Hölder Continuity of the Spectral Measures for One-Dimensional Schrödinger Operator in Exponential Regime

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Abstract

Avila and Jitomirskaya prove that the spectral measure $\mu_{f,\lambda,v,\alpha,x}$ of quasi-periodic Schrödinger operator is $1/2$-Hölder continuous with appropriate initial vector $f$, if $\alpha$ satisfies Diophantine condition and $\lambda$ is small. In the present paper, the conclusion is extended to that for all $\alpha$ with $\beta(\alpha) < \infty$, the spectral measure $\mu_{f,\lambda,v,\alpha,x}$ is $1/2$-Hölder continuous with small $\lambda$, if $v$ is real analytic in a neighbor of $\{ |\Im x| \leq C \beta \}$, where $C$ is a large absolute constant. In particular, the spectral measure $\mu_{f,\lambda,v,\alpha,x}$ of almost Mathieu operator is $1/2$-Hölder continuous if $|\lambda| < e^{-C\beta}$ with $C$ a large absolute constant.

1 Introduction and the Main results

In the present paper, we study the quasi-periodic Schrödinger operator $H = H_{f,\lambda,v,\alpha,x}$ on $\ell^2(\mathbb{Z})$:

$$(H_{f,\lambda,v,\alpha,x}u)_n = u_{n+1} + u_{n-1} + \lambda v(x + n\alpha)u_n, \quad (1.1)$$

where $v : T = \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}$ is the potential, $\lambda$ is the coupling, $\alpha$ is the frequency, and $x$ is the phase. In particular, the almost Mathieu operator (AMO) is given by (1.1) with $v(x) = 2 \cos(2\pi x)$, denoted by $H_{\lambda,\alpha,x}$.

Below, we always assume $\alpha \in \mathbb{R}\setminus\mathbb{Q}$, and the potential $v$ is real analytic in a strip of the real axis.
The quasi-periodic Schrödinger operator is not only related to some fundamental problems in physics [23], but also is fascinating because of its remarkable richness of the related spectral theory. In Barry Simon’s list of Schrödinger operator problems for the twenty-first century [28], there are three problems about AMO. The problems of quasi-periodic Schrödinger operator have attracted many authors, for instance, Avila-Jitomirskaya [3],[4],[5], Avron-Simon [6], Bourgain-Goldstein-Schlag [9],[10], Goldstein-Schlag [15],[16],[17] and Jitomirskaya-Last[20],[21].

For \( \lambda = 0 \), it is easy to verify that Schrödinger operator (1.1) has purely absolutely continuous spectrum \([-2,2]\) by Fourier transform. We expect the property ( of purely absolutely continuous spectrum) preserves under sufficiently small perturbation, i.e., \( \lambda \) is small. Usually there are two smallness about \(|\lambda|\). One is perturbative, meaning that the smallness \(|\lambda|\) depends not only on the potential \(v\), but also on the frequency \(\alpha\); the other is non-perturbative, meaning that the smallness condition only depends on the potential \(v\), not on \(\alpha\).

Recall that averaging the spectral measure \(\mu_{v,\alpha,x}^0\) with respect to \(x\) (see (2.8)) yields the integrated density of states (IDS), whose Hölder continuity is critical to the purely absolutely continuous spectrum. In the present paper, we concern the Hölder continuity of IDS, and generally, of the individual spectral measures \(\mu_{v,\alpha,x}^f\). In our another paper [26], we will investigate the persistence of the purely absolutely continuous spectrum under small perturbation by the Hölder continuity of IDS and some additional results in [2],[25].

The following notions are essential in the study of equation (1.1).

We say \(\alpha \in \mathbb{R}\backslash \mathbb{Q}\) satisfies a Diophantine condition \(DC(\kappa,\tau)\) with \(\kappa > 0\) and \(\tau > 0\), if

\[
||k\alpha||_{\mathbb{R}/\mathbb{Z}} > \kappa|k|^{-\tau}\text{ for any } k \in \mathbb{Z}\backslash \{0\},
\]

where \(||x||_{\mathbb{R}/\mathbb{Z}} = \min_{\ell \in \mathbb{Z}} |x - \ell|\). Let \(DC = \bigcup_{\kappa > 0, \tau > 0} DC(\kappa, \tau)\). We say \(\alpha\) satisfies Diophantine condition, if \(\alpha \in DC\).

Let

\[
\beta(\alpha) = \limsup_{n \to \infty} \frac{\ln q_{n+1}}{q_n},
\]

(1.2)

where \(\frac{p_n}{q_n}\) is the continued fraction approximants to \(\alpha\). One usually calls set \(\{\alpha \in \mathbb{R}\backslash \mathbb{Q} | \beta(\alpha) > 0\}\) exponential regime and set \(\{\alpha \in \mathbb{R}\backslash \mathbb{Q} | \beta(\alpha) = 0\}\) sub-exponential regime. Notice that the set DC is a real subset of the sub-exponential regime, i.e., \(DC \subsetneq \{\alpha : \beta(\alpha) = 0\}\).

Here we would like to talk about some history on Hölder continuity of IDS, and generally, of the individual spectral measures \(\mu_{v,\alpha,x}^f\).

In [14], Eliasson treats (1.1) as a dynamical systems problem– reducibility of associated cocycles. He shows that such cocycles are reducible for a.e. spectrum, and gives out useful estimates for the non-reducible ones via a sophisticated KAM-type methods, which breaks
the limitations of the earlier KAM methods, for instance, the work of Dinaburg and Sinai (13) (they need exclude some parts of the spectrum). As a result, Eliasson proves that \( H = H_{\lambda, \alpha, x} \) has purely absolutely continuous spectrum for \( \alpha \in DC \) and \( |\lambda| < \lambda_0(\alpha, v, \nu) \). His student Amor uses the sophisticated KAM iteration to establish the 1/2-Hölder continuity of IDS in a similar regime: \( \alpha \in DC \) and \( |\lambda| < \lambda_0(\alpha, v, \nu) \) (1). Amor’s arguments also apply to quasi-periodic Schrödinger operator in multifrequency.

Both of Eliasson and Amor’s results are perturbative (i.e., the smallness of \( \lambda \) depends on \( \alpha \)). Such limitation are inherent to traditional KAM theory. The other stronger results, i.e., non-perturbative results, will be introduced next.

Bourgain proves that for a.e. \( \alpha \) and \( x \), \( H = H_{\lambda, \alpha, x} \, (H_{\lambda, \alpha, x}) \) has purely absolutely continuous spectrum if \( |\lambda| < \lambda_0(v) \) (\( \lambda < 1 \)). Bourgain approaches this by classical Aubry duality and the sharp estimate of Green function in the regime of positive Lyapunov exponent (7), (11). By the way, in the regime of positive Lyapunov exponent, he sets up the Hölder continuity of \( IDS \) by the Hölder continuity of Lyapunov exponent \( L(E) \) and Thouless formula (6):

\[
L(E) = \int \ln |E - E'| dN(E').
\] (1.3)

This is because, by Hilbert transform and some theories of singular integral operators, the Hölder continuity passes from \( L(E) \) to \( N(E) \) (15). Note that both \( L(E) \) and \( N(E) \) depend on \( v, \lambda, \alpha, \) and \( \nu \), we sometimes drop the parameters dependence for simplicity. Earlier, Goldstein and Schlag (15) have already obtained Bourgain’s results. Concretely, \( L(E) \) and \( N(E) \) are Hölder continuous in the interval \([E_1, E_2]\) for strong Diophantine condition \( k \)-frequency \( \alpha \) if \( L(E) > 0 \) in \([E_1, E_2]\). They all approach their results by the avalanche principle and sharp large deviation theorems (7), (15). Notice that \( L(E) > 0 \) when \( |\lambda| \) is large in non-perturbative regime by the subharmonic methods (p.17, (7)).

Here we would point out some other meaningful results. Suppose \( v \) is a trigonometric polynomial of degree \( k_0 \), and assume positive Lyapunov exponents and Diophantine \( \alpha \). Goldstein-Schlag (16) shows that \( N(E) \) is \((1/2 - \epsilon)\)-Hölder continuous for any \( \epsilon > 0 \). As for AMO, combining with Aubry duality, Goldstein and Schlag’s arguments suggest the IDS is \((1/2 - \epsilon)\)-Hölder continuous for all \( \lambda \neq \pm 1 \) and \( \alpha \in DC \). Their approach is via finite volume bounds, i.e., investigating the eigenvalue problem \( H\phi = E\phi \) on a finite interval \([1, N] \)

\(^1\lambda_0(\ast)\) means \( \lambda_0 \) depends on \( \ast \).

\(^2\)Quasi-periodic Schrödinger operator in multifrequency \((k \text{ dimension})\) is given by \( (H_{\lambda, \alpha, x})_n = u_{n+1} + u_{n-1} + Av(x + na)u_n \), where \( v : \mathbb{T}^k = \mathbb{R}^k / \mathbb{Z}^k \to \mathbb{R} \) is the potential.

\(^3\)We say \( \alpha \) satisfies strong Diophantine condition if there exist some \( \kappa > 0, \tau > 1 \) such that

\[
\|k\alpha\|_{\mathbb{R}/\mathbb{Z}} > \frac{\kappa}{|k|(\ln(1 + |k|))^\tau} \text{ for } k \in \mathbb{Z}\setminus\{0\}.
\]
with Dirichlet boundary conditions. The tools in [16] have been already turned out to be an effective way in dealing with the quasi-periodic Schrödinger operators, see [12] and [17] for example. Before [16], Bourgain has set up $(1/2 - \epsilon)$-Hölder continuity for AMO with $\alpha \in DC$ and large (small) $|\lambda|$ perturbatively [8].

Avila and Jitomirskaya address this issue by firstly developing the quantitative version of Aubry duality (§1.1). They establish the $1/2$-Hölder continuity of IDS if $\lambda$ is small non-perturbatively in sub-exponential regime (i.e., $\alpha$ satisfies $\beta(\alpha) = 0$), and for AMO, $N(E)$ is $1/2$-Hölder continuous for all $\lambda \neq \pm 1$ in sub-exponential regime. Note that Avila and Jitomirskaya use the quantitative version of Aubry duality to obtain many other results of spectral theory, for example, solving the sixth problem in [28] entirely and the dry version of Ten Martini Problem partly. We refer the reader to [2] and [4] for more discussion.

Avila and Jitomirskaya’s analysis also allows to investigate a more delicate question: Hölder continuity of the individual spectral measures. This is quite different from previous work. They show that for all $x$ and vectors $f \in \ell^1 \cap \ell^2$, the spectral measures $\mu_{f,v,x,\alpha}^{(J)}$ is $1/2$-Hölder continuous uniformly in $x$, if $\lambda$ is small non-perturbatively and $\alpha \in DC$ [5]. Avila and Jitomirskaya approach this by the sharp estimate for the dynamics of Schrödinger cocycles in [20], [21], [24].

In the present paper, we extend the quantitative version of Aubry duality to all $\alpha$ with $\beta(\alpha) < \infty$. Together with Avila and Jitomirskaya’s arguments in [5], we obtain the following results.

**Theorem 1.1.** For irrational number $\alpha$ such that $\beta(\alpha) < \infty$, if $v$ is real analytic in a neighbor of $|\Im x| \leq C\beta$, where $C$ is a large absolute constant, then there exists some $\lambda_0 = \lambda_0(v, \beta) > 0$ such that $\mu_{f,v,x,\alpha}^{(J)}(J) \leq C(\lambda(v, \alpha)/|J|^{1/2})^2||f||_1^2$, for all intervals $J$ and all $x$ if $|\lambda| < \lambda_0$, where $\mu_{f,v,x,\alpha}^{(J)}$ is the associated spectral measure with $f \in \ell^1 \cap \ell^2$. In particular, $\lambda_0 = e^{-C\beta}$ for AMO.

**Remark 1.1.** If $\beta(\alpha) = 0$ and $v$ is real analytic in a strip of real axis, then by Theorem [7] $\lambda_0 = \lambda_0(v)$, and $\lambda_0 = 1$ for AMO. Those results are non-perturbative. Clearly, if $0 < \beta(\alpha) < \infty$, the results obtained by Theorem [7] are perturbative.

### 1.1 Quantitative Aubry Duality and Outline of the present paper

In the present paper, we deal with the Hölder continuity of individual spectral measure as the program of Avila and Jitomirskaya [4], [5]. Thus it is necessary to introduce Avila and Jitomirskaya’s main contribution-quantitative Aubry duality more details.

Classical Aubry duality (§2.2) suggests that Anderson localization (only pure point spectrum with exponentially decaying eigenfunctions) for the dual model $H_{v,x,0}$ leads to
reducibility for almost every energy [27]. A more subtle duality theory is that pure point spectrum for almost every $\theta$ in the dual model allows to conclude purely absolutely continuous spectrum for almost every $x$ [18]. However, localization in general does not hold for every $\theta$ [22]. This of course fits with the fact Schrödinger cocycles are not reducible for all energies [14]. Thus the classical Aubry duality can not deal with all energies.

It is therefore natural to introduce a weakened notion of localization that could be expected to hold for every phase, and to develop some way to link the reducibility. Avila and Jitomirskaya make this idea come true. Namely, they introduce a new concept: almost localization of the dual model $\{\hat{H}_{\lambda,v,\alpha,\theta}\}_{\theta \in \mathbb{R}}$, which is a kind of weakened notion of localization, and establish a quantitative version of Aubry duality that links local exponential decay of solutions to eigenvalue problem of $\{H_{\lambda,v,\alpha,\theta}\}_{x \in \mathbb{R}}$ (Lemma 4.3). See [2] and [4] for more details.

By some sharp estimates for the dynamics of Schrödinger cocycles via the quantitative version of Aubry duality, Avila and Jitomirskaya obtain some results of the Hölder continuity of IDS non-perturbatively. Together with the dynamical reformulation of weyl-function and power-law subordinacy techniques in [20], [21], [24], they set up the $\frac{1}{2}$-Hölder continuity of individual spectral measures, which we have said before.

Avila and Jitomirskaya’s discussion is concentrated on sub-exponential regime. In [25], we have extended the quantitative version of Aubry duality to exponential regime for AMO. In the present paper, we success to generalize the results of [2], [4] and [25], and set up the quantitative version of Aubry duality for general potential $v$ in exponential regime.

In order to get sharp estimate for the dynamics of Schrödinger cocycles via the quantitative version of Aubry duality, the priori estimate of the transfer matrix $A_n(x)$ is necessary, where $A_n(x)$ is given by (2.2) with $A = S_{\lambda,v,E}$ and $E \in \Sigma_{\lambda,v,\alpha}$ (since the spectrum of $H_{\lambda,v,\alpha,\theta}$ is independent of $x$, we denote by $\Sigma_{\lambda,v,\alpha}$). In the present paper, we obtain

$$||A_n(x)|| = e^{\varphi(n)}$$

through strip $|\Im x| < \eta$ ($\eta$ will be specified later), which the proofs of Avila [2] and Avila-Jitomirskaya [4] do not apply and Avila actually make the following footnote in [2]:

In the case of the almost Mathieu operator it is possible to show that we can take $\eta = \frac{-\ln |\lambda|}{2\pi}$ in (1.4). For the generalization (i.e., general potential $v$), it is possible to show that it is enough to choose $\eta$ in (1.4) such that $v$ is holomorphic in a neighborhood of $|\Im x| \leq \eta$ and $\eta \leq \frac{1}{2\pi} \epsilon_1$, where $\epsilon_1$ is the one in the strong localization estimate.

We have confirmed (1.4) for the case of AMO [25]. In §4, we will verify the claims for general $v$ by a new method.

The present paper is organized as follows:

In §2, some basic notion will be introduced. In §3, we obtain the strong localization
estimate of the Aubry dual model $\hat{H}_{\lambda_0,\alpha,\theta}$ for all $\alpha$ with $\beta(\alpha) < \infty$. In §4, we set up the priori estimate of the transfer matrix $A_n(x)$ in a given strip. In §5, we obtain a good estimate for the dynamics of Schrödinger cocycles via the quantitative Aubry duality. In §6, combining with Avila-Jitomirskaya’s analysis in [5], we prove Theorem 1.1.

2 Preliminaries

2.1 Cocycles, Lyapunov exponent, Reducibility

Denote by $\text{SL}(2, \mathbb{C})$ the all complex $2 \times 2$-matrixes with determinant 1. We say a function $f \in C^\omega(\mathbb{R}/\mathbb{Z}, \mathbb{C})$ if $f$ is well defined in $\mathbb{R}/\mathbb{Z}$, i.e., $f(x + 1) = f(x)$, and $f$ is analytic in a strip of real axis. The definitions of $\text{SL}(2, \mathbb{R})$ and $C^\omega(\mathbb{R}/\mathbb{Z}, \mathbb{R})$ are similar to those of $\text{SL}(2, \mathbb{C})$ and $C^\omega(\mathbb{R}/\mathbb{Z}, \mathbb{C})$ respectively, except that the involved matrixes are real and the functions are real analytic.

A $C^\omega$-cocycle in $\text{SL}(2, \mathbb{C})$ is a pair $(\alpha, A) \in \mathbb{R} \times C^\omega(\mathbb{R}/\mathbb{Z}, \text{SL}(2, \mathbb{C}))$, where $A \in C^\omega(\mathbb{R}/\mathbb{Z}, \text{SL}(2, \mathbb{C}))$ means $A(x) \in \text{SL}(2, \mathbb{C})$ and the elements of $A$ are in $C^\omega(\mathbb{R}/\mathbb{Z}, \mathbb{C})$. Sometimes, we say $A$ a $C^\omega$-cocycle for short, if there is no ambiguity. Note that all functions, cocycles in the present paper are analytic in a strip of real axis. Thus we often do not mention the analyticity, for instance, we say $A$ a cocycle instead of $C^\omega$-cocycle.

The Lyapunov exponent for the cocycle $A$ is given by

$$L(\alpha, A) = \lim_{n \to \infty} \frac{1}{n} \int_{\mathbb{R}/\mathbb{Z}} \ln \|A_n(x)\| dx,$$

where

$$A_n(x) = A(x + (n - 1)\alpha)A(x + (n - 2)\alpha) \cdots A(x).$$

Clearly, $L(\alpha, A) \geq 0$ since $\det A(x) = 1$.

By the subadditivity of $L_n(\alpha, A)$, where $L_n(\alpha, A) = \int_{\mathbb{R}/\mathbb{Z}} \ln \|A_n(x)\| dx$, one has

$$L(\alpha, A) = \inf_n \frac{1}{n} L_n(\alpha, A).$$

Given two cocycles $(\alpha, A)$ and $(\alpha, A')$, a conjugacy between them is a cocycle $B \in C^\omega(\mathbb{R}/\mathbb{Z}, \text{SL}(2, \mathbb{C}))$ such that

$$B(x + \alpha)^{-1} A(x) B(x) = A'.$$

We say that cocycle $(\alpha, A)$ is reducible if it is conjugate to a constant cocycle.
2.2 Schrödinger cocycles and classical Aubry duality

We now consider the quasi-periodic Schrödinger operator $H_{\lambda,v,\alpha,x}$, the spectrum of operator $H_{\lambda,v,\alpha,x}$ does not depend on $x$, denoted by $\Sigma_{\lambda,v,\alpha}$. Indeed, shift is an unitary operator on $\ell^2(\mathbb{Z})$, thus $\Sigma_{\lambda,v,\alpha,x} = \Sigma_{\lambda,v,\alpha,x+\alpha}$, where $\Sigma_{\lambda,v,\alpha,x}$ is the spectrum of $H_{\lambda,v,\alpha,x}$. By the minimality of $x \mapsto x + \alpha$ and continuity of spectrum $\Sigma_{\lambda,v,\alpha,x}$ with respect to $x$, the statement follows.

Let $S_{\lambda,v,E} = \begin{pmatrix} E - \lambda v & -1 \\ 1 & 0 \end{pmatrix}$. We call $(\alpha, S_{\lambda,v,E})$ Schrödinger cocycle. For AMO, we call almost Mathieu cocycle, denoted by $(\alpha, S_{\lambda,v,E})$.

Note that, by dropping the symbol $v$ from a notation, we indicate the corresponding notation for AMO. For instance, denote by $\Sigma_{\lambda,\alpha}$ the spectrum of $H_{\lambda,\alpha,x}$.

Fix Schrödinger operator $H_{\lambda,v,\alpha,x}$, we define the Aubry dual model by $\hat{H} = \hat{H}_{\lambda,v,\alpha,\theta}$,

$$(\hat{H}\hat{u})_n = \sum_{k \in \mathbb{Z}} \lambda \hat{v}_k \hat{u}_{n-k} + 2 \cos(2\pi \theta + n\alpha) \hat{u}_n,$$

where $\hat{v}_k$ is the Fourier coefficients of potential $v$. In particular, for AMO, it is easy to check that $\hat{H}_{\lambda,\alpha,\theta} = \lambda H_{\lambda,\alpha,\theta}$. If $\alpha \in \mathbb{R}\setminus\mathbb{Q}$, the spectrum of $\hat{H}_{\lambda,\alpha,\theta}$ is also $\Sigma_{\lambda,\alpha}$ [18]. Classical Aubry duality expresses an algebraic relation between the families of operators $\{\hat{H}_{\lambda,v,\alpha,\theta}\}_{\theta \in \mathbb{R}}$ and $\{H_{\lambda,v,\alpha,x}\}_{x \in \mathbb{R}}$ by Bloch waves, i.e., if $u : \mathbb{R}/\mathbb{Z} \mapsto \mathbb{C}$ is an $L^2$ function whose Fourier coefficients $\hat{u}$ satisfy $\hat{H}_{\lambda,v,\alpha,\theta} \hat{u} = E \hat{u}$, then

$$U(x) = \begin{pmatrix} e^{2\pi i \theta} u(x) \\ u(x - \alpha) \end{pmatrix}$$

satisfies $S_{\lambda,v,E}(x) \cdot U(x) = e^{2\pi i \theta} U(x + \alpha)$.

2.3 Spectral measure and the integrated density of states

Let $H$ be a bounded self-adjoint operator on $\ell^2(\mathbb{Z})$. Then $(H - z)^{-1}$ is analytic in $\mathbb{C}\setminus\Sigma(H)$, where $\Sigma(H)$ is the spectrum of $H$, and we have for $f \in \ell^2$

$$\Im \langle (H - z)^{-1} f, f \rangle = \Im z \cdot \| (H - z)^{-1} f \|^2,$$

where $\langle \cdot, \cdot \rangle$ is the usual inner product in $\ell^2(\mathbb{Z})$. Thus

$$\phi_f(z) = \langle (H - z)^{-1} f, f \rangle$$

is an analytic function in the upper half plane with $\Im \phi_f \geq 0$ ($\phi_f$ is a so-called Herglotz function).
Therefore one has a representation
\[
\phi_f(z) = \langle (H - z)^{-1} f, f \rangle = \int_{\mathbb{R}} \frac{1}{x - z} d\mu^f(x),
\] (2.6)
where \(\mu^f\) is the spectral measure associated to vector \(f\). Alternatively, for any Borel set \(\Omega \subseteq \mathbb{R}\),
\[
\mu^f(\Omega) = \langle \mathbb{E}(\Omega) f, f \rangle,
\] (2.7)
where \(\mathbb{E}(\Omega)\) is the corresponding spectral projection of \(H\).

Denote by \(\mu^{e_0}_{av,\alpha, x}\) the spectral measure of Schrödinger operator \(H_{av,\alpha, x}\) and vector \(f\) as before. The integrated density of states (IDS) \(N_{av,\alpha}\) is obtained by averaging the spectral measure \(\mu^{e_0}_{av,\alpha, x}\) with respect to \(x\), i.e.,
\[
N_{av,\alpha}(E) = \int_{\mathbb{R}/\mathbb{Z}} \mu^{e_0}_{av,\alpha, x}(-\infty, E] dx,
\] (2.8)
where \(e_0\) is the Dirac mass at 0 \(\in \mathbb{Z}\).

2.4 Continued fraction expansion

Define as usual for \(0 \leq \alpha < 1\),
\[
a_0 = 0, \alpha_0 = \alpha,
\]
and inductively for \(k > 0\),
\[
a_k = \lfloor \alpha_{k-1}^{-1} \rfloor, \alpha_k = \alpha_{k-1}^{-1} - a_k,
\]
where \(\lfloor t \rfloor\) denotes the greatest integer less than or equal \(t\).

We define
\[
p_0 = 0, \quad q_0 = 1,
\]
\[
p_1 = 1, \quad q_1 = a_1,
\]
and inductively,
\[
p_k = a_k p_{k-1} + p_{k-2},
\]
\[
q_k = a_k q_{k-1} + q_{k-2}.
\]
Recall that \(\{q_n\}_{n \in \mathbb{N}}\) is the sequence of best denominators of irrational number \(\alpha\), since it satisfies
\[
\forall 1 \leq k < q_{n+1}, \|k\alpha\|_{\mathbb{R}/\mathbb{Z}} \geq \|q_n\alpha\|_{\mathbb{R}/\mathbb{Z}}. \tag{2.9}
\]
Moreover, we also have the following estimate,
\[
\frac{1}{2q_{n+1}} \leq \Delta_n = \frac{\|q_n\alpha\|_{\mathbb{R}/\mathbb{Z}}}{q_{n+1}} \leq \frac{1}{q_{n+1}}. \tag{2.10}
\]
3 Strong localization estimate

Given $\theta \in \mathbb{R}$ and $\epsilon_0 > 0$, we say $k$ is an $\epsilon_0$-resonance for $\theta$ if $\|2\theta - k\alpha\|_{\mathbb{R}/\mathbb{Z}} \leq e^{-\epsilon_0|k|}$ and $\|2\theta - k\alpha\|_{\mathbb{R}/\mathbb{Z}} = \min_{j \in \mathbb{Z}} \|2\theta - j\alpha\|_{\mathbb{R}/\mathbb{Z}}$.

Clearly, $0 \in \mathbb{Z}$ is an $\epsilon_0$-resonance. We order the $\epsilon_0$-resonances $0 = |n_0| < |n_1| \leq |n_2| \cdots$. We say $\theta$ is $\epsilon_0$-resonant if the set of $\epsilon_0$-resonances is infinite. If $\theta$ is non-resonant, with the set of resonances $\{n_0, n_1, \cdots, n_j\}$, we set $n_{j+1} = \infty$.

Below, unless stated otherwise, $C$ is a large absolute constant and $c$ is a small absolute constant, which may change through the arguments, even when appear in the same formula. However, their dependence on other parameters, will be explicitly indicated. For instance, we denote by $C(\alpha)$ a large constant depending on $\alpha$. Let $\| \cdot \|$ be the Euclidean norms, and denote $\|f\|_1 = \sup_{x \in \mathbb{R}} |f(x)|$, $\|f\|_0 = \sup_{x \in \mathbb{R}} |f(x)|$.

**Definition 3.1.** Given a self-adjoint operator $H$ on $\ell^2(\mathbb{Z})$, we say $\phi$ is an extended state of $H$, if $H\phi = E\phi$ with $\phi(0) = 1$ and $|\phi(k)| \leq 1 + |k|$, where $E \in \Sigma(H)$.

**Definition 3.2.** We say that $\hat{H}_{av,\alpha,\theta}$ is almost localized if there exists $C_0 > 1$, $\tilde{C} > 0$, $\epsilon_0 > 0$ and $\epsilon_1 > 0$ such that for any extended state $\hat{u}$, i.e., $\hat{H}_{av,\alpha,\theta}\hat{u} = E\hat{u}$ satisfying $\hat{u}_0 = 1$ and $|\hat{u}_k| \leq 1 + |k|$, where $E \in \Sigma_{av,\alpha}$, then we have $|\hat{u}_k| \leq \tilde{C} e^{-\epsilon_1|k|}$ for $C_0|n_j| < |k| < C_0^{-1}|n_{j+1}|$, where set $\{n_j\}$ is the $\epsilon_0$-resonances for $\theta$. Sometimes, we also say $\hat{H}_{av,\alpha,\theta}$ satisfies a strong localization estimate with parameters $C_0$, $\epsilon_0$, $\epsilon_1$ and $\tilde{C}$.

The next theorem is our main work in this section.

**Theorem 3.1.** Suppose irrational number $\alpha$ satisfies $0 < \beta(\alpha) < \infty$. Let $\epsilon_0 = C_1^{2}\beta$ and $\epsilon_1 = C_1^{3}\beta$, where $C_1$ is a large absolute constant such that it is much larger than any absolute constant $C$, $e^{-1}$ emerging in the present paper. There exists a absolute constant $C_2$ such that if $v$ is analytic in strip $|\Im x| < C_2\beta$, then there exists $\lambda_0 = \lambda_0(v, \beta) > 0$ such that $\hat{H}_{av,\alpha,\theta}$ satisfies a strong localization estimate with parameters $C_0 = 3$, $\epsilon_0$, $\epsilon_1$ and $\tilde{C} = C(\lambda, v, \alpha)$, for all $\lambda$ with $0 < |\lambda| < \lambda_0$. In particular, $\lambda_0 = e^{-C_2\beta}$ for AMO.

In [25], we have obtained Theorem 3.1 for AMO via estimating Green function. For general potential $v$, we also use the sharp estimate of Green function to prove Theorem 3.1 by the methods of Avila-Jitomirskaya in [4] or Bourgain-Jitomirskaya in [11]. Combining with our discussion in [25], one can obtain Theorem 3.1. Next, we will give a almost entire proof.

Without loss of generality, assume $\lambda > 0$. Let $\hat{H}_{av,\alpha,\theta} = \frac{1}{\lambda} \hat{H}_{av,\alpha,\theta}$, it suffices to prove $\hat{H}_{av,\alpha,\theta}$ is almost localized. We will sometimes drop the $E, \lambda, \alpha, \theta$-dependence from the notations if there is no ambiguity. Define $H_I = R_I \hat{H} R_I$, where $R_I$ = coordinate restriction to
I = [x_1, x_2] ⊂ \mathbb{Z}, and denote by \( G_I = (\hat{H}_I - E)^{-1} \) the associated Green function, if \( \hat{H}_I - E \) is invertible. Denote by \( G_I(x, y) \) the matrix elements of the Green function \( G_I \).

Assume \( \phi \) is an extended state of \( \hat{H}_{\lambda v, \alpha, \theta} \). Our objective is to show that \( |\phi(k)| \leq C(\lambda, v, \alpha)e^{-\epsilon_1|k|} \)
for \( 3|n_j| < |k| < \frac{1}{3}|n_{j+1}| \).

It is easy to check that (p.4, [11])

\[
\phi(x) = -\sum_{y \in I, \text{element} I} G_I(x, y)\hat{v}_{y-k}\phi(k),
\]

for \( x \in I \).

Set \( a_k = \sum_{|j| \geq |k|, j \neq 0} |\hat{v}_j| \).

Definition 3.3. Fix \( m > 0 \). A point \( x \in \mathbb{Z} \) will be called \((m, N)\)-regular if there exists an interval \([x_1 + 1, x_2 - 1]\) with \( x_2 = x_1 + N + 1 \), containing \( x \) such that

\[
\sum_{y \in I, i=1,2} |G_I(x, y)a_{y-x}| < e^{-mN} \text{ for } i = 1, 2;
\]

otherwise, \( y \) will be called \((m, N)\)-singular.

Lemma 3.1. For any \( m > 0 \), \( 0 \) is \((m, N)\)-singular if \( N > N(m) \).

Proof: Otherwise, \( 0 \) is \((m, N)\)-regular, i.e., there exists an interval \([x_1 + 1, x_2 - 1]\) with \( x_2 = x_1 + N + 1 \), containing \( 0 \) such that

\[
\sum_{y \in I, i=1,2} |G_I(0, y)a_{y-x}| < e^{-mN} \text{ for } i = 1, 2.
\]

In (3.1), let \( x = 0 \) and recall that \( |\phi(k)| \leq 1 + |k| \), then

\[
|\phi(0)| = |\sum_{y \in I, k \neq 0} G_I(0, y)\hat{v}_{y-k}\phi(k)| \\
\leq \sum_{y \in I, k \neq 0} |G_I(0, y)\hat{v}_{y-k}|(1 + |k|) \\
\leq 2N \sum_{y \in I, k \neq 0} |G_I(0, y)\hat{v}_{y-k}||y - k| \\
\leq 2N \sum_{y \in I, i=1,2} |G_I(0, y)a_{y-x}| \\
\leq 2Ne^{-mN} < 1
\]

for \( N > N(m) \), which is contradicted to the hypothesis \( \phi(0) = 1 \). □

Let us denote

\[
P_N(\theta) = \det((\hat{H}_{\lambda v, \alpha, \theta} - E)|_{[0,N-1]}).
\]

\( ^4 \text{N > N(m) means N is large enough depending on m.} \)
Following [19], \( P_N(\theta) \) is an even function of \( \theta + \frac{1}{2}(N - 1)\alpha \) and can be written as a polynomial of degree \( N \) in \( \cos 2\pi(\theta + \frac{1}{2}(N - 1)\alpha) \):

\[
P_N(\theta) = \sum_{j=0}^{N} c_j \cos^j 2\pi(\theta + \frac{1}{2}(N - 1)\alpha) = Q_N(\cos 2\pi(\theta + \frac{1}{2}(N - 1)\alpha)).
\] (3.5)

Let \( A_k,r = \{ \theta \in \mathbb{R} \mid Q_k(\cos 2\pi\theta) \leq e^{(k+1)r} \} \) with \( k \in \mathbb{N} \) and \( r > 0 \).

**Lemma 3.2.** Suppose \( \beta(\alpha), \epsilon_0 \) and \( \epsilon_1 \) satisfy the hypothesis of Theorem 3.1. Let \( C_3 \) be a large absolute constant. There exists a absolute constant \( C_2 \) such that if \( v \) is analytic in strip \( |Im| < C_2\beta \), then there exists \( \lambda_0 = \lambda_0(v,\beta) > 0 \) such that if \( y \in \mathbb{Z} \) is \( (C_1\epsilon_1, N) \)-singular, \( N > N(\lambda, v, \alpha) \), and \( x \in [y - (1 - \delta)N, y - \delta N] \) \( \cap \mathbb{Z} \) with \( \delta \in (\frac{1}{40}, \frac{1}{2}) \), we have \( \theta + (x + \frac{N-1}{2})\alpha \) belongs to \( A_{N,-\ln,\lambda-C_2\epsilon_0,} \) for all \( \lambda \in (0, \lambda_0) \).

**Proof:** Otherwise, there exist \( \delta \in (\frac{1}{40}, \frac{1}{2}) \) and \( x \in [y - (1 - \delta)N, y - \delta N] \) \( \cap \mathbb{Z} \) such that \( \theta + (x + \frac{N-1}{2})\alpha \notin A_{N,-\ln,\lambda-C_2\epsilon_0,} \). Without loss of generality, assume \( x = 0 \). Thus \( \theta + \frac{N-1}{2}\alpha \notin A_{N,-\ln,\lambda-C_2\epsilon_0,} \), that is \( P_N(\theta) > e^{-C_2\epsilon_0 N} \) by (3.5). Set \( x_1 = -1 \), \( x_2 = N \). It is enough to show that for \( y \in [x_1 + 1, x_2 - 1] = I \) with \( \text{dist}(y, \partial[x_1, x_2]) \geq \delta N \), one has

\[
(\ast) = \sum_{z \in \mathcal{I}, r = 1, 2} |G_I(y, z) \lambda_{z-x}| < e^{-C_1\epsilon_1 N}.
\] (3.6)

By Cramer’s rule \( G_I(\gamma, z) = \frac{\mu_{z,z}}{P_N(\theta)} \), where \( \mu_{y,z} \) is the corresponding minor. Together with the estimate of \( \mu_{y,z} \) in Lemma 3.3 and 3.4 below, we have

\[
(\ast) \leq (\lambda e^{C_1\epsilon_0})^{-1} \sum_{n=1}^{N} \sum_{\gamma \in [\gamma_n, \gamma_n+1]} |\det R_{\gamma,\gamma} (\hat{\mathcal{H}} - E) R_{\gamma,\gamma}^* \lambda_{\gamma_n,\gamma_n+1}| \parallel \lambda_{\gamma_n,\gamma_n+1} \parallel \parallel v \parallel_{0}^{1} \parallel C_1(\gamma, \sigma) \parallel (\parallel v \parallel_{0}^{1} + C_1^{-1} A^{-1} (n + 1)^{2})^{-\frac{(n+1)}{2}} e^{-\sigma\parallel v \parallel_{0}^{1}} \parallel \lambda_{\gamma_n,\gamma_n+1} \parallel \parallel v \parallel_{0}^{1}.
\] (3.7)

where \( \sigma > 0 \) is such that

\[
\parallel \lambda \parallel \leq C(\gamma, \sigma) e^{-2k\parallel v \parallel_{0}^{1}}
\] (3.8)

and \( (\gamma, \gamma') = |y_{\gamma'+1} - x_{\gamma+1}| + \sum_{i=1}^{\parallel y \parallel} |y_{\gamma'+1} - y_{\gamma+1}| \). Let \( G_{b,n} = \{ \gamma, \parallel y \parallel = n \text{ and } b(\gamma, \gamma') = b \} \), thus

\[
(\ast) \leq (\lambda e^{C_1\epsilon_0})^{-1} \sum_{n=1}^{N} \sum_{b} C_1(\gamma, \sigma) (\parallel v \parallel_{0}^{1} + C_1^{-1} A^{-1} (n + 1)^{2})^{-\frac{(n+1)}{2}} e^{-\sigma\parallel v \parallel_{0}^{1}} \# G_{b,n}
\] (3.9)
If $G_{h,n} \neq 0$, then $\delta N \leq \max\{\text{dist}(y, \partial I), n + 1\} \leq b \leq (n + 1)N \leq N^2$. By Stirling formula, setting $b = rN, n + 1 = sb$, we have $b(n) \leq CrNe^{\phi(s)rN}$, where $\phi(s) = -s \ln s - (1 - s) \ln(1 - s)$ with $0 < s \leq 1$. Thus we have

$$\tag{3.10} (\ast) \leq e^{C_3\epsilon_0 + C\|v\|_{L^1}^{1/2}\lambda^{1/2}N^2} \sup_{0 < s \leq 1 \atop \delta N \leq s} \left( \frac{\lambda^{-1}}{C(v, \sigma)} \right)^{2/s^2} e^{-s \ln N \cdot e^{\phi(s)rN}}.$$

To prove (3.6), it suffices to show

$$\tag{3.11} (\ast \ast) = \sup_{0 < s \leq 1} C_3\epsilon_0 + C\|v\|_{L^1}^{1/2}\lambda^{1/2} + \left( \ln C(v, \sigma) + \ln \lambda - 2 \ln rs - \frac{\sigma}{s} + \frac{\phi(s)}{s} \right) rs$$

for any $r \in [\delta, n + 1]$.

Using that $\|v\|_{L^1} \leq \frac{C(c, \sigma)}{\sigma}$, and that $\phi(s)/s \leq 1 - \ln s$, one has

$$\tag{3.12} (\ast \ast) \leq C_3\epsilon_0 + \left( Crc_0^{1/2} - \frac{r}{2} \right) \sigma + \left( C + \ln c_0 + 3 \ln \frac{\sigma}{s} - \frac{\sigma}{2s} \right) rs,$$

where $c_0 = r^{-2}\lambda C(v, \sigma)\sigma^{-3}$. It is easy to verify that $3 \ln \frac{\sigma}{s} - \frac{\sigma}{2s} \leq C$, then

$$\tag{3.13} (\ast \ast) \leq C_3\epsilon_0 + \left( Crc_0^{1/2} - \frac{r}{2} \right) \sigma + (C + \ln c_0) rs.$$

Thus to show $(\ast \ast) \leq -2C_1\epsilon_1$, it is enough to estimate (3.12) at $r = \delta$, that is

$$\tag{3.14} (\ast \ast) \leq C_3\epsilon_0 + \left( Cc_0^{1/2} - \frac{1}{2} \right) \delta \sigma + (\ln C + \ln c_0) \delta s \leq -2C_1\epsilon_1,$$

with $c_0 = \delta^{-2}\lambda C(v, \sigma)\sigma^{-3}$.

If $v$ is analytic in $|\phi x| < C_2\beta$, then

$$\tag{3.15} |\hat{v}_x| \leq C(v, \sigma)e^{-2\sigma|x|},$$

with $\sigma = \frac{C_0\beta}{4}$.

If $|\lambda| < \lambda_0(v, \beta)$ such that

$$\tag{3.16} Cc_0^{1/2} - 1/2 < -1/4, C + \ln c_0 < 0,$$

then we have

$$\tag{3.17} (\ast \ast) < C_3\epsilon_0 - \frac{C_2}{640} \beta \leq -2C_1\epsilon_1,$$

since $\delta \geq 1/40$ and $C_2$ is large enough. □
Lemma 3.3. (Lemma 10, [7])
\[
\mu_{\gamma,z} = \sum_{\gamma} \alpha_{\gamma} \det R_{i\gamma}(\tilde{H} - E)R_{i\gamma}^* \prod_{i=1}^{\gamma} |\hat{\nu}_{\gamma_{i+1}-\gamma_i}|,  \tag{3.18}
\]
where the sum is taken over all ordered subsets $\gamma = (\gamma_1, \ldots, \gamma_n)$ of $I$ with $\gamma_1 = y$ and $\gamma_n = z$, $|\gamma| = n - 1$, and $\alpha_{\gamma} \in \{-1, 1\}$.

Lemma 3.4. (Lemma 5.6, [8]) For any $\Lambda \subset I$ and $N > N(\lambda, v, \alpha)$,
\[
|\det R_{i\Lambda}(\tilde{H} - E)R_{i\Lambda}^*| \leq \lambda^{-N} e^{C\|\nu\|^2_0 + 1/2\|\nu\|^2_1 (\|\nu\|_0 + C^{-1} \lambda^{-1} \frac{\#_\Lambda^2}{N^2})^\#_\Lambda}.  \tag{3.19}
\]

Definition 3.4. We say that the set $\{\theta_1, \cdots, \theta_{k+1}\}$ is $\xi$-uniform if
\[
\max_{x \in [-1, 1]} \max_{i = 1, \cdots, k+1} \frac{|x - \cos 2\pi \theta_i|}{\cos 2\pi \theta_i - \cos 2\pi \theta_j} < \epsilon^\xi.  \tag{3.20}
\]

Lemma 3.5. (Lemma 9.3, [3]) Let $\xi_1 < \xi$. If $\theta_1, \cdots, \theta_{k+1} \in A_{k-\ln \lambda - \xi}$, then $\{\theta_1, \cdots, \theta_{k+1}\}$ is not $\xi_1$-uniform for $k > k(\xi, \xi_1, \lambda)$.

Without loss of generality, assume $3|n_j| < k < \frac{|n_{j+1}|}{3}$. Select $n$ such that $q_n < \frac{\xi}{k} < q_{n+1}$ and let $s$ be the largest positive integer satisfying $sq_n \leq \frac{k}{8}$. Set $I_1, I_2 \subset \mathbb{Z}$ as follows
\[
I_1 = [-2sq_n + 1, 0] \text{ and } I_2 = [k - 2sq_n + 1, k + 2sq_n], \text{ if } n_j < 0,  \tag{3.21}
\]
\[
I_1 = [0, 2sq_n - 1] \text{ and } I_2 = [k - 2sq_n + 1, k + 2sq_n], \text{ if } n_j \geq 0.  \tag{3.22}
\]
In either case, the total number of elements in $I_1 \cup I_2$ is $6sq_n$. Let $\theta_j = \theta + j^\alpha$ for $j' \in I_1 \cup I_2$.

Lemma 3.6. (Lemma 3.9, [25]) The set $\{\theta_{j'}\}_{j' \in I_1 \cup I_2}$ constructed as (3.21) or (3.22) is $\xi_0$-uniform for $k > k(\alpha)$ (or equivalently $n > n(\alpha)$).

We can now finish the proof of Theorem 3.1. By Lemma 3.5 and 3.6 there exists some $j_0 \in I_1 \cup I_2$ such that $\theta_{j_0} \notin A_{6sq_n - 1, -\ln \lambda - C_3 \xi_0}$ for some absolute constant $C_3$ ($C_3$ is larger than the absolute constant $C$ emerging in Lemma 3.6). Notice that $y = 0$ is $(C_1 \epsilon_1, N)$-singular by Lemma 3.1. If we let $y = 0, N = 6sq_n - 1, \delta = \frac{90}{600}$ in Lemma 3.2 then for all $j' \in I_1,$ $\theta_j \in A_{6sq_n - 1, -\ln \lambda - C_3 \xi_0}$ if $n > n(\lambda, v, \alpha)$ (or equivalently $k > k(\lambda, v, \alpha)$). Let $j_0 \in I_2$ be such that $\theta_{j_0} \notin A_{6sq_n - 1, -\ln \lambda - C_3 \xi_0}$. Again by Lemma 3.2, $k$ is $(C_1 \epsilon_1, 6sq_n - 1)$-regular. By the proof of Lemma 3.1 and noting $sq_n \geq \frac{k}{16}$, we obtain
\[
|\phi(k)| \leq e^{-\epsilon_1 k}  \tag{3.23}
\]
for $k > k(\lambda, v, \alpha)$ and $3|n_j| < k < \frac{1}{2} |n_{j+1}|$. For $k < 0$, the proof is similar. Thus
\[
|\phi(k)| \leq e^{-|k|}.  \tag{3.23}
\]
if $|k| > C(\lambda, v, \alpha)$ and $3|n_j| < |k| < \frac{1}{3}|n_{j+1}|$. That is
\[ |\phi(k)| \leq C(\lambda, v, \alpha)e^{-\epsilon_1|k|} \] (3.24)

for all $k$ with $3|n_j| < |k| < \frac{1}{3}|n_{j+1}|$. \Box

For frequency $\alpha$ with $\beta(\alpha) = 0$, $\hat{H}_{\lambda, \alpha, \theta}$ also satisfies strong localization estimate with small $\lambda$. This has been proved by Avila and Jitomirskaya in [4].

**Theorem 3.2.** (Theorem 5.1, [4]) Assume $v$ is real analytic in a strip of real axis and $\beta(\alpha) = 0$. There exists $\lambda_0(v) > 0$ such that if $0 < |\lambda| < \lambda_0$, $C_0 > 1$, there exist $\epsilon_0 = \epsilon_0(v, \lambda) > 0$, $\epsilon_1 = \epsilon_1(v, \lambda, C_0) > 0$ such that $\hat{H}_{\lambda, \alpha, \theta}$ satisfies strong localization estimate with parameters $C_0, \epsilon_0, \epsilon_1$ and $\tilde{C} = C(\lambda, v, \alpha, C_0)$. More precisely, for any extended state $\hat{u}$ of $\hat{H}_{\lambda, \alpha, \theta}$, we have $|\hat{u}_k| \leq C(\lambda, v, \alpha, C_0)e^{-\epsilon_1|k|}$ for all $k$ with $C_0|n_j| < |k| < C_0^{-1}|n_{j+1}|$, where $\{n_j\}$ is the $\epsilon_0$-resonances for $\theta$. In particular, $\lambda_0 = 1$ for AMO.

After carefully checking the details of the proof of Theorem 3.2 we can obtain another version.

**Theorem 3.3.** Assume $v$ is real analytic in a strip of real axis and $\beta(\alpha) = 0$. There exists $\lambda_0(v) > 0$ such that if $0 < |\lambda| < \lambda_0$, there exist $\epsilon_0 = \epsilon_0(v, \lambda) > 0$, $\epsilon_1 = C_1\epsilon_0$, where $C_1$ is a large absolute constant, such that $\hat{H}_{\lambda, \alpha, \theta}$ satisfies strong localization estimate with parameters $C_0 = 3, \epsilon_0, \epsilon_1$ and $\tilde{C} = C(\lambda, v, \alpha)$. More precisely, for any extended state $\hat{u}$ of $\hat{H}_{\lambda, \alpha, \theta}$, we have $|\hat{u}_k| \leq C(\lambda, v, \alpha)e^{-\epsilon_1|k|}$ for all $k$ with $3|n_j| < |k| < 3^{-1}|n_{j+1}|$, where $\{n_j\}$ is the $\epsilon_0$-resonances for $\theta$. In particular, $\lambda_0 = 1$ for AMO.

### 4 The proof of a claim from Avila

To set up the sharp estimates for the dynamics of Schrödinger cocycles via the quantitative version of Aubry duality, the priori estimate of transfer matrix $A_n(x)$ in given strip is of importance, where $A_n(x)$ is given by (2.2) with $A = S_{\lambda, v, E}$.

**Theorem 4.1.** Suppose $\hat{H}_{\lambda, \alpha, \theta}$ satisfies a strong localization estimate with parameters $C_0 > 1, \epsilon_0, \epsilon_1 = 2\pi\eta$ and $\tilde{C}$. If $v$ is real analytic in a neighbor of $|\Re x| \leq \eta$, then $\sup_{|\Re x| \leq \eta} \|A_k(x)\| \leq C(\lambda, v, \alpha, \eta, \delta)\epsilon_0^k$ for any $\delta > 0$, where $A(x) = \begin{pmatrix} E - \lambda v(x) & -1 \\ 1 & 0 \end{pmatrix}$ with $E \in \Sigma_{\lambda, v, \alpha}$.

**Remark 4.1.** In footnote 5 of [2], Avila think Theorem 4.7 is right, which we have mentioned in §1.1. We will confirm the statements in this section.

In this section, fix $\eta = \frac{\epsilon_1}{2\pi}$. If we can prove that the Lyapunov exponent is vanishing in the strip $|\Re x| \leq \eta$, by Furman’s uniquely ergodic theorem, Theorem 4.1 is easy to set up (see the proof of Theorem 4.7 in [25]). Thus it suffices to prove the following lemma.
**Lemma 4.1.** Under the hypotheses of Theorem 4.1, let \( \alpha \in \mathbb{R} \setminus \mathbb{Q} \) and \(-\eta \leq \epsilon \leq \eta\), then \( L(\alpha, \epsilon) = 0 \), where \( L(\alpha, \epsilon) = L(\alpha, A_{\epsilon}) \) and

\[
A_{\epsilon} = \begin{pmatrix}
E - \lambda v(x + i\epsilon) & -1 \\
1 & 0
\end{pmatrix}
\]

with \( E \in \Sigma_{\nu, \alpha} \).

Following (2.3), the Lyapunov exponent \( L(\alpha, \epsilon) \) is lower semi-continuous with respect to \( \alpha \in \mathbb{R} \setminus \mathbb{Q} \) and \( \epsilon \), thus it is enough to show that, for any \( \kappa, \tau > 0 \), \( L(\alpha, \epsilon) = 0 \) if \( \alpha \in DC(\kappa, \tau) \).

In this section, \( \hat{C} \) is a large constant and \( \hat{c} \) is a small constant. They are allowed to depend on parameters \( \nu, \lambda, \alpha, C_0, \hat{C}, \epsilon_0, \epsilon_1, \kappa, \tau \), which may change through the arguments, even when appear in the same formula. Further dependence on other parameters, will be explicitly indicated. For instance, we will use \( \hat{C}(\delta) \) for a large constant depending on \( \delta \), and \( \nu, \lambda, \alpha, C_0, \hat{C}, \epsilon_0, \epsilon_1, \kappa, \tau \).

For the proof of vanishing Lyapunov exponent, a couple of lemmata and theorems are necessary.

We will say that a trigonometrical polynomial \( p : \mathbb{R}/\mathbb{Z} \mapsto \mathbb{C} \) has essential degree at most \( k \) if its Fourier coefficients outside an interval \( I \) of length \( k \) (i.e., \( k = b - a \) for \( I = [a, b] \)) are vanishing.

**Lemma 4.2.** (Theorem 6.1, [2]) Let \( 1 \leq r \leq \lfloor q_{n+1}/q_n \rfloor \). If \( p \) has essential degree at most \( k = rq_n - 1 \) and \( x_0 \in \mathbb{R}/\mathbb{Z} \), then

\[
||p||_0 \leq Cq_{n+1}^{Cr_1} \sup_{0 \leq j \leq k} |p(x_0 + ja)|. \tag{4.1}
\]

If \( \alpha \in DC(\kappa, \tau) \), then \( q_{n+1} \leq \frac{4}{\epsilon} q_n \) by (2.9) and (2.10), and (4.1) becomes

\[
||p||_0 \leq Ce^{C\ln q_{n+1}} \sup_{0 \leq j \leq k} |p(x_0 + ja)| \leq Ce^{C(k)} \sup_{0 \leq j \leq k} |p(x_0 + ja)|. \tag{4.2}
\]

**Lemma 4.3.** (Theorem 3.3, [2]) If \( E \in \Sigma_{\nu, \alpha} \), then there exists \( \theta \in \mathbb{R} \) and a bounded solution of \( \dot{\hat{u}}_{\nu, \alpha, \theta} = E\hat{u} \) with \( \hat{u}_0 = 1 \) and \( |\hat{u}_k| \leq 1 \).

Given \( E \in \Sigma_{\nu, \alpha} \), let \( \theta = \theta(E) \) and solution \( \hat{u}_k \) be given by Lemma 4.3 and \( \{n_j\} \) be the set of \( \epsilon_0 \)-resonances for \( \theta(E) \).

**Lemma 4.4.** (Lemma 3.1, [2]) If \( \alpha \in DC(\kappa, \tau) \), then \( |n_{j+1}| \geq a||2\theta - n_j\alpha||^{-\eta} \geq a e^{a\eta|n_j|} \), where \( a = a(\kappa, \tau) \).

**Lemma 4.5.** (Theorem 2.6, [2]) Let \( U : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{C}^2 \) be analytic in \( |\Re x| < \eta \). Assume that \( \delta_1 < ||U(x)|| < \delta_2^{-1} \) for all \( x \) in strip \( |\Re x| < \eta \). Then there exists \( B : \mathbb{R}/\mathbb{Z} \rightarrow SL(2, \mathbb{C}) \) analytic in \( |\Re x| < \eta \) with first column \( U \) and \( ||B||_\eta \leq C\delta_1^{-2}\delta_2^{-1}(1 - \ln(\delta_1\delta_2)) \).
Lemma 4.6. (Theorem 6.2, [2]) Let \( L(\alpha, \epsilon) = 0 \) for \( \epsilon = 0 \).

Proof of Lemma 4.1

Let
\[
\eta_1 = \sup_{\epsilon} \{ \epsilon \in [0, \eta] \mid L(\alpha, \xi) = 0 \text{ for any } |\xi| \leq \epsilon \}.
\]

By the lower semi-continuity, \( L(\alpha, \epsilon) = 0 \) for \( |\epsilon| \leq \eta_1 \). Suppose Lemma 4.1 does not hold, then \( \eta_1 < \eta \). Take \( 3\eta_2 = \eta - \eta_1 \). Let \( n = rq_k - 1 < q_k + 1 \) be the maxima with \( n < \frac{1}{4}|n_j| + 1 \) (if \( \theta \) is non-resonant, take any \( n = r^j \gamma > C(\eta_2) e^{C(\eta_2)n} \)), and let \( u'(x) = \sum_{k \in I} \hat{u}_k e^{2\pi ikx} \) with \( I = [-[\frac{\pi}{2}], n - [\frac{\pi}{2}] \} \). Define \( U(x) = \left\{ g(x) \begin{cases} u'(x - \alpha) \\ 0 \end{cases} \right. \), by direct computation
\[
AU(x) = e^{2\pi i\theta} U(x + \alpha) + e^{2\pi i\theta} \left( g(x) \begin{array}{c} 0 \\ \cdots \end{array} \right),
\]
and the Fourier coefficients of \( g(x) \) satisfy
\[
\hat{g}_k = \chi_I(k)(E - 2 \cos 2\pi(\theta + k\alpha))\hat{u}_k - \lambda \sum \chi_I(k - j)\hat{u}_{k-j},
\]
where \( \chi_I \) is the characteristic function of \( I \). Since \( \hat{H}u = E\hat{u} \), one also has
\[
-\hat{g}_k = \chi_{\mathbb{Z} \setminus I}(k)(E - 2 \cos 2\pi(\theta + k\alpha))\hat{u}_k - \lambda \sum \chi_{\mathbb{Z} \setminus I}(k - j)\hat{u}_{k-j}.
\]
Notice that \( |\hat{u}_k| < \tilde{C} e^{-2\eta \vert k \vert} \) for \( \frac{1}{4\tilde{C}} |n_j| < |k| < C_0 |n_j + 1| \) and \( |\hat{u}_k| < 1 \) for others. Thus \( |\hat{u}_k| < \tilde{C} e^{-2\eta \vert k \vert} \) for \( \tilde{C} \ln n < |k| < \tilde{C}n \) by Lemma 4.4 and \( |\hat{u}_k| < 1 \) for all \( k \). It is easy to check that \( \|g\|_{\eta_1 + \eta_2} \leq \tilde{C}(\eta_2)e^{-\eta_1 n} \) and \( \|U(x)\|_{\eta_1 + \eta_2} \leq \tilde{C}(\eta_2)e^{\eta_1 n} \), since \( v \) is analytic in a neighbor of \( \{\mathbb{F}_x \mid \eta \} \).

Fix \( \delta = \frac{1}{C_1}\eta_2 \), where \( C_1 \) is given by Theorem 5.1. Then there exists \( \xi(\lambda, \nu, \alpha, \eta_1, \delta) \) with \( 0 < \xi < \eta_2 \) such that
\[
\sup_{|\mathbb{F}_x| < \eta_1 + \xi} \|A_k(x)\| \leq \tilde{C}(\eta_2, \delta)e^{\xi k},
\]
since \( L(\alpha, \epsilon) = 0 \) for \( |\epsilon| \leq \eta_1 \) (Theorem 4.7, [25]).

Next we will prove that the following estimate holds,
\[
\inf_{|\mathbb{F}_x| < \eta_1 + \xi} \|U(x)\| \geq \tilde{C}(\eta_2, \delta)e^{-C\delta n}.
\]

Otherwise, let \( x_0 \) with \( \mathbb{F}_x = t \) and \( |t| < \eta_1 + \xi \) such that \( \|U(x_0)\| \leq \tilde{C}(\eta_2, \delta)e^{-C\delta n} \). By (4.3) and (4.6), \( \|U(x_0 + j\alpha)\| \leq \tilde{C}(\eta_2, \delta)e^{-C\delta n}, 0 \leq j \leq n, \) since \( \|g\|_{\eta_1 + \eta_2} \leq \tilde{C}(\eta_2)e^{\eta_1 n} \). This implies \( \|u'(x_0 + j\alpha)\| \leq \tilde{C}(\eta_2, \delta)e^{-C\delta n}, 0 \leq j \leq n \). Thus \( \|u'_n\| \leq \tilde{C}(\eta_2, \delta)e^{-C\delta n} \) by (4.2), where \( u'_n(x) = u'(x + t_i) \), contradicting to \( \int_{R/Z} u'_n(x)dx = 1 \) (since \( \hat{u}_0 = 1 \)).

Let \( B(x) \in SL(2, \mathbb{C}) \) be the matrix, whose first column is \( U(x) \), given by Lemma 4.5, then \( \|B\|_{\eta_1 + \xi} \leq \tilde{C}(\eta_2, \delta)e^{-C\delta n} \). Combining with (4.3), it is easy to check that

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\[
B(x + \alpha)^{-1}A(x)B(x) = \begin{pmatrix} e^{2\pi i\theta} & 0 \\ 0 & e^{-2\pi i\theta} \end{pmatrix} + \begin{pmatrix} \beta_1(x) & b(x) \\ \beta_2(x) & \beta_3(x) \end{pmatrix},
\]
(4.8)

where \( \|b\|_{\eta,\xi} \leq \tilde{C}(\eta_2, \delta)e^{C\xi} \), and \( \|\beta_1\|_{\eta,\xi}, \|\beta_2\|_{\eta,\xi}, \|\beta_3\|_{\eta,\xi} \leq \tilde{C}(\eta_2, \delta)e^{-c\eta_2n} \). Taking \( \Phi = DB(x)^{-1} \), where \( D = \begin{pmatrix} d & 0 \\ 0 & d^{-1} \end{pmatrix} \) with \( d = e^{-c\eta_2n} \), we get

\[
\Phi(x + \alpha)A(x)\Phi(x)^{-1} = \begin{pmatrix} e^{2\pi i\theta} & 0 \\ 0 & e^{-2\pi i\theta} \end{pmatrix} + Q(x)
\]
(4.9)

where \( \|Q\|_{\eta,\xi} \leq \tilde{C}(\eta_2, \delta)e^{-c\eta_2n} \) and \( \|\Phi\|_{\eta,\xi} \leq \tilde{C}(\eta_2, \delta)e^{c\eta_2n} \). Thus

\[
\sup_{0 \leq s \leq \tilde{C}(\eta_2, \delta)e^{c\eta_2n}} \|A_s\|_{\eta,\xi} \leq \tilde{C}(\eta_2, \delta)e^{c\eta_2n},
\]
(4.10)

that is

\[
\|A_k\|_{\eta,\xi} \leq \tilde{C}(\eta_2, \delta)k^C
\]
(4.11)

with \( k = \tilde{C}(\eta_2, \delta)e^{c\eta_2n} \). It follows that \( L(\alpha, \epsilon) = 0 \) for any \( |\epsilon| < \eta_1 + \xi \), which is contradicted to the definition of \( \eta_1 \). \( \square \)

5 Sharp estimate for the dynamics of Schrödinger cocycles

In section §4, we set up the priori estimate of the transfer matrix \( A_n(x) \) in a given strip \( |\Im x| < \eta \). In this section, we will set up sharp estimate for the dynamics of Schrödinger cocycles.

We first concern the exponential regime. For \( \alpha \) with \( 0 < \beta(\alpha) < \infty \), let \( \epsilon_0, \epsilon_1, C_0 \) and \( \lambda_0(\nu, \beta) \) be given by Theorem 5.1. Fix \( \lambda \) with \( 0 < |\lambda| < \lambda_0 \). Given \( E \in \Sigma_{\nu,\alpha} \), let \( \theta = \theta(E) \) and solution \( \tilde{u}_k \) be given by Lemma 4.5 and \( \{n_j\} \) be the set of resonances for \( \theta(E) \).

Below, let \( A = S_{\lambda\nu, E} = \begin{pmatrix} E - \lambda \nu & -1 \\ 1 & 0 \end{pmatrix} \). For simplicity, set \( h_1 = C_1\beta, h_2 = \epsilon_0, h = \epsilon_1 \).

Before our main work, we first give some simple facts.

**Lemma 5.1.** (Lemma 4.2, [25]) For \( \{n_j\} > C(\alpha) \),

\[
\|\theta - n_j\alpha\|_{\mathbb{R}/\mathbb{Z}} \geq e^{-3\beta|n_j+1|},
\]
(5.1)

in particular, \( |n_{j+1}| > c_1^2 |n_j| \).

**Lemma 5.2.** (Lemma 3.1, [25]) The following small divisor condition holds,

\[
\|k\alpha\|_{\mathbb{R}/\mathbb{Z}} \geq c(\alpha)e^{-2\beta|k|}, \text{ for any } k \in \mathbb{Z}\setminus\{0\}.
\]
(5.2)
Lemma 5.3. For any $k$ with $|k| \leq |n_j|$ and $k \neq n_j$, the following holds,
\[ \|2\theta - ka\|_{\mathbb{R}/\mathbb{Z}} \geq c(\alpha)e^{-\beta|n_j|}. \] (5.3)

**Proof:** If $\|2\theta - n_j\alpha\|_{\mathbb{R}/\mathbb{Z}} \geq c(\alpha)e^{-\beta|n_j|}$, by the definition of resonance,
\[ \|2\theta - ka\|_{\mathbb{R}/\mathbb{Z}} \geq \|2\theta - n_j\alpha\|_{\mathbb{R}/\mathbb{Z}} \geq c(\alpha)e^{-\beta|n_j|}. \] (5.4)

If $\|2\theta - n_j\alpha\|_{\mathbb{R}/\mathbb{Z}} \leq c(\alpha)e^{-\beta|n_j|}$,
\[ \|2\theta - ka\|_{\mathbb{R}/\mathbb{Z}} \geq \|(n_j - k)a\|_{\mathbb{R}/\mathbb{Z}} - \|2\theta - n_j\alpha\|_{\mathbb{R}/\mathbb{Z}} \geq c(\alpha)e^{-\beta|n_j|} - c(\alpha)e^{-\beta|n_j|} \geq c(\alpha)e^{-\beta|n_j|}, \] (5.5)
where the second inequality holds by (5.2). □

Lemma 5.4. For any $k$ with $|k| \leq C_1|n_j|$ and $k \neq n_j$, the following holds,
\[ \|2\theta - ka\|_{\mathbb{R}/\mathbb{Z}} \geq e^{-CC_1\beta|n_j|}, \] (5.6)
if $|n_j| > C(\alpha)$.

**Proof:** By (5.2)
\[ \|2\theta - ka\|_{\mathbb{R}/\mathbb{Z}} \geq \|(n_j - k)a\|_{\mathbb{R}/\mathbb{Z}} - \|2\theta - n_j\alpha\|_{\mathbb{R}/\mathbb{Z}} \geq c(\alpha)e^{-CC_1\beta|n_j|} - e^{-CC_1\beta|n_j|} \geq e^{-CC_1\beta|n_j|}, \] (5.7)
if $|n_j| > C(\alpha)$. □

Fix some $n = |n_j|$ if defined, otherwise let $N = \infty$. Let $u(x) = u^{\delta}(x)$ and $U^{\delta}(x) = \left( \begin{array}{c} e^{2\pi i \delta} u^{\delta}(x) \\ u^{\delta}(x - \alpha) \end{array} \right)$ with $I_1 = [-\frac{N}{2}, -\frac{N}{2}]$ as in §4.

For simplicity, denote by $C_\ast(c_\ast)$ a large/small constant depending on $\lambda, \nu, \alpha$. Clearly, by strong localization estimate $\|U^{\delta}\|_{c_\ast} < C_\ast e^{C_\ast N}, i = 1, 2$.

Following (4.3)-(4.5), it is easy to verify that
\[ AU^{\delta}(x) = e^{2\pi i \delta} U^{\delta}(x + \alpha) + \begin{pmatrix} g(x) \\ 0 \end{pmatrix} \text{ with } \|g\|_{c_\ast} \leq C_\ast e^{-c\delta N}. \] (5.8)

Lemma 5.5. For $i = 1, 2$,
\[ \inf_{|g|<c_\ast} \|U^{\delta}(x)\| \geq c_\ast e^{-C_\ast |n_j|}. \] (5.9)
Proof: Following Theorem 4.1 and (5.8), we can prove the lemma. See Theorem 4.13 in [25] for details. □

**Theorem 5.1.**

$$\sup_{0 \leq x \leq e^{ch_2}} \|A_x\|_{ch_2} \leq C_\ast e^{C_{\beta N}}.$$  \hfill (5.10)

**Proof:** It suffices to prove that if $N < \infty$, then

$$\sup_{0 \leq x \leq e^{ch_2}} \|A_x\|_{ch_2} \leq C_\ast e^{C_{\beta N}}.$$  \hfill (5.11)

Let $B(x) \in \text{SL}(2, \mathbb{C})$ be the matrix, whose first column is $U^1(x)$, given by Theorem 4.5 with $\eta = ch_2$, then $\|B\|_{ch_2} \leq C_\ast e^{Ch_2}$ by (5.9) and a simple fact $\|U^1\|_{ch_2} < C_\ast e^{Ch_2}$. Combining with (5.8), one easily verifies that

$$B(x + \alpha)^{-1}A(x)B(x) = \begin{pmatrix} e^{2\pi i\theta} & 0 \\ 0 & e^{-2\pi i\theta} \end{pmatrix} + \begin{pmatrix} \beta_1(x) & b(x) \\ \beta_2(x) & \beta_3 x \end{pmatrix}$$  \hfill (5.12)

where $\|b\|_{ch_2} < C_\ast e^{Ch_2}$, and $\|\beta_1\|_{ch_2}$, $\|\beta_2\|_{ch_2}$, $\|\beta_3\|_{ch_2} < C_\ast e^{-ch_2}$.

By Lemma 5.1

$$\|b\|_{ch_2} < C_\ast e^{Ch_2} < C_\ast e^{C_{\beta N}}.$$  

Solving the following equation (by comparing the Fourier coefficients)

$$W(x + \alpha)^{-1} \begin{pmatrix} e^{2\pi i\theta} & b(x) \\ 0 & e^{-2\pi i\theta} \end{pmatrix} W(x) = \begin{pmatrix} e^{2\pi i\theta} & b_1(x) \\ 0 & e^{-2\pi i\theta} \end{pmatrix},$$

where $b_\ell = \sum_{|k| \leq N} b_k e^{2\pi i k x}$ and

$$W(x) = \begin{pmatrix} 1 & w(x) \\ 0 & 1 \end{pmatrix},$$  \hfill (5.13)

we obtain

$$\hat{w}_k = -\hat{b}_k \frac{e^{-2\pi i\theta}}{1 - e^{-2\pi i(2\theta - ka)}}$$  \hfill (5.14)

for $|k| < N$, and $\hat{w}_k = 0$ for $|k| \geq N$. By small divisor condition (5.3) (replacing $n_j$ with $n_{j+1}$ in Lemma 5.3)

$$\|2\theta - ka\|_{\mathbb{R}/\mathbb{Z}} > c(\alpha) e^{-C_{\beta N}}$$

for $|k| < N$,

one has $\|W\|_{ch_2} < C_\ast e^{C_{\beta N}}$.

Let $B_1(x) = BW$, noting that $\|B_1\|_{ch_2} < C_\ast e^{-ch_2 N}$, then $\|B_1\|_{ch_2} < C_\ast e^{C_{\beta N}}$ and

$$B_1(x + \alpha)^{-1}A(x)B_1(x) = \begin{pmatrix} e^{2\pi i\theta} & 0 \\ 0 & e^{-2\pi i\theta} \end{pmatrix} + \begin{pmatrix} \beta_1'(x) & b'(x) \\ \beta_2'(x) & \beta_3' x \end{pmatrix}$$  \hfill (5.15)

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where \( \|b'\|_{ch_2} < C_* e^{-ch_2 n} \), and \( \|\beta_1'\|_{ch_2}, \|\beta_2'\|_{ch_2}, \|\beta_3'\|_{ch_2} < C_* e^{-ch_2 n} \). It follows that

\[
\sup_{0 \leq s \leq c_* e^{by_2 n}} \|A_s\|_{ch_2} \leq C_* e^{CBN}.
\]  

We finish the proof. \( \Box \)

**Theorem 5.2.** There exists \( B : \mathbb{R}/\mathbb{Z} \to SL(2, \mathbb{C}) \) analytic with \( \|B\|_{ch_1} < C_* e^{Ch_1 n} \) such that

\[
B(x + \alpha)^{-1} A(x) B(x) = \begin{pmatrix} e^{2\pi i \theta} & 0 \\ 0 & e^{-2\pi i \theta} \end{pmatrix} + \begin{pmatrix} \beta_1(x) & b(x) \\ \beta_2(x) & \beta_3(x) \end{pmatrix}
\]  

where \( \|b\|_{ch_1} < C_* e^{-ch_2 n} \), and \( \|\beta_1\|_{ch_1}, \|\beta_2\|_{ch_1}, \|\beta_3\|_{ch_1} < C_* e^{-ch_2 n} \).

**Proof:** Let \( B_1(x) \in SL(2, \mathbb{C}) \) be the matrix, whose first column is \( U(x) \), given by Theorem 4.5 with \( \eta = ch_1 \), then \( \|B_1\|_{ch_1} \leq C_* e^{Ch_1 n} \) and

\[
B_1(x + \alpha)^{-1} A(x) B_1(x) = \begin{pmatrix} e^{2\pi i \theta} & 0 \\ 0 & e^{-2\pi i \theta} \end{pmatrix} + \begin{pmatrix} \beta_1'(x) & b'(x) \\ \beta_2'(x) & \beta_3'(x) \end{pmatrix}
\]  

where \( \|b'\|_{ch_1} < C_* e^{Ch_1 n} \), and \( \|\beta_1'\|_{ch_1}, \|\beta_2'\|_{ch_1}, \|\beta_3'\|_{ch_1} < C_* e^{-ch_2 n} \).

Let

\[
\hat{w}_k = -\hat{b}'_k e^{-2\pi i \theta} \left( \frac{1}{1 - e^{-2\pi i (2\theta - k \alpha)}} \right)
\]  

for \( |k| < C_1 n \) and \( k \neq n_j \), and \( \hat{w}_k = 0 \) for \( |k| \geq C_1 n \) or \( k = n_j \).

If \( n \leq C(\alpha) \), it is easy to see that Theorem 5.2 has already held by (5.18). Thus we assume \( n > C(\alpha) \) so that the small divisor condition (5.6) holds, that is

\[
\|2\theta - k\alpha\|_{\mathbb{R}/\mathbb{Z}} > C(\alpha) e^{-CC_1 \beta_0},
\]  

for \( |k| < C_1 n \) and \( k \neq n_j \). By (5.19) and (5.20), we have \( \|W\|_{ch_1} \leq C_* e^{Ch_1 n} \), where \( w(x) = \sum_{k \in \mathbb{Z}} \hat{w}_k e^{2\pi i k x} \) and

\[
W(x) = \begin{pmatrix} 1 & w(x) \\ 0 & 1 \end{pmatrix}.
\]  

Let \( B(x) = B_1 W \), then \( \|B\|_{ch_1} \leq C_* e^{Ch_1 n} \) and

\[
B(x + \alpha)^{-1} A(x) B(x) = \begin{pmatrix} e^{2\pi i \theta} & 0 \\ 0 & e^{-2\pi i \theta} \end{pmatrix} + \begin{pmatrix} \beta_1(x) & b'(x) \\ \beta_2(x) & \beta_3(x) \end{pmatrix},
\]  

where \( \|b'\|_{ch_1} < C_* e^{-ch_1 C_1 n} < C_* e^{-ch_2 n} \), \( b'(x) = \hat{b}'_n e^{2\pi i n j x} \), and \( \|\beta_1\|_{ch_1}, \|\beta_2\|_{ch_1}, \|\beta_3\|_{ch_1} < C_* e^{-ch_2 n} \).

Thus to prove Theorem 5.2 it suffices to verify

\[
\|\hat{b}'_n\| \leq C_* e^{-ch_2 n}.
\]  

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Let

\[ W'(x) = \begin{pmatrix} e^{2ni\theta} & b'_j(x) \\ 0 & e^{-2ni\theta} \end{pmatrix} \quad (5.24) \]

We can compute exactly

\[ W'_j(x) = \begin{pmatrix} e^{2ni\theta} & b'_j(x) \\ 0 & e^{-2ni\theta} \end{pmatrix}, \quad (5.25) \]

where \( |b'_j(x)| = |\hat{b}'_j| \sum_{k=0}^{j-1} e^{-2\pi i k (2\theta - n_j \alpha)} \) if \( \sin \pi (2\theta - n_j \alpha) \neq 0 \), and \( |b''_j(x)| = s|\hat{b}''_j| \) otherwise. Therefore one has

\[ ||W'_j||_0 \geq \frac{s|\hat{b}'_j|}{100}, 0 \leq s \leq ||2\theta - n_j \alpha||_{R/Z}/10. \quad (5.26) \]

On the other hand,

\[ ||W'_j||_0 \leq 1 + s|\hat{b}'_j| \leq C_*(1 + s)e^{Ch_1 n}, s \geq 0. \quad (5.27) \]

since \( ||b'||_{Ch_1} < C_* e^{Ch_1 n} \).

Since \( A = B(x + \alpha)(W'(x) + Z(x))B(x)^{-1} \), where

\[ Z(x) = \begin{pmatrix} \beta_1(x) & b'(x) \\ \beta_2(x) & \beta_3 x \end{pmatrix}, \quad (5.28) \]

after careful computation,

\[ ||A_s||_0 \geq ||B||^{-2}_0 \left( ||W'_j||_0 - \sum_{k=1}^{j} \left( \sum_{k=1}^{j} ||Z||_0 \left( \max_{0 \leq j < s} ||W'_j||_0 \right)^{1+k} \right) \right), \quad (5.29) \]

Clearly, \( ||Z||_0 \leq C_* e^{-Ch_1 n} \) by the estimates of elements of \( Z \), thus

\[ ||A_s||_0 \geq C_* e^{-Ch_1 n} (||W'_j||_0 - C_* e^{-Ch_1 n}), 0 \leq s \leq C_* e^{-Ch_1 n}. \quad (5.30) \]

Combining with (5.10), \( ||W'_{s+1}||_0 \leq C_* e^{Ch_1 n}, 0 \leq s \leq C_* e^{Ch_1 n} < ||2\theta - n_j \alpha||_{R/Z}/10. \) By (5.26), we get the estimate

\[ |\hat{b}'_{n_j}| \leq C_* e^{-Ch_1 n}. \quad (5.31) \]

We finish the proof. \( \Box \)

If frequency \( \alpha \) satisfies \( \beta(\alpha) = 0 \), by Theorem 5.3 there exists \( \lambda_0(\nu) > 0 \) such that if \( 0 < |\lambda| < \lambda_0(\nu) \), \( \hat{H}_{\lambda,\nu,\alpha,0} \) satisfies a strong localization estimate with parameters \( \epsilon_0(\lambda, \nu), \epsilon_1(\lambda, \nu), C_0 = 3, \tilde{C} = C(\lambda, \nu, \alpha) \), where \( \epsilon_1 = C_1 \epsilon_0 \) with \( C_1 \) large enough. Let \( h' = \epsilon_1, h'' = \epsilon_0 \) and \( h'_1 = \frac{\epsilon_0}{h'} \). As the proof of Theorem 5.2, we have the following theorem. In order to avoid repetition, we omit the proof.
Theorem 5.3. Fix some $n = |n|_1$ and let $N = |n_{i+1}|$ if defined, otherwise let $N = \infty$. Then there exists $B: \mathbb{R}/\mathbb{Z} \rightarrow SL(2, \mathbb{C})$ analytic with $\|B\|_{ch_1} < C_\ast e^{C|n|}$ such that

$$B(x + \alpha)^{-1}A(x)B(x) = \begin{pmatrix} e^{2\pi i\theta} & 0 \\ 0 & e^{-2\pi i\theta} \end{pmatrix} + \begin{pmatrix} \beta_1(x) & b(x) \\ \beta_2(x) & \beta_3(x) \end{pmatrix}$$

(5.32)

with $\|b\|_{ch_1} < C_\ast e^{-C|n|}$, and $\|\beta_1\|_{ch_1}$, $\|\beta_2\|_{ch_1}$, $\|\beta_3\|_{ch_1} < C_\ast e^{-C|n|}$.

6 Proof of Theorem 1.1

Let $\mu_{\lambda;v;x} = \mu^0_\lambda + \mu^0_\lambda$, where $\epsilon_i$ is the Dirac mass at $i \in \mathbb{Z}$. For simplicity, sometimes we drop some parameters dependence, for example, replacing $\mu_{\lambda;v;x}$ with $\mu_x$ or $\mu$.

Our main theorem is:

Theorem 6.1. For every $0 < \epsilon < 1$ and $E \in \Sigma_{\lambda;v;x}$, $\mu_x(E - \epsilon, E + \epsilon) \leq C_\ast \epsilon^{1/2}$.

The proof of Theorem 6.1 will be given later. Theorem 1.1 can be immediately derived from Theorem 6.1.

Proof of Theorem 1.1. Since spectral measure $\mu_x$ vanishes on $\mathbb{R}\setminus \Sigma_{\lambda;v;x}$, by Theorem 6.1

$$\mu_x(J) \leq C_\ast |J|^{1/2} \text{ for any interval } J \subset \mathbb{R}.$$  \hspace{1cm} (6.1)

Let $\sigma: \ell^2(\mathbb{Z}) \rightarrow \ell^2(\mathbb{Z})$ be the shift $f(i + 1) = \sigma f(i)$, then $\sigma H_{\lambda;v;x} \sigma^{-1} = H_{\lambda;v;x+\alpha}$. Thus $\mu^\sigma f = \mu_x$ and $\mu^\sigma_x = \mu^{\sigma_0}_{x+\alpha} \leq \mu_{x+\alpha}$. By (2.7), $(\mu_x(J))^{1/2}$ defines a semi-norm on $\ell^2(\mathbb{Z})$. Therefore, by the triangle inequality,

$$(\mu_x(J))^{1/2} \leq \sum_{k \in \mathbb{Z}} |f(k)||(\mu_x(J))^{1/2} \leq C_\ast |J|^{1/4} \|f\|_{\ell^2}.$$  \hspace{1cm} (6.2)

This implies Theorem 1.1.

Here we list two direct corollaries from Theorem 1.1.

Corollary 6.1. For $\alpha \in \mathbb{R}\setminus \mathbb{Q}$ with $0 < \beta(\alpha) < \infty$, if potential $v$ is real analytic in strip $|\Im x| < C\beta$, where $C$ is a large absolute constant, then there exists $\lambda_0 = \lambda_0(v, \beta) > 0$ such that the integrated density of states of $H_{\lambda;v;x}$ is $1/2$-Hölder for $|\lambda| < \lambda_0$. In particular, $\lambda_0 = e^{-C\beta}$ for AMO.

Corollary 6.2. If irrational number $\alpha$ satisfies $\beta(\alpha) = 0$, then for any $v \in C^\omega(\mathbb{R}/\mathbb{Z}, \mathbb{R})$, there exists $\lambda_0 = \lambda_0(v) > 0$ such that the integrated density of states of $H_{\lambda;v;x}$ is $1/2$-Hölder for $|\lambda| < \lambda_0$. In particular, $\lambda_0 = 1$ for AMO.

Remark 6.1. For AMO, by Aubry duality, the integrated density of states of $H_{\lambda;v;x}$ is also $1/2$-Hölder if $|\lambda| > e^{C\beta}$.
6.1 Weyl function

We will use Weyl function to estimate spectral measure. For this reason, we give some simple facts of Weyl function firstly.

Given $E + i\epsilon$ with $E \in \mathbb{R}$ and $\epsilon > 0$, there exists a non-zero solution $u^+ \neq 0$ of $H_{\lambda, v, x}u^+ = (E + i\epsilon)u^+$ which is $\ell^2$ at $+\infty$. The Weyl function is given by

$$m^+ = -\frac{u^+_1}{u^+_0}. \quad (6.3)$$

Let

$$M(E + i\epsilon) = \int \frac{1}{E' - (E + i\epsilon)} d\mu(E'), \quad (6.4)$$

where $\mu = \mu_{\lambda, v, x} = \mu_{\lambda, v, x}^{\epsilon-1} + \mu_{\lambda, v, x}^\epsilon$. Clearly, $M(z)$ is a Herglotz function. It is immediate from the definition that

$$\Im M(E + i\epsilon) \geq \frac{1}{2\epsilon} \mu(E - \epsilon, E + \epsilon). \quad (6.5)$$

Recall the usual action of $\text{SL}(2, \mathbb{C})$,

$$\left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \cdot z = \frac{az + b}{cz + d}.$$

We define $z_\gamma = R_\gamma z$ with $\gamma \in \mathbb{R}/\mathbb{Z}$, where

$$R_\gamma = \left( \begin{array}{cc} \cos 2\pi \gamma & -\sin 2\pi \gamma \\ \sin 2\pi \gamma & \cos 2\pi \gamma \end{array} \right),$$

and let $\psi(z) = \sup_\gamma |z_\gamma|$.

**Lemma 6.1.** The following inequality holds (p. 573, [5]),

$$|M(z)| \leq \psi(m^+(z)) \text{ for } \Im z > 0. \quad (6.6)$$

For $k \in \mathbb{N}$, let

$$P_{(k)} = \sum_{j=1}^{k} A_{2j-1}(x + \alpha)A_{2j-1}(x + \alpha). \quad (6.7)$$

Then $P_{(k)}$ is an increasing family of positive self-adjoint operators. In addition, $\|P_{(k)}\|$, $\det P_{(k)}$, and $\det P_{(k)}$ are all increasing positive functions of $k$. Note that $\text{tr}(A_{2j-1}^{\epsilon}A_{2j-1}) \geq 2$, then $\|P_{(k)}\|$ (and hence $\det P_{(k)}$) is unbounded (since $\text{tr}P_{(k)} \geq 2k$).

**Lemma 6.2.** (Lemma 4.2, [5]) Let $\epsilon$ be such that $\det P_{(k)} = \frac{1}{4\epsilon^2}$, then

$$C^{-1} < \frac{\psi(m^+(E + i\epsilon))}{2\epsilon\|P_{(k)}\|} < C. \quad (6.8)$$
Theorem 6.2. For $k \geq 1$, we have $\| P(k) \| \leq C_* (P(k))^{-1/3}$.

The proof of Theorem 6.2 will be given in the end.

Set $\epsilon_k = \sqrt{\frac{1}{4 \det P(k)}}$, i.e., $\det P(k) = \frac{1}{4 \epsilon_k^2}$.

Lemma 6.3. We have $\psi(m^* (E + i\epsilon_k)) \leq C_* \epsilon_k^{-1/2}$.

Proof: By Theorem 6.2, $\| P(k) \| = \det P(k)(P(k))^{-1/3} < C_* \epsilon_k^{-1/3}$. Thus $\| P(k) \| \leq C_* \epsilon_k^{-3/2}$ and the statement follows from (6.8).

Proof of Theorem 6.1: Clearly, $\lim_{k \to \infty} \epsilon_k = 0$. Following the proof of Theorem 4.1 in [5] (p. 580), $\epsilon_{k+1} > c \epsilon_k$. Combining with (6.5), it is enough to show that

$$\Im M(E + i \epsilon) \leq C_* \epsilon^{-1/2}$$

(6.9) holds for $\epsilon = \epsilon_k$. This follows immediately from (6.6) and Lemma 6.3.

6.2 Proof of Theorem 6.2

We give two lemmata first.

Lemma 6.4. (Lemma 4.3, [1]) Let

$$T(x) = \begin{pmatrix} e^{2ni0} & t(x) \\ 0 & e^{-2ni0} \end{pmatrix}$$

where $t$ has a single non-zero Fourier coefficient, i.e., $t(x) = \hat{t} e^{2nirx}$. Let $X(x) = \sum_{j=1}^k T_{2j-1}(x)^* T_{2j-1}(x)$, then

$$\| X \|_0 \approx k (1 + |\hat{t}|^2 \min\{k^2, \| 2\theta - r\sigma \|_{\mathbb{R}/\mathbb{Z}} \})$$

(6.10)

$$\| X^{-1} \|_0^{-1} \approx k$$

(6.11)

where the notation $a \approx b (a, b > 0)$ denotes $C^{-1}a \leq b \leq Ca$.

Lemma 6.5. (Lemma 4.4, [1]) Let $t, T$ and $X$ be as in the Lemma 6.4. Let $\tilde{T} : \mathbb{R}/\mathbb{Z} \to \text{SL}(2, \mathbb{C})$, and set $\tilde{X}(x) = \sum_{j=1}^k \tilde{T}_{2j-1}(x)^* \tilde{T}_{2j-1}(x)$. Then

$$\| X - \tilde{X} \|_0 \leq 1$$

(6.12)

provided that

$$\| T - \tilde{T} \|_0 \leq ck^{-2} (1 + 2k \| t \|_0)^{-2}.$$  

(6.13)

To prove Theorem 6.2, it is enough to show the following lemma holds.
Lemma 6.6. For $\alpha$ with $0 < \beta(\alpha) < \infty$, then

\[
\frac{\|P_k\|}{\|P_k^{-1}\|^{-1}} \leq C_\star, \text{ if } C_\star e^{C_1 n} \leq k \leq C_\star e^{C_2 N}.
\] (6.14)

For $\alpha$ with $\beta(\alpha) = 0$, then

\[
\frac{\|P_k\|}{\|P_k^{-1}\|^{-1}} \leq C_\star, \text{ if } C_\star e^{C_1 n} \leq k \leq C_\star e^{C_2 N}.
\] (6.15)

Proof: We only give the proof of the case $0 < \beta(\alpha) < \infty$, the other case is similar. Set $\Delta > n$. Let $|r| \leq \Delta$ be such that $\|2\theta - ra\| = \min_{|j| \leq \Delta} \|2\theta - ja\|$, then $|r| \geq n$. Following the proof of Lemma 5.3

\[
\|2\theta - ja\|_{\mathbb{R}/\mathbb{Z}} \geq c(\alpha)e^{-C|\beta||r|}, \text{ for } |j| \leq |r|, j \neq r;
\] (6.16)

\[
\|2\theta - ja\|_{\mathbb{R}/\mathbb{Z}} \geq c(\alpha)e^{-C|\beta||j|}, \text{ for } |r| < |j| \leq \Delta.
\] (6.17)

Using theorem 5.2 decompose $b = t + g + q$ so that $t$ has only the Fourier coefficient $r$, i.e., $t(x) = \hat{\beta}_r e^{2\pi i r x}$, $g$ has only the Fourier coefficients $j \neq r$ with $|j| \leq \Delta$ and $q$ is the rest. Then

\[
B(x + \alpha)^{-1}A(x)B(x) = T + G + H,
\] (6.18)

where

\[
T = \begin{pmatrix} e^{2\pi i \theta} & t \\ 0 & e^{-2\pi i \theta} \end{pmatrix}, \quad G = \begin{pmatrix} 0 & g \\ 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} \beta_1 & q \\ \beta_3 & \beta_4 \end{pmatrix}.
\]

Thus

\[
\|H\|_0 \leq C_\star e^{-C_1 n} e^{-C_1 \Delta} + C_\star e^{-C_1 N}.
\] (6.19)

Solving the following equation

\[
W(x + \alpha)^{-1}(T + G)(x)W(x) = T(x),
\] (6.20)

with $W(x) = \begin{pmatrix} 1 & w(x) \\ 0 & 1 \end{pmatrix}$, then we have

\[
\hat{w}_j = -\hat{\beta}_j \frac{e^{-2\pi i \theta}}{1 - e^{-2\pi i (2\theta - ja)}}, \text{ for } j \neq r, |j| \leq \Delta.
\] (6.21)

and $\hat{w}_j = 0$ for others. Thus

\[
\|W - id\|_0 \leq \|w\|_0 \leq \sum_{|j| \leq |r|} |\hat{w}_j| + \sum_{|r| < |j| \leq \Delta} |\hat{w}_j| \leq C_\star e^{C_1 \beta r - C_1 n}.
\] (6.22)
Let $\Psi = BW$,

$$||\Psi||_0 \leq C_* e^{Ch_1 n} + C_* e^{C\bar{\beta}r-chn}.$$  \hspace{1cm} (6.23)

Let $k_\Delta \geq 0$ be maximal such that for $1 \leq k < k_\Delta$, if we let

$$\tilde{T}(x) = \Psi(x + \alpha)^{-1} A(x) \Psi(x)$$  \hspace{1cm} (6.24)

and

$$\tilde{X}(x) = \sum_{j=1}^{k} \tilde{T}_{2j-1}(x)^* \tilde{T}_{2j-1}(x), X = \sum_{j=1}^{k} T_{2j-1}(x)^* T_{2j-1}(x),$$  \hspace{1cm} (6.25)

then

$$||X - \tilde{X}||_0 \leq 1.$$  \hspace{1cm} (6.26)

Notice that

$$\tilde{T}(x) - T(x) = W(x + \alpha)^{-1} H(x) W(x),$$

then

$$||\tilde{T} - T||_0 \leq ||W||_0^2 ||H||_0.$$  \hspace{1cm} (6.27)

Following Lemma \ref{lemma},

$$||W||_0^2 ||H||_0 \geq c k_\Delta^{-2}(1 + 2k_\Delta|\hat{b}_i|)^{-2} \geq c_* k_\Delta^{-4},$$  \hspace{1cm} (6.28)

since $|\hat{b}_i| < C_*$.

Thus

$$k_\Delta \geq \frac{c_*}{||\Psi||_0^4 ||H||_0^4} \geq \frac{c_*}{(1 + C_* e^{C\bar{\beta}r-chn})(C_* e^{-chn} e^{C\bar{\beta}r\Delta} + C_* e^{-chn})} \geq c_* \min(e^{chn}, e^{C\bar{\beta}r\Delta} e^{chn}).$$  \hspace{1cm} (6.29)

Notice that

$$||P_k|| \leq ||\Psi||_0^2 ||\tilde{X}(x + \alpha)||$$

and

$$||P_k^{-1}||^{-1} \geq ||\Psi||_0^{-4} ||\tilde{X}(x + \alpha)^{-1}||^{-1}. $$

Since $||\tilde{X}|| \leq ||X|| + 1$ and $||\tilde{X}^{-1}|| \geq ||X^{-1}||^{-1} - 1$ for $1 \leq k < k_\Delta$. By Lemma \ref{lemma},

$$||P_k|| \leq C_* k(1 + |\hat{b}_i|^2 k^2) (e^{Ch_1 n} + e^{C\bar{\beta}r-chn})$$

and

$$||P_k^{-1}||^{-1} \geq c_* (e^{Ch_1 n} + e^{C\bar{\beta}r-chn})^{-1} k.$$  \hspace{1cm} (6.30)
Thus,

\[ \frac{\|P_k\|}{\|P_k^{-1}\|^{-3}} < C_\ast |\hat{b}_r|^2(e^{Ch_1 n} + e^{C_\beta r - ch_2 n})^4 + C_\ast \frac{1}{k^3}(e^{Ch_1 n} + e^{C_\beta r - ch_2 n}) \leq C_\ast, \quad (6.31) \]

provided that \( k \geq k_\Delta^- \), where \( k_\Delta^- = (e^{Ch_1 n} + e^{C_\beta r - ch_2 n})^{1/2} \), since \( |\hat{b}_r|^2(e^{Ch_1 n} + e^{C_\beta r - ch_2 n})^4 < C_\ast \) by theorem [5.2]. We obtain that

\[ \frac{\|P_k\|}{\|P_k^{-1}\|^{-3}} \leq C_\ast, \text{ for } k^-_\Delta < k < k_\Delta. \quad (6.32) \]

In order to prove the Lemma, we have to show that for any \( k \) with \( C_\ast e^{Ch_1 n} \leq k < \epsilon e^{Ch_2 N} \), there exists \( \Delta > n \) such that \( k^-_\Delta < k < k_\Delta \). This is easy to satisfy by setting \( \Delta = \epsilon \frac{\ln k}{\beta} \). \( \square \)

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