DIVISOR SUMS REPRESENTABLE AS THE SUM OF TWO SQUARES

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Abstract. Let \( s(n) \) denote the sum of the proper divisors of the natural number \( n \). We show that the number of \( n \leq x \) such that \( s(n) \) is a sum of two squares has order of magnitude \( x/\sqrt{\log x} \), which agrees with the count of \( n \leq x \) which are a sum of two squares. Our result confirms a special case of a conjecture of Erdős, Granville, Pomerance and Spiro, who in a 1990 paper asserted that if \( A \subset \mathbb{N} \) has asymptotic density zero (e.g. if \( A \) is the set of \( n \leq x \) which are a sum of two squares), then \( s^{-1}(A) \) also has asymptotic density zero.

1. Introduction

For a natural number \( n \), define \( s(n) \) to be the sum of the proper divisors of \( n \). In a 1990 paper [2], Erdős, Granville, Pomerance, and Spiro propose the following conjecture, which has motivated a substantial amount of research concerning \( s(n) \).

Conjecture 1.1. Let \( A \) be a subset of the natural numbers of asymptotic density zero. Then \( s^{-1}(A) \) also has asymptotic density zero.

Certain special cases of this conjecture are known to be true. If \( A \) is the set of prime numbers, Pollack shows in [7] that the number of \( n \leq x \) such that \( s(n) \in A \) is \( O(x/\log x) \).

In [8], the same author shows that if \( A \) is the set of palindromes in any given base, then \( s^{-1}(A) \) has density zero. If one wishes to consider sets \( A \) without a prescribed structure, work of Pollack, Pomerance and Thompson [9] shows that the conjecture holds for any set \( A \) whose counting function is \( \ll x^{1/2+\epsilon(x)} \), where \( \epsilon(x) \) is any fixed function tending to 0 as \( x \to \infty \).

Let \( B(x) \) denote the count of \( n \leq x \) such that \( n \) can be represented as a sum of two squares. A classical result of Landau [5] states that \( B(x) \sim Cx/\sqrt{\log x} \), where \( C \) is an explicit constant. In particular, the set of \( n \leq x \) which can be represented as the sum of two squares has asymptotic density zero, and so, in light of Conjecture 1.1, it is natural to wonder whether the preimage of this set under \( s(n) \) has density zero. It turns out that we can prove something stronger.

Theorem 1.2. Let \( B_s(x) \) denote the count of \( n \leq x \) such that \( s(n) \) can be written as a sum of two squares. Then

\[
B_s(x) \asymp \frac{x}{(\log x)^{3/2}},
\]

where the constants implied by the \( \asymp \) symbol are absolute.

In other words, not only does Conjecture 1.1 hold in this particular case, but in fact the order of magnitude of \( B_s(x) \) is the same as that of \( B(x) \).

In Section 2, we give several lemmas which lay the groundwork for the proof of Theorem 1.2. In Section 3, we establish a crude upper bound on the number of \( n \leq x \) such that \( s(n) \) is a sum of two squares, which we leverage to show that we can impose a number of additional
conditions on \( n \). We prove the upper bound implicit in Theorem 1.2 in Section 4, and we prove the lower bound in Section 5.

**Notation.** For a positive integer \( k \), we let \( \log^k x \) denote the \( k \)th iterate of the natural logarithm. In particular, \( \log_k x \) is not to be confused with the base-\( k \) logarithm, and at each occurrence we assume that \( x \) is large enough so that \( \log_k x \) is a well-defined quantity. For a natural number \( n \), we let \( P(n) \) denote the largest prime factor of \( n \). The function \( \tau(n) \) represents the number of divisors of \( n \), and \( \varphi(n) \) is Euler’s function, as usual. Other notation may be defined as necessary.

\section{Preliminaries}

Let \( P(n) \) denote the largest prime factor of \( n \). We define
\[
\mathcal{E}(x) := \{ n \leq x : P(n) \leq x^{1/\log_2 x} \text{ or } P(n)^2 \mid n \}. 
\]
(1)

Notice that each integer \( n \notin \mathcal{E}(x) \) can be written in the form \( n = mP \), where \( P = P(n) \) and \( P \nmid m \). We also have \( s(n) = Ps(m) + \sigma(m) \), where \( \sigma(m) \) denotes the sum of all divisors of \( m \). It is convenient to work with such representations of \( s(n) \), as they are amenable to sieve methods (see Section 3). The following lemma shows that considering only those \( n \notin \mathcal{E}(x) \) does not discard too many values of \( n \).

**Lemma 2.1.** For sufficiently large \( x \), we have \( \#\mathcal{E}(x) \ll x / (\log x)^2 \).

To prove Lemma 2.1, we require the following result of de Bruijn [1, Theorem 2].

**Proposition 2.2.** Let \( x \geq y \geq 2 \) satisfy \( (\log x)^2 \leq y \leq x \). Whenever \( u := \frac{\log x}{\log y} \to \infty \), we have
\[
\Psi(x, y) \leq x/u + o(u).
\]

**Proof of Lemma 2.1.** If \( n \in \mathcal{E}(x) \), then either \( P(n) \leq x^{1/\log_2 x} \) or \( P(n)^2 \mid n \). By Proposition 2.2, the number of \( n \leq x \) for which the former possibility holds is \( \ll x / (\log x)^2 \). The number of \( n \leq x \) for which the latter possibility holds is
\[
\ll x \sum_{p > x^{1/\log_2 x}} \frac{1}{p^2} \ll xe^{-\log x / \log_2 x},
\]
and so the count of these \( n \) is also \( \ll x / (\log x)^2 \).

In the proof of Theorem 1.2, we make frequent use of a generalization of Mertens’s theorem to arithmetic progressions. The version stated below follows from a result proved by Mertens himself (cf. [8, pp. 41–43, 449–450]). Note that although there is a dependence on the modulus \( b \) and the residue class \( a \pmod{b} \) in the constants in the following two results, we will work exclusively in the case \( b = 4 \) and \( a = 1 \) or \( a = 3 \).

**Theorem 2.3.** For fixed \( a, b \in \mathbb{Z} \),
\[
\sum_{p \leq x \atop p \equiv a \pmod{b}} \frac{1}{p} = \frac{1}{\varphi(b)} \log \log x + c_{a,b} + O_b \left( \frac{1}{\log x} \right),
\]
where subscripts indicate dependence upon \( a \) and \( b \).
Using the fact that $\log(1 - 1/p) = -\frac{1}{p} + O(1/p^2)$, one quickly deduces the following corollary. Though stronger versions of this theorem exist in the literature (e.g. [12]), this weak result is sufficient for us.

**Corollary 2.4.** For fixed $a, b \in \mathbb{Z}$,

$$\prod_{p \leq x \atop p \equiv a \pmod{b}} \left(1 - \frac{1}{p}\right) \asymp \frac{1}{(\log x)^{1/\varphi(b)}},$$

where the constants implied by the $\asymp$ symbol depend only on $a$ and $b$.

Let $n$ be a natural number. Suppose $s(n)$ is a sum of two squares. Then by a classical theorem of Fermat, if $R$ is the largest factor of $s(n)$ supported on the primes $p \equiv 3 \pmod{4}$, then $R$ is a square. The following lemma plays a crucial role in the proof of the upper bound implicit in Theorem 1.2. It provides an upper bound on the number of $n \leq x$ whose prime factors belong to a certain interval $[z, y]$, with the possible exception of primes $p \leq z$ dividing this square factor $R$.

**Lemma 2.5.** Let $x \geq y \geq z \geq 2$, and let $R$ be a positive integer. Let $\Psi(x)$ denote the count of numbers $n \leq x$ such that $n$ is $y$-smooth and such that, if $p \mid n$ with $p \leq z$ and $p \equiv 3 \pmod{4}$, then $p \mid R$. Then

$$\Psi(x) \ll \frac{x}{\sqrt{\log z}} e^{-u/2} \frac{R}{\varphi(R)}, \quad \text{where} \quad u := \frac{\log x}{\log y}.$$

A key ingredient in the proof of Lemma 2.5 is a result of Halberstam and Richert [4], which we state in precisely the same form as [7, Lemma 2.4] (which itself is derived from [10, Corollary 5.1, p. 309]).

**Lemma 2.6.** Let $f$ be a real-valued, nonnegative multiplicative function. Suppose there are positive constants $\lambda_1$ and $\lambda_2$, with $\lambda_2 < 2$, so that $f(p^k) \leq \lambda_1 \lambda_2^{k-1}$ for all prime powers $p^k$. Then for all $x \geq 1$,

$$\sum_{n \leq x} f(n) \ll_{\lambda_1, \lambda_2} x \exp \left( \sum_{p \leq x} \frac{f(p) - 1}{p} \right).$$

**Proof of Lemma 2.5.** Let $\chi(n)$ denote the characteristic function of those $n$ counted by $\Psi$. Then for any $\alpha > 0$,

$$\Psi(x) = \sum_{n \leq x} \chi(n) \leq x^{3/4} + \sum_{x^{3/4} < n \leq x} \chi(n) \leq x^{3/4} + x^{-3\alpha/4} \sum_{n \leq x} \chi(n)n^\alpha.$$

Let $\alpha = 2/(3 \log y)$. We estimate this final sum by applying Lemma 2.6 with $f(n) = \chi(n)n^\alpha$. Notice that $f(p^k) = 0$ if $p > y$ or if $p < z$ with $p \equiv 3 \pmod{4}$ and $p \not| R$, and $f(p^k) = p^{k\alpha}$ if $p \in [z, y]$ or if $p < z$ with $p \equiv 3 \pmod{4}$ and $p \mid R$. Since $p^{k\alpha} \leq \exp(2k/3)$, the hypotheses
of Lemma 2.6 are satisfied with $\lambda_1 = \lambda_2 = \exp(2/3) < 2$. Thus,

$$\sum_{n \leq x} \chi(n)n^\alpha \ll x \exp \left( - \sum_{p \equiv 3 \pmod{4}} \frac{1}{p} \right) \exp \left( \sum_{p \leq y} \frac{p^\alpha - 1}{p} \right)$$

$$\ll \frac{x}{\sqrt{\log z}} \exp \left( \sum_{p \mid R} \frac{1}{p} \right) \exp \left( \sum_{p \leq y} \frac{p^\alpha - 1}{p} \right)$$

(2)

using Theorem 2.3. Now,

$$\exp \left( \sum_{p \mid R} \frac{1}{p} \right) \leq \exp \left( - \sum_{p \mid R} \log \left( 1 - \frac{1}{p} \right) \right) = \prod_{p \mid R} \left( 1 - \frac{1}{p} \right)^{-1} = \frac{R}{\varphi(R)}.$$

For the second exponential factor in (2), we have that $\alpha \log p \ll 1$ for $p \leq y$, and so $p^\alpha - 1 \ll \alpha \log p$. Therefore, $\sum_{p \leq y} \frac{p^\alpha - 1}{p} \ll \alpha \sum_{p \leq y} \frac{\log p}{p} \ll \alpha \log y \ll 1$. Putting it all together, we have shown

$$\Psi(x) \ll x^{3/4} + x^{-3\alpha/4} \frac{x}{\sqrt{\log z}} \frac{R}{\varphi(R)}. \quad (3)$$

With $u$ and $\alpha$ as previously defined, we have $x^{-3\alpha/4} = e^{-u/2}$. Also, if $y \geq 11$, then since $x \geq z$,

$$\frac{x}{\sqrt{\log z}} e^{-u/2} \geq \frac{x}{\log x} x^{-1/(2 \log 11)} \gg x^{0.79}.$$  

(If $y \leq 11$, then $\Psi(x) = O((\log x)^4).$) Thus the second term in (3) dominates, and the proof is complete. \qed

Next, we present a lemma which facilitates the proof of the upper bound implicit in Theorem 1.2 by allowing us to assume that the square factor $R$ of $s(n)$ is not too big.

**Lemma 2.7.** Let $n \leq x$ with $n \notin \mathcal{E}(x)$, so that $n = mP$ with $P > x^{1/\log_2 x}$ and $P \nmid m$. Let $R$ be the largest divisor of $s(n)$ supported on the primes $p \equiv 3 \pmod{4}$ not exceeding $x/m^{1/10}$. The number of such $n \leq x$ with $R > (\log x)^2$ is $o(x/\sqrt{\log x})$.

**Proof.** Suppose $n = mP$ is as above with $R > (\log x)^2$. Then $R$ is a square. Since every prime factor of $R$ is at most $(x/m)^{1/10}$, if $R > (x/m)^{1/4}$, then by removing factors of the form $p^2$ from $R$ one at a time, we will eventually obtain a square factor of $s(n)$ that is at most $(x/m)^{1/4}$ but greater than $(x/m)^{1/4 - 2/10} = (x/m)^{1/20} > (\log x)^2$. So, if $R > (\log x)^2$, there is a square factor $S$ of $s(n)$ with $(\log x)^2 < S \leq (x/m)^{1/4}$.

Write $S = S_1S_2$, where $S_1 = (S, s(m))$ and $S_2 = S/S_1$. We count the number of $n \leq x$ corresponding to a fixed factorization $S = S_1S_2$, then sum on possible factorizations. Since $S \mid s(n) = Ps(m) + \sigma(m)$, $S_1 \mid \sigma(m)$, and so $S_1 \mid \sigma(m) - s(m) = m$. Also, $S_2 = S/(S, s(m))$ is coprime to $s(m)/(S, s(m)) = s(m)/S_1$, and so the congruence

$$Ps(m) \equiv -\frac{\sigma(m)}{S_1} \pmod{S_2}$$

places $P$ in a uniquely determined residue class modulo $S_2$. Note also that $P \leq x/m$, which means that $x/m > x^{1/\log_2 x}$. By the Brun–Titchmarsh inequality \cite[Theorem 3.8, p. 110]{4},
the number of choices for $P$ given $m$ is
\[
\ll \frac{x/m}{\varphi(S_2) \log\left(\frac{x}{mS_2}\right)} \ll \frac{x \log x}{m \varphi(S_2) \log x},
\]
since $\frac{x}{mS_2} \geq \frac{x}{mS} \geq (\frac{x}{m})^{3/4} \geq x^{3/4 \log_2 x}$ (here we use the fact that $S \leq (x/m)^{1/4}$). Summing on $m \leq x^{1-1/\log_2 x}$ that are multiples of $S$, shows that the number of possible values of $n = mP$ corresponding to the decomposition $S = S_1 S_2$ is
\[
\ll \frac{x \log_2 x}{S_1 \varphi(S_2)}.
\]
We sum now on decompositions $S = S_1 S_2$. Using the fact that $S_2/\varphi(S_2) \leq S/\varphi(S)$,
\[
\sum_{S_1 S_2 = S} \frac{1}{S_1 \varphi(S_2)} = \frac{1}{q} \sum_{s_2|S} \frac{S_2}{s_2 \varphi(S_2)} \leq \frac{1}{S} \left( \tau(S) \frac{S}{S_2 \varphi(S_2)} \right) = \frac{\tau(S)}{\varphi(S)},
\]
for a total count of $n \leq x$ that is
\[
\ll x \log_2 \frac{x \tau(S)}{\varphi(S)}.
\]
Summing this quantity over squares $S > (\log x)^2$ gives a final upper bound of
\[
\ll \frac{x}{(\log x)^{1+o(1)}} = o(x/\sqrt{\log x}).
\]

We conclude Section 2 with three lemmas, which are Lemmas 2.1, 2.2 and 2.7, respectively, of [7]. We refer the reader to that paper for their proofs.

**Lemma 2.8.** Let $x \geq 3$, and let $q$ be a positive integer. The number of $n \leq x$ for which $q \nmid \sigma(n)$ is
\[
\ll \frac{x}{(\log x)^{1/\varphi(q)}}
\]
uniformly in $q$.

**Lemma 2.9.** For each prime $p$, the number of $n \leq x$ for which $p \mid \sigma(n)$ is
\[
\ll \frac{x \log_2 x}{p^{1/2}}.
\]

**Lemma 2.10.** Let $q$ be a natural number with $q \leq x^{1/\log_3 x}$. The number of $n \leq x$ not belonging to $E(x)$ for which $q \mid s(n)$ is
\[
\ll \frac{\tau(q)}{\varphi(q)} \cdot x \log_3 x.
\]

### 3. A crude, but useful, upper bound

For $n \leq x$ with $n \notin \mathcal{E}(x)$, we have $n = mP$ where $P > x^{1/\log_2 x}$ and $P \not\mid m$. The purpose of the present section is to establish the following proposition, which allows us to impose a number of additional conditions on $m$ without sacrificing too many values of $n$. Our proof borrows many ideas from an argument of Pollack [7, Section 5.2].
Proposition 3.1. For each \( n \leq x \), write \( n = mP \), where \( P := P(n) \). Consider those \( n \leq x \) such that \( s(n) \) can be written as a sum of two squares. For all but \( o(x/\sqrt{\log x}) \) such values of \( n \), all of the following conditions hold:

(i) \[ \prod_{p \leq \sqrt{\log x}} p \] \( \text{divides} \) \( \sigma(m) \)

(ii) \[ \sum_{p|\sigma(m)} \frac{1}{p} \leq 1 \]

(iii) \[ \sum_{p|s(m)} \frac{1}{p} \leq 1. \]

Proof. First, let us assume that \( n \not\in \mathcal{E}(x) \); by Lemma 2.1, this discards \( O(x/(\log x)^2) \) values of \( n \). Suppose \( s(n) \) can be written as the sum of two squares. If \( Q \) is the largest divisor of \( s(n) \) supported on the primes \( p \equiv 3 \pmod{4} \), then \( Q \) must be a square. Write \( n = mP \), with \( P = P(n) \) and \( P \nmid m \); then \( s(n) = Ps(m) + \sigma(m) \).

We begin by bounding the number of \( n \) corresponding to a given \( m \). By an application of Selberg’s upper bound method (specifically [4, Theorem 4.2]), the number of possibilities for the prime \( P \leq x/m \) is

\[ \ll \frac{x/m}{\log x/m} \prod_{p \leq x/m} \left( 1 - \frac{\rho(p)}{p} \right) \prod_{p \mid \sigma(m)} \left( 1 - \frac{1}{p} \right)^{-1}, \tag{4} \]

where \( \rho(p) = \#\{n \pmod{p} : ns(m) + \sigma(m) \equiv 0 \pmod{p}\} \).

We observe that \((m, \sigma(m))\) is not divisible by any primes \( p \equiv 3 \pmod{4}, p \nmid Q \). Indeed, \( s(n)/Q \) is free of such prime factors. But

\[ (m, \sigma(m)) \mid (P + 1)\sigma(m) - mP = s(n), \tag{5} \]

so that \( s(n) \) cannot be free of prime factors \( p \equiv 3 \pmod{4}, p \nmid Q \) unless the same is true for \((m, \sigma(m))\). Now, for a prime \( p \equiv 3 \pmod{4} \) and \( p \nmid Q \), if \( p \mid s(m) \) and \( p \mid \sigma(m) \), then \( p \mid m \), which contradicts \((m, \sigma(m))\) being free of such prime factors. Thus, for such primes \( p \), \( \rho(p) = 0 \) if \( p \mid s(m) \) and \( \rho(p) = 1 \) otherwise. Therefore, the expression (4) is

\[ \ll \frac{x/m}{\log x/m} \prod_{p \leq x/m} \left( 1 - \frac{1}{p} \right) \prod_{p \mid \sigma(m)} \left( 1 + \frac{1}{p} \right) \]

\[ \ll \frac{x/m}{\log x/m} \prod_{p \leq x/m} \left( 1 - \frac{1}{p} \right) \prod_{p \mid \sigma(m)} \left( 1 + \frac{1}{p} \right), \tag{6} \]

where we have removed the condition \( p \nmid Q \) from the first product at the cost of allowing \( p \mid Q \) in the second product. By Corollary 2.4, the first product is \( \ll 1/\sqrt{\log x/m} \), and so (6) becomes
\[ \ll \frac{x/m}{(\log \frac{x}{m})^{3/2}} \prod_{p|Qs(m)\sigma(m), p \equiv 3 \pmod{4}} \left(1 + \frac{1}{p}\right). \quad (7) \]

Note that \( \prod_{p|Qs(m)\sigma(m)}(1 + 1/p) \leq \frac{\sigma(Qs(m)\sigma(m))}{Qs(m)\sigma(m)} \). Since \( Qs(m)\sigma(m) \leq s(n)^3 < x^3 \), we have \( \frac{\sigma(Qs(m)\sigma(m))}{Qs(m)\sigma(m)} \ll \log_2(x) \) by maximal order results for the \( \sigma \)-function. Also, since \( P > x^{1/\log_2 x} \), we have that \( m < x/\exp(\log x/\log_2 x) \). Therefore \( \log(x/m) > \log x/\log_2 x \). Using these facts together with Corollary 2.4, the number of possibilities for the prime \( P \leq x/m \) satisfies

\[ \ll \frac{x(\log_2 x)^{5/2}}{m(\log x)^{3/2}}. \quad (8) \]

Using this upper bound, Proposition 3.1 follows from the same arguments appearing in the proof of Theorem 1.11 in [7]; for completeness, we repeat these arguments here. First, we may assume

\[ m \geq \exp(\log x/(\log_2 x)^4), \quad (9) \]

since summing (8) over those \( m \) less than this bound gives a count of corresponding \( n \) that is \( o(x/\sqrt{\log x}) \). Let \( L = [\log x] \), and choose \( \lambda \in L = \{1/L, \ldots, (L - 1)/L\} \) as large as possible so that \( m > x^\lambda \); a simple calculation shows that \( m \in (x^\lambda, ex^\lambda] \). By Lemma 2.8 the number of \( m \in (x^\lambda, ex^\lambda] \) for which (i) fails is

\[ \ll \sum_{d \leq \sqrt{\log_2 x}} \frac{x^\lambda}{(\log x^\lambda)^{1/\varphi(d)}} \ll x^\lambda \sum_{d \leq \sqrt{\log_2 x}} \exp(-\log_2(x^\lambda)/\sqrt{\log_2 x}) \]

\[ \ll x^\lambda/\exp\left(\frac{1}{2}\sqrt{\log_2 x}\right) \ll x^\lambda/(\log_2 x)^4. \]

Here, we use that \( m \asymp x^\lambda \) and that \( m \) satisfies (9). From Lemma 2.9 the count of \( m \in (x^\lambda, ex^\lambda] \) for which (ii) fails is

\[ \ll \sum_{m \leq x^\lambda} \prod_{p|\sigma(m), p>(\log_2 x)^{10}} \frac{1}{p} \ll \sum_{p>(\log_2 x)^{10}} \sum_{m \leq x^\lambda} \frac{1}{p} \]

\[ \ll x^\lambda \log_2 x \sum_{p>(\log_2 x)^{10}} \frac{1}{p^{3/2}} \ll x^\lambda/(\log_2 x)^4. \]

For (iii), since \( s(m) < \sigma(m) < x \),

\[ \sum_{p|s(m)} \frac{1}{p} \leq \frac{1}{\log x} \sum_{p|s(m)} \frac{1}{p} \leq \frac{1}{\log x} \cdot \frac{\log s(m)}{\log_2 x} \leq \frac{1}{\log_2 x}. \]

Hence, if \( m \) fails (iii), then for large enough \( x \)

\[ \sum_{p|s(m)} \frac{1}{p} \geq \frac{1}{2}. \quad (10) \]
The size of the set $E(x^\lambda)$ is $O(x^\lambda/(\log x)^4)$, while the number of $m \in (x^\lambda, ex^\lambda]$ not belonging to $E(x^\lambda)$ and satisfying (10) is at most

$$2 \sum_{m \leq x^\lambda} \sum_{p \mid s(m)} \frac{1}{p} = 2 \sum_{(\log_2 x)^{10} < p \leq \log x} \sum_{m \leq x^\lambda} \frac{1}{p} \sum_{m \in E(x^\lambda)} 1 \leq x^\lambda \log_3 x \sum_{p > (\log_2 x)^{10}} \frac{1}{p^2} \ll \frac{x^\lambda}{(\log_2 x)^9}.$$ 

Here we applied Lemma 2.10 with $x$ replaced by $ex^\lambda$ and $q$ replaced by $p$. Collecting our estimates, we see that the number of $m \in (x^\lambda, ex^\lambda]$ failing one of (i), (ii), or (iii) is

$$O\left(\frac{x^\lambda}{(\log_2 x)^4}\right).$$

(11)

Summing the bound (11) over these $m$, we see that the number of corresponding $n$ is

$$\frac{x(\log_2 x)^{5/2}}{(\log x)^{3/2}} \cdot \frac{1}{x^\lambda} \cdot \frac{x^\lambda}{(\log_2 x)^4} \ll \frac{x}{(\log x)^{3/2}(\log_2 x)^{5/2}}.$$ 

Summing over the $O(\log x)$ values of $\lambda$ completes the proof.

4. PROOF OF UPPER BOUND IN THEOREM 1.2

In this section, we establish an upper bound for the number of $n \leq x$, $n = mP$, where $P > x^{1/\log_2 x}$ and $m$ satisfies conditions (i) – (iii) from Proposition 3.1, such that $s(n)$ is the sum of two squares. Let $\lambda \in \{0, 1/L, \ldots, (L - 1)/L\}$, where $L = \lceil \log x \rceil$. Then $\lambda < \log m/\log x \leq \lambda + 1/L$, so that $x^\lambda < m \leq ex^\lambda$.

Let $R$ be the largest divisor of $s(n)$ supported on the primes $p \equiv 3 \pmod{4}$ not exceeding $(x/m)^{1/10}$. Note that this means $R$ is a square. By Lemma 2.7 we can assume that $R \leq (\log x)^2$, as this discards a negligible number of $n$. We will count those $n \leq x$ corresponding to a given $R$ and, at the end, sum on square numbers $R$ with the aforementioned property.

If $R \mid s(n)$ and $s(n) = Ps(m) + \sigma(m)$, then $\gcd(R, s(m)) \mid \sigma(m)$. Fix a decomposition $R = R_1R_2$, where $R_1 = \gcd(R, s(m))$ and $R_2 = R/R_1$; then

$$R_2 \mid Ps(m)/R_1 + \sigma(m)/R_1.$$ 

Therefore

$$Ps(m)/R_1 \equiv -\sigma(m)/R_1 \pmod{R_2}.$$ 

Since $s(m)/R_1$ and $R_2$ are coprime, $P$ belongs to a uniquely determined residue class $a \pmod{R_2}$, where $0 \leq a < R_2$ and $a$ depends on $m$ and $R$. So we count the number of $P \equiv a \pmod{R_2}$ for which $Ps(m) + \sigma(m)$ is free of prime factors $q \equiv 3 \pmod{4}$, $q \leq (x/m)^{1/10}$, $q$ not dividing $R$.

Note that $P = a + R_2u$, where $u \leq x/mR_2$. We now claim that for each prime $q \leq (x/m)^{1/10}$ with $q \nmid R$ and $q \nmid s(m)\sigma(m)$, the number $u$ avoids two distinct residue classes modulo $q$. Indeed, if $P > (x/m)^{1/10}$, then $P = a + R_2u$ is not divisible by any primes $\leq (x/m)^{1/10}$. Thus, for each prime $q \leq (x/m)^{1/10}$ with $q \nmid R$, we have that $u$ avoids the residue class
\[-R_2^{-1}a \pmod{q}.\] In addition, for \(n\) to correspond to \(R\), we need that if \(q \leq (x/m)^{1/10}\), with \(q \nmid R\) and \(q \equiv 3 \pmod{4}\),
\[(a + R_2u)s(m) + \sigma(m)\]
is not divisible by \(q\). This means that \(u\) avoids the residue class \(R_2^{-1}(\sigma(m)s(m)^{-1} - a) \pmod{q}\) – which is distinct from the residue class \(-R_2^{-1}a \pmod{q}\) that \(u\) already avoids – provided \(q \nmid s(m)\sigma(m)\).

Since \(R_2 \leq R \leq (\log x)^2\), the number of such \(P\) is, by Brun’s sieve,
\[
\ll \frac{x}{mR_2 \log(x/m)^{3/2}} \left(\frac{R}{\varphi(R)}\right)^2 \prod_{\substack{q|m(s(m)\sigma(m)) \\ q \equiv 3 \pmod{4}} \atop q>R} \left(1 + \frac{1}{q}\right).
\]

Now, we work with the product. We can write this product as
\[
\prod_{\substack{q|m(s(m)\sigma(m)) \\ q \equiv 3 \pmod{4}} \atop q>R} \left(1 + \frac{1}{q}\right) = P_1 P_2 P_3,
\]
where
\[
P_1 = \prod_{\substack{q|m(s(m)\sigma(m)) \\ q \equiv 3 \pmod{4} \atop q \leq \sqrt{\log_2 x}} \atop q \nmid R} \left(1 + \frac{1}{q}\right), \quad P_2 = \prod_{\substack{q|m(s(m)\sigma(m)) \\ q \equiv 3 \pmod{4} \atop q \geq (\log_2 x)^{10} \atop q \nmid R}} \left(1 + \frac{1}{q}\right), \quad P_3 = \prod_{\substack{q|m(s(m)\sigma(m)) \\ q \equiv 3 \pmod{4} \atop \sqrt{\log_2 x} \leq q \leq (\log_2 x)^{10} \atop q \nmid R}} \left(1 + \frac{1}{q}\right).
\]

By taking logarithms and using Mertens’ theorem, conditions (ii) and (iii) give \(P_2 \ll 1\), and the same technique shows \(P_3 \ll 1\) as well. It remains to estimate \(P_1\). We have
\[
P_1 = \prod_{\substack{q|m(s(m)\sigma(m)) \\ q \equiv 3 \pmod{4} \atop q \leq \sqrt{\log_2 x} \atop q \nmid R}} \left(1 + \frac{1}{q}\right) \leq \prod_{\substack{q|m(s(m)\sigma(m)) \\ q \equiv 3 \pmod{4} \atop q \leq \sqrt{\log_2 x}}} \left(1 + \frac{1}{q}\right) \prod_{\substack{q|m(s(m)\sigma(m)) \\ q \equiv 3 \pmod{4} \atop \sqrt{\log_2 x} \leq q \leq \sqrt{\log_2 x}}} \left(1 + \frac{1}{q}\right).
\]

By condition (i) above, the second product on the right-hand side runs over all primes \(q \leq \sqrt{\log_2 x}\). Upon taking logarithms and using Mertens’s theorem for progressions, one sees that this product is \(\ll \sqrt{\log_3 x}\). As for the first product on the right-hand side, recall that \((m, \sigma(m)) = (s(m), \sigma(m))\) is free of prime factors \(q \equiv 3 \pmod{4}\); hence, this first product is empty, by (i). Therefore
\[
\prod_{\substack{q|m(s(m)\sigma(m)) \atop q \equiv 3 \pmod{4} \atop q \nmid R}} \left(1 + \frac{1}{q}\right) \ll \sqrt{\log_3 x},
\]
and so the number of such \(P\) is
\[
\ll \frac{x^{1-\lambda} \sqrt{\log_3 x}}{(1 - \lambda)^{3/2}(\log x)^{3/2}} \left(\frac{R}{\varphi(R)}\right)^2 \frac{1}{R_2}. \hspace{1cm} (12)
\]
Now we count \( m \) corresponding to \( \lambda \). Since \( R_1 \mid s(m) \) and \( R_1 \mid \sigma(m) \), we have \( R_1 \mid m \). Writing \( m = R_1m' \), it suffices to count \( m' \). We have \( m' \leq ex^{\lambda}/R_1 \), \( m' \) is \( x^{1-\lambda} \)-smooth, and \( m' \) has no prime factors \( q \equiv 3 \pmod{4} \) with \( q \leq \sqrt{\log x} \), except possibly those such \( q \) with \( q \mid R \). By Lemma \ref{lem:smooth-count}, the number of such \( m' \) is

\[
\ll \frac{x^{\lambda}}{R_1 \sqrt{\log x}} \left(1 - \frac{\lambda}{R} \right)^2 \frac{R}{\varphi(R)}.
\]

(13)

By multiplying (12) and (13), we see that the overall count of \( n \) is

\[
\ll \frac{x(1 - \lambda)^{1/2}}{R_1 R_2 (\log x)^{3/2}} \left( \frac{R}{\varphi(R)} \right)^3.
\]

Summing on decompositions \( R = R_1 R_2 \) and then on square numbers \( R < (\log x)^2 \) gives

\[
\sum_{R < (\log x)^2} \sum_{R_1 R_2 = R} \frac{x(1 - \lambda)^{1/2}}{R_1 R_2 (\log x)^{3/2}} \left( \frac{R}{\varphi(R)} \right)^3 = \sum_{R < (\log x)^2} \frac{x(1 - \lambda)^{1/2}}{(\log x)^{3/2}} \left( \frac{R}{\varphi(R)} \right)^3 \tau(R) R
\]

\[
\ll \frac{x}{(\log x)^{3/2}}.
\]

Finally, summing on the \( O(\log x) \) values of \( \lambda \) yields the desired bound.

\[\square\]

5. PROOF OF LOWER BOUND IN THEOREM 1.2

The aim of this section is to establish a lower bound of the form

\[
\sum_{n \leq x \atop s(n) = \square + \square} 1 \gg \frac{x}{\sqrt{\log x}}.
\]

We achieve this lower bound by imposing a number of conditions on \( n \leq x \). We assume that \( n \notin \mathcal{E}(x) \), so that \( n = mP \) where \( P := P(n) > x^{1/\log_2 x} \) and \( P \mid m \). We also require that \( m \) satisfies the bound (11), condition (i) of Proposition 3.1, and that if a prime \( p \) divides \((m, \sigma(m))\), then \( p \equiv 1 \pmod{4} \). Further, we count only those \( m \) that can be decomposed as \( m = m_1 m_2 \), where \( m_1 \) and \( m_2 \) satisfy:

- \( m_1 \) squarefree
- \( m_1 \) is \( \sqrt{\log_2 x} \)-smooth
- \( p \mid m_1 \implies p \equiv 1 \pmod{4} \)
- \( m_2 \equiv 3 \pmod{4} \)
- \( p \mid m_2 \implies p > (\log_2 x)^{10} \).

With this setup, we bound from below the sum

\[
\sum_{m \leq x^{1/500}} \sum_{\substack{x^{1/2} < P \leq x/m \\
\text{\emph{p} | P(s(m) + \sigma(m)) \implies p \equiv 1 \pmod{4}}}} 1,
\]

where the ' on the sum indicates that \( m \) satisfies all conditions mentioned above. Note that \((m, \sigma(m)) = (s(m), \sigma(m))\) can be written as a sum of two squares, as \((m, \sigma(m))\) is free of
prime factors \( q \equiv 3 \pmod{4} \). Since the set of numbers which are a sum of two squares is closed under multiplication, it suffices to estimate
\[
\sum_{m \leq x^{1/500}} \sum_{x^{1/2} < P \leq x/m, p | P \alpha(m) + \beta(m) \implies p \equiv 1 \pmod{4}} 1
\]
from below, where \( \alpha(m) = s(m)/(m, \sigma(m)) \) and \( \beta(m) = \sigma(m)/(m, \sigma(m)) \).

We handle the inner sum by appealing to a theorem tailor-made for our problem, namely [3, Theorem 14.8], stated below.

**Theorem 5.1.** Let \( \alpha \geq 1, \beta \neq 0, (\alpha, \beta) = 1, \) and \( \alpha + \beta \equiv 1 \pmod{4} \). Then, for \( x \geq 2|\beta|\alpha^{-1}, x \geq \alpha^{68} \), we have
\[
\sum_{\substack{p \leq x \ni \alpha + \beta \equiv 1 \pmod{4} \implies q \equiv 1 \pmod{4} \implies p \mid q}} 1 \sim \frac{x}{(\log x)^{3/2}} \prod_{p \mid \alpha \beta, p \equiv 3 \pmod{4}} \left( 1 + \frac{1}{p} \right).
\]

Since \( m = m_1 m_2 \equiv 3 \pmod{4} \), we have that \( \alpha(m) + \beta(m) \equiv 1 \pmod{4} \). We also have \( (\alpha(m), \beta(m)) = 1, \) and \( \alpha(m) \leq s(m) \leq m^2 \leq x^{1/250} \). Hence, all of the hypotheses of Theorem 14.8 are satisfied, and so
\[
\sum_{m \leq x^{1/500}} \sum_{x^{1/2} < P \leq x/m, p | P \alpha(m) + \beta(m) \implies p \equiv 1 \pmod{4}} 1 \gg \frac{x}{(\log x)^{3/2}} \prod_{p \mid (\alpha(m)\beta(m)), p \equiv 3 \pmod{4}} \left( 1 + \frac{1}{p} \right).
\]

Consider the product in the above display. Noting that \( (\alpha(m), \beta(m)) = 1 \) and \( (s(m), \sigma(m)) \) is free of prime factors congruent to 3 \( \pmod{4} \), we can write
\[
\prod_{p \mid \alpha(m)\beta(m), p \equiv 3 \pmod{4}} \left( 1 + \frac{1}{p} \right) = \prod_{p \mid \alpha(m), p \equiv 3 \pmod{4}} \left( 1 + \frac{1}{p} \right) \prod_{p \mid \beta(m), p \equiv 3 \pmod{4}} \left( 1 + \frac{1}{p} \right)
\]
\[
= \prod_{p \mid s(m), p \equiv 3 \pmod{4}} \left( 1 + \frac{1}{p} \right) \prod_{p \mid \sigma(m), p \equiv 3 \pmod{4}} \left( 1 + \frac{1}{p} \right).
\]

Now, since \( \sigma(m) \) is divisible by all primes \( p \leq \sqrt{\log_2 x} \) by property (i) from Proposition 3.1, it follows from Corollary 2.4 that
\[
\prod_{p \mid \sigma(m), p \equiv 3 \pmod{4}} \left( 1 + \frac{1}{p} \right) \gg \frac{\sqrt{\log_3 x}}{x}.
\]

By simply ignoring the product over \( p \mid s(m) \), we obtain a lower bound of the form
\[
\sum_{m \leq x^{1/500}} \sum_{x^{1/2} < P \leq x/m, p \mid P \alpha(m) + \beta(m) \implies p \equiv 1 \pmod{4}} 1 \sim \frac{x^{3/2}}{(\log x)^{3/2}} \sum_{m \leq x^{1/500}} \frac{1}{m}.
\]

(14)

For the sum, note that
\[
\sum_{m \leq x^{1/500}} \frac{1}{m} \gg \sum_{m \leq x^{1/500}} \frac{1}{m} - \sum_{m \leq x^{1/500}} \frac{1}{m}.
\]
where the $\flat$ symbol indicates that the sum is over those $m = m_1m_2$ satisfying only the decomposition conditions in the bulleted list above, and the $\sharp$ symbol indicates a sum over those $m$ (satisfying the decomposition conditions) for which:

(a) there exists a prime $q \mid (m, \sigma(m))$, $q \equiv 3 \pmod{4}$;
(b) $m$ fails condition (i) from Proposition 3.1 or
(c) $m$ fails to satisfy the bound (9).

We bound the $\sharp$ sum from above, to show we are not subtracting too much. We deal first with condition (a). If a prime $q \equiv 3 \pmod{4}$ divides $(m, \sigma(m))$, then $q \mid m_1^2$, and hence $m \geq q > (\log x)^{10}$. For $t \geq (\log x)^{10}$, let $A(t)$ denote the number of $m \leq t$ such that there exists such a prime $q \mid (m, \sigma(m))$. From the proof of Lemma 2.2 of [11], $A(t) \ll t/(\log x)^{4}$.

An application of partial summation and integration by parts yields

$$\sum_{(m, \sigma(m))} \frac{1}{m} \ll \int_{(\log x)^{10}}^{x^{1/500}} \frac{A(t)}{t^2} \, dt \ll \frac{1}{(\log x)^{4}} \int_{(\log x)^{10}}^{x^{1/500}} \frac{1}{t} \, dt \ll \frac{\log x}{(\log x)^{4}}.$$

Now suppose $m$ falls into group (b), but not (c). As before, we set $L = \lceil \log x \rceil$ and consider $\lambda \in \mathcal{L} = \{1/L, \ldots, (L - 1)/L\}$. The number of $m$ in the interval $(x^\lambda, ex^\lambda]$ for which one of the properties (i) – (iii) fails is $O(x^\lambda/(\log x)^{4})$ (cf. equation (11)), and hence, summing over $m$ satisfying (b) but not (c) yields (with $y = \exp(\log x/(\log x)^{4})$)

$$\sum_{y \leq m \leq x^{1/500}} \frac{1}{m} \ll \sum_{\lambda \in \mathcal{L}} \sum_{x^\lambda \leq m \leq ex^\lambda} \frac{1}{m} \ll \sum_{\lambda \in \mathcal{L}} \frac{x^\lambda}{(\log x)^{4}} \ll \frac{\log x}{(\log x)^4}.$$

Finally, if $m$ is counted by (c), we have the easy bound

$$\sum_{m \leq \exp(\log x/(\log x)^4)} \frac{1}{m} \ll \frac{\log x}{(\log x)^{4}}.$$

Thus, we have shown that

$$\sum_{m \leq x^{1/500}} \frac{\sharp}{m} \ll \frac{\log x}{(\log x)^{4}}.$$

This upper bound, combined with the factor of $x(\log x)^{1/2}/(\log x)^{3/2}$ from [11], results in a term of the shape

$$x(\log x)^{1/2} \int_{x^{1/500}}^{x} \frac{\sharp}{m} \ll x(\log x)^{1/2} \int_{x^{1/500}}^{x} \frac{\log x}{(\log x)^{3/2}} = o\left(\frac{x}{(\log x)^{1/2}}\right),$$

which is negligible.

It remains to bound the quantity

$$\frac{x(\log x)^{1/2}}{(\log x)^{3/2}} \sum_{m \leq x^{1/500}} \frac{1}{m}.$$
from below. It is clear that
\[ \sum_{m \leq x^{1/500}} \frac{1}{m} \gg \left( \sum_{m_1 \leq x^{1/1000} \text{ squarefree}} \frac{1}{m_1} \right) \left( \sum_{m_2 \leq x^{1/1000}} \frac{1}{m_2} \right). \]

We first consider the sum over \( m_1 \). Notice that
\[ \prod_{p \leq \sqrt{\log_2 x} \atop p \equiv 1 \pmod{4}} p = \exp \left( \sum_{p \leq \sqrt{\log_2 x} \atop p \equiv 1 \pmod{4}} \log p \right) \leq \exp(c \sqrt{\log_2 x}) < x^{1/1000} \]
for some absolute constant \( c > 0 \) and sufficiently large \( x \). Therefore, the sum over \( m_1 \) is
\[ \sum_{m_1 \leq x^{1/1000} \text{ squarefree}} \frac{1}{m_1} = \prod_{p \leq \sqrt{\log_2 x} \atop p \equiv 1 \pmod{4}} \left( 1 + \frac{1}{p} \right) \asymp (\log_3 x)^{1/2}, \]
again by Corollary 2.4. For the sum over \( m_2 \), a quick application of Brun’s sieve shows that
\[ \# \{ m_2 \leq z : p | m_2 \implies p \in [ (\log_2 x)^{10}, z], m_2 \equiv 3 \pmod{4} \} \gg \frac{z}{\log_3 x} \]
for all \( z \geq \log x \), say. Then, by partial summation,
\[ \sum_{m_2 \leq x^{1/1000}} \frac{1}{m_2} \gg \frac{\log x}{\log_3 x}. \]
Combining all of our estimates, we obtain
\[ \frac{x(\log_3 x)^{1/2}}{(\log x)^{3/2}} \sum_{m \leq x^{1/500}} \frac{1}{m} \gg \frac{x(\log_3 x)^{1/2}}{(\log x)^{3/2}} \cdot (\log_3 x)^{1/2} \cdot \frac{\log x}{\log_3 x} = \frac{x}{\sqrt{\log x}}, \]
as desired.

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