A convenient basis for the Izergin-Korepin model

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Abstract

We propose a convenient orthogonal basis of the Hilbert space for the Izergin-Korepin model (or the quantum spin chain associated with the $A_2^{(2)}$ algebra). It is shown that the monodromy-matrix elements acting on the basis take relatively simple forms (c.f. acting on the original basis), which is quite similar as that in the so-called F-basis for the quantum spin chains associated with $A$-type (super)algebras. As an application, we present the recursive expressions of Bethe states in the basis for the Izergin-Korepin model.

Keywords: Spin chain; Bethe Ansatz; Izergin-Korepin model; F-basis.

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1 Introduction

The quantum inverse scattering method (QISM) (or the algebraic Bethe Ansatz method (ABA)) provides a powerful method of solving eigenvalue problems for quantum integrable systems [1]. In this framework, the quasi-particle creation and annihilation operators are constructed by the off-diagonal matrix elements of the monodromy-matrix. The Bethe states (eigenstates) are obtained by acting the creation operators on a reference state [1, 2]. However, the apparently simple action of creation operators is intricate on the level of the local operators by non-local effects arising from polarization clouds or compensating exchange terms [3]. This makes the exact and explicit computation of correlation functions very involved (if not impossible). It was shown [3] that for the inhomogeneous XXX and XXZ spin chains there does exist a particular basis (the so-called F-basis [4]), in which the actions of the monodromy matrices can be simplified dramatically. This leads to the analysis of these models in the F-basis [5]. Since then such a basis has been constructed for other models only related to the A-type algebras: the high-spin XXX spin chains [6], the quantum integrable spin chains [7] associated with \( gl(m) \) algebra and their elliptic generalizations [8, 9], and the supersymmetric Fermionic models related to the superalgebras \( gl(m|n) \) [10, 11]. Whether this kind of basis does exist for other quantum integrable systems (especially for those related to the non \( A \)-type (super)algebras) is still an interesting open problem. The aim of this paper focuses on this problem for the first simplest quantum spin chain beyond \( A \)-type, namely, the Izergin-Korepin (IK) model [12].

The IK model has played a fundamental role in quantum integrable models associated with algebras beyond \( A \)-type. It was introduced as a quantum integrable model related to the Dodd-Bullough-Mikhailov or Jiber-Mikhailov-Shabat model [13, 14], one of two integrable relativistic models containing one scalar field (the other is sine-Gordon model). The \( R \)-matrix of the model corresponds to the simplest twisted affine algebra \( A_2^{(2)} \). Moreover, it also has many applications in the studies of the loop models [15] and self-avoiding walks [16]. The Bethe Ansatz solution for eigenvalues of the IK model with the periodic boundary condition was first given by Reshetikhin with his elegant analytical Bethe Ansatz method [17]. The corresponding Bethe states was then constructed by Tarasov [18], which initiated the way to construct Bethe states for quantum integrable models beyond \( A \)-type [15, 18, 19, 20, 21, 22, 23]. The purpose of the present paper is to propose a representation basis for the
IK model with periodic boundary condition, which would play a similar role as that of the F-basis for quantum integrable systems related to the $A$-type.

The paper is organized as follows. Section 2 serves as an introduction to our notations for the IK model with the periodic boundary condition. In section 3, we propose an orthogonal basis of the Hilbert space of the model. It is shown that the matrix elements of the monodromy matrix acting on this basis take simple forms, comparing with those in the original basis. In section 4, we give the recursive relations of the vector components of Bethe states in this basis, which can determine the explicit expressions of the states. We give the solution of the quantum inverse scattering problem for the IK model. The concluding remarks are given in section 5. Some detailed technical calculations are given in Appendices A-C.

2 IK model

Throughout, $V$ denotes a three-dimensional linear space with an orthonormal basis $\{|i\rangle| i = 1, 2, 3\}$. We shall adopt the standard notations: for any matrix $A \in \text{End}(V)$, $A_j$ is an embedding operator in the tensor space $V \otimes V \otimes \cdots$, which acts as $A$ on the $j$-th space and as identity on the other factor spaces; For $B \in \text{End}(V \otimes V)$, $B_{ij}$ is an embedding operator of $B$ in the tensor space, which acts as identity on the factor spaces except for the $i$-th and $j$-th ones.

The $R$-matrix $R(u) \in \text{End}(V \otimes V)$ of the IK model is given by

$$
R_{12}(u) = \begin{pmatrix}
\begin{array}{ccc}
c(u) & b(u) & d(u) \\
\bar{a}(u) & g(u) & f(u) \\
\bar{f}(u) & \bar{g}(u) & \bar{c}(u)
\end{array}
\end{pmatrix},
$$

(2.1)

The general method to solve the quantum inverse problem for an integrable spin chain was given in [27, 28]. Here we just list the results for this particular model.
where the matrix elements are

\[ a(u) = \sinh(u - 3\eta) - \sinh 5\eta + \sinh 3\eta + \sinh \eta, \]
\[ b(u) = \sinh(u - 3\eta) + \sinh 3\eta, \]
\[ c(u) = \sinh(u - 5\eta) + \sinh \eta, \quad d(u) = \sinh(u - \eta) + \sinh \eta, \]
\[ e(u) = -2e^{-\frac{u}{2}} \sinh 2\eta \cosh(u - 3\eta), \quad \bar{e}(u) = -2e^{\frac{u}{2}} \sinh 2\eta \cosh(u - 3\eta), \]
\[ f(u) = -2e^{-u+2\eta} \sinh \eta \sinh 2\eta - e^{-\eta} \sinh 4\eta, \]
\[ \bar{f}(u) = 2e^{u-2\eta} \sinh \eta \sinh 2\eta - e^\eta \sinh 4\eta, \]
\[ g(u) = 2e^{-\frac{u}{2}+2\eta} \sinh \frac{u}{2} \sinh 2\eta, \quad \bar{g}(u) = -2e^{\frac{u}{2}-2\eta} \sinh \frac{u}{2} \sinh 2\eta. \quad (2.2) \]

The \( R \)-matrix satisfies the quantum Yang-Baxter equation (QYBE)

\[ R_{12}(u_1 - u_2)R_{13}(u_1 - u_3)R_{23}(u_2 - u_3) = R_{23}(u_2 - u_3)R_{13}(u_1 - u_3)R_{12}(u_1 - u_2). \quad (2.3) \]

For convenience, in the following parts of this paper, let us introduce some functions

\[ \omega(u) = \frac{c(u)d(u)}{a(u)d(u) - g(u)\bar{g}(u)}, \quad y(u) = \frac{d(u)}{\bar{g}(u)}, \quad \bar{g}(u) = \frac{d(u)}{g(u)}, \quad z(u) = \frac{c(u)}{b(u)}. \quad (2.4) \]

The monodromy-matrix \( T(u) \) is an \( n \times n \) matrix with operator-valued elements acting on \( V^\otimes N \) as

\[ T_0(u) = R_{0N}(u - \theta_N)R_{0N-1}(u - \theta_{N-1}) \cdots R_{01}(u - \theta_1), \quad (2.5) \]

where \( \{ \theta_j | j = 1, \cdots, N \} \) are generic free complex parameters which are usually called the inhomogeneous parameters. The QYBE \[23\] implies that the monodromy-matrix \( T(u) \) satisfies the exchange relations (or the Yang-Baxter relations)

\[ R_{12}(u - v) T_1(u) T_2(v) = T_2(v) T_1(u) R_{12}(u - v). \quad (2.6) \]

The corresponding transfer matrix \( t(u) \) can be constructed by the standard way \[1\] as

\[ t(u) = Tr_0 \{ T_0(u) \}. \quad (2.7) \]

The IK model with periodic boundary condition is a quantum spin chain described by the Hamiltonian

\[ H = \frac{\partial}{\partial u} \{ \ln t(u) \} |_{u=0, \{ \theta_i \}=0} = \frac{1}{\sinh \eta - \sinh 5\eta} \sum_{i=1}^{N} H_{i,i+1}, \quad (2.8) \]
where the local Hamiltonian $H_{i,i+1}$ is

$$H_{i,i+1} = \frac{\partial}{\partial u} \{ P_{i,i+1} R_{i,i+1}(u) \} |_{u=0}. \quad (2.9)$$

The periodic boundary condition for the Hamiltonian (2.8) reads

$$H_{N,N+1} = H_{N,1}. \quad (2.10)$$

The QYBE leads to the fact that the transfer matrices $t(u)$ given by (2.7) with different spectral parameters are mutually commuting:

$$[t(u), t(v)] = 0.$$

This ensures the integrability of the IK model with periodic boundary described by the Hamiltonian (2.8) and (2.10).

### 3 Orthogonal basis for the IK model

It was shown [3] that for the inhomogeneous XXX and XXZ spin chains there does exist a particular basis (the so-called F-basis [4]), in which the actions of the monodromy matrices can be simplified dramatically. Since then such a basis has been constructed for other models only related to the $A$-type algebras [6, 7, 8, 9, 10, 11]. This leads to the F-basis analysis of these models [5, 11].

In this section, we propose a convenient basis of the Hilbert space parameterized by the $N$ generic inhomogeneity parameters $\{ \theta_j | j = 1, \cdots, N \}$. It is found that the actions of monodromy-matrix elements on this basis take drastically simple forms like those in the so-called F-basis [3, 4, 7, 9] for the models related to the A-type (super)algebras. For convenience, let us introduce the notations

$$A_i(u) = T^i_1(u), \quad B_1(u) = T^1_2(u), \quad B_2(u) = T^1_3(u), \quad B_3(u) = T^2_3(u), \quad \text{for } i = 1, 2, 3,$$

$$C_1(u) = T^2_1(u), \quad C_2(u) = T^3_1(u), \quad C_3(u) = T^3_2(u). \quad (3.1)$$

The monodromy-matrix becomes

$$T(u) = \begin{pmatrix}
A_1(u) & B_1(u) & B_2(u) \\
C_1(u) & A_2(u) & B_3(u) \\
C_2(u) & C_3(u) & A_3(u)
\end{pmatrix}. \quad (3.2)$$
These operators satisfy the quadratic commutation relation (2.6) (or the Yang-Baxter algebra) whose structure constants are given by the matrix elements of the $R$-matrix. The commutation relation allows us to derive the exchange relations among the operators in (3.2). Some relevant exchange relations for our purpose among the operators are given in Appendix A.

Let us introduce the left quasi-vacuum state $\langle 0 |$ and the right quasi-vacuum state $| 0 \rangle$ as follows

$$\langle 0 | = (1, 0, 0)_{[1]} \otimes \cdots \otimes (1, 0, 0)_{[N]}, \quad | 0 \rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_{[1]} \otimes \cdots \otimes \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_{[N]}.$$ (3.3)

The operators (3.1) acting on the quasi-vacuum states give rise to

$$| 0 \rangle A_i(u) = \alpha_i(u) | 0 \rangle, \quad i = 1, 2, 3,$$ (3.4)

$$| 0 \rangle B_i(u) = 0, \quad | 0 \rangle C_i(u) \neq 0, \quad i = 1, 2, 3,$$ (3.5)

$$A_i(u) | 0 \rangle = \alpha_i(u) | 0 \rangle, \quad i = 1, 2, 3,$$ (3.6)

$$C_i(u) | 0 \rangle = 0, \quad B_i(u) | 0 \rangle \neq 0, \quad i = 1, 2, 3,$$ (3.7)

where the functions $\alpha_i(u)$ are

$$\alpha_1(u) = \prod_{l=1}^{N} c(u - \theta_l) = \prod_{l=1}^{N} \{ \sinh(u - \theta_l - 5\eta) + \sinh \eta \},$$

$$\alpha_2(u) = \prod_{l=1}^{N} b(u - \theta_l) = \prod_{l=1}^{N} \{ \sinh(u - \theta_l - 3\eta) + \sinh 3\eta \},$$

$$\alpha_3(u) = \prod_{l=1}^{N} d(u - \theta_l) = \prod_{l=1}^{N} \{ \sinh(u - \theta_l - \eta) + \sinh \eta \}.$$ (3.8)

For convenience, we introduce two functions

$$\xi(u) = e^{\eta} \frac{\alpha_3(u^{(2)})}{\alpha_2(u^{(1)})}, \quad \bar{\xi}(u) = e^{-\eta} \frac{\alpha_3(u^{(2)})}{\alpha_2(u^{(1)})},$$ (3.9)

where we have used the convention: $u^{(1)} = u + 4\eta$; $u^{(2)} = u + 6\eta + i\pi$.

3.1 A convenient basis for the IK model

In this subsection, we construct a convenient basis for the IK model, and parameterize it as follows. For two non-negative integers $m_2$ and $m$ such that $m_2 \leq m \leq N$, let us introduce a
m-tuple positive integers \( P = \{p_1, \cdots, p_m\} \), which satisfy the relation

\[
1 \leq p_1 < p_2 < \cdots < p_{m_2} \leq N, \quad 1 \leq p_{m_2+1} < \cdots < p_m \leq N, \quad \text{and} \quad p_j \neq p_i.
\] (3.10)

For each \( P \), let us introduce a left state \( \langle \theta_{p_m}^{(2)}, \cdots, \theta_{p_{m_2+1}}^{(2)}; \theta_{p_{m_2}}^{(1)}, \cdots, \theta_{p_1}^{(1)} \rangle \) and a right state \( |\theta_{p_1}^{(1)}, \cdots, \theta_{p_{m_2}}^{(1)}; \theta_{p_{m_2+1}}^{(2)}, \cdots, \theta_{p_m}^{(2)} \rangle \) parameterized by the \( N \) inhomogeneity parameters \( \{\theta_j\} \) as follows:

\[
\langle \theta_{p_m}^{(2)}, \cdots, \theta_{p_{m_2+1}}^{(2)}; \theta_{p_{m_2}}^{(1)}, \cdots, \theta_{p_1}^{(1)} \rangle = \langle 0|C_2(\theta_{p_m}^{(2)}) \cdots C_2(\theta_{p_{m_2+1}}^{(2)}) C_1(\theta_{p_{m_2}}^{(1)}) \cdots C_1(\theta_{p_1}^{(1)}) \rangle,
\] (3.11)

\[
|\theta_{p_1}^{(1)}, \cdots, \theta_{p_{m_2}}^{(1)}; \theta_{p_{m_2+1}}^{(2)}, \cdots, \theta_{p_m}^{(2)} \rangle = B_1(\theta_{p_1}^{(1)}) \cdots B_1(\theta_{p_{m_2}}^{(1)}) B_2(\theta_{p_{m_2+1}}^{(2)}) \cdots B_2(\theta_{p_m}^{(2)})|0\rangle,
\] (3.12)

where \( m_2 \) (resp. \( m - m_2 \)) is the number of the operators \( C_1(u) \) or \( B_1(u) \) (resp. \( C_2(u) \) or \( B_2(u) \)), and \( \theta_i^{(1)} = \theta_i + 4\eta \); \( \theta_i^{(2)} = \theta_i + 6\eta + i\pi \). It is easy to check that the states (3.11) and (3.12) are common eigenstates of the operator \( A_1(u) \) with different \( u \), namely,

\[
\langle \theta_{p_m}^{(2)}, \cdots, \theta_{p_{m_2+1}}^{(2)}; \theta_{p_{m_2}}^{(1)}, \cdots, \theta_{p_1}^{(1)} \rangle A_1(u) = \alpha_1(u) \prod_{i=1}^{m_2} z(\theta_{p_i}^{(1)} - u) \prod_{l=m_2+1}^{m} \frac{c(\theta_{p_l}^{(2)} - u)}{d(\theta_{p_l}^{(2)} - u)} \times \langle \theta_{p_m}^{(2)}, \cdots, \theta_{p_{m_2+1}}^{(2)}; \theta_{p_{m_2}}^{(1)}, \cdots, \theta_{p_1}^{(1)} \rangle,
\] (3.13)

\[
A_1(u) |\theta_{p_1}^{(1)}, \cdots, \theta_{p_{m_2}}^{(1)}; \theta_{p_{m_2+1}}^{(2)}, \cdots, \theta_{p_m}^{(2)} \rangle = \alpha_1(u) \prod_{i=1}^{m_2} z(\theta_{p_i}^{(1)} - u) \prod_{l=m_2+1}^{m} \frac{c(\theta_{p_l}^{(2)} - u)}{d(\theta_{p_l}^{(2)} - u)} \times |\theta_{p_1}^{(1)}, \cdots, \theta_{p_{m_2}}^{(1)}; \theta_{p_{m_2+1}}^{(2)}, \cdots, \theta_{p_m}^{(2)} \rangle,
\] (3.14)

where the functions \( z(u) \) and \( \alpha_i(u) \) are given by (2.4) and (3.8). From the exchange relations given by (A.1)-(A.22) below, we can verify the above relations. It is easy to show that the states (3.11) and (3.12) are non-zeros thanks to the orthogonal relations (see below (3.22) and (3.23)).

### 3.2 Orthogonality and other properties of the basis

With help of the exchange relations given by (A.1)-(A.22) below, we can derive some quasi-symmetry properties of the left states \( \{\langle \theta_{p_m}^{(2)}, \cdots, \theta_{p_{m_2+1}}^{(2)}; \theta_{p_{m_2}}^{(1)}, \cdots, \theta_{p_1}^{(1)} \rangle\} \)

\[
\langle \theta_{p_m}^{(2)}, \cdots, \theta_{p_{m_2+1}}^{(2)}; \theta_{p_{m_2}}^{(1)}, \cdots, \theta_{p_1}^{(1)} \rangle = w(\theta_{p_{i+1}}^{(1)} - \theta_{p_i}^{(1)}) \langle \theta_{p_m}^{(2)}, \cdots, \theta_{p_{m_2+1}}^{(2)}; \theta_{p_{m_2}}^{(1)}, \cdots, \theta_{p_1}^{(1)} \rangle,
\] (3.15)

\[\text{Similar results can also be obtained for the right states.}\]
\begin{align*}
\langle \theta^{(2)}_{p_{m}}, \ldots, \theta^{(2)}_{p_{m+1}}, \theta^{(1)}_{p_{m}}, \ldots, \theta^{(1)}_{p_{1}} \rangle &= z(\theta^{(2)}_{p_{m+1}} - \theta^{(1)}_{p_{m}})\langle \theta^{(2)}_{p_{m}}, \ldots, \theta^{(2)}_{p_{m+2}}, \theta^{(1)}_{p_{m}}, \theta^{(2)}_{p_{m+1}}, \theta^{(1)}_{p_{m-1}}, \ldots, \theta^{(1)}_{p_{1}} \rangle, \\
&= \langle \theta^{(2)}_{p_{m}}, \ldots, \theta^{(2)}_{p_{i+1}}, \theta^{(2)}_{p_{i}}, \theta^{(2)}_{p_{i+1}}, \ldots, \theta^{(2)}_{p_{m+1}}, \theta^{(1)}_{p_{m}}, \ldots, \theta^{(1)}_{p_{1}} \rangle, \\
&= \langle \theta^{(2)}_{p_{m}}, \ldots, \theta^{(2)}_{p_{i+1}}, \theta^{(2)}_{p_{i}}, \theta^{(2)}_{p_{i+1}}, \ldots, \theta^{(1)}_{p_{m}}, \ldots, \theta^{(1)}_{p_{1}} \rangle. 
\end{align*}

(3.16)

Noting the fact that \( \alpha_1(\theta^{(i)}_l) = 0 \), for \( l = 1, \ldots, N; \ i = 1, 2 \), we can also obtain some useful identities

\begin{align*}
\langle \theta^{(2)}_{p_{m}}, \ldots, \theta^{(2)}_{p_{m+1}}, \theta^{(1)}_{p_{m}}, \ldots, \theta^{(1)}_{p_{1}} | T^i_{j}(\theta^{(1)}_{p_{i}}) = 0, T^i_{j} = B_1, B_2, A_1, \\
\langle \theta^{(2)}_{p_{m}}, \ldots, \theta^{(2)}_{p_{m+1}}, \theta^{(1)}_{p_{m}}, \ldots, \theta^{(1)}_{p_{1}} | T^i_{j}(\theta^{(2)}_{p_{i}}) = 0, T^i_{j} = B_1, B_2, B_3, C_1, A_1, A_2, \\
T^i_{j}(\theta^{(1)}_{p_{i}})\theta^{(1)}_{p_{m}}, \ldots, \theta^{(1)}_{p_{m+1}}, \theta^{(2)}_{p_{m}}, \ldots, \theta^{(2)}_{p_{1}} = 0, T^i_{j} = C_1, C_2, A_1, \\
T^i_{j}(\theta^{(2)}_{p_{i}})\theta^{(1)}_{p_{m}}, \ldots, \theta^{(1)}_{p_{m+1}}, \theta^{(2)}_{p_{m}}, \ldots, \theta^{(2)}_{p_{1}} = 0, T^i_{j} = C_1, C_2, C_3, B_1, A_1, A_2. 
\end{align*}

(3.17)

(3.18)

(3.19)

(3.20)

(3.21)

It should be emphasized that in the above identities \( l \) takes the value of \( m + 1, \ldots, N \). As an example, a brief proof for the identity (3.21) is given in Appendix B. These properties and the exchange relations of the operators allow us to derive the orthogonal relations between the left states and the right states

\begin{align*}
\langle \theta^{(2)}_{q_{m}}, \ldots, \theta^{(2)}_{q_{m+1}}, \theta^{(1)}_{q_{m-1}}, \ldots, \theta^{(1)}_{q_{1}} | \theta^{(1)}_{p_{1}}, \ldots, \theta^{(1)}_{p_{m}}, \theta^{(2)}_{p_{m-1}}, \ldots, \theta^{(2)}_{p_{m}} \rangle = \delta_{m,m^{'}} \delta_{m_{2},m_{2}^{'}} \prod_{k=1}^{m} \delta_{p_{k},q_{k}} G_{m}(\theta^{(1)}_{p_{1}}, \ldots, \theta^{(1)}_{p_{m}}, \theta^{(2)}_{p_{m+1}}, \ldots, \theta^{(2)}_{p_{m}}), 
\end{align*}

(3.22)

where the factor \( G_{m}(\theta^{(1)}_{p_{1}}, \ldots, \theta^{(1)}_{p_{m}}, \theta^{(2)}_{p_{m+1}}, \ldots, \theta^{(2)}_{p_{m}}) \) is given by

\begin{align*}
G_{m}(\theta^{(1)}_{p_{1}}, \ldots, \theta^{(1)}_{p_{m}}, \theta^{(2)}_{p_{m+1}}, \ldots, \theta^{(2)}_{p_{m}}) &= \prod_{k=1}^{m_2} \{ 2 \cosh \eta \sinh(2\eta) \alpha^{(1)}_{p_{k}}(\theta^{(1)}_{p_{k}}) \alpha^{(1)}_{p_{k}}(\theta^{(1)}_{p_{k}}) \\
&\times \prod_{i=1}^{m_2} \prod_{i \neq k} z(\theta^{(1)}_{p_{1}} - \theta^{(1)}_{p_{k}}) \prod_{l=k+1}^{m_2} \omega(\theta^{(1)}_{p_{l}} - \theta^{(1)}_{p_{k}}) \prod_{j=m_2+1}^{m} c(\theta^{(2)}_{p_{j}} - \theta^{(1)}_{p_{j}})z(\theta^{(2)}_{p_{j}} - \theta^{(1)}_{p_{j}})z(\theta^{(2)}_{p_{j}} - \theta^{(1)}_{p_{j}}) \} \\
&\times \prod_{k=m_2+1}^{m} \{ f(0)\alpha^{(1)}_{p_{k}}(\theta^{(2)}_{p_{k}}) \alpha^{(2)}_{p_{k}}(\theta^{(2)}_{p_{k}}) \prod_{i=m_2+1}^{m} c(\theta^{(2)}_{p_{k}} - \theta^{(2)}_{p_{k}}) \prod_{i \neq k} d(\theta^{(2)}_{p_{k}} - \theta^{(2)}_{p_{k}}) \}.
\end{align*}

(3.23)
The functions \( \{ \alpha_i^{(1)}(u) \} \) are
\[
\alpha_i^{(1)}(u) = \prod_{k=1, k \neq i}^{N} c(u - \theta_k) = \prod_{k=1, k \neq i}^{N} \{ \sinh(u - \theta_k - 5\eta) + \sinh \eta \}, \quad i = 1, \cdots, N. \tag{3.24}
\]

On the other hand, we know that the total number of the linear-independent left (right) states given in (3.11) and (3.12) is
\[
\sum_{m=0}^{N} \frac{N!}{(N-m)!m!} \sum_{m=0}^{m} \frac{m!}{(m-m_2)!m_2!} = \sum_{m=0}^{N} \frac{N!}{(N-m)!m!} 2^m = 3^N.
\]

Thus these right (left) states form an orthogonal right (left) basis of the Hilbert space, namely,
\[
\text{id} = \sum_{m=0}^{N} \sum_{m_2=0}^{m} \sum_{P} \frac{1}{G_m(\theta^{(1)}_{p_1}, \cdots, \theta^{(1)}_{p_{m_2}}, \theta^{(2)}_{p_{m_2+1}}, \cdots, \theta^{(2)}_{p_{m_2}})} \\
\times |\theta^{(1)}_{p_1}, \cdots, \theta^{(1)}_{p_{m_2}}, \theta^{(2)}_{p_{m_2+1}}, \cdots, \theta^{(2)}_{p_{m_2}}\rangle \langle \theta^{(2)}_{p_{m_2}}, \cdots, \theta^{(2)}_{p_{m_2+1}}, \theta^{(1)}_{p_1}, \cdots, \theta^{(1)}_{p_{m_2}}|, \tag{3.25}
\]

where the notation \( \sum_P \) indicates the sum over all possible combination \( P \) satisfying the condition (3.10).

Some remarks are in order. The states given by (3.11) (resp. (3.12)) are eigenstates of the commutative family \( A_1(u) \) and serve as the basis of the left (right) Hilbert space for generic inhomogeneous parameters \( \{ \theta_j \} \). These kind of states are relevant to the separation of variables (SoV) \[24\] states and the F-basis \[3\] for the quantum spin chain associated with the \( A \)-type algebra. For the \( su(2) \) case, the corresponding states are the SoV states for the XXZ spin chain, and was shown in \[25\] that it coincides with the so-called F-basis \[3\]. For the \( su(n) \) case, the corresponding states are the nested generalization of the SoV states \[26\] for the trigonometric \( su(n) \) spin chain and coincide with the associated F-basis \[7, 8, 9, 10, 11\].

### 3.3 Operators in the basis

The exchange relations (A.1)-(A.22) and the identities (3.18)-(3.21) enable us to calculate the actions of the operators \( A_i(u) \), \( B_i(u) \) and \( C_i(u) \) on the basis given by (3.11) and (3.12). Direct calculation shows that the resulting actions on this basis become much simpler, comparing with those on the original basis. Here we list some of them relevant for us to obtain the
explicit expressions of Bethe states

\[
\langle \theta_{p_{m}}^{(2)}, \ldots, \theta_{p_{m+1}}^{(2)}; \theta_{p_{m2}}^{(1)}, \ldots, \theta_{p_{1}}^{(1)} | \ A_1(u) = \alpha_1(u) \prod_{i=1}^{m_2} z(\theta_{p_i}^{(1)} - u) \prod_{l=m_2+1}^{m} c(\theta_{p_l}^{(2)} - u) \\
\times \langle \theta_{p_{m}}^{(2)}, \ldots, \theta_{p_{m+1}}^{(2)}; \theta_{p_{m2}}^{(1)}, \ldots, \theta_{p_{1}}^{(1)} | \ 
\]

\[
\langle \theta_{p_{m}}^{(2)}, \ldots, \theta_{p_{m+1}}^{(2)}; \theta_{p_{m2}}^{(1)}, \ldots, \theta_{p_{1}}^{(1)} | B_1(u) = \sum_{i=1}^{m_2} \frac{\bar{e}(\theta_{p_i}^{(1)} - u)}{b(\theta_{p_i}^{(1)} - u)} \alpha_1(u) \alpha_2(\theta_{p_i}^{(1)}) \prod_{h=1}^{i-1} \omega(\theta_{p_i}^{(1)} - \theta_{p_h}^{(1)}) \\
\times \prod_{j=1, j \neq i}^{m_2} z(\theta_{p_j}^{(1)} - u) \frac{z(\theta_{p_i}^{(1)} - \theta_{p_j}^{(1)})}{\omega(\theta_{p_i}^{(1)} - \theta_{p_j}^{(1)})} \prod_{k=m_2+1}^{m} \frac{c(\theta_{p_k}^{(2)} - u)}{d(\theta_{p_k}^{(2)} - u)} \bar{d}(\theta_{p_k}^{(2)} - u) \bar{z}(\theta_{p_i}^{(1)} - \theta_{p_k}^{(1)}) \bar{z}(\theta_{p_i}^{(1)} - \theta_{p_j}^{(1)}) \\
\times \langle \theta_{p_{m}}^{(2)}, \ldots, \theta_{p_{m+1}}^{(2)}; \theta_{p_{m2}}^{(1)}, \ldots, \theta_{p_{1}}^{(1)} | 
\]

\[
+ \sum_{i=m_2+1}^{m} \left[ \frac{\bar{e}(\theta_{p_i}^{(2)} - u)}{d(\theta_{p_i}^{(2)} - u)} - \sum_{l=1}^{m_2} \frac{\bar{e}(\theta_{p_l}^{(1)} - \theta_{p_i}^{(1)}) \bar{e}(\theta_{p_i}^{(1)} - u) c(\theta_{p_i}^{(2)} - u)}{b(\theta_{p_l}^{(1)} - \theta_{p_i}^{(1)}) b(\theta_{p_i}^{(1)} - u) d(\theta_{p_i}^{(2)} - u)} \prod_{j=1, j \neq l}^{m_2} z(\theta_{p_j}^{(1)} - u) z(\theta_{p_i}^{(1)} - \theta_{p_j}^{(1)}) \right] \\
\times \prod_{k=m_2+1, k \neq i}^{m} \frac{b(\theta_{p_k}^{(2)} - \theta_{p_k}^{(1)}) c(\theta_{p_k}^{(2)} - u)}{c(\theta_{p_k}^{(2)} - \theta_{p_k}^{(1)}) d(\theta_{p_k}^{(2)} - u)} \bar{z}(\theta_{p_i}^{(1)} - \theta_{p_k}^{(2)}) \alpha_1(u) \bar{c}(\theta_{p_i}) \\
\times \langle \theta_{p_{m}}^{(2)}, \ldots, \hat{\theta}_{p_i}^{(2)}, \ldots, \theta_{p_{m+1}}^{(2)}; \theta_{p_{m2}}^{(1)}, \theta_{p_{m2}}^{(1)}, \ldots, \theta_{p_{1}}^{(1)} |, \right. \] (3.27)
\[
\langle \theta^{(2)}_{p_m}, \ldots, \theta^{(2)}_{p_{m+1}}; \theta^{(1)}_{p_{m+2}}, \ldots, \theta^{(1)}_{p_1} \rangle \mid B_2(u) = \\
\sum_{l=m+1}^{m} \frac{f(\theta^{(2)}_{p_l} - u)}{d(\theta^{(2)}_{p_l} - u)} m \prod_{i=1}^{m} \frac{c(\theta^{(2)}_{p_i} - u)c(\theta^{(2)}_{p_i} - \theta^{(2)}_{p_{j}})}{d(\theta^{(2)}_{p_i} - u)d(\theta^{(2)}_{p_i} - \theta^{(2)}_{p_{j}})} \times \alpha_1(u) \alpha_3(\theta^{(2)}_{p_l}) \\
\times \langle \theta^{(2)}_{p_m}, \ldots, \theta^{(2)}_{p_{m+1}}; \theta^{(1)}_{p_{m+2}}, \ldots, \theta^{(1)}_{p_1} \rangle \\
\sum_{l=m+1}^{m} \sum_{i\neq l}^{m} \left\{ \frac{g(\theta^{(1)}_{p_i} - u)\bar{e}(\theta^{(1)}_{p_i} - u)}{d(\theta^{(1)}_{p_i} - u)d(\theta^{(1)}_{p_i} - \theta^{(1)}_{p_{j}})} \times \frac{f(\theta^{(1)}_{p_l} - u)c(\theta^{(1)}_{p_l} - \theta^{(1)}_{p_{j}})}{d(\theta^{(1)}_{p_l} - u)d(\theta^{(1)}_{p_l} - \theta^{(1)}_{p_{j}})} \right\} w(\theta^{(1)}_{p_l} - \theta^{(1)}_{p_{j}}) \\
\times \prod_{j=m+1}^{m} \frac{c(\theta^{(2)}_{p_j} - u)z(\theta^{(2)}_{p_j} - \theta^{(2)}_{p_{j}})z(\theta^{(2)}_{p_j} - \theta^{(2)}_{p_{j}})}{d(\theta^{(2)}_{p_j} - u)z(\theta^{(2)}_{p_j} - \theta^{(2)}_{p_{j}})z(\theta^{(2)}_{p_j} - \theta^{(2)}_{p_{j}})} \prod_{k=1}^{m} d(\theta^{(2)}_{p_k} - u) \times \alpha_1(u) \tilde{\xi}(\theta^{(1)}_{p_j}) \\
\times \langle \theta^{(2)}_{p_m}, \ldots, \theta^{(2)}_{p_{m+1}}; \theta^{(1)}_{p_{m+2}}, \ldots, \theta^{(1)}_{p_1} \rangle \\
+ \sum_{l=1}^{m} \sum_{i\neq l}^{m} \left\{ \frac{\bar{g}(\theta^{(1)}_{p_i} - u)e(\theta^{(1)}_{p_i} - u)}{\bar{b}(\theta^{(1)}_{p_i} - u)\bar{b}(\theta^{(1)}_{p_i} - \theta^{(1)}_{p_{j}})} \times \frac{\bar{f}(\theta^{(1)}_{p_l} - u)g(\theta^{(1)}_{p_l} - \theta^{(1)}_{p_{j}})}{\bar{b}(\theta^{(1)}_{p_l} - u)g(\theta^{(1)}_{p_l} - \theta^{(1)}_{p_{j}})} \right\} \alpha_1(u) \alpha_2(\theta^{(1)}_{p_l}) \alpha_2(\theta^{(1)}_{p_i}) \\
\times \prod_{h=1}^{m} w(\theta^{(1)}_{p_l} - \theta^{(1)}_{p_{h}}) w(\theta^{(1)}_{p_l} - \theta^{(1)}_{p_{h}}) \prod_{j=1}^{m} w(\theta^{(1)}_{p_l} - \theta^{(1)}_{p_{j}}) w(\theta^{(1)}_{p_l} - \theta^{(1)}_{p_{j}}) \\
\times \prod_{k=m+1}^{m} \frac{c(\theta^{(1)}_{p_k} - u)z(\theta^{(1)}_{p_k} - \theta^{(1)}_{p_k})z(\theta^{(1)}_{p_k} - \theta^{(1)}_{p_k})z(\theta^{(1)}_{p_k} - \theta^{(1)}_{p_k})z(\theta^{(1)}_{p_k} - \theta^{(1)}_{p_k})}{d(\theta^{(1)}_{p_k} - u)z(\theta^{(1)}_{p_k} - \theta^{(1)}_{p_k})z(\theta^{(1)}_{p_k} - \theta^{(1)}_{p_k})z(\theta^{(1)}_{p_k} - \theta^{(1)}_{p_k})z(\theta^{(1)}_{p_k} - \theta^{(1)}_{p_k})} \\
\times \langle \theta^{(2)}_{p_m}, \ldots, \theta^{(2)}_{p_{m+1}}; \theta^{(1)}_{p_{m+2}}, \ldots, \theta^{(1)}_{p_1} \rangle \\
+ \sum_{l=1}^{m} \sum_{i=1}^{m} \sum_{j \neq l}^{m} \left\{ \frac{\bar{g}(\theta^{(2)}_{p_i} - u)e(\theta^{(2)}_{p_i} - u)}{\bar{b}(\theta^{(2)}_{p_i} - u)\bar{b}(\theta^{(2)}_{p_i} - \theta^{(2)}_{p_{j}})} \times \frac{\bar{f}(\theta^{(2)}_{p_l} - u)c(\theta^{(2)}_{p_l} - \theta^{(2)}_{p_{j}})}{\bar{b}(\theta^{(2)}_{p_l} - u)c(\theta^{(2)}_{p_l} - \theta^{(2)}_{p_{j}})} \right\} \alpha_1(u) \alpha_2(\theta^{(1)}_{p_l}) \alpha_2(\theta^{(1)}_{p_i}) \\
\times \prod_{h=1}^{m} w(\theta^{(2)}_{p_l} - \theta^{(2)}_{p_{h}}) w(\theta^{(2)}_{p_l} - \theta^{(2)}_{p_{h}}) \prod_{j=1}^{m} w(\theta^{(2)}_{p_l} - \theta^{(2)}_{p_{j}}) w(\theta^{(2)}_{p_l} - \theta^{(2)}_{p_{j}}) \\
\times \prod_{k=m+1}^{m} \frac{c(\theta^{(2)}_{p_k} - u)z(\theta^{(2)}_{p_k} - \theta^{(2)}_{p_k})z(\theta^{(2)}_{p_k} - \theta^{(2)}_{p_k})z(\theta^{(2)}_{p_k} - \theta^{(2)}_{p_k})z(\theta^{(2)}_{p_k} - \theta^{(2)}_{p_k})}{d(\theta^{(2)}_{p_k} - u)z(\theta^{(2)}_{p_k} - \theta^{(2)}_{p_k})z(\theta^{(2)}_{p_k} - \theta^{(2)}_{p_k})z(\theta^{(2)}_{p_k} - \theta^{(2)}_{p_k})z(\theta^{(2)}_{p_k} - \theta^{(2)}_{p_k})} \\
\times \langle \theta^{(2)}_{p_m}, \ldots, \theta^{(2)}_{p_{m+1}}; \theta^{(1)}_{p_{m+2}}, \ldots, \theta^{(1)}_{p_1} \rangle \\
(3.28)
where the parameter with a hat \( \hat{\theta}_p^{(i)} \) means this parameter is absent and the functions \( \xi(u) \) and \( \bar{\xi}(u) \) are given by (3.9).

Expanding the operators \( A_i(u), B_i(u) \) and \( C_i(u) \) in terms of the local operators \( \{i\}(j) |i, j = 1, 2, 3; l = 1, \cdots , N\} \) (i.e., in original basis) gives rise to that the total number of all summing terms in the decomposition for each operator may increase exponentially with \( N \) (which was shown even for the very simple case of the XXZ chain [3]). In contrast, the total number of summing terms for each decomposition in (3.26)-(3.28) only increases as a polynomial of \( N \). This leads to the fact that the actions of the monodromy matrices in the very basis provided by (3.11)-(3.12) can be simplified dramatically. It is believed that such a basis would play the same role for the IK model as that of the F-basis for the quantum spin chains related to the \( A \)-type (super)algebras [3, 7, 8, 9, 10, 11]. Moreover, such simplified actions of the creation operators further allow us to construct the recursive relations of the Bethe states, which uniquely determine the state.

4 Bethe states in the basis

4.1 Bethe states

The off-shell Bethe states of the IK model can be constructed by the recursive relation [18]

\[
\langle \phi_n(u_1, \cdots , u_n) \rangle = B_1(u_1)\langle \phi_{n-1}(u_2, \cdots , u_n) \rangle - B_2(u_1)\sum_{i=2}^n \frac{\alpha_1(u_i)}{y(u_1 - u_i)} \prod_{j=2}^{i-1} \omega(u_i - u_j) \prod_{k=2}^n \omega(u_k - u_j) \langle \phi_{n-2}(u_2, \cdots , \hat{u}_i, \cdots , u_n) \rangle, \tag{4.1}
\]

where the parameter with a hat \( \hat{u}_i \) means this parameter is absent and the initial conditions of the above recursive relations are

\[
\langle \phi_0 \rangle = |0\rangle, \quad \langle \phi_1(u) \rangle = B_1(u)|0\rangle. \tag{4.2}
\]

These states become the eigenstates of the transfer matrix \( t(u) \) (or on-shell) if the parameters satisfy the Bethe Ansatz equations (BAEs) [18]

\[
\frac{\alpha_1(u_j)}{\alpha_2(u_j)} = \prod_{k=1}^n \frac{z(u_j - u_k)}{z(u_k - u_j)} w(u_k - u_j), \quad j = 1, \cdots , n. \tag{4.3}
\]

Using (3.27) and (3.28) we can calculate the expressions of the Bethe states in terms of the basis (3.11) as follows. Let us define scalar products of the Bethe state \( \langle \phi_n(u_1, \cdots , u_n) \rangle \) with

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vectors in the basis

\[ \langle \theta^{(2)}_{p_m}, \ldots, \theta^{(2)}_{p_{m+1}}, \theta^{(1)}_{p_{m+1}}, \ldots, \theta^{(1)}_{p_1} | \phi_n(u_1, \ldots, u_n) \rangle = F_{2m-m_2,n}(\theta^{(2)}_{p_m}, \ldots, \theta^{(2)}_{p_{m+1}}, \theta^{(1)}_{p_{m+1}}, \ldots, \theta^{(1)}_{p_1} | u_1, \ldots, u_n) \] (4.4)

It is easy to verify that

\[ F_{2m-m_2,n}(\theta^{(2)}_{p_m}, \ldots, \theta^{(2)}_{p_{m+1}}, \theta^{(1)}_{p_{m+1}}, \ldots, \theta^{(1)}_{p_1} | u_1, \ldots, u_n) = \delta_{2m-m_2,n} F_n(\theta^{(2)}_{p_m}, \ldots, \theta^{(2)}_{p_{m+1}}, \theta^{(1)}_{p_{m+1}}, \ldots, \theta^{(1)}_{p_1} | u_1, \ldots, u_n) \] (4.5)

With the help of the relations (1.1, 3.27 and 3.28) and following the method in (10), we can derive some recursive relations among these scalar products

\[ F_n(\theta^{(2)}_{p_m}, \ldots, \theta^{(2)}_{p_{m+1}}, \theta^{(1)}_{p_{m+1}}, \ldots, \theta^{(1)}_{p_1} | u_1, \ldots, u_n) = \]

\[ C_1 F_{n-1}(\theta^{(2)}_{p_m}, \ldots, \theta^{(2)}_{p_{m+1}}, \theta^{(1)}_{p_{m+1}}, \ldots, \theta^{(1)}_{p_1} | u_2, \ldots, u_n) + C_2 F_{n-1}(\theta^{(2)}_{p_m}, \ldots, \theta^{(2)}_{p_{m+1}}, \theta^{(1)}_{p_{m+1}}, \ldots, \theta^{(1)}_{p_1} | u_2, \ldots, u_n) + C_3 F_{n-1}(\theta^{(2)}_{p_m}, \ldots, \theta^{(2)}_{p_{m+1}}, \theta^{(1)}_{p_{m+1}}, \ldots, \theta^{(1)}_{p_1} | u_2, \ldots, \hat{u}_j, \ldots, u_n) + C_4 F_{n-1}(\theta^{(2)}_{p_m}, \ldots, \theta^{(2)}_{p_{m+1}}, \theta^{(1)}_{p_{m+1}}, \ldots, \theta^{(1)}_{p_1} | u_2, \ldots, \hat{u}_j, \ldots, u_n) + C_5 F_{n-1}(\theta^{(2)}_{p_m}, \ldots, \theta^{(2)}_{p_{m+1}}, \theta^{(1)}_{p_{m+1}}, \ldots, \theta^{(1)}_{p_1} | u_2, \ldots, \hat{u}_j, \ldots, u_n) + C_6 F_{n-1}(\theta^{(2)}_{p_m}, \ldots, \theta^{(2)}_{p_{m+1}}, \theta^{(1)}_{p_{m+1}}, \ldots, \theta^{(1)}_{p_1} | u_2, \ldots, \hat{u}_j, \ldots, u_n) \] (4.6)

where the concrete form of the coefficients \( C_i (i = 1, \ldots, 6) \) are given in Appendix C. The above recursive relation allows one to determine each scalar products in (4.4) uniquely. Here we give the explicit expressions of the first two \( F_1 \) and \( F_2 \) of the functions

\[ F_1(\theta^{(1)}_{p_1} | u_1) = \frac{\bar{e}(\theta^{(1)}_{p_1} - u_1)}{b(\theta^{(1)}_{p_1} - u_1)} \alpha_1(u_1) \alpha_2(\theta^{(1)}_{p_1}), \]

\[ F_2(\theta^{(2)}_{p_1} | u_1, u_2) = \left[ \frac{\bar{e}(\theta^{(2)}_{p_1} - u_1)\bar{e}(\theta^{(1)}_{p_1} - u_2)}{d(\theta^{(2)}_{p_1} - u_1)b(\theta^{(1)}_{p_1} - u_2)} e^{-\eta} - \frac{\bar{f}(\theta^{(2)}_{p_1} - u_1)}{y(u_1 - u_2)d(\theta^{(2)}_{p_1} - u_1)} \right] \alpha_1(u_1) \alpha_1(u_2) \alpha_3(\theta^{(1)}_{p_1}), \]

\[ F_2(\theta^{(1)}_{p_2}, \theta^{(1)}_{p_1} | u_1, u_2) = \left\{ \frac{\bar{e}(\theta^{(1)}_{p_1} - u_1)z(\theta^{(1)}_{p_1} - \theta^{(2)}_{p_2})\bar{e}(\theta^{(2)}_{p_1} - u_2)z(\theta^{(1)}_{p_2} - u_1)}{b(\theta^{(1)}_{p_1} - u_1)w(\theta^{(1)}_{p_1} - \theta^{(2)}_{p_2})b(\theta^{(2)}_{p_2} - u_2)} \right\} \alpha_1(u_1) \alpha_1(u_2) \alpha_2(\theta^{(1)}_{p_1}) \alpha_2(\theta^{(1)}_{p_2}). \] (4.7)
According to (3.25), (4.4) and (4.5), we can expand the Bethe states (4.1) as
\[
|\phi_n(u_1, \ldots, u_n)\rangle = \text{id} \times |\phi_n(u_1, \ldots, u_n)\rangle
\]
\[
= \sum_{m=0}^{N} \sum_{m_2=0}^{m} \sum_{P} \frac{1}{G_m(\theta_{p_1}^{(1)}, \ldots, \theta_{p_{m_2+1}}^{(1)}, \ldots, \theta_{p_m}^{(2)} | \theta_{p_1}^{(1)}, \ldots, \theta_{p_{m_2+1}}^{(1)} ; \theta_{p_{m_2+1}}^{(2)}, \theta_{p_m}^{(2)})}
\times \langle \theta_{p_{m_2+1}}^{(1)}, \ldots, \theta_{p_m}^{(2)} | \phi_n(u_1, \ldots, u_n) \rangle
\]
\[
= \sum_{m=0}^{N} \sum_{m_2=0}^{m} \sum_{P} \frac{F_n(\theta_{p_1}^{(1)}, \ldots, \theta_{p_{m_2+1}}^{(1)} ; \theta_{p_{m_2}}^{(2)}, \ldots, \theta_{p_m}^{(2)})}{G_m(\theta_{p_1}^{(1)}, \ldots, \theta_{p_{m_2+1}}^{(1)} ; \theta_{p_{m_2}}^{(2)}, \theta_{p_m}^{(2)})}
\times |\theta_{p_1}^{(1)}, \ldots, \theta_{p_{m_2+1}}^{(1)} ; \theta_{p_{m_2}}^{(2)}, \ldots, \theta_{p_m}^{(2)}\rangle,
\]\(4.8\)
where the notation \(\sum_{2m-m_2=n}\) indicates the sum over all integers \(\{0 \leq m_2 \leq m \leq N\}\) satisfying the condition: \(2m - m_2 = n\). Thanks to the fact that the scalar products \(F_i(u)\) defined by (4.4) can be determined by the very recursive relations (4.6). This allows us to give the explicit expressions of the Bethe states of the IK model with the periodic boundary condition.

4.2 Inverse Problem

The important problem in the theory of quantum integrable models, after diagonalizing the corresponding Hamiltonians, is to solve the corresponding quantum inverse scattering problem. Namely, local operators are reconstructed in terms of the matrix elements of the monodromy-matrix. The general method to solve the problem for a quantum integrable spin chain was given in [27, 28]. It is easy to check that the \(R\)-matrix (2.1)-(2.2) of the IK model possesses the required properties:

\[
\text{Initial condition : } R_{12}(0) = \varphi \mathbb{P}_{12},
\]
\[
\text{Unitarity relation : } R_{12}(u)R_{21}(-u) = \phi(u) \times \text{id},
\]
where
\[
\varphi = \sinh \eta - \sinh 5\eta, \quad \phi(u) = [\sinh \eta + \sinh(u - 5\eta)][\sinh \eta - \sinh(u + 5\eta)].
\]
The \(\mathbb{P}_{ij}\) is the permutation operator. As shown in [28], these properties of the \(R\)-matrix directly indicate the identity:

\[
\text{tr}_0 \{x_0 T_0(\theta_i)\} = \prod_{j=1}^{i-1} t^{-1}(\theta_j) x_i \prod_{j=1}^{i} t(\theta_j),
\]
where \( \{ x_j | j = 0, \ldots, N \} \) are local operators acting on the \( j \)-th space and \( t(u) \) is the transfer matrix. Define the local operator \( e^{ij} = |i \rangle \langle j | \), \( 1 \leq i, j \leq 3 \) and let \( x_0 = e_0^{ij} \), and then we can express the local spin operators in terms of the operator entries of the monodromy-matrix. As an example, here we list some of them

\[
e^{11}_i = \prod_{i=1}^{N-1} \prod_{j=i+1}^{N} \phi_{ij}^{-1} \varphi_{ij}^{-N} \prod_{j=1}^{i-1} t(\theta_j) A_1(\theta_i) \prod_{j=i+1}^{N} t(\theta_j),
\]
\[
e^{12}_i = \prod_{i=1}^{N-1} \prod_{j=i+1}^{N} \phi_{ij}^{-1} \varphi_{ij}^{-N} \prod_{j=1}^{i-1} t(\theta_j) C_1(\theta_i) \prod_{j=i+1}^{N} t(\theta_j),
\]
\[
e^{13}_i = \prod_{i=1}^{N-1} \prod_{j=i+1}^{N} \phi_{ij}^{-1} \varphi_{ij}^{-N} \prod_{j=1}^{i-1} t(\theta_j) C_2(\theta_i) \prod_{j=i+1}^{N} t(\theta_j),
\]
\[
e^{21}_i = \prod_{i=1}^{N-1} \prod_{j=i+1}^{N} \phi_{ij}^{-1} \varphi_{ij}^{-N} \prod_{j=1}^{i-1} t(\theta_j) B_1(\theta_i) \prod_{j=i+1}^{N} t(\theta_j),
\]
\[
e^{22}_i = \prod_{i=1}^{N-1} \prod_{j=i+1}^{N} \phi_{ij}^{-1} \varphi_{ij}^{-N} \prod_{j=1}^{i-1} t(\theta_j) A_2(\theta_i) \prod_{j=i+1}^{N} t(\theta_j),
\]
\[
e^{31}_i = \prod_{i=1}^{N-1} \prod_{j=i+1}^{N} \phi_{ij}^{-1} \varphi_{ij}^{-N} \prod_{j=1}^{i-1} t(\theta_j) B_2(\theta_i) \prod_{j=i+1}^{N} t(\theta_j),
\]
\[
e^{32}_i = \prod_{i=1}^{N-1} \prod_{j=i+1}^{N} \phi_{ij}^{-1} \varphi_{ij}^{-N} \prod_{j=1}^{i-1} t(\theta_j) A_3(\theta_i) \prod_{j=i+1}^{N} t(\theta_j),
\]

where

\[
\phi_{ij} = [\sinh \eta + \sinh(\theta_i - \theta_j - 5\eta)][\sinh \eta - \sinh(\theta_i - \theta_j + 5\eta)].
\]

Due to the fact that the Bethe states \([4.11]\) are obtained by acting the creation operators \( B_1(u) \) and \( B_2(u) \) (for the left Bethe state, by the acting the creation operators \( C_1(u) \) and \( C_2(u) \)) on the corresponding reference state and that all the local operators \( \{ e^{ij}_l | l = 1, \ldots, N \} \) have been reconstructed in terms of the operators \( A_i(u), B_i(u) \) and \( C_i(u) \) as \([4.10]\), one can perform the corresponding F-basis analysis of correlation functions \([11]\) of the IK model like those in the quantum integrable spin chains associated with the \( A \)-type (super)algebras \([5, 6, 11]\).
5 Conclusions

We have introduced a convenient basis (3.11) and (3.12) of the Hilbert space for the IK model with the periodic boundary condition, which is the quantum spin chain associated with the $A_2^{(2)}$ algebra. It is shown that matrix elements of the monodromy matrix acting on the very basis take simple forms (3.26)-(3.28), which is quite similar as that in the F-basis for a quantum spin chain associated with $A$-type (super)algebra. As an application, we have obtained the recursive relations (4.6) of vector components of the Bethe states of the model in the very basis, which allow us uniquely to determine the states. With the explicit expressions (4.8) of the Bethe states and the solution of quantum inverse problem, one can further calculate the correlation functions of the IK model with the periodic boundary condition.

It is well-known [17] that taking the rational limit (i.e., $\eta \to 0$) the IK model becomes the $su(3)$-invariant spin chain. It is easy to show that in this limit the resulting basis of (3.11) and (3.12) is exactly the rational version of the basis given recently in [26] which coincides with the F-basis [7] of the $su(3)$-invariant closed chain.

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Appendix A: Exchange relations

The quadratic commutation relation (2.6) allows us to derive the exchange relations among the operators (3.2) given by the matrix elements of the monodromy matrix $T(u)$. Here we list some of the exchange relations among the monodromy-matrix elements which have been used in our calculation.
\[
[A_1(u), A_1(v)] = [B_2(u), B_2(v)] = [C_2(u), C_2(v)] = [A_3(u), A_3(v)] = 0, \quad (A.1)
\]

\[
B_1(u)B_1(v) = w(v - u) [B_1(v)B_1(u) - \frac{1}{y(v - u)} B_2(v)A_1(u)] + \frac{1}{y(u - v)} B_2(u)A_1(v), \quad (A.2)
\]

\[
A_1(u)B_1(v) = z(v - u) B_1(v)A_1(u) - \frac{e(v - u)}{b(v - u)} B_1(u)A_1(v), \quad (A.3)
\]

\[
A_1(u)B_2(v) = \frac{c(v - u)}{d(v - u)} B_2(v)A_1(u) - \frac{g(v - u)}{d(v - u)} B_1(u)B_1(v) - \frac{f(v - u)}{d(v - u)} B_2(u)A_1(v), \quad (A.4)
\]

\[
B_1(u)B_2(v) = z(v - u) B_2(v)B_1(u) - \frac{e(v - u)}{b(v - u)} B_2(u)B_1(v), \quad (A.5)
\]

\[
B_2(u)B_1(v) = \frac{1}{z(u - v)} B_1(v)B_2(u) + \frac{e(u - v)}{c(u - v)} B_2(v)B_1(u), \quad (A.6)
\]

\[
C_1(u)B_1(v) = B_1(v)C_1(u) - \frac{e(v - u)}{b(v - u)} \left[ A_2(u)A_1(v) - A_2(v)A_1(u) \right], \quad (A.7)
\]

\[
B_1(u)B_3(v) = B_3(v)B_1(u) + \frac{\tilde{e}(v - u)}{b(v - u)} B_2(v)A_2(u) - \frac{e(v - u)}{b(v - u)} B_2(u)A_2(v), \quad (A.8)
\]

\[
B_2(u)B_3(v) = \frac{1}{z(v - u)} B_3(v)B_2(u) + \frac{e(u - v)}{c(u - v)} B_2(v)B_3(u), \quad (A.9)
\]

\[
B_3(u)B_2(v) = \frac{1}{z(v - u)} B_2(v)B_3(u) + \frac{e(u - v)}{b(u - v)} B_3(v)B_2(u), \quad (A.10)
\]

\[
A_2(u)B_2(v) = z(u - v)z(v - u)B_2(v)A_2(u) + \frac{\tilde{e}(u - v)}{b(u - v)} \left[ B_3(u)B_1(v) - B_1(u)B_3(v) + \frac{\tilde{e}(u - v)}{b(u - v)} B_2(u)A_2(v) \right], \quad (A.11)
\]

\[
A_3(u)B_1(v) = \frac{b(u - v)}{d(u - v)} B_1(v) A_3(u) - \frac{1}{y(u - v)} B_3(u)A_2(v) + \frac{e(u - v)}{d(u - v)} B_2(v)C_3(u) - \frac{\tilde{f}(u - v)}{d(u - v)} B_2(u)C_3(v), \quad (A.12)
\]

\[
A_3(u)B_2(v) = \frac{c(u - v)}{d(u - v)} B_2(v)A_3(u) - \frac{1}{y(u - v)} B_3(u)B_3(v) - \frac{\tilde{f}(u - v)}{d(u - v)} B_2(u)A_3(v), \quad (A.13)
\]

\[
C_1(u)B_2(v) = \frac{b(v - u)}{d(v - u)} B_2(v)C_1(u) + \frac{e(v - u)}{d(v - u)} B_3(v)A_1(u) - \frac{f(v - u)}{d(v - u)} B_3(u)A_1(v) - \frac{g(v - u)}{d(v - u)} A_2(u)B_1(v), \quad (A.14)
\]

\[
C_3(u)B_2(v) = \frac{d(v - u)}{b(v - u)} B_2(v)C_3(u) + \frac{g(v - u)}{b(v - u)} B_3(v)A_2(u) + \frac{f(v - u)}{b(v - u)} A_3(v)B_1(u) - \frac{e(v - u)}{b(v - u)} A_3(u)B_1(v), \quad (A.15)
\]

\[
C_2(u)B_1(v) = \frac{d(v - u)}{b(v - u)} B_1(v)C_2(u) + \frac{g(v - u)}{b(v - u)} A_2(v)C_1(u) + \frac{f(v - u)}{b(v - u)} C_3(v)A_1(u) - \frac{e(v - u)}{b(v - u)} C_3(u)A_1(v), \quad (A.16)
\]

\[
C_4(u)A_1(v) = \frac{b(u - v)}{d(u - v)} A_1(v)C_3(u) + \frac{e(u - v)}{d(u - v)} B_1(v)C_2(u) - \frac{1}{y(u - v)} A_2(u)C_1(v) - \frac{\tilde{f}(u - v)}{d(u - v)} B_1(u)C_2(v), \quad (A.17)
\]

\[
C_2(u)B_2(v) = B_2(v)C_2(u) + \frac{1}{y(u - v)} \left[ B_3(v)C_1(u) - C_3(u)B_1(v) \right] + \frac{f(v - u)}{d(v - u)} \left[ A_3(v)A_1(u) - A_3(u)A_1(v) \right], \quad (A.18)
\]
Appendix B: Proof of the vanishing properties

As a typical example, we give a brief proof of the identity (3.21), namely,

\[ B_1(\theta^{(2)}_{\pi_l})|0\rangle = 0, \quad l = m + 1, \ldots, N. \]  

We prove the above identity by the induction. First we need to prove

\[ B_1(\theta^{(2)}_{\pi_l})|0\rangle = 0. \]  

It is easy to check that it is true for the \( N = 1 \) case. The proof goes by induction in the number of particles starting from \( N = 1 \). Assume that \( (B.2) \) were also true for the cases of \( N = 1, \ldots, L \), which can be denoted as

\[ B_1^L(\theta^{(2)}_{\pi_l})|0\rangle^L = 0. \]
where the operator $X^L$ means $X$ is embedded in the $L$ tensor space. We show that it is valid for $N = L + 1$

$$
B_1^{L+1}(\theta_{p_1}^{(2)})|0\rangle^{L+1} = \left\{ A_1^{L+1}(\theta_{p_1}^{(2)}) \otimes B_1^L(\theta_{p_1}^{(2)}) + B_1^{L+1}(\theta_{p_1}^{(2)}) \otimes A_2^L(\theta_{p_1}^{(2)}) + B_2^{L+1}(\theta_{p_1}^{(2)}) \otimes C_3^L(\theta_{p_1}^{(2)}) \right\} \left\{ |0\rangle^{(L+1)} \otimes |0\rangle^L \right\} = 0,
$$

(B.4)

where the operator $X^{(L+1)}$ means $X$ is embedded in the $(L+1)$-th space. Thus, the relation (B.2) is proven. Using (B.2) and the exchange relations (A.5), we can easily get

$$
B_1(\theta_{p_1}^{(2)})|\theta_{p_{m+1}}^{(2)}, \ldots, \theta_{p_m}^{(2)}\rangle = 0.
$$

(B.5)

Finally, the exchange relations (A.2) and (B.5) allow us to derive

$$
B_1(\theta_{p_1}^{(2)})|\theta_{p_1}^{(1)}, \ldots, \theta_{p_{m+2}}^{(2)}, \theta_{p_{m+1}}^{(2)}, \ldots, \theta_{p_m}^{(2)}\rangle = \prod_{j=1}^{m_2} w(\theta_{p_j}^{(1)} - \theta_{p_j}^{(2)}) B_1(\theta_{p_1}^{(1)}), \ldots, B_1(\theta_{p_{m+2}}^{(2)}) B_1(\theta_{p_1}^{(2)})|\theta_{p_{m+1}}^{(2)}, \ldots, \theta_{p_m}^{(2)}\rangle = 0.
$$

(B.6)

Thus, the relation (B.1) has been proved.

**Appendix C: Coefficients of the recursive relation (4.6)**

By using the relations (3.27), (3.28) and (4.1), we can derive the recursive relations (4.6) among the functions $\{F_i(u)\}$, with their coefficients being given as follows:

$$
C_1 = \sum_{i=1}^{m_2} \frac{c(\theta_{p_i}^{(2)} - u_1)}{d(\theta_{p_i}^{(2)} - u_1)} \prod_{h=1}^{i-1} \omega(\theta_{p_h}^{(1)} - \theta_{p_i}^{(1)}) \prod_{j=1}^{m_2} \frac{z(\theta_{p_i}^{(1)} - u_1)}{\omega(\theta_{p_i}^{(1)} - \theta_{p_j}^{(1)})} \times \alpha_1(u_1) \alpha_2(\theta_{p_i}^{(1)})
$$

$$
\times \prod_{k=m_2+1}^{m} \frac{c(\theta_{p_k}^{(2)} - u_1)}{d(\theta_{p_k}^{(2)} - u_1)} z(\theta_{p_i}^{(1)} - \theta_{p_k}^{(2)}) z(\theta_{p_k}^{(2)} - \theta_{p_i}^{(1)}),
$$

(C.1)
\[ C_2 = \sum_{i=m_2+1}^{m} \left[ \frac{\bar{e}(\theta^{(2)} - u_1)}{d(\theta^{(2)} - u_1)} - \sum_{l=1}^{m_2} \frac{e(\theta^{(1)} - \theta^{(2)}_l)e(\theta^{(1)}_l - u_1)c(\theta^{(2)}_l - u_1)}{b(\theta^{(1)}_l - \theta^{(2)}_l)b(\theta^{(1)}_l - u_1)d(\theta^{(2)}_l - u_1)} \prod_{j=1}^{m_2} z(\theta^{(1)}_j - u_1)z(\theta^{(1)}_j - \theta^{(2)}_j) \right] \]
\[ \times \prod_{k=m_2+1}^{m} \frac{b(\theta^{(2)}_k - \theta^{(1)}_k)c(\theta^{(2)}_k - u_1)}{c(\theta^{(2)}_k - \theta^{(1)}_k)d(\theta^{(2)}_k - u_1)} z(\theta^{(2)}_k - \theta^{(2)}_k) \times \alpha_1(u_1)\bar{\xi}(\theta_p), \quad (C.2) \]

\[ C_3 = -\sum_{i=m_2+1}^{m} \frac{\bar{f}(\theta^{(2)}_i - u_1)}{d(\theta^{(2)}_i - u_1)} \prod_{j=m_2+1}^{m} \frac{c(\theta^{(2)}_j - u_1)c(\theta^{(2)}_j - \theta^{(2)}_i)}{d(\theta^{(2)}_j - u_1)d(\theta^{(2)}_j - \theta^{(2)}_i)} \prod_{i=1}^{m_2} \frac{d(\theta^{(1)}_i - u_1)}{b(\theta^{(1)}_i - u_1)} \times \alpha_1(u_1)\alpha_3(\theta^{(2)}_i) \]
\[ \times \left\{ \sum_{j=2}^{n} \frac{\alpha_1(u_j)}{g(u_1 - u_j)} \prod_{i=2}^{j-1} \omega(u_j - u_i) \prod_{k=2}^{n} z(u_k - u_j) \right\}, \quad (C.3) \]

\[ C_4 = -\sum_{i=m_2+1}^{m} \sum_{j=m_2+1}^{m} \left\{ \frac{\bar{g}(\theta^{(2)}_j - u_1)e(\theta^{(2)}_j - u_1)}{d(\theta^{(2)}_j - u_1)d(\theta^{(2)}_j - u_1)} - \frac{\bar{f}(\theta^{(2)}_i - u_1)c(\theta^{(2)}_i - u_1)}{d(\theta^{(2)}_i - u_1)d(\theta^{(2)}_i - u_1)}\bar{g}(\theta^{(2)}_i - \theta^{(2)}_j) \right\} w(\theta^{(1)}_i - \theta^{(1)}_j) \]
\[ \times \prod_{j=m_2+1}^{m} \frac{c(\theta^{(2)}_j - u_1)z(\theta^{(2)}_j - \theta^{(2)}_j)z(\theta^{(2)}_j - \theta^{(2)}_i)}{d(\theta^{(2)}_j - u_1)z(\theta^{(2)}_j - \theta^{(1)}_i)z(\theta^{(2)}_j - \theta^{(1)}_i)} \prod_{k=1}^{m_2} \frac{d(\theta^{(2)}_k - u_1)}{b(\theta^{(2)}_k - u_1)} \times \alpha_1(u_1)\bar{\xi}(\theta_p)\bar{\xi}(\theta_p) \]
\[ \times \left\{ \sum_{j=2}^{n} \frac{\alpha_1(u_j)}{g(u_1 - u_j)} \prod_{i=2}^{j-1} \omega(u_j - u_i) \prod_{k=2}^{n} z(u_k - u_j) \right\}, \quad (C.4) \]

\[ C_5 = -\sum_{l=1}^{m_2} \sum_{i=m_2+1}^{m_2} \left\{ \frac{\bar{g}(\theta^{(1)}_i - u_1)e(\theta^{(1)}_i - u_1)}{b(\theta^{(1)}_i - u_1)b(\theta^{(1)}_i - u_1)} - \frac{\bar{f}(\theta^{(1)}_i - u_1)g(\theta^{(1)}_i - \theta^{(1)}_i)}{b(\theta^{(1)}_i - u_1)b(\theta^{(1)}_i - \theta^{(1)}_i)}\bar{g}(\theta^{(1)}_i - \theta^{(1)}_l) \right\} \]
\[ \times \prod_{h=1}^{i-1} \frac{w(\theta^{(1)}_h - \theta^{(1)}_h)}{w(\theta^{(1)}_h - \theta^{(1)}_h)} \prod_{h=1}^{i} \frac{w(\theta^{(1)}_h - \theta^{(1)}_h)}{w(\theta^{(1)}_h - \theta^{(1)}_h)} \prod_{j=1}^{m_2} \frac{z(\theta^{(1)}_j - u_1)}{z(\theta^{(1)}_j - \theta^{(1)}_j)z(\theta^{(1)}_j - \theta^{(1)}_j)} \]
\[ \times \prod_{k=m_2+1}^{m} \frac{c(\theta^{(2)}_k - u_1)}{d(\theta^{(2)}_k - u_1)} z(\theta^{(2)}_k - \theta^{(2)}_k)z(\theta^{(2)}_k - \theta^{(2)}_k)z(\theta^{(2)}_k - \theta^{(2)}_k)z(\theta^{(2)}_k - \theta^{(1)}_k) \times \alpha_1(u_1)\alpha_2(\theta^{(1)}_p)\alpha_2(\theta^{(1)}_p) \]
\[ \times \left\{ \sum_{j=2}^{n} \frac{\alpha_1(u_j)}{g(u_1 - u_j)} \prod_{i=2}^{j-1} \omega(u_j - u_i) \prod_{k=2}^{n} z(u_k - u_j) \right\}, \quad (C.5) \]
\[ C_6 = - \sum_{l=1}^{m_2} \sum_{i=m_2+1}^{m} \left\{ \frac{e(\theta_{pi}^{(2)} - u_1) g(\theta_{pi}^{(1)} - u_1) z(\theta_{pi}^{(1)} - \theta_{pi}^{(1)})}{d(\theta_{pi}^{(2)} - u_1) b(\theta_{pi}^{(1)} - u_1) w(\theta_{pi}^{(1)} - \theta_{pi}^{(1)})} \prod_{h \neq l} z(\theta_{pi}^{(1)} - \theta_{pi}^{(1)}) \right. \\
\left. + \frac{\bar{g}(\theta_{pi}^{(2)} - \theta_{pi}^{(1)})}{b(\theta_{pi}^{(2)} - \theta_{pi}^{(1)})} \frac{\bar{f}(\theta_{pi}^{(2)} - \theta_{pi}^{(1)})}{b(\theta_{pi}^{(2)} - \theta_{pi}^{(1)}) g(\theta_{pi}^{(2)} - \theta_{pi}^{(1)})} \right. \\
\left. \frac{\bar{f}(\theta_{pi}^{(1)} - u_1) c(\theta_{pi}^{(2)} - u_1)}{b(\theta_{pi}^{(1)} - u_1) d(\theta_{pi}^{(2)} - u_1) w(\theta_{pi}^{(1)} - \theta_{pi}^{(1)})} \prod_{j \neq \bar{i},k} z(\theta_{pi}^{(1)} - \theta_{pi}^{(1)}) z(\theta_{pi}^{(1)} - \theta_{pi}^{(1)}) \right\} \\
\times \prod_{j=m_2+1}^{m} \frac{c(\theta_{pj}^{(2)} - u_1) z(\theta_{pj}^{(2)} - \theta_{pj}^{(2)}) z(\theta_{pj}^{(1)} - \theta_{pj}^{(2)}) z(\theta_{pj}^{(2)} - \theta_{pj}^{(1)}) \times \alpha_1(u_1) \alpha_2(\theta_{pj}^{(1)}) \bar{\xi}(\theta_{pj})}{d(\theta_{pj}^{(2)} - u_1) z(\theta_{pj}^{(2)} - \theta_{pj}^{(1)}) z(\theta_{pj}^{(1)} - \theta_{pj}^{(2)})} \\
\times \left\{ \sum_{j=2}^{n} \frac{\alpha_1(u_j)}{g(u_1 - u_j)} \prod_{i=2}^{j-1} \omega(u_j - u_i) \prod_{k=2}^{n} z(u_k - u_j) \right\}, \tag{C.6} \]

where the functions \( \xi(u) \) and \( \bar{\xi}(u) \) are given by (3.9).

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