TRACE-BASED CRYPTOANALYSIS OF CYCLOTOMIC PL WE
FOR THE NON-SPLIT CASE

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Abstract. We provide an attack against the decision version of PL WE over
the cyclotomic ring $\mathbb{F}_q[x]/(\Phi_p^k(x))$ with $k > 1$ in the case where $q \equiv 1 \pmod{p}$
but $\Phi_p^k(x)$ is not totally split over $\mathbb{F}_q$. Our attack uses that the roots of
$\Phi_p^k(x)$ over suitable extensions of $\mathbb{F}_q$ have zero-trace and has overwhelming
success probability in function of the number of samples taken as input. An
implementation in Maple and some examples of our attack are also provided.

1. Introduction

Lattice-based cryptography has achieved an extraordinary success in the last five
years, specially since the National Institute for Standards and Technology launched
a public contest in 2017 to standardise different sets of quantum-resistant primi-
tives. With four candidates proposed for standardisation the past 5th of July, three
of them belong to the lattice-based category (Crystals-Kyber for Encryption and
Key Encapsulation, Crystals-Dilithium and Falcon for Digital Signatures). These
three suites are based, among other items, on several learning problems over certain
large dimensional vector spaces over finite fields, some of them with an extra ring
structure. Although in some early proposals before the NIST contest there were
considered quotient rings attached to cyclotomic polynomials, several concerns due
to an increasing number of attacks against ad-hoc instances have sparked an un-
derstandable scepticism. However, cyclotomic rings over finite fields still play a
prominent role in lattices-based primitives. For instance, in both Crystals pack-
ages, the NTT transform for the quotient ring $\mathbb{F}_q[x]/(x^{2^8}+1)$ is still used. Moreover,
some current proposals for homomorphic encryption like PALISADE are built upon
a learning problem backed on quotient cyclotomic rings.

The Polynomial Learning With Errors problem (PLWE from now on) was intro-
duced in [Stehlé et al., 2009] and the Ring Learning With Errors problem (RLWE
from now on) in [Lyubashevsky et al., 2013]. To briefly describe the first problem,
let us denote by $R$ the quotient ring $R := \mathbb{Z}[x]/(f(x))$ with $f(x) \in \mathbb{Z}[x]$ of degree $n$,
monic and irreducible over $\mathbb{Z}[x]$ and let $q \geq 2$ be a prime. Set $R_q := \mathbb{F}_q[x]/(f(x))$
and consider a random variable $U$, uniformly distributed over $R_q$ and $\chi_R$, a dis-
crete Gaussian distribution over $R_q$ (precise definitions will be provided in the
next section). The PLWE problem (in decision form, the version we investigate
in this work) consists in guessing, with non-negligible advantage, if a set of pairs
$\{(a_i(x), b_i(x))\}_{i \geq 1} \subseteq R_q \times R_q$ of arbitrary size has been sampled either from $U$ or
from $\chi_R$. Similarly, for the second problem, we consider the ring of integers $\mathcal{O}_K$
of a number field $K$ of degree $n$ and letting again $q \geq 2$ be a prime, we consider a

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random variable $U$, uniformly distributed over $O_K/qO_K$ and $\chi_K$, a discrete Gaussian distribution over $O_K/qO_K$. The RLWE problem consists now in guessing, with non-negligible advantage, if a set of pairs $\{(\alpha_i, \beta_i)\}_{i \geq 1} \subseteq O_K/qO_K \times O_K/qO_K$ has been sampled either from $U$ or from $\chi_K$.

In both works, the polynomial $f(x)$ defining $R_q$ and $K$ is considered to be cyclotomic, due to the distinctive arithmetic properties enjoyed by these fields, which are used in the security proofs backing the corresponding cryptosystems as well as in practical implementations. However, some other works like [Rosca et al., 2018] do without the cyclotomic character of the underlying number field as long as this is Galoisian. Moreover, both problems are shown to enjoy a reduction from a supposedly intractable problem for the family of ideal lattices (again, precise definitions in the next section) whenever the prime $q$ is of the form $q = a(n)$ with $a(x) \in R[x]$ of degree independent of $n$, the lattice dimension (see Lemma 4.1 in [Stehle et al., 2009] for PLWE and Theorem 3.6 in [Lyubashevsky et al., 2013] for RLWE). As a matter of fact, in most applications the degree of $a(x)$ is considered to be 2 and this is what we will also assume here.

Despite the fact that the aforementioned ideal-lattice based problems are supposed to be intractable in the worst case, some particular instantiations of PLWE have been encountered to be insecure. For instance, in [Elias et al., 2016] and [Elias et al., 2015], the authors provide an attack towards PLWE if there exists a simple root $\alpha \in \mathbb{F}_q$ of the polynomial $f(x)$ such that either

a) $\alpha = 1$, or
b) $\alpha$ has small order modulo $q$, or
c) $\alpha$ has small residue modulo $q$.

The present work deals exclusively with cyclotomic rings. We will denote by $\Phi_n(x)$ the $n$-th cyclotomic polynomial, whose degree is $\phi(n)$, as very well known, where $\phi(\cdot)$ is Euler’s totient function. It is easy to check that $\alpha = \pm 1$ is never a root of $\Phi_n(x)$ modulo $q$ unless $q \mid n$ and all the roots of $\Phi_n(x)$ have maximal order $n$, hence cyclotomic polynomials are protected against conditions a) and b) above. Furthermore, in [Lyubashevsky et al., 2013] it is assumed that $q$ is totally split in $R$, or equivalently, that $q \equiv 1 \pmod{n}$. The reason is that the $\phi(n)$ different ideals over $qR$ have all of them norm $q$, are permuted transitively by the field automorphisms and the ring operations required to apply the Chinese Remainder Theorem become very efficient. But again, in [Rosca et al., 2018] the authors introduce a clearing-the-ideal technique which allows to generalise the arguments previously developed for cyclotomic fields to any abelian number field. In any case, the requirement that the prime $q$ is totally split in $R$ has been, until now, essentially a condition which affects the efficiency of the computations on the quotient ring, but it has not been linked in a crucial manner to the security of the cryptosystem.

The main contribution of the present work is precisely to highlight that the totally split behaviour, beyond facilitating the algebraic and arithmetic manipulations, is an essential feature also in terms of the security, at least in this cyclotomic setting. In particular, we develop a new attack against cyclotomic PLWE for $n = p^k$ in the case in which $q = 1 + p^\ell u$, with $A \ll k$ (we will assume for simplicity that $A = 2$) and $u$ coprime to $p$. We still assume, for the security reduction to hold, that $q = O(n^2)$.

Our attack proceeds by recursively evaluating an input set of samples in $R_q \times R_q$ at a root $\alpha \in \mathbb{F}_{q^{\phi(n)}}$ of $\Phi_n(x)$. After each evaluation, we take the trace of the result and check, as in [Elias et al., 2016], whether this trace belongs to a certain distinguished region, of cardinal smaller than $q$. The main key for our attack to work is the fact that, for the considered root of $\Phi_n(x)$, most of its powers have zero trace, what allows for a very efficient definition of the small region to which the
errors should belong to. The fact that these traces are zero follows from Lemma 11 in [Wu et al., 2017], which exhibits the splitting behaviour of $\Phi_n(x)$ over $\mathbb{F}_q$. Even more, this zero trace property allows for very fast computations with the aid of the automorphic evaluation algorithm described in [Elia et al., 2012], a very helpful tool to evaluate polynomials over finite fields which significantly outperforms Horner’s method.

We have structured our paper in four sections: the second recalls several definitions and notations, as well as some facts on random variables over finite fields, which we will need later on. The third introduces our attack in two stages: the first stage is an attack against samples whose uniform first component belong to a certain distinguished subspace and the second stage is a general attack which reduces to the previous one in probabilistic polynomial time on the size of the set of samples and the parameters of the quotient ring. Finally, Section 4 is for simulations of our attack using Maple. The Maple code and for our examples has been located in a github domain referenced therein.

2. Preliminary facts and notations

Before recalling the definition of the RLWE and PLWE problems, we find it convenient to explicitly mention and prove a few facts about uniform and discrete Gaussian random variables over finite fields, as they are at the very background of the learning problems under study and, moreover, some of our arguments in Section 3 will involve certain manipulations of uniform distributions supported over finite dimensional vector spaces over $\mathbb{F}_q$.

Definition 2.1. Let $E$ be an $\mathbb{F}_q$-vector space of dimension $d$ so that $|E| = q^d$. A random variable $X$ with values over $E$ is said to be uniform if for every $v \in E$ we have that $P[X = v] = 1/q^d$.

We will use the following fact:

Lemma 2.2. If $X_1, X_2, ..., X_n$ are independent uniform distributions over $\mathbb{F}_q$ then, for each $\lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{F}_q$, not all of them zero, the variable $\sum_{i=1}^n \lambda_i X_i$ is uniform.

Proof. First, it is obvious that given a uniform distribution $X$ over $\mathbb{F}_q$ and given $\lambda \in \mathbb{F}_q^*$, the variable $\lambda X$ is uniform. So, it is enough to prove that if $X_1$ and $X_2$ are uniform, then the variable $X_1 + X_2$ is uniform. But for $i \in \mathbb{F}_q$, using the Total Probability Theorem we have

$$P[X_1 + X_2 = i] = \sum_{j \in \mathbb{F}_q} P[X_1 + X_2 = i|X_2 = j]P[X_2 = j] = \frac{1}{q} \sum_{j \in \mathbb{F}_q} P[X_1 = i-j] = \frac{1}{q}.$$

The second statistical component of our learning problems is the notion of a discrete Gaussian random variable. We follow closely [Lyubashevsky et al., 2013] Section 2.

First, let $n, s_1$ and $s_2$ be non-negative integers such that $n = s_1 + 2s_2$. Consider the following $\mathbb{R}$-vector subspace of $\mathbb{C}^n$:

$$\Lambda_n = \{(x_1, ..., x_n) \in \mathbb{R}^{s_1} \times \mathbb{C}^{2s_2} : x_{s_1+i} = \overline{\tau}_{s_1+i}, \text{ for } 1 \leq i \leq s_2\},$$

which, endowed with the restricted Hermitian metric in $\mathbb{C}^n$, is a Euclidean space of dimension $n$.

As well known, for $r > 0$, the Gaussian function $\rho_r(x) = \exp(-\pi||x||^2/r^2)$ defines, after normalising, the density function of a continuous Gaussian random variable of zero mean and variance-covariance matrix $rI$.
\{h_i\}_{i=1}^n$ of $\Lambda_n$ and given a vector $r = (r_1, ..., r_n) \in \mathbb{R}_+^n$ such that $r_{s_1+i} = r_{s_1+s_2+i}$ for $1 \leq i \leq s_2$, if $D_{r_i}$ is a 1-dimensional continuous Gaussian random variable of zero mean and variance $r_i^2$ and if these $n$ random variables are independent, then the variable $D_r = \sum_{i=1}^n D_{r_i}$ is a continuous elliptic $n$-dimensional Gaussian variable of zero mean and whose variance-covariance matrix has $r$ as main diagonal and 0 elsewhere. Denote by $\rho_r(x)$ the density function of $D_r$.

Now, suppose that $\mathcal{L}$ is a full-rank lattice contained in $\Lambda_n$ (namely, an abelian and finitely generated over $\mathbb{Z}$ additive subgroup of $\Lambda_n$ of rank $n$) and suppose that $\{h_i\}_{i=1}^n$ is a $\mathbb{Z}$-basis of $\mathcal{L}$, hence a basis of $\Lambda_n$ as a vector space.

**Definition 2.3.** A discrete random variable $X$ supported on $\mathcal{L}$ is called a discrete elliptic Gaussian random variable if its probability function is given by

$$P[X = x] = \frac{\rho_r(x)}{\rho_r(\mathcal{L})} \text{ for } x \in \mathcal{L}.$$  

2.1. The R/P-LWE problems. Let $K$ be a Galois number field of degree $n$ over $\mathbb{Q}$, namely, suppose that $K$ is the splitting field of a monic irreducible polynomial $f(x) \in \mathbb{Z}[x]$ of degree $n$ and let $\alpha \in K$ be a root of $f(x)$. The Galois group $\text{Gal}(K/\mathbb{Q})$ consists of $n$ automorphisms of $K$ fixing $\mathbb{Q}$, each of them determined by its value at $\alpha$. Denote these automorphisms by $\{\sigma_i\}_{i=1}^n$ with $\sigma_1 = \text{Id}$. We can label the roots of $f(x)$ (and hence the Galois group) in such a way that $\{\alpha_i\}_{i=1}^n$ are real roots and $\{\alpha_i\}_{i=s_1+1}^{s_1+s_2}$ are $s_2$ pairs of complex roots such that $\alpha_i = \overline{\alpha_{i+s_2}}$ for $s_1 + 1 \leq i \leq s_1 + s_2$. Of course, it might perfectly happen that $s_1 = 0$ (we say that $K$ is a totally complex or CM field) or that $s_2 = 0$ (we say then that $K$ is totally real).

Denote by $O_K$ the ring of integers of $K$. Along this communication we will assume that $K$ is monogenic, namely, that $O_K = \mathbb{Z}[\alpha]$ for an algebraic integer $\alpha \in O_K$. Denote also, as in the introduction, $R := \mathbb{Z}[x]/(f(x)) \simeq \mathbb{Z}[\alpha]$ and for a prime $q \in \mathbb{Z}$, let us set $\mathbb{R}_q := R/qR \cong \mathbb{F}_q[x]/(f(x))$.

We can naturally endow the rings $R$ and $O_K$ with a lattice structure over $\mathbb{R}^n$. Indeed, for $R$, we can just embed it into $\mathbb{R}^n$ by the coefficient embedding $\sigma_{\text{coef}}$, an orthonormal basis of this lattice being just the canonical basis:

$$\sigma_{\text{coef}} : R \rightarrow \mathbb{R}^n, \quad \sum_{i=0}^{n-1} a_i x^i \mapsto (a_0, ..., a_{n-1}),$$

where $\mathbb{F}_q^d$ stands for the class of $x^i$ modulo the ideal $(f(x))$.

As for $O_K$, recall that this ring is finitely generated over $\mathbb{Z}$ of rank $n$, hence we can embed it by the canonical embedding $\sigma_{\text{can}}$ into $\Lambda_n$, an orthonormal basis of this lattice being the set of vectors:

$$h_j = \begin{cases} e_j & \text{if } 1 \leq j \leq s_1, \\ \frac{1}{\sqrt{2}} (e_j + e_{s_2+j}) & \text{if } s_1 + 1 \leq j \leq s_1 + s_2, \\ \frac{1}{\sqrt{2}} (e_{j-s_2} - e_j) & \text{if } s_1 + s_2 + 1 \leq j \leq n, \end{cases}$$

where $\{e_j\}_{j=1}^n$ is the canonical basis of $\mathbb{C}^n$ and $i$ denotes the imaginary unit:

$$\sigma_{\text{can}} : O_K \rightarrow \mathbb{C}^n, \quad \alpha \mapsto (\sigma_1(\alpha), ..., \sigma_n(\alpha)).$$

The lattices $\sigma_{\text{can}}(O_K)$ and $\sigma_{\text{coef}}(R)$ inherit, from $O_K$ and $R$ respectively, an extra multiplicative structure and deserve a special terminology in the lattice-based cryptography literature:

**Definition 2.4.** An ideal lattice is a lattice $\mathcal{L}$ such that there exists a ring $R$, an ideal $I \subseteq R$, and an additive group monomorphism $\sigma : R \rightarrow \mathbb{R}^n$ such that $\mathcal{L} = \sigma(I)$. 


Now, for a prime $q \geq 2$, if $\chi$ is a discrete Gaussian distribution with values on $O_K$ (or, rather, in its image $\sigma_{\text{can}}(O_K) \subseteq \Lambda_n$ under the canonical embedding), we can further reduce the outputs of $\chi$ modulo $q$. Likewise, if $\chi$ is a discrete Gaussian distribution on $R$ (or, rather, in its image $\sigma_{\text{coeff}}(R) \subseteq \mathbb{R}^n$ by the coefficient embedding) we can also reduce its outputs modulo $q$. These are random variables with finite supports and we refer to them as discrete Gaussian variables modulo $q$.

The RLWE (for $O_K/qO_K$) problem and the PLWE (for $R_q$) are defined in terms of their corresponding oracles, which we recall next:

For a fixed prime $q \in \mathbb{Z}$, let $\chi_K$ be a discrete Gaussian random variable of $0$ mean and variance-covariance matrix $\Sigma_K$ taking values in $O_K/qO_K$ (or, rather, in the $n$-dimensional torus $T_K = (K \otimes \mathbb{R})/O_K$) and let $\chi_R$ a discrete Gaussian random variable of $0$ mean and variance-covariance matrix $\Sigma_R$, with values in $R_q$, embedded in the torus $T_R = (R \otimes \mathbb{R})/R$.

**Definition 2.5 (RLWE/PLWE oracles).** For a fixed element $s \in O_K/qO_K$ (resp. $R_q$), an RLWE oracle attached to the triple $(O_K/qO_K, s, \chi_K)$ (resp. a PLWE oracle attached to the triple $(R_q, s, \chi_R)$) is a probabilistic algorithm $A_{s,\chi_K}$ (resp. $A_{s,\chi_R}$) that works as follows:

1. Samples an element $a \in O_K/qO_K$ (resp. in $R_q$) from a uniform random sampler.
2. Samples an element $e \in O_K/qO_K$ from $\chi_K$ (resp. in $R_q$ from $\chi_R$).
3. Returns the element $(a, b = as + e)$.

Denoting by $(O_K/qO_K)^2$ the Cartesian product $(O_K/qO_K) \times (O_K/qO_K)$ and by $R_q^2$ the Cartesian product $R_q \times R_q$, the R/P-LWE problems are defined as follows. We give the decision version, since that is all we need for our purpose.

**Definition 2.6 (RLWE/PLWE decision problems).** Let $\chi_K$ and $\chi_R$ be defined as before. The R/P-LWE problem consists in deciding with non-negligible advantage, for a set of samples of arbitrary size $(a_i, b_i) \in (O_K/qO_K)^2$ (resp. in $R_q^2$), whether they are sampled from the R/P-LWE oracle or from the uniform distribution.

Until Section 4, unless stated otherwise, we will exclusively deal with the PLWE problem and hence, the ring $R$ will be embedded into $\mathbb{R}^n$ by the coefficient embedding. In particular, our discrete Gaussian distribution will take values over the quotient ring $R_q$, as in [Elias et al., 2016] and [Elias et al., 2015].

### 2.2. The cyclotomic polynomial and its splitting behaviour over finite fields.

In our setting, we will deal with $K = K_n := \mathbb{Q}(\zeta_n)$, the $n$-th cyclotomic field (where $\zeta_n$ denotes a primitive complex $n$-root of unity). It is well known that $K_n$ is the splitting field of the $n$-th cyclotomic polynomial, which we will denote by $\Phi_n(x)$. In particular, $K_n/\mathbb{Q}$ is a Galois extension of degree $m := \phi(n)$, where $\phi$ stands for the Euler’s totient function. It is also well known that $K_n$ is monogenic, in particular $O_K = \mathbb{Z}[\zeta_n]$.

When $q \equiv 1 \pmod{n}$, the prime $q$ is totally split in $O_K$ and hence $\Phi_n(x)$ has $m$ different roots in $\mathbb{F}_q$, all of them of maximal multiplicative order $n$. We will deal, however, with the non-totally split case and moreover, we will suppose that $n = p^h$ for a prime $p$. The following result addresses the factorisation of $\Phi_n(x)$ into irreducible factors in $\mathbb{F}_q[x]$:

**Theorem 2.7 ([Wu et al., 2017]).** Let $q = 1 + p^A u$, with $A \geq 1$, and $p, q$ primes. Suppose that $(a, p) = 1$ and denote by $\Omega(p^A)$ the group of primitive $p^A$-th roots of unity in $\mathbb{F}_q$. Assume $n > A$. Then, we have:

$$\Phi_{p^A}(x) = \prod_{\rho \in \Omega(p^A)} \left( x^{p^{A-n}} - \rho \right).$$
where the polynomials $x^{p^{n-A}} - \rho$ are irreducible over $\mathbb{F}_q$.

We have the following straightforward consequence which will be useful later on:

**Corollary 2.8.** Notations as in Theorem 2.4, for every $v \in \mathbb{N}$ such that $(v, p) = 1$, for each $\rho \in \Omega(p^k)$ and for each $0 \leq k < n - A$, the polynomial $x^{p^{n-k-A}} - \rho^v$ is irreducible over $\mathbb{F}_q[x]$.

**Proof.** First, notice that $\rho^v$ is also a primitive $p^k$-th root of unity, hence if we could express $x^{p^{n-k-A}} - \rho^v = f(x)g(x)$ with $\deg(f(x)), \deg(g(x)) \geq 1$, it would follow that $x^{p^{n-A}} - \rho^v = f(x^p)g(x^p)$, a contradiction. □

### 2.3. Fast evaluation of polynomials over finite fields

One of the issues we must confront is to evaluate polynomial expressions over a finite field at elements of certain extensions, in the most efficient possible manner. In particular, for cyclotomic prime conductors of almost-cryptographic size, the well-known Horner’s algorithm might easily become inefficient. The method that we will use, due to Elia, Rosenthal and Schipani, is called automorphic evaluation drastically reduces the number of $\mathbb{F}_q$-products by a square root factor.

**Theorem 2.9** ([Elia et al., 2012] Theorem 3). The minimum number of $\mathbb{F}_q$-products required to evaluate a polynomial of degree $n$ with coefficients in $\mathbb{F}_{q^n}$ at an element of $\mathbb{F}_{q^n}$ with $m \geq s$, is upper bounded by

$$2s(\sqrt{n(q-1)} + 1/2).$$

### 3. An attack based on traces over finite extensions of $\mathbb{F}_q$

In [Elia et al., 2016] and [Elia et al., 2015], the authors give an attack against the PLWE problem attached to a quotient ring $R_q = \mathbb{F}_q[x]/(f(x))$ and a prime $q$, if there exists a simple root $\alpha \in \mathbb{F}_q$ such that either

a) $\alpha = 1$, or
b) $\alpha$ has small order modulo $q$, or
c) $\alpha$ has small residue modulo $q$.

The expressions small order and small residue are not made fully precise herein, but the examples provided by the authors show that by such a term we must understand orders of up to 5 while by small residue the authors understand $\alpha = 2, 3$.

By the Chinese remainder theorem, we can write

$$R_q \simeq \mathbb{F}_q[x]/(x - \alpha) \times \mathbb{F}_q[x]/(h(x)),$$

with $h(x)$ coprime to $x - \alpha$. Then we have the ring homomorphism

$$\psi_\alpha: R_q \to \mathbb{F}_q[x]/(x - \alpha) \simeq \mathbb{F}_q,$$

which is precisely the evaluation map, namely, $\psi_\alpha(g(x)) = g(\alpha)$, for any $g(x) \in R_q$.

Let $n$ be the degree of $f(x)$ and assume that $\alpha = 1$ is a root of $f(x)$. Given a PLWE sample $(a(x), b(x) = a(x)s(x) + e(x))$, the error term, $e(x) = \sum_{i=0}^{n-1} e_i x^i$, has its coefficients $e_i \in \mathbb{F}_q$ sampled from a discrete Gaussian variable of small enough standard deviation $\sigma$ (the authors set $\sigma = 8$, as suggested by applications). For an element $s \in \mathbb{F}_q$, writing $s = s(1)$ and applying the evaluation map, we have

$$b(1) - a(1)s = e(1) = \sum_{i=0}^{n-1} e_i,$$

and the sum $\sum_{i=0}^{n-1} e_i$ is hence sampled from a discrete Gaussian variable of standard deviation $\sqrt{n\sigma}$, which according to practical specifications, is of order $O(q^{1/4})$. 
For a right guess \( s = s(1) \), the value \( b(1) - a(1)s \) will belong to the set of integers \([-2\sqrt{n}\sigma, 2\sqrt{n}\sigma] \cap \mathbb{Z}\) (which can be easily enumerated) with probability about 0.95. Hence, we will refer to the set \([-2\sqrt{n}\sigma, 2\sqrt{n}\sigma] \cap \mathbb{Z}\) as the smallness region for this attack.

In the same paper, the authors develop a more subtle attack, based on the multiplicative order of the roots of \( f(x) \). Suppose that \( r \) is the order of a root \( \alpha \in \mathbb{F}_q, \alpha \neq 1 \). Again given a PLWE sample \((a(x), b(x) = a(x)s(x) + e(x))\), for \( s = s(\alpha) \in \mathbb{F}_q \), we have

\[
b(\alpha) - a(\alpha)s = e(\alpha) = \sum_{j=0}^{r-1} \sum_{i=0}^{n/r-1} e_{ir+j} \alpha^i,
\]

assuming without loss of generality that \( r \mid n \).

Now the terms \( e_j = \sum_{i=0}^{n/r-1} e_{ir+j} \) come from a Gaussian distribution of 0 mean and standard deviation \( \sqrt{n/r}\sigma \), hence they will belong to the set of integers \([-2\sqrt{n/r}\sigma, 2\sqrt{n/r}\sigma] \cap \mathbb{Z}\) with probability 0.95. This leads us to consider in this case the smallness region as the set \( \Sigma \) of all possible values for \( e(\alpha) \), which can be precomputed and stored in a look-up table. Observe that

\[
|\Sigma| \leq \left( 4\sqrt{n/r}\sigma + 1 \right)^r.
\]

**Input:** A collection of samples \( C = \{(a_i(x), b_i(x))\}_{i=1}^M \subseteq \mathbb{F}_q^2 \)

A look-up table \( \Sigma \) of all possible values for \( e(\alpha) \)

**Output:** A guess \( g \in \mathbb{F}_q \) for \( s(\alpha) \),
or NOT PLWE,
or NOT ENOUGH SAMPLES

Algorithm 1. Algorithm solving PLWE decision problem

These ideas turn in [Elias et al., 2015] into the Algorithm 1 whose probability of success is given by the following result:

**Proposition 3.1** ([Elias et al., 2015], Proposition 3.1). Assume \( |\Sigma| < q \). If Algorithm 1 returns NOT PLWE, then the samples come from the uniform distribution. If it outputs anything other than NOT PLWE, then the samples are valid PLWE samples with probability \( 1 - (|\Sigma|/q)^M \). In particular, this probability tends to 1 as \( M \) grows.

**Remark 3.2.** Notice that the cyclotomic polynomial \( \Phi_n(x) \) is protected against these attacks. Indeed, \( \alpha = 1 \) is never a root modulo \( q \neq p \). Moreover, for \( q \equiv 1 \pmod{n} \) the order of each of the \( m \) different roots of \( \Phi_n(x) \) is precisely \( n \).
3.1. Our method. Preliminary facts. In this section we present an attack against PLWE for $\Phi_m(x)$ and for non totally-split primes $q$ by using roots of $\Phi_m(x)$ over finite degree extensions of $\mathbb{F}_q$. To brief notation, let $m = p^n$ and set $N = \phi(m)$. Setting now $R_q := \mathbb{F}_q[x]/(\Phi_m(x))$, we will assume, as in Theorem 2.7 that $q = 1 + p^A u$, with $A \geq 1$, $(u, p) = 1$ and that $n > A$.

Our attack starts with a primitive $p^A$-th root $\rho$ of unity modulo $q$, for which we take $\alpha \in \mathbb{F}_{q^n-A} \setminus \mathbb{F}_q$, a $p^{n-A}$-th root of $\rho$. Due to Theorem 2.7 we have $Tr(\alpha) = 0$, where $Tr$ stands for the trace of $\mathbb{F}_{q^n-A}$ over $\mathbb{F}_q$.

Now, if $(a(x), b(x) = a(x)s(x) + e(x)) \in R_q^2$ is a PLWE sample attached to a secret $s(x)$ and an error term $e(x) = \sum_{i=0}^{N-1} e_i x^i$, then

$$b(\alpha) - a(\alpha)s = e(\alpha),$$

with $s := s(\alpha) \in \mathbb{F}_{q^n-A}$ and

$$(3.1) \quad Tr(b(\alpha) - a(\alpha)s) = Tr(e(\alpha)) = \sum_{i=0}^{N-1} e_i t_i,$$

where $t_i = Tr(\alpha^i)$.

If $(i, p) = 1$ then $t_i = 0$ since $\alpha$ is a root of $x^{p^n-A} - \rho$ and $ord(\alpha^i) = ord(\alpha) = m$.

More in general, we will make use of the following

Lemma 3.3. Notations as before, for $i = p^k v$ with $(v, p) = 1$ and $0 \leq k < n - A$, then $t_i = 0$.

Proof. For $i = p^k v$ with $(v, p) = 1$ and $0 \leq k < n - A$, the element $\alpha^{p^k v}$ is a root of the polynomial $x^{p^n-k-A} - \rho^v$ and since $\rho^v$ is also a primitive $p^A$-th root of unity, this polynomial is irreducible according to Corollary 2.8. Hence $Tr(\alpha^{p^k v}) = 0$. □

Applying Lemma 3.3 to the the right hand side of Equation 3.1 we are left with

$$(3.2) \quad Tr(b(\alpha) - a(\alpha)s) = p^{n-A} \sum_{j=0}^{p^{A-1} - 1} e_{j p^{n-A}} \rho^j.$$

But, again, the coefficients $e_{j p^{n-A}}$ are sampled from a discrete Gaussian $N(0, \sigma^2)$ and we can list those elements which occur with probability beyond 0.95, namely, the integer values in the interval $[-2\sigma, 2\sigma]$.

From now on we will suppose that $A = 2$ and $\sigma = 8$ so that in $[-2\sigma, 2\sigma]$ there are 32 integers. We can construct a look-up table where the expression 3.2 takes on values with large probability, namely, the smallness region, $\Sigma$. Observe that

$$(3.3) \quad |\Sigma| \leq (4\sigma + 1)^{p(p-1)}. $$

To construct $\Sigma$ requires $32^p(p-1)$ multiplications in $\mathbb{F}_q$, which is feasible for not very large values of $p$.

3.2. The trace map. In order to compute the trace of an element $\theta \in \mathbb{F}_{q^{p^n-2}}$, we can proceed by fixing an $\mathbb{F}_q$-basis of $\mathbb{F}_{q^{p^n-2}}$. For instance, we will stick to the power-basis $\{1, \alpha, ..., \alpha^{p^{n-2} - 1}\}$. Now we identify $\mathbb{F}_{q^{p^n-2}} \cong \mathbb{F}_q^{p^n-2}$ and we can write

$$\theta = \sum_{i=0}^{p^{n-2} - 1} a_i \alpha^i.$$
where \( a_i \in \mathbb{F}_q \) and \( \alpha^{p^{n-2}} = \rho \), our chosen \( p^2 \)-th root of unity in \( \mathbb{F}_q \). Taking trace, which is an \( \mathbb{F}_q \)-linear map, we have, as explained in Subsection 2.1:

\[
(3.4) \quad Tr(\theta) = \sum_{i=0}^{p^{n-2}-1} a_i Tr(\alpha^i) = p^{n-2}a_0.
\]

A first tentative approach to exploit a root \( \alpha \in \mathbb{F}_{q^{p^{n-2}}} \) for an attack would be to run over the elements \( s \) in this field as putative guesses for \( s(\alpha) \) and to decide for each sample \( (a(x), b(x)) \in \mathbb{R}_q^2 \) whether \( Tr(b(\alpha) - a(\alpha)s) \) belongs or not to the smallness region \( \Sigma \). One would need to evaluate \( Tr(b(\alpha)) \) and \( Tr(a(\alpha)s) \) for each \( s \in \mathbb{F}_q^{p^{n-2}} \). Leaving aside that running through all the elements of this large field is definitely unfeasible, we can evaluate \( Tr(b(\alpha)) \), which is independent of \( s \), by applying Lemma 3.3:

\[
Tr(b(\alpha)) = p^{n-2} \sum_{j=0}^{p(p-1)-1} b_{jp^{n-2}} \rho^j.
\]

Hence, evaluating \( Tr(b(\alpha)) \) takes about \( 2\sqrt{p(p-1)(q-1)} \mathbb{F}_q \)-products.

As for \( Tr(a(\alpha)s) \), notice that the map \( s \mapsto T_a(\alpha)(s) := Tr(a(\alpha)s) \) is also \( \mathbb{F}_q \)-linear, hence identifying \( s \in \mathbb{F}_{q^{p^{n-2}}} \) with its coordinates \( (s_0, s_1, \ldots, s_{p^{n-2}-1}) \), we can write:

\[
(3.5) \quad T_a(\alpha)(s) = Tr \left( \sum_{i=0}^{N-1} a_i \alpha^i \sum_{j=0}^{p^{n-2}-1} s_j \alpha^j \right).
\]

Since \( 0 \leq i \leq N - 1 \) and \( 0 \leq j \leq p^{n-2} - 1 \), the terms for which the trace do not vanish are those of the form \( a_i x_j \alpha^{i+j} \) with \( i + j = vp^{n-2} \), with \( 0 \leq v \leq p(p-1) \). Namely:

\[
(3.6) \quad T_a(\alpha)(s) = p^{n-2}a_0 s_0 + p^{n-2} \sum_{i=1}^{p(p-1)-1} \left( \sum_{j=0}^{p^{n-2}-1} s_j a_{xvp^{n-2}-j} \right) \rho^i.
\]

Since for each \( 0 \leq j \leq p^{n-2} - 1 \) we have to evaluate a polynomial of degree \( p(p-1) \) over \( \mathbb{F}_q \), which takes about \( 2\sqrt{p(p-1)(q-1)} \), evaluating \( Tr(a(\alpha)s) \) takes \( 2p^{n-2}\sqrt{p(p-1)(q-1)} \) per sample. However, as we can see, the expression for the trace in Equation (3.6) is rather complicated and computationally far from optimal, specially if we have to perform it for each sample and for each guess. For this reason, our attack will start by taking samples in a certain distinguished subspace and then showing that, essentially, the original set of input samples is either uniform or PLWE if and only if a new set of samples on this subspace manufactured from the original input is so. The fact that the set of samples belong to this subspace will also grant that we can reduce the set of putative guesses for \( s(\alpha) \) to just \( \mathbb{F}_q \).

### 3.3. A distinguished subspace.

Instead of in \( R_q^2 \), we will start considering samples in \( R_{q,0} \times R_q \) where

\[
R_{q,0} = \{ p(x) \in R_q : p(\alpha) \in \mathbb{F}_q \}.
\]

Observe that, for \( a(x) \in R_{q,0} \), it holds that \( Tr(a(\alpha)s) = a(\alpha) Tr(s) = p^{n-2}a(\alpha)s_0 \), which requires only two \( \mathbb{F}_q \)-multiplications to compute.

**Proposition 3.4.** The set \( R_{q,0} \) is an \( \mathbb{F}_q \)-vector subspace of \( R_q \) with dimension \( p^{n-1}(p-1) - p^{n-2} + 1 \).
Proof. It is obvious that $R_{q,0}$ is an $\mathbb{F}_q$ vector subspace of $R_q$. As for the dimension, notice that for $p(x) = \sum_{i=0}^{N-1} p_i x^i$, we have, by dividing each index $i$ by $p^{n-2}$:

$$ p(\alpha) = \sum_{j=0}^{p^{n-2}-1} \left( \sum_{v=0}^{(p-1)\alpha} p_{vp^{n-2+j}} \right) \alpha^j, $$

hence $p(\alpha) \in \mathbb{F}_q$ if and only if $\sum_{v=0}^{(p-1)\alpha-1} x_{vp^{n-2+j}} \rho_v = 0$ for each $0 < j \leq p^{n-2} - 1$. These are $p^{n-2} - 1$ linearly independent equations, hence the result follows. \hfill \Box

3.4. The attack. Stage 1. Next, we turn the previous ideas into an attack. Let us set $S := \mathbb{F}_{q^{n-2}}$ and assume that we are given a set of samples from $R_{q,0} \times R_q$. The goal is to distinguish whether these samples come from the $R_{q,0}$-PLWE distribution or from a uniform distribution with values in $R_{q,0} \times R_q$. To that end, given a sample $(a_i(x), b_i(x))$, we pick a guess $s \in S$ for $s(\alpha)$ and check whether $c_i := \frac{1}{p^{n-2}} Tr(b_i(\alpha) - a_i(\alpha)s)$ belongs to the look-up table $\Sigma$ defined in Equation 3.3. If this is not the case, we can safely remove from $S$ not only $s$, but also all the elements $t \in \mathbb{F}_{q^{n-2}}$ with the same trace as $s$. But notice that if $s = \sum_{j=0}^{p^{n-2}-1} s_j \alpha^j$, then an element $t = \sum_{j=0}^{p^{n-2}-1} t_j \alpha^j$ has the same trace as $s$ if and only if $t_0 = s_0$. Hence, given an $s \in S$, if we find a sample $(a_i(x), b_i(x))$ for which $c_i \notin \Sigma$, then we can delete $q^{p^{n-2}-1}$ elements of $S$.

Since $a(\alpha) \in \mathbb{F}_q$, then

$$ \frac{1}{p^{n-2}} Tr(b(\alpha) - a(\alpha)s) = \frac{1}{p^{n-2}} Tr(b(\alpha)) - \frac{1}{p^{n-2}} a(\alpha) Tr(s). $$

Therefore, it is enough just to check, for each $g \in \mathbb{F}_q$ (so that $g$ is a putative value for $Tr(s)$), whether or not

$$ \frac{1}{p^{n-2}} Tr(b(\alpha)) - \frac{1}{p^{n-2}} a(\alpha) g \in \Sigma. $$

This is at the price that if the algorithm returns just an element $g \in \mathbb{F}_q$, we should understand that this element is just the trace of one of the $q^{p^{n-2}-1}$ possible guesses for $s(\alpha)$. However, this is (even if weaker than Algorithm 1) enough as a decision attack.

\begin{center}
\begin{tabular}{ll}
\textbf{Input:} & A set of samples $C = \{(a_i(x), b_i(x))\}_{i=1}^M \in R_{q,0} \times R_q$ \hfill \(\text{A look-up table } \Sigma \text{ of all possible values for } Tr(e(\alpha))\) \\
\textbf{Output:} & \text{PLWE}, or NOT PLWE, or NOT ENOUGH SAMPLES \\
& \hfill \text{set } G := \emptyset \hfill \text{or NOT ENOUGH SAMPLES} \\
& \hfill \text{for } g \in \mathbb{F}_q \text{ do} \hfill \text{(or NOT ENOUGH SAMPLES)} \\
& \hfill \quad \text{for } (a_i(x), b_i(x)) \in C \text{ do} \hfill \text{if } G = \emptyset \text{ then return NOT PLWE} \\
& \hfill \quad \quad \text{if } p^{n-2} (Tr(b(\alpha)) - a(\alpha)g) \notin \Sigma \text{ then} \hfill \text{if } |G| = 1 \text{ then return PLWE} \\
& \hfill \quad \quad \quad \text{next } g \hfill \text{if } |G| > 1 \text{ then return NOT ENOUGH SAMPLES} \\
& \hfill \quad \text{set } G := G \cup \{g\} \hfill \text{set } G := G \cup \{g\} \\
& \hfill \text{if } G = \emptyset \text{ then return NOT PLWE} \hfill \text{if } G = \emptyset \text{ then return NOT PLWE} \\
& \hfill \text{if } |G| = 1 \text{ then return PLWE} \hfill \text{if } |G| = 1 \text{ then return PLWE} \\
& \hfill \text{if } |G| > 1 \text{ then return NOT ENOUGH SAMPLES} \hfill \text{if } |G| > 1 \text{ then return NOT ENOUGH SAMPLES} \\
\end{tabular}
\end{center}

Algorithm 2. Decision attack against $R_{q,0}$-PLWE

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As pointed out above, notice that if $|G| = 1$, say $G = \{g\}$, unlike Algorithm, our Algorithm does not output a guess for $s(\alpha)$; indeed, all we can venture is that it likely exists some $\hat{s} \in \mathbb{F}_{q}^* \setminus \{z\}$ such that $Tr(\hat{s}) = g$ with $s(\alpha) = \hat{s}$.

Next, we evaluate the complexity of our attack in terms of $\mathbb{F}_q^*$-multiplications:

**Proposition 3.5.** Given $M$ samples in $R_{q,0} \times R_q$, the number of $\mathbb{F}_q^*$-multiplications required for Algorithm is, at worst, of order $O((\sqrt{p(p-1)(q-1)}Mq)$.

**Proof.** To begin with, given $g \in \mathbb{F}_q$:

- For each sample $(a(x), b(x))$, evaluating $Tr(b_i(\alpha))$, by automorphic evaluation requires $2\sqrt{p(p-1)(q-1)}$ multiplications in $\mathbb{F}_q$. Hence checking whether $\frac{1}{p}Tr(b(\alpha)) - \frac{1}{p}Tr(a(\alpha))g \in \Sigma$ requires $2\sqrt{p(p-1)(q-1)} + 2$ multiplications in $\mathbb{F}_q$.
- In the worst case, the condition will fail for all the samples, in which case we will perform $(2\sqrt{p(p-1)(q-1)} + 2)M$ multiplications in $\mathbb{F}_q$ for each $g \in \mathbb{F}_q$.

Since the previous steps must be performed for every $g \in \mathbb{F}_q$, the number of multiplications for the worst case will be $(2\sqrt{p(p-1)(q-1)} + 2)M$. \hfill $\square$

Next, to obtain the the probability of success of our attack, we start making the following:

**Remark 3.6.** Given an input sample $((a(x), b(x)) \in R_{q,0} \times R_q$ for Algorithm and given $g \in \mathbb{F}_q$, with $a(x) = \sum_{j=0}^{p^n-1} a_j \cdot \alpha^j$ and $b(x) = \sum_{j=0}^{p^n-1} b_j \cdot \alpha^j$, notice that checking whether $\frac{1}{p}Tr(b(\alpha)) - \frac{1}{p}Tr(a(\alpha))g \in \Sigma$ is exactly the same as checking whether $\mu' = \mu g'(\mu) \in \Sigma$ where $\mu'(x) = \sum_{j=0}^{p^n-1} a_j \cdot \mu^{p^n-2} - x^j$ and $\mu'(x) = \sum_{j=0}^{p^n-1} b_j \cdot \mu^{p^n-2} - x^j$, with $\mu'(x), \mu'(x) \in \mathbb{F}_q[x]/(\Phi,p(x))$. Thus, the result of Algorithm 2 on samples $(a_i(x), b_i(x)) \in R_q^2$ is exactly the result of Algorithm applied to the samples $(a'_i(x), b'_i(x)) \in (R_q^2)^2$.

The following result will also be useful in our proof:

**Lemma 3.7.** Let $\{(a_i(x), b_i(x))\}_{i=1}^M$ be a set of input samples for Algorithm where, as usual, the $a_i(x)$ are taken uniformly from $R_{q,0}$ with probability $q^{-d}$. Then, for the corresponding input samples $(a'_i(x), b'_i(x))$ for Algorithm the elements $a'_i(x)$ are taken uniformly from $R_q$ with probability $q^{-p(p-1)}$.

**Proof.** For every sample $a(x)$ taken uniformly from $R_{q,0}$, if we write, as in Proposition 3.4

$$a(x) = \sum_{j=0}^{p^n-1} \left( \sum_{v=0}^{p(1-1)} a_{vp^n-2+j} \cdot x^{vp^n-2} \right) \cdot x^j,$$

we observe that the polynomial $a_0(x) = \sum_{v=0}^{p(p-1)-1} a_{vp^n-2+j} \cdot x^{vp^n-2}$ will be sampled from $R_{q,0}$ with probability $q^{-d}$ where $d = p^n/(p-1) - p^{n-2} + 2$. But for each $j \in \{1, ..., p^{n-2} - 1\}$ and for each $p(p-1)$-tuple $(a_j, a_{p^n-2+j}, ..., a_{p(p-1)-1})$ such that $\sum_{v=0}^{p(p-1)-1} a_{vp^n-2+j} \cdot p^v = 0$, the polynomial

$$a_0(x) + \sum_{j=1}^{p^{n-2}-1} \sum_{v=0}^{p(p-1)-1} a_{vp^n-2+j} \cdot x^{vp^n-2+j}$$

is also sampled with probability $q^{-d}$. These tuples form a vector space of dimension $p(p-1) - 1$, hence, there are $q^{p(p-1)-1}$ of such tuples for every $j$. Hence, for Algorithm the input sample $a'_i(x) = \sum_{v=0}^{p(p-1)-1} a_{vp^n-2+j} \cdot x^v$ (notice that $a_0(x) =$
\[ a'(x'^{n-2}) \] will occur with probability \( q^{-dP} \), where \( P \) is the number of joint samples for all the \( j \)'s together, namely \( P = q^{(p(p-1)-1)(n-2^{-1})} \), hence, the sample \( a'(x) \) for Algorithm 1 will occur with probability

\[ q^{-d+(p(p-1)-1)(n-2^{-1})} = q^{-p(p-1)}. \]

\[ \square \]

We can now study the probability of success of our attack:

**Proposition 3.8.** Assume that \( |\Sigma| < q \). If Algorithm 2 returns NOT PL WE, then the samples come from the uniform distribution on \( R_q \). If it outputs anything else than NOT PL WE, then the samples are valid PLWE samples with probability \( 1 - (|\Sigma|/q)^M \). In particular, this probability tends to 1 as \( M \) grows.

**Proof.** Set input samples \( S = \{(a_i(x), b_i(x))\}_{i=1}^M \) and \( S' = \{(a'_i(x), b'_i(x))\}_{i=1}^M \), and let us define the following events:

- \( E_q = \) The input samples \( S \) for Algorithm 2 are uniform,
- \( E'_q = \) The input samples \( S' \) for Algorithm 1 are uniform,
- \( rP = \) Algorithm 2 returns PL WE on input samples \( S \),
- \( rP' = \) Algorithm 1 returns PL WE on input samples \( S' \),
- \( rNE = \) Algorithm 2 returns NOT PL WE on input samples \( S \),
- \( rNE' = \) Algorithm 1 returns NOT PL WE on input samples \( S' \).

We clearly have \( rP \cup rNE \subseteq rP' \cup rNE' \). On the other hand, if \( rP' \cup rNE' \) holds, it is because the set \( G \) of guesses for \( s'(\rho) \) in Algorithm 1 on input samples \( S' \) has at least one element, hence, this element will also be a guess for \( Tr(s(\alpha)) \) in Algorithm 2 on input samples \( S \) and hence \( rP \cup rNE \) will also hold. Henceforth

\[ rP \cup rNE = rP' \cup rNE' \] and \( rNP = rNP' \).

On the other hand, as we have pointed out in Remark 3.6, we have that \( E_q \subseteq E'_q \).

Further, if \( E'_q \) holds then, given \( s \in \mathbb{F}_p \), by using Lemma 2.2, the elements \( b'_i(\rho) - sa'_i(\rho) \) are uniformly taken on \( \mathbb{F}_q \). This fact implies that the input samples for Algorithm 2 cannot come from the PLWE distribution: Otherwise, if \( (a_i(x), b(x) = a_i(x)s(x) + e_i(x)) \) is a PLWE sample for Algorithm 2 with \( e_i(x) = \sum_{j=0}^{p^{n-1}(p-1)-1} e_{ij}x^j \), then the terms \( e_{ij} \) are taken from an \( \mathbb{F}_q \)-valued Gaussian \( N(0, \sigma) \) and so are taken, in particular, those of the form \( e_{jnp^{-2}} \). Hence for \( s = Tr(s(\alpha)) \) we have

\[ \frac{1}{p^{n-2}} Tr(b(\alpha)) - \frac{1}{p^{n-2}} Tr(a(\alpha)) s = b'(\rho) - sa'(\rho) = \sum_{j=0}^{p(p-1)-1} e_{jnp^{-2}} \rho^j, \]

which is a contradiction. Hence the input samples \( S \) for Algorithm 2 should be uniform and \( E_q = E'_q \).

Hence

\[ E_q \cap (rP \cup rNE) = E'_q \cap (rP' \cup rNE') \] and \( E_q \cap rNP = E'_q \cap rNP' \).

Hence if the algorithm returns NOT PL WE then

\[ P[E_q | rNP] = \frac{P[E_q \cap rNP]}{P[rNP]} = \frac{P[E'_q \cap rNP']}{P[rNP']} = P[E'_q | rNP']. \]
On the other hand, if the algorithm returns anything else than NOT PLWE, then:

\[
P[E_q | rP \cup rNE] = \frac{P[E_q \cap (rP \cup rNE)]}{P[rP \cup rNE]} = \frac{P[E_q' \cap (rP' \cup rNE')]}{P[rP' \cup rNE']}
\]

which equals \((|\Sigma|/q)^M\) due to Proposition 3.1.

3.5. The attack. Stage 2. Suppose we are given access to arbitrarily many samples \(\{(a_i(x), b_i(x))\}_{i \geq 1} \subseteq R_q^2\) taken, all of them, either from the uniform distribution or from the PLWE distribution. We claim that we can decide, with non-negligible advantage, and in probabilistic polynomial time, which distribution do they come from.

First of all, since we can sample arbitrarily many times, we keep only those samples \((a_i(x), b_i(x)) \in R_{q,0} \times R_q\). For a given sample in \(R_{q,0}^2\), the probability that it actually belongs to \(R_{q,0} \times R_q\) is \(q^{d-N}\), where \(d = \dim(R_{q,0})\) and \(N = p^{n-1}(p-1)\).

Hence, the probability that for \(l\) samples in \(R_{q,0}^2\), at least \(k\) of them belong to \(R_{q,0} \times R_q\) can be approximated (if \(lq^{d-N} > 5\)) as

\[
P[B(l, q^{d-N}) \geq k] \cong P\left[ N(lq^{d-N}, \sqrt{lq^{d-N}(1-q^{d-N})}) \geq k \right],
\]

where \(B(l, q^{d-N})\) stands for a binomial distribution of parameters \(l\) and \(q^{d-N}\) and \(N(lq^{d-N}, \sqrt{lq^{d-N}(1-q^{d-N})})\) is a Gaussian distribution of mean \(lq^{d-N}\) and standard deviation \(\sqrt{lq^{d-N}(1-q^{d-N})}\). But the right hand side of Equation 3.8 can be written as

\[
P \left[ Z \geq \frac{k - lq^{d-N}}{\sqrt{lq^{d-N}(1-q^{d-N})}} \right],
\]

where \(Z\) is a standard Gaussian distribution. Hence, the probability that, out from \(l\) samples in \(R_{q,0}^2\) we obtain \(k\) of them from \(R_{q,0} \times R_q\) is about 1/2 if \(k = lq^{d-N} > 5\).

\begin{itemize}
  \item \textbf{Input:} An oracle \(X\) over \(R_q^2\) which can be either uniform or PLWE \\
  \hspace{1cm} A value \(k\)
  \\
  \textbf{Output:} \textbf{UNIFORM}, \\
  \hspace{1cm} or PLWE, \\
  \hspace{1cm} or \textbf{NOT ENOUGH SAMPLES}
\end{itemize}

\begin{algorithm}
  \begin{itemize}
    \item \textbf{set} \(C := \emptyset\)
    \item \textbf{do}
      \begin{itemize}
        \item \(\times (a(x), b(x)) \in R_q^2\)
        \item \textbf{if} \(a(x) \in R_{q,0}\) \textbf{then}
          \begin{itemize}
            \item \textbf{set} \(C := C \cup \{(a(x), b(x))\}\)
          \end{itemize}
        \end{itemize}
      \item \textbf{until} \(|C| = k\)
    \item \textbf{set} \(\omega := \text{Algorithm} (\text{2}) (C)\)
    \item \textbf{return} \(\omega\)
  \end{itemize}
\end{algorithm}

\textbf{Algorithm 3. Decision attack on PLWE}

Secondly, we must choose \(k\) so that \(1 - (|\Sigma|/q)^k \geq \theta\), where \(\theta\) is our desired success probability threshold. Now, we can use Algorithms 3 whose validity can be justified as follows: Suppose that \(X\) is our random distribution with values in \(R_q^2\), which can be either uniform or a PLWE-oracle and we want to distinguish which kind is it. We can define the following random algorithm \(X_0\), which can also be thought of as a random variable with values on \(R_{q,0} \times R_q\):
Proof. First, notice that for each $P_{14}$

\begin{align}
\text{where } B \text{ hand side of Equation 3.9 equals } P \text{ independence, and taking into account that } i \text{ } X \text{-th running of the variable } X \text{, and secret } s(x), \text{ then the right hand side of Equation 3.9 equals }
\end{align}

Hence, if $X$ is uniform, $P[C_{i,v}] = q^{-2N}$ and therefore $P[X_0 = v] = q^{-(d+2)}$. On the other hand, if $\chi_R$ is a PLWE-oracle attached to a discrete $R_q$-valued Gaussian variable $N$ and secret $s(x)$, then the right hand side of Equation 3.9 reads

\begin{align}
q^{N-d} P[C_{i,v}], \text{ with } v = (a(x), a(x)s(x) + e(x))
\end{align}

and where $e(x)$ is sampled from $\chi_R$, hence the value of Equation 3.10 is precisely $P[\chi_R = e(x)]$ and thus $X_0$ is a PLWE-oracle with values on $R_{q,0} \times R_q$. \hfill \Box

4. Coding examples

We have developed a Maple program in order to check the real applicability and performance of our Algorithms. The code can be found at GitHub. We have tested the code in Maple version 10.00 (a pretty old version), so it should be also executable over virtually any other higher version. As a disclaimer, keep in mind that we made no attempt at optimising the code: our aim was just presenting a "proof of concept" of the validity and feasibility of the algorithms contributed in this work. Depending on the available hardware and software, the execution may appear a little bit "sluggish".

The code implements a number of procedures to carry out the different needed tasks. Maple has an internal uniform sampler (called rand) that can be easily adjusted to give samples over any finite set of numbers. In particular, we use it over the base field $\mathbb{F}_q$. We implement an approximation of discrete Gaussian based on the regular normal Gaussian distribution provided directly by Maple.

Regarding the provided Maple sheet, running each example just boils down to selecting the desired parameters (in the so-called "main section") and executing the

https://github.com/raul-duran-diaz/PLWE-TraceAttack
The execution comprises the following steps:

1. Obtaining samplers for the uniform distribution \( \text{rollq} \) and for the discrete Gaussian \( X \).
2. Obtaining a prime of the desired size (by tweaking the initial value for \( u \)) satisfying the conditions for Theorem 2.7 to hold.
3. Computing the cyclotomic polynomial, its roots in \( \mathbb{F}_q \), and assigning to the variable \( \rho \) any one of them.
4. Computing the set \( \Sigma \) as a function of the parameters.
5. Selecting a number of executions (variable \( \text{n tests} \)) for Algorithm 2, and a number of samples for each execution (variable \( M \)). Then
   - First, a loop is executed \( \text{n tests} \) times and, for each turn, \( M \) samples from the PLWE oracle are generated, and passed to Algorithm 2. If it returns anything different from a set containing only one element, the execution is recorded as a failure.
   - Second, another loop is executed, but generating the samples from the uniform oracle, and passing each set of samples to Algorithm 2. If it returns anything other than an empty set, the execution is recorded as a failure.

We tested the implementation using two examples with different parameters that we summarise in the following table:

| Parameter | Example 1 | Example 2 |
|-----------|-----------|-----------|
| \( p \)   | 2         | 2         |
| \( n \)   | 10        | 11        |
| \( A \)   | 2         | 2         |
| \( q \)   | 24029     | 40013     |
| \( \sigma \) | 8        | 8         |
| \( \text{n tests} \) | 5        | 5         |
| \( M \)   | 10        | 10        |

The following table presents the dependent parameters, namely, those depending on the selected parameters shown above.

| Dependent param | Example 1 | Example 2 |
|-----------------|-----------|-----------|
| Polynomial \( \Phi \) | \( x^{512} + 1 \) | \( x^{1024} + 1 \) |
| \( m \)         | 1024      | 2048      |
| \( N \)         | 512       | 1024      |
| Factors of \( \Phi \) over \( \mathbb{F}_q \) | \((x^{256} + 11937)(x^{256} + 12092)\) | \((x^{512} + 27481)(x^{512} + 12532)\) |
| \( \rho \)      | -11937    | -27481    |

The executions of the two examples gave no failures both for PLWE oracle and for the uniform oracle, which gives us the confidence that the attack is correctly designed and at the same time is feasible for not too large parameter values.

Regarding execution times, the most time consuming part (by far) is the process of sampling generation. For Example 2, our hardware platform (Virtual Box configured with 1 GB of main memory running over Intel CORE i5, @2.2 GHz) needs about 200 seconds for generating the set of 10 samples of any kind, and just about 1 second for running Algorithm 2 over the same set.
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