Correlation Inequality for Formal Series

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Abstract

We extend the considerations of the paper [1] and prove two correlation inequalities for totally ordered set.

Introduction

We extend the considerations of the paper [1] and prove two correlation inequalities (statement of Lemma below and inequality (12)) for totally ordered set. From other side in [2] was made two conjectures (statement of Lemma and inequality (12)) for poset 2^X of subsets of finite set X with FKG condition on probability measure (see (2)). In [1] was stated that considerations from it lead to the proof of the Lemma under these FKG conditions on measure μ, but it turns out that that considerations are not sufficient for the proof and the problem is still open. To solve these conjectures (as we show here it is sufficient to prove Lemma and then (12) follows), if they are true, one need to make some additional efforts.

Main Text

First we introduce class of correlation inequalities.
Assume that $f_1, \ldots, f_n$ are nonnegative nondecreasing functions $2^X \to R$. The expectation of a random variable $f : 2^X \to R$ with respect to $\mu$ we denote by $\langle f \rangle_\mu$. For a subset $\delta \in [n]$ define

$$E_\delta = \left\langle \prod_{i \in \delta} f_i \right\rangle_\mu.$$ 

Let

$$\sigma = \{\sigma_1, \ldots, \sigma_\ell\}$$

be a partition of $[n]$ into disjoint subsets. Define

$$E_\sigma = \prod_{i=1}^\ell E_{\sigma_i}.$$ 

Let $\lambda_1 = |\sigma_i|$. We have $\sum_{i=1}^\ell \lambda_i = n$. Let $\lambda(\sigma) = (\lambda_1, \ldots, \lambda_\ell)$ and $\lambda_1 \geq \ldots \geq \lambda_\ell$. For a partition $\lambda$ of number $n$ define

$$E_\lambda = \sum_{\sigma : \lambda(\sigma) = \lambda} E_\sigma.$$ 

We need the following

**Lemma 1** Consider the totally ordered set $1, \ldots, N$ with probability measure $\mu$ on it and let’s functions $f_i$, $i = 1, \ldots, n$ are nonnegative and monotone nondecreasing. Functional

$$E_n(f_1, \ldots, f_n) = \sum_{\lambda \vdash n} c_\lambda E_\lambda \tag{1}$$

where

$$c_\lambda = (-1)^{\ell+1} \prod_{i=1}^\ell (\lambda_i - 1)!$$

is nonnegative.

In [2] was conjectured that statement of Lemma (along with (12)) is true when probability measure $\mu$ on $2^X$ satisfies FKG conditions

$$\mu(A \cap B) \mu(A \cup B) \geq \mu(A) \mu(B) \tag{2}$$

and functions $f_i$ are nonnegative and monotone.
Note that under conditions from the Lemma in particular case \( n = 2 \) Lemma gives Chebyshev inequality

\[
\langle f_1 f_2 \rangle_\mu \geq \langle f_1 \rangle_\mu \langle f_2 \rangle_\mu
\]

Hence our proof can be considered as extension of Chebyshev inequality to multiple variables. For monotone functions \( f_i(j), \ i = 1, \ldots, n; \ j = 1, \ldots, N \) we put

\[
f_i(1) = a_{i,1}, \ f_i(j) = f_i(j - 1) + a_{i,j}, \ j = 2, \ldots, N, \ a_{i,j} \geq 0. \tag{3}
\]

Then substituting in the formula

\[
E_n(f_1, \ldots, f_n) = \sum_{\lambda \vdash n} c_\lambda \sum_{\sigma: \lambda(\sigma) = \lambda} \prod_{i=1}^\ell \langle \prod_{j \in \sigma_i} f_j \rangle_\mu \tag{4}
\]

coefficients \( c_\lambda \) one can easily check that coefficient of the monomial

\[
\prod_{j=0}^{N-1} \prod_{i=\sum_j m_s+1}^{j+1} f_i(j+1)
\]

in the rhs of (4) is as follows

\[
F_{m_1, \ldots, m_N}(\mu) = \sum_{k_{i,j}: \sum_i k_{i,j} = m_j} \prod_{j=1}^N \mu^{k_j}(j) \frac{\Pi_{j=1}^N m_j! (\sum_{j=1}^N \kappa_j - 1)}{\Pi_{j=1}^N \Pi_i (i^{k_{i,j}} k_{i,j}!)} \tag{5}
\]

where \( \kappa_j = \sum_i k_{i,j} \). Analogous monomials with \( m_j \) factors \( f_i p(j) \) have the same coefficients. Indeed

\[
\sum_{\lambda \vdash n} c_\lambda \sum_{\sigma: \lambda(\sigma) = \lambda} \prod_{i=1}^\ell \langle \prod_{j \in \sigma_i} f_j \rangle
\]

\[
= \sum_{\lambda \vdash n} \sum_{\sigma: \lambda(\sigma) = \lambda} (-1)^{\sum_{j=1}^N \kappa_j - 1} \prod ((i - 1)!)^{\sum_{j=1}^N \kappa_j} \prod_{j=1}^N \mu(j) \prod_{s \in \sigma_i} f_s(j),
\]

where \( \{k_{i,j}\} \) are number of occurrence of the sets of cardinality \( i \) in the projection of partition \( \sigma \) onto \( m_j \). Number of partitions of \( \sigma \) with given \( \{k_{i,j}\} \) is

\[
\frac{\Pi_{j=1}^N m_j!}{\Pi_{j=1}^N i^{(k_{i,j} - 1)} (k_{i,j}!)}. 
\]
Using identity ([7, p.181])
\[
\sum_{\{k_i\}} \frac{y^k}{\prod(k_i!^{i_k_i})} = \frac{y(y + 1) \ldots (y + n - 1)}{n!}
\]
from (5) we obtain formula
\[
F_{m_1, \ldots, m_N}(\mu) = -(-\mu(1))(1 - \mu(1)) \ldots (m_1 - 1 - \mu(1))
\times (-\mu(2))(1 - \mu(2)) \ldots (m_2 - 1 - \mu(2)) \ldots
\times (-\mu(N))(1 - \mu(N)) \ldots (m_n - 1 - \mu(N)) = -\prod_{j=1}^N \prod_{i=1}^{m_j} (i - 1 - \mu(j))
\]
Using decomposition (3) it is easy to see that to prove Lemma it is sufficient to prove the inequality
\[
\sum \{m_s\} F_{m_1, \ldots, m_N}(\mu) \prod_{j=0}^{N-1} \prod_{i=1}^{\sum_{s=1}^{j} m_s \geq 1} \left( \sum_{t=1}^{i} a_{i,t} \right) \geq 0. \tag{6}
\]
One can check that coefficient before the monomial
\[
\prod_{j=0}^{N-1} \prod_{i=1}^{\sum_{s=1}^{j} m_s + 1} a_{i,j+1}
\]
in the lhs of (6) is
\[
B(m_1, \ldots, m_{N-1}) \triangleq -\sum_{\{i_j\}} \prod_{j=1}^{N-1} \left( \sum_{s=1}^{j} m_s - \sum_{s=1}^{j-1} i_s \right) \prod_{i=1}^{i_j} (i-1-\mu(j)) \times \prod_{i=1}^{n-\sum_{s=1}^{N-1} i_s} (i-1-\mu(N)). \tag{7}
\]
Thus to prove (7) and complete the proof of Lemma it is sufficient to prove the inequality
\[
B(m_1, \ldots, m_{N-1}) \geq 0. \tag{8}
\]
We prove this inequality by induction on \(m_j\). Let’s (8) is true for \(m_N = n - \sum_{s=1}^{N-1} m_s \geq 1\), then for \(m_N + 1\) we have the expression for \(B(m_1, \ldots, m_{N-1})\) (we use the identity \(\binom{\ell}{p} = \binom{\ell-1}{p} + \binom{\ell-1}{p-1}\))
\[
- \sum_{\{i_k\}} \prod_{j=1}^{N-1} \left( \sum_{s=1}^{j} m_s - \sum_{s=1}^{j-1} i_s \right) \prod_{i=1}^{i_j} (i - 1 - \mu(j)) \prod_{j=1}^{n-\sum_{s=1}^{N-1} i_s} (i-1-\mu(N)). \tag{9}
\]
\[
\times \prod_{i=1}^{n-N-1} (i-1 - \mu(N))(n - \sum_{s=1}^{N-1} i_s - \mu(N)).
\]

(10)

Last expression is nonnegative due to the induction proposal and the fact that (because \(m_N \geq 1\))

\[
n - \sum_{s=1}^{N-1} i_s - \mu_N \geq 0.
\]

Step by step using induction we come to the expression for \(B\) with \(m_N = 1\) (we assume at first that \(m_N > 0\)) and start induction on \(m_{N-1}\). Let (8) is true for \(m_N\), then the expression (9) for \(m_N + 1\)

\[
- \sum_{\{i_s\}}^{N-1} \prod_{j=1}^{i_s} \left( \sum_{s=1}^{j-1} m_s - \sum_{s=1}^{j-1} i_s \right) \prod_{j=1}^{i_s} (i-1 - \mu(j))
\]

\[
\times \prod_{i=1}^{n-N-1} (i-1 - \mu(N)) \left( n - \sum_{s=1}^{N-2} i_s - \mu(N-1) - \mu(N) \right).
\]

(11)

Because \(m_N = 1\), then

\[
n - \sum_{s=1}^{N-2} i_s - \mu(N-1) - \mu(N) \geq 0
\]

and thus by induction hypothesis expression (11) is nonnegative. Continuing this process to other \(m_j, j = N-3, \ldots, 1\) we come to the situation when \(m_{N_0} = 1, m_j = 0, j \neq N_0\) for some \(N_0 \in [N]\). Thus to complete the induction we need to prove that expression (8) in the case, when \(m_j = 1\) only for one value of \(j\), and all other \(m_s = 0\). But this negativeness immediately follows from the relation

\[
B(0, \ldots, 0, 1, 0, \ldots, 0) = \mu(j).
\]

This proves Lemma.

Next we consider the set of formal series \(P[[t]]\), whose coefficients are monotone nondecreasing nonnegative functions on \(2^X\). Then \(p(A) = p_1(A)t + p_2(A)t^2 + \ldots \in P[[t]]\). In [2] was formulated the following

**Conjecture 1** For FKG probability measure \(\mu\) the following inequality is true

\[
1 - \prod_{A \in 2^X} (1 - p(A))^\mu(A) \geq 0.
\]

(12)
The inequality (12) is understood as non-negativeness of coefficients of formal series obtained by series expansion of the product on the left-hand side of this inequality.

We will prove, that inequality (12) follows from inequalities

\[ E_n(f_1, \ldots, f_n) \geq 0 \quad (13) \]

for all \( n \) and hence it is sufficient to prove last inequalities and then inequality (12) follows under the same conditions on \( \mu \).

We make some transformations of the expression in the lhs of (12). We have

\[
1 - \prod_{A \in 2^X} (1 - p(A))^{\mu(A)} = 1 - \exp \left\{ \langle \ln(1 - p) \rangle \mu \right\} \\
= 1 - \exp \left\{ - \sum_{i=1}^{\infty} \frac{1}{i} \langle p^i \rangle \mu \right\} \\
= \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j!} \left( \sum_{i=1}^{\infty} \frac{1}{i} \langle p^i \rangle \mu \right)^j \\
= \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j!} \sum_{\{q_s\} : \sum q_s = j} \left( \sum_{\{i_s\}} \left( \prod_{j=1}^{j} (i_1)^{q_1} (i_2)^{q_2} \ldots (i_j)^{q_j} \right) \right) \sum_{\{i_s\}} \left( \prod_{j=1}^{j} \langle p^i \rangle \mu \right)^{q_s}.
\]

Next remind that the number of partitions of \( n \) with given set \( \{q_i\} \) of occurrence of \( i \) is equal to

\[
\frac{n!}{\prod_i (i!)^{q_i} q_i!}.
\]

Continuing the last chain of identities and using last formula we obtain

\[
1 - \prod_{A \in 2^X} (1 - p(A))^{\mu(A)} \quad (14)
\]

\[
= \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\lambda \vdash n} \sum_{\lambda(\sigma) = \lambda} (-1)^{\ell(\lambda) + 1} \prod_{i} (\lambda_i - 1)! \sum_{\sigma: \lambda(\sigma) = \lambda} E_\sigma(p, \ldots, p) \\
= \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\lambda \vdash n} \sum_{\lambda(\sigma) = \lambda} c_\lambda E_\lambda(p, \ldots, p) \\
= \sum_{n=1}^{\infty} \frac{1}{n!} E_n(p, \ldots, p)
\]
Hence now to prove the conjecture we need to show that
\[ E_n(p, \ldots, p) \geq 0. \] (15)

But the coefficients of the formal series \( E_n(p, \ldots, p) \) are the sums of \( E_n(p_{i_1}, \ldots, p_{i_n}) \) for multisets \( \{i_1, \ldots, i_n\} \). This completes the proof that inequality (12) follows from inequalities (13) under the same conditions on \( \mu \).

Thus because we prove Lemma, we prove inequality (12) for totally ordered lattice and this is our main result.

**Remark**

To extend Lemma for the conditions (2) one can try to find proper expansion for the monotone functions \( f_i \) which extend expansion (3) to the case of poset \( 2^X \).

**References**

[1] Blinovsky, V., A Proof of One Correlation Inequality, Problems of Inform. Transm., 2009, vol. 45, no. 3, pp. 264- 269.

[2] Sahi, S., Higher Correlation Inequalities, Combinatorica, 2008, vol.28, no. 2, pp. 209- 227.

[3] Riordan J., Combinatorial identities, Wiley, Ney York, 1968