Photon modulated coherent states of a generalized isotonic oscillator by Weyl ordering and their non-classical properties

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Abstract We construct photon modulated coherent states of a generalized isotonic oscillator by expanding the newly introduced superposed operator through Weyl ordering method. We evaluate the parameter $A_3$ and the $s$-parameterized quasi probability distribution function to confirm the non-classical nature of the states. We also calculate the identities related with the quadrature squeezing to explore the squeezing property of the states. Finally, we investigate the fidelity between the photon modulated coherent states and coherent states to quantify the non-Gaussianity of the states. The constructed states and their associated non-classical properties will add further knowledge on the potential.

1 Introduction

In quantum optics, the ordering of non-commuting operators, say annihilation ($\hat{a}$) and creation ($\hat{a}^{\dagger}$) operators, with $[\hat{a}, \hat{a}^{\dagger}] = 1$, play a central role since most of the physical quantities are calculated through the expectation values of various operator valued functions of these two non-commuting operators. The rules which are being widely used to express these two operators in ordered form are (i) normal, (ii) antinormal and (iii) Weyl ordering \cite{1}. Unlike the normal/antinormal ordering in the Weyl ordering one enlists all possible combinations of these two operators \cite{1}. These operators perform the action of subtraction/addition of a photon from/to a field and are represented by the equations $\hat{a}|n\rangle = \sqrt{n}|n - 1\rangle$ and $\hat{a}^{\dagger}|n\rangle = \sqrt{n + 1}|n + 1\rangle$ respectively. To explore the annihilation and creation operators, which obey the relation $[\hat{a}, \hat{a}^{\dagger}] = 1$, of some exactly solvable potentials that possess linear energy spec-
trum (besides the harmonic oscillator), one may utilize the shape invariance property [2][3][4][5].

The photon added coherent state was first introduced as an intermediate state by Agarwal and Tara [6]. This coherent state can be obtained by repeatedly operating the photon creation operator on a coherent state. The resultant state is shown to exhibit non-zero amplitude and admits certain non-classical properties such as phase squeezing and sub-Poissonian statistics. Such an intermediate state can be experimentally realized by the passage of two-level atoms, which are kept in the excited state, through a cavity that encloses an \( m \)-photon medium [6]. The photon added coherent state can also be generated through the parametric down conversion process [6]. The photon subtraction from a field has been demonstrated experimentally by Wenger and his coworkers by observing some photons which split from the initial field when it passes through a beam splitter of high transmittivity [7]. The single-photon addition to the coherent field using a ‘non-degenerate parametric down-converter’ has also been demonstrated recently [8]. It is clear from the experimental evidences as well as theoretical arguments that the photon addition and subtraction can be used as probes to generate various non-classical states. The photon addition and subtraction are important tools to test certain fundamental quantum phenomena. The experimental generation of the photon added and subtracted states helps us to verify quantitatively the bosonic commutation relation between the annihilation and creation operators, that is \([\hat{a}, \hat{a}^\dagger] = 1\) [9]. Besides this, the photon addition and subtraction can be applied to improve entanglement between Gaussian states [10]. They also serve as a promising tool for quantum-state engineering [11].

Inspired by the wide number of applications, in recent years, attempts have also been made to construct new quantum states through the elementary action of coherent superposition of photon addition and subtraction operators, that is \( \mu \hat{a} + \nu \hat{a}^\dagger \), to act upon a coherent state [12][13]. In this direction, very recently, Xu et al have introduced a new generalized photon modulated thermal state by considering the generalized superposed operator \((\mu \hat{a} + \nu \hat{a}^\dagger)^N\) [14]. The authors have derived normal ordering form of \((\mu \hat{a} + \nu \hat{a}^\dagger)^N\) using the generating function of a Hermite polynomial. In this paper, we consider this generalized superposed operator and expand it through Weyl ordering. We then obtain an explicit expression of this superposed operator in terms of \( :\hat{a}^m \hat{a}^n :\), where the symbol \( : \) denotes normal ordered product form of bosonic operators and \( m \) and \( n \) are arbitrary integers. We use this expanded form of the generalized operator to construct the photon modulated coherent states of the generalized isotonic oscillator potential (1). We investigate the non-classical properties of the photon modulated states by evaluating the parameter \( A_3 \) [15], quadrature squeezing [16] and \( s \)-parameterized function [17]. Our results confirm the non-classical nature and squeezing property of the photon modulated coherent states. The photon modulated coherent states constructed in this paper to the potential (1) is new and add additional results to this system.

The photon modulated coherent states can be generated experimentally through the interference between the coherent and Fock states using “quan-
tum catalysis” \cite{18}. In a recently proposed quantum catalysis technique \cite{19}, the interference is achieved through the variable beam splitter that comprises a polarizing beam splitter, a half-wave plate and interference beam splitter. Though this technique, the probability amplitude of a coherent state is modulated by means of photons (Fock states). Hence, the conditional preparation of this kind of multi-photon states shows the tunability into the non-classical regime which finds applications in quantum metrology, computation and communication \cite{19}. Motivated by this experimental progress, we intend to construct photon modulated coherent states for the generalized isotonic oscillator.

In the following section, we recall the construction of coherent states for the generalized isotonic oscillator. In Sec. 3, we express the superposed operator \((\hat{\mu} + \hat{v})^N\) through Weyl ordering. We construct the photon modulated coherent states for the generalized isotonic oscillator in Sec. 4. In Sec. 5, we evaluate the parameter \(A_3\), quadrature squeezing identities, \(s\)-parameterized quasi-probability distribution function and the fidelity. Finally, we summarize our results in Sec. 6.

2 Generalized isotonic oscillator

Very recently we have constructed the annihilation and creation operators satisfying the Heisenberg-Weyl algebra for the following generalized isotonic oscillator potential \cite{3, 4}

\[
V(x) = x^2 + 8 \left( \frac{2x^2 - 1}{x^2 + 1} \right) ^2. \tag{1}
\]

The Schrödinger equation associated with this potential admits the following eigenfunctions and eigenvalues \cite{20},

\[
\psi_n(x) = N_n \frac{P_n(x)}{(1 + 2x^2)} e^{-x^2/2}, \tag{2}
\]

\[
E_n = -\frac{3}{2} + n, \quad n = 0, 3, 4, \ldots \tag{3}
\]

respectively, where the normalization constant is given by

\[
N_n = \left[ \frac{(n-1)(n-2)}{2^{n+1}\sqrt{n!}} \right]^{1/2}, \quad n = 0, 3, 4, \ldots \tag{4}
\]

The newly defined \(P\)-Hermite polynomials are given by

\[
P_n(x) = \begin{cases} 
1, & \text{if } n = 0, \\
H_n(x) + 4nH_{n-2}(x) + 4(n-3)H_{n-4}(x), & \text{if } n = 3, 4, 5, \ldots \tag{5}
\end{cases}
\]

While deriving the expressions \cite{2} and \cite{3}, Planck’s constant \((\hbar)\) and the mass \((m_0)\) are absorbed suitably.
In our earlier studies, besides the number states \( |n\rangle \), we have shown that the potential \( (1) \) admits coherent states and various non-classical states including nonlinear coherent states and nonlinear squeezed states \([3,4,21]\). While exploring the ladder operators from the recurrence relations, involving the eigenfunctions \( |2\rangle \), we obtained them only as \( f \)-deformed ones \([3]\). In other words, these deformed operators act on the number states to yield

\[
\hat{N}_-|n\rangle = \sqrt{n} f(n) |n-1\rangle, \\
\hat{N}_+|n\rangle = \sqrt{n+1} f(n+1) |n+1\rangle,
\]

with \( f(n) = \sqrt{(n-1)(n-3)} \). We considered these operators as \( f \)-deformed partner of the generalized isotonic oscillator \( (1) \). By transforming them into new ones, namely

\[
\hat{a} = \sqrt{\frac{n-2}{N_- N_+}} \hat{N}_-, \\
\hat{a}^\dagger = \sqrt{\frac{n-2}{N_- N_+}} \hat{N}_+,
\]

so that the new ones are self-adjoint to each other and their combination satisfy the Heisenberg-Weyl algebra (for more details one may refer Refs. \([3,4]\)), that is

\[
[\hat{a}, \hat{a}^\dagger]|n\rangle = |n\rangle, \\
[\hat{a}^\dagger \hat{a}, \hat{a}]|n\rangle = -\hat{a}|n\rangle, \\
[\hat{a}^\dagger, \hat{a}^\dagger]|n\rangle = \hat{a}^\dagger|n\rangle.
\]

We then proved that the new operators factorize the Hamiltonian \( \hat{H}' = \hat{H} + E_0 \) as \( \hat{H} = \hat{a}^\dagger \hat{a} \), where \( \hat{H} \) is the Hamiltonian associated with the potential \( (1) \).

In this way we have obtained the annihilation and creation operators for the generalized isotonic oscillator potential \( (1) \) that possesses linear energy spectrum as in the case of harmonic oscillator. We have also obtained the coherent states,

\[
|\zeta\rangle = e^{-|\zeta|^2/2} \sum_{n=0}^{\infty} \frac{\zeta^n}{\sqrt{n!}} |n+3\rangle.
\]

In Sec. 4, we consider this coherent state, \( |\zeta\rangle \), as input state to construct photon modulated coherent state.

### 3 Weyl-ordered form of \((\mu \hat{a} + \nu \hat{a}^\dagger)^N\)

We construct photon modulated coherent states of the generalized isotonic oscillator potential \( (1) \) by expanding the superposed operator \( (\mu \hat{a} + \nu \hat{a}^\dagger)^N \) through Weyl ordering method. Here \( \mu \) and \( \nu \) are complex parameters and \( N \) is a real integer. We then obtain an explicit expression of this superposed operator in terms of : \( \hat{a}^m \hat{a}^n : \), where the symbol \( : \) denotes normal ordered product of bosonic operators and \( m \) and \( n \) are real integers. We mention here that this generalized superposed operator \( (\mu \hat{a} + \nu \hat{a}^\dagger)^N \) is not a Hermitian.
To begin with, we expand this function in terms of power series \(^{22}\), that is

\[
[(\mu \hat{a} + \nu \hat{a}^\dagger)^N]_W = \sum_{k=0}^N \binom{N}{k} \left[ (\mu \hat{a})^k (\nu \hat{a}^\dagger)^{N-k} \right]_W,
\]

(11)

where \(W\) denotes Weyl ordering and \(N\) is an integer. Recalling the Weyl ordering method for the operator \((\hat{a}^n \hat{a}^m)^W\) \(^{15}\)

\[
(\hat{a}^n \hat{a}^m)^W = (-1)^m \mathcal{H}_{m,n}(z, -z^*),
\]

(12)

is the two parameter Hermite polynomial and \(z\) is the coherent state eigenvalue. Equation (11) can now be expanded to yield

\[
[(\mu \hat{a} + \nu \hat{a}^\dagger)^N]_W = \nu^N N! \sum_{k=0}^N \binom{N}{k} \left( \frac{\mu}{\nu} \right)^k \sum_{l=0}^{\min(m,n)} \frac{(-1)^l (-z^*)^{m-k} z^{n-k}}{l! (N-k)! (N-k-l)!},
\]

(13)

We have derived an expression which is stated in terms of normal ordering \(\hat{a}^\dagger \hat{a}\). We consider (14) as the desired expression because the physical quantities can now be expressed in terms of the coherent state eigenvalues. Using this expression, we construct the photon modulated coherent states of (1) for the coherent state \(|\zeta\rangle\) given in (10).

4 Photon modulated coherent states

Photon modulated coherent states of the generalized isotonic oscillator can be obtained by letting the operator \((\mu \hat{a} + \nu \hat{a}^\dagger)^N\), given in (14), to act on its coherent states, that is

\[
|N, \zeta\rangle = N_{\mu,\nu,N} (\mu \hat{a} + \nu \hat{a}^\dagger)^N |\zeta\rangle
\]

\[
= N_{\mu,\nu,N} \nu^N N! \sum_{k=0}^N \binom{N}{k} \sum_{l=0}^{\min(N-k,k)} \frac{\left( \frac{\mu}{\nu} \right)^k (-1)^l \mathcal{H}_{N-k-l,k-l}(z, -z^*)}{l! (N-k-l)!},
\]

(15)

where \(N_{\mu,\nu,N}\) is the normalization constant. We fix the constant, \(N_{\mu,\nu,N}\), from the normalization condition \(\langle N, \zeta| N, \zeta\rangle = 1\) which in turn provides

\[
N_{\mu,\nu,N}^{-2} = |\nu|^{2N} (N!)^2 \sum_{k=0}^N \binom{N}{k} \left( \frac{|\mu|}{|\nu|} \right)^{2k} \sum_{l=0}^{\min(N-k,k)} \frac{(\frac{\mu}{\nu})^l (-1)^l \mathcal{H}_{N-k-l,k-l}(z, -z^*)}{l! (N-k-l)! (N-k-l)!}.
\]

(16)
To evaluate the above expression (16) we insert the complete relation of the coherent states, \( \frac{1}{\pi} \int_{-\infty}^{\infty} |\alpha\rangle \langle \alpha| \ d^2 \alpha = 1 \), so that Eq. (16) becomes

\[
N_{\mu,\nu,N}^{-2} = |\nu|^{2N} (N!)^2 \sum_{k=0}^{N} \left( \frac{|\mu|}{|\nu|} \right)^{2k} \frac{\min(N-k,k)}{l! (k-l)! (N-k-l)!} \left( \frac{1}{\pi (l! (k-l)! (N-k-l)!)^2} \right) \times \\
\int_{-\infty}^{\infty} \langle \zeta : \hat{a}^{k-l} \hat{a}^{N-k-l} : |\alpha\rangle \langle \alpha| : \hat{\alpha}^{N-k-l} \hat{\alpha}^{l-1} : \zeta \rangle \ d^2 \alpha.
\]

We first evaluate the integral (which we call it as \( G \)) appearing in (17). Evaluating the matrix elements of \( \hat{\alpha}^{k-l} \hat{\alpha}^{N-k-l} \) and \( \hat{\alpha}^{N-k-l} \hat{\alpha}^{l-1} \) of the coherent states, we get

\[
G = e^{-\frac{|\zeta|^2}{4}} |\zeta|^{2(k-l)} \int_{-\infty}^{\infty} e^{-|\alpha|^2 + \zeta^* \alpha + \alpha^* \zeta} |\alpha|^{2(N-k-l)} d^2 \alpha
\]

which can be re-expressed as

\[
G = e^{-\frac{|\zeta|^2}{4}} |\zeta|^{2(k-l)} \int_{-\infty}^{\infty} e^{-|\alpha|^2 + \zeta^* \alpha + \alpha^* \zeta} d^2 \alpha.
\]

Now employing the integral formula \( J_n e^{-ax} \), we find

\[
G = \pi e^{-\frac{|\zeta|^2}{4}} |\zeta|^{2(k-l)} \left( e^{\frac{|\zeta|^2}{4}} \right)
\]

To obtain the last expression in (20) we have used the Rodrigues’ formula

\[
L_n(x) = \frac{e^x}{n!} \frac{\partial^n}{\partial x^n} (e^{-x} x^n),
\]

where \( L_n(x) \) is the Laguerre polynomial of order \( n \).

Substituting (20) in (17), we obtain the normalization constant for the photon modulated coherent states \(|N, \zeta\rangle\) of the form

\[
N_{\mu,\nu,N}^{-2} = |\nu|^{2N} (N!)^2 \sum_{k=0}^{N} \left( \frac{|\mu|}{|\nu|} \right)^{2k} \frac{\min(N-k,k)}{l! (k-l)! (N-k-l)!} \left( \frac{1}{\pi (l! (k-l)! (N-k-l)!)^2} \right) \times \\
\int_{-\infty}^{\infty} |\zeta|^{2(k-l)} (N-k-l)! L_{N-k-l} \left( -|\zeta|^2 \right).
\]

In the case \( N = 0 \) and \( \mu = \nu = 0 \), we find \( N_{0,0,N}^{-2} = 1 \). When \( \mu = 0, \nu = 1 \) and \( N \neq 0 \), we obtain

\[
N_{0,1,N}^{-2} = N! L_N \left( -|\zeta|^2 \right),
\]

which is the normalization constant of the photon added coherent state \[6\].

When \( \mu = 1, \nu = 0 \) and \( N \neq 0 \), all the terms in the expression (21) vanish except \( k = N \) which in turn yield \( |\zeta|^{2N} \). Our result confirms that the normalization constant (21) generalizes the normalization constant of the photon added coherent state (vide (22)) and photon subtracted coherent state.
5 Photon statistical properties of the photon modulated coherent states

In this section, we investigate the classical and non-classical properties of the constructed photon modulated coherent states. For this purpose, we calculate the quantities (i) $A_3$-parameter, (ii) quadrature squeezing and (iii) $s$-parameterized quasi probability distribution function.

5.1 $A_3$-parameter

To test the non-classical character of the photon modulated coherent states $|N,\zeta\rangle$, we investigate the parameter $A_3$. The parameter $A_3$ can be calculated from the expression (23).

$$A_3 = \frac{\det m^{(3)}}{\det \mu^{(3)} - \det m^{(3)}},$$

(23)

where

$$m^{(3)} = \begin{pmatrix} 1 & m_1 & m_2 \\ m_1 & m_2 & m_3 \\ m_2 & m_3 & m_4 \end{pmatrix} \quad \text{and} \quad \mu^{(3)} = \begin{pmatrix} 1 & \mu_1 & \mu_2 \\ \mu_1 & \mu_2 & \mu_3 \\ \mu_2 & \mu_3 & \mu_4 \end{pmatrix}.$$

(24)

In the expression (24), $m_j = \langle \hat{a}^\dagger \hat{a}^j \rangle$ and $\mu_j = \langle (\hat{a}^\dagger \hat{a})^j \rangle$, $j = 1, 2, 3, 4$. This parameter was introduced as a counterpart to the Mandel’s parameter $Q$. To test the non-classical character of the field even if it does not exhibit the squeezing and sub-Poissonian statistics, the Mandel’s parameter $Q$ is generalized to a quantity $m_n$ formed from the moments of Glauber-Sudarshan function $P$, $m_n = \hat{b}^n \hat{b}^\dagger n$, where $\hat{b}$ and $\hat{b}^\dagger$ are the annihilation and creation operators of the harmonic oscillator. To normalize $m_n$ another quantity $\mu_n$, where $\mu_n = (\hat{b}^\dagger \hat{b})^n$, which is formed from the moments of number distribution has also been introduced. The normalized quantity with $n = 3$ obtained from $m_n$ and $\mu_n$ is termed as parameter $A_3$ (vide Eq. (23)).

For the coherent and vacuum states $\det m^{(3)} = 0$ and for a Fock state $\det m^{(3)} = -1$ and $\det \mu^{(3)} = 0$. For the non-classical states $\det m^{(3)} < 0$ and since $\det \mu^{(3)} > 0$, it follows that the parameter $A_3$ lies between 0 and -1.

To calculate the parameter $A_3$, we evaluate $m_j$’s and $\mu_j$’s, $j = 1, 2, 3, 4$, for the photon modulated coherent states $|N,\zeta\rangle$ as

$$m_j = N^{2\mu,\nu,N} \frac{|\mu|^2 (\mu!)^2}{\pi^2} \sum_{k=0}^{N} \left( \frac{|\mu|}{|\nu|} \right)^{2k} \sum_{l=0}^{\min(N-k,k)} \frac{\left( \frac{1}{2} \right)^l}{(l!(k-l)!(N-k-l)!)^2} \times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \langle \zeta | : \hat{a}^{k-l} \hat{a}^{N-k-l} : | \alpha \rangle \langle \alpha | : \hat{a}^{\dagger k-l} \hat{a}^{\dagger N-k-l} : | \beta \rangle \langle \beta | : \hat{a}^{l N-k-l} \hat{a}^{l k-l} : | \zeta \rangle \times d^2 \alpha d^2 \beta. \tag{25}$$
The double integral appearing in (25) can be evaluated in the same manner as in the case of finding the normalization constant. We skip this rather lengthy and straightforward derivation and the final form of the expression reads

\[
m_j = N^{p,v,N}_k |\nu|^{2N} (N!)^2 \sum_{k=0}^{N} \left( \frac{|\mu|}{|\nu|} \right)^{2k} \sum_{l=0}^{\min(N-k,k)} \frac{1}{l!} \frac{1}{(l!)^2} \frac{(N-k-l+j)!}{(N-k-l)! (N-k-l+1)!^2} \times \mathcal{L}_{N-k-l+1}(-|\zeta|^2), \quad j = 1, 2, 3, 4. \tag{26}
\]

Equation (26) provides explicit expressions for the moments \(m_j\) of the Glauber-Sudarshan function \(P\).

To evaluate the expression \(\mu_j = \langle (\hat{a}^\dagger \hat{a})^j \rangle\) we again express the quantity \((\hat{a}^\dagger \hat{a})^j\) in Weyl ordering

\[
(\hat{a}^\dagger \hat{a})^j_W = \sum_{i=0}^{j} \sum_{r=0}^{i} \frac{(-1)^r (i-r)^j}{r!} : \hat{a}^{r+i} \hat{a}^{j-r} :.
\tag{27}
\]

Using (27), we find

\[
\mu_j = N^{p,v,N}_k |\nu|^{2N} (N!)^2 \sum_{k=0}^{N} \left( \frac{|\mu|}{|\nu|} \right)^{2k} \sum_{l=0}^{\min(N-k,k)} \frac{1}{l!} \frac{1}{(l!)^2} \frac{(N-k-l+j)!}{(N-k-l)! (N-k-l+1)!^2} \times \mathcal{L}_{N-k-l+1}(-|\zeta|^2), \tag{28}
\]

The double integral can be evaluated in the same manner as we done previously. The final result shows

\[
\mu_j = N^{p,v,N}_k |\nu|^{2N} (N!)^2 \sum_{k=0}^{N} \left( \frac{|\mu|}{|\nu|} \right)^{2k} \sum_{l=0}^{\min(N-k,k)} \frac{1}{l!} \frac{1}{(l!)^2} \frac{(N-k-l+j)!}{(N-k-l)! (N-k-l+1)!^2} \times \mathcal{L}_{N-k-l+1}(-|\zeta|^2), \tag{29}
\]

which can be used to calculate \(\mu^{(3)}\). Using the expressions (26) and (29), we can determine the parameter \(A_3\).

From the expectation values, we work out the parameter \(A_3\), for the states \(|N,\zeta\rangle\) with \(\zeta = re^{i\theta}\), numerically. We plot the results in figure 1 where we have drawn the parameter \(A_3\) against \(r = |\zeta|\). The figures (a) and (b) demonstrate that the parameter \(A_3\) lies in-between 0 to -1 which ensures that the states given in (13) are non-classical. Figure (b) also reveals the fact that for large values of \(|\zeta|\) the parameter \(A_3\) goes to zero. We also observe that the negativity of the parameter \(A_3\) reduces while we increase the number \(N\) (figure (b)).
5.2 Quadrature squeezing

The non-classical nature of a quantum state can also be confirmed by examining the degree of squeezing it possesses. We analyze the squeezing in two new conjugate variables, namely deformed position ($\hat{X}$) and momentum ($\hat{Y}$) coordinates, which are defined as

$$\hat{X} = \frac{1}{\sqrt{2}}(\hat{a}^\dagger + \hat{a}), \quad \hat{Y} = \frac{i}{\sqrt{2}}(\hat{a}^\dagger - \hat{a}).$$

(30)

To analyze the squeezing in the quadratures $\hat{X}$ and $\hat{Y}$ in which the Heisenberg uncertainty relation holds, $(\Delta \hat{X})^2 (\Delta \hat{Y})^2 \geq \frac{1}{4}$, where $\Delta \hat{X}$ and $\Delta \hat{Y}$ denote uncertainties in $\hat{X}$ and $\hat{Y}$ respectively, we evaluate the following two inequalities, that is

$$I_1 = \langle \hat{a}^2 \rangle + \langle \hat{a}^\dagger \hat{a} \rangle - \langle \hat{a} \rangle^2 - \langle \hat{a}^\dagger \rangle^2 - 2\langle \hat{a} \rangle \langle \hat{a}^\dagger \rangle + 2\langle \hat{a}^\dagger \hat{a} \rangle < 0,$$

(31)

$$I_2 = -\langle \hat{a}^2 \rangle - \langle \hat{a}^\dagger \hat{a} \rangle + \langle \hat{a} \rangle^2 + \langle \hat{a}^\dagger \rangle^2 - 2\langle \hat{a} \rangle \langle \hat{a}^\dagger \rangle + 2\langle \hat{a}^\dagger \hat{a} \rangle < 0,$$

(32)

which can be derived from the squeezing condition $(\Delta \hat{X})^2 < \frac{1}{2}$ or $(\Delta \hat{Y})^2 < \frac{1}{2}$ by implementing the expressions given in (30). The expectation values should be calculated with respect to the photon modulated coherent states $|N, \zeta\rangle$ for which the squeezing property has to be examined.

The identities $I_1$ and $I_2$ are calculated numerically and plotted in figure 2. We have drawn the curves for $N = 1, 2, 3, 6$ with $\mu = \frac{1}{3}$ and $\nu = \frac{2}{3}$. The curves above and below 0 are associated with $I_1$ and $I_2$ respectively. The squeezing in $Y$ can be observed for all values of $\mu, \nu$ and $N$ which is clearly illustrated in figure 2. From our analysis we conclude that the states $|N, \zeta\rangle$ are non-classical.

5.3 $s$-parameterized quasi-probability function

We also confirm the non-classicality nature of the photon modulated coherent states by studying the $s$-parameterized quasi-probability distribution function.
for the states \[ |N, \zeta \rangle \] with \( \mu = \frac{1}{3}, \nu = \frac{2}{3} \) and \( N = 1, 2, 3, 6 \).

The s-parameterized quasi-probability distribution function, \( F(\gamma, s) \), is the Fourier transform of the s-parameterized characteristic function \[ \hat{T}(\gamma, s) = \frac{2}{1-s} \hat{D} \exp \left( \frac{\ln 1 + s \hat{a}^\dagger \hat{a}}{s-1} \right) \hat{D}^{-1}(\gamma), \] where

\[ \hat{D}(\gamma) \] is the displacement operator.

We express the operator function \( \exp (\lambda \hat{a}^\dagger \hat{a}) \), where \( \lambda = \ln \left( \frac{s+1}{s-1} \right) \), appearing in Eq. (34), in terms of Weyl ordered form through the identity

\[ e^{-\lambda \hat{a}^\dagger \hat{a}} = \frac{2}{1+e^{-\lambda}} \exp \left( -\frac{2(1-e^{-\lambda})}{1+e^{-\lambda}} : \hat{a}^\dagger \hat{a} : \right). \] (35)

To obtain the s-parameterized quasi-probability distribution function for the photon modulated coherent states, \( \hat{\rho} = |N, \zeta \rangle \langle N, \zeta | \), we first determine the operator \( \hat{T}(\gamma, s) \) by substituting the expression (35) in (34). We then insert the obtained expression in (33). Doing so we find

\[ F(\gamma, s) = \frac{2^{N^2}}{\pi^{2N}} \frac{|\nu|^{2N}}{(1-s)^{N}} \frac{ \sum_{k=0}^{N} \left( \frac{|\mu|}{|\nu|} \right)^{2k} \min(N-k, k) \left( \frac{2}{1+e^{-\lambda}} \right)^l (l! (k-l)! (N-k-l)!)^2 } \]

\[ \times \oint \oint \oint \oint \langle \zeta | \hat{G}_N^\dagger | \alpha \rangle \langle \alpha | D(\gamma) | \beta \rangle \langle \beta | \exp \left( -\frac{2(1-e^{-\lambda})}{1+e^{-\lambda}} : \hat{a}^\dagger \hat{a} : \right) | \xi \rangle \langle \xi | D^{-1}(\gamma) | \eta \rangle \hat{G}_N | \zeta \rangle d^2 \alpha d^2 \beta d^2 \xi d^2 \eta, \] (36)
As usual, we first evaluate the integrals appearing in (36). After a very lengthy calculation we arrive at

\[
\hat{F}(\gamma, s) = \frac{2 N_{\mu,\nu,N}^2 |\nu|^2(N!)^2}{\pi^2 (1-s)} \exp \left[ -\frac{2 + s}{s} (|\gamma|^2 + |\zeta|^2) + \left( \frac{s + 1}{s} \right) (\gamma^* \zeta + \gamma \zeta^*) \right]
\times \sum_{k=0}^{N} \binom{|\mu|}{|\nu|}^{2k} \sum_{l=0}^{min(N-k,k)} \binom{2}{l} |\zeta|^{2(k-l)} (N-k-l)! \left( \frac{s}{2} \right)^{N-k-l} \sum_{|l|} \left[ \frac{2|\gamma|^2}{s} - \frac{2 + s}{s} |\gamma|^2 + \frac{s + 1}{s} (\gamma^* \zeta + \gamma \zeta^*) \right]
\times \mathcal{L}_{N-k-l} \frac{2|\gamma|^2}{s+2} - \frac{2 + s}{s} |\gamma|^2 + \frac{s + 1}{s} (\gamma^* \zeta + \gamma \zeta^*) ,
\]

where \( \mathcal{L}_n \) is the Laguerre polynomial of order \( n \).

Using the expression (37), we determine the \( s \)-parameterized quasi-probability distribution function numerically for two sets of values, namely (i) \( s = 1.2, \mu = 0.001, \nu = 1.2 \) and \( N = 2 \) and (ii) \( s = 1.2, \mu = 0.001, \nu = 1.2, \zeta = -1e^{0.1} \) and \( \gamma = i \).

Fig. 3 The plot of the quasi distribution function \( F(\gamma, s) \) for photon modulated coherent states \( |N, \zeta⟩ \) with (a) \( s = 1.2, \mu = 0.001, \nu = 1.2 \) and \( N = 2 \) and (b) \( s = 1.2, \mu = 0.001, \nu = 1.2, \zeta = -1e^{0.1} \) and \( \gamma = i \).

5.4 Fidelity

In this sub-section we examine the fidelity between modulated coherent states and the original coherent states. The fidelity is a non-Gaussianity measure
able to quantify the non-Gaussian character of a quantum state [26]. Using this we confirm the non-classicality of the constructed state.

To calculate the fidelity, we evaluate

\[ F_{\mu,\nu,N} = \frac{|\langle \zeta | N, \zeta \rangle|^2}{|\langle \zeta | \zeta \rangle|^2}. \] (38)

Substituting (15) in the above definition with \(|\langle \zeta | \zeta \rangle|^2 = 1\) and evaluating the resultant expression, we arrive at the following expression for the fidelity between photon modulated coherent state and its coherent state, namely

\[ F_{\mu,\nu,N} = N_{\mu,\nu,N} \left( \frac{\mu}{\nu} \right)^2 N \left( N! \right)^2 \times \sum_{k=0}^{N} \left( \frac{\mu}{\nu} \right)^{2k} \frac{\min(N-k,k)}{\min(k,l)} \sum_{l=0}^{\min(N-k,k)} \frac{(l)! \left( \frac{1}{2} \right)^l \left( N-2l \right)}{(k-l)! (N-k-l)!} \left. \right|^2. \] (39)

In the limit \( N = 0 \), we should get \( F_{\mu,\nu,0} = 1 \) which can be directly verified from (39).

![Fig. 4](image-url) The plot of fidelity of photon modulated coherent states \(|N, \zeta\rangle\) with \( \mu = \frac{1}{3}, \nu = \frac{2}{3} \) for different values of \( N \) as (a) \( N = 0, 1, 3 \) and 10.

We plot the fidelity (39) between photon modulated coherent state and its original state as a function of \(|\zeta| = r\) for four different values of \( N \) say \( N = 0, 1, 2 \) and 10. We fix the parameters \( \mu = \frac{1}{3} \) and \( \nu = \frac{2}{3} \). For \( N = 0 \), photon modulated coherent state reduces to the coherent state which is again confirmed by getting a straight line at \( F_{\mu,\nu,N} = 1 \). The fidelity should decrease when we increase the number of photons. This has also been clearly illustrated in figure 4. This test also validates the non-classicality of the states \(|N, \zeta\rangle\).

6 Conclusion

In this paper, we have constructed photon modulated coherent states of a generalized isometric oscillator potential (1) by considering a generalized superposed operator \((\mu \hat{a} + \nu \hat{a}^\dagger)^N\). We implemented Weyl ordering to expand this generalized operator. The resultant expression comes out with two parameter Hermite polynomial, \( H_{m,n}(\hat{a}, \hat{a}^\dagger) \), with a finite series in \( a^m \hat{a}^n \). Since we have already known the ladder operators, we plugged them in the superposed
operator \((\mu \hat{a} + \nu \hat{a}^\dagger)^N\) and obtained the photon modulated coherent states of the potential under investigation. We have evaluated the parameter \(A_3\), quadrature squeezing identities \(I_1\) and \(I_2\) and the \(s\)-parameterized function for the constructed photon modulated coherent states. All our results confirm the non-classical nature of the photon modulated coherent states. Since the coherent states of the generalized isotonic oscillator is Gaussian with respect to new canonical variable \(X^3\), we have also analyzed the transition from Gaussianity to non-Gaussianity by evaluating the fidelity between photon modulated coherent states and coherent states. This result also confirms the non-classicality nature of the photon modulated coherent states. The conclusions present in this paper will add further knowledge on this potential.

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