K-theory and K-homology of the wreath products of finite with free groups

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Abstract

Consider the wreath product $\Gamma = F \wr F_n = \bigoplus_{F_n} F \rtimes F_n$, with $F$ a finite group and $F_n$ the free group on $n$ generators. We study the Baum-Connes conjecture for this group. Our aim is to explicitly describe the Baum-Connes assembly map for $F \wr F_n$. To this end, we compute the topological and the analytical K-groups and exhibit their generators. Moreover, we present a concrete 2-dimensional model for $E \Gamma$. As a result of our K-theoretic computations, we obtain that $K_0(C^*_r(\Gamma))$ is the free abelian group of countable rank with a basis consisting of projections in $C^*_r\left(\bigoplus_{F_n} F\right)$ and $K_1(C^*_r(\Gamma))$ is the free abelian group of rank $n$ with a basis consisting of the unitaries coming from the free group.

1 Introduction

The Baum-Connes conjecture, at the fascinating intersection of several areas in mathematics, was stated in 1982 by P. Baum and A. Connes. For a group $G$, it proposes a link via a certain assembly map between K-theory of the reduced group $C^*$-algebra $C^*_r(G)$ and the classifying space for proper actions $E G$. Formally, it states that the assembly map

$$\mu^G_i : K^G_i(E G) \to K_i(C^*_r(G)) \quad i = 0, 1$$

is an isomorphism of two abelian groups.

The conjecture has been verified in a variety of cases including the huge class of a-T-menable groups (e.g. amenable groups and free groups), due to Higson and Kasparov [5]. Among many as yet unanswered questions about different aspects of this conjecture, we aim at understanding its stability under semidirect product. Indeed, we would like to understand the conjecture for a group $G = N \rtimes Q$ in terms of the status of the conjecture for $N$ and $Q$. In this respect, at the very beginning of our way, we try to elucidate the isomorphism for some groups for which the conjecture is satisfied. We have started our investigation in [11] and [3]. This work can be considered a generalisation of the latter.

Keywords: Wreath product, Baum-Connes conjecture

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The group under consideration in this article is the wreath product $\Gamma = F \wr F_n = \bigoplus_{\mathbb{F}_n} F \rtimes F_n$, where $F$ is a finite group and $\mathbb{F}_n$ is the free group on $n$ generators. The group $F \wr \mathbb{F}_n$ is a-T-menable as the groups $F$ and $\mathbb{F}_n$ are so, see Theorem 1.1 in [2]. Therefore a result by Higson-Kasparov in [3] guarantees that the conjecture holds for this group. The aim of the present article is to describe this isomorphism explicitly. To this end, we compute all involved K-groups and find their generators. The computations on the topological side become possible, thanks to the existence of a 2-dimensional model for $\mathcal{E}\Gamma$. We add that part of our explicit approach to the conjecture is to shed light on its topological side by providing a concrete 2-dimensional model. Lastly, we show that the assembly map sends generators to generators, as desired. Therefore, the Baum-Connes conjecture is explicitly proved for these groups.

Our main tool on the analytical side is the Pimsner-Voiculescu 6-term exact sequence and on the topological side the Martínez spectral sequence. We note that the existence of a huge torsion subgroup led us to consider the equivariant K-homology via its link to Bredon homology.

In order to formally state our main results, we fix some notation. Let $\text{Min} F$ denote the set of (Murray-von Neumann equivalence classes of) minimal projections in $C^*_r(F) = \mathbb{C}F$ and let $\hat{F}$ denote the set of unitary equivalence classes of irreducible representations of $F$. Moreover, denote by $\text{Min} F^{(\mathbb{F}_n)}$ and by $\hat{F}^{(\mathbb{F}_n)}$, respectively, the set of finitely supported maps from $\mathbb{F}_n$ to $\text{Min} F$ and to $\hat{F}$.

In the theorems below, corresponding to Theorem 3.3 and Theorem 4.2, respectively, we describe the K-theory and K-homology of $F \wr \mathbb{F}_n$.

**Theorem A.** Let $\Gamma = F \wr \mathbb{F}_n$ with $F$ a non-trivial finite group. Write $\mathbb{F}_n = \langle a_1, \ldots, a_n \rangle$. The K-groups of $C^*_r(\Gamma)$ can be described as two free abelian groups.

\[ K_0(C^*_r(\Gamma)) = \mathbb{Z}R \text{ with } R \text{ a countable basis indexed by representatives for } \mathbb{F}_n\text{-orbits in } \text{Min } F^{(\mathbb{F}_n)}. \]

\[ K_1(C^*_r(\Gamma)) = \mathbb{Z}[a_1] \oplus \cdots \oplus \mathbb{Z}[a_n]. \]

**Theorem B.** Let $\Gamma = F \wr \mathbb{F}_n$ with $F$ a non-trivial finite group. Write $\mathbb{F}_n = \langle a_1, \ldots, a_n \rangle$. The topological K-groups of $\mathcal{E}\Gamma$ can be described as two free abelian groups.

\[ K^0_0(\mathcal{E}\Gamma) = \mathbb{Z}R' \text{ with } R' \text{ a countable basis indexed by representatives for } \mathbb{F}_n\text{-orbits in } \hat{F}^{(\mathbb{F}_n)}. \]

\[ K^1_0(\mathcal{E}\Gamma) = \mathbb{Z}[v_1] \oplus \cdots \oplus \mathbb{Z}[v_n], \text{ where } [v_i] \text{ is the canonical generator of } K_1^\mathbb{Z}(\mathcal{E}\mathbb{Z}) \text{ via the identification } \mathbb{Z} = \langle a_i \rangle. \text{ Indeed, the inclusions } \langle a_i \rangle \hookrightarrow \mathbb{F}_n \hookrightarrow \Gamma \text{ give rise to an inclusion } \langle v_i \rangle = K^\mathbb{Z}_1(\mathcal{E}\mathbb{Z}) \hookrightarrow K^\mathbb{F}_n_1(\mathcal{E}\mathbb{F}_n) \cong K_1^\mathbb{Z}(\mathcal{E}\Gamma). \]

Comparing these two sides via the assembly map, the isomorphism for $F \wr \mathbb{F}_n$ is elementarily demonstrated in Theorem 5.1. Additionally, due to K-amenability, we implicitly compute
the K-theory of $C^*_r(\Gamma)$. We conclude the article by a remark on the modified trace conjecture. Assuming $\tau: C^*_r(\Gamma) \to \mathbb{C}$ to be the canonical trace on $C^*_r(\Gamma)$, in our case we have that

$$\text{Im } \tau_*(K_0(C^*_r(\Gamma))) = \mathbb{Z}\left[\frac{1}{[F]}\right].$$

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2 Preliminaries

In this section, we introduce some notations and recall some facts that we will use in the rest of the article. In particular, we introduce our main tools for the topological computations relevant for the left-hand side of the assembly map.

2.1 Topological tools: exact sequences

We start with the definition of the main object appearing on the left-hand side of the Baum-Connes assembly map.

Definition 2.1. For a group $G$, let $\mathcal{F}$ be the family of finite subgroups of $G$. The classifying space for proper action denoted by $EG$ is a $G$-CW-complex such that for all elements in $\mathcal{F}$, the isotropy group is contractible and for all other subgroups this is empty.

Note that $EG$ is unique up to $G$-homotopy equivalence. Moreover, an infinite dimensional ($G$-CW-complex) model for $EG$ always exists. However, the most interesting one is the one with the minimal dimension as it simplifies the homological computations.

One of the goals of the Baum-Connes conjecture is to compute the K-theory of $C^*_r(G)$ via the $G$-equivariant K-homology of $EG$. The homology theory used for the left-hand side of the Baum-Connes conjecture is called Bredon homology, which we briefly recall here. In the context of Baum-Connes conjecture, this is defined in terms of the family $\mathcal{F}$ of finite subgroups of $G$ and its coefficients are certain functors. Now, a few words on the nature of these functors. The orbit category $\mathcal{O}_G^G$ is the category whose objects are the homogeneous spaces $G/H$ for $H \in \mathcal{F}$ and the morphisms are $G$-maps $G/H \to G/K$. The category of covariant functors from $\mathcal{O}_G^G$ to the category $\mathbf{Ab}$ of abelian groups is denoted by $G\text{-Mod}_{\mathcal{F}}$. The objects of this category are called $\mathcal{O}_G^G$-modules. The Bredon homology groups of $EG$ with coefficient $M$ in $G\text{-Mod}_{\mathcal{F}}$ is denoted by $H^i_{\mathcal{F}}(EG; M)$. We remark that $H^i_{\mathcal{F}}(EG; M)$ is isomorphic to $H^i_{\mathcal{F}}(G; M)$.

The appropriate coefficient module in this context is the $\mathcal{O}_G^G$-module $R_C \in G\text{-Mod}_{\mathcal{F}}$. The value of $R_C$ on $G/H$ is the complex representation ring $R_C(H)$. We recall that $R_C(H)$ is the free abelian group on $\hat{H}$, the dual of $H$. For more details on Bredon homology see [8].
Despite the fact that the Baum-Connes’ philosophy suggests that computations on the topological side should be easier because we could use standard methods from algebraic topology, such computations can turn to be hard. However, for \( \dim(\mathbb{E}G) \leq 2 \) we have more explicit results.

**Theorem 2.2.** [8, Theorem I.3.17] Suppose \( \mathbb{E}G \) has a model with \( \dim \mathbb{E}G = 1 \), i.e. a tree. Then \( H^i_F(G; R_C) = 0 \) for \( i > 1 \), and there is an exact sequence

\[
0 \rightarrow H^1_F(G; R_C) \rightarrow \bigoplus_{[e]} R_C(G_e) \rightarrow \bigoplus_{[v]} R_C(G_v) \rightarrow H^0_F(G; R_C) \rightarrow 0,
\]

where the direct sums are respectively taken over the orbits of edges and vertices of the tree \( \mathbb{E}G \), and \( G_e \) and \( G_v \) denote the stabilisers of the edges and the vertices, respectively.

**Theorem 2.3.** [8, Theorem I.5.27] Suppose there exists a model for \( \mathbb{E}G \) with \( \dim(\mathbb{E}G) \leq 2 \). There is a short exact sequence

\[
0 \rightarrow H^0_F(G; R_C) \rightarrow K^G_0(\mathbb{E}G) \rightarrow H^2_F(G; R_C) \rightarrow 0,
\]

and an isomorphism \( H^1_F(G; R_C) \cong K^G_1(\mathbb{E}G) \).

We proceed by introducing another tool, a spectral sequence for group extensions, which will play an important role in our homological computations. This spectral sequence is an analogue of the Lyndon-Hochschild-Serre spectral sequence in group homology, which was developed by Martínez in [7]. Consider the group \( G = N \rtimes Q \) associated with a split short exact sequence \( 0 \rightarrow N \rightarrow G \rightarrow Q \rightarrow 0 \). Denote by \( \mathfrak{F} \) the family of finite subgroups of \( G \) and by \( \mathfrak{F} \) the family of finite subgroups of \( Q \). Set \( \mathfrak{F}_N = \mathfrak{F} \cap N \) to be the family of finite subgroups of \( N \). Consider the pull back of family \( \mathfrak{F} \), that is \( \mathfrak{F} \)\( = \{ H \leq G : N \leq H \text{ and } H/N \in \mathfrak{F} \} \). Theorem 5.1 in [7] provides us with a spectral sequence whose second page satisfies

\[
E^{2}_{p,q} = H^p_F(Q; R_Q^\mathfrak{F}_q(\mathbb{Z}; D)) \Rightarrow E^{\infty}_{p,q} = H^p_F(G/D).
\]

2.2 Invariants and co-invariants

For a set \( X \), we denote by \( Z^X \) the free abelian group generated by \( X \). It can be identified with the group of almost everywhere zero functions \( X \rightarrow \mathbb{Z} \).

Let \( M \) be a \( G \)-module. We denote by \( M^G \) the sub-module of \( G \)-invariants of the action and by \( M_G = M/\langle m - g.m : m \in M, g \in G \rangle \) the module of \( G \)-co-invariants. In fact, \( M^G \) and \( M_G \) are respectively the largest \( G \)-invariant submodule and \( G \)-invariant quotient. We recall Lemma 1 in [3].

**Remark 2.4.** Let \( G \) be a countable group, \( X \) be a countable \( G \)-set, and \( Y \) be the set of finite \( G \)-orbits in \( X \). The space of invariants and co-invariants of \( G \acts Z^X \) can be described as

\[
(Z^X)^G \cong ZY,
\]

\[
(Z^X)_G \cong Z(G \setminus X).
\]
Let $R$ be a set of representatives for the orbit space of $G$ on $X$. Since $(ZX)_G$ is a free abelian group, the quotient map $ZX \to (ZX)_G$ splits, hence we get

$$ZX = \langle m - g \cdot m : m \in ZX, \ g \in G \rangle \oplus ZR.$$ 

Let $(F,p)$ be a finite pointed set. We denote by $F^{(F_n)}$ the countable set of maps $f : F_n \to F$ that are almost everywhere $p$, or equivalently have finite support. In the above description take $X = F^{(F_n)}$ with the action of $F_n$ by left multiplication, here, we describe a set of representatives for $F_n$-orbits in $F^{(F_n)}$.

Given $f \in F^{(F_n)}$, we consider the convex hull of its support which is a finite sub-tree $S$ of the Cayley graph of $F_n$. We present such $f$ by $\chi_f^S \in F^{(F_n)}$ in order to take into account the sub-tree $S$ associated to its support. Therefore, for $w \in F_n$

$$\chi_f^S(w) = \begin{cases} f(w) & w \in \text{support} \ f \\ p & \text{otherwise} \end{cases}.$$ 

For a tree $T$, we define its barycentre as either a vertex or an edge remaining after removing successively the terminal vertices and corresponding edges.

Now we say that a tree $T$ is admissible if it is a finite sub-tree of the Cayley graph of $F_n = \langle a_1, \ldots, a_n \rangle$ with its barycentre either $e$ or an edge $[e, a_i]$ for some $i = 1, \ldots, n$.

The next lemma, which is a generalisation of Lemma 2 in [3], will help us to describe the $\mathbb{K}_0$-groups as co-invariants of certain actions. See Theorem 3.3 and Theorem 4.2.

**Lemma 2.5.** Let $F_n = \langle a_1, \ldots, a_n \rangle$ and let $(F,p)$ be a finite pointed set. Consider the action $F_n \curvearrowright F^{(F_n)}$ by left multiplication. A countable set $R$ of representatives for $F_n$-orbits is

$$R = \{ \chi_f^S : f \in F^{(F_n)}, S \text{ is admissible} \}.$$ 

In particular,

$$Z(F^{(F_n)}) = \langle m - w \cdot m : m \in Z(F^{(F_n)}), w \in F_n \rangle \oplus ZR$$

$$= \langle m - a_i \cdot m : m \in Z(F^{(F_n)}), 1 \leq i \leq n \rangle \oplus ZR.$$ 

**Proof.** One sees that no two distinct elements of $R$ belong to the same orbit of this action. We next show that all elements of $F^{(F_n)}$ lie in the orbit of some element in $R$. Take $f \in F^{(F_n)}$. Let $S$ be the sub-tree associated to its support. Its barycentre is either a vertex $w$ or an edge $[w, wa_i]$ for $i \in \{1, \ldots, n\}$. In either cases, the sub-tree $\hat{S} := w^{-1}S$ is admissible. Therefore, $f$ belongs to the orbit of $\chi_{\hat{S}}^f$, where $\hat{f} \in F^{(F_n)}$. 

We close this section by describing the Baum-Connes assembly map for the locally finite group $\bigoplus F_n F$. See Section 4 in [3] for more details.
2.3 The Baum-Connes assembly map for the locally finite group \( \bigoplus F_n \)

Consider the group \( \Gamma = \bigoplus F_n \rtimes F_n \) with the action \( F_n \rtimes \bigoplus F_n \) by left multiplication on the indices. Let \( B_n = \{ w \in F_n : |w| \leq n \} \) denote the balls of radius \( n \) (with respect to the word metric) on the Cayley graph of \( F_n \). Write \( B = \bigoplus F_n \). Let \( B_n = \{ f \in B : \text{supp} f \subset B_n \} \) correspond to \( B_n = \bigoplus m_n F_n \), where \( m_n = |B_n| \). We may express the group \( B \) as the co-limit of the increasing sequence of \( B \), and \( B_n \)’s. It turns out that the subgroup \( B \) plays an essential role in our K-theoretic computations. In particular, as we will see, \( C^*(B) \) provides us with sufficiently many projections to generate \( K_0 (C^*(\Gamma)) \).

The conjecture holds for locally finite groups as co-limits are preserved by K-theory and by the assembly map. See Corollary I.5.2 and Theorem I.5.10 in [8]. However, we need a more descriptive picture of its assembly map. Let us first recall the Baum-Connes assembly map for a finite group \( F \).

Let \( \pi \in \hat{F} \) and let \( e_\pi \) denote the minimal projection in \( M_{\dim \pi}(\mathbb{C}) \). We then have

\[
\mu^F_0 : R_C (F) \to \text{Min } F \text{ with } \pi \mapsto e_\pi.
\]

Analogue to Corollary 1 in [3], we have the following proposition.

**Proposition 2.6.** Let \( F \) be a non-trivial finite group, and \( B = \bigoplus F_n \).

- The free abelian group \( \mathbb{Z}(\hat{F}) \) is isomorphic to \( K_0 (\mathbb{C}B) \) via
  \[
  \pi \in F_n \mapsto \bigotimes_{w \in \text{supp } \pi} \pi w \in R_C (\bigoplus_{w \in \text{supp } \pi} F_w) \subset K_0 (\mathbb{C}B)
  \]
  (where \( \pi = (\pi w)_w \), the group \( F_w \) is the corresponding copy of \( F \) in \( B \) to the index \( w \in F_n \) and \( \pi w \) is an irreducible representation of \( F_w \)).

- The free abelian group \( \mathbb{Z}(\text{Min } F) \) is isomorphic to \( K_0 (C^* B) \) via
  \[
  p \in \text{Min } F \mapsto \bigotimes_{w \in \text{supp } p \pi w} \pi w \in \text{Min } (\bigoplus_{w \in \text{supp } p \pi w} F_w) \subset K_0 (C^* B),
  \]
  and in particular the trivial map \( 1_{p_F} \in \text{Min } F \) with the constant value \( p_F \) is mapped to 1, the \( K_0 \)-class of 1 \( \in C^* (B) \).

- For \( \pi \in \hat{F} \), we have \( \mu_B (\pi) = \bigotimes_{w \in \text{supp } \pi} \mu_F (\pi w) \in K_0 (C^* B) \).

We remark that, through the article, we consider in particular the finite pointed sets \( (\hat{F}, 1) \) and \( (\text{Min } F, p_F) \), where 1 denotes the trivial representation of \( F \), and \( p_F \) denotes the projection \( p_F = \frac{1}{|F|} \sum_{f \in F} 1 f \).

3 K-theory of \( C^*_r (\Gamma) \)

In this section, we explicitly describe the analytical side of the Baum-Connes conjecture for \( F \rtimes F_n \).
Let $A$ be a $C^*$-algebra and as before let $F_n = \langle a_1, \ldots, a_n \rangle$. For $i = 1, \ldots, n$, let $\alpha_i \in \text{Aut}(A)$ such that the $\alpha_i$'s define an action $\alpha$ of $F_n$ on $A$. For $i = 1, \ldots, n$, denote by $u_i$ the unitary $u_i \in C^*_r(F_n) \subset B(\ell^2(F_n))$. We assume $A$ is faithfully represented on a Hilbert space $\mathcal{H}$, i.e. $A \subset B(\mathcal{H})$. The reduced crossed product $A \rtimes_r F_n$ is generated by

$$\langle A, u_1, \ldots, u_n : \alpha_i(a) = u_iau_i^{-1}, 1 \leq i \leq n, a \in A \rangle \subset B(\ell^2(F_n, \mathcal{H})).$$

Let $\sigma = \sum_{i=1}^n \text{Id} - \alpha_i$. The Pimsner-Voiculescu $6$-term exact sequence for reduced crossed products with free groups [10], Theorem 3.5] is a very convenient tool in K-theory. It provides us with information about the K-theory of such a crossed product in terms of the K-theory of the initial $C^*$-algebra

$$\begin{array}{ccc}
\bigoplus_{i=1}^n K_0(A) & \xrightarrow{\sigma^*} & K_0(A) \\
\partial_1 & & \partial_0 \\
K_1(A \rtimes_r F_n) & \leftarrow & K_1(A) \xrightarrow{\sigma^*} \bigoplus_{i=1}^n K_1(A).
\end{array}$$

In order to describe our analytical K-groups, we need to understand the kernel of the homomorphism $\sigma^*$ appearing above. We start with a lemma which is essential for the later computations.

**Lemma 3.1.** Let $(F, p)$ be a finite pointed set and let $F_n = \langle a_1, \ldots, a_n \rangle$. For $X = F^{(F_n)}$, consider

$$\psi: \bigoplus_{i=1}^n \mathbb{Z}X \rightarrow \mathbb{Z}X \quad \text{via} \quad \psi((f_i)_{1 \leq i \leq n}) = \sum_{i=1}^n f_i - a_i \cdot f_i,$$

where $a_i \cdot f_i(x) = f_i(a_i^{-1}x)$. The kernel of $\psi$ is described by $\text{Ker}(\psi) = \mathbb{Z}1_p \oplus \ldots \oplus \mathbb{Z}1_p$, where $1_p$ denotes the constant map with the value $p$.

**Proof.** Choose $(f_i)_{1 \leq i \leq n} \in \bigoplus_{i=1}^n \mathbb{Z}X$. If all $f_i$’s are $F_n$-invariant, then obviously $(f_i)_{1 \leq i \leq n} \in \text{Ker}(\psi)$. In particular, $1_p$ is $F_n$-invariant, hence $\bigoplus_{i=1}^n \mathbb{Z}1_p \subset \text{Ker}(\psi)$. To conclude the statement of the lemma we need to show that these are in fact the only elements in the kernel. In order to do so we need some preparation.

We may write $X = \bigcup_{x \in X} F_n \cdot x$. For $x \in X \setminus \{1_p\}$, we have that $F_n \cdot x \cong F_n$ as $F_n$ acts freely on $X \setminus \{1_p\}$. For $f \in \mathbb{Z}X$ and $x \in X \setminus \{1_p\}$, we define $\tilde{f}_x: F_n \rightarrow \mathbb{Z}$ by $\tilde{f}(w) = f(w \cdot x)$. Note that for $0 \neq f \in \mathbb{Z}X$, if $\tilde{f}_x = 0$ for all $x \in X \setminus \{1_p\}$, then $f \in \mathbb{Z} \cdot 1_p$. Therefore the new statement to prove is

$$0 \neq (f_i)_{1 \leq i \leq n} \in \text{Ker}(\psi) \quad \text{implies} \quad \forall x \in X \setminus \{1_p\}, \quad (\tilde{f}_x)_{1 \leq i \leq n} = 0.$$

For $x \in X \setminus \{1_p\}$, we define

$$S = \bigcup_{1 \leq i \leq n} \text{supp} \tilde{f}_i \cup \bigcup_{1 \leq i \leq n} a_i \text{supp} \tilde{f}_i.$$

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We assume $S$ is non-empty. Note that for $(f_i)_{1 \leq i \leq n} \in \text{Ker}(\psi)$ and for $w \in S$, there have to be at least two functions such that $w$ belongs to their supports, otherwise $(f_i)_{1 \leq i \leq n}$ can not belong to the kernel. Choose a word $w \in S \subset F_n$ with the maximum length. The requirements above lead us to consider the following possibilities, for $i \neq j \in \{1, \ldots, n\}$:

1. $w \in \text{supp} \tilde{f}_i \cap \text{supp} \tilde{f}_j$,
2. $w \in \text{supp} \tilde{f}_i \cap \text{supp} \tilde{f}_j$,
3. $w \in a_i \text{supp} \tilde{f}_i \cap a_j \text{supp} \tilde{f}_j$, and
4. $w \in \text{supp} \tilde{f}_i \cap a_j \text{supp} \tilde{f}_j$.

Claim: We can deduce contradictions from all these cases. Accepting this claim immediately implies that $\tilde{f}_{ix} = 0$ for $i = 1, \ldots, n$. As $x \in X \setminus \{1_p\}$ is chosen arbitrarily, we have that $(\tilde{f}_{ix})_{1 \leq i \leq n} = 0$ for all such $x$. Therefore $(f_i)_{1 \leq i \leq n} \in \bigoplus_{i=1}^n \mathbb{Z} \cdot 1_p$, which finishes the proof.

Proof of the claim: By an argument on the length of words in $F_n$, we successively show that all of the above possibilities lead to a contradiction.

Case 1: If $w \in \text{supp} \tilde{f}_i$, then $a_i w \in a_i \text{supp} \tilde{f}_i \subset S$ (†). Moreover, by the other assumption in 1, $w \in a_i \text{supp} \tilde{f}_i$. Hence there exists $u \in \text{supp} \tilde{f}_i$ such that $w = a_i u$, equivalently $u = a_i^{-1} w$ (‡). Now on the one hand, (†) forces $w$ to start with $a_i^{-1}$ (otherwise, $|a_i w| > |w|$ which is impossible). On the other hand (‡) implies that $w$ starts with any letter except $a_i^{-1}$ (if not, then $|u| = |a_i^{-1} w| > |w|$ which is a contradiction). Therefore this case can not happen.

Case 2: Let $w$ be in the supports of $\tilde{f}_i$ and $\tilde{f}_j$. Denote by $a_l$ the starting letter of $w$. Either $l \in \{i, j\}$ or $l \neq i, j$. If $l = i$ (respectively, $l = j$), then $a_j w \in a_j \text{supp} \tilde{f}_j$ (respectively, $a_i w \in a_i \text{supp} \tilde{f}_i$) provides a longer word in $S$, which is impossible. Now if $l \neq i, j$, then $a_i w \in a_i \text{supp} \tilde{f}_i$ provides a longer word in $S$ which is impossible. Hence this case can not happen.

Case 3: If $w$ is in the intersection of the shifted supports, then there exists $u \in \text{supp} \tilde{f}_i$ and $v \in \text{supp} \tilde{f}_j$ such that $a_i u = w = a_j v$. Equivalently, we have that $a_i^{-1} w = u$ and $a_j^{-1} w = v$. If $w$ starts with either $a_i$ or $a_j$, then $v$ or $u$ respectively gives us a longer word in $S$. If $w$ starts in any other letter than these two, then both $u$ and $v$ give a longer word in $S$. That is a contradiction.

Case 4: Let $w \in \text{supp} \tilde{f}_i$ and $w \in a_j \text{supp} \tilde{f}_j$. Then there exists $u \in \text{supp} \tilde{f}_j$ such that $a_j^{-1} w = u$. This implies that $w$ has to start with $a_j$. Moreover, the assumption at the beginning that $w \in \text{supp} \tilde{f}_i$, together with the fact that $w$ starts with $a_j$ guarantee that $a_i w \in a_i \text{supp} \tilde{f}_i \subset S$ has a longer length than the maximum. This is impossible. \[\square\]
In the next proposition, we identify the image of the unitary \( u_i \in A \rtimes_r \mathbb{F}_n \), \( i = 1, \ldots, n \), under \( \partial_1 \) in the Pimsner-Voiculescu 6-term exact sequence. This generalises Lemma 2 in \([11]\).

**Proposition 3.2.** Let \( A \) be a unital \( C^* \)-algebra and write \( \mathbb{F}_n = \langle a_1, \ldots, a_n \rangle \). For \( i \in \{1, \ldots, n\} \), let \( \alpha_i \in \text{Aut}(A) \) define an action \( \alpha \) of \( \mathbb{F}_n \) on \( A \). The boundary map \( \partial_1 : K_1(A \rtimes_r \mathbb{F}_n) \to \bigoplus_{i=1}^n K_0(A) \) in the Pimsner-Voiculescu 6-term exact sequence behaves in the following way with respect to the unitaries \( u_i \in \mathbb{C}_r^*(\mathbb{F}_n) \subset A \rtimes_r \mathbb{F}_n \)

\[
\partial_1([u_i]) = (0, \ldots, 0, -[1], 0, \ldots, 0).
\]

In our proof, we use the original proof of Pimsner-Voiculescu in \([10]\).

**Proof.** Let \( i \in \{1, \ldots, n\} \). Consider the following subset in \( \mathbb{F}_n \)

\[
\mathcal{W}_i = \{ \text{reduced words in } \mathbb{F}_n \text{ that end with } a_i \}.
\]

Note that for \( i \neq j \in \{1, \ldots, n\} \), we have that \( e \in \mathcal{W}_i \), \( a_j \mathcal{W}_i = \mathcal{W}_i \) and \( a_i \mathcal{W}_i = \mathcal{W}_i \setminus \{e\} \).

Assume \( A \subset \mathbb{B}(\mathcal{H}) \). Let \( \mathbb{K} \subset \mathbb{B}(\mathcal{H}) \) denote the \( C^* \)-algebras of the compact operators on \( \mathcal{H} \). We recall that for \( j = 1, \ldots, n \) we have that \( u_j \in \mathbb{C}_r^*(\mathbb{F}_n) \subset A \rtimes_r \mathbb{F}_n \subset \mathbb{B}(\ell^2(\mathbb{F}_n, \mathcal{H})) \).

Consider the compression of these unitaries to \( \ell^2(\mathcal{W}_i, \mathcal{H}) \subset \ell^2(\mathbb{F}_n, \mathcal{H}) \). These provide us with \( n - 1 \) unitaries \( U_j \), for \( i \neq j \in \{1, \ldots, n\} \), and one non-unitary isometry \( S_i \). We consider the following \( C^* \)-algebra so-called Toeplitz algebra

\[
\mathcal{T}_{n,i} = \mathbb{C}^*(\{A, U_1, \ldots, U_{i-1}, S_i, U_{i+1}, \ldots, U_n\}) \subset \mathbb{B}(\ell^2(\mathcal{W}_i, \mathcal{H})).
\]

Let \( P_e = I - S_i S_i^* \) be the projection to \( \delta_e \otimes \mathcal{H} \). The ideal generated by \( P_e \) in \( \mathcal{T}_{n,i} \) is isomorphic to \( A \otimes \mathbb{K} \). Now we consider the Toeplitz extension

\[
0 \to A \otimes \mathbb{K} \to \mathcal{T}_{n,i} \xrightarrow{P_{n,i}} A \rtimes_r \mathbb{F}_n \to 0.
\]

The surjection \( P_{n,i} : \mathcal{T}_{n,i} \to A \rtimes_r \mathbb{F}_n \) is defined by

\[
\begin{align*}
a &\mapsto a, \quad a \in A \\
U_j &\mapsto u_j, \quad j \in \{1, \ldots, n\}, \quad j \neq i \\
S_i &\mapsto u_i.
\end{align*}
\]

Let \( B_n = \{(t_1, \ldots, t_n) \in \bigoplus_{i=1}^n \mathcal{T}_{n,i} : P_{n,1}(t_1) = \ldots = P_{n,n}(t_n)\} \) be the fibred product of the \( C^* \)-algebras \( \mathcal{T}_{n,i} \) over \( A \rtimes_r \mathbb{F}_n \). Consider the exact sequence introduced on page 153 in \([10]\)

\[
0 \to (A \otimes \mathbb{K})^n \to B_n \to A \rtimes_r \mathbb{F}_n \to 0.
\]
Restrict the action $\alpha$ of $F_n$ to the action by its $i$-th generator, $\alpha_i$, on $A$ and write $\langle a_i \rangle \cong \mathbb{Z}$. Together with the Toeplitz extension, this exact sequence fits into a commuting diagram

\[
\begin{array}{ccccccccc}
0 & \rightarrow & (A \otimes \mathbb{K})^n & \rightarrow & B_n & \rightarrow & A \rtimes_r F_n & \rightarrow & 0 \\
\uparrow{\iota_i} & & & & & \uparrow & & \uparrow \\
0 & \rightarrow & A \otimes \mathbb{K} & \rightarrow & T_{n,i} & \rightarrow & A \rtimes \mathbb{Z} & \rightarrow & 0,
\end{array}
\]

with $\iota_i$ embedding to the $i$-th component and the middle homomorphism maps $a \in T_{n,i}$ to $(a, \ldots, a)$ and maps $S_i$ to $(U_1, \ldots, U_i, S_i, U_i, \ldots, U_i)$. Once we have such a commuting diagram the naturality of the 6-term exact sequence provides us with a commutative diagram in $K$-theory

\[
\begin{array}{ccccccccc}
K_0((A \otimes \mathbb{K})^n) & \rightarrow & K_0(B_n) & \rightarrow & K_0(A \rtimes_r F_n) & \rightarrow & & & \\
\uparrow K_0(A \otimes \mathbb{K}) & & & & & \uparrow & & \uparrow \\
K_0(T_{n,i}) & \rightarrow & K_0(A \rtimes \mathbb{Z}) & \rightarrow & & & & & \\
\uparrow K_1(A \rtimes_r F_n) & & & & & \uparrow K_1(B_n) & & \uparrow K_1((A \otimes \mathbb{K})^n) \\
K_1(A \rtimes \mathbb{Z}) & \leftarrow & K_1(T_{n,i}) & \leftarrow & K_1(A \otimes \mathbb{K}). & & & & \\
\end{array}
\]

The left vertical square fits into the diagram

\[
\begin{array}{ccccccccc}
K_0(A) & \rightarrow & K_0(A \otimes \mathbb{K}) & \rightarrow & K_0((A \otimes \mathbb{K})^n) & \rightarrow & \oplus_{i=1}^n K_0(A \otimes \mathbb{K}) & \rightarrow & \oplus_{i=1}^n K_0(A) \\
\uparrow & & & & & \uparrow & & \uparrow \\
K_1(A \rtimes \mathbb{Z}) & \rightarrow & K_1(A \rtimes_r F_n). & & & & & & \\
\end{array}
\]

The triangles at the sides are the natural identifications in $K$-theory. Hence we get a commutative diagram

\[
\begin{array}{ccccccccc}
K_0(A) & \rightarrow & \oplus_{i=1}^n K_0(A) & \rightarrow & K_0(A) & \rightarrow & & & \\
\uparrow & & & & & \uparrow & & \uparrow \\
\partial_{1,i} & & & & & \partial_1 & & \\
K_1(A \rtimes \langle u_i \rangle) & \rightarrow & K_1(A \rtimes_r F_n). & & & & & & \\
\end{array}
\]

We remark that $\partial_{1,i}$ is the boundary map in the Pimsner-Voiculescu 6-term exact sequence for $\mathbb{Z}$, that is $n = 1$ in $F_n$. Moreover, we have that $\partial_1 = \bigoplus_{i=1}^n \partial_{1,i}$ as mentioned in the statement of Theorem 3.5 in [10]. Due to Lemma 2 in [11], we have that $\partial_{1,i}(\langle u_i \rangle) = -[1] \in K_0(A)$ which is mapped by $\iota_i$ to the $i$-th component of the $n$-tuple $(0, \ldots, 0, -[1], 0, \ldots, 0)$. This finishes the proof. \qed
In the next theorem, we explicitly describe the right-hand side of the Baum-Connes conjecture for $F \wr F_n$.

**Theorem 3.3.** Let $\Gamma = F \wr F_n$ with $F$ a non-trivial finite group. Write $F_n = \langle a_1, \ldots, a_n \rangle$.

The $K$-groups of $C^*_r(\Gamma)$ can be described as the free abelian groups

$$K_0(C^*_r(\Gamma)) = \mathbb{Z}R$$
$$K_1(C^*_r(\Gamma)) = \mathbb{Z}[u_1] \oplus \cdots \oplus \mathbb{Z}[u_n].$$

**Proof.** Let $B = \bigoplus_{F_n} F$. We may write $C^*_r(\Gamma) = C^*_r(B) \rtimes F_n$. Due to Proposition 2.6, we have that $K_0(C^*(B)) = \mathbb{Z}(\text{Min } F(F_n))$. Moreover, $C^*(B)$ is an AF-algebra as $C^*(B) = \bigotimes_{F_n} \mathbb{C}F$ hence $K_1(C^*(B)) = 0$. Substituting these $K$-groups in the Pimsner-Voiculescu 6-term exact sequence, we get the diagram

$$
\begin{array}{c}
\bigoplus_{i=1}^{n} \mathbb{Z}(\text{Min } F(F_n)) \\
\downarrow \partial_1 \\
K_1(C^*_r(\Gamma)) \\
\downarrow \partial_0 \\
0 \\
\end{array}
\xrightarrow{\sigma^*} \mathbb{Z}(\text{Min } F(F_n)) \xrightarrow{\iota^*} K_0(C^*_r(\Gamma)) \xrightarrow{\delta_0} 0.
$$

We start from the left-hand side of the diagram. Injectivity of $\partial_1$ implies that $K_1(C^*_r(\Gamma)) = \text{Im } \partial_1$. By exactness of the diagram at $\bigoplus_{i=1}^{n} \mathbb{Z}(\text{Min } F(F_n))$, we have that $\text{Im } \partial_1 = \text{Ker } (\sigma^*)$. Due to Lemma 3.1, for $X = \text{Min } F(F_n)$, $p = p_F$ and the operator $\psi = \sigma^*$, the kernel is generated by $n$ copies of $1_{p_F} \in \text{Min } F(F_n)$. Thus we have

$$\text{Ker } (\sigma^*) = \mathbb{Z}1_{p_F} \oplus \cdots \oplus \mathbb{Z}1_{p_F}.$$ 

We recall from Proposition 2.6 that the element $1_{p_F}$ corresponds to $[1] \in \mathbb{Z}(\text{Min } F(F_n))$. Moreover, we observed in Proposition 3.2 that $\partial_1([u_i]) = (0, \ldots, -[1], 0, \ldots, 0)$, therefore

$$K_1(C^*_r(\Gamma)) = \mathbb{Z}[u_1] \oplus \cdots \oplus \mathbb{Z}[u_n].$$

In order to compute $K_0(C^*_r(\Gamma))$, we focus on the right-hand side of the diagram. Surjectivity of $\iota^*$ implies that $K_0(C^*_r(\Gamma)) = \mathbb{Z}(\text{Min } F(F_n))/\text{Ker } \iota^*$.

By exactness of the diagram at $\mathbb{Z}(\text{Min } F(F_n))$, we have that $\text{Ker } \iota^* = \text{Im } (\sigma^*)$. Therefore

$$K_0(C^*_r(\Gamma)) = \mathbb{Z}(\text{Min } F(F_n))/\text{Im } (\sigma^*).$$

Furthermore, we identify $\text{Im } (\sigma^*) = \langle f - a_i \cdot f : f \in \mathbb{Z}(\text{Min } F(F_n)), 1 \leq i \leq n \rangle$. Using Remark 2.4 and Lemma 2.5, we can then express this quotient as the free abelian group on the (countable) set of representatives for $F_n$-orbits, that is $K_0(C^*_r(\Gamma)) = \mathbb{Z}R$ with the described basis. 

\[ \square \]
4 K-homology of $E\Gamma$

In this section, we first present a suitable chain complex in order to define our homological groups. Later, we construct an explicit 2-dimensional model for $E\langle F \wr \mathbb{F}_n \rangle$ and finally, we compute K-homology of $E\Gamma$.

Let $e \in \mathbb{F}_n$ denote the neutral element. Consider the free right $\mathbb{F}_n$-module $\bigoplus_{i=1}^{n} \mathbb{Z}[F]_n$. The elements $e_j = (0, \ldots, 0, \underbrace{e}_{\text{j-th slot}}, 0, \ldots, 0) \in \bigoplus_{i=1}^{n} \mathbb{Z}[F]_n$, for $1 \leq j \leq n$, form the canonical basis for this free module. Example I.4.3 in [1] provides us with the free resolution

$$0 \to \bigoplus_{i=1}^{n} \mathbb{Z}[F]_n \xrightarrow{\delta} \mathbb{Z}[F]_n \xrightarrow{\epsilon} \mathbb{Z} \to 0,$$

with the augmentation $\epsilon$ and the boundary map $\delta$ satisfying $j \in \{1, \ldots, n\}$,

$$\delta(e_j) = e - a_j.$$

Let $M$ be a left $\mathbb{F}_n$-module. We recall that $\mathbb{Z}[F]_n \otimes_{\mathbb{Z}[F]_n} M \cong M$. Applying the functor $- \otimes_{\mathbb{Z}[F]_n} M$ to the above resolution provides us with the chain complex

$$0 \to \bigoplus_{i=1}^{n} M \xrightarrow{\delta} M \to 0.$$

Note that in the complex above, by abusing the notation of $\delta$, we have that $\delta(m_1, \ldots, m_n) = \sum_{j=1}^{n} m_j - a_j m_j$.

Therefore, we can write the first two homology groups

$$H_0(\mathbb{F}_n; M) = M/\text{Im}(\delta) \quad \text{and} \quad H_1(\mathbb{F}_n; M) = \text{Ker}(\delta) \leq \bigoplus_{i=1}^{n} M.$$  

4.1 A 2-dimensional model for $E\Gamma$

In this part, we construct a concrete 2-dimensional model for $E\Gamma$. As we will see, this model comes from the model for $E\langle B \rangle$. This construction generalises the one for $E(\langle F \wr \mathbb{Z} \rangle)$ in [3] and it is derived from [4] for infinite cyclic extensions.

Let $B$ be our locally finite group represented as a co-limit of an increasing sequence $B_n$. We know that $E\langle B \rangle$ has a one dimensional model which is a tree $T$. Let $V$ and $E$ respectively denote the set of vertices and edges of $T$. The tree $T$ can be described as follows:

- $V = E = \bigsqcup_{n>0} B/B_n$
- For $b \in B$, the vertices $bB_n$, $bB_{n+1} \in V$ are connected via the edge labelled by $bB_n \in E$. 

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Consider the Cayley graph of $F_n$. Intuitively speaking, the idea of construction is to install the tree $T$ coming from a model for $E_B$, over all vertices of the Cayley graph of $F_n$ and then identify certain subcomplexes of these trees in a compatible way that the resulting complex meets all requirements of being a model for $E\Gamma$.

Denote by $T_w$ the tree $T$ over the word $w \in F_n$, and by $bB_{n,w}$ a vertex on the $n$-th level of the filtration of the tree $T_w$. For $w \in F_n$, we denote by $[w, wa_j]$ the edge from $w$ to $wa_j$ on the Cayley graph of $F_n$.

We define a $B$-action on $T_w$: for $b, f \in B$ and $w \in F_n$

\[ f \cdot_w bB_{n,w} := (w^{-1} \cdot f)bB_{n,w}, \quad n \in \mathbb{N}. \]

Note that each $T_w$ is a model for $E_B$ as well.

For $j = 1, \ldots, n$, we define the gluing maps between neighbouring trees

\[ \varphi^a_{aj} : T_w \to T_{wa_j} : bB_{n,w} \mapsto (a^{-1}_j \cdot b)B_{n+1,wa_j}. \]

It can easily be checked that the $\varphi^a_{aj}$ are $B$-equivariant, that is

\[ \varphi^a_{aj}(f \cdot_w bB_{n,w}) = f \cdot_{wa_j} \varphi^a_{aj}(bB_{n,wa_j}). \]

For $w \in F_n$, we may identify an edge $[w, wa_i]$ with the interval $[0, 1]$. We define

\[ \tilde{Z} := \bigcup_{j=1}^{n} \bigcup_{w \in F_n \mid |wa_j| > |w|} (T_w \times [w, wa_j]) \cup \bigcup_{j=1}^{n} \bigcup_{w \in F_n \mid |wa_j| > |w|} (T_w \times [w, wa_j^{-1}]). \]

Each non-trivial word on the Cayley graph of $F_n$ has $2^n - 1$ possibilities of increasing its length. We then identify points on the boundaries of edges in $\tilde{Z}$. More explicitly, if we assume that $w$ ends with $a_i$, then for $k, j \in \{1, \ldots, n\}$ with $k \neq j \neq i$ we identify

\[ T_w \times [w, wa_j] \ni (bB_{n,w}, wa_k^{j+1}) \sim (\varphi^{a_{j+1}}_{w}(bB_{n,w}), wa_k^{j+1}) \in T_{wa_j} \times [wa_k, wa_k^{j+1}a_k], \]

\[ T_w \times [w, wa_i] \ni (bB_{n,w}, wa_i) \sim (\varphi^a_{w}(bB_{n,w}), wa_i) \in T_{wa_i} \times [wa_i, wa_i^2]. \]

For the trivial word we hence consider all $2^n$ identifications. We define the mapping telescope $Z := \tilde{Z}/\sim$.

This quotient space is a candidate to be the desired model for $E\Gamma$. The group $\Gamma$ acts on it. Indeed, the actions of $B$ and $F_n$ on $Z$ combine into the conjugation action $\beta$ of $\Gamma$ on $Z$. To see this, we define the following actions:

\[ B \overset{\beta}{\curvearrowright} Z : \theta(f)(bB_{n,w}, t_{[w, wa_j]}) = ((w^{-1} \cdot f)bB_{n,w}, t_{[w, wa_j]}), \]

\[ F_n \overset{\eta}{\curvearrowright} Z : \eta(a_j)(bB_{n,w}, t_{[w, wa_j]}) = (bB_{n,a_jw}, t_{[a_jw, a_jwa_j]}), \]
where \( t_{[w,wa_i]} \) denotes a point on the edge \([w,wa_i]\), hence \( t_{[a_jw,a_jwa_i]} \) is the point moved on the shifted edge. Moreover,

\[
\eta(a_j)\theta(f)\eta(a_j)^{-1}(bB_{n,w}, t_{[w,wa_i]}) = \eta(a_j)\theta(f)(bB_{n,a_j^{-1}w}, t_{[a_j^{-1}w,a_j^{-1}wa_i]})
\]

\[
= \eta(a_j)(\theta(f)(bB_{n,a_j^{-1}w}, t_{[a_j^{-1}w,a_j^{-1}wa_i]}))
\]

\[
= \eta(a_j)((w^{-1} \cdot f)(bB_{n,a_j^{-1}w}, t_{[a_j^{-1}w,a_j^{-1}wa_i]}))
\]

\[
= \eta(a_j)((w^{-1} \cdot (a_j \cdot f))(bB_{n,w}, t_{[w,wa_i]}))
\]

\[
= \beta_{a_j}(f)(bB_{n,w}, t_{[w,wa_i]}).
\]

Some observations are pertinent here.

- \( F_n \) acts freely.
- The action \( \eta \) has a fundamental domain \( D = \bigcup_{j=1}^{n} T_e \times [e,a_j] \).
- Vertex stabilisers are finite, hence \( \Gamma \) acts properly on \( Z \).

Combining these observations, we have the following proposition.

**Proposition 4.1.** The topological space \( Z \) is a 2-dimensional model for \( E\Gamma \).

**Proof.** It is verbatim the proof of Proposition 2 in [3]. \( \square \)

In the rest of this section we compute the K-homology of \( E\Gamma \), which is the Bredon homology of \( \Gamma = B \rtimes F_n \) with coefficients in \( R_C \). For this we appeal to the Martinez spectral sequence recalled in Preliminary Section. Consider the split exact sequence \( 0 \to B \to \Gamma \to F_n \to 0 \) associated to the group \( \Gamma = B \rtimes F_n \). Obviously, the family \( \mathcal{S} \) of finite subgroups of \( F_n \) only consists of the trivial group. Therefore the pull-back of this family only consists of the group \( B \). Hence the second page of this spectral sequence is

\[
E_2^{p,q} = H^\mathcal{S}_p(F_n; H^B_q(B; R_C)) = H_p(F_n; H^B_q(B; R_C)).
\]

Our aim is to compute these groups.

We recall that the dimension of models for \( E\mathcal{B} \) and \( E\mathcal{F}_n \) is one. Hence on the one hand, \( E_{p,q}^2 \) is trivial for \( p \geq 2 \), and on the other hand, by Theorem 2.2, Bredon homology of locally finite group \( B \) is trivial for \( q \geq 1 \). Therefore, the only non-zero terms are \( p = 0, 1 \) and \( q = 0 \). Particularly, \( E^2_{0,0} \) and \( E^2_{1,0} \) are the only non-trivial terms of the \( E^2 \)-page. This means that the \( E^2 \)-page is concentrated in horizontal axis and the spectral sequence collapses in this page as there is no differential. Accordingly, we have \( E^{\infty}_{p,q} = E^{2}_{p,q} \) for \( p, q \geq 0 \). We recall that by Martinez’s result, the spectral sequence converges to \( H^\mathcal{S}_{p+q}(\Gamma; R_C) \). Together with the discussion at the beginning of this part, we may identify Bredon homology groups with
homology groups of $F_n$ with coefficient in the free abelian group $H_0^\delta(B; R_C)$. In particular, we need to compute

$$H_0^\delta(\Gamma; R_C) = E^\infty_{0,0} = E^2_{0,0} = H_0(F_n; H_0^\delta(B; R_C)), \quad (1)$$

and

$$H_1^\delta(\Gamma; R_C) = E^\infty_{1,0} = E^2_{1,0} = H_1(F_n; H_0^\delta(B; R_C)). \quad (2)$$

In the next theorem we explicitly describe the left-hand side of the Baum-Connes assembly map for $F \wr \mathbb{F}_n$.

**Theorem 4.2.** Let $\Gamma = F \wr \mathbb{F}_n$ with $F$ a non-trivial finite group. Write $\mathbb{F}_n = \langle a_1, \ldots, a_n \rangle$. The topological $K$-groups of $E \Gamma$ can be described as two free abelian groups.

$$K^i_0(E \Gamma) = \mathbb{Z}R^i$$

with $R^i$ a countable basis indexed by representatives for $\mathbb{F}_n$-orbits in $\hat{F}(\mathbb{F}_n)$.\[K^i_0(E \Gamma) = \mathbb{Z}[v_1] \oplus \cdots \oplus \mathbb{Z}[v_n], \text{ where } [v_i] \text{ is the canonical generator of } K^i_0(\mathbb{E} \mathbb{Z}) \text{ via the identification } \mathbb{Z} = \langle a_i \rangle. \text{ Indeed, the inclusions } \langle a_i \rangle \hookrightarrow \mathbb{F}_n \hookrightarrow \Gamma \text{ give rise to an inclusion } \langle v_i \rangle = K^i_0(\mathbb{E} \mathbb{Z}) \hookrightarrow K^i_0(E \mathbb{F}_n) \cong K^i_0(E \Gamma).]

**Proof.** Due to Theorem 2.3 and equations in (1) and (2), we have that $K^i_0(E \Gamma) \cong H^i_0(\Gamma; R_C) = H_i(F_n; H_0^\delta(B; R_C))$ for $i = 0, 1$. In order to compute the homological groups with the appropriate coefficients, we tensor the free resolution, at the beginning of this section, with $H_0^\delta(B; R_C)$. We recall that by Theorem 2.2 and Proposition 2.6 we have that $H_0^\delta(B; R_C) \cong K_0^B(E \mathbb{B}) \cong \hat{Z}(\mathbb{F}_n)$. Therefore, we have

$$K^i_0(E \Gamma) \cong H^i_0(\Gamma; R_C) = \frac{\mathbb{Z}\hat{F}(\mathbb{F}_n)}{\text{Im}(\delta)} = \frac{\mathbb{Z}\hat{F}(\mathbb{F}_n)}{\langle f - a_i, f \in \mathbb{Z}\hat{F}(\mathbb{F}_n), 1 \leq i \leq n \rangle}. \quad (3)$$

Due to Remark 2.4 and Lemma 2.5 $K_0^i(E \Gamma)$ is a free abelian group on the orbit space of the action $\mathbb{F}_n \curvearrowright \hat{F}(\mathbb{F}_n)$ with the described basis.

For computing $K^i_0(E \Gamma)$, we appeal to Lemma 3.1. In view of that lemma for $X = \hat{F}(\mathbb{F}_n)$ and $\mathfrak{p} = 1$, the trivial representation of $F$, we may describe the kernel as the fixed points of the action. More precisely,

$$K^i_0(E \Gamma) \cong H_0^\delta(\Gamma; R_C) = \text{Ker}(\delta) = \mathbb{Z}1_1 \oplus \cdots \oplus \mathbb{Z}1_1, \quad (4)$$

in order to identify $1_1$ in $i$-th copy with $[v_i]$, we need to make some observations. As the groups $\mathbb{Z}$ and $\mathbb{F}_n$ are torsion-free, we have that

$$K^\mathbb{Z}_1(E \mathbb{Z}) \cong K_0(B\mathbb{Z}) \cong K_1(\mathbb{S}^1) \quad \text{and} \quad K^{\mathbb{F}_n}_1(E \mathbb{F}_n) \cong K_0(B\mathbb{F}_n) \cong K_1(\bigvee_{n} \mathbb{S}^1),$$
where \( BG \) stands for the classifying space and \( \bigvee_n S^1 \) denotes the wedge of \( n \)-circles. We recall that \( H_1(F_n) = \mathbb{Z} \oplus \ldots \oplus \mathbb{Z} \) (\( n \)-times), and by Theorem 2.3, \( K_f^n(E F_n) \cong H_1(F_n; \mathbb{Z}) \).

Moreover, due to functionality of K-theory and homology theory we have that \( S^1 \hookrightarrow \bigvee_n S^1 \) induces \( K_0(S^1) \hookrightarrow K_0(\bigvee_n S^1) \), and \( \mathbb{Z} \to \mathbb{Z} \) induces \( H_1(F_n; Z) \to H_1(F_n; \hat{Z}^F(F_n)) \).

Now consider the following composition

\[
K_0^\Gamma(E \Gamma) \xrightarrow{\cong} K_0(S^1) \xrightarrow{\sigma} K_1(\bigvee_n S^1) \xrightarrow{\tau} H_1(F_n; Z) \xrightarrow{\cong} K_1(\hat{Z}^F(F_n)) \xrightarrow{\cong} K_1(E \Gamma).
\]

Choose \( a_i \in F_n \), and consider the identification \( \langle a_i \rangle \cong \mathbb{Z} \). By Theorem 2.2 we have that \( K_0^\Gamma(E \mathbb{Z}) \cong H_1(\mathbb{Z}; \mathbb{Z}) = \mathbb{Z}[v_i] \). According to the composition above, we can see that \( [v_i] \mapsto (0, \ldots, 0, 1, 0, \ldots, 0) \in K_1^\Gamma(E \Gamma) \). This finishes the proof.

\[ \square \]

5 The Baum-Connes assembly map for \( F \wr F_n \)

In this section, we explicitly describe the Baum-Connes assembly map for \( F \wr F_n \), where \( F \) is a non-trivial finite group.

**Theorem 5.1.** Let \( F \) be a non-trivial finite group. The Baum-Connes assembly map for \( \Gamma = F \wr F_n \) can be described as:

- The assembly map \( \mu_0^\Gamma : K_0^\Gamma(E \Gamma) \to K_0(C^*_r(\Gamma)) \) is an isomorphism between two countably generated free abelian groups.
- The assembly map \( \mu_1^\Gamma : K_1^\Gamma(E \Gamma) \to K_1(C^*_r(\Gamma)) \) is an isomorphism between two free abelian groups of rank \( n \).

**Proof.** \( \mu_0^\Gamma \) is an isomorphism. Let \( \iota : B \hookrightarrow \Gamma \) be an inclusion. Consider the following diagram

\[
\begin{array}{c}
\bigoplus_{i=1}^n K_0^B(E B) \xrightarrow{\sigma^*} K_0^B(E B) \xrightarrow{\iota^*} K_0^\Gamma(E \Gamma) \xrightarrow{\nu_0^\Gamma} 0 \\
\bigoplus_{i=1}^n \mu_0^B \\
\bigoplus_{i=1}^n K_0(C^*(B)) \xrightarrow{\sigma^*} K_0(C^*(B)) \xrightarrow{\iota^*} K_0(C_r^*(\Gamma)) \xrightarrow{\nu_0^\Gamma} 0.
\end{array}
\]

Theorem 3.3 together with Theorem 1.2 imply that top and bottom sequences are exact. Moreover, functoriality of the assembly map (Corollary II.1.3 in [8]) yields the commutativity of the whole diagram. By Proposition 2.6, \( \mu_0^B \) and hence \( \bigoplus_{i=1}^n \mu_0^B \) are isomorphisms. The Five Lemma then implies that \( \mu_0^\Gamma \) is an isomorphism.
\( \mu_1^\Gamma \) is an isomorphism. Consider the comparison diagram

\[
\begin{array}{ccc}
K_1^F(E\Gamma) & \xrightarrow{\mu_1^F} & K_1(C^*_r(\Gamma)) \\
\downarrow & & \downarrow \\
K_1^F(E\mathbb{F}_n) & \xrightarrow{\mu_1^F} & K_1(C^*_r(\mathbb{F}_n)) \\
\downarrow & & \downarrow \\
K_1^F(E\mathbb{Z}) & \xrightarrow{\mu_1^F} & K_1(C^*_r(\mathbb{Z})).
\end{array}
\]

Due to Theorem 3.3 and Theorem 4.2, we know that \( K_1^F(E\Gamma) \cong K_1(C^*_r(\Gamma)) \cong \mathbb{Z}^n \) and \( K_1^F(E\Gamma) = K_1^F(E\mathbb{F}_n) \cong \bigoplus_{i=1}^n K_1^F(E\mathbb{Z}) \). For \( i = 1, \ldots, n \), take the generator \([v_i]\) of the \( i \)-th summand in \( K_1^F(E\Gamma) \). Write \( (a_i) \cong \mathbb{Z} \). The \( n \)-tuple \((0, \ldots, 0, [v_i], 0, \ldots, 0) \in K_1^F(E\mathbb{F}_n) \) maps to \([v_i]\in K_1^F(E\mathbb{Z})\). By the explicit description in [8, Section II.2.4], we know that the assembly map \( \mu_1^F \) transfers the generator on one side to the other. Moreover, by functoriality of K-theory we have that \( K_1(C^*_r(\mathbb{F}_n)) \cong \bigoplus_{i=1}^n K_1(C^*_r(\mathbb{Z})) \). Therefore, \((0, \ldots, 0, [v_i], 0, \ldots, 0) \) maps to \((0, \ldots, 0, [u_i], 0, \ldots, 0) \).

6 Trace

We close the article by a remark on the image of the induced (canonical) trace on \( K_0(C^*_r(\Gamma)) \), that is \( \text{Im} \, \tau_*(K_0(C^*_r(\Gamma))) \). This is relevant in the context of the modified Trace conjecture formulated by Lück in [6].

Let \( \tau : C^*_r(G) \to \mathbb{C} \) be the canonical trace on \( C^*_r(G) \). Concerning the (modified) Trace conjecture, it is predicted that for the induced homomorphism \( \tau_* : K_0(C^*_r(G)) \to \mathbb{R} \) we have that

\[
\text{Im} \, \tau_* \subset \mathbb{Z}\left[\left\{\frac{1}{|H|} : H \leq G, |H| < \infty\right\}\right] \subset \mathbb{Q}.
\]

In the case of \( \Gamma = F \wr \mathbb{F}_n \), thanks to the surjectivity of \( \iota_* : K_0(C^*(B)) \to K_0(C^*_r(\Gamma)) \), we only need to consider \( \text{Im} \, \tau_*(C^*(B)) = \text{Im} \, \tau_*(\mathbb{Z}(\text{Min}F(\mathbb{F}_n))) \). Therefore the computations in Proposition 5 in [3] implies the predicted result

\[
\text{Im} \, \tau_*(K_0(C^*_r(\Gamma))) = \mathbb{Z}\left[\frac{1}{|F|}\right].
\]

References

[1] K.S. Brown Cohomology of groups Springer, 1982

[2] Y. Cornulier, Y. Stalder and A. Valette Proper actions of wreath products and generalizations Trans. Amer. Math. Soc. 364 (2012), 3159-3184
[3] R. Flores, S. Pooya and A. Valette *K-theory and K-homology for the lamplighter groups of finite groups* arXiv:1610.02798, to appear in Proc. London Math. Soc.

[4] M. Fluch *On Bredon (co-)homological dimensions of groups* PhD Thesis, 2011. http://www.fluch.pl/docs/phdthesis.pdf

[5] N. Higson and G. Kasparov *E-theory and KK-theory for groups which act properly and isometrically on Hilbert space* Invent. Math. 144 (2001), no. 1, 23–74

[6] W. Lück *The relation between the Baum-Connes conjecture and the trace conjecture* Invent. Math. 149 (2002), 123-152

[7] C. Martínez *A spectral sequence in Bredon co(homology)* J. Pure Appl. Alg., 176 (2002), 161-173

[8] G. Mislin and A. Valette *Proper group actions and the Baum-Connes conjecture*, Advanced Courses in Mathematics - CRM Barcelona, Birkhäuser, 2003

[9] M. Pimsner and D. Voiculescu *Exact sequences for K-groups and Ext- groups of certain crossed products C∗-algebras* J. Operator theory 4 (1980), 93-118

[10] M. Pimsner and D. Voiculescu *K-groups of reduced crossed products by free groups* J. Operator theory 8 (1982), 131-156

[11] S. Pooya and A. Valette *K-theory for the C∗-algebras of the solvable Baumslag-Solitar groups* arXiv:1604.05607

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