$
\Lambda_i(z_1)\Lambda_i(z_2) =: \Lambda_i(z_1)\Lambda_i(z_2) :
\times \exp \left( \sum_{m=1}^{\infty} \frac{[r-1]}{m} [(r+1)m]_x \right) (i = 1, 3),
\Lambda_2(z_1)\Lambda_2(z_2) =: \Lambda_2(z_1)\Lambda_2(z_2) :
\times \exp \left( \sum_{m=1}^{\infty} \frac{[2m]}{m} [(r+1)m]_x \right) (i = 1, 3),
\Lambda_i(z_1)\Lambda_j(z_2) =: \Lambda_i(z_1)\Lambda_j(z_2) :
\times \exp \left( - \sum_{m=1}^{\infty} \frac{1}{m} [(r-1)m]_x \right) (1 \leq i < j \leq 3),
\Lambda_j(z_1)\Lambda_i(z_2) =: \Lambda_j(z_1)\Lambda_i(z_2) :
\times \exp \left( - \sum_{m=1}^{\infty} \frac{1}{m} [(r+1)m]_x \right) (1 \leq i < j \leq 3). \quad (5.12)
$$

6 Proof of the main theorem

In this Section we prepare fusion relations, which are used for proof of the main theorem. By direct calculation, we have following Propositions.

Proposition 6.1 The structure functions $f_{i,j}(z)$ defined in (6.2) satisfies the following fusion relations.

$$
f_{i,j}(w) \times f_{i,j}(x^{\mp(j+1)}w) = \begin{cases} 
  f_{j+1,i}(x^{\mp j}w) \Delta_1(x^{\mp j}w) & (1 \leq i \leq j) \\
  f_{j+1,i}(x^{\mp j}w) & (1 \leq j < i)
\end{cases}, \quad (6.1)
$$

$$
f_{i,j}(w) f_{i,j}(x^{\mp(i+j)}w) = f_{i,j}(x^{\mp j}w) \Delta_1(x^{\mp j}w) \quad (i,j \geq 1),
\quad (6.2)
$$

$$
f_{i,j}(w) f_{i,j}(x^{\mp(i-j-2k)}w) = f_{i,j}(x^{\mp k}w) f_{j+1,i}(x^{\mp j+k}w) \quad (i,j,i-k,j+k \geq 1). \quad (6.3)
$$

Proposition 6.2 The functions $\Delta_i(z)$ defined in (6.3) and $f_{i,j}(z)$ defined in (6.2) satisfy the following fusion relations.

$$
\Delta_{i+1}(z) = \frac{\prod_{k=1}^{i} \Delta_2(x^{-i-1+2k}z)}{\prod_{k=1}^{i} \Delta_1(x^{-i+2k}z)} \quad (i = 1, 2, 3, \cdots), \quad (6.4)
$$

$$
f_{i,j}(z) = \frac{\prod_{k=1}^{i} f_{j+1,i}(x^{-j+2k}z)}{\prod_{k=1}^{i} \Delta_1(x^{-i+2k}z)} \quad (i = 1, 2, 3, \cdots). \quad (6.5)
$$
Proposition 6.3 In the case of both one-fermion diagram and two-fermion diagram, for $0 < x < 1$ the generating functions $T_i(z)$ defined in (3.15) and (3.17) satisfy the following fusion relations.

$$
\lim_{w_1 \to x^{(i+j)i+1}} \left( 1 - x^{\pm(i+j)i+1} w_2 \right) f_{i,j}(w_2/w_1)T_i(w_1)T_j(w_2) = \mp \frac{|x|^2}{|x|} (x - x^{-1}) \prod_{k=1}^{\min(i,j)-1} \Delta_i(x^{2k+1})T_i+x+j(x^{\pm i}w_2) \quad (i, j \geq 1). \tag{6.6}
$$

Proof of Proposition 6.3 We show the case of one-fermion diagram. Proof for two-fermion diagram is given in the same way. First we study the simplest case $i = j = 1$. By Proposition 3.1 we have the following relations.

$$
\lim_{w_1 \to x^{2}w_2} \left( 1 - \frac{x^{2}w_2}{w_1} \right) f_{1,1}(w_2/w_1)\Lambda_k(w_1)\Lambda_k(w_2) = 0 \quad (k = 1, 2),
$$

$$
\lim_{w_1 \to x^{-2}w_2} \left( 1 - \frac{x^{-2}w_2}{w_1} \right) f_{1,1}(w_2/w_1)\Lambda_1(w_1)\Lambda_1(w_2) = 0,
$$

$$
\lim_{w_1 \to x^{2}w_2} \left( 1 - \frac{x^{2}w_2}{w_1} \right) f_{1,1}(w_2/w_1)\Lambda_1(w_1)\Lambda_k(w_2) = -\frac{|x|}{|x|} (x - x^{-1}) : \Lambda_k(w_2)\Lambda_1(x^{2}w_2) :,
$$

$$
\lim_{w_1 \to x^{-2}w_2} \left( 1 - \frac{x^{-2}w_2}{w_1} \right) f_{1,1}(w_2/w_1)\Lambda_1(w_1)\Lambda_k(w_2) = -\frac{|x|}{|x|} (x - x^{-1}) : \Lambda_k(x^{-2}w_2)\Lambda_1(w_2) :.
$$

For $1 \leq k < l \leq 3$ we have

$$
\lim_{w_1 \to x^{2}w_2} \left( 1 - \frac{x^{2}w_2}{w_1} \right) f_{1,1}(w_2/w_1)\Lambda_k(w_1)\Lambda_l(w_2) = 0,
$$

$$
\lim_{w_1 \to x^{-2}w_2} \left( 1 - \frac{x^{-2}w_2}{w_1} \right) f_{1,1}(w_2/w_1)\Lambda_1(w_1)\Lambda_l(w_2) = 0,
$$

$$
\lim_{w_1 \to x^{2}w_2} \left( 1 - \frac{x^{2}w_2}{w_1} \right) f_{1,1}(w_2/w_1)\Lambda_1(w_1)\Lambda_k(w_2) = \frac{|x|}{|x|} (x - x^{-1}) : \Lambda_k(w_2)\Lambda_1(x^{2}w_2) :,
$$

$$
\lim_{w_1 \to x^{-2}w_2} \left( 1 - \frac{x^{-2}w_2}{w_1} \right) f_{1,1}(w_2/w_1)\Lambda_1(w_1)\Lambda_k(w_2) = \frac{|x|}{|x|} (x - x^{-1}) : \Lambda_k(x^{-2}w_2)\Lambda_1(w_2) :.
$$

Summing up the above relations we have (6.6) for $i = j = 1$. In the case of one-fermion diagram and $i + j \geq 3$, summing up the relations summarized in Appendix B.1 we have (6.6) for $i + j \geq 3$. Proof for two-fermion diagram is given in the same way. □

6.2 First generating function

We prove a simple case of the quadratic relations (3.14) by direct calculation.

Proposition 6.4 In the case of both one-fermion diagram and two-fermion diagram, for $0 < x < 1$ the generating functions $T_i(z)$ $(i = 1, 2, 3, \cdots)$ defined in (3.15) satisfy the following quadratic relations.

$$
f_{i,1}(z_2/z_1)T_1(z_1)T_1(z_2) - f_{i,1}(z_1/z_2)T_1(z_2)T_1(z_1) = [r]_x [r - 1]_x (x - x^{-1}) \left( \delta \left( \frac{x^{i-1}z_2}{z_1} \right) T_{i+1}(x z_2) - \delta \left( \frac{x^{i+1}z_2}{z_1} \right) T_{i+1}(x z_2) \right). \tag{6.7}
$$

Proof of Proposition 6.4 We prove this in the case of one-fermion diagram. Proof for two-fermion diagram is given in the same way. First we prove this in the simplest case $i = j = 1$. Using Propositions 3.1 and relations (3.4), we have the following normal ordering formulae. For $i = 1, 2$ we have

$$
f_{1,1}(z_2/z_1)\Lambda_i(z_1)\Lambda_i(z_2) - f_{1,1}(z_1/z_2)\Lambda_i(z_2)\Lambda_i(z_1) = 0, \tag{6.8}
$$

$$
f_{1,1}(z_2/z_1)\Lambda_3(z_1)\Lambda_1(z_2) - f_{1,1}(z_1/z_2)\Lambda_3(z_2)\Lambda_1(z_1) = \frac{|x|^2}{|x|} (x - x^{-1}) \left( \delta \left( \frac{x^{-2}z_2}{z_1} \right) : \Lambda_3(x^{-2}z_2)\Lambda_3(z_2) : - \delta \left( \frac{x^{2}z_2}{z_1} \right) : \Lambda_3(z_2)\Lambda_3(x^{2}z_2) : \right). \tag{6.9}
$$

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For $1 \leq i < j \leq 3$ we have
\[
\begin{align*}
f_{1,1}(z_2/z_1)\Lambda_i(z_1)\Lambda_j(z_2) - f_{1,1}(z_1/z_2)\Lambda_j(z_2)\Lambda_i(z_1) = [r]_z[r-1]_z(x - x^{-1}) & \left( \delta \left( \frac{x^{-j+i-2k}z_2}{z_1} \right) f_{i-k,j+k}(x^{-j}z_1)T_{i-k}(x^kz_1)T_{j+k}(x^{-j}z_2) ight. \\
& \; - \Delta_1(x^{2j+i}z_2) \right) \times \delta \left( \frac{x^{-j-i}z_2}{z_1} \right) T_{j+i}(x^{-j}z_2) - \Delta_1(x^{j+2i}z_2) T_{j+i}(x^jz_2).
\end{align*}
\]
(6.10)

Summing up the above relations, we have the quadratic relation (6.7) for $i = 1$. Next we study the case $i \geq 2$. Using Propositions 3.1 and relations (3.4), (6.4), (6.5), we have the normal ordering formulae (6.14) and use the quadratic relation (6.16) from left to $\text{LHS}_{i,j}$.

\section{Proof of the main theorem}

In this Section we prove Theorems 3.2 and 3.3. Define $\text{LHS}_{i,j}$ and $\text{RHS}_{i,j}(k)$ ($1 \leq k \leq i \leq j$) by
\[
\begin{align*}
\text{LHS}_{i,j} &= f_{i,j}(z_2/z_1)T_i(z_1)T_j(z_2) - f_{j,i}(z_1/z_2)T_j(z_2)T_i(z_1), \\
\text{RHS}_{i,j}(k) &= \prod_{l=1}^{k-1} \Delta_1(x^{2l+i+1}[r]_z[r-1]_z(x - x^{-1})
\times \left( \delta \left( \frac{x^{-j+i-2k}z_2}{z_1} \right) f_{i-k,j+k}(x^{-j}z_1)T_{i-k}(x^kz_1)T_{j+k}(x^{-j}z_2) \right)
\times \Delta_1(x^{j+2i}z_2) T_{j+i}(x^jz_2) \right).
\end{align*}
\]
(6.12)

We will prove the following relation by induction of $i$ ($1 \leq i \leq j$).
\[
\text{LHS}_{i,j} = \sum_{k=1}^{i} \text{RHS}_{i,j}(k).
\]
(6.15)

Starting point $i = 1 \leq j$ has already been proved in Proposition 6.4. We assume the relation (6.15) holds for $i$ ($1 \leq i < j$). Then we will show $\text{LHS}_{i+1,j} = \sum_{k=1}^{i+1} \text{RHS}_{i+1,j}(k)$ from this assumption. Multiply $f_{1,i}(z_1/z_3)f_{1,j}(z_2/z_3)T_1(z_3)$ from left to $\text{LHS}_{i,j}$ and use the quadratic relation $f_{1,j}(z_2/z_3)T_j(z_2) = f_{j,1}(z_3/z_2)T_j(z_2)T_1(z_3) + \cdots$. Using fusion relation (6.1), we have the following.
\[
\begin{align*}
f_{1,j}(z_2/z_3)f_{i,j}(z_2/z_1)f_{1,i}(z_1/z_3)T_1(z_3)T_i(z_1)T_j(z_2)
& \quad - f_{j,1}(z_3/z_2)f_{j,i}(z_1/z_2)T_j(z_2)f_{1,i}(z_1/z_3)T_1(z_3)T_i(z_1)
& \quad - [r]_z[r-1]_z(x - x^{-1}) \Delta_1(x^{j-i}z_1/z_3)f_{j+1,i}(x^{j-i}z_1/z_3)T_{j+1}(x^jz_3)T_1(z_1)
& \quad + [r]_z[r-1]_z(x - x^{-1}) \Delta_1(x^jz_1/z_3)f_{j+1,i}(x^jz_1/z_3)T_{j+1}(x^{-j}z_3)T_i(z_1).
\end{align*}
\]
(6.16)
Multiply \( \frac{1}{[r]_x[r-1]_x(x-x^{-1})} \) \((1 - x^{-i-1} z_1 / z_3)\) and take the limit \( z_3 \to x^{-i-1} z_1 \). Use the fusion relation (6.6) and \( \lim_{z_3 \to x^{-i-1} z_1} (1 - x^{-i-1} z_1 / z_3) \Delta_1(x^{-i} z_1 / z_3) = [r]_x[r-1]_x(x-x^{-1}) \). Using the fusion relation (6.1) and the commutation relation:

\[
\begin{aligned}
& f_{j+1,i}(x^{j-i+1}) T_{j+1}(x^{-i-1} z_1) T_i(z_1) = f_{i,j+1}(x^{j-i-1}) T_i(z_1) T_{j+1}(x^{j-i-1} z_1),
\end{aligned}
\]

we have the following.

\[
\begin{aligned}
& f_{i+1,j}(x^{j-i-1} z_1) T_{i+1}(x^{-i-1} z_1) T_j(z_2) = f_{j,i+1}(x^{-i-1} z_1 / z_2) T_j(z_2) T_{i+1}(x^{-i-1} z_1) \\
& - [r]_x[r-1]_x(x-x^{-1}) \delta \left( \frac{x^{-i-j} z_2}{z_1} \right) f_{i+1,j}(x^{-i-j+1} z_1) T_i(z_1) T_{j+1}(x^{-i-1} z_2) \\
& + [r]_x[r-1]_x(x-x^{-1}) \delta \left( \frac{x^{i+j} z_2}{z_1} \right) \prod_{l=1}^{i-1} \Delta_1(x^{2l+1}) T_{i+j+1}(x^{i+1} z_2). \quad (6.17)
\end{aligned}
\]

Multiply \( f_{i,i}(z_1 / z_3) f_{i,j}(z_2 / z_3) T_i(z_3) \) from left to RHS_{i,j}(i) and use the fusion relation (6.2). Then we have the following.

\[
\begin{aligned}
& \prod_{l=1}^{i-1} \Delta_1(x^{2l+1}) [r]_x[r-1]_x(x-x^{-1}) \\
& \times \left( \delta \left( \frac{x^{-i-j} z_2}{z_1} \right) f_{i,i+1}(x^{j-i} z_1 / z_3) \Delta_1(x^{-i} z_1 / z_3) T_i(z_3) T_{i+j}(x^{j} z_1) \\
& - \delta \left( \frac{x^{i+j} z_2}{z_1} \right) f_{i+1,i}(x^{-i-j} z_1 / z_3) \Delta_1(x^{-i} z_1 / z_3) T_i(z_3) T_{i+j}(x^{-j} z_1) \right). \quad (6.18)
\end{aligned}
\]

Multiply \( \frac{1}{[r]_x[r-1]_x(x-x^{-1})} \) \((1 - x^{-i-1} z_1 / z_3)\) and take the limit \( z_3 \to x^{-i-1} z_1 \). Use the fusion relation (6.6) and \( \lim_{z_3 \to x^{-i-1} z_1} (1 - x^{-i-1} z_1 / z_3) \Delta_1(x^{-i} z_1 / z_3) = [r]_x[r-1]_x(x-x^{-1}) \). Then we have the following.

\[
\begin{aligned}
& [r]_x[r-1]_x(x-x^{-1}) \delta \left( \frac{x^{-i-j} z_2}{z_1} \right) \prod_{l=1}^{i-1} \Delta_1(x^{2l+1}) T_{i+j+1}(x^{i+1} z_2) \\
& - [r]_x[r-1]_x(x-x^{-1}) \delta \left( \frac{x^{i+j} z_2}{z_1} \right) \prod_{l=1}^{i-1} \Delta_1(x^{2l+1}) f_{i,i+j}(x^{-i-j+1}) T_i(x^{-i-1} z_1) T_{i+j}(x^{j} z_2). \quad (6.19)
\end{aligned}
\]

Multiply \( f_{i,i}(z_1 / z_3) f_{i,j}(z_2 / z_3) T_i(z_3) \) from left to RHS_{i,j}(k) (\( 1 \leq k \leq i-1 \)) and use the fusion relation (6.3). Using the commutation relation:

\[
\begin{aligned}
f_{i-k,k+j}(x^{j-i}) T_{i-k}(x^{k} z_1) T_{j+k}(x^{j-i+k} z_1) = f_{j+k,i-k}(x^{i-j}) T_{j+k}(x^{j-i+k} z_1) T_{i-k}(x^{k} z_1),
\end{aligned}
\]

we have the following.

\[
\begin{aligned}
& \prod_{l=1}^{k-1} \Delta_1(x^{2l+1}) [r]_x[r-1]_x(x-x^{-1}) \\
& \times \left( \delta \left( \frac{x^{-i-j-i+2k} z_2}{z_1} \right) f_{i-k,k+i}(x^{k} z_1 / z_3) f_{j+k,i-k}(x^{i-j}) f_{j,i+k}(x^{i+j+k} z_1 / z_3) T_i(z_3) T_{j+k}(x^{j-i+k} z_1) T_{i-k}(x^{k} z_1) \\
& - \delta \left( \frac{x^{j+i+2k} z_2}{z_1} \right) f_{i-k,k+j}(x^{k} z_1 / z_3) f_{i-k,j+k}(x^{j-i}) f_{j,i+k}(x^{i-j-k} z_1 / z_3) T_i(z_3) T_{i-k}(x^{k} z_1) T_{j+k}(x^{k} z_2) \right). \quad (6.20)
\end{aligned}
\]

Multiply \( \frac{1}{[r]_x[r-1]_x(x-x^{-1})} \) \((1 - x^{-i-1} z_1 / z_3)\) and take the limit \( z_3 \to x^{-i-1} z_1 \). Use the fusion relations (6.6) and (6.1). Using the commutation relation:

\[
\begin{aligned}
f_{i-k+1,j+k}(x^{j-i+1}) T_{i-k+1}(x^{-k-1} z_1) T_{j+k}(x^{j-i-k} z_1) = f_{j+k,i-k+1}(x^{i-j-1}) T_{j+k}(x^{j-i-k} z_1) T_{i-k+1}(x^{k} z_1),
\end{aligned}
\]
we have the following.

\[
\prod_{i=1}^{k} \Delta_i(x^{2i+1}) \left[ r, x \right]_{k-1} \delta \left( x^{2i+1} \right) f_{i+k-1, i-k} \left( x^{i-1} z_1 \right) T_{i-k+1} \left( x^{k-1} z_2 \right)
\]

\[
- \prod_{i=1}^{k-1} \Delta_i(x^{2i+1}) \left[ r, x \right] \delta \left( x^{2i+1} \right) f_{i+k+1, i+k} \left( x^{i-1} z_1 \right) T_{i+k+1} \left( x^{k-1} z_2 \right)
\]

\[
(6.21)
\]

Sum up (6.17), (6.19), (6.21) for \( 1 \leq k \leq i - 1 \), and shift the variable \( z_1 \to x z_1 \). Then we have \( \text{LHS}_{i+1, j} = \sum_{k=1}^{i+1} \text{RHS}_{i+1, j}(k) \). By induction of \( i \) we have shown the quadratic relations (3.14).

7 Concluding remarks

In this Section we discuss two special limits, some related works, and some open problems.

- The deformed \( W \)-algebra \( \mathcal{W}_q(\mathfrak{g}) \) is a two-parameter deformation of the classical \( W \)-algebra \( \mathcal{W}(\mathfrak{g}) \) and includes the \( q \)-Poisson \( W \)-algebra and the \( W \)-algebra as special case. In what follows we set parameters \( q = x^2 \), \( t = q^\beta \), \( \beta = \frac{r-1}{r} \). We study two special limits. One is the classical limit : \( \beta \to 0, q \) : fixed and the other is the conformal limit : \( q \to 1, \beta : \) fixed.

(1) We study the \( q \)-Poisson \( W \)-algebra \([3, 25, 26]\) in the classical limit : \( \beta \to 0 \) and \( q \) fixed. The defining relation of the deformed \( W \)-superalgebra \( \mathcal{W}_q(\mathfrak{g}(2|1)) \) is given by

\[
[T_i[m], T_j[n]] = - \sum_{l=1}^{m+1} f_{i,j}^l (T_i[m-l]T_j[n+l] - T_j[n-l]T_i[m+l])
\]

\[
+[r]_x [r-1]_x (x-x^{-1}) \sum_{k=1}^{m} \prod_{l=1}^{k-1} \Delta_l(x^{2l+1})
\]

\[
\times \sum_{m \in \mathbb{Z}} \left( f_{i-k,j+k}(x^{i-j})x^{j-l+k(m-n)+k(m-n)+4kl}T_{i-k}[m-l]T_j[k+l] \right)
\]

\[
-f_{i-k,j+k}(x^{i-j})x^{j-l+k(m-n)-4kl}T_{i-k}[m-l]T_{j+k}[k+l]. \quad (7.1)
\]

Here we define \( f_{i,j}^l \) by \( f_{i,j}(z) = \sum_{l=0}^{\infty} f_{i,j}^l z^l \) where the structure functions \( f_{i,j}(z) \) are given in (3.33). Define the \( q \)-Poisson bracket \( \{, \} \) by taking the classical limit \( \beta \to 0 \) with \( q \) fixed as follows.

\[
\{ T_i^{PB}(m), T_j^{PB}(n) \} = - \lim_{\beta \to 0} \frac{1}{\beta \log q} [T_i(m), T_j(n)]. \quad (7.2)
\]

Here we set \( T_i^{PB}(m) \) by

\[
T_i(z) = \sum_{m \in \mathbb{Z}} T_i(m) z^{-m} \to T_i^{PB}(z) = \sum_{m \in \mathbb{Z}} T_i^{PB}(m) z^{-m} \quad (\beta \to 0, q : \text{fixed}). \quad (7.3)
\]

\( \beta \)-expansions of the structure functions are given as follows.

\[
f_{i,j}(z) = 1 + \beta \log q \sum_{m=1}^{\infty} \left( \frac{1}{4} \text{Min}(i, j) m \right)_q \left( \frac{1}{4} \text{Max}(i, j) - 1 \right)_m z^{-m} (q - q^{-1}) + O(\beta^2) \quad (i, j \geq 1), \quad (7.4)
\]

\[
[r]_x [r-1]_x (x-x^{-1}) = - \beta \log q + O(\beta^2). \quad (7.5)
\]
Hence we have the following defining relations of the $q$-Poisson $W$-superalgebra.

\[
\{ T_i^{PB}(z_1), T_j^{PB}(z_2) \} = (q - q^{-1}) C_{i,j}(z) T_i^{PB}(z_1) T_j^{PB}(z_2) \\
+ \sum_{k=1}^{i} \delta \left( \frac{q^{-i+1} - k}{z_1} \right) T_{i-k}^{PB}(q^{-k} z_1) T_j^{PB}(q^{-k} z_2) \\
- \sum_{k=1}^{i} \delta \left( \frac{q^{-i+1} + k}{z_2} \right) T_{i-k}^{PB}(q^{-k} z_2) T_j^{PB}(q^{-k} z_1) (1 \leq i \leq j).
\]

(7.6)

Here we set the structure functions $C_{i,j}(z)$ $(i, j \geq 1)$ by

\[
C_{i,j}(z) = \sum_{m \in \mathbb{Z}} \left[ \frac{1}{2} \text{Min}(i,j)m \right]_q \left[ \frac{1}{2} \text{Max}(i,j) - 1 \right] m \frac{z^m}{[m]_q} (i, j \geq 1).
\]

(7.7)

We note degeneration of the structure functions.

\[
C_{1,2}(z) = C_{2,1}(z) = 0 \quad (i = 1, 2).
\]

(7.8)

(2) The deformed $W$-algebra $W_{qt}(\mathfrak{g})$ includes the $W$-algebra $W_{\beta}(\mathfrak{g})$ in the conformal limit: $q \rightarrow 1$ and $\beta$ fixed. Here we set $t = q^\beta$. Define the dressed boson $b_i(m) = m s_i^+(m) a_i(m)$. Introduce the boson $b_i^{\text{FFT}}(m)$ in the conformal limit.

\[
[b_i^{\text{FFT}}(m), b_j^{\text{FFT}}(n)] = \lim_{\beta \text{ fixed}} [b_i(m), b_j(n)] = m B_{i,j}(0) \delta_{m+n} (i, j = 1, 2),
\]

(7.9)

Introduce the ordinary screening charge $\tilde{Q}_i^\beta = \int e^{Q_i + a_i(0) \log z} \exp \left( \sum_{m \neq 0} \frac{1}{m} b_i^{\text{FFT}}(m) z^{-m} \right) :dz : (i = 1, 2)

in the same notation of [27]. We define the $W$-superalgebra $W_{\beta}(\mathfrak{sl}(2|1))$ as intersection of the kernel of the screening charge $\tilde{Q}_i^\beta$ $(i = 1, 2)$:

\[
W_{\beta}(\mathfrak{sl}(2|1)) = \bigcap_{i=1,2} \text{Ker } \tilde{Q}_i^\beta.
\]

(7.10)

We expect that, in the conformal limit, our deformed $W$-superalgebra $W_{qt}(\mathfrak{sl}(2|1))$ becomes isomorphic to the $W$-superalgebra $W_{\beta}(\mathfrak{sl}(2|1))$, because we define $W_{qt}(\mathfrak{sl}(2|1))$ as the subspace of $H_{qt}(2|1)$ of elements that commute with the deformed screening currents $S_i^+(z)$ up-to total difference. We expect that our $W_{\beta}(\mathfrak{sl}(2|1))$ coincides with construction by homology of BRST complex that we call quantum Drinfeld-Sokolov reduction [27] [28]. We expect that our $W_{qt}(\mathfrak{sl}(2|1))$ gives a one-parameter deformation of the $N = 2$ superconformal algebra.

- In Appendix A we checked Ding-Feigin’s construction reproduce the same $W_{qt}(\mathfrak{sl}(N))$ given in [6] [7] [8]. It seems to be possible to extend Ding-Feigin’s construction to the case of many fermions, which will give us higher-rank generalization $W_{qt}(\mathfrak{so}(M|N))$ and its quadratic relations. We expect to report it in the near future. It seems to be possible to extend Ding-Feigin’s construction to other superalgebras. We comment some related works, and open problems.
In [29, 30], a deformation of the $\mathfrak{sl}(N)$-KdV theory were aimed by Feigin, Shiraishi, Watanabe, and the author. Two classes of infinitely many commutative operators were constructed in terms of the deformed $W$-algebra $W_{\text{qt}}(\mathfrak{sl}(N))$. We call one of them local integrals of motion, and the other nonlocal integrals of motion, since they can be regarded as a one-parameter deformation of local and nonlocal integrals of motion for the $W$-algebra $W_\beta(\mathfrak{sl}(N))$. Local integrals of motion are given as integrals involving a product of generating functions $T_i(w)$ of the deformed $W$-algebra and theta functions. Nonlocal integrals of motion are given as integrals involving a product of affinization of the screening currents and theta functions. To show commutativity of local integrals of motion, quadratic relations of $W_{\text{qt}}(\mathfrak{sl}(N))$ were used directly. To show commutativity of nonlocal integrals of motion, commutation relations of the deformed screening currents were used directly. It is interesting to construct the local integrals of motion and nonlocal integrals of motion for the deformed $W$-superalgebra $W_{\text{qt}}(\mathfrak{sl}(M|N))$. In [31], Feigin, Jimbo, and Mukhin constructed the same integrals of motion for the deformed $W$-algebra $W_{\text{qt}}(\mathfrak{sl}(N))$ as trace of the universal-$\mathcal{R}$ of the quantum toroidal algebras $\mathfrak{gl}_1, \mathfrak{gl}_N$. In [32], Feigin, Jimbo, Mukhin, and Vilkovisky constructed the integrals of motion for the deformed $W$-superalgebra $W_{\text{qt}}(\mathfrak{sl}(M|N))$, as an image of ”universal integrals of motion” in quantum toroidal algebra [31] on a tensor product of two kinds of the Fock spaces $F_1^{\otimes M} \otimes F_2^{\otimes N}$.

In [33], bosonizations of vertex operators associated with the $\mathfrak{sl}(N)$ SOS model was constructed by Asai, Jimbo, Miwa, and Pugai. Bosonizations of vertex operators are given as integrals involving a product of the screening currents, theta functions and highest element of vertex operators. In [35, 36, 37], the deformed $W$-algebra $W_{\text{qt}}(\mathfrak{sl}(N))$ was obtained from fusion of vertex operators associated with the $\mathfrak{sl}(N)$ SOS model. It is interesting to construct the deformed $W$-algebra $W_{\text{qt}}(\mathfrak{sl}(M|N))$ by fusion of vertex operators associated with the $\mathfrak{sl}(M|N)$ SOS model [24]. Direct proof of commutation relations of vertex operators for the quantum group $U_q(\mathfrak{sl}(M|N))$ is very useful for bosonizations of vertex operators associated with the $\mathfrak{sl}(M|N)$ SOS model. In [39], the $L$-operator associated with the elliptic algebra $U_{qp}(\mathfrak{sl}(N))$ was constructed by another fusion of vertex operators associated with the $\mathfrak{sl}(N)$ SOS model. It is interesting to construct the $L$-operator associated with the elliptic superalgebra $U_{qp}(\mathfrak{sl}(M|N))$.

In [5, 6], a deformation of the correspondence between conformal field theory and the Calogero-Sutherland model were constructed by Awata, Kubo, Odake, and Shiraishi. Singular vectors of the $W$-algebra $W_\beta(\mathfrak{sl}(N))$ are realized by the Jack polynomials. They proposed a bosonization of the deformed $W$-algebra $W_{\text{qt}}(\mathfrak{sl}(N))$ in order to make singular vectors be the Macdonald polynomials that are a one-parameter deformation of the Jack polynomials. They constructed integral representations of singular vectors using bosonization of the screening currents. It is interesting to study special functions of singular vectors of $W_{\text{qt}}(\mathfrak{sl}(M|N))$. It is interesting check whether singular vectors are the super-Macdonald polynomials introduced in [40], or not.

In [41, 42], Arnaudon, Avan, Frappat, Ragoucy, and Shiraishi tried to make ”deformed Sugawara...
construction” of the deformed Virasoro algebra $\mathcal{W}_{qt}(\mathfrak{sl}(2))$ from the vertex-type elliptic algebra $\mathcal{A}_{qt}(\mathfrak{sl}(2))$.

Though they have not yet completed their project, as a by-product they obtained the structure functions $f_{i,j}(z)$ of the deformed Virasoro algebra. It is interesting to calculate the structure functions $f_{i,j}(z)$ of the deformed $W$-superalgebra $\mathcal{W}_{qt}(\mathfrak{sl}(M|N))$ from the vertex-type elliptic algebra $\mathcal{A}_{qt}(\mathfrak{sl}(M|N))$ [33].

Acknowledgements. The author would like to thank Professor Michio Jimbo, Professor Kenji Iohara, and Professor Ryusuke Endo for interest and many valuable discussions. This work is supported by the Grant-in-Aid for Scientific Research C (26400105) and C (19K03509) from Japan Society for the Promotion of Science.

A Deformed $W$-algebra $\mathcal{W}_{qt}(\mathfrak{sl}(N))$

In this Appendix we reproduce the same deformed $W$-algebra $\mathcal{W}_{qt}(\mathfrak{sl}(N))$ given in [6, 7, 8] by Ding and Feigin’s approach [1]. Let $N = 2, 3, 4, \ldots$. Let $\varepsilon_j$ ($j = 1, 2, \ldots, N$) be a basis with inner product $\langle \varepsilon_i, \varepsilon_j \rangle$ such that

$$\langle \varepsilon_i, \varepsilon_j \rangle = \delta_{i,j} \quad (1 \leq i, j \leq N). \quad (A.1)$$

The root system of $\mathfrak{sl}(N)$ and the corresponding Dynkin-diagram are given as follows.

$$\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \ldots \ldots \quad \alpha_{N-2} \quad \alpha_{N-1}$$

Simple root system = \{ $\alpha_1 = \varepsilon_1 - \varepsilon_2$, $\alpha_2 = \varepsilon_2 - \varepsilon_3$, $\ldots$, $\alpha_{N-1} = \varepsilon_{N-1} - \varepsilon_N$ \}

We introduce the Heisenberg algebra $\mathcal{H}_{qt}(N)$ generated by $a_i(m), Q_i$ ($m \in \mathbb{Z}, 1 \leq i \leq N-1$) satisfying

$$[a_i(m), a_j(n)] = \frac{1}{m} A_{i,j}^{\mathfrak{sl}_N}(m) \delta_{m+n,0} \quad (m, n \neq 0, 1 \leq i, j \leq N-1),$$

$$[a_i(0), Q_j] = A_{i,j}^{\mathfrak{sl}_N}(0) \quad (1 \leq i, j \leq N-1). \quad (A.2)$$

The remaining commutators vanish. Here we assume

$$A_{i,i+1}^{\mathfrak{sl}_N}(m) = A_{i+1,i}^{\mathfrak{sl}_N}(-m) \quad (m \neq 0, 1 \leq i \leq N-2),$$

$$A_{i,i}^{\mathfrak{sl}_N}(m) = 1 \quad (m \neq 0, 1 \leq i \leq N-1),$$

$$A_{i,j}^{\mathfrak{sl}_N}(m) = 0 \quad (m \neq 0, |i-j| \geq 2, 1 \leq i, j \leq N-1). \quad (A.3)$$

We define $A_{i,j}^{\mathfrak{sl}_N}(0)$ in [A.8]. In what follows we work on Fock space of the Heisenberg algebra.

We introduce the deformed screening currents $S_i^{\mathfrak{sl}_N}(z)$ ($1 \leq i \leq N-1$) as follows.

$$S_i^{\mathfrak{sl}_N}(z) = \exp (Q_i + a_i(0) \log z) : \exp \left( \sum_{m \neq 0} s_i^{\mathfrak{sl}_N}(m) a_i(m) z^{-m} \right) : \quad (1 \leq i \leq N-1). \quad (A.4)$$

Later we will find parameters $s_i^{\mathfrak{sl}_N}(m)$ ($1 \leq i \leq N-1$) from the ansatz [A.13]. The normal ordering formulae are given by

$$S_i^{\mathfrak{sl}_N}(z_1) S_j^{\mathfrak{sl}_N}(z_2) = z_2^{s_i^{\mathfrak{sl}_N}(0) h_{i,j}^{\mathfrak{sl}_N}(z_2/z_1)} : S_i^{\mathfrak{sl}_N}(z_1) S_j^{\mathfrak{sl}_N}(z_2) : \quad (1 \leq i, j \leq N-1), \quad (A.5)$$
where we set

\[ h_{i,j}^{s_{1N}}(z) = \exp \left( \sum_{m=1}^{\infty} \frac{1}{m} \lambda_{i,j}^{s_{1N}}(m) A_{i,j}^{s_{1N}}(m) s_{j}^{s_{1N}}(-m) z^m \right) \quad (1 \leq i, j \leq N - 1). \]  

(A.6)

We assume

\[ h_{i,j}^{s_{1N}}(z) = h_{j,i}^{s_{1N}}(z) \quad (1 \leq i, j \leq N - 1). \]  

(A.7)

We impose the following conditions.

\[ A_{i,i}^{s_{1N}}(0) = 2\alpha \quad (1 \leq i \leq N - 1), \]

\[ A_{i,i+1}^{s_{1N}}(0) = A_{i+1,i}^{s_{1N}}(0) = -\beta \quad (1 \leq i \leq N - 2), \]

\[ A_{i,j}^{s_{1N}}(0) = 0 \quad (|i - j| \geq 2, 1 \leq i, j \leq N - 1). \]  

(A.8)

Here \( \alpha, \beta \ (\alpha \neq \frac{1}{2}) \) are a generic parameter.

We introduce the first generating function \( T_{1}^{s_{1N}}(z) \) of the deformed \( W \)-algebra \( W_{qt}(\mathfrak{sl}(N)) \) by

\[ T_{1}^{s_{1N}}(z) = \sum_{j=1}^{N} g_{j}^{s_{1N}} \Lambda_{j}^{s_{1N}}(z), \]  

(A.9)

where we set

\[ \Lambda_{i}^{s_{1N}}(z) = \exp \left( \sum_{j=1}^{N-1} \lambda_{i,j}^{s_{1N}}(0) a_{j}(0) \right) : \exp \left( \sum_{j=1}^{N-1} \sum_{m \neq 0} \lambda_{i,j}^{s_{1N}}(m) s_{j}^{s_{1N}}(m) a_{j}(m) z^{-m} \right) : \quad (1 \leq i \leq N). \]  

(A.10)

Later we will determine parameters \( \lambda_{i,j}^{s_{1N}}(m) \) \( (1 \leq i \leq N, 1 \leq j \leq N - 1, m \in \mathbb{Z}) \) and parameters \( g_{i}^{s_{1N}} \) \( (1 \leq i \leq N) \) from the ansatz (A.13).

We assume the normal ordering formulae of \( \Lambda_{i}^{s_{1N}}(z) \) and \( S_{j}^{s_{1N}}(w) \) have at most one pole and one zero. For \( 1 \leq i \leq N \), we assume

\[ \Lambda_{i}^{s_{1N}}(z_{1}) S_{j}^{s_{1N}}(z_{2}) =: \Lambda_{i}^{s_{1N}}(z_{1}) S_{j}^{s_{1N}}(z_{2}) := S_{j}^{s_{1N}}(z_{2}) \Lambda_{i}^{s_{1N}}(z_{1}) \quad (1 \leq j \leq i - 2), \]

\[ \Lambda_{i}^{s_{1N}}(z_{1}) S_{i}^{s_{1N}}(z_{2}) = \frac{q_{i}(z_{1} - p_{i} z_{2})}{p_{i}(z_{1} - q_{i} z_{2})} : \Lambda_{i}^{s_{1N}}(z_{1}) S_{i}^{s_{1N}}(z_{2}) : \quad (|z_{1}| >> |z_{2}|, 1 \leq i \leq N - 1), \]

\[ S_{i}^{s_{1N}}(z_{2}) \Lambda_{i}^{s_{1N}}(z_{1}) = \frac{q_{i}(z_{1} - p_{i} z_{2})}{p_{i}(z_{1} - q_{i} z_{2})} : \Lambda_{i}^{s_{1N}}(z_{1}) S_{i}^{s_{1N}}(z_{2}) : \quad (|z_{2}| >> |z_{1}|, 1 \leq i \leq N - 1), \]

\[ \Lambda_{i+1}^{s_{1N}}(z_{1}) S_{i}^{s_{1N}}(z_{2}) = \frac{q_{i}(z_{1} - p_{i} z_{2})}{p_{i}(z_{1} - q_{i} z_{2})} : \Lambda_{i+1}^{s_{1N}}(z_{1}) S_{i}^{s_{1N}}(z_{2}) : \quad (|z_{1}| >> |z_{2}|, 1 \leq i \leq N - 1), \]

\[ S_{i}^{s_{1N}}(z_{2}) \Lambda_{i+1}^{s_{1N}}(z_{1}) = \frac{q_{i}(z_{1} - p_{i} z_{2})}{p_{i}(z_{1} - q_{i} z_{2})} : \Lambda_{i+1}^{s_{1N}}(z_{1}) S_{i}^{s_{1N}}(z_{2}) : \quad (|z_{2}| >> |z_{1}|, 1 \leq i \leq N - 1), \]

\[ \Lambda_{i}^{s_{1N}}(z_{1}) S_{j}^{s_{1N}}(z_{2}) =: \Lambda_{i}^{s_{1N}}(z_{1}) S_{j}^{s_{1N}}(z_{2}) := S_{j}^{s_{1N}}(z_{2}) \Lambda_{i}^{s_{1N}}(z_{1}) \quad (i + 1 \leq j \leq N - 1). \]  

(A.11)

Here parameters \( p_{i}, p'_{i}, q_{i}, q'_{i} \) \( (1 \leq i \leq N - 1) \) satisfy

\[ p_{i}, p'_{i}, q_{i}, q'_{i} > 0, \quad q_{i} \neq q_{i}, \quad p_{i} \neq q_{i}, \quad p'_{i} \neq q'_{i} \quad (1 \leq i \leq N - 1). \]  

(A.12)

**Ansatz** We assume that the first generating function \( T_{1}^{s_{1N}}(z) \) of the deformed \( W \)-algebra \( W_{qt}(\mathfrak{sl}(N)) \)
commutes with the deformed screening currents $S_i^{slN}(w)$ up-to total difference.

$$[T_i^{slN}(z), S_i^{slN}(w)] = g_i^{slN} \left( \frac{q_i}{p_i} - 1 \right) : A_i^{slN}(z) S_i^{slN}(w) : \left( \delta \left( \frac{q_i w}{z} \right) - \delta \left( \frac{q_i w}{z} \right) \right) \quad (1 \leq i \leq N - 1). (A.13)$$

In what follows we introduce the parameters $x, r, s > 0$ as follows.

$$\frac{q'_i}{q_i} = x^{2r}, \quad \beta = \frac{r - 1}{r}, \quad s = q_1. \quad (A.14)$$

We will write al parameters $p_i, p'_i, q_i, q'_i, A_{i,j}^{slN}(m), s_i^{slN}(m), \lambda_{i,j}^{slN}(m), g_i^{slN}$ by only three parameters $x, r, s$.

As the same arguments about the structure functions in Section 3 we have followings.

**Proposition A.1** The parameters $q_i, q'_i, p_i, p'_i$ $(1 \leq i \leq N - 1)$, $A_{i,j}^{slN}(0)$ $(1 \leq i, j \leq N - 1)$, and $g_i^{slN}$ $(1 \leq i \leq N)$ are given by

$$A_{i,j}^{slN}(0) = \frac{2(r - 1)}{r}, \quad A_{i+1,i}^{slN}(0) = A_{i,i+1}^{slN}(0) = \frac{1 - r}{r} \quad (1 \leq i \leq N - 1),$$

$$g_{i,j} = s x^{j - 1}, \quad q'_i = s x^{2r + j - 1}, \quad p_j = s x^{2r + j - 3}, \quad p'_j - s x^{j + 1} \quad (1 \leq j \leq N - 1),$$

$$\sum_{m=1}^{\infty} \frac{1}{m} \left[ \frac{1}{[r - 1]_m} x [2m]_x \right] z^m \quad (1 \leq i \leq N - 1).$$

We define $B_{i,j}^{slN}(m)$ $(1 \leq i, j \leq N - 1, m \neq 0)$ by

$$B_{i,j}^{slN}(m) = B_{i,j}^{slN}(m) \iff A_{i,j}^{slN}(m) = A_{i,j}^{slN}(m), \quad B_{i,j}^{slN}(m) = B_{i,j}^{slN}(m) \quad (1 \leq i, j \leq N - 1).$$

By definitions of $A_{i,j}^{slN}$ and $B_{i,j}^{slN}$ we have

$$B_{i,j}^{slN}(m) = B_{i,j}^{slN}(m) \iff A_{i,j}^{slN}(m) = A_{i,j}^{slN}(m), \quad B_{i,j}^{slN}(m) = B_{i,j}^{slN}(m) \quad (1 \leq i, j \leq N - 1).$$

**Proposition A.3** For $N \geq 2$, $B_{i,j}^{slN}(m)$ $(1 \leq i, j \leq N - 1, m \neq 0)$ are given by

$$B_{i,i}^{slN}(m) = \frac{1}{B_{i+1,i}^{slN}(m)} \quad (1 \leq i \leq N - 1),$$

$$B_{i+1,i}^{slN}(m) = \frac{1}{B_{i,i}^{slN}(m)} \quad (1 \leq i \leq N - 2),$$

$$B_{i,j}^{slN}(m) = 0 \quad (|i - j| \geq 2, 1 \leq i, j \leq N - 1).$$

(A.19)
There is ambiguity for the choice of $A_{i,j}^{s\lambda N}(m)$ and $s_{i}^{s\lambda N}(m)$. For $m \neq 0$, we have a solution as follows.

$$A_{i,i}^{s\lambda N}(m) = 1 \quad (1 \leq i \leq N-1), \quad A_{i,j}^{s\lambda N}(m) = 0 \quad (|i - j| \geq 2, 1 \leq i, j \leq N-1),$$
$$A_{i,i+1}^{s\lambda N}(m) = -x^{-m}\frac{|m|_{x}}{2m|_{x}}, \quad A_{i+1,i}^{s\lambda N}(m) = -x^{-m}\frac{|m|_{x}}{2m|_{x}} \quad (1 \leq i \leq N-2).$$

(A.20)

The remaining elements vanish. For $m > 0$ and $1 \leq i \leq N-1$, we have

$$s_{i}^{s\lambda N}(m) = x^{-m}\sqrt{\frac{(r-1)m|_{x}}{2m|_{x}}} \quad s_{i}^{s\lambda N}(-m) = -x^{m}\sqrt{\frac{(r-1)m|_{x}}{2m|_{x}}}.$$  

(A.21)

**Proposition A.4** For $N \geq 2$, a bosonization of the deformed screening currents $S_{i}^{s\lambda N}(z)$ ($1 \leq i \leq N-1$) introduced in (A.4) is given by $A_{i,j}^{s\lambda N}(m)$ ($1 \leq i, j \leq N-1, m \in Z$) in (A.18), (A.20) and $s_{i}^{s\lambda N}(m)$ ($1 \leq i \leq N-1, m \in Z_{\neq 0}$) in (A.21).

Next, we construct bosonization of $\Lambda_{i}^{s\lambda N}(z)$ ($1 \leq i \leq N$). We consider the assumption (A.11) upon the parameterization of $q_{i}, q'_{i}, p_{i}, p'_{i}$ given in Proposition A.1. They are equivalent to the following set of equations. Parameter $\lambda_{i}^{s\lambda N}(m)$ ($1 \leq i \leq N, 1 \leq j \leq N-1, m \in Z$) in (A.10) are determined as follows.

For $N = 2$, parameters $\lambda_{1,1}^{s\lambda N}(m), \lambda_{2,1}^{s\lambda N}(m)$ ($m \neq 0$) are determined by

$$B_{1,1}^{s\lambda N}(m)\lambda_{1,1}^{s\lambda N}(m) + B_{2,1}^{s\lambda N}(m)\lambda_{2,1}^{s\lambda N}(m) = -s^{m}x^{(r-1)m}[(r-1)m|_{x}(x-x^{-1}),$$
$$B_{1,1}^{s\lambda N}(m)\lambda_{1,1}^{s\lambda N}(m) + B_{2,2}^{s\lambda N}(m)\lambda_{2,2}^{s\lambda N}(m) = 0,$$

$$B_{1,1}^{s\lambda N}(m)\lambda_{2,1}^{s\lambda N}(m) + B_{2,2}^{s\lambda N}(m)\lambda_{2,2}^{s\lambda N}(m) = s^{m}x^{(r+1)m}[(r-1)m|_{x}(x-x^{-1}),$$

$$B_{1,2}^{s\lambda N}(m)\lambda_{1,2}^{s\lambda N}(m) + B_{2,2}^{s\lambda N}(m)\lambda_{2,2}^{s\lambda N}(m) = -s^{m}x^{r+m}[(r-1)m|_{x}(x-x^{-1}),$$

$$B_{1,2}^{s\lambda N}(m)\lambda_{3,1}^{s\lambda N}(m) + B_{2,2}^{s\lambda N}(m)\lambda_{3,2}^{s\lambda N}(m) = 0,$$

$$B_{1,3}^{s\lambda N}(m)\lambda_{3,1}^{s\lambda N}(m) + B_{2,2}^{s\lambda N}(m)\lambda_{3,2}^{s\lambda N}(m) = s^{m}x^{(r+2)m}[(r-1)m|_{x}(x-x^{-1}).$$

(A.22)

For $N = 3$ parameters $\lambda_{i,j}^{s\lambda N}(m)$ ($m \neq 0$) are determined by

$$B_{1,1}^{s\lambda N}(m)\lambda_{1,1}^{s\lambda N}(m) + B_{2,1}^{s\lambda N}(m)\lambda_{2,1}^{s\lambda N}(m) + B_{3,1}^{s\lambda N}(m)\lambda_{3,1}^{s\lambda N}(m) = -s^{m}x^{(r-1)m}[(r-1)m|_{x}(x-x^{-1}),$$
$$B_{1,1}^{s\lambda N}(m)\lambda_{2,1}^{s\lambda N}(m) + B_{2,2}^{s\lambda N}(m)\lambda_{2,2}^{s\lambda N}(m) = s^{m}x^{(r+1)m}[(r-1)m|_{x}(x-x^{-1}),$$

$$B_{1,1}^{s\lambda N}(m)\lambda_{3,1}^{s\lambda N}(m) + B_{2,2}^{s\lambda N}(m)\lambda_{3,2}^{s\lambda N}(m) + B_{3,1}^{s\lambda N}(m)\lambda_{3,1}^{s\lambda N}(m) = 0 \quad (3 \leq i \leq N),$$

$$B_{j-1,j-1}^{s\lambda N}(m)\lambda_{i,j-1}^{s\lambda N}(m) + B_{j-1,j+1}^{s\lambda N}(m)\lambda_{i,j+1}^{s\lambda N}(m) + B_{j+1,j+1}^{s\lambda N}(m)\lambda_{i,j+1}^{s\lambda N}(m) = 0 \quad (1 \leq i < j \leq N-2),$$

$$B_{j-1,j-1}^{s\lambda N}(m)\lambda_{i,j-1}^{s\lambda N}(m) + B_{j+1,j+1}^{s\lambda N}(m)\lambda_{i,j+1}^{s\lambda N}(m) + B_{j+1,j+1}^{s\lambda N}(m)\lambda_{i,j+1}^{s\lambda N}(m) = 0 \quad (2 \leq j \leq N-2),$$

$$B_{j-1,j-1}^{s\lambda N}(m)\lambda_{i,j-1}^{s\lambda N}(m) + B_{j+1,j+1}^{s\lambda N}(m)\lambda_{i,j+1}^{s\lambda N}(m) + B_{j+1,j+1}^{s\lambda N}(m)\lambda_{i,j+1}^{s\lambda N}(m) = 0 \quad (2 \leq j < i \leq N-1).$$

(A.23)
Theorem A.5 follows. Parameters $N \geq 2$, parameters $\lambda_{i,1}^{slN} (0), \lambda_{2,1}^{slN} (0)$ are determined by

$$2 \lambda_{1,1}^{slN} (0) = -2r \log x, \quad -\lambda_{2,1}^{slN} (0) = 2r \log x. \quad (A.25)$$

For $N = 3$, parameters $\lambda_{i,j}^{slN} (0)$ are determined by

$$2 \lambda_{1,1}^{slN} (0) - \lambda_{1,2}^{slN} (0) = -2r \log x, \quad -\lambda_{1,1}^{slN} (0) + 2 \lambda_{1,2}^{slN} (0) = 0,$$
$$2 \lambda_{2,1}^{slN} (0) - \lambda_{2,2}^{slN} (0) = 2r \log x, \quad -\lambda_{2,1}^{slN} (0) + 2 \lambda_{2,2}^{slN} (0) = -2r \log x,$$
$$2 \lambda_{3,1}^{slN} (0) - \lambda_{3,2}^{slN} (0) = 0, \quad -\lambda_{3,1}^{slN} (0) + 2 \lambda_{3,2}^{slN} (0) = 2r \log x. \quad (A.26)$$

For $N \geq 4$, parameter $\lambda_{i,j}^{slN} (0)$ are determined by

$$2 \lambda_{1,1}^{slN} (0) - \lambda_{1,2}^{slN} (0) = -2r \log x,$$
$$2 \lambda_{2,1}^{slN} (0) - \lambda_{2,2}^{slN} (0) = 2r \log x,$$
$$2 \lambda_{3,1}^{slN} (0) - \lambda_{3,2}^{slN} (0) = 0 \quad (3 \leq i \leq N),$$
$$-\lambda_{i,j}^{slN} (0) + 2 \lambda_{i,j}^{slN} (0) - \lambda_{i,j+1}^{slN} (0) = 0 \quad (1 \leq i < j \leq N - 2),$$
$$-\lambda_{j,j}^{slN} (0) + 2 \lambda_{j,j}^{slN} (m) - \lambda_{j,j+1}^{slN} (0) = -2r \log x \quad (2 \leq j \leq N - 2),$$
$$-\lambda_{i,j}^{slN} (0) + 2 \lambda_{i,j}^{slN} (0) - \lambda_{i,j+1}^{slN} (0) = 2r \log x \quad (2 \leq j < i - 1 \leq N - 1),$$
$$-\lambda_{i,N-2}^{slN} (0) + 2 \lambda_{i,N-1}^{slN} (0) = 0 \quad (1 \leq i \leq N - 2),$$
$$-\lambda_{i,N-1}^{slN} (0) + 2 \lambda_{i,N-2}^{slN} (0) + 2 \lambda_{i,N-1}^{slN} (0) = -2r \log x,$$
$$-\lambda_{i,N-2}^{slN} (0) + 2 \lambda_{i,N-1}^{slN} (0) = 2r \log x. \quad (A.27)$$

Solving the above equations, we have the following Theorem.

**Theorem A.5** For $N \geq 2$, a bosonization of $\Lambda_{i}^{slN} (z) \quad (1 \leq i \leq N)$ introduced in (A.10) is given as follows. Parameters $\lambda_{i,j}^{slN} (m) \quad (1 \leq i \leq N, 1 \leq j \leq N - 1, m \in \mathbb{Z})$ are given by

$$\lambda_{i,j}^{slN} (m) = -s m x(N+r-1)^m \frac{[rm]x j^m m}{[Nm]x} (x - x^{-1}) \quad (1 \leq j < i \leq N, m \neq 0),$$
$$\lambda_{i,j}^{slN} (m) = s m x(r-1)^m \frac{([N-j]m) x}{[Nm]x} (x - x^{-1}) \quad (1 \leq i \leq j \leq N - 1, m \neq 0), \quad (A.28)$$
$$\lambda_{i,j}^{slN} (0) = \frac{2rj}{N} \log x \quad (1 \leq j < i \leq N),$$
$$\lambda_{i,j}^{slN} (0) = -\frac{2r(N-j)}{N} \log x \quad (1 \leq i \leq j \leq N - 1). \quad (A.29)$$

A set of solutions of $A_{i,j}^{slN} (m) \quad (1 \leq i, j \leq N - 1, m \in \mathbb{Z})$ and $s_{i}^{slN} (m) \quad (1 \leq i \leq N - 1, m \in \mathbb{Z}_{\neq 0})$ is given in (A.15), (A.20), and (A.21).
Proposition A.6 For $N \geq 2$, the first generating function $T_i^{slN}(z)$ commutes with the deformed screening currents $S_i^{slN}(w)$ up-to total difference.

\[
[T_i^{slN}(z), S_i^{slN}(w)] = S_i^{slN}(x^{2(1-r)} - 1) : \Lambda_i^{slN}(z) S_i^{slN}(w) : \left( \delta \left( \frac{sx^{r-1}w}{z} \right) - \delta \left( \frac{sx^{2r+i-1}w}{z} \right) \right) (1 \leq i \leq N - 1).
\] (A.30)

From the bosonization of $\Lambda_i^{slN}(z)$, we have the following Proposition.

Proposition A.7 For $N \geq 2$, the normal ordering rules of $\Lambda_i^{slN}(z) (1 \leq i \leq N)$ are given as follows.

\[
\begin{align*}
\Lambda_i^{slN}(z_1) \Lambda_i^{slN}(z_2) &= : \Lambda_i^{slN}(z_1) \Lambda_i^{slN}(z_2) : \\
\times &\exp \left( \sum_{m=1}^{\infty} \frac{1}{m} \frac{[rm]_x[(r-1)m]_x[(N-1)m]_x}{[Nm]_x} (x - x^{-1})^2(z_2/z_1)^m \right) (1 \leq i \leq N), \\
\Lambda_i^{slN}(z_1) \Lambda_j^{slN}(z_2) &= : \Lambda_i^{slN}(z_1) \Lambda_j^{slN}(z_2) : \\
\times &\exp \left( - \sum_{m=1}^{\infty} \frac{1}{m} \frac{[m]_x[rm]_x[(r-1)m]_x}{[Nm]_x} x^{-N}(x - x^{-1})^2(z_2/z_1)^m \right) (1 \leq i < j \leq N), \\
\Lambda_j^{slN}(z_1) \Lambda_i^{slN}(z_2) &= : \Lambda_i^{slN}(z_1) \Lambda_j^{slN}(z_2) : \\
\times &\exp \left( - \sum_{m=1}^{\infty} \frac{1}{m} \frac{[m]_x[rm]_x[(r-1)m]_x}{[Nm]_x} x^{N}(x - x^{-1})^2(z_2/z_1)^m \right) (1 \leq i < j \leq N). \quad (A.31)
\end{align*}
\]

In what follows we focus our attention to the case of $0 < x < 1$.

Proposition A.8 In the case of $N \geq 2$ and $0 < x < 1$, the deformed screening currents $S_i^{slN}(z)$ satisfy

\[
\begin{align*}
S_i^{slN}(z_1) S_i^{slN}(z_2) &= \frac{[u_1 - u_2 - 1]}{[u_1 - u_2 + 1]} S_i^{slN}(z_2) S_i^{slN}(z_1) \quad (1 \leq i \leq N - 1), \\
S_i^{slN}(z_1) S_{i+1}^{slN}(z_2) &= - \frac{[u_1 - u_2 + \frac{1}{2}]}{[u_1 - u_2 - \frac{1}{2}]} S_i^{slN}(z_2) S_i^{slN}(z_1) \quad (1 \leq i \leq N - 2), \\
S_i^{slN}(z_1) S_j^{slN}(z_2) &= S_j^{slN}(z_2) S_i^{slN}(z_1) \quad (|i - j| \geq 2, 1 \leq i, j \leq N - 1), \quad (A.32)
\end{align*}
\]

where we set $z_i = x^{2u_i} (1 \leq i \leq 2)$. Here we used the theta function $[u]$ defined in (2.3).

We introduce the structure functions $f_{i,j}^{slN}(z) (1 \leq i,j \leq N - 1)$ by

\[
f_{i,j}^{slN}(z) = \exp \left( - \sum_{m=1}^{\infty} \frac{1}{m} \frac{[\min(i,j)]_x[(N - \max(i,j))m]_x}{[Nm]_x[m]_x} z^m \right).
\] (A.33)

Proposition A.9 In the case of $N \geq 2$ and $0 < x < 1$, the normal ordering rules of $\Lambda_i^{slN}(z) (1 \leq i \leq N)$ introduced in (A.11) are given as follows.

\[
\begin{align*}
f_{1,i}^{slN}(z_2/z_1) \Lambda_i^{slN}(z_1) \Lambda_i^{slN}(z_2) &= : \Lambda_i^{slN}(z_1) \Lambda_i^{slN}(z_2) : \quad (1 \leq i \leq N), \\
f_{1,i}^{slN}(z_2/z_1) \Lambda_i^{slN}(z_1) \Lambda_j^{slN}(z_2) &= \Delta_1(x^{-1}z_2/z_1) : \Lambda_i^{slN}(z_1) \Lambda_j^{slN}(z_2) : \quad (1 \leq i < j \leq N), \\
f_{1,i}^{slN}(z_2/z_1) \Lambda_j^{slN}(z_1) \Lambda_i^{slN}(z_2) &= \Delta_1(xz_2/z_1) : \Lambda_j^{slN}(z_1) \Lambda_i^{slN}(z_2) : \quad (1 \leq i < j \leq N). \quad (A.34)
\end{align*}
\]

Here we set $\Delta_1(z)$ and $f_{1,1}^{slN}(z)$ in (3.24) and (A.25), respectively.
In this Appendix we summarize some normal ordering rules in the case of one-fermion diagram.

Here we set $\Delta_T$.

In the case of introduced in (A.10) are given as follows.

\[
T_i^{slN}(z) = \sum_{1 \leq s_1 < s_2 < \cdots < s_i \leq N} : \Lambda_{s_1}^{slN}(x^{-i+1}z) \Lambda_{s_2}^{slN}(x^{-i+3}z) \cdots \Lambda_{s_i}^{slN}(x^{i-1}z) : \quad (1 \leq i \leq N). \tag{A.35}
\]

Here we set $g_1^{slN} = g_2^{slN} = \cdots = g_N^{slN} = 1$. We have $T_N^{slN}(z) = 1$.

In what follows we focus our attention to the regime $0 < y = x^{-1} < 1$. We define the structure functions $f_{i,j}^{slN*}(z)$ by

\[
f_{i,j}^{slN*}(z) = f_{i,j}^{slN}(z) \big|_{x \to y} \quad (1 \leq i, j \leq N). \tag{A.36}
\]

Here we used $f_{i,j}^{slN}(z)$ defined in (A.33).

**Theorem A.10** In the case of $N \geq 2$ and $0 < y < 1$, the deformed screening currents $S_i^{slN}(z)$ satisfy

\[
S_i^{slN}(z_1)S_i^{slN}(z_2) = \frac{[u_1 - u_2 - 1]^*}{[u_1 - u_2 + 1]^*} S_i^{slN}(z_2)S_i^{slN}(z_1) \quad (1 \leq i \leq N - 1),
\]

\[
S_i^{slN}(z_1)S_{i+1}^{slN}(z_2) = \frac{[u_1 - u_2 + 1]^*}{[u_1 - u_2 - 1]^*} S_{i+1}^{slN}(z_2)S_i^{slN}(z_1) \quad (1 \leq i \leq N - 2),
\]

\[
S_i^{slN}(z_1)S_j^{slN}(z_2) = S_j^{slN}(z_2)S_i^{slN}(z_1) \quad (|i - j| \geq 2, 1 \leq i, j \leq N - 1), \tag{A.37}
\]

where we set $z_1 = y^{2u_1}(1 \leq i \leq 2)$. Here we used $[u]^*$ defined in (A.4).

**Proposition A.11** In the case of $N \geq 2$ and $0 < y < 1$, the normal ordering rules of $\Lambda_i^{slN}(z)(1 \leq i \leq N)$ introduced in (A.10) are given as follows.

\[
f_{i,1}^{slN*}(z_2/z_1)\Lambda_i^{slN}(z_1)\Lambda_j^{slN}(z_2) =: \Lambda_i^{slN}(z_1)\Lambda_j^{slN}(z_2) \quad (1 \leq i \leq N),
\]

\[
f_{i,1}^{slN*}(z_2/z_1)\Lambda_i^{slN}(z_1)\Lambda_j^{slN}(z_2) = \Delta_i^*(y_2/z_1) : \Lambda_i^{slN}(z_1)\Lambda_j^{slN}(z_2) : \quad (1 \leq i < j \leq N),
\]

\[
f_{i,1}^{slN*}(z_2/z_1)\Lambda_j^{slN}(z_1)\Lambda_i^{slN}(z_2) = \Delta_j^*(y_1/z_2/z_1) : \Lambda_j^{slN}(z_1)\Lambda_i^{slN}(z_2) : \quad (1 \leq i < j \leq N). \tag{A.38}
\]

Here we set $\Delta_i^*(z)$ and $f_{i,1}^{slN*}(z)$ in (C.7) and (A.30), respectively.

From the normal ordering rules of $\Lambda_i^{slN}(z)$, in the case of $0 < y = x^{-1} < 1$, we define the generating functions $T_i^{slN}(z)$ ($1 \leq i \leq N$) as follows.

\[
T_i^{slN}(z) = \sum_{1 \leq s_1 < s_2 < \cdots < s_i \leq N} : \Lambda_{s_1}^{slN}(y^{i-1}z)\Lambda_{s_2}^{slN}(y^{i-3}z) \cdots \Lambda_{s_i}^{slN}(y^{-i+1}z) : \quad (1 \leq i \leq N). \tag{A.39}
\]

Here we set $g_1^{slN} = g_2^{slN} = \cdots = g_N^{slN} = 1$. We have $T_N^{slN}(z) = 1$.

The formulae for $0 < y < 1$ are different from those for $0 < x < 1$ in power index $y^{\pm 1}$, because $\mathcal{W}q_l(sl(N))$ is not invariant under $x \to x^{-1}$. $T_i^{slN}(z)$ in the regime $0 < x < 1$ coincides with those of [6, 7]. $T_i^{slN}(z)$ in the regime $0 < y = x^{-1} < 1$ coincides with those of [8].

**B Normal ordering rule**

In this Appendix we summarize some normal ordering rules in the case of one-fermion diagram.
B.1 Fusion

In this Appendix we summarize some fusion formulae in the case of one-fermion diagram.

- In the case of \( \left( 1 - \frac{x^{i+1}w_2}{w_1} \right) f_{1,i}(w_2/w_1)T_1(w_1)T_i(w_2) \) \( (i \geq 2) \):
  
  We take the limit \( w_1 \to x^{i+1}w_2 \).

  \[
  \left( 1 - \frac{x^{i+1}w_2}{w_1} \right) f_{1,i}(w_2/w_1)\Lambda_3(w_1) : \Lambda_1(x^{-i+1}w_2)\Lambda_2(x^{-i+3}w_2) \prod_{l=1}^{i-2} \Lambda_3(x^{-i+3+2l}w_2) :
  \]

  \[
  \to - \frac{[r]_x[r-i+1]_x}{[i-1]_x} (x - x^{-1}) : \Lambda_1(x^{-i+1}w_2)\Lambda_2(x^{-i+3}w_2) \prod_{l=1}^{i-1} \Lambda_3(x^{-i+3+2l}w_2) : , \quad (B.1)
  \]

  \[
  \left( 1 - \frac{x^{i+1}w_2}{w_1} \right) f_{1,i}(w_2/w_1)\Lambda_3(w_1) : \Lambda_k(x^{-i+1}w_2) \prod_{l=1}^{i-1} \Lambda_3(x^{-i+2l+1}w_2) :
  \]

  \[
  \to - \frac{[r]_x[r-i]_x}{[i]_x} (x - x^{-1}) : \Lambda_k(x^{-i+1}w_2) \prod_{l=1}^{i} \Lambda_3(x^{-i+2l+1}w_2) : , \quad (k = 1, 2), \quad (B.2)
  \]

  \[
  \left( 1 - \frac{x^{i+1}w_2}{w_1} \right) f_{1,i}(w_2/w_1)\Lambda_3(w_1) : \Lambda^i_l \Lambda_3(x^{-i+2l-1}w_2) :
  \]

  \[
  \to - \frac{[r]_x[r-i-1]_x}{[i+1]_x} (x - x^{-1}) : \prod_{l=1}^{i+1} \Lambda_3(x^{-i+2l+1}w_2) : . \quad (B.3)
  \]

  The remaining normal orderings vanish.

- In the case of \( \left( 1 - \frac{x^{-(i+1)}w_2}{w_1} \right) f_{1,i}(w_2/w_1)T_1(w_1)T_i(w_2) \) \( (i \geq 2) \):
  
  We take the limit \( w_1 \to x^{-(i+1)}w_2 \).

  \[
  \left( 1 - \frac{x^{-(i+1)}w_2}{w_1} \right) f_{1,i}(w_2/w_1)\Lambda_1(w_1) : \Lambda_2(x^{-i+1}w_2) \prod_{l=1}^{i-1} \Lambda_3(x^{-i+2l+1}w_2) :
  \]

  \[
  \to \frac{[r]_x[r-1]_x}{[1]_x} (x - x^{-1}) : \Lambda_1(x^{-i+1}w_2)\Lambda_2(x^{-i+1}w_2) \prod_{l=1}^{i-1} \Lambda_3(x^{-i+2l+2}w_2) : , \quad (B.4)
  \]

  \[
  \left( 1 - \frac{x^{-(i+1)}w_2}{w_1} \right) f_{1,i}(w_2/w_1)\Lambda_k(w_1) \prod_{l=1}^{i} \Lambda_3(x^{-i+2l-1}w_2) :
  \]

  \[
  \to \frac{[r]_x[r-1]_x}{[1]_x} (x - x^{-1}) : \Lambda_k(x^{-i+1}w_2) \prod_{l=1}^{i} \Lambda_3(x^{-i+2l-1}w_2) : , \quad (k = 1, 2), \quad (B.5)
  \]

  \[
  \left( 1 - \frac{x^{-(i+1)}w_2}{w_1} \right) f_{1,i}(w_2/w_1)\Lambda_3(w_1) : \Lambda^i_l \Lambda_3(x^{-i+2l-1}w_2) :
  \]

  \[
  \to \frac{[r]_x[r-i-1]_x}{[i+1]_x} (x - x^{-1}) : \prod_{l=1}^{i+1} \Lambda_3(x^{-i+2l-1}w_2) : . \quad (B.6)
  \]

  The remaining normal orderings vanish.
• In the case of \( 1 - \frac{x^{i+1}w_2}{w_1} \) \( f_{i,1}(w_2/w_1)T_i(w_1)T_1(w_2) \) \( (i \geq 2) \):

We take the limit \( w_1 \to x^{i+1}w_2 \).

\[
\left(1 - \frac{x^{i+1}w_2}{w_1} \right) f_{i,1}(w_2/w_1) : \Lambda_2(x^{-i+1}w_1) \prod_{l=1}^{i-1} \Lambda_3(x^{-i+2l+1}w_1) : \Lambda_1(w_2)
\]

\[
\to - \frac{[r]_x[r-1]_x}{[1]_x}(x-x^{-1}) : \Lambda_1(w_2) \Lambda_2(x^2w_2) \prod_{l=1}^{i-1} \Lambda_3(x^{2+2l}w_2) : \text{, (B.7)}
\]

\[
\left(1 - \frac{x^{i+1}w_2}{w_1} \right) f_{i,1}(w_2/w_1) : \prod_{l=1}^{i} \Lambda_3(x^{-i+1+2l}w_1) : \Lambda_k(w_2)
\]

\[
\to - \frac{[r]_x[r-1]_x}{[1]_x}(x-x^{-1}) : \Lambda_k(w_2) \prod_{l=1}^{i} \Lambda_3(x^{2l}w_2) : \text{ (k = 1, 2), (B.8)}
\]

\[
\left(1 - \frac{x^{i+1}w_2}{w_1} \right) f_{i,1}(w_2/w_1) : \prod_{l=1}^{i} \Lambda_3(x^{-i+1+2l}w_1) : \Lambda_3(w_2)
\]

\[
\to - \frac{[r]_x[r-i]_x}{[i+1]_x}(x-x^{-1}) : \prod_{l=1}^{i+1} \Lambda_3(x^{2l-2}w_2) : \text{. (B.9)}
\]

The remaining normal orderings vanish.

• In the case of \( 1 - \frac{x^{-(i+1)}w_2}{w_1} \) \( f_{i,1}(w_2/w_1)T_i(w_1)T_1(w_2) \) \( (i \geq 2) \):

We take the limit \( w_1 \to x^{-(i+1)}w_2 \).

\[
\left(1 - \frac{x^{-(i+1)}w_2}{w_1} \right) f_{i,1}(w_2/w_1) : \Lambda_1(x^{-i+1}w_1) \Lambda_2(x^{-i+3}w_1) \prod_{l=1}^{i+2} \Lambda_3(x^{-i+2l+3}w_1) : \Lambda_3(w_2)
\]

\[
\to \frac{[r]_x[r-i+1]_x}{[i-1]_x}(x-x^{-1}) : \Lambda_1(x^{-2i}w_2) \Lambda_2(x^{-2i+2}w_2) \prod_{l=1}^{i-1} \Lambda_3(x^{-2i+2l+2}w_2) : \text{, (B.10)}
\]

\[
\left(1 - \frac{x^{-(i+1)}w_2}{w_1} \right) f_{i,1}(w_2/w_1) : \Lambda_k(x^{-i+1}w_1) \prod_{l=1}^{i-1} \Lambda_3(x^{-i+1+2l}w_1) : \Lambda_3(w_2)
\]

\[
\to \frac{[r]_x[r-i]_x}{[i]_x}(x-x^{-1}) : \Lambda_k(x^{-2i}w_2) \prod_{l=1}^{i} \Lambda_3(x^{-2i+2l}w_2) : \text{ (k = 1, 2), (B.11)}
\]

\[
\left(1 - \frac{x^{-(i+1)}w_2}{w_1} \right) f_{i,1}(w_2/w_1) : \prod_{l=1}^{i} \Lambda_3(x^{-i+1+2l}w_1) : \Lambda_3(w_2)
\]

\[
\to \frac{[r]_x[r-i-1]_x}{[i+1]_x}(x-x^{-1}) : \prod_{l=1}^{i+1} \Lambda_3(x^{-2i+2l}w_2) : \text{. (B.12)}
\]

The remaining normal orderings vanish.
We take the limit \( w_1 \rightarrow x^{i+j}w_2 \).

\[
1 - \frac{x^{i+j}w_2}{w_1} f_{i,j}(w_2/w_1) : \prod_{l=1}^{i} \Lambda_l(x^{-i+2l-1}w_1) : \Lambda_1(x^{-j+1}w_2) \Lambda_2(x^{-j+3}w_2) \prod_{l=1}^{j-2} \Lambda_3(x^{-j+3+2l}w_2) : \\
\rightarrow - \frac{[r]_x[r-1]_x}{[1]_x} (x - x^{-1}) \prod_{l=1}^{i+j-2} \frac{[r-l]_x}{[l]_x} \prod_{l=1}^{i} \frac{[l]_x}{[r-l]_x} \prod_{l=1}^{j-2} \frac{[l]_x}{[r-l]_x}
\]

\[
\times \prod_{l=1}^{\text{Min}(i,j)-1} \Delta_1(x^{2l+1}) : \Lambda_1(x^{-j+1}w_2) \Lambda_2(x^{-j+3}w_2) \prod_{l=1}^{i+j-2} \Lambda_3(x^{-j+3+2l}w_2) :, \quad \text{(B.13)}
\]

\[
\left(1 - \frac{x^{i+j}w_2}{w_1}\right) f_{i,j}(w_2/w_1) : \prod_{l=1}^{i} \Lambda_l(x^{-i+2l-1}w_1) : \Lambda_k(x^{-j+1}w_2) \prod_{l=1}^{i-1} \Lambda_3(x^{-j+2l+1}w_2) :\\
\rightarrow - \frac{[r]_x[r-1]_x}{[1]_x} (x - x^{-1}) \prod_{l=1}^{i+j-1} \frac{[r-l]_x}{[l]_x} \prod_{l=1}^{i} \frac{[l]_x}{[r-l]_x} \prod_{l=1}^{j-1} \frac{[l]_x}{[r-l]_x}
\]

\[
\times \prod_{l=1}^{\text{Min}(i,j)-1} \Delta_1(x^{2l+1}) : \Lambda_k(x^{-j+1}w_2) \prod_{l=1}^{i+j-1} \Lambda_3(x^{-j+2l+1}w_2) : \quad (k = 1, 2), \quad \text{(B.14)}
\]

\[
\left(1 - \frac{x^{i+j}w_2}{w_1}\right) f_{i,j}(w_2/w_1) : \prod_{l=1}^{i} \Lambda_l(x^{-i+2l-1}w_1) : \prod_{l=1}^{j} \Lambda_3(x^{-j+2l-1}w_2) :\\
\rightarrow - \frac{[r]_x[r-1]_x}{[1]_x} (x - x^{-1}) \prod_{l=1}^{i+j} \frac{[r-l]_x}{[l]_x} \prod_{l=1}^{i} \frac{[l]_x}{[r-l]_x} \prod_{l=1}^{j} \frac{[l]_x}{[r-l]_x}
\]

\[
\times \prod_{l=1}^{\text{Min}(i,j)-1} \Delta_1(x^{2l+1}) : \prod_{l=1}^{i+j} \Lambda_3(x^{-j+2l-1}w_2) :, \quad \text{(B.15)}
\]

The remaining normal orderings vanish.

- In the case of \( \left(1 - \frac{x^{-(i+j)}w_2}{w_1}\right) f_{i,j}(w_2/w_1) T_i(w_1) T_j(w_2) \) \((i, j \geq 2)\):

  We take the limit \( w_1 \rightarrow x^{-(i+j)}w_2 \).

\[
\left(1 - \frac{x^{-(i+j)}w_2}{w_1}\right) f_{i,j}(w_2/w_1) : \Lambda_1(x^{-i+1}w_1) \Lambda_2(x^{-i+3}w_1) \prod_{l=1}^{i-2} \Lambda_3(x^{-i+2l+3}w_1) : \prod_{l=1}^{j} \Lambda_3(x^{-j+2l-1}w_2) :\\
\rightarrow - \frac{[r]_x[r-1]_x}{[1]_x} (x - x^{-1}) \prod_{l=1}^{i+j-2} \frac{[r-l]_x}{[l]_x} \prod_{l=1}^{i-2} \frac{[l]_x}{[r-l]_x} \prod_{l=1}^{j} \frac{[l]_x}{[r-l]_x}
\]

\[
\times \prod_{l=1}^{\text{Min}(i,j)-1} \Delta_1(x^{2l+1}) : \Lambda_1(x^{-2i-j+1}w_2) \Lambda_2(x^{-2i-j+3}w_2) \prod_{l=1}^{i+j-2} \Lambda_3(x^{-2i-j+2l+3}w_2) :, \quad \text{(B.16)}
\]

\[
\left(1 - \frac{x^{-(i+j)}w_2}{w_1}\right) f_{i,j}(w_2/w_1) : \Lambda_k(x^{-i+1}w_1) \prod_{l=1}^{i-1} \Lambda_3(x^{-i+2l+1}w_1) : \prod_{l=1}^{j} \Lambda_3(x^{-j+2l-1}w_2) :\\
\rightarrow - \frac{[r]_x[r-1]_x}{[1]_x} (x - x^{-1}) \prod_{l=1}^{i+j-1} \frac{[r-l]_x}{[l]_x} \prod_{l=1}^{i-1} \frac{[l]_x}{[r-l]_x} \prod_{l=1}^{j} \frac{[l]_x}{[r-l]_x}
\]

\[
\times \prod_{l=1}^{\text{Min}(i,j)-1} \Delta_1(x^{2l+1}) : \Lambda_k(x^{-2i-j+1}w_2) \prod_{l=1}^{i+j-1} \Lambda_3(x^{-2i-j+2l+1}w_2) : \quad (k = 1, 2), \quad \text{(B.17)}
\]
\[
\left(1 - \frac{x^{-(i+j)}w_2}{w_1}\right)f_{i,j}(w_2/w_1) : \prod_{l=1}^{i} A_3(x^{-i+2l-1}w_1) :: \prod_{l=1}^{j} A_3(x^{-j+2l-1}w_2) : \\
\rightarrow \frac{[r]_x[r-1]_x}{[1]_x}(x - x^{-1})^{-i+j} \prod_{l=1}^{i} \frac{[r-l]_x}{[l]_x} \prod_{l=i+1}^{j} \frac{[r-l]_x}{[r-l]_x} \\
\times \prod_{l=1}^{\text{Min}(i,j)-1} \Delta_1(x^{2l+1}) : \prod_{l=1}^{i} A_3(x^{-2l-i+2l-1}w_2) : \frac{(B.18)}{.}
\]

The remaining normal orderings vanish.

### B.2 Exchange relation

In this Appendix we summarize some exchange relations in the case of one-fermion diagram.

- In the case of \( f_{i,1}(z_2/z_1)T_1(z_1)T_1(z_2) - f_{i,1}(z_1/z_2)T_1(z_2)T_1(z_1) \):
  
  For \( i \geq 2 \) we have followings.

\[
f_{i,1}(z_2/z_1)A_k(z_1) : A_1(x^{-i+1}z_2)A_2(x^{-i+3}z_2) \prod_{j=1}^{i-2} A_3(x^{-i+2j}z_2) : \ \\
- f_{i,1}(z_1/z_2) : A_1(x^{-i+1}z_2)A_2(x^{-i+3}z_2) \prod_{j=1}^{i-2} A_3(x^{-i+2j}z_2) : A_2(z_1) = 0 \ (k = 1, 2), \quad (B.19)
\]

\[
f_{i,1}(z_2/z_1)A_k(z_1) : A_k(x^{-i+1}z_2) \prod_{j=1}^{i-1} A_3(x^{-i+2j}z_2) : \ \\
- f_{i,1}(z_1/z_2) : A_k(x^{-i+1}z_2) \prod_{j=1}^{i-1} A_3(x^{-i+2j}z_2) : A_k(z_1) = 0 \ (k = 1, 2), \quad (B.20)
\]

\[
f_{i,1}(z_2/z_1)A_1(z_1) : A_2(x^{-i+1}z_2) \prod_{j=1}^{i-1} A_3(x^{-i+2j}z_2) : \ \\
- f_{i,1}(z_1/z_2) : A_2(x^{-i+1}z_2) \prod_{j=1}^{i-1} A_3(x^{-i+2j}z_2) : A_1(z_1) \ \\
= \frac{[r]_x[r-1]_x}{[1]_x}(x - x^{-1}) \left( \delta \left( \frac{x^{-i+1}z_2}{z_1} \right) - \delta \left( \frac{x^{-i+3}z_2}{z_1} \right) \right) : A_1(z_1)A_2(x^{-i+1}z_2) \prod_{j=1}^{i-1} A_3(x^{-i+2j}z_2) :,
\]

\[
f_{i,1}(z_2/z_1)A_k(z_1) : \prod_{j=1}^{i} A_3(x^{-i+2j-1}z_2) : - f_{i,1}(z_1/z_2) : \prod_{j=1}^{i} A_3(x^{-i+2j-1}z_2) : A_k(z_1) \ \\
= \frac{[r]_x[r-1]_x}{[1]_x}(x - x^{-1}) \left( \delta \left( \frac{x^{-i+1}z_2}{z_1} \right) - \delta \left( \frac{x^{-i+3}z_2}{z_1} \right) \right) : A_k(z_1) \prod_{j=1}^{i} A_3(x^{-i+2j}z_2) \ (k = 1, 2), \quad (B.22)
\]

\[
f_{i,1}(z_2/z_1)A_3(z_1) : \prod_{j=1}^{i} A_3(x^{-i+2j-1}z_2) : - f_{i,1}(z_1/z_2) : \prod_{j=1}^{i} A_3(x^{-i+2j-1}z_2) : A_3(z_1) \ \\
= \frac{[r]_x[r-i-1]_x}{[i+1]_x}(x - x^{-1}) \left( \delta \left( \frac{x^{-i+1}z_2}{z_1} \right) - \delta \left( \frac{x^{-i+3}z_2}{z_1} \right) \right) : A_3(z_1) \prod_{k=1}^{i} A_3(x^{-i+2j-1}z_2) :, \quad (B.23)
\]
\[
\begin{align*}
\text{In this Appendix we study } & \mathcal{W}_{\text{qt}}(sl(2|1)) \text{ in another regime. We focus our attention to the regime } 0 < y = x^{-1} < 1. \text{ Define the theta function } [u]^* \text{, the functions } \Delta_i^*(z) (i = 0, 1, 2, \ldots) \text{ and the structure functions } f_{i,j}^*(z) (i, j = 1, 2, 3, \ldots) \text{ by}
\end{align*}
\]
\[
[u]^* = [u]|_{x \to y}, \quad \Delta_i^*(z) = \Delta_i(z)|_{x \to y}, \quad f_{i,j}^*(z) = f_{i,j}(z)|_{x \to y}. \tag{C.1}
\]

Here $[u]$, $\Delta_i(z)$, and $f_{i,j}(z)$ are defined in (2.3), (3.2), and (4.3), respectively.

**Proposition C.1** In the case of one-fermion diagram and $0 < y = x^{-1} < 1$, the deformed screening currents $S_i^+(z)$ introduced in (2.8) satisfy the following commutation relations.

\[
\begin{align*}
S_i^+(z_1)S_j^+(z_2) &= \frac{[u_1 - u_2 - 1]^*}{[u_1 - u_2 + 1]^*} S_i^+(z_2)S_j^+(z_1), \\
S_i^+(z_1)S_j^+(z_2) &= -S_j^+(z_2)S_i^+(z_1), \\
S_i^+(z_1)S_j^+(z_2) &= -\frac{[u_1 - u_2 + \frac{1}{2}]}{[u_1 - u_2 - \frac{1}{2}]} S_j^+(z_2)S_i^+(z_1). \tag{C.2}
\end{align*}
\]
where we set \( z_i = y^{2n_i} \) (\( i = 1, 2 \)). Here we used the theta function \([u]^*\) defined in \( \text{(C.1)} \).

**Proposition C.2** In the case of one-fermion diagram and the regime \( 0 < y = x^{-1} < 1 \), the normal ordering rules of \( \Lambda_i(z) \) (\( i = 1, 2, 3 \)) are given by

\[
\begin{align*}
\Delta_i^*(z_1) \Lambda_i(z_1) \Lambda_j(z_2) &= \Delta_i^*(y_{z_2/z_1}) : \Lambda_i(z_1) \Lambda_j(z_2) : \quad (1 \leq i < j \leq 3), \\
\Delta_i^*(z_2) \Lambda_i(z_1) \Lambda_j(z_2) &= \Delta_i^*(y_{z_2/z_1}^{-1}) : \Lambda_j(z_1) \Lambda_i(z_2) : \quad (1 \leq i < j \leq 3), \\
\Delta_i^*(z_1) \Lambda_i(z_1) \Lambda_j(z_2) &= : \Lambda_i(z_1) \Lambda_i(z_2) : \quad (i = 1, 2), \\
\Delta_i^*(z_2) \Lambda_i(z_1) \Lambda_j(z_2) &= : \Lambda_i(z_1) \Lambda_i(z_2) : . \quad (C.3)
\end{align*}
\]

Here \( \Delta_i^*(z) \) (\( i = 1, 2 \)) and \( f_{i,1}^*(z) \) are defined in \( \text{(C.1)} \).

**Proposition C.3** In the case of two-fermion diagram and \( 0 < y = x^{-1} < 1 \), the deformed \( S_i^+ \) (\( i = 1, 2 \)) introduced in \( \text{(2.8)} \) satisfy the following commutation relations.

\[
\begin{align*}
S_i^+(z_1) S_i^+(z_2) &= - S_i^+(z_2) S_i^+(z_1) \quad (i = 1, 2), \\
S_i^+(z_1) S_i^{+\ast}(z_2) &= - \left[ \frac{u_1 - u_2 - r_i^+}{[u_1 - u_2 + r_i^+]^*} \right] S_i^+(z_2) S_i^{+\ast}(z_1), \quad (C.4)
\end{align*}
\]

where we set \( z_i = y^{2n_i} \) (\( i = 1, 2 \)). Here we used the theta function \([u]^*\) defined in \( \text{(C.1)} \).

**Proposition C.4** In the case of the two-fermion diagram and the regime \( 0 < y = x^{-1} < 1 \). The normal ordering rules of \( \Lambda_i(z) \) (\( i = 1, 2, 3 \)) are given by

\[
\begin{align*}
\Delta_i^*(z_1) \Lambda_i(z_1) \Lambda_j(z_2) &= \Delta_i^*(y_{z_2/z_1}) : \Lambda_i(z_1) \Lambda_j(z_2) : \quad (1 \leq i < j \leq 3), \\
\Delta_i^*(z_2) \Lambda_i(z_1) \Lambda_j(z_2) &= \Delta_i^*(y_{z_2/z_1}^{-1}) : \Lambda_j(z_1) \Lambda_i(z_2) : \quad (1 \leq i < j \leq 3), \\
\Delta_i^*(z_1) \Lambda_i(z_1) \Lambda_j(z_2) &= : \Lambda_i(z_1) \Lambda_i(z_2) : \quad (i = 1, 3), \\
\Delta_i^*(z_2) \Lambda_i(z_1) \Lambda_j(z_2) &= : \Lambda_i(z_1) \Lambda_i(z_2) : . \quad (C.5)
\end{align*}
\]

Here \( \Delta_i^*(z) \) (\( i = 1, 2 \)) and \( f_{i,1}^*(z) \) are defined in \( \text{(C.1)} \).

The formula for \( 0 < y < 1 \) is different from those for \( 0 < x < 1 \) in power index \( y^{\pm 1} \), because \( \mathcal{W}_{qt}(\mathfrak{sl}(2|1)) \) is not invariant under \( x \to x^{-1} \). The formula for \( 0 < x < 1 \) gives \( \mathcal{W}_{qt}(\mathfrak{sl}(2|1)) \)-version of \( \mathcal{W}_{qt}(\mathfrak{sl}(N)) \)-realization given in \( \text{[6,7]} \). The formula for \( 0 < y < 1 \) gives \( \mathcal{W}_{qt}(\mathfrak{sl}(2|1)) \)-version of \( \mathcal{W}_{qt}(\mathfrak{sl}(N)) \)-realization given in \( \text{[8]} \).

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