VANISHING THEOREMS AND SYZYGIES
FOR K3 SURFACES AND FANO VARIETIES

F. J. GALLEGRO AND B. P. PURNAPRAJNA

May 26, 1996

Abstract. In this article we prove some strong vanishing theorems on K3 surfaces. As an application of them, we obtain higher syzygy results for K3 surfaces and Fano varieties.

1. Introduction

In this article we prove some vanishing theorems on K3 surfaces. An application of the vanishing theorems is a result on higher syzygies for K3 surfaces and Fano varieties.

One part of our results fits a meta-principle stating that if $L$ is a line bundle that is a product of $(p+1)$ ample and base point free line bundles satisfying certain conditions, then $L$ satisfies the condition $N_p$ (a condition on the free resolution of the homogeneous coordinate ring of $X$ embedded by $L$). Other illustrations of this meta-principle have been given in [GP1], [GP2] and [GP3]. The condition $N_p$ may be interpreted, through Koszul cohomology, as a vanishing condition on a certain vector bundle.

The other part of our results provides strong vanishing theorems that imply, in particular, the vanishing needed for $N_p$. We also prove stronger variants of the principle stated above for K3 surfaces and Fano varieties.

Before stating our results in detail, we recall some key results in this area, namely the normal generation and normal presentation on K3 surfaces due to Mayer and St. Donat. Mayer and St. Donat proved that if $L$ is a globally generated line bundle on a K3 surface $X$ such that the general member in the linear system is a non hyperelliptic curve of genus $g \geq 3$, then $L$ is normally generated (in other words, the homogeneous coordinate ring of $X$ in projective space $\mathbb{P}(H^0(L))$ is projectively normal). If one further assumes that the general member in the linear system is a non-trigonal curve which is not a plane quintic, then $L$ is normally presented (in other words, the homogeneous ideal $I_X$ defining $X$ in $\mathbb{P}(H^0(L))$ is generated...
by quadrics). Regarding results on higher syzygies, it follows from a more general result of Ein and Lazarsfeld [EL] that, if \( L \) is a very ample bundle on a K3 surface then \( L^{\otimes (p+2)} \) satisfies \( N_p \). There are results on normal generation for Fano threefolds due to Iskovskih. For details we refer the reader to [I].

In order to state our main theorem on syzygies we require the following:

Let \( X \) be an irreducible projective variety and \( L \) a very ample line bundle on \( X \), whose complete linear series defines the morphism

\[ \Phi_L : X \rightarrow \mathbf{P}(H^0(L)). \]

Let \( S = \bigoplus_{m=0}^{\infty} S^m H^0(X, L) \) and \( R(L) = \bigoplus_{m=0}^{\infty} H^0(X, L^{\otimes m}) \). Since \( R(L) \) is a finitely generated module over \( S \), it has a minimal graded free resolution. We say that the line bundle \( L \) satisfies \( N_p \), if \( I_X \) is generated by quadrics and the matrices in the free resolution of \( R \) over \( S \) have linear entries until the \((p-1)th \) stage. Using this notation one says that a normally generated line bundle satisfies \( N_0 \) and that a normally presented line bundle satisfies \( N_1 \).

We prove that if \( L \) is a globally generated bundle on a K3 surface \( X \) such that the general member in the linear system \( C \in | L | \) is a non-hyperelliptic curve of genus \( g \geq 4 \), then \( L^{\otimes (p+1)} \) satisfies \( N_p \). If we further assume that the general member in the linear system is a non-trigonal curve of genus \( g \geq 5 \) which is not a plane quintic, then \( L^{\otimes p} \) satisfies \( N_p \). We also show that on a Fano threefold of index 1, Picard number 1 and very ample anticanonical bundle of sectional genus \( g \geq 4 \), \( L^{\otimes (p+1)} \) satisfies \( N_p \) for any ample line bundle \( L \). If we further assume that the general member in \( | L \otimes O_S | \) is a non-trigonal curve of genus \( g \geq 5 \) which is not a plane quintic curve (where \( S \) is a general, hence smooth, member of \( | L \otimes O_S | \)), then \( L^{\otimes p} \) satisfies \( N_p \). We generalize these results to Fano varieties of dimension \( n \) with index \((n-2)\). Our results generalize the results of St. Donat and Iskovskih and improve the bound given by Ein and Lazarsfeld for K3 surfaces and Fano varieties. Our results for K3 surfaces do not assume that \( L \) is ample.

We refer the reader to Sections 4 and 6 for the statements of our vanishing theorems on K3 surfaces and Fano varieties.

## 2. Normal Generation and Normal Presentation on K3 Surfaces

In this section we prove some vanishing theorems on K3 surfaces. As a consequence of them we recover well known results on normal generation and normal presentation due to Mayer and St. Donat. On the other hand the proofs of these vanishing theorems will serve as a warm-up for the sequel, letting us introduce part of the machinery and ideas used in Sections 4 and 6 to prove results regarding higher syzygies. Moreover, Proposition 2.2 and 2.4 will be the first steps (and indeed, the key steps) of the inductive process leading towards our higher syzygy results.

First of all, we will introduce the setting in which we will work and some elementary facts about line bundles on K3 surfaces. Throughout this article we will work over the field of complex numbers.

In Sections 2, 4 and 5, \( X \) will be a smooth K3 surface and \( L = O_X(C) \) will be a globally generated line bundle on \( X \) (for example, if \( C^2 > 0 \), it follows from [St.D] that \( L \) is globally generated). Also, by taking \( C \) general in \(| L | \), we can assume by Bertini’s theorem that \( C \) is smooth (See [St.D]). For a globally generated line bundle \( C \) one has the associated vector bundle \( M_0 \) given by the short exact sequence:
0 \longrightarrow M_G \longrightarrow H^0(G) \otimes \mathcal{O}_X \longrightarrow G \longrightarrow 0. \quad (2.1.1)

We will constantly abuse the notation and write \( C \) in place of \( L = \mathcal{O}_X(C) \) (for example, we will write \( M_C \) instead of \( M_L, pC \) instead of \( L^{\otimes p} \)).

(2.1.2) An elementary and useful fact is that \( H^1(\mathcal{O}_X(C)) = 0 \) for any irreducible curve on \( X \). If, in addition, the genus \( g \) of \( C \) is bigger than or equal to 2, \( H^1(\mathcal{O}_X(rC)) = 0 \) for all \( r \geq 1 \).

A typical situation we will often encounter is the following: We have a vector bundle \( E \) and a line bundle \( L \) and we want the multiplication map

\[ \varphi: H^0(E) \otimes H^0(L^{\otimes r}) \longrightarrow H^0(E \otimes L^{\otimes r}) \]

to surject. We make the following

**Remark 2.1.** Let \( E \) be a coherent sheaf and let \( L \) be a line bundle on a variety \( X \). Consider the following commutative diagram

\[
\begin{array}{ccc}
H^0(E) \otimes H^0(L^{\otimes r}) & = & H^0(E) \otimes H^0(L^{\otimes r}) \\
\downarrow \alpha_r & & \downarrow \\
H^0(E \otimes L) \otimes H^0(L^{\otimes r-1}) & \cong & H^0(E \otimes L^{\otimes r-1}) \\
\downarrow \alpha_{r-1} & & \downarrow \\
H^0(E \otimes L^{\otimes 2}) \otimes H^0(L^{\otimes r-2}) & \cong & H^0(E \otimes L^{\otimes r-2}) \\
\downarrow \alpha_{r-2} & & \downarrow \\
\vdots & & \vdots \\
H^0(E \otimes L^{\otimes r-2}) \otimes H^0(L^{\otimes 2}) & \cong & H^0(E \otimes L^{\otimes r-2}) \\
\downarrow \alpha_{r-3} & & \downarrow \\
H^0(E \otimes L^{\otimes r-1}) \otimes H^0(L) & \xrightarrow{\alpha_1} & H^0(E \otimes L^{\otimes r}) \\
\end{array}
\]

The multiplication map \( \psi \) is surjective if the maps \( \alpha_1, \alpha_2, ..., \alpha_r \) are surjective.

We will use the following lemma very often:

**Lemma 2.1.** Let \( X \) be a regular variety (i.e., a variety such that \( H^1(\mathcal{O}_X) = 0 \)). Let \( E \) be a vector bundle on \( X \) and \( L = \mathcal{O}_X(C) \) a globally generated line bundle such that \( H^1(E \otimes L^{-1}) = 0 \). If the multiplication map

\[ H^0(E \otimes \mathcal{O}_C) \otimes H^0(L \otimes \mathcal{O}_C) \xrightarrow{\alpha} H^0(E \otimes L \otimes \mathcal{O}_C) \]
surjects, then the map

\[ H^0(E) \otimes H^0(L) \xrightarrow{\beta} H^0(E \otimes L) \]

also surjects.

**Proof:** We have the sequence

\[ 0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(C) \longrightarrow \mathcal{O}_X(C) \longrightarrow 0. \]
Taking the global sections of the short exact sequence above and then tensoring with $H^0(E)$ yields the following commutative diagram:

$$
\begin{array}{ccc}
H^0(E) \otimes H^0(O_X) & \hookrightarrow & H^0(E) \otimes H^0(L) \\
\downarrow & & \downarrow \\
H^0(E) & \hookrightarrow & H^0(E \otimes L)
\end{array}
\rightarrow H^0(\bigotimes pC \otimes rC) \rightarrow 0
$$

The vertical left hand side arrow is surjective for trivial reasons. The vertical right hand side arrow is surjective because of the surjectivity of $\alpha$ and the vanishing of $H^1(E \otimes L \otimes rC) = 0$. The exactness at the right of the top horizontal sequence follows from the vanishing of $H^1(O_X)$. The surjectivity of $\beta$, which is the vertical middle arrow, is then obtained by chasing the diagram. □

**Proposition 2.2.** Let $X$ be a K3 surface and $L = O_X(C)$ be a line bundle on it. Assume further that the general member $C \in |L|$ is a smooth non-hyperelliptic curve of genus $g \geq 3$. Then

$$H^1(\bigotimes pC \otimes rC) = 0 \text{ for all } r \text{ and } p \geq 1$$

**Proof:** We consider the sequence (2.1.1) with $G = pC$, we tensor it with $rC$ and take global sections. Since $H^1(rC) = 0$ by (2.1.2), we obtain

$$H^0(pC) \otimes H^0(rC) \xrightarrow{\psi} H^0((p + r)C) \rightarrow H^1(\bigotimes pC \otimes rC) \rightarrow 0,$$

so the vanishing of $H^1(\bigotimes pC \otimes rC)$ is equivalent to the surjectivity of $\psi$. To show the surjectivity of $\psi$ we use Remark 2.1. According to the remark we need to check the surjectivity of several maps. Here we will only show the surjectivity of

$$H^0(pC) \otimes H^0(C) \xrightarrow{\alpha} H^0((p + 1)C);$$

(note that this map corresponds to $\alpha_r$ in Remark 2.1. Similar arguments work for the rest of the maps $\alpha_{r-1}, ... \alpha_1$). By (2.1.2) and Lemma 2.1, it is enough to check that

$$H^0(pC \otimes O_C) \otimes H^0(C \otimes O_C) \rightarrow H^0((p + 1)C \otimes O_C)$$

surjects. Since $C \otimes O_C$ is the canonical divisor $K_C$ on $C$ the required surjection follows from Noether’s theorem. □

**Corollary 2.3.** (St. Donat, A. Mayer): Let $X$ be a K3 surface and $L = O_X(C)$ be a line bundle on it. Assume further that the general member $C \in |L|$ is a smooth non-hyperelliptic curve of genus $g \geq 3$. Then $L = O_X(C)$ is normally generated.

**Proof:** From (2.1.1) it suffices to prove that $H^1(M_L \otimes L \otimes rC) = 0$ for all $r \geq 1$. This follows from the above lemma by taking $p = 1$. □

If we impose extra conditions on $C$ we obtain a stronger vanishing result:
Proposition 2.4. Let $X$ be a K3 surface and $L = \mathcal{O}_X(C)$ be the line bundle corresponding to the divisor $C$, where $C$ is a non-trigonal, non-plane quintic curve of genus $g \geq 4$. Then,

$$H^1(M_C^{\otimes 2} \otimes rC) = 0 \text{ for all } k \text{ and } r \geq 1$$

We recover from this proposition the following result of St. Donat:

Corollary 2.5. (St. Donat): Let $X$ be a K3 surface and $L$ a line bundle as in the above proposition. Then, $L$ is normally presented.

Proof: By Corollary 2.3, $L$ is normally generated. Then by [GL1], it is enough to prove that $H^1(\wedge^2 M_L \otimes L^{\otimes r}) = 0$ for all $r \geq 1$. Since we are working over the field of complex numbers, it is enough to prove that $H^1(M \otimes C \otimes L^{\otimes r}) = 0$. This follows from the Propositions 2.2 and 2.4 by taking $k = 1$. □

Proposition 2.4 gives us a stronger vanishing result than the one needed to prove Corollary 2.5, which is $H^1(M_C^{\otimes 2} \otimes rC) = 0$. We prove Proposition 2.4 in several steps, starting by proving the vanishing just mentioned.

Lemma 2.6. Let $X$ be a K3 surface and $L = \mathcal{O}_X(C)$ be the line bundle corresponding to the divisor $C$, where $C$ is a non-trigonal curve of genus $g \geq 4$, which is not isomorphic to a smooth plane quintic. Then,

$$H^1(M_C^{\otimes 2} \otimes rC) = 0 \text{ for all } r \geq 1$$

Proof: We prove the lemma when $r = 1$. The proof when $r \geq 2$ is less complicated and follows from similar arguments. Tensoring (2.2.1) with $M_C \otimes C$ and taking global sections yields

$$H^0(M_C \otimes C) \otimes H^0(C) \xrightarrow{\varphi} H^0(M_C \otimes 2C) \rightarrow H^1(M_C^{\otimes 2} \otimes C)$$

$$\rightarrow H^1(M_C \otimes C) \otimes H^0(C).$$

The group $H^1(M_C \otimes C)$ vanishes by Proposition 2.2. Therefore the vanishing of $H^1(M_C^{\otimes 2} \otimes C)$ is equivalent to the surjectivity of $\varphi$. To show the surjectivity of $\varphi$ we use Lemma 2.1. We need to see that $H^1(M_C) = 0$ and that

$$H^0(M_C \otimes C \otimes \mathcal{O}_C) \otimes H^0(C \otimes \mathcal{O}_C) \xrightarrow{\alpha} H^0(M_C \otimes 2C \otimes \mathcal{O}_C)$$

surjects. The former follows from the vanishing of $H^1(\mathcal{O}_X)$. To prove the later we consider the following sequence (see [GP2] for details):

$$0 \rightarrow \mathcal{O}_C \rightarrow M_C \otimes \mathcal{O}_C \rightarrow M_C \otimes \mathcal{O}_C \rightarrow 0 \text{ (2.6.1)}. $$

If we tensor (2.6.1) by $C$ and take global sections, we obtain

$$0 \rightarrow H^0(K_C) \rightarrow H^0(M_C \otimes C \otimes \mathcal{O}_C) \xrightarrow{\delta} H^0(M_{KC} \otimes \mathcal{O}_C) \rightarrow H^1(M_C \otimes K_C) \rightarrow 0.$$
The map $\beta$ is surjective because $h^1(M_C \otimes C \otimes O_C) = g + 1$ and $h^1(M_{K_C} \otimes K_C) = g$, where $K_C = C \otimes O_C$ is the canonical bundle on $C$. We can therefore write the following commutative diagram (we denote $E = M_C \otimes C$ and $F = M_{K_C} \otimes K_C$):

\[
\begin{array}{cccccc}
0 & \rightarrow & H^0(K_C) \otimes H^0(K_C) & \rightarrow & H^0(E \otimes O_C) \otimes H^0(K_C) & \overset{\beta}{\rightarrow} & H^0(F) \otimes H^0(K_C) \\
& & \downarrow \alpha & & \downarrow \nu & & \\
0 & \rightarrow & H^0(K_C^2) & \rightarrow & H^0(E \otimes K_C) & \rightarrow & H^0(F \otimes K_C).
\end{array}
\]

The left hand side vertical arrow is surjective by Noether’s theorem and the right hand side vertical arrow is surjective by Petri’s theorem. Thus $\alpha$ also surjects as we wished. □

We used Petri’s theorem in the course of the last proof. We would like now to give an alternative proof in the case of non-bielliptic curves without using Petri’s theorem, which fits in a more general context. It suggests that we can get “more” than just the surjectivity we need. In fact using the technique in our proof presented below and building upon the work of [GL2], the second author and G. Pareschi prove that the canonical ring of a curve $C$ satisfying the hypothesis in Lemma 2.6 is Koszul. We would like to sketch the proof of the surjectivity of the map under consideration (readers who are familiar with Koszul conditions in terms of surjectivity of global sections of bundles will recognize that the surjection in the statement of Lemma 2.7 is the first step needed to show that the canonical ring of the curve under consideration is Koszul).

The alternative proof therefore follows the same steps as the previous one, except that we show the surjectivity of $\nu$ in the following way:

**Lemma 2.7.** Let $C$ be a smooth curve of genus $g \geq 6$ which is neither trigonal nor bielliptic. Assume further that it is not isomorphic to a smooth plane quintic. Then the multiplication map :

\[
H^0(M_{K_C} \otimes K_C) \otimes H^0(K_C) \rightarrow H^0(M_{K_C} \otimes K_C^2)
\]

is surjective.

**Proof:** It follows from the hypothesis on $C$ and Mumford-Martens theorems (cf. [ACGH]; see also [GL2]) that there exists a divisor $D = x_1 + x_2 + \cdots + x_{g-1}$ with $h^0(D) = 2$ such that $D$ and $K_C(-D)$ are globally generated.

Also by [GL2] we have the following exact sequences:

\[
0 \rightarrow M_{K(-D)} \rightarrow M_K \rightarrow \Sigma_D \rightarrow 0 \quad (1)
\]

\[
0 \rightarrow O_C(-x_{g-2} - x_{g-1}) \rightarrow \Sigma_D \rightarrow \bigoplus_{i=1}^{g-3} O_C(-x_i) \rightarrow 0 \quad (2).
\]

If we tensor (1) and (2) by $K_C$ and take global sections, we obtain

\[
0 \rightarrow H^0(M_{K(-D)} \otimes K_C) \rightarrow H^0(M_K \otimes K_C) \rightarrow H^0(\Sigma_D \otimes K_C) \rightarrow 0 \quad (3)
\]

If we tensor (1) and (2) by $K_C$ and take global sections, we obtain
0 \rightarrow H^0(K_C(-x_{g-2} - x_{g-1})) \rightarrow H^0(K_C \otimes D) \rightarrow H^0(\bigoplus_{i=1}^{g-3} K_C(-x_i)) \rightarrow 0 \quad (4).

The exactness of (3) and (4) at the right hand side is deduced as follows: the key point is to see that \( H^0(K_C \otimes D) = g - 2 \). For this, one compares the bounds of \( h^1(K_C \otimes \Sigma_D) \) obtained from the long exact sequences of cohomology following (3) and (4), keeping into account that \( h^0(M_{K_C} \otimes K_C) = g \) by Noether’s theorem and that \( h^1(M_{K_C}(-D) \otimes K_C) = 2 \) by the base-point-free pencil trick (cf. [ACGH], Section 3). The rest is just a matter of adding and subtracting dimensions of vector spaces.

Now denote \( K_C \otimes \Sigma_D \) by \( \Gamma \) and \( M_{K_C}(-D) \otimes K_C \) by \( P \). If we tensor (3) by \( H^0(K_C) \) and consider the obvious multiplication maps, we obtain

\[
\begin{array}{cccccc}
0 & \rightarrow & H^0(P) \otimes H^0(K_C) & \rightarrow & H^0(F) \otimes H^0(K_C) & \rightarrow & H^0(\Gamma) \otimes H^0(K_C) & \rightarrow & 0 \\
\downarrow \delta & & \downarrow \nu & & \downarrow \epsilon & & \downarrow \nu & & \downarrow \epsilon \\
0 & \rightarrow & H^0(P \otimes K_C) & \rightarrow & H^0(F \otimes K_C) & \rightarrow & H^0(\Gamma \otimes K_C) & \rightarrow & 0.
\end{array}
\]

Therefore, to obtain the surjectivity of \( \nu \), it suffices to check that both \( \delta \) and \( \epsilon \) surject. Since \( M_{K_C}(-D) = K_C(-D)^* \), the map \( \delta \) is in fact the multiplication map

\[
H^0(D) \otimes H^0(K_C) \rightarrow H^0(K_C \otimes D)
\]

which surjects by base-point-free pencil trick and the fact that \( h^0(D) = 2 \).

To see the surjectivity of \( \epsilon \) we argue as follows: we tensor (4) by \( H^0(K_C) \) and considering the corresponding multiplication maps we obtain the commutative diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & H^0(T) \otimes H^0(K_C) & \rightarrow & H^0(\Gamma) \otimes H^0(K_C) & \rightarrow & \bigoplus_{i=1}^{g-3} (H^0(\Lambda) \otimes H^0(K_C)) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & H^0(T \otimes K_C) & \rightarrow & H^0(\Gamma \otimes K_C) & \rightarrow & \bigoplus_{i=1}^{g-3} H^0(\Lambda \otimes K_C) & \rightarrow & 0
\end{array}
\]

where \( T = K_C(-x_{g-2} - x_{g-1}) \) and \( \Lambda = K_C(-x_i) \). We want the vertical arrows at the sides to be surjective. Since \( C \) is non-hyperelliptic, non-trigonal, non-bielliptic curve which is not isomorphic to a plane quintic, Mumford-Martens says that the complete linear series associated to \( K_C(-x_i) \) and \( K_C(-x_{g-2} - x_{g-1}) \) give a birational map from \( C \) onto its image. The surjectivity of the vertical arrows on the sides follows from the following result of Castelnuovo (see [ACGH], page 151), for \( D' = K_C(-x_i), K_C(-x_{g-2} - x_{g-1}) \) and \( l = 1 \): Let \( |D'| \) be a complete base point free linear series of dimension \( r = r(D') \geq 3 \) on a smooth curve \( C \), and assume that
the mapping
\[ \Phi_{D'} : C \longrightarrow P^n \]
is birational onto its image. Then the natural map
\[ \text{Sym}^l H^0(C, \mathcal{O}(D')) \otimes H^0(C, K_C) \longrightarrow H^0(C, K_C(lD')) \]
is surjective for \( l \geq 0 \). □

Now we combine all these elements to give the proof of Proposition 2.4:

**Proof** (of Proposition 2.4). We will prove the result when \( r = 1 \) (the arguments for \( r \geq 2 \) are similar). Thus we want to show that \( H^1(M \otimes^p C) = 0 \) for all \( k \geq 1 \). Since \( H^1(M \otimes C) = 0 \) by Proposition 2.2, \( H^1(M \otimes^p C) \) appears as the cokernel of
\[ H^0(M \otimes C) \otimes H^0(kC) \xrightarrow{\varphi} H^0(M \otimes (k + 1)C). \]
To show that \( \varphi \) is surjective we again use Remark 2.1. We will only show here the surjectivity of one of the maps (in fact the first step, corresponding to \( \alpha_r \)) appearing in the statement of the remark, namely \( H^0(M \otimes C) \otimes H^0(C) \xrightarrow{\lambda} H^0(M \otimes 2C). \)

From (2.1.1) it follows that the vanishing of \( H^1(M \otimes M \otimes C) \) would imply the surjectivity of \( \lambda \). On the other hand, since \( H^1(kC) = 0 \) by (2.1.2),
\[ H^1(M \otimes M \otimes C) \]
is also the cokernel of
\[ H^0(kC) \otimes H^0(M \otimes C) \xrightarrow{\mu} H^0(M \otimes (k + 1)C). \]
Using again Remark 2.1, one concludes from (2.1.1) and Lemma 2.6 the surjectivity of \( \mu \).

3. General lemmas

In this section we develop some homological tools necessary to prove the vanishing theorems in the next section. We recall two lemmas from [GP2]. The first is connected to the following problem: Consider two globally generated line bundles \( L_1 \) and \( L_2 \). We would like to relate the vanishing of the cohomology of \( H^1(M \otimes^{p+1} L_1 \otimes L_2) \) to the vanishing of the cohomology of a similar bundle on a divisor \( Y \) of \( X \), obtained by restricting \( L_1 \) and \( L_2 \) to \( Y \). One sees in the next section that Lemma 3.1 plays a crucial role in proving the vanishing theorems for K3 surfaces. When one is working over an algebraic surface, the lemma transfers the problem of computing cohomology of an unstable bundle to that of a semistable bundle, which in general is easier to compute.
Lemma 3.1. Let $X$ be a projective variety, let $q$ be a nonnegative integer and let $F_i$ be a globally generated line bundle on $X$ for all $1 \leq i \leq q + 1$. Let $Q$ be an effective line bundle on $X$ and let $C$ be a reduced and irreducible member of $|Q|$. Let $R$ be a line bundle on $X$ such that

\begin{align}
(3.1.1) H^1(F_i \otimes Q^*) &= 0 \\
(3.1.2) H^1(R \otimes O_C) &= 0 \\
(3.1.3) H^1(M(F_{i_1} \otimes O_C) \otimes \cdots \otimes M(F_{i_{q'}} \otimes O_C) \otimes R) &= 0 \quad \text{for all } 0 \leq q' \leq q.
\end{align}

Then, for all $-1 \leq q'' \leq q$ and any subset $\{j_k\} \subseteq \{i\}$ with $\# \{j_k\} = q'' + 1$ and for all $0 \leq k' \leq q'' + 1$

\[ H^1(M_{F_{j_1}} \otimes M_{F_{j_2}} \otimes \cdots \otimes M_{F_{j_{k'} + 1}} \otimes O_C) \otimes \cdots \otimes M(F_{i_{q'' + 1}} \otimes O_C) \otimes R \otimes O_C) = 0. \]

The next lemma deals roughly with the following situation: Consider in addition to $L_1$ and $L_2$, two more “positive” line bundles $L_1'$ and $L_2'$ (in the sense that $L_i \otimes L_i'$ is an effective line bundle). We would like to relate the vanishing of the cohomology of $M_{L_1}^{\otimes p + 1} \otimes L_2$ and $M_{L_1}^{\otimes p + 1} \otimes L_2'$ to the vanishing of the cohomology of $M_{L_1}^{\otimes p + 1} \otimes L_2$. The usefulness of these constructions is quite clear. For example, they give us a way to prove that if a line bundle $L$ satisfies the property $N_p$, then so does the tensor product of $L$ with certain effective line bundle.

Lemma 3.2. Let $X$ be a projective variety, let $q$ be a nonnegative integer and let $F_i$ be a globally generated line bundle on $X$ for all $1 \leq i \leq q + 1$. Let $Q$ be an effective line bundle on $X$ and let $C$ be a reduced and irreducible member of $|Q|$. Let $R$ be a line bundle on $X$ such that

\begin{align}
(3.2.1) H^1(F_i \otimes Q^*) &= 0 \\
(3.2.2) H^1(R \otimes Q \otimes O_C) &= 0 \\
(3.2.3) H^1(M(F_{i_1} \otimes O_C) \otimes \cdots \otimes M(F_{i_{q'}} \otimes O_C) \otimes R \otimes Q) &= 0 \quad \text{for all } 0 \leq q' \leq q.
\end{align}

If $H^1(M_{F_1} \otimes M_{F_2} \otimes \cdots \otimes M_{F_{q + 1}} \otimes R) = 0$, then

\[ H^1(M_{F_1} \otimes M_{F_2} \otimes \cdots \otimes M_{F_{q+1}} \otimes R \otimes Q) = 0. \]

We now prove a general lemma that will be used to prove vanishing theorems for K3 surfaces in the next section.

Lemma 3.3. Let $X$ be a projective variety and $L$ a globally generated line bundle. Assume that

\begin{align}
(3.3.1) H^1(M_{L}^{\otimes j} \otimes M_{L}^{\otimes i} \otimes L^{\otimes p}) &= 0 \quad \text{for all } i, j \text{ such that } i + j = p \text{ and for all } r \geq 1. \\
(3.3.2) H^1(M_{L}^{\otimes (p+1)} \otimes L^{\otimes (p+k)}) &= 0 \quad \text{for all } k \geq q_0 \geq 0.
\end{align}

Then $H^1(M_{L}^{\otimes i} \otimes M_{L}^{\otimes j} \otimes L^{\otimes (p+k)}) = 0$ for all $i, j$ such that $i + j = p + 1$, $k \geq q_0 \geq 0$ and $r \geq 1$.

Proof: By induction on $i$: If $i = 0$, we have $H^1(M_{L}^{\otimes (p+1)} \otimes L^{\otimes (p+k)}) = 0$ by hypothesis. Assume the lemma to be true for $i - 1$ and we will prove it for $i$. So we want

\[ H^1(M_{L}^{\otimes i} \otimes M_{L}^{\otimes i} \otimes L^{\otimes (p+k)}) = 0 \quad \text{where } i + j = p + 1, k \geq q_0 \geq 0 \text{ and } r \geq 1. \]
By induction hypothesis, one has the vanishing $H^1(M_{L}^j \otimes 1 M_{L}^{i+1} \otimes 1 L^{(p+k)}) = 0$ for all $\otimes M_{L^{(p+k)} \otimes L^{(p+k)}}$ sits in the following exact sequence obtained from (2.1.1):

$$H^0(M_{L}^{j} \otimes 1 M_{L^{(p+k)} \otimes L^{(p+k)}} \otimes H^0(L^{(p+k)}) \rightarrow H^0(M_{L}^{j} \otimes 1 M_{L^{(p+k)} \otimes L^{(p+k)}} \otimes H^0(L^{(p+k)}) \rightarrow H^1(M_{L}^{j} \otimes 1 M_{L^{(p+k)} \otimes L^{(p+k)}} \otimes H^0(L^{(p+k)}).$$

The last term of the above sequence is zero by hypothesis, since $i + j - 1 = p$. In view of Remark 2.1, it is enough to prove that

$$H^0(M_{L}^{j} \otimes 1 M_{L^{(p+k)} \otimes L^{(p+k)}} \otimes H^0(L^{(p+k)}) \rightarrow H^0(M_{L}^{j} \otimes 1 M_{L^{(p+k)} \otimes L^{(p+k)}} \otimes H^0(L^{(p+k)}) \rightarrow 0.$$

We will show the first step, namely that

$$H^0(M_{L}^{j} \otimes 1 M_{L^{(p+k)} \otimes L^{(p+k)}} \otimes H^0(L) \rightarrow H^0(M_{L}^{j} \otimes 1 M_{L^{(p+k)} \otimes L^{(p+k)}} \otimes H^0(L))$$

surjects; the others are similar. Observe that the cokernel of the above multiplication map is

$$H^1(M_{L}^{j+1} \otimes 1 M_{L^{(p+k)} \otimes L^{(p+k)}}),$$

which is zero by induction assumption. □

**Lemma 3.4.** Let $X$ be a variety such that $H^1(O_X) = 0$. Let $L = O_X(C)$ be a globally generated line bundle. If

1. $H^1(M_{L}^{p} \otimes 1 L^{(p+k)}) = 0$
2. $H^1(M_{L}^{p} \otimes 1 L^{(p+1+k)}) = 0$
3. $H^1(M_{L}^{p} \otimes M_{L^{(p+k)} \otimes O_C} \otimes L^{(p+k)} \otimes O_C) = 0$

then $H^1(M_{L}^{(p+1)} \otimes L^{(p+k)}) = 0$.

**Proof:** Observe that $H^1(M_{L}^{(p+1)} \otimes L^{(p+k)})$ sits in the following exact sequence:

$$H^0(M_{L}^{p} \otimes L^{(p+k)}) \otimes H^0(L) \rightarrow H^0(M_{L}^{p} \otimes L^{(p+k+1)}) \rightarrow H^1(M_{L}^{(p+1)} \otimes L^{(p+k)} \otimes L^{0}(L).$$

Since $H^1(M_{L}^{p} \otimes L^{p}) = 0$ by hypothesis, it is enough to prove that the multiplication map $\varphi$ surjects. In view of Lemma 2.1, it is enough to check the vanishing of $H^1(M_{L}^{p} \otimes L^{(p+k+1)})$ and $H^1(M_{L}^{p} \otimes M_{L^{(p+k)} \otimes O_C} \otimes L^{(p+k)} \otimes O_C)$, which follows from hypothesis. □

**4. Vanishing theorems on K3 surfaces**

In this section we prove some strong vanishing theorems, namely Theorems 4.1 and 4.6, regarding certain vector bundles associated to a globally generated line bundle on a K3 surface. As an application of Theorem 4.1, we obtain higher syzygy results for K3 surfaces.
**Theorem 4.1.** Let $X$ be a K3 surface and $L$ a line bundle on $X$. Assume that
the general member $C \in |L|$ is a smooth non-trigonal, non-plane quintic curve of
genus $g \geq 5$. Then

$$H^1(M_C^i \otimes M_C^j \otimes (p+k)C) = 0 \text{ for all } i + j = p + 1, \ k \geq 0 \text{ and } r \geq 1.$$ 

To prove this theorem we need a number of intermediate results.

**Lemma 4.2.** Let $X$ be a K3 surface and $L = O_X(C)$ be a line bundle such that
the smooth general member $C \in |L|$ is a non-hyperelliptic curve of genus $g > 3$. Then
the cohomology group $H^1(M_C^p \otimes (p+k)C) = 0$ for all $k \geq 0$ and $p \geq 1$.

**Proof:** The proof is by induction on $p$. If $p = 1$ the vanishing holds by Proposition 2.2. Assume it is true for $p-1$; then we have $H^1(M_C^{p-1} \otimes (p-1+k)C) = 0$
for all $k \geq 0$. We want to show that $H^1(M_C^p \otimes (p+k)C) = 0$. Tensoring (2.1.1)
(substitute $G = C$) with $M_C^{(p-1)} \otimes (p+k)C$ and taking global sections yields the
long exact sequence:

$$H^0(M_C^{p-1} \otimes (p+k)C) \otimes H^0(C) \xrightarrow{\mu} H^0(M_C^{p-1} \otimes (p+1+k)C) \rightarrow$$

$$\rightarrow H^1(M_C^p \otimes (p+k)C) \rightarrow H^1(M_C^{p-1} \otimes (p+k)C) \otimes H^0(C).$$

The last term is zero by induction assumption. So it is enough to prove that the
multiplication map $\mu$ surjects.

Since $H^1(M_C^{p-1} \otimes (p-1+k)C) = 0$ by induction, we may use Lemma 2.1 to
reduce the problem of seeing the surjectivity of the map $\mu$, which is a multiplication
map of global sections of vector bundles on a K3 surface, to the problem of checking
the surjectivity of a multiplication map of sections of bundles on a curve. According
to Lemma 2.1, it suffices to show that

$$H^0(M_C^{p-1} \otimes (p+k)C \otimes O_C) \otimes H^0(C \otimes O_C) \rightarrow H^0(M_C^{p-1} \otimes (p+1+k)C \otimes O_C)$$

surjects. This will follow from the vanishing

$$H^1(C, M_C^{p-1} \otimes O_C \otimes M_KC \otimes (p+k)K_C) = 0,$$

where $C \otimes O_C = K_C$ is the canonical bundle on $C$. To see the vanishing, we use
Lemma 3.1. The first condition required by Lemma 3.1 is the vanishing of $H^1($
$O_X)$. The second is the vanishing of $H^1((p+k)K_C)$, which occurs because $p \geq 2$.
Finally, to verify the third condition one has to check the vanishing of

$$H^1(M_K^{p-1} \otimes M_KC \otimes (p+k)K_C).$$

Since the bundle $M_KC$ is stable (see [PR]) and the tensor product of semistable
bundles is semistable (see [Mi]), $M_C^{p-1} \otimes (p+k)C \otimes O_C$ is semistable. Therefore,
it is enough to check that the slope $\mu(M_C^{p-1} \otimes (p+k)C \otimes O_C) > 2g - 2$, where $g$
is the genus of $C$ and $\mu(E) = \deg E / \rank E$ for a vector bundle $E$. We have

$$\mu(M_K^{p-1} \otimes (p+k)K_C) = p\mu(M_KC) + (p+k)\deg(K_C),$$

so we need $p(-2) + (p-1+k)(2g-2) > 0$, which is true for $p \geq 2$ and $g > 3$. □
Lemma 4.3. Let $X$ be a K3 surface and let $L = \mathcal{O}_X(C)$ be a globally generated line bundle such that the smooth general member $C \in |L|$ is a non-hyperelliptic curve of genus $g > 3$. Then the cohomology group $H^1(M_C^{\otimes i} \otimes M_{rC}^{\otimes j} \otimes (p+k)C) = 0$ for all $i+j = p$ and for all $k \geq 0$ and $r \geq 1$.

Proof: The proof is by induction on $i+j$. If $i+j = 1$ we want $H^1(M_C \otimes (p+k)C)$ and $H^1(M_{rC} \otimes (p+k)C)$ to vanish, which is true by Proposition 2.2. Let us assume that the result is true for $p-1$. So we have $H^1(M_C^{\otimes i} \otimes M_{rC}^{\otimes j} \otimes (p-1+k)C) = 0$ for $i+j = p-1$. We want to prove the result for $p$. Let $i+j = p$. We need to show that

$$H^1(M_C^{\otimes i} \otimes M_{rC}^{\otimes j} \otimes (p+k)C) = 0.$$ 

We prove this by induction on $j$. For $j = 0$, the vanishing is the conclusion of Lemma 4.2. For $j > 0$, it is enough to prove that the following multiplication map $\mu$, which sits in the following exact sequence

$$H^0(M_C^{\otimes i} \otimes M_{rC}^{\otimes j-1} \otimes (p+k)C) \otimes H^0(rC) \xrightarrow{\mu} H^0(M_C^{\otimes i} \otimes M_{rC}^{\otimes j-1} \otimes (p+r+k)C) \rightarrow H^1(M_C^{\otimes i} \otimes M_{rC}^{\otimes j-1} \otimes (p+k)C) \otimes H^0(rC),$$

surjects. The last term is zero by induction assumption, since $i+j-1 = p-1$. In view of Remark 2.1, it is enough to show that the following multiplication map $\lambda$ sitting in the following exact sequence

$$H^0(M_C^{\otimes i} \otimes M_{rC}^{\otimes j-1} \otimes (p+k)C) \otimes H^0(C) \xrightarrow{\lambda} H^0(M_C^{\otimes i} \otimes M_{rC}^{\otimes j-1} \otimes (p+k+1)C) \rightarrow H^1(M_C^{\otimes i+1} \otimes M_{rC}^{\otimes j-1} \otimes (p+k)C),$$

surjects for all $k \geq 0$. This surjection follows from the vanishing of

$$H^1(M_C^{\otimes i+1} \otimes M_{rC}^{\otimes j-1} \otimes (p+k)C)$$

which follows in turn by induction assumption on $j$. \( \square \)

Lemma 4.4. Let $X$ be a K3 surface and $L$ a line bundle on $X$. Assume that the general member $C \in |L|$ is a smooth non-trigonal, non-plane quintic curve of genus $g \geq 5$. Then for any integer $p \geq 2$,

$$H^1(M_C^{\otimes i} \otimes (i-1+k)C) = 0$$

for all $i = 2, ..., p$ and $k \geq 0$.

Proof: We will prove the lemma by induction on $i$. If $i = 2$, we want

$$H^1(M_C^{\otimes 2} \otimes (2-1+k)C) = 0.$$ 

This is true by Proposition 2.4. Assume the statement of the lemma for $p-1$. So we have $H^1(M_C^{\otimes (p-1)} \otimes (p-2+k)C) = 0$ for all $k \geq 0$.

We want to show that $H^1(M_C^{\otimes p} \otimes (p-1+k)C) = 0$. Tensoring (2.1.1) with $M_C^{\otimes (p-1)} \otimes (p-1+k)C$ and taking global sections yields the following sequence:

$$H^0(M_C^{\otimes (p-1)} \otimes (p-1+k)C) \otimes H^0(C) \rightarrow H^0(M_C^{\otimes (p-1)} \otimes (p+k)C)$$

$$H^1(M_C^{\otimes p} \otimes (p-1+k)C) \rightarrow H^1(M_C^{\otimes (p-1)} \otimes (p-1+k)C) \otimes H^0(C).$$
The last term is zero by induction assumption. Since

\[ H^1(M_C^p \otimes (p - 2 + k)C) = 0 \]

by induction, we may use Lemma 2.1 to reduce the problem of seeing the surjectivity of the map \( \alpha \), which is a multiplication map of global sections of vector bundles on a K3 surface, to the problem of checking the surjectivity of a multiplication map of sections of bundles on a curve. According to Lemma 2.1, it suffices to show that

\[ H^0(M_C^p \otimes (p - 1 + k)C \otimes O_C) \otimes H^0(C \otimes O_C) \longrightarrow H^0(M_C^p \otimes (p + k)C \otimes O_C) \]

surjects. This will follow from the vanishing

\[ H^1(C, M_C^p \otimes O_C \otimes M_KC \otimes (p - 1 + k)K_C) = 0, \]

where \( C \otimes O_C = K_C \) is the canonical bundle on \( C \). To see the vanishing, we use Lemma 3.1. The first condition required by Lemma 3.1 is the vanishing of \( H^1(O_X) \).

The second is the vanishing of \( H^1((p - 1 + k)K_C) \), which occurs because \( p \geq 3 \). Finally, to verify the third condition one has to check the vanishing of

\[ H^1(M_KC \otimes (p - 1 + k)K_C). \]

Since the bundle \( M_{C \otimes O_C} \) is stable (see [PR]) and the tensor product of semistable bundles is semistable (See [Mi]), \( M_{C \otimes O_C} \otimes (p - 1 + k)C \otimes O_C \) is semistable so it is enough to check that \( \mu(M_{C \otimes O_C} \otimes (p - 1 + k)C \otimes O_C) > 2g - 2 \), where \( g \) is the genus of \( C \). We have

\[ \mu(M_{C \otimes O_C} \otimes (p - 1 + k)C \otimes O_C) = p\mu(M_KC) + (p - 1 + k)\deg(K_C), \]

since \( C \otimes O_C = K_C \). Therefore we need \( p(-2) + (p - 2 + k)(2g - 2) > 0 \), which is true for \( p \geq 3 \) and \( g \geq 5 \). \( \square \)

We are now ready to prove Theorem 4.1:

**Proof** (of Theorem 4.1): We want to apply Lemma 3.3. For that we have to check the two hypotheses in the statement of the lemma. The first follows from Lemma 4.3 and the second follows from Lemma 4.4. \( \square \)

We now use the above results to prove more general vanishing theorems on K3 surfaces.

**Lemma 4.5.** Let \( X \) be a K3 surface, \( L = O_X(C) \) a globally generated line bundle such that the smooth general member \( C \in |L| \) is a non-hyperelliptic curve of genus \( g > 3 \). Then the cohomology groups

\[ H^1(M_C^j \otimes M_{i_1C}^{i_1} \otimes M_{i_2C}^{i_2} \otimes \cdots \otimes M_{i_rC}^{i_r} \otimes (p + k)C) = 0 \]

for all \( i_s \geq 1 \)

and \( j + i_1 + i_2 + \cdots + i_r = p \), where \( s = 1, \ldots, r \).
**Proof:** We will prove the theorem only for \( r = 2 \) (for the sake of simplicity). The proof is very similar to the proof of Lemma 4.3. The proof is by induction on \( j + j_1 + j_2 \). If \( j + j_1 + j_2 = 1 \), then the theorem is true by Proposition 2.2. Assume the theorem to be true for \( p - 1 \). Now let \( j + j_1 + j_2 = p \). So we want

\[
H^1(M_C^{\otimes j} \otimes M_{i_1C}^{\otimes j_1} \otimes M_{i_2C}^{\otimes j_2} \otimes (p + k)) = 0.
\]

We will prove the vanishing by induction on \( j_1 \). If \( j_1 = 0 \), the result follows from Lemma 4.3. To prove for \( j_1 > 0 \), it is enough to prove that \( \mu \), which sits in the exact sequence

\[
H^0(E \otimes (p + k)C) \otimes H^0(i_1C) \xrightarrow{\delta} H^0(E \otimes (p + k + i_1)C) \rightarrow H^1(F \otimes (p + k)) \rightarrow H^1(M_C^{\otimes j} \otimes M_{i_1C}^{\otimes j_1-1} \otimes M_{i_2C}^{\otimes j_2} \otimes (p + k)C) \otimes H^0(i_1C),
\]

surjects, where \( E = M_C^{\otimes j} \otimes M_{i_1C}^{\otimes j_1-1} \otimes M_{i_2C}^{\otimes j_2} \) and \( F = M_C^{\otimes j} \otimes M_{i_1C}^{\otimes j_1} \otimes M_{i_2C}^{\otimes j_2} \). The last term is zero by induction assumption, since \( j + j_1 - 1 + j_2 = p - 1 \).

In view of Remark 2.1, it is enough to show the surjectivity of the multiplication map \( \lambda \) sitting in the following exact sequence:

\[
H^0(E \otimes (p + k)C) \otimes H^0(C) \xrightarrow{\lambda} H^0(E \otimes (p + k + 1)C) \rightarrow H^1(G \otimes (p + k)C),
\]

where \( G = M_C^{\otimes j+1} \otimes M_{i_1C}^{\otimes j_1-1} \otimes M_{i_2C}^{\otimes j_2} \). The surjection follows from the vanishing of

\[
H^1(M_C^{\otimes j+1} \otimes M_{i_1C}^{\otimes j_1-1} \otimes M_{i_2C}^{\otimes j_2} \otimes (p + k)C),
\]

which in turn follows from induction assumption on \( j_1 \). \( \square \)

**Theorem 4.6.** Let \( X \) be a K3 surface and \( L = \mathcal{O}_X(C) \) be a line bundle such that the general member \( C \in |L| \) is a smooth non-trigonal, non-plane quintic curve of genus \( g \geq 5 \). Then

\[
H^1(M_C^{\otimes j} \otimes M_{i_1C}^{\otimes j_1} \otimes M_{i_2C}^{\otimes j_2} \otimes \cdots \otimes M_{i_rC}^{\otimes j_r} \otimes (p + k)C) = 0 \text{ for all } j + j_1 + \cdots + j_r = p + 1 \text{ and } i_t \geq 1 \text{ where } t = 1, \ldots, r \text{ and } k \geq 0.
\]

**Proof:** We will prove only the case \( r = 2 \), the general case is exactly similar. So we have to show that

\[
H^1(M_C^{\otimes j} \otimes M_{i_1C}^{\otimes j_1} \otimes M_{i_2C}^{\otimes j_2} \otimes (p + k)C) = 0 \text{ for all } j + j_1 + j_2 = p + 1, \ i_s \geq 1.
\]

The proof follows from induction on \( j_1 + j_2 \). If \( j_1 + j_2 = 1 \), the result is true by Theorem 4.1. We assume the result for \( j_1 + j_2 - 1 \) we want now to prove it for \( j_1 + j_2 \). We need only to prove that the multiplication map in the following long exact sequence surjects:

\[
H^0(E \otimes (p + k)C) \otimes H^0(i_2C) \rightarrow H^0(E \otimes (p + k + i_1)C) \rightarrow H^1(F \otimes (p + k)) \rightarrow H^1(M_C^{\otimes j} \otimes M_{i_1C}^{\otimes j_1} \otimes M_{i_2C}^{\otimes j_2-1} \otimes (p + k)C) \otimes H^0(i_1C).
\]
where \( E = M_C^{\otimes j} \otimes M_{i_1 C}^{\otimes j_1} \otimes M_{i_2 C}^{\otimes j_2 - 1} \) and \( F = M_C^{\otimes j} \otimes M_{i_1 C}^{\otimes j_1} \otimes M_{i_2 C}^{\otimes j_2} \). The last term in the above exact sequence is zero by previous lemma, since \( j + j_1 + j_2 - 1 = p \).

In the light of Remark 2.1 and arguments used repeatedly throughout this article it is enough to prove that the multiplication map below is surjective:

\[
H^0(E \otimes (p+k)C) \otimes H^0(C) \rightarrow H^0(E \otimes (p+k+1)C) \rightarrow H^1(R \otimes (p+k)C)
\]

where \( R = E \otimes M_C = M_C^{\otimes j+1} \otimes M_{i_1 C}^{\otimes j_1} \otimes M_{i_2 C}^{\otimes j_2 - 1} \). By induction assumption the result is true for \( j_1 + j_2 - 1 \), hence we have the vanishing of

\[
H^1(M_C^{\otimes j+1} \otimes M_{i_1 C}^{\otimes j_1} \otimes M_{i_2 C}^{\otimes j_2 - 1} \otimes (p+k)C).
\]

□

5. Higher syzygies of K3 surfaces

In this section we give an application of the vanishing theorems proved in Section 4. In particular, we show that the vanishing theorems imply a result on higher syzygies for K3 surfaces.

Note that we are all along abusing the notation by writing \( M_C \) instead of \( M_L \), where \( L = \mathcal{O}_X(C) \). We now revert back to the notation \( M_L \) ! So for instance \( M_L \otimes r \) corresponds to the notation \( M_r \) which we have been using above.

**Theorem 5.1.** Let \( X \) be a K3 surface and \( L = \mathcal{O}_X(C) \) be a line bundle such that the smooth general member \( C \in |L| \) is a non-hyperelliptic curve of genus \( g > 3 \). Then \( L^{\otimes (p+1)} \) satisfies \( N_p \).

**Proof:** Let \( L' = L^{\otimes (p+1)} \). By Proposition 2.2 \( L' \) satisfies \( N_0 \). By [GL1], it is enough to prove

\[
H^1(\bigwedge^i M_{L'} \otimes (L')^{\otimes s}) = 0 \text{ for all } 1 \leq i \leq p + 1 \text{ and } s \geq 1.
\]

Since we are working over the complex numbers, it is enough to prove \( H^1(M_{L'}^{\otimes i} \otimes (L')^{\otimes s}) = 0 \) for all \( 1 \leq i \leq p + 1 \) and \( s \geq 1 \). The vanishing follows from Lemma 4.3. □

**Theorem 5.2.** Let \( X \) be a K3 surface and \( L \) a line bundle on \( X \). Assume further that the general member \( C \in |L| \) is a smooth non-trigonal, non-plane quintic curve of genus \( g \geq 5 \). Then \( L^{\otimes p} \) satisfies \( N_p \).

**Proof:** We need only to prove \( H^1(M_{L'}^{\otimes i} \otimes (L')^{\otimes s}) = 0 \), for all \( 1 \leq i \leq p + 1 \), where now \( L' = L^{\otimes p} \). By Lemma 4.3 and Theorem 4.1 we have the vanishing of

\[
H^1(M_{L'}^{\otimes i} \otimes M_L^{\otimes j} \otimes L^{\otimes (p+k)})
\]

for all \( i + j \leq p + 1 \) and for all \( k \geq 0 \). By letting \( j = 0 \) and \( r = p \), we get the desired vanishing. □
6. SYZYGIES OF FANO VARIETIES

In this section we prove a vanishing theorem on Fano threefolds of index one, namely Theorem 6.6, and we generalize the result to Fano varieties of dimension \( n \) with index \( (n - 2) \). These results have as consequence results on higher syzygies of Fano threefolds and, generally, of Fano varieties of dimension \( n \) with index \( (n - 2) \). The techniques and method of the proofs are almost entirely analogous to the case of K3 surfaces, so most of the time we will just sketch the proof and leave the details to the reader.

Along the first part of this section (until Theorem 6.8), the variety \( X \) will be a Fano threefold with very ample anticanonical bundle. We will write \( L = -K_X \). Under these hypotheses a general member \( S \) of \( |L| \) is a smooth K3 surface. Note that if \( X \) has index 1, then \( L \) is the ample primitive line bundle generating \( \text{Pic}(X) \).

For interesting examples of Fano 3-folds of index 1, we refer the reader to [C].

**Lemma 6.1.** Let \( X \) be a Fano threefold, \( L = -K_X \) very ample and \( S \) a general member of \( |L| \). Then

\[
H^1(M_{pS} \otimes rS) = 0 \quad \text{for all} \quad r, \ p \geq 1
\]

**Proof:** First we remark that, since \( X \) is a Fano threefold, we have \( H^1(\mathcal{O}_X(rS')) = 0 \) for all \( r \geq 1 \), for any smooth surface \( S' \subset X \). So tensoring the sequence

\[
0 \rightarrow M_{pS} \rightarrow H^0(pS) \otimes \mathcal{O}_X \rightarrow pS \rightarrow 0
\]

by \( rS \) yields:

\[
H^0(pS) \otimes H^0(rS) \xrightarrow{\varphi} H^0((p + r)S) \rightarrow H^1(M_{pS} \otimes rS) \rightarrow 0.
\]

Thus it is enough to prove that \( \varphi \) is surjective. In view of Remark 2.1, it is enough to prove that the map

\[
H^0(pS) \otimes H^0(S) \rightarrow H^0((p + 1)S)
\]

is surjective for all \( p \geq 1 \). Since \( H^1(\mathcal{O}_X) = 0 \) and \( H^1(p'S) = 0 \) for all \( p' \geq 1 \), by Lemma 2.1 it is enough to show that the map below is surjective:

\[
H^0(pS \otimes \mathcal{O}_S) \otimes H^0(S \otimes \mathcal{O}_S) \rightarrow H^0((p + 1)S \otimes \mathcal{O}_S).
\]

This follows from Proposition 2.2 (Note that the general member \( C \in |\mathcal{O}_S(S)| \) is a non-hyperelliptic curve of genus \( g \geq 3 \)).

If we allow \( p = 1 \), we obtain Iskovskih’s result (see [I]).

**Remark 6.1.** Lemma 6.1 admits a slight variant in its statement: we could just assume that \( L \) is globally generated and that \( C \) is non-hyperelliptic of genus \( g \geq 3 \). Under these assumptions 6.1 would imply in particular that \( L \) is very ample. All the results that follow can be reformulated as well in the same fashion, thus indicating how our syzygy results on Fano varieties fit in our meta-principle.
**Lemma 6.2.** Let $X$ be a Fano threefold, $L = -K_X$ very ample and $S$ a general member of $|L|$. Assume further that the smooth general member $C \in |O_S(S)|$ is a curve of genus $g > 3$. Then the cohomology group

$$H^1(X, M_S^p \otimes (p + k)S) = 0 \text{ for all } k \geq 0, \ p \geq 1.$$

**Proof:** The proof as usual is by induction on $p$. If $p = 1$ the vanishing holds by Lemma 6.1. Assume that the result is true for $p - 1$; we have

$$H^1(M_S^{p - 1} \otimes (p - 1 + k)S) = 0 \text{ for all } k \geq 0.$$

We want to show that $H^1(M_S^p \otimes (p + k)S) = 0$. For that it is enough to prove that the multiplication map $\alpha$ in the long exact sequence

$$H^0(M_S^{p - 1} \otimes (p + k)S) \otimes H^0(S) \xrightarrow{\alpha} H^0(M_S^{p - 1} \otimes (p + 1 + k)S) \rightarrow H^1(M_S^{p - 1} \otimes (p + k)S) \otimes H^0(S),$$

surjects, since the last term of the sequence is zero by induction.

By Lemma 2.1 it is enough to show that $H^1(M_S^{p - 1} \otimes (p - 1 + k)S) = 0$ in order to reduce the problem of checking the surjectivity of the multiplication map $\alpha$ on the Fano threefold $X$ to the problem of checking the surjectivity of a multiplication map on the K3 surface $S$. The required vanishing follows from induction hypothesis. So we need to prove that the following multiplication map on $S$ surjects:

$$H^0(M_S^{p - 1} \otimes (p + k)S \otimes O_S) \otimes H^0(S \otimes O_S) \rightarrow H^0(M_S^{p - 1} \otimes (p + 1 + k)S \otimes O_S)$$

the above map fits in the long exact sequence, whose next term is

$$H^1(M_S^{p - 1} \otimes M_S \otimes O_S \otimes (p + k)S \otimes O_S).$$

In view of Lemma 3.1 we need only to show that $H^1(S, M_C^p \otimes (p + k)C) = 0$. The vanishing follows from Lemma 4.2. □

**Lemma 6.3.** Let $X$ be a Fano threefold, $L$ and $S$ as above with the additional assumption that the smooth general member $C \in |O_S(S)|$ is a non-trigonal, non-plane quintic curve of genus $g \geq 5$. Then

$$H^1(X, M_S^{2} \otimes kS) = 0 \text{ for all } k \geq 1.$$

**Proof:** We will prove the Lemma for $k = 1$, the rest are similar and easier. The cohomology group $H^1(M_S^2 \otimes S)$ is the cokernel of the multiplication map

$$H^0(M_S \otimes S) \otimes H^0(S) \xrightarrow{\varphi} H^0(M_S \otimes 2S),$$

because $H^1(M_S \otimes S) = 0$ by Lemma 6.1. Hence it is enough to prove that $\varphi$ surjects. Let $E = M_S \otimes S$. Consider the commutative diagram:
The left hand side of the above diagram surjects. In order to prove that the multiplication map on the right hand side surjects, we use Lemma 2.1, since $H^1(\mathcal{O}_X) = 0$ and $H^1(M_S) = 0$. Thus we reduce the problem of checking the surjectivity of the multiplication map on the threefold to checking on the K3 surface. By tensoring the sequence (see [GP2])

$$0 \to \mathcal{O}_S \otimes H^0(\mathcal{O}_X) \to M_S \otimes \mathcal{O}_S \to M_{(S \otimes \mathcal{O}_S)} \to 0$$

by $S$ and taking global sections we have the following exact sequence:

$$0 \to H^0(S \otimes \mathcal{O}_S) \to H^0(M_S \otimes S \otimes \mathcal{O}_S) \to H^0(M_{(S \otimes \mathcal{O}_S)} \otimes S \otimes \mathcal{O}_S) \to 0.$$ 

Tensoring the above sequence with $H^0(S \otimes \mathcal{O}_S)$ and considering the obvious multiplication maps yields the following commutative diagram:

$$
\begin{array}{cccc}
0 & \to & H^0(C) \otimes H^0(C) & \to & H^0(E \otimes \mathcal{O}_S) \otimes H^0(C) & \to & H^0(F) \otimes H^0(C) & \to & 0 \\
0 & \to & H^0(2C) & \to & H^0(E \otimes C \otimes \mathcal{O}_S) & \to & H^0(F \otimes C) & \to & 0
\end{array}
$$

where $C \in \mathcal{O}_S(S)$ and $F = M_{(S \otimes \mathcal{O}_S)} \otimes S \otimes \mathcal{O}_S$. The maps on the left hand side and right hand side are surjective by Propositions 2.2 and 2.4. □

**Lemma 6.4.** Let $X$ and $L = \mathcal{O}_X(S)$ be as in Lemma 6.3. Then for any integer $p \geq 2$,

$$H^1(M_S^{\otimes i} \otimes (i - 1 + k)S) = 0 \text{ for all } i = 2, ..., p \text{ and } k \geq 0.$$

**Proof:** We will prove the lemma by induction on $i$. If $i = 2$, we want

$$H^1(M_S^{\otimes 2} \otimes (2 - 1 + k)S) = 0 \text{ for all } k \geq 0.$$

This is true by Lemma 6.3. Assume the statement of the lemma for $p-1$. So we have $H^1(M_S^{\otimes (p-1)} \otimes (p-2 + k)S) = 0$. We want to show that $H^1(M_S^{\otimes p} \otimes (p-1 + k)S) = 0$.

The above group fits in the following long exact sequence:

$$H^0(M_S^{\otimes (p-1)} \otimes (p - 1 + k)S) \otimes H^0(S) \xrightarrow{\beta} H^0(M_S^{\otimes (p-1)} \otimes (p + k)S) \to H^1(M_S^{\otimes p} \otimes (p - 1 + k)S) \otimes H^0(S).$$

Since the last term in the above sequence is zero by induction, it is enough to show that $\beta$ is surjective. By Lemma 2.1 we can reduce the problem of checking the surjectivity of $\beta$ restricting it to the K3 surface $S$, provided $H^1(M_S^{\otimes (p-1)} \otimes (p - 2 + k)S) = 0$, which is true again by induction. So we want the following multiplication map on $S$ to surject:

$$0 \to H^0(E) \to H^0(E \otimes S) \to H^0(E \otimes S \otimes \mathcal{O}_S) \to 0.$$
The map fits in a long exact sequence whose next term is

\[ H^1(M_S^{(p-1)} \otimes (p-1+k)S \otimes O_S \otimes M_S \otimes O_S) \]

In order to show that the above cohomology group vanishes, it is enough, by Lemma 3.1 to show that

\[ H^1(S, M_C^{(p-1)} \otimes (p-1+k)C) = 0, \]

but this holds by Lemma 4.4. □

**Lemma 6.5.** Let \( X \) be a Fano threefold, \( L = -K_X \) very ample and \( S \) a general member of \( |L| \). Assume further that the smooth general member \( C \) of \( |O_S(S)| \) is a curve of genus \( g > 3 \). Then

\[ H^1(M_S^{(i)} \otimes M_{rS}^{(j)} \otimes (p+k)S) = 0 \]

for all \( i + j = p, k \geq 0 \) and \( r \geq 1 \).

**Proof:** Mimic the proof of Lemma 4.3 word by word. □

**Theorem 6.6.** Let \( X \) be a Fano threefold, \( L = -K_X \) very ample and \( S \) a general member of \( |L| \). Assume further that the smooth general member \( C \) of \( |O_S(S)| \) is a non-trigonal curve of genus \( g \geq 5 \), which is not isomorphic to a smooth plane quintic. Then \( H^1(M_S^{(i)} \otimes M_{rS}^{(j)} \otimes (p+k)S) = 0 \) for all \( i + j = p+1, k \geq 0 \) and \( r \geq 1 \).

**Proof:** We want to apply Lemma 3.3 for the Fano threefold \( X \). The hypothesis (3.3.1) and (3.3.2) needed in Lemma 3.3 follows from Lemmas 6.5 and 6.4 respectively. □

Now we can use the vanishing results obtained so far to prove results on the syzygies of Fano threefolds as announced:

**Theorem 6.7.** Let \( X \) be a Fano threefold, \( L = -K_X \) very ample and \( S \) a general member of \( |L| \). Assume further that the smooth general member \( C \) of \( |O_S(S)| \) is a curve of genus \( g > 3 \). Then \( L^{(p+1)} \) satisfies \( N_p \).

**Proof:** Denote \( L' = L^{(p+1)} \). By Lemma 6.1 \( L' \) satisfies \( N_0 \). Then by [GL1] it is enough to prove \( H^1(M_L^{(i)} \otimes (L')^{(s)}) = 0 \) for all \( 1 \leq i \leq p+1 \) and \( s \geq 1 \). Since we are working over the complex numbers, it is enough to prove \( H^1(M_L^{(i)} \otimes (L')^{(s)}) = 0 \) for all \( 1 \leq i \leq p+1 \). But the needed vanishing follows from Lemma 6.5. □

**Theorem 6.8.** Let \( X \) be a Fano threefold, \( L = -K_X \) very ample and \( S \) a general member of \( |L| \). Assume further that the smooth general member \( C \) of \( |O_S(S)| \) is a non-trigonal curve of genus \( g \geq 5 \), which is not isomorphic to a smooth plane quintic. Then \( H^1(M_L^{(i)} \otimes (L')^{(s)}) = 0 \) for all \( 1 \leq i \leq p+1 \). But the needed vanishing follows from Lemma 6.5. □
is a non-trigonal curve of genus \( g \geq 5 \) which is not isomorphic to a smooth plane quintic curve. Then \( L^{\otimes p} \) satisfies \( N_p \).

**Proof:** Denote \( L' = L^{\otimes p} \). By Lemma 6.1 \( L' \) satisfies \( N_0 \). We have proved the vanishing

\[
H^1(X, M_S^{\otimes i} \otimes M_S^{\otimes j} \otimes (p + k)S) = 0 \quad \text{for all } i + j \leq p + 1, k \geq 0 \text{ and } r \geq 1
\]

By letting \( i = 0 \), and \( r = p \), we have

\[
H^1(X, M_S^{\otimes p} \otimes (p + k)S) = 0 \quad \text{for all } 1 \leq j \leq p + 1.
\]

Then the theorem follows from the result in [GL1] mentioned in the previous proof. □

As indicated in the introduction of this section the statements of the above theorems and lemmas are in particular statements on the primitive ample bundle of an index 1, Picard number 1 Fano threefold with very ample anticanonical bundle. For that reason they hold more generally for the primitive bundle \( L = \mathcal{O}_X(H) \) (and indeed, any ample bundle) of any Fano variety of dimension \( n \), index \( n - 2 \) and Picard number 1 for which \( L \) is very ample. In such a situation \(-K_X = (n - 2)H\).

The basic observation is this one:

**Observation:** Let \( X \) be a Fano \( n \)-fold of index \( (n - 2) \) and \( H \) a primitive, very ample line bundle on \( X \). Then a smooth member in the linear system of \(| H |\) is a Fano \((n - 1)\) fold of index \((n - 3)\); let us call the smooth member also \( H \), then by adjunction

\[
K_H = (H - (n - 2)H) \otimes \mathcal{O}_H = -(n - 3)H \otimes \mathcal{O}_H.
\]

Therefore all the vanishing theorems and syzygy results which follow will be obtained by induction on the dimension of the Fano variety. We will just sketch the proofs and leave the details to the reader.

**Lemma 6.9.** Let \( X \) be a Fano \( n \)-fold of index \( (n - 2) \) and \( H \) a primitive, very ample line bundle on \( X \). Assume that the general \( n - 1 \) section \( H^{(n-1)} \) is a curve of genus \( \geq 3 \). Then,

\[
H^1(M_{pH} \otimes rH) = 0 \quad \text{for all } r, p \geq 1
\]

**Proof:** Note first that on a Fano \( n \)-fold, for any smooth irreducible divisor \( H' \subset X \), we have \( H^1(\mathcal{O}_X(H')) = 0 \). Then since \( L \) is very ample, \( H^{(n-1)} \) is a non-hyperelliptic curve, for the general member of the linear system corresponding to \(| H^{(n-1)} |\) on the surface obtained by intersecting \( H \) \((n - 2)\) times is a K3 surface. This allows us to use the results proved for K3 surfaces as first step of the induction on the dimension of \( X \).

Tensoring the sequence

\[
0 = M_{rH} \rightarrow H^0(pH) \otimes \mathcal{O}_X(-rH) \rightarrow H_{rH} \rightarrow 0
\]
by $rH$ yields:

$$H^0(pH) \otimes H^0(rH) \xrightarrow{\varphi} H^0((p + r)H) \longrightarrow H^1(M_{pH} \otimes rH) \longrightarrow 0$$

so it is enough to prove that $\varphi$ is surjective. In view of Remark 2.1, it is enough to prove that the map

$$H^0(pH) \otimes H^0(H) \otimes H^0((p + r)H)$$

is surjective and by the same remark it is enough to show that

$$H^0(pH) \otimes H^0((p + 1)H)$$

is surjective for any $p \geq 1$. By Lemma 2.1 it is enough to show that the map

$$H^0(pH \otimes O_H) \otimes H^0(H \otimes O_H) \longrightarrow H^0((p + 1)H \otimes O_H)$$

surjects, since being $X$ a Fano $n-$fold, $H^1(nH) = 0$ for all $n \geq 0$. The above surjection follows from the induction assumption on the dimension of the Fano variety. □

**Lemma 6.10.** Let $X$ be a Fano $n-$fold of index $(n - 2)$ and Picard number 1 having a primitive and very ample bundle $L = O_X(H)$. Assume that the general $n - 1$ section $H^{(n-1)}$ is a curve of genus $g > 3$. Then the cohomology group

$$H^1(X, M_H^{\otimes p} \otimes (p + k)H) = 0 \text{ for all } k, p \geq 0.$$  

**Proof:** Mimic the proof of Lemma 6.2 word for word by replacing $S$ with $H$. As in the last paragraph of Lemma 6.2 we need the vanishing of

$$H^1(S, M_S^{\otimes p-1} \otimes M_{S\otimes O_S} \otimes (p + k)S \otimes O_S),$$

here by Lemma 3.1 we need only to show that

$$H^1(H, M_H^{\otimes p-1} \otimes M_{H\otimes O_H} \otimes (p + k)H \otimes O_H) = 0.$$  

The needed vanishing follows from the induction assumption on the dimension. □

**Lemma 6.11.** Let $X$ and $L$ be as above with the additional assumption that $H^{(n-1)}$ is a non-trigonal, non-plane quintic curve of genus $g \geq 5$. Then

$$H^1(X, M_H^{\otimes 2} \otimes H) = 0.$$  

**Proof:** Mimic the proof of Lemma 6.3 and use the induction assumption on the dimension. Note that if one member in the linear system of $| H^{(n-1)} |$ is non-trigonal, the general member is also non-trigonal. □
Lemma 6.12. Let $X$ and $L$ be as in the above lemma. Then for any integer $p \geq 2$, 
\[ H^1(M_H^{\otimes i} \otimes (i-1+k)H) = 0 \text{ for all } i = 2, \ldots, p \text{ and } k \geq 0. \]

Proof: We may assume by induction the vanishing for Fano varieties of dimension $(n-1)$ with index $(n-3)$. Mimic the proof of Lemma 6.4. All the needed vanishings for Fano varieties of one dimension less follows from induction assumption. □

Note that by mimicking the arguments in Lemma 4.4 (as was done in Lemma 6.5) we obtain,
\[ H^1(X, M_H^{\otimes i} \otimes M_rH^{\otimes j} \otimes (p+k)H) = 0 \text{ for all } i + j = p, \text{ for all } k \geq 0, \text{ and } r \geq 1 \]
\[ (\star) \]

Then by applying Lemma 3.3 we obtain,
\[ H^1(X, M_H^{\otimes i} \otimes M_rH^{\otimes j} \otimes (p+k)H) = 0 \text{ for all } i + j \leq p+1, \text{ for all } k \geq 0, \text{ and } r \geq 1 \]
\[ (\bullet) \]

Theorem 6.13. Let $X$ be a Fano $n-$fold of index $(n-2)$ and $L = \mathcal{O}_X(H)$ be the primitive bundle. Assume that $L$ is very ample and $H^{(n-1)}$ is a smooth non-hyperelliptic curve of genus $g > 3$. Then $L^{\otimes(p+1)}$ satisfies $N_p$.

Proof: By [GL1] it is enough to prove $H^1(\bigwedge^i M_L' \otimes (L')^{\otimes s}) = 0$ for all $1 \leq i \leq p+1$ and $s \geq 1$. Since we are working over the complex numbers, it is enough to prove $H^1(M_L^{\otimes i} \otimes (L')^{\otimes s}) = 0$ for all $1 \leq i \leq p+1$, where $L' = L^{\otimes(p+1)}$. The theorem follows from Lemma 6.10. □

Theorem 6.14. Let $X$ be a Fano $n-$fold of index $(n-2)$ and $L = \mathcal{O}_X(H)$ be the primitive bundle. Assume further that $H^{(n-1)}$ is a non-trigonal, non-plane quintic curve of genus $g \geq 5$. Then $L^{\otimes p}$ satisfies $N_p$.

Proof: We have proved the vanishing $H^1(X, M_H^{\otimes i} \otimes M_rH^{\otimes j} \otimes (p+k)H) = 0$ for all $i + j \leq p+1$, for all $k \geq 0$ and $r \geq 1$. By letting $i = 0$, and $r = p$, we have
\[ H^1(X, M_H^{\otimes i} \otimes (p+k)H) = 0 \text{ for all } 1 \leq j \leq p+1. \]
So by [GL1] $L^{\otimes p}$ satisfies $N_p$. □

References

[C] S.D. Cutkosky, On Fano 3-folds, Manuscripta math. 64, 189-204 (1989)
[EL] L. Ein & R. Lazarsfeld, Koszul cohomology and Syzygies of Projective varieties, Inv Math 111 (1993), no1, 51-67.
[GP1] F. Gallego & B.P. Purnaprajna, Normal presentation on elliptic ruled surfaces, (to appear in J of Algebra).
[GP2] Higher Surveys of elliptic ruled surfaces, (to appear in J of Algebra).
[GP3] ____, *Syzygies of Surfaces and Calabi-Yau 3-folds*, Preprint.

[GL1] M. Green & R. Lazarsfeld, *Some results on the syzygies of finite sets and algebraic curves*, Compositio Math. 67 (1989) 301-314.

[GL2] ____, *A simple proof of Petri’s theorem on canonical curves*, Geometry Today (129-142), 1985, Birkhauser, Ed: Arbarello et al

[I] V.A. Iskovskih, *Fano 3-folds 1*, Izv.Akad.Nauk SSSR, Vol 11 (1977) no.3.

[M] A. Mayer, *Families of K3 surfaces*, Nagoya Mathematical Journal (1972)

[Mi] Y. Miyaoka, *The chern class and Kodaira dimension of a minimal variety*, Alg Geo-Sendai 1985, Adv. Studies in Pure Math., Vol 10, 449-476.

[PR] K. Paranjape & S. Ramanan, *On the canonical ring of a curve*, Alg. Geo. and Commutative Algebra in honor of Nagata, Vol 2, 503-516.

[Pa-Pu] G. Pareschi & B. Purnaprajna, *Canonical ring of a curve is Koszul: A Simple proof*, (to appear in Illinois J. of Math.)

[StD] B. Saint-Donat, *Projective models of K3 surfaces*, Amer. J. of Math. 96, (1974) 602-639.

F.J. Gallego: Dpto. de Álgebra, Facultad de Matemáticas, Universidad Complutense de Madrid, 28040 Madrid, Spain

E-mail address: gallego@sunali.mat.ucm.es

B.P. Purnaprajna: Dept. of Mathematics, Brandeis University, Waltham MA 02254-9110, USA

E-mail address: purna@max.math.brandeis.edu