Collinear libration points in the elliptic restricted three body problem (ER3BP) under radiating and triaxial primaries with gravitational potential from the belt

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ABSTRACT

This paper examines the effects of radiation pressure and triaxiality of two stars (primaries) surrounded by a belt (circumbinary disc) on the positions and stability of a third body of an infinitesimal mass in the neighbourhood of collinear libration points in the framework of elliptic restricted three body problem (ER3BP). We have found the solutions to the location of collinear points \( L_i \) \( (i = 1, 2, 3) \). We have investigated these collinear points numerically and graphically using radiating binary system (FL virginis and Procyon). The positions and stability of these points are found to be affected by triaxiality, radiation and the gravitational potential from the belt. The collinear libration points are found to be unstable.

1. Introduction

It is well known that the Circular Restricted Three-Body Problem (CR3BP) describes the motion of an infinitesimal mass under the gravitational effect of two finite masses (primaries) moving in the circular orbits. Although the exact solution to this problem is unknown, a number of special solutions can be obtained from the rotating frame where the infinitesimal mass has zero velocity and acceleration. These solutions correspond to equilibrium positions, which are five in the R3BP, three of them are called collinear points namely; \( L_1, L_2, L_3 \) and the remaining two are triangular points named \( L_4, L_5 \).

Due to perturbations, the position of the third-body (infinitesimal mass) is slightly changed, such that the particles resultant motion may lead to a rapid departure from the vicinity of these points; if this occurs, such a point is said to be unstable. If however, the body returns to its original position after mere oscillations, such a position of equilibrium point is said to be stable.

The study of the Circular Restricted Three-body Problem (CR3BP) is generalized by some authors who amended the potential function, for instance to include oblateness instead of spherical objects, triaxiality, radiation of the primaries and other perturbing forces.

The R3BP has received considerable attention from astronomers and physicists because of its applicability in the study of stellar and solar dynamics. It is known that orbits of most celestial bodies are elliptic rather than circular; hence we study the motion of the binary system (FL virginis and Procyon) in elliptic orbits, because CR3BP is inadequate in describing the dynamics of a particle emitting radiation. The gravitational force alone cannot be considered in studying the dynamics of a stellar system (Singh, 2013). For example, gravity is not the major force present when a star collides with a gas and dust particles, but the repelling forces of radiation pressure (Radzievskii, 1950). This attracted (Kumar and Ishwar, 2011) to include radiation pressure in their study. In (Singh, 2013), the collinear libration points were investigated under small perturbations of the Corolis and centrifugal forces, triaxiality and radiation pressure of the primaries and they were found to be unstable. Considering the effect of triaxiality of the bigger primary and oblateness of the companion on the location and stability of the collinear equilibrium points (Singh and Umar, 2012) observed that the positions and stability of the collinear libration points are affected by the perturbations in addition to the eccentricity and semi-major axis of orbits of the primaries.

In studying the Elliptical Restricted Three body Problem (Danby, 1964) used numerical integration to obtain the linear stability of the position of equilibrium points. He used the mass ratio \( \mu \) and eccentricity \( \epsilon \) to obtain a stability diagram in the \( \mu-\epsilon \) plane. In their study (Kumar and Narayan, 2012), examined the effects of photogravitation and oblateness of the primaries on the existence and stability of the third body around the collinear point \( L_3 \) and they concluded that the third body oscillation...
around $L_1$ is unstable. Investigation by (Sultan et al., 2018), of a test particle in the vicinity of collinear equilibrium points under the influence of triaxial primaries, despite using a series form approach to obtain the location of collinear equilibrium points the three collinear equilibrium points were all found to be unstable. The same results were obtained by (Singh and Tyokyaa, 2017) when they studied the stability of collinear points in ER3BP with oblateness up to zonal harmonic $J_4$.

Several authors have examined the nature of stability of collinear points in the framework of ER3BP using different characterizations. For example (Alzahrani et al., 2017), considered the third body to be moving in the gravitational field of an irregular asteroid, they found a necessary and sufficient condition for finding the three collinear points and proved the existence of these points and triangular equilibrium points. In their model (Aliroma et al., 2019) showed that the stability region could depend mainly on the eccentricity of the orbits in addition to considered perturbations. Exploiting Vinti’s method in which an oblate-spheroidal coordinate system was used to describe the orbital motion and obtaining x-coordinates in the form of series solution (Chakraborty et al., 2016) investigated the linear stability of the collinear points using two binary system Luyten-726 and Kruger-60 and found them to be unstable.

### Table 1. Numerical data for the binary system.

| Binary system | Masses $M_\text{s}$ | $a$ | $e$ | Luminosity $L_\text{s}$ | Spectral type |
|---------------|-------------------|-----|-----|------------------------|---------------|
| FL virginis   | 0.076 0.067       | 0.3 | 0.9306$^+$ | 1.372 x 10^{-3} 1.078 x 10^{-3} | M5/M7         |
| Procyon       | 1.53 0.617        | 0.4 | 4.3$^+$  | 6.93 0.00049          | F5/DA         |

Source: Stellar-DataBase/The American Astronomical Society/SIMBAD.

### Table 2. The effect of triaxiality on the positions of collinear equilibrium points $L_i (i=1,2,3)$ for the binary system FL virginis and Procyon (when the arbitrary values of coefficient of triaxility are considered).

| Binary system | Triaxiality | Location |
|---------------|-------------|----------|
| FL virginis   | $\sigma_1$  | $L_1$ | $L_2$ | $L_3$ |
|               | $\sigma_2$  |        |       |       |
|               | $\sigma_3$  |        |       |       |
|               | $\sigma_4$  |        |       |       |
| 0.004         | 0.02        | 0.003  | 0.011 | 1.187357 0.59363926 | -1.4208216 |
| 0.005         | 0.03        | 0.004  | 0.012 | 1.187463 0.53690786 | -1.4205792 |
| 0.0051        | 0.031       | 0.0041 | 0.0121| 1.187474 0.53661700 | -1.4205494 |
| 0.0052        | 0.032       | 0.0042 | 0.0122| 1.187484 0.53631497 | -1.4205306 |
| Procyon       | 0.04        | 0.02   | 0.003 | 0.011 1.441199 0.192524 | -1.2733849 |
|               | 0.005       | 0.03   | 0.004 | 0.012 1.441287 0.178198 | -1.2723137 |
|               | 0.0055      | 0.035  | 0.0045| 0.0125 1.441296 0.176627 | -1.2722063 |
|               | 0.0056      | 0.0356 | 0.0046| 0.0126 1.441305 0.175053 | -1.2720989 |

For FL virginis $M_\text{B} = 0.01, \mu = 0.4685, e = 0.3, a = 0.219$.

For Procyon $M_\text{B} = 0.01, \mu = 0.2874, e = 0.4, a = 0.714$. 

![Figure 1. The configuration of the problem.](image)

![Figure 2. Position of the test particle at collinear equilibrium point $L_1$ located after the smaller.](image)

![Figure 3. Position of the test particle at collinear equilibrium point $L_2$ lying between the bigger primary ($M_1$) and the smaller Primary ($M_2$) at a distance $\rho$ from $M_2$.](image)

![Figure 4. Position of the test particle at collinear equilibrium point $L_3$ lying after the Bigger.](image)
In this work we studied the locations and stability of collinear equilibrium points, when both primaries are triaxial and radiating with gravitational potential from the belt using two radiating binary stars (FL virginis and Procyon). FL virginis is a binary star system that consist of two red dwarf stars at a distance of approximately 14.2 light years away from the sun with moderate eccentric orbit while Procyon is the brightest star in the constellation minor and usually the eight brightest star in the night sky consisting of a white hued main-sequence star and a faint white star in the constellation minor and usually the eight brightest star in the night sky consisting of a white hued main-sequence star and a faint white dwarf companion. This paper is organized as follows: Section 1 is introduction, the equations of motion are presented in Section 2, we introduce computations relating to the location of equilibrium points and their stability together with the perturbations in Section 3, while in Section 4

Table 3. The effect of Gravitational potential from the belt on the position of collinear equilibrium points L(i = 1, 2, 3) for the binary system FL virginis and Procyon.

| Binary system | q1   | q2   | M_L | L1   | L2   | L3   |
|---------------|------|------|-----|------|------|------|
| FL virginis   | 0.999981 | 0.999983 | 0.01 | 1.187357 | 0.539839 | -1.4208216 |
|               | 0.012 | 1.187345 | 0.539213 | -1.4208201 |
|               | 0.012 | 1.187333 | 0.538581 | -1.4208186 |
|               | 0.012 | 1.187330 | 0.538455 | -1.4208183 |
| Procyon       | 0.9952 | 0.9999 | 0.01 | 1.441199 | 0.192524 | -1.2733849 |
|               | 0.001 | 1.441178 | 0.190772 | -1.2733801 |
|               | 0.012 | 1.441156 | 0.189048 | -1.2733530 |
|               | 0.012 | 1.441152 | 0.188707 | -1.2733743 |

For FL virginis M_L = 0.01, μ = 0.4685, ε = 0.3, a = 0.219.
For Procyon M_L = 0.01, μ = 0.2874, ε = 0.4, a = 0.714.

Table 4. The characteristic roots (λ_1, 2, λ_3, 4) of collinear points for the system FL virginis.

| L1       | λ_1, 2 | λ_3, 4 | Stability Behaviour |
|----------|--------|--------|---------------------|
| 1.187357 | ±0.772008 ±0.45379i | Unstable |
| 1.187463 | ±0.773183 ±1.45204i | Unstable |
| 1.187474 | ±0.773299 ±0.45187i | " |
| 1.187484 | ±0.773418 ±1.45168i | " |
| L2       | λ_1, 2 | λ_3, 4 | |
| 0.52316074 | (591.91) ±0.644039 0.797181i | Unstable |
| 0.526992139 | (15670.3-17545.3i) | " |
| 0.5263883 | (17810.2-20147.9i) | " |
| 0.52668503 | (20410.4-23337.2i) | " |
| L3       | λ_1, 2 | λ_3, 4 | |
| -1.4208216 | (74.8461-69.842i) | Unstable |
| -1.4205792 | (71.884-59.9976i) | " |
| -1.4205549 | (71.5622-58.9124i) | " |
| -1.4205306 | (71.2346-57.8039i) | " |

M_L = 0.01.

Table 5. The characteristic roots (λ_1, 2, λ_3, 4) of collinear points for the system Procyon.

| L1       | λ_1, 2 | λ_3, 4 | Stability Behaviour |
|----------|--------|--------|---------------------|
| 1.441199 | ±0.838287 ±1.357741 | Unstable |
| 1.441287 | ±0.839167 ±1.35494 | " |
| 1.441296 | ±0.839254 ±1.35461i | " |
| 1.441305 | ±0.839342 ±1.35432i | " |
| L2       | λ_1, 2 | λ_3, 4 | |
| 0.192524 | (0.699956-0.797181i) | Unstable |
| 0.178198 | (0.617039-0.746857i) | " |
| 0.176627 | (0.606967-0.741765i) | " |
| 0.175035 | (0.596454-0.736477i) | " |
| L3       | λ_1, 2 | λ_3, 4 | |
| -1.2733849 | (0.644039-1.53291i) | Unstable |
| -1.2723137 | (411.49-360.325i) | " |
| -1.2722063 | (406.36-351.777i) | " |
| -1.2720889 | (401.274-343.264i) | " |

M_L = 0.1.
we give the numerical explanation of our analysis, section 5 contains discussion and finally conclusion is in Section 6.

2. Equations of motion

The equations of motion of a test particle in the ER3BP when both primaries are triaxial and radiating with gravitational potential from the belt are presented here in dimensionless-pulsating coordinates system $\{\xi, \eta, \zeta\}$ as given in (Singh and Umar, 2012) as follows:

$$\dot{\xi} - 2\dot{\eta} = \Omega \xi$$
$$\eta + 2\dot{\zeta} = \Omega \eta$$

(1)

$$\Omega = \left(1 - e^2\right)^{-\frac{1}{2}} \left[\frac{1}{2} \left(\xi^2 + \eta^2\right) + \frac{1}{2} \left(1 - \mu\right) \frac{q q_{1}}{r_1} + \frac{1}{2} \left(1 - \mu\right) \frac{\left(2\sigma_1 - \sigma_2\right)}{2r_1^2} - \frac{1}{2} \left(1 - \mu\right) \left(\sigma_1 - \sigma_3\right) q_1 \eta^2}{2r_1^2} + \frac{\mu q_2}{r_2^2} + \frac{\mu \left(2\sigma_1 - \sigma_2\right) q_2}{2r_2^2} - \frac{3\mu \left(\sigma_1 - \sigma_3\right) q_2 \eta^2}{2r_2^5} + \frac{Mb}{\left(r^2 + T^2\right)^{\frac{3}{2}}}}\right]$$

(2)

$$r_1^2 = (\xi + \mu)^2 + \eta^2 + \zeta^2$$

$$r_2^2 = (\xi + \mu - 1)^2 + \eta^2 + \zeta^2$$

(3)

Mean motion:

In the two-body problem, the orbits of the bigger primary $m_1$ and the smaller primary $m_2$ with the semi-major axis $a = a_1 = a m_2$ and $a_2 = a m_1$, respectively, having the same eccentricity $e$. 

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![Figure 5](image1.png)

Figure 5. a) The effect of potential from the belt on the first collinear point (L1) of FL virginis. (b) The effect of potential from the belt on the second collinear point (L2) of FL virginis. (c) The effect of potential from the belt on the third collinear point (L3) of FL virginis.

![Figure 6](image2.png)

Figure 6. a) The effect of potential from the belt on the first collinear point (L1) of Procyon. (b) The effect of potential from the belt on the second collinear point (L2) of Procyon. (c) The effect of potential from the belt on the third collinear point (L3) of Procyon.

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Their equation, when they are triaxial rigid bodies surrounded by a belt, can be express as:

\[ n^2 a \left( \frac{1 + e^2}{1 - e^2} \right) = k^2 m_2 \left( 1 + \frac{3}{2} (2\sigma_1 - \sigma_2) + \frac{3}{2} (2\sigma_3 - \sigma_1) + \frac{k^2 \text{M}_{\text{bc}}}{(r_s^2 + T^2)^{3/2}} \right) \]

and

\[ n^2 a \left( \frac{1 + e^2}{1 - e^2} \right) = k^2 m_1 \left( 1 + \frac{3}{2} (2\sigma_1 - \sigma_2) + \frac{3}{2} (2\sigma_3 - \sigma_1) + \frac{k^2 \text{M}_{\text{bc}}}{(r_s^2 + T^2)^{3/2}} \right) \]

where \( k^2 \) is the gravitational constant. Adding these two equations and considering \( m_1, m_2 = 1, k^2 = 1 \) we obtain:

\[ n^2 a \left( \frac{1 + e^2}{1 - e^2} \right) = 1 + \frac{3}{2} (2\sigma_1 - \sigma_2) + \frac{3}{2} (2\sigma_3 - \sigma_1) + \frac{2M_{\text{bc}}}{(r_s^2 + T^2)^{3/2}} \]

Neglecting the higher powers of \( e^2 \) and its product with \( M_0 \) and \( \sigma_i \) (i=1,2,3,4) we have:

\[ n^2 \frac{a}{1 + \frac{3}{2} (2\sigma_1 - \sigma_2) + \frac{3}{2} (2\sigma_3 - \sigma_1) + \frac{2M_{\text{bc}}}{(r_s^2 + T^2)^{3/2}}} = \frac{\sigma_i^3 - \sigma^2}{5\sigma_i^3} = \frac{\sigma_i^3 - \sigma^2}{5\sigma_i^3} = \frac{\sigma_i^3 - \sigma^2}{5\sigma_i^3}, \sigma_i = \frac{\sigma_i^3 - \sigma^2}{5\sigma_i^3}, \mu = \frac{M_2}{M_1 + M_2} \leq \frac{1}{2}, \sigma_i \ll 1, \langle i = 1, 2, 3, 4 \rangle \]

where \( \mu \) is the mass parameter, \( n \) is the mean motion of the primaries, \( r_1 \) and \( r_2 \) represent distances of the third body from the primaries, \( \sigma_1 \) and \( \sigma_2 \) denote the triaxiality of the bigger primary, while \( \sigma_3 \) and \( \sigma_4 \) denote the triaxiality of the smaller primary.

The lengths of the axes are denoted by \( a, b, c \) for the bigger primary and \( a', b', c' \) for the smaller primary, \( r_1 (i=1,2) \) are the distances of the infinitesimal mass from the bigger and smaller primaries respectively, while \( q_i \) is the radiation factor of the bigger primary, \( q_2 \) is the radiation factor of the smaller primary, \( \rho \) is the semi-major axis of the orbits of the primaries and \( e \) is the eccentricity. \( M_{\text{bc}} \ll 1 \) is the total mass of the belt, \( r \) is the radial distance of the infinitesimal mass given by \( r^2 = x^2 + y^2, T = A + B.A \) and \( B \) are the parameters which determine the density profile of the belt (Miyamoto and Nagai, 1975; Jiang and Yeh, 2003; Kushvah, 2008). The parameter \( B \) controls the size of the core of the density profile and it is known as the core parameter while \( r_s \) is the radial distance of the infinitesimal body through the collar libration points in the classical R3BP. The configuration of the system shows both primaries being surrounded by circumbinary disc in the \( \xi-n \) plane (Figure 1).

3. Positions of the collar libration points

To obtain the collar libration points, we obtain the first partial derivatives of Eq. (2) with respect to \( \xi \) and \( \eta \) respectively, and equate them to zero. That is, \( \Omega_\xi = \Omega_\eta = 0 \).

\[ \Omega_\xi = (1 - e^2)^{-1/2} \left[ 1 - \frac{1}{n^2} \left( (1 - \mu)(\xi + \mu)q_1 \frac{1}{r_1^2} + 3(1 - \mu)(\xi + \mu)(2\sigma_1 - \sigma_2)q_1 \frac{1}{2r_1^2} \right) - \frac{15(1 - \mu)(\xi + \mu)(\sigma_1 - \sigma_2)q_1 r_1}{2r_1^2} + \frac{\mu(\xi + \mu - 1)q_2}{r_1^2} + 3\mu(\xi + \mu - 1)(\sigma_1 - \sigma_2)q_2 \frac{1}{2r_2^2} - \frac{15\mu(\xi + \mu - 1)(\sigma_1 - \sigma_2)q_2 r_2}{2r_2^2} \right] = 0 \]

\[ \Omega_\eta = (1 - e^2)^{-1/2} \left[ 1 - \frac{1}{n^2} \left( (1 - \mu)q_1 \frac{1}{r_1^2} + 3(1 - \mu)(2\sigma_1 - \sigma_2)q_1 \frac{1}{2r_1^2} \right) - \frac{15(1 - \mu)(\sigma_1 - \sigma_2)q_1 r_1}{2r_1^2} + \frac{\mu(\sigma_1 - \sigma_2)q_2}{r_1^2} + 3\mu(\sigma_1 - \sigma_2)q_2 \frac{1}{2r_2^2} - \frac{15\mu(\sigma_1 - \sigma_2)q_2 r_2}{2r_2^2} \right] = 0 \]

\[ (5) \]

Since the collar libration points lies on the \( \xi - \) axis, it implies that \( \eta = 0 \), using Eq. (3), the first equation of (5) reduces to

\[ n^2 \xi - \left( \frac{1 - \mu}{\xi + \mu} \right) q_1 \frac{1}{r_1^2} - \frac{3(1 - \mu)(\xi + \mu)(2\sigma_1 - \sigma_2)q_1}{2r_1^2} - \frac{\mu(\xi + \mu - 1)q_2}{r_1^2} + \frac{3\mu(\sigma_1 - \sigma_2)q_2}{r_1^2} \right] = 0 \]

\[ (6) \]

The distance between the primaries is unity, therefore:

\[ \xi_2 - \xi_1 = 1, \mu_2 = -\mu, \xi_2 = 1 - \mu \]

Rewriting Eq. (6) using Eq. (7) we obtain:

\[ n^2 \xi - \left( \frac{1 - \mu}{\xi + \mu} \right) q_1 \frac{3(1 - \mu)(2\sigma_1 - \sigma_2)q_1}{2(\xi + \mu)^2} + \xi_2 \right] = 0 \]

\[ (8) \]

In order to locate the collar points \( L_1, L_2, L_3 \) we divide the orbital plane into three parts \( \xi > \xi_2, \xi_1 < \xi < \xi_2 \) and \( \xi_2 < \xi \) with respect to the primaries.

3.1. Position of \( L_1 (\xi > \xi_2) \)

Let the collar libration point \( L_1 \) be on the right hand side of the smaller primary at a distance \( \rho \) from it on the \( \xi \) axis.

Primary \( (M_2) \) at a distance \( \rho \) from \( M_2 \).

In the interval \( (\xi > \xi_2) \) we let \( \xi_2 = \rho_1, \xi_1 = 1 + \rho \Rightarrow \xi = 1 + \rho + \xi_2 - \xi_1 = 1 \)

\[ \Rightarrow \xi = 1 + \rho - \mu, \xi_2 = 1 + \rho, \xi_3 = \rho \]

Thus by substituting Eq. (9) in Eq. (8) we have:

\[ 2n^2 \rho(1 - \mu)q_1 + 3(1 - \mu)q_1 r_2 - 2(1 - \mu)(1 + \rho)^2 q_1 \rho^2 - 3(1 - \mu)(2\sigma_1 - \sigma_2)q_1 \rho^2 \frac{M_{\text{bc}}}{(r_s^2 + T^2)^{3/2}} = 0 \]

\[ (10) \]
After expansion we obtain:

\[
2n^2\rho^9 + 2n^2(5 - \mu)^9 + 2n^2(2(5 - 2\mu))\rho^9 + 2n^2(5 - 3\rho) - (q_1 - \rho_1 - \mu q_1)\rho^9
\]

\[
+ \left(2n^2(5 - 4\rho) - (q_1 - \rho_1 - \mu q_1)\rho^9 + 2n^2(1 - \mu - 2\rho - q_1 + \rho_2 - 6\mu q_1)\rho^9
\]

\[
- 3(2\sigma, q_1 - \sigma, q_1) + 3(2\mu_\sigma, \mu_\sigma, \mu_\sigma, q_2) - 3(2\mu_\sigma, \mu_\sigma, q_2)\rho^9
\]

\[
- 4(\mu_\sigma + 6\mu_\sigma_2 - 3\mu_\sigma q_2)\rho^9 - 2(\mu_2 + 18\mu_\sigma_2 - 9\mu_\sigma q_2)\rho^9
\]

\[
- 3(8\mu_\sigma q_2 - 4\mu_\sigma_2 q_2)\rho - 3(2\mu - \mu_\sigma)q_2 + \frac{Mb(1 + \rho - \mu)}{r^2 + T^2} \right)^{3/2} = 0
\]

(11)

3.2. Position of \( L_2 (\xi < \xi_2) \)

Let the collinear libration point \( L_2 \) be on the left hand side of the smaller primary at a distance \( \rho \) from it on the \( \xi \)-axis.

In the interval \( L_2 (\xi < \xi_2) \), we get

\[
\xi_1 - \xi = \rho; \xi_2 - \xi = 1 - \rho \Rightarrow \xi = 1 - \rho - \mu \text{ and } r_1 = 1 - \rho, \; r_2 = \rho
\]

Using Eq. (12) in Eq. (8) we get:

\[
2n^2(1 - \rho - \mu)(1 + \rho)^2 + 2n^2(1 - \rho - \mu)^2 q_1 - 2n^2(1 - \rho - \mu)^2 q_1
\]

\[
+ 2n^2(1 - \rho - \mu)^2 q_1 - 3n^2(\sigma_1 - \sigma_2)q_1
\]

\[
+ 2n^2(1 - \rho - \mu)^2 q_1 - 3n^2(\sigma_1 - \sigma_2)q_1
\]

\[
\frac{Mb(1 + \rho - \mu)}{r^2 + T^2} \right)^{3/2} = 0
\]

(13)

Hence, expanding Eq. (13) we obtain:

\[
-2n^2\rho^9 + 2n^2(5 - \mu)^9 - 2n^2(2(5 - 2\mu))\rho^9 + 2n^2(5 - 3\rho) - (q_1 - \rho_1 - \mu q_1)\rho^9
\]

\[
- (q_1 - \rho_1 - \mu q_1)\rho^9 + \left(-2n^2(5 - 4\rho) - 4(q_1 - \rho_1 - 2q_1 - 2\mu q_1)\rho^9
\]

\[
+ (2n^2(1 - \mu) - 2q_1 - 2q_1 - 6\mu_q_1) - 2n^2\sigma_1 - \sigma_2 q_1) + 2n^2\sigma_1 q_1 - \mu_\sigma q_2
\]

\[
+ 3(2\sigma_1 q_2 - \mu_\sigma q_2)\rho^9 - 4(2\mu_\sigma q_2 + 6\mu_\sigma q_2 - 3\mu_\sigma q_2)\rho^9
\]

\[
+ (2\mu_\sigma q_2 + 18\mu_\sigma q_2 - 9\mu_\sigma q_2)\rho^9
\]

\[
\frac{Mb(1 + \rho - \mu)}{r^2 + T^2} \right)^{3/2} = 0
\]

(14)

3.3. Position of \( L_3 (\xi_1 > \xi) \)

Let the collinear libration point \( L_3 \) be on the left hand side of the bigger primary at a distance \( 1 - \rho \) from it on the \( \xi \)-axis.

Primary \( (M_4) \) at a distance \( 1 - \rho \) from \( M_4 \).

Finally, in the interval \((\xi_1 > \xi)\) we get

\[
\xi_1 - \xi = 1 - \rho; \xi_2 - \xi = 2 - \rho, \text{ and } r_1 = 1 - \rho; r_2 = 2 - \rho, \xi
\]

\[
- 1 - \mu + \rho
\]

Using Eq. (15) in Eq. (8) we get:

\[
2n^2(\rho - 1 - \mu)^2 q_1 + 2n^2(1 - \rho - \mu)^2 q_1
\]

\[
+ 3(1 - \mu)(2\sigma_1 - \sigma_2)(2 - \rho)^4 q_1 + 2n^2(1 - \mu)^2 q_1
\]

\[
- 3n(2\sigma_1 - \sigma_2)(1 - \rho)^4 q_1 + \frac{Mb(1 + \rho - \mu)}{r^2 + T^2} \right)^{3/2} = 0
\]

Expanding Eq. (16) we get:

\[
+ 2n^2\rho^9 + 2n^2(1 - \mu)^9 + 2n^2(2(37 + 6\mu))^9 + 2(1 - 2(121 - 31\mu)
\]

\[
+ 2n^2(501 - 180\mu) - 24(401 - 54\mu) + 2\sigma_1 - q_1 - \mu q_1
\]

\[
- 2\sigma_1 q_1 - \mu q_1 + 2\mu_\sigma q_1 - \mu q_1)\rho^9 + (2n^2(680 + 360\mu)
\]

\[
- 88(2\sigma_1 q_1 - 2\mu_\sigma q_2 + 12 - 4\rho_1 q_1, 2\rho_1 q_1, 4\rho_1 q_1, 2\mu_\sigma q_2
\]

\[
- 2\sigma_1 q_1 - \mu q_1 + 2\mu_\sigma q_1 - \mu q_1)\rho^9 + (2n^2(112 + 96\mu)
\]

\[
- 8(1 - 6\rho_1 q_1, 16 \rho_1 q_1, 24 \rho_1 q_1, 12 \rho_1 q_1
\]

\[
+ 12 \rho_1 q_1 - 5 \rho_1 q_1 - 3 \rho_1 q_2) - 12 \rho_1 q_2)\rho^9 + (2n^2(16 + 16\mu)
\]

\[
+ 8(4 \rho_1 q_1 - 4 \rho_1 q_1, 16 \rho_1 q_2, 6 \rho_1 q_2, 24 \rho_1 q_2
\]

\[
+ (2n^2(2 - 3 \rho_1 q_2 - 2 \rho_1 q_2) = 0
\]

(17)

Eqs. (11), (14), and (17) are 9th degree equations with the root \( \rho \). We shall further solve Eqs. (11), (14), and (17) numerically for the real values of \( \rho \). Then using its values we shall find the positions of \( L_{1,2,3} \).

3.4. Stability of the collinear equilibrium points

We use the characteristic equation of the system as given by Singh and Umar (2012), to determine the stability of the collinear libration point \( L_i \) (\( i=1,2,3 \)) which is:

\[
\lambda^4 - (\Omega_0^2 + \Omega_0^2 - 4)\lambda^2 + \Omega_0^2 \Omega_0^2 - (\Omega_0^2)^2 = 0
\]

(18)

The second partial derivative of Eq. (2), with \( \eta = 0 \) can be written as:

\[
\Omega_0^2 = (1 - \epsilon)^{-\frac{1}{2}} \left[ 1 + \frac{2}{\mu_2} \left( \frac{\mu_2}{|\xi + \mu|} + \frac{3(1 - \mu)(2\sigma_1 - \sigma_2)}{|\xi + \mu|^3} \right) + \frac{\mu_2}{|\xi + \mu - 1|} \right]
\]

(19)

It is obvious that

\[
\Omega_0^2 > 0
\]

(20)

3.4.1. Stability of collinear point \( L_i \) in the interval \( \xi > \xi_2 \)

In the first interval we have \( \xi > \xi_2 \)

\[
r_1 = (\xi + \mu) \Rightarrow \xi = (r_1 - \mu) \text{ and } r_2 = (\xi + \mu - 1)
\]

Substituting Eq. (20) in Eq. (6), we get:

\[
\frac{(1 - \mu)}{r^2} = r^2 \frac{3(1 - \mu)(2\sigma_1 - \sigma_2)q_1}{2r^2} \frac{\mu q_2}{r^2} \frac{3(2\sigma_1 - \sigma_2)q_1}{2r^2} + \frac{Mb(r_1 - \mu)}{(r_1^2 + T^2)} \right)^{3/2}
\]

Using Eq. (21) in the second equation of (19), we obtain

(22)
The effect of gravitational potential from the belt (\(M_b\)) on the collinear equilibrium points of Procyon is the same with its effect on the collinear points of FL Virginis even though the orbital properties are different. As shown in Figure 5a, L1 slopes downwards towards the x-axis in the negative direction so also is L2 in Figure 6b. In the case of L3 (Figure 6c) the negative slope is towards x-axis in the positive direction showing movement of the particle towards the bigger primary or barycentre (see Figures 2, 3, and 4) for clarification. The inverse relationship existing between the collinear points of the primaries (FL Virginis and Procyon) and \(M_b\) lead to changes from their locations with any increase of \(M_b\). It is worth to note that we adopted the same procedure used in obtaining the graphs in section 4.1 in generating the graphs in section 4.2, using the same values of triaxiality and \(M_b\) but different orbital properties – we use orbital properties of Procyon.

5. Discussion

In Table 2 we have used Eqs. (11), (14), and (17) to compute the positions of the collinear equilibrium points for the increasing values of triaxiality of the two binaries (FL Virginis and Procyon) with the aid of the software MATHEMATICA. The triaxiality factors \(\sigma_i < 1\) (i=1,2,3,4) are small quantities therefore we choose arbitrary values within the range (0 < 0.011 < 0.0356) for each of the triaxiality factors \(\sigma_i\) (i=1,2,3,4) in Table 2. We can see from Table 2, that the positions of collinear points are affected by the increase of triaxiality. In the case of FL Virginis as the triaxial effect increases the position of \(L_1\) shifts away from the smaller primary while \(L_2\) shifts towards the bigger primary, and \(L_3\) which lies close to the more massive primary shifts towards it. In the case of Procyon \(L_1\) shows the same behavior with \(L_1\) of FL Virginis as it shifts away from the smaller primary, also in this case \(L_2\) moves towards the bigger primary, while \(L_3\) shifts towards the bigger primary or the barycentre. In Table 3, we examine the dynamical effect of the belt on the collinear points \(L_{1,2,3}\) of FL Virginis by increasing the values of the belt. The increase moves \(L_1\) towards the smaller primary, \(L_2\) towards the larger primary i.e towards the barycentre while the third collinear point \(L_3\) moves closer to the bigger primary and for Procyon as can be seen in Table 3, the effect of the increase cause \(L_1\) to shift towards the smaller primary, \(L_2\) moves towards the larger primary or the barycentre and \(L_3\) moves towards the bigger primary.

The stability of collinear equilibrium points are obtained by substituting Eqs. (19) and (20) in (18). The characteristic roots obtained are shown in Tables 4 and 5 for the systems FL Virginis and Procyon. The characteristic roots obtained in Tables 4 and 5 using the eccentricity, semi-major axis and radiation of the two binary systems with arbitrarily chosen triaxial values are either positive, negative or complex.
presence of positive real parts in the roots shows that the collinear libration points are unstable. This instability behaviour agrees with those of (Singh and Umar, 2014; Kumar and Narayan, 2012; Singh and Tokyaa, 2017). We have also shown geometrically, the effect on the belt on the positions of collinear points (Figures 5a,5b,5c and 6a,6b and 6c) of the binaries where the graphs show a downward slope. Hence an increase in \( M_b \) will shift any of the collinear points towards the barycentre. The implication of this will result in \( L_1 \) shifting towards the smaller primary, \( L_2 \) will also shift towards the smaller primary and \( L_3 \) will shift towards the more massive primary (see Figures 2, 3, and 4) for confirmation.

6. Conclusion

We have examined the positions and stability of collinear equilibrium points in the elliptic restricted three-body problem under the influence of triaxiality, radiation and gravitational potential from the belt. We found that three collinear points \((L_{1,2,3})\) exist and their position and linear triaxiality, radiation and gravitational potential from the belt. We found points in the elliptic restricted three body problem under the influence of triaxiality, radiation and gravitational potential from the belt. The implication of this will result in \( L_1 \) shifting towards the smaller primary, \( L_2 \) will also shift towards the smaller primary and \( L_3 \) will shift towards the more massive primary (see Figures 2, 3, and 4) for confirmation.

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Data availability statement

Data will be made available on request.

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Additional information

No additional information is available for this paper.

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