ON SHIFT HARNACK INEQUALITIES FOR SUBORDINATE SEMIGROUPS AND MOMENT ESTIMATES FOR LÉVY PROCESSES

CHANG-SONG DENG AND RENÉ L. SCHILLING

ABSTRACT. We show that shift Harnack type inequalities (in the sense of F.-Y. Wang [14]) are preserved under Bochner’s subordination. The proofs are based on two types of moment estimates for subordinators. As a by-product we establish moment estimates for general Lévy processes.

1. INTRODUCTION AND MOTIVATION

Subordination in the sense of Bochner is a method to generate new (‘subordinate’) stochastic processes from a given process by a random time change with an independent one-dimensional increasing Lévy process (a ‘subordinator’). A corresponding notion exists at the level of semigroups. If the original process is a Lévy process, so is the subordinate process. For instance, any symmetric $\alpha$-stable Lévy process can be regarded as subordination of a Brownian motion, cf. [10]. This provides us another approach to investigate jump processes via the corresponding results for diffusion processes. See [5] for the dimension-free Harnack inequality for subordinate semigroups, [11] for subordinate functional inequalities and [3] for the quasi-invariance property under subordination. In this paper, we will establish shift Harnack inequalities, which were introduced in [14], for subordinate semigroups.

Let $(S_t)_{t \geq 0}$ be a subordinator. Being a one-sided Lévy process, it is uniquely determined by its Laplace transform which is of the form

$$E e^{-u S_t} = e^{-t \phi(u)}, \quad u > 0, \quad t \geq 0.$$  

The characteristic exponent $\phi : (0, \infty) \to (0, \infty)$ is a Bernstein function having the following Lévy–Khintchine representation

$$\phi(u) = bu + \int_{(0,\infty)} (1 - e^{-ux}) \nu(dx), \quad u > 0.$$  

The drift parameter $b \geq 0$ and the Lévy measure $\nu$—a measure on $(0, \infty)$ satisfying $\int_{(0,\infty)} (x \wedge 1) \nu(dx) < \infty$—uniquely characterize the Bernstein function. The corresponding transition probabilities $\mu_t := P(S_t \in \cdot)$ form a vaguely continuous convolution semigroup of probability measures on $[0, \infty)$, i.e. one has $\mu_{t+s} = \mu_t * \mu_s$ for all $t, s \geq 0$ and $\mu_t \to \mu_0 := 0$ weakly as $t \to 0$.

If $(X_t)_{t \geq 0}$ is a Markov process with transition semigroup $(P_t)_{t \geq 0}$, then the subordinate process is given by the random time-change $X_t^\phi := X_{S_t}$. The process $(X_t^\phi)_{t \geq 0}$ is again a

Date: December 23, 2014.

2010 Mathematics Subject Classification. 60J75, 47G20, 60G51.

Key words and phrases. Shift Harnack inequality, subordination, subordinate semigroup, Lévy process.

The first-named author gratefully acknowledges support through the Alexander-von-Humboldt foundation, the National Natural Science Foundation of China (11401442) and the International Postdoctoral Exchange Fellowship Program (2013).
Markov process, and it is not hard to see that the subordinate semigroup is given by the Bochner integral
\begin{equation}
{P_t^\phi f} := \int_{[0, \infty)} P_s f \mu_t(ds), \quad t \geq 0, \ f \text{ bounded, measurable.}
\end{equation}

The formula (1.2) makes sense for any Markov semigroup \((P_t)_{t \geq 0}\) on any Banach space \(E\) and defines again a Markov semigroup. We refer to [10] for details, in particular for a functional calculus for the generator of \((P_t^\phi)_{t \geq 0}\). If \((S_t)_{t \geq 0}\) is an \(\alpha\)-stable subordinator \((0 < \alpha < 1)\), the dimension-free Harnack type inequalities in the sense of [12] were established in [5], see [13] for more details on such Harnack inequalities. For example, if \((P_t)_{t \geq 0}\) satisfies the log-Harnack inequality
\[ P_t \log f(x) \leq \log P_t f(y) + \Phi(t, x, y), \quad x, y \in E, \ t > 0, \ f \in \mathcal{B}_b(E), \ f \geq 1, \]
for some function \(\Phi : (0, \infty) \times E \times E \to [0, \infty)\), then a similar inequality holds for the subordinate semigroup \((P_t^\phi)_{t \geq 0}\); that is, the log-Harnack inequality is preserved under subordination. For the stability of the power-Harnack inequality, we need an additional condition on \(\alpha\): if the following power-Harnack inequality holds
\[ (P_t f(x))^p \leq (P_t^\Phi f(y)) \exp[\Phi(t, p, x, y)], \quad x, y \in E, \ t > 0, \ f \in \mathcal{B}_b(E), \ f \geq 0, \]
where \(p > 1\) and \(\Phi(\cdot, p, x, y) : (0, \infty) \to [0, \infty)\) is a measurable function such that for some \(\kappa > 0\)
\[ \Phi(t, p, x, y) = O(t^{-\kappa}) \quad \text{as} \ t \to 0, \]
then \((P_t^\phi)_{t \geq 0}\) satisfies also a power-Harnack inequality provided that \(\alpha \in (\kappa/(1 + \kappa), 1)\), see [5] Theorem 1.1. We stress that the results of [5] hold for any subordinator whose Bernstein function satisfies \(\phi(u) \geq Cu^\alpha\) for large values of \(u\) with some constant \(C > 0\) and \(\alpha \in (0, 1)\) (as before, \(\alpha \in (\kappa/(1 + \kappa), 1)\) is needed for the power-Harnack inequality), see [15] Proof of Corollary 2.2.

Recently, new types of Harnack inequalities, called shift Harnack inequalities, have been proposed in [14]: A Markov semigroup \((P_t)_{t \geq 0}\) satisfies the shift log-Harnack inequality, if for some fixed element \(e \in E\)
\begin{equation}
P_t \log f(x) \leq \log P_t[f(\cdot + e)](x) + \Psi(t, e), \quad x \in E, \ t > 0, \ f \in \mathcal{B}_b(E), \ f \geq 1,
\end{equation}
and the shift power-Harnack inequality with power \(p > 1\), if
\begin{equation}
(P_t f(x))^p \leq (P_t[f^p(\cdot + e)](x)) \exp[\Phi(t, p, e)], \quad x \in E, \ t > 0, \ f \in \mathcal{B}_b(E), \ f \geq 0;
\end{equation}
here, \(\Psi(\cdot, e), \Phi(\cdot, p, e) : (0, \infty) \to [0, \infty)\) are measurable functions.

These new Harnack type inequalities can be applied to heat kernel estimates and quasi-invariance properties of the underlying transition probability under shifts, see [14] [13] for details. Therefore, it is natural to consider the stability of the shift Harnack inequality under subordination.

In many specific cases, see Example 2.4 in Section 2 below, we have \(\Psi(s, e)\) and \(\Phi(s, p, e)\) are of the form \(C_1 s^{-\kappa_1} + C_2 s^{-\kappa_2} + C_3\), with constants \(C_i \geq 0\), \(i = 1, 2, 3\), depending only on \(e \in E\) and \(p > 1\), and exponents \(\kappa_1, \kappa_2 > 0\). As it turns out, this means that we have to control \(E \delta_{S_t}^e\) for \(\kappa \in \mathbb{R} \setminus \{0\}\) for the shift log-Harnack inequality and \(E \delta_{S_t}^e, \kappa \in \mathbb{R} \setminus \{0\}\) for the shift power-Harnack inequality. Throughout the paper, we use the convention that \(\frac{1}{s} := \infty\).

Since moment estimates for stochastic processes are interesting on their own, we study such (exponential) moment estimates first for general Lévy processes, and then for subordinators. For real-valued Lévy processes without Brownian component estimates for the \(p\)th \((p > 0)\) moment were investigated in [7] and [9] via the Blumenthal–Getoor index introduced in [2]. While the focus of these papers were short time asymptotics, we
need estimates also for $t \gg 1$ which requires a different set of indices of the underlying processes.

Let us briefly indicate how the paper is organized. In Section 2 we establish the shift Harnack type inequalities for the subordinate semigroup $P_t^\phi$ from the corresponding inequalities for $P_t$. Some practical conditions are presented to ensure the stability of the shift Harnack inequality under subordination; in Example 2.4 we illustrate our results using a class of stochastic differential equations. Section 3 is devoted to moment estimates of Lévy processes: Subsection 3.1 contains, under various conditions, several concrete (non-)existence results and estimates for moments, while Subsection 3.2 provides the estimates for $E S_t^\kappa$ and $E e^{\delta S_t^\kappa}$ which were used in Section 2. As usual, we indicate by subscripts that a constant $C = C_{\alpha, \beta, \gamma, \ldots}$ depends on the parameters $\alpha, \beta, \gamma, \ldots$.

2. A shift Harnack inequality for subordinate semigroups

In this section, we use the moment estimates for subordinators from Subsection 3.2 to establish shift Harnack inequalities for subordinate semigroups. Let $(P_t)_{t \geq 0}$ be a Markov semigroup on a Banach space $E$ and $S = (S_t)_{t \geq 0}$ be a subordinator whose characteristic (Laplace) exponent $\phi$ is the Bernstein function given by (1.1). Recall that the subordinate semigroup $(P_t^\phi)_{t \geq 0}$ is defined by (1.2).

Before we can state our main results, we need to introduce two indices for subordinators:

$$
\sigma_0 := \sup \left\{ \alpha : \lim_{u \to 0} \frac{\phi(u)}{u^\alpha} = 0 \right\},
$$

$$
\sigma_1 := \sup \left\{ \alpha : \liminf_{u \to \infty} \frac{\phi(u)}{u^\alpha} > 0 \right\}.
$$

Since

$$
\lim_{u \to 0} \frac{\phi(u)}{u} = \lim_{u \to 0} \phi'(u) = b + \lim_{u \to 0} \int_{(0, \infty)} xe^{-ux} \nu(dx) = b + \int_{(0, \infty)} x \nu(dx) \in (0, \infty]
$$

it is clear that $\sigma_0 \in [0, 1]$. Moreover, the following formula holds, see [3]:

$$
\sigma_0 = \sup \left\{ \alpha : \limsup_{u \to 0} \frac{\phi(u)}{u^\alpha} < \infty \right\} = \sup \left\{ \alpha \leq 1 : \int_{y > 1} y^\alpha \nu(dy) < \infty \right\}.
$$

For any $\epsilon > 0$, noting that

$$
0 \leq \frac{\phi(u)}{u^{1+\epsilon}} \leq \frac{b}{u^\epsilon} + \frac{1}{u^\epsilon} \int_{(0, 1)} x \nu(dx) + \frac{1}{u^{1+\epsilon}} \nu(x \geq 1), \quad u > 0,
$$

yields

$$
\lim_{u \to \infty} \frac{\phi(u)}{u^{1+\epsilon}} = 0,
$$

one has $\sigma_1 \leq 1 + \epsilon$. Since $\epsilon > 0$ is arbitrary, we conclude that $\sigma_1 \in [0, 1]$.

**Remark 2.1.** We will frequently use the condition that $\liminf_{u \to \infty} \phi(u)u^{-\alpha} > 0$ for some $\alpha > c \geq 0$. This is clearly equivalent to either $c < \alpha < \sigma_1$ or $\alpha = \sigma_1 > c$ and $\liminf_{u \to \infty} \phi(u)u^{-\sigma_1} > 0$.

Assume that $P_t$ satisfies the following shift log-Harnack inequality

$$
P_t \log f(x) \leq \log P_t[f(\cdot + e)](x) + \frac{C_1(e)}{t^{\kappa_1}} + C_2(e)t^{\kappa_2} + C_3(e)
$$

for all $t > 0$, $f \in \mathcal{B}_b(E)$ with $g \geq 1$ and $x \in E$; here $e \in E$ is a fixed point, $\kappa_1 > 0$, $\kappa_2 \in (0, 1]$, and $C_i(e) \geq 0$, $i = 1, 2, 3$, are constants depending only on $e$. 


We are going to show that the subordinate semigroup $P_t^\phi$ satisfies a similar shift log-Harnack inequality. The following assumptions on the subordinator will be important:

(H1) $\kappa_1 > 0$ and $\phi(u) \geq c_1 \log(1 + u)$ holds for all $u \geq c_2$ and suitable constants $c_1 > 0$ and $c_2 \geq 0$.

(H2) $\sigma_1 > 0$ and $\kappa_1 > 0$.

(H3) $\kappa_2 \in (0, 1]$ satisfies $\int_{y \geq 1} y^{\kappa_2} \nu(dy) < \infty$.

(H4) $\kappa_2 \in (0, 1]$ satisfies $\inf_{\theta \in [\kappa_2, 1]} \int_{y > 0} y^\theta \nu(dy) < \infty$.

(H5) $\kappa_2 \in (0, \beta)$ where $\beta > 0$ and $\limsup_{u \downarrow 0} \phi(u) u^{-\beta} < \infty$.

**Theorem 2.2.** Suppose that $P_t$ satisfies (2.3). In each of the following cases the subordinate semigroup $P_t^\phi$ satisfies also a shift log-Harnack inequality

\[ P_t^\phi \log f(x) \leq \log P_t^\phi[f(\cdot + e)](x) + \Psi(t, e). \tag{2.4} \]

\[ P_t^\phi \log f(x) \leq \log P_t^\phi[f(\cdot + e)](x) + \Psi(t, e). \]

a) Assume (H1) and (H3). Then (2.4) holds for all $t > c_1/\kappa_1$, $f \in \mathcal{B}_b(E)$, $f \geq 1$, and $x \in E$ with $\Psi(t, e)$ of the form

\[ C_1(e) \left( \frac{c_1^{\kappa_1}}{\kappa_1 \Gamma(\kappa_1)} + \frac{\Gamma(c_1 t - \kappa_1)}{\Gamma(c_1 t)} \right) + C_2(e) \left[ \left( b + \int_{0 < y < 1} y \nu(dy) \right) t^{\kappa_2} + \left( \int_{y \geq 1} y^{\kappa_2} \nu(dy) \right) t \right] + C_3(e). \]

b) Assume (H1) and (H4). Then (2.4) holds for all $t > c_1/\kappa_1$, $f \in \mathcal{B}_b(E)$, $f \geq 1$, and $x \in E$ with $\Psi(t, e)$ of the form

\[ C_1(e) \left( \frac{c_1^{\kappa_1}}{\kappa_1 \Gamma(\kappa_1)} + \frac{\Gamma(c_1 t - \kappa_1)}{\Gamma(c_1 t)} \right) + C_2(e) \inf_{\theta \in [\kappa_2, 1]} \left[ b^\theta t^\theta + \left( \int_{y > 0} y^\theta \nu(dy) \right) t \right] + C_3(e). \]

c) Assume (H1) and (H5). Then (2.4) holds for all $t > c_1/\kappa_1$, $f \in \mathcal{B}_b(E)$, $f \geq 1$ and $x \in E$ with $\Psi(t, e)$ of the form

\[ C_1(e) \left( \frac{c_1^{\kappa_1}}{\kappa_1 \Gamma(\kappa_1)} + \frac{\Gamma(c_1 t - \kappa_1)}{\Gamma(c_1 t)} \right) + C_2(e) C_{\kappa_2, \beta}(t \lor 1)^{\kappa_2/\beta} + C_3(e), \]

where $C_{\kappa_2, \beta} > 0$ is some constant.

d) Assume (H2) and (H3). If $\liminf_{u \to \infty} \phi(u) u^{-\alpha} > 0$ for some $\alpha > 0$ \(^{1}\) then (2.4) holds for all $t > 0$, $f \in \mathcal{B}_b(E)$, $f \geq 1$, and $x \in E$ with $\Psi(t, e)$ of the form

\[ C_1(e) \left( \frac{c_1^{\kappa_1}}{(t \wedge 1)^{\kappa_1/\alpha}} + C_2(e) \left[ \left( b + \int_{0 < y < 1} y \nu(dy) \right) t^{\kappa_2} + \left( \int_{y \geq 1} y^{\kappa_2} \nu(dy) \right) t \right] + C_3(e). \]

where $C_{\alpha, \kappa_1} > 0$ is some constant.

\(^{1}\)In analogy to Remark 2.1 this is equivalent to either $0 < \beta < \sigma_0$ or $\beta = \sigma_0 > 0$ and $\limsup_{u \downarrow 0} \phi(u) u^{-\alpha_0} < \infty$.

\(^{2}\)This is equivalent to either $0 < \alpha < \sigma_1$ or $\alpha = \sigma_1 > 0$ and $\liminf_{u \to \infty} \phi(u) u^{-\sigma_1} > 0$, see Remark 2.1.
Therefore, and the desired estimates follow from the corresponding moment bounds in Section 3.2. By Jensen’s inequality we find for all \( t > 0 \), \( f \in \mathcal{B}_b(E) \), \( f \geq 1 \), and \( x \in E \) with \( \Psi(t, e) \) of the form
\[
C_1(e) \frac{C_{\alpha, \kappa_1}}{(t \wedge 1)^{\kappa_1/\alpha}} + C_2(e) \inf_{\theta \in [\kappa_2, 1]} \left[ t^\theta \int_{y > 0} y^\theta \nu(dy) \right] t^{\kappa/\theta} + C_\beta(e),
\]
where \( C_{\alpha, \kappa_1} > 0 \) is some constant.

f) Assume (H2) and (H5). If \( \liminf_{u \to \infty} \phi(u)u^{-\alpha} > 0 \) for some \( \alpha > 0 \), then (2.4) holds for all \( t > 0, f \in \mathcal{B}_b(E), f \geq 1 \), and \( x \in E \) with \( \Psi(t, e) \) of the form
\[
C_1(e) \frac{C_{\alpha, \kappa_1}}{(t \wedge 1)^{\kappa_1/\alpha}} + C_2(e)C_{\kappa_2, \beta}(t \wedge 1)^{\kappa_2/\beta} + C_3(e),
\]
where \( C_{\alpha, \kappa_1} > 0 \) and \( C_{\kappa_2, \beta} > 0 \) are some constants.

Proof. Because of (2.3) the shift log-Harnack inequality (1.3) for \( P_t \) holds with \( \Psi(s, e) = C_1(e)s^{-\kappa_1} + C_2(e)s^{\kappa_2} + C_3(e) \). Note that each of (H1) and (H2) implies \( \phi(u) \to \infty \) as \( u \to \infty \), hence excluding the compound Poisson subordinator, so
\[
\mu_t(\{0\}) = P(S_t = 0) = 0, \quad t > 0.
\]
By Jensen’s inequality we find for all \( t > 0 \)
\[
P_t^\phi \log f(x) = \int_{(0, \infty)} P_s \log f(x) \mu_t(ds)
\leq \int_{(0, \infty)} \left[ \log P_s[f(\cdot + e)](x) + \Psi(s, e) \right] \mu_t(ds)
\leq \log \int_{(0, \infty)} P_s[f(\cdot + e)](x) \mu_t(ds) + \int_{(0, \infty)} \Psi(s, e) \mu_t(ds)
= \log P_t^\phi[f(\cdot + e)](x) + \int_{(0, \infty)} \Psi(s, e) \mu_t(ds).
\]
Therefore,
\[
P_t^\phi \log f(x) \leq \log P_t^\phi[f(\cdot + e)](x) + C_1(e)ES_t^{-\kappa_1} + C_2(e)ES_t^{\kappa_2} + C_3(e)
\]
and the desired estimates follow from the corresponding moment bounds in Section 3.2.

Now we turn to the shift power-Harnack inequality for \( P_t^\phi \). Given a Lévy measure \( \nu \) on \( \mathbb{R}^d \), define
\[
K_{\epsilon, \delta, \kappa, d} := \epsilon^{\kappa/2} \int_{|y| \geq 1} e^{\delta|y|^\kappa} \nu(dy) - \nu(|y| \geq 1)
+ \frac{d}{2} \epsilon^{\kappa/2 + \delta} \left( \delta^2 \kappa^2 e^{\kappa - 1} + \delta \kappa (3 - \kappa) \epsilon^{\kappa/2 - 1} \right) \int_{0 < |y| < 1} |y|^2 \nu(dy)
\]
for \( \epsilon > 0, \delta \geq 0 \) and \( \kappa \in (0, 1] \). If \( d = 1 \), we simply write \( K_{\epsilon, \delta, \kappa} := K_{\epsilon, \delta, \kappa, 1} \).

Theorem 2.3. Let \( p > 1 \) and assume that \( P_t \) satisfies the following shift power-Harnack inequality
\[
(\text{2.6}) \quad (P_t f(x))^p \leq (P_t[f^p(\cdot + e)](x)) \exp \left[ \frac{H_1(p, e)}{t^{\kappa_1}} + H_2(p, e)t^{\kappa_2} + H_3(p, e) \right]
\]
for all \( t > 0, f \in \mathcal{B}_b(E), f \geq 0, \) and \( x \in E \), where \( e \in E \) is fixed, \( \kappa_1 > 0, \kappa_2 \in (0, 1] \) and \( H_i(p, e) \geq 0, i = 1, 2, 3, \) are constants depending on \( p \) and \( e \).\footnote{This is equivalent to either 0 < \( \alpha < \sigma_1 \) or \( \alpha = \sigma_1 > 0 \) and \( \liminf_{u \to \infty} \phi(u)u^{-\sigma_1} > 0 \).}
Assume that $q > 1$, and $\liminf_{u \to \infty} \phi(u) u^{-\alpha} > 0$ for some $\alpha > \kappa_1/(1 + \kappa_1)$\footnote{This is equivalent to either $\kappa_1/(1 + \kappa_1) < \alpha < \sigma_1$ or $\alpha = \sigma_1 > \kappa_1/(1 + \kappa_1)$ and $\liminf_{u \to \infty} \phi(u) u^{-\sigma_1} > 0$, see Remark 2.1}. Then there exists some constant $C_{\alpha, \kappa_1}$ such that:

a) If

$$\int_{y > 1} \exp \left[ \frac{qH_2(p, e)}{p - 1} y^{\kappa_2} \right] \nu(dy) < \infty,$$
then the subordinate semigroup $P^\phi_t$ satisfies the shift power-Harnack inequality \[^{1.4}\] for all $t > 0$, $f \in \mathcal{B}_b(E)$, $f \geq 0$, and $x \in E$ with an exponent $\Phi(t, p, e)$ given by

$$C_{\alpha, \kappa_1} H_1(p, e) + C_{\alpha, \kappa_1} \left( \frac{q}{(q - 1)(p - 1)} \right)^{\frac{(1-\alpha)\kappa_1}{\alpha(1-\alpha)\kappa_1}} \left( \frac{H_1(p, e)}{t^{\kappa_1/\alpha}} \right)^{\frac{\alpha - (1-\alpha)\kappa_1}{\alpha(1-\alpha)\kappa_1}} + H_1(p, e) \right)^{\kappa_2} + \inf_{\epsilon > 0} \left( \frac{H_2(p, e)}{\epsilon^{\kappa_2/2}} + \frac{p - 1}{q} K_{\epsilon^{1/2}/p^{1/2}} \right) + H_3(p, e).$$

b) If

$$\int_{0 < y < 1} y^{\kappa_2} \nu(dy) < \infty \quad \text{and} \quad \int_{y > 1} \exp \left[ \frac{qH_2(p, e)}{p - 1} y^{\kappa_2} \right] \nu(dy) < \infty,$$
then the subordinate semigroup $P^\phi_t$ satisfies the shift power-Harnack inequality \[^{1.4}\] for all $t > 0$, $f \in \mathcal{B}_b(E)$, $f \geq 0$, and $x \in E$ with an exponent $\Phi(t, p, e)$ given by

$$C_{\alpha, \kappa_1} H_1(p, e) + C_{\alpha, \kappa_1} \left( \frac{q}{(q - 1)(p - 1)} \right)^{\frac{(1-\alpha)\kappa_1}{\alpha(1-\alpha)\kappa_1}} \left( \frac{H_1(p, e)}{t^{\kappa_1/\alpha}} \right)^{\frac{\alpha - (1-\alpha)\kappa_1}{\alpha(1-\alpha)\kappa_1}} + H_1(p, e) \right)^{\kappa_2} + \int_{y > 0} \left( \exp \left[ \frac{qH_2(p, e)}{p - 1} y^{\kappa_2} \right] - 1 \right) \nu(dy) \right) t + H_3(p, e).$$

Proof. As in the proof of Theorem 2.2 we see $\mu_t(\{0\}) = 0$ for any $t > 0$. By (2.6) and the Hölder inequality one has

$$(P^\phi_t f(x))^p = \left( \int_{0, \infty} P_s f(x) \mu_t(ds) \right)^p \leq \left( \int_{0, \infty} (P_s[F^p(\cdot, +)](x))^{\frac{1}{p}} \exp \left[ \frac{H_1(p, e)}{p^{\kappa_1}} + \frac{H_2(p, e)}{p} \sigma_{\kappa_2} + \frac{H_3(p, e)}{p} \right] \mu_t(ds) \right)^p \leq \left( \int_{0, \infty} P_s[F^p(\cdot, +)](x) \mu_t(ds) \right)^p \times \left( \int_{0, \infty} \exp \left[ \frac{H_1(p, e)}{p^{\kappa_1}} + \frac{H_2(p, e)}{p - 1} \sigma_{\kappa_2} + \frac{H_3(p, e)}{p - 1} \right] \mu_t(ds) \right)^{p-1} = \left( P^\phi_t[F^p(\cdot, +)](x) \right) e^{H_3(p, e)} \left( E \exp \left[ \frac{H_1(p, e)}{p - 1} \sigma_{\kappa_2} + \frac{H_2(p, e)}{p - 1} \sigma_{\kappa_2} \right] \right)^{p-1} \leq \left( P^\phi_t[F^p(\cdot, +)](x) \right) e^{H_3(p, e)} \times \left( E \exp \left[ \frac{qH_1(p, e)}{(q - 1)(p - 1)} \sigma_{\kappa_2} \right] \right)^{(q-1)(p-1)} \left( E \exp \left[ \frac{qH_2(p, e)}{p - 1} \sigma_{\kappa_2} \right] \right)^{\frac{q-1}{q}}.
Consider the following stochastic differential equation on 

\[ (2.11) \]

\[ \int 1 - \left( \frac{q}{(q-1)(p-1)} \right)^{\alpha} \left( \frac{H_1(p, e)}{t^{\alpha/2}} \right)^{\alpha} + \left( 1 + t^{-\alpha} \right) H_1(p, e) \right]. \]

On the other hand, it follows from (2.7) and Corollary 3.12(c) that

\[ (2.12) \]

\[ (2.13) \]

It is well known that (A1)

There exists a locally bounded measurable function \( K : [0, \infty) \to [0, \infty) \) such that

\[ |l_t(x) - l_t(y)| \leq K_t |x - y|, \quad x, y \in \mathbb{R}^d, \quad t \geq 0. \]

(A2) For each \( t \geq 0 \), the matrix \( \Sigma_t \) is invertible and there exists a measurable function \( \lambda : [0, \infty) \to (0, \infty) \) such that \( \lambda \in L^2_{\text{loc}}([0, \infty)) \) and \( \| \Sigma_t^{-1} \| \leq \lambda_t \) for all \( t \geq 0 \).

It is well known that (A1) ensures that (2.11) has for each starting point \( X_0 = x \in \mathbb{R}^d \) a unique solution \( (X_t^x)_{t \geq 0} \) with infinite life-time. By \( P_t \) we denote the associated Markov semigroup, i.e.

\[ P_t f(x) = \mathbb{E} f(X_t^x), \quad t \geq 0, \quad f \in \mathcal{B}_b(\mathbb{R}^d), \quad x \in \mathbb{R}^d. \]

Let \( e \) be a fixed point in \( \mathbb{R}^d \). Assume that for some \( \kappa_1 > 0 \) and \( \kappa_2 \in (0, 1] \)

\[ (2.12) \]

\[ (2.13) \]

Typical examples for \( K \) and \( \lambda \) satisfying (2.12) and (2.13) are

- \( \lambda_s = 1 \) and \( K_s = s^\theta \wedge 1 \) for \( \theta \leq 0 \). Then it is easy to see that (2.12) is fulfilled with \( \kappa_1 > 0 \) and (2.13) is satisfied with \( \kappa_2 \in [(2\theta + 1) \vee 0, 1] \setminus \{0\} \).

- \( \lambda_s = s^\theta \) for \( -1/2 < \theta \leq 0 \) and \( K_s = 1 \). Then (2.12) holds with \( \kappa_1 \geq 1 - 2\theta \) and (2.13) holds with \( \kappa_2 \in (0, 1 + 2\theta] \).

We are going to show that there exists a constant \( C > 0 \) such that for all \( t > 0 \) and \( x \in \mathbb{R}^d \) the following shift log- and power-(\( p > 1 \))-Harnack inequalities hold:

\[ P_t \log f(x) \leq \log P_t \left[ f(\cdot + e) \right](x) + C|e|^2 \left( \frac{1}{\kappa_1} + t^{\kappa_2} + 1 \right), \quad f \in \mathcal{B}_b(\mathbb{R}^d), \quad f \geq 1, \]
(\(P_t f(x)\))^p \leq (\(P_t[f^p(\cdot + e)](x)\)) \exp \left[ \frac{C p |e|^2}{p - 1} \left( \frac{1}{t^{\kappa_1}} + t^{\kappa_2} + 1 \right) \right], \quad f \in \mathcal{B}_b(\mathbb{R}^d), \quad f \geq 0.

In particular, Theorems 2.2 and 2.3 can be applied.

Although the proof of Example 2.4 relies on known arguments, see e.g. [14, 13], we include the complete proof for the convenience of the reader.

**Proof of Example 2.4.** Fix \(t > 0\) and \(x \in \mathbb{R}^d\). We adopt the new coupling argument from [14] (see also [13]) to construct another process \((Y^x_t)_{t \geq 0}\) also starting from \(x\) such that \(Y^x_t - X^x_t = e\) at the fixed time \(t\). The process \((Y^x_s)_{s \geq 0}\) is the solution of the following equation

\[
(2.14) \quad dY^x_s = l_s(X^x_s) \, ds + \Sigma_s \, dW_s + \frac{e}{t} \, ds, \quad Y^x_0 = x.
\]

Clearly,

\[
(2.15) \quad Y^x_t - X^x_t = \int_0^t \frac{e}{t} \, ds = \frac{r}{t} e, \quad 0 \leq r \leq t,
\]

and, in particular, \(Y^x_t - X^x_t = e\). Rewrite (2.14) as

\[
dY^x_s = l_s(Y^x_s) \, ds + \Sigma_s \, d\tilde{W}_s, \quad Y^x_0 = x,
\]

where

\[
\tilde{W}_s := W_s + \int_0^s \Sigma_r^{-1} \left( l_r(X^x_r) - l_r(Y^x_r) + \frac{e}{t} \right) \, dr, \quad 0 \leq s \leq t.
\]

Let

\[
M_t := \int_0^t \Sigma_r^{-1} \left( l_r(X^x_r) - l_r(Y^x_r) + \frac{e}{t} \right) \, dW_r.
\]

Since it follows from (A2), (A1) and (2.15) that

\[
\left| \Sigma_r^{-1} \left( l_r(X^x_r) - l_r(Y^x_r) + \frac{e}{t} \right) \right| \leq \lambda_r \left| l_r(X^x_r) - l_r(Y^x_r) + \frac{e}{t} \right| \\
\leq \lambda_r \left( \frac{K_r |X^x_r - Y^x_r| + |e|}{t} \right) \\
= \frac{|e|}{t} \lambda_r (rK_r + 1), \quad 0 \leq r \leq t,
\]

the compensator of the martingale \(M\) satisfies

\[
(2.16) \quad \langle M \rangle_t = \int_0^t \left| \Sigma_r^{-1} \left( l_r(X^x_r) - l_r(Y^x_r) + \frac{e}{t} \right) \right|^2 \, dr \leq \frac{|e|^2}{t^2} \int_0^t \lambda_r^2 (rK_r + 1)^2 \, dr.
\]

Set

\[
R_t := \exp \left[ -M_t - \frac{1}{2} \langle M \rangle_t \right].
\]

Novikov’s criterion shows that \(ER = 1\). By the Girsanov theorem, \((\tilde{W}_s)_{0 \leq s \leq t}\) is a \(d\)-dimensional Brownian motion under the probability measure \(R_t \mathbb{P}\).

To derive the shift log-Harnack inequality for \(P_t\), we first note that (2.16) implies

\[
\log R_t = -M_t - \frac{1}{2} \langle M \rangle_t
\]

\[
= -\int_0^t \Sigma_r^{-1} \left( l_r(X^x_r) - l_r(Y^x_r) + \frac{e}{t} \right) \, d\tilde{W}_r + \frac{1}{2} \langle M \rangle_t
\]

\[
\leq -\int_0^t \Sigma_r^{-1} \left( l_r(X^x_r) - l_r(Y^x_r) + \frac{e}{t} \right) \, d\tilde{W}_r + \frac{|e|^2}{2t^2} \int_0^t \lambda_r^2 (rK_r + 1)^2 \, dr.
\]
Since $E R_t = 1$, we find with the Jensen inequality for any random variable $F \geq 1$
\[
\int R_t \log \frac{F}{R_t} \, dP \leq \log \int F \, dP, \quad \text{hence,} \quad \int R_t \log F \, dP \leq \log \int F \, dP + \int R_t \log R_t \, dP.
\]
Thus, we get for any $f \in \mathcal{B}_b(\mathbb{R}^d)$ with $f \geq 1$
\[
P_t \log f(x) = E R_t \log f(Y_t^x)
= E \{ R_t \log f(X_t^x + e) \}
\leq \log E f(X_t^x + e) + E \{ R_t \log R_t \}
= \log P_t[f(\cdot + e)](x) + E R_t \log R_t
\leq \log P_t[f(\cdot + e)](x) + \frac{1}{2t^2} \int_0^t \lambda_t^2(rK_r + 1)^2 \, dr.
\]
(2.17)

On the other hand, for any $p > 1$ and $f \in \mathcal{B}_b(\mathbb{R}^d)$ with $f \geq 0$, we deduce with the Hölder inequality that
\[
(P_t f(x))^p = (E R_t f(Y_t^x))^p
= (E \{ R_t f(X_t^x + e) \})^p
\leq (E f^p(X_t^x + e)) \left( E R_t^{p^{-1}} \right)^{p^{-1}}
= (P_t[f^p(\cdot + e)](x)) \left( E R_t^{p^{-1}} \right)^{p^{-1}}.
\]
Because of (2.16), it follows that
\[
E R_t^{p^{-1}} = E \exp \left[ \frac{p}{2(p-1)^2} \langle M \rangle_t - \frac{p}{p-1} M_t - \frac{p^2}{2(p-1)^2} \langle M \rangle_t \right]
\leq \exp \left[ \frac{p}{2(p-1)^2} \frac{|e|^2}{t^2} \int_0^t \lambda_t^2(rK_r + 1)^2 \, dr \right] \exp \left[ - \frac{p}{p-1} M_t - \frac{p^2}{2(p-1)^2} \langle M \rangle_t \right]
= \exp \left[ \frac{p|e|^2}{2(p-1)^2 t^2} \int_0^t \lambda_t^2(rK_r + 1)^2 \, dr \right].
\]
In the last step we have used the fact that $\exp \left[ - \frac{p}{p-1} M_t - \frac{p^2}{2(p-1)^2} \langle M \rangle_t \right]$ is a martingale; this is due to (2.16) and Novikov’s criterion. Therefore,
\[
(P_t f(x))^p \leq (P_t[f^p(\cdot + e)](x)) \exp \left[ \frac{p|e|^2}{2(p-1)^2 t^2} \int_0^t \lambda_t^2(rK_r + 1)^2 \, dr \right]
\]
holds for all $p > 1$ and non-negative $f \in \mathcal{B}_b(\mathbb{R}^d)$.

Finally, it remains to observe that (2.12) and (2.13) imply that there exists a constant $C > 0$ such that
\[
\frac{1}{t^2} \int_0^t \lambda_t^2(rK_r + 1)^2 \, dr \leq 2C \left( \frac{1}{t^{\alpha_1}} + t^{\alpha_2} + 1 \right) \quad \text{for all} \quad t > 0.
\]
Substituting this into (2.17) and (2.18), respectively, we obtain the desired shift log- and power-Harnack inequalities. \qed

3. Moment estimates for Lévy processes

3.1. General Lévy processes. A Lévy process $X = (X_t)_{t \geq 0}$ is a $d$-dimensional stochastic process with stationary and independent increments and almost surely càdlàg (right-continuous with finite left limits) paths $t \mapsto X_t$. As usual, we assume that $X_0 = 0$. Our standard references are [8, 4]. Since Lévy processes are (strong) Markov processes,
they are completely characterized by the law of $X_t$, hence by the characteristic function of $X_t$. It is well known that

$$E e^{i \xi \cdot X_t} = e^{-t \psi(\xi)}, \quad t > 0, \; \xi \in \mathbb{R}^d,$$

where the characteristic exponent $\psi : \mathbb{R}^d \to \mathbb{C}$ is given by the Lévy–Khintchine formula

$$\psi(\xi) = -i \ell \cdot \xi + \frac{1}{2} \xi \cdot Q \xi + \int_{|y| > 0} \left(1 - e^{i \xi \cdot y} + i \xi \cdot y \mathbb{I}_{(0,1)}(|y|)\right) \nu(dy),$$

where $\ell \in \mathbb{R}^d$ is the drift coefficient, $Q$ is a non-negative semidefinite $d \times d$ matrix, and $\nu$ is the Lévy measure on $\mathbb{R}^d \setminus \{0\}$ satisfying $\int_{y \neq 0} (1 \wedge |y|^2) \nu(dy) < \infty$. The Lévy triplet or characteristics $(\ell, Q, \nu)$ uniquely determine $\psi$, hence $X$ and the infinitesimal generator of $X$ is given by

$$\mathcal{L} f = \ell \cdot \nabla f + \frac{1}{2} \nabla \cdot Q \nabla f + \int_{|y| > 0} \left(f(y + \cdot) - f - y \cdot \nabla f \mathbb{I}_{(0,1)}(|y|)\right) \nu(dy), \quad f \in C^2_c(\mathbb{R}^d).$$

Recall, cf. [8, Theorem 25.3], that the Lévy process $X$ has a $\kappa$th ($\kappa > 0$) moment, i.e. $E|X_t|^\kappa < \infty$ for some (hence, all) $t > 0$, if and only if

$$\int_{|y| \geq 1} |y|^\kappa \nu(dy) < \infty.$$  \tag{3.1}

**Theorem 3.1.** Let $X$ be a Lévy process in $\mathbb{R}^d$ with characteristics $(\ell, 0, \nu)$ and $\kappa \in (0, 1]$. If (3.1) holds, then for any $t > 0$

$$E|X_t|^\kappa \leq |\ell|^\kappa t^\kappa + \left(\int_{|y| \geq 1} |y|^\kappa \nu(dy)\right) t$$

$$+ 2 \left(\frac{d}{2} \kappa (3 - \kappa) \int_{0<|y|<1} |y|^2 \nu(dy)\right)^{\kappa/2} \left[1 + \nu(|y| \geq 1) t\right]^{1-\kappa/2} t^{\kappa/2}.$$

**Proof.** Rewrite $X_t$ as $X_t = \ell t + \hat{X}_t$, $t \geq 0$, where $\hat{X} = (\hat{X}_t)_{t \geq 0}$ is a Lévy process in $\mathbb{R}^d$ generated by

$$\mathcal{L} f = \int_{y \neq 0} \left(f(y + \cdot) - f - y \cdot \nabla f \mathbb{I}_{(0,1)}(|y|)\right) \nu(dy), \quad f \in C^2_c(\mathbb{R}^d).$$

Noting that

$$(x+y)^\gamma \leq x^\gamma + y^\gamma, \quad x, y \geq 0, \; \gamma \in [0, 1],$$

it suffices to show that

$$E|\hat{X}_t|^\kappa \leq \left(\int_{|y| \geq 1} |y|^\kappa \nu(dy)\right) t$$

$$+ 2 \left(\frac{d}{2} \kappa (3 - \kappa) \int_{0<|y|<1} |y|^2 \nu(dy)\right)^{\kappa/2} \left[1 + \nu(|y| \geq 1) t\right]^{1-\kappa/2} t^{\kappa/2}$$

for all $t > 0$.

Fix $\epsilon > 0$ and $t > 0$. Let

$$f(x) := (\epsilon + |x|^2)^{\kappa/2}, \quad x \in \mathbb{R}^d,$$

and

$$\tau_n := \inf \left\{ s \geq 0 : |\hat{X}_s| > n \right\}, \quad n \in \mathbb{N}.$$
By the Dynkin formula we get for any $n \in \mathbb{N}$
\[
E \left[ \left( \epsilon + |\hat{X}_{t \land \tau_n}| \right)^{\kappa/2} \right] - \epsilon^{\kappa/2} = E \left[ \int_{(0,t \land \tau_n)} \mathcal{L}f(\hat{X}_s) \, ds \right]
\]
(3.5)
\[
= E \left[ \int_{(0,t \land \tau_n)} \left( \int_{|y| \geq 1} \left( f(\hat{X}_s + y) - f(\hat{X}_s) \right) \nu(dy) \right) \, ds \right] + E \left[ \int_{(0,t \land \tau_n)} \left( \int_{0 < |y| < 1} \left( f(\hat{X}_s + y) - f(\hat{X}_s) - y \cdot \nabla f(\hat{X}_s) \right) \nu(dy) \right) \, ds \right].
\]
We estimate the two terms separately. For the first expression we have
\[
\int_{|y| \geq 1} \left( f(\hat{X}_s + y) - f(\hat{X}_s) \right) \nu(dy) \leq \int_{|y| \geq 1} \left( \epsilon^{\kappa/2} + |y|^\kappa \right) \nu(dy)
\]
(3.6)
\[
= \epsilon^{\kappa/2} \nu(|y| \geq 1) + \int_{|y| \geq 1} |y|^\kappa \nu(dy).
\]
For the second term, we observe that for any $x \in \mathbb{R}^d$
\[
\left| \frac{\partial^2 f}{\partial x_j \partial x_i}(x) \right| = |\kappa(\kappa - 2) (\epsilon + |x|^2)^{\kappa/2 - 2} x_i x_j + \kappa \left( \epsilon + |x|^2 \right)^{\kappa/2 - 1} \mathbf{1}_{\{i = j\}}| \\
\leq \kappa(2 - \kappa) \left( \epsilon + |x|^2 \right)^{\kappa/2 - 2} |x|^2 + \kappa \left( \epsilon + |x|^2 \right)^{\kappa/2 - 1} \\
\leq \kappa(2 - \kappa) \epsilon^{\kappa/2 - 1} + \kappa \epsilon^{\kappa/2 - 1} \\
= \kappa(3 - \kappa) \epsilon^{\kappa/2 - 1}.
\]
By Taylor’s theorem,
\[
f(\hat{X}_s + y) - f(\hat{X}_s) - y \cdot \nabla f(\hat{X}_s) = \frac{1}{2} \sum_{i,j=1}^d \frac{\partial^2 f}{\partial x_j \partial x_i}(\hat{X}_s + \theta_y \hat{X}_s,y) y_i y_j
\]
\[
\leq \frac{1}{2} \sum_{i,j=1}^d \left| \frac{\partial^2 f}{\partial x_j \partial x_i}(\hat{X}_s + \theta_y \hat{X}_s,y) \right| y_i y_j \\
\leq \frac{1}{2} \kappa(3 - \kappa) \epsilon^{\kappa/2 - 1} \sum_{i,j=1}^d |y_i y_j| \\
\leq \frac{d}{2} \kappa(3 - \kappa) \epsilon^{\kappa/2 - 1} |y|^2,
\]
where $\theta_y \in [-1,1]$ depends on $\hat{X}_s$ and $y$. Thus, we get
\[
\int_{0 < |y| < 1} \left( f(\hat{X}_s + y) - f(\hat{X}_s) - y \cdot \nabla f(\hat{X}_s) \right) \nu(dy) \leq \frac{d}{2} \kappa(3 - \kappa) \epsilon^{\kappa/2 - 1} \int_{0 < |y| < 1} |y|^2 \nu(dy).
\]
Combining this with (3.6) and (3.5), we arrive at
\[
E \left[ (\epsilon + |\hat{X}_{t \land \tau_n}|)^{\kappa/2} \right] \leq \epsilon^{\kappa/2} + \left( \epsilon^{\kappa/2} \nu(|y| \geq 1) + \int_{|y| \geq 1} |y|^\kappa \nu(dy) \right) E[t \land \tau_n]
\]
\[
+ \left( \frac{d}{2} \kappa(3 - \kappa) \epsilon^{\kappa/2 - 1} \int_{0 < |y| < 1} |y|^2 \nu(dy) \right) E[t \land \tau_n]
\]
\[
\leq \epsilon^{\kappa/2} + \left( \epsilon^{\kappa/2} \nu(|y| \geq 1) + \int_{|y| \geq 1} |y|^\kappa \nu(dy) \right)
\]
\[
+ \frac{d}{2} \kappa (3 - \kappa) e^{\kappa/2 - 1} \int_{0 < |y| < 1} |y|^2 \nu(dy) t.
\]

Since \( \tau_n \uparrow \infty \) as \( n \uparrow \infty \), we can let \( n \uparrow \infty \) and use the monotone convergence theorem to obtain
\[
E|\hat{X}_t|^\kappa \leq E \left[ (\epsilon + |\hat{X}_t|^2)^{\kappa/2} \right]
\leq \epsilon^{\kappa/2} + \left( \epsilon^{\kappa/2} \nu(|y| \geq 1) + \int_{|y| \geq 1} |y|^\kappa \nu(dy) \frac{d}{2} \kappa (3 - \kappa) e^{\kappa/2 - 1} \int_{0 < |y| < 1} |y|^2 \nu(dy) \right) t
= \left( \int_{|y| \geq 1} |y|^\kappa \nu(dy) \right) t
+ \left[ 1 + \nu(|y| \geq 1) t \right] \epsilon^{\kappa/2} + \left[ \frac{d}{2} \kappa (3 - \kappa) \left( \int_{0 < |y| < 1} |y|^2 \nu(dy) \right) t \right] \epsilon^{\kappa/2 - 1}.
\]

Since \( \epsilon > 0 \) is arbitrary, we can optimize in \( \epsilon > 0 \), i.e. let
\[
\epsilon \downarrow \frac{\frac{d}{2} \kappa (3 - \kappa) \left( \int_{0 < |y| < 1} |y|^2 \nu(dy) \right) t}{1 + \nu(|y| \geq 1) t},
\]
to get (3.3). This completes the proof. \( \square \)

**Theorem 3.2.** Let \( X \) be a Lévy process in \( \mathbb{R}^d \) with characteristics \( (\ell, 0, \nu) \) and \( \kappa \in (0, 1] \). If
\[
(3.7) \quad \inf_{\theta \in [\kappa, 1]} \int_{y \neq 0} |y|^\theta \nu(dy) < \infty,
\]
then for any \( t > 0 \)
\[
E|X_t|^\kappa \leq \inf_{\theta \in [\kappa, 1]} \left[ |\hat{\ell}|^{\theta} t^{\frac{\kappa}{\theta}} + \left( \int_{y \neq 0} |y|^\theta \nu(dy) \right) t \right]^{\frac{\kappa}{\theta}},
\]
where
\[
(3.8) \quad \hat{\ell} := \ell - \int_{0 < |y| < 1} y \nu(dy).
\]

**Proof.** By assumption, \( \int_{0 < |y| < 1} |y| \nu(dy) < \infty \), i.e. \( X \) has bounded variation. Therefore, \( X_t = \hat{\ell} t + \tilde{X}_t, \ t \geq 0 \), where \( (\tilde{X}_t)_{t \geq 0} \) is a drift-free Lévy process with generator
\[
\tilde{\mathcal{L}} f = \int_{y \neq 0} (f(y + \cdot) - f) \nu(dy), \quad f \in C^2_b(\mathbb{R}^d).
\]
Let
\[
\tau_n := \inf \{ t : |\tilde{X}_t| > n \}, \quad n \in \mathbb{N}.
\]
It follows from Dynkin’s formula and (3.2) that for any \( \theta \in [\kappa, 1] \) and \( n \in \mathbb{N} \)
\[
E|\tilde{X}_{t \land \tau_n}|^\theta = E \left[ \int_{[0, t \land \tau_n]} \left( \int_{y \neq 0} (|\tilde{X}_s + y|^\theta - |\tilde{X}_s|^\theta) \nu(dy) \right) ds \right] \leq \left( \int_{y \neq 0} |y|^\theta \nu(dy) \right) t.
\]
Since \( \tau_n \uparrow \infty \) as \( n \uparrow \infty \), we can let \( n \uparrow \infty \) and use the monotone convergence theorem to get
\[
E|\tilde{X}_t|^\theta \leq \left( \int_{y \neq 0} |y|^\theta \nu(dy) \right) t.
\]
Using (3.2) again, we obtain that

\[(3.9) \quad E|X_t|^\theta \leq |\hat{\ell}|^\theta t^\theta + E|\hat{X}_t|^\theta \leq |\hat{\ell}|^\theta t^\theta + \left( \int_{y \neq 0} |y|^\theta \nu(dy) \right) t. \]

Together with Jensen’s inequality, this yields for any \( \theta \in [\kappa, 1] \) and \( t > 0 \)

\[E|X_t|^\kappa \leq \left[ E|X_t|^\theta \right]^{\kappa/\theta} \leq \left[ |\hat{\ell}|^\theta t^\theta + \left( \int_{y \neq 0} |y|^\theta \nu(dy) \right) t \right]^{\kappa/\theta}, \]

which finishes the proof. \( \square \)

In Section 2 we have introduced the index \( \sigma_0 \) for subordinators using the characteristic Laplace exponent (Bernstein function) \( \phi \). A similar index exists for a general Lévy process \( X \)—it is the counterpart at the origin of the classical Blumenthal–Getoor index—but its definition is based on the characteristic (i.e. Fourier) exponent \( \psi \):

\[(3.10) \quad \beta_0 := \sup \left\{ \alpha \geq 0 : \liminf_{|\xi| \to 0} \frac{\psi(\xi)}{|\xi|^\alpha} = 0 \right\} = \sup \left\{ \alpha \geq 0 : \limsup_{|\xi| \to 0} \frac{\psi(\xi)}{|\xi|^\alpha} = 0 \right\}. \]

Let us prove that \( \beta_0 \in [0, 2] \). Without loss of generality we may assume that \( \nu \neq 0 \); otherwise, the assertion is trivial. It is obvious that \( \beta_0 \geq 0 \). Since

\[1 - \cos u \geq (1 - \cos 1)u^2, \quad 0 \leq u \leq 1,\]

we have for all \( \xi \in \mathbb{R}^d \) that

\[|\psi(\xi)| \geq \text{Re} \psi(\xi) = \frac{1}{2} \xi \cdot Q\xi + \int_{y \neq 0} (1 - \cos |\xi \cdot y|) \nu(dy) \]
\[\geq \int_{y \neq 0, |\xi \cdot y| \leq 1} (1 - \cos |\xi \cdot y|) \nu(dy) \]
\[\geq (1 - \cos 1) \int_{y \neq 0, |\xi \cdot y| \leq 1} |\xi \cdot y|^2 \nu(dy). \]

Because \( \nu \neq 0 \), it is not hard to see that there exists a unit vector \( x_0 \in \mathbb{R}^d \) such that \( \nu_D := \mathbb{1}_D \nu \neq 0 \), where

\[D := \left\{ z \in \mathbb{R}^d \setminus \{0\} : \arccos \frac{x_0 \cdot z}{|z|} \in \left[0, \frac{\pi}{8}\right] \right\}. \]

Since \( \xi, y \in D \) satisfy

\[0 \leq \arccos \frac{\xi \cdot y}{|\xi||y|} \leq \arccos \frac{\xi \cdot x_0}{|\xi|} + \arccos \frac{x_0 \cdot y}{|y|} \leq \frac{\pi}{8} + \frac{\pi}{8} = \frac{\pi}{4}, \]

we see

\[\frac{\xi \cdot y}{|\xi||y|} \geq \frac{1}{\sqrt{2}}, \quad \xi, y \in D. \]

Thus, we get for all \( \xi \in D \) that

\[|\psi(\xi)| \geq (1 - \cos 1) \int_{y \in D, |\xi \cdot y| \leq 1} |\xi \cdot y|^2 \nu(dy) \]
\[\geq \frac{1 - \cos 1}{2} |\xi|^2 \int_{y \in D, |\xi \cdot y| \leq 1} |y|^2 \nu(dy), \]

and, by Fatou’s lemma,

\[\liminf_{\xi \in D, |\xi| \to 0} \frac{|\psi(\xi)|}{|\xi|^2} \geq \frac{1 - \cos 1}{2} \liminf_{\xi \in D, |\xi| \to 0} \int_{\mathbb{R}^d} |y|^2 \mathbb{1}_{(|\xi \cdot y| \leq 1)} \nu_D(dy) \]
This shows that \( \beta_0 \leq 2 \). Moreover, we have

\[
\beta_0 = \sup \left\{ \alpha \leq 2 : \int_{|y|>1} |y|^\alpha \nu(dy) < \infty \right\} = \sup \left\{ \alpha \geq 0 : \limsup_{|\xi| \to 0} \frac{\psi(\xi)}{|\xi|^\alpha} < \infty \right\}.
\]

The first equality is a special case of [8, Proposition 5.4]. The second equality follows immediately from the fact that

\[
\limsup_{|\xi| \to 0} \frac{\psi(\xi)}{|\xi|^\alpha} = 0 \iff \limsup_{|\xi| \to 0} \frac{\psi(\xi)}{|\xi|^\alpha} = 0.
\]

**Theorem 3.3.** Let \( X \) be a Lévy process in \( \mathbb{R}^d \). Assume that \( \limsup_{|\xi| \to 0} |\psi(\xi)||\xi|^{-\beta} < \infty \) for some \( \beta > 0 \). If \( \kappa \in (0, \beta) \), then

\[
\mathbb{E}|X_t|^\kappa \leq C_{\kappa, \beta, d} t^{\kappa/\beta}, \quad t \geq 1,
\]

holds for some constant \( C_{\kappa, \beta, d} > 0 \).

**Remark 3.4.** Let \( X \) be a symmetric \( \alpha \)-stable Lévy process in \( \mathbb{R}^d \) with \( 0 < \alpha < 2 \). Then \( \psi(\xi) = |\xi|^\alpha \) and we can choose \( \beta = \beta_0 = \alpha \). For \( t > 0 \) it is well known that \( \mathbb{E}|X_t|^\kappa \) is finite if, and only if, \( \kappa \in (0, \alpha) = (0, \beta) \), in which case \( \mathbb{E}|X_t|^\kappa = t^{\kappa/\alpha} \mathbb{E}|X_1|^\kappa \). This means that Theorem 3.3 is sharp for symmetric \( \alpha \)-stable Lévy processes.

**Proof of Theorem 3.3.** Since \( 0 < \kappa < \beta \leq \beta_0 \leq 2 \), we have, see e.g. [11, III.18.23],

\[
|x|^\kappa = c_{\kappa, d} \int_{\mathbb{R}^d \setminus \{0\}} (1 - \cos(x \cdot \xi)) |\xi|^{-\kappa-d} d\xi, \quad x \in \mathbb{R}^d,
\]

where

\[
c_{\kappa, d} := \frac{\kappa 2^{\kappa-1} \Gamma(-\frac{\kappa+d}{2})}{\pi d^{\frac{d+\kappa}{2}}},
\]

By Tonelli’s theorem, we get

\[
\mathbb{E}|X_t|^\kappa = c_{\kappa, d} \mathbb{E} \left[ \int_{\mathbb{R}^d \setminus \{0\}} (1 - \cos(X_t \cdot \xi)) |\xi|^{-\kappa-d} d\xi \right]
\]

\[
= c_{\kappa, d} \int_{\mathbb{R}^d \setminus \{0\}} (1 - \Re e^{-iX_t \cdot \xi}) |\xi|^{-\kappa-d} d\xi
\]

\[
= c_{\kappa, d} \int_{\mathbb{R}^d \setminus \{0\}} (1 - \Re e^{-it\psi(\xi)}) |\xi|^{-\kappa-d} d\xi.
\]

Since \( \Re \psi \geq 0 \), we have

\[
|1 - \Re e^{-it\psi(\xi)}| \leq |1 - e^{-it\psi(\xi)}| \leq 2 \wedge \left( t|\psi(\xi)| \right), \quad \xi \in \mathbb{R}^d \setminus \{0\},
\]

and, by our assumption,

\[
|\psi(\xi)| \leq C|\xi|^\beta, \quad 0 < |\xi| \leq 1
\]

\footnote{In analogy to Remark 2.1 this is equivalent to either \( 0 < \beta < \beta_0 \) or \( \beta = \beta_0 > 0 \) and \( \limsup_{|\xi| \to 0} |\psi(\xi)||\xi|^{-\beta_0} < \infty \).}
for some constant $C_\beta > 0$. Thus, we find for all $t \geq 1$

$$E|X_t|^\kappa \leq c_{\kappa,d} \int_{\mathbb{R}^d \setminus \{0\}} \left[ 2 \wedge (t |\psi(\xi)|) \right] |\xi|^{-\kappa - d} \, d\xi$$

$$\leq c_{\kappa,d} \int_{0<|\xi|\leq t^{-1/\beta}} t |\psi(\xi)||\xi|^{-\kappa - d} \, d\xi + c_{\kappa,d} \int_{|\xi|> t^{-1/\beta}} 2 |\xi|^{-\kappa - d} \, d\xi$$

$$\leq c_{\kappa,d} C_\beta t \int_{0<|\xi|\leq t^{-1/\beta}} |\xi|^\beta |\xi|^{-\kappa - d} \, d\xi + 2c_{\kappa,d} \int_{|\xi|> t^{-1/\beta}} |\xi|^{-\kappa - d} \, d\xi$$

$$= c_{\kappa,d} C_\beta t^{\kappa/\beta} \int_{0<|\xi|\leq 1} |\xi|^\beta |\xi|^{-\kappa - d} \, d\xi + 2c_{\kappa,d} t^{\kappa/\beta} \int_{|\xi|> 1} |\xi|^{-\kappa - d} \, d\xi.$$

Now the estimate follows with the constant

$$C_{\kappa,\beta,d} := c_{\kappa,d} C_\beta \int_{0<|\xi|\leq 1} |\xi|^\beta |\xi|^{-\kappa - d} \, d\xi + 2c_{\kappa,d} \int_{|\xi|> 1} |\xi|^{-\kappa - d} \, d\xi. \quad \square$$

**Proposition 3.5.** Let $X$ be a Lévy process in $\mathbb{R}^d$. If $\nu \neq 0$ and $\kappa > 1$, then

$$E e^{\delta |X_t|^\kappa} = \infty$$

for all $\delta > 0$ and $t > 0$.

**Remark 3.6.** It is well known, see [8, Theorem 26.1(ii)], that

$$E e^{\delta |X_t| \log |X_t|} = \infty, \quad \delta > 0,$$

for any Lévy process with Lévy measure that has an unbounded support $\text{supp} \nu$. Since for any $\kappa > 1$ there exists a constant $C_\kappa > 0$ such that

$$e^{\delta |x| \log |x|} \leq C_\kappa e^{\delta |x|^\kappa}, \quad \delta > 0, x \in \mathbb{R}^d,$$

this implies Proposition 3.5 if $\text{supp} \nu$ is unbounded; our proposition, however, is valid for all non-zero $\nu$.

**Proof of Proposition 3.5** Since $\nu \neq 0$ we may, without loss of generality, assume that there exist some Borel set $A \subset \mathbb{R}$ with either $\inf A > 0$ or $\sup A < 0$ and Borel sets $B_2, \ldots, B_d \subset \mathbb{R} \setminus \{0\}$ such that

$$\eta := \nu(\Lambda) \in (0, \infty),$$

where $\Lambda := A \times B_2 \times \cdots \times B_d$. The jump times of jumps with size in the set $\Lambda$ define a Poisson process, say $(N_t)_{t \geq 0}$, with intensity $\eta$. Note that $X$ can be decomposed into two independent Lévy processes

$$X_t = X^1_t + X^2_t, \quad t \geq 0,$$

where $X^1$ is a compound Poisson process with Lévy measure $\nu|_\Lambda$, and $X^2$ is a Lévy process with Lévy measure $\nu - \nu|_\Lambda$; moreover, $X^1$ and $X^2$ are independent processes. Set

$$r := (\inf A) \wedge (\sup A) \in (0, \infty).$$

By the triangle inequality we find for any $y \in \mathbb{R}^d$,

$$|X^1_t + y| \geq |X^1_t| - |y| \geq r N_t - |y|.$$ 

Using Stirling's formula

$$n! \leq \sqrt{2\pi n^{n+1/2} e^{-n+1/2}} \leq \sqrt{2\pi n^{n+1} e^{-n+1}} \quad n \in \mathbb{N},$$
we obtain that for any $\delta, t > 0$
\begin{equation}
Ee^{\delta|X_t^1+y|^n} \geq \mathbb{E} \left[ e^{\delta(rN_t-|y|)^n} \mathbb{1}_{\{rN_t>|y|\}} \right] \\
= \sum_{n: rn>|y|} e^{\delta(\eta t)^n} \frac{(\eta t)^n e^{-\eta t}}{n!} \\
\geq \frac{e^{-\eta t}}{\sqrt{2\pi} e} \sum_{n: rn>|y|} \frac{(\eta t)^n e^{\delta(rn-|y|)^n}}{n^{n+1}} = \infty,
\end{equation}
where the last equality is due to $\kappa > 1$. Combining this with Tonelli’s theorem, we get
\begin{equation}
Ee^{\delta|X_t|^n} = \int_{\mathbb{R}^d} e^{\delta|X_t^1+y|^n} \mathbb{P}(X_t^2 \in dy) = \infty.
\end{equation}

**Proposition 3.7.** Let $X$ be a Lévy process in $\mathbb{R}^d$. If the characteristic exponent $\psi$ is real-valued, then for any $t > 0$
\begin{equation}
E|X_t|^{-\theta} = \infty \text{ for all } \theta > d,
\end{equation}
and so
\begin{equation}
Ee^{\delta|X_t|^{-n}} = \infty \text{ for all } \delta > 0, \kappa > 0.
\end{equation}

**Remark 3.8.** The characteristic exponent $\psi$ of a Lévy process is real-valued if, and only if, the process is symmetric, i.e. it has characteristics $(0, Q, \nu)$ with $\nu(A) = \nu(-A)$ for all $A \in \mathcal{B}(\mathbb{R}^d \setminus \{0\})$.

**Proof of Proposition 3.7.** The second assertion follows from the first one as we may choose $n \in \mathbb{N}$ such that $n\kappa \geq d$ and
\begin{equation}
Ee^{\delta|X_t|^{-n}} \geq \frac{\delta^n}{n!} E|X_t|^{-n\kappa}.
\end{equation}

Recall that
\begin{equation}
\frac{1}{y^\theta} = \frac{1}{\Gamma(r)} \int_0^\infty e^{-uy} u^{r-1} du, \quad r > 0, \quad y \geq 0,
\end{equation}
and
\begin{equation}
e^{-u|x|} = c_d \int_{\mathbb{R}^d} e^{i(x, \xi)} \frac{u}{(u^2 + |\xi|^2)^{d+1}} d\xi, \quad x \in \mathbb{R}^d, \quad u > 0,
\end{equation}
where
\begin{equation}
c_d := \Gamma \left( \frac{d+1}{2} \right) / \pi^{\frac{d+1}{2}}.
\end{equation}

Using Tonelli’s theorem, we get
\begin{equation}
E|X_t|^{-\theta} = \frac{1}{\Gamma(\theta)} \mathbb{E} \left[ \int_0^\infty e^{-u|X_t|} u^{\theta-1} du \right] \\
= \frac{c_d}{\Gamma(\theta)} \int_0^\infty \mathbb{E} \left[ \int_{\mathbb{R}^d} e^{i(X_t, \xi)} \frac{u}{(u^2 + |\xi|^2)^{d+1}} d\xi \right] u^{\theta-1} du.
\end{equation}

Since $\psi(\xi) \in \mathbb{R}$ for all $\xi \in \mathbb{R}$, $0 < e^{-t\psi(\xi)} \leq 1$, and we can use Fubini’s theorem for the inner integrals and then Tonelli’s theorem for the two outer integrals to get
\begin{equation}
E|X_t|^{-\theta} = \frac{c_d}{\Gamma(\theta)} \int_{\mathbb{R}^d} \left( \int_0^\infty e^{-u|X_t|} \frac{u}{(u^2 + |\xi|^2)^{d+1}} du \right) u^{\theta-1} du \\
= \frac{c_d}{\Gamma(\theta)} \int_{\mathbb{R}^d} \left( \int_0^\infty \frac{u^\theta}{(u^2 + |\xi|^2)^{d+1}} du \right) e^{-t\psi(\xi)} d\xi.
\end{equation}
\begin{align*}
\int_{|y| \geq 1} \exp[\delta |y|^\kappa] \nu(dy) < \infty,
\end{align*}

where the last equality follows from \( \theta \geq d \).

Let \( \delta > 0 \) and \( \kappa \in (0, 1] \). Since the function \( \mathbb{R}^d \ni x \mapsto \exp[\delta |x|^\kappa] \in \mathbb{R} \) is submultiplicative, \( \mathbb{E} \exp[\delta |X_t|^\kappa] < \infty \) for some (hence, all) \( t > 0 \) if, and only if,

\begin{equation}
\int_{|y| \geq 1} \exp[\delta |y|^\kappa] \nu(dy) < \infty,
\end{equation}

see e.g. [8, Theorem 25.3].

**Theorem 3.9.** Let \( X \) be a Lévy process in \( \mathbb{R}^d \) with characteristics \( (\ell, 0, \nu) \), \( \delta > 0 \) and \( \kappa \in (0, 1] \). If (3.14) holds, then for any \( t > 0 \)

\[ \mathbb{E} \exp[\delta |X_t|^\kappa] \leq \exp \left[ \delta |\ell|^\kappa t^\kappa + \inf_{\epsilon > 0} \left( \delta \epsilon^{\kappa/2} + K_{\epsilon, \delta, \kappa, d} \right) \right], \]

where \( K_{\epsilon, \delta, \kappa, d} \) is given by (2.5).

**Proof.** As in the proof of Theorem 3.1 we only need to show that

\begin{equation}
\mathbb{E} \exp[\delta |\hat{X}_t|^\kappa] \leq \mathbb{E} \exp \left( \delta \left( \epsilon + |\hat{X}_t|^2 \right)^{\kappa/2} \right) \leq \exp \left[ \delta \epsilon^{\kappa/2} + K_{\epsilon, \delta, \kappa, d} \right]
\end{equation}

for all \( \epsilon > 0 \) and \( t > 0 \). Fix \( \epsilon > 0 \) and \( t > 0 \). Let

\[ g(x) := \exp \left( \delta \left( \epsilon + |x|^2 \right)^{\kappa/2} \right), \quad x \in \mathbb{R}^d, \]

and define \( \tau_n \) by (3.4). By Dynkin’s formula,

\begin{equation}
\mathbb{E} g(\hat{X}_{t \wedge \tau_n}) e^{\delta \epsilon^{\kappa/2}} = \mathbb{E} \left[ \int_{[0, t \wedge \tau_n]} \hat{L} g(\hat{X}_s) \, ds \right], \quad n \in \mathbb{N}.
\end{equation}

Let us estimate \( \hat{L} g(\hat{X}_s) \) for \( s < t \wedge \tau_n \). First,

\begin{align}
& \int_{|y| \geq 1} \left( g(\hat{X}_s + y) - g(\hat{X}_s) \right) \nu(dy) \\
\leq & \exp \left[ \delta |\hat{X}_s|^\kappa \right] \int_{|y| \geq 1} \left( \exp \left[ \delta \epsilon^{\kappa/2} + \delta |y|^\kappa \right] - 1 \right) \nu(dy) \\
\leq & g(\hat{X}_s) \left( e^{\delta \epsilon^{\kappa/2}} \int_{|y| \geq 1} \exp \left[ \delta |y|^\kappa \right] \nu(dy) - \nu(|y| \geq 1) \right).
\end{align}

On the other hand, since for any \( x \in \mathbb{R}^d \)

\begin{align*}
& \left| \frac{\partial^2 g}{\partial x_j \partial x_i}(x) \right| \\
= & \left| g(x) \left( \delta^2 \kappa^2 \left( \epsilon + |x|^2 \right)^{\kappa/2 - 2} x_i x_j + \delta \kappa (2 - \kappa) \left( \epsilon + |x|^2 \right)^{\kappa/2 - 1} \mathbb{1}_{i=j} \right) \right| \\
\leq & g(x) \left( \delta^2 \kappa^2 \left( \epsilon + |x|^2 \right)^{\kappa/2 - 2} |x|^2 + \delta \kappa (2 - \kappa) \left( \epsilon + |x|^2 \right)^{\kappa/2 - 1} \right) \\
\leq & g(x) \left( \delta^2 \kappa^2 \epsilon^{\kappa-1} + \delta \kappa (2 - \kappa) \epsilon^{\kappa-1} + \delta \kappa \epsilon^{\kappa-1} \right) \\
= & g(x) \left( \delta^2 \kappa^2 \epsilon^{\kappa-1} + \delta \kappa (3 - \kappa) \epsilon^{\kappa-1} \right),
\end{align*}
it follows that for any \( z \in \mathbb{R}^d \) with \( |z| \leq 1 \)
\[
\left| \frac{\partial^2 g}{\partial x_i \partial x_j}(\hat{X}_s + z) \right| \leq \exp \left[ \delta e^{\kappa/2} + \delta |\hat{X}_s|^\kappa + \delta |z|^\kappa \right] \left( \delta^2 \kappa^2 e^{\kappa-1} + \delta \kappa (3 - \kappa) e^{\kappa/2-1} \right)
\leq g(\hat{X}_s) \exp \left[ \delta e^{\kappa/2} + \delta \right] \left( \delta^2 \kappa^2 e^{\kappa-1} + \delta \kappa (3 - \kappa) e^{\kappa/2-1} \right).
\]
By Taylor’s theorem, one has
\[
g(\hat{X}_s + y) - g(\hat{X}_s) - y \cdot \nabla g(\hat{X}_s)
= \frac{1}{2} \sum_{i,j=1}^{d} \frac{\partial^2 g}{\partial x_i \partial x_j}(\hat{X}_s + \theta_{\hat{X},y}) y_i y_j
\leq \frac{1}{2} \sum_{i,j=1}^{d} \left| \frac{\partial^2 g}{\partial x_i \partial x_j}(\hat{X}_s + \theta_{\hat{X},y}) \right| |y_i y_j|
\leq \frac{1}{2} g(\hat{X}_s) \exp \left[ \delta e^{\kappa/2} + \delta \right] \left( \delta^2 \kappa^2 e^{\kappa-1} + \delta \kappa (3 - \kappa) e^{\kappa/2-1} \right) \sum_{i,j=1}^{d} |y_i y_j|
\leq g(\hat{X}_s) \frac{d}{2} \exp \left[ \delta e^{\kappa/2} + \delta \right] \left( \delta^2 \kappa^2 e^{\kappa-1} + \delta \kappa (3 - \kappa) e^{\kappa/2-1} \right) |y|^2,
\]
where \( \theta_{\hat{X},y} \in [-1, 1] \) depends on \( \hat{X}_s \) and \( y \). Thus,
\[
\int_{0 < |y| \leq 1} \left( g(\hat{X}_s + y) - g(\hat{X}_s) - y \cdot \nabla g(\hat{X}_s) \right) \nu(dy)
\leq g(\hat{X}_s) \frac{d}{2} \exp \left[ \delta e^{\kappa/2} + \delta \right] \left( \delta^2 \kappa^2 e^{\kappa-1} + \delta \kappa (3 - \kappa) e^{\kappa/2-1} \right) \int_{0 < |y| \leq 1} |y|^2 \nu(dy).
\]
Combining this with (3.17), we obtain
\[
\hat{\mathcal{L}}g(\hat{X}_s) \leq K_{\epsilon, \delta, \kappa, d} g(\hat{X}_s), \quad s < t \wedge \tau_n.
\]
This, together with (3.15) and Tonelli’s theorem, yields
\[
\mathbb{E} \left[ g(\hat{X}_t) \mathbb{1}_{\{t \leq \tau_n\}} \right] - \exp \left[ \delta e^{\kappa/2} \right] \leq \mathbb{E} g(\hat{X}_{t \wedge \tau_n}) - \exp \left[ \delta e^{\kappa/2} \right]
\leq K_{\epsilon, \delta, \kappa, d} \mathbb{E} \left[ \int_{[0, t \wedge \tau_n)} g(\hat{X}_s) \, ds \right]
= K_{\epsilon, \delta, \kappa, d} \mathbb{E} \left[ \int_0^t g(\hat{X}_s) \mathbb{1}_{\{s \leq \tau_n\}} \, ds \right]
= K_{\epsilon, \delta, \kappa, d} \int_0^t \mathbb{E} \left[ g(\hat{X}_s) \mathbb{1}_{\{s \leq \tau_n\}} \right] \, ds.
\]
We deduce from Gronwall’s inequality that
\[
\mathbb{E} \left[ g(\hat{X}_t) \mathbb{1}_{\{t \leq \tau_n\}} \right] \leq \exp \left[ \delta e^{\kappa/2} + K_{\epsilon, \delta, \kappa, d} t \right],
\]
for all \( n \in \mathbb{N} \). Finally, (3.15) follows as \( n \uparrow \infty \). \( \square \)

**Theorem 3.10.** Let \( X \) be a Lévy process in \( \mathbb{R}^d \) with characteristics \((\ell, 0, \nu)\), \( \delta > 0 \) and \( \kappa \in (0, 1) \). If (3.14) and
\[
(3.18) \quad \int_{0 < |y| < 1} |y|^\kappa \nu(dy) < \infty
\]
hold, then for any \( t > 0 \)

\[
\mathbb{E} \exp [\delta |X_t|^{\kappa}] \leq \exp \left[ \delta \hat{\ell} t^\kappa + M_{\delta, \kappa} t \right],
\]

where \( \hat{\ell} \) is given by (3.8) and

\[
M_{\delta, \kappa} := \int_{y \neq 0} \left( \exp [\delta |y|^{\kappa}] - 1 \right) \nu(dy).
\]

**Remark 3.11.** Since \( \nu \) is a Lévy measure, it is easy to see that (3.14) and (3.18) imply \( M_{\delta, \kappa} < \infty \).

**Proof of Theorem 3.10.** As in the proof of Theorem 3.2, we use Dynkin’s formula, (3.2) and Tonelli’s theorem to obtain that for all \( n \in \mathbb{N} \)

\[
\mathbb{E} \left[ \exp \left[ \delta |\tilde{X}_t|^{\kappa} \right] \mathbb{1}_{\{t < \tau_n\}} \right] - 1 \\
\leq \mathbb{E} \left[ \exp \left[ \delta |\tilde{X}_{t \wedge \tau_n}|^{\kappa} \right] \right] - 1 \\
= \mathbb{E} \left[ \int_{(0, t \wedge \tau_n)} \left( \int_{y \neq 0} \left( \exp [\delta |\tilde{X}_s + y|^{\kappa}] - \exp [\delta |\tilde{X}_s|^{\kappa}] \right) \nu(dy) \right) ds \right] \\
\leq \mathbb{E} \left[ \int_{(0, t \wedge \tau_n)} \left( \int_{y \neq 0} \exp [\delta |\tilde{X}_s|^{\kappa}] \left( \exp [\delta |y|^{\kappa}] - 1 \right) \nu(dy) \right) ds \right] \\
= M_{\delta, \kappa} \mathbb{E} \left[ \int_{(0, t \wedge \tau_n)} \exp [\delta |\tilde{X}_s|^{\kappa}] \right] ds \\
= M_{\delta, \kappa} \mathbb{E} \left[ \int_0^t \exp [\delta |\tilde{X}_s|^{\kappa}] \mathbb{1}_{\{s < \tau_n\}} ds \right] \\
= M_{\delta, \kappa} \int_0^t \mathbb{E} \left[ \exp [\delta |\tilde{X}_s|^{\kappa}] \mathbb{1}_{\{s < \tau_n\}} \right] ds.
\]

This together with Gronwall’s inequality yields that

\[
\mathbb{E} \left[ \exp \left[ \delta |\tilde{X}_t|^{\kappa} \right] \mathbb{1}_{\{t < \tau_n\}} \right] \leq e^{M_{\delta, \kappa} t}, \quad n \in \mathbb{N}.
\]

Letting \( n \uparrow \infty \), we conclude that

\[
\mathbb{E} \exp [\delta |\tilde{X}_t|^{\kappa}] \leq e^{M_{\delta, \kappa} t}.
\]

It remains to use (3.2) to complete the proof. \( \square \)

### 3.2. Subordinators

Subordinators are increasing Lévy processes. Let \( S = (S_t)_{t \geq 0} \) be a subordinator with Bernstein function \( \phi \) given by (1.1). In this section, we consider estimates for \( ES_t^{\kappa} \) and \( \mathbb{E} e^{\delta S_t^{\kappa}} \), where \( \kappa \in \mathbb{R} \setminus \{0\} \) and \( \delta > 0 \).

The following result is a direct consequence of the Theorems 3.1, 3.2, 3.9 and 3.10.

**Corollary 3.12.** Let \( S \) be a subordinator with Lévy measure \( \nu \) and \( \kappa \in (0, 1] \).

a) If \( \nu \) satisfies (3.1), then for every \( t > 0 \)

\[
ES_t^{\kappa} \leq \left( b + \int_{0<y<1} y \nu(dy) \right)^\kappa t^\kappa + \left( \int_{y \geq 1} y^{\kappa} \nu(dy) \right) t \\
+ 2 \left( \frac{1}{2} \kappa (3 - \kappa) \int_{0<y<1} y^2 \nu(dy) \right)^{\kappa/2} \left( 1 + \nu(y \geq 1) t \right)^{1-\kappa/2} t^{\kappa/2}.
\]
b) If \( \nu \) satisfies (3.17), then for every \( t > 0 \)
\[
\mathbb{E} S_t^\kappa \leq \inf_{\theta \in [\kappa,1]} \left[ b^\theta t^\theta + \left( \int_{y>0} y^\theta \nu(dy) \right) t \right]^{\kappa/\theta}.
\]

c) Let \( \delta > 0 \). If \( \nu \) satisfies (3.14), then for every \( t > 0 \)
\[
\mathbb{E} \exp \left[ \delta S_t^\kappa \right] \leq \exp \left[ \delta \left( b + \int_{0<y<1} y \nu(dy) \right) t^\kappa + \inf_{\epsilon>0} \left( \delta \epsilon^{\kappa/2} + K_{\epsilon,\delta,\kappa} t \right) \right],
\]
where \( K_{\epsilon,\delta,\kappa} = K_{\epsilon,\delta,\kappa,1} \) is given by (2.5) with \( d = 1 \).

d) Let \( \delta > 0 \). If \( \nu \) satisfies (3.18) and (3.14), then for every \( t > 0 \)
\[
\mathbb{E} \exp \left[ \delta S_t^\kappa \right] \leq \exp \left[ \delta b^\kappa t^\kappa + \left( \int_{y>0} \left( \exp \left[ \delta y^\kappa \right] - 1 \right) \nu(dy) \right) t \right].
\]

The following example shows that the result in Corollary 3.12(b) is sharp.

**Example 3.13.** Let \( S = (S_t)_{t \geq 0} \) be the Gamma process with parameters \( \alpha, \beta > 0 \); this is a subordinator with
\[
b = 0, \quad \nu(dy) = \alpha y^{-1} e^{-\beta y} I_{(0,\infty)}(y) dy.
\]

It is known that the distribution of \( S_t \) at time \( t > 0 \) is a \( \Gamma(\alpha t, \beta) \)-distribution, i.e.
\[
\mathbb{P}(S_t \in dx) = \frac{\beta^{\alpha t}}{\Gamma(\alpha t)} x^{\alpha t-1} e^{-\beta x} I_{(0,\infty)}(x) dx.
\]

Let \( \kappa \in (0,1] \). Then we have
\[
G(t) := \mathbb{E} S_t^\kappa = \frac{\Gamma(\alpha t + \kappa)}{\beta^\kappa \Gamma(\alpha t)}
\]
and
\[
H(t) := \inf_{\theta \in [\kappa,1]} \left[ t \int_{y>0} y^\theta \nu(dy) \right]^{\kappa/\theta} = \frac{1}{\beta^\kappa} \inf_{\theta \in [\kappa,1]} \left[ \alpha t \Gamma(\theta) \right]^{\kappa/\theta}.
\]

Since
\[
\lim_{t \to \infty} \frac{G(t)}{H(t)} = \lim_{t \to \infty} \frac{\Gamma(\alpha t + \kappa)}{\Gamma(\alpha t) \alpha t} = \frac{1}{\Gamma(\kappa)} \lim_{t \to \infty} \frac{\Gamma(\alpha t + \kappa)}{\Gamma(\alpha t + 1)} = 1,
\]
the upper bound in Corollary 3.12(b) is sharp for small \( t \). Moreover, by Stirling’s formula
\[
\Gamma(x) \sim \sqrt{2\pi x} x^{x-\frac{1}{2}} e^{-x}, \quad x \to \infty,
\]
once has
\[
\lim_{t \to \infty} \frac{G(t)}{H(t)} = \lim_{t \to \infty} \frac{\Gamma(\alpha t + \kappa)}{\Gamma(\alpha t) (\kappa/\alpha t)^{\kappa/\alpha t}} = \frac{1}{e^{\kappa}} \lim_{t \to \infty} \left( 1 + \frac{\kappa}{\alpha t} \right)^{\alpha t + \kappa - \frac{1}{2}} = 1.
\]

This means that Corollary 3.12(b) is also sharp as \( t \to \infty \).

Recall that the beta function is given by
\[
B(x,y) = \int_0^\infty u^{x-1} (1+u)^{-x-y} du = \frac{\Gamma(x) \Gamma(y)}{\Gamma(x+y)}, \quad x,y > 0.
\]

**Theorem 3.14.** Let \( S \) be a subordinator with Bernstein function \( \phi \) and \( \kappa > 0 \). If
\[
\phi(u) \geq c_1 \log(1+u), \quad u \geq c_2,
\]
holds for some constants \( c_1 > 0 \) and \( c_2 \geq 0 \), then for every \( t > \kappa/c_1 \)
\[
\mathbb{E} S_t^{-\kappa} \leq \frac{c_2^k}{\kappa \Gamma(\kappa)} + \frac{\Gamma(c_1 t - \kappa)}{\Gamma(c_1 t)}.
\]
If \( \phi(u) = c_1 \log(1 + u) \) for some \( c_1 > 0 \) and all \( u \geq 0 \) (i.e. \( S \) is the Gamma subordinator with parameters \( c_1 \) and 1), then \( c_2 = 0 \) and the equality holds in the last line.

**Proof.** By (3.13), Tonelli’s theorem and (3.20), we obtain for \( t > \kappa/c_1 \)

\[
E S_{t}^{-\kappa} = \frac{1}{\Gamma(\kappa)} E \left[ \int_0^\infty u^{\kappa-1} e^{-uS_t} \, du \right] = \frac{1}{\Gamma(\kappa)} \int_0^\infty u^{\kappa-1} e^{-t\phi(u)} \, du
\]

\[
\leq \frac{1}{\Gamma(\kappa)} \left( \int_0^{c_2} u^{\kappa-1} \, du + \int_0^\infty u^{\kappa-1} (1 + u)^{-c_1 t} \, du \right) = \frac{1}{\Gamma(\kappa)} \left( \frac{c_2^\kappa}{\kappa} + B(\kappa, c_1 t - \kappa) \right) = \frac{c_2^\kappa}{\kappa \Gamma(\kappa)} + \frac{\Gamma(c_1 t - \kappa)}{\Gamma(c_1 t)}.
\]

If \( \phi(u) = c_1 \log(1 + u) \), then it is clear that \( c_2 = 0 \) and the above inequality becomes an equality. \( \square \)

Recall that the subordinator index \( \sigma_0 \in [0, 1] \) was defined in (2.1).

**Theorem 3.15.** Let \( S \) be a subordinator. Assume that \( \limsup_{u \downarrow 0} \phi(u)u^{-\beta} < \infty \) for some \( \beta > 0 \). If \( \kappa \in (0, \beta) \), then

\[
E S_{t}^{-\kappa} \leq C_{\kappa, \beta}(t \vee 1)^{\kappa/\beta}, \quad t > 0,
\]

holds for some constant \( C_{\kappa, \beta} > 0 \).

**Remark 3.16.** As in Remark 3.4, it is easy to see that Theorem 3.15 is sharp for the \( \alpha \)-stable subordinator, \( 0 < \alpha < 1 \).

**Proof of Theorem 3.15.** Since \( S_t \) is increasing in \( t \), it suffices to prove the statement for \( t \geq 1 \). Since \( 0 < \kappa < \beta \leq \sigma_0 \leq 1 \), we know that

\[
x^{\kappa} = \frac{\kappa}{\Gamma(1 - \kappa)} \int_0^\infty (1 - e^{-xu}) u^{-\kappa-1} \, du, \quad x \geq 0,
\]

and using Tonelli’s theorem, we get

\[
E S_{t}^{-\kappa} = \frac{\kappa}{\Gamma(1 - \kappa)} E \left[ \int_0^\infty (1 - e^{-S_t u}) u^{-\kappa-1} \, du \right] = \frac{\kappa}{\Gamma(1 - \kappa)} \int_0^\infty (1 - e^{-t\phi(u)}) u^{-\kappa-1} \, du.
\]

By our assumptions, there exists a constant \( C_\beta > 0 \) such that

\[
\phi(u) \leq C_\beta u^\beta, \quad 0 \leq u \leq 1.
\]

Combining this with (3.21) and the elementary estimate

\[
1 - e^{-t\phi(u)} \leq 1 \wedge [t\phi(u)].
\]

we obtain for every \( t \geq 1 \)

\[
E S_{t}^{-\kappa} \leq \frac{\kappa}{\Gamma(1 - \kappa)} \int_0^\infty (1 \wedge [t\phi(u)]) u^{-\kappa-1} \, du
\]

\[\text{This is equivalent to either } 0 < \beta < \sigma_0 \text{ or } \beta = \sigma_0 > 0 \text{ and } \limsup_{u \downarrow 0} \phi(u)u^{-\sigma_0} < \infty.\]
Let $\sigma$ and large 22 C.-S. DENG AND R. L. SCHILLING argument. The proof is based on the fact that the functions $x$ The following result is essentially due to [5]. For the sake of completeness, we present the holds for some constant $\tilde{\kappa}$, $\delta, x > 0$, are completely monotone functions, cf. [10, Chapter 1].

Recall that $\sigma_1$ is defined by $[22]$. Let

$$\sigma_\infty := \inf \left\{ \alpha \geq 0 : \lim_{u \to \infty} \frac{\phi(u)}{u^{\alpha}} = 0 \right\}.$$ 

It is easy to see that $0 \leq \sigma_1 \leq \sigma_\infty \leq 1$; the index $\sigma_\infty$ is often called the Blumenthal–Getoor index of the subordinator $S$, cf. [2], and it is well known that

$$\sigma_\infty \geq \inf \left\{ \alpha \geq 0 : \int_{(0,1)} y^\alpha \nu(dy) < \infty \right\}$$

and the equality holds provided that the subordinator has no drift, i.e. $b = 0$ in (1.1). It is also not hard to see, cf. [11,11], that

$$\sigma_\infty = \inf \left\{ \alpha \geq 0 : \limsup_{u \to \infty} \frac{\phi(u)}{u^{\alpha}} < \infty \right\}.$$ 

The following result is essentially due to [3]. For the sake of completeness, we present the argument. The proof is based on the fact that the functions $x \mapsto x^{-\kappa}$ and $x \mapsto \exp[\delta x^{-\kappa}]$, $\kappa, \delta, x > 0$, are completely monotone functions, cf. [10] Chapter 1.

**Theorem 3.17.** Let $S$ be a subordinator with index $\sigma_1 > 0$.

a) Let $\kappa > 0$. If $\liminf_{u \to \infty} \phi(u)u^{-\alpha} > 0$ for some $\alpha > 0$ then there exists a constant $C_{\alpha,\kappa} > 0$ such that for all $t > 0$

$$E S_t^{-\kappa} \leq \frac{C_{\alpha,\kappa}}{(t \wedge 1)^{\kappa/\alpha}}.$$ 

b) Let $\kappa \in (0, \sigma_1/(1 - \sigma_1))$. If $\liminf_{u \to \infty} \phi(u)u^{-\alpha} > 0$ for some $\alpha > \kappa/(1 + \kappa)$ then there exists a constant $\bar{C}_{\alpha,\kappa} > 0$ such that for all $\delta, t > 0$

$$E \exp \left[ \delta S_t^{-\kappa} \right] \leq \exp \left[ \bar{C}_{\alpha,\kappa} \left( \delta + \frac{\delta}{t^{\kappa/\alpha}} \frac{(t - \sigma_1)}{\sigma_1^{\kappa/\alpha}} + \frac{\delta}{t^{\kappa/\alpha}} \right) \right].$$

c) Let $\sigma_1 < 1$ and $\delta, t > 0$. If

$$\phi(u) \geq \zeta u^{\sigma_1}$$

holds for some constant

$$\zeta > \frac{1}{t} \delta^{1 - \sigma_1} \left( 1 - \sigma_1 \right) \left( \frac{\sigma_1}{\sigma_1 - \sigma_1} \right)^{\sigma_1 - 1}$$

and large $u$, then there exists a positive constant $C_\zeta$ such that

$$E \exp \left[ \delta S_t^{-\frac{\sigma_1}{1 - \sigma_1}} \right] \leq e^{C_\zeta \delta} + \frac{C_\zeta \delta}{\sqrt{2\pi \sigma_1}} \frac{\delta}{(1 - \sigma_1)^{\frac{1}{2} + \frac{1}{\sigma_1}} (\zeta t)^{\frac{1}{2} + \frac{1}{\sigma_1}} - \delta}.$$

---

7This is equivalent to either $0 < \alpha < \sigma_1$ or $\alpha = \sigma_1 > 0$ and $\liminf_{u \to \infty} \phi(u)u^{-\sigma_1} > 0$.

8This is equivalent to either $\kappa/(1 + \kappa) < \alpha < \sigma_1$ or $\alpha = \sigma_1 > \kappa/(1 + \kappa)$ and $\liminf_{u \to \infty} \phi(u)u^{-\sigma_1} > 0$. 

---
Proof. a) By our assumption, there exist constants $C_1 = C_1(\alpha) > 0$ and $C_2 = C_2(\alpha) \geq 0$ such that

\begin{equation}
(3.23) \quad \phi(u) \geq C_1 u^\alpha, \quad u \geq C_2.
\end{equation}

Combining this with \([3.13]\), we obtain

$$
E_S^{-\kappa} = \frac{1}{\Gamma(\kappa)} \int_0^\infty u^{\kappa-1} e^{-t\phi(u)} \, du
$$

\begin{align*}
&\leq \frac{1}{\Gamma(\kappa)} \int_0^{C_2} u^{\kappa-1} \, du + \frac{1}{\Gamma(\kappa)} \int_0^\infty u^{\kappa-1} e^{-tC_1 u^\alpha} \, du \\
&= \frac{C_2^{\kappa}}{\kappa \Gamma(\kappa)} + \frac{\Gamma(\frac{\kappa}{\alpha})}{\alpha \Gamma(\kappa)(tC_1)^{\frac{\alpha}{\kappa}}}
\end{align*}

\begin{align*}
&\leq \frac{C_{\alpha,\kappa}}{(t \wedge 1)^{\frac{\alpha}{\kappa}}},
\end{align*}

where

$$
C_{\alpha,\kappa} := \frac{C_2^{\kappa}}{\kappa \Gamma(\kappa)} + \frac{\Gamma(\frac{\kappa}{\alpha})}{\alpha \Gamma(\kappa)C_1^{\frac{\alpha}{\kappa}}}.
$$

b) It follows from \([3.13]\) that for $x \geq 0$

$$
e^{\delta x^{-\kappa}} = 1 + \sum_{n=1}^\infty \frac{\delta^n}{n!} x^{nk} = 1 + \sum_{n=1}^\infty \frac{\delta^n}{n!} \Gamma(n\kappa) \int_0^\infty u^{nk-1} e^{-ux} \, du
$$

\begin{align*}
&= 1 + \int_0^\infty e^{-ux} k(u) \, du,
\end{align*}

where

$$
k(u) := \sum_{n=1}^\infty \frac{\delta^n}{n!\Gamma(n\kappa)} u^{nk-1}, \quad u > 0.
$$

Now we can use Tonelli’s theorem to obtain

\begin{equation}
(3.24) \quad E \exp \left[\delta S_t^{-\kappa}\right] = 1 + E \left[ \int_0^\infty e^{-uS_t} k(u) \, du \right] = 1 + \int_0^\infty e^{-t\phi(u)} k(u) \, du.
\end{equation}

Note that under the assumptions in b), \([3.23]\) also holds. Then we get

$$
E \exp \left[\delta S_t^{-\kappa}\right] \leq 1 + \int_0^{C_2} k(u) \, du + \int_0^\infty \exp \left[ -C_1 t u^\alpha \right] k(u) \, du
$$

\begin{align*}
&= 1 + \sum_{n=1}^\infty \frac{\delta^n C_{\alpha,\kappa}^{2n}}{n!\Gamma(n\kappa)n\kappa} + \frac{1}{\alpha} \sum_{n=1}^\infty \frac{\delta^n \Gamma(\frac{nk}{\alpha})}{n!\Gamma(n\kappa)(tC_1)^{\frac{n\kappa}{\alpha}}}.
\end{align*}

Combining this with the inequalities

$$
\sqrt{2\pi} \, x^{-\frac{1}{2}} e^{-x} \leq \Gamma(x) \leq \sqrt{2\pi} \, x^{-\frac{1}{2}} e^{-x + \frac{1}{12x}}, \quad x > 0,
$$

$$
n! \geq \sqrt{2\pi} \, n^{n+\frac{1}{2}} e^{-n}, \quad n \in \mathbb{N},
$$
we arrive at
\[
E \exp \left[ \delta S_t^- \right] \leq 1 + \frac{1}{2\pi \sqrt{\kappa}} \sum_{n=1}^{\infty} \frac{(\delta C_2^k e^{\kappa+1-k})^n}{n!n^{(1+k)n}} + \frac{1}{\sqrt{2\pi \alpha}} \sum_{n=1}^{\infty} \frac{e^{\alpha/(12\kappa)}}{\sqrt{n}} n^{(\frac{\kappa}{\alpha}-k-1)n} \left( \frac{e^{k+1}}{\kappa} \left( \frac{\kappa}{\alpha e C_1} \right)^{\frac{\kappa}{\alpha}} \frac{\delta}{\kappa^{\frac{\kappa}{\alpha}}} \right)^n.
\]
(3.25)
\[
\leq 1 + \frac{1}{2\pi \sqrt{\kappa}} \sum_{n=1}^{\infty} \frac{(\delta C_2^k e^{\kappa+1-k})^n}{n!n^{(1+k)n}} + \frac{e^{\alpha/(12\kappa)}}{\sqrt{2\pi \alpha}} \sum_{n=1}^{\infty} n^{(\frac{\kappa}{\alpha}-k-1)n} \left( \frac{e^{k+1}}{\kappa} \left( \frac{\kappa}{\alpha e C_1} \right)^{\frac{\kappa}{\alpha}} \frac{\delta}{\kappa^{\frac{\kappa}{\alpha}}} \right)^n.
\]

Set \( G := \delta C_2^k e^{\kappa+1-k} \); because of
\[
G^n/n^{(1+k)n} \leq G^n/n^n \leq G^n/n! = e^G - 1.
\]
(3.27)

Set
\[
\epsilon := -\frac{\kappa}{\alpha} + (0, 1) \quad \text{and} \quad H := \frac{e^\kappa}{\kappa} \left( \frac{\kappa}{\alpha e C_1} \right)^{\frac{\kappa}{\alpha}} \frac{\delta}{\kappa^{\frac{\kappa}{\alpha}}} > 0.
\]
Using Jensen’s inequality and (3.26), it holds that
\[
\sum_{n=1}^{\infty} \frac{H^n}{n^{\epsilon n}} = \sum_{n=1}^{\infty} \left( \frac{(2H)^\frac{\kappa}{\alpha}}{n^n} \right) \leq \left( \sum_{n=1}^{\infty} \frac{(2H)^\frac{\kappa}{\alpha}}{n^n} \right)^\epsilon \leq \left( \sum_{n=1}^{\infty} \frac{1}{n!} \left( \frac{(2H)^\frac{\kappa}{\alpha}}{2} \right)^n \right)^\epsilon = \left( \exp \left[ \frac{(2H)^\frac{\kappa}{\alpha}}{2} \right] - 1 \right)^\epsilon.
\]

Combining the above estimates and using the following elementary inequalities
\[
1 + z(e^x - 1)^y \leq e^{2xy + xz + yz} \quad \text{and} \quad \frac{1}{2} e^x + \frac{1}{2} e^y \leq e^{x+y}, \quad x, y, z \geq 0,
\]
we obtain
\[
E \exp \left[ \delta S_t^- \right] \leq 1 + \frac{1}{2\pi \sqrt{\kappa}} (e^G - 1) + \frac{e^{\alpha/(12\kappa)}}{\sqrt{2\pi \alpha}} \left( \exp \left[ \frac{(2H)^\frac{\kappa}{\alpha}}{2} / 2 \right] - 1 \right)^\epsilon
= \frac{1}{2} \left( 1 + \frac{1}{\pi \sqrt{\kappa}} (e^G - 1) + \frac{1}{\pi \sqrt{\kappa}} \right) + \frac{e^{\alpha/(12\kappa)}}{\sqrt{2\pi \alpha}} \left( \exp \left[ \frac{(2H)^\frac{\kappa}{\alpha}}{2} / 2 \right] - 1 \right)^\epsilon
\leq \frac{1}{2} \exp \left[ \left( 2 + \frac{1}{\pi \sqrt{\kappa}} + \frac{1}{\pi \sqrt{\kappa}} \right) G \right] + \frac{1}{2} \exp \left[ (2H)^\frac{\kappa}{\alpha} + \frac{2e^{\alpha/(12\kappa)}}{\sqrt{2\pi \alpha}} \left( \frac{2e^{\alpha/(12\kappa)}}{\sqrt{2\pi \alpha}} + 1 \right) \frac{2H}{2} \right]
\leq \exp \left[ \left( 2 + \frac{1}{\pi \sqrt{\kappa}} + \frac{1}{\pi \sqrt{\kappa}} \right) G + \epsilon(2H)^\frac{\kappa}{\alpha} + 2^{-\epsilon} e^{\alpha/(12\kappa)} \left( \frac{2e^{\alpha/(12\kappa)}}{\sqrt{2\pi \alpha}} + 1 \right) H \right].
\]
\[
\leq \exp \left[ \tilde{C}_{\alpha, \kappa} \left( \delta + \left( \frac{\delta}{t^{\kappa/\alpha}} \right)^{\frac{\alpha}{1-\alpha}} + \frac{\delta}{t^{\kappa/\alpha}} \right) \right],
\]

with a suitable constant \( \tilde{C}_{\alpha, \kappa} \) depending only on \( \alpha \) and \( \kappa \).

c) Let \( \kappa := \sigma_1/(1 - \sigma_1) \). By our assumption, there exists a constant \( \hat{\zeta} \geq 0 \) such that \( \phi(u) \geq \zeta u^{\sigma_1}, \quad u \geq \hat{\zeta} \).

Using (3.25) where we replace \( c \) by \( \sigma_1 \), \( \zeta \) and \( \hat{\zeta} \), respectively, we get

\[
E \exp \left[ \delta S_t^{-1/2} \right] = E \exp \left[ \delta S_t^{-\kappa} \right]
\]

\[
\leq 1 + \frac{1}{2\pi \sqrt{\kappa}} \sum_{n=1}^{\infty} \left( \frac{\delta \hat{\zeta} e^{\sigma_1/(1+\kappa)}}{n^{1/(1+\kappa)}} \right)^n + \frac{e^{\sigma_1/(12\kappa)}}{2\pi \sigma_1} \sum_{n=1}^{\infty} \left( \frac{e^{\kappa/(12\kappa)}}{\kappa^\kappa} \left( \frac{\kappa}{\sigma_1 e^\zeta} \right)^{\kappa/\sigma_1} \frac{\delta}{t^{\kappa/\sigma_1}} \right)^n.
\]

By (3.27)—with \( G \) replaced by \( \delta \hat{\zeta} e^{\sigma_1/(1+\kappa)} \)—and (3.28), we get

\[
1 + \frac{1}{2\pi \sqrt{\kappa}} \sum_{n=1}^{\infty} \left( \frac{\delta \hat{\zeta} e^{\sigma_1/(1+\kappa)}}{n^{1/(1+\kappa)}} \right)^n \leq 1 + \frac{1}{2\pi \sqrt{\kappa}} \left( \exp \left[ \delta \hat{\zeta} e^{\sigma_1/(1+\kappa)} \right] - 1 \right)
\]

\[
\leq \exp \left[ \left( 2 + \frac{1}{4\pi^2 \kappa} + \frac{1}{2\pi \sqrt{\kappa}} \right) \delta \hat{\zeta} e^{\sigma_1/(1+\kappa)} \right] = e^{C_\zeta \delta},
\]

where the constant \( C_\zeta \) depends on \( \sigma_1 \) and \( \hat{\zeta} \). Note that

\[
\frac{e^{\sigma_1/(12\kappa)}}{2\pi \sigma_1} \sum_{n=1}^{\infty} \left( \frac{e^{\kappa/\sigma_1}}{\kappa^\kappa} \left( \frac{\kappa}{\sigma_1 e^\zeta} \right)^{\kappa/\sigma_1} \frac{\delta}{t^{\kappa/\sigma_1}} \right)^n = \frac{e^{1-\sigma_1/12}}{2\pi \sigma_1} \sum_{n=1}^{\infty} \frac{\delta^n}{(1 - \sigma_1)\sigma_1^{1/\sigma_1} (\zeta t)^{1-\sigma_1}},
\]

which is, due to

\[
\frac{\delta}{(1 - \sigma_1)\sigma_1^{1/\sigma_1} (\zeta t)^{1-\sigma_1}} < 1,
\]

convergent with sum

\[
\frac{e^{1-\sigma_1/12}}{\sqrt{2\pi \sigma_1}} \frac{\delta}{(1 - \sigma_1)\sigma_1^{1/\sigma_1} (\zeta t)^{1-\sigma_1}}.
\]

This implies the estimate. \( \square \)

**Proposition 3.18.** Let \( S \) be a subordinator and assume that \( \sigma_\infty < 1 \).

a) If \( \kappa > \sigma_\infty/(1 - \sigma_\infty) \), then \( E \exp \left[ \delta S_t^{-\kappa} \right] = \infty \) for all \( \delta, t > 0 \).

b) If \( \sigma_\infty > 0 \) and there is a constant \( C > 0 \) such that

\[
(3.29) \quad \frac{\phi(u)}{u^{\sigma_\infty}} \leq \frac{1}{t} \left( \frac{\delta}{(1 - \sigma_\infty)\sigma_\infty^{1/\sigma_\infty}} \right)^{1-\sigma_\infty}, \quad u \geq C
\]

holds for some \( t, \delta > 0 \), then \( E \exp \left[ \delta S_t^{-\sigma_\infty/1-\sigma_\infty} \right] = \infty \).
Proof. a) Pick $\alpha \in (\sigma_\infty, \kappa/(1 + \kappa))$. By the definition of $\sigma_\infty$, there exist two positive constants $C_1 = C_1(\alpha)$ and $C_2 = C_2(\alpha)$ such that

\begin{equation}
\phi(u) \leq C_1 u^\alpha, \quad u \geq C_2.
\end{equation}

Together with (3.29), this yields that

\begin{equation}
E \exp \left[ \delta S_t^{1 - \sigma_\infty} \right] \geq \int_{C_2}^{\infty} e^{-t\phi(u)} k(u) \, du \quad \geq \int_{C_2}^{\infty} e^{-C_1 t u^\alpha} k(u) \, du \quad = \sum_{n=1}^{\infty} \frac{\delta^n}{n! \Gamma(n \kappa)} \left( C_t \right)^{\frac{n \alpha}{\alpha}} \int_{C_1 C_2 t}^{\infty} u^{\frac{n \alpha}{\alpha} - 1} e^{-u} \, du
\end{equation}

(3.31)

\begin{equation}
\geq \sum_{n=1}^{\infty} \frac{\delta^n}{n! \Gamma(n \kappa)} \left( C_t \right)^{\frac{n \alpha}{\alpha}} \left[ \Gamma \left( \frac{n \kappa}{\alpha} \right) - \int_0^{C_1 C_2 t} u^{\frac{n \alpha}{\alpha} - 1} e^{-u} \, du \right]
\end{equation}

(3.32)

By (3.19) and

\( n! \sim \sqrt{2\pi n} n^{\frac{1}{2}} e^{-n}, \quad n \to \infty, \)

we have for large $n \in \mathbb{N}$

\begin{equation}
\frac{\delta^n}{n! \Gamma(n \kappa)} \left( C_t \right)^{\frac{n \alpha}{\alpha}} \left[ \Gamma \left( \frac{n \kappa}{\alpha} \right) - \frac{\alpha}{n \kappa} \left( C_1 C_2 t \right)^{\frac{n \alpha}{\alpha}} \right] \sim \frac{\delta^n}{2\pi \kappa^{-\frac{1}{2}} \gamma(n(1 + \kappa)) \left( e^{-\kappa - 1} \kappa \left( C_t \right)^{\frac{n \alpha}{\alpha}} \right)^n} \left[ \sqrt{\frac{2\pi \alpha}{n \kappa}} n^{\frac{n \alpha}{\alpha}} \left( C_1 C_2 t \right)^{\frac{n \alpha}{\alpha}} \right] \end{equation}

(3.32)

which, because of $-1 - \kappa + \frac{\kappa}{\alpha} > 0$, tends to infinity as $n \to \infty$.

b) Using (3.32), and using similar arguments as in (3.31) and (3.32), we obtain that

\begin{equation}
E \exp \left[ \delta S_t^{1 - \sigma_\infty} \right]
\geq \sum_{n=1}^{\infty} \left( 1 - \sigma_\infty \right)^{\sigma_\infty} \frac{\sigma_\infty^{1 - \sigma_\infty}}{n! \Gamma \left( \frac{n \sigma_\infty}{1 - \sigma_\infty} \right)} \left[ \Gamma \left( \frac{n}{1 - \sigma_\infty} \right) - \frac{1 - \sigma_\infty}{n} \left( C^{\sigma_\infty} \left( \frac{\delta}{(1 - \sigma_\infty)\sigma_\infty^{1 - \sigma_\infty}} \right) \right)^{1 - \sigma_\infty} \right] \frac{n^{\sigma_\infty}}{1 - \sigma_\infty}
\end{equation}

and

\begin{equation}
\frac{\left( 1 - \sigma_\infty \right)^{\sigma_\infty} \sigma_\infty^{1 - \sigma_\infty}}{n! \Gamma \left( \frac{n \sigma_\infty}{1 - \sigma_\infty} \right)} \left[ \Gamma \left( \frac{n}{1 - \sigma_\infty} \right) - \frac{1 - \sigma_\infty}{n} \left( C^{\sigma_\infty} \left( \frac{\delta}{(1 - \sigma_\infty)\sigma_\infty^{1 - \sigma_\infty}} \right) \right)^{1 - \sigma_\infty} \right] \frac{n^{\sigma_\infty}}{1 - \sigma_\infty}
\end{equation}

\begin{equation}
\sim \sqrt{\frac{\sigma_\infty}{2\pi \sqrt{n}}} \end{equation}

as $n \to \infty$. This finishes the proof. \( \square \)
References

[1] C. Berg, G. Forst: Potential Theory on Locally Compact Abelian Groups. Springer, Ergebnisse der Mathematik und ihrer Grenzgebiete. II. Ser. Bd. 87, Berlin 1975.
[2] R.M. Blumenthal, R.K. Getoor: Sample functions of stochastic processes with stationary independent increments. J. Math. Mech. 10 (1961) 493–516.
[3] C.-S. Deng, R.L. Schilling: On a Cameron–Martin type quasi-invariance theorem and applications to subordinate Brownian motions. Preprint 2014.
[4] N. Jacob: Pseudo Differential Operators and Markov Processes (Volume I). Imperial College Press, London 2001.
[5] M. Gordina, M. Röckner, F.-Y. Wang: Dimension-independent Harnack inequalities for subordinated semigroups. Potential Anal. 34 (2011) 293–307.
[6] H. Luschgy, G. Pagès: Moment estimates for Lévy processes. Electron. Commun. Probab. 13 (2008) 422–434.
[7] P.W. Millar: Path behaviour of processes with stationary independent increments. Z. Wahrscheinlichkeitstheorie verw. Geb. 17 (1971) 53–73.
[8] K. Sato: Lévy Processes and Infinitely Divisible Distributions, Cambridge University Press, Cambridge 1999.
[9] R.L. Schilling: Growth and Hölder conditions for the sample paths of Feller processes. Probab. Theor. Rel. Fields 112 (1998) 565–611.
[10] R.L. Schilling, R. Song, Z. Vondraček: Bernstein Functions. Theory and Applications (2nd Edn). De Gruyter, Studies in Mathematics 37, Berlin 2012.
[11] R.L. Schilling, J. Wang: Functional inequalities and subordination: stability of Nash and Poincaré inequalities. Math. Z. 272 (2012) 921–936.
[12] F.-Y. Wang: Logarithmic Sobolev inequalities on noncompact Riemannian manifolds. Probab. Theor. Rel. Fields 109 (1997) 417–424.
[13] F.-Y. Wang: Harnack Inequalities for Stochastic Partial Differential Equations. Springer, New York 2013.
[14] F.-Y. Wang: Integration by parts formula and shift Harnack inequality for stochastic equations. Ann. Probab. 42 (2014) 994–1019.
[15] F.-Y. Wang, J. Wang: Harnack inequalities for stochastic equations driven by Lévy noise. J. Math. Anal. Appl. 410 (2014) 513–523.

(C.-S. Deng) School of Mathematics and Statistics, Wuhan University, Wuhan 430072, China
Current address: TU Dresden, Fachrichtung Mathematik, Institut für Mathematische Stochastik, 01062 Dresden, Germany
E-mail address: dengcs@whu.edu.cn

(R.L. Schilling) TU Dresden, Fachrichtung Mathematik, Institut für Mathematische Stochastik, 01062 Dresden, Germany
E-mail address: rene.schilling@tu-dresden.de