Multi-agent Systems with Compasses

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Abstract

In this paper, we first study agreement protocols for coupled continuous-time nonlinear dynamics over cooperative multi-agent networks. To guarantee convergence for such systems, it is common in the literature to assume that the vector field of each agent is pointing inside the convex hull formed by the states of the agent and its neighbors. This convexity condition is relaxed in this paper, as we show that it is enough that the vector field belongs to a strict tangent cone based on a local supporting hyperrectangle. The new condition has the natural physical interpretation of a compass, as each agent needs only to know in which orthant each of its neighbor is. It is proven that the multi-agent system achieves exponential state agreement if and only if the time-varying communication topology is uniformly jointly quasi-strongly connected. Cooperative–antagonistic multi-agent networks are also considered. For these systems, the (cooperative–antagonistic) relation matrix has a negative element for arcs corresponding to antagonistic interactions. State agreement may not be achieved for cooperative–antagonistic multi-agent systems. Instead it is shown that asymptotic absolute state agreement is achieved if the time-varying communication topology is uniformly jointly strongly connected.

1 Introduction

In the last decade, coordinated control of multi-agent systems has attracted extensive attention due to its broad applications in engineering, physics, biology and social sciences, e.g., [6, 16, 19, 23, 26, 27, 37, 38]. A fundamental idea is that by carefully implementing distributed control protocols for each agent, collective tasks can be reached for the overall system using only

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neighboring information exchange. Several important results have been established for instance in the area of mobile systems, e.g., spacecraft formation flying, rendezvous of multiple robots, and animal flocking [10, 30, 33, 35].

Agreement protocols, where the goal is to drive the states of the agents to reach a common value using local interactions, play a basic role in coordination of multi-agent systems. The idea of state agreement protocol and its fundamental convergence properties were established for linear systems in the classical work [36]. The convergence of linear agreement-seeking model has been widely studied since then for both continuous-time and discrete-time models, e.g., [3, 16, 31]. Much understandings have been established, such as the explicit convergence rate in many cases [7, 21, 28, 29]. A major challenge is on how to quantitatively characterize the influence of a time-varying communication graph on the agreement convergence. Agreement protocols with nonlinear dynamics have also drawn much attention in the literature, e.g., [4, 15, 20, 25, 32, 34]. Due to the complexity of nonlinear dynamics, it is in general difficult to obtain explicit convergence rates for these systems. All the above studies on linear or nonlinear multi-agent dynamics are based on the standing assumption that agents in the network are cooperative. Recently, motivated from opinion dynamics evolving over social networks [12, 39], state agreement problems over cooperative–antagonistic networks were introduced in [1, 2]. In such networks, antagonistic neighbors exchange their states with opposite signs compared to cooperative neighbors.

In most of the work discussed above, convexity assumption plays an essential role in the local interaction rule for reaching state agreement. For discrete-time models, it is usually assumed that each agent updates its state as a convex combination of its neighbors’ states [5, 16]. A precise characterization of this convexity condition guaranteeing asymptotic agreement was established in [25]. For continuous-time models, an interpretation of this assumption becomes that the vector field for each agent must fall into the relative interior of a tangent cone formed by the convex hull of the relative state vectors in its neighborhood [20]. The recent work [22] generalized agreement protocols to convex metric spaces, but the same convexity assumption for the local dynamics continued to play an important role in ensuring agreement convergence.

The main contribution of this paper is to relax the convexity condition needed for agreement convergence for nonlinear multi-agent systems, at the cost of equipping each agent with a “compass” with the help of which the direction of each axis can be observed for a prescribed global coordinate system. We do not require that each agent has access to its own or its neighbors’
states, but the information exchange is as usual based on relative states of the agents. Using the compass, each agent can derive a strict tangent cone from a local supporting hyperrectangle. This cone defines the feasible set of local control actions for the agent to guarantee convergence to state agreement. It is argued that the vector field of an agent can be outside of the convex hull formed by the states of the agent and its neighbors, and thus provides a relaxed condition for agreement.

The two main results of the paper are the following:

• A necessary and sufficient condition is established for exponential state agreement under general nonlinear dynamics. We show that the underlying communication graph associated with the nonlinear interactions being uniformly jointly quasi-strongly connected is necessary and sufficient for exponential agreement. The convergence rate is explicitly given.

• A general framework is proposed for characterizing cooperative–antagonistic networks. Following [2], we assume that there is a sign reverse along any arc to an antagonistic agent. Absolute agreement is defined in the sense that the absolute value of each agent state component reaches an agreement asymptotically. We show that the underlying graph being uniformly jointly strongly connected, irrespectively of the number of antagonistic agents, is sufficient for asymptotic absolute agreement.

The remainder of the paper is organized as follows. In Section 2, we give some mathematical preliminaries on convex analysis [3], graph theory [14], and Dini derivatives [13]. The nonlinear multi-agent dynamics, the interaction graph and the compass are presented in Section 3. The main result on agreement for cooperative multi-agent system is presented in Section 4. The main result on absolute agreement for cooperative–antagonistic network is given in Section 5. A brief concluding remark is given in Section 6.

2 Primaries

In this section, we introduce some mathematical preliminaries on convex analysis [3], graph theory [14], and Dini derivatives [13].
2.1 Convex analysis

For any nonempty set $S \subseteq \mathbb{R}^d$, $\|x\|_S = \inf_{y \in S} \|x - y\|$ is called the distance between $x \in \mathbb{R}^d$ and $S$, where $\| \cdot \|$ denotes the Euclidean norm. A set $S \subset \mathbb{R}^d$ is called convex if $(1 - \zeta)x + \zeta y \in S$ when $x \in S$, $y \in S$, and $0 \leq \zeta \leq 1$. A convex set $S \subset \mathbb{R}^d$ is called a convex cone if $\zeta x \in S$ when $x \in S$ and $\zeta > 0$. The convex hull of $S \subset \mathbb{R}^d$ is denoted $\text{co}(S)$ and the convex hull of a finite set of points $x_1, x_2, \ldots, x_n \in \mathbb{R}^d$ denoted $\text{co}\{x_1, x_2, \ldots, x_n\}$.

Let $S$ be a convex set. Then there is a unique element $P_S(x) \in S$, called the convex projection of $x$ onto $S$, satisfying $\|x - P_S(x)\| = \|x\|_S$ associated to any $x \in \mathbb{R}^d$. It is also known that $\|x\|_S^2$ is continuously differentiable for all $x \in \mathbb{R}^d$, and its gradient can be explicitly computed \[\nabla \|x\|_S^2 = 2(x - P_S(x))\].

Let $S \subset \mathbb{R}^d$ be convex and closed. The interior and boundary of $S$ is denoted by $\text{int}(S)$ and $\partial S$, respectively. If $S$ contains the origin, the smallest subspace containing $S$ is the carrier subspace denoted by $\text{cs}(S)$. The relative interior of $S$, denoted by $\text{ri}(S)$, is the interior of $S$ with respect to the subspace $\text{cs}(S)$ and the relative topology used. If $S$ does not contain the origin, $\text{cs}(S)$ denotes the smallest subspace containing $S - z$, where $z$ is any point in $S$. Then, $\text{ri}(S)$ is the interior of $S$ with respect to the subspace $z + \text{cs}(S)$. Similarly, we can define the relative boundary $\text{rb}(S)$.

Let $S \subset \mathbb{R}^d$ be a closed convex set and $x \in S$. The tangent cone to $S$ at $x$ is defined as the set

$$\mathcal{T}(x, S) = \left\{ z \in \mathbb{R}^d : \liminf_{\zeta \to 0} \frac{\|x + \zeta z\|_S}{\zeta} = 0 \right\}.$$ 

Note that if $x \in \text{int}(S)$, then $\mathcal{T}(x, S) = \mathbb{R}^d$. Therefore, the definition of $\mathcal{T}(x, S)$ is essential only when $x \in \partial S$. Figure 1 gives an example of two tangent cones of boundary points of a rectangle. Here $\mathcal{T}(x_i, S)$ and $\mathcal{T}(x_j, S)$ are the tangent halfspace and the quarter space shifted to the origin and restricted in $S$, respectively. The following lemma will be used.

**Lemma 1.** [3] Let $S_1, S_2 \subset \mathbb{R}^d$ be convex sets. If $x \in S_1 \subset S_2$, then $\mathcal{T}(x, S_1) \subset \mathcal{T}(x, S_2)$.

2.2 Graph Theory

A directed graph $G$ consists of a pair $(V, E)$, where $V = \{1, 2, \ldots, n\}$ is a finite, nonempty set of nodes and $E \subseteq V \times V$ is a set of ordered pairs of nodes denoted arcs. The set of neighbors of node $i$ is denoted $\mathcal{N}_i := \{j : (j, i) \in E\}$. A directed path in a directed graph is a sequence of
arcs of the form \((i, j), (j, k), \ldots\). If there exists a path from node \(i\) to \(j\), then node \(j\) is said to be reachable from node \(i\). If for node \(i\), there exists a path from \(i\) to any other node, then \(i\) is called a root of \(G\). \(G\) is said to be strongly connected if each node is reachable from any other node. \(G\) is said to be quasi-strongly connected if \(G\) has a root.

### 2.3 Dini derivatives

Consider the differential equation

\[ \dot{x} = f(t, x), \]

where \(f : \mathbb{R} \times \mathcal{M} \rightarrow \mathbb{R}^d\) is continuous in \(x \in \mathcal{M} \subset \mathbb{R}^d\) for fixed \(t\) and piecewise continuous in \(t\) for fixed \(x\). Let \(V(t, x) : \mathbb{R} \times \mathcal{M} \rightarrow \mathbb{R}\) be locally Lipschitz with respect to \(x\) and uniformly continuous with respect to \(t\). Define

\[ D^+_f V(t, x) = \lim_{\tau \to 0^+} \sup \frac{V(t + \tau, x + \tau f(t, x)) - V(t, x)}{\tau}. \]

The function \(D^+_f V\) is called the upper Dini derivative of \(V\) along the trajectory of \(\dot{x} = f(t, x)\). Suppose that for an initial condition \(x(t_0)\), \(\dot{x} = f(t, x)\) has a solution \(x(t)\) defined on an interval \([0, \epsilon)\) and let \(D^+ V(t, x(t))\) be the upper Dini derivative of \(V(t, x(t))\) with respect to \(t\), i.e.,

\[ D^+ V(t, x) = \lim_{\tau \to 0^+} \sup \frac{V(t + \tau, x(t + \tau)) - V(t, x(t))}{\tau}. \]

Let \(t^* \in [0, \epsilon)\) and put \(x(t^*) = x^*.\) Then we have that

\[ D^+ V(t^*, x(t^*)) = D^+_f V(t^*, x^*). \]
The following lemma can be found in [11].

**Lemma 2.** Suppose for each $i \in V$, $V_i : \mathbb{R} \times M \to \mathbb{R}$ is continuously differentiable. Let $V(t,x) = \max_{i \in V} V_i(t,x)$, and let $\mathcal{V}_1(t) = \{ i \in V : V_i(t,x(t)) = V(t,x(t)) \}$ be the set of indices where the maximum is reached at time $t$. Then

$$D^+ V(t,x(t)) = \max_{i \in \mathcal{V}_1(t)} \dot{V}_i(t,x(t)).$$

3 Multi-agent System

In this section, we present the model of the considered multi-agent systems, introduce the corresponding interaction graph, and define some useful geometric concept used in the control laws.

Consider a multi-agent system with agent set $V = \{1, \ldots, n\}$. Let $x_i \in \mathbb{R}^d$ denote the state of agent $i$. Let $x = (x_1^T, x_2^T, \ldots, x_n^T)^T$ and denote $D = \{1, 2, \ldots, d\}$.

3.1 Nonlinear multi-agent dynamics

Let $\mathfrak{P}$ be a given (finite or infinite) set of indices. An element in $\mathfrak{P}$ is denoted by $p$. For any $p \in \mathfrak{P}$, we define a function $f_p(x_1, x_2, \ldots, x_n) : \mathbb{R}^{dn} \to \mathbb{R}^d$ associated with $p$, where

$$f_p(x_1, x_2, \ldots, x_n) = \begin{pmatrix} f^1_p(x_1, x_2, \ldots, x_n) \\ \vdots \\ f^n_p(x_1, x_2, \ldots, x_n) \end{pmatrix}$$

with $f^i_p : \mathbb{R}^d \to \mathbb{R}^d$, $i = 1, 2, \ldots, n$.

Let $\sigma(t) : [t_0, \infty) \to \mathfrak{P}$ be a piecewise constant function, so, there exists a sequence of increasing time instances $\{t_l\}_0^\infty$ such that $\sigma(t)$ remains constant for $t \in [t_l, t_{l+1})$ and switches at $t = t_l$.

The dynamics of the multi-agent systems is described by the switched nonlinear system

$$\dot{x}(t) = f_{\sigma(t)}(x(t)).$$

(3)

We place some mild assumptions on this system.

**Assumption 1** (Dwell time). *There exists a lower bound $\tau_d > 0$, such that $\inf_l (t_{l+1} - t_l) \geq \tau_d$.***
Assumption 2 (Uniformly locally Lipschitz). $f_p(x)$ is uniformly locally Lipschitz on $\mathbb{R}^d$, i.e., for every $x \in \mathbb{R}^d$, we can find a neighborhood $U_\alpha(x) = \{y \in \mathbb{R}^d : \|y - x\| \leq \alpha\}$ for some $\alpha > 0$ such that there exists a real number $L(x) > 0$ with $\|f_p(a) - f_p(b)\| \leq L(x)\|a - b\|$ for any $a, b \in U_\alpha(x)$ and $p \in \mathcal{P}$.

Under Assumptions 1 and 2, the Caratheodory solutions of (3) exist for arbitrary initial conditions, and they are absolutely continuous functions for almost all $t$ on the maximum interval of existence \cite{8,13}. All our further discussions will be on the Caratheodory solutions of (3) without specific mention.

3.2 Interaction graph

Having the dynamics defined for the considered multi-agent system, we introduce next its interaction graph.

Definition 1 (Interaction graph). The graph $G_p = (V, E_p)$ associated with $f_p$ is the directed graph on node set $V = \{1, 2, \ldots, n\}$ and arc set $E_p$ such that $(j, i) \in E_p$ if and only if $f_p^i$ depends on $x_j$, i.e., there exist $x_j, \overline{x}_j \in \mathbb{R}^d$ such that $f_p^i(x_1, \ldots, x_j, \ldots, x_n) \neq f_p^i(x_1, \ldots, \overline{x}_j, \ldots, x_n)$.

The set of neighbors of node $i$ in $G_p$ is denoted by $N_i(p)$. The dynamic interaction graph associated with system (3) is denoted by $G_{\sigma(t)} = (V, E_{\sigma(t)})$. The joint graph of $G_{\sigma(t)}$ during time interval $[t_1, t_2)$ is defined by $G_{\sigma(t)}([t_1, t_2)) = \bigcup_{t \in [t_1, t_2)} G(t) = (V, \bigcup_{t \in [t_1, t_2)} E_{\sigma(t)})$. We impose the following definition on the connectivity of $G_{\sigma(t)}$.

Definition 2 (Joint connectivity). (i) $G_{\sigma(t)}$ is uniformly jointly quasi-strongly connected if there exists a constant $T > 0$ such that $G([t, t + T))$ is quasi-strongly connected for any $t \geq t_0$.

(ii) $G_{\sigma(t)}$ is uniformly jointly strongly connected if there exists a constant $T > 0$ such that $G([t, t + T))$ is strongly connected for any $t \geq t_0$.

3.3 Compass, hyperrectangle, and tangent cone

We assume that each agent has access to a compass corresponding to a common Cartesian coordinate system. We use $(\overrightarrow{r}_1, \overrightarrow{r}_2, \ldots, \overrightarrow{r}_d)$ to represent the basis of the $\mathbb{R}^d$ Cartesian coordinate
system. Here $\vec{r}_k = (0, \ldots, 0, 1, 0, \ldots, 0)$ denotes the unit vector in the direction of axis $k \in D$. Obviously, a point in $\mathbb{R}^d$ can be described by $z = z_1 \vec{r}_1 + z_2 \vec{r}_2 + \cdots + z_d \vec{r}_d$, where $z_k$ is a real number for all $k \in D$.

A hyperrectangle is the generalization of a rectangle to higher dimensions. An axis-aligned hyperrectangle is a hyperrectangle subject to the constraint that the edges of the hyperrectangle are parallel to the Cartesian coordinate axes.

**Definition 3** (Supporting hyperrectangle). Let $C \subset \mathbb{R}^d$ be a bounded set. The supporting hyperrectangle $\mathcal{H}(C)$ is the axis-aligned hyperrectangle

$$\mathcal{H}(C) = [\min(C)_1, \max(C)_1] \times [\min(C)_2, \max(C)_2] \times \cdots \times [\min(C)_d, \max(C)_d],$$

where $\min(C)_k := \min\{y_k : y_k$ is the $k$th entry of $y \in C\}$, and $\max(C)_k := \max\{y_k : y_k$ is the $k$th entry of $y \in C\}$.

In other words, a supporting hyperrectangle of a bounded set $C$ is an axis-aligned minimum bounding hyperrectangle.

**Definition 4** (Strict tangent cone). Let $A \subset \mathbb{R}^d$ be an axis-aligned hyperrectangle and $\gamma > 0$ a constant. The $\gamma$-strict tangent cone to $A$ at $x \in \mathbb{R}^d$ is the set

$$\mathcal{T}_\gamma(x, A) = \begin{cases} \text{cs}(A); & \text{if } x \in \text{ri}(A) \\ \mathcal{T}(x, A) \cap \{z \in \mathbb{R}^d : |\langle z, \vec{r}_k \rangle| \geq \gamma D_k(A)\}; & \text{if } x \in \text{rb}_k(A), \end{cases}$$

where $\text{rb}_k(A)$ denotes one of the two facets of $A$ perpendicular to the axis $\vec{r}_k$, and $D_k(A) = |\max(A)_k - \min(A)_k|$ denotes the side length parallel to the axis $\vec{r}_k$.

### 4 Cooperative Multi-agent Systems: Exponential Agreement

In this section, we study the convergence property of the nonlinear switched system (3) over a cooperative network defined by an interaction graph. Introduce $C_i^p(x) = \text{co}\{x_i, x_j : j \in \mathcal{N}_i(p)\}$. We impose the following assumption.

**Assumption 3** (Vector field). For all $i \in \mathcal{V}$, $p \in \Psi$, and $x \in \mathbb{R}^{dn}$, it holds that $f_i^p(x) \in \mathcal{T}_\gamma(x_i, \mathcal{H}(C_i^p(x)))$.

**Remark 1.** In Assumption 3, the vector field $f_i^p$ can be chosen freely from the set $\mathcal{T}_\gamma(x_i, \mathcal{H}(C_i^p(x)))$. Hence, the assumption specifies constraints on the feasible controls for the considered multi-agent
system. Here $C^i_p(x)$ denotes the convex hull formed by agent $i$ and its neighbors, $\mathcal{H}(C^i_p(x))$ (defined in Section 3.3) denotes the supporting hyperrectangle of the set $C^i_p(x)$, and $T_{\gamma}(x_i, \mathcal{H}(C^i_p(x)))$ (also defined in Section 3.3) denotes the $\gamma$-strict tangent cone to $\mathcal{H}(C^i_p(x))$ at $x_i$. Figure 2 gives an example of the convex hull and the supporting hyperrectangle formed by agent $i$ and its’ neighbors. Three feasible vectors $f^i_p$ are presented.

**Remark 2.** It is essential to capture what information exchange is required in a multi-agent system implementing a control law fulfilling Assumption 3. Each agent uses its own coordinate system to locate in which orthant each of its neighbor is. Then the agent constructs the supporting hyperrectangle based on the relative states between itself and its neighbors, similarly to conventional agreement protocols for multi-agent systems. When the agent is inside its supporting hyperrectangle, the vector field for the agent can be chosen arbitrary. When the agent is on the boundary of its supporting hyperrectangle, the feasible control is just any direction pointing inside the halfspace of its supporting hyperrectangle. Note that the absolute state of the agents is not needed, but each agent needs to identify $d - 1$ absolute directions such that it can identify the direction of its neighbors with respect to itself. For example, for the planar case $d = 2$, each agent just needs to be equipped with a compass (providing direction information) to implement this protocol. The compass provides the quadrant location information of its neighbors. For
\[ d > 2, \text{ the (generalized) compass gives information on which orthant the neighbors belong to.} \]

The linear consensus algorithm is an example satisfying Assumptions 2 and 3. More specifically, consider the following system on \( \mathbb{R}^n \),

\[
\dot{x}_i = f_p(x) = -\sum_{j \in \mathcal{N}_i(t)} a_{ij}(t)(x_i - x_j), \quad i = 1, 2, \ldots, n,
\]

where the continuous function \( a_{ij}(t) > 0 \) is the weight of arc \((j, i)\) for \( i, j \in \mathcal{V} \). In addition, we assume that there exists \( a^* > 0 \) and \( a_* > 0 \) such that \( a_* \leq a_{ij}(t) \leq a^*, \forall t \geq 0 \).

It is easy to check that Assumption 2 is satisfied when \( a_{ij}(t) \leq a^*, \forall t \geq 0 \) and Assumption 3 is satisfied when \( a_{ij}(t) \geq a_*, \forall t \geq 0 \). From this example, we can clearly see that the vector field assumption only requires state differences available, rather than absolute states.

Define the agreement manifold as \( \mathcal{J} = \{ x \in \mathbb{R}^{dn} : x_1 = x_2 = \cdots = x_n \} \).

**Definition 5** (Exponential agreement). The switched coupled system (3) is said to achieve exponential state agreement on \( \mathcal{S}_0 \subseteq \mathbb{R}^d \) if there exist positive constants \( k(\mathcal{S}_0), \lambda(\mathcal{S}_0) \), such that for all \( t_0 \geq 0 \) and \( x(t_0) \in \mathcal{S}_0^n \),

\[
\| x(t) \|_{\mathcal{J}} \leq k e^{-\lambda(t-t_0)} \| x(t_0) \|_{\mathcal{J}}.
\]

**Remark 3.** Definition 5 is a set attraction definition. However, exponential convergence guarantees that set attraction implies set stability, where set stability with respect to \( \mathcal{J} \) can be defined as: \( \forall \varepsilon > 0, \exists \delta > 0 \) such that \( \forall t_0 \geq 0 \),

\[
\| x(t_0) \|_{\mathcal{J}} \leq \delta \implies \| x(t) \|_{\mathcal{J}} \leq \varepsilon, \forall t \geq t_0.
\]

This implication does not hold for asymptotic stability, as stability and attraction are two independent concepts in this case. See [17] for more details.

The main result for the agreement seeking of the nonlinear multi-agent dynamics over cooperative networks is given as follows.

**Theorem 1.** Suppose \( \mathcal{S}_0 \) is compact and that Assumptions 1–3 hold. The multi-agent system (3) achieves exponential state agreement on \( \mathcal{S}_0 \) if and only if its interaction graph \( \mathcal{G}_\sigma(t) \) is uniformly jointly quasi-strongly connected.

**Remark 4.** Theorem 1 is consistent with the main results in [21, 22, 23]. Our analysis relies on some techniques developed in [13]. Theorem 1 allows that the vector field belongs to a larger
convex set compared with the convex hull condition proposed in [20, 22, 25]. In addition, we allow the agent dynamics to switch over a possibly infinite set and we derive in the proof the explicit exponential rate for the convergence.

**Remark 5.** General consensus functions are constructed in [9] to guarantee asymptotic or finite-time consensus. Compared with [9], our algorithm is based on Lipschitz continuous functions and focuses on the exponential convergence speed. Moreover, we allow the communication topology to be jointly quasi-strongly connected instead of weakly connected and weight-balanced.

We next prove the theorem by analyzing a contraction property of (3).

### 4.1 Invariant Set

**Definition 6 (Invariant set).** A set $\mathcal{M} \subset \mathbb{R}^{dn}$ is an invariant set for the system (3) if for all $t_0 \geq 0$,

$$x(t_0) \in \mathcal{M} \implies x(t) \in \mathcal{M}, \forall t \geq t_0.$$ 

For all $k \in \mathcal{D}$, define

$$M_k(t) = \max_{i \in \mathcal{V}} \{x_{ik}(t)\}, \quad m_k(t) = \min_{i \in \mathcal{V}} \{x_{ik}(t)\},$$

where $x_{ik}$ denotes $k$th entry of $x_i$. In addition, define the supporting hyperrectangle by the initial states of all agents as $\mathcal{H}_0 := \mathcal{H}(\mathcal{C}(x(t_0)))$, where $\mathcal{C}(x) = \text{co}\{x_1, x_2, \ldots, x_n\}$.

In the following lemma, we show that the supporting hyperrectangle formed by the initial states of all agents is non-expanding over time.

**Lemma 3.** Let Assumptions 1–3 hold. Then, $\mathcal{H}_0^n$ is an invariant set, i.e., $x_i(t) \in \mathcal{H}_0^n, \forall i \in \mathcal{V},$ $\forall t \geq t_0$.

**Proof.** We first show that $D^+ M_k(t) \leq 0, \forall k \in \mathcal{D}$. Let $\mathcal{V}_1(t) = \{i \in \mathcal{V} : x_{ik}(t) = M_k(t)\}$ be the set of indices where the maximum is reached at $t$. It then follows from Lemma 2 that for all $k \in \mathcal{D}$,

$$D^+ M_k(t) = \max_{i \in \mathcal{V}_1(t)} \dot{x}_{ik} = \max_{i \in \mathcal{V}_1(t)} f_p^{ik}(x(t)),$$

where $f_p^{ik}$ denotes $k$th entry of the vector $f_p^i$. Consider any initial state $x(t_0) \in \mathcal{H}_0^n$ and any initial time $t_0$. For any point $x \in \mathcal{H}_0^n$, since $\mathcal{H}_0^n$ is convex, it follows that $\mathcal{H}(\mathcal{C}^i_p(x)) \subseteq \mathcal{H}_0^n, \forall i \in \mathcal{V}$, and $\forall p \in \mathcal{P}$. Therefore, it follows from Definition 3 and Lemma that

$$f_p^i(x) \in T_\gamma(x_i, \mathcal{H}(\mathcal{C}^i_p(x))) \subseteq T(x_i, \mathcal{H}(\mathcal{C}^i_p(x))) \subseteq T(x_i, \mathcal{H}_0^n).$$
It follows from the definition of tangent cone that \( \langle f^i_p(x), -r^k \rangle \geq 0 \) for \( x_{ik} = M_k \). This shows that \( f^i_p(x) \leq 0 \) for \( i \in V \) satisfying \( x_{ik} = M_k \). It follows that for all \( k \in D \) and any \( x \in H_0^0 \),

\[
D^+ M_k(t) \leq 0.
\]

We can similarly show that for all \( k \in D \), \( D^+ m_k(t) \geq 0 \).

Therefore, it follows that for all \( k \in D \) and any \( x \in H_n^0 \),

\[
D^+ M_k(t) \leq 0.
\]

Remark 6. In light of Lemma 3, every agent converges to a finite limit as long as state agreement is achieved asymptotically. In other words, the statement of Theorem 1 can be slightly enhanced in the way that for any initial time \( t_0 \) and initial value \( x(t_0) \in S_0^0 \), there exist \( \eta_*(x(t_0)) \in S_0 \) and two constants \( k_*(S_0), \lambda_*(S_0) \), such that

\[
\| x_i(t) - \eta_* \| \leq k_* e^{-\lambda_*(t-t_0)} \| x_i(t_0) - \eta_* \|
\]

for all \( t \geq t_0 \).

4.2 Interior agents

In this subsection, we study the state evolution of the agents whose states are interior points of \( \mathcal{H}(\mathcal{C}(x)) \). We first state a comparison lemma, which will be used repeatedly later.

Lemma 4. (Lemma 3.4 of [17]): Consider the scalar differential equation

\[
\dot{\chi} = f(t, \chi), \quad \chi(t_0) = \chi_0,
\]

where \( f(t, \chi) \) is continuous in \( t \) and locally Lipschitz in \( \chi \), for all \( t \geq 0 \) and all \( \chi \in U \subset \mathbb{R} \). Let \([t_0, T)\) be the maximal interval of existence of the solution \( x(t) \), and suppose \( \chi(t) \in U \) for all \( t \in [t_0, T) \). Let \( v(t) \) be a continuous function whose upper right-hand derivative \( D^+ v(t) \) satisfies the differential inequality

\[
D^+ v(t) \leq f(t,v(t)), \quad v(t_0) \leq \chi_0
\]

with \( v(t) \in U \) for all \( t \in [t_0, T) \). Then, \( v(t) \leq \chi(t) \) for all \( t \in [t_0, T) \).

A function \( h(\cdot) \) is called Lipschitz continuous on set \( U \) if there exists a constant \( L_U \) such that \( \| h(a) - h(b) \| \leq L_U \| a - b \| \) for all \( a, b \in U \). We also need the following lemma on uniformly locally Lipschitz functions.
**Lemma 5.** Suppose Assumption A holds, i.e., $f_p, p \in \mathcal{P}$ is uniformly locally Lipschitz. Assume that there exists a point $z_0 \in \mathbb{R}^{dn}$ such that $\sup_{p \in \mathcal{P}} f_p(z_0)$ (or $\inf_{p \in \mathcal{P}} f_p(z_0)$) is finite. Then $g(x) := \sup_{p \in \mathcal{P}} f_p(x)$ (or $\inf_{p \in \mathcal{P}} f_p(x)$) is well defined and it is Lipschitz continuous on every compact set $U$.

**Proof.** Let $U$ be a given compact set. Define $\mathcal{U}_{z_0} = \overline{\text{co}(\{z_0\} \cup \mathcal{U})}$. Based on Theorem 1.14 of [21], a locally Lipschitz function is Lipschitz continuous on every compact subset. Plugging in the fact that $f_p, p \in \mathcal{P}$ is uniformly locally Lipschitz, there is $L_{\mathcal{U}_{z_0}} > 0$ such that $\|f_p(a) - f_p(b)\| \leq L_{\mathcal{U}_{z_0}} \|a - b\|$ for all $a, b \in \mathcal{U}_{z_0}$ and $p \in \mathcal{P}$. It becomes straightforward that $g(x)$ is finite at every point in $\mathcal{U}_{z_0}$ and $L_{\mathcal{U}_{z_0}}$ is a Lipschitz constant of $g$ on $\mathcal{U}_{z_0}$. Therefore, the lemma holds.

The following lemma is from [18].

**Lemma 6.** Suppose that $f(x_1, y) : \mathbb{R} \times \mathcal{M} \to \mathbb{R}$ is locally Lipschitz with respect to $[x_1, y]^T$, where $\mathcal{M} \subset \mathbb{R}^q$ is compact. Then $g(x_1) = \max_{y \in \mathcal{M}} f(x_1, y)$ is locally Lipschitz.

In the following lemma, we show that the projection of the state on any coordinate axis is strictly less than an upper bound as long as it is initially strictly less than this upper bound. The proof follows from a similar argument used in the proof Lemma 4.9 in [18].

**Lemma 7.** Let Assumptions B and C hold and assume that $\mathcal{G}_{\sigma(t)}$ is uniformly jointly quasi-strongly connected. Then, for any $(t_1, x(t_1)) \in \mathbb{R} \times \mathcal{H}_0^q$, any $\varepsilon > 0$ sufficient small, and any $T^* > 0$, if $x_{ik}(t_2) \leq M_k(x(t_1)) - \varepsilon$ at some $t_2 \geq t_1$ for some $k \in \mathcal{D}$, then $x_{ik}(t) \leq M_k(x(t_1)) - \delta$, where $\delta = e^{-L_1 T^*} \varepsilon$ for all $t \in [t_2, t_2 + T^*]$, and $L_1^*$ is a positive constant related to $\mathcal{H}_0$.

**Proof.** Fix $(t_1, x(t_1)) \in \mathbb{R} \times \mathcal{H}_0^q$. Denote $\psi = x(t_1)$ and

$$
\mathcal{M} = \mathcal{H}_0^q(\psi) \times \mathcal{H}(\psi) \times \cdots \times \mathcal{H}(\psi),
$$

where $\mathcal{H}(\psi) = [m_1(\psi), M_1(\psi)] \times \cdots \times [m_d(\psi), M_d(\psi)]$, and $\mathcal{H}_0^q(\psi) = [m_1(\psi), M_1(\psi)] \times \cdots \times [m_{k-1}(\psi), M_{k-1}(\psi)] \times [m_{k+1}(\psi), M_{k+1}(\psi)] \times \cdots \times [m_d(\psi), M_d(\psi)]$. The rest of the proof will be divided in three steps.

**(Step 1).** Define the following nonlinear function

$$
g_\psi(\cdot) : [m_k(\psi), M_k(\psi)] \to \mathbb{R}
$$

$$
x_{ik} \mapsto \sup_{p \in \mathcal{P}} \max_{k \in \mathcal{D}} \{\max_{i \in \mathcal{V}} \{\max_{y \in \mathcal{M}} f_p^k(x_{ik}, y)\}\},
$$

13
where \( f^i_p(x_{ik}, y) \) denotes the \( k \)th entry of the vector \( f^i_p(x) \). In this step, we establish some useful properties of \( g_\psi(\cdot) \) based on Lemmas 9 and 11. We make the following claim.

**Claim A:** (i) \( g_\psi(\chi) = 0 \) if \( \chi = M_k(\psi) \); (ii) \( g_\psi(\chi) > 0 \) if \( \chi \in [m_k(\psi), M_k(\psi)) \); (iii) \( g_\psi(\chi) \) is Lipschitz continuous with respect to \( \chi \) on \([m_k(\psi), M_k(\psi)]\).

It follows from Lemma 3 that \( \forall t \geq t_1, f^i_p(x) \in \mathcal{T}_t(x_i, \mathcal{H}(C^i_p(x))) \subseteq \mathcal{T}(x_i, \mathcal{H}(C^i_p(x))) \), \( \mathcal{T}(x_i, \mathcal{H}(C^i_p(x))) \subseteq \mathcal{T}(x_i, \mathcal{H}(C^i_p(x))) \), \( \forall i \in \mathcal{V}, \forall \theta \in \mathcal{P} \). Then, it follows from Definition 4 that \( f^i_p(x), -\frac{r_k}{\theta} \geq 0 \) if \( x_{ik} = M_k(\psi) \), which shows that \( f^i_p(x) \leq 0 \) when \( x_{ik} = M_k(\psi) \). This implies that \( g_\psi(\chi) \leq 0 \) when \( \chi = M_k(\psi) \) based on the definition of \( g_\psi(\chi) \). We next show that actually \( g_\psi(\chi) = 0 \) when \( \chi = M_k(\psi) \). Since \( \mathcal{G}_{\sigma(t)} \) is uniformly jointly quasi-strongly connected, there must exist \( i \in \mathcal{V} \) and \( p \in \mathcal{P} \) such that \( x_j = x_i \in \mathcal{H}(C^i(x)) \), for all \( j \in \mathcal{N}_i(p) \) and \( \mathcal{N}_i(p) \) is not empty. This shows that \( f^i_p(x) = 0 \) from Assumption 3 which prove that \( g_\psi(\chi) = 0 \) if \( \chi = M_k(\psi) \). This proves (i).

Next, for any \( \chi \in [m_k(\psi), M_k(\psi)] \), based on the assumption that \( \mathcal{G}_{\sigma(t)} \) is uniformly jointly quasi-strongly connected, there exists \( i \in \mathcal{V} \), and \( p \in \mathcal{P} \) such that \( x_{ik} = M_k(\psi), \forall k^\sigma \in \{1, \ldots, k-1, k+1, \ldots, d\} \) and \( x_{ik} = M_k(\psi), \forall k \in \mathcal{D}, \) for all \( j \in \mathcal{N}_i(p) \) and \( \mathcal{N}_i(p) \) is not empty. In such case, \( \mathcal{H}(C^i_p(x)) \) is a line from point \((M_1(\psi), M_2(\psi), \ldots, M_d(\psi))\) to

\[
(M_1(\psi), \ldots, M_{k-1}(\psi), \chi, M_{k+1}(\psi), \ldots, M_d(\psi)).
\]

It then follows from Assumption 3 that \( f^i_p(x) \geq \gamma(M_k(\psi) - \chi) > 0 \). This verifies that \( g_\psi(\chi) > 0 \), \( \forall \chi \in [m_k(\psi), M_k(\psi)] \). This proves (ii).

Finally, it follows from Lemma 3 that \( g^i_p(x_{ik}) : [m_k(\psi), M_k(\psi)] \to \mathbb{R}, x_{ik} \to \max_{y \in \mathcal{M}} f^i_p(x_{ik}, y) \) is locally Lipschitz with respect to \( x_{ik}, \forall k \in \mathcal{D}, \forall i \in \mathcal{V} \) and \( \forall \theta \in \mathcal{P} \). Then, it follows from Theorem 1.14 of [21] that \( g^i_p(x_{ik}) \) is Lipschitz continuous with respect to \( x_{ik} \) on \([m_k(\psi), M_k(\psi)]\). From the first property of \( g_\psi(\chi) \), it follows that \( g_\psi(0) = 0 \). Therefore, based on Lemma 5, it follows that \( g_\psi(\chi) \) is Lipschitz continuous with respect to \( \chi \) on \([m_k(\psi), M_k(\psi)]\). This proves (iii) and the claim holds.

**Step II.** In this step, based on the definition of \( g_\psi(\cdot) \), we construct and investigate the nonlinear function \( h_{\mathcal{H}_\phi}(\cdot) \), which measures the distance between \( x_{ik} \) and the upper boundary \( M_k(x(t_1)) \).

Denote \( \varphi = \chi - M_k(\psi) \) and define

\[
h_{\mathcal{H}_\phi}(\varphi) : [\bar{a} - \bar{a}, 0] \to \mathbb{R},
\]
\[ \varphi \mapsto \begin{cases} g_\psi(\varphi + M_k(\psi)); & \text{if } \varphi \in [m_k(\psi) - M_k(\psi), 0] \\ g_\psi(m_k(\psi)); & \text{if } \varphi \in [\hat{a} - \tilde{a}, m_k(\psi) - M_k(\psi)) \end{cases} \]

where \( \hat{a} = \min_{k \in \mathcal{D}} m_k(x(t_0)) \) and \( \tilde{a} = \max_{k \in \mathcal{D}} M_k(x(t_0)) \) are constants determined by \( \mathcal{H}_0 \).

Obviously, \( h_{\mathcal{H}_0}(\varphi) \) is continuous. We make the following claim.

Claim B: (i) \( h_{\mathcal{H}_0}(\varphi) \) is Lipschitz continuous with respect to \( \varphi \) on \( [\hat{a} - \tilde{a}, 0] \), where the Lipschitz constant is denoted by \( L_1^* \) and \( L_1^* \) is related to the initial bounded set \( \mathcal{H}_0 \); (ii) \( h_{\mathcal{H}_0}(\varphi) > 0 \) if \( \varphi \in [\hat{a} - \tilde{a}, 0) \); (iii) \( h_{\mathcal{H}_0}(\varphi) = 0 \) if \( \varphi = 0 \).

Note that \( h_{\mathcal{H}_0}(\varphi) = g_\psi(\varphi + M_k(\psi)) \) is compact on the compact set \( [m_k(\psi) - M_k(\psi), 0] \). It follows that \( h_{\mathcal{H}_0}(\varphi) \) is Lipschitz continuous with respect to \( \varphi \) on the compact set \( [\hat{a} - \tilde{a}, 0] \). This shows that (i) holds and properties (ii) and (iii) follow directly from the definition of \( h_{\mathcal{H}_0}(\varphi) \).

(Step III). In this step, we take advantage of \( g_\psi(\chi) \) and \( h_{\mathcal{H}_0}(\varphi) \) to show that \( x_{ik} \) will be always strictly less than the upper boundary \( M_k(\psi) \) as long as it is initially strictly less than \( M_k(\psi) \).

Suppose \( x_{ik}(t_2) \leq M_k(\psi) - \varepsilon \) at some \( t_2 \geq t_1 \) and let \( T^* > 0 \). Based on the definition of \( g_\psi(\chi) \), it follows that

\[ x_{ik}(t) = f_{\sigma(t)}^k(x(t)) \leq g_\psi(x_{ik}(t)), \quad \forall t \geq t_2. \]

Let \( \chi(t) \) be the solution of \( \dot{\chi} = g_\psi(\chi) \) with initial condition \( \chi(t_2) = x_{ik}(t_2) \). Based on Lemma 4, it follows that \( x_{ik}(t) \leq \chi(t), \forall t \geq t_2. \)

Note that \( \varphi = \chi - M_k(\psi) \) and \( \dot{\varphi} = g_\psi(\chi) = h_{\mathcal{H}_0}(\varphi) \). It follows from the first property of \( h_{\mathcal{H}_0}(\varphi) \) that \( |h_{\mathcal{H}_0}(\varphi) - h(0)| \leq L_1^*|\varphi|, \forall \varphi \in [\hat{a} - \tilde{a}, 0] \). This shows that \( h_{\mathcal{H}_0}(\varphi) \leq -L_1^*\varphi \) based on the second and the third properties of \( h_{\mathcal{H}_0}(\varphi) \). Thus, the solution of \( \dot{\varphi} = h_{\mathcal{H}_0}(\varphi) \) satisfies

\[ \varphi(t) \leq e^{-L_1^*(t-t_2)}\varphi(t_2), \quad \forall t \geq t_2 \] based on Lemma 4.

Therefore, \( x_{ik}(t) \leq \chi(t) = \varphi(t) + M_k(\psi) \leq e^{-L_1^*(t-t_2)}(\chi(t_2) - M_k(\psi)) + M_k(\psi) \leq e^{-L_1^*T^*}(x_{ik}(t_2) - M_k(\psi)) + M_k(\psi) \leq M_k(\psi) - e^{-L_1^*T^*}\varepsilon \) for all \( t \in [t_2, t_2 + T^*] \).

The following lemma is a symmetrical result with Lemma 7 whose proof can be obtained using Lemma 7 under transformation \( z_i = -x_i, i = 1, \ldots, n \) and is therefore omitted.

**Lemma 8.** Let Assumptions and assume that \( \mathcal{G}_{\sigma(t)} \) is uniformly jointly quasi-strongly connected. For any \((t_1, x(t_1)) \in \mathbb{R} \times \mathcal{H}_0^0\), and any \( \varepsilon > 0 \) sufficient small, and any \( T^* > 0 \), if \( x_{ik}(t_2) \geq m_k(x(t_1)) + \varepsilon \) at some \( t_2 \geq t_1 \) for some \( k \in \mathcal{D} \), then \( x_{ik}(t_2) \geq m_k(x(t_1)) + \delta \), where \( \delta = e^{-L_2^*T^*}\varepsilon \) for all \( t \in [t_2, t_2 + T^*] \), where \( L_2^* \) is a positive constant related to \( \mathcal{H}_0 \).
4.3 Boundary agents

In the following lemma, we show that any agent that is attracted by an interior agent will become an interior agent after a time period.

**Lemma 9.** Let Assumptions 1–3 hold and assume that $G_{\sigma(t)}$ is uniformly jointly quasi-strongly connected. For any $(t_1, x(t_1)) \in \mathbb{R} \times H_0^n$, any $\delta_1 > 0$ sufficiently small and any $T^* > 0$, if there is an arc $(j, i)$ and a time $t_2 \geq t_1$ such that $j \in N_i(\sigma(t))$, and $x_{jk}(t) \leq M_k(x(t_1)) - \delta_1$ for all $t \in [t_2, t_2 + \tau_d]$, then there exists a $t_3 \in [t_1, t_2 + \tau_d]$ such that $x_{ik}(t) \leq M_k(x(t_1)) - \delta_2$, for all $t \in [t_3, t_3 + T^*]$, where $\delta_2 = \frac{\gamma_T}{L_1^+ \tau_d + \gamma_T + 1} e^{-L_1^+ T^*} \delta_1$ for some constants $L_1^+$ and $L_1^+$ related to $H_0$.

**Proof.** We first show that there exists $t_3 \in [t_1, t_2 + \tau_d]$ such that $x_{ik}(t_3) \leq M_k(x(t_1)) - \varepsilon$, where $\varepsilon = \frac{\gamma_T}{L_1^+ \tau_d + \gamma_T + 1} \delta_1$ for some constant $L_1^+$. This is equivalent to show that $\|x_i(t_3)\|_B = 0$, where $B := H_k^+(C(x(t_1)))$ and an axis-aligned hyperrectangle $H_k^+$ defined as $H_k^+(C(x)) = \{y \in H(C(x)) : y_k \leq M_k(x) - \varepsilon\}$. Obviously, $B$ is compact convex set. Suppose $\|x_i(t_3)\|_B \neq 0$. It then follows that $0 < \|x_i(t_3)\|_B \leq \varepsilon$ for all $t \in [t_1, t_2 + \tau_d]$.

Considering the time interval $t \in [t_2, t_2 + \tau_d]$, we define $\mathbf{x}(t) = [\mathbf{x}_{11}, \ldots, \mathbf{x}_{1d}, \mathbf{x}_{21}, \ldots, \mathbf{x}_{2d}, \ldots, \mathbf{x}_{n1}, \ldots, \mathbf{x}_{nd}]$, $\mathbf{x}_{ik}(t) = \max_{i \in V} \{x_{ik}(t)\} = M_k(x(t))$ for certain $i$ and certain $k$, and $\mathbf{x}_{i'k}(t) = x_{i'k}(t)$ for other $i' \in V \setminus i$ and $k' \in D \setminus k$. Let $V_i(t) = \{i \in V : x_{ik}(t) = M_k(x(t))\}$. It then follows from Lemma 2 that

$$D^+ \|\mathbf{x}(t)\|_B = \max_{i \in V_i(t)} \langle (\text{sgn}(x_i(t) - P_B(x_i(t))), f^i_{\sigma(t)}(\mathbf{x})) \rangle = \langle (\text{sgn}(\mathbf{x}_i - P_B(\mathbf{x}_i)), f^i_{\sigma}(\mathbf{x})) \rangle = \langle \mathbf{n}_k, f^i_{\sigma}(\mathbf{x}) \rangle,$$

where the componentwise sign function $\text{sgn}(\cdot)$ is defined as $\text{sgn}(z) = [\text{sgn}(z_1), \text{sgn}(z_2), \ldots, \text{sgn}(z_d)]$ for a vector $z = [z_1, z_2, \ldots, z_d]$ and $\text{sgn}(z_1)$ is the sign function: $\text{sgn}(z_1) = 1$ if $z_1 > 0$, $\text{sgn}(z_1) = 0$ if $z_1 = 0$, and $\text{sgn}(z_1) = -1$ if $z_1 < 0$. Note that $\text{sgn}(\mathbf{x}_i - P_B(\mathbf{x}_i)) = \mathbf{n}_k$ whenever $\|x_i\|_B > 0$.

The rest of the proof will be divided into three steps.

**(Step 1).** Define

$$g^{ik}(x(t)) = \inf_{p \in \mathfrak{P}} \langle \mathbf{n}_k, f^{ik}_p(x(t)) \rangle = \inf_{p \in \mathfrak{P}} f^{ik}_p(x(t)),$$

where $\mathfrak{P}_1 := \{p \in \mathfrak{P} :$ there is an arc $(j, i)$ such that $j \in N_i(p)$, and $x_{jk} \leq M_k(x(t_1)) - \delta_1\}$. In this step, we establish some useful properties of $g^{ik}(\cdot)$ based on Lemmas 3 and 5.
It has been shown that $f^i_k(x(t))$ is uniformly locally Lipschitz with respect to $x$ and compact on $\mathcal{H}_0^n$, $\forall i \in \mathcal{V}$, $\forall p \in \mathcal{P}$ based on Assumption 2 and Lemma 3. Also, $f^i_k(x_1, x_2, \ldots, x_n) = 0$, $\forall p \in \mathcal{P}$ when $x_1 = x_2 = \cdots = x_n = \eta$, where $\eta \in \mathbb{R}^d$ is a constant. It then follows from Lemma 5 that $g^i_k(x(t))$ is Lipschitz continuous with respect to $x \in \mathcal{H}_0^n$. Therefore, there exists a positive constant $L_1^+$ related to $\mathcal{H}_0$ such that $|g^i_k(\overline{x})| - |g^i_k(x)| \leq L_1^+ \|x(t) - \overline{x}(t)\| \leq L_1^+ \varepsilon, \forall x, \overline{x} \in \mathcal{H}_0^n$.

(Step II). In this step, we show that the derivative of $\|x_i(t)\|_B$ along the solution of (3) has a lower bound. For any $p \in \mathcal{P}$ such that there is an arc $(j, i)$ such that $j \in N_i(p)$, and $x_{jk} \leq M_k(x(t_1)) - \delta_1$, it follows from Assumption 3 of $f^j_p(\overline{x}) \in T_j(\overline{x}, \mathcal{H}(C^1_p(\overline{x})))$ and $\overline{x}(t) = M_k(x(t)) \geq x_{ik}(t) > M_k(x(t_1)) - \varepsilon$ that

$$\langle \overline{r}_k, f^j_p(\overline{x}) \rangle \geq \gamma D_k(\mathcal{H}(\text{co}\{x_i, x_j : j \in N_i(p)\})) \geq \gamma D_k(\mathcal{H}(\text{co}\{x_i, x_j\})) > \gamma (\delta_1 - \varepsilon),$$

where the first inequality is based on Assumption 3 by noting that $\overline{x}_i \in rb_k \mathcal{H}(\text{co}\{x_i, x_j : j \in N_i(p)\})$, where $rb_k \mathcal{H}(\text{co}\{x_i, x_j : j \in N_i(p)\})$ is the facet of $\mathcal{H}(\text{co}\{x_i, x_j : j \in N_i(p)\})$ perpendicular to $\overline{r}_k$. Also note that $\varepsilon < \delta_1$ based on the definition of $\varepsilon$.

It follows that for all $t \in [t_2, t_2 + \tau_d]$,

$$|g^j_k(x)| \geq |g^j_k(\overline{x})| - L_1^+ \varepsilon = \inf_{p \in \mathcal{P}_i} \langle \overline{r}_k, f^j_p(\overline{x}) \rangle - L_1^+ \varepsilon > \gamma (\delta_1 - \varepsilon) - L_1^+ \varepsilon.$$

Therefore,

$$|D^\|x_i(t)\|_B| = |f^i_k(\sigma(t))| \geq |g^i_k(x(t))| > \gamma (\delta_1 - \varepsilon) - L_1^+ \varepsilon.$$

(Step III). In this step, we show that there exists a $t_3 \in [t_1, t_2 + \tau_d]$ such that $x_{ik}(t_3) \leq M_k(x(t_1)) - \varepsilon$ and finally prove this lemma by using Lemma 7.

Define $\varepsilon = \frac{\gamma \tau_d}{L_1^+ \tau_d + \gamma \tau_d + 1} \delta_1$. It follows that $(\gamma (\delta_1 - \varepsilon) - L_1^+ \varepsilon) \tau_d = \varepsilon$. This shows that $|g^i_k(x)| > \varepsilon$. Moreover,

$$\left\| \|x_i(t_2 + \tau_d)\|_B - \|x_i(t_2)\|_B \right\| = \int_{t_2}^{t_2 + \tau_d} \left| D^\|x_i(\tau)\|_B \right| d\tau > \frac{\tau_d \varepsilon}{\tau_d} = \varepsilon.$$

This contradicts the assumption that $0 < \|x_i(t)\|_B \leq \varepsilon$ for all $t \in [t_1, t_2 + \tau_d]$. Thus, there exists a $t_3 \in [t_1, t_2 + \tau_d]$ such that $x_{ik}(t_3) \leq M_k(x(t_1)) - \varepsilon$.

Finally, based on Lemma 7, we obtain $x_{ik}(t) \leq M_k(x(t_1)) - \delta_2$ for all $t \in [t_3, T^*]$, where $\delta_2 = e^{-L_1^+ T^*} \varepsilon$. This completes the proof of the lemma.

The following lemma is a symmetrical result with Lemma 9.
Lemma 10. Let Assumptions 2, 3 hold and assume that $G_{\sigma(t)}$ is uniformly jointly quasi-strongly connected. For any $(t_1, x(t_1)) \in \mathbb{R} \times H_0$, any $\delta_1 > 0$ sufficient small and any $T^* > 0$, if there is an arc $(j, i)$ and a time $t_2 \geq t_1$ such that $j \in N_i(\sigma(t))$, and $x_{jk}(t) \geq m_k(x(t_1)) + \delta_1$, then there exists a $t_3 \in [t_1, t_2 + \tau_d]$ such that $x_{ik}(t) \geq m_k(x(t_1)) + \delta_2$, for all $t \in [t_3, t_3 + T^*]$, where
\[
\delta_2 = \frac{\tau_d}{L_1^2 \eta + \gamma \tau_d + 1} e^{-L_1^2 T^*} \delta_1
\]
for some constants $L_1^2$ and $L_1^+$ related to $H_0$.

4.4 Proof of Theorem 1

The necessity proof follows a similar argument as the proof of Theorem 3.8 of [20]. It is therefore omitted. We prove the sufficiency.

Define
\[
V(x) = \rho(\mathcal{H}(C(x)))
\]
where $\rho(\mathcal{H}(C(x)))$ denotes the diameter of the hyperrectangle $\mathcal{H}(C(x))$. Clearly, it follows from Lemma 4 that $V(x)$ is nonincreasing along (3) and $x_i(t) \in H_0, \forall i \in V, \forall t \geq t_0$. We prove this theorem by showing that $V(x)$ is strictly shrinking over the time.

Since $G_{\sigma(t)}$ is uniformly jointly quasi-strongly connected, there is a $T > 0$ such that the union graph $\mathcal{G}([t_0, t_0 + T])$ is quasi-strongly connected. Define $T_1 = T + 2\tau_d$, where $\tau_d$ is the dwell time. Denote $\kappa_1 = t_0 + \tau_d$, $\kappa_2 = t_0 + T_1 + \tau_d$, \ldots, $\kappa_{n^2} = t_0 + (n^2 - 1)T_1 + \tau_d$. Thus, there exists a node $i_0 \in V$ such that $i_0$ has a path to every other nodes jointly on time interval $[\kappa_{l_1}, \kappa_{l_1} + T]$, where $i = 1, 2, \ldots, n$ and $1 \leq l_1 \leq l_2 \leq \cdots \leq l_n \leq n^2$. Denote $\overline{T} = n^2 T_1$.

We divide the rest of the proof into three steps.

(Step I). Consider the time interval $[t_0, t_0 + \overline{T}]$ and $k = 1$. In this step, we show that an agent that does not belong to the interior set will become an interior agent due to the attraction of interior agent $i_0$.

More specifically, define $\varepsilon_1 = \frac{M_1(x(t_0)) - m_1(x(t_0))}{2}$. It is trivial to show that $M_1(x(t)) = m_1(x(t)), \forall t \geq t_0$ when $M_1(x(t_0)) = m_1(x(t_0))$ based on Definition 4. Therefore, we assume that $M_1(x(t_0)) \neq m_1(x(t_0))$ without loss of generality. Split the node set into two disjoint subsets $\mathcal{V}_1 = \{j | x_{j1}(t_0) \leq M_1(x(t_0)) - \varepsilon_1\}$ and $\overline{\mathcal{V}}_1 = \{j | j \notin \mathcal{V}_1\}$.

Assume that $i_0 \in \mathcal{V}_1$. This implies that $x_{i_01}(t_0) \leq M_1(x(t_0)) - \varepsilon_1$. It follows from Lemma 7 that $x_{i_01}(t) \leq M_1(x(t_0)) - \delta_1, \forall t \in [t_0, t_0 + \overline{T}]$, where $\delta_1 = e^{-L_1^2 T} \varepsilon_1$. Considering the time interval $[\kappa_{l_1}, \kappa_{l_1} + T]$, we can show that there is an arc $(i_1, j_1) \in \mathcal{V}_1 \times \overline{\mathcal{V}}_1$ such that $i_1$ is a neighbor of $j_1$ because otherwise there is no arc $(i_1, j_1)$ for any $i_1 \in \mathcal{V}_1$ and $j_1 \in \overline{\mathcal{V}}_1$ (this contradicts the fact $i_0 \in \mathcal{V}_1$ has a path to every other nodes jointly on time interval $[\kappa_{l_1}, \kappa_{l_1} + T]$). Therefore, there
exists a time $\tau \in [\kappa_1, \kappa_1 + T] = [t_0 + (l_1 - 1)T + \tau_d, t_0 + l_1 T - \tau_d]$ such that $j_1 \in N_l(\sigma(\tau))$. Based on Assumption 1, it follows that there is time interval $[\tau_1, \tau_1 + \tau_d] \subset [t_0 + (l_1 - 1)T, t_0 + l_1 T]$ such that $j_1 \in N_l(\sigma(\tau))$, for all $t \in [\tau_1, \tau_1 + \tau_d]$.

Also note that $i_1 \in \mathcal{V}_1$ implies that $x_{i1}(t_0) \leq M_1(x(t_0)) - \varepsilon_1$. This further shows that $x_{i1}(t) \leq M_1(x(t)) - \delta_1$, $\forall t \in [t_0, t_0 + \mathcal{T}]$ based on Lemma 4. Therefore, it follows from Lemma 9 that there exists a $i \in \mathcal{V}_1$ such that $x_{i1}(t_0) \leq M_1(x(t_0)) - \varepsilon_2$ and $x_{i1}(t) \leq M_1(x(t)) - \delta_2$, $\forall t \in [t_2, t_2 + \mathcal{T}]$, where $\varepsilon_2 = \frac{\gamma_1 \tau_d}{L_{11\gamma_1 + \gamma_1 + 1}} e^{-L_1 \mathcal{T}} \varepsilon_1$ and $\delta_2 = \frac{\gamma_1 \tau_d}{L_{11\gamma_1 + \gamma_1 + 1}} e^{-L_1 \mathcal{T}} \delta_1$.

To this end, we have shown that at least two agents are not on the upper boundary at $t_0 + l_1 T$.

(Step II). In this step, we show that the side length of the hyperrectangle $\mathcal{H}(\mathcal{C}(x))$ parallel to the $k$th axis $\mathbf{r}_k$ at $t_0 + \mathcal{T}$ is strictly less than that at $t_0$.

We can now redefine two disjoint subsets $\mathcal{V}_2 = \{ j \mid x_{j1}(t_0) \leq M_1(x(t_0)) - \varepsilon_2 \}$ and $\mathcal{V}_2 = \{ j \mid j \notin \mathcal{V}_2 \}$. It then follows that $\mathcal{V}_2$ has at least two nodes. By repeating the above analysis, we can show that $x_{i1}(t) \leq M_1(x(t)) - \delta_n$, $\forall i \in \mathcal{V}$, $\forall t \in [t_n, t_n + \mathcal{T}]$ by noting that $\delta_n = \min_{i \in \mathcal{V}} \{ \delta_i \}$, where $t_n \in [t_0, \tau_n + \tau_d] \subseteq [t_0 + (l_n - 1)T_1, t_0 + l_n T_1]$ and $\delta_n = e^{-nL_1 \mathcal{T}} \frac{(\gamma_1 \tau_d)^{n-1}}{L_{11\gamma_1 + \gamma_1 + 1}^{n-1}} \varepsilon_1$.

Instead, if $i_0 \in \mathcal{V}_1$, or what is equivalent, $x_{i01}(t_0) \geq m_1(x(t_0)) + \varepsilon_1$, we can similarly show that $x_{i1}(t) \geq m_1(x(t)) - \overline{\delta}_n$, $\forall i \in \mathcal{V}$, $\forall t \in [t_n, t_n + \mathcal{T}]$, where $t_n \in [t_0, \tau_n + \tau_d] \subseteq [t_0 + (l_n - 1)T_1, t_0 + l_n T_1]$ and $\overline{\delta}_n = e^{-nL_2 \mathcal{T}} \frac{(\gamma_1 \tau_d)^{n-1}}{L_{21\gamma_1 + \gamma_1 + 1}^{n-1}} \varepsilon_1$ using Lemmas 8 and 10.

Therefore, it follows that $D_1(\mathcal{H}(x(t_0 + \mathcal{T}))) \leq D_1(\mathcal{H}(x(t_0))) - \beta_1 D_1(\mathcal{H}(x(t_0)))$, where $\beta_1 = e^{-nL_1 \mathcal{T}} \frac{(\gamma_1 \tau_d)^{n-1}}{2(L + \tau_d + \gamma_1 \tau_d + 1)^{n-1}}$ and $L^* = \max\{L_1, L_2\}$.

(Step III). In this step, we show that $\rho(\mathcal{H}(\mathcal{C}(x)))$ at $t_0 + d\mathcal{T}$ is strictly less than at $t_0$ and thus prove the theorem by showing that $V$ is strictly shrinking.

We consider the time interval $[t_0 + \mathcal{T}, t_0 + 2\mathcal{T}]$ and $k = 2$. Following similar analysis as of Step I and Step II, we can show that $D_2(\mathcal{H}(x(t_0 + 2\mathcal{T}))) \leq D_2(\mathcal{H}(x(t_0))) - \beta_2 D_2(\mathcal{H}(x(t_0)))$, where $\beta_2 = e^{-nL_2 \mathcal{T}} \frac{(\gamma_1 \tau_d)^{n-1}}{2(L + \tau_d + \gamma_1 \tau_d + 1)^{n-1}}$.

By repeating the above analysis, it follows that

$$V(x(t_0 + d\mathcal{T})) - V(x(t_0)) \leq -\beta V(x(t_0)),$$

where $\beta = e^{-nL_2 \mathcal{T}} \frac{(\gamma_1 \tau_d)^{n-1}}{2(L + \tau_d + \gamma_1 \tau_d + 1)^{n-1}}$. 

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Then, let \( N \) be the smallest positive integer such that \( t \leq t_0 + NdT \). It then follows that

\[
V(x(t)) \leq (1 - \beta)^{N-1}V(x(t_0)) \\
\leq \frac{1}{1 - \beta} \left(1 - \beta \right)^{\frac{t-t_0}{d_T}}V(x(t_0)) \\
= \frac{1}{1 - \beta} e^{-\beta^*(t-t_0)}V(x(t_0)),
\]

where \( \beta^* = \frac{1}{d_T} \ln \frac{1}{1 - \beta} \). Denote \( \mathcal{H}(S_0) \) as the supporting hyperrectangle of \( S_0 \). Since \( x(t_0) \in \mathcal{H}_0^n \subseteq \mathcal{H}^n(S_0) \), it follows that the above inequality holds for any \( x(t_0) \in \mathcal{H}^n(S_0) \) or any \( x(t_0) \in S_0 \). By choosing \( k = \frac{1}{1 - \beta} \) and \( \lambda = \beta^* \), we have that exponential state agreement is achieved on \( S_0 \).

**Remark 7.** The convergence is semi-global since the selection of \( \lambda \) and \( k \) depends on that the initial common space is given in advance and compact, i.e., the assumption that \( S_0 \) is compact is necessary to guarantee exponential state agreement.

### 5 Cooperative–antagonistic Multi-agent Systems: Absolute Agreement

In this section, we study state agreement over cooperative–antagonistic networks. We still focus on the nonlinear dynamics (3), but we use the model for antagonistic agent interactions introduced in [2], where an agent receives the state with opposite sign from its antagonistic neighbors. We introduce the following definition of the relation matrix, where each arc is assigned a plus or a minus sign.

**Definition 7 (Relation matrix).** Let \( G_p \) be the interaction graph associated with \( f_p \). The relation matrix \( R_p \in \mathbb{R}^{n \times n} \) associated with \( G_p \) is defined as \([R_p]_{ij} = 1 \) if \((i, j) \in E_p \) and \( i \) is cooperative to \( j \), \([R_p]_{ij} = -1 \) if \((i, j) \in E_p \) and \( i \) is antagonistic to \( j \), and \([R_p]_{ij} = 0 \) if \((i, j) \notin E_p \).

Define \( \widehat{\mathcal{C}}_i^p(x) := co\{x_i, [R_p]_{ij}x_j : j \in N_i(p)\} \). For cooperative–antagonistic networks, we impose the following assumption, instead of Assumption 3.

**Assumption 4 (Vector field).** For all \( i \in \mathcal{V} \), \( p \in \mathcal{P} \) and \( x \in \mathbb{R}^{dn} \), it holds that \( f_p^i(x) \in \mathcal{T}_i(x_i, \mathcal{H}(\widehat{\mathcal{C}}_i^p(x))) \).

Simple examples (see, e.g., [2]) can be found that state agreement cannot be achieved for cooperative–antagonistic networks. Instead, it is possible that different agents hold different values with opposite signs, which is known as bipartite consensus [2]. Therefore, we are interested
in the absolute agreement in this section. Introduce $\mathcal{J} = \{x \in \mathbb{R}^{dn} : |x_1|_* = |x_2|_* = \cdots = |x_n|_*\}$, where the componentwise absolute value $|\cdot|_*$ is defined as $|z|_* = [|z_1|, |z_2|, \ldots, |z_d|]^T$ for the vector $z = [z_1, z_2, \ldots, z_d]^T$. Asymptotic absolute agreement of system (3) is defined as follows.

**Definition 8** (Asymptotic absolute agreement). System (3) achieves asymptotic absolute agreement for initial time $t_0 \geq 0$ and initial state $x(t_0) \in \mathbb{R}^{nd}$ if

$$\lim_{t \to \infty} \|x(t)\|_{\mathcal{J}} = 0.$$  

Absolute agreement means that the absolute values of the node states eventually reach an agreement. In this case it is possible that different agents converge to zero state, an non-zero same state or split into two different states.

We present the following main result on absolute agreement of cooperative–antagonistic networks.

**Theorem 2.** Let Assumptions 1, 2 and 4 hold. Then system (3) achieves asymptotic absolute agreement for all initial time $t_0 \geq 0$ and all initial state $x(t_0) \in \mathbb{R}^{nd}$ if the interaction graph $G_{\sigma(t)}$ is uniformly jointly strongly connected.

**Remark 8.** The state agreement result in Theorem 1 relies on uniformly jointly quasi-strong connectivity, while the absolute state agreement result in Theorem 2 needs uniformly jointly strong connectivity. In fact, we conjecture that strong connectivity is essential for absolute state agreement in the sense that uniformly jointly quasi-strong connectivity might not be enough. The reason is that although Lemmas 7 and 9 can be rebuilt for the upper bound of the node absolute values for cooperative–antagonistic networks, the corresponding Lemmas 8 and 10 no longer hold.

**Remark 9.** For cooperative networks, we established the exponential rate for the convergence in the proof of Theorem 1. In contrast, for cooperative–antagonistic networks in Theorem 2, the convergence speed is unclear. We conjecture that exponential convergence might not hold under the conditions of Theorem 2. The reason is again that Lemmas 8 and 10 cannot be recovered for cooperative–antagonistic networks.

We believe that these differences reveal some important nature of cooperative and cooperative–antagonistic networks.

**Remark 10.** Compared to the results given in [2], Theorem 2 requires no conditions on the structural balance of $R_p$. In other words, Theorem 2 shows that every positive or negative
arc contributes to the convergence of the absolute values of the nodes’ states, even for general nonlinear multi-agent dynamics.

**Remark 11.** Definition 8 is again a set attraction definition. We can similarly define set stability with respect to $\mathcal{J}$: $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\forall t_0 \geq 0$,

$$\|x(t_0)\|_{\mathcal{J}} \leq \delta \quad \Rightarrow \quad \|x(t)\|_{\mathcal{J}} \leq \varepsilon, \quad \forall t \geq t_0.$$  

However, set stability might not hold under the conditions of Theorem 2.

The proof of Theorem 2 will be given using a contradiction arguments, with the help of a series of preliminary lemmas.

### 5.1 Invariant Set

In this section, we construct an invariant set for the dynamics under the cooperative–antagonistic networks. For all $k \in \mathcal{D}$, define

$$M_k^i(x(t)) = \max_{i \in \mathcal{V}} |x_{ik}(t)| \in \mathbb{R}.$$  

In addition, define an origin-symmetric supporting hyperrectangle $\mathcal{H}(\hat{C}(x)) \subset \mathbb{R}^d$ as

$$\mathcal{H}(\hat{C}(x)) := [-M_1^i(x), M_1^i(x)] \times \cdots \times [-M_d^i(x), M_d^i(x)].$$  

The origin-symmetric supporting hyperrectangle formed by the initial states of all agents $\mathcal{H}_0$ is given by

$$[-\max_{i \in \mathcal{V}} |x_{i1}(t_0)|, \max_{i \in \mathcal{V}} |x_{i1}(t_0)|] \times \cdots \times [-\max_{i \in \mathcal{V}} |x_{id}(t_0)|, \max_{i \in \mathcal{V}} |x_{id}(t_0)|].$$  

Introduce the state transformation

$$y_{ik} = x_{ik}^2, \quad \forall i \in \mathcal{V}, \quad \forall k \in \mathcal{D}.$$  

The analysis will be carried out on $y_{ik}$, instead of $x_{ik}$ to avoid non-smoothness. The following lemma establishes an invariant set for system (3).

**Lemma 11.** Let Assumptions 1, 2 and 4 hold. Then, for system (3), $\mathcal{H}_0$ is an invariant set, i.e., $x_i(t) \in \mathcal{H}_0$, $\forall i \in \mathcal{V}, \forall t \geq t_0$.\[22\]
Proof. Let \( y_k = \max_{i \in V} y_{ik} \), for all \( k \in D \). We first show that \( D^+ y_k \leq 0 \), for all \( k \in D \). It follows from (1) that

\[
\dot{y}_{ik} = 2(x_{ik}, f_{ik}^p(x)), \quad \forall i \in V, \forall k \in D.
\]

Let \( V_1(t) = \{ i \in V : y_{ik}(x_i(t)) = y_k(x(t)) \} \) be the set of indices where the maximum is reached at \( t \). It then follows from Lemma 11 that for all \( k \in D \),

\[
D^+ y_k = 2 \max_{i \in V_1(t)} |\langle x_{ik}, f_{ik}^p(x) \rangle|.
\]

Consider any \( x(t_0) \in \mathcal{H}_0^n \) and any initial time \( t_0 \). For any point \( x \in \mathcal{H}_0^n \), since \( \mathcal{H}_0^n \) is convex, closed, and symmetric to the origin, it follows that \( \mathcal{H}(\hat{C}_p(x)) \subseteq \mathcal{H}_0, \forall i \in V, \forall p \in \mathfrak{P} \). Therefore, it follows from Definition 4 and Lemma 11 that

\[
f_{ik}^p(x) \in \mathcal{T}_f(x_i, \mathcal{H}(\hat{C}_p(x))) \subseteq \mathcal{T}(x_i, \mathcal{H}(\hat{C}_p(x))) \subseteq \mathcal{T}(x_i, \mathcal{H}_0), \quad \forall i \in V, \forall p \in \mathfrak{P}.
\]

Based on Definition 4 it follows that \( \langle f_{ik}^p(x), -\gamma_k \rangle \geq 0 \) for \( x_{ik} = \sqrt{y_k} \geq 0 \) and \( \langle f_{ik}^p(x), \gamma_k \rangle \geq 0 \) for \( x_{ik} = -\sqrt{y_k} \leq 0 \). This shows that \( \langle x_{ik}, f_{ik}^p(x) \rangle \leq 0 \) for \( i \in V_1 = \{ i \in V : y_{ik} = y_k \} \). It follows that for all \( \forall k \in D \), and \( x \in \mathcal{H}_0^n \),

\[
D^+ y_k \leq 0, \quad \forall k \in D.
\]

Therefore, \( x_{ik}(t) \leq \max_{i \in V} x_{ik}(t_0), \forall i \in V, \forall k \in D \), which shows that

\[
-\max_{i \in V} |x_{ik}(t_0)| \leq x_{ik}(t) \leq \max_{i \in V} |x_{ik}(t_0)|, \quad \forall i \in V, \forall k \in D, \forall t \geq t_0.
\]

This implies that \( \mathcal{H}_0^n \) is an invariant set.

\[ \square \]

Remark 12. In Figures 3–4, we highlight the different invariant sets for cooperative and cooperative–antagonistic networks. The supporting hyperrectangle \( \mathcal{H}(\mathcal{C}(x)) \) given in Lemma 3 is illustrated in Figure 3 and the origin-symmetric supporting hyperrectangle \( \mathcal{H}(\hat{C}(x)) \) given in Lemma 11 is illustrated in Figure 4.

5.2 Interior agents

Before moving on, we need the following technical lemma.

Lemma 12. Suppose that \( f(x_1, y) : M_1 \times M \to \mathbb{R} \) is locally Lipschitz with respect to \( [x_1, y]^T \), where \( M_1 \subset \mathbb{R} \) and \( M \subset \mathbb{R}^q \) are compact. Then \( g(x_1) : M_1 \to \mathbb{R}, \ x_1 \mapsto \max_{y \in M} x_1 f(x_1, y) \) is Lipschitz continuous with respect to \( x_1 \) on \( M_1 \).
Figure 3: An example of the supporting hyperrectangle of $\mathcal{H}(\mathcal{C}(x))$.

Figure 4: An example of the origin-symmetric supporting hyperrectangle $\mathcal{H}(\hat{\mathcal{C}}(x))$. 
Proof. Because $f(x_1, y)$ is locally Lipschitz with respect to $[x_1, y]^T$ and $\mathcal{M}_1$ and $\mathcal{M}$ are compact, it follows that $f(x_1, y)$ is Lipschitz continuous with respect to $[x_1, y]^T$. Therefore, there exists a constant $L$ such that

$$
\|f(x_1, y) - f(\bar{x}, y)\| \leq L\|x_1 - \bar{x}\|, \ \forall x_1, \bar{x} \in \mathcal{M}_1, \ \forall y \in \mathcal{M}.
$$

Also, since $f(x_1, y)$ is continuous and the continuous function on the compact set is compact, there exist constants $L_x$ and $L_f$ such that $L_x = \max_{x_1 \in \mathcal{M}_1} \|x_1\|$ and $L_f = \max_{x_1 \in \mathcal{M}_1, y \in \mathcal{M}} \|f(x_1, y)\|.

Let $y_x$ and $\bar{y}_x$ be the points satisfying $g(x_1) = \max_{y \in \mathcal{M}} \{x_1 f(x_1, y)\} = x_1 f(x_1, y_x)$ and $g(\bar{x}_1) = \max_{y \in \mathcal{M}} \{\bar{x}_1 f(\bar{x}_1, y)\} = \bar{x}_1 f(\bar{x}_1, y_\bar{x})$. It is trivial to show that $x_1 f(x_1, y_x) \geq x_1 f(x_1, y_\bar{x})$ and $\bar{x}_1 f(\bar{x}_1, y_x) \geq \bar{x}_1 f(\bar{x}_1, y_\bar{x})$. Therefore, there exists $\bar{x} = (1 - \lambda)x_1 + \lambda\bar{x}_1$, where $0 \leq \lambda \leq 1$ such that $\bar{x}_1 f(\bar{x}, y_x) = \bar{x}_1 f(\bar{x}, y_\bar{x})$. Thus,

$$
\|g(x_1) - g(\bar{x}_1)\| = \|x_1 f(x_1, y_x) - \bar{x}_1 f(\bar{x}, y_x)\| + \|\bar{x}_1 f(\bar{x}, y_x) - \bar{x}_1 f(\bar{x}, y_\bar{x})\|
$$

It then follows that

$$
\|g(x_1) - g(\bar{x}_1)\| \leq L\|x_1\|\|x_1 - \bar{x}\| + \|f(\bar{x}, y_x)\|\|x_1 - \bar{x}\|
$$

Therefore, $g(x_1)$ is Lipschitz continuous with respect to $x_1$ on $\mathcal{M}_1$. \qed

In the following lemma, we show that the projection of the state on any axis is strictly less than a certain upper bound as long as it is initially strictly less than this upper bound.

Lemma 13. Let Assumptions $\square$ and $\square$ hold and assume that $\mathcal{G}_{\sigma(t)}$ is uniformly jointly strongly connected. For any $(t_1, x(t_1)) \in \mathbb{R} \times \mathcal{H}^0_k$, any $\varepsilon > 0$ sufficient small and any $T^* > 0$, if $y_{i_k}(t_2) \leq y^* - \varepsilon$ at some $t_2 \geq t_1$ for some $k \in \mathcal{D}$, where $y^* \geq y_k(x(t_1))$ is a constant. Then $y_{i_k}(t) \leq y^* - \delta$ for all $t \in [t_2, t_2 + T^*]$, where $\delta = e^{-L^* T^*} \varepsilon$, and $L^*$ is a positive constant related to $\mathcal{H}_0$.

Proof. Consider any $(t_1, x(t_1)) \in \mathbb{R} \times \mathcal{H}^0_k$, and let $\psi = x(t_1)$ and $\mathcal{M} = \mathcal{H}^0_k(\psi) \times \mathcal{H}(\psi) \times \cdots \times \mathcal{H}(\psi)$, where $\mathcal{H}(\psi) = [\mathcal{H}^i_k(\psi), M^i_k(\psi)] \times \cdots \times [\mathcal{H}^j_k(\psi), M^j_k(\psi)]$, and $\mathcal{H}^0_k(\psi) = [\mathcal{H}^i_k(\psi), M^i_k(\psi)] \times \cdots$
\[\times [-M_{k-1}^1(\psi), M_{k-1}^1(\psi)] \times [-M_{k+1}^1(\psi), M_{k+1}^1(\psi)] \times \cdots \times [-M_d^1(\psi), M_d^1(\psi)]\]. Again, for clarity we divide the rest of the proof into three steps.

(Step I). Define the following function

\[g_{H_0}(\chi) : [-\bar{a}, \bar{a}] \to \mathbb{R},\]

\[x_{ik} \to \begin{cases} 
\sup_{p \in \mathcal{P}} \{\max_{k \in D} \{\max_{y \in \mathcal{M}} x_{ik} f_p^{ik}(x_{ik}, y)\}\}, & \text{if } x_{ik} \in [-M_k^1(\psi), M_k^1(\psi)], \\
g_{H_0}(M_k^1(\psi)), & \text{if } x_{ik} \in (M_k^1(\psi), \bar{a}], \\
g_{H_0}(-M_k^1(\psi)), & \text{if } x_{ik} \in [-\bar{a}, -M_k^1(\psi)),
\end{cases}\]

where \(\bar{a} = \max_{k \in D} M_k^1(x(t_0))\) is a constant. Obviously, \(g_{H_0}(\chi)\) is continuous with respect to \(\chi\). In this step, we establish some useful properties of \(g_{H_0}(\cdot)\) based on Lemmas 6 and 12. We make the following claim.

Claim A: (i) \(g_{H_0}(\chi) > 0\) if \(\chi \in (-M_k^1(\psi), M_k^1(\psi))\); (ii) \(g_{H_0}(\chi) = 0\) if \(\chi \in [-\bar{a}, -M_k^1(\psi)] \cap [M_k^1(\psi), \bar{a}]\); (iii) \(g_{H_0}(\chi)\) is Lipschitz continuous with respect to \(\chi\) on \([-\bar{a}, \bar{a}]\).

The first and second properties of Claim A can be obtained following a similar analysis to the proof of Lemma 7.

For the third property of Claim A, it follows from Lemma 12 that \(g_p^{ik}(x_{ik}) : [-M_k^1(\psi), M_k^1(\psi)] \to \mathbb{R}, x_{ik} \mapsto \max_{y \in \mathcal{M}} x_{ik} f_p^{ik}(x_{ik}, y)\) is Lipschitz continuous with respect to \(x_{ik}\), \(\forall k \in D\), \(\forall i \in \mathcal{V}\), and \(\forall p \in \mathcal{P}\). Also note that \(g_{H_0}(M_k^1(\psi)) = 0\). Then, it follows from Lemma 5 that \(g_{H_0}(\chi)\) is Lipschitz continuous with respect to \(\chi\) on \([-M_k^1(\psi), M_k^1(\psi)]\). By noting that \(g_{H_0}(\chi)\) is compact on the compact set \([-M_k^1(\psi), M_k^1(\psi)]\) and from the definition of \(g_{H_0}(\chi)\), it follows that \(g_{H_0}(\chi)\) is Lipschitz continuous with respect to \(\chi\) on \([-\bar{a}, \bar{a}]\).

(Step II). In this step, based on the definition of \(g_{H_0}(\cdot)\), we construct another nonlinear function \(h_{H_0}(\cdot)\) that can be used to measure the distance between \(y_{ik}\) and the upper boundary \(y_k(\psi)\).

First, considering any \(T^* > 0\), we suppose that \(y_{ik}(t_2) \leq y^* - \varepsilon\) at some \(t_2 \geq t_1\). Based on the definitions of \(g_{H_0}(\chi)\), it follows that

\[y_{ik}(t) = 2(x_{ik}, f_{g_{H_0}(\chi)}^{ik}(x(t))) \leq 2g_{H_0}(x_{ik}(t)), \quad \forall t \geq t_1.\]

It also follows from the properties of \(g_{H_0}(\chi)\) that there exists a Lipschitz continuous constant \(L_1\) such that \(g_{H_0}(\chi) = |g_{H_0}(\chi) - g_{H_0}(M_k^1(\psi))| \leq L_1|\chi - M_k^1(\psi)| = L_1(M_k^1(\psi) - \chi), \forall \chi \in [-M_k^1(\psi), M_k^1(\psi)]\) and \(g_{H_0}(\chi) = |g_{H_0}(\chi) - g_{H_0}(-M_k^1(\psi))| \leq L_1|\chi + M_k^1(\psi)| = L_1(M_k^1(\psi) + \chi), \forall \chi \in [-M_k^1(\psi), M_k^1(\psi)],\) where \(L_1\) is related to \(H_0\).
Therefore, for the case of $x_{ik} \geq 0$, we have that \( \dot{y}_{ik}(t) \leq 2L_{1}(M_{k}^{1}(\psi) - x_{ik}) = 2L_{1}(M_{k}^{1}(\psi) - \sqrt{y_{ik}(t)}) \). For the case of $x_{ik} < 0$, we have that \( \dot{y}_{ik}(t) \leq 2L_{1}(M_{k}^{1}(\psi) + x_{ik}) = 2L_{1}(M_{k}^{1}(\psi) - \sqrt{y_{ik}(t)}) \). Overall, it follows that

\[
\dot{y}_{ik}(t) \leq 2L_{1}(M_{k}^{1}(\psi) - \sqrt{y_{ik}(t)}), \quad \forall t \geq t_{2}.
\]

Denote $\phi(t)$ be the solution of $\dot{\phi} = \overline{h}_{\psi}(\phi)$ with initial condition $\phi(t_{2}) = y_{ik}(t_{2})$, where $\overline{h}_{\psi}(\phi) : [0, y_{k}(\psi)] \to \mathbb{R}$, $\phi \mapsto 2L_{1}(M_{k}^{1}(\psi) - \sqrt{\phi})$. It follows from Lemma 4 that $y_{ik}(t) \leq \phi(t)$, $\forall t \geq t_{2}$.

Next, by letting $\varphi = y_{k}(\psi) - \phi$, we define the following function

\[
h_{H_{0}}(\varphi) : [0, \hat{a}^{2}] \to \mathbb{R}
\]

\[
\varphi \mapsto \begin{cases} 
\overline{h}_{\psi}(y_{k}(\psi) - \varphi); & \text{if } \varphi \in [0, y_{k}(\psi)], \\
h_{H_{0}}(y_{k}(\psi)); & \text{if } x_{ik} \in (y_{k}(\psi), \hat{a}^{2}].
\end{cases}
\]

We make the following claim.

Claim B: (i) $h_{H_{0}}(\varphi)$ is Lipschitz continuous with respect to $\varphi$ on $[0, \hat{a}^{2}]$; (ii) $h_{H_{0}}(\varphi) = 0$ if $\varphi = 0$; (iii) $h_{H_{0}}(\varphi) > 0$ if $\varphi \in (0, \hat{a}^{2}]$.

These three properties can be easily checked based on the definition of $h_{H_{0}}(\varphi)$.

(Step III). In this step, we take advantage of function $h_{H_{0}}(\cdot)$ to show that $y_{ik}$ will be always strictly less than the upper boundary $y^{*}$ as long as it is initially strictly less than $y^{*}$.

It follows from the first property of $h_{H_{0}}(\varphi)$ that there exists a constant $L^{*}$ related to $\mathcal{H}_{0}$ such that $|h_{H_{0}}(\varphi) - h_{H_{0}}(0)| \leq L^{*} \varphi$, $\forall \varphi \in [0, \hat{a}^{2}]$. From the second and third properties of $h_{H_{0}}(\varphi)$, it follows that $h_{H_{0}}(\varphi) \leq L^{*} \varphi$, $\forall \varphi \in [0, \hat{a}^{2}]$ and $\varphi = -\hat{\phi} = \overline{h}_{\psi}(\phi) = -h_{H_{0}}(\phi)$, $\forall \varphi \in [0, y_{k}(\psi)]$. It follows from Lemma 4 that the solution of $\dot{\varphi} = -h_{H_{0}}(\varphi)$ satisfies $\varphi(t) \geq e^{-L^{*}(t-t_{2})}\varphi(t_{2})$, $\forall t \geq t_{2}$ since $-h_{H_{0}}(\varphi) \geq -L^{*} \varphi$.

Therefore, $y_{ik}(t) \leq \phi(t) = y_{k}(\psi) - \varphi(t) \leq y_{k}(\psi) - e^{-L^{*}(t-t_{2})}(y_{k}(\psi) - \phi(t_{2})) \leq y_{k}(\psi) - e^{-L^{*}T^{*}}(y_{k}(\psi) - y_{ik}(t_{2})) = y^{*} + y_{k}(\psi) - y^{*} - e^{-L^{*}T^{*}}(y^{*} + y_{k}(\psi) - y^{*} - y_{ik}(t_{2})) = y^{*} - e^{-L^{*}T^{*}}(y^{*} - y_{ik}(t_{2})) + (y_{k}(\psi) - y^{*})(1 - e^{-L^{*}T^{*}}) \leq y^{*} - e^{-L^{*}T^{*}} \varepsilon$ for all $t \in [t_{2}, t_{2} + T^{*}]$ since $y^{*} \geq y_{k}(\psi)$. This proves the lemma by letting $\delta = e^{-L^{*}T^{*}} \varepsilon$. \(\square\)

### 5.3 Boundary Agents

A technical lemma is first given.
Lemma 14. Suppose that \( f(x) : \mathcal{M} \to \mathbb{R} \) is locally Lipschitz with respect to \( x \), where \( \mathcal{M} \subset \mathbb{R}^d \) is compact. Then \( g(x) : \mathcal{M} \to \mathbb{R} \), \( x \mapsto x_1 f(x) \) is Lipschitz continuous with respect to \( x \) on \( \mathcal{M} \), where \( x_1 \) denotes a certain element of \( x \).

Proof. Because \( f(x) \) is locally Lipschitz with respect to \( x \) and \( \mathcal{M} \) is compact, it follows that \( f(x) \) is Lipschitz continuous with respect to \( x \). Therefore, there exists a constant \( L \) such that

\[
\|f(x) - f(\overline{x})\| \leq L\|x - \overline{x}\|, \quad \forall x, \overline{x} \in \mathcal{M}.
\]

Also, since \( f(x) \) is continuous and the continuous function on the compact set is still compact, there exist constants \( L_x \) and \( L_f \) such that \( L_x = \max_{x \in \mathcal{M}} \|x\| \) and \( L_f = \max_{x \in \mathcal{M}} \|f(x)\| \).

It then follows that \( \forall x, \overline{x} \in \mathcal{M} \),

\[
\|g(x) - g(\overline{x})\| = \|x_1 f(x) - x_1 f(\overline{x})\| \leq LL_x \|x - \overline{x}\| + L_f \|x - \overline{x}\| = (LL_x + L_f)\|x - \overline{x}\|.
\]

Therefore, \( g(x) \) is Lipschitz continuous with respect to \( x \) on \( \mathcal{M} \).

In the following lemma, we show that any agent that is attracted by an agent strictly inside the upper boundary is drawn strictly inside that boundary.

Lemma 15. For any \( (t_1, x(t_1)) \in \mathbb{R} \times \mathcal{H}_0^n \) and any \( \delta > 0 \) sufficient small, if there is an arc \((j, i)\) and a time \( t_2 \geq t_1 \) such that \( j \in N_i(\sigma(t)), \) and \( y_{jk}(t) \leq y^* - \delta \) for all \( t \in [t_2, t_2 + \tau_d] \) for some \( k \in \mathcal{D} \), where \( y^* \geq y_k(x(t_1)) \) is a constant. Then there exists a \( t_3 \in [t_1, t_2 + \tau_d] \) such that \( y_{ik}(t_3) \leq y^* - \varepsilon \), where \( \varepsilon = \frac{\gamma \tau_d \delta}{2(L^+ \tau_k + \gamma t_d + 1)} \) and \( L^+ \) is a constant related to \( \mathcal{H}_0 \).

Proof. We prove this lemma by contradiction. Suppose that there does not exist a \( t_3 \in [t_1, t_2 + \tau_d] \) such that \( |x_{ik}(t_3)| \leq \sqrt{y^* - \varepsilon_1} \), where \( \varepsilon_1 \) is a positive constant. Then it follows that \( \sqrt{y^* - \varepsilon_1} < |x_{ik}(t)| \leq M^i_k(x(t_1)) \) for all \( t \in [t_1, t_2 + \tau_d] \).

We focus on the time interval \( t \in [t_2, t_2 + \tau_d] \). Define \( \overline{x}(t) \) by only replacing \( x_{ik}(t) \) in \( x(t) \) with \( \overline{x}_{ik}(t) = \max_{i \in \mathbb{V}} \{ x_{ik}(t) \} = M^i_k(x(t_1)). \) It follows from (2) that

\[
\frac{d}{dt} g_{ik}(\overline{x}(t)) = 2\langle \overline{x}_{ik}(t), f^i_{\sigma(t)}(\overline{x}(t)) \rangle = 2\|\overline{x}_{ik}(t)\| \langle \text{sgn}(M^i_k(x(t))), f^i_{\sigma(t)}(\overline{x}(t)) \rangle = 2\|\overline{x}_{ik}(t)\| \langle \overrightarrow{r^i_k}, f^i_{\sigma(t)}(\overline{x}(t)) \rangle,
\]

where the third equality is due to \( \text{sgn}(M^i_k(x)) = 1 \) when \( M^i_k(x) > 0 \). This shows that

\[
D^+ |\overline{x}_{ik}(t)| = \langle \overrightarrow{r^i_k}, f^i_{\sigma(t)}(\overline{x}(t)) \rangle.
\]

Next, we define

\[
g^{ik}(x(t)) = \inf_{p \in \mathbb{V}_1} \langle r^i_k, f^i_{\sigma(t)}(x(t)) \rangle = \inf_{p \in \mathbb{V}_1} f^i_{\sigma(t)}(x(t)),
\]
where $\mathcal{P}_1 := \{p \in \mathcal{P} : \text{there is an arc } (j, i) \text{ such that } j \in N_i(p), \text{ and } y_{jk} \leq y^* - \delta\}$. Following a similar analysis as in the proof of Lemma 9, we can first show that $g^{ik}(x)$ is Lipschitz continuous with respect to $x \in \mathcal{H}_0^n$. By noting the fact that $M_k^1(x(t_1)) - \varepsilon_1 \leq \sqrt{y^*} - \varepsilon_1 < |x_{ik}(t)|$, it follows that there exists a positive constant $L^+$ related to $\mathcal{H}_0$ such that $|g^{ik}(x) - g^{ik}(x)| \leq |g^{ik}(x) - g^{ik}(x)| \leq L^+\|x(t) - x(t)\| \leq L^+\varepsilon_1, \forall x, x \in \mathcal{H}_0^n$.

The condition $y_{jk}(t) \leq y^* - \delta$ is equivalent to $\sqrt{y^*} - |x_{jk}(t)| \geq \frac{\delta}{2\sqrt{y^*}}$. Therefore, for any $p \in \mathcal{P}$ such that there is an arc $(j, i)$ such that $j \in N_i(p)$ and $\sqrt{y^*} - |x_{jk}(t)| \geq \frac{\delta}{2\sqrt{y^*}}$, it follows from Assumption 1 of $f^k_p(x) \in \mathcal{T}_1(x, \mathcal{H}(\mathcal{C}_p(x)))$ and the fact that $M_k^1(x) \geq |x_{ik}(t)| > \sqrt{y^*} - \varepsilon_1$ that

$$\left|\langle \tilde{r}^k_{f^k_p(x)} \rangle \right| \geq \gamma D_h(\mathcal{H}(\text{co} \{x, [R_p]_{ij} x_j, j \in N_i(p)\})) \geq \gamma D_h(\mathcal{H}(\text{co} \{x, [R_p]_{ij} x_j\})) > \gamma \left(\frac{\delta}{2\sqrt{y^*}} - \varepsilon_1\right).$$

Note that $\frac{\delta}{2\sqrt{y^*}} > \varepsilon_1$ based on the definition of $\varepsilon_1$.

It follows that for all $t \in [t_2, t_2 + \tau_d]$,

$$|g^{ik}(x)| \geq |g^{ik}(x)| - L^+\varepsilon_1 = \inf_{p \in \mathcal{P}_1} \langle \tilde{r}^k_{f^k_p(x)} \rangle - L^+\varepsilon_1 \geq \gamma \left(\frac{\delta}{2\sqrt{y^*}} - \varepsilon_1\right) - L^+\varepsilon_1.$$

We thus arrive at

$$|D^+|x_{ik}(t)|| = |f^k_{\sigma(t)}(x(t))| \geq |g^{ik}(x(t))| > \gamma \left(\frac{\delta}{2\sqrt{y^*}} - \varepsilon_1\right) - L^+\varepsilon_1.$$

Choose $\varepsilon_1$ sufficient small such that $\varepsilon_1 = \frac{\gamma \tau_d \delta}{2(L^+ \tau_d + \gamma \tau_d + 1) \sqrt{y^*}}$. Such $\varepsilon_1$ exists for every $y^* > 0$. It follows that $\gamma \left(\frac{\delta}{2\sqrt{y^*}} - \varepsilon_1\right) - L^+\varepsilon_1 > \frac{\delta}{2\tau_d}$. This shows that $|g^{ik}(x(t))| > \frac{\delta}{2\tau_d}, \forall t \in [t_2, t_2 + \tau_d]$. Also,

$$|x_{ik}(t_2 + \tau_d) - x_{ik}(t_2)| = \int_{t_2}^{t_2 + \tau_d} |D^+|x_{ik}(\tau)|| d\tau \geq \frac{\tau_d \varepsilon_1}{\tau_d} = \varepsilon_1.$$

This contradicts the assumption that $\sqrt{y^*} - \varepsilon_1 < |x_{ik}(t)| \leq M_k^1(x(t_1))$ for all $t \in [t_2, t_2 + \tau_d]$. It then follows that $y_{jk}(t_3) \leq y^* - \varepsilon$, where $\varepsilon = \sqrt{y^*}\varepsilon_1 = \frac{\gamma \tau_d \delta}{2(L^+ \tau_d + \gamma \tau_d + 1) \sqrt{y^*}}$.

\section{5.4 Proof of Theorem 2}

The theorem is proved using a contradiction argument.

Lemma 11 implies that for any initial time $t_0$ and initial value $x(t_0)$, there exist $y^*_k, k \in \mathcal{D}$ such that

$$\lim_{t \to \infty} y_k = y^*_k, \quad k \in \mathcal{D}.$$

Define $h_{ik} = \lim_{t \to \infty} \sup y_{ik}(t)$ and $\ell_{ik} = \lim_{t \to \infty} \inf y_{ik}(t), \forall i \in \mathcal{V}, \forall k \in \mathcal{D}$. Clearly, $0 \leq \ell_{ik} \leq h_{ik} \leq y^*_k$. Based on Definition 8, absolute state agreement is achieved if and only if $h_{ik} = \ell_{ik} = y^*_k$.
\[ \forall i \in \mathcal{V}, \forall k \in \mathcal{D}. \] The desired conclusion holds trivially if \( y_k^* = 0, k \in \mathcal{D}. \) Therefore, we assume that \( y_k^* > 0 \) for some \( k \in \mathcal{D} \) without loss of generality.

Suppose that there exists a node \( i_1 \in \mathcal{V} \) such that \( 0 \leq \ell_{i_1k} < h_{i_1k} \leq y_k^* \). Based on the fact that \( \lim_{t \to \infty} y_k(t) \to y_k^* \), it follows that for any \( \varepsilon > 0 \), there exists a \( \hat{t}(\varepsilon) > t_0 \) such that

\[ 0 \leq y_k(t) \leq y_k^* + \varepsilon, \quad \forall i \in \mathcal{V}, \quad t \geq \hat{t}(\varepsilon). \]

Take \( \alpha_{1k} = \sqrt{\frac{1}{2}(\ell_{i_1k} + h_{i_1k})} \). Therefore, there exists a time \( t_1 \geq \hat{t}(\varepsilon) \) such that \( |x_{i_1k}(t_1)| = \alpha_{1k} \).

This shows that

\[ x_{i_1k}^2(t_1) = h_{i_1k} - (h_{i_1k} - \alpha_{1k}^2) \leq y_k^* + \varepsilon - (h_{i_1k} - \alpha_{1k}^2) = y_k^* + \varepsilon - \varepsilon_1, \]

where \( \varepsilon_1 = h_{i_1k} - \alpha_{1k}^2 > 0 \). The first inequality is based on the definition of \( h_{i_1k} \).

Since \( \mathcal{G}_\mathcal{D}(t) \) is uniformly jointly strongly connected, there is a \( T > 0 \) such that the union graph \( \mathcal{G}([t_1, t_1 + T]) \) is jointly strongly connected. Define \( T_1 = T + 2\tau_d \), where \( \tau_d \) is the dwell time. Denote \( \kappa_1 = t_1 + \tau_d, \kappa_2 = t_1 + T_1 + \tau_d, \ldots, \kappa_n = t_1 + (n - 1)T_1 + \tau_d \). For each node \( i \in \mathcal{V}, i \) has a path to every other nodes jointly on time interval \([\kappa_l, \kappa_l + T]\), where \( l = 1, 2, \ldots, n \).

Denote \( T = nT_1 \).

Consider time interval \([t_1, t_1 + T]\). Based on the fact that \( y_k(x(t_1)) \leq y_k^* + \varepsilon \) and considering \( y_k^* + \varepsilon \) as the role of \( y^* \) in Lemma 13 it follows that \( y_k(t) \leq y_k^* + \varepsilon - \delta_1, \forall t \in [t_1, t_1 + T] \), where \( \delta_1 = e^{-LT}\varepsilon_1 \).

Since for each node \( i \in \mathcal{V}, i \) has a path to every other nodes jointly on time interval \([\kappa_l, \kappa_l + T]\), where \( l = 1, 2, \ldots, n \), there exists \( i_2 \in \mathcal{V} \) such that \( i_1 \) is a neighbor of \( i_2 \) during the time interval \([\kappa_l, \kappa_l + T]\). Based on Lemma 15 it follows that there exists \( t_2 \in [t_1, \tau_1 + \tau_d] \subset [t_1 + T, t_1 + 2T] \) such that \( x_{i_2k}^2(t_2) \leq y_k^* + \varepsilon - \varepsilon_2, \forall t \in [t_2, t_1 + T] \), where \( \varepsilon_2 = \frac{\varepsilon_1}{2(L + \tau_d + \gamma_{\tau_d + 1})}\delta_1 \). This further implies that \( x_{i_2k}^2(t) \leq y_k^* + \varepsilon - \delta_2, \forall t \in [t_2, t_1 + T] \), where \( \delta_2 = e^{-LT}\varepsilon_2 \). By repeating the above analysis, we can show that \( y_{ik}(t) \leq y_k^* + \varepsilon - \delta_n, \forall t \in [t_n, t_1 + T], \forall i \in \mathcal{V} \), where \( t_n \in [t_1, \tau_n + \tau_d] \subset [t_1 + (n - 1)T, t_1 + nT] \), and \( \delta_n \) can be iteratively obtained as \( \delta_n = e^{-nLT}\varepsilon_1^{n-1}\varepsilon_2^{n-1}\varepsilon_{n-1}^{n-1} \). This is indeed true because \( \delta_i \leq \delta_{i-1}, \forall i = 2, 3, \ldots, n \).

This shows that \( y_k(t_1 + T) = \max_{i \in \mathcal{V}} y_{ik} < y_k^* \) for sufficient small \( \varepsilon \) satisfying \( \varepsilon < \delta_n \). It follows from Lemma 11 that \( y_k(t) \leq y_k(t_1 + T) < y_k^*, \forall t \geq t_1 + T \), which contradicts the definition of \( y_k^* \). Therefore, \( h_{ik} = \ell_{ik} = y_k^*, \forall i \in \mathcal{V}, \forall k \in \mathcal{D} \). This proves absolute state agreement and the theorem holds.
6 Conclusions

Agreement protocols for nonlinear multi-agent dynamics over cooperative or cooperative–
antagonistic networks were investigated. A class of nonlinear control laws were introduced
based on a relaxed convexity conditions. The price was that each agent must get access to a
common coordinate system, a compass, but the precise state is not assumed to be available.
Each agent specified a local supporting hyperrectangle with the help of the compass, and then
a strict tangent cone was determined based on which local control can be found. Under mild
conditions on the nonlinear dynamics and the interaction graph, we proved that for cooperative
networks, exponential state agreement is achieved if and only if the communication topology
is uniformly jointly quasi-strongly connected. For cooperative–antagonistic networks, absolute
state agreement is achieved asymptotically if the time-varying communication topology is
uniformly jointly strongly connected. The results generalized the existing studies on agreement
seeking of multi-agent systems. Future works include higher-order agent dynamics and other
dynamics on antagonistic arcs.

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