The Berkovits Method for Conformally Invariant Non-linear $\sigma$-Models on $G/H$

Shogo Aoyama*
Department of Physics
Shizuoka University
Ohya 836, Shizuoka
Japan
March 27, 2022

Abstract

We discuss 2-dimmensional non-linear $\sigma$-models on the Kähler manifold $G/H$ in the first order formalism. Using the Berkovits method we explicitly construct the $G$-symmetry currents and primaries, when $G/H$ are irreducible. It is a variant of the Wakimoto realization of the affine Lie algebra using a particular reducible Kähler manifold $G/U(1)^r$ with $r$ the rank of $G$.

PACS: 02.20.Tw, 11.25.Hf, 11.30.Na
Keywords: Kähler coset space, Conformal field theory, Affine Lie algebra

*e-mail: spsaoya@ipc.shizuoka.ac.jp
The Berkovits formalism for the superstring\cite{1} is expected to be a new way which overcomes the long-standing problem of the Rammond-Neveu-Schwarz(RNS) and Green-Schwarz(GS) formalisms. The right-moving contribution of the GS superstring action was put in the linealized form

\[ S = \int d^2z (\frac{1}{2} \partial x^m \partial x_m + \rho_\alpha \partial \theta^\alpha). \]  

(1)

After the Wick rotation the \( SO(10) \) currents are given by \( M^{mn} = \frac{1}{2} \rho \gamma^{mn} \) in this formalism, while they are given by \( \hat{M}^{mn} = \psi^m \psi^n \) in the RNS formalism. There is a crucial difference between the OPE’s of both currents. To cope with this discrepancy Berkovits added a \( bc \) system\cite{2} taking the form

\[ S' = \int d^2z (v_{ab} \partial u_{ab} + \beta \partial \gamma). \]  

(2)

Here \( u_{ab} \) are the coordinates parametrizing the coset space \( SO(10)/U(5) \) and \( \gamma \) is a bosonic ghost. \( v_{ab} \) and \( \beta \) are their canonical conjugate momenta for the respective quantity. He constructed new \( SO(10) \) currents \( N^{mn} \) by using this \( bc \) system, so that the modified \( SO(10) \) currents \( M'^{mn} = \frac{1}{2} \rho \gamma^{mn} \theta + N^{mn} \) have the same OPE’s as \( \hat{M}^{mn} = \psi^m \psi^n \). Moreover the combined theory given by \( S + S' \) is free of conformal anomaly. Quantization of the superstring is done by studying the BRST cohomology. To this end the pure spinor \( \lambda^\alpha \in 16 \) of \( SO(10) \) satisfying

\[ \lambda \gamma^m \lambda = 0 \]  

(3)

plays an essential role. Decomposing \( 16 \) under \( U(5) \) as \( 1 + 10 + 5 \) we can solve this equation\cite{1, 3} by using the fields of the \( bc \) system (2) as

\[ \lambda^\alpha = \begin{pmatrix} \gamma \\ \gamma_{u_{ab}} \\ -\frac{1}{8} \gamma e_{abcde} u_{bc} u_{de} \end{pmatrix}. \]  

(4)

The OPE’s \( N^{mn}(z) \lambda^\alpha (w) \) yield the correct \( SO(10) \) algebra only if the bosonic ghosts are fermionized as \( \beta = \partial \xi e^{-\varphi} \) and \( \gamma = \eta e^\varphi \).

In short, the point of the Berkovits formalism is to construct the currents \( N^{mn} \) of weight 1 and the primaries \( \lambda^\alpha \) of weight 0, belonging to \( 45 \) and \( 16 \) of \( SO(10) \) respectively, by using the \( bc \) system which manifests the \( U(5) \) symmetry alone. We note that \( SO(10)/U(5) \) is a Kähler coset space. The construction may be generalized with the general Kähler coset space \( G/H \), though our concern deviates from the central issue of the formalism about the superstring. It is a variant of the free field realization of the WZWN model with \( G \) symmetry, which is called the Wakimoto realization of the affine Lie algebra of \( G \). The Wakimoto realization itself was well studied by many people\cite{4, 5}. Nonetheless in this letter we pursue the study using the Berkovits method for the following reasons. Firstly in \cite{4, 5} the affine Lie algebra of \( G \) is realized by using the \( bc \) system

\[ S'' = \int d^2z \left( \sum_{\alpha \in \Delta_+} p_\alpha \partial q^\alpha + \sum_{i=1}^r \varphi^i \right), \]  

(5)
in which $\Delta_+$ denotes the set of positive roots for the Lie algebra and $r$ is the rank of $G$. The fields $q^\alpha$ parametrize the Kähler coset space $G/U(1)^r$. Instead we use the $bc$ system (2) in an extended form so as to parametrize the general Kähler coset space $G/H$

Secondly in [4, 5] the explicit formulae for the $G$-symmetry currents were given for those corresponding to the positive or negative simple roots, but for other currents they were yet implicit. Thirdly the $G$-symmetry primaries were not discussed in [4]. In [5] they were discussed, but the construction was based on $G/U(1)^r$ and not so explicit as (4). The final reason of the study is that we consider the $bc$ system (2) or (5) as a non-linear $\sigma$-model on $G/H$ formulated in the first order. Then we can see a close relationship between the $G$-symmetry currents and the Killing potential which exists for the general Kähler coset space $G/H$. The Kähler geometry underlying in the the affine Lie algebra of $G$ is different depending on which subgroup $H$ is taken.

The aim of this letter is to give an explicit and simple construction of the $G$-symmetry currents and primaries, by parametrizing the Kähler coset space $G/H$ in the case where it is irreducible. As for the $G$-symmetry primaries we are interested in those of weight 0. It suffices to find them in the fundamental representation (or the spinorial representation for $G = SO(n)$). Then those in any other representation can be constructed by tensoring them. We would like to stress on the fact that the resulting primaries satisfy the $G$-symmetry algebra only if the bosonic ghosts are fermionized according to Berkovits. At the end of the letter it will be pointed out that the Kähler geometry of the affine Lie algebra is useful to study the non-commutative geometry.

We start with a brief summary of the Kähler geometry. Consider a real $2N$-dimentional symplectic manifold $\mathcal{M}$ with local coordinates $(x^1, x^2 \cdots, x^{2N})$. The line-element and the symplectic 2-form respectively given by

$$ds^2 = \frac{1}{2} g_{ij} dx^i dx^j, \quad \omega = \frac{1}{2} \omega_{ij} dx^i \wedge dx^j. \quad (6)$$

We write the world-sheet action of a non-linear $\sigma$-model on $\mathcal{M}$ as

$$S = \frac{1}{2} \int d^2 \xi [\eta^{ab} g_{ij} + \varepsilon^{ab} \omega_{ij}] \partial_a x^i \partial_b x^j. \quad (7)$$

$\mathcal{M}$ is a Kähler manifold if the 2-form $\omega$ is closed and there exists a complex structure $J^i_j$ such that $J^i_j J^j_k = -\delta^i_k$ and $\omega_{ij} = g_{ik} J^k_j$. We locally set $J^i_j$ to be

$$J^i_j = \begin{pmatrix} -i\delta^\alpha_\beta & 0 \\ 0 & i\delta^{\bar{\alpha}}_{\bar{\beta}} \end{pmatrix}. $$

Then (6) and (7) are reduced to

$$ds^2 = g_{\alpha\bar{\beta}} dx^\alpha dx^\bar{\beta}, \quad \omega = ig_{\alpha\bar{\beta}} dq^\alpha \wedge dq^\bar{\beta}, \quad (8)$$

and

$$S = \int d^2 z g_{\alpha\bar{\beta}} \partial q^\alpha \bar{\partial} q^\bar{\beta}. \quad (9)$$

3
in which \( q^\alpha \) and \( q^{\bar{\alpha}} \) are complex coordinates obtained by complexifying \( x^i \) by the projectors \((1 \pm J)^i_\alpha\) respectively. The closure of \( \omega \) given by (8) implies the existence of a Kähler potential \( K \) such that

\[
g_{\alpha\bar{\beta}} = \frac{\partial^2 K}{\partial q^\alpha \partial q^{\bar{\beta}}}.
\]

If the Kähler manifold \( \mathcal{M} \) is a coset space \( G/H \), there exists a set of holomorphic Killing vectors \( R^A\alpha \) satisfying

\[
\mathcal{L}_{R^A} R^{B\alpha} = f^{ABC} R^{C\alpha},
\]

(10)

\[
\mathcal{L}_{R^A} g_{\alpha\bar{\beta}} = 0.
\]

(11)

Here \( \mathcal{L}_{R^A} \) is the Lie-derivative with respect to \( R^A\alpha \) and \( f^{ABC} \) are the structure constants of the symmetry group \( G \). Owing to (11) the action (9) is invariant under \( \delta q^\alpha = \epsilon^A R^A\alpha \) and \( \delta q^{\bar{\alpha}} = \epsilon^A R^{\bar{\alpha}} \) with infinitesimal global parameters \( \epsilon^A \) of the \( G \)-symmetry. But it is not conformally invariant. This model may be put in the first order formalism as

\[
S = \int d^2 z [p_\alpha \delta q^\alpha + \text{c.c.}],
\]

(12)

by setting \( g_{\alpha\bar{\beta}} \partial q^{\bar{\beta}} \) to be a world-sheet vector \( p_\alpha \). Then the action (12) is conformally invariant. It has also the \( G \)-symmetry under \( \delta q^\alpha = \epsilon^A R^A\alpha \) together with

\[
\delta p_\alpha = -\epsilon^A p_\beta R^{A\beta}_\alpha \left( \equiv -\epsilon^A p_\beta \frac{\partial R^{A\beta}}{\partial q^\alpha} \right).
\]

The Noether currents for the \( G \)-symmetry take the form

\[
J^A = p_\alpha R^{A\alpha}, \quad \bar{J}^A = p_\alpha R^{\bar{A}\alpha}.
\]

(13)

Owing to (10) they transform as the adjoint representation of \( G \):

\[
\delta J^A = \epsilon^A f^{ABC} J^C,
\]

by the \( G \)-symmetry transformations \( \delta q^\alpha \) and \( \delta p_\alpha \) above mentioned. This classical argument no longer holds at the quantum level. Namely, if the \( G \)-symmetry is correctly realized at the quantum level, with the free field OPE

\[
p_\alpha(z) q^\beta(w) \sim \delta^\beta_\alpha \frac{1}{z-w},
\]

(14)

one should check that

\[
J^A(z) J^B(w) \sim \frac{f^{ABC} J^C(w)}{z-w} + \frac{k g^{AB}}{(z-w)^2} + O((z-w)^{-2}).
\]

(15)

Here \( k \) is some constant and \( g^{AB} \) is the Killing metric of the symmetry group \( G \). But the OPE of the currents (13) takes the form

\[
J^A(z) J^B(w) \sim \frac{f^{ABC}}{z-w} + \frac{R^{A\alpha}\beta(z) R^{B\bar{\alpha}}(w)}{(z-w)^2} + O((z-w)^{-2}),
\]

4
with the simplified notation $R^{A\alpha}(z) \equiv R^{A\alpha}(q(z))$. In general we have neither
\[ \lim_{z \to w} R^{A_{,\beta}(z) R^{B_{,\beta}}(w)} \neq k\delta^{AB} \quad \text{nor} \quad \lim_{z \to w} R^{A_{,\beta}(z) \partial q^\gamma(z) R^{B_{,\beta}}(w)} \neq 0, \]
so that the $G$-symmetry is broken at the quantum level. To recover the $G$-symmetry we use the Berkovits method. Namely we generalize the action (12) introducing the bosonic ghosts fields $\beta$ and $\gamma$
\[ S = \int d^2z [p_{\alpha} \bar{\partial} q^\alpha + \beta \bar{\partial} \gamma + c.c.]. \]
We modify the $G$-symmetry currents by adding terms of weight $(1, 0)$ as
\[ J^A = p_{\alpha} R^{A\alpha} + F^A_{,\beta} \gamma + G^A_{,\alpha} \partial q^\alpha. \tag{16} \]
Here $F^A$ and $G^A$ are holomorphic functions of $q^\alpha$. The question is whether they can be determined so that the modified currents satisfy the algebra (15). Let us check it using (14) and
\[ \beta(z) \gamma(w) \sim \frac{1}{z - w}. \]
We then find that
\[ J^A(z) J^B(w) \sim \frac{\Lambda^{AB}(w)}{z - w} + \frac{\Theta^{AB}(z, w)}{(z - w)^2} + O((z - w)^{-2}), \]
in which
\[ \Lambda^{AB}(z) = f^{ABC} p_{\alpha} R^{C_{,\alpha}} + R^{A\alpha} F_{,\alpha}^B - R^{B_{,\alpha}} F_{,\alpha}^A, \]
\[ \Theta^{AB}(z, w) = R^{A_{,\beta}(z) R^{B_{,\beta}}(w)} + F^A(z) F^B(w) + \{R^{A_{,\alpha}}(z) G^B_{,\alpha}(w) + G^A_{,\alpha}(z) R^{B_{,\alpha}}(w)\}. \]
The condition for this to satisfy (15) is
\[ R^{A\alpha} F_{,\alpha}^B - R^{B_{,\alpha}} F_{,\alpha}^A = f^{ABC} F^A, \tag{17} \]
\[ \lim_{z \to w} \Theta^{AB}(z, w) = k g^{AB}, \tag{18} \]
\[ \lim_{z \to w} \frac{\partial \Theta^{AB}(z, w)}{\partial q(z)^\alpha} = f^{ABC} G^C_{,\alpha}(w). \tag{19} \]
We may identify the holomorphic functions $F^A$ with those appearing in the Lie-variation of the Kähler potential
\[ \mathcal{L}_{R^A K} = F^A + c.c., \tag{20} \]
modulo a multiplicative constant. Such functions indeed satisfy the condition (17) for the Kähler coset space in general[6]. As for the other holomorphic functions $G^A$, we propose that
\[ G^A_{,\alpha} \propto \delta^A_{\alpha}. \tag{21} \]
We now show that the conditions (18) and (19) are also satisfied when the Kähler coset space $G/H$ is irreducible.

The irreducible Kähler coset space $G/H$ is defined as follows. The generators of $G$ are decomposed as

$$\{X^\alpha, \bar{X}_\alpha, H^i, Y\},$$

in which $X^\alpha$ and their conjugates $\bar{X}_\alpha$ are coset generators and $Y$ is a $U(1)$ generator. Then $X^\alpha (\bar{X}_\alpha)$ belong to an irreducible representation under the subgroup $H$ generated by $H^i$ and $Y$. Such a Kähler coset space is called the hermitian symmetric space and is characterized by the Lie algebra of the form

$$[X^\alpha, \bar{X}_\beta] = t (\Sigma^i)^\alpha_\beta H^i, \quad [X^\alpha, X^\beta] = 0,$$
$$[H^i, X^\alpha] = (\Sigma^i)^\alpha_\beta X^\beta, \quad [Y, X^\alpha] = X^\alpha, \quad \text{c.c.}, \quad (22)$$

with some constants $t$ and $s$ depending on the representation of $G$. The local coordinates of the coset space $q^\alpha$ and $\bar{q}^\alpha$ correspond to the generators $X^\alpha$ and $\bar{X}_\alpha$ respectively. From now on we change the notation of $\bar{q}^\alpha$ as $\bar{q}^\alpha$ in accordance with that of $X^\alpha$. Therefore raising or lowering the tensor indices should be done by writing the metrics $g^\beta_\alpha$ or $(g^{-1})^\beta_\alpha$ explicitly. Simple algebra gives

$$[X^\alpha, [X^\beta, \bar{X}_\gamma]] = -\{t (\Sigma^i)^\gamma_\beta (\Sigma^i)^\delta_\alpha + s \delta^\alpha_\beta \delta^\gamma_\delta \} X^\delta \equiv M^\alpha_\beta X^\delta. \quad (23)$$

The quantity $M^\alpha_\beta$ plays a key role in the method and has a remarkable property. It is summarized by the statement that $M^\alpha_\beta$ is completely symmetric in the indices $(\alpha_1, \alpha_2, \ldots, \alpha_n, \beta_1, \ldots, \beta_n)$, whenever it is completely symmetrized in the indices $(\gamma_0, \gamma_1, \ldots, \gamma_n)$, and vice versa. For the case of $n = 1$ we have $M^\alpha_\beta = M^\beta_\alpha = M^\beta_\gamma$. With this quantity the Killing vectors $R^{A\alpha}$ are given[7] by

$$R^\gamma_\alpha = i \delta^\gamma_\alpha, \quad R^\alpha_\alpha = \frac{i}{2} M^\alpha_\beta q^\beta \bar{q}^\gamma, \quad R^\alpha_i = i (\Sigma^i)^\alpha_\beta q^\beta, \quad R^\gamma = iq^\gamma. \quad (24)$$

The Kähler potential is also given by

$$K = \frac{1}{Q} \log(1 + Q)q,$$

with $Q^\alpha_\beta = -\frac{1}{2} M^\beta_\alpha q^\gamma \bar{q}^\delta[8]$. Then the Lie-variation of the Kähler potential (20) takes the form

$$\varepsilon^A \mathcal{L}_R^A K = i \varepsilon_\alpha q^\alpha - i \varepsilon^\alpha \bar{q}_\alpha.$$

From this we can find the holomorphic functions $F^A$ in (16) to be

$$F^A \equiv \{F^\alpha, \bar{F}_\alpha, F^i, F\} \propto \{iq^\alpha, 0, 0, ir\}. \quad (25)$$
Here $r$ is a real constant. There is a classical argument to determine this $U(1)$ part for the general Kähler coset space[9]. But at the quantum level we let it be arbitrary as suggested by the Berkovits method[1]. With (21), (24) and (25) the currents $J^A \equiv \{J^\alpha, J_\alpha, J^i, J^j\}$, proposed by (16), take the form

$$J^\alpha = \frac{i}{2} M^\alpha_{\beta\gamma} q^\beta p^\gamma + A q^\alpha \beta \gamma + C \partial q^\alpha,$$

$$J = i q p + B \beta \gamma,$$

$$\bar{J}_\alpha = i p_\alpha \quad J^i = i p^\Sigma q.$$  

(26)

Here $A$, $B$ and $C$ are constants to be determined from the current algebra (15). We have checked that these currents indeed satisfy (15) with

$$A = -s B, \quad k = \frac{C}{t} = s N - \frac{1}{s} A^2, \quad tr \Sigma^i \Sigma^j = k \delta^{ij},$$

and the Killing metric $g^{AB}$ read from the Casimir of the Lie algebra (22)

$$\bar{X}_\alpha X^\alpha + X^\alpha \bar{X}_\alpha + t H^i H^i + s Y^2.$$  

Now we shall find a set of primaries $\phi^I$ with conformal weight 0 which transforms as the fundamental representation of $G$

$$J^A(z)\phi^I(w) \sim \frac{f^{AI} \phi^I(w)}{z-w}.  \quad (27)$$

Primaries in any other representation can be constructed by tensoring them. Our conjecture for $\phi^I$ is that

$$\phi^I = \begin{pmatrix} \gamma \phi^{a_1} \\ \gamma \phi^{a_2} \\ \vdots \\ \gamma \phi^{a_n} \end{pmatrix},$$

(28)

in which $\phi^{a_i}$’s are holomorphic functions $q^a$ and irreducible components in the decomposition under the subgroup $H$:

$$J^i(z)\phi^a(w) \sim \frac{f^{ia} \phi^a(w)}{z-w}.$$  

(27)

It is a homogeneous part of the algebra (27). Such irreducible components can be easily constructed as $J^i$ is given linearly in $p_a$ and $q^a$. But they fail to satisfy the coset part of the algebra (27). Here we again have recourse to the Berkovits method. We fermionize the ghost pair as $\beta = \partial \xi e^{-\varphi}$ and $\gamma = \eta e^\varphi[2]$ with

$$\xi(z)\eta(w) \sim \frac{1}{z-w}, \quad \varphi(z)\varphi(w) \sim -\log(z-w).$$  

7
In (26) and (26) we replace the quantity $\beta \gamma$ by
\[ \beta \gamma \Rightarrow a \xi \eta + b \partial \varphi, \]
with the constants obeying
\[ a^2 - b^2 = -1. \] (30)
Then the replacement does not change the OPE
\[ \beta \gamma(z) \cdot \beta \gamma(w) \sim -\frac{1}{(z-w)^2}. \]
Consequently the $G$-symmetry currents algebra equivalently holds because $\beta \gamma$ in the currents (26) takes part in the algebra through this OPE. However in the algebra (27) it does through $\beta \gamma(z) \cdot \gamma(w)$. The fermionization changes this OPE as
\[ \beta \gamma(z) \cdot \gamma(w) \Rightarrow (a \xi \eta + b \partial \varphi) (z) \cdot \eta e^\varphi (w) \sim -(a + b) \frac{\gamma(w)}{z-w}. \] (31)
Note that the condition (30) leaves a freedom still to fix the factor $a + b$ in this equation.

We claim that the algebra (27) with $J^A = J^\alpha, J^\beta$ is recovered by choosing it appropriately. Indeed Berkovits constructed the $SO(10)$ pure spinor $\lambda^\alpha$ on the Kähler coset space $SO(10)/U(5)$ in the form (4) by using this fermionization of $\beta$ and $\gamma$.[1]

In this paper we give other examples to support the conjecture (28). The first example is the coset space $SU(n+1)/\{SU(n) \otimes U(1)\}$[10]. The generators of $SU(n+1)$ are decomposed as $\{X^\alpha, \bar{X}^\alpha, H^\alpha_\beta, Y\}$
\[ \{X_I, J\} = \{X^\alpha, \bar{X}^\alpha, X^\alpha_\beta, X^\alpha_n\} \equiv \{X^\alpha, \bar{X}^\alpha, H^\alpha_\beta, Y\}. \] (32)
The Lie algebra takes the form
\[ [X^\alpha, \bar{X}^\beta] = H^\alpha_\beta - (1 + \frac{1}{n}) \delta^\alpha_\beta Y, \quad [X^\alpha, X^\beta] = 0, \]
\[ [H^\alpha_\beta, X^\gamma] = \delta^\alpha_\beta X^\gamma - \frac{1}{n} \delta^\alpha_\gamma X^\beta, \quad [Y, X^\alpha] = -X^\alpha. \] (33)
From which we find $M^\alpha_\beta = -\delta^\alpha_\delta \delta^\beta_\gamma - \delta^\alpha_\gamma \delta^\beta_\delta$. The quantity $\phi^\iota$ in the fundamental representation of $SU(n+1)$ transforms as
\[ [X^\alpha, \begin{pmatrix} \phi \\ \phi^\beta \end{pmatrix}] = \begin{pmatrix} \phi^\alpha \\ 0 \end{pmatrix}, \quad [X^\alpha, \begin{pmatrix} \phi \\ \phi^\beta \end{pmatrix}] = \begin{pmatrix} 0 \\ \delta^\beta_\delta \phi \end{pmatrix}, \]
\[ [H^\alpha_\gamma, \begin{pmatrix} \phi \\ \phi^\beta \end{pmatrix}] = \begin{pmatrix} \delta^\gamma_\delta \phi^\alpha - \frac{1}{n} \delta^\gamma_\delta \phi^\beta \\ 0 \end{pmatrix}, \quad [Y, \begin{pmatrix} \phi \\ \phi^\beta \end{pmatrix}] = \begin{pmatrix} \frac{n}{n+1} \phi \\ -\frac{n+1}{n} \phi^\beta \end{pmatrix}. \] (34)
The currents corresponding to (32) are given by
\[ J^\alpha = -iq^\alpha qp + Aq^\alpha \beta \gamma + C \partial q^\alpha, \]
\[ J = -iqp + B \beta \gamma, \quad J^\alpha = iq^\alpha p_\beta - \frac{i}{n} \delta^\alpha_\gamma qp, \]
\[ \bar{J}_\alpha = ip^\alpha. \]
They satisfy (26) with

\[ A = \sqrt{n+1}, \quad B = \frac{n}{\sqrt{n+1}}, \quad iC = 1, \quad k = 1, \]

and the Killing metric \( g^{AB} \) taken from the Casimir of the Lie algebra (33)

\[ \bar{X}_\alpha X^\alpha + X^\alpha \bar{X}_\alpha + H^\alpha_\beta H^\beta_\alpha + (1 + \frac{1}{n})Y^2. \]

We propose the form of the primary fields in the fundamental representation of \( SU(n+1) \) as

\[ \phi^I = \left( \begin{array}{c} \gamma \\ \gamma q^\alpha \end{array} \right). \]  

We find that they satisfy (27) with the coefficients taken from (34) if \(- (a + b) A = i\) in the replacement (29).

The second example is the coset space \( E_6/\{SO(10) \otimes U(1)\} \). The generators of \( E_6 \) are decomposed as \( \{X^\alpha, \bar{X}_\alpha, H^{mn}, Y\} \). The Lie algebra takes the form

\[ [H^{mn}, H^{kl}] = -i(\delta^{mk}H^{nl} + \delta^{nl}H^{mk} - \delta^{ml}H^{nk} - \delta^{nk}H^{ml}), \]

\[ [H^{mn}, X^\alpha] = -i(\gamma^{mn})^\alpha_\beta X^\beta, \quad [Y, X^\alpha] = -\frac{\sqrt{3}}{2}X^\alpha, \]

\[ [X^\alpha, \bar{X}_\beta] = -i(\gamma^{mn})^\alpha_\beta H^{mn} - \sqrt{3}\delta^\alpha_\beta Y, \quad [X^\alpha, X^\beta] = 0. \]

Here \( \gamma^{mn} \) are the \( SO(10) \) generators in the spinor representation. In the Majorana-Weyl representation the \( SO(10) \) Dirac matrices \( \Gamma^m \) are given by

\[ \Gamma^m = \begin{pmatrix} 0 & \gamma^m \\ \bar{\gamma}^m & 0 \end{pmatrix}. \]

The Clifford algebra takes the form

\[ (\gamma^m \bar{\gamma}^n + \gamma^n \bar{\gamma}^m)^\alpha_\beta = 2\delta^{mn}\delta^\alpha_\beta, \quad (\bar{\gamma}^m \gamma^n + \bar{\gamma}^n \gamma^m)^\alpha_\beta = 2\delta^{mn}\delta^\alpha_\beta, \]

and the \( SO(10) \) generators are given by either of

\[ (\gamma^m \bar{\gamma}^n - \gamma^n \bar{\gamma}^m)^\alpha_\beta = 2(\gamma^{mn})^\alpha_\beta, \quad (\bar{\gamma}^m \gamma^n - \bar{\gamma}^n \gamma^m)^\alpha_\beta = 2(\gamma^{mn})^\alpha_\beta. \]

From (23) we find that

\[ M^\alpha_\beta = \frac{1}{4}(\gamma^{mn})^\alpha_\gamma(\gamma^{mn})^\beta_\delta - \frac{3}{2}\delta^\alpha_\gamma\delta^\beta_\delta. \]

The \( SO(10) \) currents are given by

\[ J^\alpha = \frac{i}{8}(\gamma^{mn})^\alpha_\gamma p^{\gamma mn} - \frac{3i}{4}q^\alpha qp + Aq^\alpha \beta \gamma + C\partial q^\beta, \]

\[ J = -\frac{\sqrt{3}i}{2}qp + B\beta \gamma, \quad J_\alpha = ip_\alpha, \]

\[ J^{mn} = \frac{1}{2}p^{\gamma mn}q. \]
We can check that they satisfy the algebra (15) with
\[ A = 2\sqrt{6}, \quad B = 2\sqrt{2}, \quad C = -8i, \quad k = 4, \]
and the Killing metric \( g^{AB} \) read from the Casimir of the algebra (36).
\[
\frac{1}{2}(\bar{X}_\alpha X^\alpha + X^\alpha \bar{X}_\alpha) + \frac{1}{2}H_{mn}H^{mn} + Y^2.
\]
The fundamental representation of \( E_6 \) is 27, which is decomposed under \( SO(10) \) as 1+16+10. Correspondingly we have the the primaries in this representation as \( \phi^I = \{ \phi, \phi^\alpha, \phi^k \} \).
The transformation law under \( E_6 \) is given\[11\] by
\[
\begin{align*}
[X^\alpha, \begin{pmatrix} \phi \\ \phi^\beta \\ \phi^k \end{pmatrix}] &= \begin{pmatrix} \sqrt{2}\phi^\alpha \\ (\bar{\gamma}^m)_{\alpha\beta} \phi^m \\ 0 \end{pmatrix}, \\
[X_\alpha, \begin{pmatrix} \phi \\ \phi^\beta \\ \phi^k \end{pmatrix}] &= \begin{pmatrix} 0 \\ \sqrt{2}\delta^\beta_\alpha \phi \\ (\bar{\gamma}^k \phi)_{\alpha} \end{pmatrix}, \\
[H^{mn}, \begin{pmatrix} \phi \\ \phi^\beta \\ \phi^k \end{pmatrix}] &= \begin{pmatrix} 0 \\ -i(\bar{\gamma}^{mn} \phi)^\beta \\ -i(\delta^{mk} \phi^m - \delta^{nk} \phi^m) \end{pmatrix}, \\
[Y, \begin{pmatrix} \phi \\ \phi^\beta \\ \phi^k \end{pmatrix}] &= \begin{pmatrix} 2\sqrt{3}\phi \\ 2\sqrt{3}\phi^\beta \\ -\sqrt{3}\phi^k \end{pmatrix}.
\end{align*}
\]
The Jacobi identity for the transformation law can be checked by means of the formula
\[
M^{\alpha\beta}_{\gamma\delta} = (\bar{\gamma}^m)_{\gamma\delta}(\bar{\gamma}^m)^{\alpha\beta} - 2\delta^{[\alpha}_{\gamma}\delta^{\beta]}_{\delta}.
\]
We find that the set of primaries
\[
\phi^I = \begin{pmatrix} \frac{1}{\sqrt{2}}\gamma \\ \gamma q^\alpha \\ \frac{1}{2}\gamma q^m q \end{pmatrix}
\]
satisfies the algebra (27) with the coefficients \( f^{AI}_J \) taken from (37) if \(-(a + b)A = 2i\) in the replacement (29). This expression itself was found in \[12\] without quantization for discussing other physics.

To conclude our arguments some comments are in order. The form of the \( G \)-symmetry currents in (16) was inferred from the Killing potentials
\[
-iM^A = K_\alpha R^{A\alpha} - F^A,
\]
which exist for the general Kähler coset space and satisfy the Lie algebra of \( G \)
\[
\mathcal{L}_{R^A}M^B = f^{ABC}M^C.
\]
Here $K$ is the Kähler potential and $F^A$ are the holomorphic functions $F^A$ found from (20)\[6\]. The Kähler 2-form (8) can be put in the form
\[ \omega = dp_\alpha \wedge dq^\alpha, \]
with $p_\alpha = iK_\alpha$. The Killing potentials $M^A$ are rewritten in terms of the so-called Darboux coordinates $(p_\alpha, q^\alpha)$\[7\] as
\[ M^\alpha = \frac{i}{2} M^\alpha_{\gamma \delta} q^\gamma q^\delta + q^\alpha, \quad \bar{M}_\alpha = i p_\alpha, \]
\[ M^i = i p \Sigma^i, \quad M = i pq - 1. \]

Here the $U(1)$ part in $F^A$, which was left arbitrary as $r$ in (25), was fixed so that (39) is fulfilled. This way of fixing the $U(1)$ part\[6\] is alternative to the one discussed in \[9\] for the general Kähler coset space. It is now clear that there is a close relationship between these Killing potentials and the $G$-symmetry currents (26). In \[7\] it was discussed that the introduction of the Darboux coordinates simplifies quantum deformation of the Kähler manifold. Namely quantum deformation of the symplectic manifold defined by (6) is done through the non-commutative $\star$ product
\[ f(x) \star g(x) = \sum_n \frac{1}{n!} \left( -\frac{i\hbar}{2} \right)^n \omega^{i_1 j_1} \omega^{i_2 j_2} \cdots \omega^{i_n j_n} \partial_{i_1} \partial_{i_2} \cdots \partial_{i_n} f(x) \partial_{j_1} \partial_{j_2} \cdots \partial_{j_n} g(x), \]
according to Fedosov\[13\], in which $\omega^{ij} = g^{ik} g^{jl} \omega_{kl}$. When the symplectic manifold is kählerian, the $\star$ product becomes the well-known Moyal product by using the Darboux coordinates. Then it is easy to study the non-commutative algebras for the Killing potentials
\[ [M^A(q, \bar{q}), M^B(q, \bar{q})]_\star = -i(c_1 \hbar + c_3 \hbar^3 + c_5 \hbar^5 + \cdots) f^{ABC} M^C, \]
\[ \bar{M}^A(q, \bar{q}) \star M^A(q, \bar{q}) = c_0 + c_2 \hbar^2 + c_4 \hbar^4 + \cdots. \]

In \[7, 14\] the numerical constants $c_j, (j \geq 0)$ were found to be
\[ c_0 = R \quad (\text{Riemann scalar}), \quad c_1 = 1, \]
\[ c_2 = -\frac{1}{2} (tr \Sigma^i \Sigma^i + N), \quad c_i = 0 \quad \text{for} \quad i \geq 3. \]

The Berkovits method discussed in this letter indicates that the fuzzy Kähler coset space may be studied by incorporating the bosonic ghosts and generalizing the the Moyal product
\[ f(p, q, \beta, \gamma) \star g(p, q, \beta, \gamma) = \sum_n \frac{1}{n!} f(p, q, \beta, \gamma) \left( \frac{i\hbar}{2} \partial_{\beta} \partial_{\gamma} \partial_{\beta} \partial_{\gamma} - \partial_{\beta} \partial_{\gamma} \partial_{\beta} \partial_{\gamma} - \partial_{\beta} \partial_{\gamma} \partial_{\beta} \partial_{\gamma} \right)^n f(p, q, \beta, \gamma). \]

The study in this direction is expected to shed a new light on the non-commutative geometry of the Kähler coset space.
In this letter the $G$-symmetry currents have been explicitly given for the irreducible Kähler coset space $G/H$ as (26). The $G$-symmetry primaries could not be given in such a general form because the decomposition (28) under the subgroup $H$ varies from case to case. But we are sure of being able to construct them explicitly for other types of the irreducible Kähler coset space. The construction may be extended to the reducible Kähler coset space $G/\{S \otimes U(1)^k\}$, of which extreme case $G/U(1)^r$ is the flag manifold used for the Wakimoto realization of the Lie algebra of $G$. The study is undergoing.

Acknowledgements

The work was supported in part by the Grant-in-Aid for Scientific Research No. 13135212.

References

[1] N. Berkovits, “Super-Poincaré covariant quantization of the superstring”, JHEP 0004(2000)018, hep-th/0001035.

[2] D. Friedan, E. Martinec and S. Shenker, “Conformal invariance, supersymmetry, and string theory”, Nucl. Phys. B271(1986)93.

[3] Y. Aisaka and Y. Kazama, “A new first class algebra, homological perturbation and extention of pure spinor formalism for superstring”, JHEP 0302(2003)017, hep-th/0212316.

P.A. Grassi, G. Policastro, M. Porrati and P. van Nieuwenhuizen, “Covariant quantization of superstrings without pure spinor constraints”, JHEP 0210(2002)054, hep-th/0112162;

N. Berkovits and N.A. Nekrasov, “The character of pure spinors”, Lett. Math. Phys. 74(2005)75, hep-th/0503075.

[4] M. Wakimoto, “Fock representations of the affine Lie algebra $A_1^{(1)}$”, Commun. Math. Phys. 104(1986)605;

B. Feigin and F. Frenkel, “A family of representations of affine Lie algebras”, Russ. Math. Surv. 43, No. 5(1988)221;

K. Ito and S. Komata, “Feigin-Fuchs representations of arbitrary affine Lie algebras”, Mod. Phys. Lett. A6(1991)581;

J. de Boer and L. Fehér, “Wakimoto realizations of current algebras: an explicit construction”, Commun. Math. Phys. 189(1997)759, hep-th/9611083.

[5] J.L. Petersen and J. Ramussen, “Free field realizations of 2D current algebras, screening currents and primary fields”, Nucl. Phys. B502(1997)649, hep-th/9704052.
[6] J. Bagger and E. Witten, “Gauge invariant supersymmetric nonlinear sigma model”, Phys. Lett. B118 (1982)103.

[7] S. Aoyama and T. Masuda, “The fuzzy Kähler coset space with the Darboux coordinates”, Phys. Lett. B521 (2001)376.

[8] Y. Achiman, S. Aoyama and J.W. van Holten, “The non-linear supersymmetric σ model on $E_6/\text{SO}(10) \times U(1)$”, Phys. Lett. 141B (1984)64; “Gauged supersymmetric σ models and $E_6/\text{SO}(10) \times U(1)$”, Nucl. Phys. B258B (1985)179.

[9] K. Itoh, T. Kugo and H. Kunitomo, “Supersymmetric nonlinear realization for arbitrary Kählerian coset space $G/H$”, Nucl. Phys. B263 (1986)295.

[10] Y. Achiman, S. Aoyama and J.W. van Holten, “Symmetry breaking in gauged supersymmetric sigma models”, Phys. Lett. 150B (1985)153.

[11] K. Itoh, T. Kugo and H. Kunitomo, “Supersymmetric non-linear Lagrangians of Kählerian coset spaces $G/H$: $G = E_6, E_7$ and $E_8$”, Prog. Theor. Phys. 75 (1986)386.

[12] K. Higashijima and M. Nitta, “Supersymmetric nonlinear sigma models as gauge theories”, Prog. Theor.Phys. 103 (2000)635, hep-th/9911139.

[13] B.V. Fedosov, “A simple geometrical construction of deformation quantization”, J. Differential Geom. 40 (1994)213; “Deformation quantization and index theorem”, in: Mathematical Topics, Vol. 9, Akademie-Verlag, Berlin, 1996.

[14] S. Aoyama and T. Masuda, “The fuzzy Kähler coset space by the Fedosov formalism”, Phys. Lett. B514 (2001)385, hep-th/0105271.

[15] S. Aoyama and T. Masuda, “The fuzzy $S^4$ by quantum deformation”, Nucl. Phys. B656 (2003)325, hep-th/0212214.

[16] S. Aoyama, “Four-fermi coupling of the supersymmetric non-linear σ-model on $G/S \otimes \{U(1)\}^k$, Nucl. Phys. B578 (2000)449, hep-th/0001160.