REEB DYNAMICS DETECTS ODD BALLS

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ABSTRACT. We give a dynamical characterisation of odd-dimensional balls within the class of all contact manifolds whose boundary is a standard even-dimensional sphere. The characterisation is in terms of the non-existence of short periodic Reeb orbits.

1. Introduction

1.1. Definitions and the main result. Let $(M,\alpha)$ be a compact, connected contact manifold (with a fixed choice of contact form $\alpha$) of dimension $2n+1$, $n \in \mathbb{N}$, whose boundary $\partial M$ is diffeomorphic to $S^{2n}$.

We write $\inf_0(\alpha)$ for the infimum of all positive periods of contractible closed orbits of the Reeb vector field $R_\alpha$. When there are no closed contractible Reeb orbits, we have $\inf_0(\alpha) = \infty$, otherwise $\inf_0(\alpha)$ is a minimum and in particular positive.

Our main result will be a criterion for $M$ to be diffeomorphic to a ball in terms of $\inf_0(\alpha)$ and an embeddability condition on $\partial M$. To formulate this condition, we introduce the following terminology.

Definition. (a) Write $D^{2n}$ for the closed unit disc in $\mathbb{R}^{2n}$. The $(2n+1)$-dimensional manifold (with boundary)

$$Z := \mathbb{R} \times D^{2n}$$

with contact form

$$\alpha_{\text{cyl}} := db + \frac{1}{2} \sum_{j=1}^{n} (x_j dy_j - y_j dx_j)$$

will be referred to as the contact cylinder.

(b) We say that $\partial M$ admits a contact embedding into the contact cylinder $Z$ if there is an embedding $\varphi$ of a collar neighbourhood of $\partial M \subset M$ into $\text{Int}(Z)$ with $\varphi^* \alpha_{\text{cyl}} = \alpha$ and with the image of the collar under $\varphi$ contained in the interior of $\varphi(\partial M)$.

Theorem 1. Assume that the boundary $\partial M \cong S^{2n}$ of a contact manifold $(M,\alpha)$ as above admits a contact embedding into the contact cylinder, and $\inf_0(\alpha) \geq \pi$. Then $M$ is diffeomorphic to a ball.

This theorem has been proved for $\dim M = 3$ by Eliashberg and Hofer [5]. In that paper, they also announced the theorem for the higher-dimensional case, but a proof has never been published. They formulated the higher-dimensional case...
under the additional homological assumption $H_2(M; \mathbb{R}) = 0$; this condition, as we shall see, is superfluous.

For simplicity, we shall assume throughout that $n \geq 2$, although a large part of our argument also works for $n = 1$. Our proof shows that $(M, \alpha)$ is diffeomorphic to a ball whenever $\partial M \cong S^{2n}$ admits a contact embedding into the cylinder $Z_r := \mathbb{R} \times D_r^{2n}$ of radius $r$, and $\inf_0(\alpha) > \pi r^2$. Given a contact embedding into $Z = Z_1$, it may be regarded as an embedding into a cylinder of slightly smaller radius. Hence, even though the proof below will be based on the assumption $\inf_0(\alpha) > \pi$, the result holds under the weaker assumption $\inf_0(\alpha) \geq \pi$.

1.2. **Idea of the proof.** The contact embedding $\varphi$ of $\partial M$ into the contact cylinder $Z$ allows us to form a new contact manifold $\hat{\mathbb{R}}^{2n+1}$ by removing the bounded component of $\mathbb{R} \times \mathbb{R}^{2n} \setminus \varphi(\partial M)$ and gluing in $M$ instead. Similarly, we write $\hat{Z}$ for the cylinder $Z$ with $M$ glued in. We shall be studying the moduli space $W$ of holomorphic discs $u = (a, f) : D \to \mathbb{R} \times \hat{\mathbb{R}}^{2n+1} =: W$ in the symplectisation $\hat{W}$ of $\hat{\mathbb{R}}^{2n+1}$, where the discs are subject to certain boundary and homological conditions. (We always write $\mathbb{D}$ for the closed unit disc in $\mathbb{C}$ when regarded as the domain of definition of our holomorphic discs.) It will turn out that $f(D)$ is always contained in $\hat{Z}$. We then have the following dichotomy. Either the evaluation map

$$ev : W \times D \rightarrow \hat{Z} \quad \quad (a, f, z) \mapsto f(z)$$

is proper and surjective, i.e. gives a filling, in which case topological arguments involving the $h$-cobordism theorem can be used to show that $M$ must be a ball. (Strictly speaking, a topologically inessential subset of $\partial Z$ has to be removed on the right-hand side for $ev$ to be surjective.) Otherwise there will be breaking of holomorphic discs, which entails the existence of short contractible periodic Reeb orbits as in Hofer’s paper [15].

1.3. **Remarks.**

1. The bound $\pi$ in Theorem 1 is optimal. Inside $Z$ one can form the connected sum as described by Weinstein [23], cf. [10, Section 6.2], with any contact manifold, producing a belt sphere of radius $r_0$ smaller than, but arbitrarily close to 1. Inside this belt sphere one finds a periodic orbit of length $\pi r_0^2$.

2. In the 3-dimensional case, Theorem 1 can be strengthened. If $\inf_0(\alpha) \geq \pi$, then there are in fact no closed Reeb orbits at all. (This was part of the formulation of the theorem in [5].) In this 3-dimensional case, the holomorphic discs project to embedded discs in $\hat{Z}$, where they produce a foliation by discs transverse to the Reeb direction, see [5, Section 2]. This precludes closed orbits.

3. The existence of a foliation by discs as in (2) implies that there cannot even be trapped Reeb orbits, i.e. orbits that are bounded in forward or backward time. In [11] we show in joint work with Nena Röttgen that this is a strictly 3-dimensional phenomenon. In higher dimensions it is possible to have a Reeb dynamics on Euclidean space, standard outside a compact set, with trapped orbits but no periodic ones.

4. One may consider manifolds $M$ with disconnected boundary (and boundary components different from $S^{2n}$). The requirement of a contact embedding into the contact cylinder $Z$ is made for each component of $\partial M$ individually. By translating the images of these components in the $\mathbb{R}$-direction one may then assume without loss of generality that they are not nested. The collection $\varphi(\partial M)$ of these images
is contained in a large ellipsoid $E$ inside $\text{Int}(Z)$. The manifold obtained from $E$ by removing the interiors of the components of $\varphi(\partial M)$ and gluing in $M$ instead has non-trivial fundamental group: by taking a path in $M$ joining two boundary components, and a second path joining these two boundary points in the exterior of $\varphi(\partial M) \subset E$, one creates an essential loop. It follows that this manifold contains a contractible Reeb orbit of period smaller than $\pi$. This orbit must in fact be contained entirely in $M$, since the Reeb flow on $Z$ is positively transverse to any hypersurface $\{b\} \times D^{2n}$.

In other words, Theorem 1 provides a means of detecting contractible periodic orbits on non-compact manifolds or manifolds with boundary. See [2, 3, 22] for related work.

(5) For $n = 2$ the compactness argument for the moduli space simplifies, see Remark 12.

2. Symplectisations of contactisations

The contact cylinder $Z$ may be regarded as the contactisation of the exact symplectic manifold $D^{2n} \subset \mathbb{C}^n$. In the latter, we have the obvious holomorphic discs $D^{2n} \cap \{(z_2, \ldots, z_n) = \text{const.}\}$. In order to lift these to holomorphic discs in the symplectisation of $Z$, it is advantageous to proceed in two steps: first lift them to holomorphic discs in $\mathbb{C} \times D^{2n}$, and then transform them to holomorphic discs in the symplectisation $\mathbb{R} \times Z$ using an explicit biholomorphism

$$\Phi: \mathbb{R} \times \mathbb{R} \times D^{2n} \to \mathbb{C} \times D^{2n}.$$  

The desired boundary condition for the holomorphic discs on the left-hand side gives us the boundary conditions for the holomorphic discs on the right.

This allows one to transform a Cauchy–Riemann problem on the left with respect to a ‘twisted’ almost complex structure (which preserves the contact hyperplanes and pairs the Reeb with the symplectisation direction) into a Poisson problem on a single real-valued function.

This idea is implicit in [5, p. 1320] and has also been used in [21, Proposition 5]. Before we turn to our specific situation, we discuss this transformation in slightly greater generality.

2.1. Lifting holomorphic discs. Let $(V, J_V)$ be a Stein manifold of complex dimension $n - 1$. We write $\psi$ for a plurisubharmonic potential on $V$, so that $\omega_V := -d(\psi \circ J_V)$ is a Kähler form on $V$. In fact, what is really relevant for the following discussion is the existence of such a potential, not the integrability of $J_V$, cf. [13, Section 3.1]. Write $\lambda := -d\psi \circ J_V$ for the primitive 1-form of the symplectic form $\omega_V$.

The contactisation of $V$ is $(\mathbb{R} \times V, \alpha := db + \lambda)$, where $b$ denotes the $\mathbb{R}$-coordinate. Notice that $\partial_b$ is the Reeb vector field of the contact form $\alpha$. A symplectisation of this manifold is

$$(\mathbb{R} \times \mathbb{R} \times V, \omega := d(\tau \alpha)),$$

where $\tau$ is a strictly increasing smooth real-valued function on the first $\mathbb{R}$-factor (whose coordinate we shall denote by $a$). A compatible almost complex structure $J$ on this symplectic manifold, which in addition preserves the contact hyperplanes

$$\ker \alpha = \{v - \lambda(v)\partial_b : v \in TV\}$$
on \( \{a\} \times \mathbb{R} \times V \), is given by
\[
J(\partial_a) = \partial_b \quad \text{and} \quad J(v - \lambda(v)\partial_b) = J_V v - \lambda(J_V v)\partial_b.
\]
If \( J_V \) is not integrable, then \( J \) may only be tamed by \( \omega \).

A straightforward calculation gives the following generalisation of [21, Proposition 5]:

**Proposition 2.** The map
\[
\Phi: (\mathbb{R} \times \mathbb{R} \times V, J) \to (\mathbb{C} \times V, i \oplus J_V)
\]
\[
(a, b, z) \mapsto (a - \psi(z) + ib, z)
\]
is a biholomorphism. \( \square \)

Given a holomorphic disc \( \mathbb{D} \ni z \mapsto h(z) \in V \), we want to lift this to a holomorphic disc
\[
\mathbb{D} \ni z \mapsto (a(z), b(z), h(z))
\]
in the symplectisation, with boundary in the zero level of the symplectisation, i.e. \( a|_{\partial \mathbb{D}} \equiv 0 \). By Proposition 2, the functions \( a \) and \( b \) are found as follows. Let \( a: \mathbb{D} \to \mathbb{R} \) be the unique solution, smooth up to the boundary, of the Poisson problem
\[
\begin{cases}
\Delta a = \Delta(\psi \circ h) & \text{on } \text{Int}(\mathbb{D}), \\
a = 0 & \text{on } \partial \mathbb{D}.
\end{cases}
\]
Then \( a - \psi \circ h \) is harmonic, and we may choose the function \( b \) (unique up to adding a constant) such that \( a - \psi \circ h + ib \) is holomorphic. Notice that the function \( a \) is subharmonic.

**2.2. Examples.** (1) Our first example shows how to derive the set-up of [5] in this general context. We take \( V = \mathbb{C} \) with plurisubharmonic potential \( \psi(x + iy) = x^2/2 \). This yields the contact form \( db + x dy \) on \( \mathbb{R} \times \mathbb{C} \). Start with the holomorphic disc \( h: \mathbb{D} \to \mathbb{C} \) given by inclusion. The solution \( a \) of the corresponding Poisson problem — this is equation (52) in [5] — is given by \( a(x, y) = (x^2 + y^2 - 1)/4 \). For \( b \) one obtains \( b(x, y) = b_0 - xy/2 \). Notice that \( a - \psi \circ h + ib \) is the holomorphic function \( z \mapsto -(z^2 + 1)/4 + ib_0 \).

(2) For our second example we take \( V = \mathbb{C} \) with plurisubharmonic potential \( \psi(z) = |z|^2/4 \). This gives rise to the contact form \( db + (x dy - y dx)/2 \) on \( \mathbb{R} \times \mathbb{C} \). The solution \( a \) of the Poisson problem is unchanged, but \( b \) is now simply a constant function. The example in [21] is obtained by crossing this \( V \) with a cotangent bundle \( T^*Q \), on which one takes the plurisubharmonic potential \( ||p||^2/2 \), with \( p \) denoting the fibre coordinate, corresponding to the canonical Liouville 1-form on \( T^*Q \).

**2.3. The contact cylinder.** The contact form \( \alpha_{cyl} \) on the contact cylinder \( Z = \mathbb{R} \times D^{2n} \) derives from the plurisubharmonic potential
\[
\psi(z_1, \ldots, z_n) := \frac{1}{4} \sum_{j=1}^{n} |z_j|^2
\]
on \( D^{2n} \), where \( z_j = x_j + iy_j \).
Similar to Example 2.2(2), for any choice of parameters $b \in \mathbb{R}$, $s, t \in \mathbb{R}^{n-1}$ with $|t| < 1$ and $|s|^2 < 1 - |t|^2$, we have the obvious holomorphic discs
\[
\begin{align*}
D & \rightarrow \mathbb{C} \times D^{2n} \\
\quad \quad z & \mapsto \left(-\frac{1}{2} + ib, v^t_s(z)\right),
\end{align*}
\]
with
\[
\gamma_s^t(z) := (\sqrt{1 - |s + it|^2} z, s + it).
\]
Under the biholomorphism $\Phi$, these pull back to the holomorphic discs
\[
\begin{align*}
u^t_{s,b} : \quad \mathbb{D} & \rightarrow \mathbb{R} \times \mathbb{R} \times D^{2n} \\
\quad \quad z & \mapsto \left(\frac{1}{2}(1 - |s + it|^2) \cdot (|z|^2 - 1), b, \gamma_s^t(z)\right).
\end{align*}
\]
The disc $\nu^t_{s,b}$ has boundary on the totally real cylinder
\[
R^t := \{0\} \times \mathbb{R} \times \{z \in S^{2n-1} : z_1 \neq 0, (y_2, \ldots, y_n) = t\}.
\]
These $R^t$ foliate the subset $\{z_1 \neq 0, |t| < 1\}$ of $\partial \{0\} \times Z$.

Later on, when we describe the holomorphic discs we wish to consider, it is convenient to think of the $n$-sphere
\[
S^t := \{z \in S^{2n-1} : (y_2, \ldots, y_n) = t\}
\]
as coming equipped with an open book decomposition. The subset $\{z_1 = 0\}$ of this $n$-sphere is an $(n-2)$-sphere; it constitutes the binding of the open book. The fibration of its complement over $S^1$ is given by the map $(z_1, x_2, \ldots, x_n) \mapsto z_1/|z_1|$. The fibre over $\theta \in S^1$ is an open $(n-1)$-discs parametrised by
\[
s \mapsto \gamma^t_s(e^{i\theta}), \quad |s|^2 < 1 - |t|^2;
\]
we shall refer to this as the $\theta$-page $P_\theta$. The last factor in the description of $R^t$ is the union over all these pages.

3. The moduli space of holomorphic discs

We now form the contact manifold $(\mathbb{R}^{2n+1}, \alpha)$ as explained in Section 1.2. Let
\[
(W := \mathbb{R} \times \mathbb{R}^{2n+1}, \omega := d(\tau \alpha))
\]
be its symplectisation, where $\tau : \mathbb{R} \rightarrow \mathbb{R}^+$ is a smooth function with $\tau' > 0$ and $\tau(a) = e^a$ for $a \geq 0$. The freedom of choosing $\tau$ on $\{a < 0\}$ is required for the asymptotic analysis cited in Section 4.

3.1. The almost complex structure. Choose $b_0, r \in \mathbb{R}^+$, $r < 1$, such that $\varphi(\partial M)$ is contained in the interior of the box
\[
B := [-b_0, b_0] \times D^2_r \subset Z,
\]
where $D^2_r \subset \mathbb{C}^n$ denotes a closed $2n$-disc of radius $r$. We write $\hat{B}$ for the result of gluing $M$ into this box, in other words,
\[
\hat{\mathbb{R}}^{2n+1} = \hat{B} \cup_{\partial B} \left((\mathbb{R} \times \mathbb{C}^n) \setminus \text{Int}(B)\right).
\]
We shall also have occasion to use the notation $\hat{Z}$ for the cylinder $Z$ with $M$ glued in, that is,
\[
\hat{Z} = \hat{\mathbb{R}}^{2n+1} \setminus (\mathbb{R} \times (\mathbb{C}^n \setminus \text{Int}(D^{2n}))).
\]
On the symplectic manifold $(W, \omega)$ we choose an almost complex structure $J$ compatible with $\omega$ subject to the following conditions:
(J1) On the complement of $\mathbb{R} \times \text{Int}(\tilde{B})$, the almost complex structure $J$ equals the one described in Section 2.1.

(J2) On $\mathbb{R} \times \text{Int}(\tilde{B})$, we make a generic choice (in a sense explained in Section 5.2) of an $\mathbb{R}$-invariant almost complex structure $J$ preserving $\ker \hat{\alpha}$ and satisfying $J(\partial_\alpha) = R \hat{\alpha}$.

Condition (J1) will allow us to prove that holomorphic discs in the relevant region are standard. Condition (J2) implies that the breaking of holomorphic discs corresponds to cylindrical ends asymptotic to Reeb orbits.

3.2. The moduli space. We now consider holomorphic discs (smooth up to the boundary) of the form

$$u = (a, f): (\mathbb{D}, \partial \mathbb{D}) \mapsto (W = \mathbb{R} \times \mathbb{R}^{2n+1}, R^k),$$

i.e. with totally real boundary condition, where $t$ is allowed to vary over the open unit disc in $\mathbb{R}^{n-1}$. We shall call the value of $t$ corresponding to a given $u$ the ‘boundary level’ of the holomorphic disc.

We define $W$ to be the moduli space of such discs $u$, which are supposed to satisfy the following conditions:

(M1) The relative homology class $[u] \in H_2(W, R^k)$, with $t$ equal to the boundary level of $u$, equals that of $u^*_{a,b}$ for some $b \in \mathbb{R}$, $s \in \mathbb{R}^{n-1}$, where $|b|$ is large (such that $u^*_{a,b}$ may be regarded as a holomorphic disc in $W$).

(M2) For $k = 0, 1, 2$, the $S^n$-component of $f(e^{i k \pi / 2}) \in R^k \subset \mathbb{R} \times S^n \subset \mathbb{R} \times S^{2n-1}$ lies in the page $P_{k \pi / 2}$ in the terminology of Section 2.3.

Notice that the boundary of a holomorphic disc with boundary on $R^k$ need not be transverse to the pages of the open book decomposition of $S^n$. In particular, there may be holomorphic discs which intersect the pages $P_{k \pi / 2}$, $k \in \{0, 1, 2\}$, several times; such a disc admits multiple parametrisations in accordance with requirement (M2).

3.3. Properties of the holomorphic discs. Here we collect some basic properties of the discs $u \in W$.

Lemma 3. The Maslov index $\mu$ of any disc $u \in W$, i.e. the index of the bundle pair $(u^*TW, (u|_{\partial \mathbb{D}})^*TR^k)$, equals 2.

Proof. We appeal to the axiomatic definition of the Maslov index in [19] Section C.3. For the disc $u_0 := u_{b,0}$ in $\mathbb{R} \times \mathbb{R}^{2n+1}$, the bundle $u_0^*T(\mathbb{R} \times \mathbb{R}^{2n+1})$ is a trivial $\mathbb{C}^{n+1}$-bundle. The fibre of the totally real subbundle $(u_0|_{\partial \mathbb{D}})^*TR^k$ over $e^{i \theta} \in \partial \mathbb{D}$ is given by $\mathbb{R} i \oplus \mathbb{R} i e^{i \theta} \oplus \mathbb{R}^{n-1}$. So the normalisation property of the Maslov index implies $\mu(u_{b,0}) = 2$.

By the homotopy invariance of the Maslov index, we have $\mu(u^*_{a,b}) = 2$ for all standard discs $u^*_{a,b}$ in $W$. Finally, given any $u \in W$, we may choose $u^*_{a,b}$ in the same relative homology class, so that $u - u^*_{a,b}$ is a boundary. This implies $\mu(u) = 2$. □

Lemma 4. Each disc $u \in W$ has symplectic energy $\int_0^1 u^* \omega$ less than or equal to $\pi$.

Proof. Since $u = (a, f)$ maps the boundary of the disc $\mathbb{D}$ into the 0-level of the symplectisation, the theorem of Stokes gives $\int_0^1 u^* \omega = \int_{\partial \mathbb{D}} f^* \hat{\alpha}$. Next we observe that the boundary is mapped to $R^k \subset \{0\} \times \mathbb{R} \times \mathbb{C}^n$, where $\hat{\alpha} = \alpha_{cyl}$. Also, the map $f$ can be written near the boundary in components as $f = (b, \mathbf{h}) = (b, h_1, \ldots, h_n)$,
with every \( h_j \) holomorphic. The boundary condition translates into \( \text{Im} \, h_j|_{\partial \mathbb{D}} = t_j \) for \( j = 2, \ldots, n \), so for these \( j \) we have \( h^* \, dy_j = 0 \) and \( y_j \, dx_j = t_j \, dx_j \) is exact. The exact parts of \( \alpha_{cy} \) integrate to zero, so we obtain
\[
\int_{\partial \mathbb{D}} f^* \alpha = \frac{1}{2} \int_{\partial \mathbb{D}} h_1^* (x_1 \, dy_1 - y_1 \, dx_1).
\]
In polar coordinates, that last integrand equals \( h_1^* (r_1^2 \, d\theta_1) \). Along the image of \( h_1 \) we have \( r_1 \leq 1 \), and by the homological condition (M1) the winding number of \( h_1|_{\partial \mathbb{D}} \) equals 1. This gives the claimed upper bound on \( \int_{\mathbb{D}} u^* \omega \).

**Lemma 5.** All discs \( u \in \mathcal{W} \) are holomorphically indecomposable and simple.

*Proof.* By ‘holomorphically indecomposable’ we mean that \( u \in \mathcal{W} \) cannot be decomposed into more than one holomorphic disc \( w \) in \( \mathbb{W} \) with boundary in \( \mathbb{R}^k \).

Indeed, given any such disc \( w \), an estimate as in the preceding lemma and the positivity of the energy of \( w \) imply that \( w|_{\partial \mathbb{D}} \) has positive winding in the \((x_1, y_1)\)-plane. Since for \( u \in \mathcal{W} \) the corresponding winding number is one, there can be no non-trivial decomposition of \( u \).

Now, according to \([17] \text{ Theorem A}\), the homology class \([u] \in H_2(\mathbb{W}; \mathbb{L}^1)\) of a holomorphic disc with totally real boundary condition can be decomposed into positive multiples of homology classes represented by simple discs, which are obtained from a decomposition of \( \mathbb{D} \). Since, as just shown, \( u \) is indecomposable, the disc \( u \) itself must be simple.

Simplicity of the discs \( u = (a, f) \) will not be quite enough for our purposes. We shall also need simplicity of \( f \) in the sense of the following lemma, cf. \([16] \text{ Theorem 1.14}\). Here \( \pi \) denotes the projection of \( TM \) onto \( \ker \hat{\alpha} \) along the Reeb vector field \( R_{\alpha} \).

**Lemma 6.** For each \( u = (a, f) \in \mathcal{W} \), the set
\[
F_{\text{inj}} := \{ z \in \mathbb{D} : \pi \circ T_z f \neq 0, \, f^{-1}(f(z)) = \{z\} \}
\]
of ‘\( f \)-injective points’ is open and dense in \( \mathbb{D} \).

*Proof.* The combination of defining conditions for \( F_{\text{inj}} \) is open, so we need only show that \( F_{\text{inj}} \) is dense in \( \mathbb{D} \). We begin with three observations about the behaviour of the holomorphic discs \( u \).

First of all, in a neighbourhood of the boundary \( \partial \mathbb{D} \subset \mathbb{D} \) we can write \( f = (b, h) \) as in the proof of Lemma 4. The strong maximum principle and the boundary lemma of E. Hopf imply that \( h|_{\partial \mathbb{D}} \) is an immersion, hence \( \pi \circ T f|_{\partial \mathbb{D}} \neq 0 \). Moreover, a variant of the Carleman similarity principle \([5] \) pp. 1315/6] implies that the set \( \{ z \in \mathbb{D} : \pi \circ T_z f = 0 \} \) is finite.

Secondly, the boundary \( \partial \mathbb{D} \) maps under \( f \) to \( \mathbb{R} \times S^{2n-1} \). Near any point in \( \text{Int}(\mathbb{D}) \) that putatively maps to \( \mathbb{R} \times (\mathbb{C}^n \setminus \text{Int}(D^{2n})) \), we could write \( f = (b, h) \) as above, and we would find that \( h \) violates the maximum principle. We conclude in particular that there are no mixed intersections of the holomorphic disc \( u \), i.e. pairs of an interior and a boundary point with the same image.

Thirdly, from the work in \([25] \) it follows that the immersion \( u|_{\partial \mathbb{D}} = (0, f|_{\partial \mathbb{D}}) \) has at most finitely many double points. Otherwise the respective preimages would accumulate in two separate points — for in a common limit point the differential \( Tu \) would be singular — and \([25] \text{ Lemma 4.2}\) would imply that the differentials \( Tu \) in the two limit points are collinear over \( \mathbb{R} \). Furthermore, by Lemma 4(i) below,
the collinearity factor would have to be positive. Then [25] Lemma 4.3 would imply that \( u \) is not simple, contradicting the preceding lemma.

From these last two observations we infer that \( F_{\text{inj}} \) contains \( \partial \mathbb{D} \) with the exception of at most finitely many points, and in particular is non-empty.

Now we prove that \( F_{\text{inj}} \) is dense, arguing by contradiction. If \( F_{\text{inj}} \) were not dense, the set \( \text{Int}(\mathbb{D}) \setminus F_{\text{inj}} \) would have non-empty interior. By the preceding observations we can find an open subset \( U \subset \text{Int}(\mathbb{D}) \) such that for each \( z \in U \) the set \( f^{-1}(f(z)) \subset \text{Int}(\mathbb{D}) \) contains more than just the point \( z \), and such that \( \pi \circ T_u f \neq 0 \) in all points \( w \in f^{-1}(f(U)) \). The latter implies that the points in \( f^{-1}(f(z)) \) are isolated, and hence finite in number.

What follows is an explication of an argument in [16, p. 459]. Fix a point \( z_0 \in U \) and write \( f^{-1}(f(z_0)) = \{ z_0, z_1, \ldots, z_N \} \). Choose pairwise disjoint (and disjoint from \( U \)) open neighbourhoods \( U_k \subset \text{Int}(\mathbb{D}) \) of \( z_k, k = 1, \ldots, N \), such that \( f|_{U_k} \) is an embedding. By a compactness argument, \( U \) can be chosen so small that

\[
\begin{align*}
  f(U) \subset \bigcup_{k=1}^{N} f(U_k),
\end{align*}
\]

and such that \( f|_U \) is likewise an embedding. Choose relatively compact neighbourhoods \( U'_k \subset U_k \) of \( z_k, k = 1, \ldots, N \). By shrinking \( U \) to a smaller neighbourhood of \( z_0 \), we can ensure that

\[
\begin{align*}
  f(U) \subset \bigcup_{k=1}^{N} f(U'_k).
\end{align*}
\]

Set \( A_k := (f|_U)^{-1}(f(U'_k)) \subset U \). If \( A_1 \) has non-empty interior, we can shrink \( U \) such that \( f(U) \subset f(U'_1) \) (but \( U \) need no longer be a neighbourhood of \( z_0 \)). The argument then concludes as in [16] pp. 459/60, leading to a contradiction to \( u \) being simple. If \( A_1 \) has empty interior, so that \( U \setminus A_1 \) is dense in \( U \), we find that

\[
\begin{align*}
  f(U) \subset \bigcup_{k=2}^{N} f(U'_k) \subset \bigcup_{k=2}^{N} f(U_j).
\end{align*}
\]

The argument concludes inductively. \( \square \)

3.4. Bounds on the holomorphic discs. In the next lemma we collect some restrictions on the image \( u(\mathbb{D}) \) of the holomorphic discs \( u \in \mathcal{W} \).

Lemma 7. For \( u = (a, f) \in \mathcal{W} \) we have:

(i) \( a < 0 \) on \( \text{Int}(\mathbb{D}) \).

(ii) \( f(\text{Int}(\mathbb{D})) \) is contained in the interior of \( \tilde{Z} \), i.e.

\[
\begin{align*}
  f(\text{Int}(\mathbb{D})) \cap (\mathbb{R} \times (\mathbb{C}^n \setminus \text{Int}(D^{2n}))) = \emptyset.
\end{align*}
\]

Proof. (i) The holomorphicity of \( u = (a, f) \) (with respect to an almost complex structure preserving \( \ker \hat{\alpha} \) and satisfying \( J(\partial_a) = R_a \)) implies \( f^*\hat{\alpha} = -da \circ 1 \), so \( a \) is subharmonic. We have \( a|_{\partial \mathbb{D}} \equiv 0 \), but \( a \) cannot be identically zero on all of \( \mathbb{D} \), for otherwise we would have \( f^*\alpha \equiv 0 \) and \( f^*da \equiv 0 \), which would imply that \( u \) has zero symplectic energy density and hence is constant, contradicting (M1). The strong maximum principle for \( a \) then implies the claim.

(ii) Near the points of \( \mathbb{D} \) mapping under \( f \) to \( \mathbb{R} \times (\mathbb{C}^n \setminus \text{Int}(D^{2n})) \) we can write this map in components as \( f = (b, h) \). Since the level sets of the plurisubharmonic potential \( \psi \) (defined in Section 2.3) are strictly pseudoconvex, they cannot
be touched from the inside by a non-constant holomorphic disc at an interior point of that disc. Hence, if \( f(\text{Int}(\mathbb{D})) \) were not contained in \( \text{Int}(Z) \), we would find that the map \( h \) is defined and locally constant on a non-empty open and closed subset of \( \mathbb{D} \), and hence on all of \( \mathbb{D} \), contradicting the homological assumption (M1).

Since a generic choice of the almost complex structure \( J \) is only allowed on \( \mathbb{R} \times \text{Int}(\hat{B}) \), this can be used to guarantee regularity in the sense of \cite{19} Definition 3.1.4 only for those holomorphic discs that pass through this ‘perturbation domain’, see \cite{19} Remark 3.2.3. We therefore want to show that all other discs belong to the standard family \( u^t_{s,b} \), where transversality is obvious. This will be used below to show that \( W \) is actually a manifold.

**Lemma 8.** Let \( u = (a, f) \in W \). If \( f(\mathbb{D}) \subset \hat{\mathbb{R}}^{2n+1} \setminus \text{Int}(\hat{B}) \), then \( u = u^t_{s,b} \) for some \( s \in \mathbb{R}^{n-1} \) with \(|s|^2 < 1 - |t|^2\), \( b \in \mathbb{R} \), and \( t \) equal to the boundary level of \( u \).

**Proof.** Since \( f \) maps to the complement of \( \text{Int}(\hat{B}) \), we can write it globally as \( f = (b, h) \), with every component \( h_j \) of \( h \) a holomorphic map \( \mathbb{D} \to \mathbb{C} \). The boundary condition for \( u \) means that for \( j = 2, \ldots, n \) we have \( \text{Im} h_j = t_j \) on \( \partial \mathbb{D} \). The minimum and maximum principle for harmonic functions implies that \( \text{Im} h_j = t_j \) on all of \( \mathbb{D} \). Hence, by the open mapping theorem, \( \text{Re} h_j =: s_j \) is likewise constant on \( \mathbb{D} \) for \( j = 2, \ldots, n \).

The component \( h_1 \) is then a holomorphic disc in \( \mathbb{C} \) with boundary on the circle of radius \( r_1 := \sqrt{1 - |s + it|^2} \). Moreover, again by the boundary lemma of E. Hopf and condition (M2), the restriction \( h_1|_{\partial \mathbb{D}} \) is an orientation-preserving diffeomorphism of \( \partial \mathbb{D} \) onto that circle. The argument principle implies that \( h_1 \) is a biholomorphism of \( \mathbb{D} \) onto \( \mathbb{D}_{r_1} \), and then (M2) forces \( h_0 = r_1 \cdot \text{id}_\mathbb{D} \).

By Proposition 2 the function
\[
z \mapsto a(z) - \frac{1}{4}|h(z)|^2 = a(z) - \frac{1}{4}(r_1^2|z|^2 + |s + it|^2)
\]
on \( \mathbb{D} \) is harmonic, taking the constant value \(-1/4\) on \( \partial \mathbb{D} \), hence it is constant on \( \mathbb{D} \). This means that the imaginary part \( b \) that makes this into a holomorphic function must also be constant. Solving for \( a(z) \) we get
\[
a(z) = \frac{1}{4}(1 - |s + it|^2) \cdot (|z|^2 - 1) \quad \text{on} \quad \mathbb{D},
\]
i.e. \( u = u^t_{s,b} \).

The next lemma will allow us to control the degree of the evaluation map \( \text{ev} \). It says that non-standard disc can never reach \( b \)-levels with \(|b| > b_0 \). This is also relevant for compactness.

**Lemma 9.** Let \( u = (a, f) \in W \). On the closed set \( A := f^{-1}(\hat{\mathbb{R}}^{2n+1} \setminus \text{Int}(\hat{B})) \subset \mathbb{D} \), which includes the whole boundary \( \partial \mathbb{D} \) in its interior, we write \( f = (b, h) \). If the function \( b \) takes values outside \([-b_0, b_0] \), then \( f \) maps to a \( b \)-level set \( \{b_1\} \times D^{2n} \) with \(|b_1| > b_0 \) and hence, by the preceding lemma, the holomorphic curve \( u \) equals \( u^t_{s,b_1} \) for some \((s, t) \in \text{Int}(D^{2n-2}) \).

**Proof.** Choose \( z_* \in A \) with \( b_* := b(z_*) \) of maximal absolute value. Notice that \( z_* \) is an interior point of \( A \).

By Proposition 2 the function
\[
g := a - |h|^2/4 + ib
\]
is holomorphic on \( \text{Int}(A) \). We should now like to argue with the maximum principle that the imaginary part \( b \) of \( g \) has to be constant equal to \( b_* \) on an open and closed subset of \( D \). If \( z_* \in \text{Int}(D) \), this inference is indeed conclusive, just as in part (ii) of Lemma \( \text{[7]} \). If \( z_* \in \partial D \), we reason as follows.

The real part of the holomorphic function \( g \) takes the constant value \(-1/4\) on \( \partial D \subset \text{Int}(A) \). It follows that the function can be extended by Schwarz reflection to the complementary set \( \overline{A} \) of \( A \) in \( \mathbb{C} \setminus D \), with \( \mathbb{C} \) denoting the Riemann sphere. Indeed, the holomorphic function \( i(g + 1/4) \) takes real values on \( \partial D \), so the Schwarz reflection principle applies to this function, and we simply transform the extension via the map \( w \mapsto -iw - 1/4 \) to a holomorphic extension of \( g \). Now \( z_* \) is an interior point of \( A \cup \overline{A} \), and we conclude as before with the maximum principle. \( \square \)

Write \( A_r \) for the annulus \( D^{2n} \setminus D^{2n}_r \), with \( r < 1 \) chosen as in Section \( \text{[2.3]} \). For \( 0 < \delta < 1 \), write \( V_\delta \) for the image of the map

\[
\mathbb{D} \times (D^{2n-2} \setminus D^{2n-2}_1) \quad \longrightarrow \quad D^{2n} \\
(z, s + it) \quad \longmapsto \quad v_b^s(z),
\]

with \( v_b^s(z) \) defined as in Section \( \text{[2.2]} \) (and with \( |s|^2 + |t|^2 \) allowed to take the value 1). Select \( \delta \) small enough such that the closure of \( V_\delta \) is contained in \( A_r \). Observe that the part of this image that lies in \( S^{2n-1} = \partial D^{2n} \subset A_r \), which is the same as the image of the points \( z \in \partial \mathbb{D} \) under the above map, constitutes for each fixed \( t \) a neighbourhood of the binding of the open book decomposition of the \( n \)-sphere

\[
\{ z \in S^{2n-1} : (y_2, \ldots, y_n) = t \}
\]
described in Section \( \text{[2.3]} \). Our next lemma says that any holomorphic disc whose \( D^{2n} \)-component intersects one of these neighbourhoods is of necessity standard.

**Lemma 10.** Let \( u = (a, f) \in \mathcal{W} \). If \( f(\mathbb{D}) \) intersects \( \mathbb{R} \times V_\delta \), then the holomorphic disc \( u \) equals one of the standard discs \( u^A_{s,b} \).

**Proof.** On the non-empty open set \( U := f^{-1}(\mathbb{R} \times V_\delta) \) we can write \( f = (b, h) \) as previously. The function \( (z_2, \ldots, z_n) \) attains its norm minimum on \( V_\delta \setminus V_\delta \), so we find a point \( z_* \in U \) where the plurisubharmonic function \( |h_2|^2 + \cdots + |h_n|^2 \) attains its maximum. Since the imaginary part of \( h_j \) equals \( t_j \) on \( \partial \mathbb{D} \) for \( j = 2, \ldots, n \), we may argue as in the preceding proof that \( (h_2, \ldots, h_n) \) is defined and constant on all of \( \mathbb{D} \), and we conclude as before with Lemma \( \text{[8]} \). \( \square \)

4. **Compactness**

In this section we establish, under the assumption \( \inf_0(\alpha) > \pi \), a \( C^0 \)-bound (in \( a \)-direction) on any sequence of holomorphic discs \( u_\nu \) in the moduli space \( \mathcal{W} \) with convergent boundary levels \( t_\nu \to t_0 \). In \( \text{[8]} \) it is shown that this yields a Gromov-convergent subsequence; one can also deduce this from \( \text{[24]} \). With Lemma \( \text{[5]} \) this implies compactness of the truncated moduli space

\[
\mathcal{W}_\delta := \{ u = (a, f) \in \mathcal{W} : f(\mathbb{D}) \subset \hat{B} \setminus ([-b_0, b_0] \times V_\delta) \},
\]
i.e. the space obtained from \( \mathcal{W} \) by cutting off the ends corresponding to standard discs.
4.1. **Variable boundary condition.** The holomorphic discs $u \in W$ have boundary on the totally real cylinder $R^k$, which varies with the parameter $t \in \mathbb{R}^{n-1}$, $|t| < 1$. It is possible to fix the boundary condition, at the cost of allowing the almost complex structure to vary. This is done with the help of a flow that enables us to identify different copies of $R^k$, at least in a neighbourhood of a reference parameter $t_0$. That flow will also provide explicit charts when we discuss transversality.

We begin with an elementary geometric construction. Consider the unit sphere

$$S^{m-1} \subset \mathbb{R}^m = \{(x, y): x \in \mathbb{R}^{m-k}, y \in \mathbb{R}^k\}$$

and the projection $\pi: \mathbb{R}^m \to \mathbb{R}^k$, $(x, y) \mapsto y$. Then the vector field

$$-\langle v, y \rangle \cdot x/\|x\|^2, v$$

is the unique tangent vector field to $S^{m-1} \setminus \{\|x\| = 0\}$ that projects onto a given vector field $v$ on the open unit ball in $\mathbb{R}^k$ and is orthogonal to the level sets of $\pi$. In particular, the flow of this vector field will move such level sets to level sets. Observe also that the $\mathbb{R}^{m-k}$-component of this vector field points in radial direction.

Given a reference point $y_0 \in \text{Int}(D^k) \subset \mathbb{R}^k$ and a constant $\rho > 0$ such that the closed disc $D^k_\rho(y_0)$ of radius $\rho$ around $y_0$ is contained in the open unit ball $\text{Int}(D^k)$, choose a bump function $\chi$ on this unit ball, identically 1 on the $\rho$-ball around $y_0$, identically 0 near the boundary of the unit ball. Then, for any $v \in \mathbb{R}^k$ of length smaller than $\rho$, the flow $\psi^v_t$ of the lift of $\chi \cdot v$ to $S^{m-1} \setminus \{\|x\| = 0\}$ has the following properties:

(i) $\psi^v_t$ depends smoothly on $v$,

(ii) $\psi^v_t$ fixes a neighbourhood of $\{\|x\| = 0\}$,

(iii) $\psi^v_t$ is radial in $\mathbb{R}^{m-k}$-direction,

(iv) $\psi^v_t$ sends the level set $\pi^{-1}(y_0)$ onto $\pi^{-1}(y_0 + v)$.

In our context, we apply this construction to the map $\pi: S^{2n-1} \to \mathbb{R}^{n-1}$ sending $(z_1, \ldots, z_n)$ to $(y_2, \ldots, y_n)$, where the level sets are the spheres $S^1$. We extend the flow $\psi^v_t$ to a compactly supported flow on $\mathbb{R} \times \mathbb{R}^{2n+1}$ by first extending the lift of $\chi \cdot v$ as a vector field not depending on $a, b$ or the radial parameter in $D^{2n}$, and then cutting it off with a bump function supported near

$$\{0\} \times [-b_0, b_0] \times S^{2n-1}$$

and identically 1 in a smaller neighbourhood of that set.

For a sequence $u_\nu$ of holomorphic discs of level $t_\nu \to t_0$, we can then use the maps $\psi^{1-t_0}_t$ (with reference point $t_0$) to pull back the $u_\nu$ to $J_\nu$-holomorphic discs of level $t_0$, where $J_\nu := (\psi^{1-t_0}_t)^* J$ is $C^\infty$-convergent to $J$ and coincides with $J$ outside the neighbourhood described in the preceding paragraph.

Notice that by condition (iii), the diffeomorphisms $\psi^{1-t_0}_t$ are supported outside a neighbourhood of $\{|t| = 1\}$, and they preserve the pages and the binding of the open book $z_1/|z_1|: S^{2n-1} \setminus \{z_1 = 0\} \to S^1$. This means that the pulled-back discs still satisfy condition (M2).

4.2. The $C^0$-bound. Now we apply this construction to the truncated moduli space $W_\delta$. Consider a sequence $(u_\nu)$ of holomorphic discs $u_\nu = (a_\nu, f_\nu) \in W_\delta$. Then, in particular the levels $t_\nu$ will be contained in a compact subset of $\text{Int}(D^{n-1})$. Hence, after passing to a subsequence, we may assume that $t_\nu \to t_0$ for some $t_0 \in \text{Int}(D^{n-1})$. With the construction from the preceding section we may take the $u_\nu$ to be $J_\nu$-holomorphic discs of fixed boundary level $t_0$. The almost complex
structures $J_\nu$ equal $J$ outside a compact neighbourhood of $\{0\} \times [-b_0,b_0] \times S^{2n-1}$ and converge to $J$ in the $C^\infty$-topology. For the symplectic energy this implies by Lemma 4 that we have an estimate of the form $\int_D u^* \omega \leq \pi + e_\nu$ with $e_\nu \to 0$ for $\nu \to \infty$.

**Proposition 11.** In the situation just described, the sequence $(a_\nu)$ of real numbers is bounded from below.

The proof of this proposition will take up the remainder of this section. Arguing by contradiction, we assume that there is a sequence $(u_\nu = (a_\nu, f_\nu))$ as just described with $\min_D a_\nu \to -\infty$. The mean value theorem implies that there is no uniform bound on $\max_D |\nabla u_\nu|$. Here $|.|$ denotes the norm corresponding to an $\mathbb{R}$-invariant metric on $W$ of the form $d\alpha^2 + g_{\mathbb{R}^{2n+1}}$, with $g_{\mathbb{R}^{2n+1}}$ any Riemannian metric on $\mathbb{R}^{2n+1}$.

Bubbling off analysis as in [14, Section 6] shows that, a priori, the following phenomena might occur:
- bubbling of spheres
- bubbling of finite energy planes
- breaking
- bubbling of discs (this can only happen at boundary points).

The first is impossible in an exact symplectic manifold. The second and third phenomenon are precluded by the assumption $\inf_0 (\alpha) > \pi$ and the energy estimate from Lemma 4, cf. [14, p. 584], since a finite energy plane in a symplectisation is asymptotic to a contractible Reeb orbit. Notice that this rules out any kind of bubbling at interior points.

This leaves the bubbling of discs. We are going to show presently that any disc that bubbles off takes away a minimal amount $\hbar$ of energy. Hence, again by Lemma 4 there exists a finite subset $\Gamma \subset \partial D$ such that the sequence $(u_\nu)$ converges in $C^\infty_{\text{loc}}(D \setminus \Gamma)$ to a holomorphic disc (after removal of singularities), see [19, Theorem 4.6.1]. Proposition 11 will then be proved by showing that the assumption $a_\nu \to -\infty$ would lead to an infinite collection of bubble discs, contradicting our energy estimates.

**Remark 12.** The compactness result in [8] or [24] shows that the $C^0$-bound on $a$, together with the finiteness of the set $\Gamma$ of bubbling points, is enough to conclude that $\Gamma$ is actually the empty set, which means that we have compactness for $W_\delta$.

In [12] pp. 559/60 and [14] p. 544 we were able to use a simpler argument for showing that $\Gamma$ is empty. This relied on the fact that the boundary of the discs under consideration there had to be transverse to the (integrable) characteristic foliation of a 2-sphere in a 3-dimensional contact manifold. In the present set-up, one could use the same argument in the case $n = 2$. For $n > 2$, the characteristic foliation on the $n$-sphere $S^t$ is not integrable.

Before continuing with the proof of Proposition 11 we derive the necessary lower estimate on the energy of bubble discs. Here we use the symplectic energy $e$ with respect to the symplectic form $\omega = d(e^a \hat{\alpha})$, which is the same as the Dirichlet energy with respect to the norm $|.|_{\text{exp}}$ coming from the metric $g_{\text{exp}} := d(e^a \hat{\alpha})(.,.)$:

$$e(u) := \int_D u^* d(e^a \hat{\alpha}) = \frac{1}{2} \int_D |\nabla u|_{\text{exp}}^2.$$
Lemma 13. There is a constant \( h = h(t, \delta) \) such that any non-constant holomorphic disc
\[ u = (a, f): (\mathbb{D}, \partial\mathbb{D}) \rightarrow (W, R^t) \]
with image of \( f \) as in the definition of \( W_\delta \) satisfies \( e(u) \geq h \).

Proof. By the conformal invariance of the Dirichlet energy it suffices to prove this estimate with the domain of definition replaced by \( (\mathbb{H}, \partial\mathbb{H}) \), where \( \mathbb{H} \) denotes the closed upper half-plane in \( \mathbb{C} \). Write \( B_r \) for the open unit disc of radius \( r \) in \( \mathbb{C} \).

According to the mean value inequality [19, Lemma 4.3.1] there are constants \( h \) and \( C \) such that
\[ \sup_{B_r \cap \mathbb{H}} |\nabla u|_{\exp}^2 \leq \frac{C}{r^2} \int_{B_{2r} \cap \mathbb{H}} |\nabla u|_{\exp}^2 \]
whenever the estimate
\[ e(u|_{B_{2r} \cap \mathbb{H}}) < h \]
is satisfied. The proof of the mean value inequality in [19] requires a modification of the metric in a compact neighbourhood of
\[ \{0\} \times [-b_0, b_0] \times (S^{2n-1} \setminus V_\delta), \]
relying on a lemma of Frauenfelder [7, Lemma A.3]. This compact modification of the metric can be absorbed into the constants in the mean value inequality.

A more subtle issue is that in [19] the mean value inequality is only stated for compact manifolds. However, the crucial ingredients in the proof are a \( C^2 \)-bound on the almost complex structure \( J \), which in our case is guaranteed by \( \mathbb{R} \)-invariance, and a \( C^0 \)-bound on the curvature of the metric. The latter also exists in our non-compact situation, since the Christoffel symbols of the metric \( g_{\exp} \) do not depend on \( a \).

If \( e(u) < h \), we see that \( u \) is constant by taking the limit \( r \to \infty \) in the mean value inequality. \( \square \)

Continuing with the proof of Proposition [11] choose points \( \zeta_\nu \in \mathbb{D} \) with \( a_\nu(\zeta_\nu) = \min_{\mathbb{D}} a_\nu \). Because of the \( C^\infty \)-convergence on \( \mathbb{D} \setminus \Gamma \) we may assume, after passing to a subsequence, that \( \zeta_\nu \) converges to a point in \( \Gamma \). We choose a neighbourhood of this point in \( \mathbb{D} \) (not containing any other points of \( \Gamma \)) conformally equivalent to the closed upper half-plane \( \mathbb{H} \subset \mathbb{C} \). This conformal identification is made in such a way that the limit point of the \( \zeta_\nu \) in \( \Gamma \subset \partial\mathbb{D} \) is identified with \( 0 \in \mathbb{R} \). We then regard the \( u_\nu \) as maps
\[ u_\nu = (a_\nu, f_\nu): (\mathbb{H}, \mathbb{R}) \rightarrow (W, R^t). \]
For this sequence, we have \( C^\infty \)-convergence on \( \mathbb{H} \setminus \{0\} \), so by the mean value theorem we may choose a sequence \( (z_\nu = x_\nu + iy_\nu) \) in \( \mathbb{H} \) with
\[ |\nabla u_\nu(z_\nu)| = \max_{\mathbb{D}} |\nabla u_\nu| \rightarrow \infty \quad \text{and} \quad z_\nu \rightarrow 0. \]

In the usual bubbling off analysis, a rescaling argument (going back to Hofer) is used to produce a sequence with a uniform gradient bound. This sequence is then shown to be \( C^\infty \)-convergent, and its limit, after removal of singularities, is a bubble with potentially contradictory properties. Here we use a variant of soft rescaling that does not directly produce a convergent sequence, but rather a sequence that would produce further bubbling with contradictory properties, cf. [7, Section 3.2.1].

Set \( R_\nu = 1/|x_\nu - \zeta_\nu| \). So we have sequences \( y_\nu \to 0 \) and \( R_\nu \to \infty \). We now distinguish two cases, according to the behaviour of the sequence \( (y_\nu, R_\nu) \).
**Case 1.** \((y_\nu R_\nu)\) is unbounded, and without loss of generality \(y_\nu R_\nu \to \infty\). In this case we consider the rescaled sequence \((v_\nu)\) of holomorphic maps on \(\mathbb{H}\) defined by
\[
v_\nu(z) := u_\nu(x_\nu + y_\nu z).
\]
The \(a\)-component of \(v_\nu\) goes to \(-\infty\) in the points \((\zeta_\nu - x_\nu)/y_\nu\), whose distance from zero equals
\[
\left| \frac{1}{y_\nu(\zeta_\nu - x_\nu)} \right| = \frac{1}{y_\nu R_\nu} \to 0.
\]
The gradient of \(v_\nu\) attains its maximum in \((z_\nu - x_\nu)/y_\nu = i\). This would imply bubbling at the interior point \(i\), which we have ruled out.

**Case 2.** \((y_\nu R_\nu)\) is bounded. Here we look at the rescaled sequence \((v_\nu)\) defined by
\[
v_\nu(z) := u_\nu(x_\nu + z/R_\nu).
\]
The \(a\)-component of \(v_\nu\) goes to \(-\infty\) in the points \(R_\nu(\zeta_\nu - x_\nu)\), which have distance 1 from 0, so the gradient maximum of the sequence \((v_\nu)\) must blow up. The gradient of \(v_\nu\) attains its maximum in \(iy_\nu R_\nu\), and since there is no interior bubbling, we have \(iy_\nu R_\nu \to 0\).

Now we can repeat the bubbling off analysis for the sequence \((v_\nu)\). Because of the \(C^\infty_{loc}\)-convergence outside a finite set of points on the boundary, the sequence of points \(R_\nu(\zeta_\nu - x_\nu)\) has to converge to a point in \(\partial \mathbb{H}\), i.e. to \(\pm 1\). Then, once again, we choose a neighbourhood of this limit point conformally equivalent to \(\mathbb{H}\) and not containing other bubble points, and rescale as before inside this neighbourhood. Since the sequence \((v_\nu)\) also has a bubble point at 0, we know that for sufficiently large \(\nu\) the Dirichlet energy of \(v_\nu\) on the neighbourhood of the bubble point \(\pm 1\) is bounded above by \(\pi - h/2\).

Iterating this argument, we eventually arrive at a curve with negative energy. This contradiction proves Proposition \[1\]

5. Transversality

The purpose of this section is to show that the truncated moduli space \(W^t\) is a smooth, oriented manifold with boundary. As usual, this is achieved by proving transversality results in the setting of \(W^{1,p}\)-maps for some \(p > 2\). Smoothness of the holomorphic discs is then implied by elliptic regularity.

Let \(\mathcal{B}\) denote the space of \(W^{1,p}\)-maps
\[
u: \left(\mathbb{D}, \partial \mathbb{D}\right) \longrightarrow (W, \{0\} \times \mathbb{R}^{2n+1}),
\]
where \(u(\partial \mathbb{D})\) is supposed to be contained in \(R^t\) for some \(t \in \mathbb{R}^{n-1}\) with \(|t| < 1\), and \(u\) is required to satisfy the homological condition (M1) from Section \[3.2\]. Write \(\mathcal{B}_t \subset \mathcal{B}\) for the subspace of discs corresponding to a fixed boundary level \(t\).

The space \(\mathcal{B}_t\) is a (separable) Banach manifold modelled on the Banach space of \(W^{1,p}\)-sections of \(u^*(TW, TR^t)\) (i.e. vector fields along \(u\) that are tangent to \(R^t\) along the boundary); charts are obtained from such vector fields along \(u\) by choosing a metric for which the submanifold \(R^t\) is totally geodesic and then applying the exponential map, see \[0\]. The construction from Section \[4.1\] shows that the map sending a disc \(u \in \mathcal{B}\) to its level \(t\) gives \(\mathcal{B}\) the structure of a locally trivial fibration over the open unit disc in \(\mathbb{R}^{n-1}\) with fibre \(\mathcal{B}_t\). Tangent vectors at \(u \in \mathcal{B}\) can be written uniquely as \(u + \nu|_u\), where \(u \in T_u \mathcal{B}_t\), and \(\nu\) is a vector field as in Section \[4.1\] corresponding to a small shift \(\mathbf{\nu}\) around the level \(t\) of \(u\).
5.1. The linearised Cauchy–Riemann operator. Over $\mathcal{B}$ we have a Banach space bundle $\mathcal{E}$ whose fibre over the point $u \in \mathcal{B}$ is the space $L^p(u^*TW)$ of $L^p$-vector fields along $u$; see for instance [1] Proposition 6.13 for the construction of the bundle structure. This bundle inherits the local product structure from $\mathcal{B}$.

Fix an almost complex structure $J$ on $W$ satisfying the conditions $(J1)$ and $(J2)$. The Cauchy–Riemann operator $u \mapsto u_x + J(u)u_y$ defines a section of $\mathcal{E}$. In order to discuss transversality, we need to compute the vertical differential $D_u$ of this section at $u \in \mathcal{B}$. To this end, consider a path of holomorphic curves

$$u^s := \psi^s \circ \exp_u(su)$$

for $s$ in some small interval around 0, where $\psi$ denotes the flow as in Section 4.1. This path is tangent to $u + v|_u$ in $s = 0$. Let $\nabla$ be a torsion-free connection on $TW$. Write

$$\nabla_s = (\nabla_{\partial u^s/\partial s})|_{s=0}, \quad \nabla_x = (\nabla_{\partial u^s/\partial x})|_{s=0},$$

and likewise $\nabla_y$. Since the torsion of $\nabla$ vanishes, we have

$$\nabla_s \frac{\partial u^s}{\partial x} = \nabla_x \frac{\partial u^s}{\partial s} = \nabla_x (u + v),$$

and similarly for $\partial u^s/\partial y$. Hence

$$D_u(u + v|_u) = \nabla_s (u^s_x + J(u^s)u^s_y)$$

$$= \nabla_x (u + v) + J(u) \nabla_y (u + v) + (\nabla_{u+y}J)(u) u_y$$

$$= D^k_u u + K_u \overline{\mathcal{F}},$$

where

$$D^k_u u := \nabla_x u + J(u) \nabla_y u + (\nabla_u J)(u) u_y,$$

$$K_u \overline{\mathcal{F}} := \nabla_x v + J(u) \nabla_y v + (\nabla_v J)(u) u_y.$$
is transverse to \((\mathbb{R} \times P_0) \times (\mathbb{R} \times P_{\pi/2}) \times (\mathbb{R} \times P_{\pi})\).

If the first condition is satisfied, \(\tilde{W}\) will be a manifold of the expected dimension \(2n + 2\); if in addition (ii) holds, then \(\mathcal{W}\) will be a manifold of dimension \(2n - 1\).

The proof that the set of regular \(J\) is non-empty, in fact of second Baire category, follows the standard line of reasoning as in the proof of Theorems 3.1.5 and 3.4.1 of [19]. Selecting such a regular \(J\) is the generic choice we make in (J2). For the standard discs \(u_{s,a}^t\), transversality is obvious. By Lemma 8 all discs that are not standard pass through the region where \(J\) may be chosen generically, which is sufficient to achieve transversality by [19, Remark 3.2.3]. In contrast with the set-up in [19], we are only allowed to perturb \(J\) along \(\xi\), keeping it compatible with \(d\tilde{\alpha}\). But this is exactly the situation dealt with by Bourgeois in the appendix of [4]. The proof given there carries over to our situation; the essential ingredient of Bourgeois’s argument is that the set of \(f\)-injective points is open and dense, which is precisely our Lemma 6.

5.3. Orientation. In order to speak of the degree of the evaluation map \(ev\) on \(\mathcal{W} \times \mathbb{D}\), we need to put an orientation on the moduli space \(\tilde{\mathcal{W}}\). Given the relation between \(\mathcal{W}\) and \(\tilde{\mathcal{W}}\) described in the preceding section, it suffices to orient \(\tilde{\mathcal{W}}\), and that in turn amounts to showing that the determinant line bundle \(\text{det} D\) over \(\tilde{\mathcal{W}}\) is oriented, since \(\ker D = T_u \tilde{\mathcal{W}}\).

Recall that the determinant line \(\text{det} F\) is defined for any Fredholm operator \(F\) as \(\text{det} F = \det \ker F \otimes (\text{det} \text{coker} F)^*\). Since \(D_u\) is surjective for all \(u \in \tilde{\mathcal{W}}\), the determinant line bundle is simply \(\text{det} D = \bigwedge^{2n+2} \ker D\). In the arguments that follow, however, we use deformations through not necessarily surjective Fredholm operators, so we need to work with determinant lines, in general.

As we have seen, the operator \(D_u\) splits (by slight abuse of notation) as \(D_u = D_u^t + K_u\). The linear interpolation of \(D_u\) to \(D_u^t + O_{\mathcal{W}_u}\) is via Fredholm operators, since \(K_u\) is compact. It follows that \(\text{det}(D_u) = \text{det}(D_u^t + O_{\mathcal{W}_u})\), see [9, p. 680], whence

\[
\text{det}(D_u) = \text{det} D_u^t \otimes \text{det} \mathfrak{M}_u.
\]

The second factor inherits a natural orientation from the orientation of \(\mathbb{R}^{n-1}\). The first factor is naturally oriented by the construction in [9, Section 8.1]. Our situation is a particularly simple one, since \(TR^t\) is a trivial bundle. This implies that any bundle pair \((u^*TW, (u|_{\partial D})^*TR^t)\) comes with a natural trivialisation of the boundary bundle, and this suffices for the construction of a natural orientation of the determinant line bundle.

6. Proof of Theorem 1

By Sections 4 and 5 (notably Lemma 9), the assumption \(\inf_\theta(\alpha) > \pi\) of Theorem 1 implies that the evaluation map

\[
ev: \mathcal{W} \times \mathbb{D} \rightarrow \tilde{Z} \setminus (\mathbb{R} \times \{z \in S^{2n-1}: z_1 = 0\})
\]

is a proper map of degree 1. By Lemmata 8 and 10 we may pretend that \(\mathcal{W} \times \mathbb{D}\) and \(\tilde{Z}\) are — after smoothing corners — compact, oriented manifolds with boundary, without changing the homotopy type of these spaces, and that \(\ev\) is a smooth degree 1 map between these manifolds.
Homotopical and homological arguments similar to the ones that follow were used by McDuff [18].

**Proposition 14.** The manifold $\hat{Z}$ is simply connected.

**Proof.** Given a loop in $\hat{Z}$, we homotope it to an embedded circle $C$ inside $\text{Int}(\hat{Z})$ that intersects the complement of $\hat{B}$, in other words, such that it passes through the region where all holomorphic discs (more precisely, their $f$-components) are standard. We can make the evaluation map $W \times D \to \hat{Z}$ transverse to $C$ by a perturbation compactly supported in $\text{Int}(\hat{B})$. The preimage of $C$ under this perturbed map will then be a single circle $C' \subset W \times \{1\}$ mapping with degree 1 onto $C$. The homotopy of $C'$ to a loop in $W \times \{1\}$ induces a homotopy of $C$ to a loop in the cell $(\mathbb{R} \times \mathbb{R}^+ \times \mathbb{C}^{n-1}) \cap \partial Z$. 

**Lemma 15.** Let $\phi : (P, \partial P) \to (Q, \partial Q)$ be a degree 1 map between compact, oriented $m$-dimensional manifolds with boundary. Then the induced homomorphism $\phi_* : H_k(P; \mathbb{F}) \to H_k(Q; \mathbb{F})$ in singular homology with coefficients in a field $\mathbb{F}$ is surjective in each degree $k \in \mathbb{N}_0$.

**Proof.** Over a field, the Kronecker pairing between homology and cohomology is non-degenerate, so equivalently we need to show injectivity of the induced homomorphism $\phi^*$ in cohomology.

Given a non-zero class $\beta \in H^k(Q)$, Poincaré duality allows us to find a class $\gamma \in H^{m-k}(Q, \partial Q)$ such that $\beta \cup \gamma$ is the orientation generator of $H^m(Q, \partial Q)$. Since $\phi$ is of degree 1, we have

$$0 \neq \phi^*(\beta \cup \gamma) = \phi^* \beta \cup \phi^* \gamma,$$

which forces $\phi^*$ to be injective on $H^k(Q)$. 

**Proposition 16.** The manifold $\hat{Z}$ has the integral homology of a point.

**Proof.** With the preceding lemma this follows with an argument completely analogous to the proof of Proposition 14.

**Proof of Theorem 4.** Since $2n \neq 3$, the smooth Schoenflies theorem tells us that the subset of $\hat{Z}$ bounded by $\varphi(\partial M)$ and a standard ellipsoid surrounding $\varphi(\partial M)$ is diffeomorphic to a collar of $\partial M$. Hence $M$ is a strong deformation retract of $\hat{Z}$. So by Propositions 14 and 10 the manifold $M$ is a simply connected homology ball with boundary diffeomorphic to $S^{2n}$. It follows that $M$ is diffeomorphic to a ball: for $n \geq 3$ we appeal to Proposition A on page 108 of Milnor’s lectures [20]; for $n = 2$, to Proposition C on page 110.

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