Degeneracy of Landau levels and quantum group $sl_q(2)$

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Abstract

We show that there is a kind of quantum group symmetry $sl_q(2)$ in the usual Landau problem and it is this quantum group symmetry that governs the degeneracy of Landau levels. We find that under the periodic boundary condition, the degree of degeneracy of Landau levels is finite, and it just equals the dimension of the irreducible cyclic representation of the quantum group $sl_q(2)$. 
It is well known that the symmetry play an important role in physics. Sometimes we need not solve the problem explicitly, we can obtain much important and interesting information of a physical system or simplify our calculation by analyzing the symmetry of system. In particular, the degeneracy of energy levels is often related to some dynamical symmetries of a system [1,2]. In general, for a system with Hamiltonian $H$, let $\hat{F}$ and $\hat{G}$ denote two operators corresponding to physical quantities of the system, if they do not commute with each other, i.e., $[\hat{F}, \hat{G}] \neq 0$, and both are the conservative quantities, then energy levels of the system must be degenerate except for few special levels. For example, for a spinless particle moving in a plane, this system has the Euclidian group symmetry whose generators are momentum operator $\hat{p}_i (i = x, y)$ and the $z$-component of the angular momentum $\hat{L}_z$. Since $\hat{p}_i (i = x, y)$ and $\hat{L}_z$ do not commute with each other, and both of them are the conservative quantities of the system, energy levels of this system exhibit infinite-fold degeneracy which comes from the infinity of the dimension of the irreducible representation of the Euclidian group [2].

On the other hand, in the past ten years, the so-called quantum group symmetry (QGS) and its representation theory has attracted the attention of the physicists and mathematicians [8,12-14]. Needless to say, the mathematical structure of the QGS is very beautiful, however, to the physicist, they are more interested in its application in the physics, In this respect, P. B. Wiegmann et al. [16] and Y. Hatsugai et al. [17] has completed their original exploration. Certainly, it is still very interesting to look for more applications of the QGS in physics.

In this letter, we will show that the QGS also can be found even in the simplest system of quantum mechanics and it is the origin of the degeneracy of energy levels in our problem. In more detail, with the help of the representation theory of quantum group, we determined the degree of the degeneracy of Landau energy levels for such a system which a spinless particle moves in a plane and experiences a uniform external magnetic field $\vec{B}$.

We consider a spinless particle which moves in a plane and experiences an uniform external magnetic field along $z$-direction, $\vec{B} = B\hat{e}_z$. The Hamiltonian of system can be written as

$$H = \frac{1}{2m} (\vec{p} + e\vec{A})^2$$

where $m, e$ are the mass and charge of particle, respectively. $\vec{A}$ is vector potential which
satisfy
\[ \nabla \times \vec{A} = B \vec{e}_z \tag{2} \]

The above problem can be easily solved in a proper gauge [1,3]. In present paper, we study the gauge-independent case with a periodical boundary condition (PBC). It is well known that in this system there does not exist the translational invariance, however, it can exhibits magnetic translation invariance which generated by the magnetic translation operator [4] defined by
\[ t(\vec{a}) = \exp\left[ \frac{i}{\hbar} \vec{a} \cdot (\vec{p} + e\vec{A} + e\vec{r} \times \vec{B}) \right] \tag{3} \]
where \( \vec{a} = a_x \hat{e}_x + a_y \hat{e}_y \) is an arbitrary two-dimentional vector. The magnetic translation operator \( t(\vec{a}) \) satisfies the following group property [5,6]:
\[ t(\vec{a}) t(\vec{b}) = \exp\left[ -\frac{i}{\hbar} \vec{e}_z \cdot (\vec{a} \times \vec{b}) \right] t(\vec{b}) t(\vec{a}) \tag{4} \]
where \( a_0 \equiv \sqrt{\frac{\hbar}{eB}} \) is the magnetic length.

Let
\[ \vec{\kappa} = \vec{p} + e\vec{A} + e\vec{r} \times \vec{B} \tag{5} \]

It is easy to prove that
\[ [t(\vec{a}), H] = 0, \quad [\vec{\kappa}, H] = 0 \tag{6} \]
which means that the system under consideration is invariant under the magnetic translation transformation Eq.(3), and \( \vec{\kappa} \) is a conservative quantity.

With the help of the magnetic translation operator, one can construct the following operators [7]:
\[
\begin{align*}
J_+ & = \frac{1}{q - q^{-1}}[t(\vec{a}) + t(\vec{b})], \\
J_- & = \frac{-1}{q - q^{-1}}[t(-\vec{a}) + t(-\vec{b})], \\
q^{2J_3} & = t(\vec{b} - \vec{a}), \\
q^{-2J_3} & = t(\vec{a} - \vec{b})
\end{align*}
\tag{7}
\]
with
\[ q = \exp(\frac{i2\pi \Phi}{\Phi_0}) \tag{8} \]
where \( \Phi = \frac{1}{2} \vec{B} \cdot (\vec{a} \times \vec{b}) \) is magnetic flux through the triangle enclosed by vector \( \vec{a} \) and \( \vec{b} \), \( \Phi_0 = \frac{\hbar}{2} \) is magnetic flux quanta. A straightforward calculation shows that these operators \( J_+, J_- \) and \( J_3 \) satisfy the algebraic relation of the quantum group \( sl_q(2) \) [8] as follows:

\[
[J_+, J_-] = [2J_3]_q
\]

\[
q^{J_3} J_\pm q^{-J_3} = q^\pm J_\pm
\]

(9)

where we have used the following notation:

\[
[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}
\]

(10)

From Eqs.(6) and (7) it follows that:

\[
[J_\pm, H] = 0, \quad [q^{\pm J_3}, H] = 0
\]

(11)

which indicates that \( J_\pm \) and \( J_3 \) are conservative quantities of the system. Therefore, there is the quantum group \( sl_q(2) \) in the Landau problem under our consideration.

Let \( \Psi \) be wave function of system in Schrödinger picture. In order to calculate explicitly the degree of degeneracy of Landau levels, we impose the following PBC on the wave function[9]:

\[
t(\vec{L}_1)\Psi = \Psi, \quad t(\vec{L}_2)\Psi = \Psi
\]

(12)

where \( \vec{L}_1 = L_1\hat{e}_x \), and \( \vec{L}_2 = L_2\hat{e}_y \). This boundary condition means that the particle is confined in a rectangular area of size \( L_1 \times L_2 \). From Eq.(12) it follows that the operators \( t(\vec{L}_1) \) and \( t(\vec{L}_2) \) commute with each other. That is,

\[
t(\vec{L}_1)t(\vec{L}_2) = t(\vec{L}_2)t(\vec{L}_1)
\]

(13)

however, from Eq.(4) we have

\[
t(\vec{L}_1)t(\vec{L}_2) = \exp[-i\frac{\vec{e}_z \cdot (\vec{L}_1 \times \vec{L}_2)}{a_0^2}]t(\vec{L}_2)t(\vec{L}_1)
\]

(14)

Combining Eq.(13) with Eq.(14) yields that

\[
\exp(i2\pi \frac{\Phi}{\Phi_0}) = 1
\]

(15)

where \( \Phi = \frac{1}{2}BL_1L_2 \) is the magnetic flux through the triangle enclosed by \( \vec{L}_1 \) and \( \vec{L}_2 \). Eq.(15) implies that

\[
\Phi = N_s\Phi_0
\]

(16)
where $N_s$ is a positive integer. Therefore, the periodic boundary condition Eq.(12) is equivalent to the magnetic flux quantization.

Notice that not all the translation operators $t(\vec{a})$ can keep the boundary condition Eq.(12) invariant. In other words,

$$t(\vec{L}_i)t(\vec{a})\Psi = t(\vec{a})\Psi$$  \hspace{1cm} (i = 1, 2)  \hspace{1cm} (17)

can not be satisfied by an arbitrary magnetic translation $t(\vec{a})$. However, if we define two primitive magnetic translation operators in the following way [9]:

$$T_x \equiv t(\frac{\vec{L}_1}{N_s}), \hspace{1cm} T_y \equiv t(\frac{\vec{L}_2}{N_s})$$  \hspace{1cm} (18)

One can find that only $T_x, T_y$ and their integer powers can make Eq.(17) hold.

By a straightforward calculation, it can be checked that the following relations hold

$$T_y T_x = \exp(i\frac{2\pi}{N_s})T_x T_y,$$

$$T_y T_{-x} = \exp(-i\frac{2\pi}{N_s})T_{-x} T_y$$  \hspace{1cm} (19)

$$T_{-y} T_x = \exp(-i\frac{2\pi}{N_s})T_x T_{-y},$$

$$T_{-y} T_{-x} = \exp(i\frac{2\pi}{N_s})T_{-x} T_{-y}$$  \hspace{1cm} (20)

$$T_{-x} T_x = T_{-y} T_y = 1$$  \hspace{1cm} (21)

Making use of the operators $T_{\pm x}, T_{\pm y}$ and the above commutation relations, we can construct a basic quantum group with the generators as follows:

$$J_+ = \frac{-i}{q - q^{-1}}(T_{-x} + T_{-y}), \hspace{1cm} J_- = \frac{-i}{q - q^{-1}}(T_x + T_y)$$  \hspace{1cm} (22)

$$K^{+2} = qT_{-y}T_x, \hspace{1cm} K^{-2} = q^{-1}T_{-x}T_y$$  \hspace{1cm} (23)

where the deformation parameter is given by

$$q = \exp(i\frac{\pi}{N_s})$$  \hspace{1cm} (24)

It is easy to check that these generators obey the standard commutation relations of the quantum group $sl_q(2)$ [8]:

$$[J_+, J_-] = \frac{K^{+2} - K^{-2}}{q - q^{-1}}, \hspace{1cm} K^{+} J_{\pm} K^{-} = q^{\pm 1} J_{\pm}$$  \hspace{1cm} (25)
We can also find that the generators $J_\pm$ and $K^\pm$ are conservative quantities of the system under our consideration, namely

$$[J_\pm, H] = 0, \quad [K^\pm, H] = 0 \quad (26)$$

The above analysis indicates that $sl_q(2)$ is the basic symmetry in our system. Furthermore, according to the fundamental principle of quantum mechanics, Eq.(25) and Eq.(26) imply that there is degeneracy of Landau levels in the system.

In what follows we will discuss the relation between degeneracy of Landau levels and the cyclic representation of $sl_q(2)$. Since $N_s$ is an integer, from Eq.(24) we see that

$$q^{2N_s} = 1 \quad (27)$$

which means that $q$ is a root of unity. In this case, the representation of quantum group has many exotic properties [10,11]. Typically, it has the cyclic representation, which implies that there is neither highest weight nor the lowest weight [10,11], and the dimension of the irreducible representation is $2N_s$ in the case under our consideration.

Furthermore, without loss of generality, according to Eq.(26) we can simultaneously diagonalize $H$ and $K^\pm$. In other words, one can choose a set of basis vectors $|n,k\rangle = |n\rangle \otimes |k\rangle$ to be the simultaneous eigenvectors of operators $H$ and $K^\pm$. That is, we can take

$$H |n,k\rangle = E_n |n,k\rangle \quad (28)$$

and

$$K^\pm |n,k\rangle = q^{\pm(\lambda-2k-2\mu)} |n,k\rangle \quad (29)$$

where $n = 0, 1, ..., \infty$ is the symbols of the energy level, and $k = 0, 1, ..., 2N_s - 1$ is the new quantum numbers which distinguish the different quantum states in the same degenerate energy level.

According to the representation theory of quantum group at root of unity [12,13,14], the actions of the $sl_q(2)$ generators on these basis vectors are given by

$$J_\pm |n,k\rangle = [\lambda - \mu - k + 1] |n,k-1\rangle, \quad (1 \leq k \leq 2N_s - 1) \quad (30)$$

$$J_+ |n,0\rangle = \xi^{-1}[\lambda - \mu + 1] |n,2N_s - 1\rangle \quad (31)$$

$$J_- |n,k\rangle = |n,k+1\rangle, \quad (0 \leq k \leq 2N_s - 2) \quad (32)$$
\[ J_- |n, 2N_s - 1\rangle = \xi |n, 0\rangle \]  

(33)

where \( \lambda, \xi, \mu \) are constants determined by the cyclic properties of the representation of quantum group and the notation \([x] = \frac{q^x - q^{-x}}{q - q^{-1}}\) has been used.

Since the dimension of the irreducible representation space \(|n, k\rangle\) is \(2N_s\), from Eqs.(28), (29) and (30) we can see that the degree of degeneracy of Landau levels is just \(2N_s\)[15]. This is one of the main conclusions of this paper. In particular, when the boundary of the system approaches to the infinity (i.e. \(L_1 \rightarrow \infty, L_2 \rightarrow \infty\)), we can see that \(2N_s \rightarrow \infty\). In this case, the system exhibits the continous degeneracy, this is a well known result.

In summary, we have shown that there is quantum group symmetry in the Landau problem, and the existence of the quantum group symmetry is independent of the choice of the gauge, and the degeneracy of Landau levels in the system under our consideration originates from the quantum group symmetry \(sl_q(2)\). We have found that under the PBC, the degree of the degeneracy of Landau levels is finite, and it is just the dimension of the irreducible cyclic representation of the quantum group \(sl_q(2)\). When the boundary approaches to infinity, the usual result on the degeneracy of Landau level can be recovered. It is worth mentioning that for a particle with spin, for instance, an electron moving in a plane, although each energy level will split into two due to the additional Zeeman’s energy, However, the degree of the degeneracy of its energy levels still keep \(2N_s\) except the ground state for which the degree of the degeneracy is \(N_s\). The reason is that energy levels overlap between the upper level and lower level except the lowest one, which is just the ground state.

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