PAINLEVÉ IV: ROOTS AND ZEROS

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Abstract. We consider the (real) fourth Painlevé equation in which both parameters vanish, analyzing the square-roots of its solutions and paying special attention to their zeros.

INTRODUCTION

In [2] we offered elementary proofs for fundamental properties of the unique triple-zero solution to the first Painlevé equation. In [3] we treated in a similar fashion all solutions to the second Painlevé equation whose graphs pass through the origin. Here we consider aspects of what is arguably the next case: the fourth Painlevé equation, which was discovered by Gambier. The general form of this equation is

\[
\frac{d^2w}{dz^2} = 1 + \frac{3}{2}w^3 + 4zw^2 + 2(z^2 - \alpha)w + \frac{\beta}{w}
\]

where \( \alpha \) and \( \beta \) are parameters. As is suggested by its form, this equation is properly complex: in fact, each of its solutions is meromorphic in the plane, with simple poles of residue \( \pm 1 \); see [1] and references therein. Note the presence of the dependent variable in denominators: this separates the fourth Painlevé equation from the first and second; of course, it engenders some complications.

In line with the setting of our previous papers, we shall consider the fourth Painlevé equation in purely real terms; moreover, we shall only consider the case in which \( \alpha = 0 \) and \( \beta = 0 \). Accordingly, our version of the fourth Painlevé equation (P IV) is

\[
\dddot{s} = \frac{1}{2s}s^2 + \frac{3}{2}s^3 + 4ts^2 + 2t^2s
\]

to be solved for real \( s \) as a function of real \( t \); our preference here for \( \dddot{s} \) over \( s' \) as notation for the derivative is largely on account of the otherwise awkward \( s'^2 \) or \( (s')^2 \) for its square. If the solution \( s \) is strictly positive then its (positive) square-root \( \sqrt{s} \) satisfies a second-order equation that is simpler than P IV in having no first-derivative term and no awkward denominators; in the opposite direction, the squares of nowhere-zero solutions to this simpler equation satisfy P IV. Those circumstances in which solutions to P IV or the simpler equation have (isolated) zeros call for separate handling. All of these matters are discussed under the sections Square-roots (on the differential equations themselves) and Isolated zeros (on the special handling of zeros); in our final section on Remarks we address some related issues without proof.

Square-roots

We begin with some elementary observations regarding our version of the fourth Painlevé equation, which we restate for the record as

\[
\ddot{s} = \frac{1}{2s}s' + \frac{3}{2}s^3 + 4ts^2 + 2t^2s,
\]

where the ratio on the right side is to be understood as a limit when necessary.
Observe that $\mathbb{P}$ may be reformulated in a number of ways. First we may clear the awkward denominator, thus:

$$2s^{\dddot{y}} - \dot{s}^2 = 3s^4 + 8ts^3 + 4t^2s^2.$$ 

Further, we may factor the right side, thus:

$$2s^{\ddot{y}} - \gamma^2 = s^2(3s + 2t)(s + 2t).$$

Observe also that reversing the sign of the dependent variable leads to the equation

$$\mathbb{P}: \quad 2s^{\ddot{y}} - \gamma^2 = \frac{1}{2}s^2 + 3s^3 - 4ts^2 + 2t^2s;$$

of course, sign-reversal in $\mathbb{P}$ leads to $\mathbb{P}$ likewise. Incidentally, passage between $\mathbb{P}$ and $\mathbb{P}$ may also be effected by reversal of the independent variable. Of course, $\mathbb{P}$ admits reformulations akin to those for $\mathbb{P}$ itself.

Let us agree to write $\mathbb{P}$ for the set comprising all solutions to the Painlevé equation $\mathbb{P}$; when extra clarity is called for, we may write $\mathbb{P}(I)$ for the set comprising all solutions to $\mathbb{P}$ on the open interval (more generally, open set) $I \subseteq \mathbb{R}$. Similarly, we write $\mathbb{P}$ for the set of all solutions to $\mathbb{P}$ (on some interval, which we may indicate for clarity). We observed above that multiplication by $-1$ yields a bijection $\mathbb{P} \to \mathbb{P}: s \mapsto -s$;

also, that reversal of the independent variable yields a bijection from $\mathbb{P}(I)$ to $\mathbb{P}(-I)$.

Now, let $s \in \mathbb{P}$ be a strictly positive solution to $\mathbb{P}$ and write $\sigma := \sqrt{s} = s^{1/2}$ for its positive square-root. Certainly, $\sigma$ is twice-differentiable. Further, from $s = \sigma^2$ it follows that

$$\dot{s} = 2\sigma \dot{\sigma}$$

so that

$$\sigma^2 = \frac{s^2}{4\sigma^2} = \frac{s^2}{4s}$$

and

$$\ddot{s} = 2\sigma^2 + 2\dot{\sigma} = \frac{s^2}{2s} + 2\sigma \ddot{\sigma}$$

so that

$$\ddot{s} - \frac{s^2}{2s} = 2\sigma \ddot{\sigma}.$$ 

All of this requires only that the twice-differentiable function $s$ be strictly positive. Recalling that $s$ is a solution to $\mathbb{P}$ we deduce that

$$2\sigma \dddot{s} = \dddot{s} - \frac{s^2}{2s} = \frac{3}{2}s^3 + 4ts^2 + 2t^2s = \frac{1}{2}s(3s + 2t)(s + 2t)$$

or

$$\dddot{s} = \frac{1}{4}\sigma(3\sigma^2 + 2t)(\sigma^2 + 2t).$$

This finding prompts us to formalize the auxiliary differential equation

$$\mathbb{P}^{1/2}: \quad 4\dddot{\sigma} = \sigma(3\sigma^2 + 2t)(\sigma^2 + 2t)$$

alongside its companion

$$\mathbb{P}^{1/2}: \quad 4\dddot{\sigma} = \sigma(3\sigma^2 - 2t)(\sigma^2 - 2t).$$

It also prompts us to introduce $\mathbb{P}^{1/2}$ and $\mathbb{P}^{1/2}$ for the corresponding spaces of solutions. Notice that the map $\sigma \mapsto -\sigma$ preserves the spaces $\mathbb{P}^{1/2}$ and $\mathbb{P}^{1/2}$ while the map $t \mapsto -t$ interchanges them.

The following result was established in the motivating lead-up to equation $\mathbb{P}^{1/2}$.
The hypotheses ensure that not only

\[ \text{Proof.} \]

\[ \text{Theorem 4.} \]

with a zero would vanish throughout its interval of definition.

\[ \sigma \text{ now ensures that} \]

\[ \text{vanishing of} \]

\[ s \]

\[ \text{Painlevé uniqueness theorem. On the other hand,} \]

\[ q \text{ automatically} \]

\[ \text{We wish to explore the extendibility of Theorem 1 and Theorem 2 to this context.} \]

\[ \text{rational but has} \]

\[ s \]

\[ \text{between a polynomial; consequently, the initial value problem for} \]

\[ \text{a} \]

\[ \text{for initial data involving a zero of} \]

\[ s \]

\[ \text{Theorem 3.} \]

\[ \text{Theorem 2.} \]

\[ \text{or} \]

\[ s \]

\[ \text{We shall address such situations carefully in the next section; naturally,} \]

\[ \text{we may ignore the identically zero function.} \]

\[ \text{Isolated zeros} \]

As announced, we here consider situations in which \( s \in \mathbb{P} \) or \( \sigma \in \mathbb{P}^{1/2} \) has a zero. Specifically, we shall assume that such a function has an isolated zero at the point \( a \) in the open interval \( I \): more specifically, we shall assume that the function vanishes at \( a \) but at no other point of \( I \). We wish to explore the extendibility of Theorem 1 and Theorem 2 to this context.

Observe at once from \( [\mathbb{P}] \) (say in a reformulation) that if \( s \in \mathbb{P} \) satisfies \( s(a) = 0 \) then automatically \( \dot{s}(a) = 0 \). In like but more straightforward manner, \( [\mathbb{P}^{1/2}] \) tells us that if \( \sigma \in \mathbb{P}^{1/2} \) satisfies \( \sigma(a) = 0 \) then automatically \( \dot{\sigma}(a) = 0 \). We shall use these observations throughout the subsequent discussion, perhaps without comment.

Before proceeding further, it is convenient to draw attention to an important difference between \( [\mathbb{P}] \) and \( [\mathbb{P}^{1/2}] \). On the one hand, \( [\mathbb{P}^{1/2}] \) has the form \( \ddot{\sigma} = \Phi(t, \sigma) \) in which \( \Phi(t, \sigma) \) is a polynomial; consequently, the initial value problem for \( [\mathbb{P}^{1/2}] \) has a standard local existence-uniqueness theorem. On the other hand, \( [\mathbb{P}] \) has the form \( \ddot{s} = F(t, s, \dot{s}) \) in which \( F(t, s, \dot{s}) \) is rational but has \( s \) in the denominator: the standard local existence-uniqueness theorem breaks down for initial data involving a zero of \( s \). In fact, we have seen that if \( s \) satisfies \( [\mathbb{P}] \) then the vanishing of \( s(a) \) forces that of \( \dot{s}(a) \); were standard local uniqueness to apply, a solution to \( [\mathbb{P}] \) with a zero would vanish throughout its interval of definition.

**Theorem 3.** Let \( \sigma \in \mathbb{P}^{1/2} \) and let \( \sigma \geq 0 \) on \( I \ni a \). If \( \sigma(a) = 0 \) then \( \sigma = 0 \).

**Proof.** The hypotheses ensure that not only \( \sigma(a) = 0 \) but also \( \dot{\sigma}(a) = 0 \). The identically zero function satisfies \( [\mathbb{P}^{1/2}] \) on \( I \) with the same initial data. The local uniqueness theorem for \( [\mathbb{P}^{1/2}] \) now ensures that \( \sigma = 0 \).

It follows at once that Theorem 1 has no direct extension allowing an isolated zero.

**Theorem 4.** If \( s \in \mathbb{P} \) is strictly positive except for an isolated zero at \( a \in I \) then \( \sqrt{s} \notin \mathbb{P}^{1/2} \).

**Proof.** The (positive) square-root \( \sqrt{s} \) is zero at \( a \in I \) but strictly positive on \( I \setminus \{a\} \); Theorem 3 therefore excludes \( \sqrt{s} \) from \( \mathbb{P}^{1/2} \).

Notwithstanding this negative result, we have the following.
**Theorem 5.** If \( s \in \mathbb{P} \) is strictly positive except for an isolated zero at \( a \in I \) then there exists \( \sigma \in \mathbb{P}^{1/2} \) such that \( s = \sigma^2 \).

*Proof.* To the left of \( a \) there are only two continuous square-roots of \( s \), namely \( \pm \sqrt{s} \); likewise to the right of \( a \). Since the taking of like signs on each side of \( a \) leads to failure, we mix signs: thus, well-define \( \sigma \) on \( I \) by
\[
\sigma(t) = \begin{cases} 
-\sqrt{s(t)} & \text{if } I \ni t \leq a, \\
+\sqrt{s(t)} & \text{if } I \ni t \geq a.
\end{cases}
\]

Theorem 4 easily places \( \pm \sigma \) in \( \mathbb{P}^{1/2} \) on \( I \setminus \{a\} \); we must show that \( \sigma \) is twice-differentiable at its zero \( a \) with \( \ddot{\sigma}(a) = 0 \). Let \( I \ni t \neq a \); as \( s = \sigma^2 \),
\[
\dot{\sigma}(t) = \frac{\tilde{s}(t)}{2\sigma(t)}
\]
while as \( s \in \mathbb{P} \) and \( s(a) = 0 \),
\[
\frac{\tilde{s}(t)^2}{2s(t)} = \tilde{s}(t) - \frac{1}{2}s(t)\left(3s(t) + 2t\right)(s(t) + 2t)
\]
and
\[
\lim_{t \to a} \frac{\tilde{s}(t)^2}{4s(t)^2} = \lim_{t \to a} \frac{\tilde{s}(t)^2}{4s(t)} = \frac{1}{2} \tilde{s}(a).
\]
Checking signs, the taking of square-roots yields
\[
\lim_{t \to a} \ddot{\sigma}(t) = \sqrt{\frac{1}{2} \tilde{s}(a)};
\]
as \( \sigma \) is continuous on \( I \), it follows that \( \sigma \) is (continuously) differentiable throughout \( I \) by an application of the mean value theorem. As \( \sigma \) satisfies \( \mathbb{P}^{1/2} \) on \( I \setminus \{a\} \), it follows that
\[
\lim_{t \to a} \dddot{\sigma}(t) = \lim_{t \to a} \frac{1}{4}\sigma(t)\left(3\sigma(t)^2 + 2t\right)(\sigma(t)^2 + 2t) = 0
\]
whence a further application of the mean value theorem to the continuous function \( \ddot{\sigma} \) shows that \( \sigma \) is twice-differentiable at \( a \) with \( \dddot{\sigma}(a) = 0 \) as required. \( \square \)

Of course, a similar argument shows that if \( s \in \mathbb{P} \) is strictly negative except for an isolated zero at \( a \in I \) then there exists \( \sigma \in \mathbb{P}^{1/2} \) such that \( s = -\sigma^2 \); again, \( \sigma \) takes opposite signs on opposite sides of \( a \).

Theorem 4 and Theorem 5 are complements to Theorem 1 for cases in which \( s \in \mathbb{P} \) has an isolated zero. There are analogous complements to Theorem 2 for cases in which \( \sigma \in \mathbb{P}^{1/2} \) has an isolated zero.

The appropriate counterpart of Theorem 4 is immediate.

**Theorem 6.** If \( \sigma \in \mathbb{P}^{1/2} \) is strictly positive except for an isolated zero at \( a \in I \) then \( \sigma^2 \notin \mathbb{P} \).

*Proof.* If \( \sigma^2 \) were to lie in \( \mathbb{P} \) then its positive square-root would lie outside \( \mathbb{P}^{1/2} \) according to Theorem 4 but this positive square-root is \( \sigma \) itself. \( \square \)

The appropriate counterpart of Theorem 5 requires just a little more work.

**Theorem 7.** If \( \sigma \in \mathbb{P}^{1/2} \) takes opposite signs on opposite sides of \( a \in I \) then \( \sigma^2 \in \mathbb{P} \).

*Proof.* The twice-differentiable square \( s := \sigma^2 \) satisfies \( \mathbb{P} \) on \( I \setminus \{a\} \) by Theorem 2. Notice that if \( t \in I \) then \( \ddot{s}(t) = 2\sigma(t)\dot{\sigma}(t) \) and
\[
\dddot{s}(t) = 2\sigma(t)\ddot{\sigma}(t) + 2\dot{\sigma}(t)^2.
\]
Consequently, as $\sigma(a) = 0$ it follows that
\[ \ddot{s}(a) - \frac{1}{2} s(a)(3s(a) + 2a)(s(a) + 2a) = 2\dot{\sigma}(a)^2 \]
and
\[ \lim_{t \to a} \frac{\ddot{s}(t)^2}{2s(t)} = \lim_{t \to a} \frac{(2\sigma(t)\dot{\sigma}(t))^2}{2\sigma(t)^2} = \lim_{t \to a} 2\dot{\sigma}(t)^2 = 2\dot{\sigma}(a)^2. \]
This shows that $s$ satisfies $[P]$ at $a$ also and concludes the demonstration. □

We close by remarking that the case of $s \in \mathbb{P}$ with an isolated zero at which the second derivative also vanishes is as tidy as can be: in fact, the case does not arise! Our first step towards this result is perhaps a little peculiar in hindsight.

**Theorem 8.** Let $s \in \mathbb{P}$ have an isolated zero at $a$. If $\dddot{s}(a) = 0$ then each derivative of $s$ vanishes at $a$.

**Proof.** The second reformulation of $[P]$ informs us that
\[ 2s \dddot{s} - s^2 = s^2(3s + 2t)(s + 2t) = s^2Q \]
say, where $Q$ is quadratic in $s$ and $t$. Away from the isolated zero, we may differentiate: the resulting terms $\pm 2s \dddot{s}$ on the left cancel, to yield
\[ 2s \dddot{s} = 2s\dot{\sigma}Q + s^2\dot{Q} \]
whence
\[ 2s \dddot{s} = 2\dot{s}Q + s\dot{Q} \]
away from $a$ and hence at $a$ also. All that remains is to differentiate inductively. □

We can now see that this case is indeed vacuous: taking the (difficult!) meromorphicity of $s$ for granted, the identity theorem implies that $s$ is zero throughout the open interval in which $a$ is an isolated zero; this is absurd!

In particular, it follows that $s \in \mathbb{P}$ cannot change sign at an isolated zero.

**Remarks**

We round out our account with some miscellaneous comments on related topics of interest.

Recall that the fourth Painlevé equation is properly a complex differential equation. The process of passing to a square-root is naturally more elaborate in the complex setting: as we mentioned, solutions to the fourth Painlevé equation are meromorphic, with simple poles; square-roots of such functions cannot be meromorphic! Nonetheless, there is sufficient reason for further study of the relevant auxiliary equation
\[ 4\frac{d^2 \omega}{dz^2} = \omega(3\omega^2 + 2z)(\omega^2 + 2z). \]

We have seen in our study of the fourth Painlevé equation $[P]$ that the auxiliary differential equation $[P^{1/2}]$ is of definite theoretical interest. In fact, this auxiliary equation is also of considerable practical help, aside from its ability to handle initial data involving a zero. We began exploring solutions of the fourth Painlevé equation with the aid of WZGrapher, a valuable freeware program developed by Walter Zorn. Quite early in our explorations, we noticed apparent graphical instabilities: for example, solutions of $[P]$ with certain initial data would at first appear to be oscillatory; upon zooming out, such a solution might seem to suffer a catastrophe, oscillations disappearing and being replaced by a blow-up or spike; upon zooming out further, oscillations might reappear; and so on. Not surprisingly, such catastrophic behaviour manifests itself at a zero of the solution and so involves the awkward denominator in $[P]$. These apparent graphical instabilities seem to be removed by passage to the corresponding solutions of $[P^{1/2}]$ as the reader may care to see using WZGrapher.
The factorized form of the fourth Painlevé equation \[ P \]

\[ \ddot{s} - \frac{\dot{s}^2}{2s} = \frac{1}{2} s(3s + 2t)(s + 2t) \]

indicates that the lines ‘\( s = 0 \)’, ‘\( s = -2t/3 \)’ and ‘\( s = -2t \)’ have geometric significance for its solutions. Similarly, ‘\( \sigma = 0 \)’ and the parabolas ‘\( \sigma^2 = -2t/3 \)’ and ‘\( \sigma^2 = -2t \)’ have geometric significance for solutions to the auxiliary equation \[ P^{1/2} \]

\[ 4\dot{\sigma} = \sigma(3\sigma^2 + 2t)(\sigma^2 + 2t). \]

This geometric significance can be seen in a concavity diagram. The curves ‘\( \sigma = 0 \)’, ‘\( \sigma^2 = -2t/3 \)’ and ‘\( \sigma^2 = -2t \)’ divide the (\( t, \sigma \))-plane into regions. The sign of \( \sigma \) is negative/positive in the regions directly above/below the half-parabola ‘\( \sigma = +\sqrt{-2t/3} \)’ so that solutions to \[ P^{1/2} \] have the opportunity to oscillate about this half-parabola; similarly, solutions to \[ P^{1/2} \] may oscillate about ‘\( \sigma = -\sqrt{-2t/3} \)’.

In fact, experimentation with WZGrapher reveals that solutions to \[ P^{1/2} \] that do not suffer blow-up in both time directions tend to display steadily decaying oscillations about the upper or lower half of the parabola ‘\( \sigma^2 = -2t/3 \)’ as \( t \to -\infty \); and that solutions often tend to linger alongside ‘\( \sigma = 0 \)’ or ‘\( \sigma^2 = -2t \)’ as they make more or less extended approaches to tangency. Also, it not infrequently happens that a minuscule change in initial data causes a solution ‘\( \sigma \)’ to flip its oscillations from one half-parabola to the other, or to flip the direction of its finite-time blow-up, in such a way that the sudden transition is not detectable in \( \sigma^2 \). On a more aesthetic note, when oscillations of ‘\( \sigma \in P^{1/2} \)’ occur about a half-parabola ‘\( \sigma = \pm\sqrt{-2t/3} \)’ they are quite evenly balanced. By contrast, when oscillations of ‘\( s \in P \)’ occur about the line ‘\( s = -2t/3 \)’ they are uneven, displaying larger arches on the side of the line away from ‘\( s = 0 \)’. Of course, squaring accounts for the difference.

One relatively simple family of illustrative examples takes ‘\( \sigma \in P^{1/2} \)’ with ‘\( \sigma(0) = 0 \)’ and ‘\( \dot{\sigma}(0) \)’ strictly positive. As ‘\( \dot{\sigma}(0) \)’ increases from 0 to a little beyond 1.169868591, two gradual changes to the solution ‘\( \sigma \)’ take place simultaneously: on the one hand, ‘\( \sigma \)’ oscillates about ‘\( \sigma = -\sqrt{-2t/3} \)’, the amplitude of the oscillations initially decreasing and finally increasing; on the other hand, ‘\( \sigma \)’ lingers initially along ‘\( \sigma = 0 \)’ and finally along ‘\( \sigma = -\sqrt{-2t} \)’; when ‘\( \dot{\sigma}(0) \)’ is around 0.65 the oscillations have their least amplitude and there is no lingering along either curve. As ‘\( \dot{\sigma}(0) \)’ increases from 1.169868591 to 1.169868592 the oscillations disappear, to be replaced by a negative blow-up in finite negative time; thereafter, the lingering along ‘\( \sigma = -\sqrt{-2t} \)’ gradually disappears and the finite-time blow-up accelerates. Throughout, ‘\( \sigma \in P^{1/2} \)’ has a unique zero, at which it changes sign; accordingly, its square lies in \( P \).

We leave to the reader the pleasure of exploring this family of examples in WZGrapher (or some similar program). Among many other families to explore, we recommend the following: take ‘\( \sigma(0) = 1 \)’ and let ‘\( \dot{\sigma}(0) \)’ run from ‘\( -0.933899363 \)’ to ‘\( 1.579186627 \)’, noting the several transitions with reference to ‘\( \sigma = 0 \)’, ‘\( \sigma^2 = -2t/3 \)’ and ‘\( \sigma^2 = -2t \)’; take ‘\( \sigma(-6) = 2 \)’ and let ‘\( \dot{\sigma}(-6) \)’ run from ‘\( -0.170889967 \)’ to ‘\( -0.170889968 \)’ (!).

REFERENCES

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[3] P.L. Robinson, *Homogeneous Painlevé II Transcendent*, arXiv 1608.02139 (2016).