Research Article

Wenhui Luo*

Further extensions of Hartfiel’s determinant inequality to multiple matrices

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Abstract: Following the recent work of Zheng et al., in this paper, we first present a new extension Hartfiel’s determinant inequality to multiple positive definite matrices, and then we extend the result to a larger class of matrices, namely, matrices whose numerical ranges are contained in a sector. Our result complements that of Mao.

Keywords: determinant inequality, positive definite matrix, numerical range, sector matrix

MSC: 15A45, 15A60

1 Introduction

Throughout the paper, we denote by $\mathbb{M}_n$ the set of $n \times n$ complex matrices. Recall that the numerical range (see, e.g., [3]) of $A \in \mathbb{M}_n$ is defined as the set on the complex plane

$$W(A) = \{v^*Av : v \in \mathbb{C}^n, \ v^*v = 1\}.$$ 

For a fixed $\theta \in [0, \pi/2)$, obviously the set on the complex plane

$$S_\theta = \{z \in \mathbb{C} : \Re z > 0, |\Im z| \leq (\Re z) \tan \theta\}$$

is a sector excluding the vertex. We shall mainly consider matrices whose numerical range is contained in $S_\theta$, the so called sector matrices [11]. For any $A \in \mathbb{M}_n$, its real (or Hermitian) part is denoted by $\Re A := (A + A^*)/2$. Clearly, if $W(A) \subset S_\theta$, then $\Re A$ is positive definite.

A fundamental determinant inequality states that if $A, B \in \mathbb{M}_n$ are positive definite, then

$$\det(A + B) \geq \det A + \det B. \quad (1)$$

In [5], Haynsworth proved the following improvement of (1).

**Theorem 1.1.** [5, Theorem 3] Suppose $A, B \in \mathbb{M}_n$ are positive definite. Let $A_k$ and $B_k$, $k = 1, \ldots, n - 1$, denote the $k$th leading principal submatrices of $A$ and $B$, respectively. Then

$$\det(A + B) \geq \left(1 + \sum_{k=1}^{n-1} \frac{\det B_k}{\det A_k}\right) \det A + \left(1 + \sum_{k=1}^{n-1} \frac{\det A_k}{\det B_k}\right) \det B.$$

Using a clever argument, Hartfiel [4] refined Haynsworth’s result by adding a nonnegative term on the right side of the inequality.

*Corresponding Author: Wenhui Luo: Department of Education and Educational Technology, Jiangmen Polytechnic, Jiangmen, Guangdong, China, E-mail: Lwh-822@163.com

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Theorem 1.2. [4] Under the same condition as in Theorem 1.1,
\[
\det(A + B) \geq \left(1 + \sum_{k=1}^{n-1} \frac{\det B_k}{\det A_k}\right) \det A + \left(1 + \sum_{k=1}^{n-1} \frac{\det A_k}{\det B_k}\right) \det B + (2^n - 2n)\sqrt{\det AB}.
\]

Hartfiel’s inequality (2) has been extended to sector matrices by a number of authors; see [8, 10, 14, 17]. In [8], Hou and Dong extended Hartfiel’s inequality to a triple of matrices. By making use of Hou and Dong’s result, Zheng et al. [17] improved and extended the main result in [10], moreover, they obtained the following two theorems.

Theorem 1.3. [17, Theorem 2.6] Suppose \(A, B \in M_n\) such that \(W(A), W(B) \subset S_0\). Let \(A_k\) and \(B_k\), \(k = 1, \ldots, n - 1\), denote the \(k\)th leading principal submatrices of \(A\) and \(B\), respectively. Then
\[
\sec^{2n-1}(\theta) \det(A + B) \geq \left(1 + \sum_{k=1}^{n-1} \frac{\det B_k}{\det A_k}\right) |\det A| + \left(1 + \sum_{k=1}^{n-1} \frac{\det A_k}{\det B_k}\right) |\det B| + (2^n - 2n)\sqrt{|\det AB|}.
\]

Remark 1.4. In [10], Lin proved a weaker result with the first coefficient \(\sec^{3n-2}(\theta)\) instead of \(\sec^{2n-1}(\theta)\).

Theorem 1.5. [17, Theorem 2.8] Let \(M, N, L \in M_n\) such that \(W(M), W(N), W(L) \subset S_0\). Then it holds
\[
|\det(M + N + L)| \geq \prod_{j=1}^{n} \left(\frac{|\det N_j|}{\det N_{j-1}} + \frac{|\det L_j|}{\det L_{j-1}}\right) \cos^j(\theta) + \prod_{j=1}^{n} \left(\frac{|\det L_j|}{\det L_{j-1}} + \frac{|\det M_j|}{\det M_{j-1}}\right) \cos^j(\theta) + \prod_{j=1}^{n} \left(\frac{|\det M_j|}{\det M_{j-1}} + \frac{|\det N_j|}{\det N_{j-1}}\right) \cos^j(\theta) - (|\det M| + |\det N| + |\det L|),
\]
where by convention, \(\det M_0 = \det N_0 = \det L_0 = 1\).

Very recently, Mao in [14] extended Theorem 1.3 to any number of sector matrices. More precisely, she obtained the following result.

Theorem 1.6. Let \(A_j \in M_n\) with \(W(A_j) \subset S_0\), and let \(A_{jk}\), \(k = 1, \ldots, n - 1\), denote the \(k\)th leading principal submatrix of \(A_j\), \(j \in M := \{1, \ldots, m\}\). Then
\[
\sec^{2n-1}(\theta) \left|\det\left(\sum_{j=1}^{m} A_j\right)\right| \geq \sum_{j=1}^{m} \left(1 + \sum_{k=1}^{n-1} \frac{\sum_{i \notin j} |\det A_{jk}|}{\det A_{jk}}\right) |\det A_j| + (2^n - 2n) \sum_{1 \leq i \neq j \leq m} \sqrt{|\det A_i A_j|}.
\]

The main goal of the present paper is to extend Theorem 1.5 to any number of sector matrices. To this end, we first present a relevant result for positive definite matrices. Some corollaries are included.

2 Main Results

Before stating our results, we need to present some lemmas that are useful in our proofs.
The first lemma is folklore in matrix analysis.

**Lemma 2.1.** [5, Lemma 2] Let $A, B \in M_n$ be positive definite and let $A_j, B_j$ denote the $k$th leading principal submatrix of $A, B$, respectively. Then

$$\frac{\det(A_j + B_j)}{\det(A_{j-1} + B_{j-1})} \geq \frac{\det A_j}{\det A_{j-1}} + \frac{\det B_j}{\det B_{j-1}}, \quad j = 1, \ldots, n,$$

where by convention $\det A_0 = \det B_0 = 0$.

The second lemma is known as the Ostrowski-Taussky inequality (see [7, p. 510]).

**Lemma 2.2.** Let $A \in M_n$ with $A$ positive definite. Then

$$\det(A) \leq |\det A|.$$

The third lemma gives a reverse of the Ostrowski-Taussky inequality.

**Lemma 2.3.** [10, Lemma 2.6] Let $A \in M_n$ with $W(A) \subset S_\theta$. Then

$$\det(A) \geq \cos^n(\theta) \cdot |\det A|.$$

The fourth lemma pays a key role in our new extension of Hartfiel’s inequality to multiple positive definite matrices.

**Lemma 2.4.** (ref. [1, Corollary 4.4]) Suppose $A_j \in M_n, j \in 1, \ldots, m$, are positive definite. Then

$$\det \left( \sum_{j=1}^m A_j \right) + (m - 2) \sum_{j=1}^m \det A_j \geq \sum_{1 \leq i \neq j \leq m} \det(A_i + A_j).$$

It is worthy to mention that in [1, Corollary 4.4], the authors stated their result for the generalized matrix functions, which includes the determinant as a special case. The inequality in Lemma 2.4 is a direct extension of the main result in [13].

Below is our extension of Hartfiel’s inequality to multiple positive definite matrices.

**Proposition 2.5.** Let $A_j \in M_n, j = 1, \ldots, m$, be positive definite, and let $A_{jk}, k = 1, \ldots, n - 1$, denote the $k$th leading principal submatrix of $A_j$. Then

$$\det \left( \sum_{j=1}^m A_j \right) \geq \sum_{k=1}^n \prod_{1 \leq i \neq j \leq m} \left( \frac{\det A_{jk}}{\det A_{i(k-1)}} + \frac{\det A_{jk}}{\det A_{j(k-1)}} \right) - (m - 2) \sum_{j=1}^m \det A_j,$$

where by convention $\det A_{j0} = 1$ for all $j$.

**Proof.** Applying Lemma 2.1 to $A_1, A_j$ gives

$$\frac{\det(A_{jk} + A_{jk})}{\det(A_{i(k-1)} + A_{j(k-1)})} \geq \frac{\det A_{jk}}{\det A_{i(k-1)}} + \frac{\det A_{jk}}{\det A_{j(k-1)}}, \quad j = 1, \ldots, n.$$

Taking products for $k$ from 1 to $n$ gives

$$\det(A_j + A_j) \geq \prod_{k=1}^n \left( \frac{\det A_{jk}}{\det A_{i(k-1)}} + \frac{\det A_{jk}}{\det A_{j(k-1)}} \right).$$

Now by Lemma 2.4, we have

$$\det \left( \sum_{j=1}^m A_j \right) \geq \sum_{1 \leq i \neq j \leq m} \det(A_i + A_j) - (m - 2) \sum_{j=1}^m \det A_j.$$
This completes the proof.

The following result extends Theorem 1.5.

**Theorem 2.6.** Let \( A_j \in \mathbb{M}_n, j = 1, \ldots, m, \) such that \( W(A_j) \subset S_\theta, \) and let \( A_{jk}, k = 1, \ldots, n - 1, \) denote the \( k \)th leading principal submatrix of \( A_j. \) Then it holds

\[
\left| \det \left( \sum_{j=1}^{m} A_j \right) \right| \geq \sum_{1 \leq i < j \leq m} \prod_{k=1}^{n} \left( \frac{\det A_{ik}}{\det A_{(k-1)i}} + \frac{\det A_{jk}}{\det A_{(k-1)j}} \right) - (m - 2) \sum_{j=1}^{m} \det A_j,
\]

where by convention \( \det A_{j0} = 1 \) for all \( j.\)

**Proof.** First of all, since \( W(\sum_{j=1}^{m} A_j) \subset S_\theta, \) then by Lemma 2.2,

\[
\left| \det \left( \sum_{j=1}^{m} A_j \right) \right| \geq \det \left( \sum_{j=1}^{m} A_j \right) = \det \left( \sum_{j=1}^{m} \Re A_j \right).
\]

As \( \Re A_j \) are positive definite for all \( j, \) we can apply Proposition 2.5 to get

\[
\left| \det \left( \sum_{j=1}^{m} A_j \right) \right| \geq \sum_{1 \leq i < j \leq m} \prod_{k=1}^{n} \left( \frac{\det \Re A_{ik}}{\det \Re A_{(k-1)i}} + \frac{\det \Re A_{jk}}{\det \Re A_{(k-1)j}} \right) - (m - 2) \sum_{j=1}^{m} \det \Re A_j
\]

\[
\geq \sum_{1 \leq i < j \leq m} \prod_{k=1}^{n} \left( \frac{\det \Re A_{ik}}{\det A_{(k-1)i}} + \frac{\det \Re A_{jk}}{\det A_{(k-1)j}} \right) - (m - 2) \sum_{j=1}^{m} \det A_j
\]

\[
\geq \sum_{1 \leq i < j \leq m} \prod_{k=1}^{n} \left( \frac{\det A_{ik}}{\det A_{(k-1)i}} + \frac{\det A_{jk}}{\det A_{(k-1)j}} \right) \cos^k \theta - (m - 2) \sum_{j=1}^{m} \det A_j,
\]

in which the second inequality is by Lemma 2.2 and the third inequality is by Lemma 2.3, respectively. This completes the proof. \( \square \)

A matrix \( A \in \mathbb{M}_n \) is called accretive-dissipative if both real part \( \Re A \) and imaginary part \( \Im A := (A - A^*)/2i \) (in the sense of Cartesian decomposition) are positive definite. This class of matrices has appeared in numerical linear algebra [2, 6, 12] and has been studied recently by a number of authors [9, 15, 16]. Note that \( M \) is accretive-dissipative if and only if \( W(e^{-i\pi/4}M) \subset S_{n/4}. \) This observation enables us to state the following two corollaries.

**Corollary 2.7.** Let \( A_j \in \mathbb{M}_n, j = 1, \ldots, m, \) be accretive-dissipative, and let \( A_{jk}, k = 1, \ldots, n - 1, \) denote the \( k \)th leading principal submatrix of \( A_j. \) Then it holds

\[
\left| \det \left( \sum_{j=1}^{m} A_j \right) \right| \geq \sum_{1 \leq i < j \leq m} \prod_{k=1}^{n} \left( \frac{\det A_{ik}}{\det A_{(k-1)i}} + \frac{\det A_{jk}}{\det A_{(k-1)j}} \right) 2^{-k/2} - (m - 2) \sum_{j=1}^{m} \det A_j,
\]

where by convention \( \det A_{j0} = 1 \) for all \( j.\)
Proof. Since $A_j \in \mathbb{M}_n$ are accretive-dissipative, we have $W(e^{-it/\theta}A_j) \subset S_{n/\theta}$ for all $j$. Then we apply Theorem 2.6 to the matrices $e^{-it/\theta}A_j$ to get the result, because in this case $\cos^k \theta = 2^{-k/2}$.

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