CONVERGENCE OF HEISENBERG MODULES OVER QUANTUM 2-TORI FOR THE MODULAR GROMOV-HAUSDORFF PROPINQUITY

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Abstract. The modular Gromov-Hausdorff propinquity is a distance on classes of modules endowed with quantum metric information, in the form of a metric form of a connection and a left Hilbert module structure. This paper proves that the family of Heisenberg modules over quantum two tori, when endowed with their canonical connections, form a continuous family for the modular propinquity.

1. Introduction

The modular Gromov-Hausdorff propinquity [15] is a distance on modules over C*-algebras endowed with some quantum metric information, designed to advance our project of constructing an analytic framework for the study of classes of C*-algebras as geometric objects. While convergence for compact quantum metric has been an active area of our research with many developments, built on top of our noncommutative analogue of the Gromov-Hausdorff metric called the dual propinquity [16, 12, 20, 17, 13, 11, 3, 14], a natural and crucial question in noncommutative metric geometry concerned the behavior of modules over convergent sequences of quantum metric spaces. Indeed, modules encode much geometric information and are key ingredients for theories in mathematical physics, and thus for our research program to advance toward its goal, it is essential to develop a metric geometric theory for modules over C*-algebras. Remarkably, there seems to be no classical model to work from for such a new distance. We propose our answer, called the modular Gromov-Hausdorff propinquity, as a deep extension of the quantum propinquity in [15], and this paper provides the first deep example of convergence for our new metric. We proved in [11] that quantum tori form a continuous family for the quantum propinquity, and we now prove that Heisenberg modules over quantum two-tori also form continuous families for our new modular propinquity. As all finitely generated, projective modules over quantum two tori are sums of Heisenberg modules and free modules, and free modules were handled in [15], we thus provide in this paper a detailed picture of the metric geometry of the class of finitely generated, projective modules over quantum two-tori.
Noncommutative metric geometry [5, 28, 30] studies noncommutative generalizations of Lipschitz algebras, defined as:

**Definition 1.1.** An ordered pair \((\mathfrak{A}, L)\) is a Leibniz quantum compact metric space when \(\mathfrak{A}\) is a unital C*-algebra, whose norm we denote as \(\| \cdot \|_{\mathfrak{A}}\) and whose unit is denoted as \(1_{\mathfrak{A}}\), and \(L\) is a seminorm defined on a dense Jordan-Lie subalgebra \(\text{dom}(L)\) of the space of self-adjoint elements \(\text{sa}(\mathfrak{A})\) of \(\mathfrak{A}\) such that:

1. \(\{ a \in \text{dom}(L) : L(a) = 0 \} = \mathbb{R}1_{\mathfrak{A}}\),
2. the Monge-Kantorovich metric \(m_{\mathfrak{A}}L\) defined on the state space \(\mathcal{S}(\mathfrak{A})\) of \(\mathfrak{A}\) by setting, for any two \(\varphi, \psi \in \mathcal{S}(\mathfrak{A})\):
   \[
   m_{\mathfrak{A}}L(\varphi, \psi) = \sup \{ |\varphi(a) - \psi(a)| : a \in \text{dom}(L), L(a) \leq 1 \}
   \]
   metrizes the weak* topology restricted to \(\mathcal{S}(\mathfrak{A})\),
3. \(L\) is lower semi-continuous,
4. \(\max \{ L\left(\frac{ab+ba}{2}\right), L\left(\frac{ab-ba}{2i}\right)\} \leq \|a\|_{\mathfrak{A}} L(b) + \|b\|_{\mathfrak{A}} L(a)\).

Leibniz quantum compact metric spaces, and more generally quasi-Leibniz quantum compact metric spaces (a generalization we will not need in this paper), form a category with the appropriate notion of Lipschitz morphisms [22], containing such important examples as quantum tori [28], Connes-Landi spheres [23], group C*-algebras for Hyperbolic groups and nilpotent groups [29, 24], AF algebras [11], Podlès spheres [2], certain C*-crossed-products [1], among others. Any compact metric space \((X, d)\) give rise to the Leibniz quantum compact metric space \((C(X), \text{Lip})\) where \(C(X)\) is the C*-algebra of \(\mathbb{C}\)-valued continuous functions over \(X\), and \(\text{Lip}\) is the Lipschitz seminorm induced by \(d\).

Our research aims at providing a new analytical framework based on metric geometry for the study of quantum compact metric spaces, motivated by approximations problems from mathematical physics, and by the exploration of the metric aspects of noncommutative geometry [6]. To this end, we have developed an analogue of the Gromov-Hausdorff distance between quantum compact metric spaces [16, 12, 20], applied it to obtain various new continuity and approximation results for quantum metric spaces [10, 17, 13, 11, 14, 25], discovered analogues of important results in metric geometry such as Gromov compactness theorem [17, 14], showed new applications of our metric to group actions and approximations of symmetries in noncommutative geometry [19]. This paper is part of our discovery of a generalization of Leibniz quantum compact metric spaces to quantum vector bundles [15, 18], modeled after certain Hilbert modules over quantum metric spaces, and of the Gromov-Hausdorff propinquity to these metrized quantum vector bundles. This new metric, called the modular Gromov-Hausdorff propinquity, endows the class of Hilbert modules, with some metric information in the form of a generalized connection, with a topology which then enables us to discuss continuity of certain families of modules. Our modular propinquity is an additional step toward putting a topology on the sort of structures which arise in the description of theories in mathematical physics, as our main purpose for our research.

The present manuscript provides our first, principal example of convergence for the modular propinquity: the family of Heisenberg modules over quantum 2-tori. Finitely generated projective modules over irrational rotation algebras can be described, up to module isomorphism, as either free — a case with which we dealt in [15] — or constructed through a projective representation of \(\mathbb{R}^2\), as shown in [26]. The latter type of modules were introduced by Connes [4] and provided the
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background for the study of noncommutative geometry of quantum tori. The construction of these modules was later extended by Rieffel to all quantum tori in [27], where they provide a large class of (though in general, not all) projective finitely generated modules over quantum tori. As projective representations of \( \mathbb{R}^2 \) are in fact obtained from representations of the Heisenberg group, the modules constructed from these representations are called Heisenberg modules in [27], and we shall follow this terminology. In fact, the Heisenberg group action will prove essential to our construction.

The first step in bringing Heisenberg modules over quantum 2-tori into our framework is to turn them into metrized quantum vector bundles, which means that we must endow them with a certain type of norm called a D-norm. A natural way to construct D-norms involve the noncommutative analogues of connections. This delicate task was undertaken in the companion paper [18]. The metric structure of Heisenberg modules is actually provided by the connection introduced on these modules by Connes [4], which is also the connection solving the Yang-Mills problem over the quantum 2-tori in [7]. We recall the results of [18] at the beginning of this paper, including several lemmas and propositions which will prove helpful in our current work.

Once we bring Heisenberg modules within the realm of our modular propinquity, we become capable of discussing the problem of convergence of such modules. Heisenberg modules are parametrized by the quantum torus acting on them and by a pair of integers \( p, q \) which, in particular, relate to a projective representation of the finite group of the form \( \mathbb{Z}_q \), where \( \mathbb{Z}_q = \mathbb{Z} / q \mathbb{Z} \). We shall prove in this section, as our main application of the modular propinquity for this paper, that for a fixed pair \( p, q \) of integers, and thus in particular, for a fixed projective representation of some \( \mathbb{Z}_q \), the family of Heisenberg modules over varying quantum tori, form a continuous family for the modular propinquity. Informally, and a matter worthy of further investigation, our result can be seen as a sort of continuity of K-theory.

The strategy of this paper, which is reflected in its structure, begins with proving a continuous field type of result for the D-norms defined on Heisenberg modules. We then prove a form of uniform approximation for elements in Heisenberg modules using certain convolution-type operators. This involve the use of some harmonic analysis based upon the Hermite functions. We then bring all of this together in our main result.

As a matter of convention throughout this paper, we will use the following notations.

Notation 1.2. By default, the norm of a normed vector space \( E \) is denoted by \( \| \cdot \|_E \). When \( \mathfrak{A} \) is a C*-algebra, the space of self-adjoint elements of \( \mathfrak{A} \) is denoted by \( \mathfrak{sa}(\mathfrak{A}) \). The state space of \( \mathfrak{A} \) is denoted by \( \mathcal{S}(\mathfrak{A}) \). In this work, all C*-algebras \( \mathfrak{A} \) will always be unital with unit \( 1_\mathfrak{A} \).

Convention 1.3. If \( P \) is some seminorm on a vector subspace \( D \) of a vector space \( E \), then for all \( x \in E \setminus D \) we set \( P(x) = \infty \). With this in mind, the domain \( D \) of \( P \) is the set \( \{ x \in E : P(x) < \infty \} \), with the usual convention that \( 0 \cdot \infty = 0 \) while all other operations involving \( \infty \) give \( \infty \).

2. The modular Propinquity

At the core of our geometry for modules is quantum metric information, encoded on a module via the combination of left Hilbert structure and a special kind of norm
which generalizes the idea of a connection. The prototype for such a structure is the module \( IV \) of continuous sections of a vector bundle \( V \) over a compact Riemannian manifold \( M \) with metric \( g \), endowed with a hermitian metric \( h \) and some associated metric connection \( \nabla \). For any two \( \omega, \eta \in \Gamma V \), we then set \( \langle \omega, \eta \rangle_V : x \in M \mapsto \int_X h_x(\omega_x, \eta_x) \, d\text{Vol}(x) \) where \( \text{Vol} \) is the volume form over \( M \) for \( g \), which turns \( \Gamma V \) into a \( C(M) \)-left Hilbert module. We also define, for all \( \omega \in \mathcal{M} \), the norm \( D(\omega) \) as the operator norm for the operator \( \nabla \omega : X \in \Gamma(TM) \mapsto \nabla_X \omega \in \Gamma V \) — noting that the space of vector fields \( \Gamma TM \) of \( M \) has a norm induced by the metric \( g \). Our general definition for a metrized quantum vector bundle abstracts this picture. We will only encounter Leibniz metrized quantum vector bundles in this work (see [11, 3] for more general examples of metrized quantum vector bundles). We thus define:

**Definition 2.1** ([15, Definition 3.8]). A Leibniz metrized quantum vector bundle \( (\mathcal{M}, \langle \cdot, \cdot \rangle_{\mathcal{M}}, D, \mathfrak{A}, L) \) consists of:

1. a Leibniz quantum compact metric space \( (\mathfrak{A}, L) \) called the base space,
2. a \( \mathfrak{A} \)-left Hilbert module \( (\mathcal{M}, \langle \cdot, \cdot \rangle_{\mathcal{M}}) \),
3. a norm \( D \) defined on a dense subspace of \( \mathcal{M} \) such that \( D(\omega) \geq \sqrt{\langle \omega, \omega \rangle_{\mathcal{M}}} \) for all \( \omega \in \mathcal{M} \), and such that the set:
   \[ \{ \omega \in \mathcal{M} : D(\omega) \leq 1 \} \]
   is compact in \( \mathcal{M} \),
4. for all \( a \in \mathfrak{sa}(\mathfrak{A}) \) and for all \( \omega \in \mathcal{M} \), we have the inner Leibniz inequality:
   \[ D_{\mathcal{M}}(a\omega) \leq (\|a\|_\mathfrak{A} + L_{\mathfrak{A}}(a)) D_{\mathcal{M}}(\omega), \]
5. for all \( \omega, \eta \in \mathcal{M} \), we have the modular Leibniz inequality:
   \[ \max \{ L_{\mathfrak{A}}(R(\omega, \eta)_{\mathcal{M}}), L_{\mathfrak{A}}(\mathfrak{A}(\omega, \eta)_{\mathcal{M}}) \} \leq 2D_{\mathcal{M}}(\omega)D_{\mathcal{M}}(\eta). \]

We refer to [15] for a discussion of these objects. [15, Example 3.10] shows that the prototype of a hermitian vector bundle over a compact Riemannian manifold, as sketched above, is indeed an example of a metrized quantum vector bundle. We however emphasize that metrized quantum vector bundles contain their base space. We recall from [15] the type of morphisms between metrized quantum vector bundles which we will use in this paper. We begin with the definition we employ for module morphisms, which is designed to accommodate different base spaces.

**Definition 2.2** ([15, Definition 3.5]). Let \( (\mathcal{M}, \langle \cdot, \cdot \rangle_{\mathcal{M}}) \) and \( (\mathcal{N}, \langle \cdot, \cdot \rangle_{\mathcal{N}}) \) be two left Hilbert modules over, respectively, two C*-algebras \( \mathfrak{A} \) and \( \mathfrak{B} \). A module morphism \( (\Theta, \theta) \) is given by a \( \mathfrak{A} \)-algebra morphism \( \theta : \mathfrak{A} \to \mathfrak{B} \), and a \( \mathfrak{C} \)-linear map \( \Theta : \mathcal{M} \to \mathcal{N} \), such that for all \( a \in \mathfrak{A} \) and \( \omega, \eta \in \mathcal{M} \), we have:

1. \( \Theta(a\omega) = \theta(a)\Theta(\omega) \),
2. \( \langle \Theta(\omega), \Theta(\eta) \rangle_{\mathcal{N}} = \langle \omega, \eta \rangle_{\mathcal{M}} \).

The module morphism \( (\Theta, \theta) \) is unital when \( \theta \) is a unital \( \mathfrak{A} \)-morphism.

We thus have the following notion of morphism of metrized quantum vector bundles, designed to encompass the entirety of the underlying structure. In particular, the modular propinquity will be null exactly when two metrized quantum vector bundles will be isometric in the sense of the following definition.
Definition 2.3 ([15, Definition 3.18]). Let: 
\[ \Omega_\Theta = (\mathcal{M}, \langle \cdot, \cdot \rangle_\mathcal{M}, D_\mathcal{M}, \mathfrak{A}, L_\mathfrak{A}) \] and 
\[ \Omega_\Theta = (\mathcal{N}, \langle \cdot, \cdot \rangle_\mathcal{N}, D_\mathcal{N}, \mathfrak{B}, L_\mathfrak{B}) \]
be two metrized quantum vector bundles. A morphism \((\Theta, \theta)\) from \(\Omega_\Theta\) to \(\Omega_\Theta\) is a unit module morphism \((\Theta, \theta)\) such that:

1. \(\theta\) is continuous from \((\text{dom}(L_\mathfrak{A}), L_\mathfrak{A})\) to \((\text{dom}(L_\mathfrak{B}), L_\mathfrak{B})\), i.e. there exists \(C > 0\) such that \(L_\mathfrak{B} \circ \theta \leq C L_\mathfrak{A}\) on \(\text{dom}(L_\mathfrak{A})\),

2. \(\Theta\) is continuous from \((\text{dom}(D_\mathcal{M}), D_\mathcal{M})\) to \((\text{dom}(D_\mathcal{N}), D_\mathcal{N})\), i.e. there exists \(M > 0\) such that for all \(\omega \in \mathcal{M}\) we have \(D_\mathcal{N}(\Theta(\omega)) \leq M D_\mathcal{M}(\omega)\).

Such a morphisms is an epimorphism when both \(\theta\) and \(\Theta\) are surjective, and a monomorphism when both \(\theta\) and \(\Theta\) are both monomorphisms.

A isomorphism is thus given by a morphism \((\Theta, \theta)\) where \(\theta\) is a \(*\)-isomorphism, \(\Theta\) is a bijection and \((\Theta^{-1}, \theta^{-1})\) is a morphism from \(\Omega_\Theta\) onto \(\Omega_\Theta\).

We constructed in [15] a metric, called the modular Gromov-Hausdorff propinquity, on the class of Leibniz metrized quantum vector bundles. The construction of the modular Gromov-Hausdorff propinquity is involved, though for practical purposes, the key ingredient to understand is the notion of a modular bridge, which encodes the idea of a kind of noncommutative metric embedding of metrized quantum vector bundles. A modular bridge generalizes the notion of a bridge between Leibniz quantum compact metric spaces as proposed in [16] in the construction of the quantum propinquity. Thus, we shall extend this idea for our present construction.

Notation 2.4. If \(D\) is a norm on a vector space \(E\), we denote the unit ball in \(E\) for \(D\) by \(D_1(E)\).

Definition 2.5. Let \(\Omega_\mathcal{M} = (\mathcal{M}, \langle \cdot, \cdot \rangle_\mathcal{M}, D_\mathcal{M}, \mathfrak{A}, L_\mathfrak{A})\) and \(\Omega_\mathcal{N} = (\mathcal{N}, \langle \cdot, \cdot \rangle_\mathcal{N}, D_\mathcal{N}, \mathfrak{B}, L_\mathfrak{B})\) be two metrized quantum vector bundles. A modular bridge
\[ \gamma = (\mathfrak{D}, x, \pi_\mathfrak{A}, \pi_\mathfrak{B}, (\omega_j)_{j \in J}, (\eta_j)_{j \in J}) \]
from \(\Omega_\mathcal{M}\) to \(\Omega_\mathcal{N}\) is given by:

1. a unital \(C^*\)-algebra \(\mathfrak{D}\),
2. an element \(x \in \mathfrak{D}\), called the pivot of \(\gamma\), such that:
\[ \mathfrak{S}_1(\mathfrak{D}|\omega) := \{ \varphi \in \mathfrak{S}(\mathfrak{D}) | \forall a \in \text{sa}(\mathfrak{D}) \quad \varphi(a \omega) = \varphi(\omega a) = \varphi(a) \}
\]
is not empty,
3. two unital *-isomorphisms \(\pi_\mathfrak{A} : \mathfrak{A} \hookrightarrow \mathfrak{D}\) and \(\pi_\mathfrak{B} : \mathfrak{B} \hookrightarrow \mathfrak{D}\),
4. an index set \(J\) and two families \((\omega_j)_{j \in J}, (\eta_j)_{j \in J}\) in \(\mathcal{M}, \mathcal{N}\) such that for all \(j \in J\), we have \(\omega_j \in \mathfrak{S}_1(D_\mathcal{M}(\omega_j))\) and \(\eta_j \in \mathfrak{S}_1(D_\mathcal{N}(\eta_j))\).

The modular Gromov-Hausdorff distance is constructed from numbers attached to modular bridges. There are two numbers related to modules, and two more which come from the bridge between the quantum base spaces. The numbers related to the base space are simply inherited from our construction of the quantum propinquity [16], to which we refer for an explanation of their meaning.

Notation 2.6. The Hausdorff distance over the hyperspace of closed subsets of a metric space \((X, d)\) is denoted by \(\text{Haus}_d\), and if \(X\) is a normed vector space with norm \(\| \cdot \|_X\), the Hausdorff distance for the metric induced by \(\| \cdot \|_X\) is simply denoted by \(\text{Haus}_{\| \cdot \|_X}\).
**Definition 2.7.** Let \( \Omega_\mathfrak{A} = (\mathcal{M}, \langle \cdot, \cdot \rangle_\mathcal{M}, D_{\mathcal{M}}, \mathfrak{A}, L_\mathfrak{A}) \) and \( \Omega_\mathfrak{B} = (\mathcal{N}, \langle \cdot, \cdot \rangle_\mathcal{N}, D_{\mathcal{N}}, \mathfrak{B}, L_\mathfrak{B}) \) be two metrized quantum vector bundles and let:

\[
\gamma = (\mathcal{D}, x, \pi_\mathfrak{A}, \pi_\mathfrak{B}, (\omega_j)_{j \in J}, (\eta_j)_{j \in J})
\]

be a bridge from \( \Omega_\mathfrak{A} \) to \( \Omega_\mathfrak{B} \).

1. The **bridge seminorm** \( b_\gamma, (\cdot, \cdot) \) of \( \gamma \) is the seminorm defined on \( \mathfrak{A} \oplus \mathfrak{B} \) by setting, for all \( a \in \mathfrak{A} \) and \( b \in \mathfrak{B} \):

\[
b_\gamma (a, b) = \| \pi_\mathfrak{A}(a) x - x \pi_\mathfrak{B}(b) \|_\mathcal{D}.
\]

2. The **basic reach** of \( \gamma \) is the nonnegative number:

\[
g_b (\gamma) = \max \left\{ \sup_{a \in \mathfrak{A}} \inf_{b \in \mathfrak{B}} b_\gamma (a, b), \sup_{b \in \mathfrak{B}} \inf_{a \in \mathfrak{A}} b_\gamma (a, b) \right\}.
\]

3. The **height** of \( \gamma \) is the nonnegative number:

\[
\varsigma (\gamma) = \max \left\{ \text{Haus}_{\text{mk}_\mathfrak{A}} (\mathcal{F}(\mathfrak{A}), \{ \varphi \circ \pi_\mathfrak{A} : \varphi \in \mathcal{F}_1 (\mathcal{D}) \}), \right. \\
\left. \text{Haus}_{\text{mk}_\mathfrak{B}} (\mathcal{F}(\mathfrak{B}), \{ \varphi \circ \pi_\mathfrak{B} : \varphi \in \mathcal{F}_1 (\mathcal{D}) \}) \right\}.
\]

The data from modules enter in the following measurements associated with a modular bridge:

**Definition 2.8.** Let \( \Omega_\mathfrak{A} = (\mathcal{M}, \langle \cdot, \cdot \rangle_\mathcal{M}, D_{\mathcal{M}}, \mathfrak{A}, L_\mathfrak{A}) \) and \( \Omega_\mathfrak{B} = (\mathcal{N}, \langle \cdot, \cdot \rangle_\mathcal{N}, D_{\mathcal{N}}, \mathfrak{B}, L_\mathfrak{B}) \) be two metrized quantum vector bundles and let:

\[
\gamma = (\mathcal{D}, x, \pi_\mathfrak{A}, \pi_\mathfrak{B}, (\omega_j)_{j \in J}, (\eta_j)_{j \in J})
\]

be a bridge from \( \Omega_\mathfrak{A} \) to \( \Omega_\mathfrak{B} \).

1. The modular reach \( \gamma \) is:

\[
g^\mathcal{M} (\gamma) = \max \left\{ b_\gamma (\omega_j, \omega_k), \langle \eta_j, \eta_k \rangle_\mathcal{N} : j, k \in J \right\}.
\]

2. Let \( k_{D_{\mathcal{M}}} \) be the metric defined on \( \mathcal{M} \) by:

\[
k_{D_{\mathcal{M}}} : \omega, \eta \in \mathcal{M} \mapsto \sup \{|\langle \omega - \eta, \xi \rangle_\mathcal{M} | : \xi \in \mathcal{M}, D_{\mathcal{M}} (\xi) \leq 1 \}.
\]

and similarly for \( k_{D_{\mathcal{N}}} \). The imprint of \( \gamma \) is:

\[
\varpi (\gamma) = \max \left\{ \text{Haus}_{k_{D_{\mathcal{M}}}} (\{ \omega_j : j \in J \}, \{ \omega \in \mathcal{M} : D_{\mathcal{M}} (\omega) \leq 1 \}), \right. \\
\left. \text{Haus}_{k_{D_{\mathcal{N}}}} (\{ \eta_j : j \in J \}, \{ \eta \in \mathcal{N} : D_{\mathcal{N}} (\eta) \leq 1 \}) \right\}.
\]

3. The reach of \( \gamma \) is:

\[
g (\gamma) = \max \{ g_b (\gamma), g^\mathcal{M} (\gamma) + \varpi (\gamma) \}.
\]

4. The **length** of \( \gamma \) is:

\[
\lambda (\gamma) = \max \{ \varsigma (\gamma), g (\gamma) \}.
\]

We summarize the key ingredients needed for our work with the following characterization of the modular Gromov-Hausdorff propinquity:

**Theorem-Definition 2.9.** There exists a class function \( \Lambda^\text{mod} \) which, to any pair of \( (F, G, H) \)-metrized quantum vector bundles \( \Omega_\mathfrak{A}, \Omega_\mathfrak{B} \), associated a nonnegative number \( \Lambda^\text{mod} (\Omega_\mathfrak{A}, \Omega_\mathfrak{B}) \), such that:
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\[ \Lambda^{\text{mod}}(\Omega_A, \Omega_B) < \infty, \]

\[ \Lambda^{\text{mod}}(\Omega_A, \Omega_B) \leq \lambda(\gamma) \text{ for any modular bridge } \gamma \text{ from } \Omega_A \text{ to } \Omega_B, \]

\[ \Lambda^{\text{mod}}(\Omega_A, \Omega_B) = \Lambda^{\text{mod}}(\Omega_B, \Omega_A), \]

\[ \Lambda^{\text{mod}}(\Omega_A, \Omega_B) \leq \Lambda^{\text{mod}}(\Omega_A, \Omega_D) + \Lambda^{\text{mod}}(\Omega_D, \Omega_A) \text{ for any } (F, G, H) \text{-metrized quantum vector bundle } \Omega_D, \]

\[ \Lambda^{\text{mod}}(\Omega_A, \Omega_B) = 0 \text{ if and only if } \Omega_A \text{ and } \Omega_B \text{ are isomorphic as metrized quantum vector bundles.} \]

\( \Lambda^{\text{mod}} \) is the largest class function satisfying all the above conditions.

The notable properties of our metric is that it is indeed null only between fully quantum isometric modules, and satisfy the triangle inequality. Upper bounds for the propinquity between two metrized quantum vector bundles can be computed using modular bridges. In this paper, we will indeed construct bridges between Heisenberg modules, which will provide us with our main continuity result.

3. Background on Heisenberg modules as metrized quantum vector bundles

The Heisenberg group is the Lie group given by:

\[ H_3 = \left\{ \begin{pmatrix} 1 & x & u \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, u \in \mathbb{R} \right\}. \]

We shall identify \( H_3 \) with \( \mathbb{R}^3 \) via the natural map \( (x, y, u) \in \mathbb{R}^3 \mapsto \begin{pmatrix} 1 & x & u \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \),

which is a Lie group isomorphism once we equip \( \mathbb{R}^3 \) with the multiplication:

\[ (x_1, y_1, u_1)(x_2, y_2, u_2) = (x_1 + x_2, y_1 + y_2, u_1 + u_2 + x_1 y_2) \]

for all \( (x_1, y_1, u_1), (x_2, y_2, u_2) \in \mathbb{R}^3 \).

The Heisenberg group \( H_3 \) is a Lie group with Lie algebra isomorphic (as a Lie algebra) to the span of:

\[ P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } T = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \]

We note that the set \( \{P, Q\} \) actually generated the Lie algebra of \( H_3 \) since \([P, Q] = T\) and \( T \) is central. Moreover, a quick computation shows that for all \( x, y \in \mathbb{R} \):

\[ \exp_{H_3}(xP + yQ) = \begin{pmatrix} 1 & x & u \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \]

where \( \exp_{\text{Heisenberg Group}} \) is the exponential map from the Heisenberg Lie algebra to \( H_3 \).

If we set, for any \( \bar{\sigma} \in \mathbb{R} \setminus \{0\} \) and \( \xi \in L^2(\mathbb{R}) \):

\[ \alpha_{\bar{\sigma},1}^{x,y,u} : s \in \mathbb{R} \mapsto \exp(2i\pi(\bar{\sigma}u + sx))\xi(s + \bar{\sigma}y) \]

then we define a unitary representation of \( H_3 \), and any nontrivial irreducible unitary representations of the Heisenberg group is unitarily equivalent to \( \alpha_{\bar{\sigma},1} \) for some \( \bar{\sigma} \neq 0 \) [8]. We note that they all are infinite dimensional (the other, trivial, unitary representations of \( H_3 \) are one-dimensional).
Let $\bar{\theta} \in \mathbb{R} \setminus \{0\}$. For all $(x, y) \in \mathbb{R}^2$ and for all $\xi \in L^2(\mathbb{R})$, set:

$$\sigma_{\bar{\theta}, 1}^{x, y} \xi = \exp_{\bar{\theta}, 1}(xP + yQ) \xi = \sigma_{\bar{\theta}, 1}^{x, y} : s \in \mathbb{R} \mapsto \exp(i\bar{\theta}xy + 2i\pi sx)\xi(s + \bar{\theta}y).$$

The map $\sigma_{\bar{\theta}, 1}^{x, y}$ is a unitary on $L^2(\mathbb{R})$ for all $(x, y) \in \mathbb{R}^2$. Moreover, for all $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$, we note that:

$$\sigma_{\bar{\theta}, 1}^{x_1, y_1} \sigma_{\bar{\theta}, 1}^{x_2, y_2} = e_\theta((x_1, y_1), (x_2, y_2)) \sigma_{\bar{\theta}, 1}^{x_1 + x_2, y_1 + y_2},$$

where:

$$e_\theta : ((x_1, y_1), (x_2, y_2)) \in \mathbb{R}^2 \times \mathbb{R}^2 \mapsto \exp(i\pi \theta(x_2y_1 - x_1y_2)).$$

Therefore, $\sigma_{\bar{\theta}, 1}$ is a projective representation of $\mathbb{R}^2$ on $L^2(\mathbb{R})$ for the bicharacter $e_\theta$, namely the Schrödinger representation of “Plank constant” $\bar{\theta}$. Moreover, every nontrivial irreducible unitary projective representation of $\mathbb{R}^2$ is unitarily equivalent to one of $\sigma_{\bar{\theta}, 1}$ for some $\bar{\theta} \neq 0$ (here, by nontrivial, we mean associated with a nontrivial cocycle).

The quantum 2-torus $A_\theta$ is the universal $C^*$-algebras generated by two unitaries $U, V$ such that $UV = \exp(2i\pi \theta)VU$, for some $\theta \in \mathbb{R}$. These $C^*$-algebras are equivalently described as the twisted convolution $C^*$-algebras $C^*(\mathbb{Z}^2, e_\theta)$. They are the prototypes of a noncommutative manifold $[5, 6]$, and have been extensively studied. In particular, the class of finitely generated projective modules over a quantum 2-torus consists, up to module isomorphism, of free finitely generated modules, and Heisenberg modules, which we now describe. We will use the notations of Theorem-Definition (3.1) all throughout this paper.

**Theorem-Definition 3.1 ( [4], [26], [7] ).** Let $\theta \in \mathbb{R}$ and $q \in \mathbb{N} \setminus \{0\}$. Let $p \in \mathbb{Z}$, $q \in \mathbb{N} \setminus \{0\}$, and let $d \in q\mathbb{N} \setminus \{0\}$. The Heisenberg module $\mathcal{H}_\theta^{p, q, d}$ is the module over $A_\theta$ defined as follows.

Let $\mathbb{Z}_q = \mathbb{Z}/q\mathbb{Z}$, and let:

$$u_{p, q} = \begin{pmatrix} 1 & z & z^2 & \cdots & z^{q-1} \\ 1 & 0 & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 1 \end{pmatrix}$$

and

$$v_{p, q} = \begin{pmatrix} 0 & \cdots & \cdots & \cdots & 1 \\ 1 & 0 & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 1 \end{pmatrix},$$

with $z = \exp\left(\frac{2i\pi p}{q}\right)$. Since $u_{p, q}^q = v_{p, q}^q = 1$, the map:

$$\rho_{p, q, 1} : (z, w) \in \mathbb{Z}_q^2 \mapsto \rho_{p, q, 1}^{z, w} = \exp\left(\frac{i\pi pmn}{q}\right)u_{p, q}^n v_{p, q}^m$$

where $|n| = z$ and $|m| = w$

is a well-defined projective representation of $\mathbb{Z}_q^2$.

For all $d \in q\mathbb{N}$, $d > 0$, we now set:

$$\rho_{p, q, d}^{n, m} = \rho_{p, q, 1}^{n, m} \otimes \text{id}_{\mathbb{C}_d^2}$$

where $\text{id}_{\mathbb{C}}$ is the identity map on $\mathbb{C}_d^2$.

Let:

$$\bar{\theta} = \theta - \frac{p}{q}.$$
Let \( \alpha_{0,1} \) be the action of the Heisenberg group \( \mathbb{H}_3 \) on \( L^2(\mathbb{R}) \) given by Expression (3.1).

For \((n,m) \in \mathbb{Z}^2\), denoting the class of \( n \) and \( m \) in \( \mathbb{Z}/q\mathbb{Z} \), respectively, by \([n]\) and \([m]\), we set:

\[
\varpi^{n,m}_{p,q,d} = \sigma^{n,m}_{0,1} \otimes \rho^{[n],[m]}_{p,q,d}.
\]

For all \( n,m \in \mathbb{Z} \), the map \( \varpi^{n,m}_{p,q,d} \) is a unitary of \( L^2(\mathbb{R}) \otimes \mathbb{C}^d \), and moreover \( \varpi^{n,m}_{p,q,d} \) is an \( \mathbb{C}_q \)-projective representation of \( \mathbb{Z}^2 \).

By universality, the Hilbert space \( L^2(\mathbb{R}) \otimes \mathbb{C}^d \) is a module over \( \mathcal{A}_\theta \), with, in particular, for all \( f \in \ell^1(\mathbb{Z}^2) \) and \( \xi \in L^2(\mathbb{R}, \mathbb{C}^d) = L^2(\mathbb{R}) \otimes \mathbb{C}^d \):

\[
f \xi = \sum_{n,m \in \mathbb{Z}} f(n,m) \varpi^{n,m}_{p,q,d} \xi.
\]

Let \( \mathcal{H}^{p,q,d}_{\theta} = \mathcal{S}(\mathbb{C}^d) \subseteq L^2(\mathbb{R}) \otimes \mathbb{C}^d \) be the space of all \( \mathbb{C}^d \)-valued Schwarz functions on \( \mathbb{R} \), i.e.:

\[
\mathcal{H}^{p,q,d}_{\theta} = \left\{ \xi : \mathbb{R} \to \mathbb{C}^d \mid \xi \text{ is infinitely differentiable on } \mathbb{R} \text{ and } \forall p \in \mathbb{R} [X], n \in \mathbb{N} \lim_{s \to \pm \infty} p(s) \xi^{(n)}(s) = 0 \right\}.
\]

For all \( \xi, \omega \in \mathcal{H}^{p,q,d}_{\theta} \), define \( (\xi, \omega)_{\mathcal{H}^{p,q,d}_{\theta}} \) as the function in \( \ell^1(\mathbb{Z}^2) \) given by:

\[
(\xi, \omega)_{\mathcal{H}^{p,q,d}_{\theta}} : (n,m) \in \mathbb{Z}^2 \mapsto \left( \varpi^{n,m}_{p,q,d} \xi, \omega \right)_{L^2(\mathbb{R}) \otimes E}.
\]

The Heisenberg module \( \mathcal{H}^{p,q,d}_{\theta} \) is the completion of \( \mathcal{H}^{p,q,d}_{\theta} \) for the norm associated with the \( \mathcal{A}_\theta \)-inner product \( \langle \cdot, \cdot \rangle_{\mathcal{H}^{p,q,d}_{\theta}} \).

In [18], we proved that the norms of Heisenberg modules have a natural continuity property, a fact which will be important for our present work.

**Proposition 3.2** (Proposition 3.5)). Let \( p, q \in \mathbb{N} \) and \( d \in q\mathbb{N} \) with \( d > 0 \). Let \( (\xi_k)_{k \in \mathbb{N}} \) be a family in \( \mathcal{S}(\mathbb{C}^d) \) such that \( (k,t) \in \mathbb{N} \times \mathbb{R} \mapsto \xi_k(t) \) is (jointly) continuous and there exists \( M > 0 \) such that \( \|\xi_k^{(s)}(t)\|_{\mathbb{C}^d} \leq \frac{M}{1 + t^2} \) for all \( k \in \mathbb{N} \), \( t \in \mathbb{R} \) and \( s \in \{0, 1, 2\} \).

If \( (\theta_k)_{k \in \mathbb{N}} \) is a sequence in \( \mathbb{R} \) converging to \( \theta_{\infty} \) and such that \( \theta_k - \theta_q = 0 \) for all \( k \in \mathbb{N} \), then:

\[
\lim_{k \to \infty} \|\xi_k\|_{\mathcal{H}^{p,q,d}_{\theta_k}} = \|\xi_{\infty}\|_{\mathcal{H}^{p,q,d}_{\theta_{\infty}}}.
\]

We also proved in [18] that the action of the Heisenberg group \( \mathbb{H}_3 \) on Heisenberg modules is strongly continuous, as part of the following Proposition which will also help us with our current work.

**Proposition 3.3** (Proposition 4.8)). Let \( p \in \mathbb{Z}, q \in \mathbb{N} \setminus \{0\} \) and \( d \in q\mathbb{N} \) with \( d > 0 \). Let \( C > 0 \) and \( M > 0 \) some constant. Let \( 0 < \partial_- < \partial_+ \). There exists \( K > 0 \) such that for all \( \xi \in \mathcal{S}(\mathbb{C}^d) \) satisfying:

\[
\max \left\{ \|\xi(s)\|_{\mathbb{C}^d}, \|s\xi(s)\|_{\mathbb{C}^d}, \|\xi'(s)\|_{\mathbb{C}^d}, \|s\xi'(s)\|_{\mathbb{C}^d}, \|\xi''(s)\|_{\mathbb{C}^d}, \|s\xi''(s)\|_{\mathbb{C}^d}, \right\} \leq \frac{M}{1 + s^2},
\]
the following holds for all \( s \in \mathbb{R}, \bar{\theta} \in [\bar{\theta}_-, \bar{\theta}_+] \) and \((x, y, u) \in \mathbb{R}^3\) with \(|x| + |y| + |u| \leq C\):

\[
\max \left\{ \left\| \alpha_{\bar{\theta}, d}^{x,y,u} \xi(n)(s) - \xi(n)(s) \right\|_{C^2} : n \in \{0, 1, 2\} \right\} \leq \frac{K(|x| + |y| + |u|)}{1 + s^2}.
\]

In particular, for all \( \bar{\theta} \neq 0 \) and \( \theta = \bar{\theta} + \frac{p}{q} \):

\[
\lim_{(x, y, u) \to 0} \left\| \alpha_{\bar{\theta}, d}^{x,y,u} \xi - \xi \right\|_{\mathcal{M}^{p,q,d}} = 0.
\]

Both Propositions (3.2) and (3.3) employ a particular lemma from [21] which we will use in our proof of the main theorem of this paper as well; we therefore include it here for our reader’s convenience.

**Lemma 3.4** ([18, Lemma 3.2]). Let \( p, q \in \mathbb{N} \) with \( q > 0 \) and \( d \in q\mathbb{N} \) with \( d > 0 \). If \((\xi_k)_{k \in \mathbb{N}}\) is a family of \( \mathbb{C}^d\)-valued \( C^2\)-functions over \( \mathbb{R} \) such that:

1. there exists \( M > 0 \) such that for all \( k \in \mathbb{N} \) and \( t \in \mathbb{R} \):

\[
\max \{ \| \xi_k(t) \|_{C^2}, \| \xi'_k(t) \|_{C^2}, \| \xi''_k(t) \|_{C^2} \} \leq \frac{M}{1 + t^2},
\]

2. \((t, k) \in \mathbb{R} \times \mathbb{N} \to \xi_k(t)\) is continuous,

and if \((\theta_k)_{k \in \mathbb{R}}\) is a sequence converging to \( \theta_\infty \) such that \( \theta_k - \frac{p}{q} \neq 0 \) for all \( k \in \mathbb{N} \), then we have:

\[
\lim_{k \to \infty} \left\| \langle \xi_k, \xi_k \rangle_{\mathcal{M}^{p,q,d}} - \langle \xi_\infty, \xi_\infty \rangle_{\mathcal{M}^{p,q,d}} \right\|_{L^1(\mathbb{Z}^2)} = 0.
\]

Quantum 2-tori are naturally endowed with a Leibniz quantum compact metric space structure. Fix \( \theta \in \mathbb{R} \). By universality of the quantum 2-tori, for any \((z_1, z_2) \in \mathbb{T}^2\), the following relations define a unique \(*\)-automorphism of \( A_\theta\):

\[
\beta^{z_1,z_2}(U) = z_1 U \quad \text{and} \quad \beta^{z_1,z_2}(V) = z_2 V
\]

and \( \beta : (z_1, z_2) \in \mathbb{T}^2 \to \beta^{z_1,z_2} \) thus defined is a strongly continuous action of the Lie group \( T^2 \) on \( A_\theta \) by \(*\)-automorphisms. For any continuous length function \( \ell \) on \( T^2 \), if we set:

\[
\forall a \in A_\theta \quad L(a) = \sup \left\{ \frac{\| a - \beta^z(a) \|_{A_\theta}}{\ell(z)} : z \in \mathbb{T}^2, z \neq (1, 1) \right\}
\]

then by [28], the pair \((A_\theta, L)\) is an Leibniz quantum compact metric space.

The principal result of our work in [18] is that, putting the above ingredients together, we proved that Heisenberg modules are indeed metrized quantum vector bundles over quantum 2-tori, for the appropriate choice of a D-norm.

**Definition 3.5** ([18, Definition 6.1]). Let \( p \in \mathbb{Z}, q \in \mathbb{N} \setminus \{0\} \) and \( d \in q\mathbb{N} \) with \( d > 0 \). Let \( \theta \in \mathbb{R} \setminus \left\{ \frac{p}{q} \right\} \). Let \( \| \cdot \| \) be a norm on \( \mathbb{R}^2 \). We endow the Heisenberg module \( \mathcal{M}_{\theta}^{p,q,d} \) with the norm:

\[
D_{\theta}^{p,q,d}(\xi) = \sup \left\{ \| \xi \|_{\mathcal{M}_{\theta}^{p,q,d}}, \frac{\| \alpha_{\bar{\theta}, d}^{x,y}(\exp^{P+yQ}_\theta) \xi - \xi \|_{\mathcal{M}_{\theta}^{p,q,d}}}{2\pi |\bar{\theta}| \| (x, y) \|} : (x, y) \in \mathbb{R}^2 \setminus \{0\} \right\}
\]

where \( \bar{\theta} = \theta - \frac{p}{q} \).
We will employ the following equivalent formulation of our D-norm in this work.

**Lemma 3.6 ([18]).** For all \( p, q \in \mathbb{N}, d \in q\mathbb{N} \) with \( d > 0 \), \( \theta \in \mathbb{R} \setminus \{pq^{-1}\} \) and \( \xi \in \mathcal{H}^{p,q,d}_\theta \), the following identities hold:

\[
D^{p,q,d}_\theta(\xi) = \sup \left\{ \|\xi\|_{\mathcal{H}^{p,q,d}_\theta}, \frac{\|x^\theta + y\xi\|_{\mathcal{H}^{p,q,d}_\theta}}{2\pi|\partial|(x,y)} : (x,y) \in \mathbb{R}^2 \setminus \{0\} \right\}
\]

\[
= \sup \left\{ \|\xi\|_{\mathcal{H}^{p,q,d}_\theta}, \frac{\|x^\theta + y\xi\|_{\mathcal{H}^{p,q,d}_\theta}}{2\pi|\partial|(x,y)} : (x,y) \in \mathbb{R}^2 \setminus \{0\} \right\}.
\]

**Theorem 3.7 ([18, Theorem 6.11]).** Let \( \mathcal{H}^{p,q,d}_\theta \) be the Heisenberg module over \( \mathcal{A}_\theta \) for some \( \theta \in \mathbb{R}, p \in \mathbb{Z}, q \in \mathbb{N} \setminus \{0\} \) and \( d \in q\mathbb{N} \setminus \{0\} \). Let \( \tilde{\sigma} = \theta - \frac{p}{q} \) and assume \( \tilde{\sigma} \neq 0 \). Let \( \|\cdot\| \) be a norm on \( \mathbb{R}^2 \). If we set, for all \( \xi \in \mathcal{H}^{p,q,d}_\theta \):

\[
D^{p,q,d}_\theta(\xi) = \sup \left\{ \|\xi\|_{\mathcal{H}^{p,q,d}_\theta}, \frac{\|\xi\|_{\mathcal{H}^{p,q,d}_\theta}}{2\pi|\partial|(x,y)} : (x,y) \in \mathbb{R}^2 \setminus \{0\} \right\},
\]

and for all \( a \in \mathcal{A}_\theta \):

\[
L_\theta(a) = \sup \left\{ \frac{\|\beta^{\exp(ix),\exp(iy)}a - a\|_{\mathcal{A}_\theta}}{\|(x,y)\|} : (x,y) \in \mathbb{R}^2 \setminus \{0\} \right\}
\]

then \( (\mathcal{H}^{p,q,d}_\theta, \langle \cdot , \cdot \rangle_{\mathcal{H}^{p,q,d}_\theta}, D^{p,q,d}_\theta, \mathcal{A}_\theta, L_\theta) \) is a Leibniz metrized quantum vector bundle.

**Theorem** (3.7) utilizes several lemmas from [18] which we will need for our current work. At the core of the proof of Theorem (3.7) are certain convolution-style operators which will act as “approximation operators” by giving a systematic way to approximate in norm elements in Heisenberg modules by finite dimensional vector spaces, with the approximation being controlled by the D-norm.

**Hypothesis 3.8.** Let \( p \in \mathbb{Z}, q \in \mathbb{N} \setminus \{0\} \), and let \( d \in q\mathbb{N} \) with \( d > 0 \). Let \( \theta \in \mathbb{R} \setminus \{\frac{p}{q}\} \). We write \( \tilde{\sigma} = \theta - \frac{p}{q} \).

We shall employ the notations of Theorem-Definition (3.1).

**Lemma 3.9 ([18, Lemma 6.5]).** Assume Hypothesis (3.8). If \( f \in L^1(\mathbb{R}^2) \) and:

\[
\sigma^f_{\tilde{\sigma},d} = \int_{\mathbb{R}^2} f(x,y)\alpha_{\tilde{\sigma},d}^{x,y} \frac{x}{\pi} \, dx \, dy
\]

then \( \sigma^f_{\tilde{\sigma},d} \) is a well-defined operator on \( \mathcal{H}^{p,q,d}_\theta \) and \( \|\sigma^f_{\tilde{\sigma},d}\|_{\mathcal{H}^{p,q,d}_\theta} \leq \|f\|_{L^1(\mathbb{R}^2)} \).

**Lemma 3.10 ([18, Lemma 6.6]).** Assume Hypothesis (3.8). Let \( \varepsilon > 0 \). If \( f : \mathbb{R}^2 \to [0, \infty) \) is measurable and satisfies:

1. \( \int_\mathbb{R}^2 f = 1 \),
2. \( \int f(x,y)(x,y) \, dx \, dy \leq \frac{\varepsilon}{2\pi} \),
Proof. Let \( r \in \mathbb{R} \) such that, if \( r \in \mathbb{R} \mapsto rf(r) \) is Lebesgue integrable, and if we set:

\[
f^\circ : (x, y) \in \mathbb{R}^2 \mapsto f \left( \sqrt{x^2 + y^2} \right),
\]

then the operator \( \sigma_{\alpha, d}^f \) is a compact operator for the Banach space \( \mathcal{H}_d^{p,q,d} \).

We note that while in general, the action of the Heisenberg groups on Heisenberg modules is not by isometry of our D-norms, we can use our approximation operators as near isometries. This begins the new material for this paper.

Lemma 3.12. Let \( p \in \mathbb{Z}, q \in \mathbb{N} \) and \( d \in \mathbb{N} \setminus \{0\} \). For all \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that, if \( f : \mathbb{R}^2 \to \mathbb{R} \) is an integrable function supported on a \( \mathbb{R}^2[0, \delta] \), and if \( \delta \neq 0 \), then for all \( \xi \in \mathcal{S}(\mathbb{C}^d) \), we have:

\[
D_{\mathcal{H}_d^{p,q,d}}^{p,q,d}(\sigma_{\alpha, d}^f \xi) \leq (1 + \varepsilon)D_{\mathcal{H}_d^{p,q,d}}^{p,q,d}(\xi).
\]

Proof. Let \( \delta \in \mathbb{R} \setminus \{0\} \) and let \( \theta = \frac{q}{q} + \delta \). We first record that for all \( (x, y, u), (z, w, v) \in \mathbb{H}_3 \):

\[
\alpha_{\delta, d}^x, \alpha_{\delta, d}^z = \exp \left( 2i\pi \delta (xw - yz) \right) \alpha_{\delta, d}^x, \alpha_{\delta, d}^z.
\]

Next, we denote by \( \| \cdot \|_2 \) the standard Euclidean norm on \( \mathbb{R}^2 \) and, since \( \mathbb{R}^2 \) is finite dimensional, we can find \( k > 0 \) such that \( \| \cdot \|_2 \leq k \| \cdot \| \). For all \( x, y, z, w \in \mathbb{R} \), we then compute:

\[
|\exp(2i\pi \delta (xw - yz)) - 1| = 2|\sin(\pi(xw - yz))| \leq 2|\delta||xw - yz| \leq 2|\delta||x, y_2||z, w_2| \leq 2|\delta||x, y_2||z, w_2|.
\]

Let \( \varepsilon > 0 \). Let \( f \) be an integrable function supported on \( \mathbb{R}^2[0, \delta] \). For all \( (x, y, u) \in \mathbb{R}^2[0, \delta] \) and \( (x, y) \in \mathbb{R}^2 \) with \( \|(x, y)\| \leq \delta = \frac{2}{\pi} \), we compute:

\[
\left\| \alpha_{\delta, d}^x \frac{x, y, z}{\mathbb{R}} \left( \int_K f(z, w) \alpha_{\delta, d}^z, \frac{z}{\mathbb{R}} \xi \, dx \, dy \right) - \int f(z, w) \alpha_{\delta, d}^z, \frac{z}{\mathbb{R}} \xi \, dx \, dy \right\| \leq \int_K f(z, w) |\exp(2i\pi \delta (xy - zw)) - 1| \left\| \alpha_{\delta, d}^z, \frac{z}{\mathbb{R}} \xi \right\| \, dw \, dz + \int_K f(z, w) \left\| \alpha_{\delta, d}^z, \frac{z}{\mathbb{R}} \left( \alpha_{\delta, d}^x, \frac{x}{\mathbb{R}} \xi - \xi \right) \right\| \, dw \, dz \leq 2k|\delta| \|\alpha, y_2\| \int_K f(z, w) \xi \, dx \, dy + \int f(2i\pi \delta ||(x, y)|| D_{\mathcal{H}_d^{p,q,d}}^{p,q,d}(\xi) \leq (k\delta + 1)2\pi|\delta| ||(x, y)|| D_{\mathcal{H}_d^{p,q,d}}^{p,q,d}(\xi) \leq (\varepsilon + 1)2\pi|\delta| ||(x, y)|| D_{\mathcal{H}_d^{p,q,d}}^{p,q,d}(\xi).
\]
By Definition (3.5), we now have:

$$D(\sigma_{\delta,d}^f, \xi) = \sup \left\{ \left\| \sigma_{\delta,d}^f - \sigma_{\delta,d}^f \xi - \sigma_{\delta,d}^f \right\|_{d} : x, y \in \mathbb{R} \setminus \{0\} \right\}$$

$$\leq (\varepsilon + 1) D_{\delta,d}^{p,q} (\xi).$$

Our lemma is now proven. \(\square\)

4. A CONTINUOUS FIELD OF D-NORMS

Our first step in establishing a continuity result for D-norms on Heisenberg modules is to reformulate the expression of our D-norms.

Lemma 4.1. Let \(p \in \mathbb{Z}, q \in \mathbb{N} \setminus \{0\}\) and \(d \in q\mathbb{N} \setminus \{0\}\). Let \(r : \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R} \setminus \{0\}\) be a continuous function.

If \(\xi \in \mathcal{S}(\mathbb{C}^d)\), then for all \((x, y) \in \mathbb{R}^2\) with \(\|(x, y)\| = 1\), \(\delta \in \mathbb{R} \setminus \{0\}\), the function:

$$t \in (0, \infty) \mapsto \omega_{x,y,t,\delta} = \sigma_{\delta,d}^{r(x,y)\xi - \xi_{r(\delta)}} / 2\pi t$$

where \(\xi_{r(\delta)} : t \in \mathbb{R} \mapsto \xi(r(\delta)t)\), can be extended by continuity at 0. Moreover, for all \(\delta \in \mathbb{R} \setminus \{0\}\):

$$D_{\delta,d}^{p,q,d} (\xi_{r(\delta)}) = \sup \left\{ \|\xi_{r(\delta)}\|_{\mathcal{H}_{p,q}^{d}}, \|\omega_{x,y,t,\delta}\|_{\mathcal{H}_{p,q}^{d}} : (x, y, t) \in \mathbb{R}^2, \|(x, y)\| = 1, t \in [0, 1] \right\}$$

and

$$(x, y, t, \delta) \in \mathbb{R}^2 [0, 1] \times \mathbb{R} \times (\mathbb{R} \setminus \{0\}) \mapsto \langle \omega_{x,y,t,\delta}, \omega_{x,y,t,\delta} \rangle_{\mathcal{H}_{p,q}^{d}}$$

is continuous to \(\ell^1(\mathbb{Z}^2), \|\cdot\|_{\ell^1(\mathbb{Z}^2)}\).

Proof. We begin by setting a domain over which we shall study our functions \(\omega\). For our purpose, we choose some arbitrary \(\delta_{\infty} \neq 0\) and then \(0 < \delta_- < \delta_+\) such that \(\|\delta_{\infty}\| \in (\delta_-, \delta_+).\) We set:

$$\Omega = \{(x, y, \delta) \in \mathbb{R}^3 : \|(x, y)\| = 1, |\delta| \in [\delta_-, \delta_+]\}$$

while:

$$\Sigma = \{(x, y, t, \delta) \in \mathbb{R}^4 : (x, y, \delta) \in \Omega, t \in [0, 1]\}$$

and:

$$\Sigma_* = \{(x, y, t, \delta) \in \mathbb{R}^4 : (x, y, \delta) \in \Omega, t \in [0, 1]\}.$$ 

Let \(\xi \in \mathcal{S}(\mathbb{C}^d)\) and let \(M_0 > 0\) be chosen so that for all \(s \in \mathbb{R}\):

$$\max\{\|\xi^{(n)}(s)\|_{\mathbb{C}^d}, \|s\xi^{(n)}(s)\|_{\mathbb{C}^d} : n \in \{0, 1, 2, 3, 4\}\} \leq \frac{M_0}{1 + s^2}.$$ 

Now, \(r\) is continuous on \([\delta_-, \delta_+]\), and thus there exists \(R_-, R_+ > 0\) such that \(R_- \leq r(\delta) \leq R_+\) for all \(\delta \in [\delta_-, \delta_+].\) Thus for all \(s \in \mathbb{R}\):

$$\max\{\|\xi_{r(\delta)}^{(n)}(s)\|_{\mathbb{C}^d} : n \in \{0, 1, 2, 3, 4\}\} \leq \frac{M_0}{1 + R^2 s^2}$$

and

$$\max\left\{\|s\xi_{r(\delta)}^{(n)}(s)\|_{\mathbb{C}^d} : n \in \{0, 1, 2, 3, 4\}\right\} \leq \frac{M_0}{R^2 (1 + R^2 s^2)}.$$
By the same reasoning as we have already seen in our paper, we thus conclude that there exists \( M > 0 \) such that for all \( \bar{\partial} \in \mathbb{R} \) with \( |\bar{\partial}| \in [\bar{\partial}_-, \bar{\partial}_+] \) and for all \( s \in \mathbb{R} \):

\[
\max\{\|\xi_r^{(n)}(s)\|_{C^4}, \|s^{(n)}(s)\|_{C^4} : n \in \{0, 1, 2, 3, 4\}\} \leq \frac{M}{1 + s^2}.
\]

We first extend \( (x, y, t, \bar{\partial}) \in \Sigma_* \rightarrow \omega_{x,y,t,\bar{\partial}} \) to \( \Sigma \) by continuity. For all \( (x, y, t, \bar{\partial}) \in \Sigma_* \), we observe that for all \( s \in \mathbb{R} \):

\[
\omega_{x,y,t,\bar{\partial}}(s) = \frac{\exp(i\pi (t^2 \partial xy + 2txs))\xi_{r(\partial)}(s + \bar{\partial}ty) - \xi_{r(\partial)}(s)}{2\pi \bar{\partial}t} \\
\quad + \frac{\xi_{r(\partial)}(s + \bar{\partial}ty) - \xi_{r(\partial)}(s)}{2\pi \bar{\partial}t} \exp(i\pi(\bar{\partial} t^2 x y + 2txs)) - 1 + \frac{\xi_{r(\partial)}(s + \bar{\partial}ty) - \xi_{r(\partial)}(s)}{2\pi \bar{\partial}t}.
\]

Since \( \xi \) is a Schwarz function, thus differentiable, we have for all \( (x, y) \in \mathbb{R}^2 \) with \( \|(x, y)\| = 1 \) and \( \bar{\partial} \in \mathbb{R} \setminus \{0\} \):

\[
\lim_{t \to 0^+} \omega_{x,y,t,\bar{\partial}}(s) = x \frac{is}{\bar{\partial}} \xi_{r(\partial)}(s) + yr(\bar{\partial}) \frac{\xi_{r(\partial)}'(s)}{2\pi}.
\]

Thus we set, for all \( (x, y) \in \mathbb{R}^2 \) with \( \|(x, y)\| = 1 \) and \( \bar{\partial} \in \mathbb{R} \setminus \{0\} \):

\[
\omega_{x,y,0,\bar{\partial}} : s \in \mathbb{R} \mapsto x \frac{is}{\bar{\partial}} \xi_{r(\partial)}(s) + yr(\bar{\partial}) \frac{\xi_{r(\partial)}'(s)}{2\pi}.
\]

We observe that our statement thus far is about pointwise convergence of the family of functions \( (\omega_{x,y,t,\bar{\partial}})_{t > 0} \rightarrow \omega_{x,y,0,\bar{\partial}} \) for fixed \( x, y \in \mathbb{R}^2 \) with \( \|(x, y)\| = 1 \) and \( \bar{\partial} \neq 0 \). This is different from the notion of convergence in the \( C^* \)-Hilbert norm. To obtain convergence for the Heisenberg \( C^* \)-Hilbert norm, and more information, we now proceed to establish some regularity properties for \( \omega \), in order to apply Lemma (3.4).

By Proposition (3.3), we already know that there exists \( M_1 > 0 \) such that for all \( (x, y, t, \bar{\partial}) \in \Sigma_* \) and \( s \in \mathbb{R} \):

\[
\|\sigma_{t,xy}^{tx,ty} \xi_{r(\partial)}(s) - \xi_{r(\partial)}(s)\|_{C^4} \leq \frac{M_1 t\|(x, y, \frac{1}{2}txy)\|_1}{1 + s^2},
\]

where \( \| \cdot \|_1 \) is the usual 1-norm on \( \mathbb{R}^3 \).

The map \( (x, y, t) \in \mathbb{R}^3 \mapsto (x, y, \frac{1}{2}tx) \) is continuous from \( \mathbb{R}^3 \) to itself. The set \( K = \{(x, y) \in \mathbb{R}^2 : \|(x, y)\| = 1\} \times [0, 1] \) is compact and thus there exists \( M_2 > 0 \) such that:

\[
\sup \left\{ \left\| \left( x, y, \frac{tx}{2} \right) \right\|_1 : (x, y, t) \in K \right\} = M_2.
\]

Thus:

\[
\|\omega_{x,y,t,\bar{\partial}}(s)\|_{C^4} \leq \frac{t\|(x, y, \frac{1}{2}txy)\|_1}{2\pi |\bar{\partial}|t} \frac{M_1}{1 + s^2} \leq \frac{M_1 M_2}{2\pi |\bar{\partial}|} \frac{M_1 M_2}{1 + s^2}.
\]
On the other hand, by assumption:
\[
\|\omega_{x,y,0}(s)\|_{C^4} \leq |x| \frac{M}{|\partial|(1 + s^2)} + |y| \frac{R_+ M}{2\pi(1 + s^2)} \leq \left( \frac{1}{\partial} + \frac{R_+}{2\pi} \right) \frac{MM_2}{1 + s^2}.
\]

In summary, there exists \( M_3 = \max \left\{ \frac{M_1 M_2}{\partial}, MM_2 \left( \frac{1}{\partial} + \frac{R_+}{2\pi} \right) \right\} > 0 \) such that for all \((x, y, \partial, t) \in \Sigma\) and all \( s \in \mathbb{R}\):
\[
\|\omega_{x,y,t,\partial}(s)\|_{C^4} \leq \frac{M_3}{1 + s^2}.
\]

By construction, \((\omega_{x,y,t,\partial})_{t>0}\) converges pointwise to \(\omega_{x,y,0,\partial}\) as \(t \to 0\). We now prove that this convergence is indeed uniform.

We begin with the following computation for all \((x, y, t, \partial) \in \Sigma_*\) and for all \( s \in \mathbb{R}\):
\[
\begin{aligned}
\|\omega_{x,y,t,\partial}(s) - \omega_{x,y,0,\partial}(s)\|_{C^4} &= \left| \sigma(t,s,\partial) \frac{\xi_{r(\partial)}(s) - \xi_{r(\partial)}(s)}{2\pi t \partial} - \left( \frac{ix}{\partial} \xi_{r(\partial)}(s) + \frac{y}{2\pi} \xi'_{r(\partial)}(s) \right) \right| \\
&\leq \exp \left( i\pi(\partial t^2 xy + 2stx) \right) - 1 - \frac{ix}{\partial} \xi_{r(\partial)}(s) \\
&\quad + \frac{1}{2\pi} \left| \frac{\xi_{r(\partial)}(s + \partial ty) - \xi_{r(\partial)}(s)}{\partial t} - y r(\partial) \xi'_{r(\partial)}(s) \right|.
\end{aligned}
\]

We first note that for all \((x, y, t, \partial) \in \Sigma_*\), by the mean value theorem, if \(g : t > 0 \mapsto \exp(i\pi(t^2 xy + 2txs))\), then there exists \(t_c \in [0, t]\) such that:
\[
\left| \frac{\exp(i\pi(\partial t^2 xy + 2txs)) - 1}{2\pi t \partial} \xi_{r(\partial)}(s + \partial ty) - \frac{ix}{\partial} \xi_{r(\partial)}(s + \partial ty) \right| \\
\leq \frac{M}{1 + (s + \partial ty)^2} \left| \frac{\exp(i\pi(t^2 xy + 2txs)) - 1}{2\pi t \partial} - \frac{ix}{\partial} \right| \\
= \frac{M}{2\pi |\partial| (1 + (s + \partial ty)^2)} \left| \frac{t}{2} g''(t_c) \right| \\
= \frac{M |t|}{2\pi |\partial| (1 + (s + \partial ty)^2)} \left| \left( i\pi \partial xy - \frac{\pi^2}{2} (2xs + 2t_c \partial xy)^2 \right) \exp(2i\pi st_c x) \right| \\
= \left| t \right| \frac{M}{2|\partial| (1 + (s + \partial ty)^2)} \left| \left( \partial xy + \frac{\pi}{2} |2xs + 2t_c \partial xy|^2 \right) \right| \\
\leq \left| t \right| \frac{M}{2|\partial| (1 + (s + \partial ty)^2)} \left| \left( \partial xy + \frac{\pi}{2} (4x^2 s^2 + 4t^2 \partial^2 x^2 y^2 + 8t_c \partial x^2 |y|) \right) \right|.
\]

Once again, by continuity over the compact \(\Omega \times [0, 1]\), there exists \(M_4 > 0\) such that for all \((x, y, \partial, t) \in \Omega \times [0, 1]\):
\[
\max \left\{ |\partial xy|, 2\pi x^2, 2\pi t^2 \partial^2 x^2 y^2, 2\pi \partial t_c x^2 |y| \right\} \leq M_4.
\]

Therefore, for all \(s \in \mathbb{R}\) and \((x, y, \partial, t) \in \Omega \times [0, 1]$: 

Consequently, for all $s > M_4$ we have:

$$
\frac{MM_4(1 + |s|^2)}{2\delta_-(1 + (s + \delta ty)^2)} \leq \frac{MM_4(1 + |s|^2)}{2\delta_-(1 + (-M_4)^2)}
$$

while for all $s < -M_4$ we have:

$$
\frac{MM_4(1 + |s|^2)}{2\delta_-(1 + (s + \delta ty)^2)} \leq \frac{MM_4(1 + |s|^2)}{2\delta_-(1 + (s + M_4)^2)}
$$

Thus, there exists $M_5 > 0$ such that if $|s| > M_4$ then:

$$
\frac{MM_4(1 + |s|^2)}{2\delta_-(1 + (s + \delta ty)^2)} \leq M_5.
$$

The function $(x, y, t, s) \in \mathbb{R}^4 \mapsto \frac{MM_4(1 + |s|^2)}{2\delta_-(1 + (s + \delta ty)^2)}$ is continuous, and thus it is bounded by some $M_6 > 0$ on the compact $\Sigma \times [-M_4, M_4]$.

Letting $M_7 = \max\{M_5, M_6\}$, we conclude that for all $(x, y, \delta, t) \in \Sigma_*$ and $s \in \mathbb{R}$, we have:

$$
\left| \frac{\exp(2i\pi txs)}{\delta t} - 1 \right|_{C^4} \leq M_7|t|.
$$

Consequently, if $|t| < \delta_1 = \frac{\varepsilon}{3M_7}$, then:

$$
\left| \frac{\exp(2i\pi txs)}{\delta t} - 1 \right|_{C^4} \leq M_7|t| < \frac{\varepsilon}{3}.
$$

Now, since $s \in \mathbb{R} \mapsto s\xi(s)$ is uniformly continuous on $\mathbb{R}$ as Schwarz function, there exists $\delta_2 > 0$ such that for all $0 < \delta < \delta_2$ and for all $s \in \mathbb{R}$, we have:

$$
|s(r + t)s - s\xi(s)| < \frac{\varepsilon R_0 x}{6M_4}.
$$

Moreover, since $\xi$ is bounded on $\mathbb{R}$, we may choose $\delta_2 > 0$ small enough so that:

$$
\sup_{s \in \mathbb{R}} \|\xi(s)\|_{C^4} < \frac{\varepsilon R_0 x}{6M_4}.
$$

Let now $\delta_3 = \frac{\delta_2}{R_0 x M_4}$. If $|t| < \delta_3$ then $r(\delta)\delta ty < \delta_2$ and therefore:

$$
\left| \frac{i s}{\delta} \xi_\delta(s + \delta ty) - \frac{i s}{\delta} \xi_\delta(s) \right|_{C^4} \leq \frac{\varepsilon R_0 x}{6M_4} + \frac{\varepsilon}{6M_4} \sup_{s \in \mathbb{R}} \|\xi(s)\|_{C^4} = \frac{\varepsilon}{6} + \frac{\varepsilon}{6} = \frac{\varepsilon}{3}.
$$
We thus deduce that for all \((x, y, t, \delta) \in \Sigma\), if \(|t| < \delta_3\), then:

\[
\sup_{s \in \mathbb{R}} \left| \frac{is}{\partial} \xi_r(\partial)(s + \partial t) - \frac{is}{\partial} \xi_r(\partial)(s) \right| < \frac{\varepsilon}{3}.
\]

(4.2)

Last, since \(\xi^t\) is also a Schwarz function and in particular, also uniformly continuous on \(\mathbb{R}\), there exists \(\delta_4 > 0\) such that \(|\xi^t(s + r) - \xi^t(s)| < \frac{\varepsilon}{3M}\) for all \(0 \leq r < \delta_4\).

Thus for all \((x, y, t, \delta) \in \Sigma\) and \(s \in \mathbb{R}\), if \(|t| < \delta_5 = \frac{\delta}{M_x M_y}\), then:

\[
\left| (\xi_r(\partial)(s + t\partial y) - \xi_r(\partial)(s)) - y\xi^t_r(\partial)(s) \right| \leq |r(\partial)| \int_0^t |\xi^t_r(\partial)(s + r\partial y) - \xi^t_r(\partial)(s)| \, dr \leq t \frac{\varepsilon}{3}.
\]

Thus for all \((x, y, t, \delta) \in \Sigma_\ast\) with \(|t| < \delta_4\), we have:

\[
\sup_{s \in \mathbb{R}} \left| \frac{\xi_r(\partial)(s + t\partial y) - \xi_r(\partial)(s)}{t} - \xi^t_r(\partial)(s) \right| < \frac{\varepsilon}{3}.
\]

In conclusion, for all \((x, y, t, \delta) \in \Sigma_\ast\) with \(0 < t < \min\{\delta_1, \delta_3, \delta_5\}\) and for all \(s \in \mathbb{R}\), we have established:

\[
\sup_{s \in \mathbb{R}} \left| \omega_{x,y,t,\delta}(s) - \omega_{x,y,0,\delta}(s) \right| < \varepsilon.
\]

In other words, setting for all \(t \in (0, 1)\):

\[
f_t : (x, y, \delta, s) \in \Omega \times \mathbb{R} \mapsto \omega_{x,y,t,\delta}(s)
\]

converges uniformly on \(\Omega \times \mathbb{R}\) to:

\[
f_0 : (x, y, \delta, s) \in \Omega \times \mathbb{R} \mapsto \omega_{x,y,0,\delta}(s)
\]

when \(t\) goes to 0.

Since \((x, y, t, \delta, s) \in \Sigma_\ast \times \mathbb{R} \mapsto f_t(x, y, \delta, s)\) and \(f_0\) are both continuous, we deduce, in particular, that:

\[(x, y, t, \delta, s) \in \Sigma \times \mathbb{R} \mapsto \omega_{x,y,t,\delta}(s)\]

is (jointly) continuous.

The entire reasoning up to now may be applied equally well to \(\xi^{(n)}\) for \(n \in \{0, 1, 2\}\) — as one may check that \(\omega^{(n)}\) is indeed obtained by substituting \(\xi\) with \(\xi^{(n)}\).

Therefore, we are now able to apply Lemma (3.4) to conclude that:

\[(x, y, t, \theta) \in \Sigma \mapsto \langle \omega_{x,y,t,\theta}, \omega_{x,y,t,\theta} \rangle_{\mathcal{M}_{p,q,d}} \in \left(\ell^1(\mathbb{Z}), \| \cdot \|_{\ell^1(\mathbb{Z})}\right)\]

is continuous as desired (to make notations clear: we pick a sequence \((\theta_n)_{n \in \mathbb{N}}\) converging to some \(\theta\), and we choose \((x_n)_{n \in \mathbb{N}} \in \mathbb{R}^\mathbb{N}\), \((y_n)_{n \in \mathbb{N}} \in \mathbb{R}^\mathbb{N}\), and \(t_n \in [0, 1]^\mathbb{N}\) such that for all \(n \in \mathbb{N}\), we have \((x_n, y_n, t_n, \theta_n - \frac{\theta_n}{q}) \in \Sigma\), and then we set, in the notations of Lemma (3.4), the functions \(\xi_n = f_{t_n}(x_n, y_n, \delta_n, \cdot)\) and \(\xi_\infty = f_{t_\infty}(x_\infty, y_\infty, \delta_\infty, \cdot)\).

We conclude our proof by observing that by Theorem (3.6):

\[
\mathcal{D}_{p,q,d}^\theta(\xi_r(\partial)) = \sup \left\{ \left\| \frac{x,y}{\partial} \xi_r(\partial) - \xi_r(\partial) \right\|_{\mathcal{M}_{p,q,d}} : (x, y) \in \mathbb{R}^2, \| (x, y) \| \leq 1 \right\}
\]

\[
= \sup \left\{ \left\| \xi_r(\partial) \right\|_{\mathcal{M}_{p,q,d}}, \left\| \omega_{x,y,t,\delta} \right\|_{\mathcal{M}_{p,q,d}} : \| (x, y) \| = 1, t \in [0, 1] \right\}
\]
as stated.

We now prove that $D$-norms on Heisenberg modules form continuous fields.

**Proposition 4.2.** Let $p \in \mathbb{Z}$, $q \in \mathbb{N} \setminus \{0\}$ and $d \in q\mathbb{N} \setminus \{0\}$. Let $\xi \in \mathcal{S}(\mathbb{C}^d)$. Let $r : \mathbb{R} \setminus \{\frac{p}{q}\} \to \mathbb{R} \setminus \{0\}$ be a continuous function. If $(\theta_k)_{k \in \mathbb{N}}$ is a sequence in $\mathbb{R}$ converging to $\theta_\infty$ and such that $\theta_k \neq \frac{p}{q}$ for all $k \in \mathbb{N}$, then:

$$\lim_{k \to \infty} D^{p,q,d}_{\theta_k}(\xi_{r(\theta_k)}) = D^{p,q,d}_{\theta_\infty}(\xi_{r(\theta_\infty)}),$$

where $\xi_{r(\theta)} : t \in \mathbb{R} \mapsto \xi(r(\theta)t)$ for all $\theta \neq \frac{p}{q}$.

**Proof.** The result is trivial if $\xi = 0$, which is equivalent to $D^{p,q,d}_{\theta}(\xi) = 0$ for all $\theta \in \mathbb{R}$ with $\theta \neq \frac{p}{q}$.

Now, fix $\theta \in \mathbb{R} \setminus \{\frac{p}{q}\}$. We shall prove that $\vartheta \mapsto D^{p,q,d}_{\theta}(\xi_{r(\vartheta)})$ is continuous at $\vartheta$.

Let $\delta_1 > 0$ such that $I = [\vartheta - \delta_1, \vartheta + \delta_1] \subseteq \mathbb{R} \setminus \{\frac{p}{q}\}$.

Let:

$$\Upsilon = \{(x, y, t) \in \mathbb{R}^3 : \| (x, y) \| = 1, t \in [0, 1] \}.$$  

Let $\xi \in \mathcal{S}(\mathbb{C}^d)$. We set:

$$t \in (0, \infty) \mapsto \omega_{x,y,t,\vartheta} = \frac{\sigma^{tx,ty}_{\vartheta,q} \xi_{r(\vartheta + p/q)} - \xi_{r(\vartheta + p/q)}}{2\pi \| \vartheta \| t}.$$  

By Lemma (4.1), $D^{p,q,d}_{\theta}(\xi_{r(\vartheta)})$ is the maximum of $\| \xi_{r(\vartheta)} \|_{\mathbb{F}^{p,d}}$ and the square root of:

$$E^{p,q,d}_{\theta}(\xi_{r(\vartheta)})^2$$

$$= \sup \left\{ \left\| \omega_{x,y,t,\vartheta} - \frac{p}{q}, \omega_{x,y,t,\vartheta} - \frac{p}{q} \right\|_{\mathbb{F}^{p,q,d}} : (x, y, t) \in \Upsilon \right\}$$

$$\leq \sup \left\{ \left\| \omega_{x,y,t,\vartheta} - \frac{p}{q}, \omega_{x,y,t,\vartheta} - \frac{p}{q} \right\|_{\mathbb{F}^{p,q,d}} : (x, y, t) \in \Upsilon \right\}$$

where $\Upsilon$ is a compact subset of $\mathbb{R}^3$, independent of $\vartheta$. By Proposition (3.2), the function $\vartheta \in I \mapsto \| \xi_{r(\vartheta)} \|_{\mathbb{F}^{p,q,d}}$ is continuous, so it is sufficient to show that $\vartheta \in I \mapsto E_{\theta}^{p,q,d}(\xi_{r(\vartheta)})$.

Now, since:

$$\nu : (x, y, t, \vartheta) \in \Upsilon \times I \mapsto \left\langle \omega_{x,y,t,\vartheta} - \frac{p}{q}, \omega_{x,y,t,\vartheta} - \frac{p}{q} \right\rangle_{\mathbb{F}^{p,q,d}}$$

is continuous in $\left(\ell^1(\mathbb{Z}^2), \| \cdot \|_{\ell^1(\mathbb{Z}^2)} \right)$ by Lemma (4.1), it is uniformly continuous on the compact $\Upsilon_2 = \Upsilon \times I$.

Let $\| (z, w, s, h) \|_{\infty} = \max \{ |z|, |w|, |s|, |h| \}$ for all $(z, w, s, h) \in \mathbb{R}^4$.

Let $\delta_2 > 0$ be chosen so that for all $(x, y, t, \vartheta), (z, w, s, h) \in \Upsilon_2$ with $\|(x, y, t, \vartheta) - (z, w, s, h)\|_{\infty} < \delta_2$ we have:

$$|\nu(x, y, t, \vartheta) - \nu(z, w, s, h)| < \frac{\varepsilon}{4}.$$

Let $G \subseteq \Upsilon_2$ be a $\delta_2$-dense finite subset of $\Upsilon_2$ in the sense of the norm $\| \cdot \|_{\infty}$.

Let:

$$F = \{ (z, w, r) \in \mathbb{R}^3 : \exists h \in \mathbb{R} \ (z, w, r, h) \in G \}.$$
By construction, $F$ is finite and $\delta_2$-dense in $\mathcal{Y}$ for the restriction of $\| \cdot \|_\infty$ to $\mathbb{R}^3 \sim \mathbb{R}^3 \times \{0\}$.

Fix any $\vartheta \in [\theta - \frac{\xi}{q} - \delta_1, \theta - \frac{\xi}{q} + \delta_1]$ and set $\vartheta = \vartheta + \frac{\xi}{q}$. Now, let $(x, y, t) \in \mathcal{Y}$. There exists $(z, w, r) \in F$ with $\max\{|x - z|, |y - w|, |t - r|\} < \delta_2$. We then observe:

\[
\begin{align*}
\|trxy, rz, rw, \frac{r^2}{2} \xi_r(\vartheta) - \xi_r(\vartheta)\|_{\mathcal{G}^{p,q,d}}^2 & \leq \frac{\|trxy, rz, rw, \frac{r^2}{2} \xi_r(\vartheta) - \xi_r(\vartheta)\|_{\mathcal{G}^{p,q,d}}^2}{2\pi|\vartheta|t} + \frac{\|trxy, rz, rw, \frac{r^2}{2} \xi_r(\vartheta) - \xi_r(\vartheta)\|_{\mathcal{G}^{p,q,d}}^2}{2\pi|\vartheta|r} + \frac{\|trxy, rz, rw, \frac{r^2}{2} \xi_r(\vartheta) - \xi_r(\vartheta)\|_{\mathcal{G}^{p,q,d}}^2}{2\pi|\vartheta|r} \\
& \leq \frac{\|t\omega_{x,y,t,3,\mathcal{Z}^2}\|_{\mathcal{G}^{p,q,d}} - \|t\omega_{x,y,t,3,\mathcal{Z}^2}\|_{\mathcal{G}^{p,q,d}}}{2\pi|\vartheta|t} + \frac{\|t\omega_{x,y,t,3,\mathcal{Z}^2}\|_{\mathcal{G}^{p,q,d}} - \|t\omega_{x,y,t,3,\mathcal{Z}^2}\|_{\mathcal{G}^{p,q,d}}}{2\pi|\vartheta|r} + \frac{\|t\omega_{x,y,t,3,\mathcal{Z}^2}\|_{\mathcal{G}^{p,q,d}} - \|t\omega_{x,y,t,3,\mathcal{Z}^2}\|_{\mathcal{G}^{p,q,d}}}{2\pi|\vartheta|r} \\
& \leq \frac{\|\omega_{x,y,t,3,\mathcal{Z}^2}(\xi_r(\vartheta) - \xi_r(\vartheta))\|_{\mathcal{G}^{p,q,d}}^2}{2\pi|\vartheta|r}.
\end{align*}
\]

Let $F^{p,q,d}_{\vartheta}(\eta)$ be given by:

\[
F^{p,q,d}_{\vartheta}(\eta) = \max \left\{ \frac{\|r_{z,w,\frac{r^2}{2}} \eta - \eta\|_{\mathcal{G}^{p,q,d}}}{2\pi(\vartheta + \frac{\xi}{q})r} : (z, w, r) \in F \right\}
\]

for all $\eta \in \mathcal{S}(\mathbb{C}^d)$.

We thus have proven:

\[
F^{p,q,d}_{\vartheta}(\xi_r(\vartheta))^2 \leq E^{p,q,d}_{\vartheta}(\xi_r(\vartheta))^2 \leq \frac{\varepsilon}{4} + F^{p,q,d}_{\vartheta}(\xi_r(\vartheta))^2.
\]

Therefore:

\[
\left| E^{p,q,d}_{\vartheta}(\xi_r(\vartheta))^2 - E^{p,q,d}_{\vartheta}(\xi_r(\vartheta))^2 \right| \leq \frac{\varepsilon}{2} + |F^{p,q,d}_{\vartheta}(\xi_r(\vartheta))^2 - F^{p,q,d}_{\vartheta}(\xi_r(\vartheta))^2|.
\]
Note that for any \( \eta \in S(C^d) \), the quantity \( F_{\vartheta}^{p,q,d}(\eta) \) is finite as the maximum of finitely many values. Also note that the set \( F \) does not change with \( \vartheta \in I \) — the only dependence of \( F_{\vartheta}^{p,q,d} \) on \( \vartheta \) is via the choice of the quantum torus norm \( \| \cdot \|_{\mathcal{H}_{\vartheta}^{p,q,d}} \).

Now the key observation is that \( \vartheta \in I \mapsto F_{\vartheta}^{p,q,d}(\xi) \) is continuous. Fix \( (z,w,r) \in F \). By Proposition (3.2), the function:

\[
\vartheta \in I \mapsto \left\| \frac{\alpha_{\vartheta,d}^{z,w} \xi_r(\vartheta) - \xi_r(\theta)}{2\pi |\bar{\vartheta}|r} \right\|_{\mathcal{H}_{\vartheta}^{p,q,d}}
\]

is continuous (we note that \( \vartheta \in I \mapsto \alpha_{\vartheta,d}^{z,w} \xi_r(\vartheta) - \xi_r(\theta) \) satisfies the necessary hypothesis, owing to \( \xi \) being a Schwarz function and \( r \) being continuous. The details follow similar lines to our proof of Lemma (4.1) and we shall omit them this time around).

Thus \( \vartheta \in I \mapsto F_{\vartheta}^{p,q,d}(\xi_r(\vartheta)) \) is the maximum of finitely many continuous functions, and is therefore continuous as well.

Thus there exists \( \delta_3 > 0 \) such that for all \( \vartheta \in [\theta - \delta_3, \theta + \delta_3] \) we have:

\[
|F_{\vartheta}^{p,q,d}(\xi_r(\vartheta)) - F_{\theta}^{p,q,d}(\xi_r(\theta))|^2 \leq \frac{\varepsilon}{2}
\]

Thus if \( \delta = \min\{\delta_1, \delta_3\} > 0 \) then for all \( \vartheta \in [\theta - \delta, \theta + \delta] \) we have:

\[
|E_{\vartheta}^{p,q,d}(\xi_r(\vartheta)) - E_{\theta}^{p,q,d}(\xi_r(\theta))|^2 \leq \varepsilon.
\]

Since \( E_{\vartheta}(\xi_r(\vartheta)) \geq 0 \) for all \( \vartheta \in [\theta - \delta, \theta + \delta] \) and \( \sqrt{\cdot} \) is a continuous function on \([0, \infty)\), we have shown that:

\[
\vartheta \in I \mapsto E_{\vartheta}^{p,q,d}(\xi_r(\vartheta))
\]

is continuous at \( \theta \). Therefore, as the maximum of two continuous functions by Lemma (4.1) and Proposition (3.2):

\[
\vartheta \in I \mapsto D_{\vartheta}^{p,q,d}(\xi_r(\vartheta))
\]

is continuous at \( \theta \) as well.

\[\square\]

**Corollary 4.3.** Let \( p \in \mathbb{Z} \), \( q \in \mathbb{N} \setminus \{0\} \) and \( d \in q\mathbb{N} \setminus \{0\} \). If \( \xi \in S(C^d) \), then:

\[
\theta \in \mathbb{R} \setminus \left\{ \frac{p}{q} \right\} \mapsto D_{\theta}^{p,q,d}(\xi)
\]

is continuous.

**Proof.** This follows from Proposition (4.2) using \( r : x \in \mathbb{R} \mapsto 1 \). \[\square\]

### 5. Convergence

We now present our main convergence result for the modular pr opinquity. Our first step consists in finding an appropriate choice of anchors. We establish two lemmas to this end. The first lemma extends Lemma (3.11) by proving that while the range of the operators involved in Lemma (3.11) depends on the parameters used to define the Heisenberg modules, its dimension does not. The second lemma then uses the particular basis of Hermite functions obtained in the first lemma to construct our anchors.
Lemma 5.1. For all $j \in \mathbb{N}$ and $\bar{\delta} > 0$, let:

$$\psi_j^\delta : x \in [0, \infty) \mapsto \bar{\delta} \exp\left(-\frac{\pi \bar{\delta} r^2}{2}\right) L_j^\delta \left(\pi \bar{\delta} r^2\right)$$

where $L_j^\delta : t \in \mathbb{R} \mapsto \sum_{k=0}^{n} \left(\frac{t}{\bar{\delta}}\right)^k \frac{(-1)^k}{k!} \frac{1}{j-k}$ is the $j$th Laguerre polynomial.

Note that $\psi_j^\delta(t) = \bar{\delta} \psi_j^1(\sqrt{\bar{\delta}} t)$ for all $t \geq 0$, with $\psi_j^1$ the $j$-Laguerre function.

Let $f$ be compactly supported continuous. For all $j \in \mathbb{N}$ and $\delta \neq 0$, we set:

$$C_j^\delta(f) = \frac{1}{j} \sum_{k=0}^{j} \frac{j+1-k}{j+1} \langle f \psi_j^\delta, \psi_j^\delta \rangle_{L^2(\mathbb{R}, r dr)} \psi_j^\delta.$$ 

For all $\varepsilon > 0$ and $\partial_0 \neq 0$, there exists $N \in \mathbb{N}$ and $\delta \in (0, |\partial_0|)$ such that, for all $\bar{\delta} \in [\partial_0 - \delta, \partial_0 + \delta]$, we have:

$$\left\| f - \sum_{j=0}^{N} C_j^\delta(f) \right\|_{L^1(\mathbb{R}+, r dr)} \leq \varepsilon.$$

Proof. We fix $\partial_0 \neq 0$. By [31, Theorem 6.2.1], as in the proof of Lemma (3.11), there exists $N > 0$ such that:

$$\left\| f - \sum_{j=0}^{N} C_j^\delta(f) \right\|_{L^1(\mathbb{R}+, r dr)} \leq \varepsilon/2.$$

Let $Q = \sum_{j=0}^{N} \frac{N+1-j}{N+1}$. Let $K_1 > 0$ be chosen so that $f(x) = 0$ whenever $x \geq K_1$ (as $f$ is compactly supported by assumption). Let $M_1 = \int_{0}^{\infty} |f(r)| r dr$.

Let $M_2 = \max\{\|\psi_j^\delta\|_{L^1(\mathbb{R}, r dr)} : j \in \{0, \ldots, N\}\}$. Now, there exists $C > 0$ and $K_2 > 0$ such that for all $x \geq K_2$ and for all $j \in \{0, \ldots, N\}$:

$$|\psi_j^1(r)| \leq C \exp\left(-\frac{j^2}{4}\right).$$

Indeed, one checks trivially that $\lim_{r \to \infty} \exp\left(\frac{1}{4} j^2\right) \psi_j^1(r) = 0$ and once again, we work with finitely many functions.

Let $K_3 \geq K_2$ be chosen so that $\int_{K_3}^{\infty} \exp\left(-\frac{j^2}{4}\right) r dr \leq \frac{\varepsilon}{16CM_2QN}.$

Last, let $M_3 = \max\left\{\langle f \psi_j^\delta, \psi_j^\delta \rangle_{L^2(\mathbb{R}, r dr)} : j \in \{0, \ldots, N\}\right\}$. For all $j \in \{0, \ldots, N\}$, the function $\psi_j^1$ is continuous on $\mathbb{R}_+$, so $(\psi_j^1)^2$ is continuous on $[0, K_3]$, and thus the family $\{\psi_j^1, (\psi_j^1)^2 : j \in \{0, \ldots, N\}\}$ is uniformly equicontinuous on this compact interval. Thus there exists $\delta_1 > 0$ such that if $|x-y| \leq \delta_1$ then $(\psi_j^1)^2(x) - (\psi_j^1)^2(y) \leq \frac{\varepsilon}{8QM_2M_3N}$ and $|\psi_j^1(x) - \psi_j^1(y)| \leq \frac{\varepsilon}{16QM_2N}$. For all $j \in \{0, \ldots, N\}$ (note that of course, it is important here that we work with finitely many functions, so we trivially have a uniformly equicontinuous family).

Using the continuity of the square root function and the square function, there exists $\delta_2 \in (0, \delta_1)$ such that if $|\partial - \partial_0| \leq \delta_2$ then $|\sqrt{\partial} - \sqrt{\partial_0}| \leq \frac{\delta_2}{2}$ and $|\partial^2 - \partial_0^2| \leq \frac{\delta_2^2}{2}$. Therefore, $|\sqrt{\partial} - \sqrt{\partial_0}| \leq \delta_1$ for all $x \in \mathbb{R}_+$ with $|x| \leq K_1$, so that for
all \(j \in \{0, \ldots, N\}:\)

\[
\left| \int_0^\infty f(r) \left( (\psi^2_0)(r) - (\psi^2_{\partial_0})(r) \right) r dr \right| \leq M_1 \sup_{|r| \leq K_1} |\partial^2 (\psi^2_1)(\sqrt{\delta r}) - \partial^2 (\psi^2_0)(\sqrt{\delta_0 r})| + 2M_2 |\delta^2 - \delta^2_0| \\
\leq M_1 \sup_{|r| \leq K_1} |\partial^2 (\psi^2_1)(\sqrt{\delta r}) - (\psi^2_1)(\sqrt{\delta_0 r})| + 2M_2 |\delta^2 - \delta^2_0| \\
\leq \frac{\varepsilon}{4QM_2N}.
\]

We thus conclude that for all \(\delta \in [\delta_0 - \delta_2, \delta_0 + \delta_2]\) and \(j \in \{0, \ldots, N\}:\)

\[
\left| \left\langle f\psi^2_\delta, \psi^2_\delta \right\rangle_{L^2(R_+, r dr)} - \left\langle f\psi^2_{\partial_0}, \psi^2_{\partial_0} \right\rangle_{L^2(R_+, r dr)} \right| \leq \frac{\varepsilon}{4QM_2N}
\]

Moreover, for \(\delta \in [\delta_0 - \delta_2, \delta_0 + \delta_2]\):

\[
\left| \int_0^\infty |\psi^2_\delta(r)| - |\psi^2_{\partial_0}(r)| r dr \right| \\
\leq \int_0^{K_3} |\partial \psi^2_\delta(\sqrt{\delta r}) - \partial_0 \psi^2_1(\sqrt{\delta_0 r})| r dr + \int_{K_3}^\infty |\partial \psi^2_\delta(\sqrt{\delta r}) - \partial_0 \psi^2_1(\sqrt{\delta_0 r})| r dr \\
\leq 2\delta_0 \left( \frac{\varepsilon}{16\delta_0 QM_3N} + \int_{K} 2C \exp(-Dr^2) r dr \right) \\
\leq \frac{\varepsilon}{4QM_3N}.
\]

Therefore, for all \(j \in \{0, \ldots, N\}\) and for all \(\delta \in [\delta_0 - \delta_2, \delta_0 + \delta_2]\):

\[
\left| \left\langle f\psi^2_\delta, \psi^2_\delta \right\rangle_{L^2(R_+, r dr)} - \left\langle f\psi^2_{\partial_0}, \psi^2_{\partial_0} \right\rangle_{L^2(R_+, r dr)} \right| \\
\leq \left| \left\langle f\psi^2_\delta, \psi^2_\delta \right\rangle_{L^2(R_+, r dr)} \psi^2_\delta - \left\langle f\psi^2_{\partial_0}, \psi^2_{\partial_0} \right\rangle_{L^2(R_+, r dr)} \psi^2_{\partial_0} \right|_{L^1(R_+, r dr)} \\
\leq \left| \left\langle f\psi^2_\delta, \psi^2_\delta \right\rangle_{L^2(R_+, r dr)} \psi^2_\delta - \left\langle f\psi^2_{\partial_0}, \psi^2_{\partial_0} \right\rangle_{L^2(R_+, r dr)} \psi^2_{\partial_0} \right|_{L^1(R_+, r dr)} \\
+ \left| \left\langle f\psi^2_{\partial_0}, \psi^2_{\partial_0} \right\rangle_{L^2(R_+, r dr)} \psi^2_{\partial_0} - \left\langle f\psi^2_{\partial_0}, \psi^2_{\partial_0} \right\rangle_{L^2(R_+, r dr)} \psi^2_{\partial_0} \right|_{L^1(R_+, r dr)} \\
\leq \pi \delta_0 \left| f_{\partial_0} \right|_{L^1(R_+, r dr)} \left| \left\langle f\psi^2_\delta, \psi^2_\delta \right\rangle_{L^2(R_+, r dr)} - \left\langle f\psi^2_{\partial_0}, \psi^2_{\partial_0} \right\rangle_{L^2(R_+, r dr)} \right| \\
+ M_3 \left| \partial \psi^2_\delta - \partial \psi^2_{\partial_0} \right|_{L^1(R_+, r dr)} \\
\leq \frac{\varepsilon}{4QM_2 N_2}M_2 + M_3 \frac{\varepsilon}{4QM_2 N_3} \\
\leq \frac{\varepsilon}{2Q N}.
\]

Consequently, for all \(\delta \in [\delta_0 - \delta_2, \delta_0 + \delta_2]\) and \(j \in \{0, \ldots, N\}:\)

\[
\left| C^2_\delta(f) - C^2_{\partial_0}(f) \right|_{L^1(R_+, r dr)} 
\]
Let \( \delta \) be chosen as fixed elements in the space \( S \).

Proof. Let \( \delta \) such that, for all \( f \) function \( \epsilon \) is a \( \epsilon \)-dense subset of \( D \).

Thus for all \( \delta \in [\delta_0 - \delta_2, \delta_0 + \delta_2] \):

\[
\left\| f - \sum_{j=1}^{N} C^j_0(f) \right\|_{L^1(\mathbb{R}^+, r dr)} \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\]

This concludes our lemma. \( \square \)

We recall the following lemma from [18], as it will provide us with

**Lemma 5.2** ([18, Lemma 6.8]). For all \( n \in \mathbb{N} \), let \( \psi_n : \mathbb{R}^2 \to [0, \infty) \) be an integrable function supported on \( [0, \frac{1}{n+1}] \) and with \( \int_{\mathbb{R}^2} \psi_n = 1 \).

If \( f : \mathbb{R}^2 \to [0, \infty) \) is integrable on some ball centered at 0 in \( (\mathbb{R}^2, \| \cdot \|) \), and \( f \) continuous at 0, then:

\[
\lim_{n \to \infty} \int_{\mathbb{R}^2} \psi_n(x, y) f(x, y) \, dx dy = f(0).
\]

We are now in a position to prove the existence of a good choice of anchors, which will be chosen as fixed elements in the space \( S(\mathbb{C}^d) \) of Schwarz functions which will give good approximations in norms in a whole family of Heisenberg modules.

**Lemma 5.3.** Let \( p \in \mathbb{Z} \), \( q \in \mathbb{N} \setminus \{0\} \) and \( d \in q\mathbb{N} \setminus \{0\} \). Let \( \delta_0 \neq 0 \). For all \( \epsilon > 0 \), there exist \( \delta \in \left(0, \frac{|B_0|}{\epsilon}\right) \) and a finite subset \( \mathcal{F} \) of \( D \left( D_{\mathcal{F} + q \delta}^p \right) \) \( \{0\} \) for \( \| \cdot \|_{\mathcal{F} + q \delta} \)

such that, for all \( \delta \in [\delta_0 - \delta, \delta_0 + \delta] \), the set:

\[
\left\{ \frac{D^p_{\mathcal{F} + q \delta}}{\mathcal{F} + \delta}(\omega) : \omega \in \mathcal{F} \right\}
\]

is a \( \epsilon \)-dense subset of \( D \left( D_{\mathcal{F} + q \delta}^p \right) \) for \( \| \cdot \|_{\mathcal{F} + q \delta} \).

Proof. Let \( \epsilon > 0 \) be given. By Lemmas (5.2), (3.10) and (3.12), there exists a function \( f : \mathbb{R}^+ \to \mathbb{R}^+ \) such that, if \( g : (x, y) \in \mathbb{R}^2 \to f \left( \sqrt{x^2 + y^2} \right) \) then:

1. \( \int_{\mathbb{R}^+} f(r) \, r \, dr = \frac{1}{2\pi} \),
2. \( \int_{\mathbb{R}^2} g(x, y) \| (x, y) \| \, dx dy \leq \frac{\epsilon}{48 \pi \delta_0} = \frac{\epsilon}{2 \pi \delta_0} \),
3. \( D^p_{\mathcal{F} + q \delta}(\sigma^q_{\mathcal{F}, \delta} \xi) \leq (1 + \frac{\epsilon}{\delta_0}) D^p_{\mathcal{F} + q \delta}(\xi) \) for all \( \xi \neq \frac{p}{q} \) and all \( \xi \in S(\mathbb{C}^d) \).
as in [18, Lemma 6.10], for instance given by Lemma (5.2). By Lemma (3.10), we have that for all \( \omega \in \mathcal{S}(\mathbb{C}^d) \), we have:

\[
\left\| \omega - \sigma_{\partial, \delta}^g \omega \right\|_{\mathcal{H}_{p,q,d}^{r+\delta}} \leq \frac{\varepsilon}{16} D_{p,q,d}^{r+\delta}(\omega),
\]

for all \( \delta \in \left[ \frac{\delta_0}{2}, \frac{3\delta_0}{2} \right] \).

We apply Lemma (5.1) to obtain some \( N \in \mathbb{N} \) and \( \delta_0 \in \left( 0, \frac{|\delta_0|}{2} \right) \) such that:

\[
\left\| f - \sum_{j=0}^{N} C_j^g(f) \right\|_{L^1(\mathbb{R}^+, rdr)} \leq \frac{\varepsilon}{8}.
\]

We now note that thanks to a change of variable in the definition of the operator \( \sigma_{\partial, \delta}^g \), it is sufficient to prove our result for \( \delta > 0 \). We shall henceforth assume \( \delta > 0 \).

Let \( h_\delta : (x,y) \in \mathbb{R}^2 \mapsto \sum_{j=0}^{N} C_j^g(f) (\sqrt{x^2 + y^2}) \) for all \( \delta \neq 0 \).

For each \( \delta \in [\delta_0 - \delta_0, \delta_0 + \delta_0] \), we then have, in a manner similar to the proof of Lemma (3.11):

\[
\left\| \sigma_{\partial, \delta}^g - \sigma_{\partial, \delta}^{h_\delta} \right\|_{\mathcal{H}_{p,q,d}^{r+\delta}} \leq \left\| f - \sum_{j=0}^{N} C_j^g(f) \right\|_{L^1(\mathbb{R}^+, rdr)} \leq \frac{\varepsilon}{8}.
\]

Consequently, for all \( \delta \in [\delta_0 - \delta_0, \delta_0 + \delta_0] \) and for all \( \omega \in \mathcal{D} \left( D_{p,q,d}^{r+\delta} \right) \), we have:

\[
\left\| \omega - \sigma_{\partial, \delta}^{h_\delta} \omega \right\|_{\mathcal{H}_{p,q,d}^{r+\delta}} \leq \frac{3\varepsilon}{16},
\]

therefore:

\[
\left\| \omega - \frac{1}{1 + \frac{\varepsilon}{16}} \sigma_{\partial, \delta}^{h_\delta} \omega \right\|_{\mathcal{H}_{p,q,d}^{r+\delta}} \leq \frac{\varepsilon}{16} \left\| \omega \right\|_{\mathcal{H}_{p,q,d}^{r+\delta}} + \frac{3\varepsilon}{16} \leq \frac{\varepsilon}{4}
\]

while \( D_{p,q,d}^{r+\delta} \left( \frac{1}{1 + \frac{\varepsilon}{16}} \sigma_{\partial, \delta}^{h_\delta} \omega \right) \leq 1 \).

Our efforts thus far show that the range of \( \sigma_{\partial, \delta}^{h_\delta} \) is of dimension \( N + 1 \), spanned by:

\[
\left\{ \mathcal{H}_j^\delta = \partial^j \mathcal{H}_1^\delta \left( \sqrt{\delta} \right) : j \in \{ 0, \ldots, N \} \right\}
\]

where:

\[
\mathcal{H}_1^\delta : t \in \mathbb{R} \mapsto \frac{(2)^{j}}{\sqrt{j!2^j}} \exp \left( t \frac{2\sqrt{2\pi}}{2} \right) H_j \left( t \sqrt{2\pi} \right)
\]

and \( H_j \) is the \( j \)th Hermite polynomial, as seen in Lemma (3.11).

Let \( V = \mathbb{C}^N \). For any \( \delta > 0 \) we define \( \eta_\delta : (c_j)_{j \in \{1, \ldots, N\}} \in V \mapsto \sum_{j=1}^{N} c_j \mathcal{H}_j^\delta \).

The map \( \eta_\delta \) is a linear injection from \( V \) to \( \mathcal{S} \otimes \mathbb{C}^d \). For each \( \delta > 0 \) and \( c \in V \), we set \( ||c||_3 = D_{p,q,d}^{r+\delta}(\eta_\delta(c)) \); of course \( ||c||_3 \) is a norm on \( V \).

We now set \( ||c||_V = \sup_{\delta \in \left[ \delta_0 - \delta_0, \delta_0 + \delta_0 \right]} ||c||_3 \). By construction, it is sufficient to check that \( || \cdot ||_V \) is valued in \( \mathbb{R}_+ \) (i.e. is never infinite) to conclude that \( || \cdot ||_V \) is a norm on \( V \).
Let $c = (c_j)_{j \in \{0, \ldots, N\}} \in V$. Note that for all $t \in \mathbb{R}$ and $\delta > 0$:

$$\eta_0(c)(t) = \sum_{j=0}^{N} c_j \mathcal{H}_t^0(t) = (\delta)^{\frac{1}{4}} \sum_{j=0}^{N} \mathcal{H}_t^1(\sqrt{\delta} t) = (\delta)^{\frac{1}{4}} \eta_1(c)(\sqrt{\delta} t),$$

(5.1)

and of course, $\eta_1 \in \mathcal{S}(\mathbb{C}^{d})$. Thus by Proposition (4.2), we conclude that $\bar{c} \in [\bar{\delta}_0 - \delta_0, \bar{\delta}_0 + \delta_0] \rightarrow D_{\frac{p}{2} + \hat{q}}^{p,q,d}(\eta_0(c))$ is continuous as well as the product of the two continuous functions $\bar{c} \in [\bar{\delta}_0 - \delta_0, \bar{\delta}_0 + \delta_0] \rightarrow D_{\frac{p}{2} + \hat{q}}^{p,q,d}(\eta_1(c)(\sqrt{\bar{\delta}} t))$ (by Proposition (4.2)), and $\bar{c} \in [\bar{\delta}_0 - \delta_0, \bar{\delta}_0 + \delta_0] \rightarrow \sqrt{\bar{\delta}}$.

Therefore, it reaches its maximum on the compact $[\bar{\delta}_0 - \delta_0, \bar{\delta}_0 + \delta_0]$, which is by definition the number $\|c\|_{V}$.

Thus $\|\cdot\|_{V}$ is a norm on $V$.

We now make another observation. We have, for all $\delta \in [\bar{\delta}_0 - \delta_0, \bar{\delta}_0 + \delta_0]$:

$$\|c\|_{\delta} - \|d\|_{\delta_0} \leq \|c\|_{a} - \|c\|_{a_0} + \|d\|_{\delta_0} - \|d\|_{a_0} \leq \|c\|_{a} - \|c\|_{a_0} + \|d\|_{\delta_0} - \|d\|_{a_0} \leq \|c\|_{a} - \|c\|_{a_0} + \|d\|_{\delta_0} - \|d\|_{a_0} \leq \|c\|_{\delta} - \|c\|_{\delta_0}.$$

Thus the function:

$$n : (\bar{c}, c) \in [\bar{\delta}_0 - \delta_0, \bar{\delta}_0 + \delta_0 \times V \mapsto \|c\|_{\bar{\delta}}$$

is continuous. It is in particular continuous on the compact $[\bar{\delta}_0 - \delta_0, \bar{\delta}_0 + \delta_0] \times B$ where $B$ is the closed unit ball for $\|\cdot\|_{V}$.

Therefore there exists $k > 0$ such that for all $c \in V$, $\bar{c} \in [\bar{\delta}_0 - \delta_0, \bar{\delta}_0 + \delta_0]$, we have:

$$k \|c\|_{V} \leq \|c\|_{a} \leq \|c\|_{V}.$$

Let now $E = \{c \in V : \|c\|_{V} \leq \frac{1}{2}\}$. Since $V$ is finite dimensional, $E$ is compact. Therefore, the function $n$ is uniformly continuous on the compact $[\bar{\delta}_0 - \delta_0, \bar{\delta}_0 + \delta_0] \times E$. Let $\delta_1 \in (0, \delta_0)$ be chosen so that, for all $\delta \in [\bar{\delta}_0 - \delta_1, \bar{\delta}_0 + \delta_1]$, and for all $c, d \in E$ with $\|c - d\|_{V} \leq \delta_1$, we have:

$$|n(\bar{c}, c) - n(\bar{c}, d)| \leq \frac{k \varepsilon}{8}.$$

In particular, for all $\delta \in [\bar{\delta}_0 - \delta_1, \bar{\delta}_0 + \delta_1]$ and all $c \in E$, we have:

$$\|c\|_{a} - \|c\|_{a_0} \leq \frac{k \varepsilon}{8}.$$

Let $E = \{c \in V : \|c\|_{V} \leq \frac{1}{2}\}$. By definition, $E$ is a bounded subset of $V$ which is finite dimensional. Thus $E$ is totally bounded for $\|\cdot\|_{V}$. Let $\mathcal{E}$ be a finite $\frac{\varepsilon}{8}$-dense subset of $E$ for $\|\cdot\|_{V}$. We assume $0 \not\in \mathcal{E}$ (we can simply pick a $\frac{\varepsilon}{16}$-dense subset of $E$ and then remove 0 from it if needed).
For all \( c \in \mathfrak{F} \), the function:

\[
l_c : \eta \in [\delta_0 - \delta_1, \delta_0 + \delta_1] \mapsto \frac{D_{p,q,d}^{p,q,d}(\eta \circ_h (c)) - D_{p,q,d}^{p,q,d}(\eta \circ_h (0))}{D_{p,q,d}^{p,q,d}(\eta \circ_h (c))}
\]

is continuous on a compact, and it is null at \( \delta_0 \); hence there exists \( \delta_2 \in (0, \delta_1) \) such that for all \( \delta \in [\delta_0 - \delta_2, \delta_0 + \delta_2] \) and \( c \in \mathfrak{F} \), we have:

\[
l_c(\delta) \leq \frac{\epsilon}{4}.
\]

We emphasize that in the definition of \( l_c \), for any \( c \in \mathfrak{F} \), only involves the element \( \eta \circ_h (c) \), and the only dependence on the variable is through the choice of D-norm.

Last, for all \( d \in \mathfrak{F} \), the function \( \delta \in [\delta_0 - \delta_2, \delta_0 + \delta_2] \mapsto \|\eta \circ_h (d) - \eta \circ_h (0)\|_{\mathcal{W}_{p,q,d}} \) is continuous by Proposition (3.2) and Expression (5.1). Therefore, since \( \mathfrak{F} \) is finite, there exists \( \delta_3 \in (0, \delta_2) \) such that for all \( \delta \in [\delta_0 - \delta_3, \delta_0 + \delta_3] \) and for all \( d \in \mathfrak{F} \):

\[
\|\eta \circ_h (d) - \eta \circ_h (0)\|_{\mathcal{W}_{p,q,d}} \leq \frac{\epsilon}{4}.
\]

Fix now \( \delta \in [\delta_0 - \delta_3, \delta_0 + \delta_3] \). Let now \( \eta \in \mathcal{D}_1 \left( \mathcal{D}_{p,q,d} \right) \). Let \( c \in V \) so that \( \frac{1}{1 + \frac{1}{8n}} \sigma_{\delta, d}(\eta) = \eta \circ_h (c) \) and note that by construction, \( \|\eta - \eta \circ_h (c)\|_{\mathcal{W}_{p,q,d}} \leq \frac{\epsilon}{4} \) while \( D_{p,q,d}^{p,q,d}(\eta \circ_h (c)) \leq 1 \).

Since \( D_{p,q,d}^{p,q,d}(\eta \circ_h (c)) \leq 1 \) so \( \|c\|_V \leq \frac{1}{2} \). Thus, \( \|\eta \circ_h (c)\| \leq 1 + \frac{\epsilon}{8} \).

Thus, \( \frac{1}{1 + \frac{1}{8n}} \eta \circ_h (c) \in \mathfrak{C} \), and thus there exists \( d \in \mathfrak{F} \) such that \( \|\frac{1}{1 + \frac{1}{8n}} c - d\|_V \leq \frac{\epsilon}{8} \).

Thus:

\[
\|c - d\|_V \leq \frac{\frac{\epsilon}{4} \|c\|_V + \frac{\epsilon}{8} \leq \frac{\epsilon}{4}.
\]

We conclude by observing that:

\[
\left| \frac{D_{p,q,d}^{p,q,d}(\eta \circ_h (d))}{D_{p,q,d}^{p,q,d}(\eta \circ_h (0))} \right| \eta \circ_h (d) - \eta \circ_h (0) \leq \left| \frac{D_{p,q,d}^{p,q,d}(\eta \circ_h (d))}{D_{p,q,d}^{p,q,d}(\eta \circ_h (0))} \right| \eta \circ_h (d) - \eta \circ_h (c) \leq \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{4} + \frac{\epsilon}{4} + \|\eta \circ_h (d) - \eta \circ_h (0)\|_{\mathcal{W}_{p,q,d}} + \|\eta \circ_h (d) - \eta \circ_h (c)\|_{\mathcal{W}_{p,q,d}} + \frac{\epsilon}{4} = \frac{\epsilon}{4} + \frac{\epsilon}{4} + \|c - d\|_V \leq \frac{\epsilon}{4} \leq \epsilon.
\]

This concludes our lemma.

\( \square \)

We now summarize, in the following lemma, all the elements of the proof of convergence for quantum tori, as worked in [10], which we will employ in the current paper.
Notation 5.4. Let $\mu$ be the probability Haar measure on the 2-torus $\mathbb{T}^2$.

For any $f \in L^1(\mathbb{T}^2, \mu)$ and $\vartheta \in \mathbb{R}$, we denote by $\beta^f_\vartheta$ the operator on $A_\vartheta$ defined for all $a \in A_\vartheta$ by:

$$\beta^f_\vartheta(a) = \int_{\mathbb{T}^2} f(z) \beta^z_\vartheta(a) \, d\mu(z)$$

which is continuous with $\|\beta^f_\vartheta\|_{A_\vartheta} \leq \|f\|_{L^1(\mathbb{T}^2, \mu)}$.

Lemma 5.5. Let $\ell$ be a continuous length function on $\mathbb{T}^2$. Let $\vartheta \in \mathbb{R}$ and $\varepsilon > 0$. There exists $\delta_\varepsilon > 0$, a trace-class operator $T$ on $\ell^2(\mathbb{Z}^2)$ with nonempty 1-level set and operator norm equals to 1, a finite dimensional subspace $V \subseteq \ell^1(\mathbb{Z}^2)$ and a nonnegative continuous function $F_{\varepsilon} : \mathbb{T}^2 \to [0, \infty)$ such that, for all $\vartheta \in [\vartheta - \delta_\varepsilon, \vartheta + \delta_\varepsilon]$:

1. if $a \in V$ then $a^* \in V$,
2. $\beta^f_\vartheta$ is a finite rank operator and $\beta^f_\vartheta(sa(A_\vartheta)) = V$,
3. $1_{A_\vartheta} \in V$,
4. the function:

$$(\vartheta, a) \in \left( \mathbb{R} \setminus \left\{ \frac{p}{q} \right\} \right) \times V \mapsto L_\vartheta(a)$$

is continuous,
5. if $\tau : f \in \ell^1(\mathbb{Z}^2) \to f(0)$, i.e. the restriction of the unique $\beta_\vartheta$-invariant tracial state of $A_\vartheta$ to $\ell^1(\mathbb{Z}^2)$ (noting $\tau$ does not depend on $\vartheta$), and if $E = V \cap \ker(\tau)$ while $\Sigma$ is the unit sphere in $E$ for $\| \cdot \|_{\ell^1(\mathbb{Z}^2)}$, then for any $a \in \Sigma$, if $s(a, \vartheta) = \frac{L_\vartheta(a)}{L_\vartheta(a)} > 0$ then:

$$\|\pi_\vartheta(a)T - T\pi_\vartheta(s(a)a)\|_{\ell^2(\mathbb{Z}^2)} \leq L_\vartheta(a)\varepsilon,$$

while:

$$|1 - s(a, \vartheta)| < \varepsilon;$$

6. the length of the bridge $(\mathfrak{B}(\ell^2(\mathbb{Z}^2)), T, \pi_\vartheta, \pi_\vartheta)$ is no more than $\varepsilon$, where $\mathfrak{B}(\ell^2(\mathbb{Z}^2))$ is the C*-algebra of all bounded linear operators on $\ell^2(\mathbb{Z}^2)$.

Proof. The construction of the bridges in this lemma is the matter of [10] — including the construction of $T$. We will only need its existence and the properties listed here, which involve all the work in [10] to be established.

We note that Assertions (1), (2), (3) and (4) were established in [30]; a summary is presented in [10, Theorem 3.19] (all these assertions are extended to fuzzy tori in [9]).

Assertion (5) is established as part of [10, Claim 5.15]. Assertion (6) is [10, Claim 5.15]. Of course, the computation of the length of the bridges defined in Assertion (6) provides the upper bound on the quantum propinquity between quantum tori in [10].

\[\square\]

Corollary 5.6. Let $\varepsilon > 0$; let $\delta > 0$ be given by Lemma (5.5). If for all $a \in E \setminus \{0\}$ and $\vartheta \in [\vartheta - \delta, \vartheta + \delta]$, we set $s(a, \vartheta) = \frac{L_\vartheta(a)}{L_\vartheta(a)} > 0$, then:

$$\|\pi_\vartheta(a)T - T\pi_\vartheta(s(a)a)\|_{\ell^2(\mathbb{Z}^2)} \leq L_\vartheta(a)\varepsilon.$$  

Proof. Fix $\vartheta \in [\vartheta - \delta, \vartheta + \delta]$. If $L_\vartheta(a) = 0$ then $a \in R_{1a_\vartheta}$; as $a \in E$ we conclude that $a = 0$. Thus $s(a, \vartheta)$ is well-defined.
We then note that \( s(a) = s(ra) \) for any \( r > 0 \) by definition. Moreover, if \( a \in E \) and \( a \neq 0 \), then 
\[
\frac{1}{\|a\|_{\ell^1(\mathbb{Z}^2)}} a \in \Sigma
\]
and thus by Lemma (5.5):
\[
\|\pi_0(a)T - T\pi_0(s(a))\|_{\ell^2(\mathbb{Z}^2)} = \|a\|_{\ell^1(\mathbb{Z}^2)} \pi_0 \left( \frac{1}{\|a\|_{\ell^1(\mathbb{Z}^2)}} a \right) T - T\pi_0 \left( \frac{s(\|a\|_{\ell^1(\mathbb{Z}^2)}^{-1} a)}{\|a\|_{\ell^1(\mathbb{Z}^2)}} a \right) \|_{\ell^2(\mathbb{Z}^2)} 
\leq \|a\|_{\ell^1(\mathbb{Z}^2)} \mathcal{L}_\vartheta \left( \frac{1}{\|a\|_{\ell^1(\mathbb{Z}^2)}} a \right) \varepsilon = \mathcal{L}_\vartheta(a) \varepsilon.
\]
This concludes our corollary.

We now conclude our paper with the main result of its second part, which demonstrates that the modular propinquity endows the moduli space of Heisenberg modules over quantum 2-tori with a nontrivial geometry.

**Theorem 5.7.** Let \( \| \cdot \| \) be a norm on \( \mathbb{R}^2 \). For all \( \theta \in \mathbb{R} \), we equip the quantum torus \( \mathcal{A}_\theta \) with the \( L \)-seminorm:
\[
\mathcal{L}_\vartheta : a \in sa(\mathfrak{A}) \mapsto \sup \left\{ \frac{\|e^{\exp(ix)\theta, \exp(iy)\theta} a - a\|_{\mathcal{A}_\theta}}{|(x, y)|} : (x, y) \in \mathbb{R}^2 \setminus \{0\} \right\}.
\]
For all \( p \in \mathbb{Z} \), \( q \in \mathbb{N} \setminus \{0\} \) and \( d \in q\mathbb{N} \setminus \{0\} \) and for all \( \theta \in \mathbb{R} \setminus \{ \frac{\pi}{q} \} \), we endow the Heisenberg module \( \mathcal{H}_\theta^{p,q,d} \) with the \( D \)-norm:
\[
\mathcal{D}_\theta^{p,q,d} : \xi \in \mathcal{H}_\theta^{p,q,d} \mapsto \sup \left\{ \|\xi\|_{\mathcal{H}_\theta^{p,q,d}}, \frac{\|\sigma_{\theta,d}^x \xi - \xi\|_{\mathcal{H}_\theta^{p,q,d}}}{2\pi(\theta - \frac{\pi}{q}) |(x, y)|} : (x, y) \in \mathbb{R}^2 \setminus \{0\} \right\}.
\]
Let \( p \in \mathbb{Z} \) and \( q \in \mathbb{N} \setminus \{0\} \). Let \( d \in q\mathbb{N} \setminus \{0\} \). For any \( \theta \in \mathbb{R} \setminus \{ \frac{\pi}{q} \} \), we have:
\[
\lim_{\delta \to 0} \Lambda^{\text{mod}} \left( \left( \mathcal{H}_\theta^{p,q,d}, \mathcal{H}_\theta^{p,q,d}, \mathcal{D}_\theta^{p,q,d}, \mathcal{A}_\theta, \mathcal{L}_\vartheta \right), \mathcal{H}_\theta^{p,q,d}, \mathcal{D}_\theta^{p,q,d}, \mathcal{A}_\theta, \mathcal{L}_\vartheta \right) = 0.
\]
**Proof.** Let \( p \in \mathbb{Z} \) and \( q \in \mathbb{N} \setminus \{0\} \). Let \( d \in q\mathbb{N} \setminus \{0\} \). Let \( X = \mathbb{R} \setminus \{ \frac{\pi}{q} \} \).
Let \( \theta \in X \) and let \( \varepsilon > 0 \).
We shall apply Lemma (5.5) and use its notations for \( \frac{\pi}{q} > 0 \) (rather than \( \varepsilon \)).
To begin with, for all \( \theta \in \mathbb{R} \), we note that if \( a \in \ell^1(\mathbb{Z}^2) \), then \( \beta_\vartheta^z(a) = \beta^z(a) \) does not depend on \( \vartheta \in \mathbb{R} \) for any \( z \in \mathbb{T}^2 \). Thus, the restriction of \( \beta_\vartheta^z \) to \( \ell^1(\mathbb{Z}^2) \) is independent of \( \vartheta \), valued in \( V \), and will be denoted by \( \beta^z \).
By Lemma (5.3), there exists a finite subset \( F = \{ \omega_j : j \in \{1, \ldots, N\} \} \) of \( \mathcal{D}_1 \left( \mathcal{H}_\theta^{p,q,d} \right) \setminus \{0\} \) for some \( N \in \mathbb{N} \) and \( \delta_0 > 0 \) such that, for all \( \theta \in [\theta - \delta_0, \theta + \delta_0] \), the set:
\[
\left\{ \frac{\mathcal{D}_\theta^{p,q,d}(\omega_j)}{\mathcal{D}_\theta^{p,q,d}(\omega_j)} : j \in \{1, \ldots, N\} \right\}
\]
is \( \frac{\varepsilon}{16} \)-dense in \( \mathcal{D}_1(\mathbb{D}^{p,q,d}_{\theta}) \).

We thus record:

**Summary 5.8.** Any modular bridge from \( \left( \mathcal{H}^{p,q,d}_{\theta}, \langle \cdot, \cdot \rangle_{\mathcal{H}^{p,q,d}_{\theta}}, \mathbb{D}^{p,q,d}_{\theta}, \mathcal{A}_{\theta}, L_{\theta} \right) \) to \( \left( \mathcal{H}^{p,q,d}_{\theta}, \langle \cdot, \cdot \rangle_{\mathcal{H}^{p,q,d}_{\theta}}, \mathbb{D}^{p,q,d}_{\theta}, \mathcal{A}_{\theta}, L_{\theta} \right) \) whose anchors are \( (\omega_j)_{j \in \{1, \ldots, N\}} \) and co-anchors are \( \left( \frac{D^{p,q,d}_{\theta}(\omega_j)}{D^{p,q,d}_{\theta}(\omega_j)} \right)_{j \in \{1, \ldots, N\}} \), has imprint at most \( \frac{\varepsilon}{16} \).

For each \( \omega \in F \), the map \( \vartheta \in X \mapsto D^{p,q,d}_{\theta}(\omega) \) is continuous by Proposition (4.2). The function \( \vartheta \mapsto L_{\vartheta}(\omega, \eta)_{\mathcal{H}^{p,q,d}_{\eta}} \) is also continuous for all \( \omega, \eta \in F \) (see Lemma (5.5)). Last, for any \( \omega \in F \), we note that the continuous function \( \vartheta \in X \mapsto D^{p,q,d}_{\theta}(\omega) \), reaches its minimum on the compact \( [\theta - \delta_{\mathbb{F}}, \theta + \delta_{\mathbb{F}}] \), and thus in particular, since \( \omega \neq 0 \) and \( D^{p,q,d}_{\theta} \) is a norm, this minimum is not zero (note that \( \delta_{\mathbb{F}} > 0 \) is given by Lemma (5.5)).

Thus the functions:

\[
y_{j,k}^r : \vartheta \in X \mapsto \frac{L_{\vartheta} \left( \mathbb{R}^d \rho \varphi (\omega_j, \omega_k)_{\mathcal{H}^{p,q,d}_{\theta}} \right)}{L_{\vartheta} \left( \mathbb{R}^d \rho \varphi (\omega_j, \omega_k)_{\mathcal{H}^{p,q,d}_{\theta}} \right)} \frac{D^{p,q,d}_{\theta}(\omega_j)D^{p,q,d}_{\theta}(\omega_k)}{D^{p,q,d}_{\theta}(\omega_j)D^{p,q,d}_{\theta}(\omega_k)}
\]

and

\[
y_{j,k}^3 : \vartheta \in X \mapsto \frac{L_{\vartheta} \left( \mathbb{Z}^d \rho \varphi (\omega_j, \omega_k)_{\mathcal{H}^{p,q,d}_{\theta}} \right)}{L_{\vartheta} \left( \mathbb{Z}^d \rho \varphi (\omega_j, \omega_k)_{\mathcal{H}^{p,q,d}_{\theta}} \right)} \frac{D^{p,q,d}_{\theta}(\omega_j)D^{p,q,d}_{\theta}(\omega_k)}{D^{p,q,d}_{\theta}(\omega_j)D^{p,q,d}_{\theta}(\omega_k)}
\]

are continuous as well for all \( j, k \in \{1, \ldots, N\} \). Consequently, the function:

\[
y = \max_{j,k \in \{1, \ldots, N\}} \{ |y_{j,k}^r|, |y_{j,k}^3| \}
\]

is continuous as the maximum of finitely many continuous functions. We also note that \( y(\theta) = 0 \).

Thus there exists \( \delta_2 > 0 \) such that:

\[
|y| < \frac{\varepsilon}{16} \quad \text{on} \quad [\theta - \delta_2, \theta + \delta_2].
\]

For each \( j \in \{1, \ldots, N\} \), let:

\[
\eta_j = \frac{D_{\theta}(\omega_j)}{D_{\theta}(\omega_j)} \omega_j.
\]

By construction, we have:

\[
D_{\theta}(\eta_j) = D_{\theta}(\omega_j).
\]

Last, by Lemma (3.4), there exists \( \delta_3 > 0 \) such that for all \( \vartheta \in [\theta - \delta_3, \theta + \delta_3] \) we have, for all \( j, k \in \{1, \ldots, N\} \):

\[
\left\| \langle \eta_j, \eta_k \rangle_{\mathcal{H}^{p,q,d}_{\eta}} - \langle \eta_j, \eta_k \rangle_{\mathcal{H}^{p,q,d}_{\eta}} \right\|_{L_1(\mathbb{Z}^d)} \leq \frac{\varepsilon}{16}.
\]

Let \( \delta_4 = \min\{\delta_{\mathbb{F}}, \delta_2, \delta_3\} \) and \( \vartheta \in [\theta - \delta_4, \theta + \delta_4] \).

We now begin a string of inequalities for two given \( j, k \in \{1, \ldots, N\} \). To begin with, we apply Lemma (5.5) to obtain for all \( a \in \mathcal{A}_{\theta} \):

\[
\|a - \beta \varphi a\|_{\mathcal{A}_{\theta}} \leq \|Ra - \beta \rho \varphi Ra\|_{\mathcal{A}_{\theta}} + \|3a - \beta \rho \varphi 3a\|_{\mathcal{A}_{\theta}} \leq \frac{\varepsilon}{16} (L_{\theta}(Ra) + L_{\theta}(3a)) \leq \frac{\varepsilon}{8} L_{\theta}(a).
\]
Therefore, using the inner Leibniz inequality:

\[(5.2) \quad \| \pi_\theta \left( \langle \omega_j, \omega_k \rangle_{q,p-\alpha} \right) T - T \pi_\theta \left( \langle \eta_j, \eta_k \rangle_{q,p-\alpha} \right) \|_{L^2(\mathbb{Z}^2)} \leq \| \langle \omega_j, \omega_k \rangle_{q,p-\alpha} - \beta^{Fe}(\omega_j, \omega_k) \|_{A_\theta} + \| \langle \eta_j, \eta_k \rangle_{q,p-\alpha} - \beta^{Fe}(\eta_j, \eta_k) \|_{A_\theta} + \| \pi_\theta \left( \beta^{Fe}(\omega_j, \omega_k) - \beta^{Fe}(\eta_j, \eta_k) \right) \|_{L^2(\mathbb{Z}^2)} \]

\[\leq \frac{\varepsilon}{8} \left( L_\theta \left( \langle \omega_j, \omega_k \rangle_{q,p-\alpha} \right) + L_\theta \left( \langle \eta_j, \eta_k \rangle_{q,p-\alpha} \right) \right) + \| \pi_\theta \left( \beta^{Fe}(\omega_j, \omega_k) - \beta^{Fe}(\eta_j, \eta_k) \right) \|_{L^2(\mathbb{Z}^2)} \]

\[\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{16}. \]

Our next step is to replace \(\beta^{Fe}(\eta_j, \eta_k)\) with \(\beta^{Fe}(\eta_j, \eta_k)\), because our work in [10] follows a single element in \(V\) from \(A_\theta\) to \(A_0\).

Now, noting that \(\beta^{Fe}\) has norm 1:

\[\| \beta^{Fe}(\eta_j, \eta_k) \|_{A_0} \leq \| \beta^{Fe}(\eta_j, \eta_k) - \beta^{Fe}(\eta_j, \eta_k) \|_{A_\theta} + \| \beta^{Fe}(\eta_j, \eta_k) - \beta^{Fe}(\eta_j, \eta_k) \|_{A_0} \leq \| \beta^{Fe}(\eta_j, \eta_k) - \beta^{Fe}(\eta_j, \eta_k) \|_{A_0} + \| \beta^{Fe}(\eta_j, \eta_k) - \beta^{Fe}(\eta_j, \eta_k) \|_{A_1(\mathbb{Z}^2)}. \]

Thus we conclude, as \(\| T \|_{L^2(\mathbb{Z}^2)} = 1:\)

\[(5.3) \quad \| \pi_\theta \left( \beta^{Fe}(\omega_j, \omega_k) \right) T - T \pi_\theta \left( \beta^{Fe}(\eta_j, \eta_k) \right) \|_{L^2(\mathbb{Z}^2)} \leq \| \pi_\theta \left( \beta^{Fe}(\omega_j, \omega_k) - \beta^{Fe}(\eta_j, \eta_k) \right) \|_{L^2(\mathbb{Z}^2)} + \frac{\varepsilon}{16}. \]

Inserting Inequality (5.3) in Inequality (5.2) we thus have:

\[(5.4) \quad \| \pi_\theta \left( \langle \omega_j, \omega_k \rangle_{q,p-\alpha} \right) T - T \pi_\theta \left( \langle \eta_j, \eta_k \rangle_{q,p-\alpha} \right) \|_{L^2(\mathbb{Z}^2)} \leq \frac{9\varepsilon}{16} + \| \pi_\theta \left( \beta^{Fe}(\omega_j, \omega_k) - \beta^{Fe}(\eta_j, \eta_k) \right) \|_{L^2(\mathbb{Z}^2)}. \]
We now insert the factor \( s(\cdot) \) of Lemma (5.5) into our inequality. To make notations somewhat easier to read, we set:

\[
\begin{align*}
\tau_{j,k}^R &= s \left( \mathcal{R} \langle \omega_j, \omega_k \rangle, \mathcal{R}^{p,q,d} \right) = \frac{L_0 \left( \mathcal{R} \langle \omega_j, \omega_k \rangle, \mathcal{G}^{p,q,d} \right)}{L_0 \left( \mathcal{R} \langle \omega_j, \omega_k \rangle, \mathcal{G}^{p,q,d} \right)} \\
\end{align*}
\]

and:

\[
\begin{align*}
\tau_{j,k}^\mathcal{G} &= s \left( \mathcal{G} \langle \omega_j, \omega_k \rangle, \mathcal{R}^{p,q,d} \right) = \frac{L_0 \left( \mathcal{G} \langle \omega_j, \omega_k \rangle, \mathcal{G}^{p,q,d} \right)}{L_0 \left( \mathcal{G} \langle \omega_j, \omega_k \rangle, \mathcal{G}^{p,q,d} \right)}
\end{align*}
\]

We thus compute:

\[
\begin{align*}
(5.5) \quad \left\| \mathcal{R} \left( \pi_0 \left( \beta \mathcal{Fe} \langle \omega_j, \omega_k \rangle, \mathcal{G}^{p,q,d} \right) \right) - T \pi_0 \left( \beta \mathcal{Fe} \langle \eta_j, \eta_k \rangle, \mathcal{G}^{p,q,d} \right) \right\|_{L^2(\mathbb{Z}^2)} \\
&\leq \left\| \pi_0 \left( \mathcal{R} \beta \mathcal{Fe} \langle \omega_j, \omega_k \rangle, \mathcal{G}^{p,q,d} \right) T - T \pi_0 \left( \beta \mathcal{Fe} \langle \eta_j, \eta_k \rangle, \mathcal{G}^{p,q,d} \right) \right\|_{L^2(\mathbb{Z}^2)} \\
&\leq \left\| \pi_0 \left( \mathcal{R} \beta \mathcal{Fe} \langle \omega_j, \omega_k \rangle, \mathcal{G}^{p,q,d} \right) T - T \pi_0 \left( \tau_{j,k}^R \mathcal{R} \beta \mathcal{Fe} \langle \omega_j, \omega_k \rangle, \mathcal{G}^{p,q,d} \right) \right\|_{L^2(\mathbb{Z}^2)} \\
&\quad + \left\| \pi_0 \left( \tau_{j,k}^R - \mathcal{D}_0^{p,q,d}(\omega_j) \mathcal{D}_0^{p,q,d}(\omega_k) \right) \mathcal{R} \beta \mathcal{Fe} \langle \omega_j, \omega_k \rangle, \mathcal{G}^{p,q,d} \right\|_{L^2(\mathbb{Z}^2)} \\
&\leq \left\| \pi_0 \left( \mathcal{R} \beta \mathcal{Fe} \langle \omega_j, \omega_k \rangle, \mathcal{G}^{p,q,d} \right) T - T \pi_0 \left( s_{j,k}^R \mathcal{R} \beta \mathcal{Fe} \langle \omega_j, \omega_k \rangle, \mathcal{G}^{p,q,d} \right) \right\|_{L^2(\mathbb{Z}^2)} \\
&\quad + \left\| \left( s_{j,k}^R - \mathcal{D}_0^{p,q,d}(\omega_j) \mathcal{D}_0^{p,q,d}(\omega_k) \right) \mathcal{R} \beta \mathcal{Fe} \langle \omega_j, \omega_k \rangle, \mathcal{G}^{p,q,d} \right\|_{L^2(\mathbb{Z}^2)} \cdot \left| q_\theta \right|.
\end{align*}
\]

By assumption on \( \vartheta \), we have:

\[
\begin{align*}
(5.6) \quad \left\| \left( s_{j,k}^R - \mathcal{D}_0^{p,q,d}(\omega_j) \mathcal{D}_0^{p,q,d}(\omega_k) \right) \mathcal{R} \beta \mathcal{Fe} \langle \omega_j, \omega_k \rangle, \mathcal{G}^{p,q,d} \right\|_{A_\vartheta} \leq y(\vartheta) < \frac{\varepsilon}{16},
\end{align*}
\]

and thus, plugging Inequality (5.6) in Inequality (5.5), we obtain:

\[
\begin{align*}
(5.7) \quad \left\| \pi_0 \left( \mathcal{R} \beta \mathcal{Fe} \langle \omega_j, \omega_k \rangle, \mathcal{G}^{p,q,d} \right) T - T \pi_0 \left( \mathcal{R} \beta \mathcal{Fe} \langle \eta_j, \eta_k \rangle, \mathcal{G}^{p,q,d} \right) \right\|_{L^2(\mathbb{Z}^2)} \\
&\leq \left\| \pi_0 \left( \mathcal{R} \beta \mathcal{Fe} \langle \omega_j, \omega_k \rangle, \mathcal{G}^{p,q,d} \right) T - T \pi_0 \left( s_{j,k}^R \mathcal{R} \beta \mathcal{Fe} \langle \omega_j, \omega_k \rangle, \mathcal{G}^{p,q,d} \right) \right\|_{L^2(\mathbb{Z}^2)} + \frac{\varepsilon}{16}.
\end{align*}
\]

The elements \( \mathcal{R} \beta \mathcal{Fe} \langle \omega_j, \omega_k \rangle, \mathcal{G}^{p,q,d} \), for all \( \vartheta \in X \), lie in \( V \). We now wish them to lie in \( E = \ker \tau \cap V \) with \( \tau : f \in L^2(\mathbb{Z}^2) \mapsto f(0) \) to use Lemma (5.5). Again to ease notations, let:

\[
\tau_{j,k}^\mathcal{R} = \tau \left( \beta \mathcal{Fe} \langle \omega_j, \omega_k \rangle, \mathcal{G}^{p,q,d} \right).
\]

Of course, \( \tau_{j,k}^\mathcal{R} = \tau \left( \mathcal{R} \beta \mathcal{Fe} \langle \omega_j, \omega_k \rangle, \mathcal{G}^{p,q,d} \right) = \tau \left( \mathcal{G} \beta \mathcal{Fe} \langle \omega_j, \omega_k \rangle, \mathcal{G}^{p,q,d} \right) \).

We thus evaluate:

\[
\begin{align*}
\pi_0 \left( \mathcal{R} \beta \mathcal{Fe} \langle \omega_j, \omega_k \rangle, \mathcal{G}^{p,q,d} \right) T - T \pi_0 \left( s_{j,k}^R \mathcal{R} \beta \mathcal{Fe} \langle \omega_j, \omega_k \rangle, \mathcal{G}^{p,q,d} \right) \right\|_{L^2(\mathbb{Z}^2)}
\end{align*}
\]
\[ \begin{align*}
&\leq \left\| \pi_\theta \left( \Re \beta^{Fe} \langle \omega_j, \omega_k \rangle_{\mathcal{G}^p,q,d} - \tau_{th} \right) T - T \pi_\theta \left( s_{j,k} \Re \beta^{Fe} \langle \omega_j, \omega_k \rangle_{\mathcal{G}^p,q,d} - s_{j,k} \tau_{th} \right) \right\|_{\ell^2(\mathbb{Z}^2)} \\
&\quad + \left| \tau_{th} - s_{j,k} \tau \right|.
\end{align*} \]

Now \(|\tau_{th} - s_{j,k} \tau| \leq |1 - s_{j,k} \tau| \leq |1 - s_{j,k}| < \frac{\varepsilon}{16} \) since \(\tau_{j,k} \leq \|\langle \omega_j, \omega_k \rangle_{\mathcal{G}^p,q,d}\|_{A_0} \leq 1\). We thus have:

\[\begin{align*}
(5.8) \quad &\|\pi_\theta \left( \Re \beta^{Fe} \langle \omega_j, \omega_k \rangle_{\mathcal{G}^p,q,d} - \tau_{th} \right) T - T \pi_\theta \left( s_{j,k} \Re \beta^{Fe} \langle \omega_j, \omega_k \rangle_{\mathcal{G}^p,q,d} - s_{j,k} \tau_{th} \right) \|_{\ell^2(\mathbb{Z}^2)} \\
&\leq \left\| \pi_\theta \left( \Re \beta^{Fe} \langle \omega_j, \omega_k \rangle_{\mathcal{G}^p,q,d} - \tau_{j,k} \right) T - T \pi_\theta \left( s_{j,k} \Re \beta^{Fe} \langle \omega_j, \omega_k \rangle_{\mathcal{G}^p,q,d} - s_{j,k} \tau_{j,k} \right) \right\|_{\ell^2(\mathbb{Z}^2)} \\
&\quad + \frac{\varepsilon}{16}.
\end{align*}\]

We are now in the position to apply Lemma (5.5) and conclude:

\[\begin{align*}
(5.9) \quad &\left\| \pi_\theta \left( \Re \beta^{Fe} \langle \omega_j, \omega_k \rangle_{\mathcal{G}^p,q,d} - \tau_{j,k} \right) T - T \pi_\theta \left( s_{j,k} \Re \beta^{Fe} \langle \omega_j, \omega_k \rangle_{\mathcal{G}^p,q,d} - s_{j,k} \tau_{j,k} \right) \right\|_{\ell^2(\mathbb{Z}^2)} \\
&\leq \frac{\varepsilon}{16}.
\end{align*}\]

We now insert Inequality (5.9) into Inequality (5.8) and the result in Inequality (5.7) to conclude:

\[\begin{align*}
(5.10) \quad &\left\| \Re \left( \pi_\theta \left( \beta^{Fe} \langle \omega_j, \omega_k \rangle_{\mathcal{G}^p,q,d} - \tau_{j,k} \right) T - T \pi_\theta \left( \beta^{Fe} \langle \eta_j, \eta_k \rangle_{\mathcal{G}^p,q,d} \right) \right) \right\|_{\ell^2(\mathbb{Z}^2)} \leq \frac{3\varepsilon}{16}.
\end{align*}\]

We get the same inequality as Inequality (5.10) for \(\Im\) in place of \(\Re\) by the same reasoning, so we get:

\[\begin{align*}
(5.11) \quad &\left\| \pi_\theta \left( \beta^{Fe} \langle \omega_j, \omega_k \rangle_{\mathcal{G}^p,q,d} - \tau_{j,k} \right) T - T \pi_\theta \left( \beta^{Fe} \langle \eta_j, \eta_k \rangle_{\mathcal{G}^p,q,d} \right) \right\|_{\ell^2(\mathbb{Z}^2)} \\
&\leq \left\| \Re \left( \pi_\theta \left( \beta^{Fe} \langle \omega_j, \omega_k \rangle_{\mathcal{G}^p,q,d} - \tau_{j,k} \right) T - T \pi_\theta \left( \beta^{Fe} \langle \eta_j, \eta_k \rangle_{\mathcal{G}^p,q,d} \right) \right) \right\|_{\ell^2(\mathbb{Z}^2)} \\
&\quad + \left\| \Im \left( \pi_\theta \left( \beta^{Fe} \langle \omega_j, \omega_k \rangle_{\mathcal{G}^p,q,d} - \tau_{j,k} \right) T - T \pi_\theta \left( \beta^{Fe} \langle \eta_j, \eta_k \rangle_{\mathcal{G}^p,q,d} \right) \right) \right\|_{\ell^2(\mathbb{Z}^2)} \\
&\leq \frac{3\varepsilon}{8}.
\end{align*}\]

Thus inserting Inequality (5.11) in Inequality (5.4), we conclude:

\[\begin{align*}
(5.12) \quad &\left\| \pi_\theta \left( \langle \omega_j, \omega_k \rangle_{\mathcal{G}^p,q,d} - \tau_{j,k} \right) T - T \pi_\theta \left( \langle \eta_j, \eta_k \rangle_{\mathcal{G}^p,q,d} \right) \right\|_{\ell^2(\mathbb{Z}^2)} \leq \frac{9\varepsilon}{16} + \frac{3\varepsilon}{8} = \frac{15\varepsilon}{16}.
\end{align*}\]

By construction, the following is a modular bridge (note that \(\|T\|_{\ell^2(\mathbb{Z}^2)} = 1\): \(\gamma_\theta = (\mathcal{B}(\ell^2(\mathbb{Z}^2)), T, \pi_\theta, \pi_\theta, \langle \omega_j \rangle_{j \in \{1, \ldots, n\}}, \langle \eta_j \rangle_{j \in \{1, \ldots, n\}})\).

By Lemma (5.5), the length of the basic bridge \(\gamma_\theta\) is no more than \(\frac{\varepsilon}{16}\), so the basic reach and the height of \(\gamma\) are bounded by \(\frac{\varepsilon}{16}\). Now, Expression (5.12) states that the modular reach of \(\gamma\) is bounded above by \(\frac{15\varepsilon}{16}\). Thus by Definition (2.8), the reach of \(\gamma\) is no more than \(\frac{9\varepsilon}{16} + \frac{15\varepsilon}{16} = \varepsilon\).

By Summary (5.8), the imprint of \(\gamma\) is no more than \(\frac{\varepsilon}{16}\).
Thus by Definition (2.8), the length of $\gamma$ is no more than $\varepsilon = \max\{\varepsilon, \varepsilon_1\}$. If we identify $\gamma$ with the modular trek $(\gamma)$, we conclude by Theorem-Definition (2.9) that:

$$\Lambda_{\text{mod}}\left(\left(H_{p,q,d}^{\varphi}, \langle \cdot, \cdot \rangle_{H_{p,q,d}^{\varphi}}, D_{p,q,d}^{\varphi}, A^{\varphi}, L^{\varphi}\right), \left(H_{p,q,d}^{\varphi}, \langle \cdot, \cdot \rangle_{H_{p,q,d}^{\varphi}}, D_{p,q,d}^{\varphi}, A^{\varphi}, L^{\varphi}\right)\right) \leq \varepsilon.$$ 

This concludes our proof. □

We conclude with an interesting observation. The proof of Theorem (5.7) reveals that the class of Heisenberg modules over all quantum 2-tori is actually iso-pivotal. Trivially, we can include in this class all the free modules over quantum 2-tori and keep the class iso-pivotal. Hence we conclude that the direct sum of metrized quantum vector bundles on the class of all free modules of finite rank and all Heisenberg modules over all quantum 2-tori is in fact jointly continuous with respect to the modular propinquity by [15, Theorem 8.2]. We thus get some additional convergence results from our work for our new metric.

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