Extracting Common Time Trends from Concurrent Time Series: Maximum Autocorrelation Factors with Application to Tree Ring Time Series Data

Matz A. Haugen,* Bala Rajaratnam, and Paul Switzer

Department of EESS and Statistics, Stanford University, Stanford, CA, 94305

February 12, 2015

Abstract

Concurrent time series commonly arise from monitoring of the environment such as air quality measurement networks, weather stations, oceanographic buoys, or in paleo form such as lake sediments, tree rings, ice cores, or coral isotopes, with each monitoring or sampling site providing one of the time series. The goal is to extract a common time trend or signal in the observed data. Other examples where the goal is to extract a common time trend for multiple time series are in stock price time series, neurological time series, and quality control time series. For this purpose we develop properties of MAF [Maximum Autocorrelation Factors] that linearly combines time series in order to maximize the resulting signal-to-noise-ratio [SNR]. We quantify the SNR advantages of MAF in comparison with PCA [Principal Components Analysis], a commonly used method for linearly combining time series. We compare statistical sample properties of MAF and PCA. Then, we apply both MAF and PCA to 21 concurrent tree-ring time series from the western US and compare the extracted common time trends covering the period 1850-1998.

1 Introduction and Preliminaries

A common goal in the analysis of a collection of \( p \) concurrent time series \( Z_j(t), j = 1, \ldots, p \), observed at times \( t = 1, \ldots, n \), is to extract a common time trend which we refer to as the signal. Specifically, we look at optimizing a linear combination \( Y(t) = w'Z(t), t = 1, \ldots, n \), where \( w \) is an optimized coefficient \( p \)-vector.

For example, if the goal is to maximize variance over time of the combined series, \( Y(t) \), then this is equivalent to finding the first principal component in a PCA (Principal Component Analysis). Then the coefficient vector \( w_{PCA} \) is the principal eigenvector of the cross-covariance matrix, \( S \), where \( S_{ij} \) is the covariance over time between the pair of time series \( Z_i(t) \) and \( Z_j(t) \). The idea of PCA is that a time series with more variability over time is believed to contain more of the time trend information. Some applications of PCA to multiple time series analysis are given in Janson and Rajaratnam [2014], McShane and Wyner [2011], Briffa et al. [2008], Li et al. [2007]. However, maximizing variance across time, as PCA seeks to do, will not necessarily be well suited to revealing coherent underlying latent time trends because PCA does not make use of the specific time order of the data. If the time order of the time series were permuted, say, then the covariance matrix \( S \) and the coefficient vector \( w_{PCA} \) are unchanged.

*Electronic address: mahaugen@stanford.edu
1.1 MAF - Maximum Autocorrelation Factors

Arguably, an optimization criterion for the coefficient vector $w$ for combining the $p$ concurrent time series should specifically maximize a measure of temporal coherence of the transformed time series, rather than the time variance used in PCA. An alternative to PCA is MAF [Maximum Autocorrelation Factors] where variance maximization is replaced by autocorrelation maximization, which explicitly does depend on the time ordering of the $p$-variate observations. The motivation for MAF is that smoothly evolving time trends contained in time series data will enhance autocorrelation. We show in Appendix B that the MAF-optimized coefficient vector $w_{MAF}$ is obtained as the leading eigenvector of the matrix

$$S^{-1/2}S_{\Delta}S^{-1/2}, \quad (1.1)$$

where $S_{\Delta}$ is the $p \times p$ covariance matrix of the time-differenced time series. Any rescaling of the original time series, $Z(t)$, will preserve the MAF time series. This invariance property for MAF is derived in Appendix B. On the other hand, PCA component time series are not invariant to rescaling or recombining of the original data.

Some applications of MAF to multiple time series analysis are given in Switzer and Green [1984], Shapiro and Switzer [1989], Gallagher et al. [2014]. Our interest in MAF derives from applications to the analysis of multiple time series of climate proxy data from tree ring measurements, described in Section 5. A fuller discussion of the analysis of tree ring data will be presented in a separate paper.

To intuitively appreciate the difference between MAF and PCA, suppose we have $p = 2$ time series, one that is pure white noise and the other that is a linear time trend without noise, with both series having unit variance over time. Then PCA is indifferent between the noisy time series with zero autocorrelation and the clean time series with unit autocorrelation, whereas MAF will put all its weight on the noiseless linear time trend. If the two original time series contained each a mixture of time trend and noise, then the MAF time series will amplify the time trend relative to the noise.

1.2 An Illustration

Figure 1 shows an example with four parallel time series, rescaled to have zero mean and unit variance. These 150-year time series are extracted from the database used in Mann et al. [2008] and represent tree-ring time series. A visual inspection suggests that the first two time series exhibit more evident temporal structure than the last two time series. The corresponding PCA and MAF time series are shown in Figure 2 and these are also rescaled to have zero mean and unit variance. The MAF time series appears to concentrate the temporal structure whereas PCA seems to exhibit more temporal noise. The autocorrelation of the MAF time series is 0.758 while that of the PCA time series is 0.582. PCA and MAF coefficient matrices are shown in Table 1 and we see that the MAF time series up-weights the first two data time series and down-weights the last two data time series.
Figure 1: Four tree ring time series, each one scaled to have unit variance and zero mean.

Figure 2: MAFs and PCs of the time series shown in Figure 1.

Table 1: MAF and PCA coefficients of 4 time series.

|   | MAF | PCA |
|---|-----|-----|
| 1 |  0.80 |  0.59 |
| 2 |  0.30 |  0.58 |
| 3 |  0.24 |  0.42 |
| 4 | -0.47 |  0.37 |

1.3 Summary of results

In Section 2 we introduce the signal-plus-noise model and show that under general conditions, the MAF time series maximizes the signal-to-noise ratio (SNR) of a combination of the original time series. Equivalently, MAF also maximizes the correlation between the combined time series and the underlying signal time series. To compare we also show that the PCA time series maximizes signal plus noise variance rather than the ratio. We show that the MAF time series SNR is equal or superior to the PCA time series SNR in all situations. When the noise is iid, i.e. with zero cross-correlation and equal variance, the two methods are equivalent, and otherwise MAF improves the SNR compared to PCA.

In Section 3 a specific illustration is given where two groups of time series are considered, each with different signal strengths present in combination with noise. Explicit expressions are given for both MAF and PCA. We then derive the explicit form of the MAF and PCA coefficients vectors in certain other cases.
Section 4 explores the statistical properties of MAF and shows that MAF coefficient estimates are consistent as the number of time steps are increased while keeping the number of time series constant. Illustrations are also given to quantify the difference between MAF and PCA regarding their correlations with the underlying time trend. To determine the presence of a signal in the data, a hypothesis testing procedure is presented where the null hypothesis is a pure noise time series. Using resampling, we use autocorrelation as a test statistic and illustrate the power of the test at different sample sizes and significance levels.

Application to tree ring time series in western North America is shown in Section 5. We illustrate MAF and PCA for these time series. Both MAF and PCA suggest underlying common time trends, but MAF appears to show these trends more clearly. A null hypothesis test is highly significant and suggests the presence of time trends in the data. Concluding remarks are presented in Section 6.

2 The signal-plus-noise model

Suppose \( f(t) \) is a fixed but unknown normalized underlying signal time series with zero mean and average mean square equal to 1 over the observation period \( t = 1, ..., n \). We have \( p \) observed concurrent time series, \( Z(t) \), that are represented as

\[
Z(t) = f(t) \cdot b + \varepsilon(t)
\]

\[
\sum_t f(t) = 0 \quad \text{and} \quad \sum_t f^2(t) = 1,
\]

with \( E[\varepsilon(t)] = 0 \) and \( Var[\varepsilon(t)] = \Sigma_\varepsilon, \forall t \)

where \( \varepsilon(t) \) is the \( p \)-variate time-stationary residual noise time series and \( b = (b_1, b_2, \ldots, b_p) \) is a coefficient vector. \( f(t), \varepsilon(t) \) and \( b \) are all unobserved and unknown. We call this the S+N model. A linear combination of the \( p \) observed times series \( Z(t) \) is another time series \( w'Z(t) \). The signal-to-noise ratio for the combined time series is denoted \( SNR(w) \).

\[
SNR(w) \equiv \frac{\text{signal mean square}}{\text{noise variance}} = \frac{(w'b)^2}{w'^\Sigma_\varepsilon w}.
\]

The MAF and PCA time series are examples of such linearly combined time series with particular choices for \( w \). We now show conditions under which the MAF time series maximizes \( SNR(w) \) over \( w \).

Proposition 1. Suppose that the random stationary time series model for the residual noise is such that the \( p \times p \) residual autocovariance matrix has the proportional form,

\[
Cov(\varepsilon(t), \varepsilon(t+1)) = k_\varepsilon Cov(\varepsilon(t)) = k_\varepsilon \Sigma_\varepsilon,
\]

as would be the case for iid or autoregressive noise time series, such that

\[
Var(w'Z(t), w'Z(t+1)) = (w'b)^2 k_f + k_\varepsilon w'^\Sigma_\varepsilon w, \quad \text{with} \; k_f > k_\varepsilon,
\]

where \( k_f \) is the lag-1 autocorrelation of a normalized signal, \( f(t) \). Then MAF maximizes \( S/N \) and PCA maximizes \( S+N \).
Proof. We show that maximizing $\text{SNR}(w)$ over linear combinations if $w$ is equivalent to maximizing the lagged autocorrelation, denoted $r(w)$, of the combined time series $wZ(t)$

$$r(w) = \frac{\text{Cov}(w'Z(t), w'Z(t+1))}{\text{Cov}(w'Z(t))} = 1 - \frac{\text{Var}(w'\Delta Z(t))}{2\text{Var}(w'Z(t))},$$

$$\Delta Z(t) = Z(t) - Z(t-1) = b'\Delta f(t) + \Delta \varepsilon(t)$$

is the time differenced data vector,

$$\Delta f(t) = f(t) - f(t-1)$$

is the time-differenced signal,

$$\Delta \varepsilon(t) = \varepsilon(t) - \varepsilon(t-1)$$

is the time differenced noise vector. (2.5)

We can write

$$\text{Var}(wZ(t)) = (w'b)^2 + w'S_\varepsilon w. \quad (2.6)$$

Using (2.2), (2.5), (2.6) we can express the model autocorrelation, $r(w)$, of the combined time series $w'Z(t)$ as

$$r(w) = \frac{\text{SNR}(w) \cdot k_f + k_\varepsilon}{\text{SNR}(w) + 1} \quad (2.7)$$

which is a monotone function of $\text{SNR}(w)$, if $k_f > k_\varepsilon$. The quantity $k_f$ is the mean square of the time-differenced signal time series,

$$k_f = \frac{1}{t} \sum_t (\Delta f(t))^2. \quad (2.8)$$

$k_f$ may be regarded as a measure of signal coherence. Hence, maximizing $r(w)$ is equivalent to maximizing $\text{SNR}(w)$. Since MAF maximizes autocorrelation, MAF will also maximize the signal-to-noise variance ratio over combinations of $p$ observable cross-correlated time series, where each observable time series is a sum of a signal contribution and a random noise contribution.

PCA, on the other hand, maximizes the total variance of the combined time series, i.e. PCA maximizes, with respect to $w$, the quantity

$$\text{Var}[w'Z(t)] = \text{Var}[w'(b'f(t) + \varepsilon(t))] = (w'b)^2 + w'S_\varepsilon w. \quad (2.9)$$

Switzer and Green [1984] show that MAF and PCA are equivalent if the noise covariance matrix is given by

$$\text{Cov}(\varepsilon(t)) = \sigma^2 I, \quad (2.10)$$

i.e. the noise component of each input has the same variance and these $p \times p$ noise components have no cross-correlation. However, this equivalence between MAF and PCA does not hold when there is noise cross-correlation or heterogeneous noise variance. In signal extraction, maximizing $\text{SNR}(w)$ is arguably more desirable than maximizing overall variance of a linear combination of the input time series as in PCA. It is also important to note that the MAF time series is invariant to any rescaling of the input time series, shown in Appendix B, whereas the PCA time series is scale dependent. The MAF optimization criterion is clearly more suited to the goal of extracting a common signal component from multiple time series.

Further desirable properties of the MAF time series are

- The SNR of the MAF time series under the S+N model is proportional to the expected value of a likelihood ratio statistic for a gaussian noise specification. This is demonstrated in Appendix B.
• The MAF time series is maximally correlated with the underlying signal time series under the S+N model. This follows by noting that the squared cross-correlation

\[
\text{Cor}[f(t), w'Z(t)]^2 = \frac{\text{SNR}(w)}{\text{SNR}(w) + 1},
\]

which is monotonic in \(\text{SNR}(w)\).

• The MAF coefficient vector can be expressed as

\[
w_{\text{MAF}} = \Sigma^{-1}_e b,
\]

which we show in Appendix B.

We can generalize a S+N model to allow for multiple underlying signal time series. Specifically, we can express the model for the data as

\[
Z(t) = Bf(t) + \epsilon(t),
\]

where \(f(t)\) is a vector of \(r\) mutually orthogonal signal time series, and \(B\) is a \(p \times r\) matrix of signal strength coefficients. This extension will be more fully explored in a later paper. To allow for the possibility of multiple signal time series, we maximize autocorrelation in the observation space orthogonal to the leading MAF time series. This extends to higher order MAFs that maximize autocorrelation while being uncorrelated to the lower ordered MAFs. Computationally, this leads to extraction of higher order eigenvectors of (1.1). Extraction of higher order MAFs is analogous to extraction of higher order principal components. In section 5 we show an example of higher order MAF computations with tree ring data.

3 Illustrations of MAF/PCA comparisons in S+N model

3.1 Illustration: Two groups of time series

We consider a scenario of \(p = 2q\) concurrent time series under the S+N model, with iid noise. In addition we assume that the noise time series have common cross-correlation \(\rho > \frac{1}{p-1}\). The first \(q\) time series each has \(\text{SNR} = b_1\); the remaining \(q\) time series each has \(\text{SNR} = b_2 < b_1\). A linear combination \(w'Z(t)\) of the \(2q\) time series assign the same weight \(w_1\) to each of the first \(q\) time series and a different weight \(w_2\) to each of the remaining \(q\) time series.

Lemma 1. The \(\text{SNR}(w_1, w_2)\) of a linear combination of the \(2q\) time series is

\[
\text{SNR}(w_1, w_2) = \frac{b_2^2 q (1 + \nu \gamma)^2}{(1 - \rho)(1 + \nu^2) + pq(1 + \nu)^2}
\]

\[
\nu = \frac{w_2}{w_1}, \quad \gamma = \frac{b_2}{b_1}.
\]

whose maximum occurs at

\[
\nu_{\text{MAF}} = \frac{w_2}{w_1} = \frac{\gamma(1 - \rho + \rho q) - \rho q}{1 - \rho + \rho q - \gamma \rho q},
\]

which corresponds to the MAF coefficient ratio.

Similarly, the PCA coefficients are determined by maximizing total variance,

\[
\max_{\nu} \{(qw_1 b_1 + qw_2 b_2)^2 + (1 - \rho) + \rho(qw_1 + qw_2)^2\}.
\]

which is maximized when

\[
\nu_{\text{PCA}} = \frac{w_2}{w_1} = \sqrt{\alpha^2 + 1 - \alpha}, \text{ with } \alpha = \frac{b_1^2 - b_2^2}{2(b_1 b_2 + \rho)}.
\]
The proof follows by substituting the specific parameter values into the general expression for SNR($w$) in (2.2) and then considering roots of quadratic expressions in (3.3) and (3.1).

Consider what happens to both SNRs when changing the number of time series in each group, $q$. Note that $\nu_{PCA}$ does not depend on $q$, while $\nu_{MAF}$ does. The associated SNR of PCA approaches a constant, while SNR of MAF will grow linearly with $q$. In fact, $\lim_{q \to \infty} \nu_{MAF} = -1$ and thus

$$
\lim_{q \to \infty} \frac{1}{q} \frac{S}{N}_{|MAF} = \frac{b_1^2(1 - \gamma)^2}{2(1 - \rho)}. \quad (3.5)
$$

Similarly,

$$
\lim_{q \to \infty} \frac{S}{N}_{|PCA} = \frac{b_1^2(1 + \nu_{PCA}\gamma)}{\rho(1 + \nu_{PCA})^2}. \quad (3.6)
$$

In Figure 3, MAF and PCA SNR values are compared as $\gamma$ and $\rho$ are changed. The ratios of the SNRs is plotted in a contour plot. Each panel shows a different $p$, the total number of time series. We see that increasing the cross-correlation, $\rho$, increases the difference between MAF and PCA SNR while increasing $\gamma$ has the opposite effect. Increasing the number of time series will exacerbate the difference between the SNRs.

![Figure 3](image)

Figure 3: The SNR of MAF divided by the SNR of PCA when $q = 1, 5, 25$, with $\gamma$ and $\rho$ changing.

### 3.2 Characterization of MAF coefficients for time series with common cross-correlation

Consider the multivariate time series model,

$$ Z(t) = f(t)b + \varepsilon(t), \quad (3.7) $$

where $f(t)$ is the signal time series, $b$ is the vector of signal strengths for each time series, and with $\text{Cor}[^{\varepsilon_i(t)}\varepsilon_j(t)] = \rho$ and $\text{Var}[\varepsilon_i(t)] = \sigma_i^2$.

**Lemma 2.** The MAF coefficient vector $w = w_1, \ldots, w_p$, is given by

$$ w_i \propto \frac{b_i}{\sigma_i^2} - \frac{\rho}{1 + \rho(p-1)} \sum_{j=1}^{p} \frac{b_j}{\sigma_i\sigma_j}, \quad i = 1 \ldots p. \quad (3.8) $$
We prove this lemma by expressing the noise covariance matrix as

\[ \Sigma_\varepsilon = A \left[ \rho 1_p 1_p + (1 - \rho) I \right] A \]  

(3.9)

where \( A_{ii} = \sigma_i \) and \( A_{ij} = 0 \) for \( i \neq j \), thus,

\[ \Sigma_\varepsilon^{-1} = \frac{1}{1 - \rho} \left[ A^{-2} - \frac{\rho}{1 + \rho(p-1)} A^{-1} 1_p 1_p' A^{-1} \right] . \]  

(3.10)

From Appendix B, we have that \( w_{MAF} = \Sigma_\varepsilon^{-1} b \), and the lemma follows by substitution.

We now consider the special case where all input time series have the same noise variance. Substitution into (3.8) gives the MAF coefficient vector

\[ w_i \propto b_i - \frac{\rho}{1 + \rho(p-1)} \sum_{j=1}^{p} b_j, \quad i = 1 \ldots p. \]  

(3.11)

In this special case of common noise variance for all time series, we show in Appendix A that the MAF coefficient vector in equation (3.11) is a linear combination of the first two PC coefficient vectors. However, when the noise variances are not all equal, this will not be the case. In fact, no linear combination of the PCs can be used to obtain the MAF time series.

If furthermore, \( \rho = 0 \), i.e. no cross-correlation between noise time series, then the MAF and PCA coefficient vectors are the same and are proportional to signal strength vector \( b \).

4 Sampling properties

Having looked at the model properties of MAF and PCA, we now turn to their sample properties. The section is divided into three parts. In the first part, we show that the sample covariance and lagged covariance yield consistent estimates of MAF coefficients and factors under the S+N model as the number of time steps grows. PCA estimates are treated similarly. In the second part, an example is shown to compare MAF and PCA as a signal recovery technique. We use the signal cross-correlation with MAF and PC time series as the metric of comparison, and find that MAF is both more resilient to increased noise and more suitable when the noise has cross-correlation. In the third section, we introduce a hypothesis testing framework to test if an underlying time trend extracted by MAF is statistically significant.

4.1 Consistency

The following theorem shows that as the number of time steps grows for a \( p \)-variate time series, the MAF and PCA coefficients will converge to their model values.

**Theorem 1.** Consider a set of time series \( Z_n(t) \in \mathbb{R}^p \), such that

\[ Z_n(t) = f_n(t) b + \varepsilon_n(t) \quad t = 1, \ldots, n \]

\[ \Delta Z_n(t) = Z_n(t) - Z_n(t+1) = \Delta f_n(t) b + \Delta \varepsilon_n(t) \]  

(4.1)

with \( f_n(t) \in \mathbb{R}, b, \varepsilon_n(t) \in \mathbb{R}^p, \Delta \varepsilon_n = \varepsilon_n(t) - \varepsilon_n(t+1), \) and \( \Delta f_n(t) = f_n(t) - f_n(t+1) \). \( \varepsilon_n \) is a weakly stationary \( p \)-variate time series whose autocovariance is absolutely summable, e.g. iid or
autoregressive noise. The signal time series is such that

\[ \frac{1}{n} \sum_{t=1}^{n} f_n(t) = 0, \quad \forall n, \]

\[ \frac{1}{n} \sum_{t=1}^{n} f_n^2(t) = 1, \quad \forall n, \]

\[ \frac{1}{n-1} \sum_{t=1}^{n-1} [\Delta f_n(t) - \Delta f_n(t)]^2 = a, \quad \forall n, \quad (4.2) \]

where \( \Delta f_n(t) = \frac{1}{n} (f_n(1) - f_n(n)) \).

Then, the sample MAF and PCA coefficients converge in probability to their model counterparts.

The proof is divided into three parts: 1. Stationarity of differenced time series, 2. Convergence in probability of \( S_n \) and \( S_{\Delta n} \), the sample cross-correlation and lagged cross-correlation to their model counterparts. 3. Consistency of MAF and PCA coefficients to their model counterparts. Details are given in Appendix B.

4.2 Simulation study

We do 100 simulations of \( p = 3 \) parallel time series of length \( n = 150 \), using the S+N model of Section 2 viz.

\[ Z(t) = f(t)b + \varepsilon(t) \quad (4.3) \]

where \( f(t) \) is a specified underlying signal time series shown in Figure 4. This time series is a rescaled and interpolated version of the mean annual surface time series for the northern hemisphere for the years 1850-2007, taken from Mann et al. [2008]. \( b \) is the \( p \)-vector of signal strengths. \( \varepsilon(t) \) is and iid zero-mean gaussian noise \( p \)-vector time series. The \( 3 \times 3 \) cross-covariance matrix for \( \varepsilon \) has a unit diagonal and common value \( \rho \) in the off-diagonal entries.

![Figure 4: Signal used in the signal recovery process, shown with a mean zero and unit variance.](image)

For each of the 100 realizations of the three parallel time series we compute the combined MAF\((t)\) time series and the combined PCA\((t)\) time series. The cross-correlations of the MAF time series and the PCA time series with the signal time series are used as a metric for comparison.
Two simulations of the data are shown in Figure 5 with their associated smoothed MAF and PCA time series on the right. The first row of Figure 5 shows the three parallel time series with $b = (0.8, 0.4, 0.2)$ and cross-correlation $\rho = 0.25$. The second row shows a parallel time series with a weaker signal strength with $b = (0.4, 0.2, 0.1)$ and $\rho = 0.25$. Note the similarity of MAF and PCA when the signal strength is stronger and the disparity when the signal strength is weaker. The cross-correlation of the MAF time series with the underlying signal for the first row of Figure 5 is 0.71 and the PCA equivalent is 0.61. For the second row of Figure 5 the signal cross-correlations for MAF and PCA 0.28 and 0.15 respectively.

![Figure 5](image)

Figure 5: Top: One realization of the data with the signal shown in bold on top of each time series. The Signal-to-Noise Ratio (SNR), corresponding to $b = (0.8, 0.4, 0.2)$, is annotated above each figure and each time series has been scaled to have unit variance and zero mean. The smoothed MAF1 and PC1 are shown on the right. Bottom: Same as top with different SNR, $b = (0.4, 0.2, 0.1)$. Lastly, noise cross-correlation, $\rho = 0.25$.

The full set of scenarios that we consider in this illustration are:

- A fixed signal strength vector, $b = (0.8, 0.4, 0.2)$, with changing noise cross-correlation $\rho$.
- A fixed noise cross-correlation $\rho = 0.25$ with changing $b = (0.8c, 0.4c, 0.2c)$ for $c \in [0.5, 2.5]$.

Figure 6 contains plots of signal cross-correlations with MAF and PCA time series for each of the scenario parameter combinations. Each plotted point represents an average over 100 simulations together with the standard error. We see that MAF takes advantage of cross-correlation in the noise and uses it to amplify the signal, while PCA fails to exploit this property in the noise and thus under-performs compared to MAF.
Figure 6: Left: Correlation of signal estimate, using MAF or PCA, with signal while changing cross-correlation of noise. Right: Correlation of signal estimate, using MAF or PCA, with signal while multiplying the signal strength vector by a factor as shown on the x-axis. The error bars show twice the standard error from the mean after 100 repetitions.

4.3 Inference

Consider the same \( p \)-variate time series of length \( n \) as described in Equation 4.3. One might want to test whether a time signal is indeed present in the data or not. We consider the following hypotheses:

\[
H_0 : Z_n(t) = \varepsilon(t) \\
H_A : Z_n(t) = f(t)b + \varepsilon(t), \quad f(t) \neq \text{constant},
\]

where \( \varepsilon(t) \) is an iid zero-mean gaussian noise \( p \)-vector time series with cross-correlation \( \rho \) and unit variance.

Let the test statistic be the maximized autocorrelation of the data, i.e. the autocorrelation of the combined time series with the highest temporal structure.

To obtain a \( p \)-value from simulations, we permute the original data time steps. The corresponding data set’s leading MAF autocorrelation can be compared with that of the original data. We then repeat the procedure \( B \) times to calculate the \( p \)-value.

Permuting the data allows for exact type 1 error control because we sample from the population as opposed to an estimate of the population which is the case for the bootstrap. Furthermore, the validity of the bootstrap depends on the empirical distribution’s asymptotic convergence to the population distribution, but the permutation test does not have this requirement.

We continue using the two examples presented in Figure 5. The top panels show three time series where the signal strength vector \( b_A = (0.8, 0.4, 0.2) \), while the lower panels contain a weaker signal strength, \( b_B = (0.4, 0.2, 0.1) \). Furthermore, the smooth line in each panel is the underlying signal before the noise is added, while the gray lines show the raw observations used to calculate the MAF transformation.

The MAF autocorrelation distributions are shown in Figure 7, where the solid vertical line is the autocorrelation of the original observations while the histograms represent the autocorrelations of 1000 sets of permuted observations. The \( p \)-value represents the probability of the observed MAF autocorrelation under the null hypothesis, i.e. the absence of a signal.
the $p$-value is 0, corresponding to the left panel, while the weak-signal case has a $p$-value of 0.58, corresponding to the right panel of Figure 7.

![Figure 7](image)

**Figure 7:** The distributions of the maximized autocorrelations under the null hypothesis, where the original data has been resampled by permuting the time steps. The black vertical line shows the original autocorrelation, with the strong-signal example on the left and the weak-signal example on the right.

We proceed by calculating the power of the test under various signal strengths. To calculate the power we do the following,

1. For a given signal strength vector, $b$, simulate a data set and calculate the MAF autocorrelation.
2. Permute the data set $B$ times to obtain $B$ new MAF autocorrelations and the associated $p$-value.
3. Repeat steps 1-2 $K$ times to obtain a distribution of $p$-values.
4. The power as a function of the significance level, $\alpha$, will be the number of $p$-values less than $\alpha$ as a fraction of $K$.
5. Repeat steps 1-4 for a different signal strength.

A plot of the power as a function of the significance level for a number of signal strengths is shown in Figure 8 with $B = 400$, $K = 200$, and $\rho = 0.25$. Each line represents a different $b$ vector obtained by multiplying a base vector, $(0.8, 0.4, 0.2)$, with a signal-multiplication factor $c \in [1, 0.2]$. The top-most line in the left panel, corresponding to $c = 1$ shows a power of approximately 1 at all significance levels. The bottom most line corresponds to $c = 0.2$ and shown a power less than 0.2 for the given range of $\alpha$. In the right panel, the time series have been subsampled at regular intervals to give 50 time steps, naturally decreasing the power of the test.
Figure 8: The power of the test as a function of the significance level, $\alpha$, for multiple signal-multiplication factors. Starting with a signal strength vector $b = (0.8, 0.4, 0.2)$, the factor for each line is $c = \{1, 0.5, 0.2\}$, which corresponds to decreasing position of the lines.

## 5 Tree ring data example

To show an application of MAF, we look at tree ring data from western United States. The data is obtained from Mann et al. [2008] and show the annual growth of tree rings, and has been pre-processed as described in Mann et al’s Supplemental section. We selected 21 concurrent tree ring time series for the period 1850-1999, of which 4 were already shown in Figure 1. A figure of these 21 time series, scaled, centered, and annotated by their names, are shown in Figure 9. Some time series show more temporal coherence than others. The goal here is to extract a common underlying temporal signal.

The 3 first MAFs and PCs are shown in Figure 10 where each time series is annotated by its sample autocorrelation. A smooth version of each time series is shown in bold with 30 years per knot starting at the last year. MAF produces time series that (1) are more autocorrelated than PCA and (2) are sorted in decreasing autocorrelation. Time series could be calibrated to temperature records and used to backcast temperature for earlier time periods where only the tree ring data is available.

---

1The raw data was download from the supplemental information from Mann at http://www.meteo.psu.edu/holocene/public_html/supplements/MultiproxyMeans07/
Figure 9: 21 tree ring time series from the western United States, centered and scaled.
Figure 10: Centered and scaled MAFs and PCs of the tree rings with smoothed equivalents shown in bold.

Sampling uncertainty estimates of the MAF factors can also be obtained through the block bootstrap. For example, if the signal of interest is a smoothly varying time series one can smooth the sample data accordingly to obtain a smooth set of time series. These can be subtracted from the un-smoothed sample data and the associated residuals can be resampled and added to the smoothed time series to obtain a new sample data set, say \( Z^* (t) \), that can be used to calculate new MAF coefficients and MAF factors. A plot of this is shown in Figure 11 where 100 resampled datasets are created by doing a block bootstrap with a block size of 5 years. The block bootstrap is implemented to account for non-parametric time dependencies in the residuals, also included in Section 2 as part of the S+N model.

Each new MAF is created by using normalized MAF coefficients with a \( l_2 \)-norm of 1. The smooth applied is a spline with knots every 30 years. We see a clear signal present in the first two MAFs with the confidence bands bounding the original smooth MAFs. However, the third MAF time series’ (MAF3) original estimate can be seen partially outside the confidence interval. This suggests that MAF3 is mainly composed of noise such that when the tree ring data is resampled and the MAF is recalculated the trend associated with MAF3 disappears.

Hypothesis testing is performed using the MAF SNR estimate as the test statistic. We compute the original SNR using the smooth MAF time series as a signal estimate, and the residuals as the noise time series. The ratio of the signal and residual variances becomes the SNR estimate. A similar hypothesis test can be performed on MAF2 to test for the presence of multiple signals. This is a natural extension of the methodology as each MAF factor is the linear combination of time series which maximizes autocorrelation orthogonal to the previous MAF factors, a property that follows from the eigenvalue/eigenvector nature of the MAF formulation, mentioned in Section 1. Of course one could test for the presence of an arbitrary number of orthogonal signals.

Under the alternative hypothesis, we assume that at least one signal is present, while under the null hypothesis, no signal is present, and so the data can be block re-sampled without changing the SNR. The resultant empirical SNRs are shown in Figure 12 where we see that MAF 1 and 2 have a significant signal while MAF3 does not, with an associated \( p \)-value of 0, 0, and 0.1. For PCA, all three factors are significant, with \( p \)-values all 0, possibly because the signals are distributed across more PCs. Note also that the SNR for MAF is higher than PCA for the first 2 factors but not
for the third, again suggesting that MAF has isolated the signal in the first two factors while PCA requires more factors to convey the same information.

Figure 11: A set of 100 resampled MAF factors using the block bootstrap. The thick green line shows the original MAF with a smoother applied, while the grey lines are the un-smoothed resampled MAFs. Dashed lines show the 95th confidence bands.

Figure 12: A computational hypothesis test using SNR as the test statistic. 1000 block bootstrapped time series with block size 5 are plotted in histograms for each MAF and PCA 1-3.
6 Discussion

We demonstrated advantages of the MAF optimization criterion in comparison with PCA for the purpose of extracting a common time trend component from multiple concurrent time series. In particular, under a model where each time series is a combination of the underlying time trend with additive noise, we showed that the MAF-optimized linear combination of time series, i.e., maximizing autocorrelation, also maximizes the signal-to-noise ratio among all possible linear combinations. The sub-optimality of PCA can become worse as the number of available time series grows, as the cross-correlation between time series increases, and as the noise levels increase. We also investigated some sampling properties of the MAF analysis and showed through simulations that the MAF-optimized combined time series can be statistically more stable than the corresponding PCA-optimized time series obtained from the same set of concurrent time series data.

We illustrated some initial applications of MAF applied to combining 21 concurrent annual tree ring time series for a region in western North America, covering the period 1850-1999, with the goal of extracting common time trend information. Regional tree ring time series data are believed to be imperfect proxies for regional weather time series, such as average annual temperature, and in a subsequent paper we are investigating the calibration between regional temperature time series and regional tree ring proxy time series for these and other regions of the globe. An important step in the calibration is the extraction of common time trend information from the proxy data.

While the focus of this paper is on extraction of a time trend where each of the observed time series is regarded as a noisy version of the time trend, our continuing research will explore more complex factor model situations where there are superposed time trend components in each of the observed time series. MAF generalizes in the same way that PCA generalizes for the extraction of multiple factors. Some indication of multiple superposed time trends can be seen in the tree ring proxy data that are illustrated in this paper. For purposes of calibrating the tree ring data to temperature data, better calibration may result from considering two or more extracted time trend components.

References

Peter Arbenz and Gene H Golub. On the spectral decomposition of hermitian matrices modified by low rank perturbations with applications. *SIAM Journal on Matrix Analysis and Applications*, 9(1):40–58, 1988.

Keith R Briffa, Vladimir V Shishov, Thomas M Melvin, Eugene A Vaganov, Håken Grudd, Rashit M Hanemirov, Matti Eronen, and Muktar M Naurzbaev. Trends in recent temperature and radial tree growth spanning 2000 years across northwest eurasia. *Philosophical Transactions of the Royal Society B: Biological Sciences*, 363(1501):2269–2282, 2008. doi: 10.1098/rstb.2007.2199.

R. Bunch, James, P. Nielsen, Christopher, and C. Sorensen, Danny. Rank-one modification of the symmetric eigenproblem. *Numerische Mathematik*, 31(1):31–48–, 1978. ISSN 0029-599X. URL http://dx.doi.org/10.1007/BF01396012.

Joseph L Doob. *Stochastic processes*, volume 101. New York Wiley, 1953.

Rick Durrett. *Probability: theory and examples*. Cambridge university press, 2010.

Neal B. Gallagher, Jeremy M. Shaver, Randall Bishop, Robert T. Roginski, and Barry M. Wise. Decompositions using maximum signal factors. *J. Chemometrics*, 28(8):663–671, August 2014. ISSN 1099-128X. URL http://dx.doi.org/10.1002/cem.2634.
L. Janson and B. Rajaratnam. Robust reconstructions with temporal dependencies. *Journal of the American Statistical Association* (in press), 2014.

B. Li, D. W. Nychka, and C.M. Ammann. The 'hockey stick' and the 1990s: a statistical perspective on reconstructing hemispheric temperatures. *Tellus A*, 59(5):591–598, 2007.

Michael E. Mann, Zhihua Zhang, Malcolm K. Hughes, Raymond S. Bradley, Sonya K. Miller, Scott Rutherford, and Fenbiao Ni. Proxy-based reconstructions of hemispheric and global surface temperature variations over the past two millennia. *Proceedings of the National Academy of Sciences*, 2008. doi: 10.1073/pnas.0805721105.

B.B. McShane and J. Wyner. A statistical analysis of multiple temperature proxies: Are reconstructions of surface temperatures over the last 1000 years reliable? *The Annals of Applied Statistics*, 2011.

R.J. Muirhead. *Aspects of Multivariate Statistical Theory*. Wiley, 2005.

D.E. Shapiro and P. Switzer. Extracting time trends from multiple monitoring sites. Technical report, Stanford University, 1989.

P. Switzer and A. A. Green. Min/max autocorrelation factors for multivariate spatial imagery. Technical report, Stanford University, 1984.

### A Getting general MAF coefficients

We present an alternative method for deriving the MAF coefficients under the general model given in Equation 3.7. To do this, we first develop the case where all input time series have the same noise level we get the following covariance for \( \mathbf{Z}(t) \),

\[
\Sigma_Z = \mathbf{b}\mathbf{b}' + \rho \mathbf{1}_p\mathbf{1}'_p + (1 - \rho)I
\]  

(A.1)

where \( \rho \) is the common cross-correlation across all the time series. The lagged covariance structure is given in 2.6. The PCs are given by the eigenvectors of \( \Sigma_Z \), whereas the MAFs are the eigenvectors of \( \Sigma_Z^{-1/2}\Sigma \Delta Z \Sigma_Z^{-1/2} \).

First consider the special case where \( \bar{b} = \sum_{i=1}^{p} b_i = 0 \). This implies that \( \mathbf{b}'\mathbf{1}_p = 0 \) and the eigenvectors are \( \mathbf{b}, \mathbf{1}_p \) and all the vectors perpendicular to these two. The corresponding eigenvalues are \( (|\mathbf{b}|^2, \rho p, 0) + 1 - \rho \) where the last eigenvalue is repeated \( p - 2 \) times. If \( |\mathbf{b}|^2 > \rho p \), PC1 will be \( \mathbf{b} \). PC2 will then be \( \mathbf{1}_p \), unless \( \rho < 0 \) in which case PC2 will be in the aforementioned nullspace. Lastly, if \( |\mathbf{b}|^2 < \rho p \), PC1 will be \( \mathbf{1}_p \).

Another special case if where \( \rho = 0 \). Here the highest eigenvalue and corresponding eigenvector will be proportional to \( \mathbf{b} \) while all the others will be perpendicular to \( \mathbf{b} \) for both MAF and PCA.

In the general case where \( \bar{b} \neq 0 \) and \( \rho \neq 0 \), notice that \( \mathbf{b}\mathbf{b}' + \rho \mathbf{1}_p\mathbf{1}'_p \) is rank 2. So the dimensionality of that nullspace is \( p - 2 \). All vectors in this nullspace will have an eigenvalue of \( 1 - \rho \). The remaining two eigenvectors are found by assuming a general structure of the eigenvectors, \( \mathbf{v} = a_1\mathbf{1}_p + a_2\mathbf{b} \). It then follows that the eigenvectors/eigenvalues of \( \Sigma_Z \) are given by

\[
v_1 = \left( \frac{\rho p - |\mathbf{b}|^2 + \Delta}{2\mathbf{b}\mathbf{b}'} \right) \mathbf{1}_p + \mathbf{b}, \quad \lambda_1 = \frac{\rho p + |\mathbf{b}|^2 + \Delta}{2} + 1 - \rho
\]

\[
v_2 = \left( \frac{\rho p - |\mathbf{b}|^2 - \Delta}{2\mathbf{b}\mathbf{b}'} \right) \mathbf{1}_p + \mathbf{b}, \quad \lambda_2 = \frac{\rho p + |\mathbf{b}|^2 - \Delta}{2} + 1 - \rho \]  

(A.2)
where
\[ \Delta^2 = (|b|^2 - \rho p)^2 + 4(\hat{b}p)^2 \rho, \]  
(A.3)

where $|b|^2$ is the squared sum of the SNRs of the input time series.

The vectors in the nullspace have mean equal to zero. This can be seen by considering any eigenvector $v \in \mathcal{N}(bb' + \rho 1_p 1_p')$,
\[
(bb' + \rho 1_p 1_p')v = 0
\]
\[
(bb' + \rho 1_p 1_p')\bar{v} = 0.
\]
(A.4)

Because $b$ is in general not equal to $1_p$, we have $b'v = \bar{v} = 0$

To get the eigenvectors corresponding to the MAFs, consider also the lagged covariance matrix, $\Sigma_{\Delta} = k_f bb' + k_e (\rho 1_p 1_p' + (1 - \rho)I)$

Letting $\Sigma_{\Delta}^{-1} = \Gamma D^{-1} \Gamma' = HH'$, where $H = \Gamma D^{-1/2}$. Furthermore, because the optimal SNR in Equation (2.2) does not depend on $k_f$ as long as $k_f < k_e$, we can set $k_f = 0$, w.l.o.g. This is because the optimal SNR coefficients for each time series is equivalent to the MAF1 coefficients, which is the eigenvector corresponding to the smallest eigenvalue of
\[ \bar{\Sigma}_{\Delta} = H' (\rho 1_p 1_p' + (1 - \rho)I) H, \]  
(A.6)

where the coordinate system has been rotated such that $\Sigma_{\Delta} = I$. The MAF coefficients will change but the resulting MAF factors will not under this rotation, as shown in Property 4.

Now, let $\bar{u}_i$ be the mean of the normalized version of the vectors in Equation A.2, $\lambda_i$ be the corresponding eigenvectors. By letting $\Lambda$ be the diagonal matrix with $\frac{1}{\sqrt{\lambda_i}}$ along the diagonal and $c_i = \frac{\bar{u}_i}{\sqrt{\lambda_i}}$, we can recast this in a more familiar form,
\[ (\bar{\Sigma}_{\Delta})_{ij} = \Lambda + cc'. \]  
(A.7)

A closer look at this matrix will reveal that $c_i = 0$, $\forall i > 2$. Furthermore, $\Lambda_{ii} = 1$, $\forall i > 2$. This means that we can decompose the matrix as follows,
\[
\begin{bmatrix}
A & 0_{2 \times (p-2)} \\
0_{(p-2) \times 2} & 1_{(p-2) \times (p-2)}
\end{bmatrix}
\]  
(A.8)

And because $A$ is symmetric and $2 \times 2$, its eigenvalues/eigenvectors can be found in closed form. The remaining eigenvectors can be made the standard basis vectors $e_i = (0_1, ..., 0_{i-1}, 1, 0_{i+1}, ..., 0_p)$, $\forall i > 2$. In particular, by solving
\[
A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \lambda_1^{-1} (1 - \rho + \rho \hat{p}^2 \bar{u}_1^2) \\ \frac{\rho \hat{p}^2}{\sqrt{\lambda_1 \lambda_2}} \bar{u}_1 \bar{u}_2 \\ \frac{\rho \hat{p}^2}{\sqrt{\lambda_1 \lambda_2}} \bar{u}_1 \bar{u}_2 \end{bmatrix}, \begin{bmatrix} \mu - d \\ b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \mu \begin{bmatrix} x \\ y \end{bmatrix},
\]  
(A.9)

we find that the eigenvalues/eigenvectors are
\[
\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{\sqrt{b^2 + (\mu - d)^2}} \begin{bmatrix} \mu - d \\ b \end{bmatrix}, \quad \mu_1, \mu_2 = \frac{a + d \pm \sqrt{(a - d)^2 + 4b^2}}{2},
\]  
(A.10)

where we are interested in the smallest eigenvalue, i.e. where we subtract the term involving the discriminant.
Now, let \( \tilde{w} \) be the full vector in \( \mathbb{R}^p \) with zeroes everywhere except in the first two entries which take the values \( x \) and \( y \). To get the values of each coefficient in the basis of the original time series, we do an inverse transformation,

\[
w = (H')^{-1} \tilde{w} = H \tilde{w} = [h_1 \ h_2 \ \ldots \ h_p] \begin{bmatrix} x \\ y \\ 0 \\ \vdots \\ 0 \end{bmatrix} = xh_1 + yh_2.
\]  

(A.11)

Note that this expression is a linear combination of PC1 and PC2. Furthermore, the largest eigenvalue of the two in Equation A.10 is equal to 1, just like the other degenerate eigenvalues. This leaves only one non-degenerate eigenvalue, which can be interpreted as there being only one signal present in different strengths.

The result in Equation A.11 can be used to obtain MAF1 in a more general setting, where the noise is of unequal variance,

\[
z_t = bf_t + \varepsilon_t, \quad \text{with } (\varepsilon_t)_i \sim (0, \sigma_i^2) \quad \forall t,
\]  

(A.12)

by taking advantage of the fact that MAF is preserved under linear transformations. We can write the modified covariance matrix as

\[
A'\Sigma_Z A = A' \left( \tilde{b}\tilde{b}' + \rho I_p \right) A,
\]  

(A.13)

where \( \tilde{b} = b/\sigma \) and \( \sigma^2 \) is the vector is noise variances. Similarly for the lagged covariance matrix.

It then follows that the eigenvalue equation to be solved is

\[
\Sigma_Z^{-1/2} \Sigma_\Delta \Sigma_Z^{-1/2} \tilde{w} = \lambda \tilde{w},
\]  

(A.14)

where \( \tilde{w} = Aw \). But \( \tilde{w} \) is already given in equation (A.11), and thus \( w = A^{-1} \tilde{w} \), which are the MAF1 coefficients in the original coordinate system.

This this general case with unequal noise variance, the MAF will not be a linear combination of PC1 and PC2. The reason for this is that the vectors in the nullspace of the new covariance matrix’ rank-2 update will not in general be an eigenvector of the full covariance matrix, with the unequal variance terms in the diagonal. This means that the eigenvalue problem cannot be rewritten in a form similar to the one given in equation (A.8). In fact, the PCA eigenvectors do not even exist in closed form, but must be obtained by solving a determinant equation for the eigenvalues. This problem is explored in Arbenz and Golub [1988], Bunch et al. [1978] and the references therein.

### B Proofs

**Proposition 2.** Consider the following set of hypotheses,

\[
H_0: Z_n(t) = b + \varepsilon(t) \\
H_A: Z_n(t) = f(t)b + \varepsilon(t), \quad f(t) \neq \text{constant},
\]  

(B.1)

such that, \( \sum_t f(t) = 0 \) and \( \sum_t f^2(t) = 1 \) as defined in (4.2), and \( \varepsilon(t) \sim N(0, \Sigma) \).

Then, the model MAF autocorrelation is proportional to the expected value of the maximized likelihood ratio test statistic under the alternative model.
Proof of Proposition 2. Under $H_0$ and the Gaussian assumption, the log-likelihood

$$\ln(L_0) = -\frac{1}{2} \ln(|\Sigma_{\epsilon}|) - \frac{1}{2} \sum_{t=1}^{n} z(t)' \Sigma_{\epsilon}^{-1} z(t) - \frac{p}{2} \ln(2\pi),$$  \hspace{1cm} (B.2)

where we assume that the mean of $z_t$ is zero without loss of generality.

Now, the likelihood under the alternative hypothesis

$$\ln(L_A) = -\frac{1}{2} \ln(|\Sigma_{\epsilon}|) - \frac{1}{2} \sum_{t=1}^{n} (z(t) - bf(t))' \Sigma_{\epsilon}^{-1} (z(t) - bf(t)) - \frac{p}{2} \ln(2\pi).$$  \hspace{1cm} (B.3)

The likelihood ratio is

$$\ln(L_A) - \ln(L_0) = -\frac{1}{2} \sum_{t=1}^{n} f^2(t) bf(t)' \Sigma_{\epsilon}^{-1} + \sum_{t=1}^{n} f(t) z(t)' \Sigma_{\epsilon}^{-1} bf(t)$$

$$= -\frac{1}{2} b' \Sigma_{\epsilon}^{-1} b + \sum_{t=1}^{n} f(t) z(t)' \Sigma_{\epsilon}^{-1} b$$  \hspace{1cm} (B.4)

where the last equality holds because of the unit squared sum of $f(t)$. Now, if we substitute $z(t)$ for the alternative model and taking expectations under either model, we get

$$E[\ln(L_A) - \ln(L_0)] = -\frac{1}{2} b' \Sigma_{\epsilon}^{-1} b + E \left[ \sum_{t=1}^{n} f(t) (bf(t) - \epsilon(t))' \Sigma_{\epsilon}^{-1} b \right]$$

$$= -\frac{1}{2} b' \Sigma_{\epsilon}^{-1} b,$$  \hspace{1cm} (B.5)

where the term involving $\epsilon$ is zero after taking the expectation.

Now, we already proved that MAF1 maximizes the model signal-to-noise ratio,

$$SNR = \frac{(w' b)^2}{w' \Sigma_{\epsilon} w}.$$  \hspace{1cm} (B.6)

Making the change of coordinates $u = \Sigma_{\epsilon}^{1/2} w$, using the familiar spectral decomposition for the square root, gives the SNR representation

$$\frac{(u' \Sigma_{\epsilon}^{-1/2} b)^2}{u' u}.$$  \hspace{1cm} (B.7)

Using a geometric argument, the normalized vector, $u$, which maximizes SNR is parallel to $\Sigma_{\epsilon}^{-1/2} b$. And so in the original coordinate system,

$$w = \Sigma_{\epsilon}^{-1} b.$$  \hspace{1cm} (B.8)

Substituting the expression for these MAF1 coefficients in our definition for SNR gives

$$SNR_{optimal} = \frac{(b' \Sigma_{\epsilon}^{-1} b)^2}{b' \Sigma_{\epsilon}^{-1} \Sigma_{\epsilon} \Sigma_{\epsilon}^{-1} b} = b' \Sigma_{\epsilon}^{-1} b,$$  \hspace{1cm} (B.9)

which is proportional to the expected likelihood ratio test statistic. \hfill \square

Proposition 3. The MAF-transformation matrix, $W_{MAF} = [w_1, w_2, ..., w_p]$, contains the eigenvectors of $S^{-1/2} S\Delta S^{-1/2}$. 

21
Proof of Proposition 3. By definition, the MAF1 vector of coefficients, $w_1$, minimizes

$$\frac{w_1'S\Delta w_1}{w_1'Sw_1}. \quad (B.10)$$

By letting $S^{1/2}w_1 = u_1$ we get

$$\frac{u_1'S^{-1/2}S\Delta S^{1/2}u_1}{u_1'u_1}, \quad (B.11)$$

which is equivalent to minimizing

$$u_1'S^{-1/2}S\Delta S^{1/2}u_1$$
subject to $u_1'u_1 = 1$. \quad (B.12)

Following Muirhead [2005], the minimizing vector is the eigenvector with the lowest eigenvalue. Furthermore, the eigenvector with the $k^{th}$ smallest eigenvalue minimizes

$$u_k'S^{-1/2}S\Delta S^{1/2}u_k$$
subject to $u_k'u_k = 1$
and $u_k'u_i = 0 \ \forall i < k$. \quad (B.13)

$u_k$ corresponds to MAF$k$, after MAF$i, \forall i < k$, has been projected out of the data. The linear transformation $w_k = S^{-1/2}u_k$ gives the eigenvectors in the original coordinate system. \qedsymbol

**Proposition 4.** MAF factors are invariant to linear transformations of the data matrix, $Z$.

Proof of Proposition 4. Consider an arbitrary linear transformation $\tilde{Z} = ZA$ where $A$ is invertible and where each column of $Z$ contains one time series. Under this transformation the covariance matrix and lagged covariance matrix become $A'SA$ and $A'S\Delta A$ and we now want to minimize

$$\frac{\tilde{w}_1'A'S\Delta A\tilde{w}_1}{\tilde{w}_1'A'SA\tilde{w}_1}. \quad (B.14)$$

Now we see that $\tilde{w} = A^{-1}w$. So, in the transformed system, the MAF1 factor is

$$\tilde{Z}\tilde{w} = ZAA^{-1}w = Zw, \quad (B.15)$$
and thus equivalent to the original MAF1 factor. \qedsymbol

**Proof of Theorem 1.** We begin by recognizing that since $\varepsilon_n$ is a weakly stationary $p$-variate time series, we have

\begin{align*}
E[\varepsilon_n(t)] &= \mathbf{0} \\
\text{Cov}[\varepsilon_n(t)] &= \Sigma_{\varepsilon} \\
\text{Cov}[\Delta \varepsilon_n(t)] &= \Sigma_{\Delta \varepsilon}, \quad (B.16)
\end{align*}

with the assumption of lagged summability,

$$\sum_{\tau=0}^{\infty} |\gamma_{\tau,i}| < \infty. \quad (B.17)$$

where the lagged autocovariance for noise component $i$ is

$$\gamma_{\tau,i} = \text{Cov}[\varepsilon_{n,i}(t), \varepsilon_{n,i}(t+\tau)]. \quad (B.18)$$
Furthermore, let
\[ S_n = \frac{1}{n} \sum_{t=1}^{n} [Z_n(t) - \bar{Z}_n(t)][Z_n(t) - \bar{Z}_n(t)]', \]
\[ S_{\Delta n} = \frac{1}{n} \sum_{t=1}^{n} [\Delta Z_n(t) - \bar{\Delta Z}_n(t)][\Delta Z_n(t) - \bar{\Delta Z}_n(t)]', \] (B.19)

where \( \bar{Z}_n(t) = \sum_{i=1}^{t} Z_n(t) \), similarly for \( \Delta Z_n(t) \).

**Part I: Stationarity of differenced time series:**
We now show that \( \Delta \varepsilon_n(t) - \Delta \bar{\varepsilon}_n \) is a zero mean weakly stationary time series, using the weak stationarity of \( \varepsilon_n(t) \). Let
\[ \text{Cov}[\varepsilon(t), \varepsilon(t+h)] = \gamma(h), \] (B.20)
which is by definition not a function of \( t \). Then
\[ \text{Cov}[\Delta \varepsilon(t), \Delta \varepsilon(t+h)] = \text{Cov}[\varepsilon(t), \varepsilon(t+h)] - \text{Cov}[\varepsilon(t+1), \varepsilon(t+h)] + \text{Cov}[\varepsilon(t+1), \varepsilon(t+h+1)] - \text{Cov}[\varepsilon(t), \varepsilon(t+h+1)] \]
\[ = 2\gamma(h) - \gamma(h+1) - \gamma(h-1) \quad \text{for} \ h > 0, \] (B.21)
is not a function of \( t \) and is thus also weakly stationary. It is trivial to show that the differenced time series has zero mean.

**Part II: Consistency of \( S_n \) and \( S_{\Delta n} \)**
Substituting Equation 2.11 in Equation 2.11 we get
\[ S_n = \frac{1}{n} \sum_{t=1}^{n} [(f_n(t) - \bar{f}_n) b + \varepsilon_n(t) - \bar{\varepsilon}_n][(f_n(t) - \bar{f}_n) b + \varepsilon_n(t) - \bar{\varepsilon}_n]', \]
\[ = \frac{1}{n} \sum_{t=1}^{n} [bb' f_n^2(t) + 2f_n(t) b(\varepsilon_n(t) - \bar{\varepsilon}_n)' + (\varepsilon_n(t) - \bar{\varepsilon}_n)(\varepsilon_n(t) - \bar{\varepsilon}_n)']. \] (B.22)

As \( n \to \infty \), The first term equals \( bb' \) by definition, the second term goes to zero in probability because the \( p \)-vector of the time-averaged residuals goes to the zero vector in probability, and the third term goes to \( \Sigma_\varepsilon \) because \( \varepsilon_n(t) \) is a weakly stationary time series with zero mean [Doob 1953, Durrett 2010].

Thus,
\[ S_n = \frac{1}{n} \sum_{t=1}^{n} [Z_n(t) - \bar{Z}_n(t)][Z_n(t) - \bar{Z}_n(t)]' \xrightarrow{p} bb' + \Sigma_\varepsilon = E[S_n] \] (B.23)

Now, consider
\[ S_{\Delta n} = \frac{1}{n} \sum_{t} [bb'(\Delta f_n - \Delta \bar{f}_n)^2(t) + 2(\Delta f_n(t) - \Delta \bar{f}_n) b(\Delta \varepsilon_n(t) - \Delta \bar{\varepsilon}_n)'(t) + (\Delta \varepsilon_n(t) - \Delta \bar{\varepsilon}_n)(\Delta \varepsilon_n(t) - \Delta \bar{\varepsilon}_n)']. \] (B.24)

Applying the same arguments and using the weak stationarity of the differenced time series, we get
\[ S_{\Delta n} = \frac{1}{n} \sum_{t} [\Delta Z_n(t) - \bar{\Delta Z}_n(t)][\Delta Z_n(t) - \bar{\Delta Z}_n(t)]' \xrightarrow{p} a bb' + \Sigma_{\Delta \varepsilon} = E[S_{\Delta n}]. \] (B.25)
To wit,

\[ S_n \xrightarrow{p} E[S_n] = \Sigma = bb' + \Sigma_\epsilon \text{ as } n \to \infty \]
\[ S_{\Delta n} \xrightarrow{p} E[S_{\Delta n}] = \Sigma_\Delta = a bb' + \Sigma_\Delta_\epsilon \text{ as } n \to \infty, \]  

(B.26)

**Part III: Consistency of MAF and PCA coefficients**

PCA and MAF coefficients are the eigenvectors of \( S \) and \( S^{-1/2} S_\Delta S^{-1/2} \) respectively, using a spectral decomposition \( S^{-1/2} = HL^{-1/2} H' \). The continuous mapping theorem ensures that consistent estimates of the covariance and lagged covariance matrices implies consistent estimates of MAF and PCA coefficients, i.e. the coefficients will also converge to their model values, the eigenvectors of the model covariance matrix, \( \Sigma \) for PCA and \( \Sigma^{-1/2} \Sigma_\Delta \Sigma^{-1/2} \), with \( \Sigma^{-1/2} = \Gamma D^{-1/2} \Gamma' \). \( \square \)