Linear stochastic wave-equations for continuously measured quantum systems

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Abstract

While the linearity of the Schrödinger equation and the superposition principle are fundamental to quantum mechanics, so are the backaction of measurements and the resulting nonlinearity. It is remarkable, therefore, that the wave-equation of systems in continuous interaction with some reservoir, which may be a measuring device, can be cast into a linear form, even after the degrees of freedom of the reservoir have been eliminated. The superposition principle still holds for the stochastic wave-function of the observed system, and exact analytical solutions are possible in sufficiently simple cases. We discuss here the coupling to Markovian reservoirs appropriate for homodyne, heterodyne, and photon counting measurements. For these we present a derivation of the linear stochastic wave-equation from first principles and analyze its physical content.
I. INTRODUCTION

Perhaps the most fundamental principle underlying quantum mechanics is the superposition principle for quantum amplitudes which is equivalent to the linearity of the Schrödinger equation. However, as is well known, the time-evolution of the state vector $|\psi\rangle$ of any system is governed by the linear Schrödinger equation only as long as the system is decoupled from its environment, and, in particular, from any extraneous measuring apparatus. E.g. if during a short time interval such an extraneous coupling is introduced, which performs a measurement of an observable $X$ without destroying the system, the linear time-evolution is interrupted; the wave-function is replaced by a new wave-function $|\psi'\rangle$ chosen at random among the eigenfunctions $|\psi_n\rangle$ satisfying $X|\psi_n\rangle = X_n|\psi_n\rangle$ with the eigenvalues $X_n$; the probability for $|\psi_n\rangle$ is given by

$$p_n = |\langle\psi_n|\psi\rangle|^2, \quad n \in I$$

(1.1)

where $I$ is the set of all possible quantum numbers, here taken as discrete, for simplicity. This step in the time-evolution is therefore random and nonlinear. In order to formulate it explicitly let us introduce a classical random variable $N$, which only takes the values $N = n, \ n \in I$ with probabilities

$$P(N = n) = p_n, \quad \sum_{n \in I} p_n = 1.$$ (1.2)

Then we may write for a measurement

$$|\psi\rangle \rightarrow |\psi'\rangle = |\psi_N\rangle.$$ (1.3)

If similar measurements are performed repeatedly in time, say at the discrete time $t_i$, then the linear deterministic time-evolution between successive measurements is repeatedly interrupted by steps like (1.3) with more random variables $N_i$ whose probabilities follow again from eq. (1.1) and are therefore correlated, in general.

The possibility to describe measured systems, or more generally systems coupled to their environment, by stochastic wave-functions similar to (1.3) is of course well known. By
the work of many authors over the years this possibility has been turned into an efficient tool, useful in particular for numerical simulations, by deriving equations of motion for the wave-functions of systems continuously coupled to their environment. All of the early and some of the more recent work in this area is purely phenomenological (see e.g. [1–5]) and often motivated by the desire to find generalizations of the Schrödinger equation on a fundamental level including wave-function collapse. Some of the more recent work, however, stresses microscopic derivations, in certain weak coupling and Markovian limits, from the Schrödinger equation for the coupled system and reservoir (see e.g. [6–9]). As one may expect from eq. (1.3), the resulting wave-equations generalize the Schrödinger equation for isolated systems in two ways: they contain in an explicit way classical stochastic variables analogous to \( N \) in eq. (1.3), and they are nonlinear, analogous to the nonlinear dependence of \( N \) on \( |\psi\rangle \) in eq. (1.3) via its distribution (1.2), (1.1).

The appearance of stochastic elements in the generalized Schrödinger equation may look unusual at first sight but should not cause surprise, as it corresponds directly to the observed stochastic behaviour e. g. of measured quantum systems. However, the appearance of a nonlinearity in quantum measurements, even though well known to all physicists, may still be considered as alarming, because it seems to change the foundations of quantum mechanics: In the real world, perhaps with the exception of the universe as a whole, there aren’t any systems which are completely decoupled from their environment. This seems to lead to the alarming conclusion that linearity and the superposition principle are only approximate concepts for any system in the real world. Indeed, any finite real-world system in an excited state will sooner or later decay via one or several random nonlinear steps similar to eq. (1.3). But if the nonlinearity of the environment-coupled reduced time-evolution must be accepted, the role of the superposition principle becomes obscure. E. g. it may no longer be possible to analyze the fate of an initially given linear superposition of states by studying the fate of each of its constituents separately.

In view of this fundamental problem raised by the nonlinear stochastic evolution (1.3) it is very remarkable that the stochastic nonlinear wave-equations, describing a broad class
of systems in continuous interaction with their environment, can actually be cast into a linear form, containing the same information. These linear equations still contain classical stochastic variables, however now distributed independently of the wave-function. The noise they describe is therefore independent of the system under study and may be considered a property of the environment alone.

The existence of these linear versions of the stochastic wave-equations has repeatedly been noted in the literature [8–12], however their fundamental relevance in connection with the superposition principle has, to our knowledge, not been stressed. In the present paper we wish to derive and study in a unified manner the linear stochastic wave-equations corresponding to homodyne, heterodyne and photon-count measurements in quantum optics.

These measurements have been considered before by Wiseman and Milburn [6,7] using nonlinear stochastic wave-equations. A microscopic derivation of a linear stochastic wave-equation corresponding to homodyne measurements was presented by Belavkin [8]. Our derivation starts from the same assumptions, but appears to be simpler, at least to us. Photon-counting has been discussed in the framework of phenomenological nonlinear and a phenomenological linear stochastic wave-equations in papers by Barchielli [12,13] and by Belavkin [10]. The linear stochastic wave-equation for photon-counting, which we shall derive in the present paper is different from the phenomenological version used in their work. Stochastic Schrödinger equations with quantized noise terms have been discussed in detail in the work of Gardiner et al [14]. While these authors consider equations in the large Hilbert space of system and environment, our work here is concerned with wave-equations in the Hilbert space of the system without environment. An extensive review of the use of nonlinear stochastic wave-equations in quantum optics and a large list of references is given in [15].

The derivation and use of stochastic wave-equations, both nonlinear and linear, is so far restricted to Markovian reservoirs and we shall also adopt this framework here. It remains an interesting open problem for future work to examine also non-Markovian extensions.
II. LINEAR STOCHASTIC DESCRIPTION OF A GENERAL QUANTUM MEASUREMENT

Here we shall present the general idea to be applied in the subsequent sections for the case of a general quantum measurement. It is our aim to show how the stochastic description to be applied in subsequent sections is firmly rooted in the basic formalism established by von Neumann (1931) [16], Wigner (1963) [17] and others.

The nonlinear stochastic change of a wave-function due to the idealized instantaneous measurement of a quantum observable was recalled in eq. (1.3). The nonlinearity resides in the dependence of the probability \( p(N = n) = |\langle \psi | \psi_n \rangle|^2 \) on \( |\psi\rangle \). We now turn to a stochastically equivalent linear description, i.e. one which gives the same results for physical probabilities.

We may begin by noting that the transformation (1.3) would trivially be linear if the distribution of the random variable \( N \) would be independent of the state \( |\psi\rangle \). Let us therefore introduce a new classical random variable \( \tilde{N} \) also distributed over all quantum numbers \( n \in I \) with some fixed but arbitrary distribution \( \tilde{p}_n \) satisfying

\[
p(\tilde{N} = n) = \tilde{p}_n > 0, \quad \tilde{p}_n \neq 0 \forall n, \quad \sum_{n \in I} \tilde{p}_n = 1.
\]

Let us then consider the replacement of the nonlinear transformation (1.3) by the linear one

\[
|\psi\rangle \rightarrow |\tilde{\psi}'_{\tilde{N}}\rangle = \frac{1}{\sqrt{\tilde{p}_{\tilde{N}}}}|\psi_{\tilde{N}}\rangle\langle \psi_{\tilde{N}}|\psi\rangle
\]

(2.2)

where \( \tilde{N} \) is distributed according to (2.1). We note that \( |\tilde{\psi}'_{\tilde{N}}\rangle \) is not normalized,

\[
\langle \tilde{\psi}'_{\tilde{N}}|\tilde{\psi}'_{\tilde{N}}\rangle = \frac{1}{\tilde{p}_{\tilde{N}}}|\langle \psi_{\tilde{N}}|\psi\rangle|^2 \neq 1
\]

(2.3)

i.e. the transformation (2.2) is not unitary. However, we have instead

\[
\sum_{n \in I} \tilde{p}_n \langle \tilde{\psi}'_n|\tilde{\psi}'_n\rangle = 1.
\]

(2.4)

The probability \( P_X(X_n) \) to observe the eigenvalue \( X_n \) of the measured observable \( X \) is fully encapsulated in the linear stochastic transform \( |\tilde{\psi}'_{\tilde{N}}\rangle \) of \( |\psi\rangle \) via the formula
\[ P_X(X_n) = \tilde{p}_n \langle \tilde{\psi}'_n | \tilde{\psi}'_n \rangle. \]  

Using eq. (2.2) in eq. (2.5) it can be seen that the usual formula \( P_X(X_n) = |\langle \psi_n | \psi \rangle|^2 = p_n \) results and, furthermore, that \( \tilde{p}_n \) drops out completely and may remain arbitrary, as long as the conditions (2.1) are satisfied. E.g. one may replace \( \tilde{p}_n \) by a constant and interpret the corresponding equidistribution of \( \tilde{N} \) over the set I as reflecting our complete ignorance of the outcome of the measurement prior to the specification of \( |\psi\rangle \) and prior to the measurement. Correspondingly, eq. (2.5) represents the transformation from the 'input' distribution \( \tilde{p}_n \), reflecting our ignorance, to the 'output'-distribution \( p_n \) produced by the measurement. This particular way of thinking about eq. (2.5) will be useful later on, where descriptions analogous to eq. (2.2) are applied to various concrete measurements which are performed continuously in time. In fact, concrete expressions for the probability \( \tilde{p}_\tilde{N} \) are obtained in these concrete models.

From eq. (2.5) all moments of the measured observable may be calculated. Furthermore, it is straightforward to check that for measurements performed successively in time multi-time joint probabilities of the observables and the correlation functions associated to them may also be calculated by the repeated use of eq. (2.2) for each measurement. Hence, eq. (2.2) provides a description of the measurement of \( X \) which is just as complete as the nonlinear description (1.3), but linear. E.g. if the transformation (2.2) is known for two wave-functions \( |\psi_A\rangle \) and \( |\psi_B\rangle \) then we have for their linear superposition

\[
C_A |\psi_A\rangle + C_B |\psi_B\rangle \rightarrow \langle C_A |\psi'_A\rangle + C_B |\psi'_B\rangle = \frac{1}{\sqrt{\tilde{p}_{\tilde{N}}}} (C_A \langle \tilde{\psi}_{\tilde{N}} | \psi_A \rangle + C_B \langle \tilde{\psi}_{\tilde{N}} | \psi_B \rangle). \tag{2.6}
\]

We note that this expression still describes the stochastic 'collapse' of the superposition to a single state \( |\tilde{\psi}_{\tilde{N}}\rangle \). Yet the use of eq. (2.6) in the formula (2.5) correctly predicts the interferences of the superposition observed in the measurement of \( X \) based on the behavior of the individual components of the mixture.

We may ask now whether the wave-function \( |\tilde{\psi}'_{\tilde{N}}\rangle \) is physical, i.e. whether it corresponds in any sense to the actual state of the reduced system. This question is difficult or perhaps
impossible to answer, even for the wave-function of closed systems, which may well be considered as being devoid of physical reality and no more than a mere computational tool to obtain physical probabilities and their moments via some definite algorithm. The same may be said for the wave-function $|\tilde{\psi}_{N}\rangle$, the only difference being that the usual algorithm for obtaining probabilities is slightly changed according to eq. (2.3).

In order to derive eq. (2.2) from the Schrödinger equation we may use the method of von Neumann (1931) [16] and Wigner (1963) [17]. They show that an uncorrelated initial state $|\psi\rangle$ of the measured system and $|\phi\rangle$ of the measuring apparatus evolves according to

$$|\psi\rangle \otimes |\phi\rangle \rightarrow \sum_{n} |\psi_{n}\rangle |\psi_{n}\rangle \otimes |\phi_{n}\rangle$$

(2.7)

into an entangled state. The entanglement in eq. (2.7) is real and could in principle be observed if further joint measurements on system and apparatus are made. However, as long as further measurements are made only on the system or only on the apparatus the relative phases between the components of eq. (2.7) for different $n$ are unobservable, because both the $|\psi_{n}\rangle$ and the $|\phi_{n}\rangle$ are orthogonal for different $n$, the $|\psi_{n}\rangle$ because they correspond to different eigenvalues $X_{n}$, the $|\phi_{n}\rangle$ because they are macroscopically distinguishable states. Hence, as far and only as far as such restricted further measurements are concerned, a super-selection rule holds for $n$, (i.e. states of different $n$ are not coherently superposed) and the right hand side of eq. (2.7) is equivalent to a mixture

$$\sum_{n} |\langle \psi_{n} | \psi \rangle|^{2} |\psi_{n}\rangle \otimes |\phi_{n}\rangle$$

(2.8)

This restriction of the type of further measurements allows to circumvent von Neumann’s and Wigner’s argument that a pure state cannot evolve into a mixture, an old stumbling block in quantum measurement theory. Tracing out the apparatus we end up with a mixed state, which is equivalent to the description (2.2) and (2.4) for arbitrary $\tilde{p}_{n}$. 

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Let us briefly recall the principle of homodyne measurements. A quantum oscillator or mode, say the electromagnetic mode of a cavity at a given frequency but with (in comparison slowly varying) time-dependent amplitude, radiates into the vacuum, where the classical field of a local oscillator at the same frequency with stable phase and time-independent amplitude is superimposed and the beat intensity is detected. In balanced homodyning a difference scheme is employed to get rid of the dominating field intensity of the local oscillator in the output.

The input-output formalism [18] is perfectly adapted to describe this situation: The interaction of the system, i.e. the quantum oscillator, with an input field generates an output field depending linearly on the amplitude of the oscillator via a coupling constant. In the case at hand the input field are just vacuum fluctuations entering the quantum oscillator by the time-reverse of the channel through which it radiates into the vacuum. For a differential time-step we may write

\[
\text{dB}^{\text{out}}(t) = \sqrt{\gamma}a(t) + \text{dB}^{\text{in}}(t).
\]

Here \(a(t), a^\dagger(t)\) are the mode operators of the oscillator in the interaction picture (i.e. with the oscillator frequency \(\omega_0\) split off); \(\sqrt{\gamma}\) is a coupling constant which we take positive by appropriately adjusting the origin of the phase of \(a\); \(\text{dB}^{\text{in}}(t)\) represents the future directed increments of the quantized input-field which is taken as quantized Gaussian white noise [19]. Therefore

\[
[dB^{\text{in}}(t), dB^{\text{in}}(t')] = 0 = [dB^{\dagger}_{\text{in}}(t), dB^{\dagger}_{\text{in}}(t')]
\]

\[
[dB^{\text{in}}(t), dB^{\dagger}_{\text{in}}(t')] = 0 \text{ for } t \neq t'
\]

\[
[dB^{\text{in}}(t), dB^{\text{in}}(t)] = dt
\]

\[
[dB^{\text{in}}(t), a(t')] = [dB^{\dagger}_{\text{in}}(t), a(t')] = 0 \text{ for } t \geq t'.
\]

Acting on the vacuum we have \(dB^{\dagger}_{\text{in}}(t)dB^{\dagger}_{\text{in}}(t) = 0\). As the increments \(dB^{\text{in}}(t), dB^{\dagger}_{\text{in}}(t)\) are causally disconnected with the quantum oscillator at time \(t\) all their commutors with

\[
\text{dB}^{\text{out}}(t) = \sqrt{\gamma}a(t) + \text{dB}^{\text{in}}(t).
\]
the oscillator variables at the same time vanish. The operator $dB_{\text{out}}(t)$ in eq. (3.1) is the output field generated in $dt$ by the interaction of the system oscillator with the vacuum. The Hamiltonian for this interaction is

$$H_{\text{int}}(t) dt = H_0(a, a^\dagger, t) dt + i\sqrt{\gamma}(a d\sqrt{\gamma}B_{\text{in}}(t) - d\sqrt{\gamma}B_{\text{in}}(t) a^\dagger).$$  \hspace{1cm} (3.3)$$

Here $H_0(a, a^\dagger, t)$ is the part of the Hamiltonian responsible for the generation of the quantum oscillation, in the interaction picture. With the unitary operator

$$U_{\text{int}}(t) = \left[ \exp \left( -i \int_0^t ds H_{\text{int}}(s) \right) \right]_+$$ \hspace{1cm} (3.4)

where $[\ldots]_+$ denotes time-ordering, we may then write

$$dB_{\text{out}}(t) = U_{\text{int}}^\dagger(t) d\sqrt{\gamma}B_{\text{in}}(t) U_{\text{int}}(t)$$ \hspace{1cm} (3.5)

which gives back eq. (3.1). The interaction Hamiltonian and eq. (3.1) are therefore indeed consistent. As $dB_{\text{in}}(t)$ and $dB_{\text{out}}(t)$ are connected by a unitary transformation, the commutation relations of $dB_{\text{out}}(t)$, $dB_{\text{out}}^\dagger(t)$ at equal time coincide with those of $dB_{\text{in}}(t)$, $dB_{\text{in}}^\dagger(t)$.

The signal measured in the homodyne experiment may now be expressed in terms of the output field and its adjoint as

$$|\beta|\Theta(t) = |\beta| \int_0^t d\Theta(s)
$$

$$|\beta|d\Theta(t) \equiv |\beta| \left( dB_{\text{out}}(t) e^{-i\varphi} + dB_{\text{out}}^\dagger(t) e^{i\varphi} \right).$$ \hspace{1cm} (3.6)

Here $\beta e^{i\varphi}$ is the complex amplitude of the classical local oscillator. If this is expressed in terms of the input field we find

$$d\Theta(t) = 2 \sqrt{\gamma} \chi(t) dt + d\Xi(t)$$ \hspace{1cm} (3.7)

with the input field

$$d\Xi(t) = (dB_{\text{in}}(t) e^{-i\varphi} + dB_{\text{in}}^\dagger(t) e^{i\varphi} )$$ \hspace{1cm} (3.8)

and the system observable
\[ X(t) = \frac{1}{2}(ae^{-i\varphi} + a^\dagger e^{i\varphi}) \] (3.9)

measured in this experiment.

The commutation relations (3.2) imply that

\[ [d\Xi(t), d\Xi(t')] = 0 \] (3.10)

and it follows from eq. (3.7) and eq. (3.2) that

\[ [d\Theta(t), d\Xi(t)] = 0 = [d\Theta(t), X(t)]. \] (3.11)

Therefore, there is a representation in which all operators in eq. (3.7) are simultaneously diagonal and may be replaced by their simultaneous eigenvalues, which we shall denote by corresponding lower case letters. Thus

\[ d\theta(t) = 2\sqrt{\gamma}x(t)dt + d\xi(t). \] (3.12)

The measured output field is therefore generated by the incoming vacuum fluctuations and the instantaneous eigenvalue of the \( X \)-quadrature component of the quantum oscillator. The relation (3.12) implies that the output process \( d\theta(t) \), like the input process obeys the rule of Ito calculus \((d\theta(t))^2 = dt\).

Eq. (3.12) defines the output-process \( d\theta(t) \) conditioned on the input-process \( d\xi(t) \) and the value of \( x \) at time \( t \). In particular, if the distributions of \( d\xi(t) \), \( p_\xi(d\xi(t)) \), and of \( x \) at \( t \) are independent, as will turn out to be the case lateron, the distribution of \( d\theta(t) \) conditioned on \( x(t) \) is given by

\[ p_\theta(d\theta(t)) = p_\xi(d\theta(t) - 2\sqrt{\gamma}x(t)dt) \] (3.13)

It remains to derive the stochastic Schrödinger equation. The generator of the total time-evolution follows from

\[ d|\psi(t)\rangle = dU_{\text{int}}(t)|\psi(0)\rangle = dU_{\text{int}}(t)U_{\text{int}}^\dagger(t)|\psi(t)\rangle \] (3.14)

as
\[ L(t)dt = dU_{\text{int}}(t)U_{\text{int}}^\dagger(t) \]  

(3.15)

where \(dU_{\text{int}}(t)\) is calculated from \(U_{\text{int}}(t)\) by using the Ito calculus for the stochastic differential \(dB(t)\) and its adjoint. The result is

\[
L(t)dt = -iH_0(a,a^\dagger,t)dt + \sqrt{\gamma}(adB_{\text{in}}^\dagger(t) - dB_{\text{in}}(t)a^\dagger) - \frac{\gamma}{2}a^\dagger adt \\
-\frac{\gamma}{2}(a^\dagger a + aa^\dagger)dB_{\text{in}}^\dagger(t)dB_{\text{in}}(t).
\]

(3.16)

Initially the total wave-function \(|\psi(0)\rangle \otimes |\{0\}\rangle\) is the vacuum state \(|\{0\}\rangle\) of the field. The total wave function at time \(t\) may be represented as the time-ordered product over discretized time

\[
\left[ \prod_{t_i=0}^{t} (1 + L(t_i)dt_i) \right] |\psi(0)\rangle \otimes |\{0\}\rangle.
\]

(3.17)

Then we may use the fact \([dB_{\text{in}}(t),dB_{\text{in}}^\dagger(t')] = 0\) for \(t \neq t'\) to bring all \(dB(t_i)\) to the right where they are annihilated when hitting \(|\{0\}\rangle\). By the same token we may replace \(dB_{\text{in}}^\dagger(t_i)\) by

\[
d\Xi(t_i)e^{-i\varphi} = dB_{\text{in}}^\dagger(t_i) + e^{-2i\varphi}dB_{\text{in}}(t_i)
\]

(3.18)

which may be further replaced by \(d\xi(t_i)e^{-i\varphi}\) in the \(\xi\)-representation in which all \(d\Xi(t)\) are simultaneously diagonal. Thus \(L(t)\) in eq. (3.17) is reduced to

\[
L(t)dt = -iH_0(a,a^\dagger,t)dt - \frac{\gamma}{2}a^\dagger adt + \sqrt{\gamma}ae^{-i\varphi}d\xi(t).
\]

(3.19)

At this stage there are no operators left in eq. (3.17) which act on \(|\{0\}\rangle\), which may therefore be divided out. The essential point here is that the component of the wave-function representing the environment (i.e. the vacuum) is not eliminated by taking a trace, but rather by division. Therefore no information is lost in this step.

For a single infinitesimal time-step the expression (3.17) now reduces to a linear stochastic wave-equation

\[
d|\psi_\xi(t)\rangle = \left[ -iH_0(a,a^\dagger,t)dt - \frac{\gamma}{2}a^\dagger adt + \sqrt{\gamma}ae^{-i\varphi}d\xi(t) \right]|\psi_\xi(t)\rangle.
\]

(3.20)
In order to recover the \textit{full} wave-equation, including the vacuum which was eliminated, both sides of eq. (3.20) have to be multiplied by $\varphi_0(\{\xi\})$, the vacuum $|\{0\}\rangle$ in the $\xi$-representation. This functional is Gaussian and its absolute square

$$|\varphi_0(\{\xi\})|^2d\{\xi\} = d\mu^W(\{\xi\})$$

(3.21)
gives the classical Wiener measure of the white noise process $\xi(s)$ with $\xi(t) = \int_0^t d\xi(s)$, $(d\xi(s))^2 = ds$. We shall denote by $d\mu^W_t(\{\xi\})$ the Wiener measure of $\xi(s)$ for $0 < s \leq t$. It should be noted that $|\psi(\xi(t))|$ is not normalized. However, the statistical operator defined by $\rho(t) = \int d\mu^W_t(\{\xi\}) |\psi(\xi(t))\rangle\langle\psi(\xi(t))|$ is correctly normalized. If one wishes the usual master equation for $\rho(t)$ can be derived from eq. (3.20), but this is of no concern here.

From the unitarity of the total time-evolution with $H_{\text{int}}$ it follows that all expectation values of oscillator variables $Y$ are given by

$$\langle Y(t) \rangle = \int |\varphi_0(\{\xi\})|^2d\{\xi\} \langle \psi(\xi(t))|Y|\psi(\xi(t)) \rangle$$

$$= \int d\mu^W_t(\{\xi\}) \langle \psi(\xi(t))|Y|\psi(\xi(t)) \rangle. \quad (3.22)$$

More generally all time-ordered multi-time correlation functions of not necessarily hermitian oscillator variables $Y_1, Y_2, Y_3$ etc. are given by

$$\langle Y_1^{+}(t'_1)Y_2^{+}(t'_2)\cdots Y_n^{+}(t'_n)Y_m(t_m)\cdots Y_2(t_2)Y_1(t_1) \rangle$$

$$= \int d\mu^W_t(\{\xi\}) \langle \psi(\xi(t'_1))|Y_1^{+}(t'_2 - t'_1)Y_2^{+}(t'_2 - t'_1)K_\xi^{+}(t'_n - t'_1)Y_n^{+}(t'_n - t'_1) \rangle$$

$$\times Y_mK_\xi(t_m - t_{m-1})\cdots Y_2K_\xi(t_2 - t_1)Y_1|\psi(\xi(t_1))\rangle$$

(3.23)

where $t'_n > t'_{n-1} > \cdots > t'_1$ and $t_m > t_{m-1} > \cdots > t_1$. Here $K_\xi(t)$ is the \textit{(non-unitary)} time-evolution generated by eq. (3.20) which may be written as

$$K_\xi(t) = \left[ \exp \left\{ -i \int_0^t \left( H_0 - i\frac{\gamma}{2}a^+a - \frac{a^++a}{2}a^2e^{-2i\varphi} \right) ds + \sqrt{\gamma}a e^{-i\varphi} \int_0^t d\xi(s) \right\} \right]_+. \quad (3.24)$$

$K_\xi^{+}(t)$ is the adjoint of $K_\xi(t)$. The kernel $L(t)$ in the middle of eq. (3.23) is

$$L_\xi(t) = \begin{cases} K_\xi(t) & \text{if } t > 0 \\ K_\xi^{+}(t) & \text{if } t < 0 \\ 1 & \text{if } t = 0. \end{cases} \quad (3.25)$$
Thus, the full information about the reduced dynamics of the quantum oscillator is contained in the linear stochastic wave-equation (3.20). Even though $K_\xi(t)$ is nonunitary, it is unitary under the integral over $d\mu^W_t(\xi)$,

$$\int d\mu^W_t(\xi) K^\dagger_\xi(t) K_\xi(t) = 1.$$  \hspace{1cm} (3.26)

This may be checked explicitly but also follows immediately from the unitarity of the time-evolution of the total system.

The full information about the output process $d\theta(t)$, regardless of the internal dynamics of the system, is also contained in eq. (3.20). Indeed, the measure $d\mu_t(\{\theta\})$ defined by putting $Y_i = 1 = Y_i^\dagger$ in eq. (3.23)

$$d\mu_t(\{\theta\}) = d\mu^W_t(\{\theta\}) \langle \psi_\theta(t) | \psi_\theta(t) \rangle$$  \hspace{1cm} (3.27)

can be identified with the measure of the output process for $0 < s \leq t$. We note that eq. (3.27) is just a variant of eq. (2.5) of section 2. To see the relation with the output process one computes the conditional measure

$$p(d\theta(t)) = p^W(d\theta(t)) \left( 1 + \frac{d\langle \psi_\theta(t) | \psi_\theta(t) \rangle}{\langle \psi_\theta(t) | \psi_\theta(t) \rangle} \right).$$  \hspace{1cm} (3.30)

From eq. (3.20) we obtain

$$\frac{d\langle \psi_\theta(t) | \psi_\theta(t) \rangle}{\langle \psi_\theta(t) | \psi_\theta(t) \rangle} = \sqrt{\frac{\langle \psi_\theta(t) | a e^{-i\varphi} + a^\dagger e^{i\varphi} | \psi_\theta(t) \rangle}{\langle \psi_\theta(t) | \psi_\theta(t) \rangle}} d\theta(t).$$  \hspace{1cm} (3.31)

Recalling that $p^W(d\theta(t))$ is just a Gaussian, formally written as

$$p^W(d\theta(t)) = \frac{d\theta}{\sqrt{2\pi dt}} \exp \left( -\frac{1}{2} \frac{d\theta(t)^2}{dt} \right),$$  \hspace{1cm} (3.32)
we obtain equally formally, using the Ito calculus with \((d\theta(t))^2 = dt\)

\[
p(d\theta(t)) = \frac{d\theta}{\sqrt{2\pi dt}} \exp \left[ -\frac{1}{2dt} (d\theta - 2\sqrt{\gamma} \langle X(t) \rangle d\theta)^2 \right].
\] (3.33)

Therefore

\[
d\theta(t) = 2\sqrt{\gamma} \langle X(t) \rangle d\theta + d\xi(t)
\] (3.34)

where \(d\xi(t)\) is the Wiener increment and where we defined

\[
\langle X(t) \rangle_\theta = \frac{\langle \psi(t)|X|\psi(t) \rangle}{\langle \psi(t)|\psi(t) \rangle}.
\] (3.35)

Eqs. (3.34) and (3.33) may be compared with eqs. (3.12) and (3.13), respectively. In the latter equations the output process \(d\theta(t)\) was conditioned on the simultaneous eigenvalue \(x(t)\), while in eqs. (3.33), (3.34) it is conditioned, via \(\langle X(t) \rangle_\theta\), on the values of \(d\theta(s)\) for preceding times \(s < t\). Therefore, because their probability distributions differ, the increments defined by eq. (3.12) and (3.34) are not the same. \(\langle X(t) \rangle_\theta\) is the quantum expectation value of the \(X\)-quadrature component for a given realization of the output process at times prior to \(t\).

Let us evaluate the multi-time correlation functions \(\langle d\theta(t_n) \ldots d\theta(t_1) \rangle\) of the output process implied by eq. (3.27) and (3.34). In the measure (3.27) of \(\theta(t)\) we use the representation

\[
|\psi(t)\rangle = K_\theta(t)|\psi(0)\rangle.
\] (3.36)

In (3.27) the integration is performed over the Wiener measure. In order to achieve this explicitly, for each time \(t_i\) appearing in the multi-time correlation functions the infinitesimal step \(dt_i\) in the functional (3.36) of \(d\theta(t_i)\) is represented explicitly as \((1 + L(t_i))dt_i\) like in eq. (3.17) with

\[L(t_i)\] from eq. (3.19), making the dependence on \(d\theta(t_i)\) explicit. The Wiener integrals over all the \(d\theta(t_i)\) can then easily be performed with the result

\[
\langle d\theta(t_n) \ldots d\theta(t_1) \rangle = (2\sqrt{\gamma})^{n} dt_n \ldots dt_1 \langle : X(t_n) \ldots X(t_1) : \rangle
\] (3.37)

where \(:\ldots:\) denotes normal and time-ordering. The result (3.37) holds for arbitrary correlation functions as long as \(t_i \neq t_j\). For all pairs \(t_i, t_j\) with \(t_i = t_j\) one has to take \(d\theta(t_i) d\theta(t_i) = dt_i\).
summing over all possible pairings and consider the correspondingly reduced correlation function.

In summary, the linear stochastic wave-equation has the following uses:

(i) Solving it with $d\xi(t)$ representing Wiener noise it can be used to obtain all correlation functions of system (i.e. oscillator) variables via eqs. (3.23).

(ii) Solving it after replacing $d\xi(t) \rightarrow d\theta(t)$ representing the yet unknown output noise (i.e. taking $d\theta(t)$ arbitrary) one obtains the conditional quantum expectation (3.35) of the $X$-quadrature as a functional of the output noise from eq. (3.35). All correlation functions of the measured $X$-quadrature are then given by the correlation functions (3.37) of (3.34).

(iii) Finally the solution of (i) gives the complete measure of the output field when inserted in eq. (3.27).

If one is interested in generating stochastic numerical samples of $\langle X(t)\rangle_\theta$ and hence $d\theta(t)$ one first makes the replacement $d\xi(t) \rightarrow d\theta(t)$ in the linear stochastic wave-equation and then reexpresses the unknown output process $d\theta(t)$ by the known Wiener process $d\xi(t)$ via eq. (3.34). However, this step generates a nonlinearity. The superposition principle is then lost in these numerical simulations. For nonlinear stochastic wave equations and numerical examples see [4,15,20,21].

IV. EXACTLY SOLVABLE EXAMPLES

We consider two examples where eq. (3.20) can be exactly solved. The first example is a harmonic oscillator periodically driven by some external field with amplitude $F$, and frequency $\omega = \omega_0 + \delta$ in the rotating wave approximation. Thus

$$H_0(a, a^\dagger, t) = \frac{\gamma}{2i} \left( F^* e^{i\delta t} a - F e^{-i\delta t} a^\dagger \right).$$  \hspace{1cm} (4.1)$$

With a coherent initial state
\[ |\psi_\xi(0)\rangle = |\alpha\rangle, \]  
\[ (4.2) \]

where
\[ a|\alpha\rangle = \alpha|\alpha\rangle, \]  
\[ (4.3) \]

the solution of eq. (3.20), after some calculation, is obtained as
\[ |\psi_\xi(t,\alpha)\rangle = \exp \left[ \frac{1}{2} \left( |\alpha(t)|^2 - |\alpha(0)|^2 + \alpha^2(t) - \alpha^2(0) \right) + \gamma \int_0^t ds \alpha(s) \text{Re} \left( F e^{-i\delta s} \right) \right. \]
\[ + \sqrt{7} \int_0^t ds \alpha(s) e^{-i\varphi} \left. \right] |\alpha(t)\rangle, \]  
\[ (4.4) \]

with
\[ \alpha(t) = \left( \alpha - \frac{F}{1 - \frac{2\delta}{\gamma}} \right) e^{-\frac{\gamma}{2}t} + \frac{F}{1 - \frac{2\delta}{\gamma}} e^{-i\delta t}. \]  
\[ (4.5) \]

As any initial state can be expanded in coherent states, with 
\[ f(\alpha) = \frac{1}{\pi} \langle \alpha | \psi_\xi(0) \rangle, \]
\[ |\psi_\xi(0)\rangle = \int d^2 \alpha f(\alpha)|\alpha\rangle \]
\[ (4.6) \]

and due to the linearity of eq. (3.20), the general solution is simply
\[ |\psi_\xi(t)\rangle = \int d^2 \alpha f(\alpha)|\psi_\xi(t, \alpha)\rangle. \]  
\[ (4.7) \]

For a single coherent state as initial condition the measured quadrature
\[ \langle X(t) \rangle_\xi = \frac{\langle \psi_\xi(t, \alpha) | X | \psi_\xi(t, \alpha) \rangle}{\langle \psi_\xi(t, \alpha) | \psi_\xi(t, \alpha) \rangle} = \frac{1}{2} \left[ \alpha(t) e^{-i\varphi} + \text{c. c.} \right] \]  
\[ (4.8) \]

turns out to be independent of the measurement noise. This is no longer true if a superposition of coherent states is present initially.

Using the present method the decay of the initial superposition \( |\psi_\xi(0)\rangle = \text{const.} \left( |\alpha\rangle + | - \alpha\rangle \right) \) is studied in [22]. See also [23,24] where this problem is studied using the nonlinear stochastic wave-equation.

Finally, we determine the measure of the output process. Using eq. (3.27) with \( |\psi_\theta(t)\rangle \) given by eq. (4.7) where \( \xi \) is replaced by \( \theta \), one obtains a rather complicated expression.
Unless \( f(\alpha) \) is an exponential linear, or quadratic plus linear, in the real- and imaginary parts of \( \alpha \), the resulting measure will be non-Gaussian. However, if \( \langle X(t) \rangle_\theta \) is known as a functional of \( \theta \), as for the example (4.8) where it is even independent of \( \theta \), one may use eq. (3.33) to obtain

\[
d\mu_t(\{\theta\}) = d\mu_t^W(\{\theta\}) \exp \left( 2\sqrt{\gamma} \int_0^t \langle X(s) \rangle_\theta d\theta(s) - 2\gamma \int_0^t \langle X(s) \rangle_\theta^2 ds \right). \tag{4.9}
\]

For the example (4.8) the result (4.9) is just a shifted Wiener measure.

As a second example we turn to subharmonic generation from the vacuum, where

\[
H_0(a, a^\dagger) = \frac{\kappa}{2i}(a^2 - a^\dagger a^\dagger) \tag{4.10}
\]

and

\[
|\psi_\xi(0)\rangle = |0\rangle. \tag{4.11}
\]

The ansatz

\[
|\psi_\xi(t)\rangle = \exp \left( g(t) + \alpha(t) a^\dagger + \frac{1}{2} \beta(t) a^\dagger a^\dagger \right) |0\rangle \tag{4.12}
\]

solves eq. (3.20), provided the differential equations

\[
d\beta(t) = -\gamma \beta(t) dt + \left( \kappa - (\kappa + \gamma e^{-2i\varphi}) \beta^2(t) \right) dt
\]

\[
d\alpha(t) = -\left( \frac{\gamma}{2} + (\kappa + \gamma e^{-2i\varphi}) \beta(t) \right) \alpha(t) dt + \sqrt{\gamma} \beta(t) e^{-i\varphi} d\xi(t)
\]

\[
dg(t) = -\frac{\kappa}{2} \left( \alpha^2(t) + \beta(t) \right) dt - \frac{\gamma}{2} \alpha^2(t) e^{-2i\varphi} dt + \sqrt{\gamma} \alpha(t) e^{-i\varphi} d\xi(t) \tag{4.13}
\]

are satisfied with the initial conditions \( \alpha(0) = \beta(0) = g(0) = 0 \). The integration of eqs. (4.13) yields

\[
\beta(t) = \frac{1}{\kappa + \gamma e^{-2i\varphi}} \left( -\frac{\gamma}{2} + \sqrt{\frac{\gamma^2}{4} + \kappa^2 + \kappa \gamma e^{-2i\varphi} \tanh \sqrt{\frac{\gamma^2}{4} + \kappa^2 + \kappa \gamma e^{-2i\varphi} (t - t_0)}} \right)
\]

\[
\alpha(t) = \sqrt{\gamma} \int_0^t e^{-\frac{\gamma}{2}(t-\tau) - (\kappa + \gamma e^{-2i\varphi}) \int_\tau^t \beta(s) ds} \beta(\tau) e^{-i\varphi} d\xi(\tau)
\]

\[
g(t) = -\frac{1}{2} \int_0^t d\tau \left( \kappa \beta(\tau) + (\kappa + \gamma e^{-2i\varphi}) \alpha^2(\tau) \right) + \sqrt{\gamma} \int_0^t \alpha(\tau) e^{-i\varphi} d\xi(\tau). \tag{4.14}
\]
The coefficient $\beta(t)$ describes squeezing. It contains the constant of integration $t_0$, which is fixed by the initial condition $\beta(0) = 0$. The squeezing coefficient $\beta(t)$ is influenced by the measurement through $\gamma$, but unlike the coefficient $\alpha(t)$ which determines the amplitude and is completely generated from the output-noise, the squeezing coefficient $\beta(t)$ is independent of the noise.

It is easy to calculate the prediction for the measured quadrature $\langle X(t) \rangle_\xi$ as a functional of the output noise, here still written as $\xi$, for simplicity. We obtain

$$\langle X(t) \rangle_\xi = \frac{\text{Re}[(\alpha(t) + \beta(t)\alpha^*(t))e^{-i\varphi}]}{1 - |\beta(t)|^2}$$

(4.15)

where $\alpha$ is a complex linear functional of $\xi$. In fact $\langle X(t) \rangle_\xi$ is also a linear functional of $\xi$. The measure for the output noise is given by eq. (3.27) and may be written as

$$d\mu_t(\{\xi\}) = d\mu_t^W(\{\xi\})e^{g(t)+g^*(t)+|\alpha(t)|^2} \langle \alpha(t)|e^{\frac{1}{2}\beta^*(t)a^2}e^{\frac{1}{2}\beta(t)a^\dagger} \rangle \langle \alpha(t) \rangle.$$

(4.16)

The expectation value with respect to coherent states can be evaluated explicitly by the general form of the Baker Haussdorff formula, but we will not go into the necessary algebra here. In fact, like in the first example there is an easier way directly based on eq. (3.33) and the formula (4.15), directly yielding the Gaussian measure (4.9) with $\langle X(t) \rangle_\theta$ given by (4.15), after $\xi$ is replaced by $\theta$.

V. HETERODYNE MEASUREMENTS

In heterodyne measurements the frequency $\omega_{lo}$ of the local oscillator differs from that of the quantum oscillator, $\Omega = \omega_0 - \omega_{lo} \neq 0$. Therefore the phase $\varphi$ in eq. (3.6) must be replaced by $\varphi + \Omega t$. In particular we now define

$$d\Theta(t) = \left(dB_{out}(t)e^{-i\varphi-i\Omega t} + dB_{out}^\dagger(t)e^{i\varphi+i\Omega t}\right)$$

$$d\Xi(t) = \left(dB_{in}(t)e^{-i\varphi-i\Omega t} + dB_{in}^\dagger(t)e^{i\varphi+i\Omega t}\right)$$

$$X(t) = \frac{1}{2}\left( ae^{-i\varphi-i\Omega t} + a^\dagger e^{i\varphi+i\Omega t} \right)$$

(5.1)
and eqs. (3.7), (3.12) hold unchanged. The entire derivation of the preceding section leading up to eq. (3.20) may then be taken over, and we obtain

\[ d|\psi_\xi(t)\rangle = \left[ -iH_0(a, a^\dagger, t)dt - \frac{\gamma}{2}a^\dagger ada t + \sqrt{\gamma}\ae^{-i\varphi-t\Omega t}d\xi(t) \right]|\psi_\xi(t)\rangle \]

which differs from eq. (3.20) only by the replacement

\[ d\xi(t) \to d\xi(t)e^{-i\Omega t}. \]

The examples of section 4 can be extended to the case of heterodyning. However, it is not enough to make the replacement (5.3) in the solutions, because the Ito-rule \((d\xi(t))^2 = dt\) is not analytic in \(d\xi(t)\).

The wave-function (4.4) of the harmonic oscillator is extended to the case of heterodyning by the replacement \(\varphi \to \varphi(s) = \varphi + \Omega s\). As the measured quadrature is independent of the noise it changes only because of the changed definition (5.1), again leading to the replacement \(\varphi \to \varphi + \Omega t\) in eq. (4.8).

In the example of subharmonic generation, the generalization to the case of heterodyning is less straightforward. In eqs. (4.13) it leads to the same replacements \(\varphi \to \varphi + \Omega t\) as before. As a consequence the equation for the squeezing parameter \(\beta(t)\) can no longer be integrated by a separation of variables. However, while \(\beta(t)\) is changed it still remains unaffected by the measurement noise. Furthermore, the solutions for \(\alpha(t)\) and \(g(t)\) in eq. (4.14) remain valid if the change \(\varphi \to \varphi + \Omega t\) is made there. In the case where \(|\Omega|\) is large compared to the characteristic inverse time-scales of the system it is possible to average over many cycles of \(\Omega \) (6,7). Then

\[ e^{-i\varphi}e^{-\alpha t}d\xi = d\xi_1 + id\xi_2 \]

is a complex white noise with real- and imaginary parts \(d\xi_1, d\xi_2\) satisfying

\[ d\xi_1^2 = d\xi_2^2 = \frac{1}{2}dt, \quad d\xi_1d\xi_2 = 0. \]

The dependence on \(\varphi\) and \(\Omega\) in eq. (5.2) then disappears. The two examples of section 4 then simplify again: In the oscillator wave-function (4.4) the term \(\alpha^2(s)e^{-2i\varphi(s)}\) averages out and
disappears, and $d\xi(s)e^{-i\varphi(s)} = d\xi_1(s) + id\xi_2(s)$ is now complex white noise. The measured quadrature (4.8) is unaffected by these simplifications and produces a signal at frequencies $\pm \Omega$.

In the example of subharmonic generation all terms with $e^{-2i\varphi(t)}$ in eq. (4.13) disappear by averaging out and $e^{-i\varphi(s)}d\xi(s)$ becomes complex white noise. The solution is then given by (4.14) without the $e^{-2i\varphi}$-terms and with complex noise. The phase $\varphi(t) = \varphi + \Omega t$ then only remains in $\langle X(t) \rangle_\xi$ in eq. (4.13) and again produces a signal at $\pm \Omega$.

VI. PHOTON COUNTING

In this section we consider the case of photon counting by an ideal photodetector placed behind the partially transmitting mirror of a cavity. The concept of input and output processes used in the preceding section to describe homodyne and heterodyne measurements can be generalized to this case. Let us introduce the input photon number process $\Lambda_{\text{in}}(t)$ defined by (see e.g. [24-26])

$$\Lambda_{\text{in}}(t) = \int_0^t b_{\text{in}}^\dagger(s)b_{\text{in}}(s)ds$$ (6.1)

where formally

$$b_{\text{in}}(s) = \frac{dB_{\text{in}}(s)}{ds}.$$ (6.2)

The increment $d\Lambda_{\text{in}}(t)$ can be expressed in terms of the white-noise creation and annihilation operators as

$$d\Lambda_{\text{in}}(t) = \frac{dB_{\text{in}}^\dagger(t)dB_{\text{in}}(t)}{dt}.$$ (6.3)

It satisfies the commutation relations

$$[d\Lambda_{\text{in}}(t), d\Lambda_{\text{in}}(t')] = 0, \quad [d\Lambda_{\text{in}}(t), X(s)] = 0 \quad \text{for} \ t \geq s$$ (6.4)

for any system variable $X$. We shall denote the eigenvalues of $d\Lambda_{\text{in}}(t)$ by $d\lambda(t)$.
The output number process can be calculated using the Hamiltonian $H_{\text{int}}$ in the interaction representation of eq. (3.3) as a unitary transformation

$$\Lambda_{\text{out}}(t) = U_{\text{int}}^\dagger(t)\Lambda_{\text{in}}(t)U_{\text{int}}(t)$$

(6.5)

of the input number process, with $U_{\text{int}}$ defined in eq. (3.4). The increment $d\Lambda_{\text{out}}$ of the output process satisfies the quantum stochastic differential equation [24-26]

$$d\Lambda_{\text{out}}(t) = d\Lambda_{\text{in}}(t) + \sqrt{\gamma}(a(t)dB_{\text{in}}^\dagger(t) + dB_{\text{in}}(t)a^\dagger(t)) + \gamma a^\dagger(t)a(t)dt$$

(6.6)

where all system operators are in the Heisenberg picture. Note that the output photon number process satisfies the same equal-time commutation relations and multiplication rule as the input process [25,27]. Eq. (6.6) is the analogue of eq. (3.7). The difference lies in the fact that the output number process does not commute with the input number process for equal times as can be seen from eq. (6.6) and the commutation relation $[dB_{\text{in}}(t), dB_{\text{in}}^\dagger(t)] = dt$; as a consequence the input and output number processes at equal time can not be diagonalized simultaneously, i.e. it is not possible to express the eigenvalues of the output process in terms of the $d\lambda(t)$.

Let us consider now the time evolution of the state of the total system. The generator $L(t)dt$ is unchanged and still given by eq. (3.16). Up to now we have not specified the initial state of the bath which is necessary to evaluate the term $dB_{\text{in}}^\dagger(t)dB_{\text{in}}(t)$. Assuming the bath to be in a vacuum state as in section 3 we immediately see from eq. (6.3) that the input number process vanishes, which is clear from a physical point of view: due to its absorbing nature a photodetector cannot ‘see’ the vacuum fluctuations. The mean of the output process is given by $\langle d\Lambda_{\text{out}}(t) \rangle = \gamma \langle a^\dagger a \rangle(t)dt$ as can be seen by taking the expectation value of eq. (6.6) in the vacuum state of the bath. However, in this case it is not clear how to apply the input-output formalism to derive the equation governing the time evolution of the state of the system. A calculation of this time evolution can still be performed in a manner similar to the preceding section if we take the bath to be in a coherent state $|\{\beta\}\rangle$ with
\[ dB_{in}(t)\{\beta\} = \sqrt{\gamma} \epsilon(t) dt \{\beta\} \]  

(6.7)

where \( \sqrt{\gamma} \epsilon(t) \) is the arbitrary complex amplitude of the coherent state. At the end of the calculation we may take the limit \( \epsilon \to 0 \). Physically, this assumption means that not only the vacuum fluctuations but also a small classical field enter the cavity of the quantum oscillator. According to eq. (6.7) the product \( dB_{\text{in}}(t)^\dagger dB_{\text{in}}(t) \) is of second order in \( dt \) and therefore can be set equal to zero in the following (this substitution can be justified rigorously in terms of quantum stochastic integration; see e. g. \( \text{[28]} \)). We then obtain the following multiplication table

\[
\begin{align*}
DB_{\text{in}}(t)DB_{\text{in}}(t) &= 0 = DDB_{\text{in}}(t)^\dagger DDB_{\text{in}}(t) \\
DB_{\text{in}}(t)DB_{\text{in}}(t)^\dagger &= dt \\
DB_{\text{in}}(t)dt &= 0 = DDB_{\text{in}}(t)^\dagger dt \\
D\Lambda_{\text{in}}(t)DB_{\text{in}}(t) &= 0 \\
D\Lambda_{\text{in}}(t)D\Lambda_{\text{in}}(t) &= D\Lambda_{\text{in}}(t) \\
D\Lambda_{\text{in}}(t)^\dagger dt &= 0.
\end{align*}
\]

(6.8)

For the eigenvalues of \( D\Lambda_{\text{in}}(t) = d\lambda(t) \) the condition

\[ (d\lambda(t))^2 = d\lambda(t) \]  

(6.9)

implies

\[ d\lambda(t) = \begin{cases} 
0 \\
1
\end{cases} \]

(6.10)

The \( d\lambda(t) \) at different times are statistically independent due to the statistical independence of the underlying quantum Poisson process at different times.

Proceeding as in section 3 we compute the wave function of the total system at time \( t \) as

\[
|\psi_{\text{tot}}(t)\rangle = \left[ \prod_{t_i=0}^{t} (1 + L(t_i)dt_i) \right] + |\psi(0)\rangle \otimes |\{\beta\}\rangle
\]

(6.11)
where we assume that initially the state of the total system factorizes. Now the operators $dB_{in}(t_i)$ can be commuted to the right where they are replaced by $\sqrt{\gamma} \epsilon(t_i) dt_i$ when acting on $|\{\beta\}\rangle$. Then the Schrödinger equation for the total wave function is given by

$$
\left( d + iH_0(a, a^\dagger, t) dt + \frac{\gamma}{2} a^\dagger a dt - \sqrt{\gamma}(aB_{in}^{\dagger}(t) - \sqrt{\gamma} \epsilon(t)a^\dagger dt) \right) |\psi(t)\rangle \otimes |\{\beta\}\rangle = 0.
$$

(6.12)

Now we make the transformations

$$
\sqrt{\gamma} aB_{in}^{\dagger}(t) |\psi(t)\rangle \otimes |\{\beta\}\rangle = \sqrt{\gamma} aB_{in}^{\dagger}(t) dB_{in}(t) |\psi(t)\rangle \otimes |\{\beta\}\rangle = \frac{1}{\epsilon(t)} ad\lambda_{in}(t) |\psi(t)\rangle \otimes |\{\beta\}\rangle
$$

(6.13)

where we have used eq. (6.3) in the last step. In the representation where the input number process $d\Lambda_{in}(t)$ can be replaced by the eigenvalue $d\lambda(t)$ the Schrödinger equation (6.12) is therefore given by

$$
\left( d + iH_0(a, a^\dagger, t) dt + \frac{\gamma}{2} a^\dagger a dt + \gamma \epsilon(t)a^\dagger dt - \frac{1}{\epsilon(t)} ad\lambda(t) \right) |\psi_\lambda(t)\rangle \langle \lambda |\{\beta\}\rangle = 0
$$

(6.14)

so that the bath state can be divided out. In this way we obtain a linear stochastic Schrödinger equation for the state of the system alone

$$
d|\psi_\lambda(t)\rangle = \left\{ -iH_0(a, a^\dagger, t) dt - \frac{\gamma}{2} a^\dagger a dt - \gamma \epsilon(t)a^\dagger dt + \frac{1}{\epsilon(t)} ad\lambda(t) \right\} |\psi_\lambda(t)\rangle
$$

(6.15)

with the measure

$$
d\mu^P(|\lambda\rangle) = |\langle \lambda |\{\beta\}\rangle|^2.
$$

(6.16)

Here and in the following $|\lambda\rangle$ denotes the number states of the bath satisfying

$$
d\Lambda_{in}(s)|\lambda\rangle = d\lambda(s)|\lambda\rangle
$$

(6.17)

for all $s$. Discretizing time $t \rightarrow t_i$ and using the fact that $d\lambda(t_i)$ is 0 or 1 we obtain the formal expression

$$
d\mu_t^P(|\lambda\rangle) = \prod_{i=1}^{N-1} \left[ (1 - \gamma|\epsilon(t_i)|^2 dt_i)(1 - d\lambda(t_i)) + \gamma|\epsilon(t_i)|^2 dt_i d\lambda(t_i) \right].
$$

(6.18)
Thus $d\lambda(t)$ is an independent Poisson process with mean value

$$\langle d\lambda(t) \rangle = \langle \{\beta\} | d\Lambda_{\text{in}}(t) | \{\beta\} \rangle = \gamma |\epsilon(t)|^2 dt. \quad (6.19)$$

Using eq. (6.15) and the fact that $d\lambda(t) dt = 0$ we can write the wave function of the system at time $t + dt$ in the form

$$|\psi_\lambda(t + dt)\rangle = |\psi_\lambda(t)\rangle + d|\psi_\lambda(t)\rangle$$

$$= (1 - d\lambda(t)) \left( 1 - iH_0(a, a^\dagger, t) dt - \frac{\gamma}{2} a^\dagger a dt - \gamma \epsilon(t) a^\dagger dt \right) |\psi_\lambda(t)\rangle$$

$$+ d\lambda(t) \left( \frac{a}{\epsilon(t)} + 1 \right) |\psi_\lambda(t)\rangle. \quad (6.20)$$

This expression clearly shows how the wave function evolves during the time interval $dt$ depending on the result of the photon counting: if no photon is detected (with probability $1 - \gamma |\epsilon(t)|^2 dt$ according to eq. (6.18)), i.e. $d\lambda(t) = 0$, this time evolution, according to eq. (6.20), is governed by the non-hermitian Hamiltonian $\tilde{H} = H_0 - i\frac{\gamma}{2} a^\dagger a - i\gamma \epsilon(t) a^\dagger$; if a photon is detected (with probability $\gamma |\epsilon(t)|^2 dt$), i.e. $d\lambda(t) = 1$, the wave function jumps according to the second term in eq. (6.20). This time evolution of the wave function due to the linear stochastic Schrödinger equation (6.15) is analogous to the nonlinear prescription introduced phenomenologically by Carmichael [13] to generate numerically what he has called 'quantum trajectories' for the wave function of the system conditioned on the measured photon number (see also [14]). Applying eq. (6.20) repeatedly in time the formal solution of the linear stochastic Schrödinger equation (6.15) given by eq. (6.11) can be written in the more elegant form, with discretized time $t_i$, $t_0 = 0$, $t_N = t$

$$|\psi_\lambda(t)\rangle = \left[ \prod_{i=0}^{N-1} \left( (1 - d\lambda(t_i)) \left( 1 - iH_0(a, a^\dagger, t) dt - \frac{\gamma}{2} a^\dagger a dt_i - \gamma \epsilon(t_i) a^\dagger dt_i \right) \right) \right]_+ |\psi(0)\rangle$$

$$\equiv K_\lambda(t, 0) |\psi(0)\rangle \quad (6.21)$$

which will be useful for the calculation of correlation functions. It is important to note that as a consequence of factoring out the state of the bath in eq. (6.14) the time-evolution operator
\( K_\lambda(t, 0) \) is non-unitary. It is now straightforward to obtain the analogue of eq. (3.23) for arbitrary multi-time correlation functions with \( d\mu^W \) replaced by \( d\mu^P \).

Let us show now how the complete information on the output process is contained in the solution of eq. (6.15). To this end we define a jump process \( \kappa(t) \), with \( (d\kappa(t))^2 = d\kappa(t) \), which is distributed according to the measure

\[
d\mu_t(\{\kappa\}) = d\mu_t^P(\{\kappa\}) \langle \psi_\kappa(t)|\psi_\kappa(t) \rangle
\]

(6.22)

where \( d\mu_t^P(\{\kappa\}) \) is the Poisson measure (6.18) of the input process. The measure (6.22) is normalized, i. e.

\[
\int d\mu_t(\{\kappa\}) = \int d\{\kappa\} |\langle \{\beta\}|\kappa \rangle|^2 \langle \psi_\kappa(t)|\psi_\kappa(t) \rangle = \int d\{\kappa\} \langle \psi_\kappa(t)|\kappa \rangle \otimes \langle \{\beta\}|\kappa \rangle \langle \kappa|\beta \rangle \otimes |\psi_\kappa(t)\rangle
\]

(6.23)

because the total wave-function of system and environment is normalized. Going through the same argument as in section 3, eq. (3.27)-(3.35), we obtain for the conditional measure of a single time step

\[
p(d\kappa(t)) = p^P(d\kappa(t)) \left( 1 + \frac{d\langle \psi_\kappa(t)|\psi_\kappa(t) \rangle}{\langle \psi_\kappa(t)|\psi_\kappa(t) \rangle} \right)
\]

(6.24)

where

\[
p^P(d\kappa(t)) = (1 - \gamma|\epsilon|^2dt)(1 - d\kappa(t)) + \gamma|\epsilon|^2dt d\kappa(t).
\]

(6.25)

The increment of the norm of \( |\psi_\kappa(t)\rangle \) can be calculated from eq. (6.24) and we obtain

\[
p(d\kappa(t) = 0) = (1 - \gamma|\epsilon|^2dt) \left( 1 - \gamma\epsilon(t)a^\dagger + \epsilon^*(t)a_\kappa dt - \gamma\langle a^\dagger a_\kappa \rangle dt \right)
\]

(6.26)

\[
= 1 - \gamma|\epsilon(t)|^2dt - \gamma\epsilon(t)a^\dagger + \epsilon^*(t)a_\kappa dt - \gamma\langle a^\dagger a_\kappa \rangle dt
\]

\[
p(d\kappa(t) = 1) = \gamma|\epsilon(t)|^2dt \left( 1 + \frac{\langle a_\kappa \rangle}{\epsilon(t)} + \frac{\langle a^\dagger_\kappa \rangle}{\epsilon^*(t)} + \frac{\langle a^\dagger_\kappa a_\kappa \rangle}{|\epsilon(t)|^2} \right) - \gamma\epsilon(t)a^\dagger + \epsilon^*(t)a_\kappa dt - \gamma\langle a^\dagger a_\kappa \rangle dt
\]

(6.26)

where we defined for an arbitrary operator \( \Omega \).
The expectation value of $d\kappa(t)$ according to this normalized probability distribution is simply given by

$$\langle d\kappa(t) \rangle = p(d\kappa(t) = 1).$$  (6.28)

Comparing eq. (6.26) with the expectation value of eq. (6.6) in the coherent state of the bath we see that the expectation value of the process $d\kappa(t)$ introduced by eq. (6.22) is equal to that of the output process $d\Lambda_{\text{out}}(t)$.

We can go even further and show the equality of all correlation functions of $d\kappa(t)$ and the corresponding correlation functions of $d\Lambda_{\text{out}}(t)$, which means that $d\kappa(t)$ is stochastically equivalent to the output process. The calculation of correlation functions of $d\kappa(t)$ is greatly simplified by the use of the formal solution (6.21) of the linear stochastic Schrödinger equation (6.15) (see also section 3). Let us consider first the two-time correlation function

$$\int d\mu t + \tau (\{\kappa\})(d\kappa(t) d\kappa(t)) = \langle d\kappa(t + \tau)d\kappa(t) \rangle$$ (6.29)

with $\tau > 0$. According to eq. (6.22) we can reexpress eq. (6.29) in terms of the two-time correlation function of the input process as

$$\langle d\kappa(t + \tau)d\kappa(t) \rangle = \int d\mu_{t+\tau}^P(\{\lambda\}) d\lambda(t + \tau)d\lambda(t)(\psi_\lambda(t + \tau + d\tau)|\psi_\lambda(t + \tau + d\tau))$$ (6.30)

where $|\psi_\lambda(t + \tau + d\tau))$ is given by eq. (6.21). Now we assume that the wave function at time $t$ is given and not conditioned on the input process, so that $|\psi(t))$ is simply the initial state of the system. The components of $|\psi_\lambda(t + \tau + d\tau))$ with $d\lambda(t) = 0$ or $d\lambda(t + \tau) = 0$ cannot contribute in the integral (6.30). Therefore we can rewrite eq. (6.30) as

$$\langle d\kappa(t + \tau)d\kappa(t) \rangle = \int d\mu_{t+\tau}^P(\{\lambda\}) d\lambda(t)d\lambda(t)(\psi_\lambda(t + \tau + d\tau)|\psi_\lambda(t + \tau + d\tau)) \times pP(d\lambda(t) = 1)pP(d\lambda(t + \tau) = 1)$$ (6.31)

with the Poisson measure conditioned on $d\lambda(t) = d\lambda(t + \tau) = 1$ and
\[ |\psi_{\lambda}(t + \tau + d\tau)\rangle = \left(\frac{a}{\epsilon(t + \tau)} + 1\right) K_{\lambda}(t + \tau, t + dt) \left(\frac{a}{\epsilon(t)} + 1\right) |\psi(t)\rangle. \] (6.32)

The non-unitary evolution operator \( K_{\lambda}(t + \tau, t + dt) \) is defined in eq. (6.21) and we have used again the fact that \( d\lambda(t) = d\lambda(t + \tau) = 1 \). Since the probability \( p^{\lambda}(d\lambda(t) = 1) \) is independent of the state \( |\psi_{\lambda}(t)\rangle \) of the system we obtain

\[
\langle d\kappa(t + \tau)d\kappa(t)\rangle = \gamma^2 |\epsilon(t)|^2 |\epsilon(t + \tau)|^2 dt^2 \int d\mu_{i+\tau}^P(\{\lambda\}) \langle \psi(t)\rangle \left(\frac{a^\dagger}{\epsilon^*(t + \tau)} + 1\right) K_{\lambda}^\dagger(t + \tau, t + dt) \times
\]
\[
\times \left(\frac{a^\dagger}{\epsilon^*(t + \tau)} + 1\right) \left(\frac{a}{\epsilon(t + \tau)} + 1\right) K_{\lambda}(t + \tau, t + dt) \left(\frac{a}{\epsilon(t)} + 1\right) |\psi(t)\rangle
\]
\[
= \gamma^2 dt^2 \int d\mu_{i+\tau}^P(\{\lambda\}) \langle \psi(t)\rangle |(a^\dagger + \epsilon^*(t))K_{\lambda}^\dagger(t + \tau, t + dt)(a^\dagger + \epsilon^*(t + \tau))| \times
\]
\[
\times (a + \epsilon(t + \tau))K_{\lambda}(t + \tau, t + dt)|\psi(t)\rangle. \] (6.33)

Due to the non-unitarity of \( K_{\lambda}(t + \tau, t + dt) \) it is not possible, in general, to insert \( K_{\lambda}K_{\lambda}^\dagger \) between factors at will. However, because the total time evolution of system and reservoir is unitary, we still have unitarity under the integral

\[
\int d\mu_{i+\tau}^P(\{\lambda\}) K_{\lambda}^\dagger(t + \tau, t + dt)K_{\lambda}(t + \tau, t + dt) = 1 \] (6.34)

as may be checked by using eq. (6.21). Therefore under the integral over \( d\mu^P(\{\lambda\}) \) the stochastic time-evolution operator \( K_{\lambda}(t + \tau, t + dt) \) may be treated as unitary, and in this sense \( K_{\lambda}(t) \) can be used to define a stochastic Heisenberg picture,

\[
\Omega_{\lambda}(t) = K_{\lambda}^\dagger(t, 0 + dt)\Omega K_{\lambda}(t, 0 + dt). \] (6.35)

Using this we may now rewrite eq. (6.33) as

\[
\langle d\kappa(t + \tau)d\kappa(t)\rangle = \gamma^2 dt^2 \int d\mu_{i+\tau}^P(\{\lambda\}) \langle \psi|\tilde{a}_{\lambda}^\dagger(t)\tilde{a}_{\lambda}(t + \tau)\tilde{a}_{\lambda}(t + \tau)\tilde{a}_{\lambda}(t)|\psi\rangle \] (6.36)

where we used \( \tilde{a}(t) = a(t) + \epsilon(t) \). The right-hand side is the standard form for the degree of second order coherence \( g^{(2)}(t, t + \tau) \) to be found in textbooks on quantum optics (see e. g. [29]).

The above procedure can be generalized in a straightforward manner for the calculation of correlation functions of higher order. The general \( n \)-time correlation function for unequal times \( t_i \neq t_j \) for all \( i, j \) is thereby obtained as
\[ \langle d\kappa(t_n) \ldots d\kappa(t_1) \rangle = \gamma^n d^n \int d\mu_{t_n}(\{\lambda\}) \langle \psi| \tilde{a}^\dagger(\lambda)(t_1) \ldots \tilde{a}^\dagger(\lambda(t_n)) \tilde{a}(\lambda(t_n)) \ldots \tilde{a}(\lambda(t_1))|\psi \rangle \]  \hspace{1cm} (6.37)

where we have assumed that \( t_n > t_{n-1} > \ldots t_1 \). Eq. (6.37) is equal to the degree of \( n \)-th order coherence.

To complete the comparison of the process \( d\kappa(t) \) with the output process \( d\Lambda_{\text{out}}(t) \) we compute the general normal ordered \( n \)-time correlation function of the output process

\[ \langle :d\Lambda_{\text{out}}(t_n) \ldots d\Lambda_{\text{out}}(t_1) : \rangle = \langle \psi(t_1)| \otimes (\{\beta\}| :d\Lambda_{\text{out}}(t_n) \ldots d\Lambda_{\text{out}}(t_1) : |\{\beta\} \otimes |\psi(t_1)\rangle \]  \hspace{1cm} (6.38)

where \( : : \) denotes normal and time-ordering. Note that the expectation value is taken in the state of the total system because the output process \( d\Lambda_{\text{out}}(t) \) acts in the Hilbert space of the total system. Expressing \( d\Lambda_{\text{out}}(t) \) in terms of the input field by eq. (6.6) and using the statistical independence of \( d\Lambda_{\text{in}}(t), dB_{\text{in}}(t) \) and \( dB_{\text{in}}^\dagger(t) \) at different times we finally obtain

\[ \langle :d\Lambda_{\text{out}}(t_n) \ldots d\Lambda_{\text{out}}(t_1) : \rangle = \gamma^n dt^n \langle \tilde{a}^\dagger(t_1) \ldots \tilde{a}^\dagger(t_n) \tilde{a}(t_n) \ldots \tilde{a}(t_1) \rangle \]  \hspace{1cm} (6.39)

where the expectation value on the right side of eq. (6.39) is again taken in the state of the total system. Now comparing eq. (6.39) with eq. (6.37) we see that both expressions are equal and \( d\kappa(t) \) is indeed statistically equivalent to the output process.

Let us now make contact to the nonlinear stochastic Schrödinger equation used by others. This is done by replacing the input process \( d\lambda(t) \) in eq. (6.15) by the output process \( d\kappa(t) \), i.e.

\[ \frac{d\lambda(t)}{\sqrt{\langle d\lambda(t) \rangle}} \rightarrow \frac{d\kappa(t)}{\sqrt{(\gamma|\epsilon(t)|^2 + \langle \epsilon(t) a^\dagger + e^{*}(t)a_{\kappa} + \langle a^\dagger a \rangle_{\kappa}) \rangle(dt) \sqrt{P(t)\, dt}}} \]  \hspace{1cm} (6.40)

where \( P(t)dt = \langle d\kappa(t) \rangle \). This replacement is phenomenological in distinction to the microscopic derivation of eq. (6.15) we have given above. It corresponds to the transition from the linear equation (2.2) to the nonlinear equation (1.3) by taking \( \tilde{P}_{\hat{N}} \) as equal to the output probability \( |\langle \psi_{\hat{N}}|\psi \rangle|^2 \). In this way we obtain a nonlinear stochastic Schrödinger equation for the non-normalized wave function \( |\psi_{\kappa}(t)\rangle \)

\[ d|\psi_{\kappa}(t)\rangle = \left\{ -iH_0(a, a^\dagger, t)dt - \frac{\gamma}{2} a^\dagger adt - \gamma \epsilon(t)a^\dagger dt + \sqrt{\gamma} a \frac{\sqrt{P(t)\, dt}}{\sqrt{P(t)}} d\kappa(t) \right\} |\psi_{\kappa}(t)\rangle \]  \hspace{1cm} (6.41)
The nonlinearity arises because $P(t)$ depends on the wave-function due to the appearance of expectation values in its definition (6.40). The superposition principle is thereby destroyed. Performing the limit $\epsilon(t) \rightarrow 0$, i.e. setting the coherent driving field equal to zero this stochastic Schrödinger equation takes on the form

$$
d|\psi_\kappa(t)\rangle = \left\{ -iH_0(a, a^\dagger, t)dt - \frac{\gamma}{2}a^\dagger adt + \frac{a}{\sqrt{\langle a^\dagger a \rangle_\kappa}}d\kappa(t) \right\} |\psi_\kappa(t)\rangle \quad (6.42)
$$

and for the normalized $|\phi_\kappa(t)\rangle = |\psi_\kappa(t)\rangle/\langle \psi_\kappa(t) | \psi_\kappa(t) \rangle^{1/2}$

$$
d|\phi_\kappa(t)\rangle = \left\{ -iH_0(a, a^\dagger, t)dt - \frac{\gamma}{2}(a^\dagger a - \langle a^\dagger a \rangle)dt + \left( \frac{a}{\sqrt{\langle a^\dagger a \rangle_\kappa}} - 1 \right) d\kappa(t) \right\} |\phi_\kappa(t)\rangle \quad (6.43)
$$

which is identical to the nonlinear stochastic Schrödinger equation introduced by Carmichael [15] (see also [4]). Note that the classical output process cannot be reexpressed in terms of the classical input process as was already stated in the beginning of this section. It should also be noted that the limit $\epsilon \rightarrow 0$ obviously cannot be taken in the linear equation (6.15). However, even in eq. (6.41) this limit cannot be taken in a strict form, because the resulting eqs. (6.42), (6.43) are ill defined for the vacuum state, where $\langle a^\dagger a \rangle_\kappa = 0$.

Finally, for illustration we apply the linear stochastic Schrödinger equation (6.15) to the periodically driven harmonic oscillator already studied in section 4, i.e. we solve the linear stochastic Schrödinger equation

$$
d|\psi_\lambda(t)\rangle = \left\{ -\frac{\gamma}{2} \left( F^* e^{i\delta t} a - F e^{-i\delta t} a^\dagger \right) dt - \frac{\gamma}{2} a^\dagger adt - \gamma \epsilon(t) a^\dagger dt + \frac{a}{\epsilon(t)} d\lambda(t) \right\} |\psi_\lambda(t)\rangle \quad (6.44)
$$

where initially $|\psi(0)\rangle = |\alpha_0\rangle$. Inserting the ansatz $|\psi_\lambda(t)\rangle = g(t)|\alpha(t)\rangle$ in eq. (6.44) gives the following two differential equations

$$
d\alpha(t) = -\left( \frac{\gamma}{2} \alpha(t) + \gamma \epsilon(t) - \frac{\gamma}{2} F e^{-i\delta t} \right) dt
$$

$$
dg(t) = \left[ -\frac{\gamma}{2} \left\{ |\alpha(t)|^2 + i\text{Im} \left( F^* \alpha(t) e^{i\delta t} \right) + 2\text{Re}(\alpha^*(t) \epsilon(t)) \right\} dt + \frac{\alpha(t)}{\epsilon(t)} d\lambda(t) \right] g(t) \quad (6.45)
$$

with initial conditions $g(0) = 1$, $\alpha(0) = \alpha_0$. The integration of the differential equation for $\alpha(t)$ yields
\[ \alpha(t) = \left( \alpha_0 - \frac{F}{1 - \frac{2i\delta}{\gamma}} \right) e^{-\frac{2i\delta}{\gamma} t} + \frac{F}{1 - \frac{2i\delta}{\gamma}} e^{-i\delta t} - \gamma e^{-\frac{2i\delta}{\gamma} t} \int_0^t ds \, \epsilon(s) e^{\frac{2i\delta}{\gamma} s} \] (6.46)

while the solution for \( g(t) \) can be written in the manner of eq. (6.21) as

\[ g(t) = \prod_{i=0}^{N-1} \left[ (1 - d\lambda(t_i)) \left( -\frac{\gamma}{2} \left\{ |\alpha(t_i)|^2 + i \text{Im} \left( F^* \alpha(t_i) e^{i\delta t_i} \right) + 2 \text{Re}(\alpha^*(t_i) \epsilon(t_i)) \right\} \right) + d\lambda(t_i) \frac{\alpha(t_i)}{\epsilon(t_i)} \right] \] (6.47)

with \( t_0 = 0, t_N = t \).

**VII. CONCLUSION**

Traditionally the time evolution of quantum systems comes in two forms, one is linear, unitary, and deterministic, and describes systems which are completely isolated, even from any external measuring apparatus; the other is nonlinear, non-unitary and random and describes the results of measurements. In the present paper we have discussed how the non-linearity in the description of measurements can be avoided, first for the usual schematic and idealized general description of quantum measurements, and then more concretely for the well-known quantum optical measurement schemes of homodyne- and heterodyne measurements and photon-counting. In each of these concrete examples we have given microscopic derivations of the linear wave-equations, which govern the measured systems. In the limit of vanishing coupling with the measurement devices (\( \gamma \to 0 \)) they reduce to the Schrödinger equation for closed systems. The existence of these equations should dispel the old believe that the two forms of quantum dynamics mentioned above are incompatible. Rather they appear here as limiting forms of the microscopically founded description we have given. This is only possible because a *linear* description of non-isolated measured quantum systems exists, which preserves the validity of the superposition principle also for this class of systems. Besides some practical advantages in certain special cases this is the fundamental reason why we have put all emphasis on the linear form of the wave-equations and mentioned their equivalent nonlinear counterparts only in passing. In this linear description the effects of
the environment or the measurement apparatus resides only in the non-unitary dissipative terms and the non-unitary stochastic terms of the wave-equation. These alone are sufficient to produce the familiar disappearance of interference terms as the coupling of the system to the external world is increased, a nonlinearity of any form is not required to produce this effect.

For the three measurement schemes which we have discussed in detail we have shown that the solutions of the linear wave-equations yield:

- expectation values and correlation functions of all system variables from formulas like eq. (3.23), where the unnormalized stochastic expectation value taken with the stochastic wave-function is averaged over the Wiener noise, in the case of homodyne or heterodyne measurements, or over the Poisson noise in the case of photon counting;

- stochastic realizations according to their correct measure of the measured output field \( \Theta(t) \) given by eqs. (3.6) or (5.1), or \( \Lambda_{\text{out}}(t) \) of eqs. (5.3), (5.6) in the case of photon-counting, via the normalized stochastic expectation values \( \langle X(t) \rangle_\theta \) or \( \langle \hat{a}^\dagger(t)\hat{a}(t) \rangle_\kappa \) in eqs. (3.33) or (6.26), respectively; this result is particularly important because it allows to evaluate all correlation-functions, power spectra, etc. of the measured quantities from a single, sufficiently long time series via eqs. (3.37) or (6.37), respectively.

- the complete measure of the output field of the measurement via eq. (3.27), or eq. (6.22), respectively.

The practical advantage of the linear description we have illustrated by giving exact solutions for a number of examples. Of course many more examples for exact solutions can and undoubted will now be given: all examples of nonlinear quantum optics which allow for a linearization (e.g. linear parametric amplifiers, Raman amplifiers, nonlinear quantum oscillators like lasers if operating far above threshold) fall in this class. However, the linear stochastic wave-equations present crucial advantages even if intrinsically nonlinear physical processes are considered whose quantum effects under measurements or interactions with
reservoirs are of interest — e. g. the important class of nonlinear mesoscopic systems exhibiting at the same time quantum interference effects like dynamical localization or Ericson-type fluctuations and traces of classical chaos. Obviously, if stochastic wave-equations are chosen to describe such systems it is only their linear version which will permit to give a transparent description of such quantum interference effects. For instance path-integral solutions of the linear wave-equations exist and semi-classical approximations can be based on them.

So far the theory has only been developed within the Markovian framework. In fact, the reliance on the Markovian limit is heavy and one may suspect that this limit is really essential in order to achieve the description of the subsystem by a stochastic wave-function. Although this may be possible, it has not been shown, and non-Markovian generalizations of the formalism we have presented remain a challenge.

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