THE “FUNDAMENTAL THEOREM” FOR THE
ALGEBRAIC K-TH EORY OF STRONGLY Z-GRAD ED RINGS

THOMAS HÜTTEMANN

Abstract. The “fundamental theorem” for algebraic K-theory expresses the
K-groups of a Laurent polynomial ring \(L[t, t^{-1}]\) as a direct sum of two copies
of the K-groups of \(L\) (with a degree shift in one copy), and certain “nil” groups
of \(L\). It is shown here that a modified version of this result generalises to
strongly Z-graded rings; rather than the algebraic K-groups of \(L\), the splitting
involves groups related to the shift actions on the category of \(L\)-modules com-
ing from the graded structure. (These actions are trivial in the classical case).
The nil groups are identified with the reduced K-theory of homotopy nilpotent
twisted endomorphisms, and analogues of Mayer-Vietoris and localisation
sequences are established.

Contents

Introduction 2

· The fundamental theorem for the algebraic K-theory of rings 2
· Relation with other work 3
· Structure of the paper 3
· Acknowledgements 3

1. Notation and main results 3

· Notation and conventions 3
· Strongly Z-graded rings 4
· The groups \(R\text{NK}^+(R_0)\) 4
· Shifts and shift differences 5
· The first negative K-group 5
· The fundamental theorem 5
· The Mayer-Vietoris sequence 6
· Homotopy nilpotent twisted endomorphisms and the localisation
sequence 6
· Noetherian regular rings 7

2. Induced modules and chain complexes 7

· The Grothendieck group of a ring 7
· Induced and stably induced modules 7
· Stabilisation of chain complexes 8
· Induced and stably induced chain complexes. The chain complex
lifting problem 8
· The chain complex lifting problem for acyclic complexes 9
· The chain complex lifting problem for well-behaved ring
homomorphisms 10

3. Finite domination 11

4. The projective line associated with a strongly Z-graded ring 12

5. The algebraic K-theory of the projective line 13

Date: April 1, 2020.

2010 Mathematics Subject Classification. Primary 19D50; Secondary 19D35 16E20 18G35.

1
The fundamental theorem for the algebraic K-theory of rings. The “fundamental theorem”, also known as the Bass-Heller-Swan formula, expresses the algebraic K-groups of a Laurent polynomial ring $R_0[t, t^{-1}]$ as a direct sum

$$K_q(R_0[t, t^{-1}]) \cong K_q R_0 \oplus K_{q-1}(R_0) \oplus RNK^+_q(R_0) \oplus RNK^-_q(R_0).$$

It will be shown that this result largely depends on the structure of $R_0[t, t^{-1}]$ as a $\mathbb{Z}$-graded ring, and that a similar splitting can be established in much greater generality.

By way of analogy we think of any $\mathbb{Z}$-graded ring $R = \bigoplus_{k \in \mathbb{Z}} R_k$ as a substitute for a Laurent polynomial ring, and consider the subrings $R_{\leq 0} = \bigoplus_{k \leq 0} R_k$ and $R_{\geq 0} = \bigoplus_{k \geq 0} R_k$ as substitutes for the polynomial rings $R_0[t^{-1}]$ and $R_0[t]$. If the ring $R$ is strongly graded in the sense that $R_k R_{\ell} = R_{k+\ell}$ for all integers $k$ and $\ell$, the analogy is appropriate, and many results known for (Laurent) polynomial rings can be proved to hold for the more general setting, e.g., the characterisation of finite domination via Novikov homology [HS17], the splitting for the algebraic K-theory of the projective line [HM18], and the connection between finite domination and non-commutative localisation [Hüt18].

This is not idle play: the class of strongly $\mathbb{Z}$-graded rings is much bigger than the class of Laurent polynomial rings. One specific example of a strongly $\mathbb{Z}$-graded rings is $K[A, B, C, D]/(AB + CD = 1)$ where $K$ is a field, $\deg(A) = \deg(C) = 1$ and $\deg(B) = \deg(D) = -1$. (The only units in this ring are the non-zero elements of $K$ in degree 0.) A natural infinite family of examples is formed by the Leavitt path algebras associated with row-finite directed graphs without sink satisfying a certain condition $Y$ [NÖ20, Theorem 1.3]; this includes all Leavitt path algebras associated with finite directed graphs without sink.
Back to the fundamental theorem, the graded structure of \( R = \bigoplus_{k \in \mathbb{Z}} R_k \) induces “shift functors” \( s_k : M \mapsto M \otimes_{R_0} R_k \) on categories of \( R_0 \)-modules; in case of a strongly graded ring, these functors preserve projectivity and satisfy the relation \( s_k \circ s_\ell \equiv s_{k+\ell} \). The key point is to measure how non-trivial the resulting \( \mathbb{Z} \)-action on algebraic \( K \)-groups is, which is done by considering the kernel \( A_q \) and cokernel \( B_q \) of the shift difference map

\[
\text{sd}_* = \text{id} - s_{-1} : K_q R_0 \longrightarrow K_q R_0.
\]

It turns out that the groups \( A_{q-1} \) and \( B_q \) play the role of \( K_{q-1} R_0 \) and \( K_q R_0 \) in the classical formulation (0.1) of the fundamental theorem. More precisely, \( K_q R \) will be shown to be an extension of \( A_{q-1} \) by a direct sum of \( B_q \) with two appropriately defined nil terms (Theorem 1.7); the nil terms are identified as the reduced algebraic \( K \)-theory of categories of homotopy nilpotent endomorphisms (Theorem 11.1).

**Relation with other work.** The “classical” fundamental theorem for the higher algebraic \( K \)-theory of rings has been proved by QUILLEN and GRAYSON [Gra76]. It has been extended from the \( K \)-theory of rings to the \( K \)-theory of schemes by THOMASON and TROBAUGH [TT90, Theorem 6.6], and to the algebraic \( K \)-theory of spaces [HKV+01] by KLEIN, VOGELL, WALDHAUSEN, WILLIAMS and the author. A version for skew LAURENT polynomial rings has been discussed by YAO [Yao95]. More recently, the result has been established by LÜCK and STEIMLE for skew LAURENT extensions of additive categories [LS16], and by FONTES and OGLE [FO18] in the context of \( S \)-algebras. Most of the recent accounts follow the pattern laid out in [HKV+01], which is in turn loosely based on [Gra76].

The present paper goes beyond previous generalisations inasmuch as it moves the focus away from the very special case of (skew) LAURENT polynomial rings to the essential information contained in the graded structure of the ring extension \( R_0 \subseteq R \).

**Structure of the paper.** The paper is structured in a way that avoids forward references in proofs. §1 provides an overview of notation and main results. §2 discusses induced chain complexes. §3 introduces finite domination, an important finiteness condition for chain complexes. In §§4–5 the “projective line” and its \( K \)-theory are reviewed. §6 is devoted to an analysis of the “nil terms” in the fundamental theorem. §7 contains the proof that the “fundamental square” of the projective line (roughly speaking relating the \( K \)-groups of \( R_{\leq 0} \), \( R \) and \( R_{\geq 0} \) with those of the projective line) is homotopy cartesian, which leads to a proof of the MAYER-VIETORIS sequence in §§ and a proof of the fundamental theorem in §9. In §10 we establish a “localisation sequence” for algebraic \( K \)-theory, and finish the paper by identifying the nil groups as the reduced \( K \)-theory of categories of homotopy nilpotent twisted endomorphisms in §11.

**Acknowledgements.** The basic ideas for this paper were developed during a research visit of the author to Beijing Institute of Technology. Their hospitality and financial support is greatly appreciated.

1. NOTATION AND MAIN RESULTS

**Notation and conventions.** The word “ring” will always refer to an associative unital ring, homomorphisms of rings respect the unit, and “modules” are understood to be unital and right, unless otherwise specified. Let \( R = \bigoplus_{k \in \mathbb{Z}} R_k \) be a \( \mathbb{Z} \)-graded unital ring, so that \( R_k R_\ell \subseteq R_{k+\ell} \) for all \( k, \ell \in \mathbb{Z} \). (Here \( R_k R_\ell \) is the set of finite sums of products \( xy \) with \( x \in R_k \) and \( y \in R_\ell \).) The component \( R_0 \) is a subring
of $R$ with the same unit element \cite[Proposition 1.4]{Dad80}. Two further subrings of note are

$$R_{\leq 0} = \bigoplus_{k \leq 0} R_k \quad \text{and} \quad R_{\geq 0} = \bigoplus_{k \geq 0} R_k .$$

There are ring inclusions

$$R_{\leq 0} \xrightarrow{i^-} R_0 \xleftarrow{i^+} R_{\geq 0} \quad \text{and} \quad R_{\leq 0} \xleftarrow{j^-} R \xrightarrow{j^+} R_{\geq 0} ,$$

and ring homomorphisms given by projection

$$R_{\leq 0} \xrightarrow{p^-} R_0 \xrightarrow{p^+} R_{\geq 0} ;$$

they satisfy the relations

$$j^- \circ i^- = j^+ \circ i^+ , \quad p^- \circ i^- = \text{id}_{R_0} \quad \text{and} \quad p^+ \circ i^+ = \text{id}_{R_0} .$$

These various maps are used to define induction functors for modules,

$$i^-_* : P \mapsto P \otimes_{R_0} R_{\leq 0}$$

and its relatives $i^+_* , j^-_*$ and $p^-_*$; the resulting maps on algebraic $K$-groups are denoted by the same symbols as the functors.

**Strongly $\mathbb{Z}$-graded rings.** The $\mathbb{Z}$-graded ring $R$ is called *strongly graded* if $R_k R_\ell = R_{k+\ell}$ for all $k, \ell \in \mathbb{Z}$, or equivalently, if $R_0 R_{-1} = R_0 = R_{-1} R_1$. This ensures that the ring multiplication yields $R_0$-bimodule isomorphisms $R_k \otimes_{R_0} R_{-k} \cong R_0$ and, more generally, $R_k \otimes_{R_0} R_{-\ell} \cong R_{k+\ell}$; consequently, each $R_k$ is an invertible $R_0$-bimodule, and hence a finitely generated projective (left and right) $R_0$-module \cite[Proposition 1.6]{HS17}. Similarly, one verifies \cite[Lemma 1.9]{HS17} that

$$R_{\leq q} = \bigoplus_{k \leq q} R_k \quad \text{and} \quad R_{-p} = \bigoplus_{k \geq -p} R_k$$

are finitely generated projective (left and right) modules over $R_{\leq 0}$ and $R_{\geq 0}$, respectively, for all $p, q \in \mathbb{Z}$.

**The groups $^{R}NK_q^\pm (R_0)$.** Let $R$ be a $\mathbb{Z}$-graded ring.

**Definition 1.1.** We define the nil groups of $R_0$ relative to $R$ as

$$^{R}NK_q^-(R_0) = \text{coker} \left( i^-_* : K_q(R_0) \longrightarrow K_q(R_{\leq 0}) \right)$$

and

$$^{R}NK_q^+(R_0) = \text{coker} \left( i^+_* : K_q(R_0) \longrightarrow K_q(R_{\geq 0}) \right) .$$

Thus we have a split short exact sequence

$$0 \longrightarrow K_q(R_0) \xrightarrow{i^+_* K_q(R_{\geq 0}) \xrightarrow{p^+_*}} ^{R}NK_q^+(R_0) \longrightarrow 0$$

resulting in isomorphisms

$$K_q(R_{\geq 0}) \cong K_q(R_0) \oplus ^{R}NK_q^+(R_0) \quad \text{and} \quad ^{R}NK_q^+ \cong \ker \left( p^+_* : K_q(R_{\geq 0}) \longrightarrow K_q(R_0) \right) . \tag{1.2}$$

Of course these remarks hold for $^{R}NK_q^-(R_0)$ and $K_q(R_{\leq 0})$ *mutatis mutandis*.
Shifting and Shift Differences. The \( \mathbb{Z} \)-graded over-ring \( R \) of the ring \( R_0 \) defines endofunctors of the category of \( R_0 \)-modules, the shift functors, by
\[
s_j : P \mapsto P \otimes_{R_0} R_j \quad (j \in \mathbb{Z}).
\]
If \( R \) is strongly graded then \( R_j \) is an invertible \( R_0 \)-bimodule whence \( s_j \) defines an auto-equivalence of the category of finitely generated projective \( R_0 \)-modules, with resulting isomorphisms on algebraic \( K \)-groups
\[
s_j : K_q R_0 \longrightarrow K_q R_0.
\]

**Definition 1.3.** Suppose that \( R \) is a strongly \( \mathbb{Z} \)-graded ring. The \( q \)th shift difference map of \( R_0 \) relative to \( R \) is defined as
\[
\text{sd}_q = \text{id} - s_{-1} : K_q R_0 \longrightarrow K_q R_0.
\]
The kernel of \( \text{sd}_q \) is called the \( q \)th shift kernel of \( R_0 \) relative to \( R \), and is denoted by the symbol \( \ker R K_q R_0 \). The cokernel of this map is called the \( q \)th shift cokernel of \( R_0 \) relative to \( R \), and is denoted by the symbol \( \coker R K_q R_0 \).

It might be worth pointing out that we could just as well use the shift functor \( s_1 \) in place of \( s_{-1} \); in view of the \( R_0 \)-bimodule isomorphism \( R_{-1} \otimes_{R_0} R_1 \cong R_0 \), this would lead to the same description of the shift kernel and of an isomorphic description of the shift cokernel. In any case, the exact sequence
\[
0 \longrightarrow \ker R K_q R_0 \longrightarrow K_q R_0 \xrightarrow{\text{sd}_q} K_q R_0 \longrightarrow \coker R K_q R_0 \longrightarrow 0
\]
determines the \( q \)th shift kernel and \( q \)th shift cokernel up to canonical isomorphism.

**Remark 1.5.** If there is an \( R_0 \)-bimodule isomorphism \( R_{-1} \cong R_0 \) then \( \text{sd}_q \) is the zero map whence \( \ker R K_q R_0 = \coker R K_q R_0 = K_q R_0 \). This happens, for example, in case \( R \) is a Laurent polynomial ring \( R = R_0[t, t^{-1}] \) with a central indeterminate \( t \).

**The First Negative \( K \)-group.** Let \( R = \bigoplus_{k \in \mathbb{Z}} R_k \) be a \( \mathbb{Z} \)-graded ring.

**Definition 1.6.** We define the first negative \( K \)-group of \( R_0 \) relative to \( R \) as
\[
R K_{-1}(R_0) = \coker(j^-_q + j^+_q),
\]
where the ring inclusions \( j^- \) and \( j^+ \) induce the map
\[
j^- + j^+ : K_0(R_{\leq 0}) \oplus K_0(R_{\geq 0}) \longrightarrow K_0(R).
\]

In the case of a Laurent polynomial ring \( R_0 \subseteq R_0[t, t^{-1}] \) this recovers the Bass \( K \)-group \( K_{-1}(R_0) = R_0[t, t^{-1}] K_{-1}(R_0) \).

**The Fundamental Theorem.** The fundamental theorem expresses \( K_q(R_0)[t, t^{-1}] \) as a direct sum of groups \( K_q(R_0), K_{q-1}(R_0), R^+ K_q(R_0) \) and \( R^- K_q(R_0) \). In the more general context of \( R_0 \subseteq R \) with \( R \) strongly \( \mathbb{Z} \)-graded, the fundamental result reads as follows:

**Theorem 1.7** (The fundamental theorem for the algebraic \( K \)-theory of strongly \( \mathbb{Z} \)-graded rings). Let \( R \) be a strongly \( \mathbb{Z} \)-graded ring. There are short exact sequences of abelian groups
\[
0 \longrightarrow R^- K_q(R_0) \oplus \coker R K_q(R_0) \oplus R^+ K_q(R_0) \longrightarrow K_q(R) \longrightarrow R K_{q-1}(R_0) \longrightarrow 0 \quad (\text{for } q > 0)
\]
and

\[ 0 \longrightarrow R\text{NK}_0^-(R_0) \oplus \text{coker} R\text{K}_0(R_0) \oplus R\text{NK}_0^+(R_0) \longrightarrow R\text{K}_0(R) \longrightarrow R\text{K}_1R_0 \longrightarrow 0. \]  

(1.7b)

The Mayer-Vietoris sequence. Let \( R \) be a strongly \( \mathbb{Z} \)-graded ring. By analogy with algebraic geometry, one can consider a “projective line” which is obtained by “gluing spec \( R_{\leq 0} \) and spec \( R_{\geq 0} \) along their intersection spec \( R \); more precisely, one can define the analogue of the category of quasi-coherent sheaves on the projective line as the category of certain diagrams of modules. The projective line \( \mathbb{P}^1 \) in this sense was introduced and its \( K \)-theory computed by Montgomery and the author [HM18]; the relevant parts of the theory will be surveyed in §4 below.

**Theorem 1.8 (Mayer-Vietoris sequence).** Let \( R \) be a strongly \( \mathbb{Z} \)-graded ring. There is a long exact sequence of algebraic \( K \)-groups

\[ \ldots \longrightarrow K_{q+1}(R) \longrightarrow K_q(R_{\leq 0}) \oplus K_q(R_{\geq 0}) \longrightarrow K_q(R) \longrightarrow \]  

\[ \delta \]  

\[ \delta \]  

\[ \delta \]  

Homotopy nilpotent twisted endomorphisms and the localisation sequence. Let \( R \) be a strongly \( \mathbb{Z} \)-graded ring. We define \( R\text{Nil}^+(R_0) \) to be the category of pairs \((Z, \zeta)\), with \( Z \) an \( R_0 \)-finitely dominated bounded chain complexes of projective \( R_0 \)-modules, and \( \zeta : Z \otimes_{R_0} R_1 \longrightarrow Z \otimes_{R_0} R_0 \) a homotopy nilpotent twisted endomorphism, cf. §6. Let \( R\text{Nil}^+_q(R_0) \) denote the \( q \)th algebraic \( K \)-group of \( R\text{Nil}^+(R_0) \) (with respect to the quasi-isomorphisms as weak equivalences and the levelwise split monomorphisms as cofibrations).

**Theorem 1.9.** Let \( R \) be a strongly \( \mathbb{Z} \)-graded ring. The forgetful functor \((Z, \zeta) \mapsto Z\), defined on \( R\text{Nil}^+(R_0) \), induces a homomorphism \( R\text{Nil}^+_q(R_0) \longrightarrow K_qR_0 \) with kernel \( R\text{NK}_q(R_0) \). The groups \( R\text{Nil}^+_q(R_0) \) fit into a long exact sequences of algebraic \( K \)-groups

\[ \ldots \longrightarrow K_{q+1}R \longrightarrow R\text{Nil}^+_q(R_0) \longrightarrow K_qR \longrightarrow \]  

\[ \phi \]  

with \( \phi \) induced by the functor sending \((Z, \zeta)\) to the \( R_{\geq 0} \)-module complex \( Z \) with \( R_{\geq 0} \) acting through \( \zeta \).

There is a symmetric version involving a category \( R\text{Nil}^-(R_0) \), using \( R_{\leq 1} \) in place of \( R_1 \), and its \( K \)-groups \( R\text{Nil}^-_q(R_0) \). The complete version of the result is formulated and proved in Theorems 10.1 and 11.1 below.
Noetherian regular rings. Suppose that \( R \) is a \( \mathbb{Z} \)-graded ring (possibly not strongly graded). Suppose further that \( R \) is right noetherian and regular. For this situation, van den Bergh [VdB86] established an exact sequence
\[
\ldots \rightarrow K_{q+1}R \rightarrow K_q^\text{gr} \rightarrow K_q^\text{gr} \rightarrow K_qR \rightarrow \ldots ,
\]
where \( K_q^\text{gr} \) denotes the “graded \( K \)-theory of \( R \).” That is, the \( K \)-theory of the category of finitely generated projective \( \mathbb{Z} \)-graded \( R \)-modules; the map \( \sigma \) is the difference of the maps induced by the identity and the shift map \( M \mapsto M(1) \). In case \( R \) is strongly graded, the categories of finitely generated projective \( R_q \)-modules and of finitely generated projective graded \( R \)-modules are equivalent via the functor \( P \mapsto P \otimes R_0 \) (DADE [Dad80, Theorem 2.8]). Thus we have the long exact sequence
\[
\ldots \rightarrow K_{q+1}R \rightarrow K_qR_0 \rightarrow K_qR_0 \rightarrow K_qR \rightarrow \ldots
\]
which we can split at the shift difference map to obtain:

**Theorem 1.10.** Suppose that \( R \) is a strongly \( \mathbb{Z} \)-graded right noetherian regular ring. There are short exact sequences (for \( q > 0 \))
\[
0 \rightarrow \ker K_q(R_0) \rightarrow K_q(R) \rightarrow \ker K_{q-1}(R_0) \rightarrow 0 .
\]

Comparison with Theorem 1.7 suggests that \( R\text{NK}_q^\text{gr}(R_0) = 0 \) in this situation. We will not pursue the issue here.

## 2. Induced Modules and Chain Complexes

**The Grothendieck group of a ring.** Let \( L \) be a rings. The group \( K_0(L) \) is, by definition, the Grothendieck group of the category \( \text{P}(L) \) of finitely generated projective \( L \)-modules; we denote the element corresponding to the module \( P \) by the symbol \([P]\). The cokernel of the map \( K_0(\mathbb{Z}) \rightarrow K_0(L) \) induced by the induction functor \( M \mapsto M \otimes \mathbb{Z} L \) is the reduced Grothendieck group \( K_0(\mathbb{Z} \downarrow L) \) of \( L \), more usually denoted by the symbol \( K_0(L) \). We write the element corresponding to the module \( P \) by \( \alpha([P]) \). The following equivalences are well known:
\[
[P] = [Q] \text{ in } K_0(L) \Leftrightarrow \exists k \geq 0: P \oplus L^k \cong Q \oplus L^k \quad (2.1a)
\]
and
\[
\alpha([P]) = \alpha([Q]) \text{ in } K_0(\mathbb{Z} \downarrow L) \Leftrightarrow \exists k, \ell \geq 0: P \oplus L^k \cong Q \oplus L^\ell \quad (2.1b)
\]
In particular, \( \alpha([P]) = 0 \) if and only if \( P \) is stably free.

We will repeatedly make use of the fact that \( K_0(L) \) can be described in other ways, for example using the machinery of Waldhausen \( K \)-theory applied to the category \( \text{Ch}^\text{gr} \text{P}(L) \) of bounded complexes of finitely generated projective \( L \)-modules (with quasi-isomorphisms as weak equivalences, and levelwise split monomorphisms as cofibrations). This description is such that a chain complex \( C \) in \( \text{Ch}^\text{gr} \text{P}(L) \) gives rise to the element \([C] = \chi(C) = \sum_k (-1)^k [C_k] \) of \( K_0(L) \). If \( C \) is contractible then \([C] = 0 \).

**Induced and stably induced modules.** Let \( f: L \rightarrow S \) be a ring homomorphism, with induced map
\[
f_*: K_0(L) \rightarrow K_0(S) , \quad [P] \mapsto [P \otimes L S] .
\]
We will use the notation \( f_* \) also for the induction functor \(- \otimes_L S \) so that \( f_*([P]) = [f_*(P)] \).
Definition 2.2. Let \( Q \) be a finitely generated projective \( S \)-module.

1. We say that \( Q \) is induced from \( L \) if there exists a finitely generated projective \( L \)-module \( P \) such that \( f_*(P) \cong Q \).
2. The module \( Q \) is called stably induced from \( L \) if there exist a number \( q \geq 0 \) and a finitely generated projective \( L \)-module \( P \) such that \( f_*(P) \cong Q \oplus S^q \).

Algebraic \( K \)-theory provides an obstruction to modules being stably induced, the obstruction group being \( K_0(L \downarrow S) = \text{coker}(f_*) \). It comes equipped with a canonical map
\[
\alpha: K_0(S) \longrightarrow K_0(L \downarrow S) .
\] (2.3)

Proposition 2.4. Let \( Q \) be a finitely generated projective \( S \)-module. The following are equivalent:

1. The module \( Q \) is stably induced from \( L \).
2. The element \( \alpha([Q]) \) of \( K_0(L \downarrow S) \) is trivial.

For example, if \( Q \) is a stably free \( S \)-module then \( Q \) is certainly stably induced from \( L \) so that \( \alpha([Q]) = 0 \) in \( K_0(L \downarrow S) \).

Proof of Proposition 2.4. If \( Q \oplus S^s = P \oplus _L S \) then \([Q] = f_*([P]) - f_*([L^s]) \) lies in the image of \( f_* \), whence \( \alpha([Q]) = 0 \in K_0(L \downarrow S) \).

Conversely, suppose that \( \alpha([Q]) = 0 \). Then there exists \( a \in K_0(L) \) such that \([Q] = f_*(a) \) in \( K_0(S) \). We can find a finitely generated projective \( L \)-module \( M \) and a number \( r \geq 0 \) such that \( a = [M] - [L^r] \) in \( K_0(L) \). By applying \( f_* \) and re-arranging we find the equality
\[
[Q \oplus S^s] = [Q] + [S^s] = f_*(a) + f_*([L^r]) = f_*([M]) \in K_0(S) .
\]
This in turn implies that there exists \( k \geq 0 \) with
\[
Q \oplus S^s \oplus S^k \cong f_*(M) \oplus S^k = f_*(M \oplus L^k) .
\]
This shows \( Q \) to be stably induced. \( \square \)

Stabilisation of chain complexes. As a matter of notation, we write \( D(k, M) \) for the chain complex concentrated in degrees \( k \) and \( k - 1 \) with non-trivial entries \( M \) and differential the identity map of \( M \).

Definition 2.5. Let \( D \) be a chain complex of \( S \)-modules. We say that the chain complex \( D' \) is a stabilisation of \( D \) if \( D' = D \oplus \bigoplus_k D(k, F_k) \) for finitely many finitely generated free \( S \)-modules \( F_k \).

If \( D' \) is a stabilisation of \( D \) then there are mutually homotopy inverse chain homotopy equivalences \( s: D \longrightarrow D' \) (the inclusion) and \( r: D' \longrightarrow D \) (the projection) such that \( rs = \text{id}_D \), and such that \( \text{coker}(s) = \ker(r) = \bigoplus_k D(k, F_k) \) is a contractible bounded complex with finitely generated free chain, boundary and cycle modules.

Induced and stably induced chain complexes. The chain complex lifting problem. Let \( f: L \longrightarrow S \) be a ring homomorphism.

Definition 2.6. Let \( D \) be a bounded complex of finitely generated projective \( S \)-modules.

1. We say that \( D \) is induced from \( L \) if there exists a bounded complex of finitely generated projective \( L \)-module \( C \) such that \( f_*(C) \cong D \).
(2) The complex $D$ is called \textit{stably induced from} $L$ if there exist a bounded complex $C$ of finitely generated projective $L$-modules such that $f_*(C)$ is isomorphic to a stabilisation of $D$.

Again, let $D$ be a bounded complex of finitely generated projective $S$-modules; suppose that all chain modules $D_k$ are induced from $L$. The \textit{chain complex lifting problem} is to decide whether $D$ is homotopy equivalent to a complex induced from $L$. We will address two variations of the theme below and show that

- if $D$ is acyclic, then there exists an \textit{acyclic} bounded complex $C$ of finitely generated projective $L$-modules such that $f_*(C)$ is isomorphic to a stabilisation of $D$;
- if $f$ satisfies a certain strong flatness condition, then $D$ is stably induced from $L$.

The chain complex lifting problem for acyclic complexes.

\textbf{Theorem 2.7.} Every acyclic complex of stably induced modules is stably induced from an acyclic complex. — More precisely, let $D$ be an acyclic bounded chain complex of finitely generated projective $S$-modules concentrated in chain levels 0 to $n$ such that each chain module $D_k$ is stably induced from $L$. Then there exist a stabilisation $D'$ of $D$ and an acyclic bounded complex $C'$ of finitely generated projective $L$-modules, both concentrated in chain levels 0 to $n$, such that $f_*(C')$ is isomorphic to $D'$.

\textit{Proof.} As $D$ is acyclic there are finitely generated projective $S$-modules $E_k$, for $1 \leq k \leq n$, such that

$$D \cong \bigoplus_{k=1}^{n} D(k, E_k).$$

(2.8)

By construction we have $E_1 = D_0$ and consequently $\alpha([E_1]) = 0$, since $D_0$ is stably induced from $L$ by hypothesis. By iteration, assuming that $\alpha([E_{k-1}]) = 0$ is known, we infer from $D_{k-1} = E_{k-1} \oplus E_k$ that $\alpha([E_k]) = 0$ as well.

Thus for $1 \leq k \leq n$ we can choose numbers $j_k \geq 0$ and finitely generated projective $L$-modules $P_k$ such that

$$E_k \oplus S^{j_k} \cong f_*(P_k).$$

We define chain complexes

$$D' = \bigoplus_{k=1}^{n} D(k, E_k \oplus S^{j_k}) \quad \text{and} \quad C' = \bigoplus_{k=1}^{n} D(k, P_k);$$

thus $C'$ and $D'$ are acyclic complexes concentrated in chain levels 0 to $n$. The computation

$$D' = \bigoplus_{k=1}^{n} D(k, E_k \oplus S^{j_k}) = \bigoplus_{k=1}^{n} \left( D(k, E_k) \oplus D(k, S^{j_k}) \right) \cong D \oplus \bigoplus_{k=1}^{n} D(k, S^{j_k})$$

(2.8)

confirms $D'$ as a stabilisation of $D$. Finally, by construction we have isomorphisms of chain complexes

$$f_*(C') = \bigoplus_{k=1}^{n} D(k, f_*(P_k)) \cong \bigoplus_{k=1}^{n} D(k, E_k \oplus S^{j_k}) = D'$$

as required. \qed
The chain complex lifting problem for well-behaved ring homomorphisms.
As before, let \( f: L \longrightarrow S \) be a ring homomorphism. We consider \( S \) as an \( L\text{-}L\)-bimodule, with \( L \) acting via \( f \). To solve the chain complex lifting problem, we will assume that the following condition is satisfied:

The \( L\text{-}L\)-bimodule \( S \) is a filtered colimit of \( L\text{-}L\)-sub-bimodules \( S_j \) which are finitely generated projective right \( L\)-modules such that \( S_j \otimes_L S = S \). \((2.9)\)

In particular, \( S \) is a flat right \( L\)-module in this case.

**Example 2.10.** The ring inclusion \( f: L = R_{\leq 0} \longrightarrow R = S \) with \( R \) a strongly \( \mathbb{Z}\)-graded ring satisfies condition \((2.9)\) since \( R = \bigcup_{k \geq 0} R_{\leq k} \) with \( R_{\leq k} \) a finitely generated projective (left and right) \( R_{\leq 0}\)-module \([HM18, \text{Lemma I.2.2}]\) such that \( R_{\geq k} \otimes_{R_{\leq 0}} R = R \) \([HM18, \text{Lemma I.2.4}]\).

**Proposition 2.11.** Let \( D \) be a chain complex consisting of finitely generated projective \( S\)-modules concentrated in chain levels 0 to \( n \). Suppose that each chain module \( D_k \) is induced from \( L \) so that \( D_k \cong f_*(C'_k) \) for a finitely generated projective \( L\)-module \( C'_k \). Suppose further that the ring homomorphism \( f \) satisfies condition \((2.9)\). Then there exists a chain complex \( C \) of finitely generated projective \( L\)-modules, concentrated in chain levels 0 to \( n \), such that \( f_*(C) \cong D \).

**Proof.** We denote the differentials of \( D \) by \( d_k: D_k \longrightarrow D_{k-1} \). Set \( C_k = \{0\} \) for \( k < 0 \) and for \( k > n \), and choose \( C_n = C'_n \).

Since \( C_n \) is finitely generated, the composite map

\[
C_n \longrightarrow C_n \otimes_L S \cong D_n \xrightarrow{d_n} D_{n-1} = f_*(C'_{n-1}) \xrightarrow{(2.9)} \text{colim}_j C'_{n-1} \otimes_L S_j
\]

factors through some stage \( j_n \) of the colimit system, resulting in a map

\[
\partial_n: C_n \longrightarrow C'_{n-1} \otimes_L S_{j_n} =: C_{n-1}
\]

with target a finitely generated projective \( L\)-module, by our hypotheses on \( f \), such that \( f_*(\partial_n) \equiv d_n \).

Since \( C_{n-1} \) is finitely generated, the composite map

\[
C_{n-1} \longrightarrow C'_{n-1} \otimes_L S \cong D_{n-1} \xrightarrow{d_{n-1}} D_{n-2} = f_*(C'_{n-2}) \xrightarrow{(2.9)} \text{colim}_j C'_{n-2} \otimes_L S_j
\]

factors through some stage \( j \) of the colimit system, resulting in a map

\[
\tilde{\partial}_{n-1}: C_{n-1} \longrightarrow C'_{n-2} \otimes_L S_j.
\]

By replacing \( j \) with a larger index \( j_{n-1} \) (where “larger” refers to the filtered properties of the indexing category), we may assume that the composite with \( \partial_n \) is the zero map. In other words, we have a map

\[
\tilde{\partial}_{n-1}: C_{n-1} \longrightarrow C'_{n-2} \otimes_L S_{j_{n-1}} =: C_{n-2}
\]

with target a finitely generated projective \( L\)-module, by our hypotheses on \( f \), such that \( f_*(\tilde{\partial}_{n-1}) \equiv d_n \) and \( \tilde{\partial}_{n-1} \circ \partial_n = 0 \).

The process is repeated iteratively, until we have constructed \( C_0 \) and \( \partial_1 \). \( \square \)

**Proposition 2.12.** Let \( D \) be a chain complex consisting of finitely generated projective \( S\)-modules concentrated in chain levels 0 to \( n \), such that each chain
module $D_k$ is stably induced from $L$. Suppose that the ring homomorphism $f$ satisfied condition (2.9). Then there exist a stabilisation $D'$ of $D$ and a chain complex $C'$ of finitely generated projective $L$-modules, both concentrated in chain levels $0 \leq k \leq n + 1$, such that $f_*(C') \cong D'$.

Proof. For $0 \leq k \leq n$ choose a number $s_k \geq 0$ such that $D_k \oplus S^{s_k}$ is induced from $L$, and choose a finitely generated projective $L$-module $C_k$ with $f_*(C_k) \cong D_k \oplus S^{s_k}$. Set $D' = D \oplus \bigoplus_k D(k + 1, S^{s_k})$. By construction, $D'$ is concentrated in chain levels 0 to $n + 1$, and each chain module $D'$ is induced from $L$ (as it is the direct sum of a module induced from $L$ with a finitely generated free module). Hence Proposition 2.11 yields a bounded complex $C'$ of finitely generated projective $L$-modules, concentrated in chain levels 0 to $n + 1$, such that $f_*(C') \cong D'$.

\[ \begin{array}{c}
\Delta & \rightarrow & \cdots & \rightarrow & \Delta & \rightarrow & 0 \\
\cup & & & & & & \\
0 & \rightarrow & C & \xrightarrow{f} & D & \xrightarrow{g} & E & \rightarrow & 0 \\
\end{array} \]

3. Finite domination

Let $C$ be a (possibly unbounded) chain complex of $K$-modules, for some unital ring $K$. We say that $C$ is $K$-finitely dominated, or of type (FP) over $K$, if $K$ is homotopy equivalent to a bounded complex of finitely generated projective $K$-modules.

Finite domination satisfies the “2-of-3 property” with respect to short exact sequences of chain complexes. We include a proof for completeness.

Lemma 3.1. Let $C$, $D$ and $E$ be bounded below complexes of projective $K$-modules, and suppose that there is a short exact sequence

\[ 0 \rightarrow C \xrightarrow{f} D \xrightarrow{g} E \rightarrow 0. \]  

If any two of the complexes are $K$-finitely dominated then so is the third.

Proof. The given sequence gives rise to a short exact sequence of bounded complexes of projective $K$-modules

\[ 0 \rightarrow D \rightarrow \text{cyl}(g) \xrightarrow{h} \text{cone}(g) \rightarrow 0, \]

together with homotopy equivalences $E \rightarrow \text{cyl}(g)$ and $q: C[1] \rightarrow \text{cone}(g)$. By iteration, there is a short exact sequence

\[ 0 \rightarrow \text{cyl}(g) \rightarrow \text{cyl}(h) \rightarrow \text{cone}(h) \rightarrow 0, \]

together with homotopy equivalences $C[1] \rightarrow \text{cyl}(h)$ and $D[1] \rightarrow \text{cone}(h)$. As finite domination is invariant under suspension and homotopy equivalences, it suffices to prove that if $C$ and $D$ are $K$-finitely dominated so is $E$.

Since we are dealing with bounded below complexes of projective modules, the canonical map $\text{cone}(f) \rightarrow E$ is a homotopy equivalence. As $C$ and $D$ are $K$-finitely dominated there exist bounded complexes $C'$ and $D'$ of finitely generated projective $K$-modules and chain homotopy equivalences $\alpha: C \rightarrow C'$ and $\beta: D \rightarrow D'$. Choose a homotopy inverse $\alpha'$ of $\alpha$, and let $h: \text{id}_C \simeq \alpha' \alpha$ be a homotopy. The chain map

\[ \begin{pmatrix} \alpha \\ \beta f h \\ \beta \end{pmatrix} : \text{cone}(f) \rightarrow \text{cone}(\beta f \alpha') \]

is a quasi-isomorphism (since $\alpha$ and $\beta$ are) and hence a homotopy equivalence; its target is a bounded complex of finitely generated projective $K$-modules. Thus $E \simeq \text{cone}(f) \simeq \text{cone}(\beta f \alpha')$ shows that $E$ is $K$-finitely dominated. \qed
4. The projective line associated with a strongly $\mathbb{Z}$-graded ring

The projective line associated with a strongly $\mathbb{Z}$-graded ring has been introduced by Montgomery and the author [HM18]. We recall definitions and $K$-theoretical results which will take a central place when establishing the fundamental theorem.

From now on we will assume throughout that $R = \bigoplus_{k \in \mathbb{Z}} R_k$ is a strongly $\mathbb{Z}$-graded ring unless other hypotheses are specified.

**Definition 4.1** (Sheaves and vector bundles on the projective line [HM18, Definitions II.1.1 and II.2.1]). A quasi-coherent sheaf on $\mathbb{P}^1$, or just sheaf for short, is a diagram

$$\mathcal{Y} = \begin{pmatrix} Y^- & v^- & Y^0 & v^+ & Y^+ \end{pmatrix}$$

(4.2)

where $Y^-$, $Y^0$ and $Y^+$ are modules over $R_{\leq 0}$, $R$ and $R_{\geq 0}$, respectively, with an $R_{\leq 0}$-linear homomorphisms $v^-$ and an $R_{\geq 0}$-linear homomorphism $v^+$, such that the diagram of the adjoint $R$-linear maps

$$Y^- \otimes_{R_{\leq 0}} R \xrightarrow{v^-} Y^0 \xleftarrow{v^+} Y^+ \otimes_{R_{\geq 0}} R$$

(4.3)

consists of isomorphisms. This latter condition will be referred to as the sheaf condition. A morphism $f = (f^-, f^0, f^+): \mathcal{Y} \to \mathcal{Z}$ between sheaves is a commutative diagram of the form

$$\begin{array}{ccccccc}
Y^- & & Y^0 & & Y^+ \\
\downarrow f^- & & \downarrow f^0 & & \downarrow f^+ \\
Z^- & & Z^0 & & Z^+
\end{array}$$

with $f^-$, $f^0$ and $f^+$ homomorphisms of modules over $R_{\leq 0}$, $R$ and $R_{\geq 0}$, respectively.

We call the sheaf $\mathcal{Y}$ a vector bundle if its constituent modules are finitely generated projective modules over their respective ground rings. The category of vector bundles (and all morphisms of sheaves between them) is denoted by $\text{Vect}(\mathbb{P}^1)$.

Specific examples of vector bundles are the twisting sheaves

$$\mathcal{O}(k, \ell) = \begin{pmatrix} R_{\leq k} & \subseteq & R & \supseteq & R_{\geq -\ell} \end{pmatrix}$$

where $k$ and $\ell$ are integers. (Note that the diagrams $\mathcal{O}(k, \ell)$ may not be sheaves if the $\mathbb{Z}$-graded ring $R$ fails to be strongly graded.) — Taking tensor product with twisting sheaves defines an interesting operation on the category of sheaves on $\mathbb{P}^1$.

**Definition 4.4** (Twisting [HM18, Definition II.2.4]). Let $\mathcal{Y}$ be a sheaf, and let $k, \ell \in \mathbb{Z}$. We define the $(k, \ell)$th **twist of $\mathcal{Y}$**, denoted $\mathcal{Y}(k, \ell)$, to be the sheaf

$$\mathcal{Y}(k, \ell) = \begin{pmatrix} Y^- \otimes_{R_{\leq 0}} R_{\leq k} & \to & Y^0 \otimes_R R & \to & Y^+ \otimes_{R_{\geq 0}} R_{\geq -\ell} \end{pmatrix},$$

with structure maps induced by those of $\mathcal{Y}$ and the inclusion maps.

**Definition 4.5.** Let $k, \ell \in \mathbb{Z}$. The $(k, \ell)$th **canonical sheaf functor** $\Psi_{k,\ell}$ is defined by

$$\Psi_{k,\ell}: P(R_0) \to \text{Vect}(\mathbb{P}^1), \quad P \mapsto P \otimes \mathcal{O}(k, \ell)$$
where $P(R_0)$ is the category of finitely generated projective $R_0$-modules, and the symbol $P \otimes \mathcal{O}(k, \ell)$ denotes the sheaf

\[ P \otimes_{R_0} R_{\leq k} \rightarrow P \otimes_{R_0} R \leftarrow P \otimes_{R_0} R_{\geq -\ell}. \]

**Definition 4.6.** The sheaf cohomology modules of the sheaf $\mathcal{Y}$ of (4.2) are defined by the exact sequence

\[ 0 \rightarrow H^0 \mathcal{Y} \rightarrow Y^- \oplus Y^+ \xrightarrow{\nu^- - \nu^+} Y^0 \rightarrow H^1 \mathcal{Y} \rightarrow 0, \]

that is, $H^0 \mathcal{Y} = \ker(\nu^- - \nu^+)$ and $H^1 \mathcal{Y} = \text{coker}(\nu^- - \nu^+)$. We will also use the notation $\Gamma \mathcal{Y}$ for $H^0 \mathcal{Y}$ and speak of global sections of $\mathcal{Y}$.

The $R_0$-modules $H^0 \mathcal{Y}$ and $H^1 \mathcal{Y}$ depend functorially on $\mathcal{Y}$. Considering $\mathcal{Y}$ as a diagram of $R_0$-modules we have isomorphisms $H^q \mathcal{Y} = \lim_{\leftarrow q \mathcal{Y}}$ for $q = 0, 1$.

One can explicitly compute the sheaf cohomology of twisting sheaves by direct inspection [HM18, Proposition II.3.4]. Similarly, one can show:

**Proposition 4.7.** For $P \in P(R_0)$ and $k, \ell \in \mathbb{Z}$ there is an isomorphism

\[ \Gamma \Psi_{k,\ell} P \cong \bigoplus_{j=-\ell}^k s_j P \]

where $s_j P = P \otimes_{R_0} R_j$. The isomorphism is natural in $P$ so that there are isomorphisms of functors

\[ \Gamma \circ \Psi_{k,\ell} \cong 0 \quad \text{if } k + \ell < 0, \]

\[ \Gamma \circ \Psi_{0,0} \cong \text{id}, \]

\[ \Gamma \circ \Psi_{k,-k} \cong s_k. \]

Moreover, if $k + \ell \geq -1$ then $H^1 \circ \Psi_{k,\ell} \mathcal{Y} = 0$. \hfill \Box

5. The algebraic $K$-theory of the projective line

We let $Ch^b \text{Vect}(\mathbb{P}^1)$ denote the category of bounded chain complexes of vector bundles; similarly, we denote by $Ch^b \text{Vect}(\mathbb{P}^1)_0$ the category of bounded chain complexes in the category $\text{Vect}(\mathbb{P}^1)_0$. A map $f$ of vector bundles will be called an $h$-equivalence, or a quasi-isomorphism, if $f^*$ is a quasi-isomorphism of chain complexes of modules for each decoration $? \in \{-, 0, +\}$. As weak equivalences are defined homologically, they satisfy the saturation and extension axioms. All categories mentioned have a cylinder functor given by the usual mapping cylinder construction which satisfies the cylinder axiom.

**Definition 5.1** ([HM18, Definition III.1.1]). The $K$-theory space of the projective line is defined to be

\[ K(\mathbb{P}^1) = \Omega[h\mathcal{S} \cdot Ch^b \text{Vect}(\mathbb{P}^1)], \]

where “$h$” stands for the category of $h$-equivalences.

It is technically more convenient to use a certain subcategory of $\text{Vect}(\mathbb{P}^1)$, on which the global sections functor $\Gamma$ is exact:

**Definition 5.2.** The category $\text{Vect}(\mathbb{P}^1)_0$ is the full subcategory of $\text{Vect}(\mathbb{P}^1)$ consisting of those objects $\mathcal{Y}$ satisfying

\[ H^1 \mathcal{Y}(k, \ell) = 0 \quad \text{for all } k, \ell \in \mathbb{Z} \text{ with } k + \ell \geq 0. \]
Lemma 5.3 ([HM18, Corollary III.1.4]). The inclusion $\operatorname{Vect}(\mathbb{P}^1)_0 \subseteq \operatorname{Vect}(\mathbb{P}^1)$ induces a homotopy equivalence

$$hS_\ast \operatorname{Ch}^\ast \operatorname{Vect}(\mathbb{P}^1)_0 \xrightarrow{\cong} hS_\ast \operatorname{Ch}^\ast \operatorname{Vect}(\mathbb{P}^1) ,$$

and hence a homotopy equivalence $\Omega|hS_\ast \operatorname{Ch}^\ast \operatorname{Vect}(\mathbb{P}^1)_0| \xrightarrow{\cong} K(\mathbb{P}^1)$.

For $k + \ell \geq -1$ the functor $\Psi_{k,\ell} : \operatorname{Ch}^k \mathbb{P}(R_0) \rightarrow \operatorname{Ch}^\ell \operatorname{Ch}^\ast \operatorname{Vect}(\mathbb{P}^1)_0$ is an exact functor between WALDHAUSEN categories (with quasi-isomorphisms and $h$-equivalences as weak equivalences, and cofibrations the monomorphisms with cokernel an object of the category under consideration).

**Theorem 5.4 ([HM18, Theorem III.5.1]).** Suppose that $R = \bigoplus_{k \in \mathbb{Z}} R_k$ is a strongly $\mathbb{Z}$-graded ring. There is a homotopy equivalence of $K$-theory spaces

$$K(R_0) \times K(R_0) \xrightarrow{\cong} K(\mathbb{P}^1)$$

induced by the functor

$$\Psi_{-1,0} + \Psi_{0,0} : \operatorname{Ch}^k \mathbb{P}(R_0) \times \operatorname{Ch}^\ell \mathbb{P}(R_0) \xrightarrow{\cong} \operatorname{Ch}^\ast \operatorname{Vect}(\mathbb{P}^1)_0$$

$$(C, D) \mapsto \Psi_{-1,0}(C) \oplus \Psi_{0,0}(D) .$$

6. The nil terms

**The category of twisted endomorphisms.** Let $M$ be an $R_0$-module. A (positive) twisted endomorphism of $M$ is an $R_0$-linear map

$$\alpha : M \otimes_{R_0} R_1 \rightarrow M \otimes_{R_0} R_0 .$$

The collection of such pairs $(M, \alpha)$, for various modules $M$ and their twisted endomorphisms, forms a category $\operatorname{Tw}^+ \operatorname{End}(R_0)$; a morphism $f : (M, \alpha) \rightarrow (N, \beta)$ is an $R_0$-linear map $f : M \rightarrow N$ with $\beta \circ (f \otimes \operatorname{id}_{R_1}) = (f \otimes \operatorname{id}_{R_0}) \circ \alpha$. — The category $\operatorname{Tw}^- \operatorname{End}(R_0)$ of (negative) twisted endomorphisms of the form $M \otimes_{R_0} R_{-1} \rightarrow M \otimes_{R_0} R_0$ is defined analogously.

In case of a strongly $\mathbb{Z}$-graded ring $R$ we have the equality $R_{n+1} = R_n R_1$, or (what is the same) the isomorphism $R_{n+1} \cong R_n \otimes_{R_0} R_1$. Given an object $(M, \alpha)$ of $\operatorname{Tw}^+ \operatorname{End}(R_0)$, we can thus recursively define the $n$th iteration of $\alpha$ to be the map

$$\alpha^{(n)} : M \otimes_{R_0} R_n \rightarrow M \otimes_{R_0} R_0 \quad (n \geq 0)$$

determined by

$$\alpha^{(0)} = \operatorname{id} \quad \text{and} \quad \alpha^{(1)} = \alpha ,$$

with $\alpha^{(n+1)}$ being the composition

$$M \otimes_{R_0} R_{n+1} \cong M \otimes_{R_0} R_n \otimes_{R_0} R_1 \xrightarrow{\alpha^{(n)} \otimes \operatorname{id}_{R_1}} M \otimes_{R_0} R_0 \otimes_{R_0} R_1 \xrightarrow{\cong} M \otimes_{R_0} R_1 \xrightarrow{\alpha} M \otimes_{R_0} R_0 .$$

We say that the twisted endomorphism $\alpha$ of $M$ is nilpotent if $\alpha^{(n)} = 0$ for $n \gg 0$.

Every $R_{\geq 0}$-module $M$ determines an object $(M, \mu)$ of $\operatorname{Tw}^+ \operatorname{End}(R_0)$, with $M$ considered as an $R_0$-module by restriction of scalars, and $\mu$ defined by $m \otimes r \mapsto mr \otimes 1$ for $m \in M$ and $r \in R_1$. There results a functor

$$\Phi = \Phi^+ : \operatorname{Mod}-R_{\geq 0} \rightarrow \operatorname{Tw}^+ \operatorname{End}(R_0) ;$$
an analogous construction provides us with a functor

\[ \Phi^\sim : \text{Mod-}R_{\leq 0} \longrightarrow \text{Tw}^\sim \text{End}(R_0). \]

**Lemma 6.1.** The functors \( \Phi = \Phi^+ \) and \( \Phi^- \) are isomorphisms of categories.

**Proof.** This is just the theory of modules over tensor rings, since (thanks to the strong grading) \( R_{\geq 0} \) is the tensor ring of \( R_1 \) over \( R_0 \), and \( R_{\leq 0} \) is the tensor ring of \( R_{-1} \) over \( R_0 \).

As a matter of notation, for \( (M, \alpha) \in \text{Tw}^+ \text{End}(R_0) \) we will denote the \( R_{\geq 0} \)-module \( \Phi^-_1(M, \alpha) \) simply by the symbol \( M \); if we wish to stress the twisted endomorphism, we shall write \( M^\alpha \). The module structure of \( M^\alpha \) and the maps \( \alpha^{(n)} \) are related by the square diagram (6.2) below; the vertical maps, induced by inclusions, are isomorphisms of \( R_0 \)-modules ([HM18, Proposition I.2.12]), and the bottom horizontal map is the obvious one, mapping \( x \otimes r \in M^\alpha \otimes R_{\geq n} R_{\geq 0} \) to \( x \otimes r \in M^\alpha \otimes R_{\geq 0} R_{\geq 0} \).

\[
\begin{array}{ccc}
M \otimes_{R_0} R_{\geq n} & \overset{\alpha^{(n)}}{\longrightarrow} & M \otimes_{R_0} R_{\geq 0} \\
\cong & & \cong \\
M^\alpha \otimes_{R_{\geq 0}} R_{\geq n} & \longrightarrow & M^\alpha \otimes_{R_{\geq 0}} R_{\geq 0}
\end{array}
\] (6.2)

The diagram commutes.

**The characteristic sequence of a twisted endomorphism.** For a strongly \( \mathbb{Z} \)-graded ring \( R \) we have the equality \( R_1 R_{-1} = R_0 \), by definition of strongly graded. In particular, there exist finitely many elements \( x^{(1)}_j \in R_1 \) and \( y^{(-1)}_j \in R_{-1} \) so that \( \sum_j x^{(1)}_j y^{(-1)}_j = 1 \). For any \( R_{\geq 0} \)-module \( M \) there results a well-defined map of \( R_{\geq 0} \)-modules

\[
\chi_M = \chi : M \otimes_{R_0} R_{\geq 1} \longrightarrow M \otimes_{R_0} R_{\geq 0}, \quad m \otimes r \mapsto m \otimes r - \sum_j mx^{(1)}_j y^{(-1)}_j r
\] (6.3)

which does not depend on the specific choice of elements \( x^{(1)}_j \) and \( y^{(-1)}_j \) ([Hüt18, §6]).

**Proposition 6.4** (Characteristic sequence of a twisted endomorphism). Let \( (M, \alpha) \) be an object of \( \text{Tw}^+ \text{End}(R_0) \). There is a short exact sequence of \( R_{\geq 0} \)-modules, natural in \( (M, \alpha) \),

\[
0 \longrightarrow M^\alpha \otimes_{R_0} R_{\geq 1} \overset{\chi_M}{\longrightarrow} M^\alpha \otimes_{R_0} R_{\geq 0} \overset{\pi}{\longrightarrow} M^\alpha \longrightarrow 0,
\] (6.5)

with \( \pi(m \otimes r) = mr \). The sequence is split exact as a sequence of \( R_0 \)-modules.

This generalises the usual characteristic sequence of a module equipped with an endomorphism ([Bas68, Proposition XII.1.1]).

**Proof.** The sequence (6.5) is nothing but the “canonical resolution” of ([Hüt18, Lemma 6.2] for the \( R_{\geq 0} \)-module \( M^\alpha \) associated with \( (M, \alpha) \).
Chain complexes of twisted endomorphisms. A bounded chain complex in the category $\text{Tw}^+\text{End}(R_0)$, that is, an object of the category $\text{Ch}^+\text{Tw}^+\text{End}(R_0)$, consists of a pair $(C, \alpha)$ where $C$ is a bounded chain complex of $R_0$-modules, and

$$\alpha : C \otimes_{R_0} R_1 \longrightarrow C \otimes_{R_0} R_0$$

is a map of $R_0$-module complexes.

Definition 6.6. The mapping half-torus of $(C, \alpha) \in \text{Ch}^+\text{Tw}^+\text{End}(R_0)$ is the complex of $R_{\geq 0}$-modules

$$\mathcal{H}(C, \alpha) = \text{cone} \left( \chi_C : C \otimes_{R_0} R_\geq 1 \longrightarrow C \otimes_{R_0} R_\geq 0 \right),$$

with $\chi_C = \chi$ defined in (6.3).

The mapping half-torus provides an exact additive functor defined on the category $\text{Tw}^+\text{End}(R_0)$ to the category of chain complexes of $R_{\geq 0}$-modules. It comes equipped with a natural $R_{\geq 0}$-linear map $\mathcal{H}(C, \alpha) \longrightarrow C^\alpha$ induced from the short exact sequence (6.5).

Proposition 6.7 ([Hüt18, Corollaries 6.5 and 6.8]). Suppose that $(C, \alpha)$ is a bounded chain complex in $\text{Tw}^+\text{End}(R_0)$.

1. The map $\mathcal{H}(C, \alpha) \longrightarrow C^\alpha$ is a quasi-isomorphism.
2. If $C$ is a complex of projective $R_0$-modules, the map $\mathcal{H}(C, \alpha) \longrightarrow C^\alpha$ is a chain homotopy equivalence of $R_0$-module complexes.
3. If $C$ is an $R_0$-finitely dominated complex of projective $R_{\geq 0}$-modules, then $\mathcal{H}(C, \alpha)$ is $R_{\geq 0}$-finitely dominated. \hfill $\Box$

Homotopy nilpotent twisted endomorphisms. Let $(C, \alpha)$ be a chain complex in the category $\text{Tw}^+\text{End}(R_0)$. We say that $\alpha$ is homotopy nilpotent if the chain map

$$\alpha^{(n)} : C \otimes_{R_0} R_n \longrightarrow C \otimes_{R_0} R_0$$

is null homotopic for some (equivalently, all) sufficiently large $n \geq 0$. Using diagram (6.2), this amounts to saying that the map of $R_0$-module complexes

$$C^\alpha \otimes_{R_{\geq 0}} R_{\geq n} \longrightarrow C^\alpha \otimes_{R_{\geq 0}} R_{\geq 0} = C^\alpha$$

(induced by the inclusion map $R_{\geq n} \subseteq R_{\geq 0}$) is null homotopic for $n \gg 0$. Since $R_{\geq n}$ is an invertible $R_{\geq 0}$-bimodule with inverse $R_{\geq -n}$, this in turn is equivalent to

$$C^\alpha = C^\alpha \otimes_{R_{\geq 0}} R_{\geq 0} \longrightarrow C^\alpha \otimes_{R_{\geq 0}} R_{\geq -n}$$

(6.9)

being null homotopic for $n \gg 0$. Under suitable finiteness assumptions, this is equivalent to the stronger condition that the map (6.9) is null homotopic for $n \gg 0$ as a map of $R_{\geq 0}$-module complexes:

Lemma 6.10. Let $C$ be an $R_{\geq 0}$-finitely dominated complex of $R_{\geq 0}$-modules. The following statements are equivalent:

1. The induced complex $C \otimes_{R_{\geq 0}} R$ is acyclic.
2. For $n \gg 0$ the obvious map $\nu_{0,n} : C \otimes_{R_{\geq 0}} R_{\geq 0} \longrightarrow C \otimes_{R_{\geq 0}} R_{\geq -n}$, considered as a map of $R_{\geq 0}$-module chain complexes, is null homotopic.
3. For $n \gg 0$ the obvious map $\nu_{0,n} : C \otimes_{R_{\geq 0}} R_{\geq 0} \longrightarrow C \otimes_{R_{\geq 0}} R_{\geq -n}$, considered as a map of $R_0$-module chain complexes, is null homotopic.

If these equivalent conditions hold, the complex $C$ is $R_0$-finitely dominated.
Proof. All three statements and the conclusion on finite domination are invariant under homotopy equivalence of complexes of $R_{\geq 0}$-modules. As $C$ is $R_{\geq 0}$-finitely dominated we may, and will, assume from the outset that $C$ is a bounded complex of finitely generated projective $R_{\geq 0}$-modules.

We use the notation 

$$\nu: C \otimes R_{\geq 0} R_{\geq 0} \longrightarrow C \otimes R_{\geq 0} R_{\geq 0}$$

and

$$\nu_{k,n}: C \otimes R_{\geq 0} R_{\geq 0} R_{\geq 0} \longrightarrow C \otimes R_{\geq 0} R_{\geq 0} R_{\geq 0}$$

for the obvious $R_{\geq 0}$-linear maps sending $x \otimes r$ to $x \otimes r$.

(1) $\Rightarrow$ (2): The map $\nu$ is certainly null homotopic as its target is contractible by hypothesis, and a choice of null homotopy yields a map $H: \text{cyl}(\text{id}_C) \longrightarrow C \otimes R_{\geq 0} R_{\geq 0}$

restricting to $\nu$ and 0, respectively, on the “ends” of the cylinder. Since the source is a bounded complex of finitely generated projective modules, the map $H$ factors through a finite stage of the increasing union, resulting in a number $N \geq 0$ and a map

$$H': \text{cyl}(\text{id}_C) \longrightarrow C \otimes R_{\geq 0} R_{\geq 0}.$$  \hspace{1cm} (6.11)

By construction, $H'$ is a null homotopy of $\nu_{0,N}$, and hence $\nu_{0,n} \simeq 0$ for all $n \geq N$.

(2) $\Rightarrow$ (3): This is a tautology.

(3) $\Rightarrow$ (1): Let $n$ be sufficiently large so that $\nu_{0,n} \simeq 0$ as a map of $R_0$-module complexes. As $\nu_{k,n}$ is isomorphic to $\nu_{0,n} \otimes R_0 R_{\geq 0}$, as a map of $R_0$-modules ([HM18, Proposition 1.2.12]), we conclude that $\nu_{k,n} \simeq 0$ for all $k \geq 0$. In particular, the maps $\nu_{k,n}$ induce the trivial map on homology. Since $C \otimes R_{\geq 0} R = \colim_{n \geq 0} C \otimes R_{\geq 0} R_{\geq 0}$, and since homology commutes with filtered colimits, we conclude by cofinality that

$$H_*(C \otimes R_{\geq 0} R) = \colim_{n \geq 0} H_*(C \otimes R_{\geq 0} R_{\geq 0}) = \colim_{k \geq 0} H_*(C \otimes R_{\geq 0} R_{\geq 0})$$

the last colimit taken with respect to the maps $H_*(\nu_{k,n})$ which are trivial. It follows that the colimit is trivial, whence $C \otimes R_{\geq 0} R$ is acyclic.

Suppose now that the equivalent conditions of the Lemma are satisfied. Going back to (6.11) and the notation used there, we know that the chain map

$$\nu_{0,N}: C \cong C \otimes R_{\geq 0} R_{\geq 0} \longrightarrow C \otimes R_{\geq 0} R_{\geq 0}$$

is null homotopic. Hence its mapping cone is homotopy equivalent to the mapping cone $M = C \oplus \Sigma(C \otimes R_{\geq 0} R_{\geq 0})$ of the zero map between the same complexes, which contains $C$ as a direct summand. On the other hand, the map $\nu_{0,N}$ is injective (since $C$ is assumed to consist of projective $R_{\geq 0}$-modules), hence its mapping cone is quasi-isomorphic to the cokernel $K$ of $\nu_{0,N}$. The $\ell$th chain module $K_\ell$ of $K$ is isomorphic, as an $R_{\geq 0}$-module, to $C_\ell \otimes R_{\geq 0} (R_{\geq 0}/R_{\geq 0})$. Since $C_\ell$ is a direct
summand of $R^n_{>0}$, for suitable $n \geq 0$, the module $K_\ell$ is a direct summand of
\[ R^n_{>0} \otimes R_{>0} (R_{>0}/R_{>0}) \approx \left( R_{>0} \otimes R_{>0} (R_{>0}/R_{>0}) \right)^{n_\ell}, \]
the last isomorphism being $R_0$-linear. Since $R$ is strongly graded each $R_q$ is a finitely generated projective $R_0$-module so that $K_\ell$ is a finitely generated projective $R_0$-module as well. Thus the (bounded) complex $K$ is $R_0$-finitely dominated. Since $C$ consists of projective $R_0$-modules so does $M$, and as $M$ is quasi-isomorphic to $K$ by the above arguments there is in fact an $R_0$-linear homotopy equivalence $M \approx K$. It follows that $M$ is $R_0$-finitely dominated, hence so is its direct summand $C$. — The finiteness result also follows from Theorem 8.1 of [Hüt18]; in the notation used there the ring $R_*((t^{-1}))$ contains $R$ so that $C \otimes_{R_{>0}} R_*((t^{-1})) \approx C \otimes_{R_{>0}} R \otimes R_*((t^{-1}))$ is acyclic as required. □

From Lemma 6.10, and from the fact that the maps (6.8) and (6.9) are isomorphic as $R_0$-linear maps, we conclude:

**Corollary 6.12.** Let $(C, \alpha)$ be an object of $Ch^B \text{Tw}^+ \text{End}(R_0)$. Suppose that the associated $R_{>0}$-module complex $C^\alpha$ is $R_{>0}$-finitely dominated. The twisted endomorphism $\alpha$ is homotopy nilpotent if and only if $C^\alpha \otimes_{R_{>0}} R$ is acyclic. □

For later use we record the following useful fact:

**Lemma 6.13.** Let $(Z, \zeta)$ be an object of $Ch^B \text{Tw}^+ \text{End}(R_0)$. Suppose that the associated $R_{>0}$-module complex $Z^\alpha$ is $R_{>0}$-finitely dominated complex of projective $R_{>0}$-modules. Suppose further that $Z^\alpha \otimes_{R_{>0}} R$ is acyclic. Then $Z$ is an $R_0$-finitely dominated bounded complex of projective $R_0$-modules.

**Proof.** Since $R$ is strongly $Z$-graded the complex $Z$ consists of projective $R_0$-modules. By hypothesis, $Z^\alpha$ is homotopy equivalent to a bounded complex $C$ of finitely generated projective $R_{>0}$-modules. Since $Z^\alpha \otimes_{R_{>0}} R$ is acyclic so is $C \otimes_{R_{>0}} R$. This forces $C$, hence $Z^\alpha$, to be $R_0$-finitely dominated in view of Lemma 6.10. □

**The nil category.** We are now in a position to define the nil category, the category of homotopy nilpotent endomorphisms of chain complexes satisfying a suitable finiteness constraint. In fact there are two variants, corresponding to the two subrings $R_{>0}$ and $R_{\geq 0}$ of $R$.

**Definition 6.14.** The positive nil category $^RR\text{Nil}^+(R_0)$ of $R_0$ relative to $R$ is the full subcategory of $Ch^B (\text{Tw}^+ \text{End}(R_0))$ consisting of those chain complexes $(C, \alpha)$ such that

(1) the underlying $R_0$-module complex $C$ consists of projective $R_0$-modules, and is $R_0$-finitely dominated;
(2) the chain map $\alpha : C \otimes_{R_0} R_1 \longrightarrow C \otimes_{R_0} R_0$ is homotopy nilpotent in the sense that $\alpha^{(n)} \simeq 0$ for $n \gg 0$.

The negative nil category $^RR\text{Nil}^-(R_0)$ of $R_0$ relative to $R$ is defined analogously, using the category $\text{Tw}^- \text{End}(R_0)$ of negative twisted endomorphisms in place of $\text{Tw}^+ \text{End}(R_0)$.
Remark 6.15. Let \((C, \alpha)\) be an object of \(R\text{Nil}^+(R_0)\). By Proposition 6.7, the \(R_{\geq 0}\)-linear map \(\delta(C, \alpha) \rightarrow C^\alpha\) is a quasi-isomorphism with \(R_{\geq 0}\)-finitely dominated source. The same map is an \(R_0\)-linear homotopy equivalence. Thus the second condition in the definition of \(R\text{Nil}^+(R_0)\) is equivalent to the condition \(\delta(C, \alpha) \otimes_{R_{\geq 0}} R \cong 0\), by Corollary 6.12.

A morphism in the category \(R\text{Nil}^+(R_0)\) is called a cofibration if it is injective and its cokernel consists of projective \(R_0\)-modules. We say the morphism is a weak equivalence, or a q-equivalence, if it is a quasi-isomorphism of \(R_0\)-module complexes.

Lemma 6.16. These definitions equip \(R\text{Nil}^+(R_0)\) with the structure of a category with cofibrations and weak equivalences.

Proof. This is mostly straightforward; the gluing lemma holds, for example, since it holds in the category of bounded complexes of projective \(R_0\)-modules. What needs explicit verification is that the requisite pushouts exist within the category \(R\text{Nil}^+(R_0)\). Using the functor \(\Phi\) and its inverse \((A, \alpha) \mapsto A^\alpha\) we identify \(R\text{Nil}^+(R_0)\) with a full subcategory of the category \(\mathcal{K}\) of bounded chain complexes of \(R_{\geq 0}\)-modules. Let \((A, \alpha), (B, \beta)\) and \((C, \gamma)\) be objects in \(R\text{Nil}^+(R_0)\), let \(f: (A, \alpha) \rightarrow (B, \beta)\) be a cofibration, and let \(g: (A, \alpha) \rightarrow (C, \gamma)\) be an arbitrary morphism in \(R\text{Nil}^+(R_0)\). We can then form the pushout diagram

\[
\begin{array}{ccc}
A^\alpha & \xrightarrow{f} & B^\beta \\
\downarrow{g} & & \downarrow{f'} \\
C^\gamma & \xrightarrow{\pi} & P^\pi \\
\end{array}
\]

in the category \(\mathcal{K}\), and claim that \((P, \pi)\) is an object of \(\text{Ch}^\mathbb{Z}(\text{Tw}^+\text{End}(R_0))\).

As \(f\) is a cofibration in \(R\text{Nil}^+(R_0)\) it is an injective map, and the same is thus true for its pushout \(f'\). Let \(K^\kappa\) denote the cokernel of \(f\), associated with \((K, \kappa) \in \text{Ch}^\mathbb{Z}(\text{Tw}^+\text{End}(R_0))\). From general properties of pushout squares we know that \(\text{coker}(f') \cong K^\kappa\). Hence we obtain a commutative ladder diagram of short exact sequences of chain complexes:

\[
\begin{array}{ccc}
0 & \rightarrow & A^\alpha & \rightarrow & B^\beta & \rightarrow & K^\kappa & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & C^\gamma & \rightarrow & P^\pi & \rightarrow & K^\kappa & \rightarrow & 0 \\
\end{array}
\]

Again, since \(f\) is a cofibration its cokernel \(K^\kappa\) consists of projective \(R_0\)-modules. Thus both sequences are levelwise split short exact sequences of \(R_0\)-module complexes, and \(P\) consists of projective \(R_0\)-modules. By Lemma 3.1, applied to the top row, the complex \(K^\kappa\) is \(R_0\)-finitely dominated, which implies, by applying Lemma 3.1 to the bottom row, that so is \(P^\pi\).
Application of the exact half-torus function yields another commutative ladder diagram of short exact sequences,

\[
\begin{array}{ccccccc}
0 & \rightarrow & H(A, \alpha) & \rightarrow & H(B, \beta) & \rightarrow & H(K, \kappa) & \rightarrow & 0 \\
0 & \rightarrow & H(C, \gamma) & \rightarrow & H(P, \pi) & \rightarrow & H(K, \kappa) & \rightarrow & 0 \\
\end{array}
\]

consisting of bounded complexes of finitely generated projective \( R \geq 0 \)-modules; in particular, both short exact sequences are levelwise split. By Remark 6.15, the chain complexes \( H(A, \alpha) \), \( H(B, \beta) \) and \( H(C, \gamma) \) are \( R \geq 0 \)-finitely dominated. Hence \( H(K, \kappa) \) and \( H(P, \pi) \) are \( R \geq 0 \)-finitely dominated as well, by two applications of Lemma 3.1.

The bottom row yields a short exact sequence

\[
0 \rightarrow H(C, \gamma) \otimes_{R \geq 0} R \rightarrow H(P, \pi) \otimes_{R \geq 0} R \rightarrow H(K, \kappa) \otimes_{R \geq 0} R \rightarrow 0 .
\]

The first and third entry are acyclic, by Corollary 6.12, hence so is the middle entry. Applying the Corollary again leads us to conclude that \((P, \pi)\) is an object of \( R \Nil^+ (R_0) \).

With this lemma in place, we can now introduce the notation

\[
R \Nil^+_q (R_0) = K_q (\text{FD}^+ (R_0)) = \pi_q \Omega [q \mathcal{S} \cdot R \Nil^+ (R_0)]
\]  

and

\[
R \Nil^-_q (R_0) = K_q (\text{FD}^- (R_0)) = \pi_q \Omega [q \mathcal{S} \cdot R \Nil^- (R_0)] .
\]

For \( L \) a unital ring write \( \text{FD}(L) \) for the category of \( L \)-finitely dominated bounded complexes of projective \( L \)-modules. It is well-known, and can be verified without difficulty using WALDHAUSEN’s approximation theorem, that the inclusion \( \text{Ch}^\ell (L) \subset \text{FD}(L) \) induces a homotopy equivalence on K-theory spaces

\[
K(L) = \Omega[q \mathcal{S} \cdot \text{Ch}^\ell (L)] \xrightarrow{\epsilon} \Omega[q \mathcal{S} \cdot \text{FD}(L)] ,
\]

where “\( q \)” stands for quasi-isomorphisms as usual, and the cofibrations are the injective maps with levelwise projective cokernel. Hence the forgetful functors

\[
o^\pm : R \Nil^\pm (R_0) \rightarrow \text{FD}(R_0) , \quad (Z, \zeta) \mapsto Z
\]

yield group homomorphisms

\[
o^- : R \Nil^- (R_0) \rightarrow K_q (R_0) \quad \text{and} \quad o^+ : R \Nil^+ (R_0) \rightarrow K_q (R_0) ,
\]

and we will establish in Theorem 11.1 that there are isomorphisms

\[
R \NK^-_{q+1} (R_0) \cong \ker(o^-) \quad \text{and} \quad R \NK^+_{q+1} (R_0) \cong \ker(o^+) ,
\]

identifying the groups \( R \NK^\pm_{q+1} (R_0) \) with the \( q \)th reduced algebraic K-group of the category \( R \Nil^\pm (R_0) \).
7. The fundamental square

7.1. The fundamental square of the projective line. Let \( f = (f^-, f^0, f^+) \) be a morphism \( \mathcal{Y} \to \mathcal{Z} \) in the category \( \text{Ch}^\bullet \text{Vect}(\mathbb{P}^1) \) of bounded chain complexes of vector bundles on the projective line associated with a strongly \( \mathbb{Z} \)-graded ring \( R \). We say that \( f \) is an \( h^{-} \)-equivalence, for the decoration \( ? \in \{ -, 0, + \} \), if the component \( f^? \) is a quasi-isomorphism of chain complexes. Together with the previous notion of cofibrations, viz., levelwise split injections in each component, this equips \( \text{Ch}^\bullet \text{Vect}(\mathbb{P}^1) \) with three new structures of a category with cofibrations and weak equivalences. Note that an \( h^{-} \)-equivalence is automatically an \( h^{0} \)-equivalence as well since in the commutative square

\[
\begin{array}{ccc}
Y^- \otimes_{R_{\leq 0}} R & \stackrel{v^-}{\longrightarrow} & Y^0 \\
| & & | \\
| & & | \\
f^- \otimes R & \longrightarrow & f^0 \\
| & & | \\
| & & | \\
Z^- \otimes_{R_{\leq 0}} R & \stackrel{v^-}{\longrightarrow} & Z^0
\end{array}
\]

the horizontal maps are isomorphisms (sheaf condition), and the left vertical map is a quasi-isomorphism because \( R \) is a flat left \( R_{\leq 0} \)-module thanks to the strong grading. — Similarly, every \( h^{+} \)-equivalence is an \( h^{0} \)-equivalence.

Thus the identity functor yields a commutative square of \( K \)-theory spaces

\[
\begin{array}{ccc}
h_{-}S_{\bullet} \text{Ch}^\bullet \text{Vect}(\mathbb{P}^1) & \longrightarrow & h_{+}S_{\bullet} \text{Ch}^\bullet \text{Vect}(\mathbb{P}^1) \\
| & & | \\
| & & | \\
h_{-}S_{\bullet} \text{Ch}^\bullet \text{Vect}(\mathbb{P}^1) & \longrightarrow & h_{0}S_{\bullet} \text{Ch}^\bullet \text{Vect}(\mathbb{P}^1)
\end{array}
\]

which we call the fundamental square of the projective line. Associated with it, by taking vertical homotopy fibres, is the map

\[
\alpha: h_{-}S_{\bullet} \text{Ch}^\bullet \text{Vect}(\mathbb{P}^1)^{h-} \longrightarrow h_{+}S_{\bullet} \text{Ch}^\bullet \text{Vect}(\mathbb{P}^1)^{h0}
\]

where the notation is as in WALDHAUSEN’S fibration theorem [Wal85, Theorem 1.6.4]. Explicitly, \( \text{Ch}^\bullet \text{Vect}(\mathbb{P}^1)^{h-} \) is the full subcategory of \( \text{Ch}^\bullet \text{Vect}(\mathbb{P}^1) \) consisting of those objects \( \mathcal{Y} \) for which \( \mathcal{Y} \longrightarrow 0 \) is an \( h_{-} \)-equivalence, that is, which satisfy \( Y^- \cong * \) and \( Y^0 \cong * \), and \( \text{Ch}^\bullet \text{Vect}(\mathbb{P}^1)^{h0} \) is the full subcategory of \( \text{Ch}^\bullet \text{Vect}(\mathbb{P}^1) \) consisting of those objects \( \mathcal{Y} \) for which \( \mathcal{Y} \longrightarrow 0 \) is an \( h_{0} \)-equivalence, that is, which satisfy \( Y^0 \cong * \).

The fibres of the fundamental square. The (homotopy) fibres of the maps in the fundamental square can be identified explicitly: they are homotopy equivalent to \( K \)-theory spaces of categories of homotopy nilpotent twisted endomorphisms. We will use this identification presently to conclude that the fundamental square is homotopy cartesian.

**Theorem 7.3** (Fibres of the fundamental square). *The functor*

\[
F: \text{Ch}^\bullet \text{Vect}(\mathbb{P}^1)^{h0} \longrightarrow \mathcal{R} \text{Nil}^+ (R_0), \quad \mathcal{Y} = (Y^- \longrightarrow Y^0 \longleftarrow Y^+) \mapsto \Phi(Y^+),
\]
defined on the category of bounded chain complexes of vector bundles $\mathcal{Y}$ with $Y^0 \simeq *$, induces a homotopy equivalence

$$h_{+}S_{\bullet} Ch^{3} \operatorname{Vect}(P^{1})^{h_{0}} \longrightarrow qS_{\bullet} R Nil^{+}(R_{0}) , \tag{7.3a}$$

and, by restriction, a homotopy equivalence

$$hS_{\bullet} Ch^{3} \operatorname{Vect}(P^{1})^{h_{-}} \longrightarrow qS_{\bullet} R Nil^{+}(R_{0}) \tag{7.3b}$$

where the letter “$q$” stands for weak equivalences in the category $R Nil^{+}(R_{0})$.

By symmetry, there are analogous homotopy equivalences

$$h_{-}S_{\bullet} Ch^{3} \operatorname{Vect}(P^{1})^{h_{0}} \longrightarrow qS_{\bullet} R Nil^{-}(R_{0}) , \tag{7.4a}$$

and, by restriction,

$$hS_{\bullet} Ch^{3} \operatorname{Vect}(P^{1})^{h_{+}} \longrightarrow qS_{\bullet} R Nil^{-}(R_{0}) . \tag{7.4b}$$

Proof. The functor $F$ is well defined. Indeed, the component $Y^{+}$ of $\mathcal{Y}$ is a bounded complex of finitely generated projective $R_{\geq 0}$-modules, hence $Y^{+}$ consists of projective $R_{0}$-modules. The hypothesis $Y^{+} \otimes_{R_{\geq 0}} R \simeq Y^{0} \simeq *$ implies that $Y^{+}$ is $R_{0}$-finitely dominated by Lemma 6.13. Moreover, the associated twisted endomorphism

$$Y^{+} \otimes_{R_{0}} R_{1} \longrightarrow Y^{+} \otimes_{R_{0}} R_{0}$$

is homotopy nilpotent by Corollary 6.12.

We will employ WALDHAUSEN’s approximation theorem [Wal85, Theorem 1.6.7], combined with the standard observation that the map required by property (App 2) does not have to be a cofibration (since it can be replaced by one in the presence of cylinder functors). — We can in fact treat cases (7.3a) and (7.3b) at the same time as the proofs are almost identical.

To start with, a morphism $a = (a^{-}, a^{0}, a^{+})$ in $Ch^{3} \operatorname{Vect}(P^{1})^{h_{0}}$ is an $h_{+}$-equivalence, by definition, if and only if $F(a) = a^{+}$ is a quasi-isomorphism. A morphism $b = (b^{-}, b^{0}, b^{+})$ in $Ch^{3} \operatorname{Vect}(P^{1})^{h_{-}}$ is an $h$-equivalence if and only if $F(b) = b^{+}$ is a quasi-isomorphism since $b^{-}$ and $b^{0}$ are maps between acyclic complexes, hence are quasi-isomorphisms in any case. In other words, the functor $F$ satisfies property (App 1) in both cases under consideration.

Let $(Z, \zeta) \in R Nil^{+}(R_{0})$ and $Y^{+} \in Ch^{3} \operatorname{Vect}(P^{1})^{h_{0}}$, and let $f : \Phi(Y^{+}) \longrightarrow (Z, \zeta)$ be a morphism in $R Nil^{+}(R_{0})$. We can equivalently consider $f$ as an $R_{\geq 0}$-linear map of chain complexes $Y^{+} \longrightarrow Z^{\zeta}$. By Remark 6.15, the chain complex $Z^{\zeta}$ is quasi-isomorphic to the $R_{\geq 0}$-finitely dominated complex $\mathcal{F}(Z, \zeta)$ so that there exists a bounded complex $E$ of finitely generated projective $R_{\geq 0}$-modules and a quasi-isomorphism $e : E \longrightarrow Z^{\zeta}$. We can lift $f$ up to homotopy to an $R_{\geq 0}$-linear map $g : Y^{+} \longrightarrow E$ so that $eg$ is homotopic to $f$. A choice of homotopy $f \simeq eg$ determines a map $h' : \operatorname{cyl}(g) \longrightarrow Z^{\zeta}$ such that the composite $Y^{+} \longrightarrow \longrightarrow Z^{\zeta}$ coincides with $f$, and such that the composite $E \longrightarrow \longrightarrow Z^{\zeta}$ coincides with $e$. In particular, $h'$ is a quasi-isomorphism of $R_{\geq 0}$-module complexes. (Here $i_{0}$ and $i_{1}$ are the front and back inclusion into the mapping cylinder.)

We can now form a complex $X^{+}$ by attaching to $\operatorname{cyl}(g)$ a direct sum of contractible complexes of the type $D(\ell, M)$, with $M$ a finitely generated projective $R_{\geq 0}$-module, so that all chain modules of $X^{+}$ except possibly one are finitely
generated free modules. We let \( a^+ \) denote the composition of \( i_0 \) with the inclusion \( \text{cyl}(g) \hookrightarrow X^+ \), and let \( h^+ \) denote the composition of the projection \( X^+ \twoheadrightarrow \text{cyl}(g) \) with \( h' \). We have constructed a commutative diagram

\[
\begin{array}{ccc}
Y^+ & \xrightarrow{a^+} & X^+ \\
\downarrow \cong & & \downarrow h^+ \\
Z^\varsigma & \xrightarrow{a^0} & X^0
\end{array}
\]

where \( X^+ \) is a bounded complex of finitely generated projective \( R_{\geq 0} \)-modules, and \( h^+ \) is a quasi-isomorphism.

We will now extend \( X^+ \) to a vector bundle \( \mathcal{X} = (X^-, X^0, X^+) \) and \( a^+ \) to a map of vector bundles \( a = (a^-, a^0, a^+): Y \rightarrow \mathcal{X} \). This may involve enlarging \( X^+ \) further, by taking the direct sum with contractible complexes of the form \( D(k, R^k_{\geq 0}) \) and composing \( a^+ \) and \( h^+ \) with the obvious inclusion and projection maps. We shall keep the notation \( X^+ \) and \( h^+ \) even for the modified data.

Let \( X^0 = X^+ \otimes_{R_{\geq 0}} R \). Recall that \( Z^\varsigma \) is quasi-isomorphic to \( X^+ \), via the map \( h^+ \), and to the half-torus \( \mathcal{H}(Z, \zeta) \), by Remark 6.15. Both \( X^+ \) and \( \mathcal{H}(Z, \zeta) \) are bounded complexes of finitely generated projective \( R_{\geq 0} \)-modules, so these complexes are homotopy equivalent. It follows that \( X^0 \cong \mathcal{H}(Z, \zeta) \otimes_{R_{\geq 0}} R \) is contractible, by Remark 6.15 again; this implies that \( [X^0] = \sum_{k} (-1)^k[X^0_k] = 0 \in K_0(R) \). As all chain modules of \( X^0 \) are free with the possible exception of a single module, we conclude that \( X^0 \) consists of stably free modules, and hence consists of modules which are stably induced from \( R_{\leq 0} \). We now appeal to Theorem 2.7: we can modify \( X^+ \) by taking direct sum with finitely many contractible complexes of the form \( D(k, R^k_{\geq 0}) \), thereby modifying \( X^0 \) in an analogous manner, so that \( X^0 \) is isomorphic to \( X^- \otimes_{R_{\leq 0}} R \) for a bounded acyclic complex \( X^- \) of finitely generated projective \( R_{\leq 0} \)-modules.

We have thus constructed a vector bundle

\[
\mathcal{X} = (X^-, X^0, X^+)
\]

together with a quasi-isomorphism \( h^+: X^+ \twoheadrightarrow Z^\varsigma \) and a map \( a^+: Y^+ \twoheadrightarrow X^+ \) such that \( h^+ \circ a^+ = f \). The map \( a^+ \) induces a compatible map \( a^0 = \xi^+_i \circ (a^+ \otimes \text{id}) \circ (\xi^+_i)^{-1} \), see the diagram in Fig. 1. For each chain level \( \ell \) the composite map

\[
\begin{array}{ccc}
Y^- & \xrightarrow{v^-} & Y^0 \\
\downarrow a^0 & & \downarrow a^+ \otimes \text{id} \\
X^- & \xrightarrow{\xi^-} & X^0 \\
\end{array}
\]

\[
\begin{array}{ccc}
Y^0 & \xrightarrow{\xi^+_i} & Y^+ \otimes_{R_{\geq 0}} R \\
\downarrow a^+ \otimes \text{id} & & \downarrow a^+ \otimes \text{id} \\
X^0 & \xrightarrow{\xi^+_i} & X^+ \otimes_{R_{\geq 0}} R
\end{array}
\]

**Figure 1.** Diagram used in proof of Theorem 7.3
The map \( Y^{-} \xrightarrow{\nu^{-}} Y^{0} \xrightarrow{a^{0}} X_{\ell}^{0} \cong X_{\ell}^{-} \otimes_{R \leq 0} R = \bigcup_{q \geq 0} X_{\ell}^{-} \otimes_{R \leq q} R \leq q \)

factors through some term \( X_{\ell}^{-} \otimes_{R \leq 0} R \leq q \) so that, by choosing \( q \gg 0 \), the map \( a^{0} \circ \nu^{-} \) factorises as

\[
Y^{-} \xrightarrow{\alpha^{-}} X^{-} \otimes_{R \leq 0} R \leq q \xrightarrow{a} X^{0},
\]

the second map in this composition given by \( x \otimes r \mapsto \xi^{-}(x) \cdot r \). That is, there exists \( q \geq 0 \) and \( a^{-} : Y^{-} \longrightarrow X^{-} \otimes_{R \leq 0} R \leq q \) such that \( a = (a^{-}, a^{0}, a^{+}) : Y \longrightarrow \mathcal{X}(q, 0) \) is a map of vector bundles, and such that \( h^{+} \circ F(a) = f \). Since \( X^{-} \) and \( X^{0} \) are contractible by construction, \( \mathcal{X}(q, 0) \) is an object of \( \text{Ch}^{b} \text{Vect}(\mathbb{P}^{1})_{b}^{-} \subseteq \text{Ch}^{b} \text{Vect}(\mathbb{P}^{1})_{b}^{h_{0}} \).

This proves that \( F \) satisfies property (App 2) in addition to (App 1), and the approximation theorem applies. \( \square \)

**Corollary 7.5.** The map \( \alpha \) of (7.2) is a homotopy equivalence. Thus, the fundamental square (7.1) is homotopy cartesian.

**Proof.** The previous Theorem asserts that in the chain of maps

\[
h_{\mathcal{S}} \text{Ch}^{b} \text{Vect}(\mathbb{P}^{1})_{b}^{-} \xrightarrow{\alpha} h_{\mathcal{S}} \text{Ch}^{b} \text{Vect}(\mathbb{P}^{1})_{b}^{h_{0}} \xrightarrow{F} q_{\mathcal{S}} \text{RNil}^{+}(R_{0})
\]

both \( F \) and \( F \circ \alpha \) are homotopy equivalences. It follows that \( \alpha \) is a homotopy equivalence as well. Since the fundamental square consists of connected spaces it is homotopy cartesian. \( \square \)

**Auxiliary categories.** We let \( \mathcal{C}_{+} \) denote the category of bounded chain complexes of “vector bundles on \( \text{spec}(R_{\geq 0}) \)”, that is, diagrams of the form

\[
Y^{0} \xrightarrow{\nu^{+}} Y^{+}
\]

with \( Y^{0} \) and \( Y^{+} \) being bounded complexes of finitely generated projective over \( R \) and \( R_{\geq 0} \), respectively, subject to the condition that the associated adjoint map \( Y^{0} \xrightarrow{\nu^{+}} Y^{+} \otimes_{R \geq 0} R \) be an isomorphism. A morphism \( g = (g^{0}, g^{+}) \) from \( Y^{0} \xrightarrow{\nu^{+}} Y^{+} \) to \( Z^{0} \xrightarrow{\zeta^{+}} Z^{+} \) consists of an \( R \)-linear map \( g^{0} : Y^{0} \longrightarrow Z^{0} \) and an \( R_{\geq 0} \)-linear map \( g^{+} : Y^{+} \longrightarrow Z^{+} \) such that \( \zeta^{+} \circ g^{+} = g^{0} \circ \nu^{+} \).

By \( \mathcal{D}_{+} \) we denote the full subcategory of \( \mathcal{C}_{+} \) consisting of objects \( Y^{0} \xrightarrow{\nu^{+}} Y^{+} \) such that all chain modules \( Y^{0}_{k} \) are stably induced from \( R \leq 0 \), that is, such that

\[
[Y^{0}_{k}] \in \text{im} \left( K_{0}(R_{\leq 0}) \longrightarrow K_{0}(R) \right),
\]

or equivalently, such that \( \alpha([Y^{0}_{k}]) = 0 \) in \( K_{0}(R_{\leq 0} \downarrow R) \) for all \( k \).

**Lemma 7.6.** Every object of \( \mathcal{D}_{+} \) stably extends to an object of \( \text{Ch}^{b} \text{Vect}(\mathbb{P}^{1}) \). That is, given an object \( Y^{0} \xrightarrow{\nu^{+}} Y^{+} \) of \( \mathcal{D}_{+} \) there exist an object

\[
Z^{-} \xrightarrow{\zeta^{-}} Z^{0} \xrightarrow{\zeta^{+}} Z^{+}
\]
of \( \text{Ch}^\circ \text{Vec}(\mathbb{P}^1) \) and finitely many numbers \( j_k \geq 0 \) together with a commutative diagram

\[
\begin{array}{ccc}
Y^0 \oplus \bigoplus_k D(k, R^{j_k}) & \xrightarrow{v^+ \oplus \text{inc}} & Y^+ \oplus \bigoplus_k D(k, R^{j_k}) \\
\downarrow & & \downarrow \\
Z^0 & \xrightarrow{\zeta^+} & Z^+ 
\end{array}
\]

where “inc” denotes the obvious inclusion map based on the ring inclusion \( R \supseteq R_{\geq 0} \).

**Proof.** We apply Proposition 2.12 to the ring inclusion \( f : R \leq 0 \longrightarrow R \) and the chain complex \( D = Y^0 \). We obtain a stabilisation

\[ D' = Y^0 \oplus \bigoplus_k D(k, R^{j_k}) \]

of \( D = Y^0 \) and a bounded complex of finitely generated projective \( R_{\leq 0} \)-modules \( Z^- = C' \) such that there is an isomorphism \( i : Z^- \otimes_{R_{\leq 0}} R \longrightarrow D' \). Write \( Z^0 \) in place of \( D' \), let \( \zeta^- : Z^- \longrightarrow Z^0 \) be the \( R_{\leq 0} \)-linear map adjoint to \( i \), and define

\[ Z^+ = Y^+ \oplus \bigoplus_k D(k, R^{j_k}) \]

with \( \zeta^+ = v^+ \oplus \text{inc} \) the data satisfies all the requirements of the Lemma. \( \square \)

A morphism \( g = (g^0, g^+) \) in \( C_+ \) is called a cofibration if both \( g^0 \) and \( g^+ \) are levelwise injections such that \( \text{coker}(g) \) is an object of \( C_+ \). We call \( g \) an \( h_+ \)-equivalence if \( g^+ \) is a quasi-isomorphism (and hence so is \( g^0 \cong g^+ \otimes_{R_{\geq 0}} R \)).

**Lemma 7.7.** With these definitions, both \( C_+ \) and \( D_+ \) are categories with cofibrations and weak equivalences.

**Proof.** This is clear for \( C_+ \) since this category is equivalent to the category of bounded chain complexes of finitely generated projective \( R_{\geq 0} \)-modules, via the functors

\[
(Y^0 \longleftarrow Y^+) \quad \text{and} \quad (M^+ \rightarrow (M^+ \otimes_{R_{\geq 0}} R \longleftarrow M^+) .
\]

It remains to observe that the cokernel of a cofibration \( g \) in \( D_+ \) is automatically an object of \( D_+ \) since, given \( g^0_k : Y^0_k \longrightarrow Z^0_k \), we have the equality

\[ \alpha([\text{coker} g^0_k]) = \alpha([Z^0_k]) - \alpha([Y^0_k]) = 0 \]

in the group \( K_0(R_{\leq 0} \downarrow R) \). \( \square \)

We make the analogous symmetric definitions for \( C_- \) and \( D_- \). The category \( C_0 \) is defined to be the category of bounded chain complexes of finitely generated projective \( R \)-modules, with \( D_0 \) the full subcategory of those objects \( Y^0 \) such that all chain modules \( Y^0_k \) are stably induced from both \( R_{\leq 0} \) and \( R_{\geq 0} \), that is, such that

\[ [Y^0_k] \in \text{im} (K_0 R_{\leq 0} \\ K_0 R) \cap \text{im} (K_0 R_{\geq 0} \longrightarrow K_0 R) \quad \text{for all } k .
\]

Lemmas 7.6 and 7.7 are valid \textit{mutatis mutandis} for these categories.
Lemma 7.8. The forgetful functor
\[(Y^0 \longleftrightarrow Y^+) \mapsto Y^+\]
induces isomorphisms on algebraic \(K\)-groups
\[K_q\mathcal{D}_+ \mapsto K_qR_{\geq 0} \quad (q > 0)\]
and an injection \(K_0\mathcal{D}_+ \subseteq K_0R_{\geq 0}\). — These statements hold mutatis mutandis for the analogous maps \(K_q\mathcal{D}^- \subseteq K_qR_{\leq 0}\) and \(K_q\mathcal{D}_0 \subseteq K_qR\).

Proof. As remarked before, the functor \(Y \mapsto Y^+\) establishes an equivalence of \(\mathcal{D}_+\) with the category of bounded chain complexes of finitely generated projective \(R_{\geq 0}\)-modules, and \(\mathcal{D}_+\) corresponds to the subcategory of complexes \(Y^+\) such that \(Y^+ \otimes R_{\geq 0} \subseteq \mathcal{D}_+\). Let \(\mathcal{D}'\) and \(\mathcal{C}'\) denote the full subcategories of complexes concentrated in chain level 0. Then the map of \(K\)-groups in question can be computed using QUILLEN’s \(Q\)-construction, applied to the inclusion of exact categories \(\mathcal{D}' \subseteq \mathcal{C}'\). The result is now an immediate consequence of GRAYSON cofinality \([\text{Gra79, Theorem 1.1}]\) since \(\mathcal{D}'\) is closed under extension in \(\mathcal{C}'\); indeed, an object \(Y \in \mathcal{C}'\) is in \(\mathcal{D}'\) if and only if \(\alpha([Y^0]) = 0 \in K_0(R_{\leq 0} \downarrow R)\), and \(K\)-theory is additive on short exact sequences. To see that the former category is cofinal in the latter it suffices to observe that every finitely generated projective module can be complemented to a finitely generated free one, which is automatically stably induced. □

The corners of the fundamental square. We can now identify the corners of the fundamental square as the algebraic \(K\)-theory spaces of the categories \(\mathcal{D}', \mathcal{D}_0\) and \(\mathcal{D}^+\).

Lemma 7.9. The forgetful functor \(\Phi^+: \text{Ch}^p \text{Vect}(\mathbb{P}^1) \longrightarrow \mathcal{D}_+\), given by
\[\mathcal{Y} = (Y^- \longleftrightarrow Y^0 \longleftrightarrow Y^+) \mapsto \Phi^+(\mathcal{Y}) = (Y^0 \longleftrightarrow Y^+)\]
induces a homotopy equivalence on \(S_\bullet\)-constructions with respect to \(h_+\)-equivalences:
\[h_+S_\bullet \text{Ch}^p \text{Vect}(\mathbb{P}^1) \xrightarrow{\cong} h_+S_\bullet \mathcal{D}_+\]
The statement holds mutatis mutandis for \(\mathcal{D}_0\) and \(\mathcal{D}_-\) as well.

Proof. By definition of \(h_+\)-equivalences the forgetful functor respects and detects weak equivalences. Let
\[\mathcal{X} = (X^- \longleftrightarrow X^0 \longleftrightarrow X^+)\]
be an object of \(\text{Ch}^p \text{Vect}(\mathbb{P}^1)\), and let
\[\hat{\mathcal{Y}} = (Y^0 \longleftrightarrow Y^+)\]
be an object of \(\mathcal{D}_+\). Given a morphism
\[g = (g^0, g^+): \Phi^+(\mathcal{X}) \longrightarrow \hat{\mathcal{Y}}\]
we construct the object \(\mathcal{Z} \in \text{Ch}^p \text{Vect}(\mathbb{P}^1)\) associated with \(\hat{\mathcal{Y}}\) as described in Lemma 7.6: denote the composition of \(g\) with the inclusion \(\hat{\mathcal{Y}} \longrightarrow \Phi^+(\mathcal{Z})\) by \(h = (h^0, h^+)\). The composite map
\[X^- \longleftrightarrow X^0 \xrightarrow{h^0} Z^0\]
factors as
\[ X^- \xrightarrow{h^-} Z^- \otimes_{R_{\leq 0}} R_{\leq q} \subseteq Z^- \otimes_{R_{\leq 0}} R \cong Z^0 \]
for sufficiently large \( q \geq 0 \); this is because \( X^- \) is a bounded complex of finitely generated \( R_{\leq 0} \)-modules, and because
\[ Z^0 \cong Z^- \otimes_{R_{\leq 0}} R = Z^- \otimes_{R_{\leq 0}} \bigcup_{k \geq 0} R_{\leq k} = \bigcup_{k \geq 0} Z^- \otimes_{R_{\leq 0}} R_k . \]
This results in a map
\[ h = (h^-, h^0, h^+) : X \longrightarrow Z(q, 0) \]
in \( \text{Ch}^b \text{Vect}(\mathbb{P}^1) \) (we identify the canonically isomorphic complexes \( \Phi^+(Z) = Z^+ \) and \( \Phi^+(Z(q, 0)) = Z^+ \otimes_{R_{\geq 0}} R \) here). The projection map \( p : \Phi^+(Z(q, 0)) \longrightarrow \overline{Y} \) is an \( h_+ \)-equivalence, and satisfies the condition \( p \circ \Phi^+(h) = g \). By Waldhausen’s approximation theorem [Wal85, Theorem 1.6.7], \( \Phi^+ \) induces a homotopy equivalence on \( S \)-constructions.

8. Establishing the Mayer-Vietoris sequence

We will now establish the MAYER-VIETORIS sequence of Theorem 1.8. As before let \( R \) be a strongly \( \mathbb{Z} \)-graded ring. The induction functors
\[ j^-_* = (- \otimes_{R_{\leq 0}} R) \quad \text{and} \quad j^+_* = (- \otimes_{R_{\geq 0}} R) \]
give rise to maps \( \gamma = j^-_* - j^+_* : K_q(R_{\leq 0}) \oplus K_q(R_{\geq 0}) \longrightarrow K_q(R) \), for \( q \geq 0 \).

In view of Lemma 7.9, the homotopy cartesian fundamental square of the projective line (7.1) yields a MAYER-VIETORIS sequence of algebraic \( K \)-groups
\[ \ldots \xrightarrow{\gamma} K_{q+1}D_0 \xrightarrow{\delta} K_q\mathbb{P}^1 \xrightarrow{\beta=(\beta_-, \beta_+)} K_qD_- \oplus K_qD_+ \xrightarrow{\gamma=\gamma_- - \gamma_+} K_qD_0 \xrightarrow{\delta} \ldots , \]
for \( q \geq 0 \), ending with a surjective homomorphism \( K_0D_- \oplus K_0D_+ \longrightarrow K_0D_0 \). By Lemma 7.8 this sequence coincides, for \( q > 0 \), with the one of Theorem 1.8, so we only need to look at its tail end. Using Lemma 7.8 again we construct the following commutative diagram:

\[ \begin{array}{ccc}
K_1(R) & \longrightarrow & K_0(\mathbb{P}^1) \\
\downarrow & = & \downarrow \\
K_1D_0 & \longrightarrow & K_0(\mathbb{P}^1) \\
\end{array} \]

\[ \begin{array}{ccc}
\subseteq & = & \subseteq \\
K_1D_0 & \longrightarrow & K_0D_- \oplus K_0D_+ \\
\end{array} \]

\[ \begin{array}{ccc}
K_1D_0 & \longrightarrow & K_0D_0 \\
\gamma & = & \gamma \\
\end{array} \]

\[ \begin{array}{ccc}
& \longrightarrow & \longrightarrow \\
K_1D_0 & \longrightarrow & K_0D_- \oplus K_0D_+ \\
\beta & = & \beta \\
\end{array} \]

\[ \begin{array}{ccc}
& \longrightarrow & \longrightarrow \\
K_1D_0 & \longrightarrow & K_0D_0 \\
\overline{\gamma} & = & \overline{\gamma} \\
\end{array} \]

We know that the bottom row is exact, and argue that this remains true for the top row. — Replacing the target of \( \beta \) by a larger group does not change \( \text{ker}(\beta) \), so the top row is exact at \( K_0(\mathbb{P}^1) \). It is exact at \( K_0(R) \) and \( R^{-1}K_1(R_0) \) by definition of the latter group. Thus it remains to verify exactness at the third entry.

So let \( (x, y) \in \text{ker} \overline{\gamma} \) be given. We can find finitely generated projective modules \( P \) and \( Q \) over \( R_{\leq 0} \) and \( R_{\geq 0} \), respectively, and numbers \( p, q \geq 0 \), such that
\[ x = [P] - [R^p_{\leq 0}] \in K_0(R_{\leq 0}) \quad \text{and} \quad y = [Q] - [R^q_{\geq 0}] \in K_0(R_{\geq 0}) \, . \]
The condition \( \overline{\gamma}(x, y) = 0 \) translates into the equality
\[ [P \otimes_{R_{\leq 0}} R] - [R^p] = [Q \otimes_{R_{\geq 0}} R] - [R^q] \]
9. Proof of the fundamental theorem

The $K$-theory of the projective line revisited. Recall from Theorem 5.4 that there are isomorphisms of $K$-groups

\[ \tilde{\alpha} = (\Psi_{-1,0} \Psi_{0,0}): K_q R_0 \oplus K_q R_0 \to K_q \mathbb{P}^1, \quad ([P],[Q]) \mapsto [\Psi_{-1,0}(P)] + [\Psi_{0,0}(Q)], \]

where $\Psi_{k,t}$ stands for the canonical sheaf functors (Definition 4.5). By pre-composing $\tilde{\alpha}$ with the invertible map $\left( \begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix} \right)$, we obtain modified isomorphisms

\[ \alpha = \left( - \Psi_{-1,0} + \Psi_{0,0} \Psi_{0,0} \right): K_q R_0 \oplus K_q R_0 \to K_q \mathbb{P}^1, \]

\[ ([P],[Q]) \mapsto -[\Psi_{-1,0}(P)] + [\Psi_{0,0}(P)] + [\Psi_{0,0}(Q)]. \quad (9.1) \]

(This map cannot be confused with the map $\alpha$ from (2.3).) As we have a cylinder functor at our disposal, the additivity theorem implies that we can model the minus sign by taking suitable “homotopy cofibres” of maps of functors. Concretely, the isomorphisms $\alpha$ are induced by the functor

\[ A: \text{Ch}^b \mathbb{P}(R_0) \times \text{Ch}^b \mathbb{P}(R_0) \to \text{Vect}(\mathbb{P}^1)_0, \]

\[ (C,D) \mapsto \text{cone}(\Psi_{-1,0}(C) \to \Psi_{0,0}(D)) \oplus \Psi_{0,0}(D) \]

and the ensuing homotopy equivalence of $K$-theory spaces

\[ \text{cone}(\Psi_{-1,0} \to \Psi_{0,0}) + \Psi_{0,0}: K(R_0) \times K(R_0) \to K(\mathbb{P}^1), \]

where “+” refers to the $H$-space structure given by direct sum. Alternatively, we can use the functor

\[ A': \text{Ch}^b \mathbb{P}(R_0) \times \text{Ch}^b \mathbb{P}(R_0) \to \text{Vect}(\mathbb{P}^1)_0, \]

\[ (C,D) \mapsto \Sigma \Psi_{-1,0}(C) \oplus \Psi_{0,0}(C) \oplus \Psi_{0,0}(D); \]

the maps induced by $A$ and $A'$ are homotopic, by the additivity theorem.

It will be convenient to have an explicit homotopy inverse for $\alpha$. We record the following fact:

Lemma 9.3. The functor

\[ \Xi: \text{Vect}(\mathbb{P}^1)_0 \to \text{Ch}^b \mathbb{P}(R_0) \times \text{Ch}^b \mathbb{P}(R_0), \quad \mathcal{V} \mapsto \left( \begin{array}{c} \Sigma \Gamma \mathcal{V}(1,0) \oplus \Gamma \mathcal{V} \oplus \Gamma \mathcal{V}(1,-1) \\ \Sigma \Gamma \mathcal{V}(1,-1) \oplus \Gamma \mathcal{V}(1,0) \end{array} \right) \]

induces a homotopy inverse on the level of $K$-theory spaces; here $\Gamma$ is the “global sections” functor from Definition 4.6.
Recalling that suspension represents the $\Xi$-spaces, this shows that the composition from Proposition 4.7, where $s_j P = P \otimes R_0 R_j$ is the $j$th shift of $P$, one calculates readily that the first component of $\Xi \circ A'(C, D)$ is
\[
\Sigma^2 \Psi_{0,0}(C) \oplus \Sigma \Gamma \Psi_{1,0}(C) \oplus \Sigma \Gamma \Psi_{1,0}(D) \\
\oplus \Sigma \Gamma \Psi_{-1,0}(C) \oplus \Gamma \Psi_{0,0}(C) \oplus \Gamma Y_{0,0}(D) \\
\oplus \Sigma \Gamma \Psi_{0,-1}(C) \oplus \Gamma \Psi_{1,-1}(C) \oplus \Gamma \Psi_{1,-1}(D)
\]
\[
\cong \Sigma^2 C \oplus \Sigma C \oplus \Sigma s_1 C \oplus \Sigma D \oplus \Sigma s_1 D \\
\oplus C \oplus D \\
\oplus s_1 C \oplus s_1 D,
\]
while the second component is
\[
\Sigma^2 \Psi_{0,-1}(C) \oplus \Sigma \Gamma \Psi_{1,-1}(C) \oplus \Sigma \Gamma \Psi_{1,-1}(D) \\
\oplus \Sigma \Gamma \Psi_{0,0}(C) \oplus \Gamma \Psi_{1,0}(C) \oplus \Gamma Y_{1,0}(D)
\]
\[
\cong \Sigma s_1 C \oplus \Sigma s_1 D \\
\oplus \Sigma C \oplus C \oplus s_1 C \oplus D \oplus s_1 D.
\]
Recalling that suspension represents the $H$-space structure inverse on $K$-theory spaces, this shows that the composition $\Xi \circ A'$ induces a map that is homotopic to the identity as required. \hfill \Box

**The modified Mayer-Vietoris sequence.** On a much more elementary level, we have automorphisms
\[
\eta = \begin{pmatrix} \text{id} & -i_\ast^{-1} p_j^1 \\ 0 & \text{id} \end{pmatrix} : K_0(R_{\leq 0}) \oplus K_0(R_{\geq 0}) \to K_0(R_{\leq 0}) \oplus K_0(R_{\geq 0}) \quad (q \geq 0)
\]
with inverse given by
\[
\eta^{-1} = \begin{pmatrix} \text{id} & i_\ast^{-1} p_j^1 \\ 0 & \text{id} \end{pmatrix} : K_0(R_{\leq 0}) \oplus K_0(R_{\geq 0}) \to K_0(R_{\leq 0}) \oplus K_0(R_{\geq 0}).
\]
Both maps are induced by functors, with the minus sign modelled by suspension (making implicit use of the additivity theorem and the presence of a cylinder functor again). Explicitly, $\eta$ is induced by the functor
\[
\text{Ch}_\ast^0 \mathbf{P}(R_{\leq 0}) \times \text{Ch}_\ast^0 \mathbf{P}(R_{\geq 0}) \to \text{Ch}_\ast^0 \mathbf{P}(R_{\leq 0}) \times \text{Ch}_\ast^0 \mathbf{P}(R_{\geq 0}) ,
\]
\[
(C, D) \mapsto (C \oplus \Sigma i_\ast^{-1} p_j^1 D, D).
\]
We use the map $\eta$ and its inverse to construct a modified **Mayer-Vietoris sequence**
\[
\ldots \xrightarrow{\gamma^{-1}} K_{q+1} R \xrightarrow{\alpha^{-1} \delta} K_q R_0 \oplus K_q R_0 \xrightarrow{\eta \delta} K_q R_{\leq 0} \oplus K_q R_{\geq 0} \xrightarrow{\gamma^{-1}} K_q R \xrightarrow{\alpha^{-1} \delta} \ldots ,
\]
(9.4)
ending with the exact sequence
\[ K_0 R_0 \oplus K_0 R_0 \xrightarrow{\eta \alpha} K_0 R_{\leq 0} \oplus K_0 R_{\geq 0} \xrightarrow{\gamma \eta^{-1}} K_0 R \xrightarrow{} R K_{-1} K_0 R \xrightarrow{} 0. \]

Note that any exact sequence of the form \( A \xrightarrow{f} B \xrightarrow{g} C \) gives rise to a short exact sequence
\[ 0 \xrightarrow{} A/\ker(f) \xrightarrow{} B \xrightarrow{} \im(g) \xrightarrow{} 0; \]
in this way, sequence (9.4) can be split into short exact sequences
\[ 0 \xrightarrow{} (K_q R_{\leq 0} \oplus K_q R_{\geq 0})/\ker(\gamma \eta^{-1}) \xrightarrow{\gamma \eta^{-1}} K_q R \xrightarrow{\im(\alpha^{-1} \delta)} \xrightarrow{} 0 \quad (q > 0) \]
and
\[ 0 \xrightarrow{} (K_0 R_{\leq 0} \oplus K_0 R_{\geq 0})/\ker(\gamma \eta^{-1}) \xrightarrow{\gamma \eta^{-1}} K_0 R \xrightarrow{} \xrightarrow{} R K_{-1} K_0 R \xrightarrow{} 0. \]

From exactness of the modified MAYER-VIETORIS sequence (9.4) again we have the equalities \( \ker(\gamma \eta^{-1}) = \im(\eta \alpha) \) and \( \im(\alpha^{-1} \delta) = \ker(\eta \beta \alpha) \), so we obtain short exact sequences
\[ 0 \xrightarrow{} \coker(\eta \beta \alpha) \xrightarrow{} K_q R \xrightarrow{} \ker(\eta \beta \alpha) \xrightarrow{} 0 \quad (q > 0) \quad (9.5) \]
and
\[ 0 \xrightarrow{} \coker(\eta \beta \alpha) \xrightarrow{} K_0 R \xrightarrow{} R K_{-1} K_0 R \xrightarrow{} 0. \quad (9.6) \]

The map \( \eta \beta \alpha \). On the level of categories the effect of the map \( \beta \alpha \) is to send \( (P, Q) \in \text{Ch}^i \text{P}(R_0) \times \text{Ch}^i \text{P}(R_0) \) to \( (C, D) \in \text{Ch}^i \text{P}(R_{\leq 0}) \times \text{Ch}^i \text{P}(R_{\geq 0}) \) where
\[
C = \text{cone}(P \otimes_{R_0} R_{\leq -1} \xrightarrow{\epsilon} P \otimes_{R_0} R_{\leq 0}) \oplus (Q \otimes_{R_0} R_{\leq 0}) = i^*_\epsilon \text{cone}(s_{-1} P \xrightarrow{} P) \oplus i^*_\epsilon Q
\]
and
\[
D = \text{cone}(P \otimes_{R_0} R_{\geq 0} \xrightarrow{=} P \otimes_{R_0} R_{\geq 0}) \oplus (Q \otimes_{R_0} R_{\geq 0}) = \approx Q \otimes_{R_0} R_{\geq 0} = i^*_\epsilon Q;
\]

here \( s_{-1} \) denotes the shift functor
\[ s_{-1} : Q \mapsto Q \otimes_{R_0} R_{-1} \]
which maps the category of finitely generated projective \( R_0 \)-modules to itself. —
To determine the effect of the map \( \eta \) on \( (C, D) \), recall first that weakly equivalent functors induce homotopic maps on \( K \)-theory spaces. So we can, for example, replace \( D \) by \( i^*_\epsilon Q \). The first component of \( \eta \beta \alpha(P, Q) \) then is, up to homotopy,
\[
i^*_\epsilon \text{cone}(s_{-1} P \xrightarrow{} P) \oplus i^*_\epsilon Q \oplus \Sigma i^*_\epsilon p^+_Q i^*_Q Q.
\]
Since \( p^+_Q i^*_Q Q = Q \), and since suspension represents a homotopy inverse for direct sum [Wal85, Proposition 1.6.2], we can thus model the effect of \( \eta \beta \alpha \) by the functor
\[
(P, Q) \mapsto (i^*_\epsilon \text{cone}(s_{-1} P \xrightarrow{} P), i^*_\epsilon Q).
\]

On the level of \( K \)-groups, this means that the effect of \( \eta \beta \alpha \) is described by the diagonal matrix
\[
\eta \beta \alpha = \begin{pmatrix} i^*_\epsilon (\id - s_{-1}) & 0 \\ 0 & i^*_\beta \end{pmatrix} : K_q R_0 \oplus K_q R_0 \xrightarrow{} K_q R_{\leq 0} \oplus K_q R_{\geq 0}. \quad (9.7)
\]
The kernel of $\eta\beta$. In view of (9.7), $\ker(\eta\beta) = \ker(i^-_s(s_{-1} - \text{id})) \oplus \ker(i^+_s)$.

Now the maps $i^-_s$ are split injective (with left inverse $p^+_s$), thus

\[
\ker(\eta\beta) \cong \ker(\text{sd}_*) \oplus \{0\} = K'_q R_0 \tag{9.8}
\]

where $\text{sd}_*$ denotes the shift difference map $\text{sd}_* = \text{id} - s_{-1}: K_q R_0 \longrightarrow K_q R_0$

from (1.4) with kernel $\ker K_q R_0$ (Definition 1.3).

The cokernel of $\eta\beta$. From the specific representation of $\eta\beta$ in (9.7) we read off that

\[
\text{coker}(\eta\beta) = \text{coker}(i^-_s \circ \text{sd}_*) \oplus \text{coker}(i^+_s),
\]

the second summand being nothing but $R\text{Nil}_q^+ R_0$ by definition. To identify the first summand we compose $i^-_s \circ \text{sd}_*$ with the splitting isomorphism (1.2):

\[
K_q(R_0) \xrightarrow{\text{sd}_*} K_q(R_0) \xrightarrow{i^-_s} K_q(R_{<0}) \xrightarrow{S = (c, p^-_s)} R\text{Nil}_q^- \oplus K_q(R_0)
\]

(Here $c: K_q(R_{\leq 0}) \longrightarrow R\text{Nil}_q^-$ is the canonical projection onto the cokernel of $i^-_s$.)

Thus $\text{coker}(i^-_s \circ \text{sd}_*) \cong \text{coker}(S \circ i^-_s \circ \text{sd}_*)$. Since $c \circ i^-_s = 0$, and since $p^-_s \circ i^-_s$ is the identity, we conclude that this group is isomorphic to the cokernel of the composition

\[
K_q R_0 \xrightarrow{\text{sd}_*} K_q R_0 \xrightarrow{i^-_s} \text{coker} R\text{Nil}_q^- \oplus K_q R_0,
\]

which is $R\text{Nil}_q^- \oplus \text{coker} R\text{Nil}_q^- \oplus K_q R_0$. In total, this results in an isomorphism

\[
\text{coker}(\eta\beta) \cong R\text{Nil}_q^- R_0 \oplus \text{coker} R\text{Nil}_q^- R_0 \oplus R\text{Nil}_q^+ R_0 \tag{9.9}
\]

Putting the identifications (9.8) and (9.9) of the kernel and cokernel of $\eta\beta$ into the sequences (9.5) and (9.6) gives precisely the advertised Fundamental Theorem 1.7. \hfill \Box

10. The localisation sequence

In this section we will establish the long exact “localisation” sequence of Theorem 1.9. There are in fact two “mirror-symmetric” versions, which we state in full for completeness.

**Theorem 10.1 (“Localisation sequence”).** Let $R$ be a strongly $\mathbb{Z}$-graded ring. There are long exact sequences of algebraic $K$-groups

\[
\ldots \longrightarrow K_{q+1} R \longrightarrow R\text{Nil}_q^+(R_0) \xrightarrow{\phi} K_q R_{\geq 0} \longrightarrow K_q R \tag{10.1a}
\]

and

\[
\ldots \longrightarrow K_{q+1} R \longrightarrow R\text{Nil}_q^-(R_0) \xrightarrow{\phi} K_q R_{\leq 0} \longrightarrow K_q R \tag{10.1b}
\]

with $\phi$ induced by the forgetful functor $(Z, \zeta) \mapsto Z$ on the category $R\text{Nil}_q^\pm(R_0)$. The groups $R\text{Nil}_q^\pm(R_0)$ are as defined in (6.17a) and (6.17b).
Proof. We will show that the sequence (10.1a) is exact; the argument for sequence (10.1b) is similar.

We apply Waldhausen’s fibration theorem to the right-hand vertical map in the fundamental square (7.1). In view of Lemma 7.9 this results in exact sequences of $K$-groups

\[ \cdots \to K_{q+1} \to K_q \to \pi_k\Omega h^+ \to \pi_k\Omega h^- \to K_{q-1} \to K_q \to \cdots \]

for $q \geq 0$. In view of Lemma 7.8 and Theorem 7.3, this proves exactness of the sequence (10.1a) in the $q \geq 1$ range down to the term $K_1 R_{\geq 0}$. Lemma 7.8 also states that in the commutative diagram in Fig. 2, the two vertical maps labelled $f$ and $g$

\[
\begin{array}{ccc}
K_1 R & \to & K_0 R \\
\downarrow & & \downarrow f \\
K_1 & \to & K_0
\end{array}
\]

are injective. As the bottom row is exact, the top row is automatically exact as well except possibly at $K_0 R_{\geq 0}$. Let $x \in K_0 R_{\geq 0}$ be an element with $j^+_1(x) = 0$. We can write $x = [P] - [Q]$, for finitely generated projective $R_{\geq 0}$-modules $P$ and $Q$. The condition $j^+_1(P) = j^+_1(Q) = j^+_1(x) = 0 \in K_0 R$, that is, $[P \oplus R_{\geq 0}] = [Q \oplus R_{\geq 0}] \in K_0 R$, means that there exists a number $\ell \geq 0$ together with an isomorphism

\[ j^+_1(P) \oplus R^\ell = (P \oplus R_{\geq 0}) \oplus R^\ell \cong (Q \oplus R_{\geq 0}) \oplus R^\ell = j^+_1(Q) \oplus R^\ell. \]

Let $P'$ be a complement of $P$ so that $P' \oplus P$ is a finitely generated free $R_{\geq 0}$-module. Then $j^+_1(P' \oplus P) \oplus R^\ell$ is a finitely generated free $R$-module, and

\[ j^+_1(P' \oplus P) \oplus R^\ell = j^+_1(P') \oplus j^+_1(P) \oplus R^\ell \cong j^+_1(P') \oplus j^+_1(Q) \oplus R^\ell = j^+_1(P' \oplus Q) \oplus R^\ell, \]

so that $j^+_1(P' \oplus Q) \oplus R^\ell$ is a finitely generated free $R$ module as well. Consequently, both $j^+_1(P' \oplus P)$ and $j^+_1(P' \oplus Q)$ stably extend to $R_{\leq 0}$; thus the two diagrams

\[ p = \begin{array}{ccc} j^+_1(P' \oplus P) & \to & P' \oplus P \\
\end{array} \]

\[ q = \begin{array}{ccc} j^+_1(P' \oplus Q) & \to & P' \oplus Q \\
\end{array} \]

are objects of $D_+$. Let us consider the element $z = [p] - [q] \in K_0 D_+$. We calculate

\[ f(z) = f([p] - [q]) = [P' \oplus P] - [P' \oplus Q] = [P] - [Q] = x \]

(see Lemma 7.8 for the effect of $f$). As $x \in \ker(j^+_1)$, and as $g$ is an injection, this implies $z \in \ker(j^+_1)$, and by exactness of the lower horizontal sequence we infer that $z = k'(y)$ for some $y \in \pi_0\Omega h^+ \to \pi_0\Omega h^- \to K_{q-1} \to K_q \to \cdots$. This proves the top row to be exact at $K_0 R_{\geq 0}$, and establishes together with Theorem 7.3 the tail end of the long exact sequence.

It remains to identify the map $\phi$ in the sequence (10.1a). To this end, let $\text{FD}(R_{\geq 0})$ denote the category of bounded complexes of $R_{\geq 0}$-modules which are
quasi-isomorphic to a bounded complex of finitely generated projective \( R_{\geq 0} \)-modules, and consist of projective \( R_0 \)-modules. The inclusion functor
\[
\mathrm{Ch}^\oplus P(R_{\geq 0}) \longrightarrow \mathcal{FD}(R_{\geq 0})
\]
is exact and yields isomorphisms on algebraic \( K \)-groups (with respect to weak equivalences the quasi-isomorphisms, and cofibrations the monomorphisms with levelwise \( R_0 \)-projective cokernel) as can be checked with the help of Waldhausen’s approximation theorem. This inclusion is the right hand vertical map in the square diagram of Fig. 3. The top horizontal arrow maps the complex of vector bundles \( \mathbb{Z} \) to its component \( \mathbb{Z}^+ \). The left-hand vertical map is the one discussed in Theorem 7.3, identifying the group \( \pi_q \Omega|_{h_0} \mathbb{P}^1 \mathrm{Vect}(\mathbb{P}^1)^{h_0} \) with \( R\mathbb{Nil}_q^+(R_0) \); it is given by the assignment
\[
\mathbb{Z} = (\mathbb{Z}^+ \rightarrow \mathbb{Z}_0 \xleftarrow{\zeta^+} \mathbb{Z}^+) \mapsto (\mathbb{Z}^+, \zeta)
\]
with \( \zeta \) the composition \( \mathbb{Z}^+ \otimes_{R_0} R_1 \cong \mathbb{Z}^+ \otimes_{R_{\geq 0}} R_{\geq 1} \rightarrow \mathbb{Z}^+ \otimes_{R_{\geq 0}} R_{\geq 0} \cong \mathbb{Z}^+ \otimes_{R_0} R_0 \).

Note that \( \mathbb{Z}^+ \) consists of projective \( R_0 \)-modules since \( R \) is strongly \( \mathbb{Z} \)-graded, that \( \mathbb{Z}^+ \) is \( R_0 \)-finitely dominated by Lemma 6.13, and that \( \zeta \) is homotopy nilpotent by Corollary 6.12. Using the functor \( (Z, \zeta) \mapsto Z^\zeta \) in the lower horizontal position renders the square diagram commutative; the complex \( Z^\zeta \) is indeed an object of \( \mathcal{FD}(R_{\geq 0}) \) by Remark 6.15. The induced map on algebraic \( K \)-groups is the map \( \phi \) occurring in the sequence (10.1a). — The proof is complete. \( \Box \)

11. The \( K \)-theory of homotopy nilpotent twisted endomorphisms

The nil groups \( R\mathbb{Nil}_q^\pm R_0 \) can be interpreted as obstruction groups for finitely generated projective modules over \( R_{\leq 0} \) or \( R_{\geq 0} \), respectively, to be stably induced from \( R_0 \) (Proposition 2.4). For \( q > 0 \) the groups \( R\mathbb{Nil}_q^\pm R_0 \) are isomorphic to the (reduced) algebraic \( K \)-groups of the category of nilpotent twisted endomorphisms, as will be shown in this section. This will also complete the proof of Theorem 1.9.

**Theorem 11.1 (Nil groups and \( K \)-theory of nilpotent endomorphisms).** For \( q > 0 \) there are natural isomorphisms of algebraic \( K \)-groups
\[
R\mathbb{Nil}_q^\pm(R_0) \xrightarrow{\cong} \ker \left( o^\pm : R\mathbb{Nil}_{q-1}^\pm R_0 \longrightarrow K_{q-1} R_0 \right)
\]
and
\[
R\mathbb{Nil}_q^-(R_0) \xrightarrow{\cong} \ker \left( o^- : R\mathbb{Nil}_{q-1}^+ R_0 \longrightarrow K_{q-1} R_0 \right),
\]
with the groups \( R\mathbb{Nil}_{q-1}^\pm R_0 \) from (6.17a) and (6.17b), and maps \( o^\pm \) as introduced in (6.10) induced by the forgetful functors \( o^\pm : (Z, \zeta) \mapsto Z \).
Proof. Upon taking the homotopy fibre of the top horizontal map in the fundamental square (7.1) we obtain a long exact sequence of $K$-groups containing the snippet

\[ K_q \mathbb{P}^1 \longrightarrow K_q \mathbb{D}_+ \xrightarrow{\nabla} K_{q-1} \text{Vect}(\mathbb{P}^1)^{h+} \xrightarrow{\epsilon} K_{q-1} \mathbb{P}^1 \longrightarrow K_{q-1} \mathbb{D}_+, \]

with $\epsilon$ induced by the inclusion functor $\text{Ch}^0 \text{Vect}(\mathbb{P}^1)^{h+} \longrightarrow \text{Ch}^0 \text{Vect}(\mathbb{P}^1)$; the symbol $\nabla$ stands for a connecting homomorphism in the long exact sequence of homotopy groups associated with the fibration. — Using the isomorphism $\alpha : K_q R_0 \oplus K_q R_0 \longrightarrow K_q \mathbb{P}^1$ from (9.1), and with $K_k(R_{\geq 0})$ in place of $K_k \mathbb{D}_+$ (Lemma 7.8), we obtain the exact sequence

\[ K_q R_0 \oplus K_q R_0 \xrightarrow{\epsilon} K_q R_{\geq 0} \xrightarrow{\nabla} K_{q-1} \text{Vect}(\mathbb{P}^1)^{h+} \]

\[ \xrightarrow{\alpha^{-1}} K_{q-1} R_0 \oplus K_{q-1} R_0 \xrightarrow{\epsilon} K_{q-1} R_{\leq 0}. \]

The map $\epsilon$ is the difference of maps induced by the functors

\[(P, Q) \mapsto (P \otimes R_0 R_{\geq 0}) \oplus (Q \otimes R_0 R_{\geq 0}) \quad \text{and} \quad (P, Q) \mapsto P \otimes R_0 R_{\geq 0}, \]

thus $\epsilon$ is the usual split injection induced from $i^+ : Q \mapsto Q \otimes R_0 R_{\geq 0}$ on the second summand, and is the zero map on the first. Hence the image of the map $\alpha^{-1} i$, which equals the kernel of $\epsilon$, is $K_{q-1} R_0 \oplus \{0\}$, and there results another exact sequence

\[ 0 \longrightarrow \{0\} \oplus K_q R_0 \xrightarrow{i^+} K_q R_{\geq 0} \xrightarrow{\nabla} K_{q-1} \text{Vect}(\mathbb{P}^1)^{h+} \xrightarrow{pr_1 \alpha^{-1}} K_{q-1} R_0 \longrightarrow 0. \]

By Theorem 7.3 we have an isomorphism $\omega : K_{q-1} \text{Vect}(\mathbb{P}^1)^{h+} \cong K_{q-1} ^{\text{RNil}^-}(R_0)$. — The composition $pr_1 \alpha^{-1} \omega^{-1}$ is induced

by the forgetful functor $o^{-} : (Y, v) \mapsto Y$. (11.2)

Assuming the claim for the moment, we obtain an exact sequence

\[ K_q R_0 \xrightarrow{i^+} K_q R_{\geq 0} \xrightarrow{\omega \nabla} K_{q-1} ^{\text{RNil}^-} R_0 \xrightarrow{o^{-}} K_{q-1} R_0, \]

giving the isomorphism

\[ ^{\text{RNil}^+} K_q = \text{coker} i^+ = K_q R_{\geq 0} / \ker(\omega \nabla) \cong \text{im}(\omega \nabla) = \ker(o^{-}). \]

This establishes (11.1a). The isomorphism of (11.1b) is verified using a symmetric argument, employing the isomorphism

\[ K_q R_0 \oplus K_q R_0 \longrightarrow K_q \mathbb{P}^1, \]

\[ ([P], [Q]) \mapsto [\Psi_{0,0}(P)] - [\Psi_{0,-1}(P)] + [\Psi_{0,0}(Q)], \quad \text{in place of } \alpha. \]

It remains to verify Claim (11.2). By Lemma 9.2 the isomorphism $\alpha^{-1}$ is induced by the functor

\[ \Xi : \text{Vect}(\mathbb{P}^1)_{0} \longrightarrow \text{Ch}^0 \mathbb{P}(R_0) \times \text{Ch}^0 \mathbb{P}(R_0), \quad Y \mapsto \left( \Sigma \Gamma \mathbb{Y}(1,0) \oplus \Gamma \mathbb{Y} \oplus \Gamma \mathbb{Y}(1,-1) \right) \]

\[ \quad \left( \Sigma \Gamma \mathbb{Y}(1,-1) \oplus \Gamma \mathbb{Y}(1,0). \right) \]

Now consider the diagram in Fig. 4. It is not a commutative diagram of functors, but on the level of $K$-theory spaces it is homotopy commutative. Since all vertical arrows marked as inclusion maps induce identity maps on $K$-groups, and since $\omega$ induces an isomorphism on $K$-groups, this establishes Claim (11.2). — The vertical arrows $a$ and $b$ result in homotopy equivalences on $K$-theory spaces with
respect to $h$-equivalences; this is analogous to Lemma 5.3), and the arguments of Lemma III.1.3 and Corollary III.1.4 of [HM18] carry over verbatim. For the arrow $i$ see (6.18). As to the alleged homotopy commutativity, recall that for \( Z \in \text{Vect}(\mathbb{P}^1)_0 \) we have \( \lim_{\leftarrow} Z(k, \ell) = H^1 Z(k, \ell) = 0 \) when \( k + \ell \geq 0 \), by definition of the category \( \text{Vect}(\mathbb{P}^1)_0 \); consequently, the map \( \Gamma Z(k, \ell) \rightarrow \text{holim} Z(k, \ell) \) is a quasi-isomorphism when \( k + \ell \geq 0 \), where “holim” stands for the homotopy limit or homotopy pullback construction, which is the dual of the double mapping cylinder. Furthermore, for \( Z \in \text{Vect}(\mathbb{P}^1)^{h+}_0 \) we have \( Z(k, \ell)^{+} = Z^{+} \otimes_{R \geq 0} R_{\geq -\ell} \simeq * \) and \( Z(k, \ell)^0 = Z^{0} \otimes_{R} R \simeq * \), which for \( k + \ell \geq 0 \) implies
\[
\Gamma Z(k, \ell) \simeq \text{holim} \left( Z^{-} \otimes_{R \leq 0} R_{\leq k} \rightarrow Z^{0} \otimes_{R} R \leftarrow Z^{+} \otimes_{R \geq 0} R_{\geq -\ell} \right) \simeq \text{holim} \left( Z^{-} \otimes_{R \leq 0} R_{\leq k} \rightarrow * \leftarrow * \right) \simeq Z^{-} \otimes_{R \leq 0} R_{\leq k}.
\]
Thus the composition \( i \circ \text{pr}_1 \Xi \), that is, the functor that sends \( Y \in \text{Vect}(\mathbb{P}^1)^{h+}_0 \) to \( \Sigma \Gamma Y(1, 0) \oplus \Gamma Y \oplus \Gamma Y(1, -1) \), is weakly equivalent to the functor that sends \( Y \) to
\[
(\Sigma Y^{-} \otimes_{R \leq 0} R_{\leq 1}) \oplus Y^{-} \oplus (Y^{-} \otimes_{R \leq 0} R_{\leq 1})
\]
by the additivity theorem, the induced map on $K$-theory spaces is homotopic to the map induced by \( Y \mapsto Y^{-} \) since suspension models an inverse for the $H$-space structure. But the formula \( Y \mapsto Y^{-} \) describes the effect of \( o^{-} \circ \omega \) as well. This finishes the proof. \hfill \Box

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Thomas Hüttemann, Queen’s University Belfast, School of Mathematics and Physics, Mathematical Sciences Research Centre, Belfast BT7 1NN, UK

\textit{E-mail address}: t.huettemann@qub.ac.uk

\textit{URL}: https://t-huettemann.github.io/

\textit{URL}: https://pure.qub.ac.uk/en/persons/thomas-huettemann