On the Universal Scheme of $r$-relative clusters of a family of surfaces

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1 Introduction

Kleiman introduced in (Kleiman, 1981) the iterated blowups, varieties $X_r$ which naturally parametrize ordered $r$-point clusters of a variety $X$, for each $r$. These have proven to be very useful in enumerative geometry, and in other areas of algebraic geometry, especially in the case when $X$ is a surface. See (Alberich-Carramiñana and Roé, 2005), (De Poi, 2003), (Fernández de Bobadilla, 2005), (Kleiman, 1981), (Kleiman and Piene, 1999), (Kleiman and Piene, 2004), (Ran, 2005), (Roé, 2001a), and (Roé, 2001b).

Following the general philosophy that Grothendieck expounds in his “Éléments de Géometrie Algébrique”, given a scheme $B$, we generalize the concept of point cluster of a surface to the $B$-points cluster of $B$-surface $S$, or relative clusters of a family of surfaces $\pi: S \to B$. We define schemes $Cl_r$ that generalize the iterated blow ups in the sense that they naturally parametrize $B$-points clusters of $r$ points of a $B$-surface $S$, or $r$-relative clusters of a family of surfaces $\pi: S \to B$, and we prove their existence if $B$ is proper and $S$ quasiprojective (Theorem 2.4).

An explicit construction of $Cl_{r+1}$ is possible, in terms of iterated blow ups centered at suitable sheaves of ideals (analogous to Kleiman (Kleiman, 1981)) with support at a subscheme of $Cl_r^2 := Cl_r \times_{Cl_{r-1}} Cl_r$, which fails to be Cartier only along the diagonal. More precisely, we show that there is an open $V \subseteq Cl_{r+1}$, consisting of the clusters of sections in which the image of last section is not contained in the exceptional divisor, and a stratification $\bigsqcup_{p \in P} Cl_p \hookrightarrow Cl_r^2$, by locally closed subschemes, such that $V$ is isomorphic to a subset of strata $\bigsqcup_{p \in \text{adm}} Cl_p$ (Theorems 4.8 and 4.9). For some strata whose closure intersects the diagonal, the corresponding component of $Cl_{r+1}$ is (an open subset of) a blowup as above; since strata are often singular along the diagonal, the center of the blowup is not always uniquely determined (but the blowup map is uniquely determined). In section 5 we show a few simple examples illustrating the situations that may occur.

2 Preliminaries

We fix an algebraically closed base field $K$, consider all morphisms as morphisms of $K$-schemes and, also, fix a separated surjective morphism $\pi: S \to B$. 

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For a morphism $f : X \to Y$ and a subscheme $Z \subseteq Y$, we denote $f^{-1}(Z)$ the schematic pre-image (i.e. $f^{-1}(Z) \cong X \times_Y Z$, which is a subscheme of $X$). We use $f : X \hookrightarrow Y$ to denote a morphism which is an isomorphism with its image. Given a scheme $Y$ and $y \in Y$, we call $k(y)$ the residue field of $y$, $im_y$ the immersion $\text{Spec}(k(y)) \hookrightarrow Y$ of $y$ and, if $pr_Y : Y \times B \to Y$ is the projection, $B_y := pr_Y^{-1}(y)$ and $pr_y := pr_Y|_{B_y}$.

**Lemma 2.1.** Any section $a : A \to C$ of a separated morphism $\alpha : C \to A$, is a closed immersion.

*Proof.* It is easy to check that the following diagram is a cartesian square.

$$
\begin{array}{ccc}
A & \xrightarrow{a} & C \\
\downarrow{a} & & \downarrow{\text{id}_C \times_A (a \circ \alpha)} \\
C & \hookrightarrow & C \times_A C
\end{array}
$$

where $\Delta$ is the diagonal, which is a closed immersion by definition, and then, so is the morphism $a$.

**Lemma 2.2.** If the following diagram is a cartesian square,

$$
\begin{array}{ccc}
C & \xrightarrow{\alpha} & A \\
\downarrow{g} & & \downarrow{f} \\
D & \xrightarrow{\beta} & B
\end{array}
$$

and $b : B \to D$ is a section of $\beta$, then there is a unique section $a : A \to C$ of $\alpha$, which makes the following diagram a cartesian square,

$$
\begin{array}{ccc}
A & \xrightarrow{\alpha} & C \\
\downarrow{f} & & \downarrow{g} \\
B & \xrightarrow{b} & D
\end{array}
$$

*Proof.* It is clear that the right diagram below, is the pull back of the left diagram.

$$
\begin{array}{ccc}
A & \xrightarrow{\text{id}_A} & A \\
\downarrow{\text{id}_B} & & \downarrow{\text{id}_B} \\
B & \xrightarrow{\text{id}_B} & B
\end{array}
$$

So, by the transitivity of the base extension, see [Hartshorne 1977, II.3(p.89)], in the following sequence of cartesian squares,

$$
\begin{array}{ccc}
X & \xrightarrow{a} & C & \xrightarrow{\alpha} & A \\
\downarrow{F} & & \downarrow{g} & & \downarrow{f} \\
B & \xrightarrow{b} & D & \xrightarrow{\beta} & B
\end{array}
$$

$X = A$, $F = f$ and $a$ is a section of $\alpha$. 

\[2\]
Remark 1. Under conditions of previous lemma, by lemma 2.1 if the morphisms \(\alpha\) and \(\beta\) are separated, the morphisms \(a\) and \(b\) are closed immersions.

Lemma 2.3. If the following diagram is a cartesian square,

\[
\begin{array}{ccc}
A & \xrightarrow{a} & C \\
\downarrow f & & \downarrow g \\
B & \xrightarrow{b} & D
\end{array}
\]

with \(a\) and \(b\) closed immersions and the morphisms \(\text{bl}_A : \text{Bl}_A(C) \to C\) and \(\text{bl}_B : \text{Bl}_B(D) \to D\) are the blow up of \(C\) and \(D\) at \(a(A)\) and \(b(B)\) respectively. Then there is a unique morphism \(g' : \text{Bl}_A(C) \to \text{Bl}_B(D)\) such that the following diagram is a cartesian square.

\[
\begin{array}{ccc}
\text{Bl}_A(C) & \xrightarrow{\text{bl}_A} & C \\
\downarrow g' & & \downarrow g \\
\text{Bl}_B(D) & \xrightarrow{\text{bl}_B} & D
\end{array}
\]

Proof. The schemes \(a(A)\) and \(g^{-1}(b(B))\) are the same because \(A\) and \(B\) are isomorphic to \(a(A)\) and \(b(B)\) respectively, and then \(a(A)\) is isomorphic to the pull back of,

\[
\begin{array}{ccc}
C & \xrightarrow{g} & D \\
\downarrow b(B) \leftarrow & & \downarrow b(B)
\end{array}
\]

which is the definition of \(g^{-1}(b(B))\). So, the pre-image of the center of the blow up \(\text{bl}_B\), the closed subscheme \(b(B)\), by the morphism \((g \circ \text{bl}_A)\) is exactly the exceptional divisor of the blow up \(\text{bl}_A\). By the universal property of the blow up, there exist a unique morphism \(g' : \text{Bl}_A(C) \to \text{Bl}_B(D)\) such that \(\text{bl}_B \circ g' = g \circ \text{bl}_A\) and the Blow up closure lemma of [Vakil, 2012, p.422 19.2.6] give the cartesianity.

Remark 2. Under conditions of previous lemma, by the transitivity of the base extension, if the right diagram of the following diagrams is a cartesian square, then so is the left diagram.

\[
\begin{array}{ccc}
C & \xrightarrow{\alpha} & A \\
\downarrow g & & \downarrow f \\
D & \xrightarrow{\beta} & B
\end{array} \quad \Rightarrow \quad \begin{array}{ccc}
\text{Bl}_A(C) & \xrightarrow{\alpha \circ \text{bl}_A} & A \\
\downarrow g' & & \downarrow f \\
\text{Bl}_B(D) & \xrightarrow{\beta \circ \text{bl}_B} & B
\end{array}
\]

Definition 2.1. \((X, \psi)\) is a sections family (or sf) of \(\pi\) if \(X\) is a scheme and \(\psi : X \times B \to S\) is a morphism that makes the following diagram commute.
Let \((X, \psi)\) a sf of \(\pi\), we call \((X, \psi)\) an Universal family of sections (or Usf) of \(\pi\) if it satisfies the following universal property: for every sf \((Y, \rho)\) of \(\pi\) there is a unique morphism \(f: Y \to X\) which makes the following diagram commute.

If the Usf of \(\pi\) exists, then is unique up to unique isomorphism, and we fix the notation \((X, \psi)\) for it.

**Theorem 2.4.** If \(B\) is proper and \(S\) quasiprojective then the Usf \((X, \psi)\) of \(\pi\) exist and \(X\) is quasiprojective and locally noetherian \([\text{Nitsure, 2003, ch 6, ex 2}].\)

For the morphism \(\pi\) there is a contravariant functor \(S._{-}: \text{Sch} \to \text{Set}\) defined as follows. For every scheme \(Y\), let

\[
S.Y := \{\rho \in \text{Hom}_{\text{Sch}}(B \times Y, S) \text{ such that } (Y, \rho) \text{ is a sf of } \pi\}
\]

and given \(f\) a morphism, \(S.f := (Id_B \times f)^*\).

**Notation** Consider \((Y, \rho)\) a sf of \(\pi\) and \(y \in Y\), if \(y\) is a closed point, \(B \cong B_y\) and we denote \(y\) as the section of \(\pi\).

If \(y\) is not closed \(B \not\cong B_y\), but we can construct \(S^y := \text{Spec}(k(y)) \times S\), \(\pi^y := Id_{\text{Spec}(k(y))} \times \pi: S^y \to B_y\) and \(y := S.im_y(\rho) \times pr_y\) as a section of \(\pi^y\).

**Remark 3.** If \(S\) is a blow up, by the Blow up closure lemma of \([\text{Vakil, 2013, p.422 19.2.6}],\) so is \(S^y\), and its exceptional divisor is the pre-image of the exceptional divisor of \(S\) by the projection.

**Proposition 2.5.** A scheme \(X\) represents \(S._{-}\) if and only if \(X\) is the Usf of \(\pi\)

**Proof.** We just sketch how the proof works. If \((X, \psi)\) is the Usf of \(\pi\) then we can construct a natural isomorphism \(\mu: S._{-} \to \text{Hom}_{\text{Sch}}(_{-}, X)\). Given \(Y\) a scheme, for each \(\rho \in S.Y\), by the universal property of \((X, \psi)\), we define \(\mu_Y(\rho) \in \text{Hom}_{\text{Sch}}(Y, X)\) as the unique morphism such that \(\rho = S.(\mu_Y(\rho))(\psi)\).

If \(X\) represents \(S._{-}\), fix a natural isomorphism \(\eta: S._{-} \to \text{Hom}_{\text{Sch}}(_{-}, X)\), then, if \(\psi := \eta_X^{-1}(Id_X) \in S.X\), the couple \((X, \psi)\) is the Usf of \(\pi\).

**Corollary 2.6.** If the Usf \((X, \psi)\) of \(\pi\) exists, then the following map is bijective.

\[
\Phi : X(K) \to \{\text{sections of } \pi: S \to B\}
\]
3 Clusters of sections

Given a smooth projective surface $S$ a \textit{cluster} is a finite set of points $K$ of $S$ or a finite sequence of blow ups of $S$ centred at points, such that, for every $p \in K$, if $q$ is a point such that $p$ is infinitely near to $q$, then $q \in K$, see Casas-Alvero [2000]. For each $n \geq 1$ there is $X_n$ a family of smooth projective surfaces which parametrizes ordered $n$-point clusters of $S$. See J.Roé [Roé, 2012], and in greater generality, see Kleiman [1981].

\textbf{Definition 3.1.} $\pi$ is a \textit{family} if it is flat and finite type, $B$ is irreductible, regular and the generic fibre is integral. $\pi$ is a family of surfaces if it is a family of relative dimension 2.

Given $\pi: S \to B$ a family of surfaces with $(X, \psi)$ Usf, $\sigma \in X(K)$ and $\pi_\sigma: S_{\sigma} \to B$ the family of surfaces obtained by blowing up the image of the section $\sigma$ in $S$, i.e. $\pi_\sigma := \text{bl}_\sigma \circ \pi$, we denote by $E_\sigma$ the exceptional divisor of $S_{\sigma}$ and $(X_\sigma, \psi_\sigma)$ the Usf. of $\pi_\sigma$.

Sections of $\pi_\sigma$ whose image is contained in $E_\sigma$ are said to belong to the first infinitesimal neighbourhood of section $\sigma$. Inductively, a section $\tau$ belongs to the $k$-th infinitesimal neighbourhood of $\sigma$ if belongs to the first infinitesimal neighbourhood of a section in the $(k-1)$-th infinitesimal neighbourhood of $\sigma$. A section \textit{infinitely near} to $\sigma$ is a section in some infinitesimal neighbourhood of $\sigma$.

\textbf{Definition 3.2.} An \textit{ordered relative cluster of $\pi$ without base} or simply an \textit{ordered relative cluster} is a finite sequence of morphisms $\{\sigma_n\}_n$ such that for all $n \geq 0$, $\sigma_n$ is section of $\pi_n: S_n \to B$, where, for $n > 0$, $\pi_n$ makes the following diagram commute,

$$
\begin{array}{ccc}
S_n := \text{Bl}_{\sigma_{n-1}(B)}(S_{n-1}) & \xrightarrow{\text{bl}_{\sigma_{n-1}(B)}} & S_{n-1} \cup \sigma_{n-1}(B) \\
\begin{array}{c}
\pi_n \\
\pi_{n-1}
\end{array} & & \downarrow \\
& B &
\end{array}
$$

with $S_0 := S$ and $\pi_0 := \pi$.

\textbf{Definition 3.3.} Given $\sigma$, section of $\pi$, an \textit{ordered relative cluster of $\pi$ with base $\sigma$} is $\{\sigma_n\}_n$ an ordered relative cluster of $\pi$ such that for all $n$, $\sigma_n$ is infinitely near to $\sigma$.

\textbf{Definition 3.4.} For every $r > 0$, $(C, \beta)$ is a \textit{$r$-relative clusters family} (or $r$-rcf) of $\pi$ if $C$ is a scheme and each $\beta_i: C \times B \to S^C_i$, for $i = 0, \ldots, r-1$, is a section of each $\pi^C_i$ where $S^C_0 := C \times S$, $\text{bl}^C_i: S^C_{i+1} \to S^C_i$ is the blow up at the image of $\beta_i$, $\pi^C_0 := (\text{Id}_C \times \pi)$ and $\pi^C_{i+1} := \text{bl}^C_i \circ \pi^C_i$.

\textbf{Definition 3.5.} Given a $r$-rcf $(C, \beta)$ and $c \in C(K)$, for each $i = 0, \ldots, r-1$, we define $c^C$ and $(\pi^C_i)_c$ by base change, according to
Proposition 3.1. Given a r-ref $(C, \beta)$ and $c \in C(K)$, the sequence of morphisms $(\beta^0, \ldots, \beta^{r-1})$ is a relative cluster of $\beta$.

Proof. It is clear from the definition that $\beta^0$ is section of $(\pi_C)^c$. The scheme $(S^C_0)^c_c$ is isomorphic to $S$, so $\beta^0$ is a section of $\beta$.

The question now is if for each $i$, $(S^C_i)^c_c$ is $B$-isomorphic to the blow up of $(S^C_i)^c$ at $\beta_i(B)$. We can apply the lemma 2.3 to the lefthand side square of the previous diagram, which says that there is a unique morphism that makes the following diagram a cartesian square.

$$
\begin{array}{ccc}
B & \xrightarrow{\beta_i} & (S^C_i)^c_c \xleftarrow{(\pi_C)^c} B & \xrightarrow{\text{Spec}(K(c)) \cong \text{Spec}(K)} \\
\downarrow & & \downarrow & \\
C \times B & \xrightarrow{\beta_i} & S^C_i & \xleftarrow{\pi_C} C \times B & \xrightarrow{\text{pre}} C
\end{array}
$$

By the transitivity of the base change, $(S^C_i)^c_c$ is the previous pull back too, so $\text{Bl}_{C(B)}((S^C_i)^c_c) \cong_B (S^C_{i+1})^c_c$. \qed

Remark 4. Given $(C, \beta)$ a r-ref of $\pi$ there are a $S^C_i$, the blow up of $S^C_{i-1}$ at the image of $\beta_{i-1}$, and $\pi_i^C := \pi_{i-1} \circ \text{bl}_{i}^C$. But it is possible that there are no sections of $\pi_i^C$.

Definition 3.6. A morphism between two r-ref $(C, \beta)$ and $(C', \beta')$, or r-ref-morphism is a morphism $f:C \rightarrow C'$ such that for all $i = 0, \ldots, r-1$, $\beta_i$ is the base change of $\beta'_i$, i.e. the following diagram is a cartesian square,

$$
\begin{array}{ccc}
C \times B & \xrightarrow{\beta_i} & S^C_i \\
\downarrow & & \downarrow \beta_i \\
C' \times B & \xrightarrow{\beta_i'} & S'^C_i
\end{array}
$$

where $f_0 = f \times \text{Ids}$ and $f_i$, for $i = 1, \ldots, r-1$, is defined iteratively. Suppose $f_i$ defined and satisfy the hypothesis that previous diagram is a cartesian square, so lemma 2.3 give the existence of the morphism $f_{i+1}: S^C_{i+1} \rightarrow S'^C_{i+1}$.

Remark 5. There is a morphism $f_r: S^C_r \rightarrow S'^C_r$ such that $f_0 \circ \text{bl}_{r}^C = \text{bl}_{r}^{C'} \circ f_r$.

Notation Given a r-ref $(C, \beta)$ with $r \geq k$ and $\beta_r = (\beta_0, \ldots, \beta_{r-1})$, we denote $\beta_{|k} = (\beta_0, \ldots, \beta_{k-1})$, so $(C, \beta_{|k})$ is a k-ref.

Remark 6. Given a r-ref $(C, \beta)$ with $r \geq k$, the target schemes of $\beta_i$ and $(\beta_{|k})_i$ are the same.

Definition 3.7. A Universal r-relative clusters family (or r-Urcf) of $\pi$ is a r-ref $(C, \beta)$ such that for every r-ref $(C', \beta')$ there is a unique r-ref-morphism $f:(C', \beta') \rightarrow (C, \beta)$.  

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If a $r$-$\text{Urcf}$ of $\pi$ exists, by abstract nonsense, it is uniquely determined, up to unique isomorphism. We will use $(C_l, \alpha')$ to denote it, and $\pi_0' := \pi^{C_l}$, $\text{bl}_r' := \text{bl}_{C_l}'$, and $S_l' := S_l^{C_l}$.

**Remark 7.** $(Y, \rho)$ is a sf of $\pi$ if and only if $(\text{Id}_Y \times Y): Y \times B \to Y \times S$ is a section of $\pi_0'$, i.e. if and only if $(Y, \text{Id}_Y \times Y)$ is a 1-ref of $\pi$. Therefore, if $Y$ is separated, the image of $(\text{Id}_Y \times Y)$ is a closed subscheme, and, if they exist, $X \cong C_l$, and $\alpha_0' \cong \text{Id}_X$.

**Definition 3.8.** Assuming $(r + 1)$-$\text{Urcf}$ and $r$-$\text{Urcf}$ of $\pi$ exist, $(C_{l+1}, \alpha'^{+1})$ and $(C_{l+1}, \alpha^{+1}_{l+1}, \alpha^{+1}_{l+2}, \text{bl}_{l+1}r \circ \alpha^{+1}_{l+1})$ are $r$-$\text{Urcf}$ of $\pi$. So, there are two $r$-$\text{Urcf}$-morphisms

$$p'^{r+1} : (C_{l+1}, \alpha'^{+1}) \to (C_l, \alpha')$$

$$f'^{r+1} : (C_{l+1}, \alpha^{+1}_{l+1}, \alpha^{+1}_{l+2}, \text{bl}_{l+1}r \circ \alpha^{+1}_{l+1}) \to (C_l, \alpha')$$

Observe that, if $(r - 1)$-$\text{Urcf}$ of $\pi$ exists, $p' \circ p'^{+1} = p' \circ f'^{+1}$ as a $r - 1$-$\text{Urcf}$-morphisms.

In forthcoming sections we always consider $C_{l+1}$ to be a $C_l$-scheme with structure morphism $p'^{+1}$; in particular when we consider some fibred product of some $r$-$\text{Urcf}$ it will be understood with respect to morphisms $p'^{+1}$.

For each $r = 1, 2, \ldots$ we define a contravariant functor $Cl^r : K-Sch \to \text{Set}$. It sends each scheme $C$ to the set of sequences of morphisms:

$$Cl^r C := \{\beta : C \times B \to SC \text{ such that } (C, \beta) \text{ is a } r \text{-rcf of } \pi\}.$$

Given a morphism $f : C' \to C$ and a sequence of morphisms $\beta : Cl^r C$ the image $\beta' := Cl^r f(\beta)$ will have to be a sequence of morphisms such that $(C', \beta')$ is a $r$-$\text{Urcf}$ of $\pi$. We construct $\beta_{l+1}'$ iteratively and at the same time that the sequence of morphisms $f_0 : S_{0}' \to S_{0}^C$ of the $r$-$\text{Urcf}$-morphism $f : (C', \beta') \to (C, \beta)$. Starting with $f_0 = f \times \text{Id}_S$, the following diagram is a cartesian square,

$$\begin{array}{ccc}
S_0^{C'} & \xrightarrow{\beta_{0}'} & C' \times B \\
\downarrow{f_0 := f \times \text{Id}_S} & & \downarrow{f \times \text{Id}_B} \\
S_0^C & \xrightarrow{\beta_0} & C \times B \\
\end{array}$$

because $(C' \times S) \cong (C \times S) \times_C C'$, $S_0^{C'} \cong S_0^C \times_C C'$ and, by the transitivity of the base extension, $S_0^{C'} \cong S_0^C \times_{C \times B} (C' \times B)$. The morphism $\beta_0$ is a section of $\pi_0'$, a separated morphism, so by lemma 2.2 and the following remark, there is a unique section $\beta_0' : C' \times B \to S_0^C$ of $\pi_0^{C'}$ that makes the following diagram a cartesian square

$$\begin{array}{ccc}
C' \times B & \xrightarrow{\beta_0'} & S_0^C \\
\downarrow{f \times \text{Id}_B} & & \downarrow{f_0 := f \times \text{Id}_S} \\
C \times B & \xrightarrow{\beta_0} & S_0^C \\
\end{array}$$
and it is a closed immersion. Applying the lemma 2.3 and follow remark we construct the morphism $f_i : \mathcal{S}_i^{C'} \to \mathcal{S}_i^{C}$ that makes the following diagram a cartesian square.

$$
\begin{array}{ccc}
\mathcal{S}_i^{C'} & \xrightarrow{\pi_i^{C'}} & C' \times B \\
\downarrow f_i & & \downarrow f \times \text{Id}_B \\
\mathcal{S}_i^{C} & \xrightarrow{\pi_i^{C}} & C \times B \\
\end{array}
$$

Now we can iterate the process to obtain $\beta_i'$ and $f_{i+1}$ for $i = 1, \ldots, r - 1$.

This functor shows that with these definitions, given two $r$-rcf $(C, \beta)$ and $(C', \beta')$, a morphism $f : C \to C'$ is a $r$-rcf-morphism between $(C, \beta)$ and $(C', \beta')$ if and only if $\beta = Cl^r f(\beta')$, which gives an explanation for the (maybe surprising at first sight) definition of $r$-rcf-morphism.

Theorem 3.2 is a natural generalization of proposition 2.5 and is proved essentially the same way.

**Theorem 3.2.** A scheme $C$ represents $Cl^r$ if and only if $C$ is the $r$-Urcf of $\pi$.

**Corollary 3.3.** If $r$-Urcf of $\pi$ exists, then the following map is bijective.

$$
\Psi : Cl_r(K) \to \{ \text{ordered clusters of } r\text{-sections of } \pi \} \\
c \mapsto (c^0, \ldots, c^{r-1})
$$

**Theorem 3.4.** If $B$ is proper and $S$ quasi-projective, then the $r$-Urcf of $\pi$ exist and $Cl_r$ is quasi-projective.

**Proof.** By remark 7 and theorem 2.4 the $1$-Urcf of $\pi$ exists, it is the Usf of $\pi$, and it is a quasi-projective scheme.

Now, we work by induction over $r$. Suppose that $(Cl_{r-1}, \alpha_{r-1})$ exists and $Cl_{r-1}$ is quasi-projective. There are the Usf $(X_r, \psi_r)$ of $pr_B \circ \pi_{r-1} : \mathcal{S}_{r-1} \to B$ and $(BCL_{r-1}, \varphi)$ Usf of $pr_B : Cl_{r-1} \times B \to B$. Now $(Cl_{r-1}, \text{Id}_{Cl_{r-1} \times B})$ and $(X_r, \pi_{r-1} \circ \psi_r)$ are families of sections of $pr_B : B \times Cl_{r-1} \to B$, by the universal property of $(BCL_{r-1}, \varphi)$, there are unique morphisms $i_{r-1} : Cl_{r-1} \to BCL_{r-1}$ and $g_r : X_r \to BCL_{r-1}$ such that the following diagrams are commutative.

Consider the fibred product $Cl_r := Cl_{r-1} \times BCL_{r-1}$, $X_r$. We can give to $Cl_r$ structure of $(r-1)$-rcf with the morphisms $Cl^{r-1} f_r(\alpha_{r-1})$. So, we define $\alpha_{r-1}$ in this way, furthermore, we have defined $\mathcal{S}_r$ and the following diagram is a cartesian square, for $i = 0, \ldots, r-1$. 


\[ \begin{array}{ccc}
S_i^r & \xrightarrow{p_i^r} & S_i^{r-1} \\
\pi_i^r & & \pi_i^{r-1} \\
Cl_r \times B & \xrightarrow{\rho^r \times 1_{dB}} & Cl_{r-1} \times B
\end{array} \]

Observe that

\[ \pi_{r-1}^r \circ \psi_r \circ (j_r \times 1_{dB}) = \rho^r \times 1_{dB}, \]

so \( \alpha_{r-1}^r := 1_{dB} \times (Cl_r \times B) \circ (\psi_r \circ (j_r \times 1_{dB})): Cl_r \times B \to S_i^{r-1} \) is well defined, which is the unique section of \( \pi_{r-1}^r \) such that \( \rho_{r-1}^r \circ \alpha_{r-1}^r = \psi_r \circ (j_r \times 1_{dB}) \).

The next commutative diagram illustrates in part the situation.

\[ \begin{array}{ccc}
Cl_r \times B & \xrightarrow{j_r \times 1_{dB}} & X_r \times B & \xrightarrow{\psi_r} & S_i^{r-1} \\
\rho^r \times 1_{dB} & & \psi_r & & \pi_i^{r-1} \\
Cl_{r-1} \times B & \xrightarrow{\beta_{r-1} \times 1_{dB}} & BCl_{r-1} \times B & \xrightarrow{\varphi} & Cl_{r-1} \times B
\end{array} \]

It is clear that \( (Cl_r, \alpha^r) \) is a r-rcf. Let us check that it satisfies the required universal property.

Consider \((C, \beta)\) another r-rcf of \( \pi \). By the universal property of \((Cl_{r-1}, \alpha_{r-1}^r)\), there is a unique \((r-1)\)-rcf morphism \( p_i^C : (C, \beta_{|r-1}) \to (Cl_{r-1}, \alpha_{r-1}^r) \) with morphisms \( p_i^C : S_i^C \to S_i^{r-1} \) for all \( i = 0, \ldots, r-2 \), and \( \beta_{|r-1} = Cl_{r-1} \times p_i^C(\alpha_{r-1}^r) \). Furthermore, \((C, \beta_{r-1} \circ \beta_{r-1})\) and \((C, \pi_{r-1}^r \circ p_i^C \circ \beta_{r-1})\) are families of sections of \( pr_B \circ \pi_{r-1}^r \) and \( pr_B : B \times Cl_{r-1} \to B \) respectively, therefore there are two unique morphisms \( k : C \to X_r \) and \( h : C \to BCl_{r-1} \) such that \( \pi_{r-1}^r \circ \beta_{r-1} = \psi_r \circ (1_{dB} \times k) \) and \( \pi_{r-1}^r \circ p_i^C \circ \beta_{r-1} = \varphi \circ (1_{dB} \times h) \). Consider the following diagram,

\[ \begin{array}{ccc}
C \times B & \xrightarrow{k \times 1_{dB}} & Cl_{r-1} \times B \\
\pi_{r-1}^r \circ p_i^C \circ \beta_{r-1} & & \pi_{r-1}^r \circ \beta_{r-1} \\
Cl_{r-1} \times B & \xrightarrow{1_{dB}} & Cl_{r-1} \times B \\
BCl_{r-1} \times B & \xrightarrow{\varphi} & Cl_{r-1} \times B
\end{array} \]

which commutes by the unicity of \( h \). The following diagram is commutative as well,

\[ \begin{array}{ccc}
C \times B & \xrightarrow{h \times 1_{dB}} & Cl_{r-1} \times B \\
\pi_{r-1}^r \circ \beta_{r-1} & & \pi_{r-1}^r \\
Cl_{r-1} \times B & \xrightarrow{1_{dB}} & Cl_{r-1} \times B \\
BCl_{r-1} \times B & \xrightarrow{\varphi} & Cl_{r-1} \times B
\end{array} \]
again by the unicity of \( h, \ i_{r-1} \circ p^C = g_r \circ k \), and by the universal property of the fibred product there is a unique morphism \( G := p^C \times B Cl_{r-1}, k : C \to Cl_r \) such that \( k = j_r \circ G \) and \( p^C = p^r \circ G \).

Now it is clear that 
\[
\beta_{|r-1} = Cl^r p^C (\alpha_{r-1}^{-1}), = Cl^{r-1} G \circ Cl^{r-1} p^r (\alpha_{r-1}^{-1}). = Cl^{r-1} G (\alpha^r_{|r-1}).
\]

Then, if the following diagram is a cartesian square,
\[
\begin{array}{ccc}
C \times B & \xrightarrow{G \times (Id_B)} & Cl_r \times B \\
\downarrow{\beta_{r-1}} & & \downarrow{G_{r-1}} \\
S_{r-1}^C & \xrightarrow{\alpha_{r-1}^r} & S_{r-1}^{Cl_r}
\end{array}
\]

we are done, because \( \beta_r = Cl^r G (\alpha^r) \), and this condition is sufficient to imply \( k = j_r \circ G \) and \( p^C = p^r \circ G \). Then by the universal property of the product, the morphism \( G \) is unique with this property.

It is long but standard to check that the previous diagram is a cartesian square. \( \square \)

**Remark 8.** The morphism \( i_{r-1} : Cl_{r-1} \to B Cl_{r-1} \) has inverse morphism \( pr_{Cl_{r-1}} \circ \varphi \circ \lambda_{t_{|r-1}(Cl_{r-1})} \) where \( t \in B \) is a closed point and \( \lambda_t \) is the inclusion \( B Cl_{r-1} \cong \{ t \} \times B Cl_{r-1} \leftarrow B \times B Cl_{r-1} \). Then \( i_{r-1} \) is a closed immersion because \( i_{r-1} \) is a section of \( pr_{Cl_{r-1}} \circ \varphi \circ \lambda_t \), a separated surjective morphism.

**Remark 9.** Because \( i_{r-1} \) is an isomorphism with its image, so is \( j_r \). We define \( \gamma_r := p^r_{r-1} \circ \alpha_{r-1}^r \), then \((Cl_r, p^r, \gamma_r)\) satisfy the following universal property.

For each \((C, \alpha, \beta)\) such that \((C, \beta)\) is a sf of \( pr_B \circ \pi^{r-1}_1, \alpha : C \to Cl_{r-1}\) and \( \pi^{r-1}_1 \circ \beta = \alpha \times Id_B \), there is a unique morphism \( g : C \to Cl_r \) with \( \beta = \gamma_r \circ (g \times Id_B) \).

These remark give a natural alternative definition of \( Cl_r \), but the existence of \( Cl_{r-1} \) is required. Actually, whenever we know of the existence of \( Cl_{r-1} \), we will use this definition and we refer it as the remark universal property of \((Cl_r, p^r, \gamma_r)\).

### 4 \( Cl_{r+1} \) as a blowup scheme

In this section assume \( B \) proper and \( S \) quasiprojective, so that the existence result of the \((Cl_r, \alpha^r)\) is holds for \( r \geq 1 \).

Consider \( Cl_2 : = Cl_r \times Cl_{r-1}, \Delta_r \) the image of its diagonal morphism, the projections \( q_1, q_2 : Cl_2 \to Cl_r \), \( q_1 \) over the first \( Cl_r \) and \( q_2 \) over the second factor, and the morphisms \( p_i := q_i \times Id_B \), then \((p^r \times Id_B) \circ p_1 = (p^r \times Id_B) \circ p_2 \).

**Proposition 4.1.** There is a unique morphism \( \rho : Cl_2 \times B \to S_{r-1}^{Cl_r} \) which makes the following diagram commute.

\[
\begin{array}{ccc}
Cl_r \times B & \xrightarrow{\rho_2} & Cl_r \times B \\
\downarrow{\gamma_r} & & \downarrow{\rho} \\
S_{r-1}^C & \xrightarrow{\pi_{r-1}^C} & S_{r-1}^{Cl_r}
\end{array}
\]

\[
\begin{array}{ccc}
Cl_r \times B & \xrightarrow{p_1} & Cl_r \times B \\
\downarrow{\rho_1} & & \downarrow{p_1} \\
S_{r-1}^C & \xrightarrow{\pi_{r-1}^C} & S_{r-1}^{Cl_r}
\end{array}
\]

\[
\begin{array}{ccc}
Cl_r \times B & \xrightarrow{\rho_2} & Cl_r \times B \\
\downarrow{\gamma_r} & & \downarrow{\rho} \\
S_{r-1}^C & \xrightarrow{\pi_{r-1}^C} & S_{r-1}^{Cl_r}
\end{array}
\]

\[
\begin{array}{ccc}
Cl_r \times B & \xrightarrow{\rho_1} & Cl_r \times B \\
\downarrow{\gamma_r} & & \downarrow{\rho} \\
S_{r-1}^C & \xrightarrow{\pi_{r-1}^C} & S_{r-1}^{Cl_r}
\end{array}
\]

\[
\begin{array}{ccc}
Cl_r \times B & \xrightarrow{p_1} & Cl_r \times B \\
\downarrow{\gamma_r} & & \downarrow{\rho} \\
S_{r-1}^C & \xrightarrow{\pi_{r-1}^C} & S_{r-1}^{Cl_r}
\end{array}
\]

\[
\begin{array}{ccc}
Cl_r \times B & \xrightarrow{\rho_2} & Cl_r \times B \\
\downarrow{\gamma_r} & & \downarrow{\rho} \\
S_{r-1}^C & \xrightarrow{\pi_{r-1}^C} & S_{r-1}^{Cl_r}
\end{array}
\]
Proof. The remark 5 shows that $S_r^{r-1}$ is the pull back of 

\[ \pi_r^{r-1} \]

Then, the morphism $\rho$ is just the product morphism $p_1 \times (\pi_r^{r-1} \times B) \gamma_r \circ p_2$, which exists by definition of $p_1$ and $p_2$.

Explicitly, over the closed points, $\rho$ sends $(c,d,t)$ to $(c,d \pi_r^{r-1}(t))$, which is possible because $p_1(c) = p_1(d)$, and observe that $\rho|_{\Delta_r} = \alpha_r^{r-1}$.

**Proposition 4.2.** If $F := p^{r+1} \times_{Cl_r^{r-1}} f^{r+1}: Cl_{r+1} \to Cl_2$, the following diagram commutes.

\[
\begin{array}{ccc}
Cl_{r+1} \times B & \xrightarrow{\gamma_{r+1}} & S_r^r \\
F \times Id_B & & \downarrow \rho \\
Cl_2 \times B & \xrightarrow{bl_r^r} & S_r^{r-1}
\end{array}
\]

If $(Y_r, \rho_r)$ is the Usf of $pr_B \circ \pi_r^{r-1}$, then there exists a unique morphism $L: Cl_2 \to Y_r$ such that $\rho = \rho_r \circ (L \times Id_B)$ because $(Cl_2^r, \rho)$ is a sf of $pr_B \circ \pi_r^{r-1}$.

**Lemma 4.3.** $L$ is a closed immersion.

Proof. By the universal property of $(B Cl_r, \varphi_r)$, there exists a unique morphism $f_1: Y_r \to B Cl_r$ such that the following diagram commutes.

\[
\begin{array}{ccc}
Y_r \times B & \xrightarrow{\rho_r} & S_r^{r-1} \\
f_1 \times Id_B & & \downarrow \pi_r^{r-1} \\
B Cl_r \times B & \xrightarrow{\varphi_r} & Cl_r \times B
\end{array}
\]

Consider the following cartesian square,

\[
\begin{array}{ccc}
Y_r' & \xleftarrow{i_r'} & Y_r \\
p_1 & & \downarrow f_1 \\
Cl_r & \xleftarrow{i_r} & B Cl_r
\end{array}
\]

and the morphism $L' := q1 \times_{B Cl_r} L: Cl_2 \to Y_r'$. Since $i_r$ is a closed immersion, so is $i_r'$.

If we find a morphism $p_2: Y_r' \to Cl_r$ such that the following diagram commutes

\[
\begin{array}{ccc}
Cl_r & \xleftarrow{p_1} & Y_r' & \xrightarrow{p_2} & Cl_r \\
& \downarrow q1 & & \downarrow q2 \\
& L' & \xrightarrow{q2} & Cl_r
\end{array}
\]
and \( p' \circ p_1 = p' \circ p_2 \), we are done because, by the universal property of the fibred product \( \text{Cl}_{r+2} := \text{Cl}_r \times \text{Cl}_{r-1}, \text{Cl}_r \), there is a unique morphism \( Y_r' \to \text{Cl}_{r+2} \) which commute with \( p_i \) and \( q_i \) and it is the invers morphism of \( L' \), so \( L = i'_r \circ L' \) a composition of a isomorphism and a closed immersion.

We define \( \beta := p'_r - 1 \circ \rho \circ (i'_r \times \text{Id}_{B'}) \), observe that \( p'_r - 1 \circ \rho = \beta \circ (L' \times \text{Id}_{B'}) \).

By the commutativity of the following diagram we can apply remark \( \ref{remark-1} \) universal property of \((\text{Cl}_r, \rho, \gamma_r)\) over \((Y_r', p', p_1, \beta)\).

\[
\begin{array}{ccc}
Y_r' \times B & \xrightarrow{i'_r \times \text{Id}_{B'}} & Y_r \times B \\
\downarrow{p_1 \times \text{Id}_{B'}} & & \downarrow{f_1 \times \text{Id}_{B'}} \\
\text{Cl}_r \times B & \xrightarrow{\gamma_r} & \text{Cl}_r \times B
\end{array}
\]

it says that there is a unique morphism \( p_2 : Y_r' \to \text{Cl}_r \) such that \( p' \circ p_1 = p' \circ p_2 \) and \( \beta = \gamma_r \circ (p_2 \times \text{Id}_{B'}) \). We only need \( p_2 \circ L' = q_2 \), but, by definition of \( \rho \), \( p'_r - 1 \circ \rho = \gamma_r \circ (q_2 \times \text{Id}_{B'}) \), and \( p'_r - 1 \circ \rho = \gamma_r \circ ((p_2 \circ L') \times \text{Id}_{B'}) \) too. So, by remark \( \ref{remark-1} \) universal property of \((\text{Cl}_r, p', \gamma_r)\) applied to \((\text{Cl}_{r+2}, p', q_1, p'_r - 1 \circ \rho)\), these two morphisms are the same.

We call \( E \) the exceptional divisor of \( \text{bl}'_{r+1} \), the blow up of \( S'_{r+1} \) at the image of \( \alpha'_r - 1 \), and \( A := \gamma_{r+1}^{-1}(E) \). Note that \((\text{Cl}_{r+1}, \gamma_{r+1})\) is a sf of \( p_{rB} \circ \pi_{r-1}' \) and, using the notation of remark \( \ref{remark-1} \) for each \( c \in \text{Cl}_{r+1} \), we define \( A_c := \gamma_{r+1}^{-1}(E^c) \subseteq B_c \) where \( E^c \) is the exceptional divisor of \((S'_{r+1})^c \).

**Proposition 4.4.** Given a point \( c \in \text{Cl}_{r+1} \), its image \( F(c) \) belong to \( \Delta_r \) if and only if \( A_c = B_c \), furthermore, given \( t \in B_c \), \( (c, t) \in A \) if and only if \( \rho \circ (F \times \text{Id}_{B'})(c, t) = \alpha'_r - 1(p' - 1(c), t) \).

**Proof.** It’s clear by the commutativity of the following diagram.

\[
\begin{array}{ccc}
\text{Cl}_{r+1} \times B & \xrightarrow{\gamma_{r+1}} & S'_r \\
\downarrow{\text{Id}_{\text{Cl}_{r+1}}} \downarrow{f_{R \times \text{Id}_{B'}}} & & \downarrow{\text{Id}_{S'_r}} \\
\text{Cl}_{r-1} \times \text{Cl}_r \times B & \xrightarrow{\rho} & S'_{r-1}
\end{array}
\]

**Proposition 4.5.** \( A \) is a locally principal closed subscheme of \( \text{Cl}_{r+1} \times B \) and, given \( c \in \text{Cl}_{r+1}, A_c \subseteq B_c \) is an effective Cartier divisor if and only if \( F(c) \notin \Delta_r \).

**Proof.** \( A := \gamma_{r+1}^{-1}(E) \) is locally principal by definition because \( E \) is a Cartier divisor. The second statement follows from the following commutative diagram.
Given \( c \in C_{l+1} \), then in the ordered relative cluster \((\mathfrak{c}_0, \ldots, \mathfrak{c}_m)\) the section \(\mathfrak{c}_r^+\) is in the first infinitesimal neighbourhood of section \(\mathfrak{c}_r^{-1}\) exactly when \(F(c) \in \Delta_r\) (if and only if \(A_c = B_c\)).

Define the open set \( V := Cl_{l+1} \setminus F^{-1}(\Delta_r) \), the closed \( I := p^{-1}(\alpha_{r-1}(Cl_r \times B)) \subseteq Cl_2 \times B \) (note that \(\alpha_{r-1}\) is a section so its image is closed) and a scheme \( J \) by the following cartesian square,

\[
\begin{array}{ccc}
J & \longrightarrow & I \\
\downarrow & & \downarrow \\
V & \longrightarrow & Cl_2 \\
\end{array}
\]

\[
F|_V : \quad Cl_r \longrightarrow Cl_2 
\]

**Proposition 4.6.** The morphism \( J \to V \), of the previous definition, is flat.

**Proof.** By transitivity of the base extension,

\[
J = ((F|_V \times 1d_B) \circ \rho)^{-1}(\alpha_{r-1}(Cl_r \times B))
\]

and, if \( J' := A \cap (V \times B) = (\gamma_{r+1}|_{V \times B})^{-1}(E) \),

\[
J' = (bl'_{r} \circ \gamma_{r+1}|_{V \times B})^{-1}(\alpha_{r-1}(Cl_r \times B))
\]

so, by proposition 4.2 \( J \cong J' \).

Now, for every \( dd \in V \), the fibre \( J_{dd} \) of \( J \to V \) at \( dd \in V \) is isomorphic to \( A_{dd} \), an effective Cartier divisor of the fibre of \( V \times B \to V \) at \( dd \in V \), \( B_{dd} \). \( J \) is included in \( V \times B \) and the morphism \( J \to V \) is the composition \( J \leftarrow V \times B \to V \).

Now, we are done by lemma [Stacks Project Authors, 2013, Tag 062Y].

**Remark 10.** Given \( c \in V \), the morphism \( bl'_{r} \circ \mathfrak{c}_r^+ \) is a section of \( \rho_{rB} \circ \pi_{r-1}^{-1} \) and the strict transform of its image by the blow up \( bl'_{r} \) is exactly the image of \( \mathfrak{c}_r^+ \) because \( (bl'_{r} \circ \mathfrak{c}_r^+)(B_c) \cap \rho_{r-1}^{-1}(B_c) \subseteq bl'_{r} \circ \mathfrak{c}_r^+(B_c) \) is isomorphic to \( \mathfrak{c}_c^+ \cap E_c \subseteq \mathfrak{c}_r^+ \) and to \( A_r \subseteq B_c \), which is an effective Cartier divisor.

**Definition 4.1.** Given closed \((d, d') \in Cl_d^2\), we say that \((d, d')\) is a pair of admissible \(r\)-relative clusters or \(d\) is admissible with respect to \(d\), if \( pr_B \circ \pi_{r-1}(d'^{-1}(B) \cap (d'^{-1}(B)) \) is an effective Cartier divisor of \( B \). And denote \( Cl^\text{adm}_d \) the set of all pair of admissible \(r\)-relative clusters and \( Cl^\text{adm}_d \) the set of all admissible \(r\)-relative clusters with respect to \(d\).

**Corollary 4.7.** For each \( r \geq 1 \) and closed \( d \in Cl_r \), the sets \( Cl^\text{adm}_d \) and \( Cl^\text{adm}_d \) are constructible.

**Proof.** By remark 10 \( CL^\text{adm}_d = F(Cl_{r+1}) \setminus \Delta_d \) and \( Cl^\text{adm}_d = f'^{r+1}(Cl_{r+1}) \setminus \{d\} \).

**Theorem 4.8.** If \( B \) is regular, there is a stratification, \( \sqcup p Cl^p \hookrightarrow Cl^2_2 \) by locally closed subscheme, such that
1) $\Delta_r$ is a stratum itself.

2) There are two kinds of stratum. Type I with all closed $(d, d') \in Cl^p$ an admissible pair of $r$-relative clusters and type II with all closed $(d, d') \in Cl^p$ not admissible pair of $r$-relative clusters.

3) $V$ is isomorphic to the disjoint union of Type I strata, which we call $\sqcup_{adm} Cl^p$.

Proof. The flatness stratification of $Cl^p_{r+1}$ by the surjective morphism

$$I \hookrightarrow Cl^p_{r+1} \times B$$

$\sqcup_p Cl^p \hookrightarrow Cl^p_{r+1}$, is the stratification that we want.

(1) For every $dd \in Cl^p_{r+1}$, the dimension of the fibre $I_{dd}$ is equal to $\dim(B)$ if and only if $dd \in \Delta_r$, then, by the equidimensionality of the fibres of a flat morphism, the closed set $\Delta_r$ is a stratum.

(2) Given $CIP$ a stratum, by Grothendieck 1962, VI, Théorème 2.1 (i) and Kleiman 2005, 3.4, if there is $dd \in Cl^p$ such that $(\varphi|_I)^{-1}(dd) \cong I_{dd} \cong p^r(B(I_{dd})) \subset B$ is an effective Cartier divisor then for all $dd' \in Cl^p_{r+1}$, $I_{dd'}$ is an effective Cartier divisor. Here we use the hypothesis that $B$ is regular.

(3) Define $\sqcup_{adm} I^p$ by the following cartesian square.

By Stacks Project Authors 2013, Tag 062Y, $\sqcup_{adm} I^p \hookrightarrow \sqcup_{adm} Cl^p \times B$ is an effective relative Cartier divisor, and it is the pre-image of the centre of blow up $bl^r_{\gamma}$ by $\rho$ restricted to $\sqcup_{adm} Cl^p \times B$. So, by the universal property of blow up $bl^r_{\gamma}$ and remark universal property of $(Cl^p_{r+1+p^r+1}, \gamma_{r+1})$, there is a unique morphism $G: \sqcup_{adm} Cl^p \to Cl_{r+1}$ which makes the following diagram commute.

The following diagram commutes.

$$\begin{array}{ccc}
Cl_{r+1} \times B & \xrightarrow{\gamma_{r+1}} & S^r \\
G \times 1_{B} & \downarrow & \\
\sqcup_{adm} Cl^p \times B & \xrightarrow{\rho|_{\sqcup_{adm} Cl^p \times B}} & S^r_{r-1}
\end{array}$$
By the universal property of \((Y_r, \rho_r)\), the morphisms \(L|_{\sqcup_{\text{adm}} Cl^p} \times B\) and \(L \circ F \circ G\) are the same, so \(F \circ G = i|_{\sqcup_{\text{adm}} Cl^p}\) because \(L\) is an isomorphism with its image. Since \(\Delta_r\) is not a Type I stratum, \(G|_{\sqcup_{\text{adm}} Cl^p} \subseteq F^{-1}(\sqcup_{\text{adm}} Cl^p) \subseteq V\), we define a morphism \(G': \sqcup_{\text{adm}} Cl^p \to V\) as \(G\) with the image in \(V\).

By proposition 4.6 and the universal property of the flatness stratification there is a unique morphism \(F': V \to \sqcup_p Cl^p\) which makes the following diagram commute.

\[
\begin{array}{ccc}
V & \overset{F|_V}{\longrightarrow} & Cl^p_2 \\
\downarrow & & \downarrow \\
\sqcup_p Cl^p & \overset{\sqcup_p Cl^p}{\longrightarrow} & Cl^p_2
\end{array}
\]

The following diagram commutes.

\[
\begin{array}{ccc}
\sqcup_{\text{adm}} Cl^p & \overset{G'}{\longrightarrow} & V \\
\downarrow & & \downarrow \\
\sqcup_{p} Cl^p & \overset{\sqcup_p Cl^p}{\longrightarrow} & Cl^p_2
\end{array}
\]

Furthermore, by remark 10 \(F'(V) \subseteq \sqcup_{\text{adm}} Cl^p\), so we define the morphism \(F'' : V \to \sqcup_{\text{adm}} Cl^p\) as \(F'\) with the image in \(\sqcup_{\text{adm}} Cl^p\). It is clear \(F'' \circ G' = Id|_{\sqcup_{\text{adm}} Cl^p}\), and \(G' \circ F'' = Id_V\) because \(V \hookrightarrow Cl^p_{r+1}\) is injective and the remark 9 universal property of \((Cl^p_{r+1}, p^{r+1}, \gamma_{r+1})\) says that \(G' \circ F''\) composed with \(V \hookrightarrow Cl^p_{r+1}\) is this immersion.

The following theorem is an immediate consequence of the previous 4.8.

**Theorem 4.9.** Each irreducible component \(Z\) of \(Cl^p_{r+1}\) is either

1. composed entirely of clusters whose \(r+1\)th section is infinitely near to the \(r\)th (and \(F(Z) \subseteq \Delta_r\)).
2. isomorphic to a type I stratum whose closure does not intersect \(\Delta_r\).
3. birational to a type I stratum whose closure intersects \(\Delta_r\), in this case \(F|_Z\) is a blowup map whose center fails to be Cartier only on \(\Delta_r\).
5 Examples

5.1 Example I

Consider a smooth family $\pi: S \to C$ with $S$ quasiprojective and $C$ a smooth curve. The dimension of the base is 1, so there are no restrictions to the pairs of admissible sections. The 1-Urcf of $\pi$ is its Usf, $(X, \psi)$ with $S_0^1 := X \times S$ and $\sigma_0 := Id_X \times X \psi$. We know that it exists by theorem 2.4.

The irreducible components of $X$ are finite or countable $\{X_d\}_{d \in \mathbb{N} \cap \mathbb{C}}$. For each pair of irreducible components $X_d$ and $X'_d$ of $X$ there is a $n_{d,d'} \in \mathbb{N}$ which bound the degree of the 0-cycle intersection of $\sigma(P^1)$ and $\tau(P^1)$ for all pair of sections $(\sigma, \tau) \in X_d(K) \times X'_d(K)$.

We are going to define the flatness stratification of $X \times X$ by the morphism,

$$I \longrightarrow X \times X \times P^1 \quad \longrightarrow \quad X \times X$$

If $X_i^2$ is the locally closed subset consisting of the pairs of sections which intersect at a 0-cycle of degree $i$, for each $i \geq 0$, with 0 the degree of empty set, then the stratification is

$$\bigsqcup_{i \geq 0} X_i^2 \sqcup \Delta_X,$$

and for each $i$, the stratum $X_i^2$ decomposes as $\bigsqcup_{d,d' \geq 1} X_{i,d,d'}^2$ with $X_{i,d,d'}^2$ consisting of the pairs of sections $(\sigma, \tau)$ of $X_i^2$ which $\sigma \in X_d$ and $\tau \in X_{d'}$.

Consider $S_i^1 := Bl_{\text{obj}}(X \times Y)(S_0^i)$. $E$ its exceptional divisor and $V$ and $\rho: X \times X \times P^1 \to S_0^i$ as in section 4 for $r = 0$. On every section of $S_i^1 \to B$ we are only interested in whether its image is contained in some fibre $(S_i^1)_\sigma$ of the family $S_i^1 \to X$.

For every pair of distinct sections $(\sigma, \tau)$ of $X(K)$ the strict transform of $\rho(\sigma, \tau, P^1)$ is the image of one of these sections of $S_i^1$ not contained in the exceptional divisor. Moreover, it is clear that the image of one of these sections of $S_i^1$ not contained in the exceptional divisor is the strict transform of $\rho(\sigma, \tau, P^1)$ for some pair of distinct sections $(\sigma, \tau)$ of $X(K)$.

So, $V \cong \bigsqcup_{i \geq 0} X_i^2 = \bigsqcup_{i \geq 0} \bigsqcup_{d,d' \geq 1} X_{i,d,d'}$ and they are all clusters of two sections with the second section not infinitely near to the first one. We call $X_{i,d,d'}$ the component of $V$ isomorphic to $X_{i,d,d'}$.

Now assume for simplicity $C = P^1$, for each $\sigma \in X(K)$ the exceptional divisor of the fiber $(S_i^1)_\sigma$ is a rational surface isomorphic to $\mathbb{P}(N_{\mathbb{P}^1/S}(\sigma)) \cong \mathbb{P}(O_{\mathbb{P}^1}(a) \oplus O_{\mathbb{P}^1}(b))$, for some $a, b$ in $\mathbb{Z}$, which is the Hirzebruch surface, $F_{[a-b]}$. If $e := b - a \geq 0$,

$$\text{Pic}(E) = \mathbb{Z}[C] + \mathbb{Z}[F]$$

with $C^2 = -e$, $F^2 = 0$ and $CF = 1$. So, an irreducible curve linearly equivalent to $D = nC + mF$ intersects each fibre $(S_i^1)_\sigma$ at exactly one point, for each
ample I, that the center ideal of a blow up ($F$) Let us now illustrate, with an explicit computation in a particular case of ex-

5.3 Example III

So, $C$ class components of $E$

The previous example shows a situation in which there are infinitely many com-

5.2 Example II

Then, $Z$

The only pairs of admissible sections of $\pi$ are the constant ones.

Lemma 5.1. Given $\sigma = [P_1 : P_2 : P_3]$ and $\tau = [Q_1 : Q_2 : Q_3]$ two morphisms of $\mathbb{P}^2 \rightarrow \mathbb{P}^2$, with $\sigma$ non constant and $Z := \sigma(\mathbb{P}^2) \cap \tau(\mathbb{P}^2)$ is not equal to $\sigma(\mathbb{P}^2)$. Then, $Z$ is not a Cartier divisor in $\sigma(\mathbb{P}^2)$ (and $\pi^{-1}(Z)$ is not a Cartier divisor in $\mathbb{P}^2$).

Corollary 5.2. The only pairs of admissible sections of $\pi$ are the constant ones.

It also possible to see that there are no other pairs of infinitely near sections. So,

5.3 Example III

Let us now illustrate, with an explicit computation in a particular case of exam-

$\pi \in \mathbb{P}^1$, if and only if $1 = D \cdot F = n$. The curve $D$ is the image of a section if and only if it is irreducible and $D = C + mF$ with $m = 0$ or $m \geq e$. The Usf

$X_E$ of $E \rightarrow \mathbb{P}^1$ has a component $(X_E)_m$ for each $m \geq e$.

It is clear that there are a finite number of $X_{i,d,d'}$ such that $0 \neq F(X_{i,d,d'}) \cap \Delta_X = X_{i,d,d'} \cap \Delta_X$. So, there are infinitely many $(X_E)_m$ with $F((X_E)_m) \subseteq \Delta_X$ which are sections of $E$ that are not a limit of sections of some $X_{i,d,d'}$.

In each component of $X$ there is a locally closed subset $N(a,b)$, for each possible pair of integers $(a,b)$, formed by the sections $\sigma$ with $\mathcal{N}_{\mathbb{P}^2/S} \cong \mathcal{O}_{\mathbb{P}^2}(a) \oplus \mathcal{O}_{\mathbb{P}^2}(b)$. Over $N$, for each $m >> 0$, there is a family of rational quasiprojective varieties parametrizing all sections infinitely near to sections in $N(a,b)$ and of class $C + mF$. These families form irreducible components of $Cl_2$.

The previous example shows a situation in which there are infinitely many com-

Consider $\pi : \mathbb{P}^2 \times \mathbb{P}^2 \rightarrow \mathbb{P}^2$ the projection over the second factor. Its $1$-Urcf

$(X, \psi)$, the Usf, is an union of irreducible connected components to the reader.

The component $Cl^0_2$ of $Cl_2$ with the clusters whose first section is constant is isomorphic to

$\text{Bl}_{\Delta_{2,2}}(\mathbb{P}^2 \times \mathbb{P}^2) \cup \left( \left( X_0 \times \bigcup_{d>0} X_d \right) \cap Cl^\text{adm}_1 \right)$.

In $\text{Bl}_{\Delta_{2,2}}(\mathbb{P}^2 \times \mathbb{P}^2)$ there are all clusters for which the second section is constant too, even if they are in the exeptional divisor. If $(\sigma, \tau) \in Cl^0_2$, $\text{Bl}_{\Delta_{2,2}}(\mathbb{P}^2 \times \mathbb{P}^2) \cong \text{Bl}_{\phi}(\mathbb{P}^2) \times \mathbb{P}^2$, for some $q \in \mathbb{P}^2$, and the exeptional divisor is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^2 \rightarrow \mathbb{P}^2$, with only the constant sections. So, in $\left( X_0 \times \bigcup_{d>0} X_d \right) \cap Cl^\text{adm}_1$ there are all the other clusters of $Cl^0_2$.

The following easy lemma is left to the reader.

Lemma 5.1. Given $\sigma = [P_1 : P_2 : P_3]$ and $\tau = [Q_1 : Q_2 : Q_3]$ two morphisms of $\mathbb{P}^2 \rightarrow \mathbb{P}^2$, with $\sigma$ non constant and $Z := \sigma(\mathbb{P}^2) \cap \tau(\mathbb{P}^2)$ is not equal to $\sigma(\mathbb{P}^2)$. Then, $Z$ is not a Cartier divisor in $\sigma(\mathbb{P}^2)$ (and $\pi^{-1}(Z)$ is not a Cartier divisor in $\mathbb{P}^2$).

Corollary 5.2. The only pairs of admissible sections of $\pi$ are the constant ones.

It also possible to see that there are no other pairs of infinitely near sections. So,

$Cl_2 = Cl^0_2 \cong \text{Bl}_{\Delta_{2,2}}(\mathbb{P}^2 \times \mathbb{P}^2)$.

5.3 Example III

Let us now illustrate, with an explicit computation in a particular case of exam-

example I, that the center ideal of a blow up $\left( F \right)_{\mathbb{P}^2}$ in the notation of theorem 5.9.
Given a line $R$ in $\mathbb{P}^3$, choose coordinates $[x : y : z : t]$ such that $R = V(x, y)$. We are interested in the family given by the pencil of planes $\{\alpha y - \beta x = 0\}$, where $[\alpha : \beta] \in \mathbb{P}^1 =: B$ and whose total space is $S := B|_R(\mathbb{P}^3)$.

Any line $E$ in $\mathbb{P}^3$ which does not meet $R$ determines a section of $\pi$. So, by Plücker coordinates, there is an open $U_1$ of $X$ isomorphic to the affine chart $\{p_1 \neq 0\}$ of the quadric $V := V(p_1p_4 + p_2p_5 + p_3p_6)$ of $\mathbb{P}^3$ with coordinates $[p_1 : p_2 : p_3 : p_4 : p_5 : p_6]$, if we send each point of $V$ to the line

$$V \begin{pmatrix} p_0x - p_2y + p_1z \\ p_5x + p_3y - p_1t \\ p_4x - p_3z + p_2t \\ p_4y + p_5z + p_6t \end{pmatrix}$$

and each line $V(a_0x + a_1y + a_2z + a_3t, b_0x + b_1y + b_2z + b_3t)$ to the point $[a_2b_3 - a_3b_2 : a_3b_1 - a_1b_3 : a_1b_2 - a_2b_1 : a_0b_1 - a_1b_0 : a_0b_2 - a_2b_0 : a_0b_3 - a_3b_0]$ of $V$.

If we take affine coordinates $(a, b, c, d) = (p_6/p_1, -p_2/p_1, -p_5/p_1, -p_3/p_1,)$ of $\mathbb{A}^4$ to each point of $\mathbb{A}^4$ corresponds the line

$$L_{(a,b,c,d)} := V \begin{pmatrix} ax + by + z \\ cx + dy + t \end{pmatrix}$$

which are all lines that do not meet $R$.

Now, if $(a, b, c, d) \neq (a', b', c', d')$, $L_{(a,b,c,d)} \cap L_{(a',b',c',d')} = \begin{cases} \{p\} & \text{if } (a - a')(d - d') - (b - b')(c - c') = 0 \\ \emptyset & \text{else} \end{cases}$

So, $\rho_1 := \rho|_{U_1 \times U_1 \times \mathbb{P}^1} : U_1 \times U_1 \times \mathbb{P}^1 \to U_1 \times S,$

$$\rho_1 ((a, b, c, d), (a', b', c', d'), [u : v]) = \begin{align*} & = ((a, b, c, d), [u : v : -a'u - b'v : -c'u - d'v], [u : v]) \end{align*}$$

and the flatness stratification of $U_1 \times U_1$ is, if

$$Y := V((a - a')(d - d') - (b - b')(c - c')) \subset \mathbb{A}^4 \times \mathbb{A}^4,$$

$$U_1 \times U_1 = \Delta_1 \cup (Y \setminus \Delta_1) \cup Y^c.$$ 

We now focus on the component $Z$ of $C_{l_2}$ dominating the stratum $Y \setminus \Delta_1$. In fact $F(Z) = Y$ is a singular quadric in $\mathbb{A}^4 \times \mathbb{A}^4 \cong \mathbb{A}^8$, with singular locus $\Delta_1$. The blow up $S^1_1$ of $U_1 \times S$ at the image of $a_0^1$, $V(ax + by + z, cx + dy + t)$, is add a $\mathbb{P}^1$ with coordinates $[\nu : \mu]$ and the equation

$$\mu(ax + by + z) - \nu(cx + dy + t) = 0.$$ 

The morphism $\rho$ restricted to $(Y \setminus \Delta_1) \times \mathbb{P}^1$ extends to $S^1_1$ and over the coordinates $[\nu : \mu]$ is well defined if it is $[a - a' : b - b']$ or $[c - c' : d - d']$, because
if both of two are well defined they are equal since \((a, b, c, d, a', b', c', d')\) belongs in \(Z\).

Now, consider the strict transform \(\tilde{Y}\) of \(Y\) under the blow up of \(U_1 \times U_1\) at the ideal \((a-a', b-b')\), which add a \(\mathbb{P}^1\) with coordinates \([\omega : \eta]\) and the equation

\[\eta(a-a') - \omega(b-b') = 0.\]

We can extend \(\rho\) over all \(\tilde{Y} \times \mathbb{P}^1 \rightarrow S_1\) with coordinates \([\nu : \mu] = [\omega : \eta]\), that imply \(\tilde{Y} \cong Z\), because any two distinct points of \(\tilde{Y}\) gives two distinct sections of \(S_1\). \(\tilde{Y}\) is a small resolution of \(Y\); observe that the ideal is not unique, the ideal \((c-c', d-d')\) works too.

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