Self-Inductance and the Mass of Current Carriers in a Circuit

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Abstract

In this article, the self-inductance of a circular circuit is treated from an untraditional, particle-based point of view. The electromagnetic fields of Faraday induction are calculated explicitly from the point-charge fields derived from the Darwin Lagrangian for particles confined to move in a circular orbit. For a one-particle circuit (or for $N$ non-interacting particles), the induced electromagnetic fields depend upon the mass and charge of the current carriers while energy is transferred to the kinetic energy of the particle (or particles). However, for an interacting multiparticle circuit, the mutual electromagnetic interactions between particles can dominate the behavior so that the mass and charge of the individual particles becomes irrelevant; the induced fields are then comparable to the inducing fields and energy goes into magnetic energy. In addition to providing a deeper understanding of self-inductance, the example suggests that the claims involving “hidden mechanical momentum” in connection with momentum balance for interacting multiparticle systems are unlikely to be accurate.
A. Introduction

When the self-inductance of a circuit is discussed in electromagnetism textbooks,[1][2] the masses and the charges of the current carriers in the circuit are never mentioned. Yet clearly, the charge carriers play a crucial role in the circuit’s electromagnetic behavior. Why is it that the self-inductance of a circuit can be discussed without any mention of the details of the charge carriers?

In this article, we take an untraditional viewpoint and treat self-inductance as arising from the electromagnetic fields of point charges as derived from the Darwin Lagrangian.[3] In the analysis, we illustrate the roles of particle mass and particle charge in a simple electromagnetic situation. The analysis makes clear just when and why the details of the charge carriers disappear from self-inductive behavior. The unusual perspective presented here seems of interest not only as an analysis providing a deeper understanding of electromagnetic theory, but also as a suggestive commentary on the controversial topic of ”hidden mechanical momentum.” Writings in the current literature of electromagnetism[4] (including some excellent text books,[1][2]) insist that the mechanical momentum of the carriers of charge is the basis for a ”hidden mechanical momentum” which must be involved in the momentum balance of electromagnetic systems. In contradiction to this widely-quoted view, we will suggest that our example illustrates that ”hidden mechanical momentum” is a concept which is unlikely to play any role in multiparticle electromagnetism.

B. Textbook Discussion of Self-Inductance

According to the standard textbooks of electromagnetism, if an external emf $emf_{ext}$ is present in a continuous circuit with a self-inductance $L$ and resistance $R$, the current $i$ in the circuit is given by the differential equation

\[ emf_{ext} = L \frac{di}{dt} + iR \] (1)

Here the self-inductance $L$ is a quantity which depends upon the geometry of the circuit. For example, the self-inductance per unit length $L/l$ of a long solenoid[5] of $n$ turns per unit length and cross-section area $A$ is given in gaussian units by

\[ L/l = 4\pi n^2 A/c^2 \] (2)
where \( c \) is the speed of light in vacuum. The energy balance for the circuit is found by multiplying Eq. (1) by the current \( i \)

\[
emf_{\text{ext}} \times i = \frac{d}{dt} \left( \frac{1}{2} Li^2 \right) + i^2 \mathcal{R}
\]  

(3)

corresponding to a power \( emf_{\text{ext}} \times i \) delivered by the external emf going into the time-rate-of-change of magnetic energy \((1/2)Li^2\) stored in the inductor and the power \( i^2 \mathcal{R} \) lost in the resistor. In the standard textbook discussion, the mass of the current carriers is never mentioned; it appears neither in the current calculation nor in the energy balance.

C. Simple Model for a Particle-Based Analysis

There is an alternative point of view regarding self-inductance which directs attention to the charges which carry the current rather than to the geometry of the electromagnetic circuit. The appearance of the speed of light \( c \) in the expression (2) for the self-inductance per unit length of a long solenoid reminds us that self-inductance involves a relativistic effect, and hence our model must be relativistic at least through order \( v^2/c^2 \). Accordingly, the analysis makes use of the Darwin Lagrangian and the derived expressions for the electromagnetic fields of interacting point charges through order \( 1/c^2 \). We will illustrate this alternative point of view for the case of \( N \) equally-spaced particles, all of mass \( m \) and charge \( e \), which are held by external centripetal forces in a circular orbit of radius \( R \) centered on the origin in the \( xy \)-plane. There is no frictional force and hence no resistance \( \mathcal{R} \) in the model. Rather, the system may be thought of as consisting of charged beads sliding on a frictionless ring.

If an axially symmetric magnetic field is applied perpendicular to the plane of the circular orbit in the \(-\hat{z}\)-direction and is increasing in magnitude, then the time-rate-of-change of the external magnetic field will produce an external electric field \( E_{\text{ext}} \) in a circular pattern in the \( \hat{\phi} \)-direction \( E_{\text{ext}}(r) = \hat{\phi}E_{\text{ext}}(r) \). The external emf around the circular orbit is given by

\[
emf_{\text{ext}} = \oint E_{\text{ext}}(r) \cdot dr = 2\pi RE_{\text{ext}}(R)
\]  

(4)

The external electric field \( E_{\text{ext}}(r) \) places a tangential force \( F_i = e_i\hat{\phi}E_{\text{ext}}(R) \) on the \( i \)th particle located at \( r_i \). The self-inductance of the charged-particle system is determined by the response of all the particles \( e_i \) in the circular orbit.
D. Electromagnetic Fields of Particles from the Darwin Lagrangian

In order to determine the behavior of the charged particles $e_i$, we turn to the Darwin Lagrangian\[6\] which represents the behavior of interacting charged particles through order $1/c^2$

$$\mathcal{L} = \sum_{i=1}^{i=N} m_i c^2 \left( -1 + \frac{v_i^2}{2c^2} + \frac{(v_i^2)^2}{8c^4} \right) - \frac{1}{2} \sum_{i=1}^{i=N} \sum_{j \neq i} \frac{e_i e_j}{|r_i - r_j|} \right)$$

$$+ \frac{1}{2} \sum_{i=1}^{i=N} \sum_{j \neq i} \frac{e_i e_j}{2c^2} \left[ \frac{v_i \cdot v_j}{|r_i - r_j|} + \frac{v_i \cdot (r_i - r_j) v_j \cdot (r_i - r_j)}{|r_i - r_j|^3} \right]$$

$$- \sum_{i=1}^{i=N} e_i \Phi_{ext}(r_i, t) + \sum_{i=1}^{i=N} \frac{e_i v_i}{c} \cdot A_{ext}(r_i, t)$$

(5)

where the last line includes the scalar potential $\Phi_{ext}$ and vector potential $A_{ext}$ associated with the external electromagnetic fields. The Lagrangian equations of motion can be rewritten in the form of Newton’s second law $dp/dt = d(m\gamma \mathbf{v})/dt = \mathbf{F}$ with $\gamma = (1 - v^2/c^2)^{-1/2}$. In this Newtonian form, we have

$$\frac{d}{dt} \left[ \frac{m_i v_i}{(1 - v_i^2/c^2)^{1/2}} \right] \approx \frac{d}{dt} \left[ m_i \left( 1 + \frac{v_i^2}{2c^2} \right) v_i \right] = e_i \mathbf{E} + e_i \frac{v_i}{c} \times \mathbf{B}$$

$$= e_i \left( \mathbf{E}_{ext}(r_i, t) + \sum_{j \neq i} \mathbf{E}_j(r_i, t) \right) + e_i \frac{v_i}{c} \times \left( \mathbf{B}_{ext}(r_i, t) + \sum_{j \neq i} \mathbf{B}_j(r_i, t) \right)$$

(6)

with the Lorentz force on the $i$th particle arising from the external electromagnetic fields and from the electromagnetic fields of the other particles. The electromagnetic fields due to the $j$th particle are given through order $v^2/c^2$ by\[7\]

$$\mathbf{E}_j(r, t) = e_j \left( \frac{r - r_j}{|r - r_j|^3} \right) \left[ 1 + \frac{v_j^2}{2c^2} - \frac{3}{2} \left( \frac{v_j \cdot (r - r_j)}{c|r - r_j|} \right)^2 \right]$$

$$- \frac{e_j}{2c^2} \left( \frac{a_j}{|r - r_j|} + \frac{a_j \cdot (r - r_j)(r - r_j)}{|r - r_j|^3} \right)$$

(7)

and

$$\mathbf{B}_j(r, t) = e_j \frac{v_j}{c} \times \frac{(r - r_j)}{|r - r_j|^3}$$

(8)

where in Eq. (7) the quantity $a_j$ refers to the acceleration of the $j$th particle.
E. One-Particle Model for a Circuit

1. Motion of the Charged Particle

We start with the case when there is only one charged particle of mass $m$ and charge $e$ in the circular orbit. In this case, the tangential acceleration $a_\phi$ of the charged particle $e$ in the circular orbit arises from the force of only the external electric field $E_{\text{ext}}$. From Eq. (6), written for a single particle and with $d(m\gamma v)/dt = m\gamma^3 a_\phi$ where $\gamma = (1 - v^2/c^2)^{-1/2}$, we have

$$a_\phi = \frac{eE_{\text{ext}}(R)}{m\gamma^3}$$

where $E_{\text{ext}}(R)$ is the magnitude of the tangential electric field due to the external emf $\text{emf}_{\text{ext}}$ at the position of the charge $e$.

2. Magnetic Field of the Charged Particle

The magnetic field $B_e$ at the center of the circular orbit due to the accelerating charge $e$ is given by Eq. (8)

$$B_e(0, t) = \kappa e \frac{v}{cR^2}$$

where the velocity $v$ is increasing since the external electric field $E_{\text{ext}}$ gives a positive charge $e$ a positive acceleration in the $\hat{\phi}$-direction. This magnetic field $B_e$ produced by the orbiting charge $e$ is increasing in the $\hat{z}$-direction, which is in the opposite direction from the increasing external magnetic field which created the external electric field $E_{\text{ext}}(r)$ and the external emf $\text{emf}_{\text{ext}}$ in Eq. (4).

3. Induced Electric Field

Associated with this changing magnetic field $B_e$ created by the orbiting charge $e$, there should be an induced electric field $E_e(r, t)$ according to Faraday’s law. Thus averaging over the circular motion of the charge $e$, we expect an average induced tangential electric field $\langle E_{\phi}(r) \rangle$ at a distance $r$ from the center of the circular orbit (where $r << R$ so that the
magnetic field $B_e$ has approximately the value $B(0, t)$ at the center) given from Eq. (10) by

$$2\pi r \langle E_{e\phi}(r) \rangle = \frac{e}{2} \frac{d\Phi_e}{dt} = -\frac{1}{c} \frac{d}{dt} \left( B_e(0, t) \pi r^2 \right)$$

$$= -\frac{1}{c} \frac{d}{dt} \left[ \frac{e}{c R^2} \pi r^2 \right] = -\frac{1}{c} \left( e \frac{a_\phi}{c R^2} \right) \pi r^2$$

since $dv/dt = a_\phi$. Using Eq. (9), the average tangential electric field follows from Eq. (11) as

$$\langle E_{e\phi}(r, t) \rangle = -\frac{e^2 R E_{ext}(R)}{2mc^2\gamma^3 R^2}$$

We will now show that this induced average tangential electric field $\langle E_{e\phi}(r, t) \rangle$ is exactly the average electric field due to the charge $e$ obtained by use of the electric field expression given in Eq. (7). Thus we assume that the charge $e$ is located momentarily at $r_e = \hat{x} R \cos \phi_e + \hat{y} R \sin \phi_e$ and average over the phase $\phi_e$. Since the entire situation is axially symmetric when averaged over $\phi_e$, we may take the field point along the $x$-axis at $r = \hat{x} R$, and later generalize to cylindrical coordinates. The velocity fields given in the first line of Eq. (7) point from the charge $e$ to the field point. Also, the velocity fields are even if the sign of the velocity $v_e$ is changed to $-v_e$. Thus the velocity fields when averaged over the circular orbit can point only in the radial direction. The accelerations fields arising from the centripetal acceleration of the charge will also point in the radial direction. Since we are interested in the average tangential component of the field $E_e$, we need to average over only the tangential acceleration terms in the second line of Eq. (7). If the field point is close to the center of the circular orbit so that $r << R$, then we may expand in powers of $r/R$; we retain only the first-order terms, giving $|\hat{x} r - r_e|^2 \approx R^{-1} (1 + \hat{x} r \cdot r_e / R^2)$ and $|\hat{x} r - r_e|^{-3} \approx R^{-3} (1 + 3\hat{x} r \cdot r_e / R^2)$. Then the average tangential component of the electric field due to the charge $e$ can be written as

$$\langle E_{e\phi}(\hat{x} r, t) \rangle = \left\langle -\frac{e}{2c^2} \left( \frac{a_{e\phi}}{|\hat{x} r - r_e|} + \frac{a_{e\phi} \cdot (\hat{x} r - r_e)(\hat{x} r - r_e)}{|\hat{x} r - r_e|^3} \right) \right\rangle$$

$$= \left\langle -\frac{e}{2c^2} \left( \frac{a_{e\phi}}{R} \left( 1 + \frac{\hat{x} r \cdot r_e}{R^2} \right) + \frac{a_{e\phi} \cdot (\hat{x} r - r_e)(\hat{x} r - r_e)}{R^3} \right) \cdot \left( 1 + \frac{3\hat{x} r \cdot r_e}{R^2} \right) \right\rangle$$

$$\approx \left\langle -\frac{e}{2c^2} \left( \frac{a_{e\phi}}{R} \left( 1 + \frac{\hat{x} r \cdot r_e}{R^2} \right) \right) \right\rangle$$

$$= -\frac{e a_{e\phi}}{2c^2 R^2} r_e$$

Now we average over the phase $\phi_e$ with $r_e = \hat{x} R \cos \phi_e + \hat{y} R \sin \phi_e$ and $a_{e\phi} = a_{e\phi}(-\hat{x} \sin \phi_e + \hat{y} \cos \phi_e)$. We note that $\langle a_{e\phi} \rangle = 0$, $a_{e\phi} \cdot r_e = 0$, $\langle a_{e\phi} (\hat{x} r - r_e) \rangle = \hat{y} a_{e\phi} / 2 = -\langle (a_{e\phi} \cdot \hat{x}) r_e \rangle$. After averaging and retaining terms through order $r/R$, equation (13) becomes

$$\langle E_{e\phi}(r, t) \rangle = -\frac{\hat{y} e a_{e\phi} R}{2c^2 R^2}$$

which is in agreement with our earlier results in Eqs. (11) and (12).
4. Limit on the Induced Electric Field

We are now in a position to comment on the average response of our one-particle circuit to the applied external emf. For this one-particle example, the response depends crucially upon the mass \( m \) and charge \( e \) of the particle. When the mass \( m \) is large, the acceleration of the charge is small; therefore the induced tangential electric field \( E_{e\phi} \) in Eq. (12) is small. This large-mass situation is what is usually assumed in examples of charged rings responding to external emfs. On the other hand, if we try to increase the induced electromagnetic field \( E_{e\phi} \) by making the mass \( m \) small, we encounter a fundamental limit of electromagnetic theory. The allowed mass \( m \) is limited below by considerations involving the classical radius of the electron \( r_{cl} = e^2/(mc^2) \). Classical electromagnetic theory is valid only for distances large compared to the classical radius of the electron. Thus in our example where the radius \( R \) of the orbit is a crucial parameter, we must have \( R >> r_{cl} \). Thus we require the mass \( m >> e^2/(Rc^2) \) and so \( e^2/(mc^2R) << 1 \). Combing this limit with \( r/R < 1 \), and \( 1 < \gamma \) leads to a limit on the magnitude of the induced electric field in Eq. (12)

\[
\langle E_{e\phi}(r, t) \rangle << E_{ext}(R) \quad \text{for} \quad r < R
\]

(15)

The induced electric field is small compared to the external electric field associated with the external emf.

5. Energy Balance

We also note that the power delivered by the external electric field goes into kinetic energy of the orbiting particle. Thus if take the Newton’s-second-law equation giving Eq. (9) and multiply by the speed \( v \) of the particle, we have

\[
\frac{d}{dt}(m\gamma v) = \frac{d}{dt}(m\gamma c^2) = m\gamma^3a_{\phi} = eE_{ext}(R)v
\]

(16)

so that the power \( eE_{ext}(R)v \) delivered to the charge \( e \) by the external electric field goes into kinetic energy of the particle.

The situation of a one-particle circuit can be summarized as follows. For the one-particle circuit, the induced electric field is small compared to the external electric field and depends explicitly upon the particle’s mass and charge, while the energy transferred by the external
field goes into kinetic energy of the one charged particle. Clearly this is not the situation
which we usually associate with electromagnetic induction for circuit problems.

F. Multi-Particle Model for a Circuit

Motion of the Charged Particles

In order to make contact with the usual discussion of self-inductance in electromagnetism,
we must go to the situation of many electric charges. The force on any charge in the circular
orbit is now the sum of the forces due to the original external electric field plus that due to
the fields of all the other charged particles in the circular orbit as given in Eq. (6). The
magnetic force $e_i v_i \times B/c$ is simply a deflection and does not contribute the tangential
acceleration. Thus the equation of motion for the $i$th particle becomes

$$\frac{d}{dt}(m_i \gamma_i v_i) \cdot \hat{\phi} = m_i \gamma_i^3 R \frac{d^2 \phi_i}{dt^2}$$

$$= \hat{\phi}_i \cdot e_i \left\{ E_{ext}(r_i) + \sum_{j \neq i} e_j \frac{(r_i - r_j)}{|r_i - r_j|^3} - \sum_{j \neq i} \frac{e_j}{2c^2} \left( \frac{a_j \cdot (r_i - r_j)(r_i - r_j)}{|r_i - r_j|^3} + \frac{a_j \cdot (r_i - r_j)}{|r_i - r_j|^3} \right) \right\}$$

(17)

Since the particles are equally spaced is around the circular orbit and all have the same
charge $e$ and mass $m$, the situation is axially symmetric. The equation of motion for every charge takes the same form, and the angular acceleration of each charge is the same, $d^2 \phi_i/dt^2 = d^2 \phi/dt^2$. For simplicity of calculation, we will take the $N$th particle along the $x$-axis so that $\phi_N = 0$, $r_N = \hat{x} R$, and $\hat{\phi}_N = \hat{y}$. The other particles are located at $r_j = \hat{x} R \cos(2\pi j/N) + \hat{y} R \sin(2\pi j/N)$, corresponding to an angle $\phi_j = 2\pi j/N$ for $j = 1, 2, ..., N$. The tangential acceleration of the $j$th particle is given by $a_{j\phi} = (d^2 \phi/dt^2) [-\hat{x} R \sin(2\pi j/N) + \hat{y} R \cos(2\pi j/N)]$. By symmetry, it is clear that the electrostatic fields, the velocity fields, and the centripetal acceleration fields of the other particles can not contribute to the tangential electric field at particle $N$. The equation of motion for the tangential acceleration for each charge in the circular orbit is the same as that for the $N$th particle, which from Eq. (17) is

$$m \gamma^3 R \frac{d^2 \phi}{dt^2} = \left\{ e E_{ext}(R) - \sum_{j=1}^{j=N-1} e_j \frac{c^2}{2c^2} \left( \frac{\hat{y} \cdot a_{j\phi}}{|\hat{x} R - r_j|} + \frac{a_{j\phi} \cdot (\hat{x} R - r_j)}{|\hat{x} R - r_j|^3} \right) \right\}$$

(18)
Now we evaluate the distance between the \( j \)th particle and the \( N \)th particle in the circular orbit as

\[
|xR - r_j| = [2R^2 - 2R^2 \cos(2\pi j/N)]^{1/2} = [4R^2 \sin^2(\pi j/N)]^{1/2} = |2R \sin(\pi j/N)|
\]

while

\[
\hat{y} \cdot \mathbf{a}_{j\phi} = (d^2 \phi / dt^2) R \cos(2\pi j/N)
\]

and

\[
\mathbf{a}_j \cdot (\hat{x}R - r_j) \hat{y} \cdot (\hat{x}R - r_j) = (d^2 \phi / dt^2) R [-R \sin(2\pi j/N)] [-R \sin(2\pi j/N)]
\]

Then equation (18) becomes

\[
m\gamma^3 R \frac{d^2 \phi}{dt^2} = eE_{\text{ext}}(R) - \frac{d^2 \phi}{dt^2} \sum_{j=1}^{j=N-1} \frac{e^2}{2c^2} \left( \frac{R \cos(2\pi j/N)}{2R \sin(\pi j/N)} + \frac{R[R \sin(2\pi j/N)][R \sin(2\pi j/N)]}{|2R \sin(\pi j/N)|^3} \right)
\]

or, solving for \( d^2 \phi / dt^2 \),

\[
\frac{d^2 \phi}{dt^2} = eE_{\text{ext}}(R) \left[ m\gamma^3 R + \sum_{j=1}^{j=N-1} \frac{e^2}{2c^2} \frac{\cos(2\pi j/N)}{2 \sin(\pi j/N)} + \frac{\sin^2(2\pi j/N)}{2 \sin(\pi j/N)^3} \right]^{-1}
\]

If there is only one particle on the frictionless ring so that \( N = 1 \), the sum disappears, and the tangential acceleration corresponds to the result obtained earlier in Eq. (9) above with \( Rd^2 \phi / dt^2 = a_\phi \). We note that the mass term in Eq. (23) remains unchanged by the number of particles while the sum increases with each additional particle. Thus if there are many particles, then the electric field at particle \( i \) due to the other particles \( j \) can lead to so large a sum in Eq. (23) that the mass contribution \( m\gamma^3 R \) becomes insignificant. In this case, the common angular acceleration of each particle becomes from Eq. (23)

\[
\frac{d^2 \phi}{dt^2} \approx \frac{2e^2}{e} E_{\text{ext}}(R) \left[ \sum_{j=1}^{j=N-1} \left( \frac{\cos(2\pi j/N)}{2 \sin(\pi j/N)} + \frac{\sin^2(2\pi j/N)}{2 \sin(\pi j/N)^3} \right) \right]^{-1}
\]

We see that in this multiparticle situation the angular acceleration no longer depends upon the mass \( m \) of the charge carriers.

2. Induced Electric Field

Furthermore, in this multiparticle situation where the particle mass becomes insignificant, the left-hand side of Eq. (18) is negligible, so that the sum \( \sum_j E_{ej}(\mathbf{r}_i) \) of the acceleration
fields of all the other charges $e_j$ cancels the external electric field $E_{ext}(r_i)$ of the external emf at the position $r_i$ of each charge in the circular orbit

$$- E_{ext}(r_i) \approx E_e(r_i) = \sum_{j \neq i} E_{ej}(r_i)$$

(25)

Thus analogous to the situation in electrostatics where the fields of the charges in a conductor move to new positions so as to cancel the external electric field at the position of each charge, here the charges accelerate so that the acceleration fields cancel the external field at the position of each charge. Now the induced electric field $E_e(r) = \sum_i E_{ei}(r)$ at a general field point due to the orbit particles is independent of the charge $e$ of the charge carriers, since the angular acceleration in Eq. (24) depends inversely as the charge $e$, and this inverse dependence upon $e$ cancels with the $e$ appearing in Eq. (7) so as to give an induced electric field which is independent of the charge on the charge carriers.

3. **Self-Inductance of the Circuit**

The self-inductance of the multiparticle circuit can be obtained from Eq. (11) when the circuit resistance vanishes. If the resistance vanishes, the external emf $emf_{ext} = 2\pi RE_{ext}(R)$ around the ring equals the self-inductance multiplied by the time-rate-of-change of the current $i = Ne(d\phi/dt)/(2\pi)$

$$emf_{ext} = 2\pi RE_{ext}(R) = L \frac{di}{dt} = L \left( \frac{Ne}{2\pi} \frac{d\phi}{dt^2} \right)$$

(26)

giving the self-inductance $L$ of the multi-charge ring system (where we can ignore the negligible mass contribution) from Eq. (24) as

$$L = \frac{2\pi RE_e(R)}{[Ne(d^2\phi/dt^2)/(2\pi)]} = \frac{(2\pi)^2 R}{2e^2N} \sum_{j=1}^{N-1} \left( \frac{\cos(2\pi j/N)}{2\sin(\pi j/N)^2} + \frac{\sin^2(2\pi j/N)}{2\sin(\pi j/N)^3} \right)$$

(27)

We see that the self-inductance of this multiparticle, circular-orbit circuit is now independent of the mass $m$ or the charge $e$ of the current carriers.

4. **Magnetic Energy of the Current Carriers**

The magnetic energy $U_{mag} = \int d^3r B^2/(8\pi)$ stored in the circular-orbit circuit is given by the cross-terms (but not the self-terms) when the magnetic field is squared, and corresponds
to the velocity-dependent double sum in the Darwin Lagrangian Eq. (5). Thus we have

\[ U_{\text{mag}} = \frac{1}{2} \sum_{i=1}^{N} \sum_{j \neq i} \frac{1}{8\pi} \int d^3r \ 2\mathbf{B}_{ei}(r) \cdot \mathbf{B}_{ej}(r) \]

\[ = \frac{1}{2} \sum_{i=1}^{N} \sum_{j \neq i} \frac{e^2}{2c^2} \left( \frac{\mathbf{v}_i \cdot \mathbf{v}_j}{|\mathbf{r}_i - \mathbf{r}_j|} + \frac{\mathbf{v}_i \cdot (\mathbf{r}_i - \mathbf{r}_j) \mathbf{v}_j \cdot (\mathbf{r}_i - \mathbf{r}_j)}{|\mathbf{r}_i - \mathbf{r}_j|^3} \right) \]

\[ = N \sum_{j=1}^{N-1} e^2 \left( \frac{v_{\hat{y}} \cdot \mathbf{v}_j}{|\hat{x}R - \mathbf{r}_j|} + \frac{v_{\hat{y}} \cdot (-\mathbf{r}_j) \mathbf{v}_j \cdot (\hat{x}R)}{|\hat{x}R - \mathbf{r}_j|^3} \right) \]

(28)

where in the last line of Eq. (28) we have used \( \mathbf{v}_i \cdot \mathbf{r}_i = 0 \) and have taken advantage of the symmetry to evaluate the magnetic energy when the \( N \)th particle is located on the \( x \)-axis at \( \mathbf{r}_N = \hat{x}R \) and is moving with velocity \( \mathbf{v}_N = \hat{y}v = \hat{y}Rd\phi/dt \). The \( j \)th particle is located at \( \mathbf{r}_j = R[\hat{x}\cos(2\pi j/N) + \hat{y}\sin(2\pi j/N)] \) with velocity \( \mathbf{v}_j = R(d\phi/dt)[-\hat{x}\sin(2\pi j/N) + \hat{y}\cos(2\pi j/N)] \).

Introducing these expressions along with the distance given in Eq. (19), the magnetic energy of Eq. (28) is

\[ U_{\text{mag}} = \frac{N}{2} \sum_{j=1}^{N-1} \frac{e^2}{2c^2} \left( \frac{R^2}{R} \frac{d\phi}{dt} \right)^2 \left( \frac{\cos(2\pi j/N)}{2\sin(\pi j/N)} + \frac{\sin^2(2\pi j/N)}{|2\sin(\pi j/N)|^3} \right) \]

\[ = \frac{1}{2} \left( \frac{2\pi}{2c^2N} \right) \sum_{j=1}^{N-1} \left( \frac{\cos(2\pi j/N)}{|2\sin(\pi j/N)|} + \frac{\sin^2(2\pi j/N)}{|2\sin(\pi j/N)|^3} \right) \left( \frac{eN}{2\pi} \frac{d\phi}{dt} \right)^2 \]

(29)

We recognize the current \( i = eN(d\phi/dt)/(2\pi) \) and so can read off the self-inductance of the circuit from \( U_{\text{mag}} = (1/2)Li^2 \). The expression for the self-inductance \( L \) is the same as in Eq. (27). Now the power \( P = eE_{\text{ext}}(R)v_i \) delivered to the \( i \)th charge by the external electric field \( E_{\text{ext}} \) associated with the original emf does not go into particle kinetic energy but rather is converted into electromagnetic energy associated with the particles on the ring. This electromagnetic energy is exactly the stored magnetic energy given by the expression \( (1/2)Li^2 \).

For our example of an external emf acting on a circular charged-particle circuit, the situation with many particles is totally transformed from the situation with only one particle. As charged particles are added to the circuit, the mechanical inertia increases linearly with the number of particles \( N \) while the inertia associated with the mutual electromagnetic interactions increases quadratically with \( N \). In the multiparticle case, the mutual interaction between the particles overwhelms the single-particle behavior so that the mass and charge of the individual charge carriers is no longer of significance. The particles move so that the sum of the induced electric fields at each particle cancels the external electric field. [15]
G. Discussion

1. Summary of the Calculations

In the analysis above, we have discussed the self-inductance of a very simple circuit from an unfamiliar point of view. We have evaluated the electric fields of individual electric charges and shown how a system involving a single charge is transformed over to a familiar electromagnetic system when the number of charges is increased. Our simple example involves charges under centripetal constraining forces giving a circular orbit but allowing tangential acceleration along the circular orbit. In the one particle example, the induced electric field depends upon both the charge $e$ and the mass $m$ of the charge carrier with the induced field proportional to $e^2/m$. In the one-particle case, the energy transferred to the particle goes into the kinetic energy of the ring particle. However, when we deal with a multi-particle case, then the electromagnetic forces between the charges are such as to transform the behavior over to the familiar behavior of a conducting circuit where the charge and mass of the current carriers are of no significance. When there are a large number of charged particles, the acceleration of each charge is determined by the requirement that the sum of the acceleration fields of all the other charges should cancel the external electric field which produces the external emf around the circuit.

2. Comments on "Hidden Mechanical Momentum"

We suggest that the simple example discussed here is not only an interesting approach to ideas of self-inductance, but also is a reminder that it is unlikely that effects of particle mass will be of importance in multiparticle electromagnetic systems. Some forty years ago, the claim was made that relativistic effects of mechanical particle momentum were needed to account for the conservation of linear momentum in the poorly-understood interaction of a magnet and a charged particle.[10] This perplexing magnet-charge interaction is the basis for controversies associated with the Aharonov-Bohm effect,[11] the Aharonov-Casher effect,[12] and Mansuripur’s paradox.[13] The associated idea of “hidden mechanical momentum” has become entrenched in the research and textbook literature of electromagnetism.[1] [2] [4] However, the only valid calculation of “hidden mechanical momentum” involves particles which have negligible mutual interactions; these calculations are equivalent to a one-particle
All of the other cases of “hidden mechanical momentum” are simply empty claims without valid calculations. In the present transparent analysis, the one-particle example where particle inertia dominates is completely different from the mutually-interacting multiparticle limit where electromagnetic inertia dominates. The one-particle example contributes no understanding and has no relevance to the multiparticle case; it simply misleads physicists. We suggest that the idea of “hidden mechanical momentum” is an error based on invalid extrapolations from a one-particle example. We believe that “hidden mechanical momentum” is extraordinarily unlikely to have any relevance as a valid physical idea within multiparticle electromagnetism.

[1] See, for example, the undergraduate textbook by D. J. Griffiths, *Introduction to Electrodynamics* 3rd edn (Prentice-Hall, Upper Saddle River,NJ 1999).

[2] See, for example, the graduate-level textbook by J. D. Jackson, *Classical Electrodynamics* 3rd edn (John Wiley & Sons, New York, 1999).

[3] I am not aware of any such treatment carried out in the literature.

[4] See the listings under “Hidden Momentum” in the resource letter by D. J. Griffiths, “Resource Letter EM-1: Electromagnetic Momentum,” Am. J. Phys. 80, 7-18 (2012).

[5] See, for example, Ref. 1, p. 316, Problem 7.22.

[6] See, for example, Ref. 2, pp. 596-598.

[7] For example, the fields corresponding to the Darwin Lagrangian are given by L. Page and N. I. Adams, “Action and reaction between moving charges,” Am. J. Phys. 13, 141-147 (1945).

[8] The situation of our one-particle example contains elements of what is often termed “the Feynman disk paradox.” See R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures on Physics* (Addison-Wesley, Reading MA, 1964), Vol. II, p. 17-5 and p. 27-6. See also, Ref. 1, pp. 359-361. In the present article, we evaluate the induced electromagnetic field arising from the accelerating disk charges by use of the Darwin Lagrangian fields.

[9] It seems interesting that for this circular charge arrangement, the typical multiparticle behavior requires at least four charges. The summation in Eq. (27) is steadily increasing with increasing particle number $N$. However, the summation starts out negative ($-0.5$) for $N = 2$, and is still negative ($-0.289$) for $N = 3$. Only for $N = 4$, does the summation become...
positive (+0.207).

[10] See, for example, S. Coleman and J. H. van Vleck, “Origin of ‘hidden momentum forces’ on magnets,” Phys. Rev. 171, 1370-1375 (1968).

[11] Y. Aharonov and D. Bohm, “Significance of electromagnetic potentials in quantum theory,” Phys. Rev. 115, 485-491 (1959).

[12] Y. Aharonov and A. Casher, “Topological quantum effects for neutral particles,” Phys. Rev. Lett. 53, 319-321 (1984).

[13] M. Mansuripur, “Trouble with the Lorentz law of force: Incompatibility with special relativity and momentum conservation,” Phys. Rev. Lett. 108, 193901 (2012).

[14] See, for example, Ref. 1, pp. 560-521, where the current-carrying charges are treated as though they were isolated particles with no mutual interactions; thus the behavior is equivalent to a one-particle example. See also, T. H. Boyer, “Interaction of a point charge and a magnet: Comments on ‘Hidden Mechanical Momentum Due to Hidden Nonelectromagnetic Forces’,” e-print arXiv:0708.3367v1 (2007).

[15] The present article is the first in a series of four articles by T. H. Boyer using the magnet model consisting of mutually-interacting point charges moving on a circular path. Article 2: “Interaction of a Magnet and a Point Charge: Unrecognized Internal Electromagnetic Momentum Eliminates the Myth of Hidden Mechanical Momentum.” Article 3: “Classical Interaction of a Magnet and a Point Charge: The Shockley-James Paradox.” Article 4: “Classical Interaction of a Magnet and a Point Charge: The Classical Electromagnetic Forces Responsible for the Aharonov-Bohm Phase Shift.”