THE GEOMETRY OF HIDA FAMILIES AND Λ-ADIC HODGE THEORY

BRYDEN CAIS

To Haruzo Hida, on the occasion of his 60th birthday.

Abstract. We construct Λ-adic de Rham and crystalline analogues of Hida’s ordinary Λ-adic étale cohomology, and by exploiting the geometry of integral models of modular curves over the cyclotomic extension of $\mathbb{Q}_p$, we prove appropriate finiteness and control theorems in each case. We then employ integral $p$-adic Hodge theory to prove Λ-adic comparison isomorphisms between our cohomologies and Hida’s étale cohomology. As applications of our work, we provide a “cohomological” construction of the family of $(\varphi, \Gamma)$-modules attached to Hida’s ordinary Λ-adic étale cohomology by [Dee01], and we give a new and purely geometric proof of Hida’s finiteness and control theorems. We are also able to prove refinements of the main theorems in [MW86] and [Oht95]; in particular, we prove that there is a canonical isomorphism between the module of ordinary Λ-adic cuspforms and the part of the crystalline cohomology of the Igusa tower on which Frobenius acts invertibly.

1. Introduction

1.1. Motivation. In his landmark papers [Hid86a] and [Hid86b], Hida proved that the $p$-adic Galois representations attached to ordinary cuspidal Hecke eigenforms by Deligne ([Del71a], [Car86]) interpolate $p$-adic analytically in the weight variable to a family of $p$-adic representations whose specializations to integer weights $k \geq 2$ recover the “classical” Galois representations attached to weight $k$ cuspidal eigenforms. Hida’s work paved the way for a revolution—from the pioneering work of Mazur on Galois deformations to Coleman’s construction of $p$-adic families of finite slope overconvergent modular forms—and began a trajectory of thought whose fruits include some of the most spectacular achievements in modern number theory.

Hida’s proof is constructive and has at its heart the étale cohomology of the tower of modular curves $\{X_1(Np^r)\}_r$ over $\mathbb{Q}$. More precisely, Hida considers the projective limit $H_1^{\text{ét}} := \varprojlim_r H_1^{\text{ét}}(X_1(Np^r)_{\mathbb{Q}}, \mathbb{Z}_p)$ (taken with respect to the trace mappings), which is naturally a module for the “big” $p$-adic Hecke algebra $\mathcal{H}^* := \varprojlim_r \mathcal{H}^*_r$, which is itself an algebra over the completed group ring $\Lambda := \mathbb{Z}_p[1+p\mathbb{Z}_p] \simeq \mathbb{Z}_p[T]$ via the diamond operators. Using the idempotent $e^* \in \mathcal{H}^*$ attached to the (adjoint) Atkin operator $U^*_p$ to project to the part of $H_1^{\text{ét}}$ where $U^*_p$ acts invertibly, Hida proves in [Hid86a, Theorem 3.1] (via the comparison isomorphism between étale and topological cohomology and explicit calculations in group cohomology) that $e^*H_1^{\text{ét}}$ is finite and free as a module over $\Lambda$, and that the resulting Galois representation

$$\rho : G_\mathbb{Q} \longrightarrow \text{Aut}_\Lambda(e^*H_1^{\text{ét}}) \simeq \text{GL}_m(\mathbb{Z}_p[T])$$

$p$-adically interpolates the representations attached to ordinary cuspidal eigenforms.
By analyzing the geometry of the tower of modular curves, Mazur and Wiles [MW86] were able to relate the inertial invariants of the local (at $p$) representation $\rho_p$ to the étale cohomology of the Igusa tower studied in [MW83], and in so doing proved\(^1\) that the ordinary filtration of the Galois representations attached to ordinary cuspidal eigenforms interpolates: both the inertial invariants and covariants are free of the same finite rank over $\Lambda$ and specialize to the corresponding subquotients in integral weights $k \geq 2$. As an application, they provided examples of cuspforms $f$ and primes $p$ for which the specialization of the associated Hida family of Galois representations to weight $k = 1$ is not Hodge-Tate, and so does not arise from a weight one cuspform via the construction of Deligne-Serre [DS74]. Shortly thereafter, Tilouine [Til87] clarified the geometric underpinnings of [Hid86a] and [MW86], and removed most of the restrictions on the $p$-component of the nebentypus of $f$. Central to both [MW86] and [Til87] is a careful study of the tower of $p$-divisible groups attached to the “good quotient” modular abelian varieties introduced in [MW84].

With the advent of integral $p$-adic Hodge theory, and in view of the prominent role it has played in furthering the trajectory initiated by Hida’s work, it is natural to ask if one can construct Hodge–Tate, de Rham and crystalline analogues of $e^*H_{\text{ét}}^1$ and if so, to what extent the integral comparison isomorphisms of $p$-adic Hodge theory can be made to work in $\Lambda$-adic families. In [Oht95], Ohta has addressed this question in the case of Hodge cohomology. Using the invariant differentials on the tower of $p$-divisible groups studied in [MW86] and [Til87], Ohta constructs a $\Lambda\widehat{\otimes}\mathbb{Z}_p\mathbb{Z}_p[\mu_{p\infty}]$-module from which, via an integral version of the Hodge–Tate comparison isomorphism [Tat67] for ordinary $p$-divisible groups, he is able to recover the semisimplification of the “semilinear representation” $\rho_p\widehat{\otimes}\mathcal{O}_{C_p}$, where $C_p$ is, as usual, the $p$-adic completion of an algebraic closure of $\mathbb{Q}_p$. Using Hida’s results, Ohta proves that his Hodge cohomology analogue of $e^*H_{\text{ét}}^1$ is free of finite rank over $\Lambda\widehat{\otimes}\mathbb{Z}_p\mathbb{Z}_p[\mu_{p\infty}]$ and specializes to finite level exactly as one expects. As applications of his theory, Ohta provides a construction of two-variable $p$-adic $L$-functions attached to families of ordinary cuspidal cuspforms differing from that of Kitagawa [Kit94], and, in a subsequent paper [Oht00], provides a new and streamlined proof of the theorem of Mazur–Wiles [MW84] (Iwasawa’s Main Conjecture for $\mathbb{Q}$: see also [Wil90]). We remark that Ohta’s $\Lambda$-adic Hodge-Tate isomorphism is a crucial ingredient in the forthcoming proof of Sharifi’s conjectures [Sha11], [Sha07] due to Fukaya and Kato [FK12].

1.2. Results. In this paper, we construct the de Rham and crystalline counterparts to Hida’s ordinary $\Lambda$-adic étale cohomology and Ohta’s $\Lambda$-adic Hodge cohomology, and we prove appropriate control and finiteness theorems in each case via a careful study of the geometry of modular curves and abelian varieties. We then prove a suitable $\Lambda$-adic version of every integral comparison isomorphism one could hope for. In particular, we are able to recover the entire family of $p$-adic Galois representations $\rho_p$ (and not just its semisimplification) from our $\Lambda$-adic crystalline cohomology. As a byproduct of our work, we provide geometric constructions of several of the “cohomologically elusive” semi-linear algebra objects in $p$-adic Hodge theory, including the family of étale $(\varphi, \Gamma)$-modules attached to $e^*H_{\text{ét}}^1$ by Dee [Dec01]. As an application of our theory, we give a new and purely geometric proof of Hida’s freeness and control theorems for $e^*H_{\text{ét}}^1$.

In order to survey our main results more precisely, we introduce some notation. Fix an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ as well as a $p$-power compatible sequence $\{\varepsilon^{(r)}\}_{r \geq 0}$ of primitive $p^r$-th roots of unity in $\overline{\mathbb{Q}}_p$. We set $K_r := \mathbb{Q}_p(\mu_{p^r})$ and $K'_r := K_r(\mu_N)$, and we write $R_r$ and $R'_r$ for the rings of integers in $K_r$ and $K'_r$, respectively. Denote by $\mathcal{G}_{\mathbb{Q}_p} := \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ the absolute Galois group and by $\mathcal{H}$ the kernel of the $p$-adic cyclotomic character $\chi : \mathcal{G}_{\mathbb{Q}_p} \to \mathbb{Z}_p^\times$. We write $\Gamma := \mathcal{G}_{\mathbb{Q}_p}/\mathcal{H} \simeq \text{Gal}(K_\infty/K_0)$ for

\(^1\)Mazur and Wiles treat only the case of tame level $N = 1$. 
the quotient and, using that $K_0^\prime/Q_0^p$ is unramified, we canonically identify $\Gamma$ with $\text{Gal}(K_0^\prime/K_0^\prime)$. We will denote by $\langle u \rangle$ (respectively $\langle v \rangle_N$) the diamond operator$^2$ in $\mathcal{H}_n$ attached to $u^{-1} \in \mathbb{Z}_p^\times$ (respectively $v^{-1} \in (\mathbb{Z}/N\mathbb{Z})^\times$) and write $\Delta_\omega$ for the image of the restriction of $\langle \cdot \rangle : \mathbb{Z}_p^\times \to \mathcal{H}_n$ to $1 + p^3\mathbb{Z}_p \subseteq \mathbb{Z}_p^\times$. For convenience, we put $\Delta := \Delta_1$, and for any ring $A$ we write $A_A := \lim_{\to} A[\Delta/\Delta_r]$ for the completed group ring on $\Delta$ over $A$; if $\varphi$ is an endomorphism of $\Lambda_A$, we again write $\varphi$ for the induced endomorphism of $A_A$ that acts as the identity on $\Delta$. Finally, we denote by $X_r := X_1(Np^r)$ the usual modular curve over $\mathbb{Q}$ classifying (generalized) elliptic curves with a $[\mu_{Np^r}]$-structure, and by $J_r := J_1(Np^r)$ its Jacobian.

Our first task is to construct a de Rham analogue of Hida’s $e^*H^1_{dR}$. A naive idea would be to mimic Hida’s construction, using the (relative) de Rham cohomology of $\mathbb{Z}_p$-integral models of the modular curves $X_r$ in place of $p$-adic étale cohomology. However, this approach fails due to the fact that $X_r$ has bad reduction at $p$, so the relative de Rham cohomology of integral models does not provide good $\mathbb{Z}_p$-lattices in the de Rham cohomology of $X_r$ over $\mathbb{Q}_p$. To address this problem, we use the canonical integral structures in de Rham cohomology studied in [Cai09] and the canonical integral model $X_r$ of $X_r$ over $\mathcal{O}_r$ associated to the moduli problem $([\text{bal. } \Gamma_1(p^r)])^{[\text{can.}; \ [\mu_N]}$ to construct well-behaved integral “de Rham cohomology” for the tower of modular curves. For each $r$, we obtain a short exact sequence of free $\Lambda_{\text{R}_r}$-modules with semilinear $\Gamma$-action and commuting $\mathcal{H}_n$-action

\[
\begin{array}{cccccc}
0 & \to & H^0(X_r, \omega_{X_r/\mathcal{O}_r}) & \to & H^1(X_r/\mathcal{O}_r) & \to & H^1(X_r, \mathcal{O}_{X_r}) & \to & 0 \\
\end{array}
\]

which is (co)variently functorial in finite $\Lambda_{\text{R}_r}$-morphisms of the generic fiber $X_r$, and whose scalar extension to $\Lambda$ recovers the Hodge filtration of $H^1_{\text{dR}}(X_r/\mathcal{O}_r)$. Extending scalars to $\mathcal{O}_\infty$ and taking projective limits, we obtain a short exact sequence of $\Lambda_{\text{R}_\infty}$-modules with semilinear $\Gamma$-action and commuting linear $\mathcal{H}_n$-action

\[
\begin{array}{cccccc}
0 & \to & H^0(\omega) & \to & H^1_{\text{dR}} & \to & H^1(\mathcal{O}) \\
\end{array}
\]

Our first main result (see Theorem 5.2.3) is that the ordinary part of (1.2.2) is the correct de Rham analogue of Hida’s ordinary $\Lambda$-adic étale cohomology:

**Theorem 1.2.1.** There is a canonical short exact sequence of finite free $\Lambda_{\text{R}_\infty}$-modules with semilinear $\Gamma$-action and commuting linear $\mathcal{H}_n$-action

\[
\begin{array}{cccccc}
0 & \to & e^*H^0(\omega) & \to & e^*H^1_{\text{dR}} & \to & e^*H^1(\mathcal{O}) & \to & 0 \\
\end{array}
\]

As a $\Lambda_{\text{R}_\infty}$-module, $e^*H^1_{\text{dR}}$ is free of rank $2d$, while each of the flanking terms in (1.2.3) is free of rank $d$, for $d = \sum_{k=3}^{p+1} \dim F_p S_k(\Gamma_1(N); F_p)^{\text{ord}}$. Applying $\otimes_{\Lambda_{\text{R}_\infty}} \mathcal{O}_{\infty}[\Delta/\Delta_r]$ to (1.2.3) recovers the ordinary part of the scalar extension of (1.2.1) to $\mathcal{O}_{\infty}$.

We then show that the $\Lambda_{\text{R}_\infty}$-adic Hodge filtration (1.2.3) is very nearly “auto dual”. To state our duality result more succinctly, for any ring homomorphism $A \to B$, we will write $\langle \cdot \rangle_B := (\cdot) \otimes_A B$ and $\langle \cdot \rangle_B^\vee := \text{Hom}_B((\cdot) \otimes_A B, B)$ for these functors from $A$-modules to $B$-modules. If $G$ is any group of automorphisms of $A$ and $M$ is an $A$-module with a semilinear action of $G$, for any “crossed” homomorphism$^3$ $\psi : G \to A^\times$ we will write $M(\psi)$ for the $A$-module $M$ with “twisted” semilinear $G$-action given by $g \cdot m := \psi(g)gm$. Our duality theorem is (see Proposition 5.2.4):

$^2$Note that we have $\langle u^{-1} \rangle = \langle u \rangle^*$ and $\langle v^{-1} \rangle_N = \langle v \rangle_N$, where $\langle \cdot \rangle^*$ and $\langle \cdot \rangle_N$ are the adjoint diamond operators; see §2.3.

$^3$That is, $\psi(\sigma \tau) = \psi(\sigma) \cdot \sigma \psi(\tau)$ for all $\sigma, \tau \in \Gamma$. 

Theorem 1.2.2. The natural cup-product auto-duality of (1.2.1) over $R'_r := R_r[\mu_N]$ induces a canonical $\Lambda_{R'_\infty}$-linear and $\mathfrak{S}^*$-equivariant isomorphism of exact sequences

$$
0 \longrightarrow e^* H^0(\omega)(\langle \chi \rangle a)_{N})_{\Lambda_{R'_\infty}} \longrightarrow e^* H^1_{dR}(\langle \chi \rangle a)_{N})_{\Lambda_{R'_\infty}} \longrightarrow e^* H^1(\theta)(\langle \chi \rangle a)_{N})_{\Lambda_{R'_\infty}} \longrightarrow 0
$$

that is compatible with the natural action of $\Gamma \times \text{Gal}(K'_0/K_0) \simeq \text{Gal}(K'_\infty/K_0)$ on the bottom row and the twist of the natural action on the top row by the $\mathfrak{S}^*$-valued character $\langle \chi \rangle a_{\gamma}$, where $a(\gamma) \in (\mathbb{Z}/N\mathbb{Z})^\times$ is determined for $\gamma \in \text{Gal}(K'_0/K_0)$ by $\zeta^a(\gamma) = \gamma \zeta$ for every $N$-th root of unity $\zeta$.

We moreover prove that, as one would expect, the $\Lambda_{R'_\infty}$-module $e^* H^0(\omega)$ is canonically isomorphic to the module $eS(N, \Lambda_{R'_\infty})$ of ordinary $\Lambda_{R'_\infty}$-adic cusp forms of tame level $N$; see Corollary 5.3.5.

To go further, we study the tower of $p$-divisible groups attached to the “good quotient” modular abelian varieties introduced by Mazur-Wiles [MW84]. To avoid technical complications with logarithmic $p$-divisible groups, following [MW86] and [Oht95], we will henceforth remove the trivial tame character by working with the sub-idempotent $e^* \gamma$ corresponding to projection to the part where $\mu_{p-1} \subseteq \mathbb{Z}_p^\times \simeq \Delta$ acts non-trivially. As is well-known (e.g. [Hid86a, §9] and [MW84, Chapter 3, §2]), the $p$-divisible group $G_r := e^* J_r[p^\infty]$ over $\mathbb{Q}$ extends to a $p$-divisible group $\mathfrak{G}_r$ over $R_r$, and we write $\mathfrak{G}_r := \mathfrak{G}_r \times_{R_r} \mathbb{F}_p$ for its special fiber. Denoting by $\mathbf{D}(\cdot)$ the contravariant Dieudonné module functor on $p$-divisible groups over $\mathbb{F}_p$, we form the projective limits

$$
D^*_\infty := \lim_{\rightarrow} D(\mathfrak{G}_r^*) \quad \text{for} \quad \star \in \{\text{ét}, m, \text{null}\},
$$

taken along the mappings induced by $\mathfrak{G}_r \to \mathfrak{G}_{r+1}$. Each of these is naturally a $\Lambda$-module equipped with linear (!) Frobenius $F$ and Verschelbing $V$ morphisms satisfying $FV = VF = p$, as well as a linear action of $\mathfrak{S}^*$ and a “geometric inertia” action of $\Gamma$, which reflects the fact that the generic fiber of $\mathfrak{G}_r$ descends to $\mathbb{Q}_p$. The $\Lambda$-modules (1.2.4) have the expected structure (see Theorem 5.5.2):

Theorem 1.2.3. There is a canonical split short exact sequence of finite and free $\Lambda$-modules

$$
0 \longrightarrow D^\text{ét}_\infty \longrightarrow D_\infty \longrightarrow D^m_\infty \longrightarrow 0.
$$

with linear $\mathfrak{S}^*$ and $\Gamma$-actions. As a $\Lambda$-module, $D_\infty$ is free of rank $2d'$, while $D^\text{ét}_\infty$ and $D^m_\infty$ are free of rank $d'$, where $d' := \sum_{k=3}^p \dim \mathbb{F}_p S_k(\Gamma_1(N); \mathbb{F}_p)^\text{ord}$. For $\star \in \{m, \text{ét}, \text{null}\}$, there are canonical isomorphisms

$$
D^\star_\infty \otimes_{\Lambda} \mathbb{Z}_p[\Delta/\Delta_r] \simeq D(\mathfrak{G}^*_r)
$$

which are compatible with the extra structures. Via the canonical splitting of (1.2.5), $D^\text{ét}_\infty$ for $\star = \text{ét}$ (respectively $\star = m$) is identified with the maximal subspace of $D_\infty$ on which $F$ (respectively $V$) acts invertibly. The Hecke operator $U^*_p \in \mathfrak{S}^*$ acts as $F$ on $D^\text{ét}_\infty$ and as $(p)_N V$ on $D^m_\infty$, while $\Gamma$ acts trivially on $D^\text{ét}_\infty$ and via $(\chi(\cdot))^{-1}$ on $D^m_\infty$.

We likewise have the appropriate “Dieudonné” analogue of Theorem 1.2.2 (see Proposition 5.5.3):
Theorem 1.2.4. There is a canonical $\mathcal{H}^\ast$-equivariant isomorphism of exact sequences of $\Lambda_{R_0}$-modules

$$
\begin{array}{cccccc}
0 & \xrightarrow{\cong} & \mathbf{D}_\infty^\text{et}(\langle \chi \rangle(a)_{R_0}) & \xrightarrow{\cong} & \mathbf{D}_\infty^\text{dR}(\langle \chi \rangle(a)_{R_0}) & \xrightarrow{\cong} & \mathbf{D}_\infty^\text{m}(\langle \chi \rangle(a)_{R_0}) & \xrightarrow{\cong} & 0 \\
0 & \xrightarrow{} & (\mathbf{D}_\infty^\text{m})_\Lambda^\vee & \xrightarrow{} & (\mathbf{D}_\infty^\text{dR})_\Lambda^\vee & \xrightarrow{} & (\mathbf{D}_\infty^\text{m})_\Lambda^\vee & \xrightarrow{} & 0 \\
\end{array}
$$

that is $\Gamma \times \text{Gal}(K_0'/K_0)$-equivariant, and intertwines $F$ (respectively $V$) on the top row with $V^\vee$ (respectively $F^\vee$) on the bottom.\footnote{Theorem 1.2.5.}

Just as Mazur-Wiles are able to relate the ordinary-filtration of $e^sH^1_{\text{et}}$ to the étale cohomology of the Igusa tower, we can interpret the slope filtration (1.2.5) in terms of the crystalline cohomology of the Igusa tower as follows. For each $r$, let $I_r$ and $I_r'$ be the two “good” irreducible components of $X_r \times_R \mathbf{F}_p$ (see Remark 2.3.12), each of which is isomorphic to the Igusa curve $\text{Ig}(p^r)$ of tame level $N$ and $p$-level $p^r$. For $s \in \{0, \infty\}$ we form the projective limit

$H^1_{\text{cris}}(I^s) := \lim_{r \to \infty} H^1_{\text{cris}}(I^*_r/\mathbf{Z}_p)$

with respect to the trace mappings on crystalline cohomology induced by the canonical degeneracy maps on Igusa curves. Then $H^1_{\text{cris}}(I^s)$ is naturally a $\Lambda$-module with linear Frobenius $F$ and Verschiebung $V$ endomorphisms. Letting $f'$ be the idempotent of $\Lambda$ corresponding to projection to the part where $\mu_{p-1} \subseteq \Delta \hookrightarrow \Lambda$ acts nontrivially, we prove (see Theorem 5.5.4):

Theorem 1.2.5. There is a canonical isomorphism of $\Lambda$-modules, compatible with $F$ and $V$,

\begin{equation}
\mathbf{D}_\infty = \mathbf{D}_\infty^\text{m} \oplus \mathbf{D}_\infty^\text{et} \cong f^1H^1_{\text{cris}}(I^0)_{\text{ord}} \oplus f^1H^1_{\text{cris}}(I^\infty)_{F_{\text{ord}}},
\end{equation}

which preserves the direct sum decompositions of source and target. This isomorphism is Hecke and $\Gamma$-equivariant, with $U^s_\rho$ and $\Gamma$ acting as $\langle p \rangle_{N,V} \oplus F$ and $(\chi(\cdot))^{-1} \oplus \text{id}$, respectively, on each direct sum.

We note that our “Dieudonné module” analogue (1.2.6) is a significant sharpening of its étale counterpart [MW86, §4], which is formulated only up to isogeny (i.e. after inverting $p$). From $\mathbf{D}_\infty$, we can recover the $\Lambda$-adic Hodge filtration (1.2.3), so the latter is canonically split (see Theorem 5.5.7):

Theorem 1.2.6. There is a canonical $\Gamma$ and $\mathcal{H}^\ast$-equivariant isomorphism of exact sequences

\begin{equation}
\begin{array}{cccccc}
0 & \xrightarrow{\cong} & e^{s'}H^0(\omega) & \xrightarrow{\cong} & e^{s'}H^1_{\text{dR}}(\partial) & \xrightarrow{\cong} & 0 \\
0 & \xrightarrow{} & \mathbf{D}_\infty^\text{m} \otimes_{\Lambda} \Lambda_{R_{\infty}} & \xrightarrow{} & \mathbf{D}_\infty \otimes_{\Lambda} \Lambda_{R_{\infty}} & \xrightarrow{} & \mathbf{D}_\infty^{\text{et}} \otimes_{\Lambda} \Lambda_{R_{\infty}} & \xrightarrow{} & 0 \\
\end{array}
\end{equation}

where the mappings on bottom row are the canonical inclusion and projection morphisms corresponding to the direct sum decomposition $\mathbf{D}_\infty = \mathbf{D}_\infty^\text{m} \oplus \mathbf{D}_\infty^\text{et}$. In particular, the Hodge filtration exact sequence (1.2.3) is canonically split, and admits a canonical descent to $\Lambda$.\footnote{Here, $F^\vee$ (respectively $V^\vee$) is the map taking a linear functional $f$ to $\varphi^{-1} \circ f \circ F$ (respectively $\varphi \circ f \circ V$), where $\varphi$ is the Frobenius automorphism of $R_0' = \mathbf{Z}_p[\mu_N]$.}
We remark that under the identification (1.2.7), the Hodge filtration (1.2.3) and slope filtration (1.2.5) correspond, but in the opposite directions. As a consequence of Theorem 1.2.6, we deduce (see Corollary 5.5.8 and Remark 5.5.9):

**Corollary 1.2.7.** There is a canonical isomorphism of finite free \(\Lambda\) (respectively \(\Lambda_{R_0}\))-modules \[ e^! S(N, \Lambda) \simeq D^m_{\Lambda} \] respectively \[ e^! S_\Lambda(\lambda) \simeq D^et_{\Lambda}((\alpha)_{N}) \otimes \Lambda_{R_0} \]
that intertwines \(T \in \mathcal{S} = \lim \mathcal{S}_p\) with \(T^* \in \mathcal{S}^*\), where we let \(U^*_{p}\) act as \((p) NV\) on \(D^m_{\Lambda}\) and as \(F\) on \(D^et_{\Lambda}\). The second of these isomorphisms is in addition \(Gal(K'/K_0)\)-equivariant.

We are also able to recover the semisimplification of \(e^* H^1_{\text{et}}\) from \(D^m_{\Lambda}\). Writing \(\mathcal{S} \subseteq \mathcal{G}_p\) for the inertia subgroup at \(p\), for any \(Z_p[\mathcal{G}_p]\)-module \(M\), we denote by \(M^{\mathcal{S}}\) (respectively \(M_\mathcal{S} := M/M^{\mathcal{S}}\)) the sub (respectively quotient) module of invariants (respectively covariants) under \(\mathcal{S}\).

**Theorem 1.2.8.** There are canonical isomorphisms of \(\Lambda_{W((\mathbb{F}_p))}\)-modules with linear \(\mathcal{S}^*\)-action and semilinear actions of \(F, V\), and \(\mathcal{G}_p\),
\[
D^et_{\Lambda} \otimes \Lambda_{W((\mathbb{F}_p))} \simeq (e^* H^1_{\text{et}})^{\mathcal{S}} \otimes \Lambda_{W((\mathbb{F}_p))}
\]
and
\[
D^m_{\Lambda}(-1) \otimes \Lambda_{W((\mathbb{F}_p))} \simeq (e^* H^1_{\text{et}})^{\mathcal{S}} \otimes \Lambda_{W((\mathbb{F}_p))}.
\]
Writing \(\sigma\) for the Frobenius automorphism of \(W((\mathbb{F}_p))\), the isomorphism (1.2.8a) intertwines \(F \otimes \sigma\) with \(id \otimes \sigma\) and \(id \otimes g\) with \(g \otimes g\) for \(g \in \mathcal{G}_p\), whereas (1.2.8b) intertwines \(V \otimes \sigma^{-1}\) with \(id \otimes \sigma^{-1}\) and \(g \otimes g\) with \(g \otimes g\), where \(g \in \mathcal{G}_p\) acts on the Tate twist \(D^m_{\Lambda}(-1) := D^m_{\Lambda} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(-1)\) as \((\chi(g)^{-1}) \otimes \chi(g)^{-1}\).

Theorem 1.2.8 gives the following “explicit” description of the semisimplification of \(e^* H^1_{\text{et}}\):

**Corollary 1.2.9.** For any \(T \in (\mathcal{S}_{\text{ord}})^\times\), let \(\lambda(T) : \mathcal{G}_p \to \mathcal{S}_{\text{ord}}\) be the unique continuous (for the \(p\)-adic topology on \(\mathcal{S}_{\text{ord}}\)) unramified character whose value on \((any lift of)\ Frob\_p\) is \(T\). Then \(\mathcal{G}_p\) acts on \((e^* H^1_{\text{et}})^{\mathcal{S}}\) through the character \(\lambda(U^*_{p})^{-1}\) and on \((e^* H^1_{\text{et}})^{\mathcal{S}}\) through \(\chi^{-1} \cdot (\chi^{-1}) \lambda((p) N U^*_{p})\).

We remark that Corollary 1.2.7 and Theorem 1.2.8 combined give a refinement of the main result of [Oht95]. We are furthermore able to recover the main theorem of [MW86] (that the ordinary filtration \(I\_\text{ord}\) intertoplates \(p\)-adic analytically):

**Corollary 1.2.10.** Let \(d\) be the integer of Theorem 1.2.3. Then each of \((e^* H^1_{\text{et}})^{\mathcal{S}}\) and \((e^* H^1_{\text{et}})^{\mathcal{S}}\) is a free \(\Lambda\)-module of rank \(d\), and for each \(r \geq 1\) there are canonical \(\mathcal{S}^*\) and \(\mathcal{G}_p\)-equivariant isomorphisms of \(\mathbb{Z}_p[\Delta/\Delta_r]\)-modules
\[
(e^* H^1_{\text{et}})^{\mathcal{S}} \otimes_{\Lambda} \mathbb{Z}_p[\Delta/\Delta_r] \simeq e^* H^1_{\text{et}}(X_\mathcal{G}_p, \mathbb{Z}_p)^{\mathcal{S}}
\]
\[
(e^* H^1_{\text{et}})^{\mathcal{S}} \otimes_{\Lambda} \mathbb{Z}_p[\Delta/\Delta_r] \simeq e^* H^1_{\text{et}}(X_\mathcal{G}_p, \mathbb{Z}_p)^{\mathcal{S}}
\]
To recover the full \(\Lambda\)-adic local Galois representation \(e^* H^1_{\text{et}}\), rather than just its semisimplification, it is necessary to work with the full Dieudonné crystal of \(\mathcal{S}\) over \(R_r\). Following Faltings [Fal99] and Breuil (e.g. [Bre00]), this is equivalent to studying the evaluation of the Dieudonné crystal of \(\mathcal{S}_r \times_{R_r} \mathbb{Z}_p/pR_r\) on the “universal” divided power thickening \(S_r \to R_r/pR_r\), where \(S_r\) is the \(p\)-adically completed PD-hull of the surjection \(\mathbb{Z}_p[u_r] \to R_r\) sending \(u_r\) to \(\varepsilon(r) - 1\). As the rings \(S_r\) are too unwieldy to
directly construct a good crystalline analogue of Hida’s ordinary étale cohomology, we must functorially
descend the “filtered $S_r$-module” attached to $\mathfrak{S}_r$ to the much simpler ring $\mathfrak{S}_r \coloneqq \mathbb{Z}_p[[u_r]]$. While such
descent is provided (in rather different ways) by the work of Breuil–Kisin and Berger–Wach, neither
of these frameworks is suitable for our application: it is essential for us that the formation of this
descent to $\mathfrak{S}_r$ commute with base change as one moves up the cyclotomic tower, and it is not at all
clear that this holds for Breuil–Kisin modules or for the Wach modules of Berger. Instead, we use
the theory of [CL12], which works with frames and windows à la Lau and Zink to provide the desired
functorial descent to a “$(\varphi, \Gamma)$-module” $\mathcal{M}_r(\mathfrak{S}_r)$ over $\mathfrak{S}_r$. We view $\mathfrak{S}_r$ as a $\mathbb{Z}_p$-subalgebra of $\mathfrak{S}_{r+1}$ via
the map sending $u_r$ to $\varphi(u_{r+1}) := (1 + u_{r+1})^p - 1$, and we write $\mathfrak{S}_\infty := \varprojlim \mathfrak{S}_r$ for the rising union
of the $\mathfrak{S}_r$, equipped with its Frobenius automorphism $\varphi$ and commuting action of $\Gamma$ determined by $\gamma u_r := (1 + u_r)^{\chi(\gamma)} - 1$. We then form the projective limits
\[
\mathcal{M}_*^\infty := \varprojlim (\mathcal{M}_r(\mathfrak{S}_r^*) \otimes \mathfrak{S}_\infty) \quad \text{for} \quad * \in \{\text{ét}, m, \text{null}\}
\]
taken along the mappings induced by $\mathfrak{S}_r \times_R R_{r+1} \to \mathfrak{S}_{r+1}$ via the functoriality of $\mathcal{M}_r(\cdot)$ and its
compatibility with base change. These are $\Lambda_{\mathfrak{S}_\infty}$-modules equipped with a semilinear action of $\Gamma$, a
linear and commuting action of $\mathfrak{S}_*^\infty$, and a commuting $\varphi$ (respectively $\varphi^{-1}$) semilinear endomorphism $F$
(respectively $V$) satisfying $FV = VF = \omega$, for $\omega := \varphi(u_1)/u_1 = u_0/\varphi^{-1}(u_0) \in \mathfrak{S}_\infty$, and they provide our crystalline analogue of Hida’s ordinary étale cohomology (see Theorem 5.6.2):

**Theorem 1.2.11.** There is a canonical short exact sequence of finite free $\Lambda_{\mathfrak{S}_\infty}$-modules with linear $\mathfrak{S}_*^\infty$-
action, semilinear $\Gamma$-action, and commuting semilinear endomorphisms $F, V$ satisfying $FV = VF = \omega$
\[
\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{M}_\infty^{\text{ét}} & \longrightarrow & \mathcal{M}_\infty & \longrightarrow & \mathcal{M}_\infty^m \longrightarrow & 0.
\end{array}
\]

Each of $\mathcal{M}_*^\infty$ for $* \in \{\text{ét}, m\}$ is free of rank $d'$ over $\Lambda_{\mathfrak{S}_\infty}$, while $\mathcal{M}_\infty$ is free of rank $2d'$, where $d'$ is
as in Theorem 1.2.3. Extending scalars on (1.2.10) along the canonical surjection $\Lambda_{\mathfrak{S}_\infty} \twoheadrightarrow \mathfrak{S}_\infty[\Delta/\Delta_r]$
yields the short exact sequence
\[
\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{M}_r(\mathfrak{S}_r^{\text{ét}}) \otimes \mathfrak{S}_\infty & \longrightarrow & \mathcal{M}_r(\mathfrak{S}_r) \otimes \mathfrak{S}_\infty & \longrightarrow & \mathcal{M}_r(\mathfrak{S}_r^m) \otimes \mathfrak{S}_\infty & \longrightarrow & 0
\end{array}
\]
compatibly with $\mathfrak{S}_*^\infty$, $\Gamma$, $F$ and $V$.

Again, in the spirit of Theorems 1.2.2 and 1.2.4, there is a corresponding “autoduality” result for
$\mathcal{M}_\infty$ (see Theorem 5.6.4). To state it, we must work over the ring $\mathfrak{S}' := \varprojlim \mathbb{Z}_p[\mu_N][[u_r]]$, with the
inductive limit taken along the $\mathbb{Z}_p$-algebra maps sending $u_r$ to $\varphi(u_{r+1})$.

**Theorem 1.2.12.** Let $\mu : \Gamma \to \Lambda_{\mathfrak{S}_\infty}^\times$ be the crossed homomorphism given by $\mu(\gamma) := \frac{m}{\gamma u_1} \chi(\gamma)\langle \chi(\gamma) \rangle$.
There is a canonical $\mathfrak{S}_*^\infty$- and $\text{Gal}(K'/K_0)$-compatible isomorphism of exact sequences
\[
\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{M}_\infty^{\text{ét}}(\mu(\alpha)_N)_{\Lambda_{\mathfrak{S}_\infty}} & \longrightarrow & \mathcal{M}_\infty^{\text{ét}}(\mu(\alpha)_N)_{\Lambda_{\mathfrak{S}_\infty}} & \longrightarrow & \mathcal{M}_\infty^{\text{ét}}(\mu(\alpha)_N)_{\Lambda_{\mathfrak{S}_\infty}} & \longrightarrow & 0
\end{array}
\]
that intertwines $F$ (respectively $V$) on the top row with $V^V$ (respectively $F^V$) on the bottom.

---

5As explained in Remark 4.1.4, the $p$-adic completion of $\mathfrak{S}_\infty$ is actually a very nice ring: it is canonically and Frobenius
equivariantly isomorphic to $W(F_p[u_0])$, for $F_p[u_0]$ the perfection of the $F_p$-algebra $F_p[u_0]$. 

The $\Lambda_{\mathcal{S}_\infty}$-modules $\mathcal{M}^\text{et}_\infty$ and $\mathcal{M}^\text{m}_\infty$ have a particularly simple structure (see Theorem 5.6.5):

**Theorem 1.2.13.** There are canonical $\mathcal{S}^\ast$, $\Gamma$, $F$ and $V$-equivariant isomorphisms of $\Lambda_{\mathcal{S}_\infty}$-modules

\[(1.2.11a) \quad \mathcal{M}^\text{et}_\infty \simeq D^\text{et}_\infty \otimes \Lambda_{\mathcal{S}_\infty},\]

intertwining $F$ (respectively $V$) with $F \otimes \varphi$ (respectively $F^{-1} \otimes \omega \cdot \varphi^{-1}$) and $\gamma \in \Gamma$ with $\gamma \otimes \gamma$, and

\[(1.2.11b) \quad \mathcal{M}^\text{m}_\infty \simeq D^\text{m}_\infty \otimes \Lambda_{\mathcal{S}_\infty},\]

intertwining $F$ (respectively $V$) with $V^{-1} \otimes \omega \cdot \varphi$ (respectively $V \otimes \varphi^{-1}$) and $\gamma$ with $\gamma \otimes \chi(\gamma)^{-1} \gamma u_1 / u_1$.

In particular, $F$ (respectively $V$) acts invertibly on $\mathcal{M}^\text{et}_\infty$ (respectively $\mathcal{M}^\text{m}_\infty$).

From $\mathcal{M}_\infty$, we can recover $D_\infty$ and $e^s H^1_{\text{dR}}$, with their additional structures (see Theorem 5.6.6):

**Theorem 1.2.14.** Viewing $\Lambda$ as a $\Lambda_{\mathcal{S}_\infty}$-algebra via the map induced by $u_r \mapsto 0$, there is a canonical isomorphism of short exact sequences of finite free $\Lambda$-modules

\[
\begin{array}{cccccc}
0 & \longrightarrow & \mathcal{M}^\text{et}_\infty \otimes \Lambda & \longrightarrow & \mathcal{M}_\infty \otimes \Lambda & \longrightarrow & \mathcal{M}^\text{m}_\infty \otimes \Lambda & \longrightarrow & 0 \\
& \simeq & \downarrow & \simeq & \downarrow & \simeq & \downarrow & \simeq & \\
0 & \longrightarrow & D^\text{et}_\infty & \longrightarrow & D_\infty & \longrightarrow & D^\text{m}_\infty & \longrightarrow & 0
\end{array}
\]

which is $\Gamma$ and $\mathcal{S}^\ast$-equivariant and carries $F \otimes 1$ to $F$ and $V \otimes 1$ to $V$. Viewing $\Lambda_{R_\infty}$ as a $\Lambda_{\mathcal{S}_\infty}$-algebra via the map $u_r \mapsto (\varepsilon^{(r)})^p - 1$, there is a canonical isomorphism of short exact sequences of $\Lambda_{R_\infty}$-modules

\[
\begin{array}{cccccc}
0 & \longrightarrow & \mathcal{M}^\text{et}_\infty \otimes \Lambda_{R_\infty} & \longrightarrow & \mathcal{M}_\infty \otimes \Lambda_{R_\infty} & \longrightarrow & \mathcal{M}^\text{m}_\infty \otimes \Lambda_{R_\infty} & \longrightarrow & 0 \\
& \simeq & \downarrow & \simeq & \downarrow & \simeq & \downarrow & \simeq & \\
0 & \longrightarrow & e^s H^1(\Theta) & \longrightarrow & e^s H^1_{\text{dR}} & \longrightarrow & e^s H^0(\omega) & \longrightarrow & 0
\end{array}
\]

with $i$ and $j$ the canonical sections given by the splitting in Theorem 1.2.6.

To recover Hida’s ordinary étale cohomology from $\mathcal{M}_\infty$, we introduce the “period” ring of Fontaine\footnote{Though we use the notation introduced by Berger and Colmez.} $\hat{\mathbb{E}}^+ := \varprojlim \mathcal{O}_{C_p}/(p)$, with the projective limit taken along the $p$-power mapping; this is a perfect valuation ring of characteristic $p$ equipped with a canonical action of $\mathcal{G}_{Q_p}$ via “coordinates”. We write $\hat{\mathbb{E}}$ for the fraction field of $\hat{\mathbb{E}}^+$ and $\hat{\mathbb{A}} := W(\hat{\mathbb{E}})$ for its ring of Witt vectors, equipped with its canonical Frobenius automorphism $\varphi$ and $\mathcal{G}_{Q_p}$-action induced by Witt functoriality. Our fixed choice of $p$-power compatible sequence $\{\varepsilon^{(r)}\}$ determines an element $\varepsilon := (\varepsilon^{(r)} \mod p)_{r \geq 0}$ of $\hat{\mathbb{E}}^+$, and we $\mathbb{Z}_p$-linearly embed $\mathcal{S}_\infty$ in $\hat{\mathbb{A}}$ via $u_r \mapsto \varphi^{-r}(\varepsilon - 1)$ where $[\varepsilon]$ is the Teichmüller section. This embedding is $\varphi$ and $\mathcal{G}_{Q_p}$-compatible, with $\mathcal{G}_{Q_p}$ acting on $\mathcal{S}_\infty$ through the quotient $\mathcal{G}_{Q_p} \twoheadrightarrow \Gamma$. 

Theorem 1.2.15. Twisting the structure map $\mathfrak{S}_\infty \to \tilde{A}$ by the Frobenius automorphism $\varphi$, there is a canonical isomorphism of short exact sequences of $\Lambda_\tilde{A}$-modules with $\mathfrak{H}^*$-action

$$0 \longrightarrow \mathfrak{M}^\ell_\infty \otimes \Lambda_\tilde{A} \longrightarrow \mathfrak{M}_\infty \otimes \Lambda_{\tilde{A}} \longrightarrow \mathfrak{M}^m_\infty \otimes \Lambda_{\tilde{A}} \longrightarrow 0$$

(1.2.12)

That is $\mathfrak{G}_{\mathfrak{Q}_p}$-equivariant for the “diagonal” action of $\mathfrak{G}_{\mathfrak{Q}_p}$ (with $\mathfrak{G}_{\mathfrak{Q}_p}$ acting on $\mathfrak{M}_\infty$ through $\Gamma$) and intertwines $F \otimes \varphi$ with $\text{id} \otimes \varphi$ and $V \otimes \varphi^{-1}$ with $\text{id} \otimes \varphi^{-1}$. In particular, there is a canonical isomorphism of $\Lambda$-modules, compatible with the actions of $\mathfrak{H}^*$ and $\mathfrak{G}_{\mathfrak{Q}_p}$,

$$e^*H^1_{\text{ét}} \simeq \left( \mathfrak{M}_\infty \otimes \Lambda_{\tilde{A}} \right)^{F \otimes \varphi = 1}. $$

Theorem 1.2.15 allows us to give a new proof of Hida’s finiteness and control theorems for $e^*H^1_{\text{ét}}$:

Corollary 1.2.16 (Hida). Let $d'$ be as in Theorem 1.2.3. Then $e^*H^1_{\text{ét}}$ is free $\Lambda$-module of rank $2d'$. For each $r \geq 1$ there is a canonical isomorphism of $\mathbb{Z}_p[\Delta/\Delta_r]$-modules with linear $\mathfrak{H}^*$ and $\mathfrak{G}_{\mathfrak{Q}_p}$-actions

$$e^*H^1_{\text{ét}} \otimes \mathbb{Z}_p[\Delta/\Delta_r] \simeq e^*H^1_{\text{ét}}(X, \mathfrak{Q}_p, \mathbb{Z}_p)$$

which is moreover compatible with the isomorphisms (1.2.9a) and (1.2.9b) in the manner that

We also deduce a new proof of the following duality result [Oht95, Theorem 4.3.1] (cf. [MW86, §6]):

Corollary 1.2.17 (Ohta). Let $\nu : \mathfrak{G}_{\mathfrak{Q}_p} \to \mathfrak{H}^*$ be the character $\nu := \chi(\chi)\lambda((p)N)$. There is a canonical $\mathfrak{H}^*$ and $\mathfrak{G}_{\mathfrak{Q}_p}$-equivariant isomorphism of short exact sequences of $\Lambda$-modules

$$0 \longrightarrow (e^*H^1_{\text{ét}})^{\mathfrak{H}^*}(\nu) \longrightarrow e^*H^1_{\text{ét}}(\nu) \longrightarrow (e^*H^1_{\text{ét}})^{\mathfrak{H}^*}(\nu) \longrightarrow 0$$

$$0 \longrightarrow \text{Hom}_\Lambda((e^*H^1_{\text{ét}})^{\mathfrak{H}^*}(\nu), \Lambda) \longrightarrow \text{Hom}_\Lambda(e^*H^1_{\text{ét}}(\nu), \Lambda) \longrightarrow \text{Hom}_\Lambda((e^*H^1_{\text{ét}})^{\mathfrak{H}^*}(\nu), \Lambda) \longrightarrow 0$$

The $\Lambda$-adic splitting of the ordinary filtration of $e^*H^1_{\text{ét}}$ was considered by Ghate and Vatsal [GV04], who prove (under certain technical hypotheses of “deformation-theoretic nature”) that if the $\Lambda$-adic family $\mathcal{F}$ associated to a cuspidal eigenform $f$ is primitive and $p$-distinguished, then the associated $\Lambda$-adic local Galois representation $\rho_{\mathcal{F}, p}$ is split split if and only if some arithmetic specialization of $\mathcal{F}$ has CM [GV04, Theorem 13]. We interpret the $\Lambda$-adic splitting of the ordinary filtration as follows:

Theorem 1.2.18. The short exact sequence (1.2.10) admits a $\Lambda_\mathfrak{S}_\infty$-linear splitting which is compatible with $F$, $V$, and $\Gamma$ if and only if the ordinary filtration of $e^*H^1_{\text{ét}}$ admits a $\Lambda$-linear splitting which is compatible with the action of $\mathfrak{G}_{\mathfrak{Q}_p}$.

1.3. Overview of the article. Section 2 is preliminary: we review the integral $p$-adic cohomology theories of [Cai09] and [Cai10], and summarize the relevant facts concerning integral models of modular curves from [KM85] that we will need. Of particular importance is a description of the $U_p$-correspondence in characteristic $p$, due to Ulmer [Ulm90], and recorded in Proposition 2.3.20.
In §3, we study the de Rham and crystalline cohomology of the Igusa tower, and prove the key “freeness and control” theorems that form the technical characteristic $p$ backbone of this paper. Via an almost combinatorial argument using the description of $U_p$ in characteristic $p$, we then relate the cohomology of the Igusa tower to the mod $p$ reduction of the ordinary part of the (integral $p$-adic) cohomology of the modular tower.

Section 4 is a summary of the theory developed in [CL12], which uses Dieudonné crystals of $p$-divisible groups to provide a “cohomological” construction of the $(\varphi, \Gamma)$-modules attached to potentially Barsotti–Tate representations. It is precisely this theory which allows us to construct our crystalline analogue of Hida’s ordinary $\Lambda$-adic étale cohomology.

Section 5 constitutes the main body of this paper, and the reader who is content to refer back to §2-4 as needed should skip directly there. In §5.1, we develop a commutative algebra formalism for working with projective limits of “towers” of cohomology that we use frequently in the sequel. Using the canonical lattices in de Rham cohomology studied in [Cai09] (and reviewed in §2.1), we construct our $\Lambda$-adic de Rham analogue of Hida’s ordinary $\Lambda$-adic étale cohomology in §5.2, and we show that the expected freeness and control results follow by reduction to characteristic $p$ from the structure theorems for the de Rham cohomology of the Igusa tower established in §3. Using work of Ohta [Oht95], in §5.3 we relate the Hodge filtration of our $\Lambda$-adic de Rham cohomology to the module of $\Lambda$-adic cuspforms. In section 5.4, we study the tower of $p$-divisible groups whose cohomology allows us to construct our $\Lambda$-adic Dieudonné and crystalline analogues of Hida’s étale cohomology in §5.5 and §5.6, respectively. We establish $\Lambda$-adic comparison isomorphisms between each of these cohomologies using the integral comparison isomorphisms of [Cai10] and [CL12], recalled in §2.2 and §4.1, respectively. This enables us to give a new proof of Hida’s freeness and control theorems and of Ohta’s duality theorem in §5.6.

As remarked in §1.2, and following [Oht95] and [MW86], our construction of the $\Lambda$-adic Dieudonné and crystalline counterparts to Hida’s étale cohomology excludes the trivial eigenspace for the action of $\mu_{p-1} \subseteq \mathbb{Z}_p^\times$ so as to avoid technical complications with logarithmic $p$-divisible groups. In [Oht00], Ohta uses the “fixed part” (in the sense of Grothendieck [Gro72, 2.2.3]) of Néron models with semiabelian reduction to extend his results on $\Lambda$-adic Hodge cohomology to allow trivial tame nebentype character. We are confident that by using Kato’s logarithmic Dieudonné theory [Kat89] one can appropriately generalize our results in §5.5 and §5.6 to include the missing eigenspace for the action of $\mu_{p-1}$.

1.4. Notation. If $\varphi : A \to B$ is any map of rings, we will often write $M_B := M \otimes_A B$ for the $B$-module induced from an $A$-module $M$ by extension of scalars. When we wish to specify $\varphi$, we will write $M \otimes_{A, \varphi} B$. Likewise, if $\varphi : T' \to T$ is any morphism of schemes, for any $T$-scheme $X$ we denote by $X_{T'}$ the base change of $X$ along $\varphi$. If $f : X \to Y$ is any morphism of $T$-schemes, we will write $f_{T'} : X_{T'} \to Y_{T'}$ for the morphism of $T'$-schemes obtained from $f$ by base change along $\varphi$. When $T = \text{Spec}(R)$ and $T' = \text{Spec}(R')$ are affine, we abuse notation and write $X_{R'}$ or $X \times_R R'$ for $X_{T'}$.

We will frequently work with schemes over a discrete valuation ring $R$. We will often write $X, Y, \ldots$ for schemes over $\text{Spec}(R)$, and will generally use $X, Y, \ldots$ (respectively $\overline{X}, \overline{Y}, \ldots$) for their generic (respectively special) fibers.

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2. Preliminaries

This somewhat long section is devoted to recalling the geometric background we will need in our constructions. Much (though not all) of this material is contained in [Cai09], [Cai10] and [KM85].

2.1. Dualizing sheaves and de Rham cohomology. We begin by describing a certain modification of the usual de Rham complex for non-smooth curves. The hypercohomology of this (two-term) complex is in general much better behaved than algebraic de Rham cohomology and will enable us to construct our $\Lambda$-adic de Rham cohomology. We largely refer to [Cai09], but remark that our treatment here is different in some places and better suited to our purposes.

Definition 2.1.1. A curve over a scheme $S$ is a morphism $f : X \to S$ of finite presentation which is a flat local complete intersection\(^7\) of pure relative dimension 1 with geometrically reduced fibers. We will often say that $X$ is a curve over $S$ or that $X$ is a relative $S$-curve when $f$ is clear from context.

Proposition 2.1.2. Let $f : X \to S$ be a flat morphism of finite type. The following are equivalent:

\(^7\)That is, a syntomic morphism in the sense of Mazur [FM87, II, 1.1]. Here, we use the definition of l.c.i. given in [SGA71, Exp. VIII, 1.1].
The morphism $f : X \to S$ is a curve.

(2) For every $s \in S$, the fiber $f_s : X_s \to \text{Spec } \kappa(s)$ is a curve.

(3) For every $x \in X$ with $s = f(x)$, the local ring $\mathcal{O}_{X_s, x}$ is a complete intersection\(^8\) and $f$ has geometrically reduced fibers of pure dimension 1.

Moreover, any base change of a curve is again a curve.

**Proof.** Since $f$ is flat and of finite presentation, the definition of local complete intersection that we are using (i.e. [SGA71, Exp. VIII, 1.1]) is equivalent to the definition given in [DG 7, IV.4, 19.3.6] by [SGA71, Exp. VIII, 1.4]; the equivalence of (1)–(3) follows immediately. The final statement of the proposition is an easy consequence of [DG 7, IV.4, 19.3.9]. ■

**Corollary 2.1.3.** Let $f : X \to S$ be a finite type morphism of pure relative dimension 1.

(1) If $f$ is smooth, then it is a curve.

(2) If $X$ and $S$ are regular and $f$ has geometrically reduced fibers then $f$ is a curve.

(3) If $f$ is a curve then it is Gorenstein and hence also Cohen Macaulay.

**Proof.** The assertion (1) is obvious, and (2) follows from the fact that a closed subscheme of a regular scheme is regular if and only if it is defined (locally) by a regular sequence; cf. [Liu02, 6.3.18]. Finally, (3) follows from Proposition 2.1.2 (3) and the fact that every local ring that is a complete intersection is Gorenstein and hence Cohen Macaulay (see, e.g., Theorems 18.1 and 21.3 of [Mat89]). ■

Fix a relative curve $f : X \to S$. We wish to apply Grothendieck duality theory to $f$, so we henceforth assume that $S$ is a noetherian scheme of finite Krull dimension\(^9\) that is Gorenstein and excellent, so that $\mathcal{O}_S$ is a dualizing complex for $S$ [Har66, V, §10]. Since $f$ is CM by Corollary 2.1.3 (3), by [Con00, Theorem 3.5.1] the relative dualizing complex $f^! \mathcal{O}_S$ has a unique nonzero cohomology sheaf, which is in degree $-1$, and we define the relative dualizing sheaf for $X$ over $S$ (or for $f$) to be:

$$\omega_f = \omega_{X/S} := H^{-1}(f^! \mathcal{O}_S).$$

Since the fibers of $f$ are Gorenstein, $\omega_{X/S}$ is an invertible $\mathcal{O}_X$-module by [Har66, V, Proposition 9.3, Theorem 9.1]. The formation of $\omega_{X/S}$ is compatible with arbitrary base change on $S$ and étale localization on $X$ [Con00, Theorem 3.6.1].

**Remark 2.1.4.** Since $S$ is Gorenstein and of finite Krull dimension and $f^!$ carries dualizing complexes for $S$ to dualizing complexes for $X$ (see [Har66, V, §8]), the sheaf $\omega_{X/S}$ (thought of as a complex concentrated in some degree) is a dualizing complex for the abstract scheme $X$.

**Proposition 2.1.5.** Let $X \to S$ be a relative curve. There is a canonical map of $\mathcal{O}_X$-modules

$$c_{X/S} : \Omega^1_{X/S} \longrightarrow \omega_{X/S}$$

whose formation commutes with any base change $S' \to S$, where $S'$ is noetherian of finite Krull dimension, Gorenstein, and excellent. Moreover, the restriction of $c_{X/S}$ to any $S$-smooth subscheme of $X$ is an isomorphism.

**Proof.** See [AEZ78], especially Théorème III.1, and cf. [Liu02, 6.4.13]. ■

---

\(^8\)That is, the quotient of a regular local ring by a regular sequence.

\(^9\)Nagata gives an example [Nag62, A1, Example 1] of an affine and regular noetherian scheme of infinite Krull dimension, so this hypotheses is not redundant.
Definition 2.1.6. We define the two-term $\mathcal{O}_S$-linear complex (of $\mathcal{O}_S$-flat coherent $\mathcal{O}_X$-modules) concentrated in degrees 0 and 1

\begin{equation}
\omega^\bullet_f = \omega^\bullet_{X/S} := \mathcal{O}_X \xrightarrow{d_S} \omega_{X/S}
\end{equation}

where $d_S$ is the composite of the map (2.1.1) and the universal $\mathcal{O}_S$-derivation $\mathcal{O}_X \to \Omega^1_{X/S}$. We view $\omega^\bullet_{X/S}$ as a filtered complex via "la filtration bête" [Del71b], which provides an exact triangle

\begin{equation}
\omega_{X/S}[-1] \xrightarrow{d_S} \omega^\bullet_{X/S} \to \mathcal{O}_X
\end{equation}

in the derived category that we call the Hodge Filtration of $\omega^\bullet_{X/S}$.

Since $c_{X/S}$ is an isomorphism over the $S$-smooth locus $X^{sm}$ of $f$ in $X$, the complex $\omega^\bullet_{X/S}$ coincides with the usual de Rham complex over $X^{sm}$. Moreover, it follows immediately from Proposition 2.1.5 that the formation of $\omega^\bullet_{X/S}$ is compatible with any base change $S' \to S$ to a noetherian scheme $S'$ of finite Krull dimension that is Gorenstein and excellent.

Definition 2.1.7. Let $f : X \to S$ relative curve over $S$. For each nonnegative integer $i$, we define

$$\mathcal{H}^i(X/S) := R^if_!\omega^\bullet_{X/S}.$$ 

When $S = \text{Spec} R$ is affine, we will write $H^i(X/R)$ for the global sections of the $\mathcal{O}_S$-module $\mathcal{H}^i(X/S)$.

The complex $\omega^\bullet_{X/S}$ and its filtration (2.1.3) behave extremely well with respect to duality:

Proposition 2.1.8. Let $f : X \to S$ be a proper curve over $S$. There is a canonical quasi-isomorphism

\begin{equation}
\omega^\bullet_{X/S} \simeq R^i\mathcal{H}\text{om}^\bullet_X(\omega^\bullet_{X/S}, \omega_{X/S}[-1])
\end{equation}

which is compatible with the filtrations on both sides induced by (2.1.3). In particular:

1. There is a natural quasi-isomorphism
   $$R^if_*\omega^\bullet_{X/S} \simeq R^i\mathcal{H}\text{om}^\bullet_X(R^if_*\omega^\bullet_{X/S}, \mathcal{O}_S)[-2]$$
   which is compatible with the filtrations induced by (2.1.3).

2. If $\rho : Y \to X$ is any finite morphism of proper curves over $S$, then there is a canonical quasi-isomorphism
   $$R^\rho_*\omega^\bullet_{Y/S} \simeq R^i\mathcal{H}\text{om}^\bullet_X(R^\rho_*\omega^\bullet_{Y/S}, \omega_{X/S}[-1]).$$
   that is compatible with filtrations.

Proof. For the first claim, see the proofs of Lemmas 4.3 and 5.4 in [Cai09], noting that although $S$ is assumed to be the spectrum of a discrete valuation ring and the definition of curve in that paper differs somewhat from the definition here, the arguments themselves apply verbatim in our context. The assertion (1) (respectively (2)) follows from this by applying $R^if_*$ (respectively $R^\rho_*$) to both sides of (2.1.4) and appealing to Grothendieck duality [Con00, Theorem 3.4.4] for the proper map $f$ (respectively $\rho$); see the proofs of Lemma 5.4 and Proposition 5.8 in [Cai09] for details.

In our applications, we need to understand the cohomology $H^i(X/S)$ for a proper curve $X \to S$ when $S$ is either the spectrum of a discrete valuation ring $R$ of mixed characteristic $(0, p)$ or the spectrum of a perfect field. We now examine each of these situations in more detail.
First suppose that $S := \text{Spec}(R)$ is the spectrum of a discrete valuation ring $R$ having field of fractions $K$ of characteristic zero and perfect residue field $k$ of characteristic $p > 0$, and fix a normal curve $f : X \to S$ that is proper over $S$ with smooth and geometrically connected generic fiber $X_K$. This situation is studied extensively in [Cai09], and we content ourselves with a summary of the results we will need. To begin, we recall the following “concrete” description of the relative dualizing sheaf:

**Lemma 2.1.9.** Let $i : U \hookrightarrow X$ be any Zariski open subscheme of $X$ whose complement consists of finitely many points of codimension $2$ (necessarily in the closed fiber of $X$). Then the canonical map

$$\omega_{X/S} \longrightarrow i_* i^* \omega_{X/S} \cong i_* \omega_{U/S}$$

is an isomorphism. In particular, $\omega_{X/S} \cong i_* \Omega^1_{U/S}$ for any Zariski open subscheme $i : U \hookrightarrow X^{\text{sm}}$ whose complement consists of finitely many points of codimension two.

**Proof.** The first assertion is [Cai10, Lemma 3.2]. The second follows from this, since $X^{\text{sm}}$ contains the generic fiber and the generic points of the closed fiber by our definition of curve. \hfill \blacksquare

**Proposition 2.1.10.** Let $\rho : Y \to X$ be a finite morphism of normal and proper $S$-curves.

1. Attached to $\rho$ are natural pullback and trace morphisms of complexes

$$\rho^* : \omega_{X/S}^\bullet \longrightarrow \rho_* \omega_{Y/S}^\bullet \text{ and } \rho_* : \rho_! \omega_{Y/S}^\bullet \longrightarrow \omega_{X/S}^\bullet$$

which are of formation compatible with étale localization on $X$ and flat base change on $S$ and are dual via the duality of Proposition 2.1.8 (2).

2. For any $S$-smooth point $y \in Y^{\text{sm}}$ with image $x := \rho(y)$ that lies in $X^{\text{sm}}$, the induced mappings of complexes of $\mathcal{O}_{X,x}$-modules $\omega_{X,S,x}^\bullet \to \omega_{Y,S,y}^\bullet$ and $\omega_{Y,S,y}^\bullet \to \omega_{X,S,x}^\bullet$ coincide with the usual pullback and trace mappings on de Rham complexes attached to the finite flat morphism of smooth schemes $\text{Spec}(O_{Y,y}) \to \text{Spec}(O_{X,x})$.

**Proof.** The assertions of (1) follow from the proofs of Propositions 4.5 and 5.5 of [Cai09], while (2) is a straightforward consequence of the very construction of $\rho_*$ and $\rho^*$ as given in [Cai09, §4]. \hfill \blacksquare

Since the generic fiber of $X$ is a smooth and proper curve over $K$, the Hodge to de Rham spectral sequence degenerates [DI87], and there is a functorial short exact sequence of $K$-vector spaces

$$(2.1.5) \quad 0 \longrightarrow H^0(X_K, \Omega^1_{X_K/K}) \longrightarrow H^1_{\text{dR}}(X_K/K) \longrightarrow H^1(X_K, \mathcal{O}_{X_K}) \longrightarrow 0$$

which we call the Hodge filtration of $H^1_{\text{dR}}(X_K/K)$.

**Proposition 2.1.11.** Let $f : X \to S$ be a normal curve that is proper over $S = \text{Spec}(R)$.

1. There are natural isomorphisms of free $R$-modules of rank $1$

$$H^0(X/R) \cong H^0(X, \mathcal{O}_X) \text{ and } H^2(X/R) \cong H^1(X, \omega_{X/S}),$$

which are canonically $R$-linearly dual to each other.

2. There is a canonical short exact sequence of finite free $R$-modules, which we denote $H(X/R)$,

$$0 \longrightarrow H^0(X, \omega_{X/S}) \longrightarrow H^1(X/R) \longrightarrow H^1(X, \mathcal{O}_X) \longrightarrow 0$$

that recovers the Hodge filtration $(2.1.5)$ of $H^1_{\text{dR}}(X_K/K)$ after extending scalars to $K$.

3. Via the canonical cup-product auto-duality of $(2.1.5)$, the exact sequence $H(X/R)$ is naturally isomorphic to its $R$-linear dual.
(4) The exact sequence $H(X/R)$ is contravariantly (respectively covariantly) functorial in finite morphisms $\rho : Y \to X$ of normal and proper $S$-curves via pullback $\rho^*$ (respectively trace $\rho_*$); these morphisms recover the usual pullback and trace mappings on Hodge filtrations after extending scalars to $K$ and are adjoint with respect to the canonical cup-product autoduality of $H(X/R)$ in (3).

**Proof.** By Raynaud’s “critère de platitude cohomologique” [Ray74, Théorème 7.2.1] (see also [Cai09, Proposition 2.7]), our requirement that curves have geometrically reduced fibers implies that $f : X \to S$ is cohomologically flat. The proposition now follows from Propositions 5.7–5.8 of [Cai09]. □

We now turn to the case that $S = \text{Spec}(k)$ for a perfect field $k$ and $f : X \to S$ is a proper and geometrically connected curve over $k$. Recall that $X$ is required to be geometrically reduced, so that the $k$-smooth locus $U := X^\text{sm}$ is the complement of finitely many closed points in $X$.

**Proposition 2.1.12.** Let $X$ be a proper and geometrically connected curve over $k$.

1. There are natural isomorphisms of 1-dimensional $k$-vector spaces

$$H^0(X/k) \simeq H^0(X, \mathcal{O}_X) \quad \text{and} \quad H^2(X/k) \simeq H^1(X, \omega_{X/k}),$$

which are canonically $k$-linearly dual to each other.

2. There is a natural short exact sequence, which we denote $H(k)$

$$0 \longrightarrow H^0(X, \omega_{X/k}) \longrightarrow H^1(X/k) \longrightarrow H^1(X, \mathcal{O}_X) \longrightarrow 0$$

which is canonically isomorphic to its own $k$-linear dual.

**Proof.** Consider the long exact cohomology sequence arising from the exact triangle (2.1.3). Since $X$ is proper over $k$, geometrically connected and reduced, the canonical map $k \to H^0(X, \mathcal{O}_X)$ is an isomorphism, and it follows that the map $d : H^0(X, \mathcal{O}_X) \to H^0(X, \omega_{X/k})$ is zero, whence the map $H^0(X/k) \to H^0(X, \mathcal{O}_X)$ is an isomorphism. Thanks to Proposition 2.1.8 (1), we have a canonical quasi-isomorphism

$$\mathbb{R}\Gamma(X, \omega_{X/k}^\bullet) \simeq \mathbb{R}\text{Hom}_k^\bullet(\mathbb{R}\Gamma(X, \omega_{X/k}^\bullet), k)[-2]$$

that is compatible with the filtrations induced by (2.1.3). Using the spectral sequence

$$E_2^{m,n} = \text{Ext}_k^m(H^{-n}(X, \omega_{X/k}^\bullet)) \implies H^{m+n}(\mathbb{R}\text{Hom}_k^\bullet(\mathbb{R}\Gamma(X, \omega_{X/k}^\bullet), k))$$

and the vanishing of $\text{Ext}_k^m(\cdot, k)$ for $m > 0$, we deduce that $H^2(X/k) \simeq H^0(X/k)^\vee$ is 1-dimensional over $k$. Since Grothendieck’s trace map $H^1(X, \omega_{X/k}) \to k$ is an isomorphism, we conclude that the surjective map of 1-dimensional $k$-vector spaces $H^1(X, \omega_{X/k}) \to H^2(X/k)$ must be an isomorphism. It follows that the map $d : H^1(X, \mathcal{O}_X) \to H^1(X, \omega_{X/k})$ is zero as well, as desired. The fact that that the resulting short exact sequence in (2) is canonically isomorphic to its $k$-linear dual, and the fact that the isomorphisms in (1) are $k$-linearly dual are now easy consequences of the isomorphism (2.1.6). □

We now suppose that $k$ is algebraically closed, and following [Con00, §5.2], we recall Rosenlicht’s explicit description [Ros58] of the relative dualizing sheaf $\omega_{X/k}$ and of Grothendieck duality.

Denote by $k(X)$ the “function field” of $X$, i.e. $k(X) := \prod_i k(\xi_i)$ is the product of the residue fields at the finitely many generic points of $X$, and write $j : \text{Spec}(k(X)) \to X$ for the canonical map. By definition, the sheaf of meromorphic differentials on $X$ is the pushforward $\Omega^1_k(X/k) := j_*\Omega^1_{k(X)/k}$. Our
hypothesis that $X$ is reduced implies that it is smooth at its generic points, so $j$ factors through the open immersion $i : U := X^{\text{sm}} \hookrightarrow X$. By [Con00, Lemma 5.2.1], the canonical map of $\mathcal{O}_X$-modules

$$\omega_{X/k} \longrightarrow i_* i^* \omega_{X/k} \simeq i_* \Omega^1_{U/k} \tag{2.1.7}$$

is injective, and it follows that $\omega_{X/k}$ is a subsheaf of $\Omega^1_{k(X)/k}$. Rosenlicht’s theory gives a concrete description of this subsheaf, as we now explain.

Let $\pi : X^n \rightarrow X$ be the normalization of $X$. We have a natural identification of “function fields” $k(X^n) = k(X)$ and hence a canonical isomorphism $\pi_* \Omega^1_{k(X^n)/k} \simeq \Omega^1_{k(X)/k}$ of sheaves on $X$.

**Definition 2.1.13.** Let $\omega^\text{reg}_{X/k}$ be the sheaf of $\mathcal{O}_X$-modules whose sections over any open $V \subseteq X$ are those meromorphic differentials $\eta$ on $\pi^{-1}(V) \subseteq X^n$ which satisfy

$$\sum_{y \in \pi^{-1}(x)} \text{res}_y(s\eta) = 0 \tag{2.1.8}$$

for all $x \in V(k)$ and all $s \in \mathcal{O}_{X,x}$, where $\text{res}_y$ is the classical residue map on meromorphic differentials on the smooth (possibly disconnected) curve $X^n$ over the algebraically closed field $k$.

**Remark 2.1.14.** Let $\text{Irr}(X)$ be the set of irreducible components of $X$. Since $\pi$ is an isomorphism over $U$ and $X$ is smooth at its generic points, $X^n$ is the disjoint union of the smooth, proper, and irreducible $k$-curves $I^n$ for $I \in \text{Irr}(X)$. Therefore, a meromorphic differential $\eta$ on $X^n$ may be viewed as a tuple $\eta = (\eta^n)_{I \in \text{Irr}(X)}$, with $\eta^n$ a meromorphic differential on the smooth and irreducible curve $I^n$. The condition for a meromorphic differential $\eta$ on $\pi^{-1}(V)$ to be a section of $\omega^\text{reg}_{X/k}$ over $V$ is then

$$\sum_{y \in \pi^{-1}(x)} \text{res}_y(s_y\eta^n_y) = 0$$

for all $x \in V(k)$ and all $s \in \mathcal{O}_{X,x}$, where $I^n_y$ is the unique connected component of $X^n$ on which $y$ lies and $s_y$ is the image of $s$ under the canonical map $\mathcal{O}_{X,x} \rightarrow \mathcal{O}_{I^n_y,y}$.

As any holomorphic differential on $X^n$ has zero residue at every closed point, the pushforward $\pi_* \Omega^1_{X^n/k}$ is naturally a subsheaf of $\omega^\text{reg}_{X/k}$, and this inclusion is an equality at every $x \in U(k)$ since $\pi$ is an isomorphism over $U$. It likewise follows from the definition that any section of $\omega^\text{reg}_{X/k}$ must be holomorphic at every smooth point of $X$, so there is a natural inclusion

$$\omega^\text{reg}_{X/k} \hookrightarrow i_* \Omega^1_{U/k} \tag{2.1.9}$$

which is an isomorphism over $U$. Moreover, by [Con00, Lemma 5.2.2], any section of $\omega^\text{reg}_{X/k}$ has poles at the finitely many non-smooth points of $X$ with order bounded by a constant depending only on $X$, and it follows that $\omega^\text{reg}_{X/k}$ is a coherent sheaf on $X$.

Since (2.1.9) is an isomorphism at the generic points of $X$, we have a quasi-coherent flasque resolution

$$0 \longrightarrow \omega_{X/k}^\text{reg} \longrightarrow \Omega^1_{k(X)/k} \longrightarrow \bigoplus_{x \in X^0} i_x_* \left( \Omega^1_{k(X)/k,x} / \omega_{X/k,x}^\text{reg} \right) \longrightarrow 0.$$
where $X^0$ is the set of closed points of $X$ and $i_x : \text{Spec}(O_{X,x}) \to X$ is the canonical map. The associated long exact cohomology sequence yields an exact sequence of $k$-vector spaces

\begin{equation}
\Omega^1_{k(X)/k} \longrightarrow \bigoplus_{x \in X^0} \left( \Omega^1_{k(k(x)/k,x)/\omega^\text{reg}_{X/k,x}} \right) \longrightarrow H^1(X,\omega^\text{reg}_{X/k}) \longrightarrow 0.
\end{equation}

For $x \in X^0$, the $k$-linear “residue” map

$$\text{res}_x : \Omega^1_{k(X)/k,x} \longrightarrow k$$

defined by $\text{res}_x(\eta) := \sum_{y \in \pi^{-1}(x)} \text{res}_y(\eta)$

kills $\omega^\text{reg}_{X/k,x}$, and the induced composite map

$$\Omega^1_{k(X)/k} \longrightarrow \bigoplus_{x \in X^0} \left( \Omega^1_{k(k(x)/k,x)/\omega^\text{reg}_{X/k,x}} \right) \stackrel{\sum \text{res}_x}{\longrightarrow} k$$

is zero by the residue theorem on the (smooth) connected components of $X$. Thus, from (2.1.10) we obtain a $k$-linear “trace map”

\begin{equation}
\text{res}_X : H^1(X,\omega^\text{reg}_{X/k}) \longrightarrow k
\end{equation}

which coincides with the usual residue map when $X$ is smooth. Rosenlicht’s explicit description of the relative dualizing sheaf and of Grothendieck duality for $X/k$ is:

**Proposition 2.1.15** (Rosenlicht). Let $X$ be a proper and geometrically connected curve over $k$ with $k$-smooth locus $U$. Viewing $\omega_{X/k}$ and $\omega^\text{reg}_{X/k}$ as subsheaves of $i_*\Omega^1_{U/k}$ via (2.1.7) and (2.1.9), respectively, we have an equality

$$\omega_{X/k} = \omega^\text{reg}_{X/k} \text{ inside } i_*\Omega^1_{U/k}.$$ 

Under this identification, Grothendieck’s trace map $H^1(X,\omega_X) \to k$ coincides with $-\text{res}_X$.

**Proof.** See [Con00, Theorem 5.2.3].

We now return to the situation that $S = \text{Spec}(R)$ for a discrete valuation ring $R$ with fraction field $K$ of characteristic zero and perfect residue field $k$ of characteristic $p > 0$.

**Lemma 2.1.16.** Let $X$ be a normal and proper curve over $S = \text{Spec}(R)$ with smooth and geometrically connected generic fiber, and denote by $\overline{X} := X_k$ the special fiber of $X$; it is a proper and geometrically connected curve over $k$ by Proposition 2.1.2 (2).

1. The canonical base change map

\[
\begin{array}{cccccc}
0 & \longrightarrow & H^0(X,\omega_X/S) \otimes_R k & \longrightarrow & H^1(X/R) \otimes_R k & \longrightarrow & H^1(X,\mathcal{O}_X) \otimes_R k & \longrightarrow & 0 \\
\downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
0 & \longrightarrow & H^0(\overline{X},\omega_{\overline{X}}/k) & \longrightarrow & H^1(\overline{X}/k) & \longrightarrow & H^1(\overline{X},\mathcal{O}_{\overline{X}}) & \longrightarrow & 0
\end{array}
\]

is an isomorphism.
Let $\rho : Y \to X$ be a finite morphism of normal and proper curves over $S$ with smooth and geometrically connected generic fibers. The canonical diagrams (one for $\rho^\ast$ and one for $\rho_\ast$)

\[
\begin{align*}
H^0(Y, \omega_{Y/S}) \otimes k & \xrightarrow{\rho^\ast \otimes 1} H^0(X, \omega_{X/S}) \otimes k \\
\approx & \quad \approx \\
H^0(\overline{Y}, \omega_{\overline{Y}/k}) & \xrightarrow{(2.1.7)} H^0(\overline{Y}, \omega_{\overline{Y}/k}) \\
(2.1.7) & \quad (2.1.7)
\end{align*}
\]

commute, where $\overline{\rho}^\ast$ and $\overline{\rho}_\ast$ are the usual pullback and trace morphisms on meromorphic differential forms associated to the finite flat map $\overline{\rho} : Y \to X$ of smooth curves over $k$.

**Proof.** Since $X$ is of relative dimension 1 over $S$, the cohomologies $H^1(X, \mathcal{O}_X)$ and $H^1(X, \omega_{X/S})$ both commute with base change, and they are both free over $R$ by Proposition 2.1.11. We conclude that $H^i(X, \mathcal{O}_X)$ and $H^i(X, \omega_{X/S})$ commute with base change for all $i$ and hence that the left and right vertical maps in the base change diagram (1) (whose rows are exact by Propositions 2.1.11 and 2.1.12) are isomorphisms. It follows that the middle vertical map in (1) is an isomorphism as well. The compatibility of pullback and trace under base change to the special fibers, as asserted by the diagram in (2), is a straightforward consequence of Proposition 2.1.10 (2), using the facts that $X$ and $Y$ are smooth at generic points of closed fibers and that $\rho : Y \to X$ takes generic points to generic points as noted in the proof of Lemma 2.1.9.

\[\square\]

### 2.2. Universal vectorial extensions and Dieudonné crystals

There is an alternate description of the short exact sequence $H(X/R)$ of Proposition 2.1.11 (2) in terms of Lie algebras and Néron models of Jacobians that will allow us to relate this cohomology to Dieudonné modules. To explain this description and its connection with crystals, we first recall some facts from [MM74] and [Cai10].

Fix a base scheme $T$, and let $G$ be an fppf sheaf of abelian groups over $T$. A **vectorial extension** of $G$ is a short exact sequence (of fppf sheaves of abelian groups)

\[
(2.2.1) \quad 0 \to V \to E \to G \to 0.
\]

with $V$ a vector group (i.e. an fppf abelian sheaf which is locally represented by a product of $\mathbb{G}_a$’s). Assuming that $\text{Hom}(G, V) = 0$ for all vector groups $V$, we say that a vectorial extension (2.2.1) is **universal** if, for any vector group $V'$ over $T$, the pushout map $\text{Hom}_T(V, V') \to \text{Ext}_T^1(G, V')$ is an isomorphism. When a universal vectorial extension of $G$ exists, it is unique up to canonical isomorphism and covariantly functorial in morphisms $G' \to G$ with $G'$ admitting a universal extension.

**Theorem 2.2.1.** Let $T$ be an arbitrary base scheme.

1. If $A$ is an abelian scheme over $T$, then a universal vectorial extension $\mathcal{E}(A)$ of $A$ exists, with $V = \omega_A^1$, and is compatible with arbitrary base change on $T$.
2. If $p$ is locally nilpotent on $T$ and $G$ is a $p$-divisible group over $T$, then a universal vectorial extension $\mathcal{E}(G)$ of $G$ exists, with $V = \omega_G^1$, and is compatible with arbitrary base change on $T$. 


(3) If $p$ is locally nilpotent on $T$ and $A$ is an abelian scheme over $T$ with associated $p$-divisible group $G := A[p^\infty]$, then the canonical map of fppf sheaves $G \to A$ extends to a natural map

$$
\begin{array}{c}
0 \longrightarrow \omega_{G^t} \longrightarrow \mathcal{E}(G) \longrightarrow G \longrightarrow 0 \\
0 \longrightarrow \omega_A \longrightarrow \mathcal{E}(A) \longrightarrow A \longrightarrow 0
\end{array}
$$

which induces an isomorphism of the corresponding short exact sequences of Lie algebras.

Proof. For the proofs of (1) and (2), see [MM74, I, §1.8 and §1.9]. To prove (3), note that pulling back the universal vectorial extension of $A$ along $G \to A$ gives a vectorial extension $\mathcal{E}'$ of $G$ by $\omega_A$. By universality, there then exists a unique map $\psi : \omega_{G^t} \to \omega_A$ with the property that the pushout of $\mathcal{E}(G)$ along $\psi$ is $\mathcal{E}'$, and this gives the map on universal extensions. That the induced map on Lie algebras is an isomorphism follows from [MM74, II, §13]. ■

For our applications, we will need a generalization of the universal extension of an abelian scheme to the setting of Néron models; in order to describe this generalization, we first recall the explicit description of the universal extension of an abelian scheme in terms of rigidified extensions.

For any commutative $T$-group scheme $F$, a rigidified extension of $F$ by $G_m$ over $T$ is a pair $(E, \sigma)$ consisting of an extension (of fppf abelian sheaves)

$$
(2.2.2) \quad 0 \longrightarrow G_m \longrightarrow E \longrightarrow F \longrightarrow 0
$$

and a splitting $\sigma : \text{Inf}^1(F) \to E$ of the pullback of (2.2.2) along the canonical closed immersion $\text{Inf}^1(F) \to F$. Two rigidified extensions $(E, \sigma)$ and $(E', \sigma')$ are equivalent if there is a group homomorphism $E \to E'$ carrying $\sigma$ to $\sigma'$ and inducing the identity on $G_m$ and on $F$. The set $\text{Extrig}_T(F, G_m)$ of equivalence classes of rigidified extensions over $T$ is naturally a group via Baer sum of rigidified extensions [MM74, I, §2.1], so the functor on $T$-schemes $T' \rightsquigarrow \text{Extrig}_T(F_T, G_m)$ is naturally a group functor that is contravariant in $F$ via pullback (fibered product). We write $\mathcal{E}xtrig_T(F, G_m)$ for the fppf sheaf of abelian groups associated to this functor.

**Proposition 2.2.2 (Mazur-Messing).** Let $A$ be an abelian scheme over an arbitrary base scheme $T$. The fppf sheaf $\mathcal{E}xtrig_T(A, G_m)$ is represented by a smooth and separated $T$-group scheme, and there is a canonical short exact sequence of smooth group schemes over $T$

$$
(2.2.3) \quad 0 \longrightarrow \omega_A \longrightarrow \mathcal{E}xtrig_T(A, G_m) \longrightarrow A^t \longrightarrow 0.
$$

Furthermore, (2.2.3) is naturally isomorphic to the universal extension of $A^t$ by a vector group.

Proof. See [MM74], I, §2.6 and Proposition 2.6.7. ■

In the case that $T = \text{Spec} R$ for $R$ a discrete valuation ring of mixed characteristic $(0, p)$ with fraction field $K$, we have the following generalization of Proposition 2.2.2:

**Proposition 2.2.3.** Let $A$ be an abelian variety over $K$, with dual abelian variety $A^t$, and write $A$ and $A^t$ for the Néron models of $A$ and $A^t$ over $T = \text{Spec}(R)$. Then the fppf abelian sheaf $\mathcal{E}xtrig_T(A, G_m)$ on the category of smooth $T$-schemes is represented by a smooth and separated $T$-group scheme. Moreover, there is a canonical short exact sequence of smooth group schemes over $T$

$$
(2.2.4) \quad 0 \longrightarrow \omega_A \longrightarrow \mathcal{E}xtrig_T(A, G_m) \longrightarrow A^{t0} \longrightarrow 0
$$
which is contravariantly functorial in $A$ via homomorphisms of abelian varieties over $K$. The formation of (2.2.4) is compatible with smooth base change on $T$; in particular, the generic fiber of (2.2.4) is the universal extension of $A^1$ by a vector group.

Proof. Since $R$ is of mixed characteristic $(0, p)$ with perfect residue field, this follows from Proposition 2.6 and the discussion following Remark 2.9 in [Cai10].

In the particular case that $A$ is the Jacobian of a smooth, proper and geometrically connected curve $X$ over $K$ which is the generic fiber of a normal proper curve $X'$ over $R$, we can relate the exact sequence of Lie algebras attached to (2.2.4) to the exact sequence $H(X/R)$ or Proposition 2.1.11 (2):

**Proposition 2.2.4.** Let $X$ be a proper relative curve over $T = \text{Spec}(R)$ with smooth generic fiber $X$ over $K$. Write $J := \text{Pic}^0_{X/K}$ for the Jacobian of $X$ and $J'$ for its dual, and let $\mathfrak{g}, \mathfrak{g}'$ be the corresponding Néron models over $R$. There is a canonical homomorphism of exact sequences of finite free $R$-modules

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \text{Lie} \omega_{\mathfrak{g}} & \longrightarrow & \text{Lie} \mathscr{E}_{\text{trig}}(\mathfrak{g}, G_m) & \longrightarrow & \text{Lie} \mathfrak{g}'^0 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^0(X, \omega_{X/T}) & \longrightarrow & H^1(X/R) & \longrightarrow & H^1(X, O_X) & \longrightarrow & 0
\end{array}
$$

(2.2.5)

that is an isomorphism when $X$ has rational singularities.\(^{11}\) For any finite morphism $\rho : Y \to X$ of $S$-curves satisfying the above hypotheses, the map (2.2.5) intertwines the $\rho_*$ (respectively $\rho^*$) on the bottom row with $\text{Pic}(\rho)^*$ (respectively $\text{Alb}(\rho)^*$) on the top.

Proof. See Theorem 1.2 and (the proof of) Corollary 5.6 in [Cai10].

\(^{11}\)Recall that $X$ is said to have rational singularities if it admits a resolution of singularities $\rho : X' \to X$ with the natural map $R^1\rho_*O_{X'} = 0$. Trivially, any regular $X$ has rational singularities.
Let $G$ be a $p$-divisible group over $T$, considered as an fppf abelian scheme on $T$. As in [BBM82], we define the (contravariant) *Dieudonné crystal of $G$ over $T$* to be

$$D(G) := \mathcal{E}xt^1_{T/\Sigma}(G, \mathcal{O}_{T/\Sigma}).$$

It is a locally free crystal in $\mathcal{O}_{T/\Sigma}$-modules, which is contravariantly functorial in $G$ and of formation compatible with base change along PD-morphisms $T' \to T$ of $\Sigma$-schemes thanks to 2.3.6.2 and Proposition 2.4.5 (ii) of [BBM82]. If $T' = \text{Spec}(A)$ is affine, we will simply write $D(G)_A$ for the finite locally free $A$-module associated to $D(G)_{T'}$.

The structure sheaf $\mathcal{O}_{T/\Sigma}$ is canonically an extension of $G_\alpha$ by the PD-ideal $\mathfrak{d}_{T/\Sigma} \subseteq \mathcal{O}_{T/\Sigma}$, and by applying $\mathcal{H}om_{T/\Sigma}(G, \cdot)$ to this extension one obtains (see Propositions 3.3.2 and 3.3.4 as well as Corollaire 3.3.5 of [BBM82]) a short exact sequence (the *Hodge filtration*)

$$0 \longrightarrow \mathcal{E}xt^1_{T/\Sigma}(G, \mathfrak{d}_{T/\Sigma}) \longrightarrow D(G) \longrightarrow \mathcal{E}xt^1_{T/\Sigma}(G, G_\alpha) \longrightarrow 0 \tag{2.2.7}$$

that is contravariantly functorial in $G$ and of formation compatible with base change along PD-morphisms $T' \to T$ of $\Sigma$-schemes. The following “geometric” description of the value of (2.2.7) on a PD-thickening of the base will be essential for our purposes:

**Proposition 2.2.6.** Let $G$ be a fixed $p$-divisible group over $T$ and let $T'$ be any $\Sigma$-PD thickening of $T$. If $G'$ is any lifting of $G$ to a $p$-divisible group on $T'$, then there is a natural isomorphism

$$0 \longrightarrow \omega_{G'} \longrightarrow \mathcal{L}ie(\mathcal{E}(G')) \longrightarrow \mathcal{L}ie(G') \longrightarrow 0$$

that is moreover compatible with base change in the evident manner.

**Proof.** See [BBM82, Corollaire 3.3.5] and [MM74, II, Corollary 7.13].

**Remark 2.2.7.** In his thesis [Mes72], Messing showed that the Lie algebra of the universal extension of $G'$ is “crystalline in nature” and used this as the *definition* of $D(G)$. (See chapter IV, §2.5 of [Mes72] and especially 2.5.2.) Although we prefer the more intrinsic description (2.2.6) of [MM74] and [BBM82], it is ultimately Messing’s original definition that will be important for us.

### 2.3. Integral models of modular curves.

We record some basic facts about integral models of modular curves that will be needed in what follows. We assume that the reader is familiar with [KM85], and will freely use the notation and terminology therein. Throughout, we fix a prime $p$ and a positive integer $N$ not divisible by $p$.

**Definition 2.3.1.** Let $r$ be a nonnegative integer and $R$ a ring containing a fixed choice $\zeta$ of primitive $p^r$-th root of unity in which $N$ is invertible. The moduli problem $\mathcal{P}_r^N := (\text{bal. } \Gamma_1(p^r)[\zeta^{\text{can}}; [\mu_N]])$ on $(\text{Ell} / R)$ assigns to $E / S$ the set of quadruples $(\phi : E \to E', P, Q; \alpha)$ where:

1. $\phi : E \to E'$ is a $p^r$-isogeny.
2. $P \in \ker \phi(S)$ and $Q \in \ker \phi'(S)$ are generators of $\ker \phi$ and $\ker \phi'$, respectively, which pair to $\zeta$ under the canonical pairing $\langle \cdot, \cdot \rangle_\phi : \ker \phi \times \ker \phi' \to \mu_{\deg \phi}$ [KM85, §2.8].
3. $\alpha : \mu_N \hookrightarrow E[N]$ is a closed immersion of $S$-group schemes.

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12Noting that it suffices to define the crystal $D(G)$ on $\Sigma$-PD thickenings $T'$ of $T$ to which $G$ admits a lift.
Proposition 2.3.2. If \( N \geq 4 \), then the moduli problem \( \mathcal{P}_r^\gamma \) is represented by a regular scheme \( \mathbf{M}(\mathcal{P}_r^\gamma) \) that is flat of pure relative dimension 1 over Spec(\( R \)). The moduli scheme \( \mathbf{M}(\mathcal{P}_r^\gamma) \) admits a canonical compactification \( \overline{\mathbf{M}}(\mathcal{P}_r^\gamma) \), which is regular and proper flat of pure relative dimension 1 over Spec(\( R \)).

Proof. Using that \( N \) is a unit in \( R \), one first shows that for \( N \geq 4 \), the moduli problem \([\mu_N]\) on \((\text{Ell} / \mathcal{O}_\mathbb{I})\) is representable over Spec(\( R \)) and finite étale; this follows from 2.7.4, 3.6.0, 4.7.1 and 5.1.1 of [KM85], as \([\mu_N]\) is isomorphic to \([\Gamma_1(N)]\) over any \( R \)-scheme containing a fixed choice of primitive \( N \)-th root of unity (see also [KM85, 8.4.11]). By [KM85, 4.3.4], to prove the first assertion it is then enough to show that \([\text{bal.}\, \Gamma_1(p^\gamma)]\) on \((\text{Ell} / \mathcal{O}_\mathbb{I})\) is relatively representable and regular, which (via [KM85, 9.1.7]) is a consequence of [KM85, 7.6.1 (2)]. For the second assertion, see [KM85, §8].

Recall that we have fixed a compatible sequence \( \{\varepsilon_r\}_{r \geq 1} \) of primitive \( p^r \)-th roots of unity in \( \overline{\mathbb{Q}}_p \).

Definition 2.3.3. We set \( \mathcal{X}_r := \overline{\mathbf{M}}(\mathcal{P}_r^{(\varepsilon_r)}) \), viewed as a scheme over \( T_r := \text{Spec}(\mathcal{O}_\mathbb{I}) \).

There is a canonical action of \( \mathbb{Z}_p^\times \times (\mathbb{Z} / N\mathbb{Z})^\times \) by \( R_r \)-automorphisms of \( \mathcal{X}_r \), defined at the level of the underlying moduli problem by
\[
(u, v) \cdot (\phi : E \to E', P, Q; \alpha) := (\phi : E \to E', uP, u^{-1}Q; \alpha \circ v)
\]
as one checks by means of the computation \((uP, u^{-1}Q)_{\phi} = (P, Q)_{\phi}^\gamma = (P, Q)_{\phi}\). Here, we again write \( v : \mu_N \to \mu_N \) for the automorphism of \( \mu_N \) functorially defined by \( \zeta \mapsto \zeta^v \) for any \( N \)-th root of unity \( \zeta \). We refer to this action of \( \mathbb{Z}_p^\times \times (\mathbb{Z} / N\mathbb{Z})^\times \) as the diamond operator action, and will denote by \( \langle u \rangle \) (respectively \( \langle v \rangle \)) the automorphism induced by \( u \in \mathbb{Z}_p^\times \) (respectively \( v \in (\mathbb{Z} / N\mathbb{Z})^\times \)).

There is also an \( R_r \)-semilinear “geometric inertia” action of \( \Gamma := \text{Gal}(K_{\infty} / K_0) \) on \( \mathcal{X}_r \), which allows us to descend the generic fiber of \( \mathcal{X}_r \) to \( K_0 \). To explain this action, for \( \gamma \in \Gamma \) and any \( T_r \)-scheme \( T' \), let us write \( T'_r \) for the base change of \( T' \) along the morphism \( T_r \to T_r \) induced by \( \gamma \in \text{Aut}(\mathcal{O}_\mathbb{I}) \). There is a canonical functor \((\text{Ell} / (T'_r)_{\gamma}) \to (\text{Ell} / T_r)\) obtained by viewing an elliptic curve over a \( (T'_r)_{\gamma} \)-scheme \( T' \) as the same elliptic curve over the same base \( T' \), viewed as a \( T_r \)-scheme via the projection \((T_r)_{\gamma} \to T_r \).

For a moduli problem \( \mathcal{P} \) on \((\text{Ell} / T_r)\), we denote by \( \gamma^* \mathcal{P} \) the moduli problem on \((\text{Ell} / (T_r)_{\gamma})\) obtained by composing \( \mathcal{P} \) with this functor; see [KM85, 4.1.3]. Each \( \gamma \in \Gamma \) gives rise to a morphism of moduli problems \( \gamma : \mathcal{P}_r^{(\varepsilon_r)} \to \gamma^* \mathcal{P}_r^{(\varepsilon_r)} \) via
\[
\gamma(\phi : E \to E', P, Q; \alpha) := (\phi_{\gamma} : E_{\gamma} \to E'_{\gamma}, \chi(\gamma)^{-1}P_{\gamma}, Q_{\gamma}; \alpha_{\gamma})
\]
where the subscript of \( \gamma \) means “base change along \( \gamma \)” (see §1.4). Since
\[
\langle \chi(\gamma)^{-1}P_{\gamma}, Q_{\gamma} \rangle_{\phi_{\gamma}} = \gamma(\langle P, Q \rangle)^{\chi(\gamma)^{-1}} = \langle P, Q \rangle_{\phi}
\]
this really is a morphism of moduli problems on \((\text{Ell} / T_r)\). We thus obtain a morphism of \( T_r \)-schemes
\[
\gamma : \mathcal{X}_r \longrightarrow (\mathcal{X}_r)_{\gamma}
\]
for each \( \gamma \in \Gamma \), compatibly with change in \( \gamma \). The induced semilinear action of \( \Gamma \) on the generic fiber of \( \mathcal{X}_r \) provides a descent datum with respect to the canonical map \( \text{Spec}(K_r) \to \text{Spec}(K_0) \), which is necessarily effective as this map is étale. Thus, there is a unique scheme \( X_r \) over \( K_0 = \mathbb{Q}_p \) with \((X_r)_K_r \cong (\mathcal{X}_r)_{K_r} ;\) as the diamond operators visibly commute with the action of \( \Gamma \), they act on \( X_r \) by \( \mathbb{Q}_p \)-automorphisms in a manner that is compatible with this identification.

Remark 2.3.4. We may identify \( X_r \) with the base change to \( \mathbb{Q}_p \) of the modular curve \( X_1(Np^r) \) over \( \mathbb{Q} \) classifying pairs \((\mathcal{E}, \alpha)\) of a generalized elliptic curve \( \mathcal{E}/S \) together with an embedding of \( S \)-group
schemes $\alpha : \mu_{Np^r} \hookrightarrow E^{\text{sm}}$ whose image meets each irreducible component in every geometric fiber. If instead we were to use the geometric inertia action on $X_r$ induced by
\[
\gamma(\phi : E \to E', P, Q; \alpha) := (\phi_\gamma : E_\gamma \to E'_\gamma, P_\gamma, \chi(\gamma)^{-1}Q_\gamma ; \alpha_\gamma),
\]
then the resulting descent $X'_r$ of the generic fiber of $X_r$ to $\mathbb{Q}_p$ would be canonically isomorphic to the base change to $\mathbb{Q}_p$ of the modular curve $X_1(Np^r)'$ over $\mathbb{Q}$ classifying generalized elliptic curves $E/S$ with an embedding of $S$-group schemes $\mathbb{Z}/Np^r\mathbb{Z} \hookrightarrow E^{\text{sm}}[Np^r]$ whose image meets each irreducible component in every geometric fiber. Of course, $X_1(Np^r)'$ (respectively $X_1(Np^r)$) is the canonical model of the upper half-plane quotient $\Gamma_1(Np^r)\backslash \mathbb{H}^*$ with $\mathbb{Q}_p$-rational cusp $\text{cusp } i\infty$ (respectively 0).

Recall \cite[(6.7)]{KM85} that over any base scheme $S$, a cyclic $p^{r+1}$-isogeny of elliptic curves $\phi : E \to E'$ admits a “standard factorization” (in the sense of \cite[6.7.7]{KM85})
\[
(2.3.4) \quad E := E_0 \xrightarrow{\phi_{0,1}} E_1 \cdots \xrightarrow{\phi_{r,r+1}} E_r \xrightarrow{\phi_{r+1,r}} E_{r+1} := E'.
\]

For each pair of nonnegative integers $a < b \leq r + 1$ we will write $\phi_{a,b}$ for the composite $\phi_{a,a+1} \circ \cdots \circ \phi_{b-1,b}$ and $\phi_{b,a} := \phi_{a,b}'$ for the dual isogeny. Using this notion, we define “degeneracy maps” $\rho, \sigma : X_{r+1} \hookrightarrow X_r$ (over the map $T_{r+1} \to T_r$) at the level of underlying moduli problems as follows (cf. \cite[11.3.3]{KM85}):
\[
\begin{align*}
(2.3.5) \quad \rho(\phi : E_0 \to E_{r+1}, P, Q; \alpha) & := (\phi_{0,r} : E_0 \to E_r, pP, \phi_{r+1,r}(Q); \alpha) \\
\sigma(\phi : E_0 \to E_{r+1}, P, Q; \alpha) & := (\phi_{1,r+1} : E_1 \to E_{r+1}, \phi_{0,1}(P), pQ; \phi_{0,1} \circ \alpha).
\end{align*}
\]

By the universal property of fiber products, we obtain morphisms $T_{r+1}$-schemes
\[
(2.3.6) \quad X_{r+1} \xrightarrow{\rho} X_r \times_{T_r} T_{r+1}.
\]

that are compatible with the diamond operators and the geometric inertia action of $\Gamma$. 

**Remark 2.3.5.** On generic fibers, the morphisms (2.3.6) uniquely descend to degeneracy mappings $\rho, \sigma : X_{r+1} \hookrightarrow X_r$ of smooth curves over $\mathbb{Q}_p$. Under the identification $X_r \simeq X_1(Np^r) \mathbb{Q}_p$ of Remark 2.3.4, the map $\rho$ corresponds to the “standard” projection, induced by $\tau \mapsto \tau$ on the complex upper half-plane, whereas $\sigma$ corresponds to the morphism induced by $\tau \mapsto p\tau$.

Recall that we have fixed a choice of primitive $N$-th root of unity $\zeta_N$ in $\overline{\mathbb{Q}}_p$. The Atkin Lehner “involution” $w_{\zeta_N}$ on $X_r \times_{R_r} R'_r$ is defined as in [Col94, §8]. Following [KM85, 11.3.2], we define the Atkin Lehner automorphism $w_{\zeta_N}$ of $X_r$ over $R_r$ on the underlying moduli problem $S^\zeta_{r+1}$ as
\[
w_{\zeta_N}(\phi : E \to E', P, Q; \alpha) := (\phi' : E' \to E, -Q, P; \phi \circ \alpha).
\]

We then define $w_r := w_{\zeta_N} \circ w_{\zeta_N} = w_{\zeta_N} \circ w_{\zeta_N}$; it is an automorphism of $X_r \times_{R_r} R'_r$ over $R'_r := R_r[\mu_N]$.

**Proposition 2.3.6.** For all $(u, v) \in \mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times$ and all $\gamma \in \text{Gal}(K'_\infty/K_0)$, the identities
\[
w_r(u)v = \langle v \rangle_N^{-1}\langle u \rangle_N^{-1}w_r
\]
\[
(\gamma^sw_r)\gamma = \gamma w_r(\chi(\gamma))^{-1}\langle a(\gamma) \rangle_N^{-1}
\]
\[
w_r^2 = (-p^r)N\langle -N \rangle
\]
\[
\rho w_{r+1} = w_r\sigma
\]
\[
\sigma w_{r+1} = \langle \rho \rangle_N w_r \rho
\]
hold, with $a : \text{Gal}(K'_\infty/K_0) \to (\mathbb{Z}/N\mathbb{Z})^\times$ the character determined by $\gamma \zeta = \zeta^a(\gamma)$ for all $\zeta \in \mu_N(\overline{\mathbb{Q}}_p)$. 

Proof. This is an easy consequence of definitions. ■

In order to describe the special fiber of $X_r$, we must first introduce Igusa curves:

**Definition 2.3.7.** Let $r$ be a nonnegative integer. The moduli problem $\mathcal{X}_r := ([\text{Ig}(p^r)]; [\mu_N])$ on $(\text{Ell}/F_p)$ assigns to $(E/S)$ the set of triples $(E, P; \alpha)$ where $E/S$ is an elliptic curve and

1. $P \in E(p^r)(S)$ is a point that generates the $r$-fold iterate of Verschiebung $V^r : E(p^r) \rightarrow E$.
2. $\alpha : \mu_N \hookrightarrow E[N]$ is a closed immersion of $S$-group schemes.

**Proposition 2.3.8.** If $N \geq 4$, then the moduli problem $\mathcal{X}_r$ on $(\text{Ell}/F_p)$ is represented by a smooth affine curve $\mathcal{M}(\mathcal{X}_r)$ over $F_p$ which admits a canonical smooth compactification $\overline{\mathcal{M}(\mathcal{X}_r)}$.

**Proof.** One argues as in the proof of Proposition 2.3.2, using [KM85, 12.6.1] to know that $[\text{Ig}(p^r)]$ is relatively representable on $(\text{Ell}/F_p)$, regular 1-dimensional and finite flat over $(\text{Ell}/F_p)$. ■

**Definition 2.3.9.** Set $\text{Ig}_r := \overline{\mathcal{M}(\mathcal{X}_r)}$; it is a smooth, proper, and geometrically connected $F_p$-curve.

There is a canonical action of the diamond operators $\mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times$ on the moduli problem $\mathcal{X}_r$ via $(u, v) \cdot (E, P; \alpha) := (E, uP; v \circ \alpha)$; this induces a corresponding action on $\text{Ig}_r$ by $F_p$-automorphisms. We again write $\langle u \rangle$ (respectively $\langle v \rangle_N$) for the action of $u \in \mathbb{Z}_p^\times$ (respectively $v \in (\mathbb{Z}/N\mathbb{Z})^\times$). Thanks to the “backing up theorem” [KM85, 6.7.11], one also has natural degeneracy maps

$$\rho : \text{Ig}_{r+1} \longrightarrow \text{Ig}_r \quad \text{induced by} \quad \rho(E, P; \alpha) := (E, VP, \alpha)$$

on underlying moduli problems. This map is visibly equivariant for the diamond operator action on source and target. Let $\text{ss}_r$ be the (reduced) closed subscheme of $\text{Ig}_r$ that is the support of the coherent ideal sheaf of relative differentials $\Omega_{\text{Ig}_r/\text{Ig}_0}$; over the unique degree 2 extension of $F_p$, this scheme breaks up as a disjoint union of rational points—the supersingular points. The map (2.3.7) is finite of degree $p$, generically étale and totally (wildly) ramified over each supersingular point.

We can now describe the special fiber of $X_r$:

**Proposition 2.3.10.** The scheme $\overline{X}_r := X_r \times_{T_r} \text{Spec}(F_p)$ is the disjoint union, with crossings at the supersingular points, of the following proper, smooth $F_p$-curves: for each pair $a, b$ of nonnegative integers with $a + b = r$, and for each $u \in (\mathbb{Z}/p^{\min(a,b)}\mathbb{Z})^\times$, one copy of $\text{Ig}_{\text{max}(a,b)}$.

We refer to [KM85, 13.1.5] for the definition of “disjoint union with crossings at the supersingular points”. Note that the special fiber of $X_r$ is (geometrically) reduced; this will be crucial in our later work. We often write $I_{(a,b,u)}$ for the irreducible component of $\overline{X}_r$ indexed by the triple $(a, b, u)$ and will refer to it as the $(a, b, u)$-component (for fixed $(a, b)$ we have $I_{(a,b,u)} = \text{Ig}_{\text{max}(a,b)}$ for all $u$).

For the proof of Proposition 2.3.10, we refer the reader to [KM85, 13.11.2–13.11.4], and content ourselves with recalling the correspondence between (non-cuspidal) points of the $(a, b, u)$-component and $[\text{bal.}\Gamma_1(p^r)]^1$-can-structures on elliptic curves.\(^{13}\)

Let $S$ be any $F_p$ scheme, fix an ordinary elliptic curve $E_0$ over $S$, and let $(\phi : E_0 \rightarrow E_r, P, Q; \alpha)$ be an element of $\mathscr{D}_r^1(E_0/S)$. By [KM85, 13.11.2], there exist unique nonnegative integers $a, b$ with the property that the cyclic $p^a$-isogeny $\phi$ factors as a purely inseparable cyclic $p^a$-isogeny followed by an

\(^{13}\)Note that under the canonical ring homomorphism $R_r \rightarrow F_p$, our fixed choice $\varepsilon^{(r)}$ of primitive $p^r$-th root of unity maps to $1 \in F_p$, which is a primitive $p^r$-th root of unity by definition [KM85, 9.1.1], as it is a root of the $p^r$-th cyclotomic polynomial over $F_p$!
étale $p^b$-isogeny (this is the standard factorization of $\phi$). Furthermore, there exists a unique elliptic curve $E$ over $S$ and $S$-isomorphisms $E_0 \simeq E(p^a)$ and $E_r \simeq E(p^a)$ such that the cyclic $p^r$ isogeny $\phi$ is:

$$
E_0 \simeq E(p^a) \xrightarrow{\sigma} E(p^r) \xrightarrow{V^b} E(p^a) \simeq E_r
$$

and $P \in E(p^a)(S)$ (respectively $Q \in E(p^a)$) is an Igusa structure of level $p^b$ (respectively $p^a$) on $E$ over $S$. When $a \geq b$ there is a unique unit $u \in (\mathbb{Z}/p^{a} \mathbb{Z})^\times$ such that $V^{a-b}(Q) = uP$ in $E(p^a)(S)$ and when $b \geq a$ there is a unique unit $u \in (\mathbb{Z}/p^{b} \mathbb{Z})^\times$ such that $uV^{b-a}(P) = Q$ in $E(p^a)(S)$. Thus, for $a \geq b$ (respectively $b \geq a$) and fixed $u$, the data $(E, Q; p^{-b}V^{b} \circ \alpha)$ (respectively $(E, P; p^{-b}V^{b} \circ \alpha)$) gives an $S$-point of the $(a, b, u)$-component $I_{g\text{max}(a,b)}$. Conversely, suppose given $(a, b, u)$ and an $S$-valued point of $I_{g\text{max}(a,b)}$ which is neither a cusp nor a supersingular point (in the sense that it corresponds to an ordinary elliptic curve with extra structure). If $a \geq b$ and $(E, Q; \alpha)$ is the given $S$-point of $I_{g_a}$ then we set $P := u^{-1}V^{a-b}(Q)$, while if $b \geq a$ and $(E, P; \alpha)$ is the given $S$-point of $I_{g_b}$ then we set $Q := uV^{b-a}P$. Due to [KMS85, 13.11.3], the data

$$(E(p^a) \xrightarrow{\sigma} E(p^r) \xrightarrow{V^b} E(p^a), P, Q; F^b \circ \alpha)$$

gives an $S$-point of $\mathbf{M}(\mathcal{P}^1_r)$. These constructions are visibly inverse to each other.

**Remark 2.3.11.** When $r$ is even and $a = b = r/2$, there is a choice to be made as to how one identifies the $(r/2, r/2, u)$-component of $\overline{\mathcal{X}}_r$ with $I_{g_{r/2}}$: if $(\phi : E_0 \to E_r, P, Q; \alpha)$ is an element of $\mathcal{P}^1_r(E_0/S)$ which corresponds to a point on the $(r/2, r/2, u)$-component, then for $E$ with $E_0 \simeq E(p^{r/2}) \simeq E_r$, both $(E, P; p^{-r/2}V^{r/2} \circ \alpha)$ and $(E, Q; p^{-r/2}V^{r/2} \circ \alpha)$ are $S$-points of $I_{g_{r/2}}$. Since $uP = Q$, the corresponding closed immersions $I_{g_{r/2}} \xrightarrow{\phi} \overline{\mathcal{X}}_r$ are twists of each other by the automorphism $\langle u \rangle$ of the source. We will consistently choose $(E, Q; p^{-r/2}V^{r/2} \circ \alpha)$ to identify the $(r/2, r/2, u)$-component of $\overline{\mathcal{X}}_r$ with $I_{g_{r/2}}$.

**Remark 2.3.12.** As in [MW86, pg. 236], we will refer to $I_{r}^\infty := I_{(r,0,1)}$ and $I_{r}^0 := I_{(0,r,1)}$ as the two “good” components of $\overline{\mathcal{X}}_r$. The $\mathbb{Q}_p$-rational cusp $\infty$ of $X_r$ induces a section of $\mathcal{X}_r \to T_r$ which meets $I_{r}^\infty$, while the section induced by the $K_1$-rational cusp $0$ meets $I_{r}^0$. It is precisely these irreducible components of $\overline{\mathcal{X}}_r$ which contribute to the “ordinary” part of cohomology. We note that $I_{r}^\infty$ corresponds to the image of $I_{g_r}$ under the map $i_1$ of [MW86, pg. 236], and corresponds to the component of $\overline{\mathcal{X}}_r$ denoted by $C_{\infty}$ in [Til87, pg. 343], by $C_{\infty}^c$ in [Sab96, pg. 231] and, for $r = 1$, by $I$ in [Gro90, §7].

By base change, the degeneracy mappings (2.3.6) induces morphisms $\overline{\rho}, \overline{\sigma} : \overline{\mathcal{X}}_{r+1} \to \overline{\mathcal{X}}_r$ of curves over $\mathbf{F}_p$, which admit the following descriptions on irreducible components:

**Proposition 2.3.13.** Let $a, b$ be nonnegative integers with $a + b = r + 1$ and $u \in (\mathbb{Z}/p^{\min(a,b)} \mathbb{Z})^\times$. The restriction of the map $\overline{\sigma} : \overline{\mathcal{X}}_{r+1} \to \overline{\mathcal{X}}_r$ to the $(a, b, u)$-component of $\overline{\mathcal{X}}_{r+1}$ is:

$$
\begin{align*}
I_{g_a} &= I_{(a,b,u)} \xrightarrow{F_{a/\rho}} I_{(a-1,b,u)} = I_{g_{a-1}} & \quad & b < a \leq r + 1 \\
I_{g_b} &= I_{(a,b,u)} \xrightarrow{\langle u \rangle^{-1}F} I_{(a-1,b,u \mod p^{a-1})} = I_{g_{b}} & \quad & a = b = r/2 \\
I_{g_b} &= I_{(a,b,u)} \xrightarrow{F} I_{(a-1,b,u \mod p^{a-1})} = I_{g_{b}} & \quad & a < b < r + 1 \\
I_{g_{r+1}} &= I_{(0,r+1,1)} \xrightarrow{(p)\times F} I_{(0,r,1)} = I_{g_{r}} & \quad & (a, b, u) = (0, r + 1, 1)
\end{align*}
$$
and the restriction of the map \( \bar{\rho} : \overline{X}_{r+1} \to \overline{X}_r \) to the \((a, b, u)\)-component of \( \overline{X}_{r+1} \) is:

\[
\begin{align*}
\text{Ig}_{r+1} &= I_{(r+1,0,1)} \circ \bar{\rho} I_{(r,0,1)} = \text{Ig}_r : (a, b, u) = (r+1, 0, 1) \\
\text{Ig}_a &= I_{(a,b,u)} F - I_{(a,b-1,u \mod p^{b-1})} = \text{Ig}_a : b < a + 1 \leq r + 1 \\
\text{Ig}_b &= I_{(a,b,u)} \langle u \rangle F^{a_0} I_{(a,b-1,u)} = \text{Ig}_{b-1} : a + 1 = b = r/2 + 1 \\
\text{Ig}_b &= I_{(a,b,u)} F^{a_0} I_{(a,b-1,u)} = \text{Ig}_{b-1} : a + 1 < b \leq r + 1
\end{align*}
\]

Here, for any \( \mathbb{F}_p \)-scheme \( I \), the map \( F : I \to I \) is the absolute Frobenius morphism.

**Proof.** Using the moduli-theoretic definitions \((2.3.5)\) of the degeneracy maps, the proof is a pleasant exercise in tracing through the functorial correspondence between the points of \( \overline{X}_r \) and points of \( \text{Ig}_{(a,b,u)} \). We leave the details to the reader. \( \blacksquare \)

We likewise have a description of the automorphism of \( \overline{X}_r \) induced via base change by the geometric inertia action\(^{14}\) \((2.3.2)\) of \( \Gamma \):

**Proposition 2.3.14.** Let \( a, b \) be nonnegative integers with \( a + b = r \) and \( u \in (\mathbb{Z}/p^{\min(a,b)}\mathbb{Z})^\times \). For \( \gamma \in \Gamma \), the restriction of \( \pi : \overline{X}_r \to \overline{X}_r \) to the \((a, b, u)\)-component of \( \overline{X}_r \) is:

\[
\begin{align*}
\text{Ig}_a &= I_{(a,b,u)} \text{id} I_{(a,b,\chi(\gamma)u)} = \text{Ig}_a : b \leq a \leq r \\
\text{Ig}_b &= I_{(a,b,u)} \langle \chi(\gamma) \rangle^{-1} I_{(a,b,\chi(\gamma)u)} = \text{Ig}_b : a < b \leq r
\end{align*}
\]

Following [Ulm90, §7–8], we now define a correspondence \( \pi_1, \pi_2 : Y_r \to X_r \) on \( X_r \) over \( R_r \) which naturally extends the correspondence on \( X_r \) giving the Hecke operator \( U_p \) (see below for a brief discussion of correspondences).

**Definition 2.3.15.** Let \( r \) be a nonnegative integer and \( R \) a ring containing a fixed choice \( \zeta \) of primitive \( p^r \)-th root of unity in which \( N \) is invertible. The moduli problem \( \mathcal{D}_r^\zeta := ([\Gamma_0(p^r+1); r, r]^{\zeta\text{-can}}; [\mu_N]) \) on \((\text{Ell}/R)\) assigns to \( E/S \) the set of quadruples \((\phi : E \to E', P, Q, \alpha)\) where:

1. \( \phi \) is a cyclic \( p^r+1 \)-isogeny with standard factorization

\[
E = : E_0 \xrightarrow{\phi_{0,1}} E_1 \cdots \xrightarrow{\phi_{r,r+1}} E_r \xrightarrow{\phi_{r+1,1}} E_{r+1} = E'
\]

2. \( P \in E_1(S) \) and \( Q \in E_r(S) \) are generators of \( \ker \phi_{1,r+1} \) and \( \ker \phi_{r,0} \), respectively, and satisfy

\[
(\phi_{r+1,1}(Q), \phi_{1,r+1}(P), \phi_{1,0}(P), Q) \phi_{0,r} = \zeta.
\]

3. \( \alpha : \mu_N \to E[N] \) is a closed immersion of \( S \)-group schemes.

**Proposition 2.3.16.** If \( N \geq 4 \), then the moduli problem \( \mathcal{D}_r^\zeta \) is represented by a regular scheme \( \mathcal{M}(\mathcal{D}_r^\zeta) \) that is flat of pure relative dimension \( 1 \) over \( \text{Spec}(R) \). This scheme admits a canonical compactification \( \overline{\mathcal{M}}(\mathcal{D}_r^\zeta) \), which is regular and proper flat of pure relative dimension \( 1 \) over \( \text{Spec}(R) \).

\(^{14}\) Since \( \Gamma \) acts trivially on \( \mathbb{F}_p \), for each \( \gamma \in \Gamma \) the base change of the \( R_r \)-morphism \( \gamma : X_r \to (X_r)_\gamma \) along the map induced by the canonical surjection \( R_r \to \mathbb{F}_p \) is an \( \mathbb{F}_p \)-morphism \( \gamma : X_r \to (X_r)_\gamma \simeq X_r \).
Proof. As in the proof of Proposition 2.3.2, it suffices to prove that \([\Gamma_0(p^{r+1}); r, r]^{\text{can}}\) is relatively representable and regular, which follows from [KM85, 7.6.1]; see also §7.9 of op. cit.

Definition 2.3.17. We set \(Y_r := \mathcal{M}(\mathcal{Z}_r^{(r)}),\) viewed as a scheme over \(T_r = \text{Spec}(R_r).\)

The scheme \(Y_r\) is equipped with an action of the diamond operators \(\mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times,\) as well as a “geometric inertia” action of \(\Gamma\) given moduli-theoretically exactly as in (2.3.1) and (2.3.2). The “semilinear” action of \(\Gamma\) is equivalent to a descent datum—necessarily effective—on the generic fiber of \(Y_r,\) and we denote by \(Y_r\) the resulting unique \(\mathbb{F}_p\)-descent of \((Y_r)_{K_r}.\)

Remark 2.3.18. We may identify \(Y_r\) with the base change to \(\mathbb{Q}_p\) of the modular curve \(X_1(Np^r; Np^{r-1})\) over \(\mathbb{Q}\) classifying triples \((E_1, \alpha, C)\) where \(E_1\) is a generalized elliptic curve, \(\alpha : \mu_{Np^r} \hookrightarrow E_1^m[Np^r]\) is an embedding of group schemes whose image meets each irreducible component in every geometric fiber, and \(C\) is a locally free subgroup scheme of rank \(p\) in \(E_1^m[p]\) with the property that \(C \cap \im \alpha = 0.\) Note that \(X_1(Np^r; Np^{r-1})\) is the canonical model over \(\mathbb{Q}\) with rational cusps \(i_\infty\) of the modular curve \(\Gamma_{r+1}^\ast \backslash \mathcal{M}\), for \(\Gamma_{r+1} := \Gamma_1(p^r) \cap \Gamma_0(p^{r+1}).\)

There is a canonical morphism of curves \(\pi : X_{r+1} \rightarrow Y_r\) over \(T_{r+1} \rightarrow T_r\) induced by the morphism

\[
\mathcal{Z}_r^{(r)} \rightarrow \mathcal{Z}_r^{(r)} \quad \text{given by} \quad \pi(\phi : E \rightarrow E', P, Q; \alpha) := (\phi : E \rightarrow E', \phi_{0,1}(P), \phi_{r+1,r}(Q); \alpha).
\]

One checks that \(\pi\) is equivariant with respect to the action of the diamond operators and of \(\Gamma,\) and so descends to a map \(\pi : Y_r \rightarrow X_r\) of smooth curves over \(\mathbb{Q}_p.\) It is likewise straightforward to check that the two projection maps \(\sigma, \rho : X_{r+1} \rightarrow X_r\) of (2.3.5) factor through \(\pi\) via unique maps of \(T_r\)-schemes \(\pi_1, \pi_2 : Y_r \rightarrow X_r,\) given as morphisms of underlying moduli problems on \((\text{Ell}/R_r)\)

\[
\begin{align*}
\pi_1(\phi : E_0 \rightarrow E_{r+1}, P, Q; \alpha) & := (E_1 \xrightarrow{\phi_{1,r+1}} E_{r+1}, P, \phi_{r+1,r}(Q); \phi_{0,1} \circ \alpha) \\
\pi_2(\phi : E_0 \rightarrow E_{r+1}, P, Q; \alpha) & := (E_0 \xrightarrow{\phi_{0,r}} E_r, \phi_{1,0}(P), Q; \alpha)
\end{align*}
\]

That these morphisms are well defined and that one has \(\rho = \pi \circ \pi_2\) and \(\sigma = \pi \circ \pi_1\) is easily verified (see [Ulm90, §7] and compare to [KM85, §11.3.3]). They are moreover finite of generic degree \(p,\) equivariant for the diamond operators, and \(\Gamma\)-compatible; in particular, \(\pi_1, \pi_2\) descend to finite maps \(\pi_1, \pi_2 : Y_r \rightarrow X_r\) over \(\mathbb{Q}_p.\) Via our identifications in Remarks 2.3.4 and 2.3.18, the map \(\pi_1\) corresponds to the usual “forget \(C\)” map, while \(\pi_2\) corresponds to “quotient by \(C\)” map. We stress that the “standard” degeneracy map \(\rho : X_{r+1} \rightarrow X_r\) factors through \(\pi_2\) (and not \(\pi_1\)).

Proposition 2.3.19. The scheme \(\widetilde{Y}_r := Y_r \times_{T_r} \text{Spec}(\mathbb{F}_p)\) is the disjoint union, with crossings at the supersingular points, of the following proper, smooth \(\mathbb{F}_p\)-curves: for each pair of nonnegative integers \(a, b\) with \(a + b = r + 1\) and for each \(u \in (\mathbb{Z}/p^{\min(a,b)}\mathbb{Z})^\times,\) one copy of

\[
\begin{cases}
\mathcal{I}_{\text{rm}((a,b)} & \text{if } ab \neq 0 \\
\mathcal{I}_g & \text{if } (a, b) = (r + 1, 0) \text{ or } (0, r + 1)
\end{cases}
\]

We will write \(J_{(a,b,u)}\) for the irreducible component of \(\widetilde{Y}_r\) indexed by \((a, b, u),\) and refer to it as the \((a, b, u)\)-component; again, \(J_{(a,b,u)}\) is independent of \(u.\) The proof of Proposition 2.3.19 is a straightforward adaptation of the arguments of [KM85, 13.11.2–13.11.4] (see also [Ulm90, Proposition 8.2]). We recall the correspondence between non-cuspidal points of the \((a, b, u)\)-component and \([\Gamma_0(p^{r+1}); r, r]^{\text{can}}\)-structures on elliptic curves.
Fix an ordinary elliptic curve $E_0$ over an $\mathbf{F}_p$-scheme $S$, and let $(\phi : E_0 \to E_{r+1}, P, Q; \alpha)$ be an element of $\mathcal{O}^1_0(E_0/S)$. As before, there exist unique nonnegative integers $a, b$ with $a + b = r + 1$ and a unique elliptic curve $E/S$ with the property that the cyclic $p^{r+1}$-isogeny $\phi$ factors as

$$E_0 \simeq E^{(p^b)} \xrightarrow{F^a} E^{(p^{r+1})} \xrightarrow{V^b} E^{(p^a)} \simeq E_{r+1}.$$  

First suppose that $ab \neq 0$. Then the point $P \in E^{(p^{b+1})}(S)$ (respectively $Q \in E^{(p^{a+1})}(S)$) is an \([\text{Ig}(p^b)]\) (respectively \([\text{Ig}(p^a)]\)) structure on $E^{(p^b)}$ over $S$. If $a \geq b$, there is a unit $u \in (\mathbf{Z}/p^b\mathbf{Z})^\times$ such that $V^{a-b}(Q) = uP$ in $E^{(p^{b+1})}(S)$, while if $a \leq b$ then there is a unique $u \in (\mathbf{Z}/p^b\mathbf{Z})^\times$ with $uV^{b-a}(P) = Q$ in $E^{(p^{a+1})}(S)$. For $a \geq b$ (respectively $a < b$), and fixed $u$, the data $(E^{(p^b)}, Q; p^{1-b}V^{b-1} \circ \alpha)$ (respectively $(E^{(p^a)}, P; p^{1-a}V^{a-1} \circ \alpha)$) gives an $S$-point of the $(a, b, u)$-component $\text{Ig}_{\max(a,b)}$. If $b = 0$ (respectively $a = 0$), then $Q \in E^{(p^a)}(S)$ (respectively $P \in E^{(p^b)}(S)$) is an \([\text{Ig}(p^a)]\)-structure on $E = E_0$ (respectively $E = E_{r+1}$). In these extremal cases, the data $(E, Q; \alpha)$ (respectively $(E, P; p^{-r-1}V^{r+1} \circ \alpha)$) gives an $S$-point of the $(r + 1, 0, 1)$-component (respectively $(0, r + 1, 1)$-component) $\text{Ig}_r$.

Conversely, suppose given $(a, b, u)$ and an $S$-point of $\text{Ig}_{\max(a,b)}$ which is neither cuspidal nor supersingular. If $r + 1 > a \geq b$ and $(E, Q; \alpha)$ is the given point of $\text{Ig}_a$, then we set $P := uV^{a-b}(Q) \in E^{(p^b)}(S)$, while if $r + 1 > b \geq a$ and $(E, P; \alpha)$ is the given point of $\text{Ig}_b$, we set $Q := uV^{b-a}P \in E^{(p^a)}(S)$. Then

$$(E^{(p^{b-1})} \xrightarrow{F} E^{(p^b)} \xrightarrow{F^{a-1}} E^{(p^a)} \xrightarrow{V^{b-1}} E^{(p^a)} \xrightarrow{V} E^{(p^{a-1})}, P, Q; F^{b-1} \circ \alpha)$$

is an $S$-point of $\mathcal{M}(\mathcal{O}^1_r)$. If $b = 0$ and $(E, Q, \alpha)$ is an $S$-point of $\text{Ig}_r$, then we let $P \in E^{(p)}(S)$ be the identity section and we obtain an $S$-point $(F^{r+1} : E \to E^{(p^{r+1})}, P, Q; \alpha)$ of $\mathcal{M}(\mathcal{O}^1_r)$. If $a = 0$ and $(E, P, \alpha)$ is an $S$-point of $\text{Ig}_r$, then we let $Q \in E^{(p)}(S)$ be the identity section and we obtain an $S$-point $(V^{r+1} : E^{(p^{r+1})} \to E, P, Q; F^{r+1} \circ \alpha)$ of $\mathcal{M}(\mathcal{O}^1_r)$.

Using the descriptions of $\overline{\mathcal{X}}_r$ and $\overline{\mathcal{Y}}_r$ furnished by Propositions 2.3.10 and 2.3.19, we can calculate the restrictions of the degeneracy maps $\overline{\pi}_1, \overline{\pi}_2 : \overline{\mathcal{Y}}_r \to \overline{\mathcal{X}}_r$ to each irreducible component of the special fiber of $\mathcal{Y}_r$. The following is due to Ulmer.\footnote{We warn the reader, however, that Ulmer omits the effect of the degeneracy maps on $[\mu_N]$-structures, so his formulæ are slightly different from ours.} [Ulm90, Proposition 8.3]:

**Proposition 2.3.20.** Let $a, b$ be nonnegative integers with $a + b = r + 1$ and $u \in (\mathbf{Z}/p^{\min(a,b)}\mathbf{Z})^\times$. The restriction of the map $\overline{\pi}_1 : \overline{\mathcal{Y}}_r \to \overline{\mathcal{X}}_r$ to the $(a, b, u)$-component of $\overline{\mathcal{Y}}_r$ is:

$$\begin{cases} 
\text{Ig}_r = J_{(r+1,0,1)} \xrightarrow{\rho} I_{(r,0,1)} = \text{Ig}_r : (a, b, u) = (r + 1, 0, 1) \\
\text{Ig}_a = J_{(a,b,u)} \xrightarrow{\rho} I_{(a-1,b,u)} = \text{Ig}_{a-1} : b < a < r + 1 \\
\text{Ig}_b = J_{(a,b,u)} \xrightarrow{(u^{-1})} I_{(a-1,b,u \mod p^{a-1} = 1)} = \text{Ig}_b : a = b = (r + 1)/(2) \\
\text{Ig}_b = J_{(a,b,u)} \xrightarrow{\text{id}} I_{(a-1,b,u \mod p^{a-1} = 1)} = \text{Ig}_b : a < b < r + 1 \\
\text{Ig}_r = J_{(0,r+1,1)} \xrightarrow{(p)} I_{(0,r,1)} = \text{Ig}_r : (a, b, u) = (0, r + 1, 1) 
\end{cases}$$
and the restriction of the map \( \pi_2 : \overline{Y}_r \to \overline{X}_r \) to the \((a, b, u)\)-component of \( \overline{Y}_r \) is:

\[
\begin{aligned}
\text{Ig}_r &= J_{(r + 1, 0, 1)} \circ \text{id} \circ \pi_{(r, 0, 1)} = \text{Ig}_r : (a, b, u) = (r + 1, 0, 1) \\
\text{Ig}_a &= J_{(a, b, u)} \circ \text{id} \circ \pi_{(a, b - 1, u \text{ mod } p^{b - 1})} = \text{Ig}_a : b < a + 1 \leq r + 1 \\
\text{Ig}_b &= J_{(a, b, u)} \circ \rho \circ \pi_{(a, b - 1, u)} = \text{Ig}_{b - 1} : a + 1 = b = r/2 + 1 \\
\text{Ig}_b &= J_{(a, b, u)} \circ \rho \circ \pi_{(a, b - 1, u)} = \text{Ig}_{b - 1} : a + 1 < b < r + 1 \\
\text{Ig}_r &= J_{(0, r + 1, 1)} \circ F \circ \pi_{(0, r, 1)} = \text{Ig}_r : (a, b, u) = (0, r + 1, 1)
\end{aligned}
\]

Proof. The proof is similar to the proof of Proposition 2.3.13, using the correspondence between irreducible components of \( \overline{Y}_r \), \( \overline{X}_r \) and Igusa curves that we have explained, together with the moduli-theoretic definitions (2.3.9) of the degeneracy mappings. We leave the details to the reader. \( \blacksquare \)

We end this section with a brief discussion of correspondences on curves and their induced action on cohomology and Jacobians, which we then apply to the specific case of modular curves. Fix a ring \( R \) and a proper normal curve \( X \) over \( S = \text{Spec} R \). Throughout this discussion, we assume either that \( R \) is a discrete valuation ring of mixed characteristic \((0, p)\) with perfect residue field, or that \( R \) is a perfect field (and hence the normal \( X \) is smooth).

**Definition 2.3.21.** A correspondence \( T := (\pi_1, \pi_2) \) on \( X \) is an ordered pair \( \pi_1, \pi_2 : Y \to X \) of finite \( S \)-morphisms of normal and \( S \)-proper curves. The transpose of a correspondence \( T := (\pi_1, \pi_2) \) on \( X \) is the correspondence on \( X \) given by the ordered pair \( T^* := (\pi_2, \pi_1) \).

Thanks to Proposition 2.1.11 (4), any correspondence \( T := (\pi_1, \pi_2) \) on \( X \) induces an \( R \)-linear endomorphism of the short exact sequence \( H(X/R) \) via \( \pi_1^* \pi_2^* \). By a slight abuse of notation, we denote this endomorphism by \( T \); as endomorphisms of \( H(X/R) \) we then have

\[
T = \pi_1^* \pi_2^* \quad \text{and} \quad T^* = \pi_2^* \pi_1^*.
\]

Given a finite map \( \pi : X \to X \), we will consistently view \( \pi \) as a correspondence on \( X \) via the association \( \pi \rightsquigarrow (\text{id}, \pi) \). In this way, we may think of correspondences on \( X \) as “generalized endomorphisms.” This perspective can be made more compelling as follows.

First suppose that \( R \) is a field, and fix a correspondence \( T \) given by an ordered pair \( \pi_1, \pi_2 : Y \to X \) of finite morphisms of smooth and proper curves. Then \( T \) and its transpose \( T^* \) induce endomorphisms of the Jacobian \( J_X := \text{Pic}^0_{X/R} \) of \( X \), which we again denote by the same symbols, via

\[
T := \text{Alb}(\pi_2) \circ \text{Pic}^0(\pi_1) \quad \text{and} \quad T^* := \text{Alb}(\pi_1) \circ \text{Pic}^0(\pi_2)
\]

Note that when \( T = (\text{id}, \pi) \) for a morphism \( \pi : X \to X \), the induced endomorphisms (2.3.11) of \( J_X \) are given by \( T = \text{Alb}(\pi) \) and \( T^* := \text{Pic}^0(\pi)^* \).\(^{16}\) Abusing notation, we will simply write \( \pi \) for the endomorphism \( \text{Alb}(\pi) \) of \( J_X \) induced by the correspondence \((1, \pi)\), and \( \pi^* \) for the endomorphism \( \text{Pic}^0(\pi)^* \) induced by \((\pi, 1) = (1, \pi)^*\). When \( \pi : X \to X \) is an automorphism, an easy argument shows that \( \pi^* = \pi^{-1} \) as automorphisms of \( J_X \).

\(^{16}\)Because of this fact, for a general correspondence \( T = (\pi_1, \pi_2) \) the literature often refers to the induced endomorphism \( T \) (respectively \( T^* \)) of \( J_X \) as the Albanese (respectively Picard) or covariant (respectively contravariant) action of the correspondence \((\pi_1, \pi_2)\). Since the definitions (2.3.11) of \( T \) and \( T^* \) both literally involve Albanese and Picard functoriality, we find this old terminology confusing, and eschew it in favor of the consistent notation we have introduced.
With these definitions, the canonical filtration compatible isomorphism $H^1_{dR}(X/R) \cong H^1_{dR}(J_X/R)$ is $T$ (respectively $T^*$)-equivariant with respect to the action (2.3.10) on $H^1_{dR}(X/R)$ and the action on $H^1_{dR}(J_X/R)$ induced by pullback along the endomorphisms (2.3.11); see [Cai10, Proposition 5.4].

Now suppose that $R$ is a discrete valuation ring with fraction field $K$ and fix a correspondence $T$ on $X$ given by a pair of finite morphisms of normal curves $\pi_1, \pi_2 : Y \to X$. Let us write $T_K$ for the induced correspondence on the (smooth) generic fiber $X_K$ of $X$. Via (2.3.11) and the Néron mapping property, $T_K$ and $T^*_K$ induces endomorphisms of the Néron model $J_X$ of the Jacobian of $X_K$, which we simply denote by $T$ and $T^*$, respectively. Thanks to Proposition 2.2.4, the filtration compatible morphism (2.2.5) is $T$- and $T^*$-equivariant for the given action (2.3.10) on $H^1(X/R)$ and the action on $\text{Lie} \mathcal{E}_{\text{trig}}(J_X, G_m)$ induced by (2.3.11) and the (contravariant) functoriality of $\mathcal{E}_{\text{trig}}(\cdot, G_m)$.

\textbf{Remark 2.3.22.} As in Remark 2.2.5, if $X$ is a normal proper curve over $R$ with rational singularities, then any correspondence on $X_K$ induces a filtration compatible endomorphism of $H^1(X/R)$ via its action on $J_{X_K}$, the Néron mapping property, and the isomorphism (2.2.5) of Proposition 2.2.4.

We now specialize this discussion to the case of the modular curve $X_1(Np^r)$ over $Q$. For any prime $\ell$, one defines the Hecke correspondences $T_\ell$ for $\ell \nmid Np$ and $U_\ell$ for $\ell | Np$ on $X_1(Np^r)$ as in [Col94, §8] (cf. also [Gro90, §3] and [MW84, Chapter 2, §5.1–5.8], though be aware that the latter works instead with the modular curves $X_1(Np^r)$ of Remark 2.3.4). If $\ell \neq p$, we have similarly defined correspondences $T_\ell$ and $U_\ell$ on $\text{Ig}_r$ over $F_p$ (see [MW84, Chapter 2, §5.4–5.5]). For $\ell \neq p$, the Hecke correspondences extend to correspondences on $X_r$ over $R_r$, essentially by the same definition, while for $\ell = p$ the correspondence $U_p := (\pi_1, \pi_2)$ on $X_r$ is defined using the maps (2.3.9). We use the same symbols to denote the induced endomorphisms (2.3.11) of the Jacobian $J_1(Np^r)$.

\textbf{Definition 2.3.23.} We write $\mathfrak{H}_r(Z)$ (respectively $\mathfrak{H}_r^*(Z)$) for the $Z$-subalgebra of $\text{End}_Q(J_1(Np^r))$ generated by the Hecke operators $T_\ell$ (respectively $T^*_\ell$) for $\ell \nmid Np$ and $U_\ell$ (respectively $U^*_\ell$) for $\ell | Np$, and the diamond operators $\langle u \rangle$ (respectively $\langle u \rangle^*$) for $u \in Z_p^\times$ and $\langle v \rangle_N$ (respectively $\langle v \rangle^*_N$) for $v \in (Z/NZ)^\times$. For any commutative ring $A$, we set $\mathfrak{H}_r(A) := \mathfrak{H}_r(Z) \otimes_Z A$ and $\mathfrak{H}_r^*(A) := \mathfrak{H}_r^*(Z) \otimes_Z A$, and for ease of notation we set $\mathfrak{H}_r := \mathfrak{H}_r(Z_p)$ and $\mathfrak{H}_r^* := \mathfrak{H}_r^*(Z_p)$.

The relation between the Hecke algebras $\mathfrak{H}_r$ and $\mathfrak{H}_r^*$ is explained by the following:

\textbf{Proposition 2.3.24.} Denote by $w_r$ the automorphism of $(J_r)_{/K_r'}$ induced by the correspondence $(1, w_r)$ on $(X_r)_{/K_r'}$ over $K_r'$. Viewing $\mathfrak{H}_r$ and $\mathfrak{H}_r^*$ as $Z_p$-subalgebras of $\text{End}_{K_r'}((J_r)_{/K_r'}) \otimes_Z Z_p$, conjugation by $w_r$ carries $\mathfrak{H}_r$ isomorphically onto $\mathfrak{H}_r^*$: that is, $w_rT = T^*w_r$ for all Hecke operators $T$.

\textbf{Proof.} This is standard; see, e.g., [Til87, pg. 336], [Oht95, 2.1.8], or [MW84, Chapter 2, §5.6 (c)].

3. Differentials on modular curves in characteristic $p$

We now analyze the “modified de Rham cohomology” (§2.1) of the special fibers of the modular curves $X_r/R_r$, and we relate its ordinary part to the de Rham cohomology of the “Igusa Tower.”

3.1. The Cartier operator. Fix a perfect field $k$ of characteristic $p > 0$ and write $\varphi : k \to k$ for the $p$-power Frobenius map. In this section, we recall the basic theory of the Cartier operator for a smooth and proper curve over $k$. As we will only need the theory in this limited setting, we will content ourselves with a somewhat \textit{ad hoc} formulation of it. Our exposition follows [Ser58, §10], but the reader may consult [Oda69, §5.5] or [Car57] for a more general treatment.

Let $X$ be a smooth and proper curve over $k$ and write $F : X \to X$ for the absolute Frobenius map; it is finite and flat and is a morphism over the endomorphism of $\text{Spec}(k)$ induced by $\varphi$. Let $D$ be an
effective Cartier (=Weil) divisor on $X$ over $k$, and write $\mathcal{O}_X(-D)$ for the coherent (invertible) ideal sheaf determined by $D$. The pullback map $F^\ast : \mathcal{O}_X \to F_\ast \mathcal{O}_X$ carries the ideal sheaf $\mathcal{O}_X(-nD) \subseteq \mathcal{O}_X$ into $F_\ast \mathcal{O}_X(-npD)$, so we obtain a canonical $\varphi$-semilinear pullback map on cohomology

\begin{equation}
F^\ast : H^1(X, \mathcal{O}_X(-nD)) \to H^1(X, \mathcal{O}_X(-npD)).
\end{equation}

By Grothendieck–Serre duality, (3.1.1) gives a $\varphi^{-1}$-semilinear “trace” map\footnote{This map coincides with Grothendieck’s trace morphism on dualizing sheaves attached to the finite map $F$.} of $k$-vector spaces

\begin{equation}
V := F_\ast : H^0(X, \Omega^1_{X/k}(npD)) \to H^0(X, \Omega^1_{X/k}(nD))
\end{equation}

**Proposition 3.1.1.** Let $X/k$ be a smooth and proper curve, $D$ an effective Cartier divisor on $X$, and $n$ a nonnegative integer.

1. There is a unique $\varphi^{-1}$-linear endomorphism $V := F_\ast$ of $H^0(X, \Omega^1_{X/k}(nD))$ which is dual, via Grothendieck-Serre duality, to pullback by absolute Frobenius on $H^1(X, \mathcal{O}_X(-nD))$.
2. The map $V$ “improves poles” in the sense that it factors through the canonical inclusion

$$H^0(X, \Omega^1_{X/k}([\frac{n}{p}]D)) \to H^0(X, \Omega^1_{X/k}(nD)).$$

3. If $\rho : Y \to X$ is any finite morphism of smooth proper curves over $k$, and $\rho^*D$ is the pullback of $D$ to $Y$, then the induced pullback and trace maps

$$H^0(Y, \Omega^1_{Y/k}(n\rho^*D)) \xrightarrow{\rho^\ast} H^0(X, \Omega^1_{X/k}(nD))$$

commute with $V$.

4. Assume that $k$ is algebraically closed. Then for any meromorphic differential $\eta$ on $X$ and any closed point $x$ of $X$, the formula

$$\text{res}_x(V \eta)^p = \text{res}_x(\eta)$$

holds, where $\text{res}_x$ is the canonical “residue at $x$ map” on meromorphic differentials.

**Proof.** Both (1) and (2) follow from our discussion, while (3) follows (via duality) from the fact that the $p$-power map commutes with any ring homomorphism. Finally, (4) follows from the fact that the canonical isomorphism $H^1(X, \Omega^1_{X/k}) \to k$ induced by the residue map coincides with the negative of Grothendieck’s trace isomorphism (cf. Proposition 2.1.15), together with the fact that Grothendieck’s trace morphism is compatible with compositions; see Appendix B and Corollary 3.6.6 of [Con00].

**Remark 3.1.2.** Quite generally, if $\rho : Y \to X$ is any finite morphism of smooth curves over $k$ and $y$ is any $k$-point of $Y$ with $x = \rho(y) \in X(k)$, then for any meromorphic differential $\eta$ on $Y$ we have

\begin{equation}
\text{ord}_x(\rho_\ast \eta) \leq \left\lfloor \frac{\text{ord}_y(\eta)}{e} \right\rfloor
\end{equation}

where $e$ is the ramification index of the extension of discrete valuation rings $\mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$. Indeed, if $\mathcal{I}_x$ and $\mathcal{I}_y$ denote the ideal sheaves of the reduced closed subschemes $x$ and $y$, then the pullback map $\mathcal{O}_X \to \rho_\ast \mathcal{O}_Y$ carries $\mathcal{I}_x^n$ into $\rho_\ast \mathcal{I}_y^{ne}$. Passing to the map on $H^1$’s and using Grothendieck duality, we
deduce that ρ∗ carries $H^0(Y, \Omega^1_{\mathcal{O}/k} \otimes \mathcal{I}_y^{-\neq})$ into $H^0(X, \Omega^1_X \otimes \mathcal{I}_x^{-\neq})$, whence the estimate (3.1.3). If moreover $k$ is algebraically closed, then we have (cf. [Tat68, Theorem 4])

(3.1.4) \[ \text{res}_x(\rho_\eta) = \text{res}_y(\eta). \]

We recall the following (generalization of a) well-known lemma of Fitting:

\textbf{Lemma 3.1.3.} Let $A$ be a commutative ring, ϕ an automorphism of $A$, and $M$ be an $A$-module equipped with a ϕ-semilinear endomorphism $F : M \to M$. Assume that one of the following holds:

(1) $M$ is a finite length $A$-module.
(2) $A$ is a complete noetherian adic ring, with ideal of definition $I \subsetneq A$, and $M$ is a finite $A$-module.

Then there is a unique direct sum decomposition

(3.1.5) \[ M = M^{\text{ord}} \oplus M^{\text{nil}}, \]

where $M^{\text{ord}}$ is the maximal $\varphi$-stable submodule of $M$ on which $F$ is bijective, and $M^{\text{nil}}$ is the maximal $F$-stable submodule of $M$ on which $F$ is (topologically) nilpotent. The assignment $M \rightsquigarrow M^\varphi$ for $\varphi = \text{ord, nil}$ is an exact functor on the category of (left) $A[F]$-modules verifying (1) or (2).

\textbf{Proof.} For the proof in case (1), we refer to [Laz75, VI, 5.7], and just note that one has:

\[ M^{\text{ord}} := \bigcap_{n \geq 0} \text{im}(F^n) \quad \text{and} \quad M^{\text{nil}} := \bigcup_{n \geq 0} \ker(F^n), \]

where one uses that $\varphi$ is an automorphism to know that the image and kernel of $F^n$ are $A$-submodules of $M$. It follows immediately from this that the association $M \rightsquigarrow M^\varphi$ is a functor from the category of left $A[F]$-modules of finite $A$-length to itself. It is an exact functor because the canonical inclusion $M^\varphi_\ast \to M$ is an $A[F]$-direct summand. In case (2), our hypotheses ensure that $M/I^n M$ is a noetherian and Artinian $A$-module, and hence of finite length, for all $n$. Our assertions in this situation then follow immediately from (1), via the uniqueness of (3.1.5), together with fact that $M$ is finite as an $A$-module, and hence $I$-adically complete (as $A$ is). \[ \square \]

We apply 3.1.3 to the $k$-vector space $M := H^0(X, \Omega^1_{X/k})$ equipped with the $\varphi^{-1}$ semilinear map $V$:

\textbf{Definition 3.1.4.} The $k[V]$-module $H^0(X, \Omega^1_{X/k})^V_{\text{ord}}$ is called the $V$-ordinary subspace of holomorphic differentials on $X$. It is the maximal $k$-subspace of $H^0(X, \Omega^1_{X/k})$ on which $V$ is bijective. The nonnegative integer $\gamma_X := \dim_k H^0(X, \Omega^1_{X/k})^V_{\text{ord}}$ is called the Hasse-Witt invariant of $X$.

\textbf{Remark 3.1.5.} Let $D$ be any effective Cartier divisor. Since $V := F_*$ and $F := F^*$ are adjoint under the canonical perfect $k$-pairing between $H^0(X, \Omega^1_{X/k}(D))$ and $H^1(X, \mathcal{O}_X(\mathcal{O}_X(D)))$, this pairing restricts to a perfect duality pairing

(3.1.6) \[ H^0(X, \Omega^1_{X/k}(D))^V_{\text{ord}} \times H^1(X, \mathcal{O}_X(\mathcal{O}_X(D)))^{F_\text{ord}} \to k. \]

In particular (taking $D = 0$) we also have $\gamma_X = \dim_k H^1(X, \mathcal{O}_X)^{F_\text{ord}}$.

The following “control lemma” is a manifestation of the fact that the Cartier operator improves poles (Proposition 3.1.1, (2)):

\textbf{Lemma 3.1.6.} Let $X$ be a smooth and proper curve over $k$ and $D$ an effective Cartier divisor on $X$. Considering $D$ as a closed subscheme of $X$, we write $D_\text{red}$ for associated reduced closed subscheme.
(1) For all positive integers $n$, the canonical morphism
\[ H^0(X, \Omega^1_{X/k}(D_{\text{red}})) \to H^0(X, \Omega^1_{X/k}(nD)) \]
induces a natural isomorphism on $V$-ordinary subspaces.
(2) For each positive integer $n$, the canonical map
\[ H^1(X, \mathcal{O}_X(-nD)) \to H^1(X, \mathcal{O}_X(-D_{\text{red}})) \]
induces a natural isomorphism on $F$-ordinary subspaces.
(3) The identifications in (1) and (2) are canonically $k$-linearly dual, via Remark 3.1.5.

Proof. This follows immediately from Proposition 3.1.1, (2) and Remark 3.1.5.

Now let $\pi : Y \to X$ be a finite branched covering of smooth, proper and geometrically connected curves over $k$ with group $G$ that is a $p$-group. Let $D_X$ be any effective Cartier divisor on $X$ over $k$ with support containing the ramification locus of $\pi$, and put $D_Y = \pi^* D_X$. As in Lemma 3.1.6, denote by $D_{X,\text{red}}$ and $D_{Y,\text{red}}$ the underlying reduced closed subschemes; as $D_{Y,\text{red}}$ is $G$-stable, the $k$-vector spaces $H^0(Y, \Omega^1_{Y/k}(nD_{Y,\text{red}}))$ and $H^1(Y, \mathcal{O}_Y(-nD_{Y,\text{red}}))$ are canonically $k[G]$-modules for any $n \geq 1$.

The following theorem of Nakajima is the key to the proofs of our structure theorems for $\Lambda$-modules:

**Proposition 3.1.7** (Nakajima). Assume that $\pi$ is ramified, let $\gamma_X$ be the Hasse-Witt invariant of $X$ and set $d := \gamma_X - 1 + \deg(D_{X,\text{red}})$. Then for each positive integer $n$:

1. The $k[G]$-module $H^0(Y, \Omega^1_{Y/k}(nD_{Y,\text{red}}))^\text{Vord}$ is free of rank $d$ and independent of $n$.
2. The $k[G]$-module $H^1(Y, \mathcal{O}_Y(-nD_{Y,\text{red}}))^{\text{Ford}}$ is naturally isomorphic to the contragredient of $H^0(Y, \Omega^1_{Y/k}(nD_{Y,\text{red}}))^\text{Vord}$; as such, it is $k[G]$-free of rank $d$ and independent of $n$.

Proof. The independence of $n$ is simply Lemma 3.1.6; using this, the first assertion is then equivalent to Theorem 1 of [Nak85]. The second assertion is immediate from Remark 3.1.5, once one notes that for $g \in G$ one has the identity $g_* = (g^{-1})^*$ on cohomology (since $g_* g^* = \deg g = \text{id}$), so $g^*$ and $(g^{-1})^*$ are adjoint under the duality pairing (3.1.6).

We end this section with a brief explanation of the relation between the de Rham cohomology of $X$ over $k$ and the Dieudonné module of the $p$-divisible group of the Jacobian of $X$. This will allow us to give an alternate description of the $V$-ordinary (respectively $F$-ordinary) subspace of $H^0(X, \Omega^1_{X/k})$ (respectively $H^1(X, \mathcal{O}_X)$) which will be instrumental in our applications.

Pullback by the absolute Frobenius gives a semilinear endomorphism of the Hodge filtration $H(X/k)$ of $H^1_{\text{dR}}(X/k)$ which we again denote by $F = F^*$. Under the canonical autoduality of $H(X/k)$ provided by Proposition 2.1.12 (2), we obtain $\varphi^{-1}$-semilinear endomorphism

\[
\begin{align*}
V := F_* : H^1_{\text{dR}}(X/k) &\to H^1_{\text{dR}}(X/k) \\
\end{align*}
\]

whose restriction to $H^0(X, \Omega^1_{X/k})$ coincides with (3.1.2). Let $A$ be the “Dieudonné ring”, i.e. the (noncommutative if $k \neq \mathbb{F}_p$) ring $A := W(k)[F, V]$, where $F, V$ satisfy $FV = VF = p, F\alpha = \varphi(\alpha) F$, and $V\alpha = \varphi^{-1}(\alpha) V$ for all $\alpha \in W(k)$. We view $H^1_{\text{dR}}(X/k)$ as a left $A$-module in the obvious way.

**Proposition 3.1.8** (Oda). Let $J := \text{Pic}^0_{X/k}$ be the Jacobian of $X$ over $k$ and $G := J[p^\infty]$ its $p$-divisible group. Denote by $\mathbf{D}(G)$ the contravariant Dieudonné crystal of $G$, so the Dieudonné module $\mathbf{D}(G)_W$ is naturally a left $A$-module, finite and free over $W := W(k)$. 


(1) There are canonical isomorphisms of left $A$-modules

$$H^1_{\text{dR}}(X/k) \simeq D(J)_k \simeq D(G)_k.$$  

(2) For any finite morphism $\rho : Y \to X$ of smooth and proper curves over $k$, the identification of (1) intertwines $\rho_*$ with $D(\text{Pic}^0(\rho))$ and $\rho^*$ with $D(\text{Alb}(\rho))$.

(3) Let $G = G^{\text{et}} \times G^m \times G^{\text{nil}}$ be the canonical direct product decomposition of $G$ into its maximal étale, multiplicative, and local-local subgroups. Via the identification of (1), the canonical mappings in the exact sequence $H(X/k)$ induce natural isomorphisms of left $A$-modules

$$H^0(X, \Omega^1_{X/k} \otimes_{\kappa} \kappa) \simeq D(G^m)_k \quad \text{and} \quad H^1(X, \mathcal{O}_X) \otimes_{\kappa} \kappa \simeq D(G^{\text{et}})_k$$

(4) The isomorphisms of (3) are dual to each other, using the duality pairing of Remark 3.1.5 together with the canonical isomorphism $D(G^i) \simeq D(G^i)_k$ and the autoduality of $G$ resulting from the autoduality of $J$.

Proof. Using the characterizing properties of the Cartier operator defined by Oda [Oda69, Definition 5.5] and the explicit description of the autoduality of $H^1_{\text{dR}}(X/k)$ in terms of cup-product and residues, one checks that the endomorphism of $H^1_{\text{dR}}(X/k)$ in [Oda69, Definition 5.6] is adjoint to $F^*$, and therefore coincides with the endomorphism $V := F_*$ in (3.1.7); cf. the proof of [Ser58, Proposition 9].

We recall that one has a canonical isomorphism

$$(3.1.8) \quad H^1_{\text{dR}}(X/k) \simeq H^1_{\text{dR}}(J/k)$$

which is compatible with Hodge filtrations and duality (using the canonical principal polarization to identify $J$ with its dual) and which, for any finite morphism of smooth curves $\rho : Y \to X$ over $k$, intertwines $\rho_*$ with $\text{Pic}^0(\rho)^*$ and $\rho^*$ with $\text{Alb}(\rho)^*$; see [Cai10, Proposition 5.4], noting that the proof given there works over any field $k$, and cf. Proposition 2.2.4. It follows from these compatibilities and the fact that the Cartier operator as defined in [Oda69, Definition 5.5] is functorial that the identification (3.1.8) is moreover an isomorphism of left $A$-modules, with the $A$-structure on $H^1_{\text{dR}}(J/k)$ defined as in [Oda69, Definition 5.8].

Now by [Oda69, Corollary 5.11] and [BBM82, Theorem 4.2.14], for any abelian variety $B$ over $k$, there is a canonical isomorphism of left $A$-modules

$$(3.1.9) \quad H^1_{\text{dR}}(B/k) \simeq D(B)_k$$

Using the definition of this isomorphism in Proposition 4.2 and Theorem 5.10 of [Oda69], it is straightforward (albeit tedious\footnote{Alternately, one could appeal to [MM74], specifically to Chapter I, 4.1.7, 4.2.1, 3.2.3, 2.6.7 and to Chapter II, §13 and §15 (see especially Chapter II, 13.4 and 1.6). See also §2.5 and §4 of [BBM82].}) to check that for any homomorphism $h : B' \to B$ of abelian varieties over $k$, the identification (3.1.9) intertwines $h^*$ and $D(h)$. Combining (3.1.8) and (3.1.9) yields (1) and (2).

Now since $V = F_*$ (respectively $F = F^*$) is the zero endomorphism of $H^1(X, \mathcal{O}_X)$ (respectively $H^0(X, \mathcal{O}_X)$), the canonical mapping

$$H^0(X, \Omega^1_{X/k}) \longrightarrow H^1_{\text{dR}}(X/k) \simeq D(G)_k \quad \text{respectively} \quad D(G)_k \simeq H^1_{\text{dR}}(X/k) \longrightarrow H^1(X, \mathcal{O}_X)$$

induces an isomorphism on $V$-ordinary (respectively $F$-ordinary) subspaces. On the other hand, by Dieudonné theory one knows that for any $p$-divisible group $H$, the semilinear endomorphism $V$
(respectively $F$) of $D(H)_W$ is bijective if and only if $H$ is of multiplicative type (respectively étale). The (functorial) decomposition $G = G^\text{et} \times G^m \times G^\text{II}$ yields a natural isomorphism of left $A$-modules

$$D(G)_W \simeq D(G^\text{et})_W \oplus D(G^m)_W \oplus D(G^\text{II})_W,$$

and it follows that the natural maps $D(G^m)_W \to D(G)_W$, $D(G)_W \to D(G^\text{et})_W$ induce isomorphisms

$$(3.1.10) \quad D(G^m)_W \simeq D(G)^{\text{ord}} \quad \text{and} \quad D(G)^{\text{ord}} \simeq D(G^\text{et})_W,$$

respectively, which gives (3). Finally, (4) follows from Proposition 5.3.13 and the proof of Theorem 5.1.8 in [BBM82], using Proposition 2.5.8 of op. cit. and the compatibility of the isomorphism (3.1.8) with duality (for which see [Col98, Theorem 5.1] and cf. [Cai10, Lemma 5.5]).

### 3.2. The Igusa tower.

We apply Proposition 3.1.7 to the Igusa tower (Definition 2.3.9). The canonical degeneracy map $\rho : I_r \to I_1$ defined by (2.3.7) is finite étale outside$^{19} s \equiv s \mod{ss}$ and totally (wildly) ramified over $ss_{s1}$, and so makes $I_r$ in to a branched cover of $I_1$ with group $\Delta/\Delta_r$. The cohomology groups $H^0(I_r, \Omega^1_{I_r/F_p(ss)})$ and $H^1(I_r, \partial I_r(-ss))$ are therefore naturally $F_p[\Delta/\Delta_r]$-modules.

**Proposition 3.2.1.** Let $r$ be a positive integer, write $\gamma$ for the $p$-rank of $J_1(N)_{F_p}$, and set $\delta := \deg ss$. Then

1. The $F_p[\Delta/\Delta_r]$-modules $H^0(I_r, \Omega^1_{I_r/F_p(ss)})^{\text{ord}}$ and $H^1(I_r, \partial I_r(-ss))^{\text{ord}}$ are both free of rank $d := \gamma + \delta - 1$. Each is canonically isomorphic to the contragredient of the other.

2. For any positive integer $s \leq r$, the canonical trace mapping associated to $\rho : I_r \to I_s$ induces natural isomorphisms of $F_p[\Delta/\Delta_s]$-modules

$$\rho_* : H^0(I_r, \Omega^1_{I_r/F_p(ss)})^{\text{ord}} \otimes_{F_p[\Delta/\Delta_r]} F_p[\Delta/\Delta_s] \xrightarrow{\simeq} H^0(I_s, \Omega^1_{I_s/F_p(ss)})^{\text{ord}}$$

$$\rho_* : H^1(I_r, \partial I_r(-ss))^{\text{ord}} \otimes_{F_p[\Delta/\Delta_r]} F_p[\Delta/\Delta_s] \xrightarrow{\simeq} H^1(I_s, \partial I_s(-ss))^{\text{ord}}$$

**Remark 3.2.2.** Using the moduli interpretation of $I_r$ and calculations on formal groups of universal elliptic curves, one can show [KM85, Lemma 12.9.3] that pullback induces a canonical identification

$$\rho^* \Omega^1_{I_s/k} = \Omega^1_{I_r/k}(p^{r-1}(p^r - p^s) \cdot ss).$$

If $n$ is any positive integer, it follows easily from this that $\rho^*$ identifies $H^0(I_s, \Omega^1_{I_s/k}(n \cdot ss))$ with the $\Delta_s/\Delta_r$-invariant subspace of $H^0(I_r, \Omega^1_{I_r/k}(-N_{r,s}(n) \cdot ss))$, for $N_{r,s}(n) = p^{r-1}(p^r - p^s) - p^r - n$. In particular, via pullback, $H^0(I_1, \Omega^1_{I_1/k}(p^r - p))$ is canonically identified with the $\Delta/\Delta_r$-invariant subspace of $H^0(I_r, \Omega^1_{I_r/k})$, so the $k$-dimension of this subspace grows exponentially with $r$. In this light, it is remarkable that the $V$-ordinary subspace has controlled growth. We will not use these facts in what follows, though see Remark 3.2.4.

In order to prove Proposition 3.2.1, we require the following Lemma (cf. [MW83, p. 511]):

**Lemma 3.2.3.** Let $\tau : Y \to X$ be a finite flat and generically étale morphism of smooth and geometrically irreducible curves over a field $k$. If there is a geometric point of $X$ over which $\tau$ is totally ramified then the induced map of $k$-group schemes $\operatorname{Pic}(\tau) : \operatorname{Pic}_X/k \to \operatorname{Pic}_Y/k$ has trivial scheme-theoretic kernel.

---

$^{19}$We will frequently write simply $ss$ for the divisor $ss_{s1}$ on $I_r$ when $r$ is clear from context.
Proof. The hypotheses and the conclusion are preserved under extension of $k$, so we may assume that $k$ is algebraically closed. We fix a $k$-point $x ∈ X(k)$ over which $π$ is totally ramified, and let $y ∈ Y(k)$ be the unique $k$-point of $Y$ over $x$. To prove that $\operatorname{Pic}_{X/k} → \operatorname{Pic}_{Y/k}$ has trivial kernel, it suffices to prove that the map of groups $π^*_R : \operatorname{Pic}(X_R) → \operatorname{Pic}(Y_R)$ is injective for every Artin local $k$-algebra $R$. We fix such a $k$-algebra, and denote by $x_R ∈ X(R)$ and $y_R ∈ Y(R)$ the points obtained from $x$ and $y$ by base change. Let $\mathcal{L}$ be a line bundle on $X_R$ whose pullback to $Y_R$ is trivial; our claim is that we may choose a trivialization $π^* \mathcal{L} ≅ \mathcal{O}_{Y_R}$ of $π^* \mathcal{L}$ over $Y_R$ which descends to $X_R$. In other words, by descent theory, we assert that we may choose a trivialization of $π^* \mathcal{L}$ with the property that the two pullback trivializations under the canonical projection maps

$$Y_R ×_{X_R} Y_R \xrightarrow{\rho_1} \rho_2 \xrightarrow{Y_R}$$

coincide.

We first claim that the $k$-scheme $Z := Y ×_X Y$ is connected and generically reduced. Since $π$ is totally ramified over $x$, there is a unique geometric point $(y, y)$ of $Z$ mapping to $x$ under the canonical map $Z → X$. Since this map is moreover finite flat (because $π : Y → X$ is finite flat due to smoothness of $X$ and $Y$), every connected component of $Z$ is finite flat onto $X$ and so passes through $(y, y)$. Thus, $Z$ is connected. On the other hand, $π : Y → X$ is generically étale by hypothesis, so there exists a dense open subscheme $U ⊆ X$ over which $π$ is étale. Then $Z ×_X U$ is étale—and hence smooth—over $U$ and the open immersion $Z ×_X U → Z$ is schematically dense as $U → X$ is schematically dense and $π$ is finite and flat. As $Z$ thus contains a $k$-smooth and dense subscheme, it is generically reduced.

Fix a choice $e$ of $R$-basis of the fiber $\mathcal{L}(x_R)$ of $\mathcal{L}$ at $x_R$. As any two trivializations of $π^* \mathcal{L}$ over $Y_R$ differ by an element of $R^*$, we may uniquely choose a trivialization which on $x_R$-fibers

$$\mathcal{L}(x_R) ≅ π^* \mathcal{L}(y_R) \xrightarrow{\sim} \mathcal{O}_{Y_R}(y_R) ≅ R$$

carries $e$ to 1. The obstruction to the two pullback trivializations under (3.2.2) being equal is a global unit on $Y_R ×_{X_R} Y_R$. But since $Y_R ×_{X_R} Y_R = (Y ×_X Y)_R$, we have by flat base change

$$H^0(Y_R ×_{X_R} Y_R, \mathcal{O}_{Y_R ×_{X_R} Y_R}) = H^0(Y ×_X Y, \mathcal{O}_{Y ×_X Y}) ⊗_k R = R$$

where the last equality rests on the fact that $Y ×_X Y$ is connected, generically reduced, and proper over $k$. Thus, the obstruction to the two pullback trivializations being equal is an element of $R^*$, whose value may be calculated at any point of $Y_R ×_{X_R} Y_R$. By our choice (3.2.3) of trivialization of $π^* \mathcal{L}$, the value of this obstruction at the point $(y_R, y_R)$ is 1, and hence the two pullback trivializations coincide as desired. 

Proof of Proposition 3.2.1. Since $ρ : I_r → I_s$ is a finite branched cover with group $Δ_s/Δ_r$ and totally wildly ramified over $\mathfrak{ss}_s$, we may apply Proposition 3.1.7, which gives (1).

To prove (2), we work over $k := \overline{\mathbb{F}}_p$ and argue as follows. Since $ρ : I_r → I_s$ is of degree $p^{r−s}$ and totally ramified over $\mathfrak{ss}_s$, we have $ρ^* \mathfrak{ss}_s = p^{r−s} · \mathfrak{ss}$; it follows that pullback induces a map

$$H^1(I_s, \mathcal{O}_{I_s}(−\mathfrak{ss})) \xrightarrow{ρ^*} H^1(I_r, \mathcal{O}_{I_r}(−\mathfrak{ss}))$$

which we claim is injective. To see this, we observe that the long exact cohomology sequence attached to the short exact sequence of sheaves on $I_r$

$$0 → \mathcal{O}_{I_r}(−\mathfrak{ss}) → \mathcal{O}_{I_r} → \mathcal{O}_{\mathfrak{ss}} → 0$$

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$$0 → \mathcal{O}_{I_r}(−\mathfrak{ss}) → \mathcal{O}_{I_r} → \mathcal{O}_{\mathfrak{ss}} → 0$$

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$$H^1(I_s, \mathcal{O}_{I_s}(−\mathfrak{ss})) \xrightarrow{ρ^*} H^1(I_r, \mathcal{O}_{I_r}(−\mathfrak{ss}))$$

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$$0 → \mathcal{O}_{I_r}(−\mathfrak{ss}) → \mathcal{O}_{I_r} → \mathcal{O}_{\mathfrak{ss}} → 0$$

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$$H^1(I_s, \mathcal{O}_{I_s}(−\mathfrak{ss})) \xrightarrow{ρ^*} H^1(I_r, \mathcal{O}_{I_r}(−\mathfrak{ss}))$$

which we claim is injective. To see this, we observe that the long exact cohomology sequence attached to the short exact sequence of sheaves on $I_r$

$$0 → \mathcal{O}_{I_r}(−\mathfrak{ss}) → \mathcal{O}_{I_r} → \mathcal{O}_{\mathfrak{ss}} → 0$$
(with $\mathcal{O}_{\text{ss}}$ a skyscraper sheaf supported on $\text{ss}$) yields a commutative diagram with exact rows

$$
\begin{array}{c}
0 \to H^0(I_s, \mathcal{O}_{I_s}) \to H^0(I_s, \mathcal{O}_{\text{ss}}) \to H^1(I_s, \mathcal{O}_{I_s}(-\text{ss})) \to H^1(I_s, \mathcal{O}_{I_s}) \to 0 \\
0 \to H^0(I_r, \mathcal{O}_{I_r}) \to H^0(I_r, \mathcal{O}_{\text{ss}}) \to H^1(I_r, \mathcal{O}_{I_r}(-\text{ss})) \to H^1(I_r, \mathcal{O}_{I_r}) \to 0
\end{array}
$$

(3.2.5)

The left-most vertical arrow are is an isomorphism because $I_r$ is geometrically connected for all $r$. Since $\text{ss}$ is reduced, we have $H^0(I_r, \mathcal{O}_{\text{ss}}) = k^{\deg \text{ss}}$ for all $r$, so since $\rho : I_r \to I_s$ totally ramifies over every supersingular point, the second left-most vertical arrow in (3.2.5) is also an isomorphism. Now the rightmost vertical map in (3.2.5) is identified with the map on Lie algebras $\text{Lie} \text{Pic}^0_{I_s/k} \to \text{Lie} \text{Pic}^0_{I_r/k}$ induced by $\text{Pic}^0(\rho)$, which is injective thanks to Lemma 3.2.3 and the left-exactness of the functor $\text{Lie}$. An easy diagram chase using (3.2.5) then shows that (3.2.4) is injective, as claimed.

Using again the equality $\rho^* (\mathcal{O}_{\text{ss}}) = p^{-s} \cdot \text{ss}$, pullback of meromorphic differentials yields a mapping

$$
H^0(I_s, \Omega^1_{\text{ss}/k} \mathcal{O}_{\text{ss}}) \longrightarrow H^0(I_r, \Omega^1_{\text{ss}/k} (p^{-s} \cdot \text{ss}))
$$

which is injective since $\rho : I_r \to I_s$ is separable.

Dualizing the injective maps (3.2.4) and (3.2.6), we see that the canonical trace mappings

$$
H^0(I_r, \Omega^1_{I_r/k}) \to H^0(I_s, \Omega^1_{I_s/k})\quad (3.2.7a)
$$

$$
H^1(I_r, \mathcal{O}_{I_r}(-p^{-s} \cdot \text{ss})) \to H^1(I_s, \mathcal{O}_{I_s}(-\text{ss}))\quad (3.2.7b)
$$

are surjective for all $r \geq s \geq 1$. Passing to $V$- (respectively $F$-) ordinary parts and using Lemma 3.1.6 (1), we conclude that the canonical trace mappings attached to $I_r \to I_s$ induce surjective maps as in Proposition 3.2.1 (2). By (1), these mappings are then surjective mappings of free $F_p [\Delta/\Delta_s]$-modules of the same rank, and are hence isomorphisms.

**Remark 3.2.4.** If $G$ is any cyclic group of $p$-power order, then the representation theory of $G$ is rather easy, even over a field $k$ of characteristic $p$. Denoting by $\gamma$ any fixed generator of $G$, for each integer $d$ with $1 \leq d \leq \#G$, there is a unique indecomposable representation of $G$ of dimension $d$, given explicitly by the $k[G]$-module $V_d := k[G]/(\gamma - 1)^d$. By using Artin-Schreier theory for a $G$-cover of proper smooth curves $Y \to X$ over $k$, for any $G$-stable Cartier divisor $D$ on $Y$ it is possible to determine the multiplicity of $V_d$ in the $k[G]$-module $H^0(Y, \Omega^1_{Y/k}(D))$ purely in terms of the ramification data of $Y \to X$. This is carried out for $D = \emptyset$ in [VM81]. For the $G := \Delta/\Delta_s$-cover $I_r \to I_1$, one finds

$$
H^0(I_r, \Omega^1_{I_r/k}) \simeq k[G]^{\text{g}(I_1)} \oplus \left( k[G]/(\gamma - 1)^{p^{-1}-1} \right)^{p(\deg \text{ss})} \oplus \bigoplus_{d=1}^{p^{-1}-2} \left( k[G]/(\gamma - 1)^d \right)^{p(\deg \text{ss})}
$$

as $k[G]$-modules, where $g(I_1)$ is the genus of $I_1$.

The space of meromorphic differentials $H^0(I_1, \Omega^1_{I_1/k} \mathcal{O}_{\text{ss}})$ has a natural action of $F_p^\times$ via the diamond operators $(\cdot)$, and the eigenspaces for this action are intimately connected with mod $p$ cusp forms:

**Proposition 3.2.5.** Let $S_k(N; F_p)$ be the space of weight $k$ cuspforms for $\Gamma_1(N)$ over $F_p$, and denote by $H^0(I_r, \Omega^1_{I_1/k} \mathcal{O}_{\text{ss}})(k-2)$ the subspace of $H^0(I_r, \Omega^1_{I_1/k} \mathcal{O}_{\text{ss}})$ on which $F_p^\times$ acts through the character
(u) \mapsto u^{k-2}. For each k with 2 < k < p + 1, there are canonical isomorphisms of $F_p$-vector spaces

\[ S_k(N; F_p) \simeq H^0(I_1, \Omega^1_{I_1/F_p})(k-2) \simeq H^0(I_1, \Omega^1_{I_1/F_p}(ss))(k-2) \]

which are equivariant for the Hecke operators, with $U_p$ acting as usual on modular forms and as the Cartier operator $V$ on differential forms. For $k = 2, p+1$, we have the following commutative diagram:

\[
\begin{array}{ccc}
S_2(N; F_p) & \xrightarrow{\simeq} & H^0(I_1, \Omega^1_{I_1/F_p})(0) \\
A & \downarrow & \\
S_{p+1}(N; F_p) & \xrightarrow{\simeq} & H^0(I_1, \Omega^1_{I_1/F_p}(ss))(0)
\end{array}
\]

where $A$ is the Hasse invariant.

**Proof.** This follows from Propositions 5.7–5.10 of [Gro90], using Lemma 3.3.5; we note that our forward reference to Lemma 3.3.5 does not result in circular reasoning. \[ \square \]

**Remark 3.2.6.** For each $k$ with $2 \leq k \leq p + 1$, let us write $d_k := \dim_{F_p} S_k(N; F_p)^{\text{ord}}$ for the $F_p$-dimension of the subspace of weight $k$ level $N$ cuspforms over $F_p$ on which $U_p$ acts invertibly. As in Proposition 3.2.1 (1), let $\gamma$ be the $p$-rank of the Jacobian of $X_1(N)_{F_p}$ and $\delta := \deg_{ss}$. It follows immediately from Proposition 3.2.5 that we have the equality

\[ d := \gamma + \delta - 1 = \sum_{k=3}^{p+1} d_k. \]

### 3.3. Structure of the ordinary part of $H^0(\overline{X}_r, \omega_{\overline{X}_r/F_p})$

Keep the notation of §3.2 and let $\mathcal{X}_r/R_r$ be as in Definition 2.3.3. As before, we denote by $\overline{X}_r := \mathcal{X}_r \times_{R_r} F_p$ the special fiber of $\mathcal{X}_r$; it is a curve over $F_p$ in the sense of Definition 2.1.1. In this section, using Rosenlicht’s theory of the dualizing sheaf as explained in §2.1 and the explicit description of $\overline{X}_r$ given by Proposition 2.3.10, we will compute the ordinary part of the cohomology $H(\overline{X}_r/F_p)$ in terms of the de Rham cohomology of the Igusa tower.

For notational ease, as in Remark 2.3.12 we write $I^{\infty}_r := I_{(r,0,1)}$ and $I^0 := I_{(0,r,1)}$ for the two “good” components of $\overline{X}_r$. Each of these components is abstractly isomorphic to the Igusa curve $\operatorname{Ig}(p')$ of level $p'$ over $X_1(N)_{F_p}$, and we will henceforth make this identification; for $s \leq r$, we will write simply $\rho : I^*_r \rightarrow I^*_s$ for the canonical degeneracy map induced by (2.3.7). Using Proposition 2.3.20, one checks that the $\delta_r$-correspondences on $\mathcal{X}_r$ restrict to the $\delta_r^*$-correspondences on $I^{\infty}_r$, (the point is that the degeneracy maps defining $U_p$ on $\mathcal{X}_r$ restrict to a correspondence on $I^{\infty}_r$), while the $\delta_r^*$-correspondences on $\mathcal{X}_r$ restrict to the $\delta_r^*$-correspondences on $I^0$. In particular, $U_p = (F, (p)_{N})$ on $I^{\infty}_r$ and $U^*_p = (F, \text{id})$ on $I^0_r$. For $s = 0, \infty$, we denote by $i^*_s : I^{\infty}_r \hookrightarrow \overline{X}_r$ the canonical closed immersion.

**Proposition 3.3.1.** For each positive integer $r$, pullback of differentials along $i^0_r$ (respectively $i^\infty_r$) induces a natural and $S^*_r$ (resp. $\delta^*_r$)-equivariant isomorphism of $F_p[\Delta/\Delta_r]$-modules

\[ e^*_r H^0(\overline{X}_r, \omega_{\overline{X}_r}) \xrightarrow{\simeq} H^0(I^0_r, \Omega^1_{I^0_r(ss)})^{\text{ord}}, \quad \text{resp.} \quad e^*_r H^0(\overline{X}_r, \omega_{\overline{X}_r}) \xrightarrow{\simeq} H^0(I^{\infty}_r, \Omega^1_{I^{\infty}_r(ss)})^{\text{ord}}. \]

which is $\Gamma$-equivariant for the “geometric inertia action” (2.3.3) on $\overline{X}_r$ and the action $\gamma \mapsto (\chi(\gamma))^{-1}$ on $I^0_r$ (respectively the trivial action on $I^{\infty}_r$). The isomorphisms (3.3.1) induce identifications that are
compatible with change in $r$: the four diagrams formed by taking the interior or the exterior arrows

\begin{equation}
\begin{array}{c}
e_r^* H^0(\overline{X}_r, \omega_{\overline{X}_r}) \xrightarrow{F_2'(\eta)^*} H^0(I^0_{\overline{X}_r}, \Omega^1_{I^0_{\overline{X}_r}(ss)}) \underset{V_{\text{ord}}}{\longrightarrow} e_r H^0(\overline{X}_r, \omega_{\overline{X}_r}) \xrightarrow{F_2'(\eta)^*} H^0(I^\infty_{\overline{X}_r}, \Omega^1_{I^\infty_{\overline{X}_r}(ss)}) \underset{V_{\text{ord}}}{\longrightarrow} e_r^* H^0(\overline{X}_r, \omega_{\overline{X}_r}) \xrightarrow{(p_2)^*} H^0(I^0_{\overline{X}_r}, \Omega^1_{I^0_{\overline{X}_r}(ss)}) \underset{V_{\text{ord}}}{\longrightarrow}
\end{array}
\end{equation}

(3.3.2) 

\begin{align*}
\rho^* & \quad \sigma^* & \quad \rho^* & \quad \rho^* \\
\sigma^* & \quad \rho^* & \quad \rho^* & \quad \rho^*
\end{align*}

and are all commutative for $s \leq r$. Via the automorphism $\overline{w}_r$ of $\overline{X}_r$ and the identification $I^0_r \simeq I^\infty_r$, the first diagram of (3.3.2) is carried isomorphically and compatibly on to the second. The same assertions hold true if we replace $\overline{X}_r$ with $\overline{X}^n_r$ and $\Omega^1_{I^0_r}$ with $\Omega^1_{I^\infty_r}$ throughout.

**Proof.** We may and do work over $k := \overline{\mathbb{F}}_p$, and we abuse notation slightly by writing $\overline{X}_r$ for the geometric special fiber of $X_r$. If $X$ is an $\mathbb{F}_p$-scheme, we likewise again write $X$ its base change to $k$, and we write $F : X \to X$ for the base change of the absolute Frobenius of $X$ over $\mathbb{F}_p$ to $k$. Let $\overline{X}_r \to \overline{X}_r$ be the normalization map; by Proposition 2.3.10, we know that $\overline{X}_r$ is the disjoint union of proper smooth and irreducible Igusa curves $I_{(a,b,u)}$ indexed by triples $(a, b, u)$ with $a, b \neq 0$ nonnegative integers satisfying $a + b = r$ and $u \in (\mathbb{Z}/p\mathbb{N}(a,b)\mathbb{Z})^\times$. Via Proposition 2.1.15, we identify $\omega_{\overline{X}_r/k}$ with Rosenlicht’s sheaf of regular differentials, and we simply write $\omega_{\overline{X}_r}$ for this sheaf. By Definition 2.1.13 and Remark 2.1.14, we have a functorial injection of $k$-vector spaces

\begin{equation}
H^0(\overline{X}_r, \omega_{\overline{X}_r})^\vee \longrightarrow H^0(\overline{X}^n_r, \Omega^1_{k(\overline{X}_r)}) \simeq \prod_{(a,b,u)} \Omega^1_{k(I_{(a,b,u)})}
\end{equation}

(3.3.3)

with image precisely those elements $(\eta_{(a,b,u)})$ of the product that satisfy $\sum \text{res}_{x_{(a,b,u)}}(s_\eta_{(a,b,u)}) = 0$ for each supersingular point $x \in X_r(k)$ and all $s \in \mathcal{O}_{X_r,x}$, where $x_{(a,b,u)}$ is the unique point of $I_{(a,b,u)}$ lying over $x$ and the sum is over all (a, b) as above. We henceforth identify $\eta \in H^0(\overline{X}_r, \omega_{\overline{X}_r})$ with its image under (3.3.3), and we denote by $\eta_{(a,b,u)}$ the $(a, b, u)$-component of $\eta$.

Recall from (2.3.10) that the correspondence $U_p := (\pi_1, \pi_2)$ on $X_r$ given by the degeneracy maps $\pi_1, \pi_2 : y_r \to X_r$ of (2.3.9) yields endomorphisms $U_p := (\pi_1)_* \circ \pi_2^*$ and $U_p^* := (\pi_2)_* \circ \pi_1^*$ of $H^0(X_r, \omega_{X_r/r_e})$; we will again denote by $U_p$ and $U_p^*$ the induced endomorphisms $U_p \otimes 1$ and $U_p^* \otimes 1$ of

$$H^0(\overline{X}_r, \omega_{\overline{X}_r}) \simeq H^0(X_r, \omega_{X_r/r_e}) \otimes_{R_e} k,$$

where the isomorphism is the canonical one of Lemma 2.1.16 (1). By the functoriality of normalization, we have an induced correspondence $U_p := (\pi_1, \pi_2)$ on $\overline{X}_r^n$, and we write $U_p$ and $U_p^*$ for the resulting endomorphisms (2.3.10) of $H^0(\overline{X}_r^n, \Omega^1_{k(\overline{X}_r^n)})$. By Lemma 2.1.16 (2), the map (3.3.3) is then $U_p$ and $U_p^*$ equivariant. The Hecke correspondences away from $p$ and the diamond operators act on the source of (3.3.3) via “reduction modulo $p$” and on the target via the induced correspondences in the usual way (2.3.10), and the map (3.3.3) compatible with these actions thanks to Lemma 2.1.16 (2). Similarly, the semilinear “geometric inertia” action of $\Gamma := \text{Gal}(K_\infty/K_0)$ on $X_r$ induces a linear action on $\overline{X}_r^n$ as in Proposition 2.3.14 (2.1.14), and the map (3.3.3) is equivariant with respect to these actions.
We claim that for any meromorphic differential \( \eta = (\eta_{(a,b,u)}) \) on \( \mathcal{X}_r^n \), we have

\[
(U_p \eta)_{(a,b,u)} = \begin{cases}
F_* \eta_{(r,0,1)} : (a, b, u) = (r, 0, 1) \\
\rho_* \eta_{(a+1,b-1,u)} : 0 < b < a \\
\sum_{u' \in (\mathbb{Z}/p^a \mathbb{Z})^r} \langle u' \rangle \eta_{(a+1,b-1,u')} : r \text{ odd, } a = b - 1 \\
\sum_{u' \in (\mathbb{Z}/p^a \mathbb{Z})^r} \rho^* (u') \eta_{(a+1,b-1,u')} : r \text{ even, } a = b - 1 \\
\sum_{u' \equiv u \mod p^b} \rho^* \eta_{(a+1,b-1,u')} : 0 < a < b - 2 \\
\sum_{u' \equiv u \mod p^b} \rho^* \eta_{(a+1,b-1,u')} : 0 < a < b - 2
\end{cases}
\]

The proof of this claim is an easy exercise using the definition of \( U_p \), the explicit description of the maps \( \pi_1^n \) and \( \pi_2^n \) given in Proposition 2.3.20, and the fact that \( F^* \) kills any global meromorphic differential form on a scheme of characteristic \( p \). In a similar manner, one derives the explicit description

\[
(U_p^* \eta)_{(a,b,u)} = \begin{cases}
\langle p \rangle_N^{-1} F_* \eta_{(0,r,1)} : (a, b, u) = (0, r, 1) \\
\rho_* \eta_{(a-1,b+1,u)} : 0 < a < b \\
\langle u' \rangle^{-1} \rho_* \eta_{(a-1,b+1,u') : r \text{ even, } b = a} \\
\sum_{u' \equiv u \mod p^b} \rho^* \eta_{(a-1,b+1,u')} : r \text{ odd, } b = a - 1 \\
\sum_{u' \equiv u \mod p^b} \rho^* \eta_{(a-1,b+1,u')} : 0 < b < a - 1
\end{cases}
\]

The crucial observation for our purposes is that for \( 0 < b \leq r \), the \((a, b, u)\)-component of \( U_p \eta \) depends only on the \((a + 1, b - 1, u')\)-components of \( \eta \) for varying \( u' \), and similarly for \( 0 < a \leq r \) the \((a, b, u)\)-component of \( U_p^* \eta \) depends only on the \((a - 1, b + 1, u')\)-components of \( \eta \). By induction, we deduce

\[
(U_p^n \eta)_{(a,b,u)} = \begin{cases}
\rho^a(p)\eta_{(0,r,1)} F_*^{a-n} \eta_{(0,r,1)} : b \leq a \\
\sum_{u' \equiv u \mod p^b} \langle u' \rangle \rho^a(p) \eta_{(0,r,1)} : a < b
\end{cases}
\]

and

\[
(U_p^* \eta)_{(a,b,u)} = \begin{cases}
\rho^a(p)\eta_{(0,r,1)} F_*^{a-n} \eta_{(0,r,1)} : a < b \\
\sum_{u' \equiv u \mod p^b} \langle u' \rangle^{-1} \rho^a(p) \eta_{(0,r,1)} : b \leq a
\end{cases}
\]

for any \( n \geq r \geq 1 \).

For any \( r > 0 \) and for \( * = \infty \), 0 we define maps

\[
\gamma^*_p : H^0(I^*_r, \Omega^1_k(\mathcal{X}_r^n)) \longrightarrow H^0(\mathcal{X}_r^n, \Omega^1_k(\mathcal{X}_r^n))
\]
by

\[(\gamma^\infty_r(\eta))(a,b,u) := \begin{cases} 
\rho_s^b F^{-b}_s \eta & : b \leq a \\
\sum_{a' \in (Z/p^bZ)^\times} \langle u' \rangle \rho_s^{a'} F^{-b}_s \eta & : a < b \\
\sum_{u' \equiv u \mod p^a} \langle u' \rangle \rho_s^{a'} F^{-b}_s \eta & : u' \in (Z/p^bZ)^\times
\end{cases}
\]

and

\[(\gamma^0_r(\eta))(a,b,u) := \begin{cases} 
\rho_s^b(p)^a_N F^{-a}_s \eta & : a < b \\
\sum_{a' \in (Z/p^aZ)^\times} \langle u' \rangle \rho_s^{a'}(p)^a_N F^{-a}_s \eta(0,r,1) & : b \leq a \\
\sum_{u' \equiv u \mod p^b} \langle u' \rangle \rho_s^{a'}(p)^a_N F^{-a}_s \eta & : u' \in (Z/p^aZ)^\times
\end{cases}
\]

These maps are well-defined because $F_s = V$ is invertible on the $V$-ordinary subspace, and they are immediately seen to be injective by looking at $(r,0,1)$-components. Note moreover that the $(a,b,u)$-component of $\gamma^\infty_r(\eta)$ is independent of $u$.

We claim that the maps $\gamma^\infty_r$ have image in $H^0(\overline{X}_r, \omega_{\overline{X}_r})$ (i.e. that they factor through (3.3.3)). To see this, we proceed as follows. Suppose that $x$ is any supersingular point on $\overline{X}_r$, and $s \in \mathcal{O}_{\overline{X}_r,x}$ is arbitrary. By Proposition 2.1.15 and Definition 2.1.13, we must check that the sum of the residues of $s\gamma^\infty_r(\eta)$ at all $k$-points of $\overline{X}_r$ lying over $x$ is zero. Using (3.3.6a), we calculate that this sum is equal to

\[(\gamma^\infty_r(\eta))(a,b,u) := \sum_{b \leq a} \sum_{u \in (Z/p^bZ)^\times} \text{res}_{x(a,b,u)}(s\rho_s^b F^{-b}_s \eta) + \sum_{a < b} \sum_{u \in (Z/p^aZ)^\times} \text{res}_{x(a,b,u)}(s\langle u \rangle \rho_s^b F^{-b}_s \eta)
\]

where $x(a,b,u)$ denotes the unique point of the $(a,b,u)$-component of $\overline{X}_r$ over $x$, and the outer sums range over all nonnegative integers $a, b$ with $a + b = r$. We claim that for any meromorphic differential $\omega$ on $I_{(a,b,u)}$ and any supersingular point $y$ of $I_{(a,b,u)}$ over $x$, we have

\[\text{res}_y(\omega) = \text{res}_y(\langle u \rangle \omega)\]

for all $u \in Z_p^\times$, and, if in addition $\omega$ is $V$-ordinary,

\[\text{res}_y(s\omega) = s(x) \text{res}_y(\omega)\]

Indeed, (3.3.8a) is a consequence of (3.1.2), using the fact that the automorphism $\langle u \rangle$ of $I_{(a,b,u)}$ fixes every supersingular point, while (3.3.8b) is deduced by thinking about formal expansions of differentials at $y$ and using the fact that a $V$-ordinary meromorphic differential has at worst simple poles thanks to Lemma 3.1.6. Via (3.3.8a)–(3.3.8b), we reduce the sum (3.3.7) to

\[\sum_{a+b=r} \sum_{u \in (Z/p^bZ)^\times} s(x) \text{res}_{x(a,b,u)}(\rho_s^{\min(a,b)} F^{-b}_s \eta) = \sum_{a+b=r} \varphi(p^b) s(x) \text{res}_{x(a,b,u)}(\rho_s^{\min(a,b)} F^{-b}_s \eta)\]

where the first equality above follows from the fact that for fixed $a, b$, the points $x(a,b,u)$ for varying $u \in (Z/p^{\min(a,b)}Z)^\times$ are all identified with the same point on $I_g(p^{\max(a,b)})$, and the second equality is a consequence of (3.1.2), since $\rho(x(r,0,1)) = x(r-1,1,1)$. As $\eta$ is $V$-ordinary, there exists a $V$-ordinary meromorphic differential $\xi$ on $I^0$ with $\eta = F_s \xi$; substituting this expression for $\eta$ in to (3.3.9) and applying (3.1.2) once more, we conclude that (3.3.9) is zero, as desired. That $\gamma^\infty_r$ has image in $H^0(\overline{X}_r, \omega_{\overline{X}_r})$ follows from a nearly identical calculation, and we omit the details.

It follows immediately from our calculations (3.3.4a)–(3.3.4b) and the definitions (3.3.6a)–(3.3.6b) that the relations $U_p \circ \gamma^\infty_r = \gamma^\infty_r \circ F_s$ and $U_p \circ \gamma^0_r = \gamma^0_r \circ (p)_N^{-1} F_s$ hold. Since $F_s$ is invertible on the
source of $\gamma_\ast^r$, it follows immediately that $\gamma_\ast^0$ has image contained in $e_\ast^r H^0(\overline{X}_r, \omega_{\overline{X}_r})$ and that $\gamma_\ast^\infty$ has image contained in $e_\ast^r H^0(\overline{X}_r, \omega_{\overline{X}_r})$.

To see that these containment are equalities, we proceed as follows. Suppose that $\xi \in e_\ast^r H^0(\overline{X}_r, \omega_{\overline{X}_r})$ is arbitrary. We claim that the meromorphic differential $\xi_{(r,0,1)}$ on $I^0_r$ has at worst simple poles along $s\bar{s}$ (and is holomorphic outside $s\bar{s}$). Indeed, for each $n > 0$ we may find $\xi^{(n)} \in e_\ast^r H^0(\overline{X}_r, \omega_{\overline{X}_r})$ with $\xi = U^n_p \xi^{(n)}$. As discussed in $\S 2.1$, when viewed as a meromorphic differential on $\overline{X}_r^n$ any section of $\omega_{\overline{X}_r}$ has poles of order bounded by a constant depending only on $r$ (see [Con00, Lemma 5.2.2]). Since $F : I^\infty_r \to I^\infty_r$ is inseparable of degree $p$ (so totally ramified over every supersingular point), it follows from Remark 3.1.2 that there exists $n > r$ such that the meromorphic differential $F^\ast_n \xi^{(n)}_{(r,0,1)}$ has at worst simple poles along $s\bar{s}$; by the formula (3.3.10) for $U^n_p$, we conclude that the same is true of

$$\xi_{(r,0,1)} = (U^n_p \xi^{(n)})_{(r,0,1)} = F^\ast_n \xi^{(n)}_{(r,0,1)}.$$  

Applying this with $\xi^{(r)}$ in the role of $\xi$, and using (3.3.5a) and (3.3.6a) we calculate

$$\xi = U^r_p \xi^{(r)} = \gamma^\infty_r (F^\ast_r \xi^{(r)}_{(r,0,1)}),$$  

so $\gamma^\infty_r$ surjects onto $e_\ast^r H^0(\overline{X}_r, \omega_{\overline{X}_r})$ and is hence an isomorphism onto this image. A nearly identical argument shows that $\gamma_\ast^0$ is an isomorphism onto $e_\ast^r H^0(\overline{X}_r, \omega_{\overline{X}_r})$.

Since pullback of meromorphic differentials along $i^\infty_r : I^\infty_r \hookrightarrow \overline{X}_r$ is given by projection

$$H^0(\overline{X}_r^n, \Omega^1_\mathbb{k}(\overline{X}_r)) \sim \prod_{(a,b,u)} H^0(I_{(a,b,u)}, \Omega^1_\mathbb{k}(I_{(a,b,u)})) \xrightarrow{\text{proj}_{(r,0,1)}} H^0(I^\infty_r, \Omega^1_\mathbb{k}(I^\infty_r))$$  

onto the $(r,0,1)$-component, the composition of $\gamma^\infty_r$ and (the restriction of) $(i^\infty_r)^\ast$ in either order is the identity map. Since $i^\infty_r$ is compatible with the $\mathcal{H}_r$-correspondences, the resulting isomorphism (3.3.1) is $\mathcal{H}_r$-equivariant (with $U_p$ acting on the target via $F_\ast$). Similarly, since the “geometric inertia” action (2.3.3) of $\Gamma$ on $X_r$ is compatible via $i^\infty_r$ with the trivial action on $I^\infty_r$ by Proposition 2.3.14, the isomorphism (3.3.1) is equivariant for these actions of $\Gamma$. A nearly identical analysis shows that $(i^\infty_r)^\ast$ is $\mathcal{H}_r^\ast$-compatible (with $U^\ast_p$ acting on the target as $\langle p \rangle^\ast N F^\ast_\mathcal{H}_r$) and $\Gamma$-equivariant for the action of $\Gamma$ on $I^\infty_r$ via $\langle \chi(\cdot) \rangle^{-1}$. The commutativity of the four diagrams in (3.3.2) is an immediate consequence of the descriptions of the degeneracy mappings $\bar{p}, \bar{\sigma}$ on $\overline{X}_r^n$, furnished by Proposition 2.3.13 and the explication (3.3.11) of pullback by $i^\ast_r$ in terms of projection. That $\pi_r$ interchanges the two diagrams in (3.3.2) is an immediate consequence of Proposition 2.3.6.

Finally, that the assertions of Proposition 3.3.1 all hold if $\overline{X}_r$ and $\Omega^1_{I^\ast_r}(s\bar{s})$ are replaced by $\overline{X}_r^n$ and $\Omega^1_{I^\ast_r}$, respectively, follows from a a similar—but much simpler—argument. The point is that the maps $\gamma_\ast^r$ of (3.3.6a)–(3.3.6b) visibly carry $H^0(I^\ast_r, \Omega^1_{I^\ast_r})^{\text{Var}}$ into $H^0(\overline{X}_r^n, \Omega^1_{\overline{X}_r^n})$, from which it follows via our argument that they induce the claimed isomorphisms.

Since $\overline{X}_r$ is a proper and geometrically connected curve over $F_p$, Proposition 2.1.12 (2) provides short exact sequences of $F_p[\Delta/\Delta_r]$-modules with linear $\Gamma$ and $\mathcal{H}_r^\ast$ (respectively $\mathcal{H}_r$)-action

$$0 \longrightarrow e_\ast^r H^0(\overline{X}_r, \omega_{\overline{X}_r}/F_p) \longrightarrow e_\ast^r H^1(\overline{X}_r/F_p) \longrightarrow e_\ast^r H^1(\overline{X}_r, \mathcal{O}_{\overline{X}_r}) \longrightarrow 0$$ 

(3.3.12a)
respectively
\[(3.3.12b) \quad 0 \rightarrow e_r^*H^0(\overline{X}_r, \omega_{\overline{X}_r}/F_p) \rightarrow e_r^*H^1(\overline{X}_r/F_p) \rightarrow e_r^*H^1(\overline{X}_r, \mathcal{O}_{\overline{X}_r}) \rightarrow 0 \]
which are canonically $F_p$-linearly dual to each other. We likewise have such exact sequences in the
\[ \text{case of } \overline{X}_n, \text{ note that since } \overline{X}_n \text{ is smooth, the short exact sequence } H(\overline{X}_n/F_p) \text{ is simply the Hodge}
\]
filtration of $H^1_{\text{DR}}(\overline{X}_n/F_p)$.

Corollary 3.3.2. The absolute Frobenius morphism of $\overline{X}_r$ over $F_p$ induces a natural $F_p[\Delta/\Delta_r]$-linear, 
\[ \Gamma \text{-compatible, and } \mathfrak{H}_r^* \text{ (respectively } \mathfrak{H}_r) \text{ equivariant splitting of } (3.3.12a) \text{ (respectively } (3.3.12b)). \]
Furthermore, for each $r$ we have natural isomorphisms of split short exact sequences
\[(3.3.13a) \quad 0 \rightarrow e_r^*H^0(\overline{X}_r, \omega_{\overline{X}_r}/F_p) \rightarrow e_r^*H^1(\overline{X}_r/F_p) \rightarrow e_r^*H^1(\overline{X}_r, \mathcal{O}_{\overline{X}_r}) \rightarrow 0
\]
\[ F_r^*(\rho)^* \cong \left( \begin{array}{ccc} \cong \end{array} \right) \]
\[ 0 \rightarrow H^0(I^0_r, \Omega^1_{\text{ss}})^{\text{Vord}} \rightarrow H^0(I^0_r, \Omega^1_{\text{ss}})^{\text{Vord}} \oplus H^1(I^0_r, \mathcal{O}(\text{ss}))^{\text{Vord}} \rightarrow H^1(I^0_r, \mathcal{O}(\text{ss}))^{\text{Vord}} \rightarrow 0 \]
\[(3.3.13b) \quad 0 \rightarrow e_r^*H^0(\overline{X}_r, \omega_{\overline{X}_r}/F_p) \rightarrow e_r^*H^1(\overline{X}_r/F_p) \rightarrow e_r^*H^1(\overline{X}_r, \mathcal{O}_{\overline{X}_r}) \rightarrow 0
\]
\[ F_r^*(\rho)^* \cong \left( \begin{array}{ccc} \cong \end{array} \right) \]
\[ 0 \rightarrow H^0(I^0_r, \Omega^1_{\text{ss}})^{\text{Vord}} \rightarrow H^0(I^0_r, \Omega^1_{\text{ss}})^{\text{Vord}} \oplus H^1(I^0_r, \mathcal{O}(\text{ss}))^{\text{Vord}} \rightarrow H^1(I^0_r, \mathcal{O}(\text{ss}))^{\text{Vord}} \rightarrow 0 \]
which are compatible with the extra structures. The identification $(3.3.13a)$ (respectively $(3.3.13b)$) is
\[ \text{moreover compatible with change in } r \text{ using the trace mappings attached to } \rho : I^r_r \rightarrow I^{r-1}_r \text{ and to } \overline{\rho} : \overline{X}_r \rightarrow \overline{X}_{r-1} \text{ (respectively } \overline{\sigma} : \overline{X}_r \rightarrow \overline{X}_{r-1}). \]
The same statements hold true if we replace $\overline{X}_r$, $\Omega^1_{I^r_r}(\text{ss})$, and $\mathcal{O}_{I^r_r}(\text{ss})$ with $\overline{X}_n$, $\Omega^1_{I^r_n}(\text{ss})$, and $\mathcal{O}_{I^r_n}(\text{ss})$, respectively.

Proof. Pullback by the absolute Frobenius endomorphism of $\overline{X}_r$ induces an endomorphism of $(3.3.12a)$
\[ e_r^*H^1(\overline{X}_r, \mathcal{O}_{\overline{X}_r}) \rightarrow e_r^*H^1(\overline{X}_r/F_p) \]
that is $\Gamma$ and $\mathfrak{H}_r^*$-compatible and projects to the endomorphism $F^*$ of $e_r^*H^1(\overline{X}_r, \mathcal{O}_{\overline{X}_r})$. On the other
\[ \text{hand, Proposition 3.3.1 gives a natural } \Gamma \text{ and } \mathfrak{H}_r^* \text{-equivariant isomorphism of } F_p[\Delta/\Delta_r]-\text{modules}
\]
\[ (3.3.14) \quad e_r^*H^1(\overline{X}_r, \mathcal{O}_{\overline{X}_r}) \rightarrow e_r^*H^1(\overline{X}_r/F_p) \]
As this isomorphism intertwines $F^*$ on source and target, we deduce that $F^*$ acts invertibly on 
\[ e_r^*H^1(\overline{X}_r, \mathcal{O}_{\overline{X}_r}) \]. We may therefore pre-compose $(3.3.14)$ with $(F^*)^{-1}$ to obtain a canonical splitting of
\[ \text{of } (3.3.12a), \text{ which by construction is } F_p[\Delta/\Delta_r]-\text{linear and compatible with } \Gamma \text{ and } \mathfrak{H}_r^*. \]
The existence of $(3.3.13a)$ as well as its compatibility with $\Gamma$, $\mathfrak{H}_r^*$ and with change in $r$ follows immediately from Proposition 3.3.1 and duality (see Remark 3.1.5). The corresponding assertions for the exact sequence
\[ (3.3.12b) \text{ and the diagram } (3.3.13b) \text{ are proved similarly, and we leave the details to the reader. A}
\]
\[ \text{nearly identical argument shows that the same assertions hold true when } \overline{X}_r, \Omega^1_{I^r_r}(\text{ss}), \text{ and } \mathcal{O}_{I^r_r}(\text{ss}) \]
are replaced by $\overline{X}_n$, $\Omega^1_{I^r_n}(\text{ss})$, and $\mathcal{O}_{I^r_n}(\text{ss})$, respectively. \( \blacksquare \)
Corollary 3.3.3. The exact sequences (3.3.12a) and (3.3.12b) are split short exact sequences of free \( F_p[\Delta/\Delta_v] \)-modules whose terms have \( F_p[\Delta/\Delta_v] \)-ranks \( d, 2d, \) and \( d \), respectively, for \( d \) as in Remark 3.2.6. For \( s \leq r \), the degeneracy maps \( \rho, \sigma : \mathcal{X}_r \twoheadrightarrow \mathcal{X}_s \) induce natural isomorphisms of exact sequences

\[
\rho_s : e^*_r H(\mathcal{X}_r / F_p) \otimes_{F_p[\Delta/\Delta_s]} F_p[\Delta/\Delta_r] \xrightarrow{\sim} e^*_s H(\mathcal{X}_s / F_p)
\]

\[
\sigma_s : e^*_r H(\mathcal{X}_r / F_p) \otimes_{F_p[\Delta/\Delta_s]} F_p[\Delta/\Delta_r] \xrightarrow{\sim} e^*_s H(\mathcal{X}_s / F_p)
\]

that are \( \Gamma \) and \( \mathcal{S}_r^* \) (respectively \( \mathcal{S}_r \)) equivariant.

Proof. This follows immediately from Proposition 3.2.1 and Corollary 3.3.2. \( \square \)

Remark 3.3.4. We warn the reader that the naïve analogue of Corollary 3.3.3 in the case of \( \mathfrak{F}_{\mathcal{X}_r}^n \) is false: while \( H^0(I_r, \Omega^1_{I_r/k})_{V^\text{ord}} \) is a free \( F_p[\Delta/\Delta_r] \)-module, the submodule of holomorphic differentials need not be. Over \( k = F_p \), the residue map gives a short exact sequence of \( k[\Delta/\Delta_r] \)-modules

\[
0 \rightarrow H^0(I_r, \Omega^1_{I_r/k})_{V^\text{ord}} \rightarrow H^0(I_r, \Omega^1_{I_r/k}(\mathbb{S}))_{V^\text{ord}} \rightarrow \ker \left( k^d \sum_r k \right) \rightarrow 0
\]

with middle term that is free over \( k[\Delta/\Delta_r] \); see Theorem 2 of [Nak85]. The splitting of this exact sequence is then equivalent to the projectivity—hence freeness—of \( H^0(I_r, \Omega^1_{I_r/k})_{V^\text{ord}} \) over \( k[\Delta/\Delta_r] \).

In order to formulate the correct analogue of Corollary 3.3.3 in the case of \( \mathfrak{F}_{\mathcal{X}_r}^n \), we proceed as follows. Denote by \( \tau : F_p^\times \rightarrow \mathbb{Z}_p^\times \) the Teichmüller character, and for any \( \mathbb{Z}_p \)-module \( M \) with a linear action of \( F_p^\times \) and any \( j \in \mathbb{Z}/(p-1)\mathbb{Z} \), let

\[
M(j) := \{ m \in M : d \cdot m = \tau(d)^j m \text{ for all } d \in F_p^\times \}
\]

be the subspace of \( M \) on which \( F_p^\times \) acts via \( \tau^j \). As \( \# F_p^\times = p - 1 \) is a unit in \( \mathbb{Z}_p^\times \), the submodule \( M(j) \) is a direct summand of \( M \). Explicitly, the identity of \( \mathbb{Z}_p[F_p^\times] \) admits the decomposition

\[
1 = \sum_{j \in \mathbb{Z}/(p-1)\mathbb{Z}} f_j \quad \text{with} \quad f_j := \frac{1}{p-1} \sum_{g \in F_p^\times} \tau^{-j}(g) \cdot g
\]

into mutually orthogonal idempotents \( f_j \), and we have \( M(j) = f_j M \). In applications, we will consistently need to remove the trivial eigenspace \( M(0) \) from \( M \), as this eigenspace in the \( p \)-adic Galois representations we consider is not potentially crystalline at \( p \). We will write

\[
f' := \sum_{j \in \mathbb{Z}/(p-1)\mathbb{Z}, j \neq 0} f_j
\]

for the idempotent of \( \mathbb{Z}_p[F_p^\times] \) corresponding to projection away from the 0-eigenspace for \( F_p^\times \).

Applying these considerations to the identifications of split exact sequences in Corollary 3.3.2, which are compatible with the canonical diamond operator action of \( \mathbb{Z}_p^\times \simeq F_p^\times \times \Delta \) on both rows, we obtain a corresponding identification of split exact sequences of \( \tau^j \)-eigenspaces, for each \( j \) mod \( p - 1 \). The following is a generalization of [Gro90, Proposition 8.10 (2)]:
Lemma 3.3.5. Let $j$ be an integer with $j \not\equiv 0 \mod p-1$. For each $r$, there are canonical isomorphisms

\[(3.3.18) \quad H^0(I_r, \Omega^1_{\mathfrak{X}_r})(j) \xrightarrow{\sim} H^0(I_r, \Omega^1_{\mathfrak{X}_r}(ss))(j) \quad \text{and} \quad H^1(I_r, \mathcal{O}(-ss))(j) \xrightarrow{\sim} H^1(I_r, \mathcal{O})(j)\]

The normalization map $\nu : \mathfrak{X}^n_r \to \mathfrak{X}_r$ induces a natural isomorphism of split exact sequences

\[(3.3.19) \quad 0 \to e_r^* H^0(\mathfrak{X}_r, \Omega^1_{\mathfrak{X}_r})(j) \xrightarrow{\nu^*} e_r^* H^0_{dR}(\mathfrak{X}_r/F_p)(j) \to e_r^* H^1(\mathfrak{X}_r, \mathcal{O}_{\mathfrak{X}_r})(j) \to 0\]

where the central vertical arrow is deduced from the outer two vertical arrows via the splitting of both rows by the Frobenius endomorphism. The same assertions hold if we replace $e_r^*$ with $e_r$.

Proof. The first map in (3.3.18) is injective, as it is simply the canonical inclusion. To see that it is an isomorphism, we may work over $k := \overline{F}_p$. If $\eta$ is any meromorphic differential on $I_r$ on which $\mathbf{F}_p^\times$ acts via the character $\tau^j$, then since the diamond operators fix every supersingular point on $I_r$ we have

$$\text{res}_x(\eta) = \text{res}_x((u)\eta) = \tau^j(u) \text{res}_x(\eta)$$

for any $x \in ss(k)$ and all $u \in \mathbf{F}_p^\times$. As $j \not\equiv 0 \mod p-1$, so $\tau^j$ is nontrivial, we must therefore have $\text{res}_x(\eta) = 0$ for all supersingular points $x$. If in addition $\eta$ is holomorphic outside $ss$ with at worst simple poles along $ss$, then $\eta$ must be holomorphic everywhere, so the first map in (3.3.18) is surjective, as desired. The second mapping in (3.3.18) is dual to the first, and hence an isomorphism as well.

Now for each $j \not\equiv 0 \mod p-1$, we have a commutative diagram

\[(3.3.20) \quad \begin{array}{ccc}
   e_r^* H^0(\mathfrak{X}_r, \Omega^1_{\mathfrak{X}_r})(j) & \xrightarrow{\nu^*} & e_r^* H^0(\mathfrak{X}_r, \omega_{\mathfrak{X}_r})(j) \\
   (q^\varphi_j)^* \downarrow \cong & & \downarrow (q^\varphi_j)^* \\
   H^0(I_r, \Omega^1_{I_p})(j)^{\text{ord}} & \xrightarrow{\cong} & H^0(I_r, \Omega^1_{I_p}(ss))(j)^{\text{ord}}
\end{array}\]

of $\mathbf{F}_p[\Delta/\Delta_*]$-modules with $\Gamma$ and $\mathfrak{S}_r^*$-action in which the two vertical arrows are isomorphisms by Proposition 3.3.1 and the bottom horizontal mapping is an isomorphism as we have just seen. We conclude that the top horizontal arrow of (3.3.20) is an isomorphism as well. Thus, the left vertical map in (3.3.19) is an isomorphism, so the same is true of the right vertical map by duality. The diagram (3.3.19) then follows at once from the fact the both rows are canonically split by the Frobenius endomorphism, thanks to Corollary 3.3.2. A nearly identical argument shows that the same assertions hold if we replace $e_r^*$ with $e_r$ throughout. \[\square\]

If $A$ is any $\mathbf{Z}_p[\mathbf{F}_p^\times]$-algebra and $a \in A$, we will write $a' := f'a$ for the product of $a$ with the idempotent $f'$ of (3.3.17), or equivalently the projection of $a$ to the complement of the trivial eigenspace of $\mathbf{F}_p^\times$. We will apply this to $A = \mathfrak{S}_r$, $\mathfrak{S}_r^*$, viewed as $\mathbf{Z}_p[\mathbf{F}_p^\times]$-algebras in the usual manner, via the diamond operators and the Teichmüller section $\tau : \mathbf{F}_p^\times \to \mathbf{Z}_p^\times$. 

Proposition 3.3.6. For each \( r \) there are natural isomorphisms of split short exact sequences

\[
0 \longrightarrow e_r^{s'} H^0(\overline{X}_r, \Omega^1_{\overline{X}_r}) \longrightarrow e_r^{s'} H^1_{dR}(\overline{X}_r/F_p) \longrightarrow e_r^{s'} H^1(\overline{X}_r, \mathcal{O}_{\overline{X}_r}) \longrightarrow 0
\]

(3.3.21a)

\[
0 \longrightarrow f^r H^0(I^0_r, \Omega^1_{\overline{X}_r}) \longrightarrow f^r H^0(I^0_r, \Omega^1_{\overline{X}_r}) \oplus f^r H^1(I^\infty_r, \mathcal{O})_{\text{F}ord} \longrightarrow f^r H^1(I^\infty_r, \mathcal{O})_{\text{F}ord} \longrightarrow 0
\]

(3.3.21b)

Setting \( d' := \sum_{k=3}^d d_k \) where \( d_k := \dim F_p S_k(N; F_p)^{\text{ord}} \) as in Remark 3.2.6, the terms in the top rows of (3.3.21a) and (3.3.21b) are free \( F_p[\Delta/\Delta_r] \)-modules of ranks \( d', 2d', \) and \( d' \). The identification (3.3.21a) (respectively (3.3.21b)) is \( \Gamma \) and \( \mathfrak{H}_r^* \) (respectively \( \mathfrak{H}_r \))-equivariant, and compatible with change in \( r \) using the trace mappings attached to \( \rho : I^*_r \rightarrow I^*_r \) and to \( \rho : \overline{X}_r \rightarrow \overline{X}_s \) (respectively \( \rho : \overline{X}_r \rightarrow \overline{X}_s \)).

Proof. This follows immediately from Corollaries 3.3.2–3.3.3 and Lemma 3.3.5, using the fact that the group ring \( F_p[\Delta/\Delta_r] \) is local, so any projective \( F_p[\Delta/\Delta_r] \)-module is free. \( \blacksquare \)

As usual, we write \( \text{Pic}^0_{\overline{X}_r/F_p}[p^\infty] \) for the \( p \)-divisible group of the Jacobian of \( \overline{X}_r \) over \( F_p \); it is equipped with canonical actions of \( \mathfrak{H}_r^* \) and \( \mathfrak{H}_r \), as well as a “geometric inertia” action of \( \Gamma \) over \( F_p \).

Definition 3.3.7. We define \( \Sigma_r := e_r^{s'} \text{Pic}^0_{\overline{X}_r/F_p}[p^\infty] \), equipped with the induced actions of \( \mathfrak{H}_r^* \) and \( \Gamma \).

We will employ Proposition 3.3.6 and Oda’s description (Proposition 3.1.8) of Dieudonné modules in terms of de Rham cohomology to analyze the structure of \( \Sigma_r \).

Proposition 3.3.8. For each \( r \), there is a natural isomorphism of \( A := \mathbb{Z}_p[F,V] \)-modules

\[
D(\Sigma_r)_{\text{F}_p} \simeq e_r^{s'} H^1_{dR}(\overline{X}_r/F_p) \simeq f^r H^0(I^\infty_r, \Omega^1_{\overline{X}_r}) \oplus f^r H^1(I^0_r, \mathcal{O})_{\text{F}ord},
\]

which is compatible with \( \mathfrak{H}_r^* \), \( \Gamma \), and change in \( r \) and which carries \( D(\Sigma^m_r)_{\text{F}_p} \) (respectively \( D(\Sigma^{et}_r)_{\text{F}_p} \)) isomorphically onto \( f^r H^0(I^0_r, \Omega^1_{\overline{X}_r}) \) (respectively \( f^r H^1(I^\infty_r, \mathcal{O})_{\text{F}ord} \)). In particular, \( \Sigma_r \) is ordinary.

Proof. First note that since the identifications (3.3.21a) and (3.3.21b) are induced by the canonical closed immersions \( i_r^* : I^*_r \hookrightarrow \overline{X}_r \), they are compatible with the natural actions of Frobenius and the Cartier operator. The isomorphism (3.3.22) is therefore an immediate consequence of Propositions 3.1.8 and 3.3.6. Since this isomorphism is compatible with \( F \) and \( V \), we have

\[
D(\Sigma^m_r)_{\text{F}_p} \simeq D(\Sigma^m_r)^{\text{ord}}_{\text{F}_p} \simeq f^r H^0(I^0_r, \Omega^1_{\overline{X}_r})_{\text{ord}}
\]

and

\[
D(\Sigma^{et}_r) \otimes_{\mathbb{Z}_p} \text{F}_p \simeq D(\Sigma^{et}_r)^{\text{F}ord}_{\text{F}_p} \simeq f^r H^1(I^\infty_r, \mathcal{O})_{\text{F}ord}
\]

and we conclude that the canonical inclusion \( D(\Sigma^m_r)_{\text{Z}_p} \oplus D(\Sigma^{et}_r)_{\text{Z}_p} \hookrightarrow D(\Sigma_r)_{\text{Z}_p} \) is surjective, whence \( \Sigma_r \) is ordinary by Dieudonné theory. \( \blacksquare \)
We now analyze the ordinary $p$-divisible group $\Sigma_r$ in more detail. Since $\overline{X}_r$ is the disjoint union of proper smooth and irreducible Igusa curves $I_{(a,b,u)}$ (see Proposition 2.3.10) with $I_r^0 := I_{(0,r,1)}$ and $I_r^\infty = I_{(r,0,1)}$, we have a canonical identification

$$\text{Pic}^0_{\overline{X}_r/F_p} = \prod_{(a,b,u)} \text{Pic}^0_{I_{(a,b,u)}/F_p}. \tag{3.3.24}$$

For $* = 0, \infty$ let us write $j_r^* := \text{Pic}^0_{\overline{X}_r/F_p}$ for the Jacobian of $I_r^*$ over $F_p$. The canonical closed immersions $i_r^* : I_r^* \hookrightarrow \overline{X}_r$ yield (by Picard and Albanese functoriality) homomorphisms of abelian varieties over $F_p$

$$\text{Alb}(i_r^*) : j_r^* \longrightarrow \text{Pic}^0_{\overline{X}_r/F_p} \text{ and } \text{Pic}^0(i_r^*) : \text{Pic}^0_{\overline{X}_r/F_p} \longrightarrow j_r^*. \tag{3.3.25}$$

Via the identification (3.3.24), we know that $j_r^*$ is a direct factor of $\text{Pic}^0_{\overline{X}_r/F_p}$; in these terms $\text{Alb}(i_r^*)$ is the unique mapping which projects to the identity on $j_r^*$ and to the zero map on all other factors, while $\text{Pic}^0(i_r^*)$ is simply projection onto the factor $j_r^*$. As $\Sigma_r$ is a direct factor of $f' \text{Pic}^0_{\overline{X}_r/F_p} [p^\infty]$, these mappings induce homomorphisms of $p$-divisible groups over $F_p$

\begin{align*}
&f' j_r^*[p^\infty]^m \xrightarrow{\text{Alb}(i_r^*)} f' \text{Pic}^0_{\overline{X}_r/F_p} [p^\infty]^m \xrightarrow{\text{proj}} \Sigma_r \tag{3.3.26a} \\
&\Sigma_r \xrightarrow{\text{incl}} f' \text{Pic}^0_{\overline{X}_r/F_p} [p^\infty]^\text{et} \xrightarrow{\text{Pic}^0(i_r^*)} f' j_r^*[p^\infty]^\text{et} \tag{3.3.26b}
\end{align*}

which we (somewhat abusively) again denote by $\text{Alb}(i_r^*)$ and $\text{Pic}^0(i_r^*)$, respectively. The following is a sharpening of [MW84, Chapter 3, §3, Proposition 3] (see also [Til87, Proposition 3.2]):

**Proposition 3.3.9.** The mappings (3.3.26a) and (3.3.26b) are isomorphisms. They induce a canonical split short exact sequences of $p$-divisible groups over $F_p$

$$0 \longrightarrow f' j_r^*[p^\infty]^m \xrightarrow{\text{Alb}(i_r^*)} \Sigma_r \xrightarrow{\text{Pic}^0(i_r^*)} f' j_r^*[p^\infty]^\text{et} \longrightarrow 0 \tag{3.3.27}$$

which is:

1. $\Gamma$-equivariant for the geometric inertia action on $\Sigma_r$, the trivial action on $f' j_r^*[p^\infty]^\text{et}$, and the action via $\langle \chi(\cdot) \rangle^{-1}$ on $f' j_r^*[p^\infty]^m$.
2. $S_r^*$-equivariant with $U^*_p$ acting on $f' j_r^*[p^\infty]^\text{et}$ as $F$ and on $f' j_r^*[p^\infty]^m$ as $\langle p \rangle N V$.
3. Compatible with change in $r$ via the mappings $\text{Pic}^0(p)$ on $j_r^*$ and $\Sigma_r$.

**Proof.** It is clearly enough to prove that the sequence (3.3.27) induced by (3.3.26a) and (3.3.26b) is exact. Since the contravariant Dieudonné module functor from the category of $p$-divisible groups over $F_p$ to the category of $A$-modules which are $\mathbb{Z}_p$ finite and free is an exact anti-equivalence, it suffices to prove such exactness after applying $\mathbf{D}(\cdot)\mathbb{Z}_p$. As the resulting sequence consist of finite free $\mathbb{Z}_p$-modules, exactness may be checked modulo $p$ where it follows immediately from Propositions 3.3.6 and 3.3.8. The claimed compatibility with $\Gamma$, $S_r^*$, and change in $r$ is deduced from Propositions 2.3.14, 2.3.20, and 2.3.13, respectively. \hfill \blacksquare

**Remark 3.3.10.** It is possible to give a short proof of Proposition 3.3.9 along the lines of [MW84] or [Til87] by using Proposition 2.3.20 directly. We stress, however, that our approach via Dieudonné modules gives more refined information, most notably that the Dieudonné module of $\Sigma_r[p]$ is free as an $F_p[\Delta/\Delta_r]$-module. This fact will be crucial in our later arguments.
4. Dieudonné crystals and $(\varphi, \Gamma)$-modules

In this section, we summarize the main results of [CL12], which provides a classification of $p$-divisible groups over $R_r$ by certain semi-linear algebra structures. These structures—which arise naturally via the Dieudonné crystal functor—are cyclotomic analogues of Breuil and Kisin modules, and are closely related to Wach modules.\footnote{See [CL12] for the precise relationship.}

4.1. $(\varphi, \Gamma)$-modules attached to $p$-divisible groups. Fix a perfect field $k$ of characteristic $p$. Write $W := W(k)$ for the Witt vectors of $k$ and $K$ for its fraction field, and denote by $\varphi$ the unique automorphism of $W(k)$ lifting the $p$-power map on $k$. Fix an algebraic closure $\bar{K}$ of $K$, as well as a compatible sequence $\{\varepsilon^{(r)}\}_{r \geq 1}$ of primitive $p$-power roots of unity in $\bar{K}$, and set $\mathcal{G}_K := \text{Gal}(\bar{K}/K)$. For $r \geq 0$, we put $K_r := K(\mu_p^r)$ and $R_r := W[\mu_p^r]$, and we set $\Gamma_r := \text{Gal}(K_\infty/K_r)$, and $\Gamma := \Gamma_0$.

Let $\mathcal{G}_r := W[[u_r]]$ be the power series ring in one variable $u_r$ over $W$, viewed as a topological ring via the $(p, u_r)$-adic topology. We equip $\mathcal{G}_r$ with the unique continuous action of $\Gamma$ and extension of $\varphi$ determined by
\[
(4.1.1) \quad \gamma u_r := (1 + u_r)^{\chi(\gamma)} - 1 \quad \text{for } \gamma \in \Gamma \quad \text{and} \quad \varphi(u_r) := (1 + u_r)^p - 1.
\]

We denote by $\mathcal{O}_{\mathcal{G}_r} := \mathcal{G}_r[\frac{1}{u_r}]$ the $p$-adic completion of the localization $\mathcal{G}_r((p))$, which is a complete discrete valuation ring with uniformizer $p$ and residue field $k((u_r))$. One checks that the actions of $\varphi$ and $\Gamma$ on $\mathcal{G}_r$ uniquely extend to $\mathcal{O}_{\mathcal{G}_r}$.

For $r > 0$, we write $\theta : \mathcal{G}_r \rightarrow R_r$ for the continuous and $\Gamma$-equivariant $W$-algebra surjection sending $u_r$ to $\varepsilon^{(r)} - 1$, whose kernel is the principal ideal generated by the Eisenstein polynomial $E_r := \varphi^r(u_r)/\varphi^{r-1}(u_r)$, and we denote by $\tau : \mathcal{G}_r \rightarrow W$ the continuous and $\varphi$-equivariant surjection of $W$-algebras determined by $\tau(u_r) = 0$. We lift the canonical inclusion $R_r \hookrightarrow R_{r+1}$ to a $\Gamma$- and $\varphi$-equivariant $W$-algebra injection $\mathcal{G}_r \hookrightarrow \mathcal{G}_{r+1}$ determined by $u_r \mapsto \varphi(u_{r+1})$; this map uniquely extends to a continuous injection $\mathcal{O}_{\mathcal{G}_r} \hookrightarrow \mathcal{O}_{\mathcal{G}_{r+1}}$, compatibly with $\varphi$ and $\Gamma$. We will frequently identify $\mathcal{G}_r$ (respectively $\mathcal{O}_{\mathcal{G}_r}$) with its image in $\mathcal{G}_{r+1}$ (respectively $\mathcal{O}_{\mathcal{G}_{r+1}}$), which coincides with the image of $\varphi$ on $\mathcal{G}_{r+1}$ (respectively $\mathcal{O}_{\mathcal{G}_{r+1}}$). Under this convention, we have $E_r(u_r) = E_1(u_1) = u_0/u_1$ for all $r > 0$, so we will simply write $\omega := E_r(u_r)$ for this common element of $\mathcal{G}_r$ for $r > 0$.

**Definition 4.1.1.** We write $\mathcal{B}^\varphi_{\mathcal{G}_r}$ for the category of Barsotti-Tate modules over $\mathcal{G}_r$, i.e. the category whose objects are pairs $(M, \varphi_M)$ where

- $M$ is a free $\mathcal{G}_r$-module of finite rank,
- $\varphi_M : M \rightarrow M$ is a $\varphi$-semilinear map whose linearization has cokernel killed by $\omega$,

and whose morphisms are $\varphi$-equivariant $\mathcal{G}_r$-module homomorphisms. We write $\mathcal{B}^\varphi_{\mathcal{G}_r}^{\Gamma}$ for the subcategory of $\mathcal{B}^\varphi_{\mathcal{G}_r}$ consisting of objects $(M, \varphi_M)$ which admit a semilinear $\Gamma$-action (in the category $\mathcal{B}^\varphi_{\mathcal{G}_r}$) with the property that $\Gamma_r$ acts trivially on $M/u_rM$. Morphisms in $\mathcal{B}^\varphi_{\mathcal{G}_r}^{\Gamma}$ are $\varphi$ and $\Gamma$-equivariant morphisms of $\mathcal{G}_r$-modules. We often abuse notation by writing $M$ for the pair $(M, \varphi_M)$ and $\varphi$ for $\varphi_M$.

If $(M, \varphi_M)$ is any object of $\mathcal{B}^\varphi_{\mathcal{G}_r}^{\Gamma}$, then $1 \otimes \varphi_M : \varphi^*M \rightarrow M$ is injective with cokernel killed by $\omega$, so there is a unique $\mathcal{G}_r$-linear homomorphism $\psi_M : M \rightarrow \varphi^*M$ with the property that the composition of $1 \otimes \varphi_M$ and $\psi_M$ (in either order) is multiplication by $\omega$. Clearly, $\varphi_M$ and $\psi_M$ determine eachother.
**Definition 4.1.2.** Let \( \mathcal{M} \) be an object of \( \text{BT}^{\varphi, \Gamma} \). The dual of \( \mathcal{M} \) is the object \( (\mathcal{M}^t, \varphi_{\mathcal{M}^t}) \) of \( \text{BT}^{\varphi, \Gamma} \) whose underlying \( \mathcal{G}_r \)-module is \( \mathcal{M}^t := \text{Hom}_{\mathcal{G}_r}(\mathcal{M}, \mathcal{G}_r) \), equipped with the \( \varphi \)-semilinear endomorphism

\[
\varphi_{\mathcal{M}^t} : \mathcal{M}^t \xrightarrow{1 \otimes \varphi_{\mathcal{M}}} \mathcal{M}^t \cong (\varphi^* \mathcal{M})^t \xrightarrow{\psi_{\mathcal{M}^t}^\vee} \mathcal{M}^t
\]

and the commuting action of \( \Gamma \) given for \( \gamma \in \Gamma \) by

\[
(\gamma f)(m) := \chi(\gamma)^{-1} \varphi^{-1}(\gamma u_r/u_r) \cdot \gamma(f(\gamma^{-1}m)).
\]

There is a natural notion of base change for Barsotti–Tate modules. Let \( k'/k \) be an algebraic extension (so \( k' \) is automatically perfect), and write \( W' := W(k') \), \( R'_r := W'[\mu_{p'}] \), \( \mathcal{G}'_r := W'[u_r] \), and so on. The canonical inclusion \( W \hookrightarrow W' \) extends to a \( \varphi \) and \( \Gamma \)-compatible \( W \)-algebra injection \( \iota_r : \mathcal{G}_r \hookrightarrow \mathcal{G}'_{r+1} \), and extension of scalars along \( \iota_r \) yields a canonical canonical base change functor \( \iota_{rs} : \text{BT}^{\varphi, \Gamma} \rightarrow \text{BT}^{\varphi, \Gamma}_{\mathcal{G}'_{r+1}} \) which one checks is compatible with duality.

Let us write \( p\text{div}^\Gamma_{R_r} \) for the subcategory of \( p \)-divisible groups over \( R_r \) consisting of those objects and morphisms which descend (necessarily uniquely) to \( K = K_0 \) on generic fibers. By Tate’s Theorem, this is of course equivalent to the full subcategory of \( p \)-divisible groups over \( K_0 \) which have good reduction over \( K_r \). Note that for any algebraic extension \( k'/k \), base change along the inclusion \( \iota_r : R_r \hookrightarrow R'_{r+1} \) gives a covariant functor \( \iota_{rs} : p\text{div}^\Gamma_{R_r} \rightarrow p\text{div}^\Gamma_{R'_{r+1}} \).

The main result of [CL12] is the following:

**Theorem 4.1.3.** For each \( r > 0 \), there is a contravariant functor \( \mathcal{M}_r : p\text{div}^\Gamma_{R_r} \rightarrow \text{BT}^{\varphi, \Gamma} \) such that:

1. The functor \( \mathcal{M}_r \) is an exact equivalence of categories, compatible with duality.
2. The functor \( \mathcal{M}_r \) is of formation compatible with base change: for any algebraic extension \( k'/k \), there is a natural isomorphism of composite functors \( \iota_{rs} \circ \mathcal{M}_r \cong \mathcal{M}_{r+1} \circ \iota_{rs} \) on \( p\text{div}^\Gamma_{R_r} \).
3. For \( G \in p\text{div}^\Gamma_{R_r} \), put \( \overline{G} := G \times_{R_r} k \) and \( G_0 := G \times_{R_r} R_r/pR_r \).
   a. There is a functorial and \( \Gamma \)-equivariant isomorphism of \( W \)-modules
      \[
      \mathcal{M}_r(G) \otimes_{\mathcal{G}_r, \varphi} W \cong \text{D}(\overline{G})_W,
      \]
      carrying \( \varphi \otimes \varphi \) to \( F : \text{D}(\overline{G})_W \rightarrow \text{D}(\overline{G})_W \) and \( \psi_{\mathcal{M}} \otimes 1 \) to \( V \otimes 1 : \text{D}(\overline{G})_W \rightarrow \varphi^* \text{D}(\overline{G})_W \).
   b. There is a functorial and \( \Gamma \)-equivariant isomorphism of \( R_r \)-modules
      \[
      \mathcal{M}_r(G) \otimes_{\mathcal{G}_r, 0, \varphi} R_r \cong \text{D}(G_0)_{R_r}.
      \]

We wish to explain how to functorially recover the \( \mathcal{G}_r \)-representation afforded by the \( p \)-adic Tate module \( T_p G_K \) from \( \mathcal{M}_r(G) \). In order to do so, we must first recall the necessary period rings; for a more detailed synopsis of these rings and their properties, we refer the reader to [Col08, §6–§8].

As usual, we put\(^{21}\)

\[
\overline{E}^+ := \lim_{\rightarrow_{x \rightarrow 2x}} \mathcal{O}_C_K / (p),
\]

equipped with its canonical \( \mathcal{G}_K \)-action via “coordinates” and \( p \)-power Frobenius map \( \varphi \). This is a perfect (i.e. \( \varphi \) is an automorphism) valuation ring of characteristic \( p \) with residue field \( \overline{K} \) and fraction field \( \overline{E} := \text{Frac}(\overline{E}^+) \) that is algebraically closed. We view \( \overline{E} \) as a topological field via its valuation topology, with respect to which it is complete. Our fixed choice of \( p \)-power compatible sequence

\(^{21}\)Here we use the notation introduced by Berger and Colmez; in Fontaine’s original notation, this ring is denoted \( R \).
 induces an element \( \xi := (\xi \mod p) \), and we set \( E_K := k(\xi) \), viewed as a topological
subring of \( \hat{E} \); note that this is a \( \varphi \)- and \( \mathcal{G}_K \)-stable subfield of \( \hat{E} \)
that is independent of our choice of \( \xi \). We write \( E := E_K^{\text{sep}} \) for the
separable closure of \( E_K \) in the algebraically closed field \( \hat{E} \). The natural
\( \mathcal{G}_K \)-action on \( \hat{E} \) induces a canonical identification \( \text{Gal}(E/E_K) = \mathcal{H} := \ker(\chi) \subseteq \mathcal{G}_K \),
so \( E^{\mathcal{H}} = E_K \). If \( E \) is any subring of \( \hat{E} \), we write \( E^+ := E \cap \hat{E}^+ \) for the intersection
(taken inside \( \hat{E} \)).

We now construct Cohen rings for each of the above subrings of \( \hat{E} \). To begin with, we put
\[
\hat{A}^+ := W(\hat{E}^+), \quad \text{and} \quad \hat{A} := W(\hat{E}),
\]
each of these rings is equipped with a canonical Frobenius automorphism \( \varphi \) and action of \( \mathcal{G}_K \)
via Witt functoriality. Set-theoretically identifying \( W(\hat{E}) \) with \( \prod_{m=0}^\infty \hat{E} \) in the usual way, we endow each
factor with its valuation topology and give \( \hat{A} \) the product topology.\(^{22}\) The \( \mathcal{G}_K \) action
on \( \hat{A} \) is then continuous and the canonical \( \mathcal{G}_K \)-equivariant \( W \)-algebra surjection \( \theta : \hat{A}^+ \to \mathcal{O}_{C_K} \) is continuous when
\( \mathcal{O}_{C_K} \) is given its usual \( p \)-adic topology. For each \( r \geq 0 \), there is a unique continuous \( W \)-algebra map
\( j_r : \mathcal{O}_{C_r} \to \hat{A} \) determined by \( j_r(u_r) := \varphi^{-r}(\xi) - 1 \). These maps are moreover \( \varphi \) and \( \mathcal{G}_K \)-equivariant,
with \( \mathcal{G}_K \) acting on \( \mathcal{O}_{C_r} \) through the quotient \( \mathcal{G}_K \to \Gamma \), and compatible with change in \( r \). We define
\( A_{K,r} := \text{im}(j_r : \mathcal{O}_{C_r} \to \hat{A}) \), which is naturally a \( \varphi \) and \( \mathcal{G}_K \)-stable subring of \( \hat{A} \)
that is independent of our choice of \( \xi \). We again omit the subscript when \( r = 0 \). Note that \( A_{K,r} = \varphi^{-r}(A_K) \) inside \( \hat{A} \),
and that \( A_{K,r} \) is a discrete valuation ring with uniformizer \( p \) and residue field \( \varphi^{-r}(E_K) \) that is purely
inseparable over \( E_K \). We define \( A_{K,\infty} := \bigcup_{r \geq 0} A_{K,r} \) and write \( A_K \) (respectively \( \hat{A}_K \)) for the closure
of \( A_{K,\infty} \) in \( \hat{A} \) with respect to the weak (respectively strong) topology.

Let \( A_{K,\infty}^\text{sh} \) be the strict Henselization of \( A_{K,r} \) with respect to the separable closure of its residue
field inside \( \hat{E} \). Since \( \hat{A} \) is strictly Henselian, there is a unique local morphism \( A_{K,\infty}^\text{sh} \to \hat{A} \)
recovering the given inclusion on residue fields, and we henceforth view \( A_{K,\infty}^\text{sh} \) as a subring of \( \hat{A} \). We denote by
\( A_r \), the topological closure of \( A_{K,\infty}^\text{sh} \) inside \( \hat{A} \) with respect to the strong topology, which is a \( \varphi \) and
\( \mathcal{G}_K \)-stable subring of \( \hat{A} \), and we note that \( A_r = \varphi^{-r}(A) \) and \( A_{r,\mathcal{H}} = A_{K,r} \) inside \( \hat{A} \). We note also that
the canonical map \( \mathbb{Z}_p \to \hat{A}^{\varphi=1} \) is an isomorphism, from which it immediately follows that the same
is true if we replace \( \hat{A} \) by any of its subrings constructed above. If \( A \) is any subring of \( \hat{A} \), we define
\( A^+ := A \cap \hat{A}^+ \), with the intersection taken inside \( \hat{A} \).

Remark 4.1.4. We will identify \( \mathcal{G}_r \) and \( \mathcal{O}_{C_r} \) with their respective images \( A_{K,r}^+ \) and \( A_{K,r} \) in \( \hat{A} \) under
\( j_r \). Writing \( \mathcal{G}_\infty := \lim_{\rightarrow} \mathcal{G}_r \) and \( \mathcal{O}_{\infty} := \lim_{\rightarrow} \mathcal{G}_r \), we likewise identify \( \mathcal{G}_\infty \) with \( A_{K,\infty}^+ \) and \( \mathcal{O}_{\infty} \) with
\( A_{K,\infty} \). Denoting by \( \hat{\mathcal{G}}_\infty \) (respectively \( \hat{\mathcal{G}}_\infty \)) the \( p \)-adic (respectively \( (p, u_0) \)-adic) completions, one has
\[
\hat{\mathcal{G}}_\infty = A_{K,r}^+ = W(E_K^{\text{rad},+}) \quad \text{and} \quad \hat{\mathcal{G}}_\infty = A_{K} = W(\hat{E}_K),
\]
for \( E_K^{\text{rad}} := \bigcup_{r \geq 0} \varphi^{-r}(E_K) \) the radiciel (=perfect) closure of \( E_K \) in \( \hat{E} \) and \( \hat{E}_K \) its topological completion.
Via these identifications, \( \omega := u_0/u_1 \in A_{K,1}^+ \) is a principal generator of \( \ker(\theta : A^+ \to \mathcal{O}_{C_K}) \).

We can now explain the functorial relation between \( \mathcal{M}_r(G) \) and \( T_pG_K \):

\(^{22}\) The valuation \( v_K \) on \( \hat{E} \) induces the usual discrete valuation on \( E_{K,r} \), with the unusual normalization \( 1/p^{r-1}(p-1) \).
\(^{23}\) This is what is called the \textit{weak topology} on \( \hat{A} \). If each factor of \( \hat{E} \) is instead given the discrete topology, then the
product topology on \( \hat{A} = W(\hat{E}) \) is the familiar \( p \)-adic topology, called the \textit{strong topology}. 
Theorem 4.1.5. Let $G \in \text{pdiv}^\Gamma_R$, and write $H^1_{\text{c}}(G_K) := (T_pG_K)^{\vee}$ for the $\mathbb{Z}_p$-linear dual of $T_pG_K$. There is a canonical mapping of finite free $A_r^+$-modules with semilinear Frobenius and $G_K$-actions
\begin{equation}
\mathcal{M}_r(G) \otimes_{\mathcal{E}_r, \varphi} A_r^+ \longrightarrow H^1_{\text{c}}(G_K) \otimes_{\mathbb{Z}_p} A_r^+
\end{equation}
that is injective with cokernel killed by $u_1$. Here, $\varphi$ acts as $\varphi_{\mathcal{M}_r(G)} \otimes \varphi$ on source and as $1 \otimes \varphi$ on target, while $G_K$ acts diagonally on source and target through the quotient $G_K \to \Gamma$ on $\mathcal{M}_r(G)$. In particular, there is a natural $\varphi$ and $G_K$-equivariant isomorphism
\begin{equation}
\mathcal{M}_r(G) \otimes_{\mathcal{E}_r, \varphi} A_r \simeq H^1_{\text{c}}(G_K) \otimes_{\mathbb{Z}_p} A_r.
\end{equation}
These mappings are compatible with duality and with change in $r$ in the obvious manner.

Corollary 4.1.6. For $G \in \text{pdiv}^\Gamma_R$, there are functorial isomorphisms of $\mathbb{Z}_p[G_K]$-modules
\begin{align}
&\mathcal{M}_r(G) \otimes_{\mathcal{E}_r, \varphi} (\mathcal{M}_r(G), A_r^+) \\
&H^1_{\text{c}}(G_K) \cong (\mathcal{M}_r(G) \otimes_{\mathcal{E}_r, \varphi} A_r^+)^{\varphi_{\mathcal{M}_r(G)} \otimes_{\mathbb{Z}_p} \varphi = 1}
\end{align}
which are compatible with duality and change in $r$. In the first isomorphism, we view $A_r^+$ as a $\mathcal{E}_r$-algebra via the composite of the usual structure map with $\varphi$.

Remark 4.1.7. By definition, the map $\varphi^r$ on $\mathcal{E}_r$ is injective with image $\mathcal{E}_r := \mathcal{E}_r^{\text{reg}}$, and so induces a $\varphi$-semilinear isomorphism of $W$-algebras $\varphi^r : \mathcal{E}_r^{\text{reg}} \xrightarrow{\cong} \mathcal{E}_r$. It follows from (4.1.4b) of Corollary 4.1.6 and Fontaine’s theory of $(\varphi, \Gamma)$-modules over $\mathcal{E}$ that $\mathcal{M}_r(G) \otimes_{\mathcal{E}_r, \varphi^r} \mathcal{E}_r$ is the $(\varphi, \Gamma)$-module functorially associated to the $\mathbb{Z}_p[G_K]$-module $H^1_{\text{c}}(G_K)$.

For the remainder of this section, we recall the construction of the functor $\mathcal{M}_r$, both because we shall need to reference it in what follows, and because we feel it is enlightening. For details, including the proofs of Theorems 4.1.3–4.1.5 and Corollary 4.1.6, we refer the reader to [CL12].

Fix $G \in \text{pdiv}^\Gamma_R$, and set $G_0 := G \times_{R_r} R_r/pR_r$. The $\mathcal{E}_r$-module $\mathcal{M}_r(G)$ is a functorial descent of the evaluation of the Dieudonné crystal $\mathbf{D}(G_0)$ on a certain “universal” PD-thickening of $R_r/pR_r$, which we now describe. Let $S_r$ be the $p$-adic completion of the PD-envelope of $\mathcal{E}_r$ with respect to the ideal $\ker \theta$, viewed as a (separated and complete) topological ring via the $p$-adic topology. We give $S_r$ its PD-filtration: for $q \in \mathbb{Z}$ the ideal $\text{Fil}^q S_r$ is the topological closure of the ideal generated by $\{\alpha^n : \alpha \in \ker \theta, n \geq q\}$. By construction, the map $\theta : \mathcal{E}_r \to R_r$ uniquely extends to a continuous surjection of $\mathcal{E}_r$-algebras $S_r \to R_r$, which we again denote by $\theta$ and whose kernel $\text{Fil}^1 S_r$ is equipped with topologically PD-nilpotent divided powers. One shows that there is a unique continuous endomorphism $\varphi$ of $S_r$ extending $\varphi$ on $\mathcal{E}_r$, and that $\varphi(\text{Fil}^1 S_r) \subseteq pS_r$; in particular, we may define $\varphi_1 : \text{Fil}^1 S_r \to S_r$ by $\varphi_1 := \varphi/p$, which is a $\varphi$-semilinear homomorphism of $S_r$-modules. Note that $\varphi_1(E_r)$ is a unit of $S_r$, so the image of $\varphi_1$ generates $S_r$ as an $S_r$-module.

Since the action of $\Gamma$ on $\mathcal{E}_r$ preserves $\ker \theta$, it follows from the universal mapping property of divided power envelopes and $p$-adic continuity considerations that this action uniquely extends to a continuous and $\varphi$-equivariant action of $\Gamma$ on $S_r$ which is compatible with the PD-structure and the filtration. Similarly, the transition map $\mathcal{E}_r \hookrightarrow \mathcal{E}_{r+1}$ uniquely extends to a continuous $\mathcal{E}_r$-algebra homomorphism $S_r \to S_{r+1}$ which is moreover compatible with filtrations (because $E_r(u_r) = E_{r+1}(u_{r+1})$ under our identifications), and for nonnegative integers $s < r$ we view $S_s$ as an $S_r$-algebra via these maps.

\footnotetext{24}{Here we use our assumption that $p > 2$.}
Definition 4.1.8. Let $\text{BT}^G_{S_r}$ be the category of triples $(\mathcal{M}, \text{Fil}^1 \mathcal{M}, \varphi, \#_1)$ where

- $\mathcal{M}$ is a finite free $S_r$-module and $\text{Fil}^1 \mathcal{M} \subseteq \mathcal{M}$ is an $S_r$-submodule.
- $\text{Fil}^1 \mathcal{M}$ contains $(\text{Fil}^1 S_r) \mathcal{M}$ and the quotient $\mathcal{M} / \text{Fil}^1 \mathcal{M}$ is a free $S_r / \text{Fil}^1 S_r = R_r$-module.
- $\varphi, \#_1 : \text{Fil}^1 \mathcal{M} \to \mathcal{M}$ is a $\varphi$-semilinear map whose image generates $\mathcal{M}$ as an $S_r$-module.

Morphisms in $\text{BT}^G_{S_r}$ are $S_r$-module homomorphisms which are compatible with the extra structures. As per our convention, we will often write $\mathcal{M}$ for a triple $(\mathcal{M}, \text{Fil}^1 \mathcal{M}, \varphi, \#_1)$, and $\#_1$ for $\varphi, \#_1$ when it can cause no confusion. We denote by $\text{BT}^G_{S_r} \Gamma$ the subcategory of $\text{BT}^G_{S_r}$ consisting of objects $\mathcal{M}$ that are equipped with a semilinear action of $\Gamma$ which preserves $\text{Fil}^1 \mathcal{M}$, commutes with $\varphi, \#_1$, and whose restriction to $\Gamma_r$ is trivial on $\mathcal{M} / u_r \mathcal{M}$; morphisms in $\text{BT}^G_{S_r} \Gamma$ are $\Gamma$-equivariant morphisms in $\text{BT}^G_{S_r}$.

The kernel of the surjection $S_r / p^n S_r \to R_r / p R_r$ is the image of the ideal $\text{Fil}^1 S_r + p S_r$, which by construction is equipped topologically PD-nilpotent divided powers. We may therefore define

$$ (4.1.5) \mathcal{M}_r(G) = D(G_0)_S := \lim_{n \to} D(G_0)_{S/p^n S} , $$

which is a finite free $S_r$-module that depends contravariantly functorially on $G_0$. We promote $\mathcal{M}_r(G)$ to an object of $\text{BT}^G_{S_r} \Gamma$ as follows. As the quotient map $S_r \to R_r$ induces a PD-morphism of PD-thickenings of $R_r / p R_r$, there is a natural isomorphism of free $R_r$-modules

$$ (4.1.6) \mathcal{M}_r(G) \otimes_{S_r} R_r \simeq D(G_0)_{R_r} . $$

By Proposition 2.2.6, there is a canonical “Hodge” filtration $\omega_G \subseteq D(G_0)_{R_r}$, which reflects the fact that $G$ is a $p$-divisible group over $R_r$ lifting $G_0$, and we define $\text{Fil}^1 \mathcal{M}_r(G)$ to be the preimage of $\omega_G$ under the composite of the isomorphism $(4.1.6)$ with the natural surjection $\mathcal{M}_r(G) \to \mathcal{M}_r(G) \otimes_{S_r} R_r$; note that this depends on $G$ and not just on $G_0$. The Dieudonné crystal is compatible with arbitrary base change, so the relative Frobenius $F_{G_0} : G_0 \to G_0^{(p)}$ induces an canonical morphism of $S_r$-modules

$$ \varphi^* D(G_0)_{S_r} \simeq D(G_0^{(p)})_{S_r} \xrightarrow{D(F_{G_0})} D(G_0)_{S_r} , $$

which we may view as a $\varphi$-semilinear map $\varphi, \#_1(G) : \mathcal{M}_r(G) \to \mathcal{M}_r(G)$. As the relative Frobenius map $\omega_{G_0^{(p)}} \to \omega_{G_0}$ is zero, it follows that the restriction of $\varphi, \#_1(G)$ to $\text{Fil}^1 \mathcal{M}_r(G)$ has image contained in $p, \mathcal{M}_r(G)$, so we may define $\varphi, \#_1(G) \otimes_{S_r} R_r \simeq \mathcal{M}_r(G) / p$, and one proves as in [Kis06, Lemma A.2] that the image of $\varphi, \#_1(G) \otimes_{S_r} R_r$ generates $\mathcal{M}_r(G)$ as an $S_r$-module.

It remains to equip $\mathcal{M}_r(G)$ with a canonical semilinear action of $\Gamma$. Let us write $G_{K_r}$ for the generic fiber of $G$ and $G_K$ for its unique descent to $K = K_0$. The existence of this descent is reflected by the existence of a commutative diagram with cartesian square

$$ (4.1.7) \begin{array}{ccc}
G_K \times K_r & \xrightarrow{1 \times \gamma} & G_K \times K_r \\
\downarrow{\rho_1} & & \downarrow{\rho_2} \\
\Spec(K) & \xrightarrow{\gamma} & \Spec(K_r)
\end{array} $$

$$ (G_K \times K_r) \xrightarrow{\gamma} G_K \times K_r $$

$$ \rho_1 \square \rho_2 $$

$$ \Spec(K_r) \xrightarrow{\gamma} \Spec(K_r) $$
for each $\gamma \in \Gamma$, compatibly with change in $\gamma$; here, the subscript of $\gamma$ denotes base change along the map of schemes induced by $\gamma$. Since $G$ has generic fiber $G_{K_r} = G_K \times_K K_r$, Tate’s Theorem ensures that the dotted arrow above uniquely extends to an isomorphism of $p$-divisible groups over $R_r$

$G \longrightarrow \gamma \longrightarrow G_{\gamma}$

compatibly with change in $\gamma$. By assumption, the action of $\Gamma$ on $S_r$ commutes with the divided powers on $\Fil^1 S_r$ and induces the given action on the quotient $S_r \to R_r$; in other words, $\Gamma$ acts by automorphisms on the object $(\Spec(R_r/pR_r) \to \Spec(S_r/p^n S_r))$ of $\Cris((R_r/pR_r)/W)$. Since $D(G_0)$ is a crystal, each $\gamma \in \Gamma$ therefore gives an $S_r$-linear map

$\gamma^* D(G_0)_{S_r} \simeq D((G_0)_\gamma)_{S_r} \longrightarrow D(G_0)_{S_r}$

and hence an $S_r$-semilinear (over $\gamma$) endomorphism $\gamma$ of $\mathcal{M}(G)$. One easily checks that the resulting action of $\Gamma$ on $\mathcal{M}(G)$ commutes with $\varphi_{\mathcal{M},1}$ and preserves $\Fil^1 \mathcal{M}(G)$. By the compatibility of $D(G_0)$ with base change and the obvious fact that the $W$-algebra surjection $S_r \to W$ sending $u_r$ to 0 is a PD-morphism over the canonical surjection $R_r/pR_r \to k$, there is a natural isomorphism

$\mathcal{M}(G) \otimes_{S_r} W \simeq D(G_W)$

It follows easily from this and the diagram (4.1.7) that the action of $\Gamma_r$ on $\mathcal{M}(G)/u_r \mathcal{M}(G)$ is trivial.

To define $\mathcal{M}_r(G)$, we functionally descend the $S_r$-module $\mathcal{M}(G)$ along the structure morphism $\alpha_r : \mathcal{S}_r \to S_r$. More precisely, for $\mathcal{M} \in \BT_{\mathcal{S}_r}^{\varphi, \Gamma}$, we define $\alpha_{r*}(\mathcal{M}) := (M, \Fil^1 M, \Phi_1) \in \BT_{\mathcal{S}_r}^{\varphi, \Gamma}$ via:

$M := \mathcal{M} \otimes_{\mathcal{S}_r, \alpha_r \circ \varphi} S_r$ with diagonal $\Gamma$-action

$\Fil^1 M := \{m \in M : \varphi_{\mathcal{M}} \otimes \id(m) \in \mathcal{M} \otimes_{\mathcal{S}_r} \Fil^1 S_r \subseteq \mathcal{M} \otimes_{\mathcal{S}_r} S_r\}$

$\Phi_1 : \Fil^1 M \varphi_{\mathcal{M}} \otimes \id \mathcal{M} \otimes \Fil^1 S_r \otimes \varphi \longrightarrow \mathcal{M} \otimes_{\mathcal{S}_r, \varphi} S_r = M$.

The following is the key technical point of [CL12], and is proved using the theory of windows:

**Theorem 4.1.9.** For each $r$, the functor $\alpha_{r*} : \BT_{\mathcal{S}_r}^{\varphi, \Gamma} \to \BT_{\mathcal{S}_r}^{\varphi, \Gamma}$ is an equivalence of categories, compatible with change in $r$.

**Definition 4.1.10.** For $G \in \pddiv_R^{\varphi, \Gamma}$, we write $\mathcal{M}_r(G)$ for the functorial descent of $\mathcal{M}(G)$ to an object of $\BT_{\mathcal{S}_r}^{\varphi, \Gamma}$ as guaranteed by Theorem 4.1.9. By construction, we have a natural isomorphism of functors $\alpha_{r*} \circ \mathcal{M}_r \simeq \mathcal{M}_r$ on $\pddiv_{R_r}^{\varphi, \Gamma}$.

**Example 4.1.11.** Using Messing’s description of the Dieudonné crystal of a $p$-divisible group in terms of the Lie algebra of its universal extension (cf. remark 2.2.7), one calculates that for $r \geq 1$

$\mathcal{M}_r(Q_p/Z_p) = \mathcal{S}_r$, $\varphi_{\mathcal{M}_r}(Q_p/Z_p) : = \varphi$, $\gamma := \gamma$

$\mathcal{M}_r(\mu_{p^{\infty}}) = \mathcal{S}_r$, $\varphi_{\mathcal{M}_r}(\mu_{p^{\infty}}) : = \omega \cdot \varphi$, $\gamma := \chi(\gamma)^{-1} \varphi^{r-1}(\gamma u_r/u_r) \cdot \gamma$

with $\gamma \in \Gamma$ acting as indicated. Note that both $\mathcal{M}_r(Q_p/Z_p)$ and $\mathcal{M}_r(\mathbf{G}_m[p^{\infty}])$ arise by base change from their incarnations when $r = 1$, as follows from the fact that $\omega = \varphi(u_1)/u_1$ and $\varphi^{r-1}(\gamma u_r/u_r) = \gamma u_1/u_1$ via our identifications.
4.2. The case of ordinary $p$-divisible groups. When $G \in \text{pdiv}^\Gamma_{R_\ell}$ is ordinary, one can say significantly more about the structure of the $\mathcal{G}_\tau$-module $\mathfrak{M}_r(G)$. To begin with, we observe that for arbitrary $G \in \text{pdiv}^\Gamma_{R_\ell}$, the formation of the maximal étale quotient of $G$ and of the maximal connected and multiplicative-type sub $p$-divisible groups of $G$ are functorial in $G$, so each of $G^{\acute{e}t}$, $G^0$, and $G^m$ is naturally an object of $\text{pdiv}^\Gamma_{R_\ell}$ as well. We thus (functorially) obtain objects $\mathfrak{M}_r(G^*)$ of $\text{BT}_{\mathcal{G}_\tau}$ which admit particularly simple descriptions when $\ast = \acute{e}t$ or $m$, as we now explain.

As usual, we write $\overline{G}^\ast$ for the special fiber of $G^*$ and $\textbf{D}(\overline{G}^\ast)_W$ for its Dieudonné module. Twisting the $W$-algebra structure on $\mathcal{G}_\tau$ by the automorphism $\varphi^{-1}$ of $W$, we define objects of $\text{BT}_{\mathcal{G}_\tau}$

\begin{equation}
\mathfrak{M}^{\acute{e}t}_r(G) := \textbf{D}(\overline{G}^{\acute{e}t})_W \otimes_{W,\varphi^{-1}} \mathcal{G}_\tau, \quad \varphi^{\acute{e}t}_\mathfrak{M} := F \otimes \varphi, \quad \gamma := \gamma \otimes \gamma
\end{equation}

\begin{equation}
\varphi^{m}_\mathfrak{M} := V^{-1} \otimes E_r \otimes \varphi, \quad \gamma := \gamma \otimes \chi(\gamma)^{-1}\varphi^{-1}(\gamma u_r/u_r) \cdot \gamma
\end{equation}

with $\gamma \in \Gamma$ acting as indicated. Note that these formulae make sense and do indeed give objects of $\text{BT}_{\mathcal{G}_\tau}$ as $V$ is invertible\footnote{A $\varphi^{-1}$-semilinear map of $W$-modules $V : D \rightarrow D$ is invertible if there exists a $\varphi$-semilinear endomorphism $V^{-1}$ whose composition with $V$ in either order is the identity. This is easily seen to be equivalent to the invertibility of the linear map $V \otimes 1 : D \rightarrow \varphi^*D$, with $V^{-1}$ the composite of $(V \otimes 1)^{-1}$ and the $\varphi$-semilinear map $\text{id} \otimes 1 : D \rightarrow \varphi^*D$.} on $\textbf{D}(\overline{G}^\ast)_W$ and $\gamma u_r/u_r \in \mathcal{G}_\tau$. It follows easily from these definitions that $\varphi^{\acute{e}t}_\mathfrak{M}$ linearizes to an isomorphism when $\ast = \acute{e}t$ and has image contained in $\omega \cdot \mathfrak{M}^{\acute{e}t}_r(G)$ when $\ast = m$ Of course, $\mathfrak{M}^{\acute{e}t}_r(G)$ is contravariant functorial in—and depends only on—the closed fiber $\overline{G}^\ast$ of $G^*$.

**Proposition 4.2.1.** Let $G$ be an object of $\text{pdiv}^\Gamma_{R_\ell}$ and let $\mathfrak{M}^{\acute{e}t}_r(G)$ and $\mathfrak{M}^m_r(G)$ be as above. The map $F^r : G_0 \rightarrow G_0^{(p^r)}$ (respectively $V^r : G_0^{(p^r)} \rightarrow G_0$) induces a natural isomorphism in $\text{BT}_{\mathcal{G}_\tau}$

\begin{equation}
\mathfrak{M}_r(G^{\acute{e}t}) \simeq \mathfrak{M}^{\acute{e}t}_r(G) \quad \text{respectively} \quad \mathfrak{M}_r(G^m) \simeq \mathfrak{M}^m_r(G)
\end{equation}

These identifications are compatible with change in $r$ in the sense that for $\ast = \acute{e}t$ (respectively $\ast = m$) there is a canonical commutative diagram in $\text{BT}_{\mathcal{G}_\tau}$

\begin{equation}
\begin{array}{c}
\mathfrak{M}_{r+1}(G^* \times_{R_\ell} R_{r+1}) \xrightarrow{\text{(4.2.2)}} \mathfrak{M}^{\acute{e}t}_{r+1}(G \times_{R_\ell} R_{r+1}) \xrightarrow{\text{(4.2.3)}} \mathfrak{M}^m_{r+1}(G \times_{R_\ell} R_{r+1}) \\
\xrightarrow{\text{(4.2.4a)}} \text{D}(\overline{G}^*)_W \otimes_{W,\varphi^{-1}} \mathcal{G}_{r+1} \xrightarrow{\text{F} \otimes \text{id}} \xrightarrow{\text{V}^{-1} \otimes \text{id}} \text{D}(\overline{G}^*)_W \otimes_{W,\varphi^{-1}} \mathcal{G}_{r+1}
\end{array}
\end{equation}

where the left vertical isomorphism is deduced from Theorem 4.1.3 (2).

**Proof.** For ease of notation, we will write $\mathfrak{M}_r^*$ and and $\textbf{D}^*$ for $\mathfrak{M}^*_r(G)$ and $\textbf{D}(\overline{G}^*_W)$, respectively. Using (4.1.10), one finds that $\mathfrak{M}^{\acute{e}t}_r := \alpha_{r+1}(\mathfrak{M}^{\acute{e}t}_r) \in \text{BT}^\Gamma_{\mathcal{G}_\tau}$ is given by the triple

\begin{equation}
\mathfrak{M}^{\acute{e}t}_r := (\textbf{D}^{\acute{e}t} \otimes_{W,\varphi^r} S_r, \textbf{D}^{\acute{e}t} \otimes_{W,\varphi^r} \text{Fil}^1 S_r, F \otimes \varphi_1)
\end{equation}

with $\Gamma$ acting diagonally on the tensor product. Similarly, $\alpha_r(\mathfrak{M}^m_r)$ is given by the triple

\begin{equation}
(\textbf{D}^m \otimes_{W,\varphi^r} S_r, \textbf{D}^m \otimes_{W,\varphi^r} \text{Fil}^1 S_r, V^{-1} \otimes v_r : \varphi)
\end{equation}

where $v_r = \varphi(E_r)/p$ and $\gamma \in \Gamma$ acts on $\textbf{D}^m \otimes_{W,\varphi^r} S_r$ as $\gamma \otimes \chi(\gamma)^{-1}\varphi^{-1}(\gamma u_r/u_r) \cdot \gamma$. Put $\lambda := \log(1+u_0)/u_0$, where $\log(1+X) : \text{Fil}^1 S_r \rightarrow S_r$ is the usual (convergent for the $p$-adic topology) power series and
$u_0$ is viewed as an element of $S_r$ via the structure map $S_0 \to S_r$ (concretely, $u_0 = \varphi^r(u_r)$). For each \( r \geq 0 \), one checks that $\lambda$ admits the convergent product expansion $\lambda = \prod_{i \geq 0} \varphi^i(v_r)$, so $\lambda \in S_r^\times$ and

\[
(4.2.5) \quad \frac{\lambda}{\varphi(\lambda)} = \varphi(E_r)/p = v_r \quad \text{and} \quad \frac{\lambda}{\gamma \lambda} = \chi(\gamma)^{-1}\varphi^r(\gamma u_r/u_r) \quad \text{for} \quad \gamma \in \Gamma.
\]

It follows from (4.2.5) that the $S_r$-module automorphism of $\mathbf{D}^m \otimes_{W,\varphi^r} S_r$ given by multiplication by $\lambda$ carries (4.2.4b) isomorphically onto the object of $\text{BT}^\varphi_{S_r}$ given by the triple

\[
(4.2.6) \quad \mathcal{M}^m_r := \left( \mathbf{D}^m \otimes_{W,\varphi^r} S_r, \mathbf{D}^m \otimes_{W,\varphi^r} S_r, V^{-1} \otimes \varphi \right)
\]

with $\Gamma$ acting diagonally on the tensor product.

On the other hand, since $G^\text{ét}_0$ (respectively $G^m_0$) is étale (respectively of multiplicative type) over $R_r/pR_r$, the relative Frobenius (respectively Verschiebung) morphism of $G_0$ induces isomorphisms

\[
(4.2.7a) \quad G^\text{ét}_0 \xrightarrow{F^r} (G^\text{ét}_0)^{(p^r)} \simeq \varphi^r \mathcal{G}^\text{ét} \times_k R_r/pR_r
\]

respectively

\[
(4.2.7b) \quad G^m_0 \xrightarrow{V^r} (G^m_0)^{(p^r)} \simeq \varphi^r \mathcal{G}^m \times_k R_r/pR_r
\]

of $p$-divisible groups over $R_r/pR_r$, where we have used the fact that the map $x \mapsto x^{p^r}$ of $R_r/pR_r$ factors as $R_r/pR_r \to k \hookrightarrow R_r/pR_r$ in the final isomorphisms of both lines above. Since the Dieudonné crystal is compatible with base change and the canonical map $W \to S_r$ extends to a PD-morphism $(W, p) \to (S_r, pS_r + \text{Fil}^1 S_r)$ over $k \to R_r/pR_r$, applying $\mathbf{D}(\cdot)_{S_r}$ to (4.2.7a)–(4.2.7b) yields natural isomorphisms $\mathbf{D}(G^\text{ét}_0)_{S_r} \simeq \mathbf{D}^* \otimes_{W,\varphi^r} S_r$ for $* = \text{ét}$, $\text{m}$ which carry $F$ to $F \otimes \varphi$. It is a straightforward exercise using the construction of $\mathcal{M}_r(G^*)$ given in §4.1 to check that these isomorphisms extend to give isomorphisms $\mathcal{M}_r(G^\text{ét}) \simeq \mathcal{M}^\text{ét}_r$ and $\mathcal{M}_r(G^m) \simeq \mathcal{M}^m_r$ in $\text{BT}^\varphi_{S_r}$. By Theorem 4.1.9, we conclude that we have natural isomorphisms in $\text{BT}^\varphi_{S_r}$ as in (4.2.2). The commutativity of (4.2.3) is straightforward, using the definitions of the base change isomorphisms.

Now suppose that $G$ is ordinary. As $\mathcal{M}_r$ is exact by Theorem 4.1.3 (1), applying $\mathcal{M}_r$ to the connected-étale sequence of $G$ gives a short exact sequence in $\text{BT}^\varphi_{S_r}$

\[
(4.2.8) \quad 0 \to \mathcal{M}_r(G^\text{ét}) \to \mathcal{M}_r(G) \to \mathcal{M}_r(G^m) \to 0
\]

which is contravariantly functorial and exact in $G$. Since $\varphi_{2n}$ linearizes to an isomorphism on $\mathcal{M}_r(G^\text{ét})$ and is topologically nilpotent on $\mathcal{M}_r(G^m)$, we think of (4.2.8) as the “slope filtration” for Frobenius acting on $\mathcal{M}_r(G)$. On the other hand, Proposition 2.2.6 and Theorem 4.1.3 (3b) provide a canonical “Hodge filtration” of $\mathcal{M}_r(G) \otimes R_r \simeq \mathbf{D}(G_0)_{R_r}$:

\[
(4.2.9) \quad 0 \to \omega_G \to \mathbf{D}(G_0)_{R_r} \to \text{Lie}(G^\text{ét}) \to 0
\]

which is contravariant and exact in $G$. Our assumption that $G$ is ordinary yields (cf. [Kat81]):

**Lemma 4.2.2.** With notation as above, there are natural and $\Gamma$-equivariant isomorphisms

\[
(4.2.10) \quad \text{Lie}(G^\text{ét}) \simeq \mathbf{D}(G^\text{ét}_0)_{R_r} \quad \text{and} \quad \mathbf{D}(G^m_0)_{R_r} \simeq \omega_G.
\]
Composing these isomorphisms with the canonical maps obtained by applying $D(\cdot)_{R_r}$ to the connected-étale sequence of $G_0$ yield functorial $R_r$-linear splittings of the Hodge filtration (4.2.9). Furthermore, there is a canonical and $\Gamma$-equivariant isomorphism of split exact sequences of $R_r$-modules

\[
0 \longrightarrow \omega_G \longrightarrow D(G_0)_{R_r} \longrightarrow \text{Lie}(G^t) \longrightarrow 0
\]  
(4.2.11)

\[
0 \longrightarrow D(G^\text{et}_0)_{W} \otimes_{W,\varphi^r} R_r \longrightarrow D(G)_{W} \otimes_{W,\varphi^r} R_r \longrightarrow D(G^\text{et}_0)_{W} \rightarrow 0
\]

with $i, j$ the inclusion and projection mappings obtained from the canonical direct sum decomposition $D(G)_{W} \simeq D(G^\text{et}_0)_{W} \oplus D(G^\text{mt}_0)_{W}$.

**Proof.** Applying $D(\cdot)_{R_r}$ to the connected-étale sequence of $G_0$ and using Proposition 2.2.6 yields a commutative diagram with exact columns and rows

\[
0 \longrightarrow \omega_G \longrightarrow \omega_{G^m} \longrightarrow 0
\]  
(4.2.12)

\[
0 \longrightarrow D(G^\text{et}_0)_{R_r} \longrightarrow D(G_0)_{R_r} \longrightarrow D(G^\text{m}_0)_{R_r} \longrightarrow 0
\]

\[
0 \longrightarrow \text{Lie}(G^\text{et}_t) \longrightarrow \text{Lie}(G^t) \longrightarrow 0
\]

where we have used the fact that that the invariant differentials and Lie algebra of an étale $p$-divisible group (such as $G^{\text{et}}$ and $G^{\text{mt}} \simeq G^{\text{et}}$) are both zero. The isomorphisms (4.2.10) follow at once. We likewise immediately see that the short exact sequence in the center column of (4.2.12) is functorially and $R_r$-linearly split. Thus, to prove the claimed identification in (4.2.11), it suffices to exhibit natural isomorphisms of free $R_r$-modules with $\Gamma$-action

\[
D(G^\text{et}_0)_{R_r} \simeq D(G^\text{et}_0)_{W} \otimes_{W,\varphi^r} R_r \quad \text{and} \quad D(G^\text{m}_0)_{R_r} \simeq D(G^\text{m}_0)_{W} \otimes_{W,\varphi^r} R_r,
\]

both of which follow easily by applying $D(\cdot)_{R_r}$ to (4.2.7a) and (4.2.7b) and using the compatibility of the Dieudonné crystal with base change as in the proof of Proposition (4.2.1).  

From the slope filtration (4.2.8) of $\mathfrak{M}_r(G)$ we can recover both the (split) slope filtration of $D(G)_{W}$ and the (split) Hodge filtration (4.2.9) of $D(G_0)_{R_r}$.
Proposition 4.2.3. There are canonical and $\Gamma$-equivariant isomorphisms of short exact sequences

$$
\begin{align*}
0 \rightarrow & \mathcal{M}_r(G^\text{sh}) \otimes W \rightarrow \mathcal{M}_r(G) \otimes W \rightarrow \mathcal{M}_r(G^m) \otimes W \rightarrow 0 \\
& (4.2.14a) \\
0 \rightarrow & \mathcal{D}(G^\text{sh})_W \rightarrow \mathcal{D}(G)_W \rightarrow \mathcal{D}(G^m)_W \rightarrow 0 \\
& (4.2.14b) \\
0 \rightarrow & \mathcal{M}_r(G^\text{sh}) \otimes R_r \rightarrow \mathcal{M}_r(G) \otimes R_r \rightarrow \mathcal{M}_r(G^m) \otimes R_r \rightarrow 0 \\
& (4.2.14c)
\end{align*}
$$

Here, $i : \text{Lie}(G^t) \hookrightarrow \mathcal{D}(G_0)_{R_r}$ and $j : \mathcal{D}(G_0)_{R_r} \rightarrow \omega_G$ are the canonical splittings of Lemma 4.2.2, the top row of (4.2.14b) is obtained from (4.2.8) by extension of scalars, and the isomorphism (4.2.14a) intertwines $\varphi_{\mathcal{M}_r(-)} \otimes \varphi$ with $F \otimes \varphi$ and $\psi \otimes 1$ with $V \otimes 1$.

Proof. This follows immediately from Theorem 4.1.3 (3a) and Lemma 4.2.2. \hfill $\square$

5. Results and Main Theorems

In this section, we will state and prove our main results as described in §1.2. Throughout, we will keep the notation of §1.2 and of §4.1 with $k := F_p$.

5.1. The formalism of towers. In this preliminary section, we set up a general commutative algebra framework for dealing with the various projective limits of cohomology modules that we will encounter.

Definition 5.1.1. A tower of rings is an inductive system $\mathcal{A} := \{A_r\}_{r \geq 1}$ of local rings with local transition maps. A morphism of towers $\mathcal{A} \rightarrow \mathcal{A}'$ is a collection of local ring homomorphisms $A_r \rightarrow A'_r$ which are compatible with change in $r$. A tower of $\mathcal{A}$-modules $\mathcal{M}$ consists of the following data:

1. For each integer $r \geq 1$, an $A_r$-module $M_r$.
2. A collection of $A_r$-module homomorphisms $\varphi_{r,s} : M_r \rightarrow M_s \otimes_{A_r} A_r$ for each pair of integers $r \geq s \geq 1$, which are compatible in the obvious way under composition.

A morphism of towers of $\mathcal{A}$-modules $\mathcal{M} \rightarrow \mathcal{M}'$ is a collection of $A_r$-module homomorphisms $M_r \rightarrow M'_r$ which are compatible with change in $r$ in the evident manner. For a tower of rings $\mathcal{A} = \{A_r\}$, we will write $A_{\infty}$ for the inductive limit, and for a tower of $\mathcal{A}$-modules $\mathcal{M} = \{M_r\}$, we set

$$M_B := \lim_r (M_r \otimes_{A_r} B)$$

and write simply $M_{\infty} := M_{A_{\infty}}$, for any $A_{\infty}$-algebra $B$, with the projective limit taken with respect to the induced transition maps.

Lemma 5.1.2. Let $\mathcal{A} = \{A_r\}_{r \geq 0}$ be a tower of rings and suppose that $I_r \subseteq A_r$ is a sequence of proper principal ideals such that $A_r$ is $I_r$-separated and the image of $I_r$ in $A_{r+1}$ is contained in $I_{r+1}$ for all $r$. Write $I_{\infty} := \lim I_r$ for the inductive limit, and set $\overline{A}_r := A_r/I_r$ for all $r$. Let $\mathcal{M} = \{M_r, p_{r,s}\}$ be a tower of $\mathcal{A}$-modules equipped with an action\footnote{That is, a homomorphism of groups $\Delta \rightarrow \text{Aut}_{\mathcal{A}}(\mathcal{M})$, or equivalently, an $A_r$-linear action of $\Delta$ on $M_r$ for each $r$ that is compatible with change in $r$.} of $\Delta$ by $\mathcal{A}$-automorphisms. Suppose that $M_r$ is free of finite rank over $A_r$ for all $r$, and that $\Delta_r$ acts trivially on $M_r$. Let $B$ be an $A_{\infty}$-algebra, and observe...
that $M_B$ is canonically a module over the completed group ring $\Lambda_B$. Assume that $B$ is either flat over $A_\infty$ or that $B$ is a flat $\bar{A}_\infty$-algebra, and that the following two conditions hold for all $r > 0$

(5.1.1a) $\bar{M}_r := M_r/I_rM_r$ is a free $\bar{A}_r[\Delta/\Delta_r]$-module of rank $d$ that is independent of $r$.

(5.1.1b) For all $s \leq r$ the induced maps $\bar{\rho}_{r,s} : \bar{M}_r \longrightarrow \bar{M}_s \otimes_{\bar{\Pi}_s} \bar{A}_r$ are surjective.

Then:

1. $M_r$ is a free $A_r[\Delta/\Delta_r]$-module of rank $d$ for all $r$.

2. The induced maps of $A_r[\Delta/\Delta_s]$-modules

$$M_r \otimes_{A_r[\Delta/\Delta_s]} A_r[\Delta/\Delta_s] \longrightarrow M_s \otimes_{A_s} A_r$$

are isomorphisms for all $r \geq s$.

3. $M_B$ is a finite free $\Lambda_B$-module of rank $d$.

4. For each $r$, the canonical map

$$M_B \otimes_{\Lambda_B} B[\Delta/\Delta_r] \longrightarrow M_r \otimes_{A_r} B$$

is an isomorphism of $B[\Delta/\Delta_r]$-modules.

5. If $B'$ is any $B$-algebra which is flat over $A_\infty$ or $\bar{A}_\infty$, then the canonical map

$$M_B \otimes_{\Lambda_B} \Lambda_{B'} \longrightarrow M_{B'}$$

is an isomorphism of finite free $\Lambda_{B'}$-modules.

Proof. For notational ease, let us put $\Lambda_{A_r,s} := A_r[\Delta/\Delta_s]$ for all pairs of nonnegative integers $r, s$. Note that $\Lambda_{A_r,s}$ is a local $A_r$-algebra, so the principal ideal $I_r := I_r\Lambda_{A_r,s}$ is contained in the radical of $\Lambda_{A_r,s}$.

Let us fix $r$ and choose a principal generator $f_r \in A_r$ of $I_r$ (hence also of $\bar{I}_r$). The module $M_r$ is obviously finite over $\Lambda_{A_r,r}$ (as it is even finite over $A_r$), so by hypothesis (5.1.1a) we may choose $m_1, \ldots, m_d \in M_r$ with the property that the images of the $m_i$ in $\bar{M}_r = M_r/\bar{I}_rM_r$ freely generate $\bar{M}_r$ as an $\bar{A}_r[\Delta/\Delta_r] = \Lambda_{A_r,r}/\bar{I}_r$-module. By Nakayama’s Lemma [Mat89, Corollary to Theorem 2.2], we conclude that $m_1, \ldots, m_d$ generate $M_r$ as a $\Lambda_{A_r,r}$-module. If

$$\sum_{i=1}^d x_i m_i = 0$$

is any relation on the $m_i$ with $x_i \in \Lambda_{A_r,r}$, then necessarily $x_i \in \bar{I}_r\Lambda_{A_r,r}$, and we claim that $x_i \in \bar{I}_r^j$ for all $j \geq 0$. To see this, we proceed by induction and suppose that our claim holds for $j \leq N$. Since $\bar{I}_r$ is principal, for each $i$ there exists $x_i' \in \Lambda_{A_r,r}$ with $x_i = f_r^N x_i'$, and the relation (5.1.2) reads $f_r^N m = 0$ with $m \in M_r$ given by $m := \sum_{i=1}^d x_i' m_i$. Since $M_r$ is free as an $A_r$-module, it is in particular torsion free, so we conclude that $m = 0$. Since the images of the $m_i$ freely generate $M_r/\bar{I}_rM_r$, it follows that $x_i' \in \bar{I}_r$ and hence that $x_i \in \bar{I}_r^{N+1}$, which completes the induction. By our assumption that $A_r$ is $\bar{I}_r$-adically separated, we must have $x_i = 0$ for all $i$ and the relation (5.1.2) is trivial. We conclude that $m_1, \ldots, m_d$ freely generate $M_r$ over $\Lambda_{A_r,r}$, giving (1).

To prove (2), note that our assumption (5.1.1b) that the maps $\bar{\rho}_{r,s}$ are surjective for all $r \geq s$ implies that the same is true of the maps $\rho_{r,s}$ (again by Nakayama’s Lemma) and hence that the induced map of $\Lambda_{A_r,s}$-modules in (2) is surjective. As this map is then a surjective map of free $\Lambda_{A_r,s}$-modules of the same rank $d$, it must be an isomorphism.
Since the kernel of the canonical surjection $\Lambda_{A_r,r} \to \Lambda_{A_r,s}$ lies in the radical of $\Lambda_{A_r,r}$, we deduce by Nakayama’s Lemma that any lift to $M_r$ of a $\Lambda_{A_r,s}$-basis of $M_s \otimes_{A_r} A_r$ is a $\Lambda_{A_r,r}$-basis of $M_r$. It follows easily from this that the projective limit $M_B$ is a free $\Lambda_B$-module of rank $d$ for any flat $\Lambda_{A_r}$-algebra $B$. The corresponding assertions for any flat $\Lambda_{A_r}$-algebra $B$ follow similarly, using the hypotheses (5.1.1a) and (5.1.1b) directly, and this gives (3).

Observe that the mapping of (4) is obtained from the canonical surjection $M_B \to M_r \otimes_{A_r} B$ by extension of scalars, keeping in mind the natural identification $M_r \otimes_{A_r} B \otimes_{\Lambda_B} B[\Delta/\Delta_r] \simeq M_r \otimes_{A_r} B$. It follows at once that this mapping is surjective. By (1) and (3), we conclude that the mapping in (4) is a surjection of free $B[\Delta/\Delta_r]$-modules of the same rank and is hence an isomorphism as claimed.

It remains to prove (5). Extending scalars, the canonical maps $M_B \to M_r \otimes_{A_r} B$ induce surjections

$$M_B \otimes_{\Lambda_B} \Lambda_{B'} \longrightarrow (M_r \otimes_{A_r} B) \otimes_{\Lambda_B} \Lambda_{B'} \simeq M_r \otimes_{A_r} B'$$

that are compatible in the evident manner with change in $r$. Passing to inverse limits gives the mapping $M_B \otimes_{\Lambda_B} \Lambda_{B'} \to M_{B'}$ of (5). Due to (3), this is then a map of finite free $\Lambda_{B'}$-modules of the same rank, so to check that it is an isomorphism it suffices by Nakayama’s Lemma to do so after applying $\otimes_{\Lambda_B} B'[\Delta/\Delta_r]$, which is an immediate consequence of (4).  

We record the following elementary commutative algebra fact, which will be extremely useful to us:

**Lemma 5.1.3.** Let $A \to B$ be a local homomorphism of local rings which makes $B$ into a flat $A$-algebra, and let $M$ be an arbitrary $A$-module. Then $M$ is a free $A$-module of finite rank if and only if $M \otimes_A B$ is a free $B$-module of finite rank.

**Proof.** First observe that since $A \to B$ is local and flat, it is faithfully flat. We write $M = \varprojlim M_\alpha$ as the direct limit of its finite $A$-submodules, whence $M \otimes_A B = \varprojlim(M_\alpha \otimes_A B)$ with each of $M_\alpha \otimes_A B$ naturally a finitely generated $B$-submodule of $M \otimes_A B$. Assume that $M \otimes_A B$ is finitely generated as a $B$-module. Then there exists $\alpha$ with $M_\alpha \otimes_A B \to M \otimes_A B$ surjective, and as $B$ is faithfully flat over $A$, this implies that $M_\alpha \to M$ is surjective, whence $M$ is finitely generated over $A$. Suppose in addition that $M \otimes_A B$ is free as a $B$-module. In particular, $M \otimes_A B$ is $B$-flat, which implies by faithful flatness of $B$ over $A$ that $M$ is $A$-flat (see, e.g. [Mat89, Exercise 7.1]). Then $M$ is a finite flat module over the local ring $A$, whence it is free as an $A$-module by [Mat89, Theorem 7.10].  

Finally, we analyze duality for towers with $\Delta$-action.

**Lemma 5.1.4.** With the notation of Lemma 5.1.2, let $\mathscr{M} := \{M_r, \rho_{r,s}\}$ and $\mathscr{M}' := \{M'_r, \rho'_{r,s}\}$ be two towers of $\mathscr{A}$-modules with $\Delta$-action satisfying (5.1.1a) and (5.1.1b). Suppose that for each $r$ there exist $A_r$-linear perfect duality pairings

$$\langle \cdot, \cdot \rangle_r : M_r \times M'_r \longrightarrow A_r$$

with respect to which $\delta$ is self-adjoint for all $\delta \in \Delta$, and which satisfy the compatibility condition\(^{27}\)

$$\langle \rho_{r,s} m, \rho'_{r,s} m' \rangle_s = \sum_{\delta \in \Delta_s/\Delta_r} \langle m, \delta^{-1} m' \rangle_r$$

\(^{27}\)By abuse of notation, for any map of rings $A \to B$ and any $A$-bilinear pairing of $A$-modules $\langle \cdot, \cdot \rangle : M \times M' \to A$, we again write $\langle \cdot, \cdot \rangle : M_B \times M'_B \to B$ for the $B$-bilinear pairing induced by extension of scalars.
for all \( r \geq s \). Then for each \( r \), the pairings \( (\cdot, \cdot)_r : M_r \times M'_r \to \Lambda_{A_r, r} \) defined by
\[
(m, m')_r := \sum_{\delta \in \Delta/\Delta_r} \langle m, \delta^{-1} m' \rangle_r \cdot \delta
\]
are \( \Lambda_{A_r, r} \)-bilinear and perfect, and compile to give a \( \Lambda_B \)-linear perfect pairing
\[
(\cdot, \cdot)_{\Lambda_B} : M_B \times M'_B \to \Lambda_B.
\]

In particular, \( M_B' \) and \( M_B \) are canonically \( \Lambda_B \)-linearly dual to eachother.

**Proof.** An easy reindexing argument shows that \( (\cdot, \cdot)_r \) is \( \Lambda_{A_r, r} \)-linear in the right factor, from which it follows that it is also \( \Lambda_{A_r, r} \)-linear in the left due to our assumption that \( \delta \in \Delta \) is self-adjoint with respect to \( (\cdot, \cdot)_r \). To prove that \( (\cdot, \cdot)_r \) is a perfect duality pairing, we analyze the \( \Lambda_{A_r, r} \)-linear map
\[
(5.1.5) \quad M_r \xrightarrow{m \mapsto (m, \cdot)_r} \text{Hom}_{\Lambda_{A_r, r}}(M'_r, \Lambda_{A_r, r}).
\]

Due to Lemma 5.1.2, both \( M_r \) and \( M'_r \) are free \( \Lambda_{A_r, r} \)-modules, necessarily of the same rank by the existence of the perfect \( A_r \)-duality pairing (5.1.3). It follows that (5.1.5) is a homomorphism of free \( \Lambda_{A_r, r} \)-modules of the same rank. To show that it is an isomorphism it therefore suffices to prove it is surjective, which may be checked after extension of scalars along the augmentation map \( \Lambda_{A_r, r} \to A_r \) by Nakayama’s Lemma. Consider the diagram
\[
\begin{align*}
M_r \otimes_{\Lambda_{A_r, r}} A_r &\xrightarrow{(5.1.5) \otimes 1} \text{Hom}_{\Lambda_{A_r, r}}(M'_r, \Lambda_{A_r, r}) \otimes_{\Lambda_{A_r, r}} A_r \\
&\xrightarrow{\xi} \text{Hom}_{A_r}(M'_r \otimes_{\Lambda_{A_r, r}} A_r, A_r) \\
M_1 \otimes_{\Lambda_1} A_r &\xrightarrow{\rho_{r,1} \otimes 1} \text{Hom}_{A_r}(M'_1 \otimes_{\Lambda_1} A_r, A_r)
\end{align*}
\]

where \( \xi \) is the canonical map sending \( f \otimes \alpha \) to \( \alpha(f \otimes 1) \), and the bottom horizontal arrow is obtained by \( A_r \)-linearly extending the canonical duality map \( m \mapsto \langle m, \cdot \rangle_1 \). On the one hand, the vertical maps in (5.1.6) are isomorphisms thanks to Lemma 5.1.2 (2), while the map \( \xi \) and the bottom horizontal arrow are isomorphisms because arbitrary extension of scalars commutes with linear duality of free modules.\(^\dagger\)

On the other hand, this diagram commutes because (5.1.4) guarantees the relation
\[
\langle \rho_{r,1} m, \rho'_{r,1} m'_1 \rangle_{(5.1.4)} \sum_{\delta \in \Delta/\Delta_r} \langle m, \delta^{-1} m' \rangle_r \equiv (m, m')_r \text{ mod } I_\Delta
\]

where \( I_\Delta = \ker(\Lambda_{A_r, r} \to A_r) \) is the augmentation ideal We conclude that (5.1.5) is an isomorphism, as desired. The argument that the corresponding map with the roles of \( M_r \) and \( M'_r \) interchanged is an isomorphism proceeds *mutatis mutandis*.

Using the definition of \( (\cdot, \cdot)_r \) and (5.1.4), one has more generally that
\[
(\rho_{r,s} m, \rho'_{r,s} m')_s \equiv (m, m')_r \text{ mod } \ker(\Lambda_{A_r, r} \to \Lambda_{A_r, s})
\]

\(^\dagger\)Quite generally, for any ring \( R \), any \( R \)-modules \( M, N \), and any \( R \)-algebra \( S \), the canonical map
\[
\xi_M : \text{Hom}_R(M, N) \otimes_R S \to \text{Hom}_S(M \otimes_R S, N \otimes_R S)
\]
sending \( f \otimes s \) to \( s(f \otimes \text{id}_S) \) is an isomorphism if \( M \) is finite and free over \( R \). Indeed, the map \( \xi_R \) is visibly an isomorphism, and one checks that \( \xi_{M_1 \oplus M_2} \) is naturally identified with \( \xi_{M_1} \oplus \xi_{M_2} \).
for all $r \geq s$. In particular, the pairings $(\cdot, \cdot)_r$ induce, by extension of scalars, a $\Lambda_B$-bilinear pairing

$$(\cdot, \cdot)_{\Lambda_B} : M_B \times M'_B \longrightarrow \Lambda_B$$

which satisfies the specialization property

$$(\cdot, \cdot)_{\Lambda_B} \equiv (\cdot, \cdot)_r \mod \ker(\Lambda_B \rightarrow \Lambda_{B,r}).$$

From $(\cdot, \cdot)_{\Lambda_B}$ we obtain in the usual way duality morphisms

$$(5.1.8) \quad M_B \xrightarrow{\text{hom-}(\cdot, \cdot)_{\Lambda_B}} \text{Hom}_{\Lambda_B}(M'_B, \Lambda_B) \quad \text{and} \quad M_B \xrightarrow{\text{hom-}(\cdot, \cdot)_{\Lambda_B}} \text{Hom}_{\Lambda_B}(M_B, \Lambda_B)$$

which we wish to show are isomorphisms. Due to Lemma 5.1.2 (3), each of (5.1.8) is a map of finite free $\Lambda_B$-modules of the same rank, so we need only show that these mappings are surjective. As the kernel of $\Lambda_B \rightarrow \Lambda_{B,r}$ is contained in the radical of $\Lambda_B$, we may by Nakayama’s Lemma check such surjectivity after extension of scalars along $\Lambda_B \rightarrow \Lambda_{B,r}$ for any $r$, where it follows from (5.1.7) and the fact that $M_r$ and $M_s$ are free $\Lambda_{A_r}$-modules, so that the extension of scalars of the perfect duality pairing $(\cdot, \cdot)_r$ along the canonical map $\Lambda_{A_r} \rightarrow \Lambda_{B,r}$ is again perfect.

5.2. Ordinary families of de Rham cohomology. Let $\{X_r/T_r\}_{r \geq 0}$ be the tower of modular curves introduced in §2.3. As $X_r$ is regular and proper flat over $T_r = \text{Spec}(R_r)$ with geometrically reduced fibers, it is a curve in the sense of Definition 2.1.1 (thanks to Corollary 2.1.3) which moreover satisfies the hypotheses of Proposition 2.1.11. Abbreviating

$$(5.2.1) \quad H^0(\omega_r) := H^0(X_r, \omega_{X_r/S_r}), \quad H^1_{dR,r} := H^1(X_r/R_r), \quad H^1(\theta_r) := H^1(X_r, \theta_{X_r}),$$

Proposition 2.1.11 (2) provides a canonical short exact sequence $H(X_r/R_r)$ of finite free $R_r$-modules

$$(5.2.2) \quad 0 \longrightarrow H^0(\omega_r) \longrightarrow H^1_{dR,r} \longrightarrow H^1(\theta_r) \longrightarrow 0$$

which recovers the Hodge filtration of $H^1_{dR}(X_r/K_r)$ after inverting $p$.

The Hecke correspondences on $X_r$ induce, via Proposition 2.1.11 (4) (or by Proposition 2.2.4 and Remark 2.2.5), canonical actions of $\delta_r$ and $\delta_r^\ast$ on $H(X_r/R_r)$ via $R_r$-linear endomorphisms. In particular, $H(X_r/R_r)$ is canonically a short exact sequence of $\mathbb{Z}_p[[\mathbb{Z}/N^r\mathbb{Z}]^\times]$-modules via the diamond operators. Similarly, pullback along (2.3.3) yields $R_r$-linear morphisms $H((X_r)_/\gamma) \rightarrow H(X_r/R_r)$ for each $\gamma \in \Gamma$; using the fact that hypercohomology commutes with flat base change (by Čech theory), we obtain an action of $\Gamma$ on $H(X_r/R_r)$ which is $R_r$-semilinear over the canonical action of $\Gamma$ on $R_r$ and which commutes with the actions of $\delta_r$ and $\delta_r^\ast$ as the Hecke operators are defined over $K_0 = \mathbb{Q}_p$.

For $r \geq s$, we will need to work with the base change $X_s \times_{T_s} T_r$, which is a curve over $T_r$ thanks to Proposition 2.1.2. Although $X_s \times_{T_s} T_r$ need no longer be regular as $T_r \rightarrow T_s$ is not smooth when $r > s$, we claim that it is necessarily normal. Indeed, this follows from the more general assertion:

**Lemma 5.2.1.** Let $V$ be a discrete valuation ring and $A$ a finite type Cohen-Macaulay $V$-algebra with smooth generic fiber and geometrically reduced special fiber. Then $A$ is normal.

**Proof.** We claim that $A$ satisfies Serre’s “$R_1 + S_2$”-criterion for normality [Mat89, Theorem 23.8]. As $A$ is assumed to be CM, by definition of Cohen-Macaulay $A$ verifies $S_i$ for all $i \geq 0$, so we need only show that each localization of $A$ at a prime ideals of codimension 1 is regular. Since $A$ has geometrically reduced special fiber, this special fiber is in particular smooth at its generic points. As $A$ is flat over $V$ (again by definition of CM), we deduce that the (open) $V$-smooth locus in Spec $A$ contains the generic points of the special fiber and hence contains all codimension-1 points (as the generic fiber of Spec $A$ is assumed to be smooth). Thus $A$ is $R_1$, as desired. ■
We conclude that $X_s \times_{T_s} T_r$ is a normal curve, and we obtain from Proposition 2.1.11 a canonical short exact sequence of finite free $R_r$-modules $H(X_s \times_{T_s} T_r/R_r)$ which recovers the Hodge filtration of $H^1_{dR}(X_s/K_r)$ after inverting $p$. As hypercohomology commutes with flat base change and the formation of the relative dualizing sheaf and the structure sheaf are compatible with arbitrary base change, we have a natural isomorphism of short exact sequences of free $R_r$-modules

\[(5.2.3)\]

\[H(X_s \times_{T_s} T_r/R_r) \simeq H(X_s/R_s) \otimes_{R_s} R_r.\]

In particular, we have $R_r$-linear actions of $\delta^*_s$, $\delta_r$ and an $R_s$-semilinear action of $\Gamma$ on $H(X_s \times_{T_s} T_r/R_r)$. These actions moreover commute with one another.

Consider now the canonical degeneracy map $\rho : X_r \to X_s \times_{T_s} T_r$ of curves over $T_r$ induced by (2.3.6). As $X_r$ and $X_s \times_{T_s} T_r$ are normal and proper curves over $T_r$, we obtain from Proposition 2.1.11 (4) canonical trace mappings of short exact sequences

\[(5.2.4)\]

\[\rho_* : H(X_r/R_r) \longrightarrow H(X_s \times_{T_s} T_r/R_r) \simeq H(X_s/R_s) \otimes_{R_s} R_r\]

which recover the usual trace mappings on de Rham cohomology after inverting $p$; as such, these mappings are Hecke and $\Gamma$-equivariant, and compatible with change in $r, s$ in the obvious way. Tensoring these maps (5.2.4) over $R_r$ with $R_{\infty}$, we obtain projective systems of free $R_{\infty}$ with semilinear $\Gamma$-action and commuting, linear $\delta^*_s$-action:

**Definition 5.2.2.** We write

\[H^0(\omega) := \varinjlim_r \left( H^0(\omega_r) \otimes R_{\infty} \right), \quad H^1_{dR} := \varinjlim_r \left( H^1_{dR,r} \otimes R_{\infty} \right), \quad H^1(\Theta) := \varinjlim_r \left( H^1(\Theta_r) \otimes R_{\infty} \right)\]

for the projective limit with respect to the maps induced by $\rho_*$, each of which is naturally a module for $\Lambda_{R_{\infty}} = R_{\infty}[\Delta]$, and is equipped with a semilinear $\Gamma$-action and a linear $\delta^*_s$-action.

Although we have a left exact sequence of $\Lambda_{R_{\infty}}$-modules with semilinear $\Gamma$-action and $\delta^*_s$-action

\[0 \longrightarrow H^0(\omega) \longrightarrow H^1_{dR} \longrightarrow H^1(\Theta)\]

this sequence is almost certainly not right exact. It is moreover unlikely that any of the $\Lambda_{R_{\infty}}$-modules in Definition 5.2.2 are finitely generated. The situation is much better if we pass to ordinary parts:

**Theorem 5.2.3.** Let $e^*$ be the idempotent of $\delta^*_s$ associated to $U^*_p$ and let $d$ be the positive integer defined as in Proposition 3.2.1 (1). Then $e^*H^0(\omega)$, $e^*H^1_{dR}$ and $e^*H^1(\Theta)$ are free $\Lambda_{R_{\infty}}$-modules of ranks $d$, $2d$, and $d$ respectively, and there is a canonical short exact sequence of free $\Lambda_{R_{\infty}}$-modules with linear $\delta^*_s$-action and $R_{\infty}$-semilinear $\Gamma$-action

\[(5.2.5)\]

\[0 \longrightarrow e^*H^0(\omega) \longrightarrow e^*H^1_{dR} \longrightarrow e^*H^1(\Theta) \longrightarrow 0.\]

For each positive integer $r$, applying $\otimes_{\Lambda_{R_{\infty}}} R_{\infty}[\Delta/\Delta_r]$ to (5.2.5) yields the short exact sequence

\[(5.2.6)\]

\[0 \longrightarrow e^*H^0(\omega_r) \otimes_{R_r} R_{\infty} \longrightarrow e^*H^1_{dR} \otimes_{R_r} R_{\infty} \longrightarrow e^*H^1(\Theta) \otimes_{R_r} R_{\infty} \longrightarrow 0,\]

compatibly with the actions of $\delta^*_s$ and $\Gamma$.

**Proof.** Applying $e^*$ to the short exact sequence $H(X_r/R_r)$ yields a short exact sequence

\[(5.2.7)\]

\[0 \longrightarrow e^*H^0(\omega) \longrightarrow e^*H^1_{dR,r} \longrightarrow e^*H^1(\Theta_r) \longrightarrow 0\]
of $R_r[\Delta/\Delta_r]$-modules with linear $S_r^\ast$-action and $R_r$-semilinear $\Gamma$-action in which each term is free as an $R_r$-module.\footnote{Indeed, $e^*M$ is a direct summand of $M$ for any $S_r^\ast$-module $M$, and hence $R_r$-projective ($= R_r$-free) if $M$ is.}

Similarly, for each pair of nonnegative integers $r \geq s$, the trace mappings (5.2.4) induce a commutative diagram with exact rows

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & e^*H^0(\omega_r) & \longrightarrow & e^*H^1_{\text{dr},r} & \longrightarrow & e^*H^1(\mathcal{O}_r) & \longrightarrow & 0 \\
& & \rho_s \downarrow & & \rho_s \downarrow & & \rho_s \downarrow & & \\
0 & \longrightarrow & e^*H^0(\omega_s) \otimes_{R_s} R_r & \longrightarrow & e^*H^1_{\text{dr},s} \otimes_{R_s} R_r & \longrightarrow & e^*H^1(\mathcal{O}_s) \otimes_{R_s} R_r & \longrightarrow & 0
\end{array}
\] (5.2.8)

We will apply Lemma 5.1.2 with $A_r = R_r$, $I_r = (\pi_r)$, $B = R_\infty$ nd with $M_r$ each one of the terms in (5.2.7). In order to do this, we must check that the hypotheses (5.1.1a) and (5.1.1b) are satisfied.

Applying $\otimes_{R_r} \mathbf{F}_p$ to the short exact sequence (5.2.7) and using the fact that the idempotent $e^*$ commutes with tensor products, we obtain, thanks to Lemma 2.1.16 (1), the short exact sequence of $\mathbf{F}_p$-vector spaces (3.3.12a). By Corollary 3.3.3, the three terms of (3.3.12a) are free $\mathbf{F}_p[\Delta/\Delta_r]$-modules of ranks $d, 2d$, and $d$ respectively, so (5.1.1a) is satisfied for each of these terms. Similarly, applying $\otimes_{R_r} \mathbf{F}_p$ to the diagram (5.2.8) yields a diagram which by Corollary 3.3.2 is naturally isomorphic to the diagram of $\mathbf{F}_p[\Delta/\Delta_r]$-modules with split-exact rows

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & H^0(I^\infty_r, \Omega^1_{\mathcal{O}_r})_{\text{ord}} & \longrightarrow & H^0(I^\infty_s, \Omega^1_{\mathcal{O}_s})_{\text{ord}} \oplus H^1(I^0_r, \mathcal{O}(-\omega))_{\text{ord}} & \longrightarrow & H^1(I^0_r, \mathcal{O}(-\omega))_{\text{ord}} & \longrightarrow & 0 \\
& & \rho_s \downarrow & & \rho_s \downarrow & & \rho_s \downarrow & & \\
0 & \longrightarrow & H^0(I^\infty_s, \Omega^1_{\mathcal{O}_s})_{\text{ord}} & \longrightarrow & H^0(I^\infty_s, \Omega^1_{\mathcal{O}_s})_{\text{ord}} \oplus H^1(I^0_s, \mathcal{O}(-\omega))_{\text{ord}} & \longrightarrow & H^1(I^0_s, \mathcal{O}(-\omega))_{\text{ord}} & \longrightarrow & 0
\end{array}
\]

Each of the vertical maps in this diagram is surjective due to Proposition 3.2.1 (2), and we conclude that the hypothesis (5.1.1b) is satisfied as well. Furthermore, the vertical maps in (5.2.8) are then surjective by Nakayama’s Lemma, so applying $\otimes_{R_r} R_\infty$ yields an inverse system of short exact sequences in which the first term satisfies the Mittag-Leffler condition. Passing to inverse limits is therefore (right) exact, and we obtain the short exact sequence (5.2.5).

Due to Proposition 2.1.11 (3), the short exact sequence (5.2.2) is auto-dual with respect to the canonical cup-product pairing $(\cdot, \cdot)$ on $H^1_{\text{dr},r}$. We extend scalars along $R_r \to R'_{\infty} := R_r[\mu_N]$, so that the Atkin-Lehner “involution” $w_r$ is defined, and consider the “twisted” pairing on ordinary parts

\[
\langle \cdot, \cdot \rangle_r : (e^*H^1_{\text{dr},r})_{R'_r} \times (e^*H^1_{\text{dr},r})_{R'_r} \longrightarrow R'_r \quad \text{given by} \quad \langle x, y \rangle_r := (x, w_r U^y_{p} y).
\] (5.2.9)

It is again perfect and satisfies $\langle T^s x, y \rangle = \langle x, T^s y \rangle$ for all $x, y \in (e^*H^1_{\text{dr},r})_{R'_r}$ and $T^s \in S_r^\ast$.

**Proposition 5.2.4.** The pairings (5.2.9) compile to give a perfect $\Lambda_{R'_\infty}$-linear duality pairing

\[
\langle \cdot, \cdot \rangle_{\Lambda_{R'_\infty}} : (e^*H^1_{\text{dr}})_{\Lambda_{R'_\infty}} \times (e^*H^1_{\text{dr}})_{\Lambda_{R'_\infty}} \longrightarrow \Lambda_{R'_\infty} \quad \text{given by} \quad \langle x, y \rangle_{\Lambda_{R'_\infty}} := \lim_{r \to \infty} \sum_{\delta \in \Delta/\Delta_r} \langle x_r, \delta^{-1} y_r \rangle_r \cdot \delta
\]
for \( x = \{ x_r \}_r \) and \( y = \{ y_r \}_r \) in \( (e^*H^1_{\text{dR}})_\Lambda_{R^\infty} \). The pairing \( \langle \cdot, \cdot \rangle_{\Lambda_{R^\infty}} \) induces a canonical isomorphism

\[
0 \longrightarrow e^*H^0(\omega)(\langle \chi \rangle (a)_N)_{\Lambda_{R^\infty}} \longrightarrow e^*H^1_{\text{dR}}(\langle \chi \rangle (a)_N)_{\Lambda_{R^\infty}} \longrightarrow e^*H^1(\Theta)(\langle \chi \rangle (a)_N)_{\Lambda_{R^\infty}} \longrightarrow 0
\]

that is \( \mathcal{H}^* \)-equivariant and compatible with the natural action of \( \Gamma \times \text{Gal}(K'_0/K_0) \simeq \text{Gal}(K'_\infty/K_0) \) on the bottom row and the twist \( \gamma \cdot m := \langle \chi(\gamma) \rangle (a(\gamma))_N N^{-\gamma} m \) of the natural action on the top, where \( a(\gamma) \in (\mathbb{Z}/N\mathbb{Z})^\times \) is determined by \( \zeta^{\mu(\gamma)} = \gamma \zeta \) for every \( \zeta \in \mu_N(\overline{\mathbb{Q}}) \).

Proof of Proposition 5.2.4. That \( \langle \cdot, \cdot \rangle_{\Lambda_{R^\infty}} \) is a perfect duality pairing follows easily from Lemma 5.1.4, using Theorem 5.2.3 and the formalism of §5.1, once we check that the twisted pairings (5.2.9) satisfy the hypothesis (5.1.3). By the definition (5.2.9) of \( \langle \cdot, \cdot \rangle_r \), this amounts to the computation

\[
\langle \rho_1 x, w_r U^r p_1 y \rangle_r = \langle x, \rho_1 w_r U^r p_1 y \rangle_{r+1} = \langle x, w_r U^r_\rho p_1 y \rangle_{r+1} = \sum_{\delta \in \Delta_r/\Delta_{r+1}} \langle x, w_r U^r_\rho \delta^{r+1} \rangle_{r+1}
\]

where we have used Proposition 2.3.6 and the identity \( \rho_2^* p_1 = U^r p_1 \rho_2^* \) on \( H^1_{\text{dR}, r+1} \), which follows from Lemma 5.4.1 by using Lemma 5.4.5 and Proposition 2.2.4. We obtain an isomorphism of short exact sequences of \( \Lambda_{R^\infty} \)-modules as in (5.2.4), which it remains to show is \( \Gamma \times \text{Gal}(K'_0/K_0) \)-equivariant for the specified actions. For this, we compute that for \( \gamma \in \text{Gal}(K'_\infty/K_0) \),

\[
\langle \gamma x, \gamma y \rangle_r = \langle \gamma x, w_r U^r \gamma y \rangle_r = \langle \gamma x, \gamma w_r U^r (\chi(\gamma)^{-1})(a(\gamma)^{-1})_N y \rangle_r = \gamma(x, \chi(\gamma)^{-1})(a(\gamma)^{-1})_N y)_r
\]

where we have used Proposition 2.3.6 and the fact that the cup product is Galois-equivariant. It now follows easily from definitions that

\[
\langle \gamma x, \gamma y \rangle_{\Lambda_{R^\infty}} = \langle \chi(\gamma)^{-1} \rangle \gamma \langle x, (a(\gamma)^{-1})_N y \rangle_{\Lambda_{R^\infty}},
\]

and the claimed \( \Gamma \times \text{Gal}(K'_0/K_0) \)-equivariance of (5.2.4) is equivalent to this.

Remark 5.2.5. For an open subgroup \( H \) of \( \mathcal{G}_K \) and any \( H \)-stable subfield \( F \) of \( C_K \), denote by \( \text{Rep}_F(H) \) the category of finite-dimensional \( F \)-vector spaces that are equipped with a continuous semilinear action of \( H \). Recall [Sen81] that classical Sen theory provides a functor \( D_{\text{Sen}} : \text{Rep}_{C_K}(\mathcal{G}_K) \to \text{Rep}_{K_\infty}(\Gamma) \) which is quasi-inverse to \( (\cdot) \otimes_{K_{\infty}} C_K \). Furthermore, for any \( W \in \text{Rep}_{K_\infty}(\mathcal{G}_K) \), there is a unique \( K_\infty \)-linear operator \( \Theta_D \) on \( D := D_{\text{Sen}}(W) \) with the property that \( \gamma x = \exp(\log \chi(\gamma) \cdot \Theta_D)(x) \) for all \( x \in D \) and all \( \gamma \) in a small enough open neighborhood of \( 1 \) in \( \Gamma \).

We expect that for \( W \) any specialization of \( e^*H^1_{\text{et}} \) along a continuous homomorphism \( \Lambda \to K_\infty \), there is a canonical isomorphism between \( D := D_{\text{Sen}}(W \otimes C_K) \) and the corresponding specialization of \( e^*H^1_{\text{dR}} \), with the Sen operator \( \Theta_D \) induced by the Gauss-Manin connections on \( H^1_{\text{dR}} \). In this way, we might think of \( e^*H^1_{\text{dR}} \) as a \( \Lambda \)-adic avatar of \( D_{\text{Sen}}(e^*H^1_{\text{et}} \otimes \Lambda_{\text{dC}_K}) \). We hope to pursue these connections in future work.

\[\text{■}\]

\[\text{The reader will check that our forward reference to §5.4 does not involve any circular reasoning.}\]
5.3. Ordinary $\Lambda$-adic modular forms. In this section, we discuss the relation between $e^*H^0(\omega)$ and ordinary $\Lambda_{k_{\infty}}$-adic cuspforms as defined by Ohta [Oht95, Definition 2.1.1].

We begin with some preliminaries on modular forms. For a ring $A$, a congruence subgroup $\Gamma$, and a nonnegative integer $k$, we will write $S_k(\Gamma; A)$ for the space of weight $k$ cuspforms for $\Gamma$ over $A$; we put $S_k(\Gamma) := S_k(\Gamma; \overline{Q})$. If $\Gamma'$, $\Gamma$ are congruence subgroups and $\gamma \in \text{GL}_2(\mathbb{Q})$ satisfies $\gamma^{-1}\Gamma\gamma \subseteq \Gamma$, then there is a canonical injective “pullback” map on modular forms $\iota_\gamma : S_k(\Gamma) \rightarrow S_k(\Gamma')$ given by $\iota_\gamma(f) := f|_{\gamma^{-1}\Gamma\gamma}$. When $\Gamma' \subseteq \Gamma$, unless specified to the contrary, we will always view $S_k(\Gamma)$ as a subspace of $S_k(\Gamma')$ via $\iota_{id}$. As $\gamma^{-1}\Gamma\gamma$ is necessarily of finite index in $\Gamma$, one also has a canonical “trace” mapping

\begin{equation}
\text{tr}_\gamma : S_k(\Gamma') \rightarrow S_k(\Gamma) \quad \text{given by} \quad \text{tr}_\gamma(f) := \sum_{\delta \in \gamma^{-1}\Gamma\gamma \setminus \Gamma} (f|_{\gamma\delta})
\end{equation}

with the property that $\text{tr}_\gamma \circ \iota_\gamma$ is multiplication by $[\Gamma : \gamma^{-1}\Gamma\gamma]$ on $S_k(\Gamma)$.

We define

\begin{align*}
S^\infty_2(\Gamma; R_r) &:= S_2(\Gamma; R_r) \quad \text{and} \quad S^0_2(\Gamma; R_r) := \{ f \in S_2(\Gamma; \mathbb{Q}_p) : f|_{w_r} \in S^\infty_2(\Gamma; R_r) \},
\end{align*}

By definition, $S^*_2(\Gamma; R_r)$ for $* = 0, \infty$ are $R_r$-submodules of $S_2(\Gamma; K^*_r)$ that are carried isomorphically onto each other by the automorphism $w_r$ of $S_2(\Gamma; K^*_r)$. Note that $S^*_2(\Gamma; R_r)$ is precisely the $R_r$-submodule consisting of cusps whose formal expansion at the cusp $\ast$ has coefficients in $R_r$. As the Hecke algebra $\mathfrak{H}_r$ stabilizes $S^\infty_2(\Gamma; R_r)$, it follows immediately from Proposition 2.3.24 that $S^0_2(\Gamma; R_r)$ is stable under the action of $\mathfrak{H}_r^*$. Furthermore, $\text{Gal}(K_{0}/K_0)$ acts on $S_2(\Gamma; K^*_r)$ and $S^\infty_2(\Gamma; R_r)$ through the second tensor factor, and this action leaves stable the $R_r$-submodule $S^\infty_2(\Gamma; R_r)$. The second equality of Proposition 2.3.6 then implies that $S^0_2(\Gamma; R_r)$ is also $\text{Gal}(K^*_r/K_0)$-stable $R_r$-submodule of $S_2(\Gamma; K^*_r)$. A straightforward computation shows that the direct factor $\text{Gal}(K^*_0/K_0)$ of $\text{Gal}(K^*_r/K_0)$ acts trivially on $S^\infty_2(\Gamma; R_r)$ and through $\langle a \rangle_{N}^{-1}$ on $S^0_2(\Gamma; R_r)$.

We can interpret $S^*_{2}(\Gamma; R_r)$ geometrically as follows. As in Remark 2.3.12, for $* = \infty, 0$ let $I_r^\ast$ be the irreducible component of $X_r$ passing through the cusp $\ast$, and denote by $X_r^\ast$ the complement in $X_r$ of all irreducible components of $X_r$ distinct from $I_r^\ast$. By construction, $X_r$ and $X_r^\ast$ have the same generic fiber $X_r \times_{\mathbb{Q}_p} K_r$. Using Proposition 2.3.10, it is not hard to show that the diamond operators induce automorphisms of $X_r^\ast$, and one checks via Proposition 2.3.14 that the “semilinear” action (2.3.3) of $\gamma \in \Gamma$ on $X_r^\ast$ carries $X_r^\ast$ to $(X_r^\ast)^{\gamma}$ for all $\gamma$.

**Lemma 5.3.1.** Formal expansion at the $R_r$-point $\infty$ (respectively $R'_r$-point 0) of $X_r^\ast$ induces an isomorphism of $R_r$-modules

\begin{equation}
H^0(X_r^\infty, \Omega^1_{X_r^\infty/R_r}) \simeq S^\infty_2(\Gamma; R_r) \quad \text{respectively} \quad H^0(X_r^0, \Omega^1_{X_r^0/R_r}) \langle \langle a \rangle_N^{-1} \rangle \simeq S^0_2(\Gamma; R_r)
\end{equation}

which is equivalent for the natural action of $\Gamma$ and $\mathfrak{H}_r$ (respectively $\mathfrak{H}_r^*$) on source and target and, in the case of the second isomorphism, intertwines the action of $\text{Gal}(K_{0}/K_0)$ via $\langle a \rangle^{-1}_N$ on source with the natural action on the target.

**Proof.** The proof is a straightforward adaptation of the proof of [Edi06, Proposition 2.5].

Now $X_r \rightarrow S_r$ is smooth outside the supersingular points, so there is a canonical closed immersion $\iota_\ast^* : X_r^\ast \hookrightarrow X_r^{sm}$. Using Lemmas 2.1.9 and 5.3.1, pullback of differentials along $\iota_\ast^*$ gives a natural map

\begin{equation}
H^0(X_r, \omega_{X_r/T_r}) \simeq H^0(X_r^{sm}, \Omega^1_{X_r^{sm}/T_r}) \xrightarrow{(\iota_\ast^*)^*} H^0(X_r^\ast, \Omega^1_{X_r^\ast/T_r}) \simeq S^2_2(\Gamma; R_r)
\end{equation}
which is an isomorphism after inverting $p$. In particular, the map (5.3.3) is injective, $\Gamma$-equivariant, and compatible with the natural action of $\mathcal{F}_r$ (respectively $\mathcal{F}_r^\ast$) on source and target for $* = \infty$ (respectively $* = 0$), and in the case of $* = 0$ intertwines the action of $\text{Gal}(K_0'/K_0)$ via the character $\langle a \rangle^{-1}_N$ on source with the natural action on the target.

**Remark 5.3.2.** The image of (5.3.3) for $* = \infty$ is naturally identified with the space of weight 2 cuspforms for $\Gamma_r$ whose formal expansion at every cusp has $R_r$-coefficients.

Applying the idempotent $e$ (respectively $e^*$) to (5.3.3) with $* = \infty$ (respectively $* = 0$) gives an injective homomorphism

\begin{equation}
(5.3.4a)
\xymatrix{ e H^0(\mathcal{X}_r, \omega_{\mathcal{X}_r}/T_r) \ar[r] & e S^\infty_2(Np^r; R_r)}
\end{equation}

respectively

\begin{equation}
(5.3.4b)
\xymatrix{ e^* H^0(\mathcal{X}_r, \omega_{\mathcal{X}_r}/T_r)((\langle a \rangle^{-1}_N) \ar[r] & e^* S^0_2(Np^r; R_r)}
\end{equation}

which is compatible with the canonical actions of $\Gamma$ and of $\mathcal{F}_r$ (respectively $\mathcal{F}_r^\ast$) on source and target and in the case of (5.3.4a) is $\text{Gal}(K_0'/K_0)$-equivariant.

**Proposition 5.3.3.** The mappings (5.3.4a) and (5.3.4b) are isomorphisms.

**Proof.** We treat the case of (5.3.4a); the proof that (5.3.4a) is an isomorphism goes through mutatis mutandis. We must show that (5.3.4a) is surjective. To do this, let $\nu \in e H^0(\mathcal{X}_r, \omega_{\mathcal{X}_r}/R_r)$ be arbitrary. Since (5.3.4a) is an isomorphism after inverting $\pi_r$, there exists a least nonnegative integer $d$ such that $\pi^d_r \nu$ is in the image of (5.3.4a). Assume that $d \geq 1$, and let $\eta \in e H^0(\mathcal{X}_r, \omega_{\mathcal{X}_r}/R_r)$ be any element mapping to $\pi^d_r \nu$. For an irreducible component $I$ of $\mathcal{X}_r$, write $I^h$ for the complement of the supersingular points in $I$, and denote by $i^\infty_r : I_r^\infty \hookrightarrow \mathcal{X}_r^\infty$ the canonical immersion. We then have a commutative diagram

\begin{equation}
(5.3.5)
\xymatrix{ H^0(\mathcal{X}_r, \omega_{\mathcal{X}_r}/R_r) \ar[r]^{(5.3.3) \text{ mod } \pi_r} \ar[d]_{\prod_{I \in \text{Irr}(\mathcal{X}_r)} H^0(I^h, \Omega^1_{I^h}/F_p)} & H^0(\mathcal{X}_r^\infty, \Omega^1_{\mathcal{X}_r^\infty}/R_r) \otimes_{R_r} F_p \ar[d]^{(\langle \nu \rangle^*)} \\
H^0(I^h, \Omega^1_{I^h}/F_p) \ar[r]_{\text{proj}_\infty,} & H^0(I_r^\infty, \Omega^1_{I_r^\infty}/F_p)}
\end{equation}

where the left vertical mapping follows from Definition 2.1.13 and Remark 2.1.14 (cf. the proof of Proposition 3.3.1), while the bottom map is simply projection. Our assumption that $d \geq 1$ implies that the image of $\overline{\eta} := \eta \mod \pi_r$ under the composite of the right vertical and top horizontal maps in (5.3.5) is zero and hence, viewing $\overline{\eta} = (\eta_{a,b,u})$ as a meromorphic differential on the normalization of $\overline{\mathcal{X}}_r$, we have $\eta_{(r,0,1)} = \text{proj}_\infty(\overline{\eta}) = 0$. Using the formula (3.3.5a), we deduce that $U^a_r \overline{\eta} = 0$ for $n$ sufficiently large. But $U^a_r$ acts invertibly on $\eta$ (and hence on $\overline{\eta}$) so we necessarily have that $\overline{\eta} = 0$ or what is the same thing that $\eta \mod \pi_r = 0$. We conclude that $\pi^{d-1} \nu$ is in the image of (5.3.4a), contradicting the minimality of $d$. Thus $d = 0$ and (5.3.4a) is surjective. ■

For $s \leq r$, Ohta shows [Oht95, 2.3.4] that the trace mapping $\text{tr}_{id} : S_k(\Gamma_r; K_r) \to S_k(\Gamma_s; K_s) \otimes_{K_s} K_r$ attached to the inclusion $\Gamma_r \subseteq \Gamma_s$ carries $S^0_k(\Gamma_r; R_r)$ into $S^0_k(\Gamma_s; R_s) \otimes_{R_s} R_r$, so that the projective
limit
\[ G^*_k(N, R_\infty) := \lim_{\rightarrow} S^0_k(\Gamma_r; R_r) \otimes_{R_r} R_\infty \]

makes sense. It is canonically a \( \Lambda_{R_\infty} \)-module, equipped with an action of \( \mathcal{H}_* \), a semilinear action of \( \Gamma \), and a natural action of \( \text{Gal}(K'_\infty/K_0) \). On the other hand, let \( eS(N; \Lambda_{R_\infty}) \subset \Lambda_{R_\infty}[q] \) be the space of ordinary \( \Lambda_{R_\infty} \)-adic cuspforms of level \( N \), as defined in [Oht95, 2.5.1]. This space is equipped with an action of \( \mathcal{H}_* \) via the usual formulae on formal \( q \)-expansions (see, for example [Wil88, §1.2]), as well as an action of \( \Gamma \) via its \( q \)-coefficient-wise action on \( \Lambda_{R_\infty}[q] \).

**Theorem 5.3.4** (Ohta). Then there is a canonical isomorphism of \( \Lambda_{R_\infty} \)-modules

\[
(5.3.6) \quad eS(N; \Lambda_{R_\infty}) \xrightarrow{\simeq} e^*G^*_k(N, R_\infty)
\]

that intertwines the action of \( T \in \mathcal{H}_* \) on the source with that of \( T^* \in \mathcal{H}_* \) on the target, for all \( T \in \mathcal{H}_* \). This isomorphism is \( \text{Gal}(K'_\infty/K_0) \)-equivariant for the natural action of \( \text{Gal}(K'_\infty/K_0) \) on \( e^*G^*_k(N, R_\infty) \) and the twisted action \( \gamma \cdot \mathcal{F} := \langle (\chi(\gamma))^{-1} (\alpha(\gamma))^{-1} \mathcal{F} \rangle_{N} \) on \( eS(N; \Lambda_{R_\infty}) \).

**Proof.** For the definition of the canonical map (5.3.6), as well as the proof that it is an isomorphism, see Theorem 2.3.6 and its proof in [Oht95]. With the conventions of [Oht95], the claimed compatibility of (5.3.6) with Hecke operators is a consequence of [Oht95, 2.5.1], while the \( \text{Gal}(K'_\infty/K_0) \)-equivariance of (5.3.6) follows from [Oht95, Proposition 3.5.6].

**Corollary 5.3.5.** There is a canonical isomorphism of \( \Lambda_{R_\infty} \)-modules

\[
(5.3.7) \quad eS(N; \Lambda_{R_\infty}) \langle (\chi)^{-1} \rangle \simeq e^*H^0(\omega)
\]

that intertwines the action of \( T \in \mathcal{H}_* \) on the source with \( T^* \in \mathcal{H}_* \) on the target and is \( \Gamma \)-equivariant for the canonical action of \( \Gamma \) on \( e^*H^0(\omega) \) and the twisted action \( \gamma \cdot \mathcal{F} := \langle (\chi(\gamma))^{-1} \mathcal{F} \rangle_{N} \) on \( eS(N; \Lambda_{R_\infty}) \).

**Proof.** This follows immediately from Proposition 5.3.3 and Theorem 5.3.4.

### 5.4. \( \Lambda \)-adic Barsotti-Tate groups.

In order to define a crystalline analogue of Hida’s ordinary \( \Lambda \)-adic étale cohomology, we will apply the theory of §4 to a certain “tower” \( \{ G_r/R_r \}_{r \geq 1} \) of \( p \)-divisible groups (a \( \Lambda \)-adic Barsotti Tate group in the sense of Hida [Hid05a], [Hid05b]) whose construction involves artfully cutting out certain \( p \)-divisible subgroups of \( J_r[p^\infty] \) over \( \mathbb{Q} \) and the “good reduction” theorems of Langlands-Clay–Ray-Saito. The construction of \( \{ G_r/R_r \}_{r \geq 1} \) is certainly well-known (e.g. [MW86, §1], [MW84, Chapter 3, §1], [Til87, Definition 1.2] and [Oht95, §3.2]), but as we shall need substantially finer information about the \( G_r \) than is available in the literature, we devote this section to recalling their construction and properties.

For nonnegative integers \( i \leq r \), write \( \Gamma^i_r := \Gamma_1(Np^i) \cap \Gamma_0(p^r) \) for the intersection (taken inside \( \text{SL}_2(\mathbb{Z}) \)), so \( \Gamma_r = \Gamma^r_r \). We will need the following fact (cf. [Til87, pg. 339], [Oht95, 2.3.3]) concerning the trace mapping (5.3.1) attached to the canonical inclusion \( \Gamma_r \subset \Gamma_i \) for \( r \geq i \); for notational clarity, we will write \( \text{tr}_{r,i} : S_k(\Gamma_r) \rightarrow S_k(\Gamma_i) \) for this map.

**Lemma 5.4.1.** Fix integers \( i \leq r \) and let \( \text{tr}_{r,i} : S_k(\Gamma_r) \rightarrow S_k(\Gamma_i) \) be the trace mapping (5.3.1) attached to the inclusion \( \Gamma_r \subset \Gamma_i \). For \( \alpha := \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \), we have an equality of \( \overline{\mathbb{Q}} \)-endomorphisms of \( S_k(\Gamma_r) \)

\[
(5.4.1) \quad \iota_{\alpha^{r-i}} \circ \text{tr}_{r,i} = (U_p^*)^{-i} \sum_{\delta \in \Delta_i/\Delta_r} \langle \delta \rangle.
\]
Proof. We have index $p^{r-i}$ inclusions of groups $\Gamma_r \subseteq \Gamma_i^j \subseteq \Gamma_i$ with $\Gamma_r$ normal in $\Gamma_i^j$, as it is the kernel of the canonical surjection $\Gamma_i^j \rightarrow \Delta_i/\Delta_r$. For each $\delta \in \Delta_i/\Delta_r$, we fix a choice of $\sigma_\delta \in \Gamma_r^i$ mapping to $\delta$ and calculate that

$$
\Gamma_i = \prod_{\delta \in \Delta_i/\Delta_r} \prod_{j=0}^{p^{r-i}-1} \Gamma_r \sigma_\delta \varrho_j
$$

where $\varrho_j := \begin{pmatrix} 1 & 0 \\ jNp^i & 1 \end{pmatrix}$.

On the other hand, for each $0 \leq j < p^{r-i}$ one has the equality of matrices in $GL_2(\mathbb{Q})$

$$
p^{r-i} \varrho_j \alpha^{-(r-i)} = \tau_r \begin{pmatrix} 1 & -j \\ 0 & p^{r-i} \end{pmatrix} \tau_r^{-1}
$$

for $\tau_r := \begin{pmatrix} 0 & -1 \\ Np^r & 0 \end{pmatrix}$.

The claimed equality (5.4.1) follows easily from (5.4.2) and (5.4.3), using the equalities of operators

$$
(\cdot)|_{\sigma_\delta} = (\delta) \quad \text{and} \quad U^*_p = v_r U_p w_r^{-1} \quad \text{on} \quad S_k(\Gamma_r)
$$

(see Proposition 2.3.24). \hfill \blacksquare

Perhaps the most essential “classical” fact for our purposes is that the Hecke operator $U_p$ acting on spaces of modular forms “contracts” the $p$-level, as is made precise by the following:

**Lemma 5.4.2.** If $f \in S_k(\Gamma_i^j)$ then $U^*_p f$ is in the image of the canonical map $\nu_{id} : S_k(\Gamma_{r-d}) \hookrightarrow S_k(\Gamma_r)$ for each integer $d \leq r - i$. In particular, $U^*_p = w_r U_p w_r^{-1}$ on $S_k(\Gamma_r)$ (see Proposition 2.3.24).

Certainly Lemma 5.4.2 is well-known (e.g. [Til87], [Hid05a], [Oht99]), because of its importance in our subsequent applications, we sketch a proof (following the proof of [Oht99, Lemma 1.2.10]; see also [Hid05a, §2]). We note that $\Gamma_r \subseteq \Gamma_i^j$ for all $i \leq r$, and the resulting inclusion $S_k(\Gamma_i^j) \hookrightarrow S_k(\Gamma_r)$ has image consisting of forms on $\Gamma_r$ which are eigenvectors for the diamond operators and whose associated character has conductor with $p$-part dividing $p^i$.

**Proof of Lemma 5.4.2.** Fix $d$ with $0 \leq d \leq r - i$ and let $\alpha := \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$ be as in Lemma 5.4.1; then $\alpha^d$ is an element of the commeasurable of $\Gamma_i^j$ in $SL_2(\mathbb{Q})$. Consider the following subgroups of $\Gamma_{r-d}^i$:

$$
H := \Gamma_{r-d}^i \cap \alpha^{-d} \Gamma_r^i \alpha^d \\
H' := \Gamma_{r-d}^i \cap \alpha^{-d} \Gamma_{r-d}^i \alpha^d,
$$

with each intersection taken inside of $SL_2(\mathbb{Q})$. We claim that $H = H'$ inside $\Gamma_{r-d}^i$. Indeed, as $\Gamma_i^j \subseteq \Gamma_{r-d}^i$, the inclusion $H \subseteq H'$ is clear. For the reverse inclusion, if $\gamma := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \Gamma_{r-d}^i$, then we have $\alpha^{-d} \gamma \alpha^d = \begin{pmatrix} p^{-d} & 0 \\ 0 & p^d \end{pmatrix}$, so if this lies in $\Gamma_{r-d}^i$ we must have $x \equiv 0 \mod p^r$ and hence $\gamma \in \Gamma_r^i$. We conclude that the coset spaces $H/\Delta_{r-d}^i$ and $H'/\Delta_{r-d}^i$ are equal. On the other hand, for any commeasurable subgroups $\Gamma, \Gamma'$ of a group $G$ and any $g$ in the commeasurable of $\Gamma$ in $G$, an elementary computation shows that we have a bijection of coset spaces

$$
(\Gamma' \cap g^{-1} \Gamma g) \gamma \simeq \Gamma \Gamma' \gamma
$$

via $(\Gamma' \cap g^{-1} \Gamma g) \gamma \mapsto \Gamma g \gamma$. Applying this with $g = \alpha^d$ in our situation and using the decomposition

$$
\Gamma_{r-d}^i \alpha^d \Gamma_{r-d}^i = \prod_{j=0}^{p^{r-d}-1} \Gamma_{r-d}^i \begin{pmatrix} 1 & j \\ 0 & p^d \end{pmatrix}
$$

(see, e.g. [Shi94, proposition 3.36]), we deduce that we also have

$$
\Gamma_r^i \alpha^d \Gamma_r^i = \prod_{j=0}^{p^{r-d}-1} \Gamma_r^i \begin{pmatrix} 1 & j \\ 0 & p^d \end{pmatrix}.
$$
Writing \( U : S_k(\Gamma^i_r) \to S_k(\Gamma^i_{r-d}) \) for the “Hecke operator” given by (e.g. [Oht99, \S 3.4]) \( \Gamma^i_r \alpha^d \Gamma^i_{r-d} \), an easy computation using 5.4.4 shows that the composite
\[
S_k(\Gamma^i_r) \xrightarrow{U} S_k(\Gamma^i_{r-d}) \xrightarrow{\epsilon} S_k(\Gamma^i_r)
\]
coincides with \( U^d_p \) on \( q \)-expansions. By the \( q \)-expansion principle, we deduce that \( U^d_p \) on \( S_k(\Gamma^i_r) \) indeed factors through the subspace \( S_k(\Gamma^i_{r-d}) \), as desired. \( \blacksquare \)

For each integer \( i \) and any character \( \varepsilon : (\mathbb{Z}/Np^i)^\times \to \overline{\mathbb{Q}}^\times \), we denote by \( S_2(\Gamma_i, \varepsilon) \) the \( \mathcal{H}_r \)-stable subspace of weight 2 cusp forms for \( \Gamma_i \) over \( \overline{\mathbb{Q}} \) on which the diamond operators act through \( \varepsilon(\cdot) \). Define
\[
(5.4.5) \quad \nabla_r := \bigoplus_{\varepsilon \in \mathcal{H}_r} S_2(\Gamma_i, \varepsilon)
\]
where the inner sum is over all Dirichlet characters defined modulo \( Np^i \) whose \( p \)-parts are primitive (i.e. whose conductor has \( p \)-part exactly \( p^i \)). We view \( \nabla_r \) as a \( \overline{\mathbb{Q}} \)-subspace of \( S_2(\Gamma_r) \) in the usual way (i.e. via the embeddings \( \iota_{\text{id}} \)). We define \( \nabla_r^* \) as the direct sum (5.4.5), but viewed as a subspace of \( S_2(\Gamma_r) \) via the “nonstandard” embeddings \( \iota_{\text{nr}^{-i}} : S_2(\Gamma_i) \to S_2(\Gamma_r) \).

As in (3.3.17), we write \( f' \) for the idempotent of \( \mathbb{Z}_p[\mu_{p-1}] \) corresponding to “projection away from the trivial \( \mu_{p-1} \)-eigenspace.” From the formulae (3.3.16) we see that \( h' := (p-1)f' \) lies in the subring \( \mathbb{Z}[\mu_{p-1}] \) of \( \mathbb{Z}_p[\mu_{p-1}] \) and satisfies \( h'^{2} = (p-1)h' \). We define endomorphisms of \( S_2(\Gamma_r) \):
\[
(5.4.6) \quad U^*_r := h' \circ (U^*_p)^{r+1} = (U^*_p)^{r+1} \circ h' \quad \text{and} \quad U_r := h' \circ (U_p)^{r+1} = (U_p)^{r+1} \circ h'.
\]

Corollary 5.4.3. As subspaces of \( S_2(\Gamma_r) \) we have \( w_r(\nabla_r^*) = \nabla_r \). The space \( \nabla_r \) (respectively \( \nabla_r^* \)) is naturally an \( \mathcal{H}_r \) (resp. \( \mathcal{H}_r^* \))-stable subspace of \( S_2(\Gamma_r) \), and admits a canonical descent to \( \overline{\mathbb{Q}} \). Furthermore, the endomorphisms \( U_r \) and \( U^*_r \) of \( S_2(\Gamma_r) \) factor through \( \nabla_r \) and \( \nabla_r^* \), respectively.

Proof. The first assertion follows from the relation \( w_r \circ \iota_{\alpha_{r^{-i}}} = \iota_{\text{id}} \circ w_i \) as maps \( S_2(\Gamma_i) \to S_2(\Gamma_r) \), together with the fact that \( w_i \) on \( S_2(\Gamma_i) \) carries \( S_2(\Gamma_i, \varepsilon) \) isomorphically onto \( S_2(\Gamma_i, \varepsilon^{-1}) \). The \( \mathcal{H}_r \)-stability of \( \nabla_r \) is clear as each of \( S_2(\Gamma_i, \varepsilon) \) is an \( \mathcal{H}_r \)-stable subspace of \( S_2(\Gamma_r) \); that \( \nabla_r^* \) is \( \mathcal{H}_r^* \)-stable follows from this and the commutation relation \( T^*w_r = w_T T \) of Proposition 2.3.24. That \( \nabla_r \) and \( \nabla_r^* \) admit canonical descents to \( \overline{\mathbb{Q}} \) is clear, as \( \mathcal{H}_r \)-conjugate Dirichlet characters have equal conductors. The final assertion concerning the endomorphisms \( U_r \) and \( U^*_r \) follows easily from Lemma 5.4.2, using the fact that \( h' : S_2(\Gamma_r) \to S_2(\Gamma_r) \) has image contained in \( \bigoplus_{i=1}^r S_k(\Gamma^i_r) \). \( \blacksquare \)

Definition 5.4.4. We denote by \( V_r \) and \( V^*_r \) the canonical descents to \( \overline{\mathbb{Q}} \) of \( \nabla_r \) and \( \nabla_r^* \), respectively.

Following [MW84, Chapter III, \S 1] and [Til87, \S 2], we recall the construction of certain “good” quotient abelian varieties of \( J_r \) whose cotangent spaces are naturally identified with \( V_r \) and \( V^*_r \). In what follows, we will make frequent use of the following elementary result:

Lemma 5.4.5. Let \( f : A \to B \) be a homomorphism of commutative group varieties over a field \( K \) of characteristic 0. Then:

1. The formation of Lie and Cot commutes with the formation of kernels and images: the kernel (respectively image) of \( \text{Lie}(f) \) is canonically isomorphic to the Lie algebra of the kernel (respectively image) of \( f \), and similarly for cotangent spaces at the identity. In particular, if \( A \) is connected and \( \text{Lie}(f) = 0 \) (respectively \( \text{Cot}(f) = 0 \)) then \( f = 0 \).
(2) Let \( i : B' \hookrightarrow B \) be a closed immersion of commutative group varieties over \( K \) with \( B' \) connected. If \( \text{Lie}(f) \) factors through \( \text{Lie}(i) \) then \( f \) factors (necessarily uniquely) through \( i \).

(3) Let \( j : A \to A'' \) be a surjection of commutative group varieties over \( K \) with connected kernel. If \( \text{Cot}(f) \) factors through \( \text{Cot}(j) \) then \( f \) factors (necessarily uniquely) through \( j \).

Proof. The key point is that because \( K \) has characteristic zero, the functors \( \text{Lie}(\cdot) \) and \( \text{Cot}(\cdot) \) on the category of commutative group schemes are exact. Indeed, since \( \text{Lie}(\cdot) \) is always left exact, the exactness of \( \text{Lie}(\cdot) \) follows easily from the fact that any quotient mapping \( G \to H \) of group varieties in characteristic zero is smooth (as the kernel is a group variety over a field of characteristic zero and hence automatically smooth), so the induced map on \( \text{Lie} \) algebras is a surjection. By similar reasoning one shows that the right exact \( \text{Cot}(\cdot) \) is likewise exact, and the first part of (1) follows easily. In particular, if \( \text{Lie}(f) \) is the zero map then \( \text{Lie}(\text{im}(f)) = 0 \), so \( \text{im}(f) \) is zero-dimensional. Since it is also smooth, it must be étale. Thus, if \( A \) is connected, then \( \text{im}(f) \) is both connected and étale, whence it is a single point; by evaluation of \( f \) at the identity of \( A \) we conclude that \( f = 0 \). The assertions (2) and (3) now follow immediately by using universal mapping properties.

To proceed with the construction of good quotients of \( J_r \), we now consider the diagrams of “degeneracy mappings” of curves over \( \mathbb{Q} \) for \( i = 1, 2 \)

\[
\begin{array}{ccc}
X_r & \xrightarrow{\pi} & Y_r \\
\pi_i & \nearrow & X_{r-1}
\end{array}
\]

where \( \pi \) and \( \pi_i \) are the maps induced by (2.3.8) and (2.3.9), respectively. These mappings covariantly (respectively contravariantly) induce mappings on the associated Jacobians via Albanese (respectively Picard) functoriality. Writing \( JY_r := \text{Pic}^0_Y/\mathbb{Q} \) and setting \( K^i_r := JY_1 \) for \( i = 1, 2 \) we inductively define abelian subvarieties \( \iota_r : K^i_r \hookrightarrow JY_r \) and abelian variety quotients \( \alpha^i_r : J_r \to B^i_r \) as follows:

\[
\begin{align*}
(5.4.7_i) & \\
B^i_{r-1} & := J_{r-1}/\text{Pic}^0(\pi)(K^i_{r-1}) & \text{and} & \\
K^i_r & := \ker(JY_r \xrightarrow{\alpha_{r-1} \circ \text{Alb}(\pi)} B^i_{r-1})^0
\end{align*}
\]

for \( r \geq 2, i = 1, 2 \), with \( \alpha^i_{r-1} \) and \( \iota_r \) the obvious mappings; here, \( (\cdot)^0 \) denotes the connected component of the identity of \( (\cdot) \). As in [Oht95, §3.2], we have modified Tilouine’s construction [Til87, §2] so that kernel of \( \alpha_r \) is connected; i.e. is an abelian subvariety of \( J_r \) (cf. Remark 5.4.8). Note that we have a commutative diagram of abelian varieties over \( \mathbb{Q} \) for \( i = 1, 2 \)

\[
\begin{array}{ccc}
J_{r-1} & \xrightarrow{\alpha^i_{r-1}} & B^i_{r-1} \\
\uparrow & & \uparrow \\
\text{Alb}(\pi_i) & & \text{Alb}(\pi_i)
\end{array}
\]

\[
\begin{array}{ccc}
K^i_r & \xrightarrow{\iota_r} & JY_r \\
\uparrow & & \uparrow \\
\text{Pic}^0(\pi) & & \text{Pic}^0(\pi)
\end{array}
\]

\[
\begin{array}{ccc}
K^i_r & \xrightarrow{\iota_r} & J_r \\
\downarrow & & \downarrow \\
\text{Pic}^0(\pi) & & \text{Pic}^0(\pi)
\end{array}
\]

\[
\begin{array}{ccc}
J_r & \xrightarrow{\alpha^i_r} & B^i_r \\
\downarrow & & \downarrow \\
B^i_r & & B^i_r
\end{array}
\]

with bottom two horizontal rows that are complexes.

Warning 5.4.6. While the bottom row of (5.4.9_i) is exact in the middle by definition of \( \alpha^i_r \), the central row is not exact in the middle: it follows from the fact that \( \text{Alb}(\pi_i) \circ \text{Pic}^0(\pi) \) is multiplication by \( p \) on \( J_{r-1} \) that the component group of the kernel of \( \alpha^i_{r-1} \circ \text{Alb}(\pi) : JY_r \to B^i_{r-1} \) is nontrivial with order divisible by \( p \). Moreover, there is no mapping \( B^i_{r-1} \to B^i_r \), which makes the diagram (5.4.9_i) commute.
In order to be consistent with the literature, we adopt the following convention:

**Definition 5.4.7.** We set $B_r := B_r^2$ and $B_r^\alpha := B_r^1$, with $B_r^i$ defined inductively by (5.4.8). We likewise set $\alpha_r := \alpha_r^2$ and $\alpha_r^\alpha := \alpha_r^1$.

**Remark 5.4.8.** We briefly comment on the relation between our quotient $B_r$ and the “good” quotients of $J_r$ considered by Ohta [Oht95], by Mazur-Wiles [MW84], and by Tilouine [Til87]. Recall [Til87, §2] that Tilouine constructs an abelian variety quotient $\alpha_r^\prime : J_r \rightarrow B_r^\prime$ via an inductive procedure nearly identical to the one used to define $B_r = B_r^2$: one sets $K_1^r := JY_1$, and for $r \geq 2$ defines

$$B_{r-1}^\prime := J_{r-1}/\text{Pic}^0(\pi) (K_{r-1}^{\prime}) \quad \text{and} \quad K_r^\prime := \ker(JY_r \overset{\alpha_{r-1} \circ \text{Alb}(\pi_2)}{\longrightarrow} B_{r-1}^\prime).$$

Using the fact that the formation of images and identity components commutes, one shows via a straightforward induction argument that $\alpha_r : J_r \rightarrow B_r$ identifies $B_r$ with $J_r/(\ker \alpha_r)^0$; in particular, our $B_r$ is the same as Ohta’s [Oht95, §3.2] and Tilouine’s quotient $\alpha_r^\prime : J_r \rightarrow B_r^\prime$ uniquely factors through $\alpha_r$ via an isogeny $B_r \rightarrow B_r^\prime$ which has degree divisible by $p$ by Warning 5.4.6. Due to this fact, it is essential for our purposes to work with $B_r$ rather than $B_r^\prime$. Of course, following [Oht95, 3.2.1], we could have simply defined $B_r$ as $J_r/(\ker \alpha_r)^0$, but we feel that the construction we have given is more natural.

On the other hand, we remark that $B_r$ is naturally a quotient of the “good” quotient $\tilde J_r \rightarrow A_r$ constructed by Mazur-Wiles in [MW84, Chapter III, §1], and the kernel of the corresponding surjective homomorphism $A_r \rightarrow B_r$ is isogenous to $J_0 \times J_0$.

**Proposition 5.4.9.** Over $F := \mathbb{Q}(\mu_{Np^\prime})$, the automorphism $w_r$ of $J_r F$ induces an isomorphism of quotients $B_{r,F} \simeq B_{r,F}^\star$. The abelian variety $B_r$ (respectively $B_r^\star$) is the unique quotient of $J_r$ by a $\mathbb{Q}$-rational abelian subvariety with the property that the induced map on cotangent spaces

$$\text{Cot}(B_r) \overset{\text{Cot}(\alpha_r)}{\longrightarrow} \text{Cot}(J_r) \simeq S_2(\Gamma_r; \mathbb{Q}) \quad \text{respectively} \quad \text{Cot}(B_r^\star) \overset{\text{Cot}(\alpha_r^\star)}{\longrightarrow} \text{Cot}(J_r) \simeq S_2(\Gamma_r; \mathbb{Q})$$

has image precisely $V_r$ (respectively $V_r^\star$). In particular, there are canonical actions of the Hecke algebras $\tilde H_r(\mathbb{Z})$ on $B_r$ and $\tilde H_r^\star(\mathbb{Z})$ on $B_r^\star$ for which $\alpha_r$ and $\alpha_r^\star$ are equivariant.

**Proof.** By the construction of $B_r^2$ and Proposition 2.3.6, the automorphism $w_r$ of $J_{r,F}$ carries $\ker(\alpha_r)$ to $\ker(\alpha_r^\star)$ and induces an isomorphism $B_{r,F} \simeq B_{r,F}^\star$ over $F$ that intertwines the action of $\tilde H_r$ on $B_r$ with $\tilde H_r^\star$ on $B_r^\star$. The isogeny $B_r \rightarrow B_r^\prime$ of Remark 5.4.8 induces an isomorphism on cotangent spaces, compatibly with the inclusions into $\text{Cot}(J_r)$. Thus, the claimed identification of the image of $\text{Cot}(B_r)$ with $V_r$ follows from [Til87, Proposition 2.1] (using [Til87, Definition 2.1]). The claimed uniqueness of $J_r \rightarrow B_r$ follows easily from Lemma 5.4.5 (3). Similarly, since the subspace $V_r$ of $S_2(\Gamma_r)$ is stable under $\tilde H_r$, we conclude from Lemma 5.4.5 (3) that for any $T \in \tilde H_r(\mathbb{Z})$, the induced morphism $J_r \overset{T}{\longrightarrow} J_r \rightarrow B_r$ factors through $\alpha_r$, and hence that $\tilde H_r(\mathbb{Z})$ acts on $B_r$ compatibly (via $\alpha_r$) with its action on $J_r$. ■

---

31The notation Tilouine uses for his quotient is the same as the notation we have used for our (slightly modified) quotient. To avoid conflict, we have therefore chosen to alter his notation.

32We must warn the reader that Tilouine [Til87] writes $\tilde H_r(\mathbb{Z})$ for the $\mathbb{Z}$-subalgebra of $\text{End}(J_r)$ generated by the Hecke operators acting via the $(\cdot)^\star$-action (i.e. by “Picard” functoriality) whereas our $\tilde H_r(\mathbb{Z})$ is defined using the $(\cdot)^{\alpha}$-action. This discrepancy is due primarily to the fact that Tilouine identifies tangent spaces of modular abelian varieties with spaces of modular forms, rather than cotangent spaces as is our convention. Our notation for regarding Hecke algebras as sub-algebras of $\text{End}(J_r)$ agrees with that of Mazur-Wiles [MW84, Chapter II, §5], [MW86, §7] and Ohta [Oht95, 3.1.5].
Lemma 5.4.10. There exist unique morphisms $B^*_r \cong B^*_{r-1}$ of abelian varieties over $\mathbb{Q}$ making

$$
\begin{array}{ccc}
J_r & \xrightarrow{\alpha^*_r} & B^*_r \\
\text{Alb}(\sigma) & \downarrow & \downarrow \\
J_{r-1} & \xrightarrow{\alpha^*_{r-1}} & B^*_{r-1}
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
J_r & \xrightarrow{\alpha^*_r} & B^*_r \\
\text{Pic}^0(\rho) & \downarrow & \downarrow \\
J_{r-1} & \xrightarrow{\alpha^*_{r-1}} & B^*_{r-1}
\end{array}
$$

commute; these maps are moreover $\mathcal{H}^*_r(\mathbb{Z})$-equivariant. By a slight abuse of notation, we will simply write $\text{Alb}(\sigma)$ and $\text{Pic}^0(\rho)$ for the induced maps on $B^*_r$ and $B^*_{r-1}$, respectively.

Proof. Under the canonical identification of $\text{Cot}(J_r) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ with $S_2(\Gamma_r)$, the mapping on cotangent spaces induced by $\text{Alb}(\sigma)$ (respectively $\text{Pic}^0(\rho)$) coincides with $\iota_\alpha : S_2(\Gamma_{r-1}) \to S_2(\Gamma_r)$ (respectively $\text{tr}_{r,r-1} : S_2(\Gamma_r) \to S_2(\Gamma_{r-1})$). As the kernel of $\alpha^*_r : J_r \to B^*_r$ is connected by definition, thanks to Lemma 5.4.5 (3) it suffices to prove that $\iota_\alpha$ (respectively $\text{tr}_{r,r-1}$) carries $V^*_{r-1}$ to $V^*_r$ (respectively $V^*_r$ to $V^*_{r-1}$). On one hand, the composite $\iota_\alpha \circ \text{tr}_{r,r-1} : S_2(\Gamma_1, \varepsilon) \to S_2(\Gamma_r)$ coincides with the embedding $\iota_{\alpha^*_{r-1}}$, and it follows immediately from the definition of $V^*_r$ that $\iota_\alpha$ carries $V^*_{r-1}$ into $V^*_r$. On the other hand, an easy calculation using (5.4.1) shows that one has equalities of maps $S_2(\Gamma_1, \varepsilon) \to S_2(\Gamma_r)$

$$
\iota_\alpha \circ \text{tr}_{r,r-1} \circ \iota_{\alpha(r-1)} = \begin{cases} 
\iota_\alpha(r-0)pU^*_p & \text{if } i < r \\
0 & \text{if } i = r.
\end{cases}
$$

Thus, the image of $\iota_\alpha \circ \text{tr}_{r,r-1} : V^*_r \to S_2(\Gamma_r)$ is contained in the image of $\iota_\alpha : V^*_{r-1} \to S_2(\Gamma_r)$; since $\iota_\alpha$ is injective, we conclude that the image of $\text{tr}_{r,r-1} : V^*_r \to S_2(\Gamma_{r-1})$ is contained in $V^*_{r-1}$ as desired. ■

Proposition 5.4.11. The abelian varieties $B_r$ and $B^*_r$ acquire good reduction over $\mathbb{Q}_p(\mu_{p^r})$.

Proof. See [MW84, Chap III, §2, Proposition 2] and cf. [Hid86a, §9, Lemma 9]. ■

As in §3.3, we denote by $e^s := f'e' \in \mathcal{H}^s$ and $e' := f'e \in \mathcal{H}$ the sub-idempotents of $e^s$ and $e$, respectively, corresponding to projection away from the trivial eigenspace of $\mu_{p-1}$.

Proposition 5.4.12. The maps $\alpha_r$ and $\alpha^*_r$ induce isomorphisms of $p$-divisible groups over $\mathbb{Q}$

$$
e' J_r[p^\infty] \simeq e^s B^*_r[p^\infty] \quad \text{and} \quad e' J_r[p^\infty] \simeq e' B_r[p^\infty],
$$

respectively, that are $\mathcal{H}^s (\text{respectively } \mathcal{H})$ equivariant and compatible with change in $r$ via $\text{Alb}(\sigma)$ and $\text{Pic}^0(\rho)$ (respectively $\text{Alb}^0(\sigma)$).

We view the maps (5.4.6) as endomorphisms of $J_r$ in the obvious way, and again write $U^*_r$ and $U_r$ for the induced endomorphism of $B^*_r$ and $B_r$, respectively. To prove Proposition 5.4.12, we need the following geometric incarnation of Corollary 5.4.3:

Lemma 5.4.13. There exists a unique $\mathcal{H}^s_r(\mathbb{Z})$ (respectively $\mathcal{H}_r(\mathbb{Z})$)-equivariant map $W^*_r : B^*_r \to J_r$ (respectively $W_r : B_r \to J_r$) of abelian varieties over $\mathbb{Q}$ such that the diagram

$$
\begin{array}{ccc}
J_r & \xrightarrow{\alpha^*_r} & B^*_r \\
\downarrow & U^*_r & \downarrow \\
J_r & \xrightarrow{\alpha^*_r} & B^*_r
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
J_r & \xrightarrow{\alpha_r} & B_r \\
\downarrow & U_r & \downarrow \\
J_r & \xrightarrow{\alpha_r} & B_r
\end{array}
$$

commutes.
Proof. Consider the endomorphism of $J_r$ given by $U_r$. Due to Corollary 5.4.3, the induced mapping on cotangent spaces factors through the inclusion $\text{Cot}(B_r) \to \text{Cot}(J_r)$. Since the kernel of the quotient mapping $\alpha_r : J_r \to B_r$ giving rise to this inclusion is connected, we conclude from Lemma 5.4.5 (3) that $U_r$ factors uniquely through $\alpha_r$ via an $\mathfrak{H}_r$-equivariant morphism $W_r : B_r \to J_r$. The corresponding statements for $B^*_r$ are proved similarly. 

Proof of Proposition 5.4.12. From (5.4.11) we get commutative diagrams of $p$-divisible groups over $\mathbf{Q}$

\[
\begin{align*}
\varepsilon'_r J_r[p^\infty] & \xrightarrow{\alpha_r^*} \varepsilon'_r B^*_r[p^\infty] \\
U_r \downarrow \cong W_r \downarrow \cong U_r & \quad \text{and} \quad \varepsilon' J_r[p^\infty] \xrightarrow{\alpha_r^*} \varepsilon' B^*_r[p^\infty]
\end{align*}
\]

(5.4.12)

in which all vertical arrows are isomorphisms due to the very definition of the idempotents $\varepsilon'$ and $\varepsilon'$. An easy diagram chase then shows that all arrows must be isomorphisms. 

We will write $\mathfrak{B}_r$, $\mathfrak{B}^*_r$, and $\mathfrak{J}_r$, respectively, for the Néron models of the base changes $(B_r)_K$, $(B^*_r)_K$ and $(J_r)_K$ over $T_r := \text{Spec}(R_r)$; due to Proposition 5.4.12, both $\mathfrak{B}_r$ and $\mathfrak{B}^*_r$ are abelian schemes over $T_r$. By the Néron mapping property, there are canonical actions of $\mathfrak{J}_r(\mathbf{Z})$ on $\mathfrak{B}_r$, $\mathfrak{J}_r$ and of $\mathfrak{J}^*_r(\mathbf{Z})$ on $\mathfrak{B}^*_r$, $\mathfrak{J}^*_r$ over $R_r$ extending the actions on generic fibers as well as “semilinear” actions of $\Gamma$ over the $\Gamma$-action on $R_r$ (cf. (4.1.7)). For each $r$, the Néron mapping property further provides diagrams

\[
\begin{align*}
\varepsilon'_r \mathfrak{J}_r \times_{T_r} T_{r+1} & \xrightarrow{\alpha^*_r} \varepsilon'_r \mathfrak{B}^*_r \times_{T_r} T_{r+1} \\
\mathfrak{J}_{r+1} & \xrightarrow{\alpha^*_r} \mathfrak{B}^*_r \times_{T_{r+1}} \mathfrak{J}_{r+1} \quad \text{respectively} \quad \varepsilon'_r \mathfrak{J}_r \times_{T_r} T_{r+1} & \xrightarrow{\alpha^*_r} \varepsilon'_r \mathfrak{B}^*_r \times_{T_r} T_{r+1} \\
\mathfrak{J}_{r+1} & \xrightarrow{\alpha^*_r} \mathfrak{B}^*_r \times_{T_{r+1}} \mathfrak{J}_{r+1}
\end{align*}
\]

(5.4.13)

of smooth commutative group schemes over $T_{r+1}$ in which the inner and outer rectangles commute, and all maps are $\mathfrak{J}_{r+1}(\mathbf{Z})$ (respectively $\mathfrak{J}_{r+1}(\mathbf{Z})$) and $\Gamma$ equivariant. 

Definition 5.4.14. We define $\mathfrak{S}_r := \varepsilon'_r (\mathfrak{B}^*_r[p^\infty])$ and we write $\mathfrak{S}^*_r$ for its Cartier dual, each of which is canonically an object of $\text{pdiv}_{R_r}^\Gamma$. For each $r \geq s$, noting that $U^s_p$ is an automorphism of $\mathfrak{S}_r$, we obtain from (5.4.13) canonical morphisms

\[
\begin{align*}
\rho_{r,s} : \mathfrak{S}_s \times_{T_s} T_r & \xrightarrow{\varepsilon'_s (\mathfrak{B}^*_s[p^\infty])} \mathfrak{S}_r \\
\rho'_r : \mathfrak{S}^*_s \times_{T_s} T_r & \xrightarrow{(U^r_p - 1 \text{Alb}(\sigma))^*} \mathfrak{S}^*_r
\end{align*}
\]

(5.4.14)

in $\text{pdiv}_{R_r}^\Gamma$, where $(\cdot)^i$ denotes the $i$-fold composition, formed in the obvious manner. In this way, we get towers of $p$-divisible groups $\{\mathfrak{S}_r, \rho_{r,s}\}$ and $\{\mathfrak{S}^*_r, \rho'_r\}$; we will write $G_r$ and $G'_r$ for the unique descent of the generic fibers of $\mathfrak{S}_r$ and $\mathfrak{S}^*_r$ to $\mathbf{Q}_p$, respectively. We let $T^* \in \mathfrak{H}^*_r$ act on $\mathfrak{S}_r$ through the action of $\mathfrak{H}^*_r(\mathbf{Z})$ on $\mathfrak{B}^*_r$, and on $\mathfrak{S}^*_r = \mathfrak{H}^*_r$ by duality (i.e. as $(T^*)^\vee$). The maps (5.4.14) are then $\mathfrak{H}^*_r$-equivariant. 

By Proposition 5.4.12, $G_r$ is canonically isomorphic to $\varepsilon'_r J_r[p^\infty]$, compatibly with the action of $\mathfrak{H}^*_r$. Since $J_r$ is a Jacobian—hence principally polarized—one might expect that $\mathfrak{S}_r$ is isomorphic to its dual in $\text{pdiv}_{R_r}^\Gamma$. However, this is not quite the case as the canonical isomorphism $J_r \simeq J_r^\vee$ intertwines the actions of $\mathfrak{H}_r$ and $\mathfrak{H}^*_r$, thus intertwaching the idempotents $\varepsilon'$ and $\varepsilon'$. To describe the precise relationship

\[\text{Of course, } G'_r = G^\vee_r. \text{ Our non-standard notation } \mathfrak{S}^*_r \text{ for the Cartier dual of } \mathfrak{S}_r \text{ is preferable, due to the fact that } \rho'_r \text{ is not simply the dual of } \rho_{r,s}; \text{ indeed, these two mappings go in opposite directions!}\]
between $\mathcal{G}_r$ and $\mathcal{G}_r$, we proceed as follows. For each $\gamma \in \text{Gal}(K_r'/K_0) \simeq \Gamma \times \text{Gal}(K_0'/K_0)$, let us write $\phi_{\gamma} : G_rK_r' \rightarrow \gamma^*(G_rK_r')$ for the descent data isomorphisms encoding the unique $Q_p = K_0$-descent of $G_rK_r'$ furnished by $G_r$. We “twist” this descent data by the $\text{Aut}_{Q_p}(G_r)$-valued character $\langle \chi \rangle \langle a \rangle_N$ of $\text{Gal}(K_0'/K_0)$: explicitly, for $\gamma \in \text{Gal}(K_r'/K_0)$ we set $\psi_{\gamma} := \phi_{\gamma} \circ \langle \chi \rangle \langle a \rangle_N$ and note that since $\langle \chi \rangle \langle a \rangle_N$ is defined over $Q_p$, the map $\gamma \rightsquigarrow \psi_{\gamma}$ really does satisfy the cocycle condition. We denote by $G_r(\langle \chi \rangle \langle a \rangle_N)$ the unique $p$-divisible group over $Q_p$ corresponding to this twisted descent datum. Since the diamond operators commute with the Hecke operators, there is a canonical induced action of $\mathcal{G}_r$ on $G_r(\langle \chi \rangle \langle a \rangle_N)$. By construction, there is a canonical $K_r'$-isomorphism $G_r(\langle \chi \rangle \langle a \rangle_N)K_r' \simeq G_rK_r'$. Since $G_r$ acquires good reduction over $K_r$ and the $\mathcal{G}_r$-representation afforded by the Tate module of $G_r(\langle \chi \rangle \langle a \rangle_N)$ is the twist of $T_pG_r$ by the unramified character $\langle a \rangle_N$, we conclude that $G_r(\langle \chi \rangle \langle a \rangle_N)$ also acquires good reduction over $K_r$, and we denote the resulting object of $\text{pd} \text{div}^\Gamma_{R_r}$ by $\mathcal{G}_r(\langle \chi \rangle \langle a \rangle_N)$.

**Proposition 5.4.15.** There is a natural $\mathcal{G}_r$-equivariant isomorphism of $p$-divisible groups over $Q_p$

\[(5.4.15) \quad G_r' \simeq G_r(\langle \chi \rangle \langle a \rangle_N)\]

which uniquely extends to an isomorphism of the corresponding objects in $\text{pd} \text{div}^\Gamma_{R_r}$ and is compatible with change in $r$ using $\rho_{r,s}$ on $G_r'$ and $p_{r,s}$ on $G_r$.

**Proof.** Let $\varphi_r : J_r \rightarrow J_r'$ be the canonical principal polarization over $Q_p$; one then has the relation $\varphi_r \circ T = (T')^\vee \circ \varphi_r$ for each $T \in \mathcal{H}_r(Z)$. On the other hand, the $K_r'$-automorphism $\psi_r : J_rK_r' \rightarrow J_rK_r'$ intertwines $T \in \mathcal{H}_r(Z)$ with $T^* \in \mathcal{J}_r(Z)$. Thus, the $K_r'$-morphism

\[
\psi_r : J_rK_r' \xrightarrow{(U_p)^\vee} J_rK_r' \xrightarrow{\varphi^{-1}_r} J_rK_r' \xrightarrow{w_r} J_rK_r'
\]

is $\mathcal{J}_r(Z)$-equivariant. Passing to the induced map on $p$-divisible groups and applying $e^{*'}$, we obtain from Proposition 5.4.12 an $\mathcal{J}_r'$-equivariant isomorphism of $p$-divisible groups $\psi_r : G_rK_r' \simeq G_rK_r'$. As

\[
\begin{array}{ccc}
J_rK_r' \xrightarrow{\langle \chi \rangle \langle a \rangle_N w_r} J_rK_r' \\
\downarrow 1 \times \gamma & & \downarrow 1 \times \gamma \\
(J_rK_r')_{\gamma} \xrightarrow{\gamma^* \langle a \rangle N} (J_rK_r')_{\gamma}
\end{array}
\]

commutes for all $\gamma \in \text{Gal}(K_r'/K_0)$ by Proposition 2.3.6, the $K_r'$-isomorphism $\psi_r$ uniquely descends to an $\mathcal{J}_r'$-equivariant isomorphism (5.4.15) of $p$-divisible groups over $Q_p$. By Tate’s Theorem, this identification uniquely extends to an isomorphism of the corresponding objects in $\text{pd} \text{div}^\Gamma_{R_r}$. The asserted compatibility with change in $r$ boils down to the commutativity of the diagrams

\[
\begin{array}{ccc}
& e^{*'}J_r[p] \xrightarrow{(U_p)^\vee} e^{*'}J_r[p] \xrightarrow{(U_p)^\vee} J_rK_r' & \\
& \downarrow (U_p^{-1} \text{Alb}(\sigma))^{r-s} & \quad \text{and} & \quad \downarrow \text{Alb}(\sigma)^{r-s} & \text{Pic}^0(\sigma)^{r-s} \\
& e^{*'}J_r[p] \xrightarrow{(U_p)^\vee} e^{*'}J_r[p] \xrightarrow{(U_p)^\vee} J_rK_r' & \\
& \downarrow (U_p^{-1} \text{Alb}(\sigma))^{r-s} & \quad \downarrow \text{Alb}(\sigma)^{r-s} & \text{Pic}^0(\sigma)^{r-s} \\
& J_sK_r \xrightarrow{\psi^{-1}_r} J_sK_r \xrightarrow{w_s} J_sK_r & & J_rK_r' \xrightarrow{\psi^{-1}_r} J_rK_r' \xrightarrow{w_r} J_rK_r
\end{array}
\]

for all $s \leq r$. The commutativity of the first diagram is clear, while that of the second follows from Proposition 2.3.6 and the fact that for any finite morphism $f : Y \rightarrow X$ of smooth curves over a field $K$, one has $\varphi_Y \circ \text{Pic}^0(f) = \text{Alb}(f)^{r} \circ \varphi_X$, where $\varphi_Y : J_Y \rightarrow J_Y'$ is the canonical principal polarization on Jacobians for $* = X, Y$ (see, for example, the proof of Lemma 5.5 in [Cai10]).
We now wish to relate the special fiber of $\mathcal{G}_r$ to the $p$-divisible group $\Sigma_r := e^* \text{Pic}_0^r/F_p \langle p^\infty \rangle$ of Definition 3.3.7. In order to do this, we proceed as follows. Since $\mathcal{X}_r$ is regular, and proper flat over $R_r$ with (geometrically) reduced special fiber, $\text{Pic}_0^r/\mathcal{X}_r/R_r$ is a smooth $R_r$-scheme by §8.4 Proposition 2 and §9.4 Theorem 2 of [BLR90]. By the Néron mapping property, we thus have a natural mapping $\text{Pic}_0^r/\mathcal{X}_r/R_r \rightarrow \mathcal{P}_r$ that recovers the canonical identification on generic fibers, and is in fact an isomorphism by [BLR90, §9.7, Theorem 1]. Composing with the map $\alpha_r : \mathcal{P}_r \rightarrow \mathbf{B}_r^*$ and passing to special fibers yields a homomorphism of smooth commutative algebraic groups over $F_p$.

\begin{equation}
\text{Pic}_0^r/\mathcal{X}_r/F_p \xrightarrow{\sim} \mathcal{P}_r \xrightarrow{\mathcal{G}_r^*} \mathbf{B}_r^*
\end{equation}

Due to [BLR90, §9.3, Corollary 11], the normalization map $\overline{\mathcal{X}_r} \rightarrow \overline{\mathcal{X}}$ induces a surjective homomorphism $\text{Pic}_0^r/\overline{\mathcal{X}_r}/F_p \rightarrow \text{Pic}_0^r/\overline{\mathcal{X}}/F_p$ with kernel that is a smooth, connected linear algebraic group over $F_p$. As any homomorphism from an affine group variety to an abelian variety is zero, we conclude that (5.14.6) uniquely factors through this quotient, and we obtain a natural map of abelian varieties:

\begin{equation}
\text{Pic}_0^r/\overline{\mathcal{X}_r}/F_p \xrightarrow{\sim} \mathcal{G}_r^* \xrightarrow{\mathbf{B}_r^*}
\end{equation}

that is necessarily equivariant for the actions of $\mathcal{G}_r^*(\mathbb{Z})$ and $\Gamma$. As in 3.3.25, we write $j_r^* := \text{Pic}_0^r/\mathcal{X}_r/F_p$ the Jacobian of $I^*_r$ for $* = 0, \infty$. The following Proposition relates the special fiber of $\mathcal{G}_r$ to the $p$-divisible group $\Sigma_r$ of Definition 3.3.7, and thus enables an explicit description of the special fiber of $\mathcal{G}_r$ in terms of the $p$-divisible groups of $j_r^*$ (cf. §3 and §4, Proposition 1 of [MW86] and pgs. 267–274 of [MW84]).

**Proposition 5.4.16.** The mapping (5.14.7) induces an isomorphism of $p$-divisible groups over $F_p$

\begin{equation}
\mathcal{G}_r := e^* \mathbf{B}_r^*[p^\infty] \simeq e^* \text{Pic}_0^r/\mathcal{X}_r/F_p \langle p^\infty \rangle := \Sigma_r
\end{equation}

that is $\mathcal{G}_r^*$ and $\Gamma$-equivariant and compatible with change in $r$ via the maps $\rho_{r,s}$ on $\mathcal{G}_r$ and the maps $\text{Pic}_0^r(\rho)^{r-s}$ on $\Sigma_r$. In particular, $\mathcal{G}_r/R_r$ is an ordinary $p$-divisible group, and for each $r$ there is a canonical exact sequence, compatible with change in $r$ via $\rho_{r,s}$ on $\mathcal{G}_r$ and $\text{Pic}_0^r(\rho)^{r-s}$ on $j_r^*[p^\infty]$

\begin{equation}
0 \xrightarrow{\alpha} \text{Pic}_0^r(\mathcal{G}_r^*)^{\text{et}} \xrightarrow{\gamma} \mathcal{G}_r \xrightarrow{\rho_{r,s}} \text{Et}(\mathcal{G}_r^*)^{\text{et}} \xrightarrow{\beta} 0
\end{equation}

where $\alpha_r : I^*_r \hookrightarrow \mathcal{G}_r^*$ are the canonical closed immersions for $* = 0, \infty$. Moreover, (5.4.19) is compatible with the actions of $\mathcal{G}_r^*$ and $\Gamma$, with $U_p^*$ (respectively $\gamma \in \Gamma$) acting on $\text{Pic}_0^r(\mathcal{G}_r^*)^{\text{et}}$ as $(p)_N V$ (respectively $\langle \chi(\gamma) \rangle^{-1}$) and on $f' j_r^*[p^\infty]^{\text{et}}$ as $F$ (respectively id).

**Proof.** The diagram (5.4.11) induces a corresponding diagram of Néron models over $R_r$ and hence of special fibers over $F_p$. Arguing as above, we obtain a commutative diagram of abelian varieties

\begin{equation}
\begin{array}{ccc}
\text{Pic}_0^r/\mathcal{X}_r/F_p & \xrightarrow{\pi_r^*} & \mathbf{B}_r^* \\
U_r^* & \searrow & U_r^* \\
\text{Pic}_0^r/\overline{\mathcal{X}_r}/F_p & \xrightarrow{\pi_r^*} & \mathbf{B}_r^*
\end{array}
\end{equation}

over $F_p$. The proof of 5.4.12 now goes through *mutatis mutandis* to give the claimed isomorphism (5.4.18). The rest follows immediately from Proposition 3.3.9.
5.5. **Ordinary families of Dieudonné modules.** Let \( \{ \mathcal{G}_r / R_r \}_{r \geq 1} \) be the tower of \( p \)-divisible groups given by Definition 5.4.14. From the canonical morphisms \( p_{r,s} : \mathcal{G}_s \times_{T_s} T_r \to \mathcal{G}_r \), we obtain a map on special fibers \( \mathcal{G}_s \to \mathcal{G}_r \) over \( F_p \) for each \( r \geq s \); applying the contravariant Dieudonné module functor \( D(\cdot) := D(\cdot)Z_p \) yields a projective system of finite free \( Z_p \)-modules \( \{ D(\mathcal{G}_r) \}_r \) with compatible linear endomorphisms \( F, V \) satisfying \( FV = VF = p \).

**Definition 5.5.1.** We write \( D_\infty := \lim_{\mathcal{G}_r} D(\mathcal{G}_r) \) for the projective limit of the system \( \{ D(\mathcal{G}_r) \}_r \). For \( \star \in \{ \text{ét}, m \} \) we write \( D^\star_\infty := \lim_{\mathcal{G}_r} D(\mathcal{G}_r^\star) \) for the corresponding projective limit.

Since \( \mathcal{G}_r^\star \) acts by endomorphisms on \( \mathcal{G}_r \), compatibly with change in \( r \), we obtain an action of \( \mathcal{G}^\star \) on \( D_\infty \) and on \( D^\star_\infty \). Likewise, the “geometric inertia action” of \( \Gamma \) on \( \mathcal{G}_r \) by automorphisms of \( p \)-divisible groups over \( F_p \) gives an action of \( \Gamma \) on \( D_\infty \) and \( D^\star_\infty \). As \( \mathcal{G}_r \) is ordinary by Proposition 5.4.16, applying \( D(\cdot) \) to the (split) connected-étale sequence of \( \mathcal{G}_r \) gives, for each \( r \), a functorially split exact sequence

\[
0 \longrightarrow D(\mathcal{G}_r^\text{ét}) \longrightarrow D(\mathcal{G}_r) \longrightarrow D(\mathcal{G}_r^m) \longrightarrow 0
\]

with \( Z_p \)-linear actions of \( \Gamma \), \( F \), \( V \), and \( \mathcal{G}^\star \). Since projective limits commute with finite direct sums, we obtain a split short exact sequence of \( \Lambda \)-modules with linear \( \mathcal{G}^\star \) and \( \Gamma \)-actions and commuting linear endomorphisms \( F, V \) satisfying \( FV = VF = p \):

\[
0 \longrightarrow D_\infty \longrightarrow D^{\text{ét}}_\infty \longrightarrow D^m_\infty \longrightarrow 0.
\]

**Theorem 5.5.2.** As in Proposition 3.3.6, set \( d' := \sum_{k=3}^p \dim_{F_p} S_k(N; F_p)_{\text{ord}} \). Then:

1. \( D_\infty \) is a free \( \Lambda \)-module of rank \( 2d' \), and \( D^\star_\infty \) is free of rank \( d' \) over \( \Lambda \) for \( \star \in \{ \text{ét}, m \} \).
2. For each \( r \geq 1 \), applying \( \otimes_{\Lambda} Z_p[\Delta/\Delta_r] \) to (5.5.2) yields the short exact sequence (5.5.1), compatibly with \( \mathcal{G}^\star \), \( \Gamma \), \( F \) and \( V \).
3. Under the canonical splitting of (5.5.2), \( D^{\text{ét}}_\infty \) is the maximal subspace of \( D_\infty \) on which \( F \) acts invertibly, while \( D^m_\infty \) corresponds to the maximal subspace of \( D_\infty \) on which \( V \) acts invertibly.
4. The Hecke operator \( U^\star_r \) acts as \( F \) on \( D^m_\infty \) and as \( (p)_N V \) on \( D^m_\infty \).
5. \( \Gamma \) acts trivially on \( D^{\text{ét}}_\infty \) and via \( (\chi)^{-1} \) on \( D^m_\infty \).

**Proof.** We apply Lemma 5.1.2 with \( A_r = Z_p \), \( I_r = (p) \), and with \( M_r \) each one of the terms in (5.5.1). Due to Proposition 3.3.8, there is a natural isomorphism of split short exact sequences

\[
\begin{array}{c}
0 \longrightarrow D(\mathcal{G}_r^\text{ét})_{F_p} \longrightarrow D(\mathcal{G}_r)_{F_p} \longrightarrow D(\mathcal{G}_r^m)_{F_p} \longrightarrow 0 \\
\downarrow \cong \\
0 \longrightarrow f' H^1(I_r^\text{ét}, \mathcal{O})_{F_p} \longrightarrow f' H^0(I_r^\infty, \Omega^1)_{\text{Ord}} \oplus f' H^1(I_r^\text{ét}, \mathcal{O})_{F_p} \longrightarrow f' H^0(I_r^\infty, \Omega^1)_{\text{Ord}} \longrightarrow 0
\end{array}
\]

that is compatible with change in \( r \) using the trace mappings attached to \( \rho : I^\star_r \to I_s \) and the maps on Dieudonné modules induced by \( p_{r,s} : \mathcal{G}_s \to \mathcal{G}_r \). The hypotheses (5.1.1a) and (5.1.1b) of Lemma 5.1.2 are thus satisfied with \( d' \) as in the statement of the theorem, thanks to Proposition 3.2.1 (1)–(2) and Lemma 3.3.5. We conclude from Lemma 5.1.2 that (1) and (2) hold. As \( F \) (respectively \( V \)) acts invertibly on \( D(\mathcal{G}_r^\text{ét}) \) (respectively \( D(\mathcal{G}_r^m) \)) for all \( r \), assertion (3) is clear, while (4) and (5) follow immediately from Proposition 5.4.16.

As in Proposition 5.2.4, the short exact sequence (5.5.2) is very nearly “auto dual”:
Proposition 5.5.3. There is a canonical isomorphism of short exact sequences of $\Lambda R'_0$-modules

$$0 \to \mathbf{D}_\infty^A((\langle \chi \rangle \langle a \rangle_N)_{\Lambda R'_0} \to \mathbf{D}_\infty((\langle \chi \rangle \langle a \rangle_N)_{\Lambda R'_0} \to \mathbf{D}_\infty^A((\langle \chi \rangle \langle a \rangle_N)_{\Lambda R'_0} \to 0$$

(5.5.3)

$$0 \to (\mathbf{D}_\infty^A)_{\Lambda R'_0} \to (\mathbf{D}_\infty)_{\Lambda R'_0} \to (\mathbf{D}_\infty^A)_{\Lambda R'_0} \to 0$$

that is $\mathcal{C}^*$ and $\Gamma \times \Gal(K'_0/K_0)$-equivariant, and intertwines $F$ (respectively $V$) on the top row with $V^\vee$ (respectively $F^\vee$) on the bottom.

Proof. We apply the duality formalism of Lemma 5.1.4. Let us write $Q$ so our claim follows from the equality in $\End_{\mathbb{Q}}(5.5.6)$ $\Pic(\rho)$, which, as in the proof of Proposition 5.2.4, follows from Lemma 5.4.1 via Lemma 5.4.5. Again, by the functor with duality.

(5.5.5)

$D(\mathcal{G}_r)((\langle \chi \rangle \langle a \rangle_N) \otimes R'_0 \simeq D(\mathcal{G}_r)((\langle \chi \rangle \langle a \rangle_N) \otimes Z_p \otimes R'_0 \simeq D(\mathcal{G}_r) \otimes Z_p \otimes R'_0 \simeq D(\mathcal{G}_r) \otimes Z_p \otimes R'_0$ that are $\mathcal{C}^*$-equivariant, $\Gal(K'_0/K_0)$-compatible for the standard action $\sigma \cdot f(m) := \sigma f(\sigma^{-1}m)$ on the $R'_0$-linear dual of $D(\mathcal{G}_r) \otimes Z_p \otimes R'_0$, and compatible with change in $r$ using $\rho_{r,s}$ on $D(\mathcal{G}_r)$ and $\rho'_{r,s}$ on $D(\mathcal{G}_r)$.

We claim that the resulting perfect “evaluation” pairings

$$\langle \cdot, \cdot \rangle_r : D(\mathcal{G}_r)((\langle \chi \rangle \langle a \rangle_N) \otimes Z_p \otimes R'_0 \otimes D(\mathcal{G}_r) \otimes Z_p \otimes R'_0 \longrightarrow R'_0$$

satisfy the compatibility hypothesis (5.1.4) of Lemma 5.1.4. Indeed, the stated compatibility of (5.5.4) with change in $r$ and the very definition (5.4.14) of the transition maps $\rho'_{r,s}$ implies that for $r \geq s$

$$\langle D(Pic^0(\rho)^{r-s})x, y \rangle_s = \langle x, D(U_p^{r-s}) Alb(\sigma)^{r-s})y \rangle_r,$$

so our claim follows from the equality in $\End_{\mathbb{Q}}(J_{r+1})$

$$\Pic(\rho) \circ Alb(\sigma) = U_p^s \sum_{\delta \in \Delta_r/\Delta_{r+1}} \langle \delta^{-1} \rangle,$$

which, as in the proof of Proposition 5.2.4, follows from Lemma 5.4.1 via Lemma 5.4.5. Again, by the $\mathcal{C}^*$-compatibility of (5.5.4), the action of $\mathcal{C}^*$ is self-adjoint with respect to (5.5.5), so Lemma 5.1.4 gives a perfect $\Gal(K'_0/K_0)$-compatible duality pairing $\langle \cdot, \cdot \rangle : D(\mathcal{G}_r)((\langle \chi \rangle \langle a \rangle_N) \otimes \Lambda R'_0 \times D(\mathcal{G}_r) \otimes \Lambda R'_0 \to \Lambda R'_0$ with respect to which $T^*$ is self-adjoint for all $T^* \in \mathcal{C}^*$. That the resulting isomorphism (5.5.3) intertwines $F$ with $V^\vee$ and $V$ with $F^\vee$ is an immediate consequence of the compatibility of the Dieudonné module functor with duality.

We can interpret $D^*_\infty$ in terms of the crystalline cohomology of the Igusa tower as follows. Let $I'^0_r$ and $I'^\infty_r$ be the two “good” components of $\overline{\mathcal{G}}_r$ as in Remark 2.3.12, and form the projective limits

$$H_{cris}^1(I'^*) := \lim_{\to} H_{cris}^1(I'^*_r)$$

for $\ast \in \{\infty, 0\}$, taken with respect to the trace maps on crystalline cohomology (see [Ber74, VII, §2.2]) induced by the canonical degeneracy mappings $\rho : I'^*_r \to I'^*_s$. Then $H_{cris}^1(I'^*)$ is naturally a $\Lambda$-module (via the diamond operators), equipped with a commuting action of $F$ (Frobenius) and $V$.
(Verschiebung) satisfying \( FV = VF = p \). Letting \( U^* \) act as \( F \) (respectively \( \langle p \rangle \) \( N \)) on \( H^1_{\text{cris}}(I^*) \) for \( * = \infty \) (respectively \( * = 0 \)) and the Hecke operators outside \( p \) (viewed as correspondences on the Igusa curves) act via pullback and trace at each level \( r \), we obtain an action of \( \mathcal{S}^* \) on \( H^1_{\text{cris}}(I^*) \). Finally, we let \( \Gamma \) act trivially on \( H^1_{\text{cris}}(I^*) \) for \( * = \infty \) and via \( \langle \chi^{-1} \rangle \) for \( * = 0 \).

**Theorem 5.5.4.** There is a canonical \( \mathcal{S}^* \) and \( \Gamma \)-equivariant isomorphism of \( \Lambda \)-modules

\[
D_\infty = D_\infty^m \oplus D_\infty^{\text{et}} \simeq \mathfrak{f}' H^1_{\text{cris}}(I^0)^{\text{ord}} \oplus \mathfrak{f}' H^1_{\text{cris}}(I^\infty)^{\text{ord}}
\]

which respects the given direct sum decompositions and is compatible with \( F \) and \( V \).

**Proof.** From the exact sequence (5.4.19), we obtain for each \( r \) isomorphisms

\[
(5.5.7) \quad D(G^m_r) \xrightarrow{\sim} f' D(j^0_{p^\infty})^{\text{ord}} \quad \text{and} \quad f' D(j^\infty_{p^\infty})^{\text{ord}} \xrightarrow{\sim} D(G^{\text{et}}_r)
\]

that are \( \mathcal{S}^* \) and \( \Gamma \)-equivariant (with respect to the actions specified in Proposition 5.4.16), and compatible with change in \( r \) via the mappings \( D(\rho_{r,s}) \) on \( D(G^m_r) \) and \( D(\rho) \) on \( D(j^\infty_{p^\infty}) \). On the other hand, for any smooth and proper curve \( X \) over a perfect field \( k \) of characteristic \( p \), thanks to [MM74] and [III79, II, §3 C Remarque 3.11.2] there are natural isomorphisms of \( W(k)[F, V] \)-modules

\[
(5.5.8) \quad D(J_X[p^\infty]) \simeq H^1_{\text{cris}}(J_X/W(k)) \simeq H^1_{\text{cris}}(X/W(k))
\]

that for any finite map of smooth proper curves \( f : Y \to X \) over \( k \) intertwine \( D(\text{Pic}(f)) \) and \( D(\text{Alb}(f)) \) with trace and pullback by \( f \) on crystalline cohomology, respectively. Applying this to \( X = I^* \) for \( * = 0, \infty \), appealing to the identifications (5.5.7), and passing to inverse limits completes the proof. \( \blacksquare \)

Applying the idempotent \( f' \) of (3.3.17) to the Hodge filtration (5.2.5) yields a short exact sequence of free \( \Lambda_{R_{\infty}} \)-modules with semilinear \( \Gamma \)-action and linear commuting action of \( \mathcal{S}^* \):

\[
(5.5.9) \quad 0 \longrightarrow \omega_{\mathcal{S}_r} \longrightarrow D(G_{r,0})_{R_r} \longrightarrow \text{Lie}(\mathcal{S}^r) \longrightarrow 0.
\]

The key to relating (5.5.9) to the slope filtration (5.5.2) is the following comparison isomorphism:

**Proposition 5.5.5.** For each positive integer \( r \), there is a natural isomorphism of short exact sequences

\[
(5.5.10) \quad 0 \longrightarrow \omega_{\mathcal{S}_r} \longrightarrow D(G_{r,0})_{R_r} \longrightarrow \text{Lie}(\mathcal{S}^r) \longrightarrow 0
\]

that is compatible with \( \mathcal{S}^r_\infty \), \( \Gamma \), and change in \( r \) using the mappings (5.4.14) on the top row and the maps \( \rho_* \) on the bottom. Here, the bottom row is obtained from (5.2.2) by applying \( e'' \) and the top row is the Hodge filtration of \( D(G_{r,0})_{R_r} \) given by Proposition 2.2.6.

**Proof.** Let \( \alpha^* : J_r \to B^*_r \) be the map of Definition 5.4.7. We claim that \( \alpha^* \) induces a canonical isomorphism of short exact sequences of free \( R_r \)-modules

\[
(5.5.11) \quad 0 \longrightarrow \omega_{\mathcal{S}_r} \longrightarrow D(G_{r,0})_{R_r} \longrightarrow \text{Lie}(\mathcal{S}^r) \longrightarrow 0
\]

that is compatible with \( \mathcal{S}^r_\infty \), \( \Gamma \), and change in \( r \) using the mappings (5.4.14) on the top row and the maps \( \rho_* \) on the bottom. Here, the bottom row is obtained from (5.2.2) by applying \( e'' \) and the top row is the Hodge filtration of \( D(G_{r,0})_{R_r} \) given by Proposition 2.2.6.
that is $\delta_r^*$ and $\Gamma$-equivariant and compatible with change in $r$ using the map on Néron models induced by $\Pic^0(\rho)$ and the maps (5.4.14) on $\mathcal{S}_r$. Granting this claim, the proposition then follows immediately from Proposition 2.2.4.

To prove our claim, we introduce the following notation: set $V := \Spec(R_r)$, and for $n \geq 1$ put $V_n := \Spec(R_r/p^nR_r)$. For any scheme (or $p$-divisible group) $X$ over $V$, we put $X_n := X \times_V V_n$. If $\mathcal{A}$ is a Néron model over $V$, we will write $H(\mathcal{A})$ for the short exact sequence of free $R_r$-modules obtained by applying Lie to the canonical extension (2.2.4) of $\mathcal{A}^{(0)}$. If $G$ is a $p$-divisible group over $V$, we similarly write $H(G_n)$ for the short exact sequence of Lie algebras associated to the universal extension of $G_n^\vee$ by a vector group over $V_n$ (see Theorem 2.2.1, (2)). If $\mathcal{A}$ is an abelian scheme over $V$ then we have natural and compatible (with change in $n$) isomorphisms

$$(5.5.12) \quad H(\mathcal{A}_n[p^\infty]) \simeq H(\mathcal{A}_n) \simeq H(\mathcal{A})/p^n,$$

thanks to Theorem 2.2.1, (3) and (1); in particular, this justifies our slight abuse of notation.

Applying the contravariant functor $e^sH(\cdot)$ to the diagram of Néron models over $V$ induced by (5.4.11) yields a commutative diagram of short exact sequences of free $R_r$-modules

$$(5.5.13) \quad \begin{array}{ccc}
\ e^sH(J_r) & \xrightarrow{u_1^r} & e^sH(B_r^*) \\
\ e^sH(J_r) & \xleftarrow{u_1^r} & e^sH(B_r) \\
\end{array}$$

in which both vertical arrows are isomorphisms by definition of $e^s$. As in the proofs of Propositions 5.4.12 and 5.4.16, it follows that the horizontal maps must be isomorphisms as well:

$$(5.5.14) \quad e^sH(J_r) \simeq e^sH(B_r^*)$$

Since these isomorphisms are induced via the Néron mapping property and the functoriality of $H(\cdot)$ by the $\delta_r^*(\mathbb{Z})$-equivariant map $\alpha_r^* : J_r \to B_r^*$, they are themselves $\delta_r^*$-equivariant. Similarly, since $\alpha_r^*$ is defined over $\mathbb{Q}$ and compatible with change in $r$ as in Lemma 5.4.10, the isomorphism (5.5.14) is compatible with the given actions of $\Gamma$ (arising via the Néron mapping property from the semilinear action of $\Gamma$ over $K_r$ giving the descent data of $J_r/K_r$ and $B_r/K_r$ to $\mathbb{Q}_p$) and change in $r$. Reducing (5.5.14) modulo $p^n$ and using the canonical isomorphism (5.5.12) yields the identifications

$$(5.5.15) \quad e^sH(J_r)/p^n \simeq e^sH(B_r^*)/p^n \simeq e^sH(B_{r,n}^*[^{\infty}]p^n) \simeq H(e^sB_{r,n}[^{\infty}]p^n) =: H(\mathcal{S}_r,n)$$

which are clearly compatible with change in $n$, and which are easily checked (using the naturality of (5.5.12) and our remarks above) to be $\delta_r^*$ and $\Gamma$-equivariant, and compatible with change in $r$. Since the surjection $R_r \to R_r/pR_r$ is a PD-thickening, passing to inverse limits (with respect to $n$) on (5.5.15) and using Proposition 2.2.6 now completes the proof.

**Corollary 5.5.6.** Let $r$ be a positive integer. Then the short exact sequence of free $R_r$-modules

$$(5.5.16) \quad 0 \longrightarrow e^sH^0(\omega_r) \longrightarrow e^sH^1_{dR,r} \longrightarrow e^sH^1(\mathcal{O}_r) \longrightarrow 0$$
is functorially split; in particular, it is split compatibly with the actions of $\Gamma$ and $\mathcal{S}_r^*$. Moreover, (5.5.16) admits a functorial descent to $\mathbb{Z}_p$: there is a natural isomorphism of split short exact sequences

$$
0 \longrightarrow e^*H^0(\omega_r) \longrightarrow e^*H^1_{dR,r} \longrightarrow e^*H^1(\mathcal{O}_r) \longrightarrow 0
$$

(5.5.17)

$$
0 \longrightarrow D(\overline{\mathcal{S}_r^m}) \otimes R_r \longrightarrow D(\overline{\mathcal{G}_r}) \otimes R_r \longrightarrow D(\overline{\mathcal{G}_r}^\text{et}) \otimes R_r \longrightarrow 0
$$

that is $\mathcal{S}_r^*$ and $\Gamma$ equivariant, with $\Gamma$ acting trivially on $\overline{\mathcal{G}_r}^\text{et}$ and through $\langle \chi \rangle^{-1}$ on $\overline{\mathcal{S}_r^m}$. The identification (5.5.17) is compatible with change in $r$ using the maps $\rho_*$ on the top row and the maps induced by $G_r = G_m \times \mathcal{G}_r$ on the bottom row.

**Proof.** Consider the isomorphism (5.5.10) of Proposition 5.5.5. As $\mathcal{G}_r$ is an ordinary $p$-divisible group by Proposition 5.4.16, the top row of (5.5.10) is functorially split by Lemma 4.2.2, and this gives our first assertion. Composing the inverse of (5.5.10) with the isomorphism (4.2.11) of Lemma 4.2.2 gives the claimed identification (5.5.17). That this isomorphism is compatible with change in $r$ via the specified maps follows easily from the construction of (4.2.11) via (4.2.13). $\blacksquare$

We can now prove Theorem 1.2.6. Let us recall the statement:

**Theorem 5.5.7.** There is a canonical isomorphism of short exact sequences of finite free $\Lambda_{R_\infty}$-modules

$$
0 \longrightarrow e^*H^0(\omega) \longrightarrow e^*H^1_{dR} \longrightarrow e^*H^1(\mathcal{O}) \longrightarrow 0
$$

(5.5.18)

$$
0 \longrightarrow D^m_{\Lambda} \otimes \Lambda_{R_\infty} \longrightarrow D_{\Lambda} \otimes \Lambda_{R_\infty} \longrightarrow D^{\text{et}}_{\Lambda} \otimes \Lambda_{R_\infty} \longrightarrow 0
$$

that is $\Gamma$ and $\mathcal{S}_r^*$-equivariant. Here, the mappings on bottom row are the canonical inclusion and projection morphisms corresponding to the direct sum decomposition $D_{\Lambda} = D^m_{\Lambda} \oplus D^{\text{et}}_{\Lambda}$. In particular, the Hodge filtration exact sequence (5.5.9) is canonically split, and admits a canonical descent to $\Lambda$.

**Proof.** Applying $\otimes_{R_r} R_\infty$ to (5.5.17) and passing to projective limits yields an isomorphism of split exact sequences

$$
0 \longrightarrow e^*H^0(\omega) \longrightarrow e^*H^1_{dR} \longrightarrow e^*H^1(\mathcal{O}) \longrightarrow 0
$$

$$
0 \longrightarrow \lim_{\rho \circ V^{-1}} D(\overline{\mathcal{S}_r^m}) \otimes R_\infty \longrightarrow \lim_{\rho \circ (V^{-1} \times F)} D(\overline{\mathcal{G}_r}) \otimes R_\infty \longrightarrow \lim_{\rho \circ F} D(\overline{\mathcal{G}_r}^\text{et}) \otimes R_\infty \longrightarrow 0
$$
On the other hand, the isomorphisms
\[ \overline{\mathfrak{G}}_r = \overline{\mathfrak{G}}^m_r \times \overline{\mathfrak{G}}^{et}_r \xrightarrow{V^{-r} \times F^r} \overline{\mathfrak{G}}^m_r \times \overline{\mathfrak{G}}^{et}_r = \mathfrak{G}_r \]
duced an isomorphism of projective limits
\[ \varprojlim_{\rho} \left( \mathbf{D}(\overline{\mathfrak{G}}_r) \otimes R_{\infty} \right) \xrightarrow{\simeq} \varprojlim_{\rho(V^{-1} \times F)} \left( \mathbf{D}(\overline{\mathfrak{G}}_r) \otimes R_{\infty} \right) \]
which is visibly compatible with the the canonical splittings of source and target. The result now follows from Lemma 5.1.2 (5) and the proof of Theorem 5.5.2, which guarantee that the canonical mapping \( \mathbf{D}_\infty \otimes_\Lambda \Lambda_{R_\infty} \to \varprojlim_{\rho} (\mathbf{D}(\overline{\mathfrak{G}}_r) \otimes R_{\infty}) \) is an isomorphism respecting the natural splittings. \( \blacksquare \)

As in §5.3, for any subfield \( K \) of \( \mathbf{C}_p \) with ring of integers \( R \), we denote by \( eS(N; \Lambda_R) \) the module of ordinary \( \Lambda_R \)-adic cuspforms of level \( N \) in the sense of [Oht95, 2.5.5]. Following our convention of §3.3, we write \( eS(N; \Lambda_R) \) for the direct summand of \( eS(N; \Lambda_R) \) on which \( \mu_{p-1} \hookrightarrow \mathbf{Z}_p^\times \subseteq \mathfrak{N} \) acts nontrivially.

**Corollary 5.5.8.** There is a canonical isomorphism of finite free \( \Lambda \)-modules
\[ (5.5.19) \quad e^*S(N; \Lambda) \simeq \mathbf{D}_\infty^m \]
that intertwines \( T \in \mathfrak{N} \) on \( e^*S(N; \Lambda) \) with \( T^* \in \mathfrak{N}^* \) on \( \mathbf{D}_\infty^m \), where \( U^*_p \) acts on \( \mathbf{D}_\infty^m \) as \( (p)_N V \).

**Proof.** We claim that there are natural isomorphisms of finite free \( \Lambda_{R_\infty} \)-modules
\[ (5.5.20) \quad \mathbf{D}_\infty^m \otimes_\Lambda \Lambda_{R_\infty} \simeq e^*H^0_\gamma(\omega) \simeq e^*S(N, \Lambda_{R_\infty}) \simeq e^*S(N, \Lambda) \otimes_\Lambda \Lambda_{R_\infty} \]
and that the resulting composite isomorphism intertwines \( T^* \in \mathfrak{N}^* \) on \( \mathbf{D}_\infty^m \) with \( T \in \mathfrak{N} \) on \( e^*S(N; \Lambda) \) and is \( \Gamma \)-equivariant, with \( \gamma \in \Gamma \) acting as \( \langle \chi(\gamma) \rangle^{-1} \otimes \gamma \) on each tensor product. Indeed, the first and second isomorphisms are due to Theorem 5.5.7 and Corollary 5.3.5, respectively, while the final isomorphism is a consequence of the definition of \( e^*S(N; \Lambda_R) \) and the facts that this \( \Lambda_R \)-module is free of finite rank [Oht95, Corollary 2.5.4] and specializes as in [Oht95, 2.6.1]. Twisting the \( \Gamma \)-action on the source and target of the composite \( (5.5.20) \) by \( \langle \chi \rangle \) therefore gives a \( \Gamma \)-equivariant isomorphism
\[ (5.5.21) \quad \mathbf{D}_\infty^m \otimes_\Lambda \Lambda_{R_\infty} \simeq S(N, \Lambda) \otimes_\Lambda \Lambda_{R_\infty} \]
with \( \gamma \in \Gamma \) acting as \( 1 \otimes \gamma \) on source and target. Passing to \( \Gamma \)-invariants on \( (5.5.21) \) yields \( (5.5.19) \). \( \blacksquare \)

**Remark 5.5.9.** Via Proposition 5.5.3 and the natural \( \Lambda \)-adic duality between \( e\mathfrak{N} \) and \( eS(N; \Lambda) \) [Oht95, Theorem 2.5.3], we obtain a canonical \( \text{Gal}(K_0'/K_0) \)-equivariant isomorphism of \( \Lambda_{R'_0} \)-modules
\[ e^*\mathfrak{N} \otimes_\Lambda \Lambda_{R'_0} \simeq \mathbf{D}_\infty^{et}(\langle a \rangle_N) \otimes_\Lambda \Lambda_{R'_0} \]
that intertwines \( T \otimes 1 \) for \( T \in \mathfrak{N} \) acting on \( e^*\mathfrak{N} \) by multiplication with \( T^* \otimes 1 \), with \( U^*_p \) acting on \( \mathbf{D}_\infty^{et}(\langle a \rangle_N) \) as \( F \). From Theorem 5.5.4 and Corollary 5.5.8 we then obtain canonical isomorphisms
\[ e^*S(N; \Lambda) \simeq f'H^1_{\text{cris}}(I^0)_{\text{ord}} \text{ respectively } e^*\mathfrak{N} \otimes_\Lambda \Lambda_{R'_0} \simeq f'H^1_{\text{cris}}(I^\infty)_{\text{ord}}(\langle a \rangle_N) \otimes_\Lambda \Lambda_{R'_0} \]
interwining \( T \) (respectively \( T \otimes 1 \)) with \( T^* \) (respectively \( T^* \otimes 1 \)) where \( U^*_p \) acts on crystalline cohomology as \( (p)_N V \) (respectively \( F \otimes 1 \)). The second of these isomorphisms is moreover \( \text{Gal}(K_0'/K_0) \)-equivariant.

In order to relate the slope filtration \( (5.5.2) \) of \( \mathbf{D}_\infty \) to the ordinary filtration of \( e^*H^1_{\text{et}} \), we require:

**Lemma 5.5.10.** Let \( r \) be a positive integer let \( G_r = e^*J_r[p^\infty] \) be the unique \( \mathbf{Q}_p \)-descent of the generic fiber of \( \mathfrak{G}_r \), as in Definition 5.4.14. There are canonical isomorphisms of free \( W(\overline{\mathbf{F}}_p) \)-modules
\[ (5.5.22a) \quad \mathbf{D}(\overline{\mathfrak{G}}^r)_{Z_p} \otimes W(\overline{\mathbf{F}}_p) \simeq \text{Hom}_{\mathbf{Z}_p}(T_pG_r^{et}, Z_p) \otimes W(\overline{\mathbf{F}}_p) \]
(5.5.22b) \[ D(\mathfrak{G}_r^\text{et})(-1) \otimes W(F_p) \simeq \text{Hom}_{\mathbb{Z}_p}(T_p \mathfrak{G}_r^\text{et}, \mathbb{Z}_p) \otimes W(F_p). \]

that are \( \mathfrak{G}_r^\text{et} \)-equivariant and \( \mathcal{G}_{Q_p} \)-compatible for the diagonal action on source and target, with \( \mathcal{G}_{Q_p} \) acting trivially on \( D(\mathfrak{G}_r^\text{et}) \) and via \( \chi^{-1} \cdot (\chi^{-1}) \) on \( D(\mathfrak{G}_r^\text{et})(-1) := D(\mathfrak{G}_r^\text{et}) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(-1). \) The isomorphism (5.5.22a) intertwines \( F \otimes \sigma \) with \( 1 \otimes \sigma \) while (5.5.22b) intertwines \( V \otimes \sigma^{-1} \) with \( 1 \otimes \sigma^{-1}. \)

Proof. Let \( \mathfrak{S} \) be any object of \( \text{pdiv}_r \) and write \( G \) for the unique descent of the generic fiber \( \mathfrak{S}_K \) to \( \mathbb{Q}_p. \) We recall that the semilinear \( \Gamma \)-action on \( D(\mathfrak{G}_r^\text{et}) \) and via \( \chi^{-1} \cdot (\chi^{-1}) \) on \( D(\mathfrak{G}_r^\text{et})(-1) := D(\mathfrak{G}_r^\text{et}) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(-1). \) The isomorphism (5.5.22a) intertwines \( F \otimes \sigma \) with \( 1 \otimes \sigma \) while (5.5.22b) intertwines \( V \otimes \sigma^{-1} \) with \( 1 \otimes \sigma^{-1}. \)

For any étale \( p \)-divisible group \( H \) over a perfect field \( k, \) one has a canonical isomorphism of \( W(\overline{k}) \)-modules with semilinear \( \mathcal{G}_k \)-action

\[
D(H) \otimes W(\overline{k}) \simeq \text{Hom}_{\mathbb{Z}_p}(T_p H, \mathbb{Z}_p) \otimes W(\overline{k})
\]

that intertwines \( F \otimes \sigma \) with \( 1 \otimes \sigma \) and \( 1 \otimes g \) with \( g \otimes g \) for \( g \in \mathcal{G}_k; \) for example, this can be deduced by applying [BM79, §4.1 a)] to \( \mathfrak{T}_F \) and using the fact that the Dieudonné crystal is compatible with base change. In our case, the étale \( p \)-divisible group \( \mathfrak{G}_r^\text{et} \) lifts \( \mathfrak{G}_r^\text{et} \) over \( R_r, \) and we obtain a natural isomorphism of \( W(\overline{\mathbb{F}}_p) \)-modules with semilinear \( \mathcal{G}_r \)-action

\[
D(\mathfrak{G}_r^\text{et}) \otimes W(F_p) \simeq \text{Hom}_{\mathbb{Z}_p}(T_p \mathfrak{G}_r^\text{et}, \mathbb{Z}_p) \otimes W(F_p).
\]

By naturality in \( \mathfrak{S}_r, \) this identification respects the semilinear \( \Gamma \)-actions on both sides (which are trivial, as \( \Gamma \) acts trivially on \( \mathfrak{S}_r^\text{et}); \) as explained in our initial remarks, it is precisely this action which allows us to view \( T_p \mathfrak{S}_r^\text{et} \) as a \( \mathbb{Z}_p[\mathcal{G}_{Q_p}] \)-module, and we deduce (5.5.22a). The proof of (5.5.22b) is similar, using the natural isomorphism (proved as above) for any multiplicative \( p \)-divisible group \( H/k \)

\[
D(H) \otimes W(\overline{k}) \simeq T_p H^t \otimes W(\overline{k})
\]

which intertwines \( V \otimes \sigma^{-1} \) with \( 1 \otimes \sigma^{-1} \) and \( 1 \otimes g \) with \( g \otimes g, \) for \( g \in \mathcal{G}_k. \)

Proof of Theorem 1.2.8 and Corollary 1.2.10. For a \( p \)-divisible group \( H \) over a field \( K, \) we will write \( H^1_{\text{ét}}(H) := \text{Hom}_{\mathbb{Z}_p}(T_p H, \mathbb{Z}_p); \) our notation is justified by the standard fact that, for \( J_X \) the Jacobian of a curve \( X \) over \( K, \) there is a natural isomorphisms of \( \mathbb{Z}_p[\mathcal{G}_K] \)-modules

\[
(5.5.23) \quad H^1_{\text{ét}}(J_X[p^\infty]) \simeq H^1_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_p).
\]

It follows from (5.5.22a)–(5.5.22b) and Theorem 5.5.2 (1)–(2) that \( H^1_{\text{ét}}(G_r^\text{et}) \otimes_{\mathbb{Z}_p} W(\overline{\mathbb{F}}_p) \) is a free \( W(\overline{\mathbb{F}}_p)[\Delta/\Delta_r] \)-module of rank \( d' \) for \( * \in \{ \text{ét}, m \}, \) and hence that \( H^1_{\text{ét}}(G_r^*) \) is a free \( \mathbb{Z}_p[\Delta/\Delta_r] \)-module of rank \( d' \) by Lemma 5.1.3. In a similar manner, using the faithful flatness of \( W(\overline{\mathbb{F}}_p)[\Delta/\Delta_r] \) over \( \mathbb{Z}_p[\Delta/\Delta_r], \) we deduce that the canonical trace mappings

\[
(5.5.24) \quad H^1_{\text{ét}}(G_r^*) \longrightarrow H^1_{\text{ét}}(G_r^*)
\]
are surjective for all $r \geq r'$. By Lemma 5.1.2, we conclude that $H_1^1(G_\infty^\ast) := \lim_{\leftarrow r} H_1^1(G^\ast_r)$ is a free $\Lambda$-module of rank $d'$ and that there are canonical isomorphisms of $\Lambda_W(\mathbb{F}_p)$-modules

$$H_1^1(G_\infty^\ast) \otimes_{\Lambda} \Lambda_W(\mathbb{F}_p) \simeq \lim_{\leftarrow r} \left( H_1^1(G^\ast_r) \otimes_{\mathbb{Z}_p} W(\mathbb{F}_p) \right)$$

for $\ast \in \{\text{ét}, m\}$. Since we likewise have canonical identifications

$$D^\ast \otimes_{\Lambda} \Lambda_W(\mathbb{F}_p) \simeq \lim_{\leftarrow r} \left( D(G^\ast_r) \otimes_{\mathbb{Z}_p} W(\mathbb{F}_p) \right)$$

thanks to Lemma 5.1.2 and (the proof of) Theorem 5.5.2, passing to inverse limits on (5.5.22a)–(5.5.25) gives a canonical isomorphism of $\Lambda_W(\mathbb{F}_p)$-modules

$$\text{(5.5.25)}
D^\ast \otimes_{\Lambda} \Lambda_W(\mathbb{F}_p) \simeq H_1^1(G_\infty^\ast) \otimes_{\Lambda} \Lambda_W(\mathbb{F}_p)$$

for $\ast \in \{\text{ét}, m\}$.

Applying the functor $H_1^1(\cdot)$ to the connected-étale sequence of $G_r$ yields a short exact sequence of $\mathbb{Z}_p[\mathcal{H}_{\mathbb{Q}_p}]$-modules

$$0 \longrightarrow H_1^1(G^\text{ét}_r) \longrightarrow H_1^1(G_r) \longrightarrow H_1^1(G_m^\text{ét}_r) \longrightarrow 0$$

which naturally identifies $H_1^1(G^\ast_r)$ with the invariants (respectively covariants) of $H_1^1(G_r)$ under the inertia subgroup $\mathcal{I} \subseteq \mathcal{H}_{\mathbb{Q}_p}$ for $\ast = \text{ét}$ (respectively $\ast = m$). As $G_r = e^s J_r[p^\infty]$ by definition, we deduce from this and (5.5.23) a natural isomorphism of short exact sequences of $\mathbb{Z}_p[\mathcal{H}_{\mathbb{Q}_p}]$-modules

$$\text{(5.5.26)}
0 \longrightarrow H_1^1(G^\text{ét}_r) \longrightarrow H_1^1(G_r) \longrightarrow H_1^1(G_m^\text{ét}_r) \longrightarrow 0$$

where for notational ease abbreviate $H_1^1(G_r^\ast) := H_1^1(X_{r[\mathcal{H}_{\mathbb{Q}_p}], \mathbb{Z}_p})$. As the trace maps (5.5.24) are surjective, passing to inverse limits on (5.5.26) yields an isomorphism of short exact sequences

$$\text{(5.5.27)}
0 \longrightarrow \lim_{\leftarrow r} (e^s H_1^1(G_r^\ast)) \longrightarrow \lim_{\leftarrow r} e^s H_1^1(G_r) \longrightarrow \lim_{\leftarrow r} (e^s H_1^1(G_m^\text{ét}_r)) \longrightarrow 0$$

Since inverse limits commute with group invariants, the bottom row of (5.5.27) is canonically isomorphic to the ordinary filtration of Hida’s $e^s H_1^1$, and Theorem 1.2.8 follows immediately from (5.5.25). Corollary 1.2.10 is then an easy consequence of Theorem 1.2.8 and Lemma 5.1.3; alternately one can prove Corollary 1.2.10 directly from Lemma 5.1.2, using what we have seen above.

5.6. Ordinary families of $\mathcal{O}$-modules. We now study the family of Dieudonné crystals attached to the tower of $p$-divisible groups $\{\mathcal{G}_r/R_r\}_{r \geq 1}$. For each pair of positive integers $r \geq s$, we have a morphism $\rho_{r,s} : \mathcal{G}_s \times_{\mathcal{T}_s} T_r \rightarrow \mathcal{G}_r$ in $p\text{div}_{R_r}^+$; applying the contravariant functor $\mathcal{M}_r : p\text{div}_{R_r}^+ \rightarrow \mathcal{B}_S$, we
studied in §4.1 to the map on connected-étale sequences induced by $\rho_{r,s}$ and using the exactness of $\mathcal{M}_r$ and its compatibility with base change (Theorem 4.1.3), we obtain maps of exact sequences in $\mathcal{B}T_{s}^Γ$

$$\begin{array}{cccc}
0 & \longrightarrow & \mathcal{M}_r (\mathcal{G}_r^\text{ét}) & \longrightarrow & \mathcal{M}_r (\mathcal{G}_r) & \longrightarrow & \mathcal{M}_r (\mathcal{G}_r^m) & \longrightarrow & 0 \\
(5.6.1) & & \mathcal{M}_r (\rho_{r,s}) & & \mathcal{M}_r (\rho_{r,s}) & & \mathcal{M}_r (\rho_{r,s})
\end{array}$$

for each integer $0 < m < N_r$.

\begin{equation}
\begin{array}{cccc}
0 & \longrightarrow & \mathcal{M}_r (\mathcal{G}_r^\text{ét}) \otimes \mathcal{G}_r & \longrightarrow & \mathcal{M}_r (\mathcal{G}_r) \otimes \mathcal{G}_r & \longrightarrow & \mathcal{M}_r (\mathcal{G}_r^m) \otimes \mathcal{G}_r & \longrightarrow & 0 \\
(5.6.2) & & \mathcal{M}_r (\rho_{r,s}) & & \mathcal{M}_r (\rho_{r,s}) & & \mathcal{M}_r (\rho_{r,s})
\end{array}
\end{equation}

**Definition 5.6.1.** Let $\star = \text{ét}$ or $\star = \text{m}$ and define

$$\begin{array}{c}
\mathcal{M}_\infty := \lim_{r} \left( \mathcal{M}_r (\mathcal{G}_r) \otimes \mathcal{G}_r \right) \\
\mathcal{M}_\infty^\star := \lim_{r} \left( \mathcal{M}_r (\mathcal{G}_r^\star) \otimes \mathcal{G}_r \right)
\end{array}$$

with the projective limits taken with respect to the mappings induced by (5.6.1).

Each of (5.6.2) is naturally a module over the completed group ring $\Lambda_{\mathcal{G}_\infty}$ and is equipped with a semilinear action of $\Gamma$ and a $\phi$-semilinear Frobenius morphism defined by $F := \lim \left( \phi_{\mathcal{M}_r} \otimes \phi \right)$. Since $\phi$ is bijective on $\mathcal{G}_\infty$, we also have a $\phi^{-1}$-semilinear Verscheibung morphism defined as follows. For notational ease, we provisionally set $M_r := \mathcal{M}_r (\mathcal{G}_r) \otimes \mathcal{G}_r \mathcal{G}_\infty$ and we define

$$\begin{array}{c}
V_r : M_r \overset{m \mapsto 1 \otimes m}{\longrightarrow} \phi^{-1} M_r \underset{\phi^{-1} (\phi_{\mathcal{M}_r} \otimes 1)}{\longrightarrow} \phi^{-1} \phi^* M_r \simeq M_r
\end{array}$$

with $\phi_{\mathcal{M}_r}$ as above Definition 4.1.2. It is easy to see that the $V_r$ are compatible with $r$, and we put $V := \lim V_r$ on $\mathcal{M}_\infty$. We define Verscheibung morphisms on $\mathcal{M}_\infty^\star$ for $\star = \text{ét}, \text{m}$ similarly. As the composite of $\psi_{\mathcal{M}_r}$ and $1 \phi_{\mathcal{M}_r}$ in either order is multiplication by $E_r (u_r) = u_0 / u_1 = : \omega$, we have $FV = VF = \omega$.

Due to the functoriality of $\mathcal{M}_r$, we moreover have a $\Lambda_{\mathcal{G}_\infty}$-linear action of $\mathcal{G}_r^\star$ on each of (5.6.2) which commutes with $F$, $V$, and $\Gamma$.

**Theorem 5.6.2.** As in Proposition 3.3.6, set $d' := \sum_{k=3}^p \dim_{\mathbb{F}_p} S_k (N; \mathbb{F}_p)^{\text{ord}}$. Then $\mathcal{M}_\infty$ (respectively $\mathcal{M}_\infty^\star$ for $\star = \text{ét}, \text{m}$) is a free $\Lambda_{\mathcal{G}_\infty}$-module of rank $2d'$ (respectively $d'$) and there is a canonical short exact sequence of $\Lambda_{\mathcal{G}_\infty}$-modules with linear $\mathcal{G}_r^\star$-action and semi linear actions of $\Gamma$, $F$ and $V$

$$\begin{array}{c}
0 \longrightarrow \mathcal{M}_\infty^\text{ét} \longrightarrow \mathcal{M}_\infty \longrightarrow \mathcal{M}_\infty^\text{m} \longrightarrow 0 \\
(5.6.4)
\end{array}$$

Extension of scalars of (5.6.4) along the quotient $\Lambda_{\mathcal{G}_\infty} \rightarrow \mathcal{G}_\infty [\Delta / \Delta_r]$ recovers the exact sequence

$$\begin{array}{c}
0 \longrightarrow \mathcal{M}_r (\mathcal{G}_r^\text{ét}) \otimes \mathcal{G}_r \longrightarrow \mathcal{M}_r (\mathcal{G}_r) \otimes \mathcal{G}_r \longrightarrow \mathcal{M}_r (\mathcal{G}_r^m) \otimes \mathcal{G}_r \longrightarrow 0 \\
(5.6.5)
\end{array}$$

for each integer $r > 0$, compatibly with $\mathcal{G}_r^\star$, $\Gamma$, $F$, and $V$.

**Proof.** Since $\phi$ is an automorphism of $\mathcal{G}_\infty$, pullback by $\phi$ commutes with projective limits of $\mathcal{G}_\infty^\star$-modules. As the canonical $\mathcal{G}_\infty^\star$-linear map $\phi^* \Lambda_{\mathcal{G}_\infty} \rightarrow \Lambda_{\mathcal{G}_\infty}$ is an isomorphism of rings (even of $\mathcal{G}_\infty$-algebras), it therefore suffices to prove the assertions of Theorem 5.6.2 after pullback by $\phi$, which will be more convenient due to the relation between $\phi^* \mathcal{M}_r (\mathcal{G}_r)$ and the Dieudonné crystal of $\mathcal{G}_r$. 
Pulling back (5.6.1) by \( \varphi \) gives a commutative diagram with exact rows

\[
0 \longrightarrow \varphi^* \mathcal{M}_r(G^\Delta_{s^\text{et}}) \longrightarrow \varphi^* \mathcal{M}_r(G_s) \longrightarrow \varphi^* \mathcal{M}_r(G^\Delta_{s^m}) \longrightarrow 0
\]

(5.6.6)

\[
0 \longrightarrow \varphi^* \mathcal{M}_s(G^\Delta_{s^\text{et}}) \otimes \mathcal{S}_r \longrightarrow \varphi^* \mathcal{M}_r(G_s) \otimes \mathcal{S}_r \longrightarrow \varphi^* \mathcal{M}_r(G^\Delta_{s^m}) \otimes \mathcal{S}_r \longrightarrow 0
\]

and we apply Lemma 5.1.2 with \( A_r := \mathcal{S}_r, I_r := (u_r), B = \mathcal{S}_\infty \), and with \( M_r \) each one of the terms in the top row of (5.6.6). The isomorphism (4.2.14a) of Proposition 4.2.3 ensures, via Theorem 5.5.2 (1), that the hypothesis (5.1.1a) is satisfied.

Due to the functoriality of (4.2.14a), for any \( r \geq s \), the mapping obtained from (5.6.6) by reducing modulo \( I_r \) is identified with the mapping on (5.5.1) induced (via functoriality of \( \mathcal{D}(\cdot) \)) by \( \overline{\rho}_{r,s} \). As was shown in the proof of Theorem 5.5.2, these mappings are surjective for all \( r \geq s \), and we conclude that hypothesis (5.1.1b) holds as well. Moreover, the vertical mappings of (5.6.6) are then surjective by Nakayama’s Lemma, so as in the proof of Theorems 5.2.3 and 5.5.2 (and keeping in mind that pullback by \( \varphi \) commutes with projective limits of \( \mathcal{S}_\infty \)-modules), we obtain, by applying \( \otimes \mathcal{S}_r \mathcal{S}_\infty \) to (5.6.6), passing to projective limits, and pulling back by \((\varphi^{-1})^*\), the short exact sequence (5.6.4).

Remark 5.6.3. In the proof of Theorem 5.6.2, we could have alternately applied Lemma 5.1.2 with \( A_r = \mathcal{S}_r \) and \( I_r := (E_r) \), appealing to the identifications (4.2.14b) of Proposition 4.2.3 and (5.5.10) of Proposition 5.5.5, and to Theorem 5.2.3.

The short exact sequence (5.6.4) is closely related to its \( \Lambda_{\mathcal{S}_\infty} \)-linear dual. In what follows, we write \( \mathcal{S}_\infty' := \varprojlim_r \mathbb{Z}_p[\mu_N][u_r] \), taken along the mappings \( u_r \mapsto \varphi(u_{r+1}) \); it is naturally a \( \mathcal{S}_\infty \)-algebra.

**Theorem 5.6.4.** Let \( \mu : \Gamma \to \Lambda_{\mathcal{S}_\infty}^\times \) be the crossed homomorphism given by \( \mu(\gamma) := \frac{m}{\gamma m_1} \chi(\gamma) \langle \chi(\gamma) \rangle \). There is a canonical \( \mathcal{H}^* \) and \( \text{Gal}(K_{\infty}'/K_0) \)-equivariant isomorphism of exact sequences of \( \Lambda_{\mathcal{S}_\infty} \)-modules

\[
0 \longrightarrow \mathcal{M}_\infty^\Delta(\mu(a)N)_{\Lambda_{\mathcal{S}_\infty}'} \longrightarrow \mathcal{M}_\infty(\mu(a)N)_{\Lambda_{\mathcal{S}_\infty}'} \longrightarrow \mathcal{M}_\infty^m(\mu(a)N)_{\Lambda_{\mathcal{S}_\infty}'} \longrightarrow 0
\]

(5.6.7)

\[
0 \longrightarrow \mathcal{M}_\infty^\Delta(\mu(a)N)_{\Lambda_{\mathcal{S}_\infty}'} \longrightarrow \mathcal{M}_\infty(\mu(a)N)_{\Lambda_{\mathcal{S}_\infty}'} \longrightarrow \mathcal{M}_\infty^m(\mu(a)N)_{\Lambda_{\mathcal{S}_\infty}'} \longrightarrow 0
\]

that intertwines \( F \) (respectively \( V \)) on the top row with \( V^\vee \) (respectively \( F^\vee \)) on the bottom.

**Proof.** We first claim that there is a natural isomorphism of \( \mathcal{S}_\infty' \)[\( \Delta/\Delta_r \)]-modules

\[
\mathcal{M}_r(G_r)(\mu(a)N) \otimes_{\mathcal{S}_r} \mathcal{S}_\infty' \simeq \text{Hom}_{\mathcal{S}_\infty'}(\mathcal{M}_r(G_r) \otimes_{\mathcal{S}_r} \mathcal{S}_\infty', \mathcal{S}_\infty')
\]

(5.6.8)

that is \( \mathcal{H}^* \)-equivariant and \( \text{Gal}(K_{\infty}'/K_0) \)-compatible for the standard action \( \gamma \cdot f(m) := \gamma f(\gamma^{-1}m) \) on the right side, and that intertwines \( F \) and \( V \) with \( V^\vee \) and \( F^\vee \), respectively. Indeed, this follows immediately from the identifications

\[
\mathcal{M}_r(G_r)(\mu(a)N) \otimes_{\mathcal{S}_r} \mathcal{S}_\infty' \simeq \mathcal{M}_r(G_r') \otimes_{\mathcal{S}_r} \mathcal{S}_\infty' \simeq \mathcal{M}_r(G_r') \otimes_{\mathcal{S}_r} \mathcal{S}_\infty' =: \mathcal{M}_r(G_r') \otimes_{\mathcal{S}_r} \mathcal{S}_\infty' \simeq \mathcal{M}_r(G_r') \otimes_{\mathcal{S}_r} \mathcal{S}_\infty' =: \mathcal{M}_r(G_r) \otimes_{\mathcal{S}_r} \mathcal{S}_\infty'
\]

and the definition (Definition 4.1.2) of duality in \( \text{BT}_{\mathcal{S}_r}^\Delta \); here, the first isomorphism in (5.6.9) results from Proposition 5.4.15 and Theorem 4.1.3 (2), while the final identification is due to Theorem 4.1.3 (1). The identification (5.6.8) carries \( F \) (respectively \( V \)) on its source to \( V^\vee \) (respectively \( F^\vee \)) on its target due to the compatibility of the functor \( \mathcal{M}_r(\cdot) \) with duality (Theorem 4.1.3 (1)).
From (5.6.8) we obtain a natural Gal($K_r'/K_0$)-compatible evaluation pairing of $\mathcal{G}'_\infty$-modules

\[(5.6.10) \quad \langle \cdot, \cdot \rangle_r : \mathcal{M}_r(\mathcal{G}_r)(\mu(a)_N) \otimes \mathcal{G}'_\infty \times \mathcal{M}_r(\mathcal{G}_r) \otimes \mathcal{G}'_\infty \longrightarrow \mathcal{G}'_\infty\]

with respect to which the natural action of $\mathcal{H}$ is self-adjoint, due to the fact that (5.6.9) is $\mathcal{H}^*$-equivariant by Proposition 5.4.15. Due to the compatibility with change in $r$ of the identification (5.4.15) of Proposition 5.4.15 together with the definitions (5.4.14) of $\rho_{r,s}$ and $\rho'_{r,s}$, the identification (5.6.9) intertwines the map induced by $\text{Pic}^0(\rho)$ on its source with the map induced by $U_p^{-1} \text{Alb}(\sigma)$ on its target. For $r \geq s$, we therefore have

\[
\langle \mathcal{M}_r(\rho_{r,s})x, \mathcal{M}_r(\rho_{r,s})y \rangle_s = \langle x, \mathcal{M}_r(U_p^{s-r} \text{Pic}^0(\rho)^{r-s} \text{Alb}(\sigma)^{r-s})y \rangle_r = \sum_{\delta \in \Delta_s/\Delta_r} \langle x, \delta^{-1}y \rangle_r,
\]

where the final equality follows from (5.5.6). Thus, the perfect pairings (5.6.10) satisfy the compatibility condition (5.1.4) of Lemma 5.1.4 which, together with Theorem 5.6.2, completes the proof. ■

The $\Lambda_{\mathcal{E}_\infty}$-modules $\mathcal{M}_{\mathcal{E}_\infty}^\text{et}$ and $\mathcal{M}_{\mathcal{E}_\infty}^\text{m}$ admit canonical descents to $\Lambda$:

**Theorem 5.6.5.** There are canonical $\mathcal{H}^*$, $\Gamma$, $F$ and $V$-equivariant isomorphisms of $\Lambda_{\mathcal{E}_\infty}$-modules

\[
(5.6.11a) \quad \mathcal{M}_{\mathcal{E}_\infty}^\text{et} \simeq D_{\mathcal{E}_\infty}^\text{et} \otimes _{\Lambda} \Lambda_{\mathcal{E}_\infty},
\]

intertwining $F$ (resp. $V$) with $F \otimes \varphi$ (resp. $V^{-1} \otimes \varphi^{-1}$) and $\gamma \in \Gamma$ with $\gamma \otimes \gamma$, and

\[
(5.6.11b) \quad \mathcal{M}_{\mathcal{E}_\infty}^\text{m} \simeq D_{\mathcal{E}_\infty}^\text{m} \otimes _{\Lambda} \Lambda_{\mathcal{E}_\infty},
\]

intertwining $F$ (resp. $V$) with $V^{-1} \otimes \varphi \cdot \gamma$ (resp. $V \otimes \varphi^{-1}$) and $\gamma$ with $\gamma \otimes \chi(\gamma)^{-1}u_1/u_1$.

In particular, $F$ (resp. $V$) acts invertibly on $\mathcal{M}_{\mathcal{E}_\infty}^\text{et}$ (resp. $\mathcal{M}_{\mathcal{E}_\infty}^\text{m}$).

**Proof.** We twist the identifications (4.2.2) of Proposition 4.2.1 to obtain natural isomorphisms

\[
\begin{align*}
\mathcal{M}_r(\mathcal{G}_r) & \underset{F^\text{et}(4.2.2)}{\simeq} D(\mathcal{G}_r^\text{et})_{\mathcal{Z}_p} \otimes_{\mathcal{Z}_p} \mathcal{E}_r \\
\mathcal{M}_r(\mathcal{G}_r) & \underset{V^{-1} \text{et}(4.2.2)}{\simeq} D(\mathcal{G}_r^\text{m})_{\mathcal{Z}_p} \otimes_{\mathcal{Z}_p} \mathcal{E}_r
\end{align*}
\]

that are $\mathcal{H}^*$-equivariant and, Thanks to 4.2.3, compatible with change in $r$ using the maps on source and target induced by $\rho_{r,s}$. Passing to inverse limits and appealing to Lemma 5.1.2 and (the proof of) Theorem 5.5.2, we deduce for $* = \text{et}$, $\text{m}$ natural isomorphisms of $\Lambda_{\mathcal{E}_\infty}$-modules

\[
\mathcal{M}_r^* \simeq \lim_{\longrightarrow r} \left(D(\mathcal{G}_r^*)_{\mathcal{Z}_p} \otimes_{\mathcal{Z}_p} \mathcal{E}_\infty) \right) \simeq D^* \otimes _{\Lambda} \Lambda_{\mathcal{E}_\infty}
\]

that are $\mathcal{H}^*$-equivariant and satisfy the asserted compatibility with respect to Frobenius, Verscheibung, and the action of $\Gamma$ due to Proposition 4.2.1 and the definitions (4.2.1a)–(4.2.1b). ■

We can now prove Theorem 1.2.14, which asserts that the slope filtration (1.2.14) of $\mathcal{M}_\infty$ specializes, on the one hand, to the slope filtration (5.5.2) of $D_\infty$, and on the other hand to the Hodge filtration (5.5.9) (in the opposite direction!) of $\epsilon^* H^1_{dR}$. We recall the precise statement:
Theorem 5.6.6. Let $\tau : \Lambda_{\infty} \to \Lambda$ be the $\Lambda$-algebra surjection induced by $u_r \mapsto 0$. There is a canonical $\Gamma$ and $\Gamma^*$-equivariant isomorphism of split exact sequences of finite free $\Lambda$-modules

$$0 \rightarrow \mathcal{M}^{\text{ét}}_{\infty} \otimes_{\Lambda_{\infty}} \Lambda \rightarrow \mathcal{M}_{\infty} \otimes_{\Lambda_{\infty}} \Lambda \rightarrow \mathcal{M}^{\text{m}}_{\infty} \otimes_{\Lambda_{\infty}} \Lambda \rightarrow 0$$

(5.6.12)

which carries $F \otimes 1$ to $F$ and $V \otimes 1$ to $V$.

Let $\theta \circ \varphi : \Lambda_{\infty} \to \Lambda_{\infty}$ be the $\Lambda$-algebra surjection induced by $u_r \mapsto (\varepsilon(r))^{\mu} - 1$. There is a canonical $\Gamma$ and $\Gamma^*$-equivariant isomorphism of split exact sequences of finite free $\Lambda_{\infty}$-modules

$$0 \rightarrow \mathcal{M}^{\text{ét}}_{\infty} \otimes_{\Lambda_{\infty}} \Lambda_{\infty} \rightarrow \mathcal{M}_{\infty} \otimes_{\Lambda_{\infty}} \Lambda_{\infty} \rightarrow \mathcal{M}^{\text{m}}_{\infty} \otimes_{\Lambda_{\infty}} \Lambda_{\infty} \rightarrow 0$$

(5.6.13)

where $i$ and $j$ are the canonical sections given by the splitting in Theorem 1.2.6.

Proof. To prove the first assertion, apply Lemma 5.1.2 with $A_r = \mathcal{S}_r$, $I_r = (u_r)$, $B = \mathfrak{S}_{\infty}$, $B' = \mathbf{Z}_p$ (viewed as a $B$-algebra via $\tau$) and $M_r = \mathcal{M}^\text{ét}_r$ for $r \in \{\text{ét}, m, \text{null}\}$. Thanks to (4.2.14a) in the case $G = \mathcal{G}_r$, we have a canonical identification $\overline{M}_r := M_r/I_r$ is identified with that of Definition 5.5.1. It follows from this and Theorem 5.5.2 (1)–(2) that the hypotheses (5.1.1a)–(5.1.1b) are satisfied, and (5.6.12) is an isomorphism by Lemma 5.1.2 (5).

In exactly the same manner, the second assertion follows by appealing to Lemma 5.1.2 with $A_r = \mathcal{S}_r$, $I_r = (E_r)$, $B = \mathfrak{S}_{\infty}$, $B' = \Lambda_{\infty}$ (viewed as a $B$-algebra via $\theta \circ \varphi$) and $M_r = \mathcal{M}^\text{ét}_r$, using (4.2.14b) and Theorem 5.2.3 to verify the hypotheses (5.1.1a)–(5.1.1b).

Proof of Theorem 1.2.15 and Corollary 1.2.16. Applying Theorem 4.1.5 to (the connected-étale sequence of) $\mathcal{G}_r$ gives a natural isomorphism of short exact sequences

$$0 \rightarrow \mathcal{M}_r(\mathcal{G}^{\text{ét}}_r) \otimes_{\mathcal{S}_r, \phi} A_r \rightarrow \mathcal{M}_r(\mathcal{G}_r) \otimes_{\mathcal{S}_r, \phi} A_r \rightarrow \mathcal{M}_r(\mathcal{G}^{\text{m}}_r) \otimes_{\mathcal{S}_r, \phi} A_r \rightarrow 0$$

(5.6.14)

$$0 \rightarrow \mathcal{H}^1_{\text{ét}}(\mathcal{G}^{\text{ét}}_r) \otimes_{\mathbf{Z}_p} A_r \rightarrow \mathcal{H}^1_{\text{ét}}(\mathcal{G}_r) \otimes_{\mathbf{Z}_p} A_r \rightarrow \mathcal{H}^1_{\text{ét}}(\mathcal{G}^{\text{m}}_r) \otimes_{\mathbf{Z}_p} A_r \rightarrow 0$$

Due to Theorem 5.6.2, the terms in the top row of 5.6.14 are free of ranks $d', 2d'$, and $d'$ over $\overline{\mathbf{A}}_r[\Delta/\Delta_r]$, respectively, so we conclude from Lemma 5.1.3 (with $A = \mathbf{Z}_p[\Delta/\Delta_r]$ and $B = \mathbf{A}_r[\Delta/\Delta_r]$) that $H^1_{\text{ét}}(\mathcal{G}^\ast)$ is a free $\mathbf{Z}_p[\Delta/\Delta_r]$-module of rank $d'$ for $\ast \in \{\text{ét}, m, \text{null}\}$ and that $H^1_{\text{ét}}(\mathcal{G}_r)$ is free of rank $2d'$ over $\mathbf{Z}_p[\Delta/\Delta_r]$. Using the fact that $\mathbf{Z}_p \rightarrow A_r$ is faithfully flat, it then follows from the surjectivity of the vertical maps in (5.6.6) (which was noted in the proof of Theorem 5.6.2) that the canonical trace mappings $H^1_{\text{ét}}(\mathcal{G}^\ast) \rightarrow H^1_{\text{ét}}(\mathcal{G}_r)$ for $\ast \in \{\text{ét}, m, \text{null}\}$ are surjective for all $r \geq r'$. Applying Lemma 5.1.2 with $A_r = \mathbf{Z}_p$, $M_r := H^1_{\text{ét}}(\mathcal{G}_r)$, $I_r = (0)$, $B = \mathbf{Z}_p$ and $B' = \overline{\mathbf{A}}$, we conclude that $H^1_{\text{ét}}(\mathcal{G}^{\text{m}}_r)$ is free of rank
$d'$ (respectively $2d'$) over $\Lambda$ for $\ast \neq \text{ét}$, $m$ (respectively $\ast \neq \text{null}$), that the specialization mappings
\[ H^1_{\text{ét}}(\mathcal{G}_r^\ast) \otimes \mathbb{Z}_p[\Delta/\Delta_r] \longrightarrow H^1_{\text{ét}}(\mathcal{G}_r^\ast) \]
are isomorphisms, and that the canonical mappings for $\ast \in \{\text{ét}, m, \text{null}\}$
\[ (\mathcal{M}_r^\ast \otimes \Lambda \tilde{\Lambda}) \longrightarrow \lim_{\leftarrow r} \left( H^1_{\text{ét}}(\mathcal{G}_r^\ast) \otimes \tilde{\Lambda} \right) \]
are isomorphisms. Applying $\otimes_{\Lambda} \tilde{\Lambda}$ to the diagram (5.6.14), passing to inverse limits, and using the isomorphisms (5.6.15) and (5.6.16) gives again invoking (5.5.27)) the isomorphism (1.2.12). Using the fact that the inclusion $\mathbb{Z}_p \hookrightarrow \tilde{\Lambda}^{\varphi=1}$ is an equality, the isomorphism (1.2.13) follows immediately from (1.2.12) by taking $F \otimes \varphi$-invariants.

Using Theorems 1.2.15 and 5.6.4 we can give a new proof of Ohta’s duality theorem [Oht95, Theorem 4.3.1] for the $\Lambda$-adic ordinary filtration of $e'' H^1_{\text{ét}}$ (see Corollary 1.2.17):

**Theorem 5.6.7.** There is a canonical $\Lambda$-bilinear and perfect duality pairing
\[ \langle \cdot, \cdot \rangle_{\Lambda} : e'' H^1_{\text{ét}} \times e'' H^1_{\text{ét}} \rightarrow \Lambda \]
determined by $\langle x, y \rangle_{\Lambda} \equiv \sum_{\delta \in \Delta/\Delta_r} (x, w_r U_p^{\nu}(\delta^{-1})^* y)_r \delta \mod I_r$
with respect to which the action of $\mathcal{G}_r^\ast$ is self-adjoint; here, $(\cdot, \cdot)_r$ is the usual cup-product pairing on $H^1_{\text{ét}, r}$ and $I_r := \ker(\Lambda \rightarrow \mathbb{Z}_p[\Delta/\Delta_r])$. Writing $\nu : \mathcal{G}_p \rightarrow \mathcal{G}_r^\ast$ for the character $\nu := \chi(\varphi)^{\Lambda/(p)_N}$, the pairing (5.6.17) induces a canonical $\mathcal{G}_p$ and $\mathcal{G}_r^\ast$-equivariant isomorphism of exact sequences
\[
\begin{array}{cccccc}
0 & \longrightarrow & (e'' H^1_{\text{ét}})^{\varphi} & \longrightarrow & e'' H^1_{\text{ét}}(\nu) & \longrightarrow & (e'' H^1_{\text{ét}})^{\varphi}(\nu) & \longrightarrow & 0 \\
\downarrow & \simeq & \downarrow & \simeq & \downarrow & \simeq & \downarrow & \simeq & \\
0 & \longrightarrow & \text{Hom}_\Lambda((e'' H^1_{\text{ét}})^{\varphi}, \Lambda) & \longrightarrow & \text{Hom}_\Lambda(e'' H^1_{\text{ét}}, \Lambda) & \longrightarrow & \text{Hom}_\Lambda((e'' H^1_{\text{ét}})^{\varphi}, \Lambda) & \longrightarrow & 0 
\end{array}
\]

**Proof.** The proof is similar to that of Proposition 5.2.4, using Corollary 1.2.16 and applying Lemma 5.1.4 (cf. the proof of [Oht95, Theorem 4.3.1] and of [Sha11, Proposition 4.4]). Alternatively, one can prove Theorem 5.6.7 by appealing to Theorem 5.6.4 and isomorphism (1.2.13) of Theorem 1.2.15.

**Proof of Theorem 1.2.18.** Suppose first that (5.6.4) admits a $\Lambda_{\mathcal{G}_p^\ast}$-linear splitting $\mathcal{M}_m^\ast \rightarrow \mathcal{M}_r$ which is compatible with $F$, $V$, and $\Gamma$. Extending scalars along $\Lambda \rightarrow \Lambda \tilde{\Lambda} \rightarrow \Lambda \tilde{\Lambda}$ and taking $F \otimes \varphi$-invariants yields, by Theorem 1.2.15, a $\Lambda$-linear and $\mathcal{G}_p$-equivariant map $(e'' H^1_{\text{ét}})^{\varphi} \rightarrow e'' H^1_{\text{ét}}$ whose composition with the canonical projection $e'' H^1_{\text{ét}} \rightarrow (e'' H^1_{\text{ét}})^{\varphi}$ is necessarily the identity.

Conversely, suppose that the ordinary filtration of $e'' H^1_{\text{ét}}$ is $\Lambda$-linearly and $\mathcal{G}_p$-equivariantly split. Applying $\otimes_{\Lambda} \mathbb{Z}_p[\Delta/\Delta_r]$ to this splitting gives, thanks to Corollary 1.2.16 and the isomorphism (5.5.26),
a $\mathbb{Z}_p[\mathbb{G}_{\mathbb{Q}_p}]$-linear splitting of

$$0 \longrightarrow T_p G^m_r \longrightarrow T_p G_r \longrightarrow T_p G^et_r \longrightarrow 0$$

which is compatible with change in $r$ by construction. By $\Gamma$-descent and Tate’s theorem, there is a natural isomorphism

$$\text{Hom}_{p\text{div}\Gamma R_r}(\mathbb{G}^et_r, \mathbb{G}_r) \simeq \text{Hom}_{\mathbb{Z}_p[\mathbb{G}_{\mathbb{Q}_p}]}(T_p G^et_r, T_p G_r)$$

and we conclude that the connected-étale sequence of $\mathbb{G}_r$ is split (in the category $p\text{div}\Gamma R_r$), compatibly with change in $r$. Due to the functoriality of $M_r(\cdot)$, this in turn implies that the top row of (5.6.1) is split in $B^\Gamma_{\mathfrak{S}_r}$, compatibly with change in $r$, which is easily seen to imply the splitting of (5.6.4). ■

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**University of Arizona, Tucson**

*Current address*: Department of Mathematics, 617 N. Santa Rita Ave., Tucson AZ. 85721

*E-mail address*: cais@math.arizona.edu