Duality in $N = 2, 4$ supersymmetric gauge theories

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Abstract

In these lectures we present a detailed description of various aspects of
gauge theories with extended $N = 2$ and $N = 4$ supersymmetry that are
at the basis of recently found exact results. These results include the exact
calculation of the low energy effective action for the light degrees of freedom
in the $N = 2$ super Yang-Mills theory and the conjecture, supported by
some checks, that the $N = 4$ super Yang-Mills theory is dual in the sense of
Montonen-Olive.

1 Introduction

In the last few years a number of exciting results on the non-perturbative behaviour
of four-dimensional gauge theories and string theories in various dimensions have
been obtained. They are all based on the fundamental idea of duality. Duality is
a symmetry that already appears in free electromagnetism and corresponds to the
fact that the free Maxwell equations are invariant under the exchange of electric
and magnetic fields. Such a symmetry is not discussed in elementary courses on
electromagnetism because it is lost when an interaction is introduced that, due to
the absence of magnetic monopoles, has only terms with the electric current and
with the electric charge density.

If, however, we forget for a moment that no magnetic monopole has yet been
detected in experiments, and we introduce in the Maxwell equations also a magnetic

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current and a magnetic charge density, we immediately discover that the interacting Maxwell equations with these terms added preserve the invariance under the exchange of electric and magnetic fields provided that at the same time the electric and magnetic currents and charge densities are also exchanged. But, if both electric $q_i$ and magnetic $g_j$ charges are present, their quantum theory can only be consistent if they both are quantized in terms of an elementary electric and magnetic charge. This is a direct consequence of the famous Dirac quantization condition \[1\]:

$$q_i g_j = 2\pi \hbar n_{ij}$$

(1.1)

where $n_{ij}$ is an integer. This relation, in fact, implies that both the electric and magnetic charges are quantized in terms of an elementary electric $q_0$ and magnetic $g_0$ charges that satisfy themselves the Dirac quantization condition with an integer $n_0$. Consequently a theory in which the fundamental electric charge $q_0$ is small, corresponding to a perturbative electric theory, is necessarily a theory in which the magnetic charges are large, corresponding to a strongly interacting magnetic theory and vice versa. Thus we can only have a perturbative theory either in the electric or in the magnetic charge, but not in both.

Another apparently different kind of duality, the so-called Kramers-Wannier \[2\] duality, was discovered in two-dimensional Ising model. This is a model for spins $\sigma_i$, taking the values $\pm 1$, living on a square lattice and interacting according to a nearest-neighbour interaction with strength $J$. We call $Z(K)$ its partition function, that is a function of the temperature $T$ through the relation $K = J / (k_B T)$, where $k_B$ is the Boltzmann constant and $J$ is the coupling between the spins. It is known that the Ising model can be formulated either on the original lattice with coupling constant $K$ and partition function $Z(K)$ or on the dual lattice, constructed from the original lattice by selecting the central points of each square of the lattice, with coupling constant $K^*$ and partition function $Z^*(K^*)$. It turns out that the two partition functions $Z$ and $Z^*$ are equal if the two couplings constants are related by the relation

$$\sinh 2K = 1 / (\sinh 2K^*)$$

(1.2)

The formulation on the original lattice provides a good description of the system at high temperature $T$ or weak coupling $J$ (small $K$), while the one on the dual lattice gives a good description of the system at low temperature or strong coupling (small $K^*$). The relation in eq. (1.2) is also used to show that, if the system has a unique phase transition, it must occur at the self-dual point $K = K^*$.

The fact that a certain theory can be represented by two perturbatively completely different theories, as in the Ising model by expanding in $K$ or in $K^*$, became evident in the middle of the seventies with the proof of the quantum equivalence \[3, 4, 5\] between a purely bosonic field theory as the sine-Gordon one described by the Lagrangian:

$$L = \frac{1}{2} \partial_\mu \Phi \partial^\mu \Phi + \frac{M^2}{\beta^2} (\cos \beta \Phi - 1)$$

(1.3)
and a purely fermionic one as the massive Thirring model described by the Lagrangian:

\[
L = \bar{\Psi} \left( i\gamma_\mu \partial^\mu + m \right) \Psi - \frac{g^2}{2} \bar{\Psi} \gamma^\mu \Psi \bar{\Psi} \gamma_\mu \Psi
\] (1.4)

provided that the two coupling constants are related by

\[
\frac{\beta^2}{4\pi} = \frac{1}{1 + g/\pi}
\] (1.5)

As in electromagnetism and in the Ising model discussed before also in this case weak coupling in one theory (for instance in the sine-Gordon theory) corresponds to strong coupling in the other one (the Thirring theory) as expressed by the relation in eq.(1.5). It was also recognized that the sine-Gordon theory contains together with a perturbative scalar particle, corresponding to the scalar field present in the sine-Gordon Lagrangian, also a soliton solution that does not correspond to any field in the sine-Gordon Lagrangian, but that can be shown to correspond to the fermion field of the Thirring Lagrangian. The soliton has also the important property that its mass is large in sine-Gordon perturbation theory (small $\beta$). All these considerations made it soon clear that the sine-Gordon-Thirring theory was a unique theory that can be either formulated in terms of the sine-Gordon Lagrangian containing a fundamental scalar particle (we call it fundamental because it corresponds to the field $\Phi$ present in the sine-Gordon Lagrangian) or in terms of the Thirring Lagrangian containing a fermionic particle, that is fundamental from the point of view of the Thirring Lagrangian, corresponding to the fermionic field $\Psi$, but that is solitonic from the point of view of the sine-Gordon Lagrangian.

The understanding of these properties in two-dimensional theories, together with the discovery of the ’t Hooft-Polyakov monopole [6, 7] solution as a soliton in the four-dimensional Georgi-Glashow model, opened the way to the beautiful suggestion, made by Montonen and Olive [8], of duality between the original theory in terms of the fundamental particles described by the fields of the original Georgi-Glashow Lagrangian, that in this specific case were the $W$-bosons, and the dual theory in which the fundamental particles are replaced by the monopoles that are solitons of the original theory. The original formulation of Montonen and Olive implied that the two theories, the original and the dual called also electric and magnetic, were essentially described by the same Lagrangian with their gauge coupling constants related to each other as the elementary electric and magnetic charges $q_0$ and $g_0$ are related through the Dirac quantization condition (see eq.(1.1) and what follows). Although Montonen and Olive brought a number of arguments in support of their suggestion, as for instance the fact that the masses of all particles were given by the same duality invariant formula, it became soon clear that their beautiful idea could not be realized in the Georgi-Glashow model. This was mainly due to two reasons. The first one was that the mass formula was a classical formula and there was no evidence that it will keep the same form in the full quantum theory. The additional problem was how to obtain solitons with spin 1 as required by the fact that the fundamental particles of the original theory, the $W$-bosons, had spin equal to 1.
It became soon clear \[1\] that, in order to realize the beautiful duality idea of Montonen and Olive, one needed an additional ingredient, namely supersymmetry. This was the reason to study \[1\] the simplest supersymmetric theory with monopole and dyon solutions having the same structure as those in the Georgi-Glashow model, namely the \(N = 2\) super Yang-Mills theory. It was also soon recognized \[10\] that this theory had a BPS mass formula that was a direct consequence of the quantum supersymmetry algebra and not just a formula valid in the classical theory as in the Georgi-Glashow model. This solved the first problem mentioned above. Quite soon, however, it became also clear \[11\] that the supersymmetry multiplet to which the magnetic monopole belonged, did not contain a spin 1 state and consequently also the \(N = 2\) super Yang-Mills could not realize the Montonen-Olive duality. This brought Osborn \[11\] to consider \(N = 4\) super Yang-Mills in which the BPS classical mass formula is not changed by quantum corrections and the \(N = 4\) supersymmetry multiplet contains a state with spin 1. This is a theory that passed all tests for realizing the Montonen-Olive duality and in fact in the last section of these lectures we will present a reformulation of it as discussed in a recent review written by David Olive \[12\].

The Montonen-Olive duality, that by now has been extended also to string theories for space-time dimensions \(D \leq 10\) and has played a fundamental role in the understanding of the non-perturbative connections between various consistent and perturbatively inequivalent string theories, seems to be working only for theories that have enough supersymmetries to prevent classical formulas, as the BPS mass formulas, to be modified by quantum corrections and also coupling constants to run. Those theories, having potentials with flat directions, are characterized by a manifold of inequivalent vacua. Such degeneracy is sometimes modified by quantum corrections, but it is in general never wiped out.

On the other hand, these are not very interesting theories for hadron physics that at present energies is well described by QCD in which the strong fine structure constant runs and the vacuum is uniquely fixed. If we want to study theories that are closer to QCD, although still have flat directions, we must therefore go to \(N = 2\) supersymmetric theories or even better to \(N = 1\) supersymmetric theories.

In these lectures we will describe the Seiberg-Witten approach to the determination of the exact low-energy effective action for \(N = 2\) super Yang-Mills with gauge group \(SU(2)\). We have unfortunately no time to also discuss \(N = 1\) supersymmetric gauge theories. For reviews on them the reader is advised to consult Refs. \[13, 14\]. A number of reviews on duality in gauge theories and more specifically on the Seiberg-Witten approach have also appeared \[12, 13, 16, 17, 18, 19, 20, 21, 22\].

This is a much extended version of the lectures that I gave last year at the ITEP Winter School \[23\]. I have tried as much as possible to make them self-contained. The plan of the lectures is as follows.

In section \(\ref{EMDuality}\) we discuss duality in electromagnetism and in section \(\ref{DiracQuantization}\) the Dirac quantization condition. In section \(\ref{tHooftPolyakov}\) the ’t Hooft-Polyakov magnetic monopole and the Julia-Zee dyon solutions in the Georgi-Glashow model are discussed in great de-
tail, while section (5) is devoted to their semiclassical quantization. In section (6) we discuss instanton solutions in euclidean Yang-Mills theory, their difference with respect to monopoles and the introduction of the $\theta$ parameter in gauge theories. Section (7) is devoted to the Montonen-Olive duality conjecture. After the formulation of this beautiful idea it became very soon clear that this duality property cannot be satisfied in the Georgi-Glashow model where the quantum corrections invalidate the conclusions based on semiclassical considerations. The theories that have a chance to realize it were those in which the semiclassical properties are not destroyed by quantum corrections and those are the supersymmetric gauge theories. That is why in section (8) we discuss the representations of supersymmetry algebra with and without central charges, in sect. (9) we construct the supersymmetric Yang-Mills actions in four dimensions from dimensional reduction from $D = 6, 10$ and finally in section (10) we write the Lagrangians of supersymmetric gauge theories also with matter in the formalism of $N = 1$ superfields. In sect. (11) we present the semiclassical analysis of $N = 2$ super Yang-Mills theory, discuss the perturbative and instanton contributions to its low energy effective action and show that this theory has monopole and dyon solutions as the Georgi-Glashow model. Section (12) is devoted to the computation of the central charges of the supersymmetric algebra and to the derivation of the mass formula for the BPS states that is a direct consequence of the quantum supersymmetry algebra with central charges and not just a property of the classical theory as in the Georgi-Glashow model. In the second part of section (12) we show that the structure of the fermionic zero modes of $N = 2$ super Yang-Mills theory is such that Montonen-Olive duality cannot be realized in it. In sections (13) and (14) we discuss respectively the global parametrization of the moduli space in $N = 2$ super Yang-Mills with gauge group $SU(2)$ and its singularity structure, while in section (15) we explicitly construct the Seiberg-Witten solution. Section (16) is devoted to a quick review of the main properties of $N = 4$ super Yang-Mills theory. In sect. (17), following very closely Ref. [12], we riformulate the Montonen-Olive duality conjecture adapted to the $N = 4$ super Yang-Mills theory and we show that the various formulations are related to each others by the action of the modular group $SL(2, Z)$.

Many details of the calculations are presented in three appendices. Appendix A is devoted to many details concerning monopoles and dyons in the Georgi-Glashow model, Appendix B is a summary of the $N = 1$ superfield formalism and finally in Appendix C we explicitly construct the central charges of the supersymmetry algebra for $N = 2$ super Yang-Mills.

2 Electromagnetic duality

The free Maxwell equations

$$\nabla \cdot E = 0 \quad \nabla \cdot B = 0$$
\[ \nabla \wedge \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \quad \nabla \wedge \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = 0 \] (2.1)

are not only invariant under Lorentz and conformal transformations. They are also invariant under a duality transformation:

\[
\begin{align*}
\mathbf{E} &\rightarrow \mathbf{E} \cos \phi - \mathbf{B} \sin \phi \\
\mathbf{B} &\rightarrow \mathbf{B} \cos \phi + \mathbf{E} \sin \phi
\end{align*}
\] (2.2)

In particular if we take \( \phi = -\pi/2 \) one obtains from eq. (2.2) a discrete duality transformation:

\[
\mathbf{E} \rightarrow \mathbf{B} \quad \mathbf{B} \rightarrow -\mathbf{E}
\] (2.3)

This transformation is generated by the duality matrix acting on the two-vector consisting of the electric and magnetic fields as

\[
\begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix}
\] (2.4)

In terms of the complex vector \( \mathbf{E} + i\mathbf{B} \) the duality transformation in eq.(2.2) becomes

\[
\mathbf{E} + i\mathbf{B} \rightarrow e^{i\phi} (\mathbf{E} + i\mathbf{B})
\] (2.5)

Notice that the energy and momentum density of the electromagnetic field given respectively by

\[
\frac{1}{2} |\mathbf{E} + i\mathbf{B}|^2 = \frac{1}{2} \left( \mathbf{E}^2 + \mathbf{B}^2 \right)
\] (2.6)

and

\[
\frac{1}{2i} (\mathbf{E} + i\mathbf{B})^* \wedge (\mathbf{E} + i\mathbf{B}) = \mathbf{E} \wedge \mathbf{B}
\] (2.7)

are invariant under the duality transformation in eq.(2.2), while the real and imaginary part of

\[
\frac{1}{2} (\mathbf{E} + i\mathbf{B})^2 = \frac{1}{2} \left( \mathbf{E}^2 - \mathbf{B}^2 \right) + i\mathbf{E} \cdot \mathbf{B}
\] (2.8)

that are respectively the Lagrangian of the electromagnetic field and the topological charge density, transform under the duality group exactly as the doublet \( (\mathbf{E}, \mathbf{B}) \) in eq.(2.2), but with an angle equal to \( 2\phi \).

If we perform a discrete duality transformations twice, we get

\[
(\mathbf{E}, \mathbf{B}) \rightarrow (-\mathbf{E}, -\mathbf{B})
\] (2.9)

that corresponds to the charge conjugation operation.

The reason why this beautiful duality property of the free electromagnetic field is not even mentioned in the courses on electromagnetism is due to the fact that it is lost when we introduce the interaction of the electromagnetic field with matter by just adding in the right hand side of the Maxwell equations an electric current \( j_e \).
and an electric charge density $\rho_e$. If we want to keep duality we must also introduce a magnetic current $j_m$ and a magnetic charge density $\rho_m$ together with their electric counterparts. If we do so the Maxwell equations given in eq. (2.1) and written in complex notations become:

$$\nabla \cdot (E + iB) = \rho_e + i\rho_m \quad (2.10)$$

and

$$\nabla \wedge (E + iB) = i \frac{\partial}{\partial t}(E + iB) + i(j_e + ij_m) \quad (2.11)$$

The previous equations are invariant under the duality transformation given in eq.(2.5) if the electric and magnetic currents and densities transform as

$$\rho_e + i\rho_m \rightarrow e^{i\phi}(\rho_e + i\rho_m) \quad (2.12)$$

and

$$j_e + ij_m \rightarrow e^{i\phi}(j_e + ij_m) \quad (2.13)$$

In particular if we have only pointlike particles with both electric and magnetic charges $q$ and $g$ respectively, then duality implies the following transformation:

$$q + ig \rightarrow e^{i\phi}(q + ig) \quad (2.14)$$

Particles with magnetic charge are not introduced in usual electromagnetism for the very simple reason that they are not observed in the experiments. If we include them we must either think that their mass is higher than the presently available energy or find other reasons for their absence. However, if we insist in preserving duality also in the presence of interaction, as shown by Dirac [1], a theory with both electric and magnetic charges $q_i$ and $g_j$ can be consistently quantized only if the Dirac quantization condition is satisfied

$$q_i g_j = 2\pi \hbar n_{ij} \quad (2.15)$$

where $n_{ij}$ are arbitrary integers. We will derive the Dirac quantization condition in the next section.

The Dirac quantization condition is clearly not invariant under the duality transformation in eq.(2.14). It is only invariant under the discrete transformation obtained from eq.(2.14) for $\phi = -\frac{\pi}{2}$:

$$q \rightarrow g \quad \quad g \rightarrow -q \quad (2.16)$$

3 The Dirac quantization condition

In this section we show that in any consistent quantum theory containing both electric and magnetic charges the Dirac quantization condition must be satisfied.
A magnetic monopole located at the origin generates a magnetic field given by:

$$B = \frac{g}{4\pi r^3} r = |r|$$  \hspace{1cm} (3.1)

The equation of motion of a particle with mass $m$ and charge $q$ moving in the magnetic field $B$ generated by the magnetic monopole is given by:

$$m\ddot{r} = q\dot{r} \times B$$  \hspace{1cm} (3.2)

Using eqs.(3.1) and (3.2) and the general relation valid for three arbitrary vectors $A$, $B$ and $C$

$$A \times (B \times C) = B(A \cdot C) - C(A \cdot B)$$  \hspace{1cm} (3.3)

it is easy to see that

$$\frac{d}{dt} (r \times m\dot{r}) = r \times m\ddot{r} = \frac{qg}{4\pi} \frac{d\dot{r}}{dt}$$  \hspace{1cm} (3.4)

where $\dot{r} \equiv \dot{r}/r$.

The previous equation implies that the total angular momentum is conserved:

$$\frac{d}{dt} \left[ r \times m\dot{r} - \frac{qg}{4\pi} \dot{r} \right] = 0$$  \hspace{1cm} (3.5)

The first term in the bracket is the angular momentum of the particle with mass $m$ and charge $q$, while the second term is the angular momentum of the electromagnetic field generated by the electric and magnetic charges. In order to see this let us compute the angular momentum of the electromagnetic field:

$$J^{(e.m.)} = \int d^3r \, r \times (E \times B) = \int d^3r \, \frac{g}{4\pi r} \left( E - \frac{r \cdot E}{r^2} r \right) =$$

$$= \int d^3r \, E \cdot \nabla \left( \frac{g\dot{r}}{4\pi} \right) = -\frac{qg}{4\pi} \dot{r}_p$$  \hspace{1cm} (3.6)

where in the last step we have performed a partial integration and we have used the equation:

$$\nabla \cdot E = q\delta^{(3)}(r - r_p)$$  \hspace{1cm} (3.7)

Eq.(3.6) shows that the second term in eq.(3.3) is the angular momentum of the electromagnetic field.

In a quantum theory the projection of the angular momentum along a direction is quantized. This implies that:

$$\dot{r} \cdot J = -\frac{qg}{4\pi} = -\frac{1}{2} \hbar n$$  \hspace{1cm} (3.8)

that is the Dirac quantization condition in eq.(2.15).
Although the previous argument gives the correct result it is not very convincing. In particular it does not explain why we should have half-integer angular momentum without having fermions. Therefore in the following we give a more rigorous derivation of the Dirac quantization condition.

If we have a magnetic monopole located at the origin, then the divergence of the magnetic field is non zero and we cannot choose a vector potential that is regular everywhere and such that

$$B = \nabla \wedge A$$

(3.9)

We can, however, introduce a vector potential $A_N$ for the northern hemisphere and another one $A_S$ for the southern hemisphere. In the northern hemisphere we can compute the magnetic field using eq. (3.9) with $A = A_N$, while in the southern hemisphere the magnetic field is computed again from eq. (3.9) but this time with $A = A_S$. Along the equator the two vector potentials must match up to a gauge transformation:

$$A_N = A_S + \nabla \chi(\theta)$$

(3.10)

where $\theta$ is the angle around the equator.

Electrically charged particles are described by a wave function $\Psi_N(x)$ in the northern hemisphere and by another wave function $\Psi_S(x)$ in the southern hemisphere. On the equator they must be equal up to a gauge transformation:

$$\Psi_N(x) = e^{-iq\chi(\theta)/\hbar}\Psi_S(x)$$

(3.11)

where $q$ is the electric charge of the particle.

Since the wave function must be single valued when we go around the equator, we must require that the parameter of the gauge transformation satisfies the eq.

$$\chi(\theta + 2\pi) = \chi(\theta) + \frac{2\pi\hbar}{q}n$$

(3.12)

where $n$ is an integer.

Let us now compute

$$\int_{eq.} d\ell \cdot A_N - \int_{eq.} d\ell \cdot A_S = \int_0^{2\pi} d\theta \frac{d\chi}{d\theta}(\theta) = \chi(2\pi) - \chi(0) = \frac{2\pi\hbar}{q}n$$

(3.13)

where the two integrals in the l.h.s. of the previous equation are performed along the equator and we have used eqs. (3.10) and (3.12).

On the other hand the l.h.s. of eq. (3.13) can be rewritten by means of the Stokes theorem as:

$$\int_N dS \cdot B + \int_S dS \cdot B = \int_{sphere} dS \cdot B = g$$

(3.14)

where the integrals on the l.h.s. of the previous equation are performed on the northern and southern hemisphere respectively and their sum is equal to the integral over the entire sphere. In the last step in eq. (3.14) we have used the fact that inside
the sphere there is a magnetic monopole with magnetic charge $g$. Finally comparing eqs. (3.13) and (3.14) we get the Dirac quantization condition.

A consequence of the Dirac quantization condition is that both electric and magnetic charges are quantized being multiples of the elementary magnetic and electric charges $g_0$ and $q_0$ [24]:

$$q_i = n_i q_0 \quad \quad g_j = n_j g_0$$

where $g_0$ and $q_0$ satisfy the relation

$$q_0 g_0 = 2\pi \hbar n_0$$

The integer $n_0$ depends on the theory under consideration. Thus the Dirac quantization condition provides an alternative mechanism, besides the one of having the electric charge to be part of a non abelian group that is realized in grand unified theories, for the quantization of the electric charge of the elementary particles that is a phenomenon largely observed in experiments.

As noticed at the end of previous section the Dirac quantization condition is not duality invariant. In order to have duality invariance we must generalize it to particles having both electric and magnetic charges, called dyons. If we have two dyons with electric and magnetic charges equal respectively to $(q_i, g_i)$ and $(q_j, g_j)$ and if we go through an argument as the one used in the beginning of this section we get the following generalization of the Dirac quantization condition:

$$q_i g_j - q_j g_i = 2\pi \hbar n$$

that goes under the name of Dirac-Schwinger-Zwanziger [25, 26] (DSZ) quantization condition. In order to show its duality invariance we rewrite it as follows:

$$q_i g_j - q_j g_i = Q_i^T \Omega Q_j$$

where we have defined

$$Q_i^T = \begin{pmatrix} q_i & g_i \end{pmatrix} \quad \Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad Q_i = \begin{pmatrix} q_i \\ g_i \end{pmatrix}$$

Under a duality transformation the vectors $Q$ transform as follows:

$$Q \rightarrow OQ \quad Q^T \rightarrow Q^T O^T \quad O = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix}$$

The invariance under duality transformation follows easily from the identity:

$$O^T \Omega O = \Omega$$

In these first few sections we have derived a number of consequences based on the existence of magnetic monopoles. But up to now there is no evidence of their
existence in nature. The Dirac quantization condition tells us, because of relation (3.16) that in a theory in which the electric charge $q_0$ is small, the magnetic charge $g_0$ is necessarily big. But those considerations do not put any restriction on the mass of the monopoles. In the next section we will see that magnetic monopoles naturally appear in gauge theories with scalar Higgs particles transforming according to the adjoint representation of the gauge group and that in these theories their mass is big when the gauge coupling constant is small.

4 The ’t Hooft-Polyakov monopole

In this section we will discuss the monopole solution found by ’t Hooft [6] and Polyakov [7] in the Georgi-Glashow model. For more information about magnetic monopoles the reader is recommended to consult Ref. [24].

Let us consider the Georgi-Glashow model

$$L = -\frac{1}{4} F_{\mu \nu}^a F_{a \mu \nu} + \frac{1}{2} (D_{\mu} \Phi)_a (D^{\mu} \Phi)_a - V(\Phi)$$

(4.1)

where the covariant derivative of the scalar Higgs field is given by

$$(D^\mu \Phi)_a = \partial^\mu \Phi_a - e \epsilon_{abc} A_b^\mu \Phi_c$$

(4.2)

and

$$F_{a \mu \nu} = \partial^\mu A_a^\nu - \partial^\nu A_a^\mu - e \epsilon_{abc} A_b^\mu A_c^\nu$$

(4.3)

$\epsilon_{abc}$ is the Levi-Civita tensor, because the gauge group is $SU(2)$. For an arbitrary group one should substitute the Levi-Civita tensor with the structure constants $f_{abc}$ of the group. In eqs. (4.2) and (4.3) we have used the generators of the gauge group in the adjoint representation that are given by:

$$(T_a)_{bc} = -i \epsilon_{abc} \quad [T_a, T_b] = i \epsilon_{abc} T_c$$

(4.4)

See Appendix A for the explicit expressions of the gauge transformations and for a more detailed discussion of our notations.

The potential $V$ is equal to

$$V(\Phi) = \frac{\lambda}{4} \left( \Phi^2 - a^2 \right)^2$$

(4.5)

The classical equations of motion, that follow from $L$, are

$$(D_{\mu} F^{\mu \nu})_a = -e \epsilon_{abc} A_b^{\mu \nu} (D_{\mu} \Phi)_c$$

(4.6)

$$(D_{\mu} D^{\mu} \Phi)_a = -\lambda \Phi_a (\Phi^2 - a^2)$$

(4.7)

They must be considered together with the Bianchi identity.
\[ D_\mu * F^{\mu \nu} = 0 \quad \quad * F^{\mu \nu} \equiv \frac{1}{2} \epsilon^{\mu \nu \rho \sigma} F_{\rho \sigma} \]  

\( \epsilon^{\mu \nu \rho \sigma} \) is the antisymmetric Levi Civita tensor with \( \epsilon^{0123} = 1 \). We use the metric \( g_{\mu \nu} = (1, -1, -1, -1) \).

The energy is given by

\[ E \equiv \int d^3 x \, \theta_{00} = \int d^3 x \left\{ \frac{1}{2} \left[ (B_i^a)^2 + (E_i^a)^2 + (\Pi^a)^2 + [(D_i^a \Phi)^a]^2 \right] + V(\Phi) \right\} \]  

where

\[ \Pi_a = \left( D^0 \Phi \right)_a \quad F^{i0}_a = E^i_a \quad F_{a \, ij} = - \epsilon_{ijk} B^k_a \]  

with \( \epsilon_{ijk} \equiv \epsilon^{0ijk} (\epsilon_{123} = 1) \).

The energy is positive semi-definite. It vanishes if and only if

\[ F^{\mu \nu}_a = (D^\mu \Phi)_a = 0 \quad V(\Phi) = 0 \]  

These conditions are satisfied by taking

\[ \Phi_a = a \delta_{a3} \quad A_\mu^a = 0 \]  

or equivalently any gauge rotated version of them.

This field configuration corresponds to the vacuum of our model and obviously satisfies the equations of motions and the Bianchi identity in eqs.(4.6), (4.7) and (4.8).

It is easy to see that, if \( a \neq 0 \), the \( SU(2) \) gauge group is broken to \( U(1) \). With the v.e.v of the Higgs field taken along the third direction \( (\Phi_a = a \delta_{a3}) \) the \( U(1) \) gauge field \( A_3^a \) remains massless, while the two charged fields

\[ W_\pm = \frac{1}{\sqrt{2}} \left( A_1^a \pm i A_2^a \right) \]  

get a mass equal to

\[ M_W = a |q| = a e \bar{h} \]  

where \( q \) is their electric charge. They are charged with respect to the unbroken \( U(1) \). Its generator \( Q_e \) is given by the generator of the \( SU(2) \) gauge group that leaves invariant the v.e.v of the scalar field

\[ Q_e = \frac{e}{a} T_a \Phi_a \bar{h} = e T_3 \bar{h} \]  

Finally from the Higgs mechanism one gets also a neutral Higgs scalar particle with mass equal to

\[ M_H = \sqrt{2 \lambda} a h \]  

\( h \) has been explicitly written in some of the previous formulas, while has been put equal to 1 in most cases. We use also conventions where the speed of light \( c = 1 \).
In addition to the constant vacuum solution of eq.(4.12) the equations of motion admit also static (time independent) solutions. The simplest of them can be obtained starting with a radially symmetric ansatz:

$$\Phi_a = r^a, A_0^a = 0, A_i^a = -\epsilon_{aij} r^j/er^2 [1 - K(\xi)]$$ (4.17)

where $\xi \equiv acr$ is a dimensionless quantity.

Inserting this ansatz into the energy one gets:

$$E = 4\pi a e \int_0^\infty d\xi \xi^2 \left[ \xi^2 \left( \frac{dK}{d\xi} \right)^2 + K^2 H^2 + \frac{1}{2} \left( \frac{dH}{d\xi} - H \right)^2 + \frac{1}{2} (K^2 - 1)^2 + \frac{\lambda}{4e^2} (H^2 - \xi^2)^2 \right]$$ (4.18)

The insertion of the ansatz in the equations of motions (4.6) and (4.7) gives a system of coupled differential equations for the radial functions $H$ and $K$:

$$\xi^2 \frac{d^2 K}{d\xi^2} = K H^2 + K (K^2 - 1)$$ (4.19)

and

$$\xi^2 \frac{d^2 H}{d\xi^2} = 2K^2 H + \frac{\lambda}{e^2} H (H^2 - \xi^2)$$ (4.20)

They can also be obtained by minimizing the energy in eq.(4.18). In order to have a finite energy solution one must also impose boundary conditions for both $\xi = 0$ and $\xi \to \infty$. They are discussed in Appendix A. It can be shown [24] that the previous system of equations admits a finite energy solution. However, in general, it is not possible to write it down explicitly unless one takes the parameter $\lambda$ of the potential of the Higgs field equal to 0. This corresponds to the so called Bogomolny [27], Prasad, Sommerfield [28] (BPS) limit. In this limit one obtains:

$$K(\xi) = \frac{\xi}{\sinh \xi}, \quad H(\xi) = \frac{\xi}{\tanh \xi} - 1$$ (4.21)

In order to have a better understanding of this limiting case and to explicitly derive the solution in eq.(4.21) let us rewrite the sum of the two terms appearing in the energy density that involve the square of the non abelian magnetic field and the square of the space components of the covariant derivative of the Higgs field as follows

$$(B_a^a)^2 + [(D_i \Phi)^a]^2 = [B_i^a \pm (D_i \Phi)^a]^2 \mp 2B_i^a (D_i \Phi)^a$$ (4.22)

When we insert it in the energy (see eq.(4.9)) we see that all terms are positive except the last one in the r.h.s of eq.(4.22). We get therefore a lower bound for the energy

$$E \geq \mp \int d^3x B_i^a (D_i \Phi)^a$$ (4.23)
that, after a partial integration and the use of the Bianchi identity in eq. (4.8), becomes

\[ E \geq \pm \int d^3x \partial_i [B^a_i \Phi^a] \]  

(4.24)

The equality sign is obtained if and only if the following equations are satisfied:

\[ E^a_i = 0 \quad \Pi_a = 0 \quad \lambda = 0 \quad B^i_a \pm (D^i \Phi)_a = 0 \]  

(4.25)

They are first order equations that imply the second order equations of motion (4.6) and (4.7). It may be confusing to allow for a non vanishing v.e.v. \( a \) for the scalar field \( \Phi \) in the BPS limit where we put \( \lambda = 0 \). A better way of proceeding could be to start with a small, but not vanishing value of \( \lambda \), and only after we have obtained a non vanishing v.e.v. for \( \Phi \) send \( \lambda \) to zero. Another possibility is to have a potential with flat directions that allows for a non vanishing v.e.v. for \( \Phi \), but it does not fix the value of \( a \). This last case is, in fact, what happens in supersymmetric theories as we will see later on.

The ansatz in eq. (4.17) satisfies the first two equations in (4.25). In order to find the functions \( H \) and \( K \), for the BPS limit (\( \lambda = 0 \)), one must impose the last equation in (4.25) and show that one obtains the solution in eq. (4.21). For the sake of simplicity we restrict ourselves to the case of the monopole solution corresponding to the minus sign in the last eq. in (4.25). In Appendix A it is shown that the last equation in (4.25) implies the following first order equations:

\[ \xi K' = -KH \quad \xi H' = H + 1 - K^2 \]  

(4.26)

whose solution with the boundary conditions for \( \xi \to \infty \):

\[ \lim_{\xi \to \infty} K(\xi) = 0 \quad \lim_{\xi \to \infty} H(\xi) = \xi \]  

(4.27)

and with suitable boundary conditions for \( \xi \to 0 \), as discussed in the Appendix A, is given in eq. (4.21). The prime in eq. (4.26) means derivative with respect to the argument. The boundary conditions for \( \xi \to 0 \) and \( \infty \) are required in order to have a solution with finite energy (see eq. (4.18)).

Inserting the static classical solution into the energy density given by the integrand in the r.h.s of eq. (4.9), one can see that it is concentrated in a small region around the origin and goes to zero exponentially as \( r \) goes to infinity.

We will see later on that in the quantum theory the classical solution corresponds to a new particle of the spectrum that is an extended object (with size \( \sim 1/a \)) located in the region where the energy density is appreciably different from zero.

We want to show now that the soliton solution given by the ansatz in eq. (4.17) is actually a magnetic monopole with respect to the unbroken \( U(1) \) group.

It is easy to show that, in order to have a finite energy solution, for large enough values of \( r \) the following equations must be satisfied:

\[ D_\mu \Phi = 0 \quad \Phi^2 = a^2 \]  

(4.28)
apart from a small exponential correction. For instance from eq.(A.15) it easy to see that the solution in eq.(1.21) satisfies eqs.(1.28).

Corrigan et al. [29] have shown that the most general solution of the previous equations corresponds to a vector field given by:

$$A_\mu^a = \frac{1}{a^2} \epsilon_{abc} \Phi_b \partial_\mu \Phi_c + \frac{1}{a} \Phi_a B^\mu$$

(4.29)

where $B^\mu$ is arbitrary. The corresponding field strength is entirely in the direction of the Higgs field $\Phi^a$ and is equal to

$$F_{\mu\nu}^a = \frac{1}{a} \Phi_a F_{\mu\nu}$$

(4.30)

where

$$F_{\mu\nu} = \frac{1}{\epsilon a^3} \epsilon_{abc} \Phi_a \partial_\mu \Phi_b \partial_\nu \Phi_c + \partial_\mu B^\nu - \partial_\nu B^\mu$$

(4.31)

For large values of $r$ it satisfies the free Maxwell equations

$$\partial_\mu F^{\mu\nu} = 0 \quad \partial_\mu^* F^{\mu\nu} = 0$$

(4.32)

This follows from the fact that $F_{\mu\nu}^a$ in eq.(4.30) satisfies the eq. of motion (4.6) and the Bianchi identity in eq.(4.8) together with the fact that eqs.(4.28) are valid for $r \rightarrow \infty$.

We see that outside the region where the extended particle is located the non abelian field strength is aligned along the direction of the Higgs field $\Phi^a$ and is proportional to an abelian field strength $F_{\mu\nu}$ that can be interpreted as the field strength of the unbroken $U(1)$ electromagnetic.

From eq.(4.30) we can compute the non abelian magnetic field that is equal to

$$B_i^a = \frac{1}{2} \epsilon_{ijk} F_{jk}^a = \frac{1}{2a^4 e} \Phi_a \epsilon_{ijk} \epsilon^{bcd} \Phi^b \partial^j \Phi^c \partial^k \Phi^d$$

(4.33)

where we have omitted the contribution of the last two terms in the r.h.s. of eq.(1.31), containing $B_\mu$, since they will drop out in the calculation of the magnetic charge. Inserting it in eq.(4.24) one gets:

$$E \geq \pm g a \quad g = -\frac{1}{a} \int d^3 x \partial_i [B_i^a \Phi^a] = -\frac{4\pi}{e} K$$

(4.34)

where the topological charge $K$ is given by

$$K = \int d^3 x \cdot K_0$$

(4.35)

with

$$K_\mu = \frac{1}{8\pi a^3} \epsilon_{\mu\nu\rho\sigma} \epsilon^{abc} \partial^\nu \Phi^a \partial^\rho \Phi^b \partial^\sigma \Phi^c$$

(4.36)
We call it topological current because, unlike a Noether current, it is conserved independently from the equations of motion as it can be trivially checked. It can also be seen (see Appendix A and Ref. [24]) that the topological charge $K$ is an integer since it counts the number of times that the two-sphere, defined by the second equation in eq.(4.28), is covered when the two-sphere at infinity in space is covered once. The topological charge $K$, that is an integer, should not be confused with the function $K(\xi)$ introduced in the ansatz (4.17).

To summarize we get

$$E \geq \pm ag$$  \hspace{1cm} (4.37)

where $g$ is the magnetic charge of the soliton solution that is obtained by integrating the equation:

$$\partial_i B_i = \frac{4\pi}{e} K_0$$  \hspace{1cm} (4.38)

that follows from eq.(4.31). One gets:

$$g = -\frac{1}{a} \int d^3 x \partial_i (\Phi_a B_i^a) = -\int d^3 x \partial_i B_i = -\frac{4\pi}{e} K$$  \hspace{1cm} (4.39)

In the case of the static solution corresponding to the ansatz in eq.(4.17) it is easy to see that $K = \pm 1$ in such a way that $E = a|g|$. This can be seen by observing that, using the Bogomolny eq. in (4.25), one can rewrite $g$ as follows:

$$g = \pm \frac{1}{2a} \int d^3 x \partial_i \partial_i (\Phi^2) = \pm \frac{1}{2a} \int d^3 x \partial_i \partial_i \left( H^2 e^{2r^2} \right)$$  \hspace{1cm} (4.40)

Rewriting the Laplacian in polar coordinates and inserting the explicit expression for $H$ given in eq.(4.21) we get:

$$g = \pm \frac{4\pi}{2e} \int_0^\infty d\xi \frac{d}{d\xi} \left[ \xi^2 \frac{d}{d\xi} \left( \frac{1}{\tanh \xi} - \frac{1}{\xi} \right)^2 \right] = \pm \frac{4\pi}{e}$$  \hspace{1cm} (4.41)

that is obtained from the contribution of the integrand at $\xi = \infty$.

The value obtained for the magnetic charge $g = \pm \frac{4\pi}{e}$ is consistent with the Dirac quantization condition (with $n=2$) given in eq. (2.15)

$$qg = 4\pi h$$  \hspace{1cm} (4.42)

where $q$ is the charge of the $W$-boson given in eq. (4.15) for $T_3 = \pm 1$. The fact that we obtain $n = 2$ is a consequence of the fact that the gauge bosons transform according to the triplet representation of the gauge group $SU(2)$. The value $n = 1$ would have been obtained with matter fields transforming according to the fundamental doublet representation being their electric charge in this case quantized in terms of half-integers.

In conclusion we see that the Georgi-Glashow model does not contain only perturbative states as a photon, a massless Higgs field in the BPS limit and a couple...
of charged bosons, all corresponding to the fields present in the Lagrangian in eq. (4.1) and having either a zero mass or a mass proportional to the gauge coupling constant. It contains also additional particles that are soliton solutions of the classical equations of motion whose mass is instead proportional to the inverse of the gauge coupling constant as follows from eq. (4.18) and therefore are very massive in the weak coupling limit (small $e$). In particular their mass in terms of the $W$ mass is given by

$$M_{\text{sol}} = \frac{4\pi a}{e} = 4\pi \frac{M_W}{e^2}$$

(4.43)

The soliton solution following from the ansatz in eq. (4.17) has a non vanishing magnetic charge, but has zero electric charge. In order to also have a solution with a non vanishing electric charge [30] we must allow for a non vanishing electric potential of the type

$$A^0_a = \frac{r^a}{er^2}J(\xi)$$

(4.44)

instead of the vanishing ansatz given in eq. (4.17). In the case of the monopole solution the dimensionless parameter $\xi = ear$ was introduced using the dimensional parameter $a$ corresponding to the asymptotic value for $r \to \infty$ of the Higgs field. For the dyon instead we introduce $\xi = e\hat{a}r$ where $\hat{a}$ will be determined later on in terms of the asymptotic value $a$ of the Higgs field and the ratio between the electric and magnetic charge of the dyon. With this ansatz the equations of motion in eqs. (4.19) and (4.20) are modified as follows:

$$\xi^2 \frac{d^2 K}{d \xi^2} = K \left[ K^2 + \frac{\lambda}{e^2}H^2 - \xi^2 - J^2 \right]$$

(4.45)

and

$$\xi^2 \frac{d^2 H}{d \xi^2} = 2K^2H + \frac{\lambda}{e^2}H(H^2 - \xi^2), \quad \xi^2 \frac{d^2 J}{d \xi^2} = 2K^2J$$

(4.46)

In the BPS limit where $\lambda = 0$ one can obtain an analytical solution given by:

$$H(\xi) = \frac{1}{\cos \theta} \left[ \frac{\xi}{\tanh \xi} - 1 \right], \quad K(\xi) = \frac{\xi}{\sinh \xi}$$

(4.47)

and

$$J(\xi) = \tan \theta \left[ \frac{\xi}{\tanh \xi} - 1 \right]$$

(4.48)

where $\theta$ is an arbitrary constant.

Also in this case the explicit solution given in eqs. (4.47) and (4.48) can be found by minimizing the energy in eq. (4.9) as we have done in the case of the monopole. In fact by using the identity:

$$\left( B^i_a \right)^2 + \left( E^i_a \right)^2 + \left( D^i \Phi \right)_a^2 = \left[ B^i_a - (D^i \Phi)_a \cos \theta \right]^2 +$$

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\[ E^i - (D^i \Phi)_a \sin \theta \right] \] + \( 2 E^i_a (D^i \Phi)_a \sin \theta + 2 B^i_a (D^i \Phi)_a \cos \theta \) \] (4.49)

from eq.(4.49) we get

\[ E \geq \sin \theta \int d^3 x E^i_a (D^i \Phi)_a + \cos \theta \int d^3 x B^i_a (D^i \Phi)_a \] (4.50)

The identity sign in the previous eq. holds if the following eqs. are satisfied:

\[ (D^0 \Phi)_a = 0 \quad \quad \quad V(\Phi) = 0 \] (4.51)

together with

\[ B^i_a - (D^i \Phi)_a \cos \theta = 0 \quad \quad E^i_a - (D^i \Phi)_a \sin \theta = 0 \] (4.52)

The Bianchi identity in eq.(4.8) and the eq. of motion in (4.6) imply

\[ (D_i B_i)_a = 0 \quad \quad \Phi^a (D_i E_i)_a = 0 \] (4.53)

that allow to rewrite eq.(4.50) as follows:

\[ E \geq -a (\sin \theta q + \cos \theta g) \] (4.54)

where \( q \) and \( g \) are the electric and magnetic charges of the unbroken \( U(1) \) given by

\[ g = -\frac{1}{a} \int d^3 x \partial_i (\Phi^a B_i^a) \quad q = -\frac{1}{a} \int d^3 x \partial_i (\Phi^a E_i^a) \] (4.55)

The dimensional parameter \( a \) that we have introduced in eqs.(4.54) and (4.53) is the v.e.v. of the Higgs field for \( r \to \infty \), that is related to the parameter \( \hat{a} \) used for defining the dimensionless parameter \( \xi \) by:

\[ \Phi^a \rightarrow a^\frac{r^a}{r} \quad \quad a = \frac{\hat{a}}{\cos \theta} \] (4.56)

The relation between \( a \) and \( \hat{a} \) can be obtained by inserting the asymptotic behaviour for \( r \to \infty \) of \( H \) in eq.(1.17) in the ansatz for the Higgs field in eq.(4.17).

Let us now obtain directly the solution in eqs.(4.47) and (4.48) from the first order eqs.(4.51) and (4.52) and show that the parameter \( \theta \) is determined in terms of \( q \) and \( g \).

The ansatz in eqs.(4.17) and (4.44) satisfies the first eq. in (4.51), while the second equation is satisfied by requiring the vanishing of the coupling constant \( \lambda = 0 \). Inserting then the ansatz in the two eqs.(4.52) one obtains (see eq.(A.16)) the two eqs.

\[ \xi K' = -K \hat{H} \quad \quad \xi \hat{H}' = 1 - K^2 + \hat{H} \] (4.57)

where the new function \( \hat{H} \) is related to the two functions \( J(\xi) \) and \( H(\xi) \) appearing in the ansatz through the following relations

\[ \hat{H}(\xi) = \cos \theta H(\xi) \quad \quad J(\xi) = \sin \theta H(\xi) \] (4.58)
From the solution of the eqs. (4.57) for $K$ and, in this case, for $\hat{H}$ given in eq. (4.21), by means of eqs. (4.58), one can immediately obtain $H$, $K$ and $J$ given in eqs. (4.47) and (4.48). Up to now $\theta$ is an arbitrary parameter. In the following we show that $\theta$ is determined by the ratio between the electric and magnetic charges of the dyon. In fact, the magnetic charge of the dyon can be computed from the first eq. (4.55) starting from eqs. (4.28) and (4.29), proceeding as in the case of the monopole and therefore obtaining the same result as before: $g = \frac{-4\pi}{e}$. On the other hand the electric charge $q$ in the second eq. (4.55) can be computed in terms of the magnetic charge $g$ by using the eqs. (4.52). One gets

$$q = g \tan \theta$$  \hspace{1cm} (4.59)

Inserting $\theta$ determined by the previous eq. in eq. (4.54) one gets the mass of the dyon in terms of its electric and magnetic charges:

$$M = a \sqrt{q^2 + g^2}$$  \hspace{1cm} (4.60)

This formula has been deduced for the dyon soliton solution in the BPS limit, but it is actually valid for any particle of the spectrum. Notice also that it is invariant under the duality transformation in eq. (2.14).

\section{Collective coordinates}

In this section we proceed to the semiclassical quantization of the monopole solution or more generally of any classical solution of the Bogomolny equation, in the approximation in which we only allow for motions along the collective coordinates of the classical solution.

Let us consider a solution of the Bogomolny equation:

$$B_i = D_i \Phi, \quad i = 1, 2, 3.$$  \hspace{1cm} (5.1)

with a definite topological charge $K$ and let us work in the temporal gauge $A_0 = 0$. This is different from what we have done in the previous section when we constructed a classical solution corresponding to a dyon by allowing $A_0 \neq 0$. If we work in the temporal gauge $A_0 = 0$ there is no static dyon solution. We will see that, in the temporal gauge, dyons will instead emerge as time dependent solutions.

Given a solution of eq. (5.1) we can usually find an entire family of solutions with the same energy. The parameters labelling these different solutions are called collective coordinates or moduli and the space of the solutions with fixed energy or with fixed topological charge is called the moduli space of the solutions. For instance in the case of the monopole solution we have always assumed that the monopole is located at the origin, but, because of translational invariance, it could have been located in any other point. This means that, starting from the monopole solution found in Sect. 4, we could make the substitution $\mathbf{r} \rightarrow \mathbf{r} - \mathbf{R}$ obtaining another
classical solution that depends on the arbitrary three-dimensional vector $\mathbf{R}$. The three coordinates of $\mathbf{R}$ are the three translational collective coordinates.

Let us consider the Lagrangian of the Georgi-Glashow model in the BPS limit (without the $\Phi^4$ potential). In the temporal gauge it can be written in the form:

$$L = T - V$$

where the kinetic energy is given by

$$T = \frac{1}{2} \int d^3 x \left[ \dot{A}_i^a \dot{A}_i^a + \dot{\Phi}^a \dot{\Phi}^a \right]$$

and the potential energy by

$$V = \frac{1}{2} \int d^3 x \left[ B_i^a B_i^a + D_i \Phi^a D_i \Phi^a \right]$$

The dot means derivative with respect to time. For a solution of the Bogomolny eq. (5.1) the potential term is just proportional to the topological charge $K$:

$$V = \frac{1}{2} \int d^3 x \left[ B_i^a - D_i \Phi^a \right]^2 - ag = \frac{4\pi a}{e} K$$

The temporal gauge is preserved by the dynamics if we impose the Gauss’s law (corresponding to the $A_0$ equation of motion) as a constraint between physical fields:

$$D_i \dot{A}_i + ie [\Phi, \dot{\Phi}] = 0$$

where for $\Phi$ and $A_i$ we use the matrix notation discussed at the beginning of Appendix A.

In the gauge $A_0 = 0$ the configuration space of the fields is given by $\mathcal{C} = \mathcal{A}/\mathcal{G}$ where $\mathcal{A} = \{ A_i(x), \Phi(x) \}$ is the space of finite energy solutions and $\mathcal{G}$ is the set of all "small" residual gauge transformations. Residual gauge transformations are those preserving the temporal gauge $A_0 = 0$:

$$\delta A_0 = D_0 \epsilon = 0 \implies \dot{\epsilon} = 0$$

They are time independent gauge transformations. A residual gauge transformation is said to be "small" when it goes to the identity at spatial infinity. Two finite energy solutions that are related by a "small" residual gauge transformation are equivalent and should not be counted twice in the configuration space of all finite energy solutions. We distinguish "small" from "large" gauge transformations because they have very different physical meaning. A "small" gauge transformation connects two equivalent field configurations, while a "large" gauge transformation corresponds to a global symmetry, as the translational symmetry, and therefore two field configurations related by a "large" gauge transformation are physically inequivalent. For each symmetry of the theory we have a collective coordinate. In the case
of the monopole three collective coordinates correspond to the possibility of having a monopole located in any point of the three-dimensional space, while a fourth collective coordinate is related to the possibility of transforming the monopole solution by means of a ”large” gauge transformation obtaining a monopole with non zero electric charge, i.e. a dyon. Using semiclassical quantization we will show that the electric charge of the dyon is quantized as the one of the $W$-boson.

Defining the conjugate momenta:

$$ (\Pi_A)_i^a = \dot{A}_i^a \quad \Pi_\Phi^a = \dot{\Phi}^a $$

(5.8)

the hamiltonian corresponding to the Lagrangian in eq.(5.2) is given by

$$ H = T + V $$

(5.9)

where now $T$ is given in terms of the conjugate momenta:

$$ T = \frac{1}{2} \int d^3x \left[ (\Pi_A)_i^a (\Pi_A)_i^a + \Pi_\Phi^a \Pi_\Phi^a \right] $$

(5.10)

It is convenient to group together the gauge field and the Higgs field $A_I = (A_i, \Phi)$ by introducing a four-dimensional formalism where the index $I$ runs over the three-dimensional space index $i = 1, 2, 3$ and over a fourth index corresponding to the Higgs field with the extra condition that $A_I$ depends only on the first three coordinates: $\partial_4 A_I = 0$. In this notations both the kinetic energy in eq.(5.3):

$$ T = \frac{1}{2} \int d^3x \dot{A}_I^a \dot{A}_I^a $$

(5.11)

and the Gauss’s law in eq.(5.6)

$$ D_I \dot{A}_I = 0 $$

(5.12)

can be rewritten in a more compact form.

The BPS monopoles are static solutions of the eqs. of motion that minimize the potential energy in eq.(5.4). Any motion, corresponding to a dependence on the time, will give a contribution to the kinetic term increasing the energy of the solution. On the other hand when we construct a quantum theory of monopoles we must expand around the classical solutions by writing [31]:

$$ A_I(x) = A_I^{BPS}(x, z^a) + \Delta A_I(x) $$

(5.13)

where the variable $z^a$ labels the moduli space of monopoles with a specific monopole charge and $\Delta A_I$ is the quantum field. If, however, we restrict ourselves to motions with very small velocity and we start the motion along the flat directions of the potential energy i.e. in the moduli space of the static BPS monopoles, then the conservation of energy will keep the monopole close to this space pretty much in the same way as for a point-particle slowly moving along the flat directions of a potential. For very small velocity we can neglect oscillations along the transverse directions as
if we had an infinitely steep potential and limit ourselves only to a motion in the moduli space. In this approximation the expression in eq.(5.13) becomes:

\[ A_I(x, t) = A_I^{BPS}(x, z^\alpha(t)) \] (5.14)

and the dependence on the time is only through the collective coordinates that we assume to vary with time.

Since the quantity in eq.(5.14) is a solution of the BPS eq.(5.1) for any value of \( z^\alpha \), then the zero mode obtained by taking the derivative of (5.14) with respect to the collective coordinate for small values of \( z^\alpha \) satisfies the linearized Bogomolny equation. This quantity is, however, in general not orthogonal to the gauge transformation, i.e. it does not satisfy the Gauss’s law in eq.(5.12). For this reason, following Ref. [32], it is more convenient to start from a gauge rotated version of (5.14) given by

\[ A_I(x, t) = g^{-1}(x, t)A_I^{BPS}(x, z^\alpha(t))g(x, t) + \frac{1}{ie}g^{-1}(x, t)\partial_I g(x, t) \] (5.15)

and still keep \( A_0 = 0 \). This means that the transformation in eq.(5.15) is not a true gauge transformation because it acts only on the space components of the vector potential and on the Higgs field, but it leaves unchanged \( A_0 = 0 \). We are in the temporal gauge and, as explained in Ref. [32], we are not going out of it by considering time dependent gauge transformations. Eq.(5.15) should rather be understood as if we have a family of static ”gauge transformations” that are different from a time to another. This means that the function in the l.h.s of the previous equation does not only vary with the time through the time dependence of \( z^\alpha \), but also through the choice of a different ”gauge transformation”.

Let us now compute the electric field strenght in the gauge \( A_0 = 0 \):

\[ F_{0l} = \partial_0 A_l = g^{-1}z^\alpha \frac{\partial}{\partial z^\alpha}g + \frac{1}{ie} \partial_0 \left[ g^{-1}\partial_I g \right] + \left[ g^{-1}A_I^{BPS}g, g^{-1}\partial_0 g \right] \] (5.16)

Using the equation

\[ \partial_0 \left[ g^{-1}\partial_I g \right] = \partial_I \left[ g^{-1}\partial_0 g \right] + \left[ g^{-1}\partial_1 g, g^{-1}\partial_0 g \right] \] (5.17)

and remembering that the gauge functions are independent of \( z^\alpha \) we get

\[ F_{0l} = \dot{A}_I = z^\alpha \frac{\partial}{\partial z^\alpha}A_I(x, t) + \frac{1}{ie}D_I \left( g^{-1}\partial_0 g \right) \] (5.18)

Introducing the quantity

\[ \frac{1}{ie}g^{-1}\partial_0 g = -z^\alpha \epsilon_\alpha \] (5.19)

and defining

\[ \delta_\alpha A_I = \frac{\partial}{\partial z^\alpha}A_I(x, t) - D_I \epsilon_\alpha \] (5.20)
we can rewrite eq.(5.18) as follows

\[ \dot{A}_I = \dot{z}^\alpha \delta_\alpha A_I \]  

(5.21)

The parameter \( \epsilon \) and correspondently the gauge transformation \( g(x, t) \) is fixed by requiring that the quantity in eq.(5.21) satisfies the Gauss law:

\[ D_I \dot{A}_I = \dot{z}^\alpha D_I (\delta_\alpha A_I) = 0 \]  

(5.22)

An alternative but equivalent way of proceeding is to start instead from the ansatz:

\[ A_I(x, z^\alpha(t)) = A_{I}^{BPS}(x, z^\alpha(t)) \quad A_0 = \dot{z}^\alpha \epsilon_\alpha \]  

(5.23)

and compute the electric field strength obtaining

\[ F_{0I} = \partial_0 A_I - \partial_I A_0 + i e [A_0, A_I] = \dot{z}^\alpha \delta_\alpha A_I \]  

(5.24)

after the use of eq.(5.20). We want to stress that both eqs.(5.15) and (5.23) do not perform a true gauge transformation because they never transform simultaneously both \( A_I \) and \( A_0 \). In the first case only \( A_I \) is transformed, while in the second case only \( A_0 \).

Inserting the expression obtained in eq.(5.21) in the kinetic term (5.11) we get

\[ T = \frac{1}{2} G_{\alpha\beta} \dot{z}^\alpha \dot{z}^\beta \]  

(5.25)

The total Lagrangian is then the one found by Manton \[33\] and is given by:

\[ L = \frac{1}{2} G_{\alpha\beta} \dot{z}^\alpha \dot{z}^\beta - \frac{4\pi a}{e} K \]  

(5.26)

that describes precisely the motion of a free particle propagating in the moduli space of the Bogomolny solutions with magnetic charge \( K \) with metric:

\[ G_{\alpha\beta}(z) = \int d^3 x [\delta_\alpha A_i^\alpha \delta_\alpha A_i^\alpha + \delta_\alpha \Phi^\alpha \delta_\alpha \Phi^\alpha] = \int d^3 x \delta_\alpha A_i^\alpha \delta_\alpha A_i^\alpha \]  

(5.27)

Let us consider now the simplest case with \( K = 1 \) corresponding to the original \( \text{'t} \) Hooft-Polyakov solution. In this case there are four collective coordinates. Three of them are due to translational invariance, as we have already discussed, because the monopole can be centered around any point of the three-dimensional space. The time evolution of the position \( R \) is described by the Lagrangian of a free particle with mass equal to \( 4\pi a/e \). Therefore the kinetic part of the Lagrangian corresponding to these three collective coordinates is equal to:

\[ L = \frac{2\pi a}{e} R^2 \]  

(5.28)
The fourth one is more subtle and requires some discussion. We have seen that two field configurations that are related by a small gauge transformation (with gauge parameter $\epsilon$ that tends to zero at spatial infinity) are equivalent. But let us consider now the possibility of a gauge transformation whose parameter does not tend to zero at spatial infinity. As discussed above it should not be considered as a gauge transformation that connects equivalent field configurations, but rather as a symmetry transformation as in the case of the translational invariance considered above. In particular let us consider a time-dependent "gauge transformation" given by

$$A_I(t) = g^{-1} A_I^{BPS} g + \frac{1}{ie} g^{-1} \partial_I g \quad g = e^{i\chi(t)\Phi/a} \quad (5.29)$$

The dependence on time is in the parameter $\chi$ that satisfies the condition $\chi(0) = 0$. The previous transformation is a "large" gauge transformation because it does not go to the identity at spatial infinity since $\Phi^a \to a \frac{\Phi}{r}$ as $r \to \infty$. From eq.(5.29) we can easily compute

$$\dot{A}_i(t) = \frac{\dot{\chi}}{ea} D_i \Phi \quad \dot{A}_i(t) \equiv \dot{\Phi}(t) = 0 \quad (5.30)$$

Inserting the previous eqs. in the kinetic term in eq.(5.11) we get

$$T = \frac{\dot{\chi}^2}{2e^2a^2} \int d^3 x (D_i \Phi)^a B_i^a = \frac{\dot{\chi}^2}{2e^2a^2} \frac{4\pi a}{e} = \frac{2\pi}{e^3a} \chi^2 \quad (5.31)$$

where in the first step we have used the Bogomolny eq. and in the second one the fact that the three-dimensional integral gives the magnetic charge of the monopole (see eq.(4.39)).

In conclusion we get the following action for the motion in the monopole moduli space:

$$S = \frac{1}{2} \int dt \dot{z}^\alpha \dot{z}^\beta G_{\alpha\beta} \quad (5.32)$$

where

$$G_{\alpha\beta} = \frac{4\pi a}{e} \left( \begin{array}{cc} 1 & 0 \\ 0 & 1/(e^2a^2) \end{array} \right) \quad (5.33)$$

In eq.(5.32) we have neglected the second term in the r.h.s. of eq.(5.26) that is inessential being just a constant independent of $z^\alpha$. However, while the three components of $R$ are non compact variables, it turns out that $\chi$ is a compact variables varying in the interval $(0, 2\pi)$. In fact from the definition of the gauge transformation $g$ in eq.(5.24) it is easy to get:

$$g(\chi + 2\pi) = g(\chi)g(2\pi) \quad g(2\pi) = e^{2\pi i \Phi/a} \quad (5.34)$$

Since for $r \to \infty\ g(2\pi) \to 1$, $g(2\pi)$ is a "small" gauge transformation. This means that $g(\chi + 2\pi)$ and $g(\chi)$ are related by a "small" gauge transformation implying that $\chi$ is a compact variable:

$$\chi \sim \chi + 2\pi \quad (5.35)$$
The moduli space action in eq. (5.32) is then equal to the action of a free particle moving in the manifold $R^3 \otimes S^1$ with flat metric.

Let us define the conjugate momenta to the variables $R$ and $\chi$. They are given by:

$$P = \frac{4\pi a}{e} \dot{R}, \quad \Pi = \frac{4\pi a}{ae^3} \dot{\chi} \tag{5.36}$$

The Hamiltonian can be easily computed to be:

$$H = \frac{e}{8\pi a} P^2 + \frac{ae^3}{8\pi} \Pi^2 + \frac{4\pi a}{e} \tag{5.37}$$

Since $\chi$ is an angular variable the corresponding conjugate momentum $\Pi$ must be quantized: $\Pi = n_e \hbar$, where $n_e$ is an integer.

The states of minimum energy are obtained by putting $P = 0$. In this case the Hamiltonian becomes:

$$H = \frac{4\pi a}{e} \left[ 1 + \frac{e^4}{32\pi^2} (n_e \hbar)^2 \right] \sim a \sqrt{\left( \frac{4\pi}{e} \right)^2 + (n_e \hbar e)^2} = a \sqrt{g^2 + q^2} \tag{5.38}$$

where in the middle step we have used the fact that our calculations are valid for small $e^2$ and we have neglected higher orders in $e^2$ and in the last step we have used the expressions for the electric and magnetic charges of a dyon. Actually there is no doubt about the expression of the magnetic charge given in eq. (4.39). As far as the electric charge is concerned we have still to show that the electric charge of a dyon is proportional to $\Pi$. This can be seen by computing the electric field corresponding to the field configuration given in eq. (5.30). In the gauge $A_0 = 0$ we get:

$$E_i^a = F_i^a = -\frac{\dot{\chi}}{ea} (D_i \Phi)^a = -\frac{e^2}{4\pi} \Pi B_i^a \tag{5.39}$$

where we have used the Bogomolny eq. (5.1) and the explicit expression for $\Pi$ given in eq. (5.38) in terms of $\dot{\chi}$. From eq. (5.38) we can compute the electric charge:

$$-\int d^3x \partial_i (E_i^a \Phi^a / a) = \frac{e^2}{4\pi} \Pi \int d^3x \partial_i (B_i^a \Phi^a / a) = e\Pi \tag{5.40}$$

where in the last step we have used the explicit expression for the magnetic charge of a dyon. We have therefore shown that the electric charge of the dyon is proportional to $\Pi$ that is the conjugate momentum corresponding to a compact variable $\chi$. This means that, while at the classical level the charge of the dyon can assume any real value, at the quantum level instead the electric charge of a dyon is quantized as also the charge of the elementary $W^\pm$-bosons.

### 6 Instantons and Witten effect

We have seen that the monopole and more in general the dyon classical solutions of the eqs. of motion of the Georgi-Glashow model correspond to new particles of
the spectrum that are not described by any of the fields present in the Lagrangian.
In this section we want to quickly describe another kind of classical solution of the
eqs. of motion called instantons. While the monopole is a static classical solution
of the Minkowski theory, the instantons are classical solutions of the euclidean eqs.
of motion and do not correspond to new particles of the spectrum. They should,
instead, be interpreted as field configurations contributing as saddle points to the
euclidean functional integral. The simplest of these configuration was first found in
SU(2) Yang-Mills theory using an argument that is the generalization of the
Bogomolny one used earlier for the monopoles. One starts from the Yang-Mills
euclidean action:

\[ S_E = \frac{1}{4} \int d^4x F^a_{\mu \nu} F^{\mu \nu}_a \]  

(6.1)

By rewriting it as

\[ S_E = \frac{1}{8} \int d^4x \left\{ [F^a_{\mu \nu} \pm *F^a_{\mu \nu}]^2 \mp 2 F^a_{\mu \nu} * F^{\mu \nu}_a \right\} \]

(6.2)

one gets a lower bound for the action:

\[ S_E \geq \mp \frac{1}{4} \int d^4x F^a_{\mu \nu} * F^{\mu \nu}_a \]

(6.3)

The equality sign holds if

\[ F^a_{\mu \nu} \pm *F^a_{\mu \nu} = 0 \]  

(6.4)

By defining the topological charge

\[ Q = \int d^4x q(x) \quad q(x) = \frac{\epsilon^2}{32\pi^2} F^a_{\mu \nu} * F^{\mu \nu}_a \]  

(6.5)

that is an integer, since it is the topological number corresponding to the mapping of
the three-dimensional sphere into the gauge group \( SU(2) \), one can get the following
expression for the euclidean action if eq. (6.4) is satisfied

\[ S_E = \mp \frac{8\pi^2}{\epsilon^2} Q \]  

(6.6)

The simplest solution to eq. (6.4), corresponding to a value of the topological
charge \( Q = 1 \), was found in Ref. 34 and is given by:

\[ i e A_\mu = \frac{x^2}{x^2 + \lambda^2} g^{-1}(x) \partial_\mu g(x) \quad g(x) = \frac{\mathbb{1} x + i \vec{\sigma} \cdot \vec{x}}{\sqrt{x^2}} \]  

(6.7)

It can be shown that the topological charge density \( q(x) \) is a total derivative,
but, because of the existence of field configurations contributing to the functional
integral with integer non zero values of \( Q \), one can introduce in QCD or in any gauge
theory a so called \( \theta \)-term by adding to the gauge action \( S \) a term proportional to
the topological charge term:

\[ S \rightarrow S + \theta Q \]  

(6.8)
obtaining a theory that does not only depend on the gauge coupling constant $e$, but also on the parameter $\theta$. If $\theta \neq 0$ $CP$ is not a symmetry.

Since $Q$ is an integer the physics is periodic with period $2\pi$. This means that every physical quantity depends on $\theta$ through a function of $\theta$ that is periodic with period $2\pi$. Actually, because of the axial $U(1)$ anomaly, in the case of $QCD$ in presence of at least one massless quark flavour one can always cancel the dependence on $\theta$. But, if all quarks are massive, then there is an effective $\theta$ dependence. It turns out that experiments require a very small value for the $\theta$-angle in $QCD$. For a discussion of the physical consequences of the $\theta$-term in $QCD$ see Ref. [35].

In the following we will introduce a $\theta$-term in the Georgi-Glashow model studied in section (4) and, as a consequence of it, we will show that the electric charge of a dyon gets an extra contribution. This effect of the $\theta$-term is called Witten effect [36].

Let us consider a gauge rotation with a small angle $\varphi$ around the direction of the gauge field $\Phi^a$ with gauge parameter $\Lambda^a = \Phi^a/a$, where $\Phi^a$ is the Higgs field in the monopole background. At spatial infinity this is a gauge transformation corresponding to the unbroken $U(1)$. Its generator corresponds to the $U(1)$ electric charge defined in eq.(4.15). Under this transformation the Higgs field is, of course, left invariant while the vector potential gets transformed as follows:

$$\delta A^a_\mu = -\frac{\varphi}{ea}(D_\mu \Phi)^a$$ (6.9)

The generator of this transformation is obtained from the Lagrangian of the Georgi-Glashow model in eq.(4.1) with the addition of the $\theta$ term:

$$\mathcal{L}_{GG} \rightarrow \mathcal{L}_{GG} + \theta Q$$ (6.10)

and is given by:

$$N = \int d^3x \left( \frac{\delta \mathcal{L}}{\delta (\partial_\mu A^\mu_i)} \frac{\delta A^a_i}{\varphi} \right)$$ (6.11)

Using the eqs.

$$\frac{\delta F^2}{\delta (\partial_\mu A^\mu_i)} = -4E^i_a \quad \frac{\delta (F^*F)}{\delta (\partial_\mu A^\mu_i)} = -4B^i_a$$ (6.12)

one gets:

$$N = -\frac{1}{ea} \int d^3x \left\{ E^i_a(D_i \Phi)^a - \frac{\theta e^2}{8\pi^2} B^i_a(D_i \Phi)^a \right\}$$ (6.13)

Remembering the definition of the electric and magnetic charges in eq.(4.53) and using eq.(4.53) we get

$$N = -\frac{1}{e} \left( q - \frac{\theta e^2}{8\pi^2}g \right)$$ (6.14)

Since $Q_e/e$ is an integer, as follows from eq.(4.13), the finite $U(1)$ transformation generated by

$$e^{2\pi i Q_e/e} = 1$$ (6.15)
must be equal to 1. This implies that the same finite transformation generated by \( N \) must also be equal to 1 and therefore also \( N \) must be an integer:

\[
e^{2\pi i N} = 1 \implies N = -\frac{1}{e} \left( q - \frac{\theta e}{2\pi} n_m \right) = -n_e
\]

(6.16)

where \( n_e \) is an integer and we have used the Dirac quantization condition \( e g = 4\pi n_m \). We have not restricted ourselves to a monopole with topological charge \( K = 1 \), but we have allowed for any value of \( K = n_m \).

From eq. (6.16) we get finally

\[
q = n_e e + \frac{\theta e}{2\pi} n_m
\]

(6.17)

showing that, in absence of the \( \theta \)-term the electric charge is quantized in agreement with what obtained in eq. (5.38), while, in presence of a \( \theta \)-term, one gets an extra term proportional to \( \theta \) and to the magnetic charge of the dyon.

Since \( \theta \sim \theta + 2\pi n \) eq. (6.17) implies that, if a dyon with a certain value of the electric charge \( n_e \) exists, then dyons with any integer value must exist. In conclusion the electric charge of the dyon is not only quantized, but dyons with any integer value \( n_e \) of the electric charge must exist in the spectrum. This is an important consequence of \( \theta \) periodicity. Notice also that the electric charges given by formula (6.17) and the magnetic charges \( g = \frac{4\pi}{2\pi} n_m \) satisfy the DSZ quantization condition (6.17). Viceversa it can also be shown [12] that, if we require the DSZ quantization condition, the electric and magnetic charges of dyons must lie on a two-dimensional lattice given by:

\[
q + ig = q_0 (n_e + n_m \tau) \quad \quad \tau = \frac{\theta}{2\pi} + i \frac{2\pi hn_0}{q_0^2}
\]

(6.18)

with lattice periods equal to \( q_0 \) and \( q_0 \tau \). In the case of the ’t Hooft-Polyakov monopole we have \( n_0 = 2 \) as in any theory containing no field transforming according to the fundamental representation of the gauge group.

In the last part of this section we rewrite the Georgi-Glashow model with zero potential and with the addition of the \( \theta \) term in a more compact way in terms of gauge and Higgs fields normalized in such a way to include the gauge coupling constant in them. This is the formalism that is currently used in the Seiberg-Witten approach. In terms of the rescaled fields:

\[
A_\mu \rightarrow e A_\mu \quad \quad \Phi \rightarrow e \Phi
\]

(6.19)

and of the quantity

\[
\tau = \frac{\theta}{2\pi} + i \frac{4\pi}{e^2}
\]

(6.20)

one can rewrite the Lagrangian of the Georgi-Glashow model with the \( \theta \) term as follows:

\[
\mathcal{L} = -\frac{1}{16\pi} Im \left[ \tau \left( F_{a\mu}^a F_a^{\mu\nu} - i F_{a\mu}^a * F_a^{\mu\nu} \right) \right] + \frac{1}{2e^2} (D_\mu \Phi)^2
\]

(6.21)
The topological charge in eq. (6.7) becomes in this formalism:

\[ q(x) = \frac{1}{32\pi^2} F^a_{\mu\nu} \ast F^\mu
\nu_a \]  

(6.22)

If we call again \( a \) the vacuum expectation value of the rescaled field \( \Phi \) defined in eq. (6.19) we have to rewrite the mass formula in eq. (4.60) as follows:

\[ M = \frac{a}{e} |q + ig| = a |n_e + \tau n_m| \]  

(6.23)

7 Montonen-Olive duality

Leaving aside for a moment the dyon solution discussed in the previous sections we have found that the semiclassical spectrum of the Georgi-Glashow model in the BPS limit consists of two neutral particles, a massless photon and a massless Higgs particle, of an electrically charged \( W \) boson with charge equal to \( q_0 = \pm e\hbar \) and of a magnetic monopole with magnetic charge equal to \( g_0 = \pm \frac{4\pi}{e} = \frac{4\pi \hbar}{\sqrt{\varepsilon}} \).

If there is duality invariance as suggested for instance by the formula in eq. (6.23) we can make a duality transformation with angle \( \phi = -\frac{\pi}{2} \) such that

\[ q_0 \to g_0 \quad g_0 \to -q_0 \]  

(7.1)

This transformation implies that

\[ q_0 \to \frac{4\pi \hbar}{g_0} \]  

(7.2)

Based on this observation Montonen and Olive [8] suggested that there are two equivalent formulations of the same theory dual to each other. In the first one, that we call electric, the \( W \)'s are elementary particles while the magnetic monopoles are solitons. In the second one, that we call magnetic, the elementary particles are instead the magnetic monopoles while the \( W \) bosons are solitons. They also suggested that the two formulations had essentially the same Lagrangian. The only important difference between them is that the electric theory is weakly coupled when \( q_0 \to 0 \) \( (e \to 0) \) while the magnetic theory is weakly coupled when \( g_0 \to 0 \) corresponding to \( e \to \infty \). They brought the following arguments in support of their duality conjecture.

1. The mass formula in eq. (4.60), valid for all particles of the theory, is duality invariant.

2. Since there is no interaction between two monopoles, while there is a non zero interaction between a monopole and an antimonopole, if duality is correct, one must expect that the interaction between equal charge \( W \)-bosons must be zero while that between opposite charged \( W \)-bosons must be non vanishing.
This is actually verified in the BPS limit because in this limit the Higgs field is also massless and contributes with opposite sign with respect to the photon for equal charge $W$, while it contributes with the same sign for opposite charge $W$.

The Montonen-Olive duality proposal, leaves, however some unanswered questions that we list:

1. The elementary $W^\pm$ bosons have spin equal to 1. If the magnetic monopoles are dual to them they must also have spin equal to 1. But how can this happen?

2. The previous considerations are based on a mass formula that is only valid classically. How are the quantum corrections going to modify it?

3. In the previous considerations we have neglected the dyons. What is their role in the all picture?

The previous questions do not have an answer in the framework of the Georgi-Glashow model discussed in the previous section since the quantum Georgi-Glashow model is, actually, not duality invariant. But it was soon recognized that, in order to have a theory with Montonen-Olive duality, one must include supersymmetry since in a supersymmetric theory the quantum corrections coming from the bosons and the fermions tend to cancel each others preserving the structure of the classical mass formula and thus solving the second problem above. Actually the argument used in Ref. for the $N = 2$ super Yang-Mills theory is too naive and in fact wrong as pointed out in Refs. because this theory is not ultraviolet finite. In order to have a classical mass formula that is not modified by quantum corrections one must consider the $N = 4$ super Yang-Mills theory that is free from ultraviolet divergences, as it was done by Osborn who made also the important observation that in this case magnetic monopoles and dyons have also supersymmetric partners with spin equal to 1. The introduction of the $N = 4$ theory opens the way to the solution of the first two puzzles discussed above. In the meantime Witten and Olive found out that the structure of the duality invariant mass formula in eq. for a BPS state in the $N = 2$ theory is a direct consequence of the supersymmetry algebra opening the way to the quantum exact determination of the mass of the BPS states. This observation is playing an essential role also in recent developments in string theories.

In conclusion it seems that the Montone-Olive duality can be realized in a four-dimensional gauge field theory provided that the theory is supersymmetric. Therefore in the next section we discuss the representations of supersymmetry algebra and in sections and we construct the Lagrangians of supersymmetric gauge theories in four dimensions.
8 Representations of supersymmetry algebra

As in the case of the Poincaré group the representations of the supersymmetry algebra for massive particles are different from those for massless particles. The supersymmetry algebra is given in both cases by:

\[
\{Q^i_\alpha, \bar{Q}^j_\dot{\alpha}\} = 2(\sigma^\mu)_{\alpha\dot{\alpha}} P^\mu \delta^{ij} \quad i, j = 1 \ldots N \quad (8.1)
\]

\[
\{Q^i_\alpha, Q^j_\beta\} = \{\bar{Q}^i_\dot{\alpha}, \bar{Q}^j_\dot{\beta}\} = 0 \quad (8.2)
\]

The difference between the two cases is due to the fact that in the massive case one can always choose a center of mass frame where \(P_\mu = (M, \vec{0})\), while this is not possible in the massless case.

In the massive case in the center of mass frame one gets the following algebra:

\[
\{a^i_\alpha, (a^j_\beta)^\dagger\} = \delta_{\alpha\beta} \delta^{ij} \quad (8.3)
\]

and

\[
\{a^i_\alpha, a^j_\beta\} = \{(a^i_\alpha)^\dagger, (a^j_\beta)^\dagger\} = 0 \quad (8.4)
\]

where

\[
a^i_\alpha = \frac{1}{\sqrt{2M}} Q^i_\alpha \quad (a^j_\beta)^\dagger = \frac{1}{\sqrt{2M}} \bar{Q}^j_\dot{\beta} \quad (8.5)
\]

The representation of the fermionic harmonic oscillator algebra is constructed starting from a vacuum state \(|0\rangle\) satisfying the equation:

\[
a^i_\alpha |0\rangle = 0 \quad (8.6)
\]

and acting on it with the creation operators:

\[
\frac{1}{\sqrt{n!}} (a^i_{\alpha_1})^\dagger (a^i_{\alpha_2})^\dagger \ldots (a^i_{\alpha_n})^\dagger |0\rangle \quad n = 0, 1 \ldots 2N \quad (8.7)
\]

The number of states in eq. (8.7) is equal to \(\binom{2N}{n}\). Since \(n\) runs from 0 to 2\(N\) the total number of states in the representation of the massive supersymmetry algebra is equal to:

\[
\sum_{n=0}^{2N} \binom{2N}{n} = 2^{2N} \quad (8.8)
\]

The states in the representation have a maximum helicity gap \(\Delta \lambda = N\). Half of them are fermions and the other half are bosons.

In the massless case we can instead choose a frame where \(P_\mu = (E, 0, 0, -E)\). In this frame the supersymmetry algebra becomes:

\[
\{a^i, (a^j)^\dagger\} = \delta^{ij} \quad (8.9)
\]
\[ \{a^i, a^j\} = \{(a^i)^\dagger, (a^j)^\dagger\} = 0 \quad (8.10) \]

where
\[ a^i = \frac{1}{2\sqrt{E}} Q^i_1 \quad (a^j)^\dagger = \frac{1}{2\sqrt{E}} \bar{Q}^j_1 \quad (8.11) \]

The anticommutators involving the generators of the supersymmetry algebra with indices \( \alpha = 2 \) and \( \dot{\alpha} = \dot{2} \) are all vanishing and therefore they can be consistently put equal to zero:
\[ Q^i_2 = \bar{Q}^i_2 = 0 \quad (8.12) \]

Starting again from the vacuum state annihilated by the annihilation operators \( a^i \) we can construct the states of the representation acting on it with the creation operators obtaining the state:
\[ \frac{1}{\sqrt{n!}} (a^{i_1})^\dagger (a^{i_2})^\dagger \ldots (a^{i_n})^\dagger |0> \quad n = 0, 1 \ldots N \quad (8.13) \]

that contains \( \binom{N}{n} \) states. The total number of states in the massless representation is equal to:
\[ \sum_{n=1}^N \binom{N}{n} = 2^N \quad (8.14) \]

that is smaller than in the case of a massive representation. The maximum helicity in this case is \( \Delta \lambda = N/2 \). In the case \( N = 1 \) one gets only one fermionic and one bosonic state. In most cases, however, we must add another multiplet with opposite helicity in order to have a CPT invariant theory (CPT reverses the sign of helicity).

Let us finally consider the representation of the massive \( N = 2 \) algebra with non vanishing central charges \([47]\). In this case in the center of mass frame the algebra is
\[ \{Q^i_\alpha, \bar{Q}^j_\dot{\alpha}\} = 2M \delta^{ij} \delta_{\alpha\dot{\alpha}} \quad (8.15) \]
\[ \{Q^i_\alpha, Q^j_\beta\} = \epsilon^{ij} \epsilon_{\alpha\beta} \hat{Z} \quad \{\bar{Q}^i_\dot{\alpha}, \bar{Q}^j_\dot{\beta}\} = \epsilon^{ij} \epsilon_{\dot{\alpha}\dot{\beta}} \bar{\hat{Z}} \quad (8.16) \]

One can get rid of the phase in \( \hat{Z} \) by a supercharge redefinition and can rewrite the previous algebra in terms of the two quantities:
\[ a_\alpha = \frac{1}{\sqrt{2}} \left[ Q^i_\alpha + \epsilon_{\alpha\beta} \bar{Q}^3_\beta \right] \quad b_\alpha = \frac{1}{\sqrt{2}} \left[ Q^i_\alpha - \epsilon_{\alpha\beta} \bar{Q}^3_\beta \right] \quad (8.17) \]

obtaining
\[ \{a_\alpha, (a_\beta)^\dagger\} = (2M + |\hat{Z}|) \delta_{\alpha\beta} \quad \{b_\alpha, (b_\beta)^\dagger\} = (2M - |\hat{Z}|) \delta_{\alpha\beta} \quad (8.18) \]

while all the other anticommutators are vanishing.

If \( 2M = |\hat{Z}| \) all anticommutators involving the oscillators \( b \) are vanishing and therefore we can put them equal to zero. We can then use only the oscillators \( a \)
for constructing the representation, obtaining the same number of states as in the massless case. In the case $N = 2$ here considered we get the following four states:

$$
|0> \quad a^\dagger_\alpha|0> \quad a^\dagger_\alpha a^\dagger_\beta|0>
$$

(8.19)

instead of the 16 states that we found in the case without central charge (see eq. (8.8) for $N = 2$).

Extending the previous procedure to the case $N = 4$ we obtain a short representation with 16 states instead of the one with $2^8 = 256$ states obtained without central charge (See eq. (8.3) for $N = 4$).

The fact that the representations of extended supersymmetry with non vanishing central charges are shorter and have the same dimension of those for the massless case makes it possible to have a consistent supersymmetric Higgs mechanism since in this case one has the same number of degrees of freedom before and after the Higgs mechanism.

9 Supersymmetric Yang-Mills actions

In this section we construct the supersymmetric extension of Yang-Mills theory in $D = 4$ by dimensional reduction from higher dimensions [48].

Let me start from the following action in $D$ dimensions

$$
S = \int d^Dx \left\{-\frac{1}{4}F_{MN}^a F^{aMN} - \frac{i}{2}\bar{\lambda}^a \Gamma_M (D^M \lambda)^a \right\}
$$

(9.1)

If we perform the following supersymmetry transformation

$$
\delta A^a_M = \frac{i}{2} \left[\bar{\lambda}^a \Gamma_M \alpha - \bar{\alpha} \Gamma_M \lambda^a\right]
$$

(9.2)

together with

$$
\delta \lambda_a = \sigma_{RS} F_{a}^{RS} \alpha \quad \delta \bar{\lambda}_a = -\bar{\alpha} \sigma_{RS} F_{a}^{RS} \quad \sigma_{RS} = \frac{1}{4} [\Gamma_R, \Gamma_S]
$$

(9.3)

it can be seen, by using the useful identities,

$$
\Gamma^M \sigma^{RS} = \frac{1}{2} \left[ g^{MR} \Gamma^S - g^{MS} \Gamma^R - \frac{(-1)^{D/2}}{(D-3)!} \epsilon^{MRSN_1 \ldots N_{D-3}} \Gamma_{D+1} \Gamma_{N_1} \ldots \Gamma_{N_{D-3}} \right]
$$

(9.4)

and

$$
\sigma^{RS} \Gamma^M = \frac{1}{2} \left[ -g^{MR} \Gamma^S + g^{MS} \Gamma^R - \frac{(-1)^{D/2}}{(D-3)!} \epsilon^{MRSN_1 \ldots N_{D-3}} \Gamma_{D+1} \Gamma_{N_1} \ldots \Gamma_{N_{D-3}} \right]
$$

(9.5)
that the term with the $\epsilon$ tensors cancel using the Bianchi identity for $F_{\mu\nu}$, while the other terms are equal to

$$\delta S = \int \! d^D x \left\{ \frac{i}{2} \partial_M \left( \bar{\alpha} \sigma_{RS} F_{a}^{R S} \Gamma^{M} \lambda^{a} \right) + \frac{i}{2} F_{a}^{MN} \left[ \bar{\alpha} \gamma^{N} (D^{M} \lambda)^{a} - (D^{M} \bar{\lambda})^{a} \Gamma^{N} \alpha \right] \right. \\
\left. - \frac{i}{2} \bar{\alpha} (D^{M} F_{RM})^{a} \Gamma^{R} \lambda^{a} - \frac{i}{2} \bar{\lambda} (D^{M} F_{MN})^{a} \Gamma^{N} \alpha + \frac{i}{2} \bar{\lambda} (D^{M} F_{a}^{MN})^{a} (D^{R} \lambda^{a} - \bar{\lambda}^{a} \Gamma^{R} \alpha) \Gamma^{M} \alpha \right\} \quad (9.6)$$

From this eq. it follows that the action in eq. (9.1) transforms as a total derivative

$$\delta S = \frac{i}{2} \int \! d^D x \partial_M \left[ \bar{\alpha} \sigma_{RS} F_{a}^{R S} \Gamma^{M} \lambda^{a} + F_{a}^{MN} \left( \bar{\alpha} \Gamma^{N} \lambda^{a} - \bar{\lambda}^{a} \Gamma^{N} \alpha \right) \right] \quad (9.7)$$

provided that the last term in eq. (9.6) is vanishing

$$\left( \bar{\lambda}^{a} \Gamma^{M} f^{abc} \lambda^{c} \right) \left[ \bar{\alpha} \Gamma^{M} \lambda^{b} - \bar{\lambda}^{b} \Gamma^{M} \alpha \right] = 0 \quad (9.8)$$

$\alpha$ and $\lambda$ are spinors in $D$-dimensions, $\Gamma_{D+1} = \Gamma_0 \ldots \Gamma_{D-1}$ and

$$F_{a}^{MN} = \partial_{M} A_{N}^{a} - \partial_{N} A_{M}^{a} - e f^{abc} A_{M}^{b} A_{N}^{c} \quad (D_{M} \lambda)^{a} = \partial_{M} \lambda^{a} - e f^{abc} A_{M}^{b} \lambda^{c} \quad (9.9)$$

Therefore the action in eq. (9.1) is $N = 1$ supersymmetric if the equation (9.8) is satisfied. As shown in Ref. [48] this happens in the following cases:

1. $D=3$, if $\lambda$ is a Majorana spinor
2. $D=4$, if $\lambda$ is a Majorana spinor
3. $D=6$, if $\lambda$ is a a Weyl spinor
4. $D=10$, if $\lambda$ is a Weyl-Majorana spinor

There is a simple way to understand this result by noticing that, in all these cases, the number of on shell bosonic degrees of freedom, that is equal to $D - 2$, is equal to the number of on shell fermionic degrees of freedom that is equal to $2^{[D/2]}$ multiplied with a factor $x = \frac{1}{2}$ if the spinor field $\lambda$ is a Majorana or Weyl spinor and a factor $x = \frac{1}{4}$ if the spinor field is a Weyl-Majorana spinor:

$$D - 2 = x \cdot 2^{[D/2]} \quad (9.10)$$

where $[D/2] = \frac{D}{2}$ if $D$ is even and $[D/2] = \frac{D-1}{2}$ if $D$ is odd.

As a consequence of the invariance under supersymmetry one can construct a supercurrent

$$J_{a}^{M} = \sigma_{RS} F_{a}^{R S} \Gamma^{M} \lambda^{a} \quad (9.11)$$

that is conserved if the eqs. of motion are satisfied.

In particular the action in eq. (9.1) is $N = 1$ supersymmetric if $D = 6$ and 10. This fact can be used to write actions with extended $N = 2, 4$ supersymmetries
in four dimensions by the technique of dimensional reduction. Let us divide the $D$ dimensional space-time component $x^M \equiv (x^\mu, x^i)$ in a part $x^\mu$, where the index $\mu$ runs over the four-dimensional space-time, and in part $x^i$, where the index $i$ runs over the compactified $D-4$ dimensions. We assume that the various fields are independent from the compactified coordinates.

Let us start to compactify the bosonic term in the action (9.1) containing the non abelian field strenght given in eq. (5.20). The dimensional reduction of $F_{MN}$ gives respectively:

$$F^a_{\mu \nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - e f^{abc} A^b_\mu A^c_\nu \quad (9.12)$$
$$F^a_{\mu i} = \partial_\mu A^a_i - e f^{abc} A^b_\mu A^c_i \equiv (D_\mu A_i)^a \quad (9.13)$$

and

$$F^a_{ij} = - e f^{abc} A^b_i A^c_j \quad (9.14)$$

Using the previous equations one obtains immediately the compactification of the gauge kinetic term

$$-\frac{1}{4} F^a_{MN} F^{a MN} = -\frac{1}{4} F^a_{\mu \nu} F^{a \mu \nu} + \frac{1}{2} (D_\mu A_i)^a (D^\mu A_i)^a - \frac{e^2}{4} f^{abc} A^b_i A^c_i f^{ade} A^d_i A^e_i \quad (9.15)$$

where a sum over repeated indices is understood.

In order to perform the compactification of the fermionic term of the action in eq. (9.1) we have to distinguish the two cases $D=6$ and $D=10$.

A representation of the Dirac algebra for $D=6$ is given by:

$$\Gamma_\mu = \gamma_\mu \otimes 1 \quad \mu = 0, 1, 2, 3 \quad (9.16)$$
$$\Gamma_4 = \gamma_5 \otimes i \sigma_1 \quad \Gamma_5 = \gamma_5 \otimes i \sigma_2 \quad \Gamma_7 \equiv \Gamma_0 \Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4 \Gamma_5 = \gamma_5 \otimes \sigma_3 \quad (9.17)$$

where the $\sigma$-matrices are the Pauli matrices and $\gamma_5 = i \gamma^0 \gamma^1 \gamma^2 \gamma^3$.

A Weyl spinor in $D=6$ satisfies the condition:

$$(1 + \Gamma_7) \lambda = 0 \quad (9.18)$$

that is automatically satisfied if we take

$$\lambda = \left( \frac{1-\gamma_5}{2} \chi \right) \quad \bar{\lambda} = \left( \bar{\chi} \left( \frac{1-\gamma_5}{2} \right) \right) \quad (9.19)$$

where $\chi$ is a Dirac spinor in four dimensions.

Inserting it in the fermionic term in eq. (9.1) one gets:

$$i \bar{\lambda}^a \Gamma^M D_M \lambda^a = i \bar{\chi}^a \Gamma^\mu (D_\mu \chi)^a - e f^{abc} \bar{\chi}^a A^b_5 \gamma^c \chi + i e f^{abc} \bar{\chi}^a A^b_i \gamma^c \quad (9.20)$$

The Lagrangian of $N=2$ super Yang-Mills is obtained by summing the bosonic contribution in eq. (9.15) with the indices $i,j = 1,2$ to the fermionic contribution in eq. (9.20). One gets

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu \nu} F^{a \mu \nu} + \frac{1}{2} \sum_{i=1}^2 (D_\mu A_i)^a (D^\mu A_i)^a - \frac{e^2}{2} f^{abc} A^b_i A^c_i f^{ade} A^d_i A^e_i + \ldots$$

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\[ -i (\chi)^a \gamma^\mu (D_\mu \chi)^a + e f^{abc} \tilde{\chi}_a A^b_4 \gamma^c \chi^c - i e f^{abc} \chi_a A^b_5 \chi^c \]  \hspace{1cm} (9.21)

after a redefinition of the Dirac spinor \( \chi \rightarrow \sqrt{2} \chi \).

The \( N = 4 \) super Yang-Mills is instead obtained starting with a Weyl-Majorana spinor in \( D = 10 \). In \( D = 10 \) the Dirac algebra can be represented as follows:

\[
\Gamma^\mu = \gamma^\mu \otimes 1 \otimes \sigma_3 \quad \mu = 0, 1, 2, 3
\]

\[
\Gamma^{3+i} = 1 \otimes \alpha^i \otimes \sigma_1 \quad \Gamma^{6+i} = \gamma_5 \otimes \beta^i \otimes \sigma_3 \quad i = 1, 2, 3
\]  \hspace{1cm} (9.22)

where the four-dimensional internal matrices \( \alpha \) and \( \beta \) satisfy the following algebra:

\[
\{ \alpha^i, \alpha^j \} = \{ \beta^i, \beta^j \} = -2 \delta^{ij} \quad [\alpha^i, \beta^j] = 0
\]  \hspace{1cm} (9.23)

and

\[
[\alpha^i, \alpha^j] = -2 \epsilon^{ijk} \alpha^k \quad [\beta^i, \beta^j] = -2 \epsilon^{ijk} \beta^k
\]  \hspace{1cm} (9.24)

Finally the correspondent of \( \gamma_5 \) in ten dimensions is given by:

\[
\Gamma_{11} = \Gamma_0 \Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4 \Gamma_5 \Gamma_6 \Gamma_7 \Gamma_8 \Gamma_9 = 1 \otimes 1 \otimes \sigma_2
\]  \hspace{1cm} (9.25)

A Weyl-Majorana spinor satisfying the condition:

\[
(1 + \Gamma_{11}) \lambda = 0
\]  \hspace{1cm} (9.26)

can always be written as

\[
\lambda = \psi \otimes \frac{1}{\sqrt{2}} \left( \begin{array}{c} 1 \\ -i \end{array} \right) \quad \bar{\lambda} = \bar{\psi} \frac{1}{\sqrt{2}} \left( \begin{array}{c} 1 \\ -i \end{array} \right)
\]  \hspace{1cm} (9.27)

where the Majorana spinor \( \psi \) has a four dimensional space-time index on which the Dirac matrices act and another internal four dimensional index on which instead the internal matrices \( \alpha \) and \( \beta \) act.

Proceeding as in the \( N = 2 \) case we arrive at the \( N = 4 \) super Yang-Mills Lagrangian:

\[
L = -\frac{1}{4} F^a_{\mu\nu} F^{a,\mu\nu} + \frac{1}{2} \sum_{i=1}^3 (D_\mu A_i)^a (D^\mu A_i)^a + \frac{1}{2} \sum_{i=1}^3 (D_\mu B_i)^a (D^\mu B_i)^a - V(A_i, B_i) + \\
- \frac{i}{2} \left( \bar{\psi} \right)^a \gamma^\mu (D_\mu \psi)^a - \frac{\epsilon}{2} f^{abc} \bar{\psi}^a \alpha^i A^b_i \psi^c - \frac{i \epsilon}{2} f^{abc} \bar{\psi}^a \beta^i \gamma_5 B^b_i \psi^c
\]  \hspace{1cm} (9.28)

where the potential is equal to:

\[
V(A_i, B_j) = \frac{\epsilon^2}{4} f^{abc} A^b_i A^c_j f^{afg} A^f_i A^g_j + \frac{\epsilon^2}{4} f^{abc} B^b_i B^c_j f^{afg} B^f_i B^g_j + \frac{\epsilon^2}{2} f^{abc} A^b_i B^c_j f^{afg} A^f_i B^g_j
\]  \hspace{1cm} (9.29)

where

\[
36
\]
10 Supersymmetric gauge theories for D=4

In this section we start rewriting the Lagrangians for super Yang-Mills theories in four dimensions using a $N = 1$ superfield formalism that we briefly review in Appendix B in order to fix the notations. We then extend them by also including the matter superfields.

As one can see in Appendix B the most general $N = 1$ renormalizable supersymmetric gauge theory involves two kinds of superfields: a number of the chiral superfields $\Phi_i$ that describe the matter and the vector superfield $V$ that describes the gauge part of the action. In terms of those superfields the most general renormalizable supersymmetric gauge theory contains essentially three kinds of terms: a so-called $F$ term corresponding to the last component of a chiral superfield describing the super Yang-Mills part of the Lagrangian, a so-called $D$ term corresponding to the last component of a real superfield describing the kinetic term of the matter together with its interaction with the gauge field and the gaugino and another $F$ term corresponding to the superpotential that must be at most cubic in the matter fields. In conclusion the most general $N = 1$ supersymmetric gauge theory is described by the following Lagrangian by:

$$L = \int d^2 \theta d^2 \bar{\theta} \sum_i \bar{\Phi}_i e^{2V} \Phi_i + \int \left\{ d^2 \theta \left[ -\frac{1}{4} W^{\alpha} W_\alpha + W(\Phi_i) \right] + h.c. \right\}$$

(10.1)

Let us rewrite now the Lagrangians of the various supersymmetric theories with the $N = 1$ superfield formalism. Let us start from those for the pure super Yang-Mills theories. They are the following:

1. N = 1 super Yang-Mills

This theory involves only the superfield strenght $W^a_{\alpha}$ of a vector superfield $V^a$ and its Lagrangian is given by

$$L = -\frac{i}{16\pi} \int d^2 \theta \tau W^\alpha W_\alpha + h.c. = \frac{1}{8\pi} Im \left[ \tau \int d^2 \theta W^\alpha W_\alpha \right]$$

(10.2)

In terms of component fields we get

$$L = -\frac{1}{4\epsilon^2} F_{\mu \nu}^a F_{\mu \nu}^a - \frac{i}{\epsilon^2} \bar{\lambda}^a \sigma^\mu (D_\mu \lambda)^a + \frac{1}{2\epsilon^2} D^2 + \frac{\theta}{32\pi^2} F_{\mu \nu}^a F_{\mu \nu}^a$$

(10.3)

2. N = 2 super Yang-Mills

The Lagrangian of $N = 2$ super Yang-Mills contains together with the superfield strenght $W^a_{\alpha}$ of a vector superfield $V^a$, that is already present in the $N = 1$ theory, also a chiral superfield $\Phi^a$ transforming according to the adjoint representation of the gauge group. The Lagrangian of this theory is given by:

$$L = \frac{1}{\epsilon^2} \int d^2 \theta d^2 \bar{\theta} \Phi e^{2V} \Phi - \left[ \frac{i}{16\pi} \int d^2 \theta \tau W^\alpha W_\alpha + h.c. \right]$$

(10.4)
Since the first term in eq. (10.4) is real we can rewrite eq. (10.4) as follows:

$$\mathcal{L} = \frac{1}{4\pi} \text{Im} \left\{ \tau \left[ \frac{1}{2} \int d^2 \theta W^a W_a + \int d^2 \theta d^2 \bar{\theta} \bar{\Phi} e^{2V} \Phi \right] \right\}$$  \hspace{1cm} (10.5)$$

where $\tau$ is given in eq. (6.20). By expanding the superfield $\Phi$ in terms of the component fields:

$$\Phi = \phi + \sqrt{2} \theta \psi + \theta^2 F$$  \hspace{1cm} (10.6)$$

and rewriting the Lagrangian in eq. (10.5) in terms of Dirac spinors we get the same Lagrangian as in eq. (9.21) that we rewrite with all fields rescaled by the gauge coupling constant $e$ and with the addition of the $\theta$ term

$$e^2 \mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a}_{\mu\nu} + \left( D_{\mu} \phi \right)^a \left( D^a \phi \right)^a + \frac{1}{2} \left( f^{abc} \bar{\phi} \phi \right) \left[ \bar{\phi} \phi \right] + \frac{32\pi^2}{\theta e^2} F^a_{\mu\nu} F^{a}_{\mu\nu} +$$

$$-i \chi^a \gamma^\mu \left( D_{\mu} \chi \right)^a - i \sqrt{2} f^{abc} \left[ \bar{\chi}^a \left( \frac{1}{2} + \gamma_5 \right) \chi^b \right] + \frac{1}{2} \left[ \left( \frac{1}{2} - \gamma_5 \right) \bar{\phi} \phi \right] \right]$$  \hspace{1cm} (10.7)$$

provided that we make the following identifications:

$$\phi = \frac{A_5 + i A_4}{\sqrt{2}} \hspace{1cm} \chi = \left( \begin{array}{c} \psi_a \\ -i \bar{\chi}_a \end{array} \right) \hspace{1cm} \bar{\chi} = \left( \begin{array}{c} i \lambda^a \\ \bar{\psi}_a \end{array} \right)$$  \hspace{1cm} (10.8)$$

3. N = 4 super Yang-Mills

The Lagrangian of $N = 4$ super Yang-Mills contains, in addition to the vector superfield $V$ as before, also three chiral superfields $\Phi_i$ transforming according to the adjoint representation of the gauge group. It is given by:

$$\mathcal{L} = \frac{1}{e^2} \int d^2 \theta d^2 \bar{\theta} \sum_{i=1}^{3} \Phi_i e^{2V} \Phi_i + \frac{1}{8\pi} \text{Im} \left[ \int d^2 \theta \tau W^a W_a \right]$$

$$- \left[ \int d^2 \theta \sqrt{2} \Phi_1 \Phi_2 \Phi_3 + h.c. \right]$$  \hspace{1cm} (10.9)$$

If we introduce also matter superfields $Q$ and $\bar{Q}$ we can write the supersymmetric versions of QCD that are described by the following two Lagrangians.

1. N = 1 super QCD

$$\mathcal{L} = \int d^2 \theta d^2 \bar{\theta} \left[ \bar{Q} e^{2V} Q + \bar{\bar{Q}} e^{-2V} \bar{Q} \right] +$$

$$+ \frac{1}{8\pi} \text{Im} \left[ \int d^2 \theta \tau W^a W_a \right] + \left[ \int d^2 \theta m_f \bar{Q}_f Q_f + h.c. \right]$$  \hspace{1cm} (10.10)$$

2. N = 2 super QCD

$$\mathcal{L} = \int d^2 \theta d^2 \bar{\theta} \left[ \bar{Q} e^{2V} Q + \bar{\bar{Q}} e^{-2V} \bar{Q} \right] + \frac{1}{e^2} \int d^2 \theta d^2 \bar{\theta} \bar{\Phi} e^{2V} \Phi$$

$$+ \frac{1}{8\pi} \text{Im} \left[ \int d^2 \theta \tau W^a W_a \right] + \left\{ \int d^2 \theta \left[ \sqrt{2} \bar{Q} \Phi \Phi + m_f \bar{Q}_f Q_f \right] + h.c. \right\}$$  \hspace{1cm} (10.11)$$

$\dagger$ I thank Kim Splittorff for helping me in deriving eq. (10.9).
11 Semiclassical analysis of super $N = 2$ Yang-Mills theory

The structure of the bosonic part of the Lagrangian in eq.(10.7) is pretty much the same as the one of the Georgi-Glashow model in eq.(4.1). There are, however, few important differences. The first is the presence of the fermion fields. The second one is that, unlike the Georgi-Glashow model, here the potential given in the case of a $SU(2)$ gauge group by

$$V(\phi) = -\frac{1}{2e^2}[\epsilon^{abc}\phi^b\bar{\phi}^c]^2$$

(11.1)

does not fix uniquely the vacuum. In fact any field configuration of the type

$$\phi^a = (0, 0, a) = a\delta^{a3}$$

(11.2)

corresponds to a minimum of the potential with vanishing value (since supersymmetry is not broken) for any complex number $a$. The set of all values of $a$ is called the classical moduli space of the theory. Actually a better parametrization of the vacua is given in terms of the gauge invariant variable

$$u = \frac{1}{2}a^2 = Tr(\phi^2).$$

A third difference with respect to the Georgi-Glashow model is that, because of the particular structure of the potential, in the supersymmetric case the BPS limit is obtained without needing to send to zero any piece of the potential as it was instead necessary in the Georgi-Glashow model.

If $a \neq 0$, as in the Georgi-Glashow model, the $SU(2)$ gauge symmetry is broken to $U(1)$ by the supersymmetric Higgs phenomenon and the charged (with respect to the unbroken $U(1)$) components of the gauge fields $W^\pm$ together with the charged gauginos get a non vanishing mass, while the gauge field of the unbroken $U(1)$ remains massless together with its supersymmetric partners, a Dirac photino and a complex Higgs field. The massless fields belong to a massless $N = 2$ chiral supermultiplet.

Let us now list the symmetries of the Lagrangian in eq.(10.4).

1. There is a $U(1)_J$ symmetry corresponding to the following superfield transformations:

$$\Phi(\theta) \rightarrow \Phi(e^{-i\alpha}\theta) \quad W_\alpha(\theta) \rightarrow e^{i\alpha}W_\alpha(e^{-i\alpha}\theta)$$

(11.3)

This symmetry is actually part of an $SU(2)_J$ that is, however, not manifest because we are using an $N = 1$ superfield formalism in which also the second supersymmetry is not manifest. In order to have both the second supersymmetry and the entire $SU(2)_J$ manifest we must use $N = 2$ superfields.

2. The Lagrangian in eq.(10.4) is also invariant under the $U(1)_R$ transformations given by:

$$\Phi(\theta) \rightarrow e^{2i\beta}\Phi(e^{-i\beta}\theta) \quad W_\alpha(\theta) \rightarrow e^{i\beta}W_\alpha(e^{-i\beta}\theta)$$

(11.4)
In the quantum theory the $U(1)_R$ symmetry is broken by an anomaly. The corresponding Noether current satisfies the anomaly equation:

$$\partial_\mu J^\mu_R = 4N_c q(x) \quad q(x) = \frac{1}{32\pi^2} F^a_{\mu\nu} F^{\mu\nu}_a \quad (11.5)$$

where $q(x)$ is the topological charge density and $N_c$ is the number of colours. We will be mainly considering the case $N_c = 2$, but in many cases we will be writing formulas valid for an $SU(N_c)$ colour group.

There is, however, a subgroup $Z_{4N_c}$ of $U(1)_R$ that is not anomalous. It acts on the scalar field $\phi$ of the chiral superfield $\Phi$ as

$$\phi \rightarrow e^{i\pi n N_c} \phi \quad n = 1 \ldots 4N_c \quad (11.6)$$

In the case of an $SU(2)$ gauge theory it acts on $\phi$ as a $Z_4$ and on $u = Tr(\phi^2)$ as a $Z_2$ transformation:

$$\phi \rightarrow e^{\frac{i\pi n}{2}} \phi \quad u \rightarrow -u \quad (11.7)$$

An anomalous $U(1)_R$ transformation has the effect of modifying the $\theta$ angle by:

$$\theta \rightarrow \theta - 4\beta N_c \quad (11.8)$$

However if we perform an anomalous $U(1)_R$ transformation together with an opposite shift of the $\theta$ angle

$$\theta \rightarrow \theta + 4\beta N_c \quad (11.9)$$

then this is a ”symmetry” of the theory. This is not what is usually called a symmetry in field theory because we transform not only the fields but also the parameters appearing in the Lagrangian. If, however, as it happens in string theory, we consider the parameters as related to the v.e.v. of some additional field and we insert those fields instead of their v.e.v. in the Lagrangian, then the previously discussed ”symmetry” becomes a true field theory symmetry. Such a symmetry, that is a symmetry of the microscopic Lagrangian, can be used, for instance, for putting constraints on the construction of a low energy effective Lagrangian for the light degrees of freedom of the theory. This is an important observation that we will use later on.

The $N = 2$ super Yang-Mills theory is an asymptotic free theory whose $\beta$-function gets, in perturbation theory, only a non vanishing contribution from one-loop diagrams [49, 50]. As it follows from the last part of Appendix B the $\beta$-function is given by:

$$\mu \frac{\partial \mu}{\partial \mu} \equiv \beta(e) = -\frac{b_0}{(4\pi)^2} e^3 \quad b_0 = 2N_c \quad (11.10)$$

Integrating it one can compute the dependence of the coupling constant in going from the scale $M$ to a scale $Q$

$$\frac{4\pi}{e^2(M)} + \frac{N_c}{\pi} \log \frac{Q}{M} \equiv \frac{4\pi}{e^2(Q)} \quad (11.11)$$
The renormalization invariant parameter $\Lambda$ is determined from the previous equation as the value of $Q$ such that the coupling constant is divergent $\lim_{Q \to \Lambda} e(Q) = \infty$. One gets:

$$\Lambda = M e^{-\frac{4\pi^2}{N_c v^2(M)}}$$

(11.12)

In terms of $\Lambda$ one can rewrite eq.(11.11) as follows

$$\frac{e^2(Q)}{4\pi} = \frac{2\pi}{N_c \log \frac{Q^2}{\Lambda^2}},$$

(11.13)

showing that, because of asymptotic freedom, perturbation theory is good when $Q$ is large.

If we introduce in eq.(11.12) also the $\theta$ angle and we restrict ourselves to the case $N_c = 2$ we get

$$\Lambda = M e^{i\pi/2} \tau = \frac{\theta}{2\pi} + i \frac{4\pi}{e^2}$$

(11.14)

Therefore under the shift in eq.(11.9) for $N_c = 2$ we get

$$\Lambda \to \Lambda e^{2i\beta}$$

(11.15)

If we compare this equation with eq.(11.4) we see that the scalar field $\phi$ transforms under the anomalous $U(1)_R$ with the weight equal to 2 precisely as $\Lambda$ under the transformation in eq.(11.9). This implies that under an anomalous $U(1)_R$ transformation combined with a shift of $\theta$ as given in eq.(11.4) the ratio $\Lambda^2/\phi^2$ will stay invariant.

If we are interested in studying the low-energy dynamics of the $N = 2$ super Yang-Mills we can restrict ourselves to the massless fields and we can limit ourselves to a Lagrangian with at most two derivatives and with no more than four-fermion couplings. $N = 2$ supersymmetry fixes completely its form in terms of a unique function $F$. The most compact form of this low energy Lagrangian can be obtained by using the $N = 2$ chiral superfield $\Psi$, that is a function of four variables $\theta$ without depending on their complex conjugate variables $\bar{\theta}$. In the manifest $N = 2$ superfield formalism the most general low energy Lagrangian has the form:

$$L_{eff} = \frac{1}{16\pi} Im \int d^4\theta F[\Psi(\theta)]$$

(11.16)

where $F$ is a function to be determined. The $N = 2$ superfield formulation of eq.(11.16) is not discussed in these lectures. It can be found in Ref. [51] and Refs. therein. Here we just add that, under the anomalous $U(1)_R$, the $N = 2$ chiral superfield $\Psi$ is transformed as

$$\Psi(\theta) \to e^{2i\beta} \Psi(\theta e^{-i\beta})$$

(11.17)
In the $N=1$ superfield formalism Lagrangian in eq.(11.16) becomes:

$$L_{\text{eff}} = \frac{1}{4\pi} \text{Im} \left\{ \int d^4\theta \frac{\partial F}{\partial \Phi} \Phi + \int d^2\theta \frac{1}{2} \frac{\partial^2 F}{\partial \Phi^2} W^\alpha W_\alpha \right\}$$

(11.18)

In general, in a $N=1$ supersymmetric theory the coefficient of kinetic terms of the gauge fields and that of the matter fields, called Kähler potential, are completely independent. We see, instead, that the $N=2$ invariance requires that they are related being both derived from the same function $F$. In terms of the $F$ they are given by:

$$K(\Phi, \bar{\Phi}) = \frac{1}{4\pi} \text{Im} \left[ \frac{\partial F}{\partial \Phi} \bar{\Phi} \right]$$

$$\tau(\Phi) = \frac{\partial^2 F}{\partial \Phi^2}$$

(11.19)

When expressed in terms of the component fields Lagrangian in eq.(11.18) becomes

$$L = \frac{1}{4\pi} \text{Im} \left\{ \tau(\phi) \left[ \partial_\mu \bar{\phi} \partial^\mu \phi - \frac{1}{4} \left( F^2 - iF_{\mu\nu}^* F^\mu\nu \right) + \frac{1}{2} (f^{abc} \bar{\phi}_a \phi_b)^2 + \text{Fermions} \right] \right\}$$

(11.20)

The main task is to determine the explicit form of the function $F$ that in general will receive both perturbative and non-perturbative contributions.

Comparing eq. (11.18) with eq. (10.3) we see that at the tree level the function $F$ is given by

$$F_{\text{cl}} = \frac{1}{2} \tau_{\text{cl}} \Phi^2$$

$$\tau_{\text{cl}} = \frac{\theta}{2\pi} + i\frac{4\pi}{e^2}$$

(11.21)

At one loop $F$ is completely fixed by the $U(1)_R$ anomaly [51, 52]. In fact we have seen that under the $U(1)_R$ transformations the chiral superfield $\Phi$ transforms as in eq.(11.4), while the transformation of the parameter $\tau$ follows from the insertion in $\tau$ (see eq.(11.14)) of the $U(1)_R$ transformation of $\theta$ given in eq.(11.8). This implies for $F$ the following transformations

$$F''(\Phi e^{2i\beta}) - F''(\Phi) = -\frac{2\beta N_c}{\pi}$$

(11.22)

where the double prime means double derivative with respect to the argument. The solution of the previous eq. is

$$F''(\Phi) = \frac{iN_c}{\pi} \log \frac{\Phi}{M}$$

(11.23)

where $M$ is an arbitrary parameter that has been introduced in order to make the argument of the logarithm dimensionless. Integrating two times the previous equation we get [51, 52]:

$$F_1 = \frac{iN_c}{4\pi} \Phi^2 \log \frac{\Phi^2}{\Lambda^2}$$

(11.24)

that is consistent with the $U(1)_R$ and scale anomaly [51, 52].

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It can also be shown that higher loops do not give any contribution to \( F \). Only non perturbative effects, as for instance instantons, can give an additional contribution to \( F \) \[52\]. We will show now that the contribution of the instantons to \( F \) must be of the following form:

\[
F_{\text{inst.}} = \sum_{k=1}^{\infty} F_k \left( \frac{\Lambda}{\Phi} \right)^{4k} \Phi^2
\]  

(11.25)

In fact, from eq.(6.6) one can see that the contribution of an instanton with topological charge \( k \) is proportional to \( e^{-8\pi^2 k / \varepsilon^2} \). Using the explicit form of the \( \Lambda \) parameter in eq.(11.12) computed in terms of the vacuum expectation value of the scalar field \( \phi \) that we call \( a \) we get:

\[
e^{-8\pi^2 k / \varepsilon^2} = \left( \frac{\Lambda}{a} \right)^{4k}
\]  

(11.26)

The previous expression is invariant under a \( U(1)_R \) anomalous transformation acting on \( <\phi> = a \) as given in eq.(11.4) supplemented with the change in the \( \Lambda \) parameter as given in eq.(11.15). Remembering that \( \Phi \) transforms under the anomalous \( U(1)_R \) as in eq.(11.4) one gets the form of the instanton contribution as given in eq. (11.25).

In the last part of this section we show that \( N = 2 \) super Yang-Mills has solitons that are magnetic monopoles and dyons that have precisely the same structure \[9\] as those already discussed in the Georgi-Glashow model in section (4).

From the Lagrangian for the \( N = 2 \) super Yang-Mills given in eq.(10.7) one can compute the Hamiltonian that is equal to:

\[
H = \frac{1}{2\varepsilon^2} \int d^3x \left\{ (E_i^a)^2 + (B_i^a)^2 + [(D_i A_4)^a]^2 + [(D_i A_5)^a]^2 + 
+ [(D_0 A_4)_a]^2 + [(D_0 A_5)_a]^2 + (f^{abc} A_4^b A_5^c)^2 \right\} (11.27)
\]

We follow the approach of Bogomolny and rewriting the first four terms of the integrand in the previous eq. as follows:

\[
[E_i^a - \cos \theta (D_i A_4)^a - \sin \theta (D_i A_5)^a]^2 + [B_i^a + \sin \theta (D_i A_4)^a - \cos \theta (D_i A_5)^a]^2 + 
+ 2E_i^a [\cos \theta (D_i A_4)^a + \sin \theta (D_i A_5)^a] + 2B_i^a [- \sin \theta (D_i A_4)^a + \cos \theta (D_i A_5)^a] (11.28)
\]

we get a lower bound for the Hamiltonian:

\[
H \geq \frac{1}{\varepsilon^2} \int d^3x \left\{ E_i^a [\cos \theta (D_i A_4)^a + \sin \theta (D_i A_5)^a] + 
+ B_i^a [-\sin \theta (D_i A_4)^a + \cos \theta (D_i A_5)^a] \right\} (11.29)
\]

The lower bound becomes an equality when the following equations are satisfied:

\[
(D_0 A_4)_a = (D_0 A_5)_a = (f^{abc} A_4^b A_5^c)^2 = 0 \quad (11.30)
\]

\[
E_i^a = \cos \theta (D_i A_4)^a + \sin \theta (D_i A_5)^a \quad (11.31)
\]

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\[ B_i^a = -\sin \theta (D_i A_4)^a + \cos \theta (D_i A_5)^a \]  
(11.32)

and at the same time we put all fermionic fields equal to zero as it is allowed from the classical eqs. of motion.

Introducing the ansatz in eqs. (11.17) and (11.44) respectively for the space and time component of the vector potential and the following ansatz for the Higgs fields:

\[ A_4^a = \frac{r^a}{r^2} J_4(\xi) \quad A_5^a = \frac{r^a}{r^2} J_5(\xi) \]  
(11.33)

we see that eqs. (11.30) are automatically satisfied. If we impose eqs. (11.31) and (11.32), as shown in Appendix A, we get that the function \( K(\xi) \) in the ansatz in eq. (4.17) is again given by the expression in eq. (4.21), while the other functions are given in terms of a unique function \( R(\xi) \)

\[ J_4(\xi) = \alpha R(\xi) \quad J_5(\xi) = \beta R(\xi) \quad J(\xi) = \gamma R(\xi) \quad R(\xi) = \xi \coth \xi - 1 \]  
(11.34)

where the dimensionless parameter \( \xi = \hat{a} r \) and the constants \( \alpha, \beta \) and \( \gamma \) are determined in terms of \( \theta \) through the relations:

\[ \alpha \sin \theta + \beta \cos \theta = 1 \quad \gamma = \alpha \cos \theta - \beta \sin \theta \]  
(11.35)

that imply

\[ \alpha^2 + \beta^2 - \gamma^2 = 1 \]  
(11.36)

Let us see now what determines the constant \( \theta \). Let us introduce the electric and magnetic charges:

\[ q = -\frac{1}{a' e} \int d^3 x \partial_i [E_i^a A_5^a - B_i^a A_4^a] \]  
(11.37)

\[ g = -\frac{1}{a' e} \int d^3 x \partial_i [B_i^a A_5^a + E_i^a A_4^a] \]  
(11.38)

where \( a' = \sqrt{a_4^2 + a_5^2} \). \( a_4 \equiv \alpha \hat{a} \) and \( a_5 \equiv \beta \hat{a} \) are respectively the asymptotic values for \( r \to \infty \) of \( A_4 \) and \( A_5 \) and \( a' = \hat{a} \sqrt{1 + \gamma^2} \). Using the previous formulas we get

\[ E \geq -\frac{a'}{e} [q \sin \theta + g \cos \theta] \]  
(11.39)

Inserting eqs. (11.31) and (11.32) in eqs. (11.37) and (11.38) we get

\[ q = g \tan \theta \]  
(11.40)

that implies for eq. (11.33)

\[ E = \frac{a'}{e} \sqrt{q^2 + g^2} = \frac{a'}{e} |q + ig| \]  
(11.41)
In terms of the complex scalar field $\phi$ defined in eq. (10.8), calling $a$ the complex asymptotic value of $\phi$, in the BPS limit we can write the mass of the dyons in eq. (11.39) as

$$M = \sqrt{2} \frac{|a|}{e} |q + ig|$$

(11.42)

where the magnetic charge is the same as in the Georgi-Glashow model:

$$g = + \frac{4\pi}{e} n_m \quad n_m = -1$$

(11.43)

and the electric charge, after semiclassical quantization, is equal to

$$q = n_e e$$

(11.44)

Using the two previous eqs. and introducing also a $\theta$ angle we can rewrite eq. (11.42) as follows:

$$M = \sqrt{2} |a||n_e + \tau n_m|$$

(11.45)

We have been considering a $N = 2$ supersymmetric theory, but up to now the fermionic part of the Lagrangian has not played any role. All our considerations are based only on the structure of the bosonic part of the Lagrangian since all fermionic fields have been put equal to zero consistently with their classical eqs. of motion. In the following we will show that the $N = 2$ supersymmetry is essential for two reasons. The first is that the previous considerations, for instance for the calculation of the mass of the dyons, are purely classical or at most semiclassical results. We will see that they will be valid in the full quantum theory as a consequence of the supersymmetry algebra that gets modified by the presence of central charges. The second one is the presence of fermionic zero modes that in a supersymmetric theory implies that dyons and monopoles belong to supersymmetric short multiplets. As a consequence the dyons and monopoles carry a non vanishing spin. We will discuss these two aspects in the next section.

### 12 Susy algebra and fermionic zero modes

In eqs. (8.13) and (8.16) we have written the $N = 2$ supersymmetry algebra in four dimensions including a complex central charge $\hat{Z}$. The value of the central charge depends on the particular theory we are considering. In the case of $N = 2$ super Yang-Mills the central charge $Z$ has been computed by Olive and Witten in Ref. [10]. In Appendix C we give all necessary details of the calculations that bring to the result:

$$\hat{Z} = -2 \frac{q'}{e} (q - ig)$$

(12.46)

where $q$ and $g$ are given in eqs. (11.37) and (11.38). Introducing the Majorana spinors:

$$Q^i_A = \begin{pmatrix} Q^i_\alpha \\ Q^{i\dot{\alpha}} \end{pmatrix} \quad \bar{Q}^i_A = \begin{pmatrix} Q^{i\alpha} \\ \bar{Q}^{i\dot{\alpha}} \end{pmatrix}$$

(12.47)
the \( N = 2 \) algebra given in eqs. (8.15) and (8.16) can be rewritten in four-dimensional notations obtaining:

\[
\{ Q^i_A, \bar{Q}_B^j \} = 2 \gamma^\mu_{AB} P_\mu \delta^{ij} - 2 (\gamma_5)_{AB} \epsilon^{ij} V + 2 i \epsilon^{ij} \delta_{AB} U \tag{12.48}
\]

where

\[
2U = -Im \hat{Z} = -\frac{2a'g}{e} \quad 2V = -Re \hat{Z} = \frac{2a'q}{e} \tag{12.49}
\]

From the algebra in eq. (12.48) applied to a state in the center of mass frame where \( P_\mu = (M, \vec{0}) \) one gets:

\[
\{ Q^i_A, (Q^j_B)^\dagger \} = 2M \delta^{ij} \delta_{AB} - \frac{2a'}{e} L^{ij}_{AB} \tag{12.50}
\]

where \( L \) satisfies the eq.

\[
L^2 = (q^2 + g^2)\delta^{ij} \delta_{AB} \\
L^{ij}_{AB} = \epsilon^{ij} \left[ q(\gamma_5 \gamma^0)_{AB} + ig(\gamma^0)_{AB} \right] \tag{12.51}
\]

This implies that the eigenvalues of the matrix \( L \) are equal to \( \pm \sqrt{q^2 + g^2} \). Then, since the l.h.s. of eq.(12.51) is a positive definite operator, one gets a lower bound for the mass

\[
M \geq \frac{a'}{e} \sqrt{q^2 + g^2} = \frac{a'}{e} |q + ig| = \sqrt{2} |a| |q + ig| \tag{12.52}
\]

where we have introduced the complex parameter \( a \) (as in eq.(11.42)) that is the asymptotic value of the complex field \( \phi \).

We have obtained again the BPS condition, but now, unlike the case of eq.(11.42) that was derived in the classical theory, it is a direct consequence of the supersymmetry algebra that is supposed to be valid in the full quantum theory. For the BPS states, for which the equality sign holds, it is an exact mass formula. For this reason the introduction of an extended supersymmetry allows one to overcome the difficulty mentioned in the second point toward the end of section (7). Actually, as we will see in section (13), this is not quite true in \( N = 2 \) super Yang-Mills because the parameter \( a \), that describes the moduli space of the theory, provides a good description of it only in the semiclassical region where \( a \) is large. In the strong coupling region the semiclassical formula (12.52) must be modified.

In the second part of this section we will discuss the fermionic zero modes. Up to now, in order to get the monopole and dyon solutions, we have put all fermion fields equal to zero. On the other hand, if a theory is supersymmetric, by means of a supersymmetry transformation, from a classical solution in which the fermionic fields are zero we get another classical solution in which they are non zero. If we perform a supersymmetry transformation, that can be obtained by dimensional reduction from the expression in eq.(9.3) one gets

\[
\delta \chi^a = \left[ \sigma_{\mu\nu} F^{a}_{\mu\nu} - i \gamma_\mu \gamma_5 (D^\mu A^4)_a - \gamma_\mu (D^\mu A^5)_a - i f^{abc} A^b_4 A^5_5 \gamma_5 \right] \chi \tag{12.53}
\]
Assuming that the fields in the previous eq. satisfy the BPS eq. (11.32) satisfied by the monopole (taking for simplicity $\theta = 0$ and $A_4 = 0$) we can rewrite the previous supersymmetry transformation as follows:

$$\delta \chi^a = -(D^k A_5)^a \left[ \gamma_k + \frac{1}{2} \epsilon_{ijk} \gamma^i \gamma^j \right] \alpha$$

(12.54)

This implies that, if we choose the supersymmetry parameter $\alpha$ satisfying the equation:

$$\left[ 1 + \frac{1}{6} \epsilon_{ijk} \gamma^i \gamma^j \gamma^k \right] \alpha = \left[ 1 + \gamma^1 \gamma^2 \gamma^3 \right] \alpha = 0$$

(12.55)

then the supersymmetry transformation does not move the fermionic field from zero. This means that the monopole solution preserves half of the supersymmetry. On the other hand, if $\alpha$ satisfies instead the condition with the opposite sign:

$$\left( 1 - \gamma^1 \gamma^2 \gamma^3 \right) \alpha = 0$$

(12.56)

then, by means of a supersymmetry transformation we go from $\chi^a = 0$ to

$$\chi^a = -2 \gamma_k (D^k A_5)^a \alpha$$

(12.57)

that corresponds to two fermionic zero modes. In fact, since $\alpha$ is a Dirac spinor, it has four independent degrees of freedom. Half of them are killed by the condition (12.55) and therefore we are left with only two degrees of freedom that is also the expected number of zero modes for fermions transforming in the adjoint representation according to the Callias index theorem [53].

The ansatz for the monopole in eq.(4.17) is transformed if we perform a space rotation, but is left invariant if we supplement the space rotation with a global gauge transformation whose parameter is related to the one of the space rotation. In other word the generator that leaves the ansatz invariant is generated by the total angular momentum:

$$\vec{J} = \vec{L} + \vec{S} + \vec{T}$$

(12.58)

where $\vec{L}$ and $\vec{S}$ are the orbital and the spin angular momenta, while $\vec{T}$ is the generator of the $SO(3)$ gauge transformations. If we have isovector spinors as in the $N = 2$ super Yang-Mills then the total angular momentum is half-integer, since the spin is equal to $1/2$ and the orbital angular momentum is integer. In particular since $\frac{1}{2} \otimes 1 = \frac{3}{2} + \frac{1}{2}$ and since we have two zero modes the simplest possibility is that the two fermionic zero modes, given in eq.(12.57), transform according to the $\frac{1}{2}$ representation of $\vec{J}$. In the quantum theory we can expand the field $\chi$ as follows:

$$\chi = a_{1/2} \chi_0 \sqrt{\frac{1}{2}} + a_{-1/2} \chi_0^{-1/2} + \Delta \chi$$

(12.59)

where $\Delta \chi$ is the quantum field and $\chi_0^{\pm 1/2}$ are the two zero modes. The anticommutation relations satisfied by $\chi$ imply that the coefficients of the zero mode expansion in eq.(12.59) satisfy the algebra of the fermionic harmonic oscillator:

$$\{a_{1/2}, a_{1/2}^\dagger\} = \{a_{-1/2}, a_{-1/2}^\dagger\} = 0$$

(12.60)
Table 1: $N = 2$ BPS multiplet.

| STATE       | $S_z$ | # OF STATES |
|-------------|-------|-------------|
| $|0\rangle$  | 0     | 1           |
| $a_{\pm 1/2}|0\rangle$ | $\pm 1/2$ | 2           |
| $a_{1/2}a_{-1/2}|0\rangle$ | 0     | 1           |

and all other anticommutators are zero.

The states of the supersymmetric multiplet of the monopole can be constructed starting from the vacuum $|\Omega\rangle$ that is annihilated by

$$a_{1/2}|\Omega\rangle = a_{-1/2}|\Omega\rangle = 0$$  \hspace{1cm} (12.61)

and then acting with the creation operators on it. In this way we can construct the states given in Table 1.

We see that the monopole multiplet contains spin 1/2, but not spin 1. This means that $N = 2$ super Yang-Mills cannot be duality invariant à la Montonen-Olive. Notice that the hypermultiplet, that contains 4 real bosonic and 4 real fermionic states, consists of two $N = 2$ BPS multiplets.

13 Global parametrization of moduli space

In the next three sections we will be shortly describing the beautiful paper of Seiberg and Witten [54] where an exact expression for $\tau(\Phi)$ (see eq.(11.19)) in the low energy effective action of the $N = 2$ super Yang-Mills theory has been constructed.

Unlike the $N = 4$ theory which, as we will see, can be equivalently formulated either in terms of the original fundamental fields or in terms of the monopoles or more in general of the dyons of the single particle spectrum with essentially the same Lagrangian, the $N = 2$ theory cannot satisfy the Montonen-Olive duality because the fundamental fields and the magnetic monopoles belong to two different $N = 2$ superfields. The fundamental fields are in the chiral $N = 2$ vector multiplet while the magnetic monopoles and dyons are in the hypermultiplet [11].

Nevertheless the $N = 2$ theory can be formulated either in terms of the variables $\phi, A_\mu$ and $\tau(\phi)$, as we have done in eq.(11.20), or in terms of the dual variables $\phi_D, A_{D\mu}$ and $\tau_D(\phi_D)$ in pretty much the same way that free electromagnetism can be formulated either in terms of the vector potential $A_\mu$ related to the field strenght by $F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ or in terms of the dual vector potential $A_{D\mu}$ related to the dual field strenght by $^{*}F_{\mu \nu} = \partial_\mu A_{D\nu} - \partial A_{D\mu}$.

In order to explain this let us first summarize some general property of the $N = 2$ super Yang-Mills theory.
In section (11) we have seen that in this theory the low energy effective action is completely fixed by giving a holomorphic function $F(\Phi)$. In terms of $F$ we can construct the Kähler potential, as given in eq.(11.19), and the metric

$$(ds)^2 = \frac{\partial}{\partial \phi} \frac{\partial}{\partial \bar{\phi}} K(\phi, \bar{\phi}) d\phi d\bar{\phi} = \text{Im}(\tau(\phi)) d\phi d\bar{\phi} \quad \tau(\phi) = \frac{\partial^2 F(\phi)}{\partial \phi^2}$$ (13.1)

We have also seen that the moduli space of the $N = 2$ theory is in the semiclassical theory parametrized by the vacuum expectation value of the scalar field $\phi$ that we have denoted by the complex number $a$. However $a$ cannot provide a global description of the moduli space. In fact the metric $\text{Im}(\tau(a))$, that is a positive definite harmonic function divergent for $|a| \to \infty$, must have a minimum. But a globally defined harmonic function cannot have a minimum and consequently the variable $a$ cannot provide a global parametrization of the moduli space.

Therefore in Ref. [54] it was proposed to choose the gauge invariant quantity $u = \frac{1}{2} \text{Tr}(\phi^2)$ as the one that provides a global parametrization of the moduli space and to regard both $a(u)$ and the dual variable $a_D(u) \equiv \frac{\partial F}{\partial a}$ as functions of $u$. In terms of both $a$ and $a_D$ the metric in eq.(13.1) assumes the form

$$(ds)^2 = \text{Im} \left( \frac{da_D}{da} \frac{da d\bar{a}}{du d\bar{u}} \right) = \text{Im} (da_D d\bar{a}) = -\frac{i}{2} [da_D d\bar{a} - dad\bar{a}]$$ (13.2)

that is symmetric under the exchange $a \leftrightarrow a_D$.

Introducing the vector $v^\alpha = \left( \begin{array}{c} a_D \\ a \end{array} \right)$ we can rewrite the metric in the more compact form:

$$(ds)^2 = -\frac{i}{2} \epsilon_{\alpha\beta} \frac{dv^\alpha}{du} \frac{d\bar{v}^\beta}{d\bar{u}} dud\bar{u}$$ (13.3)

that clearly show its invariance under the transformation:

$$v \to Mv + c$$ (13.4)

where $M$ is a matrix of $SL(2, R)$ and $c$ is a constant vector.

An arbitrary matrix of $SL(2, R)$ is generated by the action of two independent matrices $T_b$ and $S$. The first one

$$T_b = \left( \begin{array}{cc} 1 & b \\ 0 & 1 \end{array} \right)$$ (13.5)

leaves $a$ invariant and transforms $a_D$ according to

$$a_D \to a_D + ba$$ (13.6)

This implies that $\tau(a)$ is just translated

$$\tau(a) \to \tau(a) + b$$ (13.7)
resulting in a translation for the vacuum angle $\theta$

$$\theta \to \theta + 2\pi b$$  \hspace{1cm} (13.8)

Since physical quantities are invariant when

$$\theta \to \theta + 2\pi n$$  \hspace{1cm} (13.9)

for any integer $n$, comparing eqs. (13.8) and (13.9) we deduce that $b = 1$ and consequently that the transformation associated to the matrix $T_{b=1}$ is a symmetry of the theory. By selecting $b = 1$ we have reduced the original $SL(2, R)$ symmetry group to $SL(2, Z)$.

The other independent generator

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$  \hspace{1cm} (13.10)

does not correspond to a symmetry of the theory, but provides a transformation between two different parametrizations of the theory. In fact the low energy effective Lagrangian can be represented either in terms of the variables $(A^{\mu}, \lambda, \phi; \tau(\phi))$ or in terms of the dual ones $(A^{\mu}_{D}, \lambda_{D}, \phi_{D}; \tau_{D}(\phi_{D}) = -1/\tau(\phi))$. In order to more clearly see the relation between the two formulations it is convenient to set the vacuum angle $\theta = 0$. Then we see that, if $\text{Im} \tau(\phi) = \frac{4\pi}{e^2}$, then $\text{Im} \tau_{D}(\phi_{D}) = \frac{e^2}{4\pi}$. Therefore one description may be more suitable for weak coupling, while the other for strong coupling.

In the final part of this section we discuss the exact mass formula proposed in Ref. [54] for the BPS saturated states in the $N = 2$ theory. At the semiclassical level the mass of the BPS saturated states is given by [39, 41]

$$M = \sqrt{2} |Z|$$

$$Z = a \left[ n_e + \left( \frac{\theta}{2\pi} + i \frac{4\pi}{e^2(\mu)} + i \frac{1}{\pi} \log \frac{a^2}{\mu^2C} \right) n_m \right]$$  \hspace{1cm} (13.11)

where $\mu$ is the renormalization scale and $C$ is a scheme dependent constant. The extra logarithmic term present in this formula with respect to the classical mass formula is the effect of the renormalization of the gauge coupling constant $e$. Noticing that the coefficient of $n_m$ in eq. (13.11), with a suitable choice of $C$, is equal to $a_{D} \equiv \frac{\partial F}{\partial a}$ with $F = F_{cl} + F_{1}$ given in eqs. (11.21) and (11.24), eq. (13.11) can be rewritten as follows

$$Z = an_e + a_{D}n_m = \begin{pmatrix} n_m & n_e \end{pmatrix} \begin{pmatrix} a_{D} \\ a \end{pmatrix}$$

$$M = \sqrt{2} |Z|$$  \hspace{1cm} (13.12)

Seiberg and Witten [54] proposed eq. (13.12) as an exact formula for the BPS states and made several checks for confirming its validity.

In particular $Z$ is invariant under the transformation

$$\begin{pmatrix} a_{D} \\ a \end{pmatrix} \to M \begin{pmatrix} a_{D} \\ a \end{pmatrix} \hspace{1cm} \begin{pmatrix} n_m & n_e \end{pmatrix} \to \begin{pmatrix} n_m & n_e \end{pmatrix} M^{-1}$$  \hspace{1cm} (13.13)
where $M$ is a matrix of $SL(2, Z)$ because the vector $\left( n_m \ n_e \right)$ has integer entries and its transformed must also have integer entries. This is an independent way to derive the reduction of $SL(2, R)$ to $SL(2, Z)$. Actually this procedure forces also the extra parameter $c$ in eq.(13.4) to be equal to zero.

14 Singularity structure of moduli space

In this section we study the singularity structure of $a$ and $a_D$ as functions of the variable $u$, that provides a global parametrization of the moduli space.

In the semiclassical region corresponding to a large value of $u$ we get

$$a = \sqrt{2u} \quad a_D \equiv \frac{\partial F_1}{\partial a} \sim 2ia \frac{\log a}{\pi} = i\frac{2\sqrt{2u}}{\pi} \log \sqrt{2u} \quad (14.1)$$

where at one loop $F$ is given by (see eq.(11.24)):

$$F_1 = \frac{i}{2\pi} a^2 \log \frac{a^2}{\Lambda^2} \quad (14.2)$$

and

$$\tau(u) \equiv \frac{\partial a_D}{\partial a} \sim 2i \frac{\log a}{\pi} = 2i \frac{\log \sqrt{2u}}{\pi} \quad (14.3)$$

Under a rotation around $u = \infty$ given by

$$\log u \rightarrow \log u + 2i\pi \quad (14.4)$$

$a(u)$, $a_D(u)$ and $\tau(u)$ are not monodromic functions. They transform according to

$$a \rightarrow -a \quad a_D \rightarrow -a_D + 2a \quad \tau(u) \rightarrow \tau(u) - 2 \quad (14.5)$$

The monodromy properties given in eq.(14.5) are entirely determined by the coefficient in front of the logarithm in eq.(14.3) that is related to the perturbative $\beta$-function in eq.(11.10). More precisely this coefficient is equal to a factor $(-i)$ times the value of the coefficient of the $\beta$-function in eq.(11.10), that is equal to $\frac{1}{4\pi^2}$ for $N_c = 2$, multiplied by a factor $4\pi$ that is already present in front of the effective action in eq.(11.18). The monodromy transformations in eq.(14.5) are generated by acting on the vector $\left( \begin{array}{c} a_D \\ a \end{array} \right)$ with the following monodromy matrix:

$$M_{\infty} = \left( \begin{array}{cc} -1 & 2 \\ 0 & -1 \end{array} \right) \quad (14.6)$$

The existence of a singularity requires the existence of at least another singularity. But, if we had only one additional singularity, it is easy to see that $a$ would have been a good global parameter being the monodromy group an abelian group. Since this is not possible we must require the existence of at least two additional singularities.
Following the example of what is happening in some $N = 1$ supersymmetric theories Seiberg and Witten assume that the singularities occur at those points of the moduli space where additional massless particles appear in the spectrum. In the classical theory this occurs for $a = 0$ where the $SU(2)$ symmetry is restored and $W^\pm$ become massless. They bring strong indications against this possibility in the quantum theory and instead choose the singularities at the points $m^2$ where the monopole with $(n_m, n_e) = (1, 0)$ and $(m')^2$ where the dyon with $(n_m, n_e) = (1, -1)$ become massless.

Using the exact formula in eq.(13.12) it is easy to see that this occurs at a certain value $u$ that we call $m^2$ for which $a_D(m^2) = 0$ with $a(m^2) \neq 0$ and when $a_D((m')^2) - a((m')^2) = 0$ with $a((m')^2), a_D((m')^2) \neq 0$ respectively. The existence of a $Z(2)$ symmetry that transforms $u$ in $-u$ suggests to choose $(m')^2 = -m^2$.

The monodromy around the singularity at $u = m^2$ can be easily computed by observing that the low energy theory at the point $u = m^2$ consists of a ”magnetic” $N = 2$ super QED (the matter has magnetic and non electric charge). This theory is not asymptotically free and the coefficient of the $\beta$-function, besides a sign, has a factor $1/2$ of difference with respect to the $\beta$-function previously used for studying the singularity around $u = \infty$. This means that in this case instead of the eq.(14.3) we get the following expression:

$$\tau_D \equiv -\frac{\partial a}{\partial a_D} \sim -\frac{i}{\pi} \log a_D$$  \hspace{1cm} (14.7)

Integrating the previous eq. one gets:

$$a(u) = k + \frac{ia_D}{\pi} \log a_D$$  \hspace{1cm} (14.8)

where $k \neq 0$ because we have assumed that at $u = m^2$ only the monopole and no other electrically charged particle become massless. Assuming that $a_D$ is a good coordinate near $a_D = 0$ we can write $a$ and $a_D$ as follows

$$a_D(u) \sim c_0(u - m^2) \hspace{1cm} a(u) = k + \frac{i}{\pi} c_0(u - m^2) \log [c_0(u - m^2)]$$  \hspace{1cm} (14.9)

Under the monodromy transformation:

$$\log(u - m^2) \rightarrow \log(u - m^2) + 2\pi i$$  \hspace{1cm} (14.10)

we get

$$a_D \rightarrow a_D \hspace{1cm} a \rightarrow a - 2a_D \hspace{1cm} \tau_D \rightarrow \tau_D + 2$$  \hspace{1cm} (14.11)

From the last equation we get the following transformation on $\tau$:

$$\tau(u) = -\frac{1}{\tau_D} \rightarrow \frac{\tau(u)}{1 - 2\tau(u)}$$  \hspace{1cm} (14.12)
The monodromy matrix that, acting on the vector \( \begin{pmatrix} a_D \\ a \end{pmatrix} \), generates the transformations in eqs. (14.11) and (14.12) is given by:

\[
M_{m^2} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}
\]

The singularity at \( u = -m^2 \) must be the mirror symmetric of the one at \( u = m^2 \) because the two points are related by the \( Z_2 \) symmetry that acts on \( u \) as \( u \to e^{i\pi} u \). Starting from the expression valid in the semiclassical approximation for large and positive \( u \) given in eq. (14.1) and performing the \( Z_2 \) transformation we get

\[
a \to \tilde{a} = ia \quad a_D \to \tilde{a}_D = i(a_D - a)
\]

\( \tilde{a}_D \) must have a single zero, that is the image of the zero of \( a_D \) for \( u \) near \( m^2 \): \( \tilde{a}_D = \tilde{c}_0(u + m^2) \). Then, proceeding as in the previous case, from

\[
\tilde{\tau}_D = -\frac{\partial \tilde{a}}{\partial \tilde{a}_D} = -\frac{i}{\pi} \log \tilde{a}_D = -\frac{i}{\pi} \log \left[ \tilde{c}_0(u + m^2) \right]
\]

one gets

\[
\tilde{a} = \tilde{k} + \frac{i}{\pi} \tilde{a}_D \log \tilde{a}_D
\]

In conclusion we get:

\[
\tilde{a}_D = \tilde{c}_0(u + m^2) \quad \tilde{a} = \tilde{k} + \frac{i}{\pi} \tilde{c}_0(u + m^2) \log \left[ \tilde{c}_0(u + m^2) \right]
\]

Under a monodromy transformation

\[
\log(u + m^2) \to \log(u + m^2) + 2\pi i
\]

we obtain

\[
\tilde{a}_D \to \tilde{a}_D \quad \tilde{a} \to \tilde{a} - 2\tilde{a}_D
\]

Going back to the original variables \( a \) and \( a_D \) they become:

\[
a \to 3a - 2a_D \quad a_D \to 2a - a_D \quad \tau(u) \to \frac{2 - \tau(u)}{3 - 2\tau(u)}
\]

They are generated by the monodromy matrix:

\[
M_{-m^2} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}
\]

as it can be easily checked.

If there are only three singularities the three monodromy matrices must be consistent. They in fact satisfy the consistency condition:

\[
M_{\infty} = M_{m^2} M_{-m^2}
\]

It can also be seen that the three monodromy matrices span a subgroup of the modular group, called \( \Gamma(2) \), consisting of matrices of \( SL(2, \mathbb{Z}) \) congruent to the identity matrix modulo 2.
15 Explicit solution

Having established the singularities and the monodromy transformations of \( a \) and \( a_D \) one could determine the solution algebraically apart from a regular function that can then be fixed from the semiclassical behaviour. On the other hand the following two points suggest to find the solution geometrically:

1. \( \text{Im} \tau(u) > 0 \) as the parameter \( \tau \) of a torus.

2. Under the various monodromies \( \tau(u) \) transforms pretty much in the same way as the parameter \( \tau \) of a torus transforms under a change of homology basis.

This suggests that we can define a torus for each value of \( u \) and that its \( \tau \) parameter is the desired \( \tau(u) \). A torus is usually represented as a parallelogram in the plane \( z \) with opposite sides identified. The two sides of the parallelogram go from the origin to the points 1 and \( \tau \) respectively. Functions defined on a torus are doubly periodic functions. On the torus there exists a unique holomorphic differential that in the previously described parametrization of the torus is equal to \( \omega = dz \). The parameter \( \tau \) of the torus can be obtained as the ratio of the integrals along the two cycles \( a \) and \( b \) of the torus. In the previous parametrization of the torus \( \tau \) is given by:

\[
\tau = \frac{\oint_b \omega}{\oint_a \omega} = \frac{\int_0^\tau dz}{\int_0^1 dz}
\]

A torus can also be described by a cubic equation:

\[
y^2 = x^3 + Bx^2 + Cx + D
\]

where \( B, C \) and \( D \) are arbitrary constants. The cubic defines a two sheet function. There are four branch points that we call \( x_1, x_2 \) and \( x_3 \) and \( \infty \). The first three branch points are located at the zeroes of the cubic in eq.(15.2). The cycle \( a \) corresponds to a closed path that encircles the cut between the first two branch points, while the cycle \( b \) corresponds to a path that starts from the upper side of the other cut, goes to the upper side of the first cut, continues across the cut in the second sheet and finally comes out again in the first sheet across the original cut.

In the parametrization of the torus in terms of the cubic equation the unique holomorphic differential is equal to:

\[
\omega = \frac{dx}{y(x)}
\]

and the \( \tau \) parameter of the torus is given by:

\[
\tau = \frac{\oint_b \frac{dx}{y(x)}}{\oint_a \frac{dx}{y(x)}}
\]
This form is very similar to the expression for the parameter $\tau(u)$ whose monodromy properties we have described in the previous section:

$$\tau(u) = \frac{d\alpha_D}{du}$$ (15.5)

Assuming that for each value of $u$ we can define a torus whose parameter is equal to $\tau(u)$ we get

$$\frac{d\alpha_D}{du} = \alpha \int_b^a dx \ y(x)$$
$$\frac{da}{du} = \alpha \int_a^b dy(x)$$ (15.6)

where $\alpha$ is a dimensionless constant to be determined. In order to determine the previous expressions we need to fix the coefficients $B, C$ and $D$ of the cubic in eq.(15.2) as functions of $u$ and $m^2$, that are the only dimensional parameters at our disposal. The form of the cubic can be fixed with the procedure that we now describe.

Since $a$ and $a_D$ have dimension of a mass and $u$ has dimension of a $[mass]^2$ eqs.(15.6) implies that $x$ has dimension of a $[mass]^2$ as $u$ and therefore that the parameters $B, C$ and $D$ have respectively dimension of a $[mass]^2$, a $[mass]^4$ and a $[mass]^6$. They can only be functions of the only two dimensional parameters at our disposal, namely $u$ and $m^2$. As a consequence the most general expression for $B, C$ and $D$ can be written in terms of $u$ and $m^2$ with arbitrary dimensionless coefficients:

$$B = Ru + Sm^2$$
$$C = Tu^2 + Vm^4 + Wum^2$$
$$D = Mm^6 + Nm^4u + Pu^2m^2 + Qu^3$$ (15.7)

The first requirement is that the cubic must be invariant if we perform a transformation of the $Z_2$ invariant group discussed around eq.(11.7). It acts on $x, y$ and $u$ as follows:

$$y \rightarrow \pm iy$$
$$x \rightarrow -x$$
$$u \rightarrow -u$$ (15.8)

without transforming $m^2$. The invariance of the cubic under this transformation implies that a number of coefficients must be vanishing:

$$S = W = M = P = 0$$ (15.9)

If these coefficients are vanishing it is easy to check that the cubic is also invariant under a more general $U(1)_R$ transformation, under which $x, u$ and $y$ transform respectively with weight 4, 4 and 6 provided that in addition we also transform $\theta$ as in eq.(11.9) and correspondingly also $m$ as $\Lambda$ in eq.(11.13). This is a consequence of the fact that this is a ”symmetry” of the theory as discussed after eq.(11.9).

The second requirement that restricts further the form of the cubic is the fact that the cubic in the limit of large $u$ must reproduce the semiclassical behaviour

$$\frac{d\alpha}{du} \sim 1/\sqrt{u}.$$ This implies

$$T = Q = 0$$ (15.10)
In conclusion the two previous requirements imply the following form for the cubic:

\[ y^2 = x^3 + Bux^2 + Cxm^4 + Dm^4u \quad (15.11) \]

where now \( B, C \) and \( D \) are dimensionless parameters to be determined.

A torus that is described by the cubic in eq.\((15.11)\) degenerates when the discriminant \( \Delta \) of the cubic is vanishing:

\[ \Delta \equiv 4m^4B^3Du^4 - \left[ B^2C^2 + 18BCD - 27D^2 \right] m^8u^2 + 4m^{12}C^3 = 0 \quad (15.12) \]

Our third assumption is that the points of degeneracy of the torus precisely coincide with the two points \( u = \pm m^2 \) where \( \tau(u) \) is singular with monodromy properties given respectively in eqs.\((14.12)\) and \((14.20)\). In order to impose this condition we need to distinguish two cases. The first one corresponds to a value of \( D \neq 0 \). In this case the solutions of eq.\((15.12)\) are given by:

\[ u^2 = m^4 \frac{R \pm \sqrt{R^2 - 64B^3C^3D}}{8B^3D} \quad R \equiv B^2C^2 + 18BCD - 27D^2 \quad (15.13) \]

The requirement that the discriminant in eq.\((15.12)\) is vanishing only for \( u^2 = m^4 \) gives the two eqs.

\[ R^2 = 64B^3C^3D \quad R = 8B^3D \quad (15.14) \]

They imply

\[ D = \frac{C^3}{B^3} \quad (15.15) \]

and the equation:

\[ t^4 - 8t^3 + 18t^2 - 27 = (t + 1)(t - 3)^3 = 0 \quad t = \frac{B^2}{C} \quad (15.16) \]

In conclusion, if \( D \neq 0 \), the curve is given by:

\[ y^2 = x^3 + Bux^2 + \frac{B^2m^4}{t}x + \frac{B^3m^4u}{t^3} \quad (15.17) \]

where \( t \) can only assume the two values \( t = -1 \) or \( t = 3 \).

If instead \( D = 0 \) then the vanishing of the discriminant in eq.\((15.12)\) implies:

\[ u^2 = 4\frac{C}{B^2}m^4 \Rightarrow B^2 = 4C \quad (15.18) \]

and the cubic has the form:

\[ y^2 = x^3 + Bux^2 + \frac{B^2}{4}m^4x \quad (15.19) \]

The form of the curves in eqs.\((15.17)\) and \((15.19)\) can be further simplified by a rescaling of \( x \) and \( y \)

\[ y \rightarrow \lambda^3y \quad x \rightarrow \lambda^2x \quad (15.20) \]
with an arbitrary parameter $\lambda$, without changing the curve. In this way, by choosing $\lambda^2 = -B$, we can eliminate $B$ from the two curves arriving at

$$y^2 = x^3 - ux^2 + \frac{m^4}{t} x - \frac{m^4 u}{t^3}$$  \hspace{1cm} (15.21)$$

if $D \neq 0$ and

$$y^2 = x^3 - ux^2 + \frac{m^4}{4} x$$  \hspace{1cm} (15.22)$$

if $D = 0$. This last curve is the one chosen by Seiberg and Witten in Ref. [55].

By choosing $t = -1$ in eq.(15.21) we get the curve found by Seiberg and Witten in their original paper [54]:

$$y^2 = (x - m^2)(x + m^2)(x - u)$$  \hspace{1cm} (15.23)$$

Inserting the previous curve in eq.(15.6) and integrating over $u$ we get:

$$a(u) = -2\alpha \int_a^b dx \frac{\sqrt{x - u}}{\sqrt{x^2 - m^4}} = -4\alpha \int_{-m^2}^{m^2} dx \frac{\sqrt{x - u}}{\sqrt{x^2 - m^4}}$$  \hspace{1cm} (15.24)$$

and

$$a_D(u) = -2\alpha \int_{-m^2}^{m^2} dx \frac{\sqrt{x - u}}{\sqrt{x^2 - m^4}} = -4\alpha \int_{-m^2}^{m^2} dx \frac{\sqrt{x - u}}{\sqrt{x^2 - m^4}}$$  \hspace{1cm} (15.25)$$

The factor 2 follows from the fact that the integral under the cut gives the same contribution of the one over the cut. Remembering that $a(u) \to \sqrt{2}u$ for $u \to \infty$ we get after some calculation:

$$\alpha = -\frac{1}{2\sqrt{2}\pi}$$  \hspace{1cm} (15.26)$$

arriving at the explicit solution constructed by Seiberg-Witten [54]:

$$a(u) = \frac{\sqrt{2}}{\pi} \int_{-1}^{1} dx \frac{\sqrt{x - u}}{\sqrt{x^2 - m^4}} \quad a_D(u) = \frac{\sqrt{2}}{\pi} \int_{1}^{u} dx \frac{\sqrt{x - u}}{\sqrt{x^2 - m^4}}$$  \hspace{1cm} (15.27)$$

It can be shown that both $a(u)$ and $a_D(u)$ are singular at the points $u = \pm m^2, \infty$ with exactly the monodromies discussed in the previous section. In terms of the previous functions one can construct the coefficient $\tau(u)$ of the kinetic term of the gauge field that satisfies by construction the important property: $Im\tau > 0$ for any $u$. By a shift $x \to x + u/3$ we can rewrite the curve in eq.(15.23) in the form:

$$y^2 = x^3 + ax + b$$  \hspace{1cm} (15.28)$$

where

$$a = -m^4 - u^2/3 \quad ; \quad b = -\frac{2}{3}u \left(m^4 - u^2/9\right)$$  \hspace{1cm} (15.29)$$

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Since a torus is uniquely identified by the complex parameter \(\tau\) varying in the fundamental region \(|\tau| \geq 1, -\frac{1}{2} \leq \text{Re}\tau \leq \frac{1}{2}\) or equivalently by the ratio

\[
r \equiv \frac{a^3}{b^2} = -\frac{9}{4} \frac{(m^4 + u^2/3)^3}{u^2(m^4 - u^2/9)^2}
\]

(15.30)

we see that, for each value of \(r\) corresponding to \(\tau\) in the fundamental region, we get six values of \(u\) by solving eq. (15.30). This means that \(u\) varies in a region that corresponds to 6 images of the fundamental region of \(SL(2,\mathbb{Z})\), that is, in fact, the fundamental region of \(\Gamma_2\).

Finally if we choose instead \(t = 3\) in eq. (15.21) we get the curve

\[
y^2 = x^3 - ux^2 + \frac{m^4}{3} x - \frac{m^4u}{27}
\]

(15.31)

A study of the monodromies around the points \(\pm m^2\) shows that their degrees is not infinite, as in the case of the curve in eq. (15.23) where one found logarithmic branch points. They do not seem to correspond to values of \(u\) where additional particles become massless. We do not discuss the curve in eq. (15.31) further in these lectures.

16 \(N = 4\) super Yang-Mills

In the previous sections we have seen that supersymmetry is an essential ingredient for having a dual theory in the sense of Montonen-Olive. We have also seen that \(N = 2\) super Yang-Mills is the simplest supersymmetric theory containing monopole and dyon solutions whose mass is fixed in the full quantum theory by the supersymmetry algebra and not just given by a BPS formula valid in the classical theory as in the case of the Georgi-Glashow model. \(N = 2\) super Yang-Mills has also the attractive feature of having a supersymmetry algebra that contains only two central charges, the electric and magnetic charges. On the other hand it is known from the work of Ref. [11] that the the monopole solution of the \(N = 2\) theory belongs to the hypermultiplet that does not contain a spin 1, while the \(W\)-bosons, that have spin 1, belong to the \(N = 2\) chiral multiplet. This shows immediately that the monopoles and the \(W\)-bosons of \(N = 2\) super Yang-Mills cannot be dual in the sense of Montonen-Olive. Therefore in the search of a theory in which the Montonen-Olive duality is realized one is brought to consider a theory with more supersymmetry and one is naturally led to \(N = 4\) super Yang-Mills.

We start by rewriting the Lagrangian of this theory, given in eq. (9.28), in the same notation used for the \(N = 2\) theory. We get:

\[
L = \frac{1}{4\pi} Im \left\{ \tau \left[ -\frac{1}{4} \left( F^{a}_{\mu\nu} F^{a\mu\nu} - iF^{a}_{\mu\nu} F^{a\mu\nu} * \right) + \frac{1}{2} \sum_{i=1}^{3} (D_{\mu}A_{i})^{a} (D^{\mu}A_{i})^{a} \right] + \right. \\
\left. 58 \right.
\]

\section*{58} F. Gliozzi, private communication
\[
\frac{1}{2} \sum_{i=1}^{3} (D_{\mu} B_i)^a (D^{\mu} B_i)^a - V(A_i, B_j) + 
\]
\[+ - \frac{i}{2} \bar{\psi}^a \gamma^\mu (D^\mu \psi)^a - \frac{1}{2} f^{abc} \bar{\alpha}^i A^b \bar{\psi}^c - i \frac{1}{2} f^{abc} \bar{\gamma}^5 B^b \bar{\psi}^c \}
\]
(16.1)

where the potential is equal to:
\[
V(A_i, B_j) = \frac{1}{4} f^{abc} A_i^a A_j^c f^{afg} A_f^a A_g^f + \frac{1}{4} f^{abc} B_i^b B_j^c f^{afg} B_f^b B_g^f + \frac{1}{2} f^{abc} A_i^a B_j^b f^{afg} A_f^a B_g^f
\]
(16.2)

and \(\tau\) is given in eq.(6.20).

It is known since long time that this theory, being free from ultraviolet divergences \([42, 43, 44, 45]\), has a vanishing \(\beta\)-function \([46]\) and no chiral anomaly. The vanishing of the \(\beta\)-function and the absence of the chiral anomaly follow respectively from eqs.(B.95) and (B.97) as explained at the end of Appendix B. \(N = 4\) super Yang-Mills is a conformal invariant theory at the full quantum level. Conformal invariance is spontaneously broken if some scalar field gets a non vanishing vacuum expectation value.

Many of the properties found in \(N = 2\) super Yang-Mills, as the existence of a manifold of inequivalent vacua and of monopole and dyon solutions, are also valid for \(N = 4\) super Yang-Mills. The supersymmetry algebra contains also a number of central charges \([11]\) (12 and not 2 as in the \(N = 2\) theory). As in the case of the Georgi-Glashow model all particles of the spectrum have electric and magnetic charges that lie on a two-dimensional lattice
\[
q + i g = q_0 (n_e + \tau n_m) \quad \tau = \frac{\theta}{2\pi} + i \frac{4\pi}{q_0}
\]
(16.3)
with periods \(q_0\) and \(q_0 \tau\) (\(q_0 = e\) is the electric charge of the \(W\)-boson). In this case, however, the charges are not modified by quantum corrections. Again as a consequence of the supersymmetry algebra the BPS states of the theory have a mass given by the classical formula:
\[
M = \sqrt{2}|a n_e + \tau a n_m| = \sqrt{2} \frac{|a|}{e} |q + i g|
\]
(16.4)
that, because of \(N = 4\) supersymmetry, is not modified by quantum corrections. The question now is how to select those states that are single particle states. This can be easily done if we restrict ourselves to BPS saturated states having a mass given in eq.(16.4). A single particle BPS-saturated state with mass \(M\) must be stable and this is the case if it cannot decay into a couple of BPS saturated states with mass \(M_1\) and \(M_2\), i.e.
\[
M < M_1 + M_2
\]
(16.5)
Using for the mass the expression in eq.(16.4) together with the exact expression for the charge given in eq.(16.3) one can easily see, by means of the Schwarz inequality,
Table 2: $N = 4$ monopole multiplet.

| STATE | $S_z$ | # OF STATES |
|-------|-------|-------------|
| $|0\rangle$ | 0 | 1 |
| $a_{\pm 1/2}^{(i)_\dagger} |0\rangle$ | $\pm 1/2$ | 4 |
| $a_{1/2}^{(i)} a_{1/2}^{(j)_\dagger} |0\rangle$ | 1 | 1 |
| $a_{-1/2}^{(i)_\dagger} a_{-1/2}^{(j)} |0\rangle$ | -1 | 1 |
| $a_{1/2}^{(i)} a_{-1/2}^{(j)} |0\rangle$ | 0 | 4 |
| $a_{1/2}^{(i)} a_{-1/2}^{(j)} a_{1/2}^{(k)_\dagger} |0\rangle$ | 1/2 | 2 |
| $a_{-1/2}^{(i)_\dagger} a_{1/2}^{(j)} a_{-1/2}^{(k)} |0\rangle$ | -1/2 | 2 |
| $a_{1/2}^{(i)} a_{1/2}^{(j)_\dagger} a_{-1/2}^{(k)_\dagger} |0\rangle$ | 0 | 1 |

that eq. (16.5) is satisfied if and only if the integers $(n_e, n_m)$ in eq. (16.3) are coprimes. This implies that the stable states with zero magnetic charge $(n_e, 0)$ are only the three states with $n_e = 0, \pm 1$; the states with magnetic charge corresponding to $n_m = \pm 1$ are all stable states; the states with magnetic charge corresponding to $n_m = \pm 2$ are only stable if their electric charge corresponds to odd values of $n_e$; the states with magnetic charge $n_m = \pm 3$ are stable if $n_e$ is different from 0 and is not a multiple of 3 and so on.

An explicit analysis of the fermionic zero modes, as we have done in the case of the $N = 2$ theory, shows that the number of zero modes is twice larger than that of the $N = 2$ theory. This means that, instead of the expansion given in eq. (12.59), we have in this case:

$$\chi = \sum_{i=1}^{2} \left[ a_{1/2}^{(i)} \chi_0^{1/2,i} + a_{-1/2}^{(i)} \chi_0^{-1/2,i} \right] + \Delta \chi$$

(16.6)

Therefore we get twice the number of creation and annihilation operators that we had in $N = 2$ super Yang-Mills and we can construct a bigger number of states given in Table 2. Those states fill a unique short representation of $N = 4$ supersymmetry containing one state of spin 1, four states of spin 1/2 and five states with spin 0. Both the $W$-bosons and the monopoles belong to this unique multiplet and this fact makes the realization of the Montonen-Olive duality in this theory possible.

17 Riformulation of Montonen-Olive duality

We are now in a position to riformulate the Montonen-Olive duality for $N = 4$ super Yang-Mills in a way in which the $W$-bosons, the magnetic monopoles and more in general all the dyons of the spectrum are treated in a completely democratic way. We will see that we will not just have an electric and magnetic description, but we
will have an infinite number of descriptions depending on which states of the charge lattice we are choosing as fundamental particles.

The usual formulation of the $N = 4$ super Yang-Mills is obtained by considering the states with zero magnetic charge and with electric charge equal to $\pm q_0$ corresponding to the $W$-bosons that get a mass through the Higgs mechanism, together with the massless states at the origin of the charge lattice having vanishing electric and magnetic charges and corresponding to the photon and Higgs particle. Selecting these states we have determined one of the periods of the lattice. The other period is also fixed when we specify the value of the angle $\theta$. We then ascribe a short $N = 4$ supermultiplet to each of the three states with charge equal to 0, $q_0$ and $-q_0$, and, having fixed the value of $\theta$, we can explicitly write the full Lagrangian of $N = 4$ super Yang-Mills containing only the states of the lattice that we have chosen. If the theory is dual in the sense of Montonen-Olive the other stable states of the charge lattice must appear as solitons or bound states of solitons.

On the other hand if the theory is dual in the sense of Montonen-Olive one could also start from another couple of stable states of the charge lattice corresponding to a certain dyon of the theory with a complex charge given by $\pm q'_0$ and with mass equal to $M = \sqrt{2}|a||q'_0|/e$, together with the massless photon and Higgs states located at the origin of the charge lattice and specify the vacuum angle $\theta$ by giving another vector $q'_0\tau'$ of the lattice that is not aligned with $q'_0$. We can again ascribe a $N = 4$ short multiplet to any of the states previously chosen and write, as before, a $N = 4$ super Yang-Mills Lagrangian containing the states with charges equal to 0 and $\pm q'_0$ and with a specified vacuum angle $\theta$. Also in this case the remaining stable states of the charge lattice will show up as solitons or bound states of solitons of the new Lagrangian. Duality in the sense of Montonen and Olive means that all the theories based on any pair of independent vectors of the charge lattice are equivalent.

Since the vectors $q'_0$ and $q'_0\tau'$ form an alternative basis of the charge lattice it must be possible to express them in terms of the original vectors $q_0$ and $q_0\tau$ through the relation:

$$q'_0\tau' = a q_0\tau + b q_0$$
$$q'_0 = c q_0\tau + d q_0$$  \hspace{1cm} (17.1)

with $a, b, c$ and $d$ integer numbers.

Since it must also be possible to express $q_0$ and $q_0\tau$ in terms of $q'_0$ and $q'_0\tau'$ the integer parameters of the transformation must satisfy the equation:

$$ad - bc = 1$$ \hspace{1cm} (17.2)

Therefore the transformations from a basis to another basis form the modular group $SL(2, Z)$.

Eqs. (17.1) imply a relation between $\tau$ and $\tau'$ given by

$$\tau' = \frac{a\tau + b}{c\tau + d}$$  \hspace{1cm} (17.3)

that provides a connection between the values of the parameters $(\theta, q_0)$ in the two choices of basis vectors and actions.
The modular group is generated by the two transformations:

\[ T : \quad \tau \to \tau + 1 \quad \theta \to \theta + 2\pi \quad (17.4) \]

that is a symmetry of the theory because the physics is periodic when we translate \( \theta \) by \( 2\pi \), and

\[ S : \quad \tau \to -\frac{1}{\tau} \quad q_0 \to \frac{4\pi \hbar}{q_0} \quad (i \text{f } \theta = 0) \quad (17.5) \]

that relates weak coupling with strong coupling.

The mass of the BPS-saturated states of the theory is proportional to the absolute value of the charge

\[ M \sim |q - ig| = |q_0(n_m \tau + n_e)| \quad (17.6) \]

and is left invariant if we transform \( \tau \) as in eq.(17.3) and \( q_0 \) and the charge vector \( \left(\begin{array}{c} n_m \\ n_e \end{array}\right) \) as follows

\[ q_0 \to q_0' = q_0(c\tau + d) \quad \left(\begin{array}{c} n_m \\ n_e \end{array}\right) \to \left(\begin{array}{c} n_m' \\ n_e' \end{array}\right) = \left(\begin{array}{cc} d & -c \\ -b & a \end{array}\right) \left(\begin{array}{c} n_m \\ n_e \end{array}\right) \quad (17.7) \]

with \( ad - bc = 1 \).

The modular group does not only perform a transformation from a system of basis vectors to another one, but acts also on the integer charge vector \( \left(\begin{array}{c} n_m \\ n_e \end{array}\right) \) rotating it into a new integer charge vector \( \left(\begin{array}{c} n_m' \\ n_e' \end{array}\right) \). In other words a modular transformation transforms \( q - ig \) expressed in terms of the basis vectors \( q_0 \) and \( q_0\tau \) and of the integers \( n_e \) and \( n_m \) into an expression having the same form in terms of the new basis vectors \( q_0' \) and \( q_0'\tau' \) and of the new integers \( n_e' \) and \( n_m' \) related to the old ones by eqs.(17.3) and (17.4). The invariance under the modular group requires that the presence in the spectrum of a state with a certain pair of integers implies also the presence of the state with other integers obtained from the first ones by the action of a modular transformation as in the second equation of (17.7).

In particular from eq.(17.4) it follows that, given the existence in the spectrum of the \( W^+ \)-boson corresponding to \( n_m = 0 \) and \( n_e = 1 \), the invariance under the modular group implies also the existence of the transformed state:

\[ \left(\begin{array}{c} 0 \\ 1 \end{array}\right) \to \left(\begin{array}{c} -c \\ a \end{array}\right) = \left(\begin{array}{cc} d & -c \\ -b & a \end{array}\right) \left(\begin{array}{c} 0 \\ 1 \end{array}\right) \quad (17.8) \]

Since the condition \( ad - bc = 1 \) is equivalent to require that \( c \) and \( a \) are coprimes, the existence of the \( W^+ \)-boson implies the existence in the spectrum of all stable states of the charge lattice as discussed at the end of the previous section. This is a direct consequence of the Montonen-Olive duality.
Let us consider the states with $c = -1$. They are of the type $\begin{pmatrix} 1 \\ a \end{pmatrix}$ where $a$ is an arbitrary integer. These are the dyons of $N = 4$ super Yang-Mills required, as discussed after eq.(6.17), by the $\theta$ periodicity corresponding to the generator $T$ of the modular group. The next case is $c = -2$. In this case we expect the existence of the states $\begin{pmatrix} 2 \\ a \end{pmatrix}$ where $a$ is odd. The existence of such states was shown by Sen [56]. Evidence for the existence of stable states with higher values of $c$ can be found in Ref. [57, 58, 59, 60, 61].

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A Appendix A

The action of the Georgi-Glashow model is invariant under the gauge transformations:

$$\Phi \rightarrow U\Phi U^{-1} \quad A_\mu \rightarrow UA_\mu U^{-1} + \frac{1}{ie} U \partial_\mu U^{-1}$$  \hspace{1cm} (A.1)

where

$$\Phi = \Phi_a T_a \quad A^\mu = A^\mu_a T_a$$  \hspace{1cm} (A.2)

and $T^a$ are the generator of the gauge group in the adjoint representation:

$$[T^a, T^b] = if^{abc} T^c \quad T^a_{AB} = if^{AaB}$$  \hspace{1cm} (A.3)

The covariant derivative and the Yang-Mills field strenght are given respectively by:

$$D_\mu \Phi = \partial_\mu \Phi + ie[A_\mu, \Phi] \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ie[A_\mu, A_\nu]$$  \hspace{1cm} (A.4)

Under a gauge transformation they transform as

$$D_\mu \Phi \rightarrow UD_\mu \Phi U^{-1} \quad F_{\mu\nu} \rightarrow UF_{\mu\nu} U^{-1}$$  \hspace{1cm} (A.5)
If we write $U = e^{iω^a T^a}$ we can write the action of the gauge transformation on the fields $Φ^a$ and $A^a_μ$. We get:

$$\delta Φ^a = -ε_{abc} ω^b Φ^c \quad \delta A^a_μ = -\frac{1}{e}(D_μ ω)^a$$ \quad (A.6)

In the second part of this appendix we show that the quantity $K$ defined in eq.(4.35) is an integer. In fact making use of the Stoke’s theorem we can rewrite eq.(4.35) in the following form:

$$K = \frac{1}{8πa^3} \int dΩ ε^{abc} Φ^a r^i ε_{ijk} \frac{∂Φ^b}{∂i} \frac{∂Φ^c}{∂j} \quad (A.7)$$

where $r^i = r^i/r$. Parametrizing the three components of $n^a ≡ Φ^a/a$ at spatial infinity, where $n^2 = 1$, in terms of the angles $ω$ and $χ$:

$$n^1 = \sin ω \cos χ \quad n^2 = \sin ω \sin χ \quad n^3 = \cos ω \quad (A.8)$$

and the unit sphere in three-dimensional space by:

$$\hat{r}^1 = \sin θ \cos ϕ \quad \hat{r}^2 = \sin θ \sin ϕ \quad \hat{r}^3 = \cos θ \quad (A.9)$$

the following relation can be shown

$$\frac{1}{2} ε^{abc} n^a \hat{r}^i ε_{ijk} \frac{∂n^b}{∂\hat{r}^j} \frac{∂n^c}{∂\hat{r}^k} = \hat{r}^i ϵ_{ijk} \frac{∂\cos ω}{∂\hat{r}^j} \frac{∂χ}{∂\hat{r}^k} = -\frac{1}{\sin θ} \frac{∂(\cos ω, χ)}{∂(θ, ϕ)} \quad (A.10)$$

where the last expression means the jacobian of the transformation from the variables $\cos ω$ and $χ$ to the variables $θ$ and $ϕ$. Introducing the previous identity in eq.(A.7) one gets:

$$K = -\frac{1}{4π} \int dθ dϕ \frac{∂(\cos ω, χ)}{∂(θ, ϕ)} \quad (A.11)$$

showing that $K$ just counts the number of times that one covers the two-dimensional sphere described by the variable $n^a$ when the sphere at infinity in space is covered once.

In the last part of this appendix we will explicitly solve the Bogomolny equation for the monopole and dyon.

Starting from the ansatz

$$Φ^a_0 = \frac{r^a}{er^2} H(ξ) \quad A^a_0 = \frac{r^a}{er^2} J(ξ) \quad A^a_0 = -ε_{aij} \frac{r^j}{er^2} [1 - K(ξ)] \quad (A.12)$$

it is easy to compute

$$B^a_i = -\frac{δ^a_i}{er^2} K' + \frac{r^i r^a}{er^4} \left[ξ K' + 1 - K^2 \right] \quad (A.13)$$
\(E_i^a = \frac{\delta a^i}{er^2} JK + \frac{r^i r^a}{er^4} \left[ \xi J' - J(1 + K) \right] \tag{A.14}\)

and

\[(D_i \Phi)_a = \frac{\delta a^i}{er^2} HK + \frac{r^i r^a}{er^4} \left[ \xi H' - H(1 + K) \right] \tag{A.15}\]

The ansatz in eqs.(A.12) automatically satisfies the first eq. in (4.51). The second eq. is satisfied by requiring \(\lambda = 0\). Inserting the expressions in eqs.(A.13) and (A.13) in eqs.(4.52) we get:

\[\xi K' = -K \hat{H} \quad \xi \hat{H}' = \hat{H} + 1 - K^2 \tag{A.16}\]

where \(\hat{H}(\xi) = H(\xi) \cos \theta = J(\xi) \coth \theta\).

If we insert instead eqs.(A.13), (A.14) and (A.15) in eqs.(4.31) and (4.32) and we write \(J(\xi), J_4(\xi)\) and \(J_5(\xi)\) in terms of \(R(\xi)\) as in eqs.(11.34) we get the following equations:

\[\xi K' = -K R \quad \xi R' = 1 - K^2 + R \tag{A.17}\]

where the constants \(\alpha, \beta\) and \(\gamma\) are related to \(\theta\) through the relations:

\[\alpha \sin \theta + \beta \cos \theta = 1 \quad \gamma = \alpha \cos \theta - \beta \sin \theta \tag{A.18}\]

that imply \(\alpha^2 + \beta^2 - \gamma^2 = 1\) and determine \(\alpha, \beta\) and \(\gamma\) as functions of \(\theta\).

In order to solve eqs.(A.16) we introduce the new functions \(h\) and \(k\):

\[\hat{H}(\xi) = -1 - \xi h(\xi) \quad K(\xi) = \xi k(\xi) \tag{A.19}\]

In terms of these new functions eqs.(A.16) become:

\[k' = hk \quad h' = k^2 \tag{A.20}\]

They imply

\[\frac{d}{d\xi} (k^2 - h^2) = 0 \quad \Rightarrow \quad k^2 - h^2 = \alpha \tag{A.21}\]

where \(\alpha\) is a constant that is determined by imposing the boundary conditions:

\[\lim_{\xi \to \infty} k(\xi) = 0 \quad \lim_{\xi \to \infty} h(\xi) = -1 \tag{A.22}\]

that are obtained from eqs.(4.27). Those boundary conditions require \(\alpha = -1\) and then from eq.(A.21) we get

\[h^2 - k^2 = 1 \tag{A.23}\]

Inserting it in the second eq. of (A.20) we get

\[h' = h^2 - 1 \quad \alpha \tag{A.24}\]
whose solution is
\[ h(\xi) = -\coth(\xi + \beta) \] (A.25)
where \( \beta \) is a constant to be determined. Inserting \( h(\xi) \) given in eq.(A.25) in the first eq. of (A.20) we get
\[ k' = -k [\coth(\xi + \beta)] \] (A.26)
whose solution is:
\[ k(\xi) = \gamma = \frac{1}{\sinh(\xi + \beta)} \] (A.27)

The finiteness of the energy in eq. (4.18) requires that \( \lim_{\xi \to 0} K^2 = 1 \). This limit is satisfied only if \( \beta = 0 \). Then eq.(A.23) implies \( \gamma^2 = 1 \). Choosing \( \gamma = 1 \) we arrive at the solution:
\[ h(\xi) = -\coth \xi \quad k(\xi) = \frac{1}{\sinh \xi} \] (A.28)
that, through the relations in eq.(A.19), correspond to the expressions in eqs.(4.47) and (4.48) and for \( \theta = 0 \) to those in eqs.(4.21).

## B Appendix B

In this appendix we start by introducing the Weyl spinor, the \( N = 1 \) supersymmetry transformations and we describe in some detail the chiral and vector superfields. We give also the expansion of the various terms of a \( N = 1 \) supersymmetric Lagrangian in terms of component fields. We follow the notation of the book by Wess and Bagger [62].

The generators of the Poincaré algebra are the translational generators \( P_\mu \) and the generators of the Lorentz transformations \( M_{\mu \nu} \). Under a transformation of the Poincaré group the coordinate \( x^\mu \) is transformed as
\[ x^\mu \to \Lambda^\mu_\nu x^\nu + a^\mu \] (B.1)
A particular representation of the Lorentz group is given by the Dirac spinors defined in terms of the fourdimensional \( \gamma \)-matrices
\[ \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu \nu} \] (B.2)
The quantity
\[ \frac{1}{2} \Sigma^{\mu \nu} = \frac{i}{4} (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu) \] (B.3)
satisfies the Lorentz algebra. The Poincaré generators acting on the Dirac spinors are
\[ P^\mu = i\partial^\mu \quad M^{\mu \nu} = x^\mu P^\nu - x^\nu P^\mu + \frac{1}{2} \Sigma^{\mu \nu} \] (B.4)
It is convenient to use the Weyl representation of the Dirac spinors:

\[ \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \]  
(B.5)

where

\[ \sigma^\mu = (\sigma^0, \sigma^i) \quad \bar{\sigma}^\mu = (\sigma^0, -\sigma^i) \]  
(B.6)

\( \sigma^i \) are the Pauli matrices that satisfy the relation

\[ \sigma^i \sigma^j = \delta^{ij} + i\epsilon^{ijk} \sigma^k \]  
(B.7)

Then

\[ \gamma_5 \equiv i\gamma^0 \gamma^1 \gamma^2 \gamma^3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \]  
(B.8)

In this representation the upper [bottom] two components have left [right] chirality:

\[ \Psi = \Psi_L + \Psi_R ; \quad \Psi_L = \frac{1 - \gamma_5}{2} \Psi \quad \Psi_R = \frac{1 + \gamma_5}{2} \Psi \]  
(B.9)

The left and right chirality spinor fields

\[ \Psi_L = \psi_\alpha \quad \Psi_R = \bar{\chi}^\dot{\alpha} \quad \Psi = \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^\dot{\alpha} \end{pmatrix} \]  
(B.10)

are called undotted and dotted Weyl spinors respectively. The generators of rotations and boosts are given by

\[ \frac{1}{2} \Sigma^{ij} = \frac{1}{2} \epsilon^{ijk} \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} \quad \frac{1}{2} \Sigma^{0i} = \frac{1}{2} \begin{pmatrix} -i\sigma^i & 0 \\ 0 & i\sigma^i \end{pmatrix} \]  
(B.11)

In Dirac theory the charge conjugated Dirac spinor is given by

\[ \Psi^c = C\bar{\Psi}^T \quad C = \omega \gamma^0 \gamma^2 = \omega \begin{pmatrix} -\sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix} \]  
(B.12)

with \(|\omega| = 1\) and \(\bar{\Psi} \equiv \Psi^\dagger \gamma^0\). Choosing for convenience \(\omega = -i\) we get

\[ \bar{\Psi}^T = \begin{pmatrix} \chi^\alpha \\ \bar{\psi}^\dot{\alpha} \end{pmatrix} \quad \Psi^c = \begin{pmatrix} i\sigma^2 \chi^\alpha \\ -i\sigma^2 \bar{\psi}^\dot{\alpha} \end{pmatrix} = \begin{pmatrix} \epsilon_{\alpha\beta} \chi^\beta \\ \epsilon^{\dot{\alpha}\dot{\beta}} \bar{\psi}^{\dot{\beta}} \end{pmatrix} \equiv \begin{pmatrix} \chi_\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix} \]  
(B.13)

where we have introduced the antisymmetric matrices:

\[ \epsilon_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \epsilon^{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \]  
(B.14)

In conclusion we have

\[ \Psi = \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^\dot{\alpha} \end{pmatrix} \quad \Psi^c = \begin{pmatrix} \chi_\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix} \]  
(B.15)
From eq. (B.13) one gets how to raise undotted spinors and lower dotted ones:

\[
\chi^\alpha = \epsilon^{\alpha\beta} \chi_\beta \quad \bar{\psi}_\dot{\alpha} = \epsilon_{\dot{\alpha}\dot{\beta}} \bar{\psi}^{\dot{\beta}}
\]  
(B.16)

where

\[
\epsilon^{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \epsilon_{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\]  
(B.17)

A Dirac spinor has four independent complex components. A Majorana spinor satisfies the property

\[
\Psi_M = \Psi_M^c
\]  
(B.18)

and has therefore only two independent complex components. A Dirac spinor is transformed under a Lorentz transformation by the following matrix

\[
S_{AB} = e^{-\frac{i}{4} \omega_{\mu\nu} \Sigma^{\mu\nu}}_{AB}
\]  
(B.19)

where

\[
\frac{1}{2} \Sigma^{\mu\nu} = \frac{i}{4} [\gamma^\mu, \gamma^\nu] = \begin{pmatrix} i\sigma^{\mu\nu} & 0 \\ 0 & i\bar{\sigma}^{\mu\nu} \end{pmatrix}
\]  
(B.20)

that are consistent with the following index structure

\[
(\sigma^{\mu\nu})_{\alpha\beta}^\dot{\alpha} \dot{\beta} \quad (\bar{\sigma}^{\mu\nu})_{\dot{\alpha}\dot{\beta} \alpha \beta}
\]  
(B.21)

We see that the left and right component of a Dirac spinor transform independently under a Lorentz transformation. A Dirac spinor is a reducible representation of the Lorentz group.

Under a Lorentz transformation an undotted spinor transforms as

\[
\psi_\alpha \rightarrow (S)_\alpha^\beta \psi_\beta \equiv (e^{\frac{1}{2} \omega_{\mu\nu} \sigma^{\mu\nu}})_\alpha^\beta \psi_\beta
\]  
(B.23)

while the Lorentz transformation of a dotted spinor is given by

\[
\bar{\chi}_\dot{\alpha} \rightarrow (\bar{S}^\top)^{\dot{\alpha}}_\dot{\beta} \bar{\chi}_\beta \equiv (e^{\frac{1}{2} \omega_{\mu\nu} \bar{\sigma}^{\mu\nu}})^{\dot{\alpha}}_\dot{\beta} \bar{\chi}_\beta
\]  
(B.24)

They are obtained from one another through the identity \([(\sigma^{\mu\nu})_\alpha^\beta]^{\dagger} = - (\bar{\sigma}^{\mu\nu})_{\dot{\alpha}\dot{\beta}}\). \(S\) is a matrix of \(SL(2, C)\). The Lorentz transformation of the undotted and dotted spinors obtained from the previous one by lowering or raising the index are given by:

\[
\psi^\alpha \rightarrow ([S^{-1}]^T)_\alpha^\beta \psi_\beta \quad \bar{\chi}_\dot{\alpha} \rightarrow ([\bar{S}^\top]^T)_{\dot{\alpha}}^{\dot{\beta}} \bar{\chi}_\beta
\]  
(B.25)
The superscript $T$ means the transposed matrix. Using the previous transformation rules it is easy to show that $\psi^\alpha \chi_\alpha$, $\bar{\psi} \bar{\chi}^\alpha$ and $\psi^\alpha (\sigma^\mu)_{\alpha\beta} \partial_\mu \bar{\chi}^\beta$ transform as scalars under Lorentz transformations. The following relation can also be easily shown

$$(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} = \epsilon^{\dot{\alpha}\beta} \epsilon_{\alpha\beta} \sigma^\mu_{\dot{\beta}\beta}$$

We list here a number of the useful identities

$$(\bar{\psi} \bar{\chi})^+ = \chi \psi \equiv \chi^\alpha \psi_\alpha = -\chi_\alpha \psi^\alpha = \psi^\alpha \chi_\alpha = \psi \chi$$

$$(\psi \chi)^+ = \bar{\chi} \bar{\psi} \equiv \bar{\chi}_\dot{\alpha} \bar{\psi}^{\dot{\alpha}} = -\bar{\chi}^{\dot{\alpha}} \bar{\psi}_{{\dot{\alpha}}} = \bar{\psi} \bar{\chi}$$

$$(\chi \sigma^\mu \bar{\psi})^+ = -\bar{\chi}_{\dot{\alpha}} (\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} \psi_\alpha = \psi^\alpha (\sigma^\mu)_{\alpha\beta} \bar{\chi}^\beta = - (\bar{\psi} \bar{\sigma}^\mu \chi)^+$$

where we have used $\{\psi, \chi\} = \{\bar{\psi}, \bar{\chi}\} = \{\psi, \bar{\chi}\} = 0$ and the definition $(\chi^\alpha \psi_\alpha)^+ \equiv \bar{\psi}_{\alpha} \bar{\chi}^\alpha$. Other useful formulas are

$$\theta^\alpha \theta^\beta = -\frac{1}{2} \epsilon^{\alpha\beta} \theta \theta \quad \theta_{\alpha} \theta_{\beta} = \frac{1}{2} \epsilon_{\alpha\beta} \theta \theta$$

$$\bar{\theta}^\dot{\alpha} \bar{\theta}^\dot{\beta} = \frac{1}{2} \epsilon^{\dot{\alpha}\dot{\beta}} \bar{\theta} \bar{\theta} \quad \bar{\theta}_{\dot{\alpha}} \bar{\theta}_{\dot{\beta}} = -\frac{1}{2} \epsilon_{\dot{\alpha}\dot{\beta}} \bar{\theta} \bar{\theta}$$

$$\theta^\alpha \bar{\theta}^{\dot{\alpha}} = \frac{1}{2} (\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} \theta \sigma_{\mu} \bar{\theta}$$

$$\theta \sigma^\mu \bar{\theta} \sigma^\nu \bar{\theta} = \frac{1}{2} \theta \bar{\theta} \bar{\theta} g^{\mu\nu}$$

$$\theta \psi \theta \chi = -\frac{1}{2} \psi \chi \theta \theta \quad \bar{\theta} \bar{\psi} \bar{\theta} \bar{\chi} = -\frac{1}{2} \bar{\theta} \bar{\psi} \bar{\chi}$$

$$\epsilon^{\alpha\beta} \frac{\partial}{\partial \theta^\beta} = -\frac{\partial}{\partial \theta^\alpha}$$

$$Tr(\sigma^\mu \bar{\sigma}^\nu) = 2 g^{\mu\nu}$$

$$Tr(\sigma^\mu \bar{\sigma}^\nu \sigma^\rho \bar{\sigma}^\sigma) = 2 (g^{\mu\nu} g^{\rho\sigma} + g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho} + i \epsilon^{\mu\nu\rho\sigma})$$

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Finally we list a set of relations that allow to go from a formulation in terms of Weyl spinors into a formulation in terms of Dirac spinors:

\[
\bar{\Psi}_1 \Psi_2 = \chi_1 \psi_2 + \bar{\psi}_1 \bar{\chi}_2 \\
\bar{\Psi}_1 \gamma_5 \Psi_2 = -\chi_1 \psi_2 + \bar{\psi}_1 \bar{\chi}_2 \\
\bar{\Psi}_1 \gamma^{\mu} \Psi_2 = \bar{\psi}_1 \bar{\sigma}^{\mu} \psi_2 - \bar{\chi}_2 \bar{\sigma}^{\mu} \chi_1 \\
\bar{\Psi}_1 \gamma^{\mu} \gamma_5 \Psi_2 = -\bar{\psi}_1 \bar{\sigma}^{\mu} \psi_2 - \bar{\chi}_2 \bar{\sigma}^{\mu} \chi_1 \\
\bar{\Psi}_1 \Sigma^{\mu \nu} \Psi_2 = i\chi_1 \sigma^{\mu \nu} \psi_2 + i\bar{\psi}_1 \bar{\sigma}^{\mu \nu} \bar{\chi}_2
\]

where we have used the following representation for the Dirac spinors in terms of Weyl spinors

\[
\Psi_1 = \left( \begin{array}{c} \psi_1^\alpha \\ \bar{\chi}_1^\alpha \end{array} \right) \quad \Psi_2 = \left( \begin{array}{c} \psi_2^\alpha \\ \bar{\chi}_2^\alpha \end{array} \right)
\]

Having established the formalism of Weyl spinors we introduce now the supersymmetry transformations and their action on the superfields. The supersymmetry algebra is an extension of the Poincaré algebra to include the supersymmetry generators \( Q_\alpha \) and \( \bar{Q}_{\dot{\alpha}} \). They satisfy the following (anti)commutation relations with themselves and with the generators of the Poincaré group:

\[
[P^{\mu}, Q_\alpha] = [P^{\mu}, \bar{Q}_{\dot{\alpha}}] = 0
\]

\[
[M^{\mu \nu}, Q_\alpha] = -i(\sigma^{\mu \nu})^\beta_\alpha Q_\beta
\]

\[
[M^{\mu \nu}, \bar{Q}_{\dot{\alpha}}] = -i(\sigma^{\mu \nu})^\dot{\beta}_{\dot{\alpha}} \bar{Q}_{\dot{\beta}}
\]

\[
\{Q_\alpha, Q_\beta\} = \{Q_{\dot{\alpha}}, Q_{\dot{\beta}}\} = 0
\]

\[
\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2\sigma^{\mu}_{\alpha \dot{\beta}} P^{\mu}
\]

The supersymmetry algebra can be viewed as a Lie algebra with anticommuting parameters. This observation motivates to define the corresponding group element:

\[
G(x, \theta, \bar{\theta}) = e^{ix_{\mu} P^{\mu} + \theta Q + \bar{\theta}\bar{Q}}
\]

Using the Hausdorff’s relation

\[
e^A e^B = e^{A+B+1/2[A,B]+...}
\]
where higher order terms are vanishing, we get
\[ G(a, \xi, \bar{\xi}) G(x^\mu, \theta, \bar{\theta}) = G(x^\mu + a^\mu + i \theta \sigma^\mu \xi - i \xi \sigma^\mu \bar{\theta}, \theta + \xi, \bar{\theta} + \bar{\xi}) \] (B.49)

As usual, a multiplication of two elements induces a change in the parameter space:
\[ x^\mu \rightarrow x^\mu + i \theta \sigma^\mu \xi - i \xi \sigma^\mu \bar{\theta} + a^\mu \]
\[ \theta \rightarrow \theta + \xi \]
\[ \bar{\theta} \rightarrow \bar{\theta} + \bar{\xi} \] (B.50)

This transformation for \( a^\mu = 0 \) is generated by the following operator:
\[ \xi Q + \bar{\xi} \bar{Q} \] (B.51)

They satisfy the following algebra
\[ \{ Q_\alpha, \bar{Q}_{\dot{\alpha}} \} = 2i \sigma^\mu_{\alpha \dot{\alpha}} \partial_\mu \]
\[ \{ Q_\alpha, Q_\beta \} = \{ \bar{Q}_\alpha, Q_{\dot{\beta}} \} = 0 \] (B.53)

A superfield \( \Phi(x, \theta, \bar{\theta}) \) is a function of the space-time variable \( x_\mu \) and of the two Weyl spinors \( \theta_\alpha \) and \( \bar{\theta}_{\dot{\alpha}} \). Under a supersymmetry transformation with parameters \( \xi^\alpha \) and \( \bar{\xi}^{\dot{\alpha}} \), a superfield transforms as follows
\[ \delta \Phi(x, \theta, \bar{\theta}) = \left[ (\xi Q) + (\bar{\xi} \bar{Q}) \right] \Phi(x, \theta, \bar{\theta}) \] (B.54)

It is useful to define the supersymmetric covariant derivative
\[ D_\alpha = \frac{\partial}{\partial \theta_\alpha} + i \sigma^\mu_{\alpha \dot{\alpha}} \bar{\theta}_{\dot{\alpha}} \partial_\mu \]
\[ \bar{D}_{\dot{\alpha}} = - \frac{\partial}{\partial \bar{\theta}_{\dot{\alpha}}} - i \theta_\alpha \sigma^\mu_{\alpha \dot{\alpha}} \partial_\mu \]
\[ D^\alpha = - \frac{\partial}{\partial \theta_\alpha} - i \bar{\theta}_{\dot{\alpha}} (\bar{\sigma}^\mu)^{\dot{\alpha} \alpha} \partial_\mu \]
\[ \bar{D}^{\dot{\alpha}} = \frac{\partial}{\partial \bar{\theta}_{\dot{\alpha}}} + i (\bar{\sigma}^\mu)^{\dot{\alpha} \alpha} \theta_\alpha \partial_\mu \] (B.55)

They anticommute with the supersymmetry generators given in eq. (B.52).

An arbitrary superfield can be expanded in terms of normal fields as follows
\[ F(x, \theta, \bar{\theta}) = f(x) + (\theta \varphi(x)) + (\bar{\theta} \bar{\chi}(x)) + (\theta \theta) m(x) + (\theta \bar{\theta}) n(x) + \theta \sigma^\mu \bar{\theta} v_\mu(x) + (\theta \theta) (\bar{\theta} \bar{\lambda}(x)) + (\theta \bar{\theta})(\theta \psi(x)) + (\theta \theta)(\bar{\theta} \bar{\psi}(x)) \] (B.56)

All higher powers of \( \theta \) and \( \bar{\theta} \) vanish.
From eq. (B.51) one can construct the transformations of the ordinary fields under supersymmetry. In particular it can be seen that the last component of a superfield transforms as a total derivative under supersymmetry:

\[ \delta d(x) = \frac{i}{2} \partial_\mu O^\mu \]  

(B.57)

where \( O^\mu = \xi \sigma^\mu \bar{\lambda} + \bar{\xi} \sigma^\mu \psi \). This observation will be very useful for constructing supersymmetric Lagrangians using the superfield formalism. The superfield introduced in eq. (B.56) is not reducible in general. In four dimensions the irreducible superfields are the chiral and the vector ones.

A chiral superfield is characterized by the condition

\[ \bar{D}_\alpha \Phi = 0 \]  

(B.58)

The above constraint is easily solved in terms of the two quantities

\[ y^\mu_+ = x^\mu + i \theta \sigma^\mu \bar{\theta} \quad ; \quad \theta^\alpha \]  

(B.59)

that satisfy the conditions

\[ \bar{D}_\alpha y^\mu_+ = \bar{D}_\alpha \theta = 0 \]  

(B.60)

Any function of these two variables will satisfy the condition in eq. (B.58) and therefore a chiral superfield can be written as follows

\[ \Phi = A(y_+) + \sqrt{2}(\theta \psi(y_+)) + (\theta \theta) F(y_+) = \]

\[ = A(x) + i(\theta \sigma^\mu \bar{\theta}) \partial_\mu A(x) - \frac{1}{4}(\theta \theta)(\bar{\theta} \bar{\theta}) \partial_\mu \partial^\mu A(x) + \]

\[ + \sqrt{2}(\theta \psi(x)) = \frac{i}{\sqrt{2}}(\theta \theta)(\partial_\mu \psi(x) \sigma^\mu \bar{\theta}) + (\theta \theta) F(x) \]  

(B.61)

The supersymmetry transformations of a chiral superfield in terms of the component fields are given by

\[ \delta A = \sqrt{2} \xi \psi \]

\[ \delta \psi_\alpha = \sqrt{2} \xi_\alpha F + i \sqrt{2} \sigma_{\alpha \beta} \bar{\xi} \sigma_\beta \partial_\mu A \]

\[ \delta F = i \sqrt{2} \bar{\xi} \sigma^\mu \partial_\mu \psi \]  

(B.62)

The superfield \( \bar{\Phi} \) will instead satisfy the constraint

\[ D_\alpha \bar{\Phi} = 0 \]  

(B.63)

It can be conveniently expressed in terms of the two variables

\[ y^\mu_- = x^\mu - i \theta \sigma^\mu \bar{\theta} \quad ; \quad \bar{\theta}^\alpha \]  

(B.64)

It is given by

\[ \bar{\Phi} = \bar{A}(y_-) + \sqrt{2}(\bar{\theta} \bar{\psi}(y_-)) + (\bar{\theta} \bar{\theta}) \bar{F}(y_-) = \]
\[ = \bar{A}(x) - i(\theta \sigma^\mu \bar{\theta}) \partial_\mu A(x) - \frac{1}{4}(\theta \theta)(\bar{\theta} \bar{\theta}) \partial_\mu \partial^\mu \bar{A}(x) + \]
\[ + \sqrt{2}(\bar{\theta} \bar{\psi}(x)) + \frac{i}{\sqrt{2}}(\theta \sigma^\mu \partial_\mu \bar{\psi}(x)) + (\theta \bar{\theta}) \bar{F}(x) \]  
(B.65)

It is also useful to give the transformations of the component fields belonging to the antichiral superfield

\[ \delta \bar{A} = \sqrt{2} \bar{\xi} \bar{\psi} \]
\[ \delta \bar{\psi}_i = \sqrt{2} \bar{\xi} \bar{\psi} F + i \sqrt{2} (\bar{\sigma}^\mu) \bar{\alpha} \theta \partial_\mu \bar{A} \]
\[ \delta F = i \sqrt{2} \bar{\xi} \sigma^\mu \partial_\mu \bar{\psi} \]  
(B.66)

In order to write the kinetic term of the Lagrangian of a chiral superfield in superfield notations it is useful to have the following product of superfields in terms of component fields

\[ \Phi_i \Phi_j = \bar{A}_i A_j + \sqrt{2} \theta \bar{\psi}_i \bar{A}_i + \sqrt{2} \theta \bar{\bar{\psi}} A_j + \theta^2 \bar{A}_i F_j + \bar{\theta}^2 \bar{F}_j z_j + \]
\[ + \theta \sigma^\mu \bar{\theta} \left[ i \left( \bar{A}_i \partial_\mu A_j - A_j \partial_\mu \bar{A}_i \right) - \bar{\psi} \bar{\sigma}_i \psi \right] + \sqrt{2} \theta^2 \bar{\theta} \bar{\psi} F_j + \sqrt{2} \bar{\theta}^2 \bar{\bar{\psi}} F_j + \]
\[ + \frac{i}{\sqrt{2}} \theta^2 \left[ \bar{\psi} \bar{\sigma}_i \partial_\mu \bar{A}_j - \theta \sigma^\mu \bar{\bar{\psi}} \partial_\mu \bar{A}_i \right] + \]
\[ + \frac{i}{\sqrt{2}} \bar{\theta}^2 \left[ \theta \sigma^\mu \partial_\mu \bar{\psi}_j A_i - \theta \sigma^\mu \bar{\bar{\psi}}_i \partial_\mu A_j \right] + \]
\[ + (\theta \theta)(\bar{\theta} \bar{\theta}) \left[ \bar{F}_i F_j - \frac{1}{4} \bar{A}_i \partial_\mu \partial^\mu A_j - \frac{1}{4} \partial_\mu \partial^\mu \bar{A}_i A_j + \frac{1}{2} \partial_\mu \bar{A}_i \partial^\mu A_j + \]
\[ + \frac{i}{2} \partial_\mu \bar{A}_i \partial^\mu \psi_j - \frac{i}{2} \bar{\psi}_i \bar{\sigma}^\mu \psi_j \right] \]  
(B.67)

We give also the following formulas for the term quadratic in the fields

\[ \Phi_i(x, \theta) \Phi_j(x, \theta) = \ldots + (\theta \theta) \left[ A_i F_j + F_i A_j - \psi_i \psi_j \right] \]  
(B.68)

and for the term cubic in the fields

\[ \Phi_i(x, \theta) \Phi_j(x, \theta) \Phi_k(x, \theta) = \ldots + \]
\[ + (\theta \theta) \left[ F_i A_j A_k + A_i F_j A_k + A_i A_j F_k - \psi_i \psi_j A_k - A_i \psi_j \psi_k - \psi_i A_j \psi_k \right] \]  
(B.69)

The most general renormalizable and supersymmetric action containing scalar and spinor fields is given by the sum of a kinetic term and a potential term:

\[ S = \int d^4x \left\{ \int d^2 \theta \ d^2 \bar{\theta} \ \bar{\Phi}_i \Phi_i + \left[ \int d^2 \theta W(\Phi) \right] + h.c. \right\} \]  
(B.70)
This action is automatically supersymmetric as follows from the observation (see \text{eq.}(B.57)) that the last component of a superfield transforms as a total derivative and from the fact that the integral in $d^2\theta d^2\bar{\theta}$ in the first term in eq.\,\,(B.70) selects just the last component of the real superfield in eq.\,\,(B.67), while the integral in $d^2\theta$ selects the last term of a chiral superfield. Renormalizability implies then that $W$ must contain at most a cubic power of the chiral superfields. By performing the integral over the Grassmann variables $\theta$ and $\bar{\theta}$ one gets:

\[
\int d^4x \left\{ \left[ \bar{F}_i F_i + \partial_\mu \bar{A}_i \partial^\mu A_i - i\bar{\psi}_i \bar{\sigma}^\mu \partial_\mu \psi_i \right] + \left[ F_i \frac{\partial W}{\partial A_i} - \frac{1}{2} \psi_i \psi_j \frac{\partial^2 W}{\partial A_i \partial A_j} + h.c. \right] \right\}
\]

The fields $F_i$ are non dynamical fields that can be eliminated by using their classical equation of motion:

\[
F_i = -\frac{\partial W}{\partial A_i}
\]

One gets finally the following Lagrangian

\[
L = \partial_\mu \bar{A}_i \partial^\mu A_i - i\bar{\psi}_i \bar{\sigma}^\mu \partial_\mu \psi_i - \left| \frac{\partial W}{\partial A_i} \right|^2 - \frac{1}{2} \psi_i \psi_j \frac{\partial^2 W}{\partial A_i \partial A_j} - \frac{1}{2} \bar{\psi}_i \bar{\psi}_j \frac{\partial^2 \bar{W}}{\partial \bar{A}_i \partial A_j}
\]

The vector superfield is a real superfield

\[
V = \bar{V}
\]

If we expand it in component fields we get

\[
V(x, \theta, \bar{\theta}) = f(x) + \theta \psi + \bar{\theta} \bar{\psi} + \theta \theta m(x) + \bar{\theta} \bar{\theta} \bar{m} - \theta \sigma^\mu \bar{\theta} V_\mu
\]

Under a gauge transformation the chiral superfields transform as follows

\[
\Phi \to e^{-2igA} \Phi \quad \quad \quad \Phi \to \Phi e^{2igA}
\]

where $\Lambda$ is a chiral superfield. The matrix $\Lambda$ can be written as

\[
\Lambda_{AB} = T^a_{AB} \Lambda^a \quad \quad \quad [T^a, T^b] = if^{abc} T^c
\]

where $T^a$ is the generator of the gauge group in the representation defined by the scalar superfield $\Phi$.

Therefore the kinetic term for the matter can be made supersymmetric and locally gauge invariant if we make the following substitution

\[
\int d^4\theta \bar{\Phi}_A \Phi_A \implies \int d^4\theta \bar{\Phi}_A \left( e^{2gV} \right)_{AB} \Phi_B
\]

if the vector superfield transforms as follows under a gauge transformation
\[ e^{2gV} \rightarrow e^{-2ig\bar{\Lambda}}e^{2gV}e^{2ig\Lambda} \]  

(B.79)

The previous gauge transformation for \( V^a \) is independent on the particular representation that one uses. In fact in computing the product of exponentials we encounter via the Hausdorff relation only commutators of generators. For an infinitesimal gauge transformation it can be shown that

\[
\delta(2gV) = 2ig(\Lambda - \bar{\Lambda}) + 2ig^2[V, \Lambda + \bar{\Lambda}] + \frac{2ig\bar{\alpha}}{3}[V, [V, \Lambda - \bar{\Lambda}]] + O(V^3) \]  

(B.80)

The existence of the inhomogenous term implies that we can choose the so-called Wess-Zumino gauge where we can gauge away many of the component fields present in \( V \). Since

\[
i(\Lambda - \bar{\Lambda}) = i\left\{ A - \bar{A} + i\theta\sigma^\mu\bar{\theta}\partial_\mu(A + \bar{A}) - \frac{1}{4}\theta^2\bar{\theta}^2\partial_\mu\partial^\mu(A - \bar{A}) + \right.
\]

\[
+ \sqrt{2}\theta\psi - \sqrt{2}\bar{\theta}\bar{\psi} + \frac{i}{\sqrt{2}}\theta^2\bar{\theta}\sigma^\mu\partial_\mu\bar{\psi} - \frac{i}{\sqrt{2}}\bar{\theta}^2\theta\sigma^\mu\partial_\mu\theta\psi + \theta^2F - \bar{\theta}^2\bar{F}\left\} \right. 
\]

(B.81)

By means of a gauge transformation we can choose \( A - \bar{A}, \psi, \bar{\psi}, F \) and \( \bar{F} \) in order to gauge away some of the component fields appearing in \( V \). In the Wess-Zumino gauge \( V \) can be written as follows

\[
V(x, \theta, \bar{\theta}) = -\theta\sigma^\mu\bar{\theta}A_\mu(x) + i(\theta\theta)(\bar{\theta}\lambda(x)) - i(\bar{\theta}\bar{\theta})(\theta\lambda(x)) + \frac{1}{2}(\theta\theta)(\bar{\theta}\bar{\theta})D(x) \]  

(B.82)

Remember also that

\[
V^2 = \frac{1}{2}\theta^2\bar{\theta}^2A^\mu A_\mu \]  

(B.83)

The superfield field strenght is given by

\[
W_\alpha = -\frac{1}{8g}(DD)e^{-2gV}D_\alpha e^{2gV} \quad \bar{W}_\dot{\alpha} = -\frac{1}{8g}(DD)e^{-2gV}D_{\dot{\alpha}} e^{2gV} \]  

(B.84)

They are chiral superfields

\[
D_\beta W_\alpha = D_{\dot{\beta}} \bar{W}_{\dot{\alpha}} = 0 \]  

(B.85)

In terms of component fields we get

\[
W_\alpha^a = -i\lambda_\alpha^a + \left[ \delta^\beta_\alpha D^a - i(\sigma^\mu)_\alpha^\beta F^a_{\mu} \right] \theta_\beta + (\theta\theta)(\sigma_\alpha^\mu)(D_\mu \bar{\lambda}_\alpha)^a \]  

(B.86)

and

\[
(\bar{W}_{\dot{\alpha}})^a = i(\bar{\lambda}_{\dot{\alpha}})^a + \left[ \delta^{\dot{\alpha}}_{\dot{\beta}} D^a + i(\bar{\sigma}^\mu_{\dot{\beta}})F^a_{\mu} \right] \bar{\theta}^{\dot{\beta}} - (\bar{\theta}\bar{\theta})(\bar{\sigma}^\mu)^{\dot{\alpha}}(D_\mu \lambda_\alpha)^a \]  

(B.87)
where
\[ (D_\mu \chi^\alpha)^a = \partial_\mu (\chi^\alpha)^a - gf^{abc} A_\mu^b (\chi^\alpha)^c \]  

(B.88)

For constructing a supersymmetric action the following formula is very useful
\[ W^\alpha W_\alpha = -\lambda^\alpha \lambda_\alpha - 2i\lambda^\alpha \left[ \delta_\alpha^\beta D - i(\sigma^{\mu\nu})_\alpha^\beta F_{\mu\nu} \right] \theta_\beta + \]
\[ + (\theta\theta) \left[ D^2 - 2i(\lambda^\alpha)(\sigma^{\mu\nu})_{\alpha\alpha} D_\mu \bar{\lambda}^\alpha - \frac{1}{2}(F_{\mu\nu}F^{\mu\nu} - iF_{\mu\nu}F^{\mu\nu}) \right] \]  

(B.89)

and
\[ \bar{W}_\dot{\alpha} \bar{W}^\dot{\alpha} = -\bar{\lambda}_\dot{\alpha} \bar{\lambda}^{\dot{\alpha}} + 2i\bar{\lambda}_\dot{\alpha} \left[ \delta^{\dot{\alpha}}_{\dot{\beta}} D + i(\bar{\sigma}^{\mu\nu})_{\dot{\beta}} F_{\mu\nu} \right] \bar{\theta}_{\dot{\beta}} + \]
\[ + (\bar{\theta}\bar{\theta}) \left[ D^2 - 2i(\lambda^\alpha)(\sigma^{\mu\nu})_{\alpha\dot{\alpha}} D_\mu \bar{\lambda}^{\dot{\alpha}} - \frac{1}{2}(F_{\mu\nu}F^{\mu\nu} + iF_{\mu\nu}F^{\mu\nu}) \right] \]  

(B.90)

where
\[ *F_{\mu\nu} = \frac{i}{2} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} \]  

(B.91)

The supersymmetric extension of Yang-Mills theory is given by
\[ \int d^2 \theta \frac{1}{4}[W^\alpha W_\alpha + h.c.] = -\frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a - i\bar{\lambda}_\alpha \sigma^{\mu\nu} D_\mu \lambda^\alpha + \frac{1}{2} D^2 \]  

(B.92)

while the supersymmetric extension of matter interacting with Yang-Mills theory is given by
\[ \int d^4 \theta \bar{\phi} e^{2gV} \Phi = (D_\mu A)^+(D_\mu A) - i\bar{\psi} \sigma^{\mu\nu} D_\mu \psi + \bar{F} F + \]
\[ + g\bar{A} T^a D^a A + \sqrt{2ig}\bar{A} T^a \lambda^a \psi - i\sqrt{2g}\bar{\psi} T^a \lambda^a A \]  

(B.93)

Finally by introducing a \( \theta \) term in eq.(B.92) and gauge and gaugino fields normalized in such a way to include the gauge coupling as in eq.(6.19) we can rewrite the Lagrangian of pure Yang-Mills theory as follows
\[ L = -\frac{i}{16\pi} \int d^2 \theta \tau W^2 + h.c. = \frac{1}{8\pi} Im \left\{ \int d^2 \theta \tau W^2 \right\} \]  

(B.94)

where \( \tau \) is given in eq.(B.20).

At the end of this appendix we give the one-loop formula for the \( \beta \)-function in a gauge theory containing together with the gluon also an arbitrary number of fermions and scalars. It is equal to:
\[ \beta(e) = \frac{e^3}{(4\pi)^2} \left[ -\frac{11}{3} c_G + \frac{1}{6} N_S c_S + \frac{4}{3} N_F c_F \right] \]  

(B.95)

where \( N_S \) is the number of real scalars, \( N_F \) is the number of Dirac fermions and the constant \( c \) depends on the representation of the various fields:
\[ Tr(T^a T^b) = c\delta^{ab} \]  

(B.96)
The generators $T$ are normalized in such a way to have $c = 1/2$ for the fundamental and $c = N$ for the adjoint of $SU(N)$.

From eq. (B.95) we obtain the $\beta$-function of $N = 2$ super Yang-Mills, given in eq. (11.10), if we insert $c = N_c$ for all fields and $N_S = 2$ and $N_F = 1$ since we have two real scalar fields and one Dirac fermion.

Finally inserting again $c = N_c$ and $N_S = 6$ together with $N_F = 2$ we obtain the $\beta$-function of $N = 4$ super Yang-Mills that is equal to zero.

The chiral anomaly for a system of $M$ Majorana fermions is given by:

$$\partial_\mu J^\mu_5 = 2Mc_F q_F q(x)$$  \hspace{1cm} (B.97)

where $q(x)$ is the topological charge density defined in eq. (11.5), $c_F$ is defined in eq. (B.96) and is related to the fermion representation and $q_F$ is the chiral weight of the fermions. In the case of $N = 2$ super Yang-Mills we have two Majorana fermions ($M = 2$) in the adjoint representation of $SU(N_c)$ ($c_F = N_c$) with chiral weight equal to 1 obtaining eq. (11.5). In the case of $N = 4$ super Yang-Mills it is easy to see that the Lagrangian in eq. (10.9) is invariant if the superfields $W$ and $\Phi_i$ transform under $U(1)_R$ as the fields $W$ and $\Phi$ in eq. (11.4) with weight 1 and $2/3$ (instead of 2 as in the second eq. (11.4)) respectively. As a consequence the fermionic components of the superfields $W$ and $\Phi_i$ transform with weight +1 and $-1/3$ respectively and all according to the adjoint representation of the gauge group. Adding the contributions of the four fermionic fields one gets zero and therefore $N = 4$ super Yang-Mills does not have a $U(1)_R$ anomaly.

C  Appendix C

In this Appendix we compute the central charge $Z$ of $N = 2$ super Yang-Mills given in eq. (12.46). In the case of $N = 2$ super Yang-Mills the supercharge in four dimensions can be obtained from the dimensional reduction of the supercurrent in eq. (9.11). After rescaling the fields as in eq. (6.19) one gets:

$$Q_A = \frac{1}{e^2} \int d^4x \left\{ \left[ \sigma_{\mu\nu} F_{\alpha}^{\mu\nu} - i\gamma_{\rho}(D^\rho A_4)_{\alpha}\gamma_5 + \gamma_{\rho}(D^\rho A_5)^{\alpha} + if^{abc} A_4^a A_5^c \right]_{AB} \left( \gamma^\rho \chi_\alpha \right)_B \right\}$$  \hspace{1cm} (C.1)

By saturating the supercharge in the previous equation with the supersymmetry parameter $\tilde{\alpha}_A$, introducing the Weyl spinors through the eqs.:

$$\chi = \left( \begin{array}{c} \psi_\alpha \\ \bar{\lambda}_{\dot{\alpha}} \end{array} \right) \quad \bar{\chi} = \left( \begin{array}{c} \lambda^\alpha \\ \bar{\psi}_{\dot{\alpha}} \end{array} \right)$$  \hspace{1cm} (C.2)

$$\alpha = \left( \begin{array}{c} \epsilon_{\alpha} \\ \bar{\beta}_{\dot{\alpha}} \end{array} \right) \quad \bar{\alpha} = \left( \begin{array}{c} \beta^\alpha \\ \bar{\epsilon}_{\dot{\alpha}} \end{array} \right)$$  \hspace{1cm} (C.3)

and

$$Q = \left( \begin{array}{c} Q_{\alpha}^{(1)} \\ Q_{\dot{\alpha}}^{(2)} \end{array} \right) \quad \bar{Q} = \left( \begin{array}{c} Q^{(1)}_{\alpha} \\ Q^{(2)}_{\dot{\alpha}} \end{array} \right)$$  \hspace{1cm} (C.4)
and remembering eq. (3.5) for $\gamma^\mu$ and eq. (3.8) for $\gamma_5$ together with

$$\sigma^{\mu\nu} = \frac{1}{4}[\gamma^\mu, \gamma^\nu] = \begin{pmatrix} (\sigma^{\mu\nu})^\beta_\alpha & 0 \\ 0 & (\bar{\sigma}^{\mu\nu})^\beta_\alpha \end{pmatrix}$$ (C.5)

we get

$$\alpha_A Q_A = \beta^\alpha Q^{(1)}_\alpha + \bar{\epsilon}_\tilde{\alpha} Q^{(2)}_{\tilde{\alpha}} =$$

$$= \frac{1}{e^2} \int d^3x \beta^\alpha \left[ \left(\sigma^{\mu\nu}\sigma^0\right)^{\alpha\beta}_a \psi^\alpha_\beta + ((D_\mu A_5)^a - i(D_\mu A_4)^a) \left(\sigma^{\mu\nu}\sigma^0\right)^{\beta_\alpha}_a \psi^\beta_\alpha \right] +$$

$$+ \frac{1}{e^2} \int d^3x \tilde{\epsilon}_\tilde{\alpha} \left[ \left(\bar{\sigma}^{\mu\nu}\sigma^0\right)^{\tilde{\beta}\tilde{\alpha}}_a \psi^\alpha_\beta + ((D_\mu A_5)^a + i(D_\mu A_4)^a) \left(\bar{\sigma}^{\mu\nu}\sigma^0\right)^{\tilde{\beta}_\tilde{\alpha}}_a \psi^\beta_\alpha \right]$$ (C.6)

where we have omitted the last term in eq. (C.11) because it is inessential in the calculation of the central charge of the supersymmetry algebra. From this eq. we extract

$$Q^{(1)}_\alpha = \frac{1}{e^2} \int d^3x \left\{ \left(\sigma^{\mu\nu}\sigma^0\right)^{\alpha\beta}_a \psi^\alpha_\beta + \sqrt{2} \left(\sigma^{\mu\nu}\sigma^0\right)^{\beta_\alpha}_a \psi^\beta_\alpha (\bar{D}_\mu \phi) a \right\}$$ (C.7)

and

$$\bar{Q}^{(2)}_{\tilde{\alpha}} = \frac{1}{e^2} \int d^3x \left\{ \left(\bar{\sigma}^{\mu\nu}\sigma^0\right)^{\tilde{\alpha}\tilde{\beta}}_a \psi^{\tilde{\beta}}_\tilde{\alpha} + \sqrt{2} \left(\bar{\sigma}^{\mu\nu}\sigma^0\right)^{\tilde{\beta}_{\tilde{\alpha}}}_a \psi^{\tilde{\beta}}_\tilde{\alpha} (\bar{D}_\mu \phi) a \right\}$$ (C.8)

where $\phi$ is given in eq. (10.8). On the other hand, using that $(\gamma^\mu)^\dagger = \gamma^0 \gamma^\mu \gamma^0$, we get

$$(\alpha_A Q_A)^* = \bar{Q} A =$$

$$= \frac{1}{e^2} \int d^3x \bar{\chi}^\alpha \left\{ - F^{\mu\nu}_a \gamma^0 \sigma_{\mu\nu} + (D_\mu A_5)^a \gamma^0 \gamma^\mu - i(D_\mu A_4)^a \gamma^0 \gamma_5 \gamma^\mu \right\} \alpha$$ (C.9)

Rewriting it in Weyl notation we get

$$Q_A \alpha_A = Q^{(2)}_{\alpha} \epsilon_\alpha + Q^{(1)}_{\alpha} \tilde{\beta} \tilde{\alpha} =$$

$$= \frac{1}{e^2} \int d^3x \left\{ - \lambda^a_\alpha \psi^\alpha_\beta \left(\bar{\sigma}^{\mu\nu}\sigma^0\right)^{\alpha\beta}_a + (D_\mu \phi)^a \lambda^a_\alpha \psi^\alpha_\beta \left(\sigma^{\mu\nu}\sigma^0\right)^{\beta\alpha}_a \right\}$$ (C.10)

From it we get

$$Q^{(2)}_{\alpha} = \frac{1}{e^2} \int d^3x \left\{ - \bar{\psi}^\alpha_\alpha \left(\bar{\sigma}^{\mu\nu}\sigma^0\right)^{\alpha\beta}_a + \sqrt{2} (\bar{D}_\mu \phi)^a \lambda^a_\alpha \left(\sigma^{\mu\nu}\sigma^0\right)^{\beta\alpha}_a \right\}$$ (C.11)

and

$$\bar{Q}_{\tilde{\alpha}}^{(1)} = \frac{1}{e^2} \int d^3x \left\{ - \lambda^a_\alpha \psi^\alpha_\beta \left(\bar{\sigma}^{\mu\nu}\sigma^0\right)^{\alpha\beta}_a + \sqrt{2} (D_\mu \phi)^a \lambda^a_\alpha \psi^\alpha_\beta \left(\sigma^{\mu\nu}\sigma^0\right)^{\beta\alpha}_a \right\}$$ (C.12)
The canonical equal-time anticommutation relations satisfied by $\chi$

$$\{\chi^\alpha_A(x, t), \chi^{\dagger b}_B(y, t)\} = e^2 \delta^{ab} \delta^{(3)}(\vec{x} - \vec{y}) \delta_{AB}$$  \hspace{1cm} (C.13)

imply the following anticommutation relations for $\psi$ and $\lambda$:

$$\{\psi^\alpha_a(x, t), \bar{\psi}^\dagger_b(\sigma^0)_{\hat{\alpha}\hat{\beta}}(\vec{y}, t)\} = e^2 \delta^{ab} \delta^{(3)}(\vec{x} - \vec{y}) \delta_{\hat{\alpha}\hat{\beta}}$$  \hspace{1cm} (C.14)

and

$$\{\bar{\lambda}^\hat{a}(x, t), \lambda^\alpha_b(\sigma^0)_{\alpha\hat{\beta}}(\vec{y}, t)\} = e^2 \delta^{ab} \delta^{(3)}(\vec{x} - \vec{y}) \delta_{\alpha\hat{\beta}}$$  \hspace{1cm} (C.15)

while all other anticommutators are vanishing.

Using the previous anticommutation relations one can compute

$$\{Q^{(1)}_\alpha, Q^{(2)}_\beta\} = \frac{\sqrt{2}}{e^2} \int d^3x F^a_{\rho\sigma} \left(\overrightarrow{D}_\mu \phi\right)^a \{\left(\sigma^\rho \sigma^\sigma \sigma^\mu\right)_\alpha^\gamma - \left(\sigma^\rho \sigma^0 \sigma^\sigma\right)_\alpha^\gamma\} \epsilon^{\alpha\beta\gamma}$$  \hspace{1cm} (C.16)

that can be written as

$$\{Q^{(1)}_\alpha, Q^{(2)}_\beta\} = \frac{1}{\sqrt{2}e^2} \int d^3x F^a_{\rho\sigma} \left(\overrightarrow{D}_\mu \phi\right)^a \epsilon_{\alpha\beta} T^{\rho\sigma}$$  \hspace{1cm} (C.17)

where

$$\epsilon_{\alpha\beta} T^{\rho\sigma} = \left\{\left(\sigma^\rho \sigma^\sigma \sigma^\mu\right)_\alpha^\gamma - \left(\sigma^\rho \sigma^0 \sigma^\sigma\right)_\alpha^\gamma\right\} \epsilon^{\alpha\beta\gamma}$$  \hspace{1cm} (C.18)

By saturating it with $\epsilon^{\beta\alpha}$ we get

$$2T^{\rho\sigma} = 4 \left[\eta^{\rho\mu} \eta^{\sigma0} - \eta^{\rho0} \eta^{\sigma\mu} - i\epsilon^{0\rho\sigma}\right]$$  \hspace{1cm} (C.19)

where in the last step we have used eq.(B.39). Inserting eq.(C.19) in eq.(C.17) we get

$$\{Q^{(1)}_\alpha, Q^{(2)}_\beta\} = \epsilon_{\alpha\beta} \hat{Z}$$  \hspace{1cm} (C.20)

where

$$\hat{Z} = \frac{2\sqrt{2}}{e^2} \int d^3x \partial_i \left\{\bar{\phi}_a \left[(F_a)^{0i} - i(*F_a)^{0i}\right]\right\}$$  \hspace{1cm} (C.21)

Finally remembering that

$$F_a^{0i} = E_{ai} \quad \quad *F_a^{0i} = B_{ai}$$  \hspace{1cm} (C.22)

and using eqs.(11.37) and (11.38) we arrive at eq.(12.46).

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