THE EQUIVALENCE OF TWO DISCRETENESSES OF TRANGULATED CATEGORIES

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Abstract. Given an ST-triple \((\mathcal{C}, \mathcal{D}, M)\) one can associate a co-\(t\)-structure on \(\mathcal{C}\) and a \(t\)-structure on \(\mathcal{D}\). It is shown that the discreteness of \(\mathcal{C}\) with respect to the co-\(t\)-structure is equivalent to the discreteness of \(\mathcal{D}\) with respect to the \(t\)-structure. As a special case, the discreteness of \(\mathcal{D}^b(\text{mod } A)\) in the sense of Vossieck is equivalent to the discreteness of \(K^b(\text{proj } A)\) in a dual sense, where \(A\) is a finite-dimensional algebra.

1. Introduction

Derived-discreteness of a finite-dimensional algebra was introduced by Vossieck in [20]. It is defined by counting the number of indecomposable objects in the bounded derived category. Recently this notion has been generalised by Broomhead, Pauksztello and Ploog in [10] to a notion of discreteness of a triangulated category with respect to (the heart of) a bounded \(t\)-structure. In [10] they also introduced a dual notion, namely, the notion of discreteness of a triangulated category with respect to a bounded co-\(t\)-structure (equivalently, a silting subcategory).

It turns out that ST-triples introduced in [1] provide a nice framework to study the interplay between \(t\)-structures and co-\(t\)-structures. Let \(\mathcal{C}\) and \(\mathcal{D}\) be triangulated categories and \(M\) a silting object of \(\mathcal{C}\) such that \((\mathcal{C}, \mathcal{D}, M)\) is an ST-triple. Then on \(\mathcal{D}\) there is a natural bounded \(t\)-structure, say, with heart \(\mathcal{H}\). Our main result is

Theorem (4.1). The category \(\mathcal{C}\) is \(M\)-discrete if and only if the category \(\mathcal{D}\) is \(\mathcal{H}\)-discrete.

In the literature there are another two notions of discreteness of triangulated categories, namely, silting-discreteness [2] and \(t\)-discreteness [1], defined by counting the number of silting objects and bounded \(t\)-structures, respectively. In [1] it is shown that \(\mathcal{C}\) is silting-discrete if and only \(\mathcal{D}\) is \(t\)-discrete, and that if \(\mathcal{D}\) is \(\mathcal{H}\)-discrete then \(\mathcal{D}\) is \(t\)-discrete. Together with these results Theorem [1.1] implies the following corollary, which completes the picture.

Corollary (4.2). If \(\mathcal{C}\) is \(M\)-discrete, then \(\mathcal{C}\) is silting-discrete.

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The paper is structured as follows. In Section 2 we fix the notion and briefly recall the definitions of $t$-structure, silting object and co-$t$-structure. In Section 3 we recall the definitions of ST-triple and discreteness of triangulated categories. In Section 4 we prove Theorem 4.1. In the final section we apply Theorem 4.1 to finite-dimensional algebras to recover a result of Qin [19] stating that derived-discreteness in the sense of Vossieck [20] is preserved under decollement.

Throughout let $k$ be an algebraically closed field. We use $\Sigma$ to denote the shift functors of all triangulated categories.

2. Preliminaries

The aim of this section is mainly to briefly recall the definitions of $t$-structures, silting object and co-$t$-structure and fix the notation we will use in the paper.

2.1. Triangulated categories. Let $A$ be a finite-dimensional $k$-algebra. Denote by $\text{mod} A$ the category of finite-dimensional (right) $A$-modules and by $\text{proj} A$ its full subcategory of finitely generated projective $A$-modules. Denote by $K^b(\text{proj} A)$ the bounded homotopy category of $\text{proj} A$ and by $D^b(\text{mod} A)$ the bounded derived category of $\text{mod} A$. These are two triangulated $k$-categories with shift functor being the shift of complexes.

Let $\mathcal{T}$ be a triangulated $k$-category. For two subcategories $\mathcal{A}$ and $\mathcal{B}$, let $\mathcal{A} \ast \mathcal{B}$ be the full subcategory of $\mathcal{T}$ consisting of objects $X$ with a triangle $X' \to X \to X'' \to \Sigma X'$, where $X' \in \mathcal{A}$ and $X'' \in \mathcal{B}$. We will often identify an object with the full subcategory consisting of this unique object. A full subcategory of $\mathcal{T}$ is said to be thick if it is closed under shifts, cones and direct summands. For an object $X$ of $\mathcal{T}$ denote by $\text{add}(X)$ the smallest additive subcategory of $\mathcal{T}$ containing $X$ and closed under direct summands, and by $\text{thick}(X)$ the smallest thick subcategory of $\mathcal{T}$ containing $X$. Assume that $\mathcal{T}$ has arbitrary coproducts. An object $X$ of $\mathcal{T}$ is said to be compact if the canonical map $\bigoplus_{i \in I} \text{Hom}_\mathcal{T}(X,Y_i) \to \text{Hom}_\mathcal{T}(X,\bigoplus_{i \in I} Y_i)$ is an isomorphism for any set $\{Y_i| i \in I\}$ of objects of $\mathcal{T}$; it is called a compact generator if in addition $\mathcal{T}$ coincides with its smallest triangulated category containing $X$ and closed under coproducts.

2.2. Grothendieck groups. Let $\mathcal{H}$ be an abelian $k$-category with only finitely many isoclasses (=isomorphism classes) of simple objects such that all objects of $\mathcal{H}$ are filtered by simple objects (e.g. $\text{mod} A$, where $A$ is a finite-dimensional $k$-algebra). The Grothendieck group $K_0(\mathcal{H})$ of $\mathcal{H}$ is the abelian group generated by isoclasses of objects in $\mathcal{H}$ modulo the relations $[M] + [N] - [L]$ whenever there is a short exact sequence $0 \to M \to L \to N \to 0$. For $M \in \mathcal{H}$ denote by $\text{dim}(M)$ the class of $M$ in $K_0(\mathcal{H})$. Let $K_0(\mathcal{H})^+$ be the subset of $K_0(\mathcal{H})$ consisting of classes of objects in $\mathcal{H}$. Then $K_0(\mathcal{H})$ is a free abelian group with basis the classes of simple objects, and in terms of this basis elements of $K_0(\mathcal{H})^+$ are precisely those with non-negative coefficients.
Let $\mathcal{A}$ be a Hom-finite Krull–Schmidt additive $k$-category such that $\mathcal{A} = \text{add}(M)$ for some $M \in \mathcal{A}$ (e.g. $\text{proj} \mathcal{A}$, where $A$ is a finite-dimensional $k$-algebra). The split Grothendieck group $K_0^{sp}(\mathcal{A})$ of $\mathcal{A}$ is the abelian group generated by the isoclasses of objects of $\mathcal{A}$ modulo the relations $[L] + [N] - [L \oplus N]$. For $N \in \mathcal{A}$, denote by $\text{sum}(N)$ the class of $N$ in $K_0^{sp}(\mathcal{A})$. Let $(K_0^{sp}(\mathcal{A}))^+$ be the subset of $K_0^{sp}(\mathcal{A})$ consisting of classes of objects of $\mathcal{A}$. Then $K_0^{sp}(\mathcal{A})$ is a free abelian group with basis the classes of indecomposable direct summands of $M$, and in terms of this basis elements of $K_0^{sp}(\mathcal{A})^+$ are precisely those with non-negative coefficients.

2.3. $t$-structures. Let $\mathcal{T}$ be a triangulated $k$-category.

A $t$-structure on $\mathcal{T}$ ([6]) is a pair $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ of strict (that is, closed under isomorphisms) and full subcategories of $\mathcal{T}$ such that, putting $\mathcal{T}^{p} = \Sigma^{-p} \mathcal{T}^{0}$ and $\mathcal{T}^{p} = \Sigma^{-p} \mathcal{T}^{0}$ for $p \in \mathbb{Z}$, we have

1. $\mathcal{T}^{\leq -1} \subseteq \mathcal{T}^{\leq 0}$ and $\mathcal{T}^{\geq 1} \subseteq \mathcal{T}^{\geq 0}$;
2. $\text{Hom}(X, Y) = 0$ for $X \in \mathcal{T}^{\leq 0}$ and $Y \in \mathcal{T}^{\geq 1}$,
3. for each $X \in \mathcal{T}$ there is a triangle $X' \to X \to X'' \to \Sigma X'$ in $\mathcal{T}$ with $X' \in \mathcal{T}^{\leq 0}$ and $X'' \in \mathcal{T}^{\geq 1}$.

The heart $\mathcal{T}^{0} := \mathcal{T}^{\leq 0} \cap \mathcal{T}^{\geq 0}$ is always abelian. The $t$-structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ is said to be bounded if

$$\bigcup_{p \in \mathbb{Z}} \mathcal{T}^{\leq p} = \mathcal{T} = \bigcup_{p \in \mathbb{Z}} \mathcal{T}^{\geq p},$$

or equivalently, $\mathcal{T} = \text{thick}(\mathcal{T}^{0})$.

Let $A$ be a finite-dimensional $k$-algebra. Let $\mathcal{D}^{\leq 0}$ (respectively, $\mathcal{D}^{\geq 0}$) be the full subcategory of the bounded derived category $\mathcal{D}^b(\text{mod} \ A)$ consisting of complexes with vanishing cohomologies in positive degrees (respectively, in negative degrees). Then $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is a bounded $t$-structure on $\mathcal{D}^b(\text{mod} \ A)$ with heart the full subcategory of complexes with cohomology concentrated in degree 0, which is canonically equivalent to $\text{mod} \ A$.

It is easy to see that for every integer $p$, the pair $(\mathcal{T}^{\leq p}, \mathcal{T}^{\geq p})$ is also a $t$-structure and the category $\mathcal{T}^p := \mathcal{T}^{\leq p} \cap \mathcal{T}^{\geq p}$ is the heart. By the condition (3) in the above definition, for $X \in \mathcal{T}$ there is a triangle $X' \to X \to X'' \to \Sigma X'$ with $X' \in \mathcal{T}^{\leq p}$ and $X'' \in \mathcal{T}^{\geq p+1}$. This triangle is unique up to a unique isomorphism, so the correspondences $X \mapsto X'$ and $X \mapsto X''$ extend to functors

$$\sigma^{\leq p} : \mathcal{T} \to \mathcal{T}^{\leq p} \quad \text{and} \quad \sigma^{\geq p+1} : \mathcal{T} \to \mathcal{T}^{\geq p+1},$$

respectively, called the truncation functors. Moreover, we have the set of cohomology functors

$$\{\sigma^p = \Sigma^p \sigma^{\leq p} \sigma^{\geq p} : \mathcal{T} \to \mathcal{T}^0 \mid p \in \mathbb{Z} \},$$

which is cohomological, i.e. takes triangles to long exact sequences. The next result follows directly from the definition of $\sigma^{\geq p}$ on morphisms.

**Lemma 2.1.** The map $\sigma^{\geq p}(X, Y) : \text{Hom}_\mathcal{T}(X, Y) \to \text{Hom}_\mathcal{T}(\sigma^{\geq p}X, \sigma^{\geq p}Y)$
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- is injective if $\text{Hom}_T(X, \sigma^{\leq p-1}Y) = 0$;
- is surjective if $\text{Hom}_T(X, \Sigma \sigma^{\leq p-1}Y) = 0$;
- has kernel $\{f : X \to Y | f$ factors through the morphism $\sigma^{\leq p-1}(Y) \to Y\}$.

2.4. Silting objects and co-t-structures. Let $T$ be a triangulated $k$-category.

An object $M$ of $T$ is said to be presilting if $\text{Hom}_T(M, \Sigma^p M) = 0$ for all positive integers $p$, and silting if in addition $T = \text{thick}(M)$. See [16, 5, 3]. Let $A$ be a finite-dimensional $k$-algebra. Then the free $A$-module $A_A$ of rank 1 is a silting object of the bounded homotopy category $K^b(\text{proj} A)$.

A co-t-structure on $T$ ([8, Definition 2.4]) is a pair $(T_{\geq 0}, T_{\leq 0})$ of strict and full subcategories of $T$ such that, putting $T_{\geq p} = \Sigma^{-p} T_{\geq 0}$ and $T_{\leq p} = \Sigma^{-p} T_{\leq 0}$ for $p \in \mathbb{Z}$, we have

(0) both $T_{\geq 0}$ and $T_{\leq 0}$ are additive and closed under taking direct summands;
(1) $T_{\geq 1} \subseteq T_{\geq 0}$ and $T_{\leq -1} \subseteq T_{\leq 0}$;
(2) $\text{Hom}(X, Y) = 0$ for $X \in T_{\geq 1}$ and $Y \in T_{\leq 0}$;
(3) for each $X \in T$ there is a triangle $X' \to X \to X'' \to \Sigma X'$ in $T$ with $X' \in T_{\geq 1}$ and $X'' \in T_{\leq 0}$.

The intersection $T_{\geq 0} \cap T_{\leq 0}$ is called the co-heart of the co-t-structure $(T_{\geq 0}, T_{\leq 0})$. A co-t-structure $(T_{\leq 0}, T_{\geq 0})$ is said to be bounded ([9]) if

$$\bigcup_{p \in \mathbb{Z}} T_{\leq p} = T = \bigcup_{p \in \mathbb{Z}} T_{\geq p},$$

or equivalently, $T = \text{thick}(T_{\geq 0} \cap T_{\leq 0})$.

Let $A$ be a finite-dimensional $k$-algebra. Let $P_{\geq 0}$ (respectively, $P_{\leq 0}$) be the full subcategory of $K^b(\text{proj} A)$ consisting of objects isomorphic to complexes with trivial components in negative degrees (respectively, in positive degrees). Then $(P_{\geq 0}, P_{\leq 0})$ is a bounded co-t-structure on $K^b(\text{proj} A)$ with co-heart $\text{add}(A)$, which is canonically equivalent to $\text{proj} A$.

3. ST-triples and discreteness

In this section we recall the definition of ST-triple from [11] and two notions of discreteness of triangulated categories from [10]; moreover, we show that ‘compact silting objects’ naturally produce ST-triples, and establish some auxiliary results which we will use in Section 4.

3.1. ST-triples. Let $T$ be a triangulated $k$-category.

An ST-triple inside $T$ ([11, Definition 4.3]) is a triple $(C, D, M)$, where $C$ and $D$ are thick subcategories of $T$ and $M$ is a silting object of $C$, such that

(ST1) $\text{Hom}_T(M, T)$ is finite-dimensional for any object $T$ of $T$,
(ST2) $(T_{\leq 0}, T_{\geq 0})$ is a $t$-structure on $T$, where for an integer $p$

$$T_{\leq p} := \{X \in T | \text{Hom}_T(M, \Sigma^m X) = 0 \forall m > p\},$$

$$T_{\geq p} := \{X \in T | \text{Hom}_T(M, \Sigma^m X) = 0 \forall m < p\}.$$
(ST3) \( \mathcal{T} = \bigcup_{p \in \mathbb{Z}} \mathcal{T}^{\leq p} \) and \( \mathcal{D} = \bigcup_{p \in \mathbb{Z}} \mathcal{T}^{\geq p} \).

A prototypical example of an ST-triple is the triple \((\text{R}^b(\text{proj} \ A), \text{D}^b(\text{mod} \ A), A_A)\) inside \( \text{D}^b(\text{mod} \ A) \). Note, however, that in general \( C \) and \( D \) are not comparable, see [1, the paragraph after Definition 4.3].

Let \( A \) be a triangulated \( k \)-category with arbitrary coproducts. Assume that \( M \) is a compact generator of \( A \) such that \( \text{Hom}_A(M, \Sigma^p M) \) is finite-dimensional for all \( p \in \mathbb{Z} \) and vanishes for all \( p > 0 \). Put

\[
A^c = \text{thick}(M),
\]
\[
A_{fd} = \{ X \in A \mid \bigoplus_{p \in \mathbb{Z}} \text{Hom}_A(M, \Sigma^p X) \text{ is finite-dimensional} \},
\]
\[
A^{-fd} = \{ X \in A \mid \text{Hom}_A(M, \Sigma^p X) \text{ is finite-dimensional for all } p \in \mathbb{Z}
\text{ and vanishes for } p \gg 0 \}.
\]

All \( A^c, A_{fd} \) and \( A^{-fd} \) are thick subcategories of \( A \). By [15, Theorem 3.4], the category \( A^c \) is precisely the subcategory of compact objects of \( A \). Thus it is independent of the choice of \( M \). It follows that \( A_{fd} \) and \( A^{-fd} \) are independent of the choice of \( M \) as well. Note that \( A^c \) and \( A_{fd} \) are contained in \( A^{-fd} \).

**Proposition 3.1.** Keep the notation and assumptions in the preceding paragraph. Then

(a) both \( A^c \) and \( A_{fd} \) are Hom-finite and Krull–Schmidt,

(b) \((A^c, A_{fd}, M)\) is an ST-triple inside \( A^{-fd} \).

**Proof.** (a) follows from (b) by Theorem 3.3(a) below. It is clear that \( M \) is a silting object of \( A^c \). Let us prove (b) by verifying the three conditions in the definition of an ST-triple.

(ST1) This is true by the definition of \( A_{fd} \).

(ST2) Put

\[
A^{<0} = \{ X \in A \mid \text{Hom}_A(M, \Sigma^p X) = 0 \forall p > 0 \},
\]
\[
A^{\geq 0} = \{ X \in A \mid \text{Hom}_A(M, \Sigma^p X) = 0 \forall p < 0 \}.
\]

Then by [11, Theorem 1.3] (cf. also [7, Proposition 2.8] and [3, Corollary 4.7]), \((A^{<0}, A^{\geq 0})\) is a t-structure on \( A \). For \( X \in A \), consider the triangle

\[
X' \rightarrow X \rightarrow X'' \rightarrow \Sigma X'
\]

with \( X' \in A^{<0} \) and \( X'' \in A^{\geq 1} \). Then by applying the functor \( \text{Hom}_A(M, ?) \) to this triangle we obtain isomorphisms

\[
\text{Hom}_A(M, \Sigma^p X) \cong \text{Hom}_A(M, \Sigma^p X'') \forall p \geq 1,
\]
\[
\text{Hom}_A(M, \Sigma^p X) \cong \text{Hom}_A(M, \Sigma^p X') \forall p \leq 0.
\]
As a consequence, if $X$ belongs to $\mathcal{A}_{d_t}$, so do $X'$ and $X''$. Therefore $(\mathcal{A}_{d_t}^{-\leq 0}, \mathcal{A}_{d_t}^{-\geq 0})$ is a $t$-structure on $\mathcal{A}_{d_t}$, where $\mathcal{A}_{d_t}^{-\leq 0} = \mathcal{A}_{d_t}^{-} \cap \mathcal{A}^{\leq 0}$ and $\mathcal{A}_{d_t}^{-\geq 0} = \mathcal{A}_{d_t}^{-} \cap \mathcal{A}^{\geq 0}$ are the categories defined in the definition of an ST-triple.

( ST3) This is clear from the definitions of the involved categories. \qed

For a dg (=differential graded) $k$-algebra $A$, it is known that the derived category $\mathcal{D}(A)$ of dg $A$-modules \cite{9} has arbitrary coproducts and is compactly generated by $A_A$, see \cite{15} Section 3.5. Put \[
\per(A) = \text{thick}(A_A),
\]
\[
\mathcal{D}_{fd}(A) = \{ X \in \mathcal{D}(A) \mid \bigoplus_{p \in \mathbb{Z}} H^p(X) \text{ is finite-dimensional} \},
\]
\[
\mathcal{D}_{fd}(A) = \{ X \in \mathcal{D}(A) \mid H^p(X) \text{ is finite-dimensional for all } p \in \mathbb{Z} \text{ and vanishes for } p \gg 0 \}.
\]

**Proposition 3.2** \cite{1, Proposition 6.12}. Let $A$ be a dg $k$-algebra satisfying

\begin{enumerate}
\item[(N)] $H^p(A) = 0$ for any $p > 0$,
\item[(F)] $H^p(A)$ is finite-dimensional for any $p \in \mathbb{Z}$.
\end{enumerate}

Then

\begin{enumerate}
\item[(a)] both $\per(A)$ and $\mathcal{D}_{fd}(A)$ are Hom-finite and Krull–Schmidt,
\item[(b)] $(\per(A), \mathcal{D}_{fd}(A), A_A)$ is an ST-triple inside $\mathcal{D}_{fd}(A)$.
\end{enumerate}

**Proof.** This follows from Proposition \ref{p:derived} since $H^p(M) = \text{Hom}_{\mathcal{D}(A)}(A, \Sigma^p M)$ for a dg $A$-module $M$. \qed

Let $(\mathcal{C}, \mathcal{D}, M)$ be an ST-triple inside $\mathcal{T}$. Let $\mathcal{T}_{\geq 0}$ be the smallest strict and full subcategory of $\mathcal{T}$ which contains $M$ and is closed under extensions, direct summands and negative shifts, and let

\[
\mathcal{T}_{< 0} = \Sigma^{-1} \{ X \in \mathcal{T} \mid \text{Hom}_\mathcal{T}(Y, X) = 0 \forall Y \in \mathcal{T}_{\geq 0} \}.
\]

We collect some useful results in the following theorem.

**Theorem 3.3.** Let $(\mathcal{C}, \mathcal{D}, M)$ be an ST-triple inside $\mathcal{T}$.

\begin{enumerate}
\item[(a)] \cite{1, Remark 4.4(d)] Both $\mathcal{C}$ and $\mathcal{D}$ are Hom-finite and Krull–Schmidt.
\item[(b)] \cite{1, Proposition 4.6(e)] $\bigcap_{p \in \mathbb{Z}} \mathcal{T}^{-p} = 0$.
\item[(c)] \cite{1, Proposition 4.6] $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0}) := (\mathcal{T}^{< 0} \cap \mathcal{D}, \mathcal{T}^{> 0})$ is a bounded $t$-structure on $\mathcal{D}$ with heart $\mathcal{D}^{0} = \mathcal{T}^{0}$. The object $\sigma^0(M)$ is a projective generator of $\mathcal{D}^{0}$, which is equivalent to $\text{mod End}_\mathcal{T}(M)$.
\item[(d)] $(\mathcal{T}_{\geq 0}, \mathcal{T}_{< 0})$ is a co-$t$-structure on $\mathcal{T}$ with co-heart $\text{add}(M)$ and $\mathcal{T}_{\leq 0} = \mathcal{T}^{< 0}$.
\item[(e)] \cite{1, Remark 4.18] $(\mathcal{C}_{\geq 0}, \mathcal{C}_{< 0}) := (\mathcal{T}_{\geq 0}, \mathcal{T}_{< 0} \cap \mathcal{C})$ is a bounded co-$t$-structure on $\mathcal{C}$ with co-heart $\text{add}(M)$.
\end{enumerate}

The following result is \cite{1, Proposition 4.9}. By (ST3), there exists $r \in \mathbb{Z}$ such that $Y \in \mathcal{T}^{< r}$.
Lemma 3.4. Let $r \geq l$ be integers. For $Y \in \mathcal{T}^{\leq r}$, there exist $\beta_{\geq l}(Y) \in \mathcal{T}_{\geq l}$ and $\beta_{< l-1}(Y) \in \mathcal{T}^{\leq l-1}$ and a triangle

$$\beta_{\geq l}(Y) \xrightarrow{f_Y} Y \xrightarrow{g_Y} \beta_{< l-1}(Y) \xrightarrow{\Sigma} \beta_{\geq l}(Y)$$

with the following properties:

(a) $\beta_{\geq l}(Y) \in \Sigma^{-r}M^r \cdots \Sigma^{-l}M^l$ for some $M^r, \ldots, M^l \in \text{add}(M)$;
(b) for any simple object $S$ in $\mathcal{D}^0$ and for all $p \geq l$ the map

$$\text{Hom}_\mathcal{T}(Y, \Sigma^{-p}S) \xrightarrow{f_Y} \text{Hom}(\beta_{\geq l}(Y), \Sigma^{-p}S)$$

is an isomorphism and the two spaces are isomorphic to $\text{Hom}_\mathcal{T}(M^p, S)$;
(c) for any simple object $S$ in $\mathcal{D}^0$ and for all $p \leq l-1$ the map

$$\text{Hom}_\mathcal{T}(\beta_{< l-1}(Y), \Sigma^{-p}S) \xrightarrow{g_Y} \text{Hom}_\mathcal{T}(Y, \Sigma^{-p}S)$$

is an isomorphism.

The objects $\beta_{\geq l}(Y)$ and $\beta_{< l-1}(Y)$ are constructed inductively. The first step goes as follows: Take a minimal right $\text{add}(\Sigma^{-r}M)$-approximation $f: \Sigma^{-r}M^r \to Y$ and form a triangle

$$\Sigma^{-1}Y' \xrightarrow{h} \Sigma^{-r}M^r \xrightarrow{f} Y \xrightarrow{g} Y'.$$

Then $Y' \in \mathcal{T}^{\leq r-1}$ because $M$ is silting. The ‘minimality’ and uniqueness of $\beta_{\geq l}(Y)$ is established by inductively applying the following lemma. This is crucial in the definition of $\text{Sum}$ in Section 3.2.2. The ‘limit’ of $\beta_{\geq l}(Y)$ can be considered as a generalisation of minimal projective resolutions. Note, however, that in general $\beta_{\geq l}$ and $\beta_{< l-1}$ cannot be extended to functors.

Lemma 3.5. Let $Y \in \mathcal{T}^{\leq r}$. Assume that there is a triangle

$$\Sigma^{-1}Y'' \xrightarrow{h'} \Sigma^{-r}N^r \xrightarrow{f'} Y \xrightarrow{g'} Y''$$

with $N^r \in \text{add}(M)$. Then $f'$ is a right $\text{add}(\Sigma^{-r}M)$-approximation if and only if $Y'' \in \mathcal{T}^{\leq r-1}$. If these conditions hold, then $h'$ is the direct sum of $h$ with an isomorphism in $\text{add}(\Sigma^{-r}M)$ and (3.2) is the direct sum of (3.1) with a trivial triangle.

Proof. By inspection on the long exact sequence obtained by applying $\text{Hom}_\mathcal{T}(M, ?)$ to (3.2) we obtain the first statement. If the conditions hold, then $f'$ is the direct sum of $f$ with a morphism $\Sigma^{-r}L^r \to 0$ with $L^r \in \text{add}(M)$. The second statement follows.

Repeatedly applying Lemma 3.4, we obtain the following corollary.

Corollary 3.6. Let $Y \in \mathcal{T}^{\leq r}$. Assume $N^r, \ldots, N^l \in \text{add}(M)$ and $Y'' \in \mathcal{T}^{\leq l-1}$ with $Y \in \Sigma^{-r}N^r \cdots \Sigma^{-l}N^l \ast Y''$. Then $M^r, \ldots, M^l$ and $\beta_{< l-1}(Y)$ in Lemma 3.4 are direct summands of $N^r, \ldots, N^l$ and $Y'' \in \mathcal{T}^{\leq l-1}$, respectively.
3.2. Discretenesses. Let $\mathcal{T}$ be a triangulated $k$-category and $(\mathcal{C}, \mathcal{D}, M)$ an ST-triple inside $\mathcal{T}$. We recall two notions of discreteness introduced in [10].

Recall that $M$ is a silting object of $\mathcal{C}$ and on $\mathcal{D}$ there is a bounded $t$-structure $(\mathcal{D}^{<0}, \mathcal{D}^{\geq 0})$ with heart $\mathcal{D}^0$.

3.2.1. Discreteness with respect to the $t$-structure. For $X \in \mathcal{D}$, define

$$\text{Dim}(X) = (\dim_{\sigma^p}(X))_{p \in \mathbb{Z}} \in (K_0(\mathcal{D}^0)^+)_{\leq \mathbb{Z}}.$$

Lemma 3.7. Let $X' \rightarrow X \rightarrow X'' \rightarrow \Sigma X'$ be a triangle in $\mathcal{T}$. Then

$$\text{Dim}(\sigma^{\geq p}(X)) \leq \text{Dim}(\sigma^{\geq p}(X')) + \text{Dim}(\sigma^{\geq p}(X''))$$

for any $p \in \mathbb{Z}$.

Proof. This is because $\{\sigma^p | p \in \mathbb{Z}\}$ is cohomological. \qed

For $x = (x^p)_{p \in \mathbb{Z}}, y = (y^p)_{p \in \mathbb{Z}} \in K_0(\mathcal{D}^0)^{\leq \mathbb{Z}}$ define $y \leq x$ if $x^p - y^p \in K_0(\mathcal{D}^0)^+$ for all $p \in \mathbb{Z}$. For $x \in K_0(\mathcal{D}^0)^{\leq \mathbb{Z}}$, let $\text{Ind}^x \mathcal{D}$ (respectively, $\text{Ind}^{<x} \mathcal{D}$) be the isoclasses of indecomposable objects $X$ of $\mathcal{D}$ with $\text{Dim}(X) = x$ (respectively, $\text{Dim}(X) \leq x$).

Definition 3.8 ([10, Definition 2.1]). The category $\mathcal{D}$ is called $\mathcal{D}^0$-discrete if the set $\text{Ind}^x \mathcal{D}$ is finite for any $x \in K_0(\mathcal{D}^0)^{\leq \mathbb{Z}}$.

Example 3.9. Let $A$ be a finite-dimensional $k$-algebra. For $X \in D^b(\text{mod } A)$, define $\text{Dim}(X) = (\dim H^i(X))_{i \in \mathbb{Z}}$, which belongs to the cone $(K_0(\text{mod } A)^{\leq \mathbb{Z}})$. The algebra $A$ is called derived-discrete [20] if the number of isoclasses of indecomposable objects $X$ of $D^b(\text{mod } A)$ with $\text{Dim}(X) = x$ is finite for any $x \in K_0(\text{mod } A)^{\leq \mathbb{Z}}$. It is clear that this is exactly the $(\text{mod } A)$-discreteness in the sense of Definition 3.8. There is a classification of derived-discrete algebras in [20] and a description of the AR quivers of $K^b(\text{proj})$ and $D^b(\text{mod})$ in [8] (see also [13]).

Example 3.10. Let $A = k[t]$ with $\text{deg}(t) = -1$. We consider it as a dg algebra with trivial differential. Then $A$ satisfies the conditions (N) and (F) in Proposition 3.2. Take the ST-triple $(\mathcal{C}, \mathcal{D}, M) = (\text{per } A, \mathcal{D}_{fd}(A), A_A)$ inside $\mathcal{D}_{fd}(A)$. Then $\mathcal{D}^0$ is the semisimple abelian category with a unique simple object $S = A/(t)$ (up to isomorphism).

According to [17, Theorem 4.1(ii)] (see also [12, Lemma 8.8]), all indecomposable objects of $\mathcal{D}_{fd}(A)$ are of the form $\Sigma^m A/(t^l)$ ($m \in \mathbb{Z}$ and $l \in \mathbb{N}$). Put $x(m,l) = \Sigma_{p=m}^{m+l-1} \text{Dim}(\Sigma^p S)$. Then $\text{Dim}(\Sigma^m A/(t^l)) = x(m,l)$. Therefore we have

$$\#\text{Ind}^x \mathcal{D} = \begin{cases} 1 & \text{if } x = x(m,l) \text{ for some } m \in \mathbb{Z} \text{ and } l \in \mathbb{N}, \\ 0 & \text{otherwise}. \end{cases}$$

As a consequence, $\mathcal{D}$ is $\mathcal{D}^0$-discrete.

Lemma 3.11. The category $\mathcal{D}$ is $\mathcal{D}^0$-discrete if and only if the set $\text{Ind}^{<x} \mathcal{D}$ is finite for any $x \in K_0(\mathcal{D}^0)^{\leq \mathbb{Z}}$. 

Proof. The “if” part is obvious. The “only if” part follows from the equality
\[ \text{Ind}^{x}D = \bigcup_{0 \leq y \leq x} \text{Ind}^{y}D \]
and the fact that the number of \( y \) satisfying \( 0 \leq y \leq x \) is finite. \( \square \)

3.2.2. Discreteness with respect to the silting object. Take \( Y \in \mathcal{C} \). Then there exists
\( l \in \mathbb{Z} \) such that \( \beta_{l-1}(Y) = 0 \). Take \( M^{r}, \ldots, M^{l} \in \text{add}(M) \) as in Lemma 3.4
and put \( M^{p} = 0 \) if \( p > r \) or \( p < l \). Define
\[ \text{Sum}(Y) = (\sum(M^{p}))_{p \in \mathbb{Z}} \in (K_{0}^{\text{sp}}(\text{add}(M))^{+})^{\mathbb{Z}}, \]
where \( \sum \) is defined in Section 2.2.

**Lemma 3.12.** Let \( Y' \to Y \to Y'' \to \Sigma Y' \) be a triangle in \( \mathcal{T} \). Then for any \( l \in \mathbb{Z} \)
\[ \text{Sum}(\beta_{l}(Y)) \leq \text{Sum}(\beta_{l}(Y')) + \text{Sum}(\beta_{l}(Y'')). \]

**Proof.** According to Lemma 3.3 there exist \( L^{r}, \ldots, L^{l} \) and \( N^{t}, \ldots, N^{l} \) such that \( Y' \in \Sigma^{-t}L^{r} \cdots \Sigma^{-l}L^{l} * \beta_{l-1}(Y') \) and \( Y'' \in \Sigma^{-t}N^{r} \cdots \Sigma^{-l}N^{l} * \beta_{l-1}(Y'') \). It follows by induction that
\[ Y \in Y' \ast Y'' \subseteq \Sigma^{-t}(L^{r} \oplus N^{r}) \cdots \Sigma^{-l}(L^{l} \oplus N^{l}) \ast (\beta_{l-1}(Y') \ast \beta_{l-1}(Y'')), \]
because \( \beta_{l-1}(Y') \ast \Sigma^{-p}N^{p} = \beta_{l-1}(Y') \ast \Sigma^{-p}N^{p} \subseteq \Sigma^{-p}N^{p} \ast \beta_{l-1}(Y') \) and \( L^{p} \ast N^{p} = L^{p} \oplus N^{p} \) for \( l \leq p \leq r \). By Corollary 3.6 we obtain the desired result. \( \square \)

For \( u = (w^{p})_{p \in \mathbb{Z}}, v = (v^{p})_{p \in \mathbb{Z}} \in K_{0}^{\text{sp}}(\text{add}(M))^{\mathbb{Z}} \), define \( v \leq u \) if \( w^{p} - v^{p} \in K_{0}^{\text{sp}}(\text{add}(M))^{+} \) for all \( p \in \mathbb{Z} \). For \( u \in K_{0}^{\text{sp}}(\text{add}(M))^{\mathbb{Z}} \), let \( \text{Ind}_{u}\mathcal{C} \) (respectively, \( \text{Ind}_{<u}\mathcal{C} \)) be the set of isoclasses of indecomposable objects \( Y \) of \( \mathcal{C} \) with \( \text{Sum}(Y) = u \) (respectively, \( \text{Sum}(Y) \leq u \)).

**Definition 3.13.** The category \( \mathcal{C} \) is called \( M \)-discrete if the set \( \text{Ind}_{u}\mathcal{C} \) is finite for
any \( u \in K_{0}^{\text{sp}}(\text{add}(M))^{\mathbb{Z}} \).

**Example 3.14.** Let \( A = k[t] \) with \( \text{deg}(t) = -1 \) and consider it as a dg algebra with
trivial differential. Take the ST-triple \( (\text{per}(A), D_{fd}(A), A_{A}) \) inside \( D_{fd}(A) \).

According to [17, Theorem 4.1(ii)], all indecomposable objects of \( \text{per}(A) \) are of the form \( \Sigma^{m}A/(t^{l}) \) \((m \in \mathbb{Z} \text{ and } l \in \mathbb{N}) \) or \( \Sigma^{m}A_{A} \) \((m \in \mathbb{Z}) \). Put \( u(m) = \text{Sum}(\Sigma^{m}A_{A}) \) and put \( u(m, l) = u(m) + u(m + l) \). Then \( \text{Sum}(\Sigma^{m}A/(t^{l})) = u(m, l) \). Therefore we have
\[ \#\text{Ind}_{u}\text{per}(A) = \begin{cases} 1 & \text{if } u = u(m, l) \text{ for some } m \in \mathbb{Z} \text{ and } l \in \mathbb{N}, \\ 1 & \text{if } u = u(m) \text{ for some } m \in \mathbb{Z}, \\ 0 & \text{otherwise}. \end{cases} \]

As a consequence, \( \text{per}(A) \) is \( A_{A} \)-discrete.

The following result is dual to Lemma 3.11 and its proof is similar.

**Lemma 3.15.** The category \( \mathcal{C} \) is \( M \)-discrete if and only if the set \( \text{Ind}_{<u}\mathcal{C} \) is finite
for any \( u \in K_{0}^{\text{sp}}(\text{add}(M))^{\mathbb{Z}} \).
Remark 3.16. Using Lemma 3.15 one can show that $C$ is $M$-discrete if and only if it is discrete with respect to the co-$t$-structure $(C_{\geq 0}, C_{\leq 0})$ in the sense of [10] Definition 4.1.

4. THE TWO DISCRETENESSES ARE EQUIVALENT

Let $\mathcal{T}$ be a triangulated $k$-category and $(C, D, M)$ an ST-triple inside $\mathcal{T}$. In Section 3 we recalled two notions of discreteness in [10], one for $C$ and one for $D$. The following main result of this paper states that these two notions are equivalent. This has the flavour of Koszul duality.

Theorem 4.1. The category $C$ is $M$-discrete if and only if the category $D$ is $D^0$-discrete.

Corollary 4.2. If $C$ is $M$-discrete, then it is silting-discrete.

Proof. Assume that $C$ is $M$-discrete. Then $D$ is $D^0$-discrete by Theorem 4.1. The statement then follows from [1, Theorems 7.9 and 7.1].

We split Theorem 4.1 into two propositions and prove them in Sections 4.1 and 4.2 respectively. In Section 4.3 we discuss the relation between discreteness and cone-finiteness.

Recall that there is a triple $(\mathcal{T}_{\geq 0}, \mathcal{T}_{\leq 0} = \mathcal{T}_{\leq 0}, \mathcal{T}_{\geq 0}, \mathcal{T}_{\geq 0})$, where $(\mathcal{T}_{\geq 0}, \mathcal{T}_{\leq 0})$ is a co-$t$-structure and $(\mathcal{T}_{\leq 0}, \mathcal{T}_{\geq 0})$ is a $t$-structure. The two proofs below are almost dual to each other. The subtle but serious difference comes from the fact that truncations associated to $t$-structures are functorial while truncations associated to co-$t$-structures are not. However, the interplay between these truncations is interesting and plays an important role in the proofs.

4.1. $M$-discreteness implies $D^0$-discreteness. The aim of this subsection is to prove the following implication.

Proposition 4.3. If $C$ is $M$-discrete, then $D$ is $D^0$-discrete.

The following result is a direct consequence of Lemma 2.1.

Proposition 4.4. Let $p \in \mathbb{Z}$. The functor $\sigma^{\geq p}: \mathcal{T} \to \mathcal{T}^{\geq p}$ restricts to a fully faithful functor

$$\sigma^{\geq p}: \mathcal{T}_{\geq p} \to \mathcal{T}^{\geq p}.$$

Proof. Take $X, Y \in \mathcal{T}_{\geq p}$. Since $\sigma^{\leq p-1}(Y)$ and $\Sigma \sigma^{\leq p-1}(Y)$ belong to $\mathcal{T}^{\leq p-1}$, we have $\text{Hom}_{\mathcal{T}}(X, \sigma^{\leq p-1}(Y)) = 0 = \text{Hom}_{\mathcal{T}}(X, \Sigma \sigma^{\leq p-1}(Y))$. The desired result follows from Lemma 2.1.

We immediately obtain the following corollary, taking into account that $\mathcal{T}_{\geq p} \supseteq \mathcal{T}_{\geq l}$ for $p \leq l$.

Corollary 4.5. For $Y \in \mathcal{T}_{\geq l}$, the following are equivalent:

(i) $Y$ is indecomposable,
(ii) \( \sigma^{>p}(Y) \) is indecomposable for some \( p \leq l \),
(iii) \( \sigma^{>p}(Y) \) is indecomposable for all \( p \leq l \).

Moreover, for \( Y, Z \in T_{\geq l} \), the following are equivalent:

1. \( Y \cong Z \),
2. \( \sigma^{>p}(Y) \cong \sigma^{>p}(Z) \) for some \( p \leq l \),
3. \( \sigma^{>p}(Y) \cong \sigma^{>p}(Z) \) for all \( p \leq l \).

For \( l \in \mathbb{Z} \), consider the group homomorphism

\[
\varphi_l : K_{0}^{sp}(\text{add}(M))^{\oplus \mathbb{Z}} \rightarrow K_{0}(D^{0})^{\oplus \mathbb{Z}}
\]

defined by \( \sum(\Sigma^p N) \mapsto \dim(\sigma^{>l}(\Sigma^p N)) \) for \( N \in \text{add}(M) \) and \( p \in \mathbb{Z} \). It restricts to a map

\[
\varphi_l : (K_{0}^{sp}(\text{add}(M))^{+})^{\oplus \mathbb{Z}} \rightarrow (K_{0}(D^{0})^{+})^{\oplus \mathbb{Z}}.
\]

**Proof of Proposition 4.3.** Assume that \( D \) is \( D^{0} \)-discrete.

Take \( u \in (K_{0}^{sp}(\text{add}(M))^{+})^{\oplus \mathbb{Z}} \). Then there exist \( r, l \in \mathbb{Z} \) such that \( u^p = 0 \) for \( p < l \) and for \( p > r \). Put \( x = \varphi_l(u) \in (K_{0}(D^{0})^{+})^{\oplus \mathbb{Z}} \).

Let \( Y \in \text{Ind}_u C \), i.e. \( \sum(Y) = u \). Then by Lemma 3.4 there exist \( M^r, \ldots, M^l \in \text{add}(M) \) with \( \sum(M^p) = u^p \) such that \( Y \in \Sigma^{-r}M^r \ast \cdots \ast \Sigma^{-l}M^l \). By repeatedly applying Lemma 3.7, we obtain

\[
\dim(\sigma^{>l}(X)) \leq \dim(\sigma^{>l}(\Sigma^{-r}M^r)) + \cdots + \dim(\sigma^{>l}(\Sigma^{-l}M^l)) = \varphi_l(u) = x.
\]

Therefore by Corollary 4.3, there is an injective map

\[
\text{Ind}_u C \longrightarrow \text{Ind}^{<x} D.
\]

By Lemma 3.11, the \( D^{0} \)-discreteness of \( D \) implies that \( \text{Ind}^{<x} D \) is finite. It follows that \( \text{Ind}_u D \) is finite, as desired.

\[
\square
\]

4.2. \( D^{0} \)-discreteness implies \( M \)-discreteness. The aim of this subsection is to prove the following implication.

**Proposition 4.6.** If \( D \) is \( D^{0} \)-discrete, then \( C \) is \( M \)-discrete.

The key point of our proof is the following result, which, specialising to the ST-triple \((K^b(\text{proj} A), D^b(\text{mod} A), A_A)\), strengthens \cite[Proposition 2]{21}.

**Proposition 4.7.** Let \( l \in \mathbb{Z} \). For \( X \in D^{>l} \), there is a surjective algebra homomorphism

\[
\text{End}_T(\beta_{>l-1}(X)) \longrightarrow \text{End}_T(X),
\]

whose kernel is contained in the radical of \( \text{End}_T(\beta_{>l-1}(X)) \). As a consequence, the following are equivalent:

(i) \( X \) is indecomposable,
(ii) \( \beta_{>p}(X) \) is indecomposable for some \( p \leq l - 1 \),
(iii) $\beta_{2p}(X)$ is indecomposable for all $p \leq l - 1$.
Moreover, if $X, Y \in \mathcal{D}^{\leq l}$ satisfy $\beta_{2l-1}(X) = \beta_{2l-1}(Y)$, then $X \cong Y$.

**Proof.** Rotate the triangle in Lemma 3.4 we obtain a triangle
$$\Sigma^{-1}\beta_{\leq l-2}(X) \xrightarrow{h_X} \beta_{\leq l-1}(X) \xrightarrow{f_X} X \xrightarrow{g_X} \beta_{\leq l-2}(X).$$
Since $\Sigma^{-1}\beta_{\leq l-2}(X) \in T^{\leq l-1}$ and $X \in \mathcal{D}^{\leq l} = T^{\geq l}$, this is the canonical triangle of $\beta_{\leq l-1}(X)$ associated to the $t$-structure $(T^{\leq l-1}, T^{\geq l-1})$. In particular, $X \cong \sigma^p \beta_{\leq l-1}(X)$ and $\Sigma^{-1}\beta_{\leq l-2}(X) \cong \sigma^{\leq l-2}\beta_{\leq l-1}(X)$. The ‘Moreover’ part follows immediately.

Consider the algebra homomorphism induced by the functor $\sigma^p$
$$\text{End}_T(\beta_{\leq l-1}(X)) \rightarrow \text{End}_T(\sigma^p \beta_{\leq l-1}(X)) \cong \text{End}_T(X).$$

By Lemma 2.1 this homomorphism is surjective, because $\beta_{\leq l-1}(X) \in T^{\leq l-1}$ and $\Sigma \sigma^{\leq l-1}\beta_{\leq l-1}(X) \in T^{\leq l-2}$. Moreover, the kernel of this map is
$$I := \{a: \beta_{\leq l-1}(X) \rightarrow \beta_{\leq l-1}(X) \mid a \text{ factors through } h_X\}.$$ If $a \in I$ and $S$ is a simple object of $\mathcal{D}^0$, then $\text{Hom}(a, \Sigma^p S) = 0$ for all $p \leq -l + 1$ because $\text{Hom}(h_X, \Sigma^p S) = 0$. Moreover, $\text{Hom}_T(\beta_{\leq l-1}(X), \Sigma^p S) = 0$ for all $p > -l + 1$ because $\beta_{\leq l-1}(X) \in T_{\leq l-1}$. Therefore $\text{Hom}(a, \Sigma^p S) = 0$ for all $p \in \mathbb{Z}$. We claim that $a$ belongs to the radical. Otherwise, write $\beta_{\leq l-1}(X) = Y_1 \oplus \ldots \oplus Y_s$ with $Y_1, \ldots, Y_s$ indecomposable. Then $a$ has a summand $\lambda \cdot \text{id}_{Y_i}$ with $\lambda \in k^x$ for some $i = 1, \ldots, s$. Thus restricting $\lambda^{-1}a$ to $Y_i$ we obtain $\text{id}_{Y_i}$. It follows that $\text{Hom}(\text{id}_{Y_i}, \Sigma^p S) = 0$ for all $p \in \mathbb{Z}$, which implies that $Y_i \in \bigcap_{p \in \mathbb{Z}} T^{\leq p} = 0$, a contradiction. \hfill $\square$

For $l \in \mathbb{Z}$, consider the group homomorphism
$$\psi_l: K_0(\mathcal{D}^0)^{\oplus \mathbb{Z}} \rightarrow K_0^p(\text{add}(M))^{\oplus \mathbb{Z}}$$
defined by $\text{Dim}(\Sigma^p S) \mapsto \text{Sum}(\beta_{\leq l-1}(\Sigma^p S))$ for any simple object $S$ of $\mathcal{D}^0$ and any $p \in \mathbb{Z}$. It restricts to
$$\psi_l: (K_0(\mathcal{D}^0)^{\oplus \mathbb{Z}}) \rightarrow (K_0^p(\text{add}(M))^+)^{\oplus \mathbb{Z}}.$$

**Proof of Proposition 4.6** Assume that $\mathcal{C}$ is $M$-discrete.

Take $x \in (K_0(\mathcal{D}^0)^{\oplus \mathbb{Z}})$. Let $l$ be the maximal integer such that $x^p = 0$ for all $p < l$ and put $u = \psi_l(x)$. We will define a map
$$h: \text{Ind}^x \mathcal{D} \rightarrow \text{Ind} u \mathcal{C}$$
and show that it is injective. By Lemma 3.15 the $M$-discreteness of $\mathcal{C}$ implies that $\text{Ind} u \mathcal{C}$ is finite. It follows that $\text{Ind}^x \mathcal{D}$ is finite, as desired.

**Step 1:** The definition and injectivity of $h$. Let $X \in \mathcal{D}$ be indecomposable with $\text{Dim}(X) = x$. Define $h(X) = \beta_{\leq l-1}(X)$. By Proposition 4.7, $h(X)$ is indecomposable and $h$ is injective.

**Step 2:** The well-definedness of $h$. Let $X \in \mathcal{D}$ be indecomposable with $\text{Dim}(X) = x$. We show by induction on $x$ that $\text{Sum}(\beta_{\leq l-1}(X)) \leq u$. If $X$ is a shift of a simple
object of $\mathcal{D}^0$, the inequality holds by the definition of $\psi_l$. Otherwise, take a simple subobject $S$ of $\sigma^l(X)$, consider the composition

$$\Sigma^{-l}S \rightarrow \Sigma^{-l}\sigma^l(X) \rightarrow X,$$

and form a triangle

$$\Sigma^{-l}S \rightarrow X \rightarrow X' \rightarrow \Sigma^{-l+1}S.$$ 

It follows from the octahedron axiom that $x = x' + x''$, where $x' = \text{Dim}(X')$ and $x'' = \text{Dim}(\Sigma^{-l}S)$. Thus

$$\text{Sum}(\beta_{\geq l-1}(X)) \leq \text{Sum}(\beta_{\geq l-1}(\Sigma^{-l}S)) + \text{Sum}(\beta_{l-1}(X'))$$

$$\leq \psi_l(x'') + \psi_l(x') = \psi_l(x) = u,$$

where the first inequality follows from Lemma 3.12 and the second one follows from induction hypothesis. □

4.3. Cone-finiteness. Following [10], we say that a triangulated category is \textit{cone finite} if for any two objects $X$ and $Y$, the subcategory $X \ast Y$ has only finitely many isoclasses of objects. Note that this is a property that passes to subcategories.

Corollary 4.8. The following conditions are equivalent:

(i) $C$ is $M$-discrete,
(ii) $C$ is cone finite,
(iii) $D$ is $D^0$-discrete,
(iv) $D$ is cone finite.

Proof. (i)$\iff$(ii) is [10] Theorem 4.2, (i)$\iff$(iii) is Theorem 4.1 and (iii)$\Rightarrow$(iv) is [10] Theorem 2.5(iii)].

(iv)$\Rightarrow$(iii): Assume that $D$ is cone finite. We claim that for any $x \in K_0(\mathcal{D}^0)$ the number of isoclasses of objects $X$ in $\mathcal{D}^0$ with $\text{dim}(X) = x$ is finite, \emph{i.e.} $\mathcal{D}^0$ is abelian discrete in the sense of [10] Section 2. Then it follows from [10] Corollary 2.6 that $\mathcal{D}$ is $\mathcal{D}^0$-discrete.

We prove the claim by induction on $x$. If $x$ is a standard basis element of $K_0(\mathcal{D}^0) \cong \mathbb{Z}^n$, then $X$ must be simple and the claim is true. In general, take a simple subobject $S$ of $X$ and form the short exact sequence $0 \rightarrow S \rightarrow X \rightarrow X' \rightarrow 0$. Then $x = \text{dim}(S) + \text{dim}(X')$. Moreover, the above short exact sequence yields a triangle $S \rightarrow X \rightarrow X' \rightarrow \Sigma S$ in $\mathcal{D}$, and hence $X \in S \ast X'$. Thus all objects $X$ of $\mathcal{D}^0$ with $\text{dim}(X) = x$ belong to the subcategory $\mathcal{X} = \bigcup S \ast X'$, where the union is over all isoclasses of simple objects $S$ of $\mathcal{D}^0$ and all isoclasses of objects $X'$ with $\text{dim}(X') = x - \text{dim}(S)$.

By induction hypothesis, this is a finite union. Since $\mathcal{D}$ is cone finite, each $S \ast X'$ has finitely many isoclasses of objects. It follows that $\mathcal{X}$ has only finitely many isoclasses of objects and the claim is true. □

Corollary 4.8 shows the validity of [10] Conjecture 2.7(iv)] in our setting.
5. Derived-discreteness along decollements

In this section we recall the notion of derived-discreteness of a finite-dimensional algebra due to Vossieck [20], and apply Theorem 4.1 to recover the following result due to Qin [19]. For basics on recollements, we refer to [4].

Proposition 5.1 ([19, Proposition 6]). Let $A, B, C$ be finite-dimensional $k$-algebras and assume that there is a recollement of $\mathcal{D}(A)$ by $\mathcal{D}(B)$ and $\mathcal{D}(C)$. If $A$ is derived-discrete, then so are $B$ and $C$.

5.1. Derived-discreteness. Let $A$ be a finite-dimensional $k$-algebra.

For $X \in K^b(\text{proj} A)$, take $Y$ minimal such that $Y \simeq X$ in $K^b(\text{proj} A)$. Define $\text{Sum}(X) = (\sum(Y^p))_{p \in \mathbb{Z}}$, which belongs to $(K^0(\text{proj} A)^+)_{\mathbb{Z}}$. The algebra $A$ is called $K^b(\text{proj})$-discrete if the number of isoclasses of indecomposable objects of $K^b(\text{proj} A)$ with $\text{Sum}(X) = u$ is finite for any $u \in K^0(\text{proj} A)^{\mathbb{Z}}$. It is easy to see that this is exactly the $A_A$-discreteness in the sense of Definition 3.13.

Applying Corollary 4.8 to the ST-triple $(K^b(\text{proj} A), D^b(\text{mod} A), A_A)$, we immediately obtain the following corollary which completes [10, Corollary 4.4].

Corollary 5.2. The following conditions are equivalent:

(i) $A$ is $K^b(\text{proj})$-discrete,
(ii) $K^b(\text{proj} A)$ is cone finite,
(iii) $A$ is derived-discrete,
(iv) $D^b(\text{mod} A)$ is cone finite.

5.2. Derived-discreteness is preserved along decollements. In this subsection we prove Proposition 5.1.

Proof of Proposition 5.1. Assume that $A$ is derived-discrete. By Corollary 5.2, both $D^b(\text{mod} A)$ and $K^b(\text{proj} A)$ are cone finite. In the given recollement the middle left functor restricts to a fully faithful triangle functor $D^b(\text{mod} B) \to D^b(\text{mod} A)$ and the upper right functor restricts to a fully faithful triangle functor $K^b(\text{proj} C) \to K^b(\text{proj} A)$. Hence both $D^b(\text{mod} B)$ and $K^b(\text{proj} C)$ are cone finite. By Corollary 5.2 again, both $B$ and $C$ are derived-discrete. \qed

In the rest of this subsection we give an alternative proof of Proposition 5.1 using the equivalence Corollary 5.2(i) $\iff$ (iii) only. Note that using the full Corollary 5.2 both Lemma 5.3 and Lemma 5.4 are easy to obtain.

The alternative proof for $B$ being derived-discrete is the same as that in [19], which relies on the following result appeared in the paragraph before [19, Proposition 6]. The idea of the proof is the same as Vossieck’s proof of the fact that derived-discreteness is preserved under derived equivalence ([20, Proposition 1.1]). Here we give full details. Let $A$ and $B$ be finite-dimensional $k$-algebras in the next two lemmas.

Lemma 5.3. Assume that $F : D^b(\text{mod} B) \to D^b(\text{mod} A)$ is a fully faithful triangle functor. If $A$ is derived-discrete, so is $B$. 
Proof. The triangle functor $F$ induces a group homomorphism
\[ f: K_0(\text{mod}B)\otimes\mathbb{Z} \rightarrow K_0(\text{mod}A)\otimes\mathbb{Z} \]
such that $f(\text{dim}(\Sigma^pS_i^B)) = \text{dim}(\Sigma^pF(S_i^B))$ for a complete set $\{S_i^B\}$ of simple $B$-modules and $p \in \mathbb{Z}$.

We claim that $\text{dim}F(X) \leq f(\text{dim}(X))$ for any $X \in \mathcal{D}^b(\text{mod}B)$. It follows that $F$ induces an injective map $\text{Ind}^\omega\mathcal{D}^b(\text{mod} B) \rightarrow \text{Ind}^\omega f(\mathcal{D}^b(\text{mod} A))$, which is a finite set due to the derived-discreteness of $A$ and Lemma 3.11. Thus $B$ is derived-discrete.

We prove the claim by induction on $x := \text{dim}(X)$. Recall from Step 2 of the proof of Proposition 4.6 that there is a triangle in $\mathcal{D}^b(\text{mod} A)$:
\[ \Sigma^{-1}X' \xrightarrow{g} \Sigma^{-1}S \xrightarrow{f} X \rightarrow X' \]
such that $x = x' + x''$, where $x' = \text{dim}(X')$ and $x'' = \text{dim}(\Sigma^{-1}S)$. By applying $F$ to this triangle and inspecting the associated long exact sequence of cohomologies, we obtain an inequality
\[ \text{dim}F(X) \leq \text{dim}F(\Sigma^{-1}S) + \text{dim}F(X'). \]

By induction hypothesis we have $\text{dim}F(X') \leq f(x')$. Since $\text{dim}F(\Sigma^{-1}S) = f(x'')$, it follows that $\text{dim}F(X) \leq f(x)$, as claimed. □

The following result is dual to Lemma 5.3.

Lemma 5.4. Assume that $G: K^b(\text{proj} B) \rightarrow K^b(\text{proj} A)$ is a fully faithful triangle functor. If $A$ is $K^b(\text{proj})$-discrete, so is $B$.

Proof. The triangle functor $G: K^b(\text{proj} B) \rightarrow K^b(\text{proj} A)$ induces a group homomorphism
\[ g: K^b_0(\text{proj} B)\otimes\mathbb{Z} \rightarrow K^b_0(\text{proj} A)\otimes\mathbb{Z} \]
such that $g(\text{sum}(\Sigma^pP_i)) = \text{sum}(\Sigma^pG(P_i))$ for a complete set $\{P_i^B\}$ of indecomposable projective $B$-modules and $p \in \mathbb{Z}$.

We claim that $\text{sum}G(X) \leq g(\text{sum}(X))$ for any $X \in K^b(\text{proj} B)$. It follows that $G$ induces an injective map $\text{Ind}_{\omega}K^b(\text{proj} B) \rightarrow \text{Ind}_{\omega}G(K^b(\text{proj} A))$, which is a finite set due to the $K^b(\text{proj})$-discreteness of $A$ and Lemma 3.15. Thus $B$ is $K^b(\text{proj})$-discrete.

We prove the claim by induction on $u := \text{sum}(X)$. We may assume that $X$ is minimal. Let $r$ be the minimal integer such that $X^i = 0$ for all $i > r$ and take an indecomposable direct summand $P$ of $X^r$. Then $\Sigma^{-r}P$ is a subcomplex of $X$ and there is a triangle
\[ \Sigma^{-r}P \rightarrow X \rightarrow X' \rightarrow \Sigma^{-r+1}P \]
with $\text{sum}(X') = \text{sum}(X) - \text{sum}(\Sigma^{-r}P)$. Applying $G$ to this triangle yields a triangle in $K^b(\text{proj} A)$
\[ G(\Sigma^{-r}P) \rightarrow G(X) \rightarrow G(X') \rightarrow G(\Sigma^{-r+1}P). \]
Therefore
\[
\text{Sum} G(X) \leq \text{Sum}(G(\Sigma^{-r} P)) + \text{Sum}(G(X')) \\
\leq g(\text{Sum}(\Sigma^{-r} P)) + g(\text{Sum}(X')) \\
= g(\text{Sum}(\Sigma^{-r} P) + \text{Sum}(X')) \\
= g(\text{Sum}(X)).
\]

Here the first inequality follows from Lemma 3.12 and the second one is by induction hypothesis.

**Alternative proof of Proposition 5.1.** The middle left functor in the given recollement restricts to a fully faithful triangle functor \( D^b(\text{mod } B) \to D^b(\text{mod } A) \). So by Lemma 5.3, \( B \) is derived-discrete.

Similarly, there is a fully faithful triangle functor \( K^b(\text{proj } C) \to K^b(\text{proj } A) \). By Corollary 5.2(i)⇔(iii), \( A \) is \( K^b(\text{proj}) \)-discrete. It follows from Lemma 5.4 that \( C \) is \( K^b(\text{proj}) \)-discrete. By Corollary 5.2(i)⇔(iii) again, \( C \) is derived-discrete. \( \square \)

**References**

[1] Takahide Adachi, Yuya Mizuno, and Dong Yang, *Discreteness of silting objects and t-structures of triangulated categories*, Proc. Lond. Math. Soc., in press, doi:10.1112/plms.12176.
[2] Takuma Aihara, *Tilting-connected symmetric algebras*, Algebr. Represent. Theory 6 (2013), no. 3, 873–894.
[3] Takuma Aihara and Osamu Iyama, *Silting mutation in triangulated categories*, J. Lond. Math. Soc. (2) 85 (2012), no. 3, 633–668.
[4] Lidia Angeleri Hügel, Steffen Koenig, Qunhua Liu, and Dong Yang, *Ladders and simplicity of derived module categories*, J. Algebra 472 (2017), 15–66.
[5] Ibrahim Assem, María José Souto Salorio, and Sonia Trepode, *Ext-projectives in suspended subcategories*, J. Pure Appl. Algebra 212 (2008), no. 2, 423–434.
[6] Alexander A. Beilinson, Joseph Bernstein, and Pierre Deligne, *Faisceaux pervers*, Astérisque, vol. 100, Soc. Math. France, 1982 (French).
[7] Apostolos Beligiannis and Idun Reiten, *Homological and homotopical aspects of torsion theories*, Mem. Amer. Math. Soc. 188 (2007), no. 883, viii+207.
[8] Grzegorz Bobiński, Christof Geiß, and Andrzej Skowroński, *Classification of discrete derived categories*, Cent. Eur. J. Math. 2 (2004), no. 1, 19–49 (electronic).
[9] Mikhail V. Bondarko, *Weight structures vs. t-structures; weight filtrations, spectral sequences, and complexes (for motives and in general)*, J. K-Theory 6 (2010), no. 3, 387–504.
[10] Nathan Broomhead, David Pauksztello, and David Ploog, *Discrete triangulated categories*, Bull. London Math. Soc. 50 (2018), 174–188.
[11] Mitsuo Hoshino, Yoshiaki Kato, and Jun-Ichi Miyachi, *On t-structures and torsion theories induced by compact objects*, J. Pure Appl. Algebra 167 (2002), no. 1, 15–35.
[12] Peter Jørgensen, *Auslander-Reiten theory over topological spaces*, Comment. Math. Helv. 79 (2004), no. 1, 160–182.
[13] Martin Kalck and Dong Yang, *Derived categories of graded gentle one-cycle algebras*, J. Pure Appl. Algebra 222 (2018), 3005–3035.
[14] Bernhard Keller, *Deriving DG categories*, Ann. Sci. École Norm. Sup. (4) 27 (1994), no. 1, 63–102.
[15] ______, *Tilting and derived categories*, Contribution to the Handbook of Tilting Theory, edited by L. Angeleri, D. Happel and H. Krause, to appear.

[16] Bernhard Keller and Dieter Vossieck, *Aisles in derived categories*, Bull. Soc. Math. Belg. Sér. A 40 (1988), no. 2, 239–253.

[17] Bernhard Keller, Dong Yang, and Guodong Zhou, *The Hall algebra of a spherical object*, J. Lond. Math. Soc. (2) 80 (2009), no. 3, 771–784.

[18] David Pauksztello, *Compact corigid objects in triangulated categories and co-t-structures*, Cent. Eur. J. Math. 6 (2008), no. 1, 25–42.

[19] Yongyun Qin, *Jordan–Hölder theorems for derived categories of derived discrete algebras*, J. Algebra 461 (2016), 295–313.

[20] Dieter Vossieck, *The algebras with discrete derived category*, J. Algebra 243 (2001), no. 1, 168–176.

[21] Chao Zhang and Yang Han, *Brauer-Thrall type theorems for derived module categories*, Algebr. Represent. Theory 19 (2016), no. 6, 1369–1386.

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