Vacua and walls of mass-deformed Kähler nonlinear sigma models on $Sp(N)/U(N)$

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Abstract

We study vacua and walls of mass-deformed Kähler nonlinear sigma models on $Sp(N)/U(N)$. We identify elementary walls with the simple roots of $USp(2N)$ and discuss compressed walls, penetrable walls and multiwalls by using the moduli matrix formalism.
1 Introduction

Kähler and hyper-Kähler nonlinear sigma models are studied in [1, 2, 3]. Massive hyper-Kähler nonlinear sigma models have a potential which is proportional to the square of a tri-holomorphic Killing vector field of the hyper-Kähler target space [4]. Fixed points of the Killing vector field are realized as discrete vacua. It was shown that there exist 1/2 supersymmetric kink solutions that interpolate the discrete vacua [5]. A more general potential is possible for a hyper-Kähler target space of quaternionic dimension two or more, and exact non-singular solutions representing intersecting domain walls are constructed in [6]. Multi-domain walls are studied in [7].

The moduli matrix formalism [8, 9] was proposed to construct walls systematically in non-Abelian gauge theories with $\mathcal{N} = 2$ supersymmetry in four-dimensional spacetime. The model considered in [8, 9] becomes massive hyper-Kähler nonlinear sigma models on the cotangent bundle over the Grassmann manifold $T^*G_{N_F,N_C}$. When the gauge coupling is taken to be infinity. In this limit, multiwalls are constructed as well as single walls. Multiwalls are along one spatial direction and their positions depend on moduli parameters and mass parameters. Walls can be compressed to single walls by changing moduli parameters in Abelian gauge theories and in non-Abelian gauge theories. These walls are called compressed walls. A distinguishing feature in the non-Abelian gauge theories is that walls can pass through each other [9]. Such walls are called penetrable walls. It was also shown in [9] that there is a bundle structure for nondegenerate masses, so that the vacua and the walls are on the Kähler manifold.

The walls of Kähler nonlinear sigma models on $SO(2N)/U(N)$ are studied in [10, 11]. The Hermitian symmetric space $SO(2N)/U(N)$ is realized as a quadric in the Grassmann manifold $G_{2N,N}$ in accordance with [12, 13]. As $SO(4)/U(2) \simeq \mathbb{CP}^1$ and $SO(6)/U(3) \simeq \mathbb{CP}^3$ [14], the nonlinear sigma models on $SO(2N)/U(N)$ with $N = 2$ and $N = 3$ are actually Abelian gauge theories. The walls of the nonlinear sigma models on $SO(2N)/U(N)$ with $N = 2, 3$ are studied in [10]. The walls of the nonlinear sigma models on $SO(2N)/U(N)$ for any $N$ are studied in [11]. Penetrable walls, which are related to non-Abelian nature, appear in $N \geq 4$ cases. The vacua and the walls of $N \leq 7$ cases are presented in pictorial representations where vacua and elementary walls correspond to the vertices and the segments of the representations. It is shown that there is a recurrence of a two dimensional diagram for each $N \mod 4$ in the vacuum structures that are connected to the maximum number of elementary walls. The vacuum structures are proved by induction.

The purpose of this paper is to construct walls of mass-deformed Kähler nonlinear sigma models on $G_{N_F,N_C} = SU(N_F)/SU(N_C)\times SU(N_F-N_C)\times U(1)$.

$^{1}$
models on $Sp(N)/U(N)^2$. $Sp(N) \equiv USp(2N)$, or equivalently $Sp(N) = Sp(N, \mathbb{C}) \cap U(2N)$. Unlike $SU(N)$ or $SO(2N)$, the lengths of the simple roots of $USp(2N)$ are different. Therefore the operators for the compressed walls of the nonlinear sigma models on $Sp(N)/U(N)$ should be newly defined. We discuss the definitions of the operators and show that some of multiwalls can be compressed.

Since $Sp(1)/U(1) \simeq \mathbb{C}P^1 \simeq Q^1$ and $Sp(2)/U(2) \simeq Q^3$ \cite{14}, the nonlinear sigma models on $Sp(N)/U(N)$ with $N = 1, 2$ are Abelian theories. However, the nonlinear sigma models on $Sp(N)/U(N)$ with $N \geq 3$ are non-Abelian theories, so there exist penetrable walls. We use the pictorial representations proposed in \cite{11} to investigate the vacuum structures and the recurrence of two-dimensional diagrams to prove the vacuum structures that are connected to the maximum number of elementary walls by induction.

We follow the convention of \cite{15, 16} for the description of the root systems and corresponding Lie algebras. We also identify the elementary walls with the simple root generators of $USp(2N)$ as it is done in \cite{17}. In Section 2, we discuss the nonlinear sigma models on $Sp(N)/U(N)$ and the moduli matrix formalism. In Section 3, we study walls of the Kähler nonlinear sigma models on $Sp(N)/U(N)$ with $N \leq 6$. In Section 4, we study the vacuum structures that are connected to the maximum number of elementary walls. In Section 5, we make some observations about walls of the nonlinear sigma model on $Sp(5)/U(5)$. In Section 6, we summarize our results. In Appendix A, we prove the vacuum structures that are connected to the maximum number of elementary walls.

\section{Model}

The Kähler nonlinear sigma models on $SO(2N)/U(N)$ and $Sp(N)/U(N)$ can be represented as quadrics in the Grassmann manifold $G_{2N,N}$. The Lagrangian in four dimensions is written in the $\mathcal{N} = 1$ superfield formalism \cite{13, 14, 18}:

$$
\mathcal{L} = \int d^4\theta \left( \Phi_a^i \Phi^b_i (e^V)^{ab} - \zeta V^a \right) + \int d^2\theta \left( \Phi^{ab} (\Phi^b_i J_{ij} \Phi^T_{ja}) + \text{h.c.} \right),
$$

(2.1)

where $\Phi$ is an $N \times 2N$ chiral superfield with the flavor indices $i, j = 1, \cdots, 2N$ and the color indices $a, b = 1, \cdots, N$, $V$ is an $N \times N$ matrix vector superfield in the adjoint representation.

The result of this paper is different to the result of \cite{10}. In \cite{10} we did not use the root system of $USp(2N)$ to analyse the vacua and the walls of the nonlinear sigma models on $Sp(N)/U(N)$. In this paper we identify the elementary wall operators with the simple root generators of $USp(2N)$ and find that the elementary wall operators in \cite{10} are not correct. The result of this paper seems to be consistent with the result of \cite{15} where kink monopoles are studied in similar models with $USp(2N)$ global symmetry.

\begin{table}[h]
\begin{tabular}{|c|c|}
\hline
$C_{P^{N-1}}$ & $SO(N-1) \times U(1)$ \hspace{1cm} $Q^{N-2} = SO(N-2) \times U(1)$ \hline
\end{tabular}
\end{table}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure.png}
\caption{Graphical representation of the model.}
\end{figure}
of $U(N)$ and $\Phi^a_{\alpha}0$ is a chiral superfield under a symmetric representation of $U(N)$. $\zeta$ is the Fayet-Iliopoulos parameter and we set $\zeta = 1$. $J_{ij}$ are invariant tensors defined by

$$J = \begin{cases} \sigma^1 \otimes I_N, & SO(2N)/U(N) \\ i\sigma^2 \otimes I_N, & Sp(N)/U(N) \end{cases}. \quad (2.2)$$

The superfields are written in terms of component fields:

$$\Phi^a_i (y, \theta) = \phi^a_i (y) + \sqrt{2} \theta \psi^a_i (y) + \theta \theta F^a_i (y),$$

$$V^b_a (x, \theta, \bar{\theta}) = 2 (\theta \sigma^m \bar{\theta} A^b_{ma} (x) + i(\theta)(\bar{\theta} \lambda^b_a (x) - i(\theta \bar{\theta}) (\theta \lambda)^b_a (x) + (\theta \bar{\theta}) D^b_a (x),$$

$$\Phi^a_{\alpha}0 (y, \theta) = \phi^a_{\alpha}0 (y) + \sqrt{2} \theta \psi^a_{\alpha}0 (y) + \theta \theta F^a_{\alpha}0 (y). \quad (2.3)$$

The mass-deformed Lagrangian is obtained by dimensional reduction [19]. The bosonic part of the Lagrangian in three dimensions is

$$L = -\frac{1}{2} |(D_{\mu} \phi)^a_i|^2 - |i \phi^a_i M^a_j - i \Sigma^a_i \phi^a_i|^2 + |F^a_i|^2 + (D^a_{\mu} \phi^a_i \phi^a_i - D^a_i)^2$$

$$+ (F^a_{ij} J_{ij} \phi^T_j a + \phi^a_i J_{ij} \phi^T_j a + (\phi^a_i) F^a_i J_{ij} F^T_i a + (h.c)), \quad (2.4)$$

where the Greek letter $\mu$ is a three-dimensional spacetime index. The covariant derivative is defined by $(D_{\mu} \phi)^a_i = \partial_{\mu} \phi^a_i - i A^a_{\mu} \phi^a_i$. The last term (h.c) stands for the Hermitian conjugate.

The Cartan generators of $SO(2N)$ and $USp(2N)$ are

$$H_I = e_{I,I} - e_{N+I,N+I}, \quad (I = 1, \cdots, N), \quad (2.5)$$

where $e_{I,I}(e_{N+I,N+I})$ is a $2N \times 2N$ matrix whose $(I, I)((N + I, N + I))$ component is one [15, 16]. The mass matrix can be formulated as

$$M = \vec{m} \cdot \vec{H}, \quad (2.6)$$

with vectors

$$\vec{m} := (m_1, m_2, \cdots, m_N),$$

$$\vec{H} := (H_1, H_2, \cdots, H_N). \quad (2.7)$$

The mass matrix in the basis (2.5) is

$$M = \sigma_3 \otimes \text{diag}(m_1, m_2, \cdots, m_N). \quad (2.8)$$

Since we are interested in generic mass parameters, we can set $m_1 > m_2 > \cdots > m_N$ without loss of generality.

3
Equations of motion for $D$ and $F$ yield the constraints for the Lagrangian (2.4)

\[
\begin{align*}
\phi^i_a \bar{\phi}^b_i - \delta^b_a &= 0, \\
\bar{\phi}^i_a J_{ij} \phi^j_b &= 0, \quad \text{(hermitian conjugate) = 0.}
\end{align*}
\]

We eliminate the auxiliary fields. The potential term of the model is

\[
V = |i\phi^j_a M^i_j - i \Sigma_b \phi^i_b|^2 + 4|\phi_0)^{ab} \phi^i_b|^2.
\]

The vacuum conditions are

\[
\begin{align*}
\phi^j_a M^i_j - \Sigma_b \phi^i_b &= 0, \\
(\phi_0)^{ab} \phi^i_b &= 0.
\end{align*}
\]

The condition (2.13) gives $\phi_0 = 0$ or $\phi = 0$. Since the latter solution is inconsistent with (2.9), we have $\phi_0 = 0$. The scalar field $\Sigma$ can be diagonalized by a $U(N)$ gauge transformation as

\[
\Sigma = \text{diag}(\Sigma_1, \Sigma_2, \cdots, \Sigma_N).
\]

Since $M$ and $\Sigma$ in (2.12) are both diagonal matrices the vacuum solutions to (2.12) are labelled by

\[
(\Sigma_1, \Sigma_2, \cdots, \Sigma_N) = (\pm m_1, \pm m_2, \cdots, \pm m_N).
\]

There exist $2^{N-1}$ vacua in the nonlinear sigma model on $SO(2N)/U(N)$ since the tensor (2.2) is invariant under $O(2N)$ which includes a parity transformation. On the other hand, there exist $2^N$ vacua in the nonlinear sigma model on $Sp(N)/U(N)$. The numbers are the Euler characteristics of the spaces [20].

To study wall solutions we assume that fields are static and all the fields depend only on the $x_1 \equiv x$ coordinate. We also assume that there is Poincaré invariance on the two-dimensional worldvolume of walls so we can set $A_0 = A_2 = 0$. The energy density along the $x$-direction is

\[
\mathcal{E} = \left( |(D\phi)_a^i|^2 + |\phi^j_a M^i_j - \Sigma_b \phi^i_b|^2 + 4|\phi_0)^{ab} \phi^i_b|^2 \right)
\]

\[
\geq \pm \mathcal{T},
\]

with $D \equiv D_{\mu=1}$ and

\[
\mathcal{T} = \partial(\phi^i_a M^j_i \phi^j_b) = \partial \equiv \partial_1,
\]

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which is the tension density of the wall. The tension is

\[ T = \int_{-\infty}^{+\infty} dx \partial \text{Tr}(\phi M \bar{\phi}) = \left[ \text{Tr}(\phi M \bar{\phi}) \right]_{-\infty}^{+\infty}. \]  

(2.18)

The energy density is constrained by (2.9) and (2.10).

We choose the upper sign for the BPS equation and the lower sign for the anti-BPS equation in the first squared term in (2.16). Then the BPS equation is

\[(D\phi)^{i}_{a} - (\phi^{j}_{b} M^{i}_{j} - \Sigma^{b}_{a} \phi^{i}_{b}) = 0.\]  

(2.19)

We introduce complex matrix functions \(S^{b}_{a}(x)\) and \(f^{i}_{b}(x)\), which are defined by

\[\Sigma^{b}_{a} - iA^{b}_{a} = (S^{-1} \partial S)^{b}_{a}, \quad \phi^{i}_{a} = (S^{-1})^{b}_{a} f^{i}_{b}.\]  

(2.20)

Then the equation (2.19) is solved by

\[f^{i}_{b} = H_{0b}^{j} (e^{Mx})^{i}_{j}.\]  

(2.21)

Therefore the solution to the BPS equation (2.19) is

\[\phi^{i}_{a} = (S^{-1})^{b}_{a} H_{0b}^{j} (e^{Mx})^{i}_{j}.\]  

(2.22)

The coefficient matrix \(H_{0}\) is the moduli matrix. \(\Sigma\), \(A\) and \(\phi\) are invariant under the following transformations

\[S^{b}_{a} = V^{c}_{a} S^{b}_{c}, \quad H_{0a}^{i} = V^{c}_{a} H_{0c}^{i},\]  

(2.23)

where \(V \in GL(N, \mathbb{C})\). The matrix \(V\) defines an equivalent class of \((S, H_{0})\). This is named as worldvolume symmetry in the moduli matrix formalism [8, 9]. (2.9) and (2.10) correspond to

\[H_{0a}^{i}(e^{2Mx})^{j}_{i} H_{0j}^{b} = (SS^{\dagger})^{b}_{a} \equiv \Omega^{b}_{a},\]  

(2.24)

\[H_{0a}^{i} J_{ij} H_{0j}^{Tb} = 0, \quad (\text{hermitian conjugate}) = 0.\]  

(2.25)

From (2.23) and (2.25) we can learn that moduli matrices \(H_{0}\)'s parametrize \(Sp(N)/U(N)\). The tension density (2.17) is

\[\mathcal{T} = \frac{1}{2} \partial^{2} \ln \det \Omega.\]  

(2.26)
In the moduli matrix formalism, walls are constructed from elementary walls. The elementary wall operators are the simple root generators of the flavor symmetry. So the elementary walls can be identified with the simple roots \([17]\). We summarize the simple root generators \(E_i\), \((i = 1, \cdots, N)\) and the simple roots \(\vec{\alpha}_i\) of \(SO(2N)\) and \(USp(2N)\) following the convention of \([15, 16]\). The set of vectors \(\{\hat{e}_i\}\) is the standard unit vectors 

\[
\hat{e}_i \cdot \hat{e}_j = \delta_{ij};
\]

\(\bullet SO(2N)\)

\[E_i = e_{i,i+1} - e_{i+N+1,i+N}, \quad (i = 1, \cdots, N - 1),\]
\[E_N = e_{N-1,2N} - e_{N,2N-1},\]
\[\vec{\alpha}_i = \hat{e}_i - \hat{e}_{i+1},\]
\[\vec{\alpha}_N = \hat{e}_{N-1} + \hat{e}_N.\] (2.27)

\(\bullet USp(2N)\)

\[E_i = e_{i,i+1} - e_{i+N+1,i+N}, \quad (i = 1, \cdots, N - 1),\]
\[E_N = e_{N,2N},\]
\[\vec{\alpha}_i = \hat{e}_i - \hat{e}_{i+1},\]
\[\vec{\alpha}_N = 2\hat{e}_N.\] (2.28)

The Cartan generators (2.5) and the root generators (2.28) are related and normalized by

\[
\begin{align*}
\text{Tr}(H_I H_J) &= 2\delta_{IJ}, \quad (I, J = 1, \cdots, N), \\
\text{Tr}(H_I E_i) &= 0, \\
\text{Tr}(E_i E_i^\dagger) &= \frac{4}{\vec{\alpha}_i \cdot \vec{\alpha}_i}. 
\end{align*}
\] (2.29)

In this paper, \(\langle A \rangle\) denotes a vacuum and \(\langle A \leftarrow B \rangle\) denotes a wall which connects vacuum \(\langle A \rangle\) and vacuum \(\langle B \rangle\).

The mass matrix \(M\) (2.6), which is a linear combination of the Cartan generators, and elementary wall \(\langle A \leftarrow B \rangle\), which is generated by Cartan generator \(E_i\) are related by

\[
c[M, E_i] = c(\vec{m} \cdot \vec{\alpha}_i) E_i = T_{\langle A \leftarrow B \rangle} E_i;\] (2.30)

where \(c\) is a constant and \(T_{\langle A \leftarrow B \rangle}\) is the tension of wall. The moduli matrix of elementary wall \(H_{0\langle A \leftarrow B \rangle}\), which connects \(\langle A \rangle\) and \(\langle B \rangle\) is

\[
\begin{align*}
H_{0\langle A \leftarrow B \rangle} &= H_{0\langle A \rangle} e^{E_i(r)}, \\
E_i(r) &\equiv e^r E_i, \quad (i = 1, \cdots, N),
\end{align*}\] (2.31)
where $E_i$ is an elementary wall operator and $r$ is a complex parameter with $-\infty < \text{Re}(r) < +\infty$.

Unlike $SU(N)$ and $SO(2N)$, the lengths of the simple roots of $USp(2N)$ are different. Therefore the constant $c$ in (2.30) can be different in some vacuum sectors of the nonlinear sigma models on $Sp(N)/U(N)$.

We first review the formalism for the walls of the nonlinear sigma models on $G_{N_F,N_C}$ and $SO(2N)/U(N)$. In this case $c$ is the same in all the sectors of the vacuum structure. Given the aim of the work [11], it can be fixed as $c = 1$ for convenience. Elementary walls can be compressed to single walls. In the nonlinear sigma models on $G_{N_F,N_C}$ and on $SO(2N)/U(N)$, a compressed wall of level $n$ which connects $\langle A \rangle$ and $\langle A' \rangle$ is

$$H_0(\langle A \leftarrow A' \rangle) = H_0(\langle A \rangle) e^{E_{i_1} (r_1) E_{i_2} (r_2) \cdots E_{i_n} (r_n)},$$

$$(i_m = 1, \cdots , N; \ m = 1, \cdots , n + 1). \quad (2.32)$$

A double wall moduli matrix is constructed by multiplying a single wall operator to a single wall moduli matrix. By repeating it, we get a triple wall, a quadruple wall and so on. A multiwall which interpolates $\langle A \rangle, \langle A' \rangle, \cdots$, and $\langle B \rangle$ is

$$H_{0(\langle A \leftarrow A' \leftarrow \cdots \leftarrow B \rangle)} = H_{0(\langle A \rangle)} e^{E_{i_1} (r_1) E_{i_2} (r_2) \cdots E_{i_n} (r_n)},$$

$$(i_m = 1, \cdots , N; \ m = 1, \cdots , n), \quad (2.33)$$

where parameters $r_i \ (i = 1, 2, \cdots)$ are complex parameters ranging $-\infty < \text{Re}(r_i) < \infty$.

Elementary walls pass through each other if $[E_{i_m}, E_{i_n}] = 0$, \quad (2.34)

and these walls are named as penetrable walls [9].

Elementary walls can be identified with simple roots by (2.30) [17]. Let root vector $\tilde{g}_{\langle A_1 \leftarrow A_2 \rangle}$ denote the wall which connects vacuum $\langle A_1 \rangle$ and vacuum $\langle A_2 \rangle$. The corresponding tension of the wall is $T_{\langle A_1 \leftarrow A_2 \rangle} = \tilde{m} \cdot \tilde{g}_{\langle A_1 \leftarrow A_2 \rangle}$. Then the elementary wall of (2.31) is

$$\tilde{g}_{\langle A \leftarrow B \rangle} \equiv c \tilde{\alpha}_i. \quad (2.35)$$

The compressed wall of (2.32) is

$$\tilde{g}_{\langle A \leftarrow A' \rangle} \equiv c \tilde{\alpha}_{i_1} + c \tilde{\alpha}_{i_2} + c \tilde{\alpha}_{i_3} + \cdots c \tilde{\alpha}_{i_n} + c \tilde{\alpha}_{i_{n+1}}. \quad (2.36)$$

The root vectors of the two penetrable elementary walls of (2.34) are orthogonal

$$\tilde{\alpha}_{i_m} \cdot \tilde{\alpha}_{i_n} = 0. \quad (2.37)$$
Now we study walls of the nonlinear sigma models on $Sp(N)/U(N)$. In this case, $c = 2$ for $i = 1, \cdots, N - 1$ and $c = 1$ for $i = N$ in (2.30). An elementary wall $\langle A \leftrightarrow B' \rangle$ is

$$\bar{g}_{(A \leftrightarrow B')} = c\bar{\alpha}_i.$$  \hfill (2.38)

The moduli matrix of $\langle A \leftrightarrow A'' \rangle$, which is a compressed wall of level $n$ is

$$H_{0(A \leftrightarrow A'')} = H_{0(A)}e^{[E_{i_1},[E_{i_2},[E_{i_3},\cdots,[E_{i_n},E_{i_{n+1}}]\cdots]]]}(r),$$

$$(i_m = 1, \cdots, N - 1; \ m = 1, \cdots, n + 1).$$  \hfill (2.39)

The moduli matrices and the operators are the same as (2.32) for $i = 1, \cdots, N - 1$. However, the formula should change for operator $E_N$. As an example, an elementary wall $H_{0(B \leftrightarrow B')} = H_{0(B)}e^{E_{N-1}(r)}$ and an elementary wall $H_{0(B' \leftrightarrow B'')} = H_{0(B')}e^{E_N(r)}$ are compressed to

$$H_{0(B \leftrightarrow B'')} = H_{0(B)}e^{[E_{N-1},[E_{N-1},E_N]]}(r),$$  \hfill (2.40)

or

$$H_{0(B' \leftrightarrow B'')} = H_{0(B')}e^{[[E_N,E_{N-1}],E_{N-1}]}(r).$$  \hfill (2.41)

The formulas for multiwalls (2.33) and for penetrable walls (2.34) hold for walls of nonlinear sigma models on $Sp(N)/U(N)$.

The compressed wall of (2.39) in terms of root vectors is

$$\bar{g}_{(A \leftrightarrow A'')} = 2\bar{\alpha}_{i_1} + 2\bar{\alpha}_{i_2} + 2\bar{\alpha}_{i_3} + \cdots 2\bar{\alpha}_{i_n} + 2\bar{\alpha}_{i_{n+1}},$$  \hfill (2.42)

whereas the compressed wall of (2.40) and (2.41) is

$$\bar{g}_{(B \leftrightarrow B'')} = 2\bar{\alpha}_{N-1} + \bar{\alpha}_N.$$  \hfill (2.43)

In this paper we label the moduli matrices of vacua in descending order as

$$(\Sigma_1, \Sigma_2, \cdots, \Sigma_{N-1}, \Sigma_N) = (m_1, m_2, \cdots, m_{N-1}, m_N),$$

$$(\Sigma_1, \Sigma_2, \cdots, \Sigma_{N-1}, \Sigma_N) = (m_1, m_2, \cdots, m_{N-1}, -m_N),$$

$$(\Sigma_1, \Sigma_2, \cdots, \Sigma_{N-1}, \Sigma_N) = (m_1, m_2, \cdots, -m_{N-1}, m_N),$$

$$(\Sigma_1, \Sigma_2, \cdots, \Sigma_{N-1}, \Sigma_N) = (m_1, m_2, \cdots, -m_{N-1}, -m_N),$$

$$\vdots$$

$$(\Sigma_1, \Sigma_2, \cdots, \Sigma_{N-1}, \Sigma_N) = (m_1, -m_2, \cdots, -m_{N-1}, -m_N),$$

$$(\Sigma_1, \Sigma_2, \cdots, \Sigma_{N-1}, \Sigma_N) = (-m_1, m_2, \cdots, m_{N-1}, m_N),$$

$$\vdots$$

$$(\Sigma_1, \Sigma_2, \cdots, \Sigma_{N-1}, \Sigma_N) = (-m_1, -m_2, \cdots, -m_{N-1}, -m_N).$$  \hfill (2.44)
3 Nonlinear sigma models on $Sp(N)/U(N)$ with $N \leq 6$

There are two vacua in the nonlinear sigma model on $Sp(1)/U(1)$.

$$\Phi_1 = (1, 0), \quad \Sigma = m,$$
$$\Phi_2 = (0, 1), \quad \Sigma = -m.$$  

(3.1)

The moduli matrices of the vacua are

$$H_{0(1)} = (1, 0), \quad \Sigma = m,$$
$$H_{0(2)} = (0, 1), \quad \Sigma = -m.$$  

(3.2)

There is only one wall, which is an elementary wall. The elementary wall operator is

$$E_1 = e_{1,2},$$

(3.3)

and the moduli matrix of the elementary wall is

$$H_{0(1\leftarrow 2)} = H_{0(1)} e^{E(r)} = (1, e^r).$$

(3.4)

The tension of the wall is

$$T_{(1\leftarrow 2)} = \vec{m} \cdot \vec{\alpha}_1.$$  

(3.5)

The diagram of the elementary wall is depicted in Figure 1(a).

We study walls of the nonlinear sigma model on $Sp(2)/U(2)$. The Cartan generators $H_I$, ($I = 1, 2$), the simple root generators $E_i$, ($i = 1, 2$), and the simple roots of $USp(4)$ are

$$H_1 = e_{1,1} - e_{3,3}, \quad H_2 = e_{2,2} - e_{4,4},$$
$$E_1 = e_{1,2} - e_{4,3}, \quad E_2 = e_{2,4},$$
$$\vec{\alpha}_1 = \hat{e}_1 - \hat{e}_2, \quad \vec{\alpha}_2 = 2\hat{e}_2.$$  

(3.6)

For $N = 2$ the vacuum condition (2.12) gives rise to 4 vacua, which have the following form

$$\Phi_{(1)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad (\Sigma_1, \Sigma_2) = (m_1, m_2),$$
$$\Phi_{(2)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (\Sigma_1, \Sigma_2) = (m_1, -m_2),$$
$$\Phi_{(3)} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad (\Sigma_1, \Sigma_2) = (-m_1, m_2),$$
$$\Phi_{(4)} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (\Sigma_1, \Sigma_2) = (-m_1, -m_2).$$  

(3.7)
The moduli matrices of (3.7) are

\[ H_{0(1)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad (\Sigma_1, \Sigma_2) = (m_1, m_2), \]
\[ H_{0(2)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (\Sigma_1, \Sigma_2) = (m_1, -m_2), \]
\[ H_{0(3)} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad (\Sigma_1, \Sigma_2) = (-m_1, m_2), \]
\[ H_{0(4)} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (\Sigma_1, \Sigma_2) = (-m_1, -m_2). \quad (3.8) \]

The moduli matrices of elementary walls that connect the vacua (3.8) are

\[ H_{1\leftarrow 2} = H_{0(1)} e^{E_2(r)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & e^r \end{pmatrix}, \]
\[ H_{2\leftarrow 3} = H_{0(2)} e^{E_1(r)} = \begin{pmatrix} 1 & e^r & 0 & 0 \\ 0 & 0 & -e^r & 1 \end{pmatrix}, \]
\[ H_{3\leftarrow 4} = H_{0(3)} e^{E_2(r)} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & e^r \end{pmatrix}. \quad (3.9) \]

The wall solution (2.22) with \( H_{1\leftarrow 2} \) is

\[ \phi_{12} = \begin{pmatrix} 1 \\ 0 \\ e^{m_2 x} \Delta_{-1/2}^{-1} \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ e^{-m_2 x + r} \Delta_{-1/2}^{-1} \\ 0 \end{pmatrix}, \]
\[ \Delta = e^{2m_2 x} + e^{-2m_2 x + 2Re(r)} \]
\[ \Delta_1 = e^{2m_1 x} + e^{2m_2 x + 2Re(r)}, \]
\[ \Delta_2 = e^{-2m_1 x + 2Re(r)} + e^{-2m_2 x}. \quad (3.10) \]

All the phases, which appear due to the \( U(1) \) gauge symmetry, are fixed to zero. The wall (3.10) has the limits

\[ x \rightarrow +\infty, \quad \phi_{12} \rightarrow \Phi_{(1)}, \]
\[ x \rightarrow -\infty, \quad \phi_{12} \rightarrow \Phi_{(2)}, \quad (3.11) \]
as expected. The wall solution (2.22) with \( H_{2\leftarrow 3} \) is

\[ \phi_{23} = \begin{pmatrix} e^{m_1 x} \Delta_{1}^{-1/2} & e^{m_2 x + r} \Delta_{1}^{-1/2} \\ 0 & 0 \\ -e^{-m_1 x + r} \Delta_{2}^{-1/2} & e^{-m_2 x} \Delta_{2}^{-1/2} \end{pmatrix}, \]
\[ \Delta_1 = e^{2m_1 x} + e^{2m_2 x + 2Re(r)}, \]
\[ \Delta_2 = e^{-2m_1 x + 2Re(r)} + e^{-2m_2 x}. \quad (3.12) \]
The wall (3.12) has the limits
\[
x \to +\infty, \quad \phi_{23} \to \Phi_{(2)},
\]
\[
x \to -\infty, \quad \phi_{23} \to \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{3.13}
\]
Here \( \phi_{23}(x \to -\infty) \) is related to vacuum \( \Phi_3 \) by a \( U(N) \) gauge transformation. Therefore \( \phi_{23}(x \to -\infty) \) and \( \Phi_3 \) are the same vacuum. We can also see this by making use of worldvolume symmetry. The moduli matrix of \( \phi_{23}(x \to -\infty) \) is
\[
H'_{0(3)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{3.14}
\]
which is related to \( H_{0(3)} \) by worldvolume symmetry
\[
H'_{0(3)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} H_{0(3)}. \tag{3.15}
\]
Therefore (3.12) is the elementary wall which connects vacuum \( \langle 2 \rangle \) and vacuum \( \langle 3 \rangle \). The wall solution (2.22) with \( H_{(3\leftarrow 4)} \) is
\[
\phi_{34} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} e^{m_{2x}} \Delta^{-1/2} \begin{pmatrix} 0 & 1 \\ 0 & e^{-m_{2x}+r} \Delta^{-1/2} \end{pmatrix},
\]
\[
\Delta = e^{2m_{2x}} + e^{-2m_{2x}+2\text{Re}(r)}. \tag{3.16}
\]
The wall solution (3.16) has the limits
\[
x \to +\infty, \quad \phi_{34} \to \Phi_{(3)},
\]
\[
x \to -\infty, \quad \phi_{34} \to \Phi_{(4)}. \tag{3.17}
\]
Tension \( T_{(A\leftarrow B)} \) of the wall that connects vacuum \( \Phi_{(A)} \) and vacuum \( \Phi_{(B)} \) is obtained from (3.7). The tensions of the elementary walls are
\[
T_{(1\leftarrow 2)} = \vec{m} \cdot \vec{e}_2, \tag{3.18}
\]
\[
T_{(2\leftarrow 3)} = 2\vec{m} \cdot \vec{e}_1, \tag{3.19}
\]
\[
T_{(3\leftarrow 4)} = \vec{m} \cdot \vec{e}_2. \tag{3.20}
\]
Therefore the elementary walls are identified with
\[
\vec{g}_{(1\leftarrow 2)} = \vec{g}_{(3\leftarrow 4)} = \vec{e}_2,
\]
\[
\vec{g}_{(2\leftarrow 3)} = 2\vec{e}_1. \tag{3.21}
\]
The diagram of the elementary walls are depicted in Figure 1(b). We omit the coefficients of the simple roots in elementary wall diagrams in this paper. From the diagram in Figure 1(b), one can see how a compressed walls is constructed. From (2.39), the compressed wall that interpolates (1) and (3) is

$$H_{0(1\leftarrow 3)} = H_{0(1)} e^{[E_2,E_1,E_1](r)} = \begin{pmatrix} 1 & 0 & 2e^r & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

(3.22)

and the compressed wall that interpolates (2) and (4) is

$$H_{0(2\leftarrow 4)} = H_{0(2)} e^{[E_1,E_1,E_2](r)} = \begin{pmatrix} 1 & 0 & 2e^r & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. $$

(3.23)

These are the compressed walls of level one.

It can be shown that these compressed walls can be obtained from double walls. Moduli matrices of double walls $\langle 1 \leftarrow 2 \leftarrow 3 \rangle$ and $\langle 2 \leftarrow 3 \leftarrow 4 \rangle$ are

$$H_{0(1\leftarrow 2\leftarrow 3)} = H_{0(1\leftarrow 2)} e^{E_1(r_2)} = \begin{pmatrix} 1 & e^{r_2} & 0 & 0 \\ 0 & 1 & -e^{r_1+r_2} & e^{r_1} \end{pmatrix},$$

$$H_{0(2\leftarrow 3\leftarrow 4)} = H_{0(2\leftarrow 3)} e^{E_2(r_2)} = \begin{pmatrix} 1 & e^{r_1} & 0 & e^{r_1+r_2} \\ 0 & 0 & -e^{r_1} & 1 \end{pmatrix}. $$

(3.24)

$H_{0(1\leftarrow 2\leftarrow 3)}$ can be transformed as

$$H_{0(1\leftarrow 2\leftarrow 3)} \rightarrow \begin{pmatrix} 1 & e^{r_2} & 0 & 0 \\ e^{-2r_2} & 1 & e^{r_2} & 0 \end{pmatrix} \begin{pmatrix} 1 & e^{r_1} & 0 & e^{r_1} \\ 0 & 1 & e^{r_1+r_2} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & e^{r_1+r_2+2} & 0 & e^{r+2} \\ e^{-2r_2} & 1 + e^{-r_2} & e^{r-2r_2+2} & e^{r-2r_2+2} \end{pmatrix},$$

(3.25)

where $r := r_1 + 2r_2 - \ln 2$. The limit of $H_{0(1\leftarrow 2\leftarrow 3)}$ in (3.25) as $r_2 \rightarrow +\infty$ with finite $r$ equals to $H_{0(1\leftarrow 3)}$ in (3.22). Or equivalently,

$$H_{0(1\leftarrow 2\leftarrow 3)} = H_{0(1)} e^{E_2(r_1)} e^{E_1(r_2)}$$

$$\simeq H_{0(1)} e^{E_2(r_1)} e^{[E_1,E_2](r_1+r_2+i\pi)} e^{[E_1,E_2](r_1+2r_2-\ln 2)}$$

$$= H_{0(1)} e^{E_2(r-2r_2+\ln 2)} e^{[E_1,E_2](r-r_2+2i\pi)} e^{[E_1,E_2](r)},$$

(3.26)

where $r := r_1 + 2r_2 - \ln 2$ and $\simeq$ means the following worldvolume symmetry transformation

$$H_{0(1)} e^{E_1(r_2)} = \begin{pmatrix} 1 & e^{r_2} \\ 0 & 1 \end{pmatrix} H_{0(1)} \simeq H_{0(1)}.$$ 

(3.27)
As \( r_2 \to +\infty \) with finite \( r \), \( H_{0(1\leftarrow 2\leftarrow 3)} \to H_{0(1\leftarrow 3)} \).
\( H_{0(2\leftarrow 3\leftarrow 4)} \) transforms as
\[
H_{0(2\leftarrow 3\leftarrow 4)} \to \begin{pmatrix} 1 & -e^{r_1+r_2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & e^{r_1} & 0 & e^{r_1+r_2} \\ 0 & 0 & -e^{r_1} & 1 \end{pmatrix} = \begin{pmatrix} 1 & e^{r_1} & e^{r+\ln 2} & 0 \\ 0 & 0 & -e^{r_1} & 1 \end{pmatrix}, \tag{3.28}
\]
where \( r := 2r_1 + r_2 - \ln 2 \). The limit of \( H_{0(2\leftarrow 3\leftarrow 4)} \) in (3.28) as \( r_1 \to -\infty \) with finite \( r \) equals to \( H_{0(2\leftarrow 4)} \) in (3.23). Or equivalently,
\[
H_{0(2\leftarrow 3\leftarrow 4)} = H_{0(2)} e^{E_1(r_1)} e^{E_2(r_2)}
\approx H_{0(2)} e^{[E_1, E_2] (r_1 + r_2)} e^{E_1(r_1) + [E_1, E_2] (2r_1 + r_2 - \ln 2 + i\pi)}
\approx H_{0(2)} e^{E_1(r_1) + [E_1, E_2] (2r_1 + r_2)} e^{E_1(r_1) + [E_1, E_2] (2r_1 + r_2 - \ln 2 + i\pi)}, \tag{3.29}
\]
where \( r := 2r_1 + r_2 - \ln 2 \) and \( \approx \) means the following worldvolume symmetry transformation
\[
H_{0(2)} e^{[E_1, E_2] (r_1 + r_2)} = \begin{pmatrix} 1 & e^{r_1+r_2} \\ 0 & 1 \end{pmatrix} H_{0(2)} \approx H_{0(2)}. \tag{3.30}
\]
As \( r_1 \to -\infty \) with finite \( r \), \( H_{0(2\leftarrow 3\leftarrow 4)} \to H_{0(2\leftarrow 4)} \).

Triple wall \( H_{0(1\leftarrow 2\leftarrow 3\leftarrow 4)} \) is
\[
H_{0(1\leftarrow 2\leftarrow 3\leftarrow 4)} = H_{0(1\leftarrow 2\leftarrow 3)} e^{E_2(r_3)} = \begin{pmatrix} 1 & e^{r_2} & 0 & e^{r_2+r_3} \\ 0 & 1 & -e^{r_1+r_2} & e^{r_1} \end{pmatrix}, \tag{3.31}
\]
which consists of three elementary walls \( \langle 1 \leftarrow 2 \rangle, \langle 2 \leftarrow 3 \rangle \) and \( \langle 3 \leftarrow 4 \rangle \). Since \( [E_2, [E_1, [E_1, E_2]]] = 0 \) or equivalently \( \vec{\alpha}_2 \cdot (2\vec{\alpha}_1 + \vec{\alpha}_2) = 0 \), triple wall \( \langle 1 \leftarrow 2 \leftarrow 3 \leftarrow 4 \rangle \) cannot be compressed to a compressed wall of level two. Instead, elementary wall \( \langle 1 \leftarrow 2 \rangle \) and compressed wall \( \langle 2 \leftarrow 4 \rangle \), which is a compressed wall of level one, are penetrable each other or compressed wall \( \langle 1 \leftarrow 3 \rangle \), which is a compressed wall of level one, and elementary wall \( \langle 3 \leftarrow 4 \rangle \) are penetrable each other.

We study walls of the nonlinear sigma model on \( Sp(3)/U(3) \). The simple root generators and the simple roots of \( USp(6) \) are
\[
E_1 = e_{1,2} - e_{5,4},
E_2 = e_{2,3} - e_{6,5},
E_3 = e_{3,6}, \tag{3.32}
\]

and
\[
\vec{\alpha}_1 = \hat{\varepsilon}_1 - \hat{\varepsilon}_2,
\]
\[
\vec{\alpha}_2 = \hat{\varepsilon}_2 - \hat{\varepsilon}_3,
\]
\[
\vec{\alpha}_3 = 2\hat{\varepsilon}_3.
\] (3.33)

The eight vacua of the nonlinear sigma model on $Sp(3)/U(3)$ are labelled in the descending order of (2.44):

\[
\langle 1 \rangle : (\Sigma_1, \Sigma_2, \Sigma_3) = (m_1, m_2, m_3),
\]
\[
\langle 2 \rangle : (\Sigma_1, \Sigma_2, \Sigma_3) = (m_1, m_2, -m_3),
\]
\[
\langle 3 \rangle : (\Sigma_1, \Sigma_2, \Sigma_3) = (m_1, -m_2, m_3),
\]
\[
\langle 4 \rangle : (\Sigma_1, \Sigma_2, \Sigma_3) = (m_1, -m_2, -m_3),
\]
\[
\langle 5 \rangle : (\Sigma_1, \Sigma_2, \Sigma_3) = (-m_1, m_2, m_3),
\]
\[
\langle 6 \rangle : (\Sigma_1, \Sigma_2, \Sigma_3) = (-m_1, m_2, -m_3),
\]
\[
\langle 7 \rangle : (\Sigma_1, \Sigma_2, \Sigma_3) = (-m_1, -m_2, m_3),
\]
\[
\langle 8 \rangle : (\Sigma_1, \Sigma_2, \Sigma_3) = (-m_1, -m_2, -m_3).
\] (3.34)

The tensions of elementary walls that connect vacua (3.34) are

\[
T_{\langle 1 \leftarrow 2 \rangle} = T_{\langle 3 \leftarrow 4 \rangle} = T_{\langle 5 \leftarrow 6 \rangle} = T_{\langle 7 \leftarrow 8 \rangle} = \vec{m} \cdot \vec{\alpha}_3,
\]
\[
T_{\langle 2 \leftarrow 3 \rangle} = T_{\langle 6 \leftarrow 7 \rangle} = 2\vec{m} \cdot \vec{\alpha}_2,
\]
\[
T_{\langle 3 \leftarrow 5 \rangle} = T_{\langle 4 \leftarrow 6 \rangle} = 2\vec{m} \cdot \vec{\alpha}_1.
\] (3.35)

Therefore the elementary walls are

\[
\vec{g}_{\langle 1 \leftarrow 2 \rangle} = \vec{g}_{\langle 3 \leftarrow 4 \rangle} = \vec{g}_{\langle 5 \leftarrow 6 \rangle} = \vec{g}_{\langle 7 \leftarrow 8 \rangle} = \vec{\alpha}_3,
\]
\[
\vec{g}_{\langle 2 \leftarrow 3 \rangle} = \vec{g}_{\langle 6 \leftarrow 7 \rangle} = 2\vec{\alpha}_2,
\]
\[
\vec{g}_{\langle 3 \leftarrow 5 \rangle} = \vec{g}_{\langle 4 \leftarrow 6 \rangle} = 2\vec{\alpha}_1.
\] (3.36)

There are penetrable walls since $\vec{\alpha}_1 \cdot \vec{\alpha}_3 = 0$. The diagram of the elementary walls of the nonlinear sigma model on $Sp(3)/U(3)$ are depicted in Figure 1(c). In this figure, a pair of penetrable elementary walls makes a parallelogram. A pair of facing sides of the parallelogram are the same simple roots whereas a pair of adjacent sides of the parallelogram are orthogonal simple roots.

We make some observations of walls. One can guess existence of compressed walls from the wall diagram in Figure 1(c). Since $\vec{g}_{\langle 1 \leftarrow 2 \rangle} \cdot \vec{g}_{\langle 2 \leftarrow 3 \rangle} \neq 0$, elementary wall $\langle 1 \leftarrow 2 \rangle$ and
Figure 1: Elementary walls of the nonlinear sigma models on $Sp(N)/U(N)$. (a)$N = 1$
(b)$N = 2$ (c)$N = 3$ and (d)$N = 4$. The numbers indicate the subscript $i$’s of roots $\vec{\alpha}_i$. The
left-hand side is the limit as $x \to +\infty$ and the right-hand side is the limit as $x \to -\infty$.

elementary wall $\langle 2 \leftarrow 3 \rangle$ are compressed to compressed wall $\langle 1 \leftarrow 3 \rangle$, which is a compressed wall of level one. The moduli matrix of $\langle 1 \leftarrow 3 \rangle$ is

$$H_{0(1\leftarrow 3)} = H_{0(1)} e^{[E_3,E_2](r)}.$$  \hfill (3.37)

One can also see that $\tilde{g}_{(2\leftarrow 3)} \cdot \tilde{g}_{(3\leftarrow 5)} \neq 0$. Therefore elementary wall $\langle 2 \leftarrow 3 \rangle$ and elementary wall $\langle 3 \leftarrow 5 \rangle$ are compressed to compressed wall $\langle 2 \leftarrow 5 \rangle$, which is a compressed wall of level one. The moduli matrix of compressed wall $\langle 2 \leftarrow 5 \rangle$ is

$$H_{0(2\leftarrow 5)} = H_{0(2)} e^{[E_2,E_1](r)}.$$  \hfill (3.38)

Let us consider the moduli matrix of double wall $\langle 1 \leftarrow 2 \leftarrow 3 \rangle$

$$H_{0(1\leftarrow 2\leftarrow 3)} = H_{0(1)} e^{E_3(r_1)} e^{E_2(r_2)},$$  \hfill (3.39)

and the moduli matrix of double wall $\langle 2 \leftarrow 3 \leftarrow 5 \rangle$

$$H_{0(2\leftarrow 3\leftarrow 5)} = H_{0(2)} e^{E_2(r_1)} e^{E_1(r_2)}.$$  \hfill (3.40)

Double wall $\langle 1 \leftarrow 2 \leftarrow 3 \rangle$ in (3.39) is

$$H_{0(1\leftarrow 2\leftarrow 3)} = H_{0(1)} e^{E_3(r_1)} e^{E_2(r_2)} = H_{0(1)} e^{E_2(r_2)} e^{E_3(r_1)} e^{[E_2,E_3](r_1+r_2+i\pi)} e^{[E_2,[E_2,E_3]](r_1+2r_2-\ln 2)}$$

$$\simeq H_{0(1)} e^{E_3(r_2-2\ln 2)} e^{[E_2,E_3](r_2-2\ln 2+i\pi)} e^{[E_2,[E_2,E_3]](r)},$$  \hfill (3.41)
where \( r := r_1 + 2r_2 - \ln 2 \) and \( \simeq \) means
\[
H_{0(1)} e^{E_2(r_2)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & e^{r_2} \\ 0 & 0 & 1 \end{pmatrix} H_{0(1)} \simeq H_{0(1)}. \tag{3.42}
\]

As \( r_2 \to +\infty \) with finite \( r \), the limit of \( H_{0(1-2-3)} \) equals to \( H_{0(1-3)} \). Double wall \( \langle 1 \leftarrow 2 \leftarrow 3 \rangle \) is plotted in Figure 2.

Double wall \( \langle 2 \leftarrow 3 \leftarrow 5 \rangle \) (3.40) is
\[
H_{0(2-3-5)} = H_{0(2)} e^{E_2(r_1)} e^{E_1(r_2)}
= H_{0(2)} e^{E_1(r_2)} e^{E_2(r_1)} e^{[E_2,E_1]}(r_1+r_2)
\simeq H_{0(2)} e^{E_2(r_1)} e^{[E_1,E_2]}(r), \tag{3.43}
\]
where \( r := r_1 + r_2 + i\pi \) and \( \simeq \) means
\[
H_{0(2)} e^{E_1(r_2)} = \begin{pmatrix} 1 & 0 & e^r \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} H_{0(2)} \simeq H_{0(2)}. \tag{3.44}
\]

As \( r_1 \to -\infty \) with finite \( r \), the limit of \( H_{0(2-3-5)} \) equals to \( H_{0(2-5)} \). Double wall \( \langle 2 \leftarrow 3 \leftarrow 5 \rangle \) is compressed to compressed wall \( \langle 2 \leftarrow 5 \rangle \), which is a compressed wall of level one.

Figure 2: Double wall \( \langle 1 \leftarrow 2 \leftarrow 3 \rangle \) in \( Sp(3)/U(3) \), which consists of two elementary walls \( \langle 1 \leftarrow 2 \rangle \) and \( \langle 2 \leftarrow 3 \rangle \). They are compressed to \( \langle 1 \leftarrow 3 \rangle \). \( m_1 = 8, m_2 = 4, m_3 = 2 \). (a) \( r_1 = 60, r_2 = 20 \), (b) \( r_1 = 52, r_2 = 24 \), (c) \( r_1 = 40, r_2 = 30 \).
Next we discuss penetrable walls. Since $\tilde{g}_{(3\leftarrow 5)} \cdot \tilde{g}_{(5\leftarrow 6)} = 0$, we can observe elementary wall $\langle 3 \leftarrow 5 \rangle$ and elementary wall $\langle 5 \leftarrow 6 \rangle$ pass through each other. Double wall $\langle 3 \leftarrow 5 \leftarrow 6 \rangle$ is plotted in Figure 3.

![Image of Figure 3](image)

Figure 3: Double wall $\langle 3 \leftarrow 5 \leftarrow 6 \rangle$ in $Sp(3)/U(3)$, which consists of two penetrable walls. $m_1 = 8$, $m_2 = 5$, $m_3 = 2$. (a) $r_1 = 60$, $r_2 = 50$, (b) $r_1 = 60$, $r_2 = 80$, (c) $r_1 = 60$, $r_2 = 95$.

The moduli matrix of $\langle 1 \leftarrow 5 \rangle$, which is a compressed wall of level two is

$$H_{0(1\leftarrow 5)} = H_{0(1)} e^{\left[\left[ E_3, E_2 \right], E_1 \right]} (r).$$  \hspace{1cm} (3.45)

The moduli matrix of triple wall $\langle 1 \leftarrow 2 \leftarrow 3 \leftarrow 5 \rangle$ is

$$H_{0(1\leftarrow 5)} = H_{0(1)} e^{E_3(r_1)} e^{E_2(r_2)} e^{E_1(r_3)}.$$  \hspace{1cm} (3.46)

We shall consider higher $N$. Elementary walls can be identified with the simple roots of $USp(2N)$ with proper coefficients. All the compressed single walls and multiwalls can be constructed from the elementary wall configuration. The elementary wall configuration for
$N = 4$ is

$$\bar{g}(5\leftrightarrow 9) = \bar{g}(6\leftrightarrow 10) = \bar{g}(7\leftrightarrow 11) = \bar{g}(8\leftrightarrow 12) = 2\bar{\alpha}_1,$$

$$\bar{g}(3\leftrightarrow 5) = \bar{g}(4\leftrightarrow 6) = \bar{g}(11\leftrightarrow 13) = \bar{g}(12\leftrightarrow 14) = 2\bar{\alpha}_2,$$

$$\bar{g}(2\leftrightarrow 3) = \bar{g}(6\leftrightarrow 7) = \bar{g}(10\leftrightarrow 11) = \bar{g}(14\leftrightarrow 15) = 2\bar{\alpha}_3,$$

$$\bar{g}(1\leftrightarrow 2) = \bar{g}(3\leftrightarrow 4) = \bar{g}(5\leftrightarrow 6) = \bar{g}(7\leftrightarrow 8) = \bar{g}(9\leftrightarrow 10) = \bar{g}(11\leftrightarrow 12) = \bar{g}(13\leftrightarrow 14) = \bar{g}(15\leftrightarrow 16) = \bar{\alpha}_4. \quad (3.47)$$

The diagram of the elementary walls are depicted in Figure 1(d). We leave vacuum labels out of diagrams from Figure 1(d) onwards.

While the elementary wall diagrams are planar for $N \leq 4$, the diagrams are non-planar for $N \geq 5$. The elementary wall configurations for $N = 5$ and $N = 6$ are as follows:

- $N = 5$

$$\bar{g}(9\leftrightarrow 17) = \bar{g}(10\leftrightarrow 18) = \bar{g}(11\leftrightarrow 19) = \bar{g}(12\leftrightarrow 20)$$

$$= \bar{g}(13\leftrightarrow 21) = \bar{g}(14\leftrightarrow 22) = \bar{g}(15\leftrightarrow 23) = \bar{g}(16\leftrightarrow 24) = 2\bar{\alpha}_1,$$

$$\bar{g}(5\leftrightarrow 9) = \bar{g}(6\leftrightarrow 10) = \bar{g}(7\leftrightarrow 11) = \bar{g}(8\leftrightarrow 12)$$

$$= \bar{g}(21\leftrightarrow 25) = \bar{g}(22\leftrightarrow 26) = \bar{g}(23\leftrightarrow 27) = \bar{g}(24\leftrightarrow 28) = 2\bar{\alpha}_2,$$

$$\bar{g}(3\leftrightarrow 5) = \bar{g}(4\leftrightarrow 6) = \bar{g}(11\leftrightarrow 13) = \bar{g}(12\leftrightarrow 14)$$

$$= \bar{g}(19\leftrightarrow 21) = \bar{g}(20\leftrightarrow 22) = \bar{g}(27\leftrightarrow 29) = \bar{g}(28\leftrightarrow 30) = 2\bar{\alpha}_3,$$

$$\bar{g}(2\leftrightarrow 3) = \bar{g}(6\leftrightarrow 7) = \bar{g}(10\leftrightarrow 11) = \bar{g}(14\leftrightarrow 15)$$

$$= \bar{g}(18\leftrightarrow 19) = \bar{g}(22\leftrightarrow 23) = \bar{g}(26\leftrightarrow 27) = \bar{g}(30\leftrightarrow 31) = 2\bar{\alpha}_4,$$

$$\bar{g}(1\leftrightarrow 2) = \bar{g}(3\leftrightarrow 4) = \bar{g}(5\leftrightarrow 6) = \bar{g}(7\leftrightarrow 8)$$

$$= \bar{g}(9\leftrightarrow 10) = \bar{g}(11\leftrightarrow 12) = \bar{g}(13\leftrightarrow 14) = \bar{g}(15\leftrightarrow 16)$$

$$= \bar{g}(17\leftrightarrow 18) = \bar{g}(19\leftrightarrow 20) = \bar{g}(21\leftrightarrow 22) = \bar{g}(23\leftrightarrow 24)$$

$$= \bar{g}(25\leftrightarrow 26) = \bar{g}(27\leftrightarrow 28) = \bar{g}(29\leftrightarrow 30) = \bar{g}(31\leftrightarrow 32) = \bar{\alpha}_5. \quad (3.48)$$
\[ N = 6 \]

\[
\begin{align*}
&= \vec{g}(17\leftarrow 33) = \vec{g}(18\leftarrow 34) = \vec{g}(19\leftarrow 35) = \vec{g}(20\leftarrow 36) = \vec{g}(21\leftarrow 37) = \vec{g}(22\leftarrow 38) = \vec{g}(23\leftarrow 39) = \vec{g}(24\leftarrow 40) = \vec{g}(25\leftarrow 41) = \vec{g}(26\leftarrow 42) = \vec{g}(27\leftarrow 43) = \vec{g}(28\leftarrow 44) = \vec{g}(29\leftarrow 45) = \vec{g}(30\leftarrow 46) = \vec{g}(31\leftarrow 47) = \vec{g}(32\leftarrow 48) = 2\vec{\alpha}_1; \\
&= \vec{g}(9\leftarrow 17) = \vec{g}(10\leftarrow 18) = \vec{g}(11\leftarrow 19) = \vec{g}(12\leftarrow 20) = \vec{g}(13\leftarrow 21) = \vec{g}(14\leftarrow 22) = \vec{g}(15\leftarrow 23) = \vec{g}(16\leftarrow 24) = \vec{g}(17\leftarrow 25) = \vec{g}(22\leftarrow 26) = \vec{g}(23\leftarrow 27) = \vec{g}(24\leftarrow 28) = \vec{g}(37\leftarrow 41) = \vec{g}(38\leftarrow 42) = \vec{g}(39\leftarrow 43) = \vec{g}(40\leftarrow 44) = \vec{g}(54\leftarrow 57) = \vec{g}(54\leftarrow 58) = \vec{g}(54\leftarrow 59) = \vec{g}(56\leftarrow 60) = 2\vec{\alpha}_3; \\
&= \vec{g}(3\leftarrow 5) = \vec{g}(4\leftarrow 6) = \vec{g}(11\leftarrow 13) = \vec{g}(12\leftarrow 14) = \vec{g}(19\leftarrow 21) = \vec{g}(20\leftarrow 22) = \vec{g}(27\leftarrow 29) = \vec{g}(28\leftarrow 30) = \vec{g}(35\leftarrow 37) = \vec{g}(36\leftarrow 38) = \vec{g}(43\leftarrow 45) = \vec{g}(44\leftarrow 46) = \vec{g}(51\leftarrow 53) = \vec{g}(52\leftarrow 54) = \vec{g}(59\leftarrow 61) = \vec{g}(60\leftarrow 62) = 2\vec{\alpha}_4; \\
&= \vec{g}(2\leftarrow 3) = \vec{g}(6\leftarrow 7) = \vec{g}(10\leftarrow 11) = \vec{g}(14\leftarrow 15) = \vec{g}(18\leftarrow 19) = \vec{g}(22\leftarrow 23) = \vec{g}(26\leftarrow 27) = \vec{g}(30\leftarrow 31) = \vec{g}(34\leftarrow 35) = \vec{g}(38\leftarrow 39) = \vec{g}(42\leftarrow 43) = \vec{g}(46\leftarrow 47) = \vec{g}(50\leftarrow 51) = \vec{g}(54\leftarrow 55) = \vec{g}(58\leftarrow 59) = \vec{g}(62\leftarrow 63) = 2\vec{\alpha}_5; \\
&= \vec{g}(1\leftarrow 2) = \vec{g}(3\leftarrow 4) = \vec{g}(5\leftarrow 6) = \vec{g}(7\leftarrow 8) = \vec{g}(9\leftarrow 10) = \vec{g}(11\leftarrow 12) = \vec{g}(13\leftarrow 14) = \vec{g}(15\leftarrow 16) = \vec{g}(17\leftarrow 18) = \vec{g}(19\leftarrow 20) = \vec{g}(21\leftarrow 22) = \vec{g}(23\leftarrow 24) = \vec{g}(25\leftarrow 26) = \vec{g}(27\leftarrow 28) = \vec{g}(29\leftarrow 30) = \vec{g}(31\leftarrow 32) = \vec{g}(33\leftarrow 34) = \vec{g}(35\leftarrow 36) = \vec{g}(37\leftarrow 38) = \vec{g}(39\leftarrow 40) = \vec{g}(41\leftarrow 42) = \vec{g}(43\leftarrow 44) = \vec{g}(45\leftarrow 46) = \vec{g}(47\leftarrow 48) = \vec{g}(49\leftarrow 50) = \vec{g}(51\leftarrow 52) = \vec{g}(53\leftarrow 54) = \vec{g}(55\leftarrow 56) = \vec{g}(57\leftarrow 58) = \vec{g}(59\leftarrow 60) = \vec{g}(61\leftarrow 62) = \vec{g}(63\leftarrow 64) = \vec{\alpha}_6. \tag{3.49}
\end{align*}
\]
The diagrams of the elementary walls of $N = 5$ and $N = 6$ cases are depicted in Figure 4 and Figure 5.

Figure 4: Elementary walls of the nonlinear sigma model on $Sp(5)/U(5)$. The left-hand side is the limit as $x \to +\infty$ and the right-hand side is the limit as $x \to -\infty$.

Figure 5: Elementary walls on $Sp(6)/U(6)$. The left-hand side is the limit as $x \to +\infty$ and the right-hand side is the limit as $x \to -\infty$.

4 Vacua connected to the maximum number of elementary walls

We study the vacua that are connected to the maximum number of elementary walls. We denote the vacua $\langle A \rangle$ and $\langle B \rangle$. Let $\langle A \rangle$ be the vacuum near $\langle 1 \rangle$ and $\langle B \rangle$ be the vacuum near $\langle 2^N \rangle$. From Figure 1, Figure 4 and Figure 5, we make the following observations where $\vec{m}$ denotes simple root $\vec{\alpha}_m$:

- $N = 1$
  
  \[
  \langle 1 \rangle \leftarrow \vec{1} \leftarrow \langle 2 \rangle 
  \tag{4.1}
  \]
\(N = 2\)
\[ \tilde{2} \leftarrow \langle A \rangle \leftarrow \tilde{1} \leftarrow \langle B \rangle \leftarrow \tilde{2} \] (4.2)

\(N = 3\)
\[ \tilde{3} \leftarrow \cdots \leftarrow \tilde{2} \leftarrow \langle A \rangle \leftarrow \{\tilde{1}, \tilde{3}\} \leftarrow \cdots \]
\[ \cdots \leftarrow \{\tilde{1}, \tilde{3}\} \leftarrow \langle B \rangle \leftarrow \tilde{2} \leftarrow \cdots \leftarrow \tilde{3} \] (4.3)

\(N = 4\)
\[ \tilde{4} \leftarrow \cdots \leftarrow \{\tilde{2}, \tilde{4}\} \leftarrow \langle A \rangle \leftarrow \{\tilde{1}, \tilde{3}\} \leftarrow \cdots \]
\[ \cdots \leftarrow \{\tilde{1}, \tilde{3}\} \leftarrow \langle B \rangle \leftarrow \{\tilde{2}, \tilde{4}\} \leftarrow \cdots \leftarrow \tilde{4} \] (4.4)

\(N = 5\)
\[ \tilde{5} \leftarrow \cdots \leftarrow \{\tilde{2}, \tilde{4}\} \leftarrow \langle A \rangle \leftarrow \{\tilde{1}, \tilde{3}, \tilde{5}\} \leftarrow \cdots \]
\[ \cdots \leftarrow \{\tilde{1}, \tilde{3}, \tilde{5}\} \leftarrow \langle B \rangle \leftarrow \{\tilde{2}, \tilde{4}\} \leftarrow \cdots \leftarrow \tilde{5} \] (4.5)

\(N = 6\)
\[ \tilde{6} \leftarrow \cdots \leftarrow \{\tilde{2}, \tilde{4}, \tilde{6}\} \leftarrow \langle A \rangle \leftarrow \{\tilde{1}, \tilde{3}, \tilde{5}\} \leftarrow \cdots \]
\[ \cdots \leftarrow \{\tilde{1}, \tilde{3}, \tilde{5}\} \leftarrow \langle B \rangle \leftarrow \{\tilde{2}, \tilde{4}, \tilde{6}\} \leftarrow \cdots \leftarrow \tilde{6} \] (4.6)

From Figure 1, Figure 4, Figure 5, (3.36), (3.47), (3.48) and (3.49), \(\langle A \rangle\) and \(\langle B \rangle\) are identified as follows:

\(N = 3\)
\[ \langle A \rangle = \langle 3 \rangle, \quad \langle B \rangle = \langle 6 \rangle \] (4.7)

\(N = 4\)
\[ \langle A \rangle = \langle 6 \rangle, \quad \langle B \rangle = \langle 11 \rangle \] (4.8)

\(N = 5\)
\[ \langle A \rangle = \langle 11 \rangle, \quad \langle B \rangle = \langle 22 \rangle \] (4.9)

\(N = 6\)
\[ \langle A \rangle = \langle 22 \rangle, \quad \langle B \rangle = \langle 43 \rangle \] (4.10)
The vacuum labels are not unique since we can change them as we please. Therefore let us label the vacua that are connected to the maximum number of elementary walls as $\langle A \rangle$ and $\langle B \rangle$.

The vacuum structures that are connected to the maximum number of elementary walls are as follows:

- $N = 4m - 3$ ($m \geq 2$)

\[
\begin{align*}
\vec{N} & \leftarrow \ldots \\
\ldots & \leftarrow \left\{ \underbrace{\vec{2}, \vec{4}, \ldots, \vec{4m - 6}, \vec{4m - 4}}_{2m-2} \right\} \leftarrow \langle A \rangle \leftarrow \\
& \left\{ \underbrace{1, 3, \ldots, \vec{4m - 5}, \vec{4m - 3}}_{2m-1} \right\} \leftarrow \ldots \\
\ldots & \left\{ \underbrace{1, 3, \ldots, \vec{4m - 5}, \vec{4m - 3}}_{2m-1} \right\} \leftarrow \langle B \rangle \leftarrow \\
& \left\{ \underbrace{\vec{2}, \vec{4}, \ldots, \vec{4m - 6}, \vec{4m - 4}}_{2m-2} \right\} \leftarrow \ldots \\
\ldots & \leftarrow \vec{N}
\end{align*}
\]

- $N = 4m - 2$ ($m \geq 2$)

\[
\begin{align*}
\vec{N} & \leftarrow \ldots \\
\ldots & \left\{ \underbrace{\vec{2}, \vec{4}, \ldots, \vec{4m - 4}, \vec{4m - 2}}_{2m-1} \right\} \leftarrow \langle A \rangle \leftarrow \\
& \left\{ \underbrace{1, 3, \ldots, \vec{4m - 5}, \vec{4m - 3}}_{2m-1} \right\} \leftarrow \ldots \\
\ldots & \left\{ \underbrace{1, 3, \ldots, \vec{4m - 5}, \vec{4m - 3}}_{2m-1} \right\} \leftarrow \langle B \rangle \leftarrow \\
& \left\{ \underbrace{\vec{2}, \vec{4}, \ldots, \vec{4m - 4}, \vec{4m - 2}}_{2m-1} \right\} \leftarrow \ldots \\
\ldots & \leftarrow \vec{N}
\end{align*}
\]
\[ N = 4m - 1 \quad (m \geq 2) \]

\[ \vec{N} \leftarrow \ldots \]
\[ \ldots \leftarrow \left\{ \overline{2, 4, \ldots, 4m - 4, 4m - 2} \right\} \leftarrow \langle A \rangle \leftarrow \]
\[ \left\{ \overline{1, 3, \ldots, 4m - 3, 4m - 1} \right\} \leftarrow \ldots \]
\[ \ldots \leftarrow \left\{ \overline{1, 3, \ldots, 4m - 3, 4m - 1} \right\} \leftarrow \langle B \rangle \leftarrow \]
\[ \left\{ \overline{2, 4, \ldots, 4m - 4, 4m - 2} \right\} \leftarrow \ldots \]
\[ \ldots \leftarrow \vec{N} \quad (4.13) \]

\[ N = 4m \quad (m \geq 2) \]

\[ \vec{N} \leftarrow \ldots \]
\[ \ldots \leftarrow \left\{ \overline{2, 4, \ldots, 4m - 2, 4m} \right\} \leftarrow \langle A \rangle \leftarrow \]
\[ \left\{ \overline{1, 3, \ldots, 4m - 3, 4m - 1} \right\} \leftarrow \ldots \]
\[ \ldots \leftarrow \left\{ \overline{1, 3, \ldots, 4m - 3, 4m - 1} \right\} \leftarrow \langle B \rangle \leftarrow \]
\[ \left\{ \overline{2, 4, \ldots, 4m - 2, 4m} \right\} \leftarrow \ldots \]
\[ \ldots \leftarrow \vec{N} \quad (4.14) \]

(4.11), (4.12), (4.13) and (4.14) are proved in Appendix A.

5 Walls of nonlinear sigma model on \( Sp(5)/U(5) \)

We have studied the vacuum structures that are connected to the maximum number of elementary walls for general \( N \). The elementary walls can be compressed or can pass through each other. We discuss some features of elementary walls of nonlinear sigma model on \( Sp(5)/U(5) \), which is the simplest nontrivial case. From (3.48) and (4.5), (11) is one of
the vacua that are connected to the maximum number of elementary walls. The structure near \langle 11 \rangle is

\[
\langle 7 \rangle \leftarrow 5 \leftarrow \langle 11 \rangle \leftarrow \langle 13 \rangle \leftarrow \langle 12 \rangle \leftarrow \langle 19 \rangle \leftarrow \langle 10 \rangle \leftarrow 4 \leftarrow \langle 11 \rangle \leftarrow \langle 19 \rangle \leftarrow \langle 12 \rangle \leftarrow \langle 12 \rangle.
\]

(5.1)

In (5.1), \( \vec{\alpha}_2 \cdot \vec{\alpha}_1 \neq 0 \). Therefore elementary wall \langle 7 \leftarrow 11 \rangle and elementary wall \langle 11 \leftarrow 19 \rangle are compressed to a single wall. Vacuum \langle 7 \rangle is labelled by \((\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4, \Sigma_5) = (m_1, m_2, -m_3, -m_4, m_5)\). The moduli matrix for double wall \langle 7 \leftarrow 11 \leftarrow 19 \rangle is

\[
H_{0(7\leftarrow 11\leftarrow 19)} = H_{0(7)} e^{E_2(r_1)} e^{E_1(r_2)} = H_{0(7)} e^{E_1(r_2)} e^{E_2(r_1) - [E_1, E_2](r_1 + r_2)} \approx H_{0(7)} e^{E_2(r_1) - [E_1, E_2](r_1 + r_2)}.
\]

(5.2)

where \( \approx \) means

\[
H_{0(7)} e^{E_1(r_2)} = (I_5 + e^r e_{1,2}) H_{0(7)} \approx H_{0(7)}.
\]

(5.3)

As \( r_1 \to -\infty \) with \( r_1 + r_2 = r \) (finite), \( H_{0(7\leftarrow 11\leftarrow 19)} \to H_{0(7\leftarrow 19)} \). Double wall \langle 7 \leftarrow 11 \leftarrow 19 \rangle is compressed to compressed wall \langle 7 \leftarrow 19 \rangle, which is a compressed wall of level one. This is depicted in Figure 6.

In (5.1) \( \vec{\alpha}_2 \cdot \vec{\alpha}_5 = 0 \). Therefore elementary wall \langle 7 \leftarrow 11 \rangle and elementary wall \langle 11 \leftarrow 12 \rangle are penetrable. The moduli matrix of double wall \langle 7 \leftarrow 11 \leftarrow 12 \rangle, which consists of two penetrable elementary walls \langle 7 \leftarrow 11 \rangle and \langle 11 \leftarrow 12 \rangle is

\[
H_{0(7\leftarrow 11\leftarrow 12)} = H_{0(7\leftarrow 11)} e^{E_2(r_2)} = H_{0(7)} e^{E_2(r_1)} e^{E_5(r_2)} = H_{0(7)} e^{E_5(r_2)} e^{E_2(r_1)} = H_{0(7\leftarrow 8)} e^{E_2(r_1)}.
\]

(5.4)

This is depicted in Figure 7.

6 Conclusion

We have studied the vacua and the walls of mass-deformed Kähler nonlinear sigma models on \( Sp(N)/U(N) \) by using the moduli matrix formalism. For \( N = 1 \) and \( N = 2 \), the nonlinear sigma models on \( Sp(N)/U(N) \) are Abelian theories, where single walls are compressed to compressed walls while penetrable walls are not allowed. On the other hand, for \( N \geq 3 \),
Figure 6: Double wall $\langle 7 \leftarrow 11 \leftarrow 19 \rangle$ in $Sp(5)/U(5)$. Elementary walls $\langle 7 \leftarrow 11 \rangle$ and $\langle 11 \leftarrow 19 \rangle$ are compressed to $\langle 7 \leftarrow 19 \rangle$ as $r_1 \to -\infty$ with $r_1 + r_2 = r_{\text{finite}}$. $m_1 = 12$, $m_2 = 8$, $m_3 = 6$, $m_4 = 4$, $m_5 = 2$. (a)$r_1 = 30$, $r_2 = 30$, (b)$r_1 = 22$, $r_2 = 38$, (c)$r_1 = 19$, $r_2 = 41$.

Figure 7: Double wall $\langle 7 \leftarrow 11 \leftarrow 12 \rangle$ in $Sp(5)/U(5)$. Elementary walls $\langle 7 \leftarrow 11 \rangle$ and $\langle 11 \leftarrow 12 \rangle$ pass through each other. $m_1 = 12$, $m_2 = 8$, $m_3 = 6$, $m_4 = 4$, $m_5 = 2$. (a)$r_1 = 20$, $r_2 = 25$, (b)$r_1 = 20$, $r_2 = 37$, (c)$r_1 = 20$, $r_2 = 55$.

the nonlinear sigma models on $Sp(N)/U(N)$ are non-Abelian theories so there exist pen-
etrable walls, which lead to a unique vacuum configuration for each \( N \). We have proved the vacuum structures that are connected to the maximum number of elementary walls by induction.

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Appendix A  Vacuum structures

In this appendix, we prove (4.11), (4.12), (4.13) and (4.14). The vacuum structures that are connected to the maximum number of elementary walls in the nonlinear sigma models on \( SO(2N)/U(N) \) are studied by decomposing the diagrams into two-dimensional diagrams in [11]. We use the same method in the nonlinear sigma models on \( Sp(N)/U(N) \). The rule for the decomposition is that the simple roots that have already appeared in the previous diagrams should not be repeated.

The vacuum structure of \( N = 5 \) case is depicted in Figure 4. The vacuum structure near \( \langle 1 \rangle (\langle 32 \rangle) \) decomposes into two diagrams, as is shown in Figure 8. The circle indicates \( \langle A \rangle (\langle B \rangle) \). The letter ‘X’ indicates the vacuum that is connected to the both diagrams. The left-hand side(right-hand side) of the each diagram is the limit as \( x \to +\infty (x \to -\infty) \) for the vacuum structure near \( \langle 1 \rangle \) whereas the left-hand side(right-hand side) of the each diagrams is the limit as \( x \to -\infty (x \to +\infty) \) for the vacuum structure near \( \langle 32 \rangle \).

Figure 5, which describes the vacuum structure of \( N = 6 \) case decomposes into two diagrams as is shown in Figure 9. In the same manner the vacuum structures of \( N = 7 \) and \( N = 8 \) cases are presented in Figure 10 and Figure 11. The vacuum structures repeat the four diagrams in Figure 1.

All the vacuum structures can be decomposed into two dimensional diagrams in Figure 12, where only the first two diagrams are shown and then fall into four categories. The vacuum that is connected to the maximum number of elementary walls is circled in each diagram in Figure 13.

The vacuum structures that are connected to the maximum number of elementary walls can be obtained from the repeated diagrams. \( \langle A \rangle \) denotes the vacuum near \( \langle 1 \rangle \) and \( \langle B \rangle \)
Figure 8: $N = 5$ case. The vacuum structure near $\langle 1 \rangle (\langle 32 \rangle)$ decomposes into two diagrams.

Figure 9: $N = 6$ case. The vacuum structure near $\langle 1 \rangle (\langle 64 \rangle)$ decomposes into two diagrams.

Figure 10: $N = 7$ case. The vacuum structure near $\langle 1 \rangle (\langle 128 \rangle)$ decomposes into two diagrams.

denotes the vacuum near $\langle 2^N \rangle$. The common parts of each vacuum structure near $\langle A \rangle$ and $\langle B \rangle$ are shown in Figure 14 and Figure 15. The rest of the vacuum structures are obtained from Figure 13. The remaining parts of each vacuum structure near $\langle A \rangle$ and $\langle B \rangle$ are shown in Figure 16 and Figure 17 for $N = 4m - 3$, $N = 4m - 2$, $N = 4m - 1$ and $N = 4m$.

The vacuum structure of $\langle A \rangle$ is derived from Figure 14 and Figure 16 as follows:
Figure 11: $N = 8$ case. The vacuum structure near $\langle 1 \rangle$ ($\langle 2^{56} \rangle$) decomposes into two diagrams.

Figure 12: First two diagrams of the vacuum structure near $\langle 1 \rangle$ and $\langle 2^N \rangle$.

- $N = 4m - 3$, $(m \geq 2)$
  \[
  \{ \overline{2,4,\ldots,4m-6},\overline{4m-4} \}_{2m-2} \leftarrow \langle A \rangle \leftarrow \{ \overline{1,3,\ldots,4m-5},\overline{4m-3} \}_{2m-1} \]  
  \hspace{2cm} (A.1)

- $N = 4m - 2$, $(m \geq 2)$
  \[
  \{ \overline{2,4,\ldots,4m-4,4m-2} \}_{2m-1} \leftarrow \langle A \rangle \leftarrow \{ \overline{1,3,\ldots,4m-5,4m-3} \}_{2m-1} \]  
  \hspace{2cm} (A.2)

- $N = 4m - 1$, $(m \geq 2)$
  \[
  \{ \overline{2,4,\ldots,4m-4,4m-2} \}_{2m-1} \leftarrow \langle A \rangle \leftarrow \{ \overline{1,3,\ldots,4m-3,4m-1} \}_{2m} \]  
  \hspace{2cm} (A.3)

- $N = 4m$, $(m \geq 2)$
  \[
  \{ \overline{2,4,\ldots,4m-2,4m} \}_{2m} \leftarrow \langle A \rangle \leftarrow \{ \overline{1,3,\ldots,4m-3,4m-1} \}_{2m} \]  
  \hspace{2cm} (A.4)
Figure 13: Four types of vacuum structures. The circle indicates the vacuum that is connected to the maximum number of simple roots.

Figure 14: common part near ⟨A⟩

Figure 15: common part near ⟨B⟩

Each case with $m = 1$ is shown in Figure 1. Each case with $m = 2$ is shown in Figure 8, Figure 9, Figure 10 and Figure 11. Let us assume that (A.1), (A.2), (A.3) and (A.4) are true. Then these are true for $m' = m + 1$ as it corresponds to adding one more diagram in
Figure 16: Remaining part of the vacuum structure near $\langle A \rangle$. The location of $\langle A \rangle$ is circled.

Figure 17: Remaining part of the vacuum structure near $\langle B \rangle$. The location of $\langle B \rangle$ is circled.

Figure 14. Therefore (A.1), (A.2), (A.3) and (A.4) are true.
The vacuum structure of $\langle B \rangle$ is derived from Figure 15 and Figure 17 as follows:

- $N = 4m - 3$, $(m \geq 2)$
  \[
  \{ \overbrace{1, 3, \ldots, 4m - 5, 4m - 3}^{2m-1} \} \leftarrow \langle B \rangle \leftarrow \{ \overbrace{2, 4, \ldots, 4m - 6, 4m - 4}^{2m-2} \} \quad (A.5)
  \]

- $N = 4m - 2$, $(m \geq 2)$
  \[
  \{ \overbrace{1, 3, \ldots, 4m - 5, 4m - 3}^{2m-1} \} \leftarrow \langle B \rangle \leftarrow \{ \overbrace{2, 4, \ldots, 4m - 4, 4m - 2}^{2m-1} \} \quad (A.6)
  \]

- $N = 4m - 1$, $(m \geq 2)$
  \[
  \{ \overbrace{1, 3, \ldots, 4m - 3, 4m - 1}^{2m} \} \leftarrow \langle B \rangle \leftarrow \{ \overbrace{2, 4, \ldots, 4m - 4, 4m - 2}^{2m-1} \} \quad (A.7)
  \]

- $N = 4m$, $(m \geq 2)$
  \[
  \{ \overbrace{1, 3, \ldots, 4m - 3, 4m - 1}^{2m} \} \leftarrow \langle B \rangle \leftarrow \{ \overbrace{2, 4, \ldots, 4m - 2, 4m}^{2m} \} \quad (A.8)
  \]

Each case with $m = 1$ is shown in Figure 1. Each case with $m = 2$ is shown in Figure 8, Figure 9, Figure 10 and Figure 11. Let us assume that (A.5), (A.6), (A.7) and (A.8) are true. Then these are true for $m' = m + 1$ as it corresponds to adding one more diagram in Figure 15. Therefore (A.5), (A.6), (A.7) and (A.8) are true.

For any $N$ the vacuum structure is

\[
\vec{N} \leftarrow \cdots \leftarrow \langle A \rangle \leftarrow \cdots \leftarrow \langle B \rangle \leftarrow \cdots \leftarrow \vec{N} \quad (A.9)
\]

Therefore (4.11), (4.12), (4.13) and (4.14) are proved.
References

[1] P. Di Vecchia and S. Ferrara, Nucl. Phys. B 130, 93 (1977). doi:10.1016/0550-3213(77)90394-7; E. Witten, Phys. Rev. D 16, 2991 (1977). doi:10.1103/PhysRevD.16.2991. A. D’Adda, P. Di Vecchia and M. Luscher, Nucl. Phys. B 152, 125 (1979). doi:10.1016/0550-3213(79)90083-X; B. Zumino, Phys. Lett. 87B, 203 (1979). doi:10.1016/0370-2693(79)90964-X.

[2] L. Alvarez-Gaume and D. Z. Freedman, Commun. Math. Phys. 80, 443 (1981). doi:10.1007/BF01208280

[3] T. L. Curtright and D. Z. Freedman, Phys. Lett. 90B, 71 (1980) Erratum: [Phys. Lett. 91B, 487 (1980)]. doi:10.1016/0370-2693(80)90054-4, 10.1016/0370-2693(80)91028-X; L. Alvarez-Gaume and D. Z. Freedman, Phys. Lett. 94B, 171 (1980). doi:10.1016/0370-2693(80)90850-3; M. Rocek and P. K. Townsend, Phys. Lett. 96B, 72 (1980). doi:10.1016/0370-2693(80)90215-4; U. Lindstrom and M. Rocek, Nucl. Phys. B 222, 285 (1983). doi:10.1016/0550-3213(83)90638-7 N. J. Hitchin, A. Karlhede, U. Lindstrom and M. Rocek, Commun. Math. Phys. 108, 535 (1987). doi:10.1007/BF01214418; A. S. Galperin, E. A. Ivanov, V. I. Ogievetsky and E. Sokatchev, Annals Phys. 185, 22 (1988). doi:10.1016/0003-4916(88)90257-6.

[4] L. Alvarez-Gaume and D. Z. Freedman, Commun. Math. Phys. 91, 87 (1983). doi:10.1007/BF01206053

[5] E. R. C. Abraham and P. K. Townsend, Phys. Lett. B 291, 85 (1992). doi:10.1016/0370-2693(92)90122-K; E. R. C. Abraham and P. K. Townsend, Phys. Lett. B 295, 225 (1992). doi:10.1016/0370-2693(92)91558-Q

[6] J. P. Gauntlett, D. Tong and P. K. Townsend, Phys. Rev. D 63, 085001 (2001) doi:10.1103/PhysRevD.63.085001 [hep-th/0007124].

[7] J. P. Gauntlett, D. Tong and P. K. Townsend, Phys. Rev. D 64, 025010 (2001) doi:10.1103/PhysRevD.64.025010 [hep-th/0012178].

[8] Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, Phys. Rev. Lett. 93, 161601 (2004) doi:10.1103/PhysRevLett.93.161601 [hep-th/0404198].

[9] Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, Phys. Rev. D 70, 125014 (2004) doi:10.1103/PhysRevD.70.125014 [hep-th/0405194].
[10] M. Arai and S. Shin, Phys. Rev. D **83**, 125003 (2011) doi:10.1103/PhysRevD.83.125003 [arXiv:1103.1490 [hep-th]].

[11] B. H. Lee, C. Park and S. Shin, Phys. Rev. D **96**, no. 10, 105017 (2017) doi:10.1103/PhysRevD.96.105017 [arXiv:1708.05243 [hep-th]].

[12] L. K. Hua, Amer. J. Math. 66, 470 (1944) doi:10.2307/2371910; Amer. J. Math. 66, 531 (1944) doi:10.2307/2371765; E. Calabi and E. Vesentini, On compact, locally symmetric Kahler manifolds, Ann. Math. 71, 472 (1960) doi:10.2307/1969939. ; S. Kobayashi, K. Nomizu, “Foundations of Differential Geometry Vol. II”, (Wiley Interscience, 1996).

[13] K. Higashijima and M. Nitta, Prog. Theor. Phys. **103**, 635 (2000) doi:10.1143/PTP.103.635 [hep-th/9911139].

[14] K. Higashijima, T. Kimura and M. Nitta, Nucl. Phys. B **623**, 133 (2002) doi:10.1016/S0550-3213(01)00591-0 [hep-th/0108084].

[15] M. Eto, T. Fujimori, S. B. Gudnason, Y. Jiang, K. Konishi, M. Nitta and K. Ohashi, JHEP **1112**, 017 (2011) doi:10.1007/JHEP12(2011)017 [arXiv:1108.6124 [hep-th]].

[16] A. Isaev and V. Rubakov, World Scientific Publishing Co., in press.

[17] N. Sakai and D. Tong, JHEP **0503**, 019 (2005) doi:10.1088/1126-6708/2005/03/019 [hep-th/0501207].

[18] J. Wess and J. Bagger, Princeton, USA: Univ. Pr. (1992) 259 p

[19] J. Scherk and J. H. Schwarz, Phys. Lett. **82B**, 60 (1979); doi:10.1016/0370-2693(79)90425-8 J. Scherk and J. H. Schwarz, Nucl. Phys. B **153**, 61 (1979). doi:10.1016/0550-3213(79)90592-3

[20] E. Witten, Nucl. Phys. B **202**, 253 (1982). doi:10.1016/0550-3213(82)90071-2; K. Hori and C. Vafa, hep-th/0002222. C. U. Sanchez, A. L. Cali and J. L. Moreschi, Geometriae Dedicata **64**, 261 (1997) doi:10.1023/A:1004913505961; S. B. Gudnason, Y. Jiang and K. Konishi, JHEP **1008**, 012 (2010) doi:10.1007/JHEP08(2010)012 [arXiv:1007.2116 [hep-th]].