Comment on “On the proper behavior of atoms”
by Paul Anglin

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Abstract
Paul Anglin criticised our analysis of the neoclassical theory of the
firm, but makes a number of incorrect assertions about our assumptions.
We correct these misunderstandings, but acknowledge that one criticism
he makes is correct. We correct this flaw with a new argument that
supersedes the flawed strategic reaction argument we presented in our
previous paper.

1 The profit formula
We take as our starting point, the usual profit formula of a single product market
with $n$ firms:

$$
\pi_i = q_i P(Q) - \int_0^{q_i} MC(q_i) dq_i,
$$

(1)

where $\pi_i$ is the profit obtained by firm $i$, as a function of its production $q_i$, and the total market production $Q = \sum_i q_i$. The function $P(Q)$ is the demand
curve, namely the price the good achieves when $Q$ items of the good is available
on the market. The function $MC(q_i)$ is the marginal cost of producing an extra
item of the good, given that a firm is producing $q_i$ items.

2 The trouble with derivatives
In [1], Paul Anglin critiques our paper [4]. We note a number of problems with
this critique.
Anglin’s initial proposition is that our results depend on the size of the
increment to output for each firm:
I suggest that a flawed premise is being used since it is also true that the effect on \( P \) would be about 100 times larger if the change in output by a single firm increased from \( dq_i = 1 \) to 100. So, before analyzing the effect of a change by a mass of firm, a more relevant question is: is \( dq_i = 1 \) or 100 (or \(-1\) or \(-100\))? [1, p. 278]

However, our argument was based not on discrete changes to output but on derivatives. The \( dq_i \) he mentions is “infinitesimal”: it cannot be equal to 1 or 100. In any case, we do not use “infinitesimals”, which are mathematically problematic, but regular derivatives, which in the case of a multivariate function \( y(q_1, \ldots, q_n) \) can be either partial \( \partial y / \partial q_i \) or total \( dy / dx \).

In footnote 1, Anglin conjectures that the relation \( dq_i / dQ = \sum_j \partial q_i / \partial q_j \) “seems to be a consequence of the fact that \( Q = \sum_j q_j \)” [1, p. 278]. In comments he made on a previous version of this paper, it would appear that this is the crux of his disagreement with our analysis. In [4], we effectively assumed that

\[
\frac{dq_i}{dQ} = \sum_j \frac{\partial q_i}{\partial q_j} = 1
\]  

in going from equation (4) to (6) in that paper. On reflection, we realise this criticism is correct — there is no justification for assuming \( dq_i / dQ \) has any particular value. Nevertheless, the Keen result (eq 6 of [4]) can still be derived as the system equilibrium assuming a much weaker additional condition that \( dq_i / dQ = dq_j / dQ, \forall i, j \) holds at equilibrium.

3 Symmetry of firms

In footnote 2 of [1], Anglin asserts we made a symmetry assumption \( Q = n q_i \), from which he derives an inconsistency. We did not make this assumption at any point in our paper. In the referees comments he made on an earlier version of this paper, it would appear that this is a derived consequence of our assumption that \( dq_i / dQ = 1 \). Coupled with the boundary condition \( Q = 0 \Rightarrow q_i = 0 \) and integrating, this would imply \( q_i = Q/n \).

However, since the Keen equilibrium only requires that \( dq_i / dQ = dq_j / dQ, \forall i, j \) at equilibrium, there is no specific requirement for the market to be evenly shared amongst the firms, except in the case of constant marginal cost, as detailed in section 6.

We do assume that each firm has identical marginal cost functions \( MC(q_i) \), which is also assumed in the traditional presentation of the Cournot profit maximun. This is for pedagogical convenience however, the argument presented in section 6 does not depend on this assumption, and can be easily generalised.
4 Total derivative with respect to industry output rather than single firm’s output

The traditional analysis of the Marshallian and Cournot models is to hypothesize behavior by the individual firm such that it sets the partial derivative \( \partial \pi_i / \partial q_i = 0 \) (see e.g. \[4, eq (2)\]): in the Marshallian model this is described as “atomistic” profit-maximizing behavior, while in the Cournot model it is described as a constrained profit level in response to the strategic responses of other firms. The Marshallian proposition is strictly false, since the profit of a single firm \( \pi_i \) is a function of all \( n \) firms’ outputs \( q_i \), not a single variable function, whether or not the individual firm can in fact affect the behavior of other firms. The extrema of an \( n \)-variable function is found at the zero of the derivative, i.e., when all partial derivatives \( \partial \pi_i / \partial q_j = 0 \). However

\[
\partial \pi_i / \partial q_j = \delta_{ij}(P - MC) + q_i P'
\]  

which can never be satisfied where \( q_i > 0 \) and \( P' < 0 \). The condition \( \partial \pi_i / \partial q_i = 0 \) describes an unstable equilibrium — it is vulnerable to firms pulling in the same direction, which can happen even in the absence of explicit collusion \[5\].

Instead we propose the condition that all firm’s profits are maximized with respect to total industry output \( d\pi_i / dQ = 0 \). This constrains the dynamics of firms’ outputs to an \( n - 1 \)-dimensional polyhedron, but otherwise does not specify what the individual firms should do. As an equilibrium condition, it is vulnerable to a single firm “stealing” market share. However, no firm acts in isolation. The other firms will react, negating the benefit obtained by first firm, causing the system to settle back to the \( d\pi_i / dQ = 0 \) manifold.

5 Conjectural Variation

In our paper, we introduce the idea of firms reacting to the production decisions of their competitors by introducing a dependence between our previous independent variables \( q_i \). We thank Anglin for reminding us of considerable previous history of doing this under the name of “conjectural variation”; however, this was a literature of which we were already aware, and whose use of the concept differs from our purpose in introducing it here.\[1\] Our intent was to make a mathematical argument that shows what happens in the Cournot analysis when one relaxes the assumption of atomism. We have not attempted to model any form of conjectural variation or reaction by the firms in the agent model, and in any case the agent model does not have the atomistic constraint imposed upon it.

We appreciate the reference \[2\] Anglin provided, but note that as they started from the incorrect differential condition \( (\partial \pi_i / \partial q_i = 0) \), their results are not applicable.

\[1\] This can be interpreted as firms anticipating what their competitors might do, although we tend to regard it as describing reactions to competitors in a “time-free” model, so the variation is not conjectural but reactionary.
In the next section, we present a strategic response argument that does not make use of the conjectural variation idea at all.

6 Evolution of $dq_i/dQ$

In our paper [4], we introduced a homogenous conjectural variation parameter $\partial q_i/\partial q_j = \theta$. As pointed out by Anglin, this analysis makes use of the faulty assumption $dq_i/dQ = \sum_j \partial q_i/\partial q_j$. To circumvent this problem, and generalise the argument, we take the point that $dq_i/dQ$ are unconstrained endogenous variables, and so we introduce the variables

$$\frac{dq_i}{dQ} = \theta_i. \quad (4)$$

This extends phase space from the $n$-dimensional space of firm production $q_i, i = 1 \ldots n$ to a $2n - 1$-dimensional phase space, with the constraint

$$\sum_i \frac{dq_i}{dQ} = \frac{dQ}{dQ} = 1. \quad (5)$$

The $\theta_i$ might be thought of as a firm’s response function to changing industry output.

With the usual profit formula (1) the maximum profit for a single firm obtains at the zero of

$$\frac{d\pi_i}{dQ} = P\frac{dq_i}{dQ} + q_i\frac{dP}{dQ} - MC(q_i)\frac{dq_i}{dQ}$$

$$= P\theta_i + q_iP' - MC(q_i)\theta_i \quad (6)$$

We may sum equation (6) over $i$ to obtain

$$P + QP' - \sum_i MC(q_i)\theta_i = 0. \quad (7)$$

Given a fixed market partition $\{s_i = q_i/Q\}$, the maximum profit obtains at the zero of the derivative of the total industry profit

$$\frac{d}{dQ} \left( QP - \sum_i \int_s^{q_i} MC(q) dq \right) = P + QP' - \sum_i s_iMC(q_i) = 0. \quad (8)$$

Comparing equations (7) and (8), we see that the individual firm profit is sub-maximal unless

$$\sum_i MC(q_i)(s_i - \theta_i) = 0. \quad (9)$$

The vector $(m_i = MC(q_i))$ lies in the positive cone $\mathbb{R}^{n+}$ (ie $m_i > 0, \forall i$). The vector $(t_i = s_i - \theta_i)$ lies on a hyperplane passing through the origin, and perpendicular to the unit vector $(1, 1, \ldots 1)$, since $\sum_i t_i = 0$. Condition (9) can be
thought of as a dot product $\mathbf{m} \cdot \mathbf{t} = 0$. This condition can only be satisfied if $\mathbf{m}$ is proportional to the unit vector (i.e., marginal cost is constant) or $\mathbf{t} = 0$, which implies $\theta_i = s_i, \forall i$. Given a particular partition of the market, profit of all firms will always be increased by moving the $\theta_i$ variables closer to the market share $s_i$.

Substituting this condition for variable marginal cost into (6) gives:

$$s_i P + s_i Q P' - s_i MC(s_i Q) = 0$$

which can only be simultaneously satisfied for all $i$ if the market is equipartitioned ($s_i = 1/n$).

The Keen equilibrium obtains on the manifold where $\theta_i = 1/n$. Substituting this into equation (6), one obtains

$$P - MC(q_i) + nq_i P' = 0$$

which can be rearranged to yield

$$MR_i - MC = P(Q) + q_i P'(Q) - MC(q_i) = \frac{n-1}{n}(P - MC(q_i))$$

where $MR_i$ is the marginal revenue of the firm.

When marginal cost is constant, equation (7) implies that the industry operates at the monopoly pricing at equilibrium:

$$P + QP' - MC = 0$$

and from (6) we see

$$q_i = \theta_i Q$$

Only when $\theta_i = 1/n$ does this coincide with the Keen equilibrium.

We may rearrange equation (11) to give

$$q_i = \frac{MC(q_i) - P}{nP'}$$

If the right-hand side of this equation were a monotonic decreasing function of $q_i, \forall q_j, j \neq i$, then a unique solution exists for $q_i$, the market is equipartitioned between firms and the Keen equilibrium coincides with monopoly pricing. However, if multiple solutions to (15) exist, then the market need not be equipartitioned, and in general the Keen equilibrium differs from monopoly pricing. However, in the limit $n \to \infty$, assuming finite total industry output, $q_i$ is $o(1/n)$, so $P - MC(q_i)$ tends to some positive value, differing from competitive pricing.

In the simple case of a linear demand curve, multiple solutions to $q_i$ can only exist for falling marginal cost. Such markets are dominated by a scramble.

\footnote{For example with $P(Q) = 10 - Q$ and $MC(q) = 1/q$, $(P - MC)/P'$ exhibits a peak value at an intermediate value, so is not monotonic. We thank Paul Anglin for providing this example.}
for market share, as there is a distinct “economy of scale” advantage to being market leader. The analysis presented here does not help determine what the equilibrium state will be.

If the marginal cost function differed between firms, the result \( \theta_i = s_i \) still holds. The main difference is that the corresponding equation (10) is now firm dependent

\[
P + Q P' - MC_i(s_i Q) = 0,
\]

(16)

and the market is no longer equipartitioned at equilibrium. The equivalent of (11) is

\[
MR_i - MC_i = \frac{Q - q_i}{Q}(P - MC_i(q_i)).
\]

(17)

7 Agent simulation

What evidence is there that the parameters \( \theta_i \) introduced in the previous section will undergo evolution so as to optimise the profit levels of the firms? In [5], we introduced a simple agent based model which exhibited an interesting emergent phenomenon where agents would lock into the same strategy of decreasing production to improve profits. At the start of the simulation, agents are randomly increasing or decreasing their production levels without affecting total industry production much. In terms of \( \theta_i \), this implies \( |\theta_i| \gg 1/n \), and total industry production from equation (6) is close to competitive levels. As the emergent lock in effect takes place, the firms are changing their production levels in the same way, so \( \theta_i = 1/n \), and the system converges to the Keen equilibrium.

Qualitatively, the results of the two models do differ, with the agent model exhibiting a range of convergent behaviour not seen in the differential case. In the agent model, we were also able to reproduce the neoclassical result of convergence by the firms to output levels at which each firm’s marginal cost equaled its marginal revenue in two ways. However, neither of these accord with the standard “Marshallian” or “Cournot” explanations. Convergence to the Cournot output level occurred:

1. When a fraction of firms behaved irrationally, by continuing with a strategy (for example, increasing output) when that strategy reduced profit in the previous iteration. Convergence to the neoclassical expectation was monotonic as the proportion of irrational firms was raised from zero to 25 percent; from 25 to 50 percent, the neoclassical case applied; while above 50 percent irrational behaviour, the firms and the system followed a random walk. This result was independent of the number of firms in the industry; and

2. As the standard deviation \( \sigma \) of the parameter \( \delta q_i \) rose, as shown in our paper. This result was also independent of the number of firms in the industry.
8 Conclusion

Our conclusions about the strict falsity of the Marshallian model, and the lack of content of the Cournot model—in that while it is strictly true, actual profit-maximizers would not play the Cournot-Nash game—still stand. We therefore continue to assert that economics does not have a model of price setting. Blinder et al.\cite{Blinder}, provides a good empirical survey of price setting practices in the real world, and as with our model, this survey strongly contradicts accepted neoclassical beliefs. We suggest that a good research goal for economists would be to devise a model of competition that replicates the results of this study.

References

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