Tests of AdS$_4$/CFT$_3$ correspondence for $\mathcal{N} = 2$ chiral-like theory

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Abstract

We investigate the superconformal index and the partition function for the chiral-like Chern-Simons-matter theory proposed for M2-branes probing the cones over $M^{3,2}$ and find perfect agreements with the gravity index and the gravitational free energy.
1 Introduction

There has been rapid progress in our understanding of $AdS_4/CFT_3$ correspondence in recent years. Given $AdS_4 \times Y$ with $Y$ being a Sasaki-Einstein 7-manifold, the corresponding superconformal field theory is realized as a supersymmetric Chern-Simons matter (SCSM) theory [1]. The general theories with $\mathcal{N} \geq 4$ supersymmetry were constructed in [2, 3] and a famous example out of such construction is the ABJM theory [4], which describes M2 branes on $C^4/Z_k$ with $k$ being related to the Chern-Simons level of the field theory. There have been various checks made for theories with $\mathcal{N} \geq 3$ supersymmetry such as the partition function, the famous $N^{3/2}$ behavior and the superconformal index [5, 6, 7, 8, 9, 10, 11, 12]. Some other aspects of the superconformal index are explored in, e.g., [13, 14].

If we consider theories with $\mathcal{N} = 2$ supersymmetry, we have much more diverse possibilities. It is not clear at this moment how to find field theory dual to a given gravitational background $AdS_4 \times Y$ with $\mathcal{N} = 2$ supersymmetry. For some special cases such as $M^{3,2}, Q^{1,1,1}, V^{5,2}, Y^{p,q}$, there are various proposals on the field theory duals [15, 16, 17, 18, 19, 20, 21]. A subtlety is that R-charges of the matters in such theories can have different values from the canonical one. It is proposed in [22] that the R-charge can be determined by minimizing the partition function of the field theory defined on $S^3$. Given this description, one can compute the partition function on $S^3$ of the proposed field theory and compare it with the gravitational free energy. Various impressive results are reported [23, 24, 25]. Also, for a special class of theories such as $M^{3,2}, Q^{1,1,1}$, the superconformal index is computed and comparison is made with the gravity index [26].

A curious technical aspect in computing the index and the partition function is that it is much easier to work out non-chiral theories in 4d sense. We call 3d theories ‘chiral-like theories’ if the matter fields are inherited from chiral matters in 4d. These chiral-like theories could suffer from parity anomalies and the right index computation needs to take care of this
issue properly. In the large $N$ computation of the partition function on $S^3$, one employs the saddle point approximation and it was crucial to have the vanishing long range force among the eigenvalues of the matrix model. This condition was not satisfied for chiral-like theories. However it is reasonable to expect that this obstacle is simply a technical one. The purpose of the current paper is to show that chiral-like theories can be dual to $AdS_4$ gravitational backgrounds. Specifically we consider one chiral-like theory, which is expected to be dual to $AdS_4 \times M^{3,2}$ (with nonzero discrete torsions). We compute its superconformal index and the partition function on $S^3$ to find perfect agreements with the gravity index and the gravitational free energy.

In [26], the index computation was carried out for the same theory as we considered here. Facing the parity anomaly, we have only considered the topological sectors in which the path integral is well defined. It was observed that this field theory index contains the dual gravity spectrum. However it also contains additional contribution, whose physical meaning was not clear. In [20], a method was proposed to resolve the parity anomaly. It turns out that if we introduce the off-diagonal Chern-Simons terms for $U(1)$ factors of $U(N)$ gauge groups, the theory becomes consistent. In this paper, we compute the superconformal index in the presence of the proposed off-diagonal Chern-Simons terms to find a perfect agreement with the gravity spectrum.

For the computation of the partition function, one of the crucial points of usual saddle point approximation of non-chiral theories is the cancelation of the long range forces on eigenvalues of the matrix model. Chiral-like theories do not enjoy this feature. In [27], it is suggested that if we consider the symmetrized partition function, it is possible that chiral-like theories can have the cancelation of the long range force. The underlying idea is a very simple one. If we consider the partition function, it has the form

$$Z = \int du F(u) \quad (1.1)$$

where the integration is done over the Cartan elements of the gauge group. As the measure $F(u)$ is not invariant under a $Z_2$ transformation $u \to -u$, the saddle point approximation does not exhibit this symmetry either. The proposal of [27] is to consider the symmetrized measure

$$Z = \frac{1}{2} \int du (F(u) + F(-u)) \quad (1.2)$$

which gives the same $Z$. Then the integrand of the integral has the $Z_2$ symmetry and the saddle point equation also respects this symmetry. We employ this method to the $M^{3,2}$ model to find that (log of) the partition function nicely agrees with the gravitational free energy.

The content of the paper is as follows. After the introduction, we introduce the field theory model of $M^{3,2}$ and carry out the index computation in section 2. In section 3, we work out the partition function on $S^3$ of the field theory to match the gravitational free energy.
As this work is completed, we receive the paper by Amariti et. al. [28], where similar topic is covered. However they just consider the partition function and we also consider the superconformal index for $M^{3,2}$ model to add more impressive evidences for $AdS_4/CFT_3$ correspondence.

2 Index computation

2.1 Review of previous results

The field theory proposed for M2 branes on $C(M^{3,2})$ is given by the quiver diagram in Fig 1. The Chern-Simons level for $U(N)$ is given by $(-2k, k, k)$ for $M^{3,2}/\mathbb{Z}_k$. For the brevity of notation, we shall often denote the fields by $X^I_{23} = A^I$, $X^I_{31} = B^I$, $X^I_{12} = C^I$. $I = 1, 2, 3$ is the triplet index for the $SU(3)$ global symmetry. This theory was initially studied by [18, 19]

![Figure 1: The quiver diagram of M2-branes on $C(M^{3,2})$](image)

and it was recently argued at [20] that this model is dual to $AdS_4 \times M^{3,2}$ with discrete torsion. Since the presence of the discrete torsion does not affect the gravity index in the large N limit, one can try to match the field theory index to that of the gravity side. The gravity index is given by the Plesythetic form

$$I(x, y_1, y_2, y_3) = \exp\left(\sum_n \frac{1}{n} I_{sp}(x^n, y_1^n, y_2^n, y_3^n)\right)$$

where $x, y_3$ are fugacities of the ‘energy+angular momentum’ and the Cartan of an $SU(2)$ isometry of $M^{3,2}$, respectively. $y_1, y_2$ are the fugacities of the $SU(3)$ Cartans. See [26] for more details. The $SU(2)$ Cartan is mapped to the $U(1)_B$ symmetry carried by the sum of monopole charges of the first gauge group $U(N)_{-2}$ where the subscript denotes the Chern-Simons level. For instance, at $y_1 = y_2 = 1$, the single particle index is given by

$$I_{sp}(x, y_3) = \frac{9x^2y_3}{(1 - x^2y_3)^2} + \frac{9x^2y_3^{-1}}{(1 - x^2y_3^{-1})^2}.$$
From the form of the single particle index, one can see that the large $N$ index exhibit the factorization structure

$$I(x, y_1, y_2, y_3) = I_0(x, y_1, y_2)\tilde{I}(x, y_1, y_2, y_3). \quad (2.3)$$

Here $I_0$ is the multi-particle gravity index from $U(1)_B$ neutral gravitons, which should match the large $N$ index of the field theory in the zero monopole flux sector. $\tilde{I}$ is the multi particle gravity index from gravitons carrying nonzero $U(1)_B$ charges. In the field theory side, $\tilde{I}$ should be obtained by summing all large $N$ index contribution $I'_{(p,q,r)}$ from non-zero monopole charges \{p\}_i, \{q\}_i, \{r\}_i in $U(N)_{-2},U(N)_1, U(N)_1$ gauge groups. One subtlety of the field theory of interest is that it is chiral in the 4d sense so it has odd number of bifundamental fields for each gauge group, hence it suffers from parity anomaly. It is shown in [26] that this parity anomaly makes the path integral inconsistent for certain values of monopole fluxes in the overall $U(1)$ of each $U(N)$ factor. [26] simply considered the sectors which are free of these anomaly effects.

Schematically, the large $N$ field theory index $I$ is given by

$$I = \frac{x^{e_0}}{(\text{symmetry})}\int \left[ \frac{d\alpha d\beta d\gamma}{(2\pi)^3} \right] e^{iS_{CS}^{(0)}(\alpha,\beta,\gamma)} e^{ib_0(\alpha,\beta,\gamma)} \exp \left[ \sum_{n=1}^{\infty} \frac{1}{n} f_{sp}^{(n)} \right]. \quad (2.4)$$

where the integration is done over the holonomy variables $\alpha_i, \beta_i, \gamma_i, \ i = 1 \ldots N$ of three $U(N)$s, $e_0$ is the zero point energy, $S_{CS}^{(0)}$ is the action, $b_0$ is the zero point charge contribution for a suitable saddle point configuration, which is specified by the monopole charges \{p\}_i, \{q\}_i, \{r\}_i in $U(N)_{-2}, U(N)_{1}, U(N)_{1}$ gauge groups. $f_{sp}$ is the single particle or the ‘letter’ index. Explicit expressions for $e_0, S_{CS}^{(0)}, b_0$ and $f'(^n)$ and the explanation of other variables are given in [26].

In the index formula, we take the dimensions of three chiral fields $A^I, B^I, C^I$ to be $a, b, c$ respectively. Taking into account of marginality of superpotential and discrete symmetry in the quiver diagram, one can take

$$a = 2 - 2b, \quad c = b. \quad (2.5)$$

It is shown in [26] that the zero monopole sector perfectly agrees with $I_0$ of the gravity index. The large $N$ index with nonzero monopole sectors $I'(x)$ can be divided into two pieces,

$$I'(x) = \tilde{I}(x) + I^b(x). \quad (2.6)$$

$\tilde{I}$ is the large $N$ index from factorizable monopole fluxes and $I^b$ is from non-factorizable fluxes. Factorizable fluxes satisfy $P_\pm = Q_\pm = R_\pm$ where $P_+ = \sum p_{i+}, P_- = \sum p_{i-}$ denote the sums of positive/negative fluxes in \{p\}_i and $Q_\pm, R_\pm$ are defined in a similar way. Factorizable fluxes have the following properties which distinguish them from non-factorizable fluxes.

\[Note that the large $N$ field theory index $I(x)$ is obtained by holonomy integration over the integrand containing $f_{sp}$. Similarly $f'(^n)$ denotes the single particle index with non-zero monopole flux, which gives rise to $I'(x)$ in eq. (2.6) after holonomy integration with summing over all non-zero monopole fluxes.\]
1. Factorization: \( I_{(p,q,r)} = I_{(p+,q+,r+)}I_{(p-,q-,r-)} \).

2. \( I_{(p,q,r)} \) is independent of R-charge assignment \( b \) that we have not specified.

Here \( p_\pm, q_\pm, r_\pm \) denote the positive/negative fluxes in \( \{ p_i \}, \{ q_i \} \) and \( \{ r_i \} \) respectively. In the \( \hat{I} \) part of the index, each factor with definite nonzero value of \( P_+ \) or \( P_- \) gives the same spectrum as the gravity index coming from multi-gravitons carrying positive or negative \( U(1)_B \) charges \( \pm P_\pm \), at least for a few low order terms in \( x \). \( I^b \) part gave additional spectrum which does not appear in the gravity side.

### 2.2 Index computation with mixed Chern-Simons terms

In [20], it was proposed to resolve the parity anomaly by introducing off-diagonal Chern-Simons terms for three \( U(1) \)s in \( U(N) \)s. The off diagonal CS term is given by

\[
\mathcal{L}_{\text{off-CS}} = \sum_{a,b=1}^{3} \frac{\Lambda_{ab}}{4\pi} \int \text{Tr} A_a \wedge d\text{Tr} A_b .
\]  

(2.7)

\( \Lambda_{ab} \) are symmetric \( 3 \times 3 \) matrices satisfying the following conditions

\[
\Lambda_{ab} - \frac{1}{2} A_{ab} \in \mathbb{Z} \quad \forall a, b .
\]  

(2.8)

\[
\sum_{a=1}^{3} \Lambda_{ab} = 0 \quad b = 1, 2, 3 .
\]  

(2.9)

where \( A_{ab} \) are adjacency matrix for the quiver diagram, \( A_{12} = A_{23} = A_{31} = 3 \). The first condition comes from the parity anomaly cancellation conditions and the second guarantees that chiral ring structure is not modified after introducing off diagonal CS terms. Let us call the diagonal components of \( \Lambda \) as \( l_1, l_2 \) and \( l_3 \), which determine the off-diagonal elements from (2.9):

\[
\Lambda = \begin{pmatrix}
  l_1 & \frac{1}{2}(-l_1 - l_2 + l_3) & \frac{1}{2}(-l_1 + l_2 - l_3) \\
  \frac{1}{2}(-l_1 - l_2 + l_3) & l_2 & \frac{1}{2}(-l_1 + l_2 - l_3) \\
  \frac{1}{2}(-l_1 + l_2 - l_3) & \frac{1}{2}(-l_1 - l_2 + l_3) & l_3
\end{pmatrix} .
\]  

(2.10)

From the conditions (2.8), \( l_1, l_2, l_3 \) should be integers satisfying the condition that \( \frac{1}{2}(-l_1 - l_2 + l_3) \) is half of an odd integer. As in [26], for generic monopole flux one obtains states whose energies are of order \( \mathcal{O}(N) \). As we are here interested in comparing our index with the low energy gravity index at large \( N \), let us analyze the conditions for monopole fluxes which yield \( \mathcal{O}(1) \) energy contribution in the presence of off-diagonal CS terms. With the CS levels \( k = (-2, 1, 1) \), holonomies \( (\alpha, \beta, \gamma) \) and the magnetic fluxes \( (p,q,r) \), the phase factors in the index are given
by

\[ s_{CS}^{(0)} = \sum_i (-2p_i \alpha_i + q_i \beta_i + r_i \gamma_i) + \sum_{i,j} (l_1 p_i \alpha_j + l_2 q_i \beta_j + l_3 r_i \gamma_j) \]
\[ + \frac{1}{2} \sum_{i,j} \left[ (-l_1 - l_2 + l_3)(p_i \beta_j + q_i \alpha_j) + (l_1 - l_2 - l_3)(q_i \gamma_j + r_i \beta_j) + (-l_1 + l_2 - l_3)(r_i \alpha_j + p_i \gamma_j) \right], \]

(2.11)

\[ b_0 = \frac{3}{2} \left[ \sum_{i,j} |p_i - q_j| (\alpha_i - \beta_j) + \sum_{i,j} |q_i - r_j| (\beta_i - \gamma_j) + \sum_{i,j} |r_i - p_j| (\gamma_i - \alpha_j) \right]. \]

(2.12)

Let us pick a holonomy \( \alpha_i \) whose corresponding flux \( p_i \) is zero. The phase term containing this \( \alpha_i \) is given by

\[ \left[ l_1 \sum_j p_j + \frac{1}{2} (-l_1 - l_2 + l_3) \sum_j q_j + \frac{1}{2} (-l_1 + l_2 - l_3) \sum_j r_j + \frac{3}{2} \left( \sum_j |q_j| - \sum_j |r_j| \right) \right] \alpha_i. \]

(2.13)

Since we are looking for states of \( \mathcal{O}(1) \) energy in the large \( N \) limit, only \( \mathcal{O}(1) \) number of holonomies should survive in the phase factor, as all phases have to be canceled by exciting matter fields which carry nonzero energies. This means that the coefficient in (2.13) must be zero. This argument also applies to both \( \beta \) and \( \gamma \) whose corresponding magnetic fluxes are zero. We therefore obtain the following equations

\[ \begin{pmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} |P| \\ |Q| \\ |R| \end{pmatrix}, \]

(2.14)

\[ \begin{pmatrix} l_1 \\ \frac{1}{2} (-l_1 - l_2 + l_3) \\ \frac{1}{2} (-l_1 - l_2 + l_3) \end{pmatrix} \begin{pmatrix} \frac{1}{2} (-l_1 + l_2 - l_3) \\ l_2 \\ \frac{1}{2} (l_1 - l_2 - l_3) \end{pmatrix} \begin{pmatrix} P \\ Q \\ R \end{pmatrix} = -\frac{3}{2} \begin{pmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{pmatrix}. \]

(2.15)

where \( |P| = \sum_j |p_j| = P_+ - P_- \) and \( P = \sum_j p_j = P_+ + P_- \). Note that \( \sum_j \Delta_j = 0 \). Let us also consider the zero point energy \( \epsilon_0 \),

\[ \epsilon_0 = \frac{3}{2} \left( (1 - c) \sum_{i,j} |p_i - q_j| + (1 - a) \sum_{i,j} |q_i - r_j| + (1 - b) \sum_{i,j} |r_i - p_j| \right) \]
\[ - \sum_{i < j} (|p_i - p_j| + |q_i - q_j| + |r_i - r_j|), \]

(2.16)

which could also yield an \( \mathcal{O}(N) \) energy contribution. The possible \( \mathcal{O}(N) \) contribution is

\[ N_1 \left( \frac{3}{2} (1 - c) |Q| + \frac{3}{2} (1 - b) |R| - |P| \right) + \text{cyclic} \]

(2.17)
where \( N_1 \) denotes the number \( U(1) \)'s in the Cartan of first \( U(N) \) which support zero magnetic fluxes. By substituting \(|R| = |P| + \Delta_2\) and \(|Q| = |P| - \Delta_3\) obtained from (2.14), we can check after some algebra that \( \Delta_2 \) and \( \Delta_3 \) must be the same to prevent the \( \mathcal{O}(N) \) contribution from \( \epsilon_0 \). Thus, (2.15) can be solved as

\[
Q = P + \frac{3(-l_1 + l_2 + 3l_3)}{l_1^2 + (l_2 - l_3)^2 - 2l_1(l_2 + l_3)} \Delta \tag{2.18}
\]

\[
R = P + \frac{3(-l_1 + 3l_2 + l_3)}{l_1^2 + (l_2 - l_3)^2 - 2l_1(l_2 + l_3)} \Delta \tag{2.19}
\]

where \( \Delta \equiv \Delta_2 = \Delta_3 \). As the diagonal combination of \( U(1) \subset U(N) \times U(N) \) decouples from matter fields, the fluxes should satisfy the condition

\[
2P = Q + R, \tag{2.20}
\]

which is unaffected by \( \Lambda \) due to the condition (2.8). Thus, if we choose \( \Lambda \) such that

\[
-l_1 + 2l_2 + 2l_3 \neq 0, \tag{2.21}
\]

then \( \Delta \) must be zero. In this case, \(|P| = |Q| = |R|\) and \( P = Q = R \), which implies that

\[
P_{\pm} = Q_{\pm} = R_{\pm}. \tag{2.22}
\]

This is nothing but the condition for factorizable fluxes. Since \( I^b \) comes from non-factorizable fluxes, it does not appear in the large \( N \) index if we choose off diagonal CS term satisfying (2.21). Furthermore, for factorizable fluxes, one can immediately check using (2.9) that the additional CS term with the levels \( \Lambda_{ij} \) vanishes, which means that the index contribution is not changed by this additional CS term. Thus, the large \( N \) index with (2.21) gives rise to

\[
I'(x) = \hat{I}(x). \tag{2.23}
\]

This monopole index is shown to agree with the gravity index [26]. Furthermore, now we can see the explicit decomposition of the index into \( U(1)_B \) neutral, positive and negative sectors. We already saw that \( O(1) \) contribution to the large \( N \) index is independent of the various choices \( l_1, l_2, l_3 \). However, various choices of \( l_1, l_2, l_3 \) possibly give different results for \( \mathcal{O}(N) \) contribution. Thus if we are interested in the baryon spectrum, this subtlety can play an important role. It is worthwhile to explore this issue.

So far we consider the so-called \( CP \) non-invariant model for \( M^{3,2} \), which is claimed to be dual to \( M \)-theory on \( AdS_4 \times M^{3,2} \) with discrete torsions. In [20], CP invariant field theory model dual to \( M^{3,2} \) without discrete torsion is proposed as well. The theory is given by \( U(N - 2)_0 \times U(N)_0 \times U(N)_0 \) CS theory with the matter contents given by the same quiver diagram as the CP

\[
\text{(2.21)} \quad \text{is the only extra condition on the off-diagonal Chern-Simons term that we claim should be imposed,}
\]

apart from (2.8) and (2.9) imposed by [20].
non-invariant case. We attempt to calculate the large $N$ superconformal index for the theory and compare it with the gravity spectrum. However, it is hard to see the index matching in this case. Although the $U(1)_B$ neutral sector $I_0$ is the same as that of $U(N)_{-2} \times U(N)_1 \times U(N)_1$ theory, which matches well with the gravity index, the sector with $U(1)_B$ charge = 1 with $P_+ = 1$ and $P_- = 0$ is not. For $p = q = r = 1$, we find $b_0 = 3(\beta - \gamma)$ and $\epsilon_0 = 2 - 3a$. The lowest energy contribution appears by exciting matters $A_i A_j A_k$ to screen the phase from monopoles, which have $\epsilon + j = 3a + (2 - 3a) = 2$. As $SU(3)$ indices $i, j, k$ are symmetrized, these form a 10 dimensional representation contributing to the index as $10x^2$. This is different from the gravity index $I_{grav}^{P_+ = 1} = 9x^2$. Other $U(1)_B$ charge = 1 sectors with $P_+ = n + 1$ and $P_- = n$ do not help because they seem to yield contributions to higher orders than $x^2$. We checked this by studying $b_0$ and $\epsilon_0$ in these sectors. If we check higher $U(1)_B$ charge sectors, the situation is worse. For $P_+ = n \geq 2$, monopole fluxes of the form

$$p = (2, 1, \cdots, 1, 0, \cdots), \quad q = (1, \cdots, 1, 0, \cdots), \quad r = (1, \cdots, 1, 0, \cdots)$$

all have $b_0 = 0$ and $\epsilon_0 = 2$. As there are no phases created by monopoles, they are gauge invariant without exciting matters. So they start to come with energy 2, contributing to the index as $x^2 + \cdots$. This means that there are infinite number of states at $\epsilon + j = 2$. This kind of divergence is similar to that observed in the index computation of $\mathcal{N} = 8$ super Yang-Mills [29]. At this point, it is not clear how to cure this problem in the index computation.

### 3 $S^3$ partition function and the gravitational free energy

The gravitational free energy of $AdS_4 \times Y$ with $Y$ being Sasaki-Einstein 7 manifold is given by

$$F_{grav} = N_3^2 \sqrt[2]{\frac{2\pi^6}{27\text{Vol}(Y)}},$$

in the leading order of $N$ with the normalization $R_{\hat{\mu}\hat{\nu}} = 6g_{\hat{\mu}\hat{\nu}}$ of the metric on $Y$. The volume of $M^{3,2}$ is given by [30]

$$\text{Vol}(M^{3,2}) = \frac{9\pi^4}{128}.$$  \hspace{1cm} (3.2)

If we consider $M^{3,2}/Z_k$, the volume of the space is divided by $k$ so that the free energy is given by[3]

$$F_{grav} = \frac{16\sqrt{3}\pi}{27} k^{\frac{1}{2}} N_3^2.$$  \hspace{1cm} (3.3)

\textsuperscript{3}There could be a subtlety in the formula since $M^{3,2}/Z_k$ can have orbifold singularities. However, the leading large $N$ results are expected not to be affected by the orbifold singularities.
The localization computes the partition function of the field theory
\[
Z = \int \left( \prod_{\text{Cartan}} \frac{d\sigma}{2\pi} \right) \exp \left[ i k \sum_i \left( -2\sigma_{1,i}^2 + \sigma_{2,i}^2 + \sigma_{3,i}^2 \right) + \frac{i}{4\pi} \sum_{a,b=1}^{3} \sum_{i,j} \Lambda_{ab} \sigma_{a,i} \sigma_{b,j} - \frac{3}{a=1} \sum_{i} \Delta_{a}^{m} \sigma_{a,i} \right]
\]
\[
\left( \prod_{a=1}^{3} \prod_{i,j,i \neq j} 2 \sinh \frac{\sigma_{a,i} - \sigma_{a,j}}{2} \right) \exp \left[ \sum_{I} \sum_{i,j} \ell \left( 1 - \Delta_{12}^{I} + i \frac{\sigma_{1,i} - \sigma_{2,j}}{2\pi} \right) + \text{cyclic} \right].
\]
(3.4)

where $\Delta_{a}^{m}$ is the R-charge for a bare monopole operator corresponding to a unit flux of $\text{tr} F_{a}$\[23\]. $\Delta_{ab}^{I}$ is the R-charge of the bifundamental field $X_{ab}^{I}$ and $\sigma$s are the scalars in the vector multiplets. The function $\ell(z)$ is given by
\[
\ell(z) = -z \ln \left( 1 - e^{2\pi i z} \right) + \frac{i}{2} \left( \pi z^2 + \frac{1}{\pi} \text{Li}_2 \left( e^{2\pi i z} \right) \right) - \frac{i\pi}{12}.
\]
(3.5)

Let us check the obvious flat directions of the partition function, whose consideration is crucial to the later calculation. We can see that the partition function is invariant under the following transformations up to a phase:

\[
\begin{align*}
\sigma_{a} & \rightarrow \sigma_{a} - 2\pi i \delta_{a}, \\
\Delta_{ab}^{I} & \rightarrow \Delta_{ab}^{I} + \delta_{a} - \delta_{b}, \\
\Delta_{a}^{m} & \rightarrow \Delta_{a}^{m} + k_{a} \delta_{a} + N \sum_{b} \Lambda_{ab} \delta_{b}
\end{align*}
\]
(3.6)

with 3 parameters $\delta_{a}$. Due to this symmetry, we can adjust the R-charge of the bifundamental fields as follows:

\[
\Delta_{ab}^{I} \rightarrow \Delta^{I} = \frac{\Delta_{12}^{I} + \Delta_{23}^{I} + \Delta_{31}^{I}}{3}.
\]
(3.7)

The marginality of the superpotential demands that $\sum_{I} \Delta^{I} = 2$.

One important point is that, as we consider the $Z_{2}$ transformation of the integration variables $\sigma_{i} \rightarrow -\sigma_{i}$, the integral acquires nonzero contribution only from the $Z_{2}$ even part of the integrand. We want to evaluate this partition function in the large $N$ limit. In this limit, we can evaluate the partition function by a saddle point approximation. For chiral theories in the 4-d sense, the integrand in (3.4) is not invariant under this $Z_{2}$ so that a saddle point approximation using this measure does not respect the above $Z_{2}$ symmetry of eigenvalue distribution. It is suggested in [27] that we make the integrand to be symmetric under the $Z_{2}$ symmetry by dropping the irrelevant $Z_{2}$ odd part, and then carry out the saddle point approximation. Although the partition function $Z$ after $\sigma$ integration is just the same, we call the latter expression as
\[ Z_{\text{sym}} = \frac{1}{2} \int \left( \prod_{\text{Cartan}} \frac{d\sigma}{2\pi} \right) \exp \left[ \frac{ik}{4\pi} \sum_i \left( -2\sigma_{1,i}^2 + \sigma_{2,i}^2 + \sigma_{3,i}^2 \right) + \frac{i}{4\pi} \sum_{a,b} \sum_{i,j} \Lambda_{ab}\sigma_{a,i}\sigma_{b,j} \right] \left( \prod_{a \neq i,j} 2 \sinh \frac{\sigma_{a,i} - \sigma_{a,j}}{2} \right) \left\{ \exp \left[ -\sum_a \sum_i \Delta_a^m \sigma_{a,i} + \sum_I \sum_{i,j} \ell \left( 1 - \Delta^I + i\frac{\sigma_{1,i} - \sigma_{2,j}}{2\pi} \right) + \text{cyclic} \right] \right\}.

(3.8)

For \( \sigma_{1,i} \), the saddle point equation is given by

\[
0 = -\frac{\partial F_{\text{sym}}}{\partial \sigma_{1,i}} = -\frac{ik}{\pi} \sigma_{1,i} + \frac{i}{2\pi} \sum_b \sum_j \Lambda_{b\sigma_{b,j}} + \sum_{j \neq i} \coth \frac{\sigma_{1,i} - \sigma_{1,j}}{2} \\
+ A_+ \left[ -\Delta_1^m - \frac{i}{2} \sum_I \sum_j \left( 1 - \Delta^I + i\frac{\sigma_{1,i} - \sigma_{2,j}}{2\pi} \right) \cot \pi \left( 1 - \Delta^I + i\frac{\sigma_{1,i} - \sigma_{2,j}}{2\pi} \right) \right] \\
+ \frac{i}{2} \sum_I \sum_j \left( 1 - \Delta^I + i\frac{\sigma_{3,i} - \sigma_{1,j}}{2\pi} \right) \cot \pi \left( 1 - \Delta^I + i\frac{\sigma_{3,i} - \sigma_{1,j}}{2\pi} \right) \\
+ A_- \left[ \Delta_1^m + \frac{i}{2} \sum_I \sum_j \left( 1 - \Delta^I - i\frac{\sigma_{3,i} - \sigma_{1,j}}{2\pi} \right) \cot \pi \left( 1 - \Delta^I - i\frac{\sigma_{3,i} - \sigma_{1,j}}{2\pi} \right) \right]
\]

(3.9)

where

\[
A_\pm = \exp \left[ \mp \sum_a \sum_i \Delta_a^m \sigma_{a,i} + \sum_I \sum_{i,j} \ell \left( 1 - \Delta^I \pm i\frac{\sigma_{1,i} - \sigma_{2,j}}{2\pi} \right) + \text{cyclic} \right] \right\}.
\]

\[
\exp \left[ -\sum_a \sum_i \Delta_a^m \sigma_{a,i} + \sum_I \sum_{i,j} \ell \left( 1 - \Delta^I + i\frac{\sigma_{1,i} - \sigma_{2,j}}{2\pi} \right) + \text{cyclic} \right] \\
+ \exp \left[ \sum_a \sum_i \Delta_a^m \sigma_{a,i} + \sum_I \sum_{i,j} \ell \left( 1 - \Delta^I - i\frac{\sigma_{1,i} - \sigma_{2,j}}{2\pi} \right) + \text{cyclic} \right] \right\}.
\]

(3.10)

\( \sigma_{2,i} \) and \( \sigma_{3,i} \) have similar saddle point equations. From the symmetries of these equations, we expect that the solution of these equations satisfies the following properties:

- The eigenvalue distribution is \( Z_2 \) invariant.
- The eigenvalues for \( \sigma_2 \) and \( \sigma_3 \) are the same.
In adopting the method of saddle point approximation of [23], it is crucial that the long range force on each eigenvalue should vanish. Here the long range force is the force appearing when σ_{a,i} − σ_{b,j} is very large. This can be obtained by approximating 
\coth(σ_{a,i} − σ_{b,j}) \sim \text{sgn} Re(σ_{a,i} − σ_{b,j}).

One can easily check with \( Z = Z_{\text{sym}} \) and the gauge choice (3.7) that the long range force vanishes. Since \( \sum_j σ_j = 0 \) and \( \sum_{i,j} \ell (1 − Δ^I + i \frac{σ_{1,i} − σ_{2,j}}{2π}) = \sum_{i,j} \ell (1 − Δ^I − i \frac{σ_{1,i} − σ_{2,j}}{2π}) \), the symmetrized partition function on the saddle point is reduced to

\[
Z_{\text{sym,saddle}} = \exp \left[ \frac{ik}{4π} \sum_i (−2σ_{1,i}^2 + σ_{2,i}^2 + σ_{3,i}^2) \right] \left( \prod_a \prod_{i,j,i \neq j} 2 \sinh \frac{σ_{a,i} − σ_{a,j}}{2} \right) \\
\exp \left\{ \sum_l \sum_{i,j} \left[ \ell \left( 1 − Δ^I + i \frac{σ_{1,i} − σ_{2,j}}{2π} \right) + \ell \left( 1 − Δ^I − i \frac{σ_{1,i} − σ_{2,j}}{2π} \right) \right] \right\},
\]

This saddle point value of the partition function has exactly the same form as that of a nonchiral theory. For the \( Z_2 \) invariant eigenvalue distribution, with the absence of long-range forces on the eigenvalues, we can adopt the ansatz

\[
σ_a(x) = N^α x + iy_a(x)
\]

in the large \( N \) limit. The leading order contribution of each component of the free energy is then given by

\[
F_{\text{ext}} = −\frac{ik}{4π} \sum_i (−2σ_{1,i}^2 + σ_{2,i}^2 + σ_{3,i}^2) \approx \frac{k}{2π} N^{α+1} \int_{−x_*}^{x_*} dx ρ(x) x \left[ −2y_1(x) + y_2(x) + y_3(x) \right],
\]

\[
F_{\text{int}} = −\sum_a \sum_{i,j,i \neq j} \ln \left( 2 \sinh \frac{σ_{a,i} − σ_{a,j}}{2} \right) \\
− \sum_l \sum_{i,j} \left[ \ell \left( 1 − Δ^I + i \frac{σ_{1,i} − σ_{2,j}}{2π} \right) + \ell \left( 1 − Δ^I − i \frac{σ_{1,i} − σ_{2,j}}{2π} \right) \right] + \text{cyclic}
\]

\[
= N^{2−α} \int_{−x_*}^{x_*} dx ρ(x)^2 ∑_l f^l(y_a)
\]

where \( x_* = \text{Max}(x) \) and \( ρ(x) \) is the eigenvalue density function. Detailed calculation is similar to that in [24]. The function \( f^l(y_a) \) on the last line is given by

\[
f^l(y_a) = 2 ∑_n \left[ \frac{1}{n^2} − \frac{φ^l}{2π n^2} \cos(n φ^l) \cos(n(y_2 − y_1)) + \frac{1}{2π n^2} \sin(n φ^l) \sin(n(y_2 − y_1)) + \frac{1}{π n^3} \sin(n φ^l) \cos(n(y_2 − y_1)) + \text{cyclic} \right].
\]

\[
φ^l = 2π(1 − Δ^I).
\]

Since the eigenvalue distributions of \( σ_2 \) and \( σ_3 \) are the same, we can set \( y_2 − y_1 = y_3 − y_1 = y \). In this case [24],

\[
f^l(y) = \frac{φ^l}{2π} ((2π − φ^l)^2 − y^2), \quad −2π < y < 2π.
\]
In order to have nontrivial saddle point, $F_{\text{ext}}$ and $F_{\text{int}}$ should be balanced, which implies that $\alpha = \frac{1}{2}$. Now we should determine the eigenvalue density function $\rho(x)$ and the imaginary part of the eigenvalue difference $y(x)$. Since $\rho(x)$ is constrained by the condition $\int dx \rho(x) = 1$, we write the leading order contribution of the partition function as follows:

$$F = N^{\frac{3}{2}} \left[ \frac{k}{\pi} \int_{-x_*}^{x_*} dx \rho(x) x y(x) + \sum_{I} f^I(y(x)) \right]$$

(3.17)

with the Lagrangian multiplier term. Using the variational method, we obtain equations

$$\frac{k}{\pi} x y(x) + 2 \rho(x) \sum_{I} f^I(y(x)) - \frac{\mu}{2\pi} = 0,$$

(3.18)

$$\frac{k}{\pi} \rho(x) x + \rho(x)^2 \sum_{I} f'^I(y(x)) = 0$$

(3.19)

whose solution is

$$\rho(x) = \frac{1}{2x_*}, \quad y(x) = \frac{k x_*}{\pi} x, \quad -x_* \leq x \leq x_*$$

(3.20)

Since we have not determined $x_*$ yet, we should find the value of $x_*$ that extremizes the partition function

$$F = N^{\frac{3}{2}} \left[ \frac{k^2}{6\pi^2} x_*^3 + \sum_{I} \phi^I(2\pi - \phi^I)^2 \frac{1}{4\pi} \right].$$

(3.21)

It is easy to find that $x_* = \left[ \frac{2\pi \sum_{I} \phi^I(2\pi - \phi^I)^2}{\sqrt{2k}} \right]^{\frac{1}{3}}$ extremizes $F$. Substituting this value, the partition function is given by

$$F = \sqrt{2} \frac{3}{3\pi} \left[ \left( \frac{\sum_{I} \phi^I(2\pi - \phi^I)^2}{2\pi} \right)^{\frac{3}{2}} k^{\frac{3}{2}} N^{\frac{3}{2}} \right] = \frac{4\sqrt{2}}{3} \left[ \sum_{I} (1 - \Delta^I) \Delta^{I^2} \right]^{\frac{3}{2}} k^{\frac{3}{2}} N^{\frac{3}{2}}.$$

(3.22)

This partition function is a function of the trial R-charges $\Delta^I$. Extremizing the partition function with respect to $\Delta^I$, we obtain the expected free energy

$$F = \frac{16\sqrt{3}}{27} k^{\frac{3}{2}} N^{\frac{3}{2}}.$$

(3.23)

with $\Delta^I = \frac{2}{3}$. This result is exactly the same as that of the gravity theory.

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