ON QUANTUM SEMIGROUP ACTIONS ON FINITE QUANTUM SPACES

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Abstract. We show that a continuous action of a quantum semigroup $S$ on a finite quantum space (finite dimensional $C^*$-algebra) preserving a faithful state comes from a continuous action of the quantum Bohr compactification $bS$ of $S$. Using the classification of continuous compact quantum group actions on $M_2$ we give a complete description of all continuous quantum semigroup actions on this quantum space preserving a faithful state.

1. Introduction

A quantum space is an object of the category dual to the category of $C^*$-algebras ([13, Section 0]). A morphism in the category of $C^*$-algebras from $A$ to $B$ is a non-degenerate $*$-homomorphism $\phi: A \to M(B)$, where $M(B)$ is the multiplier algebra of $B$. It is known that non-degeneracy of $\phi$ allows an extension of $\phi$ to a map $M(A) \to M(B)$ thus providing a way to compose morphisms.

For any $C^*$-algebra $A$ we write $\text{QS}(A)$ for the quantum space corresponding to $A$. Following [13] we will write $\text{Mor}(A, B)$ for the set of all morphisms from $A$ to $B$.

By a quantum semigroup we mean a quantum space $\text{QS}(A)$ endowed with additional structure described by a morphism $\Delta_A \in \text{Mor}(A, A \otimes A)$ such that

$$(\Delta_A \otimes \text{id}) \circ \Delta_A = (\text{id} \otimes \Delta_A) \circ \Delta_A.$$ 

If $\text{QS}(N)$ is another quantum space (i.e. if $N$ is a $C^*$-algebra) then an action of a quantum semigroup $S = (A, \Delta_A)$ on $\text{QS}(N)$ is a morphism $\Psi \in \text{Mor}(N, N \otimes A)$ such that

$$(\Psi \otimes \text{id}) \circ \Psi = (\text{id} \otimes \Delta) \circ \Psi.$$ 

Let $S = (A, \Delta_A)$ be a quantum semigroup and let $\text{QS}(N)$ be a quantum space. An action $\Psi_S \in \text{Mor}(N, N \otimes A)$ of $S$ on $\text{QS}(N)$ is continuous if the set

$$\{ \Psi_S(m)(a \otimes \alpha) | m \in N, a \in A \}$$

is contained and linearly dense in $N \otimes A$. This condition is also referred to as the Podleś condition. It was introduced by Piotr Podleś in his study of compact quantum group actions ([4, 5]). The term “continuity” used here comes from [9].

Let $S = (A, \Delta_A)$ and $P = (B, \Delta_B)$ be quantum semigroups. Then a quantum semigroup morphism from $P$ to $S$ is a morphism $\Lambda \in \text{Mor}(A, B)$ such that

$$(\Lambda \otimes \Lambda) \circ \Delta_A = \Delta_B \circ \Lambda.$$ 

Let $S = (A, \Delta_A)$ be a quantum semigroup and let $\text{QS}(N)$ be a quantum space such that $\dim N < \infty$ (in the terminology of [11, 12] $\text{QS}(N)$ is a finite quantum space). Let $\Psi \in \text{Mor}(N, N \otimes A)$ be an action of $S$ on $\text{QS}(N)$. Let $\omega$ be a linear functional on $N$. Then, since $M(N \otimes A) = N \otimes M(A)$, the map

$$(\omega \otimes \text{id}) \circ \Psi : N \to M(A)$$

is well defined. We say that $\omega$ is invariant for $\Psi$ (or that $\Psi$ preserves $\omega$) if

$$(\omega \otimes \text{id}) \Psi(n) = \omega(n) \mathbb{1}$$

for all $n \in N$. 

Date: 3 October 2008.

2000 Mathematics Subject Classification. 58B32, 46L85.

Key words and phrases. Quantum group, Quantum semigroup, Action, Invariant state.

Research partially supported by KBN grants nos. N201 1770 33 & 115/E-343/SPB/6.PRUE/DIE50/2005-2008.
We will now briefly describe the quantum Bohr compactification of a quantum semigroup introduced in [7]. We will freely use the terminology of the theory of compact quantum groups from [14]. Let $S = (A, \Delta_A)$ be a quantum semigroup. The quantum Bohr compactification $\mathfrak{b}S$ of $S$ is a compact quantum group $\mathfrak{b}S = (\mathfrak{A}(S), \Delta_{\mathfrak{A}(S)})$ together with a morphism $\chi_S \in \operatorname{Mor}(\mathfrak{A}(S), A)$ such that for any compact quantum group $G = (B, \Delta_B)$ and any quantum semigroup morphism $\Lambda \in \operatorname{Mor}(B, A)$ there exists a unique morphism of compact quantum groups $\mathfrak{b}\Lambda \in \operatorname{Mor}(B, \mathfrak{A}(S))$ such that

$$\Lambda = \chi_S \circ \mathfrak{b}\Lambda.$$ 

The passage from $S$ to $\mathfrak{b}S$ is a functor from the category of quantum semigroups to its full subcategory of compact quantum groups ([7]). Let us recall that $\mathfrak{A}(S)$ is defined in [7, Proposition 2.13] as the closed linear span of matrix elements of the, so called, admissible representations of $S$ ([2, Definition 2.2]).

2. Actions on finite quantum spaces and the Bohr compactification

**Theorem 2.1.** Let $N$ be a finite dimensional $C^*$-algebra and let $S = (A, \Delta_A)$ be a quantum semigroup. Let $\Psi_S \in \operatorname{Mor}(N, N \otimes A)$ be a continuous action of $S$ on $\mathcal{Q}(N)$. Let $\omega$ be a faithful state on $N$ invariant for $\Psi_S$. Then there exists an action $\mathfrak{b}\Psi_S$ of the quantum Bohr compactification $\mathfrak{b}S$ on $\mathcal{Q}(N)$ such that

$$(\operatorname{id} \otimes \chi_S) \circ \mathfrak{b}\Psi_S = \Psi_S.$$  

Moreover $\mathfrak{b}\Psi_S$ is continuous and preserves $\omega$.

The state $\omega$ defines on $N$ a scalar product $(\cdot | \cdot)_{\omega}$ given by

$$(x | y)_{\omega} = \omega(x^* y)$$

for all $x, y \in N$.

**Proof of Theorem 2.1.** Let $\{e_1, \ldots, e_n\}$ be a basis of $N$ which is orthonormal for the scalar product $(\cdot | \cdot)_{\omega}$. Let $u$ be the $n \times n$ matrix of elements $(u_{i,j})_{i,j=1,\ldots,n}$ of $M(A)$ defined by

$$\Psi_S(e_j) = \sum_{i=1}^n e_i \otimes u_{i,j}.$$  

Then $u$ is a finite dimensional representation of $S$. We have

$$u^* u = 1$$

because the basis $\{e_1, \ldots, e_n\}$ is orthonormal for $(\cdot | \cdot)_{\omega}$. Now the Podleś condition for $\Psi_S$ means that $u$ can be considered as a map $A^n \to A^n$. As such a map it is an isometry (of the Hilbert module $A^n$) and has dense range. Thus $u$ is unitary. In particular $u$ is invertible.

Let $\overline{u}$ be the matrix obtained from $u$ by taking adjoint of each matrix entry of (but not transposing the matrix). We will show that $\overline{u}$ is invertible.

Let $T$ be the (invertible) matrix $(\tau_{k,l})_{k,l=1,\ldots,n}$ such that

$$e_i^* = \sum_{p=1}^n \tau_{p,i} e_p.$$  

Then

$$\Psi_S(e_j^*) = \sum_{i=1}^n e_i^* \otimes u_{ij}^* = \sum_{i,p=1}^n \tau_{p,i} e_p \otimes u_{i,j}^*.$$  

On the other hand

$$\Psi_S(e_j^*) = \sum_{i=1}^n \tau_{i,j} \Phi(e_i) = \sum_{p,i=1}^n \tau_{i,j} e_p \otimes u_{p,i}.$$  

Therefore for each $p$ and $j$ we have

$$\sum_{i=1}^n \tau_{p,i} u_{i,j}^* = \sum_{i=1}^n u_{p,i}^* \tau_{i,j}.$$
This means that
\[(T \otimes 1)\overline{u} = u(T \otimes 1).\]
Therefore \(\overline{u}\) is invertible if and only if \(u\) is invertible and the latter fact is already established. Let \(u^T\) be the transpose matrix of \(u\). Then \(u^T = (\overline{u})^*\), so \(u^T\) also is invertible. In other words, in the terminology of \([7]\), the matrix \(u\) is a finite dimensional admissible representation of \(\mathcal{S}\).

This means that the image of the map \(\Psi_S\) is contained in \(N \otimes \mathcal{A}F(\mathcal{S}) \subset N \otimes \mathcal{M}(\mathcal{A})\). Let \(b\Psi_S\) be the map \(\Psi_S\) considered as a mapping from \(N\) to \(N \otimes \mathcal{A}F(\mathcal{S})\).

It is clear that \(\Psi_S\) is a morphism (it is a unital map of unital \(C^*\)-algebras) and satisfies \([1]\). Also it is easy to see that \(b\Psi_S\) is an action of \(b\mathcal{S}\) on \(\mathcal{Q}(\mathcal{M}_2)\) preserving \(\omega\). To see that \(b\Psi_S\) is continuous note that from \(uu^* = 1\) it follows that
\[
\sum_{j=1}^n b\Psi_S(e_j)(1 \otimes u_{i,j}^*) = e_i \otimes 1.
\]
Thus the Podleś condition is satisfied for \(b\Psi_S\).

\[\square\]

**Remark 2.2.** Let \(\mathcal{S}, N, \omega\) and \(\Psi_S\) be as in Theorem 2.1. Assume further that the action of \(\mathcal{S}\) on \(\mathcal{Q}(\mathcal{N})\) is ergodic, i.e. that for any \(n \in N\) the condition that \(\Psi_S(n) = n \otimes 1\) implies \(n \in \mathfrak{C}_1\). Then the associated action \(b\Psi_S\) of the quantum Bohr compactification of \(\mathcal{S}\) on \(\mathcal{Q}(\mathcal{N})\) is ergodic as well.

**Proposition 2.3.** Let \(N\) be a finite dimensional \(C^*\)-algebra and let \(\mathcal{G} = (\mathcal{B}, \Delta_B)\) be a compact quantum group. Let \(\Phi_G \in \text{Mor}(N, N \otimes \mathcal{B})\) be a continuous action of \(\mathcal{G}\) on \(\mathcal{Q}(\mathcal{N})\). Then there exists a faithful state on \(N\) invariant for \(\Phi_G\).

**Proof.** It is noted in \([1]\) Remark 2 after Lemma 4 that if \(\Phi_G\) is an injective and the Haar measure \(h_G\) of \(\mathcal{G}\) is faithful then the map
\[E : N \ni n \longmapsto (\text{id} \otimes h_G)\Phi_G(n) \in N\]
is faithful. Therefore if \(\varphi\) is a faithful state on \(N\) then \(\omega = \varphi \circ E\) also a faithful state. Now \(\omega\) is invariant for \(\Phi_G\) (\([8]\) Section 3).

Since \(\mathcal{G}\) might not have a faithful Haar measure, we consider the action of the reduced quantum group \(\mathcal{G}_r = (\mathcal{B}_r, \Delta_B)\) (\([12]\) Page 656). If \(\lambda \in \text{Mor}(\mathcal{B}, \mathcal{B}_r)\) is the reducing morphism, we let \(\Phi_{\mathcal{G}_r} = (\text{id} \otimes \lambda) \circ \Phi_G\). Then \(\Phi_{\mathcal{G}_r}\) is an action of \(\mathcal{G}_r\) on \(\mathcal{Q}(\mathcal{N})\) which is continuous. Note that the Haar measure of \(\mathcal{G}_r\) is faithful.

By \([8]\) Theorem 6.8 the continuity of \(\Phi_{\mathcal{G}_r}\) guarantees that there exists a subalgebra \(N\) on \(N\) such that \(\Phi_{\mathcal{G}_r}\), restricted to \(N\), is a coaction of the Hopf *-algebra \(\mathcal{B} \subset B_r\) on \(N\) which is dense in \(N\). Since \(N\) is finite dimensional, we have \(N = N\) and consequently, using the counit of \(\mathcal{B}\), we immediately find that \(\ker \Phi_{\mathcal{G}_r} = \{0\}\).

So far we have shown that there is a faithful state \(\omega\) on \(N\) invariant for \(\Phi_{\mathcal{G}_r}\) (because the Haar measure of the reduced group is faithful). All that remains it so note that \(\omega\) is also invariant for \(\Phi_G\). Indeed, for \(n \in N\) the element \(\Phi_G(n)\) belongs to \(N \otimes \mathcal{B}\). Since the reducing map \(\lambda\) is \(\Phi_G(n)\) belongs to \(N \otimes \mathcal{B}\) we have \(\Phi_{\mathcal{G}_r}(n) = \Phi_G(n)\) for each \(n\) and
\[(\omega \otimes \text{id})\Phi_G(n) = \omega(n)1.\]

\[\square\]

**Theorem 2.4.** Let \(\mathcal{S} = (\mathcal{A}, \Delta_A)\) be a quantum semigroup and let \(N\) be a finite dimensional \(C^*\)-algebra. Let \(\Psi_S \in \text{Mor}(N, N \otimes \mathcal{A})\) be a continuous action of \(\mathcal{S}\) on \(\mathcal{Q}(\mathcal{N})\). Then \(\Psi_S\) preserves a faithful state if and only if there is an action \(b\Psi_S\) of \(b\mathcal{S}\) on \(\mathcal{Q}(\mathcal{N})\) such that
\[(\text{id} \otimes \chi_S) \circ b\Psi_S = \Psi_S.\]

**Proof.** Theorem 2.1 provides the “only if” and we only need to prove the “if” part. If \(b\Psi_S\) is an action of \(b\mathcal{S}\) on \(\mathcal{Q}(\mathcal{N})\) satisfying (2), then \(b\Psi_S\) is continuous. Therefore, by Proposition 2.3 there is a faithful state \(\omega\) on \(N\) invariant for \(b\Psi_S\). Clearly \(\omega\) is also invariant for \(\Psi_S\).

\[\square\]
3. Quantum Semigroup Actions on \( \mathcal{QS}(M_2) \)

We shall now specify the situation to the case \( N = M_2 \). We will give a complete classification of all continuous quantum semigroup actions preserving a faithful state on \( \mathcal{QS}(M_2) \). To that end we will use Theorem 2.1 to reduce any such action to a continuous action of a compact quantum group. Then we will simply use the classification of such actions given in [8].

Now for \( \omega \in [0,1] \) let \( S_\omega O(3) = (C(S_\omega O(3)), \Delta_{S_\omega O(3)}) \) be the quantum \( S_\omega O(3) \) group of Podleś (Remark 3), [5 Section 3] and [8 Sections 3 & 4]). For each \( q \) there is an action \( \Psi_q \in \text{Mor}(M_2, M_2 \otimes C(S_q O(3))) \) of \( S_q O(3) \) on \( \mathcal{QS}(M_2) \) coming from the standard action of \( S_q U(2) \) on \( \mathcal{QS}(M_2) \) (Subsection 3.2, see also [2 Lemma 2.1]).

Quantum \( SO(3) \) groups can be defined as universal compact quantum groups acting on \( \mathcal{QS}(M_2) \) and preserving the Powers state
\[
\omega_q : M_2 \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{1}{1+q^2} (a + q^2d)
\]
(c.f. [8]). Up to conjugation by a unitary \( 2 \times 2 \) matrix these are all the faithful states on \( M_2 \). This fact made it possible to give in [8] a general theorem classifying all compact quantum group actions on \( M_2 \) (Theorem 6.1). Combining this result with Theorem 2.1 yields the following:

**Theorem 3.1.** Let \( \mathcal{S} = (A, \Delta_A) \) be a quantum semigroup. Let \( \Psi_S \in \text{Mor}(M_2, M_2 \otimes A) \) be a continuous action of \( \mathcal{S} \) on \( \mathcal{QS}(M_2) \) preserving a faithful state. Then there exists a number \( q \in [0,1] \), a unitary \( u \in M_2 \) and a morphism \( \Gamma \in \text{Mor}(C(S_q O(3)), A) \) such that
\[
\Psi_S(m) = (\text{id} \otimes \Gamma)((u \otimes 1)\Psi_q(u^*mu)(u^* \otimes 1)),
\]
for all \( m \in N \). Moreover \( \Gamma \) and \( u \) are unique for each \( q \).

Theorem 3.1 is just a restatement of the universal property of Podleś quantum \( SO(3) \) groups. Namely, if \( \omega \) is a faithful state on \( M_2 \), then there exist a unique \( q \) and a unique \( u \) (as in the theorem) such that \( \omega(uxu^*) = \omega_q(x) \) for all \( x \in M_2 \). Therefore if \( \Psi_S \) is a continuous action of \( \mathcal{S} = (A, \Delta_A) \) on \( \mathcal{QS}(M_2) \) preserving \( \omega \) then the associated action \( b\Psi_S \) of \( b\mathcal{S} \) on \( \mathcal{QS}(M_2) \) also preserves \( \omega \). One can check that the mapping
\[
M_2 \ni m \mapsto (u^* \otimes 1)b\Psi_S(unu^*)(u \otimes 1) \in M_2 \otimes \mathcal{AP}(\mathcal{S})
\]
is a continuous action of \( b\mathcal{S} \) on \( \mathcal{QS}(M_2) \) preserving \( \omega_q \). The universal property of the Podleś groups now guarantees the existence of a unique \( \Gamma_0 \in \text{Mor}(C(S_q O(3)), \mathcal{AP}(\mathcal{S})) \) intertwining this action with \( \Psi_q \) (c.f. [8]). The morphism \( \Gamma \) in Theorem 3.1 is simply the composition of \( \Gamma_0 \) with \( \chi_S \).

By Remark 2.2, if \( \Psi_S \) is ergodic, then so is \( b\Psi_S \). Therefore the invariant state is unique ([11 Lemma 4]). Therefore we have:

**Corollary 3.2.** Let \( \mathcal{S} = (A, \Delta_A) \) be a quantum semigroup. Let \( \Psi_S \in \text{Mor}(M_2, M_2 \otimes A) \) be a continuous ergodic action of \( \mathcal{S} \) on \( \mathcal{QS}(M_2) \) preserving a faithful state. Then there exists a unique number \( q \in [0,1] \), a unique unitary \( u \in M_2 \) and a unique morphism \( \Gamma \in \text{Mor}(C(S_q O(3)), A) \) such that
\[
\Psi_S(m) = (\text{id} \otimes \Gamma)((u \otimes 1)\Psi_q(u^*mu)(u^* \otimes 1)),
\]
for all \( m \in N \).

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