Instability of Shear Flows in Spatially Periodic Domains

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Unidirectional shear flow in a spatially periodic domain is shown to be linearly unstable with respect to both the Reynolds number and the domain aspect ratio. This finding is confirmed by computer simulations, and a simple stability condition is derived. Periodic Couette and Poiseuille flows are unstable at Reynolds numbers two orders of magnitude smaller than their aperiodic equivalents because the periodic boundaries impose fundamentally different constraints. This instability has important implications for designing computational models and experiments of nonlinear dynamic processes with periodicity.

Periodic boundary conditions (PBCs) are perhaps most popularly recognized from the PAC-MAN arcade game, where the titular character reenters the screen opposite from the side it exits, but PBCs have also long been a mainstay of many physics, chemistry, and fluid mechanics computer-simulation methods, where they are used to construct physically relevant models using finite computational domains. Particle-based molecular simulations [1, 2] are classic examples—often, only a nanoscopic volume, many orders of magnitude smaller than used in experiments, can be routinely simulated. PBCs mitigate surface effects from the small volume, and the simulated properties can agree well with experiments while modeling remarkably few particles [3, 4].

Nevertheless, PBCs affect certain processes in systematic but undesirable ways. Thermodynamic phase transitions are susceptible to finite-size effects from PBCs [5, 6]. Diffusive transport is artificially influenced by hydrodynamic interactions propagated through PBCs [7]; corrections for this effect depend on the geometry [8, 9]. In a recent study, we found secondary flows growing from a periodic Couette-like shear flow [10] in full PBCs [11]. The expected unidirectional shear flow was obtained in a cubic box, but steady vortices developed when the box was longer in the flow direction (Fig. 1). We speculated that this might indicate an underlying hydrodynamic instability, which is the subject of this article.

The field of hydrodynamic stability analysis is long-established, dating back over a century [12, 13]. One of its early achievements was Taylor’s demonstration and analysis of an instability in the Couette flow of a viscous fluid between rotating concentric cylinders [14]. The stability of many other flows and fluids has since been analyzed; Drazin and Reid give an excellent overview [13]. The stability of a shear flow within a fully periodic domain has received less attention [15–20]. This may be partially due to the fact that experiments rarely have PBCs, and so PBCs may seem to be a mathematical curiosity. However, PBCs are the norm rather than the exception in many widely used computational methods, making it of critical importance to understand their effects on flow. Additionally, shear flows that vary periodically between bounded surfaces have pseudo-PBCs if the surfaces are far apart relative to the flow’s wavelength and so may exhibit similar behavior to true PBCs [16].

Here, we carry out a linear stability analysis of a unidirectional viscous shear flow in a fully periodic domain. We identify conditions under which three different flows are linearly unstable with respect to both the Reynolds number and also, surprisingly, the aspect ratio of the domain. We perform complementary particle-based simulations that confirm the analysis and shed valuable insight on the emergence of the instability. The results indicate that many shear flows become linearly unstable due to a purely geometric effect of the periodic domain. This work provides a simple, general framework for selecting geometries with PBCs that stabilize the flow; the framework requires only the flow’s Fourier series expansion. It also underscores the need for caution when simulating nonlinear dynamic processes with PBCs, which differ in unexpected ways from bounded domains because the PBCs do not impose the same process constraints.

The starting point for the stability analysis is the

![Figure 1](image_url)
Navier–Stokes equations for the incompressible flow \( u = U + \delta u \) of a Newtonian fluid [21]. The flow and its stresses are continuous through the PBCs; this differs from bounded domains, e.g., flow between two plates, which typically have no-slip and no-penetration boundary conditions at the surfaces. The base flow \( U \) is a unidirectional shear, \( U_x(y) \), that has zero mean (no net flow) and is also a solution of the steady Navier–Stokes equations. Following the standard linear stability analysis procedure [12], we apply Squire’s theorem [22] and consider the two-dimensional disturbance in \( x \) and \( y \), \( \delta u = e^{\lambda t + ikx} v(y) \) with \( \delta u_z = 0 \); any three-dimensional disturbance will become unstable at a larger Reynolds number. The fully periodic domain has half-length \( L \) in the flow direction \( x \) and half-width \( H \) in the shear gradient direction \( y \). Linearization yields [23]

\[
\text{Re} \left[ \lambda + ikU_x \left( \frac{d^2}{dy^2} - k^2 \right) - ik \frac{d^2 U_x}{dy^2} \right] v_y = \left( \frac{d^2}{dy^2} - k^2 \right)^2 v_y, \tag{1}
\]

which has been written in dimensionless form by reducing lengths by \( H \), velocities by the maximum velocity of the shear flow \( U \), and time by \( H/U \); \( \text{Re} = UH/\nu \) is the Reynolds number with \( \nu \) being the kinematic viscosity. Because of the PBC in \( x \), the wavelength \( k = \pi m/\alpha \) is constrained by the domain aspect ratio \( \alpha = L/H \) with \( m \) being an integer. The base flow is unstable when the real part of \( \lambda \) is positive for a solution \( v_y \) that satisfies the PBC in \( y \).

In all but a handful of cases, Eq. (1) must be solved numerically, e.g., by discretization or by expanding \( v_y \) in a set of orthogonal basis functions [13]. Orszag solved Eq. (1) for plane Poiseuille flow using Chebyshev polynomials [24], but other basis functions can be used. Due to the PBCs, it is advantageous to expand the flow fields in complex finite Fourier series that are inherently periodic, \( v_y(y) = \sum_n v_n e^{i\pi ny} \) and \( U_x(y) = \sum_n U_n e^{i\pi ny} \) with \(-N \leq n, p \leq N\) for sufficiently large \( N \). These series can be directly substituted into Eq. (1) using termwise differentiation. Applying the orthogonality of the basis functions yields algebraic equations for \( v_n \) and \( \lambda \) that depend on \( U_p \) [23],

\[
-\frac{i \left( \frac{\pi m}{\alpha} \right) \sum_{p=-N}^{N} \left[ 1 - \frac{2np}{(m/\alpha)^2 + n^2} \right] U_p v_{n-p} - \frac{\pi^2[(m/\alpha)^2 + n^2]}{\text{Re}} v_n = \lambda v_n. \tag{2}
\]

This approach can be considered a generalization of prior analysis for a sinusoidal flow [15] to an arbitrary unidirectional shear flow \( U_x(y) \) in a fully periodic domain where the PBC aspect ratio \( \alpha \) constrains \( k \). To find the limit of stability for a given \( \alpha \), we numerically determined the eigenvalues of Eq. (2) [24, 25] as a function of \( \text{Re} \) and solved for \( \text{Re} \) that gave the least stable \( \lambda \) having real part \( \Re(\lambda) = 0 \) when \( m = 1 \). (Disturbances with larger \( m \) are unstable at larger \( \alpha \) because they appear as a ratio.) In doing so, we neglected the possibility of instabilities triggered by nonorthogonal eigenmodes [26].

In order to test Eq. (2), we designed a series of case-study shear flows that we simulated using the computationally efficient multiparticle collision dynamics method [27–29] for modeling fluctuating hydrodynamics. We used the stochastic rotation dynamics collision scheme [27] with cubic cells of edge length \( a \), fixed 130° rotation angle [30], and random grid shifting [31]; a Maxwell–Boltzmann rescaling thermostat to maintain constant temperature \( T \) [32]; and time 0.1\( \tau \) between collisions, where \( \tau = a\sqrt{m/k_BT} \), \( m \) is the particle mass, and \( k_B \) is Boltzmann’s constant. The particle number density was 5/\( a^3 \), giving a liquid-like Newtonian fluid with kinematic viscosity \( \nu = 0.79 a^2/\tau \) [33–35]. All simulations were performed with hoomd-blue (version 2.6.0) [36–38] using a three-dimensionally periodic simulation box.
with a square cross section \((H = 50a \text{ in } y \text{ and } z)\) and \(L\) varied in \(x\) to span \(0.6 \leq \alpha \leq 3\). The maximum velocity of the base shear flow was restricted to \(U \lesssim 0.32a/\tau\) to avoid artificial effects at large Mach numbers \([39]\), giving \(Re \leq 20\). The simulations were initialized by superimposing the base flow on the thermalized fluid, which was sustained by an external force \([23]\), and were run for \(10^7\tau\) to reach a steady state. The velocity field \(u(x, y)\) was then measured by averaging the particle velocities in square bins of size \(a^2\) every \(10\tau\) for \(0.5 \times 10^5\tau\).

The stability of the simulated base flow was systematically assessed according to three criteria: (1) deviation of the average kinetic energy \(E_k\) due to the particle \(y\)-velocities from equipartition, (2) deviation of \(u_x(y)\) from the base flow \(U_x(y)\), and (3) deviation of \(u_y(x)\) from zero. For (2) and (3), the measured velocities were first averaged along \(x\) and \(y\), respectively. The base flow was deemed unstable if all of these measures consistently exceeded the expected fluctuations \([23]\), which we found was sufficiently discriminant but tolerant of statistical noise. The base flow was classified as stable if none of these measures consistently exceeded the statistical fluctuations; otherwise, it was unclassified.

We first considered the stability of the sinusoidal shear flow, \(U_x(y) = \sin(\pi y)\) (Fig. 2a), which is well-studied \([15–18, 40]\) and so serves as a useful test of the simulations. Its Fourier coefficients are \(U_{\pm 1} = \mp i/2\) and zero otherwise. Secondary flows like those shown in Fig. 1 readily formed in the simulations at sufficiently large \(Re\) and \(\alpha\). During the accessible simulation time, the flows remained stationary, although subsequent transitions to chaotic motion have been reported \([17, 40]\). The bottom panel of Fig. 2a compares the predictions of the linear stability analysis with the simulations, which are in excellent agreement. Both indicate that the base flow remains stable for all \(\alpha \leq 1\) over this range of \(Re\), as proved theoretically by Meshalkin and Sinai \([15]\). However, Fig. 2a also shows that there is a larger window of stability with respect to \(\alpha\) as \(Re\) decreases, and over the range of \(\alpha\) tested, no flows having \(Re \lesssim 5\) were unstable.

We subsequently tested two other flows commonly used in nonequilibrium molecular simulations: a periodic plane Poiseuille flow (Fig. 2b) \([41]\) and a periodic plane Couette flow (Fig. 2c) \([10]\). Both can be generated by applying a constant force per mass \(F\) to the particles in the \(x\) direction within blocks of space centered at \(y = \pm 1/2\) and having half-width \(0 < d \leq 1/2\) \([23]\). For the periodic Poiseuille flow, \(d = 1/2\) so that each block covers a full half-space of the box. For the periodic Couette flow, we took \(d = 1/25\) to develop a linear shear flow that had only small parabolic regions near \(y = \pm 1/2\) that made the shear stress continuous. The maximum velocity of these flows is \(U = (FH^2/8\nu)(1 - (1 - 2d)^2)\), which gives the well-known plane Poiseuille result when \(d = 1/2\). In the limit \(d \to 0\), \(F\) must become large to maintain constant \(U\). The Fourier coefficients for these two flows are

\[
U_p = \frac{4 \sin(p\pi/2) \sin(prd)}{i d(1 - d)(\pi r^2)}. \tag{3}
\]

Because the block forces are only piecewise continuous, \(U_p \sim 1/p^3\) and the Fourier series for \(d^2U_p/\text{dy}^2\) converges pointwise rather than uniformly. We accordingly took \(N = 100\) and confirmed that using \(N = 200\) did not significantly affect the computed stability curves.

Like the sinusoidal flow, the simulated periodic Poiseuille and Couette flows became unstable at sufficiently large \(Re\) and \(\alpha\), in excellent agreement with the predicted stability curves (Figs. 2b-c). In our previous study using a different method \([10]\) to create Couette flow, we found stable shear flows for \(\alpha \lesssim 1.25\) \([11]\); this is in complete agreement with our new analysis for the studied parameters. We are unaware of prior reports of an instability in the periodic Poiseuille flow, but its stability was highly similar to the sinusoidal flow. This may not be surprising given that the two flows closely resemble each other (Figs. 2a-b). The periodic Couette flow (Fig. 2c) was stable over a larger parameter space than the other two flows (see below).

Given the similarity of the three stability curves, we posited that they might be dominated by only the first (long wavelength) terms in the series expansions and applied a three-mode approximation \((N = 1)\) \([18, 40]\). This drastic simplification allows the eigenvalues of Eq. (2) to be computed analytically and gives a simple relationship between the critical Reynolds number \(Re_c\) and the first Fourier modes of the base flow, \(U_1 = U_{-1}, \tag{23}\)

\[
Re_c \approx \frac{\pi}{|U_1| \sqrt{2}} \frac{1 + \alpha^{-2}}{\sqrt{1 - \alpha^{-2}}}. \tag{4}
\]

There is a minimum Reynolds number for the instability as \(\alpha \to \infty\); this result is well-known for the sinusoidal flow \([15, 18, 40]\), where all flows having \(Re < \pi \sqrt{2} \approx 4.4\) are stable, and is in good agreement with the simulations.

Remarkably, Eq. (4) approximates the solution of Eq. (2) well for all three flows when \(\alpha \geq 1.5\) (Fig. 2). This explains the similar stability of the periodic Poiseuille \((|U_1| = 16/\pi^3 \approx 0.52)\) and sinusoidal \((|U_1| = 0.5)\) flows, as their first Fourier modes are nearly identical, and the better stability of the periodic Couette flow \((|U_1| \to 4/\pi^2 \approx 0.41\text{ as }d \to 0)\). However, there are some discrepancies between Eqs. (2) and (4) at smaller \(\alpha\). In fact, Eq. (2) predicts that periodic Poiseuille flow becomes unstable for \(Re \gtrsim 5\) when \(\alpha = 1\), but Eq. (4) diverges as \(\alpha \to 1\). Certain shear flows may still be unstable in cubic domains when PBCs are used. Nevertheless, Eq. (4) provides a useful estimate for choosing domains and Reynolds numbers that keep the base flow stable.

Most of the secondary flows that developed in our simulations had two stationary vortices (Fig. 1b), although at the largest aspect ratio and Reynolds numbers simulated—\(\alpha = 3\) and \(Re = 17.5\) and 20—we found
four stationary vortices for the sinusoidal and periodic Poiseuille flows. We hypothesized that the structure of some of the secondary flows might be connected to the most unstable eigenmode, which is the fastest growing perturbation to the base flow as it evolves to its steady state. This eigenmode, \( \mathbf{w} = e^{i k x} \mathbf{v}(y) \), can be reconstructed using the Fourier coefficients for \( \mathbf{v}_y \) that comprise the eigenvector having the largest \( \mathcal{R}(\lambda) \) and computing \( v_x = (i/k) \frac{d}{dy} v_y \) based on the incompressibility of \( \mathbf{u} \). Fig. 3a shows \( \mathbf{w} \) for the periodic Couette flow at \( \text{Re} = 15 \) and \( \alpha = 1.4 \), which develops into the flow shown in Fig. 1b. It has two pairs of counterrotating vortices with streamlines primarily directed along the shear gradient \( y \), which are reminiscent of Thess’s least stable mode of the unbounded inviscid sinusoidal flow [16].

The final steady flow need not possess the same structure as either the base flow or the most unstable eigenmode. However, we considered as an ansatz that in some cases the simulated flow field might be well-approximated by a linear combination of the two, \( \mathbf{u} \approx c_1 \mathbf{U} + c_2 \mathbf{w} \), subject to a shift of the coordinates with respect to the periodic boundaries. As an example, we performed a least-squares regression to the simulated \( \mathbf{u} \) for the conditions in Fig. 3, determining optimal coefficients \( c_1 = 0.765 \) and \( c_2 = 0.373 \). The fitted flow field (Fig. 3b) bears striking similarity to the simulated flow (Fig. 1b), having a root-mean-squared error of 0.02 per velocity component. This strongly suggests that the linear analysis reasonably captures the underlying physics of the instability.

We also noted that the waiting time \( \tau_w \) for the secondary flows to emerge in the simulations was shorter for points farther from the stability curve in the \((\alpha, \text{Re})\) diagram. This is qualitatively expected because these conditions are less stable based on their eigenvalues and should require smaller fluctuations (less time) to transition. To quantify this, we computed \( \tau_w \) for the unstable flows by empirically fitting \( E_y \) to a hyperbolic tangent during the first \( 10^5 \tau \); we defined \( \tau_w \) as the first time that \( E_y \) increased by 10% of the difference between its initial and final values [23]. These times can be compared to \( 1/\mathcal{R}(\lambda) \), which is the typical timescale associated with the most unstable disturbance. We found that \( \tau_w \) had a strong linear correlation with \( 1/\mathcal{R}(\lambda) \) for all three flows (Fig. 4). Moreover, we noted that the instability could take a surprisingly long time to emerge, up to nearly \( 1.5 \times 10^4 \tau \). Having this estimate of the timescale for instability is an added benefit of our analysis.

The stability of shear flows in fully periodic domains stands in stark contrast to those in bounded ones. Plane Couette flow is linearly stable for all infinitesimal disturbances [42], while plane Poiseuille flow is linearly stable up to \( \text{Re} = 5772 \) [24]; in practice, both become unstable in the range of \( \text{Re} \approx 10^2 \) to \( 10^3 \) [43–45]. However, their periodic extensions (Figs. 2b-c) are unstable at \( \text{Re} \) up to two orders of magnitude smaller. The change of stability in PBCs is not a simple consequence of the base flow; we previously showed that an unstable periodic Couette flow could be made stable by introducing a no-penetration boundary condition at either a half period or a full period of the flow [11]. Instead, the PBCs impose fundamentally different constraints: (1) the velocity and stress are not obligated to take a certain value at a surface, expanding the types of flows that can be realized, and (2) the PBCs restrict the wavelengths of disturbances to those commensurate with the domain, introducing a strong geometric dependence.

We have given a simple recipe for determining the stability of unidirectional shear flows in spatially periodic domains that requires only their Fourier series expansions and can be used to design well-behaved computational models. This flow instability highlights the need for caution when using PBCs to simulate nonlinear processes, as the PBCs may fundamentally alter the underlying physics. Similar instabilities may exist for other coupled transport processes modeled with PBCs. We also note that spatially periodic flows can be experimen-
tally realized using, e.g., a magnetic fluid [46]. It may be possible to exploit an instability in such flows having pseudo-PBCs to induce cross-streamline mixing of a solute by toggling an appropriate external field.

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Supplemental Material for Instability of Shear Flows in Spatially Periodic Domains

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STABILITY ANALYSIS

Linearization. The domain (Fig. S1) is spatially periodic in all dimensions. The z direction (not shown) is also periodic, but it is trivially decoupled from a unidirectional shear flow Ux(y) and so will be neglected going forward. (Using Squire’s theorem, it suffices to consider only the stability of the two-dimensional problem.) The velocity field u and its derivatives (stresses) must be continuous through the periodic boundaries at x = ±L and y = ±H. Moreover, we assume that there is no net flow, so

$$\int dx \, dy \, u = 0, \quad (S1)$$

where the integral is taken over the periodic domain. The viscous incompressible flow of a Newtonian fluid is governed by the continuity equation,

$$\nabla \cdot u = 0, \quad (S2)$$

and Navier–Stokes equations,

$$\rho \left( \frac{\partial u}{\partial t} + u \cdot \nabla u \right) = -\nabla p + f + \mu \nabla^2 u, \quad (S3)$$

with density \( \rho \), viscosity \( \mu \), pressure \( p \), and body force per volume \( f \). We then assume that the velocity \( u = U + \delta u \) can be separated into two parts: the base flow \( U \) and a perturbation \( \delta u \). (The pressure field, \( p = P + \delta p \) is expressed similarly.) Both flows must satisfy the periodic boundary conditions of the domain. Inserting these into Eqs. (S2) and (S3), using that \( U \) must also be a solution of the same, and neglecting terms of \( O(\delta u^2) \) yields the linearized equations:

$$\nabla \cdot \delta u = 0 \quad (S4)$$

$$\rho \left[ \frac{\partial \delta u}{\partial t} + U \cdot \nabla \delta u + \delta u \cdot \nabla U \right] = -\nabla \delta p + \mu \nabla^2 \delta u. \quad (S5)$$

FIG. S1. Sketch of the 2L × 2H periodic domain and the base shear flow \( U_x(y) \).

Stability. We assume the base unidirectional shear flow in \( x \) is a steady state and a function of \( y \) only, \( U = U_x(y) \), and seek separable solutions \( \delta u = e^{i(kx + \ell y)} v(y) \) and \( \delta p = e^{i(kx + \ell y)} q(y) \), where \( v = (v_x, v_y) \). The continuity equation [Eq. (S4)] gives a simple relation between \( v_x \) and \( v_y \),

$$ikv_x + v'_y = 0, \quad (S6)$$
where the prime denotes the ordinary derivative, \( v'_y = \frac{dv_y}{dy} \). The Navier–Stokes equations [Eq. (S5)] give

\[
\rho(\lambda v_x + ikU_x v_x + U'_x v_y) = -ikq + \mu \left[ v'''_y - k^2 v_y \right]
\]
\[\rho(\lambda v_y + ikU_x v_y) = -q' + \mu \left[ v'''_y - k^2 v_y \right].
\]

(S7)  
(S8)

Eq. (S6) can be inserted in Eq. (S7) and solved for \( q \):

\[
-q = \frac{\rho}{k^2} (\lambda + ikU_x) v'_y + \frac{\rho}{ik} U'_x v_y - \frac{\mu}{k^2} [v'''_y - k^2 v'_y].
\]

(S9)

We substitute Eq. (S9) into Eq. (S8) and define \( \tilde{y} = y/H \) and \( \tilde{U}_x = U_x/U \), where \( U \) is the maximum of \( U_x \), which also naturally gives \( \tilde{v}_y = v_y/U \). The Reynolds number is then \( Re = UH/\nu \) with kinematic viscosity \( \nu = \mu/\rho \). This results in Eq. (1) of the main text,

\[
Re \left[ \tilde{\lambda} + ik\tilde{U}_x \right] \left( \frac{d^2}{dy^2} - k^2 \right) \tilde{v}_y = \frac{d^2}{dy^2} \tilde{U}_x \tilde{v}_y,
\]

(S10)

which is the Orr–Sommerfeld equation up to the redefinition \( \tilde{\lambda} = i\tilde{\omega} \). Note that here we explicitly denote all dimensionless variables with \( \sim \), which we omitted in the main text for clarity. The perturbation will grow (the base flow is unstable) when \( \tilde{\lambda} \) has positive real part, \( \Re(\tilde{\lambda}) > 0 \).

**Eigenvalues.** The analysis to this point has followed standard derivations. We now need to find solutions for \( \tilde{v}_y \) that satisfy both Eq. (S10) and the periodic boundary conditions. Expanding the flows in Fourier series is beneficial because of their inherent periodicity. Substituting \( \tilde{U}_x(\tilde{y}) = \sum_p \tilde{U}_p e^{i\pi n \tilde{y}} \) and \( \tilde{v}_y(\tilde{y}) = \sum_n \tilde{v}_n e^{i\pi n \tilde{y}} \) into Eq. (S10) leads to

\[
-\Re \left[ \tilde{\lambda} \sum_n \left( (\pi n)^2 + k^2 \right) \tilde{v}_n e^{i\pi n \tilde{y}} + i\tilde{k} \sum_p \sum_n \left( \pi^2(n^2 - p^2) + k^2 \right) \tilde{U}_p \tilde{v}_n e^{i\pi (n+p) \tilde{y}} \right] = \sum_n \left( (\pi n)^2 + k^2 \right)^2 \tilde{v}_n e^{i\pi n \tilde{y}}.
\]

(S11)

We replace index \( n \) by \( r = n + p \) in the double sum and simply take \( r = n \) elsewhere. We then use the orthogonality of the basis functions to integrate against \( e^{-i\pi n \tilde{y}} \) from \( \tilde{y} = -1 \) to \( \tilde{y} = 1 \) and obtain

\[
-\Re \left[ \tilde{\lambda} \sum_n \left( \frac{(\pi n)^2 + k^2}{4} \right) \tilde{v}_n + i\tilde{k} \sum_p \left( \frac{\pi^2(n^2 - 2np) + k^2}{4} \right) \tilde{U}_p \tilde{v}_{n-p} \right] = \sum_n \left( \frac{(\pi n)^2 + k^2}{4} \right)^2 \tilde{v}_n.
\]

(S12)

Note that the sum in Eq. (S12), which resulted from products between \( \tilde{v}_n, \tilde{U}_x \), and their derivatives in Eq. (S11), is a convolution in Fourier space, as expected. Equation (S12) can be rewritten in the form of an eigenvalue problem, \( \mathbf{A} \cdot \mathbf{x} = \tilde{\lambda} \mathbf{x} \), that can be solved for different values of \( k \) and \( \Re \) to find modes having unstable \( \tilde{\lambda} \). Because of the periodic boundary condition in \( x \), \( \tilde{k} \) is restricted in the values it can take since \( \tilde{k} = \pi m/L \), with \( m \) being an integer. In dimensionless variables, this gives \( \tilde{k} = \pi m/\alpha \) with \( \alpha = L/H \) being the aspect ratio of the domain. It is this constraint which introduces the aspect ratio into the stability relation,

\[
-i \left( \frac{\pi m}{\alpha} \right) \sum_p \left[ 1 - \frac{2np}{(m/\alpha)^2 + n^2} \right] \tilde{U}_p \tilde{v}_{n-p} - \frac{\pi^2[(m/\alpha)^2 + n^2]}{\Re} \tilde{v}_n = \tilde{\lambda} \tilde{v}_n.
\]

(S13)

This is Eq. (2) in the main text after truncating the expansions. Note that because \( m \) and \( \alpha \) always appear together in Eq. (S13), it is sufficient to consider the stability for \( m = 1 \).

**Three-mode approximation.** If the expansions of \( \tilde{v}_y \) and \( \tilde{U}_x \) are restricted to only three terms \((-1 \leq n, p \leq 1)\), it is possible to compute the eigenvalues of Eq. (S13) analytically:

\[
\tilde{\lambda}_0 = -\frac{\pi^2(1 + \alpha^{-2})}{\Re} 
\]
\[
\tilde{\lambda}_\pm = -\frac{\pi^2}{2\Re} \left( 1 + 2\alpha^{-2} \pm \sqrt{1 + \frac{8\Re^2}{\pi^2} \frac{1 - \alpha^{-2}}{1 + \alpha^{-2} \tilde{U}_1^2} \tilde{U}_1} \right)
\]

(S14)  
(S15)

\( \tilde{\lambda}_0 \) and \( \tilde{\lambda}_+ \) will always be negative because \( \Re > 0 \), but \( \tilde{\lambda}_- \) can be zero or positive. Solving for \( \tilde{\lambda}_- = 0 \) yields Eq. (4) of the main text after recognizing that \( \tilde{U}_1 \) and \( \tilde{U}_{-1} \) are complex conjugates, \( |\tilde{U}_1|^2 = \tilde{U}_1 \tilde{U}_{-1} \), because \( \tilde{U}_x \) is real valued.
SIMULATED FLOW FIELDS

The force per mass \( F_x(y) \) applied to the MPCD particles is connected to the flow \( U_x(y) \) by Eq. (S3) at uniform pressure because \( f_x = pF_x \). All particles had unit mass \( m \), so \( F_x \) is also the force per particle. Figure S2 shows the simulated \( F_x \) and \( U_x \) for the sinusoidal, periodic Poiseuille, and periodic Couette flows when \( Re = 20 \). All quantities in this section have dimensions with units derived from fundamental quantities in the MPCD simulations: length \( a \), mass \( m \), and energy \( k_B T \).

**Sinusoidal flow.** For the sinusoidal flow,

\[
U_x(y) = U \sin \left( \frac{\pi y}{H} \right),
\]

(S16)

the force is straightforwardly derived as

\[
F_x(y) = \frac{U \nu}{(H/\pi)^2} \sin \left( \frac{\pi y}{H} \right).
\]

(S17)

In the simulations, we chose \( U \) to obtain a certain \( Re \), setting the force amplitude \( F = (\pi \nu)^2 Re/H^3 \).

**Flow driven by a constant force in a block.** Periodic plane Poiseuille flow and Couette flow were generated by similar procedures in the simulations. We defined two blocks centered at \( y = \pm H \) of half-width \( d \leq H/2 \) each. Then, a positive, constant force \( F \) was applied to all particles in the upper block and a negative, constant force \(-F\) was applied to all particles in the lower block:

\[
F_x(y) = \begin{cases} 
-F, & |y + H/2| \geq d \\
F, & |y - H/2| \geq d \\
0, & \text{otherwise}
\end{cases}
\]

(S18)

This procedure conserves linear momentum on average, but there are instantaneous fluctuations due to the distribution of particles. A thermostat is also required to maintain constant temperature. By solving Eq. (S3) piecewise using the symmetries of the problem, the flow field can be deduced:

\[
U_x(y)/U = \begin{cases} 
-g_1(H + y), & y \leq -H/2 - d \\
-g_2(-y), & |y + H/2| < d \\
g_1(y), & |y| \leq H/2 - d \\
g_2(y), & |y - H/2| < d \\
g_1(H - y), & y \geq H/2 + d
\end{cases}
\]

(S19)

with

\[
g_1(y) = \frac{2y/H}{1 - d/H}, \quad g_2(y) = 1 - \frac{(1 - 2y/H)^2}{1 - (1 - 2d/H)^2}.
\]

(S20)

(S21)

g_1 \) is the usual linear (Couette-like) flow, and \( g_2 \) is the quadratic (Poiseuille-like) flow. The maximum velocity is

\[
U = \frac{FH^2}{8\nu} \left[ 1 - (1 - 2d/H)^2 \right].
\]

(S22)

**Periodic Poiseuille flow.** For a periodic plane Poiseuille flow, the width of the blocks was chosen to cover the full simulation box, \( d = H/2 \). Only \( g_2 \) contributes to the flow, establishing two opposing parabolic flows. The force magnitude for a given Reynolds number is \( F = 8\nu^2 Re/H^3 \).

**Periodic Couette flow.** If \( d \) is much smaller than \( H/2 \), the flow is instead dominated by \( g_1 \) and a periodic plane Couette flow can be generated having two linear regimes. \( d \) must be sufficiently large that there are enough particles in each block to reliably apply the force and achieve the targeted maximum velocity. The force magnitude for a given Reynolds number is \( F = 8\nu^2 Re/[H^3(1 - (1 - 2d/H)^2)] \). In the limit \( d \to 0 \), the shear stress has a step change at \( y = \pm H/2 \), and \( F \) approximates a delta function. For finite \( d \), \( F \) is bounded and the flow will have two small quadratic regions that keep the shear stress continuous. We chose \( d = 2a \) for our boxes having \( H = 50a \), which gave predominantly Couette-like flow fields but also blocks with sufficient numbers of particles.
FIG. S2. (a) Applied force per particle $mF_x$ and (b) resulting flow $U_x$ for the three cases simulated at $Re = 20$ with $H = 50a$ and $d = 2a$. The maximum velocity is $U \approx 0.32a/\tau$.

FLOW CLASSIFICATION

We used three criteria to systematically classify the flow field measured at steady state in the simulations. The base flow was classified as stable if:

1. The deviation of the average total kinetic energy in $y$ from equipartition ($\langle E_y \rangle = N_p k_B T/2$ for $N_p$ particles) was less than 3 standard errors of the mean as computed from its own fluctuations,

2. the absolute deviation of the flow velocity from the base flow, $|u_x(y) - U_x(y)|$, was larger than $10^{-3} a/\tau$ in fewer than 5% of the bins, and

3. the absolute value of the gradient velocity $|u_y(x)|$ was larger than $10^{-3} a/\tau$ in fewer than 5% of the bins.

If all three criteria were not met, the base flow was classified as unstable; all other cases were unclassified. The thresholds for (2) and (3) were chosen to be tolerant of the numerical and statistical fluctuations in equilibrium simulations (no flow) and so base flows with small $\alpha$ that were visibly stable were properly classified. Criterion (3) tended to be very stringent and often failed even when criteria (1) and (2) held.

TIMESCALE FITTING

In order to estimate the timescale for occurrence of the instability, we empirically fit the kinetic energy $E_y$ to a hyperbolic tangent,

$$ \frac{E_y(t)}{N_p k_B T} = \frac{1}{2} \left[ (E_0 + E_1) + (E_1 - E_0) \tanh \left( \frac{t - \tau_0}{w} \right) \right]. $$  \hspace{1cm} (S23)

We fixed $E_0 = 1/2$ because the system was initially thermalized, and so its kinetic energy must obey equipartition. We then fit the final value $E_1$, the midpoint $\tau_0$, and the width $w$, resulting in fits like those shown in Fig. S3. The waiting time was defined as $\tau_w = \tau_0 - 1.09861w$ using the “10–90 thickness”. We found it challenging to reliably fit $E_y$ for a handful of conditions close to the stability curves of Fig. 2; we accordingly neglected all points having $\Re(1/\tilde{\lambda}) > 10$ in creating and analyzing Fig. 4.
FIG. S3. Fit (red) to Eq. (S23) for kinetic energy in $y$ per particle (gray) for periodic Couette flow with $\alpha = 1.4$ and Re = 15. The dashed blue line indicates the expected value of $E_y/N$ from equipartition. The black line marks the waiting time $\tau_w$ for occurrence of the instability.