Geometric juggling with $q$-analogues

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May 13, 2014

Abstract

We derive a combinatorial equilibrium for bounded juggling patterns with a random, $q$-geometric throw distribution. The dynamics are analyzed via rook placements on staircase Ferrers boards, which leads to a steady-state distribution containing $q$-rook polynomial coefficients and $q$-Stirling numbers of the second kind. We show that the equilibrium probabilities of the bounded model can be uniformly approximated with the equilibrium probabilities of a corresponding unbounded model. This observation leads to new limit formulae for $q$-analogues.

Keywords: juggling pattern; $q$-Stirling number; Ferrers board; Markov process; combinatorial equilibrium

1 Introduction

There is rich interaction between combinatorial models of juggling with both algebraic geometry and probability theory, see e.g. [7, 8] and references therein.

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In this paper we derive an explicit equilibrium distribution for random patterns bounded to the interval \([0 \cdots m - 1] \subset \mathbb{Z}_+\) and having geometrically distributed throw heights with a parameter \(q \in (0, 1)\). The equilibrium probability of having particles at heights \(B \subset \mathbb{Z}_+\) is shown to be proportional to
\[
\prod_{x \in B} \left[ \left\lfloor x \cdots m \right\rfloor \setminus B \right] q^x, \tag{1.1}
\]
where \([k]_q = (1 - q^k)/(1 - q)\) denotes the \(q\)-analogue of an integer \(k\). This formula is a natural interpolation of the corresponding equilibria \(\prod_{x \in B} q^x\) of the unbounded system [8] and \(\prod_{x \in B} \left\lfloor x \cdots m \right\rfloor \setminus B\) of the bounded system with uniformly distributed throw heights [13]. Note that the product term in (1.1) occurs as a \(q\)-rook polynomial coefficient, see (2.10) and [6, eq. (21)]. Indeed, the key ingredient in our proof is to extend the juggling dynamics and view the extended states as rook placements on staircase Ferrers boards.

We also show that the equilibrium of the bounded geometric pattern with large \(m\) can be uniformly approximated with the Gibbs measure of the unbounded system considered in [8]. The proof utilizes the ultrafast mixing property of the unbounded geometric system along with a stochastic coupling argument.

As an application we can easily obtain the following types of convergence formulae for \(q\)-analogues and their related \(q\)-Stirling numbers. Keeping \(n\) fixed and letting \(m \to \infty\) in (1.1), we obtain
\[
\lim_{m \to \infty} Z^{-1}[m - n + 1]_q^n = (q; q)_n,
\]
where \((q; q)_n = \prod_{k=1}^{n}(1 - q^k)\) is the \(q\)-Pochhammer symbol, and
\[
Z = Z(m, n, q) = q^{-n+\left\lfloor \frac{m+1}{2} \right\rfloor} S_{1/q}(m + 1, m - n + 1). \tag{1.2}
\]
Here \(S_q(a, b)\) denotes a \(q\)-Stirling number of the second kind, defined by the recursion
\[
S_q(a + 1, b) = q^{b-1} S_q(a, b - 1) + [b]_q S_q(a, b) \quad (0 \leq b \leq a)
\]
with the initial conditions \(S_q(0, 0) = 1\) and \(S_q(a, b) = 0\) for \(b < 0\) or \(b > a\).

Further by letting \(m, n \to \infty\) in (1.1) so that \(m - n \to \infty\), we obtain
\[
\lim_{m, n \to \infty} Z^{-1}[m - n + 1]_q^n = \phi(q),
\]
where \(\phi(q) = \prod_{k=1}^{\infty}(1 - q^k)\) is the Euler function and \(Z\) is as in (1.2) above.

Because periodic juggling patterns may be viewed as affine Weyl group elements [7], our model relates to a larger family of Markov chains in algebraic
geometry whose stationary distributions have a combinatorial description. Probabilistic formulae involving $q$-combinatorics have also been found in the context of birth processes related to the number of sources and paths in directed random graphs [1] as well as approximate counting [10]. Finally, $q$-Stirling numbers of the second kind also appear in connection with periodic juggling patterns [3].

2 Bounded juggler’s exclusion process

A generic model for random juggling patterns called the juggler’s exclusion process (JEP) was recently introduced by Leskelä and Varpanen in [8]. The bounded JEP with $n$ particles and $m \geq n$ admissible particle heights $[0 \cdots m - 1]$ is a random sequence $(X_0, X_1, \ldots)$ of $n$-element subsets of $[0 \cdots m - 1]$ such that:

- $X_{t+1} = X_t - 1$ when $0 \notin X_t$. (All particles fall down by one position when the hand is empty).
- $X_{t+1} = X_t^* \cup \{\eta_t\}$ when $0 \in X_t$, where $X_t^* = (X_t \setminus 0) - 1$ and $\eta_t$ is a random integer in $[0 \cdots m - 1] \setminus X_t^*$. (The particle at hand is thrown into height $\eta_t$, not allowing collisions with the other particles.)

Analytically, the most interesting special case of the bounded model with $n$ particles and $m \geq n$ admissible heights is the bounded geometric JEP, where the throw heights $\eta_t$ are defined as follows. Denote by $\ell = m - n + 1$ the number of vacant throw heights, and let $(\xi_1, \xi_2, \ldots)$ be a sequence of independent random integers on $[0 \cdots \ell - 1]$ each having an $\ell$-truncated geometric distribution with parameter $q \in (0, 1)$ so that

$$P(\xi_t = x) = \left(1 - \frac{q}{1 - q^\ell}\right)q^x, \quad x \in [0 \cdots \ell - 1]. \tag{2.1}$$

Let $A \subset \mathbb{Z}_+$ be finite and denote by $\theta_A$ the order-preserving bijection from $\mathbb{Z}_+$ onto $\mathbb{Z}_+ \setminus A$. Then the random throw heights $\eta_t$ in the bounded geometric JEP are given by

$$\eta_t = \theta_{X_t^*}(\xi_t).$$

By [8, Thm. 2.1] we know that the bounded geometric JEP with $n$ particles and $m$ heights is ergodic and characterized by a unique equilibrium probability distribution. We denote the equilibrium probabilities by $\pi_{m,n,q}(B)$, $B \in \mathbb{Z}_+^{(n)} = \{A \subset \mathbb{Z}_+ : |A| = n\}$. 

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2.1 A combinatorial formula for the equilibrium

The probability transition matrix $P$ of an $n$-particle JEP acts on probability distributions $\mu$ on $\mathbb{Z}_{+}^{(n)}$ according to $\mu \mapsto \mu P$ as follows. Denote $B = \{i_1, \ldots, i_n\} \in \mathbb{Z}_{+}^{(n)}$ and $B + 1 = \{i_1 + 1, \ldots, i_n + 1\}$. At each step the juggler either has just waited or thrown one of the particles. Therefore

$$\mu P(B) = \mu(B + 1) + \sum_{k=1}^{n} \mu(\{0\} \cup (B \setminus \{i_k\} + 1)) h^{B \setminus \{i_k\}}(i_k),$$

where $h^{B \setminus \{i_k\}}(i_k)$ is the probability of having thrown the particle $i_k \in B$ while the other particles drifted from $B + 1$ to $B$. Hence the problem of finding a stationary distribution amounts to finding a probability distribution $\pi$ on $\mathbb{Z}_{+}^{(n)}$ such that

$$\pi(B) = \pi(B + 1) + \sum_{k=1}^{n} \pi(\{0\} \cup (B \setminus \{i_k\} + 1)) h^{B \setminus \{i_k\}}(i_k). \quad (2.2)$$

In [8] a solution was found in the one-particle general case as well as in the unbounded geometric case, and in [13] a solution was found in the bounded case with uniform throws. Solving the equation in the general unbounded case seems hard if not impossible. Also obtaining a closed-form solution in the general bounded case seems difficult.

**Theorem 2.1.** In the bounded geometric JEP the equilibrium is

$$\pi_{m,n,q}(B) = Z^{-1} \prod_{x \in B} \left[ [x \cdots m] \setminus B \right] q^x, \quad (2.3)$$

where $Z$ is as in (1.2).

We prove Theorem 2.1 by considering an extended process where the states are staircase Ferrers boards with $m + 1$ columns and with $n$ non-taking rooks and where the dynamics is best introduced by an example. Note that in the literature [4, 11] a staircase Ferrers board contains a void column; thus a board with $m + 1$ columns refers to a board with highest column length $m$. In the example below we have $m = 6$ (board with 7 columns) and $n = 3$:
The figure depicts one extension of the situation where the non-extended state \{0, 2, 3\} is followed by the non-extended state \{1, 2, 4\}. In the extended dynamics the particles drift diagonally downwards from the left to the right, and the particles are thrown from the bottom row to the leftmost column. The idea is to keep track not only of the remaining flight time (vertical axis) but also of the elapsed flight time (horizontal axis) of the particles. The ”non-taking rook” model is clearly implied by the exclusion rule that no two particles may collide. The concept of an extended state already appeared in [13], albeit in a different form.

We next introduce a \( q \)-counting statistic \( \text{circ}(C) \) for a configuration \( C \) of non-taking rooks on a staircase Ferrers board. The statistic is also best introduced by an example \((m = 6, n = 3)\):

![Diagram](image)

Given a non-attacking rook configuration \( C \), we disable all positions below and to the left of any rook, and fill circles to the remaining positions in those rows that have a rook. Then \( \text{circ}(C) \) counts the total number of circles in \( C \). In the figure \( \text{circ}(C) = 3 \).

**Definition 2.2.** In what follows, we denote by \( C_n(S_{m+1}) \) the set of all configurations of \( n \) non-taking rooks on a staircase Ferrers board \( S_{m+1} \) with \( m + 1 \) columns.

**Lemma 2.3.**

\[
\sum_{C \in C_n(S_{m+1})} q^{\text{circ}(C)} = G_q(m + 1, m - n + 1),
\]

where the numbers \( G_q(a, b) \) are defined by the recursion formula

\[
G_q(a + 1, b) = G_q(a, b - 1) + [b]_q G_q(a, b),
\]

where \( 0 \leq b \leq a \), \( G_q(0, 0) = 1 \) and \( G_q(a, b) = 0 \) for \( b < 0 \) or \( b > a \).

**Proof.** We use a method similar to [4, Thm. 1.1]. A configuration in \( C_n(S_{m+1}) \) is obtained by placing either all the \( n \) rooks in rows other than the bottom row in \( S_{m+1} \) or by placing one of the rooks in the bottom row. In the first case there are no additional circles compared to the corresponding
configuration in $C_n(S_m)$. In the second case, if the new rook in the bottom row is placed in the $i$th available square counting from right to left, there are $i - 1$ additional circles compared to the corresponding configuration in $C_{n-1}(S_m)$. Denoting

$$K_q(n, m + 1) = \sum_{C \in C_n(S_{m+1})} q^{\text{circ}(C)}$$

and summing over all configurations in $C_n(S_{m+1})$ we obtain

$$K_q(n, m + 1) = K_q(n, m) + [m - n + 1]_q K_q(n - 1, m), \quad (2.5)$$

because there is a total of $m - n + 1$ available squares in the second case considered above. Thus

$$G_q(m + 1, m + 1 - n) = K_q(n, m + 1)$$

by comparing (2.5) with (2.4).

The numbers $G_q(a, b)$ are called Gould $q$-Stirling numbers of the second kind [5] or modified $q$-Stirling numbers of the second kind [2]. They are related to the numbers $S_q(a, b)$ by

$$S_q(a, b) = q^{\binom{b}{2}} G_q(a, b). \quad (2.6)$$

Next consider the equilibrium equation for the extended process. Again the juggler either has just waited or thrown any of the particles, so the equation is similar to (2.2). We write it as

$$\mu(C) = \mu(\hat{C}) + \sum_{C^*} \mu(C^*) h(C^*, C), \quad (2.7)$$

where $\hat{C} \in C_n(S_{m+1})$ denotes the unique predecessor of $C \in C_n(S_{m+1})$ when the juggler has moved from state $\hat{C}$ to state $C$ by waiting, $C^* \in C_n(S_{m+1})$ denotes a predecessor of $C$ when the juggler has moved from state $C^*$ to state $C$ by a throw with probability $h(C^*, C)$, and the sum is taken over all predecessors of $C$. Note that $\hat{C}$ is void if there is a particle in the leftmost column of $C$, and that the sum is empty if there is no particle in the leftmost column of $C$.

**Lemma 2.4.** The equation (2.7) is solved by

$$\mu(C) = \frac{q^{-\text{circ}(C)}}{G_{1/q}(m + 1, m - n + 1)}. \quad (2.8)$$
Proof. Clearly \( \text{circ}(\hat{C}) = \text{circ}(C) \) when the juggler has waited. If the juggler has thrown a particle to the \( j \)th available square, counting from the top down in the leftmost column of \( C \), the throw probability has been \( h(C^*, C) = q^{-j-1}/(m - n + 1)_{1/q} \). Moreover, removing the particle from the leftmost column of \( C \) decreases the circle count by \( j - 1 \). If we similarly remove a particle from the \( i \)th non-taking square (counting from right to left) of the bottom row in \( C^* \), we decrease the circle count by \( i - 1 \). Summing over the \( m - n + 1 \) predecessors of \( C \) and keeping track of the circle count we obtain

\[
q^{-\text{circ}(C)} = \sum_{i=1}^{m-n+1} q^{-\text{circ}(C)-(i-1)+j-1} \frac{q^{-(j-1)}}{(m - n + 1)_{1/q}}.
\]

Hence the equation (2.7) is solved by \( \tilde{\mu}(C) = q^{-\text{circ}(C)} \), and normalization yields the denominator \( G_{1/q}(m + 1, m - n + 1) \) by Lemma 2.3.

**Lemma 2.5.** Let \( B \subset \{0 \cdots m - 1\}, |B| = n \). The sum

\[
\sum_C q^{-\text{circ}(C)}
\]

taken over all possible extensions \( C \in C_n(S_{m+1}) \) of \( B \), equals

\[
\prod_{x \in B} \left( \left| [x \cdots m] \setminus B \right| \right)_{1/q}.
\]

**Proof.** We proceed by induction on \( n \). If \( n = 1 \), there are \( m - x \) possible extensions \( C \) of \( B = \{x\} \subset \{0, m - 1\} \), and the sum (2.9) is \( 1 + q^{-1} + \cdots + q^{-m+1+x} = [m - x]_{1/q} \). When \( n \geq 2 \) consider \( x = \min B \). There are \( m - x - (n - 1) \) ways of placing a non-taking rook to row \( x \in S_{m+1} \) after \( n - 1 \) non-taking rooks have already been placed to higher rows. The claim follows.

**Proof of Theorem 2.1.** By the two previous Lemmas we have

\[
\pi_{m,n,q}(B) = \frac{\prod_{x \in B} \left( \left| [x \cdots m] \setminus B \right| \right)_{1/q}}{G_{1/q}(m + 1, m - n + 1)},
\]

i.e. by (2.6),

\[
\pi_{m,n,q}(B) = \frac{\prod_{x \in B} \left( \left| [x \cdots m] \setminus B \right| \right)_{1/q}}{q^{(m-n+1)/2}S_{1/q}(m + 1, m - n + 1)}.
\]
Denoting $B = \{i_1, \ldots, i_n\}$ with $i_1 < \cdots < i_n$, we have
\[
\prod_{x \in B} \left[ \left\lfloor x \cdots m \right\rfloor \setminus B \right]^{1/q} = \prod_{k=1}^{n} [m - n - i_k + k]^{1/q}. \tag{2.10}
\]

Using (2.10) along with the fact that $[k]_q = q^{k-1}[k]_{1/q}$ for any $k$, we obtain
\[
\pi_{m,n,q}(B) = \prod_{x \in B} \left[ \left\lfloor x \cdots m \right\rfloor \setminus B \right]^{q^x} q^{n + \binom{m+1}{2}} S_{1/q}(m+1, m-n+1)
\]
as claimed. \hfill \Box

### 3 Convergence

#### 3.1 Total variation distance and coupling

We recall some definitions from probability that are used in the sequel. A probability distribution on a countable set $S$ is a function $\mu : S \to \mathbb{R}_+$ such that $\sum_{A \in S} \mu(A) = 1$. The total variation distance between two probability distributions $\mu$ and $\nu$ on a countable set $S$ is defined by
\[
||\mu - \nu||_{TV} = \frac{1}{2} \sum_{A \in S} |\mu(A) - \nu(A)|.
\]

A coupling of $\mu$ and $\nu$ is a random vector $(X, Y)$ with values in $S \times S$ such that $P(X = A) = \mu(A)$ and $P(Y = B) = \nu(B)$ for all $A, B \in S$. A coupling $(X, Y)$ of $\mu$ and $\nu$ is maximal if
\[
P(X \neq Y) = ||\mu - \nu||_{TV}.
\]
A maximal coupling of $\mu$ and $\nu$ always exists (e.g. [9, Prop. 4.7]).

#### 3.2 Bounded geometric JEP with many vacant heights

Consider the bounded geometric JEP with $n$ particles, $m \geq n$ admissible heights, and a throw height parameter $q \in (0, 1)$. We will show that when the number of vacant throw heights $\ell = m - n + 1$ is large, the equilibrium probabilities of the bounded geometric JEP can be uniformly approximated by the equilibrium of the unbounded geometric JEP considered in [8]. The unbounded geometric JEP is defined by replacing the $\ell$-truncated geometric distribution in (2.1) by a standard geometric random distribution so that
\[
P(\xi_t = x) = (1-q)q^x, \quad x \in \mathbb{Z}_+.
\]
We denote the transition probability matrix of the bounded geometric JEP by $P_{m,n,q}$, and we view its equilibrium $\pi_{m,n,q}$ as a probability distribution on $\mathbb{Z}_+^{(n)}$, although all its mass is concentrated on $[0 \cdots m - 1]^{(n)}$. We denote by $P_{\infty,n,q}$ the transition matrix of the unbounded $q$-geometric JEP with $n$ particles, and by $\pi_{\infty,n,q}$ its equilibrium distribution given by [8, Thm. 3.3]

$$\pi_{\infty,n,q}(B) = \frac{(q;q)_n}{q^{(n)}} \prod_{x \in B} q^x, \quad B \in \mathbb{Z}_+^{(n)}. \quad (3.1)$$

To state the approximation result in its most general form, we consider a sequence of bounded geometric JEPs indexed by $k = 1, 2, \ldots$ where the $k$-th process has $n(k)$ particles, $m(k) \geq n(k)$ admissible heights, and a throw height parameter $q(k) \in (0, 1)$. For the reader’s convenience we shall omit the symbol $k$ in what follows.

**Theorem 3.1.** Assume that the number of vacant throw heights $\ell = m - n + 1$ in the sequence of bounded geometric JEPs satisfies

$$\ell \log q^{-1} - \log m \to \infty \quad \text{as } k \to \infty. \quad (3.2)$$

Then the total variation distance between the equilibria of the bounded JEP and its unbounded variant satisfies

$$||\pi_{m,n,q} - \pi_{\infty,n,q}||_{TV} \to 0 \quad \text{as } k \to \infty.$$

We first prove

**Lemma 3.2.** Consider two different $n$-particle JEPs on $\mathbb{Z}_+^{(n)}$, one with throw height distribution $\nu$ and the other with throw height distribution $\hat{\nu}$, and denote by $\mu$ and $\hat{\mu}$ their corresponding initial distributions. Then

$$||\mu \hat{P}^t - \hat{\mu} \hat{P}^t||_{TV} \leq 1 - (1 - ||\mu - \hat{\mu}||_{TV})(1 - ||\nu - \hat{\nu}||_{TV})^t$$

for all $t \geq 0$.

*Proof.* Let $(X_0, \hat{X}_0)$ be a maximal coupling of $\mu$ and $\mu'$, so that $||\mu - \hat{\mu}||_{TV} = P(X_0 \neq \hat{X}_0)$. Similarly, let $(\xi, \hat{\xi})$ be a maximal coupling of $\nu$ and $\hat{\nu}$. Let $((\xi_1, \xi_1), (\xi_2, \xi_2), \ldots)$ be an independent sequence of copies of $(\xi, \hat{\xi})$, which is also independent of $(X_0, \hat{X}_0)$.

Generate a path $(X_s)_{s \geq 0}$ of the first JEP using the initial configuration $X_0$ and throw heights $\xi_1, \xi_2, \ldots$, and a path $(\hat{X}_s)_{s \geq 0}$ of the second JEP from initial configuration $\hat{X}_0$ using throw heights $\hat{\xi}_1, \hat{\xi}_2, \ldots$. Because the pair $(X_t, \hat{X}_t)$ is a coupling of $\mu \hat{P}^t$ and $\hat{\mu} \hat{P}^t$, it follows that [12, Sec. 1.5.4]

$$||\mu \hat{P}^t - \hat{\mu} \hat{P}^t||_{TV} \leq P(X_t \neq \hat{X}_t).$$
Because $X_t = \hat{X}_t$ on the event $\{X_0 = \hat{X}_0\} \cap \left( \bigcap_{s=1}^{t} \{\xi_s = \hat{\xi}_s\} \right)$, it follows that

$$P(X_t \neq \hat{X}_t) \leq 1 - P(X_0 = \hat{X}_0) P(\xi = \hat{\xi})^t.$$ 

The claim follows after combining the above inequalities.

**Proof of Theorem 3.1.** A key ingredient of the proof is to note that the unbounded geometric JEP reaches its equilibrium exactly after all particles have been thrown [8, Thm. 3.2]. Because $\pi_{m,n,q}$ is supported on $[0 \cdots m - 1]$, all particles in the unbounded geometric JEP with initial distribution $\pi_{m,n,q}$ have been thrown by time $m$. As a consequence, $\pi_{m,n,q}P_{\infty,n,q}^t = \pi_{\infty,n,q}$ for all $t \geq m$. Because $\pi_{m,n,q}$ is invariant for $P_{m,n,q}^t$, this implies that

$$\pi_{m,n,q} - \pi_{\infty,n,q} = \pi_{m,n,q}P_{m,n,q}^t - \pi_{m,n,q}P_{\infty,n,q}^t \quad (3.3)$$

for $t \geq m$.

Because the total variation distance between the standard and the $\ell$-truncated geometric distribution equals $q^\ell$, Lemma 3.2 shows that

$$||\pi_{m,n,q}P_{\infty,n,q}^t - \pi_{m,n,q}P_{\infty,n,q}^t|| \leq 1 - (1 - q^\ell)^t.$$ 

By substituting $t = m$ into (3.3), it thus follows that

$$||\pi_{m,n,q} - \pi_{\infty,n,q}|| \leq 1 - (1 - q^\ell)^m \leq mq^\ell,$$

where the right side tends to zero due to assumption (3.2). 

### 3.3 Combinatorial limit formulae

**Corollary 3.3.** For any $n \in \mathbb{Z}_+$,

$$\lim_{m \to \infty} Z^{-1}[m - n + 1]^n_q = (q; q)_n, \quad (3.4)$$

where $Z$ is as in (1.2).

**Proof.** For the ground state $B = [0 \cdots n - 1]$, the equilibrium (3.1) for the unbounded process reduces to $\pi_{\infty,n,q}(B) = (q; q)_n$, and by (1.1) we have $\pi_{m,n,q}(B) = Z^{-1}[m - n + 1]^n_q$. The claim follows from Theorem 3.1.

**Corollary 3.4.** When both $n \in \mathbb{Z}_+$ and $m \in \mathbb{Z}_+$ approach infinity such that 3.2 holds (especially such that $m - n \to \infty$), we have

$$\lim_{m,n \to \infty} Z^{-1}[m - n + 1]^n_q = \phi(q), \quad (3.5)$$

where $Z$ is as in (1.2).
Proof. For any \( q \in (0, 1) \), \((q; q)_n\) converges to \( \phi(q) \) as \( n \to \infty \). The claim then follows as in the proof of Corollary 3.3. \( \square \)

**Remark 3.5.** Corollary 3.3 holds in a more general form, because one could have any \( B \in \mathbb{Z}_+^{(n)} \) in the proof and still obtain equilibria for random juggling processes in both bounded and unbounded settings. However, in Corollary 3.4 the right-hand side \( \phi(q) \) is not a proper JEP equilibrium, because the unbounded process fills \( \mathbb{Z}_+ \) when the number of particles approaches infinity. We will address this issue more carefully in a subsequent paper; it turns out that \( \phi(q) \) is the ground state equilibrium in a virtual setting considered in [7].

**Acknowledgements**

We thank Allen Knutson and Greg Warrington for inspiration.

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