REALIZATION SPACES OF UNIFORM PHASED MATROIDS

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Abstract

A phased matroid is a matroid with additional structure which plays the same role for complex vector arrangements that oriented matroids play for real vector arrangements.

The realization space of an oriented (resp., phased) matroid is the space of vector arrangements in $\mathbb{R}^n$ (resp., $\mathbb{C}^n$) that correspond to oriented (resp., phased) matroid, modulo a change of coordinates. According to Mnëv’s Universality Theorem, the realization spaces of uniform oriented matroids with rank greater than or equal to 3 can be as complicated as any open semi-algebraic variety.

In contrast, uniform phased matroids which are not essentially oriented have remarkably simple realization spaces if they are uniform.

We also present a criterion for realizability of uniform phased matroids that are not essentially oriented.

1 Introduction

A matroid is a combinatorial object that abstracts the notion of linear independence in a vector configuration over an arbitrary field $F$. Oriented matroids are matroids with additional structure which encodes more geometric information than matroids by assigning to each ordered basis an element of $\{-1, 0, +1\}$. The theories of matroids and oriented matroids are major branches of combinatorics with applications in many fields of mathematics, including topology, algebra, graph theory, and geometry. Their contribution to mathematics motivates the question *what is a complex matroid?* (In fact, Ziegler wrote a paper with this title [11]).

Several mathematicians have addressed the question posed by Ziegler, with varying answers. In [11], Ziegler proposed his own definition of a complex matroid. His complex matroids are discrete objects with desirable characteristics consistent with being a generalization of oriented matroids, but they lack a notion of cryptomorphisms, an important characteristic of matroids and oriented matroids.

Another approach to Ziegler’s question is to generalize oriented matroids by assigning to each ordered basis an element of the set $S^1 \cup \{0\}$. In [7], Dress and Wenzel show that this non-discrete approach corresponds to basis “orientations” over the fuzzy ring $\mathbb{C}/\mathbb{R}_{>0}$, of which $S^1 \cup \{0\}$ is a subset. Phased matroids (referred to as *complex matroids* in [11, 3]) are an answer developed by Anderson and Delucchi [1], growing out of work by Below, Krummeck and Richter-Gebert [3].
and building on Delucchi’s diploma thesis [6]. In [1], Anderson and Delucchi explore how much of the foundations of oriented matroids can be paralleled with the structure of the set $S^1 \cup \{0\}$. They give a phased analog to the chirotope and circuit axioms of oriented matroids, and show that the definitions are cryptomorphic. A recent paper of Baker and Bowler [2], yet to appear as of the writing of this introduction, generalizes oriented matroid and phased matroid properties into matroids over hyperfields, proving the definitions are cryptomorphic on the hyperfield level, and unifying the fields of oriented matroids, phased matroids, and matroids over hyperfields other than signs of real numbers and phases of complex number.

Baker and Bowler’s paper is already expected to be important by allowing common properties of phased matroids and oriented matroids to be proven once, on the hyperfield level.

In this paper, we will focus on some of the differences between phased matroids and oriented matroids that cannot be proven on at the hyperfield level, but rely on the extra degree of freedom allowed over the complex numbers that is not allowed for over the real numbers. Some long term goals of phased matroid theory are to apply it to complex vector bundles and complex hyperplane arrangements in a way analogous to the applications of oriented matroids to real hyperplane arrangements and real vector bundles.

A celebrated theorem in oriented matroid theory is Mnëv’s Universality Theorem [8], which describes how complicated the topology of realization spaces of oriented matroids (including uniform oriented matroids) can be. An obvious question for phased matroid theory, is: How does the topology of realization spaces of phased matroids compare to that of oriented matroids?

This paper sets out to answer that question in the case of uniform phased matroids that are not essentially oriented. Surprisingly, the realization spaces of uniform phased matroids that are not essentially oriented are remarkably simple. In fact, regardless of the rank of a uniform phased matroid with a groundset of size $n$, if the phased matroid is not essentially oriented then the realization space is homeomorphic to $\mathbb{R}^{n-1}$ (Theorem 3.3).

Determining realizability of an oriented matroid is NP hard ([4], Theorem 8.7.2). In contrast, in Theorem 6.3 we give a polynomial time algorithm for determining realizability of a uniform phased matroid that is not essentially oriented. This algorithm also, in the affirmative case, gives a unique canonical realization of the phased matroid.

2 Phased matroid definitions

We will use polar coordinates to denote nonzero complex numbers. Thus we will view $\mathbb{C} \setminus \{0\}$ as $\mathbb{R}_{>0} \times S^1$, where $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ is the complex unit circle.

**Definition 2.1** (Phase of a complex number [3]). Let $s \in \mathbb{R}_{>0}$, and $\alpha \in S^1$. For the complex number $z = sa \in \mathbb{C}$ the phase of $z$ is

$$\text{ph}(z) = \begin{cases} 0 & \text{if } s = 0 \\ \alpha & \text{if } s \neq 0. \end{cases}$$

Thus $\text{ph}(z) \in S^1 \cup \{0\}$. Also, $s$ is the norm of $z$.

**Definition 2.2** (Hypersum [2]). Define the hypersum $\mathbb{H}(S) = \bigoplus_{k=1}^{m} \alpha_k$ of a finite set $S = \{\alpha_1, \ldots, \alpha_m\} \subset S^1 \cup \{0\}$ to be the set of all phases of strictly positive linear combinations of $S$. Thus
• \( \emptyset(\emptyset) = \emptyset \).
• \( \emptyset(\{\mu\}) = \{\mu\} \) for all \( \mu \).
• \( \emptyset(\{\mu, -\mu\}) = \emptyset(\{\mu, 0, -\mu\}) = \{\mu, 0, -\mu\} \) for all \( \mu \).
• if \( S = \{e^{i\alpha_1}, \ldots, e^{i\alpha_k}\} \) with \( k \geq 2 \) and \( \alpha_1 < \cdots < \alpha_k < \alpha_1 + \pi \) then \( \emptyset(S) = \emptyset(S \cup \{0\}) = \{e^{i\lambda} : \alpha_1 < \lambda < \alpha_k\} \).
• if \( S = \{e^{i\alpha_1}, \ldots, e^{i\alpha_k}\} \) with \( k \geq 3 \) and \( \alpha_1 < \cdots < \alpha_k = \alpha_1 + \pi \), then \( \emptyset(S) = \emptyset(S \cup \{0\}) = \{e^{i\lambda} : \alpha_1 < \lambda < \alpha_k\} \).
• otherwise (i.e., if the nonzero elements of \( S \) do not lie in a closed half-circle of \( S^1 \)), \( \emptyset(S) = S^1 \cup \{0\} \).

Note that in [1] the hypersum is referred to as the **phased convex hull**.

**Definition 2.3** (Phirotope [1]). A rank \( r \) phirotope on the set \( E \) is a non-zero, alternating function \( \varphi : E^r \rightarrow S^1 \cup \{0\} \) satisfying the following combinatorial complex Grassmann–Plücker relations.

For any two subsets \( \{x_1, \ldots, x_{r+1}\}, \{y_1, \ldots, y_{r-1}\} \) of \( E \),

\[
0 \in \bigoplus_{k=1}^{r+1} (-1)^k \varphi(x_1, \ldots, \hat{x}_k, \ldots, x_{r+1}) : \varphi(x_k, y_1, \ldots, y_{r-1}).
\]

The phirotope definition is a generalization of a **chirotope** in which \( \text{im} (\varphi) \subseteq \{-1, 0, +1\} \).

As suggested by the name, the combinatorial complex Grassmann-Plücker relations are motivated by the Grassmann-Plücker relations, which the minors of every matrix in \( C^{r \times n} \) with \( r \leq n \) satisfy. So every such matrix gives rise to a phirotope.

**Example 2.4.** Consider the \( 3 \times 5 \) matrix

\[
M = \begin{pmatrix}
1 & 0 & 0 & \frac{1}{2} e^{i\frac{\pi}{2}} & \frac{1}{3} e^{i\frac{\pi}{3}} \\
0 & 1 & 0 & 1 & \frac{2}{3} e^{i\frac{\pi}{3}} \\
0 & 0 & 1 & 0 & -1
\end{pmatrix}
\]

with complex entries. The function \( \varphi_M : [5]^3 \rightarrow S^1 \cup \{0\} \) such that \( \varphi_M(i, j, k) = \text{ph} (\det(M_{i,j,k})) \) satisfies the combinatorial complex Grassmann-Plücker relations and is a phirotope. There is a phirotope for any \( n \times r \) matrix with \( n \geq r \).

**Proposition 2.5** (Phirotope of a matrix [3]). Every matrix \( M \in \mathbb{C}^{r \times |E|} \) with \( |E| \geq r \) gives rise to the following function:

\[
\varphi_M : E^r \rightarrow S^1 \cup \{0\}, \quad \lambda = (\lambda_1, \ldots, \lambda_r) \mapsto \text{ph}(\lambda|M).
\]

Furthermore, \( \varphi_M \) is a phirotope.

Notice that if \( M \in \mathbb{R}^{r \times |E|} \), then \( \varphi_M \) is a chirotope.

Consider \( M \in \mathbb{C}^{n \times r} \). Let \( A \in GL(r, \mathbb{C}) \). Then \( \varphi_{AM} = \text{ph}(\det(A))\varphi_M \). So multiplication on the left by \( GL(r, \mathbb{C}) \) rotates the coordinate system.
Definition 2.6 (Phased matroid [1]). The set \( M := \{ \alpha \varphi \mid \alpha \in S^1 \} \) is a phased matroid.

If \( \varphi \) is a chirotope, then \( \{-\varphi, \varphi\} \) is an oriented matroid.

Definition 2.7 (Realizable phased matroid, realization of a phased matroid). Let \( M \) be a phased matroid with a phirotope \( \varphi \). \( M \) is realizable if there exists a matrix \( M \) such that \( \varphi_M = \varphi \). The matrix \( M \) is a realization of \( M \). We can also say that the phirotope \( \varphi \) is realizable with realization \( M \).

Define the equivalence class of a matrix as \( [M] := \{ AM \mid A \in GL(r, \mathbb{C}) \} \). Notice that if \( M \) is a realization of \( M \), then every element in the equivalence class \( [M] \) is a realization of \( M \).

Example 2.8. The matrix \( M \) in Example 2.4 is a realization of the realizable phased matroid \( \{ \alpha \varphi_M \mid \alpha \in S^1 \} \) denoted \( \mathcal{M}_M \). Also, for any invertible \( 3 \times 3 \) matrix \( A \), \( AM \) is a realization of \( M \).

Definition 2.9 (Underlying matroid of a phased matroid [1]). Let \( M \) be a rank \( r \) phased matroid with phirotope \( \varphi \). Then \( M := \{ \{i_1, \ldots, i_r\} \mid \varphi(i_1, \ldots, i_r) \neq 0 \} \) is a matroid called the underlying matroid of \( M \).

If \( M \) is realizable with realization \( M \), then its underlying matroid \( \mathcal{M} \) is realizable with realization \( M \).

Example 2.10. The underlying matroid of \( \mathcal{M}_M \) obtained from the matrix \( M \) in Example 2.4 is \( \{\{1,2,3\}, \{1,2,5\}, \{1,3,4\}, \{1,3,5\}, \{1,4,5\}, \{2,3,5\}, \{3,4,5\}\} \).

2.1 Remark about uniform phased matroids

Let \( \varphi \) be a phirotope. There is an underlying matroid, \( \mathcal{M} \), with bases \( B := \{ \lambda \mid \varphi(\lambda) \neq 0 \} \). If \( \text{im}(\varphi) \subseteq S^1 \), then \( \varphi \) and \( \mathcal{M} \) are uniform.

The matrix \( M \) in Example 2.4 gives rise to a realizable phased matroid \( \mathcal{M}(M) \) which is not uniform because \( \varphi(1,2,4) = 0 \).

3 The realization space

Definition 3.1 (Realization space). The realization space over \( \mathbb{C} \) of a rank \( r \) phased matroid \( \mathcal{M} \) on the ground set \( E \) is the quotient space
\[
\mathcal{R}_\mathbb{C}(\mathcal{M}) = GL(r, \mathbb{C}) \setminus \{M \in \mathbb{C}^{r \times |E|} \mid M \text{ is a realization of } \mathcal{M}\}.
\]

If \( \mathcal{M} \) is an oriented matroid,
\[
\mathcal{R}_\mathbb{R}(\mathcal{M}) = GL(d, \mathbb{R}) \setminus \{M \in \mathbb{R}^{r \times |E|} \mid M \text{ is a realization of } \mathcal{M}\}.
\]

For the special case of oriented matroids, realization spaces have been well studied. Results from oriented matroid theory apply to the phased matroids that are also oriented matroids.

Theorem 3.2 (Mnëv’s Universality Theorem [4], Theorem 8.6.6).

1. Let \( V \subseteq \mathbb{R}^s \) be any semialgebraic variety. Then there exists a rank \( 3 \) oriented matroid \( \mathcal{M} \) whose realization space \( \mathcal{R}(\mathcal{M}) \) is homotopy equivalent to \( V \).
2. If \( V \) is an open subset of \( \mathbb{R}^s \), then \( \mathcal{M} \) may be chosen to be uniform.

However, when we consider uniform phased matroids that are not essentially oriented, the topology of the realization space is surprisingly simple.

**Theorem 3.3.** Let \( \mathcal{M} \) be a rank \( r \), uniform, not essentially oriented, realizable phased matroid on \([n]\). Then \( \mathcal{R}(\mathcal{M}) \cong \mathbb{R}^{n-1} \).

The proof of Theorem 3.3 will appear in Section 5. As a motivating Lemma, we will prove the rank 2 case (in which we can drop conditions of uniformity and not essentially orientability.)

**Lemma 3.4.** Let \( \mathcal{M} \) be a simple, rank 2, realizable phased matroid on \( n \) elements. Then \( \mathcal{R}(\mathcal{M}) \cong \mathbb{R}^{n-1+k} \) where \( k = 0 \) if \( \mathcal{M} \) is not essentially oriented and \( k > 0 \) if \( \mathcal{M} \) is essentially oriented.

This result, for rank 2 uniform phased matroids that are not essentially oriented, was previously proven in [3] using cross ratios. Since cross ratios do not generalize to higher dimensions, their proof is not generalizable to phased matroids with rank \( > 2 \). For essentially oriented phased matroids the Lemma follows from the fact that all rank-2 oriented matroids have contractible realization spaces [4]. We will show an alternative proof for the not essentially oriented case in Section 5.

First, in the following section, we will develop tools which will be useful in proving the above statements.

### 4 Useful tools

In this section, we will introduce some tools that help prove Theorem 3.3. We will see how some of the content and structure of a potential realization is determined by the phirotope. In fact, in Section 4.1 we will be on the hunt for a *canonical form* of a realization of a phased matroid. In Section 4.2 through the Triangle Lemma we see the geometry of the complex space that real space does not have. This provides insight into why the essentially oriented phased matroids have realization spaces so different from the non-essentially oriented phased matroids. The definitions in this section refer to uniform phased matroids, but they can all be generalized to the non-uniform case. See [10] for details.

#### 4.1 Canonical Form

Every phased matroid \( \mathcal{M} \) can be compared to a (possibly different) phased matroid \( \mathcal{M}' \) which is in *canonical form* (Definition 4.18). While \( \mathcal{M} \neq \mathcal{M}' \), we will show that \( \mathcal{R}(\mathcal{M}) \cong \mathcal{R}(\mathcal{M}') \). So we can look at the realization space of \( \mathcal{M}' \) in order to understand the realization space of \( \mathcal{M} \). We will find that a phased matroid in canonical form provides advantages such as easily determining whether or not the phased matroid is essentially oriented, as well as easily building a realization of a realizable phased matroid that is not essentially oriented, and computing the realization space.

In order to get a phased matroid into canonical form, there are a lot of details that we need to address. In this section we will use matroid theoretic facts about bipartite graphs, compute signs of some special permutations, and recall some basic geometry facts about similar triangles.

The next lemma will help us prove Corollary 4.2, which is an important part of the process of building up a realization of a given phased matroid containing as much information about the potential realization as we can from a phirotope. In fact, Corollary 4.2 simply allows us to
are not determined is dependent on so many things, that it is worth its own lemma.  

Let \( J = (j_1, \ldots, j_k) \) where \( j_l \in [r+1, \ldots, n] \) and \( j_l < j_{l+1} \) for all \( 1 \leq l \leq k \). Denote by \( \pi(\tilde{H}_r, J) \) the permutation of \( \tilde{H}_r, J \) where the elements of \( J \) replace (in ascending order) the elements of \( H_r \) which are missing from \( \tilde{H}_r \).

**Lemma 4.1.** Consider the matrix \((I|N)\). Let \( N' \) be a \( k \times k \) submatrix of \( N \) consisting of rows \( H = \{h_1, \ldots, h_k\} \subset [r] \) and columns \( J = \{j_1, \ldots, j_k\} \subset [n] \). Let \( \sigma = \sum_{h \in H} \sum_{k \in \tilde{H}_r, k > h} 1 \). Then det \((N') = (-1)^\sigma \det((I|N)_{\tilde{H}_r,J})\).

**Proof.** Note that \( \sigma \) is summing, for each \( h \in H \), the number of elements in \([r]\) but not in \( H \) which are greater than \( i \). The sign \((-1)^\sigma\) is the sign of the shuffle permutation, \( S(H, J) \), that replaces, in order, the elements of \( J \) for the elements of \( H \) removed from \([r]\). For example, let \( r = 7 \), \( H = \{2, 3, 5\} \), and \( J = \{8, 11, 12\} \). Then the permutation is \((1, 8, 11, 4, 12, 6, 7)\).

Since the determinant is an alternating function, the determinant of the submatrix \( N' \) is given by

\[
\det(N') = (-1)^\sigma \det((I|N)_{\tilde{H}_r,J}).
\]

**Corollary 4.2.** Let \( M \) be a rank \( r \) realizable phased matroid with realization \((I|N)\) and phirotope \( \varphi = \varphi(I|N) \). Let \( N' \) be a \( k \times k \) submatrix of \( N \) consisting of rows \( H = \{h_1, \ldots, h_k\} \) and columns \( J = \{j_1, \ldots, j_k\} \). Then, for \( \sigma \) defined as in Lemma 4.1, \( \text{ph}(\det(N')) = (-1)^\sigma \varphi(\tilde{H}_r, J) \).

**Proof.** Since \( M \) is realizable, \( \varphi(\tilde{H}_r, J) = \text{ph}(\det((I|N)_{\tilde{H}_r,J})) = (-1)^\sigma \text{ph}(\det(N')) \).

So this follows from Lemma 4.1.

Throughout this paper we are only interested in the case when \( k \in \{1, 2\} \). In the case that \( k = 1 \) we get the phases of the entries of any potential realization of \( \varphi \) because

\[
(-1)^{r-i} \text{ph}((I|N)_{\{i\},J}) = \text{ph}(\det((I|N)_{\{1, \ldots, \hat{i}, \ldots, r, j\}})) = \varphi(1, \ldots, \hat{i}, \ldots, r, j).
\]

**Corollary 4.3.** Suppose \( \{1, \ldots, r\} \) is a basis of \( M \). A phirotope of \( M \) determines the phases of \( N \) for any realization \((I|N)\) of \( M \).

For example, if we consider the matrix \( M \) from Example 2.4, we see that

\[
\varphi(1, 3, 5) = \text{ph} \left( \begin{pmatrix} 1 & 0 & \frac{1}{2} e^{\frac{i\pi}{2}} \\ 0 & 0 & \frac{3}{2} e^{\frac{i\pi}{2}} \\ 0 & 1 & -1 \end{pmatrix} \right) = -e^{\frac{i\pi}{2}} = (-1)^3 \text{ph}(M_{2,5})
\]

**Definition 4.4** (Rephasing of a phirotope \[3\]). Let \( \varphi \) be a rank \( r \) phirotope of a phased matroid \( M \). Let \( \rho = (\rho_1, \ldots, \rho_n) \in (S^1)^n \). The function

\[
\varphi^\rho : [n]^r \to S^1 \cup \{0\} \\
(\lambda_1, \ldots, \lambda_r) \to \rho_{\lambda_1} \cdots \rho_{\lambda_r} \varphi(\lambda_1, \ldots, \lambda_r)
\]

is the rephasing of \( \varphi \) by \( \rho \).
In $\mathbb{K}$ rephasing is referred to as reorienting, motivated by its similarity to reorientation of oriented matroids.

**Lemma 4.5** ([1]). The function $\varphi^\rho$ is a phirotope.

Let $\mathcal{M}^\rho$ be the phased matroid with phirotope $\varphi^\rho$. Then $\mathcal{M}^\rho$ is the rephasing of $\mathcal{M}$ by $\rho$. Notice that if we start with a chirotope $\chi$, and apply rephase by $\rho \in (S^1)^n$ then $\text{im}(\chi^\rho) \subseteq \{-\alpha_0,0,\alpha_0\}$ for some $\alpha_0 \in S^1$. So $\chi^\rho$ is a phirotope, but not a chirotope, and $\{\alpha \chi^\rho | \alpha \in S^1\}$ is a phased matroid, but not an oriented matroid.

Let $D(x_1,\ldots,x_m)$ be the $m \times m$ diagonal matrix with entries $x_1,\ldots,x_m$ on the diagonal.

**Lemma 4.6** ([2]). If $M$ is a realization of $\mathcal{M}$ then $M \cdot D(\rho_1,\ldots,\rho_n)$ is a realization of $\mathcal{M}^\rho$.

The following notion of scaling equivalence is borrowed from matroid theory.

**Definition 4.7** (Scaling equivalent matrices [3]). Let $M$ and $N$ be $r \times n$ matrices over a field $\mathbb{F}$. We say $M$ and $N$ are scaling equivalent if $N$ can be obtained by scaling rows and columns of $M$ by non-zero elements of $\mathbb{F}$.

Scaling rows and columns of a $r \times n$ matrix $M$ is algebraically equivalent to multiplying $M$ on the left by a diagonal $r \times d$ matrix and on the right by an $n \times n$ diagonal matrix (both with non-zero determinant).

In matroid theory, if two matrices are scaling equivalent, then they give rise to the same matroid. This is not the case with phased matroids. However, scaling equivalent matrices give rise to phased matroids that are rephasings of each other. The following result demonstrates why these are useful to us when studying the realization space.

**Theorem 4.8.** Let $\mathcal{M}$, $\mathcal{N}$ be phased matroids. Let $M \in \mathcal{R}(\mathcal{M})$, $N \in \mathcal{R}(\mathcal{N})$. If $M$ and $N$ are scaling equivalent, then $\mathcal{R}(\mathcal{M}) \cong \mathcal{R}(\mathcal{N})$.

**Proof.** Since $M$ and $N$ are scaling equivalent matrices, there exist diagonal matrices $D_1$ and $D_2$ such that $M = D_1ND_2$. Consider the map $f : \mathcal{R}(\mathcal{M}) \rightarrow \mathcal{R}(\mathcal{N})$ where $f([H]) = [D_1HD_2]$. Let $K \in [H]$. Then there is an $A \in GL(r,\mathbb{C})$ such that $K = AH$.

We first check that for $f$ to be well defined, we need $D_1KD_2 \in [D_1HD_2]$. So we must find a $B \in GL(r,\mathbb{C})$ such that $D_1K = BD_1HD_2$. Let $B = D_1AD_1^{-1}$. Then $D_1K = D_1AH = (D_1AD_1^{-1})D_1HD_2 = BD_1HD_2$. So $D_1K \in [D_1HD_2]$.

The inverse map is $f^{-1}([G]) = [D_1^{-1}GD_2]$. Since both $f$ and $f^{-1}$ are quotients of continuous maps, $f$ and $f^{-1}$ are continuous. Therefore, $f$ is a homeomorphism.

A consequence of Theorem 4.8 is that any phased matroid with a phirotope which is a rephasing of chirotope from an oriented matroid gives rise to a phased matroid whose realizability results defer to those of oriented matroids, including Mnëv’s Universality Theorem (3.2), and that determining realizability is NP hard [4]. Proposition 4.10 classifies such phased matroids.

Consider a matrix $M \in \mathbb{R}^{r \times n}$. $M$ gives rise to a chirotope $\chi_M$, which is also a phirotope $\varphi_M$ with $\text{im}(\varphi_M) \subseteq \{-1,0,+1\}$. We can multiply $M$ by $A \in GL(r,\mathbb{C})$ on the left to give a matrix with non-real entries which is a realization of the same phased matroid as $M$. If $\det(A) = \alpha \notin \mathbb{R}$, $\text{im}(\varphi_{AM}) \subseteq \{-\alpha,0,\alpha\} \subseteq \{-1,0,+1\}$, but $\varphi_{AM}$ is a phirotope of the same phased matroid that came from a real matrix. These special phirotopes are called complexified chirotopes. We can also complexify chirotopes that do not come from a matrix. Let $\chi$ be a chirotope and $\beta \in S^1 \setminus \mathbb{R}$. Then $\beta \chi$ is a phirotope, but is not a chirotope. The set $\{\alpha \chi | \alpha \in S^1\}$ is a phased matroid. The phased matroids of such phirotopes are called complexified oriented matroids.
matroids, and have realization spaces that defer to the oriented matroid the original chirotope comes from. Similarly, as Theorem 4.8 suggests, rephases of complexified chirotopes will give rise to a phase matroid $M'$ whose realization space defers to the realization space of the original oriented matroid.

It turns out the matrix $M$ from Example 2.4 is scaling equivalent to a real matrix.

**Example 4.9.** Let
\[
A = \begin{pmatrix}
e^{i\frac{\pi}{2}} & 0 & 0 \\
0 & 2e^{i\frac{\pi}{2}} & 0 \\
0 & 0 & 3
\end{pmatrix}
\quad \text{and} \quad
D = \begin{pmatrix}
e^{i\frac{\pi}{2}} & 0 & 0 & 0 \\
0 & \frac{1}{2}e^{i\pi} & 0 & 0 \\
0 & 0 & \frac{1}{3} & 0 \\
0 & 0 & 0 & \frac{1}{3}
\end{pmatrix},
\]
then,
\[
A \cdot \begin{pmatrix}
1 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 2 \\
0 & 0 & 1 & 0 & -1
\end{pmatrix}
\]
gives rise to a complexified oriented matroid. Furthermore,
\[
A \cdot \begin{pmatrix}
1 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 2 \\
0 & 0 & 1 & 0 & -1
\end{pmatrix} \cdot D = M.
\]
So, $M'_{M}$ is a rephase of the a complexified oriented matroid. We call such phased matroids essentially oriented.

**Proposition 4.10** (Essential orientability). Let $M$ be a phased matroid. The following are equivalent:

1. $M$ is a rephasing of a complexified oriented matroid.
2. For some phirotope $\phi$ of $M$ and some $\alpha \in S^1$, there exists $\rho \in (S^1)^n$ such that $\text{im}(\phi^\rho) \subseteq \{-\alpha, 0, \alpha\}$.
3. For every phirotope $\phi_i$ of $M$, there is an $\alpha_i \in S^1$ and $\rho^i \in (S^1)^n$ such that $\text{im}(\phi_i^\rho^i) \subseteq \{-\alpha_i, 0, \alpha_i\}$.
4. There exist a phirotope $\phi$ of $M$ and $\rho \in (S^1)^n$ such that $\text{im}(\phi^\rho) \subseteq \{-1, 0, 1\}$.

If $M$ is realizable with realization space $\mathcal{R}(M)$, the following are also equivalent to the above:

5. For some $\rho \in (S^1)^n$, there exists $M \in \mathcal{R}(M^\rho)$ with all real entries.
6. For any $M \in \mathcal{R}(M)$, there exists a matrix $N \in \mathbb{R}^{r \times n}$ such that $M$ and $N$ are scaling equivalent.

**Proof.**

(1 $\iff$ 3)

$M$ is a rephasing of a complexified oriented matroid $M'_{\chi}$
\[
\iff \quad \text{there is some } \rho \in (S^1)^n \text{ such that } M^\rho = M'_{\chi} = \{\alpha\chi \mid \alpha \in S^1\}
\]
\[
\iff \quad \text{for all phirotopes } \phi_i \text{ of } M, \text{ there is an } \alpha_i \in S^1 \text{ such that } \phi_i^\rho = \alpha_i\chi
\]
is a phirotope of $M'_{\chi}$
\[
\iff \quad \text{im}(\phi_i^\rho) = \text{im}(\alpha_i\chi) \subseteq \{-\alpha_1, 0, \alpha_1\}.
\]
(2 $\Rightarrow$ 3) Let $\varphi, \varphi'$ be phirotopes of $\mathcal{M}$. Then there is some $\beta \in S^1$ such that $\beta \varphi = \varphi'$. Suppose there is a $\rho \in (S^1)^n$ such that $\text{im}(\varphi) \subseteq \{\alpha, 0, \beta\alpha\}$. Then $\text{im}((\varphi')^\rho) = \text{im}(\beta \varphi^\rho) = \beta \text{im}(\varphi^\rho) = \{\beta(-\alpha), 0, \beta\alpha\}$.

(2 $\Rightarrow$ 4) Suppose $\varphi$ is a phirotope of $\mathcal{M}$ and there is a $\rho$ such that $\text{im}(\varphi^\rho) \subseteq \{\alpha, 0, \alpha\}$. $\alpha^{-1} \varphi$ is a phirotope of $\mathcal{M}$ and $\text{im}(\alpha^{-1} \varphi) = \alpha^{-1}(\{\alpha, 0, \alpha\} = \{1, 0, 1\}$.

(5 $\Leftrightarrow$ 6) Suppose $\mathcal{M}$ is a rank $r$ realizable phased matroid on $n$ elements. For $\rho \in (S^1)^n$, there is a matrix $N \in \mathcal{R}(\mathcal{M}^\rho)$ such that $N \in \mathbb{R}^{r \times n}$ if and only if there is $M \in \mathcal{R}(\mathcal{M})$ such that $MD(\rho) = N$. For every $M^i \in \mathcal{R}(\mathcal{M})$, there is an $A^i \in GL(r, \mathbb{C})$ such that $A^iM^i = M$. So $A^iM^iiD(\rho) = N$ if and only if $A^iM^i$ and $N$ are scaling equivalent matrices for all $A^iM^i \in \mathcal{R}(\mathcal{M})$.

(5 $\Rightarrow$ 4) Suppose there is an $M \in \mathcal{R}(\mathcal{M}^\rho)$ such that $M \in \mathbb{R}^{r \times n}$. Then every minor of $M \in \mathbb{R}$. So $\varphi^\rho(\lambda) \in \{-1, 0, 1\}$ for all $\lambda$.

(4 $\Rightarrow$ 5) Suppose there is a phirotope $\varphi$ of $\mathcal{M}$, with $\varphi(1, \ldots, r) \neq 0$, and $\rho \in (S^1)^n$ such that $\text{im}(\varphi^\rho) \subseteq \{-1, 0, 1\}$. Consider $(I|N) \in \mathcal{R}(\mathcal{M}^\rho)$. Since $\text{ph}((I|N)_{i,j}) = \varphi^\rho(1, \ldots, \hat{i}, \ldots, r, j) \in \{-1, 0, 1\}$, every entry of $(I|N) \in \mathbb{R}$.

**Definition 4.11** (Essential orientability). If any of the conditions of Proposition 4.10 hold, we say $\mathcal{M}$ (or any of its phirotopes) is essentially oriented.

Note that in previous literature, essentially oriented phased matroids and phirotopes are referred to as chirotopal [11, 13].

**Corollary 4.12.** Let $\mathcal{M}$ be a rank $r$ essentially oriented phased matroid on $[n]$. Let $\mathcal{M}_\chi$ be a rank $r$ oriented matroid on $[n]$ with chirotope $\chi$ such that for some phirotope $\varphi$ of $\mathcal{M}$, $\rho \in (S^1)^n$, and for all $\lambda \in [n]^r$, $\varphi^\rho(\lambda) = \chi(\lambda)$. Then $\mathcal{R}(\mathcal{M}_\chi) \cong \mathcal{R}(\mathcal{M})$.

**Proof.** This follows from Proposition 4.10 and Theorem 4.8.

As a consequence of Corollary 4.12, results about realizations of essentially oriented phased matroids defer to the results about their oriented matroid cousins. Our main results (Theorem 3.3 and Theorem 6.3) in this paper, refer to phased matroids that are not essentially oriented. Now that we have determined that rephasing a phased matroid results in a phased matroid with a homeomorphically equivalent realization space, we can partition phased matroids into realization classes and choose our favorite phirotope of our favorite phased matroid from the same realization class, which we will call the canonical phased matroid. We do this with the help of the associated bipartite graph, a tool also borrowed from matroid theory.

**Definition 4.13** (Associated bipartite graph $G_\mathcal{M}$ [9, 6.4]). Let $\mathcal{M}$ be a rank $r$ simple matroid on $[n]$ such that $\{1, \ldots, r\}$ is a basis. Let $\mathcal{G}_\mathcal{M}$ be the bipartite graph, with vertex set $[n]$, in which $e_{i,j}$ is an edge if and only if $\{1, \ldots, \hat{i}, \ldots, r, j\}$ is a basis of $\mathcal{M}$. The two disjoint sets of vertices of the bipartite graph $\mathcal{G}_\mathcal{M}$ are vertices $\{1, \ldots, r\}$, which are the left hand side, and vertices $\{r+1, \ldots, n\}$, which are the right hand side.

If $\mathcal{M}$ is realizable with realization $(I|N)$ then there is an edge $e_{i,j}$ in $\mathcal{G}_\mathcal{M}$ if and only if $(I|N)_{i,j} \neq 0$.

The associated bipartite graph for the phased matroid $\mathcal{M}_M$ where $M$ is the matrix from Example 2.4 is shown in Figure 1.

Notice that by Corollary 4.13 if $\mathcal{M}$ is uniform, then $\mathcal{G}_\mathcal{M}$ is a complete graph.
Theorem 4.14. Let $\mathcal{M}$ be a rank $r$ matroid on $[n]$ realizable over $\mathbb{F}$ with $(I|N) \in \mathbb{F}^{r \times n}$ a realization of $\mathcal{M}$. Let $F = \{t_1, \ldots, t_{n-k}\}$ be a spanning forest of $G_{\mathcal{M}}$. Let $(s_1, \ldots, s_{n-k})$ be an ordered $n-k$ tuple of non-zero elements of $\mathbb{F}$. Then $\mathcal{M}$ has a unique realization $(I|\tilde{N}) \in \mathbb{F}^{r \times n}$ such that for each $i \in [n-k]$, the entry of $\tilde{N}$ corresponding to $t_i$ is $s_i$.

In fact, $(I|\tilde{N})$ can be obtained from $(I|N)$ by a sequence of row and column scalings. Hence $(I|\tilde{N})$ is scaling equivalent to $(I|N)$.

As a result of Theorem 4.14, we see that given a matrix, $(I|N)$, whose associated bipartite graph has $k$ connected components, we can determine $n-k$ entries of $N$ to be any values we want. We will utilize this by choosing $(s_1, \ldots, s_{n-k})$ to be an $n-k$ tuple of ones. The next definition helps us decide which spanning tree we will use. With our choice of an $n-k$ tuple of ones, and a particular spanning tree, we can build an $r \times n$ array that might be a realization of $\mathcal{M}$, if we can determine the real lengths of the entries that are not determined by the associated bipartite graph.

Corollary 4.15. Let $\varphi$ be a rank $r$ phirotepe $\varphi : [n]^r \Rightarrow S^1 \cup \{0\}$. Let $F = \{t_1, \ldots, t_{n-k}\}$ be a spanning forest of $G_{\mathcal{M}_\varphi}$. Let $(s_1, \ldots, s_{n-k})$ be an ordered $n-k$ tuple of non-zero elements of $\mathbb{F}$. Then there exists a $\rho$ such that $(-1)^{r-i}\varphi(1, \ldots, \hat{i}, \ldots, r,j) = s_i$ for all $e_{i,j} \in T$ and for each associated $s_i \in S$.

The proof of this follows from Theorem 4.14 and Corollary 4.2.

Definition 4.16 (Canonical Spanning Tree). For each connected component in an associated bipartite graph $G$, with corresponding array $A$, we say the canonical spanning tree of the component is the tree containing all edges associated to the first non-zero entry in each row of $M$, the last non-zero entry in each column of $A$ (assuming it does not make a cycle in $G$), and any other edges of $G$ corresponding to the last non-zero, entry in each column of $A$ that has not already been included, and does not create a cycle.

Example 4.17. The canonical spanning tree for the Associated Bipartite Graph in Example 1 is seen as a collection of red dashed edges in Figure 2.

Theorem 4.14 is very useful to us because by Theorem 4.8 scaling equivalent matrices give rise to phased matroids with realization spaces with homeomorphically equivalent topology. We
will use this to distinguish a canonical realization of a phased matroid that we will use to explore the topology of the realization space of any phased matroid in its equivalence class.

**Definition 4.18** (Canonical form of a realization of \( \mathcal{M} \)). A realization \((I | \tilde{N})\) of a rank \(r\), uniform phased matroid is in canonical form if the \(n-k\) entries of \(N\) that correspond to the canonical spanning tree of \(G_M\) are 1. A uniform phirotope \(\varphi\) of \(\mathcal{M}\) is in canonical form if \(\varphi(1, \ldots, r-1, j) = 1\) for all \(j \geq d\), and \(\varphi(1, \ldots, i, \ldots, r+1) = (-1)^{i+r}\) for all \(i < r\).

**Example 4.19.** The realization of \(\mathcal{M}_M\) from Example 2.4 which is in canonical form would look like the following (we will get to the unknown entry \(*\) soon).

\[
\begin{pmatrix}
1 & 0 & 0 & 1 & * \\
0 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1
\end{pmatrix}
\]

There may not be such a realization in \(\mathcal{R}(\mathcal{M}_M)\) but there is such a realization in \(\mathcal{R}(\mathcal{M}_M^{\varphi})\) for \(\varphi = (e^{i\frac{2\pi}{3}}, e^{i\frac{4\pi}{3}}, 1, e^{i\frac{2\pi}{3}}, 1)\). The values of \(\varphi\) come from the diagonal matrix \(D\) in Example 4.9. Since this matrix is scaling equivalent to a real matrix, by Proposition 4.10 the phased matroids \(\mathcal{M}_M\) and \(\mathcal{M}_M^{\varphi}\) are essentially oriented and the unknown value \(*\) is real.

**Corollary 4.20.** Let \(\mathcal{M}\) be a rank \(r\) realizable phased matroid on \([n]\). Let \(\varphi\) be the phirotope of \(\mathcal{M}\) in canonical form. Let \((I | \tilde{N})\) be a realization of \(\mathcal{M}\) in canonical form. Then \(\mathcal{M}\) is essentially oriented if and only if \(\text{im}(\varphi) \subseteq \{-1,0,1\}\) and \((I | \tilde{N}) \in \mathbb{R}^{r \times n}\).

**Proof.** Suppose \(\mathcal{M}\) is a realizable essentially oriented phased matroid with phirotope \(\varphi\). Then \(\text{im}(\varphi) \subseteq \{-\alpha, 0, \alpha\}\) for \(\alpha \in S^1\). If \(\varphi\) is in canonical form, then \(\varphi(1, \ldots, r) = 1\). So \(\alpha = 1\) and \(\text{im}(\varphi) \subseteq \{-1,0,1\}\). In particular, \(\varphi(1, \ldots, i, \ldots, d, j) = \text{ph}((I | \tilde{N})_{i,j}) \in \{-1,0,1\}\), so \((I | \tilde{N}) \in \mathbb{R}^{r \times n}\).

The canonical form of a realization of a realizable uniform phased matroid \(\mathcal{M}\) is

\[
(I | \tilde{N}) = \begin{pmatrix}
I & \begin{pmatrix}
N \\
\vdots \\
1 & \cdots & 1
\end{pmatrix}
\end{pmatrix}.
\]

**Corollary 4.21.** Let \(\mathcal{M}\) be a uniform rank \(r\) phased matroid on \([n]\). There exists \(\rho \in (S^1)^n\) such that

\[
\mathcal{R}(\mathcal{M}) \cong \{(I | \tilde{N}) \mid (I | \tilde{N}) \text{ is a realization of } \mathcal{M}^{\rho} \text{ in canonical form}\} \times \mathbb{R}_{>0}^{n-1}.
\]
Proof. The existence of \((I|\tilde{N})\) follows from Theorem 4.14. Using Theorem 4.14 in building canonical realization \((I|\tilde{N})\), we chose the values associated with each of the \(n-1\) edges of our canonical spanning tree to be 1. It is clear that if we had chosen any other set of positive real values \((s_1, \ldots, s_{n-1}) \in \mathbb{R}_{>0}^{n-1}\), the result would be a different matrix in \(\mathcal{M}^\rho\). This \(n-1\) degrees of freedom is where the \(\mathbb{R}_{>0}^{n-1}\) comes from. The equivalence follows from Theorem 4.8. ■

4.2 Triangle Lemma

Definition 4.22 (Triangular equation). Let \(\alpha, \beta, \gamma \in S^1\) such that \(\alpha \neq \pm \beta\). An equation of the form \(\gamma = \text{ph}(s_1\alpha - s_2\beta)\), where \(s_1, s_2 \in \mathbb{R}_{>0}\), is called a triangular equation.

The following Lemma will be used often to prove later results.

Lemma 4.23 (Triangle Lemma). If \(\gamma = \text{ph}(s_1\alpha - s_2\beta)\) is a triangular equation and either \(s_1\) or \(s_2\) are known, then the equation can be solved uniquely for the unknown quantity.

Proof. Consider \(\gamma, \alpha,\) and \(\beta \in S^1\). Consider the rays through \(\alpha, \beta,\) and \(\gamma\) in the complex plane. Since \(\gamma = \text{ph}(s_1\alpha - s_2\beta)\), we can draw a triangle with interior angles \(\theta\) and \(\psi\) as seen in Figure 3. The angles \(\theta\) and \(\psi\) are determined by \(\alpha, \beta,\) and \(\gamma\). Therefore, the triangular equation determines the triangle up to similarity. Since either \(s_1\) or \(s_2\) is known, the entire triangle is determined by the triangular equation. ■

![Figure 3: The triangle with interior angles \(\theta\) and \(\psi\) is constructed from the equation \(\gamma = \text{ph}(s_1\alpha - s_2\beta)\).](image)

5 Proof of Theorem 3.3

Now that we have a canonical realization and the Triangle Lemma, we can prove our main Theorem:
Theorem 5.1 (Theorem 3.3). Let $M$ be a rank $r$, uniform, not essentially oriented, realizable phased matroid on $[n]$. Then $R(M) \cong \mathbb{R}_{>0}^{n-1}$.

As a warm-up, we will prove the case for rank 2 phased matroids, which is stated as Lemma 3.4.

5.1 Rank 2 phased matroids

For rank 2 phased matroids we can drop the condition that the phased matroid is uniform and not essentially oriented and get a similar result.

Recall, Lemma 3.4 states that for a simple, rank 2, realizable phased matroid $M$ on $n$ elements, $R(M) \cong \mathbb{R}_{>0}^{n-1} + k > 0$ where $k = 0$ if $M$ is not essentially oriented and $k > 0$ if $M$ is essentially oriented.

This result, for the uniform, rank 2 phased matroids that are not essentially oriented, was previously proven in [3] using cross ratios. Since cross ratios do not generalize to higher dimensions, their proof is not generalizable to phased matroids with rank $> 2$. For essentially oriented phased matroids the Lemma follows from the fact that all rank-2 oriented matroids have contractible realization spaces [4].

5.1.1 Proof of Lemma 3.4

Proof. If $M$ is a rank 2 phased matroid and is not uniform, then either zeros appear as entries of columns 3, ..., $n$ in the canonical realization or a pair of elements $\{i,j\}$ are parallel. But any column with a zero is either a loop in $M$, or is parallel to either $e_1$ or $e_2$. If $e$ is a loop of $M$ then $R(M/\{i\}) \cong R(M)$ and if elements $i,j$ are parallel in $M$ then $R(M/\{i\}) \times R_{>0} \cong R(M)$ [10]. For the case when $M$ is essentially oriented, the proof follows from analogous results about oriented matroids. So without loss of generality, we may assume $M$ is a uniform, not essentially oriented, realizable phased matroid.

Consider a canonical realization

$$M = \begin{pmatrix} 1 & 0 & 1 & s_4 \alpha_4 & \cdots & s_n \alpha_n \\ 0 & 1 & 1 & 1 & \cdots & 1 \end{pmatrix}$$

of $M^\rho$.

Notice that $\varphi(1,i)$ and $\varphi(2,j)$ give us the phases of each entry of $N$ so the only unknown information about the realization is the values of $s_4, \ldots, s_n \in \mathbb{R}_{>0}$. In fact

$$\{(I|\tilde{N}) | (I|\tilde{N}) \text{ is a canonical realization of } M^\rho\} \text{ can be thought of as}$$

$$\{(s_4, \ldots, s_n) \in \mathbb{R}_{>0}^{n-4} \mid s_i = m_{1,i} \text{ for a realization } M \text{ of } M^\rho\}.$$  \hspace{1cm} (2)

Since $M$ is not essentially oriented, for some $j \in \{4, \ldots, n\}$, $\alpha_j \notin \{-1, +1\}$. Therefore, by Lemma 4.23, $s_j$ is determined by the equation $\varphi(3,j) = \text{ph}(1 - s_j \alpha_j)$. For any other $k \neq j \in \{4, \ldots, n\}$, the equation $\varphi(k,j) = \text{ph}(s_k \alpha_k - s_j \alpha_j)$ determines $s_k$.

Since $s_j$ is determined by $\varphi$ for all $j \in \{4, \ldots, n\}$, the set $\{(I|\tilde{N}) | (I|\tilde{N}) \text{ is a canonical realization of } M^\rho\}$ is a single point. Therefore, by Corollary 4.21, $R(M) = \mathbb{R}_{>0}^{n-1}$. \hfill \blacksquare

5.1.2 Proof of Theorem 3.3

The previous proof for rank 2 phased matroids provides insight into the proof for rank $r$ uniform phased matroids that are not essentially oriented.
Proof. Consider $(I|\bar{N}) \in \mathcal{R}(\mathcal{M})$. Since $\mathcal{M}$ is uniform, all entries of $\bar{N}$ are non-zero. Since $\mathcal{M}$ is not essentially oriented, there is at least one non-real entry in $\bar{N}$.

Furthermore, by Corollary 4.2, for each $1 \leq i \leq r - 1$ and $r + 1 \leq j \leq n$, $\alpha_{i,j}$ is determined by $\varphi$, that is $\alpha_{i,j} = (-1)^{r-i}\varphi(1, \ldots, i-1, j, i+1, \ldots, n)$. Thus, by Corollary 4.21, it remains to find all $s_{i,j}$ such that $1 \leq i \leq r - 1$, $r + 1 \leq j \leq n$.

$$
(I|\bar{N}) = 
\begin{pmatrix}
1 & 0 & \cdots & 0 & 1 & s_{1,d+2} & \cdots & s_{1,n} \\
0 & \ddots & 0 & \vdots & 1 & s_{r-1,d+2} & \cdots & s_{r-1,n} \\
\vdots & \ddots & 0 & \vdots & 1 & 1 & \cdots & 1 \\
0 & \cdots & 0 & 1 & 1 & 1 & \cdots & 1
\end{pmatrix}
\in \mathcal{R}(\mathcal{M}).
$$

Notice, by Lemma 4.1 for $H = \{i, m\}$ where $i \leq m \leq r$, $\sigma = \sum_{k \notin H, k > i} 1 + \sum_{k \notin H, k > m} 1 = r - i - 1 + r - m = 2r - i - m - 1$, which has the same parity of $i + m + 1$. Therefore,

$$
(-1)^{i+m+1}[1, \ldots, \hat{i}, \ldots, \hat{m}, \ldots, r, k, j]_{(I|\bar{N})} = \det \begin{pmatrix}
\alpha_{i,k} & \alpha_{i,j} \\
\alpha_{m,k} & \alpha_{m,j}
\end{pmatrix}.
$$

To determine the value of $s_{i,j}$, there are four cases to consider. In each case, we find a $\lambda$-minor of $(I|\bar{N})$ that is equal to a $2 \times 2$ minor of $(I|\bar{N})$ which results in a triangular equation in which $s_{i,j}$ is the only unknown.

For Case 1, suppose $\alpha_{i,j} \neq \pm 1$. Then

$$
(-1)^{r-i+1}\varphi(1, \ldots, \hat{i}, \ldots, r - 1, j) = \text{ph} \left( \det \begin{pmatrix} 1 & s_{ij} \alpha_{ij} \\ 1 & 1 \end{pmatrix} \right) = \text{ph}(1 - s_{ij} \alpha_{ij}) \quad (3)
$$

is a triangular equation and $s_{i,j}$ is determined by the Triangle Lemma.

For the remaining three cases, we assume $\alpha_{i,j} = \pm 1$.

For the second case, suppose there exists $k > r + 1$ such that $\alpha_{i,k} \neq \pm 1$. We know $s_{ik}$ from Case 1. Without loss of generality, assume $k > j$. Then

$$
(-1)^{r-i+1}\varphi(1, \ldots, \hat{i}, \ldots, r - 1, j, k) = \text{ph} \left( \det \begin{pmatrix} s_{ij} \alpha_{ij} & \alpha_{ik} \\ 1 & 1 \end{pmatrix} \right) = \text{ph}(s_{ij} \alpha_{ij} - s_{ik} \alpha_{i,k})
$$

is a triangular equation and $s_{i,j}$ is determined by the Triangle Lemma.

For the final two cases, $s_{ik} \alpha_{i,k} \in \mathbb{R}$ for all $r + 1 < k \leq n$. This means the entire row has phase $\pm 1$.

For Case 3, we assume there is a non-real entry in the $j^{th}$ column. The equation

$$
(-1)^{i+m+1}\varphi(1, \ldots, \hat{i}, \ldots, \hat{m}, \ldots, r + 1, j) = \text{ph} \left( \det \begin{pmatrix} 1 & s_{ij} \alpha_{ij} \\ 1 & s_{mj} \alpha_{mj} \end{pmatrix} \right) = \text{ph}(s_{mj} \alpha_{mj} - s_{ij} \alpha_{ij})
$$

is triangular and $s_{i,j}$ is determined by the Triangle Lemma.

For the final case, assume all entries in row $i$ and column $j$ are real. There must be a non-real entry $s_{m,k} \alpha_{m,k} \notin \mathbb{R}$ with $m \neq i$, $k \neq j$. From previous cases, $s_{mj}$ and $s_{ij}$ are determined. Also, we know $\alpha_{ij}$, $\alpha_{i,k}$, and $\alpha_{m,j}$ are all $\pm 1$. So,

$$
(-1)^{i+m+1}\varphi(1, \ldots, \hat{i}, \ldots, \hat{m}, \ldots, k, j) = \text{ph} \left( \det \begin{pmatrix} s_{ik}(\pm 1) & s_{ij}(\pm 1) \\ s_{mk} \alpha_{mk} & s_{mj}(\pm 1) \end{pmatrix} \right) = \text{ph}(s_{ik} s_{mj} - s_{ij} s_{mk} \alpha_{mk})
$$
is a triangular equation. So \( s_{i,j} \) is determined by the Triangle Lemma.

All \( s_i \)'s are determined. So

\[
\begin{pmatrix}
1 & \cdots & 1 \\
1 & \cdots & 1 \\
\vdots & \ddots & \vdots \\
N & \cdots & N
\end{pmatrix}
\begin{pmatrix}
1 & \cdots & 1 \\
1 & \cdots & 1 \\
\vdots & \ddots & \vdots \\
N & \cdots & N
\end{pmatrix}
\begin{pmatrix}
1 & \cdots & 1 \\
1 & \cdots & 1 \\
\vdots & \ddots & \vdots \\
N & \cdots & N
\end{pmatrix}
\]

is a realiz. of \( \mathcal{M} \) \( \cong (\mathbb{R}_{>0})^0 \).

Therefore, \( \mathcal{R}(\mathcal{M}) = (\mathbb{R}_{>0})^0 \times (\mathbb{R}_{>0})^{n-1} \cong (\mathbb{R}_{>0})^{n-1} \).

6 Realizability criterion

All rank 2 oriented matroids are realizable. In contrast, not all phased matroids are realizable. In this section we give an example of a non-realizable phased matroid, and a simple realizability criterion that can be used to determine in a rank-2, or any other non-essentially oriented, uniform phased matroid is realizable.

6.1 A non-realizable rank 2 phased matroid

Example 6.1. The phased matroid \( \mathcal{M} \) with phirotope \( \varphi \) such that \( \varphi(1, 2) = \varphi(1, 3) = \varphi(1, 4) = \varphi(1, 5) = -\varphi(2, 3) = 1 \), \( \varphi(2, 4) = -e^{i\frac{\pi}{2}}, \varphi(2, 5) = -e^{i\frac{\pi}{5}}, \varphi(3, 4) = e^{i\frac{2\pi}{5}}, \varphi(3, 5) = e^{i\frac{3\pi}{5}}, \) and \( \varphi(4, 5) = e^{i\frac{4\pi}{5}} \) is not realizable.

Proof. It is not hard to check the combinatorial complex Grassman-Plücker relations to confirm \( \varphi \) is a phirotope of a phased matroid. Suppose \( \mathcal{M} \) is realizable. Since \( \varphi \) is in canonical form, the proof of Lemma 3.4 provides a construction of a potential realization of \( \varphi \).

The values of \( \varphi \) on all pairs except \( \{4, 5\} \) determine the following canonical realization of \( \mathcal{M} \):

\[
\begin{pmatrix}
1 & 0 & 1 & e^{i\frac{\pi}{2}} & e^{i\frac{4\pi}{5}} \\
0 & 1 & 1 & 1 & 1
\end{pmatrix},
\]

where \( \varphi(2, 4) \) and \( \varphi(2, 5) \) determine the phases of \( M_{1,4} \) and \( M_{1,5} \) respectively, and the norm of each entry is determined by \( \varphi(3, 4) \) and \( \varphi(3, 5) \). But \( \varphi(4, 5) = e^{i\frac{4\pi}{5}} \neq \text{ph}(e^{i\frac{\pi}{2}} - e^{i\frac{\pi}{5}}) = e^{i\frac{14\pi}{15}} \). So \( \mathcal{M} \) is not realizable.

In [3], a different method is provided to test the realizability of the above phased matroid in which a phirotope is confirmed to be realizable if an equation of 24 terms sums to 0. Their method can be generalized to uniform rank 2 phased matroids on \( n \) elements by checking the sum for all \( \binom{n}{2} \) subsets of elements of the groundset.

Example 6.1 sheds light on a two point realizability criteria for rank 2 phased matroids.

Proposition 6.2. Let \( \mathcal{M} \) be a uniform, rank 2, not essentially oriented phased matroid on \( [n] \) with phirotope \( \varphi \) in canonical form. Let \( \theta_j = \arg(\varphi(3, j)) \) and \( \psi_j = \arg(\varphi(3, j)) - \arg(\varphi(2, j)) \). Then \( \mathcal{M} \) is realizable if and only if for any pair \( j, k \in \{4, \ldots, n\} \),

\[
\varphi(j, k) = \text{ph} \left( \frac{\sin(\theta_k)}{\sin(\psi_k)} \varphi(2, k) - \frac{\sin(\theta_j)}{\sin(\psi_j)} \varphi(2, j) \right).
\]

Proof. If \( \mathcal{M} \) is realizable, a unique canonical realization \( M \) of \( \mathcal{M} \) can be constructed using \( \varphi(2,j) \) and \( \varphi(3,k) \) for all \( j, k > 3 \) where \( \frac{\sin(\theta_j)}{\sin(\psi_j)} \) is the norm of \( M_{1,j} \). Equation 4 is \( \varphi(j, k) = \text{ph}([j, k]_M) \) and must hold by definition of realizability.
The (5-point) realizability criteria for rank 2 uniform phased matroids is given in [3] depends on cross ratios, so it is not generalizable to phased matroids with rank greater than 2. However, using the Triangle Lemma as its foundation, the realizability criteria from Proposition 6.2 is easily extended to higher ranked phased matroids.

**Theorem 6.3.** Let $\mathcal{M}$ be a rank $r$, uniform, not essentially oriented phased matroid on $[n]$ with canonical phirotope $\varphi$. Let $(I|N)$ be a matrix such that $\text{ph}((I|N)_{i,j}) = \varphi(i,j)$ and $|(I|N)_{i,j}|$ is determined by $\varphi((i,r), r+1,j)$ as in Theorem 3.3. $\mathcal{M}$ is realizable if and only if for all $\lambda \in [n]^r$, $\varphi(\lambda) = \text{ph}(\lambda|_{(I|N)})$.

**Proof.** Given a uniform phased matroid with a phirotope $\varphi$ in canonical form, it is always possible to construct a potential canonical realization $(I|N)$ of $\mathcal{M}$ following the construction in the proof of Theorem 3.3. If $\varphi(\lambda) = \text{ph}(\lambda|_{(I|N)})$ for all $\lambda \in [n]^r$, then $(I|N) \in R(\mathcal{M})$. Otherwise, $\mathcal{M}$ is not realizable.

7 Conclusion

We have shown that in comparison to oriented matroids, uniform phased matroids can have remarkable simple realization space, and we can answer the realizability question for uniform not essentially oriented phased matroids in polynomial time. In light of a new umbrella theory that encompasses oriented and phased matroids called $F$-matroids, in our future work, we hope to classify which other hyperfields contain the important property described by the triangle lemma.

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