Network Marketing on a Small-World Network

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Abstract

We investigate a dynamic model of network marketing in a small-world network structure artificially constructed similarly to the Watts-Strogatz network model. Different from the traditional marketing, consumers can also play the role of the manufacturer’s selling agents in network marketing, which is stimulated by the referral fee the manufacturer offers. As the wiring probability $\alpha$ is increased from zero to unity, the network changes from the one-dimensional regular directed network to the star network where all but one player are connected to one consumer. The price $p$ of the product and the referral fee $r$ are used as free parameters to maximize the profit of the manufacturer. It is observed that at $\alpha = 0$ the maximized profit is constant independent of the network size $N$ while at $\alpha \neq 0$, it increases linearly with $N$. This is in parallel to the small-world transition. It is also revealed that while the optimal value of $p$ stays at an almost constant level in a broad range of $\alpha$, that of $r$ is sensitive to a change in the network structure. The consumer surplus is also studied and discussed.

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1 Introduction

The traditional form of the market in economics can be best described by the set of firms and consumers who sell and buy goods. In a market, a trade between a consumer and a firm occurs if her valuation $v$ exceeds the product price $p$ quoted by the firm. This description, however, hypothesizes that the product is known to consumers in the market, which is not always justified in real situations. For example, although a company produces a brand new product which can potentially attract millions of people, the company can sell very few if the information of this new product has not been propagated yet across the public. In a modernized society it costs a fortune only to make the public know that there is a new product, and companies are spending tremendous amounts of money in massive advertisement on, e.g., a half-minute TV commercial or a small portion of a nationwide newspaper. Sponsoring Olympic games or World Cup football championship costs much more.

From the above reasoning, it is natural that some companies seek other ways to make the product information available without spending the advertisement cost. One of the strategies along this direction is to motivate a buyer to recommend the product to her social surroundings. An obvious way for this is to pay the consumer if the referral induces the actual purchase of the product by her social acquaintances. This is how network marketing works.

In this paper, we introduce a way of constructing a small-world tree network \(^1\) analogous to the Watts-Strogatz network model [1,2] and use a game theoretic numerical approach to simulate network marketing in which the product is sold only through the connections in an existing social network. In parallel to the geometric small-world transition that the average distance from the manufacturer ceases to increase linearly with the network size, it is observed that manufacturer’s profit exhibits striking difference between the one-dimensional regular chain network and the small-world tree network. We also investigate the consumer surplus in network marketing, in comparison with conventional marketing.

The paper is organized as follows: In Sec. 2, the construction method of small-world tree networks is introduced. The game of consumer referrals is reviewed in Sec. 3, and extended to the case of general tree networks. Section 4 is devoted to the main result for manufacturer’s profit and also includes a discussion of the consumer surplus. Finally, a brief summary is presented in Sec. 5.

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\(^1\) In this paper, the term "small-world network" means a network with a very short characteristic path length, irrespective of the clustering property. In that sense, the tree network constructed here is a small-world network with a vanishing clustering coefficient.
Fig. 1. Construction of the network of consumer referrals. Starting from the one-dimensional chain in (a), each edge is visited and then rewired with probability $\alpha$. The edge $(j, j + 1)$ with $j = 2, 3, \text{and } 4$ in (a), (b), and (c), respectively, is rewired to a randomly selected vertex among the vertices whose ranks are less than the rank of $j$. Open square is the manufacturer, and circles represent consumers. At each step, the filled circle represents the vertex whose downward edge is under consideration of rewiring. The crossed edge denotes the situation when the rewiring is decided with probability $\alpha$. The index $i$ of each vertex and its rank $R$, which is the distance from the manufacturer, are shown for convenience. See the text for more details.

2 The Network

Social connections of people in societies have been studied in the framework of complex network [1]. In most social networks, it is now well-known that they share characteristics in common such as the small-world behavior, a high level of clustering, and so on. We here construct a directed small-world network in the same spirit as the Watts-Strogatz model [2] and then play the game of consumer referrals [3] on top of the network. To make the situation simple, we assign each consumer a unique value of the rank which is the distance (number of edges) from the company, and assume that each consumer has only one precedent consumer except the first consumer. A consumer at rank $R$ can refer all consumers at rank $R + 1$ directly connected to her. However, the reverse referral is forbidden (the product information flows only in the downward direction from the company) and accordingly, each consumer gets the referral only from one unique precedent consumer. In the graph theory, the above structure is better captured by the directed tree graph. ²

Figure 1 shows schematically the construction method of the network used in this work. We first build a one-dimensional (1D) directed chain in Fig. 1(a), where we have five consumers (circles) labelled as $i = 1, 2, \cdots, 5$ and the

² We restrict ourselves to tree graphs to avoid the situation when a consumer gets more than one referrals. See Sec. 3.
company (square). The rank $R$ of the consumer $i$ in the initial 1D chain is thus given by $R_i = i$. Starting from the consumer $j = 2$ [marked by a filled circle in Fig. 1(a)], we decide whether or not to rewire the edge $(j, j+1)$ with probability $\alpha$. If rewiring is decided [marked by the crossed edge in Fig. 1(a)], all consumers and their connections below $j = 2$ are attached to a randomly selected vertex in the set $\{i | R_i < R_j\}$. For example, in Fig. 1(a), $R_{j=2} = 2$ and there is only one vertex ($i = 1$) with $R_i < 2$, resulting in the network structure in Fig. 1(b). The process goes on to $j = 3$ in Fig. 1(b), which shows the situation as an example that the edge is not selected to be rewired. In Fig. 1(c) for $j = 4$, the edge $(j, j+1)$ can be rewired to either $i = 1$ or $i = 2$. In the case that there are more than one vertex to which the edge can be rewired, one is randomly picked: Shown in Fig. 1(d) is the situation that $i = 2$ is selected. The whole process continues subsequently from $j = 2$ to $j = N-1$.

When the rewiring probability $\alpha = 0$, the network reduces simply to a one-dimensional regular chain network, while in the opposite case of $\alpha = 1$, the network becomes a star-like network in which vertices $i = 2, 3, \ldots, N$ are all connected to $i = 1$. Similarly to the original Watts-Strogatz network [2], one can change the network structure by varying the rewiring probability $\alpha$.

In most studies of complex networks, the characteristic path length defined as the average geodesic length connecting each pair of vertices has been widely used to characterize the structural property of the network. In particular, many networks have been shown to exhibit the so-called small-world behavior that the characteristic path length increases with the network size $N$ very slowly (often in the logarithmic way). In the present study we define the average rank $\langle R \rangle$ according to

$$\langle R \rangle \equiv \frac{1}{N} \sum_i R_i,$$

which is simply the number of edges from the manufacturer, analogous to the characteristic path length in the network literature.

In Fig. 2, $\langle R \rangle$ is displayed as a function of $\alpha$ for various sizes $N = 100, 200, 400$, and $800$. As $\alpha$ is increased from zero, $\langle R \rangle$ reduces monotonically, becoming almost independent of $N$ for sufficiently large values of $\alpha$. The inset of Fig. 2 shows $\langle R \rangle$ as a function of $N$ for $\alpha = 0, 0.01, 0.02$, and $0.10$. It is clearly observed that the cases $\alpha = 0$ and $\alpha \neq 0$ exhibit quite different behavior: When $\alpha = 0$, $\langle R \rangle = (\sum_{i=1}^{N} i)/N = (N + 1)/2 \propto N$, while for $\alpha \neq 0$, $\langle R \rangle$ increases very slowly with $N$. Consequently, the network of consumer referrals constructed in the present work shows the small-world behavior at $\alpha \neq 0$, similarly to the original Watts-Strogatz network [2].
Fig. 2. The average rank $\langle R \rangle$ as a function of the rewiring probability $\alpha$ for networks of sizes $N = 100, 200, 400$, and $800$. $\langle R \rangle$ is observed to decrease with $\alpha$. decays. Inset: $\langle R \rangle$ versus $N$ for $\alpha = 0$, $0.01$, $0.02$ and $0.1$. As soon as $\alpha$ takes a nonzero finite value, $\langle R \rangle$ ceases to increase linearly with $N$, manifesting the small-world transition at $\alpha = 0$.

3 The Game of Consumer Referral

In this section, the game of consumer referrals is played on the network constructed in Sec. 2. The manufacturer produces the product at the marginal cost $c$ and sells it at the price $p$ to consumers. To make sense, $p$ should be larger than $c$ for the company tries to make a positive profit. We assume that a consumer’s valuation $v$ for the product constitutes quenched random variables following given distribution function $f(v)$. For simplicity, we use the uniform probability distribution function: $f(v) = 1$ for $v \in [0, 1]$ and $f(v) = 0$ otherwise.

The standard market works in a very simple way: The $i$th consumer has valuation $v_i$ and decides whether or not to buy according to the condition

$$v_i > p.$$  \hfill (2)

If the inequality is satisfied, the product is bought by the consumer since she thinks that the product is worth her spending. Accordingly, the probability of the purchase per consumer is simply $1 - p$ (for the uniform distribution of the valuation), yielding the total profit of the manufacturer $\Pi$

$$\Pi = N(1 - p)(p - c) - A,$$ \hfill (3)

where $A$ is the advertisement cost. The optimal price

$$p_{\text{max}} \equiv (1 + c)/2$$ \hfill (4)
yields the maximum profit

$$\Pi_{\text{max}} = \frac{N}{4} (1 - c)^2 - A.$$  \hfill (5)

In the case of network marketing, the situation becomes more complicated since each player can also make a profit if her referral induces actual purchase. In our model, we include both the referral cost $\delta$ which is the cost to make one referral, and the referral fee $r$ which is the initiative money the company pays in return for a successful referral. Each consumer pays $\delta$ irrespective of the success of her referral (e.g., the consumer has to make phone calls to persuade her social surroundings to buy the product) while $r$ is paid to her on the condition of a successful referral. As a simple example, consider the 1D regular network of two consumers 1 and 2. The second consumer has none to refer and thus her decision making is quite simple: She buys the product if $v_2 > p$. On the other hand, the first consumer buys if

$$v_1 > p - \max\{r(1 - p) - \delta, 0\},$$  \hfill (6)

where $1 - p$ is the probability that the second consumer buys the product. When this happens the first consumer earns the money $r$. Regardless of the success of her referral she should spend the cost $\delta$. If there is no worth making referrals, i.e., when $r(1 - p) - \delta$ is less than zero, she simply does not make any referral but she still buys if $v_1 > p$. 3 If there are $N$ consumers in the 1D regular chain network, the purchase condition (6) for consumer $i$ reads

$$v_i > \bar{v}_i$$  \hfill (7)

with the minimum valuation

$$\bar{v}_i = p - \max\{r(1 - \bar{v}_{i+1}) - \delta, 0\},$$  \hfill (8)

where $1 - \bar{v}_{i+1}$ is the probability that the $(i+1)$th consumer buys the product. Generalization of Eq. (8) to the network structure in Sec. 2 is straightforward:

$$\bar{v}_i = p - \sum_{j \in \Lambda_i} \max\{r(1 - \bar{v}_j) - \delta, 0\},$$  \hfill (9)

where $\Lambda_i$ is the set of neighboring consumers of $i$ in the downward direction from the manufacturer. For the consumers at the bottom ranks the minimum valuation is given by $\bar{v}_i = p$.

3 In economics, it is usually assumed that the participants in a market have unbounded rationality. In the present consumer referral model, we also assume that each player is smart, having full knowledge of the network structure and understanding game dynamics completely. From the physicist point of view, it is of interest to generalize the model towards the case of bounded rationality [4].
The consumer referral game in the present study goes as follows: (i) The network of size $N$ with the rewiring probability $\alpha$ is built. (ii) The valuation $v_i$ of each consumer is assigned from the uniform distribution in $[0,1]$. (iii) Starting from the consumers at the end of the network whose ranks take the largest value, we compute the minimum valuations as described above [$\bar{v}_i = p$ for bottom consumers and Eq. (9) for others]. (iv) After the minimum valuation is computed for every consumer, we consider consumer $i$ at rank $R_i$, starting from $i = 1$, and check the condition $v_i > \bar{v}_i$. If the inequality is fulfilled, the product is bought by $i$, and we proceed to the consumers at rank $R_i + 1$. If the inequality is not satisfied and consumer $i$ does not buy the product, the chain of buyer’s referral on the branch terminates. (v) The game ends when all chains of buy-and-refer stop. When the game ends, we compute the profit of the manufacturer

$$\Pi = N_{\text{buyer}}(p - c) - (N_{\text{buyer}} - 1)r,$$

where $N_{\text{buyer}}$ is the number of consumers who bought the product. We fix $c$ and $\delta$ as constants and obtain the profit $\Pi$ as a function of $p$ and $r$. From the point of view of the manufacturer, we aim to optimize the profit with respect to both $p$ and $r$.

In our game of consumer referrals, we assume that only the consumers who actually bought the product make referrals. We believe that this assumption is reasonable in view of that few people are willing to buy the product if the referrer herself has not bought it.

4 Results

4.1 Manufacturer’s Profit

Numerical simulations are performed at fixed values of the marginal production cost $c = 0.05$ and the referral cost $\delta = 0.01$ (the use of other values, if not too different, is not expected to change the main results). In the two-dimensional parameter space $(p, r) \in [0, 1] \times [0, 1]$, we first divide each parameter range into ten intervals of the width $\Delta = 0.1$ and then compute the profit of manufacturer $\Pi(p, r)$ at the centers of 100 square boxes in the 2D parameter space. For the box where $\Pi$ has the largest value, we repeat the above procedure four times with subsequently narrower intervals ($\Delta = 0.01, 0.001$, and 0.00001). When $\Pi(p, r)$ is computed for given values of $p$ and $r$, we make an average over 10,000 different realizations of the network structure and valuation.

Figure 3 displays the maximum profit of the manufacturer per consumer.
Fig. 3. The maximum profit of the manufacturer per consumer $\Pi_{\text{max}}/N$ versus the rewiring probability $\alpha$ for networks of sizes $N = 100, 200, 400$, and $800$. Inset: $\Pi_{\text{max}}/N$ versus $N$ for $\alpha = 0.0, 0.02$, and $0.04$.

$(\Pi_{\text{max}}/N)$ as a function of the rewiring probability $\alpha$. As $\alpha$ is increased from zero, $\Pi_{\text{max}}$ is shown to be monotonically increasing. Since all curves for different network sizes overlap in a broad range of $\alpha$ except the region where $\alpha$ is close to either zero or unity, we henceforth focus on the system size $N = 800$. One can also conclude from Fig. 3 that the profit is proportional to the network size in the broad intermediate region of $\alpha$ as expected. In the inset of Fig. 3, $\Pi_{\text{max}}/N$ is exhibited as a function of $N$ at $\alpha = 0.0, 0.02$, and $0.04$. Clearly shown is that the behavior at $\alpha = 0$ is strikingly different from those at $\alpha \neq 0$. Specifically, for $\alpha = 0$ the maximum profit $\Pi_{\text{max}}$ is constant irrespective of $N$ while it grows linearly with $N$ for sufficiently large networks as soon as $\alpha$ takes a nonzero value. This observation is in parallel to the small-world transition discussed in Sec. 2 (see Fig. 2), implying that the abrupt change in the behavior of $\Pi_{\text{max}}$ at $\alpha = 0$ is closely related with the structural small-world transition, reflected in the behavior of $\langle R \rangle$.

The behavior around $\alpha = 1$ in Fig. 3 is also interesting: As $N$ grows larger, the increase of $\Pi_{\text{max}}$ with $\alpha$ approaching unity appears to be sharper, which indicates that in the thermodynamic limit ($N \to \infty$) $\Pi_{\text{max}}$ should have discontinuity at $\alpha = 1$. When $\alpha = 1$, the network reduces to a star network, where all but one consumer ($i = 2, 3, \cdots, N$) are connected to the first consumer ($i = 1$). It is then straightforward to compute $\Pi_{\text{max}}$ as follows: The minimum valuation for the $i$th ($i > 1$) consumer simply reads $\bar{v}_i = p$ because she has none to refer, while the first consumer has $\bar{v}_1 = p - (N - 1) \max\{r(1-p)-\delta, 0\}$. The case of $r(1-p)-\delta \leq 0$ is trivial: the first consumer buys the product with probability $1-p$ but she does not make any referral. In the opposite case of $r(1-p)-\delta > 0$, the minimum valuation for the first consumer becomes negative for sufficiently large values of $N$ and consequently she always buys the product, resulting in the total number of buyers $N_{\text{buyer}} = 1+(N-1)(1-p)$.
manufacturer’s profit per consumer is thus given by $\Pi/N = (1 - p)(p - c - r)$ in the limit of large $N$, and the optimal value of $p$ is obtained from $\partial\Pi/\partial p = 0$, leading to the relation $p_{\text{max}} = (1 + c + r_{\text{max}})/2$. Since the profit is a decreasing function of $r$, $r_{\text{max}}$ is attained when $r_{\text{max}}$ satisfies $r_{\text{max}}(1 - p_{\text{max}}) = \delta$ and one finally gets $r_{\text{max}} \approx 0.02$, $p_{\text{max}} \approx 0.535$, and $\Pi_{\text{max}}/N \approx 0.216$ for the values $c = 0.05$ and $\delta = 0.01$ used in this work. This approximate value of $\Pi_{\text{max}}/N$ is in reasonably good agreement with the value at $\alpha = 1$ in Fig. 3, apparently supporting the existence of the discontinuity of $\Pi_{\text{max}}/N$ at $\alpha = 1$ in the thermodynamic limit.

In Fig. 4, the optimal values (a) $p_{\text{max}}$ and (b) $r_{\text{max}}$ are shown as functions of the rewiring probability $\alpha$ for $N = 800$. The values in Figs. 3 and 4 at $\alpha = 1$ are reasonably well described by the above approximate calculations [accurate only in the leading-order terms of $O(1/N)$]. It is very interesting to note that the optimal value of the price $p_{\text{max}}$ does not change significantly with $\alpha$ in a broad range of $\alpha$, while the optimal value of the referral fee $r_{\text{max}}$ keeps decreasing with $\alpha$. Consequently, the above result suggests that in network marketing the manufacturer needs to decrease the referral fee as the network structure becomes more complex from the 1D chain network.

We next compare the above results obtained from network marketing with the results for the conventional market. The maximum profit for the conventional market in Eq. (5) is larger for the corresponding value for network marketing shown in Fig. 3 when the advertisement cost $A$ is not so great. However, in case that $A$ grows large, e.g., for realistic values of $N$, the manufacturer can make more profit by changing to network marketing. For example, if the company spends 50% of its profit as the advertisement cost, the company can make more profit in network marketing with $\alpha = 0.7$. 

Fig. 4. The optimal price $p_{\text{max}}$ and the optimal referral fee $r_{\text{max}}$ for the network of size $N = 800$. 
4.2 Consumer Surplus

The monotonically increasing behavior of the maximum profit in Fig. 3 suggests that the manufacturer prefers the situation in which all but one consumer buy without referrals, and only one consumer makes all referrals and earns a big profit. This star-like network structure is simply equivalent to the situation when the manufacturer makes a daughter firm which takes care of all issues related with the distribution and the sale of the product.

We study the consumer surplus as follows: For given network structure parameterized by $\alpha$, the manufacturer maximizes its profit as described in Sec. 4.1 by using the optimal values $p_{\text{max}}$ and $r_{\text{max}}$. At given values of $p_{\text{max}}$ and $r_{\text{max}}$, the surplus for the $i$th consumer reads

$$\pi_i = r_{\text{max}}n_i - \delta k_i + (v_i - p_{\text{max}}),$$

where $k_i$ is the number of consumers (or potential buyers) attached to $i$, $n_i$ is the number of actual buyers among $k_i$ consumers, and $v_i - p$ is the measure of $i$’s satisfaction in buying the product (the larger $v_i - p$, the happier). When the consumer $i$ does not buy the product her consumer surplus $\pi = 0$ since she gets no referral fee from the manufacturer and does not spend any referral cost.

Figure 5 displays the averaged consumer surplus $\langle \pi \rangle \equiv (1/N) \sum \pi_i$. Very interestingly, the consumer surplus increases first, then stays almost at the same level around 0.16, and finally decreases again as $\alpha = 1$ is approached. It is thus concluded that while manufacturer’s profit becomes the highest at $\alpha = 1$, this does not lead to the highest consumer surplus.
In the conventional market described by Eqs. (4) and (5), the surplus \( v - p \) is realized only when the consumer buys, i.e., when the inequality (2) is satisfied, and accordingly the surplus per consumer is given by

\[
\langle \pi \rangle = \int_0^1 \max(v - p, 0) f(v) dv = \int_p^1 (v - p) dv, \tag{12}
\]

where the uniform distribution \( f(v) = 1 \) for \( v \in [0, 1] \) has been used. Inserting Eq. (4) to Eq. (12), one obtains

\[
\langle \pi \rangle \approx 0.11, \tag{13}
\]

where the same value of \( c = 0.05 \) has been used as in network marketing. It is interesting to see that although conventional marketing outperforms network marketing from the viewpoint of the manufacturer’s profit (unless the advertisement cost is large), the consumer surplus is much larger for network marketing in a broad range of \( \alpha \).

5 Summary

We have proposed a game theoretic way to study network marketing where consumers can also play the role of selling agents motivated by the referral fee the company offers in return for successful referrals. A simple model to build directed tree networks has been introduced to investigate the effects of the network structure on the game of network marketing. As the rewiring probability \( \alpha \) is increased from zero to unity, the network structure changes from a one-dimensional regular chain to a star network. The manufacturer’s profit is then numerically maximized by using the two parameters in the game: the price of the product and the referral fee. Observed is that the manufacturer’s profit takes the maximum value at \( \alpha = 1 \) (the star network) as expected. We have also investigated the consumer surplus and found it higher at intermediate values of \( \alpha \). Although the manufacturer’s profit is higher in conventional marketing than in network marketing in general, the consumer surplus has been found to be opposite, i.e., higher in network marketing. In real situations, the higher consumer surplus in network marketing may motivate each consumer to broaden her social acquaintances, eventually increasing the total number of potential buyers. When this happens, the manufacturer can consider changing to network marketing.
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[4] More details of the present work including extensions to other more realistic network structures with bounded rationality will be published elsewhere.