Pressure-driven instabilities in astrophysical jets

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The enormous distances over which astrophysical jets propagate without losing their coherence certainly constitute one of the most striking features of these objects. Typically, jets from Young Stellar Objects (hereafter YSOs) do reach out to a few parsecs, while the radial extent of their region of origin appears to be smaller than \( \sim 100 \) A.U, making jets extremely elongated structures.

Blandford and Rees [6] already pointed out in their 1974 pioneering work that in laboratory experiments, jets do not propagate much further than about ten times their radii, which makes the propagation lengths of astrophysical jets all the more impressive. The simplest way out of this conundrum would be to assume that jets are ballistic. Indeed, for YSO jets at least, the observed opening angle (\( \sim 5^\circ \)) is consistent with the idea that they freely expand when one compares their thermal and bulk velocities. However, this option leaves open the issue of the formation of such powerful jets in the first place. And, as critically, the ballistic hypothesis does not explain how these jets survive the development of the Kelvin-Helmholtz instability, which is now known to be quite disruptive in purely hydrodynamic jets [7] [8].

The shortcomings of the simple ballistic picture certainly motivated to some extent the elaboration of MHD jet models. Such models, however, are also prone to instabilities. The most important ones discussed in the literature can be grouped into three categories:

- MHD Kelvin-Helmholtz instability. As for its HD counterpart, the driving agent is the velocity gradient at the jet/external medium interface. This instability has received a lot of attention in the literature, as the largest source of free energy in a jet is its bulk motion (see [5] and [14] for reviews).
- Conversely, the presence of a magnetic field provides a source of instability even in the absence of bulk motion. Ideal MHD instabilities are commonly divided into current- and pressure-driven, according to the driving factor
(equilibrium parallel current in the first case, and gas pressure versus field-line curvature in the second).

- Radiative instabilities, related to the coupling of the radiation field and of the plasma dynamical quantities.

The structure of jets is not precisely known, which is one of the difficulties in analyzing their stability. Most stability analyses assume that jets can be described as some sort of cylindrical column in motion, pervaded by a magnetic field. Self-collimated jet models are not exactly cylindrical, but as the observed opening angles are small, the assumption of cylindrical shape is not expected to be a major limitation.

More critically, such jet models have a helical field structure, with the azimuthal component of the magnetic field dominating over the vertical one in the outer jet regions. This follows in most models because the magnetic tension is the confining force ensuring self-collimation. Static MHD columns (i.e., not subject to the bulk motion characterizing MHD jets) pervaded by a helical magnetic field are referred to as “screw pinches” in the fusion literature. It is also known in this context that the dominance of the azimuthal field component leads to both types of MHD instabilities mentioned above, and may cause the disruption of the plasma column itself on a few dynamical time-scales. This has long been an argument against magnetically self-confined jet models. However, recent investigations indicate that a bulk motion can play an important stabilizing role (see section 5 for pressure-driven instabilities, and e.g. [1] and references therein for current-driven instabilities). Conversely, the presence of a magnetic field can help stabilizing the Kelvin-Helmholtz modes [24], [2]. These recent advances seem to indicate that a sophisticated equilibrium jet structure is required if one is to understand jet stability properties, a state of development not yet reached by the subject, but that now appears to be within sight.

To conclude these introductory remarks, I would like to point out that, in the nonlinear phase, an instability can have three broad types of outcome: i/ disruption of the fluid configuration (in the case at hand, of the jet as a jet); ii/ internal reorganization, the flow becoming laminar again in the end; iii/ turbulence (with or without internal reorganization of the structure). The most prominent objective of the study of jet stability is to understand how the first issue is avoided in real jets; this issue may well be seen as our inability to formulate the initial value problem correctly. A second but important issue is to understand how turbulence might be driven by jet destabilization. This issue is probably more important in AGN than in YSO jets, as turbulence is often invoked in the former context as a source of high energy particle acceleration.

The object of these lecture notes is pressure-driven instabilities. As most investigations of this problem have been made in the fusion context for static columns, this essential aspect of the subject will first be reviewed, before briefly presenting the more recent (and more scant) results on moving
columns. The next section presents some general ideas about the physical origin of MHD instabilities; the concept of magnetic shear is introduced there as well, and its stabilizing role on pressure-driven instabilities, expressed by Suydam criterion, is discussed. Section 2 introduces the Lagrangian form of the perturbation equations used in static columns, as this formulation is the most powerful to derive general results, such as those derived from the Energy Principle presented in section 3. Section 4 presents the dispersion relation of pressure-driven instabilities in low magnetic shear that are expected to characterize jets outer regions. The few published results on moving columns (i.e., jets) are presented in section 5. The last section summarizes the present state of understanding of this aspect of jet stability, and outlines areas where improvement is needed.

Sections 1 and 5 are intended for a general audience, while sections 2, 3 and 4 are more theoretical in nature. The exposition is aimed at the graduate student level.

1 Heuristic description of MHD instabilities

This first section is intended to provide the reader with some qualitative and semi-quantitative ideas about the onset and characteristics of pressure-driven instabilities, leaving technical aspects of the stability analysis to the later sections.

1.1 Qualitative conditions of ideal MHD instabilities in static equilibria

The equilibrium configurations leading to an ideal MHD instability have been well investigated in the fusion literature. For current-driven instabilities, the first criterion was devised by Kruskal and Shafranov. Basically, it states that in cylindrical column of length $L$, instability follows if the magnetic field line rotates more than a certain number of times around the cylinder, from end to end. The exact number of rotations required for instability is dependent on the considered equilibrium configuration; it is usually of order unity.

Concerning pressure-driven instabilities, a more clear-cut necessary condition of instability can be stated: instability follows once the pressure force pushes the plasma outwards from the inside of the field line curvature. This condition can be derived from the Energy Principle, as will be shown later on.

These conditions of onset of instability are illustrated on Fig. 1. In an actual plasma, the origin of an instability (current- or pressure-driven) is usually not easy to pinpoint except in special instances. For example, if the plasma is cold (no pressure force), the instability is necessarily current-driven. Also the growth rates of current-driven modes are known to decrease with spatial order – e.g., they decrease with increasing azimuthal wave-number $m$ – while the most unstable pressure-driven modes have a growth rate which
is nearly independent on the wavenumber. Consequently, large wavenumber unstable modes are therefore always pressure-driven in a static, ideal MHD column. Besides these two limiting cases, an MHD instability almost always results from an inseparable mix of pressure and current driving. If the column is moving, the distinction between Kelvin-Helmholtz, current- and pressure-modes is even more blurred, except in some cases, where branches of instability can be identified by taking appropriate limits.

In terms of outcome of the instability, it is essential to know whether unstable modes are internal or external, i.e., have vanishing or substantial displacement on the plasma surface (here, the jet surface). It is well-known in the fusion context that unstable external modes are prone to disrupt the plasma, as may be the case, e.g., with the $m = 1$ ("kink") current-driven mode.

In the next sections, mostly high wavenumber modes will be examined, where the pressure-driving is most obvious, in order to best identify the characteristic features of this type of instability.

### 1.2 Magnetic shear, magnetic resonances, and Suydam’s criterion

The concept of magnetic shear plays an important role in the understanding of the stability of pressure-driven mode. The magnetic shear characterizes the change of orientation of field lines when moving perpendicularly to magnetic surfaces. In the case of cylindrical equilibria, this concept is illustrated on Fig. 2. Magnetic surfaces are cylindrical. Field lines within magnetic surfaces have an helix shape; the change of helix pitch $r B_z / B_\theta$, characterizes the magnetic shear. A quantity related to the pitch and largely used in the fusion community is the safety factor $q$:

$$q = \frac{r B_z}{R_0 B_\theta},$$

where $R_0$ is the column radius. For reasons soon to be discussed, a high enough safety factor is required for stability, hence its name. The magnetic shear $s$ is defined as
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\[ s \equiv \frac{r \, dq}{q \, dr}. \quad (2) \]

\[ \text{Fig. 2.} \] The change in the pitch of field lines between magnetic surfaces is the source of the magnetic shear (see text).

Magnetic resonances constitute another important key to the question of stability. A cylindrically symmetric equilibrium is invariant in the vertical and azimuthal direction, so that perturbations from equilibrium can without loss of generality be expanded in Fourier terms in these directions and assumed to be proportional to \( \exp(i(m\theta + kz)) \). Magnetic resonances are the (cylindrical) surfaces where the wave vector \( \mathbf{k} = m/r \mathbf{e}_\theta + k \mathbf{e}_z \) is perpendicular to the equilibrium magnetic field:

\[ \frac{\mathbf{k} \cdot \mathbf{B}_o}{B_o} = k_\parallel \equiv k_\parallel = \frac{1}{B_o} \left( \frac{m}{r} B_\theta + k B_z \right) = 0 \quad (3) \]

where \( k_\parallel \) is the component of the wave vector parallel to the equilibrium magnetic field. The significance of these surfaces stems from the fact that in general, dispersion relations incorporate a stabilizing piece of the form \( V_A^2 k_\parallel^2 \), where \( V_A \) is the Alfvén speed. This term is responsible for the propagation of Alfvén waves, and arises from the restoring force due to the magnetic tension (see section \( \square \) for the precise meaning of these statements). As such, it is always stabilizing. Obviously, this stabilization is minimal in the vicinity of a magnetic resonance for a given \( (m, k) \) mode, so that pressure-driven instabilities are preferentially triggered at magnetic resonances for any given mode.

Note however that a large magnetic shear limits the role of magnetic resonances in the destabilization of the plasma. Indeed, defining the perpendicular wave number...
\[ k_\perp = -\frac{1}{B_0} \left( \frac{m}{r} B_z - k B_\theta \right), \]  \hspace{1cm} (4) 

and designating by \( r_c \) the radial position of the magnetic resonance of the \((m,k)\) mode, one finds that

\[ k_\parallel \simeq \frac{B_\theta B_z}{B_0^2} k_\perp s \frac{r - r_c}{r_c}, \]  \hspace{1cm} (5) 

to first order in \((r - r_c)/r_c\) in the vicinity of the magnetic resonance \( r_c \). This implies that \( V_A k_\parallel \) will remain small either if \( s \ll 1 \) (small shear) or if the magnetic field is mostly perpendicular \((|B_\theta| \ll |B_z|)\) or azimuthal \((|B_z| \ll |B_\theta|)\) so that \( |B_\theta B_z|/B_0^2 \ll 1 \). However, if the field is predominantly vertical, it is little curved, and pressure destabilization is expected to be weak or non-existent according to the description of the condition on instability depicted in Fig. ??; furthermore, \( s \) being a logarithmic derivative is usually of order unity. Therefore, in practice stabilization by magnetic tension will be reduced essentially when the field is mostly azimuthal.

These features are embodied in Suydam criterion, which expresses a sufficient condition for instability:

\[ \frac{B_z^2}{8 \mu_0 r^2 s^2} + \frac{dP}{dr} > 0. \]  \hspace{1cm} (6) 

The converse of this statement is a necessary condition for stability. The origin of this criterion is briefly discussed in section ?? It turns out that this condition is both a necessary and sufficient condition of instability for large wavenumber modes [13]. The condition of instability requires \( dP/dr < 0 \), which agrees with our heuristic description of the onset of instability given above. It will be also discussed in Section ?? that the growth rates \( \gamma \) of pressure-driven instabilities are \( \gamma \sim C_S/R_o \) (\( C_S \) is the sound speed and \( R_o \) the jet radius).

Coming back to Eq. (6), the first term is stabilizing, but the stabilization will be minimal in the condition just discussed, i.e., when the field is mostly azimuthal. Indeed, in this case, the equilibrium condition Eq. (11) implies that \( dP/dr \sim B_\theta^2/\mu_0 r \gg B_z^2/r \sim B_z^2 s^2/r \). This situation is expected to hold in magnetically self-confined jets outer regions. Indeed, most such jet models (e.g., [23] and [15]) have \( |B_\theta| \gg |B_z| \) in the asymptotic jet regime to ensure confinement. This feature combined to the previous statement that MHD instabilities involving the boundary are most prone to disrupt static MHD columns makes the assessment of the role of pressure-driven instabilities in MHD jets particularly critical for the viability of such models. This viability hinges on the hopefully stabilizing role of the jet bulk motion (see section ??).
2 Ideal MHD in static columns:

The simplest framework in which the stability of jets can be investigated is ideal magnetohydrodynamics (MHD). Justifications and limitations of this approach are briefly discussed in Appendix A.

2.1 Equations

The MHD equations used in these notes are the continuity equation, the momentum equation without the viscous term, the induction equation without the resistive term, and a polytropic equation of state. Incompressibility is not assumed, as pressure-driven modes are not incompressible except at the marginal stability limit. These equations read

\[
\frac{\partial \rho}{\partial t} + \nabla \rho \mathbf{v} = 0, \quad (7)
\]

\[
\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla P_T + \frac{\mathbf{B} \cdot \nabla \mathbf{B}}{\mu_0 \rho}, \quad (8)
\]

\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}), \quad (9)
\]

\[
P = K \rho^\gamma, \quad (10)
\]

with standard notations, and where \(P_T = P + B^2/2\mu_0\) is the total (gas and magnetic) pressure, \(K\) a constant, and \(\gamma\) the polytropic index.

2.2 Equilibrium

Using a cylindrical coordinate system \((r, \theta, z)\), a static \((\mathbf{v} = 0)\) cylindrical column of axis \(z\) is described by a helical magnetic field \(B_\theta(r), B_z(r)\), and a gas pressure \(P(r)\) depending only on the cylindrical radius \(r\). The continuity and induction equations are then trivially satisfied, as well as the vertical and azimuthal component of the momentum equation, while the radial component reduces to

\[
-\frac{dP_T}{dr} - \frac{B_\theta^2}{\mu_0 r} = 0. \quad (11)
\]

This cylindrical equilibrium is best characterized by introducing a number of quantities homogeneous to an inverse length, both in vectorial (\(\mathbf{K}_B, \mathbf{K}_P\) and \(\mathbf{K}_C\)) or algebraic form (\(K_B, K_P\) and \(K_C\)). They are defined by:

\[
\mathbf{K}_B \equiv \frac{\nabla B_o}{B_o} = \frac{1}{B_o} \frac{dB_o}{dr} \mathbf{e}_r \equiv K_B \mathbf{e}_r, \quad (12)
\]
\[ \mathbf{K}_P \equiv \nabla \frac{P_o}{P_o} = \frac{1}{P_o} \frac{dP_o}{dr} \mathbf{e}_r \equiv K_P \mathbf{e}_r, \quad (13) \]

\[ \mathbf{K}_C \equiv \mathbf{e}_\parallel \cdot \nabla \mathbf{e}_\parallel = -\frac{B_o^2}{rB_o^2} \mathbf{e}_r \equiv K_C \mathbf{e}_r. \quad (14) \]

where \( B_o \) and \( P_o \) are the equilibrium distribution of magnetic field and gas pressure, and \( \mathbf{e}_\parallel = B_o/B_o \) is the unit vector parallel to the magnetic field; \( \mathbf{K}_C \) is the curvature vector of the magnetic field lines, and \( \mathbf{K}_B \) characterizes the inverse of the spatial scale of variation of the magnetic field, while \( \mathbf{K}_P \) characterizes the inverse scale of variation of the fluid pressure. The first identity in these relations is general, whereas the second one pertains to cylindrical equilibria only.

It is also convenient to introduce the plasma \( \beta \) parameter:

\[ \beta \equiv \frac{2 \mu_o P_o}{B_o^2}. \quad (15) \]

This parameter measures the relative importance of the gas and magnetic pressures.

With these definitions, the jet force equilibrium relation reads

\[ \frac{\beta}{2} \mathbf{K}_P = (\mathbf{K}_C - \mathbf{K}_B), \quad (16) \]

Both forms of the equilibrium relation, Eqs. (13) and (14), express the fact that the hoop stress due to the magnetic tension (\( \mathbf{K}_C \)) balances the gas (\( \mathbf{K}_P \)) and magnetic (\( \mathbf{K}_B \)) pressure gradient to achieve equilibrium and confine the plasma in the column. Self-confinement is achieved in this way when the external pressure is negligible at the column boundary.

2.3 Perturbations:

We want to investigate the stability with respect to deviations from equilibrium. As the background equilibrium is static, the problem is most easily formulated and analyzed in Lagrangian form: indeed, in this case, all equations but the momentum equation can be integrated with respect to time. To this effect, we introduce, for any fluid particle at position \( \mathbf{r}_o \) in the absence of perturbation, the displacement \( \xi(\mathbf{r}_o, t) \) at time \( t \) from its unperturbed position, so that its actual position is given by

\[ \mathbf{r}(\mathbf{r}_o, t) = \mathbf{r}_o + \xi(\mathbf{r}_o, t). \quad (17) \]

The unperturbed position \( \mathbf{r}_o \) is used to uniquely label all fluid elements.
Denoting by $\delta X$ the (Lagrangian) variation during the displacement of any quantity $X$, the linearized (Eulerian) equation of continuity $\partial_t \delta \rho = -\nabla (\rho_o v)$ integrates into

$$\delta \rho = -\nabla (\rho_o \xi).$$

Similarly, the linearized induction equation $\partial_t \delta B = \nabla \times (B_o \times v)$ leads to

$$\delta B = \nabla \times (B_o \times \xi).$$

From these results and the polytropic equation of state, the total pressure variation reads

$$\delta P_T = -\xi \cdot \nabla P_o - \gamma P_o \nabla \cdot \xi + \frac{B_o \cdot \delta B}{\mu_o}.$$  \hfill (20)

For static equilibria, one can without loss of generality take a Fourier transform the linearized momentum equation with respect to time. For a given Fourier mode, one can write $\xi(r, t) = \xi(r) \exp i\omega t$, so that the linearized momentum equation becomes

$$-\rho_o \omega^2 \xi = -\nabla \delta P_T + \delta T \equiv F(\xi),$$

where $\delta T = (B_o \nabla B + B_o \nabla B_o)/\mu_o$ represents the variation of the magnetic tension force. The last identity in Eq. (21) defines the linear operator $F$, operating on $\xi$ through Eqs. (19) and (20).

### 3 The Energy Principle and its consequences:

The linear operator $F$ of Eq. (21) is self-adjoint, i.e., taking into account that $F$ is real:

$$\int \eta \cdot F(\xi) d^3r = \int \xi \cdot F(\eta) d^3r.$$  \hfill (22)

A demonstration of this relation can be found, e.g., in Freidberg [16] (cf p. 242 and Appendix A of the book).

As a consequence of this property of $F$, an Energy Principle can be formulated. Defining

$$\delta W(\xi^*, \xi) = -\frac{1}{2} \int \xi^* \cdot F(\xi) d^3r,$$  \hfill (23)

1 In these expressions, the difference between the Eulerian and Lagrangian variations has been ignored as they disappear to first order in the displacement $\xi$ in the final equations. For the same reason, no distinction is made between the derivative with respect to $r$ or $r_o$.

2 Within a factor $\rho_o$. 
and

$$K(\xi^*, \xi) = \frac{1}{2} \int \rho |\xi|^2 d^3r,$$

(24)

and taking the scalar product of Eq. (21) with $\xi^*$ leads to

$$\omega^2 = \frac{\delta W}{K}.$$  \hspace{1cm} (25)

The self-adjointness of $F$ has two important consequences (Energy Principle):

- $\omega^2$ is also extremum with respect to a variation of $\xi$.
- Stability follows if and only if $\delta W > 0$ for all possible $\xi$.

Ascertaining stability through the last statement is usually an impossible task. Instead, one usually makes use of the Energy Principle in a less ambitious manner: if one can find some displacement making $\delta W < 0$ then one has a sufficient condition of instability (or, taking the converse statement, a necessary condition of stability). This is actually how Suydam criterion is demonstrated. First the expression of $\delta W$ is simplified by taking advantage of the cylindrical geometry and focusing on marginal stability and incompressible displacements (as they make $\delta W$ more easily negative; see below). Next, one chooses a particular form of displacement in the vicinity of the magnetic resonance of an $(m, k)$ mode, and looks under which conditions this displacement makes $\delta W$ negative; the condition turns out to be Suydam criterion for a well-chosen displacement. These computations are rather lengthy and the reader is referred to Freidberg’s book [16] for details.

A useful form of $\delta W$ has been derived by Bernstein et al. [4], which reads (see Freidberg [16], p. 259)

$$\delta W = \frac{1}{2} \int d^3r \left[ \frac{|Q_{\perp}|^2}{\mu_o} + \frac{B_{o}^2}{\mu_o} \nabla \cdot \xi_{\perp} + 2K_C \cdot \xi_{\perp}^2 + \gamma P_o \nabla \cdot |\xi|^2 - 2P_o(K_P \cdot \xi_{\perp})(K_C \cdot \xi_{\perp}^*) - J_{\parallel}(\xi_{\perp}^* \times e_{\parallel}) \cdot Q_{\perp} \right],$$

(26)

where $\xi_{\perp}$ is the component of the displacement perpendicular to the unperturbed field $B$, $Q_{\perp} = \nabla \times (\xi_{\perp} \times B_o)$ is the perturbation in the magnetic field, $K_C$ is the curvature vector of the magnetic field and $K_P$ the inverse pressure length-scale vector defined earlier; $J_{\parallel}$ and $e_{\parallel}$ are the current and unit vector parallel to the magnetic field, respectively. The quantities $K_P$ and $K_C$ are defined in Eqs. (13) and (14).

The first term describes the field line bending energy; it is the term responsible for the propagation of Alfvén waves, through the restoring effect of the magnetic tension, which makes field lines acting somewhat like a rubber band.

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3 The boundary term is ignored, as it is not required in this discussion.
The second term is the energy in the field compression, while the third is the energy in the plasma compression. The fourth term arises from the perpendicular current (as $\nabla P = J_\perp \times B$ in a static equilibrium), and the last one arises from the parallel current $J_\parallel$. Only these two terms can be negative, and give rise to an instability if they are large enough to make $\omega^2 < 0$. Pressure-driven instabilities are driven by the first of these two terms, while current-driven instabilities are due to the second one. Pressure-driven instabilities are further subdivided into interchange and ballooning modes, depending on the shape of the perturbation, but the basic properties of these different modes are similar, and this distinction will not be discussed further in these notes.

For our purposes here, we are mostly interested in what can be learned from the form of the fourth term. First note that this term is destabilizing in cylindrical geometry when $K_C K_P > 0$; this justifies the necessary condition of instability given in section 1.1. Furthermore, Eq. (25) and (26) imply that the pressure-driving term produces an inverse growth rate $\gamma$, of order of magnitude

$$|\gamma|^2 \sim C_S^2 K_C K_P \sim C_S^2 / R_o^2,$$

where $C_S$ is the sound speed and $R_o$ the column radius. This is quite fast, comparable to the Kelvin-Helmholtz growth rate in YSO jets. This order of magnitude will be used in the next section to set up an ordering leading to analytically tractable dispersion relations for pressure-driven unstable modes.

### 4 Dispersion relation in the large azimuthal field limit:

Most of the general results on pressure-driven instabilities were obtained in the fusion literature either from the use of the Energy Principle, or from the so-called Hain-Lüst equation (a reduced perturbation equation for the radial displacement [19] [17]). These approaches are quite powerful, but not familiar to the astrophysics community, and involve a lot of prerequisite.

It is more common in astrophysics to grasp the properties of an instability through the derivation of a dispersion relation. There are actually two papers doing this in the jet context for pressure-driven instabilities; however, the first one, by Begelman [3], focuses on the relativistic regime which brings a lot of added complexity to the discussion, and the second one [21] is partially erroneous.

Fortunately, in the limit of a near toroidal field of interest here, a dispersion relation can be derived ab initio by elementary means, and this approach is adopted here. To this effect, it is first useful to reexamine the behavior of the three MHD modes in an homogeneous medium, in the limit of quasi-perpendicular propagation. It is known that this limit allows the use of a kind of WKB type of approach in the study of interchange and ballooning...
pressure-driven modes (see, e.g., Dewar and Glasser \[12\]), a feature we shall take advantage of in these notes.

### 4.1 MHD waves in quasi-perpendicular propagation in homogeneous media:

We consider an homogeneous medium pervaded by a constant magnetic field $B_0$. The analysis of linear perturbations in such a setting leads to the well-known dispersion relation of the slow and fast magnetosonic modes and the Alfvén mode. Our purpose here is to point out useful features of these modes when the wavevector is nearly perpendicular to the unperturbed magnetic field.

![Fig. 3. Definition of the reference frame ($e_\parallel, e_l, e_A$)](image)

To this effect, let us consider plane wave solutions to Eq. (21), where $\xi \propto \exp(-i k \cdot r)$, and assume that the direction of propagation is nearly perpendicular to the magnetic field, i.e. $k_\parallel \ll k_\perp$ (defined in Eqs. (5) and (4)). The focus on quasi-perpendicular propagation comes from the remarks of section 1.2, where it was noted that instability is easier to achieve in the vicinity of magnetic resonances, i.e., where $k_\parallel \ll k_\perp$. Let us also introduce the orthogonal reference frame ($e_\parallel, e_l, e_A$) where $e_\parallel \equiv B_0 / B_0$ is parallel to the unperturbed magnetic field, $e_l \equiv k_\perp / k_\perp$, and $e_A \equiv e_\parallel \times e_l$ (see Fig. 3). With our definition of $k_\parallel$ and $k_\perp$ in Eqs. (5) and (4), $e_A = e_r$. The subscripts $l$ and $A$ stand for longitudinal and alfvénic, respectively ($e_\parallel$, $e_l$, and $e_A$ are the directions of the displacement of purely slow, fast and alfvénic modes in the limit of nearly transverse propagation adopted here, as shown below).

Denoting $(\xi_\parallel, \xi_l, \xi_A)$ the components of the lagrangian displacement $\xi$ in this reference frame, the momentum equation Eq. (21) yields the following three component equations
while the total pressure perturbation becomes
\[
\delta P_T = -i\rho_o \left[ (C_S^2 + V_A^2) k_\perp \xi_l + C_S^2 k_\parallel \xi_\parallel \right].
\]  

Eq. (30) gives the dispersion relation of Alfvén waves, \( \omega^2_A = V_A^2 k_\parallel^2 \), which decouple from the two magnetosonic modes described by the remaining two equations. The solutions of the magnetosonic modes are easily derived and possess the following important properties. Characterizing quasi-perpendicular propagation with the small parameter \( \epsilon \equiv |k_\parallel| / |k_\perp| \ll 1 \), these two equations imply \( \omega_S^2 \simeq C_S^2 V_A^2 / (C_S^2 + V_A^2) k_\parallel^2 \) and \( \xi_l \sim O(\epsilon \xi_\parallel) \) for the slow magnetosonic wave, while \( \omega_F^2 \simeq (C_S^2 + V_A^2) k_\perp^2 \) and \( \xi_\parallel \sim O(\epsilon \xi_l) \) for the fast magnetosonic one.

Furthermore, the \( \xi_l \) momentum component Eq. (29) combined with Eq. (30) and the ordering of the displacement component just pointed out implies that \( \delta P_T = 0 \) to leading order in \( \epsilon \) for the slow magnetosonic mode; note that the same property holds by construction for the Alfvén mode. The cancellation of the total pressure for these two modes is essential from a technical viewpoint, and will lead to substantial simplification in the derivation of a dispersion relation performed in the next subsection.

4.2 Dispersion relation and Kadomtsev criteria:

Let us now come back to cylindrical inhomogeneous equilibria. Remember from section 3 that the pressure-driving term will contribute a destabilizing term \( \omega^2 \sim C_S^2 / R_\parallel^2 \) to the dispersion relation. This term will be able overcome the stabilizing effect of the restoring forces of the Alfvén and slow magnetosonic modes only if \( V_A |k_\parallel|, C_S |k_\parallel| \lesssim C_S / R_\parallel \). This constraint can be achieved in the vicinity of magnetic resonance as already previously noted.

More precisely, a simplified dispersion relation can be found in the WKB limit with a displacement of the form \( \xi(r) = \xi \times \exp -i(k_r r + m\theta + k_z z) \), if the following ordering is satisfied:

- \( |k_\parallel r| \ll 1 \ll |k_r r| \ll |k_\perp r| \): the first inequality ensures that the stabilization by magnetic tension is ineffective (closeness to a resonance).

The following inequalities ensure that a WKB limit can be taken. The implied ordering \( |k_\parallel| \ll k_\perp \) ensures that \( \delta P^* \) will vanish to leading order

\footnote{For consistency with the previous sections, \( k_\perp \) is the wavenumber in the longitudinal direction; it does not include the piece in the radial direction.}
as in the homogeneous case discussed in the previous section. The last inequality allows us to neglect the contribution of the radial gradient of total pressure (which does not vanish), and greatly simplifies the analysis.

- \(|B_z/B_\theta|^2 s^2 |k_\perp| \ll |k_\parallel|\); this limit, which applies when \(|B_\theta| \gg |B_z|\), ensures that the magnetic shear is not stabilizing.
- \(|\omega|^2 \ll |\omega_F|^2\); this excludes the fast mode from the problem in the near perpendicular propagation regime considered here. As the fast mode is not expected to be destabilized in this regime (as \(|\omega_F|^2 \gg V_A^2/r^2\)), this does not limit the generality of the results while simplifying the analysis.

It turns out that the resulting dispersion relation captures most of the physics of pressure-driven instabilities; this follows because the most unstable modes have growth rates nearly independent of the azimuthal wavenumber \(m\) [13], and because current-driven instabilities are efficient only at low \(m\) and disappear from a WKB analysis.

As previously, the projection Eq. (21) on the longitudinal direction \(e_\parallel\) shows that the total pressure perturbation vanishes and that \(\xi_\parallel \sim |k_\parallel/k_\perp| \ll |\xi_\perp|\), while the components in the other two directions \((e_\parallel, e_\perp)\) are now coupled and read (some details of the derivation of these equations can be found in Appendix B)

\[
\omega^2 - V_{SM}^2 k_\parallel^2 = -i \frac{2\beta^*}{1 + \beta^*} V_A^2 K_C k_\parallel \xi_\parallel, \quad (32)
\]

\[
\left[ \omega^2 - V_A^2 (k_\parallel^2 + k_\perp^2) \right] = i \frac{2\beta^*}{1 + \beta^*} V_A^2 K_C k_\parallel \xi_\parallel, \quad (33)
\]

where \(\beta^* = C_S^2/V_A^2\), and \(V_{SM}^2 = C_S^2 V_A^2/(C_S^2 + V_A^2)\) is the slow mode speed in the near perpendicular propagation limit. The coupling of the modes blurs their character except in limiting cases.

The quantity \(k_\perp^2\) is defined as

\[
k_\perp^2 = \frac{4\beta^*}{1 + \beta^*} K_C^2 - 2\beta^* K_C K_\rho. \quad (34)
\]

Note that if \(K_C = 0\) (i.e., when reverting to an homogeneous medium), Eqs. (32) and (33) yield back the slow and Alfvén mode, respectively. The field curvature couples the two modes. The quantity \(k_\perp^2\) can be either positive or negative; the first term in Eq. (34) comes from the plasma compression, and the second one is the contribution of the pressure destabilizing term identified in section 3.

As usual, these equations possess a non-trivial solution if their determinant is non zero, which yields the following dispersion for \(\omega^2\):

\[
\omega^4 - \left[ (V_A^2 + V_{SM}^2) k_\parallel^2 + V_A^2 k_\parallel^2 \right] \omega^2 + V_A^2 V_{SM}^2 k_\parallel^2 (k_\parallel^2 - 2\beta^* K_C K_\rho) = 0. \quad (35)
\]
First note that if both $B_z = 0$ (the so-called Z-pinch configurations) and $m = 0$, this equation is degenerate: one of the roots is $\omega^2 = 0$ and the other root is $\omega^2 = V_A^2 k_o^2$. Instability then requires that $k_o^2 < 0$, as $k_o = 0$ in this case. This constrain is identical to the criterion\(^6\) derived by Kadomtsev from the Energy Principle for the $m = 0$ mode in Z pinches (see Freidberg \(\bib{16}\) p. 286).

When $m \neq 0$, Eq. (35) can be solved exactly but it is more instructive to analyze its properties. As the coefficient of $\omega^2$ is equal to the sum of the two roots, and the last term is equal to their product, one finds that if $k_o^2 > 2\beta^* K_C K_P$, the two roots are stable, and if $k_o^2 < 2\beta^* K_C K_P$, one of the roots is unstable. If $B_z = 0$ (Z pinch), this condition is identical to the criterion\(^7\) derived by Kadomtsev for $m \neq 0$ modes (see Freidberg \(\bib{16}\) pp. 284-285).

Note that all these conditions of instability require $K_C K_P > 0$, in agreement with the discussion of sections\(^8\) and \(\bib{1}\) this condition is unavoidable in magnetically self-confined jets. The analysis presented here also shows that once this condition is satisfied, instability necessarily follows in static columns where $|B_\theta| \gg |B_z|$ on some of the radial range, as the magnetic tension stabilizing effect $V_A^2 k_o^2$ is arbitrarily small in the vicinity of a magnetic resonance.

Finally, the reader may ask how the local analysis presented here informs us on the global stability properties of the column. The answer lies in in oscillation theorem of Goedbloed and Sakanaka \(\bib{18}\). The theorem states that for any $(m, k)$ unstable mode, the growth rate decreases when increasing the number of radial nodes. This implies that if an unstable mode with a large number of radial nodes is found (such as the modes considered here), an unstable nodeless mode will also exist, and this mode will have the largest growth rate. Such a mode will have a very disruptive effect on the plasma if its displacement is not vanishing on the boundary, as will be the case if the azimuthal field is dominant on the boundary.

5 Moving columns:

The previous section has shown that cylindrical columns with a predominant azimuthal magnetic field at least in some radial range are subject to pressure-driven instabilities. This situation holds in the outer region of self-confined magnetic jets, leading to a potentially disruptive configuration. However, in these regions a gradient of axial velocity due to the interaction of the moving jet with the outside medium is also expected to be present, and it is legitimate to investigate the effect of such a velocity gradient on the stability properties of pressure-driven modes.

\(^6\) Kadomtsev’s criterion for the $m = 0$ mode in a Z pinch is a necessary and sufficient condition of instability, whereas the analysis presented here shows only the sufficiency of this condition.

\(^7\) Same comment as in the previous footnote.
This problem has not yet been addressed in the astrophysics literature, but some relevant results are available in the fusion literature. In all the investigations cited below, the adopted velocity profile contains no inflexion point, in order to avoid the triggering of the Kelvin-Helmholtz instability.

It is first useful to consider what becomes of Suydam criterion in presence of background motions. This investigation has been performed by Bondeson et al. Focusing on axial flows \( U = U_z(r)e_r \), they conclude that the behavior of localized modes depends on the magnitude of

\[
M \equiv \rho^{1/2} \frac{U'_z}{q'B_z/q},
\]

where the prime denotes radial derivative, and \( q \) is the safety factor (see section 2). This quantity is a form of Alfvénic Mach number based on the velocity and magnetic shear, hence its name. When \( M^2 < \beta \), the flow shear destabilizes resonant modes. Above this limit, these modes are stable, but in this case, unstable modes exist at the edge of the slow continuum, and may be global. The authors found however that in this case the growth rates are small (comparable to the resistive instabilities growth rates). Note also that, as \( q'/q \sim 1/r \), \( M \sim (B_\phi/B_z)(r/d)(U_z/V_A) \gg 1 \) in MHD jets (\( d \) is the width of the velocity layer).

These results seem to suggest that the region where the velocity shear layer takes place at the jet boundary is substantially stabilized in MHD jets. This seems to be confirmed by global linear stability analyzes, both for interchange and ballooning modes, except possibly for the \( m = 0 \) (“saussage”) mode [10] [27] [28]. In all cases, increasing the flow Mach number efficiently reduces the amplitude of the displacement of the unstable modes at the plasma boundary, an important feature to avoid the disruption of the plasma.

An efficient stabilization mechanism has also been identified in the nonlinear regime by Hassam [29]. This author exploits an analogy between the \( m = 0 \) pressure-driven interchange mode and the Rayleigh-Taylor instability in an appropriately chosen magnetized plasma configuration. From this analysis, he concludes that the \( m = 0 \) pressure-driven mode is nonlinearly stabilized by a smooth velocity shear \( \left( dU_z/dr \sim U/R_o \right) \) if \( M_s = U_z/C_S \gtrsim \left[ \ln(\tau_d/\tau_g) \right]^{1/2} \), where \( \tau_g \) is the instability growth time-scale \( (\tau_g \sim c_s(K_{\rho c})(1/2)) \) and \( \tau_d \) the diffusion time-scale \( (\tau_d \sim \nu K_{\rho c}) \) where \( \nu \) is the viscosity, assumed comparable to the resistivity. The nonlinear evolution of an unstable, slightly viscous and resistive Z-pinch (i.e., a configuration where the field is purely azimuthal), was simulated by Desouza-Machado et al. They found that the plasma relaminarizes over almost all its volume for applied an velocity shear in good agreement with this analytic estimate. The core of the plasma still has some residual unstable “wobble”, which can apparently be stabilized by the magnetic shear if some longitudinal field \( B_z \) is added to the configuration.

---

8 This requires a generalization of Eq. (21), also, the Energy Principle no longer applies as the resulting operator is not self-adjoint.
that the large values of $\tau_d/\tau_g$ relevant to astrophysical jets lead to only weak constraints on the Mach number $M_s$, so that this nonlinear stabilization mechanism is expected to be efficient in astrophysical jets.

6 Summary and open issues:

Pressure-driven instabilities occur in static columns when the pressure force pushes the plasma out from the inside of the magnetic field lines curvature, as shown from direct inspection of the “potential energy” of the linearized displacement equation (section 3), and from the dispersion relation of local modes (section 4). When unstable modes exist, the growth rates are of the order $C_S/R_o$ where $C_S$ is the sound speed and $R_o$ the jet radius. These are very large, comparable to the Kelvin-Helmholtz growth rate (the most studied instability in jets), especially that the ratio of the magnetic energy to the gas internal energy is expected to be of order unity (within an order of magnitude or so). Such instabilities are known to be disruptive in the fusion context when the eigenmodes exhibit a substantial displacement of the plasma outer boundary; such a situation is relevant to magnetically self-confined jets, as the magnetic field in their outer region is predominantly azimuthal, a configuration most favorable to the onset of the instability (sections 1 and 4). However, the presence of a velocity gradient in the outer boundary due to the jet bulk motion is expected to have a substantial stabilizing influence, both in the linear and nonlinear regimes (section 5).

In its present state, this picture possesses a number of loose ends:

- The stabilizing role of an axial velocity gradient needs to be better understood. Not all modes may be stabilized in the linear regime, depending on the details of the equilibrium jet configuration, and the nonlinear mechanism identified in the literature is highly idealized and may not be generic. The one and only simulation of nonlinear stabilization published to date exhibits a very violent relaxation transient, which may still lead to jet disruption. On the other hand, this transient is also an indication that the initial configuration of the simulation is way out of equilibrium, a situation which may not occur in real jets.
- The role of jet rotation has not yet been correctly investigated. Preliminary results seem to indicate that it is stabilizing [21]; however, jet rotation may not be an important dynamical factor in the asymptotic jet propagation regime.
- Most investigations of pressure-driven instabilities rely on a very simple prescription of the equation of state, which raises an issue of principle. Indeed, the very large growth rates usually found for the instability indicate that it develops on time-scales much shorter than the collisional time-scale,

\[\tau_d/\tau_g = 10^{30}\]

translates into $M_s \gtrsim 8$ only; in YSO jets, this ratio is most probably significantly smaller, and the constraint even weaker.
and the use of ideal MHD as well as a polytropic equation of state may be questioned in such a context, an issue briefly addressed in Appendix A. A more complex description of the plasma is required to validate the results obtained so far.

Appendices

A On the use of ideal MHD:

In astrophysics in general, and jet stability analyzes in particular, an MHD framework is almost always adopted instead of the more precise kinetic one, due to its relative simplicity. The MHD approximation can be applied when the fluid is locally neutral, when all species can be described by a single fluid equation (i.e. when the relative drift velocity of species with respect to one another is small), and when Ohm’s law is valid. The validity of these approximations has been discussed elsewhere [22] [26] and will not be reproduced here; the interested reader is referred to these books for details.

Furthermore, MHD stability (and jet stability in particular) is often investigated within the framework of ideal MHD. Indeed, the dynamical time-scales of interest (including those of the considered instability) is almost always substantially larger than the particle collision time-scale. Moreover, an isotropic pressure is often assumed, e.g. through a barotropic equation of state, and this raises another issue of principle within the framework of ideal MHD, as, indeed, an isotropic pressure would be expected only if collisions at the particle level are not neglected.

The isotropic pressure assumption can be justified to some extent by the fact that plasma microturbulence does limit pressure anisotropy to a factor of order unity. For example, within the framework of collisionless MHD, pressure anisotropy is self-limiting (for a recent synthetic discussion of this problem within the framework of collisionless magnetorotational instability, see Sharma et al. [25] and references therein). Nevertheless, this provides little support (if any) to the adoption of a closure in the form of a barotropic or adiabatic equation of state in a collisionless setting.

Collisionless MHD approximations apply when the length-scales and frequencies under consideration are larger than the ion Larmor radius, and smaller than the ion cyclotron frequency, respectively. These conditions should be satisfied in jets, but I am not aware of any investigation of pressure-driven instabilities in this framework. Freidberg [16] argues that a simple rule of thumb to estimate the effects of the assumed closure is to replace the adiabatic index by 0, and to assume incompressibility of the motions within the framework of standard ideal MHD.
B Derivation of the dispersion relation:

Some intermediate steps in the derivation of the dispersion relation of section 4 are given here. The notations are the same as in this section.

Direct computation of the pressure and magnetic field perturbation gives

\[ \delta P = -\rho_o C_S^2 [\nabla \cdot \xi + \mathcal{K}_\rho \xi_r] , \]  
\[ \delta B = -B_o [\nabla \cdot (\xi_r + \xi_i) + \xi_r (\mathcal{K}_B + \mathcal{K}_C)] e_\parallel - ik_\parallel B_o (\xi_r + \xi_i) . \]  

This allows us to obtain the perturbation in total pressure and in magnetic tension:

\[ \delta P_T = -\rho_o \left( V_A^2 + C_S^2 \right) \nabla \cdot (\xi_r + \xi_i) + \rho_o C_S^2 k_\parallel \xi_r - \rho_o \xi_r \cdot (V_A^2 \mathcal{K}_B + V_A^2 \mathcal{K}_C + C_S^2 \mathcal{K}_\rho) , \]  
\[ \delta T = -V_A^2 \mathcal{K}_C \left( 2 \nabla \cdot (\xi_r + \xi_i) + 2 \xi_r \cdot (\mathcal{K}_B + \mathcal{K}_C) \right) + ik_\parallel V_A^2 \left[ \nabla \cdot (\xi_r + \xi_i) + 2 \xi_r \cdot \mathcal{K}_C \right] e_\parallel - k_\parallel^2 V_A^2 (\xi_r + \xi_i) . \]  

Furthermore, the equilibrium relation Eq. (16) allows us to eliminate \( \mathcal{K}_B \) in terms of \( \mathcal{K}_C \) and \( \mathcal{K}_\rho = \gamma \mathcal{K}_P \).

The longitudinal component of the linearized momentum equation reduces to \( \delta P_T = 0 \), once contributions of order \( k_\parallel \xi_i \) or \( \xi_i / r \) are neglected in front of \( k_\perp \xi_i \). This constraint shows that \( \xi_i \sim O(\xi_i / k_\parallel, k_\parallel / k_\perp) \). It also allows us to eliminate \( \nabla \cdot (\xi_r + \xi_i) \) from the remaining two component equations, which then reduce to Eqs. (32) and (33). In the process, the contribution of \( d\delta P_T/dr \) is shown to be negligible from the assumed ordering relations \( |k_r| \ll k_\perp \) and \( |B_z / B_\theta|^2 s^2 |k_\perp| \ll |k_\parallel| \), i.e., the magnetic shear stabilizing term can be neglected in this limit.

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