On rotated Schur-positive sets

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Abstract

The problem of finding Schur-positive sets of permutations, originally posed by Gessel and Reutenauer, has seen some recent developments. Schur-positive sets of pattern-avoiding permutations have been found by Sagan et al and a general construction based on geometric operations on grid classes has been given by the authors. In this paper we prove that horizontal rotations of Schur-positive subsets of permutations are always Schur-positive. The proof applies a cyclic action on standard Young tableaux of certain skew shapes and a jeu-de-taquin type straightening algorithm. As a consequence of the proof we obtain a notion of cyclic descent set on these tableaux, which is rotated by the cyclic action on them.

Keywords: Schur-positivity, cyclic descent, standard Young tableau, horizontal rotation, cyclic action

Mathematics subject classification: 05E05, 05A05, 05E18; 05E10, 05A19

1 Introduction

For each $D \subseteq [n-1] = \{1,2,\ldots,n-1\}$, define the fundamental quasisymmetric function

$$F_{n,D}(x) := \sum_{i_1 \leq i_2 \leq \cdots \leq i_n} x_{i_1}x_{i_2}\cdots x_{i_n}$$

where $\{i : \pi(i) > \pi(i+1)\}$ is the descent set of $\pi$. If $B$ is a multiset of permutations in $S_n$, we define $Q(B)$ analogously, by adding $F_{n,\text{Des}(\pi)}$ as many times as the multiplicity of $\pi$ in $B$.

We say that $B$ is symmetric if $Q(B)$ is a symmetric function. In this case, we say that $B$ is Schur-positive if the expansion of $Q(B)$ in the basis of Schur functions has nonnegative coefficients. The problem of determining whether a given subset of permutations is symmetric and Schur-positive was first posed in [11], see also [17], [10] and [19]. The search for Schur-positive subsets is an active area of research [2, 16, 15, 5, 1, 9].

For $J \subseteq [n-1]$, define the descent class

$$D_{n,J} := \{\pi \in S_n : \text{Des}(\pi) = J\}.$$
and its inverse $D_{n,j}^{-1} = \{ \pi : \pi^{-1} \in D_{n,j} \}$. It was shown in [10] that inverse descent classes are Schur-positive.

Let $c$ be the $n$-cycle $(1,2,\ldots,n)$, and let $C_n = \langle c \rangle = \{ c^k : 0 \leq k < n \}$, the cyclic subgroup of $S_n$ generated by $c$. Given $\pi \in S_n$, a permutation of the form $\pi c^k$ is called a horizontal rotation of $\pi$. Horizontal rotations played an important role in [9].

Any set $A \subseteq S_{n-1}$ can be interpreted as a subset of $S_n$ by identifying $S_{n-1}$ with the set of the permutations in $S_n$ that fix $n$. Then, $AC_n$ is the set of horizontal rotations of elements in $A$,

$$AC_n = \{ \pi(k+1)\pi(k+2)\ldots\pi(n-1)\pi(1)\pi(2)\ldots\pi(k) : \pi \in A, 0 \leq k < n \}.$$

Note in particular that all elements in $AC_n$ appear with multiplicity one.

It was recently shown that horizontally rotated inverse descent classes are always Schur-positive.

**Theorem 1.1 ([9] Theorem 7.1]).** For every $J \subseteq [n-2]$, the set $D_{n-1,J}^{-1}C_n$ is Schur-positive.

The main result of this paper is a generalization of Theorem 1.1, where $D_{n-1,J}^{-1}$ is replaced with an arbitrary Schur-positive set $A \subseteq S_{n-1}$, stated in Theorem 2.1.

The proof involves a jeu-de-taquin type algorithm for nonstandard Young tableaux, which is used to determine a $\mathbb{Z}_n$-action on standard Young tableaux of certain skew shapes. This action is analogous to the promotion cyclic action on standard Young tableaux of rectangular shape introduced by Rhoades [14]. A byproduct of the proof is a notion of cyclic descent set on standard Young tableaux of certain skew shapes, which is rotated by the cyclic action on them.

The rest of the paper is organized as follows. In Section 2 we state the main result. Section 3 introduces some tools needed for the proof, namely cyclic descent sets of permutations and rotated tableaux. The proof of the main theorem appears in Section 4. Section 5 concludes with a discussion of cyclic descents of standard Young tableaux.

## 2 Main Theorem

We write $\lambda \vdash n$ to denote that $\lambda$ is a partition of $n$, and denote by $s_\lambda$ the Schur function indexed by $\lambda$. The following is our main result. When computing $Q(A)$, we consider $A$ as a subset of $S_{n-1}$.

**Theorem 2.1.** Let $n \geq 2$. For every Schur-positive set $A \subseteq S_{n-1}$, the set $AC_n$ is Schur-positive. Additionally,

$$Q(AC_n) = Q(A)s_1.$$  \hspace{1cm} (1)

In the statement of Theorem 2.1 the fact that $AC_n$ is Schur-positive is an immediate consequence Equation (1). Indeed, if $Q(A)$ is Schur-positive, then we obtain a non-negative expansion of $Q(AC_n)$ in terms of Schur functions using Pieri’s rule [18, Theorem 7.15.7], which states that for $\lambda \vdash n-1$,

$$s_\lambda s_1 = \sum_{\mu \vdash n} s_\mu.$$

Section 4 will be devoted to proving Equation (1).

An equivalent way to write Equation (1) is

$$Q(AC_n)Q(\{id\}) = Q(A)Q(C_n),$$  \hspace{1cm} (2)

where $id$ is the identity permutation in $S_{n-1}$. To see this, note that $Q(\{id\}) = s_{n-1}$ and $Q(C_n) = s_n + s_{n-1,1} = s_1s_{n-1}$.
Remark 2.2. 1. For arbitrary subsets $A \subseteq S_{n-1}$, Equation (1) does not necessarily hold. For example, if $n = 4$ and $A = \{132\} \subset S_3$, the left-hand side $Q(AC_n) = 2s_{2,2}$ is symmetric and Schur-positive, but $Q(A)s_1$ is not symmetric.

2. Vertical rotation, i.e., left multiplication of a Schur-positive set $A \subseteq S_{n-1}$ by $C_n$ does not necessarily result in a Schur-positive set. For example, if $A = \{3142, 1423\} \subset S_4$, then $Q(A)$ and $Q(AC_5)$ are Schur-positive, but $Q(C_5A)$ is not even symmetric.

We end this section with an equivalent formulation of the main theorems in terms of characters. Recall the Frobenius characteristic map, defined by

$$\text{ch}(\chi) = \frac{1}{n!} \sum_{\pi \in S_n} \chi(\pi)p_\pi(x)$$

where $\chi : S_n \to \mathbb{C}$ is a class function, $p_\pi(x) = p_\lambda(x)$ for every permutation $\pi$ of cycle type $\lambda \vdash n$, and $p_\lambda(x)$ is a power sum symmetric function.

Using this terminology, Theorem 2.1 is equivalent to the following statement.

**Theorem 2.3.** Let $\chi$ be an $S_{n-1}$-character and $A \subseteq S_{n-1}$. If $Q(A) = \text{ch}(\chi)$, then $Q(AC_n) = \text{ch}(\chi \uparrow S_n)$.

3 Cyclic descents of rotated tableaux

In this section we introduce some tools that will be used in the proof of Theorem 2.1.

3.1 Standard Young tableaux and their rotations

For $\lambda \vdash n$, denote by $\text{SYT}(\lambda)$ the set of standard Young tableaux (SYT for short) of shape $\lambda$, and define $\text{SYT}(\lambda/\mu)$ similarly for a skew shape $\lambda/\mu$. Let $s_\lambda$ and $s_{\lambda/\mu}$ denote the corresponding Schur functions. The descent set of a standard Young tableau $T$ is defined as

$$\text{Des}(T) := \{i : i + 1 \text{ in a lower row than } i \text{ in } T\},$$

where we use the English notation, in which row indices increase from top to bottom.

Recall the Robinson–Schensted correspondence, which associates to each $\pi \in S_n$ a pair $(P_{\pi}, Q_{\pi})$ of standard Young tableaux of the same shape $\lambda$, for some $\lambda \vdash n$. The tableaux $P_\pi$ and $Q_\pi$ are called the *insertion* and *recording* tableaux of $\pi$, respectively. Inverting a permutation has the effect of switching the tableaux, namely, $P_{\pi^{-1}} = Q_\pi$ and $Q_{\pi^{-1}} = P_\pi$ for all $\pi \in S_n$. The correspondence is a Des-preserving bijection in the following sense.

**Lemma 3.1** ([18, Lemma 7.23.1]). Let $\pi \in S_n$. Then $\text{Des}(\pi) = \text{Des}(Q_\pi)$ and $\text{Des}(\pi^{-1}) = \text{Des}(P_\pi)$.

The following is a well-known result of Gessel [18, Theorem 7.19.7].

**Proposition 3.2.** For every skew shape $\lambda/\mu$,

$$\sum_{T \in \text{SYT}(\lambda/\mu)} F_{n, \text{Des}(T)} = s_{\lambda/\mu}.$$
Definition 3.3. A rotated SYT of size $n$ is a tableau on the letters $1, 2, \ldots, n$ where each letter appears exactly once, and entries are increasing along rows and columns with respect to the order $k + 1 < k + 2 < \cdots < n < 1 < 2 < \cdots < k$,

for some $1 \leq k \leq n$.

In the rest of this paper, it will be convenient to consider the entries $1, 2, \ldots, n$ as elements of $\mathbb{Z}_n$, so that $0$ is identified with $n$, and addition takes place modulo $n$. For a SYT $T$ of size $n$ and an integer $k$, denote by $k + T$ the tableau obtained by adding $k$ modulo $n$ to all its entries. Note that $R$ is a rotated tableau if and only if it is of the form $R = k + T$ for some $k$ and some SYT $T$.

Example 3.4. The rotated tableau

$$R = \begin{array}{ccc}
1 & 2 & 6 \\
5 & 1 & 3 \\
3 & 4 & 2
\end{array} = 2 + \begin{array}{ccc}
1 & 3 & 5 \\
4 & 6 & 2
\end{array}$$

is standard with respect to the order $3 < 4 < 5 < 6 < 1 < 2$.

3.2 Cyclic descents of permutations and rotated SYT

The cyclic descent set of a permutation was introduced by Cellini [6] and further studied in [12, 7].

Definition 3.5. The cyclic descent set of $\pi \in S_n$ is

$$c\text{Des}(\pi) = \begin{cases} 
\text{Des}(\pi) & \text{if } \pi(n) < \pi(1), \\
\text{Des}(\pi) \cup \{n\} & \text{if } \pi(n) > \pi(1).
\end{cases}$$

Given a subset $D \subseteq [n] \simeq \mathbb{Z}_n$, let $k + D = \{k + d : d \in D\} \subseteq [n]$, with addition modulo $n$.

Observation 3.6. For every $\pi \in S_n$ and $0 \leq k < n$,

$$c\text{Des}(\pi e^{-k}) = k + c\text{Des}(\pi).$$

A notion of cyclic descents for standard Young tableaux of rectangular shapes was introduced by Rhoades [14], see also [13]. As in the case of permutations, the cyclic descent set respects a natural $\mathbb{Z}_n$-action on the set of SYT of a given rectangular shape. This action, which was horizontal rotation in the case of permutations (Observation 3.6), is Schützenberger’s promotion operation in the case of rectangular SYT. Also, in both cases, the cyclic descent set restricts to the regular descent set when the letter $n$ is ignored.

The next definition, where again we identify $[n] \simeq \mathbb{Z}_n$, extends this concept to rotated SYT. For further discussion, see Section 5.

Definition 3.7. Let $R$ be a rotated SYT of size $n$. Define

$$c\text{Des}_{\text{rot}}(R) := \{i \in [n] : i + 1 \text{ is in a lower row than } i \text{ in } R\},$$

$$\text{Des}(R) := c\text{Des}_{\text{rot}}(R) \cap [n - 1].$$
Remark 3.8. The reading word of a SYT $T$ is the permutation obtained by reading the rows of $T$ from left to right and from bottom to top. As an alternative to Definition 3.7, we could have defined the cyclic descent set of a rotated SYT to be the cyclic descent set of the inverse of the reading word of $T$. This is equivalent to

$$cDes'_rot(T) := \{ i \in [n] : i + 1 \text{ is strictly south of } i \text{ or in the same row and west of } i \},$$

Figure 1 shows a picture of the regions where $i + 1$ has to be, relative to the location of $i$, for $i$ to be a cyclic descent in these two definitions. Even though $cDes_{rot}$ and $cDes'_rot$ do not coincide on rotated SYT in general, it can be checked that they coincide on the kind of rotated SYT considered in Section 4.

![Figure 1: Left: $i \in cDes_{rot}(T)$ if and only if $i + 1$ is in the blue region. Right: $i \in cDes'_rot(T)$ if and only if $i + 1$ is in the green region.](image)

Observation 3.9. For every SYT $T$ of size $n$ and integer $k$

$$cDes_{rot}(k + T) = k + cDes_{rot}(T).$$

Example 3.10. For the tableaux

$$T = \begin{array}{ccc} 1 & 3 & 5 \\ 2 & 4 & 6 \end{array}, \quad 2 + T = \begin{array}{ccc} 3 & 5 & 1 \\ 4 & 6 & \end{array}, \quad 3 + T = \begin{array}{ccc} 4 & 6 & 2 \\ 5 & 1 \end{array},$$

we have $cDes_{rot}(T) = \{1, 3, 6\}$, $cDes_{rot}(2 + T) = \{3, 5, 2\} = 2 + cDes_{rot}(T)$, $cDes_{rot}(3 + T) = \{4, 6, 3\} = 3 + cDes_{rot}(T)$.

4 Proof of Theorem 2.1

In this section we will prove Equation (1). We start by showing that $Q(AC_n)$ is completely determined by $Q(A)$.

Lemma 4.1. Let $A$ and $A'$ be multisets of $S_{n-1}$, and suppose that $Q(A) = Q(A')$. Then $Q(AC_n) = Q(A'C_n)$.

Proof. First, note that $Q(A) = Q(A')$ if and only if the distribution of Des is the same on $A$ as in $A'$. Indeed, the “if” direction is trivial by definition, and the converse holds because the set $\{F_{D,x} : D \subseteq [n-1]\}$ of fundamental symmetric functions forms a basis of the vector space of homogeneous quasisymmetric functions of degree $n$ (see e.g. [18, Prop. 7.19.1]).

It now remains to show that the distribution of Des on $AC_n$ is completely determined by the distribution of Des on $A$. To see this, note that for $\sigma \in S_{n-1}$, the descent set of a rotation $\sigma c^{-k}$ depends only on the descent set of $\sigma$ and on $k$, namely,

$$\text{Des}(\sigma c^{-k}) = (k + \text{Des}(\sigma)) \setminus \{n\} \cup \{k\}.$$
It follows that \( Q(AC_n) \) is completely determined by \( Q(A) \), and so \( Q(A) = Q(A') \) implies \( Q(AC_n) = Q(A'C_n) \).

Now let \( A \subseteq S_{n-1} \) be a Schur-positive set. This means that we can write \( Q(A) = \sum_{\lambda \vdash n-1} c_\lambda s_\lambda \) for some coefficients \( c_\lambda \geq 0 \). Let \( A' \) be the multiset consisting of the union of \( c_\lambda \) copies of \( A_\lambda \) for each \( \lambda \vdash n-1 \), where \( A_\lambda \subseteq S_{n-1} \) is any set satisfying \( Q(A_\lambda) = s_\lambda \). Then \( Q(A) = Q(A') \), and so \( Q(AC_n) = Q(A'C_n) \) by Lemma 4.1. We will construct appropriate sets \( A_\lambda \) and show that \( Q(AC_n) = Q(A'C_n) = s_\lambda s_1 \) for all \( \lambda \). It will then follow that

\[
Q(AC_n) = Q(A'C_n) = \sum_{\lambda} c_\lambda Q(A_\lambda C_n) = \sum_{\lambda} c_\lambda Q(A_\lambda s_1) = Q(A)s_1,
\]

proving Equation (4).

Fix \( \lambda \vdash n-1 \). Given \( Q \in SYT(\lambda) \), let \( \sigma^{-1} \) be its reading word, as described in Remark 3.8. It is easy to verify that \( Q \) is the insertion tableau of \( \sigma^{-1} \) under RSK, thus the recording tableau of \( \sigma \), hence \( \text{Des}(\sigma) = \text{Des}(Q) \) by Lemma 3.1.

Let us call \( \sigma \) the inverse reading word of \( Q \). Let \( A_\lambda \) be the set of all the permutations obtained as inverse reading words of tableaux in \( SYT(\lambda) \). The map \( Q \mapsto \sigma \) that sends each tableau to its inverse reading word is thus a \( \text{Des} \)-preserving bijection from \( SYT(\lambda) \) to \( A_\lambda \). It follows that

\[
Q(A_\lambda) = \sum_{\sigma \in A_\lambda} F_{n, \text{Des}(\sigma)} = \sum_{Q \in SYT(\lambda)} F_{n, \text{Des}(Q)} = s_\lambda,
\]

using Proposition 3.2.

Let \( \lambda^\square \) be the skew shape obtained from the Young diagram of shape \( \lambda \) by placing a disconnected box at its upper right corner. For example, the tableaux in Example 3.10 have shape \( \lambda^\square \) using Proposition 3.2.

Clearly, for every \( T \in SYT(\lambda^\square) \) with \( \delta(T) = n \), \( c_{\text{Des}_\text{rot}}(T) = \text{Des}(T) \cup \{n\} = \text{Des}(\pi) \cup \{n\} = c_{\text{Des}}(\pi) \). Observations 3.9 and 3.6 imply now that, for every \( 0 \leq k < n \),

\[
c_{\text{Des}_\text{rot}}(k + T) = k + c_{\text{Des}_\text{rot}}(T) = k + c_{\text{Des}}(\pi) = c_{\text{Des}}(\pi c^{-k}).
\]
In particular
\[ \text{Des}(k + T) = \text{Des}(\pi c^{-k}), \]  
and so \( \varphi \) is a Des-preserving bijection.

In the rest of this section we will describe another Des-preserving bijection
\[
jdt : \{ k + T : T \in \text{SYT}(\lambda^\circ), \delta(T) = n \} \rightarrow \{ P \in \text{SYT}(\lambda^\circ) : \delta(P) = k \}.
\]  
(5)

Considering the composition
\[ \{ \pi c^{-k} : \pi \in \Lambda_\lambda \} \xrightarrow{\varphi} \{ k + T : T \in \text{SYT}(\lambda^\circ), \delta(T) = n \} \xrightarrow{jdt} \{ P \in \text{SYT}(\lambda^\circ) : \delta(P) = k \}\]

and taking the union over \( k \), we will obtain a Des-preserving bijection between \( \Lambda_\lambda C_n \) and \( \text{SYT}(\lambda^\circ) \), from where it will follow that
\[
Q(\Lambda_\lambda C_n) = \sum_{k=0}^{n-1} \sum_{\pi \in \Lambda_\lambda} F_{n,\text{Des}(\pi c^{-k})} = \sum_{k=1}^{n} \sum_{\pi \in \text{SYT}(\lambda^\circ)} F_{n,\text{Des}(P)} = \sum_{P \in \text{SYT}(\lambda^\circ)} F_{n,\text{Des}(P)} = s_{\lambda^\circ} = s_{\lambda s_1},
\]
using Proposition 3.2 and the fact that the Schur function indexed by the skew shape obtained by placing a partition \( \nu \) above and to the right of \( \lambda \) is equal to \( s_{\lambda^\circ} s_{\nu} \).

Fix \( 0 \leq k < n \). Let \( T \in \text{SYT}(\lambda^\circ) \) with \( \delta(T) = n \). Define \( jdt(k + T) \) to be the result of straightening the tableau \( k + T \) by applying the following procedure, based on \textit{jeu-de-taquin}. Initialize by setting \( T_0 = k + T \), and repeat the following step —which we call \textit{elementary step}— until \( T_0 \) is a standard tableau (that is, its rows and columns are increasing):

1. Let \( i \) be the minimal entry in \( T_0 \) for which the entry immediately above or to the left of it is larger than \( i \) (such an entry exists because otherwise \( T_0 \) would be standard). Switch \( i \) with the larger of these two entries, and let \( T_0 \) be the resulting tableau.

Let \( jdt(k + T) \) be the resulting standard tableau.

**Example 4.3.** For \( k = 3 \), letting \( T \) be the tableau \( T \) in Example 3.10, we compute \( jdt(3 + T) \) as follows:

\[
3 + T = \begin{array}{ccc}
4 & 6 & 2 \\
5 & 1 &\end{array} \xrightarrow{\varepsilon} \begin{array}{ccc}
4 & 1 & 2 \\
5 & 6 &\end{array} \xrightarrow{\varepsilon} \begin{array}{ccc}
1 & 4 & 2 \\
5 & 6 &\end{array} \xrightarrow{\varepsilon} \begin{array}{ccc}
1 & 2 & 4 \\
5 & 6 &\end{array} = jdt(3 + T).
\]

In Lemmas 4.5 and 4.7 we will show that \( jdt \) is a Des-preserving bijection as described in Equation 5. First, we introduce some terminology that will be used in the proofs.

An entry in a tableau that is smaller than an entry immediately above or to the left of it will be called \textit{short}. Similarly, an entry that is larger than an entry below or to the right of it will be called \textit{tall}.

In any tableau obtained along the process that takes \( k + T \) to \( P := jdt(k + T) \), the entries \( 1, 2, \ldots, k - 1 \) will be called the \textit{moving} entries, while the entries \( k + 1, k + 2, \ldots, n \) will be called \textit{non-moving}. The moving entries in \( k + T \) are the only ones that may be short, since, relative to the corresponding entries in the standard tableau \( T \), they have decreased by \( n - k \), while the remaining entries have increased by \( k \).

If a tableau has the property that it contains no two moving entries where the smaller one is weakly south and east of the larger one, we say that this tableau restricted to the moving entries is \textit{standard}. Similarly for non-moving entries.
Lemma 4.4. Let $Q$ be a tableau obtained in an intermediate step of the jdt process applied to $k + T$. Then $Q$ restricted to the moving entries $1, 2, \ldots, k - 1$ is standard, and so is $Q$ restricted to the non-moving entries $k + 1, k + 2, \ldots, n$.

Additionally, every elementary step performed by jdt when applied to $k + T$ switches one moving entry and one non-moving entry.

Proof. The first part of the lemma is clear when $Q = k + T$, since $T$ is standard and the relative order of the moving entries (resp. the non-moving entries) does not change when adding $k$ modulo $n$.

Now let $Q$ be an intermediate tableau in the jdt process applied to $k + T$, and suppose that the restrictions of $Q$ to the moving entries and to the non-moving entries are standard. We will show that these properties are preserved when applying an elementary step to $Q$, and that this step switches a moving entry with a non-moving entry.

Let $i$ be the smallest short entry of $Q$, and note that by the assumptions on $Q$ it has to be a moving entry. Then jdt switches $i$ with some larger entry $j$ above or to the left of $i$. Since $Q$ restricted to moving entries is standard, the location of $j$ implies that $j$ is a non-moving entry.

By definition of the jdt map, the entries $1, 2, \ldots, i - 1$ in $Q$ form a left-justified standard Young tableau (otherwise $Q$ would have had a short entry smaller than $i$). Thus, after switching $i$ and $j$ in $Q'$, the resulting tableau $Q'$ still has the property of being standard when restricted to the moving entries.

To see that $Q'$ restricted to the non-moving entries is standard as well, note that the effect of an elementary step on the restriction of $Q$ to the non-moving entries is equivalent to a classical jeu-de-taquin slide in the south-east direction (see [18, Appendix A1.2]), which preserves the standard property.

Lemma 4.5. The map jdt is a bijection between $\{ k + T : T \in \text{SYT}(\lambda^{\square}), \delta(T) = n \}$ and $\{ P \in \text{SYT}(\lambda^{\square}) : \delta(P) = k \}$.

Proof. It is clear that both sets have the same cardinality $|\text{SYT}(\lambda)|$, and that the image of any rotated tableau in the in the first set is in the second set.

To prove that the map is a bijection, we will describe the inverse map. Given $P \in \text{SYT}(\lambda^{\square})$ with $\delta(P) = k$, consider the tableau $-k + P$, and define $ijdt(-k + P)$ to be the tableau obtained by applying the following straightening procedure. Initialize by setting $P_0 = -k + P$, and repeat the following elementary step until $P_0$ is a standard tableau:

(ε') Let $i$ be the maximal entry in $P_0$ for which the entry immediately below or to the right of it is smaller than $i$. Switch $i$ with the smaller of these two entries, and let $P_0$ be the resulting tableau.

Let $ijdt(-k + P)$ be the resulting standard tableau.

Let $T \in \text{SYT}(\lambda^{\square})$ with $\delta(T) = n$, and let $P = jdt(k + T)$. We will show that

$$ijdt(-k + P) = T,$$

that is, the inverse map of jdt is given by $jdt^{-1}(P) = k + ijdt(-k + P)$. This fact is represented in the following diagram.

$$
\begin{array}{c}
\begin{array}{c}
\xrightarrow{jdt} P \\
\xleftarrow{\downarrow +k} \\
\xleftarrow{\downarrow \downarrow -k}
\end{array}
\end{array}
\begin{array}{c}
\xrightarrow{ijdt} -k + P \\
\xleftarrow{\downarrow T}
\end{array}
$$
In any intermediate tableau obtained along the ijdt process applied to \(-k + P\), and also in \(T\), we define the *moving* entries to be \(n + 1 - k, n + 2 - k, \ldots, n - 1\). Note that the moving entries in \(T\) (resp. \(P\)) become the moving entries in \(k + T\) (resp. \(-k + P\)) when adding (resp. subtracting) \(k\) modulo \(n\).

When applying jdt to \(k + T\), the process starts by moving entry 1 (if it is short) in the north and/or west direction (by switching it with non-moving entries) until it is no longer short. Then the algorithm does the same to entry 2, and so on, until finally it moves entry \(k - 1\) until it is no longer short.

In \(-k + P\), the moving entries are the only ones that may be tall. When applying ijdt to \(-k + P\), the process starts by moving entry \(n - 1\) in the south and/or east direction until it is no longer tall, then it moves entry \(n - 2\) similarly, and so on, until it finally moves entry \(n - k + 1\).

We will show that each elementary step in the jdt process that takes \(k + T\) to \(P\) is reversed by an elementary step of ijdt. This is illustrated the following diagram, where we use \(\varepsilon_j\) (resp. \(\varepsilon'_j\)) to denote an elementary step of jdt (resp. ijdt) that moves the entry \(j\).

\[
\begin{array}{cccccccc}
  k + T & \xrightarrow{\varepsilon_1} & \cdots & \xrightarrow{\varepsilon_1} & \cdots & \xrightarrow{\varepsilon_2} & \cdots & \xrightarrow{\varepsilon_{k-1}} & \cdots & \xrightarrow{\varepsilon_{k-1}} & P = \text{jdt}(k + T) \\
  +k & \downarrow & +k & \downarrow & +k & \downarrow & +k & \downarrow & +k & \downarrow & -k \\
  T = \text{ijdt}(-k + P) & \xleftarrow{\varepsilon'_{n-k+1}} & \cdots & \xleftarrow{\varepsilon'_{n-k+1}} & \cdots & \xleftarrow{\varepsilon'_{n-k+2}} & \cdots & \xleftarrow{\varepsilon_{n-2}} & \cdots & \xleftarrow{\varepsilon'_{n-1}} & -k + P \\
\end{array}
\]

Consider one elementary step \(\varepsilon\) in this jdt process. Let \(Q\) be the tableau at that moment, and let \(k - i\) be the moving entry about to be switched at that step (that is, \(k - i\) is the minimal short entry of \(Q\)). Let \(a\) and \(\ell\) be the entries above and to the left of \(k - i\) in \(Q\), respectively, if they exist (otherwise, define them to be 0), and suppose that \(\ell > a\) (the case \(\ell < a\) is symmetric). Since \(k - i\) is short, we have \(\ell > k - i\), and the current elementary step \(\varepsilon\) switches \(\ell\) with \(k - i\).

Let \(Q' = \varepsilon(Q)\) be the resulting tableau. It suffices to show that one elementary step \(\varepsilon'\) of ijdt applied to \(-k + Q') reverses this switch, that is \(\varepsilon'(-k + Q') = -k + Q\). The entries \(k - i\) and \(\ell\) in \(Q\) become \(n - i\) and \(\ell - k\) in \(-k + Q'\). Since \(\ell\) was not a moving entry in \(Q\) (by Lemma 4.4, \(\ell - k\) is not a moving entry in \(-k + Q\), but \(n - i\) is, so we have \(n - i > \ell - k\), which means that \(n - i\) is a tall entry in \(-k + Q'\). We claim that \(n - i\) is the maximal tall entry in \(-k + Q'\). This is because any entry \(n - j > n - i\) corresponds to a moving entry \(k - j > k - i\) in \(Q'\) and in \(k + T\). Because moving entries are treated by jdt in increasing order, neither \(k - j\) nor the entries weakly south and east of it in \(k + T\) have been moved by jdt so far. Thus, the entry \(n - j\) in \(-k + Q'\) and the entries weakly south and east of it are the same as in \(T\), which is standard. In particular, \(n - j\) is not tall in \(-k + Q'\).

We have shown that \(n - i\) is the maximal tall entry in \(-k + Q'\).

If there is no entry below \(\ell\) in \(Q\), then there is no entry below \(n - i\) in \(-k + Q'\), so \(\varepsilon'\) switches \(n - i\) and \(\ell - k\) as desired. Otherwise, let \(s\) be the entry below \(\ell\) in \(Q\). If \(s\) is a moving entry we are done, because moving entries are never switched with each other, by Lemma 4.4 adapted to ijdt. If \(s\) is not a moving entry, then the fact that \(Q\) restricted to non-moving entries is standard implies that \(s > \ell\). It follows that the corresponding entries in \(-k + Q'\) satisfy \(s - k > \ell - k\), which implies that \(\varepsilon'\) switches \(n - i\) with \(\ell - k\) in this case as well. \(\Box\)

**Example 4.6.** For \(T\) as in Examples 3.10 and 4.3 and \(P = \text{jdt}(3 + T)\), applying ijdt to \(-3 + P\) we...
Lemma 4.7. For every row \( i \) we have

\[
3 + T = \begin{bmatrix} 3 & 1 & 2 & 6 \end{bmatrix} \uparrow \rightarrow \begin{bmatrix} 3 & 1 & 2 & 6 \end{bmatrix} \uparrow \rightarrow \begin{bmatrix} 1 & 4 & 2 & 3 \end{bmatrix} \uparrow \rightarrow \begin{bmatrix} 1 & 2 & 4 & 3 \end{bmatrix} = \text{jdt}(3 + T) = P
\]

\[
T = \text{ijdt}(-3 + P) = \begin{bmatrix} 6 & 1 & 3 & 5 \end{bmatrix} \downarrow \rightarrow \begin{bmatrix} 6 & 1 & 3 & 5 \end{bmatrix} \downarrow \rightarrow \begin{bmatrix} 4 & 1 & 5 & 6 \end{bmatrix} \downarrow \rightarrow \begin{bmatrix} 4 & 5 & 1 & 6 \end{bmatrix} = -3 + P
\]

To complete the proof of Theorem 2.11 it remains to show that \( \text{jdt} \) preserves the descent set. Recall from Definition 3.7 that, for \( 1 \leq i \leq n - 1 \), we have \( i \in \text{Des}(k + T) \) if and only if \( i + 1 \) is in a lower row than \( i \) in \( k + T \).

**Lemma 4.7.** For every \( 0 \leq k < n \) and every \( T \in \text{SYT}(\lambda^c) \) with \( \delta(T) = n \), we have

\[
\text{Des}(\text{jdt}(k + T)) = \text{Des}(k + T).
\]

**Example 4.8.** For the tableau \( T \) in Example 3.10, we have

\[
2 + T = \begin{bmatrix} 3 & 5 & 1 \end{bmatrix} \rightarrow \text{jdt}(2 + T) = \begin{bmatrix} 1 & 3 & 5 \end{bmatrix}, \quad 2 + \text{cDes}_\text{rot}(T) = \{3, 5, 2\} = \text{Des}(\text{jdt}(2 + T));
\]

\[
3 + T = \begin{bmatrix} 4 & 6 & 2 \end{bmatrix} \rightarrow \text{jdt}(3 + T) = \begin{bmatrix} 1 & 2 & 4 \end{bmatrix}, \quad 3 + \text{cDes}_\text{rot}(T) = \{4, 6, 3\} = \text{Des}(\text{jdt}(3 + T)) \cup \{6\};
\]

\[
4 + T = \begin{bmatrix} 5 & 1 & 3 \end{bmatrix} \rightarrow \text{jdt}(4 + T) = \begin{bmatrix} 1 & 3 & 5 \end{bmatrix}, \quad 4 + \text{cDes}_\text{rot}(T) = \{5, 1, 4\} = \text{Des}(\text{jdt}(4 + T)).
\]

**Proof of Lemma 4.7.** We will show that for every \( i \in [n - 1] \), \( i \in \text{Des}(k + T) \) if and only if \( i \in \text{Des}(\text{jdt}(k + T)) \). The case \( k = 0 \) is trivial, so we assume that \( k \neq 0 \). Since \( \delta(k + T) = k \), it is clear that \( k - 1 \notin \text{Des}(k + T) \) but \( k \in \text{Des}(k + T) \), and the same holds for the tableau \( \text{jdt}(k + T) \). Thus, it suffices to consider the cases \( i < k - 1 \) and \( i > k \). Recall that the entries smaller than \( k \) are called moving entries in \( k + T \).

**Case** \( i < k - 1 \). The entries \( i \) and \( i + 1 \) in \( k + T \) correspond to entries \( n - k + i \) and \( n - k + i + 1 \) in \( T \), respectively, and so we either have that \( i + 1 \) is strictly south and weakly west of \( i \) (in which case \( i \in \text{Des}(k + T) \), or \( i + 1 \) is weakly north and strictly east of \( i \) (in which case \( i \notin \text{Des}(k + T) \)). These possibilities are shown in Figure 2. Similarly, since \( \text{jdt}(k + T) \) is standard, the relative position of \( i \) and \( i + 1 \) is also given by one of the above two possibilities.

Suppose for contradiction that \( i \notin \text{Des}(k + T) \) but \( i \in \text{Des}(\text{jdt}(k + T)) \) (the case \( i \in \text{Des}(k + T) \) but \( i \notin \text{Des}(\text{jdt}(k + T)) \) is symmetric with respect to the diagonal of \( T \)). In other words, \( i + 1 \) is weakly north and strictly east of \( i \) in \( k + T \), but strictly south and weakly west of \( i \) in \( \text{jdt}(k + t) \). Consider the paths (as sequences of cells, each of them north or west from the previous one) that the entries \( i \) and \( i + 1 \) follow when \( \text{jdt} \) is applied to \( k + T \), and recall that first \( \text{jdt} \) moves \( i \) along its path, and afterwards it moves \( i + 1 \). The relative location of \( i \) and \( i + 1 \) before and after applying \( \text{jdt} \) forces these two paths to intersect, and so there must be a cell \( C \) such that the path of \( i \) leaves \( C \) by going north, while the path of \( i + 1 \) enters \( C \) by going west. Let \( a \) be the entry in \( C \) right after
Figure 2: The possible locations of $i+1$ relative to $i$ in $k+T$ and $\text{jdt}(k+T)$

$i$ leaves this cell, and let $b$ be the entry immediately northeast of $C$ at that time. Let us first argue that $a, b$ are both non-moving entries. This is clear for $a$ because it is switched with $i$, and the only switches involve a moving entry and a non-moving one by Lemma 4.4. If $b$ was moving, then we would have $i < b$, since the current tableau restricted to moving entries is standard by Lemma 4.4 but then the fact that $i + 1$ must be strictly south and weakly east of $b$ (implied by the fact that $i + 1$ enters $C$ later in the jdt process) would contradict Lemma 4.4.

Before $a$ and $i$ were switched, $a$ was immediately west of $b$, and so $a < b$, since the tableau restricted to the non-moving entries is standard by Lemma 4.4. But this contradicts that $i + 1$ arrives at cell $C$ by moving west, since at that point the entry north of $i + 1$ is $b$ and the entry west of $i + 1$ is $a$.

**Case $i > k$.** When we apply jdt to $k+T$, the effect on the subtableau consisting of the non-moving entries is the same as that of applying classical *jeu-de-taquin* slides in the south-east direction. For example, when jdt moves 1 until this entry is no longer short, the algorithm restricted to entries larger than $k$ corresponds to a *jeu-de-taquin* slide into the position that 1 occupied originally. It is known that classical *jeu-de-taquin* preserves the descent set (see [8, Lemma 3.2]), and so the descents between entries larger than $k$ are preserved by jdt.

### 5 Cyclic descents of SYT

Rhoades introduced a notion of cyclic descents on SYT of rectangular shapes, having the property that the $\mathbb{Z}_n$-action on SYT of fixed rectangular shape by promotion rotates their cyclic descent sets [14]. In our notation, the promotion operation can be described as

$$T \mapsto \text{jdt}(1+T).$$

It was noticed by Rhoades that the promotion operator does not determine a $\mathbb{Z}_n$-action on the set of SYT of a general shape. In this section we extend the concept of cyclic descents to SYT of shape $\lambda^\square$, for any partition $\lambda \vdash n - 1$, and describe a $\mathbb{Z}_n$-action on these tableaux that rotates their cyclic descent sets. This extension is a consequence of the proof presented in Section 4.

Recall from diagram (6) that, for every $P \in \text{SYT}(\lambda^\square)$,

$$\text{jdt}^{-1}(P) = \delta(P) + \text{i} \text{jdt}(\delta(P) + P),$$

where $\delta(P)$ is the entry in the upper right box of $P$.

**Definition 5.1.** For $P \in \text{SYT}(\lambda^\square)$, define its *cyclic descent set* by

$$\text{cDes}(P) := \text{cDes}_{\text{rot}}(\text{jdt}^{-1}(P)).$$
Example 5.2. As in Example 4.6 for
\[ P = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 5 & 6 \end{bmatrix}, \]
we have
\[ jdt^{-1}(P) = 3 + ijdt(-3 + P) = \begin{bmatrix} 4 & 6 & 2 \\ 5 & 1 & 3 \end{bmatrix}, \]
and so \( c\text{Des}(P) = \{3, 4, 6\} \).

Proposition 5.3. 1. For every \( P \in \text{SYT}(\lambda) \),
\[ c\text{Des}(P) \cap [n - 1] = \text{Des}(P). \]

2. The map
\[ P \mapsto jdt \left(1 + jdt^{-1}(P)\right) \]
determines a \( \mathbb{Z}_n \)-action on \( \text{SYT}(\lambda) \), which rotates the cyclic descent sets of the tableaux.

Proof. 1. Using Lemma 4.7 and Definitions 3.7 and 5.1
\[ \text{Des}(P) = \text{Des}(jdt^{-1}(P)) = c\text{Des}_{\text{rot}}(jdt^{-1}(P)) \cap [n - 1] = c\text{Des}(P) \cap [n - 1]. \]

2. By Observation 3.9 and Definition 5.1
\[ c\text{Des}(jdt(1 + jdt^{-1}(P))) = c\text{Des}_{\text{rot}}(1 + jdt^{-1}(P)) = 1 + c\text{Des}_{\text{rot}}(jdt^{-1}(P)) = 1 + c\text{Des}(P). \]

The above proposition raises the natural problem of finding a unified approach to other shapes.

The concept of a cyclic descent satisfying the properties in Proposition 5.3 can be generalized as follows.

Definition 5.4. Let \( B \) be a set of combinatorial objects carrying a descent set map \( \text{Des} : B \to 2^{[n-1]} \).
A cyclic descent extension for \( B \) is a pair \( (\psi, c\text{Des}) \) where \( \psi \) is a \( \mathbb{Z}_n \)-action on \( B \) and \( c\text{Des} \) is a map from \( B \) to \( 2^{[n]} \) such that, for all \( T \in B \),
\begin{enumerate}
  \item \( c\text{Des}(T) \cap [n - 1] = \text{Des}(T) \),
  \item \( c\text{Des}(\psi(k)T) = k + c\text{Des}(T) \) for all \( k \in \mathbb{Z}_n \).
\end{enumerate}

Two examples of cyclic descent extensions that were known before this paper are the following:

- Take \( B = S_n \), let \( c\text{Des} \) be as in Definition 3.5 and let \( \psi \) be right multiplication by \( c^{-1} \) (that is, horizontal rotation). See Observation 3.6.

- Take \( B = \text{SYT}(r^{n/r}) \), the set of SYT of given rectangular shape, let \( c\text{Des} \) be as defined by Rhoades [14], and let \( \psi \) be the promotion operation on SYT.

In this paper we have introduced two new examples of cyclic descent extensions:
• Let $B$ be the set of rotated SYT of a given shape, let $c\text{Des} = c\text{Des}_{\text{rot}}$ be as in Definition 3.7, and let $\psi$ be addition modulo $n$. See Observation 3.9.

• Take $B = \text{SYT}(\lambda^\square)$, let $c\text{Des}$ be as in Definition 5.1, and let $\psi$ be the map in Part 2 of Proposition 5.3.

**Problem 5.5.** For which skew shapes $\lambda/\mu$ does there exist a cyclic descent extension for $B = \text{SYT}(\lambda/\mu)$?

This problem, which was posed in an early version of the paper, is currently being addressed in [3, 4].

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