Approximate Eigenvalue Decompositions of Linear Transformations with a Few Householder Reflectors

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Abstract—The ability to decompose a signal in an orthonormal basis (a set of orthogonal components, each normalized to have unit length) using a fast numerical procedure rests at the heart of many signal processing methods and applications. The classic examples are the Fourier and wavelet transforms that enjoy numerically efficient implementations (FFT and FWT, respectively). Unfortunately, orthonormal transformations are in general unstructured, and therefore they do not enjoy low computational complexity properties. In this paper, based on Householder reflectors, we introduce a class of orthonormal matrices that are numerically efficient to manipulate: we control the complexity of matrix-vector multiplications with these matrices using a given parameter. We provide numerical algorithms that approximate any orthonormal or symmetric transform with a new orthonormal or symmetric structure made up of products of a given number of Householder reflectors. We show analyses and numerical evidence to highlight the accuracy of the proposed approximations and provide an application to the case of learning fast Mahalanobis distance metric transformations.

I. INTRODUCTION

The ability to efficiently perform orthonormal linear transformations of data, i.e., complexity \(O(n \log n)\) or lower given data of size \(n\), is of extreme importance in many practical applications, especially when dealing with high dimensional data or with software running on limited power devices.

When we discuss computationally efficient linear transformation, the poster algorithm is the Fast Fourier Transform (FFT) [1]. From a computational perspective [2], the FFT is an appropriate way of performing the matrix-vector multiplication between the Fourier matrix \(F \in \mathbb{C}^{n \times n}\) and a given vector. Usually, the matrix-vector multiplication between a general (even orthonormal) matrix an a given vector has quadratic complexity \(O(n^2)\) while the FFT uses properties of the highly structured Fourier matrix to reduce the complexity to \(O(n \log n)\). The FFT is also related to several other linear transformations that enjoy fast implementations like the discrete cosine transform (DCT) [3], [4], the discrete Hartley transform [5], and the Walsh-Hadamard transform [6].

Another large class of numerically efficient linear transformations is the Fast Wavelet Transforms (FWT) [7]. If we consider only orthogonal wavelets [8], [9], the first discovered and arguably the most simple and well known is the orthonormal Haar wavelet [10]. These algorithms have computational complexity \(O(n)\), with an extra speed-up when implemented via a lifting scheme [11].

Householder reflectors [12], [13] Chapter 5.1[14] are natural building blocks of orthonormal matrices: an orthonormal transformation of size \(n\) has a factorization into \(n-1\) reflectors (and a diagonal matrix with entries only in \(\{\pm 1\}\)). The idea of building orthonormal transformations that are the product of a given number of Householder reflectors (strictly less than \(n - 1\)) has been studied already in [15] in the context of learning sparsifying transforms. The other basic building blocks of orthonormal matrices have also been considered in similar problems [16], [17].

In this paper, we consider that we are given an orthonormal transformation of size \(n\) directly, and our goal is to approximately decompose it into \(h\), much smaller than \(n\), reflectors. In this case, the proposed solutions are numerically efficient; they do not involve any iterative processes, just an eigenvalue decomposition of the original orthonormal matrix. Next, we use the fact that every symmetric matrix can be diagonalized by orthonormal congruency to propose factorizations of symmetric matrices based on products of a few Householder reflectors, i.e., we factor (approximately) the orthonormal eigenspace of a symmetric matrix by a few Householder reflectors. Unfortunately, the optimization problems that arise in this case (the choice of the \(h\) Householder reflectors) are hard and have no closed form solution. Therefore, we propose iterative optimization algorithms that improve (in the sense that they lower the defined approximation error) the proposed factorization with each step.

It is worth noting that our focus is not on the numerical efficiency of the factorization algorithms for orthonormal or symmetric matrices - current decomposition algorithms exhibit very good computational properties already. Our goal is to construct new factorizations (still, in polynomial time) that approximate well the given matrices and allow for their fast manipulation: for example, after computing the factorization, we want \(O(n \log n)\) numerical complexity for matrix-vector multiplication, solving inverse problems, etc.

The paper is organized as follows: Sections II and III deal with the proposed factorizations of orthonormal and symmetric matrices, respectively; Section IV shows synthetic numerical results on random orthonormal and symmetric matrices and finally an application to the construction of fast transforms for distance metric learning.

II. APPROXIMATIONS OF ORTHONORMAL MATRICES

In this section, we describe two ways of approximating an orthonormal matrix by a product of a fixed, given, number of Householder reflectors together with theoretical insights on the accuracy of such approximations.
A. The proposed factorization

In this paper we propose methods to factorize a given orthonormal matrix $U \in \mathbb{R}^{n \times n}$ into a product of $h$ Householder reflectors such that this factorization is as close as possible to the originally given matrix. The goal is to build an approximate factorization of $U$ that is highly structured and therefore computationally efficient to use, for example in matrix-vector multiplications. We want to approximate $U$ by

$$
\tilde{U} = D \prod_{k=1}^{h} U_k = DU_h U_{h-1} \ldots U_1, \tag{1}
$$

where $D = \text{diag}(d)$ is a diagonal orthonormal matrix (i.e., $d \in \{\pm 1\}^n$) the $h$ factors are Householder reflectors

$$
U_k = I - 2u_k u_k^T, \quad \|u_k\|_2 = 1. \tag{2}
$$

Because Householder reflectors are orthonormal, we have that $\tilde{U}$ is also orthonormal. Since every $n \times n$ orthonormal matrix can be written as a product of $n - 1$ Householder reflectors (and an orthonormal diagonal), in this paper we are interested in factorizations like (1) where $h$ Householder reflectors (and an orthonormal diagonal), in this paper we are interested in factorizations like (1) where $h \ll n$, e.g., $h$ is $O(\log n)$. Because of this restriction on $h$, in general, we are not able to approximate any orthonormal $U$ (which has $O(n^2)$ degrees of freedom) by the structure $\tilde{U}$ exactly, but a non-zero error will almost always exist. Still, the goal is to reduce this error as much as possible. Because $\det(U_k) = -1$ then $\det(\tilde{U}) = (-1)^h$ and therefore we use the diagonal $D$ to ensure that the choice of $h$ does not fix the determinant value.

Matrix-vector multiplication with the matrix $U$ from (1) takes $4nh$ operations. Therefore, both the accuracy of approximating $U$ and the computational complexity of matrix-vector multiplication with $\tilde{U}$ depend on the choice of $h$. We consider the upper bound $h < \frac{n}{2}$ to keep the computational complexity of using $\tilde{U}$ strictly below the $2n^2$ operations needed for the classic, unstructured, matrix-vector multiplication.

B. Constrained Householder reflectors

Given an orthonormal matrix $U$ and the number of reflectors, in this section we analyze the following problem:

minimize $u_k, k=1,\ldots,h \|U - \tilde{U}_1\|_F^2$

subject to $\tilde{U}_1 = \prod_{k=1}^{h} U_k = I - 2 \sum_{k=1}^{h} u_k u_k^T \tag{3}$

$$
\|u_k\|_2 = 1, \quad u_k^T u_j = 0, \quad k \neq j.
$$

Because the reflector vectors are orthonormal among each other, i.e., $u_k^T u_j = 0$, the objective function in (3) is

$$
\|U - \tilde{U}_1\|_F^2 = 2n - 2\text{tr}(U) + 2 \sum_{k=1}^{h} u_k^T (U + U^T) u_k. \tag{4}
$$

Therefore, the approximation error depends on the spectral properties of $Z = U + U^T$ which is symmetric and as such has an eigenvalue decomposition $Z = V \text{diag}(z) V^T$, $z \in \mathbb{R}^n$, where $VV^T = V^TV = I$. Also notice that $\tilde{U}_1$ is always both orthonormal and symmetric. Let us assume w.l.o.g. that the real-valued eigenvalues $z$ are sorted in ascending order and that there are $n_+$ negative and $n_+$ positive eigenvalues (and we have that $n_- + n_+ = n$).

Result 1. Given $U$, in order to minimize (3) with $h \ll n$, the best $\tilde{U}_1$ is composed of $h = n_-$ Householder reflector vectors $u_k$ which are the eigenvectors corresponding to the negative eigenvalues of $Z$ and the approximation error in (4) is

$$
\|U - \tilde{U}_1\|_F^2 = 2n - 2\text{tr}(U) + 2 \sum_{k=1}^{n_-} z_k = 2n - \sum_{k=1}^{n_-} |z_k|, \tag{5}
$$

where we have used that $2\text{tr}(U) = \text{tr}(U + U^T) = \sum_{k=1}^{n} z_k$. The approximation of $U$ by $\tilde{U}_1$ is exact when all eigenvalues $z_k \in \{\pm 2\}$ and we use $h = n_-$. Householder reflectors, i.e., the number of reflectors is equal to the number of eigenvalues $z_k$ equal to negative two. If $h > n_-$ then we set $u_k = 0_{n \times 1}$, i.e., $U_k = I$, for $k = n_- + 1, \ldots, h$ because there is no reflector beyond the first $n_-$ that decreases our objective function.

Remark (the complex valued case). Given a unitary matrix $U \in \mathbb{C}^{n \times n}$ we have the decomposition $U = T \text{diag}(\lambda) T^H$, $\lambda \in \mathbb{C}^n$, and its approximation with (3) is done with reflectors constructed from the columns of $T$, denoted $t_k$. The best performance in this case is achieved as in (5) and all the reflector vectors are orthonormal to each other, i.e., $t_k^H t_j = 0, k \neq j$.

C. Unconstrained Householder reflectors

The previously imposed orthogonality condition between the reflector vectors $u_k$ can be dropped in order to achieve better approximation accuracy. Now we propose to solve the following optimization problem:

minimize $u_k, \quad k=1,\ldots,h \|U - \tilde{U}_2\|_F^2$

subject to $\tilde{U}_2 = \prod_{k=1}^{h} U_k = \prod_{k=1}^{h} (I - 2u_k u_k^T) \tag{6}$

$$
\|u_k\|_2 = 1.
$$

The goal is to update each reflector $U_k$ separately in order to reduce the value of the objective function. First, some notation. Consider $Z = U + U^T$ and eigenvalue decompositions $Z = V \text{diag}(z) V^T$ and $U = T \text{diag}(\alpha) T^H$ with $\alpha \in \mathbb{C}^n$, $T^H T = T T^H = I$ because $U$ is a normal matrix in (3) and notice that

- the spectrum of $Z$ is bounded, i.e., $-2 \leq z_k \leq 2$,
- the eigenvalues of $U$, except for $\{\pm 1\}$, come in complex conjugate pairs $\lambda_k, \lambda_{k+1} = \alpha_k \pm i \beta_k$ and therefore $Z$ has two corresponding real eigenvalues $z_{k,k+1} = 2 \alpha_k$.

To analyze the performance of $\tilde{U}_2$ let us consider a procedure that sequentially initializes $U_k$ with $k = 1, \ldots, h$. To construct an approximation as close as possible to $U$, for the first reflector $U_1$ we have to minimize:

$$
\|U - \tilde{U}_1\|_F^2 = \|U - I + 2u_1 u_1^T\|_F^2 = 2(n - \text{tr}(U)) + 2u_1^T Z u_1, \tag{7}
$$

while we add the second reflector $U_2$ the goal is to minimize:

$$
\|U - \tilde{U}_2\|_F^2 = \|U_1 - U_2\|_F^2 = \|U_1 - 1 + 2u_2 u_2^T\|_F^2 = 2n - 2\text{tr}(U_1) + 2u_2^T (U_1 + U_1^T) u_2. \tag{8}
$$

1 We use the eig function provided in Matlab® to construct this factorization. This function does not work as described in this paper: it constructs an orthogonal eigenspace only for distinct eigenvalues. If $U$ has repeated eigenvalues we explicitly orthogonalize their eigenspace using the QR algorithm.
Assuming $Z$ has at least one negative eigenvalue, we distinguish now two possibilities:

Case 1: If the lowest negative eigenvalue of $Z$ is $-2$, we initialize $u_1$ to be the eigenvector of this eigenvalue. We have that $ Zu_1 = -2u_1 $ and therefore $ Uu_1 = U^Tu_1 = -u_1 $. For $ u_2 $ we check the spectrum of

$$ UU_1 + U_1U^T = U(I - 2u_1u_1^T) + (I - 2u_1u_2^T)U^T $$

$$ = U + U^T - 2Uu_1u_1^T - 2u_1u_2^TU^T $$

$$ = Z + 2u_1u_1^T + 2u_1u_2^T = Z + 4u_1u_1^T. $$

This is a rank one update to $ Z $ that flips the $ -2 $ eigenvalue corresponding to the eigenvector $ u_1 $ to $ +2 $. The other eigenvalues/eigenvectors remain the same.

Case 2: If the lowest negative eigenvalue of $ Z $ is not $ -2 $, then we have the duplicate $ z_{k,k+1} = 2\alpha_k $ (for $ \alpha_k < 0 $) where $ \lambda_{k,k+1} = \alpha_k \pm i\beta_k $ with $ |\lambda_{k,k+1}| = 1 $ is a complex conjugate pair of eigenvalues of $ U $ with corresponding complex eigenvectors $ t_k $ and $ t_{k+1} = t_k^* $ that obey $ t_k^Ht_{k+1} = 0 $. We choose the reflector vector

$$ u_1 = \frac{1}{\sqrt{2}}(t_k + t_{k+1}) = \sqrt{2}\Re(t_k), $$

for which we achieve the minimum in $ u_1^TZu_1 = 2\alpha_k $ (with $ \|u_1\|_2 = 1 $). We now make the argument that we can always construct $ u_2 $ such that $ u_2^T(UU_1 + U_1U^T)u_2 = -2 $ which is equivalent to showing that $ u_2^TUU_1u_2 = -1 $. Consider the reflector vector

$$ u_2 = (\gamma + i\delta)t_k + (\gamma - i\delta)t_{k+1}, \gamma^2 + \delta^2 = \frac{1}{2}, \gamma, \delta \in \mathbb{R}. $$

By direct calculation we have that $ u_2^TUu_2 = \alpha_k, u_2^TUu_1 = \frac{1}{\sqrt{2}}((\gamma - i\delta)\lambda_k + (\gamma + i\delta)\lambda_{k+1}). $ Finally, expanding for $\begin{bmatrix} 1 & 1 \end{bmatrix}$ we have

$$ u_2^TUU_1u_2 = u_2^TUu_2 - 2u_2^TUu_1 + 2u_2^TUu_1 = \alpha_k - 2((\gamma - i\delta)\lambda_k + (\gamma + i\delta)\lambda_{k+1})\gamma. $$

We set the expression above to $ -1 $ by choosing $ \gamma = -\frac{\sqrt{1+\alpha_k}}{2}, \delta = -\frac{\sqrt{1-\alpha_k}}{2} $, and we finally have that

$$ u_2 = 2(\gamma \Re(t_k) - \delta \Im(t_k)). $$

Therefore, the real-valued $ u_1 $ does not just minimize the approximation error in the first step but also sets up the problem such that the reduction in the objective function will then be maximal, $ -2 $, in the second stage.

Because we found a real valued eigenvector $ u_2 $ of $ UU_1 $ with eigenvalue $ -2 $ we now find ourselves in Case 1 and therefore $ UU_1U_2 + U_2U_1^T $ has a new eigenvalue $ 2 $, the $ -2 $ eigenvalue of $ UU_1 + U_1U^T $ gets flipped.

For $ \zeta = -\frac{\sqrt{1+\alpha_k}}{2} $ and $ \eta = \frac{\sqrt{1-\alpha_k}}{2} $ we can also show that there is an eigenvector $ v = -2((\zeta \Re(t_k) - \eta \Im(t_k)) $ of $ UU_1 $ with eigenvalue $ 1 $. Notice, by direct calculation, that $ v^Tv_2 = 0 $ and therefore $ v $ is also an eigenvector of $ UU_1U_2 $ with eigenvalue $ 1 $. Finally, this means that the spectra of $ UU_1U_2 + U_2U_1^T $ and $ Z $ are identical except that the previous eigenvalues $ z_{k,k+1} = 2\alpha_k $ are now both $ 2 $ for some new eigenvectors denoted $ v $ and $ u_2 $ that are in the span of $ t_k $ and $ t_{k+1} $.

Therefore, given the orthonormal $ U $, to construct the approximation $ \hat{U}_2 $, we perform the full eigenvalue decomposition of $ U $, and we follow Case 1 and Case 2 up to the $ h $ reflectors or until we exhaust eigenvalues with negative real components. 

**Result 2.** Given $ U $, in order to minimize $ ||U - \hat{U}_2||_F^2 = 2n - 2tr(\hat{U}_2^TU) = 2n - \sum_{k=0}^{n} z_k $, where we have used $ 2tr(\hat{U}_2^TU) = 2n + \sum_{k=0}^{n} z_k $. The approximation of $ U $ by $ \hat{U}_2 $ is exact when all positive eigenvalues $ z_k $ are equal to two, and we use $ h = n $ Householder reflectors, i.e., the number of reflectors is the number of negative eigenvalues $ z_k $.

In terms of storing the factorizations (either $ U_1 $ or $ \hat{U}_2 $), we have $ n - 1 $ degrees of freedom for each Householder reflector vector and $ n $ bits for the diagonal $ D $.

### D. Bound on the expected approximation accuracy

As previously shown, for a particular orthonormal matrix $ U $, the accuracy of the approximation we construct depends on the spectrum of $ Z = U + U^T $. For the unconstrained approximation $ \hat{U}_2 $ presented in Section II-C notice that the approximation error is different if we consider $ U $ or $ -U $. This difference is caused by the fact that $ -U_2 $ is not a Householder reflector if $ U_2 $ is. Therefore, depending on the spectrum of $ Z $ we might consider and fix from the beginning $ D = -I $ in $\hat{U}_2$, i.e., we change of sign of $ U $ to maximize the summation term in (14). This discussion is mute for the constrained approximation $ U_1 $ presented in Section II-B since $\begin{bmatrix} 1 & 1 \end{bmatrix}$ depends on the absolute values of the eigenvalues of $ Z $. This is the case because we can write $ U_1 = I - 2U^T $ where $ \hat{U} \in \mathbb{R}^{n \times h} $ has orthonormal columns (the $ h $ Householder reflector vectors) and we have that $ -U_1 = I - 2\hat{U}^T $ is also a product of Householder reflectors (this time, $ n - h $) whose vectors are orthonormal, i.e., $ U \in \mathbb{R}^{n \times (n-h)} $ is such that $ [U \quad \hat{U}] $ is a full $ n \times n $ orthonormal basis.

Furthermore, given that any orthonormal matrix is diagonalized by $ n - 1 $ reflectors, Results 1 and 2 might seem counterintuitive: only $ n - 1 $ reflectors are useful (they decrease the objective function value) in the factorization. This is because when we diagonalize with $ n - 1 $ reflectors we do not have an objective function to compute because we know we have enough reflectors to perfectly diagonalize and reach zero approximation error. But if we calculate the objective function value when we diagonalize we notice that this does not monotonically decrease to zero. This explains why our proposed factorization cannot be further improved after constructing $ h = n - 1 $ reflectors.

In this section, we present a worse case result on the average performance of the approximation accuracy achieved with $ \hat{U}_2 $.

To generate a random orthonormal matrix we build a matrix with i.i.d. entries from the standard Gaussian distribution and then orthogonalize its column by the QR procedure.
Result 3. Given a random orthonormal \( U \in \mathbb{R}^{n \times n} \) we can always approximate it by \( \tilde{U}_2 D \), where \( U_2 \) is a product of \( h \) Householder reflectors as in (6) and \( D \) is a diagonal matrix with elements \( d_{ii} \in \{ \pm 1 \} \) such that

\[
\mathbb{E} \| U - \tilde{U}_2 D \|_F^2 \leq 2(n - h) - \frac{2\sqrt{2}}{\sqrt{\pi}} \sqrt{n - h}, \tag{15}
\]

Proof. We consider the Householder reflectors \( J_k \), the ones that start the diagonalization process for \( U \), but only for the first \( h \) steps of the process [13] Section 5.2.1, i.e., a partial or incomplete QR factorization applied to \( U \), as

\[
DJ_h \ldots J_1 U = \begin{bmatrix} I_{h \times h} & 0_{h \times (n-h)} \\ 0_{(n-h) \times h} & D_1 \tilde{U} \end{bmatrix}, \tag{16}
\]

with \( D = \begin{bmatrix} I_{h \times h} & 0_{h \times (n-h)} \\ 0_{(n-h) \times h} & D_1 \end{bmatrix} \) where \( \tilde{U} \in \mathbb{R}^{(n-h) \times (n-h)} \) is orthonormal and \( D_1 \tilde{U} \) has positive diagonal elements, i.e., \( d_{ii}\tilde{u}_{ii} > 0 \) for \( i = 1, \ldots, n - h \). Intuitively, each reflector \( J_k \) introduces zeros under the main diagonal on the \( k \)th column. When the reflectors introduce zeros in an orthonormal matrix (like \( U \), in our case) then we reach the block structure in (16).

The goal is to bring the block structure in (19) as close as possible to the identity matrix, i.e., \( DJ_h \ldots J_1 U \approx I \). Therefore we want to minimize the quantity

\[
\| DJ_h \ldots J_1 U - I \|_F^2 = \| U - J_1 \ldots J_h D \|_F^2, \tag{17}
\]

where we used the fact that orthonormal transformations are invariant in the Frobenius norm and that \( D \) and all the reflectors are symmetric, i.e., \( J_k^T = J_k \). Based on this result, the proposed approximation to \( U \) is \( \tilde{U}_2 D \) where

\[
\tilde{U}_2 = J_1 \ldots J_h. \tag{18}
\]

Notice that the only non-zero error term comes from the block \( D_1 \tilde{U} \) in (16). To quantify this error, using the expression of \( \tilde{U}_2 \) from (18), the Frobenius norm in (15) develops to

\[
\| U - \tilde{U}_2 D \|_F^2 = \| U - J_1 \ldots J_h D \|_F^2 = \| DJ_h \ldots J_1 U - I \|_F^2.
\]

From this observation, the second term on the right hand side of the result in (15) follows immediately.

Therefore, the result in (15) is achieved with equality when the reflectors are the ones used in the QR decomposition to introduce zeros in the first \( h \) columns of \( U \). Given that we construct \( \tilde{U}_2 \) such that it explicitly minimizes the Frobenius norm in (6) it follows that the approximation in necessarily better and therefore the upper bound in (15) holds.

We would like to note here that the reflectors \( J_k = I - 2u_ku_k^T \) used in the QR decomposition to introduce zeros below the \( k \)th diagonal element have the structure \( J_k = [0_{(k-1) \times 1}; \ j_k] \), meaning that at most \( n - k + 1 \) entries of \( J_k \) are non-zero.

Matrix-vector multiplications between the reflector \( J_k \) and a vector take \( 4(n-k+1) \) operations. This observation allows for the additional possibility of balancing the computational cost by considering sparse reflector vectors in (1). Unfortunately, in this case, no closed form solution seem to be possible and an iterative optimization problem based on the sparse-PCA approach (19) should be considered for each reflector \( U_k \).

III. APPROXIMATION OF SYMMETRIC MATRICES

Similarly to the previous section, we now describe an algorithm for the approximation of a symmetric matrix by a product of a fixed, given, number of Householder reflectors and a diagonal matrix.

A. The proposed factorization

Let us now consider fast approximations of symmetric matrices. Given a symmetric matrix \( S \in \mathbb{R}^{n \times n} \) the main result that we will use is its eigenvalue factorization as

\[
S = U \text{diag}(s)U^T, \quad UU^T = U^TU = I, \quad s \in \mathbb{R}^n, \tag{20}
\]

where \( U \) stores the orthonormal eigenvectors of \( S \) and where we assume w.l.o.g. that the real-valued entries of \( s \) (the eigenvalues of \( S \)) are stored in decreasing order of their magnitudes. Using the factorization in (1), we now propose an approximation of \( S \) as

\[
\tilde{S} = \tilde{U} \text{diag}(\tilde{s})\tilde{U}^T = D \left( \prod_{k=1}^{h} U_k \right) \text{diag}(s) \left( \prod_{k=1}^{h} U_k \right) D, \tag{21}
\]

where all matrices \( U_k \) are Householder reflectors and \( \tilde{s} \in \mathbb{R}^n \) now stored the eigenvalues of \( \tilde{S} \).

Matrix-vector multiplication with the matrix \( \tilde{S} \) from (21) takes \( (8h+1)n \) operations. We have to consider the bound \( h < \frac{n}{2} \) to keep the computational complexity of using \( \tilde{S} \) strictly below \( 2n^2 \), the regular computational complexity.

B. The proposed factorization algorithm

Given any symmetric matrix \( S \) we want to construct the factorization \( \tilde{S} \) as (21) such that it closely approximates \( S \). There are three components that we can choose in this factorization: i) the spectrum \( \tilde{s} \) of the approximation; ii) the number of Householder reflectors \( h \) and the values of the reflector vectors \( u_k, \ k = 1, \ldots, h \) and iii) the orthonormal diagonal matrix \( D \). In this section, we explain how to iteratively and separately choose these components while continuously improving the...
approximation accuracy. Based on these findings, we propose an algorithm to construct $\bar{S}$ such that $\|\bar{S} - S\|_F^2$ is reduced.

Our first goal is to choose each Householder reflector $U_k$ (with all other $h - 1$ reflectors fixed) sequentially to minimize $\|S - \bar{S}\|_F^2 = \|S - U\bar{S}U^T\|_F^2 = \|A_k - U_k B_k U_k^T\|_F^2$, (22) where we have defined the symmetric matrices

$$A_k = \left( \prod_{j=k+1}^{h} U_j \right) \left( \prod_{j=1}^{k-1} U_j \right)^T,$$

$$B_k = \left( \prod_{j=k-1}^{h} U_j \right) \text{diag}(s) \left( \prod_{j=k+1}^{h} U_j \right)^T.$$

Replacing and developing (22) for $U_k = I - 2u_k u_k^T$ we have

$$\|S - \bar{S}\|_F^2 = \|A_k - (I - 2u_k u_k^T) B_k (I - 2u_k u_k^T)|||_F^2$$

$$= \|A_k\|_F^2 + \|B_k\|_F^2 - 2tr(A_k B_k) + 4C(u_k),$$

$$= ||s||_2^2 + ||\bar{s}\||_2^2 - 2tr(A_k B_k) + 4C(u_k),$$

where we have denoted

$$C(u_k) = u_k^T (A_k B_k + B_k A_k) u_k - 2tr(A_k u_k u_k^T B_k u_k u_k^T).$$

Finding the $u_k$ with $\|u_k\|_2 = 1$ that minimizes (25) seems hard in general (making no assumptions on the spectra of $A_k$ and $B_k$) as it seems that there is no closed form solution. But notice that we can separately optimize the two parts of the expression in (26):

- Notice that the minimizer of $u_k^T (A_k B_k + B_k A_k) u_k$ is the eigenvector of the smallest, negative, eigenvalue of the symmetric matrix $A_k B_k + B_k A_k$. We denote it $\hat{u}_k$.
- To maximize the second trace term in $u_k$ we develop:
  $$\text{tr}(A_k u_k u_k^T B_k u_k u_k^T) = \text{tr}(u_k u_k^T A_k u_k u_k^T B_k)$$
  $$= \text{vec}(u_k u_k^T)^T \text{vec}(A_k u_k u_k^T B_k)$$
  $$= \text{vec}(u_k u_k^T)^T (B_k \otimes A_k) \text{vec}(u_k u_k^T)$$
  $$= u_k^T (B_k \otimes A_k) (u_k \otimes u_k).$$

We have used the cyclic property $\text{tr}(XYZ) = \text{tr}(ZXY)$ and vectorize property $\text{tr}(X^T Y) = \text{vec}(X)^T \text{vec}(Y)$ of the trace, the fact that $\text{vec}(XYZ) = (Z^T \otimes X) \text{vec}(Y)$ and $\text{vec}(X^T Y) = X \otimes Y$. We denote $v \in \mathbb{R}^{n^2}$ the eigenvector corresponding to the highest, positive, eigenvalue of the symmetric matrix $B_k \otimes A_k$. We note that the Kronecker product is never explicitly calculated in order to obtain $v$ but we compute: $\lambda_{\text{max}} A_k$, $\lambda_{\text{min}} A_k$, $\lambda_{\text{max}} B_k$, $\lambda_{\text{min}} B_k$, the pairs of lowest and highest eigenvalues of $A_k$ and $B_k$ respectively and then the highest eigenvalue of $B_k \otimes A_k$ is $\lambda_{\text{max}} \lambda_{\text{max}} B_k \otimes A_k$, and the corresponding eigenvector is $v = \lambda_{\text{max}} B_k \otimes \lambda_{\text{max}} A_k$. The Kronecker product of the eigenvectors corresponding to the eigenvalues whose product is maximum. For example, for positive semidefinite matrices $A_k$ and $B_k$ the maximum eigenvalue of $B_k \otimes A_k$ is $\lambda_{\text{max}} B_k \lambda_{\text{max}} A_k$ and therefore its corresponding eigenvector is $v = \lambda_{\text{max}} B_k \otimes \lambda_{\text{max}} A_k$, i.e., the Kronecker product of the eigenvectors corresponding to the highest eigenvalues [20, Chapter 2.1]. With the $v$ just computed we now minimize

$$\|v - u_k \otimes u_k\|_F^2 = \|v - \text{vec}(u_k u_k^T)\|_F^2 = \|V - u_k u_k^T\|_F^2,$$

where $V$ is an $n \times n$ matrix build from the re-arranged elements of $v$. Notice that because $v$ is a Kronecker product, we have that $V$ is a rank one matrix. The minimizer of (28) is the best rank one approximation of $V + V^T$ [21, Section 7]. We denote this solution $u_k^\dagger$.

Remark (initialization of $u_k^\dagger$ based on generalized Rayleigh quotient calculations when $A_k$ and $B_k$ are positive definite). The second term of $C(x)$ can be written

$$R(A, B, x) = \frac{x^T Axx^T B x}{x^T x}.$$  

Without loss of generality, for convenience, we momentarily drop the index $k$ from the notation. Assuming the matrices are positive semidefinite and $B$ is invertible, we denote the Cholesky factorization $B = LL^T$, we make the change of variable $y = L^T x$ and we have

$$R(A, B, x) = \frac{x^T Axx^T L L^T x}{x^T x} = y^T L^{-1} A L^{-T} y y^T y L^{-1} L^{-T} y.$$  

This expression is almost a generalized Rayleigh quotient [13, Chapter 8.2.3] (we have an extra multiplicative term $y^T y$). Assuming $\|y\|_2 = 1$, to maximize the last quantity in $y$ we use the generalized eigenvalue decomposition which reduces to finding the eigenvector $y$ corresponding to the highest eigenvalue of $L^T A L^{-T}$ [13, Chapter 7.7]. We recover $x = L^{-T} y$ and normalize $u_k = \|x\|_2^{-1} x$.

Finally, given the two vectors $u_k^\dagger$ and $u_k^\dagger$, we initialize the reflector vector $u_k^{(1)}$ by a two step procedure. First, we update $u_k^{(1)}$ such that $(u_k^{(1)T} u_k^{(1)} = 0$ is guaranteed

$$u_k^{(1)} = u_k - u_k^{(1)T} u_k^{(1)} u_k^{(1)}, u_k^\dagger = u_k^{(1)} \|u_k^{(1)}\|_2^{-1},$$

and then we solve the minimization problem

$$u_k^{(1)} = \arg\min_{u_k} \|u_k - (1 - \gamma/2) u_k^\dagger \pm \sqrt{(\gamma^2 - 4)/4} u_k^\dagger\|^2 C(u_k),$$

where $\gamma \in [0, \sqrt{2}]$ is chosen such that $u_k^{(1)}$ minimizes $C(u_k^{(1)})$, i.e., we sweep over the unit hypersphere between points $u_k^\dagger$ and $u_k^{(1)}$ in order to minimize $C(u_k)$. Since we optimize separately the two terms in (26) we search over linear combinations of these for a good minimizer of the overall expression. The formulation of $u_k$ in (32) guarantees that $\|u_k^{(1)}\|_2 = 1$ for any $\gamma$ while the one dimensional minimization (in $\gamma$) is done efficiently with a numerical procedure.

With the initialized reflector vector $u_k^{(1)}$ we start now an iterative gradient descent procedure to further reduce $C(u_k)$. Notice that, by the cyclic property of the trace we have $\text{tr}(A_k u_k u_k^T B_k u_k^T) = \text{tr}(u_k^T A_k u_k u_k^T B_k u_k) = (u_k^T A_k u_k)(u_k^T B_k u_k)$ is the product of two quadratic forms (the convexity of such products for optimization purposes was studied in [22]). As such, we have the gradient expression

$$\nabla C(u_k^{(i)}) = 2 (A_k B_k + B_k A_k) u_k^{(i)} - 4 ((u_k^{(i)T}) A_k u_k^{(i)} B_k + (u_k^{(i)T}) B_k u_k^{(i)} A_k) u_k^{(i)}.$$  

2We use the `fminsearch` function, initialized at $\gamma = 0$, that is provided in Matlab® to minimize $C(u_k)$ as a function of $\gamma$.
Algorithm 1 – Symmetric Householder Factorization (SHF)

Input: The symmetric matrix $S \in \mathbb{R}^{n \times n}$, the number of Householder reflectors $h$ in the factorization and the maximum number of iterations $K$.

Output: The symmetric matrix $\tilde{S}$ factored as $[21]$ such that $\|S - \tilde{S}\|_F^2$ is reduced.

1. Construct the eigenvalue decomposition of $S$ as in (20).
2. Initialize all elements of the approximate factorization:
   - The spectrum is set to $\hat{s} = s$.
   - Set all reflector vectors $u_k = 0, k = 1, \ldots, h$.
   - Update all $h$ reflector vectors according to (32).
   - The diagonal elements of $D$ are set according to (35).
3. For $1, \ldots, K$:
   - Iteratively update all reflectors, for $k = 1, \ldots, h$:
     - Construct $A_k$ and $B_k$ according to (23) and (24).
     - Starting from the current $u_k$, iteratively update the reflector vector according to (34) until convergence.
   - Update $D$ according to (35).

We update the gradient in the same fashion as (31) to obtain $g_k = \nabla C(u_k) - (u_k)^T \nabla C(u_k) u_k, \tilde{s}_k = g_k \|g_k\|_2^{-1}$, and finally we have the update equation:

$$u_k^{(i+1)} = (1 - \gamma\|g_k\|^2)(u_k^{(i)} - \sqrt{\gamma^2 - \gamma^2/4})g_k^{(i)},$$

where $\gamma$ is found by a one dimensional search such that $C(u_k^{(i+1)})$ is minimized. Since the search for $u_k^{(i+1)}$ contains the previous solution $u_k^{(i)}$ we have that this iterative procedure has a strictly monotonically descent to a local optimum point.

Finally, to update $d$ we minimize $\|S - DB_dD\|_F^2$. If we denote by $\tilde{s}_i$, $d_i \in \mathbb{R}^{n-1}$ the $i$th and rows of $S$ and $B_d$, respectively, both with the diagonal element removed then

$$d_{ii} = 1 \text{ if } \tilde{s}_i - \tilde{d}_i \geq \|s_i + \tilde{d}_i\|_2 \text{ else } d_{ii} = -1.$$  

For completeness, the full proposed learning procedure, which we call Symmetric Householder Factorization (SHF), is presented in Algorithm 1. In our discussion so far we have assumed that the spectra of the given matrix $S$ and its approximation $\tilde{S}$ are identical. We can always also optimize over the choice of the spectrum $\tilde{s}$ by minimizing

$$\|S - \tilde{S}\|_F^2 = \|U^T S \tilde{U} - \text{diag}(\tilde{s})\|_F^2,$$

which is given by $\tilde{s} = \text{diag}(U^T S \tilde{U})$. We can trivially adapt Algorithm 1 to also perform this update iteratively after the calculation of all the reflectors. We call this approach SHF with Spectrum Update (SHF–SU).

Remark (bounding $C(u_k)$). Denoting $y = \begin{pmatrix} u_k \ 1 \end{pmatrix} \in \mathbb{R}^{n^2 + n}$, an alternative way of writing the total cost is

$$C(u_k) = y^T \begin{pmatrix} -2(B_k \otimes A_k) & 0_{n^2 \times n} \\ 0_{n^2 \times n} & A_k B_k + B_k A_k \end{pmatrix} y.$$  

Therefore, if the block diagonal matrix is positive semidefinite (meaning that both $- (B_k \otimes A_k)$ and $A_k B_k + B_k A_k$ are positive semidefinite) then there is no $u_k$ that reduces the objective function in (25) (this is a sufficient condition).

Assume that the given $S$ is positive semidefinite. Notice that in our case the spectra of $A_k$ and $B_k$ are identical to the spectrum of $S$ (since $\tilde{s} = s$) and then general results from linear algebra [23, Chapter 4] show that the highest eigenvalues of $A_k B_k + B_k A_k$ are bounded by $2(\lambda_{\text{min}}^2)^{1/2} \leq \lambda_{\text{max}} A_k B_k + B_k A_k \leq 2(\lambda_{\text{max}}^2)^{1/2}$ and $(\lambda_{\text{min}}^2)^{1/2} \leq \lambda_{\text{max}} A_k B_k + B_k A_k \leq (\lambda_{\text{max}}^2)^{1/2}$, respectively, which leads to the bound

$$C(u_k) \leq \frac{C(\tilde{u}_k)}{2(\lambda_{\text{max}}^2)^{1/2} - (\lambda_{\text{min}}^2)^{1/2}} \in [-1, 1].$$

When $A_k$ and $B_k$ share the same eigenvalues we trivially have that $C(u_k) = 0$ for any $u_k$ – this reflects the situation where the $h$ reflectors exactly describe the eigenspace of $S$.

Remark (analysis of a relaxed problem when $A_k$ and $B_k$ are positive definite). Consider a similarly constrained optimization problem

$$\text{minimize } C(u_k) \text{ subject to } \|u_k\|_2 \leq 1, \quad (39)$$

The constraint is important to ensure a bounded solution, otherwise $C(u_k) = -\infty$. We now defined the Lagrangian

$$\mathcal{L}(u_k, \nu) = C(u_k) + \nu (u_k^T u_k - 1),$$

with $\nu \in \mathbb{R}^+$, a Lagrange multiplier, and define the derivative

$$\nabla \mathcal{L}(u_k, \nu) = \nabla C(u_k) + 2u_k \nu.$$

If $u_k^*$ minimizes $\mathcal{L}(u_k, \nu^*)$ then $\nabla \mathcal{L}(u_k^*, \nu^*) = 0$ (the stationarity condition) and therefore $(u_k^*)^T \nabla \mathcal{L}(u_k^*, \nu^*) = 0$ which, by rearranging terms, leads to

$$C(u_k^*) = 2(u_k^*)^T A_k u_k^* - \nu^* \|u_k^*\|_2^2.$$  

To complete the KKT conditions for (39), we also have that $\nu^* \geq 0$ (dual feasibility) and $\nu^* (\|u_k^*\|_2^2 - 1) = 0$ (complementary slackness). The optimal solution of (39) is such that $C(u_k^*) \leq 0$, since the feasible $u_k^* \leq 0$ trivially leads to $C(u_k^*) = 0$, and therefore $\nu^* \geq 2(\|u_k^*\|_2^2 - 1)$ which in the special case where $A_k$ and $B_k$ are positive definite then we necessarily have that $\nu^* > 0$ and therefore $\|u_k^*\|_2^2 = 1$.

Remark (a matrix manifold optimization approach). The problem at hand can be seen as the minimization of $C(x) \in \mathbb{R}^{n^2 + n}$ over the unit sphere manifold $S^{n-1}$ for which iterative optimization procedures are available [24, Chapters 4.6 and 6.4] that could be adapted to our case.

C. Bound on the average performance of SHF

For a particular symmetric matrix $S$, the accuracy of the approximation we construct depends on its spectrum. In this section we present a worse case result on the average performance of the approximation accuracy achieved with $\tilde{S}$.

Result 4. We generate a random symmetric matrix we build a matrix $X$ with i.i.d. entries from the standard Gaussian distribution and then extract its symmetric components $S = \frac{1}{2}(X + X^T)$. Given such a random symmetric $S \in \mathbb{R}^{n \times n}$ we can always approximate it by $\tilde{S}$ created via Algorithm 1, using $h$ Householder reflectors as in (21) such that

$$\mathbb{E}[\|S - \tilde{S}\|_F^2 \leq \sum_{i=h+1}^n \sigma_i^2 - \frac{n - h}{2}.$$


Proof. We use the eigenvalue decomposition $S = U \text{diag}(s) U^T$ and the singular value decomposition $S = U \text{diag}(\sigma) V^T$ with the vector $\sigma \in \mathbb{R}^n_+$ that has the singular values in decreasing order. We have that $V^T = D_\Sigma P U^T$ where $D_\Sigma = \text{diag}(\text{sign}(\sigma))$ because the singular values of $S$ are the absolute values of the eigenvalues $(\sigma_i = \text{sign}(s_i)s_i)$ and $P$ is a permutation matrix because while the ordering of the eigenvalues in $s$ is unimportant, the singular values are sorted in decreasing order, i.e., $\sigma_1 \geq \cdots \geq \sigma_n$.

We consider the Householder reflectors $J_h$ that start the diagonalization process for $S$, but only for the first $h$ steps of the process [13] Chapter 8], i.e., a partial or incomplete eigenvalue decomposition applied to $S$, as

$$J_h \ldots J_1 SJ_1 \ldots J_h = \begin{bmatrix} \lambda & 0_{h \times (n-h)} \\ 0_{(n-h) \times h} & \bar{S} \end{bmatrix},$$

(44)

with $\Lambda = \text{diag}(\lambda) \in \mathbb{R}^{h \times h}$ is a diagonal matrix with elements $\lambda_i$ such that $|\lambda_i| = \sigma_i$, $i = 1, \ldots, h$, and $\bar{S} \in \mathbb{R}^{(n-h) \times (n-h)}$ is a random symmetric matrix whose singular values are $\sigma_{h+1}, \ldots, \sigma_n$ (i.e., the lowest $n-h$ singular values of $S$ and therefore $|\bar{S}|_F = \sum_{i=h+1}^n \sigma_i^2$). With this choice of the reflectors, our objective function becomes

$$\|S - J_1 \ldots J_h \text{diag}(\bar{s})J_h \ldots J_1 \|_F^2 = \|J_h \ldots J_1 SJ_1 \ldots J_h - \text{diag}(\bar{s})\|_F^2,$$

(45)

that we minimize by choosing the spectra of the approximation to be $\bar{s} = \left\lfloor \text{diag}(|\bar{S}|) \right\rfloor$. With this choice we have that

$$\mathbb{E}[\|S - \bar{S}\|_F^2] = \mathbb{E}\left[\left\|\begin{bmatrix} 0_{h \times h} & 0_{h \times (n-h)} \\ 0_{(n-h) \times h} & \bar{S} - \text{diag}(\text{diag}(\bar{S})) \end{bmatrix}\right\|_F^2\right]$$

$$= \mathbb{E}[\|\bar{S} - \text{diag}(\text{diag}(\bar{S}))\|_F^2] + \mathbb{E}[\text{diag}(\bar{S})]|^2_2 - 2\text{tr}(\bar{S}^T \text{diag}(\text{diag}(\bar{S})))$$

$$= \sum_{i=h+1}^n \sigma_i^2 - \frac{n-h}{2} \approx \frac{(n-h)^2 - (n-h)}{2}.$$ 

(46)

For the final equality, which is accurate for large $n$ and $h$, we used $\mathbb{E}[\bar{S}_{ij}^2] = \frac{1}{\sqrt{2}}$ and that we are computing the Frobenius norm of a matrix with $(n-h)(n-h-1)$ non-zero elements.

This is an upper bound to the approximation accuracy of Algorithm 1 since we could initialize the procedure with the $h$ reflectors that achieve and then iteratively improve them by monotonically decreasing the objective function. The result also shows that the proposed approximation is at least as good as the low-rank approximation of $S$.

Remark (the case of positive semidefinite $S$). Given a random matrix $X$ with entries i.i.d. from the standard Gaussian distribution we call the symmetric positive definite $S = XX^T$ a Wishart matrix which is diagonally dominant (we have $\mathbb{E}[S_{ii}^2] = n^2 + 2n$ and $\mathbb{E}[S_{ij}^2] = n$, $i \neq j$). When $n$ is large and $h \ll n$ we also expect that $\bar{S}$ in (46) is diagonally dominant and therefore it is well approximated by diag(diag($\bar{S}$)) leading to a lower approximation error as compared to the indefinite case when $S = \frac{1}{2}(X + X^T)$.

IV. RESULTS

In this section we show the approximation performance of the two proposed reflector structures, for orthonormal and symmetric matrices. To measure the accuracy of our approximations, given a target matrix $X$ and its approximation $\bar{X}$ we use the normalized relative representation error:

$$\epsilon(X, \bar{X}) = \frac{1}{4} \frac{\|X - \bar{X}\|_F^2}{\|X\|_F^2}.$$ 

(47)

This quantity is normalized in such a way that given any pair of matrices $(X, \bar{X})$ with the same singular values we have that $0 \leq \epsilon(X, \bar{X}) \leq 1$.

A. Synthetic experiments: orthonormal case

In this section, we randomly generate orthonormal $U$: we construct a random matrix with entries from the standard Gaussian distribution on which we apply the QR decomposition and keep the orthonormal component.

In Figure 1 we compare the two orthonormal approximations we consider in this paper, $U_1$ and $U_2$. As discussed, even from a theoretical perspective, the approximation structure $U_2$ outperforms (or matches, in the worst case) $U_1$ since the former has more degrees of freedom in its model. We also confirmed this fact by the simulation results in Figure 1 where the gap between the two is approximately 10% in the relative representation error. Furthermore, given enough reflectors $h$, at most $n - 1$ constructed by the QR decomposition, the approximation $U_2$ can reach zero representation error while $U_1$ can rarely achieve perfect reconstruction regardless of the number of reflectors we allow.

In Figures 2 and 3 we show the relative representation error obtained when approximating a random orthonormal $U$ by $U_2$ as in (6). In Figure 3 we also show the average bound developed in Result 1, which serves here as an upper bound for the proposed approach. As the number of reflectors $h$ increases, the bound is essentially tight and matches the approximation accuracy of the $h$-step QR factorization. Also, notice that the difference between our proposed method and the bound increases with the number of reflectors $h$.

The approximations $U_1$ and $U_2$ apply also to classic transforms. If $F \in \mathbb{C}^{n \times n}$ is the Fourier matrix then $F^H F$ has $\left\lfloor \frac{n+2}{4} \right\rfloor$ eigenvalues with value $-2$ and $\left\lfloor \frac{n+2}{4} \right\rfloor$ eigenvalues with value 2 and the others up to $n$ are zero. For the Hadamard matrix $H \in \mathbb{R}^{n \times n}$ we have that $H^T H$ has half the eigenvalues with value $-2$ and half with value 2. For both these transforms, the approximation accuracies of $U_1$ and $U_2$ are identical. For other well known transforms, like the discrete cosine matrix (which is related to the Fourier) and the Haar matrix, the eigenvalues $z_k$ do not all have extremal values $\{\pm 2\}$ and therefore the approximation analyses with the proposed $U_1$ and $U_2$ cannot be easily done analytically. Regardless, we note that the proposed factorizations are not appropriate to build numerically efficient and accurate approximations of these structures (according to Results 1 and 2 that highlight the need of a small number of extreme eigenvalues in the matrices to be approximated).
B. Synthetic experiments: symmetric case

In this section, we randomly generate symmetric $S$: we construct a random matrix $X$ with entries from the standard Gaussian distribution and compute either $X + X^T$ or $XX^T$.

Figures 4 and 5 show experimental simulations for approximating random symmetric matrices indefinite and positive definite, respectively. The first observation is that, as expected, the approximation accuracy with the proposed structure is better than the eigendecomposition approach. The gap is not large due to the eigenvalue distribution of the randomly generated $S$, i.e., most of the energy is concentrated in a few (highest) eigenvalues that are also captured well in the $h$ eigenvalue decomposition.

C. Application: learning fast distance metric transformations

In the context of machine learning algorithms, the classification accuracy of many methods significantly depend on the choice of a good metric, i.e., a good distance between any two data points of the dataset. For example, the performance of the $k$-nearest neighbors algorithm ($k$-NN) [25] highly depends on using a metric that accurately reflects the relationship between data points (both, data points from the same class as well as data points from different classes). The simple, standard Euclidean distance regularly used by $k$-NN does not exploit any possible structure that might exist in the data. Given labeled data points a problem then arises is constructing a distance metric such that points from the same class are “close” and points from different classes are “far”. This is known as the distance metric learning problem [26]: given a training dataset, find a linear transformation of the input data such that points from the same class are concentrated while the separation between points of different classes increases. This technique has been shown to consistently produce improved results as compared to the Euclidean distance [27], [28], [29].

Concretely, consider a labeled training set $\{x_i, y_i\}_{i=1}^N$ with inputs $x_i \in \mathbb{R}^n$ and discrete, finite class labels $y_i$. Instead of using the Euclidean distance between points, i.e., $d(x_i, x_j) = \|x_i - x_j\|_2^2 = (x_i - x_j)^T(x_i - x_j)$, our goal is to learn a linear transformation $L \in \mathbb{R}^{n \times n}$ such that we use the new distance $d_S(x_i, x_j) = \|L(x_i - x_j)\|_2^2 = (x_i - x_j)^TL^TL(x_i - x_j)$. If we denote $S = L^TL$, then the problem of learning the symmetric positive semidefinite metric $S \in \mathbb{R}^{n \times n}$ has the name of Mahalanobis metric learning. The metric $S$ is optimized with the goal that nearest neighbors always belong to the same class while examples from different classes are maximally
Fig. 4. Relative representation error \( \| \) achieved by the proposed approximation \( \hat{S} \) as a function of the number of reflectors \( h \) for varying dimensions \( n \in \{32, 64, 128\} \). The red lines with the corresponding symbols show the same type of results achieved by the same algorithm where we also allow the spectrum update via \( \Theta \). The results are averaged over 100 realizations of random symmetric matrices \( S = \frac{1}{2}(X + X^T) \) in each case, where \( X \) is a matrix with entries i.i.d. standard Gaussian.

Fig. 5. Experimental results analogous to Figure 4 for \( S = XX^T \).

Fig. 6. Relative representation error \( \| \) achieved by SHF–SU as a function of the number of reflectors \( h \) and dimension \( n = 128 \). The dashed green line shows the same type of results achieved by partial eigendecomposition and diagonal update in [14] and the black dashed line (almost invisible due to the overlap with the previous line) shows the average bound of Result 4.

Table I

| Dataset  | Full metric [29] | \( \log_2 n \) | \( 2 \log_2 n \) | \( 3 \log_2 n \) |
|----------|-------------------|----------------|----------------|----------------|
| ISOLET   | 4.5\%             | 8.7\%          | 6.4\%          | 4.6\%          |
| NEWS     | 13.1\%            | 18.2\%         | 14.1\%         | 13.3\%         |
| MNIST    | 1.9\%             | 4.3\%          | 2.3\%          | 2.0\%          |

3http://homepage.tudelft.nl/19j49
4https://archive.ics.uci.edu/ml/datasets/isolet
5http://people.csail.mit.edu/jrennie/20Newsgroups
6http://yann.lecun.com/exdb/mnist/
V. CONCLUSIONS

In this paper, we describe a class of orthonormal matrices that are the product of just a few Householder reflectors. By controlling the number of reflectors in the factorization, we regulated the computational complexity of matrix-vector multiplications with these transformations. We perform an analysis of the proposed structures and describe algorithms that approximate (imperfectly in general) any orthonormal operator by a product of a given number of Householder reflectors. We then propose a similar factorization for symmetric matrices and we show an application in the context of a $k$-nearest neighbors classification problem where we use the proposed factorizations to approximate a learned distance metric with little performance degradation in terms of the classification accuracy and little computational overhead in the training phase but with a significant computational speedup for the testing phase.

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