Linearization of monomial ideals

Milo Orlich*

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Abstract

We introduce a construction, called linearization, that associates to any monomial ideal $I \subset \mathbb{K}[x_1, \ldots, x_n]$ an ideal $\text{Lin}(I)$ in a larger polynomial ring. The main feature of this construction is that the new ideal $\text{Lin}(I)$ has linear quotients. In particular, since $\text{Lin}(I)$ is generated in a single degree, it follows that $\text{Lin}(I)$ has a linear resolution. We investigate some properties of this construction, such as its interplay with classical operations on ideals, its Betti numbers, functoriality and combinatorial interpretations. We moreover introduce an auxiliary construction, called equification, that associates to any monomial ideal $J$ an ideal $J^\text{eq}$, generated in a single degree, in a polynomial ring with one more variable. We study some of the homological and combinatorial properties of the equification, which can be seen as a monomial analogue of the well-known homogenization construction.

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*Department of Mathematics and Systems Analysis, Aalto University, Espoo, Finland.
Email: milo.orlich@aalto.fi
Among all ideals, those with a linear resolution are somehow “simpler” and have a vast literature. There are already ways of constructing ideals (or more generally modules) with linear resolutions (see [12]). The goal of this paper is to introduce and study a new construction, called linearization, which converts an arbitrary monomial ideal $I$ in a monomial ideal $\text{Lin}(I)$ with a linear resolution. In fact, $\text{Lin}(I)$ has an even stronger property: it has linear quotients.

We start this introduction with an overview of monomial ideals and their resolutions. We then move on to outline the content of this paper.

Monomial ideals and their free resolutions

Monomial ideals in a polynomial ring $S = K[x_1, \ldots, x_n]$, that is, ideals that are generated by monomials, are a core object of study in commutative algebra. Because of their highly combinatorial structure, they are easier to understand than arbitrary ideals. For a general reference about monomial ideals, see the monograph [18].

One can reduce the study of general ideals to that of monomial ideals through the methods of Gröbner bases, see for instance Chapter 2 of [18] or Chapter 15 of [11]: one associates to an arbitrary ideal $I$ its initial ideal, which is monomial and hence easier to deal with.

A free resolution over $S$ of an ideal $I \subset S$ consists of a sequence $F_* = (F_i)_{i \in \mathbb{N}}$ of free $S$-modules and homomorphisms $d_i : F_i \to F_{i-1}$ such that there exists a surjective homomorphism $\varepsilon : F_0 \to I$ and such that $F_* \xrightarrow{\varepsilon} I \to 0$ is an exact complex. If one assumes the resolution to be minimal and graded (so that the free modules have to be shifted accordingly), one may write

$$F_i = \bigoplus_{j \in \mathbb{Z}} S(-j)^{\beta_{ij}}$$

and the natural numbers $\beta_{ij}$, called the graded Betti numbers of $I$, are uniquely determined. Kaplansky posed the problem of studying systematically the resolutions of monomial ideals in the 1960’s. A great deal of research has been...
done ever since, starting with Taylor’s PhD thesis [28], where she defined a canonical construction that works for every monomial ideal but gives a highly non-minimal resolution in general. Despite the apparently easy structure of monomial ideals, their minimal resolutions have been quite elusive for more than half a century. Over the decades, many constructions have been introduced, that either provide minimal resolutions only for certain classes of monomial ideals, or complexes that work in great generality but are not always minimal resolutions, or even resolutions. Some of these constructions have combinatorial or topological flavours that provide fruitful interpretations of what happens on the algebraic side. In particular, Hochster, Stanley and Reisner (see [26], [25] and Chapter 5 of [6]) introduced what became the Stanley–Reisner machinery, where one associates squarefree monomial ideals to simplicial complexes, understanding free resolutions in terms of simplicial homology. This beautiful bridge between commutative algebra and combinatorial topology is an instance of what has been the general trend until now: quoting Peeva’s words in [24], “introduce new ideas and constructions which either have strong applications or/and are beautiful”. Very recently, in 2019, Eagon, Miller and Ordog described a canonical minimal resolution for every monomial ideal, in [9].

**Linearization of monomial ideals**

We recall that a monomial ideal $I$ has a $d$-linear resolution if $\beta_{i,j}(I) = 0$ for $j \neq i + d$. In particular, the only $j$ for which we can have $\beta_{0,j}(I) \neq 0$ is $d$, meaning that all the minimal generators of $I$ have degree $d$. And for what concerns the higher homological positions, having a linear resolution means that when we fix bases and write the maps in the resolutions as matrices, the nonzero entries of those matrices are linear forms.

An invariant of any ideal (or module, in general), is its (Castelnuovo–Mumford) regularity, which measures how complicated the resolution is. For an ideal with all generators of degree $d$, the regularity is equal to $d$, the smallest possible, if and only if the resolution is $d$-linear. This explains why these two concepts appear together in the literature.

The literature about linear resolutions is huge. Among the many papers concerning linear resolutions we mention first of all the work [12] of Eisenbud and Goto. Another classical work is [27] by Steurich. More recent directions of research involve families of ideals such that every product of its elements has a linear resolution, for which we refer for instance to [5]. We refer in general to [24] and [18] for more information, and we provide additional references in the main body of the paper.

**Definition.** Let $I \subset \mathbb{K}[x_1, \ldots, x_n]$ be a monomial ideal with minimal set of monomial generators $G(I) = \{f_1, \ldots, f_m\}$, such that all the $f_j$’s have the same degree $d$. For all $i \in \{1, \ldots, n\}$, denote by $M_i$ the largest exponent with which $x_i$ occurs in $G(I)$. The **linearization of $I$**, inside the polynomial ring $R := \mathbb{K}[x_1, \ldots, x_n, y_1, \ldots, y_m]$, is the ideal

$$\text{Lin}(I) := (x_1^{a_1} \cdots x_n^{a_n} \mid a_1 + \cdots + a_n = d \text{ and } a_i \leq M_i \text{ for all } i) + (f_j y_j / x_k \mid x_k \text{ divides } f_j, \ k = 1, \ldots, n, \ j = 1, \ldots, m).$$
We call the first summand complete part of \( \text{Lin}(I) \) and the second summand last part of \( \text{Lin}(I) \).

Observe that the ideal \( \text{Lin}(I) \) is generated in the same degree \( d \) as the original ideal \( I \). One of the main properties of \( \text{Lin}(I) \), which is Corollary 3.8 in the main text, and the reason for the name “linearization”, is the following:

**Theorem.** The ideal \( \text{Lin}(I) \) has a linear resolution over \( R \).

This is actually implied by the stronger property of having “linear quotients”, which we now recall. Given an ideal \( I \subset S := \mathbb{K}[x_1, \ldots, x_n] \) and a polynomial \( g \in S \), the colon ideal (or quotient ideal) of \( I \) with respect to \( g \) is

\[ I : g = \{ f \in S \mid fg \in I \}. \]

An ideal \( I \subset S \) is said to have linear quotients if there exists a system of generators \( g_1, \ldots, g_m \) of \( I \) such that each colon ideal \( (g_1, \ldots, g_{k-1}) : g_k \) is generated by linear forms, for any \( k \in \{2, \ldots, m\} \). For the combinatorial meaning of linear quotients, see the next section in this introduction. The actual main result of the paper, which is Theorem 3.7 in the body of the paper and which implies the fact that \( \text{Lin}(I) \) has a linear resolution, is the following:

**Main Theorem.** List the generators of the complete part of \( \text{Lin}(I) \) in decreasing lexicographic order. Assume that \( f_1, \ldots, f_m \) are in decreasing lexicographic order. List the generators of the last part \( f_j x_k y_j \) first for increasing \( j \), and then for increasing \( k \). The ideals \( \text{Lin}(I) \) has linear quotients with respect to the given ordering of the generators.

We now continue by summarizing the paper. In Section 2 we start by giving the necessary algebraic background. We prove a result about linear quotients, namely Proposition 2.14, which says the following:

**Proposition.** Let \( I = (f_1, \ldots, f_s) \subset \mathbb{K}[x_1, \ldots, x_n] \) be a monomial ideal with linear quotients with respect to \( f_1, \ldots, f_s \). Fix \( v = (v_1, \ldots, v_n) \in \mathbb{N}^n \) and denote \( I_{\leq v} \) the ideal generated by all the generators \( f_j = x_1^{a_1} \cdots x_n^{a_n} \) of \( I \) such that \( a_i \leq v_i \) for all \( i \). Denote these generators as \( f_{b_1}, \ldots, f_{b_t} \), with \( b_1 < \cdots < b_t \). Then \( I_{\leq v} \) has linear quotients with respect to \( f_{b_1}, \ldots, f_{b_t} \).

In Section 3 we define the linearization for ideals generated in a single degree (i.e., equigenerated) and investigate some of its properties. We also consider a slightly different construction, the *-linearization \( \text{Lin}^*(I) \). For some of the results, it’s easier or more meaningful to consider \( \text{Lin}^*(I) \). We give a “more exact” functorial definition of linearization in Section 3.1.1. We prove the following two results, respectively Theorem 3.15 and Theorem 3.21:

**Theorem.** For a monomial ideal \( I \subset S = \mathbb{K}[x_1, \ldots, x_n] \) generated in degree \( d \), in the following cases \( \text{Lin}(I) \) is polymatroidal:

(a) \( d = 1 \), that is, \( I \) is generated by variables;
(b) \( d \) is arbitrary and \( I \) is principal.

In all other cases \( \text{Lin}(I) \) is not polymatroidal.
Theorem. The radical ideal of $\text{Lin}^*(I)$ has linear quotients, and hence linear resolution.

In Section 4 we focus on the case where $I$ is squarefree, which happens if and only if $\text{Lin}(I)$ is squarefree. This section is done mostly with $\text{Lin}^*(I)$ instead of $\text{Lin}(I)$ not to make the notation too heavy. The difference between the two, in the squarefree case, is anyway not meaningful. We compute the Betti numbers of $\text{Lin}^*(I)$, which depend only on how the monomials of degree $d - 1$ divide the minimal generators of $I$. In “simplicial terminology”, these monomials are called codimension-1 facets of the simplicial complex whose facet ideal is $I$. Since we have applications to hypergraphs in mind, we call them $(d - 1)$-edges instead. In Corollary 4.14 we get the Betti numbers of $\text{Lin}^*(I)$ as follows.

Let us call a $j$-cluster a set of cardinality $j$ consisting of generators of $I$ that are divided by a same monomial of degree $d - 1$. For each $j$, denote by $C_j$ the number of maximal $j$-clusters, that is, $j$-clusters that are not part of a $(j + 1)$-cluster. A closed formula for the Betti numbers is then

$$
\beta_i(\text{Lin}^*(I)) = \binom{i + d - 1}{d - 1} \binom{n}{i + d} + \binom{n - d + 1}{i} \left(m - \sum_{j \geq 2} (j - 1)C_j\right)
$$

$$
+ \sum_{j \geq 2} C_j \sum_{k=2}^j \binom{n - d + k}{i},
$$

where $m$ is the number of generators of $I$ and $d$ is their degree. In particular, if we denote $N := \max\{j \mid C_j \neq 0\}$, then we have

$$
\text{projdim}_R(\text{Lin}^*(I)) = n - d + N.
$$

In Section 4.2 we give an alternative, more conceptual proof that $\text{Lin}^*(I)$ has a linear resolution and that its Betti numbers only depend on how the monomials of degree $d - 1$ divide the generators of $I$. This proof was taught to me by Gunnar Fløystad. We conclude in Section 4.3 with an interpretation of the squarefree linearization by means of hypergraphs, related to the work of Hà and Van Tuyl in [15].

In Section 5 we introduce an auxiliary construction in order to generalize the linearization to arbitrary monomial ideals:

Definition. Let $I$ be an arbitrary monomial ideal in $\mathbb{K}[x_1, \ldots, x_n]$, with minimal system of monomial generators $G(I) = \{f_1, \ldots, f_m\}$. Denote $d_j := \deg(f_j)$ for all $j$ and $d := \max\{d_j \mid j = 1, \ldots, m\}$. We define the equification of $I$ as

$$
I^{eq} := (f_1 z^{d-d_1}, f_2 z^{d-d_2}, \ldots, f_m z^{d-d_m})
$$

in the polynomial ring $\mathbb{K}[x_1, \ldots, x_n, z]$ with one extra variable $z$. We moreover define the linearization of $I$ as

$$
\text{Lin}(I^{eq}),
$$

where $\text{Lin}$ is the linearization defined earlier for ideals generated in a single degree.
The equification construction also seems interesting in its own right, and in Section 5.1 we investigate some of its basic properties. For instance, in Proposition 5.14 we show that for the total Betti numbers of $I$ and $I^{\text{eq}}$ we have the inequalities

$$\beta_i(I) \leq \beta_i(I^{\text{eq}}) \quad \text{for all } i > 0$$

and $\beta_0(I) = \beta_0(I^{\text{eq}})$, where the resolutions are taken over the respective polynomial rings. Lastly, in Section 5.1.2 we compare $I$ and $I^{\text{eq}}$ by observing that the lcm-lattice of $I$ can be embedded in that of $I^{\text{eq}}$.

In Section 6 we discuss some open questions and possible future developments. In particular, it might be possible to generalize in some meaningful way the constructions of linearization and equification to arbitrary (at least homogeneous) ideals.

Rees algebras. We conclude this part of the introduction by drawing the reader’s attention to a well-known construction, the Rees algebra of $I$. Although it will never be used in the paper, the reason for mentioning this is that in the definition of linearization the ring is enlarged by introducing new variables, and this might seem a bit artificial on one hand, or very similar to what one does when defining Rees algebras on the other hand.

Let $I = (f_1, \ldots, f_m) \subset S := \mathbb{K}[x_1, \ldots, x_n]$ be a homogeneous ideal, where $f_1, \ldots, f_m$ are a minimal system of homogeneous generators. The Rees algebra of $I$ is the image of the $S$-algebra homomorphism

$$S[y_1, \ldots, y_m] \longrightarrow S[t]$$

$$y_i \mapsto f_i t,$$

namely the subalgebra $S[f_1 t, \ldots, f_m t] \subset S[t]$. The similarity with the linearization consists in the introduction of new variables $y_j$, as many as generators of the ideal. The similarity seems to end here: in particular, Lin($I$) has its fundamental property of of having linear resolution. However, it would be interesting to investigate in the future if there is an analogous theory of deformations for Lin($I$) as there is for the Rees algebra. See Section 6.5 of [11] for more about this.

Motivation and combinatorial interpretations

The contents of this section, in particular about Booth–Lueker graphs, are not necessary for the rest of the paper. The purpose of this section is just to provide motivation for the linearization construction, and possible combinatorial applications.

In 1975 Booth and Lueker introduced a construction, in their paper [4], that takes an arbitrary finite simple graph $G$ and returns a graph with more structure, which is denoted as $BL(G)$ in [13] and defined as follows.

**Definition 1.1.** Let $G$ be a finite simple graph on the set of vertices $\{x_1, \ldots, x_n\}$. Let $e_1, \ldots, e_m$ be the edges in $G$. We define the Booth–Lueker graph of $G$, denoted $BL(G)$, on the set of vertices $\{x_1, \ldots, x_n\} \cup \{y_1, \ldots, y_m\}$, as follows: for all $i$ and $j$, $BL(G)$ has the edge $x_i x_j$, and for each edge $e_i = x_{i_1} x_{i_2}$ of $G$, $BL(G)$ has the edges $x_{i_1} y_i$ and $x_{i_2} y_i$. 

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There is a bijection between squarefree ideals generated in degree 2 and finite simple graphs: to any graph $G$, one associates the edge ideal $I_G$, which is generated by the monomials $x_i x_j$ for all the edges $ij$ in the graph $G$. Fröberg gave the following very famous characterization:

**Theorem 1.2** (Fröberg, [14]). The edge ideal $I_G$ has 2-linear resolution if and only if the complement of $G$ is chordal (i.e., every cycle of length at least four is cut by a chord).

One can see that the complement of $BL(G)$ is chordal, and therefore the edge ideal $I_{BL(G)}$ has a linear resolution. This and related matters are addressed in [13]. After that paper was made available, it was remarked that the Booth–Lueker construction could be interpreted as a map that associates to any squarefree monomial ideal generated in degree 2 a monomial ideal, also squarefree and generated in degree 2, with linear resolution. The problem of generalizing this kind of construction to any monomial ideal was raised then by Aldo Conca, and one can see that the linearization is such a generalization.

We conclude by remarking that, just like for graphs, one can define bijections between squarefree monomial ideals of arbitrary degree and combinatorial objects (simplicial complexes or hypergraphs), and the linearity of the resolution of the ideal has a combinatorial meaning.

**Simplicial complexes.** A fundamental bridge between commutative algebra and combinatorial topology is provided by the Stanley–Reisner correspondence, a bijection between squarefree monomial ideals in $\mathbb{K}[x_1, \ldots, x_n]$ and simplicial complexes on the set $[n] = \{1, \ldots, n\}$. For the details, see for instance Chapter 1 of [22], Chapter 5 of [9], or Chapter 8 of [18]. In short, to any simplicial complex $\Delta$ one associates the Stanley–Reisner ideal $I_\Delta$, and combinatorial properties of $\Delta$ correspond to algebraic properties of $I_\Delta$. In particular, recall that Lin$(I)$ has a stronger property than linear resolution: it has linear quotients. The most relevant equivalence, from the point of view of this paper, is the following well-known result:

**Proposition 1.3** (part of Proposition 8.2.7 of [18]). The ideal $I_\Delta$ has linear quotients if and only if the Alexander dual $\Delta^\vee$ of $\Delta$ is shellable.

Hence, in the squarefree case, Lin can be interpreted as a map taking a pure simplicial complex and returning one, with more vertices, that has a shellable dual complex. Again, for the details we refer to Section 8.2 of [18].
Hypergraphs. Another bijection, which is less topological and more combinatorial in flavour, is between squarefree monomial ideals and hypergraphs. This is the topic of Section 4.3.2. To each hypergraph one can associate a monomial ideal, and in [15] the authors gave a partial characterization of the hypergraphs that have an associated ideal with linear resolution. So the Lin map can also be seen as taking an arbitrary hypergraph and returning one of those. See Section 4.3.2 for the details.

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All the experiments and computations that led to finding the results in the paper were carried out with the computer algebra system CoCoA, see [1]. For a large part of the classical results and for further reading about monomial ideals and free resolutions, we mainly refer to the excellent monographs [18], by Herzog and Hibi, and [24], by Peeva.

2 Algebraic background

Everywhere in the paper $\mathbb{K}$ will be be a field and $S := \mathbb{K}[x_1, \ldots, x_n]$ will be the polynomial ring in $n$ variables over $\mathbb{K}$ equipped with the standard grading, that is, with each variable of degree 1. Unless otherwise stated, $\mathbb{K}$ will have characteristic zero. By $\mathbb{N}$ we mean the set of non-negative integers, so that in particular $0 \in \mathbb{N}$.

2.1 Free resolutions and monomial ideals

Given a graded $S$-module $M = \bigoplus_{i \in \mathbb{Z}} M_i$ and an integer $m \in \mathbb{Z}$, we denote

$$M(-m) := \bigoplus_{i \in \mathbb{Z}} M(-m)_i,$$

where $M(-m)_i := M_i - m$,

the module $M$ shifted $m$ degrees. In particular we will consider $M = S$. We recall that a free resolution over $S$ of an $S$-module $M$ consists of a sequence $(F_i)_{i \in \mathbb{N}}$ of free $S$-modules and homomorphisms $d_i : F_i \to F_{i-1}$ such that there exists a surjective homomorphism $\varepsilon : F_0 \to M$ and such that

$$\ldots \to F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{\varepsilon} M \to 0$$

is an exact complex. The modules $M$ we will consider are finitely generated and graded. In particular, the monomial ideals of $S$ are such modules. A free resolution is graded if the maps preserve the degrees of the elements, and a it is minimal if $d_i(F_i) \subseteq mF_{i-1}$ for each $i > 0$, where $m = (x_1, \ldots, x_n)$ is the irrelevant maximal ideal of $S$. A minimal graded free resolution is unique.
up to isomorphism, and sometimes we call it “the” minimal resolution. If we
write each module of the minimal resolution as
\[ F_i = \bigoplus_{j \in \mathbb{Z}} S(-j)^{\beta_{ij}}, \]
then the natural numbers \( \beta_{ij} \), also denoted by \( \beta_{ij}(M) \), are invariants of \( M \) called the **graded Betti numbers** of \( M \). We arrange the graded Betti numbers in the so-called **Betti table** of \( M \), so that the entry in the \( i \)-th column and the \( j \)-th row is \( \beta_{i,i+j}(M) \). The **projective dimension** of \( M \) is the highest value of \( i \) such that there is a nonzero \( \beta_{i,i+j}(M) \), for some \( j \). The \( i \)-th **total Betti number** of \( M \) is \( \beta_i(M) := \sum_{j \in \mathbb{Z}} \beta_{ij}(M) \).

**Definition 2.1.** A finitely generated graded \( S \)-module \( M \) is said to have a \( d \)-linear resolution if \( \beta_{i,j}(M) = 0 \) for \( j \neq i + d \). That is, if all the nonzero entries of the Betti table of \( M \) are in the \( d \)-th row.

With few exceptions, the \( S \)-modules that we will consider in the paper will always be ideals of \( S \), and in particular monomial ideals, that is, ideals generated by monomials.

**Notation 2.2.** Recall that for a monomial ideal \( I \) there is a unique minimal system of monomial generators of \( I \), consisting of the monomials in \( I \) which are minimal with respect to the divisibility relation. We denote the minimal set of monomial generators by \( G(I) \).

Recall that a monomial \( u \) belongs to a monomial ideal \( I \) if and only if \( u \) is divided by some monomial in \( G(I) \).

**Definition 2.3.** A monomial ideal \( I \) is **equigenerated** if all the elements in \( G(I) \) have the same degree.

**Definition 2.4.** The **support** of a monomial \( u \) is the set of variables that divide \( u \). It will be denoted by \( \text{Supp}(u) \).

### 2.2 Ideals with linear quotients

**Definition 2.5.** Given two ideals \( I \) and \( J \) in any ring \( S \), we call
\[
I : J := \{ f \in S \mid fg \in I \text{ for all } g \in J \}
\]
the **colon** (or **quotient** ideal of \( I \) with respect to \( J \)). In case \( J = (g) \) is a principal ideal, we denote \( I : (g) = I : g \).

**Remark 2.6.** Given a monomial ideal \( I \subseteq S = \mathbb{K}[x_1, \ldots, x_n] \) and a monomial \( u \in S \), it is a well-known fact that
\[
I : u = \left( \frac{f}{\gcd(f,u)} \mid f \in G(I) \right),
\]
where the generators on the right-hand side might be not minimal. For a proof, see for instance Proposition 1.2.2 of [IS].
Definition 2.7. Let $I \subseteq S = \mathbb{K}[x_1, \ldots, x_n]$ be a homogeneous ideal. We say that $I$ has \textbf{linear quotients} if there exists a system of homogeneous generators $f_1, f_2, \ldots, f_m$ of $I$ such that the colon ideal $(f_1, \ldots, f_{k-1}) : f_k$ is generated by linear forms for all $k \in \{2, \ldots, m\}$.

Example 2.8. The order in which we take the generators matters. Consider this small toy example: in $\mathbb{K}[x_1, \ldots, x_5]$, take $I = (x_1x_2x_3, x_3x_4x_5, x_2x_3x_4)$. If we took the colon ideals with the generators listed in this way, then we would get in particular $(x_1x_2x_3) : x_3x_4x_5 = (x_1x_2)$, where the only generator is quadratic. On the other hand, if we order the generators as $f_1 = x_1x_2x_3$, $f_2 = x_2x_3x_4$, $f_3 = x_3x_4x_5$, then we get $(f_1) : f_2 = (x_1)$, $(f_1, f_2) : f_3 = (x_1)$. So indeed $I$ has linear quotients.

Proposition 2.9 (Proposition 8.2.1 of [18]). Let $I \subseteq S = \mathbb{K}[x_1, \ldots, x_n]$ be a homogeneous ideal equigenerated in degree $d$ and with linear quotients. Then $I$ has a $d$-linear resolution.

Corollary 2.10 (Corollary 8.2.2 of [18]). Let $I \subseteq S$ be an equigenerated homogeneous ideal with linear quotients. For each $k \in \{1, \ldots, m\}$, let $r_k$ be the number of minimal generators of $(f_1, \ldots, f_{k-1}) : f_k$. Then

$$\beta_i(I) = \sum_{k=1}^{m} \binom{r_k}{i}.$$ 

In particular, $\text{projdim}(I) = \max\{r_1, r_2, \ldots, r_m\}$.

Example 2.11. Continuing Example 2.8 with the notation of Corollary 2.10 we get $r_1 = 0$, $r_2 = 1$ and $r_3 = 1$. So the projective dimension of $I$ is 1 and we get

$$\beta_0(I) = \sum_{k=1}^{3} \binom{r_k}{0} = 1 + 1 + 1 = 3, \quad \beta_1(I) = \sum_{k=1}^{3} \binom{r_k}{1} = 0 + 1 + 1 = 2.$$ 

Next we recall a criterion that will constitute the main tool for the proof of Theorem 3.7, the main result of the paper.

Lemma 2.12 (Lemma 8.2.3 of [18]). A monomial ideal $J \subseteq \mathbb{K}[x_1, \ldots, x_n]$ has linear quotients with respect to the monomial generators $u_1, u_2, \ldots, u_t$ of $J$ if and only if for all $j$ and $i$ with $1 \leq j < i \leq t$ there exist an integer $k < i$ and an integer $\ell$ such that

$$\frac{u_k}{\gcd(u_k, u_i)} = x_\ell \quad \text{and} \quad x_\ell \text{ divides } \frac{u_j}{\gcd(u_j, u_i)}.$$ 

Proof. This follows directly from Remark 2.6. \qed
2.3 Cropping the exponents from above

Notation 2.13. Let $I \subseteq S = \mathbb{K}[x_1, \ldots, x_n]$ be a monomial ideal with minimal system of generators $G(I) = \{f_1, \ldots, f_t\}$. Write $f_i = x_1^{a_{1i}} \cdots x_n^{a_{ni}}$ for all $i$. Fix a vector of non-negative integers $v = (v_1, \ldots, v_n) \in \mathbb{N}^n$. We use $v$ to “crop” the ideal $I$ by keeping only the generators of $I$ whose vector of exponents is componentwise at most as large as the vector $v$: we denote

$$I_{\leq v} := (f_p \mid a_{pi} \leq v_i \text{ for all } i = 1, \ldots, n)$$

and we say that $I_{\leq v}$ is obtained by cropping $I$ by $v$.

This subsection is motivated by the following well-known result: if $I \subseteq \mathbb{K}[x_1, \ldots, x_n]$ is a monomial ideal which has linear resolution, then $I_{\leq v}$ still has linear resolution for any $v \in \mathbb{N}^n$ (see Section 56 of [24] for a general treatment). In what follows we prove an analogous result for linear quotients, Proposition 2.14. Before we state it, we observe that with the notation above, by Remark 2.6 we have

$$(f_1, \ldots, f_{i-1}) : f_i = \left( \frac{f_k}{\gcd(f_k, f_i)} \mid k \in \{1, \ldots, i-1\} \right),$$

where more explicitly we may write

$$\frac{f_k}{\gcd(f_k, f_i)} = x_1^{a_{1k} \min\{a_{1i}, a_{11}\}} \cdots x_n^{a_{nk} \min\{a_{ni}, a_{n1}\}} = x_1^{a_{1k} - \min\{a_{1i}, a_{11}\}} \cdots x_n^{a_{nk} - \min\{a_{ni}, a_{n1}\}}.$$

Proposition 2.14. Let $I = (f_1, \ldots, f_t) \subseteq \mathbb{K}[x_1, \ldots, x_n]$ be a monomial ideal with linear quotients with respect to the given ordering of the generators. Fix $v \in \mathbb{N}^n$ and denote $I_{\leq v} = (f_{b_1}, \ldots, f_{b_t})$, where $b_1 < \cdots < b_t$ are the indexes of the generators that survived the cropping. Then $I_{\leq v}$ has linear quotients with respect to $f_{b_1}, \ldots, f_{b_t}$.

Proof. By Lemma 2.12, $I_{\leq v}$ has linear quotients with respect to $f_{b_1}, \ldots, f_{b_t}$ if and only if, for all $1 \leq j < i \leq t$, there exist an integer $k < i$ and an integer $\ell$ such that

$$\frac{f_{b_\ell}}{\gcd(f_{b_\ell}, f_{b_i})} = x_\ell \quad \text{and} \quad x_\ell \text{ divides } \frac{f_{b_j}}{\gcd(f_{b_j}, f_{b_i})}.$$  \hspace{1cm} (1)

We will prove (1) starting from the analogous condition for $I$. So, let’s start with fixing $j < i$, namely with $b_j < b_i$. Then, since $I$ has linear quotients with respect to the given order for the generators, there exist $p < b_i$ and $\ell$ such that

$$\frac{f_p}{\gcd(f_p, f_{b_i})} = x_\ell \quad \text{and} \quad x_\ell \text{ divides } \frac{f_{b_j}}{\gcd(f_{b_j}, f_{b_i})}.$$  \hspace{1cm} (2)

This would prove (1) if we could show that $p = b_k$ for some $k$, or in other words we want to show that the vector of exponents of $f_p = x_1^{a_{1p}} \cdots x_n^{a_{np}}$ is below the cropping vector $v = (v_1, \ldots, v_n)$. The first of the two things in (2) more explicitly means that

$$\begin{cases} a_{jp} = \min\{a_{jp}, a_{jb_j}\} & \text{for } j \in [n] \setminus \{\ell\}, \\ a_{\ell p} = \min\{a_{\ell p}, a_{b_j}\} + 1, \end{cases}$$

...
or equivalently
\[
\begin{cases}
  a_{jp} \leq a_{jb_i} \leq v_j & \text{for } j \in [n] \setminus \{\ell\}, \\
  a_{\ell p} = a_{\ell b_i} + 1.
\end{cases}
\]

The second thing in (2) is more explicitly \( a_{\ell b_j} - \min\{a_{\ell b_j}, a_{\ell b_i}\} \geq 1 \), or equivalently \( a_{\ell b_i} \geq a_{\ell b_j} + 1 \). Putting these things together with \( a_{\ell p} \leq v_\ell \), which we have by construction, we get \( a_{\ell p} \leq v_\ell \), so that \( f_\ell \) actually is among the generators of \( I_{\leq v} \).

The assumption about the vector is important. One cannot simply kill some random generators. Consider for instance the following example: the ideal \((xy^2, xyz, xz^2)\) has linear quotients, but the ideal \((xy^2, xz^2)\) doesn’t. And indeed we are not cropping by any vector to get the latter ideal.

### 3 Linearization of equigenerated monomial ideals

Let \( \mathbb{K} \) be a field of characteristic zero and let \( S := \mathbb{K}[x_1, \ldots, x_n] \) be the polynomial ring in \( n \) variables over \( \mathbb{K} \).

**Notation 3.1.** Let \( I \subseteq S \) be a monomial ideal with minimal system of monomial generators \( G(I) = \{f_1, \ldots, f_m\} \) and let \( I \) be **equigenerated**, so that all the generators have the same degree \( d = \deg(f_j) \) for all \( j \). For each \( i \in \{1, \ldots, n\} \), let \( M_i \) be the highest exponent with which \( x_i \) occurs in \( G(I) \).

**Definition 3.2.** The **linearization of \( I \)**, inside the polynomial ring \( R := \mathbb{K}[x_1, \ldots, x_n, y_1, \ldots, y_m] \), is the ideal

\[
\text{Lin}(I) := \left( x_1^{a_1} \cdots x_n^{a_n} \mid a_1 + \cdots + a_n = d \text{ and } a_i \leq M_i \text{ for all } i \right) + \left( f_j y_j / x_k \mid x_k \text{ divides } f_j, \ k = 1, \ldots, n, \ j = 1, \ldots, m \right).
\]

That is, the linearization consists of two summands which we call respectively the *complete part* and the *last part* of \( \text{Lin}(I) \).

The name “complete part” comes from generalizing the Booth–Lueker construction, see Definition 1.1. There, the analogous set of generators corresponds to a complete graph. Notice moreover that we can also write the complete part as \((x_1, \ldots, x_n)^d_{\leq (M_1, \ldots, M_n, 0, \ldots, 0)}\), see Notation 2.13. The name “last part” is due to the lack of a better name.

**Remark 3.3.** Observe that the given generators of \( I \) are minimal and \( \text{Lin}(I) \) is equigenerated in the same degree \( d \) as \( I \).

**Remark 3.4.** There is a way to retrieve \( I \) from \( \text{Lin}(I) \). Actually, in most cases it’s enough to know the generators of the last part of \( \text{Lin}(I) \) in order to recover \( I \). First we discuss an intuitive way and then we make it precise. For all \( j \in \{1, \ldots, m\} \), let \( f_{j, 1}, f_{j, 2}, \ldots, f_{j, i_j} \) be the monomials such that

\[
f_{j, 1} y_j, f_{j, 2} y_j, \ldots, f_{j, i_j} y_j
\]
are the minimal generators of \( \text{Lin}(I) \) divisible by the variable \( y_j \). Then we can retrieve the generators of \( I \) as

\[
f_j = \text{lcm}(f_{j,1}, f_{j,2}, \ldots, f_{j,i}) \quad \text{for all } j \in \{1, \ldots, m\}.
\]

Now, a problem might occur if we have a generator which is a power of a single variable, say \( x_1 \). In that case, we have a generator of \( I \) that for instance is \( f_j = x_1^4 \) and then we get the generator \( x_1^4 y_j \) in \( \text{Lin}(I) \), which is the only one involving \( y_j \). Then, by taking the least common multiple as above, we don’t get the actual generator \( x_1^4 \), because we only get \( x_1^3 \). Another problem, even worse (but actually a special case of the problem just mentioned), can occur if we have a generator which is just a single variable, say \( f_j = x_1 \). Then we get the generator \( y_j \) with no \( x \)-variables in its support. This is why it’s important to know the complete part: in case we have degree \( d = 1 \), it’s enough to know the vector \( (M_1, \ldots, M_n) \) in order to deduce which generators (variable corresponding to \( M_i = 1 \)) we have. So now let’s assume we have arbitrary degree \( d \) and a generator which is a power of a variable, say \( f_j = x_1^d \).

If another generator uses the variable \( x_1 \), then of course it will be contained with exponent smaller than \( d \). All in all, we can run an algorithm that, for each \( j \), checks if \( i_j = 1 \) in \( \mathbb{K} \). In this case, it means that these \( f_j \) correspond to entries in the vector \( (M_1, \ldots, M_n) \) which are equal to \( d \). For the other \( j \)’s (if there are any) we can use the least common multiple formula above.

**Definition 3.5.** The \( \ast \)-linearization of \( I \), in the polynomial ring \( R := \mathbb{K}[x_1, \ldots, x_n, y_1, \ldots, y_m] \), is the ideal

\[
\text{Lin}^\ast(I) := (x_1^{a_1} \cdots x_n^{a_n} \mid a_1 + \cdots + a_n = d \text{ and } a_i \leq M \text{ for all } i) \\
+ (f_j y_j / x_k \mid x_k \text{ divides } f_j, k = 1, \ldots, n, j = 1, \ldots, m).
\]

That is, the \( \ast \)-linearization consists of two summands which we call respectively the complete part and the last part of \( \text{Lin}^\ast(I) \).

**Remark 3.6.** The last part of \( \text{Lin}(I) \) and \( \text{Lin}^\ast(I) \) are always equal, the only difference is in the complete part. Notice that, since \( M_i \leq M \) for each \( i \in \{1, \ldots, n\} \), we always have

\[
\text{Lin}(I) \subseteq \text{Lin}^\ast(I),
\]

with equality only if \( M_1 = M_2 = \cdots = M_n = M \). The reason for the introduction of \( \text{Lin}^\ast(I) \), which is “coarser” than \( \text{Lin}(I) \), is that given its symmetry it’s easier to understand than \( \text{Lin}(I) \) in the non-squarefree case. In the squarefree case, treated in Section 4 the difference between \( \text{Lin}(I) \) and \( \text{Lin}^\ast(I) \) is not so significant, see Remark 4.2. And moreover in that case \( \text{Lin}^\ast(I) \) is a more direct generalization of the Booth–Lueker construction in Definition 1.1.

### 3.1 Properties of the linearization

**Lexicographic order.** Let \( u = x_1^{a_1} \cdots x_n^{a_n} \) and \( v = x_1^{b_1} \cdots x_n^{b_n} \) be two monomials in \( \mathbb{K}[x_1, \ldots, x_n] \), with \( a = (a_1, \ldots, a_n) \) and \( b = (b_1, \ldots, b_n) \) in \( \mathbb{N}^n \) their respective vectors of exponents. We recall that \( u \) is larger than \( v \) in lexicographic order, written \( u \succ \text{Lex} v \), if the leftmost nonzero entry of \( a - b \) is positive. We equivalently write \( a \succ \text{Lex} b \) to mean the same thing.

The following is the main result of the paper.
Theorem 3.7. List the generators of the complete part of \( \text{Lin}(I) \), and respectively of \( \text{Lin}^*(I) \), in decreasing lexicographic order. Assume that \( f_1, \ldots, f_m \) are in decreasing lexicographic order. List the generators of the last part \( \frac{f_k}{x_k}y_j \) first for increasing \( j \), and then for increasing \( k \). The ideals \( \text{Lin}(I) \) and \( \text{Lin}^*(I) \) have linear quotients with respect to the given ordering of the generators.

Proof. First we prove that \((x_1, \ldots, x_n)^d + (\text{last part})\) has linear quotients, and then, in order to conclude that \( \text{Lin}(I) \) and \( \text{Lin}^*(I) \) have linear quotients, we apply Proposition 2.14 cropping from above by the vector \((M_1, \ldots, M_n, 1, \ldots, 1)\) and \((M_1, \ldots, M_1, \ldots, 1)\), respectively. To this end, denote \( P := (x_1, \ldots, x_n)^d \) and let \( L \) be the last part of \( \text{Lin}(I) \), which is the same as the last part of \( \text{Lin}^*(I) \). The proof will rely on Lemma 2.12. With the notation of Lemma 2.12 we will have three cases: (1) we pick both \( u_j \) and \( u_i \) in \( G(P) \), (2) we pick \( u_j \) in \( G(P) \) and \( u_i \) in \( G(L) \), (3) we pick both \( u_j \) and \( u_i \) in \( G(L) \). Of course in some special cases (like, we have only one generator for \( I \), or only one variable in \( S \)) we can’t. But anyway this covers all possibilities.

(1) Let \( u_j = x_1^{a_1} \cdots x_n^{a_n} \) and \( u_i = x_1^{b_1} \cdots x_n^{b_n} \) be in \( G(P) \), so that \( \sum_i a_i = \sum_i b_i = d \), and \( a > \text{Lex} \ b \). Let \( \ell \) be the index of the leftmost nonzero entry of \( a - b \), so that \( a_\ell - b_\ell > 0 \). Notice that \( \ell < n \), because otherwise we could not have \( \sum_i a_i = \sum_i b_i \). Then there is some \( q > \ell \) such that \( b_q > a_q \geq 0 \), which means that \( x_q \) divides \( u_i \). If we pick \( u_k := \frac{x_q}{x_\ell}u_i \), then we get
\[
\frac{u_k}{\gcd(u_k, u_i)} = x_\ell,
\]
and by construction of \( \ell \) also the second property in Lemma 2.12 holds, namely \( x_\ell \) divides \( u_j / \gcd(u_j, u_i) \).

(2) Let \( u_j = x_1^{a_1} \cdots x_n^{a_n} \in G(P) \) and \( u_i = \frac{f_y}{x_r} \in G(L) \), where \( f_t = x_1^{b_1} \cdots x_n^{b_n} \) is some generator of \( I \) and \( x_r \) is a variable dividing \( f_t \). We may write explicitly \( u_i = x_1^{b_1} \cdots x_r^{b_r-1} \cdots x_n^{b_n} y_t \). Denote
\[
S := \text{Supp}\left( \frac{u_j}{\gcd(u_j, u_i)} \right) = \left\{ x_p \mid a_p > \begin{cases} b_p & \text{for } p \neq r \\ b_r - 1 & \text{for } p = r \end{cases} \right\}.
\]
We have \( S \neq \emptyset \), because \( \sum_i a_i = \sum_i b_i \). The index \( r \) might be in \( S \) or not. We distinguish two cases:

- \( S = \{ r \} \). Then actually \( a_r > b_r \), and we can pick \( u_k = f_t \) to conclude.
- \( S \neq \{ r \} \), so let \( \ell \in S \setminus \{ r \} \). Then pick
\[
u_k := x_\ell^{b_\ell+1} x_r^{b_r-1} \prod_{p \notin \{ \ell, r \}} x_p^{b_p} = x_\ell f_t.
\]

(3) Let both \( u_j \) and \( u_i \) be in \( G(L) \). So we have
\[
u_j = \frac{f_y}{x_\alpha}y_\mu, \quad \nu_i = \frac{f_\lambda}{x_\beta}y_\lambda, \quad \text{with } x_\alpha | f_\mu, x_\beta | f_\lambda \text{ and } f_\mu, f_\lambda \in G(I).
\]
Denote \( f_\mu = x_1^{a_1} \cdots x_n^{a_n} \) and \( f_\lambda = x_1^{b_1} \cdots x_n^{b_n} \).

- \( \mu = \lambda \): Pick \( u_k := u_j \).
\(\mu \neq \lambda\): We have \(\mu < \lambda\), so that \(f_\mu > \text{Lex } f_\lambda\). Let \(\ell\) be the index of the leftmost nonzero entry of \(a - b\).

- \(\alpha = \beta\): Pick \(u_k := \frac{x_\mu}{x_\alpha} f_\lambda\), which works also in case \(\ell = \alpha\).
- \(\alpha \neq \beta\): There are still two subcases:
  * \(\ell \neq \alpha\): Pick \(u_k := \frac{x_\mu}{x_\beta} f_\lambda\), which works also in case \(\ell = \beta(\neq \alpha)\).
  * \(\ell = \alpha\): We have three final sub-subcases. In the first we have
    \[
    \begin{align*}
    a_\alpha &= b_\alpha + 1 \\
    a_\beta &= b_\beta - 1 \\
    a_p &= b_p & \text{for } p \neq \alpha, \beta,
    \end{align*}
    \]
    (so that \(\alpha < \beta\)). In this case pick \(u_k := \frac{f_\mu}{x_\alpha} y_\mu = \frac{f_\lambda}{x_\beta} y_\mu\). In the case where \(a_\alpha > b_\alpha + 1\), pick \(u_k := \frac{x_\mu}{x_\beta} f_\lambda\). In the case where \(a_\alpha = b_\alpha + 1\) and \(a_\gamma > b_\gamma\) for some \(\gamma \notin \{\alpha, \beta\}\), pick \(u_k := \frac{x_\gamma}{x_\beta} f_\lambda\).

All possible cases are included, and the proof is then complete. \(\square\)

As an immediate consequence, by Proposition 2.9, we get the following.

**Corollary 3.8.** The ideals \(\text{Lin}(I)\) and \(\text{Lin}^*(I)\) have \(d\)-linear resolution.

### 3.1.1 Functorial properties

The ideals of a fixed ring \(A\) are the objects of a category \(\text{Ideals}(A)\), with morphisms consisting of the inclusions. In particular, in this category two objects being isomorphic just means that they are equal. For a polynomial ring \(S\) we have the subcategory \(\text{MonIdeals}(S)\), where the objects are monomial ideals and the morphisms are again the inclusions. Naively speaking, we would like

\[
\text{Lin}: \text{MonIdeals}(S) \longrightarrow \text{MonIdeals}(R)
\]

to be a functor, which is just a fancy way to say that if \(I \subseteq J\) (equivalently, \(G(I) \subseteq G(J)\)) then \(\text{Lin}(I) \subseteq \text{Lin}(J)\). There is no problem for what concerns the complete part: the vector of exponents can get larger, if we enlarge the set of generators, hence inclusions are preserved.

On the other hand the last part causes trouble: the last part, and in fact the ring \(R\) itself, depends on the generators of \(I\). The variable \(y_j\) is associated to the generator \(f_j\), and this does not behave well if in the inclusion \(I \subseteq J\) some generators are listed in a different way. So, we index the \(y\)-variables \(y\) on the generators of \(I\) and not on the position of these generators in the list. The next problem is then the amount of \(y\)-variables in the ring \(R\): we can either index them on all possible monomials in \(x_1, \ldots, x_n\) and have a functor \(\text{Lin}\) that can take as input all ideals of \(S\), or we can have a functor

\[
\text{Lin}_d: \text{MonIdeals}(S)_d \longrightarrow \text{MonIdeals}(R)_d,
\]

for each \(d\) as follows.
Functorial Definition 3.9. Referring to Notation 3.1, the linearization of $I$, in the polynomial ring

$$R := \mathbb{K}[x_1, \ldots, x_n, y_u | u \text{ monomials in } x_1, \ldots, x_n \text{ of degree } d],$$

is the ideal

$$\text{Lin}_d(I) := \left( x_1^{a_1} \cdots x_n^{a_n} | a_1 + \cdots + a_n = d \text{ and } a_i \leq M_i \text{ for all } i \right)$$

$$+ \left( \frac{f_k}{x_k} y_{j_k} | x_k \text{ divides } f_j, k = 1, \ldots, n, j = 1, \ldots, m \right).$$

We stress again that this small change makes no real difference, it’s just a small technicality. When we will generalize the linearization construction to arbitrary monomial ideals in Section 5, we can consider functors a family of functors $\text{Lin}_{\leq d}: \text{MonIdeals}(S)_{\leq d} \to \text{MonIdeals}(R)_{\leq d}$, where now the $y$-variables of $R$ are indexed on all possible monomials up to degree $d$.

Remark 3.10. To sum up, if we change the notation in the definition of linearization as suggested in the discussion above, we get that $\text{Lin}(I) \subseteq \text{Lin}(J)$ if $I \subseteq J$. Same thing goes for $\text{Lin}^*(I) \subseteq \text{Lin}^*(J)$ if $I \subseteq J$.

3.1.2 Interplay of linearization and standard operations

In the category $\text{MonIdeals}(S)$, starting from two ideals $I$ and $J$ one can construct for instance $I + J$, $IJ$ and $I \cap J$, and these are all still monomial ideals and well understood. Unfortunately, it seems that Lin is really well-behaved only with respect to taking the sum.

Proposition 3.11. If we use the Functorial Definition 3.9, then the functors Lin and $\text{Lin}^*$ are such that

$$\text{Lin}(I) + \text{Lin}(J) \subseteq \text{Lin}(I + J) \quad \text{and} \quad \text{Lin}^*(I) + \text{Lin}^*(J) \subseteq \text{Lin}^*(I + J),$$

where $I$ and $J$ are equigenerated in the same degree.

Proof. We can decompose $\text{Lin}(I) = C_I + L_I$ and similarly $\text{Lin}(J) = C_J + L_J$ in complete and last part. Denote by $v_I$ the vector of highest exponents of $I$ and similarly for any other ideal involved. We have $M_i(I + J) = \max\{M_i(I), M_i(J)\}$, so that $v_{I+J} \geq v_I, v_J$ and this means that $C_I + C_J \subseteq C_{I+J}$. As for the last part, we have $L_I + L_J = L_{I+J}$. So this settles the first inclusion in the statement. For what concerns the second inclusion, the only difference is in the complete part of the $*$-linearization, which is “coarser”. In this case we simply have $M_{I+J} = \max\{M_I, M_J\}$, and this is enough.

The equality $\text{Lin}^*(I) + \text{Lin}^*(J) = \text{Lin}^*(I + J)$ holds if and only if $M_I = M_J$. The equality $\text{Lin}(I) + \text{Lin}(J) = \text{Lin}(I + J)$ holds if and only if $v_I = v_J$, where $v_I$ is the vector of maximum exponents for $I$ and similarly for $v_J$. This actually implies that we have $I \subseteq J$ or $J \subseteq I$.

Remark 3.12 (linearization and products). The behaviour of Lin (and similarly, $\text{Lin}^*$) with respect to taking the product is a bit more complicated. If $I$ and $J$ are equigenerated respectively in degrees $d_I$ and $d_J$, then both $\text{Lin}(I)\text{Lin}(J)$ and $\text{Lin}(I)$ are equigenerated in the same degree $d_I d_J$. Notice
that there cannot be an inclusion \( \text{Lin}(I)\text{Lin}(J) \subseteq \text{Lin}(IJ) \): each generator in the last part of \( \text{Lin}(IJ) \) contains only one \( y \)-variable with exponent 1, but some of the minimal generators are quadratic in the \( y \)-variables. One might then ask if there is any hope of having

\[ \text{Lin}(IJ) \subset \text{Lin}(I)\text{Lin}(J). \]

Unfortunately the “improved” version of linearization in Functorial Definition 3.9 cannot possibly work: the ring in which \( \text{Lin}(IJ) \) is defined has \( y \)-variables indexed on monomials of degree \( d_Id_J \), whereas the one in which \( \text{Lin}(I)\text{Lin}(J) \) lives has \( y \)-variables indexed on both monomials of degree \( d_I \) and \( d_J \). The problem is not in the complete part, but rather in the last part of \( \text{Lin}(IJ) \). If \( G(I) = \{ f_1, \ldots, f_m \} \) and \( G(J) = \{ g_1, \ldots, g_m \} \), in \( \text{Lin}(IJ) \) we have generators

\[
\frac{f_i g_j}{x_k} y^\ell
\]

where \( x_k \) divides \( f_i g_j \). Hence, \( x_k \) divides at least one of \( f_i \) or \( g_j \), say \( f_i \). So then we want to have an indexing of the \( y \)-variables such that \( \frac{f_i g_j}{x_k} y^\ell \) actually is in the last part of \( \text{Lin}(I) \). If we have that, then of course \( g_j \) is in the complete part of \( \text{Lin}(J) \) and then we have \( \frac{f_i g_j}{x_k} \) in \( \text{Lin}(I)\text{Lin}(J) \). There might be a way to overcome the problem, if we one comes up with a different way of indexing the variables. Or perhaps relaxing the notion of morphism in the category, making it more “loose” than set-theoretic inclusion.

**Remark 3.13** (intersections). With the intersection we have similar problems as in the case of products. It’s not clear how to define the linearization in a sensible way that allows to compare for instance \( \text{Lin}(I \cap J) \) and \( \text{Lin}(I) \cap \text{Lin}(J) \). Besides, there is an additional problem: notice that even if \( I \) and \( J \) are equigenerated in the same degree, \( I \cap J \) might not be. Consider for instance

\[
I = (x_1^2x_2, x_2x_3x_4) \quad \text{and} \quad J = (x_1x_2)
\]

in \( K[x_1, x_2, x_3] \). Then \( I \cap J = (x_1^2x_2, x_1x_2x_3x_4) \) is not equigenerated. This problem might be overcome by applying \( \text{Lin} \) or \( \text{Lin}^* \) to the equification of \( I \cap J \), introduced in Definition 5.1. This in fact is the way in which we define the linearization of an arbitrary, not necessarily equigenerated monomial ideal, in Definition 5.21. In this example we have

\[
(I \cap J)^\text{eq} = (x_1^2x_2, x_1x_2x_3x_4) \subset K[x_1, x_2, x_3, z],
\]

which looks promising, but one still needs to introduce an appropriate way to index the \( y \)-variables.

### 3.1.3 Polymatroidal ideals and linearization

The study of matroids and polymatroids is a core area in combinatorics and related fields. Polymatroidal ideals, which can be defined even without referring explicitly to any polymatroid, constitute an interesting large class of equigenerated ideals which are particularly well-behaved with respect to resolutions. For instance, all powers of a polymatroidal ideal have a linear resolution (see...
Section 5 of [7] and Section 5 of [5]). In particular, Herzog and Takayama proved in [10] that polymatroidal ideals have linear quotients. So it’s natural to ask when \( \text{Lin}(I) \) is polymatroidal. It turns out that in most cases \( \text{Lin}(I) \) is not polymatroidal. For additional information on polymatroidal ideals, see Section 12.6 of [18].

In the following, for a monomial \( u = z_1^{a_1} \cdots z_s^{a_s} \) in the variables \( z_1, \ldots, z_s \), we write \( \deg_{z_i}(u) = a_i \).

**Definition 3.14.** An equigenerated monomial ideal \( J \subseteq \mathbb{K}[z_1, \ldots, z_s] \) is **polymatroidal** if, whenever \( u \) and \( v \) belong to \( G(J) \) and \( \deg_{z_i}(u) > \deg_{z_i}(v) \) for some \( i \), there exists \( j \) such that \( \deg_{z_j}(u) < \deg_{z_j}(v) \) and \( \frac{u}{v} z_j \in G(J) \).

**Theorem 3.15.** For a monomial ideal \( I \subseteq S = \mathbb{K}[x_1, \ldots, x_n] \) equigenerated in degree \( d \), in the following cases \( \text{Lin}(I) \) is polymatroidal:

- \( d = 1 \), that is, \( I \) is generated by variables;
- \( d \) is arbitrary and \( I \) is principal.

In all other cases \( \text{Lin}(I) \) is not polymatroidal.

**Proof.** Assume that \( I \) is equigenerated in degree \( d = 1 \), so that \( I \) is generated by a set of variables in \( S \). Then \( \text{Lin}(I) \) is also generated by variables, and one can see immediately by the definition that any ideal generated by variables is polymatroidal. Assume now that \( d \) is arbitrary and \( I \) is principal, so that \( I = (f) \) and, denoting simply by \( y \) the only \( y \)-variable, we have

\[
\text{Lin}(I) = (f) + \left( \frac{f}{x_k} y \mid x_i \in \text{Supp}(f) \right).
\]

It’s immediate to see that, picking any two generators, the condition in the definition of polymatroidal ideal is satisfied.

Next we prove that in all the other cases, namely if \( d \geq 2 \) and \( I \) is not principal, \( \text{Lin}(I) \) is not polymatroidal. Let

\[
f = x_1^{e_1} \cdots x_n^{e_n} \quad \text{and} \quad g = x_1^{e_1^2} \cdots x_n^{e_n^2}
\]

be two distinct minimal generators of \( I \), so that \( \sum_i d_i = \sum_i e_i = d \). In what follows we use the notation introduced in the Functorial Definition \([3.9]\). We distinguish two cases, the second of which has several subcases:

1. If there exists an index \( k \) such that \( c_k \geq e_k + 2 \), then we pick \( u := \frac{f}{y_k} yf \) and \( v := \frac{g}{x_k} yg \) for some \( q \neq k \). We also select \( x_k \) to be the variable \( z_i \) in the definition of polymatroidal, and we see that \( \deg_{z_k}(u) > \deg_{z_k}(v) \). But there exists no variable \( x_j \) with \( \deg_{x_j}(u) < \deg_{x_j}(v) \) and such that the monomial

\[
\frac{u}{x_k} x_j = \frac{f}{x_k} x_j yf
\]

is in \( G(\text{Lin}(I)) \). Indeed, the only way this monomial could be in \( G(\text{Lin}(I)) \) would be if \( \frac{u}{x_k} x_j = \frac{f}{x_{\ell}} \) for some \( \ell \), but this is possible only if \( x_j = x_k \), and this cannot happen because of the above assumptions on \( x_k \) and \( x_j \). Therefore \( \text{Lin}(I) \) in this case is not polymatroidal.
(2) So now suppose that there is no such $k$ as above. And changing the roles of $u$ and $v$, we can actually assume that
\[ c_i \leq e_i + 1 \quad \text{and} \quad e_i \leq c_i + 1 \]
for all $i$. There must be at least one index $k$ such that $c_k = e_k + 1$. We distinguish again two subcases:

- If $e_k = 1$, then $c_k = 2$. If we pick
  \[ u := \frac{f}{x_k} y_f \quad \text{and} \quad v := \frac{g}{x_k} y_g, \]
then $1 = \deg x_k(u) > \deg x_k(v) = 0$, and of course there is no $x_j$ with $\deg x_j(v) > \deg x_j(u)$ and such that $\frac{x_k}{x_j} x_j y_f \in G(\Lin(I))$.

- Suppose that we cannot find such $k$. This means that $c_i = e_i$ for all $c_i / \in \{0, 1\}$ and $e_i / \in \{0, 1\}$.

Since $f \neq g$, we know that that there are $k$ and $p$ such that $c_k = e_k + 1 = 1$ and $c_p = e_p - 1 = 0$. We divide in two cases one last time:

* If there is $\ell \neq k$ such that $\deg x(\ell)(f) > \deg x(\ell)(g)$, meaning that $c_\ell = 1$ and $e_\ell = 0$, then we can pick
  \[ u := \frac{f}{x_\ell} y_f \quad \text{and} \quad v := \frac{g}{x_\ell} y_g, \]
and then consider $u/x_\ell$. There is no $x_j$ such that $\deg x_j(u) < \deg x_j(v)$ and $\frac{x_\ell}{x_j} x_j y_f \in G(\Lin(I))$.

* Assume then that $c_i = e_i$ for all $i \notin \{k, p\}$. Then, since $d \geq 2$, there exists some $\ell \notin \{k, p\}$ such that $\deg x(\ell)(f) = \deg x(\ell)(g) > 0$, so we can choose
  \[ u := \frac{f}{x_\ell} y_f \quad \text{and} \quad v := \frac{g}{x_\ell} y_g, \]
and consider $u/x_\ell$.

In all the possible cases we constructed two generators that show how the condition in the definition of polymatroidal ideal is not satisfied.

\[ \square \]

3.2 Radical of $\Lin(I)$ and $\Lin^*(I)$

Recall that, given an ideal $I \subseteq S = \mathbb{K}[x_1, \ldots, x_n]$, the **radical of $I$** is defined as
\[ \sqrt{I} := \{ f \in S \mid f^p \in I \text{ for some } p \in \mathbb{N} \}. \]

In case $I$ is a monomial ideal, $\sqrt{I}$ is also a monomial ideal and one has an easy way to compute generators for $\sqrt{I}$, described as follows. We use the same notation as in [IS] and, for a monomial $u = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$, we denote
\[ \sqrt{u} := \prod_{i=1, \ldots, n}^{a_i > 0} x_i. \]
Then, if \( I \) is a monomial ideal generated by monomials \( u_1, \ldots, u_s \), the radical of \( I \) is simply \( \sqrt{I} = (\sqrt{u_i} \mid i = 1, \ldots, s) \). See for instance [13], Proposition 1.2.4. We say that an ideal \( I \) is a **radical ideal** if \( I = \sqrt{I} \).

A first question that might arise is: when is \( \text{Lin}(I) \) (or \( \text{Lin}^*(I) \)) a radical ideal? For a monomial ideal, being radical is equivalent to being squarefree, that is, having all minimal monomial generators which are squarefree. Moreover, as explained in Remark 3.16 \( \text{Lin}(I) \) is squarefree, or equivalently \( \text{Lin}^*(I) \), if and only if \( I \) is squarefree. (For more details on this situation see Section 4) For this reason, we assume \( M \geq 2 \) in this subsection.

**Remark 3.16.** It is trivial to notice that applying \( \text{Lin} \) or \( \text{Lin}^* \) to the radical \( \sqrt{I} \) of some equigenerated ideal \( I \) might not make sense, because \( \sqrt{I} \) might not be equigenerated. Consider for instance \( I = (x_1^2, x_2 x_3) \), which is such that \( \sqrt{I} = (x_1, x_2 x_3) \). (A possibility could be applying the radical to the equification \( I^a \), see Definition 5.1, but we could not manage to get anything fruitful from that approach.) Our attention will then be focused on applying to \( I \) first the linearization and then the radical. It turns out that \( \sqrt{\text{Lin}(I)} \) is easier to understand and seems to have nicer properties in general than \( \sqrt{\text{Lin}(I)} \). The following example illustrates this.

**Example 3.17.** Consider \( I = (x_1^2 x_2, x_1 x_2 x_3) \subset k[x_1, x_2, x_3] \). Then one has

\[
\text{Lin}(I) = (x_1^2 x_2, x_1^2 x_3, x_1 x_2 x_3, x_1 x_2 y_1, x_1^2 y_1, x_2 x_3 y_2, x_1 x_3 y_2, x_1 x_2 y_2)
\]

and we get

\[
\sqrt{\text{Lin}(I)} = (x_1 x_2, x_1 x_3, x_1 y_1, x_2 x_3 y_2),
\]

which is somewhat difficult to describe and to control. It is not equigenerated and thus cannot have linear resolution, for instance. This is due to the lack of symmetry in the complete part of \( \text{Lin}(I) \). On the other hand one has

\[
\text{Lin}^*(I) = (x_1, x_2, x_3)^3_{\leq(2,2,2,0,0)} + (x_1 x_2 y_1, x_1^2 y_1, x_2 x_3 y_2, x_1 x_3 y_2, x_1 x_2 y_2),
\]

which has a very symmetric complete part, whose nice properties are inherited by

\[
\sqrt{\text{Lin}^*(I)} = (x_1 x_2, x_1 x_3, x_2 x_3, x_1 y_1).
\]

Computer calculations show that \( \sqrt{\text{Lin}^*(I)} \) in this example has linear quotients, and hence linear resolution. Notice moreover that \( \sqrt{\text{Lin}^*(I)} \) looks like the linearization of something but “lacks a piece”, for instance \( x_2 y_1 \) or \( x_3 y_1 \).

**Example 3.18.** Take now \( I = (x_1^2 x_2, x_2^2 x_3^2) \subset k[x_1, x_2, x_3] \). The ideal

\[
\sqrt{\text{Lin}^*(I)} = (x_1 x_2, x_1 x_3, x_2 x_3)
\]

has again linear quotients, and this time it looks even prettier, with no generators involving \( y \)-variables. More precisely, one can compute the radical ideal of \( \text{Lin}^*(I) \) by taking the radicals of the generators, and one realizes that the radicals of the generators of the last part of \( \text{Lin}^*(I) \)—which are the ones involving the \( y \)-variables—turn out to be redundant. Notice that in this case \( \text{Lin}^*(I) = \text{Lin}(I) \).
Notation 3.19. As in the rest of the section, let $d$ be the degree of all the minimal monomial generators of $I$ and $M$ be the largest exponent occurring in these generators. Then let us write

$$d = aM + b,$$

with $a, b \in \mathbb{N}$ and $b < M$,

in the “Euclidean way”. We assume that

$$a \geq 1 \quad \text{and} \quad M \geq 2. \quad (4)$$

Moreover, for a non-negative integer $b \in \mathbb{N}$, we write

$$\text{sign}(b) = \begin{cases} 1 & \text{if } b > 0, \\ 0 & \text{if } b = 0. \end{cases}$$

In the following, recall that the support $\text{Supp}(u)$ of a monomial $u$ is the set consisting of the variables which divide $u$.

Proposition 3.20. The ideal $\sqrt{\text{Lin}^*(I)}$ is generated by all squarefree monomials of degree $a + \text{sign}(b)$ in the variables $x_1, \ldots, x_n$ and by the monomials $\sqrt{f_j x_k y_j}$, where $f_j$ is a minimal generator of $I$, $x_k$ occurs with exponent 1 in $f_j$, and all the other variables occur with exponent $M$. Notice that we might possibly have such “pathological” generators, that is, generators with exactly one variable that occurs with exponent 1 and all other variables with exponent $M$, only when $b = 1$. Let $p$ be the number of “pathological” generators of $I$. Then

$$\#G(\sqrt{\text{Lin}^*(I)}) = \binom{n}{a + \text{sign}(b)} + p,$$

and all the generators have the same degree $a + \text{sign}(b)$.

Proof. When we take the radical of the complete part of $\text{Lin}^*(I)$, we find all squarefree monomials of degree $a + \text{sign}(b)$ in the variables $x_1, \ldots, x_n$. Let’s see what happens on the other hand to the last part:

$$\sqrt{\text{last part of } \text{Lin}^*(I)} = \left( \sqrt{f_j y_j / x_k} \mid x_k \text{ divides } f_j, \quad k \in \{1, \ldots, n\}, \quad j \in \{1, \ldots, m\} \right).$$

Notice that $\text{Supp}(\sqrt{u}) = \text{Supp}(u)$ by definition for any monomial $u$. Moreover, one has

$$\#\text{Supp}(f_j / x_k) = \begin{cases} \#\text{Supp}(f_j) & \text{if } x_k \text{ divides } f_j / x_k, \\ \#\text{Supp}(f_j) - 1 & \text{otherwise.} \end{cases} \quad (5)$$

More precisely, if $x_k$ occurs in $f_j$ with exponent $> 1$, then $f_j / x_k$ also contains $x_k$. If $\#\text{Supp}(f_j / x_k) \geq q + \text{sign}(r)$, then the generator $\sqrt{f_j / x_k} = \sqrt{f_j x_k y_j}$ is redundant because it’s a multiple of a generator coming from the complete part. Hence we are interested in the generators where $\#\text{Supp}(f_j / x_k) < a + \text{sign}(b)$.

Notice furthermore that we have $\#\text{Supp}(f_j) \geq a$: indeed, assume that $\#\text{Supp}(f_j) = a' < a$. Then one would have $\deg(f_j) \leq a'M < aM \leq d$, a contradiction. So, by (5), $\#\text{Supp}(f_j / x_k) < a + \text{sign}(b)$ actually happens when

$$\begin{cases} \#\text{Supp}(f_j) < a + \text{sign}(b) & \text{if } x_k \text{ divides } f_j / x_k, \\ \#\text{Supp}(f_j) - 1 < a + \text{sign}(b) & \text{otherwise.} \end{cases}$$

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The first case can happen only if \( \# \text{Supp}(f_j) = a \) and \( \text{sign}(b) = 1 \). The second case can happen only when \( \# \text{Supp}(f_j) = a \) and for any \( b \), or when \( \# \text{Supp}(f_j) = a + 1 \) and \( \text{sign}(b) = 1 \). Notice we cannot have simultaneously \( \# \text{Supp}(f_j) = a \) and \( \text{sign}(b) = 1 \), because then we would have \( \deg(f_j) \leq aM < aM + b = d \), a contradiction. So actually only the second case above can happen, when we have either \( \# \text{Supp}(f_j) = a \) and \( b = 0 \), or when \( \# \text{Supp}(f_j) = a + 1 \) and \( \text{sign}(b) = 1 \). Namely, when \( \# \text{Supp}(f_j) = a + \text{sign}(b) \).

To recap, if we have \( f_j \) and \( x_k \) such that \( x_k \) occurs with exponent 1 in \( f_j \), and such that \( \# \text{Supp}(f_j) = a + \text{sign}(b) \), then we get \( \# \text{Supp}(f_j/x_k) = a + \text{sign}(b) - 1 \) and the generator \( \sqrt{\frac{f_j}{x_k}}y_j \) is not redundant.

So, in short, \( \sqrt{\text{Lin}^+(I)} \) is generated by \( (x_1, \ldots, x_n)^{a + \text{sign}(b)} \) and by the monomials \( \sqrt{\frac{f_j}{x_k}}y_j \) where \( x_k \) occurs in \( f_j \) with exponent 1 and such that \( \# \text{Supp}(f_j) = a + \text{sign}(b) \). Now, if we have something as above, with \( x_k \) occurring with exponent 1 and such that \( \# \text{Supp}(f_j) = a + \text{sign}(b) \), then it means that \( x_k \) is the only variable with exponent 1 inside \( f_j \), and all the rest have exponent \( M \). Here’s why: Recall that we are assuming that we have \( M \geq 2 \). If we have another variable with exponent \( < M \), then we can just “move” one exponent 1 with the other exponent \( < M \), so that we get a monomial of degree \( d \), with variables occurring all with exponent \( \leq M \) and with support strictly smaller than that of \( f_j \). So in other words \( a + \text{sign}(b) \leq \# \text{Supp}(f_j/x_k) \), and the generator \( \sqrt{\frac{f_j}{x_k}}y_j \) turns out to be redundant. \( \qed \)

In the following we use same notation as in Corollary 3.10 for the numbers \( r_k \), in view of Corollary 3.22.

**Theorem 3.21.** The ideal \( \sqrt{\text{Lin}^+(I)} \) has linear quotients. Let’s order the generators so that we first have all squarefree monomials of degree \( a + \text{sign}(b) \) in decreasing lexicographic order and then we have the radicals of the “pathological” generators of \( I \). Then the numbers \( r_k \) which come from those generators range between 0 and \( n - a - \text{sign}(b) \). After that, in case \( b = 1 \), we still have \( p \) colon ideals (possibly with \( p = 0 \)). Order the pathological generators

\[
\sqrt{\frac{f_{j_1}}{x_{k_1}}y_{j_1}, \ldots, \frac{f_{j_p}}{x_{k_p}}y_{j_p}}.
\]

Let \( r_k \) be the number of generators of the colon ideal by \( \sqrt{\frac{f_{j_s}}{x_{k_s}}y_{j_s}} \). Define

\[
t_\ell := \#\left\{ s \in \{1, \ldots, \ell - 1\} \mid \frac{f_{j_s}}{x_{k_s}} = \frac{f_{j_\ell}}{x_{k_\ell}} \right\}.
\]

Then \( r_k = n - a + t_\ell \).

**Proof.** The first numbers \( r_k \) behave as in the case studied for the linearization of a squarefree ideal, for which one can see Proposition 4.7. We take the colon by \( \sqrt{\frac{f_{j_s}}{x_{k_s}}y_{j_s}} \), we get for sure as generators at least all \( x_i \notin \text{Supp}(f_{j_\ell}/x_{k_\ell}) \), and there are \( n - a \) of them. Additionally, if we have \( \frac{f_{j_s}}{x_{k_s}} = \frac{f_{j_\ell}}{x_{k_\ell}} \) for some \( s < \ell \), then it holds \( \sqrt{f_{j_s}/x_{k_s}} = \sqrt{f_{j_\ell}/x_{k_\ell}} \), so that when taking the colon we get the extra generator \( y_{j_s} \). \( \qed \)
Corollary 3.22. With the same notation as above, we have
\[ \beta_i(I) = \left( i + a + \text{sign}(b) - 1 \right) \binom{n}{i + a + \text{sign}(b) - 1} + \sum_{\ell=1}^{p} \binom{n - a + t_\ell}{i}. \]

Proof. For the complete part see Remark 4.8. The rest is from the previous theorem and Corollary 2.10. \qed

4 Linearization in the squarefree case

In this section \( \mathbb{K} \) may be any field, not necessarily of characteristic zero. Indeed, the properties here discussed are purely combinatorial and do not depend on the characteristic of \( \mathbb{K} \).

Notation 4.1. Let \( I \) be an equigenerated squarefree monomial ideal in \( S := \mathbb{K}[x_1, \ldots, x_n] \), with minimal system of monomial generators \( G(I) = \{ f_1, \ldots, f_m \} \), where \( \deg(f_j) = d \) for all \( j \).

The results in this section are written for \( \text{Lin}^*(I) \), but it’s very easy to adapt them to \( \text{Lin}(I) \), as explained in the following.

Remark 4.2. If we specify the definition of \( \text{Lin}^* \) to this case, we get, inside the polynomial ring \( R := \mathbb{K}[x_1, \ldots, x_n, y_1, \ldots, y_m] \), the ideal
\[ \text{Lin}^*(I) = (x_{i_1}x_{i_2} \cdots x_{i_d} | 1 \leq i_1 < i_2 < \cdots < i_d \leq n) \]
\[ + (f_jy_j/x_k | x_k \text{ divides } f_j, k = 1, \ldots, n, j = 1, \ldots, m). \]

That is, in the complete part we have all monomials of degree \( d \) in the \( x_i \)'s. In the last part, for each generator \( f_j \) of \( I \) we add a new variable \( y_j \) and \( d \) generators \( f_jy_j/x_k \), where we replace the only occurrence of each \( x_k \) in the support of \( f_j \) by \( y_j \). The difference between \( \text{Lin}(I) \) and \( \text{Lin}^*(I) \) is not significant in the squarefree case. In the complete part of \( \text{Lin}(I) \) we don’t have all squarefree monomials of degree \( d \) in all the \( n \) variables, but instead we have all squarefree monomials of degree \( d \) in the variables that appear in the generators of \( I \). So, what one needs to do in order to write a version for \( \text{Lin}(I) \) of the results in this section is to replace \( x_1, \ldots, x_n \) by the variables which are actually used by the generators of \( I \). Or alternatively one can assume that each of the variables \( x_1, \ldots, x_n \) appears in some generator of \( I \), in which case (in the squarefree situation!) we have \( \text{Lin}^*(I) = \text{Lin}(I) \). As a side note, observe that the slight difference makes \( \text{Lin}^*(I) \) the actual generalization of the Booth–Lueker ideals associated to the graphs introduced in Definition 1.1.

Notation 4.3. Instead of writing \( v = (1, \ldots, 1) \) in the formula \( I_{\leq v} \) introduced in Notation 2.13 for the squarefree case we simply write \( I_{\text{sqf}} \).

Remark 4.4. One can see that \( I \) is squarefree if and only if \( \text{Lin}(I) \) is squarefree. Both implications are easy: the “only if” implication is clear by construction and the “if” one follows from the fact that the complete part of \( \text{Lin}(I) \) is squarefree if and only if \( I \) is. The last part of \( \text{Lin}(I) \) would in general not be enough to detect the squarefreeness of \( I \), if for instance \( I \) had a generator like \( x_1^2 \). But for degree \( d > 2 \) it’s true that \( I \) is squarefree if and only if the last part of \( \text{Lin}(I) \) is. Same goes for \( \text{Lin}^* \).
The next is an example of the abovementioned slight difference between Lin\(^*\)(I) and Lin(I) in the squarefree case, and of how one can obtain from the results about Lin\(^*\)(I) the corresponding ones for Lin(I). The number of minimal generators of Lin\(^*\)(I) is

\[ \#G(\text{Lin}^*(I)) = \binom{n}{d} + md. \]

Let \( c := \# \bigcup_{j=1}^m \text{Supp}(f_j) \). The number of minimal generators of Lin(I) is

\[ \#G(\text{Lin}(I)) = \binom{c}{d} + md. \]

### 4.1 Betti numbers of Lin\(^*\)(I)

As proven in Theorem 3.7, Lin(I) and Lin\(^*\)(I) have linear quotients for any equigenerated ideal I, so in particular for squarefree equigenerated ideals. In this section we provide explicit formulas for the Betti numbers of Lin\(^*\)(I) in case I is squarefree. As already remarked above, one can turn the results here obtained for Lin\(^*\)(I) into analogous ones for Lin(I) simply by replacing \( n \) by the number of variables, among \( x_1, \ldots, x_n \), that actually appear in the minimal monomial generators of I.

In virtue of Corollary 3.22 and Theorem 3.7, we can use the formula

\[ \beta_i(\text{Lin}^*(I)) = \sum_{k=1}^{\#G(\text{Lin}^*(I))} \binom{r_k}{i}, \quad \text{for } i \geq 0, \]

where \( r_k \) is the number of generators of \((g_1, \ldots, g_k-1) : g_k\) and where the polynomials \( g_j \) are the generators of Lin\(^*\)(I) as listed in Theorem 3.7. We determine the numbers \( r_k \) in Proposition 4.7 and Proposition 4.13.

**Example 4.5.** Consider the ideal

\[ I = (x_1x_2x_3, x_1x_2x_4, x_1x_2x_5) \subset S = \mathbb{K}[x_1, \ldots, x_5]. \]

The hypergraph corresponding to this ideal (see Section 4.3.1) consists of three triangles that share a common edge (which is something of one dimension less!). The \( * \)-linearization of this ideal lives in \( R = \mathbb{K}[x_1, \ldots, x_5, y_1, y_2, y_3] \), and it’s the ideal

\[ \text{Lin}^*(I) = (x_1x_2x_3, \ldots, x_2x_4x_5, x_3x_4x_5) \]

\[ + (y_1x_2x_3, x_1y_1x_3, x_1x_2y_1, \ldots, x_1y_3x_5, x_1x_2y_3), \]

with \( \binom{5}{3} + md = \binom{5}{3} + 3 \times 3 = 19 \) generators. In fact, in this example one has Lin(I) = Lin\(^*\)(I). One can compute the Betti table of this ideal,

\[
\begin{array}{ccccccc}
\beta(\text{Lin}^*(I)) &=& 0 & 1 & 2 & 3 & 4 & 5 \\
3 & 19 & 45 & 43 & 21 & 6 & 1
\end{array}
\]
Next we compute the colon ideals $J_k := (g_1, \ldots, g_{k-1}) : g_k$. For $g_k$ in the complete part of $\text{Lin}^*(I)$ we get

\[
J_1 = (0) \\
J_2 = (x_3) \\
J_3 = (x_3, x_4) \\
J_4 = (x_2) \\
J_5 = (x_2, x_4) \\
J_6 = (x_2, x_3) \\
J_7 = (x_1) \\
J_8 = (x_1, x_4) \\
J_9 = (x_1, x_3) \\
J_{10} = (x_1, x_2)
\]

and, after that for $g_k$ in the last part we get

\[
J_{11} = (x_1, x_4, x_5) \\
J_{12} = (x_2, x_4, x_5) \\
J_{13} = (x_3, x_4, x_5) \\
J_{14} = (x_1, x_3, x_5) \\
J_{15} = (x_2, x_3, x_5) \\
J_{16} = (x_3, x_4, x_5, y_1) \\
J_{17} = (x_1, x_3, x_4) \\
J_{18} = (x_2, x_3, x_4) \\
J_{19} = (x_3, x_4, x_5, y_1, y_2).
\]

This example is quite small but we can observe some facts that hold in general: the colon ideals $J_k$ with $g_k$ in the complete part are very regular, they clearly don’t depend on the generators of $I$, and they have nothing to do with the variables $y_j$. They have at most $n - d$ generators, in this case $5 - 3 = 2$. On the other hand the ideals $J_k$ with $g_k$ in the last part of $\text{Lin}^*(I)$ have more generators, at least $n - d + 1$, and some of them happen to have additional generators, which are some variables $y_j$.

**Lemma 4.6.** Let $m_1, \ldots, m_{\binom{n}{d}} \in \mathbb{K}[x_1, \ldots, x_n]$ be all the squarefree monomials of degree $d$ in $n$ variables. Assume these monomials are ordered in decreasing lexicographic order. For each $k$, denote by $r_k$ the number of minimal monomial generators of $(m_1, \ldots, m_{k-1}) : m_k$. For any monomial $m$, denote $\max(m) := \max\{i \mid x_i \text{ divides } m\}$. Then

\[
r_k = \max(m_k) - d.
\]

**Proof.** For degree $d = 1$ this is trivial. For $r_1 = 0$ the formula is also clear. Fix $k > 1$ and let us consider $m_k$. The variables $x_i$ with $i < \max(m_k)$ which are not in the support of $m_k$ are

\[
\max(m_k) - 1 - (d - 1) = \max(m_k) - d.
\]
Let \( j := \max(m_k) \). For each \( i < j \) such that \( x_i \notin \text{Supp}(m_k) \), we have that \( \frac{x_i}{x_j} m_k \) has degree \( d \) and comes before \( m_k \) in the lexicographic order. So \( (m_1, \ldots, m_{k-1}) : m_k \) has as generators all such variables \( x_i \). Let us see that there cannot be more generators. There might be a monomial \( m \) that comes lexicographically before \( m_k \) and such that \( \frac{m}{\gcd(m, m_k)} \) does not consist of just one variable, but in that case this monomial \( m \) is divided by some variable \( x_i \) with \( i < j \), or else \( m \) would not be smaller than \( m_k \) in the lexicographic order.

A direct consequence is the following result. As remarked several times, to obtain its analogous version for \( \text{Lin}(I) \), one only need to replace \( n \) by the number of variables in the union of the supports of the elements of \( G(I) \).

**Proposition 4.7.** The colon ideals coming from the complete part of \( \text{Lin}^*(I) \) behave as follows. The numbers \( r_1, r_2, \ldots, r_{\binom{n}{d}} \) range between 0 and \( n - d \). Number \( j \in \{0, \ldots, n - d\} \) occurs as \( r_k \) for

\[
\binom{j + d - 1}{d - 1}
\]

values of \( k \).

**Proof.** This is immediate from Lemma \[16\]. We have \( j = r_k \) for all the \( k \)'s such that \( \max(m_k) = j + d \). Namely we have to compute how many monomials of degree \( d \) with maximum equal to \( j + d \) there are, and those are \( \binom{j + d - 1}{d - 1} \). \( \blacksquare \)

**Remark 4.8.** We recover the well-known Betti numbers of the squarefree power \( C := (x_1, \ldots, x_n)^d_{\text{sqf}} \) as

\[
\beta_i(C) = \sum_{k=1}^{\binom{n}{d}} r_k \binom{i}{j}
\]

\[
= \sum_{j=0}^{n-d} \binom{j + d - 1}{d - 1} \binom{j}{i}
\]

\[
\equiv \binom{i + d - 1}{d - 1} \sum_{j=0}^{n-d} \binom{j + d - 1}{i + d - 1}
\]

\[
= \binom{i + d - 1}{d - 1} \sum_{j=d-1}^{n-1} \binom{j}{i + d - 1}
\]

\[
= \binom{i + d - 1}{d - 1} \binom{n}{i + d},
\]

where in \((*)\) we used that, by direct calculation,

\[
\binom{j + d - 1}{d - 1} \binom{j}{i} = \binom{i + d - 1}{d - 1} \binom{j + d - 1}{i + d - 1}
\]

and in the last equality the formula

\[
\sum_{j=d-1}^{n-1} \binom{j}{i + d - 1} = \binom{n}{i + d},
\]

which is a specialization of the identity (11) in Section 1.2.6 of [20].
In order to describe what happens with the rest of the colon ideals, namely those that arise from the last part of $\text{Lin}^*(I)$, we need some definitions. There probably already is better terminology in the literature to express these things, but due to my lack of knowledge I introduce the following terms.

**Definition 4.9.** We call a monomial $u = x_{i_1} \cdots x_{i_d-1}$, with $i_1 < \cdots < i_d-1$, a 

$\text{2-edge of } I$ if $u$ divides some generator of $I$. Let us call multiplicity of a 

$\text{(d – 1)-edge } u$ of $I$ the number

$$\text{mult}(u) := \# \{ f_i \in G(I) \mid u \text{ divides } f_i \},$$

where $G(I) = \{ f_1, \ldots, f_m \}$ as in all the rest of the section.

Consider a graph and the corresponding edge ideal. Then a 1-edge would be 

a single variable dividing some generator of the edge ideal, that is, it would correspond 

to a vertex which is touched by some edge—with the ordinary meaning 

of the word “edge”—of the graph. The multiplicity of this 1-edge/vertex 

means the number of edges touching it, namely the degree of the vertex. So 

the multiplicity here defined generalises the graph-theoretical notion of degree.

**Example 4.10.** Continuing Example 4.5 for instance $x_1x_5$ is a 2-edge of $I$, 

since it divides the generator $f_3 = x_1x_2x_5$. The multiplicity of the 2-edge $x_1x_5$ 

is 1, since $x_1x_5$ divides only the generator $f_3$. Another 2-edge is $x_1x_2$, and 

this one has multiplicity 3, as it divides all three generators. The presence of 

such a 2-edge of multiplicity 3 is the reason why we have a quotient ideal with 

an extra $y$-generator and a quotient ideal with two extra $y$-generators.

**Definition 4.11.** Call a $j$-cluster a set of cardinality $j$ consisting of generators of $I$ 

that are divided by a same $(d – 1)$-edge $u$.

**Example 4.12.** Continuing the previous example, the set $\{ x_1x_2x_4, x_1x_2x_5 \}$ 

is a 2-cluster because all of its elements are divided by the same 2-edge $x_1x_2$. 

Similarly, the set $G(I)$ actually is a 3-cluster.

**Proposition 4.13.** The numbers $r_k$ coming from the last part of $\text{Lin}^*(I)$ 

range from $n – d + 1$ and above. For each integer $j \geq 2$, consider the maximal 

$j$-clusters, that is, $j$-clusters that are not part of a $(j + 1)$-cluster. For each 

maximal $j$-cluster there is a colon ideal with $n – d + 2$ generators, a colon ideal 

with $n – d + 3$ generators, . . . , up to a colon ideal with $n – d + j$ generators. 

Any maximal $j$-cluster has its own set of such ideals. All other colon ideals 

have $n – d + 1$ generators.

**Proof.** All the colon ideals of the form $(g_1, \ldots, g_{k-1}) : g_k$ with $g_k = \frac{L \ell}{x_\ell}$ in 

the last part of $\text{Lin}^*(I)$ have at least $n – d + 1$ generators. This is because 

among those generators we have at least each $x_\ell \notin \text{Supp}(\frac{L \ell}{x_\ell})$, and those are 

exactly $n – d + 1$.

Now each of these ideals might also have some “extra” $y$-generators. For each $j \geq 2$, each maximal $j$-cluster contributes to the list of ideals with an 
ideal with one extra $y$-generator, an ideal with two extra $y$-generators, . . . , up 

to an ideal with $j – 1$ extra $y$-generators. The rest of the quotient ideals coming 

from the last part of $\text{Lin}^*(I)$ have $n – d + 1$ generators, and they correspond 

to the maximal 1-clusters. This is clear by Remark 4.6 as follows. Write
for all the generators inside a maximal $j$-cluster. So then we have

$$a(d−1)$$-edge $u$ shared by all them, so that $f_{x_i}$ for all $p \in \{1, \ldots, j\}$. In the last part of $\text{Lin}^*(I)$ we have generators $\frac{f_{x_i}}{x_i}$, for each $x_i$ dividing each $f_{x_i}$, namely each $x_i$ dividing $u$ and $x_i$. Then when taking the colon by $\frac{f_{x_i}}{x_i}$ we get exactly the number of generators stated above.

**Corollary 4.14.** Let us denote by $C_j \in \mathbb{N}$ the number of maximal $j$-clusters, that is, $j$-clusters that are not part of a $(j+1)$-cluster. A closed formula for the Betti numbers is

$$\beta_i(\text{Lin}^*(I)) = \binom{i + d - 1}{d - 1} \binom{n}{i + d} + \binom{n - d + 1}{i} \left(md - \sum_{j \geq 2}(j - 1)C_j\right)$$

$$+ \sum_{j \geq 2} C_j \sum_{k = 2}^j \binom{n - d + k}{i}.$$ 

**Corollary 4.15.** Let $N := \max\{j \mid C_j \neq 0\}$. Then we get

$$\text{projdim}_R(\text{Lin}^*(I)) = n - d + N,$$

$$\text{depth}(R/\text{Lin}^*(I)) = m + d − N − 1.$$ 

**Proof.** The largest $r_k$ occurring in the sum is $n - d + N$, so the last nonzero binomial coefficient

$$\binom{r_k}{i} = \binom{n - d + N}{i}$$

corresponds to $i = n - d + N$. So we get the desired formula we wanted for the projective dimension. As for the depth, we just use the Auslander–Buchsbaum formula (see Formula 15.3 of [24]):

$$\text{depth}(R/\text{Lin}^*(I)) = \text{depth}(R) - \text{projdim}_R(R/\text{Lin}^*(I))$$

$$= n + m - (n - d + N + 1).$$ 

4.2 Conceptual proof via polarizations

What follows is a more conceptual proof, in the squarefree case, that $\text{Lin}^*(I)$ has linear resolution, a result which we already have as a special case of Corollary 3.8 and that the Betti numbers only depend on the multiplicities of the $(d − 1)$-edges, as defined in Definition 4.9. This proof and the notion of separation involved in it were very kindly explained to me by Gunnar Fløystad.

**Notation 4.16.** For any subset $\sigma \subseteq [n] := \{1, \ldots, n\}$, denote $x^\sigma := \prod_{i \in \sigma} x_i$ and $m^\sigma := (x_i \mid i \in \sigma)$.

**Definition 4.17.** Given a squarefree monomial ideal $I = (x^{\sigma_1}, \ldots, x^{\sigma_s}) \subseteq \mathbb{K}[x_1, \ldots, x_n]$, the **Alexander dual** of $I$ is the ideal

$$I^\vee := m^{\sigma_1} \cap \cdots \cap m^{\sigma_s}.$$ 

For other equivalent descriptions of $I^\vee$ and additional information, see for instance Section 62 of [24] or Section 1.5.2 of [18].
Definition 4.18. Let \( R \) be any ring and \( M \) be an \( R \)-module. An element \( a \in M \) is \( M \)-regular if the only \( m \in M \) such that \( am = 0 \) is \( m = 0 \). A sequence \( a_1, \ldots, a_n \in R \) is an \( M \)-regular sequence if the following hold:

- \( a_i \) is \( M/(a_1, \ldots, a_{i-1})M \)-regular for all \( i \in \{1, \ldots, n\} \);
- \( M/(a_1, \ldots, a_n)M \neq 0 \).

Next we recall an ubiquitous notion in commutative algebra, without specifying all the details involved because it’s just instrumental for our purposes.

Definition 4.19. For a Noetherian local ring \( R \), a finitely generated \( R \)-module \( M \neq 0 \) is a Cohen–Macaulay module if \( \text{depth}(M) = \text{dim}(M) \). If \( R \) is any Noetherian ring, \( M \) is called a Cohen–Macaulay module if the localization \( M_m \) is Cohen–Macaulay as defined in the local case, for any maximal ideal \( m \) in the support of \( M \). If \( R \) is Cohen–Macaulay as an \( R \)-module, then we say that \( R \) is a Cohen–Macaulay ring.

We never use explicitly the definition of Cohen–Macaulay ring in the proof presented in this section. The main tools that we use are the following two theorems, which are very well known. For a proof of the first theorem, see for instance Theorem 2.1.3 of [6]. For the second theorem see for instance Corollary 62.9 of [24] or the original version, which is Theorem 3 of [10].

Theorem 4.20. Let \( R \) be a Noetherian ring and \( a_1, \ldots, a_n \) be a regular sequence in \( R \). If \( R \) is a Cohen–Macaulay ring, then \( R/(a_1, \ldots, a_n) \) is a Cohen–Macaulay ring.

Theorem 4.21 (Eagon–Reiner, [10]). For a squarefree monomial ideal \( I \subseteq S \), the following are equivalent:

- \( I \) has linear resolution (see Definition 2.7);
- \( S/I^\vee \) is Cohen–Macaulay.

The very classical notion of polarization has been often used in commutative algebra and related fields, in particular to reduce the study of homological properties of any monomial ideal to the case of squarefree monomial ideals. It was originally used by Hartshorne in his proof of the connectedness of the Hilbert scheme, see Chapter 4 of Hartshorne’s paper [16]. See Section 21 of [24] for additional information. Later it became a standard tool in commutative algebra thanks to the work of Hochster. The notion in the next definition, fundamental for the proof presented in this section, is a generalization of the classical polarization. It probably first appeared in [29] and a systematic study of it is done in the recent paper [2] for powers of graded maximal ideals. Gunnar Fløystad, the second author of that paper, showed me how to prove that \( \text{Lin}^*(I) \) has a linear resolution using this framework.

Definition 4.22. Let \( p: R' \to R \) be a surjection of finite sets with the cardinality of \( R' \) one more than that of \( R \). Let \( r_1 \) and \( r_2 \) be two distinct elements of \( R' \) such that \( p(r_1) = p(r_2) \). Denote for short \( \mathbb{K}[x_R] := \mathbb{K}[x_i \mid i \in R] \). Let \( I \) be a monomial ideal in the polynomial ring \( \mathbb{K}[x_R] \) and \( J \) a monomial ideal in \( \mathbb{K}[x_{R'}] \). We say \( J \) is a separation of \( I \) if the following conditions hold:

1. The ideal \( I \) is the image of \( J \) by the map \( \mathbb{K}[x_{R'}] \to \mathbb{K}[x_R] \) induced by \( p \).
(2) Both the variables $x_{r_1}$ and $x_{r_2}$ occur in some minimal generators of $J$ (usually in distinct generators).

(3) The variable difference $x_{r_1} - x_{r_2}$ is a non-zero-divisor in the quotient ring \( \mathbb{K}[x_R]/J \).

More generally, if $p: R' \to R$ is a surjection of finite sets and $I \subseteq \mathbb{K}[x_R]$ and $J \subseteq \mathbb{K}[x_R]$ are monomial ideals such that $J$ is obtained by a succession of separations of $I$, we also call $J$ a separation of $I$. If $J$ is squarefree and a separation of $I$, then we say that $J$ is a polarization of $I$.

### 4.2.1 Preliminary constructions

Consider the following ideal, generated by all squarefree monomials of degree $d$ inside the polynomial ring $\mathbb{K}[x_0, x_1, \ldots, x_n]$, with $d \leq n$:

\[
J := (x_0, x_1, \ldots, x_n)^d_{\text{sqf}}.
\]

The ideal $J$ is the squarefree (Veronese) $n$-th power of $(x_0, \ldots, x_n)$, and it is a well-known fact that $J$ has linear resolution and is Cohen–Macaulay. Starting from this ideal $J$, we will perform some separations and take Alexander duals, and eventually get to Lin*$(I)$. (This might perhaps be surprising, since $J$ might look “much more symmetric” than Lin$(I)$: indeed, quoting the authors of [2], “a polarization is somehow a way of breaking this symmetry, but still keeping the homological properties”.) We may write

\[
J = (x_1, \ldots, x_n)^d_{\text{sqf}} + x_0(x_1, \ldots, x_n)^{d-1}_{\text{sqf}}.
\]

The ideal $J$ can be separated to

\[
H := (x_1, \ldots, x_n)^d_{\text{sqf}}
\]

\[+ (y_1 x^i \mid i = (i_1, \ldots, i_{d-1}), 1 \leq i_1 < \cdots < i_{d-1} \leq n),\]

where $x^i = x_{i_1} \cdots x_{i_{d-1}}$ in the polynomial ring $\mathbb{K}[x_1, \ldots, x_n, y_i \mid i \in \binom{n}{d-1}]$ where the new variables $y_i$’s are indexed on all the $(d - 1)$-subsets of $\{1, \ldots, n\}$. We well denote it $\mathbb{K}[x_1, \ldots, x_n, y_i]$’s for short. Order all these $(d - 1)$-subsets as $i_1, i_2, \ldots, i_{\binom{n}{d-1}}$. Then $\mathbb{K}[x_0, x_1, \ldots, x_n]/J$ is obtained from $\mathbb{K}[x_1, \ldots, x_n, y_i]/H$ by dividing out by the variable differences

\[
y_{i_1} - y_{i_2}, y_{i_2} - y_{i_3}, \ldots, y_{(d-1)}^{n-1} - y_{(d-1)}^n,
\]

so that all $y_i$’s are identified to a single variable $x_0$.

**Lemma 4.23.** The variables differences in (6) form a regular sequence in the ring $\mathbb{K}[x_1, \ldots, x_n, y_i]/H$.

**Proof.** We start by showing that $y_{i_1} - y_{i_2}$ is a regular element in the ring $\mathbb{K}[x_1, \ldots, x_n, y_i]/H$. Equivalently, if some polynomial of $\mathbb{K}[x_1, \ldots, x_n, y_i]$ multiplies $y_{i_1} - y_{i_2}$ in $H$, then we want to show that this polynomial belongs to $H$. In fact, since $H$ is a squarefree monomial ideal, it’s enough to show that if we have $u(y_{i_1} - y_{i_2}) \in H$ for a monomial $u$, then $u \in H$. Since $H$ is monomial, $u(y_{i_1} - y_{i_2}) \in H$ implies that $uy_{i_1} \in H$ and $uy_{i_2} \in H$, so that for
some generators $g_1, g_2 \in G(H)$ we have that $g_1$ divides $uy_1$ and $g_2$ divides $uy_2$. If $g_1$ or $g_2$ divides $u$, then we’re done. If not, then it means that $g_1 = y_1 x_1^i$ and $g_2 = y_2 x_1^2$. So then $\text{lcm}(x_1^i, x_1^2)$ is a monomial in at least $d$ variables, therefore in $H$, dividing $u$, and then we are done.

To continue the proof, we notice that after taking the quotient by some differences $y_1 - y_2, y_2 - y_3, \ldots, y_{i-1} - y_i$ we get a polynomial ring in less variables and an ideal $\overline{H}$ whose generators are the same as $H$, except for identifications of some $y$-variables. Those variables are not involved in proving that $y_i - y_{i+1}$ is a regular element in the new ring, hence we can simply iterate the proof above.

Hence also $H$ has linear resolution. The Alexander dual of $H$ is $$H^\vee = (x_1, \ldots, x_n)^{n-d+2} + \left(\frac{x_1 \cdots x_n}{x_1^i} y_i \mid i \in \binom{[n]}{d-1}\right).$$

By Theorem 4.23 the quotient $\mathbb{K}[x_1, \ldots, x_n, y_i]/H^\vee$ is a Cohen–Macaulay ring, since $H$ has linear resolution. Next, we will take a further quotient of the ring $\mathbb{K}[x_1, \ldots, x_n, y_i]/H^\vee$ by a regular sequence, thus preserving Cohen–Macaulayness.

**Lemma 4.24.** The differences 

$$y_{i_1} - 1, y_{i_2} - 1, \ldots, y_{(d-1)} - 1$$

form a regular sequence in $\mathbb{K}[x_1, \ldots, x_n, y_i]/H^\vee$.

**Proof.** Similarly to the case of Lemma 4.23 we start by showing that if we have $u(y_1 - 1) \in H^\vee$ for some monomial $u$, then $u \in H^\vee$. This is clear because $H^\vee$ is a monomial ideal and $u(y_1 - 1) = uy_1 - u$. Again, as in the proof of Lemma 4.23 we can iterate this and we get that the variable differences in the statement form a regular sequence.

Let $U \subseteq \binom{[n]}{d-1}$ be a set of subsets of $[n]$ of cardinality $d - 1$. We quotient out $\mathbb{K}[x_1, \ldots, x_n, y_i]/H^\vee$ by all the differences $y_i - 1$ with $i \notin U$. These differences form a regular sequence by the previous lemma, so the quotient still is a Cohen–Macaulay ring. For each $i \in U$, let $d_i$ be a positive integer, a multiplicity in the sense of Definition 1.3 for the $(d-1)$-edge $x_i$, and replace $y_i$ by the product $$y_{i, 1} y_{i, 2} \cdots y_{i, d_i},$$ where the $y_{i, t}$’s are new variables. In this way we get from $H^\vee$ the ideal $$\left(x_1, \ldots, x_n\right)^{n-d+2} + \left(\frac{x_1 \cdots x_n}{x_1^i} y_1 y_{i, 2} \cdots y_{i, d_i} \mid i \in U\right) + \left(\frac{x_1 \cdots x_n}{x_1^i} \mid i \notin U\right)$$ and the quoting by this ideal is still Cohen–Macaulay. (By the way, notice that the generators in the third summand— if there are any —, being of degree $n - d + 1$, divide some generators in the first summand, which have degree $n - d + 2$.) The Alexander dual of this ideal is $$K := \left(x_1, \ldots, x_n\right)^d + \left(x^t y_{i, t} \mid t = 1, \ldots, d_i, i \in \binom{[n]}{d-1}\right).$$

Then, again by Theorem 4.23 the ideal $K$ has a linear resolution.
4.2.2 Conclusion of the proof

Take a \(d\)-uniform hypergraph on \(\{1, \ldots, n\}\) with edge set \(E \subset \binom{[n]}{d}\). Consider the ideal \(\Lin^*(I)\) for \(I \subset k[x_1, \ldots, x_n]\) associated to \(E\), so that

\[
\Lin^*(I) = \left( x^j \mid j \in \binom{[n]}{d} \right) + \left( \frac{x^j y_j}{x_j} \mid j \in E, j \in j \right).
\]

For each \(j \in E\) we may separate the variable \(y_j\) and get monomials \(x^j y_j\), one for each \(j \in E\). This gives an ideal \(K'\) which is a separation of \(\Lin^*(I)\). For each \(i\) of cardinality \(d - 1\) we have monomials \(x^i y_i \cup \{j\}\), where each \(i \cup \{j\}\) is in \(E\). The ideal \(K'\) here constructed identifies with the ideal \(K\) above, where \(d_i\) is the number of sets \(i \cup \{j\}\) contained in \(E\). Hence \(K'\), and so \(\Lin^*(I)\), has linear resolution, and by the construction above its graded Betti numbers only depend on the multiplicities \(d_i\).

4.3 Combinatorial interpretations by means of hypergraphs

We already mentioned the connection between commutative algebra and combinatorics provided by the Stanley–Reisner correspondence. Now we change point of view and for the rest of the section we focus on another way of linking combinatorics and commutative algebra: instead of simplicial complexes, we talk about hypergraphs.

4.3.1 Background on hypergraphs

A hypergraph is a pair \(H = (V, E)\) with \(V\) a finite set and \(E \subseteq \binom{V}{d} \setminus \{\emptyset\}\) a set of nonempty subsets of \(V\). We call \(V\) the set of vertices and denote it also by \(V(H)\) and we call \(E\) the set of edges (regardless of the cardinality) and we denote it also by \(E(H)\). We say a hypergraph \((V, E)\) is \(d\)-uniform if \(E \subseteq \binom{V}{d}\), that is, if all the edges have the same cardinality \(d\). Two vertices \(v \neq w\) in \(H\) are neighbours if there is an edge \(E\) such that \(v, w \in E\). For any vertex \(v\), the neighbourhood of \(v\) is

\[
N(v) := \{ w \in X \mid w \text{ is a neighbour of } v \}.
\]

Notation 4.25. If \(H = (V, E)\) is a hypergraph and \(W \subseteq V\) is a subset, then the induced hypergraph on \(W\), denoted \(H_W\), is the subhypergraph of \(H\) whose edge set is \(\{E \in E \mid E \subseteq W\}\).

4.3.2 Hypergraphs with linear resolution

To a hypergraph \(H = (V, E)\) with \(#V = n\), we associate a squarefree monomial ideal in \(K[x_1, \ldots, x_n]\) called the edge ideal of \(H\),

\[
I_H := \left( \prod_{i \in E} x_i \mid E \in E \right).
\]

This provides a bijection between hypergraphs with \(V = \{1, \ldots, n\}\) and squarefree monomial ideals in \(K[x_1, \ldots, x_n]\). In their paper [15], Huy Tai Hà and
Adam Van Tuyl study the graded Betti numbers of $I_H$ and give a characterization, under certain assumptions on the hypergraph $H$, of the ideals $I_H$ that have a linear resolution. A special case of this characterization is Fröberg’s Theorem [12]. The following definitions and results are taken from Sections 2, 4 and 5 of [15]. We skip some assumptions which in our case are automatically satisfied because all our hypergraphs are $d$-uniform.

A chain of length $n$ in $H$ is a finite sequence

$$(E_0, v_1, E_1, v_2, E_2, v_3, \ldots, E_{n-1}, v_n, E_n)$$

where

1. $v_1, \ldots, v_n$ are pairwise distinct vertices of $H$;
2. $E_0, \ldots, E_n$ are pairwise distinct edges of $H$;
3. $v_k \in E_{k-1} \cap E_k$ for all $k \in \{1, \ldots, n - 1\}$.

Sometimes such a chain is denoted by $(E_0, \ldots, E_n)$. If $E$ and $E'$ are two edges, then $E$ and $E'$ are connected if, for some $n \in \mathbb{N}$, there is a chain $(E_0, \ldots, E_n)$ where $E = E_0$ and $E' = E_n$. If $H$ is $d$-uniform, the chain connecting $E$ to $E'$ is a proper chain if $|E_i \cap E_{i+1}| = d - 1$ for all $i \in \{0, \ldots, n - 1\}$. The (proper) chain is a (proper) irredundant chain of length $n$ if no proper subsequence is a (proper) chain from $E$ to $E'$. We define the distance between $E$ and $E'$ to be

$$\text{dist}_H(E, E') := \min\{n \mid (E_0, \ldots, E_n) \text{ is a proper irredundant chain}\}.$$

If no proper irredundant chain exists, we set $\text{dist}_H(E, E') := \infty$. A $d$-uniform hypergraph $H$ is said to be properly-connected if for any two edges $E$ and $E'$ with the property that $E \cap E' \neq \emptyset$, one has

$$\text{dist}_H(E, E') = d - |E \cap E'|.$$

A $d$-uniform properly-connected hypergraph $H = (V, \mathcal{E})$ is said to be triangulated if for every non-empty subset $W \subseteq V$, the induced subhypergraph $H_W$ contains a vertex $v \in W \subseteq V$ such that the induced hypergraph of $H_W$ on $N(v) \cup \{v\}$ is the $d$-complete hypergraph of order $|N(v)| + 1$. The edge diameter of a $d$-uniform properly-connected hypergraph $H$ is

$$\text{diam}(H) := \max\{\text{dist}_H(E, E') \mid E, E' \in \mathcal{E}\},$$

where the diameter is infinite if there exist two edges not connected by any proper chain.

Among other interesting results concerning Betti numbers and resolutions, Huy Tai Hà and Adam Van Tuyl proved the following characterization.

**Lemma 4.26** (Corollary 7.6 of [15]). Let $H$ be a $d$-uniform, properly-connected, triangulated hypergraph. Then the following are equivalent:

- $I_H$ has a linear resolution;
- $I_H$ as linear first syzygies;
- $\text{diam}(H) \leq d$. 

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Remark 4.27. It’s easy to see that the hypergraphs associated to \( \text{Lin}^*(I) \) and \( \text{Lin}(I) \) are properly-connected. They are also triangulated: given a subset of vertices \( W \), if \( W \) contains no variables \( y_j \), then pick any \( x_i \) as the vertex \( v \) in the definition of triangulated hypergraph. If \( W \) contains some \( y_j \), then pick any of those as \( v \). So then the hypergraphs associated to \( \text{Lin}^*(I) \) and \( \text{Lin}(I) \) satisfy the assumptions in Lemma 4.26. Therefore, showing that the diameter is at most \( d \) or showing that \( \text{Lin}^*(I) \) and \( \text{Lin}(I) \) have linear syzygies would be alternative ways to prove that \( \text{Lin}^*(I) \) and \( \text{Lin}(I) \) have a linear resolution.

5 Linearization for all monomial ideals

Until this point, the linearization construction has been defined only for monomial ideals which are equigenerated, that is, generated in a single degree. In order to generalize the construction to arbitrary monomial ideals, we first introduce what we call equification of a monomial ideal. This construction takes an arbitrary monomial ideal and returns an equigenerated monomial ideal. It bears some resemblance with a standard construction that goes by the name of homogenization—which gives a homogeneous ideal starting from an arbitrary ideal—and this resemblance is made exact in Remark 5.4.

As a side note, observe that the words “equification” and “equify” already exist in English, as technical terms in trading and economics. In this paper there is no relation at all to those meanings. The word “equification” was suggested to me in analogy to “sheafification”, which is a well-known process to make a presheaf into a sheaf.

5.1 Equification of a monomial ideal

Let us assume that \( \mathbb{K} \) is a field of characteristic 0.

Definition 5.1. Let \( I \) be a monomial ideal in \( S = \mathbb{K}[x_1, \ldots, x_n] \), with minimal system of monomial generators \( G(I) = \{f_1, \ldots, f_m\} \). Denote \( d_j := \deg(f_j) \) for all \( j \) and \( d := \max\{d_j \mid j = 1, \ldots, m\} \). We define the equification of \( I \) as

\[
I^{\text{eq}} := (f_1 z^{d-d_1}, f_2 z^{d-d_2}, \ldots, f_m z^{d-d_m})
\]

in the polynomial ring \( S[z] = \mathbb{K}[x_1, \ldots, x_n, z] \) with one extra variable \( z \).

Starting from some random system of generators of \( I \) would affect very much the definition of equification. For instance, if we take a redundant generator of \( I \) of high degree, the equification would be defined in that degree, much higher than if we only used the minimal generators. So this is why we take the unique minimal monomial generators in the definition.

Lemma 5.2. The generators of \( I^{\text{eq}} \) in the definition are minimal.

Proof. Assume \( f_i z^{d-d_i} \) divides \( f_j z^{d-d_j} \). Then \( f_i \) divides \( f_j \) and also \( d - d_i \leq d - d_j \). Since \( f_i z^{d-d_i} \) and \( f_j z^{d-d_j} \) have the same degree, we have \( i = j \).

Remark 5.3 (recovering \( I \) from \( I^{\text{eq}} \)). By setting \( z = 1 \), we recover \( I \). Of course this requires \( z \) to be somehow distinguishable from the rest of the variables. Consider for instance \( I = (x^3, y^2) \) in the polynomial ring \( \mathbb{K}[x, y] \). Then \( I^{\text{eq}} = (x^3 z, y^2 z) \) and \( I = (x^3, y^2) \) in the polynomial ring \( \mathbb{K}[x, y, z] \).
$(x^3, y^2z)$ in $\mathbb{K}[x, y, z]$. Clearly $x$ could not possibly be the “equifying” variable, but $y$ could be, so actually the same ideal $I^\text{eq}$ could be obtained by equifying the ideal $(x^3, z) \subseteq \mathbb{K}[x, z]$ by introducing a new variable $y$. For this reason in this section we always assume $z$ to be a well-distinguished variable.

**Remark 5.4.** The equification construction is somewhat similar to that of homogenization (see for instance Section 3.2.1 of [18]): for a polynomial $g \in S = \mathbb{K}[x_1, \ldots, x_n]$ we may write uniquely $g = g_0 + g_1 + \cdots + g_d$, where each $g_j$ is homogeneous of degree $j$ and $g_d \neq 0$. Then the homogenization of $g$ is the polynomial

$$g^\text{hom} := \sum_{j=0}^d g_j z^{d-j} \in S[z].$$

The similarity with the equification is made precise by taking the elements in the support: if we have $G(I) = \{f_1, \ldots, f_m\}$, then

$$I^\text{eq} = \{u \mid u \in \text{Supp}((f_1 + \cdots + f_m)^\text{hom})\}.$$ Here by *support* of a polynomial we mean the set of its monomials with nonzero coefficient.

**Remark 5.5.** Unlike the linearization, which is a functor as observed in Section 3.1.1, the equification map $(\cdot)^\text{eq}$ is not a functor. Take for instance, inside $\mathbb{K}[x, y]$, the ideals

$$I = (x^2, xy^3) \subseteq J = (x^2, xy^2, y^5).$$

For their equifications

$$I^\text{eq} = (x^2 z^2, xy^3) \quad \text{and} \quad J^\text{eq} = (x^2 z^3, xy^2 z^2, y^5)$$

we have $I^\text{eq} \not\subseteq J^\text{eq}$ and $J^\text{eq} \not\subseteq I^\text{eq}$.

**Remark 5.6.** Recall that a monomial ideal is prime if and only if it is generated by a bunch of variables. And also recall that a monomial ideal is radical if and only if it is squarefree. Then we have the following two equivalences:

$I^\text{eq}$ is prime $\iff$ $I$ is prime

(in which case $I^\text{eq} = I$) and, perhaps more interestingly,

$I^\text{eq}$ is radical $\iff$ $I$ is radical and generated in at most two adjacent degrees.

They are both clear. The second equivalence just means this: the original ideal can either be generated in a single degree, and in this case $I^\text{eq} = I$, or it can have generators of two distinct degrees, but they have to be in adjacent degrees so that $z$ appears with at most exponent 1 in the generators of $I^\text{eq}$.

**Remark 5.7.** A way to illustrate pictorially what happens with $I^\text{eq}$ is as follows. This works for $n = 2$. Think of the monomials in $\mathbb{K}[x, y]$ as lattice points in the plane with axes $x$ and $y$. For all $d$, consider the line $x + y = d$, which goes through all monomials of degree $d$ in $x$ and $y$. Then, what $(\cdot)^\text{eq}$ does is that we add a new axis $z$ that comes out of the plane, we take the generators of $I$ of degree $d'$ (which are the ones lying on the line $x + y = d'$) and we bring them up to level $d - d'$. So, in particular, the ones of degree $d$ stay on the plane. See Figure 2 for an example of this.
Figure 2: We draw $I = (x^3, xy, y^4)$ on the left and $I^{eq} = (x^3z, xyz^2, y^4)$ on the right. With a slight abuse of notation, the generator $x^3$ lies on the line $x + y = 3$, $xy$ on the line $x + y = 2$ and $y^4$ on the line $x + y = 4$, all of them on the plane $z = 0$. Those three parallel lines are all dashed, in the left picture. The generators of $I^{eq}$ all lie on the plane $x + y + z = 4$, in green. See Remark 5.7.

5.1.1 Betti numbers of $I^{eq}$

Unfortunately we are not able to give a complete, satisfactory description of the homological invariants of $I^{eq}$, but we do provide some partial results: Proposition 5.14 and Proposition 5.16. Before stating the results in the end of the section, we discuss some examples and issues.

**Example 5.8.** Consider, respectively in $S := \mathbb{K}[x_1, x_2, x_3]$ and in $T := S[z]$, the ideals

$I = (x_1^2, x_1x_2x_3^2, x_2^3x_3^2), \quad I^{eq} = (x_1^2z^3, x_1x_2^2x_3^2, x_2^3x_3^2)$.

Then one has

\[
\begin{align*}
\beta^S(I) &= \begin{pmatrix} 0 & 1 \\ 2 & 1 \\ 3 & - \\ 4 & - \\ 5 & 2 & 2 \end{pmatrix} \\
\beta^T(I^{eq}) &= \begin{pmatrix} 0 & 1 \\ 5 & 3 & 1 \\ 6 & - & - \\ 7 & - & - \\ 8 & - & 1 \end{pmatrix}
\end{align*}
\]

and the minimal resolutions look like

\[
0 \rightarrow S^2 \begin{pmatrix} 0 & x_2^2x_3^2 \\ x_2 & -x_1 \\ -x_1 & 0 \end{pmatrix} \rightarrow S^3 \begin{pmatrix} x_1^2 ; x_1x_2^2x_3 ; x_2^3x_3^2 \end{pmatrix} \rightarrow I \rightarrow 0
\]
and

\[
0 \to T^2 \left( \begin{array}{cc}
0 & x_2^2 x_3^2 \\
x_2 & -x_1 x_3^3 \\
-x_1 & 0
\end{array} \right) \to T^3 \langle x_1^2 x_3, x_1 x_2 x_3^2, x_2^2 x_3^3 \rangle, \quad I^{eq} \to 0.
\]

In this example we can notice that the total Betti numbers of \( I \) and \( I^{eq} \) are equal. Of course the grading cannot be the same, because the very first column of the Betti table records the degrees of the generators, and the generators of \( I^{eq} \) all have the same degree, which is the highest degree of the minimal generators of \( I \). But at least the 0-th total Betti number of \( I \) and that of \( I^{eq} \) will be the same, due to Lemma 5.2. We also notice, in this example, that the maps in the two resolutions are quite similar. They only differ for the presence of some powers of \( z \). Unfortunately this is not the case in general, as Example 5.12 will show.

**Notation 5.9.** Let us repeat the notation: let \( I \) be a monomial ideal in \( S = \mathbb{K}[x_1, \ldots, x_n] \), with \( G(I) = \{f_1, \ldots, f_m\} \). Setting \( d_j := \deg(f_j) \) for all \( j \), we denote \( d := \max\{d_j \mid j = 1, \ldots, m\} \) and \( \delta := \min\{d_j \mid j = 1, \ldots, m\} \). The equification of \( I \) is

\[
I^{eq} := (f_1 z^{d-d_1}, f_2 z^{d-d_2}, \ldots, f_m z^{d-d_m}) \subseteq S[z].
\]

**Remark 5.10.** The first step of a minimal graded free resolution is easy, and in our case we have

\[
\varepsilon: \bigoplus_{j=1}^m S(-d_j) \to I, \quad e_j \mapsto f_j,
\]

and the corresponding surjective map for \( I^{eq} \) is easily given as

\[
\zeta: \bigoplus_{j=1}^m T(-d) \to I^{eq}, \quad \eta_j \mapsto f_j z^{d-d_j}.
\]

Let’s continue by considering the syzygy module \( \text{Syz}(I) := \ker(\varepsilon) \). A syzygy of \( I \) is \((p_1, p_2, \ldots, p_m) \in S^m \) such that \( p_1 f_1 + p_2 f_2 + \cdots + p_m f_m = 0 \). Hence, by writing \( d - \delta = d_j - \delta + d - d_j \), we get

\[
0 = (p_1 f_1 + p_2 f_2 + \cdots + p_m f_m) z^{d-\delta} = p_1 z^{d_1-\delta} (f_1 z^{d_1}) + p_2 z^{d_2-\delta} (f_2 z^{d_2}) + \cdots + p_m z^{d_m-\delta} (f_m z^{d_m}),
\]

so that \((p_1 z^{d_1-\delta}, p_2 z^{d_2-\delta}, \ldots, p_m z^{d_m-\delta})\) is a syzygy for \( I^{eq} \). The problem is that the syzygies of \( I^{eq} \) obtained in this way do not generate in general all of \( \text{Syz}(I^{eq}) \), as showed in Example 5.12.

**Notation 5.11.** Denote by \( e_1, \ldots, e_m \) the standard basis of \( S^m \). For \( I \) with \( G(I) = \{f_1, \ldots, f_m\} \), we have syzygies \( f_j e_i - f_i e_j \). These clearly map to zero, but they can be refined as

\[
\sigma_{ij} := \frac{f_j}{\gcd(f_i, f_j)} e_i - \frac{f_i}{\gcd(f_i, f_j)} e_j = \frac{\lcm(f_i, f_j)}{f_i} e_i - \frac{\lcm(f_i, f_j)}{f_j} e_j,
\]

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called the **reduced trivial syzygies** of \( I \). Schreyer’s well-known theorem states that reduced trivial syzygies generate all of \( \text{Syz}(I) \). (See Theorem 15.10 of [1].)

**Example 5.12.** Consider now \( S = \mathbb{K}[x_1, \ldots, x_4] \), \( T = \mathbb{K}[z] \), and

\[
I = (x_1x_2x_4, \ x_1^2x_2^2x_3, \ x_3^3x^2_1), \quad I^{\text{eq}} = (x_1x_2x_4z^3, \ x_1^2x_2^2x_3z, \ x_3^3x^2_1).
\]

Then we get the Betti tables

\[
\beta^S(I) = \begin{pmatrix} 0 & 1 \\ 3 & 1 & - \\ 4 & - & - \\ 5 & 1 & 1 \\ 6 & 1 & - \\ 7 & - & 1 \\ 3 & 2 & \end{pmatrix} \quad \text{and} \quad \beta^T(I^{\text{eq}}) = \begin{pmatrix} 0 & 1 & 2 \\ 6 & 3 & - & - \\ 7 & - & - & - \\ 8 & - & 1 & - \\ 9 & - & - & - \\ 10 & - & 2 & - \\ 11 & - & - & 1 \\ 3 & 3 & 1 & \end{pmatrix}.
\]

As already remarked, we always have \( \beta^S_i(I) = \beta^T_i(I^{\text{eq}}) \), but in this example we have strict inequalities \( \beta^S_i(I) < \beta^T_i(I^{\text{eq}}) \) for \( i = 1, 2 \). The matrices corresponding to the first syzygies of \( I \) and \( I^{\text{eq}} \) are respectively

\[
\begin{pmatrix} x_1x_2x_3 & x_3^2x_1^2 \\ -x_4 & 0 \\ 0 & -x_1x_2 \\ \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x_1x_2x_3 & x_3^2x_1^2 & 0 \\ -x_1z^2 & 0 & x_3^2x_1^2 \\ 0 & -x_1x_2z^3 & -x_1^2x_2^2z \\ \end{pmatrix}.
\]

For \( I \) the reduced trivial syzygy \( \sigma_{23} \) is redundant, because we have \( \sigma_{23} = -x_3^2x_2^2\sigma_{12} + x_1x_2\sigma_{13} \). But for \( I^{\text{eq}} \) the corresponding syzygy, showing as third column in the matrix, is not redundant.

The following lemma is a classical result. Because its proof is beautiful and easy also in greater generality than for our purpose, we include a sketch of it for sake of completeness. Recall that, for any ring \( R \) and any \( R \)-module \( M \), an element \( a \in M \) is **\( M \)-regular** if the only \( m \in M \) such that \( am = 0 \) is \( m = 0 \). So in particular being \( R \)-regular is the same as being a non-zero-divisor of \( R \).

**Lemma 5.13.** Let \( R \) be a polynomial over a field ring and let \( M \) be a finitely generated \( R \)-module. Let \( a \in R \) be \( R \)-regular and \( M \)-regular. Let \( F \) be a free resolution of \( M \) over \( R \). Then \( F \otimes_R R/(a) \) is a free resolution of \( M/(a) \) over \( R/(a) \).

**Proof.** We may write the modules in the resolution \( F \) as \( F_i = R^{n_i} \) for some \( n_i \in \mathbb{N} \). By the distributive property of the tensor product we have

\[
\bigoplus_{j=1}^{n_i} R \otimes_R R/(a) = \bigoplus_{j=1}^{n_i} (R \otimes_R R/(a)) = \bigoplus_{j=1}^{n_i} R/(a).
\]

Moreover, since \( - \otimes_R R/(a) \) is a functor, \( F \otimes_R R/(a) \) is a complex. To conclude, we want to see that it is also exact. We have \( H_1(F \otimes_R R/(a)) \cong \text{Tor}_1^R(M, R/(a)) \) and we use the commutativity of \( \text{Tor} \). That is, by the \( R \)-regularity of \( a \), we know that

\[
G : \quad 0 \rightarrow R \xrightarrow{a} R
\]

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is a free resolution of $R/(a)$ over $R$, so that $\text{Tor}_i^R(M, R/(a)) \cong H_i(M \otimes_R G)$. But this homology is 0 for $i > 0$ because

$$M \otimes_R G : 0 \longrightarrow M \longrightarrow M$$

is exact thanks to the $M$-regularity of $a$. \hfill \Box

**Proposition 5.14.** For the total Betti numbers, we have $\beta_0^S(I) = \beta_0^T(I^{eq})$ and $\beta_i^S(I) \leq \beta_i^T(I^{eq})$ for all $i > 0$.

**Proof.** The equality $\beta_0^S(I) = \beta_0^T(I^{eq})$ is the content of Lemma 5.2. For the rest of the proof, we specify Lemma 5.13 to our setting:

$$R = T := \mathbb{K}[x_1, \ldots, x_n, z], \quad M = I^{eq}, \quad a = z - 1.$$  

We observe that $z - 1$ is clearly $T$-regular and hence $I^{eq}$-regular. Moreover, $T/(z - 1) = S$ and $I^{eq}/(z - 1)I^{eq} = I$. So, let now

$$F : 0 \longrightarrow T^{\beta_0} \longrightarrow T^{\beta_{p-1}} \longrightarrow \ldots \longrightarrow T^{\beta_0}$$

be the minimal graded free resolution of $I^{eq}$, where $\beta_i = \beta_i^T(I^{eq})$. Then we get from Lemma 5.13 we get that

$$F \otimes_T T/(z - 1) : 0 \longrightarrow S^{\beta_0} \longrightarrow S^{\beta_{p-1}} \longrightarrow \ldots \longrightarrow S^{\beta_0}$$

is a free resolution of $I$, possibly not minimal. A well-known result (see for instance Theorem 7.5 of [24]) states that any resolution contains the minimal one as a direct summand, and therefore we get the desired inequalities. \hfill \Box

Recall now the notation for reduced trivial syzygies in Notation 5.11

**Lemma 5.15.** The reduced trivial syzygy $\sigma_{ij}$ is redundant if and only if there exists $k \notin \{i, j\}$ such that $\text{lcm}(f_k, f_i)$ and $\text{lcm}(f_k, f_j)$ divide $\text{lcm}(f_i, f_j)$.

**Proof.** ($\Leftarrow$) This is a special case, with a different notation, of Proposition 8 in Section 2.9 of [8]. We include a short proof for sake of completeness: by assumption, we have the monomials $u := \text{lcm}(f_i, f_j)/\text{lcm}(f_k, f_i)$ and $v := \text{lcm}(f_i, f_j)/\text{lcm}(f_k, f_j)$, so that

$$-u\sigma_k + v\sigma_j = \frac{\text{lcm}(f_k, f_i)}{\text{lcm}(f_k, f_j)} \left( \frac{\text{lcm}(f_k, f_i)}{f_k} e_k - \frac{\text{lcm}(f_k, f_i)}{f_i} e_i \right) + \frac{\text{lcm}(f_k, f_j)}{\text{lcm}(f_k, f_j)} \left( \frac{\text{lcm}(f_k, f_j)}{f_k} e_k - \frac{\text{lcm}(f_k, f_j)}{f_j} e_j \right) = \frac{\text{lcm}(f_i, f_j)}{f_i} e_i - \frac{\text{lcm}(f_i, f_j)}{f_j} e_j = \sigma_{ij}.$$  

($\Rightarrow$) Assuming that $\sigma_{ij}$ is redundant, we may write

$$\sigma_{ij} = \sum_{\{k, \ell\} \neq \{i, j\}} p_{k\ell} \sigma_{k\ell}$$

for some $p_{k\ell}$. In particular the coefficients of $e_i$ and $e_j$ on both sides are equal, so that isolating all terms which involve $i$, and respectively $j$, we get

$$\frac{\text{lcm}(f_i, f_j)}{f_i} = p_{i\ell_1} \frac{\text{lcm}(f_i, f_{\ell_1})}{f_i} + \cdots + p_{i\ell_t} \frac{\text{lcm}(f_i, f_{\ell_t})}{f_i}$$

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and a similar expression for \( \text{lcm}(f_i, f_j)/f_j \). If we multiply by \( f_i \), and respectively by \( f_j \), and observe that the ideals generated by the least common multiples on the right-hand side are monomial ideals, this means that there exist \( \ell \) and \( k \) such that \( \text{lcm}(f_i, f_\ell) \) and \( \text{lcm}(f_k, f_j) \) divide \( \text{lcm}(f_i, f_j) \). In particular \( f_k \) divides \( \text{lcm}(f_i, f_j) \), so that indeed both \( \text{lcm}(f_k, f_j) \) and \( \text{lcm}(f_i, f_k) \) divide \( \text{lcm}(f_i, f_j) \).

Denote
\[
\sigma_{ij}^\text{eq} := \frac{\text{lcm}(g_i, g_j)}{g_i} e_i - \frac{\text{lcm}(g_i, g_j)}{g_j} e_j
\]
\[
= \frac{\text{lcm}(f_i, f_j) z^{\max\{0, d_i - d_j\}}}{f_i} e_i - \frac{\text{lcm}(f_i, f_j) z^{\max\{0, d_j - d_i\}}}{f_j} e_j
\]
the reduced trivial syzygies for \( I^\text{eq} \), where \( g_i = f_i z^{d_i} \).

**Proposition 5.16.** The reduced trivial syzygy \( \sigma_{ij}^\text{eq} \) is redundant if and only if there exists \( k \neq \{i, j\} \) such that \( \text{lcm}(f_k, f_i) \) and \( \text{lcm}(f_k, f_j) \) divide \( \text{lcm}(f_i, f_j) \) and \( \min\{d_i, d_j\} \leq d_k \).

**Proof.** This follows by Lemma 5.15. \( \text{lcm}(g_i, g_j) = \text{lcm}(f_i, f_j) z^{\max\{d - d_i, d - d_j\}} \) is divided by \( \text{lcm}(g_i, g_k) \) and \( \text{lcm}(g_k, g_j) \) if and only if \( \text{lcm}(f_i, f_k) \) and \( \text{lcm}(f_k, f_j) \) and additionally \( \max\{d - d_i, d - d_j\} \geq d - d_k \), which is equivalent to \( \min\{d_i, d_j\} \leq d_k \).

### 5.1.2 The lcm-lattice of \( I^\text{eq} \)

We recall the notion of lcm-lattice of a monomial ideal and we compare that of \( I \) and of \( I^\text{eq} \). This tool provides a way to construct a resolution of \( I^\text{eq} \), albeit not necessarily minimal. The background for this section is taken from Section 58 of [24]. We start by recalling that a lattice is a poset in which every pair of elements has a least upper bound and a greatest lower bound.

**Definition 5.17.** Let \( I \) be a monomial ideal with \( G(I) = \{f_1, \ldots, f_m\} \). The **lcm-lattice** of \( I \) is the lattice that has as elements the least common multiples of the subsets of \( \{f_1, \ldots, f_m\} \), ordered by divisibility. We denote the lcm-lattice of \( J \) by \( L_I \).

Notice in particular that the bottom element in \( L_I \) is 1, which is the least common multiple of the empty set. For more details on the notion of lcm-lattice, see Section 58 of [24].

**Example 5.18.** Consider the ideal \( I \) in Example 5.12
\[
I = (x_1 x_2 x_4, x_1^2 x_2^2 x_3, x_3^3 x_4^3) \subseteq S = \mathbb{K}[x_1, \ldots, x_4].
\]

The lcm-lattice \( L_I \) is depicted in Figure 3.

**Remark 5.19.** The lcm-lattice \( L_I \) of \( I \) is isomorphic to a sublattice of the lcm-lattice \( L_{I^\text{eq}} \) of \( I^\text{eq} \). To see this, we can set \( z = 1 \) in \( L_{I^\text{eq}} \) and observe that if we multiply any two monomials \( u \) and \( v \) in the variables \( x_1, \ldots, x_n \) by some powers of \( z \), then we simply have
\[
\text{lcm}(uz^a, vz^b) = \text{lcm}(u, v) z^{\max\{a, b\}}.
\]

The difference, as illustrated in the next example, is that we might find some redundancies after setting \( z = 1 \).
Figure 3: The lcm-lattice of $I = (x_1^2x_4, x_2^2x_3, x_3^3x_4)$ in Example 5.18.

Figure 4: Comparison of the lcm-lattices of $I = (x_1^2, x_2^2x_3^2, x_2^2x_3^2, x_2x_3^2), I^\text{eq}$, on the left, and $I^\text{eq}$, on the right. In this case we have $L_I \cong L_{I^\text{eq}}$. See Example 5.20.

**Example 5.20.** We compare the lcm-lattices of $I$ and $I^\text{eq}$: if $I$ is the ideal in Example 5.8 then $L_I \cong L_{I^\text{eq}}$. The two lattices are drawn in Figure 4. On the other hand, if $I$ is the ideal in Example 5.18 then the isomorphic copy of $L_I$ inside $L_{I^\text{eq}}$ is strictly contained inside $L_{I^\text{eq}}$. The two lattices are drawn in Figure 5. In $L_I$, the dashed part is a redundancy that we get by setting $z = 1$ in $L_{I^\text{eq}}$; compare it with the drawing in Figure 3.

The material in this last part of the section is taken from Section 62 of [24], which in turn is taken from [21] and [23].

**Definition 5.21.** Given a monomial ideal $J$, its lcm-lattice $L_J$ and some monomials $u_1, \ldots, u_s$, a map $h : L_J \setminus \{1\} \to \{u_1, \ldots, u_s\}$ is called a rooting map for $J$ if the following conditions hold:

1. For each $v \in L_J \setminus \{1\}$, $h(v)$ divides $v$;
2. If $v, v' \in L_J \setminus \{1\}$ are such that $h(v)$ divides $v'$ and $v'$ divides $v$, then $h(v) = h(v')$.

For each nonempty subset $U \subseteq \{u_1, \ldots, u_s\}$, set $h(U) := h(\text{lcm}(u \mid u \in U))$. The subset $U$ is **unbroken** if $h(U) \in U$, and $U$ is **rooted** if all nonempty
Figure 5: Comparison of the lcm-lattices of $I = (x_1x_2x_4, x_1^2x_2^3x_3, x_3^3x_4^3)$, on the left, and $I^{eq}$, on the right. See Example 5.20. Compare the lcm-lattice on the left with the one drawn in Figure 3.

Proposition 5.22 (Novik, [23]). If $J \subseteq S$ is a monomial ideal and $h$ is a rooting map for $J$, then $RC_h$ supports a simplicial free resolution of $S/J$.

For a proof and additional information, we refer to Theorem 6.0.2 of [24].

Remark 5.23 (a resolution for $I^{eq}$). Let $I \subseteq \mathbb{K}[x_1, \ldots, x_n]$ be a monomial ideal and consider the lcm-lattice of the equification $I^{eq}$. For a monomial $u \in \mathbb{K}[x_1, \ldots, x_n, z]$, denote by $\overline{u}$ the corresponding monomial in $\mathbb{K}[x_1, \ldots, x_n]$ where we set $z = 1$. The map $h : L_{I^{eq}} \setminus \{1\} \to L_I \setminus \{1\}, \quad u \mapsto \overline{u}$
is a rooting map for $I^{eq}$. Indeed, condition (1) in Definition 5.21 is satisfied because $\overline{u}$ divides $u$. As for condition (2), assume that $v$ and $v'$ in $L_{I^{eq}} \setminus \{1\}$ are such that $h(v)$ divides $v'$ and $v'$ divides $v$. These last two things, for our specific $h$, imply that $h(v)$ divides $h(v')$ and $h(v')$ divides $h(v)$, so that $h(v) = h(v')$. So $h$ constructed here is a rooting map and we can apply Proposition 5.22 to $h$.

5.2 Linearization of arbitrary monomial ideals

In this section we give a generalization of the linearization construction that works for any monomial ideal, not necessarily equigenerated.

Definition 5.24. The **linearization** of a monomial ideal $I$ is defined as

$$\text{Lin}(I) := \text{Lin}(I^{eq}),$$

where Lin on the right-hand side is the one introduced in Definition 3.2 for equigenerated monomial ideals.
Notice that, in case $I$ is equigenerated, then we get $I^{eq} = I$ and the only difference is that we consider the ideal $\text{Lin}(I)$ in a polynomial ring with one more variable. This does not affect $\text{Lin}$ in any sensible way, especially from the homological point of view. It does however interfere with $\text{Lin}^*$, and therefore we focus only on $\text{Lin}$ in this section.

**Remark 5.25.** From $\text{Lin}(I^{eq})$, as observed in Remark 5.24 one can recover $I^{eq}$. And, as discussed in Remark 5.23 from $I^{eq} \subset S[z]$ we can find $I \subset S$ simply by setting $z = 1$.

To conclude the section we examine one last matter: the presence of the variable $z$. Since the linearization in Definition 3.32 did not involve any variable $z$, is it possible to define the linearization for any monomial ideal without going through the equification? The problem is that we don’t really know how to deal with the complete part of. For what concerns the last part, one possibility is just to define it as it is defined for the case of an equigenerated ideal. The problem is that this last part alone does not seem to have any nice properties. Not even in the equigenerated case, in fact, as already discussed in Section ???. Or, assume that we do construct $\text{Lin}$ for arbitrary monomial ideals as in Definition 5.24. In that case, in analogy to setting $z = 1$ in order to get $I$ back from $I^{eq}$, what happens to $\text{Lin}(I^{eq})$ if we set $z = 1$? The answer is discussed below.

**Notation 5.26.** Let $I \subseteq S := \mathbb{K}[x_1, \ldots, x_n]$ be a monomial ideal with minimal system of monomial generators $G(I) = \{f_1, \ldots, f_m\}$. Denote $d_j := \deg(f_j)$ for all $j$ and also let $d := \max\{d_j \mid j = 1, \ldots, m\}$ and $\delta := \min\{d_j \mid j = 1, \ldots, m\}$. For each $i \in \{1, \ldots, n\}$, let $M_i$ be the highest exponent of the variable $x_i$ occurring in $G(I)$. Denote $v := (M_1, \ldots, M_n)$ the vector of highest exponents. Lastly, for each $j \in \{1, \ldots, m\}$, denote $g_j := f_j z^{d-d_j}$ the generators of $I^{eq}$.

Assume that not all generators of $I$ are in degree $d$, otherwise the equification would be the ideal itself. Namely, assume that $\delta < d$. The vector of highest exponents for $I^{eq}$ is

$$v^{eq} = (M_1, \ldots, M_n, d - \delta).$$

So we get

$$\text{Lin}(I^{eq}) = (\text{monomials in } x_1, \ldots, x_n, z \text{ of degree } d, \text{ with vector below } v^{eq})$$

$$+ \left(\frac{g_j y_j}{x_k} \mid j = 1, \ldots, m, \text{ where } x_k \text{ divides } g_j\right)$$

$$+ \left(\frac{g_j y_j}{z} \mid j = 1, \ldots, m, \text{ where } z \text{ divides } g_j\right).$$

For the last part, we could decide not to treat $z$ as one of the “normal” variables $x_i$’s, and it would not make a difference if we are going to set $z = 1$ afterwards. That is, we could decide not to put the generators of the form $\frac{f_j}{x_k} y_j$ in the last part of $\text{Lin}(I^{eq})$. Indeed, suppose $f_j = x_1^{a_1} \ldots x_n^{a_n} z^{d-d_j}$, where $d - d_j > 0$. Then we would get $\frac{f_j}{x_k} y_j$ among the generators of the last part. But when we take $z = 1$, we get $f_j y_j$, which is a redundant generator as we already have $\frac{f_j}{x_k} y_j$ for some $k$, which divides $f_j y_j$. In other words, when we take the quotient...
by \( z - 1 \), the third summand gets absorbed in the second one. And the second summand becomes

\[
\left( \frac{f_j y_j}{x_k} \mid j = 1, \ldots, m, \text{where } x_k \text{ divides } f_j \right)
\]

when taking \( z = 1 \). So the only thing left is to describe the complete part \( C \) of \( \text{Lin}(I^{eq}) \) after taking \( z = 1 \), let’s call it \( \overline{C} \).

Among the generators of \( C \) we have those where \( z \) has exponent \( d - \delta \), the highest possible. Then, when taking \( z = 1 \), from these we get all possible monomials of degree \( \delta \) with exponent vector below \( v \). All the rest of the monomials in \( \overline{C} \) (which of higher degree) are divided by such monomials. In short,

\[
\overline{C} = (x_1, \ldots, x_n)^\delta_{\leq v}.
\]

Observe that we don’t necessarily have that all monomials in \( (x_1, \ldots, x_n)^\delta \) have exponent vector below \( v \). Take for instance \( I = (x_3^3, x_1 x_2) \), so that \( d = 3 \), \( \delta = 2 \) and \( v = (3, 1) \). We have \( I^{eq} = (x_3^3, x_1 x_2 z) \) and \( v^{eq} = (3, 1, 1) \). For what concerns the complete part, we have \( C = (x_3^3, x_1^2 x_2, x_1^2 z, x_1 x_2 z) \) and \( \overline{C} = (x_3^3, x_1 x_2) \).

So now the last question is: what is the interplay of \( \overline{C} \) with the other part that survives, namely the second summand? We have the generators

\[
h_{j,k} := g_j y_j x_k^{-d - d_j y_j} x_k^{-1}, \quad \text{where } x_k \text{ divides } f_j.
\]

We have \( \deg h_{j,k} = d \) for all \( j \) and \( k \). When taking \( z = 1 \), for the residue class \( h_{j,k} \) we get \( \deg(h_{j,k}) = d_j \). So at the end of the day the only survivors are those that come from those \( f_j \)’s of degree \( \delta \). Otherwise, if \( d_j > \delta \), then \( h_{j,k} \) is divided by some monomial in \( \overline{C} = (x_1, \ldots, x_n)^\delta_{\leq v} \). Because of course the vector of exponents of \( f_j / x_k \) is below \( v \) and \( \deg(f_j / x_k) = d_j - 1 \geq \delta \).

The discussion above amounts to a proof of the following.

**Corollary 5.27.** If we set \( z = 1 \), the residue class of the ideal \( \text{Lin}(I^{eq}) \) is

\[
(x_1, \ldots, x_n)^\delta_{\leq v} + \left( \frac{f_j y_j}{x_k} \mid j = 1, \ldots, m, \text{where } x_k \text{ divides } f_j \text{ and } d_j = \delta \right),
\]

where \( \delta = \min\{d_1, \ldots, d_m\} \).

### 6 Possible future directions

Another way of defining the last part of \( \text{Lin}(I) \) in Definition 3.2 is saying that it’s generated by the monomials \( y_j \partial f_j / \partial x_k \). (One may even consider a “monic” partial derivative, as it sometimes happens.) This gives a relaxation in the definition, in that one does not need to check that the variables with respect to which one differentiates actually are in the support of the monomials, because if not one simply gets zero. This definition could also provide a possible way to generalize the linearization construction to non-monomial ideals, a possibility that has not yet been explored.

Generalizing the equification construction “as it is” to non-monomial ideals seems to fail very easily, because the definition depends very much on the system of generators one considers. Even if one takes a homogeneous ideal, where
the number of minimal homogeneous generators of each degree is invariant, the construction would still depend on the chosen system of generators: for instance, one has

\[(x + y, x^2) = (x + y, xy) \quad \text{but} \quad ((x + y)z, x^2) \neq ((x + y)z, xy).\]

It might be possible however to “improve” the definition of equification. For instance, one could fix a monomial order, and with respect to that monomial order each ideal has a canonical system of generators, its reduced Gröbner basis. In the monomial case this would reduce to our Definition 5.1.

Before we move on to the questions related to Betti splittings, we observe that one more open problem is that of understanding whether there are any nice interplays of \(\text{Lin}\) and standard operations on ideals, as briefly discussed in Section 3.1.2.

When I had the chance to present the material in this paper several questions were asked. A particularly interesting direction of investigation was suggested by Marilina Rossi. My knowledge of it relies mainly on the work of Bolognini in [3]. There he studies the concept of \(\text{Betti splitting}\), already present in the literature. A different version of this notion appears already in particular in [15], which is interesting for us because of its relation to our Section 4.3.2. Unfortunately we did not manage to find or prove anything fruitful concerning this topic, so only counterexamples are presented in this very last section.

**Definition 6.1.** Let \(I, J\) and \(K\) be monomial ideals such that \(G(I)\) is the disjoint union of \(G(J)\) and \(G(K)\), so that in particular \(I = J + K\). Then \(I = J + K\) is a **Betti splitting** if

\[\beta_{i,j}(I) = \beta_{i,j}(J) + \beta_{i,j}(K) + \beta_{i-1,j}(J \cap K)\]

for all \(i, j \in \mathbb{N}\).

This concept is intimately related to ideals with linear resolutions or a generalization of them, given by **componentwise linear ideals** (see [17] or Section 8.2 of [18]). This is why the concept seemed naturally close to the topic of this paper. The definition of \(\text{Lin}(I)\) or \(\text{Lin}^*(I)\) already provides a very natural way of partitioning the generators, namely by choosing \(J\) as the complete part \(C\) and \(K\) as the last part \(L\). So one could reasonably expect this to be a Betti splitting. But that’s not the case, at least in the following example.

**Example 6.2.** Consider \(I = (x_1^3x_2, x_2x_3^3) \subset \mathbb{K}[x_1, x_2, x_3]\). Then we have

\[
\begin{array}{c|cccc}
\beta(\text{Lin}(I)) & 0 & 1 & 2 & 4 \\
\hline
1 & 0 & 1 & 2 & 6
\end{array}
\begin{array}{c|cccc}
\beta(C) & 0 & 1 & 2 & 4 \\
\hline
1 & 0 & 1 & 2 & 6
\end{array}
\begin{array}{c|cccc}
\beta(L) & 0 & 1 & 2 & 4 \\
\hline
1 & 0 & 1 & 2 & 6
\end{array}
\begin{array}{c|cccc}
\beta(C \cap L) & 0 & 1 & 2 & 5 \\
\hline
1 & 0 & 1 & 2 & 6
\end{array}
\begin{array}{c|cccc}
\beta(L) & 0 & 1 & 2 & 5 \\
\hline
1 & 0 & 1 & 2 & 6
\end{array}
\begin{array}{c|cccc}
\beta(C \cap L) & 0 & 1 & 2 & 5 \\
\hline
1 & 0 & 1 & 2 & 6
\end{array}
\]
We do have
\begin{align*}
\beta_{0,1}(\Lin(I)) &= 11 = 4 + 7 + 0, \\
\beta_{1,5}(\Lin(I)) &= 16 = 8 + 2 + 6, \\
\beta_{2,6}(\Lin(I)) &= 6 = 2 + 0 + 4,
\end{align*}
which agree with the definition of Betti splitting, but unfortunately there is
some more rubbish in $L$ and $C \cap L$ that would give something nonzero for
$\Lin(I)$, whereas the Betti numbers of $\Lin(I)$ are zero. The same problems
occurs considering $\Lin^*(I)$ instead of $\Lin(I)$.

**Question 6.3.** When do we have that $\Lin(I)$ (or $\Lin^*(I)$), decomposed as a
sum of the complete part $C$ and the last part $L$, is a Betti splitting? Namely,
when do we have that
\[ \beta_{i,j}(\Lin(I)) = \beta_{i,j}(C) + \beta_{i,j}(L) + \beta_{i-1,j}(C \cap L) \]
for all $i, j \in \mathbb{N}$? Or is there another meaningful way of partitioning the
generators that constitutes a Betti splitting?

**Proposition 6.4** (Bolognini, Proposition 3.1 of [3]). Let $I$ be a monomial
ideal with a $d$-linear resolution, and $J, K \neq 0$ monomial ideals such that
$I = J + K$, $G(I) = G(J) \cup G(K)$ and $G(J) \cap G(K) = \emptyset$. Then the following
facts are equivalent:

(i) $I = J + K$ is a Betti splitting of $I$;

(ii) $J$ and $K$ have $d$-linear resolutions.

If this is the case, then $J \cap K$ has a $(d + 1)$-linear resolution.

Specializing this result to our notation, it means the following: $\Lin(I) = C + L$ is a Betti splitting if and only if $C$ and $L$ have a linear resolution. So,
since we already know that $C$ has a linear resolution and $L$ is generated in
degree $d$, the result tells us that $\Lin(I) = C + L$ is a Betti splitting if and only
if $L$ has $d$-linear resolution (and $C \cap L$ is automatically $(d+1)$-linear). So this
motivates the following.

**Question 6.5.** When does $L$ have a linear resolution?

In particular, for $d = 2$ we have Theorem [1,2] characterizing the squarefree
quadratic monomials with linear resolution.

**Example 6.6.** Consider the ideal $I = (x_1x_2, x_2x_3) \subset \mathbb{K}[x_1, x_2, x_3]$, which
corresponds to the path on three vertices. This ideal has linear resolution.
The last part of $\Lin(I)$ is $L = (x_1y_1, x_2y_1, x_2y_2, x_3y_2)$, and it doesn’t have a
linear resolution.

In general, it would be interesting to find properties of the last part $L$
alone. Observe that one could define it for arbitrary monomial ideals in the
same way as it is for equigenrated ideals, and investigate more in general
properties of $L$ in that case. We now conclude with a generalization of the
concept of ideal with linear resolution.
Definition 6.7. For a homogenous ideal $I \subseteq S$ we denote $I(d)$ the ideal generated by all homogeneous elements of degree $d$ in $I$. We say that $I$ is **componentwise linear** if $I(d)$ has a linear resolution for all $d$.

Componentwise linear ideals were introduced by Herzog and Hibi in [17]. Ideals with linear resolutions are componentwise linear and in particular ideals with linear quotients are componentwise linear (see [13], Lemma 8.2.10 and Theorem 8.2.15). The ideals involved in this paper are equigenerated, so being componentwise linear for them forces having a linear resolution. But perhaps this could be a meaningful concept to analyze in case the linearization can be defined in such a way that it’s not necessarily equigenerated anymore. In particular, we quote one last result from [3].

Theorem 6.8 (Bolognini, Theorem 3.3 of [3]). Let $I, J$ and $K$ be monomial ideals such that $I = J + K$ and $G(I)$ is the disjoint union of $G(J)$ and $G(K)$. If $J$ and $K$ are componentwise linear, then $I = J + K$ is a Betti splitting of $I$.

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