A Slightly Supercritical Condition of Regularity of Axisymmetric Solutions to the Navier–Stokes Equations

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Abstract. In the note, a new regularity condition for axisymmetric solutions to the non-stationary 3D Navier–Stokes equations is proven. It is slightly supercritical.

Keywords. Navier–Stokes equations, Axisymmetric solutions, Local regularity.

1. Introduction

In this note, we continue to analyse potential singularities of axisymmetric solutions to the non-stationary Navier–Stokes equations. In the previous paper [24], it has been shown that an axially symmetric solution is smooth provided a certain scale-invariant energy quantity of the velocity field is bounded. By definition, a potential singularity with bounded scale-invariant energy quantities is called the Type I blowup. It is important to notice that the above result does not follow from the so-called $\varepsilon$-regularity theory developed in [2, 16], and [10], where regularity is coming out due to smallness of those scale-invariant energy quantities.

We consider the 3D Navier–Stokes system

$$\partial_t v + v \cdot \nabla v - \Delta v = -\nabla q, \quad \text{div } v = 0$$

in the parabolic cylinder $Q = \mathbb{C} \times ]-1,0[$, where $\mathbb{C} = \{x = (x_1, x_2, x_3) : x_1^2 + x_2^2 < 1, -1 < x_3 < 1\}$. A solution $v$ and $q$ is supposed to be a suitable weak one, which means the following:

Definition 1.1. Let $\omega \subset \mathbb{R}^3$ and $T_2 > T_1$. The pair $w$ and $r$ is a suitable weak solution to the Navier–Stokes system in $Q_* = \omega \times]T_1,T_2]$ if:

1. $w \in L_{2,\infty}(Q_*), \nabla w \in L_2(Q_*), r \in L_2^2(Q_*);$  
2. $w$ and $r$ satisfy the Navier–Stokes equations in $Q_*$ in the sense of distributions;  
3. for a.a. $t \in [T_1, T_2]$, the local energy inequality

$$\int_{\omega} \varphi(x,t)|w(x,t)|^2dx + 2 \int_{T_1}^{t} \int_{\omega} \varphi|\nabla w|^2 dxdt' \leq \int_{T_1}^{t} \int_{\omega} (||w||^2(\partial_t \varphi + \Delta \varphi) + w \cdot \nabla \varphi(|w|^2 + 2r)|dxdt'$$

holds for all non-negative $\varphi \in C^1_0(\omega \times]T_1,T_2 + (T_2 - T_1)/2[).$

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In our standing assumption, it is supposed that a suitable weak solution \( v \) and \( q \) to the Navier–Stokes equations in \( Q = C \times ] -1,0[ \) is axially symmetric with respect to the axis \( x_3 \). The latter means the following: if we introduce the corresponding cylindrical coordinates \((\rho, \varphi, x_3)\) and use the corresponding representation \( v = v_\rho e_\rho + v_\varphi e_\varphi + v_3 e_3 \), then \( v_\rho = v_\varphi = v_3 = q = 0 \).

There are many papers on regularity of axially symmetric solutions. We cannot pretend to cite all good works in this direction. For example, let us mention papers: [3–5,9,11–13,19–21,26–31], and [15]. Actually, our note is inspired by the paper [20], where the regularity of solutions has been proved under a slightly supercritical assumption. We would like to consider a different supercritical assumption, to give a different proof and to get a better result.

To state our supercritical assumption, additional notation is needed. Given \( x = (x_1, x_2, x_3) \in \mathbb{R}^3 \), denote \( x' = (x_1, x_2, 0) \). Next, different types of cylinders will be denoted as \( \mathcal{C}(r) = \{ x : |x'| < r, |x_3| < r \} \), \( \mathcal{C}(0, r) = \mathcal{C}(r) + x_0 \), \( Q^{\lambda, \mu}(r) = \mathcal{C}(\lambda r) \times ] \mu r^2, 0[ \), \( Q^{1,1}(r) = Q(r) \), \( Q^{\lambda, \mu}(z_0, r) = \mathcal{C}(x_0, \lambda r) \times ]t_0 - \mu R^2, t_0[ \).

And, finally, we let

\[
f(R) := \frac{1}{\sqrt{R}} \left( \int_{-R^2}^0 \left( \int_{\mathcal{C}(r)} |v|^3 \, dx \right)^{\frac{4}{3}} \, dt \right)^{\frac{3}{4}}
\]

and

\[
M(R) := \frac{1}{\sqrt{R}} \left( \int_{Q(R)} |v|^{\frac{30}{11}} \, dz \right)^{\frac{11}{30}}
\]

for any \( 0 < R \leq 1 \) and assume that:

\[
f(R) + M(R) \leq g(R) := c_* \ln \alpha \ln^{\frac{1}{2}}(1/R)
\]

for all \( 0 < R \leq 2/3 \), where \( c_* \) and \( \alpha \) are positive constants and \( \alpha \) obeys the condition:

\[
0 < \alpha \leq \frac{1}{224}.
\]

Without loss of generality, one may assume that \( g(R) \geq 1 \) for \( 0 < R \leq \frac{2}{3} \). To ensure the above condition, it is enough to increase the constant \( c_* \) if necessary.

Our aim could be the following completely local statement.

**Theorem 1.2.** Assume that a pair \( v \) and \( q \) is an axially symmetric suitable weak solution to the Navier–Stokes equations in \( Q \) and conditions (1.2) and (1.3) hold. Then the origin \( z = 0 \) is a regular point of \( v \).

However, in this paper, we shall prove a weaker result leaving Theorem 1.2 as a plausible conjecture. We shall return to a proof of Theorem 1.2 elsewhere. In the present paper, the following fact is going to be justified.

**Theorem 1.3.** Let \( v \) be an axially symmetric solution to the Cauchy problem for the Navier–Stokes equations (1.1) in \( \mathbb{R}^3 \times ]0, T[ \) with initial divergence free field \( v_0 \) from the Sobolev space \( H^2 = W^2_2(\mathbb{R}^3) \) such that

\[
\sup_{0 < t < T - \delta} \| \nabla v(\cdot, t) \|_{L_2(\mathbb{R}^3)} \leq C(\delta) < \infty
\]

for all \( 0 < \delta < T \). Assume further that

\[
\Sigma_0 = \sup_{x \in \mathbb{R}^3} |v_{02}(x)x_1 - v_{01}(x)x_2| < \infty
\]

and

\[
\sup_{-\infty < h < \infty} f(R; (0, h, T)) + M(R; (0, h, T)) \leq g(R)
\]
for all $0 < R \leq 2/3$ (it is assumed for simplicity that $T > 1$), with some positive constants $c_\ast$ and $\alpha$, satisfying (1.3), where

$$
|\partial_t \sigma(z) - \Delta \sigma(z)| \leq (\sup_{z=(x,t) \in P(\delta,R;R) \times -R^2,0}|v(z)| + 2/\delta)|\nabla \sigma(z)|
$$

for any $0 < \delta < R < 1$, where $P(a,b;h) = \{ x : a < |x'| < b, |x_3| < h \}$. Since $v$ is axially symmetric, the first factor on the right hand side is finite. This fact, by iteration, yields

$$
\sigma \in W^{2,1}_p(P(\delta,R;R) \times -R^2,0)\}
$$

for any $0 < \delta < R < 1$ and for any finite $p \geq 2$.

It follows from the above partial regularity theory that, for any $-1 < t < 0$,

$$
\sigma(x',x_3,t) \to 0 \quad \text{as} \quad |x'| \to 0
$$

(1.7)

for all $x_3 \in [-1,1]\setminus S_\delta^R$, where $S_\delta^R = \{ x_3 \in [-1,1] : (0,x_3,t) \in S^\sigma \}$. In the same way, as it has been done in [26] and [24], one can show that $\sigma \in L_{\infty}(Q(R))$ for any $0 < R < 1$.

The main part of the proof of Theorem 1.3 is the following fact.

**Proposition 1.4.** Let $\sigma = g v_\varphi$, then

$$
\text{osc}_{z \in Q(r)} \sigma(z) \leq e^{-c\left[\ln^{1/2}(1/(2r)) - \ln^{1/2}(1/(2R))\right]} \text{osc}_{z \in Q(2R)} \sigma(z)
$$

where $c$ is a positive absolute constant and $0 < r < R \leq R_\ast(c_\ast, \alpha) \leq 1/6$.

Here, $\text{osc}_{z \in Q(r)} \sigma(z) = M_r - m_r$ and

$$
M_r = \sup_{z \in Q(r)} \sigma(z), \quad m_r = \inf_{z \in Q(r)} \sigma(z).
$$

The above statement is an improvement of the result in [20], where the bound for oscillations of $\sigma$ contains a fixed power of logarithmic factor only.
Remark 1.5. It is not so difficult to see that all results of the paper remain to be true if we replace \( v \) with \( \psi = v_0 e_\theta + v_3 e_3 \) in the definitions of quantities \( M \) and \( f \), see conditions (1.2) and (1.5).

The proof of Proposition 1.4 is based on a technique developed in [18], see also references there. We also would like to mention interesting results for the heat equation with a divergence free drift, see [1, 6, 7, 25].

2. Auxiliary Facts

Define the class \( \mathcal{V} \) of functions \( \pi : Q \to \mathbb{R} \) possessing the properties:

(i) there exists a closed set \( S^\pi \) in \( Q \), whose 1D-parabolic measure \( \mathbb{R}^3 \times \mathbb{R} \) is equal to zero and \( x' = 0 \) for any \( z = (x', x_3, t) \in S^\pi \), such that any spatial derivative is Hölder continuous in \( Q \setminus S^\pi \);

(ii) \( \pi \in W^{2,1}_2 (\mathbb{P}(\delta, R; R) \times [-R^2, 0]) \cap L^\infty(Q(R)) \)

for any \( 0 < \delta < R < 1 \).

We are going to use the following subclass \( \mathcal{V}_0 \) of the class \( \mathcal{V} \), saying that \( \pi \in \mathcal{V}_0 \) if and only if \( \pi \in \mathcal{V} \) and

\[
\partial_t \pi + \left( v + 2 \frac{x'}{|x'|^2} \right) \cdot \nabla \pi - \Delta \pi = 0
\]

in \( C \setminus \{x' = 0\} \times ]1,0[. \)

We shall also say that \( \pi \in \mathcal{V}_0 \) has the property \( (\mathcal{B}_R) \) in \( Q(2R) \) if there exists a number \( k_R > 0 \) such that \( \pi(0, x_3, t) \geq k_R \) for \( -(2R)^2 \leq t \leq 0, x_3 \in ]-2R, 2R[ \setminus S^\pi \).

Remark 2.1. Let \( 0 < r \leq R \) and \( \pi \in \mathcal{V}_0 \) have the property \( (\mathcal{B}_R) \) in \( Q(2R) \). Then \( \pi \) has the property \( (\mathcal{B}_r) \) in \( Q(2r) \) with any constant less or equal to \( k_R \).

In what follows, we always suppose that \( 0 < R \leq 1/6 \).

Proposition 2.2. Let \( \pi \in \mathcal{V}_0 \) have the property \( (\mathcal{B}_R) \). Then, for any \( 0 < k \leq k_R \), for any \( 0 < \tau_1 < \tau < 2 \), and for any \( 0 < \gamma_1 < \gamma < 4 \), the following inequality holds:

\[
\sup_{z \in Q^{\tau_1, \gamma_1}(R)} \sigma(z) \leq c_1(\tau_1, \tau, \gamma_1, \gamma, M(2R)) \left( \frac{1}{Q^{\tau, \gamma}(R)} \int_{Q^{\tau, \gamma}(R)} \sigma^{10} \pi^\frac{1}{10} \, dz \right)^{\frac{1}{10}},
\]

where \( \sigma = (k - \pi)^+ \),

\[
c_1(\tau_1, \tau, \gamma_1, \gamma, M(2R)) = \frac{c}{(\tau - \tau_1)^{10/10}} \left( 1 + \frac{\tau - \tau_1}{\sqrt{\gamma - \gamma_1}} + \left( \frac{1}{\gamma_1 \tau_1^3} \right)^{\frac{1}{10}} M(2R)^3 \right),
\]

and \( Q^{\tau, \gamma}(R) = C(\tau R) \times ]-\gamma R^2, 0[. \)

Proof. Repeating arguments in [24], we can get the following estimate of \( h = \sigma^{m} \):

\[
\left( \int_{r_2}^{0} \int_{C(r_2)} |h|^{\frac{10}{10}} \, dz \right)^{\frac{1}{10}} \leq c \left( \int_{r_1}^{0} \int_{C(r_1)} |h|^{\frac{10}{10}} \, dz \right)^{\frac{1}{10}} \left( \frac{(r_1^3 |t_1|)^{\frac{1}{10}}}{r_1 - r_2} \left( 1 + \frac{r_1 - r_2}{\sqrt{t_2 - t_1}} + \frac{|t_1|^{\frac{1}{10}}}{(r_1 - r_2)^{\frac{1}{10}}} \right) \right)^{\frac{1}{10}},
\]

(2.3)
for any $0 < r_2 < r_1 < 2R$ and $-4R^2 < t_1 < t_2 < 0$, where

$$\mathcal{M}(r_1, t_1) = \left( \frac{1}{|t_1|^2} \right)^{\frac{1}{m}} \int_0^0 \int_{C(r_1)} |v|^{\frac{10}{m}} d\sigma .$$

Next, we wish to iterate (2.3). To this end, let

$$m = m_i = \left( \frac{4}{3} \right)^i ,$$

$$r_1 = r_i = \tau_1 R + (\tau - \tau_1) R 2^{-i+1} , \quad r_2 = r_{i+1} ,$$

$$t_1 = t_i = -\gamma_1 R^2 - (\gamma - \gamma_1) R 2^{-i+1} , \quad t_2 = t_{i+1} ,$$

where $i = 1, 2, ...$. Then, we can derive from (2.3) the following inequality

$$G_{i+1} \leq \left( \frac{c^{i+1}}{\tau - \tau_1} \right)^{\frac{1}{m_i}} \left( 1 + \frac{\tau - \tau_1}{\sqrt{\gamma} - \gamma_1} + \mathcal{M}(r_i, t_i) + \frac{2(i+1)\gamma}{(\tau - \tau_1)^\gamma} \right)^{\frac{1}{m_i}} G_i , \quad (2.4)$$

where

$$G_i = \left( \frac{1}{|t_i|^2} \int_0^0 \int_{C(r_i)} \sigma^{\frac{2}{m_i}} d\sigma \right)^{\frac{2}{m_i}} .$$

Noticing that

$$\mathcal{M}(r_i, t_i) \leq c \left( \frac{1}{\gamma_1 \tau_1^2} \right)^{\frac{1}{m_i}} M(2R) ,$$

let us make use of (2.4) to obtain the estimate

$$G_{i+1} \leq \left( \frac{c^{i+1}}{\tau - \tau_1} \right)^{\frac{1}{m_i}} \left( 1 + \frac{\tau - \tau_1}{\sqrt{\gamma} - \gamma_1} + \frac{2(i+1)\gamma}{(\tau - \tau_1)^\gamma} + \left( \frac{1}{\gamma_1 \tau_1^3} \right)^{\frac{1}{m_i}} M(2R) \right)^{\frac{1}{m_i}} G_i , \quad (2.5)$$

which, after iterations, gives the following

$$G_{i+1} \leq \xi_i G_1 , \quad (2.6)$$

where

$$\xi_i = \prod_{k=1}^i \left( \frac{c^{2k+1}}{\tau - \tau_1} \right)^{\frac{1}{m_k}} \left( 1 + \frac{\tau - \tau_1}{\sqrt{\gamma} - \gamma_1} + \frac{2(k+1)\gamma}{(\tau - \tau_1)^\gamma} + \left( \frac{1}{\gamma_1 \tau_1^3} \right)^{\frac{1}{m_k}} M(2R) \right)^{\frac{1}{m_k}} .$$

Obviously,

$$\xi_i \leq \prod_{k=1}^i \left( \frac{c^{2k+1}}{\tau - \tau_1} \right)^{\frac{1}{m_k}} \left( 1 + \frac{2(k+1)\gamma}{(\tau - \tau_1)^\gamma} \right)^{\frac{1}{m_k}} \left( 1 + \frac{\tau - \tau_1}{\sqrt{\gamma} - \gamma_1} + \left( \frac{1}{\gamma_1 \tau_1^3} \right)^{\frac{1}{m_k}} M(2R) \right)^{\frac{1}{m_k}} .$$

Next,

$$\ln \xi_i \leq A_1 + A_2 + A_3 ,$$
where

\[ A_1 = \sum_{k=1}^{i} \frac{1}{m_k} (\ln c + (k + 1) \ln 2 - \ln(\tau - \tau_1)) \leq \ln c - 3 \ln(\tau - \tau_1), \]

\[ A_2 = \sum_{k=1}^{i} \frac{1}{m_k} \ln \left( 1 + \frac{2^{(k+1)\lambda}}{(\tau - \tau_1)^{\lambda}} \right) = \sum_{k=1}^{i} \frac{1}{m_k} \ln \left( \frac{2^{(k+1)\lambda}}{(\tau - \tau_1)^{\lambda}} \right) + \frac{1}{m_k} \ln \left( 1 + \frac{(\tau - \tau_1)^{\lambda}}{2^{(k+1)\lambda}} \right) \leq \ln \frac{c}{(\tau - \tau_1)^{\lambda}}, \]

\[ +(\tau - \tau_1)^{\lambda} \sum_{k=1}^{i} \frac{1}{m_k} \frac{1}{2^{(k+1)\lambda}} \leq \ln \frac{c}{(\tau - \tau_1)^{\lambda}}, \]

and

\[ A_3 = \ln \left( 1 + \frac{\tau - \tau_1}{\sqrt{\gamma - \gamma_1}} + \left( \frac{1}{\gamma_1 \tau_1^3} \right)^{\frac{1}{10}} M(2R) \right) \sum_{k=1}^{i} \frac{1}{m_k} \]

\[ \leq \ln \left( 1 + \frac{\tau - \tau_1}{\sqrt{\gamma - \gamma_1}} + \left( \frac{1}{\gamma_1 \tau_1^3} \right)^{\frac{1}{10}} M(2R) \right)^3. \]

So,

\[ \xi_i \leq \frac{c}{(\tau - \tau_1)^{\frac{10}{15}}} \left( 1 + \frac{\tau - \tau_1}{\sqrt{\gamma - \gamma_1}} + \left( \frac{1}{\gamma_1 \tau_1^3} \right)^{\frac{1}{10}} M(2R) \right)^3. \]

Passing to the limit as \( i \to \infty \) in (2.6), we complete the proof the Proposition. \( \square \)

**Remark 2.3.** If we additionally assume that \( \pi(\cdot, -\theta R^2) \geq k \) in \( B \) for some \( 0 < \theta \leq 1 \), then we do not need to use a cut-off in \( t \). So, for \( 0 < \lambda < 1 \), we have

\[ \sup_{Q^{\lambda,\sigma}(R)} \sigma \leq c'_1(\lambda, \theta, M(2R)) \left( \frac{1}{|Q^{\lambda,\theta}(R)|} \int_{Q^{\lambda,\sigma}(R)} \sigma^{\frac{10}{3}} dz \right)^{\frac{3}{10}}, \]

where

\[ c'_1(\lambda, \theta, M(2R)) = \frac{c}{(1 - \lambda)^{\frac{10}{3}}} \left( 1 + \left( \frac{1}{\theta \lambda^3} \right)^{\frac{1}{10}} M(2R) \right)^{\frac{3}{10}}. \]

**Corollary 2.4.** Let a non-negative function \( \pi \in \mathcal{V}_0 \) have the property (\( B_R \)) in \( Q(2R) \) and let \( 0 < \lambda_1 < \lambda < 2 \) and \( 0 < \theta \leq 1 \). Suppose that

\[ |\{ \pi < k \} \cap Q^{\lambda,\theta}((0, t_0), R)| < \mu |Q^{\lambda,\theta}(R)| \]

for some \( t_0 > -4R^2 \), for some \( 0 < k \leq k_R \), and for some

\[ 0 < \mu \leq \mu_*= \left( \frac{1}{2c_1(\lambda_1, \lambda, \theta/2, \theta, M(2R))} \right)^{\frac{10}{3}}. \]

Then \( \pi \geq \frac{k}{2} \) in \( Q^{\lambda_1,\theta/2}((0, t_0), R) \).

If, in addition, \( \pi(\cdot, t_0 - \theta R^2) > k \) in \( C(\lambda R) \), then \( \pi \geq \frac{k}{2} \) in \( Q^{\lambda_1,\theta}((0, t_0), R) \).

**Proof.** The first statement can be proved ad absurdum with the help of inequality (2.2) and a suitable choice of the number \( \mu_* \). The second statement is proved in the same way but with the help of the inequality of Remark 2.3. Number \( \mu_* \) is defined by the constant \( c_1 \) instead of \( c_1 \). \( \square \)
The two lemmas below are obvious modifications of the corresponding statements in the paper [18].

**Lemma 2.5.** Let $0 \leq \pi \in \mathcal{V}_0$ have the property $(\mathcal{B}_R)$ in $Q(2R)$. Given $0 < \delta_0 \leq 1$, there exists a positive number $\theta_0(\delta_0, f(2R)) \leq 1$ such that if, for $0 < \theta \leq \theta_0$, $0 < k_0 \leq k_R$, there holds

$$\left| \left\{ \pi(\cdot, t_0 - \theta R^2) \geq k_0 \right\} \cap C(R) \right| > \frac{\delta_0}{3}|C(R)|,$$

then

$$\left| \left\{ \pi(\cdot, t) \geq \frac{\delta_0}{3}k_0 \right\} \cap C(R) \right| > \frac{\delta_0}{3}|C(R)|$$

for all $t \in [t_0 - \theta R^2, t_0]$.

**Remark 2.6.** There is a formula for $\theta_0$:

$$\theta_0 = \left( \frac{cR^6}{1 + \delta_0^2 f(2R)} \right)^\frac{4}{3}.$$

**Lemma 2.7.** Let $0 \leq \pi \in \mathcal{V}_0$ have the property $(\mathcal{B}_R)$ in $Q(2R)$. Let, for any $t \in [t_0 - \theta_1 R^2, t_0]$, $k_1 \leq k_R$ for all $0 < k_1 \leq k_R$ and for some $0 < \delta_1 \leq 1$ and $0 < \theta_1 \leq 1$.

Then, for any $\mu_1 \in [0, 1]$, the following inequality is valid:

$$\left| \left\{ \pi < 2^{-s}k_1 \right\} \cap Q^{1,\theta_1}((0, t_0), R) \right| \leq \mu_1|Q^{1,\theta_1}(R)|$$

with the integer number $s$ defined as

$$s = \text{entier} \left( \frac{c}{\sqrt{\mu_1^2 \theta_1^3}(1 + f(2R))} + 1 \right).$$

**Corollary 2.8.** Let $0 \leq \pi \in \mathcal{V}_0$ have the property $(\mathcal{B}_R)$ in $Q(2R)$. If $\pi(\cdot, t) \geq k_2$ in $C(R)$, then, for any $\sigma \in [0, 1]$, the inequality $\pi \geq \beta_2 k_2$ holds in $Q^{\sigma,\theta_2}((0, t_0), R)$ with $\theta_0 = (c/g(2R))^{\frac{4}{3}}$ and $t_0 = t + \theta R^2$, where

$$\beta_2 = \frac{1}{2}2^{-c(1-\sigma)+c-\sigma-\alpha}g^{25}(2R)$$

provided $R \leq R_c(c_\ast, \alpha)$. It is supposed that $0 < k_2 \leq k_R$.

**Proof.** We apply Lemma 2.5 with $\delta_0 = 1$ and $k_0 = k_2$. Then, for $\sigma = \frac{4}{27c}$, we have

$$\theta_0 = \left( \frac{\frac{4\sigma}{1 + f(2R)}}{\frac{1}{\sigma} + f(2R)} \right)^\frac{4}{3} \geq \left( \frac{c}{g(2R)} \right)^\frac{4}{3}$$

for all $0 < R \leq R_c(c_\ast, \alpha)$ and state that the following inequality holds:

$$\left| \left\{ \pi(\cdot, t) > \frac{k_0}{3} \right\} \cap C(R) \right| \geq \frac{1}{3}|C(R)|$$

for any $t \in [t_0 - \theta_0 R^2, t_0]$, where $t_0 = t + \theta_0 R^2$. In what follows, we are going to use the quantity $(c/(g(2R)))^{\frac{4}{3}}$ as a new number $\theta_0$ instead of $\theta_0(1, f(2R))$.

Now, we are going to apply Lemma 2.7 with another set of parameters $k_1 = \frac{1}{3}k_2$, $\theta_1 = \theta_0$, $\delta_1 = \frac{1}{3}$, and

$$\mu_1 = \mu_\ast = \left( \frac{1}{2c_1} \right)^\frac{4}{3}, \quad c_1' = \frac{c}{(1-\sigma)^\frac{14}{3}} \left( 1 + \left( \frac{1}{\theta_0^2 \sigma^3} \right)^\frac{13}{3} M(2R) \right)^3 \leq \frac{c}{(1-\sigma)^\frac{14}{3}} \left( \frac{1}{\theta_0^2 \sigma^3} \right)^\frac{13}{3} g^3(2R).$$

Lemma (2.7) gives us:

$$\left| \left\{ \pi < 2^{-s}k_1 \right\} \cap Q^{1,\theta_1}((0, t_0), R) \right| < \mu_1|Q^{1,\theta_1}(R)|,$$
Lemma 2.9. Let

\[
\pi(\cdot, t_0 - \theta_0 R^2) \geq k_2 > 2^{-s}k_1 = 2^{-s} \frac{k_2}{3}.
\]

Then, from Corollary 2.4, it follows that \(\pi > \frac{1}{2} 2^{-s}k_1 = \beta_2 k_2\) with \(\beta_2 = \frac{1}{2} 2^{-s} \frac{1}{3}\) in \(Q^{\pi, \theta_0}((0, t_0), R)\). \(\square\)

Given \(\theta \in [0, 1]\), we can find an number \(0 < R_{*1}(c_*, \alpha, \theta) \leq R_*(c_*, \alpha)\) so that \(\left(\frac{c}{g(2^2)}\right)^{\frac{1}{2}} \leq \theta\) for all \(0 < r \leq R_{*1}\).

**Lemma 2.9.** Let \(0 \leq \pi \in \mathcal{V}_0\) have the property \((\mathcal{B}_R)\) in \(Q(2R)\), assuming that \(R \leq R_{*1}(c_*, \alpha, \theta)\) for some \(0 < \theta \leq 1\). Suppose further that, for some \(0 < k \leq k_R\) and for some \(-R^2 \leq \tilde{t} \leq -\theta R^2\), there holds \(\pi(\cdot, \tilde{t}) \geq k\) in \(C(R)\). Then \(\pi \geq \beta_0 k\) in \(\tilde{Q} := \mathcal{C}(\frac{2}{3} R) \times [\tilde{t}, 0]\), where

\[
\beta_0 \geq \ln^{-\frac{1}{2}}(1/R)
\]

for \(R \leq R_{*2}(c_*, \alpha, \theta)\).

**Proof.** Let

\[
N = \text{entier}\left(\frac{9}{8} \frac{\tilde{t}}{\theta_0 R^2}\right) + 1,
\]

where \(\tilde{\theta}_0 = (c/g(\frac{2}{3} 2R))^\frac{1}{2} \leq \theta\). Next, we introduce

\[
\hat{\theta}_0 = \frac{\tilde{t}}{(\frac{8N}{9} + 1/2N)R^2} \leq \tilde{\theta}_0.
\]

**Step 1.** By Corollary 2.8, the inequality \(\pi \geq \beta_2^{(1)} k\) holds at least in \(C((1 - \frac{1}{3N})R) \times [\tilde{t}_1, \tilde{t}_1 + \hat{\theta}_0 R^2]\), where \(\tilde{t}_1 = \tilde{t}, \tilde{t}_2 = \tilde{t}_1 + \hat{\theta}_0 R^2, \sigma = 1 - 1/(3N) \geq 2/3, 1 - \sigma = 1/(3N),\) and

\[
\ln \beta_2^{(1)} = -\ln 6 - cN^{40} g^{25}(2R)
\]

**Step 2.** Here, we are going to use Corollary 2.8 with \(R(1 - 1/(3N))\) instead of \(R\) and with \(\sigma = (1 - 2(3N))/(1 - 1/(3N))\). As a result, we have the estimate \(\pi \geq \beta_2^{(2)} \beta_2^{(1)} k\) at least in \(C((1 - 2/(3N))R) \times [\tilde{t}_2, \tilde{t}_2 + \hat{\theta}_0 (1 - 1/(3N))^2 R^2], \tilde{t}_3 = \tilde{t}_2 + \hat{\theta}_0 (1 - 1/(3N))^2 R^2,\) and

\[
\ln \beta_2^{(2)} = -\ln 6 - cN^{40} g^{25}(2(1 - 1/(3N))R).
\]

So, \(\pi \geq \beta_2^{(2)} \beta_2^{(1)} k\) in \(C((1 - 2(3N))R) \times [\tilde{t}, \tilde{t}_3]\).

After \(N\) steps, we shall have \(\tilde{t}_N = 0\) and

\[
\pi \geq \beta_2^{(N)} ... \beta_2^{(1)} k = \beta_0(R) k
\]

in \(C(\frac{2}{3} R) \times [\tilde{t}, 0]\), where

\[
\ln \beta_2^{(i+1)} = -\ln 6 - cN^{40} g^{25}(2(1 - i/(3N))R)
\]

for \(i = 0, 1, ..., N - 1\).

Next, according to assumption (1.2), we can have

\[
\ln \beta_0 \geq -N \ln 6 - cN^{40} \sum_{k=1}^{N-1} c_*^{25} \ln^7 \ln^\frac{1}{2} \left(\frac{1}{2(1 - i/(3N))R}\right),
\]

where \(25^\alpha < 1\). Since

\[
\ln \frac{1}{1 - x} \leq 2x
\]
provided $0 \leq x \leq 1/2$, we find, assuming that $R \leq 1/6$, the following:
\[
\ln^2 \ln^2 \left( \frac{1}{2(1 - i/(3N))R} \right) \leq \ln^2 \left( \ln \frac{1}{2R} + \frac{i}{N} \right) \leq \ln^2 \left( \ln \frac{1}{2R} \right) \leq \ln^2 \left( 1 + \frac{i}{N} \right)\]
\[
\leq \ln \left( \ln \frac{1}{2R} \right) + \frac{i}{N} \leq \ln \left( \ln \frac{1}{2R} \right) + \frac{i}{N}.\]

From the latter inequality, one can deduce the bound
\[
\ln \beta_0 \geq -N \ln 6 - cc^2_5 N^4 \ln^2 \frac{1}{2R},\]
which is valid for $0 < R \leq R_{*3}(\alpha) \leq 1/6$. Taking into account that $N \leq c(g(2R))^{1/2}$, we conclude
\[
\ln \beta_0 \geq -c_1(c_*) \ln \frac{\ln 1}{R},\]
It remains to find $R_{*4}(c_*, \alpha) \leq 1$ such that
\[
c_1(c_*) \ln \frac{\ln 1}{R} \leq 1\]
for all $0 < R \leq R_{*4}$. So, we have the required inequality provided $0 < R \leq R_{*2} = \min\{R_{*1}, R_{*3}, R_{*4}\}$. □

3. Proof of Proposition 1.4

Now, we can state an analog of Lemma 4.2 of [18] for the class $V$.

**Lemma 3.1.** Let $0 \leq \pi \in \mathcal{V}_0$ possess the property $(\mathcal{B}_R)$ in $Q(2R)$.

Suppose further that
\[
\pi \leq M_0 k_R
\]
in $Q(2R)$ for some $M_0 \geq 1$. Then, there exists $\eta \in [-R^2, -\frac{1}{2} R^2]$ such that
\[
|e_{\kappa_{0}}(\eta)| \geq \delta_0 |C(R)|\]
\[
\text{Here, } \kappa_{0} = \kappa_{0}(f(2R)) = c/(1 + f(2R)), \ e_{\kappa}(t) := \{ x \in C(R) : \pi(x, t) \geq \kappa k_R \}, \text{ and } \delta_{0}(M_{0}, f(2R)) = \left( \frac{c}{M_0(1 + f(2R))} \right)^{\frac{2}{4}}.\]

**Proof.** Here, we follow arguments of the paper [18]. They are based on the identity:
\[
\int_{Q} (-\pi \partial_t \eta - \pi \Delta \eta - (v + 2x'/|x'|^2) \cdot \nabla \eta \pi) dx dt
\]
\[
= 4\pi_0 \int_{-1}^{0} \int_{-1}^{1} \pi(0, x_3, t) \eta(0, x_3, t) dx_3 dt,\]
which is valid for any non-negative test function \( \eta \) supported in \( Q \). Here, \( \pi_0 = 3.14... \). Although a similar statement has been proven in [18] under the assumption that \( \pi \) is Lipschitz, it remains to be true for functions \( \pi \) from the class \( V_0 \) as well. Indeed, take a smooth cut-off function \( \psi = \psi(x') \) so that \( \psi(x') = \Psi(|x'|), 0 \leq \psi \leq 1, \psi(x') = 0 \) if \( |x'| \leq \varepsilon/2, \psi(x') = 1 \) if \( |x'| \geq \varepsilon, \Psi'(q) \leq c/q \) and \( \Psi''(q) \leq c/q^2 \) for some positive constant \( c \). Then, it follows from \((2.1)\) that:

\[
\int_Q \left( \pi \partial_t (\eta \psi) + \pi (u + b) \cdot \nabla (\eta \psi) + \pi \Delta (\eta \psi) \right) dz = 0.
\]

There are two difficult terms for passing to the limit as \( \varepsilon \to 0 \). The first one is as follows:

\[
I_1 := \int_Q \pi \eta \Delta \psi dx dt = J_1 + J_2,
\]

where

\[
J_1 := \int_Q (\pi \eta - (\pi \eta)|_{x'=0}) \Delta \psi dx dt,
\]

For \( J_2 \), we find

\[
J_2 := \int_Q (\pi \eta)|_{x'=0} \Delta \psi dx dt = \int_{-1}^{1} \int_{-1}^{1} (\pi \eta)|_{x'=0} dx_3 dt \int_{|x'|<1} \Delta \psi(x') dx'
\]

and

\[
\int_{|x'|<1} \Delta \psi(x') dx' = 2\pi_0 \int_{\frac{\varepsilon}{2}}^{\varepsilon} \frac{1}{\varrho} \frac{\partial}{\partial \varrho} \left( \varrho \Psi'(\varrho) \right) \varrho d\varrho = 2\pi_0 \varrho \Psi'(\varrho)|_{\frac{\varepsilon}{2}}^{\varepsilon} = 0.
\]

Now, we wish to show that

\[
J_1 := \int_Q \xi \Delta \psi dx dt \to 0
\]

as \( \varepsilon \to 0 \), where, \( \xi := \pi \eta - (\pi \eta)|_{x'=0} \). To this end, let us introduce the function

\[
H_\varepsilon(x_3, t) := \int_{\frac{\varepsilon}{2}}^{\varepsilon} \xi \Delta \psi dx'.
\]

It can be bounded from above and from below

\[
|H_\varepsilon(x_3, t)| \leq c \sup_{spt \eta} \pi \sup_{|x'|<1} \eta(x', x_3, t) \frac{1}{\varepsilon^2} \int_{\frac{\varepsilon}{2}}^{\varepsilon} \varrho d\varrho =: h(x_3, t)
\]

provided \( \varepsilon < 1 \). The function \( h \) is supported in \( |x'| < 1 \), and thus

\[
\int_{-1}^{1} \int_{-1}^{1} h(x_3, t) dx_3 dt < \infty.
\]

Now, let \((0, x_3, t)\) be a regular point of \( \pi \), i.e., \((0, x_3, t) \notin S^\pi \). Then, \( \xi(x', x_3, t) \to 0 \) as \( |x'| \to 0 \) and thus for any \( \delta > 0 \) there exists a number \( \tau(x_3, t) > 0 \) such that \( |\xi(x', x_3, t)| < \delta \) provided \( |x'| < \tau \). So,

\[
|H_\varepsilon(x_3, t)| < c \frac{\delta}{\varepsilon^2} \int_{\frac{\varepsilon}{2}}^{\varepsilon} \varrho d\varrho = c \frac{\delta}{2}.
\]
provided $\varepsilon < \tau$. Therefore, $H_\varepsilon(x_3, t) \to 0$ as $\varepsilon \to 0$ and by the Lebesgue theorem on dominated convergence, we find that

$$J_1 = \int_0^1 \int_{-1}^0 H_\varepsilon(x_3, t) dx_3 dt \to 0$$

as $\varepsilon \to 0$.

Similar arguments work for the second difficult term:

$$I := \int_Q \pi b \cdot \nabla \psi dz = J_1 + J_2,$$

where

$$J_1 = \int_Q \xi b \cdot \nabla \psi dz$$

and

$$J_2 := \int_Q (\pi \eta)|_{x'=0} b \cdot \nabla \psi dx dt = \int_{-1}^0 \int_{-1}^1 (\pi \eta)|_{x'=0} dx_3 dt 2\pi \int_{\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} \Psi'(\theta) g d\theta$$

$$= 4\pi \int_{-1}^0 \int_{-1}^1 (\pi \eta)|_{x'=0} dx_3 dt.$$

The fact that $J_1 \to 0$ as $\varepsilon \to 0$ can be justified in the same way as above, replacing $H_\varepsilon$ with the function

$$G_\varepsilon(x_3, t) := \int_\frac{\varepsilon}{2}^{\varepsilon} \xi b \cdot \nabla \psi dx'.$$

Other terms can be treated in a similar way and even easier. So, the required identity (3.3) has been proven.

Now, let us select the test function $\eta$ in (3.3), using the following notation

$$Q^{\lambda, \theta}(z_0, R) := C(x_0, \lambda R) \times [t_0 - \theta R^2, t_0]$$

so that $\eta = 1$ in $Q^{\frac{1}{2}, \frac{1}{2}}((0, -\frac{13}{16} R^2), R)$, $\eta = 0$ out of $Q^{1, \frac{1}{2}}((0, -\frac{3}{4} R^2), R)$ and $|\partial_t \eta| + |\nabla \eta|^2 + |\nabla^2 \eta| \leq c/R^2$. Taking into account that $\pi$ has the property $(B_R)$, we find

$$\frac{\pi_0}{2} k_R R^2 \leq \frac{c}{R^2} \int_{Q^{1, \frac{1}{2}}(z_R, R)} \pi dz + \frac{c}{R^2} \int_{Q^{1, \frac{1}{2}}(z_R, R)} \pi |v| dz + \frac{c}{R^2} \int_{Q^{1, \frac{1}{2}}(z_R, R)} \frac{\pi}{|x'|} dz,$$

where $z_R = (0, -\frac{3}{4} R^2)$.

Setting $E_\varepsilon = \{(x, t) : t \in ] - R^2, -\frac{3}{4} R^2[, x \in e_\varepsilon(t)\}$, we can deduce from the latter inequality

$$\frac{\pi_0}{2} k_R R^3 \leq \frac{c}{R^2} \int_{Q^{1, \frac{1}{2}}(z_R, R) \setminus E_\varepsilon} \pi dz + \frac{c}{R^2} \int_{Q^{1, \frac{1}{2}}(z_R, R) \setminus E_\varepsilon} \pi |v| dz + \frac{c}{R^2} \int_{Q^{1, \frac{1}{2}}(z_R, R) \setminus E_\varepsilon} \frac{\pi}{|x'|} dz.$$

$$+ \frac{c}{R^2} \int_{Q^{1, \frac{1}{2}}(z_R, R) \cap E_\varepsilon} \pi dz + \frac{c}{R^2} \int_{Q^{1, \frac{1}{2}}(z_R, R) \cap E_\varepsilon} \pi |v| dz + \frac{c}{R^2} \int_{Q^{1, \frac{1}{2}}(z_R, R) \cap E_\varepsilon} \frac{\pi}{|x'|} dz.$$
Applying (3.1) and recalling definitions of the sets $e_\kappa(t)$ and $E_\kappa$, we get
\[
\frac{\pi_0}{2} k_R R^3 \leq \frac{c k_R R^3}{R^2} \left\{ \left| Q^{1,\frac{1}{2}}(R) \right| + R \int_{Q^{1,\frac{1}{2}}(R) \setminus E_\kappa} |v| dz + R \int_{Q^{1,\frac{1}{2}}(R) \setminus E_\kappa} \frac{1}{|x'|} dz \right\} + \frac{c M_0 k_R}{R^2} \left\{ |E_\kappa| + R \int_{Q^{1,\frac{1}{2}}(z_R,R) \cap E_\kappa} |v| dz + R \int_{Q^{1,\frac{1}{2}}(z_R,R) \cap E_\kappa} \frac{1}{|x'|} dz \right\}.
\]

We need to estimate integrals in the above inequality. First, for integrals, containing $v$, Holder inequality gives
\[
\int_{Q^{1,\frac{1}{2}}(z_R,R) \cap E_\kappa} |v| dx \leq \left\| \frac{1}{|x'|} \right\|_{\frac{3}{4},\infty} \cdot \left\| Q^{1,\frac{1}{2}}(R) \right\|_{\frac{3}{4},\frac{3}{2},Q^{1,\frac{1}{2}}(R)} \left( \frac{-\frac{3}{4} R^2}{|x'|} \right) \left( \int_{Q^{1,\frac{1}{2}}(R)} |v|^3 dx \right)^{\frac{1}{3}} \leq f(2R) R^\frac{3}{2} \left\| \frac{1}{|x'|} \right\|_{\frac{3}{4},\infty} \cdot \left\| Q^{1,\frac{1}{2}}(R) \right\|_{\frac{3}{4},Q^{1,\frac{1}{2}}(R)} \leq f(2R) R^4
\]
and similarly
\[
\int_{Q^{1,\frac{1}{2}}(z_R,R) \cap E_\kappa} \frac{1}{|x'|} dz \leq f(2R) R^\frac{3}{2} \left\| \frac{1}{|x'|} \right\|_{\frac{3}{4},\frac{3}{2},E_\kappa}.
\]
To evaluate the last two integrals, let us take into account the fact:
\[
\left. \left\| \frac{1}{|x'|} \right\|_{\frac{3}{4},\infty} (Q^{1,\frac{1}{2}}(z_R,R)) \right. \leq c R^\frac{3}{2} R^\frac{3}{2} = c R^4
\]
and
\[
\int_{Q^{1,\frac{1}{2}}(z_R,R) \cap E_\kappa} \frac{1}{|x|} dz \leq c R^\frac{2}{3} \left\| \frac{1}{|x|} \right\|_{\frac{3}{4},1,E_\kappa}.
\]
Hence, we have
\[
\frac{\pi_0}{2} k_R R^3 \leq c k_R R^3 \left( 1 + f(2R) \right) + \frac{c M_0 k_R}{R^5} \left[ |E_\kappa| + f(2R) R^\frac{3}{2} \left\| \frac{1}{|x'|} \right\|_{\frac{3}{4},\frac{3}{2},E_\kappa} + R^\frac{5}{2} \left\| \frac{1}{|x'|} \right\|_{\frac{3}{4},1,E_\kappa} \right].
\]
So,
\[
\frac{\pi_0}{2} \leq c k \left( 1 + f(2R) \right) + \frac{c M_0}{R^5} \left[ |E_\kappa| + f(2R) R^\frac{3}{2} \left\| \frac{1}{|x'|} \right\|_{\frac{3}{4},\frac{3}{2},E_\kappa} + R^\frac{5}{2} \left\| \frac{1}{|x'|} \right\|_{\frac{3}{4},1,E_\kappa} \right].
\]
Now, one can find $\kappa = \kappa_0(f(2R)) = c/(1 + f(2R))$ such that
\[
\frac{c M_0}{R^5} \left[ |E_\kappa| + f(2R) R^\frac{3}{2} \left\| \frac{1}{|x'|} \right\|_{\frac{3}{4},\frac{3}{2},E_\kappa} + R^\frac{5}{2} \left\| \frac{1}{|x'|} \right\|_{\frac{3}{4},1,E_\kappa} \right] \geq 1.
\]
It remains to estimate two integrals on the left hand side of the latter inequality:

\[ \|\mathcal{I}\|_{L^2, E_{\kappa_0}}^{3/4} = \left( \int_{-R^2}^{-4R^2} |e_{\kappa_0}(t)|^{9} \, dt \right)^{\frac{4}{9}} \leq c |E_{\kappa_0}|^{\frac{5}{2}} R^\frac{5}{2} \]

and

\[ \|\mathcal{I}\|_{L^2, E_{\kappa_0}}^{\frac{4}{5}} \leq c |E_{\kappa_0}|^{\frac{4}{5}} R^\frac{10}{5} \].

Letting \( A = |E_{\kappa_0}| / R^5 \), we arrive at the following inequality

\[ f(A) := A + A^\frac{4}{5} + f(2R) A^\frac{2}{5} \geq \frac{1}{cM_0}. \]

Since \( f'(A) > 0 \) for \( A > 0 \), we can state that the last inequality implies

\[ \frac{|E_{\kappa_0}|}{|C(R)|^\frac{1}{2} R^2} \geq \delta_0 = \left( \frac{c}{M_0 (1 + f(2R))} \right)^{\frac{5}{4}}. \]

It is not so difficult to show the existence of \( \bar{t} \in [-R^2, -\frac{3}{4} R^2] \) with the property:

\[ |e_{\kappa_0} (\bar{t})|^\frac{1}{4} R^2 \geq |E_{\kappa_0}|. \]

So, it is proven that there exists \( \bar{t} \in [-R^2, -3R^2/4] \) such that

\[ |\{ x \in C(R) : \pi(x, \bar{t}) > \kappa_0 k_R \}| \geq \delta_0 |C(R)|, \tag{3.4} \]

which completes the proof of the lemma. \( \square \)

Now, we are able to prove Proposition 1.4.

Assume that the function \( \pi \) meets all the conditions of Lemma 3.1 and according to it, we can claim that:

\[ |e_{\kappa_0} (\bar{t})| = |\{ x \in C(R) : \pi(x, \bar{t}) \geq \kappa_0 k_R \}| \geq \delta_0 |C(R)| \]

for some \( \bar{t} \in [-R^2, -\frac{3}{4} R^2] \), \( \kappa_0 = c / g(2R) \), and \( \delta_0 = c(M_0) / g^{\frac{5}{2}}(2R) \). Now, we can calculate

\[ \theta(\delta_0(M_0, f(2R)), f(2R)) \geq c \left( \frac{\delta_0^6}{1 + \delta_0^2 f(2R)} \right)^{\frac{3}{4}} \geq c(M_0) \left( \frac{1}{g(2R)} \right)^{18}, \]

apply Lemma 2.5, and find

\[ |\{ \pi(\cdot, t) \geq \delta_0 k_R/3 \} \cap C(R) | > \delta_0/3 |C(R)| \]

for all \( t \in [\bar{t}, t_0] \) with \( t_0 = \bar{t} + \theta_0 R^2 \) and \( \theta_0 = c(M_0)(g(2R))^{-18} \).

Next, it follows from Lemma 2.7 that:

\[ |\{ \pi < 2^{-s} \delta_0 k_R/3 \} \cap Q^{1,\theta_0}(0, t_0, R) | \leq \mu_s |Q^{1,\theta_0}(R)|, \]

where

\[ s = \text{entier} \left( \frac{c}{\delta_0^6 \mu_s^2 \theta_0} (1 + f(2R)) \right) + 1 \]

and \( \mu_s \) is the number that appears in Corollary 2.4, see also Proposition 2.2. In our case,

\[ \mu_s = \left( \frac{1}{2c_1(3/4, 1, \theta_0/2, \theta_0, M(2R))} \right)^{10} \]

and, moreover

\[ c_1 (3/4, 1, \theta_0/2, \theta_0, M(2R)) \leq c \theta_0^{-\frac{4}{5}} g^3(2R) \leq c(M_0) (g(2R))^{30}. \]
Then, Corollary 2.4 implies the bound
\[
\pi \geq 2^{-s} \delta_0 \kappa_0 k_R / 6 = \hat{\beta}_2 \kappa_0 k_R
\]
in $Q^{3, \frac{1}{2}}(0, t_0, R)$. So, combining previous estimates, we find the following:
\[
\hat{\beta}_2 = \frac{1}{6} 2^{-s} \delta_0 \geq e^{-s \ln 2 - \ln \delta_0} \geq e^{-c s} \delta_0,
\]
where
\[
s \leq \frac{2g(2R)}{\delta_0^2 \mu_2^2 \theta_0} \leq c(M_0) g(2R) (g(2R))^\frac{9}{2} (g(2R))^{18} c_1 \beta^{\frac{20}{3}}
\leq c(M_0) (g(2R))^{224} (g(2R))^{30} \alpha \leq c(M_0) (g(2R))^{224}.
\]
So,
\[
\hat{\beta}_2 \geq e^{-c(M_0)(g(2R))^{224}} c(M_0) (g(2R))^{-\frac{2}{3}} \geq e^{-2c(M_0)(g(2R))^{224}}
\geq e^{-c(M_0, c_*) \ln^{224\alpha} \sqrt{\ln \frac{1}{\hat{R}}}}.
\]
Obviously, there exists a number $0 < R_{s5}(M_0, c_*, \alpha) \leq \min\{1/6, R_{s2}\}$ such that
\[
2c(M_0, c_*) \ln^{224\alpha - 1} \sqrt{\ln \frac{1}{R}} \leq \frac{1}{2}
\]
and
\[
c(M_0, c_*) \ln^{224\alpha} \sqrt{\ln \frac{1}{R}} \geq \ln g(2R)
\]
for $0 < R \leq R_{s5}(M_0, c_*, \alpha)$ and thus
\[
-c(M_0, c_*) \ln^{224\alpha} \sqrt{\ln \frac{1}{R}}
= -2c(M_0, c_*) \ln^{224\alpha} \sqrt{\ln \frac{1}{R}} + c(M_0, c_*) \ln^{224\alpha} \sqrt{\ln \frac{1}{R}}
\geq \left(2c(M_0, c_*) \ln^{224\alpha - 1} \sqrt{\ln \frac{1}{R}} \right) \ln \sqrt{\ln \frac{1}{R}} + \ln g(2R)
\geq - \ln \left( \frac{1}{R} \right)^{-\frac{1}{4}} + \ln g(2R).
\]
Now, the number $\hat{\beta}_2$ is estimated as follows:
\[
\hat{\beta}_2 \geq \left( \ln \frac{1}{R} \right)^{-\frac{1}{4}} g(2R)
\] (3.5)
for $0 < R \leq R_{s5}(M_0, c_*, \alpha)$.

Since
\[
-R^2 \leq \bar{t} + \theta_0/2R^2 = t_0 - \theta_0/2R^2 < t_0 = \bar{t} + \theta_0 R^2 \leq -\frac{3}{4} R^2 + \frac{1}{4} R^2 = -\frac{1}{2} R^2,
\]
there is $\bar{t}_1 \in [-R^2, -\frac{1}{2} R^2]$ such that
\[
\pi(\cdot, \bar{t}_1) > \hat{\beta}_2 \kappa_0 k_R
\]
in $C(\frac{1}{2} R)$. It allows us to apply Lemma 2.9 with $\theta = 1/2$, with $\frac{3}{4} R$ instead of $R$, with $\bar{t}_1$ instead of $\bar{t}$, and with $\hat{\beta}_2 \kappa_0 k_R$ instead of $k$. According to Lemma 2.9, the inequality
\[
\pi \geq \beta_0 \hat{\beta}_2 \kappa_0 k_R
\]
holds in \(Q(R/2)\). It follows from Lemma 2.9 and from (3.5) that
\[
\pi \geq \frac{ck_R}{\ln \frac{3}{4} \left( \frac{R}{2} \right)} = \beta(2R)k_R
\]
in \(Q(R/2)\).

By our assumption imposed on function \(\sigma\), we can put \(k_R = \frac{1}{2} \text{osc}_{z \in Q(2R)} \sigma(z)\). Then, either \(\pi = \sigma - m_2R\) or \(\pi = M_2R - \sigma(z)\) satisfies all the conditions of the proposition with \(M_0 = 2\). Simple arguments show that
\[
\text{osc}_{z \in Q(R/2)} \sigma(z) \leq \left( 1 - \frac{1}{2} \beta(R) \right) \text{osc}_{z \in Q(R)} \sigma(z).
\]

Now, after iterations of the latter inequality, we arrive at the following bound
\[
\text{osc}_{z \in Q(R/2^{2k+1})} \sigma(z) \leq \prod_{i=0}^{k} \left( 1 - \frac{1}{2} \beta(R/2^{2i-1}) \right) \text{osc}_{z \in Q(R)} \sigma(z) = \eta_k \text{osc}_{z \in Q(2R)} \sigma(z)
\]
being valid for any natural number \(k\).

In order to evaluate \(\eta_k\), take ln of it. As a result,
\[
\ln \eta_k = \sum_{i=0}^{k} \ln \left( 1 - \frac{1}{2} \beta(R/2^{2i-1}) \right) \leq - \sum_{i=0}^{k} \frac{1}{2} \beta(R/2^{2i-1})
\]
\[
\leq -c \sum_{i=0}^{k} (\ln(2^{2i-1}/R))^{-\frac{3}{4}} = -c \sum_{i=0}^{k} \frac{1}{(i \ln 4 + \ln(1/(2R)))^\frac{3}{4}}
\]
\[
\leq -c \int_{0}^{k+1} \frac{dx}{(x \ln 4 + \ln(1/(2R)))^\frac{3}{4}}
\]
\[
= - \frac{4c}{\ln 4} \left( x \ln 4 + \ln(1/(2R)) \right)^\frac{1}{4} \Bigg|_{0}^{k+1}
\]
\[
= -c \left( \ln \frac{3}{4} (2^{2k+1}/R) - \ln \frac{3}{4} (1/(2R)) \right).
\]

Hence, we have
\[
\text{osc}_{z \in Q(R/2^{2k+1})} \sigma(z) \leq e^{-c \left( \ln \frac{3}{4} (2^{2k+1}/R) - \ln \frac{3}{4} (1/(2R)) \right)} \text{osc}_{z \in Q(2R)} \sigma(z)
\]
and thus
\[
\text{osc}_{z \in Q(r)} \sigma(z) \leq e^{-c \left( \ln \frac{3}{4} (1/(2r)) - \ln \frac{3}{4} (1/(2R)) \right)} \text{osc}_{z \in Q(2R)} \sigma(z)
\]
for all \(0 < r < R \leq R_5(c_5, \alpha) = R_5(2, c_5, \alpha)\). So, (1.8) follows. The proof of Proposition 1.4 is complete.

4. Proof of Theorem 1.3

By the maximum principle, we have \(|\sigma| = |\varrho v_\varphi| \leq \Sigma_0\) in \(\mathbb{R}^3 \times ]0, T[\). From Proposition 1.4, it follows that
\[
|\sigma(\varrho, x_3, t)| \leq e^{-c \left( \ln \frac{3}{4} (1/(2\varrho)) - \ln \frac{3}{4} (1/(2R_*)) \right)} 2\Sigma_0.
\]
fo all $0 < \theta \leq R_\star(c_\star, \alpha)$, for all $x_3 \in \mathbb{R}$, and for $t \in [T - R_\star^2, T]$. Obviously, it remains true $\theta > R_\star$ as well. So, we have for all $x \in \mathbb{R}^3$ and for all $t \in [T - R_\star^2, T]$

$$|\sigma(\theta, x_3, t)| \leq C(c_\star, \alpha)e^{-c\ln^4(1/(2\theta))}\Sigma_0 \leq C(c_\star, \alpha)\Sigma_0 \frac{m!}{c^m \ln^\frac{m}{2}(1/(2\theta))}$$

for all natural numbers $m$.

Now, let us notice that $v(\cdot, T - R_\star^2) \in H^2$. Therefore, one can use the results of papers [12,30], and [15], see also [17], [4], on the Cauchy problem for the Navier–Stokes system (1.1) in $\mathbb{R}^3 \times [T - R_\star^2, T]$ and conclude that $v$ is a strong solution in the interval $[0, T]$.

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