Reducing SAT to Max2XOR

Carlos Ansótegui
Logic & Optimization Group (LOG), University of Lleida, Spain

Jordi Levy
IIIA-CSIC, Spain

Abstract
Representing some problems with XOR clauses (parity constraints) can allow to apply more efficient reasoning techniques. In this paper, we present a gadget for translating SAT clauses into Max2XOR constraints, i.e., XOR clauses of at most 2 variables equal to zero or to one. Additionally, we present new resolution rules for the Max2XOR problem which asks for which is the maximum number of constraints that can be satisfied from a set of 2XOR equations.

2012 ACM Subject Classification
Theory of computation → Proof theory

Keywords and phrases
Satisfiability, SAT, MaxSAT, Max2XOR, gadgets

Funding
Carlos Ansótegui: This work is supported by Grant PID2019-109137GB-C21 funded by MCIN/AEI/10.13039/501100011033.
Jordi Levy: This work is supported by Grant PID2019-109137GB-C21 funded by MCIN/AEI/10.13039/501100011033.

1 Introduction
Exclusive OR (XOR), here written ⊕, may be an alternative to the use of traditional OR to represent propositional formulas. By writing clauses $x_1 \oplus \cdots \oplus x_k$ as constraints $x_1 \oplus \cdots \oplus x_k = 1$, where 1 means true and 0 false, we can avoid the use of negation, because $\neg x \oplus C = k$ is equivalent to $x \oplus C = k \oplus 1$, where by abuse of notation, ⊕ on the right-hand side denotes the addition modulo 2. The equivalent to the resolution rule for XOR constraints, called XOR resolution rule, is

$$
\begin{align*}
X \oplus A &= k_1 \\
X \oplus B &= k_2 \\
A \oplus B &= k_1 \oplus k_2
\end{align*}
$$

where $X$, $A$ and $B$ are clauses. In the particular case of $A = B = \emptyset$ and $k_1 \neq k_2$, this rule concludes a contradiction, that we represent as ⊥. The proof system containing only this rule allows us to produce polynomial refutations for any unsatisfiable set of XOR constraint. Therefore, unless $P = NP$, we cannot express any propositional formula as an equivalent conjunction of XOR constraints. It is also well-known that a set of XOR-constraints can be solved in polynomial time using Gaussian elimination. In practice, many approaches combining CNF and XOR reasoning have been presented \cite{14, 13, 3, 15, 8, 7, 18, 10, 16, 9, 11, 17}.

In this paper, we focus on the problem of reducing the satisfiability of a CNF formula (SAT) to the satisfiability of the maximum number of XOR constraints (MaxXOR), in particular, of XOR constraints with two or less variables (Max2XOR).

We will consider weighted constraints, noted $(w) C = k$, where $w$ is a rational number that denotes the contribution of the satisfiability of the constraint to the formula. This way, we can translate every binary clause $x \lor y$ as \{$(1/2) x = 1$, $(1/2) y = 1$, $(1/2) x \oplus y = 1$\}, because when $x$ or $y$ are equal to one (i.e. $x \lor y$ is satisfied), exactly two of the XOR constraints are satisfied, which contributes $1/2 + 1/2 = 1$ to the constraint, and when $x$ and $y$ are both
equal to zero and the original clause is falsified, none of the XOR constraints are satisfied. We can also translate ternary clauses like \(x \lor y \lor z\) as \(\{(1/4) x = 1, (1/4) y = 1, (1/4) z = 1\}, \{(1/4) x = 1, (1/4) y = 1, (1/4) y \oplus z = 1\}, \{(1/4) x \oplus y = 1, (1/4) y \oplus z = 1\}, \{(1/4) x \oplus y \oplus z = 1\}\). In general, any OR clause \(x_1 \lor \cdots \lor x_k\) is equivalent to the set of (weighted) XOR constraints:

\[
\bigcup_{S \subseteq \{1, \ldots, k\}} \{(1/2^{k-1}) \bigoplus_{i \in S} x_i = 1\}
\]

This translation allows us to reduce SAT to MaxXOR. However, the reduction is not polynomial, because it translates every clause of size \(k\) into \(2^k - 1\) constraints. In Section 4, we describe a polynomial reduction that avoids this exponential explosion on expenses of introducing new variables.

In Section 5, we will discuss the possible definition of a proof system for Max2XOR, in the spirit of the MaxSAT resolution which was first defined by [12], and proven complete by [3].

## 2 Preliminaries

A \(k\)-ary constraint function is a Boolean function \(f : \{0, 1\}^k \to \{0, 1\}\). A constraint family is a set \(\mathcal{F}\) of constraint functions (with possibly distinct arities). A constraint, over variables \(V = \{x_1, \ldots, x_n\}\) and constraint family \(\mathcal{F}\), is a pair formed by a \(k\)-ary constraint function \(f \in \mathcal{F}\) and a subset of \(k\) variables, noted \(f(x_{i_1}, \ldots, x_{i_k})\), or \(f(\vec{x})\) for simplicity. A (weighted) constraint problem or (weighted) formula \(I\), over variables \(V\) and constraint family \(\mathcal{F}\), is a set of pairs (weight, constraint) over \(V\) and \(\mathcal{F}\), where the weight is a positive rational number, denoted \(I = \{(w_1) f_1(x_{i_1}, \ldots, x_{i_k}), \ldots, (w_m) f_m(x_{i_1}, \ldots, x_{i_k})\}\). The weight of a problem is \(\text{Weight}(I) = w_1 + \cdots + w_m\).

An assignment is a function \(I : \{x_1, \ldots, x_n\} \to \{0, 1\}\). We say that an assignment \(I\) satisfies a constraint \(f(x_{i_1}, \ldots, x_{i_k})\), if \(I(f(\vec{x})) = \text{def} \ f(I(x_{i_1}), \ldots, I(x_{i_k})) = 1\). The value of an assignment \(I\) is a function \(I(f) = \sum_{i=1}^{m} w_i I(f_i(\vec{x}))\), is the sum of the weights of the constraints that this assignment satisfies, i.e. \(I(P) = \sum_{i=1}^{m} w_i I(f_i(\vec{x}))\). We also define the sum of weights of unsatisfied clauses as \(\overline{T}(I) = \sum_{i=1}^{m} w_i (1 - I(f_i(\vec{x})))\). An assignment \(I\) is said to be optimal for a constraint problem \(P\), if it maximizes \(\overline{T}(P)\). We define this optimum as \(\overline{\text{Opt}}(P) = \max_I \overline{T}(P)\). We also define \(\text{Cost}(P) = \min_I \overline{T}(P)\), i.e. the minimum sum of weights of falsified constraints, that fulfills \(\overline{\text{Opt}}(P) + \text{Cost}(P) = \text{Weight}(P)\).

For convenience, we generalize constraint problems to multisets, and consider \(\{(w_1) C, (w_2) C\}\) as equivalent to \(\{(w_1) C, (w_1+w_2) C\}\). When the weight of a constraint is not explicitly specified, it is assumed to be one.

When using inference rules, we assume that they can be applied with any weight and that premises are decomposed conveniently. For instance, if we have the inference rule \(C = 0, C = 1 \vdash \Box\), and we apply it to the problem \(P = \{(w_1) A = 0, (w_2) A = 1\}\), assuming \(w_1 \geq w_2\), we proceed as follows. First, we decompose the first constraint to get \(P' = \{(w_2) A = 0, (w_1-w_2) A = 0, (w_2) A = 1\}\). Then, we apply the rule with weight \(w_2\) to get \(P'' = \{(w_2) \Box, (w_1-w_2) A = 0\}\). When applying a rule, we assume that at least one of the premises is not decomposed.

Many optimization problems may be formalized as the problem of finding an optimal assignment for a constraint problem. In the following we define some of them:

1. MaxEkSAT is the constraint family defined by the constraint functions of the form \(f(x_{i_1}, \ldots, x_{i_k}) = l_1 \lor \cdots \lor l_k\), where every \(l_i\) may be either \(x_i\) or \(\neg x_i\).
2. MaxkSAT is the union of the constraint families MaxEiSAT, for \(0 \leq i \leq k\), and MaxSAT the union, for every \(i \geq 0\).
3. MaxEkPC_0 is the constraint family defined by the constraint function \( f(x_1, \ldots, x_k) = 1 - x_1 \oplus \cdots \oplus x_k \). Similarly, MaxEkPC_1 is defined by \( f(x_1, \ldots, x_k) = x_1 \oplus \cdots \oplus x_k \), and MaxEkXOR or MaxEkPC is the union of MaxEkPC_0 and MaxEkPC_1.

4. Max2XOR or MaxPC is the union of MaxEiXOR, for \( 0 \leq i \leq k \), and MaxXOR or MaxPC the union, for every \( i \geq 0 \).

5. MaxCUT is the same as MaxE2PC_1.

In the literature, PC (similarly for PC_0 and PC_1) use to denote what we denote as MaxE3PC. However, notice that, in this paper, we are interested in the problem Max2XOR, that has got very little attention in the literature, probably because it is quite similar to MaxCUT. Notice that MaxCUT use to be defined as the problem of, given a graph, finding a partition of the vertices that maximizes the number of edges that connect vertices of distinct partition. The problem can be trivially formalized assigning a Boolean variable to each vertex, and a constraint \( x \oplus y = 1 \) to each edge connecting \( x \) and \( y \).

Next, we define gadgets as a transformation from constraints into sets of weighted constraints (or problems). These transformations can be extended to define transformations of problems into problems.

**Definition 1.** Let \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) be two constraint families. A (strict) \((\alpha, \beta)\)-gadget from \( \mathcal{F}_1 \) to \( \mathcal{F}_2 \) is a function that, for any constraint \( f(\vec{x}) \) over \( \mathcal{F}_1 \) returns a weighted constraint problem \( P = \{(w_i)g_i(\vec{x}, \vec{a})\}_{i=1, \ldots, m} \) over \( \mathcal{F}_2 \) and variables \( \{\vec{x}\} \cup \{\vec{a}\} \), where \( \vec{a} \) are fresh variables distinct from \( \vec{x} \), such that \( \beta = \sum_{i=1}^{m} w_i \) and, for any assignment \( I : \{\vec{x}\} \to \{0, 1\} : 
1. If \( I(f(\vec{x})) = 1 \), for any extension of \( I \) to \( I' : \{\vec{x}\} \cup \{\vec{a}\} \to \{0, 1\} \), \( I'(P) \leq \alpha \) and there exist one of such extension with \( I'(P) = \alpha \).
2. If \( I(f(\vec{x})) = 0 \), for any extension of \( I \) to \( I' : \{\vec{x}\} \cup \{\vec{a}\} \to \{0, 1\} \), \( I'(P) \leq \alpha - 1 \) and there exist one of such extension with \( I'(P) = \alpha - 1 \).

**Lemma 2.** The composition of a \((\alpha_1, \beta_1)\)-gadget from \( \mathcal{F}_1 \) to \( \mathcal{F}_2 \) and a \((\alpha_2, \beta_2)\)-gadget from \( \mathcal{F}_2 \) to \( \mathcal{F}_3 \) results into a \((\beta_1(\alpha_2 - 1) + \alpha_1, \beta_1\beta_2)\)-gadget from \( \mathcal{F}_1 \) to \( \mathcal{F}_3 \).

**Proof.**

The first gadget multiplies the total weight of constraints by \( \beta_1 \), and the second by \( \beta_2 \). Therefore, the composition multiplies it by \( \beta = \beta_1\beta_2 \).

For any assignment, if the original constraint is falsified, the optimal extension after the first gadget satisfies constraints with a weight of \( \alpha_1 - 1 \), and falsifies the rest \( \beta_1 - (\alpha_1 - 1) \). The second gadget satisfies constraints for a weight of \( \alpha_2 - 1 \) of the falsified plus \( \alpha_2 \) of the satisfied. Therefore, the composition satisfies constraints with a total weight \( (\alpha_2 - 1)(\beta_1 - (\alpha_1 - 1)) + \alpha_2(\alpha_1 - 1) = \beta_1(\alpha_2 - 1) + \alpha_1 - 1 \).

If the original constraint is satisfied, the optimal extension after the first gadget satisfies \( \alpha_1 \) and falsifies the rest \( \beta_1 - \alpha_1 \). After the second gadget the weight of satisfied constraints is \( (\alpha_2 - 1)(\beta_1 - \alpha_1) + \alpha_2\alpha_1 = \beta_1(\alpha_2 - 1) + \alpha_1 \).

The difference between both situations is one, hence the composition is a gadget, and \( \alpha = \beta_1(\alpha_2 - 1) + \alpha_1 \).

**Lemma 3.** If \( P \) is translated into \( P' \) using a \((\alpha, \beta)\)-gadget, then

\[
\text{Opt}(P') = (\alpha - 1)\text{Weight}(P) + \text{Opt}(P)
\]
\[
\text{Cost}(P') = (\beta - \alpha)\text{Weight}(P) + \text{Cost}(P)
\]

Previous lemma allows us to obtain an algorithm to maximize \( P \), using an algorithm to maximize \( P' \). If this algorithm is an approximation algorithm on the optimum, in order to get the minimal error, we are interested in gadgets with minimal \( \alpha \). That’s because,
traditionally, a gadget is called optimal if it minimizes $\alpha$. In our case, we will use proof systems that derive empty clauses, i.e. that prove lower bounds for the cost. Therefore, in our case, we will be interested in gadgets that minimize $\beta - \alpha$.

In the case that $P$ is a decision problem, then we can say that $P$ is unsatisfiable if, and only if, $\text{Cost}(P') \geq (\beta - \alpha) \text{Weight}(P) + 1$.

### 3 Some Simple Gadgets

As we described in the introduction, there exists a simple translation from OR clauses to XOR constraints that does not need to introduce new variables. In particular, for binary clauses it is as follows.

**Lemma 4.** The following set of transformations:

\[
\begin{align*}
  x & \rightarrow \{(1) \ x = 1\} \\
  \neg x & \rightarrow \{(1) \ x = 0\} \\
  x \lor y & \rightarrow \{(1/2) \ x = 1, \ (1/2) \ y = 1, \ (1/2) \ x \oplus y = 1\} \\
  x \lor \neg y & \rightarrow \{(1/2) \ x = 1, \ (1/2) \ y = 0, \ (1/2) \ x \oplus y = 0\} \\
  \neg x \lor y & \rightarrow \{(1/2) \ x = 0, \ (1/2) \ y = 1, \ (1/2) \ x \oplus y = 0\} \\
  \neg x \lor \neg y & \rightarrow \{(1/2) \ x = 0, \ (1/2) \ y = 0, \ (1/2) \ x \oplus y = 1\}
\end{align*}
\]

defines a \((1, 3/2)\)-gadget from Max2SAT to Max2XOR.

Notice that $x_1 \oplus \cdots \oplus x_n = 0$ and $x_1 \oplus \cdots \oplus x_n = 1$ are incompatible constraints, hence they can be canceled. Notice also that the order of the variables is irrelevant and $x \oplus x \oplus C = C$. Therefore, we can simplify any PC problem to get an equivalent problem where, for every subset of variables, there is only one constraint.

**Example 5.** Given the Max2SAT problem:

\[
\begin{align*}
  (1) \ y \\
  (2) \ x \lor y \\
  (3) \ x \lor \neg y \\
  (1) \ \neg x \lor y \\
  (2) \ y \lor \neg z \\
  (3) \ \neg y \lor z
\end{align*}
\]

the gadget described in Lemma 4, and its simplification results into the Max2XOR problem:

\[
\begin{align*}
  (1) \ x = 0 \\
  (1/2) \ y = 1 \\
  (3/2) \ z = 1
\end{align*}
\]

We can go further and reduce Max2XOR to MaxCUT. Since the constraint family MaxCUT is a subset of the constraint family Max2XOR, the most natural is to make the translation as follows.

**Lemma 6.** Given a Max2XOR problem, adding a particular variable, called $\hat{0}$, and applying the following transformations:

\[
\begin{align*}
  x = 1 & \rightarrow \{x \oplus \hat{0} = 1\} \\
  x = 0 & \rightarrow \{x \oplus a = 1, \ a \oplus \hat{0} = 1\} \\
  x \oplus y = 1 & \rightarrow \{x \oplus y = 1\} \\
  x \oplus y = 0 & \rightarrow \{x \oplus b = 1, \ b \oplus y = 1\}
\end{align*}
\]

where $a$ and $b$ are fresh variables, we get a \((2, 2)\)-gadget from Max2XOR to MaxCUT.
Notice that, strictly speaking, the transformations described in Lemma 6 do not define a gadget, since the auxiliary variable \( \hat{0} \) is not local to a constraint, but global to the whole problem.

Alternatively, we can reduce the number of auxiliary variables and constraints if we add two special auxiliary variables called \( \hat{0} \) and \( \hat{1} \) and we use the transformations:

\[
\begin{align*}
x = 1 & \rightarrow \{ x \oplus \hat{0} = 1 \} \\
x = 0 & \rightarrow \{ x \oplus \hat{1} = 1 \} \\
x \oplus y = 1 & \rightarrow \{ x \oplus y = 1 \} \\
x \oplus y = 0 & \rightarrow \{ x \oplus b = 1, \ b \oplus y = 1 \}
\end{align*}
\]

where, if \( P \) is the original Max2XOR problem, \( W = \min \left\{ \sum_{(w) \ x=0 \in P} w, \sum_{(w) \ x=1 \in P} w \right\} \).

The third constraint is added only once and depends on the whole original problem \( P \).

This special constraint ensures that optimal solutions \( I \) of the resulting MaxCUT problem satisfy \( I(\hat{0}) \neq I(\hat{1}) \). Let \( w_i = \sum_{(w) x=i \in P} w \), for \( i = 0, 1 \), i.e. \( w_i \) is the sum of the weights of constraints of the form \( (w) x = i \). With this alternative transformation, we observe that, from a set of constraints of the form \( x = 0 \) and \( x = 1 \) with total weight \( w_0 + w_1 \), we get a set of constraints of the form \( x \oplus \hat{0} = 1, x \oplus \hat{1} = 1 \) and \( \hat{0} \oplus \hat{1} = 1 \) with total weight \( w_0 + w_1 + \min\{w_0, w_1\} \leq 3/2(w_0 + w_1) \).

# From MaxSAT to Max2XOR

An easy way to reduce MaxSAT to Max2XOR is to reduce MaxSAT to Max3SAT, this to Max2SAT, and this to Max2XOR using Lemma 4. We can reduce MaxSAT to Max3SAT with the following transformation:

\[
\begin{align*}
x_1 \lor \cdots \lor x_k \rightarrow & \{ x_1 \lor x_2 \lor b_1, \ \neg b_1 \lor x_3 \lor b_2, \cdots , \neg b_{k-4} \lor x_{k-2} \lor b_{k-3}, \ \neg b_{k-3} \lor x_{k-1} \lor x_k \} \quad (1)
\end{align*}
\]

that defines a \((k-2, k-2)\)-gadget from MaxEkSAT to Max3SAT.

Later, we can use the following \((3, 5, 4)\)-gadget from Max3SAT to Max2SAT defined by Trevisan et al. [19]:

\[
(1) \quad x_1 \lor x_2 \lor x_3 \rightarrow \{ \begin{align*}
(1/2) x_1 \lor x_3, & \ (1/2) \neg x_1 \lor \neg x_3, \\
(1/2) x_1 \lor \neg b, & \ (1/2) \neg x_1 \lor b, \\
(1/2) x_3 \lor \neg b, & \ (1/2) \neg x_3 \lor b, \\
(1) x_2 \lor b \}
\end{align*} \quad (2)
\]

The concatenation of Trevisan’s gadget (2) and the gadget in Lemma 4 results into the \((2, 3)\)-gadget described in Figure 1.

## 4.1 A Sequential Translation

Trevisan’s gadget [19] is optimal. However, when concatenated with gadget [4] results into a \((2(k-2), 3(k-2))\)-gadget from MaxEkSAT to Max2XOR, that is not optimal (in the sense that it is not the one with minimal \( \alpha \)). In the following we describe a better and direct reduction from MaxEkSAT to Max2XOR.

The reduction is based on a function \( T^0 \) that takes as parameters a SAT clause and a variable. The use of this variable is for technical reasons, to make simpler the recursive
Reducing SAT to Max2XOR

\[ x_1 \oplus x_3 = 1 \]
\[ x_1 \oplus b = 0 \]
\[ x_3 \oplus b = 0 \]
\[ x_2 \oplus b = 1 \]
\[ b = 1 \]
\[ x_2 = 1 \]

**Figure 1** Graphical representation of the (2, 3)-gadget reducing Max3SAT to Max2XOR and based on [19] gadget from Max3SAT to Max2SAT. All constraints have weight 1/2. Blue edges represent equal-one constraints, and red dashed edges equal-zero constraints.

\[ b_1 \]
\[ b_2 \]
\[ \ldots \]
\[ b_{k-2} \]
\[ b_{k-1} \]

**Figure 2** Graphical representation of \( T^0(x_1 \lor \cdots \lor x_k, b_{k-1}) \). Blue edges represent \( 2PC_1 \) constraints and red dashed edges \( 2PC_0 \) constraints, all them with weight 1/2.

Definition and proof of Lemma 8 Later (see Theorem 9), it will be replaced by the constant one.

**Definition 7.** Given a SAT clause \( C \) and a variable \( b \), the translation function \( T^0(C, b) \) is defined recursively as the set of 2XOR clauses

\[ T^0(x_1 \lor x_2, b) = \{ (1/2) x_1 \oplus x_2 = 1, \ (1/2) x_1 \oplus b = 0, \ (1/2) b \oplus x_2 = 0 \} \]

for binary clauses, and

\[ T^0(x_1 \lor \cdots \lor x_k, b) = T^0(x_1 \lor \cdots \lor x_{k-1}, b') \cup \{ (1/2) b' \oplus x_k = 1, \ (1/2) b' \oplus b = 0, \ (1/2) b \oplus x_k = 0 \} \]

for \( k \geq 3 \), where \( b' \) is a fresh variable not occurring elsewhere.

**Lemma 8.** Consider the Max2XOR problem \( T^0(x_1 \lor \cdots \lor x_k, b_{k-1}) \). We have:

1. For any assignment \( I : \{ x_1, \ldots, x_k \} \rightarrow \{0, 1\} \), the extension of the assignment as \( I'(b_i) = I(x_{i+1}) \), for \( i = 1, \ldots, k-1 \), maximizes the number of satisfied 2XOR constraints, that is \( 2(k-1) \).

2. For any assignment \( I : \{ x_1, \ldots, x_k, b_{k-1} \} \rightarrow \{0, 1\} \) satisfying \( I(b_{k-1}) = 1 \), the extension of the assignment as

\[ I'(b_i) = \begin{cases} 1 & \text{if } I(x_{i+2}) = \cdots = I(x_k) = 0 \\ I(x_{i+1}) & \text{otherwise} \end{cases} \]

for \( i = 1, \ldots, k-2 \), maximizes the number of satisfied 2XOR constraints, that is \( 2(k-2) \), if \( I(x_1) = \cdots = I(x_k) = 0 \), or \( 2(k-1) \) otherwise.
Proof. For the first statement: the proof is by induction on \(k\).

For \(k = 2\), we never can satisfy the three constraints \(T^0(x_1 \lor x_2, b_1) = \{x_1 \oplus x_2 = 1, \ x_1 \oplus b_1 = 0, \ b_1 \oplus x_2 = 0\}\). We can satisfy two of them if we set \(b_1\) equal to \(x_1\) or to \(x_2\) (we choose the second possibility).

For \(k > 2\), we have:

\[
T^0(x_1 \lor \cdots \lor x_k, b_{k-1}) = T^0(x_1 \lor \cdots \lor x_{k-1}, b_{k-2}) \cup \{(1/2) b_{k-2} \oplus x_k = 1, (1/2) b_{k-2} \oplus b_{k-1} = 0, (1/2) b_{k-1} \oplus x_k = 0\}
\]

We can never satisfy all three constraints \(\{b_{k-2} \oplus x_k = 1, b_{k-2} \oplus b_{k-1} = 0, b_{k-1} \oplus x_k = 0\}\).

By setting \(I'(b_{k-1}) = I(x_k)\), it does not matter what value \(b_{k-2}\) gets, we always satisfy two of them. By induction hypothesis, we know that setting \(I'(b_i) = I(x_{i+1})\), for \(i = 1, \ldots, k-2\), we maximize the number of satisfied constraints, i.e. \(2(k-2)\), from the rest of constraints in \(T^0(x_1 \lor \cdots \lor x_{k-1}, b_{k-2})\). Therefore, we can satisfy a maximum of \(2(k-1)\) constraints from \(T^0(x_1 \lor \cdots \lor x_k, b_{k-1})\).

For the second statement: the proof is also by induction on \(k\).

For \(k = 2\) and \(I(b_1) = 0\), the number of satisfied constraints is zero, if \(I(x_1) = I(x_2) = 0\), and two, otherwise.

For \(k > 2\), there are two cases:

If \(I(x_k) = 0\) and \(I(b_{k-1}) = 1\), assigning \(I(b_{k-2}) = 1\) we maximize the number of satisfied constraints involving \(b_{k-2}\), satisfying at least two of them, whereas assigning \(I(b_{k-2}) = 0\), we can satisfy at most two of them. Hence, we take the first option. Then, we use the induction hypothesis to prove that we can satisfy other \(2(k-2)\) constraints if at least some of the \(x_i\) is one, for \(i = 1, \ldots, k-1\), or only \(2(k-3)\), if all them are zero.

If \(I(x_k) = 1\) and \(I(b_{k-1}) = 1\), it does not matter what value \(b_{k-2}\) gets, we satisfy two constraints from \(\{b_{k-2} \oplus x_k = 1, b_{k-2} \oplus b_{k-1} = 0, b_{k-1} \oplus x_k = 0\}\). We use then the first statement of the lemma, and the same assignment for \(b_i\)'s, to prove that the proposed assignment \(I'\) maximizes the satisfaction of the rest of constraints, i.e. \(2(k-2)\) of them. \(\blacksquare\)

**Theorem 9.** The translation of every clause \(x_1 \lor \cdots \lor x_k\) into \(T^0(x_1 \lor \cdots \lor x_k, \hat{1})\) defines a \((k-1, 3/2(k-1))\)-gadget from \(\text{MaxEKSAT}\) to \(\text{Max2XOR}\).

Proof. The soundness of the reduction is based on the second statement of Lemma\(^8\). Assume that we force the variable \(\hat{1}\) to be interpreted as one, i.e. \(I(\hat{1}) = 1\), or that, alternatively, we simply replace it by the constant 1. Any assignment satisfying the clause can extend to satisfy at least \(2(k-1)\) 2XOR constraints from \(T^0(x_1 \lor \cdots \lor x_k, \hat{1})\), with a total weight equal to \(k-1\). Any assignment falsifying it can only be extended to satisfy at most \(2(k-2)\) 2XOR constraints, with a total weight \(k-2\). Hence, \(\alpha = k-1\). The number of 2XOR constraints is \(3(k-1)\), all of them with weight 1/2. Therefore, \(\beta = 3/2(k-1)\).

We only describe the translation of clause \(x_1 \lor \cdots \lor x_k\) when all variables are positive. We can easily generalize the transformation, when any of the variables \(x\) are negated, simply by recalling that \(\neg x \oplus b = k\) is equivalent to \(x \oplus b = 1 - k\), and \(\neg x \oplus \neg x' = k\) is equivalent to \(x \oplus x' = k\). \(\blacksquare\)

**Remark 10.** Consider the Max2XOR problem \(P = T^0(x_1 \lor \cdots \lor x_k, b_{k-1})\).

For any assignment \(I : \{x_1, \ldots, x_k\} \to \{0, 1\}\), its extension, defined by \(I(b_i) = I(x_i) \lor \cdots \lor I(x_{i+1})\), for \(i = 1, \ldots, k-1\) (similar but not equal to the one proposed in the proof of the first statement of Lemma\(^8\) also) maximizes the number of satisfied constraints of \(P\), satisfying \(I(P) = k-1\).
For any assignment $I : \{x_1, \ldots, x_k, b_{k-1}\} \rightarrow \{0, 1\}$, where $I(b_{k-1}) = 1$, its extension $I(b_i) = I(x_1) \lor \cdots \lor I(x_{i+1})$, for $i = 1, \ldots, k - 2$ (similar but not equal to the one proposed in the proof of the second statement of Lemma 8 also) maximizes the number of satisfied constraints of $P$, satisfying $I(P) = k - 1$, when $I(x_1 \lor \cdots \lor x_k) = 1$, and $I(P) = k - 2$, when $I(x_1 \lor \cdots \lor x_k) = 0$.

### 4.2 A Parallel Translation

The previous reduction of MaxSAT into Max2XOR can be seen as a kind of sequential checker. We can use a sort of parallel checker. Before describing the reduction formally, we introduce an example.

▶ Example 11. The clause $x_1 \lor \cdots \lor x_7$ can be reduced to the following Max2XOR problem, where all XOR constraints have weight $1/2$:

```
\begin{align*}
    & x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6 \\
    & b_1 \quad b_2 \quad b_3 \\
    & b_4 \quad b_5 \\
    & \hat{1}
\end{align*}
```

▶ Definition 12. Given a clause $x_1 \lor \cdots \lor x_k$ and a variable $b$, their translation into Max2XOR constraints is defined recursively and non-deterministically as:

$$
T(x_1 \lor \cdots \lor x_k, b) =
\begin{cases}
    \{x_1 \oplus x_2 = 1, x_1 \oplus b = 0, x_2 \oplus b = 0\} & \text{if } k = 2 \\
    \{b' \oplus x_k = 1, b' \oplus b = 0, x_1 \oplus b = 0\} \cup T(x_2 \lor \cdots \lor x_{k-1}, b') & \text{if } k \geq 3, \text{ where } b' \text{ is a fresh variable} \\
    \{b' \oplus x_1 = 1, b' \oplus b = 0, x_1 \oplus b = 0\} \cup \{b'' \oplus x_k = 1, b'' \oplus b = 0, x_1 \oplus b = 0\} \cup T(x_2 \lor \cdots \lor x_r, b') \cup T(x_{r+1} \lor \cdots \lor x_k, b'') \quad \text{if } k \geq 4, \text{ where } b' \text{ and } b'' \text{ are fresh variables, and } 2 \leq r \leq k - 2
\end{cases}
$$

where all XOR constraints have weight $1/2$.

Notice that the definition of $T$ is not deterministic. When $k = 3$ we have two possible translations (applying the second or third case), and when $k \geq 4$ we can apply the second, third and forth cases, with distinct values of $r$, getting as many possible translations as possible binary trees with $k$ leaves.

Notice also that the definition of $T^0$ corresponds to the definition of $T$ where we only apply the first and second cases. Therefore, $T$ is a generalization of $T^0$.

▶ Lemma 13. Consider the set of 2XOR constraints of $T(x_1 \lor \cdots \lor x_k, b)$ without weights. We have:
1. There are $3(k-1)$ constraints.
2. Any assignment satisfies at most $2(k-1)$ of them.
3. Any assignment $I : \{x_1, \ldots, x_k\} \rightarrow \{0,1\}$ can be extended to an optimal assignment satisfying $I(b) = I(x_1 \lor \cdots \lor x_r)$ and $2(k-1)$ constraints.
4. Any assignment $I$ satisfying $I(x_1 \lor \cdots \lor x_k) = 0$ and $I(b) = 1$ can be extended to an optimal assignment satisfying $2(k-2)$ constraints.

**Proof.** The first statement is trivial.

For the second, notice that for, each one of the $k-1$ triangles of the form $\{a \oplus b = 1, a \oplus c = 0, b \oplus c = 0\}$, any assignment can satisfy at most two of the constraints of each triangle.

The third statement is proved by induction. The base case, for binary clauses is trivial. For the induction case, if we apply the second or third options of the definition of $T$ (the other cases are quite similar). We have to consider 3 possibilities:

If we extend $I(b') = I(b'') = 1$, by induction, we can extend $I$ to satisfy $2(r-2)$ constraints of $T(x_1 \lor \cdots \lor x_r, b')$ and $2((k-r)-2)$ constraints of $T(x_{r+1} \lor \cdots \lor x_k, b'')$. It satisfies 2 constraints from $\{b' \oplus b'' = 1, b' \oplus b = 0, b'' \oplus b = 0\}$, hence a total of $2(k-3)$ constraints.

If we extend $I(b') = I(b'') = 0$, by the third statement of this lemma, we can extend $I$ to satisfy $2(r-1)$ constraints of $T(x_1 \lor \cdots \lor x_r, b')$ and $2((k-r)-1)$ constraints of $T(x_{r+1} \lor \cdots \lor x_k, b'')$. It does not satisfy any constraints from $\{b' \oplus b'' = 1, b' \oplus b = 0, b'' \oplus b = 0\}$, hence a total of $2(k-2)$ constraints.

Finally, if we extend $I(b') = 1$ and $I(b'') = 0$ (or vice versa), by induction, we can extend $I$ to satisfy $2(r-2)$ constraints of $T(x_1 \lor \cdots \lor x_r, b')$ and by the third statement $2((k-r)-2)$ constraints of $T(x_{r+1} \lor \cdots \lor x_k, b'')$. It also satisfy 2 constraints from $\{b' \oplus b'' = 1, b' \oplus b = 0, b'' \oplus b = 0\}$, hence a total of $2(k-2)$ constraints.

> **Theorem 14.** The translation of every clause $x_1 \lor \cdots \lor x_k$ into $T(x_1 \lor \cdots \lor x_k, \tilde{1})$ define a $(k-1,3/2(k-1))$-gadget from MaxEkSAT into Max2XOR.

**Proof.** The theorem is a direct consequence of the two last statements of Lemma 13.
SAT instances (see [2][1]) we can determine how distant are two variables in the original SAT problem, or even better, we can obtain a dendrogram of these variables. Then, we can use this distance or the dendrogram to construct a better translation, i.e. a tree where distant variables are kept in distinct branches. This will have the effect of increasing the modularity of the problem, hence of decreasing its difficulty. We think that, using these trees for industrial SAT instances, we will obtain better experimental results than with the sequential translation.

5 On Proof Systems for Max2XOR

We can define a proof system for MaxSAT based on the MaxSAT resolution rule [12][4][5]:

\[
\begin{align*}
  x \lor A \\
  \neg x \lor B \\
  A \lor B \\
  x \lor A \lor \neg B \\
  \neg x \lor B \lor \neg A
\end{align*}
\]

where premises are replaced by the conclusions. This rule generalizes the traditional SAT resolution rule \( x \lor A, \neg x \lor B \vdash A \lor B \), but not only preserving satisfiability, but also the number of unsatisfied clauses, for any interpretation of the variables. In other words, we say that an inference rule \( P \vdash C \) is sound if \( I(P) = I(C) \), for any assignment \( I \) of the variables.

This implies that, if we can derive \( P \vdash \{m\} \cup P' \), where \( P' \) is satisfiable, then \( m \) is equal to the minimum sum of weights of unsatisfied clauses, i.e. \( \text{Cost}(P) = m \). Moreover, this rule is complete, in the sense that when \( m \) is the minimum number of unsatisfiable clauses \( m = \text{Cost}(P) \), we can construct a MaxSAT refutation proof of the form \( P \vdash \{m\} \cup P' \), where \( P' \) is satisfiable.

Gadget transformations can also be written in the form of inference rules. We know that a \((\alpha, \beta)\)-gadget from \( P \) to \( P' \) satisfies \( \text{Cost}(P') = (\beta - \alpha)\text{Weight}(P) + \text{Cost}(P) \). Therefore, unless \( \alpha = \beta \), the weight of unsatisfiable constraints is shifted. If we want to write an inference rule that preserves the sum of weights of unsatisfied constraints, we have to add an empty clause with weight \( (\beta - \alpha)\text{Weight}(P) \) to the premises, or an empty clause with the weight negated to the conclusions (on the well-understanding that this empty clause will be canceled by other proved empty clauses). For instance, the Max2SAT to Max2XOR \((1,3/2)\)-gadget that transforms \( a \lor b \) to \( \{(1/2) a = 1, (1/2) b = 1, (1/2) a \oplus b = 1\} \), and the \((1,7/4)\)-gadget from Max3SAT to Max3XOR, can be written as:

\[
\begin{align*}
  a \lor b \\
  \frac{(-1/2) \Box}{(1/2) a = 1} \quad \frac{(1/2) b = 1}{(1/2) a \oplus b = 1}
\end{align*}
\]

\[
\begin{align*}
  a \lor b \lor c \\
  \frac{(-3/4) \Box}{(1/4) a = 1} \quad \frac{(1/4) b = 1}{(1/4) c = 1} \\
  \frac{(1/4) a \oplus b = 1}{(1/4) b \oplus c = 1} \quad \frac{(1/4) a \oplus c = 1}{(1/4) a \oplus b \oplus c = 1}
\end{align*}
\]

Something similar can be done with the MaxSAT to Max2XOR gadget described in Section 4.

5.1 A Polynomial and Incomplete Proof System for Max2XOR

With respect to a proof system for Max2XOR, our first step is to use a mixture of XOR and SAT constraints. Since we are able to reduce SAT to XOR constraints of size 2, we will
only consider XOR constraints of this size. The proof system is described by the rules of Figure 3 to which we refer as Max2XOR proof system. Notice that, in all rules, the SAT clauses prevent the assignments that falsify the two premises. Notice also that, in this first version, none of the rules transforms the SAT constraints, and the SAT constraints are never resolved. Therefore, in the best case, the application of the rules to a XOR problem finishes with a set of SAT constraints.

\[ x \oplus a = 0 \]
\[ x \oplus b = 0 \]
\[ a \oplus b = 0 \]
\[ C = 0 \]

\[ (2) \ x \lor \neg a \lor \neg b \]
\[ (2) \ \neg x \lor a \lor b \]

\[ x = 0 \]
\[ x \oplus a = 0 \]
\[ a = 0 \]
\[ (2) \ \neg x \lor a \]

\[ x = 1 \]
\[ x \oplus a = 1 \]
\[ a = 1 \]
\[ (2) \ \neg x \lor \neg a \]

\[ (2) \ x \lor a \]

\[ \neg x \lor a \]
\[ \neg x \lor \neg a \]
\[ \neg a \lor b \]
\[ \neg a \lor b \]
\[ \neg a \lor b \]
\[ \neg a \lor b \]
\[ \neg a \lor b \]
\[ \neg a \lor b \]
\[ \neg a \lor b \]

\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]

\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]

\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]

\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]

\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]

\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]

\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]

\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]

\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]

\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]

\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]

\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]

\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]

\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]

\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]

\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]

\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]

\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]

\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]

\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]

\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]

\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]

\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]

\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]

\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]

\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]

\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]

\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]

\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]

\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]

\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]

\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]

\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]

\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]

\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]

\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]

\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]

\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]

\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]

\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]

\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]

\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]

\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]

\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]

\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]

\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]

\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]

\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]

\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]

\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]

\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]

\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]

\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]

\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]
\[ \neg b \lor a \]

\[ \neg b \lor a \]
For the second statement, notice that all rule applications remove at least one XOR constraint.

Notice that we cannot prove a completeness result for this proof system. In other words, unless $P = NP$, we cannot prove that any Max2XOR formula $P$, admits a derivation $P \vdash \{(m) \square\} \cup P'$, where $P'$ is satisfiable. Therefore, the proof system is polynomial, but incomplete.

5.2 Managing SAT constraints in Max2XOR Resolution rules

In our second version of the Max2XOR proof system, in contrast to the first version, we do not ignore the SAT constraints produced by the Max2XOR resolution rules.

We can simply use our gadget for SAT into Max2XOR to translate the SAT clauses produced by the Max2XOR resolution rules. We can also come up with a more compact representation with fewer fresh variables. For example, for the resolution rule on premises $x \oplus a = 0, x \oplus b = 0$ we could propose the following resolution rule:

\[
\begin{align*}
x \oplus a &= 0 \\
x \oplus b &= 0 \\
a \oplus b &= 1 \\
(2) \ x \oplus y &= 0 \\
(2) \ a \oplus y &= 0 \\
(2) \ b \oplus y &= 0
\end{align*}
\]

as an alternative to:

\[
\begin{align*}
x \oplus a &= 0 \\
x \oplus b &= 0 \\
a \oplus b &= 0 \\
(2) \ x \lor \neg a \lor \neg b \\
(2) \ \neg x \lor a \lor b
\end{align*}
\]

where $y$ is a fresh variable. (Notice that $a \oplus b = 0$ is changed by $a \oplus b = 1$ in the conclusions).

The second and third rules in Figure 3 can also be modified in a similar way.

Notice that whether we use the gadget or the more compact version provided above, we will introduce auxiliary variables. If we translate the SAT constraints with gadgets in (3), we can avoid the introduction of fresh variables. However, in this case, the conclusions are transformed in exactly the same set of XOR constraints occurring as premises. In the following, we explain further why we do need to introduce auxiliary variables while in the MaxSAT resolution rules this is not needed.

When dealing with XOR constraints, things are more complicated than with SAT clauses. The basic reason is that, given a function from a set of $n$ Boolean values to positive rational numbers $f : \{0, 1\}^n \rightarrow \mathbb{Q}^+$, representing the sum of the weights of satisfied clauses for a given assignment, we may have several ways to represent it as a SAT problem $P$, such that $f(I) = I(P)$. For instance, the function $f(a, b, c) = 2 + a(1 - c) + cb$ may be represented as $P_1 = \{\top, a \lor c, b \lor \neg c\}$, or as $P_2 = \{a \lor b, c \lor a \lor \neg b, \neg c \lor b \lor \neg a\}$.

Basically, because there are $2^n$ distinct assignments to the variables and $3^n$ distinct possible clauses (notice that a variable can appear with positive or negative polarity, or not appear in a constraint). However, using XOR constraints, since there are at most $2^n$ possible clauses (since negative polarities are not used in XOR constraints), there is only one way to represent one of such functions. In our example, $f$ can only be represented as $P_3 = \{(3/2) \top, (1/2) a = 1, (1/2) b = 1, (1/2) x \oplus a = 1, (1/2) x \oplus b = 0\}$. The same applies if we want to preserve the sum of the weights of unsatisfied clauses, as in the MaxSAT resolution rule. This means that the only way of writing an inference rule for Max2XOR, in

---

1 Notice the similarity between these problems and the premises and conclusions of the MaxSAT resolution rule.
the style of MaxSAT resolution, preserving the sum of the weights of unsatisfied clauses, is by introducing fresh variables.

Another important question is how to use the incomplete proof system on practice. Suppose that, starting with $P$ we get $\{m\} \cup P'$, and no inference rule is applicable. If $P'$ is satisfiable, we are finished (in this case, the incomplete proof system has been able to refute the formula). Otherwise, we can translate all introduced SAT clauses into 2XOR constraints, using our gadget, and repeat the process. In order to shorten this process, we have to try to get a formula $P'$ that has a small cost, or in other words, a value $m$ for the weight of $\Box$ as high as possible.

6 Conclusions

We have presented a new gadget from SAT into Max2XOR. We have also presented a first version of an incomplete proof system for Max2XOR, and pointed out some ideas to get a complete version. The final definition of this system, and the proof of completeness is left as future work. The practical implementation of a Max2XOR solver based on these ideas is also a future work.

This opens the avenue to explore the practical power of a new proof system for SAT, where the clauses in the input SAT instance are translated into Max2XOR and the Max2XOR resolution rules are applied to them.

References

1. Carlos Ansótegui, Maria Luisa Bonet, Jesús Giráldez-Cru, Jordi Levy, and Laurent Simon. Community structure in industrial SAT instances. J. Artif. Intell. Res., 66:443–472, 2019. URL: https://doi.org/10.1613/jair.1.11741
2. Carlos Ansótegui, Jesús Giráldez-Cru, and Jordi Levy. The community structure of SAT formulas. In Alessandro Cimatti and Roberto Sebastiani, editors, Theory and Applications of Satisfiability Testing - SAT 2012 - 15th International Conference, Trento, Italy, June 17-20, 2012. Proceedings, volume 7317 of Lecture Notes in Computer Science, pages 410–423. Springer, 2012. URL: https://doi.org/10.1007/978-3-642-31612-8_31
3. Peter Baumgartner and Fabio Massacci. The taming of the (X)OR. In John W. Lloyd, Verónica Dahl, Ulrich Furbach, Manfred Kerber, Kung-Kiu Lau, Catuscia Palamidessi, Luís Moniz Pereira, Yehoshua Sagiv, and Peter J. Stuckey, editors, Computational Logic - CL 2000, First International Conference, London, UK, 24-28 July, 2000, Proceedings, volume 1861 of Lecture Notes in Computer Science, pages 508–522. Springer, 2000. URL: https://doi.org/10.1007/3-540-44957-4_34
4. Maria Luisa Bonet, Jordi Levy, and Felip Manyà. A complete calculus for Max-SAT. In Armin Biere and Carla P. Gomes, editors, Theory and Applications of Satisfiability Testing - SAT 2006, 9th International Conference, Seattle, WA, USA, August 12-15, 2006, Proceedings, volume 4121 of Lecture Notes in Computer Science, pages 240–251. Springer, 2006. URL: https://doi.org/10.1007/11814948_24
5. Maria Luisa Bonet, Jordi Levy, and Felip Manyà. Resolution for Max-SAT. Artif. Intell., 171(8-9):606–618, 2007. URL: https://doi.org/10.1016/j.artint.2007.03.001
6. Jingchao Chen. Building a hybrid SAT solver via conflict-driven, look-ahead and XOR reasoning techniques. In Oliver Kullmann, editor, Theory and Applications of Satisfiability Testing - SAT 2009, 12th International Conference, SAT 2009, Swansea, UK, June 30 - July 3, 2009. Proceedings, volume 5584 of Lecture Notes in Computer Science, pages 298-
7 Marijn Heule, Mark Dufour, Joris van Zwieten, and Hans van Maaren. March_eq: Implementing additional reasoning into an efficient look-ahead SAT solver. In Holger H. Hoos and David G. Mitchell, editors, Theory and Applications of Satisfiability Testing, 7th International Conference, SAT 2004, Vancouver, BC, Canada, May 10-13, 2004, Revised Selected Papers, volume 3542 of Lecture Notes in Computer Science, pages 345–359. Springer, 2004. URL: https://doi.org/10.1007/11527695_29

8 Marijn Heule and Hans van Maaren. Aligning CNF- and equivalence-reasoning. In SAT 2004 - The Seventh International Conference on Theory and Applications of Satisfiability Testing, 10-13 May 2004, Vancouver, BC, Canada, Online Proceedings, 2004. URL: http://www.satisfiability.org/SAT04/programme/72.pdf

9 Tero Laitinen, Tommi A. Junttila, and Ilkka Niemelä. Equivalence class based parity reasoning with DPLL(XOR). In IEEE 23rd International Conference on Tools with Artificial Intelligence, ICTAI 2011, Boca Raton, FL, USA, November 7-9, 2011, pages 649–658. IEEE Computer Society, 2011. URL: https://doi.org/10.1109/ICTAI.2011.103

10 Tero Laitinen, Tommi A. Junttila, and Ilkka Niemelä. Conflict-driven XOR-clause learning. In Alessandro Cimatti and Roberto Sebastiani, editors, Theory and Applications of Satisfiability Testing - SAT 2012 - 15th International Conference, Trento, Italy, June 17-20, 2012. Proceedings, volume 7317 of Lecture Notes in Computer Science, pages 383–396. Springer, 2012. URL: https://doi.org/10.1007/978-3-642-31612-8_29

11 Tero Laitinen, Tommi A. Junttila, and Ilkka Niemelä. Extending clause learning SAT solvers with complete parity reasoning. In IEEE 24th International Conference on Tools with Artificial Intelligence, ICTAI 2012, Athens, Greece, November 7-9, 2012, pages 65–72. IEEE Computer Society, 2012. URL: https://doi.org/10.1109/ICTAI.2012.18

12 Javier Larrosa and Federico Heras. Resolution in Max-SAT and its relation to local consistency in weighted csps. In IJCAI, pages 193–198, 2005.

13 Chu Min Li. Equivalency reasoning to solve a class of hard SAT problems. Inf. Process. Lett., 76(1-2):75–81, 2000. URL: https://doi.org/10.1016/S0020-0190(00)00126-5

14 Chu Min Li. Integrating equivalency reasoning into Davis-Putnam procedure. In Henry A. Kautz and Bruce W. Porter, editors, Proceedings of the Seventeenth National Conference on Artificial Intelligence and Twelfth Conference on on Innovative Applications of Artificial Intelligence, July 30 - August 3, 2000, Austin, Texas, USA, pages 291–296. AAAI Press / The MIT Press, 2000. URL: http://www.aaai.org/Library/AAAI/2000/aaai00-045.php

15 Chu Min Li. Equivalent literal propagation in the DLL procedure. Discret. Appl. Math., 130(2):251–276, 2003. URL: https://doi.org/10.1016/S0166-218X(02)00407-9

16 Mate Soos. Enhanced gaussian elimination in DPLL-based SAT solvers. In Daniel Le Berre, editor, POS-10. Pragmatics of SAT, Edinburgh, UK, July 10, 2010, volume 8 of EPiC Series in Computing, pages 2–14. EasyChair, 2010. URL: https://easychair.org/publications/paper/j1D

17 Mate Soos and Kuldeep S. Meel. BIRD: engineering an efficient CNF-XOR SAT solver and its applications to approximate model counting. In The Thirty-Third AAAI Conference on Artificial Intelligence, AAAI 2019, The Thirty-First Innovative Applications of Artificial Intelligence Conference, IAAI 2019, The Ninth AAAI Symposium on Educational Advances in Artificial Intelligence, EAAI 2019, Honolulu, Hawaii, USA, January 27 - February 1, 2019, pages 1592–1599. AAAI Press, 2019. URL: https://doi.org/10.1609/aaai.v33i01.33011592
Mate Soos, Karsten Nohl, and Claude Castelluccia. Extending SAT solvers to cryptographic problems. In Oliver Kullmann, editor, Theory and Applications of Satisfiability Testing - SAT 2009, 12th International Conference, SAT 2009, Swansea, UK, June 30 - July 3, 2009. Proceedings, volume 5584 of Lecture Notes in Computer Science, pages 244–257. Springer, 2009. URL: https://doi.org/10.1007/978-3-642-02777-2_24, doi: 10.1007/978-3-642-02777-2_24.

Luca Trevisan, Gregory B. Sorkin, Madhu Sudan, and David P. Williamson. Gadgets, approximation, and linear programming. SIAM J. Comput., 29(6):2074–2097, 2000.