THE CONTAINMENT PROBLEM AND A RATIONAL SIMPLICIAL ARRANGEMENT

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Abstract. Since Dumnicki, Szemberg, and Tutaj-Gasińska gave in 2013 in [11] the first example of a set of points in the complex projective plane such that for its homogeneous ideal $I$ the containment of the third symbolic power in the second ordinary power fails, there has been considerable interest in searching for further examples with this property and investigating the nature of such examples. Many examples, defined over various fields, have been found but so far there has been essentially just one example found of 19 points defined over the rationals, see [18, Theorem A, Problem 1]. In [14, Problem 5.1] the authors asked if there are other rational examples. This has motivated our research. The purpose of this note is to flag the existence of a new example of a set of 49 rational points with the same non-containment property for powers of its homogeneous ideal. Here we establish the existence and justify it computationally. A more conceptual proof, based on Seceleanu's criterion [22] will be published elsewhere [19].

1. Introduction

The begin of the Millennium has brought groundbreaking results on the containment relation between symbolic and ordinary powers of homogeneous ideals established by Ein-Lazarsfeld-Smith [12] in characteristic 0 and Hochster-Huneke [17] in positive characteristic.

Let $I$ be a homogeneous ideal in the ring of polynomials $R = \mathbb{K}[x_0, \ldots, x_N]$ defined over a field $\mathbb{K}$. For a positive integer $m$ one defines the $m$th symbolic power $I^{(m)}$ of $I$ as

$$I^{(m)} = \bigcap_{P \in \text{Ass}(I)} (I^m R_P \cap R),$$

where $\text{Ass}(I)$ is the set of associated primes of $I$. A lot of research has been motivated in recent two decades by the following central question.

**Problem 1.1** (The containment problem). Decide for which $m$ and $r$ there is the containment

$$I^{(m)} \subset I^r.$$
Whereas to decide the reverse containment, i.e., to decide when

\[ I^r \subset I^{(m)} \]

holds is almost elementary: it does if and only if \( m \leq r \) [4, Lemma 8.4.1], the elegant uniform answer to Problem 1.1, provided in the following Theorem, came as a surprise.

**Theorem 1.2** (Ein-Lazarsfeld-Smith, Hochster-Huneke). Let \( I \subset \mathbb{K}[x_0, \ldots, x_N] \) be a homogeneous ideal such that every component of its zero locus \( V(I) \) has codimension at most \( e \). Then the containment

\[ I^{(m)} \subset I^r \]

holds for all \( m \geq er \).

It is natural to wonder to what extend the bound in Theorem 1.2 is optimal. A lot of attention has been given to this problem in general. Here we focus on the simplest non-trivial situation. If \( I \) is an ideal of points in \( \mathbb{P}^2 \), then Theorem 1.2 asserts that there is always the containment

\[ I^{(4)} \subset I^2. \]

It is also elementary to provide examples when the containment \( I^{(2)} \subset I^2 \) fails (three general points in \( \mathbb{P}^2 \) are sufficient). The question if in the remaining case, i.e.,

\[ I^{(3)} \subset I^2, \]

the containment always holds has been raised by Huneke and repeated in the literature by several authors who proved a considerable number of special cases. However, in 2013, Dumnicki, Szemberg, and Tutaj-Gasińska showed in [11] that the containment in (3) fails for an explicit set of 12 points in the complex projective plane, the points dual to the Hesse arrangement of lines, see [1]. These points arise also as intersection points of certain arrangement of 9 lines. These lines intersect by 3 in configuration points and there are no other intersection points among them. This the only known non-trivial arrangement of lines in characteristic 0 where each intersection point of a pair of lines lies on exactly three distinct lines.

Soon after [11] became public, series of additional examples in all characteristics have been constructed, see, e.g., [2, 7, 16, 18, 21, 23] for an overview. All these constructions were based on the same idea of checking the containment for the set of intersection points of multiplicity at least 3 (at least 3 lines pass through a point) of some special arrangements of lines. The relevance of points of multiplicity at least 3 follows from the Zariski-Nagata Theorem, see [13, Theorem 3.14], which, at least in characteristic 0 characterizes the \( m^{\text{th}} \) symbolic power of a radical ideal \( I \) as the set of polynomials vanishing to order at least \( m \) along the set of zeroes of \( I \). Among these arrangements, a prominent role is played by arrangements coming from finite reflection groups, see, e.g., [3]. Finding configurations of rational points turned out to be a little bit harder. The only known example of 19 points arising as intersection points of multiplicity at least 3 of an arrangement of 12 lines has been introduced in [10], see also the recent article [18] by Lampa-Baczyńska and the second author for detailed description. In this note we exhibit an additional example.
2. The simplicial arrangement $A(25, 2)$

An arrangement of lines is a complex generated in the real projective plane by a finite family of lines that do not form a pencil. An arrangement is simplicial if all its faces (connected components of the complement of the union of lines) are triangles. Apparently, simplicial arrangements have been first studied by Melchior in connection with his proof of the celebrated Sylvester-Gallai Theorem, see [20]. They have received considerable attention after Deligne’s paper [9]. Recently Grünbaum [15] classified all known simplicial arrangement. Cuntz pointed out that there might be gaps in Grünbaum’s list, constructed 4 new examples and provided a conceptual proof of the completeness of the classification for arrangements of up to 27 lines. The classification in general seems to be an open and challenging problem.

In the present paper we study the arrangement denoted as $A(25, 2)$ in Grünbaum’s list [15, p. 20]. The same notation is used by Cuntz [6, p. 699]. This arrangement is defined by 25 lines. They intersect altogether in 85 points of which 49 have multiplicity at least 3. This is the set $Z$ of these 49 points that we are interested in. We denote the ideal of $Z$ by $I$. The arrangement can be defined in the rational projective plane and exhibits symmetries of a square. In Figure 1 there are 24 lines from the $A(25, 2)$ arrangement visualized. An invisible line is the line at infinity. The product of the equations of all 25 lines defines an element in $I^{(3)}$ which is not contained in $I^2$, hence the containment in (3) fails for $Z$. There are 41 points from 49 points in $Z$ visible in Figure 1. The remaining 8 points belong to the line at infinity.

![Figure 1. Affine part of the simplicial arrangement $A(25, 2)$](image-url)
Below we list coordinates of all points in \( Z \). In order to alleviate notation we use the \( \pm \) notation, which indicates that all possible signs should be considered.

\[
(0 : 0 : 1), \quad (0 : \pm 12 : 1), \quad (\pm 12 : 0 : 1), \quad (\pm 6 : \pm 6 : 1), \quad (\pm 6 : \pm 18 : 1),
\]
\[
(\pm 5 : 0 : 1), \quad (0 : \pm 3 : 1), \quad (\pm 6 : 0 : 1), \quad (0 : \pm 6 : 1), \quad (\pm 2 : \pm 2 : 1),
\]
\[
(\pm 18 : \pm 6 : 1), \quad (\pm 18 : \pm 18 : 1), \quad (\pm 18 : \pm 30 : 1) \quad (\pm 30 : \pm 18 : 1).
\]

Points at infinity are

\[
(1 : 0 : 0), \quad (0 : 1 : 0), \quad (1 : -1 : 0), \quad (1 : 1 : 0),
\]
\[
(1 : -2 : 0), \quad (1 : 2 : 0), \quad (-2 : 1 : 0), \quad (2 : 1 : 0).
\]

In order to list the arrangement lines we use the same \( \pm \) convention. The line \( z = 0 \) is the line at infinity.

\[
x, \quad y, \quad z, \quad x \pm y, \quad x \pm 6z, \quad y \pm 6z, \quad x \pm 18z, \quad y \pm 18z,
\]
\[
y \pm 2x \pm 6z, \quad y \pm x \pm 12z, \quad 2y \pm x \pm 6z.
\]

It is easy to check with a symbolic algebra program, we used Singular [8], that \( I \) is generated by 3 reducible octics:

\[
f := yz(3553x^6 - 15102x^4y^2 + 11549x^2y^4 + y^6 + 385x^4z^2 - 560x^2y^2z^2
\]
\[
- 189y^4z^2 + 6300x^2z^4 + 6804y^2z^4 - 14665z^6),
\]
\[
g := xz(-11426x^6 + 4001x^4y^2 - 4002x^2y^4 + y^6 - 15874x^4z^2 + 360x^2y^2z^2
\]
\[
+ 15748y^4z^2 - 4374x^2z^4 - 4050y^2z^4 - 6568z^6),
\]
\[
h := xy(x + y)(x - y)(1819x^4 + 267x^2y^2 + 1819y^4 + 2404x^2z^2 + 2404y^2z^2 + z^4).
\]

Since \( I \) has three generators of the same degree, the containment criterion of Seceleanu [22, Theorem 3.1] applies. As the details are rather tedious we postpone them to the forthcoming note [19].

3. Bounded Negativity

The referee kindly pointed out that our result is also of interest from the point of view of the Bounded Negativity Conjecture (BNC for short), which predicts that on every smooth complex surface \( X \) the self intersection of reduced and irreducible curves cannot be arbitrarily small, i.e., for all such curves it is bounded from below by a number \( \delta(X) \) depending on the particular surface only, see [5]. In particular, they introduced the so-called \( H \)-constants of \( X \), a way to capture the relation between the number \( \delta(X) \) and \( \delta(Y) \), provided there is a birational map \( f : X \to Y \), see [2, Definition 2.1].

For an arrangement \( \mathcal{L} \) of lines [2] introduces the linear \( H \)-constant, see [2, Definition 3.1] as

\[
H_{L}(\mathcal{P}) = \frac{d^2 - \sum_{i=1}^{s} m_{P_{i}}(\mathcal{L})^2}{s},
\]

where \( \mathcal{P} = \{P_{1}, \ldots, P_{s}\} \) is the set of points where at least 2 lines from \( \mathcal{L} \) meet and \( m_{P_{i}}(\mathcal{L}) \) is the number of lines which pass through the point \( P_{i} \) for \( i = 1, \ldots, s \).

In the case of the rational example from [10] we have 12 lines which intersect in 9 double points and 19 triple points, hence \( s = 28 \) and

\[
H_{L} = \frac{-9}{4} = -2.25
\]

in this case.
In the $A(25, 2)$ arrangement, studied here, there are 25 lines which intersect in 36 double, 28 triple, 15 quadruple, and 6 sextuple points, hence $s = 85$ and
\[ H_L = -\frac{227}{85} = -2.67 \]
in this case. The lower the $H_L$-constant, the more interesting the arrangement.

It has been proved in [2, Theorem 3.15] that the least possible value of $H_L$-constants in the real projective plane is $-3$. It would be interesting to know if $-3$ can be approached by rational arrangements and, as the referee suggests, simplicial arrangements are the natural class of line arrangements to look at. We will come back to this problem in [19].

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