LETTER TO THE EDITOR

Elastic scattering losses in the four-wave mixing of Bose–Einstein condensates

J Chwedeńczuk¹, M Trippenbach¹ and K Rzązewski²

¹ Physics Department, Warsaw University, Hoża 69, PL-00-681 Warsaw, Poland
² Center for Theoretical Physics, Polish Academy of Science, Al. Lotników, PL-00-681 Warsaw, Poland

Received 27 September 2004
Published 6 December 2004
Online at stacks.iop.org/JPhysB/37/L391
doi:10.1088/0953-4075/37/24/L01

Abstract
We introduce a classical stochastic field method that accounts for the quantum fluctuations responsible for spontaneous initiation of various atom optics processes. We assume a delta-correlated Gaussian noise in all initially empty modes of atomic field. Its strength is determined by comparison with the analytical results for two colliding condensates in the low loss limit. Our method is applied to the atomic four wave mixing experiment performed at MIT (Vogels et al 2002 Phys. Rev. Lett. 89 020401), for the first time reproducing experimental data.

In recent years we observe a growing number of experiments in which the atomic Bose–Einstein condensate evolves in a nontrivial way. A whole new area of nonlinear atom optics was born. The most striking example of such a nonlinear process is the atomic four-wave mixing (4WM). In close analogy to its optical counterpart, atomic 4WM consists of generation of the new atomic beam in the nonlinear interaction of three overlapping matter waves. For the main part atomic 4WM is an example of a stimulated process. However, during this process there are also collisions between individual atoms that lead to a population of initially unoccupied atomic states. These processes have spontaneous initiation but by nature they are also examples of four particle mixing. They amount to the elastic scattering losses from the coherently evolving condensates.

The standard tool used to describe dynamics of the condensate within mean field approximation is the celebrated Gross–Pitaevskii equation (GPE). As it stands, it is capable of describing stimulated processes but not the spontaneous ones. However, at least in some experiments [1] the elastic scattering losses may become significant. There are at least two theoretical attempts to estimate such losses during the collision of condensates. In the first one [2] the authors used momentum-dependent higher-order correction to the nonlinear coupling constant in the GPE, introducing complex scattering length. In the second one [3–5] the field theoretical formulation was used. To make it effective, the authors approximate the second
quantized Hamiltonian by a quadratic form. Both methods give very similar results but are applicable only if the elastic scattering losses are merely a small correction.

It is the purpose of this letter to formulate a general method of describing elastic scattering losses in the nonlinear atom optics processes. To this end we add to the GPE a classical Gaussian noise, representing vacuum quantum fluctuations of the atomic field and an auxiliary field holding atoms scattered out from BEC. Such a technique has its roots in quantum optics.

Spontaneous optical processes have their origins in quantum fluctuations. The best known example is a process of superfluorescence [6]. In this phenomenon a sample of atoms is prepared in the internal excited state. Spontaneously emitted photons create an avalanche of electromagnetic field. However, the initiation has a quantal nature. This quantum initiation was successfully imitated by a classical noise [7–10]. There are also general methods of mapping quantum fluctuations into stochastic terms in the evolution equations of quantum optics (generalized P-representation methods) [11–13].

In optics one can find numerous other processes initiated by spontaneous emission and eventually upon populating empty modes turning into stimulated processes; e.g. spontaneous Raman scattering [14, 15], parametric down conversion [16–21], etc. There are also similar examples in atomic and molecular physics [22–29]. Our method is general and is capable of treating many of these processes. Here we demonstrate the method using the 4WM of coherent matter waves.

The first experiment demonstrating 4WM in a sodium Bose–Einstein condensate was performed at NIST [30]. This was followed by a theoretical and numerical treatment of the experiment [31–35]. In the experiment, a short time of free expansion of the condensate, after it was released from the magnetic trap, was followed by a set of two Bragg pulses [36], which created moving wavepackets. These wavepackets, together with the remaining stationary condensate, due to nonlinear interaction and under phase matching conditions created a new momentum component in the 4WM process. The standard starting point for the description of an atomic 4WM process is the Gross–Pitaevskii equation

\[
\imath\hbar \partial_t \Psi(\vec{r}, t) = \left( -\frac{\hbar^2 \nabla^2}{2m} + V + g|\Psi|^2 \right) \Psi(\vec{r}, t).
\]  

Here \( N \) is the total number of atoms, \(|\Psi|^2\) is proportional to the atomic number density and is normalized to one, \( g = 4\pi\hbar^2a/m \) is the nonlinear interaction strength, \( m \) the atomic mass, \( a \) is the scattering length and \( V \) is a confining potential. A compact ground state wavefunction \( \Psi(\vec{r}, 0) \) is created in harmonic trap potential \( V \) and centred around \( r = 0 \) with \( \Psi(0, 0) = \Psi_m \), the maximum value. Once this ground state is created, \( V \) is turned off. The development of \( \Psi(\vec{r}, t) \) is now described by equation (1) with \( V = 0 \). Later on, a set of Bragg pulses is applied and parts of the condensate begin to move.

The initial condition immediately after application of the Bragg pulses at \( t_1 \), can be approximated as being a composition of the BEC wavepackets, identical in shape to \( \Psi(\vec{r}, t_1) \) (for more details see [31]):

\[
\Phi(\vec{r}, t_1) = \Psi(\vec{r}, t_1) \sum_{i=1}^{3} f_i^{1/2} e^{i\vec{P}_i/\hbar}.
\]

Here \( f_i = N_i/N \) is the fraction of atoms in the \( i \)th wavepacket and \( \sum_{i=1}^{3} f_i = 1 \). A new wavepacket with \( \vec{P}_4 = \vec{P}_1 + \vec{P}_2 + \vec{P}_3 \) will build up, thanks to the nonlinear interactions accounted for by the last term in the Gross–Pitaevskii equation (1). After a while, the fourth wavepacket will grow to the macroscopic level. Using the de Broglie relations: \( \vec{k}_i = \vec{P}_i/\hbar \)
and \( \omega_i = \frac{\hbar k_i^2}{2m} \) we have
\[
\Phi(\vec{r}, t) = \sum_{i=1}^{4} \Phi_i(\vec{r}, t) e^{i \vec{k}_i \cdot \vec{r} - \omega_i t},
\]  
(3)
with initial conditions
\[
\Phi_i(\vec{r}, t_1) = f_i^{1/2} \Psi_i(\vec{r}, t_1), \quad i = 1, 2, 3; \quad \Phi_4(\vec{r}, t_1) = 0.
\]
Variation of the \( \Phi_i \) is assumed to be slow compared to that of the exponential in equation (3). Four equations for this slow dependence are obtained from (1) and (3). We have, when \( V \) is turned off [2]:
\[
i \hbar \partial_t \psi_i = -\frac{\hbar^2}{2m} \nabla^2 \psi_i + gN \left( |\psi_i|^2 + 2 \sum_{j \neq i} |\psi_j|^2 \right) \psi_i + 2gN \psi_{i+1} \psi_{i+2} \psi_{i+3} + (4)
\]
where we use the convention in which all indices are taken modulo 4. To account for the elastically scattered atoms we introduce an additional component of the wavefunction \( \psi_B \). It is this part of the wavefunction which will be initiated by the classical stochastic field. The stochastic field \( \xi_{ij}(\vec{r}, t) \) must be added to the equation of motion (4) to trigger the elastic scattering process of two particles from the condensate wavefunctions \( \psi_i \) and \( \psi_j \) to the background field \( \psi_B \). It is a four particle process and it must be implemented for each pair of colliding wavefunctions \( i, j \) in such a way that the total number of atoms in colliding waves + background field is still conserved. The resulting set of equations reads
\[
\left( i \hbar \partial_t + \frac{\hbar^2}{2m} \nabla^2 \right) \psi_i = gN \left( |\psi_i|^2 + 2 \sum_{j \neq i} |\psi_j|^2 \right) \psi_i + 2gN \psi_{i+1} \psi_{i+3} \psi_B \psi_{i+2} + gN \sum_{j \neq i} \psi^*_{j} \psi_B (\psi_B + \xi_{ij})
\]  
(5)
\[
\left( i \hbar \partial_t + \frac{\hbar^2}{2m} \nabla^2 \right) \psi_B = gN \left( |\psi_B|^2 + 2 \sum_{i=1}^{4} |\psi_i|^2 \right) \psi_B + gN \sum_{i=1}^{4} \psi_i \psi_j (\psi_B + \xi_{ij})^*.
\]  
(6)
On the right-hand side of equation (5) we have three terms describing the usual self and cross phase modulation, the fourth term is a source for the deterministic four wave mixing of initially macroscopically populated wavepackets and the last part represents coupling of the colliding pairs to the background field, which incorporates quantum fluctuations. Similarly, in equation (6), which defines the dynamics of the background field, we distinguish self and cross phase modulation terms as well as the coupling to the macroscopically populated wavepackets. This form of the evolution equations conserves the total number of atoms in the system and allows us to make a clear distinction between particles that are scattered and our model of the quantum noise.

For numerical calculations we assume that \( \xi_{ij}(\vec{r}, t) \) is a Gaussian stochastic process with zero mean and the only nonvanishing second-order correlation function equal to \( \langle \xi_{ij}^* (\vec{r}, t) \xi_{ij} (\vec{r}', t') \rangle = A_{ij} \delta_{r_r} \delta_{t-t'} \). Here Kronecker delta functions are assumed both in space and time since we refer to numerical simulations with spatial grid and discrete time steps. Note that we assign different stochastic process to each pair of colliding wavepackets, anticipating dependence on parameters like relative velocity.

Equations (5), (6) may be obtained from a multiatom system Hamiltonian upon using Bogoliubov decomposition of the atomic field operators into condensate parts and initially empty modes \( \psi = \sum_{i=1}^{4} \psi_i + \psi_B \) in a way analogous to that presented in [3]. We do not,
however, explicitly decompose $\psi_B$ into plane waves but obtain its Heisenberg equation of motion assuming only that it commutes with operators of macroscopically occupied modes $\psi_i$. Finally a stochastic field is added to $\psi_B$ in the source terms for 4WM as shown above. These stochastic terms mimic vacuum quantum fluctuations leading to spontaneous elastic scattering loss. But as elastically scattered atoms reside in $\psi_B$ they may eventually get amplified via bosonic stimulation when population becomes significant. This is an outline of our stochastic method; details will be presented elsewhere [37].

To fully determine equations (5), (6) we need to specify the value of constants $A_{ij}$. Just as has been done in the case of superfluorescence mentioned above, we can find $A_{ij}$ by the requirement that it reproduces known limiting analytic results [3]. In [3] the elastic scattering losses were computed analytically in the perturbative regime for two colliding Gaussian-shaped wave packets. Furthermore these wave packets were assumed to evolve without losses and without spreading. The number of elastically scattered atoms as a function of time was found in the form

$$S(t) = \left( \frac{Na}{\sigma} \right)^2 \text{Erf} \left( \frac{\sqrt{2h} Q}{m\sigma} t \right),$$

where $\sigma$ is the width of the Gaussian wave packets and $Q$ is the wave vector corresponding to the absolute value of the momentum of each of the wave packets in the centre of mass frame. The same quantity might be calculated approximately under analogous assumptions using the stochastic classical noise. The equation for $\psi_B$ in this case reads

$$i\hbar \frac{\partial}{\partial t} \psi_B = -\frac{\hbar^2}{2m} \nabla^2 \psi_B + gN(|\psi_B|^2 + 2|\psi_1|^2 |\psi_2|^2) \psi_B + 2gN \psi_1 \psi_2 (\psi_B + \xi_{12})^*, \quad (8)$$

where $\psi_{1,2}$ are two counter-propagating Gaussian wavefunctions. The approximations of [3] amount to retaining on the right-hand side of equation (8) only the last term. The approximate solution obtained this way gives the number of elastically scattered atoms as a function of time in the form

$$S_{\text{stoch}}(t) = A_{12} \Delta t \frac{4\pi \hbar}{m(2Q)} \left( \frac{Na}{\sigma} \right)^2 \text{Erf} \left( \frac{\sqrt{2h} Q}{m\sigma} t \right). \quad (9)$$

Comparing (7) with (9) we obtain $A_{12} = \frac{m(2Q)}{4\pi \hbar}$. As we anticipated, $A_{12}$ depends on the relative velocity of wave packets $\psi_1$ and $\psi_2$, which in our case is equal $(2Q)$. Note that once $A_{ij}$ are determined our numerical approach has no more adjustable parameters. In figure 1 we compare the solution of (7) with a numerical solution of the full equation (8). Note the growing discrepancy between perturbative and non-perturbative results for larger losses. This results from the bosonic enhancement present in the non-perturbative regime. We also stress that in the non-perturbative regime the stochastic noise is crucial at the early stage of evolution. It may even be dropped from equation (8) when Bose enhancement takes effect. This is why the strength of classical noise may be determined in the perturbative regime. Finally, we point out that some analogies regarding the break down of the perturbative approach were found in the study of atom–molecule conversion within the positive-P representation [38, 39].

With equations (5), (6) fully determined we turn our attention to the recent experiment from MIT [1]. This experiment, due to the large value of the ratio of collision to nonlinear timescales, had a very large number of elastically scattered atoms. The experimental configuration consists of two initial wave packets of equal strength ($\approx N/2$ atoms in each) and the third wave packet of just a tiny fraction of $N$. The magnetic trap used to generate the sodium condensate had frequencies of 80, 80 and 20 Hz in the axial direction, hence it has the shape of a cigar. Applied optical Bragg pulses to create moving wave packets propagated approximately at the
Figure 1. Number of elastically scattered atoms from the pair of counter-propagating condensate Gaussian wavefunctions, with relative velocity 1.75 mm s$^{-1}$, as a function of time. The lower curve corresponds to a small number of scattered atoms ($N = 10^4$ and Gaussian width $\sigma = 9.1 \mu m$), when the bosonic stimulation does not occur. In this limit all three methods: complex scattering length calculation [2], field theory [3] and our stochastic method, give indistinguishable results. The upper pair of curves corresponds to higher number of atoms ($N = 1.6 \times 10^5$ and Gaussian width $\sigma = 15.8 \mu m$)—the dashed line was obtained using the complex scattering length method and the solid line is a solution of equation (8).

Figure 2. Population of the third wavepacket normalized to the initial seed population as a function of time (solid line). Parameters used in the simulation correspond to the experiment performed by the Ketterle group [1]. Total number of atoms equal to 5 million. Also shown: circles—experimentally measured values, and dashed and dotted lines—solutions of the Gross–Pitaevskii equation with real and complex scattering length, respectively. The difference between population in the third and fourth waves is constant and equal to the population of the initial seed.

same angle of $\approx 0.5 \text{ rad}$ with respect to the long axis of the condensate corresponding to a relative velocity of 20 mm s$^{-1}$. In two series of 4WM measurements the chemical potential of the condensate was 2.2 and 4.4 kHz, which we identified as corresponding to 5 and 30 million atoms, respectively. In figure 2 we plot the population of the third wavepacket (the one that
was initially seeded) as a function of time. The circles are the experimental data extracted from paper [1]. Several theoretical curves are plotted. The dashed line represents results neglecting all elastic scattering losses. The dotted line accounts for the losses by means of the complex scattering length [2]. We see that neither of the curves reproduces experimental results. Our stochastic method gives the solid line which is much closer to the experimental data. It has been computed with the parameters of the experiment including the initial number of atoms inferred from the paper as being equal to 5 million.

In figure 3 a similar comparison is made for larger sample of 30 million atoms. Again our results reproduce the experimental data very well. We feel that the remaining discrepancy (our results seem to be consistently under experimental points) is due to the indistinguishability of BEC and thermal atoms in the region of the momentum space occupied by BEC.

In figure 4 we have plotted the column density momentum distribution of the sodium condensate during the 4WM process. The build-up of the third and fourth peaks is clearly seen. Also seen is the halo of elastically scattered atoms. There is a striking similarity with figure 2 of [1].
In conclusion we have formulated a classical stochastic field method that accounts for the quantum fluctuations responsible for spontaneous initiation of various atom optics processes. For instance, we can treat oscillations between atomic and molecular condensates triggered by optical or magnetic field effects [38, 39]. The method is then applied to the atomic 4WM. It gives for the first time excellent agreement with the recent MIT experiment, where the scattering losses were substantial.

We are not able to produce a rigorous derivation of our method at this point. However, we recently developed an exact treatment of the scattering of atoms from two colliding condensates, that extends the treatment of [3] beyond the perturbation regime. We obtained an excellent agreement with our stochastic filed approach [40].

Acknowledgments

We acknowledge stimulating discussions with Mariusz Gajda, Piotr Deuar and Keith Burnett. The authors would like to acknowledge support from KBN Grant 2P03 B4325 (JCh), Polish Ministry of Scientific Research and Information Technology under grant PBZ-MIN-008/P03/2003 (MT, KR).

References

[1] Vogels J M, Xu K and Ketterle W 2002 Phys. Rev. Lett. 89 020401
[2] Band Y B et al 2000 Phys. Rev. Lett. 84 5462
[3] Bach R, Trippenbach M and Rzązewski K 2002 Phys. Rev. A 65 063605
[4] Kochler T and Burnett K 2002 Phys. Rev. A 65 033601
[5] Yurovsky V A 2002 Phys. Rev. A 65 033605
[6] Vreken Q H F, Hikspoors H M J and Gibbs H M 1997 Phys. Rev. Lett. 38 764
[7] Bonifacio R and Lugjati L A 1975 Phys. Rev. A 11 1507
[8] Bonifacio R and Lugjati L A 1975 Phys. Rev. A 12 587
[9] Haake F et al 1979 Phys. Rev. A 20 2047
[10] Mostowski J and Sobolewska B 1984 Phys. Rev. A 30 1392
[11] Drummond P D and Gardner C W 1980 J. Phys. A: Math. Gen. 13 2353
[12] Carter S J et al 1987 Phys. Rev. Lett. 58 1841
[13] Gardiner C W and Zoller P 2000 Quantum Noise vol 56 (Berlin: Springer)
[14] Raymer M G and Mostowski J 1981 Phys. Rev. A 24 1980–93
[15] Mostowski J and Sobolewska B 1984 Phys. Rev. A 30 610–2
[16] Harris S E, Oshman M K and Byer R L 1967 Phys. Rev. Lett. 18 732
[17] Magde D and Mahr H 1967 Phys. Rev. Lett. 18 905
[18] Heidmann A et al 1987 Phys. Rev. Lett. 59 2555
[19] Yariv A 1985 Optical Electronics 3rd edn (New York: Holt Rinehard and Winston)
[20] Perina J, Hradil Z and Jurco B 1994 Quantum Optics and Fundamentals of Physics (Boston, MA: Kluwer)
[21] Mandel L and Wolf E 1995 Optical Coherence and Quantum Optics (New York: Cambridge)
[22] Drummond P D, Kheruntsyan K V and He H 1998 Phys. Rev. Lett. 81 3055
[23] Javanainen J and Mackie M 1999 Phys. Rev. A 59 R3186
[24] Timmermans E et al 1999 Phys. Rev. Lett. 83 2691
[25] Timmermans E et al 1999 Phys. Rep. 315 199
[26] Wynar R et al 2000 Science 287 1016
[27] Goral K, Gajda M and Rzązewski K 2001 Phys. Rev. Lett. 86 1397
[28] Vardi A, Yurovsky V A and Anglin J R 2001 Phys. Rev. A 64 063611
[29] Heinzen D J et al 2000 Phys. Rev. Lett. 84 5029
[30] Deng L et al 1999 Nature (London) 398 218
[31] Trippenbach M, Band Y B and Julienne P S 1998 Opt. Exp. 3 530
[32] Trippenbach M, Band Y B and Julienne P S 2000 Phys. Rev. A 62 023608
[33] Goldstein E V, Plüttner K and Meyestre P 1995 Quantum Semiclass. Opt. 7 743
[34] Goldstein E V and Meystre P 1995 Phys. Rev. A 59 1509
[35] Goldstein E V and Meystre P 1999 Phys. Rev. A 59 3896
[36] Kozuma M et al 2001 Phys. Rev. Lett. 82 871
[37] Chwedeńczuk J, Trippenbach M and Rzązewski K Phys. Rev. A in preparation
[38] Hope J J and Olsen M K 2001 Phys. Rev. Lett. 86 3220
[39] Poulsen U V and Moelmer K 2001 Phys. Rev. A 63 023604
[40] Zin P, Chwedeńczuk J, Perez A, Trippenbach M and Rzązewski K in preparation