Topological Currents in Neutron Stars: Kicks, Precession, Toroidal Fields, and Magnetic Helicity

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Abstract: The effects of anomalies in high density QCD are striking. We consider a direct application of one of these effects, namely topological currents, on the physics of neutron stars. All the elements required for topological currents are present in neutron stars: degenerate matter, large magnetic fields, and P-parity violating processes. These conditions lead to the creation of vector currents capable of carrying momentum and inducing magnetic fields. We estimate the size of these currents for many representative states of dense matter in the neutron star and argue that they could be responsible for the large proper motion of neutron stars (kicks), the toroidal magnetic field and finite magnetic helicity needed for stability of the poloidal field, and the resolution of the conflict between type-II superconductivity and precession. Though these observational effects appear unrelated, they likely originate from the same physics – they are all P-odd phenomena that stem from a topological current generated by P-parity violation.
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1. Introduction

It is well known that anomalies have important implications for low-energy physics: the electromagnetic decay of neutral pions $\pi^0 \rightarrow 2\gamma$ is a textbook example. Anomalies reveal intricate relationships between topological objects such as vortices, domain walls, Nambu-Goldstone bosons, and gauge fields and often result in very unusual physics. A particularly relevant example is the superconducting cosmic string on which an electric current flows without dissipation and carries momentum [1]. The effects of anomalies are well established and are reviewed in [2].

More recently the role of anomalies in QCD has studied at finite baryon density [3, 4, 5] and similar phenomena has been studied in condensed matter systems [6, 7, 8]. Since the original paper [3] many other applications of anomalies in dense QCD have been considered: an analysis of the axion physics and microscopic derivation of anomalies [4]; studying the vortex structure due to the anomalies currents in neutron stars (type-I verses type-II superconductivity) [9]; the charge separation effect at RHIC (relativistic heavy ion collider) [10, 11]; magnetism of nuclear and quark matter [12]; anomaly mediated neutrino-photon interactions at finite baryon density [13]; the chiral magnetic effect at RHIC [14] and many others.

For this paper we are interested in the result where a vector current (along with an axial current) is induced in the background of an external magnetic field when the chemical potentials of the right and left-handed fermions are not equal, $\mu_r \neq \mu_l$. As argued in [4] this phenomenon depends only on the presence of chiral symmetry and not on whether the chiral symmetry is spontaneously broken or not. Applications of this induced vector current have been discussed in neutron star physics [9] and RHIC related physics [14]. In [9] it was shown that if the left and right-handed chemical potentials are different, $\mu_r \neq \mu_l$, the resulting induced, non-dissipating vector current running along the magnetic flux tubes may change the behaviour of superconductivity from type-II to type-I, even though the Landau-Ginzburg parameter $\kappa > 1/\sqrt{2}$ suggests type-II behaviour. It was also argued that the Abrikosov lattice will be destroyed due to the helical instability in the presence of the induced vector current. This would resolve the apparent contradiction with the precession of the neutron stars [15, 16]. For another mechanisms that may resolve this contradiction see [17, 18, 19, 20].

The goal of the present study is a quantitative analysis of the conditions when P-parity violation ($\mu_r \neq \mu_l$) occurs in dense stars and a persistent, topological current is induced. We claim that topological vector currents do exist in neutron stars and we consider applications that may help explain neutron star kicks and toroidal magnetic fields.

It is well known that the weak interactions (where P-parity is strongly violated) play a dominant role in neutron star physics. Producing the asymmetry $\mu_r \neq \mu_l$ for a given microscopic, individual processes common in the bulk of neutron stars, but we are interested in coherent P-parity violating effects when the asymmetry appears in macroscopically large regions. Sections 3–7 are devoted to estimating the induced current in different environments that may exist in neutron stars, while Section 8 is devoted to possible applications of the topological currents. Readers interested in applications may go directly to Section
If non-dissipating vector currents are induced they can transfer momentum either by escaping the star or radiating photons. This would imply that a long, sustained kick, correlated with the magnetic field and the the kick axis, could be produced that explains the anomalous proper motion, $v > 800 \text{ km/s}$, observed in some stars — see [21] for examples of kick velocities. In fact, long, sustained kicks are observationally supported by the analysis in [22] and the spin-kick-magnetic field correlation is observationally supported in [22, 23, 24]. This can be considered indirect support for our proposal. The correlation between momentum and the magnetic field, $\langle \vec{P} \cdot \vec{B} \rangle$, is a P-odd effect and must be generated by a parity violating process. This mechanism is similar to neutrino emission in that we are ejecting a particle from the star, but there are some key differences. Neutrino emission is automatically asymmetric with respect to the direction of the magnetic field $\vec{B}$; however, most neutrino-based kick mechanisms have difficulty delivering the produced asymmetry to the surface of the star. Only when it can reach the surface can the asymmetry push the star. In most neutrino based mechanisms the star must be very hot ($T > \text{MeV}$) for neutrinos to be energetic enough to transfer sufficient momenta. But at such high temperatures the neutrinos can not escape the star without interacting and washing out the asymmetry. The topological nature of the induced currents may be capable of delivering the required asymmetry (produced in the interior of the star) to the surface without dissipation. In addition, our engine continues to work even when temperature drops well below $T \ll \text{MeV}$. This is because it is the chemical potential $(\mu_l - \mu_r) \neq 0$ that drives the kick, not the temperature.

We also suggest that a current running along the magnetic flux may be the source of the toroidal magnetic field and the finite magnetic helicity generally thought to be required in a neutron star [25, 26, 27, 28]. The magnetic helicity, $\mathcal{H} \equiv \int d^3 x \vec{A} \cdot \vec{B}$, that arises from the linking of toroidal and poloidal magnetic fields is a topological invariant and is a P-odd effect that must be generated though parity violation. Generating a toroidal field with currents running along the poloidal field naturally produces magnetic helicity without resorting to the breakdown of conductivity to temporarily break the invariance of the helicity.

These applications of topological currents to seemingly unrelated P-odd effects will be discussed in greater detail in Section 8 and only comprise a sample of possible applications. Remarkably, the effects of topological currents introduced in this paper can be experimentally tested in terrestrially based laboratories, particularly in some condensed matter systems and at the relativistic heavy ion collider (RHIC) at Brookhaven. This will be discussed in Section 9.

2. A Big Picture: The Basic Ingredients and Assumptions

Topological vector currents can couple to charged fermions and become real, superconducting, electromagnetic currents. Our motivation comes from the possible effects of topological currents on the physics of neutron stars. Even though axial currents exist in the star they cannot be used as an electromagnetic source and we will not consider them.
Non-dissipating, induced vector currents at \( T = 0 \) have the form \[4, 5, 9, 14\]

\[
\langle j \rangle = (\mu_l - \mu_r) \frac{e \Phi}{2\pi^2},
\]

(2.1)

where \( \mu_r \) and \( \mu_l \) are the chemical potential of the right and left-handed electrons, and \( \Phi \) is the magnetic flux. For the current to be nonzero the number of left-handed particles must be different than the number of right-handed particles. The weak interaction, which strongly violates parity, is a natural source for the required asymmetry and is where we focus our attention.

We will assume that electrons are the only reasonable charge carrier because all other charged particles in the star are too heavy. The particles in a neutron star attain equilibrium through the weak interaction, which creates predominantly left-handed particles. In an infinite system this imbalance would disappear – the average helicity of the electrons would be washed out due to the inverse weak P-violating processes. The key is that the neutron star is a finite system and electrons are removed from the star by the current before they can decay. The asymmetry that created the current is allowed to propagate to the surface and not get washed out.\(^1\)

This topological current corresponds to the lowest energy state in the thermodynamic equilibrium when \( \mu_l \neq \mu_r \) is held fixed. In reality there is a tendency for \( \mu_l \) and \( \mu_r \) to equilibrate though weak interactions; however, due to the finite size of the system a complete equilibrium can not be achieved. This is analogous to how neutrinos in cold stars can leave the system without further interactions, but unlike with neutrinos the electron chemical potential does not drop to zero. The induced current only remains non-dissipating when the system is degenerate, \( \mu \gg T \). In the star’s crust this condition becomes invalid, the current will become dissipative, and the trapped electrons and return into the system. A detailed discussion of this region when \( \mu \sim T \) is beyond the scope of the present work, and will be discussed somewhere else. If the electrons manage to escape the star the electron chemical potential will slowly decrease until the current may stop running. Charge neutrality will cause matter to accrete isotropically possibly maintaining some of the chemical potential.

In the following subsections we will discuss the structure and processes of dense matter, formally derive the topological current, and discuss in greater detail how the magnitude of the current can be estimated.

2.1 Notation

We use the convention \( \hbar = c = k_B = 1 \) unless otherwise stated. We will always denote the momentum of particle as \( p_i \), the fermi momentum as \( k_i \), and the chemical potential as

\(^1\)Here and in what follows we neglect all QED re-scattering effects, which are much stronger than weak interactions but they are P-even and, therefore, can not wash out the produced asymmetry. Because of the large magnetic field the electron only travels in the direction of the magnetic field while the motion in transverse directions confined to Landau levels. The term “mean free path” in this paper implies the weakly interacting P-odd “mean free path” when a produced asymmetry can be washed out.
µ_i. When convenience dictates, the subscript \(i\) will either be the symbol of the particle or a number, which will be labelled on the Feynman diagram. The three momentum will be bolded \(p_i\), with a magnitude denoted \(p_i\), and the four momentum will have a greek index \(p^\mu\).

### 2.2 Topological vector currents

The purpose of this section is to provide a brief introduction to topological currents and explicitly derive the form of the vector current. We will only provide a sketch here and refer the reader to [4, 5, 14] for the finer details.

Consider the vector current
\[
j^3_\nu = \bar{\psi} \gamma^3 \psi \tag{2.2}
\]

The Dirac equation in a magnetic field with potential \(A\) can be written as
\[
(H^2 + m^2)\psi_r = E^2 \psi_r, \tag{2.3}
\]

where \(\psi_l = \frac{1}{m}(E - H)\psi_r\). Eigenvalues of \(H\) acting on \(\psi_r\) are labelled \(\epsilon\), hence the Dirac equation has two solutions, \(E_\pm = \pm \sqrt{\epsilon^2 + m^2}\). The Dirac spinor can be written entirely in terms of the right-handed spinor and its eigenvalues,
\[
\psi_\pm = \left(\frac{\psi_l \psi_r}{\sqrt{4(m^2 + \epsilon^2)^{1/4}}}ight) \pm \left(\frac{\pm [(m^2 + \epsilon^2)^{1/2} \pm \epsilon]^{1/2}}{[(m^2 + \epsilon^2)^{1/2} + \epsilon]^{1/2}}\right) \psi_r \tag{2.4}
\]

We further break up the operator \(H\) into transverse and longitudinal components \(H = p_3\sigma^3 + H_\perp\), where \(p_3\) are momentum eigenstates along the magnetic field and \(H_\perp = (-i\partial_i + eA_i)\sigma_i\) is a transverse operator with eigenstates \(|\lambda\rangle\). We take the longitudinal direction to be periodic in \(L\) and take the limit \(L \to \infty\) later. One can prove that the zero modes of \(H_\perp\) are simultaneously eigenstates of \(H\) with eigenvalue \(\epsilon = p_3\sigma^3\).

The expectation value for the current is found in the usual manner by summing the current over all states weighted by the probability of each state. For fermions the probability is given by the Fermi-Dirac distribution. The derivation for the axial current assumes that the densities for the left and right-handed modes are equal. Here, in the derivation of the vector current, we consider the possibility that the densities are different and assign the left and right-handed modes each their own Dirac distribution, \(n_l\) and \(n_r\), in the expectation value,
\[
\langle j^3_\nu \rangle = \sum_E \left[ n_r(E) \bar{\psi}_l \sigma^3 \psi_r - n_l(E) \bar{\psi}_l \sigma^3 \psi_l \right]
\]
\[
= \sum_\epsilon \left[ n_r(E_+) + n_r(E_-) - n_l(E_+) - n_l(E_-) \right] \bar{\psi}_l \sigma^3 \psi_r, \tag{2.5}
\]

where we used the spinor definition above and the fact that summing over terms odd in \(\epsilon\) vanish. If write everything in terms of the eigenstates of \(H_\perp\) it can been shown that only
the zero modes of $\lambda$ survive,

$$
\langle j^3_v \rangle = \frac{1}{L} \sum_{p_3} \sum_{\lambda=0} \left[ n_r(E_+) + n_r(E_-) - n_l(E_+) - n_l(E_-) \right] \langle \lambda|\sigma^3|\lambda \rangle \tag{2.6}
$$

Following the standard arguments, the factor $\langle \lambda|\sigma^3|\lambda \rangle$ counts the difference in transverse zero modes travelling parallel to the magnetic field with positive or negative eigenvalues, $N_+$ and $N_-$. Taking $L \to \infty$ and integrating each Dirac distribution gives the number density of 1-dimensional left and right-handed fermions, $n(T,\mu)_r - n(T,\mu)_l$. With this all taken into account the current can be written as,

$$
\langle j^3_v \rangle = [n_l(T,\mu) - n_r(T,\mu)](N_+ - N_-) \tag{2.7}
$$

The difference in the positive and negative modes travelling along the magnetic field is given in physical terms by the index theorem,

$$
N_+ - N_- = \frac{e\Phi}{2\pi} \tag{2.8}
$$

where $\Phi$ is the magnetic flux. The current is then,

$$
\langle j^3_v \rangle = (n_l - n_r)\frac{e\Phi}{2\pi} \tag{2.9}
$$

where $n_r(T,\mu)$ and $n_l(T,\mu)$ are one dimensional number densities of left and right-handed Dirac fermions. Furthermore, the one-dimensional number density in the massless limit can be written in terms of the chemical potential $n = \frac{\mu}{\pi}$ in which case the expression for the current is reduced to (2.1). In this case it is purely topological.

Formulae (2.5) and (2.9) have a very simple physical meaning: to compute the current one should simply count the difference between left-handed and right-handed modes in the background of a magnetic field, see Figure 1. We assume that the modes of the current couple to electrons. In magnetic flux the spin of an electron will tend to antialign with the field. Then, if there is an excess of left-handed electrons they must move in the direction of the magnetic field and if there is an excess of right-handed electrons they must move against the field. The topological vector current acts as a pump to remove the average helicity and it stops pumping once the average helicity is zero again.

In general, the difference in left and right-handed modes is a complicated function of many parameters: magnetic field $B$, chemical potential $\mu$, temperature $T$, and mass of the particle $m$; but in the chiral limit ($m = 0$) the expression for the current takes the simple form (2.1), has pure topological character, and can be derived from an anomalous effective Lagrangian without referring to the dynamics. The current is expressed in terms of one-dimensional fermi distributions (2.7) and the physics of two other dimensions is determined by Landau levels (the lowest one for $m = T = 0$). When $T \neq 0$ and $m \neq 0$ the current will still be induced, but it will not have a simple topological form (2.1). The relevant formula in this case becomes much more complicated and is determined by the ratios of a number of dimensional parameters mentioned above, see [14] where some limiting cases have been studied.
The current is insensitive to the structure of the magnetic field and will be induced if the magnetic field is confined to magnetic flux tubes or uniformly distributed. The current is just confined to the regions where the magnetic field is present. For our purposes it is not essential whether the magnetic field is represented by magnetic flux tubes, as found in type-II superconductors, or by magnetic domains, the typical structure for the intermediate state\(^2\) argued for in [9]. The current is strongest in the degenerate regions with \(\mu \gg T\) where the background magnetic field is large and when the system becomes a less degenerate (or not degenerate at all) one should expect strong suppression [4, 14]. For numerical estimates of the effect it is convenient to count number of superconducting flux tubes in the entire star and compute the current per unit quantum flux \(\Phi_0 = \frac{2\pi}{q} = 2 \cdot 10^{-7} \text{G} \cdot \text{cm}^2\), where \(q = 2e\) is the charge of the proton Cooper pair.

Finally, we note that two different applications of the induced vector current have been previously discussed: a) neutron stars [9] and b) RHIC physics [14]. Our basic objective remains the same as in our previous paper [9]; however the main focus in this paper will be estimating the magnitude of the induced current itself while our previous paper [9] was mainly devoted to the specific application on study of the type-I vs type-II superconductivity as a result of induced vector current.

2.3 Basic weak processes

The primary composition of nuclear matter in a neutron star is constantly being debated. Fundamentally, a neutron star is made of neutrons with small, equal fractions of protons and electrons. In more exotic models hyperons may appear along with pion and kaon

\(^2\)The intermediate state is characterized by alternating domains of superconducting and normal matter where the superconducting domains exhibit the Meissner effect, while the normal domains carry the required magnetic flux.
condensates. In an effort to simplify the discussion we will constrain ourselves to four fundamental interactions that describe the majority of cases in dense stars.

These interactions have been discussed before in the context of cooling a neutron star through neutrino emission. The first two are quite closely related – the direct and modified Urca processes,

\[ e^- + P \rightarrow N + \nu_e \]  
\[ e^- + P + N' \rightarrow N' + N + \nu_e, \]

which both obey the beta equilibrium condition \( \mu_e + \mu_P = \mu_N \). The neutrinos created in these processes only interact with matter through the weak force, which is so weak that the star is transparent to the neutrinos. They leave the star and do not contribute to the equilibrium condition.

The first of these interactions, the direct Urca Process, should be the dominant process in normal nuclear matter but it is heavily suppressed because the the particles are unable to conserve momentum while remaining on their Fermi surfaces. It is possible for the direct process to conserve momentum if the proton fraction in the star is above 1/9 \([29]\), which could occur with the appearance of hyperons \([30]\). In these cases the direct Urca process dominates. The modified Urca process is able to conserve momentum through an external nucleon and was used in the first neutrino emission calculations \([31]\). This is the dominant electron producing process in normal nuclear matter, but it is very slow and the presence of exotic particles introduces processes that create and destroy electrons quicker.

As the density of matter increases it is likely that kaon \([32]\) and pion condensates will appear. We will restrict the discussion to kaon condensates, which appear at much more reasonable densities, \(3n_0\), than pion condensates, \(300n_0\), but the phenomenology of dealing with the two condensates is almost identical. Electrons are still created and destroyed by the previous interactions, but at a much slower rate than an electron decaying\(^3\) into a kaon and neutrino in the presence of nucleons,

\[ e^- + N \rightarrow \langle K \rangle^- + N + \nu_e. \]  

This interaction and its inverse process add another equilibrium condition, \( \mu_e = \mu_K \), on top of the previously mentioned beta equilibrium.

The previous three interactions encompass the creation of electrons in almost all possible neutron star interiors. The last interaction we consider is the primary source of electrons in quark stars. The direct Urca process for quarks are,

\[ e^- + u \rightarrow d + \nu_e \]  
\[ e^- + u \rightarrow s + \nu_e . \]

Unlike in normal nuclear matter there is no trouble conserving momentum in quark matter. The direct process occurs unsuppressed and there is no need to discuss a modified process \([33]\).

\(^3\)Here and in what follows the term “electron decay” means the transforming of an electron into a neutrino as a result of interactions with surrounding hadrons.
2.4 Simple models for dense matter

We will review the features of neutron matter that are required for the rest of the paper. Most importantly we will summarize the short reviews found in [33, 34, 35, 36] and state the values used in the rest of the paper. The simplest model is the non-interacting gas model where the ground state of the neutron star, \( T = 0 \), is a mixture of neutrons, protons, and electrons, that is electrically neutral. The baryon density is on the order of nuclear density \( n_0 = 0.17 \text{ fm}^{-3} \), which leads to the nucleons and electrons being highly degenerate.

The particles achieve equilibrium though the Urca processes \((2.10)\). The neutrinos produces in these reactions only react weakly in the star and easily leave it at low temperatures. Because of this neutrinos are often assumed to be non-degenerate and the chemical potentials in the star satisfy,

\[
\mu_e + \mu_P = \mu_N. \tag{2.15}
\]

Charge neutrality implies that \( n_e = n_P \), which implies that the Fermi momenta of the electrons and protons are equal, \( k_e = k_P \). This restriction has two important effects. The electrons are relativistic and the protons are non-relativistic implying that the chemical potential of the proton is much smaller that of the electron. This further implies that the density of neutrons is much higher than the electrons and protons, thus a neutron star. Assuming that the density of the neutrons is that of nuclear matter then,

\[
k_N = (3\pi^2 n_N)^{1/3} \approx 340 (n/n_0)^{1/3} \text{ MeV}. \tag{2.16}
\]

Equating the electron and neutron chemical potentials yields,

\[
k_e \approx \frac{k_N^2}{2m_N} \approx 62 (n/n_0)^{2/3} \text{ MeV}. \tag{2.17}
\]

In reality there is a correction to the non-interacting model due to the proton being more bound than the electron. It is common in literature to assume a value of

\[
k_e \approx 100 (n/n_0)^{2/3} \text{ MeV}, \tag{2.18}
\]

which is the value we will use through out the paper.

As the density of the star raises above \( 3n_0 \) there is the possibility that \( K^- \) condensates will appear [32]. As the density of the nuclear matter increases, the density of the electrons increases to a point where it becomes advantageous to decay into negatively charged kaons though the process \((2.12)\). The system reaches equilibrium through the inverse process. These processes add an additional equilibrium condition,

\[
\mu_e = \mu_K. \tag{2.19}
\]

Though negatively charged pions are lighter than kaons, they are unlikely to appear until much higher densities due to the strong interaction increasing the effective mass [30].

At around the same density as kaons appear, light hyperons and muons may appear [30]. The existence of hadrons lowers the neutron density, which lowers the ratio of neutrons
to protons, making it possible for the direct Urca process to proceed unsuppressed, and opens up processes such as \( \Lambda \rightarrow e^- + P + \bar{\nu}_e \), which are also not kinematically suppressed. These processes occur at about the same rate as the direct Urca processes, so we will take the direct Urca process as a reasonable substitute for them. The rate can then be adjusted by an integer factor to compensate for additional processes. The appearance of muons has no effect of our calculations as it is too heavy to couple to the current.

When the density gets high enough it is possible that the quarks deconfine – the hadrons break down into their constituent quarks. For this paper we will consider only the existence of light quarks in the star, which attain equilibrium through the quark Urca processes. The equilibrium conditions are

\[
\mu_u + \mu_e = \mu_d \quad (2.20)
\]

\[
\mu_u + \mu_e = \mu_s \quad (2.21)
\]

where, as in earlier cases, the neutrinos are not trapped and are not degenerate. The quark matter must also be electrically neutral,

\[
Q/e = \frac{2}{3} n_u - \frac{2}{3} n_d - \frac{1}{3} n_s - n_e = 0 \quad (2.22)
\]

The simplest models assume that the quark masses are all zero, thus their Fermi momenta are equal to their Fermi energies, and the predicted electron density is zero. This, however, is not adequate as leptons do exist in the star. Room for the electrons comes from the large mass of the strange quark. Though the up and down quarks are relatively light, \( m_u \sim m_d \sim 5 - 10 \text{ MeV} \), the strange quark mass is actually quite large, \( m_s \sim 100 - 300 \text{ MeV} \). The strange quark is nonrelativistic meaning there will be fewer of them and electrons must be present to conserve charge. For this paper will will follow [33] by assuming that the quarks are massless, \( k_u \sim k_s \sim k_d \sim k_q \), where

\[
k_q = (\pi^2 n_b)^{1/3} \sim 235 \left( \frac{n_b}{n_0} \right)^{1/3} \text{ MeV} \quad (2.23)
\]

where \( n_b \) is the baryon number density. For typical densities in the core of the neutron star the Fermi momentum is \( k_q \sim 400 \text{ MeV} \). We can approximate the electron Fermi momentum using the fraction of electrons to baryons, \( Y_e = n_e/n_b \), which yields,

\[
k_e = (3Y_e)^{1/3} k_q \quad (2.24)
\]

The typical value for the electron fraction is \( Y_e = 0.01 \).

We have discussed the nature of topological currents and determined the requirements for them to exist. We have also given an overview of many types of dense stars, which are the only viable places for these current to appear. Now we are ready to discuss the details of how the current appears in the star, and what its magnitude would be.

### 3. Estimating Currents in Dense Matter

There are three requirements for topological vector currents to be present: an imbalance in left and right-handed particles \( \mu_l \neq \mu_r \), degenerate matter \( \mu \gg T \), and the presence of the
background magnetic field $B \neq 0$. All of these are present in neutron and quark stars. The weak interaction, by which the star attains equilibrium, violates parity; particles created in this environment are primarily left-handed, see the Appendix for a quantitative estimates. The interior of the star is very dense, $\mu_e = 100$ MeV, and cold, $T = 0.1$ MeV, such that the degeneracy condition $\mu \gg T$ is met, and the star is known to have a huge magnetic field, $B \sim 10^{12}$ G.

All three criteria are met but there is a subtlety to consider. In an infinite system, a system large enough to allow the electron to decay as discussed in section 2.3, any asymmetry in left and right-handed electrons created by the weak interaction would be washed out; the creation and annihilation rates of the left-handed particles are the same. Though many more left-handed electrons are created, they are also destroyed much faster than the right-handed and no asymmetry builds. This is similar to the argument found in [37] – there is no asymmetry in equilibrium in an infinitely large system. The point is that a neutron star is a finite system, and a small asymmetry will remain in the bulk because of this, see Figure 2. To understand how this happens we shall first review what is known about structure of the magnetic field in the neutron stars.

![Figure 2](image_url)

**Figure 2:** The neutron star is a finite system – the mean free path with respect to the weak interaction is much larger than the radius of the neutron star. Electrons leave the system before they decay and contribute towards the current. The current flows because the system is not in equilibrium.

A typical neutron star has a magnetic field $B \sim 10^{12}$ G. Once it has cooled, the star is cold $T \sim 10^9$ K compared to its Fermi energy $\mu \sim 100$ MeV $\sim 10^{12}$ K, and the protons are likely superconducting. The magnetic field is large enough that it is favourable for the flux to penetrate the superconductor, rather than being completely expelled. The Meissner effect forces the flux to bundle into into either type-II vortices or type-I domains. It is generally believed that the protons form a type-II superconductor. The Landau-Ginzburg parameter $\kappa = \lambda/\xi$ determines the type of superconductivity. Typically in a neutron
star the London penetration depth is $\lambda \sim 120$ fm and the coherence length of the proton superconductor is $\xi \sim 30$ fm. This creates a ratio $\kappa > 1/\sqrt{2}$, which indicates type-II superconductivity. For a type-II superconductor the magnetic field will penetrate the star by destroying narrow regions of superconductivity that each carry a single quantum of flux, $\Phi_0 = 2\pi/q = 2 \times 10^{-7}$ G cm$^2$. But there are problems with this picture [15, 16]. It is possible that the system behaves as type-I superconductor even though the Landau-Ginzburg parameter would suggest type-II behaviour [9].

If the intermediate state is realized in neutron stars the magnetic field distribution will be again non-uniform, but the structure would be quite different. The intermediate state is characterized by alternating domains of superconducting and normal matter where the superconducting domains exhibit the Meissner effect, while the normal domains carry the required magnetic flux. The pattern of these domains is strongly related to the geometry of the problem, see [9] for details. While precise calculations are required for understanding of the magnetic structure in this case one can give the following estimation for typical size of a domain as suggested in [9, 17]

$$a \sim 10\sqrt{R\lambda},$$

where $R$ is a typical external size identified with a neutron star core ($R \sim 10$ km), while $\lambda$ is a typical microscopical scale of the problem. Numerically $a \sim 10^{-1}$ cm, which implies that a typical domain can accommodate about $10^4$ neutron vortices separated by a distance $10^{-3}$ cm. While the field distribution for the intermediate state and the type-II superconductor are very different one should anticipate that the ratio of normal to superconducting regions are the same.

Regardless of the flux structure, there are two regions of the neutron star to consider – those with magnetic flux and those without. The total units of quantum flux can be estimated as

$$N_v \sim \frac{\pi R^2 B}{\Phi_0} \sim 10^{31} B_{12}, \quad B_{12} \equiv \left( \frac{B}{10^{12} \text{G}} \right).$$

The region that a single unit of flux occupies has a radius equal to the London penetration depth of the field $\lambda \sim 100$ fm. This is multiplied by the number of vortices $N_v$ to get the total area. If we take a slice of the neutron star perpendicular to the magnetic field we find that the ratio of the area occupied by flux tubes is much smaller than the area occupied by the void

$$\frac{A_{\text{vortices}}}{A_{\text{star}}} \sim \frac{N_v \pi \lambda^2}{\pi R^2} \sim \frac{\pi \lambda^2 B}{\Phi_0} \sim 10^{-3} \cdot B_{12}.$$  

This suppression essentially reflects the difference between typical magnetic field $B \sim 10^{12}$ G and the critical magnetic field $B_{c1} \sim 10^{15}$ G when the superconductivity is destroyed.

It is clear the the magnetic flux structure inside a neutron star is non-trivial. Directly calculating the current would require careful consideration of the helicity of electrons in regions with flux, where all electrons created are left-handed, and regions without flux, where the helicity can be washed out, and how electrons diffuse from one region to the
other. These regions are shown in Figure 2. We will simplify this by considering what would happen if the flux were uniformly and continuously spread throughout the star. The magnitude of the current in the entire star depends only on the amount of flux, not its structure. The total current leaving the star would be the same as if some complex structure were present; if the flux lines are bunched it means that there are smaller regions with stronger magnetic fields.

If the magnetic field is uniformly distributed then every electron created by P-odd process in the star possibly contributes to the current. As long as the mean free path is larger than the star the produced P-asymmetry is not washed out, see Footnote 1 on pg. 4. When the electrons leave the region of degeneracy (where the non-dissipating current is produced) the current loses its quantum properties and becomes a normal dissipating current capable of transferring momentum, emitting photons, etc. In this transition region the momentum carried by the current will be transferred to the crust or into space by some means. We shall not discuss the means of this transfer in the present work. As discussed in Section 2.2 we can calculate the current by simply counting the number of left-handed electrons minus the number of right-handed created in the star. Unlike Section 2.2 there is now a large Fermi momentum which opens up many Landau levels, but only the lowest Landau level contributes to the current. We account for this by a suppression factor $P_{\text{asym}}$. The total produced current is then

$$\langle j \rangle_{\text{star}} = P_{\text{asym}}(B, \mu, T) \cdot \frac{w}{\Omega} V_{\text{star}},$$

where $w/\Omega$ is the transition rate per unit volume assuming the magnetic field is uniformly distributed and $V_{\text{star}}$ is the volume of the region of degeneracy which we assume is the same order of magnitude as the star itself. The factor $P_{\text{asym}}(\mu_e, T, B) = 2 \cdot 10^{-5} B/B_c(n/n_0)^{-4/3}$ describes the left-right asymmetry produced in magnetic field $B$ at temperature $T$ from the weak interaction and the suppression due to the Landau levels. Quantitative estimates of $P_{\text{asym}}(B, \mu, T)$ can be found in the Appendix.

For non-trivial vortex structures it is convenient to determine the current per unit fundamental flux by dividing by the total number of flux tubes. In a type-II structure this would be the current that runs along a single vortex; in a uniformly distributed environment it is simply a convenient normalization,

$$\langle j \rangle = P_{\text{asym}}(B, \mu, T) \cdot \frac{w}{\Omega} \frac{V_{\text{star}}}{N_v}.$$
not when the temperature drops which happens much earlier, see below for the numerical estimates.

The expression for the induced current, (3.4) and (3.5), naively has a different form than previously discussed (2.7), but in fact is precisely the same induced current with the same physical meaning. The separation of transverse and longitudinal (with respect to magnetic field $B_z$) degrees of freedom that is explicit in (2.7) is hidden now in formula for $P_{\text{asym}}(B, \mu, T)$ where the Landau levels (transverse degrees of freedom) are treated separately from longitudinal motion, see Appendix for details. The longitudinal degrees of freedom in eq. (2.7) are represented at $T = 0$ by the one-dimensional number density $\sim \mu/\pi$, which is the correct expression when the problem is treated as a grand-canonical ensemble with $\mu_L$ and $\mu_R$ constant due to the infinitely large bath surrounding the system. In our case the neutron star is a finite system at $T \neq 0$ where particles are continuously injected at a rate $w$. Along with $P_{\text{asym}}(B, \mu, T)$ this describes the resulting asymmetric number density as a function of external parameters $T, \mu, B$. And as they should, both equations (2.7) and (3.4) have units of current: number of particles per unit time.

The topological vector current\(^4\) arises specifically because the system is no longer in equilibrium with respect to the weak interaction and a small asymmetry $(\mu_L - \mu_R) \neq 0$ appears. The electrons are removed prematurely (before they have a chance to decay through P-odd weak interactions) either by exiting the star or by being absorbed into the crust. The P-asymmetry is preserved as electrons travel in the degeneracy region of the star because electromagnetic interactions can not wash out P-odd asymmetry. This asymmetry contributes to the difference in the chemical potential of left and right-handed electrons that is driving the current. The current is a steady state\(^5\), but constantly requires new electrons to be created in the magnetic field background where asymmetry is produced, then pushed out of the system, into the crust or into space. These electrons diffuse back into the bulk, maintaining the diffusive equilibrium of the entire star. We are now in a position to estimate the mean free paths of electrons $l \sim w^{-1}$ with respect to the weak interaction which enters our basic expression for the current (3.4). Our goal below is to give very rough, order of magnitude estimates without going into specific questions about neutron star structure and equation of state. Therefore we shall use very simplified models for dense matter systems. In addition, we present a calculation for $P_{\text{asym}}$ in the Appendix for a simplest interaction, $\beta$ decay. We assume that the asymmetry parameter $P_{\text{asym}}$ does not change much from process to process and that the entire variation in the estimates is represented by changes of the absolute value of $w$.

\section*{4. Nuclear Matter: The Direct Urca Process}

We want to determine the mean free path of the weak interaction of an electron travelling

\(^4\)The axial current will always be induced even when $\mu_L = \mu_R$, but it will not be coupled to the electromagnetic field, and can not carry the momentum. The physical consequences of this axial topological current might be quite interesting, but shall not be discussed in the present paper.

\(^5\)We assume that the variation of $(\mu_L - \mu_R)$ is adiabatically slow process, with a typical variation time to be much longer than any other time scales of the problem. It allows us to treat the system as being in the equilibrium when $(\mu_L - \mu_R)$ is assumed to be a fixed parameter.
inside a neutron star and the rate at which electrons are created. To do so we will estimate the transition rate following the standard techniques from [31]. Estimating the mean free path allows us to determine whether the helicity built from the weak interaction is washed out or if the asymmetry can escape the star\(^6\). Similar calculations have been done only for neutrinos as the electron’s mean free path is assumed to be much shorter due to electromagnetic interactions. However, as we argued above, the electromagnetic interactions do not wash out P-odd asymmetry and the electrons are allowed to propagate due to the non-dissipating, topological vector current. In order to find the mean free path we will consider the direct Urca process, equation (2.10), given by the Feynman diagram in Figure 3.

![Figure 3: The direct Urca process.](image)

Though it appears it should be, this process is not the dominant one in normal nuclear matter. It is suppressed because the particles taking part are not able to conserve momentum. If all interacting particles lie on their Fermi surface then,

\[
p_e + p_P - p_N \ll T,
\]

where \(T\) is approximately the energy of the neutrino. The suppression occurs because in order for the process to conserve momentum, the initial electron and proton, or the final neutron, must be far from their Fermi surface in a region with almost no particle occupation. Forcing this inequality become an equality introduces a suppression of order \(\sim e^{-k_N/T}\). When the proton fraction is sufficiently large, \(x_p > 1/9\), this interaction proceeds unfettered. The transition rate is given by,

\[
w = \frac{\Omega^4}{(2\pi)^{12}} \int d^3p_1 d^3p_2 d^3p_3 d^3p_4 S (2\pi)^4 \Omega \delta(p_f - p_i) |S|^2,
\]

where the \(p_i\) are respectively the electron, proton, neutron, and neutrino momentum and \(S\) is a statistical factor that takes into account the Fermi blocking.

\(^6\)Here and in what follows we do not assume that an electron physically escapes a star: we use the term “escape” to emphasize that the electron can leave the region of degeneracy without being re-scattered by dense surrounding matter. The fate of the moving electrons when they enter the non-degenerate region \(\mu_e \sim T\) from deep degenerate region depends on specific properties of matter with \(\mu_e \sim T\). In this region the current becomes dissipating, and the electrons may transfer their energy/momentum to the surrounding dense environment. This subject is beyond the interests of the present work, and shall not be discussed here.
The Fermi blocking factors limit the phase space in which the initial particles can exist and the final particles can be created. Particles that exist at the beginning of the reaction, such as the electron and proton in this case, are given a factor equal to the Fermi distribution,

$$f_i = \frac{1}{1 + e^{-(E_i - \mu_i)/T}},$$

(4.3)

which tells us where particles exist. The particles to be created, such as the neutron, are given a factor of one minus the Fermi function,

$$1 - f_i = \frac{1}{1 + e^{(E_i - \mu_i)/T}},$$

(4.4)

which restricts the phase space that particles can be created in. There is no Fermi blocking term associated with the neutrino because they leave the star without interacting and do not become dense enough to form a Fermi surface. All of these blocking factors make the statistical factor $S$.

We use the standard four Fermi scattering matrix element,

$$|S|^2 = \frac{G_F^2}{\Omega^4} (1 + 3C_A^2) \left[ 1 - \frac{1 - C_A^2}{1 + 3C_A^2} \frac{p_1 \cdot p_4}{E_1 E_4} \right],$$

(4.5)

where $C_A = 1.26$ is the Gamow-Teller coupling and $G_F = 1.17 \times 10^{-11}$ MeV$^{-2}$.

Following [31] we separate the angular and radial integrals such that the transition rate becomes,

$$w = \frac{\Omega^5}{(2\pi)^8} |S|^2 P Q,$$

(4.6)

where,

$$P = \int p_1^2 dp_1 p_2^2 dp_2 p_3^2 dp_3 p_4^2 dp_4 S \delta(E_f - E_i),$$

(4.7)

$$Q = \int d\Omega_1 d\Omega_2 d\Omega_3 d\Omega_4 \delta^{(3)}(p_f - p_i).$$

(4.8)

We start with the $Q$ integral. The contribution from the momentum of the neutrino is small compared to the rest so we neglect it in the $\delta$-function. This allows us to take an integral over all angles causing the term with angular dependence to vanish. As in [31] the angular integrals become,

$$Q = \frac{(4\pi)^5}{2p_1p_2p_3}.$$

(4.9)

We can now do the $PQ$ integral,

$$PQ = \frac{(4\pi)^5}{2} \int p_1 dp_1 p_2 dp_2 p_3 dp_3 p_4^2 dp_4 S \delta(E_f - E_i).$$

(4.10)

The first step is to change variables from energy to momentum. In doing so we will approximate the energy by the value at the Fermi surface as these are the particles most
likely to participate in the interaction. For the proton and neutron, which are nearly non-relativistic, we will approximate the energy by the effective mass, \( m_N^* \sim 0.8 m_N \) and \( m_p^* \sim 0.8 m_N \) [35]. For the electrons, which are highly relativistic, the chemical potential is equal to the Fermi momentum, \( k_e = \mu_e \). These factors can be pulled out of the integral. Doing the transformation yields,

\[
\begin{align*}
p_1 dp_1 &= \mu_e dE_1 \\
p_2 dp_1 &= m_p^* dE_2 \\
p_3 dp_1 &= m_N^* dE_3 \\
p_4^2 dp_1 &= E_4^2 dE_4.
\end{align*}
\]

Performing the neutrino integral, \( dE_4 \), over the \( \delta \)-function leaves,

\[
PQ = \frac{(4 \pi)^5}{2} \mu_e m_N^* m_P^* \int dE_1 dE_2 dE_3 (E_1 + E_2 - E_3)^2 S.
\]

The next step is to make the integral dimensionless making the substitutions,

\[
\begin{align*}
x_1 &= (E_1 - \mu_e)/T \\
x_2 &= (E_2 - \mu_P)/T \\
x_3 &= -(E_3 - \mu_N)/T,
\end{align*}
\]

such that the statistical factor accounting for the Fermi blocking becomes,

\[
S = \prod_{i=1,2,3} \frac{1}{1 + e^{x_i}}.
\]

The Jacobian of these transformations introduces a factor of \( T \) for each measure. Also, a factor of \( T^2 \) comes from the \((E_1 + E_2 - E_3)^2\) term. The chemical potentials introduced all cancel because of the equilibrium condition \( \mu_e + \mu_P - \mu_N = 0 \). The negative sign should cancel but we approximated away the factor of energy where the extra sign gets introduced. We will ignore it, transition rate is positive. We are left with,

\[
PQ = \frac{(4 \pi)^3}{2} m_N^* m_P^* \mu_e T^5 I
\]

where \( I \) is an analytic integral,

\[
I = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \int_{-(x_1+x_2)}^{\infty} dx_3 \frac{(x_1 + x_2 + x_3)^2}{(1 + e^{x_1})(1 + e^{x_2})(1 + e^{x_3})}
\]

\[
= \frac{3}{4} \left( \pi^2 \zeta(3) + 15 \zeta(5) \right)
\]

\[
\approx 20.56.
\]

Putting everything together, and using the numerical estimates from the nuclear matter discussion, the transition rate becomes,

\[
\frac{w}{\Omega} = 8 \pi^5 G_F^2 (1 + 3 C_A^2) m_N^* m_P^* \mu_e T^5 I
\]

\[
= 1.1 \times 10^{31} \left( \frac{\mu_e}{100 \text{ MeV}} \right) \left( \frac{m_N^*}{m_N} \right) \left( \frac{m_P^*}{m_P} \right) \left( \frac{n}{n_0} \right)^{2/3} (T_9)^5 \text{ s}^{-1} \text{ cm}^{-3},
\]
where $T_9 = T/(10^9 \text{ K})$ is the dimensionless, scaled temperature. A typical value for the reduced mass factor is 0.6. The temperature dependence $T^5$ is consistent with literature – remember that we are calculating the transition rate, not the luminosity. The luminosity contains an additional factor of energy in the integral that contributes an extra factor of $T$.

### 4.1 Estimate of the current from direct Urca

The first step is to ensure that the electron can actually escape the star before it decays via the weak interaction. This involves calculating the mean free path and seeing if it is larger than the radius of the star. In the literature for calculations of this type, such as neutrino luminosity for cooling, $\Omega$ is the volume of the neutron star – it is necessary to account for all the transitions that occur in the entire star. Here, we are interested in the decay rate of a single electron, so we will take $\Omega$ to be the volume in which a single electron exists, which is the inverse of the electron number density,

$$\Omega_e = \frac{1}{n_e} = \frac{3\pi^2}{\mu_e^3} = 2 \cdot 10^{-37} \left( \frac{n}{n_0} \right)^{-2} \text{ cm}^3. \quad (4.26)$$

Assuming that the electron travels at the speed of light though the protons – due to the non-dissipative nature of the current with respect to the electromagnetic interactions, the mean free path can be found using,

$$\ell_e \sim \frac{c}{w} \sim 1.2 \times 10^{11} (T_9)^{-5} \left( \frac{n}{n_0} \right)^{4/3} \text{ km} \quad (4.27)$$

The typical radius for the neutron star is $R \sim 10 \text{ km}$. We see that for $T \leq 10^{10} \text{ K}$ the electrons can easily escape the degenerate region before the P-odd asymmetry gets washed out due to the weak interactions. The counterintuitive density dependence – at higher densities the mean free path is larger – is a natural consequence of Pauli blocking. As the density of the star increases the number of protons increases and we would expect a shorter mean free path, but the number of neutrons increases as well. The suppression due to the higher neutron chemical potential is greater than the enhancement gained by increasing the proton density.

Using (3.5) we are now in a position to estimate the magnitude of the current travelling along a single quantum of flux,

$$\langle j \rangle = 9.4 \times 10^{-10} \left( \frac{n}{n_0} \right)^{-2/3} (T_9)^5 \text{ MeV}. \quad (4.28)$$

There is no dependence on the magnetic field because we have normalized per unit of quantum flux. As discussed earlier this reaction is dominant when hyperons appear at $n/n_0 \geq 3$ or when the proton fraction is large $x_p > 1/9$.

From (4.27) we see that it is much easier for an electron to escape a cool star rather than a hot, newly born star, $T \geq 10^{11} \text{ K}$. If the star is very hot the electron is not able to keep its asymmetry due to the weak rescattering as one can see from (4.27). As it cools
there will be a critical value where the electrons can escape, but are still created at a large rate, meaning the current is very large (4.28). The current is largest when the star is hot, but not so hot that the electrons are unable to escape the region of degeneracy, \( \mu_e \gg T \). This temperature is roughly determined by the condition that the electron mean free path with respect to weak interactions is approximately equal to radius for the neutron star is \( R \sim 10 \text{ km} \). It is expected that this temperature drastically depends on the equation of state and other specific properties of the environment as (4.27) suggests.

### 4.2 The effect of a large magnetic field on the transition rate

In this short subsection we will argue that the effects of the magnetic field can be safely neglected in calculating the transition rate. Of course, the magnetic field still plays a crucial role in producing the required asymmetry for the current, see the Appendix for details. There has been much consideration of Landau levels on the rates of processes that occur in neutron stars. Landau levels can have drastic effects on the spin of electrons, though these effects are suppressed by the unusually high chemical potentials found in neutron stars. An extremely large magnetic field is needed for these effects to manifest themselves in neutron stars. It must be roughly comparable to the chemical potential \( eB \sim \mu_e^2 \) in order for substantial changes for the transition rates to occur. Numerically, this corresponds to very large fields \( B \sim B_c \mu_e^2/m_e^2 \sim 10^{17} \text{G} \), which are much larger than the magnetic fields found in typical neutron stars.

To introduce Landau levels to the earlier calculation we can simply replace the electron dispersion relation with,

\[
E_e^2 = p_z^2 + m_e^2(1 + 2nb),
\]

where \( n \) are all natural numbers, but only electrons with a spin antiparallel to the magnetic field are allowed in the lowest, \( n = 0 \), level and \( b = B/B_c \) is the magnetic field normalized to the critical field, \( B_c = m_e^2/e = 4.4 \times 10^{13} \text{ G} \). The the electron phase space becomes,

\[
\Omega \int \frac{dp_z}{2\pi} \sum_{n=0}^{n_{\text{max}}} g_n m_e^2 b \frac{2}{(2\pi)^2},
\]

where the term after the sum is the degeneracy per unit area and \( g_n \), equal to 1 for \( n = 0 \) and 2 otherwise, counts the spin degeneracy. The maximum Landau level occurs for \( p_z = 0 \) when all the energy goes to putting the electron in the Landau level and none goes into the momentum. This yields,

\[
n_{\text{max}} = \frac{E_e^2 - m_e^2}{2m_e^2 b}.
\]

If the number of Landau levels is very large, \( n_{\text{max}} \sim \frac{E_e^2}{2m_e^2} \gg 1 \), which is a common case for a typical neutron star, then the phase space returns to the the one we used in the previous section (4.24). Therefore, for a typical neutron star the transition rate basically remains the same as the electron phase space essentially unchanged,

\[
\Omega \int \frac{dp_z}{(2\pi)^3} \mu_e^2,
\]
where we used $E_e \sim \mu_e$ and $\mu_e \gg m_e$. This result is in accordance with our rough arguments mentioned above that very large magnetic field is required to produce any substantial changes.

We want to contrast this generic case with a rare situation when only the lowest Landau level is accessible, $n_{\text{max}} = 0$. This occurs when the neutron star is abnormally cold, the magnetic field is unusually large, or the electron chemical potential is unnaturally small \[38\]. In this approximation we recover the usual electron dispersion relation while the available phase volume becomes,

$$\Omega \int \frac{dp_z}{(2\pi)^3} m_e^2 b.$$  \hspace{1cm} (4.33)

The entire integral takes place identically to above due to the approximation to move the momentum dependent parts of the phase space outside of the integral. We can compensate for the magnetic field in the transition rate simply by taking,

$$\mu_e \rightarrow \frac{m_e^2 b}{\mu_e}.$$  \hspace{1cm} (4.34)

In fact this transformation can be done to account for the magnetic field for any of the other rates we derive \[5.23\], \[6.12\], \[7.14\]. In particular, for the direct Urca case we are left with,

$$w \frac{\Omega}{\Omega} = \frac{G_F^2}{8\pi^5} (1 + 3C_A^2) m_N^2 m_p^* \frac{m_e^2 b}{\mu_e} T^5 f$$  \hspace{1cm} (5.1)

which we can compare to equation (4.24). This expression is valid in three cases: $T \ll 10^9$ K, $\mu_e \ll 100$ MeV, or $b \gg 1$. Only the last of these has a real chance of manifesting itself; magnetars have huge magnetic fields $B \sim 10^{16}$. Otherwise the Landau levels do not have a significant contribution and the earlier expression is valid.

5. Nuclear Matter: The Modified Urca Process

The direct Urca process is the simplest to consider, but it is unlikely to be the most common process in a star. It is much easier to conserve momentum if a nucleon is included to supplement momentum transfer. We will consider the mean free path of an electron scattering of a proton assisted by a neutron, equation (2.10). The inclusion of the nucleon-nucleon interaction into the matrix element is non-trivial and we will use the one pion exchange/landau liquid method found in \[39\], the interaction illustrated in Figure 4.

There are many other diagrams similar to the one illustrated. They include all possibilities of pion exchange between protons and nucleons as well as crossing diagrams to account for the final neutrons being indistinguishable. While summing them all the vector contributions cancel and we are left with the scattering matrix approximated to be,

$$\sum_{\text{spins}} |S|^2 = \frac{1}{\Omega^6} \frac{G_F^2}{(\mu_e)^2} C_A^2 \left( \frac{f}{m_\pi} \right)^4 \alpha_{\text{Urca}}.$$  \hspace{1cm} (5.1)
Figure 4: The modified Urca process.

where \( f \sim 1 \) is the p-wave \( \pi N \) coupling constant, \( C_A = 1.26 \) is the Gamow-Teller coupling, and \( \alpha_{\text{Urca}} \sim (0.63 - 1.76)(n_0/n)^{2/3} \) is a factor that accounts for the pion propagator and the short range Landau liquid contributions. Following [39], but somewhat preempting the calculation, we approximate the propagator of the internal nucleon using the total lepton energy \( \mu_e \).

To calculate the mean free path we will first calculate the transition rate,

\[
w = \frac{\Omega^6}{(2\pi)^{18}} \prod_{i=1}^{6} d^3p_i \ S \ (2\pi)^4 \Omega \ \delta^{(4)}(p_f - p_i) \ \sum_{\text{spins}} |S|^2,
\]

where \( S \) is the Pauli blocking factor.

We separate the transition rate into angular and radial parts leaving,

\[
w = \frac{\Omega}{(2\pi)^{14}} \ \sum_{\text{spins}} |S|^2 \ P \ Q,
\]

where,

\[
P = \prod_{i=1}^{6} \int p_i^2 dp_i \ S \ \delta(E_f - E_i)
\]

\[
Q = \prod_{i=1}^{6} \int d\Omega_i \ \delta^{(3)}(p_f - p_i).
\]

We start with the \( Q \) integral. The contribution from the momentum of the neutrino is negligible compared to the rest so we neglect it in the \( \delta \)-function. This allows us to take an integral over all angles causing the term with angular dependence to vanish. We then follow [31] in doing the angular integrals to get,

\[
Q = \frac{(4\pi)^5}{2p_1p_4p_5}.
\]
We can now do the $PQ$ integral. Changing variables from momentum to energy and approximating the energy by the particles fermi energy changes the measures to,

\[
p_1 dp_1 = m_N^* dE_1 \\
p_2^2 dp_2 = \mu_e m_N^* dE_2 \\
p_3^2 dp_3 = \mu_e^2 dE_3 \\
p_4 dp_4 = m_N^* dE_4 \\
p_5^2 dp_5 = m_N^* dE_5 \\
p_6^2 dp_6 = E_6^2 dE_6,
\]

where we took $k_P \sim \mu_e$. Then we take the $E_6$ integral over the delta function and make the following substitutions to make the remaining integral dimensionless,

\[
x_1 = \beta(E_1 - \mu_N) \quad (5.13) \\
x_2 = \beta(E_2 - \mu_P) \quad (5.14) \\
x_3 = \beta(E_3 - \mu_e) \quad (5.15) \\
x_4 = -\beta(E_4 - \mu_N) \quad (5.16) \\
x_5 = -\beta(E_5 - \mu_N). \quad (5.17)
\]

The statistical factor we are left with has two more factors, from the Fermi factor of the two extra particles, than the direct Urca case,

\[
S = \prod_{i=1}^{5} \frac{1}{1 + e^{x_i}}. \quad (5.18)
\]

Using the beta equilibrium condition, $\mu_P + \mu_e = \mu_N$, all chemical potentials cancel, leaving,

\[
PQ = \frac{(4\pi)^5}{2} (m_N^*)^2 (m_N^*)^4 \mu_e T^7 I \quad (5.19)
\]

where $I$ is the integral,

\[
I = \prod_{i=1}^{5} \left( \int_{-\infty}^{\infty} \frac{dx_i}{1 + e^{x_i}} \right) \left( \sum_{j=1}^{5} x_j \right)^2 \Theta(x_1 + x_2 + x_3 + x_4 + x_5) \quad (5.20)
\]

\[
= \prod_{i=1}^{4} \left( \int_{-\infty}^{\infty} dx_i \right) \int_{-(x_1+x_2+x_3+x_4)}^{\infty} dx_5 \left( \sum_{j=1}^{5} x_j \right)^2 \prod_{k=1}^{5} \frac{1}{1 + e^{x_k}} \quad (5.21)
\]

\[
\approx 192. \quad (5.22)
\]

Gathering all the terms together we get the transition rate per unit volume for an electron scattering off a proton assisted by a neutron,

\[
\frac{w}{\Omega} = \frac{G_F^2 C_A^2}{\pi^9} \frac{(m_N^*)^3 m_P^* \mu_e}{m_e^2} \alpha_{\text{Urca}} \frac{T^7}{T_0^7} I \quad (5.23)
\]

\[
= 9.2 \times 10^{26} \left( \frac{\mu_e}{100 \text{ MeV}} \right) \left( \frac{m_N^*}{m_N} \right)^4 T_0^7 \text{s}^{-1} \text{cm}^{-3}, \quad (5.24)
\]
where as discussed earlier \( m_N^* \sim m_p^* \sim 0.8m_N \). The rate of this process is much smaller than we found earlier in the direct case (4.24) and the temperature dependence is now \( T^7 \) rather than \( T^5 \). Also, because of the density dependence of \( \alpha_{\text{Urca}} \), the transition rate is independent of density. Both of these differences result from this being a higher order calculation, which involves two extra particles – the ingoing and outgoing neutron assisting in momentum conservation; the transition rate is suppressed by a factor of the QED coupling constant, and the two extra factors of \( T \) come from the two extra measures of integration.

5.1 Estimate of the current from modified beta decay

We are interested in the mean free path of an electron. The volume is that in which there exists a single electron \( \Omega_e \sim 10^{-36} \text{ cm}^3 \), and assume the electron moves at the speed of light, then,

\[
\ell_e \sim 1.4 \times 10^{15} \left( \frac{T}{9} \right)^{-7} \left( \frac{n}{n_0} \right)^{2} \text{ km.} \tag{5.25}
\]

At the beginning of the star’s life, when it is very hot \( T \sim 10^{12} \text{ K} \), the electrons are trapped, then as the star cools the electrons are allowed to escape. Here we get a similar counterintuitive density dependence as (4.27) but the effect is stronger because there are two neutron Pauli blocking terms to contend with.

We now follow the same prescription as earlier to derive the current,

\[
\langle j \rangle = 7.7 \times 10^{-14} \left( \frac{n}{n_0} \right)^{-4/3} T_9^{7/2} \text{ MeV.} \tag{5.26}
\]

This is by far the weakest current from any phase of matter. As before the the current is strongest, early in the star’s life when it is hot, and though it is suppressed can get quite large at high temperatures due to the \( T^7 \) temperature dependence.

6. Kaon Condensate

As the density of the star gets above three times nuclear density is it possible that a charged kaon condensate will appear [32]. It is now energetically favourable for electrons to scatter off the condensate and turn into neutrinos. We are interested in calculating the transition rate of an electron decaying in the presence of a kaon condensate, equation (2.12). The effect of condensates on the scattering matrix was first used to describe pion condensates [34] and later for kaon condensates [36, 40] and involves evaluating the process given by the Feynman diagram in Figure 5.

The process needs the aid of a neutron quasiparticle to aid it, similar to that of the modified beta decay, but not because it is required to conserve momentum. We start usual hadron and lepton currents and the kaon condensate is described by a chiral rotation in V-spin space, \( U = e^{i\gamma_5 \theta} \). For kaons, a small rotation can have a large effect so only \( \theta \ll 1 \) must be considered. The matrix element, as found in [36], is given by,

\[
\sum_{\text{spins}} |S|^2 = \frac{G_F^2 \theta^2}{4\Omega^4} (1 + 3C_A^2) \sin^2 \theta_c, \tag{6.1}
\]
where $\theta^2 \sim 0.1$ is the kaon amplitude, $C_A = 1.26$ is the Gamow-Teller coupling, and $\theta_c \sim 13^\circ$ is the Cabibbo angle.

We are interested in the regime where the kaon momentum $p_3$ is zero. The transition rate is given by,

$$w = \frac{\Omega^5}{(2\pi)^8} \sum_{\text{spins}} |\mathcal{M}|^2 PQ,$$  

(6.3)

where $S$ is the statistical factor containing Fermi blocking terms, $\Omega$ is the volume of the neutron star. The subscripts on $p_1, p_2, p_4$, and $p_5$, label the momentum of the ingoing neutron, the electron, the outgoing neutron, and the neutrino, respectively. Following the work done in [31] we can separate the angular and radial integrals so each can be evaluated independently,

$$w = \frac{\Omega^5}{(2\pi)^8} \sum_{\text{spins}} |\mathcal{M}|^2 PQ,$$  

(6.3)

where,

$$P = \int p_1^2 dp_1 p_2^2 dp_2 p_4^2 dp_4 p_5^2 dp_5 S \delta(E_f - E_i),$$  

(6.4)

$$Q = \int d\Omega_1 d\Omega_2 d\Omega_4 d\Omega_5 \delta^{(3)}(p_f - p_i).$$  

(6.5)

We start by doing the angular integral $Q$. We first rewrite the delta function in its Fourier representation,

$$Q = \frac{(4\pi)^3}{2p_1 p_2 p_4},$$  

(6.6)

The $PQ$ integral can now be started,

$$PQ = \frac{(4\pi)^3}{2p_1 p_2 p_4} \int p_1^2 dp_1 p_2^2 dp_2 p_4^2 dp_4 p_5^2 dp_5 S \delta(E_1 + E_2 - E_3 - E_4 - E_5).$$  

(6.7)

We first change the variables of integration from momentum to energy $p_i dp_i = E_i dE_i$ and perform the neutrino integral over the delta function. In an effort to make the final integral tractable, we follow [31] in approximating factors next to the measures as constant. The
neutrons are non-relativistic so their energy is just their effect mass, $m_N^*$, and the electron is ultra relativistic, because of its large Fermi momentum, and the energy is just its chemical potential, $\mu_e$, leaving,

$$E_1 dE_1 = m_N^* dE_1 \quad (6.8)$$

$$E_2 dE_2 = \mu_e dE_2 \quad (6.9)$$

$$E_4 dE_4 = m_N^* dE_4. \quad (6.10)$$

The factors can then be moved outside the integral. It is also convenient to change variables to facilitate the final integral over the Pauli blocking factors. Changing variables takes $(E_1 + E_2 - E_3 - E_4)^2$ to $(x_1 + x_2 + x_4)^2 T^2$, where we used the equilibrium condition $\mu_k = \mu_e$. Upon changing variables the measure of integration gives us a factor $T^3$ and we are left with,

$$PQ = \frac{(4\pi)^3}{2} \frac{m_N^*}{\mu_e} T^5 I \quad (6.11)$$

where $I$ is the same analytic integral from the direct Urca process (4.21).

Putting everything together we get the transition rate per unit volume of an electron decaying into a kaon,\[
\frac{w}{\Omega} = \frac{G_F^2 \theta^2}{(2\pi)^5} (1 + 3C_A^2) \sin^2 \theta_c (m_N^*)^2 \mu_e T^5 I \quad (6.12)\]

$$= 4.4 \times 10^{29} \left( \frac{\mu_e}{100 \text{ MeV}} \right) \left( \frac{m_N^*}{m_N} \right)^2 \left( \frac{n}{n_0} \right)^{2/3} (T_9)^5 \text{ s}^{-1} \text{ cm}^{-3}, \quad (6.13)$$

where $T_9 = T/(10^9 \text{ K})$ is the scaled temperature, and $m_N^* \sim 0.8 m_N$. The temperature dependance is the same as the direct Urca process (4.24), but we would expect it to be of higher order, like the modified Urca rate (5.23), because it involves an extra particle. The temperature dependence is the same because of the way we treat the kaons as a condensate, rather than an extra particle. The condensate picture is a rotation of the direct process, rather than a whole new particle interaction as in the modified Urca process.

### 6.1 Estimate of the current in a kaon condensate

Because of the high chemical potential the electrons must be relativistic leaving the mean free path to be,

$$\ell_e = 3.0 \times 10^{12} (T_9)^{-5} \left( \frac{n}{n_0} \right)^{4/3} \text{ km}. \quad (6.14)$$

Once again this mean free path larger than the radius of the neutron star and the electrons can escape. We also notice the counterintuitive, but now familiar, density dependence discussed in the direct Urca case. This effective mean free path with the helicity intrinsic in the weak interaction creates a current given by equation 3.5,

$$\langle j \rangle \simeq 3.6 \times 10^{-11} (T_9)^5 \left( \frac{n}{n_0} \right)^{-2/3} \text{ MeV}. \quad (6.15)$$
Though the generation of the current in a kaon condensate happens through a wildly different process than direct Urca, and the numbers we use are quite different, we see that after the star has cooled the numbers conspire to give currents of similar magnitude (4.28).

7. Quark Matter

The last case we consider is what would occur if the hadrons separated into their constituent quarks. This is the case in quarks stars, where degenerate quarks exist deconfined. In calculating the mean free path of electrons in quark matter there are two possible reactions we must consider, if we restrict ourselves to the lightest quarks, given in equation (2.13). Each is given by the Feynman diagram in Figure 6, where the $d$ quark can be substituted for the $s$ quark.

$$\begin{align*}
e^- & \quad p_1 \quad p_4 \\
u_e & \quad p_2 \quad p_3 \\
u_e & \quad d
\end{align*}$$

**Figure 6:** The direct Urca process for quarks.

Unlike nuclear beta decay, the lowest order quark beta processes do not require help from an external particle to conserve momentum; they proceed unsuppressed [33]. Because they are deconfined, the Fermi momentum of the quarks are much closer to each other than the Fermi surfaces of the hadrons in neutron stars. We will first consider the transition into the down quark. The transition rate is given by,

$$w = \frac{\Omega^4}{(2\pi)^{12}} \int d^3p_1 d^3p_2 d^3p_3 d^3p_4 S (2\pi)^4 \Omega \delta^{(4)}(p_f - p_i) |S|^2,$$  \hspace{1cm} (7.1)

where $S$ is the statistical factor containing Fermi blocking terms, $\Omega$ is the volume of the phase space. The subscripts on $p_1$, $p_2$, $p_3$, and $p_4$, label the momentum of the ingoing electron, the up quark, the down quark, and the neutrino, respectively. The matrix element of this process is found in [33],

$$|S|^2 = \frac{4}{\Omega^3} G_F^2 \cos^2 \theta_e \frac{4\alpha_s}{3\pi} \left[ 1 - \frac{p_3 \cdot p_4}{E_3 E_4} \right],$$  \hspace{1cm} (7.2)

which looks similar to the usual four Fermi matrix element but contains correction due to the quarks momentum that can be related to the quark gluon coupling constant by $\alpha_s = \frac{g^2}{4\pi}$. The transition rate can be split into two integrals,

$$w = \frac{\Omega}{(2\pi)^3} \frac{8}{3} G_F^2 \cos^2 \theta_e \alpha_s P Q,$$  \hspace{1cm} (7.3)
where

\[ P = \prod_{i=1}^{4} \int p_i^2 dp_i \, S \delta E_f - E_i \] (7.4)

\[ Q = \prod_{i=1}^{4} \int d\Omega_i \delta^{(3)}(p_f - p_i) \left[ 1 - \frac{p_3 \cdot p_4}{E_3 E_4} \right] . \] (7.5)

These integrals are nearly identical to the kaon case. The \( Q \) integral is the same as in the direct Urca case (4.9). We make an approximation to the remaining \( PQ \) integral when we change variables from momentum to energy. The momentum of the electron and quarks is replaced by their value at the fermi surface,

\[ p_1 dp_1 = \mu_e dE_1 \] (7.6)

\[ p_2 dp_2 = k_u dE_2 \] (7.7)

\[ p_3 dp_3 = k_d dE_3 \] (7.8)

\[ p_4^2 dp_4 = E_4^2 dE_4 . \] (7.9)

Performing the neutrino integral, \( dE_4 \), over the delta function leaves us with the same integral as equation (4.15). We perform the change of variables,

\[ x_1 = (E_1 - \mu_e) \] (7.10)

\[ x_2 = (E_1 - \mu_u) \] (7.11)

\[ x_3 = -(E_1 - \mu_d) . \] (7.12)

Following the same procedure as the quark case, the quark beta equilibrium condition \( \mu_e + \mu_u = \mu_d \) causes the chemical potentials to cancel, and the \( PQ \) integral becomes,

\[ PQ = \frac{(4\pi)^3}{2} \mu_e k_u k_d T^5 I , \] (7.13)

where \( I \) is the same integral given in equation (4.21), which is the same as the direct Urca and kaon cases. An identical calculation can be done for the electron scattering into strange quarks. Putting everything together we get the transition rate for electrons in quark matter,

\[ w_d = \Omega \frac{1}{6\pi^6} G_F^2 \cos^2 \theta_c \alpha_s k_e k_u k_d T^5 I \] (7.14)

\[ w_s = \Omega \frac{1}{6\pi^6} G_F^2 \sin^2 \theta_c \alpha_s k_e k_u k_s T^5 I . \] (7.15)

To estimate the magnitude of the transition we assume that \( k_d \approx k_u \approx k_s \) as in the massless noninteracting case and use the relationship \( k_e \approx \mu_e = (3Y_e)^{1/3} k_q \), where \( Y_e = n_e/n_b \) is the ratio of electrons to baryons in the star. The Fermi momentum of the quarks is estimated using (2.23). We follow [33] in estimating \( \alpha_s \approx 0.4 \) and \( Y_e \approx 0.01 \)

The total transition rate is the sum of \( w_s \) and \( w_d \),

\[ \frac{w}{\Omega} = 7.3 \times 10^{30} \left( \frac{n_b}{n_0} \right) (T_9)^5 \text{ s}^{-1} \text{ cm}^{-3} . \] (7.16)

We see the \( T^5 \) temperature behaviour that has become a signature for the first order, four Fermi interaction.
7.1 Estimate of the current in quark matter

As before, the first step in finding the current is estimating the mean free path of the electron in quark matter by assuming that the electron propagates at the speed of light. The same physics creates the current in quark matter and it is still non-dissipating. We now use the new definition of $n_e$ to determine the volume in which there exists a single electron, $\Omega = (Y_e n_b)^{-1}$, and we are left with,

$$\ell_e = \frac{c}{w} = 7.0 \times 10^{10} (T_9)^{-5} \text{ km}$$  \hspace{1cm} (7.17)

The radius of the star, $R \sim 10$ km, is much smaller than this and the current will propagate. In quark matter the mean free path is not dependent on density; the Pauli suppression and enhancement cancel each other – see section 4.1 for a brief discussion. When the electrons reach the crust they are removed from the system creating a new effective mean free path for the electron. This effective mean free path with the helicity intrinsic in the weak interaction creates a current,

$$\langle j \rangle = 1.2 \times 10^{-9} (T_9)^5 \left( \frac{n_b}{n_0} \right)^{1/3} \text{ MeV}.$$  \hspace{1cm} (7.18)

The typical density for quark matter is $n_b \sim 10 n_0$, but could easily be higher. Once again the numbers have conspired and the magnitude of the current is close to the value for both direct Urca ($4.28$) and kaon ($6.15$) processes. They are all first order processes, but we see a critical difference in the density dependence. Unlike the other currents the quark current actual gets larger with increasing density. This happens because quark stars remain charge neutral in a fundamentally different way than neutron stars and the electron chemical potential is defined differently.

8. Applications in Neutron Stars

We have calculated the magnitude of the topological current in four types of nuclear matter and summarize them in Table 1. Direct beta decay, which is also the preferred reaction when hyperons are present, kaon condensates, and quark matter, all are radically different, but they all make a current about the same order. There is a stark difference is in the modified Urca process, which occurs in ordinary nuclear matter where the proton fraction is below 1/9. The current is four orders of magnitude smaller than the rest. This can be accounted for by it being a higher order process involving more particles. It results in a more severe dependance on temperature which suppresses the process when the star cools.

The estimates of the current are relatively small in magnitude, but much larger currents exist on microscopical scales. The estimates represent a component of the current that produces a coherent effect in the entire region where matter is degenerate and the magnetic field all points in one direction. As a technical remark, the estimates of the current $\langle j \rangle$ are presented in MeV units. The current is defined as a number of particles crossing the surface equivalent to one unit of fundamental magnetic flux per unit time. We can obtain the electromagnetic current in conventional units by multiplying by $e = \sqrt{4\pi\alpha}$ and converting the result into Amperes: $e \langle j \rangle \sim 10^2$ A for $\langle j \rangle = 1$ MeV.
Table 1: A summary of the current per quantum unit of flux calculated from each interaction. The value is for a single quantum of flux, such as that found in a type-II vortex. The appropriate density for the kaon and the direct Urca currents is $n = 3n_0$, for quarks $n_b = 10n_0$, and for modified Urca $n = n_0$.

| Current                    | $\langle j \rangle$ (MeV)                                      |
|----------------------------|---------------------------------------------------------------|
| Direct Beta Decay          | $9.4 \times 10^{-10} \left( \frac{n}{n_0} \right)^{-2/3} (T_9)^5$ |
| Modified Beta Decay        | $7.7 \times 10^{-14} \left( \frac{n}{n_0} \right)^{-4/3} (T_9)^7$ |
| Kaon Condensate            | $3.6 \times 10^{-11} \left( \frac{n}{n_0} \right)^{-2/3} (T_9)^5$ |
| Quark                      | $1.2 \times 10^{-9} \left( \frac{n_b}{n_0} \right)^{1/3} (T_9)^5$ |

In this section we discuss five different applications of the current:

Section 8.1: Neutron star kicks
Section 8.2: Pulsar jets
Section 8.3: Toroidal magnetic field
Section 8.4: Magnetic helicity
Section 8.5: Type-II superconductivity and precession

These phenomena are observed in many neutron stars and appear to be unrelated to each other; however, we shall argue below that they might in fact originate from the same source – non-dissipating topological currents. All these phenomena have one thing in common – they are P-odd effects. There could be many more consequences of these novel, non-dissipating, topological currents introduced in the present paper that are still yet to be explored.

8.1 Topological kicks

We first consider a well established phenomenon – neutron star kicks. It is accepted that neutron stars have much higher proper velocities than their progenitors, some moving as quickly as 1000 km/s [21, 41, 42, 43, 44, 45, 46, 47]. There also appears to be a strong correlation between the direction of the kick and the spin axis of the star [23]. If the electrons carried by the current can transfer their momentum into space the current could push the star like a rocket. In typical neutron stars this is unlikely because the crust (the region where $\mu \sim T$) is thought to be very thick. But if the crust is very narrow, as it may be in bare quark stars [48] (see also recent developments [49, 50]), the electrons may leave the system or emit photons that will carry their momentum to space. In this section we will estimate the effectiveness of this as a kick mechanism.

Currently no mechanism exists that can reliably kick the star hard enough. Explosions during collapse can only reliably kick a star 50 km/s [21], asymmetric explosions can only reach 200 km/s [47], and asymmetric neutrino emission is plagued by the problem that at
temperatures high enough to produce the kick the neutrino is trapped in the star [51, 52]. We will demonstrate that topological kicks can easily provide the momentum required and do so because of the immense electron chemical potential, not because of the temperature of the star. Because of the correlation between the magnetic field and the spin axis, topological kicks naturally align with the spin axis of the star.

Though it is unclear what will happen to the electrons when they reach the surface of the star, we can estimate the size of the kick if we assume that entire momentum carried by the topological current will be transferred into the space by some means. This assumption is likely to be correct for bare quark stars when the crust is only about 1000 fm wide [48] and is most likely wrong for typical neutron stars where the crust is about 1 km thick. The energy of the electrons would be absorbed by the crust and would not contribute to the kick. As a consequence we conjecture that stars with very large kicks, \( v \gg 200 \text{ km/s} \), are quark stars and that slow moving stars, \( v \leq 200 \text{ km/s} \), are kicked by some other means [47] and are typical neutron stars. Confirmation of this classification would provide a simple and well formulated principle that distinguishes quark stars from typical neutron stars. As is known, other criteria such as mass, size, cooling rate, etc. can not easily discriminate between quark stars and neutron stars, see e.g. [53, 30] for review.

Topological kicks work by slowly and steadily pushing the star over time, much like a rocket. This is contrary to other mechanisms where the kicks happens shortly after the star’s birth. These kicks over long durations are supported by the analysis in [22]. The first step in estimating the magnitude of a topological kick is determining the total momentum transferred to the star. There are \( N_v \sim 7 \cdot 10^{33} \frac{B}{B_c} \) quantum units of flux in a star that can be distributed in either superconducting domains or vortices, see eq. (3.2). The current is independent of the internal structure of the star. If all the electrons are shot out of the star or transfer their entire momentum, then the momentum is given by the total current (3.4), the number of electrons that leave the star per unit time. The momentum a single electron transfers out of the star is equal to its Fermi momentum, \( k_e = 100 \text{ MeV} \). The fuel for the kick is the chemical potential, not the temperature. Therefore the kick may continue even when star is already cool; the kick in our mechanism is not an instant event, but is rather a long, slow, steady process that pushes the star. Putting this all together, the current transfers \( N_v k_e \langle j \rangle \) units of momentum per unit of time.

The appropriate way to estimate the magnitude of the kick is to integrate the rate of momentum transfer per unit time by taking into account the time evolution of the star as the current is very sensitive to the temperature/time as Table 1 demonstrates. The rate of momentum transfer also changes as star cools and the environment (density, phase) changes. We neglect all these complications and take a value for the current corresponding to \( T \sim 10^9 \text{ K} \), which is far from the maximum current can potentially be induced.

The momentum transfer required for the star to reach a velocity of \( v = 1000 \text{ km/s} \) is quite large. Per baryon, the momentum required is \( m_n v \sim 3 \text{ MeV} \). The total baryon number of a neutron star of one solar mass is \( B_n \sim 10^{57} \). The momentum for the entire star is then \( P \sim 3 \cdot 10^{57} \text{ MeV} \). If we choose a current \( \langle j \rangle = 10^{-10} (T_9)^5 \text{ MeV} \), corresponding
to a larger current in Table 1, the time required to attain this momentum is,

\[
t = \frac{P}{N_e k_e \langle j \rangle} \approx 3 \times 10^{10} \left( \frac{v}{1000 \text{ km/s}} \right) \left( \frac{B_e}{B} \right) \left( \frac{10^9 \text{ K}}{T} \right)^5 \text{s} \approx 5,000 \text{ years},
\]

for a star with a magnetic field \( B = 10^{12} \text{ G} \). We restore the canonical dimensionality of \( \text{MeV}^{-1} \) in seconds by multiplying \( \hbar = 6.58 \cdot 10^{-22} \text{ MeV·s} \). The current can push the star to the required velocity within the age of most neutron stars and the stronger the magnetic field the quicker it will happen. If the modified Urca process is dominant the current is much weaker and it would take much longer to push the star. If the current exists at higher temperatures it would be much stronger and even less time would be needed to reach this velocity. As electrons actually leave the star, rather than transferring their momentum through radiation, the electron chemical potential will slowly decrease and the current may stop running. Charge neutrality will cause matter to accrete isotropically and possibly maintain some of the chemical potential.

As previously mentioned it is unlikely that the electrons could escape a typical neutron star – the crust has a thickness on the order of 1 km and electrons leaving the degenerate core would quickly be absorbed. On the other hand, quarks stars have a thin interface between the degenerate matter and open space [48] that electrons might be able to easily penetrate, or transfer their momentum by emitting photons into space.\(^7\) This difference in crusts could provide a valuable distinction: stars with large kicks are quark stars, stars with small kicks are neutron stars.

Electromagnetic interactions, while much stronger than the weak interactions, can not wash out the P-odd asymmetry. This unique feature of the non-dissipating anomalous current plays a crucial role in our explanation of neutron star kicks. Indeed, the neutrino mean free path is also the same order of magnitude and the neutrino is also emitted asymmetrically with respect to \( \vec{B} \). The fundamental difference in the two carriers is that at low temperatures when the neutrino can escape the star they do not carry enough energy to explain the kick. In contrast, the electrons which make the non-dissipating current (or more precisely the quasi-particles which freely travel along the magnetic field) carry very large momentum \( \sim \mu_e \). As a result the neutrino carries too little momentum when the mean free path becomes sufficiently long, while the momentum carried by the topological current remains very high even at very low temperatures \( T \sim 10^8 \text{ K} \). This is the advantage of our kick mechanism.

\subsection*{8.2 Pulsar jets}

A different but likely related phenomena is the recent observation of pulsar jets [54] that are apparently related to neutron star kicks [55, 56]. It has been argued that spin axes and proper motion directions of the Crab and Vela pulsars are aligned. Such a correlation would follow naturally if we suppose that the kick is caused by a non-dissipating current as we

\(^7\)It has been recently argued that a crust in quark stars could be much larger in size than previously thought due to development of a new heterogeneous mixed phase [49, 50]. As we mentioned above, it is not our goal to discuss the interaction of current with the crust, however, an intense current from the core of quark star may destroy the crust in this new mixed phase in few locations similar to volcanos on earth.
mentioned above. The current, and thus the proper motion, is aligned with the magnetic field, which itself is correlated with the axis of rotation. It would be very tempting to identify the observed inner jets \cite{54} with the electrons/photons emitted as a result of the induced current. In this sense, the mechanism for the kick is similar to the electromagnetic rocket effect suggested previously \cite{56}.

8.3 Toroidal magnetic fields

There is a strong theoretical evidence for the existence of toroidal fields in neutron stars based on the stability of the poloidal magnetic field. References \cite{26, 25, 27, 59, 60} argue that toroidal and poloidal fields of similar magnitudes must be necessary to stave off hydrodynamic instabilities – the toroidal field suppresses poloidal instabilities and vice versa. Much has been done to find the observational consequences of a toroidal field, for example \cite{61}.

Estimating the toroidal magnetic field is a very complicated problem that requires a self-consistent solution of the equation of the magnetic hydrodynamics. Our induced, topological currents represent only a small part of the system. We are not attempting to solve this problem. Instead, we shall argue that the currents we estimated are more than sufficient to induce the toroidal magnetic field correlated on large scale of order 10 km.

A natural consequence of having a current running parallel to the poloidal magnetic field is that a toroidal component $H_{\text{tor}}$ will be induced. The size of the field can be calculated naively using Ampere’s law, but there is a subtlety because the magnetic field is being induced inside a superconductor. The magnetic field observed in neutron stars $B \sim 10^{12}$ G is actually induced by a much larger field $H$. The suppression comes from the perfect diamagnetism of the proton superconductor (the Meissner effect). This perfect diamagnetism is ruined at a critical field $H_{c} \sim (\Phi_{0}/4\pi\lambda^{2}) \sim 10^{15}$ G where flux penetrates the star through small regions where superconductivity has been destroyed (vortices or domains). The supercurrents responsible for the perfect diamagnetism do not flow as easily and a small field is induced.

Regardless if the flux penetrates the superconductor as single vortices or flux domains, we can assume that the superconductor is type-II.\footnote{The mechanism for type-I like superconductivity discussed in \cite{9} relies on the electromagnetic interaction between currents carrying vortices, not in fundamentally altering the Landau-Ginzburg parameter $\kappa = \lambda/\xi$. Typically the London penetration depth is $\lambda \sim 120$ fm and the coherence length is $\xi \sim 30$ fm. We then still use results from type-II superconductors when $\kappa > 1/\sqrt{2}$, but the vortices are now bunched together in large domains with higher winding numbers.} The relationship between the applied magnetic field $H$ and induced magnetic field $B$ in a type-II superconductor is very nonlinear. The details have been worked out in \cite{62} and \cite{63}, where the latter is a direct application to neutron stars. The important points are that below the first critical field $H < H_{c1}$ there is no magnetic field $B$ induced. Just above the critical field a magnetic field appears that is approximately $B \sim 10^{-3}H_{c1}$. As the applied magnetic field is increased above $H_{c1}$ the induced field starts to approach the applied field.

We want to determine if the topological current produced by the poloidal field can induce a sufficient toroidal field by finding the length scale where $H_{\text{tor}} \sim H$. Following \cite{63}...
we assume that $H \sim H_c$ and we get the relationship $B \sim 10^{-3} H$ for our magnetic field. We apply Ampere’s law for a region of size $L$,

$$H_{\text{tor}} 2\pi L = e j \cdot \left( \frac{\pi L^2 B}{\Phi_0} \right), \quad (8.2)$$

where the expression in brackets describes the number of unit fluxes bundled in the area $\pi L^2$ such that we get the total current enclosed in our loop. We take $\Phi_0 = \pi/e$ and substitute use our relationship between $H$ and $B$.

The naive estimate leads to the following expression for $H_{\text{tor}}$ in terms of magnitude of poloidal magnetic field $H$,

$$\frac{H_{\text{tor}}}{H} \sim \alpha j L \sim 4 \cdot 10^{-5} \left( \frac{j}{10^{-10} \text{ MeV}} \right) \left( \frac{L}{\text{cm}} \right). \quad (8.3)$$

This shows that a typical current from Table 1 can induce a toroidal field the same magnitude as the poloidal field on scales the order $L \sim 1 \text{ km}$, within the typical size of a neutron star. It is quite obvious that our estimate becomes unreliable when $H_{\text{tor}} \geq H$ and we can no longer ignore the current induced by the toroidal field. For $H_{\text{tor}} \geq H$ the problem requires a self consistent analysis which is beyond scope of the present paper. The point is that the toroidal field obviously develops as a result of topological currents and eq. (8.3) shows that its magnitude can easily become the same order as poloidal magnetic field. If superconductivity is completely destroyed, as we will discuss Section 8.5, $B = H$ and the toroidal field can induced on a much smaller scale, $L \sim 1 \text{ m}$.

### 8.4 Magnetic helicity

The magnetic helicity is defined as, see e.g. [64],

$$\mathcal{H} \equiv \int d^3 x \vec{A} \cdot \vec{B}. \quad (8.4)$$

The magnetic helicity is a topological object that can be expressed in terms of the linking number $n(\gamma_1, \gamma_2)$ of two curves $\gamma_1$ and $\gamma_2$. The precise relation between $\mathcal{H}$ and interlinked flux $\Phi_1$ and $\Phi_2$ is given by,

$$\mathcal{H} = 2\Phi_1 \Phi_2 = 2\Phi_0^2 N_1 N_2 \quad (8.5)$$

where $\Phi_1 = \Phi_0 N_1$ and $\Phi_2 = \Phi_0 N_2$ are expressed in terms of unit flux $\Phi_0$ and the linking number is simply reduced to $n(\gamma_1, \gamma_2) = N_1 N_2$. Therefore $\mathcal{H}$ takes integer values up to a normalization $2\Phi_0^2$. This linking number is preserved, $\frac{d\mathcal{H}}{dt} = 0$, in a magneto-fluid with zero resistivity, which is a very good approximation for neutron stars. This topological invariance provides the stability necessary for the poloidal field.

We want to emphasize that the magnetic helicity is the dot product of a vector and a pseudovector, making it a pseudoscalar. Under the parity transformation $\vec{x} \rightarrow -\vec{x}$ the magnetic helicity is P-odd: $\mathcal{H} \rightarrow -\mathcal{H}$. This implies that the magnetic helicity can be only induced if there are P-parity violating processes producing a large coherent effect on macroscopic scales.
Our observation here is that the non-dissipating topological current introduced in the present work has precisely this property: the topological current produces the P-odd correlation \( \langle \vec{P} \cdot \vec{B} \rangle \) and is capable of inducing magnetic helicity \( \mathcal{H} \sim j \). In fact, our estimate for the induced toroidal field \( H_{\text{tor}} \) unambiguously implies that the magnetic helicity will be also induced, see eq. (8.3). The magnetic flux from the toroidal and poloidal fields is always interlinked and contributes to the magnetic helicity,

\[
\mathcal{H} = 2\Phi_{\text{tor}}\Phi,
\]

where \( \Phi \) and \( \Phi_{\text{tor}} \) are the original poloidal and induced toroidal magnetic fluxes (8.3) correspondingly.

Strong observational evidence, see [61] and references therein, supporting the presence of the toroidal component unambiguously suggests that the magnetic helicity \( \mathcal{H} \) must be non-zero in neutron stars. The P-odd quality of the magnetic helicity may be strong, indirect evidence supporting our claim that P-odd topological currents have been induced at some moment in the star’s life. Otherwise, it is very difficult to understand how such a large, coherent P-odd effect could be produced.

### 8.5 The conflict between vortices and precession

It has been observed that neutron stars precess [57, 58] and that the degree of precession conflicts with the commonly held belief that the protons form a type-II superconductor in the core [15, 16]. It has been shown that if the superconductor is type-I there is no conflict [19].

When a magnetic field is applied to a type-II superconductor the flux finds it energetically favourable to penetrate it by forming many vortices each carrying a unit of quantum flux. In a neutron star the large number of these get tangled with the superfluid neutron vortices that have formed to carry angular momentum. If the star precesses with a large enough angle the superfluid vortices must break through the superconducting vortices for rotation to continue. Incredibly large amounts of energy are dissipated in this process which would cause the star to stop rotating. We conclude then that the superconductor must be type-I where the flux bunches in large groups organizing macroscopically large domains and there is room for the neutron vortices to move around.

The problem is that the Landau-Ginzburg parameter for a typical neutron star indicates the superconductor is type-II. A solution to this is suggested in [9]. If a sufficiently large current runs along type-II vortices an attractive force arises that causes the vortices to bundle together like they would in a type-I superconductor even though the Landau–Ginzburg parameters indicate type-II behaviour.

Even if the current is not strong enough to make the vortices attract each other it has also been argued [9] that the mere presence of an induced, longitudinal current, arbitrarily small, would destroy superconductivity, thus resolving the problem. In many condensed matter systems such kind of instability has been experimentally tested, see [9] for relevant references on test of this instability in condensed matter systems. This instability can be delayed for small currents or even stabilized due to the impurities. But the lesson from these
condensed matter systems is that when a current aligns with the magnetic field creating
the vortex the properties of the vortex lattice are completely changed or destroyed.

We expect similar behaviour in regions of the neutron star where both the Landau-
Ginzburg parameter suggests type-II behaviour and longitudinal currents are induced.
While many features of the system are still to be explored, the point is that that even
small topological currents along the magnetic field will likely destroy the vortex lattice by
replacing it with a new, unknown structure, similar to the condensed matter experiments
mentioned above. The exact state is not essential at the moment, only that the Abrikosov
lattice is destroyed by longitudinal currents and conflict formulated in [15, 16] is resolved.

9. Conclusion

The primary goal of this paper is to argue that a persistent, anomalous current may
be induced in neutron stars. All the crucial ingredients are present: a large degeneracy
\( \mu_e \gg T \), an approximately chiral, Dirac-like spectrum for the excitations at \( \mu_e \gg m_e \), a
large magnetic field \( B \). This induced current is very unusual and beautiful pure quantum
phenomena that does not have analogue in classical physics. It is fundamentally new,
therefore, it is desirable to know if analogous phenomena have been previously studied.

Indeed, there exist in nature systems with the potential to exhibit anomalous currents.
More so, effects analogous to those discussed in this paper have been experimentally tested
in some condensed matter systems and are going to be tested at RHIC and there is a
possibility that these effects can be experimentally tested in terrestrial laboratories.

We will start with condensed matter systems. Low temperatures and strong magnetic
fields do not present technical difficulties in the laboratory. The key is finding a system
with quasiparticles with a Dirac-like spectrum. There are such systems: superfluid \( He^3 \),
high \( T_c \) superconductors with d-wave pairing, and graphene.

Remarkably, in superfluid \( He^3 \) the current analogous to our anomalous current has
been observed, see reviews [6] and [8], and there is a potential to see evidence for it at
the relativistic heavy ion collider (RHIC) in Brookhaven. An analogue to the anomalous
current has been used to predict a charge separation effect [10] and preliminary results are
supporting it. In this analogue each requirement for the current to exist in the neutron
star has its compliment: the role of the coherent magnetic field is played by the angular
momentum \( \vec{L} \) which occurs at non-central nuclei collisions and the the role of P-parity
violating effects is played by the induced \( \theta \) vacua. The observation of the charge separa-
tion indirectly supports our prediction of induced anomalous currents. We should remark
here that very strong interactions in each nuclei-nuclei collision event can not wash out the
produced asymmetry. This is analogous to our argument in Section 2 that strong electro-
magnetic interactions do not wash out the P-odd produced asymmetry and the relevant
scale of the problem is the mean free path of the electron due to the weak interactions that
are capable of washing out P-odd effects.

If our claim that anomalous currents exist in neutron stars is confirmed it would have
an enormous effect on the physics of neutron stars. In particular, it may explain neutron
star kicks and pulsar jets (see Sections 8.1 and 8.2), it may resolve the conflict between the
observed precession of a neutron star and type-II superconductivity commonly believed
to exist in the core (see Section 8.5), and it may shed light on the nature of the toroidal
magnetic field required for stability of the poloidal field (see Section 8.3). Topological
currents also provide a source of finite magnetic helicity, a P-odd topological invariant that
does not decay in a neutron star environment. This may shed some light on the origin of
the strong, self-supporting system of toroidal and poloidal magnetic fields in neutron stars
(see Section 8.4). This necessity of finite magnetic helicity is strong, indirect evidence that
non-dissipating currents move along $\vec{B}$.

In Section 8 we mentioned many apparently unrelated observational effects: neutron
star kicks, toroidal fields, and magnetic helicity. These all have a P-odd symmetry and it is
likely that they all originate from the same P-odd physics. The topological vector current
introduced in this paper occurs because of parity violating effects. This current may be
responsible for all of these P-odd phenomena.

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A. Calculation of the Asymmetry Factor $P_{\text{asym}}(\mu, B, T)$

In order to calculate $P_{\text{asym}}(\mu, B, T)$, which is related to the average helicity of the electron,
we must consider the creation of an electron through nuclear beta decay in a large magnetic
field and chemical potential. There have been many calculations of transition rates in
magnetic fields, notably [65, 66], and more recently for both large magnetic fields and
chemical potentials [38] and references within. We are looking for a specific property of the
weak interaction, the helicity of electrons produced in a large magnetic field and chemical
potential. The helicity of a particle is given by $\Lambda = \sigma \cdot p/|p|$, where $\sigma$ are the Pauli
matrices and $p$ is the particle’s spatial momentum. The two eigenstates of the helicity
operator correspond to the values

$$ p \cdot \xi = \pm|p|, \quad (A.1) $$

where $\xi$ is the rest spin vector of the particle. The expectation value of the helicity can be
calculated by looking at the ratio of decay rates $\Gamma$ with different helicity,

$$ \langle \Lambda \rangle = \frac{\Gamma(p \cdot \xi = |p|) - \Gamma(p \cdot \xi = -|p|)}{\Gamma(p \cdot \xi = |p|) + \Gamma(p \cdot \xi = -|p|)}. \quad (A.2) $$

The small scale of the interactions and the immense magnitude of the magnetic field
make it necessary to consider the Landau levels the electrons decay into. Detailed analysis
of the affect the Landau levels on interactions while considering the complete electron wave
function has concluded that only the significant change is in the energy of the electron,

$$ E^2_c = p^2_e + m^2_e(1 + 2nb), \quad n = \begin{cases} 
1, 2, \ldots & \text{spin up} \\
0, 1, 2, \ldots & \text{spin down} \end{cases} \quad (A.3) $$
where \( b = B/B_c \) is the ratio of the magnetic field and the critical magnetic field, \( B_c = \frac{m_e^2 c^3}{\hbar e} \approx 4.4 \cdot 10^{13} \text{ G} \).

The magnetic field also affects the phase space of the electron. It can only carry linear momentum in the direction of the magnetic field. The magnetic field also causes degeneracy of levels. With this taken into account the phase space becomes,

\[
\int \frac{dp_e}{2\pi} \frac{m_e^2 b}{2\pi}.
\]  

(A.4)

The standard weak interaction matrix element for beta decay is used where we sum over everything but the electron spin,

\[
\sum |M|^2 \approx 16G_F^2 E_N E_P \left[ (1 + 3C_A^2)E_e(E_e - \mathbf{p}_e \cdot \mathbf{\xi}_e) \right. \\
\left. - (1 - C_A^2) \mathbf{p}_\nu \cdot (\mathbf{p}_e - m_e s_e) \right],
\]  

(A.5)

\( C_A \approx 1.26 \) is the axial current constant, \( G_F \) is the Fermi coupling constant,

\[
\mathbf{s} = \mathbf{\xi} + \frac{(\mathbf{p} \cdot \mathbf{\xi}) \mathbf{p}}{m(m + E)}
\]  

(A.6)

and \( \mathbf{\xi} \) is the unit polarization vector.

In order to find the decay rate we use the modified electron phase space. To find the total decay rate it is necessary to sum over the probability of an electron appearing in the each of the Landau levels. The sum truncates where the energy of a Landau level is high enough to be disallowed by conservation of energy. Specifically we let \( n_{\text{max}} \) be the largest \( n \) that satisfies \( E_e^2 > m_e^2 (1 + 2nb) \). The total decay rate is the sum of the decay rate of each level,

\[
\Gamma = \sum_{n=0}^{n_{\text{max}}} \Gamma_n,
\]  

(A.7)

where

\[
n_{\text{max}} = \frac{(E_e^2 - m_e^2)}{m_e^2 2b}.
\]  

(A.8)

The highest Landau level is reached when all of the energy goes into putting the electron in the highest Landau level and none into the momentum \( p_z \). We see that increasing the magnetic field decreases the number of levels the electron has access to. When \( b > E_e^2/2m_e^2 - 1/2 \) the electron only has access to the lowest Landau level meaning that all the electrons are spin down.

We must also account for the non-zero chemical potentials of the protons, electrons, and neutrons. Because the electron and proton have large chemical potentials their phase spaces are constricted. The presence of a Fermi surface means that only higher energy electrons and protons can be created. This is modelled by multiplying the usual phase space by Fermi blocking terms \( 1 - f(E) \), where \( f(E) \) is the Fermi distribution.

\[
1 - f_i = \frac{1}{1 + e^{-(E_i - \mu_i)/T}},
\]  

(A.9)

for particle species \( i \) where \( \mu_i = m_i + \mu_{\text{nr}} \) is the mass added to the non-relativistic chemical potential. Equation (A.9) has the property that as \( T \to 0, (1 - f_i) \to 0 \). As a consequence
any interaction that has this as a factor will stop at zero temperature because there is no thermal energy to push the process over the hump of the chemical potential.

With these considerations the decay rate into a single landau level \( n \) is,

\[
    d\Gamma_n = \frac{1}{2E_N 2E_e 2\pi 2\pi} \left[ \frac{d^3p_e}{2E_e(2\pi)^3} \frac{d^3p_P}{2E_P(2\pi)^3} \right] |M|^2 (2\pi)^4 \delta^4(p_N - p_P - p_e - p_\nu)(1 - f_P)(1 - f_e),
\]

(A.10)

There are a few quantities that naturally align themselves with the magnetic field. Firstly, all spins are either aligned or anti-aligned so if we choose \( \mathbf{B} = (0, 0, B_z) \), then

\[
    \xi_e = (0, 0, \xi_e) \quad \text{and} \quad \xi_N = (0, 0, \xi_N).
\]

(A.11)

Also, because of the Landau levels, the electrons have linear momentum only in the direction of the magnetic field,

\[
    \mathbf{p}_e = (0, 0, p_e).
\]

(A.12)

The matrix element is reduced to

\[
    \sum |M|^2 \simeq 8G_F^2 E_N E_P \left[ (1 + 3C_A^2)E_e - p_e \xi_e \right]
    \left[ (1 - C_A^2)\left|\mathbf{p}_P\right|(p_e - m_e s_e) \cos \theta \right],
\]

where \( \theta \) is the angle between the z-axis and the direction of the neutrino momentum.

The integrals up to the final electron integral are straight forward. We are left with,

\[
    d\Gamma_n = Ag_n d_{p_e}(E_N - E_P - E_e) \left[ 1 - \frac{p_e \xi_e}{E_e} \right] (1 - f_e),
\]

(A.14)

where,

\[
    A = \frac{m_e^2 C_A^2 b}{(2\pi)^3} (1 - 3C_A^2)(1 - f_P(k_{[p]})).
\]

(A.15)

The term \( p_e \xi_e = \pm |p_e| \) gives us the helicity eigenstates. We can get the combinations required for finding the average helicity,

\[
    \Gamma_n(+) - \Gamma_n(-) = -4Ag_n \int_{0}^{p_0} dp_e(E_N - E_P - E_e)^2 \left|\frac{p_e}{E_e}\right| (1 - f_e)
\]

(A.16)

\[
    \Gamma_n(+) + \Gamma_n(-) = 4Ag_n \int_{0}^{p_0} dp_e(E_N - E_P - E_e)^2 (1 - f_e).
\]

(A.17)

In order to do the final integral over the Fermi distribution we appeal to the Sommerfeld expansion,

\[
    \int_{0}^{E_0} h(E)(1 - f(E))dE = \int_{\mu}^{E_0} h(E)dE - \frac{T^2 \pi^2}{6} \left. \frac{\partial}{\partial E} h(E) \right|_{E=\mu}
\]

(A.18)

The particles in a neutron star are in chemical equilibrium. The maximum energy available for the electron is equal to its chemical potential, \( E_0 = \mu \), leaving the integral part of the Sommerfeld expansion to vanish. Physically this makes sense because the equilibrium
processes only occur thermally – the transition rate at zero temperature vanishes. Doing our integral has been reduced to taking a derivative,

\[ \Gamma_n(+) - \Gamma_n(-) = \frac{2AgnT^2\pi^2}{3} \left. \frac{p_e}{E_e} \frac{\partial}{\partial p_e} (E_N - E_P - E_e) \right|_{p_e = k_e} \]

\[ \Gamma_n(+) + \Gamma_n(-) = -\frac{2AgnT^2\pi^2}{3} \left. \frac{p_e}{E_e} \frac{\partial}{\partial p_e} (E_N - E_P - E_e) \right|_{p_e = k_e} \]

where remember that \( E_e(p_e, n) \). We can sum over each of these to get the total decay rate, then take the ratio to get the average helicity. The important values are \( E_N - E_P = \mu_e \sim k_e \) and the sum goes up to \( n_{\text{max}} = k_e^2/2m_e^2b \). The details after this are largely uninteresting and is computed by doing the sum numerically. As a check we find that the helicity at zero magnetic field is \( \langle \Lambda \rangle = -1 \) as we expect. Over the range of fields we are interested \( B = 10^{12} - 10^{15} \) G the helicity is surprisingly constant. We arrive at,

\[ \langle \Lambda \rangle = -0.84. \]  

The helicity is close to \(-1\), but not so close that it doesn’t warrant a comment. The presence of a large magnetic field in large chemical potentials ruins the chirality of the electron. This difference can be explained by appealing to conservation of momentum and the fact that spin wants to antialign in a magnetic field. In the recoiless limit the electron and neutrino have equal and opposing momenta. The neutrino being only left-handed wants to travel with the magnetic field. The electron then must travel against the field, but it also wants to have its spin aligned against the field – a right-handed configuration is formed. The electron is bullied by the neutrino into being right handed. Of course the effect is not absolute and the left-handedness intrinsic in the standard model wins out.

The last step in estimating the asymmetry \( P_{\text{asym}}(\mu_e, T, B) \) is to introduce a suppression factor proportional to the ratio of the number of states in the first Landau level in comparison with total number of states. The spin degeneracy of the lowest Landau level is one while all other Landau levels have spin degeneracy two. This implies that the produced P-asymmetry in the helicity \( \langle \vec{\sigma} \cdot \vec{P} \rangle \), given by \( (A.21) \), is not translated into P-asymmetry in the correlation of the momentum and magnetic field \( \langle \vec{B} \cdot \vec{P} \rangle \) for all Landau levels except the lowest one. Even though the P-asymmetry is present in the higher levels the spin degeneracy allows particles with the same helicity are allowed to travel in opposite directions, which results in zero current. The single spin degeneracy of the lowest Landau level means that any helicity will result in more particles moving one way than the other, thus a current. The correlation between spin and the magnetic field is a P-even effect \( \langle \vec{\sigma} \cdot B \rangle \) and together with the P-odd correlation between spin and momentum \( \langle \vec{\sigma} \cdot P \rangle \) it produces the P-odd asymmetry \( \langle \vec{B} \cdot \vec{P} \rangle \) we are interested in. Therefore, we estimate P-odd asymmetry factor as

\[ P_{\text{asym}}(\mu_e, T, B) \simeq -\langle \Lambda \rangle \cdot \left( \frac{m_e^2}{\mu_e^2} \right) \frac{B}{B_c} \left( \frac{n}{n_0} \right)^{-4/3}. \]  

(A.22)
The density dependence in the estimate is for a neutron star; in the case of quark stars we use the appropriate electron chemical potential and density dependence described in Section 2.4. The suppression factor is identical to (4.34) but appears because we are only considering the Landau levels that contribute to the current. This is different than in (4.34) where the factor appears because only one Landau level exists in the system. Numerically we see that a large chemical potential means that the number of states in the lowest Landau level is small compared to the total number of states.

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