Network-Error Correcting Codes using Small Fields

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Abstract—Recently, Ebrahimi and Fragouli proposed an algorithm to construct scalar network codes using small fields (and vector network codes of small lengths) satisfying multicast constraints in a given single-source, acyclic network. The contribution of this paper is two fold. Primarily, we extend the scalar network coding algorithm of Ebrahimi and Fragouli (henceforth referred to as the EF algorithm) to block network-error correction. Existing construction algorithms of block network-error correcting codes require a rather large field size, which grows with the size of the network and the number of sinks, and thereby can be prohibitive in large networks. We give an algorithm which, starting from a given network-error correcting code, can obtain another network code using a small field, with the same error correcting capability as the original code. Our secondary contribution is to modify the EF Algorithm itself. The major step in the EF algorithm is to find a least degree irreducible polynomial which is coprime to another large degree polynomial. We suggest an alternate method to compute this coprime polynomial, which is faster than the brute force method in the work of Ebrahimi and Fragouli.

I. INTRODUCTION

Network coding was introduced in [1] as a means to improve the rate of transmission in networks. Linear network coding was introduced in [2]. Deterministic algorithms exist [3]–[5] to construct scalar network codes (in which the input symbols and the network coding coefficients are scalars from a finite field) which achieve the maxflow-mincut capacity in the case of acyclic networks with a single source which wishes to multicast a set of finite field symbols to a set of sinks, as long as the field size $q > N$. Finding the minimum field size over which a network code exists for a given network is known to be a hard problem [6]. Most recently, an algorithm was proposed in [7] which attempts to find network codes using small field sizes, given a network coding solution for the network over some larger field size $q > N$. The algorithms of [7] also apply to linear deterministic networks [8], and for vector network codes (where the source seeks to multicast a set of vectors, rather than just finite field symbols). In this work, we are explicitly concerned about the scalar network coding problem, although the same techniques can be easily extended to accommodate for vector network coding and linear deterministic networks, if permissible, as in the case of [7].

Network-error correction, which involved a trade-off between the rate of transmission and the number of correctable network-edge errors, was introduced in [9] as an extension of classical error correction to a network setting. Along with subsequent works [10] and [11], this generalized the classical notions of the Hamming weight, Hamming distance, minimum distance and various classical error control coding bounds to their network counterparts. Algorithms for constructing network-error correcting codes which meet a generalization of the classical Singleton bound for networks can be found in [10]–[12]. Using the algorithm of [12], a network code which can correct any errors occurring in at most $\alpha$ edges can be constructed, as long as the field size $q$ is such that

$$q > N \left( \frac{|E|}{2\alpha} \right),$$

where $E$ is the set of edges in the network. The algorithms of [10], [11] have similar requirements to construct such network-error correcting codes. This can be prohibitive when $|E|$ is large, as the sink nodes and the coding nodes of the network have to perform operations over this large field, possibly increasing the overall delay in communication.

In this work, we extend the EF algorithm to block network-error correction using small fields. As in [7], we shall restrict our algorithms and analysis to fields with binary characteristic. The techniques presented can be extended to finite fields of other characteristics without much difficulty. The contributions of this work are as follows.

- We extend the EF algorithm of [7] to construct network-error correcting codes using small fields, by bridging the techniques of the EF algorithm and the network-error correction algorithm of [12].
- The major step in the EF algorithm is to compute a polynomial of least degree coprime with a polynomial, $f(X)$, of possibly large degree. While it is shown in [7] that this can be done in polynomial time, the complexity can still be large. Optimizing based on our requirement, we propose a alternate algorithm for computing the polynomial coprime with $f(X)$. This is shown to have lesser complexity than that of the EF algorithm, which simply adopts a brute force method to do the same.

The rest of this paper is organized as follows. In Section II we give the basic notations and definitions related to network coding, required for our purpose. In Section III, we review the EF algorithm briefly and then propose our modification to it, and prove that the modified algorithm has lesser complexity than the original technique in the EF algorithm. Section IV presents our algorithm for constructing network-error correcting codes using small field sizes, along with calculations of the complexity of the algorithm. Examples illustrating the
algorithm performance for network coding and error correction are presented in Section VI. Finally, we conclude the paper in Section VII with comments and directions for further research.

II. NOTATIONS AND DEFINITIONS

The model for acyclic networks considered in this paper is as in [13]. An acyclic network can be represented as a directed multi-graph $G = (V, E)$, where $V$ is the set of all nodes and $E$ is the set of all edges in the network.

We assume that every edge in the directed multi-graph representing the network has unit capacity (can carry utmost one symbol from $E$). Network links with capacities greater than unity are modeled as parallel edges. The network is assumed to be instantaneous, i.e., all nodes process the same generation (the set of symbols generated at the source at a particular time instant) of input symbols to the network in a given coding order (ancestral order [13]).

At any particular time instant) of input symbols to the network in a particular coding order (ancestral order [13]).

We define a network transfer matrix $M_T$ for a $h'$-dimensional network code, corresponding to a sink node $T$ in $G$ is a full rank $h' \times h'$ matrix as $M_T := AFD_T = AF_T$. The matrix $M_T$ governs the input-output relationship at sink $T$. The problem of designing a $h'$-dimensional network code then implies making a choice for the matrices $A, F$, and $D_T$, such that the matrices $\{M_T : T \in T\}$ have rank $h'$ each. We thus consider each element of $A, F$, and $D_T$ to be a variable $X_i$ for some positive integer $i$, which takes values from the finite field $F_q$. Let $\{X_i\}$ be the set of all variables, whose values define the network code.

The variables $X_i$s are known as the local encoding coefficients [13]. For an edge $e$ in a network with a $h'$-dimensional network code in place, the global encoding vector [13] is a $h'$-dimensional vector which defines the particular linear combination of the $h'$ input symbols which flow through $e$. It is known [13–5] that deterministic methods of constructing a $h$-dimensional network code exist, as long as $g > N$.

Let $\Lambda$ be the length of the longest path from the source to any sink. Because of the structure of the matrices $A, F$, and $D_T$, it is seen [7] that the matrix $M_T$ has degree at most $\Lambda$ in any particular variable $X_i$ and also a total degree (sum of the degrees across all variables in any monomial) of $\Lambda$. Let $f_T(X_1, X_2, \ldots, X_{|X_i|})$ be the determinant of $M_T$ and $f(X_1, X_2, \ldots, X_{|X_i|}) = \prod_{T \in T} f_T$. Then the degree in any variable (and the total degree) of the polynomials $f_T$ and $f$ are at most $h'\Lambda$ and $N h' \Lambda$ respectively.

III. MODIFYING THE EF ALGORITHM

After briefly recollecting the EF algorithm, we shall proceed to modify its key step so that the overall complexity of the algorithm is reduced.

A. The EF Algorithm - Review

**Algorithm 1:** Scalar network coding algorithm using small fields - [7]

1. Assign values $\alpha_i$s to the scalar coding coefficients $X_i$s from an appropriate field $F_{2^k}$ ($2^k = 2^{\lceil \log(N) \rceil + 1} > N$) such that the network transfer matrices $M_T$s to all the sinks are invertible.
2. Express every $X_i = \alpha_i$ as a binary polynomial $p_i(X)$ of degree at most $k - 1$ using the usual polynomial representation of the finite field $F_{2^k}$, for a particular choice of the primitive polynomial of degree $k$.
3. Substituting these polynomials representing the $X_i$s in the matrices $M_T$, calculate the determinants of $M_T$ as the polynomials $f_T(X) \in F[X]$, and also find $f(X) = \prod_{T \in T} f_T$. Then, $f(X)$ is non-zero and has degree at most $N(k - 1)\Lambda$ in the variable $X$.
4. Find an irreducible polynomial of least degree, $g(X)$, which is coprime with $f(X)$.
5. Let $X_i = p_i(X) \mod g(X)$). Thus, each $X_i$ can be viewed as an element in $\frac{F[X]}{g(X)}$. Also, for each sink $T$, the matrices $M_T$ remain invertible as $f_T(X) \mod g(X) \neq 0$, as $f(X) \mod g(X) \neq 0$.

The following lemma ensures that such a coprime $g(X)$ exists and can be found in polynomial time.

**Lemma 1:** [7] If $f(X)$ is a non-zero binary polynomial of degree $n$, there exists a coprime polynomial $g(X)$ of degree at most $\log(n + 1) - 1$, and we can identify it in polynomial time.

**Remark 1:** [7] The worst-case complexity of computing $g(X)$ is $O(n^2 \log(n))$, where $n = Nh\Lambda \log(N)$.

B. Fast algorithm for computing least degree coprime polynomial

We now present a fast algorithm for computing the least degree irreducible polynomial $g(X)$ that is coprime with $f(X)$. Note that any polynomial $g(X)$ coprime with $f(X)$ is useful only if the degree of $g(X)$ is less than $[\log(N) + 1]$, as only such a $g(X)$ can result in a network code using a smaller field than the one we started with. Using this fact, we give Algorithm [2] which computes a least degree irreducible polynomial $g(X)$ that is coprime with $f(X)$.
Algorithm 2: Fast algorithm for computing a least degree coprime polynomial to the given polynomial \( f(X) \).

(1) Let \( \mathcal{P} = \{ X^i + X : i = 1, 2, \ldots, \lceil \log(N) \rceil \} \).

(2) For each \( p_i(X) \in \mathcal{P} \) do

\[
\text{Calculate } r(X) = f(X) \pmod{p_i(X)}. \\
\text{if } r(X) \text{ is non-zero then} \\
\text{Break.}
\]

(3) Let \( p_j(X) \) be the first polynomial for which \( r(X) \) is non-zero. Note that every \( p_i(X) \in \mathcal{P} \) is the product of all irreducible polynomials whose degree divides \( i \). Also, all irreducible polynomials of degree \( i < j \) divide \( f(X) \) as all \( p_i(X) \mid f(X) \) for all \( i < j \). Therefore, at least one of the irreducible polynomials of degree \( j \) is coprime with \( f(X) \). Find one such polynomial \( g(X) \).

C. Justification for Algorithm 2

The following lemma ensures that all polynomials which are found to be coprime with \( f(X) \) by directly computing the gcd (or the remainder for irreducible polynomials) in the brute force method (as done in Algorithm 1), can also be found by running Algorithm 2 using the set of polynomials \( \mathcal{P} \)upto the appropriate degree.

Lemma 2: For some field \( \mathbb{F} \), let \( f, g \in \mathbb{F}[X] \) be two polynomials relatively prime with each other. Let \( p \in \mathbb{F}[X] \) such that \( g \mid p \). Then \( g \) is also relatively prime with the polynomial \( f \pmod{p} \).

Proof: As \( f \) and \( g \) are coprime with each other, we can obtain polynomials \( a, b \in \mathbb{F}[X] \) such that \( af + bg = 1 \). Let \( f = qp + r \) for the appropriate quotient and remainder polynomials \( q, r \in \mathbb{F}[X] \) with \( \deg(r) < \deg(p) \). Also, as \( g \mid p \), let \( p = hg \). Then,

\[
1 = a(qp + r) + bg = a(qhg + r) + bg = ar + (aqh + b)g,
\]

which means that \( g \) and \( r \) are coprime with each other, hence proving the lemma.

D. Complexity of Algorithm 2

We now prove that our method for Step 4 of Algorithm 1 has less complexity than that of [7]. Towards that end, we first prove the following lemma.

Lemma 3: Let \( f, p \in \mathbb{F}_2[X] \), be such that \( \deg(f) = M \) and \( p = X^n + X \), for some non-negative integers \( M \) and \( n \). The polynomial \( f \pmod{p} \) can be calculated using at most \( O(M) \) bit additions.

Proof: Let \( f = \sum_{i=0}^{M} f_i X^i \). We arrange the coefficients of \( f \) as follows.

\[
s_0 \quad s_1 \quad s_2 \quad \ldots \quad s_{M-1} \quad f_0 \quad f_1 \quad f_2 \quad \ldots \quad f_{n-1} \\
f_n \quad f_{n+1} \quad \ldots \quad \ldots \quad f_{2n-2} \\
f_{2n-1} \quad f_{n+1} \quad \ldots \quad \ldots \quad f_{3n-3} \\
\vdots \quad \quad \quad \vdots \quad \vdots
\]

where \( n \) is the largest positive integer such that \( an-a+1 \leq M \).

Now, note that calculating the polynomial \( f \pmod{p} \), is equivalent to adding up the rows of the arrangement, while retaining the coefficient \( f_0 \) as it is. There are \( \lceil \frac{M}{n+1} \rceil \) rows in the arrangement, and adding any two rows requires at most \( n - 1 \) additions. Thus, the total number of bit additions is \( O(M) \).

Proposition 1: The complexity of Algorithm 2 is at most \( O(N^2) + O(h N (\log(N))^2 \Lambda) \).

Proof: The worst-case for Algorithm 2 would be \( j = \lceil \log(N) \rceil \). By Lemma 3 computing \( f(X) \pmod{p_i(X)} \) for some \( p_i(X) \in \mathcal{P} \) takes at most \( O(h N \log(N)) \) operations. As there are \( \lceil \log(N) \rceil \) such \( p_i(X) \)'s, evaluating the remainders \( r(X) \)'s costs \( O(h N (\log(N)^2 \Lambda) \) operations at most. Let \( f(X) \pmod{p_j(X)} = \tilde{f}(X) \) be the polynomial of degree at most \( 2^{\lceil \log(N) \rceil} = 2^j \). Now, we have to determine the complexity in obtaining the polynomial of degree \( \lceil \log(N) \rceil \) which is coprime with \( f(X) \) (or equivalently with \( \tilde{f}(X) \)).

There are approximately \( \frac{2^j}{\sqrt{j}} \) irreducible polynomials of order \( j \). It is known (see [14], for example) that for any two polynomials \( p(X) \) and \( q(X) \) (atleast one of them of degree \( w \)), the complexity of dividing \( p(X) \) by \( q(X) \) (or equivalently, calculating \( p(X) \pmod{q(X)} \)) is \( w \log(w) \). Thus, the complexity of dividing \( f(X) \) by every possible irreducible polynomial of degree \( j = \lceil \log(N) \rceil \) is at most \( \frac{2^j}{\sqrt{j}} O(2^{j \log(2^j)}) = O(N^2) \).

Thus, the total complexity for finding the least degree polynomial \( g(X) \) coprime with \( f(X) \) (which is assured of having a coprime factor of degree \( \lceil \log(N) \rceil + 1 \)) is at most \( O(N^2) + O(h N \log(N)^2 \Lambda) \).

Remark 2: Note that the worst-case complexity of Algorithm 2 is lesser than the worst-case complexity of finding the coprime polynomial \( g(X) \) according to [7] (which assumes a direct test for coprimeness of \( f(X) \) and the candidate polynomials), indicated in Remark 1. Even if we test for coprimeness only for polynomials upto degree \( \lceil \log(N) \rceil \), the algorithm of [7] would still have a worst-case complexity of \( O(N^2 h N \log(n)) \), where \( n = NhN \log(N) \).

IV. Network-error Correcting codes using small fields

This section presents the major contribution of this work. After briefly reviewing the network-error correcting code construction algorithm in [12], we proceed to give an algorithm which can obtain network-error correcting codes using small finite fields.
A. Network-Error Correcting Codes - Approach of [12]

An edge is said to be in error if its input symbol and output symbol (both from some appropriate field \( \mathbb{F}_q \)) are not the same. We model the edge error as an additive error from \( \mathbb{F}_q \). A network-error is a \(|E|\) length vector over \( \mathbb{F}_q \), whose components indicate the additive errors on the corresponding edges. A network code which enables every sink to correct any errors in any set of edges of cardinality at most \( \alpha \) is said to be a \( \alpha \) network-error correcting code. There have been different approaches to network-error correction [9-12]. We concern ourselves with the notations and approach of [12], as the techniques of [7].

It is known [9] that the number of messages \( M \) in an \( \alpha \) network-error correcting code is upper bounded according to the network Singleton bound as

\[
M \leq q^{h-2\alpha}.
\]

Assuming that the message set is a vector space over \( \mathbb{F}_q \) of dimension \( k \), we have

\[
k \leq h - 2\alpha.
\]

This bound was later refined [11] to accommodate for the different mincuts and different error capabilities at different sinks as

\[
k \leq h_T - 2\alpha_T,
\]

where sink \( T \in T \) can correct any network-error with errors in at most \( \alpha_T \) edges.

Algorithm 3 is a brief version of the algorithm given in [12] for constructing an \( \alpha \) network-error correcting code for a given single source acyclic network that meets the network Singleton bound. The construction of [12] is based on the network code construction algorithm of [4]. The algorithm constructs a network code such that all network-errors in up to \( 2\alpha \) edges will be corrected as long as the sinks know where the errors have occurred. Such a network code is then shown [12] to be equivalent to an \( \alpha \) network-error correcting code.

It is shown in [12] that Algorithm 3 results in a network code which is a \( \alpha \) network-error correcting code meeting the network Singleton bound, as long as the field size

\[
q > |T||F| = N \left( \frac{|E|}{2\alpha} \right).
\]

B. Network Error Correction using Small Fields - Algorithm

Algorithm 4 shown in the next page, constructs a network-error correcting code using small field sizes (conditioned on the existence of an irreducible polynomial of small degree satisfying the necessary requirements).

C. Justification for Step 5 of Algorithm

In order to ensure that the error correction property of the original network code is preserved, it is sufficient if a polynomial \( g(X) \) is coprime with each polynomial \( f_i(X) \), rather than their product \( f(X) \), as shown in Step 5 of Algorithm 4.

However, the following lemma shows that both are equivalent.

**Lemma 4:** Let \( \mathcal{U} = \{ f_i : f_i \in \mathbb{F}[X], i = 1, 2, ..., n \} \) be a collection of univariate polynomials with coefficients from some field \( \mathbb{F} \). A polynomial \( g \in \mathbb{F}[X] \) is relatively prime with all the polynomials in \( \mathcal{U} \) if and only if it is relatively prime with their product.

**Proof:** *If part:* If \( g \) is relatively prime with the product of all the polynomials in \( \mathcal{U} \), then there exist polynomials \( a, b \in \mathbb{F}[X] \) such that

\[
a \left( \prod_{i=1}^{n} f_i \right) + bg = 1.
\]

For each \( j \in 1, 2, ..., n \), we can rewrite (2) as

\[
\left( a \prod_{i=1, i \neq j}^{n} f_i \right) f_j + bg = 1,
\]

which implies that \( g \) is coprime with each \( f_j \in \mathcal{U} \).

*Only if part:* Suppose \( g \) is relatively prime with all the polynomials in \( \mathcal{U} \). Then, for each \( j \in 1, 2, ..., n \), we can find
polynomials \( a_j \) and \( b_j \) such that \( a_j f_j + b_j g = 1 \). In particular,
\[
\begin{align*}
  a_1 f_1 + b_1 g &= 1, \\
  a_2 f_2 + b_2 g &= 1.
\end{align*}
\] (3)

Using (4) in (3),
\[
1 = a_1 f_1(a_2 f_2 + b_2 g) + b_1 g
\]
\[
= (a_1 a_2) f_1 f_2 + (a_1 b_2) b_1 g.
\]

Thus, \( g \) is relatively prime with \( f_1 f_2 \). Continuing with the same argument, it is clear that \( g \) is relatively prime with \( \prod_{i=1}^{r} f_i \).

\[\textbf{Algorithm 4: Network-error correcting codes under small field sizes}\]

(1) With \( q = 2^{[\log(N[F])] + 1} \), run Algorithm 3 to find an \( \alpha \) network-error correcting code meeting the network Singleton bound. Let the encoding coefficients for \( X_i \) be \( \alpha_i \).

(2) Express every \( X_i = \alpha_i \) as a binary polynomial \( p_i(X) \) of degree at most \( k - 1 \) using the usual polynomial representation of the finite field \( \mathbb{F}_{2^k} \).

(3) \textbf{foreach} \( F \in \mathcal{F} \) \textbf{do}

\begin{enumerate}
  \item \textbf{foreach} \( T \in \mathcal{T} \) \textbf{do}
    \begin{enumerate}
      \item Find a non-zero minor of the matrix \( BF_T \), obtained from a \( (k + m_T) \times (k + m_T) \) submatrix. At least one such minor exists as \( BF_T \) has rank \( k + m_T \). Let the minor be \( f_T^p(X) \), which can be of degree at most \( m_T \log(N[\mathcal{F}]) \), according to Section 11 and 12.
    \end{enumerate}
\end{enumerate}

(4) Calculate the polynomial
\[
f(X) = \prod_{F \in \mathcal{F}} \prod_{T \in \mathcal{T}} f_T^p(X).
\]

(5) Using Algorithm 2 with the set
\[
\mathcal{P} = \left\{ X^2 + X : i = 1, 2, ..., \log(N[\mathcal{F}]) \right\},
\]
find an irreducible polynomial of least degree, \( g(X) \), which is coprime with \( f(X) \).

(6) Let \( X_i = p_i(X) \mod g(X) \). Thus, each \( X_i \) can be viewed as an element in \( \mathbb{F}_{2^k} / \langle g(X) \rangle \). Because of the fact that \( f_T^p(X) \mod g(X) \neq 0 \), the new \( BF_T \) matrices obtained after the modulo operation are also full rank, which implies that the error correcting capability of the code is preserved.

are \( \left( \begin{array}{c} h \\ k + m_T \end{array} \right) \) such minors, and calculating each takes \( O \left( \left( k + m_T \right)^3 \right) \) multiplications over \( \mathbb{F}_q \). As \( (k + m_T)^3 \) can take values up to \( h \), clearly the function to be maximized is of the form \( f(m) = \left( \frac{h}{m} \right)^3 \), for \( m = 0, 1, ... , h \). Proposition 2 gives the value of \( m \) for which such a function is maximized, based on which the value in Table II has been calculated.

\[\textbf{Proposition 2:} \] For some positive integer \( n \), let \( m \) be an integer such that \( 0 \leq m \leq n \). The function
\[
f(m) = \left( \frac{n}{m} \right)^3
\]

is maximized at
\[
m = \begin{cases} \left\lceil \frac{n}{2} \right\rceil + 1 & \text{if } n \geq 2 \\ 1 & \text{if } n = 1. \end{cases}
\]

\[\textbf{Proof:} \] The statement of the theorem is easy to verify for \( n = 1 \). Therefore, let \( n \geq 2 \). Let \( g(k) = f(k) - f(k + 1) \), for some \( k \), such that \( 0 \leq k \leq n - 1 \). Then,
\[
g(k) = \left( \frac{n}{k} \right)^3 \left( \frac{n}{k+1} \right)^3
\]
\[
= \left( \frac{n}{k} \right)^3 \left( \frac{n-k}{k+1} \right)^3
\]
\[
= \left( \frac{n}{k} \right)^3 \left( 2k^3 + k^2(2-n) + k(1-2n) - n \right)
\]
\[
= \left( \frac{n}{k} \right) \tilde{g}(k),
\]

where \( \tilde{g}(k) = \left( 2k^3 + k^2(2-n) + k(1-2n) - n \right) \). Proving the statement of the theorem is then equivalent to showing that both of the following two statements are true, which we shall do separately for even and odd values of \( n \).

- \( \tilde{g}(k) \leq 0 \) for all integers \( 0 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor \).
- \( \tilde{g}(k) > 0 \) for all integers \( \left\lfloor \frac{n}{2} \right\rfloor + 1 \leq k \leq n - 1 \).

\[\textbf{Case-A (n is even):} \] Let \( k = \frac{n}{2} + i \), for some integer \( i \) such that \(-\frac{n}{2} \leq i \leq \frac{n}{2} - 1 \). Then,
\[
\tilde{g}(k) = 2k^3 + k^2(2 - 2k + 2i) + k(1 - 4k + 4i) - 2k + 2i
\]
\[
= k^3(-2 + 2i) + k(4i - 1) + 2i.
\]

For \(-\frac{n}{2} \leq i \leq 0 \), it is clear from (4) that \( \tilde{g}(k) < 0 \). If \( 1 \leq i \leq \left( \frac{n}{2} - 1 \right) \), it is clear that \( \tilde{g}(k) > 0 \). Thus, for even values of \( n \), the theorem is proved.

\[\textbf{Case-B (n is odd):} \] Let \( k = \left\lfloor \frac{n}{2} \right\rfloor + i = \left( \frac{n-1}{2} \right) + i \), for some integer \( i \) such that \(-\frac{n-1}{2} \leq i \leq \frac{n-1}{2} \). Then,
\[
\tilde{g}(k) = 2k^3 + k^2(2 - 2k + 2i + 1)
\]
\[
+ k(1 - 4k + 4i + 1) - 2k + 2i + 1
\]
\[
= k^3(-2 + 2i) + k(4i + 1) + 2i + 1.
\]

Now, for \( i = 0 \), \( k = \left( \frac{n-1}{2} \right) \geq 2 \) (as \( n \geq 2 \) and is odd). Hence, \( \tilde{g}(k) = k^3 - k + 1 < 0 \) for \( i = 0 \). If \(-\frac{n-1}{2} \leq i \leq 0 \), then by (6), it is clear that \( \tilde{g}(k) < 0 \). Thus for all \(-\frac{n-1}{2} \leq i \leq 0 \), \( \tilde{g}(k) < 0 \).

For \( 1 \leq i \leq \left( \frac{n-3}{2} \right) \), again by (6), it is clear that \( \tilde{g}(k) > 0 \), and thus the theorem holds for odd values of \( n \). This completes the proof.
Computing the non-zero minor (over $\mathbb{F}_2[X]$) of $B^e_T$ from a $(k + m^e)$ square submatrix

Calculating $f(X) = \prod_{E \in \mathcal{E}} \prod_{T \in T} f^e_T(X)$.

Computing the coprime polynomial $g(X)$

| Step(s) | Complexity | Reasoning |
|---------|------------|-----------|
| Algorithm [3] | $A := O(\sqrt{|\mathcal{E}|}N h (|\mathcal{E}| + |\mathcal{E}'| + h + 2m))$ | [12] |
| Identifying non-zero minor of matrix $B^e_T$ | $B := O\left(\left(\frac{h}{m}\right) m^3\right)$, with $m = \lceil \frac{h}{T} \rceil + 1$ | Theorem 2 |
| Computing the non-zero minor (over $\mathbb{F}_2[X]$) of $B^e_T$ | $C := O(h^2 \log(|\mathcal{E}|)) + O\left((h \log(N) |\mathcal{E}|)^3\right)$ | [12] |
| Calculating $f(X) = \prod_{E \in \mathcal{E}} \prod_{T \in T} f^e_T(X)$. | $D := O((\log(a))$, where $a = Nh |\mathcal{E}| \log(N |\mathcal{E}|)$ | [14] |
| Computing the coprime polynomial $g(X)$ | $E := O(N^2 |\mathcal{E}|^2) + O(Nh |\mathcal{E}| \log(N |\mathcal{E}|)^3)$, | Proposition 1 |

| Total complexity | $A + N |\mathcal{E}| (B + C + D) + E$ | |

V. ILLUSTRATIVE EXAMPLES

Example 1: Consider the $\binom{6}{3}$ network shown in Fig. 1

This network has 20 sinks, each of which has 3 incoming edges from some 3-combination of the 6 intermediate nodes, thus the mincut $h$ being 3. Using the methods in [3] - [5], a 3-dimensional network code can be constructed for this network as long as the field size $q > 20$. Let $q = 2^5$. Consider the following sets of vectors in $\mathbb{F}_{32}^3$, with $\beta$ being a primitive element of $\mathbb{F}_{32}$.

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & \beta \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 0 \\ \beta^6 & 1 & \beta \\ 1 & \beta \beta^6 & \beta^3 \end{bmatrix}$$

Let $b_1(X) = X^5 + X^2 + 1$ and $b_2(X) = X^5 + X^3 + X^2 + X + 1$, both of them being primitive polynomials of degree 5. Note that $A$ and $B$ are valid choices (using either $b_1(X)$ or $b_2(X)$ as the primitive) for the global encoding vectors of the 6 outgoing edges from the source, representing deterministic network coding solutions for a 3-dimensional network code for this network. We assume that the intermediate nodes simply forward the incoming symbols to their outgoing edges, i.e., their local encoding coefficients are all 1.

Example 2: Consider the network, with 18 edges, shown in Fig. 2. This network is from [15], in which a 1 network-error correcting code meeting the network Singleton bound is given by brute-force construction for this network over $\mathbb{F}_4$, which is the smallest possible field over which such a code exists. According to the algorithm in [12], a 1 network-error correcting code can be constructed deterministically if $q > 2 \binom{18}{2} = 306$. In Fig. 2 let the variable $X_1$ denote the encoding coefficient between edges $v_1 \rightarrow v_4$ and $v_4 \rightarrow v_6$. Similarly, the variable $X_2$ ($X_3$) denote the local encoding coefficients between $v_2 \rightarrow v_5$ ($v_6 \rightarrow v_7$) and
Let \( p(X) \) be the primitive polynomial of \( \mathbb{F}_9 \), where \( \beta \) is a primitive element of \( \mathbb{F}_9 \). Let \( f(X) = X^4 + X^3 + X^2 + X + 1 \) be the primitive polynomial of degree 4.

Consider two such 1 network-error correcting codes obtained using Algorithm [3] for the network of Fig. 2 as follows.

Let \( \mathcal{A} \) and \( \mathcal{B} \) be two choices for the set \( \{X_1, X_2, X_3\} \) with all the other local encoding coefficients being unity. It can be verified that these two network codes can be used to transmit one error-free \( \mathbb{F}_{2^9} \) symbol from the source to both sinks, as long as not more than single edge errors occur in the network. Table III gives the results of running Algorithm [4] for this network starting from these two codes, with \( \beta_4 \) and \( \beta_5 \) being the primitive elements of \( \mathbb{F}_4 \) and \( \mathbb{F}_8 \) respectively.

Except for \( \{X_1, X_2, X_3\} \), all the other coding coefficients remain 1 over the respective fields. As in Example [1], the initial choice of the sets \( \mathcal{A} \) and \( \mathcal{B} \) for \( \{X_1, X_2, X_3\} \) results in the final network code being over different field sizes.

With \( \mathcal{B} \), the resultant network-error correcting code is over \( \mathbb{F}_4 \), exactly the one reported in [15] by brute force construction.

VI. CONCLUSION

A new and faster method of computing a coprime polynomial to a given polynomial has been presented, thereby improving the performance of the EF algorithm, which is applicable to scalar and vector network coding. Based on the EF algorithm, a method has been presented which can obtain network-error correcting codes using small fields that meet the network Singleton bound. This technique can be adapted to obtain network-error correcting codes meeting the refined...
Singleton bound, or for linear deterministic networks which permit solutions similar to those obtained in [7].

As in the original paper [7], questions remain open about the achievability of a code using the minimal field size. As illustrated by the examples in Section V factors such as the initial choice of the network code and the primitive polynomial of the field over which the initial code is defined (using which the local encoding coefficients are represented as polynomials), control the resultant field size after the algorithm.

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