Uniqueness of Positive Solutions of the
Equation \( \Delta_g u + c_n u = c_n u^{\frac{n+2}{n-2}} \) and
Applications to Conformal Transformations

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Abstract

We study uniqueness of positive solutions to the equation \( \Delta_g u + c_n u = c_n u^{\frac{n+2}{n-2}} \) on complete Riemannian manifolds. We apply the results to show that conformal transformations on certain complete Riemannian manifolds of constant negative scalar curvature are isometries. We also study uniqueness of complete positive solutions and radial solutions.

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1. Introduction

In this paper we study uniqueness of positive solutions to the equation

\[
\Delta_g u + c_n u = c_n u^{\frac{n+2}{n-2}},
\]

where \( c_n = (n-2)/[4(n-1)] \) and \( \Delta_g \) is the Laplacian operator for a Riemannian manifold \((M, g)\). Equation (1.1) arises as in the conformal deformation of a Riemannian metric into constant negative scalar curvature. Let \( M \) be an open \( n \)-manifold with \( n \geq 3 \). Bland and Kalka \cite{bland-kalka} show that there exists a complete Riemannian metric \( g \) on \( M \) with scalar curvature equal to \( -1 \). Let \( u \) be a positive smooth function on \( M \). If the conformal metric \( g_c = u^{\frac{4}{n-2}} g \) has constant scalar curvature equal to \( -1 \), then \( u \) satisfies equation (1.1). In this paper we study when a positive solution to equation (1.1) is unique, that is, whether \( u \equiv 1 \) on \( M \).

Let \((M, g_c)\) be a complete non-compact Riemannian \( n \)-manifold with scalar curvature equal to \( -1 \).
curvature $R_{g_o}$. Aviles and McOwen study when there exists a positive smooth function $v$ on $M$ such that the conformal metric $g = v^{\frac{4}{n-2}} g_o$ has scalar curvature equal to $-1$. In this case $v$ satisfies the equation

$$\Delta_{g_o} v - c_n R_{g_o} v = c_n v^{\frac{n+2}{n-2}},$$

where $R_{g_o}$ is the scalar curvature of $(M, g_o)$. The question whether a positive solution to equation (1.2) is unique is equivalent to the uniqueness of a positive solution to equation (1.1). For if $w$ is a smooth function on $M$ which satisfies equation (1.2), then $u = w/v$ satisfies the equation (1.1). Therefore $u \equiv 1$ if and only if $v \equiv w$ on $M$.

Equation (1.1) is a semilinear elliptic equation involving the critical Sobolev exponent. For open domains in $\mathbb{R}^n$, semilinear elliptic equations which involve critical Sobolev exponents are studied by Brezis and Nirenberg, Gidas, Ni and Nirenberg, Peletier and Serrin, Ni and Serrin and Zhang. In case $M$ is a compact manifold without boundary, the maximum principle implies that equation (1.1) has only one positive solution $u \equiv 1$. If $M$ is an open manifold, then equation (1.1) may have more than one positive solution. An example can be given on the open unit ball on $\mathbb{R}^n$ with the hyperbolic metric. However if the solution is strictly positive we have the following result.

**Theorem A.** For an integer $n \geq 3$, let $(M, g)$ be a complete Riemannian $n$-manifold with scalar curvature equal to $-1$. Assume that the sectional curvature of $(M, g)$ is bounded from below and the injectivity radius of $(M, g)$ is positive. If $u \in C^\infty(M)$ is a positive solution to equation (1.1) with $\inf_{x \in M} u(x) > 0$, then $u \equiv 1$ on $M$.

Theorem A has nice applications to conformal transformations. Let $F : (M, g) \to (M, g)$ be a conformal transformation, where the scalar curvature of $(M, g)$ is equal to $-1$. Let $u = |F'|^{\frac{4-n}{2}}$, where $|F'|$ is the linear stretch factor. Then $u$ satisfies equation (1.1). Therefore uniqueness of positive solutions to (1.1) implies that $F$ is an isometry. Using theorem A we can prove the following result.

**Theorem B.** For an integer $n \geq 3$, let $(M, g)$ be a complete non-compact Riemannian $n$-manifold with constant negative scalar curvature. Assume that the sectional curvature of $(M, g)$ is bounded from below and the injectivity radius of $(M, g)$ is positive. Then any conformal transformation $F : (M, g) \to (M, g)$ is an isometry.

Kuiper proves that any conformal transformation on a complete Einstein manifold with negative scalar curvature is an isometry. Lafontaine shows that any
conformal transformation of $S^n \times \mathbb{H}^m$ ($m \geq 2$) is an isometry. As a result of theorem B, we can show that any conformal transformation of a locally symmetric space of non-compact type with dimension bigger than two is an isometry.

The assumption that $u$ is bounded away from zero may not be natural in the general content. But there may have more than one positive solution to equation (1.1) if we take away the assumption. For example, for any constant $b \geq 1$,

$$u_b(x) = \left( \frac{b(1 - |x|^2)}{b^2 - |x|^2} \right)^{\frac{n-2}{2}} \text{ for } x \in B^n$$

is a positive solution to equation (1.1) on the (scaled) hyperbolic space. A more natural assumption is that the conformal metric $g_c = u^{\frac{4}{n-2}}g$ is complete. We study the uniqueness of a positive radial solutions to equation (1.1) under this assumption. For an integer $n \geq 3$, let $(r, \Theta)$ be the polar coordinates on $\mathbb{R}^n$, where $r \geq 0$ and $\Theta \in S^{n-1}$. We consider warped product metrics on $\mathbb{R}^+ \times S^{n-1}$. Let $f$ be a smooth function on $[0, \infty)$ with $f(0) = 0$, $f'(0) = 1$ and $f > 0$ on $(0, \infty)$. The Riemannian metric

(1.3) $$g = dr^2 + f^2(r) d\Theta^2$$

is complete on $\mathbb{R}^n \cong \mathbb{R}^+ \times S^{n-1}$. The hyperbolic space is corresponding to

$$f(r) = \sinh r \text{ for } r \geq 0,$$

while the Euclidean space with the standard metric is corresponding to $f(r) = r$ for all $r \geq 0$. Let $u$ be a solution to equation (1.1) with $g$ of the form (1.3). We say that $u$ is a radial solution if $u$ is a function of $r$ only (independent on $\Theta$).

**Theorem C.** Let $u$ be a positive radial $C^2$-solution to equation (1.1) on

$$(\mathbb{R}^n, g = dr^2 + f^2(r) d\Theta^2).$$

Assume that $\lim_{r \to \infty} f(r) = \infty$ and there exist positive constants $R_o$ and $C$ such that $|f'(r)/f(r)| \leq C$ for all $r > R_o$. If the conformal metric $g_c = u^{\frac{4}{n-2}}g$ is complete, then $u \equiv 1$.

For a positive solution $v$ to equation (1.1) that may not be radial, we show that if there exists a $\delta > 0$ such that $v \leq 1 - \delta$ outside a compact set, then the conformal metric $g_c = u^{\frac{4}{n-2}}g$ is not complete. We also show that if $u$ is a positive solution to equation (1.1) on the (scaled) hyperbolic space and $u \not\equiv 1$, then $u < 1$ everywhere.
2. Bounded Solutions

Throughout this paper we let \((M, g)\) be a complete Riemannian \(n\)-manifold with \(n \geq 3\). Let \(\text{Ric}_g\) be the Ricci curvature of \((M, g)\).

**Lemma 2.1.** Assume that \(\text{Ric}_g \geq -c^2 g\) on \(M\) for some positive constant \(c\). Let \(u \in C^\infty(M)\) be a positive solution to (1.1). If \(\inf_{x \in M} u(x) > 0\) and \(\sup_{x \in M} u(x) < \infty\), then \(u \equiv 1\) on \(M\).

*Proof.* Suppose that there exists a point \(x_0 \in M\) such that \(u(x_0) > 1\). Then we have

\[
1 < \sup_{x \in M} u(x) < \infty.
\]

By the Omori-Yau maximum principle (c.f. [16]), there exists a sequence \(\{x_k\}_{k \in \mathbb{N}}\) such that

\[
\lim_{k \to \infty} u(x_k) = \sup_{x \in M} u(x), \quad |\nabla u(x_k)| \leq \frac{1}{k}
\]

and

\[
(2.2) \quad \Delta_g u(x_k) \leq \frac{1}{k}.
\]

As \(\sup_{x \in M} u(x) > 1\), there exists a number \(\epsilon > 0\) and \(N \in \mathbb{N}\) such that

\[
c_n \left( u^{\frac{n}{n-2}}(x_k) - u(x_k) \right) > \epsilon \quad \text{for all } k > N.
\]

Therefore

\[
\Delta_g u(x_k) = c_n \left( u^{\frac{n}{n-2}}(x_k) - u(x_k) \right) > \epsilon \quad \text{for all } k > N,
\]

which contradicts (2.2). Therefore \(u \leq 1\) on \(M\). Suppose that there exists a point \(y_0 \in M\) such that \(u(y_0) < 1\). Let \(v = 1 - u\) and \(a = \inf_{x \in M} u(x)\). We have \(0 < a < 1\). Then \(0 \leq v \leq 1 - a < 1\). By the Omori-Yau maximum principle, there exists a sequence \(\{y_k\}_{k \in \mathbb{N}}\) such that

\[
\lim_{k \to \infty} v(y_k) = \sup_{y \in M} v(y), \quad |\nabla v(y_k)| \leq \frac{1}{k}
\]

and

\[
(2.3) \quad \Delta_g v(y_k) \leq \frac{1}{k}.
\]

As \(1 > \sup_{y \in M} v(y) > 0\), there exists a number \(\delta > 0\) and \(N' \in \mathbb{N}\) such that

\[
\Delta_g v(y_k) = c_n \left( u(y_k) - u^{\frac{n}{n-2}}(y_k) \right) > \delta \quad \text{for all } k > N',
\]
which contradicts (2.3). Hence $u \equiv 1$ on $M$. 

**Theorem 2.4.** Let $(M, g)$ be a complete non-compact Riemannian manifold with scalar curvature equal to $-1$. Assume that the sectional curvature of $(M, g)$ is bounded from below and the injectivity radius of $(M, g)$ is positive. Let $u \in C^\infty(M)$ be a positive solution to equation (1.1). If $\inf_{x \in M} u(x) > 0$, then $u \equiv 1$ on $M$.

**Proof.** By lemma 2.1 we just need to show that $\sup_{x \in M} u(x) < \infty$. Let $i_o$ be the injectivity radius of $(M, g)$. By the assumption of the theorem, $i_o > 0$. Let $x \in M$ and $\chi \in C^\infty_0(M)$ be a non-negative function which is zero outside $B_x(i_o)$. As in [2], we multiple both sides of (1.1) by $\chi^n u$ and then integrate by parts, we obtain

$$c_n \int_M \chi^n u \frac{2n}{n-2} \, dv_g$$

$$\leq - \int_M \chi^n |\nabla u|^2 \, dv_g - n \int_M u \chi^{n-1} (\nabla \chi \cdot \nabla u) \, dv_g + c_n \int_M \chi^n u^2 \, dv_g.$$  

We have

$$-n \, u \chi^{n-1} (\nabla \chi \cdot \nabla u) = -2 \left[ \frac{n}{2} (\chi)^{\frac{n}{2}-1} u \nabla \chi \right] \cdot \left[ (\chi)^{\frac{n}{2}} \nabla u \right]$$

$$\leq \frac{n^2}{4} u^2 \chi^{n-2} |\nabla \chi|^2 + \chi^n |\nabla u|^2,$$

where the dot product and the norms are with respect to the Riemannian metric $g$.

From (2.5) we have

$$c_n \int_M \chi^n u \frac{2n}{n-2} \, dv_g \leq \frac{n^2}{4} \int_M u^2 \chi^{n-2} |\nabla \chi|^2 \, dv_g + c_n \int_M \chi^n u^2 \, dv_g.$$  

Using Young’s inequality we have

$$u^2 \chi^{n-2} |\nabla \chi|^2 \leq \frac{(n-2)\epsilon}{n} u^{\frac{2n}{n-2}} \chi^n + \frac{2\epsilon^{-\frac{n-2}{2}}}{n} |\nabla \chi|^n$$

and

$$\chi^n u^2 \leq \frac{(n-2)\epsilon}{n} u^{\frac{2n}{n-2}} \chi^n + \frac{2\epsilon^{-\frac{n-2}{2}}}{n} \chi^n ,$$

where $\epsilon$ is a positive number. If we choose $\epsilon$ to be small, then there exists a positive constant $C(n)$ depending on $n$ only such that

$$\int_M \chi^n u^{\frac{2n}{n-2}} \, dv_g \leq C(n) \int_M (|\nabla \chi|^n + \chi^n) \, dv_g.$$  

If we choose $\chi$ to be equal to one on $B_x(i_o/2)$ and zero outside $B_x(i_o)$, with $|\nabla \chi| \leq 10/i_o$, we have

$$\int_{B_x(i_o/2)} u^{\frac{2n}{n-2}} \, dv_g \leq C(n, i_o) \, \text{Vol} \, B_x(i_o).$$

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As the sectional curvature is bounded from below and the scalar curvature is equal to \(-1\), the sectional curvature is bounded from above as well. We have \(\text{Vol } B_x(i_o) \leq C(i_o)\), where \(C(i_o)\) is a positive constant depending on \(i_o\) and the upper bound for sectional curvature only \[6\]. Thus

\[
\int_{B_x(i_o)} u^{\frac{2n}{n-2}} \, dv_g \leq C
\]

and the positive constant \(C\) is independent on \(x\) and \(u\). On \((M, g)\) we have the Sobolev inequality for compactly supported functions \[1\], and \(u\) is a positive smooth function which satisfy

\[
\Delta_g u \geq -c_n u \quad \text{on } M.
\]

By using a result of Li-Schoen \[13\] and the De Giorgi-Nash-Moser regularity theory as in \[12\], we conclude that \(u\) is bounded from above. \(\Box\)

3. Conformal Transformations

**Theorem 3.1.** Let \((M, g)\) be a complete non-compact Riemannian manifold with constant negative scalar curvature. Assume that the sectional curvature of \((M, g)\) is bounded from below and the injectivity radius of \((M, g)\) is positive. Then any conformal transformation \(F : (M, g) \to (M, g)\) is an isometry.

**Proof.** By a scaling we may assume that the scalar curvature is equal to \(-1\). Let \(u = |F'|^{\frac{n}{n-2}}\), where \(|F'|\) is defined by the equation

\[
F^* g(y) = |F'(x)|^2 g(x) \quad \text{with } y = F(x).
\]

Then \(u\) satisfies the equation (c.f. \[17\])

\[
(3.2) \quad \Delta_g u(x) + c_n u(x) = c_n u^{\frac{n+2}{n-2}}(x) \quad \text{for all } x \in M.
\]

By the proof of theorem 2.4, \(u\) is bounded from above. On the other hand

\[
F^{-1} : (M, g) \to (M, g)
\]

is also a conformal diffeomorphism and

\[
(F^{-1})^* g(x) = \frac{1}{|F'(x)|^2} g(y) \quad \text{with } x = F^{-1}(y).
\]

Thus \(1/u\) also satisfies the equation

\[
\Delta_g \left( \frac{1}{u(y)} \right) + c_n \left( \frac{1}{u(y)} \right) = c_n \left( \frac{1}{u(y)} \right)^{\frac{n+2}{n-2}} \quad \text{for all } y \in M.
\]
By the proof of theorem 2.4, $1/u$ is bounded from above. Therefore $u$ is bounded away from zero. By lemma 2.1 we have $u \equiv 1$ on $M$. Therefore $F$ is an isometry.  

**Corollary 3.3.** Let $(M, g)$ be a complete Riemannian manifold with non-positive sectional curvature and constant negative scalar curvature. Then any conformal transformation $F : (M, g) \to (M, g)$ is an isometry.

**Proof.** By a scaling we may assume that the scalar curvature is equal to $-1$. Let $(\tilde{M}, \tilde{g})$ be the universal covering of $M$ with the pull-back metric $\tilde{g}$. As the scalar curvature of $\tilde{g}$ is equal to $-1$ and the sectional curvature is non-positive, therefore the sectional curvature is bounded from below. Furthermore, by the Cartan-Hadamard theorem [3], $(\tilde{M}, \tilde{g})$ has infinite injectivity radius. By pulling back the conformal transformation $F$ to a conformal transformation $\tilde{F} : \tilde{M} \to \tilde{M}$ and apply theorem 3.1, $\tilde{F}$ is an isometry. Therefore $F$ is also an isometry.  

It follows that a conformal transformation of a locally symmetric space of non-compact type is an isometry.

4. Complete Solutions

We remark that without the assumption $\inf_{x \in M} u(x) > 0$, equation (1.1) may have more than one positive solution. For example, let $n \geq 3$ be an integer and $\delta_{ij}$ be the standard Euclidean metric and

$$h_{ij}(x) = \frac{4n(n-1)}{(1 - |x|^2)^2} \delta_{ij} \quad \text{for} \quad x \in B^n$$

be the (scaled) Poincaré metric on the open unit ball $B^n$ of $\mathbb{R}^n$, scaled by the positive number $n(n-1)$ so that the scalar curvature of $h$ is $-1$. The (scaled) hyperbolic space $\mathbb{H}^n = (B^n, h)$ is conformal to the standard Euclidean metric $\delta_{ij}$ on $B^n$. Let $v$ be a positive smooth function and $g_c = v^{\frac{4}{n-2}}g$. For any smooth function $u$ we have

$$(\Delta_g - c_n R_g)(uv) = v^{\frac{4}{n-2}}(\Delta_{g_c} - c_n R_{g_c})u$$

(see, for example, [18]). Here $R_g$ and $R_{g_c}$ are the scalar curvature of $g$ and $g_c$ respectively. It follows that if $u$ is a positive solution to the equation

$$(\Delta_{g_c} - c_n R_{g_c})u = c_n u^{\frac{n+2}{n-2}},$$

then $uv$ satisfies the equation

$$(\Delta_g - c_n R_g)(uv) = c_n (uv)^{\frac{n+2}{n-2}}.$$
In particular if \( g_{ij} = \delta_{ij} \) and
\[
\frac{1}{v^{n-2}}(x) = \frac{4n(n-1)}{1-|x|^2} \quad \text{for} \quad x \in B^n,
\]
and \( u \) is a solution to equation (1.1) on the (scaled) hyperbolic space, then
\[
\Delta_v(u^n) = c_n (uv)^{n+2} \quad \text{on} \quad B^n,
\]
where \( \Delta_v \) is the Laplacian of the Euclidean space. Therefore any positive solution \( w \) to the equation
\[
\Delta_v w = c_n w^{n+2}
\]
also provides a positive solution to equation (4.4) on the (scaled) hyperbolic space. Let \((r, \Theta)\) be the polar coordinates on \( \mathbb{R}^n \), where \( r \geq 0 \) and \( \Theta \in S^{n-1} \). For a function \( w \) that depends on \( r \) only, we have
\[
\Delta_v w(r) = w''(r) + \frac{n-1}{r}w'(r) \quad \text{for} \quad r > 0.
\]
A calculation shows that equation (4.4) has positive solutions of the form
\[
w_b(x) = \frac{(4b^2n(n-1))^{\frac{n-2}{2}}}{(b^2-|x|^2)^{\frac{n-2}{2}}} \quad \text{for} \quad x \in B^n,
\]
where \( b \geq 1 \) is a constant. Thus for any number \( b \geq 1 \),
\[
u_b(x) = w(x)v^{-1}(x) = \left(\frac{b (1-|x|^2)}{b^2-|x|^2}\right)^{\frac{n-2}{2}} \quad \text{for} \quad x \in B^n
\]
is a positive solution to equation (1.1) on the (scaled) hyperbolic space. We note that if \( b > 1 \), then \( u_b < 1 \) and \( \lim_{|x| \to 1} u(x) = 0 \), and if \( b = 1 \), then \( u_b \equiv 1 \) on \( B^n \). By the proof of theorem 2.4, any positive solution \( u \) to equation (1.1) on the hyperbolic space is bounded from above. Using the maximum principle as in lemma 2.1 we have \( u \leq 1 \) on the hyperbolic space.

A positive solution to equation (1.1) is called complete if the conformal metric \( g_c = u^{\frac{4}{n-2}}g \) is a complete Riemannian metric on \( M \). In this section we discuss whether complete positive solutions to the equation (1.1) are unique. We note that the solutions given in (4.6) on the (scaled) hyperbolic space are not complete unless \( b = 1 \), that is, \( u \equiv 1 \).

**Proposition 4.7.** Let \((M, g)\) be a complete non-compact Riemannian manifold with scalar curvature equal to \(-1\). Assume that the sectional curvature of \((M, g)\) is bounded from below and the injectivity radius of \((M, g)\) is positive. Let \( u \) be a positive smooth solution to the equation (1.1). If the conformal metric \( g_c = u^{\frac{4}{n-2}}g \) is a
Proof. The Riemannian manifold \((M, g_c)\) has constant scalar curvature equal to \(-1\). Then \(v = 1/u\) satisfies the equation

\[
\Delta g_c v + c_n v = c_n v^{\frac{n+2}{n-2}} \quad \text{on } M.
\]  

(4.8)

The proof of theorem 2.4 shows that \(v\) is bounded from above, that is, \(u\) is bounded away from zero. Lemma 2.1 implies that \(u \equiv 1\) on \(M\). \(\square\)

Let us consider positive radial solutions to equation (1.1). Let \((r, \Theta)\) be polar coordinates on \(\mathbb{R}^n\), where \(r \in [0, \infty)\) and \(\Theta \in S^{n-1}\). Let

\[
g = dr^2 + f^2(r) d\Theta^2
\]

be a Riemannian metric on \(M\), where \(f \in C^\infty(0, \infty)\) is a positive function with \(f(0) = 0\) and \(f'(0) = 1\). We note that \(g\) is a complete Riemannian metric on \(\mathbb{R}^n\). The hyperbolic space is corresponding to \(f(r) = \sinh r\) for all \(r \geq 0\).

**Theorem 4.9.** Let \(u\) be a positive radial \(C^2\)-solution to equation (1.1) on \((\mathbb{R}^n, g = dr^2 + f^2(r) d\Theta^2), n \geq 3\). Assume that \(\lim_{r \to \infty} f(r) = \infty\) and there exist positive constants \(R_o\) and \(C_o\) such that \(|f'(r)/f(r)| \leq C_o\) for all \(r > R_o\). If the conformal metric \(g_c = u^{\frac{4}{n-2}} g\) is complete, then \(u \equiv 1\).

Proof. For the Riemannian metric \(g = dr^2 + f^2(r) d\Theta^2\) we have

\[
\Delta_g = \frac{\partial^2}{\partial r^2} + \frac{(n - 1)f'(r)}{f(r)} \frac{\partial}{\partial r} + \frac{1}{f^2(r)} \Delta_{\Theta},
\]

where \(\Delta_{\Theta}\) is the Laplacian for the standard unit sphere in \(\mathbb{R}^n\), see, for example, [11]. If \(u = u(r)\), then equation (1.1) becomes

\[
u''(r) + \frac{(n - 1)f'(r)}{f(r)} u'(r) = c_n (u^{\frac{n+2}{n-2}}(r) - u(r)).
\]

(4.11)

We consider the case that \(u(0) = 1\) first. If there exists \(r' \geq 0\) such that \(u(r') \geq 1\) and \(u'(r') > 0\), then \(u'(r) > 0\) for all \(r > r'\). Otherwise there exists a point \(r'' > 0\) such that \(u(r'') > 1, u'(r'') = 0\) and \(u''(r'') \leq 0\), but equation (4.11) shows that this is not possible. Similarly if there is a point \(r_o\) such that \(u(r_o) \leq 1\) and \(u'(r_o) < 0\), then \(u'(r) < 0\) for all \(r \geq r_o\). Hence if \(u \not\equiv 1\), then either \(u(r) > 1\) and \(u'(r) > 0\) or \(u(r) < 1\) and \(u'(r) < 0\) for all \(r > r_o\). Consider the first case. From equation (4.11) we have

\[
\frac{(f^{n-1}u')'}{f^{n-1}} = c_n (u^{\frac{n+2}{n-2}} - u).
\]

(4.12)
Let $\chi \in C^\infty_0([0, \infty))$ be a cut-off function. Multiplying both sides of equation (4.12) by $\chi^n u$ and then using integration by parts we obtain

$$c_n \int_0^\infty \chi^n u^2 \, dr - \int_0^\infty (f^{n-1}u') \left( \frac{\chi^n u}{f^{n-1}} \right) \, dr = c_n \int_0^\infty \chi^n u^{\frac{2n}{n-2}} \, dr.$$  

We have

$$-(f^n u') \left( \frac{\chi^n u}{f^n} \right)' = -n\chi^{n-1}u\chi' u' - \chi^n |u'|^2 + (n-1)\chi^n u u' \frac{f'}{f}.$$  

Applying the Cauchy inequality we have

$$-n\chi^{n-1}u\chi' u' = -2 \left( \frac{n}{\sqrt{2}} \chi^{\frac{n-1}{2}} u \chi' \right) \left( \frac{1}{\sqrt{2}} \chi^{\frac{n-2}{2}} u \right) \leq \frac{n^2}{2} \chi^{n-2} u^2 |\chi'|^2 + \frac{1}{2} \chi^n |u'|^2,$$

$$(n-1)\chi^n u u' \frac{f'}{f} = 2 \left( \frac{n-1}{\sqrt{2}} \chi^{\frac{n-1}{2}} u \frac{f'}{f} \right) \left( \frac{(n-1)^2}{\sqrt{2}} \chi^{\frac{n-2}{2}} u \right) \leq \frac{(n-1)^2}{2} \chi^n \left( \frac{f'}{f} \right)^2 u^2 + \frac{1}{2} \chi^n |u'|^2.$$

Together with (4.13) we obtain

$$c_n \int_0^\infty \chi^n u^2 \, dr + \frac{2(n-1)^2}{2} \int_0^\infty \left( \frac{f'}{f} \right)^2 \chi^n u^2 \, dr + \frac{n^2}{2} \int_0^\infty \chi^{n-2} u^2 |\chi'|^2 \geq c_n \int_0^\infty \chi^n u^{\frac{2n}{n-2}} \, dr.$$

Apply Young’s inequality as in (2.7) and (2.8) and using the bound $|f'/f| \leq C_o$ to have

$$\int_0^\infty \chi^n u^{\frac{2n}{n-2}} \, dr \leq C' \int_0^\infty (|\chi'|^n + \chi^n) \, dr,$$

where $C'$ is a positive constant. Let $\chi \equiv 0$ on $[0, R] \cup [R + 3, \infty)$ with $R > R_o$ and $\chi \equiv 1$ on $[R + 1, R + 2]$, $\chi \geq 0$ on $[0, \infty)$ and $|\chi'| \leq \frac{1}{2}$. From (4.15) we have

$$\int_{R+1}^{R+2} u^{\frac{2n}{n-2}} \, dr \leq C''$$

for all $R > R_o$, where $C''$ is a constant independent on $R$. Therefore $u$ is bounded from above. As $u'(r) > 0$ for all $r > 0$ and $u$ is bounded from above, we can find a sequence $\{r_k\}$ and a positive constant $\epsilon > 0$ such that $u(r_k) > 1 + \epsilon$ for all $k \in \mathbb{N}$, $\lim_{k \to \infty} u'(r_k) = 0$ and $u''(r_k) \leq 0$. But this contradicts equation (4.11). Thus we must have $u(r) < 1$ and $u'(r) < 0$ for all $r > r_o$. Assume that $u(r) \geq c$ for all $r > \infty$, where $c \in (0, 1)$ is a constant. Then we can find a sequence $\{r_k\}$ and a positive constant $\delta > 0$ such that $u(r_k) > 1 - \delta$ for all $k \in \mathbb{N}$, $\lim_{k \to \infty} u'(r_k) = 0$ and $u''(r_k) \geq 0$. But this contradicts equation (4.11). Therefore $\lim_{r \to \infty} u(r) = 0$. There exist positive constants $r' > R_o$ and $C > 0$ such that for $r \geq r'$ we have

$$f^{n-1}u'(r) = c_n f^{n-1}(r)(u^{\frac{n+2}{n-2}}(r) - u(r)) \leq -C f^{n-1}(r)u(r).$$
Integrating from \( r' \) to \( r > R' \) we have
\[
f^{n-1}(r)u'(r) \leq f^{n-1}(r')u'(r') - C \int_{r'}^r f^{n-1}(s) u(s) \, ds \leq -Cu(r) \int_{r'}^r f^{n-1}(s) \, ds ,
\]
as \( u' \leq 0 \). Therefore
\[
(4.17) \quad \frac{u'(r)}{u(r)} \leq -C \frac{\int_{r'}^r f^{n-1}(s) \, ds}{f^{n-1}(r)}.
\]
Using the bounded \( f'/f < C_o \) we have \((f^{n-1})' \leq C_o(n-1)f^{n-1} \). An integration gives
\[
(4.18) \quad f^{n-1}(r) - f^{n-1}(r') \leq C_o(n-1) \int_{r'}^r f^{n-1}(s) \, ds .
\]
As \( \lim_{r \to \infty} f(r) = \infty \), if \( r \) is large we have
\[
\frac{1}{2}f^{n-1}(r) \leq C_o \int_{r'}^r f^{n-1}(s) \, ds ,
\]
that is,
\[
(4.19) \quad \frac{\int_{r'}^r f^{n-1}(s) \, ds}{f^{n-1}(r)} \geq c'
\]
for all \( r \) large and for some positive constant \( c' \). The inequality (4.17) and (4.19) give
\[
(4.20) \quad u(r) \leq \tilde{C}e^{-cr}
\]
for all \( r \) large enough, where \( \tilde{C} \) is a positive constant. Thus the conformal metric \( g_c = u^{\frac{4}{n-2}}g \) cannot be complete. Hence \( u \equiv 1 \) on \([0, \infty)\).

Assume that \( u(0) \neq 1 \). If \( u \) is a \( C^2 \)-function on \( \mathbb{R}^n \) which depends on \( r \) only, then \( u'(0) = 0 \). If \( u(0) > 1 \), then equation (4.11) shows that there exists an \( \epsilon > 0 \) such that \( u''(r) > 0 \) for all \( r \in (0, \epsilon) \). Hence \( u'(r_o) > 0 \) for \( r_o \) small enough and hence \( u'(r) > 0 \) for all \( r > r_o \). Similarly if \( u(0) < 1 \), then \( u'(r) < 0 \) for all \( r > r_o' \). In either cases we can obtain contradiction as above. \( \square \)

We have the following result for non-radial solutions.

**Theorem 4.21.** For an integer \( n \geq 4 \) let \( g = dr^2 + f^2(r) \, d\Theta^2 \) be a Riemannian metric on \( \mathbb{R}^n \). Assume that \( \lim_{r \to \infty} f(r) = \infty \) and there exist positive constants \( R_o \) and \( C_o \) such that \( |f'(r)/f(r)| \leq C_o \) for all \( r > R_o \). Let \( u \) be a positive smooth solution to equation (1.1) on \( (\mathbb{R}^n, g) \). If there exist constants \( \delta \in (0, 1) \) and \( r_o > 0 \) such that \( u(r, \Theta) \leq 1 - \delta \) for \( r \geq r_o \) and \( \Theta \in S^{n-1} \), then the conformal metric \( g_c = u^{\frac{4}{n-2}}g \) is not complete.

**Proof.** In polar coordinates, using (4.10), equation (1.1) is given by
\[
(4.22) \quad \frac{\partial^2 u}{\partial r^2} + \frac{(n-1)f'(r)}{f(r)} \frac{\partial u}{\partial r} + \frac{1}{f^2(r)} \Delta \Theta u = c_n(u^{\frac{n+2}{n-2}} - u),
\]
where $\Delta_\Theta$ is the Laplacian for $S^{n-1}$ with the standard metric. For $r \geq r_o$, we have $u(r, \Theta) \leq 1 - \delta$ for some constant $\delta \in (0, 1)$. Therefore

$$c_n(u^{\frac{n-2}{n-1}} - u) \leq c_n[(1 - \delta)^{\frac{1}{n-2}} - 1]u = -\epsilon u,$$

where

$$\epsilon = c_n[1 - (1 - \delta)^{\frac{1}{n-2}}] > 0.$$

For a fixed number $r \geq r_o$, we integrate equation (4.22) over $S^{n-1}$ with respect to the standard measure on $S^{n-1}$ and use Green’s formula and (4.23) to obtain

$$\frac{d^2}{dr^2} \left( \int_{S^{n-1}} u d\Theta \right) + \frac{(n-1)f'(r)}{f(r)} \frac{d}{dr} \left( \int_{S^{n-1}} u d\Theta \right) \leq -\epsilon \int_{S^{n-1}} u d\Theta.$$

For $r \geq r_o$, let

$$U(r) = \int_{S^{n-1}} u(r, \Theta) d\Theta.$$

Then (4.24) can be written as

$$U''(r) + \frac{(n-1)}{f(r)} f'(r) U'(r) \leq -\epsilon U(r) \quad \text{for } r \geq r_o.$$

Therefore

$$\left( f^{n-1}(r)U'(r) \right)' \leq -\epsilon f^{n-1}U(r) \quad \text{for } r \geq r_o.$$

The argument in the proof of theorem 4.7 from (4.16) to (4.20) shows that there exist positive constants $C$ and $c$ such that

$$U(r) = \int_{S^{n-1}} u d\Theta \leq Ce^{-\sigma} \quad \text{for } r \geq r_o.$$

Assume that $n \geq 4$. We have

$$\int_{S^{n-1}} u^{\frac{2}{n-2}} d\Theta \leq \left( \int_{S^{n-1}} u d\Theta \right)^{\frac{2}{n-2}} \frac{\omega_n^{\frac{n-2}{n-1}}}{\omega_n^{\frac{2}{n-2}}},$$

where $\omega_n$ is the volume of the unit sphere in $\mathbb{R}^n$. Using (4.25) and (4.26) we have

$$\int_{r_o}^{\infty} \int_{S^{n-1}} u^{\frac{2}{n-2}} d\Theta dr < \infty.$$

We claim that there exists $\Theta_o \in S^{n-1}$ such that

$$\int_{r_o}^{\infty} u^{\frac{2}{n-2}}(r, \Theta_o) dr < \infty.$$

This means that the conformal metric $u^{\frac{2}{n-2}}(dr^2 + f^2(r) d\Theta^2)$ is not complete. To prove the claim, by (4.27) and Fubini’s theorem, there exists an positive integer $C'$ such that for any positive integer $N$ we have

$$\int_{r_o}^{N} \int_{S^{n-1}} u^{\frac{2}{n-2}}(r, \Theta) d\Theta dr = \int_{S^{n-1}} \left( \int_{r_o}^{N} u^{\frac{2}{n-2}}(r, \Theta) dr \right) d\Theta \leq C'.$$
For each integer \( N > r_o \), there exists a point \( \Theta_N \in S^{n-1} \) such that
\[
\int_{r_o}^{N} u^{\frac{2}{n-2}}(r, \Theta_N) dr \leq \frac{C'}{\omega_n} + 1.
\]
A subsequence \( \{\Theta_{N_i}\}_{i \in \mathbb{N}} \) converges to a point \( \Theta_0 \in S^{n-1} \). If
\[
\int_{r_o}^{\infty} u^{\frac{2}{n-2}}(r, \Theta_0) dr = \infty,
\]
then there exists a positive integer \( N' \) such that
\[
\int_{r_o}^{N'} u^{\frac{2}{n-2}}(r, \Theta_0) dr > \frac{C'}{\omega_n} + 2.
\]
As the function
\[
\Theta \rightarrow \int_{r_o}^{N'} u^{\frac{2}{n-2}}(r, \Theta) dr
\]
is continuous, therefore in a neighborhood of \( \Theta_0 \) we have
\[
\int_{r_o}^{N'} u^{\frac{2}{n-2}}(r, \Theta) dr > \frac{C'}{\omega_n} + \frac{3}{2}.
\]
As \( \lim_{i \to \infty} \Theta_{N_i} = \Theta_0 \) and for all \( i \) such that \( N_i > N' \), \( \Theta_{N_i} \) satisfies (4.28) and hence
\[
\int_{r_o}^{N'} u^{\frac{2}{n-2}}(r, \Theta_{N_i}) dr \leq \frac{C'}{\omega_n} + 1.
\]
This contradicts with (4.30). The proof of the claim is completed. \( \square \)

**Theorem 4.31.** For an integer \( n \geq 3 \), let \( g = dr^2 + f^2(r) \) \( d\Theta^2 \) be a Riemannian metric on \( \mathbb{R}^n \). Assume that \( \lim_{r \to \infty} f(r) = \infty \) and there exist positive constants \( R_o \) and \( C_o \) such that \( |f'(r)/f(r)| \leq C_o \) and \( |f''(r)/f(r)| < C_o \) for all \( r > R_o \). Let \( u \) be a positive smooth solution to equation (1.1) on \( (\mathbb{R}^n, g) \). If \( u \leq 1 \) on \( \mathbb{R}^n \) and \( u(0) < 1 \), then there exist positive constants \( a \) and \( C \) such that \( u(r, \Theta) \leq 1 - ae^{-Cr} \) for all \( r \geq 0 \).

**Proof.** Let \( \phi \in C^\infty(\mathbb{R}^n) \) be a positive function such that \( u \leq 1 - \phi \) on \( B_o(\epsilon) \) and \( \phi(r, \Theta) = ae^{-Cr} \) for \( r \geq \epsilon \) and \( \Theta \in S^{n-1} \), where \( \epsilon \) is a small positive constant and \( a \) and \( C \) are positive constants to be chosen later. Let \( \Phi = u - (1 - \phi) \). If there exists a point \( (r_o, \Theta_o) \in \mathbb{R}^n \) such that
\[
\Phi(r_o, \Theta_o) = \delta > 0,
\]
then \( \Phi \geq \delta \). As the solution \( u \) is bounded from above and the radial Ricci curvature of \( g \) is given by \(- (n-1)f''/f\), which is bounded from below by the assumption on \( f \), the maximum principle (c.f. [16]) implies that there exists a sequence \( \{(r_k, \Theta_k)\}_{k \in \mathbb{N}} \) such that
\[
\lim_{k \to \infty} \Phi(r_k, \Theta_k) = \sup \Phi \geq \delta, \quad |\nabla \Phi(r_k, \Theta_k)| \leq \frac{1}{k}
\]
and
\[ \Delta g \Phi(r_k, \Theta_k) \leq \frac{1}{k}. \]
If the sequence is bounded, then we may assume that
\[ \lim_{k \to \infty} (r_k, \Theta_k) = (\tilde{r}, \tilde{\Theta}). \]

At \((\tilde{r}, \tilde{\Theta})\) we have
\[ \Phi(\tilde{r}, \tilde{\Theta}) = \sup \Phi \geq \delta \quad \text{and} \quad \Delta g \Phi(\tilde{r}, \tilde{\Theta}) \leq 0. \]
The function \(s \mapsto s^{\frac{n+2}{2}} - s\) is increasing on \(\{((n-2)/(n+2))^{\frac{n-2}{4}}, \infty\}\). If we choose \(a\) to be small, then
\[
\begin{equation}
\label{eq:4.32}
\Phi(\tilde{r}, \tilde{\Theta}) \geq 1 - ae^{-C\tilde{r}} \geq \left(\frac{n-2}{n+2}\right)^{\frac{n-2}{4}}.
\end{equation}
\]
Using (4.10) to compute \(\Delta g \phi\) and (4.32), we have
\[
\begin{equation}
\label{eq:4.33}
\Delta g \Phi(\tilde{r}, \tilde{\Theta}) = c_n \left(u^{\frac{n+2}{2}} - u\right) + \Delta g \phi \geq c_n \left[(1 - ae^{-C\tilde{r}})^{\frac{n+2}{2}} - (1 - ae^{-C\tilde{r}})\right] + C^2 ae^{-C\tilde{r}} - (n-1)\frac{f'(\tilde{r})}{f(\tilde{r})} Cae^{-C\tilde{r}}.
\end{equation}
\]
As \(f'/f\) is bounded outside \(B_o(\epsilon)\), if we choose \(C\) to be large enough, then there exists a positive constant \(c\) that depends on \(C\) and \(n\) only, such that
\[
\begin{equation}
\label{eq:4.34}
c_n \left[(1 - ae^{-C\tilde{r}})^{\frac{n+2}{2}} - (1 - ae^{-C\tilde{r}})\right] + C^2 ae^{-C\tilde{r}} - (n-1)\frac{f'(\tilde{r})}{f(\tilde{r})} Cae^{-C\tilde{r}} \geq c.
\end{equation}
\]
This contradicts that \(\Delta g \Phi(\tilde{r}, \tilde{\Theta}) \leq 0\). Therefore we may assume that \(r_k \to \infty\) as \(k \to \infty\). Then
\[ \lim_{k \to \infty} \Phi(r_k, \Theta_k) \geq \delta, \]
which implies that \(u(r_k, \Theta_k) \geq 1 + \delta - ae^{-Cr_k} \geq 1 + \delta/2 > 1\) for \(k\) large. This contradicts with the assumption that \(u \leq 1\) on \(\mathbb{R}^n\). \(\square\)

**Corollary 4.35.** Let \(u\) be a positive solution to equation (1.1) on the (scaled) hyperbolic space. If \(u \neq 1\), then \(u < 1\) on \(B^n\).

**Proof.** We know that \(u \leq 1\). Assume that there is a point \(x_o \in B^n\) such that \(u(x_o) < 1\). Using an isometry we may assume that \(x_o = 0\). The (scaled) hyperbolic metric (4.1) is corresponding to a metric \(g = dr^2 + f^2(r) d\Theta^2\) with
\[ f(r) = \sqrt{n(n-1)} \sinh \frac{r}{\sqrt{n(n-1)}}. \]
We may apply theorem 4.31 to complete the proof. \(\square\)
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