MILIN’S COEFFICIENTS, COMPLEX GEOMETRY OF TEICHMÜLLER SPACES AND VARIATIONAL CALCULUS FOR UNIVALENT FUNCTIONS

SAMUEL L. KRUSHKAL

Abstract. We investigate the invariant metrics and complex geodesics in the universal Teichmüller space and Teichmüller space of the punctured disk using Milin’s coefficient inequalities. This technique allows us to establish that all non-expanding invariant metrics in either of these spaces coincide with its intrinsic Teichmüller metric.

Other applications concern the variational theory for univalent functions with quasiconformal extension. It turns out that geometric features caused by the equality of metrics and connection with complex geodesics provide deep distortion results for various classes of such functions and create new phenomena which do not appear in the classical geometric function theory.

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1. Key theorems on invariant metrics and geodesics

1.1. Preamble. The Milin coefficient inequalities arose as a generalization of the classical Grunsky inequalities but coincide with the later only for conformal maps of the unit disk.

We apply a quasiconformal variant of these inequalities to investigation of complex metric geometry and complex geodesics on two Teichmüller spaces: the universal space and Teichmüller space of the punctured disk and apply their geometry to variational calculus for univalent functions on the generic quasidisks with quasiconformal extensions. Such functions play an important role in the theory of Teichmüller spaces and also form one of the basic classes in geometric function theory.

It will be shown that the intrinsic geometric features provide deep distortion results, in particular, allow one to solve explicitly some general variational problems. On the other hand, they cause surprising phenomena which do not arise in the classical variational theory for univalent functions.

1.2. Main property of invariant metrics of Teichmüller spaces. We shall use the notations \( \hat{\mathbb{C}} = \mathbb{C} \cup \{ \infty \} \), \( \mathbb{D} = \{ |z| < 1 \} \), \( \mathbb{D}^* = \hat{\mathbb{C}} \setminus \mathbb{D} = \{ |z| > 1 \} \) and consider two Teichmüller spaces: the universal Teichmüller space \( T = T(\mathbb{D}) \) and the Teichmüller space \( T_1 = T(\mathbb{D}^0) \) of the punctured disk \( \mathbb{D}^0 = \mathbb{D} \setminus \{ 0 \} \) endowed with the homotopy class of quasiconformal homeomorphisms containing the identity map and regarded as the base point of \( T(\mathbb{D}^0) \). The space \( T_1 \) is model for Teichmüller spaces of punctured disks with arbitrary number of punctures and even for more general flat Riemann surfaces.

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Every Teichmüller space \( \tilde{T} \) is a complex Banach manifold, thus it possesses the invariant Carathéodory and Kobayashi distances (the smallest and the largest among all holomorphically non-expanding metrics). Denote these metrics by \( c_\tilde{T} \) and \( d_\tilde{T} \), and let \( \tau_\tilde{T} \) be the intrinsic Teichmüller metric of this space canonically determined by quasiconformal maps. The corresponding infinitesimal Finsler metrics (defined on the tangent bundle \( \mathcal{T}\tilde{T} \) of \( \tilde{T} \)) are denoted by \( C_\tilde{T} \) and \( K_\tilde{T} \) and \( F_\tilde{T} \), respectively. Then

\[
c_\tilde{T}(\cdot, \cdot) \leq d_\tilde{T}(\cdot, \cdot) \leq \tau_\tilde{T}(\cdot, \cdot),
\]

and by the Royden-Gardiner theorem the metrics \( d_\tilde{T} \) and \( \tau_\tilde{T} \) (and their infinitesimal forms) are equal, see, e.g. [EKK], [EM], [GL], [Ro].

In view of applications, we mainly focus on the Carathéodory metric of the space \( T_1 \) and first establish that it equals the Teichmüller metric. This yields the all non-expanding invariant metrics on \( T_1 \) agree with \( \tau_\tilde{T} \), and the Teichmüller extremal disks are geodesic with respect to all invariant metrics.

**Theorem 1.1.** The Carathéodory metric of the space \( T_1 \) coincides with its Kobayashi metric, hence all invariant non-expanding metrics on \( T_1 \) are equal its Teichmüller metric, and

\[
c_{T_1}(\varphi, \psi) = d_{T_1}(\varphi, \psi) = \tau_{T_1}(\varphi, \psi) = \inf\{d_{\mathbb{D}}(h^{-1}(\varphi), h^{-1}(\psi)) : h \in \text{Hol}(\mathbb{D}, T_1)\},
\]

where \( d_{\mathbb{D}} \) denotes the hyperbolic metric of the unit disk of curvature \(-4\).

Similarly, the infinitesimal forms of these metrics coincide with the Finsler metric \( F_{T_1}(\varphi, v) \) generating \( \tau_{T_1} \), and have holomorphic sectional curvature \(-4\).

Such a result is known only for the universal Teichmüller space and underlies various applications; its proof was given in [Kr4] (and somewhat modified in [Kr7]). In view of importance, we present this fact here as a separate theorem giving its simplified proof and new applications.

**Theorem 1.2.** All invariant non-expanding metrics on the universal Teichmüller space \( T \) are equal to its Teichmüller metric.

The proof of both theorems involves the Grunsky-Milin coefficient inequalities.

### 1.3. Complex geodesics

The equality of metrics allows one to describe complex geodesics in the spaces \( T \) and \( T_1 \). Let \( \tilde{T} \) denote either of these spaces.

Recall that if \( X \) is a domain in a complex Banach space \( E \) endowed with a pseudo-distance \( \rho_X \), then a holomorphic map \( h : \mathbb{D} \rightarrow X \) is called a complex \( \rho \)-geodesic if there exist \( t_1 \neq t_2 \) in \( \mathbb{D} \) such that

\[
d_{\mathbb{D}}(t_1, t_2) = \rho_X(h(t_1), h(t_2));
\]

one says also that the points \( h(t_1) \) and \( h(t_2) \) can be joined by a complex \( \rho \)-geodesic (see [Ve]).

If \( h \) is a complex \( c_X \)-geodesic then it also is a \( d_X \)-geodesic and the above equality holds for all points \( t_1, t_2 \in \mathbb{D} \), so \( h(\mathbb{D}) \) is a holomorphic disk in \( X \) hyperbolically isometric to \( \mathbb{D} \). As an important consequence of Theorems 1.1. and 1.2, one gets the following result where the complex geodesics are understanding in the strongest sense, i.e., as \( c_\tilde{T} \)-geodesics.

**Theorem 1.3.** (i) Any two points of the space \( \tilde{T} \) can be joined by a complex geodesic. The geodesic joining a Strebel’s point with the base point is unique and defines the corresponding Teichmüller extremal disk.

(ii) For any point \( \varphi \in \tilde{T} \) and any nonzero tangent vector \( v \) at this point, there exists at least one complex geodesic \( h : \mathbb{D} \rightarrow \tilde{T} \) such that \( h(0) = \varphi \) and \( h'(0) \) is collinear to \( v \).
1.4. **Geometric and analytic features.** The following consequence of Theorem 1.1 relates to pluripotential features of $T$ and is useful in variational problems on compact subsets of $\Sigma^0(D)$.

**Corollary 1.4.** Any non-expanding invariant metrics $\rho$ on the space $T_1$ with the base point $D^0$ relates to the similar metric $\rho_{B_k}$ on hyperbolic balls $B_k = \{ \psi \in T_1 : \tau_T(\psi, 0) < \tanh k \}$ ($0 < k < 1$) by

$$\rho_{B_k}(\psi_1, \psi_2) = \tanh^{-1}\left(\frac{l(\rho_T(\psi_1, \psi_2))}{k}\right) = d_{\overline{D}}(0, l(d_T(\psi_1, \psi_2))) = \tanh s.$$  

Similar relation holds for the pluricomplex Green functions of the space $T_1$ and its balls $B_k(T_1)$.

For the universal Teichmüller space $T$, this was established in [Kr5]. The proof for the space $T_1$ follows the same lines using Theorem 1.1.

This assertion is obtained from Theorem 1.1 using the arguments applied in [Kr5] for the Kobayashi metric of universal Teichmüller space.

It is not known, how to relate the invariant distances of the balls in generic complex manifolds $X$ with the corresponding distances on $X$.

The following corollary controls the growth of holomorphic maps of $\tilde{T}$ on geodesic disks.

**Corollary 1.5.** If a holomorphic map $J : \tilde{T} \to \mathbb{D}$ into the unit disk is such that its restriction to a geodesic disk $\mathbb{D}(\mu_0) = \{ \phi_T(t\mu_0)/\|\mu_0\|_\infty : |t| < 1 \}$ has at the origin zero of order $m$, i.e.,

$$J_{\mu_0}(t) := J \circ \phi_T(t\mu_0/\|\mu_0\|_\infty) = c_m t^m + c_{m+1} t^{m+1} + \ldots, \tag{1.3}$$

then the growth of $|J|$ on this disk is estimated by

$$|J_{\mu_0}(t)| \leq \tanh\left(\frac{|t|^m |t| + |c_m|}{1 + |c_m| |t|}\right) \leq d_{\tilde{T}}(0, \phi_T(t^m \frac{\mu_0}{\|\mu_0\|_\infty})) \tag{1.4}.$$\n
The equality in the right inequality occurs (even for one $t_0 \neq 0$) only when $|c_m| = 1$; then $J_{\mu_0}(t)$ is a hyperbolic isometry of the unit disk and all terms in (1.3) are equal.

1.5. The above theorems and corollaries have deep applications to geometric function theory. Some of those are presented in the last two sections.

2. **Background**

We recall some notions and results which will be used in the proofs of the above theorems.

2.1. **Invariant metrics on Teichmüller spaces.** Let $L$ be a bounded oriented quasicircle in the complex plane $\mathbb{C}$ with the interior and exterior domains $D$ and $D^*$ so that $D^*$ contains the infinite point $z = \infty$. Consider the unit ball of Beltrami coefficients supported on $D$,  

$$\text{Belt}(D)_1 = \{ \mu \in L_\infty(C) : \mu(z)|D^* = 0, \|\mu\|_\infty < 1 \}$$

and their pairing with $\psi \in L_1(D)$ by

$$\langle \mu, \psi \rangle_D = \int_D \mu(z)\psi(z)dx dy \ (z = x + iy).$$

The following two sets of holomorphic functions $\psi$ (equivalently, of holomorphic quadratic differentials $\psi dz^2$)

$$A_1(D) = \{ \psi \in L_1(D) : \psi \text{ holomorphic in } D \}$$

$$A_2(D) = \{ \psi = \omega^2 \in A_1(D) : \omega \text{ holomorphic in } D \}$$

are intrinsically connected with the extremal Beltrami coefficients (hence, with the Teichmüller norm) and Grunsky-Milin inequalities.

The elements of $A_2^2$ can be regarded as the squares of abelian holomorphic differentials on $D$. 


Rescaling the domain $D^*$ to have $D^* = f^{\mu_0}(D^*)$ for some $f^{\mu_0}(z) = z + b_0 + b_1 z^{-1} + \ldots$ preserving $z = 0$ with $\mu_0 \in \text{Belt}(D_1)$, one can use this domain as a new base point of the universal Teichmüller space $T$ whose points are the equivalence classes $[\mu]$ of $\mu \in \text{Belt}(D)$ so that

$$\mu_1 \sim \mu_2 \text{ if } w^{\mu_1}(z) = w^{\mu_2}(z) = f(z) \text{ on } \overline{D^*}.$$ 

We shall also denote such classes by $[f]$. This space is modeled as a bounded domain in the complex Banach space $B(D^*)$ of the Schwarzian derivatives

$$S_w = (w''/w')' - (w''/w')^2/2, \quad w = f^\mu|D^*,$$

of locally univalent functions on $D_0^*$ with norm $\|\phi\| = \sup_{D_0^*} |\phi(z)|$, where $\lambda_{D_0^*}(z)|dz|$ is the differential hyperbolic metric on $D^*$ of curvature $-4$. The modeling domain is filled by the Schwarzians of globally univalent functions on $D^*$ with quasiconformal extension.

For the unit disk $\mathbb{D}$, $\lambda_{\mathbb{D}}(z) = 1/(1 - |z|^2)$, and the global hyperbolic distance

$$d_{\mathbb{D}}(z_1, z_2) = \tanh^{-1}[(z_1 - z_2)/(1 - \overline{z}_1 z_2)].$$

The intrinsic Teichmüller metric of the space $T$ is defined by

$$\tau_T(\phi_T(\mu), \phi_T(\nu)) = \frac{1}{2} \inf \{ \log K(\mu_* \circ (\nu_*)^{-1}) : \mu_* \in \phi_T(\mu), \nu_* \in \phi_T(\nu) \},$$

where $\phi_T$ is the factorizing holomorphic projection $\text{Belt}(D) \to T$. This metric is the integral form of the infinitesimal Finsler metric (structure)

$$F_T(\phi_T(\mu), \phi_T(\nu)) = \inf \| \nu_*/(1 - |\mu|^2)^{-1} \|_{\infty} : \phi_T(\mu) \nu_* = \phi_T(\nu) \mu$$

on the tangent bundle $TT$ of $T$, which is locally Lipschitzian (see [EF]).

Note also that $\tau_T(0, S_f) = \tanh^{-1} k(f)$, where $k(f)$ is the Teichmüller norm of a univalent function $f$.

The Kobayashi and Carathéodory metrics $d_T$ and $c_T$ of a Teichmüller space $\tilde{T}$ relate to complex structure of this space and are defined, respectively, as the largest pseudometric $d$ on $\tilde{T}$ which does not get increased by the holomorphic maps $h : \mathbb{D} \to \tilde{T}$ so that for any two points $\varphi_1, \varphi_2 \in \tilde{T}$, we have

$$d_{\tilde{T}}(\varphi_1, \varphi_2) \leq \inf \{ d_{\mathbb{D}}(0, t) : h(0) = \varphi_1, h(t) = \varphi_2 \},$$

and

$$c_{\tilde{T}}(\varphi_1, \varphi_2) = \sup \{ d_{\mathbb{D}}(h(\varphi_1), h(\varphi_2)) \},$$

taking the supremum over all holomorphic maps $h : \tilde{T} \to \mathbb{D}$.

The corresponding infinitesimal forms of the Kobayashi and Carathéodory metrics are defined for the points $(\varphi, v) \in TT$, respectively, by

$$K_{\tilde{T}}(\varphi, v) = \inf \{ 1/r : r > 0, h \in \text{Hol}(\mathbb{D}_r, \tilde{T}), h(0) = \varphi, h'(0) = v \},$$

$$C_{\tilde{T}}(\varphi, v) = \sup \{ |d_f(\varphi)v| : f \in \text{Hol}(\tilde{T}, \mathbb{D}), f(\varphi) = 0 \},$$

where $\text{Hol}(X, Y)$ denotes the collection of holomorphic maps of a complex manifold $X$ into $Y$ and $\mathbb{D}_r$ is the disk $\{ |z| < r \}$.

The sectional holomorphic curvature $\kappa_F(x, v)$ of a Finsler metric $F(x, v)$ on (the tangent bundle of) a complex Banach manifold $X$ is defined as the supremum of the generalized Gaussian curvatures

$$\kappa_\lambda(t) = -\frac{\Delta \log \lambda(t)}{\lambda(t)^2} \quad \text{for} \quad \lambda(t) = F(h(t), h'(t))$$

over appropriate collections of holomorphic maps $h$ from the disk into $X$ for a given tangent direction $v$ in the image. Here $\Delta$ means the generalized Laplacian

$$\Delta \lambda(t) = 4 \lim_{r \to 0} \frac{1}{r^2} \left\{ \frac{1}{2\pi} \int_0^{2\pi} \lambda(t + re^{i\theta})d\theta - \lambda(t) \right\}.$$
(provided that \( 0 \leq \lambda(t) < \infty \)). Similar to \( C^2 \) functions, for which \( \Delta \) coincides with the usual Laplacian \( 4\partial\bar{\partial} \), one obtains that \( \lambda \) is subharmonic on a domain \( \Omega \) if and only if \( \Delta \lambda(t) \geq 0 \); hence, at the points \( t_0 \) of local maxima of \( \lambda \) with \( \lambda(t_0) > -\infty \), we have \( \Delta \lambda(t_0) \leq 0 \).

Generically, the holomorphic curvature of the Kobayashi metric \( K_X(x,v) \) of any complete hyperbolic manifold \( X \) satisfies \( K_X(x,v) \geq -4 \) at all points \( (x,v) \) of the tangent bundle \( T(X) \) of \( X \), and for the Carathéodory metric \( C_X \) we have \( K_X(x,v) \leq -4 \).

2.2. The Grunsky and Milin coefficients inequalities. Denote by \( \Sigma(D^*) \) the collection of univalent functions \( f \) in a quasidisk \( D^* \) with hydrodynamical expansion

\[
f(z) = z + b_0 + b_1z^{-1} + \ldots \text{ near } z = \infty, \tag{2.2}\]

and let \( \Sigma^0(D^*) \) denote its subset formed by functions having quasiconformal extensions across the boundary (hence to \( \mathbb{C} \)). Each \( f \in \Sigma(D^*) \) determines a holomorphic map

\[
- \log \frac{f(z) - f(\zeta)}{z - \zeta} = \sum_{m,n=1}^{\infty} \frac{\alpha_{mn}}{\chi(z)^m \chi(\zeta)^n} : (D^*_{\theta})^2 \to \hat{C} \tag{2.3}\]

where \( \chi \) is the conformal map of \( D^* \) onto the disk \( \mathbb{D}^* \) with \( \chi(\infty) = \infty, \chi'(\infty) > 0 \), and the Taylor coefficients \( \alpha_{mn} \) are called the Milin coefficients of \( f \). In the classical case \( D^* = \mathbb{D}^* \), those are the standard Grunsky coefficients.

Due to the Grunsky univalence theorem \([Gr]\) and its Milin’s extension \([Mi]\), a function \( f \) holomorphic near the infinity (with hydrodynamical normalization) is extended to a univalent function in the whole domain \( D^* \) if and only if its coefficients \( \alpha_{mn} \) satisfy the inequality

\[
\left| \sum_{m,n=1}^{\infty} \sqrt{mn} \alpha_{mn} x_m x_n \right| \leq 1
\]

for any sequence \( x = (x_n) \in l^2 \) with \( \|x\|^2 = \sum_{n=1}^{\infty} |x_n|^2 = 1 \). We denote the unit sphere of these Hilbert space by \( S(l^2) \) and call the quantity

\[
\kappa_{D^*}(f) := \sup \left\{ \left| \sum_{m,n=1}^{\infty} \sqrt{mn} \alpha_{mn} x_m x_n \right| : x = (x_n) \in S(l^2) \right\} \tag{2.4}\]

the Grunsky norm of \( f \) on \( D^* \). For \( D^* = \mathbb{D}^* \), we shall use simplified notations \( \Sigma \) and \( \kappa(f) \).

Noting that each coefficient \( \alpha_{mn}(f) \) in (1.2) is represented as a polynomial of a finite number of the initial coefficients \( b_1, b_2, \ldots, b_{m+n-1} \) of \( f \), one derives after normalizing quasiconformal extensions of \( f^\mu \) in \( D \) (for example, by \( f(0) = 0 \)) the holomorphic dependence of \( \beta_{mn}(f) \) on Beltrami coefficients \( \mu \) and on the Schwarzian derivatives \( S_f \) on \( D^* \) running over the universal Teichmüller space \( T \) with the base point \( \chi'(\infty)D^* \).

For any finite \( M, N \) and \( 1 \leq j \leq M, \ 1 \leq l \leq N \), we have

\[
\left| \sum_{m=j}^{M} \sum_{n=l}^{N} \sqrt{mn} \alpha_{mn} x_m x_n \right|^2 \leq \sum_{m=j}^{M} |x_m|^2 \sum_{n=l}^{N} |x_n|^2;
\]

this inequality is a consequence of Milin’s univalence theorems (cf. [Mi, p. 193], [Po, p. 61]). Thus for each \( x = (x_n) \in S(l^2) \), the function

\[
h_x(\mu) = \sum_{m,n=1}^{\infty} \sqrt{mn} \alpha_{mn}(f^\mu) x_m x_n \tag{2.5}\]

maps the space \( T \) holomorphically into the unit disk, and

\[
\sup_x |h_x(f^\mu)| = \kappa_{D^*}(f^\mu). \tag{2.6}\]
This implies also that the Grunsky norm $\kappa_D^*(f^\mu)$ is a continuous plurisubharmonic function on $\text{Belt}(D)_1$ and on the space $T$ (cf. [Kr2], [Kr8]).

The following key results obtained in [Kr2], [Kr8] by applying the maps (2.6) underly the proofs of Theorems 1.1 and 1.2.

**Proposition 2.1.** (a) The Grunsky norm $\kappa_D^*(f)$ of every function $f \in \Sigma^0(D^*)$ is estimated by its Teichmüller norm $k = k(f)$ by

$$\kappa_D^*(f) \leq k \frac{k + \alpha_D(f)}{1 + \alpha_D(f)k}, \quad (2.7)$$

where

$$\alpha_D(f^\mu) = \sup \{ \langle \mu, \varphi \rangle_D : \varphi \in A_1^2(D), \|\varphi\|_{A_1} = 1 \} \leq 1.$$ 

and $\kappa_D^*(f) < k$ unless $\alpha_D(f) = 1$. The last equality occurs if and only if $\kappa_D^*(f) = k(f)$.

(b) The equality $\kappa_D^* f = k(f)$ holds if and only if the function $f$ is the restriction to $D^*$ of a quasiconformal self-map $w^{\mu_0}$ of $\hat{C}$ with Beltrami coefficient $\mu_0$ satisfying the condition

$$\sup_\Sigma |\langle \mu_0, \varphi \rangle_D| = \|\mu_0\|_\infty, \quad (2.8)$$

where the supremum is taken over holomorphic functions $\varphi \in A_1^2(D)$ with $\|\varphi\|_{A_1} = 1$.

If, in addition, the equivalence class of $f$ (the collection of maps equal $f$ on $\partial D^*$) is a Strebel point, then $\mu_0$ is necessarily of the form

$$\mu_0(z) = \|\mu_0\|_\infty |\varphi_0(z)|/\varphi_0(z) \quad \text{with} \quad \varphi_0 \in A_1^2(D). \quad (2.9)$$

The condition (2.8) has a geometric nature. Its proof in [Kr2], [Kr8] shows that the functions (2.5) generate a maximizing sequence on which the Carathéodory distance $c_T(0, S_{\mu_0})$ is attained and equals the Teichmüller distance (compare with Kra’s theorem in [K] about the Carathéodory metric on Teichmüller abelian disks for Riemann surfaces with finitely generated fundamental groups).

In a special case, when the domain $D^*$ is the disk $\mathbb{D}^*$ and $f$ is analytic up to its boundary $\{|z| = 1\}$, the equality (2.9) was obtained by a different method in [Kn3].

**Remark.** The Grunsky coefficients were originally defined in [Gr] for finitely connected plane domains and have been later generalized to bordered Riemann surfaces (see [SS], [Le]). Milin’s coefficients were introduced in [Mi] also for finitely connected domains. Both types of coefficients coincide only for conformal maps of a circular disk.

2.3. **Weak**$^*$ **compactness of holomorphic families in Banach domains.** One of the main underlying facts in the proof of main theorems is the existence of holomorphic maps from the space $T$ or $T_1$ onto the unit disk on which the Carathéodory distance is attained. It relies on classical results on compactness in the dual weak$^*$ topology. The following two propositions were related to me by David Shoikhet.

An analog of Montel’s theorem for the infinite dimensional case is given by

**Proposition 2.2.** Let $X$ and $Y$ be Banach spaces and let $G$ be a domain in $X$. A bounded set $\Omega$ in $\text{Hol}(G, Y)$ is relatively compact with respect to the topology of uniform convergence on compact subsets of $D$ (compact open topology on $\text{Hol}(G, Y)$) if and only if for each $x$ in $G$ the orbit $h(x)$, where $h$ runs over $\Omega$, is relatively compact in $Y$.

In fact, this proposition is a consequence of the classical Ascoli theorem. As a consequence of the Alaoglu-Bourbaki theorem, one derives the following result.

**Proposition 2.3.** Let $Y$ be reflexive and let $\Omega$ be a bounded set in $\text{Hol}(G, Y)$. Then any sequence from $\Omega$ contains a subsequence which weakly converges to a holomorphic map from $G$ to $Y$. 
Hence, the compactness (strong or weak) is actually required only for the image of $G$ in $Y$ under a given family of maps.

In our case $Y = \mathbb{C}$, and for each $x \in \hat{T}$ its orbit $h(x)$ is located in the unit disk which is compact. This implies that for any point $x_0$ from $\hat{T}$ there exists a holomorphic map $h_0 : \hat{T} \to \mathbb{D}$ with $h_0(0) = 0$ and $d_\mathbb{D}(0, h_0(x_0)) = c_T(0, x_0)$.

3. Proof of Theorem 1.2

It suffices to establish the equality of the Carathéodory and Teichmüller metrics of $T$ for the equivalence classes $[\mu]$ which are the Strebel points of $T$. This means that $[\mu]$ contains (unique) Beltrami coefficient of the form $\mu_0(z) = k|\psi_0(z)|/|\psi_0(z)|$, where $k < 1$ and $\psi_0 \in A_1(\mathbb{D})$. Accordingly, we have in $T$ the Teichmüller geodesic disks

$$D(\mu_0) = \{ \phi_T(t\mu_0) : t \in \mathbb{D} \}.$$  

It is well known that the set of Strebel points is open and dense in any Teichmüller space (see [GL, St]); in addition, both metrics are continuous on this space.

If the defining quadratic differential $\psi_0$ has in $\mathbb{D}$ only zeros of even order, i.e. belongs to $A^2_1(\mathbb{D})$, then the equalities (1.1) follow from Proposition 2.1 with Grunsky coefficients of $f$ (with uniformly bounded ratio $O(||\mu||^2_{\infty})/||\mu||^2_{\infty}$ on compact subsets of $\mathbb{C}$), one derives that the Grunsky coefficients of $f^\mu$ are varied by

$$\alpha_{mn}(S_f^\mu) = -\frac{1}{\pi} \int_\mathbb{D} \mu(z) z^{m+n-2} dxdy + O(\|\mu\|^2_{\infty}), \quad ||\mu||_{\infty} \to 0. \quad (3.2)$$

This implies that the differential at zero of the corresponding map

$$\hat{h}_x = h_x \circ \phi_T : \text{Belt}(\mathbb{D})_1 \to T \to \mathbb{D} \quad \text{with} \quad x = (x_n) \in S(l^2)$$

is given by

$$\hat{d}\hat{h}_x[0](\mu/||\mu||_{\infty}) = -\frac{1}{\pi} \int_\mathbb{D} \mu(z) \sum_{m+n=2}^\infty \sqrt{mn} \ x_m x_n z^{m+n-2} dxdy. \quad (3.3)$$

On the other hand, as was established in [Kt2], the elements of $A^2_1(\mathbb{D})$ are represented in the form

$$\psi(z) = \omega(z)^2 = \frac{1}{\pi} \sum_{m+n=2}^\infty \sqrt{mn} \ x_m x_n z^{m+n-2}, \quad (3.4)$$

with $||\omega||_{L_2} = ||\omega||_{L_2}$. The relations (2.6), (3.3), (3.4) together with Schwarz’s lemma, imply for $\mu = \mu_0$ the desired equality

$$\text{tanh} c_T(0, S_{f^\mu_0}) = \omega(f^{\mu_0}) = k. \quad (3.5)$$

The investigation of the generic case, when $\psi_0$ has in $\mathbb{D}$ a finite or infinite number of zeros of odd order, involves the Milin coefficient inequalities.

First recall the chain rule for Beltrami coefficients: for any $\mu, \nu \in \text{Belt}(\mathbb{C})_1$, the solutions $w^\mu$ of the corresponding Beltrami equation $\partial_x w = \mu \partial_z w$ on $\hat{\mathbb{C}}$ satisfy $w^\mu \circ w^\nu = w^{\sigma_{\nu}(\mu)}$, with

$$\sigma_{\nu}(\mu) = (\nu + \mu^*)/(1 + \overline{\nu} \mu^*), \quad (3.6)$$
where
\[ \mu^*(z) = \mu \circ w^*(z) \frac{\partial x w^*(z)}{\partial z} w^*(z). \]
Thus, for \( \nu \) fixed, \( \sigma_\nu(\mu) \) depends holomorphically on \( \mu \) as a map \( L_\infty(\mathbb{C}) \to L_\infty(\mathbb{C}) \).

Without loss of generality, one can assume that the function \( \psi_0 \) does not have at \( z = 0 \) (hence at some disk \( \{ |z| < d < 1 \} \)) zero of odd order. Otherwise, after squaring
\[ f^{\mu_0} \mapsto \mathcal{R}_2 f^{\mu_0} = f^{\mu_0} (z^2)^{1/2} = z + \frac{b_0}{2} \frac{1}{z} + \frac{b_3}{2} \frac{1}{z^3} + \ldots \]
one obtains an odd function from \( \Sigma^0 \) whose Beltrami coefficient on \( \mathbb{D} \) equals \( \mathcal{R}_2 \mu_0 = \mu_0 (z^2)^2/z \) being defined by quadratic differential \( \mathcal{R}_2 \psi_0 = 4 \psi_0 (z^2)^2 \) with zero of even order at the origin.

In addition, the Taylor and Grunsky coefficients of \( \mathcal{R}_2 f^{\mu} \) are represented as polynomials of the initial Taylor coefficients \( b_1, \ldots, b_8 \) of the original function \( f^{\mu} \), thus \( \alpha_{mn}(\mathcal{R}_2 f^{\mu}) \), together with \( \alpha_{mm}(f^{\mu}) \), depend holomorphically on \( \mu \) and \( S f_\mu \).

Now fix a \( \delta > 0 \) close to 1 and delete from the disk \( \mathbb{D} \) the annulus \( A_\delta = \{ \delta < |z| < 1 \} \) and the circular triangles \( \Delta_1, \ldots, \Delta_m(\delta) \) such that the base of each \( \Delta_j \) is an arc of the circle \( \{ |z| = \delta \} \), its opposite vertex is a zero \( a_j \) of odd order satisfying \( \rho \leq |a_j| < \delta \), and two other sides of \( \Delta_j \) are the straight line segments symmetric with respect to the radial segment \( [a_j, \delta e^{i \arg a_j}] \). In the case when several zeros are located on the same radius, it suffices to take only the zero with minimal modulus. Denote
\[ E_\delta = \cup_j \Delta_j \cup A_\delta, \quad D_\delta = \mathbb{D} \setminus \overline{E_\delta} \subset \mathbb{D}, \quad D_\delta^* = \mathbb{D}^* \cup E_\delta = \mathbb{C} \setminus \overline{D_\delta} \]
and put
\[ \mu_1(z) = \begin{cases} \mu_0(z) & \text{if } z \in E_\delta, \\ 0 & \text{otherwise}. \end{cases} \]
Note that \( f^{\mu_1} \) is normalized by (2.2) and \( f^{\mu_1}(0) = 0 \), so \( S f_{\mu_1} \in \mathbb{T} \). Letting \( \mu_2 = \mu_0 - \mu_1 \), one factorizes the initial automorphism \( f^{\mu_0} \) of \( \mathbb{C} \) via
\[ f^{\mu_0} = f^{\sigma_0} \circ f^{\mu_1} \]
with
\[ \sigma_0 = (f^{\mu_1})^* \mu_0 = \left( \frac{\mu_0}{1 - \overline{\mu_1} \mu_0} \frac{\partial z f^{\mu_1}}{\partial z f^{\mu_1}} \right) \circ (f^{\mu_1})^{-1} \in \text{Belt}(f^{\mu_1}(D_\delta))_1. \]  \( \quad \) (3.7)
Since \( f^{\mu_1} \) is confluent on \( D_\delta \), the coefficient \( \sigma_0 \) is represented by \( \sigma_0 = k |\psi_\delta|/\psi_\delta \) with
\[ \psi_\delta(w) = (\psi_0 \circ \tilde{f}) (\tilde{f})^2(w) \in A_2^2(f^{\mu_1}(D_\delta)), \quad \tilde{f} = (f^{\mu_1})^{-1}; \]  \( \quad \) (3.8)
this coefficient is extremal in its class in the ball \( \text{Belt}(f^{\mu_1}(D_\delta))_1 \).

The equivalence classes of Beltrami coefficients \( \nu \in \text{Belt}(f^{\mu_1}(D_\delta))_1 \) under the relation \( \nu_1 \sim \nu_2 \) if \( w^2 = w'' \) on \( \partial f^{\mu_1}(D_\delta) \) form the quotient space \( \mathbb{T}^* = \mathbb{T}(f^{\mu_1}(D_\delta)) \) which is biholomorphically isomorphic to the universal Teichmüller space with the base point \( f^{\mu_1}(D_\delta) \). The factorizing projection \( \phi_{T^*} : \text{Belt}(f^{\mu_1}(D_\delta))_1 \to \mathbb{T}^* \) is a holomorphic split submersion, which means that it has local holomorphic sections.

The chain rule for the Schwarzians
\[ S_{f_2 \circ f_1} = (S_{f_2} \circ f_1)(f_1')^2 + S_{f_1} \]
applied to \( w^\nu \circ f^{\mu_1} \), where \( w^\nu \in \Sigma(D_\delta^*) \) creates a holomorphic map \( \eta : \mathbb{T} \to \mathbb{T}^* \) moving the base point to the base point.

Now, applying to \( w^\nu \in \Sigma^0(f^{\mu_1}(D_\delta)) \) the variation of type (3.1), one obtains the following generalizations of (3.2) to Milin’s coefficients given in [Kr8]
\[ \alpha_{mn}(S_{w^\nu}) = -\frac{1}{\pi} \int_{f^{\mu_1}(D_\delta)} \nu(w) P'_m(w) P'_n(w) du dv + O(\|\nu\|_\infty^2), \]
and accordingly, instead of (3.3),

\[ \tilde{d}h_{x} = \frac{1}{\pi} \int_{f^{-1}(D)} \frac{\nu(z)}{||\nu||_{\infty}} \sum_{m,n=1}^{\infty} x_{m}x_{n} P_{m}(z)P_{n}^{*}(z)dxdy. \]

Here \( \tilde{h}_{x} \) denotes the lifting of the maps \( h_{x} : T^{*} \rightarrow D \) (defined by (2.5)) to the ball \( \text{Belt}(f^{\mu}(D_{b})) \), and \( \{ P_{n}\}^{\infty}_{1} \) is a well-defined orthonormal polynomial basis in \( A_{1}^{1}(f^{\mu}(D_{b})) \) such that the degree of \( P_{n} \) equals \( n \) (canonically determined by the quasidisk \( f^{\mu}(D_{b}) \); cf. [Mi], [KrS]).

The quadratic differential \( \psi_{\delta} \) in (3.8) has in the domain \( f^{\mu}(D_{b}) \) only zeros of even order, thus one can again apply Proposition 2.1 with \( D_{0} = f^{\mu}(D_{b}) \) getting, similar to (3.5), the equalities

\[ \tanh c_{T^{*}}(0, S_{f^{\mu}_{0}}) = \kappa_{f^{\mu}_{1}}(f^{\mu}_{0}) = k = \tanh d_{T^{*}}(0, S_{f^{\mu}_{0}}). \]

(3.9)

On the other hand, since both Kobayashi and Carathéodory metric are contractible under holomorphic maps and from (1.1),

\[ d_{T}(0, S_{f^{\mu}_{0}}) = \tanh^{-1} k \geq c_{T^{*}}(0, S_{f^{\mu}_{0}}) \geq c_{T^{*}}(0, \eta(S_{f^{\mu}_{0}})) = c_{T^{*}}(0, S_{f^{\mu}_{0}}). \]

Comparison with (3.9) implies

\[ c_{T}(0, S_{f^{\mu}_{0}}) = d_{T}(0, S_{f^{\mu}_{0}}) = \tanh^{-1} k = \tau_{T^{*}}(0, S_{f^{\mu}_{0}}). \]

(3.10)

proving the theorem in the case when one of the points is the origin of \( T \).

The case of arbitrary two points \( \varphi = S_{f^{\mu}}, \psi = S_{f^{\nu}} \) from \( T \) is investigated in a similar way (again by applying Milin’s coefficients), or can be reduced to (3.10) by the right translations of type (3.6) moving one of these points to the origin (a new base point of \( T \).

The proof for the infinitesimal metrics is similar. This completes the proof of the theorem.

4. PROOF OF THEOREM 1.1

First recall that the elements of the space \( T_{1} = T(\mathbb{D}^{0}) \) (where \( \mathbb{D}^{0} = \mathbb{D} \setminus \{0\} \)) are the equivalence classes of the Beltrami coefficients \( \mu \in \text{Belt}(\mathbb{D}) \), so that the corresponding quasiconformal automorphisms \( w^{\mu} \) of the unit disk coincide on both boundary components (unit circle \( S^{1} = \{|z| = 1\} \) and the puncture \( z = 0 \) and are homotopic on \( \mathbb{D} \setminus \{0\} \). This space can be endowed with a canonical complex structure of a complex Banach manifold and embedded into \( T \) using uniformization.

Namely, the disk \( \mathbb{D}^{0} \) is conformally equivalent to the factor \( \mathbb{D}/\Gamma \), where \( \Gamma \) is a cyclic parabolic Fuchsian group acting discontinuously on \( \mathbb{D} \) and \( \mathbb{D}^{*} \). The functions \( \mu \in L_{\infty}(\mathbb{D}) \) are lifted to \( \mathbb{D} \) as the Beltrami \((-1,1)\)-measurable forms \( \bar{\mu}d\bar{z}/dz \) in \( \mathbb{D} \) with respect to \( \Gamma \), i.e., via \( (\bar{\mu} \circ \gamma)/\gamma' = \bar{\mu}, \gamma \in \Gamma \), forming the Banach space \( L_{\infty}(\mathbb{D}/\Gamma) \).

Extend these \( \bar{\mu} \) by zero to \( \mathbb{D}^{*} \) and consider the unit ball \( \text{Belt}(\mathbb{D}, \Gamma) \) of \( L_{\infty}(\mathbb{D}, \Gamma) \). Then the corresponding Schwarzians \( S_{w^{\mu}} \) belong to \( T \). Moreover, \( T_{1} \) is canonically isomorphic to the subspace \( T(\Gamma) = T \cap B(\mathbb{D}) \), where \( B(\Gamma) \) consists of elements \( \varphi \in B \) satisfying \((\varphi \circ \gamma)(\gamma')^{2} = \varphi \) in \( \mathbb{D}^{*} \) for all \( \gamma \in \Gamma \). Most of the results about the universal Teichmüller space presented in Section 1 extend straightforwardly to \( T_{1} \).

Due to the Bers isomorphism theorem, the space \( T_{1} \) is biholomorphically equivalent to the Bers fiber space

\[ \mathcal{F}(T) = \{ \phi_{T}(\mu), z \} \in T \times \mathbb{C} : \mu \in \text{Belt}(\mathbb{D}), z \in w^{\mu}(\mathbb{D}) \}

over the universal Teichmüller space with holomorphic projection \( \pi(\psi, z) = \psi \) (see [Ba]). This fiber space is a bounded domain in \( B \times \mathbb{C} \).

To prove the theorem, we establish the equalities (1.2) and their infinitesimal counterpart for this fiber space.

We again model the space \( T \) as a domain in the space \( B \) formed by the Schwarzians \( S_{f^{\mu}} \) of functions \( f^{\mu}(z) = z + b_{0} + b_{1}z^{-1} + \cdots \in \Sigma^{0} \) normalizing those additionally by \( f^{\mu}(1) = 1 \).
Now the quadratic differentials defining the admissible Teichmüller extremal coefficients \( \mu_0 \in \text{Belt}(\mathbb{D})_1 \) must be integrable and holomorphic only on the punctured disk \( \mathbb{D} \setminus \{0\} \) and can have simple pole at \( z = 0 \), i.e., \( \mu_0 = k|\psi_0|/\psi_0 \) with
\[
\psi_0(z) = c_0 z^{-1} + c_1 z + \ldots, \quad 0 < |z| < 1.
\]
We associate with \( f^\mu \) the odd function
\[
R_{2,0} f^\mu(z) := (f^\mu(z^2) - f^\mu(0))^{1/2} = z + \frac{b_0 - f^\mu(0)}{2z} + \frac{b_3}{z^3} + \ldots
\]  
(4.1)
whose Grunsky coefficients \( \alpha_{mn}(R_{2,0} f^\mu) \) are represented as polynomials of the first Taylor coefficients of the original function \( f^\mu \) and of \( a = f^\mu(0) \). Hence, \( \alpha_{mn}(R_{2,0} f^\mu) \) depend holomorphically on the Schwarzians \( \varphi = S f^\mu \in T \) and on values \( f^\mu(0) \), i.e., on pairs \( X = (\varphi, a) \) which are the points of the fiber space \( F(T) \). This joint holomorphy follows from Hartog’s theorem on separately holomorphic functions extended to Banach domains. It allows us to construct for \( R_{2,0} f^\mu \) the corresponding holomorphic functions (2.5) mapping the domain \( F(T) \) to the unit disk.

One can apply to \( R_{2,0} f^\mu \) the same arguments as in the proof of Theorem 1.2 and straightforwardly establish for any Teichmüller extremal disk
\[
\{ \phi T_1(t \mu_0) = X_1 := (S_{f^\mu_0}, f^\mu_0(0)) : |t| < 1 \} \quad (\mu_0 = |\psi_0|/\psi_0)
\]
in the space \( F(T) \) the key equality
\[
\sup_{x \in S(\mathbb{D})} |h_x(S_{R_{2,0} f^\mu_0})| = |t|
\]
for any \( |t| < 1 \). This equality combined with (1.1) implies
\[
c_{F(T)}(0, X_1) = \tanh^{-1} |t| = \tau_{F(T)}(0, X_1) = d_{F(T)}(0, X_1),
\]  
(4.2)
and by Bers’ biholomorphism between the spaces \( T_1 \) and \( F(T) \), the similar equalities for the corresponding metrics on the space \( T_1 \). In view of the density of Strebel’s points and continuity of metrics, these equalities extend to all extremal disks in \( T_1 \), which yields the assertion of the theorem for the distances of any point from the origin.

To establish the equality of distances between two arbitrary points \( X_1, X_2 \) in \( T_1 = T(D_\ast) \), we uniformize the base point \( \mathbb{D}_\ast = \mathbb{D} \setminus \{0\} \) (with fixed homotopy class) by a cyclic parabolic Fuchsian group \( \Gamma_0 \) acting on the unit disk (using the universal covering \( \pi : \mathbb{D} \to \mathbb{D}_\ast \) with \( \pi(0) = 0 \)) and embed the space \( T_1 \) holomorphically into \( T \) via
\[
T_1 = T \cap B(\mathbb{D}_\ast, \Gamma_0) = \text{Belt}(\mathbb{D}, \Gamma_0)/\sim
\]
(where the equivalence relation commute with the homotopy of quasiconformal homeomorphisms of the surfaces). This preserve all invariant distances on \( T_1 \).

One can use the result of the previous step which provides that for any point \( X \in T_1 \) its distance from the base point \( X_0 = \mathbb{D}_\ast \) in any invariant (no-expanding) metric is equal to the Teichmüller distance; hence,
\[
c_{T_1}(X_0, X) = \tau_{T_1}(X_0, X) = d_{T_1}(X_0, X).
\]
Now, fix a Beltrami coefficient \( \mu \in \text{Belt}(\mathbb{D}, \Gamma_0)_1 \) so that \( X_1 = w^\mu(X_0) \) as marked surfaces (i.e., with prescribed homotopy classes) and apply the change rule (3.6). It defines a holomorphic automorphism \( \sigma_{\mu} \) of the ball \( \text{Belt}(\mathbb{D}, \Gamma_0) \) which is an isometry in its Teichmüller metric. This automorphism is compatible with holomorphic factorizing projections \( \phi T_1 \) and \( \phi T_1^* \) defining the space \( T_1 \) and its copy \( T_1^* \) with the base point \( X_1 \). Thus \( \sigma_{\mu} \) it descends to a holomorphic bijective map \( \tilde{\sigma}_{\mu} \) of the space \( T_1 \) onto itself, which implies the Teichmüller isometry
\[
\tau_{T_1}(\phi T_1(\mu), \phi T_1(\nu)) = \tau_{T_1}(\phi T_1(0), \phi T_1(\sigma_{\mu}(\nu)), \quad \nu \in \text{Belt}(f^\mu(\mathbb{D}), f^\mu(\Gamma_0(f^\mu)^{-1})_1,
\]

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and similar admissible isometries for the Carathéodory and Kobayashi distances on these space. Combining this with the relations
\[ c_{T_1}(\phi_{T_1}(0), \phi_{T_1}(\sigma_{\mu}(\nu))) = \tau_{T_1}(\phi_{T_1}(0), \phi_{T_1}(\sigma_{\mu}(\nu))) = d_{T_1}(\phi_{T_1}(0), \phi_{T_1}(\sigma_{\mu}(\nu))) \]
established in the previous step, one derives the desired equalities (1.2).

The case of infinitesimal metrics is investigated in a similar way, which completes the proof of the theorem.

5. Proofs of Theorem 1.3 and Corollary 1.5

5.1. Proof of Theorem 1.3. Theorems 1.1 and 1.2 imply, together with the definition of complex geodesics, that these geodesics in \( \tilde{T} \) are the Teichmüller geodesic disks
\[ D(\mu_0) = \{ \tilde{\phi}_T(t\mu_0/||\mu_0||_\infty) : |t| < 1 \} \] (5.1)
in this space. Accordingly, the uniqueness of the complex geodesic joining a Strebel point in \( \tilde{T} \) with the origin follows from uniqueness of the disk (5.1) for such a point.

On the other hand, Tanigawa constructed in [Ta] the extremal Beltrami coefficients \( \mu_0 \) with nonconstant \( |\mu_0(z)| < ||\mu_0||_\infty \) on a set of positive measure for which there exist infinitely many distinct geodesic segments in the universal Teichmüller space \( T \) joining the points \( \phi_T(0) \) and \( \phi_T(\mu_0) \). All these segments belong to different complex geodesics joining the indicated points.

The case of geodesic disks joining two arbitrary points in \( \tilde{T} \) is investigated in similar way. This completes the proof of the theorem.

Remark. One can combine Theorems 1.1 and 1.2 with the result of [DTV] on existence of complex geodesics in convex Banach domains and get an alternative proof of Theorem 1.3. The main underlying facts ensuring the existence of geodesics are the equality of invariant metrics established for geometrically convex domains and weak* compactness.

It is well known that if a Banach space \( X \) has a predual \( Y \), then by the Alaouglu-Bourbaki theorem the closure of its open unit ball is weakly * compact. This holds, in particular, for our space \( \tilde{T} \) regarded as a bounded domain in \( B(\mathbb{D}^*, \Gamma) \), which is dual to the space \( A_1(\mathbb{D}^*, \Gamma) \) of integrable holomorphic quadratic differentials with respect to group \( \Gamma \).

Theorems 1.1 and 1.2 ensure all the needed features, and therefore one can obtain Theorem 1.3 also by applying the same arguments as in [DTV].

5.2. Proof or Corollary 1.5. We apply Golusin’s improvement of Schwarz’s lemma which asserts that a holomorphic function
\[ g(t) = c_mt^m + c_{m+1}t^{m+1} + \cdots : \mathbb{D} \to \mathbb{D} \quad (c_m \neq 0, \ m \geq 1), \]
is estimated in \( \mathbb{D} \) by
\[ |g(t)| \leq |t|^m \frac{|t| + |c_m|}{1 + |c_m||t|}, \] (5.2)
and the equality occurs only for \( g_0(t) = t^m(t + c_m)/(1 + c_mt) \); see [Go, Ch. 8].

Fix \( t_0 \neq 0 \) and denote \( \mu_0^* = \mu_0/||\mu_0||_\infty \). \( \eta(t) = |t|^m(|t| + |c_m|)/(1 + |c_m||t|) \).

By Theorems 1.1 and 1.2, there exists a holomorphic map \( j(\varphi) : \tilde{T} \to \mathbb{D} \) (the limit holomorphic function for a maximizing sequence for the Carathéodory distance) such that
\[ d_{\mathbb{D}}(0, |j \circ \tilde{\phi}_T(t_0\mu_0^*)|) = c_{\tilde{T}}(0, \tilde{\phi}_T(t_0\mu_0^*)) = d_{\tilde{T}}(0, \tilde{\phi}_T(t_0\mu_0^*)). \] (5.3)

Thus the maps
\[ h(t) = \tilde{\phi}_T(t_0\mu_0^*) : \mathbb{D} \to \mathbb{D}(\mu_0) \quad \text{and} \quad j_*(t) = j \circ \tilde{\phi}_T(t_0\mu_0^*) = j|\mathbb{D}(\mu_0) : \mathbb{D}(\mu_0) \to \mathbb{D} \]
determine two inverse hyperbolic isometries of the unit disk so that $j_\ast \circ h(t) \equiv t$.

Now, let $J$ be a holomorphic functional on $T$ with the values in $\mathbb{D}$ and its restriction $J_{\mu_0}$ to the disk $\mathbb{D}(\mu_0)$ is expanded via (1.3). Then, using the relations (5.2) and (5.3) and noting that $|\eta(t)| \leq |t|$, one derives

$$J_{\mu_0}(t_0) \leq j_\ast(\eta(t_0)) = d_T(0, h(\eta(t_0))) \leq d_T(0, h(|t_0|))$$

which implies (1.4). The case of equality easily follows from Schwarz’s lemma. This completes the proof of the corollary.

6. Applications to geometric function theory

6.1. General distortion theorem. The above theorems reveal the fundamental facts of the variational theory for univalent functions with quasiconformal extension.

Let again $L$ be a bounded oriented quasicircle in the complex plane $\mathbb{C}$ separating the origin and the infinite point, with the interior and exterior domains $D$ and $D^*$ so that $0 \in D$ and $\infty \in D^*$. Put

$$\Sigma^0(D) = \{ f \in \Sigma(D) : k(f) \leq k' \}.$$

Consider on the class $\Sigma^0(D^*)$ a holomorphic (continuous and Gateaux $\mathbb{C}$-differentiable) functional $J(f)$, which means that for any $f \in \Sigma^0(D^*)$ and small $t \in \mathbb{C}$,

$$J(f + th) = J(f) + tJ'f(h) + O(t^2), \quad t \to 0,$$

(6.1)
in the topology of uniform convergence on compact sets in $\mathbb{D}^*$. Here $J'f(h)$ is a $\mathbb{C}$-linear functional.

Assume that $J$ is lifted by $\tilde{J}(\mu) = J(f^\mu)$ to a holomorphic function on $\text{Belt}(D)_1$ and also depends holomorphically from the Schwarzian derivatives $S_{f^\mu}$ on universal Teichm"uller space $T$. Then the linear functional $J'f(h)$ in (5.1) is the strong (Fréchet) derivative of $J$ in both norms $L_\infty$ and $B(D^*)$.

Varying $f$ by (3.1), one gets the functional derivative

$$\psi_0(z) = J'_{id}(g(id, z))$$

(6.2)
for the variational kernel

$$g(w, \zeta) = 1/(w - \zeta) - 1/\zeta,$$

(6.3)
Note that any such functional $J$ is represented by a complex Borel measure on $\mathbb{C}$, which allows to extend this functional to all holomorphic functions on $D^*$ (cf. [24D]). In particular, the value $J_{id}(g(id, z))$ of $J$ on the identity map $id(z) = z$ is well-defined.

We assume that this derivative is meromorphic on $\mathbb{C}$ and has in the domain $D$ only a finite number of the simple poles (hence $\psi_0$ is integrable over $D$). All this holds, for example, in the case of the distortion functionals of the general form

$$J(f) := J(f(a_1), \ldots, f(a_m); f(z_1), f'(z_1), \ldots, f^{(\alpha_1)}(z_1); \ldots; f(z_p), f'(z_p), \ldots, f^{(\alpha_p)}(z_p)).$$

with $\tilde{J}(0) = 0$ and $\text{grad} \tilde{J}(0) \neq 0$. Here $a_1, \ldots, a_m$ are distinct fixed points in $D$, and $z_1, \ldots, z_p$ are distinct fixed points in $D^*$ with assigned orders $\alpha_1, \ldots, \alpha_p$, respectively.

To have a possibility to apply Theorems 1.1 and 1.2, we restrict ourselves by the model case $m = 1$, i.e., by the functionals

$$J(f) = J(f(a); f(z_1), f'(z_1), \ldots, f^{(\alpha_1)}(z_1); \ldots; f(z_p), f'(z_p), \ldots, f^{(\alpha_p)}(z_p))$$

(6.4)
depending on the values of maps at one point in the domain of quasiconformality. In this case,

$$\tilde{J}'_{id}(g(id, z)) = \frac{\partial \tilde{J}(0)}{\partial \omega} g(z, a) + \sum_{j=1}^p \sum_{k=0}^{\alpha_j - 1} \frac{\partial \tilde{J}(0)}{\partial \omega_{j,k}} d^k g(w, \zeta)|_{w = z, \zeta = z_k},$$

(6.5)
where $\omega = f(a)$, $\omega_{j,k} = f^{(k)}(z_j)$; hence, $\psi_0$ is a rational function.
For such functionals, Theorem 1.1 provides a general distortion theorem which shed light on underlying features and, on the other hand, implies the sharp explicit bounds.

**Theorem 6.1.** (i) For any functional $J$ of type (6.4) whose range domain $J(\Sigma^0(D^*))$ has more than two boundary points, there exists a number $\kappa_0(J) > 0$ such that for all $\kappa \leq \kappa_0(J)$, we have the sharp bound

$$
\max_{k(f) \leq \kappa} |J(f^\mu) - J(id)| \leq \max_{|t| = \kappa} |J(f^{|\psi|/\psi_0}) - J(id)|;
$$

(6.6)

in other words, the values of $J$ on the ball $\text{Belt}(D) = \{ \mu \in \text{Belt}(D) : \|\mu\|_\infty \leq \kappa \}$ are placed in the closed disk $\mathbb{D}(J(id), M_\kappa)$ with center at $J(id)$ and radius $M_\kappa = \max_{|t| = \kappa} |J(f^{|\psi|/\psi_0}) - J(id)|$. The equality occurs only for $\mu = t|\psi|/\psi_0$ with $|t| = \kappa$.

(ii) Conversely, if a functional $J$ is bounded via (6.6) for $0 < \kappa \leq \kappa_0(J)$ with some $\kappa_0(J) > 0$, then up to rescaling (multiplying $J$ by a positive constant factor),

$$
J(f^\mu) = g(S_{f^\mu}) + O(||\mu||^2_\infty) \quad \text{as} \quad \|\mu\|_\infty \to 0,
$$

(6.7)

where $g$ is holomorphic on $\mathbb{T}_1$ and its renormalization $\tilde{g}(\varphi) = g(\varphi)/\sup_{\varphi \in \mathbb{T}_1} |g(\varphi)|$ is the defining map for the disk $\mathbb{D}(\mu_0)$ as a $c\mathbb{T}_1$-geodesic in the space $\mathbb{T}_1$ with the base point representing the punctured quasidisk $D \setminus \{a\}$.

**Outline of the proof.** First note that as one can see from (6.7) that the underlying features arise from the connection between such holomorphic functionals and the corresponding $c\mathbb{T}_1$-geodesics. The proof of Theorem 1.1 implies that the restriction of $\tilde{g}(\varphi)$ to the disk formed by $\varphi = S_{f^\mu} \mathbb{R}_{2,0}^* \mu_0$, $|t| < 1$, defined by (2.5).

The results of such type were obtained in [Kr1], [Kr2] for more specific functionals which relate to complex geodesics in the universal Teichmüller space $\mathbb{T}$. The proof of Theorem 6.1 involves $c\mathbb{T}_1$-geodesics and follows the same lines. Thus we only outline the main steps.

One can replace the assumption $f^\mu(0) = 0$ for $f^\mu \in \Sigma^0(D)$ by $f^\mu(1) = 1$ and use the variation

$$
f^\mu(z) = z - \frac{1}{\pi} \int_D \mu(\zeta) \left( \frac{1}{\zeta - z} - \frac{1}{\zeta - 1} \right) d\zeta d\eta + O(||\mu||^2_\infty)
$$

(6.8)

$$
= z - \frac{1}{\pi} \int_D \frac{\mu(\zeta) d\zeta d\eta}{(\zeta - 1)(\zeta - z)} + O(||\mu||^2_\infty) \quad \text{as} \quad \|\mu\|_\infty \to 0,
$$

hence replace (6.3) by

$$
g(w, \zeta) = \frac{1}{w - \zeta} - \frac{1}{w - 1}.
$$

(6.9)

Assume also that $J(id) = 0$, and let $f_0$ be a maximizing function for $|J|$ in $\Sigma_1(D)$ (whose existence follows from compactness). Take its extremal extension to $D$, i.e., with Beltrami coefficient

$$
\|\mu_{f_0}\|_\infty = \inf \{ \|\mu\|_\infty \leq \kappa : f^\mu = f_0 \quad \text{on} \quad D^* \cup \{a_1, \ldots, a_m\} \},
$$

and suppose that

$$
\mu_{f_0} \neq \mu_0,
$$

(6.10)

where $\mu_0 = t|\psi|/\psi_0$ for some $t$ with $|t| = \kappa$. Our goal is to show that for small $\kappa$ this leads to a contradiction.

Pick in $A_1(D \setminus \{a\})$ the functions

$$
\omega_p(z) = \chi(z)^p \chi'(z) - 1 - \psi_0(z), \quad p = 1, 2, \ldots,
$$

(6.11)

where $\chi$ is a conformal map of $\mathbb{D}$ onto $D$ with $\chi(0) = 0$, $\chi'(0) > 0$ (hence, $\chi(z)^p \chi'(z) = c^p z^p + O(z^{p+1})$ as $z \to 0$), and

$$
\rho_a(z) = \frac{a - 1}{(z - 1)(z - a)}.
$$

(6.12)
For each \( z \in D \), the function
\[
r_z(\zeta) = \frac{1}{\zeta - z} - \frac{1}{\zeta - 1}
\]
is in \( A_1(D) \), hence \( r_z(\zeta) = \sum_{0}^{\infty} d_p(z)\zeta^p \), \( \zeta \in D \).

One of the main points in the proof is the following

**Lemma 6.2.** For sufficiently small \( \kappa \leq \kappa_0(J) \), the extremal Beltrami coefficient \( \mu_{f_0} \) is orthogonal in \( A_1(D) \) to all functions (6.11) and (6.12), i.e., \( \langle \mu_{f_0}, \rho_a \rangle_D = 0 \) and \( \langle \mu_{f_0}, \omega_p \rangle_D = 0 \) for all \( p \).

Its proof involves the properties of the projections of norm 1 in Banach spaces presented in [EK] and investigation of norms
\[
h(\xi) = \iint_{D} |\psi_0(z) + \xi \rho_a(z)|dxdy, \quad h_p(\xi) = \iint_{D} |\psi_0(z) + \xi \psi_p(z)|dxdy.
\]

Now apply a geodesic holomorphic map \( g : T_1 \rightarrow \mathbb{D} \) from Theorem 1.1 defining the disk \( \mathbb{D}(\mu_0) \) as \( c_{T_1} \)-geodesic; it determines a hyperbolic isometry between this disk and \( \mathbb{D} \). We lift this map onto \( \text{Belt}(\mathbb{D})_1 \) by \( \Lambda(\mu) = g \circ \phi_{T_1}(\mu) \) getting a holomorphic map of this ball onto the disk. The differential of \( \Lambda \) at \( \mu = 0 \) is a linear operator \( P : L_\infty(\mathbb{D}) \rightarrow L_\infty(\mathbb{D}) \) of norm 1 which is represented in the form
\[
P(\mu) = \beta(\mu, \psi_0)_{D} \psi_0.
\]
Let \( P(\mu_{f_0}) = \alpha(\kappa)\mu_0 \). Since, by assumption, \( f_0 \) is not equivalent to \( f^t\mu_0 \) with \( |t| = \kappa \), we have
\[
\left\{ \Lambda\left( \frac{t}{\kappa} \mu_{f_0} \right) : |t| < 1 \right\} \subseteq \left\{ |t| < 1 \right\}.
\]
Thus, by Schwarz’s lemma,
\[
|\alpha(\kappa)| < \kappa. \tag{6.13}
\]

Note that the conjugate operator
\[
P^*(\psi) = \langle \mu_0, \psi \rangle_{D} \psi_0
\]
maps \( L_1(D) \) into \( L_1(D) \) and fixes the subspace \( W = (\omega_p, \rho_a) \) of \( A_1(D \setminus \{a\}) \) spanned by functions (6.11) and (6.12).

Now consider the function
\[
\nu_0 = \mu_{f_0} - \alpha(\kappa)\mu_0 \quad \tag{6.14}
\]
which is not equivalent to zero, due to our assumption (6.10). Lemma 6.2 allows us to establish that \( \nu_0 \) annihilates all functions from \( \psi \in W \) and therefore orthogonal to all functions from the whole space \( A_1(D \setminus \{a\}) \), because \( \psi_0 \rho_a \) and \( \omega_p \), \( p = 1, 2, \ldots \) form a complete set in this space. This means that the function (6.14) belongs to the set
\[
A_1(D \setminus \{a\})^\perp = \{ \mu \in L_\infty(D) : \langle \mu, \psi \rangle_D = 0 \quad \text{for all} \quad \psi \in A_1(D \setminus \{a\}) \}.
\]

But the well-known properties of extremal quasiconformal maps imply that for any \( \nu \in A_1(D \setminus \{a\})^\perp \),
\[
\|\mu_{f_0}\|_\infty = \inf \{ \|\mu_{f_0} + \nu, \psi \|_D : \psi \in A_1(\mathbb{D} \setminus \{a\}), \|\psi\| = 1 \} \leq \|\mu_{f_0} + \nu\|_\infty.
\]
and therefore
\[
\|\mu_{f_0}\|_\infty = \kappa \leq \|\mu_{f_0} - \nu_0\|_\infty = \|\alpha(\kappa)\mu_0\|_\infty = \alpha(\kappa),
\]
which contradicts (6.13). Hence \( f_0 \) is equivalent to \( f^t|\psi_0|/\psi_0 \) and we can take \( \mu_{f_0} = t|\psi_0|/\psi_0 \) for some \( |t| = \kappa \), completing the proof of the first part of the theorem.

To prove the converse assertion (ii), we lift the original functional \( J \) to
\[
I(\mu) = \pi^{-1} \circ J(f^\nu) : \text{Belt}(D)_1 \rightarrow \mathbb{D},
\]
where $\pi$ is a holomorphic universal covering of the domain $V(J) = J(\Sigma^0(D^*)$) by a disk $D_a = \{|z| < a\}$ with $\pi(0) = 0, \pi'(0) = 1$ (the lifting is single valued, since the ball Belt($D$)$_1$ is simply connected). Let again $J(id) = 0$. The normalization of $\pi$ ensures that for sufficiently small $|\zeta|$, 

$$
\pi(\zeta) = \zeta + O(\zeta^2)
$$

(with uniform estimate of the remainder for $|\zeta| < |\zeta_0|$), which implies the asymptotic equality (6.7). The covering functional $I$ is holomorphic also in the Schwarzians $S_f$, which generates a holomorphic map $\tilde{I} : T_1 \to \mathbb{D}$ so that $I = \tilde{I} \circ \phi_{T_1}$. The above arguments provide for $I$ instead of (6.6) the bound 

$$
\max_{k(f^\mu) \leq \kappa} |I(f^\mu)| = \kappa \quad \text{for} \quad 0 < k < k_1(I).
$$

Restricting the covering map $\tilde{I}$ to the extremal disk $\{\phi_{T_1}(t\mu_0^*) : |t| < 1\} \subset T_1$ (where $\mu_0^* = |\psi_0|/\psi_0$) and applying to this restriction Schwarz’s lemma, one derives that $\tilde{I}(\phi_{T_1}(t\mu_0^*)) \equiv t$. Thus the inverse to this map must be $c_I$-geodesic, which completes the proof of the theorem.

Representing the extremal $f^\mu|\psi_0|/\psi_0$ by (6.8), one can rewrite the estimate (6.6) for $\kappa \leq \kappa_1(J)$ in the form

$$
\max_{k(f^\mu) \leq \kappa} |J(f^\mu) - J(id)| \leq \frac{\kappa}{\pi} \int_D |J'(id)(g(id, z))| dz dy = \frac{\kappa}{\pi} ||\psi_0||_1.
$$

(6.15)

6.2. A lower estimate for the bound $\kappa_0(J)$. If the functional $J$ is bounded on the whole class $\Sigma^0(D^*)$, and $J(id) = 0, \text{grad} J(id) \neq 0$, one can also derive from the above arguments also a useful lower bound for $\kappa_0(J)$, which allows one to apply Theorem 6.1 effectively. Namely, one can verify that the above proof works for

$$
\kappa \leq \kappa_0(J) = \frac{||J'_{id}||}{||J'_{id}|| + M(J) + 1},
$$

where

$$
||J'_{id}|| = \frac{1}{\pi} ||\psi_0||_1, \quad M(J) = \sup_{\Sigma^0(D^*)} |J(f)|.
$$

(6.16)

6.3. Additional remarks. 1. Similar theorem holds also for univalent functions on bounded quasidisks $D$, for example, for the canonical class $S_\kappa(D)$ of univalent functions in $D$ normalized by $f(z) = z + c_2 z^2 + \ldots$ near the origin (provided that $z = 0 \in D$) and admitting $\kappa$-quasiconformal extensions to $\hat{\mathbb{C}}$ which preserve the infinite point. Such functions are investigated in the same way.

2. Theorem 6.1 provides various explicit estimates controlling the distortion in both conformal and quasiconformal domains simultaneously (comparing the known very special results established in [GR, Kr1, Ku1, Ku2]).

3. The assumption that the distinguished point $a$ is inner, is essential, and the estimate (6.6) can fail when the functionals depend on values $f(a)$ at a prescribed point on the boundary $\partial D$. The reasons are not technical. Actually the bound $\kappa_0(J)$ depends on the distance $\text{dist}(a, \partial D)$ and generically decreases to 0 when $a$ approaches the boundary.

One can see this from the well-known result of Kühnau’ on the domain of values of $f(1)$ on $\Sigma_\kappa$ presented, for example, in [KK, Part 2]; it shows that in such a case an additional remainder $O(\kappa^2)$ can appear.
7. New phenomena

7.1. Rigidity of extremals. The intrinsic connection between the extremals of the distortion functionals on functions with quasiconformal extensions and complex geodesics causes surprising phenomena which do not appear in the classical theory concerning all univalent functions. The differences arise from the fact that in problems for the functions with quasiconformal extensions the extremals belong to compact subsets of $\Sigma^0(D^*)$ (or in other functional classes), while the maximum on the whole class is attained on the boundary functions.

We first mention the following consequence of Theorem 1.3 which provides strong rigidity of extremal maps.

Corollary 7.1. In any class of univalent functions with $\kappa$-quasiconformal extension, neither function can be simultaneously extremal for different holomorphic functionals (6.4) unless one of these functionals have equal 1-jets at the origin.

7.2. Example: the coefficient problem for functions with quasiconformal extensions. We mention here an improvement in estimating the Taylor coefficients. Though the Bieberbach conjecture for the canonical class $S$ of univalent functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in $D$ has already been proved by de Brange’s theorem, the old coefficient problem remains open for univalent functions in the disk with quasiconformal extensions. The problem was solved by the author for the functions with sufficiently small dilatations.

Denote by $S_\kappa(\infty)$ and $S_\kappa(1)$ the classes of $f \in S$ admitting $\kappa$-quasiconformal extensions $\hat{f}$ to $\hat{C}$ normalized by $\hat{f}(\infty) = \infty$ and $\hat{f}(1) = 1$, respectively. Let

$$f_{1,t}(z) = \frac{z}{(1-tz)^2}, \quad |z| < 1, \quad |t| < 1. \quad (7.1)$$

This function can be regarded as a quasiconformal counterpart of the well-known Koebe function which is extremal for many functionals on $S$.

As a special case of Theorem 6.1, we have a complete solution of the Kühnau-Niske problem [KN] given by

Theorem 7.2. [Kr3] For all $f \in S_\kappa(\infty)$ and all $\kappa \leq 1/(n^2 + 1)$,

$$|a_n| \leq 2\kappa/(n-1), \quad (7.2)$$

with equality only for the functions

$$f_{n-1,t}(z) = f_{1,t}(z^{n-1})^{1/(n-1)} = z + \frac{2t}{n-1} z^n + \ldots, \quad n = 3, 4, \ldots; \quad |t| = \kappa. \quad (7.3)$$

The estimate (7.2) also holds in the classes $S_\kappa(1)$ with the same bound for $\kappa$.

Note that every function (7.3) admits a quasiconformal extension $\hat{f}_{n-1,t}$ onto $D^* = \{|z| > 1\}$ with Beltrami coefficient $t\mu_n(z) = t|z|^{n+1}/z^{n+1}$, and $\hat{f}_{n-1,t}(\infty) = \infty$.

No estimates have been obtained for arbitrary $\kappa < 1$, unless $n = 2$; in the last case, $|a_2| \leq 2\kappa$ with equality for the function (7.1) when $|t| = k$ (cf. [Ku1], [KK], [KN]).

The rigidity provided by Corollary 7.1 yields that the function (7.1) cannot maximize $|a_n|$ in $S_\kappa(\infty)$ even for one $\kappa < 1$, unless $n = 2$. Hence, for all $\kappa < 1$,

$$\max_{f \in S_\kappa(\infty)} |a_n| > n\kappa^{n-1} \quad (n \geq 3). \quad (7.4)$$

For $n = 3$, this inequality was established in [KN] involving the elliptic integrals.
Comparing the coefficients $a_n$ of $f_{1,t}$ and $f_{n-1,t}$, one derives from (7.2) and (7.4) the rough bounds for the maximal value $\kappa_n$ of admissible $\kappa$ in (7.2):

$$\frac{1}{n^2+1} \leq \kappa_n < \left[ \frac{2}{n(n-1)} \right]^{1/(n-2)}.$$

7.3. Over-normalized functions. Another remarkable thing in the distortion theory for univalent functions with quasiconformal extension concerns over-determined normalization what reveals the intrinsic features of quasiconformality. The variational problems for such classes are originated in 1960s; the results were established mainly in terms of inverse extremal functions (see $[\text{Kr1}, \text{BK}, \text{Re}]$).

We establish here some general explicit bounds. Assume that $z = 1$ lies on the common boundary of $D$ and $D^*$ which separates the points 0 and $\infty$ and denote by $\Sigma^0(D^*, 1)$ the class of univalent functions in $D^*$ with quasiconformal extensions across $L$ which satisfy

$$f(z) = z + \text{const} + O(1/z) \text{ near } z = \infty; \quad f(1) = 1,$$

and by $\Sigma_\kappa(D^*, 1)$ its subclasses consisting of functions with $\kappa$-quasiconformal extensions. Fix in the complementary domain $D$ a finite collection of points

$$e = (e_1, \ldots, e_m)$$

and associate with this set the following subspaces of $L_1(D)$: the span $L(e)$ of rational functions

$$\rho_s(z) = \frac{e_s - 1}{(z - 1)(z - e_s)} \quad s = 1, \ldots, m,$$

the space $A_1(D_e)$ of integrable holomorphic functions in the punctured domain $D_e = D \setminus \{e_1, \ldots, e_m\}$, and

$$L_0 = L(e) \bigoplus \{c \psi_0 : c \in \mathbb{C}\},$$

where $\psi_0 = J'_{\text{id}}(g(\text{id}, \cdot))$ for $g(w, \zeta)$ given by (6.8). Let

$$\Sigma_\kappa(D^*, 1, e) = \{f \in \Sigma_\kappa(D^*, 1) : f(e_s) = e_s, \quad s = 1, \ldots, m\}, \quad \Sigma^0(D^*, 1, e) = \bigcup_\kappa \Sigma_\kappa(D^*, 1, e).$$

Note that these classes with over-determined normalization contain nontrivial maps $f^n \neq \text{id}$ for any $\kappa < 1$ what is insured, by the local existence theorem from $[\text{Kr11, Ch. 4}]$. We shall use its special case for simply connected plain domains presenting it as

**Lemma 7.3.** Let $D$ be a simply connected domain on the Riemann sphere $\hat{\mathbb{C}}$. Assume that there are a set $E$ of positive two-dimensional Lebesgue measure and a finite number of points $z_1, z_2, \ldots, z_m$ distinguished in $D$. Let $\alpha_1, \alpha_2, \ldots, \alpha_m$ be non-negative integers assigned to $z_1, z_2, \ldots, z_m$, respectively, so that $\alpha_j = 0$ if $z_j \in E$.

Then, for a sufficiently small $\varepsilon_0 > 0$ and $\varepsilon \in (0, \varepsilon_0)$, and for any given collection of numbers $w_{s,j}, s = 0, 1, \ldots, \alpha_j, \quad j = 1, 2, \ldots, m$ which satisfy the conditions $w_{0,j} \in D$,

$$|w_{0,j} - z_j| \leq \varepsilon, \quad |w_{0,j} - z_j| \leq \varepsilon, \quad |w_{s,j}| \leq \varepsilon \quad (s = 0, 1, \ldots, \alpha_j, \quad j = 1, 2, \ldots, m),$$

there exists a quasiconformal self-map $h$ of $D$ which is conformal on $D \setminus E$ and satisfies

$$h^{(s)}(z_j) = w_{s,j} \quad \text{for all } s = 0, 1, \ldots, \alpha_j, \quad j = 1, 2, \ldots, m.$$

Moreover, the Beltrami coefficient $\mu_h(z) = \partial \bar{z} h / \partial z h$ of $h$ on $E$ satisfies $\|\mu_h\|_\infty \leq M \varepsilon$. The constants $\varepsilon_0$ and $M$ depend only upon the sets $D, E$ and the vectors $(z_1, \ldots, z_m)$ and $(\alpha_1, \ldots, \alpha_m)$.

If the boundary $\partial D$ is Jordan or is $C^{l+\alpha}$-smooth, where $0 < \alpha < 1$ and $l \geq 1$, we can also take $z_j \in \partial D$ with $\alpha_j = 0$ or $\alpha_j \leq l$, respectively.
Let us estimate on such over-normalized classes the functionals
\[ J(f) = J(f(z_1), f'(z_1), \ldots, f^{(\alpha_1)}(z_1); \ldots; f(z_p), f'(z_p), \ldots, f^{(\alpha_p)}(z_p)) \] (7.6)
controlling the distortion on the domain of conformality.

Now one can use only conditional quasiconformal variations whose Beltrami coefficients are orthogonal to the rational quadratic differentials (6.12) corresponding to the fixed points. Thus the above proof of key Lemma 6.2 fails, and this Lemma and Theorem 6.1 do not work.

The following theorem provides the sharp explicit bounds for sufficiently small \( \kappa \) involving \( L_1 \)-distance between the functional derivative \( \nu_0 \) and span \( \mathcal{L}(e) \).

**Theorem 7.4.** For any functional (7.5) and any finite set \( e \) of fixed points in \( D \), there exists a positive number \( \kappa_0(J,e) < 1 \) such that for all \( \kappa \leq \kappa_0(J,e) \), we have for any function \( f \in \Sigma_\kappa(D^*, 1, e) \) the sharp bound
\[
\max_{\|\nu\| \leq \kappa} |J(f^\nu) - J(id)| = |J(f^{\nu_0}/\psi_e) - J(id)| = d\kappa + O(\kappa^2)
\] (7.7)
with uniformly bounded ratio \( O(\kappa^2)/\kappa^2 \), where
\[
\psi_e = \xi_0 \psi_0 + \sum \xi_i \rho_i
\] (7.8)
with some constants \( \xi_0, \xi_1, \ldots, \xi_m \) and
\[
d = \inf_{\mathcal{L}(e)} \|\xi_0 \psi_0 - \psi\|_1.
\] (7.9)
The constants \( \xi_0, \xi_e \) in (7.8) are determined (not necessary uniquely) by the conditions
\[
\langle |\psi_e|/\psi_e, \psi \rangle_D = 0 \quad \text{for all} \quad \psi \in \mathcal{L}(e); \quad \langle |\psi_e|/\psi_0, \psi \rangle_D = d.
\] (7.10)

**Proof.** By the Hahn-Banach theorem, there exists a linear functional \( l \) on \( L_1(D) \) such that
\[
l(\psi) = 0, \quad \psi \in \mathcal{L}(e); \quad l(\psi_0) = d,
\] (7.11)
and
\[
\|l\|_{L_1(D)} = \|l\|_{\mathcal{L}_0} = 1,
\]
and this norm is minimal on the spaces \( \mathcal{L}_0 \subset A_1(D \setminus e) \subset L_1(D) \). Hence, for any other linear functional \( \tilde{l} \) on \( L_1(D) \) satisfying (7.9) must be \( \|\tilde{l}\|_{\mathcal{L}_0} \geq 1 \) and \( \tilde{l}(\psi_0) \leq d \); otherwise, were \( \tilde{l}(\psi_0) = rd \) with \( r > 1 \), the functional \( \tilde{l}/r \) with norm less than 1 would satisfy (7.11), in contradiction to minimality.

The functional \( l \) is represented on \( L_1(D) \) via
\[
l(\psi) = \int \int_D \nu_0(z) \psi(z)dxdy, \quad \psi \in L_1,
\]
with some \( \nu_0 \in L_\infty(D) \) so that
\[
\int \int_D \nu_0(z) \psi(z)dxdy = 0, \quad \psi \in \mathcal{L}(e); \quad \int \int_D \nu_0(z) \psi_0(z)dxdy = d.
\] (7.12)
Since the norm of \( l \) on the widest space \( L_1(D) \) is attained on its subspace \( \mathcal{L}_0 \), the function \( \nu_0 \) is of the form \( \nu_0(z) = |\psi_e(z)|/\psi_e(z) \) with integrable holomorphic \( \psi_e \) on \( D \setminus e \) given by (7.8).

After extending \( \nu_0 \) by zero to \( D^* \), which yields an extremal Beltrami coefficient \( \nu_0 \in \text{Belt}(D)_1 \) for our holomorphic functional \( J \), one can represent the map \( f^{\nu_0} \) by (6.8) getting from the second equality in (7.12) and from indicated minimality of \( \|l\| \) the estimate (7.10), since for all other \( f^\nu \in \Sigma_\kappa(D^*, 1, e) \), we have \( |J(f^\nu)| \leq |J(f^{\nu_0})| \).
However, this $\kappa$-quasiconformal map can move the fixed points $e_s$ to $f_0(e_s) = e_s + O(\kappa^2)$, where $f_0 = f^{\nu_0}$. Thus one needs to apply additional $O(\kappa^2)$-quasiconformal variation $h_0$ by Lemma 7.3 to get $h_0 \circ f_0(e_s) = e_s$ (for all $s$) and preserving the values $f_0^{(\alpha_j)}(z_j)$, and then take the extremal map $\tilde{f}$ (with smallest dilatation) satisfying
\[
\tilde{f}^{(\alpha_j)}(z_j) = f_0^{(\alpha_j)}(z_j), \quad \tilde{f}(e_s) = e_s
\] (7.13)
for all given $\alpha_j$ and $e_s$ so that its defining holomorphic quadratic differential $\psi_e$ belongs to the subspace $L_0$. It can be shown, using the uniqueness of Teichmüller extremal maps generated by integrable holomorphic quadratic differential that this $\psi_e$ is unique in $A_1(D \setminus e)$.

The assertion on uniform bound for the remainder in (7.10) follows the general distortion results for quasiconformal maps. This completes the proof of the theorem.

Remarks.

1. The quadratic differential $\psi_e$ constructed in the proof depends also on $\kappa$.
2. The assumption $f(1) = 1$ can be replaced by $f(0) = 0$; then the fixed points $e_s$ must be chosen to be distinct from the origin.

Similar theorem also holds for the over-normalized functions in bounded quasidisks. We illustrate it on the coefficient problem:

**Theorem 7.5.** For any $n \geq 2$, there is a number $\kappa_n(e) < 1$ such that for $kp \leq \kappa_n$ and all $f \in S_\kappa(\infty)$, which fix a given set $e = (e_1, \ldots, e_m) \subset \mathbb{D} \setminus \{0\}$, we have the sharp bound
\[
\max_{\|\mu\| \leq \kappa} |a_n(f^\mu)| = |a_n(f^{\kappa/\psi_n}|\psi_n)| = d_n \kappa + O(\kappa^2),
\] (7.14)
where similar to (7.8) and (7.9),
\[
\psi_n(z) = cz^{-n-1} + \sum_{s=1}^{m} \xi_s \rho_s(z), \quad d_n = \inf_{\mathcal{L}(e)} \|\psi_n - \psi\|_1.
\]
The constants $c, \xi_s$ are determined from the equations of type (7.10), and the remainder in (7.14) is estimated uniformly for all $\kappa \leq \kappa_n$.

The distortion bounds of type (7.7) given by Theorem 7.4 and its corollaries hold in somewhat weakened form (up to terms $O(\kappa^2)$ for the maps preserving an infinite subset $e$ in $D$, provided that the corresponding class $\Sigma_\kappa(D^*, 1, e)$ contains the functions $f^\mu \neq \text{id}$.

The proof is similar but now the quadratic holomorphic differentials $\psi_e$ defining the extremal functions are represented instead of (7.8) in the form
\[
\psi_e = c\psi_0 + \psi, \quad \psi \in \mathcal{L}(e)
\]
and there are no variations of type Lemma 7.3 for the infinite sets.
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Department of Mathematics, Bar-Ilan University
5290002 Ramat-Gan, Israel
and Department of Mathematics, University of Virginia,
Charlottesville, VA 22904-4137, USA