Spatio-temporal dissipative Kerr solitons and regularized wave-collapses in chains of nonlinear microresonators

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We theoretically investigate the nonlinear dynamics and dissipative Kerr soliton formation in chains of equally-coupled, optically driven microresonators with Kerr nonlinearity. We show that the nonlinear dynamics of this system can be associated with an effective two-dimensional space: combined spatial and synthetic frequency dimension of each resonator. As a result, we demonstrate the existence of two fundamentally different dynamical regimes - elliptic and hyperbolic - inherent to the system. In the elliptic regime, we generate a two-dimensional dissipative Kerr soliton corresponding to the global spatio-temporal mode-locking. Moreover, we show that the presence of the second dimension leads to the observation of regularized wave collapse. Our work also elucidates the limitations of coherent soliton formation, imposed by intra-band four-wave-mixing between bands.

We show that soliton formation can both be impaired in trivial but importantly also topologically protected bands.

Introduction.— The frequency comb generation in driven nonlinear optical microresonators via dissipative Kerr soliton (DKS) formation (microcombs) has enabled compact frequency combs (Fig. 1(a)) with wide ranging applications but equally triggered the study of nonlinear dynamics in complex systems exhibiting many optical modes [1, 2]. At the leading order, DKS can be described by the 1D driven-dissipative nonlinear Schrödinger equation (NLSE) [3] known as the Lugiato-Lefever equation (LLE) [2, 4]. In this framework, a variety of the nonlinear phenomena have been observed [5–11].

Yet to date, almost all experimental and theoretical works on ‘dissipative structures’ in optically driven Kerr nonlinear resonators (be it fiber [12, 13] or microresonator based) have been focused on the single resonator case. Recent development in integrated nonlinear photonic platforms [14, 15] indicates that large-scale arrays of coupled Kerr nonlinear resonators that combine spatial and synthetic dimensions [16] are within experimental reach - yet their nonlinear dynamics under continuous wave driving remain largely unexplored. Such systems are expected to exhibit rich and unexpected nonlinear dynamics. Even the simple case of a photonic dimer has demonstrated a variety of emergent nonlinear dynamics [17]. Lattices are particularly attractive as they allow significantly more complex dispersion landscapes, which are inaccessible using traditional dispersion engineering approaches, to be synthesized. On a practical level, chains of resonators are therefore expected to provide a pathway to octave spanning dissipative Kerr solitons, which is an enduring outstanding challenge in the field. Such spectra are required for self-referencing of micro-combs. In addition, such coupled resonators can enable to create topological band structures in both 1D and 2D. Yet, to date, the dissipative Kerr soliton formation in systems that exhibit bands is not fully understood - except for the two band-case recently treated. [17] Even the simple case of a photonic dimer has demonstrated a variety of emergent nonlinear dynamics [17]. Recent investigation of a nonlinear 2D Haldane model made of coupled multi-mode optical microresonators (Fig. 1(b)) demonstrated the generation of partially coherent solitons in the topological edge state [18]. However, this study left a number of important questions open. Specifically: how does the extension of dimensionality change the four-wave mixing (FWM) processes in general and soliton generation in particular? Moreover an open question is to what extend topologically protected (i.e. robust) edge states [19] can give rise to coherent solitons, and to what extend soliton formation is robust [17, 18]. The description of the interaction of the edge state with bulk mode has not been investigated to date, which can explain the only partially coherent nature of the recently numerically predicted edge solitons in the Haldane lattice of nonlinear coupled resonators [18].

Here we theoretically analyze the dynamics of soliton formation in a 1D lattice (with periodic boundary conditions) of coupled resonators (Fig. 1(c,d)) and demonstrate that the system exhibits a rich 2D spatiotemporal nonlinear dynamics. Specifically, we demonstrate the formation of coherent spatio-temporal solitons in equally coupled 1D chains, which are formally equivalent to the solutions found to exist in the edge state of the 2D topological Haldane model [18]. Equally, we demonstrate a mechanism of breaking down of the topological edge state formed soliton using the simplest example of 1D Su-Schrieffer-Heeger (SSH) topological model. With this article, we aim at bridging the gap between the simplest and relatively well-understood case of the photonic dimer [20, 21] and the topological arrangement studied in Ref. [18], revealing the dramatic change of the dynamics caused by the increased dimensionality of the system. We demonstrate that the mean-field model describing the
Hybridized dispersion in 1D lattice of equally coupled optical resonators. (a) Schematics of a continuous laser driven single optical $\chi^{(3)}$ resonator leading to dissipative Kerr soliton formation (i.e. an optical frequency comb); (b) corresponding integrated dispersion profile which includes second-order dispersion $d_2$; (c) Schematics of 1D lattice in the ring configuration. The resonators are coupled with a rate $J$; (d) corresponding cosine band structure in the single-mode case with respect to $k_0 = N/2$. Normal ($j_2 < 0$) and anomalous ($j_2 > 0$) regions of the band structure are depicted by green and blue, respectively. Triangle, rectangle, and circle indicate the pumped supermodes for generation of a traveling soliton in Fig. 2 and investigation of the regularized wave collapse in Fig. 3. (e) Hybridized integrated dispersion of a multimode chain and modulation instability gain lobes (depicted in red) in elliptic (top panel, anomalous sGVD at $k$ and hyperbolic (bottom panel, normal sGVD at $k_0 = N/2$) regions, respectively.

Spatial eigenstates.— Assuming the equal couplings to the bus waveguides, we can consider the proposed structure as a perfect 1D photonic crystal that naturally possesses a set of collective spatial excitations or supermodes whose eigenvalues form a cosine band structure schematically shown in Fig. 1(d). The band structure describes the energy range of the excitations propagating in the crystal and imposes their dispersion law, which plays a crucial role in the context of nonlinear physics. The regions of anomalous and normal supermode GVD (sGVD) are shown in Fig. 1(d) by blue and green colors, respectively. For a given supermode index $k_0$, the linear term in the Taylor series gives the supermode FSR equal to $J_1/2\pi = 2J/N \sin(2\pi k_0/N)$ and the corresponding quadratic term yields sGVD $J_2 = 2J(2\pi/N)^2 \cos(2\pi k_0/N)$. Taking into account the resonance frequencies of each resonator with second-order dispersion coefficient $D_2$ (Fig. 1(b)), the resonance positions of the chain in the vicinity of $\omega_0$ take form

$$\omega_{\mu k} = \omega_0 + D_1 \mu + D_2 \mu^2 - 2J \cos(2\pi k/N),$$

where $D_1/2\pi$ is free spectral range (FSR) of the individual resonator. Index $k$ ($\mu$) stands for supermode (comb line) index. Now, Eq. (2) describes a hybridized 2D dispersion surface presented in Fig. 1(e). Remarkably the

dynamics can be represented in the form of elliptic or hyperbolic 2D LLE [22], thereby making a link to the prior findings in the field of spatial solitons [23, 24] and generalizing the conventional single-mode coupled resonator theory [25, 26]. We describe the principles of 2D FWM processes, the global soliton formation, and predict novel emergent nonlinear effects such as edge-to-bulk scattering and regularized wave collapse.

Model.— We theoretically describe a lattice of coupled nonlinear optical resonators using the mean-field approximation, via a system of coupled LLE. In the main text we treat the simplest case of equally coupled chain of resonators, while we present the general reasoning for an arbitrary lattice (including the 1D topological SSH model) in the Supplementary Materials.

$$\frac{\partial A_{\ell}}{\partial t} = -i\left(\kappa_{ex,\ell} + \kappa_0\right)A_{\ell} + iJ(A_{\ell-1} + A_{\ell+1}) + i\frac{D_2}{2} A_{\ell} + i\gamma_0|A_{\ell}|^2 A_{\ell} + \sqrt{\kappa_{ex,\ell}\kappa_{in,\ell}} e^{i\phi_0}. \quad (1)$$

Here $\kappa_{ex,\ell}$ is the coupling of the $\ell$-th resonator to the corresponding pump $s_{in,\ell}$ with its general phase $\phi_0$, $\omega_0$ is laser-cavity detuning, $\kappa_0$ is intrinsic linewidth of the resonators, $J$ is coupling strength between the neighboring resonators, $D_2$ is chromatic group velocity dispersion (GVD), $g_0$ is single photon Kerr frequency shift.
hybridization and corresponding 2D dispersion surface, applies to any lattice of resonators, including topologically non-trivial; and the multi-mode dynamics of the edge state of the recently studied nonlinear 2-D Haldane model [18], is described by a simple 1-D chain of resonators that mimics the two dimensional dispersion profile of the edge state (cf. SI).

However, the excitation of the individual supermode requires an accurate pump projection on its spatial profile. Though the excitation of the system via a single resonator is possible, the pump power, in this case, will be redistributed among all the supermodes within the excitation bandwidth. The number of the excited modes will depend on the local density of states within the width of the band in Fig. 1(d). Even if the resonance linewidth is small enough, so the individual resonances in Fig. 1(d) are distinguishable, the single-resonator pump scheme always leads to the excitation of supermodes in pairs due to their two-fold degeneracy, except for the modes from the very top and bottom of the band (cf. SI). For simplicity, in the following we focus on the ideal case of a single supermode excitation.

With the assumptions above, we can rewrite the system in Eq. (1) in the Fourier space for the normalized amplitude \( \psi_{\mu k} = 1/(2\pi \sqrt{N}) \int \sum_{\ell=1}^{N} \Psi_{\ell} e^{2\pi i (k/N + \mu \varphi)} d\varphi \), with \( \Psi_{\ell} = \sqrt{2g_{\ell}/\kappa \ell} \) (cf. SI), as

\[
\frac{\partial \psi_{\mu k}}{\partial \tau} = - (1 + i \zeta_0) \psi_{\mu k} - i \left[ d_2 \mu^2 - 2 \cos \frac{2\pi k}{N} \right] \psi_{\mu k} + \frac{i}{N} \sum_{k_1, k_2, k_3} \psi_{\mu_1 k_1} \psi_{\mu_2 k_2} \psi_{\mu_3 k_3} \delta_{\mu_1 + \mu_2 - \mu_3 - \mu} \delta_{k_1 + k_2 - k_3 - k} + \delta_{k_0, \mu} \delta_{\mu, 0} f_{k_0}, \tag{3}
\]

with normalized detuning \( \zeta_0 = 2\delta \omega_0/\kappa \), dispersion \( d_2 = D_2/\kappa \), coupling \( j = 2J/\kappa \), and pump \( f_{k_0} \) that is projected on \( k_0 \) and scanned across the \( \mu_0 \)-th optical resonance. The term in the square brackets is normalized integrated dispersion defined from Eq. (2) as \( d_{\mu \mu}(\mu, k) = 2(\omega_{\mu k} - \omega_0 + D_1 \mu)/\kappa \), and it incorporates the dispersion laws for resonator \( (d_2 \mu^2) \) and supermodes \( (2j \cos 2\pi k/N) \), representing the hybridized 2D dispersion surface \( d_{\mu \mu}(\mu, k) = d_2 \mu^2 - 2j \cos(2\pi k/N) \). In the case of anomalous GVD \( (d_2 > 0) \), the individual resonator, this surface with parabolic and cosine cross-sections is shown in Fig. 1(e). Local dispersion topography changes along the \( k \) axis, revealing different regions with parabolic and saddle shapes.

**Modulation instability gain lobes.**—Further, we investigate the stability of plane wave solutions \( \psi_{00} \). Considering the pump at \( \mu_0 = 0 \) and \( k - k_0 = 0 \), we investigate FWM processes between the pump mode and the modes with indexes \( \mu, k \). Linearizing the system with respect to these modes, we identify the modes with positive parametric gain. Our analysis, similar to Ref. [22], shows that the modulationally unstable solutions form an ellipse (hyperbola) in the \( \mu - k \) space (see SI for details).

\[
d_2 \mu^2 + j_2 k^2 = 4|\psi_{00}|^4 + \sqrt{|\psi_{00}|^4 - 1 - (\zeta_0 + 2j)}, \tag{4}
\]

here \(+ (-)\) stands for the excitation of \( k - k_0 = 0 \) \((k - k_0 = N/2)\). An example of the modulation instability gain lobes [Eq. (4)] is presented in Fig. 1(e) for both regions in case of \( d_2 = 0.04 \) and \( j_2 = 2|J_2|/\kappa = 1 \). Top panel in Fig. 1(e) reveals that the supermode corresponding to the excitation of all the resonators in-phase (anomalous sGVD) is unstable against small perturbations with \( \mu \) and \( k \) indexes that form an ellipse. The width and height of the ellipse are defined by pump power, \( d_2 \), and \( j_2 \) coefficients that correspond to GVD and sGVD. In contrast, the state corresponding to the excitation of the neighboring resonators in the opposite phase (normal sGVD) is unstable against the perturbations with \( \mu \) and \( k \) forming a hyperbola, showing that all the supermodes can experience positive parametric gain.

**Coherent dissipative structures.**—We continue with the simulation of the coupled LLEs in Eq. (1) for 20 resonator chain and constant normalized coupling \( j = 10.13 \) \((j_2 = 1)\). To simulate the temporal dynamics, we employ the step-adaptable Dormand-Prince Runge-Kutta method of Order 8(5,3) [27] and approximate the dispersion operator by the second-order finite difference scheme. We deliberately choose the pumping scheme allowing for exciting only a given mode. To trigger the FWM processes, we numerically scan the resonance with a fixed pump power and track field dynamics in all the resonators. We observe the formation of Turing patterns [23, 28] in both pure hyperbolic and elliptic regimes (see SI for details), and remarkably localized 2D dissipative solitons [24] traveling along the circumference of the chain, which we describe in the following.

To generate this spatio-temporal Kerr soliton (2D-DKS), we pump the 4th supermode in the elliptic regime (cf. SI) \((k - k_0 = -6)\) marked by the red triangle in Fig. 1(d)) with \(|f_{j}| = 2.35 \) and \( \zeta_0 = 10.92 \), so the local dispersion has anomalous sGVD \( j_2 = 4j(\pi/10)^2 \cos 2\pi/5 \) in addition to the non-zero supermode FSR \( j_1 = 2j\pi/5 \sin 2\pi/5 \). The obtained solution of the 2D-DKS corresponds to continuously re-circulating spatial discrete soliton that forms an ellipse with a fish-like tail in the spectral domain (cf. Fig. 2(a,b)). Similar to Cherenkov radiation for conventional DKS, the disk-shaped soliton crosses the hybridized dispersion in the vicinity of the edge of the Brillouin zone, resulting in the intensive generation of the dispersive waves, forming the fish-like spectrum (cf. SI), but preserving the soliton coherence. On the single resonator level, the optical field envelope demonstrates breathing dynamics (Fig. 2(c)) because the pulses periodically arrive in the resonator. Resolving the field envelope dynamics in time, one detects the periodic appearance of optical pulses and adjacent dispersive waves. Sampling this signal in time and computing the overall Fourier transform gives the so-called superresolution spectrum shown in Fig. 2(d). The periodic nature of the signal reveals a typical comb
FIG. 2. Localized 2D dissipative soliton in a chain of 20 resonators. Instantaneous field profile in the chain of resonators and the corresponding 2D spectral profile are shown in panels (a) and (b). Inset in (a) shows the roundtrip number of the soliton in time. Panel (c) represents the field dynamics on a single resonator level. The corresponding superresolution and averaged spectrum are presented in panel (d).

Wave collapse.—Since the LLE is the NLSE with an external forcing term and dissipation, it can possess similar features, and in particular, the effect called wave collapse [29, 30]. Wave collapses play an important role in physics. In the conservative 2D NLSE, it reveals a singularity of the model, related to a possibility of full pulse compression in a finite time. Practically, this leads to an effective mechanism of local energy dissipation. It has been shown for 2D elliptic focusing NLSE that a pulse of a finite width can implode to an infinitely small area concentrating there a finite amount of energy [31, 32] and therefore becoming ultra broad in the spectral domain. Even the presence of dissipation in 2D LLE does not restrict the wave collapses [33]. On the contrary, wave collapses do not occur in the 2D focusing NLSE with hyperbolic dispersion [32], signifying that it is the dispersion curvature that is responsible for the effect. When wave collapses happen in real systems, the corresponding spectra become too large, so the simplest approximation with second-order dispersion operator becomes not valid anymore, and higher dispersion orders must be taken into account. Consequently, the pulse width does not completely compress, and the collapse regularizes [34]. The same effect we observe in our model. Exciting incoherent dynamics by pumping the elliptic region at $|f| = 2.35$ and $\zeta_0 = 22.1$, we observe rapid formation and dissipation of narrow pulses in each cavity. A typical spatio-temporal diagram at a single resonator level is shown in Fig. 3(a). We observe the random appearance of the pulses in different parts of the cavity and further their rapid compression, during which the peak amplitude significantly exceeds the background level. However, investigating the pulse width dynamics, we find that it does not completely shrink. To find what limits the minimum pulse width, we computed the nonlinear dispersion relation (NDR) [17, 35, 36] [Fig. 3(b)] that is the 2D Fourier transform of the spatio-temporal diagram of the complex field envelope in Fig. 3(a). We observe the high photon occupancy of the region beneath the parabolas, which indicates the presence of 2D dissipative nonlinear structures. Furthermore, all the hybridized parabolas are populated by the photons, meaning that supermodes from both dispersion regions are excited. We continue by reconstructing the supermode NDR for 0th comb line ($\mu_0 = 0$) for all resonators in the following way

$$NDR(\Omega, \mu_0, k) = \frac{1}{\sqrt{N_2 N}} \sum_{\ell,n} \psi_{\mu_0}(t) e^{i(2\pi k\ell/N - \Omega t_n)}, \quad (5)$$

where $\Omega$ is slow frequency, $t_n = \Delta t n$ with $\Delta t = T/N_t$ time-step, $T$ is simulation time with $N_t$ number of discretization points. The result is shown in Fig. 3(c). The whole cosine band structure is populated, including the region of the normal dispersion that prevents the full wave collapse.

We continue the analysis by exciting the hyperbolic region under the same conditions (same pump power and relative detuning $\zeta_0 = -17.0$). As mentioned earlier, the local dispersion topography has an opposite sign of the
FIG. 3. Numerical reconstruction of the nonlinear dispersion relation in the elliptic and hyperbolic regions in the unstable regime. Panels (a-c) correspond to the elliptic region ($k_0 = 0, d_2 > 0, j_2 > 0$), (b-f) to the hyperbolic ($k_0 = N/2, d_2 > 0, j_2 < 0$). Spatiotemporal diagrams of unstable states in 0th resonator are shown in (a) and (d); The corresponding nonlinear dispersion relation (NDR) in the elliptic region (b) demonstrates excitation of all the optical and spatial modes, whereas the NDR in the hyperbolic region (e) reveals that photon transfer between the spatial supermodes is suppressed in the vicinity of the pump mode $\mu = 0$; The panes (c) and (f) represent the nonlinear supermode dispersion relation [Eq. (5)] of 0th comb line for the state in (a) and 25th comb line for the state in (d).

sGVD with respect to the elliptic region. In the conservative long-wavelength limit, this corresponds to the hyperbolic NLSE that does not have wave collapses [32]. Indeed, we observe that the spatio-temporal diagram [Fig. 3(d)] does not demonstrate any extreme events, showing slow (with respect to the elliptic case) incoherent dynamics. Further, comparing the NDR [Fig. 3(e)] with the elliptic case, we show how the mode occupancy differs. In the vicinity of $\mu = 0$, the normal sGVD suppresses the photon transfer along the $k$ axis. Nevertheless, the photon transfer to other supermodes is stimulated in the area where the line crosses the lower parabolas, resulting in the generation of dispersive waves [17, 21]. Reconstructing the supermode NDR (Fig. 3(f)) for $\mu = 25$ comb line [the average crossing position in Fig. 3(e)], we observe the predominant population of the center of the band.

In summary, our theory sheds light on nonlinear interactions in integrated photonic lattices and will be helpful for future investigations of multimode systems with complex band structures and different topological properties.

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I. COUPLED LUGIATO-LEFEVER EQUATIONS IN LATTICES OF RESONATORS

A system of weakly coupled identical optical resonators is shown to be governed by a set of linearly coupled Lugiato-Lefever equations (LLEs), which can be presented in matrix form as

$$\frac{\partial A}{\partial t} = \hat{D}A + i\hat{M}A + i|A|^2A + \hat{F},$$

(1)

where vector $A = [A_0, \ldots, A_{N-1}]^T$ contains optical field envelopes of each resonator in the lattice, matrix

$$\hat{D} = \text{diag}\left[-\frac{(\kappa_0 + \kappa_{\text{ex},0})}{2} + i\delta\omega_1 + \frac{D_2}{2}\frac{\partial^2}{\partial \varphi^2}, \ldots, -\frac{(\kappa_0 + \kappa_{\text{ex},N-1})}{2} + i\delta\omega_1 + \frac{D_2}{2}\frac{\partial^2}{\partial \varphi^2}\right]$$

contains detuning, losses, and dispersion for each resonator, the coupling between different rings is introduced in matrix $\hat{M}$, the nonlinear term $|A|^2A = [|A_0|^2A_0, \ldots, |A_{N-1}|^2A_{N-1}]^T$ describes the conventional Kerr nonlinearity, and $\hat{F} = [\sqrt{\kappa_{\text{ex},0}\delta\omega_1}, \ldots, \sqrt{\kappa_{\text{ex},N-1}\delta\omega_1}]^T$ represents the pump. In general, the coupling matrix $\hat{M}$ is diagonalizable and possesses a set of eigenvectors $\{V_j\}$ and associated eigenvalues $\lambda_i$, so any state $A$ can be represented in this basis

$$A = \sum_j c_j V_j,$$

(2)

where coefficients $c_j = \langle A|V_j\rangle$ correspond to the amplitude of the collective mode $V_j$ and $\langle \cdot | \cdot \rangle$ indicates scalar product. Therefore, Eq. (1) can be rewritten for the amplitudes $c_j$ in the basis of eigenvectors $\{V_j\}$, where the linear part of the equation will take a form of a matrix with eigenvalues $\lambda_i$ on the diagonals corresponding to the resonance frequencies of the collective excitations. However, the nonlinear term will be no longer diagonal in this basis. In the resonator index basis, the nonlinear term takes form

$$|A|^2A = \sum_{j_1,j_2,j_3} c_{j_1}c_{j_2}c_{j_3}^* V_{j_1} V_{j_2} V_{j_3}^*,$$

Projecting this expression onto the state $V_j$, one obtains the coupled-mode equations for the amplitudes $c_j$

$$\frac{\partial c_j}{\partial t} = -\frac{(\kappa_0 + \kappa_{\text{ex}})}{2} + i(\delta\omega_1 - \lambda_j)c_j + \frac{D_2}{2}\frac{\partial^2 c_j}{\partial \varphi^2} + i \sum_{j_1,j_2,j_3} c_{j_1}c_{j_2}c_{j_3}^* \langle V_{j_1} V_{j_2} V_{j_3}^* | V_j \rangle + \hat{f}_j,$$

(3)

where $\hat{f}_j = \langle F|V_j\rangle$ is projection of the pump on the eigenstate $V_j$, the nonlinear term represents the conventional four-wave mixing process with the conservation law dictated by the product $\langle V_{j_1} V_{j_2} V_{j_3}^* | V_j \rangle$. The eigenvalues $\lambda_j$, showing the dependence of supermode frequency on supermode number, naturally start to play a role of dispersion, similar to the conventional LLE in a single resonator. In general, the eigenvalues $\lambda_j$ are not equidistantly separated, and the supermode dispersion can be introduced like the integrated dispersion of a single resonator $D_{\text{int}}(k) = \lambda_k - kJ_1(k-k_0)$, where $J_1$ is the local free spectral range of the spatial supermodes in the vicinity of $k_0$.

Furthermore, this reasoning can be applied to coupled system with nontrivial topologies, including the Haldane model considered in Ref. [1] with the lattice of $21 \times 21$ resonators. Diagonalization of the coupling matrix $M$ yields

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the band structure (shown in Fig. 1(a)) with three remarkable regions: lower bulk, upper bulk, and edge states. The integrated dispersion for the edge modes, shown by blue dots in Fig. 1(b), reveals a typical dispersion curve with pronounced second and third-order dispersion coefficients. With the optical modes of each resonator, the global dispersion transforms to a two-dimensional surface with cross-sections defined by chromatic dispersion and supermode dispersion similar to the case of the one-dimensional chain considered in the main text. Pumping a given supermode \( k \) above a given threshold, four-wave mixing processes can occur and lead to generation of frequency combs, which dynamics and width will be determined by the local dispersion profile; therefore, the excitation of the supermode from the center of the edge band will be mainly determined by the neighboring edge supermodes. Remarkably, a chain of 20 equally coupled resonators (depicted by the red stars in Fig. 1(b)) has a similar profile of supermode dispersion. Neglecting the four-wave mixing induced photon transfer to the bulk modes in the Haldane lattice, the nonlinear dynamics of the edge states can be modeled as a simple chain of resonators. The simplified model provides an opportunity to analytically investigate general aspects of the dynamics and find analogies with already known effects in nonlinear physics.

II. NONLINEARLY INDUCED EDGE-TO-BULK SCATTERING IN THE SU-SCHRIEFFER–HEEGER MODEL

As we have already mentioned, a chain of microresonators can model the edge states of two-dimensional topological lattices. However, with this simplification, we neglect all the possible interactions with the bulk modes. In this section, following Ref. [2], we aim to briefly discuss limitations of our model by considering the generation of DKS at the edge state in the simplest topological model — the Su-Schrieffer–Heeger (SSH). The edge states of the SSH model are localized on the corners of the chain as shown in Fig. 2(a). The chain supports edge states in the case where inter-cell coupling \( J_{\text{inter}} \) is smaller than intra-cell coupling \( J_{\text{intra}} \) (also shown in the inset in Fig. 2(a)). In the limit \( J_{\text{inter}} \rightarrow 0 \) (trivial edge state [3]), the first resonator is completely decoupled from the chain, and its dynamics is described by conventional LLE. With the finite ratio \( J_{\text{inter}}/J_{\text{intra}} < 1 \) the formed band structure (see Fig. 2(b)) has upper and lower bulk regions with eigenmodes in the middle of the gap that correspond to the edge states. With the chromatic dispersion taken into account, the nonlinear interactions happen on the hybridized dispersion surface, presented in Fig. 2(c). Generation of the edge soliton corresponds to the formation of the dispersionless line below the edge state parabola (schematically shown in Fig. 2(c)). If the width of the bandgap is large enough (effectively corresponds to limit \( J_{\text{inter}}/J_{\text{intra}} \rightarrow 0, J_{\text{intra}} \gg \kappa \)), dynamics of the soliton will be similar to the single-resonator dynamics, because the field will be still localized in the first ring. However, if the soliton line can cross the lower bulk, additional photon transfer to the bulk modes will happen (similar effect has already been observed in the system of just two coupled resonators considered in Ref. [4]). The photons scattered to the bulk will experience now two-dimensional dynamics and drastically affect the soliton stability. This qualitative analysis allows us to extend these results to higher-dimensional topological models. For example, in a 2D lattice, a 2D soliton generated at 1D edge
state will scattered to the edge, where the dispersive waves will experience 3D nonlinear dynamics. If a corner state is realized in a 2D lattice, the corresponding corner state soliton will be similar to the conventional single-resonator DKS, however scattering to bulk will lead to 3D nonlinear dynamics of the bulk as well.

![Image of dissipative Kerr solitons at the edge state of the Su-Schrieffer–Heeger model (SSH). Panel (a) represents the spatial profile of the two edge states of the Su-Schrieffer–Heeger model (SSH) model consisting of 20 optical microresonators with the schematics of the chain in the inset. Band structure of the SSH chain of 20 resonators is shown in panel (b). The hybridized dispersion profile and schematics of the generated soliton at the edge state (black line below the edge state parabola) are shown in panel (c).](image)

### III. COUPLED LUGIATO-LEFEVER EQUATIONS FOR A CHAIN OF EQUALLY COUPLED MICRORESONATORS

In the main text, we have presented the system of coupled LLEs for the photon density $A_\ell$ in the $\ell$-th resonator. In the case of constant couplings to the bus waveguides $\kappa_{ex,\ell}$ and constant inter-resonator couplings $J$, we introduce normalized variables $d_2 = D_2/\kappa$, $\kappa = \kappa_0 + \kappa_{ex}$, $\zeta_0 = 2\delta\omega/\kappa$, $j = 2J/\kappa$, $f_\ell = \sqrt{8\kappa_{ex}g_0/\kappa^3}s_{in,\ell}e^{i\phi_\ell}$, $\Psi_\ell = \sqrt{2g_0/\kappa}A_\ell$.

The normalized set of coupled LLEs takes form

$$\frac{\partial \Psi_\ell}{\partial \tau} = -(1 + i\zeta_0)\Psi_\ell + id_2 \frac{\partial^2 \Psi_\ell}{\partial \varphi^2} + ij\left(\Psi_{\ell-1} + \Psi_{\ell+1}\right) + i|\Psi_\ell|^2\Psi_\ell + f_\ell. \tag{4}$$

The linear part can be diagonalized by the Fourier transform

$$\psi_{\mu k} = \frac{1}{2\pi\sqrt{N}} \int \frac{d\varphi}{N} \sum_{\ell=1}^{N} \Psi_\ell e^{2\pi i(k/N+\mu \varphi)} d\varphi, \tag{5}$$

where $k$ is the supermode index and $\mu$ is the comb line index. With the Kerr term, Eq. (4) transforms to

$$\frac{\partial \psi_{\mu k}}{\partial \tau} = -(1 + i\zeta_0)\psi_{\mu k} - i\left[d_2\mu^2 - 2j \cos\left(\frac{2\pi k}{N}\right)\psi_{\mu k} + \sum_{\ell=1}^{N} \sum_{k_1,k_2,k_3,\mu_1,\mu_2,\mu_3} \psi_{\mu_1 k_1} \psi_{\mu_2 k_2} \psi_{\mu_3 k_3} \delta_{\mu_1+\mu_2-\mu_3-k_1+k_2-k_3-k} + \delta_{\mu_0} \tilde{f}_k\right], \tag{6}$$

where the pump term $\tilde{f}_k$ stands for the projection of the pump on the $k$-th supermode

$$\tilde{f}_k = \frac{1}{\sqrt{N}} \sum_{\ell=1}^{N} f_\ell e^{2\pi i(k/N)}. \tag{7}$$

### IV. PUMP PROJECTION ON THE CHAIN

The pump efficiency and the number of excited supermodes in the chain of resonators depend on the spatial arrangement of the pump scheme and the density of states of supermodes. According to Eq. (7), if the resonator $\ell = 0$
is pumped, all the supermodes have a pump term with the projection amplitude $1/\sqrt{N}$. With the increasing number of resonators, pumping scheme with a single resonator excitation becomes less efficient, and more sophisticated schemes are required. To excite only one supermode with index $k_0$, one needs to adjust the relative phases of the pump lasers accurately; thus, the maximal projection on the supermode $k_0$ will be achieved for pump configuration

$$f = f^{(0)} \left[ 1, e^{-2\pi i k_0/N}, e^{-4\pi i k_0/N}, e^{-6\pi i k_0/N}, ..., e^{-2(N-1)\pi i k_0/N} \right],$$

where $f^{(0)} = \sqrt{8g_0\kappa_0 P/\pi^3\hbar\omega N}$ is normalized pump for a single resonator.

V. MODULATION INSTABILITY GAIN LOBES IN CHAINS OF COUPLED MICRORESONATORS

To investigate unstable solutions, we consider the system to be in a stable state $\psi_{k0}$ with $k_0 = 0$ or $N/2$. These two supermodes have opposite sGVD. Keeping only quadratic term in Taylor series of the cosine in Eq. (6) and performing inverse Fourier transform, we obtain the 2D LLE

$$\frac{\partial \Psi}{\partial \tau} = -(1 + i\epsilon c_{k_0}^{(0)}) \Psi + id_2 \frac{\partial^2 \Psi}{\partial \varphi^2} + i j_2 \epsilon_2 \frac{\partial^2 \Psi}{\partial \theta^2} + i |\Psi|^2 \Psi + f^{(0)} e^{-ik_0 \Theta},$$

with $\Theta = 2\pi \ell/N$, $j_2^{(0)} = \pm (2\pi/N)^2 j$, $\epsilon_{k_0} = \epsilon_0 \pm 2j$, and $+(-)$ standing for excitation of $k_0 = 0(N/2)$. Further, we investigate unstable solutions $c(t) \exp(i[\mu \varphi + k \Theta])$ [5]. The linearized system yields the following eigenvalues

$$\lambda_{1,2} = -1 \pm i \sqrt{\left( c_{k_0}^{(0)} + j_2^{(0)} k^2 - 3|\psi_{k0}|^2 \right) \left( c_{k_0}^{(0)} + j_2^{(0)} k^2 - |\psi_{k0}|^2 \right)}, \quad k \neq 0, \mu = 0$$

$$\lambda_{3,4} = -1 \pm i \sqrt{\left( c_{k_0}^{(0)} + d_2 \mu^2 - 3|\psi_{k0}|^2 \right) \left( c_{k_0}^{(0)} + d_2 \mu^2 - |\psi_{k0}|^2 \right)}, \quad k = 0, \mu \neq 0$$

$$\lambda_{5,6} = -1 \pm i \sqrt{\left( c_{k_0}^{(0)} + d_2 \mu^2 + j_2^{(0)} k^2 - 5|\psi_{k0}|^2 \right) \left( c_{k_0}^{(0)} + d_2 \mu^2 + j_2^{(0)} k^2 - 3|\psi_{k0}|^2 \right)}, \quad k \neq 0, \mu \neq 0.$$

With the stable state satisfying

$$(1 + i\epsilon c_{k_0}^{(0)}) \psi_{k0} = i |\psi_{k0}|^2 \psi_{k0} + f^{(0)} e^{-ik_0 \Theta},$$

similar to Ref. [6], we derive the position of the primary sidebands

$$d_2 \mu^2 \pm j_2 k^2 = 4|\psi_{00}|^4 + \sqrt{|\psi_{00}|^4 - 4 - c_{k_0}^{(0)}}, \quad \mu, k \neq 0,$$

$$d_2 \mu^2 = 2|\psi_{00}|^4 + \sqrt{|\psi_{00}|^4 - 1 - c_{k_0}^{(0)}}, \quad k = 0,$$

$$j_2 k^2 = 2|\psi_{00}|^4 + \sqrt{|\psi_{00}|^4 - 1 - [\zeta_0 - 2j]}, \quad \mu = 0, k_0 = 0$$

with $j_2 = (2\pi/N)^2 j$. Eq. (14) indicates that the primary combs are formed on an ellipse (hyperbola) in the vicinity $k_0 = 0$ ($N/2$), equations (15,16) reveal conventional position of the unstable solutions.

VI. TURING PATTERNS IN ELLIPTIC AND HYPERBOLIC REGIONS

[h] We simulate nonlinear dynamics and Turing patterns in hyperbolic $k_0 = N/2$ and elliptic $k_0 = 0$ regimes. To observe coherent structures, we scan the resonance by changing the normalized laser detuning $\zeta_0$ and bring the system into an unstable state. Having simulated the pattern formation, we further tune towards the monostable region ($c_{k_0}^{(0)} < \sqrt{3}$) and obtain stable coherent structures in both regimes (Fig. 3). One can see that in the elliptic regime at $|f_\ell | = 1.05$ and $\zeta_0 = 20.5$, we observe the formation of a hexagonal pattern [Fig. 3(a)] [7, 8]. On a single resonator level, this corresponds to locked pulses [Fig. 3(b)] with a typical comb spectrum shown in Fig. 3(d). The corresponding 2D $k$-$\mu$ spectral profile in Fig. 3(c) shows that the sidebands form a disk, occupying the supermodes from both anomalous ($|k - k_0| < 5$) and normal dispersion regimes.

VII. TWO-DIMENSIONAL SPATIO-TEMPORAL DISSIPATIVE KERR SOLITON

In the conventional LLE, the presence of the higher-order dispersion terms leads to the additional phase matching between the soliton and linear waves supported by the cavity, leading to the appearance of so-called Cherenkov
FIG. 3. Coherent dissipative structures in a driven nonlinear photonic ring lattice. Panels (a-d) correspond to the elliptic region ($k_0 = 0$, $d_2 > 0$, $j_2 > 0$), and panels (e-h) correspond to the hyperbolic ($k_0 = N/2$, $d_2 > 0$, $j_2 < 0$). Spatio-temporal profiles of the mode-locked structures are shown in panels (a,e) with the corresponding field profile on a single resonator level in panels (b,f). The 2D spectral profiles of the states (a) and (e) obtained via Eq. (5) are presented in (c) and (g), respectively. The spectral profile in elliptic regime (c) forms a disk, whereas the spectrum of the pattern in hyperbolic regime (g) tends to align one of the asymptotes of the hyperbola depicting modulation instability gain in Eq. (14). The Fourier spectra of the states (b) and (f) are presented in (d) and (h).

radiation (the dispersive waves) [9] as schematically depicted in Fig. 4(a). Since a soliton is a dispersionless structure, it corresponds to a line under the integrated dispersion, and once the crossing of the soliton and the dispersion curve occurs, an additional phase-matching condition is satisfied in the vicinity of the crossing, leading to the spectral enhancement of the optical harmonics. A similar effect occurs in a chain of 20 coupled resonators considered in the main text. However, since the soliton is excited in the elliptic region (pump at $k = -6$ for the 1D chain in Fig. 1), the soliton appears as a dispersionless ellipse below the hybridized dispersion surface. Due to the spatial band structure, the higher-order dispersion terms lead to the crossing of the dispersion with the soliton ellipse, creating an intersection curve of efficient phase-matching, as shown in Fig. 4(b). The modes appearing on this curve experience additional gain, creating a fish-like tail in the soliton spectrum presented in the main text.

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FIG. 4. **Cherenkov radiation in optical resonators.** (a) Schematics of Cherenkov radiation in a single resonator with third-order dispersion term. The red line represents the dispersionless DKS; blue dots depict the dispersion profile. Panel (b) corresponds to the two-dimensional case in a chain of 20 coupled resonators. The blue disk represents the spatio-temporal soliton that crosses the hybridized dispersion surface.

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