Chaos and Chaotic Phase Mixing in Galaxy Evolution and Charged Particle Beams

Henry E. Kandrup

University of Florida, Gainesville, FL 32611, USA

Abstract. This paper discusses three new issues that necessarily arise in realistic attempts to apply nonlinear dynamics to galaxy evolution, namely: (i) the meaning of chaos in many-body systems, (ii) the time-dependence of the bulk potential, which can trigger intervals of transient chaos, and (iii) the self-consistent nature of any bulk chaos, which is generated by the bodies themselves, rather than imposed externally. Simulations and theory both suggest strongly that the physical processes associated with galactic evolution should also act in nonneutral plasmas and charged particle beams. This in turn suggests the possibility of testing this physics in real laboratory experiments, an undertaking currently underway.

1 Introduction

As recently as 1990, most galactic dynamicists ignored completely the possible role of chaos in galaxies. However, the past decade has seen a growing recognition that chaos can be important in determining the structure of real galaxies. Still, much recent work involving chaos in galactic astronomy has been simplistic in that it has involved naive applications of standard techniques from nonlinear dynamics developed to analyse two- and three-degree-of-freedom time-independent Hamiltonian systems. The object here is to discuss some of the additional complications which arise if nonlinear dynamics is to be applied to real galaxies, which are many-body systems comprised of a large number of interacting masses and characterised by a self-consistently determined bulk potential which, during their most interesting phases, can be strongly time-dependent.

2 Transient chaos and collisionless relaxation

2.1 Transient chaos induced by parametric resonance

It is well known to nonlinear dynamicists that the introduction of an oscillatory time-dependence into even an otherwise integrable potential can trigger an interval of transient chaos, during which many orbits exhibit an exponentially sensitive dependence on initial conditions. Physically, this transient chaos arises from a resonance overlap between the frequencies \( \sim \Omega \) of the unperturbed orbits and the frequency or frequencies \( \sim \omega \) of the time-dependent perturbation.

In the past, the possible effects of such chaos have been considered for both nonneutral plasmas and charged particle beams. More recently, such transient chaos has also begun to be considered in the context of galactic astronomy. That work has shown that, for large fractional amplitudes, \( > 0.1 \) or
so, this resonance can be very broad, triggering significant amounts of chaos for \(10^{-1} \leq \omega/\Omega \leq 10\); and that the existence of the phenomenon is robust, comparatively insensitive to details. It will, for example, persist if one allows for damped oscillations and/or modest drifts in frequency (generated, e.g., by making \(\omega\) a random variable sampling an Ornstein-Uhlenbeck colored noise process).

The breadth of the resonance and the insensitivity to details suggest that transient chaos could well prove common, if not generic, in violent relaxation \([4]\), the collective process whereby a (nearly) collisionless galaxy or galactic halo evolves towards an equilibrium or near-equilibrium state. Violent relaxation typically involves damped oscillations triggered, e.g., by interactions with another galaxy or, in the early Universe, by the cosmological details preceding galaxy formation. However, when considering collective effects there is only one natural time scale, the dynamical time \(t_D \sim 1/\sqrt{G\rho}\), with \(\rho\) a typical mass density, which determines both the characteristic orbital time scale and (at least initially) the oscillation time scale. The exact numerical values of these time scales will involve numerical coefficients which will in general be unequal and vary as a function of location within the galaxy. If, however, one need only demand that the oscillation and orbital time scales agree to within an order of magnitude, it would seem likely that this resonance could trigger transient chaos through large parts of the galaxy. In real galaxies the oscillations will presumably damp and the frequencies drift as the density changes and, presumably, power cascades from longer to shorter scales. To the extent, however, that the details are unimportant such variations should not obviate the basic effect.

2.2 Chaotic phase mixing and collisionless relaxation

But why might such transient chaos prove important in galactic evolution? Detailed numerical simulations indicate that violent relaxation can be a very rapid and efficient process, but simple models involving regular orbits, such as Lynden-Bell’s \([4]\) balls rolling in a pig-trough, do not approach an equilibrium nearly fast enough \([5]\). The important point, however, is that allowing for the effects of chaos can in principle dramatically accelerate both the speed and efficacy of violent relaxation. An initially localised ensemble of regular orbits evolved into the future in a time-independent potential will begin by diverging as a power law in time and, only after a very long period, slowly evolve towards a time-averaged equilibrium, i.e., a uniform population of the KAM tori to which it is restricted. By contrast, a corresponding ensemble of chaotic orbits will begin by diverging exponentially at a rate that is comparable to a typical value of the largest finite time Lyapunov exponent for the orbits in the ensemble; and then converge exponentially towards an equilibrium or near-equilibrium state at a somewhat smaller, but still comparable, rate. The exponential character of this chaotic phase mixing means that the time scale associated with this process is typically far shorter than the time scale associated with regular phase mixing \([6,7,8]\).

It is evident that chaotic phase mixing in a time-independent Hamiltonian system can trigger a very rapid approach towards an equilibrium, but this does not necessarily ‘explain’ violent relaxation. If, e.g., most of the orbits in the
system are regular, it would seem unlikely that chaotic phase mixing could be sufficiently ubiquitous to explain an approach towards a (near-)equilibrium for the galaxy as a whole. Indeed, one would expect that, for a galaxy that is in or near equilibrium the relative measure of chaotic orbits should be comparatively small: If the galaxy exhibits nontrivial structures like a bar or a cusp, the types of structures which one has come to associate with chaos, one would also expect large numbers of regular orbits must be present to serve as a ‘skeleton’ to support that structure. Moreover, it is evident that, although chaotic mixing in a time-independent potential can be very efficient in mixing orbits on a constant energy surface, the energy of each particle remains conserved, so that there can be no mixing in energies. The extent to which chaotic phase mixing in a time-dependent potential will trigger an efficient shuffling of energies is not completely clear.

The important point, then, is that chaotic phase mixing associated with transient chaos in a time-dependent potential is likely to explain these remaining lacunae. At least for large amplitude perturbations, (say) 10% or more, this parametric resonance can trigger a huge increase in the relative abundance of chaotic orbits so that, for pulsation frequencies near the middle of the resonance, virtually all the orbits exhibit substantial exponential sensitivity. Moreover, given that this chaos involves a resonant coupling, it tends typically to cause a substantial shuffling of energies: those frequencies which are most apt to trigger lots of chaos are also apt to induce the largest shuffling of energies.

**Fig. 1.** (top) The emittance $\epsilon_x$ for an ensemble of initial conditions evolved in an integrable Plummer potential subjected to damped oscillations with four different frequencies. (bottom) $x$-$y$ scatter plots corresponding to the uppermost curve.
Still it should be noted that one can get a ‘near-complete’ shuffling of orbits on different constant energy surfaces even if the orbital energies are not especially well shuffled. This, however, is not necessarily a problem. Simulations of systems exhibiting efficient collective relaxation do not necessarily involve masses which completely ‘forget’ their initial conditions. Rather, comparatively efficient and complete violent relaxation is completely consistent with an evolution in which masses ‘remember’ (at least partially) their initial binding energies, i.e., in which masses that start with comparatively large (small) binding energies end up with comparatively large (small) binding energies.

That it may be possible to achieve efficient chaotic phase mixing in an oscillating galactic potential while still relaxing towards a nearly integrable state within $10t_D$ or so is illustrated in Fig. 1. This Figure was generated from orbits evolved in a time-dependent potential of the form

$$V(x, y, z, t) = -\frac{m(t)}{(1 + x^2 + y^2 + z^2)^{1/2}}; \quad m(t) = 1 + \delta m \frac{\sin \omega t}{(t_0 + t)^p}, \tag{1}$$

with $\delta m = 0.5$, $t_0 = 100$ and $p = 2$, which represents a galaxy damping towards an integrable Plummer sphere. The four curves in the top panel exhibit the $x$-component of the phase space emittance, $\epsilon_i = (\langle r_i^2 \rangle \langle v_i^2 \rangle - \langle r_i v_i \rangle^2)^{1/2}$ ($i = x, y, z$), all computed for the same localised ensemble of initial conditions, but allowing for four different frequencies $\omega$. The curves exhibit considerable structure but, at least for early times, the overall evolution is exponential. The bottom panels exhibit the $x$ and $y$ coordinates at five different times for the ensemble represented by the uppermost of the four curves. Here $t_D \sim 20$, so that $t = 256$ corresponds to roughly $12t_D$.

Intuitively, one might expect that strong oscillations, which trigger the largest finite time Lyapunov exponents and the largest number of chaotic orbits, would yield the fastest chaotic phase mixing and, hence, the most rapid and most complete violent relaxation. A time-dependence with a weaker oscillatory component, e.g., a time-dependence corresponding to a near-homologous collapse, might instead be expected to yield less chaos and, hence, less efficient and less complete violent relaxation. There is, therefore, an important need to determine the extent to which, in real simulations of violent relaxation, many/most of the orbits (or phase elements) are strongly chaotic, and the degree to which the rate and completeness of the observed violent relaxation correlate with the size of the largest finite time Lyapunov exponents and/or the relative measure of chaotic orbits. Investigations of these issues are currently underway.

3 The role of discreteness effects

3.1 Microchaos and macrochaos

The discussion in the preceding section, like most applications of nonlinear dynamics to galactic astronomy, neglects completely discreteness effects associated with the ‘true’ many-body potential, assuming that masses in a galaxy can be
approximated as evolving in a smooth, albeit time-dependent, three-dimensional potential and that ‘chaos’ has its usual meaning. That this is justified is not completely obvious. The gravitational $N$-body problem for a large number of bodies of comparable mass is strongly chaotic in the sense that individual orbits have large positive Lyapunov exponent $\chi_N$ even when there is absolutely no chaos in the continuum limit! If, e.g., a smooth density distribution corresponding to an integrable potential is sampled to generate an $N$-body density distribution, one finds that orbits evolved in this $N$-body distribution will be strongly chaotic, even for very large $N$, despite the fact that characteristics in the smooth potential generated from the same initial condition are completely integrable. Moreover, there is no sense in which the exponential sensitivity decreases with increasing $N$: if anything $\chi_N$ is an increasing function of $N$ \[1\]. In this sense, larger $N$ implies more chaos, not less!

This situation has led some astrophysicists to question, either implicitly or explicitly, the reliability of the entire smooth potential approximation. Thus, e.g., it has been suggested \[12\] that “the approximation of a smooth potential is useful for studying orbits, but not for studying their divergence.” This is of course a problematic statement in that the distinction between exponential and power law divergence, emblematic of the differences between regular and chaotic behaviour, lies at the heart of applications of nonlinear dynamics to galactic dynamics. If the Lyapunov exponents associated with the bulk potential have nothing to do with the $N$-body problem, one must perforce reject completely all conventional applications of nonlinear dynamics to galactic astronomy.

The crucial point, then, is that there does appear to be a well-defined continuum limit, even at the level of individual orbits \[13,14,15\]. Suppose that a smooth density distribution, corresponding either to an integrable potential or to a potential admitting large measures of regular orbits, is sampled to generate a fixed, i.e., frozen in time, $N$-body density distribution, and that the trajectories of test particles evolved in this frozen distribution are compared with smooth potential characteristics with the same initial conditions. In this case, there is a precise sense in which, as $N$ increases, the frozen-$N$ trajectories converge towards the smooth potential characteristics. Both visually and in terms of the complexity \[16\] of their Fourier spectra, the frozen-$N$ trajectories come to more closely resemble the smooth potential characteristics; and, viewed mesoscopically, the frozen-$N$ and smooth potential orbits remain closer in phase space for progressively longer times. In particular, a frozen-$N$ orbit corresponding to an integrable characteristic will have a large Lyapunov exponent $\chi_N$ even if, visually, it is essentially indistinguishable from the regular characteristic!

But how can this be? The key recognition here is that two ‘types’ of chaos can be present in the $N$-body problem, characterised by two different sets of Lyapunov exponents associated with physics on different scales. Close encounters between particles trigger microchaos, a generic feature of the $N$-body problem, which leads to large positive Lyapunov exponents $\chi_N$. If, however, the bulk smooth potential is chaotic, one will also observe macrochaos, which is again characterised by positive, albeit typically much smaller, Lyapunov expo-
ponents $\chi_N$. Suppose, for example, that one compares the evolution of two nearby chaotic initial conditions in a single frozen-$N$ background or the same chaotic initial condition evolved in two different frozen-$N$ realisations of the same bulk density. In this case, one typically observes a three-stage evolution, namely: (1) a rapid exponential divergence at a rate $\chi_N$ set by the true Lyapunov exponents associated with the $N$-body problem, which persists until the separation becomes large compared with a typical interparticle spacing; followed by (2) a slower exponential divergence at a rate comparable to the (typically much smaller) smooth potential Lyapunov exponent $\chi_S$, which persists until the separation becomes macroscopic; followed by (3) a power law divergence on a time scale $\propto (\ln N) t_D$. For regular initial conditions, the second stage is absent and the time scale for the third stage scales instead as $N^{1/2} t_D$.

Microchaos becomes stronger as $N$ increases in the sense that the value of $\chi_N$ increases with increasing $N$ [17]. Despite this, however, it becomes progressively less important macroscopically in that the range of the chaos, i.e., the scale on which the microchaos-driven exponential divergence of nearby orbits terminates, decreases with increasing $N$. In the limit $N \to \infty$ microchaos will become completely irrelevant but, for finite $N$, it does have an effect, at least on sufficiently short scales; and it is possible from an $N$-body simulation to extract estimates of both $\chi_N$ and the typically much smaller $\chi_S$ [15].

### 3.2 Modeling discreteness effects as friction and noise

It has been long recognised that, for sufficiently small $N$ and/or over sufficiently long times, discreteness effects will not be completely negligible. Systems like galaxies are ‘nearly collisionless’ in the sense that the stars interact primarily via collective macroscopic forces associated with the bulk density distribution; but, at least in principle, if one waits long enough discreteness effects should have an appreciable effect.

Astronomers are accustomed to modeling discreteness effects in the context of a Fokker-Planck description analogous to that formulated originally in the context of plasma physics [18]. However, it is not completely clear to what extent this is really justified. The conventional Fokker-Planck description was formulated originally to extract statistical properties of orbit ensembles and distribution functions over long time scales, assuming implicitly that the bulk potential is regular. To what extent, then, can Langevin realisations of a Fokker-Planck equation yield reliable information about individual orbits over comparatively short time scales, particularly if the orbits are chaotic?

Analyses of flows in frozen-$N$ systems indicate [14] that, at the level of both orbit ensembles and individual orbits, discreteness effects can in fact be modeled extremely well by Gaussian white noise in the context of a Fokker-Planck description, allowing for a dimensionless diffusion constant $D \propto 1/N$, consistent with the predicted scaling $D \propto \ln \Lambda/N$, with $\Lambda$ the so-called Coulomb logarithm [18]. For localised ensembles of initial conditions corresponding to both regular and chaotic orbits, phase mixing in frozen-$N$ systems and phase mixing
in smooth potentials perturbed by Gaussian white noise yield virtually identical behaviour, both in terms of the evolution of various phase space moments such as the emittance and the rate at which individual orbits in the ensemble exhibit nontrivial ‘transitions’, e.g., passing through some *entropy barrier* from one phase space region to another. And similarly, a comparison of frozen-$N$ orbits and noisy smooth potential orbits with the same initial condition reveals that their Fourier spectra typically exhibit comparable complexities. Gaussian white noise is even successful in mimicking some of the effects of microchaos. If, e.g., one tracks the divergence of two noisy orbits with the same chaotic initial condition evolved in a smooth potential, one observes the same three-stage evolution as for a pair of frozen-$N$ orbits evolved in two different frozen-$N$ potentials.

An example of this agreement is illustrated in Fig. 2, which exhibits data generated by averaging over 100 pairs of orbits evolved in frozen-$N$ density distributions which correspond in the continuum limit to a triaxial homogeneous ellipsoid with axis ratios $1.95 : 1.50 : 1.05$, perturbed by a spherically symmetric central mass spikes (‘black hole’). The top two solid curves represent (from top to bottom) results for $N = 10^{4.5}$ and $N = 10^{5.5}$. The four dotted curves represent analogous results derived for pairs of noisy orbits evolved from the same initial conditions in the smooth potential with (from top to bottom) diffusion constant $D = 10^{-4}$, $10^{-5}$, $10^{-6}$, and $10^{-7}$. The near-coincidence of the top two solid

---

**Fig. 2.** The mean spatial separation between the same initial conditions evolved in two different frozen-$N$ backgrounds (solid curves) and different noisy orbits evolved in the smooth potential from the same initial condition (dots). The solid line has a slope 0.022, equal to the mean value of the smooth potential Lyapunov exponent $\chi_S$. The dashed curve has a slope 0.75, equal to the mean value of the $N$-body Lyapunov exponent $\chi_N$. 
and dotted curves indicates that discreteness effects for $N = 10^{p+1/2}$ are well-mimicked by Gaussian white noise with $D = 10^{-p}$.

Such striking agreement suggests strongly that investigations of how orbits in smooth potentials are impacted by the introduction of friction and noise can provide important insights into the role of graininess in real galaxies. It is customary to assert that, in a system as large as a galaxy, discreteness effects reflecting close encounters between stars are unimportant because the relaxation time $t_R$ on which they can induce appreciable changes in quantities like the energy is orders of magnitude longer than the age of the Universe $[19]$. This is likely to be true if the galaxy is an exact equilibrium, especially an equilibrium characterised by an integrable potential. However, the assertion is suspect if (as must usually be the case) the system is only ‘close to’ an equilibrium or near-equilibrium, especially if the bulk potential is characterised by a phase space admitting a complex coexistence of regular and chaotic orbits.

Over the past decade, analyses of flows in time-independent Hamiltonian systems have revealed that even very weak perturbations, idealised as friction and white noise corresponding to $t_D \sim 10^6 - 10^9 t_D$ and, hence, $D \sim 10^{-6} - 10^{-9}$, can have significant effects within a time as short as $100 t_D$ or less by facilitating phase space diffusion through cantori or along the Arnold web $[20,21,22]$. The basic point is that the motions of chaotic orbits in a complex potential can be constrained significantly by topological obstructions like cantori or the Arnold web which, albeit not completely preventing motions from one phase space region to another, serve as an *entropy barrier* to impede such motions. In many respects, the physical picture is similar to the elementary problem of effusion of gas through a tiny hole in a wall. There is nothing in principle to prevent a gas molecule from passing through the hole and, hence, escaping from the region to which it is originally confined; but, if the hole is very small, the time scale associated with this effusion can be extremely long.

In the same sense, and for much the same reason, chaotic orbits trapped in one phase space region may, in the absence of perturbations, remain stuck in that region for a very long time. However, subjecting the orbits to noise will ‘wiggle’ them in such a fashion as to increase the rate at which they pass through the entropy barrier, thus accelerating phase space transport. Numerical simulations indicate that, in at least some cases, this escape process can be well approximated by a Poisson process, with the number of nonescapers decreasing exponentially at a rate $\Lambda$ that is determined by the perturbation $[23,24]$. This effect appears to result from a resonant coupling between the orbits and the noise. White noise is characterised by a flat power spectrum and, as such, will couple to more or less anything. If, however, the noise is made coloured, *i.e.*, if instantaneous kicks are replaced by impulses of finite duration, the high frequency power is reduced; and, if the autocorrelation time becomes sufficiently long that there is little power at frequencies comparable to the orbital frequencies, the effect of the noise decreases significantly. Significantly, it appears that, overall, the details of the perturbation may be largely irrelevant: additive and multiplicative Gaussian noises tend to have comparable effects and the presence or absence of friction
does not seem to matter. All that appears to matter are the amplitude and the autocorrelation time upon which there is a relatively weak, roughly logarithmic, dependence.

But what does all this imply for a real galaxy? Given that collisionless near-equilibria must be more common than true equilibria, it would seem quite possible that, during the early stages of evolution, a galaxy might settle down towards a near-equilibrium, rather than a true equilibrium, e.g., involving what have been termed [25] ‘partially mixed’ building blocks. If discreteness effects and all other perturbative effects could be ignored, such a quasi-equilibrium might persist without exhibiting significant changes over the age of the Universe. If, however, one allows for discreteness effects or, alternatively, other perturbations reflecting, e.g., a high density cluster environment, the orbits could become shuffled in such a fashion as to trigger significant changes in the phase space density and, consequently, a systematic secular evolution [26].

Such a scenario could, for example, result in the destabilisation of a bar. Many models of bars (e.g. [27]) incorporate ‘sticky’ [28] chaotic orbits as part of the skeleton of structure, replacing crucial regular orbits which can be absent near corotation and other resonances. Making these ‘sticky’ orbits become unstuck could cause the bar to dissipate. Similar effects could also cause an originally nonaxisymmetric cusp to evolve towards a more nearly axisymmetric state. To the extent that the triaxial Dehnen potentials are representative, one can argue that chaotic orbits may be extremely common near the centers of early-type galaxies, but that many of these chaotic orbits are extremely sticky [29] and, as such, could help support the nonaxisymmetric structure. Perturbations that make these sticky orbits wildly chaotic could de facto break the bones of the skeleton supporting the structure and trigger an evolution towards axisymmetry.

4 Experimental tests of galactic dynamics

4.1 Similarities between galaxies and nonneutral plasmas

Even though electrostatics and Newtonian gravity both involve $1/r^2$ forces, electric neutrality implies that the physics of neutral plasmas is very different from the physics of self-gravitating systems. Viewed over time scales $> t_R$, nonneutral plasmas and charged-particle beams are also very different from self-gravitating systems: the attractive character of gravity leads to phenomena like evaporation and core-collapse which cannot arise in a beam or a plasma. If, however, one restricts attention to comparatively short times $\ll t_R$, much of the physics should be the same. Theoretical expectations, supported by numerical simulations, suggest that it is the existence of long range order, not the sign of the interaction, which is really important; but, to the extent that this be true, collisionless plasmas and collisionless self-gravitating systems should be quite similar.

Typical sources of charged-particle beams configure the beams in trains of ‘packets’ or ‘bunches’, as they are termed by accelerator dynamicists. The objective of a good high-intensity accelerator is to generate bunches comprised
of a large total number of charges confined to a small phase space volume and then accelerate those bunches to very high energies while minimizing any growth in emittance. As one example, modern photocathode-based sources of electron beams routinely generate bunches comprised of some $10^{10} - 10^{11}$ electrons with transverse ‘emittance’ $\tilde{\epsilon}$ of a few microns. (Here $\tilde{\epsilon} = \epsilon / v_0$, where $v_0$ is the mean axial velocity of the particle distribution.) The energy relaxation time $t_R$ associated with such bunches typically corresponds to the time required for a bunch to travel a distance $\sim 1$ km or so which, in many cases, is much longer than any distance of experimental interest, so that the bunches are ‘nearly collisionless.’

Models of equilibrium configurations of nonneutral plasmas and charged-particle beams confined by electromagnetic fields can be characterized by a complex phase space quite similar to that associated with models of elliptical galaxies and, as such, have orbits with very similar properties. For example [30], the so-called ‘thermal equilibrium model’ [32] of beam dynamics, which involves a self-interacting nonneutral plasma in thermal equilibrium confined by an anisotropic harmonic oscillator potential, is strikingly similar [29] to the nonspherical generalizations of the Dehnen potential of galactic astronomy in terms of such properties as the degree of ‘stickiness’ manifested by chaotic orbits or how the relative measure of chaotic orbits and the size of the largest Lyapunov exponent vary with shape.

As in galactic dynamics, questions have been raised regarding the validity of the continuum approximation for nearly collisionless charged particle beams [31]. However, comparatively short time integrations ($t \ll t_R$) involving discreteness effects and the nature of the continuum limit in nonneutral plasmas [33] yield results essentially identical to what is observed for gravity – although the behaviour associated with neutral plasmas is very different. In particular, the macroscopic manifestations of phase mixing, for both regular and chaotic orbits, are indistinguishable, and the coexistence of microchaos and macrochaos persists unabated.

Nontrivial effects associated with a time-dependent potential have also been predicted for both nonneutral plasmas [1] and charged particle beams [2]. Although the time-dependence that is envisioned in a beam is typically less violent than that anticipated in violent relaxation within a galaxy, such is not always the case. Indeed, there is compelling experimental evidence that, in a beam, such a time-dependence can have the undesirable effect of ejecting particles from the core into an outerlying halo [34].

Perhaps most interesting, however, is the fact that numerical simulations that reproduce successfully ‘anomalous relaxation’ observed in real laboratory experiments involving accelerator beams have shown compelling evidence of chaotic phase mixing. One classic example involves the propagation of five nonrelativistic high-intensity beamlets in a periodic solenoidal transport channel, where self-consistent space-charge forces are extremely important [35]. Ideally, these beamlets should exhibit coherent periodic oscillations (quite literally disappearing and reappearing) which might be expected to decay only on a relaxation time scale $t_R$ that corresponds to a propagation distance $\sim 1$ km. However, regardless of how well the beam was matched to the transport channel, the beamlets
were seen to reappear only once, at a point $\sim 1$ m from the source, disappearing completely within 2 m or so (cf. Fig. 6.10 in [35]). Their failure to reappear again would seem to reflect some collisionless process that, in effect, causes the particles to ‘forget’ their initial conditions.

Detailed simulations using the particle-in-cell code WARP [36], which do an extremely good job of reproducing what is actually seen, demonstrate seemingly unambiguously that, because of the time-dependent space-charge potential, a large fraction of the particles in the beam experience the effects of strong, possible transient, macrochaos [37,38]. This is, e.g., evident from Fig. 3, which illustrates the evolution of representative test particles which interact with the bulk potential but not with each other. Here the left hand panel shows snapshots of the beam after it has travelled distances 0 m, 0.98 m, 2.88 m, 5.24 m, 11.52 m, and 31.68 m, with the representative ensembles superimposed. The right hand panel exhibits the evolution of the emittances $\epsilon_x$ and $\epsilon_y$ for these ensembles. It is evident that the initially localised ensembles are diverging exponentially so as to fill much of the accessible phase space, and that this exponential divergence coincides with the beamlets losing their individual identities. Also evident is the fact that the behaviour observed here is very similar to that exhibited in Fig. 1 which, recall, was generated for orbits in a perturbed Plummer potential exhibiting damped oscillations.

4.2 Testing galaxy evolution with charged particle beams

The aforementioned similarities between galactic astronomy and charged particle beams suggest the possibility of using accelerators as a laboratory for astrophysics in which one can perform experimental tests of galactic dynamics, a possibility currently being developed by a University of Florida – Fermilab/Northern Illinois University – University of Maryland collaboration. This collaboration, which has the dual aims of (1) obtaining an improved understanding of the applicability of nonlinear dynamics to nearly collisionless systems interacting via long range forces and (2) using that understanding to generate more sharply focused bunches by minimising undesirable increases in emittance, is currently planning concrete experiments which can, and presumably will, be performed on the University of Maryland Electron Ring (UMER) currently under construction. Here a number of obvious issues, all experimentally testable, come to mind:

How ubiquitous is chaotic phase mixing as a source of anomalous relaxation? Older experiments with lower-intensity beams, where the space-charge forces were comparatively unimportant, tended not to manifest extreme examples of anomalous relaxation. Anomalous relaxation appears more common in high intensity beams, especially in settings where a time-dependent density distribution generates a strongly time-dependent potential; and it is obvious to ask whether chaos is the principal culprit. The idea here is to identify the types of scenario that tend generically to yield anomalous relaxation and to determine, e.g., whether such scenarios tend typically to be associated with a bulk potential that incorporates a strong, roughly oscillatory component. Do numerical simulations of orbits ensembles evolved in such systems exhibit evidence of chaotic phase
mixing? And do individual orbits in those ensembles exhibit strong exponential sensitivity, associated, e.g., with transient chaos?

Do instabilities tend to trigger transient chaos? Instabilities in collisionless systems can exhibit behaviour qualitatively similar to that associated with turbulence in collision-dominated systems, but it is well known that turbulence is a strongly chaotic phenomenon. This possibility is especially interesting in that turbulence is another setting where different ‘types’ of chaos, characterised by wildly different time scales, can act on different length scales.

What types of geometries, both strongly time-dependent and nearly time-independent, tend to yield the most efficient chaotic phase mixing and the largest measures of chaotic orbits? Do time-dependent evolutions involving strongly convulsive oscillations tend generically to exhibit especially fast relaxation? And do they tend to yield especially large amounts of chaos, as probed by the relative measure of chaotic orbits and/or the sizes of the largest (finite time) Lyapunov exponents? To the extent that bulk properties of such ‘accelerator violent relaxation’ correlate with the degree of chaos exhibited in the evolving beam, and that the degree of chaos correlates with the form of the macroscopic time dependence, one will have a physically well-motivated explanation of which sorts of scenarios would be expected to exhibit complete and efficient violent relaxation and which would not!

Is it, e.g., true that, for nearly axisymmetric configurations, prolate (or oblate) bunches tend generically to exhibit especially large amounts of chaos? And do any such trends that are observed coincide with trends observed in models of galactic equilibria [29]? Even if a beam bunch remains nearly axisymmetric during its evolution, the acceleration mechanism can – and in general will – change its shape as it passes down an accelerator in a fashion that depends on the accelerator design. The obvious question, then, is whether the oblate or prolate phase tends generically to be especially chaotic.

Addressing this and related issues could provide important insights as to why galaxies have the detailed shapes that they do, a general question for which, at the present time, no compelling dynamical explanation exists. One knows, e.g., that elliptical galaxies tend to have isophotes that are slightly boxy or disky, and this boxiness or diskiness correlates with such properties as the rotation rate, the steepness of the central cusp, and the size of any deviations from axisymmetry [39]. Must all these effects be attributed to the detailed form of the formation scenario, or is there a clear dynamical explanation? Is it, e.g., true that the observed deviations from perfect ellipsoidal symmetry conspire to reduce the relative number of chaotic orbits or to increase the numbers of certain regular orbit types required as a skeleton to support the observed structure?

One might also use accelerator experiments to probe the role of discrete substructures and the extent to which they can be modeled as friction and noise in the context of a Fokker-Planck description [40]. If the injection of a beam involves a large mismatch, a significant charge redistribution will occur, resulting in violent relaxation, ‘turbulent’ behaviour, and the formation of substructures (‘lumps’) on a variety of scales. To the extent that such a time-dependent evo-
olution can be described in a continuum approximation, one might then expect that the bulk potential will correspond to a highly complex time-dependent phase space and that the substructures could act as a ‘noisy’ source of extrinsic diffusion, facilitating both transitions between ‘sticky’ and ‘wildly chaotic’ behaviour and, in some cases, transitions between regularity and chaos. Given the evidence (cf. [33]) that, at least over short times, discreteness effects act similarly for attractive and repulsive $1/r^2$ forces, such insights could be directly related to such issues as the destabilisation of bars in spirals and/or the secular evolution of nonaxisymmetric ellipticals towards more nearly axisymmetric states.

Do systems tend to evolve in such a fashion as to minimise the amount of chaos? There is an intuitive expectation amongst many galactic astronomers (cf. [41]) that galaxies tend to evolve towards equilibria which incorporate few if any chaotic orbits, i.e., that nature somehow favors nearly-regular equilibria. It would certainly appear true that a model must incorporate significant numbers of regular orbits to support interesting structures like bars and/or triaxiality, but this does not a priori preclude the possibility of chaotic orbits also being present. Generic time-independent three-degree-of-freedom potentials are neither completely regular nor completely chaotic, admitting instead a complex coexistence of regular and chaotic phase space regions. The obvious question then is: are galactic equilibria or near-equilibria typically well-represented by potentials which are generic in this sense; or are they, for reasons unknown, special in that they tend to be rather nearly regular?

5 Conclusions

This paper has focused on several fundamental issues that arise in attempts to apply nonlinear dynamics to real galaxies, many-body systems characterised by a self-consistently determined bulk potential which, during their most interesting phases, can be strongly time-dependent. As recently as a decade ago these issues would have been considered of largely academic, rather than practical, interest. However, recent observational advances – which facilitate improved high resolution photometry of individual objects as well as statistical analyses of large samples with varying redshift – and improved computational resources – which allow unparalleled explorations of multi-scale structure –, together with the recognition that the basic physics can be also probed in the context of charged particle beams, imply that theoretical predictions regarding these ‘academic’ issues can in fact be tested observationally, numerically, and experimentally.

Thanks to Ioannis Sideris for providing Fig. 1 and to Rami Kishek for providing Fig. 3. Thanks also to Court Bohn for his comments on a preliminary draft of the manuscript. Limited financial support was provided by NSF AST-0070809.

References

1. S. Strassburg and R. C. Davidson: Phys. Rev. E61, 5753 (2000)
2. R. Gluckstern: Phys. Rev. Lett. 73, 1247 (1994)
3. H. E. Kandrup, I. M. Vass, and I. V. Sideris: Mon. Not. R. astr. Soc.: submitted (2002) [astro-ph/0211056]
4. D. Lynden-Bell: Mon. Not. R. astr. Soc. 136, 101 (1967)
5. H. E. Kandrup: ‘Collisionless Relaxation of Stellar Systems’. In: Galaxy Dynamics, A Rutgers Symposium, ed. by D. Merritt, J. N. Sellwood, and M. Valluri (Sheridan, San Francisco 1999)
6. H. E. Kandrup and M. E. Mahon: Phys. Rev. E49, 3735 (1994)
7. M. E. Mahon, R. A. Abernathy, B. O. Bradley, and H. E. Kandrup: Mon. Not. R. astr. Soc. 275, 443 (1995)
8. D. Merritt and M. Valluri, Astrophys. J. 471, 82 (1996)
9. J. Binney: Comments Astrophys. 8, 27 (1978)
10. P. J. Quinn and W. H. Zurek: Astrophys. J. 331, 1 (1988)
11. M. Hemsendorff and D. Merritt: [astro-ph/0205533] (2002)
12. D. Heggie: ‘Chaos in the N-Body Problem of Stellar Dynamics’. In: Predictability, Stability, and Chaos in N-Body Dynamical Systems, ed. A. E. Roy (Plenum, New York 1991)
13. H. E. Kandrup and I. V. Sideris: Phys. Rev. E64, 056209-1 (2001)
14. I. V. Sideris and H. E. Kandrup: Phys. Rev. E65, 066203-1 (2002)
15. H. E. Kandrup and I. V. Sideris: Astrophys. J. in press (2003) [astro-ph/0207090]
16. H. E. Kandrup, B. L. Eckstein, and B. O. Bradley: Astron. Astrophys. 320, 65 (1997)
17. I. V. Pogorelov: Phase Space Transport and the Continuum Limit in Nonlinear Hamiltonian Systems. Ph. D. Thesis, University of Florida, Gainesville (2000)
18. M. N. Rosenbluth, W. M. MacDonald, and D. L. Judd: Phys. Rev. 107, 1 (1957)
19. J. Binney and S. Tremaine, Galactic Dynamics (Princeton University, Princeton, 1987)
20. H. E. Kandrup: Astrophys. J. 480, 155 (1997)
21. C. Siopis and H. E. Kandrup: Mon. Not. R. astr. Soc. 391, 43 (2000)
22. H. E. Kandrup and S. J. Novotny: Celestial Mechanics, submitted (2002) [astro-ph/0204019]
23. I. V. Pogorelov and H. E. Kandrup: Phys. Rev. E60, 1567 (1999).
24. H. E. Kandrup, I. V. Pogorelov, and I. V. Sideris: Mon. Not. R. astr. Soc. 311, 719 (2000)
25. D. Merritt and T. Fridman: Astrophys. J. 460, 136 (1996)
26. H. E. Kandrup: Space Science Reviews, in press (2003) [astro-ph/0011302]
27. P. A. Patsis, E. Athanassoula, and A. C. Quillen: Astrophys. J. 483, 731 (1997)
28. G. Contopoulos: Astron. J. 76, 147 (1971)
29. H. E. Kandrup and C. Siopis: Mon. Not. R. astr. Soc.; submitted (2002)
30. C. L. Bohn and I. V. Sideris: Phys. Rev. ST AB: submitted (2002)
31. J. Struckmeier, Phys. Rev. E54, 830 (1996)
32. M. Brown and M. Reiser: Phys. Plasmas 2, 965 (1995)
33. H. E. Kandrup, I. V. Sideris, and C. L. Bohn: Phys. Fluids, submitted (2002)
34. R. W. Garnett et al: In Linac 2002: Proceedings of the XXI International Linac Conference, in press.
35. M. Reiser: Theory and Design of Charged Particle Beams (Wiley, New York, 1994)
36. D. P. Grote, A. Friedman, I. Haber, and S. Yu: Fusion Eng, Design 32-33, 193 (1996)
37. R. A. Kishek, P. G. O'Shea, and M. Reiser: Phys. Rev. Lett. 85, 4514 (2000)
38. R. A. Kishek, C. L. Bohn, P. G. O'Shea, M. Reiser, and H. E. Kandrup: In: *Proceedings of the IEEE 2001 Particle Accelerator Conference*, ed P. Lucas and S. Weber (IEEE, Chicago, 2001)
39. J. Kormendy and R. Bender: Astrophys. J. Lett. 464, 119 (1996)
40. C. L. Bohn and J. R. Delayen: Phys. Rev. E50, 1516 (1994)
41. M. Schwarzschild: Astrophys. J. 232, 236 (1979)
Fig. 3. Evolution of five representative ensembles of test particles in the five-beamlet simulation. The left hand panel shows snapshots of the ensembles at (top-to-bottom left column) 0 m, 0.98 m, 2.88 m, and (top-to-bottom right column) 5.24 m, 11.52 m, and 31.68 m, with the $x$- and $y$-axes labeled in meters. The right panel shows the evolution of the logarithm of the emittances $\epsilon_x$ and $\epsilon_y$ as a function of distance $S(z)$ along the accelerator.

Generally clear exponential growth, indicating chaotic mixing is active.