Concentration of measure for the number of isolated vertices in the Erdős-Rényi random graph by size bias couplings

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Abstract

A concentration of measure result is proved for the number of isolated vertices \( Y \) in the Erdős-Rényi random graph model on \( n \) edges with edge probability \( p \). When \( \mu \) and \( \sigma^2 \) denote the mean and variance of \( Y \) respectively, \( P\left( \left( \frac{Y - \mu}{\sigma} \right) \geq t \right) \) admits a bound of the form \( e^{-kt^2} \) for some constant positive \( k \) under the assumption \( p \in (0, 1) \) and \( np \to c \in (0, \infty) \) as \( n \to \infty \). The left tail inequality

\[
P \left( \frac{Y - \mu}{\sigma} \leq -\frac{t}{\sqrt{4\mu}} \right) \leq \exp \left( -\frac{t^2\sigma^2}{4\mu} \right)
\]

holds for all \( n \in \{2, 3, \ldots\} \), \( p \in (0, 1) \) and \( t \geq 0 \). The results are shown by coupling \( Y \) to a random variable \( Y^* \) having the \( Y \)-size biased distribution, that is, the distribution characterized by \( E[f(Y^*)] = \mu E[f(Y)] \) for all functions \( f \) for which these expectations exist.

1 Introduction and main result

For some \( n \in \{1, 2, \ldots\} \) and \( p \in (0, 1) \) let \( K \) be the Erdős-Rényi random graph on the vertices \( V = \{1, 2, \ldots, n\} \) and edge success probability \( p \), that is, with edge indicators \( X_{uv} = 1(u, v \in V : uv \text{ is an edge in } K) \) independent random variables with the Bernoulli\((p)\) distribution for all \( u \neq v \). We set \( X_{vv} = 0 \) for all \( v \in V \).

Recall that the degree of a vertex \( v \in V \), denoted by \( d(v) \), is the number of edges incident on \( v \). Hence,

\[
d(v) = \sum_{u \in V} X_{uv}.
\]

Many authors have studied the distribution of

\[
Y = \sum_{v \in V} 1(d(v) = d)
\]

counting the number of vertices \( v \) of \( K \) having some fixed degree \( d \). We derive upper bounds, for fixed \( n \) and \( p \), on the tail probabilities of the number of isolated vertices of \( K \), that is, for \( Y \) in \( \mathbb{N} \) for the case \( d = 0 \), which counts the number of vertices having no incident edges.

For \( d \) in general, and \( p \) depending on \( n \), previously in Karoński and Ruciński (1987), the asymptotic normality of \( Y \) was shown when \( n^{-1/dp} \to \infty \) and \( np \to 0 \), or \( np \to \infty \) and \( np - \log n - d \log \log n \to -\infty \); see also Palka (1984) and Bollobás (1985). For the case \( d = 0 \) of isolated vertices, Barbour (1982) and Barbour et al. (1989) show that \( Y \) is asymptotic normal if and only if \( n^2p \to \infty \) and \( np - \log n \to -\infty \).

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Here we study the distribution of $Y$ using a size bias coupling that was used in Goldstein and Rinott (1996) to study the rate of convergence to the multivariate normal distribution for a vector whose components count the number of vertices of some fixed degrees. In Kordecki (1990), the mean $\mu$ and variance $\sigma^2$ of $Y$ for the particular case $d = 0$ are computed as

$$
\mu = n(1 - p)^{n-1}, \quad \text{and} \quad \sigma^2 = n(1 - p)^{n-1}(1 + np(1 - p)^{n-2} - (1 - p)^{n-2}) \quad \text{for } n \geq 2.
$$

In the same paper, Kolmogorov distance bounds to the normal of order $O(\text{Var}(Y)^{-1/2})$ were obtained.

O’Connell (1998) showed that an asymptotic large deviation principle holds for $Y$. Raic (2007) obtains non-uniform large deviation bounds for mean zero, variance one random variables in some generality, and applies his results to the case of counting the number of isolated vertices with $W = (Y - \mu)/\sigma$, yielding the bound

$$
P(W \geq t) \leq e^{\beta(t)/\sigma^2} (1 + Q(t)\beta(t)) \quad \text{for all } t \geq 0,
$$

where $\Phi(t)$ denotes the distribution function of a standard normal variate,

$$
Q(t) = \frac{12}{\sqrt{2\pi}} + \frac{23}{2} t + \frac{11\sqrt{2\pi}}{2} t^2,
$$

and

$$
\beta(t) = \frac{n}{6\sigma^3} (13 + 43np + 27(np)^2) \exp\left\{ \frac{(8 + 4np)t}{\sigma} + 2np(e^{t/\sigma} - 1) \right\}.
$$

Still from Raic (2007), when $np \to c$ as $n \to \infty$, (3) holds for all $n$ sufficiently large with

$$
\beta(t) = \frac{C_4}{\sqrt{n}} \exp\left( \frac{C_2 t}{\sqrt{n}} + C_3 \theta^{C_4 t/\sqrt{n}} - 1 \right)
$$

for some unspecified constants $C_1, C_2, C_3$ and $C_4$ depending only on $c$. For $t$ of order $\sqrt{n}$, for instance, the function $\beta(t)$ will be of order $1/\sqrt{n}$ as $n \to \infty$, allowing an asymptotic approximation of the deviation probability $P(W \geq t)$ by the normal, to within some factors.

In Theorem 1.1 we supply a bound that likewise holds also for all $n$, and that also gives somewhat more explicit information on the rate of tail decay. In particular, we see from (6) that the standardized variable $W$ has a left tail that is bounded above by $\exp(-t^2 \sigma^2/(4\mu))$. Moreover, the right tail also exhibits similar bounds over some parameter regions, with a worst case bound there of order $\exp\{-\rho t\}$ by (7) for some $\rho > 0$, but see also Corollary 1.1 for a further improvement in the regime where $np$ converges to a nonzero constant.

**Theorem 1.1.** For $n \in \{2, 3, \ldots\}$ and $p \in (0, 1)$ let $K$ denote the random graph on $n$ vertices where each edge is present with probability $p$, independently of all other edges, and let $Y$ denote the number of isolated vertices in $K$, having mean $\mu$ and variance $\sigma^2$, as given in (2). Let $M(\theta) = \exp(\theta(Y - \mu)/\sigma)$ be the moment generating function of the standardized $Y$ variable. Then, letting

$$
\gamma_s = e^s(pe^s + 1 - p)^{n-2}(npe^s + 1 - p) + (n - 1)p + 1 \quad \text{and} \quad H(\theta) = \frac{\mu}{2\sigma^2} \int_0^\infty s e^{s/\sigma} ds
$$

we have $M(\theta) \leq \exp(H(\theta))$ for all $\theta \geq 0$, and for all $t > 0$,

$$
P \left( \frac{Y - \mu}{\sigma} \geq t \right) \leq \inf_{\theta \geq 0} \exp\{-\theta t + H(\theta)\}.
$$

For all $\theta \leq 0$ we have $M(\theta) \leq \exp(\mu \theta^2/\sigma^2)$, and for all $t > 0$,

$$
P \left( \frac{Y - \mu}{\sigma} \leq -t \right) \leq \exp\left( -\frac{t^2 \sigma^2}{4\mu} \right).
$$
Though integration shows that we may explicitly write
\[
H(\theta) = \frac{\mu}{2\sigma^2} \left( np((n-1)p + 1)\theta^2 + 2\sigma^2 - 2\sigma \left( p e^{\theta/\sigma} (\sigma - n\theta) + \sigma (1-p) \right) \left( \frac{e^{\theta/\sigma} - 1}{2np} \right) \right),
\]
the integral formula for \( H(\theta) \) in the theorem appears simpler to handle.

Useful bounds for the minimization in (3) may be obtained by restricting to \( \theta \in [0, \theta_0] \) for some \( \theta_0 \). In this case, as \( \gamma_\sigma \) is an increasing function of \( s \), we have
\[
H(\theta) \leq \frac{\mu}{4\sigma^2} \gamma_\theta_0 / \theta^2 \quad \text{for } \theta \in [0, \theta_0].
\]
The quadratic \( -\theta t + \mu \gamma_\theta_0 / \theta^2 / (4\sigma^2) \) in \( \theta \) is minimized at \( \theta = 2\sigma^2 / (\mu \gamma_\theta_0 / \sigma) \). When this value falls in \( [0, \theta_0] \) we obtain the first bound in (4), and setting \( \theta = \theta_0 \) yields the second, thus,
\[
P \left( \frac{Y - \mu}{\sigma} \geq t \right) \leq \begin{cases} 
\exp \left( -\frac{\gamma_\theta^2}{\mu \gamma_\theta_0} \right) & \text{for } t \in [0, \mu \gamma_\theta_0 (2\sigma^2)] \\
\exp ( -\theta_0 t + \frac{\gamma_\theta_0}{4\sigma^2} ) & \text{for } t \in (\theta_0 \mu \gamma_\theta_0 / (2\sigma^2), \infty). 
\end{cases}
\]

Inequality (7) and the boundedness of \( Y \) yields the following useful corollary.

**Corollary 1.1.** For all \( c \in (0, \infty) \) there exists a positive constant \( k \) depending only on \( c \) such that when \( p \in (0,1) \) and \( np \to c \) as \( n \to \infty \),
\[
P( (Y - \mu) / \sigma \geq t ) \leq \exp( -kt^2 )
\]
for all \( t \geq 0 \) and all \( n \geq 2 \).

**Proof.** Since \( Y \) can be no more than \( n \), and \( \sigma^2 \) increases at rate \( n \) when \( np \to c \), there exists a positive constant \( a_0 \) such that
\[
\frac{Y - \mu}{\sigma} \leq \frac{n}{\sigma} \leq a_0 \sqrt{n}.
\]
Hence \( P( (Y - \mu) / \sigma \geq t ) = 0 \) for all \( t > a_0 \sqrt{n} \).

For any given \( n \) let \( \theta_n = a_0 \sqrt{n} \sigma^2 / \mu \). Then, as \( \gamma_\sigma \geq 2 \) for all \( s \geq 0 \), we have \( \gamma_\theta_n \mu \gamma_\theta_0 / \sigma) / (2\sigma^2) \geq a_0 \sqrt{n} \), so the first bound in (7) applies for all \( t \leq a_0 \sqrt{n} \). Note that \( \theta_n / \sigma = a_0 \sqrt{n} / \sigma / \mu \) converges to a positive constant, implying the convergence of \( \gamma_\theta_n / \sigma \), and hence that of \( \sigma^2 / (\mu \gamma_\theta_0 / \sigma) \), also to a positive constant. Since \( \sigma^2 / (\mu \gamma_\theta_0 / \sigma) \) is positive for all \( n \geq 2 \), we see that the claim of the corollary holds for all \( k \) in the nonempty interval \( (0, \inf_n \sigma^2 / (\mu \gamma_\theta_0 / \sigma)) \).

In the asymptotic of Corollary 1.1 for, say \( t = a_\sqrt{n} \), the function \( \beta(t) \) of (3) behaves like \( C / \sqrt{n} \), so the bound (3) also gives useful information for some range of positive values of \( a \) up to some upper limit. However as \( \exp(t^2 \beta(t) / 6) \) behaves like \( \exp(\sqrt{2} a^2 / 6) \), when multiplied by \( 1 - \Phi(t) \), of exponential order \( \exp(-a^2 n/2) \), the product tends to infinity for all sufficiently large \( a \), so the bound in (3) may explode before the right tail of \( W \) vanishes.

The main tool used in proving Theorem 1.1 is size bias coupling, that is, the construction of \( Y \) and \( Y^* \) on the same space where \( Y^* \) has the \( Y \)-size biased distribution characterized by
\[
E[Y f(Y)] = \mu E[f(Y^*)]
\]
for all \( f \) for which the expectations above exist. In Ghosh and Goldstein (2011a) and Ghosh and Goldstein (2011b), size bias couplings were used to prove concentration of measure inequalities when \( |Y^* - Y| \) can be almost surely bounded by a constant independent of the problem size. Here, in contrast, we apply the coupling for the number of isolated vertices of \( K \) from Goldstein and Rinott (1996), which violates the boundedness condition. Unlike the theorem used in Ghosh and Goldstein (2011a) and Ghosh and Goldstein (2011b), which can be applied to a wide variety of situations under a bounded coupling assumption, it seems
that cases where the coupling is unbounded, such as the one we consider here, need application specific treatment, and cannot be handled by one single general result.

Having its roots in the work of Baldi et al. (1989), a general prescription for constructing a variable with the size bias distribution of a sum of nonnegative variables is given in Goldstein and Rinott (1996). Helped by the fact that size biasing a nontrivial indicator random variable simply sets its value to one, specializing to nontrivial exchangeable indicators yields the following simplification as in Lemma 3.3 of Goldstein and Penrose (2010).

**Proposition 1.1.** Suppose $Y = \sum_{v \in V} X_v$, a finite sum of nontrivial exchangeable Bernoulli variables $\{X_v, v \in V\}$, and that for $w \in V$ the variables $\{X_w^v, v \in V\}$ have joint distribution

$$\mathcal{L}(X_w^v, v \in V) = \mathcal{L}(X_v, v \in V | X_w = 1).$$

Then $Y_w = \sum_{v \in V} X_w^v$ has the $Y$ size biased distribution $Y^*$, as does the mixture $Y^V$ when $V$ is a random index with values in $V$, chosen independent of all other variables.

Construction of the variable $Y^*$ is not enough for our purposes; one must couple $Y^*$ to $Y$. However, Proposition 1.1 suggests a natural coupling. Given the exchangeable indicators $\{X_v, v \in V\}$ that sum to $Y$, choose a summand uniformly and independently. If the summand value is already one, set $Y^* = Y$. Otherwise, set this variable to one, and ‘adjust’ the remaining variables to have their conditional distribution given that this variable takes on the value one. By Proposition 1.1 the sum $Y^*$ of these new variables has the $Y$-size biased distribution.

## 2 Proof of Theorem 1.1

For any graph with vertex set $V$, for $v \in V$ we let $N(v)$ denote the set of neighbors of $v$,

$$N(v) = \{w \in V : X_{vw} = 1\},$$

where $X_{vw}$ is the indicator that there exists and edge connecting vertices $v$ and $w$. We now present the proof of Theorem 1.1.

**Proof.** Following Proposition 1.1, we first construct a coupling of $Y^*$, having the $Y$-size bias distribution, to $Y$. Let $K$ be given, and let $Y$ be the number of isolated vertices in $K$. From (1) with $d = 0$ we see that $Y$ is the sum of exchangeable indicators. Let $V$ be uniformly chosen from $V$, independent of the remaining variables. If $V$ is already isolated, do nothing and set $Y^* = Y$. Otherwise, set this variable to one, and ‘adjust’ the remaining variables to have their conditional distribution given that this variable takes on the value one. By Proposition 1.1 the sum $Y^*$ of these new variables has the $Y$-size biased distribution.

Since all edges incident to the chosen $V$ are removed in order to form $K^*$, any neighbor of $V$ which had degree one thus becomes isolated, and $V$ also becomes isolated if it was not so earlier. As $1(d(w) = 0)$ is unchanged for all $w \notin \{V\} \cup N(V)$, we have

$$Y^* - Y = d_1(V) + 1(d(V) \neq 0)$$

where for any $v \in V$ we let $d_1(v) = \sum_{w \in N(v)} 1(d(w) = 1).$ (9)

In particular the coupling is monotone, that is, $Y^* \geq Y$. Further, since $d_1(V) \leq d(V)$, by (9) we have

$$0 \leq Y^* - Y \leq d(V) + 1.$$ (10)
Now note that, for real $x \neq y$, the convexity of the exponential function implies
\[
\frac{e^y - e^x}{y - x} = \int_0^1 e^{yt}(1-t)x \, dt \leq \int_0^1 (te^y + (1-t)e^x) \, dt = \frac{e^y + e^x}{2},
\]
and therefore, for all real $x, y$,
\[
|x - y| \leq |x - y| \frac{e^y + e^x}{2}.
\] (11)

Let $\theta \geq 0$. Using (11) and (10), we have
\[
E(e^{\theta Y_x} - e^{\theta Y}) \leq \frac{\theta}{2} E \left( (Y_x - Y)(e^{\theta Y_x} + e^{\theta Y}) \right)
= \frac{\theta}{2} E \left( e^{\theta Y}(Y_x - Y)(e^{\theta Y_x} + 1) \right) \leq \frac{\theta}{2} E \left( e^{\theta Y}(d(V) + 1)(e^{\theta d(V)} + 1) \right).
\] (12)

Clearly the number of isolated vertices $Y$ is a nonincreasing function of the edge indicators $X_{vw}$, while $d(V) + 1$ is a nondecreasing function of these same indicators. Hence $Y$ and $d(V) + 1$ have negative correlations, that is, by the inequality of Harris [1960],
\[
E[f(Y)g(d(V) + 1)] \leq E[f(Y)]E[g(d(V) + 1)]
\] (13)
for any two nondecreasing real functions $f$ and $g$. In particular, when $f(x) = e^{\theta x}$ and $g(x) = x(e^{\theta x} + 1)$ with $x \in [0, \infty)$, by (12) and (13) we obtain
\[
E(e^{\theta Y_x} - e^{\theta Y}) \leq \frac{\theta}{2} E e^{\theta Y} \left( (d(V) + 1)(e^{\theta d(V)} + 1) \right)
= \frac{\theta}{2} E e^{\theta Y} \left( e^{\theta} \left( d(V)e^{\theta d(V)} + e^{\theta d(V)} \right) + E(d(V)) + 1 \right).
\] (14)

To handle the terms in (14), note that for any vertex $v$ the degree $d(v)$ has the Binomial$(n - 1, p)$ distribution, and in particular
\[
E(d(v)) = (n - 1)p \quad \text{and} \quad E(e^{\theta d(v)}) = \alpha_{\theta} \quad \text{where} \quad \alpha_{\theta} = (pe^\theta + 1 - p)^{n-1}.
\]
Hence, as $V$ is chosen uniformly over the vertices $v$ of $K$,
\[
E(d(V)) = (n - 1)p \quad \text{and} \quad E(e^{\theta d(V)}) = \alpha_{\theta},
\] (15)
and now differentiation under the second expectation above, allowed since $d(V)$ is bounded, yields
\[
E(d(V)e^{\theta d(V)}) = \phi_{\theta} \quad \text{where} \quad \phi_{\theta} = (n - 1)pe^\theta(pe^\theta + 1 - p)^{n-2}.
\] (16)
Substituting (15) and (16) into (14) yields, for all $\theta \geq 0$,
\[
E(e^{\theta Y_x} - e^{\theta Y}) \leq \frac{\theta \gamma_{\theta}}{2} E(e^{\theta Y}) \quad \text{where} \quad \gamma_{\theta} = e^\theta(\phi_{\theta} + \alpha_{\theta}) + (n - 1)p + 1.
\] (17)

Letting $m(\theta) = E(e^{\theta Y})$, using that $Y$ is bounded to differentiate under the expectation, along with (8) and (17), we obtain
\[
m'(\theta) = E(Ye^{\theta Y}) = \mu E(e^{\theta Y_x}) \leq \mu \left( 1 + \frac{\theta \gamma_{\theta}}{2} \right) m(\theta).
\] (18)

Standardizing $Y$, we set
\[
M(\theta) = E(\exp(\theta(Y - \mu)/\sigma)) = e^{-\theta \mu/\sigma} m(\theta/\sigma),
\] (19)
and now by differentiating and applying (18), we obtain
\[
M'(\theta) = \frac{1}{\sigma} e^{-\theta \mu/\sigma} m'(\theta/\sigma) - \frac{\mu}{\sigma} e^{-\theta \mu/\sigma} m(\theta/\sigma)
\]
\[
\leq \frac{\mu}{\sigma} e^{-\theta \mu/\sigma} \left( 1 + \frac{\theta \gamma_0 / \sigma}{2\sigma} \right) m(\theta/\sigma) - \frac{\mu}{\sigma} e^{-\theta \mu/\sigma} m(\theta/\sigma)
\]
\[
= e^{-\theta \mu/\sigma} \frac{\mu \theta \gamma_0 / \sigma}{2\sigma^2} m(\theta/\sigma) = \frac{\mu \theta \gamma_0 / \sigma}{2\sigma^2} M(\theta).
\]

Since \( M(0) = 1 \), integrating \( M'(\theta) / M(\theta) \) over \([0, \theta]\) yields the bound
\[
\log(M(\theta)) \leq H(\theta), \text{ or that } M(\theta) \leq \exp(H(\theta)) \text{ where } H(\theta) = \frac{\mu}{2\sigma^2} \int_0^\theta s \gamma_0 / \sigma \, ds,
\]
proving the claim on \( M(\theta) \) for \( \theta \geq 0 \). Moreover, for \( \theta \) nonnegative,
\[
P \left( \frac{Y - \mu}{\sigma} \geq t \right) \leq P \left( \exp \left( \frac{\theta(Y - \mu)}{\sigma} \right) \geq e^{\theta t} \right) \leq e^{-\theta t} M(\theta) \leq \exp(-\theta t + H(\theta)).
\]
As the inequality holds for all \( \theta \geq 0 \), it holds for the infimum over \( \theta \geq 0 \), proving (19).

To demonstrate the left tail bound let \( \theta < 0 \). Since \( Y^* \geq Y \) and \( \theta < 0 \), using (11), (10) and that \( Y \) is a function of \( K \) we obtain
\[
E(e^{\theta Y} - e^{\theta Y^*}) \leq \frac{\theta}{2} E \left( (e^{\theta Y} + e^{\theta Y^*})(Y^* - Y) \right) \leq |\theta| E(e^{\theta Y} (Y^* - Y)) = |\theta| E(e^{\theta Y} E(Y^* - Y | K)).
\]
(20)

By (9) have
\[
E(Y^* - Y | K) = \frac{1}{n} \sum_{v \in V} (d_1(v) + 1(d(v) \neq 0)) \leq \frac{1}{n} \sum_{v \in V} d_1(v) + 1,
\]
(21)
and noting that
\[
\sum_{v \in V} d_1(v) = \sum_{v \in V} \sum_{w \in N(v)} 1(d(w) = 1) = \sum_{w \in V} \sum_{v \in N(w)} 1(d(w) = 1) = \sum_{w \in V} |N(w)| 1(d(w) = 1) = \sum_{w \in V} 1(d(w) = 1),
\]
the number of degree one vertices in \( K \), by (21) we find that \( E(Y^* - Y | K) \leq 2 \).

Now, by (20) and (21),
\[
E(e^{\theta Y} - e^{\theta Y^*}) \leq 2|\theta| E(e^{\theta Y})
\]
and therefore, justifying differentiating under the expectation as before, applying (9) yields
\[
m'(\theta) = E(Y e^{\theta Y}) = \mu E(e^{\theta Y^*}) \geq \mu (1 + 2\theta) m(\theta).
\]

Again with \( M(\theta) \) as in (19),
\[
M'(\theta) = \frac{1}{\sigma} e^{-\theta \mu/\sigma} m'(\theta/\sigma) - \frac{\mu}{\sigma} e^{-\theta \mu/\sigma} m(\theta/\sigma)
\]
\[
\geq \frac{\mu}{\sigma} e^{-\theta \mu/\sigma} ((1 + 2\theta / \sigma) m(\theta/\sigma)) - \frac{\mu}{\sigma} e^{-\theta \mu/\sigma} m(\theta/\sigma)
\]
\[
= \frac{2\mu \theta \gamma_0 / \sigma}{2\sigma^2} M(\theta).
\]
Dividing by \( M(\theta) \), integrating over \( [\theta, 0] \) and exponentiating yields
\[
M(\theta) \leq \exp \left( \frac{\mu \theta^2}{\sigma^2} \right),
\]
(22)
showing the claimed bound on \( M(\theta) \) for \( \theta < 0 \). The inequality in (22) implies that for all \( t > 0 \) and \( \theta < 0 \),
\[
P \left( \frac{Y - \mu}{\sigma} \leq -t \right) \leq \exp \left( \theta t + \frac{\mu \theta^2}{\sigma^2} \right).
\]
Taking \( \theta = -t \sigma^2 / (2 \mu) \) we obtain (6).
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