Intertwining operators and modular invariance

Masahiko Miyamoto *

Institute of Mathematics
University of Tsukuba
Tsukuba 305, Japan

Dedicated to Professor Hiroyoshi Yamaki on his 60th birthday

Abstract

We extend Zhu’s theory to the case of intertwining operators of vertex operator algebra $V$. Namely, we show that the space of trace functions $S^I(u, \tau)$ of intertwining operators $I$ of type $(U^W, V^W)$ satisfies modular invariance for each $U$ and $u \in U$ and we construct modular forms of vector type of rational weights. As an application, we calculate trace functions of some intertwining operators explicitly.

1 Introduction

For a rational vertex operator algebra $V$ with central charge $c$ and the set of irreducible $V$-modules $\{W^1, \ldots, W^m\}$, Zhu’s theory insists that the set of trace functions $S^I(v, \tau) = z^{wt(v)}q^{c/24}tr_{W^i} Y_{W^i}(v, z)q^{L(0)}$ for $v \in V$ satisfy some modular invariance ($SL(2, \mathbb{Z})$-invariance) if $V$ satisfies condition $C_2$ (see Def. 2.7), where $Y_{W^i}(v, z)$ is the module vertex operator of $v$ on $W^i$. Especially, if $v \in V_{[n]}$, then $SL(2, \mathbb{Z})$ acts on an $m$-dimensional vector space $\mathbb{C}S_{W^1}(v, \tau) + \cdots + \mathbb{C}S_{W^m}(v, \tau)$ and $(S_{W^1}(v, \tau), \ldots, S_{W^m}(v, \tau))$ become modular forms of vector type of integer weight $n$. This theory was extended by Dong, Li and Mason in [DLiM] to the orbifold model, where $V$ has an automorphism $g$ of finite order and one consider the trace function $\text{tr}_{W^i} g q^{L(0) - c/24}$. The author has also extended Zhu’s theory to the trace functions in many variables, [Mi1], [Mi2]. As applications of these theories, we can construct a lot of modular forms of integer weights from holomorphic vertex operator algebras. Recently, attention has come to be paid to modular forms of rational weights, see [BKMS] and [H].

*Supported by the Grants-in-Aids for Scientific Research, No. 09440004 and No. 12874001, The Ministry of Education, Science and Culture, Japan.
In this paper, we will show a new construction of modular forms of rational weights by using intertwining operators of vertex operator algebras. For example, we will construct modular forms (with a linear character) of weights \(\frac{1}{2}, \frac{1}{10}, \frac{2}{5}, \frac{1}{7}\). Actually, our proof covers the real weights, but we don’t know such a case.

An incentive of this research is Dedekind’s \(\eta\)-function
\[
\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n),
\]
which is a modular form of weight \(\frac{1}{2}\). It follows from Spinor construction [FRW] that this function is given as a trace function
\[
u - \text{wt}(u) q^{-1/24} \text{tr}|_{L(\frac{1}{2}, \frac{1}{16})} I(u, z) q^{L(0)}
\]
for some intertwining operator \(I(\ast, z) \in I \left( L(\frac{1}{2}, \frac{1}{16}), L(\frac{1}{2}, \frac{1}{16}) \right)\) of \(L(\frac{1}{2}, 0)\)-modules and some element \(u \in L(\frac{1}{2}, \frac{1}{16})\) of weight \(\frac{1}{2}\), where \(L(\frac{1}{2}, 0)\) is 2-dimensional Ising model (a rational Virasoro vertex operator algebra with central charge \(\frac{1}{2}\)) and has three modules \(L(\frac{1}{2}, 0), L(\frac{1}{2}, \frac{1}{2})\) and \(L(\frac{1}{2}, \frac{1}{16})\), where the first entry denotes central charge and the second denotes the lowest weights. Since \(L(\frac{1}{2}, \frac{1}{16})\) is the only one irreducible \(L(\frac{1}{2}, 0)\)-module \(W\) satisfying \(0 \neq I \left( W, L(\frac{1}{2}, \frac{1}{2}) \right)\) and we have \(\dim I \left( L(\frac{1}{2}, \frac{1}{16}), L(\frac{1}{2}, \frac{1}{16}) \right) = 1\), the space of trace functions
\[
< q^{-c/24} \text{tr}|_{W} I(u, z) q^{L(0)} : I \in I \left( W, L(\frac{1}{2}, \frac{1}{2}) \right), W \text{ is an } L(\frac{1}{2}, 0)\)-module >
\]
has dimension one for each \(u \in L(\frac{1}{2}, \frac{1}{16})\). This fact suggests the possibility of the extension of Zhu’s theory to the trace functions of the intertwining operators. Our main purpose in this paper is to show that this is true and to prove that the space of trace functions \(S^{I}(u, \tau)\) given by intertwining operators \(I\) of type \((W, W)\) for each \(U\) satisfies a modular invariance if \(U\) satisfies a weaker condition \(C_{[2,0]}\). See Theorem 4.15 and Theorem 5.1. Using the result (Corollary 2.13 in [Li2]) given by Li, the proofs of these theorem are essentially the same as in [Zh]. As we however are interested in the extension of Zhu’s theory and the mechanics of modular invariance of trace functions of vertex operator algebras, we will pick up and repeat the necessary parts with the suitable modifications.

The author wishes to thank E. Bannai and T. Ibukiyama for their helpful advices.

2 Preliminary results

2.1 Vertex operator algebras
**Definition 2.1** A vertex operator algebra is a $\mathbb{Z}$-graded complex vector space:

$$V = \coprod_{n \in \mathbb{Z}} V_n$$

satisfying $\dim V_n < \infty$ for all $n$ and $V_n = 0$ for $n << 0$. If $v \in V_n$ we write $\text{wt}(v) = n$ and say that $v$ is homogeneous and has (conformal) weight $n$. For each $v \in V$ there are linear operators $v_n \in \text{End}(V)$, $n \in \mathbb{Z}$ which are assemble into a vertex operator

$$Y(v, z) = \sum_{n \in \mathbb{Z}} v(n)z^{-n-1} \in (\text{End}V)[[z, z^{-1}]].$$

Various axioms are imposed:

1. For $u, v \in V$, $u(n)v = 0$ for $n$ sufficiently large.
2. There is a distinguished vacuum element $1 \in V_0$ satisfying $Y(1, z) = 1$ and $Y(v, z)1 = v + \sum_{n \geq 2} v(-n)1z^{n-1}$.
3. There is a distinguished Virasoro element $\omega \in V_2$ with generating function $Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n)z^{-n-2}$ such that the component operators generate a copy of the Virasoro algebra represented on $V$ with central charge $c$. That is

$$[L(m), L(n)] = (m - n)L(m + n) + \frac{1}{12}(m^3 - m)\delta_{m+n,0}c.$$ 

Moreover we have $V_n = \{ v \in V | L(0)v = nv \}$ and $\frac{d}{dz} Y(v, z) = Y(L(-1)v, z)$.
4. "Commutativity" holds

$$(z - w)^N[Y(v, z), Y(u, w)] = 0 \text{ for } N > > 0$$

Such a vertex operator algebra may be denoted by the 4-tuple $(V, Y, 1, \omega)$ or more usually, by $V$.

It is well known that vertex operators satisfies "Associativity".

$$(a(m)v)(r) = \sum_{i=0}^{\infty} (-1)^i \binom{m}{i} a(m - i)v(r + i) - (-1)^{m+i} \binom{m}{i} v(m + r - i)a(i) \quad (2.1)$$

for $a, v \in V$.

**Definition 2.2** A module for $(V, Y, 1, \omega)$ is a $\mathbb{Z}$-graded vector space $M = \oplus_{n \geq 0} M(n)$ with finite dimensional homogeneous spaces $M(n)$; equipped with a formal power series

$$Y^M(v, z) = \sum_{n \in \mathbb{Z}} v^M(n)z^{-n-1} \in (\text{End}(M))[[z, z^{-1}]]$$

called the module vertex operator of $v$ for $v \in V$ satisfying:

1. $Y^M(1, z) = 1_M$:
(2) $Y^M(\omega, z) = \sum L^M(n) z^{-n-1}$ satisfies:
    (2.a) the Virasoro algebra relations:
    $$[L^M(n), L^M(m)] = (n-m)L^M(n+m) + \delta_{n+m,0} \frac{n^3-n}{12} c,$$
    (2.b) the $L(-1)$-derivative property:
    $$Y^M(L(-1)v, z) = \frac{d}{dz} Y^M(v, z), \text{and}$$
    (2.c) $L^M(0)_{M(n)} = (k_n) 1_{M(n)}$ for some $k_n \in \mathbb{C}$.

(3) "Commutativity" holds:
    $$(z - w)^N [Y^M(v, z), Y^M(u, w)] = 0 \text{ for } N >> 0$$

(4) "Associativity" holds:
    $$Y^M(u_n v, z) = \text{Res}_w \left( (w - z)^n Y^M(u, w) Y^M(v, z) - (-z + w)^n Y^M(v, z) Y^M(u, w) \right),$$
where $(-z + w)^n = \sum_{i=0}^{\infty} \binom{n}{i} (-z)^{n-i} w^i$ and $\binom{n}{i} = \frac{n(n-1)\cdots(n-i+1)}{i!}$.

It follows from the definitions of modules that if $W$ is an irreducible $V$-module then $W$ has a lowest weight $r$ such that $W = \oplus_{n=0}^\infty W_{r+n}$, where $L(0)$ acts on $W_{r+n}$ as a scalar $r+n$.

**Definition 2.3** A vertex operator algebra $(V, Y, 1, \omega)$ is called "rational" if it has only finitely many irreducible modules and all modules are completely reducible. A vertex operator algebra with a unique irreducible module is called "holomorphic".

Throughout this paper, $V = \oplus_{n=0}^\infty V_n$ is a rational vertex operator algebra $(V, Y, 1, \omega)$ with central charge $c$ and $U$ is an irreducible $V$-module. We assume that $U$ is spanned by elements of the forms $L(-n_1)\cdots L(-n_t) u$ ($n_1,\ldots,n_t > 0$) for $u$ satisfying $L(n) u = 0$ ($n > 0$).

Zhu has introduced the second vertex operator algebra $(V, Y[,], 1, \tilde{\omega})$ associated to $V$ in Theorem 4.2.1 of [Zh].

**Definition 2.4** The vertex operator $Y[a, z]$ are defined for homogeneous $a$ via the equality
    $$Y[a, z] = Y(a, e^z - 1) e^{z \text{wt}(a)} \in \text{End}(V)[[z, z^{-1}]]$$
and Virasoro element $\tilde{\omega}$ is define to be $\omega - c/24$. 

4
For the proof of this fact, see [Zh] or the proof of Theorem 2.1. We should note that Zhu has used $Y[a, z]_{2\pi i} = Y(a, e^{2\pi i z} - 1)\epsilon^{2\pi i z w}(a)$ and $\tilde{\omega}_{2\pi i} = (2\pi i)^2(\omega - c/24)$ and Dong, Li and Mason have used $Y[a, z]_{1} = Y(a, e^z)\epsilon^{w(a)z}$ and $\tilde{\omega} = \omega - c/24$ because we can define it for a vertex operator algebra over the rational number field. It follows from the direct calculation that the differences are given by

$$a[n]_{2\pi i}u = (2\pi i)^{-n-1}a[n]_{1}u.$$  \hfill (2.2)

Using a change of variable we calculate that

$$v[m] = \text{Res}_z Y[v, z]z^m = \text{Res}_z Y[v, \log(1 + z)](\log(1 + z))^m(1 + z)^{-1} = \text{Res}_z Y(v, z)(\log(1 + z))^m(1 + z)^{\text{wt}(v)-1}.$$  \hfill (2.3)

In this paper, $\text{Res}_z \in \text{Hom}(V[z], V)$ is given by $\text{Res}_z(\sum_{m \in \mathbb{C}} a_m z^{-m-1}) = a_0$. In particular, for $v \in V$ and $m \in \mathbb{Z}$, there are $a_i \in \mathbb{C}$ such that

$$v[m] = v(m) + \sum_{i=1}^{\infty} a_i v(m + i).$$  \hfill (2.4)

For example we have

$$v[0] = \sum_{i=0}^{\infty} \binom{\text{wt}(v) - 1}{i} v(i).$$  \hfill (2.5)

We also write $Y[\tilde{\omega}, z] = \sum_{n \in \mathbb{Z}} L[n] z^{-n-2}$ and set $V_n = \{ v \in V \, | \, L[0]v = nv \}$. For $v \in V_n$, we denote it by $[\text{wt}(v)] = n$. For example, one has

$$L[0] = L(0) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n(n+1)} L(n)$$  \hfill (2.6)

and so $\oplus_{n \leq N} W_{r+n} = \oplus_{n \leq N} W_{r+[n]}$ for any irreducible $V$-module $W = \oplus_{n=0}^{\infty} W_{r+n}$.

The main operators in this paper are not vertex operators of elements of $V$, but intertwining operator of elements of $V$-module $U$.

**Definition 2.5** Let $(W^i, Y^i) \ (i = 1, 2, 3)$ be $V$-modules. An intertwining operator of type

$$
\begin{pmatrix}
W^3 \\
W^1 \\
W^2
\end{pmatrix}
$$

is a linear map

$$
W^1 \rightarrow \text{Hom}(W^2, W^3)[z] \\
w_1 \rightarrow I(w_1, z) = \sum_{r \in \mathbb{C}} w_1(r)z^{-r-1}
$$

such that

\begin{enumerate}
\item for $w_j \in W^j$ and $r \in \mathbb{C}$, $w_1(r + n)w_2 = 0$ for $n$ sufficiently large;
\item $\frac{d}{dz} I(w_1, z) = I(L(-1)w_1, z)$;
\item "Commutativity" holds;
\end{enumerate}

$$(z - w)^N \{ Y^3(v, z)I(u, w) - I(u, w)Y^2(v, z) \} = 0 \text{ for } N >> 0;$$
It is known (c.f. [Li1]) that (3) and (4) are possible to be replaced by the following Jacobi identity for the operators

\[
\begin{align*}
    z_0^{-1}\delta(\frac{z_1-z_2}{z_0})Y^3(v, z_1)I(w_1, z_2)w_2 - z_0^{-1}\delta(\frac{z_1-z_2}{z_0})I(w_1, z_2)Y^2(v, z_1)w_2 \\
    = z_2^{-1}\delta(\frac{z_1-z_2}{z_2})I(Y^1(v, z_0)w_1, z_2)w_2
\end{align*}
\]

(2.6)

As in the case of module actions, for \( v \in V \) and \( u \in W^1 \), we have the standard consequence from (2.6):

\[
[Y(v, z_1), I(u, z_2)] = \text{Res}_{z_0} z_2^{-1}\delta(\frac{z_1-z_2}{z_2})I(Y(v, z_0)u, z_2)
= I(Y(v, z_1-z_2) - Y(v, -z_2 + z_1))u, z_2).
\]

(2.7)

Here \( [Y(v, z_1), I(u, z_2)] \) denotes \( Y^3(v, z_1)I(u, z_2) - I(u, z_2)Y^2(v, z_1) \).

By calculating the coefficients of \( z_1^{-m-1}z_2^{-k-1} \) in (2.7), we have the following commutator formula as in the case of VOAs:

\[
[v(m), u^I(k)] = \sum_{j=0}^{\infty} \binom{m}{j} (v(j)u)^I(m + k - j),
\]

(2.8)

where \( I(u, z) = \sum_{r \in \mathbb{C}} u^I(r)z^{-r-1}, Y^i(v, z) = \sum_{i \in \mathbb{Z}} v^i(m)z^{-m-1} \) for \( i = 1, 2, 3 \) and \( [v(m), u^I(k)] \) denotes \( v^3(m)u^I(k) - u^I(k)v^2(m) \). In particular,

\[
[L(0), u^I(k)] = (\omega^1(0)u)^I(k + 1) + (\omega^1(1)u)^I(k) = (\text{wt}(u) - k - 1)u^I(k)
\]

(2.9)

and so \( u^I(\text{wt}(u) - 1) \) is a grade-preserving operator and denoted by \( a^I(u) \).

If \((W, Y^W)\) is a \((V, Y, z, 1, \omega)\)-module, then by the same arguments as in [Zi], one shows that \( W \) become a \((V, Y[,], 1, \tilde{\omega})\)-module by module vertex operator \( Y^W[z, v] = Y^W(v, e^z - 1)e^{\text{wt}(v)z} \). We will extend it to intertwining operators.

**Theorem 2.6** Let \( I(, z) \in I_{W^1W^2}^{W^3} \). Define the second intertwining operator \( I[* , z] \) by

\[
I[u, z] = I(u, e^z - 1)e^{\text{wt}(u)}
\]

(2.10)

for homogeneous \( u \in W^1 \). Then \( I[* , z] \) is an intertwining operator of type \((W^1, W^2), (W^3, W^2) \), where \( W^i \) are \((V, Y[,], 1, \tilde{\omega})\)-modules \((W^1, Y^1[, , z]), (W^2, Y^2[, , z]), (W^3, Y^3[, , z]) \).

**Proof** As mentioned at (4.2.3) in [Zi],

\[
a^I[m] = \text{Res}_z (Y^I(a, z)(\ln(1 + z))m(1 + z)^{\text{wt}(a) - 1}\]

(2.11)
for $a \in V$ and $i = 1, 2, 3$, where $Y^i[a, z] = \sum_{m \in \mathbb{Z}} a^i[m] z^{-m-1}$.

(1) Commutativity of $I[, z]$: Set $f(x, z) = (e^x - e^z)/(z - x)$. By Commutativity of $I(, z)$,

$$0 = (e^x - 1 - e^z + 1)N(Y^3(a, e^x - 1)e^{wt(a)x}I(u, e^x - 1)e^{wt(u)x}$$

$$-I(u, e^x - 1)e^{wt(u)x}Y^2(a, e^x - 1)e^{wt(a)x}$$

$$(x - z)^Nf(x, z)^N(Y^3[a, x]I[u, z] - I[u, z]Y^2[a, x])$$

Since $(e^x - e^z) = (z - x)(1 + \frac{1}{2}(x + z) + ...)$, $f(x, z)$ has an inverse in $\mathbb{C}[[x, z]]$. Hence

$$0 = (x - z)^N(Y^3[a, x]I[u, z] - I[u, z]Y^2[a, x])$$

(2) For the proofs of Associativity of $I[, z]$:

$$(a[m]u)[r] = \sum_{i=0}^{\infty} (-1)^i \binom{m}{i} a[m - i]u[r + i] - (-1)^{m+i} \binom{m}{i} u[m + r - i]a[i]$$

and $L[-1]$-derivative property:

$$I[L[-1]u, z]] = \frac{d}{dz} I[u, z],$$

see the proof of Theorem 4.2.1 in [Zh] with suitable modifications.

We next recall the so-called condition $C_2$ introduced by Zhu [Zh] and give a weaker condition.

**Definition 2.7** For a $V$-module $U$, set $C_2(U) = \langle a(-2)u : a \in V, u \in U \rangle$ and $C_{[2, 0]}(U) = \langle a[-2]u, a[0]u : a \in V, u \in U \rangle$. We call that $U$ satisfies condition $C_2$ if $\dim(U/C_2(U)) < \infty$ and $U$ satisfies condition $C_{[2, 0]}$ if $\dim(U/C_{[2, 0]}(U)) < \infty$.

We note $a(-n)u \in C_2(U)$ for any $n \geq 2$ since $(m - 1)v(-m) = (L(-1)v)(-m + 1)$. From Associativity (2.1), we easily have:

**Lemma 2.8** $V/C_2(V)$ is an associative algebra with a product given by $v \times w = v(-1)w$. Also $U/C_{[2]}(U)$ is a $V/C_{[2]}(V)$-module whose action is given by $v(-1)u + C_2(U)$ for $v \in V, u \in U$.

We also have the following from (2.3).

**Lemma 2.9** If we set $C_{[2]}(U) = \langle a[-2]u : a \in V, u \in U \rangle$, then $\dim U/C_{[2]}(U) < \infty$ if and only if $\dim U/C_2(U)$.

[Proof]
2.2 Zhu-algebra

Let $U$ be a $V$-module. Following [FZ] we define left and right actions of $V$ on $U$ as follows:

\[
a \cdot u = \text{Res}_x \frac{(1 + x)^{\text{wt}(a)}}{x} Y(a, x)u,
\]

\[
u \ast a = \text{Res}_x \frac{(1 + x)^{\text{wt}(a)-1}}{x} Y(a, x)u,
\]

for any homogeneous vector $a \in V$ and for any $u \in U$. Let $O(U)$ be the subspace of $U$ linearly spanned by all elements,

\[
\text{Res}_x \frac{(1 + x)^{\text{wt}(a)}}{x^2} Y(a, x)u,
\]

for any homogeneous $a \in V$, $u \in U$. Set $A(U) = U/O(U)$. Then it is known (Theorem 1.5.1 [FZ]) that $A(U)$ is an associative algebra with a product $\cdot$ and $A(U)$ is an $A(V)$-bimodule under the defined left and right action. Zhu showed that $\omega + O(V)$ is in the center of $A(V)$ and $A(V)$ is a finite dimensional semisimple algebra if $V$ is rational. As mentioned in Remark 2.9 in [Li2],

\[
a \cdot u - u \ast a = \text{Res}_x (1 + x)^{\text{wt}(a)-1} Y(a, x)u = \sum_{i=0}^{\infty} \binom{\text{wt}(a) - 1}{i} a_i u = a[0]u.
\]

Li recently proved the following theorem (Corollary 2.13 in [Li2]), which was mentioned in [FZ].

**Theorem 2.10** If $V$ is rational, then there is a natural linear isomorphism

\[
\pi : I \left( \begin{array}{c} W^3 \\ W^2 \end{array} \right) \rightarrow \text{Hom}_{A(V)}(A(U) \otimes_{A(V)} W^2(0), W^3(0))
\]

for irreducible $V$-modules $U, W^2, W^3$, where $W^2(0)$ and $W^3(0)$ are the top modules of $W^2$ and $W^3$, respectively. Here $\pi(I)$ is given by

\[
\pi(I)(u \otimes m) = u(\text{wt}(u) - 1 + r_2 - r_3)m
\]

for $m \in M^2(0)$, $u \in U, I(u, z) = \sum_{k \in \mathbb{C}} u(k)z^{-k-1}$ and $r_2, r_3$ are the lowest weights of $W^2$ and $W^3$, respectively.

2.3 Elliptic functions

In this section we will quote several results from [Zh]. The Eisenstein series $G_{2k}(\tau)$ ($k = 1, 2, \ldots$) are series

\[
G_{2k}(\tau) = \sum_{(m,n) \neq (0,0)} \frac{1}{(m\tau + n)^{2k}} \text{ for } k \geq 2,
\]
and
\[ G_2(\tau) = \frac{\pi^2}{3} + \sum_{m \in \mathbb{Z} - \{0\}} \sum_{n \in \mathbb{Z}} \frac{1}{(m\tau + n)^2} \text{ for } k = 1. \] (2.22)

They have the \( q \)-expansions
\[ G_{2k}(\tau) = 2\xi(2k) + \frac{2(2\pi\sqrt{-1})^{2k}}{(2k - 1)!} \sum_{n=1}^{\infty} \frac{n^{2k-1}q^n}{1 - q^n} \] (2.23)

where \( \xi(2k) = \sum_{n=1}^{\infty} \frac{1}{n^{2k}} \) and \( q = e^{2\pi\sqrt{-\tau}} \).

We make use of the following normalized Eisenstein series:
\[ E_k(\tau) = \frac{1}{(2\pi\sqrt{-1})^k} G_k(\tau) \text{ for } k \geq 2. \] (2.24)

It is clear from (2.21) and (2.22) that
\[ E_2\left( \frac{a\tau + b}{c\tau + d} \right) = (c\tau + d)^2 E_2(\tau) - \frac{c(c\tau + d)}{2\pi\sqrt{-1}} \] (2.25)

for \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \) and \( E_{2k}(\tau) \) is a modular form of weight \( 2k \) for \( k > 1 \).

Set
\[ \wp_1(z, \tau) = \frac{1}{z^2} + \sum_{w \in \mathbb{Z}\tau + \mathbb{Z} - \{0\}} \left( \frac{1}{z-w} + \frac{1}{w} + \frac{1}{zw} \right), \] (2.26)

\[ \wp_2(z, \tau) = \frac{1}{z^2} + \sum_{w \in \mathbb{Z}\tau + \mathbb{Z} - \{0\}} \left( \frac{1}{(z-w)^2} - \frac{1}{w^2} \right), \quad \text{Weierstrass } \wp \text{-function} \] (2.27)

and
\[ \wp_{k+1}(z, \tau) = -\frac{1}{k} \frac{d}{dz} \wp_k(z, \tau) \text{ for } k \geq 2. \] (2.28)

\[ \wp_k\left( \frac{2\pi\sqrt{-1}}{2\pi\sqrt{-1}} \right)(z, \tau) \quad (k = 1, 2, \ldots) \] have the Laurent expansion near \( z = 0 \),
\[ \wp_k\left( \frac{2\pi\sqrt{-1}}{2\pi\sqrt{-1}} \right)(\tau) = \frac{1}{z^k} + (-1)^k \sum_{n=1}^{\infty} \binom{2n + 1}{k - 1} E_{2n+2}(\tau) z^{2n+2-k} \] (2.29)

and the \( q \)-expansion of \( \wp_1\left( \frac{2\pi\sqrt{-1}}{2\pi\sqrt{-1}} \right) \) is
\[ \wp_1\left( \frac{2\pi\sqrt{-1}}{2\pi\sqrt{-1}} \right) = E_2(\tau)z + \frac{1}{2e^z - 1} + \sum_{n=1}^{\infty} \left( \frac{q^n}{e^z - q^n} - \frac{e^z q^n}{1 - e^z q^n} \right), \] (2.30)

see [La] p.248.
Zhu [Zh] introduced a formal power series. We adopt it by multiplying $(2\pi\sqrt{-1})^{-k}$:

$$P_k(z, q) = \frac{1}{(k-1)!} \sum_{n \neq 0} \frac{n^{k-1}z^n}{1-q^n}$$  \hspace{1cm} (2.31)

where $\frac{1}{1-q^n}$ is understood as $\sum_{i=0}^{\infty} q^{ni}$ and $\frac{1}{1-q^{-n}} = -q^n \frac{1}{1-q^n}$ for $n > 0$. Note that $z \frac{d}{dz} P_k(z, q) = kP_{k+1}(z, q)$. It is easy to prove that $P_k(z, q)$ converges uniformly and absolutely in every closed subset of the domain $\{(z, q) | |q| < |z| < 1\}$. The relation between $P_k(z, q)$ and $\varphi_k(z, \tau)$ are given by

$$P_1(e^z, q) = -\frac{1}{2\pi \sqrt{-1}} \varphi_1\left(\frac{z}{2\pi \sqrt{-1}}, \tau\right) + E_2(\tau)z - \frac{1}{2}$$  \hspace{1cm} (2.32)

$$P_2(e^z, q) = -\frac{1}{(2\pi \sqrt{-1})^2} \varphi_2\left(\frac{z}{2\pi \sqrt{-1}}, \tau\right) + E_2(\tau)$$  \hspace{1cm} (2.33)

$$P_k(e^z, q) = \left(\frac{-1}{2\pi \sqrt{-1}}\right)^k \varphi_k\left(\frac{z}{2\pi \sqrt{-1}}, \tau\right) \text{ for } k > 2.$$

### 2.4 Equations

In this paper, we will quote several equations from [Zh].

Write $(1 + z)^{wt(a)-1}(ln(1 + z)^{-1}) = \sum_{i \geq 1} c_i z^i$. Then $c_{-1} = 1$ and

$$a[-1] = \sum_{i \geq -1} c_i a(i)$$  \hspace{1cm} (2.35)

by the definition of $Y[a, z]$. Zhu proved the following equations (c.f. (4.3.8)-(4.3.11) in [Zh]):

$$\sum_{i \geq -1} c_i \text{Res}_w((w - z)^i z^{n-1-i} w^{-n} - \sum_{i \geq -1} c_i \text{Res}_w((-z + w)^i z^{n-1-i} w^{-n})$$

$$= 1,$$  \hspace{1cm} (2.36)

$$\sum_{i \geq -1} c_i \text{Res}_w((w - z)^i z^{wt(a)-1-i} w^{-wt(a)} P_1\left(\frac{z}{w}, q\right)$$

$$- \sum_{i \geq -1} c_i \text{Res}_w((-z + w)^i z^{wt(a)-1-i} w^{-wt(a)} P_1\left(\frac{zw}{w}, q\right) - 1)$$

$$= -\frac{1}{2}$$  \hspace{1cm} (2.37)

and for $m \geq 2$,

$$\sum_{i \geq -1} c_i \text{Res}_w((w - z)^i z^{wt(a)-1-i} w^{-wt(a)} P_m\left(\frac{z}{w}, q\right)$$

$$- \sum_{i \geq -1} c_i \text{Res}_w((-z + w)^i z^{wt(a)-1-i} w^{-wt(a)} P_m\left(\frac{zw}{w}, q\right)$$

$$= E_m(q).$$  \hspace{1cm} (2.38)

He also got the following equation at the end of the proof of Proposition 4.3.2 in [Zh].

$$\sum_{i \in \mathbb{N}} \sum_{k=1}^{\infty} \left((\frac{wt(a)-1+k}{i}) \frac{1}{1-q^k} x^k + (\frac{wt(a)-1-k}{i}) \frac{1}{1-q^{-k}} x^{-k}\right) a(i)b$$

$$= \sum_{m \in \mathbb{N}} P_{m+1}(x, q)a[m]b.$$  \hspace{1cm} (2.39)
[Proof] Set \( (m^{-1} + x) = \sum_{s=0}^{i} c(m, i, s)x^s \). Then since
\[
\sum_{s=0}^{\infty} \frac{1}{s!} [\ln(1 + z)]^s (1 + z)^{wt(a) - 1}w^s = (1 + z)^{wt(a) - 1} \exp(\ln(1 + z)w)
\]
\[= (1 + z)^{wt(a) - 1} (1 + z)w = \sum_{i=0}^{\infty} (wt(a) - 1 + i)z^i w^s,
\]
we have
\[
\sum_{i=s}^{\infty} c(wt(a), i, s)z^i = \frac{1}{s!} [\ln(1 + z)]^s (1 + z)^{wt(a) - 1}
\]
and so
\[
\sum_{i=s}^{\infty} c(wt(a), i, s)a(i) = \frac{1}{s!} a[s] \text{.}
\]
Therefore
\[
\sum_{m=0}^{\infty} \sum_{n \neq 0} (wt(a) - 1 + m)z^m \cdot (a(m)) = \sum_{m=0}^{\infty} \sum_{n \neq 0} \sum_{n=0}^{m} c(wt(a), m, s)z^m \cdot (a(m)) = \sum_{m=0}^{\infty} \sum_{n \neq 0} \frac{1}{s!} a[s]b(n) \cdot \frac{z^m}{1 - q^n}
\]
\[
= \sum_{s=0}^{\infty} \sum_{n \neq 0} \frac{1}{s!} a[s]b(n) \cdot \frac{z^m}{1 - q^n} = \sum_{s=0}^{\infty} P_{s+1}(x, \tau)a[s]b.
\]

3 Trace functions

Let \( W \) be a \( V \)-module. For an intertwining operator \( I(\bullet, z) \in I_U^W \), define
\[
F_{W}^I : (V \otimes H)^{\otimes (i-1)} \otimes (U \otimes H) \otimes (V \otimes H)^{\otimes (n-i)} \longrightarrow \mathbb{C}
\]
by
\[
F_{W}^I(a_1, z_1), ..., (a_{i-1}, z_{i-1}), (u, z_i), (a_{i+1}, z_{i+1}), ..., (a_n, z_n))
\]
\[
= z_1^{wt(a_1)} \cdots z_i^{wt(u)} \cdots z_n^{wt(a_n)} \text{tr}|_W Y_W(a_1, z_1) Y_W(a_2, z_2) \cdots Y_W(a_{i-1}, z_{i-1}) Y_W(a_{i+1}, z_{i+1}) \cdots Y_W(a_n, z_n) q^{L(0)}
\]
(3.1)
for homogeneous \( a_i \in V \) ( \( i = 1, ..., n \)) and \( u \in U \) and extend it linearly. In this paper, we will use \( F_{W}^I \) only for \( n = 1, 2 \). For simplicity we will often omit the lower index \( W \) in \( F_{W}^I \) when no confusion should arise. For example, \( F^I((u, z), \tau) = \text{tr}|_W o^I(u)q^{L(0)} \) does not depend on \( z \) and so we denote it \( F_{W}^I(u, \tau) \).

Since we will calculate the traces and all coefficients \( a_i(m) \) of \( Y(a_i, z) \) shift the grading by integers, it is sufficient to consider only coefficients \( u(k) \) of \( I(u, z_i) = \sum u(k)z_i^{-k-1} \) which shift the grading by integers.

The case \( U = V \) is Zhu’s theory. The arguments (circulating arguments) in Zhu’s paper depends on Commutativity among \( \{ Y_W(a_i, z) : i \} \), but not on Commutativity of \( Y_W(a_i, z) \) with itself. Hence if only one intertwining operator \( I(u, z_i) \) appears in the definition of \( F_{W}^I \), \( \{ Y_W(a_1, z_1), ..., Y_W(a_{i-1}, z_{i-1}), I(u, z_i), ..., T_W(a_n, z_n) \} \) satisfy Commutativity each other, that is, with the others, and so we can apply the circulating arguments. Therefore we have the following results.

Proposition 3.1 For any \( a \in V \) and \( u \in U \), we have
\[
F^I((a[0]u, z), q) = 0.
\]
(3.2)
Proposition 3.2

\[ F^I((a, x), (u, z), q) = z^{-wt(u)}tr_W o(a) o^I(u) q^{L(0)} + \sum_{m \in \mathbb{N}} P_{m+1}(\frac{z}{x}, q) o^I(a[m]u) q^{L(0)} \]  \hspace{1cm} (3.3)

\[ F^I((u, z), (a, x), q) = z^{-wt(u)}tr_W o(a) o^I(u) q^{L(0)} + \sum_{m \in \mathbb{N}} (P_{m+1}(\frac{z}{x}, q) - \delta_{m, 0}) o^I(a[m]u) q^{L(0)} \]  \hspace{1cm} (3.4)

[Proof] We just follow the proof of Proposition 4.3.2 in \cite{Z} with suitable modifications. For \( k \neq 0 \), set \( o_k(a) = a(wt(a) - 1 + k) \). We have

\[
tr_W o_k(a) I(u, z) q^{L(0)} = tr_W [o_k(a), I(u, z)] q^{L(0)} + tr_W I(u, z) o_k(a) q^{L(0)}
\]

\[
= \sum_{i \in \mathbb{N}} (wt(a) - 1 + k) z^{wt(a) - 1 + k - i} tr_W I(a(i) u, z) q^{L(0)} + tr_W I(u, z) q^{L(0)} o_k(a) q^k
\]

\[
= \sum_{i \in \mathbb{N}} (wt(a) - 1 + k) z^{wt(a) - 1 + k - i} tr_W I(a(i) u, z) q^{L(0)} + tr_W o_k(a) I(u, z) q^{L(0)} q^k
\]

Solving for \( tr_W o_k(a) I(u, z) q^{L(0)} \) in the above identity, we have

\[
tr_W o_k(a) I(u, z) q^{L(0)} = \frac{1}{1 - q^k} \sum_{i \in \mathbb{N}} \left( \frac{wt(a) - 1 + k}{i} \right) z^{wt(a) - 1 + k - i} tr_W I(a(i) u, z) q^{L(0)}. \]  \hspace{1cm} (3.5)

Similarly, we have

\[
tr_W I(u, z) o_k(a) q^{L(0)} = \frac{q^k}{1 - q^k} \sum_{i \in \mathbb{N}} \left( \frac{wt(a) - 1 + k}{i} \right) z^{wt(a) - 1 + k - i} tr_W I(a(i) u, z) q^{L(0)}. \]  \hspace{1cm} (3.6)

Using the above, we have

\[
F^I((a, x), (u, z), q) = z^{wt(u)} tr_W o(a) I(u, z) q^{L(0)} + z^{wt(u)} \sum_{k \neq 0} x^{-k} tr_W o_k(a) I(u, z) q^{L(0)}
\]

\[
= z^{wt(u)} tr_W o(a) I(u, z) q^{L(0)} + \sum_{i \in \mathbb{N}} (wt(a) - 1 + k) \frac{1}{1 - q^k} \sum_{j \in \mathbb{N}} (wt(a) - 1 + k - i) z^{wt(a) - 1 + k - i} tr_W I(a(i) u, z) q^{L(0)}
\]

\[
= z^{wt(u)} tr_W o(a) I(u, z) q^{L(0)} + \sum_{i \in \mathbb{N}} (wt(a) - 1 + k) \frac{1}{1 - q^k} \left( \frac{z}{x} \right)^k + \sum_{i \in \mathbb{N}} (wt(a) - 1 + k) \frac{1}{1 - q^k} \left( \frac{z}{x} \right)^k F^I((a(i) u, z), q)
\]

\[
= tr_W o(a) o^I(u) q^{L(0)} + \sum_{m \in \mathbb{N}} P_{m+1}(\frac{z}{x}, q) o^I(a[m]u) q^{L(0)} \text{ by (2.39)}.
\]

12
Similarly, we have

\[
F^I((a, u, z), (a, x), q) = z^{w(u)} tr_{|W} I(u, z) o(a) q^{L(0)} + z^{w(u)} \sum_{k \neq 0} x^{-k} tr_{|W} I(u, z) o_k(a) q^{L(0)} \\
= z^{w(u)} tr_{|W} I(u, z) o(a) q^{L(0)} + z^{w(u)} \sum_{k \neq 0} x^{-k} q_k^{L(0)} \sum_{j \in \mathbb{N}} \left( w(a) - 1 + k \right) z^{w(a) - 1 + k - i} tr_{|W} I(a(i) u, z) q^{L(0)} \\
= z^{w(u)} tr_{|W} o(a) I(u, z) q^{L(0)} - F^I((a[0] u, z), q) \\
+ \sum_{k \in \mathbb{N}} \left( \sum_{i=1}^{\infty} \left( w(a) - 1 + k \right) \frac{1}{1-q^k} \right) q_k^{L(0)} \sum_{i=1}^{\infty} \left( w(a) - 1 + k \right) \frac{1}{1-q^k} \delta_{m,0} o^i(a[m] u) q^{L(0)} \\
= tr_{|W} o(a) o^i(u) q^{L(0)} + \sum_{m \in \mathbb{N}} \left( \frac{p_{m+1}}{2\pi i} \delta_{m,0} - \delta_{m,0} \right) o^i(a[m] u) q^{L(0)} \\
\text{by (2.39).}
\]

\[\text{Proposition 3.3}\]
\[tr_{|W} o(a) o(u) q^{L(0)} = tr_{|W} o(a[-1] u) q^{L(0)} - \sum_{k=1}^{\infty} E_{2k}(\tau) tr_{|W} o(a[2k - 1] u) q^{L(0)} \]

\[\text{[Proof]}\]
Write \((1 + z)^{w(a) - 1}(\ln(1 + z))^{-1} = \sum_{i \geq -1} c_i z^i\) (note that \(c_{-1} = 1\)). Then \(a[-1] u = \sum_{i \geq -1} c_i a(i) u\). We have

\[
F^I((a[-1] u, z), q) = \sum_i c_i z^{w(a) - 1 + i} z^{w(u)} tr_{|W} I(a(i) u, z) q^{L(0)} \\
= \sum_i c_i z^{w(a) - 1 + i + w(u)} Res_{w = -z} (w - z)^i tr_{|W} I(Y(a, w - z) u, z) q^{L(0)} \\
= \sum_i c_i z^{w(a) - 1 + i + w(u)} Res_{w = -z} (w - z)^i tr_{|W} Y(a, w) I(u, z) q^{L(0)} \\
- \sum_i c_i z^{w(a) - 1 + i + w(u)} Res_{w = -z} (w - z)^i tr_{|W} Y(a, w) I(u, z) q^{L(0)} \\
= \sum_i i c_i Res_{w = -z} (w - z)^i z^{w(a) - 1 + i} w^{-w(a)} F^I((a, w), (u, z), q)) \\
- \sum_i i c_i Res_{w = -z} (w - z)^i z^{w(a) - 1 + i} w^{-w(a)} F^I((u, z), (a, w), q)).
\]

By Proposition 3.2 and (2.36)-(2.38), we have

\[
o^i(a[-1] u) q^{L(0)} = z^{w(u)} tr_{|W} o(a) I(u, z) q^{L(0)} \\
- \frac{1}{2} F^I((a[0] u, z), q) + \sum_{k=1}^{\infty} E_{2k}(\tau) F((a[2k - 1] u, z), q) \\
= tr_{|W} o(a) o^i(u) q^{L(0)} + \sum_{k=1}^{\infty} E_{2k}(\tau) o^i(a[2k - 1] u) q^{L(0)}.
\]

As an intertwining operator version of Proposition 4.3.6 in [Z1], we have proved the following:

\[\text{Theorem 3.4} \quad \text{For} \; a \in V, u \in U, \; \text{we have}\]
\[
tr_{|W} o(a[0] u) q^{L(0)} = 0, \quad \text{(3.8)}
\]
\[
tr_{|W} o(a[-2] u) q^{L(0)} + \sum_{k=2}^{\infty} (2k - 1) E_{2k}(\tau) tr_{|W} o(a[2k - 2] u) q^{L(0)} = 0. \quad \text{(3.9)}
\]
Proof] (3.8) is proved in Proposition 3.1. Replace $a$ in (3.7) by $L[-1]a$. Since $(L[-1]a)[2k - 1] = -(2k - 1)a[2k - 2]$ and $o(L[-1]a) = 0$, we have (3.9).

Lemma 3.5 For every $u \in U$,
\[\text{tr}_W o(\tilde{\omega}) o(u) q^{L(0)} = (q \frac{d}{dq} - \frac{c}{24}) \text{tr}_W o(u) q^{L(0)} \quad (3.10)\]
and so
\[\text{tr}_W o(L[-2]u) q^{L(0)} - \sum_{k=1}^{\infty} E_{2k}(\tau) \text{tr}_W o(L[2k-2]u) q^{L(0)} = (q \frac{d}{dq} - c/24) \text{tr}_W o(u) q^{L(0)} \quad (3.11)\]

[Proof] Clearly, $\text{tr}_W o(\omega) o(u) q^{L(0)} = \text{tr}_W L(0) o(u) q^{L(0)} = (q \frac{d}{dq}) \text{tr}_W o(u) q^{L(0)}$. Since $o(\tilde{\omega}) = o(\omega) - \frac{c}{24}$, we have (3.10). Substitute $\tilde{\omega}$ into $a$ of (3.7), we have (3.11) by (3.10).

4 The space of one point functions on the torus

We will use the following notation:
(a) $M(\Gamma(1))$ is the ring of holomorphic modular forms on $\Gamma(1) = SL_2(\mathbb{Z})$; it is naturally graded $M(\Gamma(1)) = \oplus_{k \geq 0} M_k(\Gamma(1))$, where $M_k(\Gamma(1))$ is the space of forms of weight $k$. It is known that $M(\Gamma(1))$ is generated by $E_4(\tau)$ and $E_6(\tau)$. (c.f. Proposition 1.3.4 in [Bu].)
(b) Set $U(\Gamma(1)) = M(\Gamma(1)) \otimes \mathbb{C} U$.
(c) $O_q(U)$ is the $M(\Gamma(1))$-subspace of $U(\Gamma(1))$ generated by the following elements:
\[v[0]u, \quad v \in V, u \in U \quad (4.1)\]
\[v[-2]u + \sum_{k=2}^{\infty} (2k - 1) E_{2k}(\tau) \otimes v[2k - 2]u \quad v \in V, u \in U \quad (4.2)\]

A crucial connection between Zhu-algebra and Eisenstein series is Lemma 5.3.2 in [Zh]. We will reform it for modules as follows:

Lemma 4.1 Set $a *_\tau u = a[-1]u - \sum_{k=1}^{\infty} E_{2k}(\tau)a[2k - 1]u$. The constant terms of $a *_\tau u$ in $U[[q]]$ for $a \in V, u \in U$ is
\[a \cdot u - \frac{1}{2} a[0]u \quad (4.3)\]
In particular, the constant term of $\tilde{\omega} *_\tau u (= L[-2]u - \sum_{k=1}^{\infty} E_{2k}(\tau)L[2k-2]u)$ is
\[\tilde{\omega} \cdot u - \frac{1}{2} \tilde{\omega}[0]u \quad (4.4)\]
Proof By (2.29), we have
\[ a^{-1} b - \sum_{k=1}^{\infty} E_{2k}(\tau) a[k-1] u = \text{Res}_z (Y[a, z] (\varphi_1(\frac{z}{2\pi i}, \tau) - \frac{G_2(\tau)}{(2\pi i)^2} z)) u \]
and its constant term is
\[ \frac{1}{2} \text{Res}_z (Y[a, z] (\frac{e^z + 1}{e^z - 1}) u) \]
\[ = \frac{1}{2} \text{Res}_z (Y(a, e^z - 1) e^{z \text{wt}(a)} (\frac{e^z + 1}{e^z - 1}) u) \]
\[ = \frac{1}{2} \text{Res}_u (Y(a, w) \frac{(1+w)w^{\text{wt}(a)} (w+2)}{w} u) \]
\[ = a \cdot u - \frac{1}{2} a[0] u \]
by (2.15) and (2.4). 

\[ \Box \]

Definition 4.2 For an irreducible $V$-module $U$, we now define the space $C_1(U)$ of one point functions on $U$ to be the $\mathbb{C}$-linear space consisting of functions
\[ S : U(\Gamma(1)) \otimes \mathcal{H} \to \mathbb{C} \]
satisfying the following conditions:

(C1) For $u \in U(\Gamma(1))$, $S(u, \tau)$ is holomorphic in $\tau$.
(C2) $S(u, \tau)$ is $M(\Gamma(1))$-linear in the sense that $S(u, \tau)$ is $\mathbb{C}$-linear in $u$ and satisfies
\[ S(f(\tau) \otimes u, \tau) = f(\tau) S(u, \tau) \]
for $f(\tau) \in M(\Gamma(1))$ and $u \in U$.
(C3) For $u \in O_q(U)$, $S(u, \tau) = 0$
(C4) For $u \in U$,
\[ S(L[-2]u, \tau) = \partial S(u, \tau) + \sum_{k=2}^{\infty} E_{2k}(\tau) S(L[2k-2]u, \tau) \]
Here $\partial$ is the operator which is linear in $u$ and satisfies
\[ \partial S(u, \tau) = \partial_k S(u, \tau) = \frac{1}{2\pi i} \frac{d}{d\tau} S(u, \tau) + g E_2(\tau) S(u, \tau) \]
(4.5)
for $u \in U[k]$ and
\[ \partial S(f \otimes u, \tau) = (\partial_h f(\tau)) S(u, \tau) + f(\tau) \partial S(u, \tau) \]
for $f(\tau) \in M_h(\Gamma(1))$ and $\partial_h f(\tau) = \frac{1}{2\pi i} \frac{df(\tau)}{d\tau} + h E_2(\tau) f(\tau)$. We note $q \frac{d}{dq} = \frac{1}{2\pi i} \frac{df(\tau)}{d\tau}$.

Set $S^I(u, \tau) = F^I(u, \tau) q^{-c/24} = \text{tr}_{W_0} o^I(u) q^{L(0) - c/24}$ for $u \in U$ and extend it linearly for $M(\Gamma(1)) \otimes U$, where $c$ is the central charge of $V$.

Proposition 4.3

$S^I(\ast, \tau) \in C_1(U)$ if $S^I(u, \tau)$ is holomorphic in $\tau$. 

15
[Proof] (C2) is clear. Theorem 3.4 implies (C3). By Lemma 3.5,

\[ F^I(L[-2]u, \tau)q^{-c/24} - \sum_{k=2}^{\infty} E_{2k}(\tau)F^I(L[2k-2]u, \tau)q^{-c/24} \]

\[ = E_2(\tau)F^I(L[0]u, \tau)q^{-c/24} + (q^{d/8} - c/24)\text{tr}_{W^0}(u)q^{L(0)}q^{-c/24} \]

\[ = E_2(\tau)F^I(L[0]u, \tau)q^{-c/24} + q^{d/8}\text{tr}_{W^0}(u)q^{L(0)-c/24} \]

\[ = \partial \left( \text{tr}_{W^0}(u)q^{L(0)-c/24} \right) \text{ by (4.5).} \]

Hence \( S^I(\ast, \tau) \) satisfies (C4).

We fix an element \( S(\ast, \tau) \in C_1(U) \) for a while.

Lemma 4.4 Let \( u \in U \). If \( U \) satisfies condition \( C_{[2,0]} \) then there are \( m \in \mathbb{N} \) and \( r_i(\tau) \in M(\Gamma) \) for \( i = 0, \ldots, m-1 \) such that

\[ S(L[-2]^mu, \tau) + \sum_{i=0}^{m-1} r_i(\tau)S(L[-2]^iu, \tau) = 0. \] (4.6)

[Proof] Since \( C_{[2,0]}(U) \) is a direct sum of homogeneous subspaces with respect to the degree \([\text{wt}]\), there exists \( N \) such that elements \( a \) satisfying \([\text{wt}]a > N \) are in \( C_{[2,0]}(U) \).

Let \( A \) be the \( M(\Gamma(1))-\)submodule of \( U(\Gamma(1)) \) generated by \( \oplus_{n \leq N} U[n] \). We claim that \( U = A + O_q(U) \). Suppose false and let \( K \) be a minimal weight of \( U/(A + O_q(U)) \) and choose \( u \in U - (A + O_q(U)) \) with weight \( K = [\text{wt}]u \). Clearly, we have \([\text{wt}]u > N \) and so

\[ u = \sum b_i[-2]w_i + \sum c_j[0]u_j \]

with \( b_i, c_j \in V, w_i, u_j \in U \). We may assume that \( u = b[-2]w \) and \([\text{wt}]b[-2]u = K \). Then

\[ b[-2]w \in -\sum_{k=2}^{\infty} (2k-1)E_{2k}(\tau)b[2k-2]w + O_q(U) \subseteq A + O_q(U) \] (4.7)

by the minimality of \( K \) since \([\text{wt}]b[2k-2]w < K \) for \( k \geq 1 \). Hence \( U(\Gamma(1))/Q_q(V) \) is a finitely generated \( M(\Gamma(1))-\)module and so there are \( m \in \mathbb{N} \) and \( r_i(\tau) \in M(\Gamma) \) such that

\[ L[-2]^mu + \sum_{i=0}^{m-1} r_i(\tau)L[-2]^iu \in O_q(U). \]

Lemma 4.5 Let \( u \in U \). Suppose that \( L[n]u = 0 \) for \( n > 0 \). Then for \( i \geq 1 \) there are elements \( f_j(\tau) \in M(\Gamma) \) for \( j = 0, \ldots, i-1 \) such that

\[ S(L[-2]^iu, \tau) = \partial^i S(u, \tau) + \sum_{j=0}^{i-1} f_j(\tau)\partial^j S(u, \tau) \] (4.8)

[Proof] We will prove (4.8) by induction on \( i \). Since \( L[n]u = 0 \) for \( n > 0 \), (C4) implies \( S(L[-2]u, \tau) = \partial S(u, \tau) \). By (C4), we have \( S(L[-2]L[-2]^iu, \tau) = \partial S(L[-2]^iu, \tau) + \).
\[ \sum_{k=2}^{\infty} E_{2k} S(L[2k-2]L[-2]^i u, \tau). \] On the other hand, since \( L[2k-2]L[-2]^i u \) is written as \( \sum_{j=0}^{i-k-1} p_j L[-2]^{j} u \) with some constants \( p_j \) for \( k \geq 2 \), we have the desired statement.

Bearing in mind the definition of \( \partial \) and using (4.8), (4.6) may be reformulated as follows:

**Proposition 4.6** Suppose that \( U \) satisfies condition \( C_{[2,0]} \) and \( u \in U \) satisfies \( L[n]u = 0 \) for \( n > 0 \). Then there are \( m \in \mathbb{N} \) and \( h_i(\tau) \in M(\Gamma) \) for \( i = 1, \ldots, m-1 \) such that

\[
(\frac{d}{d\tau})^m S(u, \tau) + \sum_{i=0}^{m-1} h_i(\tau)(\frac{d}{d\tau})^i S(u, \tau) = 0. \tag{4.9}
\]

**Theorem 4.7** If \( U \) satisfies condition \( C_{[2,0]} \), then \( S^I(u, \tau) \) is holomorphic on the upper half-plane for \( u \in U \).

**Proof** By (3.8) and (3.9), we may assume that \( L[n]u = 0 \) for \( n > 0 \). Then by the same arguments as in the proof of Proposition 4.6, we obtain that \( S^I(u, \tau) \) satisfies (4.9). Since (4.9) is regular and \( h_i(\tau) \) converges absolutely and uniformly on every closed subset of \( \{q \mid |q| < 1\} \), so does \( S^I(u, \tau) \).

As a corollary, we have:

**Corollary 4.8** Assume that \( U \) satisfies condition \( C_{[2,0]} \). Then \( S(u, \tau) \) converges absolutely and uniformly in every closed subset of the domain \( \{q \mid |q| < 1\} \) for every \( u \in U \) and the limit function can be written as \( q^h f(q) \), where \( f(q) \) is some analytic function in \( \{q \mid |q| < 1\} \).

**Proof** We will prove the assertion by induction on \( [\text{wt}]u \). Assume \( L[n]u = 0 \) for \( n > 0 \). Since \( h_i(q) \) in (4.9) converges absolutely and uniformly on every closed subset of \( \{q \mid |q| < 1\} \) and the equation (4.9) is regular, we have the desired result for \( S(u, \tau) \). If \( u = L[-1]w \), then \( S(u, \tau) = 0 \). So we may assume that \( u = L[-2]w \) since \( L[n] : (n < 0) \) is generated by \( L[-1] \) and \( L[-2] \). In this case, since

\[ S(L[-2]w, \tau) = \partial S(w, \tau) + \sum_{k=1}^{\infty} E_{2k} S(L[2k-2]w, \tau), \]

we have the desired result by induction.

Now assume that \( u \in U \) satisfies \( L[n]u = 0 \) for \( n > 0 \). (4.9) is a homogeneous linear differential equation with holomorphic coefficients and, such that 0 is a regular singular point. We are therefore in a position to apply the theory of Frobenius-Fuchs concerning
the nature of the solutions to such equations. Frobenius-Fuchs theory tells us that $S(u, \tau)$ may be expressed in the following form: for some $p \geq 0$,

$$S(u, \tau) = \sum_{i=0}^{p} (\log q)^i S_i(u, \tau)$$

(4.10)

where

$$S_i(u, \tau) = \sum_{j=1}^{b(i)} q^{\lambda_{ij}} S_{i,j}(u, \tau)$$

(4.11)

$$S_{i,j}(u, \tau) = \sum_{n=0}^{\infty} a_{i,j,n} q^n$$

(4.12)

is homomorphic on the upper half-plane, and $\lambda_{i,j1} \neq \lambda_{i,j2} \pmod{\mathbb{Z}}$ for $j_1 \neq j_2$.

We claim that $S(u, \tau)$ has the similar form for every $u \in U$. (c.f. Theorem 6.5 [DLiM]).

**Theorem 4.9** Suppose that $U$ satisfies condition $C_{[2,0]}$. For every $u \in U$, the function $S(u, \tau)$ can be expressed in the form (4.10), (4.11), (4.12).

**Proof** We will prove the theorem by induction on $[\text{wt}]u$. If $L(n)u = 0$ for $n > 0$, then we have already mentioned at (4.10)-(4.12). Since $S(L[-1]u, \tau) = 0$, we may assume $u = L[-2]w$. Then

$$S(L[-2]w, \tau) = \partial S(w, \tau) + \sum_{k=1}^{\infty} E_{2k}(\tau) S(L[2k-2]w, \tau).$$

(4.13)

By induction, $\partial S(w, \tau)$ and $S(L[2k-2]w, \tau)$ have the desired forms and so does $S(L[-2]w, \tau)$.

Before we will prove the main assertion, we will have the following lemmas:

**Lemma 4.10** Let $A$ be a semi-simple associative algebra over $\mathbb{C}$, let $\omega$ be in the center of $A$, and let $A \otimes A$ be an $A$-bimodule and let $F$ be a linear functional of $B$ satisfying $F(ab) = F(ba)$ for every $a \in A, b \in B$. Assume further that $F((\omega - r)^N u) = 0$ for every $u \in U$, where $r$ is constant and $N$ an integer. Then there are irreducible $A$-modules $M^1, ..., M^n$ on which $\omega$ acts as a scalar $r$ and $I^i(*) \in \text{Hom}_A(B \otimes_A M^i, M^i)$ for $i = 1, ..., n$ such that

$$F(b) = \sum_{i=1}^{n} \text{tr}[M^i, P_{I^i}(b)]$$

(4.14)

for every $b \in B$, where $P_{I^i}(b) \in \text{End}(M^i)$ is given by $P_{I^i}(b) m_i = I^i(b \otimes m_i)$ for $m_i \in M^i$. 

18
Lemma 4.11 We have:

\[ S_{k,j}(a[0]u, \tau) = 0 \text{ for any } a \in V, u \in U, k, j. \]  
(4.15)

\[ S_{k,j}(u_q, \tau) = 0 \text{ for any } u_q \in O_q(U), k, j. \]  
(4.16)

\[ S_{p,j}(\hat{\omega} \ast_\tau u, \tau) = \partial S_{p,j}(u, \tau) \text{ for } u \in U. \]  
(4.17)

\[ S_{k-1}(\hat{\omega} \ast_\tau u, \tau) = kS_k(u, \tau) + \partial S_{k-1}(u, \tau) \text{ for } u \in U. \]  
(4.18)

\[ (\hat{\omega} \ast_\tau - \partial)^N S_{p-k,j}(u, \tau) = 0 \text{ for } N >> 0. \]  
(4.19)

**Proof** Since \( U(\Gamma(1)) \subseteq U[[q]], \) (4.15) and (4.16) are clear. Since \( \hat{\omega} \ast_\tau u = L[-2]u - \sum_{k=1}^{\infty} E_{2k}(\tau)L[2k-2]u, \) we have

\[ S(\hat{\omega} \ast_\tau u, \tau) = \partial S(u, \tau) \]
\[ = \sum_{k=0}^{p} k(\log q)^{k-1}S_k(u, \tau) + (\log q)^k \partial S_k(u, \tau) \]  
(4.20)

for \( u \in U \) and so we have (4.18). In particular, we have (4.17). By (4.18), we have \((\hat{\omega} \ast_\tau - \partial)S_r(u, \tau) = (r+1)S_{r+1}(u, \tau).\) Repeating these steps, we obtain (4.19). 

\[ \square \]

Lemma 4.12 Assume \( T(u, \tau) = q^A \sum_{n=0}^{\infty} \alpha_n(u)q^n \) satisfies the conditions (4.15), (4.16), (4.19), then the coefficient \( \alpha_0(u) \) of leading term satisfies

\[ \alpha_0(a \cdot u) = \alpha_0(u \ast a) \text{ for } u \in U, a \in V, \]  
(4.21)

\[ \alpha_0(u) = 0 \text{ for } u \in O(U) \text{ and} \]  
(4.22)

\[ \alpha_0((\omega - c/24 - \lambda)^N \cdot u) = 0 \text{ for } u \in U. \]  
(4.23)
Lemma 4.1 and Proposition 4.14

\[ (\tilde{\omega} \ast_{\tau} - \partial) T(u, \tau) = 0 \]

Since the constant terms of elements in \( O_\n(U) \) generates \( O(U) \), we have \( \alpha_0(u) = 0 \) for \( u \in O(U) \), which proves (4.22). Since the constant term of \( \tilde{\omega} \ast_{\tau} u \) is \( \tilde{\omega} \cdot u - \frac{1}{2} \tilde{\omega}[0]u \) for \( u \in U \) by Lemma 4.1 and \( T(\tilde{\omega}[0]u, \tau) = 0 \) by Proposition 3.1, the leading terms of \( (\tilde{\omega} \ast_{\tau} - \partial) T(u, \tau) \) is \( q^\lambda \alpha_0((\omega - c/24 - \lambda) \cdot u) \). Since the operator \( \tilde{\omega} \ast_{\tau} - \partial \) does not decrease the minimal degree of elements in \( U(\Gamma(1))[[q]] \), the leading term of

\[ (\tilde{\omega} \ast_{\tau} - \partial)^N T(u, \tau) = 0 \]

is \( q^\lambda \alpha_0((\omega - c/24 - \lambda)^N \cdot u) = 0 \) for \( u \in U \).

We note that \( S^I(u, \tau) \) satisfies the same conditions.

Lemma 4.13 Assume \( T(u, \tau) = q^\lambda \sum_{n=0}^{\infty} \alpha_n(u)q^n \) satisfies the conditions (4.15), (4.16) and (4.19), then there are irreducible \( V \)-modules \( \{ W_j : j \} \) with minimal weight \( \lambda + c/24 \) and intertwining operators \( \{ I^i : j \} \) of type \((U^{W_j}, W_j)\) such that \( T(u, \tau) \) is a sum of \( S^I(u, \tau) \).

Proof Since \( \alpha_0(u) \) satisfies (4.21), (4.22), (4.23), Lemma 4.10 implies that there are irreducible \( A(V) \)-modules \( W^1(0), \ldots, W^m(0) \) on which \( \omega \) acts as a scalar \( c/24 + \lambda \) and \( I_0^i \in \text{Hom}_{A(V)}(A(U) \otimes_{A(V)} W^i(0), W^i(0)) \) such that \( \alpha_0(u) = \sum_{i=1}^{m} t_{1}M(0)P_{I_0^i}(u) \), where \( P_{I_0^i}(u)m_i = I_0^i(u \otimes m_i) \). By Theorem 2.10, there are irreducible \( V \)-modules \( M^i \) with minimal weight \( \lambda \) and intertwining operators \( I^i(\ast, z) \in I \left( \begin{array}{c} M^i \\ U M^i \end{array} \right) \) such that \( M^i(0) \) is the top levels of \( M^i \) and \( I_0^i(u \otimes m_i) = o^{I^i}(u)m_i \) for \( m_i \in M^i(0) \). Then \( T(u, \tau) - \sum S^I(u, \tau) \) satisfies the same conditions (4.15), (4.16) and (4.19) and the degree of the leading term is greater than \( \lambda \). Repeating this steps, we finally have the desired result, since there are only finitely many non-isomorphic \( V \)-modules.

In particular, we have

Proposition 4.14 \( S_i(\ast, \tau) = 0 \) if \( i > 0 \).

Proof Suppose \( p \geq 1 \). By Lemma 4.13, \( S_{p-1}(\ast, \tau) \) is a linear combination of \( S^I(\ast, \tau) \) and so \( \partial S_{p-1}(u, \tau) = S_{p-1}(\tilde{\omega} \ast_{\tau} u, \tau) \). On the other hand, we have \( S_{p-1}(\tilde{\omega} \ast_{\tau} u, \tau) = pS_p(u, \tau) + \partial S_{p-1}(u, \tau) \) by (4.18), so that \( S_p(u, \tau) = 0 \) for \( u \in U \).

So we have proved the following main theorem, which is an intertwining operator version of Theorem 5.3.1 in [24].

Theorem 4.15 Suppose \( U \) satisfies condition \( C_{[2,0]} \). Let \( \{ W^1, \ldots, W^m \} \) be the set of irreducible \( V \)-modules and let \( \{ I^k_j(\ast, z) : j = 1, \ldots, j_k \} \) be a basis of \( I \left( \begin{array}{c} W^k \\ W_k \end{array} \right) \). Then \( C_1(U) \) is spanned by

\[ \{ S_{p,j}(u, \tau) : k = 1, \ldots, m, j = 1, \ldots, j_k \} . \]
5 Modular invariance

In this section, we will show that $C_1(U)$ is invariant under the action of $SL(2, \mathbb{Z})$ if the weights of $U$ are real. We note that $SL(2, \mathbb{Z})$ is generated by \( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) and \( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \).

Since $C_1(U)$ is spanned by $S^I(\cdot, \tau)$, which have forms $q^r(\sum_{n=1}^{\infty} a_n q^n)$, the transformation of $S(u, \tau)$ by \( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \) is clear and so it is sufficient to prove the assertion for $\gamma \tau = \frac{-1}{\tau}$.

Let $t + 2m (-1 \leq t \leq 1, m \in \mathbb{Z})$ be the lowest weight of $U$. Since $U$ is an irreducible $V$-module, the weights of elements in $U$ are all in $\mathbb{Z} + t$. First we take the principal branch of $(-i\tau)^{-t}$ on $i\mathcal{H}$ by taking $(re^{2\pi\theta})^{-t} = r^{-t}e^{2\pi(-\theta)}$ for $r \geq 0$ and $-\frac{1}{2} < \theta \leq \frac{1}{2}$ and $\tau^{-t-2n}$ is understood to be $(\tau^{-t})\tau^{-2n}$ for $n \in \mathbb{Z}$.

**Theorem 5.1** For $S(\cdot, \tau) \in C_1(U)$ and $\gamma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in SL(2, \mathbb{Z})$ define

$$S|\gamma(u, \tau) = (-i\tau)^{-t}\tau^{-k}S(u, \frac{-1}{\tau})$$

for $u \in U[\mathbb{R}]$. Then $S|\gamma(\cdot, \tau) \in C_1(U)$.

**Proof** Clearly, $S|\gamma(u, \tau) = (-i\tau)^{-t}\tau^{-k}S(u, \frac{-1}{\tau})$ is clearly holomorphic and it is also clear that $S|\gamma(\cdot, \tau)$ satisfies (C2). Let $a \in V[\mathbb{R}]$. Then we have:

$$S|\gamma(a[0]u, \tau) = (-i\tau)^{-t}\tau^{-k}S(a[0]u, \frac{-1}{\tau}) = 0$$

and

$$S|\gamma(a[-2]u + \sum_{j=2}^{\infty} (2j-1)E_{2j}(\tau) \otimes a[2j-2]u, \tau)$$

$$= (-i\tau)^{-t}\tau^{-k-p-1}S(a[-2]u, \tau)$$

$$+ (-i\tau)^{-t}\sum_{j=2}^{\infty} \tau^{-k-p+2j-1}S((2j-1)E_{2j}(\frac{1}{\tau}) \otimes a[2j-2]u, \frac{-1}{\tau})$$

$$= (-i\tau)^{-t}\tau^{-k-p-1}S(a[-2]u, \tau)$$

$$+ (-i\tau)^{-t}\tau^{-k-p-1}\sum_{j=2}^{\infty} S((2j-1)E_{2j}(\tau) \otimes a[2j-1]u, \frac{-1}{\tau})$$

$$= (-i\tau)^{-t}\tau^{-k-p-1}S(a[-2]u + \sum_{j=2}^{\infty} (2j-1)E_{2j}(\tau) \otimes a[2j-1]u, \frac{-1}{\tau})$$

$$= 0.$$}

We hence have (C3). We also have

$$\frac{d}{d\tau} (S|\gamma(u, \tau)) = \frac{d}{d\tau} ((-i\tau)^{-t}\tau^{-k}S(u, \frac{-1}{\tau}))$$

$$= -k(-i\tau)^{-t}\tau^{-k-p-1}S(u, \frac{-1}{\tau}) + (-i\tau)^{-t}\tau^{-k} \frac{d}{d\tau} (S(u, \frac{-1}{\tau})),$$
Using this, we obtain:

\[
S|\gamma(L[-2]u, \tau) = (-i\tau)^{-t} \tau^{t-k} S(L[-2]u, \frac{1}{\tau}) \\
= (-i\tau)^{-t} \tau^{t-k} \left( \frac{d}{2\pi id(-1/\tau)} S(u, \frac{1}{\tau}) + kE_2(-\frac{1}{\tau}) S(u, \frac{1}{\tau}) \\
+ \sum_{n=2}^{\infty} E_{2k}(\frac{1}{\tau}) S(L[2n-2]u, \frac{1}{\tau}) \right) \\
= (-i\tau)^{-t} \tau^{t-k} \left( \frac{\tau^2}{2\pi d^2} S(u, \frac{1}{\tau}) + k\tau^2 E_2(\tau) S(u, \frac{1}{\tau}) - \frac{k\tau}{2\pi} S(u, \frac{1}{\tau}) \\
+ \tau^{2n} E_{2n}(\tau) S(L[2n-2]u, \frac{1}{\tau}) \right) \\
= (-i\tau)^{-t} \left\{ \tau^{t-k} \frac{d}{2\pi d} S(u, \frac{1}{\tau}) + \tau^{t-k} kE_2(\tau) S(u, \frac{1}{\tau}) - \tau^{t-k} \frac{k}{2\pi} S(u, \frac{1}{\tau}) \\
+ \tau^{2n} E_{2n}(\tau) S(L[2n-2]u, \frac{1}{\tau}) \right\} \\
= \frac{1}{2\pi i d^2} \left\{ (-i\tau)^{-t} \tau^{t-k} \frac{d}{d\tau} S(u, \frac{1}{\tau}) - (-i\tau)^{-t} \tau^{t-k} k\tau S(u, \frac{1}{\tau}) \right\} \\
+ (-i\tau)^{-t} \tau^{t-k} kE_2(\tau) S(u, \frac{1}{\tau}) + (-i\tau)^{-t} \tau^{2n+t-k} E_{2n}(\tau) S(L[2n-2]u, \frac{1}{\tau}) \\
= \frac{d}{2\pi i d^2} S|\gamma(u, \tau) + kE_2(\tau) S|\gamma(u, \tau) + E_{2n}(\tau) S|\gamma(L[2n-2]u, \frac{1}{\tau}) \\
\]

and so (C4). This completes the proof of Theorem 5.1.

\[\square\]

**Lemma 5.2** For \( v \in U_{[k]} \):

\[
P : \left\{ \begin{array}{c}
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix} \\
\begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}
\end{array} \right\} \rightarrow (-i\tau)^{-t} \tau^{t-k} S(u, \frac{1}{\tau})
\]

is a representation of \( SL(2, \mathbb{Z}) \).

**[Proof]** Set \( A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) and \( B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \). It is sufficient to prove \( P(A)^2 = 1 \) and \((P(A)P(B))^3 = 1\) since \( SL(2, \mathbb{Z}) \) is generated by two elements satisfying these relations freely.

\[
(P(A))^2 S(u, \tau) \\
= (-i\tau)^{-t} \tau^{t-k} (-i \frac{1}{\tau})^{-t} (-1)^{t-k} S(u, \frac{1}{\tau}) \\
= (-i\tau)^{-t} (-\frac{1}{\tau})^{-t} (-1)^{t-k} S(u, \tau) \\
= S(u, \tau) \quad (\text{since } k - t \in 2\mathbb{Z}).
\]

\[
(P(A)P(B))^3 S(u, \tau) \\
= (i)^t (-i\tau)^{-t} \tau^{t-k} (-i \frac{1}{\tau} + 1)^{-t} (\frac{1}{\tau} + 1)^{t-k} (-i(\frac{1}{\tau} + 1) + 1)^{t-k} \times \\
\times S(u, \frac{1}{\tau} + 1) \\
= (i)^t (1 - \tau)^{-t} (-\frac{1}{\tau - 1})^{-t} S(u, \tau) \\
= S(u, \tau)
\]
6 Examples

Let $L(c, h)$ denote the irreducible module of Virasoro algebra with the highest weight $h$ and the central charge $c$ and it was proved in [FZ] that $L(c, 0)$ is a VOA. The work in [FO] and [GKO] gives a complete classification of unitary highest weight representations of the Virasoro algebra. In particular, $L(c_m, 0)$ for $c = c_m = 1 - \frac{6}{(m+2)(m+3)}$ ($m = 0, 1, 2, \ldots$) are rational VOAs called discrete series and their irreducible modules are $L(c_m, h^m_{r,s})$ with $h = h^m_{r,s} = \frac{(m+3)r-(m+2)s}{4(m+2)(m+3)}$ ($r, s \in \mathbb{N}, 1 \leq s \leq r \leq m + 1$). The fusion rules among $L(c_m, h^m_{r,s})$ are all determined, see [FF], [W].

Lemma 6.1 $L(c_m, h^m_{r,s})$ satisfies condition $C_{[2,0]}$.

[Proof] Set $U = L(c_m, h^m_{r,s})$ and let $e$ be a highest weight vector of $U$. We note that $(L(1, 0), Y([-1]), 1, \omega) \cong (L(1, 0), Y([-1]), 1, \omega)$ as VOAs. Set $P = \sum L[-n]U : n = 3, 4, \ldots >$. Clearly $P \subseteq C_{[2]}(U)$ since $(m-1)a[-m] = (L[-1]a)[-m+1]$. $P$ is also invariant under the action of $Vir_+$, where $Vir_+ = \odot_{n=1}^{\infty} CL[-n]$. Since $[L[-1], L[-2]] = L[-3]$, $U$ is spanned by $L[-1^n]L[-2]^m e + P : n, m \geq 0$ and spanned by $L[-2]^m e + C_{[2]}(U) : m \geq 0$.

$L(c, 0)$ is a quotient of the corresponding verma module $M = M(c, 0)$ and we have $M \cong U(Vir_+) \cdot 1$ (cf. [FZ]). We have $L(c, 0) = M/J$ and $J$ contains a singular vectors of the form

$$\alpha = L[-2]^m 1 + \sum a_{n_1, \ldots, n_r} L[-n_1 - 2] \ldots L[-n_r - 2] 1$$

by [FF], where the sum ranges over certain $(n_1, \ldots, n_r) \in \mathbb{Z}^r_+$ with $n_1 + \ldots + n_r \neq 0, a_{n_1, \ldots, n_r} \in \mathbb{C}$. We note that $(L[-n]u)[-1]u \equiv L[-n]u[-1]u \pmod{P}$ for $w \in L(c, 0), u \in U$ and $n \geq 2$ by Associativity (2.1). Therefore, we have

$$0 = \alpha[-1] e \equiv L[-2]^m e + \sum a_{n_1, \ldots, n_r} L[-n_1 - 2] \ldots L[-n_r - 2] e \equiv L[-2]^m e \pmod{P}.$$

So $U$ is spanned by $\{L[-2]^j e + C_{[2]}(U) : j = 0, 1, \ldots, m-1\}$, which implies that $U$ satisfies condition $C_{[2,0]}$.

Lemma 6.2 Assume $U$ and $W$ are irreducible $L(c, 0)$-modules and $I(U \oplus W) \neq 0$. Let $u$ be a highest weight vector of $U$. Then $S^j(u, \tau) \neq 0$.

[Proof] Set $U = L(c, k)$ and $W = L(c, h)$. We note that $\dim W(h) = \dim U(k) = 1$. Assume $S^j(u, \tau) = 0$. We first claim that $S^j(v, \tau) = 0$ for all $v \in U$. Since $U[k] = U(k) = \mathbb{C} u, S^j(v, \tau) = 0$ for $v \in U[k]$. Assume $S^j(w, \tau) = 0$ for $[\text{wt}]w < n+k$ and $[\text{wt}]v = n+1+k$. Since $U$ is a highest weight module, we may assume $v = L[-1]w \text{ or } v = L[-2]w$ for some $w \in U$. Since $S^j(L[-1]w, \tau) = S^j(\hat{w}[0]w, \tau) = 0$ by (3.8), we may assume $v = L[-2]w$. Then (3.11) implies $S^j(v, \tau) = (q^{2r} - c/24)S^j(w, \tau) + \sum_{r=1}^\infty E_{2r}S^j(L[2r-2]w, \tau) = 0$ by induction. So we have $S^j(v, \tau) = 0$ for $v \in U$. In particular, $\text{tr}_{W(0)} o(v) = 0$ and so
Hence $S$. 

E. Bannai, M. Koike, A. Nunemasa and J. Sekiguti, Klein’s icosahedral equation

T. M. Apostol, "Modular functions and Dirichlet series in number theory", Springer

References

D. Bump, "Automorphic forms and representations", C.S.A.M 55, Cambridge 1997.

We will calculate some trace function explicitly. We always take $u \in U$ such that the
coefficient of the leading term of $S(u, \tau)$ is one.

For example, $L(\frac{1}{2}, 0)$ is the first one in the discrete series and it has three irreducible
modules $L(\frac{1}{2}, 0), L(\frac{1}{2}, \frac{1}{2}), L(\frac{1}{2}, \frac{1}{10})$ as we mentioned in the introduction. For $U = L(\frac{1}{2}, \frac{1}{2})$, $W = L(\frac{1}{2}, \frac{1}{10})$ is the only irreducible $L(\frac{1}{2}, 0)$-module satisfying $0 \neq I(U, W)$. It also satisfies $\dim I(U, W) = 1$. Hence $S^t(v, \tau)$ is a modular form (with a linear character) of weight $\frac{1}{2}$ for $v \neq u \in L(\frac{1}{2}, \frac{1}{2})(0)$. By the definition of trace function, the leading term is

$q^{-\frac{1}{31}\frac{3}{5} + \frac{2}{7}} = q^\frac{1}{31}$. Since $S(u, \tau)$ and $\eta(\tau)$ are modular forms with linear characters and same
leading terms and $S(u, \tau)/\eta(\tau)$ is holomorphic on $\mathcal{H}$, $S(u, \tau) = \eta(\tau)$.

The second one is $L(\frac{7}{10}, 0)$. It has 6 irreducible modules $L(\frac{7}{10}, 0), L(\frac{7}{10}, \frac{1}{10}), L(\frac{7}{10}, \frac{3}{5}), L(\frac{7}{10}, \frac{3}{2}), L(\frac{7}{10}, \frac{7}{16})$ and $L(\frac{7}{10}, \frac{3}{80})$. For $U = L(\frac{7}{10}, \frac{1}{10})$, $W = L(\frac{7}{10}, \frac{3}{80})$ is the only irreducible module satisfying $I(U, W) \neq 0$. It also satisfies $\dim I(U, W) = 1$. Hence $S^t(u, \tau)$ is a modular form (with a linear character) of weight $\frac{1}{10}$ for $u \in U(0)$. Its leading term

$q^{-\frac{1}{31}\frac{3}{5} + \frac{2}{7}} = q^\frac{1}{31}$ is equal to one of $(\eta(\tau))^{1/5}$. Hence $S(u, \tau) = (\eta(\tau))^{1/5}$.

$L(\frac{2}{5}, 0)$ is the third and has 10 irreducible modules. For $U = L(\frac{2}{5}, \frac{2}{5}), W = L(\frac{2}{5}, \frac{1}{15})$

is the only irreducible module satisfying $I(U, W) \neq 0$. It also satisfies $\dim I(U, W) = 1$. Hence $S^t(u, \tau)$ is a modular form (with a linear character) of weight $\frac{1}{5}$ for $u \in U(0)$

and its leading term is $q^{4\frac{3}{21} + \frac{1}{15}} = q^{\frac{1}{21}}$, which is equal to the one of $(\eta(\tau))^{4/5}$. Hence

$S(u, \tau) = (\eta(\tau))^{4/5}$.

$L(\frac{6}{7}, 0)$ is the fourth, which has 15 irreducible modules. For $U = L(\frac{6}{7}, \frac{1}{7}), W = L(\frac{6}{7}, \frac{1}{21})$

is the only irreducible module satisfying $I(U, W) \neq 0$. We also have $\dim I(U, W) = 1$. Hence $S^t(u, \tau)$ is a modular form (with a linear character) of weight $\frac{1}{7}$ for $u \in U(0)$.

The leading term is $q^{-\frac{1}{31}\frac{3}{5} + \frac{2}{7}} = q^{1/81}$, which is equal to the one of $(\eta(\tau))^{2/7}$. Hence

$S(u, \tau) = (\eta(\tau))^{2/7}$.

References

[Ap] T. M. Apostol, "Modular functions and Dirichlet series in number theory", Springer

Verlag, 1976.

[BKMS] E. Bannai, M. Koike, A. Nunemasa and J. Sekiguti, Klein’s icosahedral equation

and modular forms, preprint 1999.

[Bo] R. E. Borcherds, Vertex algebras, Kac-Moody algebras, and the Monster,

Proc. Natl. Acad. Sci. USA 83 (1986), 3068-3071.

[Bu] D. Bump, ”Automorphic forms and representations”, C.S.A.M 55, Cambridge 1997.

24
[DLiM] C. Dong, H. Li and G. Mason, Modular-invariance of trace functions in orbifold theory, preprint.

[FF] B. L. Feigin and D. B. Fuchs, Verma modules over the Virasoro algebra, Lecture Notes in Math., Vol. 1060, Springer-Verlag, Berlin and New York, 1984.

[FHL] I. B. Frenkel, Y.-Z. Huang and J. Lepowsky, "On axiomatic approaches to vertex operator algebras and modules", Memoirs Amer. Math. Soc. 104, 1993.

[FLM] I. B. Frenkel, J. Lepowsky, and A. Meurman, "Vertex Operator Algebras and the Monster", Pure and Applied Math., Vol. 134, Academic Press, 1988.

[FQS] D. Friedan, Z. Qiu and S. Shenker, Conformal invariance, unitarity and two-dimensional critical exponents, MSRI publ. # 3, Springer-Verlag, (1984), 419-449.

[FRW] A. J. Feingold, J. F. X. Ries, and M. Weiner, Spinor construction of vertex operator algebras, triality and $E\text{g}(1)$, Contemporary Math. 121, 1991.

[FZ] I. B. Frenke, Y. Zhu, Vertex operator algebras associated to representations of affine and virasoro algebras, Duke Math. J. Vol. 66 (1992) 123-168.

[GKO] P. Goddard, A. Kent and D. Olive, Unitary representations of the Virasoro algebra and super-Virasoro algebras, Commun. Math. Phys. 103 (1986), 105-119.

[Ib] T. Ibukiyama, Modular forms of rational weights and modular varieties, Abhand. Math. Sem. Univ. Hamburg, to appear.

[In] E. Ince, "Ordinary Differential Equations", Dover Publications, Inc., 1956.

[La] S. Lang, "Elliplic Functions", Springer-Verlag, 1987.

[Li1] H. Li, Local systems of vertex operators, vertex superalgebras and modules, J. Pure Appl. Alg. 109 (1996), 143-195.

[Li2] H. Li, Determining fusion rules by $A(V)$-modules and bimodules, J. Algebra 212 (1999) 515-556.

[Mi1] M. Miyamoto, A modular invariance on the theta functions defined on vertex operator algebras, Duke Math. J, Vol. 101 (2000), 221-236.

[Mi2] M. Miyamoto, Modular invariance of trace functions on VOA in many variables, preprint.

[W] W. Wang, Rationality of Virasoro vertex operator algebra, Duke Math. J. IMRN, Vol. 71, No. 1 (1993), 197-211.

[Zh] Y. Zhu, Modular invariance of characters of vertex operator algebras, J. Amer. Math. Soc. 9 (1996), 237-302.