Holography for 2d $\mathcal{N} = (0,4)$ quantum field theory

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Abstract: We study the correspondence between AdS$_3$ massive IIA supergravity vacua and two-dimensional $\mathcal{N} = (0,4)$ quiver quantum field theories. After categorizing all kinds of gravity solutions, we demystify the ones that seem to reflect anomalous gauge theories. In particular, we prove that there are bound states of D-branes on the boundary of the space which provide the dual quiver theory with exactly the correct amount of flavor symmetry in order to cancel its gauge anomalies. Then we propose that the structure of the field theory should be complemented with additional bifundamental matter, which we argue it may only be $\mathcal{N} = (4,4)$ hypermultiplets. Finally, we construct a BPS string configuration and we use the old and new supersymmetric matter to build its dual UV operator. During this holographic synthesis, we uncover some interesting features of the quiver superpotential and associate the proposed operator with the same classical mass of its dual BPS string.

Keywords: AdS$_3$, two dimensions, superfields, quivers, D-branes, bound states.

Dedicated to the memory of Muhammad Al-Arab and Muhammad Gulzar.
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1 Introduction

The AdS/CFT correspondence constitutes a primo realization of the holographic principle while it ties string theory to the most well-studied particle theories we possess. In other words, besides being a conceptual breakthrough on its own right, holography brings strong confidence that a complete quantum theory of gravity shines upon the physics of the superstring. Nonetheless, the power of this duality does not limit itself in supporting quantum gravity but also unravels the properties of certain supersymmetric quantum field theories that otherwise are yet out of our reach through the standard methods or techniques.

While over the years many type II supergravity solutions have made their appearance in the holographic arena, there is a certain kind that has recently been poping up more frequently and has become quite popular. These are supergravity backgrounds whose entirety of fields is defined by functions of the coordinates of the internal manifolds and are dual to supersymmetric quiver gauge theories. Studying those backgrounds ultimately boils down to understanding their defining functions. The dual physics of these vacua is generally described by supersymmetric conformal field theories (SCFTs), which for $d < 4$ are assumed to be strongly coupled IR fixed points that flow to better-understood UV quiver field theories through the renormalization group (RG) equations. The latter are defined on supersymmetric multiplets of fundamental fields, whose interactions are usually well-defined and provide an understandable particle theory.

SCFTs exist exclusively in $d < 7$ dimensions [1] and there has been intensive work on all of their diversity, both field theoretically and holographically. In six dimensions, an infinite family of $\mathcal{N} = (0, 1)$ theories has been discussed in [2–13]. In five dimensions, solutions in a variety of supersymmetry were analyzed in [14–21]. For $\mathcal{N} = 2$ supersymmetry in four dimensions there has been a fruitful study in [23–29], while three dimensional $\mathcal{N} = 4$ theories were discussed in [30–34].

The case of AdS$_3$ supergravity solutions is somewhat unique. Three dimensional gravity as well as the algebra of two dimensional field theory make the study of AdS$_3$ holography of particular interest and this is reflected on the rich literature regarding the subject, some representatives of which are [35–50].

Another family of such AdS$_3$ solutions was recently introduced in [50–53]. These massive IIA vacua are associated with D2-D4-D6-D8 Hanany-Witten brane set-ups [56] and were first build in [50]. The D2 and D6-branes exist as fluxes and they are dual to gauge symmetries, while the D4 and D8-branes live explicitly in the background and provide dual flavor symmetries. In [52] a particular class of them that exhibits the local geometry $\text{AdS}_3 \times S^2 \times \text{CY}_2 \times \mathbb{R}$ was distinguished and was proposed to be dual to two-dimensional quiver quantum field theories with $\mathcal{N} = (0, 4)$ supersymmetry. Some holographic aspects of these quivers were studied in [54, 55]. Those are the theories that we are about to consider.

The defining functions of a supergravity solution render the form of the fields on the gravity side of the correspondence, while they accordingly shape the exact structure of the dual quiver field theory. In order to validate the correspondence and study the whole range
of its potential, one should explore the various properties of these functions and confirm that every single time they make perfect sense on their dual field-theoretical attribution. This makes up the starting point of this article, where we take the most unusual choice of such defining functions which seems to give an anomalous dual quantum field theory. By carefully focusing on the right regions of the supergravity background we discover D-branes that are realized as global symmetries in the dual quiver structure, providing exactly the flavors needed to cancel the apparent gauge anomalies. Due to strong Ramond-Ramond (RR) fluxes on the boundary of the space these D-branes come exclusively in bound states, forming polarizations that provide flavor symmetries in an idiosyncratic way.

Observing the quiver structure of the theories under consideration, we realize that there must be some linking multiplets missing. Such multiplets bind color D2 with flavor D4-branes and color D6 with flavor D8-branes, while it is shown that those may only be \( \mathcal{N} = (4, 4) \) hypermultiplets corresponding to suspended superstrings between D2 and D4-branes or D6 and D8-branes in the ancestral Hanany-Witten set-up.

The existence of this new matter complements the quiver structure, while it seems to be also vital in the construction of the dual operator for a particular BPS string state. To be precise, after picking a semiclassical string configuration connecting two stacks of D-branes in the background, we prove that this is a BPS state and propose a string of scalar fields as its dual UV operator. We argue that this is a unique choice of a dual operator and, while two-dimensional scalars have mass dimension zero implying a vanishing conformal dimension for that operator, we conclude that the latter property is attained non-perturbatively. That is, we bring to the surface the superpotential of the UV quiver theory to find interactions between the scalars inside the operator, supporting the idea of a totally non-perturbative anomalous dimension at the IR of the RG flow.

Finally, we find that scalars inside the vector superfields should obtain a vacuum expectation value (VEV) through a Fayet-Iliopoulos term due to the U(1) theory inside each U(N) gauge group. Superpotential interactions between the vector and hypermultiplets then dictate that bifundamental matter acquires a mass, ultimately associating the dual UV operator with a classical mass equal to that of the BPS string. Since the operator mass is a sum of all the individual scalar field masses, this renders the operator very much alike to a classical bound state of particles dual to a bound string state between D-branes.

The plan of this paper is as follows. In Section 2 we review the massive IIA supergravity backgrounds and quantum field theory first constructed in [50]. We also give a brief but complete summary of two-dimensional \( \mathcal{N} = (0, 4) \) quantum field theory that is useful in understanding gauge anomalies, R-current charges and superpotentials between multiplets, all basic ingredients for the self-consistency of the present work. In Section 3 we study special solutions of vacua that naively give anomalous quiver theories and show how these are canceled by flavor symmetries produced by dielectric branes on the boundary of the space. In Section 4 we illustrate that new matter should be added in the structure of the field theory in the form of \( \mathcal{N} = (4, 4) \) hypermultiplets. Finally, in Section 5 we construct a BPS string soliton and propose a dual operator, which both seem to exhibit the same classical mass.
2 AdS\(_3\) massive IIA vacua vs \(\mathcal{N} = (0,4)\) theory

2.1 The supergravity solutions

In [50] a new family of AdS\(_3\) massive IIA supergravity solutions with \(\mathcal{N} = (0,4)\) supersymmetry was introduced. A subclass of these solutions with local geometry AdS\(_3\)\(\times\)S\(_2\)\(\times\)CY\(_2\)\(\times\)I\(_\rho\) was conjectured in [51–53] to be dual to \(\mathcal{N} = (0,4)\) quiver quantum field theories in two dimensions. These vacua have an NS NS sector, in string frame,

\[
\begin{align*}
\mathcal{F}_0 &= h_8', \\
\mathcal{F}_2 &= -\frac{1}{2} (h_8 - h_8' (\rho - 2\alpha' k\pi)) \text{vol}(S^2), \\
\mathcal{F}_4 &= \left( \partial_\rho \left( \frac{uu'}{2h_4} \right) + 2h_8 \right) d\rho \wedge \text{vol}(\text{AdS}_3) - h_4' \text{vol}(\text{CY}_2)
\end{align*}
\]

where \(u, h_4, h_8\) are functions of the coordinate \(\rho\), defining this family of supergravity backgrounds. Note that we also allow for large gauge transformations \(B_2 \rightarrow B_2 + \pi k \text{vol}_{S^2}\), every time we cross a \(\rho\)-interval \([2\pi k, 2\pi (k+1)]\), for \(k = 0, \ldots, P\). The RR sector reads

\[
\begin{align*}
\hat{F}_0 &= h_8' , \\
\hat{F}_2 &= -\frac{1}{2} (h_8 - h_8' (\rho - 2\alpha' k\pi)) \text{vol}(S^2) , \\
\hat{F}_4 &= \left( \partial_\rho \left( \frac{uu'}{2h_4} \right) + 2h_8 \right) d\rho \wedge \text{vol}(\text{AdS}_3) - h_4' \text{vol}(\text{CY}_2)
\end{align*}
\]

where \(\hat{F} = e^{-B_2} F\) is the Page flux. These functions are locally constrained as

\[
h_4'' = h_8'' = u'' = 0
\]

where the first two equations come from the Bianchi identities, while the last comes from supersymmetry. This results in piecewise linear functions

\[
h_4(\rho) = \begin{cases} 
\alpha_0 + \frac{\beta_0}{2\pi} \rho & 0 \leq \rho \leq 2\pi \\
\alpha_k + \frac{\beta_k}{2\pi} (\rho - 2\pi k) & 2\pi k \leq \rho \leq 2\pi (k+1) \ k = 1, \ldots, P - 1 \\
\alpha_P + \frac{\beta_P}{2\pi} (\rho - 2\pi P) & 2\pi P \leq \rho \leq 2\pi (P + 1)
\end{cases}
\]

\[
h_8(\rho) = \begin{cases} 
\mu_0 + \frac{\nu_0}{2\pi} \rho & 0 \leq \rho \leq 2\pi \\
\mu_k + \frac{\nu_k}{2\pi} (\rho - 2\pi k) & 2\pi k \leq \rho \leq 2\pi (k+1) \ k = 1, \ldots, P - 1 \\
\mu_P + \frac{\nu_P}{2\pi} (\rho - 2\pi P) & 2\pi P \leq \rho \leq 2\pi (P + 1)
\end{cases}
\]

while \(u = a + b\rho\) globally, for supersymmetry to be preserved. Note that \(P, \alpha_k, \mu_k\) have to be large for the supergravity limit to be trusted.
While only one of the functions $h_4, h_8$ needs to be zero at the endpoints of the $\rho$-interval — in order for the space to properly close on that coordinate — the study in [51, 52] focused exclusively on solutions where both of these defining functions vanish at the endpoints, i.e. for $\alpha_0 = \mu_0 = a = 0$ and $\nu_P = -\mu_P, \beta_P = -\alpha_P$ in the above definitions (2.4) and (2.5). It is easier to refer to the way we choose these functions schematically, the latter particular choice being represented by an example in Figure 1.

This particular choice of backgrounds — where $h_4$ and $h_8$ are both zero at the endpoints of the $\rho$-dimension — start in a smooth fashion on this coordinate as the non-Abelian T-duals of AdS$_3 \times S^3 \times$ CY$_2$. Near the endpoint $\rho = 2\pi(P+1) - x$ with $x \to 0$, on the other hand, the space becomes

$$ds^2 \sim \frac{s_1}{x} ds^2_{AdS_3} + s_3 ds^2_{CY_2} + \frac{x}{s_1} (dx^2 + s_1 s_2 ds_S^2) , \quad e^{-4\Phi} = s_4 x^2 \quad (2.6)$$

where $s_i$ are constants. According to the extremal $p$-brane solutions, classified in Appendix A, this space is a superposition of O2/O6 planes, where the O2 are smeared over O6.

Demanding that the NS NS fields are continuous across the $\rho$-intervals imposes constraining conditions on the various constants of the $h_4$ and $h_8$ functions. The simple solution to these continuity equations is

$$\mu_k = \sum_{j=0}^{k-1} \nu_j , \quad \alpha_k = \sum_{j=0}^{k-1} \beta_j \quad (2.7)$$

These conditions guarantee the continuity of the functions $h_4, h_8$. Their derivatives may, however, present jumps. These jumps imply discontinuities in the RR sector, which are interpreted as explicit branes in the background that modify the Bianchi identities.

---

**Figure 1:** An example of piecewise linear functions $h_4, h_8$ and of $u$, defining a particular supergravity background.
In order to gain a better grip on the parameters of the system, let us consider the RR charges on the intervals \([2\pi k, 2\pi(k + 1)]\). For \(\alpha' = g_s = 1\), a Dp-brane is charged under \(Q_{Dp} = (2\pi)^{p-7} \int_{\Sigma_{8-p}} \hat{F}_{8-p}\), thus in our set-up they read

\[
\begin{align*}
Q_{D8} &= 2\pi F_0 = 2\pi h'_8 = \nu_k \\
Q_{D6} &= \frac{1}{2\pi} \int_{S^2} \hat{F}_2 = h_8 - h'_8(\rho - 2\pi k) = \mu_k \\
Q_{D4} &= \frac{1}{8\pi^3} \int_{CY_2} \hat{F}_4 = \beta_k \\
Q_{D2} &= \frac{1}{32\pi^5} \int_{CY_2 \times S^2} \hat{F}_6 = h_4 - h'_4(\rho - 2\pi k) = \alpha_k
\end{align*}
\]  

(2.8)

and \(Q_{NS} = \frac{1}{4\pi^2} \int_{\rho \times S^2} H_3 = 1\), while we used that \(\text{vol}(CY_2) = 16\pi^4\). These results imply that \(\alpha_k, \beta_k, \mu_k, \nu_k\) are integers. A study of the Bianchi identities in the next section reveals that no explicit D2 and D6 branes are present in the geometry, just their fluxes\(^1\). This associates their amount, \(\alpha_k\) and \(\mu_k\) respectively, with the ranks of the (color) gauge groups in the dual field theory. On the other hand, as restated, D8 and D4 branes do exist in the geometry and modify the Bianchi identities by a delta function. Thus, \(\beta_k\) and \(\nu_k\) are associated with the ranks of the (flavor) global symmetries of the dual field theory.

2.2 Hanany-Witten brane set-up

The above story is conjectured to be generated by a certain Hanany-Witten brane set-up [56]. However, in this case the D-branes are not distributed across flat space as usual but along flat dimensions and a CY\(_2\) manifold instead, as indicated by Table 1.

Our family of supergravity backgrounds (2.1) comes to be as the near-horizon limit of this brane set-up, given always a large portion of each of the D-branes. Nevertheless, not all D-branes are explicitly present in the near-horizon limit a Hanany-Witten set-up; some are there while others exist only as RR fluxes. In order to clarify this, we turn our attention to the Bianchi identities.

We begin by noticing that \(dF_0 = h''_8 d\rho\) and \(d\hat{F}_4 = h''_4 d\rho \wedge \text{vol}(CY_2)\) where, according to the BPS equations (2.3), \(h''_4 = h''_8 = 0\) at a generic point along \(\rho\). However, \(h_4\) and \(h_8\) are piecewise functions, given by (2.4) and (2.5), which means that at the points where their slope changes we get

\[
\begin{align*}
h''_4 &= \sum_{k=1}^P \left( \frac{\beta_{k-1} - \beta_k}{2\pi} \right) \delta(\rho - 2k\pi) \\
h''_8 &= \sum_{k=1}^P \left( \frac{\nu_{k-1} - \nu_k}{2\pi} \right) \delta(\rho - 2k\pi)
\end{align*}
\]  

(2.9)

\(^1\)This is true when the worldvolume gauge field on the D8, D4 branes is absent. When it is on, as we are about to see, there is D6 and D2 flavor charge induced on the D8’s and D4’s.
Table 1: $\frac{1}{8}$-BPS brane set-up, generator of our supergravity backgrounds. The dimensions $(x_0, x_1)$ are where the 2d CFT lives. The dimensions $(x_2, ..., x_5)$ span the CY$_2$, on which the D6 and the D8-branes are wrapped. The coordinate $x_6$ is associated with $\rho$. Finally $(x_7, x_8, x_9)$ are the transverse directions realizing an SO(3)-symmetry associated with the isometries of S$^2$.

This gives the source equations

\[
dF_0 = \sum_{k=1}^{P} \left( \frac{\nu_{k-1} - \nu_k}{2\pi} \right) \delta(\rho - 2k\pi) \, d\rho
\]

\[
d\hat{F}_4 = d\hat{f}_4 = \sum_{k=1}^{P} \left( \frac{\beta_{k-1} - \beta_k}{2\pi} \right) \delta(\rho - 2k\pi) \, d\rho \wedge \text{vol}(\text{CY}_2)
\]

(2.10)

indicating that there are localized D4 and/or D8 branes at points $\rho = 2k\pi$, whenever the slope changes between the intervals $[k - 1, k]$. In fact, the D4-branes are smeared over CY$_2$, while note that we use $f_p$ for the magnetic part of a RR flux $F_p$.

The identities left read

\[
d\hat{F}_2 = d\hat{f}_2 = \frac{1}{2} h''_2 (\rho - 2k\pi) \, d\rho \wedge \text{vol}(S^2)
\]

\[
d\hat{F}_6 = d\hat{f}_6 = \frac{1}{2} h''_4 (\rho - 2k\pi) \, d\rho \wedge \text{vol}(S^2) \wedge \text{vol}(\text{CY}_2)
\]

(2.11)

where using that $x\delta(x) = 0$ on (2.9), we deduce that there are no sources present for the D6 and D2-branes. This is because of the large gauge transformations of the Kalb-Ramond field. Were it not for those, D6 and D2-charge would be induced on the D8 and D4-branes, respectively.
The above source equations suggest that the D2 and D6-branes will play the role of color branes, while the D4 and D8-branes that of flavor branes. Since gauge transformations vanish at infinity, it is the gauge fields fluctuating on the D4 or D8-branes in the bulk that are realized as global (flavor) symmetries in the dual CFT.

In the above source equations, however, we have not considered the gauge fields living on the D4 and D8 branes. Switching on a gauge field $\tilde{f}_2$ on both kinds of D-branes, we form the gauge invariant field strength $F_2 = B_2 + \lambda \tilde{f}_2$, where $\lambda = 2\pi l_s^2$, and the Bianchi identities now become

\[
\begin{align*}
\text{d} \tilde{f}_2 &= \lambda \tilde{f}_2 \wedge \text{d}F_0 \\
\text{d} \tilde{f}_4 &= h_4'' \text{d}\rho \wedge \text{vol}(CY_2) + \lambda^2 \tilde{f}_2 \wedge \tilde{f}_2 \wedge \text{d}F_0 \\
\text{d} \tilde{f}_6 &= \lambda \tilde{f}_2 \wedge (h_4'' \text{d}\rho \wedge \text{vol}(CY_2)) + \frac{\lambda^3}{3!} \tilde{f}_2 \wedge \tilde{f}_2 \wedge \tilde{f}_2 \wedge \text{d}F_0
\end{align*}
\]

(2.12)

In regard to the gauge field dynamics, it being of order $l_s^2$, we may neglect it and keep only the zeroth order contribution, that is the Bianchi identities (2.10) that give only D8 and D4-branes. Nonetheless, in the upcoming sections, we are going to encounter cases where the gauge field does become important and generates some interesting phenomena.

2.3 $\mathcal{N} = (0, 4)$ SCFT

The conjecture of [52] is that the above family of supergravity backgrounds is dual to a set of two dimensional SCFTs with $\mathcal{N} = (0, 4)$ supersymmetry. These SCFTs are considered to be the low energy fixed points on the RG flows of well defined quantum field theories. Here, we just present the basics of those theories, ultimately aiming to cancel gauge anomalies that shall arise but also to unravel some interesting properties on the structure of our particular quiver theory.

2.3.1 The UV regime

Traditionally, extended supersymmetric theories are best realized through constituent, minimal supersymmetric multiplets. $\mathcal{N} = (0, 4)$ supersymmetry is no different and boils down to $\mathcal{N} = (0, 2)$ superfields, which we now introduce.

Gauge multiplet This is a real superfield, $\mathcal{V}$, which comprises of a gauge field $A$, an adjoint-valued complex left-handed fermion $\zeta_-$ and a real auxiliary field $D$. The component expansion is

\[
\mathcal{V} = (A_0 - A_1) - i\theta^+ \bar{\zeta}_- - i\bar{\theta}^+ \zeta_- + \theta^+ \bar{\theta}^+ D
\]

(2.13)
where the bosonic and fermionic degrees of freedom (d.o.f) do not have to match (and they don’t), since the right-moving supercharges of $\mathcal{N} = (0, 2)$ supersymmetry do not act on left-handed spinors.

Having introduced the two-dimensional vector superfield, we can also define the left-handed super covariant derivative

$$D_- = D_0 - D_1 = \partial_0 - \partial_1 - i\mathcal{V}$$  \hspace{1cm} (2.14)

which may be used to write down the kinetic term for a chiral superfield and the superfield strength

$$\Upsilon = [\bar{D}_+, D_-] = -\zeta_+ - i\theta^+(D - iF_{01}) - i\theta^+\bar{\theta}^+(D_0 + D_1)\zeta_-$$  \hspace{1cm} (2.15)

This satisfies $\bar{D}_+ \Upsilon = 0$, which means that $\Upsilon$ is a spinor left-chiral superfield. Equivalently, it is a special case of a Fermi multiplet, which will be introduced below. The standard kinetic action for the gauge multiplet reads

$$S_{\text{gauge}} = \frac{1}{8g^2} \text{Tr} \int d^2 x \, d^2 \theta \, \bar{\Upsilon} \Upsilon$$

$$= \frac{1}{g^2} \text{Tr} \int d^2 x \left( \frac{1}{2} F_{01} + i\bar{\zeta}_-(D_1 + D_0)\zeta_- + D^2 \right)$$  \hspace{1cm} (2.16)

**Chiral multiplet** $\mathcal{N} = (0, 2)$ chiral multiplets contain a right-moving fermion $\psi_+$ and a single complex scalar $\phi$, each transforming in the same representation of the gauge group. These synthesize a complex-valued bosonic chiral superfield $\Phi$ with component expansion

$$\Phi = \phi + \theta^+ \psi_+ - i\theta^+ \bar{\theta}^+ (D_0 + D_1) \phi$$  \hspace{1cm} (2.17)

which, given the two dimensional supercovariant derivative $D_+ = \frac{\partial}{\partial \theta^+} - i\bar{\theta}^+ (D_0 + D_1)$ with $D_{0,1} = \partial_{0,1} + iA_{0,1}$, satisfies

$$\bar{D}_+ \Phi = 0$$  \hspace{1cm} (2.18)

yielding that $\Phi$ is a left chiral superfield. The kinetic terms for the gauged chiral multiplet are now given by

$$S_{\text{chiral}} = \int d^2 x \, d^2 \theta \, \bar{\Phi} (D_0 - D_1) \Phi$$

$$= \int d^2 x \left( -|D_\mu \phi|^2 + i\bar{\psi}_+ (D_0 - D_1) \psi_+ - i\bar{\phi} \zeta_- \psi_+ + i\bar{\psi}_+ \zeta_- \phi + \bar{\phi} D \phi \right)$$  \hspace{1cm} (2.19)
**Fermi multiplet** $\mathcal{N} = (0, 2)$ theories have the property that left-moving spinors are not necessarily accompanied by propagating, bosonic superpartners. A spinor of this kind, $\psi$, sits in an anticommuting superfield $\Psi$, obeying the condition

$$\bar{D}_+ \Psi = E$$

where, considering $\bar{D}_+^2 = 0$, $E$ is some chiral superfield

$$\bar{D}_+ E = 0$$

We call $\Psi$ a Fermi multiplet. In our context, we always take $E = E(\Phi_i)$ to be a holomorphic function of the chiral superfields $\Phi_i$. This function must be chosen so that $E(\Phi_i)$ transforms in the same manner as $\Psi$ under any symmetries. Being a chiral superfield, $E$ expands as

$$E(\Phi_i) = E(\phi_i) + \theta^+ \frac{\partial E}{\partial \phi_i} \psi_{+i} - i \theta^+ \bar{\theta}^+ (D_0 + D_1) E(\phi_i)$$

which gives the component expansion of the superfield,

$$\Psi = \psi_- - \theta^+ G - i \theta^+ \bar{\theta}^+ (D_0 + D_1) \psi_- - \bar{\theta}^+ E(\phi_i) + \theta^+ \bar{\theta}^+ \frac{\partial E}{\partial \phi_i} \psi_{+i}$$

where $G$ is a complex auxiliary field. The kinetic terms for the Fermi multiplet are given by

$$S_{\text{Fermi}} = \int d^2 x d^2 \theta \bar{\Psi} \Psi$$

$$= \int d^2 x \left( i \bar{\psi}_-(D_0 + D_1) \psi_- + |G|^2 - |E(\phi_i)|^2 - \bar{\psi}_- \frac{\partial E}{\partial \phi_i} \psi_{+i} + \bar{\psi}_{+i} \frac{\partial E}{\partial \phi_i} \psi_- \right)$$

Note that the holomorphic function $E(\phi_i)$ appears as potential terms in the Lagrangian and thus its particular choice, along with superpotential terms that are to be introduced, determines the interactions of the theory.

**Superpotentials** Each Fermi multiplet contains an auxiliary complex scalar $G$, which transforms as a total derivative, while the same is true for the $G$-term of any anticommuting chiral superfield

$$\int d^2 x d\theta^+ (\ldots) |_{\theta^+ = 0} + \text{h.c.}$$

where $(\ldots)$ is annihilated by $\bar{D}_+$. Given that a product of chiral superfields is itself a chiral
superfield and, also, that a product of an anticommuting and a commuting (scalar) superfield
is itself an anticommuting superfield, an obvious candidate for (...) is
\[
\int d^2x \, d\theta^+ (\Psi J)|_{\bar{\theta}^+ = 0} + \text{h.c.}
\]  
(2.26)

where \( J = J(\Phi_i) \) is a holomorphic function of the \( \Phi_i \)'s, while \( \mathcal{D}_+ \Psi = \mathcal{D}_+ J = 0 \). But this
only works for the special case of a chiral Fermi superfield, i.e. with \( E = 0 \). Therefore, we
consider multiple Fermi superfields \( \Psi_a \) corresponding to multiple scalar chiral superfields \( J^a \),
together building the sum \( \sum_a \Psi_a J^a \) that has a supersymmetric \( G \)-term only when it is a
chiral superfield, i.e. iff
\[
\mathcal{D}_+ (\Psi_a J^a) = 0
\]  
(2.27)

This, in turn, is satisfied if
\[
E \cdot J \equiv \sum_a E_a J^a = 0
\]  
(2.28)

which constitutes a highly strong constraint for a supersymmetric theory. This shows that
there is some tension when introducing both \( E \)-type potentials and \( J \)-type potentials associ-
ated to the same Fermi multiplet. Overall, we obtain the supersymmetric term
\[
S_J = \int d^2x \, d\theta^+ \sum_a \Psi_a J^a(\Phi_i)|_{\bar{\theta}^+} + \text{h.c.}
\]  
(2.29)

which expands in
\[
S_J = \sum_a \int d^2x \left( G_a J^a(\phi_i) + \sum_i \psi^- a \frac{\partial J^a}{\partial \phi_i} \psi^+ i \right) + \text{h.c.}
\]  
(2.30)

After integrating out the auxiliary fields \( G_a \), this results in a potential term \( \sim |J^a(\phi_i)|^2 \).
(2.30) is usually referred to as the superpotential in \( \mathcal{N} = (0, 2) \) theories. In what follows, we
will also use the notation
\[
\mathcal{W} = \Psi_a J^a(\Phi)
\]  
(2.31)

We see that there are two ways to construct potential terms in theories with \( \mathcal{N} = (0, 2) \) supers-
symmetry. Both are associated to Fermi multiplets and both involve holomorphic functions,
\( E(\phi_i) \) and \( J(\phi_i) \). The attachment between these two through \( E \cdot J = 0 \) when multiple Fermi
and chiral multiplets are present, decides for the particular interactions in the theory. But
to see how this plays out we first introduce the very protagonists of our quiver theories, that
is the \( \mathcal{N} = (0, 4) \) supersymmetric multiplets. Then, at this end of this section, we realize the
superpotential for the building blocks of our quiver theory.
Two dimensional $\mathcal{N} = (0, 4)$ supersymmetry has four real right-moving supercharges that rotate in the $(2, 2)_+ \uparrow$ representation of a $\text{SO}(4)_R \cong \text{SU}(2)_R \times \text{SU}(2)_R$ R-symmetry, where the plus sign indicates the chirality under the $\text{SO}(1, 1)$ Lorentz group. The superfields in this kind of theories are the following.

$\mathcal{N} = (0, 4)$ vector multiplet  Since in two dimensions the gauge field is not propagating it is natural that two-dimensional $\mathcal{N} = (0, 4)$ vector superfields are composed of left-handed spinors, which don’t transform under right-moving supersymmetry. Thus, a $\mathcal{N} = (0, 4)$ vector superfield consists of a $\mathcal{N} = (0, 2)$ vector and an adjoint-valued $\mathcal{N} = (0, 2)$ Fermi multiplet $\Theta$.

Besides the gauge field, there are a pair of left moving complex fermions, $\zeta^a$, $a = 1, 2$, transforming as $(2, 2)_- \uparrow$ under the R-symmetry and a triplet of auxiliary fields transforming as $(3, 1)$. The Fermi superfield obeys $\bar{D}_+ \Theta = E_\Theta$ with $E_\Theta$ depending on the matter content, i.e. the chiral superfields present in the theory.

$\mathcal{N} = (0, 4)$ hypermultiplet  There are two distinct ways to couple matter fields to a $\mathcal{N} = (0, 4)$ vector multiplet (essentially to its constituent $\mathcal{N} = (0, 2)$ Fermi multiplet). For this reason, we distinguish between hypermultiplets and twisted hypermultiplets. A $\mathcal{N} = (0, 4)$ hypermultiplet consists of a pair of $\mathcal{N} = (0, 2)$ chiral multiplets, $\Phi$ and $\tilde{\Phi}$, transforming in conjugate representations of the gauge group. The pair of complex scalars transforms as $(2, 1)$ under the R-symmetry, while the pair of right-handed spinors transforms as $(1, 2)_+ \uparrow$.

$\mathcal{N} = (0, 4)$ twisted hypermultiplet  The $\mathcal{N} = (0, 4)$ twisted hypermultiplet also consists of a pair of $\mathcal{N} = (0, 2)$ chiral multiplets, $\Sigma$ and $\tilde{\Sigma}$, transforming in conjugate representations of the gauge group. The difference from the hypermultiplet lies in the R-symmetry transformation of the fields, the different R-charge being enforced by the coupling to the Fermi field $\Theta$. The pair of scalars transform as $(1, 2)$ while the pair of right-moving fermions transforms as $(2, 1)_+ \uparrow$.

$\mathcal{N} = (0, 4)$ Fermi multiplet  We define the $\mathcal{N} = (0, 4)$ Fermi multiplets to consist of a pair of $\mathcal{N} = (0, 2)$ Fermi multiplets, $\Gamma$ and $\tilde{\Gamma}$, transforming in conjugate representations of the gauge group. The left-handed spinors transform as $(1, 1)_- \uparrow$ under the R-symmetry.

$\mathcal{N} = (0, 2)$ Fermi multiplet  Finally, we note that it is possible to have a single $\mathcal{N} = (0, 2)$ Fermi multiplet which is consistent with $\mathcal{N} = (0, 4)$ supersymmetry. For this to happen, the chiral fermion should be a singlet under the $\text{SO}(4)_R$ R-symmetry. Of course, the coupling to other matter multiplets must also respect this.
As we are about to see, our quantum field theory also contains \( \mathcal{N} = (4, 4) \) superfields that decompose under \( \mathcal{N} = (0, 4) \) supersymmetry into their \( \mathcal{N} = (0, 4) \) superfield constituents. The \( \mathcal{N} = (4, 4) \) vector multiplet splits into an \( \mathcal{N} = (0, 4) \) vector multiplet and an adjoint-valued \( \mathcal{N} = (0, 4) \) twisted hypermultiplet. The chiral superfields \( \Sigma \) and \( \tilde{\Sigma} \) inside the twisted hypermultiplet couple to the Fermi multiplet \( \Theta \) inside the \( \mathcal{N} = (0, 4) \) vector superfield. Finally, a \( \mathcal{N} = (4, 4) \) hypermultiplet decomposes into an \( \mathcal{N} = (0, 4) \) hypermultiplet, \( \Phi \) and \( \tilde{\Phi} \), and an \( \mathcal{N} = (0, 4) \) Fermi multiplet, \( \Gamma \) and \( \tilde{\Gamma} \).

From the \( SU(2)_R \times SU(2)_R \) R-symmetry of the \( \mathcal{N} = (0, 4) \) theory, we single out a \( U(1)_R \) inside one \( SU(2)_R \) and give the \( U(1)_R \) charge of each fermion in the above multiplets. This will be used below to calculate the global \( R \)-current anomaly but also in the last section, where we use fundamental fields to build an operator of certain \( R \)-charge. For the \( \mathcal{N} = (0, 4) \) vector multiplet we have that the left-handed fermion inside the vector has \( R[\zeta_-] = +1 \) while the same holds for the left-handed fermion inside the Fermi multiplet, i.e. \( R[\psi_-] = +1 \). On the contrary, both right-handed fermions inside the \( \mathcal{N} = (0, 4) \) twisted hypermultiplet have \( R[\psi_+] = 0 \). For both right-handed fermions inside the \( \mathcal{N} = (0, 4) \) hypermultiplet we have \( R[\psi_+] = -1 \). Finally, the fermion inside the \( \mathcal{N} = (0, 2) \) Fermi multiplet is uncharged under R-symmetry.

### 2.3.2 Gauge anomalies

In two dimensions quantum anomalies are one-loop-exact products of two point current correlations. The calculation is quite simple [57, 58] and, given a non-Abelian symmetry acting on \( \psi_+^a \) right-moving and \( \lambda_-^a \) left-moving fermions with charges \( Q_{Ri} \) and \( Q_{La} \) respectively, it gives

\[
< J^A_{\mu} J^B_{\nu} > \sim \text{Tr}[\gamma_3 J^A_{\mu} J^B_{\nu}] = \text{Tr}[T^A T^B] \left( \sum_i Q_{Ri}^2 - \sum_a Q_{La}^2 \right) \delta_{\mu\nu}
\]

where \( T^A \) are the non-Abelian generators. For \( SU(N) \), the algebra has a metric

\[
\text{Tr}[T^A T^B] = N\delta^{AB} \quad \text{or} \quad \text{Tr}[T^A T^B] = \frac{1}{2}\delta^{AB}
\]

depending on the generators being in the adjoint or in the (anti-) fundamental representation, respectively. Obviously, no mixing between non-Abelian currents takes place.

Since gauge anomalies need to be always canceled for a consistent quantum field theory, chiral theories like ours require us to carefully study the anomaly contribution of each multiplet. Considering the field content previously presented, the \( SU(N) \) anomaly coming from the \( \mathcal{N} = (0, 2) \) superfields comes as follows:

- Vector superfield: they are in the adjoint representation of the gauge group \( SU(N) \) and thus they contribute with a factor of \(-N\).
Figure 2: The building block of our quiver field theories. The solid black line represents a $\mathcal{N} = (4, 4)$ hypermultiplet, the maroon line a $\mathcal{N} = (0, 4)$ hypermultiplet and the dashed line represents a $\mathcal{N} = (0, 2)$ Fermi multiplet. Inside the node representing an SU($N$) gauge theory lives a $\mathcal{N} = (4, 4)$ vector multiplet. The groups SU($P$), SU($Q$) and SU($R$) can be gauge or global symmetries.

- Chiral superfield: if they are in the adjoint representation of the gauge group SU($N$) they contribute with a factor of $N$. If they are in the (anti-) fundamental representation they contribute with a factor of $\frac{1}{2}$.

- Fermi superfield: if they are in the adjoint representation of the gauge group SU($N$) they contribute with a factor of $-N$. If they are in the (anti-) fundamental representation they contribute with a factor of $\frac{1}{2}$.

Having established all the gauge anomaly contributions, the next step is to use them on our kind of quantum field theories and see if there are any restrictions coming off the anomaly cancellation condition. This is a simple task since the proposed holographic duality describes a particular building block of supersymmetric multiplets that supports our quiver field theories. That is the one on Figure 2.

Each SU($N$) gauge theory living on $N$ D2 or D6 color branes is represented by a gauge node that yields a $\mathcal{N} = (4, 4)$ vector multiplet. In $\mathcal{N} = (0, 2)$ language, each gauge node includes a vector, a Fermi and two twisted chiral multiplets in the adjoint representation of SU($N$). A gauge node connects with other (gauge or flavor) nodes which in turn represent theories of (gauge or global) symmetry groups SU($P$), SU($R$) and SU($Q$), altogether a quiver network that reflects strings stretched between branes.

In our notation of Figure 2, the SU($N$) gauge node connects to the SU($P$) (gauge or flavor) node through a $\mathcal{N} = (4, 4)$ hypermultiplet. In $\mathcal{N} = (0, 2)$ language, each such hypermultiplet includes two Fermi and two chiral multiplets. Since there are $NP$ kinds of strings between the SU($N$) and the SU($P$) brane stacks, we realize $2NP$ of each of these Fermi and chiral multiplets. The SU($N$) gauge node also connects to a SU($R$) node, through a $\mathcal{N} = (0, 4)$ hypermultiplet. That is through two $\mathcal{N} = (0, 2)$ chiral multiplets. Since there are $NR$ kinds
of strings between the SU($N$) and the SU($R$) brane stacks, we realize $2NR$ chiral multiplets connecting the two nodes. In the same manner, the SU($N$) gauge node connects to a SU($Q$) node, through $NQ \mathcal{N} = (0, 2)$ Fermi multiplets.

All that being said, we may now calculate the overall anomaly of the gauge group SU($N$) and impose that it cancels. Of course, the same job is to be done for each gauge group a quiver field theory. For SU($N$), as in Figure 2, the contributions come from the multiplets that couple to its gauge current, that is:

- $\mathcal{N} = (4, 4)$ vector multiplet: the adjoint fields contribute as $2N - N - N = 0$. This is as it must since this is a vectorial multiplet, with equal amount of right and left-moving fermions.
- $\mathcal{N} = (4, 4)$ hypermultiplet: the bifundamental fields connecting to SU($P$) contribute as $2NP(\frac{1}{2} - \frac{1}{2}) = 0$. Again, this is expected since this hypermultiplet is too vectorial.
- $\mathcal{N} = (0, 4)$ hypermultiplet: this connects to SU($R$) and contributes as $2NR\frac{1}{2} = NR$
- $\mathcal{N} = (0, 2)$ Fermi multiplet: it connects to SU($Q$) and contributes as $NQ(-\frac{1}{2})$.

Requiring the gauge anomaly cancellation, we reach the condition

$$2R = Q$$ (2.34)

which analogously must hold for each gauge group in a consistent quiver gauge theory.

If all the above is to hold, then the anomaly cancellation condition must agree with the dual situation on the supergravity side of the story. That is, since anomaly cancellation requires certain relationships between the ranks of the gauge and global symmetry groups, the amounts of branes (represented field theoretically by these ranks) in the supergravity side should be in total agreement with (2.34). This is indeed the case. Choosing an arbitrary supergravity solution where both $h_4$ and $h_8$ vanish at the end points of the $\rho$-dimension as

$$h_4(\rho) = \begin{cases} \frac{\beta_0}{2\pi} \rho & 0 \leq \rho \leq 2\pi \\ \frac{\alpha_P}{2\pi} (\rho - 2\pi k) & 2\pi k \leq \rho \leq 2\pi (k + 1) \ k = 1, \ldots, P - 1 \\ \frac{\alpha_P}{2\pi} (\rho - 2\pi P) & 2\pi P \leq \rho \leq 2\pi (P + 1) \end{cases}$$ (2.35)

$$h_8(\rho) = \begin{cases} \frac{\nu_0}{2\pi} \rho & 0 \leq \rho \leq 2\pi \\ \frac{\nu_P}{2\pi} (\rho - 2\pi k) & 2\pi k \leq \rho \leq 2\pi (k + 1) \ k = 1, \ldots, P - 1 \\ \frac{\nu_P}{2\pi} (\rho - 2\pi P) & 2\pi P \leq \rho \leq 2\pi (P + 1) \end{cases}$$ (2.36)

and $u = \frac{b_0}{2\pi} \rho$, we make use of the Page charges (2.8) to decode these functions into portions of D-branes that give a dual quiver theory as in Figure 3.
Applying the anomaly cancellation condition (2.34) on the gauge currents of the first gauge nodes of the above quiver chain we find

\[
F_0 + \nu_0 + \nu_1 = 2\nu_0 \quad \Rightarrow \quad F_0 = \nu_0 - \nu_1
\]
\[
\tilde{F}_0 + \beta_0 + \beta_1 = 2\beta_0 \quad \Rightarrow \quad \tilde{F}_0 = \beta_0 - \beta_1
\]

which are precisely the results that we get from the Bianchi identities (2.10) for the portions of the D4 and D8-branes, validating further the proposed duality.

2.3.3 U(1) R-current anomaly

Non critical for the consistency of the gauge theory but as much as instructive is the anomaly produced by the R-symmetry current. Focusing on the SU(\(N\)) gauge theory of our building block and considering the U(1)\(_R\) R-charges we found before, we consider the anomaly contributions through \(\text{Tr}[\gamma_3 Q_i^2]\) on each multiplet:

- For the fields in the adjoint representation of SU(\(N\)), the only contribution comes from the fermions inside the vector and Fermi multiplets. This amounts to a contribution of \(-2(N^2 - 1)\). This coincides with (minus) twice the number of \(\mathcal{N} = (0,4)\) vector multiples in SU(\(N\)).
• The contribution coming from the bifundamentals joining SU(N) with SU(P) is $2NP$, due to both of the right-handed fermions inside the each hypermultiplet. This is the number of $\mathcal{N} = (0,4)$ hypermultiplets in that link.

• The contribution coming from the fields running inside the maroon line, joining SU(N) with SU(R), is accordingly $2NR$, once again counting the number of $\mathcal{N} = (0,4)$ hypermultiplets running on that connection.

• Finally, the fields running over the dashed line do not contribute as the left-handed fermion is uncharged under R-symmetry.

Summarizing, we find that the total R-anomaly reads

$$\text{Tr}[\gamma r Q_r^2] \sim 2(n_{hyp} - n_{vec})$$

which is proportional to the difference between the hypermultiplets and the vector superfields of the building block. As derived in [51, 59] this anomaly is linked to the central charge of the theory

$$c = 6(n_{hyp} - n_{vec})$$

which will be useful to us in a forthcoming section, where we want to add matter in the theory while leaving this charge intact.

### 2.3.4 Quiver superpotential

As promised, we now realize a superpotential on our quiver theory by focusing on its building block, given by Figure 2. In particular, we just take one simple connection of it, that is the link between a hypermultiplet and a vector superfield. All other links can be deduced as generalizations of this connection.

Through $\mathcal{N} = (0,2)$ supersymmetric eyes, a $\mathcal{N} = (4,4)$ vector superfield breaks into a vector multiplet $\mathcal{V}$, a Fermi multiplet $\Theta$ and two (twisted) chiral multiplets $\Sigma, \tilde{\Sigma}$. On the other hand, a $\mathcal{N} = (4,4)$ hypermultiplet breaks into two chiral multiplets $\Phi, \tilde{\Phi}$ and two Fermi multiplets $\Gamma, \tilde{\Gamma}$. First things first, considering transformation properties under the R-symmetry, the Fermi multiplet $\Theta$ inside the vector superfield may only be defined by the holomorphic function

$$E_\Theta = [\Sigma, \tilde{\Sigma}]$$

and by the superpotential

$$W_\Theta = \tilde{\Phi} \Theta \Phi$$
On the contrary, the R-symmetry representations furnishing the $\mathcal{N} = (4, 4)$ hypermultiplet, define its Fermi multiplets as

$$E_{\Gamma} = \Sigma \Phi \quad E_{\tilde{\Gamma}} = -\tilde{\Phi} \Sigma$$

and let for the superpotential

$$W_{\Gamma} + W_{\tilde{\Gamma}} = \tilde{\Phi} \tilde{\Sigma} \Gamma + \tilde{\Gamma} \tilde{\Sigma} \Phi$$

In reality, it is not just the R-symmetry representations that we took into account to shape the above functions, but also the condition $E \cdot J = 0$ that constrains our theory. Here, it is obviously satisfied

$$E \cdot J = \tilde{\Phi} [\Sigma, \tilde{\Sigma}] \Phi + \tilde{\Phi} \tilde{\Sigma} \Sigma \Phi - \tilde{\Phi} \Sigma \tilde{\Sigma} \Phi = 0$$

and points out all the possible field interaction. Given the potential terms $|E_a(\phi_i)|^2$ and $|J_a(\phi_i)|^2$ in the action, all the above produce an interesting interactive sector in our theory that is going to become decisively important in the last section.

3 Dielectric branes on the boundary

The case studied in [51, 52] and in the previous section is dedicated to supergravity solutions defined by functions $h_4, h_8$ that vanish at the endpoints of the $\rho$-dimension, as in Figure 1. Nevertheless, this is just one choice among many.

To classify all other possible kinds of solutions we must first consider the restrictions that apply on the functions $h_4, h_8$ and $u$. First, at each endpoint of the $\rho$-dimension, we always need at least one of $h_4$ or $h_8$ to be zero in order for the space to properly close on that coordinate. This implies that, while one of the functions vanishes at an endpoint, the other may be non-zero. Secondly, the physics significantly changes depending on whether the function $u$ is linear or just a constant, both being legitimate solutions of the BPS equation $u''(\rho) = 0$.

While all those novel cases are totally valid as supergravity solutions (i.e. they satisfy the equations of motion (2.3)), a particular ambiguity arises in their dual quiver field theories. The ambiguity is that the gauge anomalies for these new quivers do not seem to cancel. In particular, it is the color nodes on the edges of the quivers that – naively – seem anomalous.

A promising answer to this riddle arises by focusing back on the supergravity side and observing the limiting geometry at the endpoints of the $\rho$-dimension (where the physics is dual to the aforementioned color nodes on the quiver edges). On those limiting vicinities, in contrast with the original paradigm of the previous section where the limiting space is either smooth or has O-planes, we now find D-branes. This is promising because explicit D-branes correspond to flavor symmetries (i.e. flavor nodes) that may contribute in the necessary way to cancel the gauge anomalies. Indeed, this is exactly what happens. But let us better realize all this through some solid examples.
3.1 Linear $u(\rho)$

As restated, the physics of the supergravity solutions significantly changes depending on whether the function $u$ is linear or just a constant. Therefore, we split our analysis into two discrete parts, with regards to this property. Instead of writing down all possible cases with linear $u$, we just classify them through Figures 4-9.
**Figure 10**: A simplified version of Figure 4. Here, \( h_8 \) starts and closes with a vanishing value, while \( h_4 \) starts at zero but finishes at a non-zero value.

**Figure 11**: This is the naive quiver dual to the background defined by (3.1), (3.2). In reality, there is one more flavor node, canceling the gauge anomalies for the last \( h_8 \) (D6) gauge node.

### 3.1.1 Example I

Figures 4-9 qualitatively classify all the possible aforementioned cases with linear \( u \). Nevertheless, the figures are only schematic, by which we mean that the functions \( h_4, h_8 \) are unnecessarily complicated. Thus, we shall just study simplified versions of them.

Firstly, let us study the case represented by Figure 4. For this cause, we consider Figure 10 instead which falls into the same class of backgrounds but is way simpler. This is the class of backgrounds where \( h_8 \) vanishes at the end of the \( \rho \)-interval, while \( h_4 \) does not. Therefore, according to Figure 10 the defining functions read

\[
h_4(\rho) = \begin{cases} 
\frac{\beta}{2\pi} \rho & 2\pi k \leq \rho \leq 2\pi (k+1) \\
\alpha - \frac{\beta}{2\pi} \frac{\rho}{(\rho - 2\pi(P+1))} & 2\pi P \leq \rho \leq 2\pi (P+1)
\end{cases} \quad (3.1)
\]

\[
h_8(\rho) = \begin{cases} 
\frac{\nu}{2\pi} \rho & 2\pi k \leq \rho \leq 2\pi (k+1) \\
\nu P \frac{\rho}{(2\pi(P+1) - \rho)} & 2\pi P \leq \rho \leq 2\pi (P+1)
\end{cases} \quad (3.2)
\]

The background defined by these functions is naively dual to the quiver theory given
by Figure 11. The fact that this quiver is not the right one can be easily seen by observing the last D6 gauge node, i.e. the one with gauge rank $P \nu$. For this node the gauge anomalies do not cancel. On the contrary, anomaly cancellation would occur if the gauge node was to connect with a flavor node of rank $\alpha$ (via a $\mathcal{N} = (0, 2)$ Fermi multiplet, as usual).

Through the standard brane set up, no additional branes seem to exist. However, as we explained, we shall focus on the vicinity of the supergravity background that is dual to the problematic D6 gauge node and see whether there is something interesting there. That is, we focus near the end point $\rho = 2\pi (P + 1) - x$, for $x \to 0$, where the geometry and the dilaton read

$$ds^2 = s_1 \sqrt{x} ds_{\text{AdS}_3} + \frac{1}{\sqrt{x}} \left( s_2 dx^2 + s_3 x^2 ds^2_{S^2} + s_4 ds^2_{\text{CY}_2} \right), \quad e^\Phi = s_5 x^{-\frac{1}{4}} \quad (3.3)$$

with $s_i$ real constants. As foreseen, we reached an interesting outcome since this background corresponds to D2-branes on AdS$_3$ and smeared over CY$_2$. Being explicit branes, these D2’s contribute to the flavor structure of the quiver theory and, in principle, they should cancel the gauge anomalies on the last D6 gauge node.

However, the Bianchi identities yield no explicit D2-branes in our supergravity construction. On the contrary, according to the violation of these identities, the $h_4$ function — that appears here to feed the boundary of the space with D2-branes — can only give rise to D4-branes. The situation gets demystified by studying the D4-brane sources, i.e. its WZ action. Thinking in a constructive way, since the $h_4$ function does give rise to four-branes, we look upon their WZ term

$$S_{\text{WZ}}^{D4} = \mu_4 \int \text{Tr} C_5^{el} + C_3^{el} \wedge F_2 \quad (3.4)$$

where $C^{el}$ is the electric part of a potential form and $F_2 = B_2 + \lambda \tilde{f}_2$ is the gauge invariant field strength that incorporates the D4 worldvolume gauge field. Dimensional analysis here implies $\lambda = 2\pi l_s^2$. The first term sources standard D4-branes, while the second reflects bound states of D2-branes. Taking into account the $F_4$ flux through (2.2), we deduce

$$F_6^{el} = - \frac{u^2 h_4' h_8}{h_4 (4h_4 h_8 + (u')^2)} \text{vol(AdS}_3) \wedge \text{vol(S}_2) \wedge d\rho \quad (3.5)$$

which yields that

$$C_5^{el} \to 0 \quad \text{for} \quad \rho \to \rho_f \quad (3.6)$$

leaving the second term in (3.4) as the sole player in the game. That is, the four-branes in the boundary of the space exist exclusively as bound states of D2-branes.
In fact, since $C_5^\ell \to 0$ there are no D4-branes at $\rho \to \rho_f$ and hence no D4 gauge field either. This leaves $F_2 \to B_2$, which for $h_8 \to 0$ reads

$$B_2 \to \alpha' \pi k \text{vol}(S^2)$$

(3.7)

and thus the second source term in (3.4) becomes

$$S_{WZ}^{D4} = (P + 1) \mu_2 \int \text{Tr} C^\ell_3$$

(3.8)

indicating that each four-brane on the boundary lives as a bound state of $P + 1$ D2-branes. Such a bound state is a four-brane worldvolume, which should be realized as a collective behavior of D2-branes. Hence, the gauge theory on such a four-brane originates from open strings on and between the D2-branes.

What we just studied holds for one four-brane and we may argue that the $P + 1$ D2-branes form a $U(P + 1)$ gauge theory, under certain conditions. In reality, however, we have multiple coincident four-branes. Since it is the D2-branes that condense into these four-branes, strings might end on D2-branes that belong to different four-branes. That is, despite having strings ending on D2-branes, it is the $N_4$ bound four-brane worldvolumes that coincide and form a non-Abelian $U(N_4)$ theory. These are the four-branes indicated by

$$d \hat{f}_4 = h_4'' d \rho \wedge \text{vol}(\text{CY}_2)$$

(3.9)

whose flavor gauge group is what we are after and anticipate of it canceling the gauge anomalies in the quiver theory. Their smearing over CY$_2$ also justifies the smearing of the D2-branes on the boundary. Therefore, eventually, we have to count these bound four-branes on the endpoint of the $\rho$-interval.

To this end, we have to handle things delicately. This is because the number of four-branes is associated with $h_4'$ and a derivative is not well defined on the endpoint of a closed interval. Therefore, we shall demand that $h_4|_{\rho_f} = 0$, so that the derivative becomes well defined near the endpoint $\rho_f$. This is not a physical requirement of any sort; it is just a trick to calculate the D-branes at the end of the space. Thus we now have the derivative

$$h_4'\big|_{\rho \to \rho_f} = \lim_{x \to 0} \frac{h_4(\rho_f) - h_4(\rho_f - x)}{x} = \lim_{x \to 0} \frac{-\alpha}{x}$$

(3.10)

The essence of differentiation is to realize how a function changes. In our particular context, the measure of this change is associated with the number of branes at a point. Since the background is defined on a closed interval, it makes sense to realize the absence of branes out of it as a shift of the defining function to a vanishing value. Stated otherwise, we exchange emptiness for a zero.
Figure 12: This is the actual quiver dual to the background defined by (3.1), (3.2). Here, the extra four-brane flavor node cancels the gauge anomalies for the last $h_8$ (D6) gauge node.

and, in order to calculate all the four-branes on the endpoint, the D4 Page charge in (2.8) has to be integrated\(^3\) towards $\rho_f$ as

$$N_4 = -\int_{\rho_f-x}^{\rho_f} h_4' = \alpha \quad (3.11)$$

Bottom line, we found $\alpha$ four-branes sitting on the endpoint of the $\rho$-interval, which are bound states of D2-branes.

Being a realization of the $h_4$ function, these four-branes feed the D6 color chain of the quiver with flavor. In particular, this $U(\alpha)$ is dual to a global symmetry in the quiver theory, which gives exactly the flavor needed in order to cancel the gauge anomalies of the last D6 color chain node. This is all visualized in Figure 12, where the quiver theory is now consistent.

Focusing on the starting point $\rho = 0$ of the $\rho$-interval, the background becomes the non-Abelian T-dual of $\text{AdS}_3 \times S^3 \times \text{CY}_2$, which yields no D-branes there. This is to be expected from the supergravity side, since everything is obviously smooth there. But even by just looking at the field theory, the quiver is non-anomalous at its beginning (and now everywhere for that matter), which means that no additional D-branes should be there. If there were any, these would contribute with flavor and spoil the anomaly cancellation balance.

3.1.2 Example II

Next, let us study the case represented by Figure 6. Again, we consider Figure 13 instead which falls into the same class of backgrounds but is way simpler. This is the class of backgrounds where $h_8$ vanishes at the beginning of the $\rho$-interval while $h_4$ does not and vice versa at the final end point. Therefore, according to Figure 13 the defining functions read

\(^3\)The trick we applied on the $h_4$ function, forms a situation where the branes appear smeared near the endpoint, instead of being localized with a delta function as with the rest of the D4-brane stacks along the $\rho$-dimension. This is merely an artifact of our particular handling that is resolved just by adding up (integrating over) all the branes near that endpoint.
Figure 13: A simplified version of Figure 6. Here, \( h_8 \) starts with a vanishing value and ends at a non-zero one, while the exact opposite is true for \( h_4 \).

\[
\begin{align*}
  h_4(\rho) &= \begin{cases} 
    \alpha + \frac{\beta}{2\pi} \rho & 2\pi k \leq \rho \leq 2\pi(k + 1) \quad k = 0, \ldots, P - 1 \\
    \frac{\beta P + \alpha}{2\pi} (2\pi(P + 1) - \rho) & 2\pi P \leq \rho \leq 2\pi(P + 1)
  \end{cases} \\
  h_8(\rho) &= \begin{cases} 
    \mu - \frac{\nu}{2\pi} \rho & 2\pi k \leq \rho \leq 2\pi(k + 1) \quad k = 0, \ldots, P - 1 \\
    \frac{\nu P - \mu}{2\pi} (\rho - 2\pi(P + 1)) & 2\pi P \leq \rho \leq 2\pi(P + 1)
  \end{cases}
\end{align*}
\]  

Figure 14: This is the naive quiver dual to the background defined by (3.12), (3.13). In reality, there are two more flavor nodes, canceling the gauge anomalies for the first D6 and the last D2 gauge nodes.

The background defined by these functions is – naively – dual to the quiver theory given by Figure 14. Again, this quiver cannot be the right one and this can be seen by observing the first D6 and the last D2 gauge nodes, i.e. the ones with gauge rank \( \nu \) and \( P\beta + \alpha \) respectively. For these nodes the gauge anomalies do not cancel. On the contrary, anomaly cancellation would occur if they respectively connected to a flavor node of rank \( \alpha \) and a flavor node of rank \( \mu \).
We go on and focus on the dual geometric vicinities of these "anomalous" gauge nodes, anticipating again to find the necessary portions of D-branes that cancel the gauge anomalies. We find that near the beginning of the $\rho$-interval, $\rho \to 0$, the background reads

$$ds^2 = s_1 \sqrt{x} ds^2_{AdS_3} + \frac{1}{\sqrt{x}} \left( s_2 d\rho^2 + s_3 \rho^2 ds^2_{S^2} + s_4 ds^2_{CY_2} \right), \quad e^\Phi = s_5 x^{-\frac{1}{4}} \quad (3.14)$$

with $s_i$ real constants, which corresponds to D2-branes on AdS$_3$ and smeared over CY$_2$. Near the endpoint, $\rho = 2\pi (P + 1) - x$, for $x \to 0$, the backgrounds reads

$$ds^2 = \sqrt{x} \left( m_1 ds^2_{AdS_3} + m_2 ds^2_{CY_2} \right) + \frac{1}{\sqrt{x}} \left( m_3 d\rho^2 + m_4 \rho^2 ds^2_{S^2} \right), \quad e^\Phi = m_5 x^\frac{3}{4} \quad (3.15)$$

with $m_i$ real constants, which corresponds to D6-branes on AdS$_3 \times$ CY$_2$. Being explicit branes, these D2 and D6 branes contribute to the flavor structure of the quiver theory and, in principle, they should cancel the gauge anomalies.

As with example I, the D2-branes at $\rho \to 0$ come to be as bound states of four-branes. However, concerning the background at $\rho \to \rho_f$, the Bianchi identities yield that the $h_8$ function only gives rise to D8-branes and not to D6-branes. Therefore, for the endpoint of the $\rho$-interval where the $h_8$ function is non-vanishing, we look up the D8 WZ term

$$S^{D8}_{WZ} = \mu_8 \int \text{Tr} C^e_9 + C^e_7 \wedge F_2 + C^e_5 \wedge F_2 \wedge F_2 + C^e_3 \wedge F_2 \wedge F_2 \wedge F_2 \quad (3.16)$$

where the first term sources standard D8-branes and the rest reflect eight-branes as bound states of D6, D4 and D2-branes, respectively. Taking into account the $F_0$ flux in (2.2), we deduce

$$F^{e}_{10} = F_{10} = -\frac{u^2 h_4 h_8}{h_8 (4h_4 h_8 + (u')^2)} \text{vol}(\text{AdS}_3) \wedge \text{vol}(S^2) \wedge d\rho \quad (3.17)$$

which yields that

$$C^e_9 \to 0 \quad \text{for} \quad \rho \to \rho_f \quad (3.18)$$

Since $C^e_9$ vanishes at the boundary, there are no D8-branes there and thus no D8 gauge field, reducing the WZ action to

$$S^{D8}_{WZ} = \mu_8 \int \text{Tr} C^e_7 \wedge B_2 = (P + 1) \mu_6 \int \text{Tr} C^e_7 \quad (3.19)$$
Figure 15: This is the actual quiver dual to the background defined by (3.12), (3.13). Here, the extra D2 and D6 flavor nodes cancel the gauge anomalies for the first D6 and the last D2 gauge nodes.

This is an important observation because $C_{1}^{el}$ and $C_{3}^{el}$ blow up for $h_{4} \to 0$. Which means that, if there was a D8 gauge field, the D4 and D2 bound states coming from the last two terms of (3.16) would dominate the game and we know that this is not what happens.

We conclude that every eight-brane on the boundary exists exclusively as a bound state of $P+1$ D6-branes. Again, despite having strings ending on D6-branes, it is the $N_{8}$ bound eight-brane worldvolumes that coincide and form a $U(N_{8})$ gauge theory. These are the eight-branes indicated by

$$dF_{0} = h_{8}'' d\rho$$

whose flavor gauge group should cancel the gauge anomalies in the quiver theory. Thus, it is these bound eight-branes on the boundary that we have to count.

Following the same procedure as before, we find that

$$N_{4} \big|_{\rho=0} = \alpha$$
$$N_{8} \big|_{\rho=\rho_{f}} = \mu$$

The four-branes, as bound states of D2-branes, feed the beginning of the D6 color chain of the quiver with flavor. Accordingly, the eight-branes, as bound states of D6 branes, feed with flavor the end of the D2 color chain of the quiver. As expected, they both give exactly the flavor nodes needed in order to cancel the gauge anomalies of the first D2 and of the last D6 color chain nodes. This is all visualized in Figure 15, where the quiver theory is now consistent.
3.1.3 Example III

As a final example, let us turn to the case represented by Figure 8. Again, we consider Figure 16 instead which is simpler but falls into the same class of backgrounds. For which \( h_8 \) vanishes at the beginning and at the end of the \( \rho \)-interval while \( h_4 \) is constant. Therefore, according to Figure 16 the defining functions read

\[
\begin{align*}
   h_4(\rho) &= \alpha \\
   h_8(\rho) &= \begin{cases} 
   \frac{\nu}{2\pi} \rho & 2\pi k \leq \rho \leq 2\pi(k + 1) \quad k = 0, \ldots, P - 1 \\
   \frac{\nu P}{2\pi} (2\pi(P + 1) - \rho) & 2\pi P \leq \rho \leq 2\pi(P + 1)
   \end{cases}
\end{align*}
\]

(3.22) \hspace{1cm} (3.23)

Figure 16: A simplified version of Figure 8. Here, \( h_8 \) starts and ends with a vanishing value, while \( h_4 \) is constant.

Figure 17: This is the naive quiver dual to the background defined by (3.22), (3.23). In reality, there are two more flavor nodes, canceling the gauge anomalies for the first and last D6 gauge nodes.

The background defined by these functions is — naively — dual to the quiver theory given by Figure 17. Again, this quiver cannot be the right one and this can be seen by observing the first and last D6 gauge nodes, i.e. the ones with gauge rank \( \nu \) and \( P\nu \) respectively, whose
Figure 18: This is the actual quiver dual to the background defined by (3.22), (3.23). Here, the extra D2 flavor nodes cancel the gauge anomalies for the first and last D6 gauge nodes.

gauge anomalies do not cancel. Anomaly cancellation demands each of them to connect to a flavor node of rank $\alpha$.

Indeed, near both the beginning and the end of the $\rho$-interval, the background reads

$$d\hat{s}^2 = s_1 \sqrt{x} d\hat{s}_{\text{AdS}_3}^2 + \frac{1}{\sqrt{x}} \left( s_2 d\rho^2 + s_3 \rho^2 ds_{S^2}^2 + s_4 ds_{\text{CY}_2}^2 \right), \quad e^\Phi = s_5 x^{-\frac{1}{4}}$$

with $s_i$ real constants, which corresponds to D2-branes on AdS$_3$ and smeared over CY$_2$. These D2-branes, which as we saw are bound states of four-branes ($\alpha$ of them at each endpoint), feed the D6 color chain of the quiver with exactly the flavor nodes needed in order to cancel the gauge anomalies of the first and last D6 color nodes. This is all visualized in Figure 18, where the quiver theory is now consistent.

3.2 Constant $u(\rho)$

The class of supergravity backgrounds with constant function $u(\rho)$ are slightly more subtle than the linear case. The possible kinds of backgrounds in this class are again the ones presented in Figures 4-9, but now with constant $u$. Instead of going through multiple examples again, we now pick just one that includes all the interesting behavior. Considering the bound states that occurred in the previous section, that would be Example II.

At the end of the $\rho$-dimension

Hence we revisit Example II, given by Figure 13 but now with constant $u$. The advertised background ends on its $\rho$-dimension with a vanishing $h_4$ but a non-vanishing $h_8$, giving

$$d\hat{s}^2 = \frac{1}{\sqrt{x}} \left( s_1 ds_{\text{AdS}_3}^2 + s_2 ds_{S^2}^2 \right) + \sqrt{x} \left( s_3 dx^2 + s_4 ds_{\text{CY}_2}^2 \right), \quad e^\Phi = s_5 x^{-\frac{1}{4}}$$

which corresponds to D4-branes smeared over CY$_2$. While this seems odd since $h_8$ only
produces D8-branes, our wisdom off the previous section guides us to study the source terms

\[ S_{WZ}^{D8} = \mu_8 \int C_9^{el} + C_7^{el} \wedge F_2 + C_5^{el} \wedge F_2 \wedge F_2 + C_3^{el} \wedge F_2 \wedge F_2 \wedge F_2 \] (3.26)

where the first term sources a standard D8-brane and the rest reflect eight-branes as bound states of D6, D4 and D2-branes, respectively. However, \( u \) is now constant and the D8-brane potential reads

\[ C_9^{el} = \frac{u^2}{h_8} \text{vol(AdS}_3) \wedge \text{vol(S}^2) \wedge \text{vol(CY}_2) \] (3.27)

where a convenient gauge choice is implied. This potential remains finite at \( \rho = \rho_f \) and hence there are D8-branes on that boundary, accompanied by their gauge fields. Thus, in contrast with the previous section, these D8 gauge fields may source the bound states reflected on the WZ terms in (3.26).

Studying the RR fluxes, we realize that the potentials behave at \( \rho \to \rho_f \) as

\[ C_7^{el} = \left( 2 \int h_4 d\rho \right) \text{vol(AdS}_3) \wedge \text{vol(CY}_2) \to 0 \]
\[ C_5^{el} = \left( -\frac{u^2}{h_4} \right) \text{vol(AdS}_3) \wedge \text{vol(S}^2) \to \infty \quad \text{for} \quad h_4 \to 0 \] (3.28)
\[ C_3^{el} = \left( 2 \int h_8 d\rho \right) \text{vol(AdS}_3) \to \text{const.} \]

where we again chose a convenient gauge. The fact that \( C_7^{el} \) vanishes excludes the bound state of D6-branes. Between the rest two potentials, we only have to consider \( C_5^{el} \) which blows up.

We conclude that the D8-brane gauge field couples to D4-charge through the term

\[ S_{WZ}^{D8} = \frac{\mu_4}{4\pi^2} \int C_5^{el} \wedge \tilde{f}_2 \wedge \tilde{f}_2 \] (3.29)

together forming a D8/D4 bound state. The D8 gauge flux on CY\(_2\) should be quantized as

\[ \frac{1}{4\pi^2} \int_{\text{CY}_2} \tilde{f}_2 \wedge \tilde{f}_2 = N \quad \text{for} \quad N \in \mathbb{Z} \] (3.30)

and the D4-branes are explicitly given by the Bianchi identity

\[ d\tilde{f}_4 = \tilde{f}_2 \wedge \tilde{f}_2 \wedge dF_0 = N \text{vol(CY}_2) \wedge (h_8^u d\rho) \] (3.31)

which also explains their smearing over CY\(_2\). The fact that \( C_5^{el} \) blows up makes the source
term (3.29) dominant in (3.26) and this is why the eight-branes are geometrically realized as smeared D4-branes.

Of course, the above analysis represents just one D8-brane and its bound states which makes its worldvolume theory Abelian, implying maybe a $U(N_8)$ gauge theory on the induced D4-branes. However, we instead have multiple coincident D8-branes which imply a non-Abelian D8 gauge field $\tilde{f}_a$ with $a = 1, ..., N_8^2 - 1$, where $N_8$ is the number of those D8-branes. This means that strings might end on D4-branes of different D8-brane worldvolumes, implying that the actual gauge theory exhibits the gauge group $U(N_8)$. Of course, same as in the last section, $N_8 = \mu$ and the $U(\mu)$ flavor group cancels exactly the gauge anomalies in the end of the quiver chain of the theory.

At the beginning of the $\rho$-dimension

The same background begins on its $\rho$-dimension with a vanishing $h_8$ but a non-vanishing $h_4$ function, giving

$$ds^2 = \frac{1}{\sqrt{x}} (m_1 ds_{AdS_3}^2 + m_2 ds_{S^2}^2 + m_3 ds_{CY_2}^2) + m_4 \sqrt{x} \, dx^2, \quad e^\Phi = m_5 x^{-\frac{3}{4}}$$  (3.32)

where the first term represents standard D4-branes, while the rest reflect polarized D4-branes into higher dimensional ones.

For $h_8 \to 0$, at $\rho \to 0$, the potential $C_9$ blows up and thus the third term in the above action dominates. Both the coupling to the transverse scalars and the string length order make here a more detailed treatment instructive, a calculation that takes place in Appendix B. Ultimately, the D4-branes polarize under the strong RR potential $C_9$ into an eight-brane, forming a D8/D4 bound state, giving a D8-brane background on that boundary.

Casting the usual trick on $h'_4$, we count $\alpha$ D4-branes on $\rho = 0$, on which open strings end and make up a $U(\alpha)$ gauge theory. The polarization that takes place over CY$_2$ should raise the question whether the D4-branes are enough in number to support massless string modes and thus a unitary gauge theory. The situation might seem even more foggy, considering that we do not own a metric tensor for CY$_2$. However, in reality, we are not obligated to know the precise shape of the polarized D4-branes, just that they are enough in number to be close to one another so that the modes do not get massive. And fortunately we do know that the D4-branes are a lot, since $\alpha$ must be large in the supergravity limit by construction. Therefore $U(\alpha)$ is the flavor group we anticipated in the beginning of the quiver theory, canceling exactly the gauge anomalies there.
Note that the smeared D4 and the D8-branes in this section are actually in superposition with smeared O4 and O8-planes, respectively. Of course, strings may only live on the former which is why we just consider those to find the desired flavor symmetries.

Aside from curing a problem and better realizing the way the dual field theory works, this whole section has an additional value. Since the discovery of particular flavor branes was the exact thing that made the quiver theory consistent, this calculation provides an additional validity check of the whole field theoretical structure. The further validation of the quantum quiver structure is especially important here, since the matter content of these quivers is by no means trivial. This is the subject of the following section.

4 Adding matter in the quiver field theory

The quantum quiver theory dual to the AdS$_3$ supergravity vacua we consider was presented in Section 2.3. In [52] these linear quiver theories were thoroughly analyzed and tested, while our previous section suits as further validation. Nevertheless, there is more to their story to tell. That is they are ultimately characterized by additional structure.

Let us address the problem in a constructive way. In a Hanany-Witten brane set-up, we have all possible kinds of oscillating strings stretched between the branes. In the dual quiver theory, these kinds of strings correspond to supersymmetric multiplets that bind the gauge theories (gauge nodes) together and constitute the matter content of the overall field theory. Thus, when we try to build the correct dual field theory of a particular kind of brane set-up, the problem boils down to finding all the possible matter content.

Establishing the quiver theory represented by Figure 3 as a well tested structure, we realize that there are two kinds of superfield connection missing. These are the multiplets connecting D2 gauge with D4 flavor nodes and the ones connecting D6 gauge with D8 flavor nodes, respectively representing D2-D4 and D6-D8 strings. Instead of quantizing, we may just ask what multiplets can possibly fill this gap. The problem gets quickly simplified, since we know we do not want to consider additional $\mathcal{N} = (0, 4)$ hyper multiplets nor $\mathcal{N} = (0, 2)$ Fermi multiplets. This is because their presence would spoil the fragile balance of the gauge anomaly cancellation once and for all, a balance that was further confirmed to holographically hold by the last section. Therefore, we should only consider $\mathcal{N} = (4, 4)$ hyper multiplets.

Nonetheless, our unique choice should be in harmony with the central charge of the field theory. In particular, since the central charge was found in [52] to be holographically correct for the (original) quiver theory, then the new matter content we want to add should change nothing and be entirely invisible to it. Indeed, this is exactly the case. The central charge of the quiver field theory reads

$$c = 6 (n_{hyp} - n_{vec}) = 6 \left( \sum_{j=1}^{P} (\alpha_j \mu_j - \alpha_j^2 - \mu_j^2 + 2) + \sum_{j=1}^{P-1} (\alpha_j \alpha_{j+1} + \mu_j \mu_{j+1}) \right) \quad (4.1)$$

which means that it is sensitive to the number of the hyper multiplets. This may sound
**Figure 19:** This is the new dual quiver theory, with additional $\mathcal{N} = (4, 4)$ hyper multiplets binding the D4 and D8 flavor nodes with the D2 and D6 gauge nodes, respectively. The already existing $\mathcal{N} = (4, 4)$ hyper multiplets are represented with black solid lines, while the new additional ones with orange solid lines. This figure is schematic, however, since flavor nodes are far apart in the supergravity limit.

discouraging wrt adding new $\mathcal{N} = (4, 4)$ hyper multiplets, since we want to leave the central charge intact, but it is not. This is because we work in the supergravity limit, i.e. for $P \to \infty$, which means that we are eligible to add new hyper multiplets as long as their number is sub-leading in $P$ wrt to the old ones.

In the supergravity limit the sources (flavor nodes) should exist far apart along the linear quiver, which means that the new hyper multiplets escorting them are much less than the old ones that exist between the flavor positions (connecting the gauge nodes). The proposed, enhanced quiver theory is visualized in Figure 19.

In order to prove that the new hyper multiplets are always of lower order in $P$ than the old ones, we expand the already existing ones as

$$n_{\text{hyp}} = \sum_{j=1}^{P} \left( \sum_{k=0}^{j-1} \beta_k \cdot \sum_{l=0}^{j-1} \nu_l \right) + \sum_{j=1}^{P-1} \left( \sum_{k=0}^{j-1} \beta_k \cdot \sum_{l=0}^{j} \beta_l \right) + \sum_{j=1}^{P-1} \left( \sum_{k=0}^{j-1} \nu_k \cdot \sum_{l=0}^{j} \nu_l \right) \right) \quad (4.2)$$

while the new ones, $n_{\text{hyp}}^*$, read

$$n_{\text{hyp}}^* = \sum_{j=i_1}^{i_M} \alpha_j \tilde{F}_{j-1} + \sum_{j=i_1}^{i_N} \mu_j F_{j-1}$$

$$= \sum_{j=i_1}^{i_M} \left( \sum_{k=0}^{j-1} \beta_k \left( \beta_{j-1} - \beta_j \right) \right) + \sum_{j=i_1}^{i_N} \left( \sum_{k=0}^{j-1} \nu_k \left( \nu_{j-1} - \nu_j \right) \right) \quad (4.3)$$
where \( j = i_1, \ldots, i_{M,N} \) are the \( M, N \) intervals with sources for the D4 and D8 branes, respectively. The fact that in the supergravity limit the sources (flavor nodes) should exist far apart along the linear quiver means \( M, N \ll P \).

In order to compare \( n_{\text{hyp}} \) and \( n_{\text{hyp}}^* \), we can just focus into similar terms between them. These are, for instance, the second term of (4.2) and the first of (4.3). For them, we observe that their first summation is to \( P - 1 \) and \( i_M \), respectively. Since \( M, N \ll P \), this means that the former is of order \( P \) while the latter is not. Focusing on the inner summations of the same terms, we realize that their summing products are of the same order, whatever that is. Therefore, overall, \( n_{\text{hyp}} \) is always an order higher in \( P \) than \( n_{\text{hyp}}^* \), which makes the latter invisible in the central charge for \( P \to \infty \).

The whole situation would be immediately cleared out if we quantized the system of D-branes. What is more, quantizing the D2-D4 and D6-D8 systems in flat space seems to indeed reproduce the new \( N = (4, 4) \) hypermultiplets that we just proposed to exist. However, this particular Hanany-Witten set-up is assumed to live in CY \( 2 \) dimensions as well, which makes the standard quantization techniques obscure in the case at hand and, therefore, such a study remains on the sidelines at this point.

Another link that we intentionally left out is the multiplet corresponding to superstrings between D4 and D8 flavor branes. Not giving gauge groups, these links are allowed to be any multiplet as far as the gauge anomaly balance is concerned. Therefore, this situation demands to be properly quantized and thus eludes the present work.

Truth be told, there is another path through which we might have imagined that the additional matter is an essential ingredient to our theory. This argument too surfaces from the supergravity side of the duality, but in order to illustrate it we need to consider a particular state of the string. This is what we deal with in the following section.

5 The meson string

Having worked out even the most exotic parts of the correspondence between the massive IIA vacua and the dual quantum field theory, we are certainly in desire of testing their holographic performance. In that vein, we look for a simple object to construct, starting off with the supergravity side of the story.

5.1 A BPS state

The most accessible state in our theory of gravity is a semiclassical string stretching between D-branes. That is, we consider a meson string soliton \( M_{k,m} \) on the supergravity background, that extends between stacks of flavor branes at \( \rho = 2\pi k \) and \( \rho = 2\pi m \), respectively, and which is a point on the rest of the dimensions sitting at the center \( r = 0 \) of AdS\(_3\). An analogous calculation was performed in [60].
Therefore, we let for a string embedding with $\tau = t, \sigma = \rho$, whose mass is essentially its length

$$M_M = \frac{1}{2\pi} \int d\sigma \sqrt{-\det g_{ab}} = \frac{1}{2\pi} \int_{2\pi k}^{2\pi m} d\rho \sqrt{-\det g_{ab}} = m - k \quad (5.1)$$

where $g_{ab}$ is the worldsheet pullback of the metric in (2.1). If $F_k$ and $F_m$ are the number of D-branes in the respective stacks where the string endpoints end, then this configuration transforms in the bi-fundamental representation of $SU(F_k) \times SU(F_m)$.

Since we are always interested in states that preserve some supersymmetry, we may upgrade the above configuration to a BPS state just by considering the suspended string to fluctuate on the two-sphere, whose $SU(2)$ isometry corresponds to the dual R-symmetry. This is done by including $\phi = \omega \tau$ in the above configuration, where we let this fluctuation to be small – i.e. $\omega \ll 1$ – so that the embedding simplifies still into the expression (5.1).

Picking a $U(1)_R$ inside $SU(2)_R$, we now seek the R-charge of the above state. Since the generator of the $U(1)$ on the two-sphere is associated to the 1-form $\cos \theta \, d\phi$, then we look for the string coupling terms

$$S_R \propto \int \cos \theta \, d\phi \quad (5.2)$$

As far as the R-charge is concerned, it may be read off the source terms of the form $\int J_R A_1 = Q_R \int A_1$, with $A_1 = \cos \theta \, d\phi$. The relevant term in the worldsheet action is

$$S_M = \frac{1}{2\pi} \int_{\Sigma} B_2 \quad (5.3)$$

where $\Sigma = [2\pi k, 2\pi m] \times \mathbb{R}$. Ultimately, after some manipulation given in Appendix C, this term may be actually seen as the source term

$$S_M = \frac{m - k}{2} \int \cos \theta \, d\phi \quad (5.4)$$

which yields an R-charge

$$Q_R = \frac{m - k}{2} \quad (5.5)$$

Comparing the R-charge with the string mass in (5.1), we conclude that this is indeed a BPS state.
5.2 A UV operator

Now, we want to look for the operator dual to this BPS state. To this end — since the IR SCFT is completely unknown — we consider the UV quiver theory on the $\rho$-interval $[2\pi k, 2\pi m]$ and pick the appropriate field excitations inside the supersymmetric multiplets.

Since we are dealing with a purely bosonic state, we are immediately led to consider only the complex scalars $\phi_i$ inside the $\mathcal{N} = (0, 2)$ chiral multiplets $\Phi_i$, since these are the only on-shell bosonic degrees of freedom in our theory. In particular, we choose to excite one scalar in each of the $(m - k) + 2 \mathcal{N} = (4, 4)$ hypermultiplets that connect two flavor nodes; this makes a perfect fit with the fact that string fluctuations transverse to the worldvolumes of branes are also scalar modes wrt these worldvolume theories. It also illustrates why we need the additional $\mathcal{N} = (4, 4)$ matter, as promised in the beginning of this section; if it was not for these new hypermultiplets, there would be no way to build a string of bosonic field excitations that connect two flavor nodes. And such a dual bosonic connection must somehow exist, given that the meson string we consider is a legitimate BPS state.

Shortly, however, we spot a problem. As explained in Section 2.3.1, the $\phi_i$ scalars inside any of the $\mathcal{N} = (0, 4)$ hypermultiplets are uncharged under R-symmetry, while we do need an R-charge — according to (5.5), proportional to $(m - k)$ — for our proposed operator. In fact, the only scalars that are charged under the $U(1)_R$ subgroup of the R-symmetry are the ones in the $\mathcal{N} = (0, 4)$ twisted hypermultiplets $(\Sigma_i, \tilde{\Sigma}_i)$, inside the $\mathcal{N} = (4, 4)$ vector superfields of the gauge nodes. This leads us to consider these scalars, let us call them $\sigma_i$, as well. The inclusion of these scalar fields is also somewhat compelling, since these are the ones that let the $\phi_i$ scalars interactively talk to each other; this realizes an interactive continuance among the string of fields in the operator, holographically analogous to the compactness of the string. These supersymmetric interactions will become apparent shortly.

All in all, choosing a $\sigma_i$ excitation as well in each gauge node between the $\mathcal{N} = (4, 4)$ hypermultiplets, we acquire the meson operator

$$M_{k,m} = \pi_k \left( \prod_{i=k}^{m-1} \sigma_i \phi_i \right) \sigma_m \tilde{\pi}_m$$

(5.6)

which transforms in the bifundamental representation of $SU(F_k) \times SU(F_m)$, with $F_k$ and $F_m$ the ranks of the flavor groups in the corresponding positions of the quiver chain. Here we named $\pi_i$ the scalars inside the end-point hypermultiplets connecting to the flavor nodes and also chose them to be in conjugate representations of each gauge group. Such an operator has two $\pi_i$’s, $(m - k)$ $\phi_i$’s and $(m - k + 1)$ $\sigma_i$’s, which in the supergravity limit — where sources are far apart — account for $2(m - k)$ complex scalars. Since only half of those (the $\sigma_i$’s) are R-charged, this is a promising ratio considering the string BPS condition given by (5.1) and (5.5). For clarity, the operator is highlighted in Figure 20.

The only quantities left to compare are the mass (5.1) of the BPS state and the conformal dimension of the operator $M_{k,m}$. At this point, of course, we may have an actual problem; scalar fields in two dimensions have mass dimension zero. At least classically. At first sight,
Figure 20: The meson operator $\mathcal{M}$ consists of the supersymmetric multiplets that are highlighted with blue, while the rest of the quiver structure is left blurred. If $k$ is the position of the second flavor node along the $\rho$-dimension, then this operator runs over $k + 2 \mathcal{N} = (4, 4)$ hypermultiplets and $k + 1 \mathcal{N} = (4, 4)$ vector multiplets. Notice that such an operator may also connect D4 with D8 flavors, by jumping through $\mathcal{N} = (0, 4)$ hypermultiplets.

This degrades our proposal for the operator which seems to have a vanishing scaling dimension. However, before rushing into conclusions, we should first consider two basic facts. For one, we consider the UV operator and not the actual IR situation; it is the IR operator the one that should necessarily acquire the appropriate scaling dimension. Moreover, following our arguments up to this point, our choice of component fields seems to be quite unique; there is simply no other way to build up a bosonic operator. Therefore, our only way out is the possibility of the operator acquiring an anomalous dimension through quantum effects. Whatever the case is with the IR SCFT, such quantum effects should be present in the UV Lagrangian, pointing towards an anomalous dimension $\gamma(g)$ that scales with energy.

On the other hand, studying quantum corrections is obscure in our case. This is exactly because it is the UV theory that we use to organize fields into an operator; therefore even if we assume a completely anomalous dimension $\Delta_M = \gamma(g)$, our SCFT is assumed to be strongly coupled which discredits any perturbative calculation. To be exact, it is the non-integrability of our AdS$_3$ backgrounds [61] that prohibits surfing along the range of the coupling constant, as it is possible with e.g. the work of BMN [62] in the AdS$_5 \times S^5$ correspondence. Regardless, the possibility itself of a non-perturbative anomalous dimension requires certain interactions to be there, between the fields of interest; finding whether those exist is essential to our proposal. Interestingly, such interactions indeed exist.

The interactions between the $\phi_i$’s of the hypermultiplets and the $\sigma_i$’s of the twisted hypermultiplets have actually already appeared in our study of the Fermi multiplet interactions. As seen in Section 2.3.4, Fermi multiplets defined by $\mathcal{D}_i \Gamma_a = E_a(\Phi_i, \Sigma_i)$ give a potential $|E_a(\phi_i, \sigma_i)|^2$, which for our interactive chain of multiplets exhibits quite a few components.
From those, the ones that couple $\phi_i$’s and $\sigma_i$’s are the
\[ E_{\Gamma_i}(\phi_i, \sigma_i) = \sigma_i \phi_i \] (5.7)
or $E_{\tilde{\Gamma}_i} = -\tilde{\phi}_i \sigma_i$, depending on which scalar field we excite inside a certain hypermultiplet. Accordingly, if we choose to excite $\tilde{\sigma}_i$ inside a twisted hypermultiplet, instead of its twin $\sigma_i$, then these scalars couple through the superpotential term $|J_a(\phi_i, \sigma_i)|^2$ and, in particular, through the components
\[ J_{\Gamma_i}(\phi_i, \tilde{\sigma}_i) = \tilde{\sigma}_i \phi_i \] (5.8)
or $J_{\Gamma_i} = \tilde{\phi}_i \tilde{\sigma}_i$.

These are all the interactions present between the different scalars we choose to excite and which furnish our operator (5.6) with quantum effects. We presume that those are capable of correcting it non-perturbatively to the desired conformal dimension $\Delta_M = \gamma(g) = m - k$.

### 5.3 Dual mass

While the scaling dimension of the meson operator stands as a proposal, there is another insight as to the mass of the BPS state that both enforces the proposed duality and digs out an interesting feature of the field theory.

It is simpler to explore things heuristically here. While coincident branes give massless modes, a superstring suspended between two distanced D2 or D6-branes gives a BPS hypermultiplet (in our kind of theory, presumably of $\mathcal{N} = (4, 4)$ supersymmetry) of mass $\sqrt{|\vec{x}|}$, where $\vec{x}$ is the spatial vector connecting the branes. While a hypermultiplet is massless, a mass is obtained by its coupling to a vector superfield, since the latter obtains a VEV through a Fayet-Iliopoulos $D$-term lying on the U(1) gauge theory in the brane worldvolume. That is, as seen from (2.16) and (2.19), for a U(1) vector superfield we have a $D$-related action
\[ S_D = \int \frac{1}{g^2} D^2 + \sigma D \bar{\sigma} - \xi D \] (5.9)
where the last term is the Fayet-Iliopoulos term. After integrating out the auxiliary field $D$, the potential energy $V = g^2(|\sigma|^2 - \xi)^2$ is formed which yields the new classical vacuum
\[ \langle \sigma \rangle = \sqrt{\xi} \] (5.10)
which in turn couples to the hypermultiplet and is felt as a mass.

When instead we have two stacks, one of $n_1$ and another of $n_2$ D-branes, we acquire $n_1 n_2$ hypermultiplets that transform under the $(n_1, \bar{n}_2)$ representation of $U(n_1) \times U(n_2)$. In Hanany-Witten set-ups we have parallel stacks of branes distanced and bordered by NS fivebranes, where the gauge group actually breaks down to SU($n_i) \times U(1)$; the non-trivial U(1) center
provides a Fayet-Iliopoulos $D$-term whose coupling is identified with $\xi = |\vec{x}|$. That is, the $D$-term coupling is given by the distances between the NS fivebranes [3, 56]

$$\xi = \rho_{i+1} - \rho_i$$

Each U(1) is actually the center of mass of the stack of branes and $D$ is really its Hamiltonian function, where the Fayet-Iliopoulos coupling reflects the fact that we may always add a constant to such a function. While this story is generally studied, let us bring it down onto our case and clarify how it actually works.

By adding a Fayet-Iliopoulos D-term to the $\mathcal{N} = (4, 4)$ vector superfield action and integrating out $D$, we acquire the new vacuum $\langle \sigma_i \rangle = \sqrt{\rho_{i+1} - \rho_i} = 1/\sqrt{2}$. As restated, $\sigma_i$ is one of the scalars of the $\mathcal{N} = (0, 4)$ twisted hypermultiplet inside the vector superfield on a stack of D2 or D6-branes, placed between the $(i+1)\text{th}$ and $i\text{th}$ stack of NS fivebranes. Notice here that we also normalized, by a redefinition, the fundamental $\rho$-interval distance $\rho_{i+1} - \rho_i = 2\pi$ to $1/2$, for convenience that will become apparent momentarily. Now, this VEV gives a mass to a $\mathcal{N} = (4, 4)$ hypermultiplet coupled to it and, in particular for our operator of interest, this is achieved through the interactive terms (5.7) and (5.8) that we brought up in the previous section. That is, if we choose to consider the $\sigma_i$ scalar inside the vector superfield and the $\phi_i$ scalar inside the hypermultiplet then a mass is acquired by the latter as

$$|E_{\Gamma_i}|^2 = \langle \sigma_i \rangle^2 |\phi_i|^2 = \frac{1}{2} |\phi_i|^2$$

Accordingly, for other choices of scalar fields inside those multiplets the mass is obtained through other $E$-terms or superpotential $|J|^2$ terms with $J$ as in (5.8).

Now, each such hypermultiplet is actually linked to two stacks of D-branes (gauge nodes), one on its left and one on its right along the $\rho$ dimension. This means that the mass that is gained comes from two VEV contributions, that is

$$|E_{\Gamma_i}|^2 + |E_{\Gamma_{i+1}}|^2 = (\langle \sigma_i \rangle^2 + \langle \sigma_{i+1} \rangle^2) |\phi_i|^2 = |\phi_i|^2$$

where the mass is now unity. Notice that the value of the mass comes from normalization and thus it is a matter of convention on absolute distances along the $\rho$-dimension. What really matters though is the relative positions of NS fivebranes; changing those shifts the masses of the hypermultiplets in between. Since all the NS fivebranes in our brane set-up are equally separated along $\rho$, accordingly all masses will be the same. Moreover, note that there are as many massive hypermultiplets as the U(1)’s. That is, all hypermultiplets between the gauge nodes along the quiver chain are massive. Therefore we only care about the number of those hypermultiplets that contribute to our operator.
Ultimately, the meson operator (5.6) contains $m - k$ scalar fields $\phi_i$ which are massive, associating the operator itself with a total classical mass

$$M_M = m - k$$

which exactly agrees with the mass (5.1) of the BPS string.

In regard to our particular choice of the BPS operator, besides the agreement on the dual masses it is worth emphasizing the way that this equality is supported. That is, as with the R-charge (or even the presumable anomalous dimension), it again takes both scalar fields $\phi_i$ and $\sigma_i$ to holographically reflect a dual semiclassical soliton; the $\sigma_i$’s adjust a mass (and a R-charge) and the $\phi_i$’s realize it.

Again, it is the UV particle theory that shapes the proposed meson operator $M$ and not the actual IR SCFT that sits on the dual side of our AdS$_3$ supergravity backgrounds. While this cautions us to be careful about our statements on what the actual dual BPS operator looks like, we are encouraged by the agreement in mass to make an otherwise bold conjecture: the operator mass somehow transforms into a scaling dimension. This is not as presumptuous as it may sound if we consider that the non-perturbative anomalous dimension $\Delta_M = \gamma(g) = m - k$ — that we expect — should be generated by the same interactions that produced the Fayet-Iliopoulos mass. Thus the aforementioned transformation is really thought to be a change on how we realize the same field interactions at different energy scales. That is, the interactions given by (5.7) and (5.8) may be realized as a classical mass in the UV or an anomalous dimension in the IR. This idea is strongly advocated by the fact that the gauge coupling is relevant at the IR of the two-dimensional quantum theory, where the quantum corrections should be important and the scalar masses get integrated out.

As a final comment, the BPS string is a semiclassical bound state which inspires us to assume that its dual operator should too reflect a bound state of two-dimensional fields. That being said, we notice that the operator mass is a sum of all the individual scalar field masses, a fact which renders the BPS operator indeed very much alike to a classical bound state of particles. This is a statement on classical bound states in the sense that we neglect an unimportant interaction energy, as we already did with the implicit quantum corrections between fields inside the operator or with the sphere fluctuations on the string mass. While the latter is geometrically obvious through (5.1), the former may be supported by the fact that the gauge coupling is irrelevant at the UV of two-dimensional quantum field theory.

6 Epilogue

Summarizing, in Section 3 we studied all possible categories of vacua within a particular AdS$_3$ family of massive IIA supergravity solutions, first given in [53]. Apart from the original solutions introduced there, we presented the rest kinds of vacua in the same family which
all naively seem to give anomalous dual quiver gauge theories. We proved that these erratic solutions imply D-branes on the boundary of the space, which in turn correspond to flavor symmetries that exactly cancel the apparent gauge anomalies. A special feature of the situation is that, due to strong RR fluxes on the boundary of the space, these D-branes come exclusively in bound states forming polarizations that provide the quiver with flavor in a quite idiosyncratic way.

After dealing with all possible kinds of solutions and quiver theories, in Section 4 we supplement the quiver structure with additional matter in the form of bifundamental links between color and flavor nodes. These, we argue, may only be $\mathcal{N} = (4,4)$ hypermultiplets corresponding to suspended superstrings between D2 and D4-branes or D6 and D8-branes in the ancestral Hanany-Witten set-up.

Having introduced the complementary bifundamental matter too, in Section 5 we put holography to the test by considering a semiclassical string inside the AdS$_3$ background stretched between two D-branes. We call this a meson string and by finding its mass and R-charge we show it is a BPS state. Next, we propose a UV operator dual to the soliton and we argue that there is a unique choice of fundamental scalar fields that synthesize it. Moreover, crucial to the construction of this operator is the additional bifundamental matter we have introduced. While the R-charge of the proposed operator seems to get along with our expectations, its conformal dimension is classically zero since scalar fields in two spacetime dimensions have a vanishing mass dimension. What is more, since the two-dimensional SCFT we are assuming is strongly coupled and these AdS$_3$ vacua have been proven to be non-integrable, the perturbative regime of calculations is out of our reach. Nonetheless, by bringing to the surface the superpotential of the UV quiver theory, we find interactions between the scalars inside the operator and we are led to the conclusion that the latter should acquire a totally non-perturbative anomalous dimension at the IR, equal to the mass of the BPS string.

Pursuing the holographic picture of the meson string, we focus on the quiver structure and find that scalars inside the vector superfields should obtain a VEV through a Fayet-Iliopoulos term. The latter is due to the U(1) theory inside the U($N$) gauge group of each stack of branes in the set-up. Superpotential interactions between the vector and hypermultiplets then dictate that bifundamental matter acquires a mass, ultimately associating the dual meson operator with a classical mass equal to that of the BPS string. Since the operator mass is a sum of all the individual scalar field masses, this renders the operator indeed very much alike to a classical bound state of particles dual to a bound string state between D-branes.

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A Extremal $p$-brane solutions

Extremal $p$-branes are supergravity solutions that in the context of superstring theory are identified with stacks of D$p$-branes. These are distinct from O-planes that essentially constitute boundary conditions for strings. The leading order backgrounds for all the above read

\begin{align*}
\text{p-brane} & : \quad ds^2 \sim x^{\frac{7-p}{2}} ds_{M^{1,p}}^2 + x^{\frac{p-7}{2}} (dx^2 + x^2 ds_{\Sigma^{8-p}}^2) \quad e^\Phi \sim x^{\frac{(3-p)(p-7)}{4}} \\
\text{p-brane smeared on } \Sigma^s & : \quad ds^2 \sim x^{\frac{7-p-s}{2}} ds_{M^{1,p}}^2 + x^{\frac{p+s-7}{2}} (dx^2 + +ds_{\Sigma^{s}}^2 + x^2 ds_{\Sigma^{8-p-s}}^2) \quad e^\Phi \sim x^{\frac{(3-p)(p+s-7)}{4}} \\
\text{O$p$-plane} & : \quad ds^2 \sim \frac{1}{\sqrt{x}} ds_{M^{1,p}}^2 + \sqrt{x} (dx^2 + x_0^2 ds_{\Sigma^{8-p}}^2) \quad e^\Phi \sim x^{\frac{3-p}{4}} \quad \eta_{i,j,k,l}^{D8/D4} \end{align*}

(A.1)

Here $M^{1,p}$ is a manifold that the brane fills, $\Sigma^{8-p}$ is a compact space – on which one integrates to obtain the associated charge of the brane – and $\Sigma^s$ is the manifold over which a brane may be smeared.

B The D8/D4 bound state

We consider the background of Example II with a constant $u$ function and study the beginning of its $\rho$-dimension where D4-branes seem to polarize into a D8/D4 bound state. The fact that $C^e_9$ field becomes infinitely strong at that endpoint reasonably makes the D8/D4 bound state dominant, yet a formal proof of it being the true vacuum is in order.

Our study significantly simplifies by choosing a convenient gauge for the RR potential

$$C^e_9 = \frac{u^2}{\tilde{h}_8} \text{vol(AdS}_3) \wedge \text{vol(S}_2) \wedge \text{vol(CY}_2)$$

(B.1)

On these grounds, we pick a static gauge for the D4-branes whose worldvolume fills up AdS$_3 \times S^2$, i.e. we choose worldvolume coordinates $\xi^a = (t, x, r, \theta, \phi)$, expanding the source term

$$\mathcal{S}^{D8/D4}_{WZ} = -\frac{\lambda^2}{2} \mu_8 \int \text{Tr} (i_\Phi \lambda_\Phi)^2 C^e_9 = -\frac{\lambda^2}{2} \mu_4 \int d^5 \xi \text{Tr} \Phi^i \Phi^j \Phi^k \Phi^l C_{ijkltt\theta\phi}$$

$$= -\frac{\lambda^2}{8} \mu_4 \int d^5 \xi \text{Tr} [\Phi^i, \Phi^j][\Phi^k, \Phi^l] C_0$$

(B.2)

where the Latin letters $i, j, k, l$ denote the CY$_2$ directions, while we also used that $x^i = \lambda \Phi^i$.
on dimensional grounds. In general, the transverse modes \( \Phi \) should include \( \Phi^\rho \) too, but not in our particular gauge of \( C_9 \).

Now we want to focus on \( \rho = 0 \) where all the action takes place, i.e. expand \( C_9 \) around that endpoint. It being a singular endpoint implies a Laurent expansion but, since it is also the endpoint of a closed interval, this series is not well defined around it. Thus, we just pick a point \( x \) close to \( \rho = 0 \) and expand around it, inside a circular region (of the complex domain) – of radius \( x \) too – which touches the singularity. That is, the expansion reduces to a Taylor series around \( x \) as

\[
S_{WZ}^{D8/D4} = -\frac{\lambda^2}{8} \mu_4 \int d^5 \xi \text{Tr} \left[ \Phi^i, \Phi^j \right] \left[ \Phi^k, \Phi^l \right] \left( C_9|_{\rho=x} + \lambda \Phi^\rho F_{10}|_{\rho=x} + \ldots \right) \tag{B.3}
\]

Since \( h_8 \to 0 \) for small \( x \), the RR fields \( C_9 \) and \( F_{10} \) blow up there and thus from now on we will consider them as largely valued quantities.

The above source term adds to the SYM interactions of the DBI action, which come from

\[
\sqrt{\det Q_{ij}} \quad \text{for} \quad Q_{ij} = \delta_{ij} + i\lambda [\Phi^i, \Phi^j] (G_{kj} + B_{kj}) \tag{B.4}
\]

in the flat space limit, i.e. for \( G_{\mu\nu} = \eta_{\mu\nu} \) and \( B_{\mu\nu} = 0 \). The source term (B.2) keeps terms up to third order in \( \lambda \) and hence we expand the DBI determinant accordingly as

\[
\sqrt{\det Q_{ij}} = 1 - \frac{\lambda^2}{4} \text{Tr} [\Phi^i, \Phi^j]^2 - i\frac{\lambda^3}{12} \text{Tr} [\Phi^i, \Phi^j]^3 + \ldots \tag{B.5}
\]

where the second order term is the familiar SYM scalar interaction. Taking into account the full D4-brane action \( S = S_{DBI} + S_{WZ} \), we acquire the SYM potential

\[
V(\Phi) = -\frac{\lambda^2}{4} \text{Tr} [\Phi^i, \Phi^j]^2 + \frac{\lambda^2}{8} \text{Tr} [\Phi^i, \Phi^j][\Phi^k, \Phi^l] C_9|_{\rho=x} - i\frac{\lambda^3}{12} \text{Tr} [\Phi^i, \Phi^j]^3 + \frac{\lambda^3}{8} \text{Tr} [\Phi^i, \Phi^j][\Phi^k, \Phi^l][\Phi^\rho F_{10}|_{\rho=x} \tag{B.6}
\]

where we have assumed a constant mode \( \Phi^\rho \) to simplify the game. After reparametrizing the fields conveniently to absorb some numerical factors, the SYM potential gets an order by order variation \( \frac{\partial V}{\partial \Phi} = 0 \) as

\[
\mathcal{O}(\lambda^2) : \quad [\Phi^i, \Phi^j] = [\Phi^k, \Phi^l] C_{ijkl}\ldots
\]

\[
\mathcal{O}(\lambda^3) : \quad [\Phi^i, \Phi^j][\Phi^j, \Phi^k] = -i[\Phi^i, \Phi^m] F_{iklm}\ldots \tag{B.7}
\]

which has a trivial solution \( [\Phi^i, \Phi^j] = 0 \) giving \( V_0 = 0 \), corresponding to separated D4-branes.
Alternatively, combining both of these equations, the potential also exhibits the non-trivial solution

\[ [\Phi^i, \Phi^j] = -i \epsilon^{ij} \partial_\rho \]  

(B.8)

which in momentum space reads

\[ [\Phi^i, \Phi^j] = \epsilon^{ij} p_\rho \]  

(B.9)

where we abuse the antisymmetric tensor just to sustain the antisymmetry of the commutator into the rhs. Placing this solution back into the SYM potential we get

\[ V_\star \equiv \lambda^2 p_\rho^2 C_9 |_{\rho \rightarrow 0} + O(\lambda^3) \]  

(B.10)

where we used the fact that \( C_9 \) is large at \( \rho \rightarrow 0 \).

As a matter of fact, \( C_9 \) is not only large but also negative for \( h_8 \rightarrow 0^+ \), which means that \( V_\star < 0 \). Since the separated D4-branes correspond to the null energy state \( V_0 = 0 \), the latter is unstable and condenses out into the non-trivial D8/D4 bound state with \( V_\star \) which is the true stable vacuum at \( \rho = 0 \).

As a matter of fact, \( C_9 \) is not only large but also negative for \( h_8 \rightarrow 0^+ \), which means that \( V_\star < 0 \). Since the separated D4-branes correspond to the null energy state \( V_0 = 0 \), the latter is unstable and condenses out into the non-trivial D8/D4 bound state with \( V_\star \) which is the true stable vacuum at \( \rho = 0 \).

Also, notice the fact that specifically \( V_\star \rightarrow -\infty \), due to the strong RR potential \( C_9 \rightarrow -\infty \) at \( \rho \rightarrow 0 \), which saves us from having to also investigate the D6/D4 bound state. That is, there just cannot be any lower energy than \( V_\star \).

\section{R-charge of the BPS state}

Naively, the \( B_2 \) field in (2.1) has nothing to do with the 1-form \( \cos \theta \, d\phi \). However, \( B_2 \) exhibits large gauge transformations across the \( \rho \)-intervals \([2\pi k, 2\pi (k+1)]\), which are explicitly realized through the 1-form

\[ \Lambda_1 = \Theta(\rho - 2\pi k) \Theta(2\pi (k+1) - \rho) \pi k \cos \theta \, d\phi \]  

(C.1)

Therefore, the large gauge transformations \( B_2 \rightarrow B_2 + d\Lambda_1 \) read

\[ B_2 \rightarrow B_2 + \Theta(\rho - 2\pi k) \Theta(2\pi (k+1) - \rho) \pi k \, d\Omega_2 + \left[ \delta(\rho - 2\pi k) - \delta(2\pi (k+1) - \rho) \right] \pi k \, d\rho \wedge \cos \theta \, d\phi \]  

(C.2)

where, in this explicit formulation, the only difference now is the novel delta-terms, \( B_2^\delta \). The latter, which are the ones producing the R-charge, are integrated over a \( \rho \)-interval as
\[
\frac{1}{2\pi} \int B_2^\delta = \frac{1}{2\pi} \int_R \cos \theta \, d\phi \int_{2\pi k}^{2\pi(k+1)} d\rho \left\{ \left[ \delta(\rho - 2\pi k) - \delta(2\pi(k+1) - \rho) \right] \pi k \right. \\
- \delta(2\pi k - \rho)\pi(k-1) + \delta(\rho - 2\pi(k+1))\pi(k+1) \right\}
\]  
(C.3)

where the first line is the contribution coming from \( B_2^\delta \) defined on the interval \([2\pi k, 2\pi(k+1)]\) as expected, while the second line includes the contributions coming from the intervals prior and next to that. Considering \( \int_0^\infty \delta(x)\,dx = 1/2 \), the above integral gives

\[
\frac{1}{2\pi} \int B_2^\delta = \frac{1}{2} \int_R \cos \theta \, d\phi
\]  
(C.4)

and the whole meson string \( \mathcal{M}_{k,m} \) acquires the R-charge source term

\[
S_{\mathcal{M}} = \frac{m-k}{2} \int_R \cos \theta \, d\phi
\]  
(C.5)

which yields its R-charge

\[
Q_R = \frac{m-k}{2}
\]  
(C.6)

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