RELATIVE STABILITY CONDITIONS ON FUKAYA CATEGORIES OF SURFACES

ALEX A. TAKEDA

Abstract. In this paper, we introduce the novel notion of a relative Bridgeland stability condition, in the context of a wrapped Fukaya category of a marked surface with respect to part of its boundary. This construction can be shown to have nice functorial properties and behave well under certain decompositions of surfaces. We use this method to reduce the calculation of stability conditions on the Fukaya category of any fully stopped surface into simpler cases, and in particular we show that any stability condition on such a category is of the type described by Haiden, Katzarkov and Kontsevich, i.e. given by the structure of a flat surface; there are no exotic non-geometric stability conditions.

1. Introduction

In the seminal work [8], T. Bridgeland defines a notion of stability conditions on triangulated categories, having as inspiration the stability of D-branes in string theory and SCFTs [3, 15]. This definition generalizes the concept of slope-stability for vector bundles in classical geometry and is quite remarkable; in particular the space of such structures naturally carries the structure of a complex manifold, and has an action by the group of automorphisms of the category.

The space \( \text{Stab}(\mathcal{D}) \) of stability conditions on a category \( \mathcal{D} \) has been understood for many cases of geometric interest. For instance, on the ‘B-side’ of mirror symmetry (ie. coherent sheaves), the initial example to be examined by Bridgeland is the calculation of \( \text{Stab}(\mathcal{D}) \) when \( \mathcal{D} \) is the category of coherent sheaves on the elliptic curve [8]. Following this we have Macrì’s calculation for higher-genus curves [30] and Okada’s description of \( \text{Stab} \left( \text{Coh}(\mathbb{P}^1) \right) \) [34]. The complete description of stability conditions on compact surfaces is also known, due to the work of Bridgeland [9], Toda [38], Okada [35] and others, and the difficult case of smooth projective threefolds [6, 29] has been a subject of much current interest, with a very recent announced result [28] constructing a family of stability conditions on the quintic threefold.

The analogous questions for noncompact spaces [7, 21] are often more tractable, and so are the cases of categories defined by quivers and other representation-theoretic data [10, 25, 36, 22, 14]. In these cases it is often easier to construct families of stability conditions since one has explicit exceptional collections [12]; however understanding the whole stability space requires specific knowledge of each category.

On the other side (A-side) of mirror symmetry there have been many indications that stability conditions can recover geometric data encoded by the Fukaya category, in particular regarding questions about special Lagrangian geometry [24, 37]. In the main work [20] that this paper references, Haiden, Katzarkov and Kontsevich look at stability conditions on the (wrapped) Fukaya category of a marked surface \( \Sigma \), and show that the spaces of stability conditions on \( \mathcal{F}(\Sigma) \) are related to the geometry of quadratic differentials on \( \Sigma \). The relation between moduli spaces of quadratic differentials and spaces of stability conditions already appeared in the work of Bridgeland and Smith [11].

More specifically, in [20] the authors construct a map

\[
\mathcal{M}(\Sigma) \rightarrow \text{Stab}(\mathcal{F}(\Sigma))
\]

from the moduli space \( \mathcal{M}(\Sigma) \) of “marked flat structures” on \( \Sigma \), or equivalently quadratic differentials with singularities of prescribed type. The image of this map is an union of connected components of \( \text{Stab}(\mathcal{F}(\Sigma)) \); we will call these HKK stability conditions. In some small cases (disk and annulus) they prove also that this image is the whole space, using finiteness properties of these categories. Their calculation relies on the fact that the categories are well-known and studied. This shows a recurring feature of the existing calculations of \( \text{Stab}(\mathcal{D}) \); it is much easier in general to make statements about individual components of these spaces than to know them in their entirety, as it is a priori possible that there might be exotic stability conditions that don’t correspond to intuitive geometric structures, living in components of \( \text{Stab}(\mathcal{D}) \) that are inaccessible from the known geometric components.
In this author’s opinion, one of the reasons for this recurring difficulty is that there are currently not many tools for systematically constructing stability conditions from local data. The two sides of a stability condition, the central charge and the slicing, have opposite functoriality, and it is not obvious that stability conditions should exhibit any sheaf- or cosheaf-like behavior. This means, for example, that one must have a good control of the global behavior of the geometry to study stability conditions; all the cases cited above rely heavily on complete knowledge of the global behavior of morphisms between objects.

The initial motivation for this paper is the observation that \[20\] provides an enticing counterexample to this trend, since it builds stability conditions on \(\mathcal{F}(\Sigma)\) from geometric objects with nice functorial properties, namely flat structures, which glue along nicely under a decomposition of the surface. For example, given a decomposition of \(\Sigma\) into two pieces \(\Sigma_1\) and \(\Sigma_2\) mutually overlapping along a rectangular strip \(R\), and a flat structure on \(\Sigma\), restricting the flat structure to each side gives a flat structure (with the new boundary ‘at infinity’). Moreover, once one defines the appropriate notion of compatibility between flat structures along the strip, one can glue compatible flat structures on \(\Sigma_1\) and \(\Sigma_2\) into a flat structure on \(\Sigma\).

This paper is an effort towards abstracting this idea of cutting and gluing into a construction that only makes reference to the stability conditions themselves. The appropriate local pieces of this construction are presented in Section 3 where we introduce the definition of relative stability conditions on a marked surface. A relative stability condition on \(\Sigma\) with respect to some unmarked boundary arc \(\gamma\) is an ordinary stability condition on another surface \(\Sigma,\) obtained from \(\Sigma\) by an appropriate modification along \(\gamma\). This definition behaves well under certain decompositions of surfaces. Let \(\text{RelStab}(\Sigma, \gamma)\) denote the set of relative stability conditions on \(\Sigma\) relative to \(\gamma\); we prove that this set is naturally a Hausdorff space, with a topology inherited from the topology of the spaces of (ordinary) stability conditions. Consider a decomposition \(\Sigma = \Sigma_L \cup \gamma \Sigma_R\) into two surfaces glued along boundary arcs. Our main technical result is about the existence of cutting and gluing maps relating stability conditions on \(\Sigma\) and relative stability conditions on \(\Sigma_L\) and \(\Sigma_R\).

**Theorem 1.** There is a relation of compatibility along \(\gamma\) defining a subset \(\Gamma \subset \text{RelStab}(\Sigma_L, \gamma) \times \text{RelStab}(\Sigma_R, \gamma)\) and continuous maps

\[
\text{Stab}(\mathcal{F}(\Sigma)) \xrightarrow{\text{cut}} \Gamma \xrightarrow{\text{glue}} \text{Stab}(\mathcal{F}(\Sigma))
\]

which compose to the identity.

Consider now any marked graded surface \(\Sigma\) that is ‘fully stopped’, i.e. every boundary circle has at least one marked interval. Assume also that at least one boundary circle has at least two marked intervals. In Section 5 we define a procedure for reducing the calculation of \(\text{Stab}(\mathcal{F}(\Sigma))\) to the calculation of (ordinary) stability conditions on three base cases: the disk, the annulus and the punctured torus.

In all of these cases it can be shown that every stability condition is an HKK stability condition, i.e. the map \(\mathcal{M}(\cdot) \to \text{Stab}(\mathcal{F}(\cdot))\) is an isomorphism. The cases of the disk and of the annulus are dealt with in \[20\], but the calculation for the case of the punctured torus is new. Theorem 1 implies that the gluing map \(\Gamma \to \text{Stab}(\mathcal{F}(\Sigma))\) is surjective, so knowing that all the base cases are fully described by HKK stability conditions we deduce the same for the surface \(\Sigma\).

**Theorem 2.** Every stability condition on \(\mathcal{F}(\Sigma)\) is an HKK stability condition, i.e. given by a flat structure on \(\Sigma\).

As mentioned above, this author believes that the value of this construction is not necessarily in its specific application to the case of Fukaya categories, but rather in its use for constructing and analyzing stability conditions sheaf-theoretically. It would be very fortunate if these tools could be rephrased in purely categorical terms, without direct reference to the geometry of \(\Sigma\). In general terms, the idea is to define relative stability conditions on fully faithful functors \(\mathcal{A} \to \mathcal{B}\) that can be glued to give stability conditions on pushouts of the form \(\mathcal{B} \cup_A \mathcal{B'}\).

For that purpose, we have tried to make the definitions of relative stability conditions as functorial (i.e. independent of the explicit description of the surface) as we could, but it has not yet been possible to rephrase the relevant definitions and lemmas in such terms. In particular the theorems involving the cutting and gluing maps of Section 4 still depend on the underlying topological structure of the surfaces; one of the main questions to face before generalizing them to other types categories is to find equivalents of the ‘non-crossing’ Lemma 10 which is one of the essential components of this paper.
It is likely that this kind of construction could be extended beyond Fukaya categories of surfaces; this motivates many possible directions of future study. One obvious such direction is towards extending the definition of relative stability conditions to wrapped Fukaya categories of higher-dimensional symplectic manifolds, which appear in the work of Abouzaid \[2, 1\] and others. Kontsevich \[26\] conjectured that the wrapped Fukaya category of a Weinstein manifold in any dimension can be calculated from a cosheaf of categories on its skeleton; this has been recently proven by the work of Ganatra, Pardon and Shende \[18, 19, 17\]. The description can be made more explicit by working with constructible sheaves \[33\], and the work of Nadler \[32, 31\] furnishes combinatorial models for these cosheaves of categories. This particular model applies to Weinstein manifolds with appropriately generic ‘arboreal’ skeleta, and the local data are given by quiver representation categories. Comparing to the results of this paper, this model appears very suitable to the application of relative stability conditions, since the study of stability conditions on quiver representation categories is in general much simpler than on ‘more geometric’ categories.

This paper also opens up the possibility of using these sheaf-theoretic techniques to address some questions about dynamics on surfaces; the work of Dimitrov, Haiden, Katzarkov and Kontsevich \[13, 12, 14\] investigates the relation between dynamical systems on surfaces and stability conditions on their Fukaya category. The relation between Teichm"uller theory and stability conditions was already noted in \[11, 16\], and in particular there is a close relation between the set of stable phases \(\Phi\) (which we analyze for some cases in Section 5) and measures of dynamical entropy for categories. For now, the possible applications of our methods to such questions remain topic of current and future investigations.

Acknowledgments

I would like to especially thank Tom Bridgeland, Fabian Haiden and my graduate advisor Vivek Shende for very helpful conversations and great patience in explaining technical points to me. I would also like to thank Dori Bejleri, Benjamin Gammage, Tatsuki Kuwagaki, David Nadler, Kyōji Saitō, Ryan Thorngren and Gjergji Zaimi for helpful discussions. A crucial part of this work was conducted during a working visit to the IPMU in Japan, and I would like to thank the faculty and staff of that institute for providing a great working environment. This project was supported in part by NSF CAREER DMS-1654545.

Notation

We will mostly use the conventions and notation of \[20\]. A graded marked surface (or just surface for brevity) is a smooth oriented surface \(\Sigma\) with boundary \(\partial \Sigma\) and a set of marked boundary intervals \(\mathcal{M}\), whose elements are intervals contained in \(\partial \Sigma\). The intervals in \(\partial \Sigma \setminus \mathcal{M}\) will be the unmarked boundary intervals. Throughout the paper, we will only deal with the “fully stopped” case, ie. each boundary circle in \(\partial \Sigma\) has at least one marked interval.

The grading on \(\Sigma\) is a line field \(\eta \in \Gamma(\Sigma, \mathcal{PT} \Sigma)\). The set of gradings on \(\Sigma\) up to graded diffeomorphism isotopic to the identity is a torsor over \(H^1(\Sigma, \mathbb{Z})\). Curves immersed in \(\Sigma\) are graded with respect to \(\eta\); this defines the degree of a point of intersection between curves. An arc in \(\Sigma\) is an embedded intervals with ends on marked boundary intervals system, and an arc system \(\mathcal{A}\) on \(\Sigma\) is a collection of pairwise disjoint and non-isotopic arcs.

As for (Bridgeland) stability conditions, \(\text{Stab}(\mathcal{D})\) will denote the space of stability conditions on a triangulated category \(\mathcal{D}\) satisfying the so-called support property \[27, 5\] (in the original paper \[8\] these are called full locally finite stability conditions). In all of our cases, \(K_0(\mathcal{D})\) is finite-dimensional so we will use the lattice \(\Lambda = K_0(\mathcal{D})\). As shown by Bridgeland, \(\text{Stab}(\mathcal{D})\) has the structure of a \((\text{rk} K_0(\mathcal{D}))\)-dimensional complex variety.

Fixing a stability condition and an object \(X\), we will denote by \(\text{HNEnv}(X)\) the full triangulated subcategory of \(\mathcal{D}\) generated by the semistable objects appearing in the Harder-Narasimhan filtration of \(X\). We will denote by \(\text{HNLen}(X) \in \mathbb{Z}_+\) the number of distinct phases of semistable objects appearing in the filtration; \(\text{HNLen}(X) = 1\) if and only if \(X\) is semistable.
2. Lemmas about stability conditions

In this section we collect some lemmas about stability conditions in general, and also about the specific case where $D = \mathcal{F}(\Sigma)$ is the Fukaya category of a marked surface $\Sigma$.

2.1. Stability conditions and genericity. We will make use of genericity assumptions, which will play an important role in later proofs. To state them precisely, we first recall the support property [27, 5]:

**Definition 1.** A stability condition $\sigma = (Z, \mathcal{P})$ satisfies the support property if

$$\inf_{0 \neq X \text{semistable}} \frac{|Z(X)|}{\|X\|} = C > 0,$$

where $\|\cdot\|$ is a norm on $\Lambda \otimes \mathbb{R}$.

From now on, we will only consider stability conditions satisfying the support condition above. The space $\text{Stab}(D)$ of such stability conditions is a complex manifold and the map $\text{Stab}(D) \to \text{Hom}_\mathbb{Z}(\Lambda, \mathbb{C})$, given by forgetting the slicing $\mathcal{P}$, is a local homeomorphism [8]. To express genericity we need to define walls in this space, following [11]. Let us fix a class $\gamma \in \Lambda$, and consider other classes $\alpha$ such that $\alpha$ and $\gamma$ are not both multiples of the same class in $\Lambda$.

**Definition 2.** The wall $W_\gamma(\alpha) \subset \text{Stab}(D)$ is the subset of stability conditions such that there is a phase $\phi \in \mathbb{R}$ and objects $A, G$ with respective classes $\alpha, \gamma$ such that $A \subset G$ in the abelian category $\mathcal{P}_\phi$.

Each wall $W_\gamma(\alpha)$ is contained within a codimension one subset of $\text{Stab}(D)$ where $Z(\alpha)/Z(\gamma)$ is real, and we have the following local finiteness result:

**Lemma 3.** [11, Lemma 7.7] If $B \subset \text{Stab}(D)$ is compact then for a fixed $\gamma$ only finitely many walls $W_\gamma(\alpha)$ intersect $B$.

Note that this is not true if we consider the whole collection of walls for all $\gamma$; the union of all walls can be dense in $\text{Stab}(D)$. So we will have to be specific when discussing genericity.

**Definition 3.** Let $\Xi \subset \Lambda$ be a finite subset of classes. Take

$$W_\Xi = \bigcup_{\gamma, \alpha \in \Lambda} W_\gamma(\alpha)$$

ie. the union of all closures walls for classes in $\Lambda$; we will say a stability condition $\sigma$ is $\Xi$-generic if $\sigma \in \text{Stab}(D) \setminus W_\Xi$.

By local finiteness, $W_\Xi$ is a locally-finite union of closed subsets so $\Xi$-genericity is an open condition. The connected components of $\text{Stab}(D) \setminus W_\Xi$ will be called the $\Xi$-chambers.

We will later make use of the following simple fact, which holds for any stability condition, generic or not.
Lemma 4. If \( X = E \oplus F \) then \( \text{HNLen}(X) \) is equal to the total number of distinct phases appearing among the HN decomposition of \( E \) and \( F \). In particular, \( \max(\text{HNLen}(E), \text{HNLen}(F)) \leq \text{HNLen}(X) \leq \text{HNLen}(E) + \text{HNLen}(F) \).

Proof. Follows from uniqueness of the HN decomposition, and the fact that given a HN decomposition of \( E \) and \( F \) one can algorithmically produce an HN decomposition of \( E \oplus F \). \( \square \)

It will be important for our calculations to have explicit descriptions of the indecomposable objects of \( \mathcal{F}(\Sigma) \). Fortunately, we have the following result establishing the geometricity of objects in this category.

Theorem 5. \cite[Theorem 4.3]{20} Every isomorphism class of indecomposable objects in \( \mathcal{F}(\Sigma) \) can be represented by an admissible graded curve with indecomposable local system, uniquely up to graded isotopy.

An admissible graded curve is either an either an immersed interval ending at marked intervals or an immersed circle, which does not bound a teardrop. An important role will be played by objects that can be represented by embedded curves. Let us from now on call an object an (embossed) interval object if it can be represented by an (embedded) interval, and a (embedded) circle object if it can be represented by an (embedded) circle. Note that every local system on an interval is trivial so an indecomposable embedded interval object necessarily has a rank one local system.

Another result of \cite{20} is a description of \( K_0(\mathcal{F}(\Sigma)) \) for surfaces \( \Sigma \) without unmarked boundary circles (which is the case that we are considering in this paper). The grading on \( \Sigma \) gives a double cover \( \tau \) by the orientation of the foliation lines; consider the local system of abelian groups \( \mathbb{Z}_\tau = \mathbb{Z} \otimes \mathbb{Z}/2 \tau \).

Theorem 6. \cite[Theorem 5.1]{20} There is a natural isomorphism of abelian groups \( K_0(\mathcal{F}(\Sigma)) \cong H^1(\Sigma, \mathcal{M}; \mathbb{Z}_\tau) \).

Given this description of indecomposable objects of \( \mathcal{F}(\Sigma) \), we prove the following proposition, which constrains the type of geometric objects. This will play an important role throughout this paper.

Proposition 7. For any stability condition \( \sigma \in \text{Stab}(\mathcal{F}(\Sigma)) \), every stable object is either an embedded interval object or an embedded circle object.

Proof. Since \( L \) is indecomposable its support cannot have more than one connected component. Thus the only objects we have to rule out are objects whose representatives all have self-intersections; we will call these truly immersed objects.

A stable object \( L \) must have \( \text{Ext}^i(L, L) = 0 \) for \( i < 0 \). Let \( L \) be a truly immersed objects and pick a representative of \( L \) with minimal number of self-intersections, supported on an immersed curve \( \gamma_L \). Perturbing \( L \) to calculate endomorphisms, we see that a self-intersection point \( p \) of \( \gamma_L \) contributes classes to \( \text{Ext}^i(L, L) \) in degrees \( i_p \) and \( 1 - i_p \), where \( i_p \) is the degree of intersection at \( p \). These classes are nonzero by minimality of self-intersections, so if there is a self-intersection point with \( i_p \neq 0, 1 \), one of these degrees is negative and therefore \( L \) cannot be semistable.

Figure 2. A truly immersed Lagrangian \( L \). The self-extension \( L \to E \to L \) at the self-intersection point \( p \) splits as a direct sum \( E = F \oplus G \).

The only case left to consider is when \( \gamma_L \) only has self-intersection points of degree 0 and 1; each one of these points gives nonzero classes in \( \text{Hom}(L, L) \) and \( \text{Ext}^1(L, L) \). Let us pick one of these points \( p \), and consider the corresponding nontrivial extension \( L \to E \to L \). Note that the support of \( E \) is given by two
superimposed curves so we have a direct sum decomposition $E = F \oplus G$. But by assumption $L$ is stable of phase $\phi_L$, so $E$, $F$ and $G$ are also all semistable of the same phase. Consider now the abelian category $\mathcal{P}_{\phi_L}$ of semistable objects of that phase. Since the stability condition is locally finite, this category is finite length; therefore the Jordan-Hölder theorem applies \cite{23}. Since the length of $E$ is 2, $F$ and $G$ are length one, and by uniqueness of the simple objects in the Jordan-Hölder filtration (up to permutations) we must have $F \cong G \cong L$. But this is impossible because $E$ is a nontrivial extension so $E \neq L \oplus L$. \hfill \Box

Remark. Note that the proof above does not preclude a self-intersecting object $L$ from being semistable; it just cannot be simple in $\mathcal{P}_{\phi_L}$. In fact this even happens generically: take $\Sigma$ to be the annulus with one marked interval on each boundary circle and grading such that the nontrivial embedded circle is gradable; by mirror symmetry the category $\mathcal{F}(\Sigma)$ is equivalent to $D^b(\text{Coh}(\mathbb{P}^1))$. Under this equivalence, the rank one circle object with monodromy $z \in \mathbb{C}^\times$ gets mapped to the skyscraper sheaf $\mathcal{C}_z$ on $\mathbb{P}^1$, and the interval object $I$ with both ends on the outer boundary, wrapping the annulus once, gets mapped to the skyscraper sheaf $\mathcal{C}_\infty$ on $\mathbb{P}^1$.

The space of stability conditions on this category is known to be isomorphic to $\mathbb{C}^2$ as a complex manifold \cite{34}, and there is a geometric (top dimensional) chamber in $\text{Stab}(\mathbb{P}^1)$ where all the rank one skyscraper sheaves are stable. In particular, the nontrivial extension $I \to L \to I$, represented by an immersed Lagrangian with one self-intersection as in Figure 3, is semistable. So self-intersecting objects do appear generically, but they always have Jordan-Hölder decompositions into embedded objects.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{annulus_mirror}
\caption{The annulus mirror to $D^b(\text{Coh}(\mathbb{P}^1))$. For a geometric stability condition on $\mathbb{P}^1$, the truly immersed object $L$ (corresponding to an irreducible rank 2 skyscraper sheaf $\mathcal{O}_{x_2}$) is semistable.}
\end{figure}

The result above characterizes which objects can be stable, namely embedded intervals and embedded circles with indecomposable local systems. It turns out that similar index computations also allows us to constrain the form of the HN decompositions of objects.

**Definition 4.** (Chain of stable intervals) Let us fix a stability condition $\sigma \in \text{Stab}(\mathcal{F}(\Sigma))$ and consider an indecomposable object $X$ in $\mathcal{F}(\Sigma)$. We say that $X$ has a chain of stable intervals decomposition (cosi decomposition) under $\sigma$ if there is

- A sequence of stable (therefore embedded) interval objects $X_1, \ldots, X_N$ and a sequence of marked boundary intervals $M_0, \ldots, M_N$, where the support $\gamma_i$ of the object $X_i$ has ends on $M_{i-1}$ and $M_i$,
- Extension morphisms $\eta_i \in \text{Ext}^1(X_i, X_{i+1})$ or $\eta_i \in \text{Ext}^1(X_{i+1}, X_i)$ corresponding to the shared $M_i$ marked boundary (including an extension at $M_0 = M_N$ if $X$ is a circle object),

such that the iterated extension by all the $\eta_i$ is isomorphic to $X$.

**Remark.** Note that the order $X_1, \ldots, X_N$ here is not directly related to the ordering of semistable objects in the HN decomposition of $X$; in particular there is no constraint on the phases of the $X_i$, and the extension maps can go either way.

Note that if $X$ has a cosi decomposition then its HN decomposition can be produced from it by grouping together all stable interval objects of the same phase.

**Lemma 8.** If $X$ has a cosi decomposition under $\sigma$, then it is essentially unique, ie. the sets $\{X_i\}$ and $\{M_i\}$ are uniquely defined up to isomorphism.
Proof. Follows from the uniqueness of the HN filtration and the uniqueness (up to permutation) of the Jordan-Hölder filtration on each finite-length abelian category $\mathcal{P}_\phi$. □

This decomposition also captures the isotopy class of the object $X$. Let us produce an immersed curve $\gamma$ from this data as follows: for each $i$, if the extension map $\eta_i$ belongs to $\text{Ext}^1(X_i, X_{i+1})$ we connect $\gamma_i$ to $\gamma_{i+1}$ counterclockwise (i.e. by a boundary path following $M_i$ and keeping $\Sigma$ to the right), and if $\eta_i \in \text{Ext}^1(X_{i+1}, X_i)$ we use the corresponding clockwise path from $\gamma_i$ to $\gamma_{i+1}$. From the geometricity result in Theorem 5 we can deduce that:

Lemma 9. The curve $\gamma$ is isotopic to the support $\gamma_X$ of the object $X$.

The following lemma will be central to our proofs later, and essentially means that cosi decompositions are not allowed to cross each other. From now on, we will leave the extension morphisms implicit and denote a cosi decomposition by its stable intervals.

Lemma 10. Let $X$ and $Y$ be two objects with respective cosi decompositions $(X_1, \ldots, X_m)$ and $(Y_1, \ldots, Y_n)$. We choose representatives for all the stable intervals such that the number of crossings between these two chains of intervals is minimal. Then on the surface $\Sigma$ there are none of the following polygons

1. Polygons bounded by the two chains and two transversal crossings between stable intervals.
2. Polygons bounded by the two chains and two common marked boundary intervals (with boundary paths inside the polygon).
3. Polygons bounded by the two chains, one transversal crossing and one common marked boundary interval (with a boundary path inside the polygon).

Remark. In case (2), we exclude the trivial bigon with isomorphic sides. This case is obviously allowed, and happens whenever $X$ and $Y$ share a same interval in their cosi decompositions. From all cases, we exclude the degenerate configuration where all the objects around the polygon are multiples of the same class in $K_0(\mathcal{F}(\Sigma))$. For cases (2) and (3), the parenthetical condition is there because the chains could meet at some marked boundary interval ‘on the other side of the polygon’. For instance again in the case of the annulus for an ‘algebraic’ stability condition on $\mathbb{P}^1$ the two intervals (corresponding to line bundles on $\mathbb{P}^1$) are stable and we can have the following bigon of stable objects; but the boundary path giving the extension runs outside the polygon.
Figure 6. Again, the annulus $\Delta_{(1,1)}^*$ mirror to $\mathbb{P}^1$. Under this correspondence $\mathcal{F}(\Delta_{(1,1)}^*) \cong D^b(\text{Coh}(\mathbb{P}^1))$ we have $I_1 \cong \mathcal{O}(1)$ and $I_2 \cong \mathcal{O}$ with $\text{Hom}(I_2, I_1)$ spanned by $\eta_x, \eta_y$. Note that this is not a counterexample to case (2) of Lemma 10 since the polygon doesn’t bound a disk.

Proof. Let us first prove that it is sufficient to prove the statement for adequately generic $\sigma$. By standard arguments, the locus of $\text{Stab}(\mathcal{D})$ in which the all the objects $X_i, Y_i$ are stable is open. Consider now the collection $\Xi \subset \Lambda$ containing all the classes of these objects; the corresponding union of walls $\mathcal{W}_\Xi$ is a locally-finite union of closed subsets of positive codimension. So we can find some other stability condition $\Xi'$, arbitrarily close to $\sigma$, where $X_i, Y_i$ still give cost decompositions of $X, Y$, and where the phases of any $X_i$ and $Y_j$ are pairwise distinct when $[X_i]$ and $[Y_j]$ are not proportional. If the noncrossing statement of the lemma is true for $\sigma'$ it is also true for $\sigma$ since it makes no further reference to the stability condition.

Let us start with the first type of polygon. Assume the polygon has $k$ edges on the right and $l$ edges on the left, and for ease of notation we label the intervals in this polygon starting by 1 on both sides. Without loss of generality shift the grading of $X$ such that the intersection point $p$ has index $i_p(X_1, Y_1) = 1$. By minimality of crossings $p$ contributes nonzero classes in $\text{Ext}^1(X_1, Y_1)$ and in $\text{Hom}(Y_1, X_1)$. Since both are stable objects, this implies that

$$\text{phase}(Y_1) \leq \text{phase}(X_1) \leq \text{phase}(Y_1) + 1.$$ 

Smoothing out each one of the chains of intervals separately, one gets a bigon with vertices at $p$ and $q$; the existence of the embedded bigon constrains the index of $q$ to be $i_q(X_k, Y_1) = 0$, and by the same argument we have

$$\text{phase}(X_k) \leq \text{phase}(Y_1) \leq \text{phase}(X_k) + 1.$$ 

By assumption, all the other vertices of this polygon give, on the left hand side, extension maps $X_i \overset{+1}{\rightarrow} X_{i+1}$, and on the right hand side, extension maps $Y_{i+1} \overset{+1}{\rightarrow} Y_i$. Since all these maps appear in HN decompositions we must have the following inequality between phases

$$\text{phase}(X_i) \leq \text{phase}(X_{i+1}) \text{ for all } 1 \leq i \leq k - 1, \quad \text{phase}(Y_j) \geq \text{phase}(Y_{j+1}) \text{ for all } 1 \leq j \leq l - 1$$

, which together with the previous inequality gives that the phases are all equal. But since we excluded the degenerate polygons, at least two of the $K_0$ classes of this object these objects are not multiples of the same class so by $\Xi$-genericity of $\sigma'$ they have distinct phases. The three other cases are proven by small variations of this same argument. \qed

Remark. Note that the two chains might still share a common stable interval; this is not ruled out by the argument above and in fact happens generically. Similarly, note that our definition of chain-of-intervals decomposition above does not exclude the possibility that the chain of intervals overlaps with itself. Again, in the annulus example consider some algebraic stability condition such that the stable objects are two intervals $I_1, I_2$ connecting the outer and inner boundary, and consider the embedded interval object also connecting the two boundaries but wrapping around more times; this object has a cosi decomposition given by multiple copies of $I_1$ and $I_2$.

Self-overlapping chains of intervals will pose some serious technical difficulties later on, so we will rule them out with the following criterion. Let $X$ be an indecomposable object with a cosi decomposition $(X_1, \ldots, X_N)$, with $X_i$ supported on $\gamma_i$. 
Definition 5. This is a simple cosi decomposition if all the $\gamma_i$ are in pairwise distinct isotopy classes, all the marked boundary intervals $M_1, \ldots, M_N$ are pairwise distinct and also distinct from the ends $M_0, M_{N+1}$ of $X$.

This condition implies that among the stable objects $X_i$, one does not find more than one copy of any given isomorphism class, or any of its shifts more than once. Moreover, only successive intervals share marked boundary components, so among these objects the only nontrivial degree zero homs are the self-homs and the only non-trivial extension homs are between adjacent intervals.

Lemma 11. If $X$ has a simple cosi decomposition as above, then its HN envelope $\text{HNEnv}(X)$ is equivalent to either:

- The Fukaya category of the disk $\Delta_{N+1}$ with $N+1$ marked boundary intervals, or equivalently the derived category of the $A_N$ Dynkin quiver, if $X$ is an interval object with ends on distinct marked boundary intervals, or
- The Fukaya category of the annulus $\Delta^{*}_{p,q}$ with $p$ and $q$ inner and outer boundary intervals for some $p+q = N+1$ and grading of index zero around the circle, or equivalently the derived category of the $A_N$ quiver, if $X$ is a circle object.

Proof. We can prove this constructively by giving a map of arc systems. Consider the (non-full) arc system given by all the intervals $\gamma_i$; this defines an $A_\infty$-category $\mathcal{A}$. Since this is a chain of arcs there are no polygons so all the higher structure maps $\mu^n$ between them are trivial. Note that $\text{HNEnv}(X)$ is obtained by taking the triangulated closure of $\mathcal{A}$.

If $X$ is an interval object, let us denote by $m$ the number of indices $i$ such that the extension map at $M_i$ it ‘on the left’ ie. given by an extension map in $\text{Ext}^1(X_{i+1}, X_i)$. Similarly we denote by $n$ the number of extensions ‘on the right’ ie. given by an extension map in $\text{Ext}^1(X_i, X_{i+1})$; we have $m+n = N-1$. Consider the disk $\Delta_{N+1}$ with the following arc system: position $m$ of the marked boundary intervals on the left and $n$ on the right, with the remaining two on the top and bottom. There is then a unique chain of arcs $\alpha_i$ starting from the bottom and ending at the top such that $\alpha_i$ meet on the left if the extension is in $\text{Ext}^1(X_i, X_{i+1})$ and on the right if the extension is in $\text{Ext}^1(X_{i+1}, X_i)$.

This arc system gives an $A_\infty$-category equivalent to $\mathcal{A}$, since the morphisms all agree and all the higher structure maps are zero. The argument for the circle case is similar, except we put $m$ of the marked boundary components on the inner boundary circle and $n$ on the outside (considering also the extension given by $M_0 = M_N$).

In general, objects will not have a simple cosi decomposition, but the following topological condition is sufficient.

Lemma 12. Let $X$ be an object with a cosi decomposition, supported on an embedded interval $\gamma$ separating the surface $\Sigma$ into two connected components, such that the two ends of $\gamma$ belong to distinct marked boundary intervals. Then $X$ has a simple cosi decomposition.

Proof. Let us write as before $\gamma_1, \ldots, \gamma_N$ for the intervals and $M_1, \ldots, M_{N-1}$ for the marked boundary intervals between them. We would like to rule out the possibility of having repeated intervals or marked boundary intervals.

Suppose that the subsequence

$$M_i, \gamma_{i+1}, M_{i+1}, \ldots, M_{i+k-1}, \gamma_{i+k}, M_{i+k}$$

repeats itself, ie. all those intervals and marked boundary components are isomorphic to

$$M_j, \gamma_{j+1}, M_{j+1}, \ldots, M_{j+k-1}, \gamma_{j+k}, M_{j+k}$$

for some other $j$. For simplicity assume that $j > i + k$ so there’s no overlap; and let us assume that $k$ is maximal. Let us also assume that $i > 0$ and $j + k < N$ so that we are in the middle of the chain and not at the ends, and that $j$ is the smallest index possible with these properties (because this sequence could in principle repeat many times).

There are then four possibilities for the extension maps at $M_i$ and $M_{i+k}$, as below:

If we are in the first case or third case, note that concatenating the chain by those boundary walks leads to a self-crossing of $\gamma_X$. This self-crossing cannot be eliminated by isotopy, because due to Lemma 10
Figure 7. Four possible cases for extensions within a self-overlapping chain.

are no polygons of stable intervals bound by the chain. Since we assumed that $X$ is an embedded interval object this is impossible.

As for the second case and fourth case, note that concatenating the chain by those boundary walks leads to an embedded interval that does not separate the surface into two parts, contradicting the topological condition.

The special cases to be dealt with are when this repeated sequence is at one end of the chain; in this case it is easy to see that the concatenation is always non-trivially self-intersecting, unless the overlap is just a single boundary component $M_0 = M_N$ which we also excluded by assumption. The more general case of repeated intersections, nested intersections etc. poses no essential difficulties and can be argued by repeating the argument above recursively. □

With these lemmas, we prove the following proposition constraining the form of the HN decomposition of an object.

**Proposition 13.** Let $X$ be a rank one indecomposable object of $D = F(\Sigma)$ and $\sigma \in \text{Stab}(D)$ any stability condition. Then $X$ is either a stable circle or has a chain of stable intervals decomposition under $\sigma$.

**Proof.** Suppose that $X$ is not a semistable circle. Consider the HN decomposition of $X$ under $\sigma$ and further decompose each semistable factor of phase $\phi$ using the Jordan-Hölder filtration on the abelian category $\mathcal{P}_\phi$.

We get then a total filtration

$$0 \xrightarrow{\epsilon_1=0} X_1 \xrightarrow{c_1} X_2 \xrightarrow{c_2} \cdots \xrightarrow{c_{N-1}} X_{N-1} \xrightarrow{c_N} X_N = X$$

where each factor $A_i$ is stable but the phases $\epsilon_i$ might repeat.

We will prove by induction on the total length $N$. The case $N = 1$ is obvious. Assume now that the statement is true for any object of total length $N - 1$, and take an object $X$ as above.

Consider the extension $X_{N-1} \to X_N \to A_N$. Since the object $A_N$ is stable, by Lemma 7 it is either representable either by an embedded interval or an embedded circle. We will treat these cases separately.

If $A_N$ is an interval object supported on a embedded interval $\alpha_N$, and $X_{N-1}$ is supported on some collection of immersed curves $\gamma_{N-1}$. Note that we can also express $X_{N-1}$ as an extension

$$A_N[-1] \to X_{N-1} \to X_N$$

, so we conclude that $X_{N-1}$ is either supported on a single immersed curve (interval or circle) or a direct sum of two intervals.

We choose $\alpha_N$ and $\gamma_{N-1}$ to have minimal intersections with each other. The extension map $\eta \in \text{Ext}^1(A_N, X_{N-1})$ comes from a linear combination of classes corresponding to transverse intersection points between $\alpha_N$ and $\gamma_{N-1}$, and shared marked boundary intervals; let us write

$$\eta = c_1 M_1 + c_2 M_2 + \sum_p c_p p$$

where $M_1, M_2$ are extension maps given by the marked boundary intervals at the end of $A_N$ and $p$ labels extension maps coming from intersection points. Note that the coefficients $c_1, c_2, c_p$ are not uniquely defined.
Figure 8. One example where $A_N$ extends $X_{N-1}$ with an extension map $c_2M_2 + c_p p$. Using only the extension at $p$ we obtain $X'$ which is the sum of two interval objects (each of smaller total length), which can be extended at $M_2$ to give $X$. In this case $X_{N-1}$ and $A_N$ shared the other boundary too; this does not have to be the case in general.

We see that it is impossible to have $c_1 = c_2 = 0$. If the extension happens only at transverse intersection points, then this extension is supported on two (or more) superimposed curves which is impossible since we assumed $X_N = X$ was indecomposable.

Consider then the modified extension map

$$\eta' = \sum_p c_p p$$

and the corresponding extension $X_{N-1} \to X' \to A_N$. This is supported on a set of curves that share the marked boundary intervals $M_1$ and/or $M_2$ and moreover can be extended at those to obtain the original object $X$. This topologically constrains $X'$ to be of one of three types:

1. $X' = I_1 \oplus I_2$, two intervals which can be extended at a common boundary to form the interval object $X$,
2. $X' = I_1 \oplus I_2 \oplus I_3$, three intervals which can be extended at two common boundaries to form the interval object $X$,
3. $X' = I_1 \oplus I_2$, two intervals which can be extended at both common boundaries to form a circle object $X$.

Whichever case we are in, since total length is additive, the indecomposable factors $I_1, I_2, I_3$ are all of length $\leq N - 1$ so by the induction hypothesis they have cosi decompositions, which can then be composed at the shared marked boundaries to give a cosi decomposition for $X$.

It remains to deal with the case where $A_N$ is a circle object. Since there is no boundary, the extension map $\eta \in \text{Ext}^1(A_N, X_{N-1})$ must be given by a linear combination

$$\eta = \sum_p c_p p$$

of the classes given by transverse intersections $p$ between $\alpha_N$ and $\gamma_{N-1}$. Assume first that $N \geq 3$; then $N - 1 \geq 2$ and therefore $X_{N-1}$ is not a semistable circle so by the induction hypothesis it has a cosi decomposition coming from concatenating intervals $\alpha_1, \ldots, \alpha_{N-1}$.

We see that every transverse intersection of index 1 between $\alpha_N$ and $\gamma_{N-1}$ must come from one or more transverse intersections of index 1 between $\alpha_N$ and another $\alpha_i$. However this gives a nonzero class in $\text{Hom}(A_i, A_N)$ which cannot happen if $\phi_{A_i} \geq \phi_{A_N}$, so the only possibility is that these have the same phase (ie. appear together in the HN filtration). But this is also impossible: since $A_i$ and $A_N$ are both simple objects in the abelian category $\mathcal{P}_{\phi_{A_N}}$, the existence of this nonzero morphism implies that $A_i \cong A_N$ which cannot happen since one is a circle object and another is an interval object.

The only last case to deal with is when $N = 2$ and $X$ is an extension of two stable circle objects $A_1, A_2$; by the same argument as above this can only happen if the two circles are isomorphic but then $X$ cannot be rank one. \qed
Lemma 18. \( \Sigma \) contained in that their concatenation gives the interval \( \gamma \) of triangulated categories.

Corollary 16. Let \( \text{decompositions.} \)

Proof. Suppose otherwise; then \( X' \) has a \( \text{cosi} \) decomposition. But the same chain of intervals can be concatenated to give \( X \) as well, by taking different multiples of the extension classes between the intervals in the chain, contradicting the assumption. \( \blacksquare \)

The only indecomposable objects not covered by Theorem 13 are circle objects with higher rank local systems, but this will cause no further problems:

Lemma 15. Let \( X \) be an indecomposable object supported on a circle \( \gamma \) with higher-rank local system. Then there are two possibilities for \( X \):

1. \( X \) is a semistable interval whose stable components are all rank one objects supported on \( \gamma \),
2. \( X \) has a decomposition as as chain of semistable intervals, ie. similar to a \( \text{cosi} \) decomposition except that every piece is a direct sum of stable intervals instead of a single stable interval.

Proof. Suppose \( X \) carries a rank \( r \) indecomposable local system \( \mathcal{L} \). If the rank one objects supported on \( \gamma \) are stable, then we pick \( r \) such objects with monodromies given by the eigenvalues of \( \mathcal{L} \); using the self-extension of the circle we can present \( X \) as an iterated extension of these objects, proving that \( X \) is semistable, so we are in case (1). Otherwise, these rank one objects have a \( \text{cosi} \) decomposition; again we take \( r \) copies of this chain of stable intervals and extend them appropriately to construct the local system \( \mathcal{L} \), and we are in case (2).

Combining the results above, we conclude that certain kinds of embedded intervals always have simple \( \text{cosi} \) decompositions.

Corollary 16. Let \( X \) be an object of \( \mathcal{F}(\Sigma) \) represented by an embedded interval \( \gamma_X \) with trivial rank one local system, such that \( \gamma_X \) cuts the surface into two, and has ends on distinct marked boundary intervals. Then \( X \) has a simple \( \text{cosi} \) decomposition under any stability condition, and thus there is an abstract equivalence of triangulated categories \( \text{HNE}(X) \cong D^b(A_X) \).

3. Relative stability conditions

In this section, we present a notion of stability conditions on a surface \( \Sigma \) relative to part of its boundary. This construction will exhibit functorial behavior and satisfy cutting and gluing relations. First we will give some presentations of the category \( \mathcal{F}(\Sigma) \) that will be useful in stating that definition.

3.1. Pushouts. In [20], it is shown that given a full system of arcs on \( \Sigma \), one can define a graph \( G \) dual to it and a constructible cosheaf \( \mathcal{E} \) of \( A_\infty \)-categories on \( G \) such that:

Theorem 17. [20, Theorem 3.1] The category \( \mathcal{F}(\Sigma) \) represents global sections of the cosheaf \( \mathcal{E} \), ie. is the homotopy colimit of the corresponding diagram of \( A_\infty \)-categories.

We will describe how to use this result to express \( \mathcal{F}(\Sigma) \) as certain useful homotopy colimits. Let \( \gamma \) be some embedded interval dividing \( \Sigma \) into two surfaces, \( \Sigma_L \) and \( \Sigma_R \). Suppose that we have a chain of intervals \( \gamma_1, \ldots, \gamma_N \) in distinct isotopy classes connecting \( n+1 \) distinct marked boundary intervals \( M_0, \ldots, M_n \), such that their concatenation gives the interval \( \gamma \).

Lemma 18. \( \Sigma \) admits a full system of arcs \( \mathcal{A} = \mathcal{A}_L \cup \mathcal{A}_i \cup \mathcal{A}_R \) such that every arc in \( \mathcal{A}_L \) has a representative contained in \( \Sigma_L \), every arc in \( \mathcal{A}_R \) has a representative contained in \( \Sigma_R \), and \( \mathcal{A}_\gamma = \{ \gamma_1, \ldots, \gamma_N \} \).

Proof. Consider a (non-full) system of arcs \( \overline{\mathcal{A}}_\gamma \) given by the ‘closure’ of \( \mathcal{A}_\gamma = \{ \gamma_1, \ldots, \gamma_N \} \); that is containing also a chain of arcs connecting all the marked boundary intervals to the left of the chain \( \gamma \), and the analogous chain to the right of it.

Since all the intervals in \( \overline{\mathcal{A}}_\gamma \) are non-intersecting and not pairwise isotopic there is some full arc system \( \mathcal{A} \) of \( \Sigma \) containing them; and since \( \gamma \) (and therefore the chain made by the \( \gamma_i \)) cuts the surface into two we can partition the arcs \( \mathcal{A} \) that are not among the \( \gamma_i \) into left and right subsets \( \mathcal{A}_L \) and \( \mathcal{A}_R \). By construction every arc in \( \mathcal{A}_L \) is contained in \( \Sigma_L \) and every arc in \( \mathcal{A}_R \) is contained in \( \Sigma_R \). \( \blacksquare \)
Consider this arc system $A$. Let us define $\hat{\Sigma}_L$ to be the smallest marked surface with an inclusion into $\Sigma$ that contains all the arcs in $A_L \sqcup A_\gamma$; we define $\hat{\Sigma}_R$ analogously.

We see that topologically, $\hat{\Sigma}_L, \hat{\Sigma}_R$ can be constructed from $\Sigma_L, \Sigma_R$ by attaching a disk along $\gamma$, that is
\[
\hat{\Sigma}_L = \Sigma_L \cup_{\gamma} \Delta_m, \quad \hat{\Sigma}_R = \Sigma_R \cup_{\gamma} \Delta_n
\]
where $\Delta_k$ is the disk with $k$ marked boundary intervals. By minimality of these surfaces, we must have $(m - 2) + (n - 2) = N - 1$.

Let us denote the triangulated closure of the object represented in an arc system by $\langle A \rangle$. Then we have $\mathcal{F}(\hat{\Sigma}_L) = \langle A_L \sqcup A_\gamma \rangle$ and $\mathcal{F}(\hat{\Sigma}_R) = \langle A_R \sqcup A_\gamma \rangle$. Using the cosheaf description above we can assemble all these categories into the following cube diagram:

\[
\begin{array}{c}
\langle A_\gamma \rangle \quad \xrightarrow{\mathcal{F}} \quad \mathcal{F}(\hat{\Sigma}_R) \\
\mathcal{F}(\hat{\Sigma}_L) \quad \xrightarrow{\mathcal{F}(\gamma)} \quad \mathcal{F}(\Sigma_R) \\
\mathcal{F}(\Sigma_L) \quad \xrightarrow{\mathcal{F}(\Sigma)} \quad \mathcal{F}(\Sigma)
\end{array}
\]

where the inner and outer squares, and the top and left sides are all pushouts (i.e. homotopy colimits).
3.2. Main definitions. Consider now some surface $S$ with an embedded interval $\gamma$ which connects two adjacent marked boundary intervals $M, M'$, and runs parallel to the unmarked boundary interval between them (for example we can take $(S, \gamma) = (\Sigma_L, \gamma)$ as above).

**Definition 6.** A relative stability condition on the pair $(S, \gamma)$ is the data of:

- A surface $\tilde{S} = S \cup \Delta_n$ where $\Delta_n$ is a disk with $n$ marked boundary intervals, with a given inclusion map $S \hookrightarrow \tilde{S}$,
- A stability condition $\tilde{\sigma} \in \text{Stab}(\mathcal{F}(\tilde{S}))$.

Note that the first condition implies that the embedded interval $\gamma \subset \tilde{S}$ cuts the surface into two, so by Lemma 12 any indecomposable object $C$ supported on $\gamma$ has a simple cosupport decomposition under $\tilde{\sigma}$.

Fix a relative stability condition $\sigma = (\Sigma, P)$ and let us denote by $C_1, \ldots, C_N$ the corresponding chain of stable intervals in the decomposition of $C$, supported on arcs $\gamma_1, \ldots, \gamma_N$. As in the previous subsection, we can take $(\Sigma_L, \Sigma_R) = (S, \Delta_n)$; this defines an arc system $\mathcal{A}_L \cup \mathcal{A}_\gamma \cup \mathcal{A}_R$ on $\tilde{S}$.

3.3. Restricting stability conditions and minimality. Consider now the central charges $Z_L = Z|_{\langle \mathcal{A}_L \cup \mathcal{A}_\gamma \rangle}$, $Z_R = Z|_{\langle \mathcal{A}_\gamma \cup \mathcal{A}_R \rangle}$ and the ‘candidates for slicings’ $P_L, P_R$, given by intersecting the full triangulated subcategories $P_{\varphi}$ with the full triangulated subcategories $\langle \mathcal{A}_L \cup \mathcal{A}_\gamma \rangle$, $\langle \mathcal{A}_\gamma \cup \mathcal{A}_R \rangle$, respectively.

**Lemma 19.** $\sigma|_L = (Z_L, P_L)$ and $\sigma|_R = (Z_R, P_R)$ give stability conditions on the subcategories $\langle \mathcal{A}_L \cup \mathcal{A}_\gamma \rangle$ and $\langle \mathcal{A}_\gamma \cup \mathcal{A}_R \rangle$.

**Proof.** The compatibility between the central charges and filtrations is obvious by construction; we only need to check that $P_L, P_R$ do in fact give slicings, i.e. that every object in either category has an HN decomposition by objects in each restricted slicing. This can be checked on indecomposable objects and follows from Lemma 10 every indecomposable object on either side can be represented by some immersed curve keeping to the same side of the chain $\gamma$, so therefore its HN decomposition under the original stability condition $\sigma$ cannot cross to the other side. \hfill $\square$

Note that this construction $\sigma \rightarrow (\sigma|_L, \sigma|_R)$ does not give a map from the entire stability space $\text{Stab}(\mathcal{F}(\tilde{S}))$ to any other stability space; as $\sigma$ varies, the target categories $\langle \mathcal{A}_L \cup \mathcal{A}_\gamma \rangle$ change since the decomposition of the interval object $C$ changes as we cross a wall. However, this only happens across some specific kinds of walls, defined by the following condition:

**Definition 7.** The relative stability condition $\sigma$ is non-reduced if there are two interval objects $C_i, C_{i+1}$ extended on the right (i.e. by an extension map $C_{i+1} \rightarrowtail C_i$), with the same phase.

By standard results [11], the subset of non-reduced stability conditions is contained in a locally finite union of walls of $\text{Stab}(\mathcal{F}(\tilde{S}))$ walls, so the subset of reduced stability conditions is composed of open chambers.

**Lemma 20.** Within each chamber $\mathcal{C}$ of reduced relative stability conditions, the target subcategory $\langle \mathcal{A}_L \cup \mathcal{A}_\gamma \rangle$ is constant and the map $\text{Stab}(\mathcal{F}(\tilde{S})) \rightarrow \text{Stab}(\langle \mathcal{A}_L \cup \mathcal{A}_\gamma \rangle)$ is continuous.

**Proof.** Within each reduced chamber $\mathcal{C}$, the chain $\gamma$ is constant except for the (internal) walls on which two (or more) adjacent interval objects of the same phase $C_i, C_{i+1}$ are extended on the left (i.e. by an extension map $C_i \rightarrowtail C_{i+1}$). However, though the chain $\mathcal{A}_\gamma$ changes across such a wall, by construction of $\mathcal{A}_L$ we see that $\langle \mathcal{A}_\gamma \cup \mathcal{A}_L \rangle$ stays constant. Continuity follows from the fact that a small enough neighborhood of every stability condition on some category $\mathcal{D}$ is isomorphic to $(K_0(\mathcal{D}))^\vee = \text{Hom}_{\mathbb{Z}}(K_0(\mathcal{D}), \mathbb{C})$ and in that neighborhood the map $\text{Stab}(\mathcal{F}(\tilde{S})) \rightarrow \text{Stab}(\langle \mathcal{A}_L \cup \mathcal{A}_\gamma \rangle)$ is described by the projection dual to the inclusion $K_0(\langle \mathcal{A}_L \cup \mathcal{A}_\gamma \rangle) \rightarrow K_0(\mathcal{F}(\tilde{S}))$. \hfill $\square$

For our later uses, we would like to define a notion of minimality, in the sense that the integer $n$ of marked boundary intervals of $\Delta_n$ is as small as possible.

**Definition 8.** A relative stability condition $\sigma$ on $(S, \gamma)$ minimal if every marked boundary interval of $\Delta_n$ appears in the simple chain of stable intervals decomposition of $C$.

Another way of phrasing the minimality condition is:

**Lemma 21.** $\sigma$ is minimal if and only if $\langle \mathcal{A}_R \rangle \subseteq \langle \mathcal{A}_\gamma \rangle$. 
The space of relative stability conditions. For our purposes, the part of the stability condition ‘on the disk side’ does not matter; we realize this by using an equivalence relation. Let \( \sigma \in \text{Stab}(F(\tilde{S} = S \cup \Delta_m)) \) and \( \sigma' \in \text{Stab}(F(\tilde{S}' = S' \cup \Delta_m)) \) be two relative stability conditions on \((S, \gamma)\). As above, one can (non-uniquely) pick corresponding arc systems \( A_L \sqcup A_\gamma \sqcup A_R \) and \( A'_L \sqcup A'_\gamma \sqcup A'_R \) on \( \tilde{S} \) and \( \tilde{S}' \), and restrict stability conditions to each side.

We will see that we need to be careful about genericity when defining the correct equivalence relation. For motivation let us first define a naive notion of equivalence:

**Definition 9.** (Naive equivalence) \( \sigma \sim_{\text{naive}} \sigma' \) if there is an equivalence of categories \( \langle A_L \sqcup A_\gamma \rangle \sim \langle A'_L \sqcup A'_\gamma \rangle \) (compatible with the embedding of \( F(S) \) on both sides) such that the restricted stability conditions \( \sigma|_L \) and \( \sigma'|_L \) agree.

It is clear from the definition above that \( \sim_{\text{naive}} \) defines an equivalence relation on the set of relative stability conditions on \((S, \gamma)\). We would like to define the space of relative stability conditions as the quotient of the space \( S = \bigsqcup_{n \geq 2} \text{Stab}(F(S \cup_\gamma \Delta_n)) \) by the relation \( \sim_{\text{naive}} \), but it turns out that this space is ill-behaved. For instance, it is not Hausdorff, because the graph \( \Gamma_{\sim_{\text{naive}}} \subset S \times S \) of the naive relation is not a closed subset.

**Example.** Take the simple example where \( S \cong \Delta_2 \) with unique (up to shift) indecomposable object \( C \) and \( \tilde{S} \cong \tilde{S}' \cong \Delta_3 \), with objects \( A, B, C \) as below.

![Figure 11](image.png)

**Figure 11.** The surfaces \( S \cong \Delta_2 \) and \( \tilde{S} \cong \tilde{S}' \cong \Delta_3 \). The category \( F(S) \) is equivalent to \( \text{Mod}(A_1) \) and \( F(\tilde{S}) \) is equivalent to \( \text{Mod}(A_2) \).

We have a distinguished triangle \( A \to C \to B \). Consider two infinite families of stability conditions on \( F(\Delta_3) \), \( \{\sigma_m = (Z_m, P_m)\} \) and \( \{\sigma'_m = (Z'_m, P'_m)\} \) with \( m \in \mathbb{Z}_+ \), on \( F(\Delta_3) \) given by the central charges

\[
Z_m(A) = \frac{1}{3} + \frac{1}{m}, \quad Z_m(B) = \frac{2}{3} - \frac{1}{m} \\
Z'_m(A) = \frac{2}{3} + \frac{1}{m}, \quad Z'_m(B) = \frac{1}{3} - \frac{1}{m}
\]

with \( A, B \) and \( C \) stable in all of them, picking phases for all these objects between \(-1/2\) and \(1/2\). Each one of these sequences converges in \( \text{Stab}(F(\Delta_3)) \) respectively, to the stability conditions \( \sigma_\infty, \sigma'_\infty \) with central charges

\[
Z_\infty(A) = \frac{1}{3}, \quad Z_\infty(B) = \frac{2}{3} \\
Z'_\infty(A) = \frac{2}{3}, \quad Z'_\infty(B) = \frac{1}{3}
\]

where \( A, B \) are stable but \( C \) is only semistable, with Jordan-Hölder factors \( A, B \).

Seen as relative stability conditions on \((\Delta_2, \gamma)\), all the \( \sigma_m, \sigma'_m \) for any \( m \) are equivalent under \( \sim_{\text{naive}} \); the subcategory \( \langle A_L \sqcup A_\gamma \rangle \) is \( F(\Delta_2) = \langle C \rangle \) and the central charge of \( C \) is 1 for all finite \( m \). On the other hand,
\( \sigma_\infty \) and \( \sigma'_\infty \) are not equivalent under \( \sim_{naive} \), since for those two \( \langle A_L \sqcup A'_\gamma \rangle \) is the whole category. Thus \((\sigma_\infty, \sigma'_\infty) \in \Gamma_{naive} \setminus \Gamma_{naive} \).

As in the example above, the problem always arises when we have relative stability conditions which are non-reduced. Consider a relative condition \( \sigma \) on \( (S, \gamma) \) given by a stability condition on \( F(S) \) for some \( \tilde{S} = S \cup_\gamma \Delta_n \), where the object \( C \) supported on \( \gamma \) has a HN decomposition \( C_1, \ldots, C_N \). Assume that \( \sigma \) is non-reduced; this means that there is a nonempty set of indices \( R \subset \{1, \ldots, N\} \) such that the extension map \( \gamma \) is on the right (ie. \( \in Ext^1(C_{i+1}, C_i) \)) and \( C_i \) and \( C_{i+1} \) have the same phase. Let us suppose that the set \( R \) is of the form \( j, j+1, \ldots, j+m \) for some \( 1 \leq j \leq j+m \leq N-2 \) with all objects \( C_j, \ldots, C_{j+m+1} \) having the same phase \( \delta \); the general case (where \( R \) is the disjoint union of a number of those subsets) will not be any more difficult.

Consider now the reduced arc system given by
\[
\mathcal{A}_\gamma^{\text{red}} = \{ \gamma_1, \ldots, \gamma_{j-1}, \tilde{\gamma}, \gamma_{j+m+2}, \ldots, \gamma_N \},
\]
where \( \tilde{\gamma} \) is obtained by concatenating the intervals \( \gamma_j \ldots, \gamma_{j+m+1} \) at the \( m \) marked boundaries \( M_i \) with index \( i \in R \). Let us now define a reduced restriction \( \sigma^{\text{red}} \) given by restricting the data of \( \sigma \) to the subcategory \( \langle A_L \sqcup A_\gamma^{\text{red}} \rangle \), and then adding to the category \( P_\delta \) the objects supported on \( \tilde{\gamma} \).

**Lemma 22.** \( \sigma^{\text{red}} \) is a stability condition.

**Proof.** It suffices to prove that every object in the subcategory \( \langle A_L \sqcup A_\gamma^{\text{red}} \rangle \) has an HN decomposition into stable objects also in that same subcategory. Because of Lemma 10 the only way this could fail is if there is some indecomposable object \( X \) of \( \langle A_L \sqcup A_\gamma^{\text{red}} \rangle \) in whose decomposition some but not all of the stable interval objects \( C_j, \ldots, C_{j+m+1} \) appear (if all of them appear we just replace that semistable object with the stable object \( \tilde{C} \) supported on \( \tilde{\gamma} \)). But this cannot happen for phase reasons, following a similar argument as the proof of Lemma 10. \( \square \)

For completeness let us define \( \sigma^{\text{red}} = \sigma|_L \) if \( \sigma \) is reduced. With this definition we can now state the correct notion of equivalence.

**Definition 10.** (Equivalence) \( \sigma \sim \sigma' \) if there is an equivalence of categories
\[
\langle A_L \sqcup A_\gamma^{\text{red}} \rangle \cong \langle A'_L \sqcup A'_{\gamma'}^{\text{red}} \rangle
\]
(compatible with the embedding of \( F(S) \) on both sides) such that the reduced restricted stability conditions \( \sigma^{\text{red}} \) and \( \sigma'^{\text{red}} \) agree.

It is clear from the definition that \( \sim \) is an equivalence relation on the set \( S = \bigsqcup_{n \geq 2} \text{Stab}(F(S \cup_\gamma \Delta_n)) \).

**Lemma 23.** There is a unique minimal and reduced relative stability condition in each equivalence class of the equivalence relation \( \sim \).

**Proof.** Consider some relative stability condition \( \sigma \); as above it defines a stability condition \( \sigma^{\text{red}} \) on the subcategory \( \langle A_L \sqcup A_\gamma^{\text{red}} \rangle \). Note that this subcategory is also of the form \( F(S \cup_\gamma \Delta_n) \), with \( n = |A_\gamma^{\text{red}}| + 1 \), and also by construction \( \sigma \) is equivalent to the reduced \( \sigma^{\text{red}} \) when both are viewed as relative stability conditions on \( (S, \gamma) \).

Suppose now that we have two stability conditions \( \sigma \sim \sigma' \) which are minimal and thus reduced; then the arcs in \( A_R, A'_R \) can be generated by the other arcs so by compatibility we have
\[
F(\tilde{S}) \cong \langle A_L \sqcup A_\gamma \rangle \cong \langle A'_L \sqcup A'_{\gamma'} \rangle \cong F(\tilde{S'}),
\]
but it is easy to see that no two categories \( F(S \cup_\gamma \Delta_n) \) are equivalent for different \( n \) (for example by taking \( K_0 \) so \( \tilde{S} \cong \tilde{S'} \) compatibly with the embedding of \( S \) with equivalent stability conditions. \( \square \)

**Definition 11.** (Space of relative stability conditions) Let us define \( \text{RelStab}(S, \gamma) \) as the set of minimal and reduced stability conditions; this set is given the quotient topology by the identification \( \text{RelStab}(S, \gamma) = S/ \sim \).

**Proposition 24.** The space \( \text{RelStab}(S, \gamma) \) is Hausdorff.
Proof. This is equivalent to showing that the graph $\Gamma_{\sim}$ of the equivalence relation is closed in $\mathbb{S} \times \mathbb{S}$. Since $\mathbb{S}$ is an disjoint union this is equivalent to showing $\Gamma_{\sim}$ is closed in each component $\text{Stab}(\mathcal{F}(\tilde{S})) \times \text{Stab}(\mathcal{F}(\tilde{S}'))$.

The spaces $\text{Stab}(\mathcal{F}(\tilde{S}))$ have a wall-and-chamber structure where the walls are the locus of non-reduced stability conditions. By standard arguments, the union of all walls is a locally finite union of real codimension one subsets. The complement is composed of open chambers, and by Lemma 20 the target subcategory $\mathcal{T} = (\mathcal{A}_L \sqcup \mathcal{A}_r)$ is constant on each chamber.

In the interior of each chamber

$$C = C_{\sigma} \times C_{\sigma} \subset \text{Stab}(\mathcal{F}(\tilde{S})) \times \text{Stab}(\mathcal{F}(\tilde{S}')),$$

the locus $\Gamma_{\sim}$ is the preimage of the diagonal $\Delta \subset \text{Stab}(\mathcal{T}) \times \text{Stab}(\mathcal{T})$, so it is closed by continuity.

Let us look at the walls surrounding the chamber $C$, and start with a simple codimension one wall $W$, ie. the locus at the boundary of $C$ where the phases $\phi_i, \phi_{i+1}$ of two adjacent interval objects $C_i, C_{i+1}$ (with an extension to the right) agree. There are two possibilities: $\phi_i < \phi_{i+1}$ or $\phi_i > \phi_{i+1}$ inside of $C$. In the former case, comparing the target categories we see that the reduced target category $\mathcal{T}_W^{\text{red}}$ on the wall is equal to the usual target category $\mathcal{T}_C$ in the interior of the chamber, so we can apply the same argument as inside the chamber and conclude that $\Gamma_{\sim} \cap W$ is closed.

In the latter case $\mathcal{T}_W^{\text{red}}$ is smaller than $\mathcal{T}_C$, as it doesn’t contain the objects $C_i, C_{i+1}$, only their extension. However, the closure $\Gamma_{\sim} \cap C$ meets $W$ along a closed locus contained within $\Gamma_{\sim} \cap W$, as the reduced equivalence condition is strictly weaker than the naive equivalence condition on $W$. The general case for walls of higher codimension is essentially the same and can be obtained iteratively.

Now, over the entire space $\text{Stab}(\mathcal{F}(\tilde{S})) \times \text{Stab}(\mathcal{F}(\tilde{S}'))$, since each point is surrounded by finitely many reduced chambers and $\Gamma_{\sim}$ is closed within the closure of each one of them, $\Gamma_{\sim}$ is the locally finite union of closed subsets.

Remark. Unlike the space of stability conditions $\text{Stab}(\mathcal{F}(\mathcal{S}))$, the space $\text{RelStab}(\mathcal{S},\gamma)$ is not a complex manifold; in fact it is a stratified space, with cells of unbounded dimension.

3.5. Compatibility. Consider now two surfaces $S$ and $S'$ with embedded intervals $\gamma, \gamma'$ and relative stability conditions $\sigma \in \text{RelStab}(\mathcal{S}, \gamma)$ and $\sigma' \in \text{RelStab}(\mathcal{S}', \gamma')$. Given any two such surfaces, we can glue them by identifying $\gamma = \gamma'$ and obtain a surface $S \cup_{\gamma} S'$. Since there is a full arc system on this surface containing the arc $\gamma$, one can take the ribbon graph dual to this arc system and get a pushout presentation

$$\mathcal{F}(S \cup_{\gamma} S') = \mathcal{F}(S) \cup_{\mathcal{F}(\gamma)} \mathcal{F}(S').$$

The relative stability conditions $\sigma, \sigma'$ have unique minimal and reduced representatives by Lemma 23. However they also have many minimal but non-reduced representatives.

Definition 12. A compatibility structure between $\sigma$ and $\sigma'$ is the following data:

- Minimal representatives $\tilde{\sigma} \in \text{Stab}(\mathcal{F}(\tilde{S}))$ and $\tilde{\sigma}' \in \text{Stab}(\mathcal{F}(\tilde{S}'))$ of $\sigma$ and $\sigma'$.
- Inclusions of surfaces

$$S \hookrightarrow \tilde{S} \hookrightarrow S \cup_{\gamma} S', \quad S' \hookrightarrow \tilde{S}' \hookrightarrow S \cup_{\gamma} S',$$

such that the images of the embedded intervals in the cosi decompositions of $\gamma$ and $\gamma'$ agree as an arc system $\mathcal{A}_{\gamma}$ inside of $S \cup_{\gamma} S'$, and the restrictions $\tilde{\sigma}|_{\langle \mathcal{A}_{\gamma} \rangle}$ and $\tilde{\sigma}'|_{\langle \mathcal{A}_{\gamma} \rangle}$ are the same stability condition in $\text{Stab}(\langle \mathcal{A}_{\gamma} \rangle)$.

4. Cutting and gluing relative stability conditions

In this section, we will explain how to cut (ordinary) stability conditions into relative stability conditions and glue relative stability conditions into (ordinary) stability conditions. This will allow us to reduce the calculations of stability conditions on general surfaces $\Sigma$ to the calculation of stability conditions on simpler surfaces. Before we present these results, we will need to use the following generalization of a slicing.

Definition 13. A pre-slicing $\mathcal{P}_{\text{pre}}$ on a category $\mathcal{C}$ is a choice of full triangulated subcategories $\mathcal{P}_{\text{pre}}^\phi$ for every $\phi \in \mathbb{R}$, such that $\text{Hom}(X, Y) = 0$ if $X \in \mathcal{P}_{\text{pre}}^\phi$ and $Y \in \mathcal{P}_{\text{pre}}^\psi$, $\phi > \psi$.

Remark. This is the same data as a slicing, except that we don’t require the existence of Harder-Narasimhan decompositions for objects.

Definition 14. A pre-stability condition on $\mathcal{C}$ is the data of a central charge function $Z : K_0(\mathcal{C}) \rightarrow \mathbb{C}$ and a pre-slicing $\mathcal{P}_{\text{pre}}$ satisfying the usual compatibility condition $Z(X)/|Z(X)| = e^{i\pi \phi}$ if $X \in \mathcal{P}_{\text{pre}}^\phi$. **
Let us denote by $\text{PreStab}(\mathcal{C})$ the set of all pre-stability conditions on $\mathcal{C}$. It is obvious that we have an inclusion of sets

$$\text{Stab}(\mathcal{C}) \hookrightarrow \text{PreStab}(\mathcal{C}).$$

### 4.1. Cutting stability conditions.

We return to the setting of a surface $\Sigma$ that is cut into $\Sigma_L, \Sigma_R$ by an embedded interval $\gamma$ supporting a rank one object $C$.

Consider a stability condition $\sigma \in \text{Stab}(\mathcal{F}(\Sigma))$. By Corollary 16 the object $C$ has a simple cost decomposition into objects $C_1, \ldots, C_N$ supported on arcs $\gamma_1, \ldots, \gamma_N$, which connect the marked boundary intervals $M_0, \ldots, M_N$. As in subsection 3.1 there is then a full system of arcs

$$A = A_L \sqcup A_\gamma \sqcup A_R$$

such that every arc in $A_L$ has a representative contained in $\Sigma_L$, every arc in $A_R$ has a representative contained in $\Sigma_R$, and $A_\gamma = \{\gamma_1, \ldots, \gamma_N\}$.

Each extension between $C_i$ and $C_{i+1}$ happens either on the left (ie. by an extension map $C_i \xrightarrow{+1} C_{i+1}$) or on the right (ie. by an extension map $C_{i+1} \xrightarrow{-1} C_i$). Let $m, n$ be the numbers of indices with extension on the left and right, respectively, plus 2; we have by definition $m - 2 + n - 2 = N + 1 = \text{number of marked boundary intervals along the chain}$.

Then we have surfaces $\tilde{\Sigma}_L = \Sigma_L \sqcup_\gamma \Delta_m$ and $\tilde{\Sigma}_R = \Sigma_R \sqcup_\gamma \Delta_n$ such that

$$\mathcal{F}(\tilde{\Sigma}_L) = \langle A_L \sqcup A_\gamma \rangle, \quad \mathcal{F}(\tilde{\Sigma}_R) = \langle A_R \sqcup A_\gamma \rangle.$$ 

Consider the restrictions

$$\sigma_L = \sigma|_{\langle A_L \sqcup A_\gamma \rangle}, \quad \sigma_R = \sigma|_{\langle A_R \sqcup A_\gamma \rangle}$$

that is, as in the previous section we take the data given by restricting the central charges and intersecting the slicings with each full subcategory.

**Lemma 25.** $\sigma_L, \sigma_R$ are stability conditions on $\mathcal{F}(\tilde{\Sigma}_L), \mathcal{F}(\tilde{\Sigma}_R)$.

**Proof.** The condition $Z(X) = m(X) \exp(i\pi \phi_X)$ on every semistable object $X$ is satisfied by construction, so we just need to check that every object $X \in \mathcal{F}_L$ has a HN filtration, i.e. that $\mathcal{P}_L$ indeed defines a slicing.

It is enough to check this on indecomposable objects. By geometricity, every such object $X$ is represented by an immersed curve in $\tilde{\Sigma}_L$ with indecomposable local system. Consider its image in $\mathcal{F}(\Sigma)$ which is also an immersed curve, and its chain-of-interval decomposition under $\sigma$.

If $X$ is an interval object, then both of its ends are on marked boundary components belonging to $\tilde{\Sigma}_L$, and since the associated chain of intervals is isotopic to the support of $X$, if any of those intervals in in $\Sigma_R$, then the chain must cross back to $\Sigma_L$, creating a polygon of the sort prohibited by Lemma 10. And if $X$ is a circle object then it is by definition supported on a non-nullhomotopic immersed circle, so by the same argument its chain of intervals cannot cross over to $\Sigma_R$ without also creating a prohibited polygon. Thus every stable component of the HN decomposition is in $\mathcal{F}_L$.

We then use the inclusions of marked surfaces $\Sigma_L \hookrightarrow \tilde{\Sigma}_L$ and $\Sigma_R \hookrightarrow \tilde{\Sigma}_R$ to interpret these stability conditions as relative stability conditions:

**Definition 15.** The cutting map

$$\text{cut}_\gamma : \text{Stab}(\mathcal{F}(\Sigma)) \to \text{RelStab}(\Sigma_L, \gamma) \times \text{RelStab}(\Sigma_R, \gamma)$$

sends a stability conditions $\sigma$ as above to the image of the stability conditions $(\sigma_L, \sigma_R)$.

By Lemma 23 every element of RelStab has a unique minimal and reduced representative, so we can alternatively define the cutting map by using the ‘reduced restriction’ of Lemma 22

$$\text{cut}_\gamma^\text{red} : (\sigma_L^\text{red}, \sigma_R^\text{red}).$$

**Lemma 26.** The map $\text{Stab}(\mathcal{F}(\Sigma)) \xrightarrow{\text{cut}_\gamma} \text{RelStab}(\Sigma_L, \gamma) \times \text{RelStab}(\Sigma_R, \gamma)$ is continuous.

**Proof.** We must look separately at the maps to each side; let us prove continuity of the map $\text{Stab}(\mathcal{F}(\Sigma)) \xrightarrow{\text{cut}_L} \text{RelStab}(\Sigma_L, \gamma)$. Recall that in subsection 3.4 we define the topology on the RelStab spaces as the quotient topology inherited from $S = \bigsqcup_n \text{Stab}(S \cup_\gamma \Delta_n)$. 


Note that the construction for the map $cut_L$ does not give a manifestly continuous map since the target $\mathcal{T} = \langle \mathcal{A}_L \cup \mathcal{A}_R \rangle$ changes across walls in $\text{Stab}(\mathcal{F}(\Sigma))$. We remediate this by locally defining other maps that are continuous, and which agree with $cut_L$ after identifying by the equivalence relation $\sim$.

Let $\sigma$ be a stability condition on $\mathcal{F}(\Sigma)$ such that $\sigma_L = \sigma|_{\langle \mathcal{A}_L \cup \mathcal{A}_R \rangle}$ is a non-reduced stability condition, and let us say that under $\sigma$ the object $C$ supported on $\gamma$ has a decomposition into $C_1, \ldots, C_N$ supported on embedded intervals $\gamma_1, \ldots, \gamma_N$ with respective phases $\phi_1, \ldots, \phi_N$. Non-reducedness means that there is some collection of indices $i$ such that $C_i, C_{i+1}$ have the same phase, and are extended on the right. For simplicity, suppose first that we have a single such index; the general case can be deduced by iterating this argument. Let us denote $C_{\text{bot}}$ to be the object obtained by concatenating $C_1, \ldots, C_i$, and $C_{\text{top}}$ to be the object obtained by concatenating $C_{i+1}, \ldots, C_N$.

By standard arguments, the locus on which the objects $C_1, \ldots, C_N$ are simple is open, so there is a neighborhood $U \ni \sigma$ on which all these objects are simple, and with a complex isomorphism $U \cong (\mathcal{K}_0(\mathcal{F}(\Sigma)))^\vee$. If necessary we further restrict $U$ such that on this open set the $\phi_{i-1} \neq \phi_i$ and $\phi_{i+1} \neq \phi_i$. This implies that on $U$ the chains $C_1, \ldots, C_i$ and $C_{i+1}, \ldots, C_N$ gives cosi decompositions of $C_{\text{bot}}$ and $C_{\text{top}}$, respectively.

Consider now a fixed target category $\mathcal{T}_{\text{fix}}$ given by the target $\mathcal{T}_{\sigma} = \langle \mathcal{A}_L \cup \mathcal{A}_R \rangle$ at $\sigma$. We argue that for every stability condition $\sigma' \in U$, $\sigma'|_{\mathcal{T}_{\text{fix}}}$ is a stability condition. Note that this doesn’t follow immediately from Lemma 10 since along some chambers in $U$, the pair $C_i, C_{i+1}$ is not the cosi decomposion of any object so we cannot directly use the non-crossing argument.

Nevertheless, we can use a small modification of that argument. Consider some indecomposable object $X$ in the subcategory $\mathcal{T}_{\text{fix}}$; by geometricity this can be represented by an immersed curve $\xi$ to the left of the chain of intervals, and by the results of Section 2 $X$ has a cosi decomposition into intervals $\xi_1, \ldots, \xi_M$ whose concatenation is isotopic to $\xi$.

Now, since both ends of $\xi$ are to the left of the $\gamma$ chain, and this chain is divided into two stable chains, extended on the left, the only way that the $\xi$ chain can cross the $\gamma$ chain is if it crosses the chain for $C_{\text{bot}}$ or $C_{\text{top}}$ (or both). But again this is prohibited by the noncrossing argument of Lemma 10.

Thus this defines a map $cut_{\gamma} : U \to \text{Stab}(\mathcal{T}_{\text{fix}})$ which by construction is continuous and agrees with $cut_{\gamma}$ on $U$; doing this for every wall gives continuity of $cut_{\gamma}$.

Note that by construction we have representatives $\sigma_L \in \text{Stab}(\mathcal{F}(\Sigma_L))$ and $\sigma_R \in \text{Stab}(\mathcal{F}(\Sigma_R))$ of the relative stability conditions $\sigma_{\text{red}}^L, \sigma_{\text{red}}^R$, and also inclusions of surfaces $\Sigma_L \hookrightarrow \Sigma$ and $\Sigma_R \hookrightarrow \Sigma$. It follows directly from the construction above that:

**Lemma 27.** This is a compatibility structure between $\sigma_{\text{red}}^L$ and $\sigma_{\text{red}}^R$.

### 4.2. Gluing stability conditions.

As in the previous section consider a surface cut into two parts by an embedded interval $\Sigma = \Sigma_L \cup \Sigma_R$. Suppose we have relative stability conditions $\sigma_L \in \text{RelStab}(\Sigma_L, \gamma)$ and $\sigma_R \in \text{RelStab}(\Sigma_R, \gamma)$ with some compatibility structure between them (as in Definition 12).

Unpacking this data, we have non-negative integers $m$ and $n$ and stability conditions $\sigma_L = (Z_L, \mathcal{P}_L)$ on

\[ \mathcal{F}_L = \mathcal{F}(\Sigma_L) = \mathcal{F}(\Sigma_L \cup_\gamma \Delta_m) \]

and $\sigma_R = (Z_R, \mathcal{P}_R)$ on

\[ \mathcal{F}_R = \mathcal{F}(\Sigma_R) = \mathcal{F}(\Sigma_R \cup_\gamma \Delta_n) \]

representing $\sigma_L, \sigma_R$, together with inclusions of marked surfaces $\Sigma_L \hookrightarrow \Sigma_L \hookrightarrow \Sigma$ and $\Sigma_R \hookrightarrow \Sigma_R \hookrightarrow \Sigma$.

The compatibility condition implies that the chain-of-intervals decomposition $C_1^L, \ldots, C_N^L$ of the indecomposable object $C^L \in \mathcal{F}_L$ supported on $\gamma \subset \Sigma_L$ and the chain-of-intervals decomposition $C_1^R, \ldots, C_N^R$ of the indecomposable object $C^R \in \mathcal{F}_R$ supported on $\gamma \subset \Sigma_R$ are of the same length $N$ on both sides, and that the central charges agree, ie.

\[ Z_L(C_i^L) = Z_R(C_i^R) \]

for all $i$. Also compatibility also requires that the extension maps $\eta^L_i$ and $\eta^R_i$ go the same direction, ie. either both go forward

\[ \eta^L_i \in \text{Ext}^1(C_i^L, C_{i+1}^L) \quad \text{and} \quad \eta^R_i \in \text{Ext}^1(C_i^R, C_{i+1}^R) \]

or both go backward

\[ \eta^L_i \in \text{Ext}^1(C_{i+1}^L, C_i^L) \quad \text{and} \quad \eta^R_i \in \text{Ext}^1(C_{i+1}^R, C_i^R) \]

so we have the relation $(m - 2) + (n - 2) = N - 1$ due to minimality of $\sigma_L$ and $\sigma_R$. 

The compatibility structure gives an identification between the images of $C^\gamma_1,\ldots,C^\gamma_K$ and $C^\delta_1,\ldots,C^\delta_K$ inside of $F(\Sigma)$; we denote this full subcategory spanned by these arcs $(\mathcal{A}_\gamma)$ as in previous sections. This gives a pushout presentation

$$\begin{array}{ccc}
\langle \mathcal{A}_\gamma \rangle & \longrightarrow & F_R \\
\downarrow & & \downarrow \quad j_n \\
F_L & \longrightarrow & F(\Sigma)
\end{array}$$

From this data we will produce a central charge function $K_0(F(\Sigma)) \to \mathbb{C}$ and a pre-slicing $\mathcal{P}$ on $F(\Sigma)$.

4.2.1. The central charge. Applying the functor $K_0$ to the pushout above gives us a diagram of $\mathbb{Z}$-modules

$$\begin{array}{ccc}
K_0(\langle \mathcal{A}_\gamma \rangle) & \longrightarrow & K_0(F_R) \\
\downarrow & & \downarrow \\
K_0(F_L) & \longrightarrow & K_0(F(\Sigma))
\end{array}$$

Lemma 28. This is a pushout of $\mathbb{Z}$-modules.

Proof. A priori this need not be a pushout, since $K_0$ does not necessarily commute with colimits. However note that in this case we have an explicit description of the $K_0$ groups in terms of $H^1$ groups because of Theorem 4 and the result follows from the fact that we are gluing along a single chain.

More explicitly, note that $K_0(F(S))$ for some marked surface $S$ is generated by the arcs in an arc system modulo relations coming from polygons. Completing $\mathcal{A}_\gamma$ to a full arc system $\mathcal{A}_L \sqcup \mathcal{A}_\gamma \sqcup \mathcal{A}_R$ we see that since there are no polygons crossing between the two sides of the chain, so the set of relations on $K_0(F(\Sigma))$ is the union of the sets of relations defining $K_0(F_L)$ and $K_0(F_R)$; this implies the statement above.

By compatibility of the relative stability conditions $\sigma_L$ and $\sigma_R$, the central charges on both sides agree when restricted to $K_0(\langle \mathcal{A}_\gamma \rangle)$, so we get a map $Z : K_0(F(\Sigma)) \to \mathbb{C}$; this will be our central charge.

4.2.2. The pre-slicing. We will define full subcategories $\mathcal{P}_\phi$ of semistable objects in two steps. Let us first define initial subcategories $\mathcal{P}_\phi'$ by

$$\mathcal{P}_\phi' = j_L((\mathcal{P}_L)_\phi) \cup j_R((\mathcal{P}_R)_\phi)$$

ie. we take the images of the semistable objects under $\sigma_L$ and $\sigma_R$ to be stable in $F(\Sigma)$.

Now let us algorithmically add some objects to the slicing by the following prescription. We first define a particular kind of arrangement of stable objects. Remember that $M_0,\ldots,M_N$ are boundary components of $F(\Sigma)$ appearing in a chain of intervals that compose to $\gamma$. Let us partition $\mathcal{M} = \mathcal{M}_L \sqcup \mathcal{M}_r \sqcup \mathcal{M}_R$ where $\mathcal{M}_\gamma = \{M_0,\ldots,M_N\}$, $\mathcal{M}_L$ are the other boundary components coming from $\Sigma_L$ and $\mathcal{M}_R$ are the other boundary components coming from $\Sigma_R$.

Definition 16. A lozenge of stable intervals is the following arrangement of intervals:

- Four distinct marked boundary components $M_\ell, M_r, M_{up}, M_{down}$, where $M_\ell \in \mathcal{M}_L$, $M_r \in \mathcal{M}_R$, $M_{up}, M_{down} \in \mathcal{M}_\gamma$
- A chain of intervals $\alpha_1,\ldots,\alpha_a$ linking $M_\ell$ to $M_{up}$, such that $\alpha_i$ supports a stable object $A_i \in \mathcal{P}_{\phi(\mathcal{A}_\gamma)}'$, and $\text{phase}(A_1) \leq \cdots \leq \text{phase}(A_a)$
- A chain of intervals $\beta_1,\ldots,\beta_b$ linking $M_{up}$ to $M_r$, such that $\beta_i$ supports a stable object $B_i \in \mathcal{P}_{\phi(\mathcal{B}_\gamma)}'$, and $\text{phase}(B_1) \leq \cdots \leq \text{phase}(B_b)$
- A chain of intervals $\delta_1,\ldots,\delta_d$ linking $M_\ell$ to $M_{down}$, such that $\delta_i$ supports a stable object $D_i \in \mathcal{P}_{\phi(\mathcal{D}_\gamma)}'$, and $\text{phase}(D_1) \geq \cdots \geq \text{phase}(D_d)$
- A chain of intervals $\eta_1,\ldots,\eta_d$ linking $M_{down}$ to $M_r$, such that $\eta_i$ supports a stable object $E_i \in \mathcal{P}_{\phi(\mathcal{E}_\gamma)}'$, and $\text{phase}(E_1) \geq \cdots \geq \text{phase}(E_d)$
such that the phases of these stable objects satisfy
\[
\text{phase}(D_1) \leq \text{phase}(A_1) \leq \text{phase}(D_1) + 1, \quad \text{phase}(B_1) \leq \text{phase}(A_2) \leq \text{phase}(B_1) + 1, \\
\text{phase}(B_3) \leq \text{phase}(E_2) \leq \text{phase}(B_3) + 1, \quad \text{phase}(D_3) \leq \text{phase}(E_1) \leq \text{phase}(D_3) + 1.
\]
and such that these four chain of stable intervals bound a disk. This is pictured below in Figure 12 for ease of presentation.

Consider now the complex number
\[
Z(X) := \sum_i Z(A_i) + \sum_i Z(B_i) = \sum_i Z(D_i) + \sum_i Z(E_i),
\]
which is the central charge of the object \(X\) supported on the interval from \(M_\ell\) to \(M_r\) one gets by successive extensions of the \(A_i, B_i\) or \(D_i, E_i\). The equality follows from well-definedness of \(Z\).

**Definition 17.** We call such a lozenge unobstructed if there is a choice of branch of the argument function \(\arg : \mathbb{C}^\times \to \mathbb{R}\) such that the following inequalities between the phases are satisfied:
\[
\text{phase}(D_1) \leq \arg(Z(X)) \leq \text{phase}(A_1), \quad \text{phase}(B_6) \leq \arg(Z(X)) \leq \text{phase}(E_6).
\]

It follows from the inequalities above that if a lozenge is unobstructed then there is only a single choice of \(\arg(Z(X))\) satisfying the condition; let’s call it \(\phi_X \in \mathbb{R}\). For every unobstructed lozenge we find, let us declare that the corresponding \(X\) is semistable of phase \(\phi_X\). So we define \(\mathcal{P}_\phi\) to be spanned by all objects in \(\mathcal{P}_\phi\) plus all objects of phase \(\phi\) that we obtained from unobstructed lozenges.

**Lemma 29.** The data \(Z\) and \(\mathcal{P}\) as above define a prestability condition on \(\mathcal{F}(\Sigma)\).

**Proof.** The compatibility between the argument of \(Z\) and the phase of the subcategories \(\mathcal{P}\) is automatic from the definition, since every stable object either comes directly from one side or has central charge and phase defined by the formula above. So we only have to prove that \(\mathcal{P}\) is in fact a preslicing: we must show that \(\text{Hom}(X,Y) = 0\) if \(X \in \mathcal{P}_{\phi_X}\) and \(Y \in \mathcal{P}_{\phi_Y}\) with \(\phi_X > \phi_Y\).

By definition, each full subcategory \(\mathcal{P}_\phi\) can be spanned by three full subcategories
\[
\mathcal{P}_\phi^L = j_L((\mathcal{P}_L)_\phi), \quad \mathcal{P}_\phi^R = j_R((\mathcal{P}_R)_\phi), \quad \mathcal{P}_\phi^\circ,
\]
where $P^\phi$ has all the objects of phase $\phi$ obtained from unobstructed lozenges. Note that $P^\phi$ is disjoint from the other two, but $P^L_\phi$ and $P^R_\phi$ are not disjoint; in fact their intersection is spanned by the objects supported on the chain of intervals $\{\gamma_i\}$.

Let us check vanishing of the appropriate homs. It is enough to check on stable objects. If $X,Y \in P^L$ then

$\text{Hom}(X,Y) \neq 0 \Rightarrow \phi_X \leq \phi_Y$

automatically since they’re both semistable in $\mathcal{F}_L$ and $\mathcal{F}_L \rightarrow \mathcal{F}(\Sigma)$ is fully faithful; same for the case $X,Y \in P^R$. So there are four remaining cases:

1. $X \in P^L_\phi$ and $Y \in P^R_\phi$
2. $X \in P^L_\phi$ and $Y \in P^L_\phi$
3. $X \in P^R_\phi$ and $Y \in P^\phi$
4. $X \in P^\phi$ and $Y \in P^\phi$

All the other cases can be obtained symmetrically by switching left and right. Let us treat each case separately:

1. We can find representatives of $X,Y$ contained in the images of $\tilde{\Sigma}_L, \tilde{\Sigma}_R$ respectively, such that neither intersects the chain $\{\gamma_i\}$; so there are no intersections between them. The only way we can have $\text{Hom}(X,Y) \neq 0$ is if $X$ and $Y$ are intervals sharing a common boundary component at one of the $M_i$ along the chain, with a boundary path from $X$ to $Y$.

Consider then $C_i$ and shift its grading so that the morphism $X \rightarrow C_i$ is in degree zero; then by index arguments the morphism $C_i \rightarrow Y$ is also in degree zero. But since these three objects are stable we have

$\phi_X \leq \phi_{C_i} \leq \phi(Y)$.

2. Let $X$ be obtained from an unobstructed lozenge with notation as in Definition 16, and $Y \in \mathcal{F}_L$. Consider the distinguished triangle $B \rightarrow X \rightarrow A$ and let us apply the functor $\text{Hom}(\cdot, Y)$ to get a distinguished triangle

$\text{Hom}(A,Y) \rightarrow \text{Hom}(X,Y) \rightarrow \text{Hom}(B,Y)$.

Since $Y$ comes from $\mathcal{F}_L$, it has a representative that stays to the left of the chain and therefore of $B$ so by assumption we have $\text{Hom}(B,Y) = 0$. Thus if $\text{Hom}(X,Y) \neq 0$ then $\text{Hom}(A,Y) \neq 0$. Since $A$ is given by the iterated extension of the $A_i$, there must be some $A_i$ with $\text{Hom}(A_i,Y) \neq 0$; but $A_i$ and $Y$ are both in the image of $\mathcal{F}_L$ we must have $\phi_{A_i} \leq \phi_Y$, and also by construction $\phi_X \leq \phi_{A_i}$ so we have

$\phi_X \leq \phi_{A_i} \leq \phi_Y$. 


(3) Suppose we have an unobstructed lozenge with sides $A, B, D, E$ and diagonal $Y$. A similar argument as in case (2) shows that if $\text{Hom}(X, Y) \neq 0$, then $\text{Hom}(X, D) \neq 0$, and then for some $i$ we have $\text{Hom}(X, D_i) \neq 0$

$$\phi_X \leq \phi_{D_i} \leq \phi_{D_{i+1}} \leq \phi_Y.$$ 

(4) This case can be obtained by an iterated version of the argument in case (2). Let us denote the two lozenges by $A_X, B_X, D_X, E_X$ with diagonal $X$ and $A_Y, B_Y, D_Y, E_Y$ with diagonal $Y$. Suppose that $\text{Hom}(X, Y) \neq 0$, and consider the triangle $D_X \to X \to E_X$. Consider first the case $\text{Hom}(D_X, Y) = 0$ then $\text{Hom}(E_X, Y) \neq 0$. Now consider the triangle $B_Y \to Y \to A_Y$. Since $E_X$ and $A_Y$ have representatives contained in the right and the left side, respectively, and don’t share a boundary component we have $\text{Hom}(E_X, A_Y) = 0$ so we must have $\text{Hom}(E_X, B_Y) \neq 0$. But then there must be indices $i, j$ such that $\text{Hom}((E_X)_i, (B_Y)_j) \neq 0$ so then

$$\phi_X \leq \phi_{(E_X)_i} \leq \phi_{(B_Y)_j} \leq \phi_Y.$$ 

The other case is $\text{Hom}(D_X, Y) \neq 0$. Consider the triangle $B_Y \to Y \to A_Y$. By an analogous argument we can find indices $i, j$ such that

$$\phi_X \leq \phi_{(A_X)_i} \leq \phi_{(D_Y)_j} \leq \phi_Y.$$ 

\[\square\]

4.3. Uniqueness of compatibility structure. In the same setting as the previous subsection, let $\Gamma \subset \text{RelStab}(\Sigma_L, \gamma) \times \text{RelStab}(\Sigma_R, \gamma)$ be the locus of pairs of relative stability conditions $(\sigma_L, \sigma_R)$ such that there exists a compatibility condition between $\sigma_L$ and $\sigma_R$.

**Lemma 30.** For each $(\sigma_L, \sigma_R) \in \Gamma$, there is a unique compatibility structure between $\sigma_L$ and $\sigma_R$ up to equivalence.

**Proof.** Let us first prove that the numbers $m, n$ defining $\Sigma_L, \Sigma_R$ are unique. Consider the subset

$$M_\sigma \subset S = \bigsqcup_{n \geq 2} \text{Stab}(F(\Sigma_L \cup _\gamma \Delta_n))$$

of its minimal (but possibly not reduced) representatives. Given $\tilde{\sigma} \in M_\sigma$ we consider the cost decomposition of the rank one object $C$ supported on $\gamma$ as before, and define the numbers $\text{int}(\tilde{\sigma}), \text{ext}(\tilde{\sigma})$ to be respectively the number of internal/external extensions in the $\gamma$ chain, ie. the number of indices $i$ such that the corresponding extension happens on the left/right, ie. by an extension map $i \in \text{Ext}^1(C_{i+1}, C_i)/\in \text{Ext}^1(C_i, C_{i+1})$. This defines constructible functions $\text{int}, \text{ext} : M_\sigma \to \mathbb{Z}_{\geq 0}$ such that $\text{int}(\tilde{\sigma}) + \text{ext}(\tilde{\sigma}) = N - 1$, where $N - 1$ is the total length of the object $C$ under $\tilde{\sigma}$.

We argue that the function $\text{int}$ is constant; by Lemma [23] there is a unique minimal and reduced representative $\sigma_\text{red}$ of every relative stability condition. However, reduced restriction does not change the $\text{int}$ of a stability condition, so $\text{int}(\tilde{\sigma}) = \text{int}(\sigma_\text{red}) = \text{int}(\sigma_\text{red})$ on all of $M_\sigma$. We define the same functions on the right side for the relative stability condition $\sigma' \in \text{RelStab}(\Sigma_R, \gamma)$. Compatibility implies that $\text{int}(\tilde{\sigma}) = \text{ext}(\tilde{\sigma}'), \text{ext}(\tilde{\sigma}) = \text{int}(\tilde{\sigma}')$, but since $\text{int}$ is constant there is only one possibility for the value of $\text{ext}$.

Comparing with the gluing map we have $m = \text{ext}(\tilde{\sigma}), n = \text{ext}(\tilde{\sigma}')$.

This determines the isomorphism type of the surfaces $\Sigma_L$ and $\Sigma_R$. Consider now the inclusion of marked surfaces $j_L : \Sigma_L \hookrightarrow \Sigma_L \cup _\gamma \Sigma_R$. By definition of compatibility structure, $j_L|\Sigma_L$ agrees with the inclusion $\Sigma_L \hookrightarrow \Sigma_L \cup _\gamma \Sigma_R$, so the ‘left part’ of $j_L$ is fixed; $j_L$ is determined up to equivalence by the images of the extra $m - 2$ marked boundary intervals in the disk $\Delta_m$ attached along $\gamma$ (two of the marked boundary intervals are fixed to the ends of $\gamma$).

Analogously, $j_R$ is determined up to equivalence by the image of the extra $n - 2$ marked boundary intervals of $\Delta_n$. But the images of the extra $m - 2$ marked intervals under $j_L$ is contained in the image of the marked intervals coming from $\Sigma_R$ under $j_R$, so they are fixed; the same is true for the image of the extra $n - 2$ marked intervals under $j_R$. Minimality implies that the subcategory $\langle A_L \cup A_\gamma \rangle$ is the whole category $\mathcal{F}(\Sigma_R)$ so once we fix $\sigma$, the representative $\tilde{\sigma}$ is completely determined by its restriction to $\langle A_\gamma \rangle \cong \mathcal{F}(\Delta_{N+1})$.

By the classification of stability conditions on the Fukaya category of a disk presented in [20] Section 6.2, stability conditions on $\mathcal{F}(\Delta_{N+1})$ are entirely determined by the central charges and phases of the $N + 1$ intervals in the chain. Let us label the marked boundary intervals $M_0, \ldots, M_N$ in sequence. We argue that the central charges and phases of the objects $C_1, \ldots, C_N$ are unique using the following ‘zip-up’ procedure.
Consider first the object $C_1$; since $M_0$ is in the common image of $\Sigma_L$ and $\Sigma_R$, and $M_1$ is ‘internal’ (in the subset counted by the int function) to either of those surfaces, the interval supporting $C_1$ is contained in the image of either $\Sigma_L$ or $\Sigma_R$, so its central charge $Z(C_1)$ and phase $\phi_1$ are fixed by either $\sigma_L$ or $\sigma_R$.

Suppose without loss of generality that the interval supporting $C_1$ is in the image of $\Sigma_L$, and consider now $C_2$. There are two possibilities for $M_2$; either it is internal to $\Sigma_L$ or to $\Sigma_R$. In the former case since $M_1$ and $M_2$ are in the image of the same side $\Sigma_L$, $Z(C_2)$ and $\phi_2$ are fixed by $\sigma_L$. In the latter case, $C_2$ is not in the image of either $\Sigma_L$ or $\Sigma_R$, but we consider the concatenation $C_{1+2}$ given by extending at $M_1$; both ends of this object are in the image of $\Sigma_R$ so the central charge $Z(C_{1+2})$ of this (non-stable) object is fixed by $\sigma_R$. So $Z(C_2) = Z(C_{1+2}) - Z(C_1)$ is also fixed. Moreover, among the shifts of $C_2$, there is a unique one with the extension map at $M_1$ in the correct degree, so $\phi_2$ is also fixed. Proceeding by induction we find that all $Z(C_i), \phi_i$ are fixed by the initial data $\sigma_L, \sigma_R$.

4.4. Relation between cutting and gluing maps. Because of the uniqueness of compatibility structure proven above and Lemma 29, we can define a gluing map

$$\text{RelStab}(\Sigma_L, \gamma) \times \text{RelStab}(\Sigma_R, \gamma) \supset \Gamma \xrightarrow{\text{glue}} \text{PreStab}(\mathcal{F}(\Sigma \cup \gamma \Sigma_R))$$

which produces a prestability condition.

A priori it is not obvious whether these are actual stability conditions, however this can be shown to be the case when we start with an actual stability condition $\sigma \in \mathcal{F}(\Sigma)$.

**Theorem 31.** The composition

$$\text{Stab}(\mathcal{F}(\Sigma)) \xrightarrow{\text{cut}} \Gamma \xrightarrow{\text{glue}} \text{PreStab}(\mathcal{F}(\Sigma))$$

is equal to the canonical inclusion $\text{Stab}(\mathcal{F}(\Sigma)) \hookrightarrow \text{PreStab}(\mathcal{F}(\Sigma))$.

Note that the theorem can be also stated as saying that the gluing map lands in $\text{Stab}(\mathcal{F}(\Sigma))$ and gives an right-inverse to the cutting map. It is then immediate from the definitions that this is also a left-inverse; the cutting map forgets all the stable objects coming from the lozenges so the composition

$$\Gamma \xrightarrow{\text{glue}} \text{Stab}(\mathcal{F}(\Sigma)) \xrightarrow{\text{cut}} \Gamma$$

is the identity on pairs of compatible relative stability conditions.

We will need the following lemma in the proof of 31:

**Lemma 32.** Let $X$ be a stable interval object (under $\sigma$), with a representative that crosses the interval $\gamma$ once. Then there is an unobstructed lozenge (under $\sigma_L, \sigma_R$) with diagonal $X$. Conversely, the diagonal of every unobstructed lozenge is stable under $\sigma$.

**Proof.** Let $C_1, \ldots, C_N$ be the cosi decomposition of the object $C$ supported on $\gamma$. Note that $X$ cannot cross this chain multiple times, since this would create a polygon of the sort prohibited by Lemma 10. Let us say then that $X$ intersects one $C_j$ transversely. Then we have $\text{Ext}^1(C_j, X) \cong \text{Hom}(X, C_j) \cong k$; consider the corresponding extension and cone

$$C_j \to A \oplus E \to X, \quad B \oplus D \to X \to C_j.$$ 

Each one of the objects $A, B, D, E$ is an embedded interval object and by Proposition 13 has a cosi decomposition; we denote the objects in these chains by $\{A_i\}, \{B_i\}, \{D_i\}, \{E_i\}$, respectively.
We argue that \( \{ A_i \} \) and \( \{ B_i \} \) only have extensions on the right, and \( \{ D_i \}, \{ E_i \} \) only have extensions on the left. Note first that the chains of intervals \( \{ A_i \}, \{ D_i \} \) and the interval \( \gamma \) don’t intersect mutually, since this would contradict Lemma 10. Consider the chain made up of supports of the \( A_i \) and \( D_i \). This chain together with \( \gamma \) bounds a disk, therefore every extension is on the right; this translates to extensions on the right \( \in \text{Ext}^1(A_i, A_{i+1}) \) and extensions on the left \( \text{Ext}^1(D_{i+1}, D_i) \). An analogous argument applies to \( B \) and \( E \); note that since none of these chains crosses \( \gamma \), and \( \gamma \) separates \( \Sigma \), they do not intersect one another.

Thus we have a lozenge whose diagonal is \( X \); it remains to prove it is unobstructed. Suppose that the lozenge \( A, B, D, E \) is obstructed; therefore we must have at least one of the following inequalities
\[
\phi_{A_i} \leq \phi_X, \quad \phi_{D_i} \geq \phi_X, \quad \phi_{B_i} \geq \phi_X, \quad \phi_{E_i} \leq \phi_X.
\]

Suppose first that \( \phi_{A_i} < \phi_X \). Consider then the object \( X' \) given by the iterated extension of \( A_2, \ldots A_n, B_1, \ldots B_b \), we then have a distinguished triangle
\[
X' \rightarrow X \rightarrow A_i
\]
and the map \( X \rightarrow A_i \) cannot be zero since \( X' \) is indecomposable (by Theorem 5), which cannot happen since \( \phi_X > \phi_{A_i} \). The other cases are similar; moreover, the case of coinciding phases poses no further problems since we can always take \( \sigma \) to be appropriately generic (since we need to be off of finitely many walls).

This proves one of the directions. For the converse, suppose that we have an unobstructed lozenge \( A, B, D, E \) as above, with diagonal object \( X \) which is not stable. By construction \( X \) is an embedded interval, so it has a chain-of-interval decomposition \( \{ X_i \} \) under \( \sigma \). There are two mutually exclusive cases:

1. There are representatives for all the \( X_i \) contained in the lozenge, ie. contained in the disk bounded by the lozenge or running along its sides.
2. At least one of the representatives necessarily crosses out of the lozenge.

The concatenation of the chain \( \{ X_i \} \) is isotopic to the object \( X \). Therefore in case (2), if the chain crosses out of the lozenge along one of the sides it must cross back in, and along the same side, since each of the objects \( A, B, D, E \) cuts the surface into two. Therefore we have a configuration prohibited by Lemma 10.

As for case (1), every extension between \( X_i \) and \( X_{i+1} \) must happen at one of the marked components along the boundary of the lozenge. Note that even though the chain \( \{ X_i \} \) may not be simple (intervals could in principle double back), it must not cross itself by the same lemma, and therefore there are only two options: either \( X_i \) and \( X_{i+1} \) share a boundary component along the top of the lozenge (ie. along \( A \) or \( B \) sides) and the extension happens on the right, or it is along the bottom (ie. along \( D \) or \( E \) sides) and the extension happens on the left. Suppose that at least one of the intervals \( X_i \) ends on the \( A \) side; let \( i \) be maximal among such indices. Then \( X_{i+1} \) stretches between the \( A \) side and another side of the lozenge, however its phase is smaller than \( X_i \) so this contradicts the existence of a nontrivial extension on the right \( \in \text{Ext}^1(X_i, X_{i+1}) \). The same argument can be applied along any of the other sides, in the case where no interval ends on the \( A \) side. Therefore there cannot be more than one stable interval, and \( X \) itself is stable.

The lemma above should be interpreted as stating that the unobstructed lozenges “see” all the stable interval objects that were eliminated by cutting along \( \gamma \).

Proof. (of Theorem 31) For clarity let us denote \( \sigma = (Z, \mathcal{P}) \in \text{Stab}(\mathcal{F}(\Sigma)) \), \( (\sigma_L, \sigma_R) = ((Z_L, \mathcal{P}_L), (Z_R, \mathcal{P}_R)) \) for its image under the cutting map, and \( \sigma_g = (Z_g, \mathcal{P}_g) \) for the pre-stability condition glued out of \( \sigma_L \) and \( \sigma_R \). It is clear that the central charges \( Z \) and \( Z_g \) are the same; it is enough to check on a set of generators and we can pick the arc system \( \mathcal{A}_L \cup \mathcal{A}_g \cup \mathcal{A}_R \) where the central charges agree by construction.

As for the (pre)slicings, the inclusion \( \mathcal{P}_g \subsetneq \mathcal{P} \) is a direct consequence of Lemma 32 since every diagonal of an unobstructed lozenge under \( \sigma_L, \sigma_R \) is stable under \( \sigma \). As for the inclusion \( \mathcal{P} \subsetneq \mathcal{P}_g \), by Theorem 7 every stable object is either a stable embedded interval or a stable embedded circle; again by Lemma 32 the stable embedded intervals correspond to unobstructed lozenges and appear in \( \mathcal{P}_g \), and as for the stable circles, they must not cross the chain \( \{ \gamma_i \} \) by Lemma 10 so they are either entirely contained in \( \mathcal{F}_L \) or \( \mathcal{F}_R \) and therefore also appear in \( \mathcal{P}_g \). So \( \mathcal{P}_g \) is in fact a slicing and equal to \( \mathcal{P} \).
5. Calculations

In the previous section, we outlined a procedure for cutting stability conditions on $\mathcal{F}(\Sigma)$ along some embedded interval $\gamma$ into relative stability conditions. This procedure only works when the object supported on $\gamma$ has a simple cosi decomposition, and from Lemma 12 we know that embedded intervals cutting the surface into two necessarily have this property.

Consider some general surface $\Sigma$ with genus $g$ and punctures $p_0, p_1, \ldots, p_n$ with $m_0, m_1, \ldots, m_n$ marked boundaries, respectively. Suppose that $m_0 \geq 2$. We can then decompose the surface into a disk with some number of marked boundary intervals, possibly some annuli with two marked boundary intervals on the outer boundary circle, and possibly some punctured tori with two marked boundary intervals on the boundary circle.

![Figure 14. A decomposition of the surface $\Sigma$ into a disk, possibly several annuli and possibly several punctured tori.](image)

Note that for each one of these pieces, when modified by gluing some disk $\Delta_n$ along a boundary, give rise to the following kinds of surfaces:

1. The disk $\Delta_n$ with $n \geq 2$ marked boundary intervals
2. The annulus $\Delta^*_p,q$ with $p,q$ marked boundary intervals on the outer and inner boundary circle, respectively
3. The punctured torus $T^*_n$ with $n$ marked boundary intervals

on which we need to calculate the space of (ordinary) stability conditions.

By the main theorem of [20] (Theorem 5.3) the locus of HKK stability conditions in $\text{Stab}(\mathcal{F}(\Sigma))$ is a union of connected components. Thus, if every stability condition can be continuously deformed into an HKK stability condition, then all stability conditions are HKK stability conditions.

We will use this strategy for the three base cases; in fact we will prove that every stability condition can be continuously deformed to a stability condition with finite heart. This argument already appears for the case of the disk and the annulus in [20]; we will reproduce it in greater detail so that its use in the context of the punctured torus is clearer.

5.1. Finite-heart stability conditions. The definitions and lemmas here seem to be standard in the literature to some extent and may appear with different formulations; for clarity we will assemble them here.

**Definition 18.** A stability condition $\sigma \in \text{Stab}(\mathcal{D})$ is finite-heart if the corresponding heart $\mathcal{H}$ is a finite abelian category, ie. a finite length abelian category that furthermore only has finitely many isomorphism classes of simple objects.

Note that finite-length only means that every object is finite-length but those lengths could be unbounded; this doesn’t happen in the cases we care about because of the following standard fact.

**Lemma 33.** If $\mathcal{H}$ is finite-length and $\text{rk}(K_0(\mathcal{H})) = \text{rk}(K_0(\mathcal{D})) < \infty$ then $\mathcal{H}$ is finite, and in particular the number of isomorphism classes of simple objects is equal to $\text{rk}(K_0(\mathcal{D}))$.

We have the following criterion to determine when some stability condition is finite-heart, based on the set of stable phases $\Phi \in S^1$, ie. the set of phases of stable objects.

**Lemma 34.** If $\Phi$ has a gap around zero (ie. $S^1 \setminus \Phi$ contains an open interval $I \ni 0$) and $K_0(\mathcal{D}) < \infty$ then $\sigma$ is finite-heart.
Remark. This fact is used in [20] but left unstated. The clear statement and proof of this lemma were informed to me by F. Haiden.

Proof. Note that \( \phi \) is symmetric under a \( \mathbb{Z}_2 \) rotation so having a gap around zero means that \( \Phi \) is contained in a strict cone in the upper half-plane. Thus there is \( K > 0 \) such that \(|\Im(Z(X))| > K \cdot |\Re(Z(X))| \) for any semistable object \( X \). We will argue that the set of semistable imaginary parts

\[
\{ \Im(Z(E)) | 0 \neq E \in \mathcal{P}_\phi, \phi \in \mathbb{R} \}
\]

is discrete. Suppose that there is an accumulation point, which without loss of generality we assume to be \( a > 0 \); we can then pick a sequence of pairwise non-isomorphic semistable objects \( \{ E_n \} \) such that \( \lim_{n \to \infty} |\Im(Z(E_n)) - a| = 0 \); in particular for \( \delta > 0 \) we can pick the sequence such that \(|\Im(Z(E_n)) - a| < \delta \) for every \( n \), so picking \( 0 < \delta < a \) gives \(|\Re(Z(E_n))| < K(a + \delta)\).

But since \( K \) is finite rank and the \( E_n \) are all distinct, we have \( \lim_{n \to \infty} \|E_n\| = \infty \). We then have

\[
|Z(E_n)| < |\Im(Z(E_n))| + |\Re(Z(E_n))| \leq (K + 1)|\Re(Z(E_n))| \leq (K + 1)K(a + \delta) = \text{const}.
\]

So we have \( \lim_{n \to \infty} \frac{|Z(E_n)|}{\|E_n\|} = 0 \) contradicting the support condition.

So since the set of imaginary parts of objects in the heart \( \mathcal{H} \) is discrete and bounded below by zero, any strictly descending chain of objects is finite, and therefore \( \mathcal{H} \) is finite-length, and thus \( \sigma \) is finite-heart by the assumption \( \text{rk}(K_0(D)) < \infty \).

Using the formalism of \( S \)-graphs presented in Section 6 of [20], one can prove the following lemma (which is implicitly used in the proofs of Theorems 6.1 and 6.2 of that same paper)

**Lemma 35.** If \( \sigma \) is a finite-heart stability condition on \( \mathcal{F}(\Sigma) \) then it is an HKK stability condition.

For each of the three base cases, we will see that every stability condition can be deformed to a finite-heart stability condition.

### 5.2. The disk. (Section 6.2 of [20])
We have \( \mathcal{F}(\Delta_n) \cong \text{Mod}(A_{n-1}) \), which up to shift has finitely many indecomposable objects. Thus any heart is a finite abelian category, and every stability condition is finite-heart and therefore HKK.

### 5.3. The annulus.
There are two different kinds of annulus; one where the nontrivial circle is gradable, ie. has index zero, and one where it has index nonzero. Consider first the annulus \( \Delta_{p,q,(m)} \) with \( p,q \) marked boundary components and grading \( m \neq 0 \) around the circle.

We argue that the set of stable phases is finite. Let us fix some embedded interval object \( I_0 \) to have winding number zero, and measure the winding number of every other interval or circle with reference to it. By the classification of objects, there are only finitely many primitive (ie. non multiple) classes in \( K_0(\mathcal{F}(\Delta_{p,q,(m)}^*)) \) whose winding number is less than some fixed \( N \) in absolute value, so if there are infinitely many non-isomorphic stable objects there must be a sequence of stable objects \( X_i \) with winding number \( \to \infty \).

Consider some object \( X_i \) with winding number \( N_i \) which intersects \( I_0 \) transversely \( N_i \) many times. Since the circle has index \( m \neq 0 \), this contributes classes to both \( \text{Ext}^*(I_0, X_i) \) and \( \text{Ext}^*(X_i, I_0) \) in a range spanning \( (m - 1)N_i \) degrees. But this is impossible as \( N_i \to \infty \) since the stable components of \( I_0 \) have a minimum and maximum phase.

Consider now the annulus with zero grading. We have \( \mathcal{F}(\Delta_{0,q,(0)}) \cong \text{Mod}(A_{p+q-1}) \). So we have \( \Gamma = K_0(\mathcal{F}(\Delta_{0,q,(0)})) = \mathbb{Z}^{p+q} \), and denote by \( S \subset \Gamma \) the subgroup generated by the circle around the annulus. Let \( E \subset \Gamma \) be the set of classes of indecomposable objects. By the classification of objects \( E/S \) is finite so the only possible accumulation point in the set of stable phases \( \Phi \) is \( \arg(Z(S)) \). After a rotation (which can be arbitrarily small) we can guarantee that \( \Phi \) has a gap around zero and apply Lemma 34.

### 5.4. The punctured torus.
The calculation of this case is new. From the cutting procedure we know that only need to consider the punctured torus \( T_n^* \) with \( n \geq 2 \) marked boundary components. In fact there are many inequivalent such punctured tori, with different gradings. Let us pick simple closed curves \( L \) and \( M \) as longitude and meridian, and denote by \( i_L, i_M \) the index of the grading along them. By picking different curves we get indices differing by an action of \( SL(2, \mathbb{Z}) \) so the set of distinct graded punctured tori is \( \mathbb{Z}^2/SL(2, \mathbb{Z}) \). The orbits of \( SL(2, \mathbb{Z}) \) on \( \mathbb{Z}^2 \) are labelled by gcd, so each orbit contains a unique pair of the form \((0, m)\).
Let us fix a grading such that \((i_L, i_M) = (0, m)\). It will be important for us to know what are the circle objects. The classes in \(\pi_1(T^r)\) which are representable by simple closed curves are the curves winding \((p, q)\) times around the longitude and meridian, with \(\gcd(p, q) = 1\), plus the curve \(MLM^{-1}L^{-1}\), i.e. the circle around the puncture.

For any of these tori, the index of the circle around the puncture is always 2 for topological reasons (it bounds a punctured torus) so this curve is never gradable. On the torus with \((i_L, i_M) = (0, m \neq 0)\), the index of the \((p, q)\) curve is \(mq \neq 0\) if \(q \neq 0\), so all of the embedded circle objects are supported on the longitude \(L\). On the torus with \((i_L, i_M) = (0, 0)\), every simple closed curve is gradable and supports embedded circle objects.

Remark. This is the fundamental reason why the calculation for the \((0, 0)\) will be more involved than the case of the annulus; in that case the lattice spanned by the circle objects inside of \(K_0(\mathcal{D})\) is rank one, so there can be at most one direction of phase accumulation. In the punctured torus, the central charges of stable objects could in principle occupy every direction of the lattice, making \(\Phi\) dense; we will prove that this doesn’t happen generically.

5.4.1. The \((0, m \neq 0)\) torus. Let us denote \(\mathcal{D} = \mathcal{F}(T^r_{n, (0, m)})\) where \(n\) is the number of marked boundaries. This case will be very similar to the index zero annulus. There is only one type of embedded circle object \(L\), since no other circles are gradable. Let \(\Gamma = K_0(\mathcal{D})\) and \(E \in \Gamma\) be the set of classes of stable objects.

We argue that the set \(E/\langle L\rangle\) is finite. Suppose otherwise, and note that by the classification of embedded curves, the number of embedded curves with winding numbers \((p, q)\) with \(|q| \leq N\) is infinite, but they form finitely many orbits in \(K_0(\mathcal{D})\) under the action of the subgroup \(\langle L\rangle\). Thus, if we have an infinite sequence of stable objects \(\{E_i\}\) with winding numbers \((p_i, q_i)\) and pairwise distinct classes \([E_i] \in K_0(\mathcal{D})/\langle L\rangle\), there is a subsequence with \(\lim_{i \to \infty} |q_i| = \infty\).

This is impossible in any stability condition. Note that an object with winding \(q_i\) along the meridian intersects \(L\) transversely \(|q_i|\) times; but since \(m \neq 0\) the difference in degree between each two consecutive intersections is \(|m|\), so the amplitude of nonzero degrees in both \(\text{Hom}(E_i, L)\) and \(\text{Hom}(L, E_i)\) is \(m(q_i - 1)\). Since \(|q_i| \to \infty\) we can find stable objects \(E_i\) with arbitrarily large amplitude morphisms in both directions which is impossible since \(L\) has some\(HN\) decomposition with finitely many semistable factors, having a minimum and a maximum phase.

From the fact that \(E/\langle L\rangle\) is finite we can proceed as in the annulus case, and after an infinitesimal rotation we can guarantee that any stability condition has an gap in \(\Phi\).

5.4.2. The \((0, 0)\) torus. Let us denote \(\mathcal{D} = \mathcal{F}(T^r_{n, (0, 0)})\), where \(n\) is the number of marked boundaries. We will first need some facts about \(K_0(\mathcal{D})\). By Theorem 5.1 of \([20]\) there is an isomorphism

\[
K_0(\mathcal{F}(\Sigma, M)) = H_1(\Sigma, M; \mathbb{Z}_r),
\]

where \(\mathbb{Z}_r\) is the \(\mathbb{Z}\)-local system associated to the orientation double cover of the foliation. In our case, since we are looking at the foliation with \((0, 0)\) winding, \(\mathbb{Z}_r\) is trivial.

Let us pick an explicit set of generators of \(K_0(\mathcal{D})\) as below: first choose a basis of \(H_1(T, \mathbb{Z})\) and a labeling \(M_1, \ldots, M_N\) of the marked boundary components. The classes \([L]\) and \([M]\) are represented by circles around the longitude and meridian, and \([E_i]\), \(i = 1, \ldots, N\) is represented by intervals that connect adjacent \(M_i\) and \(M_{i+1}\) along the boundary. Consider the object \(X\) winding around the longitude with ends at \(M_1, M_N\). Extending it by \(E_1, \ldots, E_{n-1}\) and by \(E_n\) both give \(L\), so in \(K_0\) we have \(\sum_{i=1}^n [E_i] = 0\).

So the classes \([L], [M], [E_1], \ldots, [E_{n-1}]\) give a basis of \(K_0(\mathcal{D})\). Since every immersed curve has well-defined winding numbers, we have a projection map

\[
w : K_0(\mathcal{D}) \to \mathbb{Z}^2 = \text{Span}([L], [M])
\]

taking a curve of \((p, q)\) winding numbers to \(p[L] + q[M]\). The following lemma tells us that the distribution of stable phases is not essentially changed by \(w\).

Lemma 36. For any sequence of stable objects \(\{X_k\}\) (with all \(X_k\) pairwise distinct) if \(\lim_{k \to \infty} \arg(Z(X_k))\) exists then

\[
\lim_{k \to \infty} \arg(Z(w([X_k]))) = \lim_{k \to \infty} \arg(Z(X_k))
\]

Proof. By the classification of indecomposables, \(X\) is represented by some circle or interval with winding \((p, q)\). If \(X\) is a circle we already have \(w([X]) = [X]\). Given embedded interval with boundaries on \(M_1, M_i\),...
one can express it as the concatenation of \( p \) copies of the interval winding along the longitude with both ends at \( M_1 \) (whose class is \([L]\), \( q \) copies of the interval winding along the meridian with both ends at \( M_1 \) (whose class is \([M]\)) and a chain of intervals \( E_1, \ldots, E_{n−1} \) connecting \( M_1 \) to \( M_i \). This chain can wind around the circle any number of times, but since \( \sum_{j=1}^{n}[E_j] = 0 \), its class is always \([E_1] + \cdots + [E_i] \). Applying \(|Z(\cdot)|\), since this sum is bounded above we have

\[
|Z(X) − Z(w(X))| ≤ C
\]

for some fixed constant \( C \).

Consider now the stable objects \( X_k \). Without loss of generality suppose that \( \lim_{k→∞} \arg(Z(X_k)) = 0 \) (ie. the positive real direction). These objects can be represented by embedded intervals; note that there are finitely many embedded intervals with fixed winding numbers. Thus in the infinite sequence of distinct objects \( \{X_k\} \) we must have \( p_k^2 + q_k^2 → ∞ \) so therefore \( |Z(X_k)| → ∞ \) and \( \Re(Z(X_k)) → +∞ \).

The triangle inequality,

\[
|Z(X_k)|−C ≤ |Z(w(X_k))| ≤ |Z(X_k)|+C
\]

also implies similar inequalities for the real and imaginary parts. Since \( |\Re(Z(X_k))| → ∞ \) we have

\[
\lim_{k→∞} |\Re(Z(w(X_k)))| ≤ \lim_{k→∞} |\Re(Z(X_k))|−C = \lim_{k→∞} |\Re(Z(X_k))|−C ≤ |\Re(Z(X_k))| = 0.
\]

so \( \lim_{k→∞} \arg(Z(w(X_k))) = \lim_{k→∞} \arg(Z(X_k)) = 0 \). \( \square \)

**Corollary 37.** If the set of stable phases \( Φ \) is dense in \( S^1 \) then the set

\[
Φ_w = \{ \arg(Z(w(X))) \mid X \text{stable} \}
\]

is also dense in \( S^1 \)

We will also need to know a bit more about which objects necessarily intersect transversely.

**Lemma 38.** Consider two embedded objects \( X \) and \( Y \) with winding numbers \((p_X, q_X)\) and \((p_Y, q_Y)\), respectively. If \(|p_Xq_Y − q_Xp_Y| ≥ 2 \) then \( X \) and \( Y \) intersect transversely.

**Proof.** If \( X \) is a circle with \((p_X, q_X) = (1, 0)\), then any \( Y \) with \(|q_Y| ≥ 1 \) intersects \( X \) transversely; if \( X \) is an embedded interval with \((p_X, q_X) = (1, 0)\), then circles with \(|q_Y| ≥ 1 \) intersect \( X \) transversely but intervals with \(|q_Y| = 1 \) may not. On the other hand, winding more times around the meridian by requiring \(|q_Y| ≥ 2 \) necessarily causes a transverse intersection. Applying the right element of \( \text{SL}(2, \mathbb{Z}) \) that sends \((1, 0) \mapsto (p_X, q_X)\) gives the statement of the lemma. \( \square \)

The following lemma gives an existence result for a certain kind of stable object.

**Lemma 39.** Let \( σ ∈ \text{Stab}(\mathcal{D}) \) be a stability condition on \( \mathcal{D} = \mathcal{F}(T_N^*) \). Then there is some stable object represented by an embedded interval with nonzero winding and ends at different marked boundaries.

**Proof.** Suppose otherwise; by the classification of embedded curves, there are three remaining possibilities for a stable object:

1. A semistable circle with winding \( \neq (0, 0) \),
2. A semistable interval with winding \( \neq (0, 0) \) both ends on the same marked boundary,
3. A semistable interval with \((0, 0)\) winding and ends possibly on different marked boundaries.

Two objects of type (2) ending on the same marked boundary \( M \) will have extension morphisms between them, but we argue that if they have different classes in \( K_0(\mathcal{D}) \) these morphisms cannot appear in the HN decomposition of any object. By keeping track of the grading with respect to the \((0, 0)\) grading on the torus, we note that if we grade the intervals such that \( \deg(f) = 1 \), then \( \deg(g) = 0 \). Thus \( φ_B ≤ φ_A \) and by genericity \( φ_B ≠ φ_A \) since \([A] ≠ [B] \), so \( f \in \text{Ext}^1(A, B) \) cannot appear in the HN decomposition.

Thus every interval with winding \((p, q)\), \( \gcd(p, q) = 1 \) and ends on the same marked boundary must be semistable, since there is no way to express it as a valid extension of the objects above. We argue that this is impossible in a generic stability condition. Take for example the semistable interval \( J \) with winding \((1, 0)\) and both ends on some marked boundary \( M \), and consider another embedded interval \( J' \) with winding \((0, 1)\), with ends on \( M \) and \( M' ≠ M \). By assumption, \( J' \) is not semistable so it must have a chain-of-intervals decomposition with at least two distinct phases; consider the interval objects in this chain that end at \( M' \); since the other end of the chain is at another marked boundary, among these objects there must be at least
one semistable interval $J'_0$ of type (3) above (ie. with zero winding). We see immediately that such an interval has an essential transversal intersection with $J$; therefore the rest of the chain (after $J'_0$) must cross $J$ as well. But this configuration is prohibited by Lemma 10.

So there must be some semistable interval object $I'$ with nonzero winding and ends on different marked boundary intervals. If $I$ is not stable, consider its Jordan-Hölder filtration into stable objects; among these there must be one stable interval object $I$ connecting two distinct marked boundaries. \hfill $\square$

Using the lemmas above, in the following calculation we show that an adequately generic stability condition does not have dense phases in $S^1$.

**Lemma 40.** Let $\sigma \in \text{Stab}(D)$ be a stability condition on $D = F(T^*_pX)$. Then possibly after a infinitesimal deformation the set of stable phases $\Phi$ has a gap, ie. $S^1 \setminus \Phi$ contains an open interval.

**Proof.** By the previous lemma, there must be some stable interval $I$ with nontrivial winding and ends on distinct marked boundary components. Applying an appropriate $\text{SL}(2,\mathbb{Z})$ automorphism, we can assume this stable interval $I$ has winding numbers $(1, 0)$, ie. winds around the longitude once. Let $L$ be the rank one trivial circle object also with winding number $(1, 0)$.

The subset of $\text{Stab}(D)$ where $I$ is stable is open by standard results [11] so there is a neighborhood $U$ of $\sigma$ where $I$ is stable. From the description of $K_0(D)$ we know that $[I] \neq [L]$, so $Z(I), Z(L)$ are not parallel in the complement of a codimension one wall. Thus, possibly after an infinitesimal deformation inside of $U$, we can guarantee that $I$ is stable and $Z(I), Z(L)$ have different arguments.

Consider the trivial rank one objects $L$ and $M$ (which may or may not be stable) supported along the longitude and meridian, with gradings so that

$$\deg(M, I) = \deg(M, L) = 0$$

and for simplicity let us rotate and scale the stability condition so that $Z(L) = 1$. Since $[L] \neq [M]$ and we fixed $Z(L) \in \mathbb{R}$, for a generic stability condition we must have $Z(M) \notin \mathbb{R}$. Let us treat the case $\Im(Z(M)) > 0$ first; the other case follows from an analogous argument.

Suppose now that $\Phi$ is dense in $S^1$; by Lemma 37, $\Phi_w$ is dense too. For a choice of winding numbers $(p, q)$, let us denote by

$$\mathcal{X}_{p,q} = \{(p', q') \mid q > 0, |pq' - qp'| \geq 2 \} \subset \mathbb{Z}^2$$

the set of winding numbers whose objects necessarily intersect transversely with objects of winding number $(p, q)$, with positive winding around the meridian.

The set $\mathcal{X}_{1,0}$ corresponding to $I$ is given by $q \geq 2$; so at infinity $\mathcal{X}_{1,0}$ approaches a sector (with angle $\pi$). Remember that for any $N$ there are only finitely many indecomposable objects with winding satisfying $p^2 + q^2 \leq N$. By density of $\Phi_w$ we can find some stable object $X_0$ with winding numbers $(p_0, q_0) \in \mathcal{X}_{1,0}$.

Consider now the set $\mathcal{X}_{1,0} \cap \mathcal{X}_{p_0,q_0}$; this set is composed of lattice points inside of two components of a subset of $\mathbb{R} \times \mathbb{R}_+$. At infinity, the right component approaches a sector with angle spanning $(0, \arctan(q_0/p_0))$ and the left component approaches a sector at $(\arctan(q_0/p_0), \pi)$. Note that here we are choosing $\arctan$ to be valued between 0 and $\pi$. Using density, let us pick some object $X_1$ with $(p_1, q_1)$ in the right component,
Figure 16. Left: the set $X_{p,q}$ for $(p,q) = (3, 4)$ is composed of the $\mathbb{Z}^2$ dots inside of the shaded area. Note that all these sets have two parts, each of which at infinity approaches a sector with finite angle. Right: after the first iteration we consider $X_{1,0} \cap X_{3,4}$. Note that after any number of iterations the each side of this set still approaches a sector with finite angle at infinity.

and $X_{-1}$ with $(p_{-1}, q_{-1})$ in the left component. Note that since the sectors span positive angles we can pick these objects with $q_{1}, q_{-1}$ arbitrarily large; since

$$|\Im(Z(X)) - \Im(Z(w(X)))| = |\Im(Z(X)) - q_X \Im(Z(M))|$$

is bounded for any indecomposable object $X$ we can also guarantee that $\Im(Z(X_1))$ and $\Im(Z(X_{-1}))$ are positive.

We would like to iterate this process; at the $n$th step we will have objects $\{X_{k}\}_{-n \leq k \leq n}$ with winding numbers $(p_k, q_k)$ running clockwise in angle, ie. $0 \leq \arctan(q_k/p_k) \leq \pi$ is decreasing. The set

$$X_{1,0} \cap X_{p_{-n},q_{-n}} \cap \cdots \cap X_{p_{0},q_{0}} \cap \cdots \cap X_{p_{n},p_{n}}$$

at infinity approaches two sectors at $(0, \arctan(q_n/p_n))$ and $(\arctan(q_{-n}, p_{-n}), \pi)$; since each of these sectors has nonzero angle we can use density and repeat the process by picking stable objects $X_{-n+1}, X_{n+1}$ in each sector, also both with central charge with positive imaginary part. Also from density of $\Phi$ it follows that we can pick objects such that

$$\lim_{k \to +\infty} \arctan(q_k/p_k) = 0, \quad \lim_{k \to -\infty} \arctan(q_k/p_k) = \pi.$$

Iterating to infinity we get stable objects $\ldots, X_{-1}, X_{0}, X_{1}, \ldots$ all mutually transversely intersecting, that also transversely intersect $I$ as well. Taking appropriate shifts we can guarantee that all these objects have phases $0 \leq \phi_k \leq 1$. We then get that

$$\lim_{k \to +\infty} \phi_k = 0, \quad \lim_{k \to -\infty} \phi_k = 1.$$

Figure 17. Stable circle $L$ and stable interval $I$ with ends on different boundary components, together with transversely intersecting stable objects $X_i, i \in \mathbb{Z}$. 
Let \( d_k \) be the degree of the intersection between \( X_k \) and \( I \), and \( f_k \) be the degree of the intersection between \( X_k \) and \( X_{k+1} \). Let us shift \( I \) such that \( d_0 = -1 \). The triangles with sides \( X_k, X_{k+1}, I \) give the relations \( d_k = d_{k+1} + f_k \). Since all the objects are stable we have inequalities for the phases
\[
\phi_k \leq \phi_I + d_k \leq \phi_k + 1, \quad \phi_k \leq \phi_{k+1} + f_k \leq \phi_k + 1.
\]
But we chose the shifts such that all the \( \phi_k \) are in \((0,1)\), so we must have \( f_k = 0 \) for all \( k \), and therefore \( d_k = -1 \) for all \( k \), so \( \phi_k - 1 \leq \phi_I \leq \phi_k \).

Taking the two limits \( k \to +\infty \) and \( k \to -\infty \) gives us \( \phi_I = \phi_L = 0 \) which contradicts the genericity of \( \sigma \).

6. Conclusions

The calculations for the three base cases above show that the every generic stability condition on those categories is an HKK stability condition; because of [20, Theorem 5.3] the image of the moduli of HKK stability conditions in \( \text{Stab}(\mathcal{D}) \) is an union of connected components, so for all these cases there are only HKK stability conditions.

The cutting and gluing procedures allow us to reduce the calculation to the three base cases, and because of Theorem [31] this proves Theorem [2]; every stability condition on a graded surface \( \Sigma \) is an HKK stability condition, i.e. given by a quadratic differential with essential singularities.

6.1. Future directions. An obvious direction of future inquiry is the extension of the definition of relative stability conditions to Fukaya categories of higher-dimensional spaces.

With inspiration in the conjectures of Kontsevich [20], the wrapped Fukaya category of a Weinstein manifold has recently been proven [18][17][19] to localize to a cosheaf of categories on the Lagrangian skeleton of the Weinstein manifold, the same way that the Fukaya categories that we considered in this paper can be calculated by a cosheaf on the dual graph.

The naive generalization of Definition 6 to this cosheaf in higher dimensions is easy to write down, but it still unclear whether one has the same nice results. We believe that the main difficulty in establishing similar results in more generality is that we lack equivalents of Lemmas 7, 10 and Theorem 13; and more fundamentally we are not aware of geometric representability results such as Theorem 5 in higher dimensions. Note that these were very important to prove our results, since even defining the cutting and gluing maps required:

1. Constraining the isomorphism type of the category \( \text{HNEnv}(X) \) for a certain class of object \( X \) (Lemma 12).
2. Having a non-crossing Lemma 10 which lets us separate the HN decomposition of some objects into a left and a right side.

We are of the opinion that answering the analogous questions for higher dimensions is the first step towards progress in that direction.

Another area of future research is to explore the relations between relative stability conditions and the work of Dimitrov, Haiden, Katzarkov and Kontsevich [13][12][14], which relates stability conditions on Fukaya categories of surfaces to questions about dynamics on the surface. In particular, it is likely that the cutting and gluing procedures of Section 4 can be used to say something about the distribution of stable phases for general surfaces; once we cut a surface \( \Sigma \) into disks, annuli and punctured tori, the collection of stable objects in \( \mathcal{F}(\Sigma) \) can be produced algorithmically from collections of stable objects on each piece. It appears that one could use this to give a partial answer to Question 4.9 of [13], about the existence of conditions on a triangulated category \( \mathcal{T} \) constraining the distribution of accumulation points in the set of stable phases; this will be a topic of future research.

References

[1] Mohammed Abouzaid. On the wrapped Fukaya category and based loops. Journal of Symplectic Geometry, 10(1):27–79, 2012.
[2] Mohammed Abouzaid and Paul Seidel. An open string analogue of Viterbo functoriality. Geometry & Topology, 14(2):627–718, 2010.
[3] Paul S Aspinwall and Michael R Douglas. D-brane stability and monodromy. Journal of High Energy Physics, 2002(05):031, 2002.
[4] Arend Bayer, Aaron Bertram, Emanuele Macrì, and Yukinobu Toda. Bridgeland stability conditions on threefolds II: an application to Fujita’s conjecture. arXiv:1106.3430, 2011.

[5] Arend Bayer and Emanuele Macrì. The space of stability conditions on the local projective plane. Duke Mathematical Journal, 160(2):263–322, 2011.

[6] Arend Bayer, Emanuele Macrì, and Yukinobu Toda. Bridgeland stability conditions on threefolds I: Bogomolov-Gieseker type inequalities. arXiv:1103.5010, 2011.

[7] Tom Bridgeland. Stability conditions on a non-compact Calabi-Yau threefold. Communications in mathematical physics, 266(3):715–733, 2006.

[8] Tom Bridgeland. Stability conditions on triangulated categories. Annals of Mathematics, pages 317–345, 2007.

[9] Tom Bridgeland. Stability conditions on K3 surfaces. Duke Mathematical Journal, 141(2):241–291, 2008.

[10] Tom Bridgeland, Yu Qiu, and Tom Sutherland. Stability conditions and the $A_2$ quiver. arXiv:1406.2566, 2014.

[11] Tom Bridgeland and Ivan Smith. Quadratic differentials as stability conditions. Publications mathématiques de l’IHÉS, 121(1):155–278, 2015.

[12] George Dimitrov. Bridgeland stability conditions and exceptional collections. PhD thesis, uniwien, 2015.

[13] George Dimitrov, Fabian Haiden, Ludmil Katzarkov, and Maxim Kontsevich. Dynamical systems and categories. The influence of Solomon Lefschetz in geometry and topology, 50:133–170, 2014.

[14] George Dimitrov and Ludmil Katzarkov. Bridgeland stability conditions on the acyclic triangular quiver. Advances in Mathematics, 288:825–886, 2016.

[15] Michael R Douglas, Bartomeu Fiol, and Christian Röhmelsberger. Stability and BPS branes. Journal of High Energy Physics, 2005(09):006, 2005.

[16] Davide Gaiotto, Gregory W Moore, and Andrew Neitzke. Wall-crossing, Hitchin systems, and the WKB approximation. Advances in Mathematics, 234:239–403, 2013.

[17] Sheel Ganatra, John Pardon, and Vivek Shende. Covariantly functorial wrapped Floer theory on Liouville sectors. arXiv preprint arXiv:1706.03152, 2017.

[18] Sheel Ganatra, John Pardon, and Vivek Shende. Microlocal Morse theory of wrapped Fukaya categories. arXiv preprint arXiv:1809.08807, 2018.

[19] Sheel Ganatra, John Pardon, and Vivek Shende. Structural results in wrapped Floer theory. arXiv preprint arXiv:1809.03427, 2018.

[20] Fabian Haiden, Ludmil Katzarkov, and Maxim Kontsevich. Flat surfaces and stability structures. Publications mathématiques de l’IHÉS, 126(1):247–318, 2017.

[21] Daniel Huybrechts, Emanuele Macrì, and Paolo Stellari. Stability conditions for generic K3 categories. Compositio Mathematica, 144(1):134–162, 2008.

[22] Akishi Ikeda. Stability conditions on CY_N categories associated to A_n-quivers and period maps. Mathematische Annalen, 367(1-2):1–49, 2017.

[23] Dominic Joyce. Configurations in abelian categoriesI: Basic properties and moduli stacks. Advances in Mathematics, 203(1):194–255, 2006.

[24] Dominic Joyce. Conjectures on Bridgeland stability for Fukaya categories of Calabi-Yau manifolds, special Lagrangians, and Lagrangian mean curvature flow. arXiv:1401.4949, 2014.

[25] Alastair King and Yu Qiu. Exchange graphs and Ext quivers. Advances in Mathematics, 285:1106–1154, 2015.

[26] Maxim Kontsevich. Homological algebra of mirror symmetry. arXiv preprint alg-geom/9411018, 1994.

[27] Maxim Kontsevich and Yan Soibelman. Stability structures, motivic Donaldson-Thomas invariants and cluster transformations. arXiv:0811.2435, 2008.

[28] Chunyi Li. On stability conditions for the quintic threefold. arXiv preprint arXiv:1810.03434, 2018.

[29] Antony Maciocia and Dulip Piyaratne. Fourier-Mukai transforms and Bridgeland stability conditions on abelian threefolds. Algebr. Geom, 2(3):270–297, 2015.

[30] Emanuele Macrì. Stability conditions on curves. Mathematical research letters, 14(4):657–672, 2007.

[31] David Nadler. Wrapped microlocal sheaves on pairs of pants. arXiv preprint arXiv:1604.00114, 2016.

[32] David Nadler. Arboreal singularities. Geometry & Topology, 21(2):1231–1274, 2017.

[33] David Nadler and Eric Zaslow. Constructible sheaves and the Fukaya category. Journal of the American Mathematical Society, 22(1):233–286, 2009.
[34] So Okada. Stability Manifold of $\mathbb{P}^1$. arXiv:math/0411220, 2004.
[35] So Okada. On stability manifolds of Calabi-Yau surfaces. International Mathematics Research Notices, 2006(9):58743–58743, 2006.
[36] Yu Qiu. Stability conditions and quantum dilogarithm identities for Dynkin quivers. Advances in Mathematics, 269:220–264, 2015.
[37] Ivan Smith. Stability conditions in symplectic topology. arXiv:1711.04263, 2017.
[38] Yukinobu Toda. Stability conditions and birational geometry of projective surfaces. Compositio Mathematica, 150(10):1755–1788, 2014.