Quantum Action Principle in Relativistic Mechanics (II)

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Quantum Action Principle formulated earlier is used as a ground for a probabilistic interpretation of one-particle relativistic quantum mechanics. In this new approach the probability "flows" in the Minkowsky space being dependent on an inner time parameter which we interpret as a particle life time. The life time is determined as a function of observable parameters of the real experiment by means of an additional condition of stationarity for the quantum action.

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I. INTRODUCTION

The subject of the present work is the problem of probabilistic interpretation of relativistic quantum mechanics (RQM) (see, for example, [1]). It arises first as the problem of probabilistic interpretation of solutions of the simplest relativistic wave equation for one particle, Klein-Gordon (KG) equation (velocity of light is equal unity)

\[ \hat{F} \psi = \left( \frac{\theta_{\mu} \partial_{\mu}^2 + m^2}{\hbar^2} \right) \psi = 0, \]  

(1)

where \( \theta_{\mu} = (+1, -1, -1, -1) \) is the signature of the Minkowsky metrics, the summation is implied over repeated indices. But for its solution one had to reject one-particle delivery of the problem. In quantum field theory, which replaces RQM, the dynamical object is a quantum field but not one particle. The reason to return to the problem in the present work is a new formulation of quantum mechanics based on a quantum action principle (QAP) [2]. In the work [3] the attention was turned to a possibility of probabilistic interpretation of one-particle RQM in the framework of QAP. Instead of a wave function \( \psi(x) \) considered as a solution of the Eq. (1), the quantum dynamics in QAP is described by a wave functional \( \Psi[x(c)] \) on world lines \( x_\mu(c), c \in [0, C] \) of a particle in the Minkowsky space with fixed end points \( x_{0\mu} \equiv x_{0\mu}(0), x_{1\mu} \equiv x_{1\mu}(C) \). The wave functional has a natural probabilistic interpretation: \( |\Psi[x(c)]|^2 \) is a density of probability of particle movement along a world line from a small neighborhood of given world line \( x_{\mu}(c) \). However, a connection of the new framework with real measurements was not established. To achieve this connection we must define the parameter \( C \) as a function of kinematical parameters of a real experiment. For this purpose, we have to change the delivery of the scattering problem by introduction of wave packets instead of plane waves for asymptotic states, and be more considerable to experimental procedures of forming of initial state and detection of particles. Notice that in non-relativistic quantum mechanics the probabilistic interpretation of one-particle scattering problem needs introduction of wave packets for asymptotic states [4]. In the new approach to RQM the initial and final states of a particle are given by wave packets in the Minkowsky space with corresponding parameters of space-time coherence. The one-particle problem considered in the present work will be a base of a more complicated problem of many-particle scattering. Consideration of this problem in the framework of QAP will need a multy-time description of dynamics [5,6].

II. QUANTUM ACTION PRINCIPLE IN RELATIVISTIC MECHANICS

We begin with a canonical form of the classical action of relativistic particle,

\[ I = \int_0^C \left( p_\mu \dot{x}_\mu - \theta_{\mu} p_\mu^2 + m^2 \right) dc, \]

(2)

where the high limit of integration \( C \) has to be considered as an independent dynamical parameter. It is defined by a condition of stationarity for the action (2) calculated on solutions of classical equations of motion and is proportional to the proper time of the particle between given end points:

\[ C = \frac{\sqrt{\theta_{\mu} (x_{1\mu} - x_{0\mu})^2}}{2m}. \]

(3)

In the new approach to RQM proposed in the present work the parameter \( C \) will be defined after solution of all dynamical equations from a condition of stationarity for a quantum action.

The difference of this new approach from the original one is, first of all, in the operator realization of basic canonical variables. Now we introduce their operator realization in a space of wave functionals as follows:

\[ \hat{x}_\mu(c) \Psi \equiv x_\mu(c) \Psi, \quad \hat{p}_\mu(c) \Psi \equiv \frac{\hbar}{i} \delta \Psi \]

(4)

\[ \delta x_\mu(c) = \frac{i}{\hbar} \frac{\delta \Psi}{\delta x_\mu(c)}. \]
where the variational derivative is defined by the equality

\[ \delta \Psi = \int_0^C \frac{\delta \Psi}{\delta x_\mu (c)} \delta x_\mu (c) \, dc, \]  
(5)

and the constant \( \hbar \) differs from the "ordinary" Plank constant. For instance, its physical dimensionality is \( \text{Joule} \times \text{sec}^2 / \text{kg} \) (in accordance with dimensionality of \( C \)).

A connection between two constants will be established later. Let us introduce in the space of wave functionals the Hermitian scalar product

\[ (\Psi_1, \Psi_2) \equiv \int d^4x (c) \bar{\Psi}_1 [x (c)] \Psi_2 [x (c)]. \]  
(6)

If the normalization condition \( ||\Psi|| \equiv \sqrt{(\Psi, \Psi)} = 1 \) is fulfilled, then the wave functional \( \Psi [x (c)] \) has a natural probabilistic interpretation, namely, \( |\Psi [x (c)]|^2 \) is a probability density of particle movement along a world line from a small neighborhood of \( x_\mu (c) \). The operators defined by the Eq. (4) are formally Hermitian with respect to the scalar product (6).

Let us turn to the formulation of QAP as a basic dynamical principle for definition of a particle wave functional and corresponding quantum action. Let us introduce an action operator by replacing in Eq. (2) the basic canonical variables by operators defined by the Eq. (1):

\[ \hat{I} \equiv \int_0^C \left[ \frac{\hbar}{i} \frac{\delta}{\delta x_\mu (c)} + \hbar^2 \theta_\mu \frac{\delta^2}{\delta x^2_\mu (c)} + m^2 \right] dc. \]  
(7)

This operator is formally Hermitian under condition that the product of noncommuting operator multipliers in the first term under the integral is symmetrized. We formulate QAP as the eigenvalue problem for the action operator,

\[ \hat{I} \Psi = \Lambda \Psi. \]  
(8)

An eigenvalue \( \Lambda \) and the corresponding eigenfunctional \( \Psi [x (c)] \) of the action operator depend on the end points \( x_1, x_0 \), the invariant parameter \( C \), and an infinite set of parameters which define the state of motion of particle. The corresponding eigenfunctional \( \Psi [x (c)] \) depends on the complete set of these parameters too.

To look for a solution of the eigenvalue problem \( \hat{I} \Psi = \Lambda \Psi \), we introduce the exponential representation of a wave functional

\[ \Psi [x (c)] = \exp \left\{ \frac{i}{\hbar} S [x (c)] + R [x (c)] \right\}, \]  
(9)

and use a so called local approximation \( \exp \) for real functionals in the exponent,

\[ S [x (c)] = \int_0^C s (c, x (c)) \, dc, \]  
(10)

\[ R [x (c)] = \int_0^C r (c, x (c)) \, dc. \]  
(11)

Notice that the local approximation is singular in QAP, in so far as the repeated variational derivative in the action operator \( \hat{I} \) is proportional to \( \delta (0) \). However, QAP in the local approximation can be reduced to Schrödinger wave equation \( \hat{I} \) by means of a proper regularization of (7). It is achieved by division of the interval \([0, C]\) into small parts of equal length \( \varepsilon = C/N \), and approximation of a world line \( x_\mu (c) \) by a broken line with vertices \( x_{n\mu} \equiv x_\mu (n \varepsilon) \). Then the exponential functional (10) can be approximated by the product

\[ \Psi [x (c)] \approx \prod_{n=1}^{N-1} \psi (n \varepsilon, x_n), \]  
(12)

where

\[ \psi (c, x) = \exp \chi (c, x), \]

\[ \chi (c, x) \equiv \frac{i}{\hbar} s (c, x (c)) + \varepsilon r (c, x (c)), \]  
(13)

and the relation

\[ \hbar = \varepsilon \hbar \]  
(14)

was introduced \( \exp \). At the final stage, the limit \( \varepsilon \to 0 \) is assumed and the product \( \varepsilon r \) has to be considered as a unit symbol. Formally, this limit is senseless, because \( \hbar \to 0 \). It is a consequence of the accepted local approximation. However, the new operator realization of basic canonical variables \( \exp \), as well as the action operator \( \hat{I} \), has definite meaning in the discrete approximation. In the discrete approximation \( \exp \), a wave functional is a function of many variables (the coordinates of vertices \( x_{n\mu} \)) and the variational derivative can be approximated by partial derivative as follows \( \exp \):

\[ \frac{\delta \Psi}{\delta x_\mu (n \varepsilon)} \simeq \frac{1}{\varepsilon} \frac{\partial \Psi}{\partial x_{n\mu}}. \]  
(15)

As a result, the action operator \( \hat{I} \) becomes a differential operator with regularized second term under the integral. Considering the discrete approximation as an intermediate stage, we obtain in the limit \( \varepsilon \to 0 \) \( \exp \)

\[ \frac{i}{\hbar} \frac{\partial \Psi (c, x (c))}{\partial c} - \hat{F} \psi (c, x (c)) \]  
(16)

where

\[ Sch \psi (c, x (c)) \equiv \int_0^C \frac{\partial \psi (c, x (c))}{\partial c} \, dc, \]  
(17)
The right hand side of the Eq. (16) is equal to an eigenvalue of the action operator if the expression (17) equals zero for arbitrary world line \( x_p (c) \) with fixed end points. It is fulfilled for any solution of Schrödinger equation

\[
i\hbar \frac{\partial \psi}{\partial c} = \hat{F} \psi. \tag{18}\]

Incidentally, if the product in the first term under the integral in the Eq. (7) is symmetrized, then the eigenvalue is real and equals

\[
\Lambda = s (c, x(c))_0^C, \tag{19}\]

where \( s (c, x(c)) \) is the (real) phase function defined by the Eq. (13) for a solution \( \psi(c, x(c)) \). The latter will be called a wave function. Therefore, QAP in the local approximation is reduced to the Schrödinger equation (18) with the time parameter \( c \in [0, C] \), and the quantum action \( \Lambda \) is defined by boundary values of real phase of the wave function. The parameters, which define a quantum state of motion of particle, mentioned above, are initial data of the Cauchy problem for Schrödinger equation (13).

The notion of quantum action was introduced first by Dirac [3] which identified it with the (complex) phase of a wave function (13) \( (\hbar/2i) \psi \). We take only the real part of the phase. In fact, now it is the ordinary definition of quantum action, but it was obtained independently from QAP. Our formulation of QAP in terms of the eigenvalue problem (13) in a space of wave functionals differs from the Schwinger quantum action principle [10] which defines a quantum action as a Hermitian operator in a space of wave functions. However, conditions of stationarity of the quantum action with respect to inner parameters of the system in both approaches are identical.

Schrödinger equation (18) can be obtained without QAP as a result of formal application of the standard quantization procedure to the action (9). Such quantum theory is formal because the parameter \( c \) is not measurable. In fact, this equation arises at an intermediate stage of definition of the Feynman propagator (11), where at the final stage integration of its solution \( \psi(C, x) \) over the length of a time interval \( C \) with the measure \( C^{-2} \) is assumed. However, in the framework of QAP the integration is senseless, as far as the result loses a direct connection with the original wave functional \( \Psi [x (c)] \), and, as a consequence, a possibility of probabilistic interpretation given by QAP. Instead of this integration we will fix the constant \( C \) by means of a quantum action stationarity condition. Therefore, QAP in this case gives a connection of solutions of Schrödinger equation (18) with real measurements.

Probabilistic interpretation of a wave functional \( \Psi [x (c)] \) follows the probabilistic interpretation of the corresponding wave function \( \psi (c, x) \). Namely, \( |\psi (c, x)|^2 \) is a probability density of a particle to be find at the moment of time \( c \) in a small neighborhood of the point \( x \) in the Minkowsky space. This probabilistic measure has no direct physical meaning until it is not connected with the real measurement. We "bind" the parameter \( c \) with measurements as follows: at the moment \( c = 0 \) a free particle springs up, and at the moment \( c = C \) the particle disappears. Before the moment \( c = 0 \) the particle was in a bound state in a source (for example, an electron in cathode), and after the moment \( c = C \) it once again comes in a bound state in a detector (Faraday cylinder). Therefore, \( C \) is a life time of a free particle. This interpretation is in accordance with the Dirac interpretation of lover and upper limits of integrations in the action (2) [13].

### III. Probabilistic Interpretation of Relativistic Quantum Mechanics

Probabilistic interpretation of a wave function in ordinary quantum mechanics is based on a differential conservation law of the probability, which follows from Schrödinger equation. In the new framework the quantity \( |\psi (c, x)|^2 \) obeys a differential conservation law, as well, which is the consequence of the Eq. (18). This probability "flows" in the Minkowsky space, but not in the real one. If we shall fix the constant \( C \) by a condition of stationarity of the quantum action, the transition amplitude between initial and final states of particle will become a function of observable quantities, and the probability will be connected with real measurements. The quantum action itself will be defined as a (real) phase of the transition amplitude.

In the preceding section we have defined the quantum action for arbitrary solution of Schrödinger equation (18) as the difference (19) of phases of the initial and final states. Here we correct this formal definition, taking in place of a solution \( \psi(C, x) \) of the Cauchy problem a complex transition amplitude from a given initial state \( \psi_{in} (x_0) \) in a source to a given final state \( \psi_{out} (x_1) \) in a detector. We consider the detection process as a reduction of the final wave function \( \psi(C, x) \) to a state corresponding to the detector:

\[
K \equiv \left( \psi_{out} e^{-\frac{i}{\hbar} \hat{F} C} \psi_{in} \right). \tag{20}\]

The symmetry between the initial and final states presenting in the amplitude (20) will be need us latter for consideration of annihilation processes in the framework of QAP.

Let us give the initial and final states of a particle. The particle springs up in a source in a process (like an electron emission in a cathode), which is localized in a finite domain of space-time. We correlate the initial state with a wave packet in the Minkowsky space:

\[
\psi_{in} (x_0) = A_0 \exp \left[ -\frac{(x_{0\mu} - X_{0\mu})^2}{2\sigma_{0\mu}^2} - \frac{i}{\hbar} \gamma \mu P_\mu x_{0\mu} \right]. \tag{21}\]

According to (21), the source has the form of an ellipsoid in the Minkowsky space (in a fixed reference frame)
centered in the point $X_{0\mu}$, and having space-time dimensions $\sigma_{0\mu}$. In the momentum representation this state is described by a wave packet. It is centered in the point $p_{0\mu}$ of a momentum space and has minimal dimensions $\hbar/\sigma_{0\mu}$ admitted by Heisenberg uncertainty principle. We shall call this state as the De-Broglie wave with the momentum $p_{0\mu}$ and the parameters of coherence $\sigma_{0\mu}$. In accordance with the symmetry demand, the final state has to be parameterized analogously. In this case for the amplitude (20) we obtain the following representation:

$$K = A_0 A_1 \int d^4p \exp \left( -\frac{i}{\hbar} \theta_\mu p_\mu^2 \mathcal{C} + \frac{i}{\hbar} m^2 \mathcal{C} \right) \times$$

$$\exp \left[ \left( \frac{\sigma_{0\mu}^2}{2\hbar^2} \right) \left( p_\mu^2 - 2\mu \sigma_{0\mu}^2 p_{0\mu} + \sigma_{1\mu}^2 p_{1\mu} \right) \right] \times$$

$$\exp \left( \frac{i}{\hbar} \theta_\mu p_\mu \Delta X_\mu \right), \quad (22)$$

where $\Delta X_\mu \equiv x_{1\mu} - x_{0\mu}$. The first exponent under the integral is the evolution operator for the equation (18) in the momentum representation. However, in the present work we destroy the symmetry between initial and final states, taking $\sigma_\mu << \sigma_{0\mu}$. The process of detection of a particle (if we speak about a particle) has not to be characterized by large space-time domain. The final expression for the amplitude (20) in this case is

$$K = A \left( \prod_\mu \left[ 1 + 2\hbar \frac{\theta_\mu \mathcal{C}}{\sigma_{0\mu}} \right]^{-1} \times$$

$$\exp \left[ \frac{\sigma_{0\mu}^2 (p_{0\mu} + i\hbar \theta_\mu \Delta X_\mu/\sigma_{0\mu}^2)^2}{2\hbar^2} \left( \frac{\sigma_0^2 + 1}{\sigma_{0\mu}^2} \right) \mathcal{C} \right] \times$$

$$\exp \left( \frac{i}{\hbar} \theta_\mu p_\mu \Delta X_\mu \right) \right). \quad (23)$$

All multipliers which do not depend on the parameters $\Delta X_\mu$, and $\mathcal{C}$ are included in a common constant $A$. Summation over the index $\mu$ in the exponent is assumed as usual.

Therefore, in our experiment the particle-wave duality is transparent: a De-Broglie wave is prepared in a source, and a particle is registered in a detector. The probability of the process equals $|K|^2$, under condition that the life time of particle $\mathcal{C}$ is determined. The phase of the transition amplitude $K$ plays the role of the quantum action. From (23) one obtains that

$$\Lambda = m^2 \mathcal{C} - \left[ \frac{\hbar}{2} \arctan \frac{2\hbar \theta_\mu \mathcal{C}}{\sigma_{0\mu}^2} +$$

$$\theta_\mu \left( \frac{\sigma_{0\mu}^2 - \hbar^2 \Delta X_\mu/\sigma_{0\mu}^2}{1 + 4\hbar^2 \mathcal{C}^2/\sigma_{0\mu}^4} \right) \mathcal{C} - p_{0\mu} \Delta X_\mu \right]. \quad (24)$$

The condition of stationarity of the classical action (25) with respect to $\mathcal{C}$ gives the classical constraint for the initial momentum of particle,

$$\theta_\mu p_{0\mu}^2 - m^2 = 0, \quad (26)$$

which was not supposed from the beginning. In classical relativistic mechanics the Eq. (20) determines $\mathcal{C}$, if we take into account the classical relation:

$$p_{0\mu} = \Delta X_\mu/2C. \quad (27)$$

In quantum theory this relation in a precise form is absent but it can be considered as an experimental result. Indeed, the real part of the transition amplitude (24) which defines the probability “flow” is proportional to the exponent

$$\exp \left[ -\frac{(\Delta X_\mu - 2p_{0\mu}C)^2}{2(\sigma_{0\mu}^2 + 4\hbar^2C^2/\sigma_{0\mu}^2)} \right], \quad (28)$$

which predicts maximal probability of the registration of a particle for detectors localized in accordance with (27). Therefore, one can expect that in the classical limit the sought-for stationary value of $\mathcal{C}$ will be equal $\mathcal{C}$.

However, the complete quantum action (24) gives us a possibility to fix $\mathcal{C}$ definitely by means of the stationarity condition,

$$\frac{\partial \Lambda}{\partial \mathcal{C}} = 0, \quad (29)$$

without any additional equations of motion. Accounting quantum corrections to the classical action,

$$\Lambda = \Lambda_0 + \hbar^2 \Lambda_2 + ..., \quad (30)$$

where

$$\Lambda_2 = - \left[ 4\theta_\mu (p_{0\mu} \Delta X_\mu - p_{0\mu}^2 C) \frac{C^2}{\sigma_{0\mu}^2} +$$

$$\theta_\mu \left( \frac{1}{\sigma_{0\mu}^2} - \frac{\Delta X_\mu^2}{\sigma_{0\mu}^4} \right) \mathcal{C} \right], \quad (31)$$

one can obtain a quasi-classical decomposition of $\mathcal{C}$. In the zero approximation ($\hbar = 0$) the stationary value of $\mathcal{C}$ equals

$$C = \frac{8 \sum_\mu \theta_\mu p_{0\mu} \Delta X_\mu/\sigma_{0\mu}^2 \pm \sqrt{D}}{24 \sum_\mu \theta_\mu p_{0\mu}^2/\sigma_{0\mu}^4}, \quad (32)$$

where

$$D = 64 \left( \sum_\mu \theta_\mu p_{0\mu} \Delta X_\mu/\sigma_{0\mu}^2 \right)^2 +$$

$$48 \left( \sum_\mu \theta_\mu p_{0\mu}^2/\sigma_{0\mu}^4 \right) \left( \sum_\mu \theta_\mu \left( \frac{1}{\sigma_{0\mu}^2} - \frac{\Delta X_\mu^2}{\sigma_{0\mu}^4} \right) \right). \quad (33)$$
Here we first write the summation symbol. To select the sign in (22) one can use the classical limit (3). If detectors are sufficiently distant from the source, so that

$$\sum_{\mu} \theta_{\mu} \frac{\Delta X_{\mu}^2}{\sigma_{0\mu}} \gg \left| \sum_{\mu} \theta_{\mu} \frac{1}{\sigma_{0\mu}} \right|,$$

(34)

and the classical relation (27) is fulfilled (that is an experimental result), then the choice of the sign + entails the correct classical limit (3). Quantum corrections to $C$ we obtain by accounting higher order terms in the decomposition (28). Substituting the stationary value $eA_\mu = \Delta X_{\mu}$, we obtain (29).

Consider the nonrelativistic limit of the amplitude $K(p_0, \Delta X)$. Taking $p_{00} \approx m >> |p_{0k}|$, and $\Delta X_0 >> |\Delta X_k|$, we obtain from (29)

$$C \approx \frac{\Delta X_0}{2m},$$

i.e. in the non-relativistic limit the life time of particle $C$ is proportional to the ordinary Newtonian time. Then the part of the exponent (28), corresponding to $\mu = 0$, may be omitted, and the remaining part of the exponent describes a packet moving with the Newtonian time $t \equiv \Delta X_0$ in the ordinary three-dimensional space. According to (25), the phase of the packet in that limit is $mt$, so that the packet obeys Schrödinger equation with the Hamiltonian

$$-\left( m + \frac{\hbar^2}{2m} \Delta \right),$$

as it must be in the non-relativistic limit.

Coming back to the main problem, let us finish description of the experiment. As it is accepted in low energy electron diffraction, let a source of particles is placed in the interior of a large screen $\Theta$, so that the relation (24) is fulfilled. Therefore, the domain of the Minkowsky space, where a particle springs up, is placed into a space-time cylinder, which space section coincides with the interior of the screen, and having infinite length in the time direction. If whole inner surface of the screen is filled by detectors, a particle springing up in the source, will be detected sooner or later by someone of detectors. A ground for our assurance is the conservation law for of probability "flow" mentioned above. Indeed, the center of a wave packet, which springs up in the source at the moment $c = 0$, in accordance with (28), moves along a straight line in the ordinary space:

$$X_k(c) = X_{0k} + \frac{p_{0k}}{m}c,$$

the packet itself expands. Therefore, the packet "leaves" the cylinder with increasing the parameter $c$. It may be expressed in another words: the probability of a world line beginning in the source to remain in the interior of the cylinder, has zero limit when $c \to \infty$. Now, let the transition amplitude $K(p_0, \Delta X)$ obeys the normalization condition

$$\int_{\Theta} d\Omega \int_{0}^{\infty} d(\Delta X_0) |K|^2 = 1,$$

(35)

where the first integral is taken over angular variables on the screen $\Theta$ surface, and the second one - over the whole waiting time of snapping into action someone of detectors. Let us stress that time is an equal (along with the space coordinates) parameter of that distribution of probability. Finiteness of the integral over the time variable $\Delta X_0$ is ensured by a multiplier in front of the exponent in (28), which decreases like $\Delta X_0^{-2}$ when $\Delta X_0 \to \infty$, so that the normalization condition (35) is meaningfull. After that, the quantity $|K(p_0, \Delta X)|^2$ will play the role of a probability density of snapping into action someone of detectors in the screen surface (some time or other).

We have considered in the present work the simplest experiment with a free relativistic particle. In the presence of an external electromagnetic field the action (2) has to be replaced by the following one:

$$I = \int_{0}^{c} \left[ p_\mu \dot{x}_\mu - \theta_{\mu} (p_\mu - eA_\mu)^2 + m^2 \right] dc.$$

(36)

Formulation of QAP and corresponding scattering problem for the action (36) has no principal difficulties, it may be realized in the framework of perturbation theory on the electric charge $e$. Principal for the new approach is the problem of interaction of a particle with quantum electromagnetic field. It will be considered in the framework of QAP in a subsequent work.

**IV. CONCLUSIONS**

Therefore, quantum action principle gives a possibility of proper probabilistic interpretation of relativistic quantum mechanics in the simple one-particle scattering experiment. We consider the present work as a ground for consideration of multi-particle scattering problem in the framework of QAP.

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