S-ARITHMETIC INHOMOGENEOUS DIOPHANTINE APPROXIMATION ON MANIFOLDS

SHREYASI DATTA AND ANISH GHOSH

Abstract. We investigate S-arithmetic inhomogeneous Khintchine type theorems in the dual setting for nondegenerate manifolds. We prove the convergence case of the theorem, including, in particular, the S-arithmetic inhomogeneous counterpart of the Baker-Sprindžuk conjectures. The divergence case is proved for \( \mathbb{Q}_p \) but in the more general context of Hausdorff measures. This answers a question posed by Badziahin, Beresnevich and Velani [4].

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1. Introduction

In this paper we are concerned with metric Diophantine approximation on nondegenerate manifolds in the p-adic, or more generally S-arithmetic setting for a finite set of primes \( S \). To motivate our results we recall Khintchine’s theorem, a basic result in metric Diophantine approximation. Let \( \Psi : \mathbb{R}^n \to \mathbb{R}_+ \) be a function satisfying

\[
\Psi(a_1, \ldots, a_n) \geq \Psi(b_1, \ldots, b_n) \quad \text{if} \quad |a_i| \leq |b_i| \quad \text{for all} \quad i = 1, \ldots, n. \tag{1.1}
\]

Such a function is referred to as a multivariable approximating function. Given such a function, define \( \mathcal{W}_n(\Psi) \) to be the set of \( x \in \mathbb{R}^n \) for which there exist infinitely many \( a \in \mathbb{Z}^n \) such that

\[
|a_0 + a \cdot x| < \Psi(a) \tag{1.2}
\]

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for some $a_0 \in \mathbb{Z}$. When $\Psi(a) = \psi(\|a\|)$ for a non-increasing function $\psi$, we write $W_n(\psi)$ for $W_n(\Psi)$. Khintchine’s Theorem ([29], [27]) gives a characterization of the measure of $W_n(\psi)$ in terms of $\psi$:

**Theorem 1.1.**

$$|W_n(\psi)| = \begin{cases} 
0 & \text{if } \sum_{k=1}^{\infty} k^{n-1} \psi(k) < \infty \\
\text{full} & \text{if } \sum_{k=1}^{\infty} k^{n-1} \psi(k) = \infty.
\end{cases} \quad (1.3)$$

Here, $\| \|$ denotes the supremum norm of a vector and $\| \|$ denotes the absolute value of a real number as well as the Lebesgue measure of a measurable subset of $\mathbb{R}^n$; the context will make the use clear. The kind of approximation considered above is called “dual” approximation in the literature as opposed to the setting of simultaneous Diophantine approximation. In this paper, we will only consider dual approximation. Given an approximation function, one can consider the corresponding $S$-arithmetic question as follows, we follow the notation of Kleinbock and Tomanov [33]. Given a finite set of primes $S$ of cardinality $l$ we set $Q_S := \prod_{\nu \in S} Q_\nu$ and denote by $|\cdot|_S$ the $S$-adic absolute value, $|x| = \max_{\nu \in S} |x(\nu)|_\nu$. For $a = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ and $a_0 \in \mathbb{Z}$ we set

$$\overline{a} := (a_0, a_1, \ldots, a_n).$$

We say that $y \in Q_S^n$ is $\Psi$-approximable ($y \in W_n(S, \Psi)$) if there are infinitely many solutions $a \in \mathbb{Z}^n$ to

$$|a_0 + a \cdot y|_S^l \leq \begin{cases} 
\Psi(\overline{a}) & \text{if } \infty \notin S \\
\Psi(a) & \text{if } \infty \in S.
\end{cases} \quad (1.4)$$

We fix Haar measure on $Q_p$, normalized to give $\mathbb{Z}_p$ measure 1 and denote the product measure on $Q_S$ by $|\cdot|_S$. Then, the following analogue of Khintchine’s theorem can be proved. Namely,

**Theorem 1.2.** $W_n(S, \psi)$ has zero or full measure depending on the convergence or divergence of the series

$$\begin{cases} 
\sum_{k=1}^{\infty} k^n \psi(k) & \text{if } \infty \notin S \\
\sum_{k=1}^{\infty} k^{n-1} \psi(k) & \text{if } \infty \in S.
\end{cases} \quad (1.5)$$

Indeed, the convergence case follows from the Borel-Cantelli lemma as usual and the divergence case can be proved using the methods in [36].
1.1. **Inhomogeneous approximation:** Given a multivariable approximating function $\Psi$ and a function $\theta : \mathbb{R}^n \to \mathbb{R}$, we set $W^\theta_n(\Psi)$ to be the set of $x \in \mathbb{R}^n$ for which there exist infinitely many $a \in \mathbb{Z}^n \setminus \{0\}$ such that
$$|a_0 + a \cdot x + \theta(x)| < \Psi(a) \quad (1.6)$$
for some $a_0 \in \mathbb{Z}$. For $\psi$ as above, the set $W^\theta_n(\psi)$ is often referred to as the (dual) set of "$(\psi, \theta)$-inhomogeneously approximable" vectors in $\mathbb{R}^n$. The following inhomogeneous version of Theorem 1.1 is established in [4]. We denote by $C^n$ the set of $n$-times continuously differentiable functions.

**Theorem 1.3.** Let $\theta : \mathbb{R}^n \to \mathbb{R}$ be a $C^2$ function. Then
$$|W^\theta_n(\psi)| = \begin{cases} 0 & \text{if } \sum_{k=1}^{\infty} k^{n-1}\psi(k) < \infty \\ \text{full} & \text{if } \sum_{k=1}^{\infty} k^{n-1}\psi(k) = \infty. \end{cases} \quad (1.7)$$
We remark that the choice of $\theta = \text{constant}$ is the setting of traditional inhomogeneous Diophantine approximation and in that case the above result was well known, see for example [19]. Similarly inhomogeneous Diophantine approximation can be considered in the $S$-arithmetic setting.

For a multivariable approximating function $\Psi$ and a function $\Theta : \mathbb{Q}^n_S \to \mathbb{Q}_S$, we say that a vector $x \in \mathbb{Q}^n_S$ is $(\Psi, \Theta)$-approximable if there exist infinitely many $(a, a_0) \in \mathbb{Z}^n \setminus \{0\} \times \mathbb{Z}$ such that
$$|a_0 + a \cdot x + \Theta(x)|_S^l \leq \begin{cases} \Psi(\tilde{a}) & \text{if } \infty \notin S \\ \Psi(a) & \text{if } \infty \in S. \end{cases} \quad (1.8)$$

The convergence case of Khintchine’s theorem in this setting again follows from the Borel Cantelli lemma. The divergence Theorem when $S = \{p\}$ comprises a single prime $p$ is a consequence of the results in this paper.

1.2. **Diophantine approximation on manifolds.** In the theory of Diophantine approximation on manifolds, one studies the inheritance of generic (for Lebesgue measure) Diophantine properties by proper submanifolds of $\mathbb{R}^n$. This theory has seen dramatic advances in the last two decades, beginning with the proof of the Baker-Sprindžuk conjectures by Kleinbock and Margulis [32] using non divergence estimates for certain flows on the space of unimodular lattices. Motivated by problems in transcendental number theory, K. Mahler conjectured in 1932 that almost every point on the curve
$$f(x) = (x, x^2, \ldots, x^n)$$
is not very well approximable, i.e. $\psi$-approximable for $\psi := \psi_{\varepsilon}(k) = k^{-n-\varepsilon}$. This conjecture was resolved by V. G. Sprindžuk [41, 42] who in turn conjectured that almost every point on a nondegenerate manifold is not very well approximable. This conjecture, in a more general, multiplicative form, was resolved by D. Kleinbock and G. Margulis in [32]. The following definition is taken from [33] and is based on [32]. Let $f : U \to F^n$ be a $C^k$ map, where $F$ is any locally compact valued field and $U$ is an open subset of $F^d$, and say that $f$ is nondegenerate at $x_0 \in U$ if the space $F^n$ is spanned by partial derivatives of $f$ at $x_0$ up to some finite order. Loosely speaking, a nondegenerate manifold is one in which is locally not contained in an affine subspace. Subsequent to the work of Kleinbock and Margulis, there were rapid advances in the theory of dual approximation on manifolds. In [11] (and independently in [1]) the convergence case of the Khintchine-Groshev theorem for nondegenerate manifolds was proved and in [6], the complementary divergence case was established.

As for the $p$-adic theory, Sprindžuk [41] himself established the $p$-adic and function field (i.e. positive characteristic) versions of Mahler’s conjectures. Subsequently, there were several partial results (cf. [34, 7]) culminating in the work of Kleinbock and Tomanov [33] where the $S$-adic case of the Baker-Sprindžuk conjectures were settled in full generality. In [23], the second named author established the function field analogue. The convergence case of Khintchine’s theorem for nondegenerate manifolds in the $S$-adic setting was established by Mohammadi and Golsefidy [37] and the divergence case for $\mathbb{Q}_p$ in [38].

In the case of inhomogeneous Diophantine approximation on manifolds, following several partial results (cf. [18] and the references in [12, 13]), an inhomogeneous transference principle was developed by Beresnevich and Velani using which they resolved the inhomogeneous analogue of the Baker-Sprindžuk conjectures. Subsequently, Badziahin, Beresnevich and Velani [4] established the convergence and divergence cases of the inhomogeneous Khintchine theorem for nondegenerate manifolds. They proved a new result even in the classical setting by allowing the inhomogeneous term to vary. The divergence theorem is established in the same paper in the more general setting of Hausdorff measures.

In this paper, we will establish the convergence case of an inhomogeneous Khintchine theorem for nondegenerate manifolds in the $S$-adic setting, as well as the divergence case for $\mathbb{Q}_p$. As in [4], the divergence case is proved in the greater generality of Hausdorff measures. Prior results in the $p$-adic theory of inhomogeneous approximation for manifolds focussed mainly on curves, cf. [14, 15, 43, 44].
1.3. **Main Results.** To state our main results, we introduce some notation following [37], recall some of the assumptions from that paper and set forth one further standing assumption. The assumptions are as follows.

(I0) $S$ contains the infinite place.

(I1) We will consider the domain to be of the form $U = \prod_{\nu \in S} U_{\nu}$ where $U_{\nu} \subset \mathbb{Q}_p$ is an open box. Here, the norm is taken to be the Euclidean norm at the infinite place and the $L^\infty$ norm at finite places.

(I2) We will consider functions $f(\mathbf{x}) = (f_\nu(x_\nu))_{\nu \in S}$ where $f_\nu = (f_\nu^{(1)}, f_\nu^{(2)}, \ldots, f_\nu^{(n)}) : U_{\nu} \rightarrow \mathbb{Q}^n_{\nu}$ is an analytic map for any $\nu \in S$, and can be analytically extended to the boundary of $U_{\nu}$.

(I3) We assume that the restrictions of $1, f_\nu^{(1)}, f_\nu^{(2)}, \ldots, f_\nu^{(n)}$ to any open subset of $U_{\nu}$ are linearly independent over $\mathbb{Q}_\nu$ and that $\|f(\mathbf{x})\| \leq 1, \|\nabla f_\nu(x_\nu)\| \leq 1$ and $|\Phi_\beta f_\nu(y_1, y_2, y_3)| \leq \frac{1}{2}$ for any $\nu \in S$, second difference quotient $\Phi_\beta$ and $x_\nu, y_1, y_2, y_3 \in U_{\nu}$. We refer the reader to Section 3 for definitions.

(I4) We assume that the function $\Psi : \mathbb{Z}^n \rightarrow \mathbb{R}$ is monotone decreasing componentwise i.e.

$$\Psi(a_1, \ldots, a_i, \ldots, a_n) \geq \Psi(a_1, \ldots, a'_i, \ldots, a_n)$$

whenever $|a_i|_S \leq |a'_i|_S$.

(I5) We assume that $\Theta(\mathbf{x}) = (\Theta_\nu(x_\nu))$ where $\Theta : U \rightarrow \mathbb{Q}$ is also analytic and can be extended analytically to the boundary of $U_{\nu}$; we will assume $\|\Theta(\mathbf{x})\| \leq 1, \|\nabla \Theta_\nu(x_\nu)\| \leq 1$ and $|\Phi_\beta \Theta_\nu(y_1, y_2, y_3)| \leq \frac{1}{2}$ for any $\nu \in S$, second difference quotient $\Phi_\beta$ and $x_\nu, y_1, y_2, y_3 \in U_{\nu}$.

We can now state the first main Theorem of the present paper.

**Theorem 1.4.** Let $S$ be as in (I0) and $U$ as in (I1). Suppose $f$ satisfies (I2) and (I3), that $\Psi$ satisfies (I4) and $\Theta$ satisfies (I5). Then

$$W_{\Psi, \Theta} := \{ \mathbf{x} \in U : f(\mathbf{x}) \text{ is } (\Psi, \Theta) - \text{approximable} \}$$

has measure zero if $\sum_{\mathbf{a} \in \mathbb{Z}^n \setminus \{0\}} \Psi(\mathbf{a}) < \infty$.

The divergence case of our Theorem is proved in the more general setting of Hausdorff measures. However, we need to impose some restrictions: we only consider the case when $S = \{ p \}$ consists of a single prime, the inhomogeneous function is assumed to be analytic, and the approximating function is not as general as in Theorem 1.4. We will denote by $H^s(X)$ the $s$-dimensional Hausdorff measure of a subset $X$ of $\mathbb{Q}_p^d$ and $\text{dim } X$ the Hausdorff dimension, where $s > 0$ is a real number.
Theorem 1.5. Let \( S \) be as in (I0) and \( U \) as in (I1). Suppose \( f : U \subset \mathbb{Q}_p^m \rightarrow \mathbb{Q}_p^n \) satisfies (I2) and (I3). Let
\[
\Psi(a) = \psi(\|a\|), \quad a \in \mathbb{Z}^{n+1}
\]
be an approximating function and assume that \( s > m - 1 \). Let \( \Theta : U \rightarrow \mathbb{Q}_p \) be an analytic map satisfying (I5). Then
\[
\mathcal{H}^s(\mathcal{W}^f_{(\Psi, \Theta)} \cap U) = \mathcal{H}^s(U) \quad \text{if} \quad \sum_{a \in \mathbb{Z}^n \setminus \{0\}} (\Psi(a))^{s+1-m} = \infty.
\]

Given an approximating function \( \psi \), the lower order at infinity \( \tau_\psi \) of \( 1/\psi \) is defined by
\[
\tau_\psi := \lim \inf_{t \rightarrow \infty} \frac{-\log \psi(t)}{\log t}.
\]
The divergent sum condition of Theorem 1.5 is satisfied whenever
\[
s < m - 1 + \frac{n + 1}{\tau_\psi}.
\]
Therefore, by the definition of Hausdorff measure and dimension, we get

Corollary 1.1. Let \( f \) and \( \Theta \) be as in Theorem 1.5. Let \( \psi \) be an approximating function as in (1.10) such that \( n + 1 \leq \tau_\psi < \infty \). Then
\[
\dim(\mathcal{W}^f_{(\Psi, \Theta)} \cap U) \geq m - 1 + \frac{n + 1}{\tau_\psi}.
\]

1.4. Remarks.

(1) We have assumed \( S \) contains the infinite place in Theorem 1.4. This is not a serious assumption, the proof in the case when \( S \) contains only finite places needs some minor modifications but follows the same outline, details will appear in [20], the PhD thesis, under preparation, of the first named author. In [37], the (homogeneous) \( S \)-adic convergence case is proved in slightly greater generality than in the present paper. Namely, instead of \( \mathbb{Q} \), the quotient field of a finitely generated subring of \( \mathbb{Q} \) is considered. This, more general formulation will also be investigated in [20].

(2) Our proof for the convergence case, namely Theorem 1.4 blends techniques from the homogeneous results, namely [33, 11, 37] and uses the transference principle developed by Beresnevich and Velani in the form used in [4]. The structure of the proof is the same as in [4]. We also take the opportunity to clarify some properties of \((C, \alpha)\)-good functions in the \( S \)-adic setting which may be of independent interest.
(3) The proof of Theorem 1.5, follows the ubiquity framework used in [4] but needs new ideas to implement in the $p$-adic setting. At present, we are unable to prove the more general $S$-adic divergence statement. We note that the $S$-adic case remains open even in the homogeneous setting.

(4) We now undertake a brief discussion of the assumptions (I1) - (I5). The conditions (I1)-(I4) are assumed in [37] and, as explained in loc. cit., are assumed for convenience. Namely, as mentioned in [37], the statement for any non-degenerate analytic manifold over $\mathbb{Q}_S$ follows from Theorem 1.4. In [4], the inhomogeneous parameter $\Theta$ is allowed to be $C^2$ when restricted to the nondegenerate manifold. However, we need to assume it to be analytic.

(5) Theorem 1.5 is slightly more general than Theorem 1.2 of [38] in the homogeneous setting. In [38], the approximating function is taken to be of the form

$$\Psi(a) = \frac{1}{\|a\|^n} \psi(\|a\|), a \in \mathbb{Z}^{n+1}$$

where $\psi$ is a more restrictive class of approximating functions. For an $n$-tuple $v = (v_1, \ldots, v_n)$ of positive numbers satisfying $v_1 + \cdots + v_n = n$, define the $v$-quasinorm $\|x\|_v$ on $\mathbb{R}^n$ by setting

$$\|x\|_v := \max |x_i|^{1/v_i}.$$ 

Following [4] we say that a multivariable approximating function $\Psi$ satisfies property $P$ if $\Psi(a) = \psi(\|a\|_v)$ for some approximating function $\psi$ and $v$ as above. As noted in loc. cit. when $v = (1, \ldots, 1)$ we have that $\|a\|_v = \|a\|$ and any approximating function $\psi$ satisfies property $P$, where $\psi$ is regarded as the function $a \rightarrow \psi(\|a\|)$. The proof of Theorem 1.5 can be modified to deal with the case of functions satisfying property $P$.

Structure of the paper. In the next section, we recall the transference principle of Beresnevich and Velani. The subsequent section studies $(C, \alpha)$-good functions in the $S$-adic setting. We then prove Theorem 1.4 and then Theorem 1.5. We conclude with some open questions.

2. INHOMOGENEOUS TRANSFERRENCE PRINCIPLE

In this section we state the inhomogeneous transference principle of Beresnevich and Velani from [12, Section 5] which will allow us to convert our inhomogeneous problem to the homogeneous one. Let $(\Omega, d)$ be a locally compact metric space. Given two countable indexing
sets $\mathcal{A}$ and $\mathcal{T}$, let $H$ and $I$ be two maps from $\mathcal{T} \times \mathcal{A} \times \mathbb{R}_+$ into the set of open subsets of $\Omega$ such that

$$H : (t, \alpha, \lambda) \in \mathcal{T} \times \mathcal{A} \times \mathbb{R}_+ \rightarrow H_t(\alpha, \lambda)$$

(2.1)

and

$$I : (t, \alpha, \lambda) \in \mathcal{T} \times \mathcal{A} \times \mathbb{R}_+ \rightarrow I_t(\alpha, \lambda)$$

(2.2)

Furthermore, let

$$H_t(\lambda) := \bigcup_{\alpha \in \mathcal{A}} H_t(\alpha, \lambda) \text{ and } I_t(\lambda) := \bigcup_{\alpha \in \mathcal{A}} I_t(\alpha, \lambda).$$

(2.3)

Let $\Psi$ denote a set of functions $\psi : \mathcal{T} \rightarrow \mathbb{R}_+ : t \rightarrow \psi_t$. For $\psi \in \Psi$, consider the limsup sets

$$\Lambda_H(\psi) = \limsup_{t \in \mathcal{T}} H_t(\psi_t) \text{ and } \Lambda_I(\psi) = \limsup_{t \in \mathcal{T}} I_t(\psi_t).$$

(2.4)

The sets associated with the map $H$ will be called homogeneous sets and those associated with the map $I$, inhomogeneous sets. We now come to two important properties connecting these notions.

**The intersection property.** The triple $(H, I, \Psi)$ is said to satisfy the intersection property if, for any $\psi \in \Psi$, there exists $\psi^* \in \Psi$ such that, for all but finitely many $t \in \mathcal{T}$ and all distinct $\alpha$ and $\alpha'$ in $\mathcal{A}$, we have that

$$I_t(\alpha, \psi_t) \cap I_t(\alpha', \psi_t) \subset H_t(\psi^*_t).$$

(2.5)

**The contraction property.** Let $\mu$ be a non-atomic finite doubling measure supported on a bounded subset $\mathcal{S}$ of $\Omega$. We recall that $\mu$ is doubling if there is a constant $\lambda > 1$ such that, for any ball $B$ with centre in $\mathcal{S}$, we have

$$\mu(2B) \leq \lambda \mu(B),$$

where, for a ball $B$ of radius $r$, we denote by $cB$ the ball with the same centre and radius $cr$. We say that $\mu$ is contracting with respect to $(I, \Psi)$ if, for any $\psi \in \Psi$, there exists $\psi^+ \in \Psi$ and a sequence of positive numbers $\{k_t\}_{t \in \mathcal{T}}$ satisfying

$$\sum_{t \in \mathcal{T}} k_t < \infty,$$
such that, for all but finitely \( t \in T \) and all \( \alpha \in A \), there exists a collection \( C_{t,\alpha} \) of balls \( B \) centred at \( S \) satisfying the following conditions:

\[
S \cap I_t(a, \psi_t) \subset \bigcup_{B \in C_{t,\alpha}} B
\]

(2.7)

\[
S \cap \bigcup_{B \in C_{t,\alpha}} B \subset I_t(a, \psi_t^+) \quad (2.8)
\]

and

\[
\mu(5B \cap I_t(a, \psi_t)) \leq k_t \mu(5B). \quad (2.9)
\]

We are now in a position to state Theorem 5 from [12]

Theorem 2.1. Suppose that \((H, I, \Psi)\) satisfies the intersection property and that \( \mu \) is contracting with respect to \((I, \Psi)\). Then

\[
\mu(\Lambda_H(\psi)) = 0 \quad \forall \ \psi \in \Psi \Rightarrow \mu(\Lambda_I(\psi)) = 0 \quad \forall \ \psi \in \Psi.
\]

(2.10)

3. \((C, \alpha)\)-good functions

In this section, we recall the important notion of \((C, \alpha)\)-good functions on ultrametric spaces. We follow the treatment of Kleinbock and Tomanov [33]. Let \( X \) be a metric space, \( \mu \) a Borel measure on \( X \) and let \((F, |\cdot|)\) be a local field. For a subset \( U \) of \( X \) and \( C, \alpha > 0 \), say that a Borel measurable function \( f : U \rightarrow F \) is \((C, \alpha)\)-good on \( U \) with respect to \( \mu \) if for any open ball \( B \subset U \) centred in \( \sup \mu \) and \( \varepsilon > 0 \) one has

\[
\mu \left( \{ x \in B \mid |f(x)| < \varepsilon \} \right) \leq C \left( \frac{\varepsilon}{\sup_{x \in B} |f(x)|} \right)^{\alpha} |B|,
\]

(3.1)

The following elementary properties of \((C, \alpha)\)-good functions will be used.

(G1) If \( f \) is \((C, \alpha)\)-good on an open set \( V \), so is \( \lambda f \ \forall \ \lambda \in F \);

(G2) If \( f_i, i \in I \) are \((C, \alpha)\)-good on \( V \), so is \( \sup_{i \in I} |f_i| \);

(G3) If \( f \) is \((C, \alpha)\)-good on \( V \) and for some \( c_1, c_2 > 0, c_1 \leq |f(x)| \leq c_2 \) for all \( x \in V \), then \( g \) is \((C(c_2/c_1)^{\alpha}, \alpha)\)-good on \( V \).

(G4) If \( f \) is \((C, \alpha)\)-good on \( V \), it is \((C', \alpha')\)-good on \( V' \) for every \( C' \geq \max\{C, 1\}, \alpha' \leq \alpha \) and \( V' \subset V \).

One can note that from (G2), it follows that the supremum norm of a vector valued function \( f \) is \((C, \alpha)\)-good whenever each of its components
is \((C, \alpha)\)-good. Furthermore, in view of (G3), we can replace the norm by an equivalent one, only affecting \(C\) but not \(\alpha\).

Polynomials in \(d\) variables of degree at most \(k\) defined on local fields can be seen to be \((C, 1/dk)\)-good, with \(C\) depending only on \(d\) and \(k\) using Lagrange interpolation. In [32], [11] and [33] (for ultrametric fields), this property was extended to smooth functions satisfying certain properties. We rapidly recall, following [40] (see also [33]), the definition of smooth functions in the ultrametric case. Let \(U\) be a non-empty subset of \(X\) without isolated points. For \(n \in \mathbb{N}\), define

\[
\nabla^n(U) = \{ (x_1, \ldots, x_n) \in U, \, x_i \neq x_j \text{ for } i \neq j \}.
\]

The \(n\)-th order difference quotient of a function \(f : U \to X\) is the function \(\Phi_n(f)\) defined inductively by \(\Phi_0(f) = f\) and, for \(n \in \mathbb{N}\), and \((x_1, \ldots, x_{n+1}) \in \nabla^n(U)\) by

\[
\Phi_n(f)(x_1, \ldots, x_{n+1}) = \frac{\Phi_{n-1}(f)(x_1, x_3, \ldots, x_{n+1}) - \Phi_{n-1}(f)(x_2, \ldots, x_{n+1})}{x_1 - x_2}.
\]

This definition does not depend on the choice of variables, as all difference quotients are symmetric functions. A function \(f\) on \(X\) is called a \(C^n\) function if \(\Phi_n(f)\) can be extended to a continuous function \(\overline{\Phi}_n(f) : U^{n+1} \to X\). We also set

\[
D_n f(a) = \overline{\Phi}_n f(a, \ldots, a), \, a \in U.
\]

We have the following theorem (c.f. [40], Theorem 29.5).

**Theorem 3.1.** Let \(f \in C^n(U \to X)\). Then, \(f\) is \(n\) times differentiable and

\[
j! D_j f = f^j
\]

for all \(1 \leq j \leq n\).

To define \(C^k\) functions in several variables, we follow the notation set forth in [33]. Consider a multiindex \(\beta = (i_1, \ldots, i_d)\) and let

\[
\Phi_\beta f = \Phi^{i_1}_1 \circ \cdots \circ \Phi^{i_d}_d f.
\]

This difference order quotient is defined on the set \(\nabla^{i_1} U_1 \times \cdots \times \nabla^{i_d} U_d\) and the \(U_i\) are all non-empty subsets of \(X\) without isolated points. A function \(f\) will then be said to belong to \(C^k(U_1 \times \cdots \times U_d)\) if for any multiindex \(\beta\) with \(|\beta| = \sum_{j=1}^d i_j \leq k\), \(\Phi_\beta f\) extends to a continuous function \(\Phi_\beta f : U_{i_1+1}^{i_1} \times \cdots \times U_{i_d+1}^{i_d}\). We then have

\[
\partial_\beta f(x_1, \ldots, x_d) = \beta! \Phi_\beta(x_1, \ldots, x_1, \ldots, x_d, \ldots, x_d)
\]

(3.2)

where \(\beta! = \prod_{j=1}^d i_j!\).
We are now ready to gather the results on ultrametric \((C, \alpha)\)-good functions that we need. We begin with Theorem 3.2 from [33].

**Theorem 3.2.** Let \(V_1, V_2, \ldots, V_d\) be nonempty open sets in \(F\), ultrametric field. Let \(k \in \mathbb{N}, A_1, \ldots, A_d > 0\) and \(f \in C^k(V_1 \times \cdots \times V_d)\) be such that

\[
|\Phi^k_f| \equiv A_j \text{ on } \nabla^{k+1} V_j \times \prod_{i \neq j} V_j, j = 1, \ldots, d. \tag{3.3}
\]

Then \(f\) is \((d k^{3-k} - \frac{k}{d k}, 1)\)-good on \(V_1 \times \cdots \times V_d\).

The following is an ultrametric analogue of Proposition 1 from [4].

**Proposition 3.1.** Let \(U_\nu\) be an open subset of \(\mathbb{Q}_\nu^d\), \(x_0 \in U_\nu\) and let \(\mathcal{F} \subset C^l(U)\) be a compact family of functions \(f : U \to \mathbb{Q}_\nu\) for some \(l \geq 2\). Also assume that

\[
\inf_{f \in \mathcal{F}} \max_{0 < |\beta| \leq l} |\partial_\beta f(x_0)| > 0. \tag{3.4}
\]

Then there exists a neighbourhood \(V_\nu \subset U_\nu\) of \(x_0\) and \(C, \delta > 0\) satisfying the following property. For any \(\Theta \in C^l(U)\) such that

\[
\sup_{x \in U_\nu} \max_{0 < |\beta| \leq l} |\partial_\beta \Theta(x_0)| \leq \delta \tag{3.5}
\]

and for any \(f \in \mathcal{F}\) we have that

1. \(f + \Theta\) is \((C, \frac{1}{d l})\)-good on \(V_\nu\).
2. \(|\nabla(f + \Theta)|\) is \((C, \frac{1}{m(l-1)})\)-good on \(V_\nu\).

**Proof.** We follow the proof of [4], which in turn is a modification of the ideas used to establish Proposition 3.4 in [11]. Here \(\nu = \infty\) is exactly Proposition 1 of [4] so we assume that \(\nu \neq \infty\). By (3.4) there exists \(C_1 > 0\) such that for any \(f \in \mathcal{F}\) there exists a multiindex \(\beta\) with \(0 < |\beta| = k \leq l\), such that

\[
|\partial_\beta f(x_0)| \geq C_1. \tag{3.6}
\]

By the compactness of \(\mathcal{F}\), \(\inf_{f \in \mathcal{F}} \max_{|\beta| \leq l} |\partial_\beta f(x_0)|\) will be actually attained for some \(f\) and we may take that value to be \(C_1\). Since there are finitely many \(\beta\), we can consider the subfamily \(\mathcal{F}_\beta := \{f \in \mathcal{F} \mid \partial_\beta f(x_0) \geq C_1\}\), which is also compact in \(C^l(U)\) and satisfies (3.4). Proving the theorem for \(\mathcal{F}_\beta\) will yield sets \(U_\beta\) where (1) and (2) above hold. Setting \(V_\nu := \bigcap_\beta U_\beta\) then proves the Proposition. We may therefore assume without loss of generality that \(\beta\) is the same for every \(f \in \mathcal{F}\).

We wish to apply Theorem 3.2 of [33] and to do so we need to satisfy (3.3). We are going to show that there exists \(A \in \text{GL}_d(\mathbb{O})\) such that
Thus we have that for $f(A^{-1}x_0)$ we have, by the chain rule that

$$\partial_i^k f \circ A(A^{-1}x_0) = \sum \sum_{i,j=k, j \geq 0} C(i_1, \ldots, i_d) a_{i_1}^{i_1} \cdots a_{d_1}^{i_d} \partial_{\beta}^k \bigg|_{\beta=(i_1, \ldots, i_d)} f(x_0)$$

$$\vdots$$

$$\partial_d^k f \circ A(A^{-1}x_0) = \sum \sum_{i,j=k, j \geq 0} C(i_1, \ldots, i_d) a_{1d}^{i_1} \cdots a_{dd}^{i_d} \partial_{\beta}^k \bigg|_{\beta=(i_1, \ldots, i_d)} f(x_0).$$

(3.7)

We want $A = (a_{ij})$ such that every element in the left side of (3.7) above is nonzero knowing that for at least one $\beta$, $\partial_{\beta}^k \bigg|_{\beta=(i_1, \ldots, i_k)} f(x_0) \neq 0$. Namely, we wish to find $A \in \text{GL}_d(\mathbb{Q})$ such that $x'_i \neq 0$ for every $i$ where

$$x'_1 = \sum C(i_1, \ldots, i_d) a_{11}^{i_1} \cdots a_{d1}^{i_1} x_{(i_1, \ldots, i_d)}$$

$$\vdots$$

$$x'_d = \sum C(i_1, \ldots, i_d) a_{1d}^{i_1} \cdots a_{dd}^{i_d} x_{(i_1, \ldots, i_d)}$$

i.e.

$$x'_1 = g(a_{11}, \ldots, a_{d1})$$

$$\vdots$$

$$x'_d = g(a_{1d}, \ldots, a_{dd})$$

and $g$ is a homogeneous polynomial of degree $k$. We already know that $\partial_{\beta}^k \bigg|_{\beta=(i_1, \ldots, i_k)} f(x_0) \neq 0$ for at least one $\beta$, so at least one $x_{(i_1, \ldots, i_k)} \neq 0$ and thus $g$ is a nonzero polynomial.

Now $g$ should have at least one nonzero value on $\{1+\pi\mathbb{Q}\} \times \{\pi\mathbb{Q}\} \times \cdots \times \{\pi\mathbb{Q}\}$, otherwise $g$ is identically zero. So take $(a_{11}, \ldots, a_{1d})$ to be the point of the aforementioned set where $g(a_{11}, \ldots, a_{1d}) \neq 0$. Then by a similar argument choose $(a_{11}, \ldots, a_{id}) \in \{\pi\mathbb{Q}\} \times \cdots \times \{1+\pi\mathbb{Q}\} \times \cdots \times \{\pi\mathbb{Q}\}$ such that $g(a_{11}, \ldots, a_{id}) \neq 0$. Choosing $A$ this way we will automatically get that $\det(A)$ is a unit, which implies that $A \in \text{GL}_d(\mathbb{Q})$.

Thus we have that for $f \in \mathcal{F}$ there exists $A_f \in \text{GL}_d(\mathbb{Q})$ depending on $f$ such that

$$\min_{i=1, \ldots, d} |\partial_i^k f \circ A_f(A_i^2(x_0))| > 0$$

(3.8)

in fact there exists a uniform $C > 0$ such that

$$\min_{i=1, \ldots, d} |\partial_i^k f \circ A_f(A_i(x_0))| > C.$$ (3.9)

This is because we can take

$$C = \inf_{f \in \mathcal{F}} \sup_{A \in \text{GL}_d(\mathbb{Q})} \min_{i=1, \ldots, d} |\partial_i^k f \circ A(A^{-1}(x_0))|,$$

which is nonzero. For if not, then there exists $\{f_n\} \in \mathcal{F}$ such that

$$\sup_{A \in \text{GL}_d(\mathbb{Q})} \min_{i=1, \ldots, d} |\partial_i^k f_n \circ A(A^{-1}(x_0))| < \frac{1}{n}.$$
Since $F$ is compact, $\{f_n\}$ has a convergent subsequence $\{f_{n_k}\} \to f \in F$. Taking limits, we get that
\[
\min_{i=1,\ldots,d} |\partial^k f \circ A(A^{-1}(x_0))| = 0 \quad \forall \ A \in \text{GL}_d(\mathbb{O}),
\]
which is a contradiction to (3.8).

Consider the following map
\[
\Phi_1 : \text{GL}_d(\mathbb{Q}_\nu) \times C^l(U_\nu) \times U_\nu \to \mathbb{Q}_\nu
\]
\[
(A, f, x) \mapsto \min_{i=1,\ldots,d} |\partial^k f \circ A(A^{-1}(x)|.
\]
It can be easily verified that $\Phi_1$ is continuous. For every $f \in F$ there exists $A_f \in \text{GL}_d(\mathbb{O})$ such that $\Phi_1(A_f, f, x_0) \geq C > C/2$, so by continuity we have an open neighbourhood $U_{A_f} \times U_f \times U_{(x_0,f)}$ of $(A_f, f, x_0)$ such that
\[
\Phi_1(A, g, x) > C/2 \quad \forall \ (A, g, x) \in U_{A_f} \times U_f \times U_{(x_0,f)}.
\]
In particular,
\[
\Phi_1(A_f, g, x) > C/2 \quad \forall g \in U_f \text{ and } \forall x \in U_{(x_0,f)}.
\] (3.10)

Now $F \subset \bigcup_f U_f$, must have a finite subcovering $\{U_{f_i}\}_{i=1}^r$. So by (3.10) we have that for every $x \in U_{x_0} = \bigcap_{i=1}^r U_{(x_0,f_i)}$ and $f \in F$ there exists $A_{f_i}$ such that
\[
\Phi_1(A_{f_i}, f, x) > C/2.
\] (3.11)

Choose $\delta = C/(4u)$ where $u$ is the constant coming from the inequality
\[
|\partial^k \Theta \circ T(T^{-1}x)| \leq u \max_{|\beta| \leq l} |\partial^\beta f(x)|
\]
for $T \in \text{GL}_d(\mathbb{O})$. Thus any $\Theta$ satisfying (3.5) will also satisfy
\[
\Phi_1(A_{f_i}, f + \Theta, x) > C/4 \quad \forall \ x \in U_{x_0}.
\]

Now consider the set $\mathcal{F}_{A_{f_i}} = \{f \in \mathcal{F} \mid \Phi_1(A_{f_i}, f, x_0) \geq C/2\}$. Clearly this is a closed subset of the compact set $\mathcal{F}$, so it is also compact. Therefore $\partial(f \circ A_{f_i})| f \in \mathcal{F}_{A_{f_i}}$ is also compact being the image of a compact set under a continuous map. Since $\mathcal{F} \subset \bigcup_{i=1,\ldots,r} \mathcal{F}_{A_{f_i}}$, we
Hence we may assume that the family does not satisfy (3.4) for every $O$. Let $O$ be compact in $O$, and assume that we want to apply the first part of this Proposition. Suppose $A$ may, without loss of generality, take the same value. This is a compact family of functions of $C$, and we may assume $A\in (3.6)$. Now this sequence $\epsilon(a)$ converges to $c\in (3.6)$. By applying the first part of the Proposition we get that for every $\nu$, $|\Phi|\geq 0$. Therefore each $\nu$ is $|\Phi|\geq 0$.

Then by compactness of $O$, we have that for some $f \in \mathcal{F}$,

$$\max_{|\beta|\leq l-1} |\partial_\beta \partial_j (f \circ A)(A^{-1}(x_0))| = 0,$$

which implies that $\Phi_1(A,f,x_0) = 0$, which is a contradiction. Thus by applying the first part of the Proposition we get that for every $j = 1, \ldots, d$, $\partial_j((f + \Theta) \circ A)$ is $(C,\frac{1}{d(l-1)})$-good on an open neighbourhood $B_{A^{-1}(x_0)}$ of $A^{-1}(x_0)$. So $(\partial_j(f + \Theta \circ A)) \circ A^{-1}$ is $(C,\frac{1}{d(l-1)})$-good on $A(B_{A^{-1}(x_0)})$. Therefore each $\partial_j(f + \Theta)$ is $(C,\frac{1}{d(l-1)})$-good on $A(B_{A^{-1}(x_0)})$ and so is $|\nabla (f + \Theta)|$. The case $|\beta| = 1$ in (3.6) is trivial (See property (G3) of $(C,\alpha)$-good functions). This completes the proof. \hfill $\square$

As a Corollary, we have,

**Corollary 3.1.** Let $U_\nu$ be an open subset of $Q_\nu$, $x_0 \in U_\nu$ be fixed and assume that $f_\nu = (f_\nu^{(1)}, f_\nu^{(2)}, \ldots, f_\nu^{(n)}) : U_\nu \to \mathbb{Q}^n$ satisfies (I2) and (I3) and that $\Theta_\nu$ satisfies (I5). Then there exists a neighbourhood $V_\nu \subset U_\nu$ of $x_0$ and positive constants $C > 0$ and $l \in \mathbb{N}$ such that for any $(a_0, a) \in \Theta^{n+1}$,

1. $a_0 + a.f_\nu + \Theta_\nu$ is $(C,\frac{1}{d(l)})$-good on $V_\nu$, and
2. $|\nabla (a.f_\nu + \Theta_\nu)|$ is $(C,\frac{1}{d(l-1)})$-good on $V_\nu$.

**Proof.** For the case $\nu = \infty$, see Corollary 3 of [4] and also [11]. So we may assume $\nu \neq \infty$. Let $\mathcal{F} := \{a_0 + a.f_\nu + \Theta_\nu \mid (a_0, a) \in \Theta^{n+1}\}$. This is a compact family of functions of $C^{l}(U_\nu)$ for every $l > 0$ since $\Theta$ is compact in $Q_\nu$. Now if this family satisfies condition (3.4) for some $l \in \mathbb{N}$, then the conclusion follows from the previous Proposition. Hence we may assume that the family does not satisfy (3.4) for every $l \in \mathbb{N}$. Then by the continuity of differential and the compactness of $\Theta$, there exists $c_l \in \Theta^n$ such that for every $2 \leq l \in \mathbb{N}$ we have

$$\max_{|\beta|\leq l} |\partial_\beta (c_l.f_\nu + \Theta_\nu)(x_0)| > 0.$$ 

Now this sequence $\{c_l\} \in \Theta^n$ has a convergent subsequence $\{c_{l_k}\}$ converging to $c \in \Theta^n$ since $\Theta^n$ is compact. By taking limits we get that

$$|\partial_\beta (c.f_\nu + \Theta_\nu)(x_0)| = 0 \forall \beta.$$
However, as each of the $f_\nu$ and $\Theta_\nu$ are analytic on $U_\nu$, there exists a neighbourhood $V_{x_0}$ of $x_0$ such that
\[
(c.f_\nu + \Theta_\nu)(x) = u \ \forall \ x \in V_{x_0},
\]
where $u \in \mathbb{Q}$ is a constant. Therefore replacing $\Theta_\nu$ by $u - c.f_\nu$, we get that
\[
\mathcal{F} = \{a_0 + u + (a - c).f_\nu \ | (a_0, a) \in \mathcal{O}^{n+1}\}.
\]
First consider the case where $|a_0 + u| < 2|a - c|$, then
\[
\mathcal{F}_1 = \left\{ \frac{a_0 + u}{|a - c|} + \frac{a - c}{|a - c|}.f_\nu \ | (a_0, a) \in \mathcal{O}^{n+1} \right\}
\]
is compact in $C'(U_\nu)$ for every $l \in \mathbb{N}$. Then by linear independence of $1, f_\nu^{(1)}, \cdots, f_\nu^{(n)}$, $\mathcal{F}_1$ satisfies (3.4) for some $l \in \mathbb{N}$. And then by Proposition 3.1 we can conclude that every element in $\mathcal{F}_1$ is $(C, \frac{1}{\alpha^l})$-good on some $V_\nu \subset V_{x_0} \subset U_\nu$ together with conclusion (2) of the Corollary above. This also implies $a_0 + u + (a - c).f_\nu$ are all $(C, \frac{1}{\alpha^l})$-good on $V_\nu$ for all $(a_0, a) \in \mathcal{O}^{n+1}$ with $|a_0 + u| < 2|a - c|$. Otherwise
\[
\sup_{x \in V_{x_0}} |a_0 + u + (a - c).f_\nu| \leq 3 \ \inf_{x \in V_{x_0}} |a_0 + u + (a - c).f_\nu|
\]
as $|a_0 + u| \geq 2|a - c|$ and it turns out to be a trivial case. This implies that for $C \geq 3$ and $0 < \alpha \leq 1$ the aforementioned functions are $(C, \alpha)$-good. 

Let us recall the following Corollary from [33] (Corollary 2.3).

**Corollary 3.2.** For $j = 1, \cdots, n$, let $X_j$ be a metric space, $\mu_j$ be a measure on $X_j$. Let $U_j \subset X_j$ be open, $C_j, \alpha_j > 0$ and let $f$ be a function on $U_1 \times \cdots \times U_n$ such that for any $j = 1, \cdots, d$ and any $x_i \in U_i$ with $i \neq j$, the function
\[
y \mapsto f(x_1, \cdots, x_{j-1}, y, x_{j+1}, \cdots, x_d)
\]
is $(C_j, \alpha_j)$-good on $U_j$ with respect to $\mu_j$. Then $f$ is $(\tilde{C}, \tilde{\alpha})$-good on $U_1 \times \cdots \times U_d$ with respect to $\mu_1 \times \cdots \times \mu_d$, where $\tilde{C} = d.\tilde{\alpha}$ are computable in terms of $C_j, \alpha_j$. In particular, if each of the functions (3.12) is $(C, \alpha)$-good on $U_j$ with respect to $\mu_j$, then the conclusion holds with $\tilde{\alpha} = \frac{\alpha}{d}$ and $\tilde{C} = dC$.

Now combining Corollary (3.1) and (3.2) we can state the following:

**Corollary 3.3.** Let $f$ and $\Theta$ be as in Corollary (3.1) and let $x_0 \in U$. Then there exists a neighbourhood $V \subset U$ of $x_0$ and $C > 0, k, k_1 \in \mathbb{N}$ such that for any $(a_0, a) \in \mathbb{Z}^{n+1}$ the following holds:

1. $x \mapsto |(a_0 + a.f + \Theta)(x)|_S$ is $(C, \frac{1}{dk})$-good on $V$. 

\[ (2) \ x \mapsto \| \nabla (a. f_{\nu} + \Theta_{\nu}) (x_{\nu}) \| \text{ is } (C, \frac{1}{d_k}) - \text{good on } V, \forall \nu \in S \]

where \( d = \max d_{\nu}. \)

4. Proof of Theorem 1.4

We set \( \phi(\nu) = \begin{cases} -\varepsilon & \text{if } \nu \neq \infty \\ 1 - \varepsilon & \text{if } \nu = \infty \end{cases} \).

From the definition, it follows that \( W_{\Psi, \Theta}^f \) admits a description as a limsup set. Namely,

\[ W_{\Psi, \Theta}^f = \limsup_{a \to \infty} W_f(a, \Psi, \Theta) \]

where

\[ W_f(a, \Psi, \Theta) = \{ x \in U : |a_0 + a \cdot f(x) + \Theta(x)|_S^f \leq \Psi(a) \text{ for some } a_0 \}. \]

We may now write

\[ W_f^\text{large}(a, \Psi, \Theta) = \{ x \in W_f(a, \Psi, \Theta) : \| \nabla (a. f_{\nu}(x_{\nu}) + \Theta_{\nu}(x_{\nu})) \|_{\nu} > \| a \|_S^{\phi(\nu)} \forall \nu \} \]

where \( 0 < \varepsilon < \frac{1}{4(n+1)^2}, \) is fixed and

\[ W_f(a, \Psi, \Theta) \setminus W_f^\text{large}(a, \Psi, \Theta) = \bigcup_{\nu \in S} W_{\nu, f}^\text{small}(a, \Psi, \Theta) \]

where

\[ W_{\nu, f}^\text{small}(a, \Psi, \Theta) = \{ x \in W_f(a, \Psi, \Theta) : \| \nabla (a. f_{\nu}(x_{\nu}) + \Theta_{\nu}(x_{\nu})) \|_{\nu} \leq \| a \|_S^{\phi(\nu)} \} \].

As the set \( S \) is finite, we have

\[ W_{\Psi, \Theta}^f = W_f^\text{large}(\Psi, \Theta) \bigcup_{\nu \in S} W_{\nu, f}^\text{small}(\Psi, \Theta) \]

where

\[ W_f^\text{large}(\Psi, \Theta) = \limsup_{a \to \infty} W_f^\text{large}(a, \Psi, \Theta) \]

and

\[ W_{\nu, f}^\text{small}(\Psi, \Theta) = \limsup_{a \to \infty} W_{\nu, f}^\text{small}(a, \Psi, \Theta). \]

To prove Theorem 1.4, we will show that each of these limsup sets has zero measure. Namely, the proof is divided into the “large derivative” case where we will show \( |W_f^\text{large}(\Psi, \Theta)| = 0 \), and the “small derivative” case which involves \( |W_{\nu, f}^\text{small}(\Psi, \Theta)| = 0 \forall \nu \in S \).
4.1. The small derivative. We begin by showing that $|W_{\nu f}(\Psi, \Theta)| = 0 \forall \nu \in S$. From the assumed property (I4) of $\Psi$, it follows that

$$\Psi(a) < \Psi_0(a) := \prod_{i=1, \ldots, n, a_i \neq 0} |a_i|^{-1}_S.$$ 

So $W_{\nu f}(\Psi, \Theta) \subset W_{\nu f}(\Psi_0, \Theta)$, which means that it is enough to show that $|W_{\nu f}(\Psi_0, \Theta)| = 0 \forall \nu \in S$. Let us take $A = \mathbb{Z} \times \mathbb{Z}^n \setminus \{0\}$ and $T = \mathbb{Z}^n_{\geq 0}$ and define the function

$$r_\nu(t) = \begin{cases} 
2^{(|t|+1)(1-\varepsilon)} & \text{if } \nu = \infty \\
2^{-(|t|+1)\varepsilon} & \text{if } \nu \neq \infty
\end{cases} \quad (4.1)$$

where $\varepsilon$ is fixed as before. Now we define sets $I'_t(\alpha, \lambda)$ and $H'_t(\alpha, \lambda)$ for every $\lambda > 0$, $t \in T$ and $\alpha = (a_0, a) \in A$ as follows:

$$I'_t(\alpha, \lambda) = \left\{ x \in U : \|\nabla(a_f(x) + \Theta(x))\|_\nu < \lambda r_\nu(t) \right\}$$

and

$$H'_t(\alpha, \lambda) = \left\{ x \in U : \|\nabla(a_f(x))\|_\nu < 2\lambda r_\nu(t) \right\} \quad (4.2)$$

and

where $2^t = (2^{t_1}, \ldots, 2^{t_n})$ and $|S| = l$. These give us the functions (2.2) and (2.1) required in the inhomogeneous transference principle. As in (2.3) and (2.4) we get $H'_t(\lambda)$, $I'_t(\lambda)$, $\Lambda'_H(\lambda)$ and $\Lambda'_I(\lambda)$. Now define $\phi_\delta : T \mapsto \mathbb{R}_+$ as $\phi_\delta(t) := 2^{|t|}$ for $\delta \in (0, \frac{\varepsilon}{2}]$. Clearly $W_{\nu f}(\Psi_0, \Theta) \subset \Lambda'_I(\phi_\delta)$ for every $\delta \in (0, \frac{\varepsilon}{2}]$. So to settle Case 2 it is enough to show that

$$|\Lambda'_I(\phi_\delta)| = 0 \text{ for some } \delta \in (0, \frac{\varepsilon}{2}]. \quad (4.4)$$

Now we recall Theorem 1.3 from [37].

**Theorem 4.1.** Let $S$ be as in (I0), $U$ be as in (I1), and assume that $f$ satisfies (I2) and (I3). Then for any $x = (x_\nu)_{\nu \in S} \in U$, one can find a neighborhood $V = \bigcap V_\nu \subseteq U$ of $x$ and $\alpha_1 > 0$ with the following property: for any ball $B \subseteq V$, there exists $E > 0$ such that for any
choice of $0 < \delta \leq 1$, $T_1, \ldots, T_n \geq 1$, and $K_\nu > 0$ with $\delta(\frac{T_1 \cdots T_n}{\text{max} T_i}) \prod K_\nu \leq 1$ one has

\[
\left\{ \begin{align*}
|\langle a, f(x) \rangle | &< \delta \\
x \in B \, \exists \, a \in \mathbb{Z}^n \setminus \{0\} : \, |a\nabla f_\nu(x_\nu)|_\nu < K_\nu, \, \nu \in S \\
|a_i| &< T_i, 1 \leq i \leq n
\end{align*} \right\} \leq E \varepsilon_1^n |B|,
\]

where $\varepsilon_1 = \max\{\delta^{\frac{1}{1}}, (\delta(\frac{T_1 \cdots T_n}{\text{max} T_i}))^{\frac{1}{(t+1)}}\}$, where $|S| = l$.

The Theorem above is an $S$-adic analogue of Theorem 1.4 in [11] and is proved using nondivergence estimates for certain flows on homogeneous spaces. We will denote the set in the LHS of (4.5) as $S(\delta, K_1, \ldots, K_L, T_1, \ldots, T_n)$ for further reference.

To show (4.4) we want to use the Inhomogeneous transference principle (2.1). Assume that $(H_\nu, I_\nu, \Phi)$ satisfies the intersection property and that the product measure is contracting with respect to $(I_\nu, \Phi)$ where, $\Phi := \{\phi_0 : 0 \leq \delta < \frac{\varepsilon}{2}\}$. Then by (2.1) it is enough to show that

\[
|\Lambda^\nu_H(\phi_0)| = 0 \text{ for some } 0 < \delta \leq \frac{\varepsilon}{2}.
\]

Note that

\[
\Lambda^\nu_H(\phi_0) = \limsup_{t \in T} \bigcup_{\alpha \in A} H^\nu_\alpha(\alpha, \phi_0(t)).
\]

Using Theorem 4.1, we will show that

\[
\sum |\bigcup_{\alpha \in A} H^\nu_\alpha(\alpha, \phi_0(t))| < \infty
\]

for some $0 < \delta < \frac{\varepsilon}{2}$. This, together with Borel-Cantelli will give us $|\Lambda^\nu_H(\phi_0)| = 0$.

By the definition 4.3 of $H^\nu_\alpha(\alpha, \phi_0(t))$, we get

\[
\bigcup_{\alpha \in A} H^\nu_\alpha(\alpha, \phi_0(t)) \subset S(2^l \phi_0(t) \Psi_0(2^t), 1, \ldots, 2, \phi_0(t)r_\nu(t), \ldots, 1, 2^{l+2}, \ldots, 2^{n+2})
\]

i.e. here $K_\nu = 2 \cdot \phi_0(t)r_\nu(t)$, $K_\omega = 1$, where $\omega \neq \nu$ and $T_i = 2^{l+2}$.

4.2. Case $1 (\nu = \infty)$. Here $r_\infty(t) = 2^{(1-\varepsilon)(|t|+1)}$. So,

\[
2^{l}2^{|t|}\Psi_0(2^t), 2^{|t|}2^{l}2^{(1-\varepsilon)(|t|+1)}1, \frac{2^{\sum_{i=1}^{n} t_i + 2}}{2^{|t|}} = 2^{2n+l+2-\varepsilon}, \frac{2^{l}(2\delta-\varepsilon)}{} \leq 1
\]
for all large $t$ as $2\delta - \varepsilon < 0$. So by Theorem 4.1 we have

$$| \bigcup_{\alpha \in A} H_t^\infty(\alpha, \phi_\delta(t)) | \leq E\varepsilon_1^{|t|}|B|,$$

where $\varepsilon_1 = \max\{2^{2\delta |t| - \sum t_i} \cdot 2^{2n + l + 1 - \varepsilon} \cdot 2^{\frac{t|2(\delta - \varepsilon)|}{l(n+1)}}, 2^{2n + l + 1 - \varepsilon} \cdot 2^{\frac{t|2(\delta - \varepsilon)|}{l(n+1)}}\}$ for all large $t \in \mathbb{Z}_{\geq 0}$. We note that $\varepsilon_1$ is ultimately the 2nd term in the parenthesis. Because if not then for infinitely many $t$,$$
\frac{\delta |t| - \sum t_i}{l} > \frac{|t|(2\delta - \varepsilon)}{l(n+1)} + O(1)
$$
which implies that$$
\sum t_i < |t| + O(1),
$$
a contradiction. Therefore we have

$$| \bigcup_{\alpha \in A} H_t^\infty(\alpha, \phi_\delta(t)) | \ll 2^{-\gamma |t|},$$

where $\gamma = \frac{(\varepsilon - 2\delta)}{l(n+1)}\alpha_1 > 0$. Hence

$$\sum_{t \in T} | \bigcup_{\alpha \in A} H_t^\infty(\alpha, \phi_\delta(t)) | \ll \sum_{t \in T} 2^{-\gamma |t|} < \infty.$$

4.3. Case 2 ($\nu \neq \infty$). The argument proceeds as in Case 1. In this case, $r_\nu(t) = 2^{-\varepsilon(|t|+1)}$. So,$$
2^l \cdot 2^\delta |t| \psi_0(2^t) \cdot 2^\delta |t| \cdot 2^{-\varepsilon(|t|+1)} \cdot \frac{2\sum_{i=1}^{n} t_i + 2}{2|t|} = 2^{2n + l + 1 - \varepsilon} \cdot 2^{\frac{t|2(\delta - \varepsilon - 1)|}{l(n+1)}} < 1
$$
for large $t$ as $2\delta - \varepsilon < 0$. Therefore, by Theorem 4.1 we have

$$| \bigcup_{\alpha \in A} H_t^\nu(\alpha, \phi_\delta(t)) | \leq E\varepsilon_1^{|t|}|B|,$$

where $\varepsilon_1 = \max\{2^{2\delta |t| - \sum t_i} \cdot 2^{2n + l + 1 - \varepsilon} \cdot 2^{\frac{t|2(\delta - \varepsilon)|}{l(n+1)}}, 2^{2n + l + 1 - \varepsilon} \cdot 2^{\frac{t|2(\delta - \varepsilon)|}{l(n+1)}}\}$ for all large $t \in \mathbb{Z}_{\geq 0}$. As in case 1, $\varepsilon_1$ is ultimately the 2nd term in the parenthesis. For if not, then for infinitely many $t$,$$
\frac{\delta |t| - \sum t_i}{l} > \frac{|t|(2\delta - \varepsilon - 1)}{l(n+1)} + O(1)
$$
which implies that$$
\sum t_i < 2|t| + O(1).
$$
This gives a contradiction. Therefore we have

$$| \bigcup_{\alpha \in A} H_t^\nu(\alpha, \phi_\delta(t)) | \ll 2^{-\gamma |t|},$$
where $\gamma = \frac{(\varepsilon - 2\delta + 1)}{l(n+1)} \alpha_1 > 0$. Hence
\[
\sum_{t \in T} \bigcup_{\alpha \in A} H^\nu_t(\alpha, \phi_\delta(t)) \ll \sum_{t \in T} 2^{-\gamma |t|} < \infty.
\]

Consequently the only thing left to verify are the intersection and contracting properties of the transference principle.

**Remark 4.1.** We will consider $|.|$ the measure to be restricted on some bounded open ball $V_{x_0}$ around $x_0 \in U$. Then we will get $|\Lambda^\nu_t(\phi_\delta) \cap V_{x_0}| = 0$. But because the space is second countable, we eventually get $|\Lambda^\nu_t(\phi_\delta)| = 0$.

### 4.4. Verifying the intersection property:

Let $t \in T$ with $|t| > \frac{l}{1 - \frac{\gamma}{2}}$. We have to show that for $\phi_\delta$ there exists $\phi_\delta^*$ such that for all but finitely many $t \in T$ and all distinct $\alpha = (a_0, a), \alpha' = (a'_0, a'_0) \in A$, we have that $I^\nu_t(\alpha, \phi_\delta(t)) \cap I^\nu_t(\alpha', \phi_\delta(t)) \subset H^\nu_t(\phi_\delta^*(t))$. Consider
\[
I^\nu_t(\alpha, \phi_\delta(t)) \cap I^\nu_t(\alpha', \phi_\delta(t)),
\]
then by Definition (4.2) we have
\[
\begin{align*}
|a_0 + a.f(x) + \Theta(x)|_S &< (\phi_\delta(t)|\Psi_0(2^t)|)^\frac{1}{2} (4.7) \\
||\nabla(a.f(x) + \Theta(x))||_\nu &< \phi_\delta(t) r_\nu(t)
\end{align*}
\]
and
\[
\begin{align*}
|a'_0 + a'.f(x) + \Theta(x)|_S &< (\phi_\delta(t)|\Psi_0(2^t)|)^\frac{1}{2} (4.8) \\
||\nabla(a'.f(x) + \Theta(x))||_\nu &< \phi_\delta(t) r_\nu(t)
\end{align*}
\]
where
\[
|a_i| < 2^{\gamma + 1} \text{ for } 1 \leq i \leq n \text{ and } |a'_i| < 2^{\gamma + 1} \text{ for } 1 \leq i \leq n.
\]

Now subtracting the respective equations of (4.8) from (4.7) we have $\alpha'' = (a_0 - a'_0, a - a')$ satisfying the following equations
\[
\begin{align*}
|a''_0 + a''.f(x)|_S &< 2^\gamma \phi_\delta(t) |\Psi_0(2^t)| \\
||\nabla(a''.f(x))||_\nu &< 2 \phi_\delta(t) r_\nu(t) (4.9) \\
|a''_i| &\leq 2^{\gamma + 2} \text{ for } 1 \leq i \leq n.
\end{align*}
\]

Observe that $a'' \neq 0$, because otherwise
\[
1 \leq |a''_0| < 2^\gamma \phi_\delta(t) |\Psi_0(2^t)| < 2^\gamma 2^{-(1-\frac{\gamma}{2})|t|},
\]
which implies that $|t| \leq \frac{l}{1 - \frac{\gamma}{2}}$, which is true for the finitely many $t$’s that we are avoiding. Therefore $\alpha'' \in A$ and $x \in H^\nu_t(\alpha'', \phi_\delta(t))$. So
here the particular choice of $\phi_\delta^*$ is $\phi_\delta$ itself. This verifies the intersection property.

4.5. **Verifying the Contraction Property**: Recall that to verify the contraction property we need to verify the following: for any $\phi_\delta \in \Phi$ we need to find $\Phi_\delta^+ \in \Phi$ and a sequence of positive numbers $\{k_t\}_{t \in T}$ satisfying

$$\sum_{t \in T} k_t < \infty$$

such that for all but finitely many $t \in T$ and all $\alpha \in A$, there exists a collection $C_{t,\alpha}$ of ball $B$ centred at a point in $S = \mathcal{V} = \overline{\mathcal{V}}$ satisfying (2.7), (2.8) and (2.9).

Let us consider the open set $5\mathcal{V}_{x_0}$ in Corollary 3.3. So we have that for any $t \in T$ and $\alpha = (a_0, a) \in A$

$$F^\nu_{t,\alpha}(x) = \max\{\Psi_0^{-1}(2^t)r_\nu(t)|a_0 + a.f(x)\} + \max\{\nu(a.f + \Theta(x))\}$$

is $(C, \frac{1}{dt})$-good on $5\mathcal{V}_{x_0}$ for some $C > 0, k \in \mathbb{N}$ and $d = \max d_\nu$. Using this new function $F^\nu_{t,\alpha}$, we can write the previous inhomogeneous sets as following :

$$I^\nu_t(\alpha, \phi_\delta(t)) = \left\{ x \in U : \begin{array}{l} F^\nu_{t,\alpha}(x) < \phi_\delta(t)r_\nu(t) \hfill \\ 2^i \leq \max\{1, |a_i|s\} < 2^{i+1} \forall 1 \leq i \leq n \end{array} \right\}.$$  \hspace{1cm} (4.11)

We also note that

$$I^\nu_t(\alpha, \phi_\delta(t)) \subset I^\nu_t(\alpha, \phi_\delta^+(t))$$

where $\phi_\delta^+(t) = \phi_{\frac{d}{2} + \frac{\varepsilon}{4}}(t) \geq \phi_\delta(t) \forall t \in T$. And $\phi_\delta^+(t) = \phi_{\frac{d}{2} + \frac{\varepsilon}{4}}(t) \in \Phi$ because $\frac{d}{2} + \frac{\varepsilon}{4} < \frac{\varepsilon}{2}$. If $I^\nu_t(\alpha, \phi_\delta(t)) = \emptyset$ then it is trivial. So without loss of generality we can assume that $I^\nu_t(\alpha, \phi_\delta(t)) \neq \emptyset$. Because for every $\phi_\delta \in \Phi \, \phi_\delta(t) \Psi_0(2^t) < 2^{-1/2||t||}$, so in particular

$$I^\nu_t(\alpha, \phi_\delta^+(t)) \subset \{ x \in U : |a_0 + a.f(x) + \Theta(x)|^t < 2^{-1/2||t||} \}.$$  \hspace{1cm} (4.12)

We recall Corollary 4 of [4],

$$\inf_{(a, a_0) \in \mathbb{R}^{n+1} \setminus \{0\}} \sup_{x \in 5\mathcal{V}_{x_0}} |a_0 + a.f_{\infty}(x_\infty) + \Theta_{\infty}(x_\infty)|_\infty > 0.$$  \hspace{1cm} \hspace{1cm} \hspace{1cm}

Therefore,

$$\inf_{(a, a_0) \in \mathbb{R}^{n+1} \setminus \{0\}} \sup_{x \in 5\mathcal{V}_{x_0}} |a_0 + a.f(x) + \Theta(x)|_s >$$
there exists a ball \( |x_0 + a \cdot f(x) + \Theta(x)|_S \) for all but finitely many \( t \). Hence (2.7) and (2.8) are satisfied. By (4.14) we have \( B(x) \cap S \subset I_t^\star(\alpha, \phi^+_\delta(t)) \) and holds for all but finitely many \( t \) . The second inequality holds because we would otherwise have \( V_{x_0} \subset I_t^\star(\alpha, \phi^+_\delta(t)) \), a contradiction holds because we would otherwise have \( V_{x_0} \subset I_t^\star(\alpha, \phi^+_\delta(t)) \), a contradiction. Then take \( C_{t,0} := \{ B(x) : x \in S \cap I_t^\star(\alpha, \phi^\delta(t)) \} \). Hence (2.7) and (2.8) are satisfied. By (4.14) we have

\[
\sup_{x \in 5B \cap I_t^\star(\alpha, \phi^\delta(t))} \mathbf{F}_{t,0}^\nu(x) \leq 2^{\left(\frac{4}{7} - \frac{2}{7}\right)\varepsilon} \sup_{x \in 5B \cap I_t^\star(\alpha, \phi^\delta(t))} \mathbf{F}_{t,0}^\nu(x) \leq 2^{\left(\frac{4}{7} - \frac{2}{7}\right)\varepsilon} \sup_{x \in 5B} \mathbf{F}_{t,0}^\nu(x).
\]

Therefore for all large \( |t| \) and \( \alpha \in \mathbb{Z}^{n+1} \) we have

\[
|5B \cap I_t^\star(\alpha, \phi^\delta(t))| \leq 2^{\left(\frac{4}{7} - \frac{2}{7}\right)\varepsilon} |5B| \sup_{x \in 5B} \mathbf{F}_{t,0}^\nu(x).
\]

Hence finally we conclude

\[
|5B \cap I_t^\star(\alpha, \phi^\delta(t))|_V \leq |5B \cap I_t^\star(\alpha, \phi^\delta(t))|_S \leq 2^{\left(\frac{4}{7} - \frac{2}{7}\right)\varepsilon} |5B| \sup_{x \in 5B} \mathbf{F}_{t,0}^\nu(x).
\]

since \( 5B \subset 5V_{x_0} \). Here we are using that the measure is doubling and the centre of the ball \( 5B \) is in \( \overline{V_{x_0}} \). So \( C_\ast \) is only dependent on \( d_\nu \). We
choose \( k_t = C_s C 2^{(\frac{3}{2} - \frac{3}{4}) \varepsilon n} \| x \| \) and as \((\frac{3}{2} - \frac{3}{4}) < 0\) we also have \( \sum k_t < \infty \) as required in (2.6). This verifies the contracting property.

4.6. **The large derivative.** In this section, we will show that \( \mathcal{W}_f^{\text{large}}(\Psi, \Theta) = 0 \). Let us recall Theorem 1.2 from [37].

**Theorem 4.2.** Assume that \( U \) satisfies (I1), \( f \) satisfies (I2), (I3) and \( 0 < \varepsilon < \frac{1}{4n|S|^2} \). Let \( \mathcal{A} \) be

\[
\begin{cases}
\mathcal{A} = \{ \mathbf{a} \in \mathbb{Z}^n, T_i \leq \frac{1}{2} |a_i|_S < T_i, \quad \| a \mathbf{f}(x) \|_\nu > \| a \|_S^{-\varepsilon}, \quad \nu \neq \infty \} \\
\mathcal{A} = \{ \mathbf{a} \in \mathbb{Z}^n, T_i \leq \frac{1}{2} |a_i|_S < T_i, \quad \| \nabla (a \mathbf{f}(x) + \Theta(x)) \|_\nu > \| a \|_S^{-\varepsilon}, \quad \nu = \infty \}
\end{cases}
\]

Then \( |\mathcal{A}| < C \delta |U| \), for large enough \( \max(T_i) \) and a universal constant \( C \).

Note that the function \((f, \Theta) : U \mapsto \mathbb{Q}_S^{n+1} \) satisfies the same properties as \( f \). So as a Corollary of the previous theorem we get,

**Corollary 4.1.** Let \( 0 < \varepsilon < \frac{1}{4(n+1)|S|^2} \) and \( \mathcal{A}_T(1) \) be the set

\[
\bigcup_{T \leq \frac{1}{2} |a_i|_S < T_i} \{ \mathbf{x} \in \mathbb{U} \mid \| \mathbf{f}(x) + \Theta(x) \|_S < \delta (\prod_{i=1}^n T_i)^{-1} \}
\]

Then \( |\mathcal{A}_T(1)| < C \delta |U| \), for large enough \( \max(T_i) \) and a universal constant \( C \).

Now take \( T_i = 2^{t_i+1} \) and \( \delta = 2^{\sum t_i+1} \Psi(2^t) \). As \( 2^{t_i} \leq |a_i|_S < 2^{t_i+1} \), this implies by (1.3) that \( \Psi(a) \geq \Psi(2^{t_i+1}) \) and we have using (4.1) that

\[
\bigcup_{2^{t_i} \leq |a_i|_S < 2^{t_i+1}} |\mathcal{W}_f^{\text{large}}(a, \Psi, \Theta) | < C 2^{\sum t_i+1} \Psi(2^t). \tag{4.21}
\]

Note that

\[
\sum \Psi(a) \geq \sum \Psi(2^{t_i+1}, \ldots, 2^{t_{n}+1}) 2^{\sum t_i},
\]

so the convergence of \( \sum \Psi(a) \) implies the convergence of the later. Therefore by (4.21) and by the Borel-Cantelli lemma we get that almost every point of \( U \) is in at most finitely many \( \mathcal{W}_f^{\text{large}}(a, \Psi, \Theta) \). Hence \( |\mathcal{W}_f^{\text{large}}(\Psi, \Theta) | = 0 \) completing the proof.
5. The divergence theorem for \( \mathbb{Q}_p \)

In this section we prove Theorem 1.5 using ubiquitous systems as in [4]. In [6], the related notion of regular systems was used. As mentioned in the introduction, the divergence case will be proved for a more restrictive choice of approximating function than the convergence case, namely for those satisfying property \( P \). Indeed a more general formulation which includes the multiplicative case of the divergence Khintchine theorem remains an outstanding open problem even for submanifolds in \( \mathbb{R}^n \). Without loss of generality, and in an effort to keep the notation reasonable, we will prove the Theorem for the usual norm, i.e. we will assume \( v = (1, \ldots, 1) \). The interested reader can very easily make the minor changes required to prove it for general \( v \). For \( \delta > 0 \) and \( Q > 1 \) we follow [4] in defining \( \Phi^f(Q, \delta) := \{ x \in U \colon \exists a = (a_0, a_1) \in \mathbb{Z} \times \mathbb{Z}^n \setminus \{0\} \text{ such that } |a_0 + a_1 \cdot f(x)|_p < \delta Q^{-(n+1)} \text{ and } \| (a_0, a_1) \| \leq Q \} \).

We now recall definition of a nice function.

**Definition 5.1 ([4], Definition 3.2).** We say that \( f \) is nice at \( x_0 \in U \) if there exists a neighbourhood \( U_0 \subset U \) of \( x_0 \) and constants \( 0 < \delta, w < 1 \) such that for any sufficiently small ball \( B \subset U_0 \) we have that
\[
\limsup_{Q \to \infty} |\Phi^f(Q, \delta) \cap B| \leq w |B|.
\]
(5.1)

If \( f \) is nice at almost every \( x_0 \) in \( U \) then \( f \) is called nice. The following Theorem from [38] plays a crucial role. It’s proof involves a suitable adaptation of the dynamical technique in [11].

**Theorem 5.1.** [38] Assume that \( f : U \to \mathbb{Q}_p^n \) is nondegenerate at \( x \in U \). Then there exists a sufficiently small ball \( B_0 \subset U \) centred at \( x \) and a constant \( C > 0 \) such that for any ball \( B \subset B_0 \) and any \( \delta > 0 \), for sufficiently large \( Q \), one has
\[
|\Phi^f(Q, \delta) \cap B| \leq C\delta |B|.
\]
(5.2)

This implies that if \( f \) is nondegenerate at \( x_0 \) then \( f \) is nice at \( x_0 \). We will now state the main two theorems of this section. Let \( \psi : \mathbb{N} \to \mathbb{R}_+ \) be a decreasing function.

**Theorem 5.2.** Assume that \( f : U \subset \mathbb{Q}_p^m \to \mathbb{Q}_p^n \) is nice and satisfies the standing assumptions (I1 and I2) and that \( s > m - 1 \). Let \( \Theta : U \to \mathbb{Q}_p \) be an analytic map satisfying assumption (I5). Let \( \Psi(a) = \psi(\|a\|), a \in \mathbb{Z}^{n+1} \) be an approximating function. Then,
\[
\mathcal{H}^s(W^f_{(\psi, \Theta)} \cap U) = \mathcal{H}^s(U) \text{ if } \sum (\Psi(a))^{s+1-m} = \infty.
\]
(5.3)
In view of Theorem 5.1, Theorem 5.2 implies Theorem 1.5. Note that condition (I3) implies the nondegeneracy of \( f \) at every point of \( U \).

5.1. Ubiquitous Systems in \( \mathbb{Q}_p^n \). Let us recall the definition of Ubiquitous systems in \( \mathbb{Q}_p^n \) following [4]. Throughout, balls in \( \mathbb{Q}_p^m \) are assumed to be defined in terms of the supremum norm \( |·| \). Let \( U \) be a ball in \( \mathbb{Q}_p^m \) and \( \mathcal{R} = (R_\alpha)_{\alpha \in J} \) be a family of subsets \( R_\alpha \subset \mathbb{Q}_p^m \) indexed by a countable set \( J \). The sets \( R_\alpha \) are referred to as resonant sets. Throughout, \( \rho : \mathbb{R}^+ \to \mathbb{R}^+ \) will denote a function such that \( \rho(r) \to 0 \) as \( r \to \infty \). Given a set \( A \subset U \), let 
\[
\Delta(A, r) := \{ x \in U : \text{dist}(x, A) < r \}
\]
where \( \text{dist}(x, A) := \inf \{ |x - a| : a \in A \} \). Next, let \( \beta : J \to \mathbb{R}^+ : \alpha \mapsto \beta_\alpha \) be a positive function on \( J \). Thus the function \( \beta \) attaches a ‘weight’ \( \beta_\alpha \) to the set \( R_\alpha \). We will assume that for every \( t \in \mathbb{N} \) the set \( J_t = \{ \alpha \in J : \beta_\alpha \leq 2^t \} \) is finite.

The intersection conditions: There exists a constant \( \gamma \) with \( 0 \leq \gamma \leq m \) such that for any sufficiently large \( t \) and for any \( \alpha \in J_t, c \in R_\alpha \) and \( 0 < \lambda \leq \rho(2^t) \) the following conditions are satisfied:
\[
|B(c, \frac{1}{2}\rho(2^t)) \cap \Delta(R_\alpha, \lambda)| \geq c_1 |B(c, \lambda)| \left( \frac{\rho(2^t)}{\lambda} \right)^\gamma (5.4)
\]
\[
|B \cap B(c, 3\rho(2^t)) \cap \Delta(R_\alpha, 3\lambda)| \leq c_2 |B(c, \lambda)| \left( \frac{r(B)}{\lambda} \right)^\gamma (5.5)
\]
where \( B \) is an arbitrary ball centred on a resonant set with radius \( r(B) \leq 3 \rho(2^t) \). The constants \( c_1 \) and \( c_2 \) are positive and absolute. The constant \( \gamma \) is referred to as the common dimension of \( \mathcal{R} \).

**Definition 5.2.** Suppose that there exists a ubiquitous function \( \rho \) and an absolute constant \( k > 0 \) such that for any ball \( B \subset U \)
\[
\liminf_{t \to \infty} \left| \bigcup_{\alpha \in J_t} \Delta(R_\alpha, \rho(2^t)) \cap B \right| \geq k |B|.
\]
Furthermore, suppose that the intersection conditions (5.4) and (5.5) are satisfied. Then the system \( (\mathcal{R}, \beta) \) is called locally ubiquitous in \( U \) relative to \( \rho \).

Let \( (\mathcal{R}, \beta) \) be a ubiquitous system in \( U \) relative to \( \rho \) and \( \phi \) be an approximating function. Let \( \Lambda(\phi) \) be the set of points \( x \in U \) such that the inequality
\[
\text{dist}(x, R_\alpha) < \phi(\beta_\alpha)
\]
(5.7)
holds for infinitely many $\alpha \in J$.

We are going to use this following ubiquity lemma from [4] in our main proof.

**Lemma 5.1.** Let $\phi$ be an approximating function and $(\mathcal{R}, \beta)$ be a locally ubiquitous system in $U$ relative to $\rho$. Suppose that there is a $0 < \lambda < 1$ such that $\rho(2^{t+1}) < \lambda \rho(2^t)$ $\forall t \in \mathbb{N}$. Then for any $s > \gamma$,

$$
\mathcal{H}^s(\Lambda(\phi)) = \mathcal{H}^s(U) \text{ if } \sum_{t=1}^{\infty} \frac{\phi(2^t)^{s-\gamma}}{\rho(2^t)^{m-\gamma}} = \infty. 
$$

We will also need the strong approximation theorem mentioned in [45].

**Lemma 5.2.** For any $\bar{\epsilon} = (\epsilon_\infty, (\epsilon_p)) \in \mathbb{R}^{2}_{>0}$ satisfying the inequality

$$
\epsilon_\infty \geq \frac{1}{2} \epsilon_p^{-1} p, 
$$

there exists a rational number $r \in \mathbb{Q}$ such that

$$
|r - \xi_\infty|_\infty \leq \epsilon_\infty, \\
|r - \xi_p|_p \leq \epsilon_p, \\
|r|_q \leq 1 \quad \forall \ q \neq p. 
$$

Before we start the proving the main theorem in this section we would like to calculate a covolume formula of certain lattices.

**Lemma 5.3.** Suppose $|y_i|_p \leq 1$ then

$$
\Gamma = \left\{ (q_0, q_1, \ldots, q_n) \in \mathbb{Z}^{n+1} : |q_i|_p \leq \frac{1}{p}, \quad i = 1, \ldots, n \right\}. 
$$

is a lattice in $\mathbb{Z}^{n+1}$ and $\text{Vol}(\mathbb{R}^{n+1}/\Gamma) = p^{j+n}$.

**Proof.** First of all $\Gamma$ is a discrete subgroup of $\mathbb{Z}^{n+1}$. Clearly $(p^i, 0, \cdots, 0) \in \mathbb{Z}^{n+1}$ is in $\Gamma$. Since $|y_i|_p \leq 1$ we may take $q_i \in \mathbb{Z}$ such that

$$
|q_i - py_i|_p \leq \frac{1}{p^i}, 
$$

which implies that $(q_i, 0, \cdots, -p, \cdots, 0) \in \Gamma$ where $-p$ is in $(i+1)$th position. We claim that

$$
\{(p^i, 0, \cdots, 0), (q_i, 0, \cdots, -p, \cdots, 0) : i = 1, \ldots, n\}$$
is a basis of $\Gamma$. The matrix comprising these elements as column vectors as follows

$$
A := 
\begin{bmatrix}
p^j & q_1 & \cdots & q_i & \cdots & q_n \\
0 & -p & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & -p & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & \cdots & -p
\end{bmatrix}.
$$

We want to show that if $m = (m_0, m_1, \cdots, m_n) \in \Gamma$ then there exists $s = (s_0, s_1, \cdots, s_n) \in \mathbb{Z}^{n+1}$ such that $As = m$. Note that

$$
A^{-1}m = \left( -\frac{m_1}{p}, \cdots, -\frac{m_n}{p} \right).
$$

As $m \in \Gamma$ we have that $p|m_i \forall i = 1, \cdots, n$, hence $-\frac{m_i}{p}$ is an integer for all $i$. Now it is enough to show that $p^{j+1}|(m_0p + q_1m_1 + \cdots + q_nm_n)$. Note that

$$
m_0p + m_1q_1 + \cdots + m_nq_n = p(m_0 + m_1y_1 + \cdots + m_ny_n) + m_1(q_1 - y_1p) + \cdots + m_n(q_n - y_np).
$$

Now conclusion follows from $m \in \Gamma$ and (5.12).

Now we will construct a ubiquitous system which will give the main result of this section.

**Theorem 5.3.** Let $x_0 \in U$ be such that $f$ is nice at $x_0$ and satisfies (I3). Then there is a neighbourhood $U_0$ of $x_0$, constants $\kappa_0 > 0$ and $\kappa_1 > 0$ and a collection $\mathcal{R} := (R_F)_{F \in \mathcal{F}_n}$ of sets $R_F \subset \tilde{R}_F \cap U_0$ such that the system $(\mathcal{R}, \beta)$ is locally ubiquitous in $U_0$ relative to $\rho(r) = \kappa_1 r^{(n+1)}$ with common dimension $\gamma := m - 1$, where

$$
\mathcal{F}_n := \left\{ F : U \to \mathbb{R} \mid \begin{array}{c}
F(x) = a_0 + a_1f_1(x) + \cdots + a_nf_n(x), \\
a = (a_0, a_1, \cdots, a_n) \in \mathbb{Z}^{n+1} \setminus 0
\end{array} \right\}
$$

and given $F \in \mathcal{F}_n$

$$
\tilde{R}_F := \{ x \in U : (F + \Theta)(x) = 0 \}
$$

and

$$
\beta : \mathcal{F}_n \to \mathbb{R}^+ : F \to \beta_F = \kappa_0|(a_0, a_1, \cdots, a_n)| = \kappa_0|a|.
$$

**Proof.** Let $\pi : \mathbb{Q}_p^m \to \mathbb{Q}_p^{m-1}$ be the projection map given by

$$
\pi(x_1, x_2, \cdots, x_m) = (x_2, \cdots, x_m),
$$
and let
\[ \tilde{\mathcal{V}} := \pi(\tilde{R}_F \cap U_0), \mathcal{V} = \bigcup_{3\rho(\beta) \text{-balls} \subset \tilde{\mathcal{V}}} \frac{1}{2}B \]  

and
\[ R_F = \begin{cases} \pi^{-1}(\mathcal{V}) \cap \tilde{R}_F & \text{if } |\partial_1(F + \Theta)(x)| > \lambda |\nabla(F + \Theta)(x)| \forall x \in U_0 \\ \emptyset & \text{otherwise.} \end{cases} \]  

where \( 0 < \lambda < 1 \) is fixed.

We claim that the \( R_F \) are resonant sets. The intersection property, namely (5.4) and (5.5) can be checked exactly as in the case of real numbers as accomplished in [4], Proposition 5. We only need to note that implicit function theorem for \( C^l(U) \) in \( \mathbb{R}^n \) was used in [4]. The Implicit function theorem in \( \mathbb{Q}_p \) holds for analytic maps and all our maps have been assumed analytic, so the proof in [4] goes through verbatim.

It remains to check the covering property (5.6) to establish ubiquity. Without loss of generality we will assume that the ball \( U_0 \) in the definition of (5.1) satisfies
\[ \text{diam } U_0 \leq \frac{1}{p}. \]  

From the Definition 5.1 of \( f \) being nice at \( x_0 \), there exist fixed \( 0 < \delta, w < 1 \) such that for any arbitrary ball \( B \subset U_0 \),
\[ \limsup_{Q \to \infty} |\Phi^f(Q, \delta) \cap \frac{1}{2}B| \leq w|\frac{1}{2}B|. \]  

So for sufficiently large \( Q \) we have that
\[ \frac{1}{2}|B \setminus \Phi^f\!\!\!: (Q, \delta)| \geq \frac{1}{2}(1-w)\frac{1}{2}|B| = 2^{-m-1}(1-w)|B|. \]

Therefore it is enough to show that
\[ \frac{1}{2}B \setminus \Phi^f\!\!\!: (Q, \delta) \subset \bigcup_{F \in \mathcal{F}, \beta \leq Q} \Delta(R_F, \rho(Q)) \cap B. \]

Suppose \( x \in \frac{1}{2}B \setminus \Phi^f\!\!\!: (Q, \delta) \). Consider the lattice
\[ \Gamma_x = \left\{ (a_0, a_1, \cdots, a_n) \in \mathbb{Z}^{n+1} : \begin{array}{l} |a_0 + a_1 f_1(x) + \cdots + a_n f_n(x)|_p < \delta Q^{-(n+1)} \\ |a_i|_p \leq \frac{1}{p} \forall 1 \leq i \leq n \end{array} \right\}, \]  

where \( 0 < \lambda < 1 \) is fixed.
and the convex set $K = [-Q, Q]^{n+1}$ in $\mathbb{R}^{n+1}$. Note that
\[ |a_o + a_1 f_1(x) + \cdots + a_n f_n(x)|_p < \delta Q^{-(n+1)} \]
if and only if
\[ |a_o + a_1 f_1(x) + \cdots + a_n f_n(x)|_p \leq p^{\lceil \log p \delta Q^{-(n+1)} \rceil}. \]
So by Lemma 5.3 we have that
\[ \text{Vol}(\mathbb{R}^{n+1}/\Gamma) = p^n p^{-\lceil \log p Q^{-(n+1)} \rceil} \leq Q^{n+1} \frac{p^{n+2}}{p^{\log p \delta Q^{-(n+1)}} - 1} \leq Q^{n+1} \frac{p^{n+2}}{\delta}. \]
\[ (5.21) \]
Using the fact that $x \not\in \Phi_{f_1(Q, \delta)}$ we get the first minima $\lambda_1 = \lambda_1(\Gamma_x, K) > 1$. Therefore using Minkowski’s theorem on successive minima, we have that
\[ 2^{n+1} Q^{n+1} \lambda_1 \lambda_2 \cdots \lambda_{n+1} \leq 2^{n+1} \text{Vol}(\mathbb{R}^{n+1}/\Gamma_x) \leq 2^{n+1} Q^{n+1} \frac{p^{n+2}}{\delta}. \]
This implies that $\lambda_{n+1} \leq \frac{p^{n+2}}{\delta}$. By the definition of $\lambda_{n+1}$ we get $n + 1$ linearly independent integer vectors $a_j = (a_{j,0}, \ldots, a_{j,n}) \in \mathbb{Z}^{n+1}(0 \leq j \leq n)$ such that the functions $F_j$ given by
\[ F_j(y) = a_{j,0} + a_{j,1} f_1(y) + \cdots + a_{j,n} f_n(y) \]
satisfy
\[ \begin{cases} 
|F_j(x)|_p < \delta Q^{-(n+1)} \\
|a_{j,i}|_\infty \leq Q \cdot \frac{p^{n+2}}{\delta} \\
|a_{j,i}|_p \leq \frac{1}{p} \text{ for } 0 \leq i, j \leq n.
\end{cases} \]
\[ (5.22) \]
As $\lambda_1 > 1$ so for every $0 \leq j \leq n$ there exists at least one $0 \leq j^* \leq n$ such that $|a_{j,j^*}|_\infty > Q$.

Now consider the following system of linear equations,
\[ \eta_0 F_0(x) + \eta_1 F_1(x) + \cdots + \eta_n F_n(x) + \Theta(x) = 0 \]
\[ \eta_0 \partial_1 F_0(x) + \eta_1 \partial_1 F_1(x) + \cdots + \eta_n \partial_1 F_n(x) + \partial_1 \Theta(x) = 1 \]
\[ (5.23) \]
\[ \eta_0 a_{0,j} + \cdots + \eta_n a_{n,j} = 0 \quad (2 \leq j \leq n). \]
Since $f_1(x) = x_1$, the determinant of this aforementioned system is
\[ \det(a_{j,i}) \neq 0. \]
Therefore there exists a unique solution to the system, say $(\eta_0, \eta_1, \ldots, \eta_n) \in \mathbb{Q}_p^n$. By the argument above, there is at least one
\(|a_{j,i}|_{\infty} > Q\). Without loss of generality assume \(|a_{0,0}|_{\infty} > Q\). Using the strong approximation Theorem 5.2 we get \(r_i \in \mathbb{Q}\) such that
\[
|r_i - 2p|_{\infty} \leq p \text{ if } a_{i,0} > 0 \text{ otherwise } |r_i + 2p|_{\infty} < p,
\]
\(|r_i - \eta_i|_p \leq 1,
\)
\(|r_i|_q \leq 1 \text{ for prime } q \neq p.\) \tag{5.24}

Now take the function
\[
F(y) = r_0 F_0(y) + r_1 F_1(y) + \cdots + r_n F_n(y)
\]
\[= a_0 + a_1 f_1(y) + \cdots + a_n f_n(y),\]
where
\[
a_i = r_0 a_{0,i} + r_1 a_{1,i} + \cdots + r_n a_{n,i}, \forall i = 0, \ldots, n. \tag{5.26}
\]

We claim that

**Claim 1.** The \(a_i\) are all integers.

From (5.24) and (5.26) we get
\[
|a_i|_q \leq 1, \forall \ i = 0, \ldots, n \text{ for } q \neq p \tag{5.27}
\]
and by (5.24), (5.23) and (5.22) we have
\[
|a_i|_p \leq \max_{j=0,\ldots,n} \{ |\eta_j - r_j|_p |a_{j,i}|_p \} \leq 1 \text{ for } i = 2, \ldots, n. \tag{5.28}
\]
So \(a_i\) are all integers for \(i = 2, \ldots, n\). Now note that
\[
F(x) + \Theta(x) = (r_0 - \eta_0) F_0(x) + \cdots + (r_n - \eta_n) F_n(x).
\]
Therefore we have
\[
|(F + \theta)(x)|_p \leq \delta Q^{-(n+1)}. \tag{5.29}
\]

Again
\[
\partial_1 (F + \Theta)(x) = (r_0 - \eta_0) \partial_1 F_0(x) + \cdots + (r_n - \eta_n) \partial_1 F_n(x) + 1.
\]
Since \(|a_{j,i}|_p \leq \frac{1}{p}\) so \(|\partial_1 F_j(x)|_p \leq \frac{1}{p}\) and thus by (5.24) we get
\[
1 - \frac{1}{p} \leq |\partial_1 (F + \Theta)(x)|_p \leq 1. \tag{5.30}
\]

Now we can show that \(a_1\) and \(a_0\) are also integers. Since \(f_1(y) = y_1\), we have
\[
a_1 = \partial_1 (F + \Theta)(x) - \partial_1 \Theta(x) - \sum_{j=2}^{n} a_j \partial_1 f_j(x) \tag{5.31}
\]
which implies that $|a_1|_p \leq 1$. This together with (5.27) proves that $a_1$ is an integer. We similarly prove that $a_0$ is an integer. We can write
\[
a_0 = (F + \Theta)(x) - \Theta(x) - \sum_{j=1}^{n} a_j f_j(x).
\] (5.32)
This implies that $|a_0|_p \leq 1$ and thus by (5.32) and (5.27) we get that $a_0$ is integer. So the first claim is proved.

Now we look at the infinity norm of the integers $a_i$. By (5.26), (5.22) and (5.24) we have
\[
|a_i|_\infty \leq |r_0 a_{0,i} + \cdots + r_n a_{n,i}|_\infty \leq 3p(n + 1)Q \frac{p^{n+2}}{\delta}
\] for $i = 0, 1, \cdots, n$. (5.33)
By the choice of $r_i$ we have $a_0 > 0$ and using the fact that $Q < |a_{0,0}|_\infty$ we get that $|a_0|_\infty > pQ$ and therefore $|a| > pQ$.

So by (5.33) and the previous observation we get
\[
\frac{1}{3p(n + 1)} p^{-(n+1)} \delta Q < \beta_F = \frac{1}{3p(n + 1)} p^{-(n+2)} \delta |a| \leq Q,
\] (5.34)
here $\kappa_0 = \frac{1}{3p(n + 1)} p^{-(n+2)} \delta$. Note that for all $y \in U_0$ we have
\[
\partial_1 (F + \Theta)(x) = \partial_1 (F + \Theta)(y) + \sum_{j=1}^{m} \Phi_j \partial_1 (F + \Theta) \star (x_j - y_j) \tag{5.35}
\]
where $\star$ is from the coefficients of $x$ and $y$. By using (5.30) and by the fact that diam($U_0$) $\leq \frac{1}{p}$ we have
\[
|\partial_1 (F + \Theta)(y)|_p \geq 1 - \frac{2}{p} \forall y \in U_0. \tag{5.36}
\]
So $F$ satisfies $|\partial_1 (F + \Theta)(x)| > (1 - \frac{2}{p}) |\nabla (F + \Theta)(x)|$ $\forall x \in U_0$ and thus by the constructions $\Delta(R_F, \rho(Q)) \neq \emptyset$.

**Claim 2.** $x \in \Delta(R_F, \rho(Q))$.

We set $r_0 := \text{diam}(B)$ and define the function
\[
g(\xi) := (F + \Theta)(x_1 + \xi, x_2, \cdots, x_d), \text{ where } |\xi|_p < r_0.
\]
Then
\[
|g(0)|_p = |(F + \Theta)(x)|_p < \delta Q^{-(n+1)}
\]
and
\[
|g'(0)|_p = |\partial_1 (F + \Theta)(x)|_p > 1 - \frac{1}{p}. \tag{5.37}
\]
Now applying Newton’s method there exists $\xi_0$ such that $g(\xi_0) = 0$ and $|\xi_0|_p < \frac{p}{(p-1)} \delta Q^{-(n+1)}$. For sufficiently large $Q$ we get $x_{\xi_0} = (x_1 + \xi_0, x_1, \cdots, x_n) \in B$, that $(F + \Theta)(x_{\xi_0}) = 0$ and that $|x - x_{\xi_0}|_p \leq \frac{p}{(p-1)} \delta Q^{-(n+1)}$. Then we will argue exactly same as in [4]. We recall the argument for the sake of completeness. By the Mean Value Theorem we will get

$$|(F + \Theta)(y)|_p \ll Q^{-(n+1)}$$

for any $|y - x_{\xi_0}|_p \ll Q^{-(n+1)}$.

Then by (5.34) and using the same argument as above tells us that for sufficiently large $Q > 0$ the ball of radius $\rho(\beta F)$ centred at $\pi x_{\xi_0}$ is contained in $\tilde{V}$. This ultimately gives $x_{\xi_0} \in R_F$. Since $|x - x_{\xi_0}|_p \leq \frac{p}{(p-1)} \delta Q^{-(n+1)}$ so $x \in \Delta(R_F, \rho(Q))$ for some $F \in \mathcal{F}_n$ such that $\beta F \leq Q$ and this completes the proof of the Theorem. □

5.2. Proof of the main divergence theorem. Now using Theorem 5.3 and lemma 5.1 we can complete the proof of Theorem 5.2.

Fix $x_0 \in U$ and let $U_0$ be the neighbourhood of $x_0$ which comes from (5.3). We need to show that

$$\mathcal{H}^s(W_{(\psi, \Theta)} \cap U_0) = \mathcal{H}^s(U_0)$$

if the series in (5.3) diverges. Consider $\phi(r) := \psi(\kappa_0^{-1} r)$. Our first aim is to show that

$$\Lambda(\phi) \subset W_{(\psi, \Theta)}^r.$$

Note that $x \in \Lambda(\phi)$ implies the existence of infinitely many $F \in \mathcal{F}_n$ such that $\text{dist}(x, R_F) < \phi(\beta F)$. For such $F \in \mathcal{F}_n$ there exists $z \in U_0$ such that $(F + \Theta)(z) = 0$ and $|x - z|_p < \phi(\beta F)$. By Mean value theorem

$$(F + \Theta)(x) = (F + \Theta)(z) + \nabla (F + \Theta)(x) \cdot (x - z) + \sum_{i,j} \Phi_{ij} (F + \Theta)(\star) (x_i - z_i) (x_j - z_j),$$

where $\star$ comes from the coefficients of $x$ and $z$. Then we have that

$$|(F + \Theta)(x)|_p \leq |x - z|_p < \phi(\beta F) = \phi(\kappa_0 |a|) = \Psi(a).$$

Hence $\Lambda(\phi) \subset W_{(\psi, \Theta)}^r$. Now the Theorem will follow if we can show that

$$\sum_{t=1}^{\infty} \frac{\phi(2^t)^{s-m+1}}{\rho(2^t)} = \infty.$$
Observe that
\[
\sum_{t=1}^{\infty} \frac{\psi(2^t)^{s-m+1}}{\rho(2^t)} \asymp \sum_{t=1}^{\infty} \frac{\psi(\kappa_0^{-1}2^t)^{s-m+1}}{\rho(2^t)} \asymp \sum_{t=1}^{\infty} \psi(\kappa_0^{-1}2^t)^{s-m+1} 2^{(n+1)}
\]
\[
\asymp \sum_{t=1, \kappa_0^{-1}2^t < |a| \leq \kappa_0^{-1}2^{t+1}} \psi(\kappa_0^{-1}2^t)^{s-m+1}.
\]

As \(\psi\) is an approximating function so we got that the above series
\[
\Rightarrow \sum_{t=1, \kappa_0^{-1}2^t < |a| \leq \kappa_0^{-1}2^{t+1}} \psi(|a|)^{s-m+1} \asymp \sum_{a \in \mathbb{Z}^{n+1} \setminus 0} \psi(|a|)^{s-m+1}
\]

\[
= \sum_{a \in \mathbb{Z}^{n+1} \setminus 0} \Psi(a)^{s-m+1} = \infty.
\]

This completes the proof of the Theorem.

6. Concluding Remarks

6.1. Some extensions. An interesting possibility is an investigation of the function field case. In [23], the function field analogue of the Baker-Sprind\'zuk conjectures were established and similarly it should be possible to prove the function field analogue of the results in the present paper.

6.2. Affine subspaces. In [30], analogues of the Baker-Sprind\'zuk conjectures were established for affine subspaces. In this setting, one needs to impose Diophantine conditions on the affine subspace in question. Subsequently, Khintchine type theorems were established (see [22, 24]), we refer the reader to [25] for a survey of results. Recently, in [10], the inhomogeneous analogue of Khintchine’s theorem for affine subspaces was established in both convergence and divergence cases. It would be interesting to consider the \(S\)-adic theory in the context of affine subspaces.

6.3. Friendly Measures. In [31] a category of measures called Friendly measures was introduced and the Baker-Sprind\'zuk conjectures were proved for friendly measures. Friendly measures include volume measures on nondegenerate manifolds, so the results of [31] generalize those of [32], but also include many other examples including measures supported on certain fractal sets. In [12], the inhomogeneous version of the Baker-Sprind\'zuk conjectures were established for a class of measures called strongly contracting which include friendly measures. It should be possible to prove an \(S\)-adic inhomogeneous analogue of the Baker-Sprind\'zuk conjectures for strongly contracting measures.
REFERENCES

[1] V. Beresnevich, A Groshrev type theorem for convergence on manifolds, Acta Math. Hungar. 94 (2002), no. 1-2, 99–130.
[2] Bernik, V., Budarina, N., Dickinson, D.: Simultaneous Diophantine approximation in the real, complex and p-adic fields. Math. Proc. Camb. Phil. Soc., 149, 193–216 (2010).
[3] V. Beresnevich, V. Bernik, H. Dickinson and M. M. Dodson, On linear manifolds for which the Khintchin approximation theorem holds, Vestsi Acad Navuk Belarusi. Ser. Fiz. - Mat. Navuk (2000), 14–17 (Belorussian).
[4] D. Badziahin, V. Beresnevich and S. Velani, Inhomogeneous theory of dual Diophantine approximation on manifolds, Advances in Mathematics 232 (2013) 1–35.
[5] V.V. Beresnevich, V.I. Bernik, E.I. Kovalevskaya, On approximation of p-adic numbers by p-adic algebraic numbers, Journal of Number Theory 111 (2005), 33–56.
[6] V. Beresnevich, V. Bernik, D. Kleinbock and G. Margulis, Metric Diophantine approximation: the Khintchine-Groshev theorem for non-degenerate manifolds, Moscow Mathematical Journal 2:2 (2002), 203–225.
[7] V.V. Beresnevich, E.I. Kovalevskaya, On Diophantine approximations of dependent quantities in the p-adic case, Mat. Zametki 73:1 (2003), 22–37; translation: Math. Notes 73:1-2 (2003), 21–35.
[8] V. Bernik, H. Dickinson, M. M. Dodson, Approximation of real numbers by values of integer polynomials, Dokl. Nats. Akad. Nauk Belarusi 42 (1998), no. 4, 51–54, 123.
[9] V. Beresnevich, D. Dickinson and S. Velani, Measure theoretic laws for lim sup sets, Mem. Amer. Math. Soc., 179 (2006).
[10] V. Beresnevich, A. Ganguly, A. Ghosh and S. Velani, Inhomogeneous dual Diophantine approximation on affine subspaces, https://arxiv.org/abs/1711.08559.
[11] V. Bernik, D. Kleinbock and G. A. Margulis, Khintchine type theorems on manifolds: the convergence case for the standard and multiplicative versions, Internat. Math. Res. Notices 9 (2001), pp. 453–486.
[12] V. Beresnevich, S. Velani, An inhomogeneous transference principle and Diophantine approximation, Proc. Lond. Math. Soc. 101 (2010) 821–851.
[13] ________, Simultaneous inhomogeneous Diophantine approximations on manifolds. Fundam. Prikl. Mat. 16 (2010), no. 5, 3–17.
[14] V. Bernik, H. Dickinson, J. Yuan, Inhomogeneous Diophantine approximation on polynomials in $Q_p$, Acta Arith. 90 (1999), no. 1, 37–48.
[15] V.I. Bernik, E.I. Kovalevskaya, Simultaneous inhomogeneous Diophantine approximation of the values of integral polynomials with respect to Archimedean and non-Archimedean valuations, Acta Math. Univ. Ostrav. 14:1 (2006), 37–42.
[16] N. Budarina, D. Dickinson, Inhomogeneous Diophantine approximation on integer polynomials with non-monotonic error function, Acta Arith. 160 (2013), no. 3, 243–257.
[17] N. Budarina and E. Zorin, Non-homogeneous analogue of Khintchine’s theorem in divergence case for simultaneous approximations in different metrics, Siaulai Math. Semin. 4(12) (2009), 21–33.
[18] Y. Bugeaud, *Approximation by algebraic integers and Hausdorff dimension*, J. Lond. Math. Soc., 65 (2002), pp. 547–559.
[19] J. W. S. Cassels, *An introduction to Diophantine Approximation*, Cambridge University Press, Cambridge, 1957.
[20] Shreyasi Datta, TIFR thesis, in preparation.
[21] H. Dickinson, M. M. Dodson, J. Yuan, *Hausdorff dimension and p-adic Diophantine approximation*, Indag. Math. (N.S.) 10 (1999), no. 3, 337–347.
[22] A. Ghosh, *A Khintchine-type theorem for hyperplanes*, J. London Math. Soc. 72, No.2 (2005), pp. 293–304.
[23] A. Ghosh, *Metric Diophantine approximation over a local field of positive characteristic*, Journal of Number Theory, 124 (2007), no. 2, 454–469.
[24] A. Ghosh, *Diophantine approximation and the Khintchine-Groshev theorem*, Monatsh. Math 163 (2011), no. 3, 281–299.
[25] A. Ghosh, *Diophantine approximation on subspaces of $\mathbb{R}^n$ and dynamics on homogeneous spaces*, to appear in the Handbook of Group Actions III/IV, Editors, L. Ji, A. Papadopoulos, S. T. Yau.
[26] A. Ghosh and A. Marnat, *On Diophantine transference principles*, https://arxiv.org/abs/1610.02161. To appear in Mathematical Proceedings of the Cambridge Philosophical Society.
[27] A. Groshev, *Une théorème sur les systèmes des formes linéaires*, Dokl. Akad. Nauk SSSR 9 (1938), pp. 151–152.
[28] Alan Haynes, *The metric theory of p-adic approximation*, Int. Math. Res. Not. IMRN 2010, no. 1, 18–52.
[29] A. Khintchine, *Einige Sätze über Kettenbrüche, mit Anwendungen auf die Theorie der Diophantischen Approximationen*, Math. Ann. 92, (1924), pp. 115–125.
[30] D. Kleinbock, *Extremal subspaces and their submanifolds*, Geom. Funct. Anal. 13, (2003), No 2, pp.437–466.
[31] D. Kleinbock, E. Lindenstrauss, B. Weiss, *On fractal measures and Diophantine approximation*, Selecta Math. (N.S.) 10 (2004), no. 4, 479–523.
[32] D. Kleinbock and G. A. Margulis, *Flows on homogeneous spaces and Diophantine Approximation on Manifolds*, Ann Math 148, (1998), pp.339–360.
[33] D. Kleinbock and G. Tomanov, *Flows on S-arithmetic homogeneous spaces and applications to metric Diophantine approximation*, Comm. Math. Helv. 82 (2007), 519–581.
[34] E.I. Kovalevskaya, *A metric theorem on the exact order of approximation of zero by values of integer polynomials in $\mathbb{Q}_p$*, Dokl. Nats. Akad. Nauk Belarusi 43:5 (1999), 34–36 (in Russian).
[35] S. Lang, *Algebra*, Second edition. Addison-Wesley Publishing Company, Advanced Book Program, Reading, MA, 1984.
[36] E. Lutz, *Sur les approximations diophantiennes linéaires P-adiques*, Actualités Sci. Ind., no. 1224, Hermann & Cie, Paris, 1955.
[37] A. Mohammadi, A. Salehi Golsefidy, *S-arithmetic Khintchine-type theorem*, Geom. Funct. Anal. 19 (2009), no. 4, 1147–1170.
[38] A. Mohammadi, A. Salehi Golsefidy, *Simultaneous Diophantine approximation on non-degenerate p-adic manifolds*, Israel J. Math. 188 (2012), 231–258.
[39] W. Schmidt, *Metrische Sätze über simultane Approximation abhängiger Grössen*, Monatsch. Math. 68 (1964), 154–166.
[40] W.H. Schikhof, Ultrametric Calculus. An Introduction to \( p \)-adic Analysis, Cambridge Studies in Advanced Mathematics 4, Cambridge University Press, Cambridge (1984).

[41] V. G. Sprindžuk, Achievements and problems in Diophantine Approximation theory, Russian Math. Surveys 35 (1980), pp. 1–80.

[42] V. G. Sprindžuk, Metric theory of Diophantine approximations, John Wiley & Sons, New York-Toronto-London, 1979.

[43] A. E. Ustinov, Inhomogeneous approximations on manifolds in \( \mathbb{Q}_p \), Vestsī Nats. Akad. Navuk Belarusī Ser. Fiz.-Mat. Navuk 2005, no. 2, 30–34, 124.

[44] A. E. Ustinov, Approximation of complex numbers by values of integer polynomials, Vestsī Nats. Akad. Navuk Belarusī Ser.Fiz.-Mat. Navuk 1 (2006) 9–14, 124.

[45] Zelo, Dmitrij Simultaneous approximation to real and \( p \)-adic numbers, Thesis (Ph.D.) University of Ottawa (Canada). 2009. 147 pp. ISBN: 978-0494-59539-8 ProQuest LLC

School of Mathematics, Tata Institute of Fundamental Research, Mumbai, 400005, India

E-mail address: shreya@math.tifr.res.in, ghosh@math.tifr.res.in