On the uniqueness of $D = 11$ interactions among a graviton, a massless gravitino and a three-form.

I: Pauli-Fierz and three-form

E. M. Cioroianu*, E. Diaconu†, S. C. Sararu‡
Faculty of Physics, University of Craiova,
13 Al. I. Cuza Street Craiova, 200585, Romania

Abstract

Cross-couplings between a massless spin-two field (described in the free limit by the Pauli-Fierz action) and an Abelian three-form gauge field in $D = 11$ are investigated in the framework of the deformation theory based on local BRST cohomology. These consistent interactions are obtained on the grounds of smoothness in the coupling constant, locality, Lorentz covariance, Poincaré invariance, and the presence of at most two derivatives in the interacting Lagrangian. Our results confirm the uniqueness of the eleven-dimensional interactions between a graviton and a three-form prescribed by General Relativity.

PACS number: 11.10.Ef

1 Introduction

A key point in the development of the BRST formalism was its cohomological understanding, which allowed, among others, a useful investigation of many interesting aspects related to the perturbative renormalization problem [1]–[5], the anomaly-tracking mechanism [5]–[10], the simultaneous study of local and rigid invariances of a given theory [11] as well as the reformulation of the construction of consistent interactions in gauge theories [12]–[16] in terms of the deformation theory [17]–[21] or, actually, in terms of the deformation of the solution to the master equation. The impossibility of cross-interactions among several Einstein (Weyl) gravitons, see Ref. [22] (or respectively Ref. [23]), and of cross-couplings among different Einstein gravitons in the presence of matter fields [22, 24]–[27] has recently been shown by means of cohomological arguments. In the same context the uniqueness of $D = 4$, $N = 1$ supergravity was proved in Ref. [28].
On the other hand, $D = 11$, $N = 1$ supergravity \cite{29, 30} has regained a central role with the advent of M-theory, whose QFT (local) limit it is. Of the many special properties of $D = 11$, $N = 1$ supergravity, one of the most striking is that it forbids a cosmological term. The proof of this result has been done in Ref. \cite{31} using a combined technique — the standard Noether current method and a cohomological approach. It is known that the field content of $D = 11$, $N = 1$ supergravity is quite simple; it comprises a graviton, a massless Majorana spin-3/2 field, and a three-form gauge field. The analysis of all possible interactions in $D = 11$ related to this field content necessitates the study of cross-couplings involving each pair of these sorts of fields and then the construction of simultaneous interactions among all the three types of fields. One of the most efficient and meanwhile elegant approaches to the problem of constructing consistent interactions in gauge field theories\footnote{By ‘consistent’ we mean that the interacting theory preserves both the field content and the number of independent gauge symmetries of the free one.} is that based on the deformation technique \cite{17} combined with local BRST cohomology \cite{32, 33}. This approach relies on computing the deformations of the solution to the master equation for the interacting theory with the help of the ‘free’ BRST cohomology. Our main aim is to construct all consistent interactions in $D = 11$ that can be added to a free theory describing a Pauli-Fierz graviton, a massless Rarita-Schwinger gravitino, and an Abelian three-form gauge field from the deformation of the ‘free’ solution to the master equation such that the interactions satisfy some general and quite natural assumptions (smoothness in the coupling constant, locality, Lorentz covariance, Poincaré invariance, and preservation of the differential order of the free field equations at the level of the coupled theory). One of the final outcomes of this procedure will be the quest for the uniqueness of $D = 11$, $N = 1$ SUGRA. In order to organize the results as logical as possible, to expose in detail the cohomological aspects involved, and (last but not least) make various comments on and comparisons with other results from the literature we chose to split our work into four main parts. The first three are dedicated to the construction of consistent interactions that involve only two of the three types of fields under considerations: i) a graviton and a three-form (present paper); ii) a three-form and massless gravitini \cite{34}; iii) massless gravitini and a graviton \cite{35}. The fourth and last part \cite{36} will put the things together and present what happens when all these fields are present: what new vertices appear, how consistent are those obtained from the previous steps, and how does the overall coupled theory looks like.

In this work we implement the first of the four steps explained in the above, namely we analyze the cross-couplings between a massless spin-two field (described in the free limit by the Pauli-Fierz action \cite{37, 38}) and an Abelian three-form gauge field in eleven spacetime dimensions. The cross-interactions are obtained under the hypotheses of smoothness of the interactions in the coupling constant, locality, Poincaré invariance, Lorentz covariance, and the presence of at most two derivatives in the Lagrangian of the interacting theory (the same number of derivatives like in the free Lagrangian). Our results
are obtained in the context of the deformation of the solution to the master equation.

We compute the interaction terms to order two in the coupling constant. In this way we obtain that the first two orders of the interacting Lagrangian resulting from our setting originate in the development of the full interacting Lagrangian (in eleven spacetime dimensions)

\[ \hat{\mathcal{L}} = \frac{2}{\lambda^2} \sqrt{g} \left( R - 2\lambda^2 \Lambda \right) + \mathcal{L}^{h-\Lambda}, \]

where the cross-coupling part reads as

\[ \mathcal{L}^{h-\Lambda} = -\frac{1}{2} \cdot \frac{1}{4!} \sqrt{g} \bar{F}_{\mu\nu\rho\lambda} \bar{F}^{\mu\nu\rho\lambda} + \lambda q \epsilon_{\mu_1 \ldots \mu_11} \bar{A}_{\mu_1 \mu_2 \mu_3} \bar{F}_{\mu_4 \ldots \mu_7} \bar{F}_{\mu_8 \ldots \mu_{11}}, \]

with \( g = \det g_{\mu\nu}, \Lambda \) the cosmological constant, \( \lambda \) the coupling constant, and \( q \) an arbitrary, real constant. Consequently, we show the uniqueness of interactions described by \( \hat{\mathcal{L}} \). The above interacting Lagrangian for \( \Lambda = 0 \) is a part of \( D = 11, N = 1 \) SUGRA Lagrangian. We note that the graviton sector is allowed at this stage to include a cosmological term, unlike \( D = 11, N = 1 \) SUGRA. This is not a surprise since it is the simultaneous presence of all fields (supplemented with massless gravitini) that ensures the annihilation of the cosmological constant, as it will be made clear in Ref. [36].

This paper is organized in six sections. In section 2 we construct the BRST symmetry of the free model, consisting in a Pauli-Fierz and an Abelian three-form gauge field. Section 3 briefly addresses the deformation procedure based on BRST symmetry. In section 4 we compute the first two orders of the interactions between the massless spin-two field and an Abelian three-form gauge field. Section 5 is devoted to analyzing the deformed theory obtained in the previous section. In this context we obtain a possible candidate that describes the interacting theory to all orders in the coupling constant. Section 6 is dedicated to the investigation of the uniqueness of interactions described by the candidate emphasized in the previous section. The last section exposes the main conclusions on this paper.

## 2 Free model: Lagrangian formulation and BRST symmetry

Our starting point is represented by a free Lagrangian action, written as the sum between the linearized Hilbert-Einstein action (also known as the Pauli-Fierz action) and the action for an Abelian three-form gauge field in eleven spacetime dimensions

\[ S_0^L [h_{\mu\nu}, A_{\mu\nu\rho}] = \int d^{11}x \left( -\frac{1}{2} \left( \partial_\mu h_{\nu\rho} \right) \left( \partial^\mu h^{\nu\rho} \right) + \left( \partial_\mu h^{\mu\rho} \right) \left( \partial^\rho h_{\nu\rho} \right) - \left( \partial_\mu h \right) \left( \partial_\nu h^{\nu\mu} \right) + \frac{1}{2} \left( \partial_\mu h \right) \left( \partial^\mu h \right) - \frac{1}{2} \cdot \frac{1}{4!} F_{\mu\nu\rho\lambda} F^{\mu\nu\rho\lambda} \right) \]
$$\equiv \int d^{11} x \left( \mathcal{L}^b + \mathcal{L}^A_0 \right).$$

(1)

Throughout the paper we work with the flat metric of ‘mostly minus’ signature, $\sigma_{\mu\nu} = (+ - \cdots -)$. In the above $h$ denotes the trace of the Pauli-Fierz field, $h = \sigma_{\mu\nu} h^{\mu\nu}$, and $F_{\mu\nu\rho\lambda}$ denotes the field-strength of the three-form gauge field ($F_{\mu\nu\rho\lambda} \equiv \partial_{[\mu} A_{\nu\rho\lambda]}$). The notation $[\mu \cdots \nu]$ (respectively $(\mu \cdots \nu)$) signifies antisymmetry (respectively symmetry) with respect to all indices between brackets without normalization factors (i.e., the independent terms appear only once and are not multiplied by overall numerical factors). The theory described by action (1) possesses an Abelian generating set of gauge transformations

$$\delta_{\epsilon,\varepsilon} h_{\mu\nu} = \partial_{[\mu} \varepsilon_{\nu]}, \quad \delta_{\epsilon,\varepsilon} A_{\mu\nu\rho} = \partial_{[\mu} \varepsilon_{\nu\rho]},$$

(2)

where the gauge parameters $\varepsilon^{\Gamma_1} = \{\epsilon_\mu, \varepsilon_{\mu\nu}\}$ are bosonic functions, with the last set completely antisymmetric. We observe that if in (2) we make the transformations

$$\varepsilon_{\mu\nu} \rightarrow \varepsilon^{(\theta)}_{\mu\nu} = \partial_{[\mu} \theta_{\nu]},$$

(3)

then the gauge variation of the three-form identically vanishes

$$\delta_{\varepsilon^{(\theta)}} A_{\mu\nu\rho} \equiv 0.$$

(4)

Moreover, if in (3) we perform the changes

$$\theta_{\mu} \rightarrow \theta^{(\phi)}_{\mu} = \partial_{\mu} \phi,$$

(5)

with $\phi$ an arbitrary scalar field, then the transformed gauge parameters from (3) identically vanish

$$\varepsilon^{(\theta^{(\phi)})}_{\mu\nu} \equiv 0.$$

(6)

Meanwhile, there is no nonvanishing local transformation of $\phi$ that annihilates $\theta^{(\phi)}_{\mu}$ of the form (5), and hence no further local reducibility identity. All these allow us to conclude that the generating set of gauge transformations given in (2) is off-shell, second-stage reducible. It is obvious that the accompanying gauge algebra is Abelian.

In order to construct the BRST symmetry for (1) we introduce the field, ghost, and antifield spectra

$$\Phi^{\Gamma_0} = (h_{\mu\nu}, A_{\mu\nu\rho}), \quad \Phi^*_{\Gamma_0} = (h^*_{\mu\nu}, A^*_{\mu\nu\rho}),$$

(7)

$$\eta^{\Gamma_1} = (\eta_\mu, C_{\mu\nu}), \quad \eta^*_{\Gamma_1} = (\eta^*_{\mu}, C^*_{\mu\nu}),$$

(8)

$$\eta^{\Gamma_2} = (C_{\mu}), \quad \eta^*_{\Gamma_2} = (C^*_{\mu}),$$

(9)

$$\eta^{\Gamma_3} = (\chi), \quad \eta^*_{\Gamma_3} = (\chi^*).$$

(10)

The fermionic ghosts $\eta^{\Gamma_1}$ respectively correspond to the bosonic gauge parameters $\varepsilon^{\Gamma_1}$ from (2), the bosonic ghosts for ghosts $\eta^{\Gamma_2}$ are associated with the first-stage reducibility parameters $\theta_{\mu}$ in (3), while the fermionic ghost for ghost for ghost $\eta^{\Gamma_3}$ is present due to the second-stage reducibility parameter $\phi$ from (4).
The star variables represent the antifields of the corresponding fields/ghosts. Their Grassmann parities are obtained via the standard rule of the BRST method
\[ \varepsilon(\chi^*) = (\varepsilon(\chi) + 1) \mod 2, \]
where we employed the notations
\[ \chi^\Gamma = (\Phi^\Gamma_0, \eta^\Gamma_1, \eta^\Gamma_2, \eta^\Gamma_3), \quad \chi^*_{\Gamma} = (\Phi^*_{\Gamma_0}, \eta^*_{\Gamma_1}, \eta^*_{\Gamma_2}, \eta^*_{\Gamma_3}). \]

Since both the gauge generators and the reducibility functions for this model are field-independent, it follows that the BRST differential \( s \) reduces to
\[ s = \delta + \gamma, \]
where \( \delta \) is the Koszul-Tate differential and \( \gamma \) denotes the exterior longitudinal derivative. The Koszul-Tate differential is graded in terms of the antighost number \( \text{agh} \) and enforces a resolution of the algebra of smooth functions defined on the stationary surface of field equations for action \( C_\infty(\Sigma) \), \( \Sigma : \delta S_L / \delta \Phi^0 = 0 \). The exterior longitudinal derivative is graded in terms of the pure ghost number \( \text{pgh} \) and is correlated with the original gauge symmetry via its cohomology in pure ghost number zero computed in \( C_\infty(\Sigma) \), which is isomorphic to the algebra of physical observables for this free theory. These two degrees of the generators \((7)-(10)\) from the BRST complex are valued as
\[ \text{pgh}(\Phi^\Gamma_0) = 0, \quad \text{pgh}(\eta^\Gamma_k) = k, \]
\[ \text{pgh}(\Phi^*_{\Gamma_0}) = 0, \quad \text{pgh}(\eta^*_{\Gamma_k}) = 0, \]
\[ \text{agh}(\Phi^\Gamma_0) = 0, \quad \text{agh}(\eta^\Gamma_k) = 0, \]
\[ \text{agh}(\Phi^*_{\Gamma_0}) = 1, \quad \text{agh}(\eta^*_{\Gamma_k}) = k + 1, \]
for \( k = 1,3 \). The actions of the differentials \( \delta \) and \( \gamma \) on the generators from the BRST complex are given by
\[ \delta h^{*\mu\nu} = 2H^{\mu\nu}, \quad \delta A^{*\mu\nu\rho} = \frac{1}{3!} \partial_\lambda F^{\mu\nu\rho\lambda}, \]
\[ \delta \eta^* = -2 \partial_\nu h^{*\mu\nu}, \quad \delta C^{*\mu\nu} = -3 \partial_\mu A^{*\mu\nu\rho}, \]
\[ \delta C^{*\mu} = -2 \partial_\nu C^{*\mu\nu}, \quad \delta C^* = -\partial_\mu A^{*\mu\nu}, \quad \delta \chi^\Gamma = 0, \]
\[ \gamma \chi^\Gamma = 0, \quad \gamma h^{*\mu\nu} = \partial_\mu (\eta^* \eta^\nu), \quad \gamma A^{*\mu\nu\rho} = \partial_\mu (\eta^* C^{\nu\rho}), \]
\[ \gamma \eta^* \mu = 0, \quad \gamma C^{*\mu\nu} = \partial_\mu (\eta^* C^{\nu}), \quad \gamma C^* = \partial_\mu C^*, \quad \gamma C = 0. \]

In the above \( H^{\mu\nu} = K^{\mu\nu} - \frac{1}{2} \sigma^{\mu\nu} K \) is the linearized Einstein tensor, with \( K^{\mu\nu} \) and \( K \) the linearized Ricci tensor and respectively the linearized scalar curvature, both obtained from the linearized Riemann tensor \( K^{\mu\nu\alpha\beta} = \frac{1}{2} \partial_\mu (h^\nu |_{\alpha\beta} \| \eta^\alpha_{\beta}) \) via its trace and respectively double trace: \( K^\alpha_{\beta} = \sigma^{\mu\beta} K^{\mu\alpha\beta} \) and respectively \( K = \sigma^{\mu\alpha} \eta^\alpha_{\beta} K^{\mu\alpha\beta} \).

The BRST differential is known to have a canonical action in a structure named antibracket and denoted by the symbol \( \{,\} \) \( (s \cdot (\cdot, S)) \), which is obtained by considering the fields/ghosts respectively conjugated to the corresponding antifields. The generator of the BRST symmetry is a bosonic functional of ghost
number zero, which is solution to the classical master equation $(S, S) = 0$. The full solution to the master equation for the free model under study reads as

$$S^{h,A} = S^L_0 + \int d^{11} x \left( h^{\mu\rho} \partial_{(\mu} \eta_{\nu)} + A^{*\mu\nu} \partial_{[\mu} C_{\nu]p} + C^{*\mu} \partial_{[\mu} C_{\nu]} + C^{*\nu} \partial_{[\mu} C_{\nu]} \right).$$

(22)

The solution to the master equation encodes all the information on the gauge structure of a given theory.

3  Deformation of the solution to the master equation: a brief review

We begin with a “free” gauge theory, described by a Lagrangian action $S^L_0 [\Phi^0]$, invariant under some gauge transformations $\delta_\epsilon \Phi^0 = Z^{\Gamma_1}_0 \epsilon^{\Gamma_1}$, i.e. $\delta S^L_0 = 0$, and consider the problem of constructing consistent interactions among the fields $\Phi^0$ such that the couplings preserve the field spectrum and the original number of gauge symmetries. This matter is addressed by means of reformulating the problem of constructing consistent interactions as a deformation problem of the solution to the master equation corresponding to the “free” theory \cite{17}. Such a reformulation is possible due to the fact that the solution to the master equation contains all the information on the gauge structure of the theory. If an interacting gauge theory can be consistently constructed, then the solution $S$ to the master equation associated with the “free” theory, $(S, S) = 0$, can be deformed into a solution $\bar{S}$

$$S \rightarrow \bar{S} = S + \lambda S_1 + \lambda^2 S_2 + \cdots = S + \lambda \int d^D x a + \lambda^2 \int d^D x b + \cdots$$

(23)

of the master equation for the deformed theory

$$(\bar{S}, \bar{S}) = 0,$$ \hspace{1cm} (24)

such that both the ghost and antifield spectra of the initial theory are preserved. Equation (24) splits, according to the various orders in the coupling constant (deformation parameter) $\lambda$, into a tower of equations:

$$\begin{align*}
(S, S) &= 0, \\
2(S_1, S) &= 0, \\
2(S_2, S) + (S_1, S_1) &= 0, \\
(S_3, S) + (S_1, S_2) &= 0, \quad \vdots
\end{align*}$$

(25) \hspace{1cm} (26) \hspace{1cm} (27) \hspace{1cm} (28)

Equation (25) is fulfilled by hypothesis. The next equation requires that the first-order deformation of the solution to the master equation, $S_1$, is a cocycle of the “free” BRST differential $s$, $sS_1 = 0$. However, only cohomologically
nontrivial solutions to (26) should be taken into account, since the BRST-exact ones can be eliminated by some (in general nonlinear) field redefinitions. This means that $S_1$ pertains to the ghost number zero cohomological space of $s$, $H^0(s)$, which is generically nonempty because it is isomorphic to the space of physical observables of the “free” theory. It has been shown (by of the triviality of the antibracket map in the cohomology of the BRST differential) that there are no obstructions in finding solutions to the remaining equations, namely (27)–(28), etc. However, the resulting interactions may be nonlocal and there might even appear obstructions if one insists on their locality. The analysis of these obstructions can be done with the help of cohomological techniques.

4 Consistent interactions between the Pauli-Fierz field and an Abelian three-form gauge field

4.1 Standard material: basic cohomologies

The aim of this section is to investigate the cross-couplings that can be introduced between a Pauli-Fierz field and an Abelian three-form gauge field. This matter is addressed in the context of the antifield-BRST deformation procedure described in the above and relies on computing the solutions to equations (26)–(28), etc., with the help of the BRST cohomology of the free theory. The interactions are obtained under the following (reasonable) assumptions: smoothness in the deformation parameter, locality, Lorentz covariance, Poincaré invariance, and the presence of at most two derivatives in the interacting Lagrangian. ‘Smoothness in the deformation parameter’ refers to the fact that the deformed solution to the master equation, (29), is smooth in the coupling constant $\lambda$ and reduces to the original solution, (22), in the free limit $\lambda = 0$. The requirement on the interacting theory to be Poincaré invariant means that one does not allow an explicit dependence on the spacetime coordinates into the deformed solution to the master equation. The requirement concerning the maximum number of derivatives allowed to enter the interacting Lagrangian is frequently imposed in the literature at the level of interacting theories; for instance, see the case of cross-interactions for a collection of Pauli-Fierz fields, Ref. [22], the couplings between the Pauli-Fierz and the massless Rarita-Schwinger fields, Ref. [28], or the direct cross-interactions for a collection of Weyl gravitons, Ref. [23]. Equation (26), which we have seen that controls the first-order deformation, takes the local form

$$sa = \partial_\mu m^\mu, \quad gh(a) = 0, \quad \varepsilon(a) = 0,$$  

(29)

for some local $m^\mu$, and it shows that the nonintegrated density of the first-order deformation pertains to the local cohomology of the free BRST differential in ghost number zero, $a \in H^0(s|d)$, where $d$ denotes the exterior spacetime differential. The solution to (29) is unique up to $s$-exact pieces plus divergences

$$a \rightarrow a + sb + \partial_\mu n^\mu;$$  

(30)
with \( gh(b) = -1 \), \( \varepsilon(b) = 1 \), \( gh(n^\mu) = 0 \), and \( \varepsilon(n^\mu) = 0 \). At the same time, if the general solution of (29) is found to be completely trivial, \( a = sb + \partial_\mu n^\mu \), then it can be made to vanish \( a = 0 \).

In order to analyze equation (29), we develop \( a \) according to the antighost number

\[
a = \sum_{i=0}^{I} a_i, \quad \text{agh}(a_i) = i, \quad \text{gh}(a_i) = 0, \quad \varepsilon(a_i) = 0,
\]

(31)

and assume, without loss of generality, that decomposition (31) stops at some finite value of \( I \). This can be shown for instance like in Appendix A of Ref. [22]. Replacing decomposition (31) into (29) and projecting it on the various values of the antighost number by means of (12), we obtain the tower of equations

\[
\gamma a_I = \partial_\mu \left( \frac{I}{m} \right)^\mu,
\]

(32)

\[
\delta a_I + \gamma a_{I-1} = \partial_\mu \left( \frac{I-1}{m} \right)^\mu,
\]

(33)

\[
\delta a_i + \gamma a_{i-1} = \partial_\mu \left( \frac{i-1}{m} \right)^\mu, \quad 1 \leq i \leq I - 1,
\]

(34)

where \( \left( \frac{i}{m} \right)^\mu \) are some local currents, with \( \text{agh}(\left( \frac{i}{m} \right)^\mu) = i \). Moreover, according to the general result from Ref. [22] in the absence of collection indices, equation (32) can be replaced in strictly positive antighost numbers by

\[
\gamma a_I = 0, \quad I > 0.
\]

(35)

Due to the second-order nilpotency of \( \gamma (\gamma^2 = 0) \), the solution to (35) is unique up to \( \gamma \)-exact contributions

\[
a_I \rightarrow a_I + \gamma b_I, \quad \text{agh}(b_I) = I, \quad \text{pgh}(b_I) = I - 1, \quad \varepsilon(b_I) = 1.
\]

(36)

Meanwhile, if it turns out that \( a_I \) reduces to \( \gamma \)-exact terms only, \( a_I = \gamma b_I \), then it can be made to vanish, \( a_I = 0 \). In other words, the nontriviality of the first-order deformation \( a \) is translated at its highest antighost number component into the requirement that \( a_I \in H^I(\gamma) \), where \( H^I(\gamma) \) denotes the cohomology of the exterior longitudinal derivative \( \gamma \) in pure ghost number equal to \( I \). So, in order to solve equation (29) (equivalent with (35) and (33)–(34)), we need to compute the cohomology of \( \gamma \), \( H(\gamma) \), and, as it will be made clear below, also the local cohomology of \( \delta \), \( H(\delta) \).

Using the results on the cohomology of \( \gamma \) in the Pauli-Fierz sector [22], as well as definitions (20) and (21), we can state that \( H(\gamma) \) is generated on the one hand by \( \chi^\ast_\mu, F_{\mu\rho\lambda}, \) and \( K_{\mu\nu\alpha\beta} \), together with their spacetime derivatives and, on the other hand, by the undifferentiated ghost for ghost for ghost as well by the ghosts \( \eta_\mu \) and their first-order derivatives \( \partial_\mu \eta_\mu \). So, the most general (and nontrivial) solution to (35) can be written, up to \( \gamma \)-exact contributions, as

\[
a^{h.a}_I = a_I \left( [F_{\mu\rho\lambda}], [K_{\mu\nu\alpha\beta}], [\chi^\ast_\mu] \right) \omega^I (C, \eta_\mu, \partial_\mu \eta_\mu),
\]

(37)
where the notation \( f ([q]) \) means that \( f \) depends on \( q \) and its derivatives up to a finite order, while \( \omega^I \) denotes the elements of a basis in the space of polynomials with pure ghost number \( I \) in the corresponding ghost for ghost for ghost, Pauli-Fierz ghosts and their antisymmetrized first-order derivatives. The objects \( \alpha_I \) (obviously nontrivial in \( H^0 (\gamma) \)) were taken to have a finite antighost number and a bounded number of derivatives, and therefore they are polynomials in the antifields \( \chi_*^\Gamma \), in the linearized Riemann tensor \( K_{\mu\nu\rho} \), and in the field-strength of the three-form \( F_{\mu\nu\rho\lambda} \) as well as in their subsequent derivatives. They are required to fulfill the property \( \text{agh} (\alpha_I) = I \) in order to ensure that the ghost number of \( a_I \) is equal to zero. Due to their \( \gamma \)-closeness, \( \gamma \alpha_I = 0 \), and to their polynomial character, \( \alpha_I \) will be called invariant polynomials. In antighost number equal to zero the invariant polynomials are polynomials in the linearized Riemann tensor, in the field-strength of the Abelian three-form, and in their derivatives.

Inserting (37) in (33), we obtain that a necessary (but not sufficient) condition for the existence of (nontrivial) solutions \( a_{I-1} \) is that the invariant polynomials \( \alpha_I \) are (nontrivial) objects from the local cohomology of the Koszul-Tate differential \( H (\delta | d) \) in antighost number \( I > 0 \) and in pure ghost number zero,

\[
\delta \alpha_I = \partial_\mu \left( \begin{array}{c}
(I-1)^\mu \\
J
\end{array}\right), \quad \text{agh} \left( \begin{array}{c}
(I-1)^\mu \\
J
\end{array}\right) = I - 1, \quad \text{pgh} \left( \begin{array}{c}
(I-1)^\mu \\
J
\end{array}\right) = 0. \tag{38}
\]

We recall that the local cohomology \( H (\delta | d) \) is completely trivial in both strictly positive antighost and pure ghost numbers (for instance, see Theorem 5.4 from Ref. [32] and also Ref. [33]). Using the fact that the Cauchy order of the free theory under study is equal to four, the general results from Refs. [32, 33], according to which the local cohomology of the Koszul-Tate differential in pure ghost number zero is trivial in antighost numbers strictly greater than its Cauchy order, ensure that

\[
H_J (\delta | d) = 0, \quad J > 4, \tag{39}
\]

where \( H_J (\delta | d) \) denotes the local cohomology of the Koszul-Tate differential in antighost number \( J \) and in pure ghost number zero. It can be shown that any invariant polynomial that is trivial in \( H_J (\delta | d) \) with \( J \geq 4 \) can be taken to be trivial also in \( H_J^{\text{inv}} (\delta | d) \). \( (H_J^{\text{inv}} (\delta | d) \) denotes the invariant characteristic cohomology in antighost number \( J \) — the local cohomology of the Koszul-Tate differential in the space of invariant polynomials.) Thus:

\[
\alpha_J = \delta b_{J+1} + \partial_\mu \left( \begin{array}{c}
(J)^\mu \\
C
\end{array}\right), \quad \text{agh} (\alpha_J) = J \geq 4 \Rightarrow \alpha_J = \delta \beta_{J+1} + \partial_\mu \left( \begin{array}{c}
(J)^\mu \\
\gamma
\end{array}\right), \tag{40}
\]

with both \( \beta_{J+1} \) and \( \left( \begin{array}{c}
(J)^\mu \\
\gamma
\end{array}\right) \) invariant polynomials. Results (39) and (40) yield the conclusion that

\[
H_J^{\text{inv}} (\delta | d) = 0, \quad J > 4. \tag{41}
\]

By proceeding in the same manner like in Refs. [22] and [39], it can be proved that the spaces \( (H_J (\delta | d))_{J \geq 2} \) and \( (H_J^{\text{inv}} (\delta | d))_{J \geq 2} \) are spanned by

\[
H_4 (\delta | d), H_4^{\text{inv}} (\delta | d) : \quad (C^*), \tag{42}
\]
\begin{align}
H_3(\delta|d), H_3^{\text{inv}}(\delta|d) : & \ (C^{*\mu}), \\
H_2(\delta|d), H_2^{\text{inv}}(\delta|d) : & \ (C^{*\mu\nu}, \eta^{*\mu}).
\end{align}

In contrast to the groups \((H_J(\delta|d))_{J\geq 2}\) and \((H_J^{\text{inv}}(\delta|d))_{J\geq 2}\), which are finite-dimensional, the cohomology \(H_1(\delta|d)\) in pure ghost number zero, known to be related to global symmetries and ordinary conservation laws, is infinite-dimensional since the theory is free. Fortunately, it will not be needed in the sequel.

The previous results on \(H(\delta|d)\) and \(H^{\text{inv}}(\delta|d)\) in strictly positive antighost numbers are important because they control the obstructions to removing the antifields from the first-order deformation. Based on formulas (39)–(41), one can successively eliminate all the pieces of antighost number strictly greater than four from the nonintegrated density of the first-order deformation by adding only trivial terms. Consequently, one can take (without loss of nontrivial objects) \(I \leq 4\) into the decomposition \((31)\). (The proof of this statement can be realized like in Appendix C from Ref. [40].) In addition, the last representative reads as in \((37)\), where the invariant polynomial is necessarily a nontrivial object from \((H_J^{\text{inv}}(\delta|d))_{2\leq J\leq 4}\) or from \(H_3(\delta|d)\) for \(J = 1\).

### 4.2 First-order deformation

Assuming \(I = 4\), the nonintegrated density of the first-order deformation, \((31)\), becomes

\begin{align}
a^h A &= a^h_0 + a^h_1 + a^h_2 + a^h_3 + a^h_4.
\end{align}

We can further decompose \(a^h\) in a natural manner as

\begin{align}
a^h A &= a^h + a^{h^{-A}} + a^A,
\end{align}

where \(a^h\) contains only fields/ghosts/antifields from the Pauli-Fierz sector, \(a^{h^{-A}}\) describes the cross-interactions between the two theories (so it effectively mixes both sectors), and \(a^A\) involves only the three-form gauge field sector. The component \(a^h\) is completely known [22] and individually satisfies an equation of the type \((29)\). It admits a decomposition similar to \((45)\)

\begin{align}
a^h &= a^h_0 + a^h_1 + a^h_2,
\end{align}

where

\begin{align}
a^h_2 &= \frac{1}{2} \eta^{*\mu} \eta^{*\nu} \partial_{[\mu} \eta_{\nu]}, \\
a^h_1 &= h^{*\mu\nu} (\partial_{[\mu} \eta^{\nu]} - \eta^{*\rho} \partial_{[\mu} h_{\nu]\rho]).
\end{align}
and $a^h_3$ is the cubic vertex of the Einstein-Hilbert Lagrangian plus a cosmological term\(^\text{2}\)

$$a^h_0 = a^h_{\text{cubic}} - 2\Lambda h,$$

with $\Lambda$ the cosmological constant. Due to the fact that $a^{h-A}$ and $a^A$ contain different sorts of fields, it follows that they are subject to two separate equations

$$s a^A = \partial^\mu m^A_{\mu},$$
$$s a^{h-A} = \partial^\mu m^{h-A}_{\mu},$$

for some local $m_{\mu}$’s. In the sequel we analyze the general solutions to these equations. The nontrivial solution $a^A$ to (50) is

$$a^A = q \varepsilon^{\mu_1...\mu_{11}} A_{\mu_1...\mu_{11}} F_{\mu_4...\mu_7} F_{\mu_8...\mu_{11}},$$

where $q$ is an arbitrary, real constant (for more details, see Ref. [41]). In the sequel we analyze the general solution to equation (51).

In agreement with (15), we can assume that the solution to (51) stops at antighost number four ($I = 4$)

$$a^{h-A} = a^h_{0-A} + a^h_{1-A} + a^h_{2-A} + a^h_{3-A} + a^h_{4-A},$$

where the components on the right-hand side of (53) are subject to equations (35) and (33)–(34) for $\omega_{\text{constant}}$ and $\omega_{\text{variable}}$. Consequently, the associated component of antighost number 0, $a^A_0$, is nevertheless the same in both formulations. Thus, the object $a^h_0$ and the first-order deformation from Ref. [22] belong to the same cohomological class from $H^0 (s|d)$.\(^\text{3}\)

\(^\text{2}\)The terms $a^h_0$ and $a^h_1$ given in (48) and (49) differ from the corresponding ones in Ref. [22] by a $\gamma$-exact and respectively a $\delta$-exact contribution. However, the difference between our $a^h_0 + a^h_1$ and the corresponding sum from Ref. [22] is a $s$-exact modulo $d$ quantity. Consequently, the associated component of antighost number 0, $a^A_0$, is nevertheless the same in both formulations. Thus, the object $a^h_0$ and the first-order deformation from Ref. [22] belong to the same cohomological class from $H^0 (s|d)$.\(^\text{3}\)
All the coefficients denoted by $f$ must be constant (neither derivative nor depending on the spacetime coordinates). Recalling that we work in $D = 11$ spacetime dimensions, we have no such constant Lorentz tensors, so $a_{3}^{h-A}$ must vanish.

Assuming now that $a_{h-A}^{2}$ stops at $I = 2$, we have that the solution to (51) reduces to

$$a_{h-A}^{2} = a_{0}^{h-A} + a_{1}^{h-A} + a_{2}^{h-A},$$

(57)

where the pieces present in (57) are subject to equations (35) and (33)–(34) for $I = 2$. The general solution to (35) (up to $\gamma$-exact contributions) can be written in $D = 11$ as

$$a_{2}^{h-A} = C^{\mu\nu} \left[ c_{1} \eta_{\mu} \eta_{\nu} + c_{2} \left( \partial_{[\mu} \eta_{\nu]} \right) \partial_{\nu} \eta_{\lambda} \sigma^{\rho\lambda} \right],$$

(58)

where $c_{1}$ and $c_{2}$ are arbitrary, real constants. Using definitions (17)–(21) we infer that

$$\delta a_{2}^{h-A} = \partial_{\rho} \left\{ -3 A^{*\mu\rho} \left[ c_{1} \eta_{\mu} \eta_{\nu} + c_{2} \left( \partial_{[\mu} \eta_{\nu]} \right) \partial_{\nu} \eta_{\lambda} \sigma^{\rho\lambda} \right] \right\}$$

$$+ \gamma \left[ 3 c_{2} A^{*\mu\rho} \left( \partial_{[\mu} \eta_{\nu]} \right) \partial_{\nu} h_{\rho]}^{\alpha} \right] - 3 c_{1} A^{*\mu\nu\rho} \eta_{\mu} \partial_{\nu} \eta_{\rho}. \quad (59)$$

Comparing (33) for $I = 2$ with the right-hand side of (59), we observe that $a_{2}^{h-A}$ of the form (58) leads to a consistent $a_{1}^{h-A}$ if and only if

$$- 3 c_{1} A^{*\mu\nu\rho} \eta_{\mu} \partial_{\nu} \eta_{\rho} = \gamma f_{1} + \partial_{\mu} t_{1}. \quad (60)$$

By taking the Euler-Lagrange derivative of both sides of (60) with respect to $A^{*\mu\nu\rho}$ and recalling that it commutes with $\gamma$, we arrive at

$$- 3 c_{1} \eta_{\mu} \partial_{\nu} \eta_{\rho} = \gamma (f_{0\mu\nu\rho}), \quad (61)$$

where

$$f_{0\mu\nu\rho} = \frac{\delta L_{f_{1}}}{\delta A^{*\mu\nu\rho}}.$$

Since $\eta_{\mu} \partial_{[\nu} \eta_{\rho]}$ is a nontrivial object from $H (\gamma)$, it results that the left-hand side of (61) is $\gamma$-exact if and only if $c_{1} = 0$. Therefore, the only consistent solution to (35) at antighost number two is

$$a_{2}^{h-A} = c_{2} C^{\mu\nu} \left( \partial_{[\mu} \eta_{\rho]} \right) \partial_{\nu} \eta_{\lambda} \sigma^{\rho\lambda}. \quad (62)$$

Inserting (62) in (33) for $I = 2$, we derive

$$a_{1}^{h-A} = - 3 c_{2} A^{*\mu\nu\rho} \left( \partial_{[\mu} \eta_{\rho]} \right) \partial_{\nu} h_{\rho]}^{\alpha} + a_{1}^{h-A}, \quad (63)$$

where $a_{1}^{h-A}$ represents the general solution to equation (35) for $I = 1$. According to (37) in pure ghost number equal to one, it results that the most general form of $a_{1}^{h-A}$ as solution to (35) for $I = 1$ that might provide effective cross-interactions can be written like

$$a_{1}^{h-A} = A^{*\mu\nu\rho} \left( M^{\lambda}_{\mu\nu\rho} \eta_{\lambda} + M^{\alpha\beta}_{\mu\nu\rho} \partial_{[\alpha} \eta_{\beta]} \right) + \tilde{h}^{*\mu\nu} \left( M^{\lambda}_{\mu\nu} \eta_{\lambda} + M^{\alpha\beta}_{\mu\nu} \partial_{[\alpha} \eta_{\beta]} \right), \quad (64)$$

12
where the $M$-like functions may depend on linearized Riemann tensor, on the field-strength of the Abelian three-form as well as on their spacetime derivatives and satisfy obvious symmetry/antisymmetry properties. Using the definitions of $\delta$ and $\gamma$, after some computations we obtain that

$$\delta a^{\lambda}_{2} = \partial_{\mu}j_{1}^{\mu} + \gamma b_{0} + c_{0},$$

(65)

where we used the notations

$$j_{1}^{\mu} = \frac{1}{2} F_{\mu \nu \rho \lambda} \left[ -c_{2} \left( \partial_{[\nu} h_{\rho] \alpha} \right) \partial_{[\mu} h_{\lambda]}^{\alpha} + \frac{1}{3} \left( M_{\nu \rho \lambda}^{\alpha} \eta_{\alpha} + M_{\nu \rho \lambda}^{\alpha \beta} \partial_{[\alpha} \eta_{\beta]} \right) \right]$$

$$- 2 \left( \partial_{\rho} \phi^{\mu \alpha \beta} \right) \left( M_{\alpha \beta}^{\lambda} \eta_{\lambda} + M_{\alpha \beta}^{\lambda \rho} \partial_{\rho} \eta_{\lambda} \right)$$

$$+ 2 \phi^{\mu \alpha \beta} \partial_{\nu} \left( M_{\alpha \beta}^{\lambda} \eta_{\lambda} + M_{\alpha \beta}^{\lambda \rho} \partial_{\rho} \eta_{\lambda} \right),$$

(66)

$$b_{0} = F_{\mu \nu \rho \lambda} \left[ \frac{c_{2}}{8} \left( \partial_{\mu} h_{\nu \alpha} \right) \partial_{[\rho} h_{\lambda]}^{\alpha} + \frac{1}{6} \left( M_{\mu \nu \rho \lambda}^{\alpha \beta} \partial_{[\alpha} h_{\beta]}^{\lambda} + \frac{1}{2} M_{\mu \nu \rho \lambda}^{\alpha} \partial_{[\alpha} h_{\lambda]}^{\beta} \right) \right]$$

$$+ 2 \phi^{\mu \alpha \beta} \left[ \frac{1}{2} h_{\beta \lambda} \partial_{\mu} M_{\alpha \nu}^{\lambda} - M_{\mu \nu}^{\lambda} \Gamma_{\lambda \alpha \beta}^{(1)} - M_{\mu \nu}^{\lambda \rho} \partial_{\rho} \Gamma_{\lambda \alpha \beta}^{(1)} \right]$$

$$+ \left( \partial_{\mu} M_{\alpha \nu}^{\lambda} \right) \left( \partial_{[\rho} h_{\lambda]}^{\beta} \right),$$

(67)

$$c_{0} = \frac{1}{4!} F_{\mu \nu \rho \lambda} \left[ \eta_{\alpha} \partial_{[\mu} M_{\nu \rho \lambda}^{\alpha \beta} + \left( \partial_{[\mu} M_{\nu \rho \lambda}^{\alpha \beta} + 2 \delta_{\lambda}^{\beta} M_{\mu \nu \rho}^{\alpha} \right) \partial_{\rho] \alpha} \eta_{\beta} \right]$$

$$+ \frac{1}{2} \phi^{\mu \alpha \beta} \left[ \partial_{[\mu} M_{\alpha \nu}^{\beta} + 2 \delta_{\beta}^{\alpha} \partial_{[\mu} M_{\alpha \nu}^{\beta} \right] \partial_{\rho] \eta_{\lambda]} + \eta_{\alpha} \partial_{[\mu} M_{\alpha \nu}^{\beta \rho}] \right],$$

(68)

$$\phi^{\mu \alpha \beta} = \frac{1}{2} \left( h^{\alpha \nu \sigma \beta} \mu - h^{\mu \nu \sigma \beta} \alpha + h^{\mu \nu \sigma} \beta \right),$$

(69)

$$\Gamma_{\lambda \alpha \beta}^{(1)} = \frac{1}{2} \left( \partial_{\alpha} h_{\beta \lambda} + \partial_{\beta} h_{\alpha \lambda} - \partial_{\lambda} h_{\alpha \beta} \right).$$

(70)

According to (54) for $I = 1$, (67) gives (up to a global factor) some of the pieces from the interacting Lagrangian at order one in the coupling constant. The hypothesis on the maximum number of derivatives in the interacting Lagrangian being equal to two induces further restrictions on the type-$M$ functions, as it will be seen below. The first term from (67) outputs an interacting vertex with three derivatives, which disagrees with this hypothesis. Therefore, we must annihilate the corresponding constant, $c_{2} = 0$. In order to provide cross-couplings, the functions $\tilde{M}_{\alpha \beta}^{\rho}$ and $\tilde{M}_{\mu \nu}^{\lambda}$ must effectively depend on the field-strength of the Abelian three-form. Consequently, the last two terms on the right-hand side of (67) will produce terms with at least three derivatives in the interacting Lagrangian, so we must discard them by setting $\tilde{M}_{\alpha \beta}^{\rho} = 0$. If we represent the functions $\tilde{M}_{\mu \nu}^{\lambda}$ as

$$\tilde{M}_{\mu \nu}^{\lambda} = f_{\mu \nu}^{\lambda \alpha \beta \gamma \delta} F_{\alpha \beta \gamma \delta},$$

13
where \( f^{\lambda\nu\beta\gamma\delta} \) are nonderivative Lorentz constants, we conclude that we have no such constant tensors in \( D = 11 \), so we must take \( M^\alpha_{\mu\nu} = 0 \). The pieces from (67) proportional with \( M^\alpha_{\mu\nu\rho} \) satisfy the assumption on the derivative order if and only if these functions are nonderivative Lorentz constants. Since in \( D = 11 \) there are no such constant tensors, we conclude that we must take \( M^\alpha_{\mu\nu\rho} = 0 \). Finally, the functions \( M^\alpha_{\mu\nu\rho} \) produce terms in the interacting Lagrangian that comply with the hypothesis on the maximum number of derivatives if and only if they are linear in the undifferentiated field-strength of the Abelian three-form.

Due to the spacetime dimension, there is just one possibility left, namely

\[
M^\alpha_{\mu\nu\rho} = k \sigma^{\alpha\beta} F_{\mu\nu\rho\beta},
\]

where \( k \) is an arbitrary, real constant.

Inserting the above results in (62) and (64), we infer

\[
a^h_{0} - A^2 = 0, \quad a^h_{1} - A^1 = k A^*_{\mu\nu\rho} F_{\mu\nu\rho\lambda} \eta^\lambda.
\]

Applying now the Koszul-Tate operator \( \delta \) on (73), we determine the interacting Lagrangian at order one in the coupling constant as

\[
a^h_{0} - A^0 = -\frac{k}{12} F^{\mu\nu\rho\lambda} \left( F_{\mu\nu\rho\sigma} h^\sigma_\lambda - \frac{1}{8} F_{\mu\nu\rho\lambda} h \right).
\]

By assembling the previous results we can state that the general solution to (51) in \( D = 11 \) reads as

\[
a^h_{0} - A^0 = k A^*_{\mu\nu\rho} F_{\mu\nu\rho\lambda} \eta^\lambda - \frac{k}{12} F^{\mu\nu\rho\lambda} \left( F_{\mu\nu\rho\sigma} h^\sigma_\lambda - \frac{1}{8} F_{\mu\nu\rho\lambda} h \right).
\]

We can still remove from (75) certain trivial, \( s \)-exact modulo \( d \) terms. Indeed, we have that

\[
\begin{align*}
\partial_{\mu} & \left[ -\frac{k}{4} F^{\mu\nu\rho\lambda} A_{\nu\rho\lambda} h^\sigma_\lambda + 3k A^*_{\mu\nu\rho} \left( A_{\nu\rho\lambda} \eta^\lambda + C_{\nu\lambda} h^\lambda_\rho \right) \\
& + k C^*_{\mu\nu\rho} \left( C^\sigma_{\nu\rho\lambda} h^\sigma_\lambda - 2 C_{\nu\rho\sigma} \eta^\sigma \right) + k C^* C^\nu \eta^\nu \right] \\
& + \frac{1}{2} A^*_{\mu\nu\rho} A_{\mu\nu\lambda} h^\lambda_\rho - k C^* \left( A_{\mu\nu\rho} \eta^\rho + C_{\mu\lambda} h^\lambda_\rho \right) \\
& + k C_{\mu\nu} \left( C_{\mu\nu} \eta^\nu - \frac{1}{2} C^\nu C_{\mu\nu} \right) - k C^* C^\mu \eta^\mu \\
& + \frac{k}{12} F^{\mu\nu\rho\lambda} \left( 3 \partial_{\mu} \left( A_{\nu\rho\lambda} h^\lambda_\sigma \right) - F_{\mu\nu\rho\sigma} h^\sigma_\lambda + \frac{1}{8} F_{\mu\nu\rho\lambda} h \right) \\
& - \frac{3k}{2} A_{\mu\nu\rho} \left( \frac{2}{3} \eta^\lambda \partial_{\lambda} A_{\mu\nu\rho} + A_{\mu\nu\lambda} \partial_{\nu} \eta^\lambda - h_{\rho\lambda} \partial^\lambda C_{\mu\nu} - C_{\mu\lambda} \partial_{\nu} h^\lambda_\rho \right) \\
& - k C^* \left( \partial_{\rho} C_{\mu\nu} \right) \eta^\rho + C_{\mu\nu} \partial_{\nu} \eta^\rho + h_{\nu\rho} \partial^\rho C_{\mu\nu} + \frac{1}{2} C^\rho \partial_{\mu} h_{\nu\rho} \right]
\end{align*}
\]
\[-\frac{k}{2} C^\mu (2\eta_\nu \partial^\nu C_\mu + C^\nu \partial_\mu \eta_\nu - h_{\mu\nu} \partial^\nu C) - kC^\mu (\partial^\nu C) \eta_\mu. \quad (76)\]

Since $S_1$ is unique up to $s$-exact modulo $d$ terms (see subsection 4.1), we can remove such terms and work, instead of (75), with

\[
a^{h-A} = \frac{k}{12} F^{\mu\nu\rho\lambda} \left[ 3\partial_\mu (A_{\nu\rho\sigma} h_\lambda^\sigma) - F_{\mu\nu\sigma} h_\lambda^\sigma + \frac{1}{8} F_{\mu\nu\rho\lambda} h \right]
- \frac{3k}{2} A^{*\mu\nu\rho} \left( \frac{2}{3} \eta^\lambda \partial_\lambda A_{\mu\nu\rho} + A_{\mu\nu}^\lambda \partial_\rho \eta_\lambda - h_{\rho\lambda} \partial^\lambda C_{\mu\nu} - C_{\mu\lambda} \partial_\rho h_\lambda^\rho \right)
- kC^{*\mu\nu} \left[ (\partial_\mu C_{\nu\rho}) \eta_\rho + C_{\mu}^\rho \partial_\nu \eta_\rho + h_{\nu\rho} \partial^\rho C_\mu + \frac{1}{2} C^\rho \partial_\mu h_{\nu\rho} \right]
- \frac{k}{2} C_{\mu}^\nu \left( 2\eta_\nu \partial^\nu C_\mu + C^\nu \partial_\mu \eta_\nu - h_{\mu\nu} \partial^\nu C \right) - kC^\mu (\partial^\nu C) \eta_\mu. \quad (77)\]

The above results can be summarized by the conclusion that the 'interacting' part of the first-order deformation of the solution to the master equation can be written as

\[
S_1^{h-A} = \int d^{11} x \left( a^{h-A} + a_0^A \right), \quad (78)\]

where $a^{h-A}$ is given in (77) and $a_0^A$ is expressed by (52).

### 4.3 Second-order deformation

Until now we have seen that the first-order deformation can be written like the sum between the Pauli-Fierz component $S_1^h$ (given in detail in Ref. [22]) and the 'interacting' part $S_1^{h-A}$, expressed by (78).

In this section we investigate the consistency of the first-order deformation, described by equation (78). Along the same line as before, we can write the second-order deformation like the sum between the Pauli-Fierz contribution and the interacting part

\[
S_2^{h,A} = S_2^h + S_2^{h-A}. \quad (79)\]

The piece $S_2^h$ can be deduced from Ref. [22], while $S_2^{h-A}$ is subject to the equation

\[
\frac{1}{2} (S_1, S_1)^{h-A} + sS_2^{h-A} = 0, \quad (80)\]

where

\[
(S_1, S_1)^{h-A} = (S_1^{h-A}, S_1^{h-A}) + 2 (S_1^h, S_1^{h-A}). \quad (81)\]

If we denote by $\Delta^{h-A}$ and $b^{h-A}$ the nonintegrated densities of the functionals $(S_1, S_1)^{h-A}$ and respectively $S_2^{h-A}$, then the local form of (80) becomes

\[
\Delta^{h-A} = -2s b^{h-A} + \partial_\mu n^\mu, \quad (82)\]

with

\[
\text{gh} (\Delta^{h-A}) = 1, \quad \text{gh} (b^{h-A}) = 0, \quad \text{gh} (n^\mu) = 1, \quad (83)\]

15
for some local currents $n^\mu$. Direct computation shows that $\Delta^{h-A}$ decomposes like

$$\Delta^{h-A} = \sum_{l=0}^{4} \Delta^{h-A}_l, \quad \text{agh} (\Delta^{h-A}_l) = I, \quad I = \overline{0, 4}, \quad (84)$$

with

$$\Delta^{h-A}_1 = \gamma \left[-k^2 C^* h_{\mu\nu} \eta^\rho \partial^\rho C + k (k+1) C^* C^\mu (\partial_{\mu} \eta_{\nu}) \eta^\nu \right] + \partial_{\mu} \tau^\mu_4, \quad (85)$$

$$\Delta^{h-A}_2 = \delta \left[-k^2 C^* h_{\mu\nu} \eta^\rho \partial^\rho C + k (k+1) C^* C^\mu (\partial_{\mu} \eta_{\nu}) \eta^\nu \right]$$

$$+ \gamma \left\{ \frac{k}{2} C^{*\mu} \left[ \frac{1-2k}{2} (\partial^\rho C) h_{\mu\rho} h_{\nu}^\rho - k C^\nu h_{\mu}^\rho \partial_{\nu} \eta_{\rho} + k C^\nu \eta^\rho \partial_{\mu} h_{\nu\rho} \right]$$

$$- \frac{1}{2} C^\nu (h_{\mu\rho} \partial_{\nu} \eta_{\rho} + h_{\nu\rho} \partial_{\mu} \eta_{\rho}) + (k+1) C^\nu \eta^\rho \left( \partial_{\mu} h_{\nu\rho} \eta_{\rho} + \partial_{\nu} h_{\rho\mu} \right)$$

$$+ 2k (\partial_{\nu} C_\mu) h^{\nu\rho} \eta_{\rho} - 2 (k+1) C_{\mu\nu} \left( \partial^{[\nu} \eta^{\rho]} \right) \eta_{\rho} \right\} \right. + \partial_{\mu} \tau^\mu_3, \quad (86)$$

$$\Delta^{h-A}_3 = \delta \left\{ \frac{k}{2} C^{*\mu} \left[ \frac{1-2k}{2} (\partial^\rho C) h_{\mu\rho} h_{\nu}^\rho - k C^\nu h_{\mu}^\rho \partial_{\nu} \eta_{\rho} + k C^\nu \eta^\rho \partial_{\mu} h_{\nu\rho} \right]$$

$$- \frac{1}{2} C^\nu (h_{\mu\rho} \partial_{\nu} \eta_{\rho} + h_{\nu\rho} \partial_{\mu} \eta_{\rho}) + (k+1) C^\nu \eta^\rho \left( \partial_{\mu} h_{\nu\rho} \eta_{\rho} + \partial_{\nu} h_{\rho\mu} \right)$$

$$+ 2k (\partial_{\nu} C_\mu) h^{\nu\rho} \eta_{\rho} - 2 (k+1) C_{\mu\nu} \left( \partial^{[\nu} \eta^{\rho]} \right) \eta_{\rho} \right\} \right. + \partial_{\mu} \tau^\mu_3, \quad (87)$$

$$\Delta^{h-A}_4 = \delta \left\{ \frac{k}{2} C^{*\mu} \left[ \frac{1-2k}{2} (\partial^\rho C) h_{\mu\rho} h_{\nu}^\rho - k C^\nu h_{\mu}^\rho \partial_{\nu} \eta_{\rho} + k C^\nu \eta^\rho \partial_{\mu} h_{\nu\rho} \right]$$

$$+ \left( \frac{k-1}{2} \right) (\partial_{\nu} C_\mu) h_{\nu\lambda} h^{\alpha\lambda} + \eta^\rho \left( \partial_{\mu} h_{\rho\nu} + \partial_{\nu} h_{\rho\mu} \sigma^{\xi\lambda} \right) C_{\nu\lambda}$$

$$+ \frac{1}{2} C_{\nu\rho} \left( h^{\mu\nu} \partial_{\mu} \eta_{\rho} + h_{\nu\rho} \partial^{[\nu} \eta^{\rho]} \right) - k C_{\nu\rho} \left( 2 \eta^\lambda \partial_{\lambda} h_{\nu\rho} + h_{\nu\lambda} \partial^{[\nu} \eta^{\rho]} \right)$$

$$+ k (\partial_{\nu} C_\mu) \eta_{\lambda} h^{\alpha\lambda} - (k+1) A_{\mu\nu\rho} \eta_{\lambda} \partial^{[\nu} \eta^{\rho]} \right\} \right. + \partial_{\mu} \tau^\mu_2, \quad (87)$$
number, we infer the following tower of equations

\[ \frac{2}{3} k (\partial_\lambda A_{\mu \nu \rho}) h^{\lambda \xi} \eta_\xi + \partial_\mu \tau_1^\mu, \]  

and

\[ \Delta^0_{h-A} = \delta \left[ \frac{3}{2} k A^{\mu \nu \rho} \left( C_\xi \partial_\mu (h_{\nu \lambda} h^{\lambda \xi}) - 2 k C_{\nu \lambda \eta} h^{\xi \eta} \partial_\xi h^\lambda_\rho \right) 
+ \frac{1}{2} h_{\rho \xi} h^{\lambda \xi} \partial_\lambda C_{\mu \nu \rho} - \frac{1}{2} A_{\mu \nu \lambda} \left( h^{\lambda \xi} \partial_\rho \eta_\xi + h_{\rho \xi} \partial_\lambda \eta_\xi \right) 
+ A_{\mu \nu \lambda \eta} \left( 2 \partial_\lambda h^{\mu \xi} - \sigma_\lambda \partial_\xi h_\rho \right) + k A_{\mu \nu \lambda} \left( 2 \partial_\xi h^{\mu \lambda} - h_{\rho \xi} \partial_\lambda \eta_\xi \right) 
+ \frac{2}{3} k (\partial_\lambda A_{\mu \nu \rho}) h^{\lambda \xi} \eta_\xi \right] 
+ \frac{1}{4} F^{\mu \nu \rho \lambda} F_{\mu \nu \xi \pi} \left( -3 h^{\xi \lambda} - \delta^\xi_\lambda h_{\mu \sigma} h^{\pi \sigma} \right) 
+ \frac{1}{8} k^2 h_{\rho \xi} h^{\lambda \xi} \partial_\lambda A_{\nu \rho \lambda} + \frac{k}{4} A_{\mu \xi \lambda} \partial_\mu h^{\xi}_\lambda \right] 
+ \frac{k}{2} A_{\mu \xi \lambda} \partial_\nu (h^{\xi}_\rho h^{\rho}_\lambda) + \frac{k}{2} h^{\xi}_\rho h^{\rho}_\lambda \partial_\xi A_{\mu \nu \lambda} \right] 
- \frac{k^2}{8} \partial_\xi \left( h^{\xi \lambda}_\mu A_{\mu \nu \lambda} \right) \left[ \partial^\mu \left( h^{\xi \mu} A_{\mu \nu \lambda} \right) - \frac{1}{3} \partial^{\xi \mu} \left( h^{\xi \mu} A_{\mu \nu \lambda} \right) \right] 
+ k q_{\mu_1 \ldots \mu_{11}} \left( h A_{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5} F_{\mu_6 \mu_7} - 8 h^{\xi}_\mu A_{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5} F_{\mu_6 \mu_7} \right) 
+ \frac{6 h^{\xi}_\mu A_{\xi \mu_1 \mu_2 \mu_3} F_{\mu_4 \mu_5} \tau_1^\mu \right] \left[ \eta_\xi \left( \frac{1}{8} h^{\xi \lambda}_\mu \partial_\xi (h^{\mu \lambda}_\xi) - \partial_\lambda (h^{\mu \lambda}_\xi) \right) + \frac{1}{2} h_{\lambda \sigma} \partial_\xi (\eta_\xi \tau_1^\mu) \right] + \partial_\mu \tau^\mu_1. \]  

Because \((S_1, S_1)^{h-A}\) contains terms of maximum antighost number equal to four, we can assume (without loss of generality) that \(b^{h-A}\) stops at antighost number five

\[ b^{h-A} = \sum_{I=0}^{5} b^I_{h-A}, \quad \text{agh} (b^I_{h-A}) = I, \quad I = 0, 5, \]  

\[ n^\mu = \sum_{I=0}^{5} n^I_\mu, \quad \text{agh} (n^I_\mu) = I, \quad I = 0, 5. \]  

By projecting equation (82) on the various (decreasing) values of the antighost number, we infer the following tower of equations

\[ \gamma b^5_{h-A} = \partial_\mu \left( \frac{1}{2} n^\mu_5 \right), \]  

17
\[ \Delta^{h-A}_I = -2 \left( \delta b^{h-A}_{I+1} + \gamma b^{h-A}_I \right) + \partial_\mu n^\mu_I, \quad I = 0, 4. \]  

Equation (92) can always be replaced with
\[ \gamma b^{h-A}_5 = 0. \]  

If we compare (85) with (93) for \( I = 4 \), then we find that \( b^{h-A}_5 \) is restricted to fulfill the equation
\[ \delta b^{h-A}_5 + \gamma b^{h-A}_4 = \partial_\mu \tilde{n}^\mu_4, \]  

where
\[ b^{h-A}_4 = -\frac{1}{2} \left[ k^2 C^* h_{\mu\nu} \eta^\mu C + k (k + 1) C^* C^\mu \left( \partial_{\mu} \eta_{\nu} \right) \eta^\nu \right] + \tilde{b}^{h-A}_4. \]  

By (37) we get that the solution to (94) reads as
\[ b^{h-A}_5 = \delta_5 \left( \left[ F_{\mu\nu\rho\lambda} \right], \left[ K_{\mu\nu\alpha\beta} \right], \left[ \chi^{\Delta} \right] \right) \omega^5 \left( C, \eta_{\mu}, \partial_{\mu} \eta_{\nu} \right). \]  

Substituting the above form of \( b^{h-A}_5 \) into (95), we infer that a necessary condition for (95) to possess solutions is that \( \tilde{b}^{h-A}_4 = 0 \), such that equation (95) reduces to
\[ \tilde{b}^{h-A}_4 = 0. \]  

Due to (98) and (99), we observe that relations (85)–(87) agree with equation (93) for \( I = 4, I = 3 \) and \( I = 2 \) respectively. On the contrary, \( \Delta^{h-A}_1 \) given in (88) cannot be written like in (93) for \( I = 1 \) unless
\[ \chi = -k (k + 1) A^{*\mu\nu\rho} F_{\mu\nu\rho\lambda} \left( \partial^{[\lambda} \eta^{\mu]} \right) \eta_{\xi}, \]  

which can be expressed like
\[ \chi = \delta \varphi + \gamma \omega + \partial_\mu \theta^\mu. \]  

Assume that (101) holds. Then, by acting with \( \delta \) on it from the left, we infer that
\[ \delta \chi = \gamma (-\delta \omega) + \partial_\mu (\delta \theta^\mu). \]  

On the other hand, using the concrete expression of \( \chi \), we have that
\[ \delta \chi = k (k + 1) \left\{ \gamma \left[ -T^{\lambda}_{\xi} \left( \eta_{\xi} \partial^{[\lambda} h^{\mu]} + \frac{1}{2} h_{\mu \xi} \partial^{[\lambda} \eta^{\mu]} \right) \right] + \partial_\mu \left( T^{\mu}_{\xi} \eta_{\xi} \partial^{[\tau} \eta^{\xi]} \right) \right\}, \]  

18
where
\[ T^{\alpha\beta} = \frac{1}{3!} F^{\mu\nu\rho}\sigma_{\mu\nu}^{\alpha\beta} - \frac{\sigma^{\alpha\beta}}{2 \cdot 4!} F^{\mu\nu\rho\lambda} F_{\mu\nu\rho\lambda} \] (104)
is the stress-energy tensor of the Abelian three-form gauge field. The right-hand side of (103) can be written like in the right-hand side of (102) if the following conditions are simultaneously satisfied
\[ -\delta\omega = -k(k + 1) T^{\chi}_\chi \left( \eta_\xi \partial^{(\chi)} h^{\xi}_\xi + \frac{1}{2} h_{\pi\xi} \partial^{(\chi)} h^{\xi}_\xi \right), \] (105)
\[ \delta l^\mu = k(k + 1) T^\mu_\lambda \eta_\xi \partial^{(\chi)} h^{\xi}_\xi. \] (106)
Since none of the quantities \( h_{\pi\xi}, \partial^{(\chi)} h^{\xi}_\xi, \eta_\xi, \) or \( \partial^{(\chi)} h^{\xi}_\xi \) are \( \delta \)-exact, we deduce that the last relations hold if stress-energy tensor of the Abelian three-form gauge field is \( \delta \)-exact
\[ T^\mu_\tau = \delta \Omega^\mu_\tau. \] (107)
Assuming that the equation (107) is valid, it further gives
\[ \partial_\mu T^\mu_\tau = \delta \left( \partial_\mu \Omega^\mu_\tau \right). \] (108)
On the other hand, by direct computation we find
\[ \partial_\mu T^\mu_\tau = \delta \left( A^{\nu\rho\lambda} F_{\nu\rho\lambda\tau} \right), \] (109)
so the right-hand side of (109) cannot be written like in the right-hand side of (108). Therefore, relation (107) is not valid, and thus neither are (105)–(106).
As a consequence, \( \chi \) must vanish, which further implies
\[ k(k + 1) = 0. \] (110)
The nontrivial solution to (110) reads as (if we take \( k = 0 \), then no interactions occur)
\[ k = -1. \] (111)
Replacing (111) in (96) (and making use of (99) and then in (86)–(89)), we identify the components of the second-order deformation as
\[ b^h_{1-\Lambda} = \frac{1}{2} C^\sigma h^{\mu\nu} \eta^\rho \partial^\sigma C, \] (112)
\[ b^h_{3-\Lambda} = \frac{1}{2} C^{\nu\mu} \left[ \frac{3}{4} h_{\mu\nu} h^{\rho} \partial^\rho C - \frac{1}{2} C^{\nu\rho} \partial_\mu h^{\rho}_\nu \right] \]
\[ + \frac{1}{4} C^{\nu} \left( h^{\rho}_\mu \partial_\nu h^{\rho}_\nu - h^{\rho}_\nu \partial_\rho h^{\nu}_\mu \right) \] (113)
\[ b^h_{2-\Lambda} = -\frac{1}{2} C^{\rho\nu} \left[ C^\nu h^\Lambda_{\rho} \partial_{\rho} h^{\Lambda}_{\nu} + \frac{1}{2} C^\rho \partial_\rho \left( h_{\nu\lambda} h^\Lambda_{\mu} - \frac{3}{2} h_{\nu\lambda} h^{\Lambda}_{\mu} \right) - \frac{3}{2} h_{\nu\lambda} h^{\Lambda} \partial_{\rho} C_{\nu} \right]. \]
With the help of (111), it results that

\[ b^{h-A}_0 = \frac{3}{4} A^{\mu\nu\rho} \left[ C_{\rho\xi} \partial_{(\mu} h_{\nu)} h^{\lambda \xi} + \frac{3}{2} h_{\rho\xi} h^{\lambda \xi} \partial_\lambda A_{\mu \nu} + 2 C_{\rho \lambda} h^{\xi \eta} \partial_{(\mu} h_{\nu)}^\lambda \right] - \frac{1}{2} A_{\mu \nu \lambda} \left( h^{\lambda \xi} \partial_{(\rho} h_{\xi)} + h_{\rho \xi} \partial^{\lambda \eta} \partial_{(\mu} h_{\nu)} + 2 \sigma^{\lambda \mu} \eta^{\alpha} \partial_{(\mu} h_{\nu)} \right) + A_{\mu \nu \lambda} h_{\rho \xi} \partial_{(\lambda} h_{\eta)} C_{\epsilon \sigma} \delta_{\mu}^{\xi} h^{\gamma \lambda} \left( 3 h^{2} + h^{\alpha \beta} h^{\alpha \beta} \right) - \frac{1}{3} \delta_{\mu}^{\xi} h_{\lambda \sigma} h^{\mu \sigma} \right]

\[ + \frac{1}{16} \left[ \frac{3}{4} A^{\mu\nu\rho} \left[ C_{\rho\xi} \partial_{(\mu} h_{\nu)} h^{\lambda \xi} + \frac{3}{2} h_{\rho\xi} h^{\lambda \xi} \partial_\lambda A_{\mu \nu} + 2 C_{\rho \lambda} h^{\xi \eta} \partial_{(\mu} h_{\nu)}^\lambda \right] - \frac{1}{2} A_{\mu \nu \lambda} \left( h^{\lambda \xi} \partial_{(\rho} h_{\xi)} + h_{\rho \xi} \partial^{\lambda \eta} \partial_{(\mu} h_{\nu)} + 2 \sigma^{\lambda \mu} \eta^{\alpha} \partial_{(\mu} h_{\nu)} \right) + A_{\mu \nu \lambda} h_{\rho \xi} \partial_{(\lambda} h_{\eta)} C_{\epsilon \sigma} \delta_{\mu}^{\xi} h^{\gamma \lambda} \left( 3 h^{2} + h^{\alpha \beta} h^{\alpha \beta} \right) - \frac{1}{3} \delta_{\mu}^{\xi} h_{\lambda \sigma} h^{\mu \sigma} \right]

and

\[ b^{h-A}_1 = \frac{1}{16} F^{\mu \nu \rho \lambda} F_{\mu \nu \rho \sigma} \left[ h_{\mu}^\xi h_{\lambda}^\xi - \frac{1}{3} \delta_{\mu}^\xi h_{\lambda}^\xi \left( \frac{1}{4} h^{2} - h_{\alpha \beta} h^{\alpha \beta} \right) - \frac{1}{3} \delta_{\mu}^{\xi} h_{\lambda \sigma} h^{\mu \sigma} \right]

Formulas (112)–(116) offer us the complete form of the interacting part from the second-order deformation of the solution to the master equation

\[ S^{h-A}_2 = \int d^{11}x \left( b^{h-A}_4 + b^{h-A}_3 + b^{h-A}_2 + b^{h-A}_1 + b^{h-A}_0 \right). \]

With the help of (111), it results that \( S^{h-A}_1 \) takes the final form

\[ S^{h-A}_1 = \int d^{11}x \left\{ -\frac{1}{12} F^{\mu \nu \rho \lambda} \left[ \partial_\mu \left( A_{\nu \rho \sigma} h_{\lambda}^\xi \right) - F_{\mu \nu \rho \sigma} h_{\lambda}^\xi + \frac{1}{8} F^{\mu \nu \rho \lambda} h \right] + \frac{3}{2} A^{\mu \nu \rho \lambda} \left[ \frac{3}{4} \eta^\xi \partial_\lambda A_{\mu \nu \rho} + A_{\mu \nu \lambda} \partial_{(\rho} h_{\sigma)} + h_{\rho} \partial_\lambda C_{\mu \nu} - C_{\mu \lambda} \partial_{(\rho} h_{\sigma)} \right] + C^{\nu \rho} \left[ \partial_\mu C_{\epsilon \sigma} \right] + C^{\nu \rho} \partial_\lambda \partial_{(\mu} h_{\nu)} + h_{\rho} \partial_\lambda C_{\mu \nu} + \frac{1}{2} C^{\nu \rho} \partial_\lambda h_{\mu \nu} \right] + \frac{1}{2} C^{\nu \rho} \left[ 2 \eta_\nu \partial_\epsilon C_{\mu} + 2 \eta_\nu \partial_{(\mu} h_{\nu)} - h_{\mu \nu} \partial_\epsilon C_{\mu} + C^\alpha \partial_\epsilon C_{\mu} \right] + \frac{1}{2} C^{\nu \rho} \left[ 2 \eta_\nu \partial_\epsilon C_{\mu} + 2 \eta_\nu \partial_{(\mu} h_{\nu)} - h_{\mu \nu} \partial_\epsilon C_{\mu} + C^\alpha \partial_\epsilon C_{\mu} \right] \}

So far, we have completely determined the first- and second-order deformations of the solution to the master equation corresponding to the free model [1].
5 Analysis of the deformed theory

In Ref. [24] (Section 5) it has been shown that the local BRST cohomologies of the Pauli-Fierz model and respectively of the linearized version of vielbein formulation of spin-two field theory are isomorphic. Because the local BRST cohomology (in ghost numbers zero and one) controls the deformation procedure, it results that this isomorphism allows one to pass in a consistent manner from the Pauli-Fierz version to the linearized version of the vielbein formulation and conversely during the deformation procedure. Nevertheless, the linearized vielbein formulation possesses more fields (the antisymmetric part of the linearized vielbein) and more gauge parameters (Lorentz parameters) than the Pauli-Fierz model, such that the switch from the former version to the latter is realized via the above mentioned isomorphism by imposing some partial gauge-fixing conditions, which come from the more general ones [42]

\[ \sigma_{\mu[a} e_{b]} = 0. \] (119)

In the context of the gauge-fixing conditions (119), simple computation leads to the vielbein fields and their inverse up to the second order in the coupling constant as

\[ e_{a}^{\mu} = (0)^{\mu}_{\mu} e_{a}^{\mu} + (1)^{\mu}_{\mu} e_{a}^{\mu} + \lambda (2)^{\mu}_{\mu} e_{a}^{\mu} + \cdots = \delta_{\mu}^{\mu} - \frac{\lambda}{2} h_{a}^{\mu} + \frac{3 \lambda^{2}}{8} h_{a}^{\rho} h_{\rho}^{\mu} + \cdots, \] (120)

\[ e^{a}_{\mu} = (0)^{a}_{\mu} e^{a}_{\mu} + (1)^{a}_{\mu} e^{a}_{\mu} + \lambda (2)^{a}_{\mu} e^{a}_{\mu} + \cdots = \delta^{a}_{\mu} + \frac{\lambda}{2} h^{a}_{\mu} - \frac{\lambda^{2}}{8} h^{a}_{\rho} h^{\rho}_{\mu} + \cdots. \] (121)

The first pieces from the expansion of the metric tensor and of its determinant \((\sqrt{g} = \sqrt{\det g_{\mu\nu}})\) in terms of the Pauli-Fierz field are written as

\[ g^{\mu\nu} = (0)^{\mu\nu} g^{\mu\nu} + (1)^{\mu\nu} g^{\mu\nu} + \lambda^{2} g^{\mu\nu} + \cdots = \sigma^{\mu\nu} - \lambda h^{\mu\nu} + \lambda^{2} h^{\mu}_{\rho} h^{\nu}_{\rho} + \cdots, \] (122)

\[ \sqrt{g} = e = \sqrt{g} + \lambda \sqrt{g} + \lambda^{2} \sqrt{g} + \cdots = 1 + \frac{\lambda}{2} h + \frac{\lambda^{2}}{8} (h^{2} - 2 h_{\mu\nu} h^{\mu\nu}) + \cdots, \] (123)

where \(e = \det e_{a}^{\mu} \).

Now, we have at hand all the ingredients required for the Lagrangian formulation of the deformed theory obtained in the previous section. The component of antighost number zero in \( S_{1}^{h-A} \) is precisely the interacting Lagrangian at order one in the coupling constant

\[ L_{1}^{h-A} = - \frac{1}{12} F_{\mu\nu\rho\lambda} \left( \frac{1}{4} F_{\mu\nu\rho\lambda} h - F_{\mu\nu\rho\sigma} h_{\rho}^{\sigma} + 3 \partial_{\mu} (A_{\nu\rho\sigma} h_{\rho}^{\sigma}) \right) \]

\[ + q e^{\mu_{1}...\mu_{11}} A_{\mu_{1} \mu_{2} \mu_{3}} F_{\mu_{4}...\mu_{11}} F_{\mu_{5}...\mu_{11}}. \] (124)

It can be put under the more suggestive form

\[ L_{1}^{h-A} = - \frac{1}{2 \cdot 4!} g^{\mu_{alpha}(0) \mu_{beta}(0) \rho_{gamma}(0)} \left( \frac{1}{8} g^{\mu_{alpha}(0) \mu_{beta}(0) \rho_{gamma}(0)} \sqrt{g} g^{\mu_{alpha}(0) \mu_{beta}(0) \rho_{gamma}(0)} \right) \]

\[ \sqrt{g} g^{\mu_{alpha}(0) \mu_{beta}(0) \rho_{gamma}(0)} \left( \frac{1}{8} g^{\mu_{alpha}(0) \mu_{beta}(0) \rho_{gamma}(0)} \right) \]
where

$$L_{2h} = - \frac{1}{2^4 \cdot 4! \sqrt{g} F_{\mu \nu \rho \lambda} \bar{F}_{\mu \nu \rho \lambda} + \lambda q e^{a_1 a_2} (0)_{a_1} A_{\mu_1 \mu_2 \mu_3} \bar{F}_{\mu_4 \mu_5 \mu_6} \bar{F}_{\mu_8 \mu_9 \mu_{10}} + \cdots}.$$
\[
\ddot{F}^{\mu \nu \rho} = g^\alpha \gamma g^\beta \gamma g^\lambda \delta \ddot{F}_{\alpha \beta \gamma \delta},
\]

\[
\epsilon_{\mu_1 \cdots \mu_{11}} = \sqrt{g} e_{a_1}^{\mu_1} e_{a_2}^{\mu_2} \cdots e_{a_{11}}^{\mu_{11}} e^{a_1 a_2 \cdots a_{11}}.
\]

The pieces from the deformed solution to the master equation that are linear in the antifields \( A^{*\alpha \beta \gamma} \) produce the deformed gauge transformations of the Abelian three-form gauge field

\[
\delta_{\epsilon,\xi} A_{\alpha \beta \gamma} = \partial_{[\alpha} \xi_{\beta] \gamma} + \lambda \left[ \epsilon^\delta \partial_\delta A_{\alpha \beta \gamma} + \frac{1}{2} A^\delta_{[\alpha \beta} \partial_{\gamma]} \xi_\delta \right]
\]

\[
+ \frac{1}{2} \left( \partial^\delta \xi_{[\alpha \beta]} h_{\gamma]} \delta + \frac{1}{2} \epsilon^\delta_{[\alpha} \partial_\delta h_{\gamma]} \delta \right) + \lambda^2 \left[ -\frac{1}{8} \epsilon^\delta_{[\alpha} \left( \partial_\delta h_{\gamma]} \right) h_{\delta \sigma} + \frac{3}{8} \epsilon^\delta_{[\alpha} h_{\gamma] \delta} \partial_\delta \epsilon_{\sigma] \gamma} \right]
\]

\[
+ \frac{3}{8} \left( \partial^\delta \xi_{[\alpha \beta]} h_{\gamma]} \delta \right) h_{\delta \sigma} - \frac{1}{4} \left( \partial_\delta h_{\gamma]} \epsilon_{\delta \sigma] \gamma} \right) + \frac{1}{4} A^\delta_{[\alpha \beta} \partial_{\gamma] \epsilon_\delta \eta_{\epsilon]} \sigma} - \frac{1}{2} \epsilon^\delta \partial_{\alpha \beta} \eta_{\gamma} \epsilon_{\sigma] \gamma} + \cdots
\]

\[
= \delta_{\epsilon,\xi} A_{\alpha \beta \gamma} + \lambda^0 \delta_{\epsilon,\xi} A_{\alpha \beta \gamma} + \lambda^2 \delta_{\epsilon,\xi} A_{\alpha \beta \gamma} + \cdots.
\]

We recall that the initial three-form gauge field possesses flat indices, i.e. \( A_{\alpha \beta \gamma} \) means \( A_{abc} \). The contributions of orders one and two to the above gauge transformations can be put under the form

\[
(1)^{\mu} \delta_{\epsilon,\xi} A_{abc} = \epsilon_{\mu} \partial_\mu A_{abc} + A_{[ab}^m \epsilon_{c]}^m + \left( \partial_{[ab} \epsilon_{c]} \right)^{\mu} + \frac{1}{2} e^{m} \omega^{(1)}_{\mu [ab} \epsilon_{c]}^m,
\]

\[
(2)^{\mu} \delta_{\epsilon,\xi} A_{abc} = (1)^{\mu} \partial_\mu A_{abc} + A_{[ab}^m \epsilon_{c]}^m + \left( \partial_{[ab} \epsilon_{c]} \right)^{\mu} + \frac{1}{2} e^{m} \omega^{(1)}_{\mu [ab} \epsilon_{c]}^m + \frac{1}{2} e^{m} \omega^{(2)}_{\mu [ab} \epsilon_{c]}^m,
\]

where we used the notations

\[
(0)^{\mu} \epsilon = \epsilon^\mu = \epsilon^a \delta^\mu_a,
\]

\[
(0)^{\mu} \epsilon_{ab} = \frac{1}{2} \partial_{[ab} \epsilon_{b]}^\mu,
\]

\[
(1)^{\mu} \epsilon_{ab} = -\frac{1}{4} \epsilon^\mu \partial_{[ab} h_{b]}^\epsilon + \frac{1}{8} h_{[a}^c \partial_{b]} \epsilon_{c} + \frac{1}{8} \left( \partial_\mu \epsilon_{[a} \right) h_{b]}^\epsilon,
\]

\[
(1)^{\mu} \omega_{\mu ab} = -\partial_{[a} h_{b]}^\epsilon \mu,
\]

\[
(2)^{\mu} \omega_{\mu ab} = -\frac{1}{4} \left( 2 h_{[a} \left( \partial_{b]} h_{[b]}^\epsilon \right) - 2 h_{[a}^\nu \partial_{b]} h_{b]}^\nu \right) \left( \partial_\mu h_{[a}^\nu \right) h_{b]}^\nu.
\]
In formulas (137) and (138) the gauge parameters \((0)\) \(\epsilon_{ab}\) and \((1)\) \(\epsilon_{ab}\) are precisely the first two terms from the Lorentz parameters expressed in terms of the flat parameters \(\epsilon^a\) via the partial gauge fixing (119). Indeed, (119) leads to

\[
\delta \epsilon \left( \sigma_{\mu[a} \epsilon_{b\mu} \right) = 0,
\]  

(141)

where

\[
\frac{1}{\lambda} \delta \epsilon_e a \mu = \bar{\epsilon}^\mu \partial_\mu e_a \mu - e_a^e \partial_\mu \bar{\epsilon}^\mu + \epsilon_b^a e_b \mu.
\]

(142)

Substituting (120) together with the expansions

\[
\bar{\epsilon}^\mu = (0)^\mu \bar{\epsilon} + \lambda (1)^\mu \bar{\epsilon} + \cdots = \left( \delta^\mu_a - \frac{\lambda}{2} h_a^\mu + \cdots \right) \epsilon^a
\]

(143)

and

\[
\epsilon_{ab} = (0)\epsilon_{ab} + \lambda (1)\epsilon_{ab} + \cdots
\]

(144)

in (141), we arrive precisely to (137)–(138). In formulas (139) and (140) \((1)\) \(\omega_{\mu ab}\) and \((2)\) \(\omega_{\mu ab}\) represent the first- and respectively second-order approximation of the spin connection

\[
\omega_{\mu ab} = \epsilon_{\mu}^\nu \partial_\nu e_{ab} - e_{\nu}^\mu \partial_\mu e_{ab} + e_{\mu}^a \partial_\nu e_{\nu b} - e_{\nu}^b \partial_\mu e_{\nu a} + e_{\mu}^a \partial_\nu e_{\nu b} - e_{\nu}^b \partial_\mu e_{\nu a} + \epsilon_{\mu}^a \omega_{[\mu} e_{\nu b]} e_{\nu} c^\rho.
\]

(145)

At this point it is easy to see that the deformed gauge transformations of the three-form gauge field (see formula (133)) come from the perturbative expansion of the full gauge transformations

\[
\bar{\delta}_{\epsilon,\chi} A_{abc} = \lambda \left( \bar{\epsilon}^\mu \partial_\mu A_{abc} + A_{[ab}^m c_m] + \left( \partial_\mu \bar{\epsilon}_{[ab]} \right) e_c^\mu + \frac{1}{2} e_{\mu}^a \omega_{[\mu\nu]} \epsilon_{\nu c}^m \right).
\]

(146)

The gauge transformations of the three-form with curved indices are obtained with the help of (142) and (146)

\[
\bar{\delta}_{\epsilon,\chi} \tilde{A}_{\mu\nu\rho} = \partial_{[\mu} \tilde{\epsilon}_{\nu\rho]} + \lambda \left( \tilde{\epsilon}^\lambda \partial_\lambda \tilde{A}_{\mu\nu\rho} + \tilde{A}_{\sigma[\mu\nu} \partial_\rho] \epsilon^\sigma \right),
\]

(147)

where

\[
\tilde{\epsilon}_{\mu\nu} = e^{a}_{\mu} e^{b}_{\nu} \epsilon_{ab}.
\]

(148)

We observe that (147) describes a set of gauge transformations that remain off-shell, second-order reducible. Indeed, if we make the transformations

\[
\tilde{\epsilon}_{\mu\nu} \rightarrow \tilde{\epsilon}_{\mu\nu}^{(\hat{\theta})} = \partial_{[\mu} \hat{\theta}_{\nu]},
\]

(149)

then the gauge variation of the three-form identically vanishes

\[
\bar{\delta}_{\epsilon,\hat{\theta}} A_{\mu\nu\rho} \equiv 0.
\]

(150)
Moreover, if in (149) we perform the changes
\[ \bar{\theta}_\mu \rightarrow \bar{\theta}^{(\phi)}_\mu = \partial_\mu \phi, \]
with \( \phi \) an arbitrary scalar field, then the transformed gauge parameters (149) identically vanish
\[ \bar{\epsilon}^{(\bar{\theta}^{(\phi)})}_{\mu \nu} = 0. \]

The results concerning the reducibility relations for the interacting theory can be read from the pieces that are simultaneously linear in the ghosts and in the antifields (with the antighost number equal to two or three from the deformed solution to the master equation).

In conclusion, under the hypotheses mentioned at the beginning of subsection 4.1, we obtained that a candidate to the Lagrangian responsible for the interactions between the spin-two field and a three-form gauge field in \( D = 11 \) is described in (129) and the deformed gauge transformations of the three-form are given by (147).

6 Uniqueness of interactions

So far, we emphasized that there exists one candidate describing the consistent interactions between one graviton and an Abelian three-form gauge field, namely
\[
\bar{\mathcal{L}} = \frac{2}{\lambda^2} e (R - 2\lambda^2 \Lambda) - \frac{1}{2} \cdot 4! e \bar{F}_{\mu \nu \rho \lambda} \bar{F}^{\mu \nu \rho \lambda} + \lambda q \varepsilon^{\mu_1 \mu_2 \cdots \mu_{11}} \bar{A}_{\mu_1 \mu_2 \mu_3} \bar{F}_{\mu_4 \cdots \mu_{7}} \bar{F}_{\mu_8 \cdots \mu_{11}},
\]
in the context of the partial gauge-fixing (141). So, the only point that remains to be done is to check that there are no other solutions.

Let us denote by \( \bar{S} \) the solution to the master equation for the theory with the standard Lagrangian (153) decomposed according to the power orders of the coupling constant \( \lambda \)
\[ \bar{S} = \bar{S}_0 + \lambda \bar{S}_1 + \lambda^2 \bar{S}_2 + \lambda^3 \bar{S}_3 + \lambda^4 \bar{S}_4 + \cdots \]
and by \( S \) the fully deformed solution of the master equation associated with the free theory (1), consistent to all orders in the coupling constant
\[ S = \bar{S} + \lambda S_1 + \lambda^2 S_2 + \lambda^3 S_3 + \cdots, \]
such that they respectively fulfill the equations
\[
\left( \bar{S}, \bar{S} \right) = 0, \quad (S, S) = 0.
\]

Until now we investigated \( \bar{S}, S_1, \) and \( S_2 \) and proved that they coincide with the standard ones
\[ \bar{S} = \bar{S}_0, \quad S_1 = \bar{S}_1, \quad S_2 = \bar{S}_2 \]
(158)
in the presence of the partial gauge-fixing \((141)\). The question is how unique are \(S_3\), \(S_4\), etc.\ given \((155)\). We will answer this question by showing that the interactions provided by our deformation procedure can always be brought to those prescribed by the usual rules from General Relativity via a suitable redefinition of the constants \(\lambda\), \(q\), and \(\Lambda\) from \((153)\). More precisely, we will prove that the fully deformed solution \((155)\) is nothing but \((154)\) up to the replacements

\[
\lambda \rightarrow \lambda \left( 1 + k_3^{(1)} \lambda^2 + k_4^{(1)} \lambda^3 + k_5^{(1)} \lambda^4 + \cdots \right),
\]
\[
\Lambda \rightarrow \Lambda \left( 1 + k_3^{(4)} \lambda^2 + k_4^{(4)} \lambda^3 + k_5^{(4)} \lambda^4 + \cdots \right),
\]
\[
q \rightarrow q \left( 1 + k_3^{(3)} \lambda^2 + k_4^{(3)} \lambda^3 + k_5^{(3)} \lambda^4 + \cdots \right),
\]

with \(k_j^{(m)}\) some arbitrary, real constants.

Our starting point is that \((154)\) and \((155)\) respectively satisfy equations \((156)\) and \((157)\) together with relations \((158)\). The projection of \((154)\) and \((155)\) on \(\lambda^3\) emphasizes that \(S_3\) and respectively \(\tilde{S}_3\) are solutions to the equations

\[
sS_3 = - (S_1, S_2), \quad s\tilde{S}_3 = - \left( \tilde{S}_1, \tilde{S}_2 \right).
\]

Recalling \((158)\) and subtracting the latter equation in \((162)\) from the former we obtain

\[
s \left( S_3 - \tilde{S}_3 \right) = 0,
\]

whose general solution, according to our results from subsection 4.2 (and to the second equality from \((158)\)), reads as

\[
S_3 - \tilde{S}_3 = \sum_{m=1}^{4} k_3^{(m)} \tilde{s}_1^{(m)}. \tag{164}
\]

In the above \(k_3^{(m)}\) are arbitrary, real constants and \(\tilde{s}_1^{(m)}\) are the independent components of the first-order deformation \(S_1 = \tilde{S}_1\) (they individually satisfy the equation \(s\tilde{s}_1^{(m)} = 0\))

\[
\tilde{s}_1^{(1)} + \tilde{s}_1^{(2)} + \tilde{s}_1^{(3)} + \tilde{s}_1^{(4)},
\]

namely, \(\tilde{s}_1^{(1)}\) represents the first-order deformation of the solution to the master equation from the Pauli-Fierz sector containing the cubic vertex of the Einstein-Hilbert Lagrangian (see \((17)\)), but not the cosmological term, \(\tilde{s}_1^{(2)}\) denotes the interacting part of the first-order deformation (see \((77)\) for \(k = -1\)), \(\tilde{s}_1^{(3)}\) stands for the first-order deformation in the three-form sector (see \((62)\), linear in \(q\), and \(\tilde{s}_1^{(4)}\) means the first-order deformation from the Pauli-Fierz sector that does
not modify the gauge transformations of the graviton (the cosmological term, linear in the cosmological constant \( \Lambda \)). By direct computation we find that the various antibrackets among \( \tilde{S}_1^{(m)} \) read as

\[
\begin{align*}
(\tilde{S}_1^{(1)}, \tilde{S}_1^{(1)}) &= -2s \tilde{S}_1^{(2)}, \\
(\tilde{S}_1^{(2)}, \tilde{S}_1^{(2)}) &= -2s \tilde{S}_1^{(2)}, \\
(\tilde{S}_1^{(2)}, \tilde{S}_1^{(4)}) &= 0,
\end{align*}
\]

(166)

\[
\begin{align*}
(\tilde{S}_1^{(3)}, \tilde{S}_1^{(3)}) &= \tilde{S}_1^{(3)} - \tilde{S}_1^{(4)}, \\
(\tilde{S}_1^{(4)}, \tilde{S}_1^{(4)}) &= 0,
\end{align*}
\]

(168)

where \( \left( \tilde{S}_2^{(m)} \right)_{m=1,2} \) are the components of the second-order deformation of the solution to the master equation \( \tilde{S}_2 = \tilde{S}_2 \) (see (165) and (171)), \( \tilde{S}_1, \tilde{S}_1 = -2s \tilde{S}_2 \), respectively induced by the decomposition (165).

\[
\tilde{S}_2 = \tilde{S}_1^{(1)} + \tilde{S}_1^{(2)} + \tilde{S}_1^{(3)} + \tilde{S}_1^{(4)}.
\]

(169)

Based on the concrete form of the various components from (165) and (169), it can be shown that their antibrackets can be expressed as

\[
\begin{align*}
(\tilde{S}_1^{(1)}, \tilde{S}_1^{(1)}) &= -2s \tilde{S}_1^{(2)}, \\
(\tilde{S}_1^{(1)}, \tilde{S}_1^{(4)}) &= 0,
\end{align*}
\]

(170)

\[
\begin{align*}
(\tilde{S}_1^{(2)}, \tilde{S}_1^{(2)}) &= -2s \tilde{S}_1^{(2)}, \\
(\tilde{S}_1^{(2)}, \tilde{S}_1^{(4)}) &= 0,
\end{align*}
\]

(172)

where \( \left( \tilde{S}_3^{(m)} \right)_{m=1,2} \) are the components of the solution to the master equation for the theory with the standard Lagrangian (158) of order three in the coupling constant, \( \tilde{S}_3 \),

\[
\tilde{S}_3 = \tilde{S}_3^{(1)} + \tilde{S}_3^{(2)} + \tilde{S}_3^{(3)} + \tilde{S}_3^{(4)}.
\]

(174)

At the same time, the various terms from (165), (169), and (174) check the individual equations

\[
\begin{align*}
(\tilde{S}_1^{(1)}, \tilde{S}_1^{(1)}) &= -2s \tilde{S}_1^{(2)}, \\
(\tilde{S}_1^{(1)}, \tilde{S}_1^{(4)}) &= 0,
\end{align*}
\]

(175)

\[
\begin{align*}
(\tilde{S}_1^{(2)}, \tilde{S}_1^{(2)}) &= -2s \tilde{S}_1^{(2)}, \\
(\tilde{S}_1^{(2)}, \tilde{S}_1^{(4)}) &= 0,
\end{align*}
\]

(177)

\[
\begin{align*}
(\tilde{S}_1^{(3)}, \tilde{S}_1^{(3)}) &= \tilde{S}_1^{(3)} - \tilde{S}_1^{(4)}, \\
(\tilde{S}_1^{(4)}, \tilde{S}_1^{(4)}) &= 0,
\end{align*}
\]

(178)
where \( (\tilde{S}_4^{(m)})_{m=1,4} \) represent the components of the solution to the master equation for the theory with the standard Lagrangian \((153)\) of order four in the coupling constant

\[
\tilde{S}_4 = \tilde{S}_4^{(1)} + \tilde{S}_4^{(2)} + \tilde{S}_4^{(3)} + \tilde{S}_4^{(4)},
\]

i.e.

\[
2 s \tilde{S}_4 + 2 \left( \tilde{S}_1, \tilde{S}_3 \right) + \left( \tilde{S}_2, \tilde{S}_2 \right) = 0. \tag{180}
\]

The fourth-order deformation of the solution of the master equation associated with the free theory \((1)\), \( S_4 \), is solution to the equation

\[
2 s S_4 + 2 (S_1, S_3) + (S_2, S_2) = 0, \tag{181}
\]

which results from \((157)\) (with \( S \) developed as in \((155)\)) projected on \( \lambda^4 \). Subtracting \((180)\) from \((181)\) and employing \((158)\) and \((164)\) we obtain

\[
s \left( S_4 - \tilde{S}_4 \right) = - \left( \tilde{S}_1, S_3 - \tilde{S}_3 \right) = - \left( \tilde{S}_1, \sum_{m=1}^4 k_3^{(m)} \tilde{S}_1^{(m)} \right). \tag{182}
\]

Inserting \((160)-(168)\) in \((182)\), we further deduce

\[
s \left( S_4 - \tilde{S}_4 \right) = s \left[ 2 k_3^{(1)} \tilde{S}_2^{(1)} + \left( k_3^{(4)} + k_3^{(1)} \right) \tilde{S}_2^{(4)} + \left( k_3^{(2)} + k_3^{(3)} \right) \tilde{S}_2^{(3)} \right] + \left( k_3^{(2)} + k_3^{(1)} \right) \left( \tilde{S}_1^{(1)}, \tilde{S}_1^{(2)} \right) + k_3^{(2)} \left( \tilde{S}_1^{(2)}, \tilde{S}_1^{(2)} \right). \tag{183}
\]

Taking into account the first relation from \((167)\), it follows that the right-hand side of \((183)\) is \( s \)-exact if and only if the constants \( k_3^{(2)} \) and \( k_3^{(1)} \) from \((164)\) are equal

\[
k_3^{(2)} = k_3^{(1)}. \tag{184}
\]

Substituting \((184)\) in \((164)\) we determine the general expression of the third-order deformation of the fully deformed solution \((155)\) of the master equation associated with the free theory \((1)\), \( S_3 \), in terms of some of the components of the solution to the master equation for the theory with the standard Lagrangian \((153)\) under the form

\[
S_3 = \tilde{S}_3 + k_3^{(1)} \left( \tilde{S}_1^{(1)} + \tilde{S}_1^{(2)} \right) + k_3^{(3)} \tilde{S}_1^{(3)} + k_3^{(4)} \tilde{S}_1^{(4)}. \tag{185}
\]

Based on the same result, namely \((184)\), from \((183)\) we infer the equation satisfied by the fourth-order deformation \( S_4 \)

\[
s \left[ S_4 - \tilde{S}_4 - 2k_3^{(1)} \tilde{S}_2^{(1)} - 2k_3^{(1)} \tilde{S}_2^{(2)} - \left( k_3^{(1)} + k_3^{(3)} \right) \tilde{S}_2^{(3)} \right] = 0,
\]

which, according to the general result from subsection \((12)\) (see also the argument leading to \((164)\) ), possesses the solution

\[
S_4 = \tilde{S}_4 + 2k_3^{(1)} \left( \tilde{S}_2^{(1)} + \tilde{S}_2^{(2)} \right) + \left( k_3^{(1)} + k_3^{(3)} \right) \tilde{S}_2^{(3)}
\]

28
fifth-order deformation $S_4$ and on the other hand we output the equation that must be fulfilled by the theory with the standard Lagrangian (153) terms of some of the components of the solution to the master equation for the free theory (1)

The last equation demands that the right-hand side of (191) is the first equation from (167), this is attained if and only if the constants $k_4^{(2)}$ and $k_4^{(1)}$ are equal

$$k_4^{(2)} = k_4^{(1)}. \quad (192)$$

Substituting (192) back in (187) and (191) respectively, on the one hand we deduce the general form of the fourth-order deformation of the fully deformed solution (155) of the master equation associated with the free theory (1), $S_4$, in terms of some of the components of the solution to the master equation for the theory with the standard Lagrangian (153)

$$S_4 = \tilde{S}_4 + 2k_3^{(1)} \left( \tilde{S}_2^{(1)} + \tilde{S}_2^{(2)} \right) + \left( k_3^{(1)} + k_3^{(3)} \right) \tilde{S}_2^{(3)} + \left( k_3^{(4)} + k_3^{(1)} \right) \tilde{S}_2^{(4)} + k_4^{(1)} \left( \tilde{S}_1^{(1)} + \tilde{S}_1^{(2)} \right) + k_4^{(3)} \tilde{S}_1^{(3)} + k_4^{(4)} \tilde{S}_1^{(4)} \quad (193)$$

and on the other hand we output the equation that must be fulfilled by the fifth-order deformation $S_5$

$$s \left[ S_5 - \tilde{S}_5 - 3k_3^{(1)} \left( \tilde{S}_3^{(1)} + \tilde{S}_3^{(2)} \right) - \left( 2k_3^{(1)} + k_3^{(3)} \right) \tilde{S}_3^{(3)} \right]$$

$$+ \left( k_3^{(4)} + k_3^{(1)} \right) \tilde{S}_3^{(4)} + \sum_{m=1}^{4} k_4^{(m)} \tilde{S}_1^{(m)} = 0, \quad (187)$$

where $\left( k_4^{(m)} \right)_{m=1}^{4}$ are some arbitrary, real constants. This ends the first step of the uniqueness procedure.

Next, we proceed like we did in the above for $S_3$ and $S_4$, but in relation with $S_4$ and $S_5$. Inserting expansions (154) and (155) respectively into equations (156) and (157) projected on $\lambda^5$, we find the equations satisfied by $S_5$ and $\tilde{S}_5$ respectively under the form

$$sS_5 + (S_4, S_4) + (S_2, S_3) = 0, \quad (188)$$

$$s\tilde{S}_5 + (\tilde{S}_4, \tilde{S}_4) + (\tilde{S}_2, \tilde{S}_3) = 0. \quad (189)$$

If we subtract (189) from (188) and recall (158), then we infer the equation

$$s \left( S_5 - \tilde{S}_5 \right) = - \left( \tilde{S}_1, S_4 - \tilde{S}_4 \right) - \left( \tilde{S}_2, S_3 - \tilde{S}_3 \right). \quad (190)$$

By replacing (185) and (187) into the right-hand side of (190) and by further calculating the resulting expression with the help of relations (166)–(168) and (170)–(173), we arrive at

$$s \left( S_5 - \tilde{S}_5 \right) = s \left[ 2k_4^{(1)} \tilde{S}_2^{(1)} + \left( k_4^{(4)} + k_4^{(1)} \right) \tilde{S}_2^{(4)} + \left( k_4^{(2)} + k_4^{(3)} \right) \tilde{S}_2^{(3)} \right.$$  
$$\left. + 3k_4^{(1)} \left( \tilde{S}_3^{(1)} + \tilde{S}_3^{(2)} \right) + \left( 2k_4^{(1)} + k_4^{(3)} \right) \tilde{S}_3^{(3)} \right.$$  
$$\left. + \left( 2k_4^{(1)} + k_4^{(4)} \right) \tilde{S}_3^{(4)} \right] + \left( k_4^{(2)} + k_4^{(1)} \right) \left( \tilde{S}_4^{(1)} + \tilde{S}_4^{(2)} \right)$$  
$$\left. + k_4^{(2)} \left( \tilde{S}_4^{(2)} - \tilde{S}_4^{(1)} \right) \right]. \quad (191)$$

The last equation demands that the right-hand side of (191) is $s$-exact. Due to the first equation from (167), this is attained if and only if the constants $k_4^{(2)}$ and $k_4^{(1)}$ are equal

$$k_4^{(2)} = k_4^{(1)}. \quad (192)$$

Substituting (192) back in (187) and (191) respectively, on the one hand we deduce the general form of the fourth-order deformation of the fully deformed solution (155) of the master equation associated with the free theory (1), $S_4$, in terms of some of the components of the solution to the master equation for the theory with the standard Lagrangian (153)
with \((175)–(178)\), we reach the equation computing its expression by means of relations \((166)–(168)\), \((170)–(173)\), and then subtract the above relations one from the other and employ \((158)\), last equation reads as

\[S_5 = \tilde{S}_5 + 3k_3^{(1)} \left( \tilde{S}_3^{(1)} + \tilde{S}_3^{(2)} \right) + \left( 2k_3^{(1)} + k_3^{(3)} \right) \tilde{S}_3^{(3)} + \left( 2k_3^{(1)} + k_3^{(4)} \right) \tilde{S}_3^{(4)} + \left( k_4^{(1)} + k_4^{(3)} \right) \tilde{S}_4^{(3)} + \sum_{m=1}^{4} k_5^{(m)} \tilde{S}_1^{(m)}, \quad (195)\]

with \(k_5^{(m)}\) some arbitrary, real constants. This completes the second step of the uniqueness procedure.

We reprise the procedure used previously for \(S_4\) and \(S_5\), but in connection with \(S_5\) and \(S_6\). In view of this, we project \((156)\) and \((157)\) on \(\lambda^6\), respectively, which provides the equations

\[2sS_6 + 2(S_1, S_5) + 2(S_2, S_4) + (S_3, S_6) = 0, \quad (196)\]
\[2s\tilde{S}_6 + 2(\tilde{S}_1, \tilde{S}_5) + 2(\tilde{S}_2, \tilde{S}_4) + (\tilde{S}_3, \tilde{S}_6) = 0, \quad (197)\]

and then subtract the above relations one from the other and employ \((158)\), obtaining

\[2s \left( S_6 - \tilde{S}_6 \right) = -2 \left( \tilde{S}_1, S_5 - \tilde{S}_5 \right) - 2 \left( \tilde{S}_2, S_4 - \tilde{S}_4 \right) + \left( \tilde{S}_3, \tilde{S}_3 \right) - (S_3, S_3). \quad (198)\]

Replacing \((183), (193),\) and \((195)\) in the right-hand side of \((198)\) and further computing its expression by means of relations \((166), (168), (170)–(173),\) and \((175)–(178),\) we reach the equation

\[2s \left( S_6 - \tilde{S}_6 \right) = s \left[ 8k_3^{(1)} \left( \tilde{S}_4^{(1)} + \tilde{S}_4^{(2)} \right) + 2 \left( 3k_3^{(1)} + k_3^{(3)} \right) \tilde{S}_4^{(3)} \right]
+ 2 \left( 3k_3^{(1)} + k_3^{(4)} \right) \tilde{S}_4^{(4)} + 6k_4^{(1)} \left( \tilde{S}_3^{(1)} + \tilde{S}_3^{(2)} \right)
+ 2 \left( 2k_4^{(1)} + k_4^{(3)} \right) \tilde{S}_3^{(3)} + 2 \left( 2k_4^{(1)} + k_4^{(4)} \right) \tilde{S}_3^{(4)}
+ 2 \left( 2k_5^{(1)} + k_5^{(3)} \right) \tilde{S}_2^{(1)} + 2 \left( k_3^{(1)} \right)^2 \tilde{S}_2^{(2)}
+ 2 \left( k_5^{(1)} + k_5^{(3)} + k_3^{(1)} k_3^{(3)} \right) \tilde{S}_2^{(3)}
+ 2 \left( k_5^{(1)} + k_5^{(4)} + k_3^{(1)} k_3^{(4)} \right) \tilde{S}_2^{(4)} \right] \]
The solution to this equation is written as

\[ +2 \left( k_5^{(2)} + k_5^{(1)} \right) \left( \tilde{S}_1^{(1)} + \tilde{S}_1^{(2)} \right) + 2k_5^{(2)} \left( \tilde{S}_1^{(2)} , \tilde{S}_1^{(2)} \right) \]  

(199)

On account of the former relation in [167], we conclude that [199] holds (i.e. its right-hand side is s-exact) if and only if the constants \( k_4^{(2)} \) and \( k_4^{(1)} \) are equal

\[ k_4^{(2)} = k_4^{(1)} . \]  

(200)

Based on the last result inserted in [195] and [199], we complete the third step of our procedure for constructing \( S \) and in fact proving the uniqueness of \( \tilde{S} \): we output the general form of the fifth-order deformation of the fully deformed \( k \)

\[
S_5 = \tilde{S}_5 + 3k_3^{(1)} \left( \tilde{S}_3^{(1)} + \tilde{S}_3^{(2)} \right) + \left( 2k_3^{(1)} + k_3^{(3)} \right) \tilde{S}_3^{(3)} + \left( 2k_3^{(1)} + k_3^{(4)} \right) \tilde{S}_3^{(4)} \\
+ 2k_4^{(1)} \left( \tilde{S}_2^{(1)} + \tilde{S}_2^{(2)} \right) + \left( k_4^{(4)} + k_4^{(1)} \right) \tilde{S}_2^{(4)} + \left( k_4^{(1)} + k_4^{(3)} \right) \tilde{S}_2^{(3)} \\
+ k_5^{(1)} \left( \tilde{S}_1^{(1)} + \tilde{S}_1^{(2)} \right) + k_5^{(3)} \tilde{S}_1^{(3)} + k_5^{(4)} \tilde{S}_1^{(4)}
\]  

(201)

and meanwhile deduce the equation verified by the deformation of the next order

\[
s \left[ S_6 - \tilde{S}_6 - 4k_3^{(1)} \left( \tilde{S}_4^{(1)} + \tilde{S}_4^{(2)} \right) - \left( 3k_3^{(1)} + k_3^{(3)} \right) \tilde{S}_4^{(3)} \\
- \left( 3k_3^{(1)} + k_3^{(4)} \right) \tilde{S}_4^{(4)} - 3k_4^{(1)} \left( \tilde{S}_3^{(1)} + \tilde{S}_3^{(2)} \right) - \left( 2k_4^{(1)} + k_4^{(3)} \right) \tilde{S}_3^{(3)} \\
- \left( 2k_4^{(1)} + k_4^{(4)} \right) \tilde{S}_3^{(4)} - \left( k_4^{(1)} \right) ^2 \tilde{S}_2^{(2)} \\
- \left( k_5^{(1)} + k_5^{(3)} + k_3^{(1)} k_3^{(4)} \right) \tilde{S}_2^{(3)} - \left( k_5^{(1)} + k_5^{(4)} + k_3^{(1)} k_3^{(4)} \right) \tilde{S}_2^{(4)} \right] = 0 
\]  

(202)

The solution to this equation is written as

\[
S_6 = \tilde{S}_6 + 4k_3^{(1)} \left( \tilde{S}_4^{(1)} + \tilde{S}_4^{(2)} \right) + \left( 3k_3^{(1)} + k_3^{(3)} \right) \tilde{S}_4^{(3)} \\
+ \left( 3k_3^{(1)} + k_3^{(4)} \right) \tilde{S}_4^{(4)} + 3k_4^{(1)} \left( \tilde{S}_3^{(1)} + \tilde{S}_3^{(2)} \right) + \left( 2k_4^{(1)} + k_4^{(3)} \right) \tilde{S}_3^{(3)} \\
+ \left( 2k_4^{(1)} + k_4^{(4)} \right) \tilde{S}_3^{(4)} + \left( k_4^{(1)} \right) ^2 \tilde{S}_2^{(2)} \\
+ \left( k_5^{(1)} + k_5^{(3)} + k_3^{(1)} k_3^{(4)} \right) \tilde{S}_2^{(3)} + \left( k_5^{(1)} + k_5^{(4)} + k_3^{(1)} k_3^{(4)} \right) \tilde{S}_2^{(4)} \\
+ \sum_{m=1}^{4} k_6^{(m)} \tilde{S}_1^{(m)} , 
\]  

(203)

with \( k_6^{(m)} \) some arbitrary, real constants, independent so far. Just like in the above it can be shown that in fact \( k_6^{(2)} = k_6^{(1)} \) (via establishing a relationship between \( S_7 \) and \( \tilde{S}_7 \)), etc.
Replacing (158), (159), (160), and (201) (for $k_6^{(2)} = k_6^{(1)}$) in (155) and regrouping the various terms according to the structure of decompositions (165), (169), (174), (179), we finally obtain

\[ S = \tilde{S}_0 + \lambda \left( 1 + k_3^{(1)} \lambda^2 + k_4^{(1)} \lambda^3 + k_5^{(1)} \lambda^4 + k_6^{(1)} \lambda^5 + \ldots \right) \left( \tilde{S}_1^{(1)} + \tilde{S}_1^{(2)} \right) \\
+ \lambda \left( 1 + k_3^{(3)} \lambda^2 + k_4^{(3)} \lambda^3 + k_5^{(3)} \lambda^4 + k_6^{(3)} \lambda^5 + \ldots \right) \tilde{S}_1^{(3)} \\
+ \lambda \left( 1 + k_3^{(4)} \lambda^2 + k_4^{(4)} \lambda^3 + k_5^{(4)} \lambda^4 + k_6^{(4)} \lambda^5 + \ldots \right) \tilde{S}_1^{(4)} \\
+ \lambda^2 \left[ 1 + 2 k_3^{(1)} \lambda^2 + 2 k_4^{(1)} \lambda^3 + \left( 2 k_5^{(1)} + \left( k_3^{(1)} \right)^2 \right) \lambda^4 + \ldots \right] \left( \tilde{S}_2^{(1)} + \tilde{S}_2^{(2)} \right) \\
+ \lambda^2 \left[ 1 + \left( k_3^{(1)} + k_3^{(3)} \right) \lambda^2 + \left( k_4^{(1)} + k_4^{(3)} \right) \lambda^3 + \ldots \right] \tilde{S}_2^{(3)} \\
+ \lambda^2 \left[ 1 + \left( k_3^{(1)} + k_3^{(4)} \right) \lambda^2 + \left( k_4^{(1)} + k_4^{(4)} \right) \lambda^3 + \ldots \right] \tilde{S}_2^{(4)} \\
+ \lambda^3 \left[ 1 + 3 k_3^{(1)} \lambda^2 + 3 k_4^{(1)} \lambda^3 + \ldots \right] \left( \tilde{S}_3^{(1)} + \tilde{S}_3^{(2)} \right) \\
+ \lambda^3 \left[ 1 + 2 k_3^{(1)} + k_3^{(3)} \right) \lambda^2 + \left( 2 k_4^{(1)} + k_4^{(3)} \right) \lambda^3 + \ldots \right] \tilde{S}_3^{(3)} \\
+ \lambda^3 \left[ 1 + 2 k_3^{(1)} + k_3^{(4)} \right) \lambda^2 + \left( 2 k_4^{(1)} + k_4^{(4)} \right) \lambda^3 + \ldots \right] \tilde{S}_3^{(4)} \\
+ \lambda^3 \left[ 1 + 4 k_3^{(1)} \lambda^2 + \ldots \right] \left( \tilde{S}_4^{(1)} + \tilde{S}_4^{(2)} \right) \\
+ \lambda^4 \left[ 1 + 3 k_3^{(1)} + k_3^{(3)} \right) \lambda^2 + \ldots \right] \tilde{S}_4^{(3)} \\
+ \lambda^4 \left[ 1 + 3 k_3^{(1)} + k_3^{(4)} \right) \lambda^2 + \ldots \right] \tilde{S}_4^{(4)} + \ldots .
\]

It is now clear that the last expression can be written as in (153) (at least in the first orders in $\lambda$) modulo the transformations (159)–(161). The conclusion of this section is that the deformation procedure for action (155) can be used at proving in an elegant manner the uniqueness of eleven-dimensional interactions between a graviton and a three-form gauge field prescribed by General Relativity.

7 Conclusion

To conclude with, in this paper we have generated the consistent interactions in eleven spacetime dimensions that can be added to a free theory describing a massless spin-two field and an Abelian three-form gauge field. Our treatment is based on the Lagrangian BRST deformation procedure, which relies on the construction of consistent deformations of the solution to the master equation with the help of standard cohomological techniques. The couplings are obtained under the hypotheses of smoothness in the coupling constant, locality, Lorentz
covariance, Poincaré invariance, and the presence of at most two derivatives in the interacting Lagrangian. Our main result is that if we decompose the metric like \( g_{\mu \nu} = \sigma_{\mu \nu} + \lambda h_{\mu \nu} \), then we can couple the Abelian three-form gauge field to \( h_{\mu \nu} \) in the space of formal series with the maximum derivative order equal to two in \( h_{\mu \nu} \) such that the resulting interactions agree with the usual couplings between the three-form and the massless spin-two field in vielbein formulation. Thus, we emphasize the uniqueness of eleven-dimensional interactions between a graviton and a three-form gauge field prescribed by General Relativity. We cannot stress enough that the cosmological term is not restricted in this context. Its presence is forbidden only if we add to the present field content other particles, such as massless gravitini.

Acknowledgments

The authors wish to thank Constantin Bizdadea and Odile Saliu for useful discussions and comments. This work is partially supported by the European Commission FP6 program MRTN-CT-2004-005104 and by the grant AT24/2005 with the Romanian National Council for Academic Scientific Research (C.N.C.S.I.S.) and the Romanian Ministry of Education and Research (M.E.C.).

References

[1] B. Voronov and I. V. Tyutin, *Theor. Math. Phys.* 50, 218 (1982).
[2] B. Voronov and I. V. Tyutin, *Theor. Math. Phys.* 52, 628 (1982).
[3] J. Gomis and S. Weinberg, *Nucl. Phys.* B469, 473 (1996).
[4] S. Weinberg, *The Quantum Theory of Fields* (Cambridge University Press, Cambridge, 1996).
[5] O. Piguet and S. P. Sorella, *Algebraic Renormalization: Perturbative Renormalization, Symmetries and Anomalies*, Lecture Notes in Physics, Vol. 28 (Springer, Berlin, 1995).
[6] P. S. Howe, V. Lindström and P. White, *Phys. Lett.* B246, 430 (1990).
[7] W. Troost, P. van Nieuwenhuizen and A. van Proeyen, *Nucl. Phys.* B333, 727 (1990).
[8] G. Barnich and M. Henneaux, *Phys. Rev. Lett.* 72, 1588 (1994).
[9] G. Barnich, *Mod. Phys. Lett.* A9, 665 (1994).
[10] G. Barnich, *Phys. Lett.* B419, 211 (1998).
[11] F. Brandt, M. Henneaux and A. Wilch, *Phys. Lett.* B387, 320 (1996).
[12] R. Arnowitt and S. Deser, *Nucl. Phys.* 49, 133 (1963).
[13] J. Farg and C. Frønsdal, *J. Math. Phys.* **20**, 2264 (1979).

[14] F. A. Berends, G. H. Burgers and H. Van Dam, *Z. Phys.* **C24**, 247 (1984).

[15] F. A. Berends, G. H. Burgers and H. Van Dam, *Nucl. Phys.* **B260**, 295 (1985).

[16] A. K. H. Bengtsson, *Phys. Rev.* **D32**, 2031 (1985).

[17] G. Barnich and M. Henneaux, *Phys. Lett.* **B311**, 123 (1993).

[18] J. D. Stasheff, Deformation theory and the Batalin-Vilkovisky master equation, in *Deformation Theory and Symplectic Geometry* (Ascona, 17–21 June 1996), eds. D. Sternheimer, J. Rawnsley and S. Gutt, Math. Phys. Stud. **20**, pp. 271–284 (Kluwer Acad. Publ., Dordrecht, 1997).

[19] J. D. Stasheff, The (secret?) homological algebra of the Batalin-Vilkovisky approach, in *Secondary Calculus and Cohomological Physics* (Moscow, 24–31 August 1997), eds. M. Henneaux, J. Krasil’shchik and A. Vinogradov, Contemp. Math. **219**, pp. 195–210 (Amer. Math. Soc., Providence, RI, 1998).

[20] J. A. Garcia and B. Knaepen, *Phys. Lett.* **B441**, 198 (1998).

[21] M. Henneaux, Consistent interactions between gauge fields: the cohomological approach, in *Secondary Calculus and Cohomological Physics* (Moscow, 24–31 August 1997), eds. M. Henneaux, J. Krasil’shchik and A. Vinogradov, Contemp. Math. **219**, pp. 93–109 (Amer. Math. Soc., Providence, RI, 1998).

[22] N. Boulanger, T. Damour, L. Gualtieri and M. Henneaux, *Nucl. Phys.* **B597**, 127 (2001).

[23] N. Boulanger and M. Henneaux, *Annalen Phys.* **10**, 935 (2001).

[24] C. Bizdadea, E. M. Cioroianu, A. C. Lungu and S. O. Saliu, *J. High Energy Phys.* JHEP **0502**, 016 (2005).

[25] C. Bizdadea, E. M. Cioroianu, D. Cornea, S. O. Saliu, S. C. Săraru, *Eur. Phys. J.* **C48**, 265 (2006).

[26] C. Bizdadea, E. M. Cioroianu, A. C. Lungu, *Int. J. Mod. Phys.* **A21**, 4083 (2006).

[27] C. Bizdadea, E. M. Cioroianu, A. C. Lungu, S. C. Săraru, *Annalen Phys.* **15**, 416 (2006).

[28] N. Boulanger and M. Esole, *Class. Quantum Grav.* **19**, 2107 (2002).

[29] E. Cremmer, B. Julia and J. Scherk, *Phys. Lett.* **B76**, 409 (1978).
[30] B. de Wit, Introduction to supergravity, in *Supersymmetry and Supergravity ’84: Proceedings of the Trieste Spring School 4–14 April, 1984*, eds. B. de Wit, P. Fayet and P. van Nieuwenhuizen, p. 49 (World Scientific, Singapore, 1984).

[31] K. Bautier, S. Deser, M. Henneaux and D. Seminara, *Phys. Lett.* **B406**, 49 (1997).

[32] G. Barnich, F. Brandt and M. Henneaux, *Commun. Math. Phys.* **174**, 57 (1995).

[33] G. Barnich, F. Brandt and M. Henneaux, *Phys. Rept.* **338**, 439 (2000).

[34] E. M. Cioroianu, E. Diaconu and S. C. Sararu, *Int. J. Mod. Phys.* **A23**, 4841 (2008).

[35] E. M. Cioroianu, E. Diaconu and S. C. Sararu, *Int. J. Mod. Phys.* **A23**, 4861 (2008).

[36] E. M. Cioroianu, E. Diaconu and S. C. Sararu, *Int. J. Mod. Phys.* **A23**, 4877 (2008).

[37] W. Pauli and M. Fierz, *Helv. Phys. Acta* **12**, 297 (1939).

[38] M. Fierz and W. Pauli, *Proc. Roy. Soc. Lond.* **A173**, 211 (1939).

[39] M. Henneaux, B. Knaepen and C. Schomblond, *Commun. Math. Phys.* **186**, 137 (1997).

[40] C. Bizdadea, C. C. Ciobirca, E. M. Cioroianu, I. Negru, S. O. Saliu and S. C. Sararu, *J. High Energy Phys.* JHEP **0310**, 019 (2003).

[41] M. Henneaux and B. Knaepen, *Phys. Rev.* **D56**, 6076 (1997).

[42] W. Siegel, Fields, hep-th/9912205