Adaptive Exact Learning in a Mixed-Up World: Dealing with Periodicity, Errors, and Jumbled-Index Queries in String Reconstruction

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Abstract. We study the query complexity of exactly reconstructing a string from adaptive queries, such as substring, subsequence, and jumbled-index queries. Such problems have applications, e.g., in computational biology. We provide a number of new and improved bounds for exact string reconstruction for settings where either the string or the queries are “mixed-up”.

Keywords: Exact Learning · String Reconstruction · Jumbled-Index Queries · Periodicity · DNA Sequencing · Stringology

1 Introduction

Exact learning involves asking a series of queries so as to learn a configuration or concept uniquely and without errors, e.g., see [12]. For example, imagine a game where a player, Alice, is trying to exactly learn a secret string, S, such as S = "rumpelstiltskin", which is known only to a magic fairy. Alice may ask the fairy questions about S, but only if they are in a form allowed by the fairy, such as “Is X a substring of S?”. Any allowable question that Alice asks must be answered truthfully by the fairy. Alice’s goal is to learn S by asking the fewest number of allowable questions. Her strategy is adaptive if her questions can depend on the answers to previous queries. This exact-learning string-reconstruction problem might at first seem like a contrived game, but it actually has a number of applications. For instance, in interactive DNA sequencing, the fairy’s string is an unknown DNA sequence, S, and allowable queries are “Is X a substring of S?” Each such question can be answered by a hybridization experiment that exposes copies of S to a mixture containing specific primers to see which ones bind to S, e.g., see [74]. Thus, we are interested in the exact-learning complexity of adaptively learning an unknown string via queries of various given types, that is, for exactly reconstructing a string from queries. Formally, we are interested in minimizing a query-complexity measure, Q(n), which, in our case, is the number of queries of certain types needed in order to exactly learn a string, S. This query-complexity concept comes from machine-learning and complexity theory, e.g., see [3,12,20,26,33,77,85].
1.1 Related Work

Motivated by DNA sequencing, Skiena and Sundaram [74] were the first to study exact string reconstruction from adaptive queries. For substring queries, of the form “Is $X$ a substring of $S$?”, they give a bound for $Q(n)$ of $(\alpha - 1)n + 2\log n + O(\alpha)$, where $\alpha$ is the alphabet size. For subsequence queries, of the form “Is $X$ a subsequence of $S$?”, they prove a bound for $Q(n)$ of $\Theta(n\log \alpha + \alpha \log n)$.

Recently, Iwama et al. [44] study the problem for binary alphabets, which removes the additive logarithmic term in this case. These papers do not consider “mixed-up” strings, however, such as strings that are periodic or periodic with errors. The abundance of repetitions and periodic runs in genomic sequences is well known and has been exploited in the last decades for biologic and medical information (see e.g. [18, 19, 31, 34, 36, 54, 66, 67, 75, 83]). It is somewhat surprising that this phenomenon has not been used to achieve more efficient algorithms. Margaritis and Skiena [61] study a parallel version of exact string reconstruction from queries, which are hybrids of adaptive and non-adaptive strategies, showing, e.g., that a length-$n$ string can be reconstructed in $O(\log^2 n)$ rounds using $n$ substring queries per round. Tsur [78] gives a polynomial approximation algorithm for the 1-round case. As in [74], these papers do not consider bounds for $Q(n)$ based on properties of the string such as its periodicity. Cleve et al. [28] study string reconstruction in a quantum-computing model, showing, for example, that a sublinear number of queries are sufficient for a binary alphabet. This result does not seem to carry over to a classical computing model, however, which is the subject of our paper.

Another type of query we consider is the jumbled (or histogram)-index query, e.g., see [4, 5, 8, 9, 53, 63]. Jumbled indexing has many applications. It can be used as a tool for de novo peptide identification (as in e.g. [45, 51, 52]), and has been used as a filter for searching an image database [27, 32, 76, 82, 87]. In this query, which has received much study of late, but has not been studied before for adaptive string reconstruction, one is given a Parikh vector, i.e., a vector of frequency counts for each character in an alphabet, and asked if there is a substring of the reference string, $S$, having these frequency counts and, if so, where it occurs in $S$. Such reconstruction may aid in narrowing down peptide identification, or focusing on image retrieval.

Another model for string reconstruction, tangential to ours and studied extensively, is the one defined by a non-adaptive oracle, e.g., see [1, 2, 14, 17, 21, 22, 24, 30, 35, 37, 38, 40–43, 47–49, 55, 57, 59, 60, 64, 65, 68–73, 79, 80, 86], where one is given a set of answers to queries in advance, and we aim to understand sufficient and necessary conditions on the answers that enable the exact reconstruction of the string. This model differs from the adaptive one considered in this paper in that it focuses on the study of combinatorial properties of strings, rather than on minimizing the number of queries. We review existing literature for non-adaptive string reconstruction in more detail in an appendix.
1.2 Our Results

We provide new and improved results for exactly reconstructing strings from adaptive substring, subsequence, and jumbled-index queries. For example, we believe we are the first to characterize query complexities for exactly reconstructing periodic strings from adaptive queries, including the following results for reconstructing a length-$n$ periodic (i.e., “mixed-up”) string, $S = p^k p'$, of smallest period $p$, where $|p'| < |p|$, with alphabet size $\alpha$:

- It requires at least $|p| \lg \alpha$ substring or subsequence queries.
- It can be done with $\alpha|p| + \lfloor \lg |p| \rfloor$ substring queries, if $n$ is known.
- It can be done with $O(\alpha|p| + \lg n)$ substring queries, if $n$ is unknown.
- It can be done with $\alpha \lceil \lg n \rceil + 2|p|\lceil \lg \alpha \rceil$ subsequence queries, for known $n$.
- It can be done with $2\alpha \lceil \lg n \rceil + 2|p|\lceil \lg \alpha \rceil$ subsequence queries, if $n$ is unknown.

Perhaps our most technical result is that we show that we can reconstruct a length-$n$ string, $S$, within Hamming distance $d$ of a periodic string $S' = p^k p'$, of smallest period $p$, using $O(\min(\alpha n, d\alpha |p| + d |p| \lg \frac{n \sqrt{d + 1}}{d + \alpha}))$ substring queries, if $n$ is unknown. We also show that we can exactly reconstruct a general length-$n$ string, $S$, using $2\alpha \lceil \lg n \rceil + n \lceil \lg \alpha \rceil$ subsequence queries, if $n$ is unknown. Such queries are another “mixed-up” setting, since there can be multiple subsequence matches for a given string. Our bound improves the previous best, decades-old result, by Skiena and Sundaram [74], who prove a query complexity of $2\alpha \lg n + 1.59n \lg \alpha + 5\alpha$ for this case. We believe we are the first to study string reconstruction using jumbled-index queries, which are yet another “mixed-up” setting, since they simply count the frequency of each character occurring in a substring. We prove the following results:

- We can reconstruct a length-$n$ string with $O(\alpha n)$ yes/no extended jumbled-index queries, which include a count for an end-of-string character, $\$.\n- For jumbled-index queries that return an index of a matching string, string reconstruction is not possible if this index is chosen adversarially, but is possible using $O(\alpha + n \lg n)$ queries if it is chosen uniformly at random.

1.3 Preliminaries

We consider strings over the alphabet $\Sigma = \{a_1, a_2, \ldots, a_\alpha\}$ of $\alpha$ letters. The size of a string $X$ is denoted by $|X|$. We use $X[i]$ to denote the $i^{\text{th}}$ letter of $X$ and $X[i..j]$ to refer to the substring of $X$ starting at its $i^{\text{th}}$ and ending at its $j^{\text{th}}$ letter (e.g., $X = X[1..|X|]$). We may ignore $i$ when expressing a prefix $X[i..j]$ of $X$. Similarly, $X[i..]$ is a suffix of $X$. Occasionally, we will express concatenation of strings $X$ and $Y$ by $X \cdot Y$ (instead of $XY$) to emphasize some property of the string. A string $X$ concatenated with itself $k$ (resp. infinitely many) times can be expressed as $X^k$ (resp. $X^\infty$). A string, $S$, has period $p$ if $S = p^k p'$, such that $k \geq 1$ is an integer and $p'$ is a prefix of $p$. We assume throughout the rest of the paper that $k > 1$, except when applying the Periodicity Lemma, below, which allows for a single occurrence of the period. Due to space constraints, we defer proofs of Lemmas/Theorems marked with ⊛ to an appendix.
Lemma 1 (Periodicity Lemma [39]). If \( p, q \) are periods of a string \( X \) of length \( |X| \geq |p| + |q| - \gcd(|p|, |q|) \), then \( X \) also has a period of size \( \gcd(|p|, |q|) \).

2 Substring Queries

In this section, we study query complexities for a string, \( S \), subject to yes/no substring queries, \( \text{IsSubstr} \), i.e. queries of “Is \( X \) a substring of \( S \)?”. We focus on the cases where \( S \) corresponds to an originally periodic string, that may have lost its periodicity property due to error corruption. The nature of the errors is context-dependent. For example, corruption may be caused by transmission errors or measurement errors. There are multiple ways to model errors in strings (see [6, 11, 23, 46, 56, 58, 81]). In this paper, we consider Hamming distance. We say that \( S \) is a \( d \)-corrupted periodic string if there exists a periodic string \( S' \) of period \( p \), such that \( |S| = |S'| \) and \( \delta(S', S) \leq d \), where \( \delta \) is the Hamming distance. We refer to \( p \) as an approximate period of \( S \). Notice that, depending on \( d \), there might exist multiple possible strings \( S' \) that originate \( S \).

Our main result in this section is the following.

Theorem 1. We can reconstruct a length-\( n \) \( d \)-corrupted periodic string \( S \) using

\[
O\left( \min\left( an, d\alpha|p| + d|p| \lg \frac{n}{d + 1} \right) \right)
\]

queries, for known \( d \), unknown \( |p| \), regardless of whether we know \( n \), where \( p \) is a smallest approximate period of \( S \).

The algorithm of Theorem 1 is a more elaborate version of a reconstruction algorithm for the special case of \( d = 0 \), i.e. when no errors occurred and \( S = S' \), and when \( n \) is not known in advance.

Theorem 2. We can reconstruct a length-\( n \) periodic string, \( S = p^k p' \), of smallest period \( p \), using \( O(\alpha|p| + \lg n) \) substring queries, assuming both \( n \) and \( |p| \) are unknown in advance.

The algorithm of Theorem 2, in turn, builds from a simple reconstruction algorithm that handles the case where \( n \) is known in advance and \( d = 0 \).

For clarity, we will present our results in increasing order of complexity, from the least general result of \( d = 0 \) and known \( n \), to the most general result of arbitrary \( d \) and unknown \( n \).

2.1 Uncorrupted Periodic Strings of Known Size

We first give a simple algorithm to reconstruct a periodic string \( S = p^k p' \) of smallest period \( p \) and known size with query complexity \( O(\alpha|p|) \), and then show how to improve this algorithm to have query complexity \( \alpha|p| \) plus lower-order terms. Our algorithms use a primitive developed by Skiena and Sundaram [74], which we call “append” (resp., prepend) a letter.” In the append (resp.,
prepend) primitive, we start with a known substring \( q \) of \( S \), and we ask queries \( \text{IsSubstr}(qa_i) \) (resp., \( \text{IsSubstr}(a_qi) \)), for each \( a_i \in \Sigma \). Note that if we know that one of the \( qa_i \) (resp., \( a_qi \)) strings must be a substring, we can save one query, so that appending or prepending a letter uses at most \( \alpha - 1 \) queries in this case.

In our simple algorithm, we iteratively grow a candidate period, \( q \), using the append primitive until \( q^{g(q)} - 1 \) is a substring, where \( g(x) = \lfloor n/|x| \rfloor \). Notice that \( q \) may be an “unlucky” cyclic rotation of \( p \), which only repeats \( g(p) - 1 \) times, and we need to account for this possibility. Thus, once we get a substring corresponding to \( q^{g(q)} - 1 \), we then append/prepend letters until we recover all of \( S \). For reference, see Algorithm 1 in Appendix B, where the number of queries is shown in parentheses for steps involving queries.

**Theorem 3.** We can reconstruct a length-\( n \) periodic string \( S = p^k p' \), of smallest period \( p \), using \( O(\alpha |p|) \) substring queries, assuming \( n \) is known in advance and \( |p| \) is unknown.

With a little more effort, we can improve the constant factor in the query complexity, by showing that, for \( k = \lfloor n/|p| \rfloor > 3 \), the following implication holds: if \( q^{g(q)} - 1 \) is a substring, then \( q \) must be a cyclic rotation of \( p \).

**Theorem 4.** We can reconstruct a length-\( n \) periodic string \( S = p^k p' \), of smallest period \( p \), using at most \( \alpha |p| + \lceil \lg |p| \rceil \) substring queries, assuming that: \( n \) is known in advance, \( k > 3 \) and \( |p| \) is unknown.

Notice that any reconstruction algorithm requires at least \( |p| \lg \alpha \) queries.

**Theorem 5.** Reconstructing a length-\( n \) string, \( S = p^k p' \), of smallest period \( p \), requires at least \( |p| \lg \alpha \) \( \text{IsSubstr} \) queries, even if \( n \) and \( |p| \) are known.

### 2.2 Uncorrupted Periodic Strings of Unknown Size

As in Section 2.1, we iteratively grow a candidate period \( q \) and attempt to recover \( S \) by concatenating \( q \) with itself in the appropriate way. The difficulty when \( n \) is unknown is that we can no longer confidently predict \( g(q) \). Thus, we can no longer issue a single query to test if \( q \) is the right period. An immediate solution is to use a doubling search\(^3\). Unfortunately, this introduces a multiplicative \( O(\lg n) \) term into the query complexity. To avoid it, we show how we can take advantage of the Periodicity Lemma (1) to amortize the extra work needed to recover \( S \).

Let us describe the algorithm (see Algorithm 3 in Appendix B for reference). We start with an empty candidate period \( q \). At each iteration, we add a letter to \( q \), using the append primitive and determine the frequency \( f \) of \( q \), i.e. the maximum integer \( f \) such that \( q^f \) is a substring of \( S \). If \( f = 1 \), we advance to the next iteration and repeat this process. If, on the other hand, \( f > 1 \), we use \( q \) to determine the largest substring \( T \) that has a period of size \( |q| \). This can be done efficiently, using doubling searches, by determining the largest suffix \( l \) of \( q \) and

\(^3\) A doubling search to determine a number, \( n \), involves doubling a query value, \( m \), until it is greater than \( n \), followed by a binary search to determine \( n \).
the largest prefix \( r \) of \( q \), such that \( \text{IsSubstr}(l \cdot q^f \cdot r) \). Once \( T \) is determined, we check whether it corresponds to \( S \) by checking if there is any letter preceding and succeeding \( T \) (see \( \text{IsValid} \) subroutine). If \( T \) corresponds to \( S \), we output it. Otherwise, we update \( q \) to be any largest substring of \( T \) whose size is assuredly less than \( |p| \): using Periodicity Lemma (1), we argue in Lemma 2 below that, if \( q \) is not a cyclic rotation of \( p \), then \( p \) must be as large as almost the entire substring \( T \); more specifically, \(|p| > |T| - |q| + 1 \). Thus, we update \( q \) to be a length-\((|T| - |q| + 1)\) prefix of \( T \). We use this fact to get a faster convergence to a cyclic rotation of \( p \), while making sure that we do not overshoot \(|p|\). Indeed, this observation will enable us to incur a \( O(\log n) \) additive factor, instead of a multiplicative one. After updating \( q \), we advance to the next iteration, where a new letter is appended to \( q \), and repeat this process until \( T = S \).

**Lemma 2.** Let \( T \) be the largest proper substring of \( S = p^k p' \), of smallest period \( p \), such that: \(|q| \) is the length of the smallest period of \( T \). Then, \(|p| > |T| - |q| + 1 \).

**Proof.** Let us assume, by contradiction, that \(|p| \leq |T| - |q| + 1 \). Then, \(|T| \geq |q| + |p| - \gcd(|q|, |p|) \). In addition, if \( p \) is a period of \( S \), then \( T \) must have a period of size \(|p| \). So, by the Periodicity Lemma, \( T \) has a period of size \( \gcd(|q|, |p|) \). Moreover, since \( T \) is the largest proper substring of \( S \), \(|p| \) is not a multiple of \(|q| \). Therefore, \( T \) must have a period smaller than \( q \), a contradiction. \( \Box \)

Let \( q_1, q_2, \ldots, q_m \) be the sequence of \( m \) candidate periods of increasing length, each of which is the result of the append/prepend primitive at the beginning of every iteration (line 3 of Algorithm 3), e.g. \(|q_1| = 1 \). Correctness of Algorithm 3 follows from the following two lemmas.

**Lemma 3.** Let \( S = p^k p' \), of smallest period \( p \), if there exists an iteration \( i \in \{1, 2, \ldots, m\} \), such that \( q_i \) is a cyclic rotation of \( p \).

**Lemma 4.** There exists an iteration \( i \in \{1, 2, \ldots, m\} \), such that \( q_i \) is a cyclic rotation of \( p \).

The following lemma shows that we can charge the logarithmic factors, incurred in each iteration \( j \), to the work that would have been required to find the letters introduced in \( q_{j+1} \). This establishes the amortization in query complexity.

**Lemma 5.** The number of queries performed in iteration \( j \) of Algorithm 3 is at most \( \alpha(|q_{j+1}| - |q_j|) + O(\alpha) \), for \( j < m \), or \( O(\alpha + \log n) \), for \( j = m \).

Theorem 2 follows from Lemmas 3 to 5. See the proof in Appendix B for details.

### 2.3 Corrupted Periodic Strings

Let us assume throughout the remainder of this section that \( S \) is a \( d \)-corrupted periodic string of approximate period \( p \). Again, the main idea of the algorithm
described in this section consists of: (1) determining a cyclic rotation of a true period (in this case, there might be multiple true periods), by iteratively growing a candidate period \( q \), and (2) using \( q \) to recover \( S \) accordingly. However, in the presence of errors, each of these steps becomes more difficult to realize efficiently. For example, in the first step, we might be growing a candidate period \( q \) that includes an error. So, in order to rightfully reject the hypothesis that \( q \) is at most as large as some approximate period \( p \), our algorithm should be able to tell the difference between (i) \(|p| = |q|\) and \( q \) includes an error and (ii) \(|p| > |q|\). Otherwise, the algorithm will keep on growing \( q \) until it is equal to \( S \), possibly incurring \( \alpha \) new queries for detecting a left or right endpoint of \( q \). In addition, the second step of using \( q \) to determine \( S \) requires more work, since the presence of errors discards the possibility of simply concatenating \( q \) with itself the required number of times. Because of these issues, it is crucial that our algorithm understands when a candidate period is or not free of errors. Thus, the algorithm relies on the following.

**Lemma 6.** Let \( A \) be any length-(2\(d+1\))\(|p|\) substring of a \(d\)-corrupted periodic string \( S \) of approximate period \( p \), corresponding to the concatenation of length-\(|p|\) substrings \( q_1, q_2, \ldots, q_{2d+1} \). Then, a cyclic rotation of \( p \) must be the only substring \( q \) appearing at least \( d+1 \) times in \( q_1, q_2, \ldots, q_{2d+1} \).

**Proof.** Clearly, there is some \( q \) that is a cyclic rotation of \( p \). Moreover, there is some \( q \) that appears at least \( d+1 \) times in \( q_1, q_2, \ldots, q_{2d+1} \), or the number of errors would exceed \( d \), by the pigeonhole principle. If \( i \neq j \), then each occurrence of \( q \), contributes at least 1 error, resulting in at least \( d+1 \) errors, a contradiction. Finally, \( q \) must be the only string with \( d+1 \) appearances in \( q_1, q_2, \ldots, q_{2d+1} \), by the pigeonhole principle. \( \square \)

We give the details for our algorithm, which is able to recover \( S \), even when its size \( n \) is unknown (see Algorithm 4 in Appendix B for reference). We maintain an initially empty substring, \( A \), of \( S \), by extending it with \( 2d+1 \) letters in each iteration, using the append and prepend primitives, potentially incurring an extra \( \alpha \) queries for detecting a left or right endpoint of \( S \). In the case that \( n = |S| < |p|(2d+1) \), the last iteration requires only \( \min(2d+1, |S| - |A|) \) new letters. Thus, after adding letters to \( A \) in the \( i \)th iteration, \( A \) is a substring of \( S \) of size at most \( i(2d+1) \). Before advancing to the next iteration, we determine the only possible length-\(i\) candidate period \( q \) that could have originated \( A \) with at most \( d \) errors (by Lemma 6). At this point we do not know if some approximate period \( p \) has size \( |p| = i \), so we try to use \( q \) to recover the rest of the string, halting whenever the total number of errors exceeds \( d \), in which case we advance to the next iteration and repeat this process for a new candidate period of size \( i+1 \). This logic is in the subroutine **Expand**(\( q \)), described next. It initializes a string \( T \) to \( q \) and expands it by doing the following at each iteration:

1. Appending to \( T \) the largest periodic substring of period \( q \). This can be done efficiently by determining the maximum value of \( r \), using a doubling search, for which \( \text{IsSubstr}(T \cdot q^{\lfloor r/|q| \rfloor} \cdot q^{..\ r \mod |q|}) \), incurring \( 2\lfloor \log r \rfloor + 1 \) queries.
2. Prepending to $T$ the largest periodic substring, of period size $|q|$. This can be done efficiently by determining the maximum value of $s$, using a doubling search, for which $\text{IsSubstr}(q[|q| + 1 - (s \mod |q|) \cdot |q|^{|s/|q|} \cdot T])$, incurring $2\lfloor \log s \rfloor + 1$ queries.

3. Determining, if they exist, the letters immediately to the left and to the right of $T$, using $2\alpha$ queries, and adding them to $T$.

The expansion process in $\text{Expand}(q)$ halts when either the total number of errors with respect to $q$, $\delta(T, q^\infty[..|T|])$, exceeds $d$ (in which case we advance to the next iteration), or when $T = S$ (in which case we return $T$).

Remark 1. $\text{Expand}(q)$ successfully returns $S$ if and only if $q$ is a cyclic rotation of some approximate period.

Lemma 7. ⊛ The number of queries performed during any call to $\text{Expand}$ is $O(d\alpha + d\log \frac{n}{d+1})$.

Correctness and query complexity of our algorithm follows from Remark 1 and Lemmas 6 and 7, giving us:

Theorem 6. ⊛ We can reconstruct a length-$n$ $d$-corrupted periodic string $S$ using $O(d\alpha |p| + d|p|\log \frac{n}{d+1})$ queries, for known $d$, unknown $|p|$, regardless of whether we know $n$, where $p$ is a smallest approximate period of $S$.

If $n$ is known, we could save the queries used to check the left and right endpoints of $S$, but this does not alter the query complexity asymptotically.

We assume a small enough number of errors, following [10]. In particular, if $d = O(k/(1 + \log n))$, our algorithm is an improvement to the $O(\alpha n)$ letter-by-letter algorithm of Skiena and Sundaram [74] for general strings, where $k = \lfloor n/|p| \rfloor$. Thus, our algorithm performs better if there is, on average, at most 1 error in every other $O(1 + \log n)^{1\text{st}}$ non-overlapping occurrence of $p$. If the number of errors is not small enough, then one should run the letter-by-letter algorithm intercalated with ours, to get an upper bound of $O(\alpha n)$ queries, giving us Theorem 1.

3 Subsequence Queries

We study the query complexity for a length-$n$ string, $S$, subject to yes/no subsequence queries, $\text{IsSubseq}$, i.e., queries of the form “Is $X$ a subsequence of $S$?” We begin with a simple lower bound.

Theorem 7. ⊛ Reconstructing a length-$n$ periodic string, $S = p^k p'$, of smallest period $p$, requires at least $|p|\log \alpha$ $\text{IsSubseq}$ queries, even if $n$ and $|p|$ are known.

Let us next describe an algorithm for reconstructing a periodic length-$n$ periodic string, $S = p^k p'$, of smallest period $p$. We begin by performing either binary searches (if $n$ is known) or doubling search (if $n$ is unknown), using queries
of the form \texttt{IsSubseq}(a^i) to determine the number of a’s in S, for each \( a \in \Sigma \). From all of these queries, we can determine the value of \( n \) if it was previously unknown. This part of our algorithm requires either \( \alpha \lfloor \log n \rfloor \) or \( 2\alpha \lceil \log n \rceil \) queries in total, depending on whether we knew \( n \) at the outset.

If the number of a’s in S is \( n \), for any \( a \in \Sigma \), then we are done, so let us assume the number of a’s in S is less than \( n \), for each \( a \in \Sigma \). Thus, when we complete all our doubling/binary searches, for each letter, \( a \in \Sigma \) that occurs a nonzero number of times in S, we have a maximal subsequence, \( S_a \), of S, consisting of a’s. Moreover, since S is periodic with a period that repeats \( k \) times, each \( S_a \) is periodic with a period that repeats \( k \) times. Unfortunately, at this point in the algorithm, we may not be able to determine \( k \). So next we create a binary merge tree, \( T \), with each of its leaves associated with a nonempty subsequence, \( S_{a_i} \), much in the style of the well-known merge-sort algorithm, so that \( T \) has height \( \lceil \log \alpha \rceil \). We then perform a bottom-up merge-like procedure in \( T \) using \texttt{IsSubseq} queries, as follows.

Let \( v \) be an internal node in \( T \), with children \( x \) and \( y \) for which we have inductively determined periodic subsequences, \( S_x \) and \( S_y \), respectively, of S. Let \( n_x = |S_x| \) and \( n_y = |S_y| \). To create the subsequence, \( S_v \), for \( v \), we need to perform a merge procedure to interleave \( S_x \) and \( S_y \). To do this, we maintain indices \( i \) and \( j \) in \( S_x \) and \( S_y \), respectively, such that we have already determined an interleaving, \( S_v[i..j] \), of \( S_x[i..i] \) and \( S_y[j..j] \). Initially, \( i = j = 0 \). We then perform the query \texttt{IsSubseq}(\( S_v[i..i+j] \cdot S_x[i+1] \cdot S_y[j+1]..n_y \)). Suppose the answer to this query is “yes”. In this case, we set \( S_v[i..i+j+1] = S_v[i..i+j] \cdot S_x[i+1] \) and we increment \( i \). If, on the other hand, the answer to the above query is “no”, then we set \( S_v[i..i+j+1] = S_v[i..i+j] \cdot S_y[j+1] \), because in this case we know that \texttt{IsSubseq}(\( S_v[i..i+j] \cdot S_y[j+1]..i+1.n_x \)) would return “yes”. If this latter condition occurs, then we increment \( j \).

Let \( q_v \) denote this new interleaving prefix, \( S_v[i..i+j] \), and let \( \hat{k} = \lfloor n/|q_v| \rfloor \). If \( q_v^kq_v' \) is a plausible interleaving of \( S_x \) and \( S_y \), where \( q_v' \) is a prefix of \( q_v \), then we next ask the query \texttt{IsSubseq}(\( q_v^kq_v' \)). If the answer is “yes”, then we set \( S_v = q_v^kq_v' \) and this completes the merge. Otherwise, we continue incrementally interleaving \( S_x \) and \( S_y \), using the current values of \( i \) and \( j \), by iterating the procedure described above. Clearly, this merge procedure asks at most \( 2|q_v| \) queries in total.

**Theorem 8.** \( \circledast \) We can determine a length-n periodic string, \( S = p^kp' \), of smallest period \( p \) of unknown size, using \( 2\alpha \lfloor \log n \rfloor + 2|p|\lceil \log \alpha \rceil \) \texttt{IsSubseq} queries, if \( n \) is unknown. If \( n \) is known, then \( \alpha \lfloor \log n \rfloor + 2|p|\lceil \log \alpha \rceil \) \texttt{IsSubseq} queries suffice.

A simple modification of our algorithm also implies the following.

**Theorem 9.** \( \circledast \) We can determine a length-n string, \( S \), using \( 2\alpha \lfloor \log n \rfloor + n\lceil \log \alpha \rceil \) \texttt{IsSubseq} queries, without knowing the value of \( n \) in advance. If \( n \) is known, then \( \alpha \lfloor \log n \rfloor + n\lceil \log \alpha \rceil \) \texttt{IsSubseq} queries suffice.

This latter theorem improves a result of Skiena and Sundaram [74], who prove a query bound of \( 2\alpha \log n + 1.59n \log \alpha + 5\alpha \) when \( n \) is unknown.
4 Jumbled-index Queries

Jumbled-indexing involves preprocessing a given string, $S$, so as to determine whether there exists a substring of $S$ whose letter frequencies match the given Parikh vector, i.e., a vector $\psi = (f_1, \ldots, f_\alpha)$ such that $f_i$ is the number of occurrences in $S$ of $a_i \in \Sigma$, e.g., see [4, 5, 8, 9, 53, 63]. In this section, we study the query complexity for reconstructing an unknown length-$n$ string, $S$, using jumbled-index queries. As observed by Acharya et al. [1, 2], strings and their reversals have the same “composition multiset”. This immediately implies the following negative result.

Lemma 8. If $S$ is not a palindrome, then $S$ cannot be reconstructed by yes/no jumbled-index queries, which return whether there’s a substring in $S$ with a given Parikh vector.

Given that simple yes/no jumbled-index queries are not sufficient for string reconstruction, let us consider an extended type of yes/no jumbled-index query.

– **Jumbled-Indexing with End-of-string symbol “$” (JIE):** given an extended Parikh vector, $\psi = (f_1, \ldots, f_\alpha, f_\$)$, for the letters in $\Sigma$ and an end-of-string symbol, $\$, which is not in $\Sigma$, this query returns a yes/no response as to whether there is a substring of $S\$ with extended Parikh vector $\psi$.

Unlike the yes/no jumbled-index queries, this variant enables full reconstruction.

Theorem 10. We can reconstruct a length-$n$ string, $S$, using $(\alpha - 1)n$ JIE queries, if $n$ is known, or $\alpha(n + 1)$ JIE queries, if $n$ is unknown.

Proof. Our method is to use a letter-by-letter reconstruction algorithm via an adaption of the prepend-a-letter primitive for substring queries. Suppose $n$ is unknown. Let $\psi$ be an extended Parikh vector for a known suffix, $s$, of $S\$; initially, $\psi = (0, 0, \ldots, 0, 1)$ and $s = \$. Then we perform a jumbled-index query for $\psi_i$, for each $a_i \in \Sigma$, where $\psi_i = \psi$ except that $\psi_i$ adds 1 to the $f_i$ value in $\psi$. If one of these, say, $\psi_i$, returns “yes”, then we prepend $a_i$ to our known suffix and we repeat this procedure using $\psi_i$ for $\psi$. If all of these queries return “no”, then we are done. If $n$ is known, on the other hand, then we can skip this last test of all-no responses and we can also save at least one query with each iteration, with the algorithm otherwise being the same. \qed

We can also consider jumbled-index queries that return an index of a matching substring for a given Parikh vector, if such a substring exists. Though related, notice that this type of query is not subsumed by the query studied in Acharya et al. [1, 2], which returns the number of occurrences (instead of position) of matching substrings in $S$. There is some ambiguity, however, if there is more than one matching substring: hence, we should consider how to handle such multiple matches. For example, if a jumbled-index query returns the indices of all matching substrings, then $\alpha$ queries are clearly sufficient to reconstruct any length-$n$ string, for any $n$, without knowing the value of $n$ in advance. Thus, let us consider two more-interesting types of jumbled-index queries.
Adaptive Exact Learning in a Mixed-Up World

- **Adversarial Jumbled-Indexing** (AJI): given a Parikh vector, $\psi = (f_1, \ldots, f_\alpha)$, this query returns, in an adversarial manner, one of the starting indices of a matching substring, if such a string exists. If there is no matching substring, this query returns **False**.

- **Random Jumbled-Indexing** (RJI): given a Parikh vector, $\psi = (f_1, \ldots, f_\alpha)$, this query returns, uniformly at random, one of the indices of a substring with Parikh vector $\psi$ if such a substring exists in $S$. If there is no such substring, this query returns **False**.

Unfortunately, for the AJI variant, there are some strings that cannot be fully reconstructed, but this is admittedly not obvious. In fact, the unreconstructability characterization of [1,2] fails for AJI queries, because the symmetry property used in their construction of pairwise “equicomposable” strings inherently yields matching substrings with symmetric (e.g. different) positions in $S$.

Nevertheless, we give a construction of an infinite family of pairwise undistinguishable strings, i.e. two strings such that, for every possible query, there exists an answer (positive or negative) that is common to both strings. Clearly, the adversarial strategy is to output these common answers when given either of these strings. In particular, for all $b \geq 1$, consider the two binary strings of length $4b + 14$ given below, which differ only in the middle section, consisting of 01 in the first string and 10 in the second:

$$S_1 = 101101(10)^b01(10)^b010010$$
$$S_2 = 101101(10)^b10(10)^b010010$$

**Theorem 11.** ⊛ The strings $S_1$ and $S_2$ cannot be distinguished using AJI queries, for $b \geq 1$.

In contrast, the query variant RJI can be used to reconstruct any length-$n$ string, $S$, without knowing the value of $n$ in advance. In particular, it is possible to reconstruct any length-$n$ string, $S$, using $O(\alpha + n \log n)$ RJI queries with high probability. Our algorithm for doing this involves a reduction to a multi-window coupon-collector problem.

Let $\psi_i$ be a Parikh vector that is all 0’s except for a count of 1 for the letter $a_i \in \Sigma$. Note that an RJI query using $\psi_i$ will return one of the $n_i$ locations in $S$ with an $a_i$ uniformly at random (if $n_i > 0$). If $n_i = 0$, for any $i = 1, 2, \ldots, \alpha$, we learn this fact immediately after one RJI query for $\psi_i$, so let us assume, w.l.o.g., that $n_i > 0$, for all $i = 1, 2, \ldots, \alpha$, after performing an initial $\alpha$ number of RJI queries.

Recall that in the **coupon-collector** problem, a collector visits a coupon window each day and requests a coupon from an agent, who chooses one of $n$ coupons uniformly at random and gives it to the collector, e.g., see [62]. The expected number of days required for the collector to get all $n$ coupons is $nH_n$, where $H_n$ is the $n$-th Harmonic number. But this assumes the collector knows when they have received all $n$ coupons (i.e., the collector knows the value of $n$).

In a coupon-collector formulation of our reconstruction problem, we instead have $\alpha$ coupon windows, one for each letter $a_i \in \Sigma$, where each window $i$ has
n_i coupons that differ from the coupons for the other windows, and we don’t
know the value of any n_i. Each day the collector must choose one of the coupon
windows, i, and request one of its coupons (corresponding to an RJI query
for ψ_i), which is chosen uniformly at random from the n_i coupons for window
i. We are interested in a strategy and analysis for the collector to collect all
n = n_1 + n_2 + ⋅⋅⋅ + n_α coupons, with high probability (i.e., with probability at
least 1 − 1/n).

Note that although we don’t know the value of any n_i, we can nonetheless
test whether collector has collected all n coupons. In particular, suppose we have
received RJI responses for all indices, 1, 2, ..., n, for letters in S, and let n_i be
the number of a_i’s we have found so far. Let ψ' = (n_1, n_2, ..., n_α), and let ψ'_i be
equal to ψ' except that we increment n_i by 1. If an RJI query for each ψ'_i returns
False, then we know we have fully reconstructed S. Thus, if n = 1, then we can
determine this and S after 2α RJI queries, so let us assume that n ≥ 2. Further,
we can assume we have a bound, N ≥ 2, which is at least n and at most twice
n, by a simple doubling strategy, where we double N any time a test for n fails
and we set N equal to any RJI query response that is larger than N. Therefore,
the remaining problem is to solve the multi-window coupon-collector problem.

Our strategy for the multi-window coupon-collector problem is simply to visit
the coupon windows in phases, so that in phase i we repeatedly visit window i
until we are confident we have all of its n_i coupons, for which the following
lemma will prove useful.

Lemma 9. ⊗ Let T_i be the number of trips to window i needed to collect all its
n_i ≥ 1 coupons. Then,

Pr (T_i > βn_i ln N) ≤ \frac{n_i}{N^β}.

Our strategy, then, is to let β ≥ 2 be constant, and in phase i, implement a
doubling strategy where we perform βN_i log N RJI queries for ψ_i, such that N_i
is an upper bound estimate for n_i, which we double each time we get more than
N_i distinct responses to our queries in this phase. So by the end of the phase i,
n_i ≤ N_i ≤ 2n_i. This gives us:

Theorem 12. ⊗ A string, S, of unknown size, n, can be reconstructed using
O(α + n log n) RJI queries, with high probability.

5 Conclusion and Open Questions

We have studied the reconstruction of strings under the following settings,
by giving efficient reconstruction algorithms and proving lower bounds: (i)
periodic strings of known and unknown sizes, with and without mismatch
errors, using substring queries; (ii) periodic strings of known and unknown sizes,
using subsequence queries and (iii) general strings, using variations of jumbled-
indexing queries. For the non-optimal algorithms given here, it would be nice to
know whether there exist matching lower bounds, or whether there exist faster
algorithms. We mention additional possible future work in Appendix E.
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A Review of Non-Adaptive String Reconstruction

In this appendix, we review work on non-adaptive string reconstruction.

Non-adaptive substring queries. There is an extensive line of work focusing on the ability to reconstruct a string given the multiset of all its length-$L$ substrings. For $L \geq a \log n$ ($a > 1$), it is shown in [24,37,41] that, as $n$ approaches infinity, almost every length-$n$ string can be recovered. The following variants have also been studied: (i) only a subset of the length-$L$ substrings is given, or each substring is subject to substitution errors of fixed Hamming distance [49,60]; (ii) the hidden string is an i.i.d. DNA string [14], combined with a random subset of the length-$L$ substrings [64], subject to probabilistic substitution errors [65] or edit errors (of fixed maximum amount) [42]; (iii) the hidden string satisfies several constraints based on its repeat statistics [21,79] and input substrings are subject to erasure errors [70]; and (iv) when partial reconstruction of the hidden string is sufficient [71]. On a different note, the authors of [22,38] consider instead the case where the input is a special set of substrings which is derived from the set of maximal substrings.

Non-adaptive subsequence queries. Perhaps the most studied problem in this category is the $k$-deck problem: given the multiset of all length-$k$ subsequences of a length-$n$ string $S$, what is the smallest value of $k$ that enables the unique reconstruction of $S$? This problem was introduced in [47], who showed an upper bound of $\lceil n/2 \rceil$. This bound was improved to $(1 + o(1))\sqrt{(n \ln n)}$ in [69] and, in the same year, to $\lceil 16/7 \sqrt{n} \rceil + 5$ in [55]. The first non-trivial lower bound, of $\log \log n$, was given in [86] and later on, was improved to $\log n$ in [59] and to $e^{O(\sqrt{\log n})}$ in [35]. Recently, Gabrys et al. [40] considered an extension of the $k$-deck problem, where one is also given a number of special subsequences of length $n - t$, $t > 0$; they provide lower and upper bounds that have a dependence on $t$. Also related to the $k$-deck problem is the work of Simon [72], on which subsequences are considered to be of length at most $k$. Another relevant problem is trace reconstruction. The input to this problem is a set of traces, distorted versions of the hidden string obtained by deletion (i.e. subsequences) or other types of errors, when sending it through a noisy channel. Similarly, the goal is to recover the hidden string $S$, either exactly or with some accuracy or probability, using the least amount of traces. To the best of our knowledge, this problem was first studied in [57], who provided bounds for the number of input traces, when subject to a worse case fixed number of substitutions, transpositions, deletions or insertion errors. In the case of exclusively dealing with deletions, where each letter is deleted with some fixed probability $q$, Batu et al. [17] showed that reconstruction is possible w.h.p. for $q = O(1/\log n)$ and $O(\log n)$ traces, when $S$ is chosen uniformly at random. Moreover, they show that, for arbitrary $S$ and for $q = O(1/n^{1/2+\epsilon})$, $O(1/\epsilon)$

4 A letter in the substring is replaced by an $\varepsilon$. 


traces are sufficient to reconstruct a close approximation of $S$ and $O(n \lg n)$ traces are sufficient to recover $S$ exactly. Later Kannan et al. [48] extended these results to the case where insertion errors are also allowed, showing that for deletion/insertion error probabilities of $q = O(1/\lg^2 n)$ and $O(\lg n)$ traces, $S$ can be recovered w.h.p., assuming it is chosen uniformly at random. Similarly, they show that an arbitrary $S$ can be recovered w.h.p., for $q = O(1/n^{3/2+\epsilon})$ and $O(1)$ traces of length at most $n^\epsilon$. Later, Viswanathan et al. [80] improved on this, by showing that deletion/insertion error probabilities of $q = O(1/\lg n)$ are sufficient to reconstruct $S$, chosen uniformly at random. They also show that $\Omega(\lg n)$ traces are necessary to reconstruct $1-o(1)$ length-$n$ strings w.h.p. In [43], the authors showed that, for the case of deletion errors only, of probability $q = O(1)$, reconstruction is possible w.h.p. using $\text{poly}(n)$ traces, when $S$ is chosen uniformly at random. Finally, Sala et al. [68] studied lower bounds on the number of input traces formed from a worst-case number of insertion errors, where $S$ is a member of specific error-correcting codes, i.e. sets of strings constructed strategically to allow recovering them from a noisy channel is modified.

**Non-adaptive jumbled-index queries.** In [1, 2], Acharya et al. study a non-adaptive version of the problem of enumerating candidate strings from the *composition multiset* of the underlying string. The composition multiset corresponds to the set of answers to all possible queries of the following type: given a Parikh vector, how many times does a matching substring occur in the hidden string? Under this model, they extend polynomial techniques used for the turnpike problem (see [30, 73]) to give: (i) sufficient (but not necessary) conditions for the ability to uniquely reconstruct a string, (ii) a sufficient characterization of unreconstructable strings and (iii) a backtracking algorithm that enumerates the set of all candidate strings, whose cardinality they lower and upper bound.

## B Omitted Algorithms and Proofs for Substring Queries

In this appendix, we give details for omitted algorithms and proofs from the section related to substring queries.

### B.1 Uncorrupted Periodic String of Known Size

Our reconstruction algorithm is described in Algorithm 1, where the number of queries is shown in parentheses for steps involving queries. We prove its correctness below. Recall that $g(x) = \lfloor n/|x| \rfloor$.

**Theorem 3.** We can reconstruct a length-$n$ periodic string $S = p^k p'$, of smallest period $p$, using $O(\alpha |p|)$ substring queries, assuming $n$ is known in advance and $|p|$ is unknown.

**Proof.** The main loop in Algorithm 1 will always terminate, because $S$ is periodic and any cyclic permutation of $p$ is a substring, when concatenated at least $g(p) - 1$
Algorithm 1: Reconstructing a periodic string \( S = p_k p' \) of known size \( n \) and smallest period \( p \), for \( k > 1 \).

1. Let \( q = \epsilon \)
2. repeat
   Append a letter to \( q \) \((\alpha - 1)\) until \( \text{IsSubstr}(q^{g(q) - 1}) \) \((1 \text{ per iteration}; |p| \text{ iterations})\)
3. Let \( T = q^{g(q) - 1} \)
4. While \( T \) is a substring of \( S \), append a letter to \( T \)
5. While \( |T| < n \) and \( T \) is a substring of \( S \), prepend a letter to \( T \) \([\alpha(2|p| - 1)]\)
6. Output \( T \) times.

It is easy to see that the procedure of iteratively appending letters to \( q \) must result in a cyclic permutation of \( p \), unless the main loop stops earlier. After the main loop, there are at most \( 2|p| - 1 \) letters left to be recovered, so the overall query complexity is at most \( \alpha|p| + \alpha(2|p| - 1) \), which is \( O(\alpha|p|) \).

Improved Method. The main challenge to achieving this improvement is that, after the main loop in Algorithm 1, \( q \) may not correspond to a cyclic rotation of \( p \). For example, in \( S = abababaab \cdot abababaab \cdot abababaab \), we may get \( q = abababa \), while the actual period is \( p = abababaab \). However, we show that, when \( k = n/|p| > 3 \), the following implication holds indeed: if \( q^{g(q) - 1} \) is a substring, then \( q \) must be a cyclic rotation of \( p \).

We begin with our proof of Theorem 4, by giving the details for our improved algorithm for reconstructing a periodic length-\( n \) string \( S \), when \( n \) is known, which is shown in Algorithm 2.

Algorithm 2: Reconstructing a periodic string \( S = p_k p' \) of known size \( n \) and smallest period \( p \), for \( k > 3 \).

start
1. Let \( q = \epsilon \)
2. repeat
   Append a letter to \( q \) \((\alpha - 1)\) until \( \text{IsSubstr}(q^{g(q) - 1}) \) \((1 \text{ per iteration}; |p| \text{ iterations})\)
3. Let \( p = \text{TrueRotation}(q) \) \((\lceil \log |q| \rceil)\)
4. Determine \( p' \) and output \( p^k p' \)

function \( \text{TrueRotation}(q) \)
   Find, using binary search, the largest suffix \( q[j..] \), such that \( \text{IsSubstr}(q[j..] \cdot q^{g(q) - 1}) \) \((\lceil \log |q| \rceil)\)
   Return \( q[j..] \cdot q[1..j - 1] \)
Remark 2. A string, $p$, is a period of a string $X$ of length $|X| \geq i|p|$ if and only if $p^j$ is a period of $X$, for all $j \in \{1, 2, \ldots, i\}$.

Theorem 4. ⊛ We can reconstruct a length-$n$ periodic string $S = p^k p'$, of smallest period $p$, using at most $\alpha|p| + \lceil \lg |p| \rceil$ substring queries, assuming that: $n$ is known in advance, $k > 3$ and $|p|$ is unknown.

Proof. Consider Algorithm 2. We claim that, immediately after the main loop, the candidate period $q$ is indeed a cyclic rotation of the true period $p$. The remainder of the proof then follows from this.

So let us prove our claim. Let $q$ be the string immediately after the main loop and let $T = q^{\lfloor n/|q| \rfloor - 1}$. If $|q| = |p|$, then $q$ is clearly a cyclic rotation of $p$. Besides, $|q|$ cannot be greater than $|p|$, because the letter-by-letter construction of $q$ would have implied a halt of the main loop when $q$ had size $|p|$: any cyclic rotation of $p$ must repeat at least $\lfloor n/|p| \rfloor - 1$ times. So let us consider the case $|q| < |p|$. Since $k > 3$, we have that $n \geq 4|p|$. Moreover, since $T = q^{\lfloor n/|q| \rfloor - 1}$, we know that $|T| \geq n - (2|q| - 1)$ and, thus, $|T| \geq 2|p|$. Since $T$ is a substring of $S$, $T$ must have a second period of size $|p|$. Moreover,

$$|T| \geq 2|p|$$

$$\geq |p| + |q|$$

$$\geq |p| + |q| - \gcd(|p|, |q|)$$

Thus, by the Periodicity Lemma (1), $T$ has a period $pr$ of size $\gcd(|p|, |q|)$. Therefore, $S$ must have a period of size $|pr|$, and thus, $S$ must have a period of size $|q|$ (by Remark 2), which contradicts the fact that $p$ is the smallest period of $S$.

Our analysis above is tight in the sense that, for $k = 3$, it no longer holds: recall the example given above, where $S = abababaab, abababaab, abababaab$ and $q = abababa$.

Lower Bound.

Theorem 5. ⊛ Reconstructing a length-$n$ string, $S = p^k p'$, of smallest period $p$, requires at least $|p| \log \alpha$ IsSubstr queries, even if $n$ and $|p|$ are known.

Proof. There are $\alpha^{|p|}$ possible periods for $S$. Since each period corresponds to a different output of a reconstruction algorithm, $A$, and each query is binary, we can model any such algorithm, $A$, as a binary decision tree, where each internal node corresponds to an IsSubstr query. Each of the $\alpha^{|p|}$ possible periods must correspond to at least one leaf of $A$; hence, the minimum height of $A$ is $\log (\alpha^{|p|})$.

B.2 Uncorrupted Periodic Strings of Unknown Size

Our reconstruction algorithm is described in Algorithm 3, where the number of queries is shown in parentheses for steps involving queries. We prove its correctness below.
Algorithm 3: Reconstructing a periodic string $S = p^k p'$, of smallest period $p$ and unknown size $n$, for $k > 1$.

\begin{algorithm}
\textbf{start}
\begin{algorithmic}
\STATE 1. Let $q = \varepsilon$
\STATE 2. \textbf{repeat}
\STATE 3. Append or prepend a letter to $q$ ($\alpha - 1$; potentially, $2\alpha - 1$ when $k \leq 2$)
\STATE 4. Determine the frequency $f$ of $q$ $(2\lceil \log f \rceil + 1)$
\STATE 5. \textbf{if} $f = 1$ \textbf{then} Let $T = q$
\STATE 6. \textbf{else}
\STATE 7. Let $l$ be the largest suffix of $q$ such that $\text{IsSubstr}(l \cdot q^f)$ $(2\lceil \log |l| \rceil + 1)$
\STATE 8. Let $r$ be the largest prefix of $q$ such that $\text{IsSubstr}(l \cdot q^f \cdot r)$ $(2\lceil \log |r| \rceil + 1)$
\STATE 9. Let $T = l \cdot q^f \cdot r$
\STATE 10. Let $q = T[\ldots |T| - |q| + 1]$
\STATE \textbf{until} $\text{IsValid}(T)$(2$\alpha$)
\STATE \textbf{Output} $T$
\end{algorithmic}
\end{algorithm}

\begin{algorithm}
\textbf{function} $\text{IsValid}(T)$
\begin{algorithmic}
\STATE Let $x$ be the letter to the left of $T$ or $\varepsilon$ if there is none $(\alpha)$
\STATE Let $y$ be the letter to the right of $T$ or $\varepsilon$ if there is none $(\alpha)$
\STATE \textbf{Return} $x = \varepsilon$ and $y = \varepsilon$
\end{algorithmic}
\end{algorithm}

Recall that we denote by $q_1, q_2, \ldots, q_m$ the sequence of $m$ candidate periods of increasing length, each of which is the result of the append/prepend primitive at the beginning of every iteration (line 3 of Algorithm 3), e.g. $|q_1| = 1$. Notice that each $q_i$ may be expanded (in line 10), so the difference $|q_i| - |q_{i-1}|$ may not necessarily be 1. In addition, let us use $f_i$ to denote the frequency of $q_i$ (computed in line 4). Correctness of Algorithm 3 follows from the following two lemmas.

Lemma 3. ⊙ Algorithm 3 successfully returns $S = p^k p'$, of smallest period $p$, if there exists an iteration $i \in \{1, 2, \ldots, m\}$, such that $q_i$ is a cyclic rotation of $p$.

Proof. If $f_i > 1$, then it is easy to see that the string $T$, computed in line 9 in iteration $i$, must correspond to $S$. If $f_i = 1$, then the algorithm essentially switches to the letter-by-letter algorithm, appending or prepending letters until the end, when $q_m = S$. Correctness of the stopping condition follows from the correctness of $\text{IsValid}$. \qed

We now show that, indeed, at some iteration $i$, the candidate period $q_i$ is a cyclic rotation of $p$.

Lemma 4. ⊙ There exists an iteration $i \in \{1, 2, \ldots, m\}$, such that $q_i$ is a cyclic rotation of $p$.  


Proof. Let us assume that there is no such iteration $i$. Then, since all the $q_i$’s are increasing in length, it must be the case that there exists an iteration $j \in \{1, 2, \ldots, m - 1\}$, such that: $|q_j| < |p|$, but $|q_{j+1}| > |p|$. However, it follows from Lemma 2 (when $f_j > 1$) and the fact that we add a single letter to $q_j$ (when $f_j = 1$) that $p$ must be at least as large as $q_{j+1}$, a contradiction.

The following lemma shows that we can charge the logarithmic factors, incurred in each iteration $j$, to the work that would have been required to find the letters introduced in $q_{j+1}$. This establishes the amortization in query complexity. We denote the number of queries in iteration $j$ of Algorithm 3 by $Q(j)$.

**Lemma 5.** The number of queries performed in iteration $j$ of Algorithm 3 is at most $\alpha(|q_{j+1}| - |q_j|) + O(\alpha)$, for $j < m$, or $O(\alpha + \lg n)$, for $j = m$.

**Proof.** Let $l_j$ and $r_j$ denote, respectively, the lengths of the prefix $l$ and suffix $r$ computed in lines 7 and 8 of Algorithm 3 in iteration $j$. The query complexity in any iteration $j$ is

$$Q(j) \leq 2[\lg f_j] + 1 + 2[\lg l_j] + 1 + 2[\lg r_j] + 1 + 4\alpha$$

Let us assume that $f_j > 1$, since otherwise the query complexity is $O(\alpha)$ and, therefore, agrees with the query complexity that is stated in the lemma.

When $j = m$, it must be the case that $q_m$ is a cyclic rotation of $p$, and therefore has size $|p|$. Thus, we spend at most: (i) $\alpha$ queries when appending the $p^{th}$ letter, (ii) $2[\lg n/|p|] + 1$ queries to determine the frequency $f_m$, and (iii) $2(2[\lg |p|] + 1)$ queries to determine the suffix and prefix of lengths $l_m$ and $r_m$, respectively. Notice that the combined log factors result in no less than $\Theta(\lg n)$. Thus, when $j = m$, the overall query complexity is $O(\alpha + \lg n)$.

When $j < m$, we have the following:

\begin{align*}
\lg f_j &\leq f_j - 1 \quad \text{ (} f_j > 1 \text{)} \\
\implies \lg f_j &\leq (f_j - 2)q_j + 1 \quad \text{ (} q_j \geq 1 \text{)} \\
\implies \lg f_j + \lg l_j + \lg r_j &\leq (f_j - 2)q_j + l + r + 1 \quad (\lg x < x) \\
\implies 2(\lg f_j + \lg l_j + \lg r_j) &\leq 2((f_j - 2)q_j + l + r + 1) \\
\implies Q(j) &\leq 2((f_j - 2)q_j + l + r) + 2 + 3 + O(\alpha) \quad \text{ (definition of} \ Q(j)\text{)} \\
\implies Q(j) &\leq 2((f_j - 2)q_j + l + r + 2) + O(\alpha) \\
\implies Q(j) &\leq 2(|q_{j+1}| - |q_j|) + O(\alpha) \quad (\ast) \\
\implies Q(j) &\leq \alpha(|q_{j+1}| - |q_j|) + O(\alpha) \quad (\alpha \geq 2),
\end{align*}

where $(\ast)$ follows from the fact that, when $f_j > 1$, $|q_{j+1}| = (f_j - 1)|q_j| + l + r + 2$.

\[\square\]

**Theorem 2.** We can reconstruct a length-$n$ periodic string, $S = p^k p'$, of smallest period $p$, using $O(\alpha |p| + \lg n)$ substring queries, assuming both $n$ and $|p|$ are unknown in advance.
Proof. Correctness follows from Lemmas 3 and 4. As for the query complexity, it follows from Lemma 5, that the overall query complexity of Algorithm 3 is
\[
\sum_{j=1}^{m} Q(j)
\]
Let \(i\) be the iteration in which \(|q_i| = |p|\) (see Lemma 4) and let us consider the queries done up to and after iteration \(i - 1\). Thus, by Lemma 5:
\[
\sum_{j=1}^{m} Q(j) = \sum_{j=1}^{i-1} \left( \alpha(|q_{j+1}| - |q_j|) + O(\alpha) \right) + \sum_{j=i}^{m} Q(j)
\]
\[
= O(\alpha|p|) + \sum_{j=i}^{m} Q(j),
\]
where the last equality follows from the telescoping nature of the first summation.
As for the second summation, regarding the queries done after iteration \(i - 1\), we consider two cases. If \(i = m\), then we spend either \(O(\alpha)\) queries if \(f_i = 1\), or \(O(\alpha + \lg n)\) queries if \(f_i > 1\), by Lemma 5. If, on the other hand, \(i < m\), then notice that it must have been the case that \(f_j = 1\) for all \(j \in \{i, i+1, \ldots, m\}\). Thus, the total number of letters in \(S\) left to recover at the end of iteration \(i - 1\) is at most \(2|q_i| - 1 = 2|p| - 1\), each of which is added during each iteration \(j \in \{i, i+1, \ldots, m\}\) using \(O(\alpha|p|)\) queries in total. Thus, whether or not \(i = m\), the overall query complexity is
\[
\sum_{j=1}^{m} Q(j) = O(\alpha|p| + \lg n)
\]
\(\square\)

B.3 Corrupted Periodic Strings

Our reconstruction algorithm is described in Algorithm 4, where the number of queries is shown in parentheses for steps involving queries. We prove its correctness below.

Lemma 7. \(\star\) The number of queries performed during any call to Expand is \(O(d\alpha + d\lg \frac{n}{d+1})\).

Proof. Each call to Expand uses at most \(2(d + 1)\alpha\) queries to determine the corrupted letters, as well as the left/right endpoints of \(S\) – the total number of iterations of the while loop in Expand is \(d + 1\), since every iteration except the last introduces at least 2 errors in \(T\), and each iteration incurs \(2\alpha\) queries.

In addition, the number of queries used by Expand\((q)\) during the doubling searches is \(\sum_{j=1}^{\ell} (2|\lg R_j| + 2|\lg L_j| + 2)\), where \(R_j\) and \(L_j\) denote, respectively, the lengths of the substrings determined via doubling searches in lines 3 and 4,
Algorithm 4: Reconstructing a $d$-corrupted periodic string $S$.

1. Let $A = \varepsilon$
2. repeat
3. Append/prepend $\min(2d + 1, |S| - |A|)$ letters to $A$ ($\alpha(2d + 2)$)
4. Let $q$ be the candidate period that is a substring of $A$, as determined by Lemma 6
5. $(\text{success, } T) = \text{Expand}(q)$ ($O(d\alpha + d\lg \frac{n}{d+1})$)
6. Output $T$

Function Expand($q$) ($O(d\alpha + d\lg \frac{n}{d+1})$)

1. Let $T = q$, done = False
2. while \(\delta(T, q^\infty[..|T|]) \leq d\) and not done do
3. Find the largest substring $R$, such that $\text{IsSubstr}(T \cdot R)$ ($2|\lg |R|| + 1$)
4. Find the largest substring $L$, such that $\text{IsSubstr}(L \cdot T \cdot R)$ ($2|\lg |L|| + 1$)
5. Let $r$ (l) be the letter to the right (left) of $L \cdot T \cdot R$ or $\varepsilon$ if there is none ($2\alpha$)
6. Let done = ($r == \varepsilon$ and $l == \varepsilon$)
7. Let $T = l \cdot L \cdot T \cdot R \cdot r$
8. if $\delta(T, q^\infty[..|T|]) > d$ then return (False, _)
9. return (True, $T$)

during the $j$th call to Expand. Since the total number of iterations is $d + 1$, there is at most $d + 2$ such $R_j$’s and $L_j$’s. Moreover, the above summation is maximized when all the $R_j$’s and $L_j$’s have the same average value of at most $(n - d)/(d + 1)$. This follows from Jensen’s inequality and concavity of log. Thus, the overall time complexity is $O(d\alpha + d\lg \frac{n}{d+1})$.

\[ \text{Theorem 6.} \quad \star \quad \text{We can reconstruct a length-$n$ $d$-corrupted periodic string $S$ using } O(d\alpha|p| + d|p|\lg \frac{n}{d+1}) \text{ queries, for known $d$, unknown $|p|$, regardless of whether we know $n$, where $p$ is a smallest approximate period of $S$.} \]

\[ \text{Proof.} \quad \text{At the $|p|$th iteration of the main loop, $A$ has size $(2d + 1)|p|$ and, by Lemma 6, $q$ must correspond to a cyclic rotation of some approximate period $p$. Correctness of reconstruction then follows from Remark 1.} \]

\[ \text{The overall query complexity consists of the queries used to expand $A$ in each iteration and the queries used in the calls to the subroutine Expand. The former requires at most $(2d + 2)\alpha|p|$ queries overall, and the latter requires at most } O(d\alpha|p| + d|p|\lg \frac{n}{d+1}), \text{ by Lemma 7. Thus, the overall query complexity is } O(d\alpha|p| + d|p|\lg \frac{n}{d+1}). \]

C Omitted Proofs for Subsequence Queries

In this appendix, we give details for omitted proofs from the section related to subsequence queries. We prove below the correctness of our reconstruction...
algorithm for periodic and general strings.

Recall that we denote by $S_v$ the subsequences associated with each node $v$ of a binary merge tree with leaves corresponding to maximal subsequences $S_a$, consisting of $a$’s for all $a \in \Sigma$. Each $S_v$ is the result merging the subsequences associated with $v$’s children and, thus, it is character disjoint with respect to every other node in the same level of the tree. Further, recall that we denote by $q_v$ the candidate period of $S_v$, constructed letter-by-letter, during the merge procedure.

**Lemma 10.** Let $p_v$ be the subsequence of $p$ consisting of the letters from $S_v$. Then $|q_v| \leq |p_v|$.

**Proof.** The letter-by-letter construction of $q_v$ ensures that $q_v$ is the smallest period of $S_v$. Since $p_v$ is itself a period of $S_v$ no smaller than the smallest, then $|p_v| \geq |q_v|$.

This gives us the following.

**Theorem 8.** We can determine a length-$n$ periodic string, $S = p^k p'$, of smallest period $p$ of unknown size, using $2\alpha \lceil \log n \rceil + 2|p|\lceil \log \alpha \rceil$ IsSubseq queries, if $n$ is unknown. If $n$ is known, then $\alpha \lceil \log n \rceil + 2|p|\lceil \log \alpha \rceil$ IsSubseq queries suffice.

**Proof.** The total query complexity includes: (i) the letter decomposition $S_a$ for all $a \in \Sigma$, during the first stage and (ii) the merge-like composition of all subsequences $S_a$, during the second stage. If $n$ is known, the first stage requires $|\Sigma|$ binary searches, incurring $\alpha \lceil \log n \rceil$ queries. Otherwise, it requires $|\Sigma|$ doubling searches, amounting to $2\alpha \lceil \log n \rceil$ queries. Regarding the second stage, we claim that any level $l$ of the binary merge tree, $T$, incurs a total of at most $2|p|$ queries, which amounts to a total of at most $2|p|\lceil \log \alpha \rceil$ queries, when taking into account all the $\lceil \log \alpha \rceil$ levels of $T$. Let $T(l)$ be the set of all nodes in $T$ at level $l$. Then, $\sum_{v \in T(l)} |q_v| \leq \sum_{v \in T(l)} |p_v| = |p|$. This follows from Lemma 10 and the fact that all $\{S_v \mid v \in T(l)\}$ are pairwise letter-disjoint. Since the merge of an internal node $v$ requires a cost of $2|q_v|$, the total cost incurred in any level $l$ of $T$ is at most $2|p|$.

**Theorem 9.** We can determine a length-$n$ string, $S$, using $2\alpha \lceil \log n \rceil + n\lceil \log \alpha \rceil$ IsSubseq queries, without knowing the value of $n$ in advance. If $n$ is known, then $\alpha \lceil \log n \rceil + n\lceil \log \alpha \rceil$ IsSubseq queries suffice.

**Proof.** Modify our subsequence-querying algorithm given in Section 3 to remove the queries for strings of the form $q_v^k q_v'$. The proof follows by an analysis similar to that for Theorem 8.

**Lower Bound.**

**Theorem 7.** We determine a length-$n$ periodic string, $S = p^k p'$, of smallest period $p$, requires at least $|p|\lceil \log \alpha \rceil$ IsSubseq queries, even if $n$ and $|p|$ are known.

**Proof.** The proof follows that of Theorem 5, which can be found in Appendix B.
D Omitted Proofs for Jumbled-Index Queries

D.1 Adversarial Jumbled-Index Queries

Theorem 11. ⊛ The strings $S_1$ and $S_2$ cannot be distinguished using AJI queries, for $b \geq 1$.

\[ S_1 = 101101(10)^b 01(10)^b 010010 \]
\[ S_2 = 101101(10)^b 10(10)^b 010010 \]

Proof. Let $n = 4b + 14$ be the size of the strings. We refer to responses that would be common to both $S_1$ and $S_2$ as helpless answers. Let us think of a positive answer $i$ to a query $(k, l)$ in terms of the space occupied by its matching substring, denoted $\langle i, i+k+l-1 \rangle$. We note that an answer that does not span the middle section or that spans it in its entirety must be helpless.

Notice that the first half of either string is the symmetric complement of the second half. This implies the following: (i) an answer $\langle i, j \rangle$ to a query $(k, l)$ exists if and only if an answer $\langle n-j+1, n-i+1 \rangle$ exists for the query $(l, k)$ and (ii) an answer is negative to $(k, l)$ if and only if an answer is negative to $(l, k)$. Therefore, we can restrict ourselves to queries of the form $(k, k+c)$, where $c \geq 0$. We consider the following cases:

1. Queries of type $(k, k)$.
   We say that an answer is $k$-centered if it is of the type $\langle n/2-(k-1), n/2+1+(k-1) \rangle$. Since any $k$-centered answer contains the middle section, it must be helpless. Thus, it is enough to show, by induction, that all queries $(k, k)$ have $k$-centered answers. Clearly, this holds for the base case $(1, 1)$, so let us assume that there exists a $(k-1)$-centered answer $a$ to the query $(k-1, k-1)$. Then, because the first half of either string is the symmetric complement of the second half, the letters preceding and succeeding $a$ must be the complement of each other. Thus, the $k$-centered answer must be valid for the query $(k, k)$.

2. Queries of type $(k, k+1)$.
   Take the $k$-centered answer and either extend it with one letter to the left, or one letter to the right. Exactly one of these options is a valid answer to $(k, k+1)$ (by the symmetric-complement property of the strings) and either are helpless, since they span the middle section.

3. Queries of type $(k, k+2)$.
   Consider, as a base case, the answer $(2, 3)$ to the query $(0, 2)$. Clearly, it is a helpless answer. Given that the letters at positions $4+2j$ and $5+2j$ are complements of each other, $(2, 3+2j)$ is a valid answer to the query $(j, j+2)$, for all $0 \leq j \leq b + 2$. For greater values of $j$, the answer is helpless regardless, since it corresponds to a substring of length greater than $2b+7$, half of the strings length and, therefore, it spans the middle section.
4. **Queries of type** \( (k, k + c) \), **for** \( c \geq 3 \).

It is enough to analyze answers that partially span the middle section (i.e. in exactly 1 letter), since otherwise answers are automatically helpless. Let \( \Delta_i \) denote the number of 1’s minus the number of 0’s for the answer \( \langle i, n/2 \rangle \), with respect to \( S_2 \), for all \( 0 \leq i \leq n/2 \) (e.g. \( \Delta_{n/2} = 1 \), corresponds to the first letter in the middle). A simple passage from right to left, for increasing values of \( i \), reveals that there exist no value of \( i \) for which \( \Delta_i = 4 \), so we do not need to handle the case \( c \geq 4 \). Moreover, the only values of \( i \) for which \( \Delta_i = 3 \) are \( i = 2 \) and \( i = 0 \), which correspond to answers for the queries \((b + 1, b + 4)\) and \((b + 2, b + 5)\), respectively. However, these queries have helpless answers: \( \langle 0, 2b + 4 \rangle \) in the former and \( \langle 2, 2b + 8 \rangle \) in the latter.

For the first string, a similar exercise reveals that there exist no answers that partially overlap the middle section and whose difference between the number of 1’s and the number of 0’s is at least 3.

\[ \square \]

D.2 **Random Jumbled-Index Queries**

**Coupon-Collector Lemma.**

**Lemma 9.** Let \( T_i \) be the number of trips to window \( i \) needed to collect all its \( n_i \geq 1 \) coupons. Then,

\[ \Pr (T_i > \beta n_i \ln N) \leq \frac{n_i}{N^\beta}. \]

**Proof.** Adapting a proof from [84], let \( Z_{j,r} \) denote the event that the \( j \)-th coupon was not picked in the first \( r \) trips to window \( i \). Then

\[ \Pr (Z_{j,r}) = \left(1 - \frac{1}{n_i}\right)^r \leq e^{-r/n_i}. \]

Thus, for \( r = \beta n_i \ln N \), we have \( \Pr(Z_{j,r}) \leq e^{-(\beta n_i \ln N)/n_i} = N^{-\beta} \). Therefore, by a union bound,

\[ \Pr (T > \beta n_i \ln N) = \Pr \left( \bigcup_j Z_{j,\beta n_i \ln N} \right) \leq n_i \cdot \Pr (Z_{1,\beta n_i \ln N}) \leq \frac{n_i}{N^\beta}. \]

\[ \square \]

**RJI Theorem**

**Theorem 12.** A string, \( S \), of unknown size, \( n \), can be reconstructed using \( O(\alpha + n \log n) \) RJI queries, with high probability.
Proof. After an initial $O(\alpha)$ queries to determine which letters from $\Sigma$ appear in $S$, the total number of remaining queries performed by our method is at most

$$2 \sum_{i=1}^{\alpha} 2 \beta N_i \ln N = 4 \sum_{i=1}^{\alpha} \beta N_i \ln N,$$

by the doubling strategy applied to each letter, $a_i \in \Sigma$, and then globally for $N$. Further,

$$\sum_{i=1}^{\alpha} \beta N_i \ln N \leq \sum_{i=1}^{\alpha} 2 \beta n_i \ln N \leq \sum_{i=1}^{\alpha} 3 \beta n_i \ln n = 3 \beta n \ln n,$$

with probability at least

$$1 - \sum_{i=1}^{\alpha} \frac{n_i}{N^\beta} = 1 - \frac{\sum_{i=1}^{\alpha} n_i}{N^\beta} = 1 - \frac{n}{N^\beta} \geq 1 - \frac{1}{n},$$

by Lemma 9, since $\beta \geq 2$. \hfill \qed

E Additional Future Work

Regarding corrupted periodic strings, different applications suffer from different types of corruption. In particular, the following error metrics have been considered in the literature: Pseudo-local metrics such as swap distance [7] or Interchange (Cayley) distance [10]; and the Levenshtein edit distance [56]. It would be interesting to see whether our reconstruction algorithms can be adapted to these more general error distances.

The next step is to reconstruct strings that have more complex syntactic regularities than periods, such as covers [13]. A length $m$ substring $C$ of a string $T$ of length $n$, is said to be a cover of $T$, if $n > m$ and every letter of $T$ lies within some occurrence of $C$. We would like to efficiently reconstruct a coverable string, without knowing its cover a-priori.

Data compression schemes such as, Lempel-Ziv [88,89] are known to compress any stationary and ergodic source down to the entropy rate of the source per source symbol, provided the input source sequence is sufficiently long. These schemes rely heavily on encoding repeated substrings by their starting index and length. In this sense, a periodic string is highly compressible. We would like to extend our ideas to reconstruct a general string in time proportional to its LZ compression.

The type of query used for reconstruction is a key factor in the reconstruction complexity. Much as the error distance, the query type is also application-dependent. A reasonable query type is the less than matching. Let $S_1$ and $S_2$ be strings of length $n$ over an ordered alphabet. We say that $S_1$ is less than $S_2$ if $S_1[i] < S_2$, $\forall i = 1, \ldots, n$. Other matchings that have been researched in the literature, are the order preserving matching [25,29,50], and the parameterized...
matching [15, 16]. In the order preserving matching, we say that two strings match if the relative order of their elements is the same, for example 1, 2, 3, 2, 1 matches such strings as 1, 100, 101, 100, 1, or 56, 61, 366, 61, 56, i.e., any string where the first element is smaller than the second, which is smaller than the third, where the fourth is equal to the second, and the fifth equals the first. Two equal-length strings $S_1, S_2$ over alphabet $\Sigma$ are said to parameterize match, if there is a bijection $f : \Sigma \rightarrow \Sigma$ such that $S_1 = f(S_2)$. Using these more powerful queries, can we reconstruct a string more efficiently?

Finally, given the impossibility result on reconstructing strings using Adversarial Jumbled-Indexing queries, it would be interesting to know whether there exists an efficient algorithm that enumerates all of the undistinguishable strings.