On Fox and augmentation quotients of semidirect products

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Abstract

Let $G$ be a group which is the semidirect product of a normal subgroup $N$ and some subgroup $T$. Let $I^n(G)$, $n \geq 1$, denote the powers of the augmentation ideal $I(G)$ of the group ring $\mathbb{Z}(G)$. Using homological methods the groups $Q_n(G, H) = I^{n-1}(G)I(H)/I^n(G)I(H)$, $H = G, N, T$, are functorially expressed in terms of enveloping algebras of certain Lie rings associated with $N$ and $T$, in the following cases: for $n \leq 4$ and arbitrary $G, N, T$ (except from one direct summand of $Q_4(G, N)$), and for all $n \geq 2$ if certain filtration quotients of $N$ and $T$ are torsion-free.

Introduction. The group ring $\mathbb{Z}(G)$ of a group $G$ is naturally filtered by the powers $I^n(G)$, $n \geq 1$, of its augmentation ideal $I(G)$. It is a long-studied problem to determine the so-called augmentation quotients $Q_n(G) = I^n(G)/I^{n+1}(G)$ in terms of the structure of $G$, also because of their close link with the dimension subgroups $D_n(G) = G \cap (1 + I^n(G))$ which can be inductively described as $D_{n+1}(G) = \text{Ker}(D_n(G) \to Q_n(G))$. The groups $Q_n(G)$ were determined for $n = 2$ by Passi [24], Sandling [28] and Losey [20] for abelian, finite and finitely generated groups $G$ and for $n = 3, 4$ and finite $G$ by Tahara [31], [32]; a functorial description for all groups was given for $n = 2$ by Bachmann and Gruenenfelder [2] and for $n = 3$ in [4], based on Quillen’s approximation of the graded ring $\text{Gr}(\mathbb{Z}(G)) = \mathbb{Z} \oplus \bigoplus_{n \geq 1} Q_n(G)$ by the enveloping ring of the Lie ring of $G$, see [20], [25] or section 1 below. With the functorial viewpoint new methods emerge in the field where most work is still based on the classical combinatorial approach:
Quillen’s computation of $\text{Gr}(Q(G))$ relies on Hopf algebra techniques, while the general results on $Q_2(G)$ and $Q_3(G)$ are obtained by homological methods as inaugurated by Passi, see [25]. Recently, Passi and Mikhailov pass from homological to homotopical methods [22], [23]. In [9] and in this paper we combine enveloping algebras and new homological observations to study the more general quotients $Q_n(G, H) = I^{n-1}(G)I(H)/I^n(G)I(H)$ for some subgroup $H$ of $G$; we call these Fox quotients because of their close relation with the classical Fox subgroups $G \cap (1 + I^{n-1}(G)I(H))$. Fox quotients (and some related groups, see [18], [11], [12]) were also extensively studied in the literature, but, except from [9], only under suitable splitting assumptions, in particular when $H$ is a semidirect factor of $G$. In fact, Sandling’s [29] and later Tahara’s work [33] on augmentation quotients of semidirect products $G = N \rtimes T$ had split the study of Fox quotients into two classes of independent problems: the study of certain filtration quotients of $\mathbb{Z}(N)$ and $\mathbb{Z}(T)$ on the one hand and of product filtrations $F_n = \sum \Delta_{n-i}I^i(T)$ on the other hand where $(\Delta_i)_{i \geq 1}$ is one of two natural filtrations of $\mathbb{Z}(N)$, see section 1. In a series of papers Khambadkone and later Karan and Vermani expressed the quotients of these product filtrations in terms of tensor products of the groups $\Delta_{n-i}/\Delta_{n-i+1}$ and $I^i(T)/I^{i+1}(T)$, for low values of $n$ and under additional assumptions, assuming either $G$ finite and $N$ finitely generated or nilpotent [17], [18], [19], or assuming torsion-freeness of sufficiently many filtration quotients of $N$ and $T$ [13], [15], [14], [34]. For a more detailed survey on Fox and augmentation quotients see Passi [25] and Vermani [35].

In this paper we treat the general case, showing that the quotients of the product filtrations above are in fact iterated amalgamated sums of tensor products of the groups $\Delta_{n-i}/\Delta_{n-i+1}$ and $I^i(T)/I^{i+1}(T)$, amalgamated along certain subgroups of torsion products of these groups. We thus completely determine the groups $Q_n(G, H)$ for arbitrary $H = G, N, T$ and $n \leq 3$, and for $n = 4$ with the exception of one of the two direct factors of $Q_4(G, N)$, see section 2. Our description is functorial and in terms of enveloping rings of certain Lie algebras associated with $N$ and $T$, see section 1. If suitable filtration quotients of $N$ and $T$ are torsion-free then our amalgamated sums degenerate to direct sum decompositions; we then express $Q_n(G, H)$ for $H = G, N, T$ in terms of tensor products of enveloping rings as above, thus improving and generalizing similar results of Karan and Vermani for $n \leq 4$ to all $n \geq 1$, see section 3.

The first three sections are devoted to a presentation of the necessary constructions and results while the proofs are deferred to sections 4 and 5.

1 Preliminary constructions and results

In this section we recall and partially generalize constructions and results from the literature which are needed in the sequel.
Let $G$ be a group. An $N$-series $\mathcal{G}$ of $G$ is a descending chain of subgroups

$$G = G_{(1)} \supset G_{(2)} \supset G_{(3)} \supset \ldots$$

such that $[G_{(i)}, G_{(j)}] \subset G_{(i+j)}$ for $i, j \geq 1$, with $[a, b] = aba^{-1}b^{-1}$. A given $N$-series $\mathcal{G}$ induces a descending chain of two-sided ideals of the group ring $\mathbb{Z}(G)$

$$\mathbb{Z}(G) = I^0_G(G) \supset I^1_G(G) \supset I^2_G(G) \supset \ldots$$

by defining $I^n_G(G)$ (for $n \geq 1$) to be the subgroup of $\mathbb{Z}(G)$ generated by the elements

$$(a_1 - 1) \cdots (a_r - 1), \quad r \geq 1, \quad a_i \in G_{(k_i)}, \text{ such that } k_1 + \ldots + k_r \geq n.$$ 

Two examples of N-series are used throughout in this paper:

- the lower central series $\gamma = (G_i)_{i \geq 1}$, $G_1 = G$ and $G_{i+1} = [G_i, G]$, where the inclusion $I(G_n) \subset I^n(G)$ implies that $I^n_G(G)$ equals $I^n(G)$, the $n$-th power of the ideal $I(G)$;

- for a normal subgroup $N$ of $G$ an $N$-series $\mathcal{N} = (N_{(i)})_{i \geq 1}$ of $N$ is defined by $N_{(i)} = [N_{(i-1)}, G]$; note that if $G = NT$ for some subgroup $T$ then $N_{(i)} = [N_{(i-1)}, N][N_{(i-1)}, T]$.

The second example was introduced by Tahara; we here gather some basic results due to him [33] and to Khambadkone [17].

**Theorem 1.1** Suppose that $G$ is the semidirect product of a normal subgroup $N$ and some subgroup $T$. Write $\Lambda_n = I^n_{N}(N)$ and let

$$\mathcal{K}_n = \sum_{i=1}^{n-1} \Lambda_{n-i} I^i(T), \quad \mathcal{K}_n^* = \sum_{i=1}^{n-1} \Lambda_{n-i} I^i(T), \quad \Gamma^*_n = \sum_{i=0}^{n-1} I^{n-i}(T) \Lambda_i.$$ 

Then

$$I(G) = I(N) \oplus I(T) \oplus I(N)I(T) \quad (1)$$

$$= I(T) \oplus I(N) \oplus I(T)I(N) \quad (2)$$

$$I^n(G) = \Lambda_n \oplus I^n(T) \oplus \mathcal{K}_n \quad (3)$$

$$I^n(G)I(T) = I^{n+1}(T) \oplus \mathcal{K}_{n+1} \quad (4)$$

$$I^n(G)I(N) = \Lambda_nI(N) \oplus \Gamma^*_n I(N) \quad (5)$$

$$I(N)I^n(G) = I(N)\Lambda_n \oplus I(N)\mathcal{K}_n^* \quad (6).$$

We need to make the obvious symmetry between the relations (5) and (6) more precise.
Remark 1.2 Let $G$ be a group with distinguished $N$-series $\mathcal{G}$ and subgroup $H$. Then the anti-ring-automorphism $(-)^*: \mathbb{Z}(G) \to \mathbb{Z}(G)$ sending $g \in G$ to $g^{-1}$, called conjugation, carries $I(H)I_{\mathcal{G}}^n(G)$ onto $I_{\mathcal{G}}^n(G)I(H)$ since the subgroups $H$ and $G_{(i)}$ are stable under inversion. Hence it induces an isomorphism of abelian groups

$$I(H)I_{\mathcal{G}}^{n-1}(G)/I(H)I_{\mathcal{G}}^n(G) \cong I_{\mathcal{G}}^{n-1}(G)I(H)/I_{\mathcal{G}}^n(G)I(H).$$

For $G = N \rtimes T$ as above, we also have

$$(I(N)K_n)^* = \left( I(N) \sum_{i=1}^n \Lambda_{n-i}^i(T) \right)^*$$

$$= \sum_{i=1}^n I^i(T) \Lambda_{n-i} I(N)$$

$$= \sum_{j=0}^{n-1} I^{n-j}(T) \Lambda_j I(N)$$

$$= \Gamma^*_n I(N). \quad (7)$$

Now recall that our aim is to determine the filtration quotients

$$Q_n(G) = I^n(G)/I^{n+1}(G)$$

$$Q_n(G, H) = I^{n-1}(G)I(H)/I^n(G)I(H)$$

for $H = N, T$; note that $Q_n(G) = Q_n(G, G)$. The relations above immediately imply the following identities.

$$Q_n(G) = \Lambda_n/\Lambda_{n+1} \oplus Q_n(T) \oplus K_n/K_{n+1} \quad (8)$$

$$Q_n(G, T) = Q_n(T) \oplus K_n/K_{n+1} \quad (9)$$

$$Q_n(G, N) = \Lambda_{n-1}I(N)/\Lambda_n I(N) \oplus \Gamma^*_{n-1}I(N)/\Gamma^*_n I(N) \quad (10)$$

It turns out that the terms on the right hand side of the above identities fall into two categories: first of all, $K_n/K_{n+1}$ and $\Gamma^*_{n-1}I(N)/\Gamma^*_n I(N)$ each of which arises from the product of a filtration of $I(N)$ with one of $I(T)$; the strategy here is to express the quotients of these product filtrations in terms of - tensor and torsion - products of the factors. Once this ”separation of the factors” is achieved (which is the main concern of this paper, see sections 2 and 3) one is left with dealing with the generalized Fox and augmentation quotients of $N$ and $T$, i.e. the groups $Q^K_n(K) = I^*_K(K)/I_{K}^{n+1}(K)$ for $K = N, T$ and $K = N, \gamma$, resp., and $Q^N_n(N, K) = I_{N}^{n-1}(N)I(K)/I_{N}^{n}(N)I(K)$ for some subgroup $K$ of $N$ (here only the case $K = N$ is needed), see the results in sections 2 and 3. The study of these groups requires the following constructions.

The basic idea, due to Quillen [20], is to approximate the groups $Q^G_n(G)$ by means of enveloping algebras. The construction for arbitrary $N$-series $\mathcal{G}$ can be
found in Passi’s book [25], but we recall it here for convenience of the reader and to fix notation.

The abelian group \( L^G = \sum_{i \geq 1} G(i)/G(i+1) \) is a graded Lie ring whose bracket is induced by the commutator pairing of \( G \). So its enveloping algebra \( UL^G \) over the integers is defined. On the other hand, the filtration quotients \( Q_n^G = I_n^G/I_{n+1}^G \) form the graded ring \( Gr^G(\mathbb{Z}(G)) = \oplus_{n \geq 0} Q_n^G \); note that one has \( Gr^n(\mathbb{Z}(G)) = \oplus_{n \geq 0} I_n^G/I_{n+1}^G \). Now the map \( L^G \to Gr^G(\mathbb{Z}(G)) \), \( aG(i+1) \mapsto a - 1 + I_{G(i+1)}^G \) for \( a \in G(i) \), is a homomorphism of graded Lie rings and hence extends to a map of graded rings

\[ \theta^G : UL^G \to Gr^G(\mathbb{Z}(G)). \]

This map is clearly surjective but rarely globally injective; for instance, \( \theta^\gamma \) is injective if \( G \) is cyclic, but is non injective for all non cyclic finite abelian groups [1]. An important favourable case is given by the following result which relies on work of Hartley [10].

**Theorem 1.3** Let \( G \) be a group and let \( G : G = G(1) \supset G(2) \supset \cdots \) be an N-series of \( G \) with torsion-free quotients \( G(i)/G(i+1) \) for all \( i \geq 1 \). Then \( \theta^G \) is an isomorphism.

**Proof:** In a first step, one adapts an argument of Quillen [26] for the case \( G = \gamma \) to show that the epimorphism

\[ \theta^G = \theta^G \otimes Q : U(L^G \otimes Q) \to Gr^G(Q(G)) \]

is an isomorphism: indeed, the image of the canonical map of graded Lie rings \( j_G : L^G \to Gr^G(Q(G)) \) consists of primitive elements (with respect to the Hopf algebra structure induced by the canonical one of \( Q(G) \)), and generates \( Gr^G(\mathbb{Z}(G)) \) as an algebra (by definition of the filtration \( (I_{Q,G}^n)_{n \geq 0} \)). So by the Milnor-Moore theorem \( Gr^G(Q(G)) \) can be identified with the enveloping algebra of the Lie algebra of its primitive elements. Hence it suffices to check that the map \( j_G \otimes Q \) is injective since then by a standard argument the Poincaré-Birkhoff-Witt theorem implies that \( \theta^G \otimes Q \) is injective, too: to recall this, let \( f : g \to g' \) be an injective map of Lie algebras over some field \( K \); we wish to show that the map of \( K \)-algebras \( U(f) : U(g) \to U(g') \) induced by \( f \) is also injective. Denote by \( \text{SP}(g) \) the symmetric algebra over the \( K \)-vector space \( g \), and by \( \text{GR}(U(g)) \) the graded algebra associated with the canonical increasing filtration of \( U(g) \), see [3]. Now under the natural Poincaré-Birkhoff-Witt isomorphism of graded algebras \( \text{SP}(g) \cong \text{GR}(U(g)) \), the map \( \text{GR}(U(f)) : \text{GR}(U(g)) \to \text{GR}(U(g')) \) corresponds to the map \( \text{SP}(f) \), which is injective since if \( r : g \to g' \) is a \( K \)-linear retraction of \( f \) then \( \text{SP}(r) \) is a retraction of \( \text{SP}(f) \). Thus \( \text{GR}(U(f)) \) is injective, too, and hence so is \( U(f) \), since any element of \( U(g) \) lies in a finite step of the increasing filtration.
Now our map $j_G$ is indeed injective by the hypothesis on $G$, as is proved in \cite{10}, see also \cite{25}. Thus also $\theta^G_1$ is injective, whence an isomorphism.

In order to descend to integral coefficients, consider the following sequence of homomorphisms of graded rings.

$$UL^G(G) \xrightarrow{\theta^G} Gr^G(\mathbb{Z}(G)) \xrightarrow{i} Gr^G(\Theta(G)) \xrightarrow{\cong} U(L^G(G) \otimes \Theta) \cong UL^G(G) \otimes \Theta$$

Here $\bar{i}$ is induced by the canonical injection $i: \mathbb{Z}(G) \hookrightarrow \Theta(G)$.

**Proof:** Just note that passing to the quotient $G \twoheadrightarrow G/G_{(m+1)}$ does affect neither $U_1L^G(G)$ nor $Gr_1^G(\mathbb{Z}(G))$ for $i \leq m$; but $G/G_{(m+1)}$ satisfies the hypothesis of Theorem \ref{13} for the N-series $\pi_G : G/G_{(m+1)} \supseteq G_{(2)}/G_{(m+1)} \supseteq \ldots$ of $G/G_{(m+1)}$. Moreover, $L^\pi_G(G/G_{(m+1)})$ being torsion-free so is its enveloping algebra (see \cite{5, Lemma 1.11}) and hence $Gr_1^G(\mathbb{Z}(G(G_{(m+1)}))) \cong Gr_1^G(\mathbb{Z}(G)) = I_G^1(G)/I_G^{i+1}(G)$ for $i \leq m$. Consequently also $I(G)/I_G^{j+1}(G)$ is torsion-free being an iterated extension of the groups $I_G^j(G)/I_G^{j+1}(G)$ for $1 \leq j \leq i$.

If $G$ is an arbitrary N-series, the kernel of $\theta^G$ is a torsion group since $\theta^G_0$ is an isomorphism, but its structure remains widely unknown, even for $G = \gamma$. At least in low degrees the problem is settled; the following result was obtained in \cite{11} for $G = \gamma$ and in \cite{3} for arbitrary N-series $G$.

**Theorem 1.5** The map $\theta_n^G$ is an isomorphism for $n = 1, 2$ and all groups $G$ and N-series $G$ of $G$.

This result is actually generalized in Theorem \ref{17} below.

We need to make Theorem \ref{15} more explicit: it says that $Q^G_2(G) \cong G/G_{(2)}$, and that $Q^G_2(G)$ can be described as follows. Recall that the exterior square $A \wedge A$ of an abelian group $A$ is defined to be the quotient of $A \otimes A$ modulo the subgroup generated by the diagonal elements. We write $G^{ab} = G/G_2$ and $G^{AB} = G/G_{(2)}$, the same for $H$; note that $G^{AB} = L^G_1(G) \cong U_1L^G(G)$. Then we have natural homomorphisms

$$G_{(2)}/G_{(3)} \leftarrow Q^G_2(G) \xrightarrow{p_2} G^{AB} \wedge G^{AB} \xrightarrow{p_2^*} G^{AB} \otimes G^{AB} \xrightarrow{p_2^*} Q^G_2(G) \xrightarrow{p_2^*} G_{(2)}/G_{(3)}$$
defined for \( a, b \in G \) by 
\[ e_2^G(aG(2) \wedge bG(2)) = [a, b]G(3), \quad b_2^G(aG(2) \wedge bG(2)) = aG(2) \otimes bG(2) - bG(2) \otimes aG(2), \quad \mu^G_2(aG(2) \otimes bG(2)) = (a - 1)(b - 1) + I^3_G(G), \] and \( p^G_2(aG(3)) = a - 1 + I^3_G(G) \). Then by Theorem 1.5 the following sequence is exact:

\[
G^{AB} \wedge G^{AB} \xrightarrow{(e_2^G - b_2^G)} G(2)/G(3) \oplus G^{AB} \otimes G^{AB} \xrightarrow{(p^G_2, \mu^G_2)} Q^G_2(G) \rightarrow 0 \quad (11)
\]

In order to describe the third Fox and augmentation quotients of \( G \) we need the following constructions. Let \( G \) be an N-series of \( G, H \) a subgroup of \( G \), and \( \mathcal{H} = (H(i))_{i \geq 1} \) be an N-series of \( H \) such that \( H(i) \subset G(i) \) for \( i \geq 1 \). These data give rise to a filtration

\[
\mathcal{F}^1 = \mathbb{Z}(G)I(H) \supset \mathcal{F}^2 = \mathbb{Z}(G)I(H(2)) + I(G)I(H) \supset \ldots
\]

of \( \mathbb{Z}(G)I(H) \) by sub-\( \mathbb{Z}(G)-\mathbb{Z}(H) \)-bimodules \( \mathcal{F}^n \) defined by

\[
\mathcal{F}^n = \sum_{i \geq 0, j \geq 1} I^i_G(H)I^j_H(H) = \sum_{i \geq 0, j \geq 1} I^i_G(H)I(H(j)).
\]

Note that if \( \mathcal{H} = \gamma \) then \( \mathcal{F}^n = I^{n-1}_G(H)I(H) \) since \( I^i_G(H) = I^i(H) \), and if \( H = G \) and \( \mathcal{H} = G \) then \( \mathcal{F}^n = I^n_G(G) \). The associated graded group \( \text{Gr}^\mathcal{G}(\mathbb{Z}(G)I(H)) = \bigoplus_{n \geq 1} \mathcal{F}^n/\mathcal{F}^{n+1} \) is a graded \( \text{Gr}^\mathcal{G}(\mathbb{Z}(G)) - \text{Gr}^\mathcal{H}(\mathbb{Z}(H)) \)-bimodule in the canonical way, and hence a \( \text{UL}^\mathcal{G}(G) - \text{UL}^\mathcal{H}(H) \)-bimodule via the maps \( \theta^\mathcal{G} \) and \( \theta^\mathcal{H} \).

We now generalize the approximation of the ring \( \text{Gr}^\mathcal{G}(\mathbb{Z}(G)) \) by \( \text{UL}^\mathcal{G}(G) \) to the bimodule \( \text{Gr}^\mathcal{G}(\mathbb{Z}(G))I(H)) \), as follows. The injection \( \iota : H \hookrightarrow G \) induces a canonical map of graded Lie rings \( \text{L}(\iota) : L^\mathcal{H}(H) \rightarrow L^\mathcal{G}(G) \) which extends to a map of graded rings \( \text{UL}(\iota) : U^\mathcal{H}(H) \rightarrow \text{UL}^\mathcal{G}(G) \). It makes \( \text{UL}^\mathcal{G}(G) \) into a \( \text{UL}^\mathcal{H}(H) \)-bimodule, whence the graded \( \text{UL}^\mathcal{G}(G)-\text{UL}^\mathcal{H}(H) \)-bimodule

\[
\text{UL}^\mathcal{G}(G, H) = \text{UL}^\mathcal{G}(G) \otimes_{\text{UL}^\mathcal{H}(H)} \text{UL}^\mathcal{H}(H)
\]

is defined where \( \text{UL}^\mathcal{H}(H) \) denotes the augmentation ideal of \( \text{UL}^\mathcal{H}(H) \). Now let the surjective map of \( \text{UL}^\mathcal{G}(G) \)-\( \text{UL}^\mathcal{H}(H) \)-bimodules

\[
\theta^\mathcal{G}: \text{UL}^\mathcal{G}(G, H) \rightarrow \text{Gr}^\mathcal{G}(\mathbb{Z}(G)I(H))
\]

be defined as follows: as \( I^i_H(H) \subset \mathcal{F}^n \) for \( n \geq 1 \), we obtain a map of graded left \( \text{Gr}^\mathcal{H}(\mathbb{Z}(H)) \)-modules \( \text{Gr}^\mathcal{H}(I(H)) \) to \( \text{Gr}^\mathcal{G}(\mathbb{Z}(G)I(H)) \) which by precomposition with \( \theta^\mathcal{H} \) gives rise to a map of graded left \( \text{UL}^\mathcal{H}(H) \)-modules \( \text{UL}^\mathcal{H}(H) \rightarrow \text{Gr}^\mathcal{G}(\mathbb{Z}(G)I(H)) \). Now \( \theta^\mathcal{G} \) is obtained by extension of scalars along the map \( \text{UL}(\iota) \). More explicitly, for \( i \geq 0, j \geq 1 \) such that \( i + j = n \), \( x \in \text{UL}^\mathcal{G}(G) \), \( y \in \text{UL}^\mathcal{H}(H) \), \( x' \in I^i_G(G) \) and \( y' \in I^j_H(H) \) such that \( \theta^\mathcal{G}(x) = x' + I^{i+1}_G(G) \) and \( \theta^\mathcal{H}(y) = y' + I^{j+1}_H(H) \), one has \( \theta^\mathcal{G}_n(x \otimes y) = x'y' + \mathcal{F}^{n+1} \). Note that for \( H = G \)
and \( H = G \), \( \theta^G = \theta^G \mu^G \) where \( \mu^G : U^G(G, G) \rightarrow U^G(G) \) is the canonical isomorphism.

We now study the map \( \theta^{GH} \) in degree \( n \leq 3 \). To start with, it is clear that
\[
U^G_1(G, H) \cong U_1L^H(H) \cong L^H_1(H) \cong H^{AB}.
\] (13)

The following convention turns out to be convenient throughout the rest of this paper.

**Convention 1.6** For a group \( K \) with \( N \)-series \( K = (K_{(i)})_{i \geq 1} \) and \( a \in K_{(i)} \) we consider the coset \( aK_{(i+1)} \in K_{(i)}/K_{(i+1)} = L^K_i(K) \) also as an element of \( U_iL^K_i(K) \), thus suppressing the canonical map \( \nu : L^K_i(K) \rightarrow UL^K_i(K) \) from the notation. Moreover, all sums, powers and products of cosets enclosed between brackets, of the type \( (aK_{(i+1)}) \) with \( a \in K_{(i)} \), are understood to be taken in the ring \( UL^K_i(K) \).

Now for \( i \geq 0 \) and \( j \geq 1 \) let
\[
\nu_{ij} : U_iL^G(G) \otimes U_jL^H(H) \rightarrow U^G_{i+j}(G, H)
\]
be the canonical map, and
\[
u : UL^H(H) \rightarrow U^G(H, H)
\]
be the map of graded \( UL^H(H) \)-bimodules given by \( \nu(x) = 1 \otimes x \). Finally, let
\[
q^G_n : G/G_n \rightarrow G/G(n)
\]
be the canonical quotient map induced by the inclusion \( G_n \subset G(n) \); and maps denoted by \( \pi_k \) for some or no index \( k \) always denote canonical quotient maps.

**Theorem 1.7** The maps \( \theta^{GH}_n \) are isomorphisms for \( n = 1, 2 \) and all \( G, H, G \) and compatible \( H \) as above.

Note that taking \( (H, H) = (G, G) \) we thus recover Theorem 1.5.

**Proof:** For \( n = 1 \), we use the classical isomorphism
\[
D_H : H^{ab} \cong \frac{\mathbb{Z}(G)I(H)}{I(G)I(H)}, \quad D_H(hH_2) = h - 1 + I(G)I(H),
\] (14)
due to Whitcomb, see [36], or [9, Proposition 3.1] for an easy homological proof of a more general fact. We obtain isomorphisms
\[
U^G_1(G, H) \cong H^{AB} \cong \frac{(H/H_2)}{\text{Im}(H_{(2)})} \overset{\nu_{11}}{\cong} \frac{\mathbb{Z}(G)I(H)}{I(G)I(H)} / \text{Im}(\mathbb{Z}(G)I(H_{(2)})) \cong \mathcal{F}_1/\mathcal{F}_2
\]
whose composite is \( \theta^{GH}_1 \).
Now let \( n = 2 \). As in [9], our arguments are most conveniently formulated in the language of pushouts of abelian groups, thereby using their elementary properties, in particular the gluing of pushouts and the link between the kernels of parallel maps in a pushout square, see [27].

Consider the following commutative diagram where \( D_1 \) is given by the projection to the cokernel of the left-hand map followed by the isomorphism \( D_1 \) in (14), and where \( j_1 \) is induced by the corresponding injection and \( D' \) is given by restriction of \( j_1 D_{H(3)} \).

\[
\begin{array}{ccc}
I(G)I(H) & \xrightarrow{\iota} & \mathbb{Z}(G)I(H) \\
I^2_G(G)I(H) + I(G)I^2_H(H) & \xrightarrow{\pi_1} & I^2_G(G)I(H) + I(G)I^2_H(H) \\
H_{(3)} \cap H_2 & \xrightarrow{D_1} & H_{(3)} \\
[\mathcal{H}_{(3)}, \mathcal{H}_{(3)}] & \xrightarrow{\pi_3} & H_{(3)} \cap H_2
\end{array}
\]

As both rows are exact, we see that

\[
\frac{I(G)I(H)}{I^2_G(G)I(H) + I(G)I^2_H(H)} \cap \operatorname{Im}(j_1) = \frac{I(H(3) \cap H_2) + I^2_G(G)I(H) + I(G)I^2_H(H)}{I^2_G(G)I(H) + I(G)I^2_H(H)}
\]

Now consider the following commutative diagram where \( \mu((gG_2) \otimes (hH_2)) = (g-1)(h-1) + I^2_2(G)I(H) \), and similarly for \( \mu' \); \( D_2(h'H_3) = h' - 1 + I^2_2(G)I(H) \) for \( h \in H_2 \), and similarly or \( D'_2 \).

\[
\begin{array}{ccc}
H^{ab} \cap H^{ab} & \xrightarrow{\iota_2} & G^{AB} \otimes H^{ab} \\
H^{ab} \otimes \mathcal{G}^{ab} \otimes 1_{\mathcal{H}}^2 & \xrightarrow{\iota_3} & G^{AB} \otimes H^{AB} \\
\mu_2 & \xrightarrow{\mu} & \mu'
\end{array}
\]

The upper left-hand square is a pushout by [9] Theorem 3.6; the upper right-hand square is a pushout since \( \pi_1 \) is surjective and \( \operatorname{Ker}(\pi_1) = \mu \operatorname{Ker}(1 \otimes q_2^H) \). By
ghuing of pushouts it follows that the upper exterior rectangle also is a pushout. Furthermore, the lower rectangle is a pushout since \( \pi_3 \) is surjective and \( \text{Ker}(\pi_3) = D_2 \pi_1 \text{Ker}(\pi_2) \) by (16), so again by gluing the two pushout rectangles the exterior square of the whole diagram also is a pushout. Thus

\[
\text{Ker}(\pi_3 \mu') = (q_2^G \iota^{ab} \otimes q_2^H) l_2^\gamma \text{Ker}(\pi_2 c_2^\gamma)
\]

We can simplify this description of \( \text{Ker}(\pi_3 \mu') \) by observing that the following diagram commutes:

\[
\begin{array}{ccc}
H^{ab} \otimes H^{ab} & \xrightarrow{(q_2^G \iota^{ab} \otimes q_2^H) l_2^\gamma} & G^{AB} \otimes H^{AB} \\
\downarrow \pi_2 c_2^\gamma & \pi_2 c_2^\gamma & \downarrow (\iota^{AB} \otimes 1) l_2^\gamma \\
H_2/\langle \bar{H}(3) \rangle & \xrightarrow{j_2} & H_2/\langle \bar{H}(3) \rangle \\
\end{array}
\]

It follows that

\[
\text{Ker}(\pi_3 \mu') = (\iota^{AB} \otimes 1) l_2^\gamma \text{Ker}(c_2^\gamma) \quad (17)
\]

Consider the following diagram of plain arrows.

\[
\begin{array}{cccc}
I(G)I(H) + \mathbb{Z}(G)I(H(3)) & \xrightarrow{j_2} & F_2/F_3 & \xrightarrow{\pi_4} & \text{Coker}(j_2) & \xrightarrow{\beta} & H_2/H(3) \\
\downarrow & \downarrow j_3 & \downarrow & \downarrow & \downarrow & \downarrow \\
I(G)I(H) + \mathbb{Z}(G)I(H(3)) & \xrightarrow{\pi_5} & \mathbb{Z}(G)I(H) & \xrightarrow{(\overline{D_H})^{-1}} & H \\
\end{array}
\]

As both rows are exact in the second group, the map \( \pi_{5,j3} \) induced by \( \pi_{5,j3} \) is injective. Moreover, \( \text{Im}(\overline{D_H})^{-1} \circ \pi_{5,j3} \) is exact, whence the map \( \beta \) exists and is an isomorphism. Together with (17) we obtain an exact sequence

\[
\text{Ker}(c_2^\gamma) \xrightarrow{(\iota^{AB} \otimes 1) l_2^\gamma} G^{AB} \otimes H^{AB} \xrightarrow{j_2 \pi_5 \mu'} F_2/F_3 \xrightarrow{\beta \pi_4} H_2/H(3) \rightarrow 0 \quad (18)
\]

As to \( U_2^{GH}(G, H) \), consider the following diagram

\[
\begin{array}{cccc}
H^{AB} \otimes H^{AB} & \xrightarrow{l_2^\gamma} & H^{AB} \otimes H^{AB} & \xrightarrow{\iota^{AB} \otimes 1} & G^{AB} \otimes H^{AB} \\
\downarrow c_2^\gamma & \downarrow \mu_2^H \iota_1^{HM} & \downarrow \iota_1^{GH} & \downarrow & \\
H_2/H(3) & \xrightarrow{\nu_2} & U_2 L^H(H) & \xrightarrow{\nu_2} & U_2^{GH}(G, H) \\
\end{array}
\]

One easily checks that both squares are pushouts, by construction of the enveloping algebra and of \( U_2^{GH}(G, H) \), resp., and taking the gradings into account. Thus the
Thus it suffices to check (21) after precomposition with \(u\). The lemma implies that it is an isomorphism. But the identity is immediate. \(\square\)

In particular, \(\nu_{11}\) is computed for \(H(2)/H_2H(3) = \Coker(c^H_2) \xrightarrow{\nu_{11}} \Coker(\nu_{11}^G)\) induced by \(u_2\nu_2\). These facts can be rewritten as an exact sequence

\[
\text{Ker}(c^H_2) \xrightarrow{(i^{AB} \otimes 1)^H} \text{G}^{AB} \otimes \text{H}^{AB} \xrightarrow{\nu_{11}^G} \text{U}_{2}^G(G, H) \xrightarrow{(\nu_{11}^G)^{-1}\pi_6} H(2)/H_2H(3) \rightarrow 0 \tag{20}
\]

where \(\pi_6: \text{U}_{2}^G(G, H) \rightarrow \Coker(\nu_{11}^G)\) is the quotient map. Now comparing sequences (18) and (20) we see that it suffices to show that the map \(\theta_2^G: \text{U}_{2}^G(G, H) \rightarrow \mathcal{F}_2/\mathcal{F}_3\) commutes with the incoming and outgoing maps, since then the five-lemma implies that it is an isomorphism. But the identity \(\theta_2^G \nu_{11}^G = j_2\pi_3\mu'\) is readily checked by going through the definitions; and to check the identity

\[
\beta\pi_4\theta_2^G = (u_2\nu_2)^{-1}\pi_6 \tag{21}
\]

first note that the two pushout squares in (19) imply that

\[
\text{U}_{2}^G(G, H) = \text{Im}(u_2) + \text{Im}(\nu_{11}^G) = \text{Im}(u_2\nu_2) + \text{Im}(u_2\mu_2^H\nu_{11}^H) + \text{Im}(\nu_{11}^G) = \text{Im}(u_2\nu_2) + \text{Im}(\nu_{11}^G)
\]

Thus it suffices to check (21) after precomposition with \(u_2\nu_2\) and \(\nu_{11}^G\), which is immediate.

As to the map \(\theta_2^G\) for \(n \geq 3\), part of its kernel is determined in [9] for \(\mathcal{H} = \gamma\); indeed, all arguments there remain valid for arbitrary \(\mathcal{H}\) as above, thus providing an explicitly defined subgroup \(\mathcal{R}_{n}^G\) of \(\text{U}_{11}^H(G, H)\) contained in \(\text{Ker}(\theta_2^G)\). In particular, one has \(\mathcal{R}_{n}^G = 0\) for \(n = 1, 2\), and \(\mathcal{R}_{3}^G\) is generated by the elements

\[
1 \otimes (cH(4)) - \sum_{q=1}^{p}(a_qG(3)) \otimes (b_qH(2)) - (b_qG(3)) \otimes (a_qH(2)) \tag{22}
\]

where \(p \geq 1\), \(a_q, b_q \in H \cap G(2)\) such that \(c = \prod_{q=1}^{p}[a_q, b_q] \in H(3)\). It is shown in [9] that \(\mathcal{R}_{n}^G \tilde{\otimes} \text{UL}^H(\mathcal{H}) = 0\), so the quotient group

\[
\tilde{\text{U}}^G(G, H) \overset{def}{=} \text{U}^G(G, H)/\text{UL}^G(G) \sum_{n \geq 3} \mathcal{R}_{n}^G
\]

is a graded \(\text{UL}^G(G)\)–\(\text{UL}^H(\mathcal{H})\)-bimodule, and \(\theta_2^G\) induces a surjective homomorphism of graded \(\text{UL}^G(G)\)–\(\text{UL}^H(\mathcal{H})\)-bimodules

\[
\tilde{\theta}_2^G: \tilde{\text{U}}^G(G, H) \twoheadrightarrow \text{Gr}^G(\mathbb{Z}(G)I(\mathcal{H})).
\]

In particular, \(\tilde{\text{U}}^G(G, G) = \text{U}^G(G, G)\) and \(\tilde{\theta}_{GG} = \theta_{GG} = \theta_G^G\mu^G\), by construction of \(\mathcal{R}_{n}^G\). While we have no information about \(\text{Ker}(\tilde{\theta}_{n}^G)\) for \(n \geq 4\) it was computed for
\(n = 3\) in the cases needed in this paper, namely for \(H = \gamma\) in [9] and for \((H, H) = (G, G)\) in [4]. Both results involve the torsion operator \(\delta_{1}^{GH}\) described below which basically is the difference between a left and a right connecting homomorphism; it also describes non trivial torsion relations in the homology of nilpotent groups (see [6]) and thus plays an important role in dimension subgroups [8], whence seems to be quite a fundamental phenomenon.

In fact, all torsion operators in this paper arise from decreasing filtrations
\[\Delta : I(K) = \Delta_{1} \supset \Delta_{2} \supset \cdots\]
of \(I(K)\) for some group \(K\) by subgroups \(\Delta_{i}\); such a filtration gives rise to short exact sequences
\[0 \to \Delta_{p+1} / \Delta_{p+2} \overset{\alpha}{\to} \Delta_{p} / \Delta_{p+2} \overset{\rho}{\to} \Delta_{p} / \Delta_{p+1} \to 0\]  
for \(p \geq 1\) where \(\alpha, \rho\) denote the canonical injection and quotient map, resp.

Moreover, we need Passi’s polynomial groups with respect to \(G\) which are denoted by \(P_{n}^{G}(G) = I_{G}^{n+1}(G) / I_{G}^{n+1}(G)\) see [25]. By Theorem 1.5 sequence (23) for \((K, \Delta_{i}) = (G, I_{G}^{i}(G))\) and \(p = 1\) gives rise to a natural exact sequence
\[0 \to U_{2}L^{G}(G) \overset{\bar{\mu}_{2}^{G}}{\to} P_{2}^{G}(G) \overset{\rho_{2}^{G}}{\to} G^{AB} \to 0 .\]  
with \(\bar{\mu}_{2}^{G} = \alpha \theta_{2}^{G}\) and \(\rho_{2}^{G} = (\theta_{1}^{G})^{-1} \rho\). Tensoring this sequence by \(H^{AB}\) and the analogous sequence for \(H\) by \(G^{AB}\) gives rise to natural exact sequences
\[\text{Tor}_{1}^{Z}(G^{AB}, H^{AB}) \overset{\tau_{G}}{\to} U_{2}L^{G}(G) \otimes U_{1}L^{H}(H) \overset{\bar{\mu}_{2}^{G} \otimes \text{id}}{\to} P_{2}^{G}(G) \otimes H^{AB} \overset{\rho_{2}^{G} \otimes \text{id}}{\to} G^{AB} \otimes H^{AB} \to 0\]  
\[\text{Tor}_{1}^{Z}(G^{AB}, H^{AB}) \overset{\tau_{H}}{\to} U_{1}L^{G}(G) \otimes U_{2}L^{H}(H) \overset{id \otimes \bar{\rho}_{1}^{H}}{\to} G^{AB} \otimes P_{2}^{H}(H) \overset{id \otimes \rho_{2}^{H}}{\to} G^{AB} \otimes H^{AB} \to 0\]  
see [21] Theorem V.6.1. Then define the torsion operator
\[\delta_{1}^{GH} = \nu_{12} \tau_{H} - \nu_{21} \tau_{G} : \text{Tor}_{1}^{Z}(G^{AB}, H^{AB}) \to U_{3}^{GH}(G, H) .\]  
To describe \(\delta_{1}^{GH}\) more explicitly we recall from [21] V.6 the description of explicit canonical generators of the torsion product \(\text{Tor}_{1}^{Z}(A, B)\) of abelian groups \(A, B\). Suppose that \(A = A_{1} / A_{2}\) and \(B = B_{1} / B_{1}\) with \(A_{1}, B_{1}\) some (non necessarily abelian) groups. Then these generators are of the form
\[\langle aA_{2}, k, bB_{2} \rangle \text{ with } a \in A_{1}, k \in \mathbb{Z}, b \in B_{1} \text{ such that } a^{k} \in A_{2} \text{ and } b^{k} \in B_{2} .\]  
As a model for all subsequent calculations, we give a detailed computation of \(\delta_{1}^{GH}\), as follows.
Lemma 1.8 Let \( \langle \bar{g}, k, \bar{h} \rangle \) be a canonical generator of \( \text{Tor}_1^\mathbb{Z}(G^{AB}, H^{AB}) \). Then

\[
\delta_1^\mathcal{G} \langle \bar{g}, k, \bar{h} \rangle = \bar{g} \otimes (h^k H_{(3)}) - (g^k G_{(3)}) \otimes \bar{h} + \left(\frac{k}{2}\right)(g^2 \otimes \bar{h} - \bar{g} \otimes \bar{h}^2) \tag{29}
\]

where we recall that the squares \( \bar{g}^2, \bar{h}^2 \) are taken in the rings \( \text{UL}^G(G) \) and \( \text{UL}^H(H) \), resp.

Proof: We give the computation of \( \nu_{21} \tau_\mathcal{G} \), the one of \( \nu_{12} \tau_\mathcal{H} \) being symmetric to this one.

Throughout in this paper, we use the following fundamental relations in \( \mathbb{Z}(G) \), for \( a \in G_{(i)}, b \in G_{(j)}, k \in \mathbb{Z} \) and \( n \geq 2 \) where \([a - 1, b - 1] \) denotes the ring commutator in \( \mathbb{Z}(G) \).

\[
ab - 1 = (a - 1) + (b - 1) + (a - 1)(b - 1) \\
\equiv (a - 1) + (b - 1) \mod I_G^{i+j}(G) \tag{30}
\]

\[
a^k - 1 = (1 + (a - 1))^k - 1 \\
\equiv \sum_{p=1}^{n-1} \binom{k}{p} (a - 1)^p \mod I_G^{ni}(G) \tag{31}
\]

\[
[a, b] - 1 = [a - 1, b - 1]a^{-1}b^{-1} \\
= [a - 1, b - 1] + [a - 1, b - 1](a^{-1}b^{-1} - 1) \\
\equiv [a - 1, b - 1] \mod I_G^{i+j+1}(G) \tag{32}
\]

Now using again \cite{21} Theorem V.6.1] one has

\[
\nu_{21} \tau_\mathcal{G} \langle \bar{g}, k, \bar{h} \rangle = \nu_{21} ((\alpha \theta_\mathcal{G}^2)^{-1} (k, \rho^{-1} \theta_\mathcal{G}^1(\bar{g})) \otimes \bar{h}) \\
= \nu_{21} ((\theta_\mathcal{G}^2)^{-1} \alpha^{-1} (k, \rho^{-1} ((g - 1) + I_G^3(G))) \otimes \bar{h}) \\
= \nu_{21} ((\theta_\mathcal{G}^2)^{-1} \alpha^{-1} (k(g - 1) + I_G^3(G)) \otimes \bar{h}) \\
= \nu_{21} ((\theta_\mathcal{G}^2)^{-1} ((g^k - 1) - \left(\frac{k}{2}\right)(g - 1)^2 + I_G^3(G)) \otimes \bar{h}) \quad \text{by (31)} \\
= \nu_{21} (\left(\frac{k}{2}\right)(gG_{(2)})^2 \otimes \bar{h}) \\
= (g^k G_{(3)}) \otimes \bar{h} - \left(\frac{k}{2}\right)(gG_{(2)})^2 \otimes \bar{h}
\]

as desired. \( \square \)

If \((H, \mathcal{H}) = (G, \mathcal{G})\) we write \( \delta_1^\mathcal{G} = \mu_3^\mathcal{G} \delta_1^{ GG} \). This map completely determines the structure of \( Q_3^\mathcal{G}(G) \), see \cite{3}. 

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Theorem 1.9 The following natural sequence of homomorphisms is exact:
\[
\text{Tor}^2(G^{AB}, G^{AB}) \xrightarrow{\delta \gamma} U_3 L^\gamma(G) \xrightarrow{\theta^\gamma} Q^\gamma_3(G) \rightarrow 0.
\]

For \( H \neq G \), or \( H = G \) but \( \mathcal{H} \neq \mathcal{G} \), however, the structure of \( \text{Ker}(\beta^\mathcal{H}_3) \) is more complicated, see \([9]\) for the case \( \mathcal{H} = \gamma \); in the special case of interest here it is described in the next section. The structure of the related groups \( U^{N\gamma}_n(N, N) \) for \( n = 2, 3 \) is determined by the following result, cf. \([9, \text{Proposition 5.2}]\). Recall that here \( N^{AB} = N/[N, G] = N/[N, N][N, T] \), and let \( \iota_2^{N\gamma} = (q_2^N \otimes id)(\iota_2^3) : N^{ab} \otimes N^{ab} \rightarrow N^{AB} \otimes N^{ab} \).

Proposition 1.10 There are canonical isomorphisms
\[
U^{N\gamma}_2(N, N) \cong (N^{AB} \otimes N^{ab}) \overset{l_2^{N\gamma}\text{Ker}(c_2^{N\gamma})}{\longrightarrow} (N^{AB} \otimes l_2^{N\gamma}\text{Ker}(c_2^{N\gamma})) \oplus l_3^N\text{Ker}(c_3^N)
\]
whose inverse maps are induced by multiplication in \( Z(G) \). Here the homomorphisms \( L^\gamma_3(N) \xleftarrow{c_3^N} (N^{ab})^\otimes_3 \xrightarrow{l_3^N} (N^{ab})^\otimes_3 \) are defined such that for \( x, y, z \in N^{ab} \), \( c_3^N(x \otimes y \otimes z) \) is the triple Lie bracket \([x, [y, z]]\) in the Lie algebra \( L^\gamma(N) \) and \( l_3^N(x \otimes y \otimes z) \) is the triple Lie bracket \([x, [y, z]]\) in the tensor algebra \( T(N^{ab}) \). Furthermore, we note \( i^{N\gamma\gamma} = id \otimes q_2^N \otimes id : N^{AB} \otimes N^{ab} \otimes N^{ab} \rightarrow N^{AB} \otimes N^{AB} \otimes N^{ab} \) and \( i^{N\gamma\gamma} = q_2^N \otimes q_2^N \otimes id : N^{ab} \otimes N^{ab} \otimes N^{ab} \rightarrow N^{AB} \otimes N^{AB} \otimes N^{ab} \).

Actually, the first isomorphism can be easily deduced from the fact that the exterior rectangle in diagram \((19)\) is a pushout: for \( \mathcal{H} = \gamma \), \( c^\mathcal{H}_2 \) is surjective, hence so is \( \nu^\mathcal{H}_1 \). Consequently, the map
\[
\nu^\mathcal{H}_1 : G^{AB} \otimes H^{ab} \bigg/ (\iota^{AB} \otimes 1)l_2^{\gamma}\text{Ker}(c_2^{\gamma}) \overset{\cong}{\longrightarrow} U_2^{\mathcal{H} \gamma}(G, H)
\]
induced by \( \nu^\mathcal{H}_1 \) is an isomorphism.

Moreover, the group \( U^{N\gamma}_2(N, N) \) can be embedded into a natural exact sequence, as follows. Consider the following part of a 6-term exact sequence
\[
\text{Tor}^2(N^{AB}, N^{AB}) \xrightarrow{\tau} N^{AB} \otimes (N_2/N_2) \xrightarrow{id \otimes i} N^{AB} \otimes N^{ab} \xrightarrow{id \otimes q_2^N} N^{AB} \otimes N^{AB} \rightarrow 0
\]
and let the map \([,]\) \(N^{AB} \otimes (N_2/N_3) \to N_2/[N_2, N_2]N_3\) be induced by the commutator pairing of \(N\), so that \([,]_{\tau}(n_1N_2, k, n_2N_2) = [n_1, n_2^k][N_2, N_2]N_3\). Furthermore, it follows from commutativity of the exterior rectangle in (19) (for \(H = G = N, \mathcal{G} = \mathcal{N}\) and \(\mathcal{H} = \gamma\)) that there is a homomorphism

\[
\overline{l}_2 : N_2/[N_2, N_2]N_3 \to U_2^{N^{\gamma}}(N, N)
\]

such that for \(n_1, n_2 \in N\) one has \(l_2([n_1, n_2]) = \nu^{N^{\gamma}}_{11}((n_1N_2) \otimes (n_2N_2) - (n_2N_2) \otimes (n_1N_2))\). Finally, for an abelian group \(A\) and \(m \geq 1\) let \(SP^m(A) = A^{\otimes m}/\Sigma_m\) denote the symmetric \(m\)-fold tensor product, and let \(\sigma_m = \sigma_m^A : A^{\otimes m} \to SP^m(A)\) be the canonical projection.

**Proposition 1.11** The following sequence of natural homomorphisms is exact:

\[
\text{Tor}_1^Z(N^{AB}, N^{AB}) \xrightarrow{[\tau]} N_2/[N_2, N_2]N_3 \xrightarrow{\overline{l}_2} U_2^{N^{\gamma}}(N, N) \xrightarrow{\overline{\sigma}_2(1 \otimes q_2^N)} SP^2(N^{AB}) \to 0 \tag{34}
\]

Moreover, \(\overline{\sigma}_2(1 \otimes q_2^N)\) has a homomorphic splitting if \(q_2^N\) does and \(N^{AB}\) is either finitely generated or uniquely 2-divisible.

Note that in the case \(\mathcal{N} = \gamma\) the assertion reduces to the well-known exact sequence

\[
0 \to N_2/N_3 \to U_2^{N}(N) \to SP^2(N^{AB}) \to 0 \tag{35}
\]

and its splitting property if \(N^{AB}\) is either finitely generated or uniquely 2-divisible, cf. [25]. For arbitrary \(\mathcal{N}\), the latter facts can in fact be deduced from the left-hand pushout square in (19), in the same way as we now deduce the proposition from the exterior pushout rectangle in (19).

**Proof of Proposition 1.11:** Recall that the exterior rectangle in (19) is a pushout. This implies the identity \(\text{Ker}(u_2\nu_2) = c_2^N\text{Ker}((1^{AB} \otimes 1)l_2)\) and an isomorphism \(\text{Coker}(u_2\nu_2) \cong \text{Coker}((1^{AB} \otimes 1)l_2)\). But here \(1^{AB} = q_2^N\), whence the exactness of sequence (34) in \(N_2/[N_2, N_2]N_3\) follows from Lemma 2.7 in [7]. Moreover, by right-exactness of the tensor product, the kernel of the composite map \(N^{AB} \otimes N^{ab} \xrightarrow{1 \otimes q_2^N} \to N^{AB} \otimes N^{AB} \xrightarrow{\overline{\sigma}_2} SP^2(N^{AB})\) equals \(\text{Im}((1^N \otimes 1)l_2)\), whence there is an isomorphism \(\text{Coker}((1^N \otimes 1)l_2) \cong SP^2(N^{AB})\) induced by \(\sigma_2(1 \otimes q_2^N)\). This implies exactness of sequence (34) in \(U_2^{N}(N, N)\). Finally, the splitting assertion follows from the well-known fact that \(\sigma_2^A\) splits if \(A\) is either finitely generated or uniquely 2-divisible: if \(s_1, s_2\) are splittings of \(q_2^N\) and of \(N^{AB}_{\mathcal{N}}\), resp., \(\nu_{11}^{N^{\gamma}}(1 \otimes s_1)s_2\) is a splitting of \(\overline{\sigma}_2(1 \otimes q_2^N)\).

\[
\square
\]

2 The first four Fox and augmentation quotients

Throughout this section \(G\) is supposed to be the semidirect product of a normal subgroup \(N\) with some subgroup \(T\). The following groups are given a complete
functorial description in terms of the Lie algebras \( L^N(N) \) and \( L^\gamma(T) \), for all \( G = N \rtimes T \):

- the quotients \( Q_n(G) \), \( Q_n(G, T) \) and \( Q_n(G, N) \) for \( n \leq 3 \);

- the direct factors \( K_4/K_5 \) of \( Q_4(G) \) and \( Q_4(G, T) \) and the direct factor \( \Gamma_3^N(I(N)/\Gamma^N_4(I(N)) \text{ of } Q_4(G, N) \), see \([8],[9]\) and \([10]\).

The groups \( Q_4^N(G) \), and hence the direct factors \( Q_4^N(N) \) and \( Q_4^N(T) \) of \( Q_4(G) \) and \( Q_4(G, T) \), were determined for finite \( G \) by Tahara \([32]\); so the only term for \( n = 4 \) whose structure remains almost completely unknown is the direct factor \( \Lambda_3 I(N)/\Lambda_4 I(N) \) of \( Q_4(G, N) \) (it is only computed under very restrictive assumptions in Corollary \([5],[6]\) below).

The proofs of all results of this section are deferred to section 4.

The groups \( Q_2(G, K) \) were determined by Tahara \([32]\) for \( K = G \) and by Karan and Vermaani \([12],[13]\) for \( K = N, T \), after partial results of Khambadkone \([17],[18]\); we quote the results here (expressed in the language of enveloping rings) for completeness but also because they are easily reproved using our general approach, see section 4.

**Theorem 2.1** There are natural isomorphisms

\[
Q_2(G) \cong U_2 L^N(N) \oplus U_2 L(T) \oplus N^{AB} \otimes T^{ab}
\]

\[
Q_2(G, T) \cong U_2 L(T) \oplus N^{AB} \otimes T^{ab}
\]

\[
Q_2(G, N) \cong U_2^\gamma(N, N) \oplus T^{ab} \otimes N^{ab}
\]

Our description of \( Q_n(G, H) \) for \( n = 3, 4 \) below involves various torsion operators coming from connecting homomorphisms as in \([21\) Theorem V.6.1]. To keep notations simple we denote by \( \tau^p_k, \tau^q_k, k = 1, 2 \) and \( \Box \) some (or no) superscript, a connecting homomorphism induced by a short exact sequence of abelian groups in the \( k \)-th variable. In particular, for \( p, q = 1, 2 \) we have maps

\[
\frac{U_{p+1} L^N(N)}{\text{Im}(\delta^p)} \otimes U_q L^\gamma(T) \xrightarrow{\tau^p_{1q}} \text{Tor}_{1}^\gamma(U_{p} L^N(N), U_q L^\gamma(T)) \xrightarrow{\tau^p_{2q}} U_{q} L^N(N) \otimes \frac{U_{q+1} L^\gamma(T)}{\text{Im}(\delta^q_2)}
\]

with \( \delta^1_1, \delta^1_2 = 0 \), \( \delta^2_1 = \delta^N_1 \) and \( \delta^2_2 = \delta_1^\gamma \), see \([27]\), where

- \( \tau^1_{1q} \) and \( \tau^2_{1q} \) are induced by the short exact sequence \([24]\) for \((G, \mathcal{G}) = (N, N)\) and \((T, \gamma)\), resp.; explicitly, we have

\[
\tau^1_{1q}(nN(2), k, x) = \left( (n^k N(3)) - \binom{k}{2} (nN(2))^2 \right) \otimes x \quad (37)
\]

\[
\tau^2_{1q}(y, k, tT_2) = y \otimes \left( (t^k T_3) - \binom{k}{2} (tT_2)^2 \right) \quad (38)
\]

for suitable \( n, t, x, y, k \); cf. the calculation of \( \tau_\mathcal{G} \) in the proof of Lemma \([1.8]\).
• $\tau_{1}^2$ and $\tau_{2}^2$ are induced by the short exact sequence

$$0 \rightarrow \text{Coker}(\delta_{1}^G) \xrightarrow{\rho_{2}^G} I_{\mathcal{G}}^2(G)/I_{\mathcal{G}}^1(G) \xrightarrow{\rho_{1}^G} U_2L^G(G) \rightarrow 0 \quad (39)$$

obtained from sequence (23) for $p = 2$ combined with Theorems 1.9 and 1.5 for $(G, \mathcal{G}) = (N, \mathcal{N})$ and $(T, \gamma)$, resp.

The maps $\tau_{1}^2$ and $\tau_{2}^2$, occurring in Theorem 2.21 below, can be made explicit if $N^{ab}$ and $T^{ab}$ are finitely generated, by the following formula (and its mirror-symmetric version, permuting $G$ and $A$, which gives the corresponding map $\tau_{2}$).

**Proposition 2.2** Let $G$ be a group with $N$-series $\mathcal{G}$ and $A$ be an abelian group such that $G^{AB}$ is finitely generated. Let $\text{Tor}(G^{AB}) = \bigoplus_{i=1}^{m} \mathbb{Z}/e_{i}\mathbb{Z}(g_{i}G_{(2)})$, $g_{i} \in G$, be a cyclic decomposition of the torsion subgroup of $G^{AB}$, and let $e_{ij}$ be the greatest common divisor of $e_{i}$ and $e_{j}$, and $p_{ij}, q_{ij} \in \mathbb{Z}$ such that $e_{ij} = e_{i}p_{ij} + e_{j}q_{ij}$. Furthermore, let $q_{3}: U_3L^G(G) \rightarrow \text{Coker}(\delta_{1}^G)$ be the quotient map. Then the connecting homomorphism

$$\tau_{1} : \text{Tor}_{1}^2(U_2L^G(G), A) \rightarrow \text{Coker}(\delta_{1}^G) \otimes A$$

induced by sequence (39) is given as follows. Let $\langle x, k, a \rangle$ be a canonical generator of $\text{Tor}_{1}^2(U_2L^G(G), A)$. According to the split exact sequence (39), $x$ can be uniquely written in the form $x = (gG_{(3)}) + \sum_{1 \leq i \leq j \leq m} l_{ij}(g_{i}G_{(j)}) \otimes (g_{j}G_{(j)})$ with $g \in G_{(2)}$, $l_{ij} \in \mathbb{Z}$ such that $g^{k} \in G_{(3)}$ and $e_{ij}$ divide $kl_{ij}$ for all $1 \leq i \leq j \leq m$. Then

$$\tau_{1}(x, k, a) = \tilde{x} \otimes a$$

with

$$\tilde{x} = q_{3} \left( (g^{k}G_{(4)}) + \sum_{1 \leq i \leq j \leq m} \frac{kl_{ij}p_{ij}}{e_{ij}} g_{i}^{e_{i}}G_{(3)}, g_{j}^{e_{j}}G_{(3)} \right) \bigoplus \sum_{1 \leq i \leq j \leq m} \frac{kl_{ij}q_{ij}}{e_{ij}} g_{i}^{e_{i}}G_{(2)}(g_{j}^{e_{j}}G_{(3)}) + \frac{kl_{ij}q_{ij}}{e_{ij}} g_{i}^{e_{i}}G_{(2)}(g_{j}^{e_{j}}G_{(3)}) \bigoplus \sum_{1 \leq i \leq j \leq m} \frac{kl_{ij}}{e_{ij}} \left( p_{ij} \left( g_{i}^{e_{i}}G_{(2)} \right)^{2} \right) \left( g_{j}^{e_{j}}G_{(2)} \right) + q_{ij} \left( \frac{e_{j}}{2} \right) \left( g_{i}^{e_{i}}G_{(2)} \right) \left( g_{j}^{e_{j}}G_{(2)} \right)^{2} \right)$$

Here the symbols $\bigoplus, \bigodot$ mean $+, -, \text{ resp.}$, but also indicate that the three summands they link together lie in the three different direct components of the decomposition

$$U_3L^G(G) \cong G_{(3)}/G_{(4)} \bigoplus (G_{(1)}/G_{(2)}) \otimes (G_{(2)}/G_{(3)}) \bigoplus \text{SP}^{3}(G_{(1)}/G_{(2)})$$

We omit the proof since it consists of a calculation which is straightforward along the same lines as the computation of $\delta_{1}^{\mathcal{H}}$ in the proof of Lemma 1.8 and of $\xi_{3}$ in the proof of Theorem 2.4 in section 4.
Moreover, we throughout identify $U_1^K(K)$ with $K^{AB}$ via the isomorphism $\theta_1^K$, see Theorem 1.3 for $(K, K) = (N, N)$ or $(T, \gamma)$. Moreover, we identify $U_3^{N\gamma}(N, N)$ with $\mu_3^{N\gamma}(\{x_{2,1}, x_{2,1}^T\}^2_{Ker(c_2^N)})$ via the isomorphism $\mu_3^{N\gamma}$ in (33).

Also recall that the structure of the group $\bar{U}_3^{N\gamma}(N, N) / U_3^{N\gamma}(N, N)$ is explicitly given by Proposition 1.10 and the generators of $R_3^{N\gamma}$ described in (22), but we will merely use the original definition of $U_3^{N\gamma}(N, N)$ in (12).

Now we are ready to describe the structure of the groups $Q_3(G, H)$ for $H = G, N, T$.

**Theorem 2.3** The terms on the right hand side of the decompositions

\[
Q_3(G) = Q_3^N(N) \oplus Q_3^T(T) \oplus \mathcal{K}_3 / \mathcal{K}_4 \\
Q_3(G, T) = Q_3^T(T) \oplus \mathcal{K}_3 / \mathcal{K}_4 \\
Q_3(G, N) = \Lambda_2 I(N) / \Lambda_3 I(N) \oplus \Gamma^*_2 I(N) / \Gamma^*_3 I(N)
\]

are determined by Theorem 1.3 and the following natural exact sequences.

\[
\begin{align*}
\text{Tor}_1^Z(N^{AB}, T^{ab}) & \xrightarrow{\delta_2} U_2 L^N(N) \otimes T^{ab} \oplus N^{AB} \otimes U_2 L^\gamma(T) \xrightarrow{\mu_3} \mathcal{K}_3 / \mathcal{K}_4 \to 0 \\
\text{Tor}_1^Z(T^{ab}, N^{ab}) & \xrightarrow{\delta_3} T^{ab} \otimes U_2 N^\gamma(N, N) \oplus U_2 L^\gamma(T) \otimes N^{ab} \xrightarrow{\mu_3} \Gamma^*_2 I(N) / \Gamma^*_3 I(N) \to 0 \\
\text{Tor}_1^Z(N^{AB}, N^{ab}) \oplus \text{Ker}\left(\left[\cdot, \cdot\right]: \text{Tor}_1^Z(N^{AB}, N^{AB}) \to N_2 / [N_2, N_2] N_3\right) & \xrightarrow{(\delta_4, \delta_5)} \bar{U}_3^{N\gamma}(N, N) \\
& \xrightarrow{\bar{\alpha}_3^{N\gamma}} \Lambda_2 I(N) / \Lambda_3 I(N)
\end{align*}
\]

Here the homomorphisms $\mu_2, \mu_3$ are induced by $\theta_2^N, \theta_2^\gamma, \theta_2^{N\gamma}$ followed by multiplication in $\mathbb{Z}(G)$, $\delta_2 = (-\tau_1^{11}, \tau_2^{11})^T$, $\delta_3, \delta_4$ are homomorphisms and $\delta_5$ is an additive relation of undeterminacy $\text{Im}(\delta_4)$, defined as follows. Using the identifications in Proposition 1.10 one has for suitable $n \in N$ and $t \in T$, see (28):

\[
\delta_2\langle nN_2, k, tT_2 \rangle = \left( - (n^kN_3) \otimes (tT_2) + \binom{k}{2} (nN_2)^2 \otimes (tT_2) , \\
(nN_2) \otimes (t^kT_3) - \binom{k}{2} (nN_2) \otimes (tT_2)^2 \right)
\]
\[ \delta_3(tT_2, k, nN_2) = (tT_2) \otimes \nu_{11}^{N_2^7} \left( \sum_{i=1}^{p} \left( (n_iN_2) \otimes (n_i'N_2) - (n_i'N_2) \otimes (n_iN_2) \right) \right) \]

\[- \left( \frac{k}{2} \right) \left( (nN_2) \otimes (nN_2) \right), \left( \frac{k}{2} (tT_2)^2 - (t^kT_3) \right) \otimes (nN_2) \]

where \( p \geq 1 \) and \( n_i, n_i' \in N \) such that \( n^k = \prod_{i=1}^{p} [n_i, n_i'] \). Furthermore, for suitable \( a, b \in N \) and denoting by \( \pi : U_3^{N_2^7}(N, N) \to \bar{U}_3^{N_2^7}(N, N) \) the quotient map,

\[ \delta_4(aN_2, k, bN_2) = \pi \left( (aN_2) \otimes (b^kN_2) - (a^kN_2) \otimes (bN_2) \right) \]

\[ + \left( \frac{k}{2} \right) \left( (aN_2)^2 \otimes (bN_2) - (aN_2) \otimes (bN_2)^2 \right) \]

Finally, for \( \sum_{r=1}^{s} \langle a_rN_2, k_r, b_rN_2 \rangle \in \text{Tor}^N_1(N^{AB}, N^{AB}) \) such that \( \prod_{r=1}^{s} [a_r, b_r^{k_r}] = e \prod_{q=1}^{r} \nu_{d_q} \) with \( c_q, d_q \in [N, G] \) and \( e \in N_3 \),

\[ \delta_5 \left( \sum_{r=1}^{s} \langle a_rN_2, k_r, b_rN_2 \rangle \right) = \sum_{r=1}^{s} (a_r^{k_r}N_3) \otimes (b_rN_2) - (b_r^{k_r}N_3) \otimes (a_rN_2) \]

\[ - \sum_{r=1}^{s} \left( \frac{k_r}{2} \right) \left( (a_rN_2) \left( (a_rN_2) - (b_rN_2) \right) \right) \otimes (b_rN_2) \]

\[ - \sum_{q=1}^{p} (c_qN_3) \otimes (d_qN_2) - (d_qN_3) \otimes (c_qN_2) \]

\[ - 1 \otimes (eN_4) + \text{Im}(\delta_4). \]

This result generalizes and extends the computation of \( Q_3(G) \) for finite \( G \) in \([33]\) and of \( Q_3(G, T) \) and \( \Gamma_2^3(I(N))/\Gamma_2^3(I(N)) \) for finite \( G \) and nilpotent \( T \) in \([19, 16]\). It seems, however, that the group \( \Lambda_2I(N)/\Lambda_3I(N) \) has not been determined before, not even in special cases.

We now turn to the case \( n = 4 \) where, apart from the direct factors \( Q_4^N(N) \) and \( Q_4^T(T) \), nothing seems to be known unless \( N \) and \( T \) satisfy certain torsion-freeness conditions, see section 3.

**Theorem 2.4** The direct factor \( K_4/K_5 \) of \( Q_4(G) \) and \( Q_4(G, T) \) (see \([33]\) and \([9]\)) is determined by the following tower of successive natural quotients where \( \text{Ker}(\pi_k) = \text{Im}(\xi_k), k = 1, 2, 3, \) and

\[ \xi_1 = \left( \delta_1^N \otimes 1 \ 0 \ 0 \ 0 \right)^t, \quad \xi_2 = \left( -\tau_1^{21} \ 0 \ \	au_2^{21} \ 0 \right)^t, \]

cf. the explicit description of these maps in \([37, 38]\) and Proposition \([2.2]\).
To describe $\xi_3$ we here suppose that $N$ and $T$ are finitely generated with cyclic decompositions of the torsion subgroups $\text{Tor}(N^{AB}) = \bigoplus_{i=1}^{r} \mathbb{Z}/a_i \mathbb{Z} \langle n_i N(2) \rangle$ and $\text{Tor}(T^{ab}) = \bigoplus_{j=1}^{s} \mathbb{Z}/b_j \mathbb{Z} \langle t_j T_2 \rangle$. Let $d_{ij}$ be the greatest common divisor of $a_i$ and $b_j$, and let $p_{ij}, q_{ij} \in \mathbb{Z}$ such that $d_{ij} = a_i p_{ij} + b_j q_{ij}$. Then an element $\omega = \sum_{i,j} \langle n_i N(2), k_{ij}, t_j T_2 \rangle \in \text{Tor}_1^\mathbb{Z}(N^{AB}, T^{ab})$ lies in $\text{Ker}(\delta_2)$ if and only if the following three conditions (i) - (iii) are satisfied:

(i) $\forall 1 \leq i \leq r, \forall 1 \leq j \leq s, \frac{k_{ij}}{d_{ij}}$ is even if $k_{ij}$ is even;

(ii) $\forall 1 \leq i \leq r, \prod_{j=1}^{s} t_j^{k_{ij}} = u_i^{a_i} v_i$ with $u_i \in T_2$ and $v_i \in T_3$;

(iii) $\forall 1 \leq j \leq s, \prod_{i=1}^{r} n_i^{k_{ij}} = y_j^{b_j} z_j$ with $y_j \in N(2)$ and $z_j \in N(3)$. 

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In this case,

\[
\xi_3(\omega) = \pi_2\pi_1 \left( -\sum_{j=1}^{s} (z_j N(4)) - \sum_{i=1}^{r} \left( \frac{p_{ij}}{d_{ij}} \binom{k_{ij}}{2} \binom{n_i N(3)}{2} \right) \right) \otimes (t_j T_2) ,
\]
\[
+ \left( \binom{k_{ij}}{3} - \frac{p_{ij}}{d_{ij}} \binom{k_{ij}}{2} \binom{a_i}{2} \right) \binom{n_i N(2)}{2} \otimes \left( u_i T_3 - \sum_{j=1}^{s} \frac{p_{ij}}{d_{ij}} \binom{k_{ij}}{2} (t_j T_2)^2 \right) ,
\]
\[
- \sum_{j=1}^{s} \left( (y_j N(3)) - \sum_{i=1}^{r} \frac{q_{ij}}{d_{ij}} \binom{k_{ij}}{2} (n_i N(2))^2 \right) \otimes \left( t_j^b T_3 - \binom{b_j}{2} (t_j T_2)^2 \right) ,
\]
\[
- \sum_{i=1}^{r} \left( n_i N(2) \right) \otimes \left( v_i T_4 - \sum_{j=1}^{s} \frac{q_{ij}}{d_{ij}} \binom{k_{ij}}{2} (t_j^b T_3) (t_j T_2) \right) ,
\]
\[
+ \left( \binom{k_{ij}}{3} - \frac{q_{ij}}{d_{ij}} \binom{k_{ij}}{2} \binom{b_j}{2} \right) \binom{t_j T_2}{2} \right) \right) \right)
\]

Similarly, the direct factor \( \Gamma^*_3 I(N)/\Gamma^*_4 I(N) \) of \( Q_4(G, N) \) can be computed by combining Theorems 4.5, 1.5, 1.9 and Remark 4.2, but the resulting description is considerably more complicated than the one of \( K_4/K_5 \) above, so we leave it to the interested reader to write it out.

In principle, one can use the key Proposition 4.3 to go on and determine \( K_n/K_{n+1} \) and \( \Gamma^*_n I(N)/\Gamma^*_n I(N) \) for \( n \geq 5 \), in terms of iterated amalgamations of tensor products of the augmentation quotients of \( N \) and \( T \) along certain torsion groups, but the results getting more and more complicated we do not attempt to make this explicit. When all these torsion terms vanish, however, the amalgamations degenerate to neat direct sum decompositions; this is described in the next section.

3 Fox and augmentation quotients under torsion-freeness assumptions

Supposing one or more among the groups \( N^{ab}, N^{AB} \) and \( T^{ab} \) torsion-free the groups \( Q_n(G, H) \) for \( H = G, N, T \) and \( n \leq 4 \) were determined by Karan and Vermani, see the precise citations below. We here generalize their results to all \( n \geq 1 \), and improve them by expressing most of the involved groups in terms of enveloping algebras. All proofs are deferred to section 5.

We formally put \( T_0 = T \) and \( N_{(0)} = N \).
Theorem 3.1 Let $n \geq 2$. Suppose that there exists $k$, $0 \leq k \leq n - 2$, such that $T_s/T_{s+1}$ is torsion-free for $0 \leq s \leq k$ and that $N(t)/N(t+1)$ is torsion-free for $0 \leq t \leq n - k - 2$. Then there are natural isomorphisms

\[
Q_n(G) \cong Q_n^N(N) \oplus Q_n(T) \oplus \bigoplus_{i=1}^{n-1} Q_i^N(N) \otimes Q_{n-i}(T)
\]

\[
Q_n(G, T) \cong Q_n(T) \oplus \bigoplus_{i=1}^{n-1} Q_i^N(N) \otimes Q_{n-i}(T)
\]

This implies the results in [13], [15] for $n = 3$ and in [14], [34] for $n = 4$ (which correspond to the case $k = 1$).

Corollary 3.2 Suppose that $T_s/T_{s+1}$ and $N(s)/N(s+1)$ are torsion-free for $1 \leq s \leq n$. Then there are natural isomorphisms

\[
Q_n(G) \cong \bigoplus_{i=0}^{n} U_i L^N(N) \otimes U_{n-i} L^\gamma(T).
\]

\[
Q_n(G, T) \cong \bigoplus_{i=0}^{n-1} U_i L^N(N) \otimes U_{n-i} L^\gamma(T).
\]

Using the Poincaré-Birkhoff-Witt theorem one deduces from 3.2 the following result which generalizes a theorem of Sandling and Tahara [30] on the augmentation quotients of an arbitrary group (which can be recovered here by taking $N = \{1\}$). Recall that by convention $\text{SP}^0(X) = \mathbb{Z}$.

Corollary 3.3 If for $1 \leq s \leq n$ the abelian groups $T_s/T_{s+1}$ and $N(s)/N(s+1)$ are free (in particular if they are torsion-free and $N$ and $T$ are finitely generated) then

\[
Q_n(G) \cong \bigoplus_{I_1} \bigotimes_{p=1}^{n} \text{SP}^{r_p}(N(p)/N(p+1)) \otimes \bigotimes_{q=1}^{n} \text{SP}^{s_q}(T_q/T_{q+1}),
\]

\[
Q_n(G, T) \cong \bigoplus_{I_2} \bigotimes_{p=1}^{n-1} \text{SP}^{r_p}(N(p)/N(p+1)) \otimes \bigotimes_{q=1}^{n} \text{SP}^{s_q}(T_q/T_{q+1}),
\]

where the index sets $I_1$ and $I_2$ are given by

\[
I_1 = \left\{(r_1, \ldots, r_n, s_1, \ldots, s_n) \mid 0 \leq r_1, \ldots, r_n, s_1, \ldots, s_n \leq n \text{ and } \sum_{p=1}^{n} r_p + \sum_{q=1}^{n} s_q = n \right\}
\]

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\[ I_2 = \left\{ (r_1, \ldots, r_{n-1}, s_1, \ldots, s_n) \mid 0 \leq r_1, \ldots, r_{n-1} \leq n-1, \ 0 \leq s_1, \ldots, s_n \leq n, \right. \\
\left. \sum_{q=1}^{n} s_q q \geq 1 \quad \text{and} \quad \sum_{p=1}^{n-1} r_p + \sum_{q=1}^{n} s_q q = n \right\}. \]

As to the quotients \( Q_n(G, N) \) we have the following results.

**Theorem 3.4** Suppose that \( \text{Tor}_1^\mathbb{Z}(I_i(T)/I_{i+1}(T), I(N)/\Lambda_{n-i-1}I(N)) = 0 \) for \( 1 \leq i \leq n-2 \). Then there is a natural isomorphism

\[
Q_n(G, N) \cong \frac{\Lambda_{n-1}I(N)}{\Lambda_n I(N)} \oplus \bigoplus_{i=1}^{n-1} Q_i(T) \otimes \frac{\Lambda_{n-i-1}I(N)}{\Lambda_{n-i}I(N)}.
\]

For \( n = 3 \) this reproduces the main result in [13], [15]; for \( n = 4 \) it implies the main result in [14], [34] since torsion-freeness of \( T_{ab} \cong I(T)/I^2(T) \) and \( N_{ab} \cong I(N)/\Lambda_1 I(N) \) imply triviality of our torsion group for \( i = 1, 2 \), resp.

**Corollary 3.5** If \( T_s/T_{s+1} \) is torsion-free for \( 1 \leq s \leq n \) then there is a natural isomorphism

\[
Q_n(G, N) \cong \bigoplus_{i=0}^{n-1} U_i L^\gamma(T) \otimes \frac{\Lambda_{n-i-1}I(N)}{\Lambda_{n-i}I(N)}.
\]

Now using [9, Proposition 2.1] we get

**Corollary 3.6** Suppose that \( N \) is a free group and that \( T_s/T_{s+1} \) and \( N_s/N_{s+1} \) are torsion-free for \( 1 \leq s \leq n \). Then there is a non-natural isomorphism

\[
Q_n(G, N) \cong \bigoplus_{i=0}^{n-1} U_i L^\gamma(T) \otimes U_{n-i-1} L^N(N) \otimes N_{ab}.
\]

### 4 Proofs for section 2

The starting point of our approach is the following elementary fact.

**Lemma 4.1** Let \( G \) be a group, \( H \) and \( K \) two subgroups of \( G \) such that \( H \cap K = \{1\} \), and let \( J \) be a left ideal of \( \mathbb{Z}(H) \) contained in \( I(H) \). Then one has a short exact sequence

\[ I(K)I(H)J \hookrightarrow I(K)J \xrightarrow{s} I(K) \otimes J/I(H)J \rightarrow 0 \]

where \( s((k-1)x) = (k-1) \otimes (x + I(H)J) \), \( k \in K \), \( x \in J \).
**Proof:** When the symbols $k$ resp. $h$ run through the nontrivial elements of $K$ resp. $H$ the elements $kh$ are distinct, and also distinct from the elements of $H$ and $K$. Thus the map $\mu : I(K) \otimes I(H) \rightarrow I(K)I(H)$ given by multiplication in $\mathbb{Z}(G)$ is an isomorphism since it sends the canonical basis $((k-1) \otimes (h-1))$ of $I(K) \otimes I(H)$ to linearly independent elements in $\mathbb{Z}(G)$. Consider the following commutative square with $j : J \rightarrow I(H)$:

$$
\begin{array}{ccc}
I(K)J & \hookrightarrow & I(K)I(H) \\
\mu' & \cong & \mu \\
I(K) \otimes J & \xrightarrow{1 \otimes j} & I(K) \otimes I(H)
\end{array}
$$

The map $1 \otimes j$ is injective as $I(K)$ is a free $\mathbb{Z}$-module, hence $\mu'$ is an isomorphism, too. So we have the following commutative diagram with exact rows where $\mu''$ is given by restriction of $\mu'$, $\bar{\mu}'$ is induced by $\mu'$, $j' : I(H)J \hookrightarrow J$ is the injection and $q$ the corresponding quotient map.

$$
\begin{array}{ccc}
I(K)I(H)J & \hookrightarrow & I(K)J \\
\mu'' & \cong & \mu' \\
I(K) \otimes I(H)J & \xrightarrow{1 \otimes j'} & I(K) \otimes J \\
\xrightarrow{1 \otimes q} & I(K) \otimes J/I(H)J & \rightarrow 0
\end{array}
$$

This shows that $\bar{\mu}'$ is an isomorphism which implies the assertion. \hfill $\square$

In the sequel, we consider an arbitrary descending filtration $\Delta : I(N) = \Delta_1 \supset \Delta_2 \supset \cdots$ of $I(N)$ by subgroups $\Delta_i$; later on, we shall specialize to the cases $\Delta = \Lambda$ or $\Delta = I(N)\Lambda$ where $(I(N)\Lambda)_i = I(N)\Lambda_{i-1}$. Let

$$
K^\Delta_n = \sum_{i=1}^{n-1} \Delta_{n-i} I^i(T).
$$

Then $K^\Delta_n = K_n$ while

$$
K_n^{I(N)\Lambda} = \sum_{i=1}^{n-1} I(N) \Lambda_{n-i-1} I^i(T)
$$

$$
= I(N) \sum_{i=1}^{n-1} \Lambda_{n-i-1} I^i(T)
$$

$$
= I(N) K^*_{n-1}. \quad (40)
$$

Thus computing filtration quotients $K_n^\Delta / K_{n+1}^\Delta$ amounts to computing the direct factors $K_n / K_{n+1}$ and $I(N)K^*_{n-1} / I(N)K^*_{n}$ of the abelian groups $Q_n(G), Q_n(G, T)$,
and of $I(N)I^{n-1}(G)/I(N)I^n(G)$ instead of $Q_n(G, N)$, see \( \text{(8), (9) and (10).} \) But the latter default is easily corrected by using the following device.

**Remark 4.2** By Remark \( \text{(12)} \) we obtain a commutative diagram

\[
\begin{array}{ccc}
\frac{I(N)I^{n-1}(G)}{I(N)I^n(G)} & \xrightarrow{(-)^*} & \frac{I^{n-1}(G)I(N)}{I^n(G)I(N)} \\
\parallel & & \parallel \\
\frac{I(N)\Lambda_{n-1}}{I(N)\Lambda_n} & \oplus & \frac{\mathcal{K}_{n}^{I(N)\Lambda}}{\mathcal{K}_{n+1}^{I(N)\Lambda}} \xrightarrow{\cong} \frac{\Lambda_{n-1}I(N)}{\Lambda_nI(N)} \oplus \frac{\Gamma_{n-1}^*I(N)}{\Gamma_{n}^*I(N)}
\end{array}
\]

Our computation of the quotients $\mathcal{K}_n^\Lambda / \mathcal{K}_{n+1}^\Lambda$ below provides a functorial computation of $K_n^I(I^N\Lambda)/K_{n+1}^I(I^N\Lambda)$ in terms of induced and connecting maps between certain tensor and torsion products, namely between quotients of the filtrations $(I^i(N)\Lambda)_{i \geq 0}$ and $(I^i(T))_{i \geq 1}$ of $I(N)$ and $I(T)$, in this order. But it is easily checked that applying the symmetry isomorphisms of the tensor and torsion product, as well as the conjugation isomorphisms $(I^N\Lambda)_i \cong (\Lambda^I(N))_i$, to our description is compatible with the conjugation isomorphism $K_n^{I(N)\Lambda}/K_{n+1}^{I(N)\Lambda} \cong \Gamma_{n-1}^*I(N)/\Gamma_{n}^*I(N)$. Thus taking the “mirror-symmetric” version of our description of the former quotients provides a description of the latter. So it finally suffices to determine the quotients $\Lambda_n/\Lambda_{n+1}$, $I^n(T)/I^{n+1}(T)$, $\Lambda_{n-1}I(N)/\Lambda_nI(N)$, and $\mathcal{K}_n^\Lambda / \mathcal{K}_{n+1}^\Lambda$ in order to determine $I^{n-1}(G)I(K)/I^n(G)I(K)$ for $K = G, N$ and $T$.

Let $1 \leq i \leq n - 1$, $1 \leq j \leq n - i$ and $i + 1 \leq m \leq \infty$. Putting $I^\infty(T) = 0$ we have connecting homomorphisms $\tau_1 = \tau_1^{n,i,j}$ and $\tau_2 = \tau_2^m = \tau_1^{n,i,j,m}$

\[
\frac{\Delta_{n-i}}{\Delta_{n-i+1}} \otimes \frac{I^i(T)}{I^{i+1}(T)} \xleftarrow{\tau_1} \text{Tor}_1 \mathbb{Z} \left( \frac{\Delta_j}{\Delta_{n-i}}, \frac{I^i(T)}{I^{i+1}(T)} \right) \xrightarrow{\tau_2} \frac{\Delta_j}{\Delta_{n-i}} \otimes \frac{I^{i+1}(T)}{I^m(T)}
\]

obtained from the short exact sequences

\[
\Delta_{n-i}/\Delta_{n-i+1} \hookrightarrow \Delta_j/\Delta_{n-i+1} \twoheadrightarrow \Delta_j/\Delta_{n-i} \to 0 \quad (41)
\]

\[
I^{i+1}(T)/I^m(T) \hookrightarrow I^i(T)/I^m(T) \twoheadrightarrow I^i(T)/I^{i+1}(T) \to 0
\]

Recall that for a canonical generator $\langle \bar{x}, k; \bar{y} \rangle$ of $\text{Tor}_1^\mathbb{Z}(\Delta_j/\Delta_{n-i}, I^i(T)/I^{i+1}(T))$, i.e. $x \in \Delta_j$, $y \in I^i(T)$, $k \in \mathbb{Z}$ such that $kx \in \Delta_{n-i}$ and $ky \in I^{i+1}(T)$, one has

\[
\tau_1(\bar{x}, k; \bar{y}) = kx \otimes \bar{y} \quad \tau_2(\bar{x}, k; \bar{y}) = \bar{x} \otimes ky.
\]

(42)
Proposition 4.3  For $1 \leq i \leq n - 1$ there is an exact sequence

$$
\text{Tor}_1^n \left( \frac{\Delta_1}{\Delta_{n-i}}, \frac{I^i(T)}{I^{i+1}(T)} \right) \xrightarrow{\tau_1, \nu_i \tau_2^n} \Delta_{n-i} \otimes \left( \frac{I^i(T)}{I^{i+1}(T)} \right) \oplus \frac{\Delta_1 I^{i+1}(T)}{\sum_{k=i+1}^n \Delta_{n-k+1} I^k(T)}
$$

where $\nu_i$ is induced by the injection $\Delta_1 I^{i+1}(T) \hookrightarrow \Delta_1 I^i(T)$, $s_i(x, y) = \bar{x} \otimes \bar{y}$ for $(x, y) \in \Delta_1 \times \Delta_1^*(T)$, $\nu_i$ is given by

$$
\nu_i : \frac{\Delta_1}{\Delta_{n-i}} \otimes \frac{I^{i+1}(T)}{I^n(T)} \cong \frac{\Delta_1 \otimes I^{i+1}(T)}{\text{Im} \left( \Delta_{n-i} \otimes I^{i+1}(T) + \Delta_1 \otimes I^n(T) \right)} \xrightarrow{\bar{\nu}_i} \frac{\Delta_1 I^{i+1}(T)}{\sum_{k=i+1}^n \Delta_{n-k+1} I^k(T)}
$$

with $\bar{\nu}_i$ being induced by multiplication in $\mathbb{Z}(G)$, and $\mu_i$ is defined in a similar way.

Proof: Consider the following diagram

$$
\begin{array}{cccccc}
\text{Tor}_1^n \left( \frac{\Delta_1}{\Delta_{n-i}}, \frac{I^i(T)}{I^{i+1}(T)} \right) & \xrightarrow{\tau_1} & \Delta_{n-i} \otimes \left( \frac{I^i(T)}{I^{i+1}(T)} \right) & \xrightarrow{\alpha \otimes id} & \Delta_1 \otimes \left( \frac{I^i(T)}{I^{i+1}(T)} \right) \\
\downarrow \nu \tau_2^n & & \downarrow \mu_i & & \\
\sum_{k=i+1}^n \Delta_{n-k+1} I^k(T) & \xrightarrow{i} & \sum_{k=i+1}^n \Delta_{n-k+1} I^k(T) & \xrightarrow{s} & \Delta_1 \otimes \left( \frac{I^i(T)}{I^{i+1}(T)} \right)
\end{array}
$$

The top row is part of a six-term exact sequence associated with the short exact sequence $\Delta_{n-i} \xrightarrow{\alpha} \Delta_1 \xrightarrow{\mu_i} \Delta_1 / \Delta_{n-i}$, and hence is exact; note that $\bar{\tau}_1$ is injective since $\Delta_1$ is a free $\mathbb{Z}$-module. The maps $\bar{\mu}_i, \bar{i}_i$ are given by multiplication and inclusion, respectively. The bottom row is induced by the exact sequence in Lemma 4.1 for $(K, H, J) = (N, T, I^i(T))$ and hence is also exact. Moreover, the diagram commutes; to see this for the left-hand square use [4.2]. Now an easy diagram chase together with right-exactness of the tensor product shows that the sequence

$$
0 \to \text{Tor}_1^n \left( \frac{\Delta_1}{\Delta_{n-i}}, \frac{I^i(T)}{I^{i+1}(T)} \right) \xrightarrow{-\tau_1, \nu_i \tau_2^n} \Delta_{n-i} \otimes \left( \frac{I^i(T)}{I^{i+1}(T)} \right) \oplus \frac{\Delta_1 I^{i+1}(T)}{\sum_{k=i+1}^n \Delta_{n-k+1} I^k(T)}
$$

26
is exact where \( q_i : \Delta_1 \rightarrow \Delta_1/\Delta_{n-i} \) is the canonical projection. Then the assertion follows by passing to the quotient modulo \( \text{Im}(\Delta_{n-i+1} \otimes (I^i(T)/I^{i+1}(T))) \) and modulo \( \tilde{\mu}_i \) \( \text{Im}(\Delta_{n-i+1} \otimes (I^i(T)/I^{i+1}(T))) = \text{Im}(\Delta_{n-i+1} \otimes I^i(T)) \), respectively. Just note that \( \tilde{\tau}_1 \) composed with the quotient map \( \Delta_{n-i} \otimes (\frac{I^i(T)}{I^{i+1}(T)}) \rightarrow (\frac{\Delta_{n-i}}{\Delta_{n-i+1}}) \otimes (\frac{I^i(T)}{I^{i+1}(T)}) \) equals \( \tau_1 \) by naturality of six-term exact sequences.

As we will see next, Proposition 4.3 allows to successively “unscrew” the filtration quotients of \( \mathcal{K}^\Delta \). The first case, however, is plain:

**Computation of \( \mathcal{K}_2^\Delta/\mathcal{K}_3^\Delta \):** For \( n = 2 \) and \( i = 1 \) Proposition 4.3 provides the exact sequence

\[
0 = \text{Tor}_1^\mathbb{Z}\left(\frac{\Delta_1}{\Delta_2}, \frac{I(T)}{I^2(T)}\right) \rightarrow \frac{\Delta_1}{\Delta_2} \otimes \frac{I(T)}{I^2(T)} \oplus \frac{\Delta_1}{\Delta_2} \frac{I^2(T)}{I^3(T)} \xrightarrow{(\mu_1, \iota_1)} \frac{\mathcal{K}_2^\Delta}{\mathcal{K}_3^\Delta} \rightarrow 0
\]

whence

\[
\frac{\mathcal{K}_2^\Delta}{\mathcal{K}_3^\Delta} \cong \frac{\Delta_1}{\Delta_2} \otimes \frac{I(T)}{I^2(T)}
\]

\[
\cong \begin{cases} 
N/N(2) \otimes T/T_2 & \text{if } \Delta = \Lambda \\
N/N_2 \otimes T/T_2 & \text{if } \Delta = I(N)\Lambda 
\end{cases}
\]

as

\[
\frac{(I(N)\Lambda)_1}{(I(N)\Lambda)_2} = \frac{I(N)}{I^2(N)} \cong N/N_2.
\]

**Proof of Theorem 2.1:** By (8) and (9) the desired computation of \( Q_2(G,T) \) and \( Q_2(G,T) \) follows from Theorem 1.5 and (15) for \( \Delta = \Lambda \). By (10), \( Q_2(G,N) \cong \Lambda_1I(N)/\Lambda_2I(N) \oplus \Gamma^*_1I(N)/\Gamma^*_2I(N) \). But \( \Lambda_1I(N)/\Lambda_2I(N) \cong U^\gamma_2(N,N) \) by Theorem 1.7 and \( \Gamma^*_1I(N)/\Gamma^*_2I(N) \cong T/T_2 \otimes N/N_2 \) by Remark 4.2 and (15). □

**Computation of \( \mathcal{K}_3^\Delta/\mathcal{K}_4^\Delta \):** Taking \( n = 3 \) and \( i = 1, 2 \) Proposition 4.3 provides the following two exact sequences

\[
\text{Tor}_1^\mathbb{Z}\left(\frac{\Delta_1}{\Delta_2}, \frac{I(T)}{I^2(T)}\right) \xrightarrow{(-\tau_1, \mu_1, \iota_1)} \frac{\Delta_2}{\Delta_3} \otimes \frac{I(T)}{I^2(T)} \oplus \frac{\Delta_1}{\Delta_2} \frac{I^2(T)}{I^3(T)} \xrightarrow{(\mu_1, \iota_1)} \frac{\mathcal{K}_3^\Delta}{\mathcal{K}_4^\Delta}
\]

\[
\text{Tor}_1^\mathbb{Z}\left(\frac{\Delta_1}{\Delta_2}, \frac{I^2(T)}{I^3(T)}\right) \rightarrow \frac{\Delta_1}{\Delta_2} \otimes \frac{I^2(T)}{I^3(T)} \oplus \frac{\Delta_1}{\Delta_2} \frac{I^3(T)}{I^4(T)} \xrightarrow{(\mu_2, \iota_2)} \frac{\Delta_1}{\Delta_2} \frac{I^2(T)}{I^3(T)} \oplus \frac{\Delta_1}{\Delta_2} \frac{I^3(T)}{I^4(T)}
\]
Using Theorem 1.5 we thus obtain the following result which requires the connecting homomorphisms

\[ \frac{\Delta_{p+1}}{\Delta_{p+2}} \otimes U_q L^\gamma(T) \xrightarrow{\tau_{pq}^1} \text{Tor}_1^Z(\frac{\Delta_p}{\Delta_{p+1}}, U_q L^\gamma(T)) \xrightarrow{\tau_{pq}^2} \frac{\Delta_p}{\Delta_{p+1}} \otimes U_{q+1} L^\gamma(T) \xrightarrow{\delta_{pq}^2} \text{Im}(\delta_2^p) \]

for \( p, q = 1, 2 \) where \( \tau_{pq}^1 \) is induced by the short exact sequence (23), and \( \tau_{pq}^2 \) is induced by sequence (24) for \( q = 1 \) and by sequence (39) for \( q = 2 \), both with \((G, \mathcal{G}) = (T, \gamma)\), compare (36).

**Theorem 4.4** For any descending subgroup filtration \( \Delta \) of \( I(N) \) there is a natural exact sequence

\[ \text{Tor}_1^Z(\Delta_1/\Delta_2, T^{ab}) \xrightarrow{\delta_2^3} (\Delta_2/\Delta_3) \otimes T^{ab} \oplus (\Delta_1/\Delta_2) \otimes U_2 L^\gamma(T) \xrightarrow{\mu} \frac{K_3^\Delta}{K_4^\Delta} \]

where \( \delta_2^3 = (-\tau_{11}^1, \tau_{11}^1)^t \) and \( \mu \) is given by the isomorphisms \( \theta_k^\gamma \), \( k = 1, 2 \), and multiplication in \( \mathbb{Z}(G) \).

**Proof of Theorem 2.3:** By (8) and (10) the desired computation of \( Q_3(G) \) and \( Q_3(G,T) \) follows from Theorem 1.4 taking \( \Delta = \Lambda \): it suffices to note that Theorem 1.5 provides an isomorphism between sequence (23) for \( p = 1 \) and sequence (24) for \((G, \mathcal{G}) = (N,N)\); this isomorphism transforms the maps \( \tau_{k1}^{11} \) in \( \tau_{k1}^{11} \), \( k = 1, 2 \), which were computed in (37) and (38).

Next by (10), \( Q_3(G,N) \cong \Lambda_2 I(N)/\Lambda_3 I(N) \oplus \frac{\Gamma_2^* I(N)}{\Gamma_3^* I(N)} \). Taking \( \Delta = I(N)\Lambda \) and using (10) Theorem 4.4 provides a presentation of \( I(N)\frac{K_2^\Delta}{I(N)K_3^\Delta} \) which turns into the desired one of \( \frac{\Gamma_2^* I(N)}{\Gamma_3^* I(N)} \) by means of Remark 4.2. To make this explicit, first note that the mirror-symmetric version of sequence (23) for \( p = 1 \) and \( \Delta = I(N)\Lambda \) is

\[ 0 \rightarrow \frac{\Lambda_1 I(N)}{\Lambda_2 I(N)} \xrightarrow{\alpha} \frac{\Lambda_1}{\Lambda_2} I(N) \rightarrow \frac{\Lambda_1}{\Lambda_1} I(N) \rightarrow 0 \tag{47} \]

Next, the mirror-symmetric versions of the maps \( \tau_{11}^{11} \) and \( \tau_{21}^{11} \) for \( \Delta = I(N)\Lambda \) are the connecting homomorphisms \( \tau_{21}^{\Lambda(N)} \) and \( \tau_{11}^{\Lambda(N)} \), resp., which form the top row of the commutative diagram

\[ T^{ab} \otimes \frac{\Lambda_1 I(N)}{\Lambda_2 I(N)} \xrightarrow{\tau_{\Lambda(N)}^{11}} \text{Tor}_1^Z(T^{ab}, \frac{\Lambda_1}{\Lambda_1} I(N)) \xrightarrow{\tau_{\Lambda(N)}^{11}} U_2 L^\gamma(T) \otimes \frac{\Lambda_1}{\Lambda_1} I(N) \]

\[ \cong \frac{\Lambda_1}{\Lambda_2} I(N) \]

and which are induced by the sequences (17), and (24) for \((G, \mathcal{G}) = (T, \gamma)\), resp.
To compute \( \tilde{\tau}^{\Lambda(N)}_2 \) let \(<tT_2, k, nN_2>\) be a canonical generator of \( \text{Tor}_1^Z(T^{ab}, N^{ab}) \). Then

\[
\tilde{\tau}^{\Lambda(N)}_2(<tT_2, k, nN_2>) = (1 \otimes \theta_2^{N(N)}(\nu_{11}))^{-1} \tilde{\tau}^{\Lambda(N)}_2(1, \theta_1^{N(N)})(tT_2, k, nN_2)
\]

\[
= (1 \otimes \theta_2^{N(N)}(\nu_{11}))^{-1} \tilde{\tau}^{\Lambda(N)}_2(tT_2, k, n - 1 + I^2(N))
\]

\[
= (1 \otimes \theta_2^{N(N)}(\nu_{11}))^{-1} ((tT_2) \otimes \alpha^{-1}(k(n - 1) + \Lambda_2 I(N)))
\]

\[
= (tT_2) \otimes (\theta_2^{N(N)}(\nu_{11}))^{-1} \left( \left(n^k - 1\right) - \left(\frac{k}{2}\right)(n - 1)^2 + \Lambda_2 I(N) \right)
\]

\[
= (tT_2) \otimes (\theta_2^{N(N)}(\nu_{11}))^{-1} \left( \left(\prod_{i=1}^{p} [n_i, n'_i] - 1\right) - \left(\frac{k}{2}\right)(n - 1)^2 + \Lambda_2 I(N) \right)
\]

\[
= (tT_2) \otimes (\theta_2^{N(N)}(\nu_{11}))^{-1} \left( \left(\sum_{i=1}^{p} (n_i - 1)(n'_i - 1) - (n'_i - 1)(n_i - 1) \right) - \left(\frac{k}{2}\right)(nN_2) + (q_2^N \otimes 1)I_2^N \text{ Ker}(c_2) \right)
\]

This provides the first component of the map \( \delta_3 = (\tilde{\tau}^{\Lambda(N)}_2, -\tilde{\tau}_1^{\Lambda(N)}) \) which is the mirror-symmetric version of \(-\delta_2^{N(N)\Lambda}\); the second component is obtained by a similar computation of \( \tilde{\tau}_1^{\Lambda(N)} \).

\[\square\]

**Computation of \( K^\Lambda_3 / K^\Lambda_2 \):** Proposition [4.3] provides the following four exact sequences extracted from sequence [4.3], taking \( n = 4 \) and \( i = 1, 2, 2, 3 \) resp.:

\[
\text{Tor}_1^Z(\frac{\Delta_1}{\Delta_3}, \frac{I(T)}{I^3(T)}) \xrightarrow{(-\tau_1, \mu_1^T \tau_1^T)} \frac{\Delta_1}{\Delta_3} \otimes \frac{I(T)}{I^3(T)} \oplus \frac{\Delta_1 I^2(T)}{\sum_{k=2}^4 \Delta_5 I^k(T)} \xrightarrow{(\mu_1, \epsilon_1)} \frac{\Delta_3 I(T) + \Delta_1 I^2(T)}{\sum_{k=2}^4 \Delta_5 I^k(T)}
\]

\[\text{(48)}\]

\[
\text{Tor}_1^Z(\frac{\Delta_2}{\Delta_3}, \frac{I^2(T)}{I^3(T)}) \xrightarrow{(-\tau_1, \mu_2^T \tau_2^T)} \frac{\Delta_2}{\Delta_3} \otimes \frac{I^2(T)}{I^3(T)} \oplus \frac{\Delta_1 I^3(T)}{\sum_{k=3}^4 \Delta_5 I^k(T)} \xrightarrow{(\mu_2, \epsilon_2)} \frac{\Delta_2 I^2(T) + \Delta_1 I^3(T)}{\sum_{k=2}^4 \Delta_5 I^k(T)}
\]

\[\text{(49)}\]

\[
\frac{\Delta_2 I^2(T) + \Delta_1 I^3(T)}{\sum_{k=2}^4 \Delta_5 I^k(T)} \xrightarrow{\epsilon_2} \frac{\Delta_1 I^2(T)}{\sum_{k=2}^4 \Delta_5 I^k(T)} \xrightarrow{\epsilon_1} \frac{\Delta_1 I^2(T)}{\sum_{k=3}^4 \Delta_5 I^k(T)}
\]

\[\text{(50)}\]

\[
\text{Tor}_1^Z(\frac{\Delta_1}{\Delta_2}, \frac{I^3(T)}{I^3(T)}) \xrightarrow{(-\tau_1, \mu_3^T \tau_3^T)} \frac{\Delta_1}{\Delta_2} \otimes \frac{I^3(T)}{I^3(T)} \oplus \frac{\Delta_1 I^4(T)}{\sum_{k=3}^4 \Delta_5 I^k(T)} \xrightarrow{(\mu_3, \epsilon_3)} \frac{\Delta_1 I^3(T)}{\sum_{k=3}^4 \Delta_5 I^k(T)}
\]

\[\text{(51)}\]
Combining sequences (49) and (51) with Theorems 1.5 and 1.9 provides an exact sequence

\[
\begin{align*}
\text{Tor}^1_Z(\Delta_2, U_2L^γ(T)) \otimes (\frac{\Delta_2}{\Delta_3} \otimes U_2L^γ(T) \oplus \frac{\Delta_3}{\Delta_2} \otimes \text{Coker}(δ_1^γ) \rightarrow \frac{\Delta_2}{\Delta_3} \otimes U_2L^γ(T) \oplus \frac{\Delta_3}{\Delta_2} \otimes \text{Coker}(δ_1^γ) \\
\rightarrow \frac{\Delta_2}{\Delta_3} \otimes U_2L^γ(T) \oplus \frac{\Delta_3}{\Delta_2} \otimes \text{Coker}(δ_1^γ)
\end{align*}
\]

Theorem 4.5 For any descending subgroup filtration \( \Delta \) of \( I(N) \) the quotient \( \mathcal{K}_4^\Delta / \mathcal{K}_3^\Delta \) is determined by the following tower of successive natural quotients where \( \text{Ker}(\pi_k) = \text{Im}(\xi_k) \), \( k = 1, 2, 3 \).

\[
\begin{align*}
\Delta_2 & \otimes \text{Tor}^2(T^{ab}, T^{ab}) \\
\rightarrow & \Delta_4 \otimes T^{ab} \oplus \Delta_3 \otimes U_2L^γ(T) \oplus \Delta_2 \otimes U_3L^γ(T) \\
\rightarrow & \Delta_4 \otimes T^{ab} \oplus \Delta_3 \otimes U_2L^γ(T) \oplus \Delta_2 \otimes \text{Coker}(δ_1^γ) \\
\rightarrow & \ker(\delta_2^\Delta) \\
\rightarrow & \text{Coker}(\xi_2) \\
\rightarrow & \mathcal{K}_4^\Delta / \mathcal{K}_3^\Delta
\end{align*}
\]

Here \( \xi_1 = (0, 0, 1 \otimes δ_1^γ)^t \), \( \xi_2 = \begin{pmatrix} -\frac{\tau_{21}^2}{\tau_1^2} & \frac{\tau_{21}^1}{\tau_1^2} & 0 \\ 0 & -\frac{\tau_{12}^1}{\tau_1^2} & \frac{\tau_{12}^2}{\tau_1^2} \end{pmatrix}^t \), \( \delta_2^\Delta = (-\frac{\tau_{11}^2}{\tau_1^2} \frac{\tau_{11}^1}{\tau_1^2})^t \) as in Theorem 4.4, and the map \( \xi_3 \) is defined in the proof below. To describe it explicitly, suppose that \( \Delta_1 / \Delta_2 \) and \( T \) are finitely generated with cyclic decomposition \( \text{Tor}(\Delta_1 / \Delta_2) = \bigoplus_{i=1}^r \mathbb{Z}/a_i \mathbb{Z} \langle x_i \rangle \), \( x_i \in \Delta_1 \), and with the remaining notations of Theorem 2.4. Then an element \( \omega = \sum_{i,j} \langle x_i, k_{ij}, t_j \rangle \in \text{Tor}^2(T^{ab}, T^{ab}) \) lies in \( \ker(\delta_2^\Delta) \) if and only if the conditions (i) and (ii) in Theorem 2.4 hold, as well as the following condition

\[(iii') \quad \forall 1 \leq j \leq s, \quad \sum_{i=1}^r k_{ij} x_i = b_j \delta_j^2 + \delta_j^3 \quad \text{with} \quad \delta_j^k \in \Delta_k, \quad k = 2, 3.\]
In this case,

$$
\xi_3(\omega) = \pi_2 \pi_1 \left( - \sum_{j=1}^{s} \delta_j^2 \otimes \bar{b}_j , \sum_{i=1}^{r} \frac{a_i}{2} \otimes \left( u_i T_3 - \sum_{j=1}^{s} \frac{p_{ij}}{d_{ij}} k_{ij}^2 \right)(t_j T_2)^2 \right)
$$

$$
- \sum_{j=1}^{s} \delta_j^2 \otimes \left( b_j^b T_3 - \left( \frac{b_j}{2} \right)(t_j T_2)^2 \right),
$$

$$
+ \sum_{i=1}^{r} \bar{x}_i \otimes \left( \bar{x}_i \otimes (u_i T_4) - \sum_{j=1}^{s} \left( \frac{q_{ij}}{d_{ij}} k_{ij}^2 \right) \left( t_j^b T_3 \right) (t_j T_2) \right)
$$

$$
+ \left( \left( \frac{k_{ij}}{3} \right) - \frac{q_{ij}}{d_{ij}} \left( \frac{k_{ij}}{2} \right) \left( \frac{b_j}{2} \right) \right)(t_j T_2)^3 \right)
$$

**Proof:** First note that

$$
\text{Coker}(\xi_1) \cong \frac{A}{A_4} \otimes T^{ab} \oplus \frac{A}{A_3} \otimes U_2 L \gamma(T) \oplus \frac{A}{A_7} \otimes \frac{I^2(T)}{P^2(T)}
$$

by Theorem 1.9. Now consider the following diagram where implicitly \( n = 4 \), \( \alpha \) and \( \beta \) are the obvious injections, \( \alpha' = \alpha \otimes 1 \), \( \rho, \rho', q \) are the obvious maps, and \( \tau_1 = \tau_1^{4,1,1}, \tau_2 = \tau_1^{4,1,1,4}, \tau'_1 = \tau_1^{4,1,2}, \tau'_2 = \tau_1^{4,1,2,4}. \)

The upper squares commute by naturality of the connecting homomorphisms; for the right-hand square we have to use this argument twice, according to the factorization \( \rho \otimes \rho' = (1 \otimes \rho')(\rho \otimes 1) \): we have \( (\rho \otimes 1) \tau_2 = \tau_2^{3,1,1,4}(\rho, 1)_\ast \) and \( (1 \otimes \rho') \tau_2^{3,1,1,4} = \tau_2^{3,1,1,3}. \) The middle squares commute by definition of the concerned
maps; together with exactness of sequence (18) this implies that the map \( \overline{1} \oplus \beta \) induced by \( 1 \oplus \beta \) is well-defined. So the whole diagram commutes. Furthermore, the first and third row are exact; in fact, the top row is part of a 6-term exact sequence, and exactness of the third row follows from sequence (50).

Now observe that \( \mathcal{K}_k^\Delta / \mathcal{K}_1^\Delta = \text{Im}(\overline{1} \oplus \beta) \). But using sequence (52) and Theorems 1.5 and 1.9 we obtain an isomorphism

\[ \Xi : \text{Coker}((-\tau_1', \nu_1(i \otimes 1)\tau_2')^l) \xrightarrow{\cong} \text{Coker}(\xi_2) . \]

Thus in order to establish the desired tower it remains to show that \( \Xi(\text{Ker}(\overline{1} \oplus \beta)) = \text{Im}(\xi_3) \) for a suitable map \( \xi_3 \). To do this we apply the snake lemma with respect to the diagram whose rows are the first and third row of diagram (53) and whose vertical maps are the compositions of the respective vertical maps in (53). Using sequence (18) we thus obtain an exact sequence

\[ \text{Im}((\rho, 1)_*) \cap \text{Ker}(\tau_2^{3,1,1,3}) \xrightarrow{\xi_3^l} \text{Coker}((-\tau_1', \nu_1(i \otimes 1)\tau_2')^l) \xrightarrow{1 \oplus \beta} \frac{\Delta_3 I(T) + \Delta_1 I^2(T)}{K_0^5} \]

where \( \xi_3^l \) is given by the switchback rule \( \xi_3^l = q(1 \oplus \beta)^{-1}(-\tau_1', \nu_1\tau_2')^{l}(\rho, 1)_*^{-1} \). But \( \text{Ker}(\tau_2^{3,1,1,3}) = (1, \theta_1')_n \) and \( \text{Im}((\rho, 1)_*) = \text{Ker}(\tau_1^{3,1,1}) = (1, \theta_1')_n \) by the corresponding 6-term exact sequence, so we get \( \text{Im}((\rho, 1)_*) \cap \text{Ker}(\tau_2^{3,1,1,3}) = (1, \theta_1')_n \). Since \( (1, \theta_1')_n \) is an isomorphism. Now define \( \xi_3 \) to be \( \Xi \xi_3^l \) pre-composed with the isomorphism \( \text{Ker}(\delta^3) \xrightarrow{\cong} \text{Im}((\rho, 1)_*) \cap \text{Ker}(\tau_2^{3,1,1,3}) \) induced by \( (1, \theta_1')_n \). Then the asserted tower is established.

It remains to prove the claims concerning the element \( \omega \). One has

\[ \tilde{\tau}_1^{11}(\omega) = \sum_{j=1}^{s} \left( \sum_{i=1}^{r} k_{ij} x_i + \Delta_3 \right) \otimes (t_j T_2^l) \in \frac{\Delta_2}{\Delta_3} \otimes T^{ab} \cong \bigoplus_{j=1}^{s} \frac{\Delta_2}{\Delta_3} \otimes \mathbb{Z}/b_j \mathbb{Z}(t_j T_2^l) , \]

whence \( \tilde{\tau}_1^{11}(\omega) = 0 \) iff \( \sum_{i=1}^{r} k_{ij} x_i + \Delta_3 = b_j \Delta_3 \), i.e., \( \sum_{i=1}^{r} k_{ij} x_i = b_j \delta_j + \delta_3^t \) with \( \delta_j \in \Delta_k \). On the other hand, noting \( \bar{x}_i = x_i + \Delta_2 \) and \( b_{jk} = \text{gcd}(b_j, b_k) \), we obtain as in the proof of Lemma 1.8

\[ \tilde{\tau}_2^{11}(\omega) = \sum_{i=1}^{r} \bar{x}_i \otimes \sum_{j=1}^{s} (\theta_2^l)^{-1} \left( (k_{ij} (t_j - 1) + l^3(T)) \right) \]

\[ = \sum_{i=1}^{r} \bar{x}_i \otimes \sum_{j=1}^{s} \left( (t_j k_{ij} T_3) - \left( \frac{k_{ij}}{2} \right) (t_j T_2)^2 \right) \]

\[ = \sum_{i=1}^{r} \bar{x}_i \otimes \left( \prod_{j=1}^{s} t_j^{k_{ij}} \right) T_3 - \sum_{i,j} \bar{x}_i \otimes \left( \frac{k_{ij}}{2} \right) (t_j T_2)^2 \]

\[ \in \frac{\Delta_1}{\Delta_2} \otimes U_2 L^7(T) \]

\[ \cong \bigoplus_{i=1}^{r} \left( \mathbb{Z}/a_i \mathbb{Z}(\bar{x}_i) \otimes (T_2/T_3) \right) \oplus \bigoplus_{1 \leq i \leq r, 1 \leq j \leq s} \mathbb{Z}/a_i \mathbb{Z}(\bar{x}_i) \otimes \mathbb{Z}/b_{jk} \mathbb{Z}((t_j T_2) \otimes (t_k T_2)) \]

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where we use the classical direct sum decomposition $U_2 L^\gamma(T) = T_2 / T_3 \oplus SP^2(T^{ab})$ for finitely generated $T^{ab}$, see also Proposition \[\ref{prop:directsum}. Thus $\hat{t}^{11}_2(\omega) = 0$ iff $\forall i$, $\prod_{j=1}^s t_{k_{ij}}^j \in T_2 a_j T_3$ and $\forall i, j$, $d_{ij} = \gcd(a_i, b_{ij})$ divides $\left(\frac{k_{ij}}{2}\right)$.

But $d_{ij}$ divides $k_{ij}$ since the latter is a common multiple of $a_i$ and $b_j$, as we suppose that the element $\langle x_i, k_{ij}, t_j \rangle$ is defined. Thus $d_{ij}$ divides $\left(\frac{k_{ij}}{2}\right)$ if and only if condition (i) is satisfied. It follows that $\omega \in \ker(\delta_3^2)$ iff the conditions (i), (ii) and (iii)' are satisfied.

To simplify the notation we henceforth identify $I^2(T)$ with $I(T)/I^2(T)$ via the isomorphism $\theta_1^\gamma$. Then for $\omega \in \ker(\delta_3^2)$,

$$\omega = \sum_{j=1}^s \sum_{i=1}^r \left( \frac{k_{ij}}{b_j} x_i, b_j, t_j T_2 \right) = \sum_{j=1}^s \sum_{i=1}^r \left( \frac{k_{ij}}{b_j} x_i, b_j, t_j T_2 \right) = (\rho, 1)_{i, j}(\omega')$$

with $\omega' = \sum_{j=1}^s \sum_{i=1}^r \left( \frac{k_{ij}}{b_j} x_i - \delta^2_j + \Delta_3, b_j, t_j T_2 \right)$. Abbreviating $A = \sum_{k=2}^4 \Delta_{5-k} I^k(T)$, we next get

$$(-\tau_1, \nu_1 \tau_2)'(\omega')$$

$$= \sum_{j=1}^s \left( - (\delta_j^3 + \Delta_4) \otimes (t_j T_2), \left( \sum_{i=1}^r \frac{k_{ij}}{b_j} x_i - \delta_j^2 \right) b_j (t_j - 1) + A \right)$$

$$= \left( - \sum_{j=1}^s \delta_j^3 \otimes \bar{t}_j, \sum_{i=1}^r x_i \left( \sum_{j=1}^s k_{ij} (t_j - 1) \right) - \sum_{j=1}^s \delta_j^2 \left( (t_j^2 - 1) - \left( \frac{b_j}{2} \right) (t_j - 1)^2 \right) + A \right)$$

(54)

The remaining calculations are again based on the identities (31) and (30). Using these and the fact that $t_{k_{ij}}^j \in T_2$ we obtain the following congruences modulo $I^4(T)$, for $1 \leq i \leq r$:

$$\sum_{j=1}^s k_{ij} (t_j - 1)$$

$$\equiv \sum_{j=1}^s \left( t_{k_{ij}}^j - 1 \right) - \left( \frac{k_{ij}}{2} \right) (t_j - 1)^2 - \left( \frac{k_{ij}}{3} \right) (t_j - 1)^3$$

$$\equiv \Pi_{j=1}^s t_{k_{ij}}^j - 1 - \sum_{j=1}^s \left( \frac{1}{d_{ij}} \left( \frac{k_{ij}}{2} \right) a_i p_{ij} (t_j - 1)^2 + b_j q_{ij} (t_j - 1)^2 \right) + \left( \frac{k_{ij}}{3} \right) (t_j - 1)^2$$

$$\equiv u_i^a v_i - a_i \sum_{j=1}^s P_{ij} d_{ij} \left( \frac{k_{ij}}{2} \right) (t_j - 1)^2 - \sum_{j=1}^s \left( \frac{q_{ij}}{d_{ij}} \left( \frac{k_{ij}}{2} \right) \left( t_{k_{ij}}^j - 1 \right) - \left( \frac{b_j}{2} \right) (t_j - 1)^2 \right)$$

$$\times (t_j - 1) + \left( \frac{k_{ij}}{3} \right) (t_j - 1)^3$$

(54)
As \( u_i a_i - v_i - 1 \equiv a_i(u_i - 1) + (v_i - 1) \) modulo \( I(T) \) since \( u_i, v_i \in T_2 \) we obtain

\[
x_i \sum_{j=1}^{n} k_{ij}(t_j - 1) \equiv a_i x_i (u_i - 1) - \sum_{j=1}^{n} p_{ij} (k_{ij}/2) (t_j - 1)^2 \\
+ x_i (v_i - 1) - \sum_{j=1}^{n} q_{ij} (k_{ij}/2) (t_j^2 - 1)(t_j - 1) \\
+ \left( \left( k_{ij}/3 - q_{ij} (k_{ij}/2) (b_j/2) \right) (t_j - 1)^3 \right) \mod A \quad (55)
\]

The formula for \( \xi_3(\omega) \) now follows by combining the identities (54) and (55).

**Proof of Theorem 2.4:** Taking \( \Delta = \Lambda \) the tower in Theorem 4.5 transforms into the one in Theorem 2.4 by use of Theorems 1.5 and 1.9. It remains to check

First of all, we may take \( x_i = n_i - 1 \) in Theorem 4.5. Now let \( \omega = \sum_i \langle n_i N(2), k_{ij}, t_j T_2 \rangle \in \text{Tor}^2(N^{AB}, T^{ab}) \). Using (31) and (30) together with the fact that \( n_i^{k_{ij}} \in N(2) \) we get the following identities for \( 1 \leq j \leq s \), denoting \( a_{ij} = \gcd(a_i, a_j) \):

\[
(\theta_2^N)^{-1} \left( \sum_{i=1}^{n} k_{ij}(n_i - 1) + \Lambda_3 \right) = (\theta_2^N)^{-1} \left( \sum_{i=1}^{r} \left( (n_i^{k_{ij}} - 1) - \left( k_{ij}/2 \right)(n_i - 1)^2 + \Lambda_3 \right) \right) \\
= \left( \prod_{i=1}^{r} n_i^{k_{ij}} \right) N(3) - \sum_{i=1}^{r} \left( k_{ij}/2 \right)(n_i N(2))^2 \\
\in U_2 L^N(N) \\
= N(2)/N(3) \oplus \bigoplus_{1 \leq i \leq j \leq r} \mathbb{Z}/a_{ij} \mathbb{Z} \langle (n_i N(2))(n_j N(2)) \rangle
\]

Here we use the standard direct sum decomposition \( U_2 L^N(N) \cong N(2)/N(3) \oplus \text{SP}^2(N^{AB}) \), cf. (35). Hence (iii)' implies (iii). Conversely, suppose that \( \omega \) satisfies conditions (i) - (iii). Then a similar calculation as in the proof of Theorem 4.5 shows that for \( 1 \leq j \leq s \) and modulo \( \Lambda_4 \),

\[
\sum_{i=1}^{r} k_{ij}(n_i - 1) \equiv b_j \left( (y_j - 1) - \sum_{i=1}^{n} q_{ij} (k_{ij}/2) (n_i - 1)^2 \right) + (z_j - 1) \\
- \sum_{i=1}^{r} \left( q_{ij} (k_{ij}/2) (n_i^{a_i} - 1)(n_i - 1) \right) \\
+ \left( \left( k_{ij}/3 - q_{ij} (k_{ij}/2) (a_i/2) \right) (n_i - 1)^3 \right)
\]

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Thus condition (iii)’ is satisfied for

\[ \delta_j^2 = (y_j - 1) - \sum_{i=1}^{r} \frac{q_{ij}}{d_{ij}} \binom{k_{ij}}{2} (n_i - 1)^2 \]

\[ \delta_j^3 = (z_j - 1) - \sum_{i=1}^{r} \left( \frac{p_{ij}}{d_{ij}} \binom{k_{ij}}{2} (n_i^a - 1)(n_i - 1) \right. \]

\[ + \left( \binom{k_{ij}}{3} - \frac{p_{ij}}{d_{ij}} \binom{k_{ij}}{2} \binom{a_i}{2} \right) (n_i - 1)^3 \left. \right) + \delta_j^4 \]

for some \( \delta_j^4 \in \Lambda_4 \). With these values of \( \delta_j^2, \delta_j^3 \) the formula for \( \xi_3 \) in Theorem 4.5 turns into the one we wished to prove. \( \square \)

5 Proofs for section 3

All results quoted in section 3 are based on the following result.

**Theorem 5.1** Let \( \Delta \) be a descending filtration of \( I(N) \) by subgroups \( I(N) = \Delta_1 \supset \Delta_2 \supset \ldots \) such that Tor\(_1^G(\frac{\Delta_{n-i}}{I_{n-i}}, \frac{I^i(T)}{I^{i+1}(T)}) = 0 \) for \( 1 \leq i \leq n - 2 \). Then there is a natural isomorphism

\[ \frac{K_n}{K_{n+1}} \cong \bigoplus_{i=1}^{n-1} \left( \frac{\Delta_{n-i}}{\Delta_{n-i+1}} \right) \otimes \left( \frac{I^i(T)}{I^{i+1}(T)} \right) \]

induced by multiplication in \( I(G) \) (from the right to the left).

**Proof:** Let \( 1 \leq i \leq n - 1 \) and consider the following diagram where we use the notation of Proposition 4.3 and where \( \hat{\mu}_i \) is the factorization of \( (\iota_1 \circ \cdots \circ \iota_{i-1}) \mu_i \) through its image.

\[
\begin{array}{ccc}
\text{Im}(\Delta_{n-i}I^i(T)) & \Rightarrow & \frac{\Delta_1}{I(T)} \\
& \uparrow \hat{\mu}_i & \downarrow (\iota_1 \circ \cdots \circ \iota_{i-1}) \\
\frac{\Delta_{n-i}}{\Delta_{n-i+1}} \otimes \frac{I^i(T)}{I^{i+1}(T)} & \xrightarrow{\mu_i} & \frac{\Delta_1}{I(T)} \\
& \sum_{k=1}^{n} \Delta_{n-k+1}I^k(T) & \sum_{k=1}^{n} \Delta_{n-k+1}I^k(T) \\
\end{array}
\]


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If $i = 1$, skip the upper part of the diagram. Note that the Tor-term in the hypothesis also vanishes for $i = n - 1$ since then $\Delta_1/\Delta_1 = 0$. Then Proposition 4.3 and the hypothesis imply that

$$\mu_i \text{ and } \nu_i \text{ are injective and } \text{Im}(\mu_i) \cap \text{Im}(\nu_i) = 0.$$  \hspace{1cm} (56)

So $(\nu_1 \circ \cdots \circ \nu_{i-1})$ is injective, and consequently the map $\hat{\mu}_i$ is an isomorphism. It remains to show that in $\Delta_1I(T)/\kappa_{n+1}^\Delta$,

$$\text{Im}(\Delta_{n-j-1}I^{j+1}(T)) \cap \sum_{l=1}^{j} \text{Im}(\Delta_{n-l}I^l(T)) = 0$$

for all $1 \leq j \leq n - 2$. We actually prove more, namely that for all $1 \leq j \leq n - 2$,

$$\text{Im}(\Delta_1I^{j+1}(T)) \cap \sum_{l=1}^{j} \text{Im}(\Delta_{n-l}I^l(T)) = 0. \quad \hspace{1cm} (57)$$

We proceed by induction on $j$. For $j = 1$, $\text{Im}(\Delta_1I^2(T)) \cap \text{Im}(\Delta_{n-1}I(T)) = \text{Im}(\nu_1) \cap \text{Im}(\mu_1) = 0$ by (56). Now suppose that relation (57) is true for $j = i - 1$. Let $x \in \Delta_1I^{i+1}(T)$ such that the coset $x + \kappa_{n+1}^\Delta$ lies in $\sum_{l=i}^{i-1} \text{Im}(\Delta_{n-l}I^l(T))$.

Then there are $y \in \frac{\Delta_{n-i}}{\Delta_{n-i+1}} \otimes \frac{I^i(T)}{I^{i+1}(T)}$, $z \in \sum_{l=i}^{i-1} \text{Im}(\Delta_{n-l}I^l(T))$ such that

$$(\nu_1 \circ \cdots \circ \nu_i)(\bar{x}) = x + \kappa_{n+1}^\Delta = (\nu_1 \circ \cdots \circ \nu_{i-1})(\mu_i y) + z.$$  \hspace{1cm} (56)

Thus

$$(\nu_1 \circ \cdots \circ \nu_{i-1})(t_i \bar{x} - \mu_i)(y) \in \text{Im}(\Delta_1I^i(T)) \cap \sum_{l=1}^{i-1} \text{Im}(\Delta_{n-l}I^l(T)) = 0$$

by the induction hypothesis. Therefore, $t_i(\bar{x}) = \mu_i(y)$ by injectivity of $\nu_1 \circ \cdots \circ \nu_{i-1}$, and whence $\bar{x} = 0$ by (56). Thus $x + \kappa_{n+1}^\Delta = (\nu_1 \circ \cdots \circ \nu_i)(\bar{x}) = 0$, so (57) also holds for $j = i$, as was to be shown. \hfill \Box

Now we are ready to prove the results stated in section 3. Starting out from (8) and (9) Theorem 3.1 is an immediate consequence of Theorem 5.1 whose hypothesis is satisfied here by Corollary 1.4; in fact, for $1 \leq i \leq k$ the group $I^i(T)/I^{i+1}(T)$ is torsion-free, while for $k + 1 \leq i \leq n - 2$ we have $n - i \leq n - k - 1$, whence $\Delta_1/\Delta_{n-i}$ is torsion-free.

Corollary 3.2 then follows, again using Corollary 1.4.

In order to prove Theorem 3.3 first use (10) to reduce to the computation of $\Gamma_{n-1}^*I(N)/\Gamma_n^*I(N)$. Now the conjugation isomorphism $I(N)/\Lambda_{n-i-1}I(N) \cong I(N)/I(N)\Lambda_{n-i-1}$ (see Remark 1.2) shows that Theorem 5.1 applies for $\Delta = I(N)\Lambda$; it provides the desired decomposition of $\Gamma_{n-1}^*I(N)/\Gamma_n^*I(N)$ by means of the mirror symmetry device in Remark 4.2.

Finally, Corollary 3.5 then follows, once again using Corollary 1.4 \hfill \Box
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