ON CHARACTERIZATION OF TORIC VARIETIES

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Abstract. We study the conjecture due to V. V. Shokurov on characterization of toric varieties. We also consider one generalization of this conjecture. It is shown that none of the characterizations holds in dimensions $\geq 3$ (yet some weaker versions of the conjecture(s) are verified). In addition, we comment on the recent paper [5], claiming a “proof” of the conjecture, in the Appendix at the end.

1. Introduction

1.1. There is an abundance of results on characterization of algebraic varieties with a transitive group action. Some of those we are familiar with are [25], [16], [36], [6] (for projective spaces and hyperquadrics) and [13], [37] (for Grassmannians and other Hermitian symmetric spaces). At the same time, not very much is known in this respect for varieties with other, less transitive group actions. Amongst the first that come into ones mind are toric varieties and, more generally, reductive varieties (see [1], [2] (and also [4], [26]) for foundations of the reductive (resp. spherical) theory). In the same spirit, there is a related Hirzebruch problem on describing all compactifications of $\mathbb{C}^n$, the topic studied in numerous papers (see [27], [8], [7] and [29] for instance). Postponing the discussion of all these matters until some other time let us focus on the toric case.

To begin with, we mention the paper [18] (which elaborates on [15]), where an arbitrary smooth complete toric variety $T$ is characterized by the property that certain sheaf $\mathcal{R}_T \in \text{Ext}^1(\mathcal{O}_T^{\oplus h+1}(X, \mathbb{C}), \Omega^1_T)$ (called potential) splits into a direct sum of line bundles $\mathcal{O}_T(-D_\alpha)$. Unfortunately, this is not quite an effective criterion, and we are up to a “numerical” one.

1.2. Let $X \to Z \ni o$ be an algebraic variety over a scheme (germ) $Z \ni o$. Put $n := \text{dim } X$ and let $n \geq 2$ in what follows.

Consider a $\mathbb{Q}$-boundary $D := \sum d_i D_i$, where $D_i$ are (not necessarily prime) effective Weil divisors on $X$, $0 \leq d_i \leq 1$. Through the rest of the paper $X$ and $D$ will be subject to the following constraints:

- the pair $(X, D)$ is log canonical (or lc for short),
- the divisor $-(K_X + D)$ is nef,
- singularities of $X$ are $\mathbb{Q}$-factorial (or $X$ is $\mathbb{Q}$-factorial).

The last condition is actually redundant (and the first one is too general) for the forthcoming considerations. Indeed, one may always apply a dlt modification $h : \tilde{X} \to X$ with $\mathbb{Q}$-factorial $\tilde{X}$, divisorially log terminal.

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1) All varieties, if not specified, are assumed to be normal, projective and defined over $\mathbb{C}$. We will also be using freely the notions and facts from the minimal model theory (see [3], [20], [19]).
(\(\tilde{X}, h_\ast^{-1}(D)\)) and \(K_\tilde{X} + h_\ast^{-1}(D) + \Sigma \equiv h^\ast(K_X + D)\) for an \(h\)-exceptional divisor \(\Sigma\) \(^2\). Replacing the pair \((X, D)\) by \((\tilde{X}, h_\ast^{-1}(D) + \Sigma)\) does not effect the arguments of the present paper.

Let \(N^1(X)\) be the Néron–Severi group of \(X\). One defines the number \(r(X, D)\) as the dimension of the \(\mathbb{Q}\)-vector subspace in \(N^1(X) \otimes \mathbb{Q}\) spanned by all the \(D_i\) (alternatively, when one drops the \(\mathbb{Q}\)-factoriality assumption, \(r(X, D)\) is defined as the dimension of the \(\mathbb{Q}\)-vector subspace spanned by all the \(D_i\) modulo algebraic equivalence).

Note that \(r(X, D) \leq \rho(X)\) is the Picard number of \(X\). A finer relation between these gadgets is provided by the following:

**Conjecture 1.3** (V.V. Shokurov). For \((X/Z \ni o, D)\) as above the estimate \(\sum d_i \leq r(X, D) + \dim X\) holds. Moreover, the equality is achieved iff the pair \((X, D')\) is (formally) toric for some \(D' \geq \Sigma D_i\), i.e. \(X\) is toric and \(D'\) is its boundary (all over \(Z \ni o\)).

Clearly, when \(X\) is a genuine toric variety and \(D\) is its boundary, with \(D_i\) being irreducible and \(d_i = 1\) for all \(i\), then \(K_X + D \sim 0\) and \(\sum d_i = r(X, D) + \dim X\), thus motivating the last statement of Conjecture 1.3. Let us give another

**Example 1.4** (cf. [30, Example 1.3]). Take \(Z := X\) for \((X/Z \ni o, D)\) being a singularity germ. Let \(X := (xy + zt = 0) \subset \mathbb{C}^4\) and \(D := (xy = 0) \cap X\). Then we have \(r(X, D) = 1\) and \(K_X = 0\). Hence \(\sum d_i = r(X, D) + \dim X\) and \(X \ni o\) is obviously toric. In fact this is not a coincidence as the local version (i.e. with \(Z = X\)) of Conjecture 1.3 has been proved in \([19, 18.22–18.23]\). Namely, given \((X/X \ni o, D)\) such that each \(D_i\) is \(\mathbb{Q}\)-Cartier, the estimate \(\sum d_i \leq \dim X\) holds. Moreover, in the case of equality one has \(X \cong (\mathbb{C}^n \ni 0)/\mathfrak{A}\), where \(\mathfrak{A}\) is a finite abelian group acting diagonally on \(\mathbb{C}^n\) and \(D_i\) correspond to \(\mathfrak{A}\)-invariant hyperplanes \((x_i = 0) \subset \mathbb{C}^n\) under the factorization morphism \(\mathbb{C}^n \rightarrow X\). This implies that the pair \((X, \cup D_i)\) is toric.

**Remark 1.5.** In view of Example 1.4, it is tempting to ask whether \(X\) (and \(D\)) in Conjecture 1.3 is by any means related to a toric variety — say, whether \((X, \cup D_i)\) is formally (not necessarily regularly or even analytically) isomorphic to a toric pair? A char \(p > 0\) version of Conjecture 1.3 might also be of some interest: for instance, when the ground field is \(\mathbb{F}_p\), is \((X, \cup D_i)\) a toric pair (up to the Frobenius twist)? Finally, one may consider a weaker version of Conjecture 1.3 with “\(X\) is toric” replaced by “\(X\) admits a torification” (compare with e.g. [21]).

**1.6.** On trying to probe Conjecture 1.3 one may assume that \(Z \neq X\) (cf. Example 1.4). Then replacing \(Z\) by a formal neighborhood of \(o\) we are led to the case of \(Z = \text{Spec} \mathbb{C}[[t]]\). Taking an embedding \(\mathbb{C}[[t]] \hookrightarrow \mathbb{C}\) one may assume that actually \(Z = o\).

The first proof of Conjecture 1.3 for \(\dim X = 2\), has appeared in [34, Theorem 6.4]. The case when \(\dim X = 3\), the pair \((X, D)\) is plt and \(K_X + D \equiv 0\) was treated in [33], and Conjecture 1.3 has been proved in full there under the stated conditions \(^3\). Let us also point out a birational (“rough”) version of Conjecture 1.3 (cf. Theorem 1.15 below) proved in [30] assuming the Weak Adjunction Conjecture. Finally, a general strategy towards the proof of

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\(^2\) \(\equiv\) stands for the numerical equivalence.

\(^3\) Note that under the extremal condition \(\sum d_i = r(X, D) + \dim X\) all (potentially toric) threefolds in [33] turned out to be actually toric and smooth (see [33, Theorem 1.2]).
Conjecture 1.3 and more, was developed in [24] and illustrated there (in passing) at some crucial points. We now recall some of the matters from [24].

First of all one generalizes Conjecture 1.3 as follows. Put
\[
c(X, D) := r(X, D) + \dim X - \sum d_i
\]
(the quantity \( c(X, D) \) was called in [24] the \textit{complexity} of \((X, D)\)). Similarly, one defines
\[
ac(X, D) := \rho(X) + \dim X - \sum d_i
\]
(the \textit{absolute complexity}), and let us also introduce
\[
c(X) := \inf \{ c(X, D) \mid D \text{ is a boundary on } X \text{ such that } - (K_X + D) \text{ is nef and the pair } (X, D) \text{ is lc} \}
\]
for consistency, and analogously \(ac(X)\) with \(c(X, D)\) replaced by \(ac(X, D)\). Then one observes that for \(d_i\) all integer the condition \(c(X, D) = 0\) (so that the pair \((X, D)\) is toric according to Conjecture 1.3) is equivalent to \(c(X, D) < 1\). This led the author of [24] to make his

Conjecture 1.7 (J. McKernan). For \((X, D)\) as above the following holds:
1) \(c(X) \geq 0\),
2) if \(ac(X, D) < 2\), then \(X\) is a rational variety,
3) if \(c(X, D) < 1\), then there is a divisor \(D'\) such that the pair \((X, D')\) is toric. Moreover, \(\cup D_\cup \subseteq D'\) and \(D' - S\) is linearly equivalent to a divisor with support in \(D\), where \(S\) is either empty or an irreducible divisor.

One obviously has the implication Conjecture 1.7 \(\Rightarrow\) Conjecture 1.3

Remark 1.8. Note that it is not possible to lose any of the assumptions in Conjectures 1.3 and 1.7. Indeed, for \(X := \mathbb{P}^1 \times \mathbb{P}^1 \times E\) and \(D := 0 \times \mathbb{P}^1 \times E + \infty \times \mathbb{P}^1 \times E + \mathbb{P}^1 \times 0 \times E + \mathbb{P}^1 \times \infty \times E\), \(E\) is an elliptic curve, we have \(c(X, D) = c(X) = 1\), but \(X\) is not even rationally connected (cf. Conjecture 1.7 [2]). Also, taking \(X := \mathbb{F}_m\) and \(D := 2E_\infty + \sum_{i=1}^{m+2} F_i\), where \(E_\infty\) is the negative section and \(F_i\) are fibers of the natural projection \(\mathbb{F}_m \twoheadrightarrow \mathbb{P}^1\) (so that \(K_X + D = 0\)), we get \(c(X, D) \leq 0\), but the pair \((X, D)\) is non-toric (cf. Conjecture 1.3 [3]). At the same time, as one immediately verifies, slightly more general versions of Conjectures 1.3 and 1.7 studied in [34], [30], [33] and [24] are equivalent to those stated above.

1.9. The aim of the present paper is to show that Conjecture 1.3 is not true as stated when \(\dim X \geq 3\) (cf. Remark 1.9). But before spelling out the details let us give a (one of many possible)

Definition 1.10. We call an algebraic variety \(X\) \textit{fake toric} (or \(\mathfrak{f}\)-\textit{toric} for short) if there exists a \(\mathbb{Q}\)-boundary \(D\) on \(X\), a \textit{quasi-toric boundary} (or \textit{qt-boundary}), such that the pair \((X, D)\) matches all the conditions in 1.2 (up to passing to a \(\mathbb{Q}\)-factorialization as usual), the pair \((X, \cup D_\cup)\) is non-toric, but \(c(X, D) < 1\). Denote by \(\mathfrak{T}^{\dim}\) the class of all \(\mathfrak{f}\)-toric varieties of dimension \(n\).

Here is our main result:
Theorem 1.11. The class $\mathcal{T}^{f,n} \neq 0$ for every $n \geq 3$. More precisely, there exists $X \in \mathcal{T}^{f,n}$ with a qt-boundary $D$ such that $K_X + D \equiv 0$, $ac(X, D) = \frac{3}{4}$.

Theorem 1.11 is proved in Section 3 below (after some simple variants of Conjecture 1.7 — as discussed in 1.13). We stress that our construction of varieties from $\mathcal{T}^{f,n}$ was very much motivated by the exposition in [24]. In fact, the arguments of [24] rely on essentially two assertions, namely that an algebraic variety isomorphic to a toric variety in codimension 1 is also toric (see [24, §3]), i.e., one can always find an integral Weil divisor $D' \geq D$ (for $D$ considered up to the linear equivalence) such that the pair $(X, D')$ is lc and $K_X + D' \equiv 0$ (see [24, §4]). Conjecture 1.7 then follows from these two assertions and a dim $X$-inductive argument.

The outlined strategy applies well when dim $X = 2$ and justifies Conjecture 1.3 in this case (cf. [34, Proposition 2.1], [32, Theorem 8.5.1]). At the same time, rational scrolls (a.k.a. $\mathbb{F}$-bundles over $\mathbb{P}^1$) are all toric, so that it is reasonable to test Conjecture 1.7 on for the 2:1 projection bundles over $\mathbb{P}$ and a dim $X = 3$ one obtains $X$ as the blowup of a quadric $Q \subset \mathbb{P}^4$ in a line $\ell \subset Q$, followed by factorizing by two commuting $\mathbb{Z}/2$-actions. Both $\mathbb{Z}/2$ act on $Q$ and are constructed as follows. One $\mathbb{Z}/2$ corresponds to the Galois action for the 2:1 projection $Q \rightarrow \mathbb{P}^3$ and another $\mathbb{Z}/2$ is a lift to $Q$ of the Galois involution on $\mathbb{P}^3$ corresponding to the quotient morphism $\mathbb{P}^3 \rightarrow \mathbb{P}(1, 1, 1, 2)$ (one can easily see such a lifting does exist). It is then an exercise to find $D$ as in Theorem 1.11 and simple fan considerations show that $X$ is non-toric (see Section 4 for further details). Finally, we observe in this way that so obtained $X$ is singular, while for smooth $X$ Conjecture 1.3 has been proved recently in [35].

Remark 1.12. Let us describe, for consistency, the above mentioned “toric-in-codimension-1” and “$\mathbb{Q}$-complementary” assertions a bit more thoroughly. Firstly, in the former case given two varieties $X_1, X_2$ (we assume both $X_i$ to be $\mathbb{Q}$-factorial), with $X_1$ toric and $\theta : X_1 \rightarrow X_2$ an isomorphism in codimension 1, it is easy to see that the indeterminacy locus of $\theta$ is torus-invariant (use [20, Lemma 6.39] for instance). Furthermore, as soon as $X_1$ is a Mori dream space (see [12]), $\theta$ can be factored into a sequence of torus-invariant $\Delta$-flips with respect to a movable divisor $\Delta$ on $X_1$, so that $\theta$ and $X_2$ are also toric. This may work for instance when both $X_i$ admit integral boundaries $D_i$ and birational contractions $f_i : X_i \rightarrow Y$ such that $c(X_i, D_i) = 0$ and $Y$ (hence also $(Y, f_{i*}(D_i))$) is toric (see the discussion in [24] right before/after Definition-Lemma 3.4).

4) Note however that there is a substantial gap in [24] at this point. Namely, let $E_i$ be the $f_i$-exceptional divisor, inducing a discrete valuation $v_i$ on the field $\mathbb{C}(Y)$. Then it is claimed in [24] that (for $(X_i, D_i)$, etc., as given, but without any assumption on $c(X_i, D_i)$) there is a sequence of toric blowups of $Y$ extracting $v_i$ (starting with the blowup of $f_i(E_i)$). But this does not occur in general (globally at least) because the scheme $f_i(E_i)$ may not be reduced (this is a popular spot of erroneous usage of [20, Lemma 2.45] — replacing the initial scheme $f_i(E_i)$ by $f_i(E_i)_{\text{red}}$). For example, consider $Y := \mathbb{C}^2$ with (toric) boundary $\Delta := (xy = 0)$, and let $f_1 : X_1 \rightarrow Y$ be the blowup of the scheme $Z := ((x + y)^2 = y^3 = 0)$ (supported at $(0, 0)$). In other words, $f_1$ is the weighted blowup with weights $(3, 2)$, so that $K_X + f^{-1}_{1*}\Delta = f^{-1}_1(K_Y + \Delta) - E_1$ and the pair $(X_1, D_1) := f^{-1}_{1*}\Delta + E_1$ is lc. However, the map $f_1$ can not be extended to a toric morphism $\tilde{f}_1 : \tilde{X}_1 \rightarrow \mathbb{P}^2$ for toric surface $\tilde{X}_1$ compactifying $X_1$ (resp. $\mathbb{P}^2$ compactifying $Y$) because $(x + y)^2, y^3$, the defining ideal of $Z$, does not coincide with $(x^2, y^2)$. One can find non-toric $\tilde{X}_1$ and $\tilde{f}_1$ though (compare with $(X, D)$ in the proof of Lemma 1.3 below).
\( f_{\ast}(D_i) \) need not be supported on the toric boundary of \( Y \), which brings us to the “\( \mathbb{Q} \)-complementary” part. In the latter case however, one may expect the exceptional complements do not occur for the \( f \)-toric pairs, as well as for the toric ones (cf. [34], [31], [22], [23], [14]), and so the boundaries \( D_i, f_{\ast}(D_i) \) can probably be made toric.

### 1.13

We conclude by stating some positive versions of Conjecture 1.7 we were able to find:

**Proposition 1.14** (Yu. G. Prokhorov). Let \( (X, D) \) be as in 1.2 \( K_X + D \equiv 0 \), \( d_i = 1 \) for all \( i \). Then

1) \( c(X, D) \geq 0 \),

2) if \( c(X, D) = 0 \), then \( X \) is rationally connected. Furthermore, if \( X \) has only terminal singularities, then \( D \) has a rationally connected component.

(Proposition 1.14 is proved in Section 2. The arguments are entirely due to Yu. G. Prokhorov.)

**Theorem 1.15.** For \( (X, D) \) as in Proposition 1.14, if \( ac(X, D) = 0 \), then \( X \) is rational. Moreover, if \( ac(X, D) = 1 \) and \( X \) is rationally connected, then \( X \) is rational as well.

**Remark 1.16.** In Theorem 1.15, it might be possible to replace \( ac(X, D) \) (resp. \( D \) being integral) by \( c(X, D) \) (resp. \( \langle D_i \rangle \neq 0 \)), but we could not extend the arguments to this setting.

It is in the proof of Theorem 1.15 where we use the nice inductive trick suggested in [24]. Namely, since \( \text{Supp} \, D \) consists of \( \geq \rho(X) + 1 \) (irreducible) divisors, one can find a map (a “Morse function”) \( f : X \rightarrow \mathbb{P}^1 \) whose fibers satisfy the hypotheses of Theorem 1.15 after what we argue by induction on \( \dim X \) (see Section 2 for further details).

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### 2. Proof of Proposition 1.14

**2.1.** Firstly, as \( K_X \equiv -D \neq 0 \), we can run the \( K_X \)-MMP (see [3, Corollary 1.3.2]). Furthermore, since \( K_X + D \equiv 0 \), the components of \( D \) are not contracted on each step (see e.g. [20] Lemma 3.38). Note also that the quantity \( c(X, D) \) does not increase on each step. Hence we may assume there is an extremal contraction \( \phi : X \rightarrow Y \) which is a Mori fiber space. Finally, as one can easily see by restricting \( D \) to a general fiber \( F \) of \( \phi \), there is a horizontal component \( D_0 \subset D \), i.e. \( \phi(D_0) = Y \). One may take \( D_0 \) to be irreducible.

**Lemma 2.2** (cf. [33 Corollary 3.7.1]). We have either \( D_0 \cap D_i \neq \emptyset \) for all \( i \) or there exists \( j \) such that \( D_0 \cap D_j = \emptyset \). In the latter case, \( \phi : X \rightarrow Y \) is a conic bundle with two (generic) sections \( D_0, D_j \).

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5) Both \( D_0 \) and \( D_j \) may contain some fiber components.
Proof. Suppose first that \( \dim X/Y > 1 \). Then, since \( D_0|_F \) is ample, one gets \( D_0 \cap D_i \neq \emptyset \) for all \( i \). Further, if \( \dim X/Y = 1 \) and \( D_0 \) does not intersect at least two of \( D_i \), say \( D_0 \cap D_1 = D_0 \cap D_2 = \emptyset \), then \( D_0, D_1, D_2 \) are multisections of \( \phi \) and we obtain
\[
0 \equiv (K_X + D)|_F = K_F + D|_F > 0
\]
on \( F = \mathbb{P}^1 \), a contradiction. Thus there is \( D_j \) with \( D_j \cap D_0 = \emptyset \). Then, since \( K_F = \mathcal{O}_F(-2) \), we have \( D_0 \cdot F = D_j \cdot F = 1 \). Hence both \( D_0, D_j \) are sections of \( \phi \).

Consider the divisor
\[
B := \sum_{i \neq 0, \ D_i \cap D_0 \neq \emptyset} D_i
\]
and the normalization \( \nu : D_0^\nu \rightarrow D_0 \). Let \( B^\nu := \text{Diff}_{D_0}(B) \) be the different of \( B \) on \( D_0^\nu \) (see e.g. [35, §3]). Note that
\[
K_{D_0^\nu} + B^\nu = \nu^*((K_X + D)|_{D_0}) \equiv 0
\]
in our case.

Further, cutting with hyperplane sections we obtain that \( B^\nu = N + \nu^*B \), where \( N \) is the preimage of the non-normal locus on \( D_0 \) (cf. [35, §3]). Also, by Inversion of Adjunction (see [17]), the pair \((D_0^\nu, B^\nu)\) is lc (because \((X, D)\) is this). Hence we may replace the pair \((X, D)\) with \((D_0^\nu, B^\nu)\) (cf. [12]). Moreover, we have
\[
B^\nu = N + \nu^*B = N + \sum_{i \neq 0} \nu^*D_i,
\]
which easily yields \( c(D_0^\nu, B^\nu) \leq c(X, D) \) (see Lemma [22]). Proposition [14] follows by induction on the dimension (note that \( B^\nu \) is integral by the construction).

Suppose now that \( c(X, D) = 0 \). Then \( c(D_0^\nu, B^\nu) = 0 \) and \( D_0 \) is rationally connected by induction. Furthermore, since the fiber \( F \) is a Fano variety with only log terminal singularities, it is rationally connected (see [11, Corollary 1.3]). This implies that \( X \) is rationally connected as well (see [10, Corollary 1.3]). Finally, if \( X \) has only terminal singularities, then its dlt modification \( \hat{X} \rightarrow X \) is a small birational morphism, and Proposition [14] follows.

**Corollary 2.3.** For \( X \) as in Proposition [14] (in fact for any rationally connected \( X \)) and any \( Q \)-Cartier divisors \( L_1, L_2 \in \text{Pic}(X) \otimes \mathbb{Q} \), we have \( L_1 \equiv L_2 \iff L_1 \sim_\mathbb{Q} L_2 \).

**Proof.** One may replace \( X \) by its resolution \( X' \). Let \( g : X' \rightarrow X \) be a birational contraction. Let us also replace each \( L_i \) by \( g^*(L_i) \). The assertion now follows from \( h^1(\mathcal{O}_{X'}) = 0 \) (\( = \dim \text{Alb}_{X'} + 1 \)) provided by Proposition [14] (2). Indeed, for \( L_1, L_2 \) both Cartier and effective the condition \( L_1 \sim L_2 \) (equivalent to \( L_1 \equiv L_2 \) due to \( \text{Alb}_{X'} = 0 \)) simply asserts that there is a rational map (a.k.a. a function in \( \mathbb{C}(X) \)) \( f : X \rightarrow \mathbb{P}^1 \) with \( f^{-1}(0) = L_1, f^{-1}(\infty) = L_2 \). The argument for arbitrary \( \mathbb{Q} \)-Cartier \( L_i \) is similar. □
3. Proof of Theorem 1.15

3.1. We will assume for what follows that \( ac(X, D) = c(X, D) \leq 1 \). The proof of Theorem 1.15 will go essentially by induction on both \( \dim X \) and \( \rho(X) \). In the former case, \( \dim X = 2 \) is the base of induction (cf. the beginning of 1.6), while in the latter case we have the following:

**Proposition 3.2.** The assertion of Theorem 1.15 holds when \( \rho(X) = 1 \).

**Proof.** After taking the cyclic coverings of \( X \) w.r.t. various \( D_i \) (see e.g. [35 §2]), the arguments in the proof of [30 Corollary 2.8] apply and show that

\[ X \cong \mathbb{P}^n / \mathfrak{A} \quad \text{or} \quad Q / \mathfrak{B}, \]

where \( Q \subset \mathbb{P}^{n+1} \) is a smooth quadric, \( \mathfrak{A} \subset \text{Aut} \mathbb{P}^n \) (resp. \( \mathfrak{B} \subset \text{Aut} Q \)) is a finite abelian group with linearized action on \( \mathbb{P}^n \) (resp. \( \mathbb{P}^{n+1} \)), and the hyperplane sections \( D_i \) correspond to the only eigen vectors in \( H^0(\mathbb{P}^n, \text{O}_{\mathbb{P}^n}(1)) \) (resp. in \( H^0(Q, \text{O}_Q(1)) \)) which \( \mathfrak{A} \) (resp. \( \mathfrak{B} \)) scales non-trivially on. Note that the case \( X = \mathbb{P}^n / \mathfrak{A} \) has been treated already in [30 Corollary 2.8] and in fact the pair \( (X, D) \) turns out to be toric.

Suppose now that \( X = Q / \mathfrak{B} \). Let \( Q^\mathfrak{B} \) be the locus of \( \mathfrak{B} \)-fixed points on \( Q \).

**Lemma 3.3.** \( Q^\mathfrak{B} \neq \emptyset \).

**Proof.** Firstly, one easily finds a \( \mathfrak{B} \)-invariant line \( l \subset \mathbb{P}^{n+1} \), so that the locus \( l \cap Q \) is also \( \mathfrak{B} \)-invariant. Now, if \( l \cap Q \) is a finite set, then \( |l \cap Q| \leq 2 \). Put \( l \cap Q = \{ P_1, P_2 \} \) for some \( P_i \in Q \). Then it is evident that \( P_i \in Q^\mathfrak{B} \) (after possibly replacing \( l \) by another \( \mathfrak{B} \)-invariant line). Finally, the case when \( |l \cap Q| = \infty \), i.e. \( l \subset Q \), is obvious. \( \square \)

Pick any \( P \in Q^\mathfrak{B} \) and consider the \( (\mathfrak{B} \)-equivariant birational) linear projection \( Q \rightarrow \mathbb{P}^n \) from \( P \). Then the \( \mathfrak{B} \)-action descends to that on \( \mathbb{P}^n \) and both \( Q / \mathfrak{B}, \mathbb{P}^n / \mathfrak{B} \) are birationally isomorphic, with rational \( \mathbb{P}^n / \mathfrak{B} \). Proposition 3.2 is proved.

**Remark 3.3.** In general, when \( X \) and each of \( D_i \) are defined over a field \( k \) (\( \text{char} k = 0 \)), one can easily adjust the arguments from the proof of Proposition 3.2 to show that \( X \) is \( k \)-rational, provided \( \rho(X \otimes_k \bar{k}) = 1 \) over the algebraic closure \( \bar{k} \).

3.5. Let \( \phi : X \rightarrow Y \) be an extremal contraction. Put \( D_Y := \phi_*(D), D_{Y,i} := \phi_*(D_i) \), and suppose that \( \phi \) is divisorial. Note that \( \phi_*K_X = K_Y \) because the singularities of \( Y \) are all rational. Hence \( K_Y + D_Y = \phi_*(K_X + D) \equiv 0 \) and \( c(Y, D_Y) \) is defined.

**Lemma 3.6.** Given \( Y, \phi \) as above, the assertion of Theorem 1.15 holds for \( X \).

**Proof.** It is clear that \( 0 \leq c(Y, D_Y) \leq c(X, D) \). Then the claim follows from Proposition 3.2 and induction on \( \rho(X) \). \( \square \)

We now turn to the construction of a pencil on \( X \) as was indicated in 1.13. Firstly, we may assume that \( D_1 \sim_\mathbb{Q} \sum \delta_i D_i =: D'_1 \) for some \( \delta_i \in \mathbb{Q} \), where \( \delta_2 \neq 0 \), say (see Corollary 2.3). Secondly, we may assume without loss of generality that all \( \delta_i \geq 0 \) (replacing, if necessary, \( D_1 \) by \( D_1 + \sum_j \delta'_j D_j \) for some \( \delta_j \in \mathbb{Q}_{\geq 0} \)), which gives a rational map \( f : X \rightarrow \mathbb{P}^1 \) with \( kD_1 = f^{-1}(0) \) and \( kD'_1 = f^{-1}(\infty) \) for some integer \( k \geq 1 \). Finally, since the
indeterminacies of $f$ are located on some of $D_i$, we will assume (for the sake of simplicity) that $f$ is undefined precisely on $D_1 \cap D_2$ (in general, one just has to consider a larger number of $D_j$, satisfying $D_1 \cap \bigcup D_j$ = the indeterminacy locus of $f$).

Further, there is a local analytic isomorphism

$$(X, D_1 + D_2, D_1 \cap D_2) \simeq \left(\mathbb{C}^n, \{x_1 x_2 = 0\}, \{x_1 = x_2 = 0\}\right) / \mathbb{Z}_m(1, q, 0, \ldots, 0)$$

at the general point of $D_1 \cap D_2$, where $m, q \in \mathbb{N}, (m, q) = 1$ (see e.g. [35, Proposition 3.9]). In particular, if $\phi : \hat{X} \to X$ is the blowup of $D_1 \cap D_2$, then we have $K_{\hat{X}} + \phi^{-1}_*(D) + E \equiv \phi^*(K_X + D)$ for the $\phi$- exceptional divisor $E$. Thus after taking a number of subsequent blowups of $D_1 \cap D_i$ we may assume that $f : X \to \mathbb{P}^1$ is regular.

3.7. Let $F$ be a fiber of $f$. Note that $K_X \cdot Z = -D \cdot Z \leq 0$ for generic $F$ and a curve $Z \subset F$.

**Lemma 3.8.** $D \cdot Z > 0$ for some $Z$ and $F$ as above.

**Proof.** Assume the contrary. Then $D_i \cdot Z \leq 0$ for all $i$ and $Z \subset F$. Hence $f(D_i) = \text{pt}$ for all $i$. But the latter is impossible because $D_i$ generate $\text{Pic}(X) \otimes \mathbb{Q}$.

Among those $Z \subset F$ with $K_X \cdot Z < 0$ (see Lemma 3.8) there is an extremal ray $R$ of the cone $\overline{NE}(X)$. The curve representing $R$ sits in some irreducible component $F' \subset F$, so that once $F$ is reducible, $F \setminus F' \neq \emptyset$, we get an extremal birational contraction $\phi : X \to Y$ over $\mathbb{P}^1$. We eliminate the case when $\phi$ is flipping by applying the $K_X$- flip and termination of MMP with scaling. On the other hand, if $\phi$ is divisorial (in which case it contracts $F'$), then we are done by Lemma 3.6.

Thus we may assume that $\rho(X) = 2$ (i.e. $f : X \to \mathbb{P}^1$ is a Mori fibration).

**Lemma 3.9.** $F$ is $\mathbb{Q}$-factorial, $\rho(F) = 1$, $D'_1 = D_2$ and $D_1, D_2$ are the only components of $D$ contracted by $f$.

**Proof.** This is proved in [30, Proposition 3.6, Lemma 2.10].

Now exactly as in [30, Lemma 2.10, (iii)], Lemma 3.9, Remark 3.4 and induction on dim imply rationality of $X$. Theorem 1.15 is completely proved.

4. **Proof of Theorem 1.11**

4.1. Let us introduce the following two automorphisms of $\mathbb{P}^4$:

$$g_1 : [x : y : z : t : w] \mapsto [y : x : z : t : w]$$

and

$$g_2 : [x : y : z : t : w] \mapsto [-x : -y : -z : t + w : -w].$$

Consider the group $G$ generated by $g_1, g_2$. We have $G = \mathbb{Z}/2 \times \mathbb{Z}/2$ (although the $G$-action is not linear). Let also

$$\ell := (x - y = t = w = 0) \subset \mathbb{P}^4.$$
be the \( G \)-invariant line. Note that \( \ell \subset Q \) for the quadric \( Q \subset \mathbb{P}^4 \) given by the equation

\[
(x + y - w)w + (x - y)^2 + (t + w)t + zw = 0.
\]

This \( Q \) is obviously \( G \)-invariant and has \( o := [1 : 1 : -2 : 0 : 0] \) as its unique singular point. Note also that \( g_1 \) (resp. \( g_2 \)) acts identically on the hyperplane \( H_1 := (x - y = 0) \) (resp. \( H_2 := (t + w/2 = 0) \)).

4.2. Let \( \phi : W \to Q \) be the blowup of \( \ell \). Then \( \phi \) is \( G \)-equivariant by construction. Set \( X := W/G \) together with the quotient map \( p : W \to X \). Let us also define \( \Delta := \phi^*\mathcal{O}_Q(1), \Sigma := \phi^{-1}\ell, R_i := \phi_i^{-1}H_i \).

**Lemma 4.3.** \( R_1 \cup R_2 \cup \Sigma \subset W \) is precisely the codimension 1 ramification locus of \( p \).

**Proof.** Threefold \( W \) (resp. \( Q \)) can be identified with the Proj of the \( \mathbb{C} \)-algebra \( A_W := \bigoplus_{k \geq 0} H^0(W, k\Delta) \oplus H^0(W, k\Sigma) \) (resp. \( A_Q := \bigoplus_{k \geq 0} H^0(W, k\Delta) \)) so that the inclusion \( A_Q \subset A_W \) corresponds to \( \phi \).

Now, considering the \( G \)-invariant subalgebras in \( A_W, A_Q \) and taking Proj we get the following commutative diagram of \( G \)-equivariant morphisms:

\[
\begin{array}{ccc}
W & \xrightarrow{p_1} & W/g_1 \\
\phi \downarrow & & \downarrow \\
Q & \xrightarrow{p_2} & X \\
\end{array}
\]

where the vertical (resp. horizontal) arrows signify birational (resp. 2 : 1) morphisms, \( p = p_2 \circ p_1 \) and \( G \) acts on \( \mathbb{P}^3 = H_1 \) via \( g_2 \) as follows:

\[
[x : z : t : w] \mapsto [-x : -z : t + w : -w].
\]

In particular, \( g_2 \) acts identically on the hyperplane \( (t + w = 0) \subset \mathbb{P}^3 \), which implies that \( \mathbb{P}^3/g_2 = \mathbb{P}(1, 1, 1, 2) \). Furthermore, the product \( g_1g_2 \) acts identically on \( \Sigma \), as can be seen directly by computing the normal bundle of \( \ell \setminus o \subset Q \setminus o \) (the latter reduces to \( \ell \subset \mathbb{P}^3 \) by noting that \( (w = 0) \) is tangent to \( Q \) along \( \ell \)).

It follows that the ramification divisor of the quotient morphism \( p_1 : W \to W/g_1 \) (resp. of \( p_2 \)) is \( R_1 \) (resp. \( p_1(R_2) \cup p_1(\Sigma) \)). This exactly means that \( R_1 \cup R_2 \cup \Sigma \) is the codimension 1 ramification locus of \( p \). \( \square \)

Let \( \alpha : X \to \mathbb{P}^3/g_2 = \mathbb{P}(1, 1, 1, 2) \) be as in (4.4). This is an extremal birational contraction of the surface \( \bar{\Sigma} := p(\Sigma) \) (scheme-theoretic image). There is another extremal contraction on \( X \) which we now describe (this will be used later when computing \( ac(X) \)).

**Remark 4.5.** After an appropriate coordinate change one may assume that \( Q = (xy - zt = 0) \) and \( \ell = (x = y = z = 0) \). Then a simple local computation shows that \( \Sigma = F_1 \cup \mathbb{P}^2 \), with \( F_1 \cap \mathbb{P}^2 = \) the \((-1)\)-curve on \( F_1 \), and \( W \) has at most ordinary double points. This implies in particular that the Picard number of a \( \mathbb{Q} \)-factorization of \( W \) equals 4. Similarly, since \( \text{Pic}(X) = \text{Pic}(W)^G \) (the \( G \)-invariant part), the Picard group of \( X \) is generated by \( p(\Delta) \) and (reducible) \( \bar{\Sigma} \), while the class group of \( X \) has rank 4.
4.6. The blowup $\phi$ resolves indeterminacies of the projection $Q \dashrightarrow \mathbb{P}^2$ from $\ell$. Then, since $p(W) = 2$, it is plain that the induced morphism $\pi : W \rightarrow \mathbb{P}^2$ is a $G$ -equivariant extremal contraction.

Further, we put $L := \Delta - \Sigma$, so that $\pi$ is given by the linear system $|L|$. It follows from the construction of $\mathbb{P}^2$ and $g_i$ in 4.1 that $G$ acts faithfully on $\mathbb{P}^2 = \pi(W)$. Then we have $\mathbb{P}^2/G \cong \mathbb{P}^2$ and the quotient morphism $\mathbb{P}^2 \rightarrow \mathbb{P}^2/G$ is given by a linear subsystem $L \subset |2\pi_*L|$. In particular, $L$ consists of certain $G$ -invariant global sections of $2\pi_*L$, and hence identifying $\pi^*L$ with a linear system on $X$ we get a morphism $\pi_X : X \rightarrow \mathbb{P}^2$ (the quotient of $\pi$) which fits into a commutative diagram

\[
\begin{array}{ccc}
W & \xrightarrow{p} & X \\
\pi & & \pi_X \\
\mathbb{P}^2 & \xrightarrow{} & \mathbb{P}^2/G = \mathbb{P}^2
\end{array}
\]

This $\pi_X$ is the second extremal contraction on $X$ (cf. Remark 4.3).

We observe next that the surface $L' := (\phi^*w = 0) - 2\Sigma$ on $W$ is $G$ -invariant and the corresponding $G$ -action is faithful on it (cf. 4.1). This implies that $\deg p|_{L'} = 4$ and thus we get $p_*L' \equiv 4\overline{L'}$ for $\overline{L'} := p(L')$. Similarly, for the surface $\Delta' := (\phi^*z = 0)$ (resp. for $\Sigma$) we have $p_*\Delta' \equiv 4\overline{\Delta'}$ (resp. $p_*\Sigma \equiv 4\overline{\Sigma}$), where $\overline{\Delta'} := p(\Delta')$ (resp. $\overline{\Sigma} := p(\Sigma)$). Note that $\Delta' \sim \Delta$ because $\ell \not\subset (z = 0)$ (cf. 4.2).

Finally, since both $R_i \in |L|$, from Lemma 4.3 we obtain $\overline{R_i} := p_*R_i \equiv 2\overline{L} := p(L)$, and the linear system $|2\overline{L}|$ is basepoint-free on $X$ (for it is the $\pi_X^*$ of a free linear system on $\mathbb{P}^2$).

Lemma 4.7. The pair $(Q, \phi_*\Delta' + \phi_*L')$ is lc.

Proof. Note that both $(Q, \phi_*\Delta')$ and $(Q, \phi_*L')$ are lc via lifting these pairs to a small resolution of $Q$. Hence by the Inversion of Adjunction it suffices to prove that $(\phi_*\Delta', \phi_*L'|_{\phi_*\Delta'})$ is lc. But it follows from 4.1 that $\phi_*\Delta'(z = 0) \cap Q$ is smooth and the pair $(\phi_*\Delta', \phi_*L'|_{\phi_*\Delta'})$ is locally analytically of the form $(\mathbb{C}^2, (xy = 0))$. □

Lemma 4.8. $ac(X, D) = \frac{3}{4}$ for an appropriate $D$ (cf. 4.6).

Proof. In the previous notation, we can write

$$-K_W = \Sigma + \Delta' + L' + \Delta'',$$

where $\Delta'' \sim \Delta$ is generic (this follows from the fact that $-K_Q = \phi_*(\Delta' + L' + \Delta'')$ and $K_W = \phi^*K_Q + \Sigma$).

Note that the pair $(W, \Sigma + \Delta' + L' + \Delta'')$ is lc, since

$$K_W + \Sigma + \Delta' + L' = \phi^*(K_Q + \phi_*\Delta' + \phi_*L')$$

by construction, the pair $(Q, \phi_*\Delta' + \phi_*L')$ is lc by Lemma 4.7, $\Delta''$ is generic and the linear system $|\Delta|$ is basepoint-free.

Further, from the Hurwitz formula (applied twice) we get

$$K_W \equiv p^*(K_X + \frac{1}{2}R_1 + \frac{1}{2}R_2 + \frac{1}{2}\overline{\Sigma})$$

6) To simplify the notation we treat $w$ and $z$ below as global sections of $O_Q(1)$.
easily derives that $X$ is non-toric. At the same time, we have $ac(X, D) = \frac{3}{4}$ (cf. Lemma 4.8), plus the sum of general divisors = pullbacks w.r.t. the projection $\mathcal{X} \rightarrow (\mathbb{P}^1)^{n-3}$, so that $K_{\mathcal{X}} + \mathcal{D} \equiv 0$.

---

7) Recall that according to [22] we do not distinguish between $X$ and its $\mathbb{Q}$-factorialization.

8) Compare with [33] Lemma 3.4.
4.11. Fix some $X \in \mathbb{T}^{f,n}$ (cf. Definition 1.10). We conclude by asking the following questions:

A) Can $X$ always be obtained as (an extremal contraction/a blowup of) the quotient $\mathbb{P}(\mathcal{E})/G$ for some toric variety $T$, (semistable) vector bundle $\mathcal{E}$ on $T$ and a finite group $G \subset \mathbb{P}(\mathcal{E})$?

B) Does $X$ admit a regular $(\mathbb{C}^*)^k$-action for some $k \geq 1$ (cf. [28])?

C) Is $X$ a compactification of the torus $(\mathbb{C}^*)^n$ (cf. 1.1 and Remark 1.5)?

D) Is $X$ a Mori dream space/an FT variety/… (see [12], [9])?

Appendix

Below are a few comments on the paper [5] (published recently in Duke Math. J.) In A.1 A.2 we show that the proof of Conjecture 1.3 given in that paper is inconsistent.

A.1. The most crucial error in [5] (destroying the whole argument basically) is at the end of Section 3 on page 25 (two last paragraphs).

Namely, the authors consider a commutative ring $R$, which is of dimension $d$ and is acted (faithfully) by the torus $(\mathbb{C}^*)^r$, some $d > r$. They claim that once there are $d$ $(\mathbb{C}^*)^r$-invariant divisors $T_i$ on $Y = \text{Spec}(R)$ such that the pair $(Y, T_1 + ... + T_d)$ is log canonical then $R$ must be the polynomial ring in $d$ variables.

This is totally wrong however.

N.B. In fact the authors consider some $R = \text{Cox}(X)$ and assume silently that $R$ is positively graded. The latter is never proved in the paper (the claim itself is wrong actually). This, at least, reveals a huge gap in their argument.

Let $\mathbb{C}^*$ act on $(x, y, z)$ as follows:

$$(x, y, z) \to (ax, a^{-1}y, z)$$

for all $a \in \mathbb{C}^*$.

Consider the surface $Y \subset \mathbb{C}^3$, given by the equation

$$xy = z^3 + z^2 + z,$$

and two divisors $T_1 = (x = z = 0), T_2 = (y = z = 0)$ on $Y$. Then all the needed conditions above are met. But $Y \neq \mathbb{C}^2$ because $\pi_1(Y) = \mathbb{Z}/3$ for example.
A.2. Another (less crucial but still) illustration is the “proof” of Proposition 7.2 in the text. There the authors appeal to rationality criterion for any (singular) conic bundle 3-fold via Prym.

N.B. Note that their conic bundle $X$ is indeed singular because $X = Y/(\mathbb{Z}/2)$ for some smooth 3-fold $Y$ and $\mathbb{Z}/2$ acting on it with some dim = 1 fixed locus.

This criterion is only known for standard (in particular smooth) conic bundles and one wonders where did the authors take from the same fact in the general case. OK, one can consider $\mathbb{Q}$-conic bundles, and then reduce everything to the standard case (Sarkisov). But this requires at most terminal singularities — which is not the case for the given $X$. (In fact the conclusion of Proposition 7.2 is false — see Theorem 1.15 above.)

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