INTEGRATING UNITARY REPRESENTATIONS OF INFINITE-DIMENSIONAL LIE GROUPS

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Abstract. We show that in the presence of suitable commutator estimates, a projective unitary
representation of the Lie algebra of a connected and simply connected Lie group \( G \) exponentiates
to \( G \). Our proof does not assume \( G \) to be finite-dimensional or of Banach–Lie type and therefore
encompasses the diffeomorphism groups of compact manifolds. We obtain as corollaries short proofs
of Goodman and Wallach’s results on the integration of positive energy representations of loop
groups and \( \text{Diff}(S^1) \) and of Nelson’s criterion for the exponentiation of unitary representations
of finite-dimensional Lie algebras.

1. Introduction

The integration of unitary representations of a finite-dimensional Lie group \( G \) from those of its Lie
algebra has been well-understood since the fundamental work of Nelson [Ne1]. Using analytic vectors,
one formally regards the unitary group \( \text{U}(H) \) of the corresponding Hilbert space as an analytic Lie
group and obtains a local homomorphism \( G \to \text{U}(H) \) via the Baker–Campbell–Hausdorff formula.
This, and more recent methods (see e.g. [Ro]), do not however apply when \( G \) is infinite-dimensional
for in the absence of a general inverse function theorem, its exponential map may fail to be locally
one-to-one, as is the case for the diffeomorphism groups of compact manifolds [Mi1].

In the present paper, we describe a method to exponentiate a unitary representation of the Lie algebra
\( L \) of \( G \) when the action of \( L \) is controlled by suitable commutator estimates. Our method does not
rely on the use of the exponential map of \( G \) and applies to finite and infinite-dimensional Lie groups,
whether of Banach–Lie type or not. It allows moreover to deal with projective representations. More
precisely, let \( \pi : L \to \text{End}(V) \) be a projective representation of \( L \) by skew-symmetric operators
acting on a dense subspace \( V \) of a Hilbert space \( H \). Thus, \( \pi \) is linear and for any \( X,Y \in L \)

\[
[\pi(X), \pi(Y)] = \pi([X,Y]) + iB(X,Y)
\]

(1.1)

for some real-valued two–cocycle \( B \) on \( L \). Our main assumption is the existence of a self–adjoint
operator \( A \geq 1 \) on \( H \) for which \( V \) is the space of smooth vectors, i.e. \( V = \bigcap_{n \geq 0} D(A^n) \)
and such that for any \( \xi \in V \) and \( n \in \mathbb{N} \)

\[
\|\pi(X)\xi\|_n \leq |X|_{n+1}\|\xi\|_{n+1}
\]

(1.2)

\[
\||A, \pi(X)\xi\|_n \leq |X|_{A,n+1}\|\xi\|_{n+1}
\]

(1.3)

where \( \|\xi\|_n = \|A^n\xi\| \) and the \( .| . \) are continuous semi–norms on \( L \).

A justification of this assumption might be in order. If \( G \) is finite–dimensional, one usually assumes
that the Laplacian \( \Delta = \sum_i \pi(X_i)^2 \) corresponding to some basis \( X_i \) of \( L \) is essentially self–adjoint
on \( V \). Nelson’s theorem then guarantees that \( \pi \) exponentiates to a unitary representation of \( G \) [Ne1]. In
this case, the action of \( L \) extends to \( \bigcap_{n \geq 0} D(\Delta^n) \) and, setting \( A = 1 - \Delta \), the estimates (1.2)–(1.3) follow
from simple intrinsic manipulations in the enveloping algebra of \( L \) [Ne1] so that Nelson’s criterion is a
special case of our assumptions. For an infinite–dimensional Lie group, no analogue of the Laplacian
exists but the above estimates were noticed by Goodman and Wallach in their work on positive energy
representations of loop groups \( LG = C^\infty(S^1, G) \) and of \( \text{Diff}(S^1) \) [GoWa1, GoWa2]. In both cases, the
‘laplacian’ \( A \) is the infinitesimal generator \( L_0 \) of rotations which embed in \( \text{Diff}(S^1) \) on the one hand

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and act automorphically on $LG$ on the other. Unlike the finite-dimensional situation, the estimates $(1.2) - (1.3)$ are a consequence of representation-dependent computations relying crucially on the fact that $L_0$ has non-negative spectrum, a defining property of positive energy representations.

The basic observation of the present paper is that the estimates $(1.2) - (1.3)$ may be used to regard the unitary group $U(H)$ as a regular Lie group. Formally then, a homomorphism $\mathcal{G} \to U(H)$ can be constructed by using a variant of Lie theory due to Thurston and Omori et al. [Om, Mi1]. Recall from [Mi1] that a Lie group $G'$ with Lie algebra $L'$ is said to be regular if for any $X \in C^\infty(I, L')$, with $I = [0, 1]$, there exists $p \in C^\infty(I, G')$ such that $\dot{p} = Xp$ and $p(0) = 1$. $p$ is then unique so that the product integral or Volterra map $X \to p(1)$ is well-defined and assumed to be smooth. This is a time-dependent exponential map, for if $X(t) \equiv X_0 \in L$, then $p(t) = \exp_{G'}(tX_0)$.

Product integrals may be used as a substitute for Baker-Campbell-Hausdorff series as follows. Let $\mathcal{G}, G'$ be Lie groups with Lie algebras $L, L'$ and assume that $\mathcal{G}$ is connected and simply connected and $G'$ regular. Then, any continuous homomorphism $F : L \to L'$ determines a unique homomorphism $\Phi : G \to G'$ with differential $F$ as follows. Let $g \in \mathcal{G}$ and $p$ a smooth path in $\mathcal{G}$ with $p(0) = 0$ and $p(1) = g$. Let $X = \dot{p}p^{-1} \in C^\infty(I, L)$ and $q \in C^\infty(I, G')$ be such that $\dot{q} = F(X)q$ and $q(0) = 1$. Set $\Phi(g) = q(1)$. To see that this is independent of $p$, pick a smooth homotopy $H : I^2 \to \mathcal{G}$ with $H(0, 0) = 1, H(1, 0) = 1, H(t, 0) = g$ and $H(t, 0) = p(t)$. The partial derivatives $\partial_t HH^{-1}$ define a flat $\mathcal{G}$-connection on $I^2$. Composing with $F$, we get a flat $G'$-connection a horizontal section $s$ of which may be constructed using product integrals. Since the connection vanishes on $\{0, 1\} \times I$, we get $s(1, 0)s(0, 0)^{-1} = s(1, 1)s(0, 1)^{-1}$ so that $\Phi$ is well-defined and is easily seen to be a homomorphism.

In our representation-theoretic context where, formally $G' = U(H)$, the rigorous definition of $\Phi : \mathcal{G} \to U(H)$ amounts to a study of the time-dependent Schrödinger equation

$$\frac{d\xi(t)}{dt} = \pi(X(t))\xi(t) \quad (1.4)$$
$$\xi(0) = \xi_0 \quad (1.5)$$

determined by $X \in C^\infty(I, L)$ and $\xi_0 \in V$ and most of the paper is devoted to proving the smooth well-posedness of $(1.4) - (1.5)$. Once that is established, the required exponentiation of $\pi$ is obtained by sending $g \in \mathcal{G}$ to the unitary operator $U(g)$ mapping $\xi_0 \in V$ to $\xi(1)$, where $\xi$ is the unique solution of $(1.4) - (1.5)$ with $X = \dot{p}p^{-1}$ and $p$ a smooth path in $\mathcal{G}$ with $p(0) = 1$ and $p(1) = g$.

The paper is structured as follows. In §3 we consider the time-independent version of $(1.4) - (1.5)$. Using Nelson’s commutator theorem, we prove that the action of $L$ on $V$ is essentially skew-adjoint and that the corresponding one-parameter groups preserve the scale defined by $A$. In §4 we prove the continuous well-posedness of $(1.4) - (1.5)$ by using product integrals with unbounded generators. The smooth well-posedness is established in §5 by studying the inhomogeneous equation obtained by formally differentiating $(1.4) - (1.5)$ with respect to $X$. §6 contains our main result (theorem 5.2.1). We define a Volterra map $: C(I, L) \to U(H)$ and show that it factors through a projective unitary representation $\rho$ of $\mathcal{G}$ the differential of which is $\pi$. We prove moreover that the central extension of $\mathcal{G}$ induced by $\rho$ is smooth. Finally, in §7 we apply our exponentiation result to positive energy representations of loop groups and Diff($S^1$) and unitary representations of finite-dimensional Lie algebras.

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Throughout this paper, $\mathcal{G}$ denotes a connected and simply connected Lie group and $\mathcal{L}$ its Lie algebra. We follow Milnor’s terminology and consider a Lie group to be a smooth manifold modelled on a complete, locally convex, real topological vector space of finite or infinite dimension, and possessing a compatible group structure. Let $\pi : \mathcal{L} \rightarrow \text{End}(V)$ be a projective representation of $\mathcal{L}$ by skew–symmetric operators acting on a dense subspace $V$ of a Hilbert space $\mathcal{H}$. We henceforth assume the existence of a self–adjoint operator $A \geq 1$ for which $V = \bigcap_{n \geq 0} \mathcal{D}(A^n)$ and such that, for any $X \in \mathcal{L}$, $\xi \in V$ and $n \in \mathbb{N}$

$$\|\pi(X)\xi\|_n \leq |X|_{n+1}\|\xi\|_{n+1} \tag{2.1}$$

$$\|A,\pi(X)\xi\|_n \leq |X|_{A,n+1}\|\xi\|_{n+1} \tag{2.2}$$

where $\|\xi\|_n = \|A^n\xi\|$ and the $|.|$ are continuous semi–norms on $\mathcal{L}$ which for convenience we take to be increasing in $n$.

For any $s \in \mathbb{R}$, let $\mathcal{H}^s$ be the completion of $V$ with respect to the inner product $(\xi,\eta)_s = (A^s\xi,A^s\eta)$ so that $A$ defines unitaries $\mathcal{H}^s \rightarrow \mathcal{H}^{s-1}$ and, if $s \geq 0$, $\mathcal{H}^s = \mathcal{D}(A^s)$. Let also $\mathcal{H}^\infty$ be $V = \bigcap_{s \geq 0} \mathcal{H}^s$ with the corresponding Fréchet topology. Since $(\xi,\eta) = (A^s\xi,A^{-s}\eta)$, $\mathcal{H}^s$ is canonically isomorphic to the (anti–)dual of $\mathcal{H}^{-s}$. In particular, by skew–symmetry of the $\pi(X)$, the estimates (2.1), (2.2) extend to any $n \in \mathbb{Z}$ provided we set $|X|_{-n} = |X|_{n+1}$ for $n \geq 0$. By (2.1) and (2.2), the operators $\pi(X) \in \text{End}(\mathcal{H}^\infty)$ extend to bounded linear maps $\mathcal{H}^n \rightarrow \mathcal{H}^{n-1}$. In particular, if $s \geq 1$, $\pi_s(X) := \pi(X)|_{\mathcal{H}^s}$ are densely defined skew–symmetric and therefore closeable operators on $\mathcal{H}$ which, by (2.1), have a common closure $\pi(X)$. We shall loosely refer to any of the $\pi_s(X)$ as $\pi(X)$, drawing a distinction only between these and $\pi(X)$.

**Proposition 2.1.** The operators $\pi(X)$, $X \in \mathcal{L}$, are essentially skew–adjoint on $\mathcal{H}^\infty$ and any of the $\mathcal{H}^n$, $n \geq 1$. Moreover, for each $n \in \mathbb{N}$, the unitaries $e^{\pi(X)}$ restrict to bounded linear maps $\mathcal{H}^n \rightarrow \mathcal{H}^n$ with

$$\|e^{\pi(X)}\|_{\mathcal{B}(\mathcal{H}^n)} \leq e^{2n|X|_{A,n}} \tag{2.3}$$

and therefore define continuous automorphisms of $\mathcal{H}^\infty$.

**Proof.** The essential skew–adjointness claim follows from Nelson’s commutator theorem [Nelson, prop. 2] since, by (2.1), (2.2) and interpolation, $\pi(X)$ and $[A,\pi(X)]$ define bounded operators $\mathcal{H}^1 \rightarrow \mathcal{H}^{-1}$. Let now $n \geq 1$ and $N = A^{2^n}$. We will show that $\mathcal{H}^s = \mathcal{D}(N^\frac{s}{2})$ is invariant under $e^{\pi(X)}$ by using an elegant trick of Faris and Lavine [Faris and Lavine, thm. 2]. We begin by establishing a simple quadratic form inequality. Let $\eta \in \mathcal{H}^\infty$, then by $[A^{2^n},\pi(X)] = \sum_{k=0}^{2^n-1} A^k[A,\pi(X)]A^{2^n-1-k}$ and (2.2), we get

$$\langle \pi(X)\eta,\pi(X)\eta \rangle + \langle N\eta,\pi(X)\eta \rangle = \langle [N,\pi(X)]\eta,\eta \rangle \leq \|N^\frac{s}{2}\eta\|\|N^{-\frac{s}{2}}[N,\pi(X)]\eta\| \leq \|N^\frac{s}{2}\eta\| \sum_{k=0}^{2^n-1} \|A^{k-n}[A,\pi(X)]A^{2^n-1-k}\eta\|$$

whence, by continuity, for any $\xi \in \mathcal{D}(N)$

$$\langle \pi(X)\eta,\pi(X)\eta \rangle + \langle N\eta,\pi(X)\eta \rangle \leq 2n|X|_{A,n}\|\eta\|^2 \tag{2.5}$$

\(^1\)We have departed from the usual convention that $\|\xi\|_n = \|A^n\xi\|$ which implies that $A$ is of order 2 with respect to its own scale. Since all operators we shall be considering are of the same order as $A$, we have preferred to normalise that order to 1.
Let now $\epsilon > 0$ and $N_\epsilon = N(\epsilon N + 1)^{-1}$, a bounded self-adjoint operator and notice that, by the spectral theorem, $D(N^{1/2}) = \{ \xi \in H| \lim_{r \to 0}(\xi, N \xi) < \infty \}$. Fix $\xi \in D(\pi(X))$ and let $\xi_t = e^{i\pi(X)\xi}$, then
\[
\frac{d}{dt}(\xi_t, N\xi_t) = (\pi(X)\xi_t, N\xi_t) + (N\xi_t, \pi(X)\xi_t)
\]
To rewrite this differently, consider
\[
(\pi(X)(\epsilon N + 1)^{-1}\xi_t, N\xi_t) + (N\xi_t, \pi(X)(\epsilon N + 1)^{-1}\xi_t)
\]
Using $(\epsilon N + 1)^{-1} = 1 - \epsilon N$ and the fact that $(\epsilon N + 1)^{-1}$ maps $H$ into $D(\pi(X))$ so that $\epsilon N\xi_t = \xi_t - (\epsilon N + 1)^{-1}\xi_t \in D(\pi(X))$, we may rewrite (2.5) as
\[
(\pi(X)\xi_t, N\xi_t) - \epsilon(\pi(X)N\xi_t, N\xi_t) + (N\xi_t, \pi(X)\xi_t) - \epsilon(N\xi_t, \pi(X)N\xi_t)
\]
which, by the skew–adjointness of $\pi(X)$ is equal to (2.6). Therefore, using (2.5)
\[
\frac{d}{dt}(\xi_t, N\xi_t) = (\pi(X)(\epsilon N + 1)^{-1}\xi_t, N(\epsilon N + 1)^{-1}\xi_t) + (N(\epsilon N + 1)^{-1}\xi_t, \pi(X)(\epsilon N + 1)^{-1}\xi_t)
\]
Integrating this inequality, we find $(e^{i\pi(X)\xi}, N e^{i\pi(X)\xi}) \leq e^{2n|X|A,n|t|} (\xi, N\xi)$ for any $\xi \in D(\pi(X))$ and therefore for any $\xi \in H\pi$. Choosing now $\xi \in D(N^{1/2})$ with $\|\xi\|_n = 1$ and letting $\epsilon \to 0$ we see that $e^{i\pi(X)\xi} \in D(N^{1/2})$ and $e^{i\pi(X)\xi}\xi_n \leq e^{2n|X|A,n}$ as claimed $\Box$

Corollary 2.2. For any $X \in L$, $\xi \in H^{\infty}$ and $k \geq 1$, we have
\[
e^{(t+h)i\pi(X)\xi} = e^{i\pi(X)\xi} + \cdot \cdot \cdot + \frac{h^k}{k!} \pi(X)^k e^{i\pi(X)\xi} + R(h)
\]
where all terms are in $H^{\infty}$ and $R(h) = o(h^k)$ in each $\| \cdot \|_n$ norm, i.e. $\| R(h) \|_n h^{-k} \to 0$ as $h \to 0$.

Proof. We have $\xi \in \cap_n D(\pi(X)^n) \subset C^\infty(\pi(X))$ and consequently, by Taylor’s theorem (2.10) holds where
\[
R(h) = \int_0^h dt_1 \cdot \cdot \cdot \int_0^{t_k} dt_{k+1} \pi(X)^{k+1} e^{t_{k+1}i\pi(X)\xi}
\]
is in $H^\infty$ since all other terms are. Moreover, by (2.4) and (2.8)
\[
\| R(h) \|_n \leq \frac{|h|^{k+1}}{(k+1)!} (|X|^{n+k+1})^{k+1} e^{2(n+k+1)|X|^A,n+k+1} \| \xi \|^{n+k+1}_n = o(h^k)
\]
$\Box$

Corollary 2.3. For any $X,Y \in L$ and $\xi \in H^{n+1}$, we have
\[
\| e^{i\pi(X)\xi} - e^{i\pi(Y)\xi} \|_n \leq |X - Y|^{n+1} e^{2(n+1)|X|^A,n+1} \| \xi \|_n
\]
Proof. Let $\xi \in H^\infty$ and set $F(t) = e^{-t\pi(X)} e^{t\pi(Y)}\xi$. By (2.10), $F$ is differentiable and $\dot{F} = e^{-t\pi(X)}(\pi(Y) - \pi(X)) e^{t\pi(Y)}\xi$. Thus,
\[
\| e^{i\pi(X)\xi} - e^{i\pi(Y)\xi} \|_n = \| \pi(X) \|_n \int_0^1 \dot{F}(t) dt \|_n
\]
\[
\leq \int_0^1 \| e^{(1-t)\pi(X)}(\pi(Y) - \pi(X)) e^{t\pi(Y)}\xi \|_n dt
\]
\[
\leq e^{2n|X|^A,n} |Y - X|^{n+1} e^{2(n+1)|X|^A,n+1} \| \xi \|_n
\]
$\Box$
3. Time–dependent ODE’s in $\mathcal{H}$

3.1. Product integrals with unbounded generators.

If $X \in C(\mathbb{R}, \mathcal{L})$ and $a < b \in \mathbb{R}$, we define below the product integral $\prod_{b \geq \tau \geq a} \exp(X(\tau)d\tau)$ by adapting the presentation of [Ne, §1.2] where the case of bounded infinitesimal generators is treated. Consider first step functions $X : [a, b] \to \mathcal{L}$, $X(\tau) = X_j$ for $\tau_j \geq \tau > \tau_{j-1}$ corresponding to subdivisions $b = \tau_n > \tau_{n-1} > \cdots > \tau_1 > \tau_0 = a$ and set

\[
\prod_{b \geq \tau \geq a} \exp(X(\tau)d\tau)\xi = e^{\Delta_n \pi(X_n)} \cdots e^{\Delta_1 \pi(X_1)}\xi
\]

(3.1.1)

where $\Delta_j = \tau_j - \tau_{j-1}$. If $X, Y$ are two step functions, which we may take as defined on a common subdivision, the identity $E_n \cdots E_1 - F_n \cdots F_1 = \sum_{k=1}^{n} E_n \cdots E_k+1(E_k - F_k)F_{k-1} \cdots F_1$ and the estimates (2.3) and (2.13) imply that

\[
\| \prod_{b \geq \tau \geq a} \exp(X(\tau)d\tau)\xi - \prod_{b \geq \tau \geq a} \exp(Y(\tau)d\tau)\xi \|_r \leq (b-a)|X - Y|_{[a, b]} e^{2(r+1)(b-a)} \max(|X|_{a, r+1}, |Y|_{a, r+1})\|\xi\|_r
\]

(3.1.2)

where $|Z|_{[a, b]} = \sup_{t \in [a, b]} |Z(t)|$. We may therefore define the product exponential as a bounded operator $\mathcal{H}^{r+1} \to \mathcal{H}^r$ for any $X \in C([a, b], \mathcal{L})$ by using a sequence of approximating step functions. However, since

\[
\|e^{\Delta_n \pi(X_n)} \cdots e^{\Delta_1 \pi(X_1)}\|_r \leq e^{2r(b-a)|X|_{a, r}} \|\xi\|_r
\]

(3.1.3)

and $\mathcal{H}^{r+1}$ is dense in $\mathcal{H}^r$, the operator extends to a bounded linear map $\mathcal{H}^r \to \mathcal{H}^r$ satisfying

\[
\| \prod_{b \geq \tau \geq a} \exp(X(\tau)d\tau)\|_{\mathcal{B}(\mathcal{H}^r)} \leq e^{2r(b-a)|X|_{a, r}}
\]

(3.1.4)

and (3.1.2). Notice that product integrals are invertible operators, in fact

\[
\prod_{b \geq \tau \geq a} \exp(X(\tau)d\tau)^{-1} = \prod_{b \geq \tau \geq a} \exp(-\tilde{X}(\tau)d\tau)
\]

(3.1.5)

where $\tilde{X}(\tau) = X(a + b - \tau)$, so that we may set, for $a > b$,

\[
\prod_{b \geq \tau \geq a} \exp(X(\tau)d\tau) = \prod_{a \geq \tau \geq b} \exp(X(\tau)d\tau)^{-1}
\]

(3.1.6)

With this convention, the following semigroup property always holds for any $a, b, c \in \mathbb{R}$,

\[
\prod_{c \geq \tau \geq b} \exp(X(\tau)d\tau) \prod_{b \geq \tau \geq a} \exp(X(\tau)d\tau) = \prod_{c \geq \tau \geq a} \exp(X(\tau)d\tau)
\]

(3.1.7)

**Lemma 3.1.1.** If $X \in C(\mathbb{R}, \mathcal{L})$, the map $t \mapsto \prod_{t \geq \tau \geq 0} \exp(X(\tau)d\tau) \in \mathcal{B}(\mathcal{H}^r)$ is strongly continuous for any $r \in \mathbb{N}$.

**Proof.** Since the operators $\prod_{t \geq \tau \geq 0} \exp(X(\tau)d\tau)$ are locally uniformly bounded in $\mathcal{B}(\mathcal{H}^r)$, it is sufficient to check strong continuity on the dense set of vectors $\xi \in \mathcal{H}^{r+1} \subset \mathcal{H}^r$. By (3.1.7)

\[
\| \prod_{t \geq \tau \geq 0} \exp(X(\tau)d\tau) \xi - \prod_{t \geq \tau \geq 0} \exp(X(\tau)d\tau) \xi \|_r
\]

\[
= \| \prod_{t \geq \tau \geq t} \exp(X(\tau)d\tau) - 1 \prod_{t \geq \tau \geq 0} \exp(X(\tau)d\tau) \xi \|_r
\]

\[
\leq \|h\| |X|_{[a, b], r+1} e^{2(r+1)|h|X|_{[a, b], r+1}} \| \prod_{t \geq \tau \geq 0} \exp(X(\tau)d\tau) \xi \|_r
\]

(3.1.8)

which tends to zero as $h \to 0$. □
3.2. Time–dependent ODE’s in $H^r$.

Let $I \ni t$ be a fixed compact interval, $r \in \mathbb{N}$, and consider the Banach space $C(I, H^{r+1}) \cap C^1(I, H^r)$ with norm $f \mapsto \|f\|_{r+1} + \|\dot{f}\|_r$ where $\|g\|_k = \sup_{t \in I} \|g(t)\|_k$.

**Theorem 3.2.1.** Let $X \in C(I, \mathcal{L})$, $\xi \in H^{r+1}$ and set

$$I(X, \xi)(t) = \prod_{t \geq \tau \geq 0} \text{Exp}(X(\tau)d\tau)\xi$$

Then, $I(X, \xi) \in C(I, H^{r+1}) \cap C^1(I, H^r)$ and is the unique solution of

$$\dot{\xi}(t) = \pi(X(t))\xi(t)$$

(3.2.2)

$$\xi(0) = \xi$$

(3.2.3)

Moreover, the map $C(I, \mathcal{L}) \times H^{r+1} \rightarrow C(I, H^{r+1}) \cap C^1(I, H^r)$, $(X, \xi) \rightarrow I(X, \xi)$ is continuous.

**Proof.** The uniqueness follows from the skew–symmetry of the $\pi(X)$ since for any solution of (3.2.2),

$$\frac{d}{dt}(\xi(t), \xi(t)) = (\pi(X(t))\xi(t), \xi(t)) + (\xi(t), \pi(X(t))\xi(t)) = 0$$

(3.2.4)

and therefore $\|\xi(t)\| = \|\xi(0)\|$. Let now $I = I(X, \xi)$. By lemma [3.1.1], $I \in C(I, H^{r+1})$. Moreover, by the semigroup property, $I(t+h) - I(t) = \prod_{t+h \geq \tau \geq t} \text{Exp}(X(\tau)d\tau)\eta - \eta$ where $\eta = I(t) \in H^{r+1}$. By (3.2.1),

$$\left\| \prod_{t+h \geq \tau \geq t} \text{Exp}(X(\tau)d\tau)\eta - \prod_{t+h \geq \tau \geq t} \text{Exp}(X(\tau)d\tau)\eta \right\|_r \leq \|h\|_r \|X - X(t)\|_{r+1}^{[t,t+h]} e^{2(r+1)||X| X|_{r+1}} \|\eta\|_{r+1} = o(h)$$

(3.2.5)

by continuity of $X$. On the other hand, in $H^r$

$$\prod_{t+h \geq \tau \geq t} \text{Exp}(X(\tau)d\tau)\eta = e^{h\pi(X(t))}\eta = \eta + h\pi(X(t))\eta + o(h)$$

(3.2.6)

since $\eta \in H^{r+1} \subset D(\pi(X(t)))$ so that $I(X, \xi)$ lies in $C^1(I, H^r)$ and satisfies (3.2.2)–(3.2.3).

To check the continuity of $I(X, \xi)$ in the first variable for the $C(I, H^{r+1})$ norm, let $X, Y \in C(I, \mathcal{L})$, $\xi \in H^{r+1}$ and $\eta \in H^{r+2}$ be an auxiliary vector. Then,

$$\|I(X, \xi) - I(Y, \xi)\|_{r+1}^{I} \leq \|I(X, \xi) - I(X, \eta)\|_{r+1}^{I} + \|I(X, \eta) - I(Y, \eta)\|_{r+1}^{I} + \|I(Y, \eta) - I(Y, \xi)\|_{r+1}^{I}$$

$$\leq (e^{2(r+1)||X| X|_{r+1} + e^{2(r+1)||Y| Y|_{r+1}}) \|\xi - \eta\|_{r+1} + |I| \|X - Y\|_{r+1} e^{2(r+2)|I| \max(\|X| X|_{r+2}, \|Y| Y|_{r+2}) \|\eta\|_{r+2}}$$

(3.2.7)

so that

$$\limsup_{Y \to X} \|I(X, \xi) - I(Y, \xi)\|_{r+1}^{I} \leq \inf_{\eta \in H^{r+2}} 2e^{2(r+1)|I| \|X| X|_{r+1} \|\xi - \eta\|_{r+1} = 0$$

(3.2.8)

Joint continuity in the $C(I, H^{r+1})$ norm now follows from

$$\|I(X, \xi) - I(Y, \psi)\|_{r+1}^{I} \leq \|I(X, \xi) - I(X, \eta)\|_{r+1}^{I} + \|I(Y, \xi) - I(Y, \eta)\|_{r+1}^{I}$$

$$\leq \|I(X, \xi) - I(Y, \xi)\|_{r+1}^{I} + e^{2(r+1)|I| \|X| X|_{r+1} \|\xi - \eta\|_{r+1} + e^{2(r+1)|I| \|Y| Y|_{r+1} \|\xi - \psi\|_{r+1}$$

(3.2.9)

and in the $C^1(I, H^r)$ from

$$\|\dot{I}(X, \xi) - \dot{I}(Y, \eta)\|_{r+1}^{I} = \|\pi(X)I(X, \xi) - \pi(X)I(Y, \eta)\|_{r+1}^{I}$$

$$\leq \|\pi(X)I(X, \xi) - \pi(Y)I(X, \xi)\|_{r+1}^{I} + \|\pi(Y)I(X, \xi) - \pi(Y)I(Y, \eta)\|_{r+1}^{I}$$

$$\leq \|X - Y\|_{r+1}^{I} \|I(X, \xi) - I(Y, \xi)\|_{r+1}^{I} + \|Y\|_{r+1}^{I} \|I(X, \xi) - I(Y, \eta)\|_{r+1}^{I}$$

(3.2.10)

$\square$
Corollary 3.2.2. The map \( C([a, b], \mathcal{L}) \to \mathcal{B}(\mathcal{H}^r), X \to \prod_{b \geq r \geq a} \exp(X(\tau)d\tau) \) is strongly continuous for any \( r \geq 0 \).

**Proof.** For \( r \geq 1 \) the claim is a direct consequence of theorem 3.2.1. If \( r = 0 \), the operators are unitaries in \( \mathcal{B}(\mathcal{H}) \) and it is sufficient to check strong continuity on the dense subspace \( \mathcal{H}^1 \subset \mathcal{H} \). This in turn follows from theorem 3.2.1.  

Corollary 3.2.3. If \( \phi : [a, b] \to [c, d] \) is a smooth map with \( \phi(a) = c \) and \( \phi(b) = d \), then, for any \( X \in C([c, d], \mathcal{L}) \),

\[
\prod_{c \geq r \geq d} \exp(X(\tau)d\tau) = \prod_{a \geq \sigma \geq b} \exp(\phi' \cdot X \circ \phi(\sigma)d\sigma) \tag{3.2.11}
\]

**Proof.** Let \( \xi_0 \in \mathcal{H}^1 \) and \( \xi : [c, d] \to \mathcal{H}^1 \) the solution of \( \dot{\xi} = \pi(X)\xi \), \( \xi(c) = \xi_0 \) so that \( \eta = \xi \circ \phi \) satisfies \( \dot{\eta} = \pi(\phi' \cdot X \circ \phi)\eta \) and \( \eta(a) = \xi_0 \). Then, by uniqueness,

\[
\prod_{c \geq r \geq d} \exp(X(\tau)d\tau)\xi_0 = \xi(d) = \eta(b) = \prod_{a \geq \sigma \geq b} \exp(\phi' \cdot X \circ \phi(\sigma)d\sigma)\xi_0 \tag{3.2.12}
\]

\( \square \)

4. **Smooth well-posedness of time-dependent ODE’s in \( \mathcal{H} \)**

4.1. **Inhomogeneous differential equations in \( \mathcal{H}^r \).**

We shall be concerned with the continuous well-posedness of the inhomogeneous linear equation

\[
\dot{\zeta} = \pi(X)\zeta + \eta \tag{4.1.1}
\]

in \( \zeta \in C(I, \mathcal{H}^{r+1}) \cap C^1(I, \mathcal{H}^r) \) with initial condition \( \zeta(0) = 0 \), \( X \in C(I, \mathcal{L}) \) and \( \eta \in C(I, \mathcal{H}^{r+1}) \) which is obtained by formally differentiating \( 3.2.2 \) with respect to \( X \). A solution is readily obtained through a variation of constants by setting \( \zeta(t) = \prod_{t \geq \tau \geq 0} \exp(X(\tau)d\tau)\zeta_0(t) \) which yields \( \dot{\zeta}_0(t) = \prod_{t \geq \tau \geq 0} \exp(X(\tau)d\tau)^{-1}\eta(t) \).

**Theorem 4.1.1.** Let \( X \in C(I, \mathcal{L}) \) and \( \eta \in C(I, \mathcal{H}^{r+1}) \), \( r \geq 0 \) and define

\[
\mathcal{J}(X, \eta)(t) = \prod_{t \geq \tau \geq 0} \exp(X(\tau)d\tau) \int_0^t \prod_{s \geq \tau \geq 0} \exp(X(\tau)d\tau)^{-1}\eta(s)ds = \int_0^t \prod_{t \geq \tau \geq s} \exp(X(\tau)d\tau)\eta(s)ds \tag{4.1.2}
\]

Then

(i) \( \mathcal{J}(X, \eta) \in C(I, \mathcal{H}^{r+1}) \cap C^1(I, \mathcal{H}^r) \) and is the unique solution of

\[
\dot{\mathcal{J}}(X, \eta) = \pi(X)\mathcal{J}(X, \eta) + \eta \tag{4.1.3}
\]

\[
\mathcal{J}(X, \eta)(0) = 0 \tag{4.1.4}
\]

(ii) The map \( \mathcal{J} : C(I, \mathcal{L}) \times C(I, \mathcal{H}^{r+1}) \to C(I, \mathcal{H}^{r+1}) \cap C^1(I, \mathcal{H}^r) \), \( (X, \eta) \to \mathcal{J}(X, \eta) \) is continuous.

**Proof.** (i) Uniqueness follows from that of solutions of the corresponding homogeneous equation, \( \mathcal{J}(X, \eta) \in C(I, \mathcal{H}^{r+1}) \) because \( t \to \prod_{t \geq \tau \geq 0} \exp(X(\tau)d\tau) \in \mathcal{B}(\mathcal{H}^{r+1}) \) is strongly continuous and so is

\[
s \to \prod_{s \geq \tau \geq 0} \exp(X(\tau)d\tau)^{-1} = \prod_{s \geq \tau \geq 0} \exp(-X(s - \tau)d\tau) \tag{4.1.5}
\]
To check the differentiability of \( J = J(X, \eta) \) in \( \mathcal{H}^r \), write
\[
J(t+h) - J(t) = \left( \prod_{t+h \geq \tau \geq t} \text{Exp}(X(\tau)d\tau) - 1 \right) J(t) + \sum_{t+h \geq \tau \geq t} \text{Exp}(X(\tau)d\tau) \int_{\tau}^{t+h} \prod_{s \geq \tau \geq t} \text{Exp}(X(\tau)d\tau)^{-1} \eta(s)ds
\]

\[
= h\pi(X(t)) J(t) + o_r(h)
\]

\[
+ \sum_{t+h \geq \tau \geq t} \text{Exp}(X(\tau)d\tau)(h \prod_{t \geq \tau \geq t} \text{Exp}(X(\tau)d\tau)^{-1} \eta(t) + o_r(h))
\]

\[
= h\pi(X(t)) J(t) + h\eta(t) + o_r(h)
\]

(4.1.6)

where the subscript in \( o_r(h) \) refers to the norm \( \| \cdot \|_k \).

(ii) Let \( X, Y \in C(I, \mathcal{L}) \) and \( \eta, \psi \in C(I, \mathcal{H}^{r+1}) \), then

\[
\| J(X, \eta)(t) - J(Y, \psi)(t) \|_{r+1} \leq \int_0^t \| \sum_{t \geq \tau \geq s} \text{Exp}(X(\tau)d\tau)\eta(s) - \sum_{t \geq \tau \geq s} \text{Exp}(Y(\tau)d\tau)\eta(s) \|_{r+1} ds + \int_0^t \| \sum_{t \geq \tau \geq s} \text{Exp}(X(\tau)d\tau)\psi(s) - \sum_{t \geq \tau \geq s} \text{Exp}(Y(\tau)d\tau)\psi(s) \|_{r+1} ds
\]

\[
\leq |I| e^{2(r+1)|I||X|_{r+1}} \eta - \xi |I|^{r+1} + |I| e^{2(r+1)|I||Y|_{r+1}} \eta - \xi |I|^{r+1}
\]

\[
\leq \limsup_{\psi \to \eta} \| J(X, \eta) - J(Y, \psi) \|_{r+1} |I| e^{2(r+1)|I||X|_{r+1}} \eta - \xi |I|^{r+1} = 0
\]

(4.1.7)

The second term is bounded by \( |I| e^{2(r+1)|I||Y|_{r+1}} \eta - \psi |I|^{r+1} \) and tends to zero uniformly in \( t \) as \( Y \to X, \psi \to \eta \). If \( \xi \in C(I, \mathcal{H}^{r+2}) \) is an auxiliary function the first term is bounded by

\[
\int_0^t \| \sum_{t \geq \tau \geq s} \text{Exp}(X(\tau)d\tau)\eta(s) - \sum_{t \geq \tau \geq s} \text{Exp}(X(\tau)d\tau)\xi(s) \|_{r+1} ds
\]

\[
+ \int_0^t \| \sum_{t \geq \tau \geq s} \text{Exp}(Y(\tau)d\tau)\xi(s) - \sum_{t \geq \tau \geq s} \text{Exp}(Y(\tau)d\tau)\eta(s) \|_{r+1} ds
\]

\[
\leq |I| e^{2(r+1)|I||X|_{r+1}} \eta - \xi |I|^{r+1} + |I| e^{2(r+1)|I||Y|_{r+1}} \eta - \xi |I|^{r+1}
\]

\[
+ |I| e^{2(r+1)|I||Y|_{r+1}} \eta - \xi |I|^{r+1}
\]

whence

\[
\limsup_{\psi \to \eta} \| J(X, \eta) - J(Y, \psi) \|_{r+1} \leq \inf_{\xi \in C(I, \mathcal{H}^{r+2})} 2|I| e^{2(r+1)|I||X|_{r+1}} \eta - \xi |I|^{r+1} = 0
\]

(4.1.8)

by density of the inclusion \( C(I, \mathcal{H}^{r+2}) \subset C(I, \mathcal{H}^{r+1}) \). The continuity of \( J \) in the \( C^1(I, \mathcal{H}^r) \) norm follows easily from the above and the fact that \( J(X, \eta) = \pi(X) J(X, \eta) + \eta \quad \Box \)

**Remark.** Similar results hold if (4.1.4) is replaced by the initial condition \( \zeta(0) = \zeta_0 \in \mathcal{H}^r \) since the solution is then given by

\[
\zeta(t) = \prod_{t \geq \tau \geq 0} \text{Exp}(X(\tau)d\tau)\zeta_0 + \int_0^t \prod_{t \geq \tau \geq s} \text{Exp}(X(\tau)d\tau)\eta(s)ds
\]

(4.1.9)

We won’t however need to work in such generality.
4.2. Differentiability properties of product exponentials.

We investigate below the smoothness of the map $C(I, \mathcal{L}) \times \mathcal{H} \to \mathcal{H}$, $(X, \xi) \to \prod_{1 \geq r \geq 0} \text{Exp}(X(\tau)d\tau)\xi$.

**Proposition 4.2.1.** For any $X \in C(I, \mathcal{L})$ and $\xi \in \mathcal{H}^{r+1}$, $r \in \mathbb{N}$, let $\mathcal{I}(X, \xi) \in C(I, \mathcal{H}^{r+1}) \cap C^1(I, \mathcal{H}^r)$ be the unique solution of

$$\dot{\mathcal{I}}(X, \xi) = \pi(X)\mathcal{I}(X, \xi)$$  \hspace{1cm} (4.2.1)

with $\mathcal{I}(X, \xi)(0) = \xi$. Then, for any $1 \leq m \leq r$ and $\delta_m, \ldots, \delta_1 \in C(I, \mathcal{L})$, the Gâteaux derivatives

$$\mathcal{I}^{(m)}(X, \xi; \delta_m, \ldots, \delta_1) = \lim_{h \to 0} \frac{\mathcal{I}^{(m-1)}(X + h\delta_m, \xi; \delta_{m-1}, \ldots, \delta_1) - \mathcal{I}^{(m-1)}(X, \xi; \delta_{m-1}, \ldots, \delta_1)}{h}$$  \hspace{1cm} (4.2.2)

with $\mathcal{I}^{(0)}(X, \xi) = \mathcal{I}(X, \xi)$, exist in $C(I, \mathcal{H}^{r-m+1}) \cap C^1(I, \mathcal{H}^{r-m})$ and are continuous in $(X, \xi, \delta_m, \ldots, \delta_1)$.

**Proof.** We shall prove inductively that $\mathcal{I}^{(m)}(X, \xi; \delta_m, \ldots, \delta_1)$ exists in $C(I, \mathcal{H}^{r-m+1}) \cap C^1(I, \mathcal{H}^{r-m})$, satisfies the inhomogeneous differential equation

$$\dot{\mathcal{I}}^{(m)}(X, \xi; \delta_m, \ldots, \delta_1) = \pi(X)\mathcal{I}^{(m)}(X, \xi; \delta_m, \ldots, \delta_1) + \sum_{i=1}^{m} \pi(\delta_i)\mathcal{I}^{(m-1)}(X, \xi; \delta_{m-1}, \ldots, \delta_i, \ldots, \delta_1)$$  \hspace{1cm} (4.2.3)

and depends jointly continuously on $X, \delta_m, \ldots, \delta_1, \xi$. For $m = 0$, this follows from theorem 3.2.1. Let now $m \geq 1$ and set, for any $h \neq 0$,

$$\psi_h = \frac{1}{h} \left( \mathcal{I}^{(m-1)}(X + h\delta_m, \xi; \delta_{m-1}, \ldots, \delta_1) - \mathcal{I}^{(m-1)}(X, \xi; \delta_{m-1}, \ldots, \delta_1) \right)$$  \hspace{1cm} (4.2.4)

By induction $\mathcal{I}^{(m)}(X, \xi; \delta_m, \ldots, \delta_1)(0) = \delta_m, 0 \xi$

and $\psi_h$ satisfies $\psi_h = \pi(X)\psi_h + \eta_h, \psi_h(0) = 0$ where

$$\eta_h = \pi(\delta_m)\mathcal{I}^{(m-1)}(X + h\delta_m, \xi; \delta_{m-1}, \ldots, \delta_1) + \sum_{i=1}^{m-1} \pi(\delta_i)\mathcal{I}^{(m-2)}(X + h\delta_m, \xi; \delta_{m-1}, \ldots, \delta_i, \ldots, \delta_1) - \mathcal{I}^{(m-2)}(X, \xi; \delta_{m-1}, \ldots, \delta_i, \ldots, \delta_1)$$  \hspace{1cm} (4.2.5)

Since the operators $\pi(\delta_m), \ldots, \pi(\delta_1)$ define bounded maps $C(I, \mathcal{H}^{r-m+2}) \to C(I, \mathcal{H}^{r-m+1})$, we see by induction that, as $h \to 0$, $\eta_h$ tends in $C(I, \mathcal{H}^{r-m+1})$ to the inhomogeneous term of (4.2.3). It follows from theorem 4.1.1 that $\psi_h$ tends in $C(I, \mathcal{H}^{r-m+1}) \cap C^1(I, \mathcal{H}^{r-m})$ to the unique solution of (4.2.3) and that the limit is continuous in $X, \delta_m, \ldots, \delta_1, \xi$.

**Corollary 4.2.2.** The map $\mathcal{I} : C(I, \mathcal{L}) \times \mathcal{H}^\infty \to \mathcal{H}^\infty$, $(X, \xi) \to \prod_{1 \geq r \geq 0} \text{Exp}(X(\tau)d\tau)\xi$ is smooth.

**Proof.** By proposition 4.2.1 all partial derivatives of $\mathcal{I}$ with respect to the first variable exist and are continuous. Since $\mathcal{I}$ and these are linear in the second variable we deduce that $\mathcal{I}$ has continuous mixed partial derivatives of all orders and hence is smooth.

**Corollary 4.2.3.** The map $C(I, \mathcal{L}) \times \mathcal{H}^{r+1} \to \mathcal{H}$, $r = 0 \ldots \infty$, $(X, \xi) \to \prod_{1 \geq r \geq 0} \text{Exp}(X(\tau)d\tau)\xi$ is of class $C^r$. In particular, any $\xi \in \mathcal{H}^{r+1}$ is of class $C^r$ for the unitary action of $C(I, \mathcal{L})$ on $\mathcal{H}$.

**Remark.** For any $h \in \mathbb{R}$ and $\xi_0 \in \mathcal{H}^{r+1}$, let $\xi_h \in C^0(I, \mathcal{H}^{r+1}) \cap C^1(I, \mathcal{H}^r)$ be the solution of $\dot{\xi}_h = h\pi(X)\xi_h, \xi_h(0) = \xi_0$. Then, in $\mathcal{H}$

$$\xi_h(t) = \xi_0 + h \int_0^t \pi(X(t_1))\xi_h(t_1)dt_1$$  \hspace{1cm} (4.2.6)

$$= \xi_0 + \sum_{k=1}^{r+1} h^k \int_{t \geq t_1 \geq \ldots \geq t_k \geq 0} \pi(X(t_1)) \ldots \pi(X(t_k))\xi_0 dt_1 \ldots dt_k + R(h)$$  \hspace{1cm} (4.2.7)

where

$$R(h) = h^{r+1} \int_{t \geq t_1 \geq \ldots \geq t_r+1 \geq 0} \pi(X(t_1)) \ldots \pi(X(t_{r+1}))\xi_h(t_{r+1}) - \xi_0)dt_1 \ldots dt_{r+1} = o(h^{r+1})$$  \hspace{1cm} (4.2.8)
since, as $h \to 0$, $\xi_h - \xi_0$ tends to zero in $C^0(I, \mathcal{H}^{r+1})$. Thus, if $\xi_0 \in \mathcal{H}^\infty$ and $X \in C(I, \mathcal{L})$ are fixed, the Taylor series of the $C^\infty(\mathbb{R}, \mathcal{H})$ function $h \mapsto \prod_{1 \leq \tau \geq 0} \text{Exp}(hX(\tau) dt)\xi_0$ at $h = 0$ is given by the Dyson expansion

$$\xi_0 + \sum_{k \geq 1} h^k \int \cdots \int \pi(X(t_1)) \cdots \pi(X(t_k))\xi_0 dt_1 \cdots dt_k$$

(4.2.9)

**Remark.** Notice that the results and proofs of sections 2-3 only depend on the fact that $\mathcal{L}$ is a vector space and $\pi(\mathcal{L}) \mapsto \text{End}(V)$ is linear, not on the fact that $\mathcal{L}$ is a Lie algebra or $\pi$ a representation.

5. **Integrating Group Representations**

5.1. **Uniqueness of exponentiation.**

Let $\pi : \mathcal{L} \mapsto \text{End}(\mathcal{D})$ be a projective representation of $\mathcal{L}$ by skew–symmetric operators acting on a dense subspace of a Hilbert space $\mathcal{H}$ and $B$ the corresponding cocycle so that, for any $X, Y \in \mathcal{L}$

$$[\pi(X), \pi(Y)] = \pi([X, Y]) + iB(X, Y)$$

(5.1.1)

**Definition.** An exponentiation of $\pi$ is a strongly continuous homomorphism

$$\rho : \mathcal{G} \mapsto PU(\mathcal{H}) = U(\mathcal{H})/\mathbb{T}$$

(5.1.2)

admitting an invariant subspace $\mathcal{D}_\rho$ with

$$\mathcal{D} \subseteq \mathcal{D}_\rho \subseteq \bigcap_{X \in \mathcal{L}} \mathcal{D}(\pi(X))$$

(5.1.3)

and such that for any $p \in C^\infty(\mathbb{R}, \mathcal{G})$ with $\rho(0) = 1$, there exists, for small $t$, a continuous lift $\tilde{\rho}(p(t))$ of $\rho(p(t))$ such that for any $\xi \in \mathcal{D}_\rho$, $t \to \tilde{\rho}(p(t))\xi$ is of class $C^1$ and satisfies

$$\frac{d}{dt} \tilde{\rho}(p)\xi = \pi(\bar{p}p^{-1})\tilde{\rho}(p)\xi$$

(5.1.4)

If $\pi$ is an ordinary representation, i.e. $B = 0$, we demand in addition that $p$ map into $U(\mathcal{H})$ and that $\tilde{\rho} = \rho$.

**Remark.** The above definition is somewhat stronger than the usual ones but avoids the use of one–parameter groups which may fail to exist or to generate $\mathcal{G}$ if the latter is an arbitrary infinite–dimensional Lie group.

**Proposition 5.1.1.** If $\mathcal{G}$ is connected, there exists at most one exponentiation of $\pi$.

**Proof.** Let $\rho_i$, $i = 1, 2$ be two exponentiations of $\pi$ with corresponding subspaces $\mathcal{D}_{\rho_1}$, $g \in \mathcal{G}$, $\rho$ a smooth path in $\mathcal{G}$ with $\rho(0) = 1$, $\rho(1) = g$ and $\xi \in \mathcal{D} \subseteq \mathcal{D}_{\rho_1} \cap \mathcal{D}_{\rho_2}$. Since (5.1.3) determines the lift uniquely up to multiplication by some $z \in \mathbb{T}$, we may assume that each $\rho_i(p)$ possesses a unitary lift $\tilde{\rho}_i(p)$ over $[0, 1]$ satisfying (5.1.4) and $\tilde{\rho}_i(\rho(0)) = 1$. Set now $F(t) = \tilde{\rho}_1(p(t))\xi - \tilde{\rho}_2(p(t))\xi$. Then, $\tilde{F} = \pi(\bar{p}p^{-1})\tilde{F}$ so that, by skew–symmetry $G(t) = \|F(t)\|^2$ satisfies $G \equiv 0$ and $F(1) = F(0) = 0$ whence $\tilde{\rho}_1(g) = \tilde{\rho}_2(g)$ $\square$

**Remark.** If $\mathcal{G}$ has an exponential map and is generated by its image $\mathcal{L}$, the above definition of exponentiation may be weakened by requiring that (5.1.4) hold only for $p(t) = \exp_G(tX)$. More precisely, since continuous one–parameter groups in $PU(\mathcal{H})$ lift to $U(\mathcal{H})$, uniquely up to multiplication by a character of $\mathbb{R}$, $\rho(\exp_G(tX))$ lifts to a one–parameter group $\tilde{\rho}(t)$ which may be normalised by demanding that

$$\frac{d}{dt} \bigg|_{t=0} \tilde{\rho}(t)\xi = \pi(X)\xi$$

(5.1.5)

It follows that $\pi(X)$ is essentially skew–adjoint [RS, thm. VIII.10] and that $\tilde{\rho}(\exp_G(tX)) = e^{\pi(tX)}$ so that $\rho$ is uniquely determined by $\pi$.

\footnote{Such is the case for Banach–Lie groups and, for example, the connected component of the identity of $\text{Diff}(M)$, $M$ a compact manifold, since it is perfect [EP1, Ep3] and therefore has no proper normal subgroups [EP2].}
5.2. The exponentiation theorem.

**Theorem 5.2.1.** Let $G$ be a connected and simply connected Lie group with Lie algebra $\mathcal{L}$ and $\pi : \mathcal{L} \to \text{End}(V)$ a projective representation of $\mathcal{L}$ by skew-symmetric operators acting on a dense subspace $V$ of a Hilbert space $\mathcal{H}$. Let $B \in H^2(\mathcal{L}, \mathbb{R})$ be the corresponding cocycle so that

$$[\pi(X), \pi(Y)] = \pi([X,Y]) + iB(X,Y)$$

(5.2.1)

Assume the existence of a self–adjoint operator $A \geq 1$ on $\mathcal{H}$ with

$$V = \bigcap_{n \geq 0} D(A^n)$$

(5.2.2)

and such that, for any $n \in \mathbb{N}$, $\xi \in V$ and $X \in \mathcal{L}$

$$\|\pi(X)\xi\|_n \leq |X|_{n+1}\|\xi\|_{n+1}$$

(5.2.3)

$$\|[A,\pi(X)]\xi\|_n \leq |X|_{A,n+1}\|\xi\|_{n+1}$$

(5.2.4)

where $\|\xi\|_n = \|A^n\xi\|$ and the $\| \|$ are continuous semi–norms on $\mathcal{L}$. Then, $\pi$ exponentiates uniquely to $G$. Moreover, $G$ leaves each $D(A^n)$ invariant and acts continuously on these.

The proof of theorem 5.2.1 depends on a number of preliminary results. Let $I = [0,1]$, then

**Lemma 5.2.2.** Let $X_i \in C^\infty(I^2, \mathcal{L})$, $i = 1,2$ satisfy the integrability condition

$$\pi(\partial_1 X_2) - \pi(\partial_2 X_1) = [\pi(X_1), \pi(X_2)]$$

(5.2.5)

Then, for any $\xi \in \mathcal{H}^\infty$, there exists $F \in C^\infty(I^2, \mathcal{H}^\infty)$ such that

$$\partial_1 F = \pi(X_1)F$$

(5.2.6)

$$F(0,0) = \xi$$

(5.2.7)

**Proof.** We proceed as in the local trivialisation of flat vector bundles. For any $(x,y) \in I$, set

$$F(x,y) = \prod_{y \geq u > 0} \text{Exp}(X_2(x,v)dv) \prod_{x \geq u > 0} \text{Exp}(X_1(u,0)du)\xi$$

(5.2.8)

We claim that $F : I^2 \to \mathcal{H}^\infty$ is smooth. To see this, use the change of variable formula (3.2.11) to rewrite $F(x,y)$ as $\mathcal{I}(Y(x,y), \mathcal{I}(Z(x,y), \xi))$ where

$$Y(x,y)(t) = yX_2(x,yt)$$

and

$$Z(x,y)(t) = xX_1(xt,0)$$

(5.2.9)

are smooth maps $I^2 \to C(I, \mathcal{L})$ and use the chain rule in conjunction with corollary 4.2.2. By construction, $F$ satisfies $\partial_2 F = \pi(X_2)F$. Set $G = (\partial_1 - \pi(X_1))F$, we wish to show that $G = 0$. This certainly holds on $I \times \{0\}$ by the very definition of $F$. Moreover, for any $H \in C^2(I^2, \mathcal{H}^\infty)$, (5.2.5) implies that $[\partial_1 - \pi(X_1), \partial_2 - \pi(X_2)]H = 0$. Thus

$$(\partial_2 - \pi(X_2))G = (\partial_2 - \pi(X_2))([\partial_1 - \pi(X_1)]F) = (\partial_1 - \pi(X_1))(\partial_2 - \pi(X_2))F = 0$$

(5.2.10)

so that, by uniqueness $G \equiv 0$.

**Lemma 5.2.3.** Let $X_i \in C^\infty(I^2, \mathcal{L})$, $i = 1,2$ be such that $[X_1, X_2] = \partial_1 X_2 - \partial_2 X_1$ and $X_2|_{\{0,1\} \times I} \equiv 0$. Then,

$$\prod_{1 \geq r \geq 0} \text{Exp}(X_1(\tau,1)dr) = e^{\int_{0}^{1} \int_{0}^{1} B(X_1, X_2) dudv} \prod_{1 \geq r \geq 0} \text{Exp}(X_1(\tau,0)dr)$$

(5.2.11)

**Proof.** Let $\tilde{\mathcal{L}} = \mathcal{L} \oplus c \cdot \mathbb{R}$ be the extension of $\mathcal{L}$ by a central element $c$ with bracket

$$[X \oplus tc, Y \oplus sc] = [X,Y] \oplus B(X,Y)c$$

(5.2.12)

$\pi$ extends to a genuine representation of $\tilde{\mathcal{L}}$ on $\mathcal{H}^\infty$ by letting $c$ act as multiplication by $i$, which moreover still satisfies $[X \oplus tc]_{n+1} = |X|_{n+1} + |t|$ and $[X \oplus tc]_{A,n+1} = |X|_{A,n+1}$. Let now $Y_i \in C^\infty(I^2, \mathcal{L})$ be given by $Y_1 = X_1$ and

$$Y_2(x,y) = X_2(x,y) + c \int_{0}^{x} B(X_1, X_2)(t,y)dt$$

(5.2.13)
Then

$$
\pi(\partial_1 Y_2 - \partial_2 Y_1) = \pi([X_1, X_2] + c B(X_1, X_2)) = [\pi(X_1), \pi(X_2)] = [\pi(Y_1), \pi(Y_2)]
$$  \hspace{1cm} (5.2.14)

By lemma \[5.2.2\], we may find \( F \in C^\infty(I^2, \mathcal{H}^\infty) \) satisfying \( \partial_1 F = \pi(Y_t) F \) and \( F(0,0) = \xi \), an arbitrary vector in \( \mathcal{H}^\infty \). Since \( Y_2(0, \cdot) \equiv 0 \), we get \( F(0, \cdot) = \xi \) and, by uniqueness of solutions of \( \partial_1 F = \pi(X_1) F \), \( F(1, y) = \prod_{1 \leq u \leq 0} \text{Exp}(X_1(u, y) du) \xi \). On the other hand, since \( Y_2(1, y) = c \int_0^1 B(X_1, X_2)(u, y) du \), both \( F(1, y) \) and \( G(y) = e^{i \int_0^1 B(X_1, X_2)(u, y) du} F(1, 0) \) are annihilated by \( (\partial_2 - i \int_0^1 B(X_1, X_2)(u, y) du) \) whence \( F(1, y) = G(y) \). Thus, \( (5.2.11) \) holds since both sides coincide on \( \mathcal{H}^\infty \).

**Definition.** For any \( p \in C^\infty(I, \mathcal{G}) \), let \( U_p = \prod_{1 \geq \tau \geq 0} \text{Exp}(\phi^{-1}(\tau) d\tau) \in U(\mathcal{H}) \).

**Proposition 5.2.4.** The unitaries \( U_p \) have the following properties

(i) Lift property: if \( X \in \mathcal{L} \) defines a one–parameter group in \( \mathcal{G} \) and \( p(t) = \exp (tX) \), then

\[
U_p = e^{i \int_I H(t) dt} \tag{5.2.15}
\]

where \( \int_I H(t) dt = \int_0^1 B(\partial_1 H \cdot H^{-1}, \partial_2 H \cdot H^{-1}) dx_1 dx_2 \).

PROOF. (i) Since \( \phi^{-1} \equiv X \), the claim follows from the definition of product integrals.

(ii) We have \( \frac{d}{dt} \phi(p(t))^{-1} = \phi'(p(t))^{-1} \phi' \) and the claim follows from the change of variable formula \( (2.11) \).

(iii) follows from \( (pg)(pg)^{-1} = pp^{-1} \).

(iv) follows from the semigroup property of product integrals.

(v) follows from the inversion formula \( (8.1.5) \).

(vi) Let \( X_1 = \partial_2 H \cdot H^{-1} \in C^\infty(I^2, \mathcal{L}) \) so that \( X_2(0, \cdot) = X_2(1, \cdot) \equiv 0 \). Since \( \partial_1 H = X_1 H \), we have

\[
\partial_1 \partial_2 H = \partial_1 X_2 H + X_2 \partial_1 H = \partial_1 X_2 H + X_2 X_1 H \tag{5.2.16}
\]

and similarly \( \partial_2 \partial_1 H = \partial_2 X_1 H + X_1 X_2 H \). Substracting and multiplying by \( H^{-1} \) to the right we get

\[
\partial_1 X_2 - \partial_2 X_1 = [X_1, X_2] \tag{5.2.17}
\]

The result now follows from lemma \( 5.2.3 \).
is a lift of $\rho(p(t))$ (resp. equals $\rho(p(t))$ if $B = 0$) which, by theorem 5.2.1 satisfies

$$\frac{d}{dt} \rho(p) \xi = \pi(\rho(p^-) \rho(p)) \xi$$

(5.2.20)

for any $\xi \in D_\rho$. Finally, $\mathcal{G}$ leaves each $D(A^n)$ invariant by $\S 3.1$ and acts continuously on them by corollary 3.2.2 $\square$

Remark. The main conclusion of theorem 5.2.1 depends only on the fact that $\mathcal{L}$ is a Lie algebra and not on the fact that it has an underlying Lie group. More precisely, to any topological Lie algebra $\mathcal{L}$, one can associate an abstract group $T(\mathcal{L})$ called the Thurston group of $\mathcal{L}$ in the following way [Mi2 5.5]. The elements of $T(\mathcal{L})$ are equivalence classes of smooth paths $I \to \mathcal{L}$ for the relation $u_0 \sim u_1$ if there exist $X_i \in C^\infty(T^2, \mathcal{L}), i = 1, 2$ satisfying $X_1(t, 0) = u_0(t), X_1(t, 1) = u_1(t), X_2|_{(0, 1) \times I} = 0$ as well as the integrability condition

$$[X_1, X_2] = \partial_1 X_2 - \partial_2 X_1$$

(5.2.21)

Any connected and simply connected Lie group $\mathcal{G}$ maps homomorphically into the Thurston group of its Lie algebra by associating to $g \in \mathcal{G}$ the class of $pp^{-1}$ where $p$ is any smooth path in $\mathcal{G}$ with $p(0) = 1, p(1) = g$ and this map is an isomorphism if $\mathcal{G}$ is regular in the sense explained in the introduction. Now if $\pi$ is a projective representation of $\mathcal{L}$ satisfies the assumptions of theorem 5.2.1, lemma 5.2.3 shows that $\pi$ yields a (projective) unitary representation of $T(\mathcal{L})$ and therefore one of the Lie group underlying $\mathcal{L}$ if one such exists.

5.3. Smoothness of central extensions of $\mathcal{G}$ arising from exponentiated representations.

A projective unitary representation $\rho : \mathcal{G} \to PU(\mathcal{H})$ lifts to a unitary representation of the continuous central extension $\rho^* U(\mathcal{H})$ of $\mathcal{G}$ obtained by pulling back the canonical central extension

$$1 \to \mathbb{T} \to U(\mathcal{H}) \xrightarrow{\rho} PU(\mathcal{H}) \to 1$$

(5.3.1)

to $\mathcal{G}$. Explicitly,

$$\rho^* U(\mathcal{H}) = \{(g, V) \in \mathcal{G} \times U(\mathcal{H}) | \rho(g) = p(V)\}$$

(5.3.2)

acts on $\mathcal{H}$ by $(g, V)\xi = V\xi$. For classification purposes, it is often useful to know that $\rho^* U(\mathcal{H})$ is a Lie group. This is so if $\mathcal{G}$ is finite-dimensional since any local continuous cocycle of $\rho^* U(\mathcal{H})$ may be regularised within its cohomology class by convolving it with a smooth function on $\mathcal{G} \times \mathcal{G}$ (see e.g. [Mi1, lemma 7.20]). In the absence of a Haar measure this ceases to be obvious but continues to hold for the class of representations considered in this paper.

Proposition 5.3.1. Let $\pi : \mathcal{L} \to End(V)$ be a projective representation of $\mathcal{L}$ satisfying the assumptions of theorem 5.2.1, $B$ its cocycle and $\rho : \mathcal{G} \to PU(\mathcal{H})$ its exponentiation. Then, $\rho^* U(\mathcal{H})$ is a smooth central extension of $\mathcal{G}$. In particular, its isomorphism class is uniquely determined by the Lie algebra cocycle of $\rho^* U(\mathcal{H})$ which is equal to $B$.

Proof. It is sufficient to exhibit a local trivialisation of $\rho^* U(\mathcal{H})$ the corresponding local multiplication and inversion of which are smooth. We begin by trivialising (5.3.1) as in [Br]. Fix $\xi \in \mathcal{H}$ of norm $1$ and consider the open set

$$U_\xi = \{[u] \in PU(\mathcal{H}) | ([u] \xi, \xi] > 0\}$$

(5.3.3)

where $[u]$ is the equivalence class of $u \in U(\mathcal{H})$ in $PU(\mathcal{H})$. Define a function

$$\alpha_\xi : p^{-1}(U_\xi) \to \mathbb{T}, \quad u \mapsto \alpha_\xi(u) = \frac{([u] \xi, \xi)]}{([u] \xi, \xi]}$$

(5.3.4)

and notice that $\alpha_\xi(e^{i\theta} u) = e^{i\theta} \alpha_\xi(u)$ so that the map $\phi : p^{-1}(U_\xi) \to U_\xi \times \mathbb{T}, \phi(u) = ([u], \alpha_\xi(u))$ is a $\mathbb{T}$-equivariant local trivialisation with inverse $\phi^{-1}([u], z) = u\alpha_\xi(u)^{-1}z$. The corresponding local multiplication and inversion on $\mathcal{G} \times \mathbb{T}$, namely

$$x \ast y = \phi((\phi^{-1}x) \cdot (\phi^{-1}y))$$

$$i(x) = \phi((\phi^{-1}x)^*)$$

(5.3.5)

(5.3.6)
read, using \( \alpha_\xi(u^*) = \overline{\alpha_\xi(u)} \),

\[
([u], z) \star ([v], w) = \left( [uv], zw \frac{\alpha_\xi(uv)}{\alpha_\xi(u)\alpha_\xi(v)} \right) \tag{5.3.7}
\]

where the quotient \( \alpha_\xi(uv)\alpha_\xi(u)^{-1}\alpha_\xi(v)^{-1} \) is independent of the choice of the lifts \( u, v \) of \([u], [v] \in PU(H)\). Pulling back by \( \rho \), we obtain a local trivialisation of \( \rho^*U(H) \) with group laws

\[
(g, v) \star (h, w) = \left( gh, zw \frac{\alpha_\xi(\rho(g)\rho(h))}{\alpha_\xi(\rho(g))\alpha_\xi(\rho(h))} \right) \tag{5.3.9}
\]

and the local adjoint action is given by

\[
\exp(\tilde{\rho}(g, \tau) \cdot \rho(g, \tau)^{-1} d\tau) \tag{5.3.11}
\]

We claim that if \( \xi \in H^\infty \), the local multiplication \( \overline{\text{(5.3.9)}} \) is smooth. To see this, let \( e : \mathcal{L} \rightarrow \mathcal{G} \) be a local chart mapping 0 to 1. \( e \) defines a local smooth embedding \( p : \mathcal{G} \rightarrow C^\infty(I, \mathcal{G}) \) mapping \( g \in \mathcal{G} \) to the path \( p(g, t) = e(t\rho^{-1}(g)) \) with endpoints 1, \( g \) and, by the construction of \( \rho \) given in theorem \( 5.2.1 \),

\[
\tilde{\rho} : g \mapsto \prod_{1 \geq \tau \geq 0} \exp(\tilde{\rho}(g, \tau) \cdot \rho(g, \tau)^{-1} d\tau) \tag{5.3.11}
\]

is a local lift of \( \rho \). By corollary \( 1.2.2 \), the map \( \mathcal{G} \times H^\infty \rightarrow H^\infty, (g, \xi) \rightarrow \tilde{\rho}(g)\xi \) is smooth and therefore so is the local multiplication \( \overline{\text{(5.3.9)}} \).

We now compute the local adjoint action and Lie bracket of \( \rho^*U(H) \). It will be more convenient to assume that the derivative of \( e \) at 0 is the identity, which may be achieved by pre–multiplying \( e \) by \((D_0e)^{-1}\). Let \( X \in \mathcal{L} \) and \( x \in \mathbb{R} \), then, identifying the Lie algebra of \( T \) with \( i\mathbb{R} \), we find

\[
\text{Ad}(g, z) \ X \oplus ix = \frac{d}{dt} \bigg|_{t=0} (g, z) \star (e(tX), e^{itx}) \star (g, z)^{-1} \tag{5.3.12}
\]

\[
= \frac{d}{dt} \bigg|_{t=0} \left( ge(tX)g^{-1}, \frac{\alpha_\xi(\tilde{\rho}(g)\tilde{\rho}(e(tX))\tilde{\rho}(g)^*)}{\alpha_\xi(\tilde{\rho}(e(tX)))} e^{itx} \right)
\]

Set \( p(t) = e(tX) \), then

\[
\tilde{\rho}(e(tX)) = \prod_{1 \geq \tau \geq 0} \exp(\tau \tilde{\rho} p^{-1}(\tau) d\tau) = \prod_{1 \geq \tau \geq 0} \exp(\tilde{\rho} p^{-1}(\tau) d\tau) \tag{5.3.13}
\]

by the change of variable formula \( \overline{\text{(3.2.11)}} \) and it follows by theorem \( 3.2.1 \) that for any \( \eta \in H^\infty \),

\[
\frac{d}{dt} \bigg|_{t=0} \tilde{\rho}(e(tX))\eta = \pi(X)\eta \tag{5.3.14}
\]

Thus

\[
\frac{d}{dt} \bigg|_{t=0} \alpha_\xi(\tilde{\rho}(e(tX))) = \frac{\partial}{\partial t} \bigg|_{t=0} \frac{(\tilde{\rho}(e(tX))\xi, \xi)}{\sqrt{\rho(e(tX))\xi, \xi(\tilde{\rho}(e(tX)))\xi}} = (\pi(X)\xi, \xi) \tag{5.3.15}
\]

and similarly,

\[
\frac{d}{dt} \bigg|_{t=0} \alpha_\xi(\tilde{\rho}(g)\tilde{\rho}(e(tX))\tilde{\rho}(g)^*) = (\tilde{\rho}(g)\pi(X)\tilde{\rho}(g)^*\xi, \xi) \tag{5.3.16}
\]

since \( \mathcal{G} \) leaves \( H^\infty \) invariant. The local adjoint action \( \overline{\text{(5.3.12)}} \) is therefore given by

\[
\text{Ad}(g, z) \ X \oplus ix = gXg^{-1} \oplus \left( ix + (\tilde{\rho}(g)\pi(X)\tilde{\rho}(g)^*\xi, \xi) - (\pi(X)\xi, \xi) \right) \tag{5.3.17}
\]

Take now \( g = e(sY), Y \in \mathcal{L} \). By \( \overline{\text{(5.3.14)}} \) and the skew–symmetry of \( \pi(Y) \),

\[
\frac{d}{ds} \bigg|_{s=0} \tilde{\rho}(e(sY))^*\xi = -\pi(Y)\xi \tag{5.3.18}
\]
so that the Lie bracket on \( L \oplus i\mathbb{R} \) is

\[
[Y \oplus iy, X \oplus ix] = \frac{d}{ds} \bigg|_{s=0} \text{Ad}(e(sY), e^{isY}) X \oplus ix
= [Y, X] \oplus ((\pi(Y), \pi(X)]\xi, \xi)
= [Y, X] \oplus \left( iB(Y, X) + (\pi([Y, X])\xi, \xi) \right)
\]

(5.3.19)

Thus, the Lie algebra cocycle of \( \rho^*U(H) \) is \( B(Y, X) - i(\pi([Y, X])\xi, \xi) \) which is cohomologous to \( B \). Finally, the fact that a smooth central extension of \( \mathcal{G} \) is uniquely determined by its Lie algebra cocycle is proved in [PS, page 54]

**Remark.** Proposition \[5.3.1\] may also be proved in the following way. By (5.2.15), \( B \), when regarded as a right–invariant, closed two–form on \( \mathcal{G} \) is integral, i.e. its value on closed two–cycles is an integral multiple of \( 2\pi \). It follows that there exists a smooth central extension \( \mathcal{G} \) of \( \mathcal{G} \) by \( \mathbb{T} \) with Lie algebra cocycle \( B \) which may be described as follows [PS, prop. 4.4.2]. Let \( \mathcal{P}\mathcal{G} \) be the space of piecewise smooth paths \( I \to \mathcal{G} \) with \( p(0) = 1 \). The concatenation of pointed paths defined by

\[
p \vee q(t) = \begin{cases} q(2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ p(2t - 1)q(1) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}
\]

(5.3.20)

induces a monoidal structure on \( \mathcal{P}\mathcal{G} \times \mathbb{T} \) and \( \tilde{\mathcal{G}} \) is the quotient of \( \mathcal{P}\mathcal{G} \times \mathbb{T} \) by the equivalence relation

\[
(p, z) \sim (q, w) \iff p(1) = q(1) \text{ and } e^{i\int_0^1 \beta} = w^z
\]

(5.3.21)

where \( \sigma \) is any two–cycle with boundary \( p \vee \bar{q} \) and \( \bar{q}(t) = q(1 - t)q(1)^{-1} \). The construction of the exponentiation \( \rho \) of \( \pi \) given in theorem \[5.2.1\] then shows that the map

\[
\mathcal{P}\mathcal{G} \times \mathbb{T} \longrightarrow \mathcal{G} \times U(H), \quad (p, z) \longrightarrow (p(1), zU_p)
\]

(5.3.22)

descends to an isomorphism of central extensions \( \tilde{\mathcal{G}} \cong \rho^*U(H) \) and in particular that \( \rho^*U(H) \) is smooth.

**Remark.** By proposition \[5.3.1\] and [Sr, prop. 7.1], the topological type of \( \rho^*U(H) \) as a principal circle bundle over \( \mathcal{G} \) is determined by the image of \( B \) in \( H^2(\mathcal{G}, \mathbb{R}) \). In particular, if \( B = dA \) for some 1–form \( A \) on \( \mathcal{G} \), then, by (5.2.15)

\[
U_p e^{i\int_p A} = U_q e^{i\int_q A}
\]

(5.3.23)

for any two homotopic paths \( p, q \) in \( \mathcal{G} \), so that the map \( g \longrightarrow (g, U_p e^{i\int_p A}) \) where \( p(0) = 1, p(1) = g \) is a section of \( \rho^*U(H) \).

6. Applications

6.1. Positive energy representations of Diff(S\(^1\)) and loop groups.

Let \( \text{Diff}^+(S^1) \) be the group of orientation–preserving diffeomorphisms of \( S^1 \). It is isomorphic to the quotient of the group \( D \) of diffeomorphisms \( \phi \in \mathbb{R} \) satisfying \( \phi(x + 2\pi) = \phi(x) + 2\pi \) by the subgroup of translations by multiples of \( 2\pi \). Since \( D \) is contractible through the map \( (\phi, t) \rightarrow t \text{id} + (1 - t)\phi \), \( \text{Diff}^+(S^1) \) is connected and \( D \) is the universal covering group of \( \text{Diff}^+(S^1) \). The Lie algebra of \( \text{Diff}^+(S^1) \) and \( D \) is \( \text{Vect}(S^1) \), the smooth real vector fields on \( S^1 \), with bracket

\[
[f, \frac{d}{d\theta}, g, \frac{d}{d\theta}] = (f' g - f g') \frac{d}{d\theta}
\]

(6.1.1)

the proof is only given for loop groups but works verbatim for any connected and simply connected Lie group.
The Virasoro algebra \( \text{Vir} \) is by definition the central extension of the Lie algebra \( \text{Vect}^\text{pol}_{\mathbb{C}}(S^1) \) of complex vector fields with finite Fourier series with respect to the cocycle
\[
\omega(f, g) = \frac{1}{12} \int_0^{2\pi} (f'' + f) g d\theta
\]

It is spanned by \( L_n = -i e^{in\theta} \frac{d}{d\theta} \), \( n \in \mathbb{Z} \) and a central element \( \kappa \) in terms of which the bracket reads
\[
[L_m, L_n] = (m-n)L_{m+n} + \delta_{m+n,0} \frac{m^3 - m}{12} \kappa
\]

A highest weight representation of \( \text{Vir} \) is a representation \( V \) such that \( \kappa \) acts as multiplication by a scalar \( c \) and \( V \) is generated over the enveloping algebra of the \( L_n \), \( n < 0 \) by an \( L_0 \)-eigenvalue \( \Omega \) annihilated by the \( L_n \), \( n > 0 \). If \( L_0 \Omega = h \Omega \), it follows from \( \omega \) that \( L_0 \) is diagonal with finite-dimensional eigenspaces and that its spectrum is contained in \( h + \mathbb{N} \). The pair \( (c, h) \) is called the highest weight of \( V \). Let \( \pi \) be the anti-linear anti-automorphism of \( \text{Vir} \) acting as \(-1\) on real vector fields so that \( \pi \mathcal{L} = \mathcal{L} \pi \) and, by \( \omega \), \( \pi = \kappa \). \( V \) is called unitarisable if it possesses an inner product \( \langle \cdot, \cdot \rangle \) such that \( \langle X \xi, \eta \rangle = \langle \xi, X \eta \rangle \) for any \( X \in \text{Vir} \) and \( \xi, \eta \in V \). In that case, \( V \) is irreducible and \( c, h \in \mathbb{R} \). In fact, \( c, h \geq 0 \) since, for any \( n \geq 1 \), \( \omega \) yields
\[
0 \leq \langle L_{-n} \Omega, L_{-n} \Omega \rangle = 2nh + \frac{n^3 - n}{12} \rho
\]
so that \( L_0 \) has non-negative spectrum. The values of \( (c, h) \) for which there exists a unitarisable module with highest weight \( (c, h) \) have been classified by the joint results of Friedan–Qiu–Shenker [FQS1, FQS2] and Goddard–Kent–Olive [GKO]. They are \( \{ (c,h) | c \geq 1, h \geq 0 \} \) together with the discrete series contained in the region \( 0 \leq c < 1 \), \( h \geq 0 \) and parametrised by
\[
c(m) = 1 - \frac{6}{(m+2)(m+3)}
\]
\[
h_{p,q}(m) = \frac{(m+3)p - (m+2)q^2 - 1}{4(m+2)(m+3)}
\]
where \( m \geq 1 \), \( p = 1 \ldots m+1 \) and \( q = 1 \ldots p \). We won’t need however to rely on this classification. The following result was conjectured by Kac and proved in special cases by Segal [Sk] and Neretin [Ner] and in the general case by Goodman and Wallach [GoWa2, thm. 4.2]

**Theorem 6.1.1.** Let \( (\pi, V) \) be a unitarisable highest weight representation of the Virasoro algebra. Then, \( \pi \) exponentiates uniquely to a projective unitary representation of \( \text{Diff}_+(S^1) \) on the Hilbert space completion \( \mathcal{H} \) of \( V \).

**Proof.** Let \( (c, h) \) be the highest weight of \( V \) and \( A = 1 + \pi(L_0) \) acting on \( \mathcal{H} \). Using simple \( \mathfrak{sl}_2(\mathbb{C}) \) arguments relying on the positivity of the spectrum of \( \pi(L_0) \), Goodman and Wallach showed [GoWa2, prop. 2.1] that for any \( t \in \mathbb{R}, \xi \in V \) and \( X = \sum_n a_n e^{in\theta} \in \text{Vect}^\text{pol}_{\mathbb{C}}(S^1) \)
\[
||\pi(X)\xi||_t \leq 2^\frac{t}{2} ||X||_{|t|} \xi ||_t + M ||X||_{|t|+\frac{1}{2}} ||\xi||_t
\]
where \( M = (c/12)^{\frac{t}{2}}, ||\xi||_t = ||A^t \xi|| \) and \( ||X||_s = \sum_n (1 + |n|)^s |a_n| \). Thus, \( \pi \) extends to a projective unitary representation of \( \text{Vect}(S^1) \) on the space of smooth vectors of \( A \) satisfying (5.2.3)–(5.2.4) where
\[
||X||_{n+1} = 2^\frac{n}{2} ||X||_{n+1} + M (||X||_{n+1} + ||X||_{n+\frac{3}{2}})
\]
\[
||X||_{A,n+1} = ||L_0, X||_{n+1}
\]
By theorem (5.2.4), \( \pi \) exponentiates to a projective unitary representation \( \rho \) of the universal cover \( \mathcal{D} \) of \( \text{Diff}_+(S^1) \). We claim that the kernel of the covering map, i.e. translations \( T_{2\pi n} = \exp_{\mathcal{D}}(2\pi inL_0) \) \( n \) by multiples of \( 2\pi \), acts by scalars. Indeed, since the spectrum of \( \pi(L_0) \) is contained in \( h + \mathbb{N} \), we have, by the definition of \( \rho \) and (i) of proposition (5.2.4)
\[
\rho(T_{2\pi n}) = e^{2\pi ini(L_0)} = e^{2\pi inh}
\]
Let now \( G \) be a compact, connected and simply connected simple Lie group and \( LG = C^\infty(S^1, G) \) its loop group. Since \( LG = \Omega G \rtimes G \) where \( \Omega G \) is the space of based loops and \( G \) that of constant ones, we get \( \pi_0(LG) = \pi_1(G) = 0 \) and \( \pi_1(LG) = \pi_2(LG) \oplus \pi_1(G) = 0 \) so that \( LG \) is connected and simply connected. The Lie algebra of \( LG \) is \( Lg = C^\infty(S^1, g) \) where \( g \) is the Lie algebra of \( G \) and has complexification \( Lg_c \). \( Lg \) has a distinguished cocycle which generates \( H^2(Lg, \mathbb{R}) \) [PS, prop. 4.2.4], namely

\[
B(X, Y) = \int_0^{2\pi} \langle X, Y' \rangle \frac{d\theta}{2\pi}
\]

where \( \langle \cdot, \cdot \rangle \) is the basic inner product, i.e. the multiple of the Killing form for which the highest root \( \theta \) has squared length 2. \( Lg \) has a dense subalgebra \( L^{pol}g \) consisting of all \( g \)-valued trigonometric polynomials and the central extension of \( L^{pol}g_c \) corresponding to \( B \) is usually denoted by \( \hat{g}_c \). It is spanned by elements \( x(n) = x \otimes e^{i\theta} \), \( x \in g_c \), \( n \in \mathbb{Z} \) and a central element \( \kappa \) with bracket

\[
[x(m), y(n)] = [x, y](m + n) + m\delta_{m+n, 0}\langle x, y \rangle \kappa
\]

Since \( B \) is invariant under the action of \( \text{Diff}_+(S^1) \) on \( Lg \) by reparametrisation, one may form the semi-direct product \( \hat{g}_c = \times \text{Vect}^{pol}_c(S^1) \). By definition, the affine Kac–Moody algebra \( \hat{g}_c \) is the subalgebra \( \hat{g}_c \times \mathbb{C} \cdot L_0 \). Let \( T \) be a maximal torus in \( G \) with Lie algebra \( t \). A highest weight representation of \( \hat{g}_c \) is a representation where the central element \( \kappa \) acts by a scalar and which is generated over the enveloping algebra of the \( x(n) \), \( n < 0 \) or \( n = 0 \) and \( x \) a root vector corresponding to a negative root, by a vector \( \Omega \) diagonalising the action of \( t_c \times \mathbb{C} \cdot L_0 \) and annihilated by the \( x(n), n > 0 \) or \( n = 0 \) and \( x \) a root vector corresponding to a positive root. Thus, for any \( t \in t_c \)

\[
\kappa \Omega = \ell \Omega
\]

\[
t\Omega = \lambda(t) \Omega
\]

\[
L_0 \Omega = h \Omega
\]

for some \( \ell, h \in \mathbb{C} \) and \( \lambda \in t^*_c \). In particular, since \([L_0, x_n] = -nx(n), L_0 \) is diagonal with finite-dimensional eigenspaces and spectrum contained in \( h + \mathbb{N} \).

A fundamental feature of highest weight representations is that the action of \( \hat{g}_c \) extends to one of \( \hat{g}_c \times \text{Vir} \), provided \( \ell + h^\vee \neq 0 \) where \( h^\vee \) is the dual Coxeter number of \( g_c \). This is obtained via the Segal–Sugawara formulæ by letting \( L_n, n \neq 0 \) act as

\[
\frac{1}{2(\ell + h^\vee)} \sum_{m \in \mathbb{Z}} x_i(-m)x^i(m + n)
\]

\[
L_0 \text{ as } \kappa
\]

\[
\frac{1}{2(\ell + h^\vee)} \left( \sum_i x_i(0)x_i(0) + 2 \sum_{n > 0} x_i(-n)x^i(n) \right)
\]

and \( \kappa \) as multiplication by \( \dim(G)\ell/(\ell + h^\vee) \) where \( x_i, x^i \) are dual basis of \( g_c \) for the basic inner product [PS, §9.4], [KR, thm. 10.1]. Let \( \tau \) be the anti-linear anti-automorphism on \( \hat{g}_c \) acting as -1 on \( L^{pol}g \), \( \kappa = \kappa \) and \( L_0 = L_0 \) and define unitarisable highest weight representations \( V \) of \( \hat{g}_c \) accordingly. If \( (\ell, \lambda) \) is the highest weight of \( V \) this is equivalent to requiring that \( V \) be irreducible and that \( \ell \in \mathbb{N} \), \( h \in \mathbb{R} \), \( \lambda \) is an integral dominant weight of \( G \) and \( (\lambda, \theta) \leq \ell \) [K, thm. 11.7]. In that case, the action of \( \text{Vir} \) given by \([6.1.10]–[6.1.11] \) is also unitarisable and \( V \) splits into an orthogonal direct sum of highest weight representations \( V_h \) of \( \text{Vir} \) of highest weights \( (\ell \dim(G)/(\ell + h^\vee), h_1) \). The following theorem was first proved in special cases by Segal [Se] and in the general case by Goodman and Wallach [GoWal] thm 6.7.

\[\text{It is easy to see that [6.1.17] differs from the original action of } L_0 \text{ by an additive constant equal to the difference of their lowest eigenvalues, namely } h - C_\lambda/2(\ell + h^\vee) \text{ where } C_\lambda \text{ is the Casimir of the irreducible } G \text{-module with highest weight } \lambda.\]
Theorem 6.1.2. Let \((\pi, V)\) be an integrable highest weight representation of \(\mathfrak{g}_c\). Then, \(\pi\) exponentiates uniquely to a projective unitary representation of \(L_G\) on the Hilbert space completion \(\mathcal{H}\) of \(V\) extending to \(L_G \rtimes \text{Diff}_+(S^1)\).

Proof. Let \((\ell, \lambda)\) be the highest weight of \(V\) and extend \(\pi\) to a representation of \(\mathfrak{g}_c \rtimes \text{Vir}\) by \((6.1.16)-(6.1.17)\). Let \(A = 1 + \pi(L_0)\) acting on \(\mathcal{H}\). As noted by Goodman and Wallach, the Segal–Sugawara formula \((6.1.17)\) for \(L_0\) readily implies that for any \(\xi \in V\), \(X = \sum_n a_n e^{int} \in \mathcal{L}^{\text{pol}} G_c\), \(f_{\mathfrak{m}} = \sum_n b_n e^{int} \mathfrak{m} \in \text{Vect}^c(S^1)\) and \(t \in \mathbb{R}\)

\[
\|\pi(X)\xi\| \leq (\ell + 1)\|X\|_{\ell + \frac{1}{2}} \|\xi\|_{\ell + \frac{1}{2}}
\]

(6.1.18)

\[
\|\pi(f_{\mathfrak{m}})\xi\| \leq \dim(G)\|f_{\mathfrak{m}}\|_{\ell + \frac{1}{2}} \|\xi\|_{\ell + 1}
\]

(6.1.19)

where \(\|\xi\| = \|A^s \xi\|\), \(\|X\| = \sum_n (1 + |n|)^s \|a_n\|\) and \(\|f_{\mathfrak{m}}\|_{\ell} = \sum_n (1 + |n|)^s \|b_n\|\) \((\text{GoWa})\) lemmas \(3.2.3.3\). By continuity, \(\pi\) extends to a projective unitary representation of \(L_G \rtimes \text{Vect}(S^1)\) on the space of smooth vectors of \(A\) satisfying \((6.2.3)-(6.2.4)\) where

\[
|X + f_{\mathfrak{m}}|_{n+1} = (\ell + 1)\|X\|_{n+\frac{1}{2}} + \dim(G)\|f_{\mathfrak{m}}\|_{n+\frac{1}{2}}
\]

(6.2.20)

\[
|X + f_{\mathfrak{m}}|_{\ell} = \|L_0, X + f_{\mathfrak{m}}\|_{n+1} = |X' + f'_{\mathfrak{m}}|_{n+1}
\]

(6.2.21)

By theorem \(5.2.1\), \(\pi\) exponentiates uniquely to a projective unitary representation \(\rho\) of \(L_G \rtimes \mathcal{D}\). Since the spectrum of \(L_0\) is contained in \(C/2(\ell + h^\vee) + \mathbb{N}\),

\[
\rho(\exp(2\pi inL_0)) = e^{2\pi i n(\pi L_0)} = e^{2\pi i \mathcal{C}_\lambda / 2(\ell + h^\vee)}
\]

(6.2.22)

and \(\rho\) factor through a representation of \(L_G \rtimes \text{Diff}_+(S^1)\) \(\square\)

6.2. Unitary representations of finite–dimensional Lie algebras.

We now derive Nelson’s exponentiation criterion from theorem \(5.2.1\). We shall need the following result which is stated as part of lemma 5.2 in \([\text{Ne}]\) but not fully proved there. I am grateful to Professor Z. Magyar for showing me how to complete its proof.

Lemma 6.2.1. Let \(\pi: \mathfrak{g} \rightarrow \text{End}(V)\) be a representation of a finite–dimensional Lie algebra by skew–symmetric operators on a dense subspace of a Hilbert space \(\mathcal{H}\), \(X_i\) a basis of \(\mathfrak{g}\) and \(\Delta = -\sum_i (X_i)^2\) the corresponding Laplacian. If \(\Delta\) is essentially self–adjoint, the closures of the operators \(\pi(X_i)\) leave \(\mathcal{H}^\infty = \bigcap_{n \geq 0} D(\nabla^n)\) invariant and

\[
\Delta|_{\mathcal{H}^\infty} = \sum_i \pi(X_i)^2
\]

(6.2.1)

Proof. By skew–symmetry of the \(\pi(X_i)\) and self–adjointness of \(\nabla\)

\[
\sum_i \pi(X_i)^2 \subseteq \sum_i \pi(X_i)^* = \nabla^* = \nabla
\]

(6.2.2)

Thus, if

\[
\mathcal{K}^\infty = \bigcap_{n \geq 0} D(\pi(X_{i_1})\cdots\pi(X_{i_n}))
\]

(6.2.3)

then \(\mathcal{K}^\infty\) is invariant under the \(\pi(X_i)\) and, by \((6.2.2)\) under \(\nabla\) so that \(\mathcal{K}^\infty \subseteq \mathcal{H}^\infty\). The converse inclusion is proved in \([\text{Ne}]\) as formula \((5.4)\) of lemma 5.2. It follows that \(\mathcal{H}^\infty = \mathcal{K}^\infty\) is invariant under the \(\pi(X_i)\) \(\square\)

Theorem 6.2.2 (Nelson). Let \(\pi: \mathfrak{g} \rightarrow \text{End}(\mathcal{D})\) be a representation of a finite–dimensional Lie algebra by skew–symmetric operators on a dense subspace of a Hilbert space \(\mathcal{H}\), \(X_i\) a basis of \(\mathfrak{g}\) and \(\Delta = \sum_i (X_i)^2\) the corresponding Laplacian. If \(\Delta\) is essentially self–adjoint, then \(\pi\) exponentiates uniquely to a unitary representation of the underlying connected and simply connected lie group \(G\).
PROOF. Let $\mathcal{H}^\infty = \bigcap_{n \geq 0} \mathcal{D}(\Delta^n)$ be as in lemma 6.2.1. For any $X = \sum c_i X_i \in \mathfrak{g}$, set $\pi(X) = \sum c_i \pi(X_i)\big|_{\mathcal{H}^\infty}$ so that $\pi$ is a linear action by skew-symmetric operators and extends $\pi$. $\pi$ is a representation since if $Z = [X,Y] \in \mathfrak{g}$, then for any $\xi \in \mathcal{H}^\infty$ and $\eta \in \mathcal{D}$,

\begin{align*}
(\pi(X), \pi(Y))|_\xi, \eta) &= -\langle [\pi(X), \pi(Y)] \eta, \xi \rangle \\
&= -\langle \xi, [\pi(X), \pi(Y)] \eta \rangle \\
&= (\pi(Z))|_\xi, \eta \rangle
\end{align*}

(6.2.4)

Set now $A = 1 - \Delta$. By (6.2.1), the Laplacian of $\pi$ is the restriction of $\Delta$ to $\mathcal{H}^\infty$ so that the estimates (6.2.2)–(6.2.4) follow from lemma 6.3 of [Ne1] and $\pi$ exponentiates to $G$ by theorem 5.2.1 □

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