Non-Commutative Mechanics as a modification of space-time

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We formulate non-relativistic classical and quantum mechanics in the non-commutative two dimensional plane. The approach we use is based on the Galilei group, where the non-commutativity is seen as a central extension upon identification of the boost generators with the position operator. We perform a systematic study of the free particle, defined by the symmetries of the space time, which include the no-commutativity. The symmetries at the classical level are analyzed in terms of Noether’s theorem. Canonical quantization is presented and the representation of the corresponding Heisenberg algebra is obtained. The path integral representation and Wigner distribution function in phase space are also discussed. We work out, both at the classical and at the quantum level, the harmonic oscillator, avoiding the use of the conventional non-canonical transformation that leads to a momentum dependent potential. We use Einstein’s model for a solid to corroborate that, according with intuition, as a consequence of the space fuzziness the entropy is a growing function of θ in the low temperature regime.

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I. INTRODUCTION

Non-commutativity (NC)\(^1\) can be interpreted either, as an intrinsic property of space time \(I\), or as a low-energy consequence of the fundamental string theory \(I\). In both cases, Non commutative quantum mechanics (NCQM) is seen as a laboratory where the properties of NC field theories can be studied. This has however lead to ambiguities of what is meant by NC mechanics, in fact different points of view exist in the literature which seem to be equally relevant. Among the more frequently used are the one based on the Moyal product, i.e. assume that the dynamics is described by the conventional equations but using everywhere Moyal products, and the second approach where one suppose that the only way NC enters is through the Poisson Brackets (commutators in QM) of the position operators. How is the NC related to the space-time, i.e. can we identify the non-commuting operators with the position? and if so, is the resulting theory consistent, do we have precise rules to describe the physics in different reference frames?. From our view point one should clearly state the assumptions made to define the NC mechanics we use, in particular, one should know what is the effect of NC on a free particle.

In this paper we adopt the first view, namely we assume NC is intrinsic property of space time and require that NC be incorporated in a consistent way. Indeed, an important property of any formulation of the mechanics is the symmetry of the underlying space-time, i.e. the relativ group on which it is based. In fact, we will follow the procedure used to formulate field theory where

the free particle is treated using the symmetries of the space time and then, once a consistent framework is available, interactions are introduced using further arguments (renormalizability, gauge invariance, SUSY). Thus, we first consider in detail the free particle taking as basis a Hamiltonian description based on a symplectic structure – determined by the symmetries of the NC space – which we assume is the Galilei group. It is important to remark that already at this level we have some constraints since consistency of NCM with Galilei group requires:

- to work in the two dimensions,
- although \(\{x_i, x_j\} \neq 0\) necessarily \(\{p_i, p_j\} = 0\).

An important point to remark is that in our approach, the Poisson Brackets (PB) (or commutators in the quantum theory) of the coordinates is always non zero, we avoid any non-canonical transformation leading to vanishing PB, otherwise it would be inconsistent with the assumed Galilei algebra. Thus, for us, the interaction of a charged particle in two dimensions with a perpendicular constant magnetic field is not a typical example of NC since in such a case the free particle reduces to the standard commutative problem which can be consistently quantized and, after that, the interaction can be introduced through minimal coupling. We will comment on these points below, but we first refer to some of the existing work in the literature. Since there are so many publications in this area, we will restraint to those of direct interest to our approach, and still, apologize for undeliberate omissions.

From our point of view the formulation of NCM that not only is consistent with, but actually is based on, the Galilei algebra is of particular interest. In such an approach – which is valid only in two dimensions \(I\) –

\(^{1}\) In the following we will use NC to stand for non-commutative and M, C and Q for mechanics classical and quantum respectively.

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the NC appears, together with the mass, as two central extensions of the algebra. This was recognized by authors in \( \mathbb{R} \) who build a model that provides a realization of the symmetry. They also discuss the quantization of the model using the formalism of constraints and a lagrangian including higher derivatives. In spite of the completeness of the work in \( \mathbb{R} \), there are several topics that deserve further discussion. Thus, in the present paper we make a summary of non-commutative classical mechanics NCCM in two dimensions. Besides the Hamiltonian formalism, the first order Lagrangian formulation of NCCM and the relation between the Hamiltonian and Lagrangian are also discussed following the approach in \( \mathbb{R} \). The equivalence between those formalism naturally leads to the conclusion that two dimensional NCCM can always be treated as a system with second class constraints, at least if interaction with gauge fields are not introduced \( \mathbb{R} \). We discuss Noether’s theorem and starting from the symmetries we derive the Hamiltonian for the free particle on the NC plane. Although this appears to be a trivial and long exercise, in fact it shed some light on some of the questions previously formulated. Indeed, besides helping in the identification of the Hamiltonian (there is not consensus on this point, for example according to the approach in \( \mathbb{S} \) the dynamics of a free particle on a NC space is equivalent to the study of the dynamics of this particle with charge \( q \) on the usual commutative space in the presence of a magnetic field), the symmetries determines the Hamiltonian as the time evolution operator, providing thus some support to the whole approach. All along the paper we emphasize the role played by symmetries, this is relevant since the proper formulation and solution of some of the problems in QM can be traced to the appropriated understanding of the classical symmetries.

With the formulation of the mechanics previously discussed the passage from classical to quantum mechanics is straightforward. At the quantum level, using canonical quantization, we analyze the representation of the Heisenberg algebra as a way to implement the canonical quantization. As explained below, the idea is to implement representations for the operators which are consistent with the commutation relations, but also determine what is the relevance of the commutation relation for the states. The advantage of this approach is that, based on the commutation relations, one can implement and compare different representations. An example where a judicious choice of the base is exploited is seen in \( \mathbb{R} \) where the periodicity of the oscillator in the non-commutative plane is analyzed, although neither the spectrum nor the eigenfunctions of the complete set of observables are derived. Previous work include also an analysis of possible realizations of the operators, which is a related but not equivalent problem \( \mathbb{R}, \mathbb{S} \). We derive expression for non-equivalent representations of the Heisenberg algebra extending previous analysis to NCQM \( \mathbb{R} \). The representations include differential representations of the operators (position and momenta) in different basis as well as gauge fields that follow from the structure of the algebra and that are relevant in the description of non trivial manifolds. As far as we know, ours is the first time a detailed analysis of the representations of the Heisenberg algebra is presented in the two dimensional NC case \( \mathbb{R} \).

An alternative procedure to study the quantum properties of a system is through the phase space Wigner distribution function. Besides the intrinsic interest on this formulation, the use of non-commutativity in the field of quantum optics \( \mathbb{R} \), where Wigner distribution function is a common tool, is a further motivation for its generalization to the non-commutative case. We are not aware of exhaustive work along this line, the difference with existing literature being again the approach \( \mathbb{R} \). We provide a compact exact expression for the Wigner distribution function, again in this case we do not need to consider an expansion in \( \theta \), the NC parameter. For completeness we include a brief discussion on the path integral formulation, where the results of the canonical quantization are used, in particular the wave functions that permit to connect different basis. Our results are analogous to those obtained in \( \mathbb{R} \).

As examples of the application of the formalism, we work out in detail two problems, both at the classical and at the quantum level: the free particle and the harmonic oscillator. The free particle can be considered the physical system where the symmetries of the space time are realized, for that reason we characterize the free particle with the Hamiltonian that is compatible with the corresponding Galilei group. On the other hand, several papers \( \mathbb{R}, \mathbb{S}, \mathbb{T} \) have dealt with the harmonic oscillator at the quantum level. The case of the two dimensional harmonic oscillator in the presence of a constant magnetic field has been considered \( \mathbb{R}, \mathbb{S} \) and it has been remarked that two phases exist, in one of these \( (B \neq 0) \) the symmetry group is SU(2) while in the other \( (B = 0) \) the symmetry group is SU(1,1). In our approach we show in CM, that although SU(2) is a symmetry of the commutative harmonic oscillator, this is not true in the NC case. We analyze the surviving symmetry and, at the quantum level, determine the eigenfunctions common to the Hamiltonian and angular momentum.

Before concluding this introduction in this paragraph we make a summary of our approach. We consider the mechanics, both classical and quantum, for a free particle imposing consistency with the symmetry of the space time, which is assumed to be Galilei group including two central extensions (the mass and the parameter of NC) \( \mathbb{R} \). We completely avoid the use of non canonical transformations and assume that the only way NC enters is through the commutation relations of the position operators plus those that the requirement of consistency demands. A potential \( V(x) \), independent of the NC parameter \( \theta \), is added to the Hamiltonian of the free particle,
which has been previously determined through the symmetr. In QM we assume the validity of the Schrödinger equation and use the time translation generator, i.e. the Hamiltonian. The differential operator associated to the Hamiltonian is obtained from the classical one plus any of the four representations of the position and momenta consistent with the Heisenberg algebra. We have tried to make the paper as self contained as possible, avoiding however unnecessary details. We organized the main body of the manuscript in six parts. Section two is devoted to classical mechanics, the third to the canonical quantization. Sections four and five deal with alternative quantization procedures. Examples in classical mechanics are discussed in section two, in QM in section six and we end with a summary of the contributions of this work.

II. CLASSICAL MECHANICS

We start with a brief summary of classical mechanics. This will allow us to introduce the notation and also to clarify the role played by symmetries in the formulation of NC mechanics. In the Hamiltonian formalism the description of a system with $n$ degrees of freedom is determined by $2n$ first order differential equations involving $2n$ independent variables that, here and thereof, we denote by $z_{\alpha}$ ($\alpha = 1, 2, \ldots, 2n$). The symplectic structure $J$ associated to this formalism defines the Poisson brackets

$$\{f, g\} = J^{\alpha \beta} \frac{\partial f}{\partial z^\alpha} \frac{\partial g}{\partial z^\beta},$$

where summation over repeated indices is understood. The parenthesis are real, antisymmetric and linear. Furthermore they satisfy the Leibniz rule and the Jacobi identity:

- $\{f, gh\} = \{f, g\}h + g\{f, h\}$
- $\{f, \{g, h\}\} = \{\{f, g\}, h\} + \{g, \{f, h\}\}$.

According to (1) the entries of $J$ are:

$$J^{\alpha \beta} = -J^{\beta \alpha} = \{z^\alpha, z^\beta\},$$

For later reference we introduce the inverse of the symplectic structure $J$ by means of the relation:

$$J^{\alpha \beta} \omega_{\beta \gamma} = \delta^\alpha_{\gamma},$$

The equations of motion for a system described by the Hamiltonian $H(z)$, in a phase space with symplectic structure $J$, are given by:

$$\dot{z}^\alpha = \{z^\alpha, H(z)\} = J^{\alpha \beta} \frac{\partial H(z)}{\partial z^\beta}, \quad \alpha, \beta = 1, \ldots, 2n.$$ (4)

If one identifies the generalized coordinates with the first $n$ variables ($q_i = z_i, i = 1, 2, \ldots, n$) of phase space and the conjugated momenta with the last $n$ ($p_i = z_{n+i}, i = n+1, n+2, \ldots, 2n$), then the information on the symplectic structure $J$ is contained in the Poisson brackets:

$$\{q_\alpha, q_\beta\} = \theta_{\alpha \beta}, \quad \{q_\alpha, p_\beta\} = \delta_{\alpha \beta}, \quad \{p_\alpha, p_\beta\} = \Theta_{\alpha \beta}. $$ (5)

So far $\theta_{\alpha \beta} = \theta_{\alpha \beta}(q, p)$ is an antisymmetric field ($\theta_{\alpha \beta} = -\theta_{\beta \alpha}$) which, in general, may depend upon the position and the momenta. However, if we set $\Theta_{\alpha \beta} = 0$, the Jacobi identity (applied to a momentum $p_i$ and two coordinates $q_j, q_k$), implies the relation:

$$\frac{\partial \theta_{ij}}{\partial q_k} = 0,$$

therefore $\theta_{\alpha \beta} = \theta_{\alpha \beta}(p)$. At this point it should be clear that, by allowing a general enough symplectic structure, NCM can be described within this formalism.

On the other hand, the lagrangian formulation is relevant in discussing the symmetries. For that reason we introduce the first order Lagrangian $L$:

$$L(z, \dot{z}, t) = K_\alpha(z) \dot{z}^\alpha - H(z, t), \quad (\alpha = 1, \cdots, 2n)$$ (7)

where $K_\alpha$ is a vector potential in phase space for the inverse of the symplectic structure, i.e.:

$$\omega_{\alpha \beta} = \frac{\partial K_\beta}{\partial z^\alpha} - \frac{\partial K_\alpha}{\partial z^\beta};$$

in order to show the equivalence of this Lagrangian to the Hamiltonian formalism previously introduced, we apply the variational principle to the action associated to this Lagrangian:

$$\delta S[z(t)] = \delta \int_{t_i}^{t_f} \left( K_\alpha(z) \dot{z}^\alpha - H(z, t) \right) dt = \int_{t_i}^{t_f} \left[ \omega_{\alpha \beta} \delta z^\beta + \frac{\partial K_\beta}{\partial z^\alpha} \delta z^\beta + \frac{\partial H}{\partial z^\beta} \delta z^\beta \right] dt$$

Therefore, if the variation at the end points are such that:

$$K_\alpha \delta z^\alpha|_{t_f} = K_\alpha \delta z^\alpha|_{t_i},$$

the variation of the action Eq.(9) implies:

$$\omega_{\beta \gamma} \delta z^\gamma = \frac{\partial H}{\partial z^\beta},$$

which are nothing but the Hamilton equations of motion Eq.(20):

$$\dot{z}^\alpha = \{z^\alpha, H(z)\} = J^{\alpha \beta} \frac{\partial H(z)}{\partial z^\beta},$$

If $\omega(z)$ has constant entries, $K_\alpha$ can be written as:

$$K_\alpha = \frac{1}{2} \dot{z}^\beta \omega_{\beta \alpha}. $$ (13)
and, under these conditions, the Lagrangian reduces to:

\[ L(z, \dot{z}, t) = \frac{1}{2} z^{\alpha} \omega_{\alpha \beta} \dot{z}^{\beta} - H(z, t), \]  

(14)

With the lagrangian formulation at hand, Noether’s theorem is formulated in the conventional way, for completeness we quote the result [3].

If under a group of order \( r \), of continuous transformations \( t \to t' = t' \), \( z^{\alpha} \to z'^{\alpha}(z, t) \), the action is invariant up to surface terms i.e.

\[ S[z'] = S[z] + \int_{t_i}^{t_f} \frac{d\Lambda(z)}{dt} dt. \]  

(15)

then, for every classical solution to the equations of motion, \( r \) functions \( \gamma_\lambda (\lambda = 1, 2, \ldots r) \) of the dynamical variables are conserved:

\[ \gamma_\lambda = K_\alpha \varphi_\alpha^\lambda - \chi_\lambda (H - z^\alpha K_\alpha) - \Lambda_\lambda. \]  

(16)

the quantities \( \phi, \chi \) and \( \Lambda \) are given by the infinitesimal transformations which are a symmetry of the action:

\[ \delta t = \varepsilon^\lambda \chi_\lambda(t), \]  
\[ \delta z^\alpha = \varepsilon^\lambda \varphi_\alpha^\lambda(z), \]  
\[ \delta \Lambda = \varepsilon^\lambda \Lambda_\lambda, \]  

(17)

where \( \varepsilon^\lambda \) \( (\lambda = 1, \ldots, r) \) are constant, infinitesimal parameters each of which is associated to a given transformation. It proofs convenient to introduce the notation \( Q = \varepsilon^\lambda \gamma_\lambda \) (no sum over \( \lambda \)) and refer to \( Q \) as the charge associated to the corresponding transformation.

It is well known [19] that the Poisson brackets of the conserved charges \( \gamma_\lambda \) define an algebra isomorphic to the global continuous symmetry group of the Lagrangian, and that the symmetry transformations can be obtained in terms of the Poisson brackets of the dynamical variables \( (z_\alpha) \) with \( \gamma_\lambda \). Instead of checking the validity of this assertion, later in this section we will use this fact to obtain the generators and, from these, the hamiltonian for a free particle.

This is as far as we can get on general grounds. To go further we need to specify the symplectic structure. We do this taking into account facts known from the mathematical literature regarding the Galilei group [21], which we assume is the symmetry group of the non-relativistic mechanics. It is known that in 3+1 dimensions, the Galilei group accepts only one central charge - to be identified with the mass of the particle - and therefore it is not possible to introduce NC in 3+1 dimensions, at least not consistently with the Galilei group. In 2+1 dimensions the Galilei algebra accepts three central extensions: the mass, the parameter associated to NC and one more that we ignore on physical grounds (we are not interested in such an extension). Note in particular that consistency with Galilei group demands the vanishing of the Poisson Bracket among the momenta\(^2\) Thus, we restrain our analysis to 2+1 dimensions, and assume the symplectic structure given by Eq. (5), where now \( \Theta = 0 \) and we further assume that:

\[ \theta_{ij} = \theta_{ij}, \]  

(18)

with \( \theta \) a constant parameter, which clearly characterize the non-commutativity. The Poisson brackets (PB) and the equations of motion read respectively:

\[ \{ f, g \} = \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} + \theta_{ij} \frac{\partial f}{\partial q_j} \frac{\partial g}{\partial q_j}, \]  

(19)

\[ \dot{q}_i = \frac{\partial H}{\partial p_i} + \theta_{ij} \frac{\partial H}{\partial q_j}, \]  

(20)

\[ \dot{p}_i = - \frac{\partial H}{\partial q_i}. \]  

Let us consider now a system invariant under spatial instantaneous transformations:

\[ \delta t = 0, \quad \delta q_i = b_i, \quad \delta p_i = 0, \]  

(21)

where \( b_i (i = 1, 2) \) are arbitrary infinitesimal parameters. In order to identify the generators of this transformations we write:

\[ \delta q_i = \{ q_i, Q_{\text{trans}} \} = \{ q_i, b_j \gamma_j \} = b_i, \]  

(22)

\[ \delta p_i = \{ p_i, Q_{\text{trans}} \} = \{ p_i, b_j \gamma_j \} = 0. \]

Thus, using (19) we conclude that the space translation generators satisfy the equations:

\[ \frac{\partial \gamma_{ij}}{\partial p_i} + \theta_{ik} \frac{\partial \gamma_{ij}}{\partial q_k} = \delta_{ij}, \]  
\[ - \frac{\partial \gamma_{ij}}{\partial q_k} = 0, \]  

(23)

and then, up to a constant, the space translation generators are the momentum components \( \gamma_i = p_i \). In a similar way we can treat the boost transformations:

\[ \delta t = 0, \quad \delta q_i = v_i t, \quad \delta p_i = mv_i, \]  

(24)

where \( v_i \) \( (i = 1, 2) \) are the infinitesimal parameters associated to boosts. In this case the transformation of the position and momentum are given by:

\[ \delta q_i = \{ q_i, Q_{\text{boost}} \} = \{ q_i, -k_j v_j \} = v_i t, \]  
\[ \delta p_i = \{ p_i, Q_{\text{boost}} \} = \{ p_i, -k_j v_j \} = mv_i. \]  

(25)

\(^2\) At this point it is not clear the relation between the Galilei algebra and the symplectic structure. The connection between these structures is seen when the generators of the Galilei group are expressed in terms of the dynamical variables of a physical system, see below.
Using (19) and (22) Eq. (26) reduces to:
\[ \frac{\partial k_j}{\partial p_i} + \theta \epsilon_{ik} \frac{\partial k_i}{\partial q_k} = -t \delta_{ij}, \tag{26} \]
\[ \frac{\partial k_j}{\partial q_i} = m \delta_{jk}, \]
therefore the boosts generators \( k_j \) are:
\[ k_i = m q_i - p_i t + m \theta \epsilon_{ij} p_j, \tag{27} \]
Likewise, we obtain for the angular momentum, the generator of spatial rotations:
\[ J = \epsilon_{ij} q_j p_i + \frac{\theta}{2} p_k p_k. \tag{28} \]
We can now determine the Hamiltonian of a system possessing all the symmetries of the Galilean group, which we take as the definition of a free particle. To this end we consider the time derivative of an arbitrary function of the phase space variables:
\[ \frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial z^\alpha} \dot{z}^\alpha. \tag{29} \]
which by using the equations of motion (19), is cast in the form:
\[ \frac{df}{dt} = \frac{\partial f}{\partial t} + \{ f, H(z) \}. \]
Applying this relation to Noether’s charges (\( \dot{Q} = 0 \)) we obtain the following relations:
\[ \{ H, H \} = 0, \quad \{ p_i, H \} = 0, \quad \{ J, H \} = 0, \tag{30} \]
\[ \{ k_i, H \} = p_i. \]
The Poisson brackets Eq. (31) after using (19, 22) and (28), amounts to the set of simultaneous equations:
\[ \frac{\partial H}{\partial q_i} = 0, \quad \frac{\partial H}{\partial p_i} = \frac{p_i}{m}. \]
Thus, we conclude that the Hamiltonian for a free particle of mass \( m \), in a non-commutative plane, is given by:
\[ H = \frac{p_i p_i}{2 m}, \tag{31} \]
This is a complicated form of deriving the conventional Hamiltonian for a free particle, in this way however we are certain that the description is consistent with the Galilean relativity. Furthermore, it is important to remark that for a explicitly time-independent Hamiltonian \( H = H(z) \), the following transformation always define a global invariance of the system
\[ \delta z^\alpha = -\frac{d z^\alpha}{dt} \delta t, \quad \delta t = \tau, \tag{32} \]
where \( \tau \) is a constant parameter. Therefore, the generator of time translations is simply
\[ \gamma = -H - \Lambda. \tag{33} \]
Therefore, for every system described by a time-independent Hamiltonian, the Hamiltonian itself is a constant of motion.
For completeness we quote the full Galilei algebra, which is obtained through the Poisson brackets and the expressions for the generators previously derived:
\[ \{ p_i, H \} = 0, \quad \{ p_i, p_j \} = 0, \quad \{ J, H \} = 0, \quad \{ J, p_i \} = \epsilon_{ij} p_j, \quad \{ k_i, p_i \} = m \delta_{ji}, \quad \{ k_i, k_j \} = \epsilon_{ij} k_j, \quad \{ k_i, k_j \} = -m^2 \theta \epsilon_{ij}. \tag{34} \]
The last two relations show the appearance of the central extensions, the mass \( m \) and the NC parameter \( \theta \). Thus, the symplectic structure used to describe the mechanics follows from the symmetry of the space-time described by the Galilei algebra.
We now consider the examples at the classical level [21]. To start let us consider the free particle. The Hamiltonian Eq. (31) leads to the equations:
\[ \dot{q}_i = \frac{p_i}{m}, \quad \dot{p}_i = 0. \]
The solution is the the conventional one:
\[ q_i = q_{i0} t + q_{i0}, \quad p_i = m q_{i0}, \]
with \( q_i(0) = q_{i0} \) and \( \dot{q}_i(0) = \dot{q}_{i0} \) given initial conditions. A more involved problem is the system, to which, for obvious reasons we will refer as the harmonic oscillator, characterized by the Hamiltonian:
\[ H = \frac{1}{2 m} p_i p_i + \frac{m \omega^2}{2} q_i q_i. \tag{35} \]
In this case the relations arising from the Hamiltonian formalism can be combined to produce the following equations for \( x = q_1 \) and \( y = q_2 \):
\[ \ddot{x} = -\omega^2 x + m \theta \omega^2 y \tag{36} \]
\[ \ddot{y} = -\omega^2 y - m \theta \omega^2 x. \]
Denoting the initial conditions by \( x(0) = x_0, \dot{x}(0) = \dot{x}_0, \)
\( y(0) = y_0 \) \( \dot{y}(0) = \dot{y}_0 \), the solution to the harmonic oscillator is given by:
\[ x(t) = T_1(t) x_0 + T_2(t) \dot{x}_0 + T_3(t) \left( m \theta \omega^2 x_0 + 2 y_0 \right) - T_4(t) \left( m \theta \dot{x}_0 + 2 \dot{y}_0 \right), \tag{37} \]
\[ y(t) = T_1(t) y_0 + T_2(t) \dot{y}_0 + T_3(t) \left( m \theta \omega^2 y_0 - 2 x_0 \right) - T_4(t) \left( m \theta \dot{y}_0 - 2 \dot{x}_0 \right). \]
where we have introduced the following definitions:

\[ T_1(t) = \frac{\cos(\phi t) + \cos(\chi t)}{2}, \quad T_4(t) = \frac{\phi \sin(\chi t) - \chi \sin(\phi t)}{2\omega^4 + m^2\omega^2\theta^2} \]

\[ T_2(t) = \frac{\phi \sin(\chi t) + \chi \sin(\phi t)}{2\omega^2}, \quad T_3(t) = \frac{\cos(\chi t) - \cos(\phi t)}{2\omega^4 + m^2\omega^2\theta^2} \]

\[ \phi, \chi = \sqrt{\omega^2 + \frac{1}{2} m^2 \theta^2 \omega^4 + \frac{1}{2} m \theta^3 \sqrt{4 + m^2 \omega^2 \theta^2}}. \]

As expected, the solution to the NC case, Eq.\((43)\), reduce to the conventional harmonic oscillator in the \(\theta \to 0\) limit. In order to asses the effect of NC on this system, it is convenient to compare the solution to the commutative case. This is easier to do in the \(m \theta \omega^2 \ll \omega\), where one shows that:

\[ x + iy \simeq (x^{(\theta=0)} + iy^{(\theta=0)}) e^{-\frac{1}{2}(m \theta \omega^2 t)}. \]  \(\text{(38)}\)

and so we conclude that the effect of NC amounts to rotate, in the \(x - y\) plane, the commutative solutions with angular velocity \(m \theta \omega^2 / 2\).

To conclude this section we consider the symmetry group of the harmonic oscillator. Since the conventional and NC cases are different, and can not be obtained one from the other (see below), for the sake of clarity we treat both of them. The procedure we follow consist in parameterizing the conserved quantities and then to obtain the parameters through the use of the Jacobi identity. In fact, if the conserved quantities do not depend explicitly of time, then the equation of motion implies:

\[ \{ H, S_i \} = 0. \]  \(\text{(39)}\)

On the other hand, using the Jacobi identity we obtain:

\[ \{ H, \{ S_i, \cdot \} \} - \{ S_i, \{ H, \cdot \} \} = \{ \{ H, S_i \}, \cdot \} = 0. \]  \(\text{(40)}\)

Let us begin with the commutative case. We will use a subscript or superscript \(\theta\) to avoid confusion with the analogous problem in the NC case. The PB of the Hamiltonian with an arbitrary function is:

\[ \{ H, \cdot \}_0 = -\frac{p_x}{m} \frac{\partial}{\partial x} - \frac{p_y}{m} \frac{\partial}{\partial y} + m \omega^2 x \frac{\partial}{\partial p_x} + m \omega^2 y \frac{\partial}{\partial p_y}. \]  \(\text{(41)}\)

We parameterize the PB of the constant of motion \(S^0_i\) with an arbitrary function in the following form:

\[ \{ S^0_i, \cdot \}_0 = \sum_{\alpha=1}^{4} \sum_{\beta=1}^{4} \left( a^0_{(i)_{\alpha \beta}} \right) \frac{\partial}{\partial z^\alpha} \]  \(\text{(42)}\)

By using this parametrization we assume the conserved quantities are bilinear in the phase space variables. Furthermore notice that we have to invert and integrate Eq.\((12)\) to obtain the \(S^0_i\). The matrix elements \(a^0_{(i)_{\alpha \beta}}\) are constant, unknown parameters, to be determined by the relations \(\text{(10)}\) and \(\text{(11)}\). After a lengthy calculation one concludes that the relations give rise to the following four linearly independent matrices \(a^0_{(i)}\):

\[ a^0_0 = \frac{1}{m} \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ m^2 \omega^2 & 0 & 0 & 0 \\ 0 & m^2 \omega^2 & 0 & 0 \end{pmatrix}, \]

\[ a^0_1 = \frac{1}{2m \omega} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ m^2 \omega^2 & 0 & 0 & 0 \\ m^2 \omega^2 & 0 & 0 & 0 \end{pmatrix}, \]

\[ a^0_2 = \frac{1}{2m \omega} \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \]

\[ a^0_3 = \frac{1}{2m \omega} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ -m^2 \omega^2 & 0 & 0 & 0 \\ 0 & m^2 \omega^2 & 0 & 0 \end{pmatrix} \]  \(\text{(43)}\)

from which we obtain the conserved quantities:

\[ S^0_0 = H, \]

\[ S^0_1 = \frac{1}{2m \omega} (p_x p_y + m^2 \omega^2 xy), \]

\[ S^0_2 = \frac{1}{4m \omega} \left[ p_x^2 - p_x^2 + m^2 \omega^2 (y^2 - x^2) \right], \]

\[ S^0_3 = \frac{J_0}{2} = \frac{1}{2} (xy + y^2 - xy). \]  \(\text{(44)}\)

It is clear that the four quantities are not independent, in fact it is easy to show that the following relation holds:

\[ (S^0_0)^2 + (S^0_2)^2 + (S^0_3)^2 = \frac{H^2}{4m^2 \omega^2}. \]  \(\text{(45)}\)

Moreover one verifies that the \(S^0_i\) \((i = 1, \ldots, 3)\) fulfill the following algebra:

\[ \{ S^0_i, S^0_j \}_0 = \epsilon_{ijk} S^0_k. \]  \(\text{(46)}\)

Thus we have re-derived the well known result that \(SU(2)\) is the symmetry group of the two dimensional commutative harmonic oscillator.

We now consider the symmetry group for the NC harmonic oscillator \(\text{[22]}\). Instead of Eq.\((41)\) we have to use:

\[ \{ H, \cdot \} = -\left( \frac{p_x}{m} + \theta m \omega^2 y \right) \frac{\partial}{\partial x} - \left( \frac{p_y}{m} - \theta m \omega^2 x \right) \frac{\partial}{\partial y} + m \omega^2 x \frac{\partial}{\partial p_x} + m \omega^2 y \frac{\partial}{\partial p_y}. \]  \(\text{(47)}\)

We assume again Eq.\((12)\) for the PB of the conserved quantities. In this case the constraints due to Eqs.\((36)\)
Eqs. (44, 49) we observe that in the intervention of the ambiguities in defining the commutative harmonic oscillator is not recovered. Operationally, it is clear that in the \( \theta \to 0 \) limit, the symmetry of the commutative harmonic oscillator is not recovered. Operationally, it is clear that in the \( \theta \to 0 \) limit, Eqs. (48, 49) reproduce only two of the three independent conserved quantities.

\[ \begin{align*}
\mathbf{a}_0 &= \frac{1}{m} \begin{pmatrix}
0 & -\theta m^2 \omega^2 & -1 & 0 \\
\theta m^2 \omega^2 & 0 & 0 & -1 \\
m^2 \omega^2 & 0 & 0 & 0 \\
0 & m^2 \omega^2 & 0 & 0
\end{pmatrix}, \\
\mathbf{a}_1 &= \begin{pmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0
\end{pmatrix},
\end{align*} \]

so that for the NC oscillator the constants of motion are:

\[ \begin{align*}
S_0 &= H, \\
S_1 &= J = xp_y - yp_x + \frac{\theta}{2} (p_x^2 + p_y^2).
\end{align*} \]

We conclude that \( SU(2) \) is not a symmetry of the harmonic oscillator on the NC plane \( 23 \). The fact that diverse conclusions regarding the symmetry of the NC harmonic oscillator is reached by different authors \( 16, 17, 18 \) is a manifestation of the ambiguities in defining the theory. A final comment is in order. Comparing Eqs. (44, 45) we observe that in the \( \theta \to 0 \) limit, the symmetry of the commutative harmonic oscillator is not recovered. Operationally, it is clear that in the \( \theta \to 0 \) limit, Eqs. (48, 49) reproduce only two of the three independent conserved quantities.

III. QUANTUM MECHANICS

In this section we discuss the problem of quantization assuming that a Hamiltonian description – as presented in the previous section – is available for the corresponding classical system. The quantization proceeds through the correspondence principle or canonical quantization \( 25 \) which associates to the classical phase space a quantum space of states described in terms of a Hilbert space. To the fundamental degrees of freedom \( z^\alpha \) the principle associates linear operators \( \hat{z}^\alpha \), which act on the Hilbert space.

The commutation relations of the quantum operators are obtained multiplying the classical PB by \( \hbar \). \( 3 \)

\[ \{ z^\alpha, z^\beta \} = J^{\alpha\beta} \to [ z^\alpha, z^\beta ] = i\hbar J^{\alpha\beta}, \]

Very much as the PB define a geometric structure on phase space, the Hilbert space possess an algebraic structure in the sense that it provides a linear representation of the algebra of the quantum operators \( 24 \).

The internal product of the Hilbert space must satisfy two requirements. First the operators associated to the classical fundamental degrees of freedom must be Hermitian and self-adjoint respect to the internal product, and second the internal product must be Hermitian, \( i.e. \)

\[ \langle \psi | \phi \rangle^* = \langle \phi | \psi \rangle, \]

the * stands for complex conjugation and \( |\psi\rangle, |\phi\rangle \) are arbitrary quantum states.

Thus, the description of the dynamics of a quantum system use the same structures as the classical one, namely the Hamiltonian and the symplectic structure. The time evolution of the system is given by the Schrödinger equation:

\[ \hat{H} |\psi; t\rangle = i\hbar \frac{d}{dt} |\psi; t\rangle, \]

where \( \hat{H} \) is the quantum Hamiltonian (Hermitian, self-adjoint, obtained from the classical time translation generator) and \( |\psi; t\rangle \) the state of the system at time \( t \).

From now on we use the notation \( x = q_1, y = q_2, p_x = p_1 \) and \( p_y = p_2 \). The commutation relations are then:

\[ \begin{cases}
[\hat{x}, \hat{y}] = [\hat{y}, \hat{p}_y] = [\hat{y}, \hat{p}_x] = 0, \\
[\hat{p}_x, \hat{p}_x] = [\hat{p}_y, \hat{p}_y] = [\hat{p}_x, \hat{p}_y] = 0 \\
\hat{x} = i\hbar \theta, \\
\hat{y} = [\hat{x}, \hat{p}_x] = [\hat{y}, \hat{p}_y] = i\hbar.
\end{cases} \]

This is the NC version of the Heisenberg algebra. In this section we will develop an abstract representation theory of this algebra. To this end we need two basic postulates:

1. There exists a base \( |x, p_y\rangle \) that diagonalizes simultaneously both the position operator \( \hat{x} \) and the momentum \( \hat{p}_y \), whose domain of eigenvalues coincides with all possible values of the coordinates \( x \) and \( p_y \), which parameterize an arbitrary connected and differentiable manifold \( M \).

\[ \hat{x} |x, p_y\rangle = x |x, p_y\rangle, \quad \hat{p}_y |x, p_y\rangle = p_y |x, p_y\rangle, \quad x, p_y \in M. \]

2. There exists an internal product \( \langle \phi | \psi \rangle \), positive definite and Hermitian for which the operators \( \hat{x}, \hat{y}, \hat{p}_x \) and \( \hat{p}_y \) are self-adjoint.

With the rules at hand, the application of the postulates in the evaluation of the matrix elements \( \langle x, p_y | \hat{x} | x', p'_y \rangle \) and \( \langle x, p_y | \hat{p}_x | x', p'_y \rangle \) imply:

\[ \begin{align*}
(x - x') \langle x, p_y | x', p'_y \rangle &= 0, \\
(p_y - p'_y) \langle x, p_y | x', p'_y \rangle &= 0.
\end{align*} \]

The general solution to these equations is:

\[ \langle x, p_y | x', p'_y \rangle = \frac{\delta(x - x')\delta(p_y - p'_y)}{\sqrt{g(x, p_y)}}, \]

\[ \frac{\delta(x - x')\delta(p_y - p'_y)}{\sqrt{g(x, p_y)}} \]

\footnote{The \( \hbar \) enters naturally in the quantization procedure, just on dimensional grounds one requires it. In the NC case the situation is different, in fact one should keep in mind the appropriated units of \( \theta \), which are not the same that at the classical level.}
where $g(x, p_y)$ is a positive definite, arbitrary function which is a a priori related to the normalization of the eigenbasis. As a consequence of this result the spectral decomposition of the identity operator in the $|x, p_y\rangle$ base is:

$$\hat{1} = \int_M dx dp_y \sqrt{g(x, p_y)} |x, p_y\rangle \langle x, p_y|.$$  

(57)

this completeness relation allow us to construct the wave function:

$$\psi(x, p_y) = \langle x, p_y| \psi \rangle,$$  

(58)

so that for any state $|x, p_y\rangle$ belonging to the representation of the algebra $g$,

$$|\psi\rangle = \int_M dx dp_y \sqrt{g(x, p_y)} \psi(x, p_y) |x, p_y\rangle,$$  

(59)

$$\langle \psi | = \int_M dx dp_y \sqrt{g(x, p_y)} \psi^*(x, p_y) \langle x, p_y|.$$  

(60)

In particular, the internal product of two arbitrary states $|\psi\rangle$ and $|\phi\rangle$, is expressed in terms of the wave functions $\psi(x, p_y)$ and $\phi(x, p_y)$:

$$\langle \psi | \phi \rangle = \int_M dx dp_y \sqrt{g(x, p_y)} \psi^*(x, p_y) \phi(x, p_y).$$  

(61)

where $A(x, p_y)$ and $B(x, p_y)$ are two real, arbitrary functions defined over $M$. Further constraints on $A$ and $B$ can arise from the hermiticity of the $\hat{y}$ operator, namely:

$$\frac{-i\hbar}{\sqrt{g(x, p_y)}} \left(-\frac{\partial}{\partial x} + \frac{\partial}{\partial p_y}\right) \delta(x - x') \delta(p_y - p_y') + \frac{[A(x, p_y) - iB(x, p_y)]}{\sqrt{g(x, p_y)}} \delta(x - x') \delta(p_y - p_y').$$

$$= \frac{i\hbar}{\sqrt{g(x', p_y')}} \left(-\frac{\partial}{\partial x'} + \frac{\partial}{\partial p_y'}\right) \delta(x - x') \delta(p_y - p_y') + \frac{[A(x', p_y') + iB(x', p_y')]}{\sqrt{g(x', p_y')}} \delta(x - x') \delta(p_y - p_y').$$

In order to solve this equation we assume the existence of a continuous distribution $T(x', p_y')$ on $M$, we multiply both sides of the last equation by $T(x', p_y')$ and integrate over the domains of $x'$ and $p_y'$

$$\int_M T(x', p_y') dx' dp_y' \left\{ \frac{-i\hbar}{\sqrt{g(x, p_y)}} \left(-\frac{\partial}{\partial x} + \frac{\partial}{\partial p_y}\right) \delta(x - x') \delta(p_y - p_y') + \frac{[A(x, p_y) - iB(x, p_y)]}{\sqrt{g(x, p_y)}} \delta(x - x') \delta(p_y - p_y') \right\}$$

$$= \int_M T(x', p_y') dx' dp_y' \left\{ \frac{i\hbar}{\sqrt{g(x', p_y')}} \left(-\frac{\partial}{\partial x'} + \frac{\partial}{\partial p_y'}\right) \delta(x - x') \delta(p_y - p_y') + \frac{[A(x', p_y') + iB(x', p_y')]}{\sqrt{g(x', p_y')}} \delta(x - x') \delta(p_y - p_y') \right\},$$

(64)

simplifying this expression and considering that $T(x, p_y)$ is an arbitrary function, $B(x, p_y)$ turns out to be:

$$B(x, p_y) = \frac{\hbar \sqrt{g(x, p_y)} }{2} \left(-\frac{\partial}{\partial x} + \frac{\partial}{\partial p_y}\right) \frac{1}{\sqrt{g(x, p_y)}},$$  

(65)

Evaluation of the matrix elements of the commutators $[[\hat{x}, \hat{y}], [\hat{y}, \hat{p}_y]]$, using the postulates and Eq. (53), lead to the relations:

$$\langle x, p_y| [\hat{x}, \hat{y}] | x', p_y'\rangle = i\hbar \frac{\delta(x - x') \delta(p_y - p_y')}{\sqrt{g(x, p_y)}} \langle x - x' | \langle x, p_y| \hat{y} | x', p_y'\rangle,$$  

(61)

$$\langle x, p_y| [\hat{y}, \hat{p}_y] | x', p_y'\rangle = i\hbar \frac{\delta(x - x') \delta(p_y - p_y')}{\sqrt{g(x, p_y)}} \langle x - x' | \langle x, p_y| \hat{p}_y | x', p_y'\rangle,$$  

(62)

Thus, matrix element of the $\hat{y}$ operator can be parameterized as follows:

$$\langle x, p_y| \hat{y} | x', p_y'\rangle$$

$$= \frac{i\hbar}{\sqrt{g(x, p_y)}} \left(-\frac{\partial}{\partial x} + \frac{\partial}{\partial p_y}\right) \delta(x - x') \delta(p_y - p_y') + \frac{[A(x, p_y) + iB(x, p_y)]}{\sqrt{g(x, p_y)}} \delta(x - x') \delta(p_y - p_y').$$

(63)
therefore the matrix elements of \( \hat{y} \) can be expressed as:

\[
\langle x, p_y | \hat{y} | x', p'_y \rangle = \frac{i\hbar}{g^{1/4}(x, p_y)} \left( -\theta \frac{\partial}{\partial x} + \frac{\partial}{\partial p_y} \right) \frac{\delta(x - x') \delta(p_y - p'_y)}{g^{1/4}(x, p_y)} + \frac{A(x, p_y)}{\sqrt{g(x, p_y)}} \delta(x - x') \delta(p_y - p'_y).
\]

(66)

The same approach can be applied to the operator \( \hat{p}_x \). To this end we consider the matrix elements of the commutators \( [\hat{x}, \hat{p}_x] \) and \( [\hat{p}_x, \hat{p}_y] \) to conclude that:

\[
\langle x, p_y | [\hat{x}, \hat{p}_x] | x', p'_y \rangle = \frac{-i\hbar}{g^{1/4}(x, p_y)} \frac{\partial}{\partial x} \left( \frac{\delta(x - x') \delta(p_y - p'_y)}{g^{1/4}(x, p_y)} \right) + \frac{C(x, p_y)}{\sqrt{g(x, p_y)}} \delta(x - x') \delta(p_y - p'_y),
\]

(67)

where \( C(x, p_y) \) is another arbitrary real function defined over \( M \).

An important consequence of the NC version of the Heisenberg algebra arises from the evaluation of

\[
\langle x, p_y | [\hat{y}, \hat{p}_x] | x', p'_y \rangle = 0.
\]

(68)

The explicit calculation leads to the compatibility restriction among the functions \( A(x, p_y) \) and \( C(x, p_y) \)

\[
\left( -\theta \frac{\partial}{\partial x} + \frac{\partial}{\partial p_y} \right) C(x, p_y) + \frac{\partial}{\partial x} A(x, p_y) = 0.
\]

(69)

If \( \chi(x, p_y) \) is a scalar function defined over \( M \), the transformation

\[
C(x, p_y) \rightarrow C(x, p_y) - \frac{\partial}{\partial x} \chi(x, p_y),
\]

\[
A(x, p_y) \rightarrow A(x, p_y) + \left( -\theta \frac{\partial}{\partial x} + \frac{\partial}{\partial p_y} \right) \chi(x, p_y),
\]

(70)

leaves the condition invariant. This strongly suggests that the functions \( A(x, p_y) \) and \( C(x, p_y) \) are associated to the phase definition of the \( |x, p_y\rangle \) basis. Under a local U(1) gauge transformation the states transform according to:

\[
|x, p_y\rangle = e^{-\hat{\theta} \chi(\hat{x}, \hat{p}_y)} |x, p_y\rangle,
\]

(71)

the matrix elements in the transformed basis are related to the original ones according to the following equations:

\[
p \langle x, p_y | \hat{y} | x', p'_y \rangle = \langle x, p_y | e^{\hat{\theta} \chi(\hat{x}, \hat{p}_y)} \hat{y} e^{-\hat{\theta} \chi(\hat{x}, \hat{p}_y)} | x', p'_y \rangle,
\]

\[
p \langle x, p_y | \hat{p}_x | x', p'_y \rangle = \langle x, p_y | e^{\hat{\theta} \chi(\hat{x}, \hat{p}_y)} \hat{p}_x e^{-\hat{\theta} \chi(\hat{x}, \hat{p}_y)} | x', p'_y \rangle.
\]

The explicit evaluation of these relations yields

\[
p \langle x, p_y | [\hat{x}, \hat{p}_x] | x', p'_y \rangle = \langle x, p_y | e^{\hat{\theta} \chi(\hat{x}, \hat{p}_y)} \hat{y} e^{-\hat{\theta} \chi(\hat{x}, \hat{p}_y)} | x', p'_y \rangle
\]

\[
= \frac{i\hbar}{g^{1/4}(x, p_y)} \left( -\theta \frac{\partial}{\partial x} + \frac{\partial}{\partial p_y} \right) \frac{\delta(x - x') \delta(p_y - p'_y)}{g^{1/4}(x, p_y)} + \frac{A(x, p_y)}{\sqrt{g(x, p_y)}} \delta(x - x') \delta(p_y - p'_y),
\]

\[
= \frac{-i\hbar}{g^{1/4}(x, p_y)} \frac{\partial}{\partial x} \delta(x - x') \delta(p_y - p'_y) + \frac{\left( C(x, p_y) - \frac{\partial}{\partial x} \chi(x, p_y) \right)}{\sqrt{g(x, p_y)}} \delta(x - x') \delta(p_y - p'_y),
\]

and

\[
p \langle x, p_y | [\hat{p}_x, \hat{p}_y] | x', p'_y \rangle = \langle x, p_y | e^{\hat{\theta} \chi(\hat{x}, \hat{p}_y)} \hat{p}_y e^{-\hat{\theta} \chi(\hat{x}, \hat{p}_y)} | x', p'_y \rangle
\]

Thus, under phase transformations of the states, the functions \( A(x, p_y) \) and \( C(x, p_y) \) behave according to \( 70 \). Therefore, the configuration space representations of the Heisenberg algebra over the manifold \( M \) are characterized, on one hand, by the function \( g(x, p_y) \), and on the other by a flat U(1) bundle defined by the fields \( A(x, p_y) \) and \( C(x, p_y) \). However, since arbitrary local gauge transformations within the U(1) bundle correspond to arbitrary local phase redefinitions of the states \( |x, p_y\rangle \), and thus relate representations of the Heisenberg algebra which are unitarily equivalent, it is clear that all inequivalent representations of the Heisenberg algebra over a manifold \( M \) are classified in terms of the topologically distinct flat U(1) bundles over that manifold, i.e. the equivalence classes under local gauge transformations of U(1) gauge fields of vanishing field strength over \( M \). In the case of a simply connected manifold, every holonomy is contractible to the identity. Then, the gauge freedom of \( |x, p_y\rangle \) can be used to remove both the \( A(x, p_y) \) and \( C(x, p_y) \) fields through the adequate choice of \( \chi(x, p_y) \). Over a simply connected manifold, the NC version of Heisenberg algebra admits only the representation in which globally \( A(x, p_y) = 0 \) and \( C(x, p_y) = 0 \). When the base manifold \( M \) is not simply connected and therefore possess topological obstructions that prevent some cycles to be contracted, rendering non trivial holonomies around them, it is not possible to completely remove both \( A(x, p_y) \) and \( C(x, p_y) \).
Let us now turn our attention to the wave functions \( \psi(x, p_y) = \langle x, p_y | \psi \rangle \). Given the parametrization of \( \langle x, p_y | \hat{y}' \rangle \) and \( \langle x, p_y | \hat{p}_x \rangle \), we can use the spectral decomposition of the identity operator in order to obtain the representation of \( \hat{y} \) and \( \hat{p}_x \) as differential operators

\[
\langle x, p_y | \hat{y} \rangle \psi = A(x, p_y) \psi(x, p_y) + \frac{i\hbar}{g^{1/4}(x, p_y)} \left( -\theta \frac{\partial}{\partial x} + \frac{\partial}{\partial p_y} \right) \left[ g^{1/4}(x, p_y) \psi(x, p_y) \right],
\]

(72)

\[
\langle x, p_y | \hat{p}_x \rangle \psi = C(x, p_y) \psi(x, p_y) + \frac{-i\hbar}{g^{1/4}(x, p_y)} \frac{\partial}{\partial x} \left[ g^{1/4}(x, p_y) \psi(x, p_y) \right].
\]

(73)

The last two expressions are the general representation of the operators which takes into account: the NC, the vanishing fields \( \langle x, p_y | \hat{y} \rangle \) and \( \langle x, p_y | \hat{p}_x \rangle \) which could arise from the topological properties of \( M \) and their possible obstruction.

The properties of the wave function \( \psi(x, p_y) \) required so that the operators \( \hat{y} \) and \( \hat{p}_x \) are both Hermitian and self-adjoint are

\[
\int_M dx dp_y \left( -\theta \frac{\partial}{\partial x} + \frac{\partial}{\partial p_y} \right) \left[ g(x, p_y) \langle x, p_y | \psi \rangle^2 \right] = 0
\]

(74)

and

\[
\int_M dx dp_y \frac{\partial}{\partial x} \left[ g(x, p_y) \langle x, p_y | \psi \rangle^2 \right] = 0.
\]

(75)

Instead of using the basis \( \{ x, p_y \} \), it is possible to work with states \( \{ y, p_x \} \) which diagonalizes simultaneously the position operator \( \hat{y} \) and the momentum component \( \hat{p}_x \) :

\[
\hat{y} | y, p_x \rangle = y | y, p_x \rangle, \quad \hat{p}_x | y, p_x \rangle = p_x | y, p_x \rangle, \quad y, p_x \in D(y, p_x).
\]

(76)

Where \( D(y, p_x) \) stands for the range of spectral values of \( y \) and \( p_x \). By analogy with the normalization of the \( \{ x, p_y \} \) eigenbasis \( \mathcal{M} \), the normalization of \( \{ y, p_x \} \) is parameterized according to the relation

\[
\langle y, p_x | y', p_x' \rangle = \frac{\delta(y - y') \delta(p_x - p_x')}{\sqrt{h(y, p_x)}},
\]

(77)

where \( h(y, p_x) \) is again an arbitrary positive definite function defined over \( D(y, p_x) \). All the results are very similar to the ones obtained in the \( \{ x, p_y \} \) basis, so instead of repeating the arguments we discuss the quantities relevant to the change among those basis, i.e. the wave functions

\[
\langle x, p_y | y, p_x \rangle.
\]

These wave functions are determined by the following set of differential equations:

\[
\left[ -i\hbar \frac{\partial}{\partial x} + C(x, p_y) \right] \left[ g^{1/4}(x, p_y) \langle x, p_y | y, p_x \rangle \right] = p_x \left[ g^{1/4}(x, p_y) \langle x, p_y | y, p_x \rangle \right],
\]

(78)

\[
\left[ -i\hbar \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial p_y} \right) + A(x, p_y) \right] \left[ g^{1/4}(x, p_y) \langle x, p_y | y, p_x \rangle \right] = y \left[ g^{1/4}(x, p_y) \langle x, p_y | y, p_x \rangle \right].
\]

(79)

Since the first order differential equations require only one an integration constant, namely the wave function \( \langle x^0, p^0_y | y, p_x \rangle \), associated to a specific point on the manifold \( M \) of coordinates \( \{ x^0, p^0_y \} \). Then, any other point of coordinates \( \{ x, p_y \} \) can be reached from \( \{ x^0, p^0_y \} \) through an oriented path \( P(x^0, p^0_y) \rightarrow (x, p_y) \) running from \( \{ x^0, p^0_y \} \) to \( \{ x, p_y \} \). The solution to the form

\[
g^{1/4}(x, p_y) \langle x, p_y | y, p_x \rangle = \left[ g^{1/4}(x_0^0, p^0_y) \langle x_0^0, p^0_y | y, p_x \rangle \right] \times \\
\Omega \left[ P(x_0^0, p^0_y) \rightarrow (x, p_y) \right] \times \\
e^{i\frac{\hbar}{\pi} (x-x^0)(p_x-p_x^0) + \theta(p_y-p^0_y) \times p_x}
\]

(80)

where \( \Omega \left[ P(x_0^0, p^0_y) \rightarrow (x, p_y) \right] \) represents an ordered holonomy along the path \( P(x_0^0, p^0_y) \rightarrow (x, p_y) \) as shown by the following formula:

\[
\Omega \left[ P(x_0^0, p^0_y) \rightarrow (x, p_y) \right] = \\
\exp \left\{ -\frac{i}{\hbar} \left[ \int_{P(x^0, p^0_y)} dx C(x, p_y) - \int_{P(p_x^0 - p_x^0)} dp_y A(x, p_y) \right] + \theta \int_{P(p_x^0 - p_x^0)} dp_y C(x, p_y) \right\}.
\]

(81)

The normalization condition of the wave function \( \langle y, p_x | y', p_x' \rangle \) also requires that

\[
\left[ g^{1/4}(x^0, p^0_y) \langle x^0, p^0_y | y, p_x \rangle \right]^2 = \frac{1}{(2\pi \hbar)^2 \sqrt{h(y, p_x)}},
\]

(82)

so that necessarily

\[
g^{1/4}(x^0, p^0_y) \langle x^0, p^0_y | y, p_x \rangle = \frac{e^{i\varphi(x^0, p^0_y, y, p_x)}}{(2\pi \hbar)h^{1/4}(y, p_x)}.
\]

(83)

where \( \varphi(x^0, p^0_y, y, p_x) \) is a specific real function independent of \( x \) and \( p_x \). Then, the wave functions \( \langle x, p_y | y, p_x \rangle \) are given by

\[
\langle x, p_y | y, p_x \rangle = \frac{e^{i\varphi(x^0, p^0_y, y, p_x)}}{(2\pi \hbar)h^{1/4}(y, p_x)h^{1/4}(x, p_y)} \times \\
e^{i\frac{\hbar}{\pi} (x-x^0)\times (p_x-p_x^0) + \theta(p_y-p^0_y) \times (p_x-p_x^0)}.
\]

(84)
The specific choice of \( \varphi(x^{(0)},p_{y}^0,y,p_x) \), such that
\[
e^{i\varphi(x^{(0)},p_{y}^0,y,p_x)} e^{-\frac{1}{2}[x^{(0)}p_x + p_{y}^0y - \theta p_{y}^0 p_x]} = 1 \tag{85}
\]
simplifies the wave function representation of NC Heisenberg algebra:
\[
\langle x, p_y | y, p_x \rangle = \frac{\Omega [P| (x^{0},p_{y}^0) \rightarrow (x, p_y)] e^{i\frac{1}{2}[x_{p_x-y} + \theta p_{y} p_x]}}{(2\pi\hbar)^{1/4}(x, p_y)\hbar^{1/4}(y, p_x)} \tag{86}
\]
This is the NC generalization of the customary wave function \((\theta = 0)\) that arises in conventional QM, and that coincides with the Fourier Transform kernel.

Another admissible basis compatible with the commutation relations \((\theta = 0)\) is \( |p\rangle = |p_x, p_{y}\rangle \), which diagonalizes simultaneously both components of the momentum operator:
\[
\hat{p}_x | p \rangle = p_x | p \rangle, \quad \hat{p}_y | p \rangle = p_y | p \rangle. \quad p_x, p_y \in D(p) \tag{87}
\]
Where \( D(p) \) is the range of spectral eigenvalues of \( p_x \) and \( p_y \). The normalization of \( |p\rangle \) can be parameterized according to
\[
|p\rangle = \frac{\delta(p_x - p_{x}')\delta(p_y - p_{y}')}{\sqrt{\gamma(p_x, p_y)}}, \tag{88}
\]
with \( \gamma(p_x, p_y) \) as a new arbitrary positive definite function, defined on the \( D(p) \) domain. This choice implies that the spectral decomposition of the unit operator, in terms of the momentum eigenbasis is of the form
\[
\hat{1} = \int_{D(p)} dp_x dp_y \sqrt{\gamma(p_x, p_y)} |p\rangle \langle p|. \tag{89}
\]
A procedure similar to the one used in the \( |x, p_y\rangle \), leads to the following matrix elements:
\[
\langle p | \hat{x} | p' \rangle = \frac{i\hbar}{\gamma^{1/4}(p_x, p_y)} \left( \frac{\partial}{\partial p_x} \right) \frac{\delta(p_x - p_x')\delta(p_y - p_y')}{\gamma^{1/4}(p_x, p_y)} \tag{90}
\]
\[
+ G_x(p_x, p_y) \frac{\delta(p_x - p_x')\delta(p_y - p_y')}{\sqrt{\gamma(p_x, p_y)}},
\]
\[
\langle p | \hat{y} | p' \rangle = \frac{i\hbar}{\gamma^{1/4}(p_x, p_y)} \left( \frac{\partial}{\partial p_y} \right) \frac{\delta(p_x - p_x')\delta(p_y - p_y')}{\gamma^{1/4}(p_x, p_y)} \tag{91}
\]
\[
+ G_y(p_x, p_y) \frac{\delta(p_x - p_x')\delta(p_y - p_y')}{\sqrt{\gamma(p_x, p_y)}}.
\]
In these equations, \( G_x(p_x, p_y) \) and \( G_y(p_x, p_y) \) are the components of a vector field defined on \( D(p) \), which by virtue of the calculation of the matrix elements of the relation \([x, y] = i\hbar \theta\), satisfy the compatibility relation
\[
\frac{\partial G_y(p)}{\partial p_x} - \frac{\partial G_x(p)}{\partial p_y} = \theta. \tag{92}
\]
This equation is invariant under the following transformation:
\[
G_i(p) \rightarrow G_i(p) + \frac{\partial \xi(p)}{\partial p_i}, \tag{93}
\]
with \( \xi(p) \) as a scalar local function defined on \( D(p) \). It is important to notice that, in contradistinction to the functions \( A(x, p_y) \) and \( C(x, p_y) \) Eqs. \((86, 87)\), in the present case \( G_x(p) = 0 \) and \( G_y(p) = 0 \) cannot be simultaneously satisfied. For this reason, it is convenient to use a slight different parametrization:
\[
\langle p | \hat{x} | p' \rangle = \frac{i\hbar}{\gamma^{1/4}(p_x, p_y)} \left( \frac{\partial}{\partial p_x} \right) \frac{\delta(p_x - p_x')\delta(p_y - p_y')}{\gamma^{1/4}(p_x, p_y)} \tag{94}
\]
\[
+ \left( -\frac{\theta}{2} p_y + F_x(p_x, p_y) \right) \frac{\delta(p_x - p_x')\delta(p_y - p_y')}{\sqrt{\gamma(p_x, p_y)}}.
\]
\[
\langle p | \hat{y} | p' \rangle = \frac{i\hbar}{\gamma^{1/4}(p_x, p_y)} \left( \frac{\partial}{\partial p_y} \right) \frac{\delta(p_x - p_x')\delta(p_y - p_y')}{\gamma^{1/4}(p_x, p_y)} \tag{95}
\]
\[
+ \left( \frac{\theta}{2} p_x + F_y(p_x, p_y) \right) \frac{\delta(p_x - p_x')\delta(p_y - p_y')}{\sqrt{\gamma(p_x, p_y)}}.
\]
With this convention, \( F_x(p_x, p_y) \) and \( F_y(p_x, p_y) \) are also the components of a vector field defined on \( D(p) \) that satisfy the condition
\[
\frac{\partial F_y(p)}{\partial p_x} - \frac{\partial F_x(p)}{\partial p_y} = 0, \tag{96}
\]
which is invariant under the transformations:
\[
F_i(p) \rightarrow F_i(p) + \frac{\partial \xi(p)}{\partial p_i}. \tag{97}
\]
The \( \hat{x} \) and \( \hat{y} \) eigenfunction representation of the operators \( \hat{x} \) and \( \hat{y} \) are:
\[
\langle p | \hat{x} | \psi \rangle = \frac{i\hbar}{\gamma^{1/4}(p)} \left( \frac{\partial}{\partial p_x} \right) \frac{\gamma^{1/4}(p)}{\gamma^{1/4}(p)} \psi(p) \tag{98}
\]
\[
+ \left( -\frac{\theta}{2} p_y + F_x(p_x, p_y) \right) \psi(p),
\]
\[
\langle p | \hat{y} | \psi \rangle = \frac{i\hbar}{\gamma^{1/4}(p)} \left( \frac{\partial}{\partial p_y} \right) \frac{\gamma^{1/4}(p)}{\gamma^{1/4}(p)} \psi(p) \tag{99}
\]
\[
+ \left( \frac{\theta}{2} p_x + F_y(p_x, p_y) \right) \psi(p).
\]
The wave function \( \langle x, p_y | p' \rangle \) can be determined through the following set of differential equations
\[
(p_y - p_y') \langle x, p_y | \hat{p}_y | p' \rangle = 0, \tag{100}
\]
\[
\begin{align*}
\left[ -i\hbar \frac{\partial}{\partial x} + C(x, p_y) \right] \left( g^{1/4}(x, p_y) \langle x, p_y | p' \rangle \right) \\
= p'_y g^{1/4}(x, p_y) \langle x, p_y | p' \rangle 
\end{align*}
\] (101)

\[
\begin{align*}
\left[ -i\hbar \frac{\partial}{\partial p_x} - \frac{\theta}{2} p_y + F_z(p_x) \right] \left( \gamma^{1/4}(p_x) \langle x, p_y | p' \rangle \right) \\
= x^2 \gamma^{1/4}(p_x) \left( x, p_y | p' \right)
\end{align*}
\] (102)

The normalized solution to these equations is

\[
\langle x, p_y | p' \rangle = \frac{\delta(p_x - p'_x) e^{\frac{i}{\hbar} \int x' p_y + \frac{\theta}{2} p_y^2}}{\sqrt{2\pi \hbar g^{1/4}(x, p_y) \gamma^{1/4}(p')}} \times
\]
\[
\times \Xi \left[ P[(p'_x, p'_y) \rightarrow (p_x, p_y)] \right] \times
\]
\[
\times \Omega \left[ P[(x', p'_y) \rightarrow (x, p_y)] \right],
\]

where \( \Omega \) and \( \Xi \) are holonomies along the ordered paths connecting the fixed points \((x'^0, p'_y(0))\) and \((p'_x(0), p'_y(0))\) to \((x, p_y)\) and \((p_x, p'_y)\), respectively. \( \Omega \) was defined before in (34), and \( \Xi \) is given by:

\[
\Xi \left[ P[(p'_x, p'_y(0)) \rightarrow (p_x, p_y)] \right] = e^{\frac{i}{\hbar} \int f(p' x - p'_x(0), dp_x F_x(p') + \int f(p' y - p'_y(0), dp_y F_y(p'))},
\]

with \( \phi(x'^0, p'_y(0), p'_x(0)) \) as a constant phase. Again, in the case of a simply connected base manifold, the gauge freedom of the eigenstates \(| p \rangle\) can be used to remove completely the vector field \( F_i(p) \) through the correct choice of the gauge transformation.

The wave function \( \langle x, p_y | p' \rangle \) can be constructed from \( \langle x, p_y | y, p_x \rangle \), given by (35), and from the spectral decomposition of the unitary operator:

\[
\begin{align*}
\langle y, p_x | p' \rangle = \int_M dxdp_y \sqrt{g(x, p_y)} \langle y, p_x | x, p_y \rangle \langle x, p_y | p' \rangle \\
= \frac{\delta(p_x - p'_x) e^{\frac{i}{\hbar} \int x' p_y + \frac{\theta}{2} p_y^2} e^{i \Phi(y, p_x, p'_x, p'_y(0), p'_y(0))}}{\sqrt{2\pi \hbar g^{1/4}(x, p_y) \gamma^{1/4}(p')}} \times
\end{align*}
\]
\[
\times \Xi \left[ P[(p'_x(0), p'_y(0)) \rightarrow (p'_x, p'_y)] \right],
\]

where \( \Phi(y, p_x, p'_x, p'_y(0), p'_y(0)) \) is an arbitrary real function.

To conclude this section we quote the change of basis among the different wave functions \( \Psi(x, p_y) = \langle x, p_y | \Psi \rangle \), \( \Psi(y, p_x) = \langle y, p_x | \Psi \rangle \) and \( \Psi(p) = \langle p | \Psi \rangle \):

\[
\Psi(x, p_y) = \int_{D(y, p_x)} dy dp_x \sqrt{h(y, p_x)} \langle x, p_y | y, p_x \rangle \Psi(y, p_x),
\]
\[
\Psi(x, p_y) = \int_{D(p_z, p_y)} dp_x dp_y \sqrt{\gamma(p'_x, p'_y)} \langle x, p_y | p' \rangle \Psi(p'),
\]
\[
\Psi(y, p_x) = \int_M dxdp_y \sqrt{g(x, p_y)} \langle y, p_x | x, p_y \rangle \Psi(x, p_y),
\]
\[
\Psi(y, p_x) = \int_{D(p_z, p_y)} dp_x dp_y \sqrt{\gamma(p'_x, p'_y)} \langle y, p_x | p' \rangle \Psi(p'),
\]
\[
\Psi(y, p_x) = \int_{D(y, p_z)} dy dp_x \sqrt{h(y, p_z)} \langle p | y, p_x \rangle \Psi(y, p_x),
\]

with \( \langle x, p_y | y, p_x \rangle \), \( \langle x, p_y | p' \rangle \) and \( \langle y, p_x | p' \rangle \) given by equations (36), (103) and (105), respectively.

### IV. PATH INTEGRAL IN PHASE SPACE

In this section, we show the relation between the canonical quantization formalism discussed in the previous part and the path integral representation of quantum amplitudes. We follow the conventional approach as well as a previous work devoted to the NC case (13).

Given a system with initial configuration \( i \), the probability associated to the evolution of this system towards a final configuration \( f \) is

\[
P_{f \rightarrow i} = | \langle f | \hat{U}(t_f, t_i) | i \rangle |^2.
\]

(107)

If we choose the eigenbasis \(| x, p_y \rangle\) to label initial and final states, the transition amplitude can be written as

\[
K(x_f, p_y, tf; xi, p_y, ti) = \langle x_f, p_y | e^{-\frac{i}{\hbar}(t_f-t_i)\hat{H}} | xi, p_y \rangle.
\]

(108)

The argument is based on the factorization of the temporal evolution operator in the form:

\[
\hat{U}(t_f, t_i) = \left\{ e^{-\frac{i}{\hbar}(t_f-t_i)\hat{H}} \right\}^N.
\]

Inserting two spectral decompositions of the unity operator (in the \(| x, p_y \rangle\) and \(| y, p_x \rangle\) basis) between each of the \( N \) factors and making use of the wave functions \( \langle x_f, p_y | y, p_x \rangle \) Eq. (36), the Kernel can be written as:
where, to avoid lengthy expressions, we introduced the

As expected, factors associated to \((109)\) cancel out among themselves, except for those

Finally, the kernel expressed as Functional Integrals over

\[ K(x_f, p_y f, t_f; x_i, p_y i, t_i) \]

\[ = \lim_{N \to \infty} \langle x_f, p_y f \rangle \left( 1 - \frac{i}{\hbar} \frac{\partial}{\partial t} \right)^N \langle x_i, p_y i \rangle \]

where the appropriate boundary conditions for the functional integrals has been indicated using the notation

\[ x(t_i) = x_i, p_y(t_i) = p_y i, x(t_f) = x_f, p_y(t_f) = p_y f \]

In this formal expression, the integration measure over phase space is the Liouville measure. One can easily identify in the last relation the classical action up to an irrelevant surface term

\[ \int_{t_i}^{t_f} dt \left[ \dot{x}p_x - \dot{p}_yp_x + \theta p_y p_x - H \right] \]

\[ = \int_{t_i}^{t_f} dt \left[ \frac{1}{2} \hat{z}^\alpha \hat{\omega}_{\alpha \beta} \hat{z}^\beta - H(z, t) + \frac{d\Lambda}{dt} \right] \]

\[ = S[z(t)] + \Lambda|_{t_i}^{t_f}. \]

It should be clear that the path integral representation can be expressed not only using the \(|x, p_y\rangle\) states but in any of the basis we analyzed.

V. WIGNER FUNCTION IN PHASE SPACE

In this section we discuss the third independent, and complete, description of QM, formally distinct to the conventional operator approach in Hilbert space and to the Path integral quantization procedure. This quantization framework is based on the Wigner quasi-distribution function \([27]\). The main feature of this formalism is the fact that interprets the coordinates of phase space \(z^\alpha\) not as operators but as c-numbers.

Wigner’s function can be built from the density matrix. The density operator is defined as the weighted sum over all possible projectors

\[ \hat{\rho} = \sum_n w_n \mid \phi_n \rangle \langle \phi_n \mid, \]

the \(|\phi_n\rangle\) form a complete set of normalized states and

\[ \sum_n w_n = 1. \]

The matrix elements of this operator, with respect to the \(|x, p_y\rangle\) basis, for instance, are

\[ \rho(x, p_y; x', p_y') = \sum_n w_n \langle x, p_y | \phi_n \rangle \langle \phi_n | x', p_y' \rangle \]

\[ = \sum_n w_n \phi_n(x, p_y) \phi_n^*(x', p_y'). \]

Due to the normalization of the states and to Eq.\([115]\):

\[ \text{Tr}(\hat{\rho}) = \int \rho(x, p_y; x, p_y) dx dp_y = 1. \]
For a given operator $\hat{A}$, the ensemble expected value is defined as

$$\langle \hat{A} \rangle = Tr(\hat{\rho}\hat{A})$$

(118)

$$= \sum_n w_n \int \phi_n(x,p_y) \left( \hat{A} \phi_n^*(x,p_y) \right) dx dp_y.$$

If $w_j = 1$ and $w_j \neq 0$ the system is in a pure state; otherwise, the system is in a mixed state. The quantity $Tr(\hat{\rho}^2) = \sum_n w_n^2 \leq 1$ (with $Tr(\hat{\rho}^2) = 1$ only possible for pure states), is called Purity.

The quasi-distribution Wigner function is defined through Wigner-Weyl prescription, which assigns a c-number function $A_W(z)$ to each operator $\hat{A}$ in Hilbert space. For the two dimensional under consideration, we have explicitly:

$$A_W(z) = \frac{1}{(2\pi \hbar)^2} \int d^2 \sigma d^2 \tau \left\{ e^{\frac{i}{\hbar}(\sigma,\tau) \times} \right\}$$

(119)

$$\times Tr \left[ \exp{\left( -\frac{i}{\hbar} (\tau \hat{p}_x + \sigma \hat{p}_y + \sigma_1 x + \sigma_2 y) \right) A \right].$$

where we introduced the notation $d^2 \sigma d^2 \tau = d\sigma_1 d\sigma_2 d\tau_1 d\tau_2$ and $\phi(\sigma,\tau) = \tau_1 p_x + \tau_2 p_y + \sigma_1 x + \sigma_2 y$. The Wigner function $W(z)$ is defined as:

$$W(z) = \frac{1}{(2\pi \hbar)^2} \int d^2 \sigma d^2 \tau \left\{ e^{\frac{i}{\hbar}(\sigma,\tau) \times} \right\}$$

(120)

$$\times Tr \left[ \exp{\left( -\frac{i}{\hbar} (\tau \hat{p}_x + \sigma \hat{p}_y + \sigma_1 x + \sigma_2 y) \right) \hat{A} \right].$$

In the following we will restrain to a cartesian and simply connected NC phase space, so that $g(x,p_y) = 1, h(y,p_y) = 1, \gamma(p) = 1$ and we can also remove the functions $A(x,p_y), C(x,p_y)$ and $F_1(p)$ by means of a local gauge transformations.

Using the commutation relations Eq.(53) and the representation of the operators Eq.(60) and analogous equations not explicitly written, it follows that the operators $\hat{x}, \hat{y}, \hat{p}_x$ and $\hat{p}_y$ are generators of translations in phase space:

$$e^{-\frac{i}{\hbar} \hat{a} \hat{x}} |x,p_y\rangle = |x+a,p_y\rangle,$$

(121)

$$e^{-\frac{i}{\hbar} \hat{b} \hat{y}} |x,p_y\rangle = |x+\theta b,p_y-b\rangle,$$

$$e^{\frac{i}{\hbar} \hat{c} \hat{p}_x} |y,p_x\rangle = |y+c,p_x\rangle,$$

$$e^{\frac{i}{\hbar} \hat{d} \hat{p}_y} |y,p_x\rangle = |y-\theta d,p_x-d\rangle,$$

$$e^{\frac{i}{\hbar} \hat{f} \hat{x}} |p_x,p_y\rangle = |p_x-f,p_y\rangle,$$

$$e^{\frac{i}{\hbar} \hat{g} \hat{y}} |p_x,p_y\rangle = |p_x,p_y-g\rangle,$$

where $a, b, c, d, f$ and $g$ are arbitrary constants. Using the Baker-Campbell-Hausdorff formula

$$e^{\hat{A}+\hat{B}} = e^{Ae^{-\frac{i}{\hbar}[\hat{A},\hat{B}]+\cdots}}$$

(122)

the Wigner function is written as:

$$W(z) = \int \frac{d^2 \sigma d^2 \tau}{(2\pi \hbar)^2} \left\{ e^{\frac{i}{\hbar}(\Phi(\sigma,\tau) - \tau_1 \sigma_1/2 - \sigma_1 \sigma_2/2 + \tau_2 \sigma_2/2)} \timesight.$$  

$$\int dx' dp_y' \langle x', p_y' | e^{-\frac{i}{\hbar} (\tau_1 \hat{p}_x + \sigma_2 \hat{p}_y + \sigma_1 x + \sigma_2 y) } e^{\frac{i}{\hbar} (\tau_1 \hat{p}_x + \sigma_2 \hat{p}_y + \sigma_1 x + \sigma_2 y) } \hat{A} | x', p_y' \rangle \}.$$  

This can be still simplified using the operators as translation generators:

$$W(z) = \int \frac{d^2 \sigma d^2 \tau}{(2\pi \hbar)^2} \left\{ e^{\frac{i}{\hbar}(\tau_1 \sigma_1 + \sigma_2 y)} \timesight.$$  

$$\langle x - \tau_1/2 - \theta \sigma_2/2, p_y + \sigma_1/2 | \hat{p} | x + \tau_1 + \theta \sigma_2, p_y - \sigma_2 \rangle \}.$$  

The change of the integration variables $\zeta = \tau_1/2 + \theta \sigma_2/2, \eta = -\sigma_2/2$ is useful to write Wigner function in a more compact way:

$$W(z) = \frac{1}{\pi \hbar^2} \int d\zeta d\eta \left\{ e^{\frac{i}{\hbar}(\zeta \sigma_1 + \eta \sigma_2)} \timesight.$$  

$$\langle x - \zeta, p_y - \eta | \hat{p} | x + \zeta, p_y + \eta \rangle \}.$$  

If the system is in a pure state, with wave function $\Psi(x,p_y; t)$, then Wigner function takes the form:

$$W(z) = \frac{1}{\pi \hbar^2} \int d\zeta d\eta \left\{ e^{\frac{i}{\hbar}(\zeta \sigma_1 + \eta \sigma_2)} \timesight.$$  

$$\Psi^*(x + \zeta, p_y + \eta; t) \Psi(x - \zeta, p_y - \eta; t) \}.$$  

It is also possible to define Winger function $W(z)$ starting from the $|y, p_x\rangle$ basis using (129) and the spectral decomposition of the unity operator:

$$W(z) = \frac{1}{\pi \hbar^2} \int du dv \left\{ e^{\frac{i}{\hbar}(xu - p_x v + \theta p_y)} \timesight.$$  

$$\langle y - v, p_x - u | \hat{p} | y + v, p_x + u \rangle \}.$$  

Similarly, in terms of the $|p\rangle$ basis the Wigner function takes the form:

$$W(z) = \frac{1}{\pi \hbar^2} \int d\eta d\eta' \left\{ e^{\frac{i}{\hbar}(xu + yv - \theta p_x - p_y)} \timesight.$$  

$$\langle p_x - u, p_y - \eta | \hat{p} | p_x + u, p_y + \eta \rangle \}.$$  

In analogy to the commutative case ($\theta = 0$), the main features of $W(z)$ are:

1. Wigner function $W(z)$ is real

$$W(z)^* = W(z).$$
2. If integrated over \( x \) and \( p_y \), \( W_\Psi(z) \) gives the correct marginal probability distribution on \( y \) and \( p_z \):

\[
|\Psi(y, p_z)|^2 = \int W_\Psi(z) dx dp_y.
\]

Similarly, if integrated over \( y \) and \( p_x \), the Wigner function reproduces the probability distribution on \( x \) and \( p_y \):

\[
|\Psi(x, p_y)|^2 = \int W_\Psi(z) dy dp_x.
\]

Finally, in order to obtain the marginal probability distribution on the momentum components, it is sufficient to integrate \( W_\Psi(z) \) over \( p_x \) and \( p_y \):

\[
|\Psi(p)|^2 = \int W_\Psi(z) dp_x dp_y.
\]

It is important to remark that wave functions \( \Psi(p) \), \( \Psi(y, p_z) \) and \( \Psi(x, p_y) \) are related by means of the transformations (106).

3. A consequence of the previous feature of Wigner function, it is evident that \( W_\Psi(z) \) is normalized

\[
\int W_\Psi(z) dx dy dp_x dp_y = 1.
\]

4. Starting from two different density operators \( \hat{\rho}_1 \) and \( \hat{\rho}_2 \), it is possible to construct two different Wigner functions \( W_1(z) \) and \( W_2(z) \). The operation \( Tr(\hat{\rho}_1 \hat{\rho}_2) \), in terms of \( W_1(z) \) and \( W_2(z) \), is given by

\[
Tr(\hat{\rho}_1 \hat{\rho}_2) = (2\pi\hbar)^2 \int W_1(z) W_2(z) dx dy dp_x dp_y.
\]

Thus, if \( A_W(z) \) is a Wigner function associated to the operator \( \hat{A} \) (119):

\[
A_W(z) = \frac{1}{\pi \hbar^2} \int d\zeta d\eta \left\{ e^{\frac{i}{\hbar} (\zeta p_x - \eta y + \theta p_z)} \times \langle x - \zeta, p_y - \eta | \hat{A} | x + \zeta, p_y + \eta \rangle \right\},
\]

then, the ensemble mean value of \( \hat{A} \) is

\[
\langle \hat{A} \rangle = Tr(\hat{\rho} \hat{A}) = (2\pi\hbar)^2 \int W(z) A_W(z) dx dy dp_x dp_y.
\]

5. If a system is in the state \( |\psi\rangle \), and a measurement that determines that the new state of the system is \( |\phi\rangle \), then the probability to obtain this result from the measurement is \( |\langle \psi | \phi \rangle|^2 \). In terms of Wigner functions, the transition probability can be written as

\[
|\langle \psi | \phi \rangle|^2 = (2\pi\hbar)^2 \int W_\psi(z) W_\phi(z) dx dy dp_x dp_y.
\]

This expression can be interpreted as the proof that Wigner function cannot be positive definite over phase space. If \( \psi \) and \( \phi \) are orthogonal, the last integral must vanish, and therefore, if \( W_\phi(z) \) is not equal to zero in a specific region of phase state, then it must take negative values in another sector.

The time dependence on Wigner function follows from:

\[
\frac{i\hbar}{\partial t} W(z, t) = \int \frac{i\hbar}{\pi \hbar^2} d\zeta d\eta \left\{ e^{\frac{i}{\hbar} (\zeta p_x - \eta y + \theta p_z)} \times \left[ \frac{\partial}{\partial t} \langle x - \zeta, p_y - \eta | \hat{A} | x + \zeta, p_y + \eta \rangle \right] - \langle x - \zeta, p_y - \eta | \hat{A} | x + \zeta, p_y + \eta \rangle \right\}.
\]

the time dependent Schrödinger equation can be used in this expression. In the \((x, p_y)\) representation the equation including a potential \( V(x, \dot{y}) \) reads:

\[
\frac{i\hbar}{\partial t} \psi(x, p_y; t) = \frac{p_y^2}{2m} \psi(x, p_y; t) - \frac{\hbar^2}{2m} \frac{\partial^2 \psi(x, p_y; t)}{\partial x^2} + V \left[ x, i\hbar \left( -\frac{\partial}{\partial x} + \frac{\partial}{\partial p_y} \right) \right] \psi(x, p_y; t).
\]

Substituting (127) in this relation one shows that, at least for potentials which are quadratic in the components of the position operator, the equation reduces to:

\[
\frac{\partial}{\partial t} W(z, t) = \{ H, W(z) \},
\]

which is nothing but the familiar Liouville equation applied to the probability distribution function in phase space. Even if the Wigner function does not satisfy all the requirements of an authentic probability distribution function, it is subject to the same mathematical relations as a real one.

VI. QM EXAMPLES ON THE NC PLANE

A. Free Particle

We begin with the simplest problem, namely the free particle in two NC dimensions. The corresponding Hamiltonian is

\[
\hat{H} = \frac{\hat{p}_x^2 + \hat{p}_y^2}{2m}.
\]
The Schrödinger equation in the momentum representation determines the spectrum of energies to be the continuum

\[ \hat{H} \psi(p; t) = i\hbar \frac{\partial}{\partial t} \psi(p; t) = \frac{p^2_x + p^2_y}{2m} \psi(p; t), \]

\[ E = \frac{p^2_x + p^2_y}{2m}. \]

The general solution is a superposition of stationary eigenfunctions of the Hamiltonian:

\[ \psi(p; t) = \int a_{E} e^{-\frac{i}{\hbar} E t} \psi_E(p) dE, \]

(129)

However, does not provide any information about the wave functions \( \psi(p) \). In order to determine these functions, we will take into account Schrödinger equation in the \( |x, p_y) \) and \( |y, p_x) \) basis;

\[ \hat{H} \psi(x, p_y) = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi(x, p_y)}{\partial x^2} + \frac{p^2_y}{2m} \psi(x, p_y), \]

(130)

\[ \hat{H} \psi(y, p_x) = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi(y, p_x)}{\partial y^2} + \frac{p^2_x}{2m} \psi(y, p_x). \]

Using separation of variables to solve those equations we obtain:

\[ \psi(x, p_y) = \frac{e^{\frac{i}{\hbar} x p_x}}{\sqrt{2\pi \hbar}} \phi(p_y), \]

(131)

\[ \psi(y, p_x) = \frac{e^{\frac{i}{\hbar} y p_y}}{\sqrt{2\pi \hbar}} \varphi(p_x). \]

where \( \varphi(p_x) \) and \( \phi(p_y) \) are undetermined functions. The wave function in momentum space is related to these by means of the following transforms:

\[ \psi(p') = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dy dp_y}{2\pi \hbar} \delta(p_x - p'_x) e^{-\frac{i}{\hbar} [y p_y - \phi'(p_y)]} e^{\frac{i}{\hbar} p_y \varphi(p_y)}, \]

\[ = \varphi(p'_x) e^{\frac{i}{\hbar} \phi'(p'_x)} \delta(p_y - p'_y), \]

(132)

\[ \psi(p') = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dx dp_x}{2\pi \hbar} \delta(p_y - p'_y) e^{-\frac{i}{\hbar} [x p_x + \phi'(p'_x)]} e^{\frac{i}{\hbar} p_x \varphi(p_y)}, \]

(133)

\[ = \phi(p'_y) e^{\frac{i}{\hbar} \phi'(p'_y)} \delta(p_x - p'_x). \]

(134)

Since \( \psi(p) \) is by assumption separable, then

\[ \psi(p) = \delta(p_x - p'_x) \delta(p_y - p'_y). \]

(135)

The same result is obtained in the conventional QM, which is not surprising, for in both cases \( \hat{p}_x \) and \( \hat{p}_y \) commute with the Hamiltonian and therefore have common eigenfunctions.

**B. Harmonic Oscillator**

Next we consider the 2D Isotropic Harmonic Oscillator, described by the Hamiltonian:

\[ \hat{H} = \frac{1}{2m} (\hat{p}_x^2 + \hat{p}_y^2) + \frac{1}{2} m \omega^2 (\hat{x}^2 + \hat{y}^2). \]

(136)

We will work in the momentum representation of the wave function. Since the potential does not depend explicitly on time, the problem reduces to the eigenvalue equation:

\[ E \psi(p) = \frac{1}{2m} (\hat{p}_x^2 + \hat{p}_y^2) \psi(p) + \frac{1}{2} m \omega^2 \left( \left( \frac{\hbar}{i} \frac{\partial}{\partial p_x} - \frac{\theta}{2} \hat{p}_y \right)^2 + \left( \frac{\hbar}{i} \frac{\partial}{\partial p_y} + \frac{\theta}{2} \hat{p}_x \right)^2 \right) \psi(p). \]

Rearranging terms, the last equation can be written as

\[ E \psi(p) = \left( 1 + \frac{m^2 \omega^2 \theta^2}{4} \right) \left( \frac{1}{2m} (\hat{p}_x^2 + \hat{p}_y^2) \psi(p) \right) - \frac{\hbar^2 m \omega^2}{4} \left( \frac{\partial^2}{\partial p_x^2} + \frac{\partial^2}{\partial p_y^2} \right) \psi(p) \]

\[ - \frac{i}{2} \hbar \theta m \omega^2 \left( \frac{\partial}{\partial p_x} p_y - \frac{\partial}{\partial p_y} p_x \right) \psi(p). \]

On the other hand, the Hamiltonian \( \hat{H} \) is rotationally invariant, that is, it commutes with the quantum version of the angular momentum Eq. [25]:

\[ \hat{J} = \hat{\hat{x}} \hat{\hat{p}}_y - \hat{\hat{y}} \hat{\hat{p}}_x + \frac{\theta}{2} (\hat{p}_x^2 + \hat{p}_y^2). \]

(139)

At this point it is convenient to recall that according to our analysis of the symmetries of the classical NC harmonic oscillator, SU(2) is not a symmetry for this system. Thus, for the harmonic oscillator, \( \hat{H} \) and \( \hat{J} \) have common eigenstates. Noticing that the angular momentum operates over the Harmonic oscillator eigenfunctions as follows:

\[ \hat{J} \psi(p) = i \hbar \left( \frac{\partial}{\partial p_x} p_y - \frac{\partial}{\partial p_y} p_x \right) \psi(p). \]

the eigenvalue equation for the Hamiltonian \( \hat{H} \) can be written in terms of \( J \) as:

\[ E \psi(p) = \left( 1 + \frac{m^2 \omega^2 \theta^2}{4} \right) \hat{H}_0 \psi(p) - \frac{1}{2} \hbar m \omega^2 \hat{J} \psi(p), \]

(141)

where we introduced the operator

\[ \hat{H}_0 = \left\{ \frac{1}{2m} (\hat{p}_x^2 + \hat{p}_y^2) - \frac{\hbar^2 m \omega^2}{2 (1 + \frac{m^2 \omega^2 \theta^2}{4})} \left[ \frac{\partial^2}{\partial p_x^2} + \frac{\partial^2}{\partial p_y^2} \right] \right\}. \]
which is nothing but the Hamiltonian of a commutative harmonic oscillator of frequency $\omega$:

$$\omega = \frac{\omega}{\sqrt{1 + m^2 \omega^2 \theta^2}}$$  \hspace{1cm} (142)

Besides $[\hat{H}, \hat{J}] = 0$, one can shown that $[\hat{H}_0, \hat{J}] = 0$ and $[\hat{H}, \hat{H}_0] = 0$ hold. Therefore $\hat{H}$, $\hat{H}_0$ and $\hat{J}$ have common eigenstates. The solution to the partial differential equation (142) is the product of harmonic oscillator eigenfunctions:

$$\psi_{n_1, n_2}(p) = e^{-\frac{p^2 + \frac{\theta^2}{m\omega^2}}{2m\omega}} \frac{1}{N} H_{n_1}(\frac{p_x}{m\omega}) H_{n_2}(\frac{p_y}{m\omega}),$$  \hspace{1cm} (143)

where $N = 2^{n_1 + n_2} \sqrt{\pi m\omega \theta} \sqrt{n_1! n_2}$. Thus, the eigenvalue equation for $H_0$ is:

$$\hat{H}_0 \psi_{n_1, n_2}(p) = \hbar \omega (n_1 + n_2 + 1) \psi_{n_1, n_2}(p),$$  \hspace{1cm} (144)

with $n_1, n_2$ positive integers. These functions form a complete orthonormal basis. The linear combination of these functions that simultaneously diagonalizes $\hat{H}$ and $\hat{J}$ is:

$$\psi_{n, j}(p) = e^{-\frac{p^2 + \frac{\theta^2}{m\omega^2}}{2m\omega}} \frac{1}{N} \sum_{r=0}^{n_1} \sum_{q=0}^{n_2} \binom{n_1 + j}{r} \binom{n_2 - j}{q} \times (-1)^q \binom{n + j}{r} \binom{n - j}{q} \binom{p_x}{m\omega} \frac{1}{m\omega} \binom{p_y}{m\omega},$$  \hspace{1cm} (145)

with $n \in \mathbb{Z}^+$, $2j \in \mathbb{Z}$, subject to the restriction $-\frac{n}{2} \leq j \leq \frac{n}{2}$ and $N = 2^n \sqrt{\pi m\omega \theta} \sqrt{(\frac{n}{2} + j)! (\frac{n}{2} - j)!}$. Putting all together the eigenvalue spectrum associated to eigenfunctions is summarized in the equations:

$$\hat{H} \psi_{n, j}(p) = E_{n, j} \psi_{n, j}(p)$$  \hspace{1cm} (146)

$$= \hbar \omega \sqrt{1 + \frac{m^2 \omega^2 \theta^2}{4}} (n + 1 - \theta m \omega^2 \hbar j) \psi_{n, j}(p),$$

$$\hat{J} \psi_{n, j}(p) = 2\hbar j \psi_{n, j}(p).$$

To conclude the discussion on the harmonic oscillator, we derive the Wigner quasi-distribution function for the ground state of the harmonic oscillator, which is described by the wave functions

$$\psi_{0, 0}(p) = \frac{e^{-\frac{p^2 + \theta^2}{2m\omega}}}{\sqrt{\pi m\omega \theta}},$$  \hspace{1cm} (147)

The Wigner distribution function is, according to Eq. (126):

$$W_{0,0}(z) = \frac{e^{-\frac{p_x^2 + p_y^2}{2m\omega}}}{\sqrt{\pi^5 m^5 \omega^5 \theta}} \times$$

$$\times \int_{-\infty}^{\infty} du du e^{-\frac{u^2 + v^2}{2m\omega}} \left\{ e^{\frac{u^2 + v^2}{2m\omega}} \left[ \frac{1}{\pi^2 \hbar^2} \right] \left[ (u + \frac{\theta}{m\omega} p_y)^2 + (v - \frac{\theta}{m\omega} p_x)^2 \right] \right\}.$$  \hspace{1cm} (148)

This function is positive definite in its entire domain. Without loss of generality let us consider for simplicity the case $m = 1, \hbar = 1, \omega = 1$ (which can be obtained through a scale transformation). The time evolution of Wigner function is determined by Eq. (128). The solution to this equation can be written operationally as:

$$W_{0,0}(z, t) = \exp \left\{ it \{ H, \cdot \} \right\} W_{0,0}(z, 0),$$  \hspace{1cm} (149)

which is nothing but a linear, time dependent transformation. Then, the complete solution is:

$$W_{0,0}(z, t) = \exp \left\{ it \{ H, \cdot \} \right\} \times$$

$$\times \frac{1}{\pi^2 \hbar^2} \exp \left\{ -\frac{\sqrt{1 + \theta^2}}{2} \left( \frac{p_x^2 + p_y^2}{m\omega} \right) \right\} \times$$

$$\times \exp \left\{ -2 \left[ \frac{(x + \frac{\theta}{m\omega} p_y)^2 + (y - \frac{\theta}{m\omega} p_x)^2}{\sqrt{1 + \theta^2}} \right] \right\}. \hspace{1cm} (150)$$

### C. 2D Solid (Einstein’s model)

The thermodynamic properties of this solid can be studied considering a canonical ensemble of $N$ distinguishable, non interacting and NC harmonic oscillators. For an oscillator of frequency $\omega$, the probability of being in the $n, j$ state, denoted $w_{n, j}$, is:

$$w_{n, j} = \frac{e^{-\frac{E_{n, j}}{kT}}}{Z(T, V, 1)},$$  \hspace{1cm} (151)

where $Z(T, V, 1)$ is the partition function

$$Z(T, V, 1) = Tr(e^{-\frac{\hat{H}}{kT}}) = \sum_{n=0}^{\infty} \sum_{j=-n/2}^{n/2} e^{-\frac{E_{n, j}}{kT}}.$$  \hspace{1cm} (152)

The calculation of this function proceeds through geometric sums, and renders the following result:

$$Z(T, V, 1) = 2 \left\{ \cos \left( \frac{\hbar \omega \Theta}{2kT} \right) - \cos \left( \frac{\hbar \omega \theta}{2kT} \right) \right\}^{-1},$$  \hspace{1cm} (153)
with
\[ \Theta = \sqrt{4 + m^2 \omega^2 \theta^2}. \] (154)

At this point, it is necessary to know the distribution function of the natural frequencies in the solid. To simplify the calculation we choose Einstein’s approximation, and set all frequencies equal \( \omega_i = \omega \). This approach, if not the better, provides a clear qualitative behavior of the system.

The partition function of the \( N \) oscillators is, then,
\[ Z(T, V, N) = Z(T, V, 1)^N. \] (155)

From this partition function, the derivation of the Free Energy is immediate
\[ A(T, V, N) = -kT \ln [Z(T, V, N)], \] (156)
\[ = NkT \ln \left[ \cos \left( \frac{\hbar \omega \Theta}{2kT} \right) - \cos \left( \frac{\hbar \omega m \theta}{2kT} \right) \right] + NkT \ln 2. \]

Thus, the entropy of the system is
\[ S = -\frac{\partial A}{\partial T}_{V,N} = -Nk \ln \left[ \cos \left( \frac{\hbar \omega \Theta}{2kT} \right) - \cos \left( \frac{\hbar \omega m \theta}{2kT} \right) \right] \]
\[ + \frac{\hbar \omega N \left[ \sqrt{4 + m^2 \omega^2 \theta^2 \sin \left( \frac{\hbar \omega \Theta}{2kT} \right) - m \omega \sin \left( \frac{\hbar \omega m \theta}{2kT} \right) } \right]}{2T \left[ \cos \left( \frac{\hbar \omega \Theta}{2kT} \right) - \cos \left( \frac{\hbar \omega m \theta}{2kT} \right) \right]} - Nk \ln 2, \] (157)

and its internal energy takes the form
\[ U = A + TS \]
\[ = \frac{\hbar \omega N \Theta \sin \left( \frac{\hbar \omega \Theta}{2kT} \right) - m \omega \Theta \sin \left( \frac{\hbar \omega m \theta}{2kT} \right)}{2T \left[ \cos \left( \frac{\hbar \omega \Theta}{2kT} \right) - \cos \left( \frac{\hbar \omega m \theta}{2kT} \right) \right]} . \] (158)

In the high temperature regime \( kT \gg \hbar \omega \), the behavior of \( U \) can be deduced expanding in power series of \( T \) :
\[ U_{kT \gg \hbar \omega} = 2NkT + \frac{\hbar^2 \omega^2 N (2 + m^2 \omega^2 \theta^2)}{12kT} + \cdots, \] (159)

We conclude that the conventional energy equipartition is also obtained in the NC case. On the other hand, in the opposite limit \( T \to 0 \), the internal energy reduces to the contribution of the minimum energy of each oscillator
\[ \lim_{T \to 0} U = N \hbar \omega \sqrt{1 + \frac{m^2 \omega^2 \theta^2}{4}}. \] (160)

Finally, the calorific capacity of this set of NC oscillators is given by
\[ C_V = \frac{\partial U}{\partial T}_{V,N} \]
\[ = \frac{\hbar^2 \omega^2 N}{2kT^2} \left[ \cos \left( \frac{\hbar \omega \Theta}{2kT} \right) - \cos \left( \frac{\hbar \omega m \theta}{2kT} \right) \right]^{-2} \]
\[ \times \left\{ (2 + m^2 \omega^2 \theta^2) \left[ \cos \left( \frac{\hbar \omega \Theta}{2kT} \right) \cos \left( \frac{\hbar \omega m \theta}{2kT} \right) - 1 \right] - m \omega \Theta \left[ \sin \left( \frac{\hbar \omega \Theta}{2kT} \right) \sin \left( \frac{\hbar \omega m \theta}{2kT} \right) \right] \right\}. \]

Again, in the high temperature regime one recovers the conventional result:
\[ C_V = 2kN - \frac{\hbar^2 \omega^2 N (2 + m^2 \omega^2 \theta^2)}{12kT^2} + \cdots. \] (162)

Finally, in figures 1 and 2 we show the behavior of the entropy \( S \) as a function of \( T \) and \( \theta \), for fixed values of \( m, \omega, \) and \( N \). In the present model, the entropy varies significatively with respect to \( \theta \), in particular if \( \theta \sim \frac{1}{(m \omega)^{-1}} \).

### VII. SUMMARY AND CONCLUSIONS

We presented a systematic study of non-commutative mechanics starting from the classical formalism and proceeding through the quantization. We emphasized the role played by symmetries, in particular we ensure that the NC free particle is consistent with Galilean relativity which follows from the well known relation among the boost generators and the position operator. The general description of NCCM in terms of second class constrained system was elaborated for Hamiltonians of the type \( H = T + V \) where \( T \) and \( V \) stand for kinetic and potential energy of the two dimensional system.

Besides providing a global view of the problem, our manuscript contains new results, in particular:

- A formulation that avoids the use of non-canonical transformations and/or expansion in the NC parameter \( \theta \).
- In classical mechanics, analytical solution for the free particle and harmonic oscillator problems. We also show that \( SU(2) \) is not the symmetry group of the isotropic harmonic oscillator, and identify the generators of the existing symmetry.
- Quantization is presented in three different frameworks: Canonical, Path Integral and Wigner Function.
• The representations of the Heisenberg Algebra in three different basis \((p_x, p_y), (x, p_y)\) and \((p_z, y)\), including the wave functions permitting the change of basis.

• Non-equivalent representations of the Heisenberg algebra characterized by gauge fields that follow naturally from the structure of the algebra. Those Fields are relevant in the description of multiply connected manifolds.

• Extension of the analysis of the fundamental properties of the Wigner function in four dimensional NC phase space.

• Analytical solution in QM for the free particle and harmonic oscillator without performing the customary non-canonical transformation, without using the structure of \(SU(2)\) generators and without assuming a priori the existence of a vacuum state.

• The thermodynamic properties of a 2D NC crystal using the 2D Einstein's solid model. The behavior of the entropy as a function of the NC parameter \(\theta\) is reported.

The proper definition of the system to be treated in NC mechanics and the certitude that it is not traded along the way are fundamental. Our approach focuses on both points, first defining a general consistent framework and second avoiding completely the use of non canonical transformations since those lead, in general, to a system whose properties are completely different from those of the original one.

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FIGURE CAPTION.

Figure 1. Normalized entropy as a function of the NC parameter \(\theta\) and temperature as obtained from Eq. (157).

Figure 2. Normalized entropy as a function of temperature, for fixed values of the NC parameter \(\theta\) as obtained from Eq. (157).
[1] A. Connes, Non commutative Geometry, Academic Press, San Diego (1994).

[2] N. Seiberg, E. Witten, JHEP 0002 (2000) 020.

[3] J. Lukierski, P.C. Stichel W.J. Zakrzewski, Annals Phys 260 (1997) 224.

[4] J.-M. Lévi-Leblond, Riv. Nuovo Cimento 4,1 (1974) 99.

[5] G. Dunne, R. Jackiw, Nucl. Phys. Proc. Suppl. 33C (1993) 114, hep-th/9204057; C. Duval, P. A. Horvthy, Phys.Lett. B479 (2000) 284, hep-th/0002233.

[6] J. Govaerts, Hamiltonian Quantisation and Constrained Dynamics, Leuven University Press, Leuven, 1991.

[7] See G. Dunne, Jackiw in Ref[5] and also A.A. Deriglasov, Noncommutative version of an arbitrary nondegenerate mechanics, hep-th/0208072.

[8] S. Chaturvedi, R. Jagannathan, R. Sridhar, V. Srinivasan, J. Phys. A: Math. Gen. 26 L105-L112.

[9] Musongela Lubo JHEP 0405 (2004) 045, hep-th/0304039.

[10] V.P. Nair, Polychronakos, Phys. Lett. B 505 (2001) 267, hep-th/0011172; see also Kang Li, J. Wang, C. Chen, Representation of Noncommutative phase space, hep-th/0407185.

[11] Y. Brihaye, C. Gonera, S. Giller, P. Kosinski, Galilean invariance in 2+1 dimensions, hep-th/9503046.

[12] J. Govaerts, V. Villanueva, Int. J. Mod. Phys. A15 (2000) 4903, quant-ph/9908014.

[13] C. Acatrinei, Comments on noncommutative particle dynamics, hep-th/0106141.

[14] See for example J.S. Bell, Speakable and Unspeakable in Quantum Mechanics, Cambridge University press, Cambridge UK, 1987; K. Banaszek, K. Wodkiewicz, Phys. Rev. A 58 (1998) 4345.

[15] A. Kokado, T. Okamura, T. Saito, Wigner’s formulation of Noncommutative Quantum Mechanics, hep-th/0208040; O.F. Dayi, L.T. Kelleyane, Mod.Phys.Lett. A17 (2002)1937 hep-th/0202062; M. Rosenbaum, J.D. Vergara, The *-value equation and Wigner distribution in noncommutative Heisenberg algebras, hep-th/0505127.

[16] S. Bellucci, A. Nersessian and C. Sochichiu, Phys.Lett. B522: 345 (2001), hep-th/0106138.

[17] A. Smilagic, E. Spallucci, Phys. Rev. D65 (2002) 107701; I. Dadic, L. Jonke, S. Meljanac, Harmonic oscillator on noncommutative spaces, hep-th/0301066.

[18] J. Gamboa, M. Loewe, C. Rojas, Int. J. Mod. Phys. A17 (2002) 2555; H.O. Girotti, Am. J. Phys. 72 (2004) 608.

[19] H. Goldstein, Classical Mechanics, Addison Wesley, Reading, Massachusetts 1980.

[20] D. R. Grigore, Journ. Math. Phys. 34 (1993) 4190, hep-th/9312048.

[21] Juan M. Romero, J.A. Santiago, J. David Vergara, Phys. Lett. A310 (2003) 9, hep-th/0211165; Juan M. Romero, J. David Vergara, Mod. Phys. Lett. A18 (2003) 1673, hep-th/0303064; A.E.F. Djemai, On noncommutative classical mechanics, hep-th/0309034.

[22] C. Vaquera-Araujo, No conmutatividad en 2 Dimensiones, Bachelor thesis, Instituto de Fisica, Universidad de Guanajuato (2005), Unpublished.

[23] O. Espinoza, P. Gaete, Symmetries in noncommutative quantum mechanics, hep-th/0206066.

[24] B. DeWitt, Rev. Mod. Phys. 29 (1957) 377.

[25] P. A. M. Dirac, Lectures on Quantum Mechanics, Belfer Graduate School of Science, Yeshiva University, New York, 1964.

[26] J. Govaerts, V. Villanueva, Int. J. Mod. Phys A15 (2000) 4903, quant-ph/9908014.

[27] E. Wigner, Phys. Rev. 40 (1932) 749.

[28] L.D. Landau, Butterworth, Heinemann, Statistical Physics, Course of theoretical Physics, Vol. 5, 2000.