Quantum error correction for non-maximally entangled states

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Quantum states have high affinity for errors and hence error correction is of utmost importance to realise a quantum computer. Laflamme showed that 5 qubits are necessary to correct a single error on a qubit. In a Pauli error model, four different types of errors can occur on a qubit. Maximally entangled states are orthogonal to each other and hence can be uniquely distinguished by a measurement in the Bell basis. Thus a measurement in Bell basis and a unitary transformation is sufficient to correct error in Bell states. However, such a measurement is not possible for non-maximally entangled states. In this work we show that the 16 possible errors for a non-maximally entangled two qubit system map to only 8 distinct error states. Hence, it is possible to correct the error without perfect knowledge of the type of error. Furthermore, we show that the possible errors can be grouped in such a way that all 4 errors can occur on one qubit, whereas only bit flip error can occur on the second qubit. As a consequence, instead of 10, only 8 qubits are sufficient to correct a single error. We propose an 8-qubit error correcting code to correct a single error in a non-maximally entangled state. We further argue that for an n-qubit non-maximally entangled state of the form $\alpha |0^n\rangle + \beta |1^n\rangle$, it is always possible to correct a single error with fewer than 5n qubits, in fact only $3n + 2$ qubits suffice.

I. INTRODUCTION

Certain problems have been shown to be solvable exponentially faster by a quantum computer than its classical counterpart. These include the factorization problem which has no known classical non-exponential algorithm. Shor showed that it is solvable in polynomial time by a quantum computer [10]. However, the main hindrance in the realization of a quantum computer is the affinity of a qubit for errors. It was Shor who showed that it is possible to correct a single error in a qubit [9] by distributing the information of a single qubit into 9 qubits. Several other studies has been performed on quantum error correction since then [1, 5, 7, 11]. Laflamme showed that 5 qubits are necessary to correct a single error in a qubit [7]. The error model considered in all of these literature are Pauli error model [8].

Entanglement is at the heart of quantum information theory. Protocols like Super-Dense Coding [3], Quantum Teleportation [2], Entanglement Assisted QKD [4] make use of quantum entanglement to achieve results superior to that of its classical counterpart. For a separable bipartite state of the form $|\psi_A\rangle \otimes |\psi_B\rangle$, from [2], it is evident that 10 qubits are necessary for error correction. However, we show that if the state is entangled, then the number of qubits required for error correction can be reduced - no encoding is required for error correction in maximally-entangled states and only 8 qubits are necessary for error correction in non-maximally entangled states. We argue that to correct an error in a non-maximally entangled state, perfect knowledge of the type of error is not necessary. We have proposed an 8-qubit quantum error correcting code to correct a single error in a non-maximally entangled state. Finally, we show that a single error in an n-partite entangled state can be corrected using $3n + 2$ qubits instead of the $5n$ qubits necessary for product states.

The rest of the paper is organized as follows - in Section II, we show that no encoding is required for maximally entangled states to correct a single error. In Section III, we argue that it is possible to correct a single error in a non-maximally entangled state with 8 qubits only and without perfect knowledge of the error type. We propose an 8-qubit error correcting code for non-maximally entangled 2 qubit states in Section IV. In section V, we argue further that for any multiparticle entangled state of the form $\alpha |0^n\rangle + \beta |1^n\rangle$, it is always possible to correct errors with less than 5n qubits, in fact only $3n + 2$ qubits. We conclude in Section V.

II. NO ENCODING REQUIRED FOR PURE MAXIMALLY ENTANGLED STATES

The notion of entanglement and the ability to determine that some states are more entangled than others leads to maximally entangled states. For a 2-qubit quantum system, the maximally entangled states are

$$|\psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$
$$|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$
$$|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$
$$|\phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$

These states, called Bell States, form a basis of the 4-dimensional Hilbert space. In Table III we show that the 16 possible types of error map the maximally entangled state $|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ to only 4 distinct error states up to an irrelevant global phase. Furthermore, these error
TABLE I. Mapping of errors in Maximally-entangled states

| Error free state | Type of error | Error state |
|------------------|---------------|-------------|
| $\frac{1}{\sqrt{2}}(00 + |11\rangle)$ | $I \otimes I$ | $|\alpha\rangle + \beta|11\rangle$ |
| | $\sigma_x \otimes \sigma_x$ | $|\alpha\rangle + \beta|00\rangle$ |
| | $\sigma_x \otimes \sigma_z$ | $|\alpha\rangle + \beta|10\rangle$ |
| | $\sigma_y \otimes \sigma_z$ | $|\alpha\rangle - \beta|01\rangle$ |
| $\frac{1}{\sqrt{2}}(00 - |11\rangle)$ | $I \otimes I$ | $|\alpha\rangle - \beta|11\rangle$ |
| | $\sigma_x \otimes \sigma_y$ | $|\alpha\rangle - \beta|00\rangle$ |
| | $\sigma_z \otimes \sigma_z$ | $|\alpha\rangle - \beta|01\rangle$ |
| | $\sigma_y \otimes \sigma_x$ | $|\alpha\rangle + \beta|10\rangle$ |

III. 8 QUBITS ARE SUFFICIENT TO CORRECT ERRORS IN NON-MAXIMALLY ENTANGLED STATES

A non-maximally entangled state can be represented mathematically as $\alpha|00\rangle \pm \beta|11\rangle$ or $|01\rangle \pm \beta|10\rangle$, where $\alpha \neq \beta$. These states are not orthogonal and hence do not form a basis. In Table III, we show that the 16 possible types of errors map an arbitrary non-maximally entangled two qubit state $|\psi\rangle = \alpha|00\rangle + \beta|11\rangle$ into 8 distinct error states up to a global phase.

From Table III, we see that the error states are not orthogonal and hence a measurement in a basis is not possible. However, similar to maximally-entangled state, it is not necessary to uniquely identify which error has occurred to correct the error. Furthermore, in Table III, we choose 8 errors of the 16 different errors from Table II which map the error free state into unique error states. Since it is not necessary to uniquely identify which error has occurred, one can consider the error to be any one of the 8 errors in Table III even when some other pattern has occurred - for example, since $\sigma_x \otimes I$ and $\sigma_y \otimes \sigma_z$ map the original state to the same error state, one can consider the error to be $\sigma_x \otimes I$ even if the actual error was $\sigma_y \otimes \sigma_z$. Error correction will be independent of the choice between these two types of error.

Table III covers all the unique error states. However, it is noticeable that the errors have been so chosen that only $I$ and $\sigma_z$ occur on the first qubit, whereas all four errors occur on the second qubit. Since any of the four types of errors are possible on the second qubit, 5 qubits are necessary to correct errors in this qubit. However, only bit-flip error ($\sigma_z$) is possible in the first qubit. So a repetitive code with 3 qubits is sufficient to correct errors in this qubit. Hence, though a non-maximally entangled state is a two qubit state, 10 qubits are not necessary for error correction. An 8-qubit code can correct a single error in a non-maximally entangled state.

In the next section, we provide an 8-qubit code for correcting a single error in a non-maximally entangled state.

IV. 8-QUBIT ERROR CORRECTING CODE FOR NON-MAXIMALLY ENTANGLED STATE

We consider the density matrix notation of a pure non-maximally entangled state of the form

$$
\rho_{AB} = (\alpha|00\rangle + \beta|11\rangle)(\alpha^*|00\rangle + \beta^*|11\rangle) = |
\alpha|^2|00\rangle \langle 00| + \alpha^* \beta|11\rangle \langle 00| + \alpha \beta^*|00\rangle \langle 11| + |\beta|^2|11\rangle \langle 11|
$$
We can obtain the density matrices of the individual states $\rho_A$ and $\rho_B$ by taking the partial trace over the density matrix of the bipartite system.

$$\rho_A = Tr_B(\rho_{AB}) = |\alpha|^2 |0\rangle \langle 0| + |\beta|^2 |1\rangle \langle 1|$$  \hspace{1cm} (1)

$$\rho_B = Tr_A(\rho_{AB}) = |\alpha|^2 |0\rangle \langle 0| + |\beta|^2 |1\rangle \langle 1|$$  \hspace{1cm} (2)

5 qubits are required to encode the state $\rho_B$, whereas 3 qubits are sufficient to encode the state $\rho_A$ for error correction.

Let us first consider the encoding of $\rho_A$. For a pure state $\alpha|0\rangle + \beta|1\rangle$, to correct only bit flip error, the following encoding with 3 qubits is necessary [5]

$$\alpha|000\rangle + \beta|111\rangle$$  \hspace{1cm} (3)

Let us consider that there is an operator $U$ which acts on the basis states as

$$U|0\rangle = |000\rangle$$
$$U|1\rangle = |111\rangle$$

Using this operator on the state $\rho_A$, we have

$$U \rho_A U^\dagger = |\alpha|^2 U|0\rangle \langle 0| U^\dagger + |\beta|^2 U|1\rangle \langle 1| U^\dagger$$

$$= |\alpha|^2 |0_L\rangle \langle 0_L| + |\beta|^2 |1_L\rangle \langle 1_L|$$  \hspace{1cm} (4)

where $|0_L\rangle = |000\rangle$ and $|1_L\rangle = |111\rangle$.

The mixed state $\rho_B$ can be similarly encoded using 5 qubits for error correction using the scheme of [3]. Using the error correcting code from [7], a quantum state $\alpha|0\rangle + \beta|1\rangle$ can be encoded as $\alpha|0_L\rangle + \beta|1_L\rangle$ where

$$|0_L\rangle = |00000\rangle + |11110\rangle - |10011\rangle - |01111\rangle + |11010\rangle + |00110\rangle + |01010\rangle + |10110\rangle$$

$$|1_L\rangle = - |00011\rangle - |11111\rangle - |10000\rangle + |01100\rangle + |11001\rangle - |00101\rangle - |01011\rangle + |10111\rangle$$

So if we consider an operator $V$ such that

$$V|0\rangle = |0_L\rangle$$
$$V|1\rangle = |1_L\rangle$$

then this operator acts on the state $\rho_A$ as follows

$$V \rho_B V^\dagger = |\alpha|^2 V|0\rangle \langle 0| V^\dagger + |\beta|^2 V|1\rangle \langle 1| V^\dagger$$

$$= |\alpha|^2 |0_L\rangle \langle 0_L| + |\beta|^2 |1_L\rangle \langle 1_L|$$  \hspace{1cm} (5)

Since the original two-qubit state $\rho_{AB}$ was a pure entangled state, it is possible to recover the state $\rho_{AB}$ from its component states $\rho_A$ and $\rho_B$ [8]. The original density matrix of interest

$$\rho_{AB}_{Encoded} = |\alpha|^2 \tilde{|0\rangle}_L \langle 0\rangle_L + \alpha^* \beta \tilde{|1\rangle}_L \langle 1\rangle_L + \alpha \beta^* \tilde{|0\rangle}_L \langle 1\rangle_L + |\beta|^2 \tilde{|1\rangle}_L \langle 0\rangle_L$$  \hspace{1cm} (6)

Hence, the state vector is

$$|\psi_{AB}\rangle_{Encoded} = \alpha \tilde{|0\rangle}_L \langle 0\rangle_L + \beta \tilde{|1\rangle}_L \langle 1\rangle_L$$  \hspace{1cm} (7)

This 8 qubit code can correct a single error in a maximally entangled state. Not only does it require two qubits less than what is expected for a bipartite state, it can also correct a single error without perfect knowledge of the error. Furthermore, since all the possible errors can occur on one state only, while only bit flip error is possible in the other qubits, if the two qubits are in a distant lab, only one of the two parties must have necessary equipment to correct all the four errors. It will be sufficient if the other party has enough equipment to correct a bit flip error only.

V. ERROR CORRECTION IN NON-MAXIMALLY ENTANGLED N-QUBIT SYSTEM

For more than 2 qubit states, the GHZ state is an entangled state [9]. For a system of $n$ qubits, the GHZ state can be mathematically denoted as

$$|GHZ\rangle = \frac{|0\rangle^\otimes n + |1\rangle^\otimes n}{\sqrt{2}}$$  \hspace{1cm} (8)

In accordance to the 2-dimensional case, we first consider a generalized version of the $|GHZ\rangle$ state where all the coefficients are equal. Such a state can be denoted as

$$|GHZ'\rangle = \alpha |0\rangle^\otimes n + \beta |1\rangle^\otimes n$$  \hspace{1cm} (9)

with $\alpha \neq \beta$. Since $|GHZ'\rangle$ is a superposition state, any operation on the state will affect both $|0\rangle^\otimes n$ and $|1\rangle^\otimes n$ equally. Hence, counting the number of possible non-identical states for only one of them is sufficient. Since this state is an $n$-qubit state and each of the qubits can take one of the two values (0 or 1), there are $2^n$ possible states. Furthermore, considering the phase, the sign of $\alpha$ and $\beta$ can be either +1 or −1. So the total number of possible non-identical states are $2 \times 2^n = 2^{n+1}$.

Since, for a Pauli error model, 4 different types of errors are possible in each qubit, for an $n$-qubit state, total number of possible errors is $4^n$. For $n > 1$, $4^n > 2^{n+1}$.

For the $|GHZ\rangle$ state, $\alpha = \beta = \frac{1}{\sqrt{2}}$. Since the coefficients are equal, half of the possible $2^{n+1}$ error states are identical. So the number of non-identical error states will be $\frac{2^{n+1}}{2} = 2^n$. Since, an $n$-dimensional
quantum state resides in a $2^n$-dimensional Hilbert Space, it is possible to find $2^n$ basis vectors. Hence, each of the error states for $|GHZ\rangle$ states complies to one of the $2^n$ orthogonal vectors of the Hilbert Space. So a measurement in that basis will be able to uniquely identify the error states, thus removing the necessity of encoding the state for error correction. This scenario reduces to the error states of Table I for 2-dimension.

We now show that the number of qubits required for an $n$-partite entangled state of the form $|GHZ\rangle$ is $5 + 3(n-1)$. For a single qubit, when $n = 1$, this case reduces to [7]. We have shown it to be true for $n = 2$ in Section III. In a two qubit non-maximally entangled state, five qubits are required to correct error in the second qubit while only 3 qubits are required to correct errors in the first qubit. Let us assume that it is true for some $m$-partite entanglement of the form $|GHZ\rangle$ for $m \geq 2$, i.e., 5 qubits are necessary for error correction of the $m$-th qubit while only 3 qubits are required to correct errors in each of the previous $m-1$ qubits. The entangled state is of the form

$$\alpha |i_1i_2\ldots i_m\rangle + \beta |\bar{i}_1\bar{i}_2\ldots \bar{i}_m\rangle$$  \hspace{1cm} (10)

where $\langle \bar{i}_j | i_j \rangle = 0 \ \forall \ j$. From our assumption, all 4 types of errors can occur on the $m$-th qubit and hence it requires five qubit for the error correction of this qubit. Now, to consider $m+1$-partite entanglement, it is not necessary to look at the first $m-1$ qubit of the state in Equation 10. We need to show that the $m$-th qubit will now require only 3 qubits for error correction while the $m+1$-th qubit will require 5 qubits.

Consider $\sigma_z$ error in the $m$-th qubit. Because of the form of an entangled state (Equation 10), this error will have similar effect on the state even if it occurs on the $m+1$-th qubit instead of the $m$-th qubit, i.e. $\sigma_z$ error in any one of these two qubits produce the same error state. So, to correct this error, it is not necessary to know whether this error occurred on the $m$-th qubit or $m+1$-th qubit. Hence, we can consider that the $\sigma_z$ error has occurred on the $m+1$-th qubit and not on the $m$-th qubit without hampering the error correction procedure.

However, the effect of $\sigma_x$ on the $m$-th qubit and the $m+1$-th qubit does not produce the same error state. So it is not possible to replace $\sigma_x$ error on the $m$-th qubit with any error on the $m+1$-th qubit. Nevertheless, $\sigma_y = \sigma_z\sigma_x$ (upto a global phase). So, for this type of error, from the previous argument, considering that the $\sigma_x$ error occurred on the $m$-th qubit and the $\sigma_z$ error occurred on the $m+1$-th qubit will keep the error state unchanged. Eventually, summing up the arguments, it is possible to consider that only $\sigma_z$ occurred on the $m$-th qubit while all four types of errors occurred on the $m+1$-th qubit. This consideration keeps the error state unchanged and hence has no effect on the error correction procedure. So, the assumption is true for $m+1$-partite entanglement when it is true for $m$-partite entanglement. Hence, for any $n$-qubit entanglement, the number of qubits required for error correction is $5 + 3(n-1) = 3n + 2$.

VI. CONCLUSION

In this paper, we have shown that though 10 qubits are necessary to correct errors in a bipartite system, a maximally entangled state need not be encoded to correct a single error, while a non-maximally entangled state requires only 8-qubits for error correction. We have also shown that any $n$-dimensional entangled state of the form $|GHZ\rangle$ will require $5 + 3(n-1)$ qubits for error correction. This follows because the information content of an entangled state is less than that of a product state. However, it shows that entangled states, which are the necessary building blocks for many quantum protocols, can be protected from errors more efficiently than product states. For large values of $n$, the savings in the number of qubits required for error correction in entangled state approaches 40% as compared to product states. Furthermore, starting with an $n$-qubit non-maximally entangled state, we have obtained that $n-1$ qubits behave classically, since only bit flip error is possible in each of them. Only the last qubit shows the quantumness of phase flip or arbitrary rotation. Hence, a classical structure emerges from a purely quantum system.

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