Oracle inequalities for square root analysis estimators with application to total variation penalties

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Abstract: We study the analysis estimator directly, without any step through a synthesis formulation. For the analysis estimator we derive oracle inequalities with fast and slow rates by adapting the arguments involving projections by Dalalyan, Hebiri and Lederer (2017). We then extend the theory to the case of the square root analysis estimator. Finally, we narrow down our attention to a particular class of analysis estimators: (square root) total variation regularized estimators on graphs. In this case, we obtain constant-friendly rates which match up to log-terms previous results obtained by entropy calculations. Moreover, we obtain an oracle inequality for the (square root) total variation regularized estimator over the cycle graph.

Keywords and phrases: Analysis, Total variation regularization, Lasso, Edge Lasso, Cycle graph, Sparsity, Trend filtering, Oracle inequality, Nullspace, Square root Lasso.

Contents

1 Introduction ........................................................................ 1
  1.1 Review of the literature ............................................. 1
    1.1.1 Synthesis and analysis .................................. 1
    1.1.2 Total variation regularized estimators .......... 2
    1.1.3 Square root regularization ........................... 3
  1.2 Contributions .......................................................... 3
  1.3 Notation ...................................................................... 4
  1.4 Model assumptions and preliminary definitions .......... 5
    1.4.1 Projection theory ........................................ 6
    1.4.2 Definitions ................................................. 6
2 Bounding the increments of the empirical process .......... 7
  2.1 Analysis estimator ................................................... 8
  2.2 Square root analysis estimator ............................... 8
3 Oracle inequalities for the analysis estimator .............. 11
  3.1 Fast rates with compatibility conditions ................. 11
  3.2 Slow rates without compatibility conditions ............ 11
  3.3 Dealing with randomness ..................................... 12
1. Introduction

1.1. Review of the literature

1.1.1. Synthesis and analysis

In the literature we find two approaches to regularized empirical risk minimization: the synthesis and the analysis approach, see Elad, Milanfar and Rubinstein (2007). Given a dictionary $X \in \mathbb{R}^{n \times p}$, the synthesis approach to the estimation of $f^0 \in \mathbb{R}^n$ is expressed by the synthesis estimator

$$\hat{f} = X \arg \min_{\beta \in \mathbb{R}^p} \left\{ \|Y - X\beta\|_n^2 + 2\lambda \|\beta\|_1 \right\}, \lambda > 0,$$

where $Y = f^0 + \epsilon, \epsilon \sim \mathcal{N}_n(0, \sigma^2 I_n), \sigma \in (0, \infty)$ and for a vector $f \in \mathbb{R}^n$ we write $\|f\|_n^2 = \sum_{i=1}^n f_i^2 / n$ and $\|f\|_n = \|f\|_2 / \sqrt{n}$. An instance of synthesis estimator is the classical Lasso (Tibshirani (1996), see Bühlmann and van de Geer (2011) and van de Geer (2016) for a thorough exposition of the theory about the Lasso).

On the other side, for an analysis operator $D \in \mathbb{R}^{m \times n}$, the analysis estimator is given by

$$\hat{f} = \arg \min_{f \in \mathbb{R}^n} \left\{ \|Y - f\|_n^2 + 2\lambda \|Df\|_1 \right\}, \lambda > 0.$$

Instances of analysis estimators are total variation regularized estimators over graphs, in particular the Fused Lasso, which corresponds to the case of the path graph. For such estimators, $D$ is taken to be the incidence matrix of some directed graph $\tilde{G} = (V, E)$. 

1. Oracle inequalities for the square root analysis estimator

4.1. Fast rates with compatibility conditions

4.2. Slow rates without compatibility conditions

4.3. Dealing with randomness

5. Total variation

5.1. Incidence matrices

5.2. Fast rates

5.2.1. Path graph

5.2.2. When $f^0$ is Lipschitz

5.2.3. Cycle graph

5.3. Slow rates

5.3.1. Comparison with other results

6. Conclusion

A. Probability inequalities

B. Proofs of Section 2

C. Proofs of Section 3

D. Proofs of Section 4

E. Proofs of Section 5

References
1.1.2. Total variation regularized estimators

Some previous studies of total variation regularized estimators (Dalalyan, Hebiri and Lederer (2017); Ortelli and van de Geer (2018)) used a step through a synthesis formulation (cf. Ortelli and van de Geer (2019)) to prove oracle inequalities for the analysis estimators. These studies were however confined to restrictive graph structures: the path in Dalalyan, Hebiri and Lederer (2017) and a class of tree graphs in Ortelli and van de Geer (2018). Other studies focusing on the Fused Lasso and not directly involving its synthesis form also implicitly relied on some kind of dictionary to handle the error term by projections onto some of the columns of this dictionary, see for instance the lower interpolant by Lin et al. (2017).

An approach handling directly the analysis formulation of the total variation regularized estimator is the approach by Hütter and Rigollet (2016), which however is not able to prove guarantees for the convergence of the mean squared error of the Fused Lasso as the other approaches do.

Let $D \in \mathbb{R}^{m \times n}$ be the incidence matrix of the directed graph $\vec{G}$. For $C > 0$, define

$$G(C) := \{ f \in \text{rowspan}(D) : \|D \vec{G} f\|_1 \leq C \} .$$

Donoho and Johnstone (1998) proved that the minimax rate of estimation over the path graph for functions $f \in G(C), C \approx 1$ is $n^{-2/3}$. Mammen and van de Geer (1997) established that the Fused Lasso tuned with $\lambda \approx n^{-2/3}C^{-1/3}$ has

$$\| \hat{f} - f^0 \|_2^2 = O_P(n^{-2/3}C^{2/3})$$

if $f^0 \in G(C)$ and thus achieves the minimax rate. Their result is based on entropy bounds (see ed. Babenko (1979); Birman and Solomjak (1967)) on the class $G(C)$, which are not constant-friendly. On the opposite side, Sadhanala, Wang and Tibshirani (2016) showed that estimators given by linear transformations of the observations are suboptimal on $G(C)$.

In a recent paper, Padilla et al. (2018) prove, by studying total variation regularized estimators over graphs combined with deep first search, that the total variation regularized estimator over any connected graph has a mean squared error of order at most $n^{-2/3}C^{2/3}$ if $f^0 \in G(C)$. This result is achieved by exploiting entropy-based bounds by Wang et al. (2016) and is not constant-friendly. Moreover, they also show that the minimax rate over $G(C)$ for $D$ being the incidence matrix of a tree graph with bounded maximal degree is $n^{-2/3}C^{2/3}$ and thus the total variation regularized estimator achieves it.

These previous results will serve as a benchmark for the evaluation of the rates obtained as corollaries to the main results of this paper, which are oracle inequalities for the generic (square root regularized) analysis estimator with fast and slow rates.
1.1.3. Square root regularization

The square root Lasso estimator, defined as
\[
\hat{\beta} := \arg \min_{\beta \in \mathbb{R}^n} \{ \|Y - X\beta\|_n + \lambda_0 \|\beta\|_1 \}, \lambda_0 > 0,
\]
was first introduced by Belloni, Chernozhukov and Wang (2011). The peculiarity of this estimator is that it allows to estimate the regression coefficients and the noise level simultaneously and thus allows to choose \(\lambda_0\) not depending on the unknown noise level \(\sigma\) when tuning the estimator to obtain oracle properties.

One can write the minimization problem in the following form
\[
(\hat{\beta}, \hat{\sigma}) := \arg \min_{\beta \in \mathbb{R}^p, \sigma > 0} \left\{ \|Y - X\hat{\beta}\|_2^n + \frac{\|Y - X\hat{\beta}\|_2^2}{\sigma} + \sigma + 2\lambda_0 \|\beta\|_1 \right\}.
\]

By differentiating the penalized loss we get the KKT conditions
\[
\hat{\sigma}^2 = \|Y - X\hat{\beta}\|_n^2 \quad \text{and} \quad \frac{X'(Y - X\hat{\beta})}{n} = \lambda_0 \hat{\sigma} \partial \|\hat{\beta}\|_1.
\]

The objective function of this second expression of the estimator is not differentiable at \(\sigma = 0\) and thus if \(\hat{\sigma} = 0\) the KKT conditions do not hold. The square root lasso estimator is studied in Belloni, Chernozhukov and Wang (2011), Sun and Zhang (2012) where it is called scaled lasso, and in van de Geer (2016) and Stucky and van de Geer (2017), between the others.

We now want to combine the results for the square root lasso exposed by van de Geer (2016) with the argument for bounding the increments of the empirical process proposed by Dalalyan, Hebiri and Lederer (2017). However, we are going to do so for the analysis estimator, motivated by the possibility to apply the results to the case of total variation regularization. We are thus going to consider the estimator
\[
\hat{f} := \arg \min_{f \in \mathbb{R}^n} \{ \|Y - f\|_n + \lambda_0 \|Df\|_1 \}, \lambda_0 > 0
\]
for a general analyzing operator \(D \in \mathbb{R}^{m \times n}\) and then narrow down to the case where \(D\) is the incidence matrix of a directed graph.

1.2. Contributions

In this article we contribute to the research on oracle inequalities for total variation regularized estimators by exposing a way of proving general oracle inequalities for an analysis estimator without transforming the analysis estimation problem into a synthesis estimation problem. In this way we generalize and connect previous approaches.

We obtain results analogous to the ones obtained by Dalalyan, Hebiri and Lederer (2017) for fast and slow rates, however starting from the consideration of a
generic analysis estimator and not of a generic synthesis estimator. This setup allows us to prove also an oracle guarantee for the total variation regularized estimator over the cycle graph, which to our knowledge is a new contribution.

Moreover, by the use of slow rates and allowing the tuning parameter \( \lambda \) to depend on some aspects of the true signal \( f^0 \in \mathbb{R}^n \), we obtain almost optimal rates for the estimation of functions of bounded total variation on the path graph without the use of the compatibility constant and without continuity assumptions on the function generating \( f^0 \). In the same way we also obtain not so slow rates for functions of bounded total variation on general graphs. This last results corresponds up to log terms to the result obtained by Padilla et al. (2018) for general graphs and by Mammen and van de Geer (1997) for the path graph by means of entropy calculations. The way of handling the increments of the empirical process by projections exposed by Dalalyan, Hebiri and Lederer (2017), in spite of being off by log-terms from the rates which can be obtained by entropy calculations, might be advantageous for a value of \( n \) not being too large. Indeed, entropy calculations are not constant-friendly, while the bounds we expose allow to easily keep track of the constants involved.

In these oracle inequalities we also introduce, inspired by some remarks by Padilla et al. (2018), a new measure for the sparsity of the signal. In Hütter and Rigollet (2016) the sparsity of the true signal was measured as \( \|Df^0\|_0 \), while we argue that a more appropriate measure is \( r_{S_0} := \dim(\mathcal{N}(D-S_0)) \) (see Subsection 1.3).

The above contributions define a framework for the derivation of oracle inequalities by the direct analysis of analysis estimators, which extends to synthesis estimators the approach involving projections introduced by Dalalyan, Hebiri and Lederer (2017) and makes it applicable in a wider range of settings.

A major contribution of this article is to show that results analogous to the ones presented in Dalalyan, Hebiri and Lederer (2017) also hold for the square root regularized analysis estimator. We adapt a lemma by van de Geer (2016) showing that the square root lasso does not overfit to the case where the increments of the empirical process for the square root analysis estimator are bounded by means of the projection arguments by Dalalyan, Hebiri and Lederer (2017). We thus show that the square root analysis estimator does not overfit on a high-probability event, if the increments of the empirical process are bounded by using projection arguments. This is a starting point for the development of oracle inequalities for the square root analysis estimator, which produce results analogous to the ones obtained for the plain analysis estimator (which match the ones found in Dalalyan, Hebiri and Lederer (2017)).

1.3. Notation

Let \( D \in \mathbb{R}^{m \times n} \) be a matrix. Let \( \{d'_i\}_{i \in [m]} \) denote the row vectors of \( D \). By \( \mathcal{N}(D) \), we denote the nullspace of \( D \), i.e.

\[
\mathcal{N}(D) := \{ x \in \mathbb{R}^n : Dx = 0 \}.
\]
By \( \mathcal{N}^\perp(D) \) we denote the orthogonal complement of \( \mathcal{N}(D) \), i.e.
\[
\mathcal{N}^\perp(D) := \{ x \in \mathbb{R}^n : x'z = 0, \forall z \in \mathcal{N}(D) \}.
\]

Note that \( \mathcal{N}^\perp(D) = \text{rowspan}(D) \).

Let \( S \subseteq [m] \) denote a subset of the row indices of \( D \). We denote the cardinality of the set \( S \) by \( s := |S| \). We write \( -S := [m] \setminus S \). Moreover, we write \( D_S = \{d'_i\}_{i \in S} \in \mathbb{R}^{s \times n} \) and \( D_{-S} = \{d'_i\}_{i \in -S} \in \mathbb{R}^{(m-s) \times n} \).

We use the shorthand notations \( \mathcal{N}_S := \mathcal{N}(D_S) \) and \( \mathcal{N}_{-S} := \mathcal{N}(D_{-S}) \). Similarly, we write \( \mathcal{N}^\perp_S := \mathcal{N}^\perp(D_S) \) and \( \mathcal{N}^\perp_{-S} := \mathcal{N}^\perp(D_{-S}) \).

Note that \( \mathcal{N}(D) = \mathcal{N}(D_S) \cap \mathcal{N}(D_{-S}) \). Moreover, if \( S, S' \subseteq [m] \) are s.t. \( S \subseteq S' \), then we have that \( \mathcal{N}(D_{S'}) \subseteq \mathcal{N}(D_S) \). In addition, if the rows of \( D_{S' \setminus S} \) can be written as linear combinations of the rows of \( D_S \), then \( \mathcal{N}(D_{S'}) = \mathcal{N}(D_S) \).

Let \( w \in \mathbb{R}^n \) be a vector. For the diagonal matrix \( W = \text{diag}\{w_i\}_{i \in [m]} \in \mathbb{R}^{m \times m} \) we write \( W_S := \text{diag}\{w_i\}_{i \in S} \in \mathbb{R}^{s \times s} \) and \( W_{-S} := \text{diag}\{w_i\}_{i \in -S} \in \mathbb{R}^{(m-s) \times (m-s)} \).

Let \( I_n \in \mathbb{R}^{n \times n} \) denote the identity matrix and let \( I_n = \{1\}^{n \times n} \).

Let \( V \subset \mathbb{R}^n \) be a linear space. By \( \Pi_V \in \mathbb{R}^{n \times n} \) we denote the orthogonal projection matrix onto \( V \) and by \( A_V := I_n - \Pi_V \) the orthogonal antiprojection matrix onto \( V \).

Let \( f \in \mathbb{R}^n \) be a column vector. We write
\[
f = (\Pi_{\mathcal{N}_{-S}} + \Pi_{\mathcal{N}^\perp_{-S}}) f := f_{\mathcal{N}_{-S}} + f_{\mathcal{N}^\perp_{-S}}.
\]

Note that we have that
\[
\Pi_{\mathcal{N}^\perp_{-S}} = I_n - \Pi_{\mathcal{N}_{-S}} = A_{\mathcal{N}_{-S}}
\]
and
\[
\Pi_{\mathcal{N}_{-S}} = I_n - \Pi_{\mathcal{N}^\perp_{-S}} = A_{\mathcal{N}^\perp_{-S}}.
\]

For \( f \in \mathbb{R}^n \) we write \( \|f\|_n^2 := \|f\|_2^2/n \) and \( \|f\|_n := \|f\|_2/\sqrt{n} \).

### 1.4. Model assumptions and preliminary definitions

Let \( f^0 \in \mathbb{R}^n \) be a signal. We observe \( Y = f^0 + \epsilon, \epsilon \sim \mathcal{N}_n(0, \sigma^2 I_n), \sigma \in (0, \infty) \).

The analysis estimator \( \hat{f} \) of \( f^0 \) is defined as
\[
\hat{f} := \arg \min_{f \in \mathbb{R}^n} \left\{ \|Y - f\|_n^2 + 2\lambda \|Df\|_1 \right\}, \lambda > 0,
\]
where \( D \in \mathbb{R}^{m \times n} \) is a so called analysing operator. Its square root version is defined as
\[
\hat{f}_\sqrt{\cdot} := \arg \min_{f \in \mathbb{R}^n} \{ ||Y - f||_n + \lambda_0 \|Df\|_1 \}, \lambda_0 > 0,
\]

In the following, for a given \( f \in \mathbb{R}^n \), we will identify the set \( S \subseteq [m] \) with
\[
S := \{ i \in [m] : d'_i f \neq 0 \}.
\]

We now introduce some quantities, which will be needed to prove oracle inequalities for \( \hat{f} \) and \( \hat{f}_\sqrt{\cdot} \).
1.4.1. Projection theory

Consider the matrix $D \in \mathbb{R}^{m \times n}$. If $D_{-S} \in \mathbb{R}^{(m-s) \times n}$ is of full rank and $m-s \leq n$. Then we have that

$$
\Pi_{N^+_{-S}} = \Pi_{\text{rowspan}(D_{-S})} = (D_{-S}D_{-S}')^{-1}D_{-S}.
$$

Let $S$ be a set of row indices of $D$. We define the set of sets $\bar{S}(S)$ as

$$
\bar{S}(S) := \text{arg max}_{\tilde{S} \supseteq S} \left\{ |\tilde{S}| : \text{rank}(D_{-\tilde{S}}) = \text{rank}(D_{-S}) \right\}.
$$

Note that the maximizer might not be unique, i.e. $\bar{S}(S)$ might contain more than one set of row indices. Moreover, it might be the case that $\bar{S}(S) = S$, e.g. in the case of $D$ being of full row rank. Thus, by $\bar{S}(S) =: \bar{S}$ we denote a general set of row indices in the set of maximizers.

Note that since $\text{rank}(D_{-\bar{S}}) = \text{rank}(D_{-S})$, we have that the rows in $D_{-\bar{S}}$ can be written as linear combinations of the rows in $D_{-S}$. Thus, we have that

$$
N(D_{-\bar{S}}) = N(D_{-S}) \text{ and therefore } \Pi_{N^+_{-S}} = \Pi_{N^+_{-\bar{S}}},
$$

Moreover note that by the definition of $\bar{S}$ it follows that $D_{-\bar{S}}$ is of full row rank. If not, then we could find a row of $D_{-\bar{S}}$ being linearly independent from the other rows of $D_{-S}$ and we could eliminate it without changing the rank of the matrix. This would lead to a contradiction, since there would be a set $\bar{S}^*$ of cardinality $|\bar{S}^*| = \bar{s} + 1$, s.t

$$
\text{rank}(D_{-\bar{S}^*}) = \text{rank}(D_{-\bar{S}})
$$

and thus $\bar{S}$ would not be a maximizer.

Hence, since $D_{-S}$ is of full row rank we have that

$$
\Pi_{N^+_{-S}} = \Pi_{N^+_{-\bar{S}}} = (D_{-\bar{S}}D_{-\bar{S}}')^{-1}D_{-\bar{S}}.
$$

1.4.2. Definitions

The matrix $D_{-\bar{S}}$ is of full rank. We can thus write its Moore-Penrose pseudoinverse $D^+_{-\bar{S}} \in \mathbb{R}^{n \times (m-\bar{s})}$ as

$$
D^+_{-\bar{S}} = D_{-\bar{S}}'(D_{-\bar{S}}D_{-\bar{S}}')^{-1}.
$$

Let $d^+_i \in \mathbb{R}^n$, $i \in [m-\bar{s}]$ denote the column vectors of $D^+_{-\bar{S}}$.

**Definition 1.1**

In analogy to Dalalyan, Hebiri and Lederer (2017), the vector $\omega \in \mathbb{R}^m$ is defined as

$$
\omega_i := \begin{cases}
|d^+_i|_n, & i \in -\bar{S}, \\
0, & i \in \bar{S}.
\end{cases}
$$

Moreover, we write $\Omega := \text{diag}(\{\omega_i\}_{i \in [m]}) \in \mathbb{R}^{m \times m}$. 

Definition 1.2 (Normalized inverse scaling factor)
In analogy to the quantity $\rho_T$ used by Dalalyan, Hebiri and Lederer (2017), the normalized inverse scaling factor $\rho = \rho(D, \bar{S}(S))$ is defined as

$$\rho := \max_{i \in \bar{S}} \omega_i.$$ 

The name of the normalized scaling factor is inspired by Hütter and Rigollet (2016), who define a similar quantity.

Definition 1.3
Let $\gamma \geq \rho$. In analogy to Dalalyan, Hebiri and Lederer (2017), the vector $w \in \mathbb{R}^m$ is defined as

$$w_i := 1 - \omega_i / \gamma, \ i \in [m].$$

Moreover, we write $W := \text{diag}(\{w_i\}_{i \in [m]}) \in \mathbb{R}^{m \times m}$.

Let $r_S := \dim(N(D_S)).$

Definition 1.4 (Weighted weak compatibility constant)
Let $W \in \mathbb{R}^{m \times m}$ be a diagonal matrix of weights with $\|W\|_\infty \leq 1$ (e.g. as in Definition 1.3). The weighted compatibility constant $\kappa^2(S, W)$ is defined as

$$\kappa^2(S, W) := r_S \min \left\{ \|f\|_n^2 : \|W_S D_S f\|_1 - \|W_{-S} D_{-S} f\|_1 = 1 \right\}.$$ 

Remark. Thus weighted compatibility constant extends the definition given by Dalalyan, Hebiri and Lederer (2017) to the case of analysis estimators. Note that an important feature is the factor $r_S$, which expresses the number of parameters to estimate if the set of nonzero linear combinations of the entries of $f$ given by the rows of $D$ is $S$.

2. Bounding the increments of the empirical process

In this section we expose the core idea behind the new results that we prove. The main points profiling our results are that:

- we study directly the analysis estimator without passing through its synthesis formulation;
- we apply the projection arguments by Dalalyan, Hebiri and Lederer (2017) to the case of square root regularization;
- to do so we use a projection theory for analysis operators.

A key step in the proof of oracle inequalities is the bound of the quantity $\epsilon' f/n$. This bound is decisive for the rate that will be obtained. In this article we extend the application field of a bound on the increments of the empirical process by means of projections, introduced by Dalalyan, Hebiri and Lederer (2017) when analysing the synthesis estimator. Indeed, we apply the same arguments to the analysis estimator and to its square root regularized version. The bound of the increments of the empirical process happens in the context of the basic inequality and is thus influenced by the assumptions that we make (in the case of the square root analysis estimator) or do not make (in the case of the analysis estimator) to obtain the basic inequality.
2.1. Analysis estimator

The case of the analysis estimator is more simple, because we have the basic inequality without assuming any extra conditions. Therefore the choice of the tuning parameter follows only from the bound on the increments of the empirical process. This case will be exposed beforehand to convey the intuition.

Lemma 2.1 (Basic inequality)

For the analysis estimator we have the so called basic inequality, i.e.

\[ \| \hat{f} - f^0 \|^2_n + \| \hat{f} - f \|^2_n \leq \| f - f^0 \|^2_n + \frac{2\epsilon'(\hat{f} - f)}{n} + 2\lambda(Df - \|D\hat{f}\|_1). \]

Proof of Lemma 2.1. See Appendix B.

The basic inequality is the starting point for the proof of oracle inequalities both with fast and slow rates. These oracle inequalities both have in common the bound on the increments of the empirical process exposed below.

For \( x, t > 0 \) define the sets

\[ T := \left\{ \| \epsilon^t \|_1 \leq \frac{\lambda n \| d_i^n \|_1}{\gamma}, \forall i \in [m - \bar{s}] \right\} \]

and

\[ X := \left\{ \| \Pi_{\mathcal{N}(D_{-\delta})} \epsilon \|_n \leq \sqrt{\frac{\sigma^2}{n}} \left( \sqrt{r_S} + \sqrt{2x} \right) \right\}. \]

Lemma 2.2

Let \( D \in \mathbb{R}^{m \times n} \) be some analysing operator. On \( X \cap T \) for the analysis estimator it holds that

\[ \frac{\epsilon'}{n} \leq \frac{\lambda}{\gamma} \| D_{-\delta} f \|_1 + \sigma \left( \sqrt{\frac{2x}{n}} + \sqrt{r_S} \right) \| f \|_n \]

\[ \leq \frac{\lambda}{\gamma} \| D_{-\delta} f \|_1 + \sigma \left( \sqrt{\frac{2x}{n}} + \sqrt{r_S} \right) \| f \|_n. \]

Proof of Lemma 2.2. See Appendix B.

2.2. Square root analysis estimator

For the square root analysis estimator the bound is more involved, since to get the basic inequality we have to assume that \( \epsilon := Y - \hat{f} \neq 0 \), i.e. that the estimator does not overfit. If the estimator does not overfit, then the KKT conditions for the square root regularized estimator hold.
Lemma 2.3 below shows that indeed one can prove that on an (high-probability) event we have that $\hat{e} \neq 0$. This proof requires that we assume that the tuning parameter $\lambda_0$ is large enough and that the penalty for the true regression function $\|Df^0\|_1$ is small enough. The proof of Lemma 2.3 requires already that we upper bound the increments of the empirical process and choose an appropriate tuning parameter. The choice on how to handle the increments of the empirical process in the proof of Lemma 2.3 is determining for how we will be allowed to handle the empirical process in the basic inequality.

Therefore, in the case of the square root regularized estimator, the first step to prove oracle inequalities is to show that the estimator does not overfit, i.e. that $\|\hat{e}\|_n > 0$.

The following lemma, showing that $\|\hat{e}\|_n > 0$, is an adaptation of Lemma 3.1 in van de Geer (2016) to the case of the analysis estimator where the increments of the empirical process are bounded by the projection arguments found in Dalalyan, Hebiri and Lederer (2017).

Define

$$\hat{R} := \max_{i \in -\bar{S}} \frac{|e \hat{d}_i^+|}{\|\hat{e}\|_n \|d^+_i\|_n n}.$$ 

For $a > 0$, $R > 0$, define the sets

$$\mathcal{A} := \left\{ \frac{\|\Pi_N(D_{-\bar{S}})\epsilon\|_2^2 / \sigma^2}{n} \in (r_S - 2\sqrt{ars} - rs + 2\sqrt{ars} + 2a) \right\} \cup \left\{ \frac{\|A_N(D_{-\bar{S}})\epsilon\|_2^2 / \sigma^2}{n} \geq n - r_S - 2\sqrt{a(n - r_S)} \right\}$$

and

$$\mathcal{R} := \left\{ \gamma \hat{R} \leq R \right\}.$$ 

**Assumption 2.1**

*Assume for some $a > 0$ that $n > 8a$ and that for some $R > 0$, $\eta \in (0, 1)$

$$\lambda_0 \geq \frac{1}{1 - \eta} R$$

and

$$\|Df^0\|_1 \leq c\sigma\sqrt{1 - \sqrt{8a/n}} / \lambda_0,$$

where

$$c = \sqrt{\frac{\eta}{2} - \frac{2\sqrt{r_S} + \sqrt{2a}}{\sqrt{n - \sqrt{8an}}}^2} + 4 - 2.$$ 

We assume that $S \subseteq [m]$ is s.t.

$$\eta > \frac{2\sqrt{r_S} + \sqrt{2a}}{\sqrt{n - \sqrt{8an}}}. $$
Lemma 2.3
Under Assumption 2.1 we have that on $\mathcal{A} \cap \mathcal{R}$

$$\frac{\left\| \hat{\varepsilon} \right\|_n}{\| \varepsilon \|_n} - 1 \leq \eta.$$  

Proof of Lemma 2.3. See Appendix B.

Remark. If compared with Lemma 3.1 by van de Geer (2016), the assumptions of Lemma 2.3 are slightly more restrictive. While Lemma 3.1 by van de Geer (2016) only requires a lower bound on $\| \varepsilon \|_n$, Lemma 2.3 requires that $\| \Pi_N(D - \bar{S})\varepsilon \|_n$ is upper and lower bounded and that $\| A_N(D - \bar{S}) \varepsilon \|_n$ is lower bounded. It is the price to pay for a more refined technique to handle the increments of the empirical process.

Corollary 2.1
Under Assumption 2.1, on $\mathcal{A} \cap \mathcal{R}$ we have that

$$\lambda_0 \| \hat{\varepsilon} \|_n \geq R \| \varepsilon \|_n.$$  

Proof of Corollary 2.1. See Appendix B.

Lemma 2.4
Under Assumption 2.1 we have that $\forall f \in \mathbb{R}^n$ on $\mathcal{A} \cap \mathcal{R}$ it holds that

$$\| \hat{f} - f^0 \|_n^2 + \| \hat{f} - f \|_n^2 \leq \| f - f^0 \|_n^2 + \frac{2\varepsilon'(\hat{f} - f)}{n} + 2\lambda_0 \| \hat{\varepsilon} \|_n (\| Df \|_1 - \| D\hat{f} \|_1).$$  

Proof of Lemma 2.4. See Appendix B.

Lemma 2.5
Under Assumption 2.1 on $\mathcal{A} \cap \mathcal{R}$ we have that

$$\frac{\varepsilon(f)}{n} \leq \lambda_0 \| \hat{\varepsilon} \|_n \| \Omega_{-S} D - S f \|_1 / \gamma + \| \Pi_N(D - \bar{S}) \varepsilon \|_n \| f \|_n.$$  

Proof of Lemma 2.5. See Appendix B.

Remark. The upper bound in Lemma 2.5 is random. On $\mathcal{A} \cap \mathcal{R}$, by Lemma 2.3 we have that

$$\| \hat{\varepsilon} \|_n \leq (1 - \eta) \| \varepsilon \|_n \leq (1 - \eta) (\| \Pi_N(D - \bar{S}) \varepsilon \|_n + \| A_N(D - \bar{S}) \varepsilon \|_n).$$  

On the set $\mathcal{A}$ we are given only an upper bound for $\| \Pi_N(D - \bar{S}) \varepsilon \|_n$ but not for $\| A_N(D - \bar{S}) \varepsilon \|_n$. Thus, when we later are going to derive nonrandom oracle upper bounds, we will have to restrict our attention to the event where also $\| A_N(D - \bar{S}) \varepsilon \|_n$ can be upper bounded in terms of $\sigma$. 
3. Oracle inequalities for the analysis estimator

In this section we are going to utilize the tools exposed in the previous sections to prove oracle inequalities for a general analysis estimator. This section reproduces the results obtained by Dalalyan, Hebiri and Lederer (2017) for the synthesis estimator. The difference is that our results are obtained by using an approach that directly handles the analysis estimator. In the next section we are going to explore how this approach translates to the case of the square root analysis estimator.

3.1. Fast rates with compatibility conditions

To obtain fast rates by using compatibility conditions one makes use of the more refined bound given by Lemma 2.2 involving $\|\Omega_D - Sf\|_1 / \gamma$. This term will flow into the weighted compatibility constant.

**Theorem 3.1** (Oracle inequality for the analysis estimator)
Assume that $\kappa(S,W) > 0$. Let $t, x > 0$. On $\mathcal{X} \cap \mathcal{T}$ for the analysis estimator it holds that
\[
\|\hat{f} - f^0\|_n^2 \leq \|f - f^0\|_n^2 + 4\lambda \|D_S f\|_1 + \left(\sigma \sqrt{\frac{2x}{n}} + \sigma \sqrt{\frac{rS}{n}} + \frac{\lambda \sqrt{rS}}{\kappa(S,W)}\right)^2.
\]

The proof combines the direct approach of analyzing the analysis estimators used by Hütter and Rigollet (2016) with the projection arguments introduced by Dalalyan, Hebiri and Lederer (2017) and later used by Ortelli and van de Geer (2018) (cf. also Lemma 2.2 in this article) for proving an oracle inequality for the Fused Lasso.

*Proof of Theorem 3.1.* See Appendix C

3.2. Slow rates without compatibility conditions

To obtain slow rates without needing compatibility conditions one utilizes the less refined version of the bound given by Lemma ?? involving $\rho \|D_S f\|_1 / \gamma$.

**Theorem 3.2** (Oracle inequality with slow rates for the analysis estimator)
On $\mathcal{X} \cap \mathcal{T}$ for the analysis estimator it holds that
\[
\|\hat{f} - f^0\|_n^2 + 2\lambda \left(1 - \frac{\rho}{\gamma}\right) \|D_S \hat{f}\|_1 + 2\lambda \|D_S f\|_1
\leq \|f - f^0\|_n^2 + \frac{\sigma^2}{n} \left(\sqrt{2x} + \sqrt{rS}\right)^2
+ 2\lambda \left(1 + \frac{\rho}{\gamma}\right) \|D_S f\|_1 + 2\lambda \|D_S f\|_1.
\]
**Proof of Theorem 3.2.** See Appendix C

**Remark.** Theorem 3.2 does not need the assumption that the (weighted) compatibility constant is bounded away from zero and thus applies also to situations where we are not able to prove such a bound.

### 3.3. Dealing with randomness

Both Theorem 3.1 and Theorem 3.2 hold on the event $\mathcal{X} \cap \mathcal{T}$. We now want to find a lower bound for the probability of that event.

Note that

$$\|\Pi_{N(D-s)}\epsilon\|_2^2 = \frac{\sigma^2}{n} \frac{\|\Pi_{N(D-s)}\epsilon\|_2^2}{\sigma^2 \sim \chi^2_{rS}},$$

where $\text{rank}(\Pi_{N(D-s)}) = r_S$. By applying Lemma A.2 for some $x > 0$ we get

$$P(\mathcal{X}) \geq 1 - e^{-x}.$$

To find a lower bound on $P(\mathcal{T})$ we apply Lemma A.1 to $\mathcal{T}$. Note that $\mathcal{T}$ can be written as

$$\mathcal{T} = \left\{ \max_{i \in [m-\bar{s}]} \left| \frac{\epsilon d_i^+}{\sigma \|d_i^+\|_2} \right| \leq \frac{\lambda \sqrt{n}}{\gamma \sigma} \right\}.$$

Since $\epsilon d_i^+ \sim \mathcal{N}(0, \sigma^2 \|d_i^+\|_2^2)$, we obtain that for the standard normal random variables $V_i = \epsilon d_i^+ / (\sigma \|d_i^+\|_2) \sim \mathcal{N}(0, 1), i \in [m-\bar{s}]$

$$\mathcal{T} = \left\{ \max_{1 \leq j \leq m-\bar{s}} |V_j| \leq \frac{\lambda \sqrt{n}}{\gamma \sigma} \right\}.$$

The moment generating function of $|V_j|$ is

$$E \left[ e^{r|V_j|} \right] = 2(1 - \Phi(-r))e^{\frac{r^2}{2}} \leq 2e^{\frac{r^2}{2}}, \forall r > 0.$$

Choosing, for some $t > 0$, $\lambda \geq \gamma \sigma \sqrt{2 \log(2n)}/\sqrt{n} + 2t/n$, e.g. $\lambda = \gamma \sigma \sqrt{2 \log(2n)}/n + 2t/n$, and applying Lemma A.1 with $p = m-\bar{s} = n-r_S$ and $t > 0$, we obtain

$$P(\mathcal{T}) \geq 1 - e^{-t}.$$

Thus we see that for the choice $\lambda \geq \gamma \sigma \sqrt{2 \log(2n)}/n + 2t/n$ we obtain that Theorem 3.1 and Theorem 3.2 hold on a set having probability at least $1 - e^{-x} - e^{-t}$, for $x, t > 0$. 

4. Oracle inequalities for the square root analysis estimator

As anticipated in Section 2, to eliminate the randomness from the upper bound on the increment of the empirical process for the case of the square root analysis estimator, we have to restrict our attention to a (high probability) set where the antiprojection of the error term is upper bounded.

We thus define the set

\[ A' := A \cap \left\{ \|A_N(D - \hat{S})\epsilon\|^2_2 / \sigma^2 \leq n - rS + 2\sqrt{a(n - rS)} + 2a \right\}. \]

Note that on \( A' \) we have that, by the Cauchy-Schwarz inequality,

\[
\|\epsilon\|^2_n = \|\Pi_{N(D - \hat{S})}\epsilon\|^2_2 + \|A_N(D - S)\epsilon\|^2_2 \\
\leq \frac{\sigma^2}{n} (n + \sqrt{8an} + 4a) \\
\leq \frac{\sigma^2}{n} (\sqrt{n} + \sqrt{4a})^2 = \sigma^2 (1 + \sqrt{4a/n})^2.
\]

4.1. Fast rates with compatibility conditions

We now expose an oracle inequality with fast rates involving compatibility conditions for the square root analysis estimator. The final result is very similar to the one obtained for the plain analysis estimator up to some constants resulting from the upper bound for \( \|\epsilon\|_n \). However there is a substantial difference in the set of assumptions one has to make due to the preliminary step of showing that the estimator does not overfit.

**Theorem 4.1** (Fast rates for the square root regularized analysis estimator)

*Under Assumption 2.1 on \( A' \cap R \) we have that*

\[
\|\hat{f}^\epsilon - f^0\|^2_n \leq \|f - f^0\|^2_n + 16\sigma\lambda_0\|D - S f\|_1 \\
+ \sigma^2 \left( \frac{2a}{n} + \sqrt{\frac{rS}{n}} + \frac{4\lambda_0 \sqrt{rS}}{\kappa(S, W)} \right)^2.
\]

*Proof of Theorem 4.1.* See Appendix D \( \Box \)

4.2. Slow rates without compatibility conditions

We expose an oracle inequality for the square root analysis estimator with slow rates not requiring compatibility conditions. The result is similar to Theorem 3.2 up to the constants and the assumptions one has to make.
Theorem 4.2 (Slow rates for the square root regularized analysis estimator)

Under Assumption 2.1 on $A' \cap R$ we have that,

$$
\|\hat{f}_\sqrt{\cdot} - f^0\|^2_n + 2(1 - \eta)\sqrt{1 - \sqrt{8a/n} \sigma \lambda_0} \left( \left( 1 - \frac{\rho}{\gamma} \right) \|D - S\hat{f}_\sqrt{\cdot}\|_1 + \|DS\hat{f}_\sqrt{\cdot}\|_1 \right)
$$

$$
\leq \|f - f^0\|^2_n + \frac{\sigma^2}{n} \left( \sqrt{2a} + \sqrt{rS} \right)^2 + 8\sigma \lambda_0 \left( \left( 1 + \frac{\rho}{\gamma} \right) \|D - S\|_1 + \|DSf\|_1 \right).
$$

Proof of Theorem 4.2. See Appendix D

Remark. The claim of Theorem 4.2 implies also the simpler inequality

$$
\|\hat{f} - f^0\|^2_n \leq \|f - f^0\|^2_n + \frac{\sigma^2}{n} \left( \sqrt{2a} + \sqrt{rS} \right)^2 + 16\sigma \lambda_0 \|DF\|_1.
$$

4.3. Dealing with randomness

We now want to find a lower bound for the probability of the events on which the results for the square root analysis estimator hold.

By Lemma A.2 (Lemma 1 in Laurent and Massart (2000)) we have that for $a > 0$

- $\mathbb{P}(A) \geq 1 - 3e^{-a}$,
- $\mathbb{P}(A') \geq 1 - 4e^{-a}$.

Moreover by Lemma A.3 (Lemma 8.1 in van de Geer (2016)) and using the union bound, we see that if we choose

$$
R \geq \gamma \sqrt{\frac{2\log(2(n - rS)) + 2t^4}{n - 1}}, t \in (0, (n - 1)/2 - \log(2(n - rS)))
$$

we have that

$$
\mathbb{P}(R) \geq 1 - e^{-t}.
$$

Thus, by such a choice of $R$ we get that

- Lemma B.1 holds on an event having probability at least $1 - e^{-t} - 3e^{-a}$.

Moreover under Assumption 2.1 we have that

- Corollary 2.1, Lemma 2.4 and Lemma 2.5 hold on an event having probability at least $1 - e^{-t} - 3e^{-a}$;
- Theorem 4.1 and Theorem 4.2 hold on an event having probability at least $1 - e^{-t} - 4e^{-a}$. 

Remark. Motivated by a more simple exposition of the results, we choose the same parameter $a$ for the upper and lower bounds for both $\|\Pi_{\mathcal{N}(D_{-S})}\epsilon\|_n$ and $\|A_{\mathcal{N}(D_{-S})}\epsilon\|_n$. However one could of course choose four different parameters, say $a_i, i \in [4]$, for the four different bounds and obtain results holding with probability $1 - e^{-t} - \sum_{i=1}^4 a_i$ resp. $1 - e^{-t} - \sum_{i=1}^4 a_i$.

5. Total variation

We present some implications of the results exposed above for the particular case of total variation regularization on graphs.

5.1. Incidence matrices

Let $\tilde{G} = (V, E)$ be a general directed graph, where the set $V = [n]$ of cardinality $n := |V|$ is the set of vertices and the set $E = \{e_1, \ldots, e_m\}$ of cardinality $m := |E|$ is the set of edges. Every edge $e_i = (e^-_i, e^+_i)$ is directed from a vertex $e^-_i \in V$ to a vertex $e^+_i \in V$, $e^-_i \neq e^+_i$.

Let $D \in \{-1, 0, 1\}^{m \times n}$ be the incidence matrix of $\tilde{G}$. Then

$$(d'_i)_j = \begin{cases} -1, & j = e^-_i, \\ 1, & j = e^+_i, \\ 0, & \text{else}. \end{cases}$$

It is known that the rank of $D$ is given by the number of vertices of the graph minus its number of connected components.

For a signal $f \in \mathbb{R}^n$ on $\tilde{G}$ let $S := \{i : d'_i f \neq 0\}$ and let $r_S := \dim(\mathcal{N}_{-S}) = n - \text{rank}(D_{-S})$. Let us define the set of edges $E_S := \{e_i \in E, i \in S\}$. We see that $r_S$ is the number of connected components that the graph $\tilde{G}_{-S} = (V, E \setminus E_S)$ has, i.e. the number of constant pieces in the signal $f$. These connected components can be any sort of graph: tree graphs as well as non-tree graphs.

As widely known, tree graphs are connected graphs without cycles. Such graphs lose the property of being connected as soon as one edge is removed from them. On the other side, we can remove an edge from a cycle (sub)graph without causing it to become disconnected. It follows that $S \setminus \mathcal{S}$, which might be empty or not unique, denotes the set of edges one can delete from $\tilde{G}_{-S}$ to get the graph $\tilde{G}_{-\mathcal{S}} = (V, E \setminus \mathcal{S})$ having the same connected components as $\tilde{G}_{-S}$ and where these connected components are all tree graphs.

It is thus easy to see that $D_{-\mathcal{S}} \in \mathbb{R}^{(n-r_S) \times n}$ is a matrix of full rank.

Let $n_1, \ldots, n_{r_S}$ be the number of vertices of each connected component $\tilde{C}_i := ([n_i], E_i), i \in [r_S]$ of $\tilde{G}_{-\mathcal{S}}$, where $|E_i| = n_i - 1, i \in [r_S]$. As explained above, $\tilde{C}_i, i \in [r_S]$ are tree graphs. Let $D_{\tilde{C}_i} \in \mathbb{R}^{(n_i-1) \times n_i}, i \in [r_S]$ be the incidence matrices of $\tilde{C}_i, i \in [r_S]$. The matrix $D_{-\mathcal{S}}$ can also be rewritten as the block
matrix
\[
\begin{pmatrix}
D_{\vec{C}_i} & \cdots & \cdots & D_{\vec{C}_rS}
\end{pmatrix} \in \mathbb{R}^{(n-r_S) \times n}
\]

by rearranging rows and columns where necessary. From now on, when writing \(D_{-S}\) we intend the matrix in its block form shown here above.

By Lemma 1 in Ijiri (1965) we have that
\[
D_{+S} = \begin{pmatrix}
D_{\vec{C}_i}^+ & \cdots & \cdots & D_{\vec{C}_rS}^+
\end{pmatrix} \in \mathbb{R}^{n \times (n-r_S)}.
\]

Let us define \(n_{\min} = \min\{n_1, \ldots, n_{r_S}\}\) and \(n_{\max} = \max\{n_1, \ldots, n_{r_S}\}\).

**Lemma 5.1** (Upper bound for the normalized inverse scaling factor)
We have \(\forall \vec{G}, \forall \vec{S}\) that
\[
\rho \leq \sqrt{\frac{n_{\max} + 1}{4n}}.
\]

**Proof of Lemma 5.1.** See Appendix E

**Remark.** Note that due to the introduction of \(\vec{S}\) the claim of Lemma 5.1 holds for any directed graph.

### 5.2. Fast rates

To prove oracle inequalities for fast rates for the analysis estimator and its square root regularized version we need to find an explicit lower bound for the weighted compatibility constant. We will now expose the tools necessary to do so in the case of the path graph and in the case of the cycle graph.

Results for the analysis estimator on the path graph have already been obtained by Ortelli and van de Geer (2018). We extend them to the square root analysis estimator. Moreover, we also show that the tools developed in Ortelli and van de Geer (2018) together with the new framework presented here, allow to handle the case of the cycle graph. We are aware of results treating the \(k^{th}\) power graphs of cycles (Hütter and Rigollet (2016)) but not of any oracle inequality implying the convergence of the mean squared error for the case of the cycle graph.

#### 5.2.1. Path graph

We now consider the elementary example of \(\vec{G} = ([n], \{(1, 2), \ldots, (n-1, n)\})\) being the path graph. It follows from the previous considerations that \(S = \vec{S}\), since a path is a tree graph.
We see that $D - S$ is a block matrix, where the blocks are again incidence matrices of some smaller path graphs. Thus, we now study $D_{\tilde{C_i}}$ for a general $i \in [r_S]$, where $\tilde{C_i} = ([n_i], \{(1, 2), \ldots, (n_i - 1, n_i)\})$. By recycling the reasoning of the Proof of Lemma 5.1 we obtain that

$$\| (D^{+}_{\tilde{C_i}})_{j} \|_2 = \sqrt{\frac{j(n_i - j)}{n_i}}.$$

The following lemma by van de Geer (2018), later also used in Ortelli and van de Geer (2018), allows us to lower bound $\kappa(S, W)$, for a diagonal matrix $W = \text{diag}([w_j]_{j \in [n-1]}) \in \mathbb{R}^{(n-1) \times (n-1)}$ with $\|W\|_\infty \leq 1$ and where by convention we choose $w_n = 1$.

**Lemma 5.2** (Theorem 6.1 and Lemma 9.1 in van de Geer (2018))
Assume that $S$ is s.t. $n_1, n_{r_S} \geq 2$ and $n_i \geq 4, \forall i \in \{n_2, \ldots, n_{r_S-1}\}$. Then

$$\frac{\sqrt{r_S}}{\kappa(S, I_{n-1})} \leq \sqrt{nK},$$

where

$$K = \frac{1}{n_1} + \sum_{i=2}^{r_S-1} \left( \frac{1}{\lfloor n_i/2 \rfloor} + \frac{1}{\lceil n_i/2 \rceil} \right) + \frac{1}{n_{r_S}}$$

and the inequality is tight.

Moreover

$$\frac{\sqrt{r_S}}{\kappa(S, W)} \leq \frac{\sqrt{r_S}}{\kappa(S, I_{n-1})} + \sqrt{n \sum_{i=2}^{n} (w_i - w_{i-1})^2}.$$

**Proof of Lemma 5.2.** The first statement follows form Theorem 6.1 and the second from Lemma 9.1 in van de Geer (2018).

The proofs are also exposed also in Ortelli and van de Geer (2018), in Lemmas 5.3-5.

Ortelli and van de Geer (2018) explain that to bound the weighted compatibility constant for the path graph one needs to cut it into $S$ smaller modules. These modules lie around an edge in $S$ and are constituted by at least one additional edge on each side of the edge in $S$, see Figure 1.

**Fig 1.** Illustration of the minimal module needed to bound the compatibility constant.

The edges not in $S$ between modules can be discarded when bounding the compatibility constant. Each module needs at least 4 vertices, s.t. we need $|S| \leq n/4$ to hope to be able to upper bound $\kappa^2(S, W)$ by using the method proposed.
by van de Geer (2018). Moreover, a vertex not involved in an edge in $S$ can not be involved in more than one module to obtain the bounds exposed in Ortelli and van de Geer (2018).

Note also that the weights in $w$ have a direct correspondence to the edges of the graph, where the edges in $S$ are s.t. $\omega_S = 0$. Moreover, the weights for the edges between modules can be chosen arbitrarily when it comes to upperbounding $\|W_SD_Sf\|_1 - \|W_-SD_-f\|_1$, even if a value for them can be also obtained by computation of the $\|\cdot\|_n$-norm of the corresponding columns of $D^S_+$. We moreover make the arbitrary decision to use the convention $w_n := 1$, as in Lemma 5.2.

**Lemma 5.3**

Assume that $n_i \geq 4, \forall i \in [r_S]$. We have that

$$\sum_{j=2}^{n}(w_i - w_{i-j})^2 \leq \frac{5}{\gamma} \frac{r_S}{n} \log \left( \frac{n}{r_S} \right),$$

**Proof of Lemma 5.3.** See the proof of Corollary 5.6 in Ortelli and van de Geer (2018).

Let $D \in \mathbb{R}^{(n-1) \times n}$ be the incidence matrix of the path graph with $n$ vertices. With the tools developed we can prove the following corollaries.

**Analysis estimator**

**Corollary 5.1**

Assume that $S$ is s.t. $n_i \geq 4, \forall i \in [r_S]$. Choose $\lambda = \sigma \sqrt{\frac{n_{\text{max}} \log(2n) + t}{n}}$. Then, with probability at least $1 - e^{-t} - e^{-x}$, it holds that

$$\|\hat{f} - f^0\|_n^2 \leq \|f - f^0\|_n^2 + 4\lambda \|D_+f\|_1$$

$$+ \sigma \left( \sqrt{\frac{2\pi}{n}} + \sqrt{\frac{r_S}{n}} + \sqrt{\frac{n_{\text{max}}K}{n}} \right) \left( \log(2n) + t \right)$$

$$+ \sqrt{\frac{10r_S}{n}} \left( \log(2n) + t \right) \log \left( \frac{n}{r_S} \right).$$

**Proof of Corollary 5.1.** See Appendix E.

Corollary 5.1 assumes some kind of minimal length condition, which however does not depend on $n$ and is needed to use the bound given by Lemma 5.3. This minimal length condition is therefore much weaker than the one found in Guntuboyina et al. (2017). Note that the choice of the tuning parameter depends both on $\sigma$ and $n_{\text{max}} = n_{\text{max}}(S)$.

The next corollary makes a stronger assumption on $S$ and allows us to obtain a more explicit rate in the oracle inequality.
Corollary 5.2
Assume that $S$ is s.t. $n_i \geq 4, \forall i \in [r_S]$. Moreover assume that $n_{\min} = n_{\max}$ and $n_{\max}$ is even. Choose $\lambda = \sigma \sqrt{\log(2n) + t}$. Then, with probability at least $1 - e^{-t} - e^{-x}$, it holds that
\[
\|\hat{f} - f^0\|_n^2 \leq \sqrt{\frac{2x}{n}} + \sqrt{\frac{r_S}{n}} + \sqrt{\frac{4r_S}{n} (\log(2n) + t)} + \sqrt{\frac{10r_S}{n} (\log(2n) + t) \log \left( \frac{n}{r_S} \right)} \cdot \lambda
\]

Proof of Corollary 5.2. If $n_{\min} = n_{\max}$, then $n_i = n/r_S, \forall i \in [r_S]$. Moreover, $K \leq 4r_S^2/n$ and the statement of Corollary 5.2 follows by plugging in these insights into Corollary 5.1.

Remark. Note that if we choose $f^0$ with the set $S_0 := \{i : d'_i f^0 \neq 0\}$ we have that $n_{\max} = n_{\max}(S_0) = n_{\max}(S_0(f^0))$ and thus the tuning parameter in Corollaries 5.1 and 5.2 depends on $f^0$.

The choice of the tuning parameter in the above oracle inequality is remarkable. Indeed, Corollary 5.2 says that, if $n_{\min} = n_{\max}$, then we can choose $\lambda$ smaller than the universal choice $\lambda \sim \sigma \sqrt{\log n/n}$. The choice of the tuning parameter in the two corollaries above assumes however the knowledge of some aspects of $f^0$ and can be seen as a motivation to choose the tuning parameter smaller than the universal choice if we know or suspect a certain specific structure for $f^0$. Another characteristic is that the tuning parameter is constant-friendly in the sense that it does not involve any unknown scaling constant and can thus be calculated if $n_{\max}$ is known. These insights were already developed by Dalalyan, Hebiri and Lederer (2017) and applied to total variation on the path graph in the case of slow rates.

Square root analysis estimator

We now extend the results obtained for the analysis estimator to the case of the square root analysis estimator.

Corollary 5.3
Under Assumption 2.1 assume that $S$ is s.t. $n_i \geq 4, \forall i \in [r_S]$. Let $a > 0$ and $t \in (0, (n - 1)/2 - \log(2(n - r_S)))$. Choose $\lambda_0 = \frac{1}{t-a} \sqrt{n_{\max} \log(2n) + t}$. Then, with probability at least $1 - e^{-t} - 4e^{-x}$, for the square root version of the total
variation regularized estimator over the path graph it holds that
\[
\| \hat{f}_{\sqrt{\cdot}} - f_0 \|^2_n \leq \| f - f_0 \|^2_n + 16\lambda_0 \sigma \| (Df)_{-S} \|_1 \\
+ \sigma \left( \sqrt{\frac{2a}{n}} + \sqrt{\frac{r_S}{n}} + \frac{4}{1-\eta} \sqrt{\frac{n_{\max} K}{n-1}} (\log(2n) + t) \right) \\
+ \frac{4}{1-\eta} \sqrt{\frac{10r_S}{n-1}} (\log(2n) + t) \log \left( \frac{n}{r_S} \right) .
\]

Proof of Corollary 5.3. The proof of Corollary 5.3 is analogous to the proof of Corollary 5.1.

Corollary 5.4
Under Assumption 2.1 assume that \( S \) is s.t. \( n_i \geq 4, \forall i \in [r_S] \). Let \( a > 0 \) and \( t \in (0, (n-1)/2 - \log(2(n-r_S))) \). Choose \( \lambda_0 = \frac{1}{1-\eta} \left( \frac{\log(2(2n)+t)}{r_S(n-1)} \right) \). Moreover assume that \( n_{\min} = n_{\max} \) and \( n_{\max} \) is even. Let \( a > 0 \) and \( t \in (0, (n-1)/2 - \log(2(n-r_S))) \).

Then, with probability at least \( 1 - e^{-t} - 4e^{-a} \), for the square root version of the total variation regularized estimator over the path graph it holds that
\[
\| \hat{f}_{\sqrt{\cdot}} - f_0 \|^2_n \leq \| f - f_0 \|^2_n + 16\lambda_0 \sigma \| (Df)_{-S} \|_1 \\
+ \sigma \left( \sqrt{\frac{2a}{n}} + \sqrt{\frac{r_S}{n}} + \frac{4}{1-\eta} \sqrt{\frac{4r_S}{n-1}} (\log(2n) + t) \right) \\
+ \frac{4}{1-\eta} \sqrt{\frac{10r_S}{n-1}} (\log(2n) + t) \log \left( \frac{n}{r_S} \right) .
\]

Proof of Corollary 5.4. If \( n_{\min} = n_{\max} \), then \( n_i = n/r_S, \forall i \in [r_S] \). Moreover, \( K \leq 4r_S^2/n \) and the statement of Corollary 5.4 follows by plugging in these insights into Corollary 5.3.

Remark. We notice that there is a tradeoff in the choice of \( \eta \). Choosing a small \( \eta \) will result in a narrower bound for \( \| \hat{\epsilon} \|_n \) in terms of \( \| \epsilon \|_n \) and in smaller constants in the tuning parameter and in the oracle bound. However, it might result in a more restrictive condition on \( S \) in Assumption 2.1.

5.2.2. When \( f^0 \) is Lipschitz

We are now going to apply the above results to the case where \( f^0 \) is Lipschitz. These results are already present in similar forms in Dalalyan, Hebiri and Lederer (2017). We extend them to the case of the square root analysis estimator.

Let \( f^0 : [0, 1] \to \mathbb{R} \). Assume that \( n \) is s.t. \( n^{1/3} \in \mathbb{N} \). Let \( \{x_i\}_{i \in [n]} = \{n^{-1}(i-1)\}_{i \in [n]} \subset [0, 1] \) be a set of cardinality \( n \) of equidistant design points placed
at distance $n^{-1}$ from each other. Let $\{x'_i\}_{i \in \mathbb{N}^{n^{1/3}}} = \{n^{-1/3}(i - 1)\}_{i \in \mathbb{N}^{n^{1/3}}} \subset \{x_i\}_{i \in \mathbb{N}}$ be a subset of cardinality $n^{1/3}$ of equidistant design points placed at distance $n^{-1/3}$ from each other. For $x \in [0, 1]$ define

$$l(x) := \max\{i \in \mathbb{N}^{n^{1/3}} : x'_i \leq x\},$$

and

$$r(x) := \max\{i \in \mathbb{N}^{n^{1/3}} : x'_i > x\}.$$

Let $f(x) = f^0(x'_{l(x)}), x \in [0, 1]$. We have that, $\forall x \in [x'_{l(x)}, x'_{r(x)}], f(x) = f^0(x'_{l(x)})$.

**Lemma 5.4**

Let $f^0 : [0, 1] \rightarrow \mathbb{R}$ be Lipschitz with constant $L$, i.e.

$$\|f^0(x) - f^0(\tilde{x})\|_2^2 \leq L\|x - \tilde{x}\|_2^2, \forall x, \tilde{x} \in [0, 1].$$

Then

$$\|f - f^0\|_n^2 \leq Ln^{-2/3}.$$

**Proof of Lemma 5.4.** See Appendix E.

Lemma 5.4 means that if $f^0 : [0, 1] \rightarrow \mathbb{R}$ is a Lipschitz continuous function, then its approximation by a piecewise constant function with $n^{1/3}$ constant pieces of length $n^{-1/3}$ incurs in a mean squared error of at most $Ln^{-2/3}$.

---

**Analysis estimator**

**Corollary 5.5**

Let $f^0$ be Lipschitz. Assume that $n \geq 8$ and that $\sigma$ does not depend on $n$. Moreover assume that $n^{1/3}$ is even. Let $x, t > 0$. Choose $\lambda = \sigma n^{-2/3} \sqrt{\log(2n) + t}$.

Then probability at least $1 - e^{-t} - e^{-t}$,

$$\|\hat{f} - f^0\|_n^2 = \mathcal{O}(n^{-2/3} \log^2 n).$$

**Proof of Corollary 5.5.** See Appendix E.

**Square root analysis estimator**

**Corollary 5.6**

Let $f^0$ be Lipschitz. Under Assumption 2.1 assume that $n \geq 8$ and that $\sigma$ does not depend on $n$. Moreover assume that $n^{1/3}$ is even. Let $a > 0$ and $t \in (0, (n - 1)/2 - \log(2(n - r_S)))$. Choose $\lambda_0 = \frac{1}{1 - \eta} \sqrt{\frac{\log(2n) + t}{n^{2/3} - n^{1/3}}}$. Then probability at least $1 - e^{-t} - 4e^{-a}$,

$$\|\hat{f}^\sqrt{\cdot} - f^0\|_n^2 = \mathcal{O}(n^{-2/3} \log^2 n).$$

**Proof of Corollary 5.6.** The proof follows from Corollary 5.4 and Lemma 5.4.
5.2.3. Cycle graph

The considerations about the path graph jointly with Theorem 3.1 (and Theorem 4.1) for the general (square root) analysis estimator allow us to obtain an oracle inequality for the total variation regularized estimator over the cycle graph.

Indeed, as we explained in Subsubsection 5.2.1, for the weighted compatibility constant to be lower bounded away from zero, we need the assumption that the set \( S \) is s.t. the graph can be cut into modules, consisting of (at least) one edge not in \( S \) followed by an edge in \( S \) followed again by (at least) one edge not in \( S \), see Figure 1.

By concatenating such modules, one can obtain a path graph. Whether or not the two ends of the path graph are joined by an edge is not relevant for the possibility to bound the compatibility constant and obtain an oracle inequality with fast rates, since the edges connecting such modules are neglected when bounding the (weighted) compatibility constant.

**Remark.** Note that for the path graph we have that \( r_S = |S| + 1 \), while for the cycle graph it holds that \( r_S = |S| \).

Let \( D \in \mathbb{R}^{n \times n} \) denote the incidence matrix of the cycle graph with \( n \) vertices.

**Corollary 5.7**

Assume that \( S \) is s.t. \( n_i \geq 4, \forall i \in [r_S] \). Then

\[
\frac{\sqrt{r_S}}{\kappa(S, I_n)} \leq \sqrt{nK'},
\]

where

\[
K' = \sum_{i=1}^{r_S} \left( \frac{1}{\lfloor n_i/2 \rfloor} + \frac{1}{\lceil n_i/2 \rceil} \right)
\]

and the inequality is tight.

Moreover

\[
\frac{\sqrt{r_S}}{\kappa(S, W)} \leq \frac{\sqrt{r_S}}{\kappa(S, I_n)} + \sqrt{n\|Dw\|_2^2}.
\]

**Proof of Corollary 5.7.** Corollary 5.7 follows from Lemma 8.2 and from the considerations above. \( \square \)

**Remark.** From Lemma 5.2 we get that, if \( n_i \geq 4, \forall i \in [r_S] \), then

\[
\|Dw\|_2^2 \leq \frac{5}{\gamma^2} \frac{r_S}{n} \log \left( \frac{n}{r_S} \right).
\]

We now have all the tools to derive an oracle inequality for the total variation regularized estimator over the cycle graph and its square root version.

**Analysis estimator**
Corollary 5.8
Assume that $S$ is s.t. $n_i \geq 4, \forall i \in [r_S]$. Let $x, t > 0$. Choose $\lambda = \sigma \sqrt{n_{\max} \log(2n) + t}$. Then, with probability at least $1 - e^{-t} - e^{-x}$, for the total variation regularized estimator over the cycle graph it holds that

$$\|\hat{f} - f^0\|_n^2 \leq \|f - f^0\|_n^2 + 4\lambda \|(Df) - S\|_1$$

$\sigma \left( \sqrt{\frac{2n}{n_{\max}}} + \sqrt{\frac{r_S}{n}} + \sqrt{\frac{n_{\max}K'}{n}(\log(2n) + t)} + \sqrt{\frac{10r_S}{n}(\log(2n) + t) \log \left( \frac{n}{r_S} \right)} \right)^2$.

**Remark.** An analogous version of Corollary 5.2 can be derived from Corollary 5.8.

**Square root analysis estimator**

Corollary 5.9
Under Assumption 2.1 assume that $S$ is s.t. $n_i \geq 4, \forall i \in [r_S]$. Let $a > 0$ and $t \in (0, (n - 1)/2 - \log(2(n - r_S)))$. Choose $\lambda_0 = \frac{1}{1 - \eta} \sqrt{n_{\max} \log(2n) + t}$. Then, with probability at least $1 - e^{-t} - 4e^{-a}$, for the square root version of the total variation regularized estimator over the cycle graph it holds that

$$\|\hat{f}^{\sqrt{\cdot}} - f^0\|_n^2 \leq \|f - f^0\|_n^2 + 16\lambda_0 \sigma \|(Df) - S\|_1$$

$\sigma \left( \sqrt{\frac{2n}{n_{\max}}} + \sqrt{\frac{r_S}{n}} + \frac{4}{1 - \eta} \sqrt{\frac{n_{\max}K'}{n - 1}(\log(2n) + t)} + \frac{4}{1 - \eta} \sqrt{10r_S \log(2n) + t \log \left( \frac{n}{r_S} \right)} \right)^2$.

**Remark.** An analogous version of Corollary 5.4 can be derived from Corollary 5.9.

5.3. **Slow rates**

Note that in the case of the so-called slow rates we do not need to lower bound the compatibility constant. A result we need is Lemma 5.1, which applies to every directed graph $\tilde{G}$. This is a key feature and allows the generalization of the results by Dalalyan, Hebiri and Lederer (2017) to a broad variety of setups.

In this subsection we identify the analysis operator $D$ with the incidence matrix of a general graph $\tilde{G}$. 
Analysis estimator

Corollary 5.10
Let $\mathbf{G}$ be any graph. Let $x, t > 0$. Choose $\lambda = \frac{\sigma}{n} \sqrt{n_{\max}(\log(2n) + 2t)}$. Then we have that with probability at least $1 - e^{-x} - e^{-t}$

$$\|\hat{f} - f^0\|_n^2 \leq \frac{\sigma^2}{n} \left( \sqrt{2x} + \sqrt{r_S} \right)^2 + 4\sigma n \sqrt{n_{\max}(\log(2n) + t)n} \|Df\|_1.$$ 

Proof of Corollary 5.10. Corollary 5.10 follows by combining Theorem 3.2 and Lemma 5.1. □

Corollary 5.11
Let $\mathbf{G}$ be any graph. Suppose that $n_{\max} = n_{\min}$. Let $x, t > 0$. Choose $\lambda = \sigma \sqrt{\frac{\log(2n) + 2t}{n_{\max}^2}}$. Then we have that with probability at least $1 - e^{-x} - e^{-t}$

$$\|\hat{f} - f^0\|_n^2 \leq \frac{\sigma^2}{n} \left( \sqrt{2x} + \sqrt{r_S} \right)^2 + 4\sigma \sqrt{n_{\max}(\log(2n) + t)n} \|Df\|_1.$$ 

Square root analysis estimator

Corollary 5.12
Let $\mathbf{G}$ be any graph. Under Assumption 2.1, let $a > 0$ and $t \in (0, (n - 1)/2 - \log(2(n - r_S)))$. Choose $\lambda_0 = \frac{1}{1 - \eta} \sqrt{n_{\max}(\log(2n) + t)}$. Then, with probability at least $1 - e^{-t} - 4e^{-a}$, it holds that

$$\|\hat{f} - f^0\|_n^2 \leq \frac{\sigma^2}{n} \left( \sqrt{2a} + \sqrt{r_S} \right)^2 + 16\sigma \frac{n_{\max}(\log(2n) + t)}{n(n - 1)} \|Df\|_1.$$ 

Proof of Corollary 5.12. Corollary 5.12 follows by combining Theorem 4.2 and Lemma 5.1. □

Corollary 5.13
Let $\mathbf{G}$ be any graph. Under Assumption 2.1, suppose that $n_{\max} = n_{\min}$. Let $a > 0$ and $t \in (0, (n - 1)/2 - \log(2(n - r_S)))$. Choose $\lambda_0 = \frac{1}{1 - \eta} \sqrt{n_{\max}(\log(2n) + t)}$. Then, with
probability at least $1 - e^{-t} - 4e^{-a}$, it holds that
\[
\| \hat{f} - f^0 \|_n^2 \leq \| f - f^0 \|_n^2 + \frac{\sigma^2}{n} \left( \sqrt{2a} + \sqrt{r_S} \right)^2 + \frac{16\sigma}{1 - \eta} \sqrt{\log(2n) + t} \| D f \|_1.
\]

5.3.1. Comparison with other results

Corollary 5.14
In Corollary 5.11 (resp. 5.13) take $f = f^0$ and $r_S \asymp n^{1/3}(\log(2n) + t)^{1/3}\| D f^0 \|_1^{2/3}$. Assume that $\sigma$ does not depend on $n$. Then we have that with probability at least $1 - e^{-x} - e^{-t}$ (resp. $1 - e^{-t} - 4e^{-a}$),
\[
\| \hat{f} - f^0 \|_n^2 = \mathcal{O}(n^{-2/3}(\log(2n) + t)^{1/3}\| D f^0 \|_1^{2/3}).
\]

Note. In Corollary 5.14 the choice of the tuning parameter would explicitly depend on $f^0$ since $r_S$ depends on $\| D f^0 \|_1$.

Corollary 5.15
In Corollary 5.11 (resp. 5.13) take $f = f^0$ and $r_S \asymp n^{1/3}(\log(2n) + t)^{1/3}$. Assume that $\sigma$ does not depend on $n$. Then we have that with probability at least $1 - e^{-x} - e^{-t}$ (resp. $1 - e^{-t} - 4e^{-a}$),
\[
\| \hat{f} - f^0 \|_n^2 = \mathcal{O}(n^{-2/3}(\log(2n) + t)^{1/3}\| D f^0 \|_1).
\]

Note. In the case of Corollary 5.15 we have that $r_S$ and thus $\lambda$ (resp. $\lambda_0$) do not explicitly depend on $f^0$.

Remark. In both cases of Corollary 5.14 and Corollary 5.15, if $\| D f^0 \| = \mathcal{O}(1)$ we obtain that
\[
\| \hat{f} - f^0 \|_n^2 = \mathcal{O}(n^{-2/3}\log^{1/3}(n)).
\]

However, it is known that the minimax rate for that case (when the graph considered is the path graph) is
\[
\| \hat{f} - f^0 \|_n^2 = \mathcal{O}(n^{-2/3})
\]
and thus our results lead to a redundant log-term.

The result about the minimax rate over the class of functions with bounded total variation can be obtained by entropy calculations (Mammen and van de Geer (1997) and references therein). These calculations are not constant-friendly, so that it may well be that, for reasonable $n$, the log-term is actually smaller than the universal constants showing up when using entropy arguments.

We thus saw that for the case of the path graph, the projection argument introduced by Dalalyan, Hebiri and Lederer (2017) to handle the increments
of the empirical process might produce **optimal rates up to a log term** not only for the total variation regularized estimator, but also for the square root total variation regularized estimator. Moreover these calculations are constant-friendly and might even result in upper bounds which, for moderate $n$, might be smaller than the ones resulting from constant-unfriendly entropy calculations.

Moreover, our results for both the classical version and the square root version of the total variation regularized estimator over a general graph match up to a log-term the result by Padilla et al. (2018), which exploits entropy calculations to establish the rate $n^{-2/3}C^{2/3}$ on the class $\{f \in \mathbb{R}^n : \|Df\|_1 \leq C\}$.

6. Conclusion

We extended the approach used to bound the increments of the empirical process by means of projections introduced by Dalalyan, Hebiri and Lederer (2017) to the cases of a general analysis estimator and of a general square root analysis estimator. By doing so we also introduced a different measure of sparsity of the signal. Indeed, instead of $\|Df^0\|_0$, which has previously been used, for instance in Hütter and Rigollet (2016), and which can be larger than $n$ if $m$ is large, we use $r_{S_0} = \dim(N(D-S_0))$. This last quantity has the advantage of being related to the effective number of parameters one needs to estimate. For instance, in the example of total variation regularization, $r_{S_0}$ denotes the number of connected components, which is a more natural measure for the number of parameters to estimate than the number of nonzero edge differences.

We then derived oracle inequalities with fast rates under some compatibility conditions and oracle inequalities with slow rates.

We obtained parallel and very similar results for both the analysis and the square root analysis estimators. The differences in these results come from the fact that for the square root analysis estimator we first have to prove that the estimator does not overfit and that the KKT conditions hold. Thus we can observe differences in the events on which the oracle inequality holds and in the constants appearing in the results. In spite of being mathematically more involved, the results for the square root analysis estimator tell us that we can get with high probability theoretical guarantees being very similar to the ones obtained for the analysis estimator by choosing a tuning parameter not depending on the unknown noise level. This fact might be helpful in practice and might speak in favor of the utilization of the square root analysis estimator.

We then narrowed down our results to (square root) total variation regularized estimators over graphs. For fast rates we considered the cases of the path graph and of the cycle graph. In these cases we were able to show that the compatibility conditions are satisfied and obtained a rate matching the minimax rate in the class of functions of bounded total variation for the path graph up to a log-term, under the condition that $f^0$ comes from a Lipschitz continuous function.

For the case of slow rates, we obtained an oracle inequality holding for all
graphs and matching the optimal rate over the path graph up to a log-term without any continuity nor compatibility assumption.

Some questions for further investigation are for instance the possibility to obtain oracle inequalities with fast rates for other graph structures (e.g. the two dimensional grid) and which rates will result from such an oracle inequality. The answer of course depends on the ability to lower bound the compatibility constant for graphs other than tree graphs and cycles. We leave this question to future research.

Appendix A: Probability inequalities

We expose three lemmas which help us to deal with the random part of the oracle inequalities.

Lemma A.1 (The maximum of $p$ random variables, Lemma 17.5 in [van de Geer, 2016])

Let $V_1, \ldots, V_p$ be real valued random variables. Assume that $\forall j \in \{1, \ldots, p\}$ and $\forall r > 0$

$$\mathbb{E}[e^{r|V_j|}] \leq 2e^{2r^2}.$$  

Then, $\forall t > 0$

$$P\left(\max_{1 \leq j \leq p}|V_j| \geq \sqrt{2\log(2p) + 2t}\right) \leq e^{-t}.$$

Lemma A.2 (The special case of $\chi^2$ random variables, Lemma 1 in [Laurent and Massart, 2000], Lemma 8.6 in [van de Geer, 2016])

Let $X \sim \chi^2_d$. Then,

$$P\left(X \geq d + 2\sqrt{dx} + 2x\right) \leq e^{-x} \text{ and } P\left(X \leq d - 2\sqrt{dx}\right) \leq e^{-x}.$$

Remark. Note that from Lemma A.2 it follows that

$$P\left(\sqrt{X} \leq \sqrt{d + \sqrt{2x}}\right) \geq P\left(X \leq d + 2\sqrt{dx} + 2x\right) \geq 1 - e^{-x}.$$

Lemma A.3 (Lemma 8.1 in [van de Geer, 2016])

For $n \geq 2$, let $\epsilon \sim N_\mathbb{R}(0, \sigma^2 I_n)$. Then, $\forall u \in \mathbb{R}^n : \|u\|_n = 1$ we have that, for $t \in (0, (n-1)/2)$,

$$P\left(\frac{u'\epsilon}{n\|\epsilon\|_n} > \sqrt{\frac{2t}{n-1}}\right) \leq 2e^{-t}.$$

Remark. Let $u_1, \ldots, u_p \in \mathbb{R}^n$ be vectors. Then by the union bound and by Lemma A.3 we have that for $t' \in (0, (n-1)/2)$

$$P\left(\max_{i \in [p]} \frac{|u_i'\epsilon|}{n\|u_i\|_n\|\epsilon\|_n} > \sqrt{\frac{2t'}{n-1}}\right) \leq 2pe^{-t'}.$$  

Now select $t = t' - \log(2p)$. Then we have that for $t \in (0, (n-1)/2 - \log(2p))$,

$$P\left(\max_{i \in [p]} \frac{|u_i'\epsilon|}{n\|u_i\|_n\|\epsilon\|_n} > \sqrt{\frac{2\log(2p) + 2t}{n-1}}\right) \leq e^{-t}.$$
Appendix B: Proofs of Section 2

Proof of Lemma 2.1. The KKT conditions for the analysis estimator write as

\[ \frac{Y - \hat{f}}{n} = \lambda D' \partial \| \hat{f} \|_1, \]

thanks to the chain rule of the subdifferential.

We have that, for \( \hat{f} \in \mathbb{R}^n \),

\[ \hat{f}'(Y - \hat{f}) = \lambda \| \hat{f} \|_1 \]

and that, for a generic \( f \in \mathbb{R}^n \),

\[ f'(Y - \hat{f}) = \lambda (Df)' \partial \| \hat{f} \|_1 \leq \lambda \| Df \|_1, \]

where the last inequality follows by the dual norm inequality and by the fact that \( \| \partial \|_{\hat{f}} \|_{\infty} \leq 1 \).

By subtracting the first of the two above expressions from the second, we find that

\[ (f - \hat{f})'(f^0 - \hat{f}) \leq \frac{\epsilon'}{n} (\hat{f} - f) + \lambda (\| f \|_1 - \| \hat{f} \|_1). \]

By polarization we obtain that

\[ \| \hat{f} - f^0 \|^2 + \| \hat{f} - f \|^2 \leq \| f - f^0 \|^2 + \frac{2\epsilon'(\hat{f} - f)}{n} + 2\lambda (\| f \|_1 - \| \hat{f} \|_1), \]

and thus have the basic inequality. \( \square \)

Proof of Lemma 2.2. We have that

\[ \epsilon' f = \epsilon' N^\perp_{\mathbb{R}^n} f + \epsilon' N_{\mathbb{R}^n} f. \]

1. We have that, since \( D_{-\mathbb{S}} \) is of full rank,

\[ \epsilon' N^\perp_{\mathbb{R}^n}(D_{-\mathbb{S}}^*) \hat{f} = \epsilon' (D'_{\mathbb{S}})(D_{\mathbb{S}}^*) D_{-\mathbb{S}} f + \epsilon' D^+_S D_{-\mathbb{S}} f. \]

On the set \( T \) we have that

\[ \frac{\epsilon' D^+_S D_{-\mathbb{S}} f}{n} \leq \frac{\lambda}{\gamma} \| \Omega_{-\mathbb{S}} D_{-\mathbb{S}} f \|_1 = \frac{\lambda}{\gamma} \| \Omega_{-\mathbb{S}} D_{-\mathbb{S}} f \|_1 \leq \frac{\lambda^2}{\gamma} \| D_{-\mathbb{S}} f \|_1. \]
2. We have that
\[
\frac{e'\Pi\chi(D_S) f}{n} \leq \|\Pi\chi(D_S)\epsilon\|_n\|f\|_n.
\]
On \(\mathcal{X}\) we have that
\[
\|\Pi\chi(D_S)\epsilon\|_n \leq \sqrt{\frac{\sigma^2}{n}} \left(\sqrt{r_S} + \sqrt{2n}\right).
\]
Thus Lemma ?? follows.

Proof of Lemma 2.3. Assumption 2.1 expresses a particular choice of the constant \(c\) in Proposition B.1 below. For \(\eta \in (0, 1)\) we have that \(\eta/2 \leq \eta/(1 + \eta)\) and thus the choice of \(c\) in Assumption 2.1 satisfies the upper bound given by Proposition B.1 (see below), which then holds, since all of its assumptions are satisfied and we consider the sets \(\mathcal{A} \cap \mathcal{R}\).

The choice of \(c\) implies that \(q = \eta/2\) and that \(c \leq \eta/2\). Thus the claim follows.

Propostion B.1 (The square root analysis estimator does not overfit)
Assume for some \(a > 0\) that \(n > 8a\) and that for some \(R > 0, \eta \in (0, 1)\)
\[
\lambda_0 \geq \frac{1}{1 - \eta} R
\]
and
\[
\|Df^0\|_1 \leq c\sigma\sqrt{1 - \sqrt{8a/n}/\lambda_0},
\]
where
\[
c < \sqrt{\left(\frac{\eta}{1 + \eta} - \frac{\sqrt{r_S} + \sqrt{2a}}{\sqrt{n - \sqrt{8an}}}\right)^2 + 4 - 2},
\]
where we assume that \(S \subseteq [m]\) is s.t.
\[
\frac{\eta}{1 + \eta} > \frac{\sqrt{r_S} + \sqrt{2a}}{\sqrt{n - \sqrt{8an}}}.\]

Let
\[
q := 2\sqrt{r_S} + \sqrt{2a} \quad \sqrt{n - \sqrt{8an}} + (c + 2)^2 - 4.
\]

Then on \(\mathcal{A} \cup \mathcal{R}\) we have that
\[
(1 + c)\|\epsilon\|_n \geq \|\tilde{\epsilon}\|_n \geq (1 - \eta q/(\eta - q))\|\epsilon\|_n.
\]

Proof of Proposition B.1, based on the proof of Lemma 3.1 by van de Geer (2016).

On the set \(\mathcal{A}\) we have that
\[
\|\tilde{\epsilon}\|_2^2/\sigma^2 \geq n - 2\sqrt{a}(\sqrt{r_S} + \sqrt{n - r_S}) \geq n - \sqrt{8an},
\]
where the last inequality follows from the Cauchy-Schwarz inequality. Thus,

\[ \| \epsilon \|_n \geq \sigma \sqrt{1 - \sqrt{8a/n}} \]

and by assumption we have that

\[ \| D f^0 \|_1 \leq c \| \epsilon \|_n / \lambda_0. \]

We show an upper and a lower bound for \( \| \hat{\epsilon} \|_n \). The proof of the upper bound does not involve the increments of the empirical process, while the proof of the lower bound does so and is therefore more delicate.

**Upper bound:**

Since the estimator \( \hat{f}_\wedge \) minimizes the objective function we have that

\[ \| Y - \hat{f}_\wedge \|_n + \lambda_0 \| D \hat{f}_\wedge \|_1 \leq \| Y - f^0 \|_n + \lambda_0 \| D f^0 \|_1. \]

It follows that

\[ \| \hat{\epsilon} \|_n \leq \| \epsilon \|_n + \lambda_0 \| D f^0 \|_1 \leq (1 + c) \| \epsilon \|_n. \]

**Lower bound:**

Note that, by the triangle inequality, we have that

\[ \| \hat{\epsilon} \|_n = \| \epsilon - (\hat{f} - f^0) \|_n \geq \| \epsilon \|_n - \| \hat{f} - f^0 \|_n. \]

Thus the lemma follows if we can prove a bound of the type

\[ \| \hat{f}_\wedge - f^0 \|_n \leq \text{const.} \| \epsilon \|_n, \]

with leading constant in \((0, 1)\).

To prove such a bound we are not allowed to use the KKT conditions. Instead we use the convexity of the loss function and of the penalty.

Define for \( t \in (0, 1) \) the convex combination

\[ \hat{f}_t := t \hat{f}_\wedge + (1 - t) f^0 \]

and its residuals

\[ \hat{\epsilon}_t := Y - \hat{f}_t = \epsilon - (\hat{f}_t - f^0) = t \hat{\epsilon} + (1 - t) \epsilon. \]

Choose

\[ t = \frac{\eta \| \epsilon \|_n}{\eta \| \epsilon \|_n + \| \hat{f}_\wedge - f^0 \|_n}. \]
Then
\[ \| \hat{f}_t - f^0 \|_n = t \| f - f^0 \|_n = \frac{\eta \| \epsilon \|_n \| \hat{f}_t - f^0 \|_n}{\eta \| \epsilon \|_n + \| f - f^0 \|_n} \leq \eta \| \epsilon \|_n. \]

We thus get that
\[ \| \hat{\epsilon}_i \|_n \geq \| \epsilon \|_n - \| \hat{f}_t - f^0 \|_n \geq (1 - \eta) \| \epsilon \|_n. \]

By the convexity of the loss and of the penalty and by the fact that \( \hat{f} \) is a minimizer of the objective function it follows that
\[ \| \hat{\epsilon}_i \|_n + \lambda_0 \| D \hat{f}_t \|_1 \leq t(\| \hat{\epsilon}_i \|_n + \lambda_0 \| D \hat{f}_t \|_1 + (1 - t)(\| \epsilon \|_n + \lambda_0 \| D f^0 \|_1)) \leq \| \epsilon \|_n + \lambda_0 \| D f^0 \|_1. \]

By squaring the inequality we get that
\[ \| \hat{\epsilon}_i \|_n^2 + 2 \lambda_0 \| \hat{\epsilon}_i \|_n \| D \hat{f}_t \|_1 \leq \| \epsilon \|_n^2 + 2 \lambda_0 \| \epsilon \|_n \| D f^0 \|_1 + \lambda_0^2 \| D f^0 \|_1^2. \]

We have that
\[ \| \hat{\epsilon}_i \|_n^2 = \| \epsilon - (\hat{f}_t - f^0) \|_n^2 = \| \epsilon \|_n^2 - \frac{2 \epsilon' (\hat{f}_t - f^0)}{n} + \| \hat{f}_t - f^0 \|_n^2. \]

By combining the squared inequality with the lower bound for \( \| \hat{\epsilon}_i \|_n \) and the expression for \( \| \hat{\epsilon}_i \|_n^2 \) we get that
\[ \| \hat{f}_t - f^0 \|_n^2 \leq 2 \lambda_0 \| \epsilon \|_n \| D f^0 \|_1 + 2 \lambda_0 (1 - \eta) \| \epsilon \|_n \| D \hat{f}_t \|_1 + \lambda_0^2 \| D f^0 \|_1^2 + \frac{2 \epsilon' (\hat{f}_t - f^0)}{n}. \]

On \( R \), for an \( S \) satisfying the assumptions of the lemma, we have that
\[ \frac{\epsilon' (\hat{f}_t - f^0)}{n} \leq \frac{\rho}{\gamma} R \| \epsilon \|_n (\| D_{-S} f \|_1 + \| D_{-S} f^0 \|_1) + \| \Pi_{N(D_{-S})} \epsilon \|_n \| \hat{f}_t - f^0 \|_n \leq R \| \epsilon \|_n (\| D f \|_1 + \| D f^0 \|_1) + \| \Pi_{N(D_{-S})} \epsilon \|_n \| \hat{f}_t - f^0 \|_n. \]
Thus
\[
\| \hat{f}_t - f^0 \|_n^2 = 2\| \Pi_{N(D_0)} \epsilon \|_n \| \hat{f}_t - f^0 \|_n \\
\leq 2(\lambda_0 + R)\| \epsilon \|_n \| DF^0 \|_1 + 2(\lambda_1 - \eta) \| \epsilon \|_n \| DF^0 \|_1 \\
+ \lambda_0 \| DF^0 \|_1^2 \\
\leq 4\lambda_0 \| \epsilon \|_n \| DF^0 \|_1 + \lambda_0^2 \| DF^0 \|_1^2 \\
= \| \epsilon \|_n^2 (\lambda_0 \| DF^0 \|_1/\| \epsilon \|_n + 2)^2 - 4) \\
\leq \| \epsilon \|_n^2 (c + 2)^2 - 4)
\]

Moreover we have that
\[
\| \hat{f}_t - f^0 \|_n^2 = 2\| \Pi_{N(D_0)} \epsilon \|_n \| \hat{f}_t - f^0 \|_n \\
= \left( \| \hat{f}_t - f^0 \|_n - \| \Pi_{N(D_0)} \epsilon \|_n \right)^2 - \| \Pi_{N(D_0)} \epsilon \|_n^2.
\]

Thus we obtain that
\[
\left( \| \hat{f}_t - f^0 \|_n - \| \Pi_{N(D_0)} \epsilon \|_n \right)^2 \leq \| \Pi_{N(D_0)} \epsilon \|_n^2 + c' \| \epsilon \|_n^2
\]

and
\[
\| \hat{f}_t - f^0 \|_n \leq \| \Pi_{N(D_0)} \epsilon \|_n + \sqrt{\| \Pi_{N(D_0)} \epsilon \|_n^2 + c' \| \epsilon \|_n^2} \\
\leq 2\| \Pi_{N(D_0)} \epsilon \|_n + \sqrt{c'} \| \epsilon \|_n.
\]

Note that
\[
\| \epsilon \|_n^2 = \| \Pi_{N(D_0)} \epsilon \|_n^2 + \| A_{N(D_0)} \epsilon \|_n^2.
\]

By using the spectral decomposition, \( \Pi_{N(D_0)} \epsilon \in \mathbb{R}^{n \times n} \) can be written as \( PP' \), where \( P \in \mathbb{R}^{n \times s} \) is s.t. \( P'P = I_s \). Moreover \( A_{N(D_0)} \epsilon \in \mathbb{R}^{n \times n} \) can be written as \( QQ' \), where \( Q \in \mathbb{R}^{n \times (n-s)} \) is s.t. \( Q'Q = I_{n-s} \) and \( Q'P = 0 \).

Let \( u := P' \epsilon \in \mathbb{R}^{s} \) and \( v := Q' \epsilon \in \mathbb{R}^{n-s} \). We have that \( u \sim N_{s}(0, \sigma^2 I_{s}) \), \( v \sim N_{n-s}(0, \sigma^2 I_{n-s}) \) and \( u \) and \( v \) are independent. We have that \( \| \Pi_{N(D_0)} \epsilon \|_n^2 = \| u \|_2^2 \) and that \( \| A_{N(D_0)} \epsilon \|_n^2 = \| v \|_2^2 \). It follows that
\[
\| \epsilon \|^2 / \sigma^2 = \| u \|_2^2 / \sigma^2 + \| v \|_2^2 / \sigma^2 \\
\sim \chi^2_{r_\sigma} + \chi^2_{n-r_\sigma}
\]

and thus the two terms are independent and can be handled separately.

On \( \mathcal{A} \) we have that
\[
\frac{\| \Pi_{N(D_0)} \epsilon \|_n^2}{\| \epsilon \|_n^2} = \frac{\| u \|_2^2}{\| u \|_2^2 + \| v \|_2^2} \leq \frac{(\sqrt{r_\sigma} + \sqrt{2n})^2}{n - \sqrt{8an}}.
\]
Therefore
\[ \| \Pi_{\mathcal{N}(D - \bar{S})} \epsilon \|_n \leq \frac{\sqrt{rS} + \sqrt{2a}}{\sqrt{n - \sqrt{8an}}} \| \epsilon \|_n =: p \| \epsilon \|_n \asymp \sqrt{\frac{rS}{n}} \| \epsilon \|_n. \]

It follows that
\[ \| \hat{f}_t - f^0 \|_n \leq (2p + \sqrt{c'})\| \epsilon \|_n =: q \| \epsilon \|_n. \]

By expressing \( \| \hat{f}_t - f^0 \|_n \) more explicitly we get that
\[ \frac{\eta \| \hat{f}_t - f^0 \|_n}{\eta \| \epsilon \|_n + \| f - f^0 \|_n} \leq q \]

and thus
\[ \| \hat{f}_t - f^0 \|_n \leq q\eta / (\eta - q) \| \epsilon \|_n. \]

We conclude that
\[ \| \hat{\epsilon} \|_n \geq \left( 1 - \frac{\eta q}{\eta - q} \right) \| \epsilon \|_n. \]

The last step is to find out how to choose \( c \) s.t. \( q \eta / (\eta - q) < 1 \). We get that
\[ q < \eta / (1 + \eta), \]

hence
\[ c' < \left( \frac{\eta}{1 + \eta} - p \right)^2 \]

and thus
\[ c < \sqrt{\left( \frac{\eta}{1 + \eta} - p \right)^2} + 4 - 2. \]

Note that we also get the assumption \( p < \eta / (1 + \eta) \), which results in the assumption
\[ \frac{\eta}{1 + \eta} > \frac{\sqrt{rS} + \sqrt{2a}}{\sqrt{n - \sqrt{8an}}}. \]

**Proof of Corollary 2.1.** Under Assumption 2.1 on \( A \cap R \) Corollary ?? holds and thus \( \| \hat{\epsilon} \|_n \geq (1 - \eta)\| \epsilon \|_n \). It follows that
\[ \frac{1}{1 - \eta} \geq \| \epsilon \|_n / \| \hat{\epsilon} \|_n. \]

By inserting this inequality into the assumption
\[ \lambda_0 \geq \frac{1}{1 - \eta} R \]

we get the claim.
Proof of Lemma 2.4. Under Assumption 2.1 we have that on $A \cap R$ the KKT conditions hold
\[ \frac{Y - \hat{f}\sqrt{n}}{n} = \lambda_0 \| \hat{\epsilon} \|_n D' \vartheta \| D\hat{f} \sqrt{n} \|_1. \]
We then obtain the basic inequality as in Lemma 2.1 (cf. also Lemma 2 in Stucky and van de Geer (2017)).

Proof of Lemma 2.5. On $R$, by using the decomposition in antiprojection and projection onto the nullspace of $D_{-S}$ and by applying the dual norm inequality to the second term we have that
\[ \epsilon^T \hat{f} \leq \epsilon^T D^+_S D_{-S} \hat{f} + \| \Pi_{N(D_{-S})} \|_n \| \hat{f} \|_n \]
\[ \leq R \| \epsilon \|_n \| \Omega_{-S} D_{-S} f \|_1 / \gamma + \| \Pi_{N(D_{-S})} \|_n \| f \|_n. \]
Moreover, on $A \cap R$, under Assumption 2.1, by Corollary 2.1 we have that $R \| \epsilon \|_n \leq \lambda_0 \| \hat{\epsilon} \|_n$ and thus the claim follows.

Appendix C: Proofs of Section 3

Proof of Theorem 3.1. By Lemma 2.1 we have the basic inequality
\[ \| \hat{f} - f^0 \|_n^2 + \| \hat{f} - f \|_n^2 \leq \| f - f^0 \|_n^2 + \frac{2\epsilon^T (\hat{f} - f)}{n} \]
\[ + 2 \lambda \| Df \|_1 - \| D\hat{f} \|_1. \]
We now see that, by the triangle inequality, we have
\[ \| Df \|_1 - \| D\hat{f} \|_1 = \| D_S f \|_1 - \| D_S \hat{f} \|_1 - (\| D_{-S} f \|_1 + \| D_{-S} \hat{f} \|_1)
+ 2 \| D_{-S} f \|_1 \]
\[ \leq \| D_S (\hat{f} - f) \|_1 - \| D_{-S} (\hat{f} - f) \|_1 + 2 \| D_{-S} f \|_1. \]
We now handle the random part, which is constituted by an increment of the empirical process, by using Lemma 2.2. By Lemma 2.2 we have that on $T \cap X$ it holds that
\[ \frac{\epsilon^T (\hat{f} - f)}{n} \leq \frac{\lambda}{\gamma} \| \Omega_{-S} D_{-S} (\hat{f} - f) \|_1 + \left( \frac{2x}{n} + \sqrt{\frac{T_S^4}{n}} \right) \| \hat{f} - f \|_n. \]
Putting the pieces together, we have that,
\[ \| \hat{f} - f \|^2_n + \| \hat{f} \|^2_n \leq \| f - f \|^2_n + 4\lambda \| D_s f \|_1 + 2 \| \hat{f} - f \|_n \left( \sqrt{\frac{2x}{n}} + \sqrt{\frac{r_s}{n}} \right) \]

If \( \kappa(S, W) > 0 \) we have that

\[ \| W_s D_s (\hat{f} - f) \|_1 - \| W_s D_s (\hat{f} - f) \|_1 \leq \sqrt{r_s} \| \hat{f} - f \|_n \]

and thus

\[ \| \hat{f} - f \|^2_n + \| \hat{f} - f \|^2_n \leq \| f - f \|^2_n + 2\lambda \| D_s (\hat{f} - f) \|_n \left( \sqrt{\frac{2x}{n}} + \sqrt{\frac{r_s}{n}} \right) \]

where the last inequality follows by \( 2ab \leq a^2 + b^2, a, b \in \mathbb{R} \).

The term \( \| f - f \|^2_n \) cancels out and we get the statement of the theorem.

Proof of Theorem 3.2. By Lemma 2.1. We have the basic inequality

\[ \| \hat{f} - f \|^2_n + \| \hat{f} - f \|^2_n \leq \| f - f \|^2_n + 2\epsilon (\hat{f} - f) \]

By Lemma 2.2 on \( \mathcal{X} \cap \mathcal{T} \) we have that

\[ \frac{\epsilon (\hat{f} - f)}{n} \leq \lambda \frac{\rho}{\gamma} \| D_s (\hat{f} - f) \|_1 + \sigma \left( \sqrt{\frac{2x}{n}} + \sqrt{\frac{r_s}{n}} \right) \| \hat{f} - f \|_n \]

\[ \leq \lambda \frac{\rho}{\gamma} \| D_s (\hat{f} - f) \|_1 + \frac{1}{2} \sigma^2 \left( \sqrt{\frac{2x}{n}} + \sqrt{\frac{r_s}{n}} \right)^2 + \frac{1}{2} \| \hat{f} - f \|^2_n \]
We thus get that
\[
\|\hat{f} - f^0\|^2_n \leq \|f - f^0\|^2_n + \sigma^2_n \left(\sqrt{2x} + \sqrt{r_S}\right)^2 + 2\lambda \left(\left(1 + \frac{\rho}{\gamma}\right)\|D_S f\|_1 - \left(1 - \frac{\rho}{\gamma}\right)\|D_S \hat{f}\|_1\right) + 2\lambda \left(\|D_S f\|_1 - \|D_S \hat{f}\|_1\right).
\]

The statement of the theorem follows. \qed

**Appendix D: Proofs of Section 4**

**Proof of Theorem 4.1.** We work under Assumption 2.1 on $A' \cap R$. By combining Lemma 2.4 and Lemma 2.5, we get that, in complete analogy to the proof of Theorem 3.1,
\[
\|\hat{f}^\sqrt{\cdot} - f^0\|^2_n \leq \|f - f^0\|^2_n + 4\lambda_0 \|\hat{\epsilon}\|_n \|D_S f\|_1 + \left(\sigma \sqrt{\frac{2a}{n}} + \sigma \sqrt{\frac{r_S}{n}} + \frac{\lambda_0 \|\hat{\epsilon}\|_n \sqrt{r_S}}{\kappa(S, W)}\right)^2.
\]

Moreover, by Corollary 2.1, we have that on $A'$
\[
\|\hat{\epsilon}\|_n \leq (1 + \eta)\|\epsilon\|_n \leq (1 + \eta)(1 + \sqrt{4a/n})\sigma.
\]
Thus we get that
\[
\|\hat{f}^\sqrt{\cdot} - f^0\|^2_n \leq \|f - f^0\|^2_n + 4(1 + \eta)(1 + \sqrt{4a/n})\sigma \lambda_0 \|D_S f\|_1 + \sigma^2 \left(\sqrt{\frac{2a}{n}} + \sqrt{\frac{r_S}{n}} + \frac{(1 + \eta)(1 + \sqrt{4a/n})\lambda_0 \sqrt{r_S}}{\kappa(S, W)}\right)^2.
\]
Since Assumption 2.1 implies that $\eta < 1$ and $n > 8a$ we get that $(1 + \eta)(1 + \sqrt{4a/n}) \leq 4$ and thus
\[
\|\hat{f}^\sqrt{\cdot} - f^0\|^2_n \leq \|f - f^0\|^2_n + 16\lambda_0 \|D_S f\|_1 + \sigma^2 \left(\sqrt{\frac{2a}{n}} + \sqrt{\frac{r_S}{n}} + \frac{4\lambda_0 \sqrt{r_S}}{\kappa(S, W)}\right)^2.
\]

**Proof of Theorem 4.2.** We work under Assumption 2.1 on $A' \cap R$. By Lemma
and Lemma 2.5 we get that, in analogy with the proof of Theorem 3.2
\[
\|\hat{f}_r - f^0\|^2_n + 2\lambda_0\|\hat{\epsilon}\|_n \left(\left(1 - \frac{\rho}{\gamma}\right)\|D_{-S}\hat{f}_r\|_1 + \|D_S\hat{f}_r\|_1\right)
\]
\[
\leq \|f - f^0\|^2_n + \frac{\sigma^2}{n} \left(\sqrt{2\sigma} + \sqrt{\sigma_S}\right)^2
\]
\[
+ 2\lambda_0\|\hat{\epsilon}\|_n \left(\left(1 + \frac{\rho}{\gamma}\right)\|D_{-S}f\|_1 + \|D_Sf\|_1\right).
\]

By Corollary 2.1 we have that
\[
2\|\epsilon\|_n \geq (1 + \eta)\|\epsilon\|_n \geq \|\hat{\epsilon}\|_n \geq (1 - \eta)\|\epsilon\|_n.
\]

Moreover on $A'$
\[
2\sigma \geq \sigma(1 + \sqrt{4\sigma/n}) \geq \|\epsilon\|_n \geq \sigma \sqrt{1 - \sqrt{8a/n}}.
\]

We thus get that
\[
\|\hat{f}_r - f^0\|^2_n + 2(1 - \eta)\sqrt{1 - \sqrt{8a/n}} \sigma\lambda_0 \left(\left(1 - \frac{\rho}{\gamma}\right)\|D_{-S}\hat{f}_r\|_1 + \|D_S\hat{f}_r\|_1\right)
\]
\[
\leq \|f - f^0\|^2_n + \frac{\sigma^2}{n} \left(\sqrt{2\sigma} + \sqrt{\sigma_S}\right)^2
\]
\[
+ 8\sigma\lambda_0 \left(\left(1 + \frac{\rho}{\gamma}\right)\|D_{-S}f\|_1 + \|D_Sf\|_1\right).
\]

\[\square\]

Appendix E: Proofs of Section 5

**Proof of Lemma 5.1.** Let $D \in \mathbb{R}^{(n-1) \times n}$ be the incidence matrix of a directed tree graph rooted at vertex 1. Let $D^+ \in \mathbb{R}^{n \times (n-1)}$ be its Moore-Penrose pseudoinverse. By Lemma 2.2 in Ortelli and van de Geer (2019) we have that $D^+$ can be obtained as
\[
D^+ = (I_n - \mathbb{I}_n/n)X_{-1},
\]
where, if we assume that $\vec{G}$ is rooted at vertex 1, $X = \left((1, 0, \ldots, 0)\right)^{-1}$. As pointed out and used in Ortelli and van de Geer (2018), $X$ has the meaning of the rooted path matrix of $\vec{G}$. Thus, the columns of $X_{-1}$ contain a minimum of 1 and a maximum of $(n - 1)$ entries having value 1, while the remaining entries are zeroes.

Let $i$ be the number of entries having value 1 of a column of $X_{-1}$. Let $v(i) \in \mathbb{R}^n$ denote any vector with $i$ entries having value 1 and $(n - i)$ entries having value 0. Define
\[
g(i, n) = \|(I_n - \mathbb{I}_n/n)v(i)\|^2_2.
\]
We have that
\[ g(i, n) = i(1 - i/n)^2 + (n - i)(i/n)^2 = \frac{i(n - i)}{n}, \quad i \in [n - 1]. \]

We have that the maximum of \( g(i, n) \) for a given \( n \) is reached at \( i = n/2 \) if \( n \) is even and at \( i \in \{ \lfloor n/2 \rfloor, \lceil n/2 \rceil \} \) if \( n \) is odd.

Let \( n \) be odd, then
\[ g \left( \frac{n - 1}{2}, n \right) = g \left( \frac{n + 1}{2}, n \right) = \frac{(n - 1)(n + 1)}{4n}. \]

If \( n \) is odd, then \( n - 1 \) and \( n + 1 \) are even and
\[ g \left( \frac{n - 1}{2}, n - 1 \right) = \frac{n - 1}{4} \]
and
\[ g \left( \frac{n + 1}{2}, n + 1 \right) = \frac{n + 1}{4}. \]

Since \( (n - 1)/n < 1 \) and \( (n + 1)/n > 1 \) we have that
\[ g \left( \frac{n - 1}{2}, n - 1 \right) < g \left( \frac{n - 1}{2}, n \right) < g \left( \frac{n + 1}{2}, n + 1 \right) \]
and thus \( \max_{i \in [n - 1]} g(i, n) \) is increasing in \( n \). We therefore have that
\[ g(i, n) \leq \frac{n + 1}{4}, \quad \forall i \in [n - 1], \forall n. \]

Since \( \max_{i \in [n - 1]} g(i, n) \) is increasing in \( n \), then the \( \ell^2 \)-norm of a column of \( D_{S_i}^+ \) will never be greater than the greatest possible \( \ell^2 \)-norm of a column of \( D_{C_i}^+ \).

We thus have that
\[ \rho = \max_{j \in [n - 1]} \| d_j^+ \|_n \leq \max_{i \in [n_{\max} - 1]} \sqrt{\frac{g(i, n_{\max})}{n}} \leq \sqrt{\frac{n_{\max} + 1}{4n}}. \]

**Proof of Corollary 5.1.** By Lemma 5.1 we have that
\[ \rho \leq \sqrt{\frac{n_{\max} + 1}{4n}}, \]
therefore we choose
\[ \gamma = \sqrt{\frac{n_{\max}}{2n}} > \rho \]
and
\[ \lambda^2 = \sigma^2 n_{\max} \frac{\log(2n) + t}{n^2}. \]
By Lemma 5.2 we have that
\[
\sqrt{\frac{r_S}{\kappa(S, W)}} \leq \sqrt{nK} + \sqrt{n \sum_{i=2}^{n} (w_i^* - w_{i-1}^*)^2}.
\]

By Lemma 5.3 we have that
\[
\sum_{i=2}^{n} (w_i^* - w_{i-1}^*)^2 \leq \frac{5}{\gamma^2} \frac{r_S}{n} \log \left( \frac{n}{r_S} \right).
\]

By combining the above results with Theorem 3.1 we get Corollary 5.1.

Proof of Lemma 5.4. Note that every interval \( [x_{l(x)}, x_{r(x)}] \) contains \( n^{2/3} \) design points, i.e.
\[
\forall x \in [0, 1], |[x_{l(x)}, x_{r(x)}] \cap \{ x_i \}_{i \in [n]} | = n^{2/3}.
\]

Note that \( x_i' = x_{n^{2/3} i} \). We have that
\[
\| f_0 - f \|_n^2 = \frac{1}{n} \sum_{i \in [n^{1/3}]} \sum_{j \in [n^{2/3}]} \| f_0(x_i') - f_0(x_{n^{2/3} i + j}) \|_2^2
\leq \frac{L}{n} \sum_{i \in [n^{1/3}]} \sum_{j \in [n^{2/3}]} \| x_{n^{2/3} i} - x_{n^{2/3} i + j} \|_2^2
= \frac{L}{n} \sum_{i \in [n^{1/3}]} \frac{1}{n^2} \sum_{j \in [n^{2/3}]} f^2
= \frac{L n^{1/3} n^{2/3} (n^{2/3} + 1)(2n^{2/3} + 1)}{6n^2} \leq 1
\leq Ln^{-2/3}.
\]

Proof of Corollary 5.5. If \( f_0 \) is Lipschitz, then by Lemma 5.4 we can find an \( f \) s.t. \( S := \{ i : d_i' f \neq 0 \} \) has equidistant elements and cardinality \( n^{1/3} \) and \( \| f_0 - f \|_n^2 \leq Ln^{-2/3} \). Note that for such an \( f \) we have that \( \| (Df)_S \|_1 = 0 \).

For such a choice of \( S \), the assumptions for Lemmas 5.2 and 5.3 to bound the weighted compatibility constant are satisfied as soon as \( n^{2/3} \geq 4 \), i.e. \( n \geq 8 \).

Since \( n^{1/3} \) is even, from Corollary 5.2 we get that
\[
\| \hat{f} - f_0 \|_n^2 \leq Ln^{-2/3} + \sigma \left( \sqrt{\frac{2x}{n}} + n^{-1/3} + 2n^{-1/3} \sqrt{\log(2n) + t} \right)
+ n^{-1/3} \sqrt{10(\log(2n) + t) \log(n^{2/3})^2},
\]
and Corollary 5.5 follows.
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