A Simplified Binet Formula for $k$-Generalized Fibonacci Numbers

Gregory P. B. Dresden
Department of Mathematics
Washington and Lee University
Lexington, VA 24450
dresdeng@wlu.edu

Abstract

In this paper, we present a particularly nice Binet-style formula that can be used to produce the $k$-generalized Fibonacci numbers (that is, the Tribonacci, Tetranacci, etc.). Furthermore, we show that in fact one needs only take the integer closest to the first term of this Binet-style formula in order to generate the desired sequence.

1 Introduction

Let $k \geq 2$ and define $F_n^{(k)}$, the $n$th $k$-generalized Fibonacci number, to satisfy the recurrence relation

$$F_n^{(k)} = F_{n-1}^{(k)} + F_{n-2}^{(k)} + \cdots + F_{n-k}^{(k)} \quad \text{($k$ terms)}$$

... and with initial conditions $0, 0, \ldots, 0, 1$ ($k$ terms) such that the first non-zero term is $F_1^{(k)} = 1$.

These numbers are also called the Fibonacci $k$-step numbers, Fibonacci $k$-sequences, or $k$-bonacci numbers. Note that for $k = 2$, we have $F_n^{(2)} = F_n$, our familiar Fibonacci numbers. For $k = 3$ we have the so-called Tribonacci (sequence number A000073 in Sloane’s Encyclopedia of Integer Sequences), followed by the Tetranacci (A000078) for $k = 4$, and so on. According to Kessler and Schiff [6], these numbers also appear in probability theory and in certain sorting algorithms. We present here a chart of these numbers for the first few values of $k$:

| $k$ | name      | i.c. | first few non-zero terms |
|-----|-----------|------|--------------------------|
| 2   | Fibonacci | 0, 1 | 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots |
| 3   | Tribonacci| 0, 0, 1 | 1, 1, 2, 4, 7, 13, 24, 44, 81, \ldots |
| 4   | Tetranacci| 0, 0, 0, 1 | 1, 1, 2, 4, 8, 15, 29, 56, 108, \ldots |
| 5   | Pentanacci| 0, 0, 0, 0, 1 | 1, 1, 2, 4, 8, 16, 31, 61, 120, \ldots |

We remind the reader of the famous Binet formula (also known as the de Moivre formula)
that can be used to calculate $F_n$, the Fibonacci numbers:

$$F_n = \frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right]$$

$$= \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

...for $\alpha > \beta$ the two roots of $x^2 - x - 1 = 0$. For our purposes, it is convenient (and not particularly difficult) to rewrite this formula as follows:

$$F_n = \frac{\alpha - 1}{2 + 3(\alpha - 2)} \alpha^{n-1} + \frac{\beta - 1}{2 + 3(\beta - 2)} \beta^{n-1}$$

(1)

We leave the details to the reader.

Our first (and very minor) result is the following representation of $F_n^{(k)}$:

**Theorem 1.** For $F_n^{(k)}$ the $n^{th}$ $k$-generalized Fibonacci number, then

$$F_n^{(k)} = \sum_{i=1}^{k} \frac{\alpha_i - 1}{2 + (k + 1)(\alpha_i - 2)} \alpha_i^{n-1}$$

(2)

for $\alpha_1, \ldots, \alpha_k$ the roots of $x^k - x^{k-1} - \cdots - 1 = 0$.

This is a new presentation, but hardly a new result. There are many other ways of representing these $k$-generalized Fibonacci numbers, as seen in the articles [2], [3], [4], [5], [7], [8], [9]. Our equation (2) of Theorem 1 is perhaps slightly easier to understand, and it also allows us to do some analysis (as seen below). We point out that for $k = 2$, equation (2) reduces to the variant of the Binet formula (for the standard Fibonacci numbers) from equation (1).

As shown in three distinct proofs ([9], [10], and [13]), the equation $x^k - x^{k-1} - \cdots - 1 = 0$ from Theorem 1 has just one root $\alpha$ such that $|\alpha| > 1$, and the other roots are strictly inside the unit circle. We can conclude that the contribution of the other roots in formula 2 will quickly become trivial, and thus:

$$F_n^{(k)} \approx \frac{\alpha - 1}{2 + (k + 1)(\alpha - 2)} \alpha^{n-1} \quad \text{... for } n \text{ sufficiently large.}$$

(3)

It’s well known that for the Fibonacci sequence $F_n^{(2)} = F_n$, the “sufficiently large” $n$ in equation (3) is $n = 0$, as shown here:

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|-----|---|---|---|---|---|---|---|
| $F_n$ | 0 | 1 | 1 | 2 | 3 | 5 | 8 |

| $\frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^n$ | 0.447 | 0.724 | 1.171 | 1.894 | 3.065 | 4.960 | 8.025 |
| error | .447 | .277 | .171 | .106 | .065 | .040 | .025 |
It is perhaps surprising to discover that a similar statement holds for all the \( k \)-generalized Fibonacci numbers. Our main result is the following:

**Theorem 2.** For \( F_n^{(k)} \) the \( n \)th \( k \)-generalized Fibonacci number, then

\[
F_n^{(k)} = \text{Round} \left[ \frac{\alpha - 1}{2 + (k + 1)(\alpha - 2)} \alpha^{n-1} \right]
\]

for all \( n \geq 2 - k \) and for \( \alpha \) the unique positive root of \( x^k - x^{k-1} - \cdots - 1 = 0 \).

We point out that this theorem is not as trivial as one might think. Note the error for \( k = 6 \), as seen in the following chart; it is not monotone decreasing.

| \( n \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|-------|---|---|---|---|---|---|---|---|
| \( F_n^{(6)} \) | 0 | 1 | 1 | 2 | 4 | 8 | 16 | 32 |
| \( \frac{\alpha - 1}{2 + 7(\alpha - 2)} \alpha^n \) | 0.263 | 0.522 | 1.035 | 2.053 | 4.072 | 8.078 | 16.023 | 31.782 |
| | .263 | .478 | .035 | .053 | .072 | .078 | .023 | .218 |

We also point out that not every recurrence sequence admits such a nice formula as seen in Theorem 2. Consider, for example, the scaled Fibonacci sequence \( 10, 10, 20, 30, 50, 80, \ldots \), which has Binet formula:

\[
F_n^{(3)} = \text{Round} \left[ \frac{10}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^n - \frac{10}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^n \right].
\]

This can be written as \( \text{Round} \left[ \frac{10}{\sqrt{5}} \left( \frac{1 \pm \sqrt{5}}{2} \right)^n \right] \), but only for \( n \geq 5 \). As another example, the sequence \( 1, 2, 8, 24, 80, \ldots \) (defined by \( G_n = 2G_{n-1} + 4G_{n-2} \)) can be written as

\[
G_n = \frac{(1 + \sqrt{5})^n}{2\sqrt{5}} - \frac{(1 - \sqrt{5})^n}{2\sqrt{5}},
\]

but because both \( 1 + \sqrt{5} \) and \( 1 - \sqrt{5} \) have absolute value greater than 1, then it would be impossible to express \( G_n \) in terms of just one of these two numbers.

## 2 Previous Results

We point out that for \( k = 3 \) (the Tribonacci numbers), our Theorem 2 was found earlier by Spickerman [11]. His formula (modified slightly to match our notation) reads as follows, where \( \alpha \) is the real root, and \( \sigma \) and \( \overline{\sigma} \) are the two complex roots, of \( x^3 - x^2 - x - 1 = 0 \):

\[
F_n^{(3)} = \text{Round} \left[ \frac{\alpha^2}{(\alpha - \sigma)(\alpha - \overline{\sigma})} \alpha^{n-1} \right] \tag{4}
\]

It is not hard to show that for \( k = 3 \), our coefficient \( \frac{\alpha - 1}{2 + 6(\alpha - 2)} \) from Theorem 2 is equal to Spickerman’s coefficient \( \frac{\alpha^2}{(\alpha - \sigma)(\alpha - \overline{\sigma})} \). We leave the details to the reader.
In a subsequent article [12], Spickerman and Joyner developed a more complex version of our Theorem 1 to represent the generalized Fibonacci numbers. Using our notation, and with \( \{\alpha_i\} \) the set of roots of \( x^k - x^{k-1} - \cdots - 1 = 0 \), their formula reads

\[
F_n^{(k)} = \sum_{i=1}^{k} \frac{\alpha_{i+1}^k - \alpha_i^k}{2\alpha_i^k - (k + 1)} \alpha_i^{n-1} \tag{5}
\]

It is surprising that even after calculating out the appropriate constants in their equation (5) for \( 2 \leq k \leq 10 \), neither Spickerman nor Joyner noted that they could have simply taken the first term in equation (5) for all \( n \geq 0 \), as Spickerman did in equation (4) for \( k = 3 \).

The Spickerman-Joyner formula (5) was extended by Wolfram [13] to the case with arbitrary starting conditions (rather than the initial sequence 0, 0, \ldots, 0, 1). In the next section we will show that our formula (2) in Theorem 1 is equivalent to the Spickerman-Joyner formula given above (and thus is a special case of Wolfram’s formula).

Finally, we note that the polynomials \( x^k - x^{k-1} - \cdots - 1 \) in Theorem 1 have been studied rather extensively. They are irreducible polynomials with just one zero outside the unit circle. That single zero is located between \( 2(1 - 2^{-k}) \) and 2 (as seen in Wolfram’s article [13]; Miles [9] gave earlier and less precise results). It is also known [13, Lemma 3.11] that the polynomials have Galois group \( S_k \) for \( k \leq 11 \); in particular, their zeros can not be expressed in radicals for \( 5 \leq k \leq 11 \). Wolfram conjectured that the Galois group is always \( S_k \). Cipu and Luca [1] were able to show that the Galois group is not contained in the alternating group \( A_k \), and for \( k \geq 3 \) it is not 2-nilpotent. They point out that this means the zeros of the polynomials \( x^k - x^{k-1} - \cdots - 1 \) for \( k \geq 3 \) can not be constructed by ruler and compass, but the question of whether they are expressible using radicals remains open.

## 3 Preliminary Lemmas

First, a few statements about the number \( \alpha \).

**Lemma 3.** Let \( \alpha > 1 \) be the real positive root of \( x^k - x^{k-1} - \cdots - x - 1 = 0 \). Then,

\[
2 \frac{1}{k} < \alpha < 2
\]

(6)

In addition,

\[
2 \frac{1}{3k} < \alpha < 2 \quad \text{for } k \geq 4
\]

(7)

**Proof.** We begin by computing the following chart for \( k \leq 5 \):

| \( k \) | \( 2 - \frac{1}{k} \) | \( 2 - \frac{1}{3k} \) | \( \alpha \) |
|---|---|---|---|
| 2 | 1.5 | 1.833\ldots | 1.618\ldots |
| 3 | 1.666\ldots | 1.889\ldots | 1.839\ldots |
| 4 | 1.75 | 1.916\ldots | 1.928\ldots |
| 5 | 1.8 | 1.933\ldots | 1.966\ldots |
It’s clear that \(2 - \frac{1}{k} < \alpha < 2\) for \(2 \leq k \leq 5\) and that \(2 - \frac{1}{3k} < \alpha < 2\) for \(4 \leq k \leq 5\). We now focus on \(k \geq 6\). At this point, we could finish the proof by appealing to \(2(1 - 2^{-k}) < \alpha < 2\) as seen in the article [13, Lemma 3.6], but we present here a simpler proof.

Let \(f(x) = (x - 1)(x^k - x^{k-1} - \ldots - x - 1) = x^{k+1} - 2x^k + 1\). We know from our earlier discussion that \(f(x)\) has one real zero \(\alpha > 1\). Writing \(f(x)\) as \(x^k(x - 2) + 1\), we have

\[
f \left(2 - \frac{1}{3k}\right) = \left(2 - \frac{1}{3k}\right)^k \left(-\frac{1}{3k}\right) + 1 \quad (8)
\]

For \(k \geq 6\), it’s easy to show

\[
3k < \left(\frac{5}{3}\right)^k = \left(2 - \frac{1}{3}\right)^k < \left(2 - \frac{1}{3k}\right)^k
\]

Substituting this inequality into the right-hand side of (8), we can re-write (8) as:

\[
f \left(2 - \frac{1}{3k}\right) < (3k) \cdot \left(-\frac{1}{3k}\right) + 1 = 0.
\]

Finally, we note that

\[
f(2) = 2^{k+1} - 2 \cdot 2^k + 1 = 1 > 0,
\]

so we can conclude that our root \(\alpha\) is within the desired bounds of \(2 - 1/3k\) and 2 for \(k \geq 6\).

We now have a lemma about the coefficients of \(\alpha^{n-1}\) in Theorems 1 and 2.

**Lemma 4.** Let \(k \geq 2\) be an integer, and let \(m^{(k)}(x) = \frac{x - 1}{2 + (k + 1)(x - 2)}\). Then,

1. \(m^{(k)}(2 - 1/k) = 1\).
2. \(m^{(k)}(2) = \frac{1}{2}\).
3. \(m^{(k)}(x)\) is continuous and decreasing on the interval \([2 - 1/k, \infty)\).
4. \(m^{(k)}(x) > \frac{1}{x}\) on the interval \((2 - 1/k, 2)\).

**Proof.** Parts 1 and 2 are immediate. As for 3, note that we can rewrite \(m^{(k)}(x)\) as:

\[
m^{(k)}(x) = \frac{1}{k + 1} \left[1 + \frac{1 - \frac{2}{k+1}}{x - (2 - \frac{2}{k+1})}\right]
\]

which is simply a scaled translation of the map \(y = 1/x\). In particular, since this \(m^{(k)}(x)\) has a vertical asymptote at \(x = 2 - \frac{2}{k+1}\), then by parts 1 and 2 we can conclude that \(m^{(k)}(x)\) is indeed continuous and decreasing on the desired interval.

To show part 4, we first note that in solving \(\frac{1}{x} = m^{(k)}(x)\), we obtain a quadratic equation with the two intersection points \(x = 2\) and \(x = k\). It’s easy to show that \(\frac{1}{x} < m^{(k)}(x)\) at \(x = 2 - 1/k\), and since both functions \(\frac{1}{x}\) and \(m^{(k)}(x)\) are continuous on the interval \([2 - 1/k, \infty)\) and intersect only at \(x = 2\) and \(x = k \geq 2\), we can conclude that \(\frac{1}{x} < m^{(k)}(x)\) on the desired interval.
Lemma 5. For a fixed value of $k \geq 2$ and for $n \geq 2 - k$, define $E_n$ to be the error in our Binet approximation of Theorem 2, as follows:

$$E_n = F_n^{(k)} - \frac{\alpha - 1}{2 + (k + 1)(\alpha - 2)} \cdot \alpha^{n-1}$$

$$= F_n^{(k)} - m^{(k)}(\alpha) \cdot \alpha^{n-1},$$

... for $\alpha$ the positive real root of $x^k - x^{k-1} - \cdots - x - 1 = 0$ and $m^{(k)}$ as defined in Lemma 4. Then, $E_n$ satisfies the same recurrence relation as $F_n^{(k)}$:

$$E_n = E_{n-1} + E_{n-2} + \cdots + E_{n-k} \quad \text{ (for } n \geq 2)$$

Proof. By definition, we know that $F_n^{(k)}$ satisfies the recurrence relation:

$$F_n^{(k)} = F_{n-1}^{(k)} + \cdots + F_{n-k}^{(k)} \quad (9)$$

As for the term $m^{(k)}(\alpha) \cdot \alpha^{n-1}$, note that $\alpha$ is a root of $x^k - x^{k-1} - \cdots - 1 = 0$, which means that $\alpha^k = \alpha^{k-1} + \cdots + 1$, which implies

$$m^{(k)}(\alpha) \cdot \alpha^{n-1} = m^{(k)}(\alpha)\alpha^{n-2} + \cdots + m^{(k)}(\alpha)\alpha^{n-(k+1)} \quad (10)$$

We combine Equations (9) and (10) to obtain the desired result. □

4 Proof of Theorem 1

As mentioned above, Spickerman and Joyner [12] proved the following formula for the $k$-generalized Fibonacci numbers:

$$F_n^{(k)} = \sum_{i=1}^{k} \frac{\alpha_i^{k+1} - \alpha_i^k}{2\alpha_i^k - (k + 1)} \alpha_i^{n-1} \quad (11)$$

Recall that the set $\{\alpha_i\}$ is the set of roots of $x^k - x^{k-1} - \cdots - 1 = 0$. We now show that this formula is equivalent to our equation (2) in Theorem 1:

$$F_n^{(k)} = \sum_{i=1}^{k} \frac{\alpha_i - 1}{2 + (k + 1)(\alpha_i - 2)} \alpha_i^{n-1} \quad (12)$$

Since $\alpha_i^k - \alpha_i^{k-1} - \cdots - 1 = 0$, we can multiply by $\alpha_i - 1$ to get $\alpha_i^{k+1} - 2\alpha_i^k = -1$, which implies $(\alpha_i - 2) = -1 \cdot \alpha_i^{-k}$. We use this last equation to transform (12) as follows:

$$\frac{\alpha_i - 1}{2 + (k + 1)(\alpha_i - 2)} = \frac{\alpha_i - 1}{2 + (k + 1)(-\alpha_i^{-k})} = \frac{\alpha_i^{k+1} - \alpha_i^k}{2\alpha_i^k - (k + 1)}$$

This establishes the equivalence of the two formulas (11) and (12), as desired. □
5 Proof of Theorem 2

Let $E_n$ be as defined in Lemma 5. We wish to show that $|E_n| < \frac{1}{2}$ for all $n \geq 2 - k$. We proceed by first showing that $|E_n| < \frac{1}{2}$ for $n = 0$, then for $n = -1, -2, -3, \ldots, 2 - k$, then for $n = 1$, and finally that this implies $|E_n| < \frac{1}{2}$ for all $n \geq 2 - k$.

To begin, we note that since our initial conditions give us that $F_n^{(k)} = 0$ for $n = 0, -1, -2, \ldots, 2 - k$, then we need only show $|m^{(k)}(\alpha) \cdot \alpha^{n-1}| < 1/2$ for those values of $n$. Starting with $n = 0$, it’s easy to check by hand that $m^{(k)}(\alpha) \cdot \alpha^{-1} < 1/2$ for $k = 2$ and $3$, and as for $k \geq 4$, we have the following inequality from Lemma 3:

$$2 - \frac{1}{3k} < \alpha,$$

which implies

$$\alpha^{-1} < \frac{3k}{6k - 1}.$$

Also, by Lemma 4,

$$m^{(k)}(\alpha) < m^{(k)}(2 - 1/3k) = \frac{3k - 1}{5k - 1},$$

so thus:

$$m^{(k)}(\alpha) \cdot \alpha^{-1} < \frac{3k - 1}{5k - 1} \cdot \frac{3k}{6k - 1} < \frac{(3k) \cdot 1}{(5k - 1) \cdot 2} < \frac{1}{2},$$

as desired. Thus, $0 < |m^{(k)}(\alpha) \cdot \alpha^{-1}| < 1/2$ for all $k$, as desired.

Since $\alpha^{-1} < 1$, we can conclude that for $n = -1, -2, \ldots, 2 - k$, then $|E_n| = m^{(k)}(\alpha) \cdot \alpha^{n-1} < 1/2$.

Turning our attention now to $E_1$, we note that $F_1^{(k)} = 1$ (again by definition of our initial conditions) and that

$$\frac{1}{2} = m(2) < m(\alpha) < m(2 - 1/k) = 1$$

which immediately gives us $|E_1| < 1/2$.

As for $E_n$ with $n \geq 2$, we know from Lemma 5 that

$$E_n = E_{n-1} + E_{n-2} + \cdots + E_{n-k} \quad \text{(for } n \geq 2)$$

Suppose for some $n \geq 2$ that $|E_n| \geq 1/2$. Let $n_0$ be the smallest positive such $n$. Now, subtracting the following two equations:

$$E_{n_0+1} = E_{n_0} + E_{n_0-1} + \cdots + E_{n_0-(k-1)}$$
$$E_{n_0} = E_{n_0-1} + E_{n_0-2} + \cdots + E_{n_0-k}$$

gives us:

$$E_{n_0+1} = 2E_{n_0} - E_{n_0-k}$$

Since $|E_{n_0}| \geq |E_{n_0-k}|$ (the first, by assumption, being larger than, and the second smaller than, $1/2$), we can conclude that $|E_{n_0+1}| > |E_{n_0}|$. In fact, we can apply this argument repeatedly to show that $|E_{n_0+i}| > \cdots > |E_{n_0+1}| > |E_{n_0}|$. However, this contradicts the observation from equation (3) that the error must eventually go to 0. We conclude that $|E_n| < 1/2$ for all $n \geq 2$, and thus for all $n \geq 2 - k$. \qed
6 Acknowledgement

The author would like to thank J. Siehler for inspiring this paper with his work on Tribonacci numbers.

References

[1] M. Cipu and F. Luca, On the Galois group of the generalized Fibonacci polynomial, *An. Științ. Univ. Ovidius Constanța Ser. Mat.* 9 (2001), 27–38.

[2] David E. Ferguson, An expression for generalized Fibonacci numbers, *Fibonacci Quart.* 4 (1966), 270–273.

[3] I. Flores, Direct calculation of $k$-generalized Fibonacci numbers, *Fibonacci Quart.* 5 (1967), 259–266.

[4] Hyman Gabai, Generalized Fibonacci $k$-sequences, *Fibonacci Quart.* 8 (1970), 31–38.

[5] Dan Kalman, Generalized Fibonacci numbers by matrix methods, *Fibonacci Quart.* 20 (1982), 73–76.

[6] David Kessler and Jeremy Schiff, A combinatoric proof and generalization of Ferguson’s formula for $k$-generalized Fibonacci numbers, *Fibonacci Quart.* 42 (2004), 266–273.

[7] Gwang-Yeon Lee, Sang-Gu Lee, Jin-Soo Kim, and Hang-Kyun Shin, The Binet formula and representations of $k$-generalized Fibonacci numbers, *Fibonacci Quart.* 39 (2001), 158–164.

[8] Claude Levesque, On $m$th order linear recurrences, *Fibonacci Quart.* 23 (1985), 290–293.

[9] E. P. Miles, Jr., Generalized Fibonacci numbers and associated matrices, *Amer. Math. Monthly* 67 (1960), 745–752.

[10] M. D. Miller, Mathematical Notes: On Generalized Fibonacci Numbers, *Amer. Math. Monthly* 78 (1971), 1108–1109.

[11] W. R. Spickerman, Binet’s formula for the Tribonacci sequence, *Fibonacci Quart.* 20 (1982), 118–120.

[12] W. R. Spickerman and R. N. Joyner, Binet’s formula for the recursive sequence of order $k$, *Fibonacci Quart.* 22 (1984), 327–331.

[13] D. A. Wolfram, Solving generalized Fibonacci recurrences, *Fibonacci Quart.* 36 (1998), 129–145.
2000 Mathematics Subject Classification: Primary 11B39, Secondary 11C08, 33F05, 65D20.

Keywords: k-generalized Fibonacci numbers, Binet, Tribonacci, Tetranacci, Pentanacci.

(Concerned with sequences A000073, A000078, and A001591.)

Received October XX, 2008.

Return to Journal of Integer Sequences home page.