Transient Heat Conduction in a Wall Exposed to a Fire: an Analytic Approach

G Casano and S Piva
ENDIF ENgineering Department In Ferrara, Università di Ferrara
via Saragat 1, 44122 Ferrara

Corresponding Author E-mail: stefano.piva@unife.it

Abstract. Fire safety engineering requires a detailed understanding of fire behaviour and of its effects on structures and people. Currently, advanced numerical codes for the prediction of the fire behaviour are available. However, they often require heavy calculations and long times. In this context analytical solutions can be useful for a fast analysis of simplified schematizations. It allows a more effective final utilization of the advanced fire codes. In this contribution, for a separation wall exposed to a fire the temperature history is analysed of the unexposed wall surface. Due to the limitations of the model, the temperature in the fire room changes stepwise, with a final value typical of a post-flashover condition. Nevertheless, with an appropriate choice of the heat transfer coefficient, the thermal action on the surface exposed to fire becomes that due to a fire following the standard temperature-time curve. The solution is then obtained by applying the separation of variables to the heat conduction equation. The problem is made dimensionless and the results are analysed in order to validate their significance. This simplified model allows to obtain useful information on the magnitude of the temperature reached.

1. Introduction
The capability of a reliable prediction of the thermal behaviour of a wall exposed to a fire is a relevant topic in Fire Safety Engineering for different reasons: structural resistance, evacuation and safety of the people, protection of stored materials. The result is that the “procedure of determining the temperature development in members on the basis of the thermal actions and the thermal material properties“ (EN 1991-1-2 [1]), synthetically defined as the temperature analysis, is a much studied subject.

For the prediction of the temperature history in load bearing or non-load bearing walls, many models, characterized by different levels of complexity, are available in the literature. For concrete walls (EN 1992-1-2 [2]) and for masonry walls (EN 1996-1-2 [3]), “the calculation methods for thermal response shall be based on the acknowledged principles and assumptions of the theory of heat transfer” and “shall include the consideration of (...) the relevant thermal actions specified in EN 1991-1-2” and “the temperature dependent thermal properties of the materials”. However, “the influence of moisture content and of migration of the moisture (...) may conservatively be neglected” and in concrete “the temperature profile in a reinforced concrete element may be assessed omitting the presence of reinforcement”. We can conclude that in the current standards for the prediction of the temperature history in a concrete or masonry wall, the solution of the Fourier equation with convective...
and radiative boundary conditions is regarded as appropriate. For complex geometries, for non linear problems (e.g. thermal properties variable with temperature) or for multiple physical processes (e.g. combustion coupled to heat transfer) different numerical codes are commercially available, while analytical solutions are usually not easily found.

At the same time, when the fire resistance of building components and constructions has to be declared, the building elements are classified in different classes, or combinations of those, by specifying different performance criteria. Normally the fire resistance classification is expressed as a time limit in minutes (15, 30, 45, 60, 90, 120, 180, 240, or 360) which shows the time the performance criteria is fulfilled during a standardized fire test. Specific criteria for the classes are: R Load bearing capacity, E Integrity, I Insulation. By following EN 1992-1-2 [2], “where compartmentation is required, the elements forming the boundaries of the fire compartment, including joints, shall be designed and constructed in such a way that they maintain their separating function during the relevant fire exposure. This shall ensure, where relevant, that:

- integrity failure does not occur (…)
- insulation failure does not occur (…)
- thermal radiation from the unexposed side is limited.”

In the present paper we concentrate our attention on insulation (Criterion “I” in the following), that is the ability of a separating element of a building construction when exposed to fire on one side, to restrict the temperature rise of the unexposed face below specified levels. In particular “Criterion “I” may be assumed to be satisfied where the average temperature rise over the whole of the non-exposed surface is limited to 140 K, and the maximum temperature rise at any point of that surface does not exceed 180 K.” (EN 1992-1-2 [2]). This temperature rise can be measured or, as previously discussed, predicted. Here, in order to verify the criterion “I”, the prediction of the time-space temperature distribution in a separating wall is obtained analytically. A large review of the analytical formulations used for the temperature prediction for walls exposed to fire is given by Wang and Tan [4].

The analytical approach necessarily limits the results for several reasons:

1) The boundary conditions are necessarily simple: imposed temperature, heat flux, third type [5]. In order to simulate an increase of temperature on the heated surface similar to that given by the standard temperature-time curve (EN 1991-1-2 [1]), the choice has fallen on a boundary condition of the third type with a stepwise change of temperature. In this case with an appropriate choice of the heat transfer coefficient it is possible to obtain the desired temperature history on the face of the wall exposed to fire.

2) The geometry are simple, mainly reduced to single layer wall of infinite extension. However, this is a largely diffused typology for the separating elements of load bearing or non-load bearing walls in fire compartments.

Nevertheless, we want to demonstrate that analytical models are important in analyzing conduction phenomena in fire systems. For this aim we investigated the thermal performance of a wall exposed to a fire. The goal is to estimate the time needed to the wall to attain the limiting value of temperature on the face unexposed to fire when the wall is used as the separation from a fire room.

2. Model
The problem is schematically shown in Fig. 1. We want to model the typical laboratory installation for testing the fire resistance of non load bearing walls, as required by EN 1363-1 [6] and EN 1364-1 [7]. A constant properties, single layer, wall is exposed to a fire on one face. On the opposite side of the wall the ambient temperature is constant. The temperature in the fire room changes stepwise, with a final value typical of a post-flashover condition. The boundary conditions on the walls are of the third type. The thickness of the wall is thin with respect of its height and width so that the wall can be considered of unlimited extension and the heat flow one-dimensional in the direction of the thickness.

The solution is obtained by applying the separation of variables to the heat conduction equation, as suggested by Carslaw and Jaeger [8] and Ozisik [9].
Figure 1. Schematic of the problem.

The problem is stated mathematically as follows:

Conservation of energy:
\[ \frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \quad (1) \]

Boundary conditions:
\[ h_l [T_{a1} - T(x = 0, t)] = -\lambda \frac{\partial T}{\partial x} (x = 0, t) \quad (2) \]
\[ -\lambda \frac{\partial T}{\partial x} (x = L, t) = h_s [T(x = L, t) - T_{a2}] \quad (3) \]

Initial condition:
\[ t = 0 \quad T(x, t = 0) = T_{a2} \quad (4) \]

For the solution of the problem the separation of variables can be applied. So the solution of Eqs. (1-4) can be decomposed as:
\[ T(x, t) = u(x) + v(x, t) \quad (5) \]

When substituting Eq. (5) in Eqs. (1-4), it follows that the problem is now given by the solution of the following differential equations:
\[ 0 = \frac{d^2 u(x)}{dx^2} \quad (6) \]
\[ \frac{\partial v(x, t)}{\partial t} = \alpha \frac{\partial^2 v(x, t)}{\partial x^2} \quad (7) \]

where Eq. (6) deals with the Steady state component of the solution and Eq. (7) with its Time dependent component.

The Steady state component of the solution is coupled to the following boundary conditions:
\[ h_l [T_{a1} - u(x = 0)] = -\lambda \frac{du}{dx} (x = 0) \quad (8) \]
\[ -\lambda \frac{du}{dx} (x = L) = h_s [u(x = L) - T_{a2}] \quad (9) \]

Equations (6) and (8-9) are then expressed in dimensionless form, according to the following dimensionless variables:
\[ U = \frac{T_{a1} - u(x)}{T_{a1} - T_{a2}} \quad \xi = \frac{x}{L} \]

Equation (6) and related boundary conditions, Eqs. (8-9), become:
\[ 0 = \frac{d^2 U}{d\xi^2} \quad (10) \]
\[ \xi = 0 \quad \frac{dU}{d\xi} \bigg|_{\xi=0} = B_i U (\xi = 0) \quad (11) \]

\[ \xi = 1 \quad \frac{dU}{d\xi} \bigg|_{\xi=1} = B_i \left( 1 - U (\xi = 1) \right) \quad (12) \]

The dimensionless parameters \( B_i \) and \( B_{i+1} \), are given by:

\[ B_i = \frac{h_i L}{\lambda} \quad B_{i+1} = \frac{h_{i+1} L}{\lambda} \]

The solution of Eqs. (10-12) is given by:

\[ U = A + B \xi \quad (13) \]

where:

\[ A = \frac{B_i}{B_i + B_{i+1} + B_i B_{i+1}} \quad (14) \]

\[ B = \frac{B_i B_{i+1}}{B_i + B_{i+1} + B_i B_{i+1}} \quad (15) \]

The Time dependent component of the solution (Eq. 7) is coupled to the following boundary conditions:

\[ x = 0 \quad -h_1 v(x = 0, t) = -\lambda \frac{\partial v(x = 0, t)}{\partial x} \quad (16) \]

\[ x = L \quad -\lambda \frac{\partial v(x = L, t)}{\partial x} = h_2 v(x = L, t) \quad (17) \]

By coupling Eq. (4) and Eq. (5) we obtain the initial condition, given by:

\[ t = 0 \quad v(x, t = 0) = T_{u_2} - u(x) \quad (18) \]

Equations (7) and related boundary and initial conditions, Eqs. (16-18), become:

\[ V = \frac{v(x, t)}{T_{u_2} - T_{u_1}} \quad \tau = \frac{t}{L^2/\alpha} \quad \xi = \frac{x}{L} \]

Equation (7) and related boundary and initial conditions, Eqs. (16-18), become:

\[ \frac{\partial V}{\partial \tau} = \frac{\partial^2 V}{\partial \xi^2} \quad (19) \]

\[ \xi = 0 \quad \frac{\partial V}{\partial \xi} \bigg|_{\xi=0} = B_i V (0, \tau) \quad (20) \]

\[ \xi = 1 \quad \frac{\partial V}{\partial \xi} \bigg|_{\xi=1} = -B_{i+1} V (1, \tau) \quad (21) \]

\[ \tau = 0 \quad V (\xi, \tau = 0) = -1 + U (\xi) \quad (22) \]

Following the method of separation of variables, the function \( V(\xi, \tau) \) can be expressed as:

\[ V(\xi, \tau) = \sum_{n=0}^{\infty} A_n X_n (\xi) \Theta_n (\tau) \quad (23) \]

\( X_n(\xi) \) and \( \Theta_n(\tau) \) are solution of:

\[ \frac{\partial \Theta_n}{\partial \tau} = -\beta_n^2 \Theta_n \quad (24) \]
\[ \frac{\partial^2 X}{\partial \xi^2} = -\beta_n^2 X_n \] (25)

A general solution of Eq.(24) can be expressed in terms of the exponential function, always tending to zero for \( t \to +\infty \):

\[ \Theta_n = \Theta_{0n} e^{-\beta_n t} \] (26)

A general solution of Eq.(25) can be expressed in terms of trigonometric functions:

\[ X_n = C_{1n} \sin(\beta_n \xi) + C_{2n} \cos(\beta_n \xi) \] (27)

By substituting Eqs. (26) and (27) in Eq. (23), it follows:

\[ V_n(\xi, \tau) = A_n \Theta_{0n} [C_{1n} \sin(\beta_n \xi) + C_{2n} \cos(\beta_n \xi)] e^{-\beta_n \tau} \] (28)

The boundary conditions (Eqs. (20) and (21)) give a relationship between \( C_{1n} \) and \( C_{2n} \):

\[ C_{1n} = \frac{B_i}{\beta_n} C_{2n} \] (29)

\[ \left[ B_i \sin(\beta_n) + \beta_n \cos(\beta_n) \right] C_{1n} + \left[ -\beta_n \sin(\beta_n) + B_i \cos(\beta_n) \right] C_{2n} = 0 \] (30)

The solution of this system of equations gives:

\[ \frac{\sin(\beta_n)}{\cos(\beta_n)} = \frac{\beta_n (B_i + B_i)}{\beta_n^2 - B_i B_i} \] (31)

It can be easily proved that, after substituting Eq. (29) in Eq. (28), the functions \( V_n(\xi, \tau) \) satisfy Eqs. (19-22):

\[ V_n(\xi, \tau) = \frac{C_{2n}}{\beta_n} \left[ B_i \sin(\beta_n \xi) + \beta_n \cos(\beta_n \xi) \right] e^{-\beta_n \tau} \] (32)

More generally:

\[ V(\xi, \tau) = \sum_{n=0}^{\infty} \frac{C_{2n}}{\beta_n} \left[ B_i \sin(\beta_n \xi) + \beta_n \cos(\beta_n \xi) \right] e^{-\beta_n \tau} \] (33)

or, more simply:

\[ V(\xi, \tau) = \sum_{n=0}^{\infty} C_{2n} X_n(\xi) e^{-\beta_n \tau} \] (34)

where:

\[ X_n = \frac{1}{\beta_n} \left[ B_i \sin(\beta_n \xi) + \beta_n \cos(\beta_n \xi) \right] \] (35)

It can be easily proved that the functions \( X_n(\xi) \), with the boundary conditions, Eqs.(26) and (27), guarantee:

\[ \int_{\xi=0}^{\xi=1} X_n(\xi) X_m(\xi) d\xi = 0, \text{ if } n \neq m \]

\[ \neq 0, \text{ if } n = m \] (36)

The admissible values of the constant of separation \( \beta_n \) are the roots of the transcendental equation Eq.(31). The search of these roots can be easily executed by means of a numerical procedure.

The constants \( C_{2n} \) that satisfy the boundary condition Eq. (20) are determined by means of the following procedure. From Eqs. (22) and(34) it follows:

\[ \sum_{n=0}^{\infty} C_{2n} X_n(\xi) = F(\xi) \] (37)
By multiplying both sides of Eq. (37) by $X_m$ and integrating from $\xi = 0$ to $\xi = 1$, on the basis of the property given by Eq. (36), it follows:

$$C_{2,n} = \frac{\int_0^1 F(\xi) X_n(\xi) d\xi}{\int_0^1 X_n^2(\xi) d\xi}$$

(38)

By calculating the integrals in Eq. (37), it follows:

$$\int_0^1 F(\xi) X_n(\xi) d\xi = \frac{1}{\beta_n} \left[ A B_i - B + \left( A + B + B_i \right) \sin(\beta_n) + \frac{B - (A + B) B_i}{\beta_n} \cos(\beta_n) \right]$$

$$\int_0^1 X_n^2(\xi) d\xi = \frac{B_i \left( B_i^2 + \beta_n^2 \right) + \left( B_i^2 + \beta_n^2 \right) \left( B_i + B_i^2 + \beta_n^2 \right)}{2 \beta_n \left( B_i^2 + \beta_n^2 \right)}$$

(39)  

(40)

From Eqs. (33, 38, 39, 40) it follows:

$$V(\xi, \tau) = 2 \sum_{n=1}^{\infty} \left[ \frac{A B_i - B}{B_i + \beta_n^2} + \left( A + B + B_i \right) \sin(\beta_n) + \frac{B - (A + B) B_i}{B_i + \beta_n^2} \cos(\beta_n) \right]$$

$$\left[ \frac{B_i \left( B_i^2 + \beta_n^2 \right) + \left( B_i^2 + \beta_n^2 \right) \left( B_i + B_i^2 + \beta_n^2 \right)}{2 B_i \beta_n \left( B_i^2 + \beta_n^2 \right)} \right] e^{-\beta_n \tau}$$

(41)

Finally, the temperature distribution is given by:

$$T(x,t) = T_{in} - T_{out} - T(x,t)\left( U - V \right)$$

(42)

3. Validation

In order to guarantee the reliability of the numerical code, two validation exercises are reported.

An important step for the analytical solution is the search of the eigenvalues of the problem, given by Eq. (31). The search of these roots is executed by means of an iterative numerical procedure. In Grigull and Erk [10] are reported the first 8 eigenvalues for a single layer wall of constant properties, where the heating scheme is similar to that proposed in the present paper. The initial condition is an assigned distribution of temperature. On the two sides of the wall we see the same heat transfer coefficient and air temperature. In Table 1 the eigenvalues calculated with our procedure are compared with those calculated by Grigull and Erk [10]. The comparison is carried out for $Bi = 0.2$. It is evident that the agreement between the results is excellent.

As the second validation exercise, a 5 mm infinite steel sheet which forms the boundary between two compartments, both initially at a temperature of 20°C, is analyzed. The temperature of the air in one of the compartments is suddenly increased to 100°C, as proposed by Drysdale [11].

![Table 1. Comparison between eigenvalues calculated and available in the literature, for the same problem.](image)
convection heat transfer coefficient is \( h = 20 \, \text{W/m}^2\text{K} \); the thermal properties are: \( \rho = 7850 \, \text{kg/m}^3 \), \( c_p = 460 \, \text{J/kgK} \), \( \lambda = 45.8 \, \text{W/mK} \). The analytical and reference solutions are compared in Fig. 2. The comparison shows a very good agreement.

4. Results and Discussion

The problem now considered is schematically shown in Fig. 1. In particular, as an example of application of the analytical solution derived in Par.2, in Fig. 3 we report the results for a concrete wall characterized by a thickness of 100 mm.

The thermal and physical properties of concrete with calcareous aggregates are those specified by the Standard EN 1992-1-2 [2]. However, in the latter the data of specific heat, thermal conductivity and density are given as a function of temperature in the range \( 20 \leq T \leq 1200 \, ^\circ\text{C} \); in our calculations we used average values in the temperature range \( 20 \div 800 \, ^\circ\text{C} \), precisely: \( \rho = 2197.2 \, \text{kg/m}^3 \), \( c_p = 1047.4 \, \text{J/kgK} \) and \( \lambda = 0.927 \, \text{Wm}^{-1}\text{K}^{-1} \). The choice of the temperature interval derives from some preliminary analyses for thicknesses in the range \( 60 \leq L \leq 175 \, \text{mm} \). For the typical thermal loading of these walls we found minimum and maximum temperature values always inside the selected temperature range.

The procedure to define the boundary conditions are different for the two faces. On the surface not exposed to fire the coefficient of heat transfer by convection and radiation is \( h (x = L) = 9 \, \text{Wm}^{-2}\text{K}^{-1} \) and the ambient temperature \( T_{\text{a2}} = 20 \, ^\circ\text{C} \). On the surface exposed to fire we used the procedure proposed by Buchanan and Rao Munukutla [12]; they suggested that “the fire-exposed face of the wall is assumed to be slightly cooler than the furnace temperature, by a factor which varies from 0.6 at time zero to 0.96 after one hour of exposure, from recent tests” [13]. Based on the latter results and on the standard temperature-time curve suggested by EN 1991-1-2 [1], incidentally in accordance with ISO 834-1 [14], we obtained for the temperature of the fire-exposed face a logarithmic time distribution:

\[
T (x = 0, t) = 15.873 + 200.8 \ln (t) \tag{43}
\]

were \( T \) is in \(^\circ\text{C}\) and \( t \) in min. The coefficient of heat transfer by convection and radiation was then chosen so to obtain, for the face exposed to fire following the ISO / EN Standard temperature-time curve, the best approximation to Eq. (43). For the case shown in Fig. 3, the result is \( h (x = 0) = 27.1 \, \text{Wm}^{-2}\text{K}^{-1} \). In Figure 3 the whole set of temperature distributions used to determine \( h (x = 0) \) are also reported. The calculated temperature on the face exposed to fire (continuous line) is a little big higher than that of Eq. (43) (line with short hatching) in the first hour, then tends to follow with a good accuracy the reference temperature.

![Graph](image-url)

**Figure 2.** Validation exercise: comparison between analytical (——) and reference (■ [11]) solutions.
In Figure 3 the temperature evolution on the non fire-exposed face shows a trend qualitatively in full accordance with those shown in the experimental investigations reported in [13]. The face is undisturbed during the initial time, then tends to exhibit a fast increase of the surface temperature. The fire resistance time of this wall, defined as the time required for the unexposed surface temperature to rise 160°C (an increase of 140 °C over the ambient temperature 20 °C), is 91.5 min.

The code based on the analytical solution proposed in Par. 2 has been used to predict the behaviour to fire, in terms of temperature trends on the unexposed surface, for non-load bearing concrete walls of different thickness, taking as the reference the limiting values proposed by EN 1992-1-2 [2]. The temperature rise of the unexposed face is the parameter used to assess the ability of a separating element of a building construction to classify the Insulation “I”, when exposed to fire on one side. By following EN 1992-1-2 [2], Criterion “T” may be assumed to be satisfied “where the average temperature rise over the whole of the non-exposed surface is limited to 140 K, and the maximum temperature rise at any point of that surface does not exceed 180 K”.

Figure 4 shows on the abscissa the wall thickness and on the ordinate axis the time required to reach the limiting temperature set by EN 1992-1-2 [2] on the surface not exposed to the fire. The Standard EN 1992-1-2 [2] provides the class wall as a function of thickness by means of tables. The range of thicknesses examined is between a minimum of 60 mm and maximum of 175 mm. The data provided by the Standard [2] are given in the form of a bar graph. Differently the results obtained from the analytical solution are provided by means of symbols interpolated by a continuous line of parabolic trend. The two distributions are in good agreement; as expected, the calculated values are always greater, even if though only slightly, than the values proposed by the standard for the different classes of wall. This agreement is more pronounced for small and medium-thick walls, the opposite for very thick walls. Necessarily, the comparison can be only qualitative, because nothing is known about the nature of the material used for the collection of the data shown in EN 1992-1-2 [2]. Nevertheless, the good agreement between two trends guarantees the reliability of the results obtained with the analytical method proposed in Par. 2, at least for a simple layer wall.

5. Concluding Remarks
Interesting analysis and characterization of conduction phenomena in structures exposed to fire can be obtained with analytical models, even if limitations of geometry, properties and boundary conditions are necessarily evidenced.
Figure 4. Criterion “I”: comparison between calculated (—♦—) and reference (bars) values for non load bearing concrete walls.

The analytical approach proposed here for the study of the performance of a wall separating a room with a fire from a different compartment, has provided useful results. With this analytic solution we are able to easily calculate the time for which the temperature of the surface unexposed to the fire reaches the limiting values given by the current standards to characterize its performance.

Trends and temperature history are those expected for these geometries and materials. A comparison of standard and predicted values for the Criterion “I”, indicating the insulation performance of the walls, shows a good agreement; thus the significance of the methodology for a simple prediction of “I” is demonstrated.

6. References
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