THE COMPARISON OF TWO CONSTRUCTIONS OF THE REFINED ANALYTIC TORSION ON COMPACT MANIFOLDS WITH BOUNDARY

RUNG-TZUNG HUANG AND YOONWEO LEE

Abstract. The refined analytic torsion on compact Riemannian manifolds with boundary has been discussed by B. Vertman ([22], [23]) and the authors ([11], [12]) but these two constructions are completely different. Vertman used a double of de Rham complex consisting of the minimal and maximal closed extensions of a flat connection and the authors used well-posed boundary conditions $P_-, L_0, P_+, L_1$ for the odd signature operator. In this paper we compare these two constructions by using the BFK-gluing formula for zeta-determinants, the adiabatic method for stretching cylinder part near boundary and the deformation method used in [6] when the odd signature operator comes from a Hermitian flat connection and all de Rham cohomologies vanish.

1. Introduction

The refined analytic torsion was introduced by M. Braverman and T. Kappeler ([4], [5]) on an odd dimensional closed Riemannian manifold with a flat bundle as an analytic analogue of the refined combinatorial torsion introduced by M. Farber and V. Turaev ([20], [21], [8], [9]). Even though these two objects do not coincide exactly, they are closely related. The refined analytic torsion is defined by using the graded zeta-determinant of the odd signature operator and is described as an element of the determinant line of the cohomologies. Specially, when the odd signature operator comes from an acyclic Hermitian connection on a closed manifold, the refined analytic torsion is a complex number, whose modulus part is the Ray-Singer analytic torsion and the phase part is the $\rho$-invariant determined by the given odd signature operator and the odd signature operator defined by the trivial connection acting on the trivial line bundle.

The refined analytic torsion on compact Riemannian manifolds with boundary has been discussed by B. Vertman ([22], [23]) and the authors ([11], [12]) but these two constructions are completely different. Vertman used a double of de Rham complex consisting of the minimal and maximal closed extensions of a flat connection. On the other hand, the authors introduced well-posed boundary conditions $P_-, L_0, P_+, L_1$ for the odd signature operator to define the refined analytic torsion on compact Riemannian manifolds with boundary. In this paper we are going to compare these two constructions when the odd signature operator comes from a Hermitian connection and all de Rham cohomologies vanish. For comparison of the Ray-Singer analytic torsion part we are going to use the BFK-gluing formula for zeta-determinants proven in [7] and the adiabatic method for stretching cylinder part near boundary. For comparison of the eta invariant part we are going to use the deformation method used in [6]. These methods were used in [12], where the authors discussed the gluing formula of the refined analytic torsion with respect to the well-posed boundary conditions $P_-, L_0, P_+, L_1$. Hence this work is a continuation of [12].

2000 Mathematics Subject Classification. Primary: 58J52; Secondary: 58J28, 58J50.
Key words and phrases. refined analytic torsion, zeta-determinant, eta-invariant, odd signature operator, well-posed boundary condition.
We now begin with the description of the odd signature operator near boundary.

2. The Refined Analytic Torsion on Manifolds with Boundary

In this section we first describe the odd signature operator $\mathcal{B}$ near boundary and introduce the well-posed boundary conditions $P_-\mathcal{L}_0, P_+\mathcal{L}_1$ for the odd signature operator. We then review the construction of the refined analytic torsions with respect to $P_-\mathcal{L}_0, P_+\mathcal{L}_1$ discussed in [11].

Let $(M, g^M)$ be a compact oriented odd dimensional Riemannian manifold with boundary $Y$, where $g^M$ is assumed to be a product metric near the boundary $Y$. We denote the dimension of $M$ by $m = 2r - 1$. Suppose that $\rho : \pi_1(M) \to GL(n, \mathbb{C})$ is a representation of the fundamental group and $E = \tilde{M} \times_{\rho} \mathbb{C}^n$ is the associated flat bundle, where $\tilde{M}$ is a universal covering space of $M$. We choose a flat connection $\nabla$ and extend it to a covariant differential $\nabla : \Omega^\bullet(M, E) \to \Omega^{\bullet+1}(M, E)$.

Using the Hodge star operator $*^M$, we define an involution $\Gamma = \Gamma(g^M) : \Omega^\bullet(M, E) \to \Omega^{m-\bullet}(M, E)$ by

$$
\Gamma \omega := i^{r-1} (\frac{m+1}{2})*^M \omega, \quad \omega \in \Omega^p(M, E), \quad (2.1)
$$

It is straightforward to see that $\Gamma^2 = \text{Id}$. We define the odd signature operator $B$ by

$$
B := \Gamma \nabla + \nabla \Gamma : \Omega^\bullet(M, E) \to \Omega^\bullet(M, E). \quad (2.2)
$$

Then $B$ is an elliptic differential operator of order 1. Let $N$ be a collar neighborhood of $Y$ which is isometric to $[0, 1) \times Y$. Then we have a natural isomorphism

$$
\Psi : \Omega^p(N, E|_N) \to C^\infty([0, 1), \Omega^p(Y, E|_Y) \oplus \Omega^{p-1}(Y, E|_Y)) \quad (2.3)
$$

Using the product structure we can induce a flat connection $\nabla^Y : \Omega^\bullet(Y, E|_Y) \to \Omega^\bullet(Y, E|_Y)$ from $\nabla$ and the Hodge star operator $*^Y : \Omega^\bullet(Y, E|_Y) \to \Omega^{m-\bullet}(Y, E|_Y)$ from $*^M$. We define two maps $\beta, \Gamma^Y$ by

$$
\beta : \Omega^p(Y, E|_Y) \to \Omega^p(Y, E|_Y), \quad \beta(\omega) = (-1)^p \omega
$$

$$
\Gamma^Y : \Omega^p(Y, E|_Y) \to \Omega^{m-1-p}(Y, E|_Y), \quad \Gamma^Y(\omega) = i^{r-1}(\frac{m+1}{2})*^Y \omega. \quad (2.4)
$$

It is straightforward that

$$
\beta^2 = \text{Id}, \quad \Gamma^Y \Gamma^Y = \text{Id}. \quad (2.5)
$$

Then simple computation shows that

$$
\Gamma = i\beta \Gamma^Y \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right), \quad \nabla = \left( \begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array} \right) \nabla_{\delta_x} + \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) \nabla^{Y}, \quad (2.6)
$$

where $\delta_x$ is the inward normal derivative to the boundary $Y$ on $N$. Hence the odd signature operator $B$ is expressed, under the isomorphism (2.3), by
We define

\[ B = -i\beta \Gamma^Y \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \nabla_{\partial_x} + \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} (\nabla^Y + \Gamma^Y \nabla^Y \Gamma^Y) \right\}. \]  

(2.7)

We denote

\[ \gamma := -i\beta \Gamma^Y \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathcal{A} := \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} (\nabla^Y + \Gamma^Y \nabla^Y \Gamma^Y) \]  

(2.8)

so that \( B \) has the form of

\[ B = \gamma (\partial_x + \mathcal{A}) \quad \text{with} \quad \gamma^2 = -\text{Id}, \quad \gamma \mathcal{A} = -\mathcal{A} \gamma. \]  

(2.9)

Since \( \nabla_{\partial_x} \nabla^Y = \nabla^Y \nabla_{\partial_x} \), we have

\[ B^2 = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \nabla^2_{\partial_x} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (\nabla^Y + \Gamma^Y \nabla^Y \Gamma^Y)^2 = (-\nabla^2_{\partial_x} + B^2_Y) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \]  

(2.10)

where

\[ B_Y = \Gamma^Y \nabla^Y + \nabla^Y \Gamma^Y. \]

We next choose a Hermitian inner product \( h^E \). All through this paper we assume that \( \nabla \) is a Hermitian connection with respect to \( h^E \), i.e., for any \( \phi, \psi \in C^\infty(E) \),

\[ dh^E(\phi, \psi) = h^E(\nabla \phi, \psi) + h^E(\phi, \nabla \psi). \]

The Green formula for \( B \) is given as follows (cf. [11]).

**Lemma 2.1.**  
(1) For \( \phi \in \Omega^q(M, E), \psi \in \Omega^{m-q}(M, E) \), \( \langle \Gamma \phi, \psi \rangle_M = \langle \phi, \Gamma \psi \rangle_M \).

(2) For \( \phi \in \Omega^q(M, E), \psi \in \Omega^{q+1}(M, E) \),

\[ \langle \nabla \phi, \psi \rangle_M = \langle \phi, \Gamma \nabla \psi \rangle_M - \langle \phi_{\text{tan}}|_Y, \psi_{\text{nor}}|_Y \rangle_Y. \]

(3) For \( \phi, \psi \in \Omega^\text{even}(M, E) \) or \( \Omega^\text{odd}(M, E) \),

\[ \langle \mathcal{B} \phi, \psi \rangle_M - \langle \phi, \mathcal{B} \psi \rangle_M = \langle \phi_{\text{tan}}|_Y, i\beta \Gamma^Y (\psi_{\text{tan}}|_Y) \rangle_Y - \langle \phi_{\text{nor}}|_Y, i\beta \Gamma^Y (\psi_{\text{nor}}|_Y) \rangle_Y = \langle \phi|_Y, \gamma(\psi|_Y) \rangle_Y. \]

**Remark:** In the assertions (2) and (3) the signs on the inner products on \( Y \) are different from those in [11] because in [11] \( \partial_x \) is an outward normal derivative.

We note that \( B_Y \) is a self-adjoint elliptic operator on \( Y \). Putting \( \mathcal{H}^\bullet(Y, E|_Y) := \ker \mathcal{B}^2_Y, \mathcal{H}^\bullet(Y, E|_Y) \) is a finite dimensional vector space and we have

\[ \Omega^\bullet(Y, E|_Y) = \text{Im} \nabla^Y \oplus \text{Im} \Gamma^Y \nabla^Y \Gamma^Y \oplus \mathcal{H}^\bullet(Y, E|_Y). \]

If \( \nabla \phi = \Gamma \nabla \Gamma \phi = 0 \) for \( \phi \in \Omega^\bullet(M, E) \), simple computation shows that \( \phi \) is expressed, near the boundary \( Y \), by

\[ \phi = \nabla^Y \phi_{\text{tan}} + \phi_{\text{tan}, h} + dx \wedge (\Gamma^Y \nabla^Y \Gamma^Y \phi_{\text{nor}} + \phi_{\text{nor}, h}), \quad \phi_{\text{tan}, h}, \phi_{\text{nor}, h} \in \mathcal{H}^\bullet(Y, E|_Y). \]  

(2.11)

We define \( \mathcal{K} \) by

\[ \mathcal{K} := \{ \phi_{\text{tan}, h} \in \mathcal{H}^\bullet(Y, E|_Y) \mid \nabla \phi = \Gamma \nabla \Gamma \phi = 0 \}, \]  

(2.12)

where \( \phi \) has the form (2.11). If \( \phi \) satisfies \( \nabla \phi = \Gamma \nabla \Gamma \phi = 0 \), so is \( \Gamma \phi \) and hence
\[ \Gamma^Y \mathcal{K} = \{ \phi_{\text{nor}, h} \in \mathcal{H}^s(Y, E|_Y) \mid \nabla \phi = \Gamma \nabla \Gamma \phi = 0 \}, \]  
where \( \phi \) has the form (2.11). The second assertion in Lemma 2.1 shows that \( \mathcal{K} \) is perpendicular to \( \Gamma^Y \mathcal{K} \).

We then have the following decomposition (cf. Corollary 8.4 in [14], Lemma 2.4 in [11]).

\[ \mathcal{K} \oplus \Gamma^Y \mathcal{K} = \mathcal{H}^s(Y, E|_Y), \]  
which shows that \( (\mathcal{H}^s(Y, E|_Y), \langle , \rangle_Y, -i \beta \Gamma^Y) \) is a symplectic vector space with Lagrangian subspaces \( \mathcal{K} \) and \( \Gamma^Y \mathcal{K} \). We denote by

\[ \mathcal{L}_0 = \frac{\mathcal{K}}{\mathcal{K}}, \quad \mathcal{L}_1 = \frac{\Gamma^Y \mathcal{K}}{\Gamma^Y \mathcal{K}}. \]  

Remark : Lemma 2.4 in [11] shows that \( \mathcal{K} \) and \( \Gamma^Y \mathcal{K} \) are the sets of all tangential and normal parts of the limiting values of extended \( L^2 \)-solutions to \( \mathcal{B}_\infty \) on \( M_\infty \), respectively. (See (3.9) below for definitions of \( \mathcal{B}_\infty \) and \( M_\infty \)).

We next define the orthogonal projections \( \mathcal{P}_{-, \mathcal{L}_0}, \mathcal{P}_{+, \mathcal{L}_1} : \Omega^s(Y, E|_Y) \oplus \Omega^s(Y, E|_Y) \to \Omega^s(Y, E|_Y) \) by

\[ \text{Im} \mathcal{P}_{-, \mathcal{L}_0} = \left( \begin{array}{c} \text{Im} \nabla^Y \mathcal{K} \\ \text{Im} \nabla^Y \mathcal{K} \end{array} \right), \quad \text{Im} \mathcal{P}_{+, \mathcal{L}_1} = \left( \begin{array}{c} \text{Im} \Gamma^Y \nabla^Y \mathcal{K} + \Gamma^Y \mathcal{K} \\ \text{Im} \Gamma^Y \nabla^Y \mathcal{K} + \Gamma^Y \mathcal{K} \end{array} \right). \]  

Then \( \mathcal{P}_{-, \mathcal{L}_0}, \mathcal{P}_{+, \mathcal{L}_1} \) are pseudodifferential operators and give well-posed boundary conditions for \( \mathcal{B} \) and the refined analytic torsion. We denote by \( \mathcal{B}_{\mathcal{P}_{-, \mathcal{L}_0}} \) and \( \mathcal{B}_{\mathcal{Q}_{-, \mathcal{L}_0}}^2 \) the realizations of \( \mathcal{B} \) and \( \mathcal{B}_{\mathcal{Q}_{-, \mathcal{L}_0}}^2 \) with respect to \( \mathcal{P}_{-, \mathcal{L}_0}, i.e. \)

\[ \text{Dom} \left( \mathcal{B}_{\mathcal{P}_{-, \mathcal{L}_0}} \right) = \{ \psi \in \Omega^s(M, E) \mid \mathcal{P}_{-, \mathcal{L}_0} (\psi|_Y) = 0 \}, \]
\[ \text{Dom} \left( \mathcal{B}_{\mathcal{Q}_{-, \mathcal{L}_0}}^2 \right) = \{ \psi \in \Omega^s(M, E) \mid \mathcal{P}_{-, \mathcal{L}_0} (\psi|_Y) = 0, \mathcal{P}_{+, \mathcal{L}_1} (\mathcal{B}\psi|_Y) = 0 \}. \]  

We define \( \mathcal{B}_{\mathcal{P}_{-, \mathcal{L}_1}}, \mathcal{B}_{\mathcal{Q}_{-, \mathcal{L}_1}}, \mathcal{B}_{\mathcal{Q}_{-, \mathcal{L}_0}}^2, \mathcal{B}_{\mathcal{Q}_{-, \mathcal{L}_0}}^2, \mathcal{B}_{\mathcal{P}_{+, \mathcal{L}_1}}, \mathcal{B}_{\mathcal{P}_{+, \mathcal{L}_0}}^2 \) and \( \mathcal{B}_{\Pi, \mathcal{L}_1}, \mathcal{B}_{\Pi, \mathcal{L}_0} \) (see Section 3) in the similar way. The following result is straightforward (Lemma 2.11 in [11]).

**Lemma 2.2.**

\[ \ker \mathcal{B}_{\mathcal{Q}_{-, \mathcal{L}_0}}^2 = \ker \mathcal{B}_{\mathcal{Q}_{-, \mathcal{L}_0}}^2 = H^s(M, Y, E), \quad \ker \mathcal{B}_{\mathcal{Q}_{+, \mathcal{L}_1}}^2 = \ker \mathcal{B}_{\mathcal{Q}_{+, \mathcal{L}_0}}^2 = H^s(M; E). \]

We choose an Agmon angle \( \theta \) by \( -\frac{\pi}{2} < \theta < 0 \). For \( \mathcal{D} = \mathcal{P}_{-, \mathcal{L}_0} \) or \( \mathcal{P}_{+, \mathcal{L}_1} \), we define the zeta function \( \zeta_{\mathcal{B}_{\mathcal{Q}_{-, \mathcal{L}_0}}^2}(s) \) and eta function \( \eta_{\mathcal{B}_{\mathcal{Q}_{-, \mathcal{L}_0}}^2}(s) \) by

\[ \zeta_{\mathcal{B}_{\mathcal{Q}_{-, \mathcal{L}_0}}^2}(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \left( \text{Tr} e^{-t \mathcal{B}_{\mathcal{Q}_{-, \mathcal{L}_0}}^2} - \dim \ker \mathcal{B}_{\mathcal{Q}_{-, \mathcal{L}_0}}^2 \right) dt = \sum_{0 \neq \lambda_j \in \text{Spec}(\mathcal{B}_{\mathcal{Q}_{-, \mathcal{L}_0}}^2)} \lambda_j^{-s}. \]
\[ \eta_{\mathcal{B}_{\mathcal{Q}_{-, \mathcal{L}_0}}^2}(s) = \frac{1}{\Gamma\left(\frac{s+1}{2}\right)} \int_0^\infty t^{-\frac{s}{2}} \text{Tr} \left( \mathcal{B} e^{-t \mathcal{B}_{\mathcal{Q}_{-, \mathcal{L}_0}}^2} \right) dt = \sum_{0 \neq \lambda_j \in \text{Spec}(\mathcal{B}_{\mathcal{Q}_{-, \mathcal{L}_0}}^2)} \text{sign}(\lambda_j)|\lambda_j|^{-s}. \]

It was shown in [11] that \( \zeta_{\mathcal{B}_{\mathcal{Q}_{-, \mathcal{L}_0}}^2}(s) \) and \( \eta_{\mathcal{B}_{\mathcal{Q}_{-, \mathcal{L}_0}}^2}(s) \) have regular values at \( s = 0 \). We define the zeta-determinant and eta-invariant by
We denote
\[ \Omega^q_+(M, E) = \text{Im} \nabla \cap \Omega^q(M, E), \quad \Omega^q_-(M, E) = \text{Im} \Gamma \nabla \cap \Omega^q(M, E), \]
and denote by \( B^+_{\text{even}} \) the restriction of \( B_{\text{even}} \) to \( \Omega^q_{\text{even}}(M, E) \). The graded zeta-determinant \( \text{Det}_{\text{gr}, \theta}(B_{\text{even}, \mathcal{D}}) \) of \( B_{\text{even}} \) with respect to the boundary condition \( \mathcal{D} \) is defined by
\[ \text{Det}_{\text{gr}, \theta}(B_{\text{even}, \mathcal{D}}) = \frac{\text{Det}_\theta B^+_{\text{even}, \mathcal{D}}}{\text{Det}_\theta (-B^-_{\text{even}, \mathcal{D}})}. \]

We next define the projections \( \tilde{\mathcal{P}}_0, \tilde{\mathcal{P}}_1 : \Omega^q(Y, E|_Y) \oplus \Omega^q(Y, E|_Y) \to \Omega^q(Y, E|_Y) \oplus \Omega^q(Y, E|_Y) \) as follows. For \( \phi \in \Omega^q(M, E) \)
\[ \tilde{\mathcal{P}}_0(\phi|_Y) = \begin{cases} \mathcal{P}_{-,-} \eta_q(\phi|_Y) & \text{if } q \text{ is even} \\ \mathcal{P}_{+,-} \eta_q(\phi|_Y) & \text{if } q \text{ is odd} \end{cases}, \quad \tilde{\mathcal{P}}_1(\phi|_Y) = \begin{cases} \mathcal{P}_{+,-} \eta_q(\phi|_Y) & \text{if } q \text{ is even} \\ \mathcal{P}_{-,-} \eta_q(\phi|_Y) & \text{if } q \text{ is odd} \end{cases}. \]

We denote by
\[ l_q := \dim \ker B^2_{Y, q}, \quad l^+_q := \dim \ker B^2_{Y, q}, \quad \text{and} \quad l^-_q := \dim \Gamma Y \cap \ker B^2_{Y, q}, \]
so that \( l_q = l^+_q + l^-_q \) and \( l^-_q = l^+_{m-1-q} \). Simple computation shows that \( \log \text{Det}_{\text{gr}, \theta}(B_{\text{even}, \mathcal{P}_{-,-}}) \) and \( \log \text{Det}_{\text{gr}, \theta}(B_{\text{even}, \mathcal{P}_{+,-}}) \) are described as follows (\([11]\)).

\[ \begin{align*}
(1) \quad \log \text{Det}_{\text{gr}, \theta}(B_{\text{even}, \mathcal{P}_{-,-}}) &= \frac{1}{2} \sum_{q=0}^{m} (-1)^{q+1} \cdot q \cdot \log \text{Det}_{2\theta} B^2_{q, \tilde{\mathcal{P}}_0} - i \pi \eta(B_{\text{even}, \mathcal{P}_{-,-}}) \\
&\quad + \frac{\pi i}{2} \left( \frac{1}{4} \sum_{q=0}^{m-1} \zeta_{B^2_{Y, q}}(0) + \sum_{q=0}^{r-2} (r - 1 - q)(l^+_q - l^-_q) \right). \\
(2) \quad \log \text{Det}_{\text{gr}, \theta}(B_{\text{even}, \mathcal{P}_{+,-}}) &= \frac{1}{2} \sum_{q=0}^{m} (-1)^{q+1} \cdot q \cdot \log \text{Det}_{2\theta} B^2_{q, \tilde{\mathcal{P}}_1} - i \pi \eta(B_{\text{even}, \mathcal{P}_{+,-}}) \\
&\quad - \frac{\pi i}{2} \left( \frac{1}{4} \sum_{q=0}^{m-1} \zeta_{B^2_{Y, q}}(0) + \sum_{q=0}^{r-2} (r - 1 - q)(l^+_q - l^-_q) \right).
\end{align*} \]

To define the refined analytic torsion we introduce the trivial connection \( \nabla^{\text{trivial}} \) acting on the trivial line bundle \( M \times \mathbb{C} \) and define the corresponding odd signature operator \( B^{\text{trivial}}_{\text{even}} : \Omega^q_{\text{even}}(M, \mathbb{C}) \to \).
Ω_{even}(M, C) in the same way as (2.2). The eta invariant \( \eta(B_{\text{trivial}}_{\text{even, } \mathcal{P}_- \mathcal{L}_0 / \mathcal{P}_+ \mathcal{L}_1}) \) associated to \( B_{\text{trivial}}_{\text{even}} \) and subject to the boundary condition \( \mathcal{P}_- \mathcal{L}_0 / \mathcal{P}_+ \mathcal{L}_1 \) is defined in the same way as in (2.19) by simply replacing \( B_{\text{even}, \mathcal{P}_- \mathcal{L}_0 / \mathcal{P}_+ \mathcal{L}_1} \) by \( B_{\text{trivial}}_{\text{even, } \mathcal{P}_- \mathcal{L}_0 / \mathcal{P}_+ \mathcal{L}_1} \). When \( \nabla \) is acyclic in the de Rham complex, the refined analytic torsion subject to the boundary condition \( \mathcal{P}_- \mathcal{L}_0 / \mathcal{P}_+ \mathcal{L}_1 \) is defined by

\[
\log \rho_{\text{an}, \mathcal{P}_- \mathcal{L}_0}(g^M, \nabla) = \log \det_{\text{gr}, \mathcal{B}_{\text{even}, \mathcal{P}_- \mathcal{L}_0}} + \frac{\pi i}{2} (\text{rank } \mathcal{E}) \eta(B_{\text{trivial}}_{\text{even, } \mathcal{P}_- \mathcal{L}_0})(0) \tag{2.24}
\]

\[
\log \rho_{\text{an}, \mathcal{P}_+ \mathcal{L}_1}(g^M, \nabla) = \log \det_{\text{gr}, \mathcal{B}_{\text{even}, \mathcal{P}_+ \mathcal{L}_1}} + \frac{\pi i}{2} (\text{rank } \mathcal{E}) \eta(B_{\text{trivial}}_{\text{even, } \mathcal{P}_+ \mathcal{L}_1})(0) \tag{2.25}
\]

The refined analytic torsion on a closed manifold is defined similarly.

On the other hand, B. Vertman also discussed the refined analytic torsion on a compact manifold with boundary in a different way. He used the minimal and maximal extensions of a flat connection, which will be explained briefly in Section 4. In this paper we are going to compare \( \rho_{\text{an}, \mathcal{P}_- \mathcal{L}_0}(g^M, \nabla) \) and \( \rho_{\text{an}, \mathcal{P}_+ \mathcal{L}_1}(g^M, \nabla) \) with the refined analytic torsion constructed by Vertman when the odd signature operator comes from an acyclic Hermitian connection. For this purpose in the next two sections we are going to compare the Ray-Singer analytic torsion and eta invariant subject to the boundary condition \( \mathcal{P}_- \mathcal{L}_0 \) and \( \mathcal{P}_+ \mathcal{L}_1 \) with those subject to the relative and absolute boundary conditions, respectively.

3. COMPARISON OF THE RAY-SINGER ANALYTIC TORSIONS

In this section we are going to compare the Ray-Singer analytic torsion subject to the boundary condition \( \mathcal{P}_- \mathcal{L}_0 \) and \( \mathcal{P}_+ \mathcal{L}_1 \) with the Ray-Singer analytic torsion subject to the relative and absolute boundary conditions. For this purpose we are going to use the BFK-gluing formula and the method of the adiabatic limit for stretching the cylinder part. In this section we do not assume the vanishing of de Rham cohomologies. We only assume that the metric is a product one near boundary. We recall that \((M, g^M)\) is a compact oriented Riemannian manifold with boundary \( Y \) with a collar neighborhood \( N = [0, 1] \times Y \) and \( g^M \) is assumed to be a product metric on \( N \). We denote by \( M_{1,1} = [0, 1] \times Y \) and \( M_2 = M - N \). To use the adiabatic limit we stretch the cylinder part \( M_{1,1} \) to the cylinder of length \( r \). We denote \( M_{1,r} = [0, r] \times Y \) with the product metric and

\[
M_r = M_{1,r} \cup_{Y_r} M_2 \quad \text{with } Y_r = \{r\} \times Y.
\]

Then we can extend the bundle \( E \) and the odd signature operator \( \mathcal{B} \) on \( M \) to \( M_r \) in the natural way and we denote these extensions by \( E_r \) and \( \mathcal{B}(r) \) (\( \mathcal{B} = \mathcal{B}(1) \)). We denote the restriction of \( \mathcal{B}(r) \) to \( M_{1,r} \) by \( \mathcal{B}_{M_{1,r}} \) and \( \mathcal{B}_{M_2} \). It is well known (cf. [2], [13]) that the Dirichlet boundary value problem for \( \mathcal{B}_q \) on \( M_2 \) has a unique solution, i.e. for \( f + dx \wedge g \in \Omega^q(M_2, E|_{M_2})|_{Y_r} \), there exists a unique \( \psi \in \Omega^q(M_2, E|_{M_2}) \) such that

\[
\mathcal{B}_q^2 \psi = 0, \quad \psi|_{Y_r} = f + dx \wedge g.
\]

Let \( \mathcal{D} \) be one of the following boundary conditions: \( \mathcal{P}_- \mathcal{L}_0 \), \( \mathcal{P}_+ \mathcal{L}_1 \), the absolute boundary condition, the relative boundary condition or the Dirichlet boundary condition. We define the Neumann jump operators \( Q_{q,1, \mathcal{D}}(r) \), \( Q_{q,2} \) and the Dirichlet-to-Neumann operator \( R_{q, \mathcal{D}}(r) \)

\[
Q_{q,1, \mathcal{D}}(r), Q_{q,2}, R_{q, \mathcal{D}}(r) : \Omega^q(Y_r, E|_{Y_r}) \oplus \Omega^{q-1}(Y_r, E|_{Y_r}) \rightarrow \Omega^q(Y_r, E|_{Y_r}) \oplus \Omega^{q-1}(Y_r, E|_{Y_r})
\]
as follows. For \( \begin{pmatrix} f \\ g \end{pmatrix} \in \Omega^q(Y, E|_{Y_r}) \oplus \Omega^{q-1}(Y, E|_{Y_r}), \) we choose \( \phi \in \Omega^q(M_{1,r}, E|_{M_{1,r}}) \) and \( \psi \in \Omega^q(M_2, E|_{M_2}) \) such that

\[
B_{q,M_1,r}^2 \phi = 0, \quad B_{q,M_2}^2 \psi = 0, \quad \phi|_{Y_r} = \psi|_{Y_r} = f + dx \wedge g, \quad \mathcal{D}(\phi|_{Y_0}) = 0. \tag{3.1}
\]

Then we define

\[
Q_{q,1,\mathcal{D}}(r)(f) = (\nabla \partial_r \phi)|_{Y_r}, \quad Q_{q,2}(f) = - (\nabla \partial_r \psi)|_{Y_r}, \quad R_{q,\mathcal{D}}(r) = Q_{q,1,\mathcal{D}}(r) + Q_{q,2}, \tag{3.2}
\]

where \( \partial_r \) is the inward unit normal vector field on \( N \subset M \).

We denote by \( B_{q,M_1,r,\mathcal{D},D}^2 (B_{q,M_2,D}^2) \) the restriction of \( B_{q}^2(r) \) to \( M_{1,r} \) (\( M_2 \)) subject to the boundary condition \( \mathcal{D} \) on \( Y_0 \) and the Dirichlet boundary condition on \( Y_r \) (the Dirichlet boundary condition on \( Y_r \)). We denote by \( B_{q,\mathcal{D}}^2(r) \) the operator \( B_{q}^2(r) \) on \( M_r \) subject to the boundary condition \( \mathcal{D} \) on \( Y_0 \). The following lemma is well known (cf. [15]).

**Lemma 3.1.** (1) \( R_{q,\mathcal{D}}(r) \) is a non-negative elliptic pseudodifferential operator of order 1 and has the form of

\[
R_{q,\mathcal{D}}(r) = 2 |A| + \text{ a smoothing operator, } \quad |A| = \begin{pmatrix} \sqrt{B_{Y,q}^2} & 0 \\ 0 & \sqrt{B_{Y,q-1}^2} \end{pmatrix}. \tag{3.3}
\]

(2) \( \ker R_{q,\mathcal{D}}(r) = \{ \phi|_{Y_r} \mid \phi \in \ker B_{q,\mathcal{D}}^2(r) \} \).

**Remark** : (1) Lemma 2.2 shows that \( \dim \ker B_{q,\mathcal{D}}^2(r) = k \) and \( \{ \varphi_1, \ldots, \varphi_k \} \) be an orthonormal basis of \( \ker B_{q,\mathcal{D}}^2(r) \). We define a positive definite \( k \times k \) Hermitian matrix \( A_{q,\mathcal{D}}(r) \) by

\[
A_{q,\mathcal{D}}(r) = (a_{ij}), \quad a_{ij} = \langle \varphi_i|_{Y_0}, \varphi_j|_{Y_0} \rangle_{Y_0}.
\]

Then the BFK-gluing formula ([7], [15], [16]) is described as follows. Setting \( l_q = \dim \ker B_{Y,q}^2 \), we have

\[
\log \det_{2\theta} B_{q,\mathcal{D}}^2(r) = \log \det_{2\theta} B_{q,M_1,r,\mathcal{D},D}^2 + \log \det_{2\theta} B_{q,M_2,D}^2 + \log \det_{2\theta} R_{q,\mathcal{D}}(r)
\]

\[
- \log 2 \cdot (\zeta_{B_{Y,q}^2}(0) + \zeta_{B_{Y,q-1}^2}(0) + l_q + l_{q-1}) - \log \det A_{q,\mathcal{D}}(r). \tag{3.4}
\]

**Comment** : (1) Lemma 2.2 shows that \( A_{q,\mathcal{D},\text{rel}}(r) = A_{q,\text{rel}}(r) \) and \( A_{q,\mathcal{D},\text{abs}}(r) = A_{q,\text{abs}}(r) \).

(2) The BFK-gluing formula was proved originally on a closed manifold in [7]. But it can be extended to a compact manifold with boundary with only minor modification when a cutting hypersurface does not intersect the boundary.

**Lemma 2.3** in [12] shows that

\[
\text{Spec} \left( B_{q,M_1,r,\mathcal{D},\text{rel}}^2 \right) \cup \text{Spec} \left( B_{q,M_1,r,\mathcal{D},\text{abs}}^2 \right) = \text{Spec} \left( B_{q,M_1,r,\mathcal{D},\text{rel}}^2 \right) \cup \text{Spec} \left( B_{q,M_1,r,\mathcal{D},\text{abs}}^2 \right),
\]

which together with (3.4) yields the following result.
Corollary 3.2.

$$\sum_{q=0}^{m} (-1)^{q+1} q \left( \log \det B_{q,\mathcal{B}^1}^2 (r) + \log \det B_{q,\mathcal{B}^1}^2 (r) - \log \det B_{q,\text{rel}}^2 (r) - \log \det B_{q,\text{abs}}^2 (r) \right)$$

$$= \sum_{q=0}^{m} (-1)^{q+1} q \left( \log \det R_{q,\mathcal{P}^-,\mathcal{C}_0} (r) + \log \det R_{q,\mathcal{P}^+,\mathcal{C}_1} (r) - \log \det R_{q,\text{rel}} (r) - \log \det R_{q,\text{abs}} (r) \right).$$

We next discuss the Dirichlet-to-Neumann operator $R_{q,\mathcal{B}} (r)$ defined by $R_{q,\mathcal{B}} (r) = Q_{q,1,\mathcal{B}} (r) + Q_{q,2}$, where $\mathcal{D}$ is one of $\mathcal{P}^-,\mathcal{C}_0$, $\mathcal{P}^+,\mathcal{C}_1$, the absolute or the relative boundary condition. The following lemma is straightforward (Lemma 2.8 in [12]).

Lemma 3.3.

$$R_{q,\mathcal{B}} (r) = Q_{q,2} + |A| + K_{q,\mathcal{B}} (r),$$

where

$$K_{q,\mathcal{P}^-,\mathcal{C}_0} (r), K_{q,\mathcal{P}^+,\mathcal{C}_1} (r) = \begin{cases} \frac{2\sqrt{B_0^2}}{e^{2\sqrt{B_0^2} r} - 1}, & \text{on } \Omega^q_+ (Y, E | Y) \oplus \Omega^{q-1}_+ (Y, E | Y) \\ \frac{1}{r}, & \text{on } \Gamma^Y \mathcal{K} \cap (\Omega^q (Y, E | Y) \oplus \Omega^{q-1} (Y, E | Y)) \\ -\frac{2\sqrt{B_1^2}}{e^{2\sqrt{B_1^2} r} + 1}, & \text{on } \Omega^q_+ (Y, E | Y) \oplus \Omega^{q-1}_+ (Y, E | Y) \\ \frac{1}{r}, & \text{on } \Gamma^Y \mathcal{K} \cap (\Omega^q (Y, E | Y) \oplus \Omega^{q-1} (Y, E | Y)) \\ 0, & \text{on } \ker B_0^2 \cap \Omega^q (Y, E | Y) \\ 0, & \text{on } \Gamma^Y \mathcal{K} \cap \Omega^{q-1} (Y, E | Y) \end{cases}$$

$$K_{q,\text{rel}} (r), K_{q,\text{abs}} (r) = \begin{cases} \frac{2\sqrt{B_0^2}}{e^{2\sqrt{B_0^2} r} - 1}, & \text{on } \Omega^q_+ (Y, E | Y) \oplus \Omega^{q-1}_+ (Y, E | Y) \\ \frac{1}{r}, & \text{on } \Gamma^Y \mathcal{K} \cap \Omega^q (Y, E | Y) \\ -\frac{2\sqrt{B_1^2}}{e^{2\sqrt{B_1^2} r} + 1}, & \text{on } \Omega^q_+ (Y, E | Y) \oplus \Omega^{q-1}_+ (Y, E | Y) \\ \frac{1}{r}, & \text{on } \Gamma^Y \mathcal{K} \cap \Omega^{q-1} (Y, E | Y) \\ 0, & \text{on } \ker B_0^2 \cap \Omega^q_+ (Y, E | Y) \\ 0, & \text{on } \Gamma^Y \mathcal{K} \cap \Omega^{q-1} (Y, E | Y) \end{cases}$$

The above lemma and (2.11) lead to the following result.

Corollary 3.4.

$$K_q (r) := R_{q,\mathcal{P}^-,\mathcal{C}_0} (r) - R_{q,\text{rel}} (r) = K_{q,\mathcal{P}^-,\mathcal{C}_0} (r) - K_{q,\text{rel}} (r) = - (R_{q,\mathcal{P}^+,\mathcal{C}_1} (r) - R_{q,\text{abs}} (r))$$

$$= \begin{cases} \frac{e^{2\sqrt{B_1^2} r} - e^{-2\sqrt{B_1^2} r}}{2\sqrt{B_1^2}}, & \text{on } \Omega^q_+ (Y, E | Y) \\ \frac{4\sqrt{B_1^2}}{e^{2\sqrt{B_1^2} r} - e^{-2\sqrt{B_1^2} r}}, & \text{on } \Omega^{q-1}_+ (Y, E | Y) \\ -\frac{1}{r}, & \text{on } \Gamma^Y \mathcal{K} \cap \Omega^q (Y, E | Y) \\ \frac{1}{r}, & \text{on } \Gamma^Y \mathcal{K} \cap \Omega^{q-1} (Y, E | Y) \\ 0, & \text{otherwise.} \end{cases}$$

In particular, if $\nabla \phi = \Gamma \nabla \Gamma \phi = 0$ for $\phi \in \Omega^q (M, E)$, then $K_q (r) (\phi | Y) = 0$.

We next discuss the limit of $\left( \log \det R_{q,\mathcal{P}^-,\mathcal{C}_0/\mathcal{P}^+,\mathcal{C}_1} (r) - \log \det R_{q,\text{rel/abs}} (r) \right)$ for $r \to \infty$. We note that
\[
\lim_{r \to \infty} R_{q, \mathcal{P}_- \mathcal{E}_0 / \mathcal{P}_+, \mathcal{E}_1}(r) = \lim_{r \to \infty} R_{q, \text{rel/abs}}(r) = Q_{q, 2} + |\mathcal{A}|.
\]

The kernel of \(Q_{q, 2} + |\mathcal{A}|\) is described as follows. For \(f \in \Omega^q(M_2, E)|_{Y_1}\), choose \(\psi \in \Omega^q(M_2, E)\) such that \(\mathcal{B}^2 \psi = 0\) and \(\psi|_{Y_1} = f\). Then,

\[
0 = \langle \mathcal{B}^2 \psi, \psi \rangle = \langle \mathcal{B} \psi, \mathcal{B} \psi \rangle + \langle (\mathcal{B} \psi)|_{Y_1}, (\gamma \psi)|_{Y_1} \rangle_{Y_1} = \langle \mathcal{B} \psi, \mathcal{B} \psi \rangle + \langle (\nabla_{\partial_s} \psi + \mathcal{A} \psi)|_{Y_1}, f \rangle_{Y_1} = \| \mathcal{B} \psi \|^2 - \langle Q_{q, 2} f, f \rangle_{Y_1} + \langle \mathcal{A} f, f \rangle_{Y_1},
\]

which leads to

\[
\langle (Q_{q, 2} + |\mathcal{A}|) f, f \rangle_{Y_1} = \| \mathcal{B} \psi \|^2 + \langle |\mathcal{A}| + \mathcal{A} f, f \rangle_{Y_1}.
\]

Hence, \(f \in \ker(Q_{q, 2} + |\mathcal{A}|)\) if and only if \(\mathcal{B} \psi = 0\) and \(|\mathcal{A}| + \mathcal{A} f = 0\), which shows that \(\psi\) is expressed, on a collar neighborhood of \(Y_1\), by

\[
\psi = \sum_{\lambda_j \in \text{spec}(\mathcal{A})} a_j e^{-\lambda_j x} \phi_j, \quad \text{where} \quad \mathcal{A} \phi_j = \lambda_j \phi_j.
\]

Let \(M_\infty := ((-\infty, 0] \times Y) \cup_Y M_2\). We can extend \(E\) and \(\mathcal{B}\) canonically to \(M_\infty\), which we denote by \(E_\infty\) and \(\mathcal{B}_\infty\). Then \(\psi\) in (3.7) can be extended to \(M_\infty\) as an \(L^2\) or extended \(L^2\)-solution of \(\mathcal{B}_\infty\) (for definitions of \(L^2\) and extended \(L^2\)-solutions we refer to \([1]\) or \([3]\)). Hence,

\[
\ker(Q_{q, 2} + |\mathcal{A}|) = \{ \psi|_{Y_1} \mid \psi \text{ is an } L^2 \text{ or extended } L^2 \text{-solution of } \mathcal{B}_\infty \text{ in } \Omega^q(M_\infty, E_\infty) \}.
\]

This fact together with Lemma 3.1 Corollary 3.4 and 2.11 leads to the following result.

**Corollary 3.5.** Let \(f \in \ker(Q_{q, 2} + |\mathcal{A}|) \) or \(f \in \ker R_{q, \mathcal{D}}(r)\), where \(\mathcal{D}\) is one of \(\mathcal{P}_0, \mathcal{P}_1\), the absolute or the relative boundary condition. Then, \(\mathcal{K}_q(r) f = 0\).

Since \(\ker R_{q, \mathcal{P}_- \mathcal{E}_0}(r) = \ker R_{q, \text{rel}}(r)\) (Lemma 2.2), we have

\[
\log \text{ Det } R_{q, \mathcal{P}_- \mathcal{E}_0}(r) - \log \text{ Det } R_{q, \text{rel}}(r) = \int_0^1 \frac{d}{ds} \log \text{ Det } R_{q, \text{rel}}(r + s \mathcal{K}_q(r)) ds = \int_0^1 \text{ Tr } \left( \left( R_{q, \text{rel}}(r + s \mathcal{K}_q(r)) - R_{q, \text{rel}}(r) \right)^{-1} \mathcal{K}_q(r) \right) ds,
\]

where \(\text{pr}_{\ker R_{q, \mathcal{P}_- \mathcal{E}_0}}(r)\) is the orthogonal projection onto \(\ker R_{q, \mathcal{P}_- \mathcal{E}_0}(r)\). We denote
Lemma 3.8. Let \( r > R \) that for \( \phi \in \mathbb{Q} \), \( r \in \mathbb{M} \), \( W \not= \phi \), we let \( X \) be the first nonzero eigenvalue of \( \mathcal{R} \). Then, we have

\[
X(r) = R_{q, rel}(r) + \text{pr}_{\ker R_{q, rel}(r)} + sK_q(r) = Q_{q, 2} + |A| + \mathcal{K}_{q, rel}(r) + \text{pr}_{\ker R_{q, rel}(r)} + sK_q(r),
\]

\[
\mathcal{M} = \{ \phi \mid K_q(r)(\phi) \neq 0 \} = \Omega_q(Y, E|Y) \oplus \Omega \Gamma^2(Y, E|Y) \oplus \mathcal{K} \oplus \Omega \Gamma^2(Y, E|Y) \oplus \mathcal{K} \oplus \Omega \Gamma^2(Y, E|Y),
\]

\[
W(r) = X(r)^{-1}(\mathcal{M}).
\]

Then, we have

\[
\text{Tr} \left( X(r)^{-1}K_q(r) \right) = \text{Tr} \left( X(r)^{-1}K_q(r) : \mathcal{M} \to W(r) \right).
\] (3.9)

Lemma 3.6.

\[
W(r) \cap \ker (Q_{q, 2} + |A|) = \{ 0 \}.
\]

\textbf{Proof.} Let \( \phi \in \ker (Q_{q, 2} + |A|) \). Corollary 3.5 shows that \( X(r) \phi = K_{q, rel}(r) \phi + \text{pr}_{\ker R_{q, rel}(r)} \phi \). Corollary 3.4 again shows \( \phi, \text{pr}_{\ker R_{q, rel}(r)} \phi \in \mathcal{M} \). From \( \phi \in \mathcal{M} \), we have \( K_{q, rel}(r) \phi \in \mathcal{M} \), which shows that \( X(r) \phi \in \mathcal{M} \). Since \( X(r) \) is invertible, this completes the proof of the lemma.

We denote by \( \mathcal{P}(r) : W(r) \to \ker (Q_{q, 2} + |A|) \) the orthogonal projection from \( W(r) \) into \( \ker (Q_{q, 2} + |A|) \). We let

\[
W_0(r) := \ker \mathcal{P}(r), \quad W_1(r) := W(r) \oplus W_0(r).
\]

Since \( \ker (Q_{q, 2} + |A|) \) is finite dimensional, so is \( W_1(r) \). Let \( \{ \phi_1, \cdots, \phi_k \} \) be an orthonormal basis for \( W_1(r) \). For each \( 1 \leq i \leq k \), \( \phi_i \) is expressed by

\[
\phi_i = \psi_i + \varphi_i,
\]

where \( 0 \neq \psi_i \in (\ker (Q_{q, 2} + |A|))^\perp \) and \( 0 \neq \varphi_i \in \ker (Q_{q, 2} + |A|) \). We put

\[
c_0 := \min \{ \| \psi_i \| \mid 1 \leq i \leq k \} > 0,
\]

which leads to the following result.

\textbf{Lemma 3.7.} For any \( \phi \in W(r) \), \( \phi \) is expressed by \( \phi = \psi + \varphi \), where \( \psi \in (\ker (Q_{q, 2} + |A|))^\perp \) and \( \varphi \in \ker (Q_{q, 2} + |A|) \). Then \( \| \psi \| \geq c_0 \| \phi \| \).

\textbf{Lemma 3.8.} Let \( \lambda_1 > 0 \) be the first nonzero eigenvalue of \( Q_{q, 2} + |A| \). Then there exists \( R_0 > 0 \) such that for \( r > R_0 \) and \( f \in \mathcal{M} \),

\[
\| X(r)^{-1}f \| \leq \frac{2}{c_0 \lambda_1} \| f \|.
\]

Hence, for \( r > R_0 \) we have

\[
\lim_{r \to \infty} \left| \text{Tr} \left( X(r)^{-1}K_q(r) \right) \right| = 0.
\]
Proof. It’s enough to prove that \( \| X(r) \phi \| \geq \frac{c_0}{r} \| \phi \| \) for \( \phi \in \mathcal{W}(r) \) and \( r \) large enough. As in Lemma 3.7, we write
\[ \| \phi \| \geq \| \psi \| \geq \| \varphi \| , \quad \varphi \in \ker (Q_{q,2} + |A|), \quad \psi \in \ker (Q_{q,2} + |A|), \] which tends to 0 as \( r \to \infty \).

Corollary 3.5 shows that
\[ X(r) \phi = (Q_{q,2} + |A|) \psi + pr_{\ker R_q, rel(r)}(r) \varphi \],
\[ \varphi \in \ker (Q_{q,2} + |A|), \quad \psi \in \ker (Q_{q,2} + |A|), \]
\[ \| \phi \| \geq \| \psi \| \geq \| \varphi \| . \]

Corollary 3.5 shows that
\[ \| \phi \| \geq \| \psi \| \geq \| \varphi \| , \quad \varphi \in \ker (Q_{q,2} + |A|), \quad \psi \in \ker (Q_{q,2} + |A|), \]
which tends to 0 as \( r \to \infty \). Similarly,
\[ \| pr_{\ker R_q, rel(r)}(r) \varphi \|^2 = \langle pr_{\ker R_q, rel(r)}(r) \varphi, \psi \rangle = \langle pr_{\ker R_q, rel(r)}(r) \varphi, \psi \rangle \]
\[ = \langle (Q_{q,2} + |A|) pr_{\ker R_q, rel(r)}(r) \varphi, (Q_{q,2} + |A|) pr_{\ker R_q, rel(r)}(r) \varphi \rangle \]
\[ = \langle -K_{q, rel}(r) pr_{\ker R_q, rel(r)}(r) \varphi, (Q_{q,2} + |A|) pr_{\ker R_q, rel(r)}(r) \varphi \rangle, \]
which tends to 0 as \( r \to \infty \). Hence, we have
\[ \| X(r) \phi \| \geq \left( \| (Q_{q,2} + |A|) \psi + pr_{\ker R_q, rel(r)}(r) \varphi \| \right) \]
\[ - \left( \| K_{q, rel}(r) \psi \| + \| pr_{\ker R_q, rel(r)}(r) \psi \| + \| sK_{q}(r) \psi \| + \| K_{q, rel}(r) \varphi \| \right) \]
\[ \geq \left( \| (Q_{q,2} + |A|) \psi \|^2 + \| pr_{\ker R_q, rel(r)}(r) \varphi \|^2 - 2 \langle (Q_{q,2} + |A|) \psi, pr_{\ker R_q, rel(r)}(r) \varphi \rangle \right)^{\frac{1}{2}} + o(r) \]
\[ \geq \left( \| (Q_{q,2} + |A|) \psi \|^2 - 2 \langle (Q_{q,2} + |A|) \psi, pr_{\ker R_q, rel(r)}(r) \varphi \rangle \right)^{\frac{1}{2}} + o(r), \quad (3.10) \]
which completes the proof of the lemma.

The above lemma with \( 3.5 \) leads to the following result.
\[ \lim_{r \to \infty} \left( \log \det R_{q, rel}(r) - \log \det R_{q, rel}(r) \right) = 0. \quad (3.11) \]

By the same method, we have
\[ \lim_{r \to \infty} \left( \log \det R_{q, abs}(r) - \log \det R_{q, abs}(r) \right) = 0. \quad (3.12) \]
Corollary 3.2 with \( 3.11 \) and \( 3.12 \) yields
\[ \lim_{r \to \infty} \sum_{q=0}^{m} (-1)^{q+1} q \left( \log \det B_{q, rel}^2(r) + \log \det B_{q, abs}^2(r) \right) = 0. \quad (3.13) \]
The proof of the following lemma is a verbatim repetition of the proof of Theorem 7.6 in [17] (cf. Theorem 2.1 in [19]).

**Lemma 3.9.** Let \( M \) be a compact manifold with boundary \( Y \) and \( N \) be a collar neighborhood of \( Y \). We suppose that \( \{g^M_t | -\delta_0 < t < \delta_0\} \) is a family of metrics such that each \( g^M_t \) is a product metric and does not vary on \( N \). Let \( \mathcal{D} \) be one of \( \mathcal{P}_0, \mathcal{P}_1 \), the absolute or the relative boundary condition. We denote by \( B^2_{q,\mathcal{D}}(t) \) the square of the odd signature operator acting on \( q \)-forms subject to \( \mathcal{D} \) and with respect to the metric \( g^M_t \). Then we have

\[
\frac{d}{dt} \left( \sum_{q=0}^{m} (-1)^{q+1} \cdot q \cdot \log \det B^2_{q,\mathcal{D}}(t) \right) = \sum_{q=0}^{m} (-1)^{q+1} \cdot q \cdot \text{Tr} \left( \text{pr}_{\mathcal{H}^q(t)} \ast_t (\frac{d}{dt} \ast_t) \right),
\]

where \( \text{pr}_{\mathcal{H}^q(t)} \) is the orthogonal projection onto the kernel of \( B^2_{q,\mathcal{D}}(t) \) and \( \ast_t \) is the Hodge star operator with respect to the metric \( g^M_t \).

Lemma 3.9 and Lemma 2.2 lead to the following corollary.

**Corollary 3.10.** We assume the same assumptions as in Lemma 3.9. Then,

\[
\frac{d}{dt} \sum_{q=0}^{m} (-1)^{q+1} \cdot q \cdot \left( \log \det B^2_{q,\mathcal{P}_0}(t) + \log \det B^2_{q,\mathcal{P}_1}(t) - \log \det B^2_{q,\text{rel}}(t) - \log \det B^2_{q,\text{abs}}(t) \right) = 0.
\]

We fix \( \delta_0 > 0 \) sufficiently small and choose a smooth function \( f(r, u) : [1, \infty) \times [0, 1] \to [0, \infty), \ (r \geq 1) \) such that for each \( r \)

\[
\text{supp}_u f(r, u) \subset [\delta_0, 1 - \delta_0], \quad \int_0^1 f(r, u)du = r - 1, \quad \text{and} \quad f(1, u) \equiv 0.
\]

Setting \( F(r, u) = u + \int_0^u f(r, t)dt, \quad F_r := F(r, \cdot) : [0, 1] \to [0, r] \) is a diffeomorphism satisfying

\[
F_r(u) = \begin{cases} 
  u & \text{for } 0 \leq u \leq \delta_0 \\
  u + r - 1 & \text{for } 1 - \delta_0 \leq u \leq 1.
\end{cases}
\]

Let \( g^M_r \) be a metric on \( M_r := ([0, r] \times Y) \cup_{r} M_2 \) which is the extension of \( g^M \) by a product one on \( [0, r] \times Y \). Then \( F^*_r g^M_r \) is a metric on \( M_r \), which is \( \begin{pmatrix} (F_r')(u)^2 & 0 \\ 0 & g_Y \end{pmatrix} \) on \( [0, 1] \times Y \). Hence, \( F^*_r g^M_r \) is a metric on \( M \) which is a product one near \( Y \). Furthermore, \( (M, F^*_r g^M_r) \) and \( (M_r, g^M_r) \) are isometric. Let \( \mathcal{B}(r) \) and \( B(r) \) be the odd signature operators defined on \( M \) and \( M_r \) associated to the metrics \( F^*_r g^M_r \) and \( g^M_r \), respectively. Then Corollary 3.10 leads to the following equalities.
\[ \sum_{q=0}^{m} (-1)^{q+1} q \cdot \left( \log \text{Det}_{2g} B_{q,\tilde{P}_0} + \log \text{Det}_{2g} B_{q,\tilde{P}_1} - \log \text{Det}_{2g} B_{q,rel} - \log \text{Det}_{2g} B_{q,abs} \right) \]

\[ = \sum_{q=0}^{m} (-1)^{q+1} q \cdot \left( \log \text{Det}_{2g} B^2_{q,\tilde{P}_0} + \log \text{Det}_{2g} B^2_{q,\tilde{P}_1} - \log \text{Det}_{2g} B^2_{q,rel} - \log \text{Det}_{2g} B^2_{q,abs} \right) \]

\[ = \sum_{q=0}^{m} (-1)^{q+1} q \cdot \left( \log \text{Det}_{2g} B^2_{q,\tilde{P}_0}(r) + \log \text{Det}_{2g} B^2_{q,\tilde{P}_1}(r) - \log \text{Det}_{2g} B^2_{q,rel}(r) - \log \text{Det}_{2g} B^2_{q,abs}(r) \right) \]

\[ = \lim_{r \to 0} \sum_{q=0}^{m} (-1)^{q+1} q \cdot \left( \log \text{Det}_{2g} B^2_{q,\tilde{P}_0}(r) + \log \text{Det}_{2g} B^2_{q,\tilde{P}_1}(r) - \log \text{Det}_{2g} B^2_{q,rel}(r) - \log \text{Det}_{2g} B^2_{q,abs}(r) \right) = 0, \]

which yields the following result. This is the main result of this section and is also interesting independently.

**Theorem 3.11.** Let \((M, g^M)\) be a compact Riemannian manifold with boundary \(Y\) and \(g^M\) be a product metric near \(Y\). Then:

\[ \sum_{q=0}^{m} (-1)^{q+1} q \cdot \left( \log \text{Det}_{2g} B^2_{q,\tilde{P}_0} + \log \text{Det}_{2g} B^2_{q,\tilde{P}_1} \right) = \sum_{q=0}^{m} (-1)^{q+1} q \cdot \left( \log \text{Det}_{2g} B^2_{q,rel} + \log \text{Det}_{2g} B^2_{q,abs} \right). \]

**Remark:** This result improves Theorem 2.12 in [12], in which the same result was obtained under the additional assumption of \(H^q(M; E) = H^q(M, Y; E) = \{0\}\) for each \(0 \leq q \leq m\).

For later use we include some result of [12] for eta invariants. We denote by \((\Omega^{even}(M, E)|_Y)^*\) the orthogonal complement of \(\left( \mathcal{H}^{even}(Y, E|_Y) \oplus \mathcal{H}^{odd}(Y, E|_Y) \right)\) in \((\Omega^{even}(M, E)|_Y)^*\), i.e.

\[ \Omega^{even}(M, E)|_Y = (\Omega^{even}(M, E)|_Y)^* \oplus \left( \mathcal{H}^{even}(Y, E|_Y) \right), \]

and denote by \(\mathcal{P}_e\) the orthogonal projection onto \((\Omega^{even}(M, E)|_Y)^*\). We define one parameter families of orthogonal projections \(\tilde{\mathcal{P}}_-(\theta), \tilde{\mathcal{P}}_+(\theta) : \Omega^{even}(M, E)|_Y \to \Omega^{even}(M, E)|_Y\) by

\[ \tilde{\mathcal{P}}_-(\theta) = \Pi_\geq \cos \theta + \mathcal{P}_- \sin \theta + \frac{1}{2}(1 - \cos \theta - \sin \theta)\mathcal{P}_* + \mathcal{P}_c, \]

\[ \tilde{\mathcal{P}}_+(\theta) = \Pi_\geq \cos \theta + \mathcal{P}_+ \sin \theta + \frac{1}{2}(1 - \cos \theta + \sin \theta)\mathcal{P}_* + \mathcal{P}_c, \quad (0 \leq \theta \leq \frac{\pi}{2}), \]

where \(\Pi_\geq\) is the orthogonal projection onto the eigenspace generated by positive eigenforms of \(\mathcal{A}\) and \(\mathcal{P}_{c_1}\) is the orthogonal projection onto \(\mathcal{L}_i (i = 1, 2)\). \(\tilde{\mathcal{P}}_-(\theta), \tilde{\mathcal{P}}_+(\theta)\) is a smooth curve of orthogonal projections connecting \(\mathcal{P}_- \mathcal{L}_0, (\mathcal{P}_+ \mathcal{L}_1)\) and \(\Pi_\geq \mathcal{L}_0, (\Pi_\geq \mathcal{L}_1)\). We denote the Calderón projector for \(B\) by \(\mathcal{C}_M\). We also denote the spectral flow for \(\left( \mathcal{B}_{\tilde{\mathcal{P}}_\pm}(\theta) \right)_{\theta \in [0, \frac{\pi}{2}]}\) and Maslov index for \(\left( \tilde{\mathcal{P}}_\pm(\theta), \mathcal{C}_M \right)_{\theta \in [0, \frac{\pi}{2}]}\) by \(\text{SF}(\mathcal{B}_{\tilde{\mathcal{P}}_\pm}(\theta))_{\theta \in [0, \frac{\pi}{2}]}\) and \(\text{Mas}(\tilde{\mathcal{P}}_\pm(\theta), \mathcal{C}_M)_{\theta \in [0, \frac{\pi}{2}]}\). We refer to [3], [14] and [18] for the definitions of the Calderón projector, the spectral flow and Maslov index. We refer to Theorem 3.12 in [12] for the proof of the following result.
We define following equalities are well known facts (cf. p.1996 in [22]).

In particular, if for each $0 \leq q \leq m$, $H^q(M; E) = H^q(M, Y; E) = \{0\}$, then $\eta(B_{P-}) - \eta(B_{P+}) \in \mathbb{Z}$.

In the remaining part of this paper we assume that $H^q(M; E) = H^q(M, Y; E) = \{0\}$ for each $0 \leq q \leq m$ so that $L_0 = L_1 = \{0\}$.

4. A FAMILY OF ODD SIGNATURE OPERATORS ON MANIFOLDS WITH BOUNDARY

In this section we construct a one parameter family of odd signature operators on manifolds with boundary connecting $\left(\begin{array}{cc} B_{P-} & 0 \\ 0 & -B_{P+} \end{array}\right)$ and the odd signature operator considered in [22], (cf. (4.3) below) by using the ideas in [6] and Section 11 in [5]. For the motivation of this work we review briefly the Vertman's construction of the refined analytic torsion discussed in [22]. We first consider a direct sum of two de Rham complexes with the chirality operator $\tilde{\Gamma}$, de Rham operator $\nabla$ and odd signature operator $B$ defined as follows.

$$L^2\Omega^\bullet(M, E) \oplus L^2\Omega^\bullet(M, E), \quad \tilde{\Gamma} = \begin{pmatrix} 0 & \Gamma \\ \Gamma & 0 \end{pmatrix}, \quad \tilde{\nabla} = \begin{pmatrix} \nabla & 0 \\ 0 & \nabla \end{pmatrix}, \quad \tilde{B} = \tilde{\Gamma}\nabla + \nabla\tilde{\Gamma}. \quad (4.1)$$

We denote by $\nabla_{\text{min}}$ and $\nabla_{\text{max}}$ the minimal and maximal closed extensions of $\nabla$ defined on smooth forms having compact supports in the interior of $M$. We refer to [22] for definitions of $\nabla_{\text{min}}$ and $\nabla_{\text{max}}$. The following equalities are well known facts (cf. p.1996 in [22]).

$$\text{Dom}(\nabla_{\text{min}}^*) = \text{Dom}(\Gamma\nabla_{\text{min}}\Gamma), \quad \text{Dom}(\nabla_{\text{max}}^*) = \text{Dom}(\Gamma\nabla_{\text{max}}\Gamma). \quad (4.2)$$

We define

$$\begin{align*}
\Omega^q_{B,\text{min}}(M, E) &= \text{Dom}(\nabla_{\text{min}}) \cap \text{Dom}(\nabla_{\text{min}}^*) \cap L^2\Omega^q(M, E), \\
\Omega^q_{B,\text{max}}(M, E) &= \text{Dom}(\nabla_{\text{max}}) \cap \text{Dom}(\nabla_{\text{max}}^*) \cap L^2\Omega^q(M, E), \\
\Omega^q_{B^2,\text{min}}(M, E) &= \{\omega \in \Omega^q_{B,\text{min}}(M, E) | \nabla_{\text{min}}\omega \in \text{Dom}(\nabla_{\text{min}}^*), \quad \nabla_{\text{min}}^*\omega \in \text{Dom}(\nabla_{\text{min}})\}, \\
\Omega^q_{B^2,\text{max}}(M, E) &= \{\omega \in \Omega^q_{B,\text{max}}(M, E) | \nabla_{\text{max}}\omega \in \text{Dom}(\nabla_{\text{max}}^*), \quad \nabla_{\text{max}}^*\omega \in \text{Dom}(\nabla_{\text{max}})\}, \quad (4.3)
\end{align*}$$

and put

$$\begin{align*}
\tilde{\nabla}_{(m)} &= \begin{pmatrix} \nabla_{\text{min}} & 0 \\ 0 & \nabla_{\text{max}} \end{pmatrix}, \\
\tilde{B}_{(m)} &= \tilde{\Gamma}\nabla_{(m)} + \nabla_{(m)}\tilde{\Gamma} = \begin{pmatrix} 0 & \Gamma\nabla_{\text{min}} + \nabla_{\text{max}}\Gamma \\ \Gamma\nabla_{\text{min}} + \nabla_{\text{max}}\Gamma & 0 \end{pmatrix}, \quad (4.4)
\end{align*}$$
where the subscript \((m)\) in \(\tilde{\nabla}_{(m)}\) and \(\tilde{B}_{(m)}\) stands for min/max. We denote by \(\tilde{\Omega}_{(m)}^{\text{even}}\) and \(\tilde{\Omega}_{(m)}^{\text{even},q}\) the restriction of \(\tilde{B}_{(m)}\) and \((\tilde{B}_{(m)})^2\) to even and \(q\)-forms, respectively. Then the domains of \(\tilde{\Omega}_{(m)}^{\text{even}}\) and \(\tilde{\Omega}_{(m)}^{\text{even},q}\) are given as follows.

\[
\text{Dom}(\tilde{\Omega}_{(m)}^{\text{even}}) = \left( \Omega_{\text{rel},(m)}^{\text{even}}(M, E) \right), \quad \text{Dom}(\tilde{\Omega}_{(m)}^{\text{even},q}) = \left( \Omega_{\text{rel},(m)}^{\text{even},q}(M, E) \right) := \tilde{\Omega}_{(m)}^{\text{even},q}(M, E \oplus E). \tag{4.5}
\]

In [22] Vertman considered the following complex

\[
0 \to \cdots \to \tilde{\nabla}_{(m)}^{q-1}(M, E \oplus E) \to \tilde{\nabla}_{(m)}^{q}(M, E \oplus E) \to \cdots \to 0. \tag{4.6}
\]

We define \(\tilde{\Omega}_{(m)}^{\text{even},\text{trivial}}\) by the same way as \(\tilde{\Omega}_{(m)}^{\text{even}}\) when \(\nabla\) is the trivial connection acting on the trivial line bundle \(M \times \mathbb{C}\). For simplicity we assume that \(H^\bullet(M; E) = H^\bullet(M, Y; E) = 0\). In this case the Vertman’s construction of the refined analytic torsion \(\rho_{\text{an},(m)}(g^M, \nabla)\) is given as follows.

\[
\log \rho_{\text{an},(m)}(g^M, \tilde{\nabla}) = \log \text{Det}_{\text{gr},\theta}(\tilde{\Omega}_{(m)}^{\text{even}}) + i\pi \text{rk}(E) \cdot \eta_{\tilde{\Omega}_{(m)}^{\text{even},\text{trivial}}}(0)
\]

\[
= \frac{1}{2} \sum_{q=0}^{m} (-1)^{q+1} q \cdot \log \text{Det}_{\theta}(\tilde{\Omega}_{(m)}^{\text{even},q}) + \frac{i\pi}{2} \sum_{q=0}^{m} (-1)^{q+1} \cdot \zeta_{\tilde{\Omega}_{(m)}^{\text{even},q}}(0)
\]

\[
- i\pi \left( \eta_{\tilde{\Omega}_{(m)}^{\text{even},\text{trivial}}(0)} - \frac{1}{2} \text{rk}(E) \cdot \eta_{\tilde{\Omega}_{(m)}^{\text{even},\text{trivial}}(0)} \right). \tag{4.7}
\]

We denote by \(\Omega^\bullet(M, E)\) the space of all smooth \(E\)-valued forms on \(M\). We define

\[
\Omega_{\text{rel}}^q(M, E) := \{ \omega \in \Omega^q(M, E) \mid \mathcal{P}_{\text{rel}}(\omega|Y) := dx \land (\omega|Y) = 0, \mathcal{P}_{\text{rel}}((\Gamma \nabla \Gamma \omega)|Y) = 0 \},
\]

\[
\Omega_{\text{abs}}^q(M, E) := \{ \omega \in \Omega^q(M, E) \mid \mathcal{P}_{\text{abs}}(\omega|Y) := dx \land (\omega|Y) = 0, \mathcal{P}_{\text{abs}}((\nabla \omega)|Y) = 0 \}, \tag{4.8}
\]

and denote by \(\mathcal{B}^2_{q,\text{rel}}\) and \(\mathcal{B}^2_{q,\text{abs}}\) the restriction of \(\mathcal{B}^2\) to \(\Omega_{\text{rel}}^q(M, E)\) and \(\Omega_{\text{abs}}^q(M, E)\), respectively. It is a well known facts (cf. Theorem 3.2 in [22], Section 2.7 in [10]) that

\[
\Omega_{\text{rel}}^q(M, E) \subset \Omega_{\mathcal{B}_{\text{rel}}}^q(M, E) \subset \Omega_{\mathcal{B}_{\text{max}}}^q(M, E),
\]

\[
\Omega_{\text{abs}}^q(M, E) \subset \Omega_{\mathcal{B}_{\text{abs}}}^q(M, E) \subset \Omega_{\mathcal{B}_{\text{max}}}^q(M, E),
\]

\[
\text{Spec}(\mathcal{B}^2_{q,\text{rel}}) = \text{Spec}(\mathcal{B}^2_{\Omega_{\mathcal{B}_{\text{rel}}}^q(M, E)}), \quad \text{Spec}(\mathcal{B}^2_{q,\text{abs}}) = \text{Spec}(\mathcal{B}^2_{\Omega_{\mathcal{B}_{\text{abs}}}^q(M, E)}), \tag{4.9}
\]

which leads to

\[
\log \text{Det}_{\theta}(\mathcal{B}^2_{q,\text{rel}}) = \log \text{Det}_{\theta}(\mathcal{B}^2_{q,\text{rel}}) + \log \text{Det}_{\theta}(\mathcal{B}^2_{q,\text{abs}}),
\]

\[
\zeta_{\mathcal{B}^2_{q,\text{rel}}}(0) = \zeta_{\mathcal{B}^2_{q,\text{rel}}}(0) + \zeta_{\mathcal{B}^2_{q,\text{abs}}}(0). \tag{4.10}
\]

The proof of the following lemma is similar to the proof of Lemma 3.4 in [11].
Lemma 4.1. Let \((M, g^M)\) be a compact oriented Riemannian manifold with boundary \(Y\) and \(g^M\) be a product metric near \(Y\). We assume that \(H^q(M, Y; E) = H^q(M; E) = 0\) for each \(0 \leq q \leq m\). Then,

\[
\zeta_{\text{rel}}^{2, q}(0) + \zeta_{\text{abs}}^{2, q}(0) = \zeta_{\text{rel}}^{2, q}(0) + \zeta_{\text{abs}}^{2, q}(0) = 0.
\]

Proof. : We denote by \(\mathcal{E}^q_{\text{double}}, \mathcal{E}^q_{\text{cyl}, \text{rel}}\) the heat kernels of \(e^{-tB^2_{\text{double}} q}\) and \(e^{-tB^2_{\text{cyl}, \text{rel}} q}\), where \(B^2_{\text{double}}\) and \(B^2_{\text{cyl}, q}\) are Laplacians acting on \(q\)-forms on the closed double \(M \cup Y M\) and the half-infinite cylinder \(Y \times (0, \infty)\) with the relative boundary condition at \(Y \times \{0\}\), respectively. Let \(\rho(a, b)\) be a smooth increasing function of real variable such that

\[
\rho(a, b)(u) = \begin{cases} 
0 & \text{for } u \leq a \\
1 & \text{for } u \geq b.
\end{cases}
\]

We put

\[
\phi_1 := 1 - \rho\left(\frac{5}{7t}, \frac{6}{7t}\right), \quad \phi_2 := \rho\left(\frac{1}{7t}, \frac{2}{7t}\right),
\]

\[
\psi_1 := 1 - \rho\left(\frac{3}{7t}, \frac{4}{7t}\right), \quad \psi_2 := \rho\left(\frac{3}{7t}, \frac{4}{7t}\right).
\]

Then a parametrix \(Q(t, (w, x), (w', y))\) of the kernel of \(e^{-tB^2_{\text{rel}} q}\) is given as follows.

\[
Q(t, (w, x), (w', y)) = \phi_1(x)\mathcal{E}^q_{\text{cyl}, \text{rel}}(t, (w, x), (w', y))\psi_1(y) + \phi_2(x)\mathcal{E}^q_{\text{double}}(t, (w, x), (w', y))\psi_2(y).
\]

It is a well known fact that

\[
\mathcal{E}^q_{\text{double}}(t, (w, x), (w, x)) \sim \sum_{j=0}^{\infty} a_{m-j}(w, x) t^{-\frac{m+j}{2}} \quad \text{with} \quad a_0(w, x) = 0, \tag{4.12}
\]

\[
\mathcal{E}^q_{\text{cyl}}(t, (w, x), (w, x)) = \frac{1}{\sqrt{4\pi t}} \left( e^{-\frac{(x-y)^2}{4t}} - e^{-\frac{(x+y)^2}{4t}} \right) e^{-tB^2_{\text{double}} q} + \frac{1}{\sqrt{4\pi t}} \left( e^{-\frac{(x-y)^2}{4t}} + e^{-\frac{(x+y)^2}{4t}} \right) e^{-tB^2_{\text{cyl}, q}}.
\]

Since \(\text{Tr} \left( e^{-tB^2_{\text{cyl}, q}} \right) \sim \sum_{j=0}^{\infty} a_{m-j}(Y, q) t^{-\frac{m-j}{2} + 1}\), we have

\[
\zeta_{\text{rel}}^{2, q}(0) = \frac{1}{4} (-a_0(Y, q) + a_0(Y, q - 1)).
\]

A similar method shows that

\[
\zeta_{\text{abs}}^{2, q}(0) = \frac{1}{4} (a_0(Y, q) - a_0(Y, q - 1)),
\]

which completes the proof of the first equality. The second equality is proven in Lemma 3.4 in [11]. \(\square\)

Remark : More generally, one can show that if \(g^M\) is a product metric near boundary, then

\[
\zeta_{\text{rel}}^{2, q}(0) + \zeta_{\text{abs}}^{2, q}(0) = \zeta_{\text{rel}}^{2, q}(0) + \zeta_{\text{abs}}^{2, q}(0) = -(\dim H^q(M, Y; E) + \dim H^q(M; E)).
\]

We recall that
\[ \tilde{B}^{\text{even}}_{(m)} : \Omega_{\text{B.min}}^{\text{even}}(M, E) \oplus \Omega_{\text{B.max}}^{\text{even}}(M, E) \to L^2 \Omega_{\text{even}}^{\text{max}}(M, E) \oplus L^2 \Omega_{\text{even}}^{\text{min}}(M, E) \] (4.13)

\[ \tilde{B}^{\text{even}}_{(r/a)} := \begin{pmatrix} 0 & \tilde{B}_{\text{abs}}^{\text{even}}(r/a) \\ \tilde{B}_{\text{rel}}^{\text{even}} & 0 \end{pmatrix} : \Omega_{\text{rel}}^{\text{even}}(M, E) \oplus \Omega_{\text{abs}}^{\text{even}}(M, E) \to \Omega_{\text{even}}(M, E) \oplus \Omega_{\text{even}}(M, E), \]

where the subscript \((r/a)\) in \(\tilde{B}^{\text{even}}_{(r/a)}\) stands for rel/abs. By the same reason as in (4.9), we have

\[ \text{Spec} \left( \tilde{B}^{\text{even}}_{(m)} \right) = \text{Spec} \left( \tilde{B}^{\text{even}}_{(r/a)} \right). \] (4.14)

By (4.10), (4.14) and Lemma 4.1 we can rewrite (4.7) as follows.

\[ \log \tilde{\rho}_{\text{an}}^{\text{even}}(g^M, \nabla) = \frac{1}{2} \sum_{q=0}^{m} (-1)^{q+1} q \cdot \left( \log \text{Det}_{2g} \tilde{B}_{\text{rel}}^{2,q} + \log \text{Det}_{2g} \tilde{B}_{\text{abs}}^{2,q} \right) - i \pi \left( \eta \left( \tilde{B}^{\text{even}}_{(r/a)} \right) - \frac{1}{2} \text{rk}(E) \cdot \eta \tilde{B}_{\text{even}, \text{trivial}}^{\text{even}}(0) \right). \] (4.15)

We next consider another complex, which is similar to (4.6). We put (cf. (2.17))

\[ \tilde{\Omega}^q \left( \tilde{P}_0/\tilde{P}_1 \right)(M, E \oplus E) := \Omega_{\tilde{P}_0}^q(M, E) \oplus \Omega_{\tilde{P}_1}^q(M, E), \]

and consider the following complex

\[ \ldots \tilde{\nabla} \to \tilde{\Omega}_{\tilde{P}_0/\tilde{P}_1}^{q-1}(M, E \oplus E) \to \tilde{\Omega}_{\tilde{P}_0/\tilde{P}_1}^q(M, E \oplus E) \to \tilde{\nabla} \to \ldots \] (4.16)

with the following operators

\[ \tilde{\Gamma}(\pm) = \begin{pmatrix} \Gamma & 0 \\ 0 & -\Gamma \end{pmatrix}, \quad \tilde{\nabla}_{(\tilde{P}_0/\tilde{P}_1)} = \begin{pmatrix} \nabla_{\tilde{P}_0} & 0 \\ 0 & \nabla_{\tilde{P}_1} \end{pmatrix}, \] (4.17)

\[ \tilde{B}_{(\tilde{P}_0/\tilde{P}_1)}^{\text{even}} := \tilde{\Gamma}(\pm) \tilde{\nabla}_{(\tilde{P}_0/\tilde{P}_1)} + \tilde{\nabla}_{(\tilde{P}_0/\tilde{P}_1)} \tilde{\Gamma}(\pm) = \begin{pmatrix} B_{\text{even}}^{\text{even}} & 0 \\ 0 & -B_{\text{even}}^{\text{even}} \end{pmatrix}, \quad \tilde{B}_{(\tilde{P}_0/\tilde{P}_1)}^{2,q} := \begin{pmatrix} B_{\text{even}}^{2,q} & 0 \\ 0 & B_{\text{even}}^{2,q} \end{pmatrix}. \]

When \(H^\bullet(M; E) = H^\bullet(M, Y; E) = 0\), we define the refined analytic torsion \(\tilde{\rho}_{\text{an}}(\mathcal{P}_-/\mathcal{P}_+)(g^M, \nabla)\) with respect to this complex and the boundary conditions \(\mathcal{P}_-, \mathcal{P}_+\) as follows (cf. (2.22)).
Corollary 4.2. Let $(M, g^M)$ be a compact oriented Riemannian manifold with boundary $Y$. We assume that $g^M$ is a product metric near $Y$ and $H^\bullet(M; E) = H^\bullet(M, Y; E) = 0$. Then,

$$
\log \tilde{\rho}_{an}(m)(g^M, \nabla) - \log \tilde{\rho}_{an,(\mathcal{P}_- / \mathcal{P}_+)}(g^M, \nabla) = -i\pi \left( \eta(\mathcal{B}_{\mathcal{P}_-}^{\text{even}}) - \eta(\mathcal{B}_{\mathcal{P}_+}^{\text{even}}) \right) + \frac{i\pi}{2} \text{rk}(E) \left( \eta_{\mathcal{B}_{\mathcal{P}_-}^{\text{even}}, \text{trivial}}(0) - \eta_{\mathcal{B}_{\mathcal{P}_+}^{\text{even}}, \text{trivial}}(0) \right).
$$

The purpose of this paper is to compare $\tilde{\rho}_{an}(m)(g^M, \nabla)$ with $\tilde{\rho}_{an,(\mathcal{P}_- / \mathcal{P}_+)}(g^M, \nabla)$. By Corollary 4.2, it’s enough to compare $\eta(\mathcal{B}_{\mathcal{P}_-}^{\text{even}})$ with $\eta(\mathcal{B}_{\mathcal{P}_-}^{\text{even}}) - \eta(\mathcal{B}_{\mathcal{P}_+}^{\text{even}})$. For this purpose we are going to use a deformation of odd signature operators and boundary conditions simultaneously. We here note that $\eta(\mathcal{B}_{\mathcal{P}_-}^{\text{even}}) - \eta(\mathcal{B}_{\mathcal{P}_+}^{\text{even}})$ and $\eta_{\mathcal{B}_{\mathcal{P}_-}^{\text{even}}, \text{trivial}}(0) - \eta_{\mathcal{B}_{\mathcal{P}_+}^{\text{even}}, \text{trivial}}(0)$ are integers by Theorem 3.11 and Lemma 4.1. We are next going to construct a one parameter family of operators connecting $\mathcal{B}_{(\mathcal{P}_-) / \mathcal{P}_+}^{\text{even}}$ and $\mathcal{B}_{(\mathcal{P}_-) / \mathcal{P}_+}^{\text{even}}$. We begin with the following de Rham complex

$$
\Omega^\bullet(M, E) \oplus \Omega^\bullet(M, E) \equiv \Omega^\bullet(M, E \oplus E).
$$

We define the de Rham operator $\nabla$ and chirality operator $\Gamma(\theta)$ (cf. 11.8, 11.9 in [5]) by

$$
\nabla = \begin{pmatrix} \nabla & 0 \\ 0 & \nabla \end{pmatrix}, \quad \Gamma(\theta) = \begin{pmatrix} \Gamma \sin \theta & \Gamma \cos \theta \\ \Gamma \cos \theta & -\Gamma \sin \theta \end{pmatrix} = \Gamma \circ \begin{pmatrix} \sin \theta & \cos \theta \\ \cos \theta & -\sin \theta \end{pmatrix}, \quad \theta \in [0, \frac{\pi}{2}).
$$

For $0 \leq \theta \leq \frac{\pi}{2}$, we define a one parameter family of odd signature operators $\mathcal{B}(\theta)$ by

$$
\mathcal{B}(\theta) : \Omega^\bullet(M, E \oplus E) \rightarrow \Omega^\bullet(M, E \oplus E),
$$

$$
\mathcal{B}(\theta) := \tilde{\mathcal{B}}(\theta) + \tilde{\Gamma}(\theta) = \begin{pmatrix} \sin \theta & \cos \theta \\ \cos \theta & -\sin \theta \end{pmatrix}.
$$

Then we have

$$
\mathcal{B}(0) = \begin{pmatrix} 0 & \mathcal{B} \\ \mathcal{B} & 0 \end{pmatrix}, \quad \mathcal{B}(\frac{\pi}{2}) = \begin{pmatrix} \mathcal{B} & 0 \\ 0 & -\mathcal{B} \end{pmatrix}, \quad \mathcal{B}(\theta)^2 = \begin{pmatrix} \mathcal{B}^2 & 0 \\ 0 & \mathcal{B}^2 \end{pmatrix}.
$$
On a collar neighborhood $N$ of the boundary, the odd signature operator $\tilde{B}(\theta)$ is expressed by

$$\tilde{B}(\theta) = \tilde{\gamma}(\theta)(\nabla_{\partial_{\epsilon}} + \tilde{A}), \quad (4.22)$$

where

$$\tilde{\gamma}(\theta) = \gamma \circ \begin{pmatrix} \sin \theta & \cos \theta \\ \cos \theta & -\sin \theta \end{pmatrix}, \quad \tilde{A} = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}. \quad (4.23)$$

Simple computation shows that

$$\tilde{\gamma}(\theta)^2 = -\text{Id}, \quad \tilde{\gamma}(\theta) \tilde{A} = -\tilde{A} \tilde{\gamma}(\theta). \quad (4.24)$$

We denote the $(-i)$-eigenspaces of $\tilde{\gamma}(\theta)$ in $(\Omega^{even}(M, E \oplus E)|_{Y})$ by $(\Omega^{even}(M, E \oplus E)|_{Y})_{\theta, \pm i}$, i.e.

$$(\Omega^{even}(M, E \oplus E)|_{Y})_{\theta, \pm i} := \frac{I \mp i \tilde{\gamma}(\theta)}{2} (\Omega^{even}(M, E \oplus E)|_{Y})). \quad (4.25)$$

Then we have

$$\Omega^{even}(M, E \oplus E)|_{Y} = (\Omega^{even}(M, E \oplus E)|_{Y})_{\theta, +i} \oplus (\Omega^{even}(M, E \oplus E)|_{Y})_{\theta, -i}. \quad (4.26)$$

For each $\theta$, $(\Omega^{even}(M, E \oplus E)|_{Y})_{\theta}$ is a symplectic vector space and each Lagrangian subspace is expressed by the graph of a unitary operator from $(\Omega^{even}(M, E \oplus E)|_{Y})_{\theta, +i}$ to $(\Omega^{even}(M, E \oplus E)|_{Y})_{\theta, -i}$.

Using the decomposition (2.3), we define two maps $U_{p_+ \oplus p_+}$ and $U_{p_{rel} \oplus p_{abs}}$ as follows (cf. (2.3)).

$U_{p_- \oplus p_+}, \quad U_{p_{rel} \oplus p_{abs}} : \Omega^{even}(M, E \oplus E)|_{Y} \rightarrow \Omega^{even}(M, E \oplus E)|_{Y}$

$$U_{p_- \oplus p_+} = (B_{Y}^{-})^{-1}((B_{Y}^{-})^{-} - (B_{Y}^{+})^{+}) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad (4.27)$$

$$U_{p_{rel} \oplus p_{abs}} = i \begin{pmatrix} -i \beta \Gamma^{Y} \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

where $B_{Y}^{-} := \nabla^{Y} \Gamma^{Y} \nabla^{Y} : \Omega^{*}_{\epsilon}(Y, E|_{Y}) \rightarrow \Omega^{*}_{\epsilon}(Y, E|_{Y})$ and $B_{Y}^{+} := \Gamma^{Y} \nabla^{Y} \Gamma^{Y} : \Omega^{*}_{\epsilon}(Y, E|_{Y}) \rightarrow \Omega^{*}_{\epsilon}(Y, E|_{Y})$. The following lemma is straightforward.

**Lemma 4.3.** (1) $U_{p_- \oplus p_+}$ and $U_{p_{rel} \oplus p_{abs}}$ are unitary operators with $(U_{p_- \oplus p_+})^{*} = U_{p_- \oplus p_+}$ and $(U_{p_{rel} \oplus p_{abs}})^{*} = -U_{p_{rel} \oplus p_{abs}}$.

(2) $(U_{p_{rel} \oplus p_{abs}})\tilde{\gamma}(0) = -\tilde{\gamma}(0)(U_{p_{rel} \oplus p_{abs}})$ and $(U_{p_- \oplus p_+})\tilde{\gamma}(\tilde{\sigma}) = -\tilde{\gamma}(\tilde{\sigma})(U_{p_- \oplus p_+}).$

(3) Im $P_{rel} \oplus$ Im $P_{abs}$ is the graph of $U_{p_{rel} \oplus p_{abs}} : (\Omega^{even}(M, E \oplus E)|_{Y})_{0, +i} \rightarrow (\Omega^{even}(M, E \oplus E)|_{Y})_{0, -i}$ and Im $P_{-} \oplus$ Im $P_{+}$ is the graph of $U_{p_- \oplus p_+} : (\Omega^{even}(M, E \oplus E)|_{Y})_{\tilde{\sigma}, +i} \rightarrow (\Omega^{even}(M, E \oplus E)|_{Y})_{\tilde{\sigma}, -i}$. 
We next define $P(\theta) : \Omega_{\text{even}}(M, E \oplus E)|_{\mathcal{Y}} \to \Omega_{\text{even}}(M, E \oplus E)|_{\mathcal{Y}}$, $(0 \leq \theta \leq \frac{\pi}{2})$ by

$$P(\theta) = (B_2^\theta)^{-1}((B_2^\theta)^{-} - (B_2^\theta)^{+}) \begin{pmatrix} \sin \theta \text{Id} & \cos \theta \text{Id} \\ \cos \theta \text{Id} & -\sin \theta \text{Id} \end{pmatrix} \sin \theta + U_{P_{\text{rel}} \oplus P_{\text{abs}}} \cos \theta,$$

where

$$\mathfrak{A}(\theta) := (B_2^\theta)^{-1}((B_2^\theta)^{-} - (B_2^\theta)^{+}) \begin{pmatrix} \sin \theta \text{Id} & \cos \theta \text{Id} \\ \cos \theta \text{Id} & -\sin \theta \text{Id} \end{pmatrix}, \quad \mathfrak{B} := U_{P_{\text{rel}} \oplus P_{\text{abs}}}. \quad (4.28)$$

Then $P(\theta)$ is a smooth path connecting $U_{P_{\text{rel}} \oplus P_{\text{abs}}}$ and $U_{P_{-} \oplus P_{+}}$. The following lemma is straightforward.

**Lemma 4.4.** (1) $\mathfrak{A}(\theta)$ and $\mathfrak{B}$ are unitary operators satisfying $\mathfrak{A}(\theta)^2 = \text{Id}$, $\mathfrak{B}^2 = -\text{Id}$, $\mathfrak{A}(\theta)^* = \mathfrak{A}(\theta)$, and $\mathfrak{B}^* = -\mathfrak{B}$.

(2) $\mathfrak{A}(\theta) \mathfrak{B}(\theta) = -\mathfrak{B}(\theta) \mathfrak{A}(\theta)$, $\mathfrak{B} \mathfrak{B}(\theta) = -\mathfrak{B}(\theta) \mathfrak{B}$ and $\mathfrak{A}(\theta)^* \mathfrak{B}(\theta) = \mathfrak{B}(\theta)^* \mathfrak{A}(\theta)$.

(3) $\mathfrak{A}(\theta)^2 \mathfrak{A}(\theta) = \mathfrak{A}(\theta) \mathfrak{A}(\theta') = \begin{pmatrix} 0 & \text{Id} \\ -\text{Id} & 0 \end{pmatrix}$.

(4) $\mathfrak{A}(\theta) \mathfrak{B} = \mathfrak{B} \mathfrak{A}(\theta)$ and $\mathfrak{A}(\theta)^* \mathfrak{B} = \mathfrak{B}^* \mathfrak{A}(\theta)$.

(5) $P(\theta)$ is a unitary operator with $P(\theta)^* = \mathfrak{A}(\theta) \sin \theta - \mathfrak{B} \cos \theta$ and $P(\theta) \mathfrak{B}(\theta) = -\mathfrak{B}(\theta) P(\theta)$.

(6) $\mathfrak{A}(\theta) \tilde{\mathfrak{A}} = -\tilde{\mathfrak{A}} \mathfrak{A}(\theta)$ and $\mathfrak{B} \tilde{\mathfrak{A}} = \tilde{\mathfrak{A}} \mathfrak{B}$ and hence $P(\theta)^* \tilde{\mathfrak{A}} = -\tilde{\mathfrak{A}} P(\theta)$.

We note that the orthogonal projections $P_{\text{rel}} \oplus P_{\text{abs}}$ and $P_{-} \oplus P_{+}$ are described as follows.

$$P_{\text{rel}} \oplus P_{\text{abs}}, \quad P_{-} \oplus P_{+} : \oplus_{k=0}^{1} \Omega_{\text{even}}(M, E \oplus E)|_{\mathcal{Y}}_{\theta,(-1)^k} \to \oplus_{k=0}^{1} \Omega_{\text{even}}(M, E \oplus E)|_{\mathcal{Y}}_{\theta,(-1)^k}$$

$$P_{\text{rel}} \oplus P_{\text{abs}} = \frac{1}{2} \begin{bmatrix} \text{Id} & \mathfrak{B}^* \\ \mathfrak{B} & \text{Id} \end{bmatrix}, \quad P_{-} \oplus P_{+} = \frac{1}{2} \begin{bmatrix} \text{Id} & \mathfrak{A}(\mathfrak{B})^* \\ \mathfrak{A}(\mathfrak{B}) & \text{Id} \end{bmatrix}. \quad (4.30)$$

We define a smooth path $\tilde{P}(\theta)$ of orthogonal projections connecting $P_{\text{rel}} \oplus P_{\text{abs}}$ and $P_{-} \oplus P_{+}$ by

$$\tilde{P}(\theta) = \frac{1}{2} \begin{bmatrix} \text{Id} & P(\theta)^* \\ P(\theta) & \text{Id} \end{bmatrix}, \quad (0 \leq \theta \leq \frac{\pi}{2}). \quad (4.31)$$

Under the decomposition (4.26) we have

$$\tilde{\gamma}(\theta) = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad \tilde{B}_Y^2 := \tilde{B}_Y^2 = \begin{bmatrix} \text{Id} & 0 \\ 0 & \text{Id} \end{bmatrix}, \quad \tilde{A} = \begin{bmatrix} 0 & \tilde{A}_- \\ \tilde{A}_+ & 0 \end{bmatrix}. \quad (4.32)$$

where $\tilde{A}_\pm = \tilde{A}|_{\Omega_{\text{even}}(M, E \oplus E)|_{\mathcal{Y}}_{\theta, \pm 1}}$. Then $\tilde{P}(\theta)$ satisfies the following properties, whose proofs are straightforward.

**Lemma 4.5.** (1) $\tilde{\gamma}(\theta) \tilde{P}(\theta) = (I - \tilde{P}(\theta)) \tilde{\gamma}(\theta)$, and $\tilde{P}(\theta) \tilde{B}_Y^2 = \tilde{B}_Y^2 \tilde{P}(\theta)$.

(2) $\tilde{P}(\theta) \tilde{A} \tilde{P}(\theta) = 0$, and $(I - \tilde{P}(\theta)) \tilde{A} (I - \tilde{P}(\theta)) = 0$. 
Remark: In this paper we are using two types of decompositions. One comes from the decomposition (2.3) and the other one comes from the decomposition (4.26). When we write a matrix form of an operator, we are going to use the notation ( ) for the decomposition (2.3) like (4.21) and use [ ] for the decomposition (4.26) like (4.30).

Let \( \tilde{B}(\theta) \tilde{P}(\theta) \) be the realization of \( \tilde{B}(\theta) \) with respect to \( \tilde{P}(\theta) \), i.e.

\[
\text{Dom} \left( \tilde{B}(\theta) \tilde{P}(\theta) \right) = \{ \phi \in H^1(\Omega^\text{even}(M, E \oplus E)) \mid \tilde{P}(\theta)(\phi|_Y) = 0 \}.
\]

Then \( \tilde{B}(\theta)^{\text{even}} \tilde{P}(\theta) \) is a smooth path of operators connecting \( \tilde{B}(\theta)^{\text{even}}(t/a) \) and \( \tilde{B}(\theta)^{\text{even}}(\tilde{P} - \tilde{P}^+ + \tilde{P} - \tilde{P}^+) \) (cf. (4.13), (4.17)).

Lemma 4.6. \( \tilde{B}(\theta) \tilde{P}(\theta) \) is essentially self-adjoint.

Proof. The Green formula for \( \tilde{B}(\theta) \) can be written as follows (cf. (3) in Lemma 2.1). For \( \tilde{\phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \tilde{\psi} = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \in \Omega^\text{even}(M, E \oplus E), \)

\[
(\tilde{B}(\theta)\tilde{\phi}, \tilde{\psi})_M = (\tilde{\phi}, \tilde{B}(\theta)\tilde{\psi})_M + (\tilde{\phi}|_Y, \tilde{\gamma}(\theta)(\tilde{\psi}|_Y))_Y.
\]

The remaining part is a verbatim repetition of the proof of Lemma 3.3 in [12].

We next define a unitary operator \( U(\theta) \) (0 \( \leq \theta \leq \frac{\pi}{2} \)) by

\[
U(\theta) : \bigoplus_{k=0}^1 (\Omega^\text{even}(M, E \oplus E)|_Y)_{\theta,(1/2)^k} \rightarrow \bigoplus_{k=0}^1 (\Omega^\text{even}(M, E \oplus E)|_Y)_{\theta,(1/2)^k}
\]

\[
U(\theta) = \begin{bmatrix} \cos \theta - \mathcal{A}^* \mathcal{A}(\theta) \sin \theta & 0 \\ 0 & \text{Id} \end{bmatrix}.
\]

(4.33)

Then \( U(\theta) \) satisfies the following equality.

\[
U(\theta) \tilde{P}(0) U(\theta)^* = \tilde{P}(\theta).
\]

(4.34)

Setting

\[
T(\theta) = -i \theta \begin{bmatrix} -\mathcal{B}^* \mathcal{A}(\theta) & 0 \\ 0 & 0 \end{bmatrix} = i \theta \mathcal{B}^* \mathcal{A}(\theta) \frac{I - i \tilde{\gamma}(\theta)}{2}.
\]

(4.35)

Lemma 4.4 shows that

\[
e^{iT(\theta)} = U(\theta).
\]

(4.36)

\( T(\theta) \) satisfies the following properties.

Lemma 4.7. \( T(\theta) \tilde{\gamma}(\theta) = \tilde{\gamma}(\theta) T(\theta) \) and \( T(\theta) \tilde{B}_T^2 = \tilde{B}_T^2 \ T(\theta) \).
Remark: Contrary to the case of [6], \( T(\theta) \) does not anticommute with \( \tilde{A} \).

Let \( \phi : [0, 1] \to [0, 1] \) be a decreasing smooth function such that \( \phi = 1 \) on a small neighborhood of 0 and \( \phi = 0 \) on a small neighborhood of 1. We use this cut-off function to extend \( T(\theta) \) defined on \( \Omega^*(M, E \oplus E) \mid Y \) to a unitary operator defined on \( \Omega^*(M, E \oplus E) \). We define \( \Psi_\theta : \Omega^*(M, E \oplus E) \to \Omega^*(M, E \oplus E) \) by

\[
\Psi_\theta(\omega)(x) = e^{i\phi(x)T(\theta)}\omega(x),
\]

where the support of \( \phi(x)T(\theta) \) is contained in \( N := [0, 1) \times Y \), a collar neighborhood of \( Y \).

**Lemma 4.8.** \( \Psi_\theta \) is a unitary operator mapping from \( \text{Dom} \left( \tilde{B}(\theta)\tilde{P}(0) \right) \) onto \( \text{Dom} \left( \tilde{B}(\theta)\tilde{P}(0) \right) \).

**Proof.** Clearly \( \Psi_\theta \) is a unitary operator. Let \( \tilde{P}(0)\omega(0) = 0 \). Then

\[
\tilde{P}(\theta)(\Psi_\theta\omega)(0) = U(\theta)\tilde{P}(0)U(\theta)^* e^{i\phi(x)T(\theta)}|_{x=0} = U(\theta)\tilde{P}(0)e^{-T(\theta)}e^{i\phi(x)T(\theta)}|_{x=0} = U(\theta)\tilde{P}(0)\omega(0) = 0,
\]

which completes the proof of the lemma.

Note that \( \text{Dom} \left( \tilde{B}(0)\tilde{P}(0) \right) \equiv \text{Dom} \left( \tilde{B}(\theta)\tilde{P}(0) \right) \subset \Omega^{\text{even}}(M, E \oplus E) \) and consider the following diagram.

\[
\begin{array}{ccc}
\text{Dom} \left( \tilde{B}(0)\tilde{P}(0) \right) & \xrightarrow{\tilde{B}(\theta)} & \Omega^{\text{even}}(M, E \oplus E) \\
\Psi_\theta \downarrow & & \Psi_\theta \\
\text{Dom} \left( \tilde{B}(\theta)\tilde{P}(0) \right) & \xrightarrow{\tilde{B}(\theta)} & \Omega^{\text{even}}(M, E \oplus E)
\end{array}
\]

Setting \( \tilde{B}(\theta) := \Psi_\theta^* \tilde{B}(\theta) \Psi_\theta |_{\text{Dom}(\tilde{B}(0)\tilde{P}(0))} \),

\( \tilde{B}(\theta) : \text{Dom} \left( \tilde{B}(0)\tilde{P}(0) \right) \to \Omega^{\text{even}}(M, E \oplus E) \)

is an elliptic \( \Psi \)DO of order 1 with a fixed domain \( \text{Dom} \left( \tilde{B}(0)\tilde{P}(0) \right) \) and has the same spectrum as \( \tilde{B}(\theta)\tilde{P}(0) \), which is a smooth path of operators connecting \( \tilde{B}^{\text{even}}_{(r/\theta)} \) at \( \theta = 0 \) and \( \tilde{B}^{\text{even}}_{(\tilde{P}_-\tilde{P}_+)} \) at \( \theta = \frac{\pi}{2} \).

5. **Comparison of the eta invariants**

In this section we discuss the variation of eta functions for \( \tilde{B}(\theta) \) to compare \( \eta \left( \tilde{B}^{\text{even}}_{(r/\theta)} \right) \) with \( \eta \left( \tilde{B}^{\text{even}}_{(\tilde{P}_-\tilde{P}_+)} \right) \). For this purpose we are going to use the deformation method in [6]. In [12] we used similar method to compare the eta invariants subject to the boundary conditions \( \tilde{P}_- \) and \( \tilde{P}_+ \) with the eta invariants subject to the APS boundary condition. We now begin with the one parameter family of the eta functions \( \eta_{\tilde{B}(\theta)}(s) \) defined by

\[
\eta_{\tilde{B}(\theta)}(s) = \frac{1}{\Gamma\left(\frac{s+1}{2}\right)} \int_0^\infty t^{-\frac{s+1}{2}} \text{Tr} \left( \tilde{B}(\theta)e^{-t(\tilde{B}(\theta))^2} \right) dt.
\]

(5.1)
If \( \eta_{\hat{B}}(s) \) has a regular value at \( s = 0 \), we define the eta invariant \( \eta(\tilde{B}(\theta)) \) by

\[
\eta(\tilde{B}(\theta)) = \frac{1}{2} \left( \eta_{\hat{B}}(0) + \dim \ker \tilde{B}(\theta) \right).
\]  

(5.2)

For a fixed \( \theta_0 \) in \( [0, \frac{\pi}{2}] \), there exist \( c(\theta_0) > 0 \) and \( \delta > 0 \) such that \( c(\theta_0) \notin \text{Spec} \left( \tilde{B}(\theta) \right) \) for \( \theta_0 - \delta < \theta < \theta_0 + \delta \). We denote by \( Q(\theta) \) the orthogonal projection onto the space spanned by eigensections of \( \tilde{B}(\theta) \) whose eigenvalues belong to \( (-c(\theta_0), c(\theta_0)) \) for \( \theta_0 - \delta < \theta < \theta_0 + \delta \). We define

\[
\eta_{\hat{B}}(s ; c(\theta_0)) = \sum_{|\lambda| > c(\theta_0)} \text{sign}(\lambda) |\lambda|^{-s} = \frac{1}{\Gamma\left(\frac{s+1}{2}\right)} \int_0^{\infty} t^{\frac{s-1}{2}} \text{Tr} \left\{ (I - Q(\theta)) \tilde{B}(\theta) e^{-t \tilde{B}(\theta)^2} \right\} dt.
\]

Then \( \eta_{\hat{B}}(s) - \eta_{\hat{B}}(s ; c(\theta_0)) \) is an entire function and

\[
\left\{ \frac{1}{2} \left( \eta_{\hat{B}}(s) + \dim \ker \tilde{B}(\theta) \right) - \frac{1}{2} \eta_{\hat{B}}(s ; c(\theta_0)) \right\}_{s=0}
\]

(5.3)

does not depend on the \( \theta \) for \( \theta_0 - \delta < \theta < \theta_0 + \delta \) up to \( \text{mod Z} \). Simple computation shows that

\[
\frac{d}{d\theta} \eta_{\hat{B}}(s ; c(\theta_0))
\]

(5.4)

\[
= \frac{1}{\Gamma\left(\frac{s+1}{2}\right)} \int_0^{\infty} t^{\frac{s-1}{2}} \text{Tr} \left( -\dot{Q}(\theta) \tilde{B}(\theta) e^{-t \tilde{B}(\theta)^2} + (I - Q(\theta)) \frac{d}{d\theta} \left( \tilde{B}(\theta) e^{-t \tilde{B}(\theta)^2} \right) \right) dt
\]

\[
= \frac{1}{\Gamma\left(\frac{s+1}{2}\right)} \int_0^{\infty} t^{\frac{s-1}{2}} \text{Tr} \left( -\dot{Q}(\theta) \tilde{B}(\theta) e^{-t \tilde{B}(\theta)^2} \right) dt - \frac{s}{\Gamma\left(\frac{s+1}{2}\right)} \int_0^{\infty} t^{\frac{s-1}{2}} \text{Tr} \left\{ (I - Q(\theta)) \left( \tilde{B}(\theta) e^{-t \tilde{B}(\theta)^2} \right) \right\} dt,
\]

where \( \dot{Q}(\theta) \) and \( \tilde{B}(\theta) \) mean the derivative of \( Q(\theta) \) and \( \tilde{B}(\theta) \) with respect to \( \theta \). Furthermore, we have (cf. Section 4.2 in [11])

\[
\text{Tr} \left( -\dot{Q}(\theta) \tilde{B}(\theta) e^{-t \tilde{B}(\theta)^2} \right) = 0, \quad \left\{ \frac{s}{\Gamma\left(\frac{s+1}{2}\right)} \int_0^{\infty} t^{\frac{s-1}{2}} \text{Tr} \left( Q(\theta) \tilde{B}(\theta) e^{-t \tilde{B}(\theta)^2} \right) dt \right\}_{s=0} = 0.
\]

These equalities imply that

\[
\frac{d}{d\theta} \eta_{\hat{B}}(s ; c(\theta_0)) = -\frac{s}{\Gamma\left(\frac{s+1}{2}\right)} \int_0^{\infty} t^{\frac{s-1}{2}} \text{Tr} \left( \tilde{B}(\theta) e^{-t \tilde{B}(\theta)^2} \right) dt + F(s)
\]

(5.5)

where \( F(s) \) is an analytic function at least for \( \text{Re } s > -1 \) with \( F(0) = 0 \). The equality (5.5) leads to the following lemma.

\textbf{Lemma 5.1.} (1) The derivative of the residue of \( \eta_{\hat{B}}(s) \) at \( s = 0 \) is given by

\[
\frac{d}{d\theta} \text{Res}_{s=0} \eta_{\hat{B}}(s) = \text{Res}_{s=0} \left( \frac{d}{d\theta} \eta_{\hat{B}}(s) \right) = \frac{4}{\sqrt{\pi}} a_{-\frac{1}{4},0} (\tilde{B}(\theta), \hat{B}(\theta)).
\]

(2) If \( \eta_{\hat{B}}(s) \) is regular at \( s = 0 \) for each \( \theta \), the derivative of \( \eta_{\hat{B}}(0) \), up to \( \text{mod Z} \), is given by

\[
\left( \frac{d}{d\theta} \eta_{\hat{B}}(0) \right)(0) = -\frac{2}{\sqrt{\pi}} a_{-\frac{1}{4},0} (\tilde{B}(\theta), \hat{B}(\theta)) \quad (\text{mod Z}).
\]
Here \( a_{\frac{1}{2},1} (\hat{B}(\theta), \hat{B}(\theta)) \) and \( a_{\frac{1}{2},0} (\hat{B}(\theta), \hat{B}(\theta)) \) are the coefficients of \( t^{-\frac{1}{2}} \log t \) and \( t^{-\frac{1}{2}} \) in the asymptotic expansion of \( \text{Tr} \left( \hat{B}(\theta) e^{-t\hat{B}(\theta)^2} \right) \) for \( t \to 0^+ \), respectively.

**Theorem 5.2.** We recall that \( \hat{B}(\theta) = \Psi_\theta^* \hat{B}(\theta) \Psi_\theta : \text{Dom}(\hat{B}(0)) \to \Omega^{\text{even}}(M, E \oplus E) \). Then :

\[
\text{Tr} \left( \hat{B}(\theta) e^{-t\hat{B}(\theta)^2} \right) \sim 0 \quad (\text{up to } e^{-\hat{t}}), \quad \text{for } t \to 0^+.
\]

Lemma 5.1 and Theorem 5.2 imply that for each \( \theta \), \( \eta_{\hat{B}(\theta)}(s) \) has a regular value at \( s = 0 \). Moreover, \( \eta_{\hat{B}(\theta)}(0) \) and \( \eta(\hat{B}(\theta)) \) do not depend on \( \theta \) up to \( \text{mod } Z \), which yields the following result.

**Corollary 5.3.**

\[
\eta \left( \hat{E}_{\text{even}}^{\text{even}}(t \cdot a) \right) = \eta \left( \hat{B}(0) \right) = \eta \left( \hat{B}(\frac{\pi}{2}) \right) = \eta \left( \hat{E}_{\text{even}}^{\text{even}}(\hat{p}_-/\hat{p}_+) \right) = \eta \left( \hat{B}_{\hat{p}_-} \right) - \eta \left( \hat{B}_{\hat{p}_+} \right) \quad (\text{mod } Z).
\]

Corollary 4.2, Corollary 5.3, Theorem 3.11, and Theorem 3.12 lead to the following result, which is the main result of this paper.

**Theorem 5.4.** Let \( (M, g^M) \) be an odd dimensional compact Riemannian manifold with boundary \( Y \) and \( g^M \) be a product metric near \( Y \). We assume that the connection \( \nabla \) is a Hermitian connection and for each \( 0 \leq q \leq m, H^q(M; E) = H^q(M, Y; E) = \{0\} \). Then :

1. \( \hat{p}_{\text{an},(m)}(g^M, \nabla) = \pm \hat{p}_{\text{an},(\hat{p}_- \cdot \hat{p}_+)}(g^M, \nabla) = \pm \hat{p}_{\text{an},\hat{p}_-}(g^M, \nabla) \cdot \hat{p}_{\text{an},\hat{p}_+}(g^M, \nabla) \in \mathbb{R} \).
2. \( \eta \left( \hat{E}_{\text{even}}^{\text{even}}(t \cdot a) \right) = \eta \left( \hat{E}_{\text{even}}^{\text{even}}(g^M, \nabla) \right) \equiv 0 \quad (\text{mod } Z) \).

6. **Proof of Theorem 5.2**

Recall that

\[
\hat{B}(\theta) = \Psi_\theta^* \hat{B}(\theta) \Psi_\theta = e^{-i\phi(x) T(\theta)} \hat{B}(\theta) e^{i\phi(x) T(\theta)}.
\]

Since \( T'(\theta) \) does not commute with \( T(\theta) \), we should be careful in computing \( \hat{B}(\theta) \). We note that

\[
\hat{B}(\theta) = \left( \frac{d}{d\theta} - \frac{1}{2} i \phi(x) T(\theta) \right) \hat{B}(\theta) e^{i\phi(x) T(\theta)} + e^{-i\phi(x) T(\theta)} \left( \frac{d}{d\theta} \hat{B}(\theta) \right) e^{i\phi(x) T(\theta)}
\]

\[
+ e^{-i\phi(x) T(\theta)} \left( \frac{d}{d\theta} e^{i\phi(x) T(\theta)} \right),
\]

which leads to

\[
\text{Tr} \left( \hat{B}(\theta) e^{-t\hat{B}(\theta)^2} \right) = \text{Tr} \left( \left( \frac{d}{d\theta} e^{-i\phi(x) T(\theta)} \right) \hat{B}(\theta) e^{-t\hat{B}(\theta)^2} e^{i\phi(x) T(\theta)} \right)
\]

\[
+ \text{Tr} \left( \left( \frac{d}{d\theta} \hat{B}(\theta) \right) e^{-t\hat{B}(\theta)^2} e^{i\phi(x) T(\theta)} \right) + \text{Tr} \left( \hat{B}(\theta) \left( \frac{d}{d\theta} e^{i\phi(x) T(\theta)} \right) e^{-i\phi(x) T(\theta)} e^{-t\hat{B}(\theta)^2} \right).
\]
Simple computation leads to the following result.

\[
\frac{d}{d\theta} e^{-i\phi(x)T(\theta)} = -e^{-i\phi(x)T(\theta)} \int_1^x e^{i\phi(u)T(\theta)} (i\phi'(u)T'(\theta)) e^{-i\phi(u)T(\theta)} du = -e^{-i\phi(x)T(\theta)} Q(x)
\]

\[
\frac{d}{d\theta} e^{i\phi(x)T(\theta)} = \int_1^x e^{i\phi(u)T(\theta)} (i\phi'(u)T'(\theta)) e^{-i\phi(u)T(\theta)} du e^{i\phi(x)T(\theta)} = Q(x)e^{i\phi(x)T(\theta)},
\]  

where

\[
Q(x) := \int_1^x e^{i\phi(u)T(\theta)} (i\phi'(u)T'(\theta)) e^{-i\phi(u)T(\theta)} du.
\]

Since \( \phi(u) \) is supported in \([0, 1]\), the support of \( Q(x) \) belongs to \([0, 1] \times Y\). Equations (6.2) and (6.3) yield the following result.

**Lemma 6.1.**

\[
\text{Tr} \left( \tilde{B}(\theta)e^{-t\tilde{B}(\theta)^2} \right) = \text{Tr} \left( \left( \frac{d}{d\theta} \tilde{B}(\theta) \right) e^{-t\tilde{B}^2(\theta)} \right) + \text{Tr} \left\{ \left( \tilde{B}(\theta)Q(x) - Q(x)\tilde{B}(\theta) \right) e^{-t\tilde{B}(\theta)^2(\theta)} \right\}.
\]

Let \( B^cyl \) be the odd signature operator defined as in (2.7) on \([0, \infty) \times Y\) and

\[
\tilde{B}(\theta)^{cyl} := B^cyl \cdot \begin{pmatrix} \sin \theta \cdot \text{Id} & \cos \theta \cdot \text{Id} \\ \cos \theta \cdot \text{Id} & -\sin \theta \cdot \text{Id} \end{pmatrix}.
\]

The heat kernel of \( \left( \tilde{B}(\theta)^{cyl}(\theta) \right)^2 \) was computed in [6] as follows.

\[
e^{-t(\tilde{B}(\theta)^{cyl}(\theta))^2}(x, y) = (4\pi t)^{-\frac{1}{2}} \left( e^{-\frac{(x-y)^2}{4t}} + (I - 2\tilde{P}(\theta)) e^{-\frac{(x+y)^2}{4t}} \right) e^{-t\tilde{A}^2} + (\pi t)^{-\frac{1}{2}} \left( I - \tilde{P}(\theta) \right) \int_0^\infty e^{-\frac{(x+y+z)^2}{4t}} \tilde{A}(\theta) e^{-t\tilde{A}^2} dz,
\]

where \( \tilde{A}(\theta) := (I - \tilde{P}(\theta))\tilde{A}(I - \tilde{P}(\theta)) \). Moreover, Lemma [4.5] shows that

\[
e^{-t(\tilde{B}(\theta)^{cyl}(\theta))^2}(x, y) = (4\pi t)^{-\frac{1}{2}} \left( e^{-\frac{(x-y)^2}{4t}} + (I - 2\tilde{P}(\theta)) e^{-\frac{(x+y)^2}{4t}} \right) e^{-t\tilde{A}^2}.
\]

**6.1. Asymptotic expansion of Tr** \( \left( \frac{d}{d\theta} \tilde{B}(\theta) \right) e^{-t\tilde{B}(\theta)^2(\theta)} \). Recall that \( N = [0, 1] \times Y \) is a collar neighborhood of \( Y \). We define cut off functions \( \phi_1, \phi_2, \psi_1 \) and \( \psi_2 \) as in (4.1). We put

\[
\mathcal{R}_{even}(t, (w, x), (w', y)) = \phi_1(x) \tilde{B}(\theta)^{cyl}_{\text{even}} e^{-t(\tilde{B}(\theta)^{cyl}(\theta))^2} (x, y) \psi_1(y) + \phi_2(x) \tilde{B}(\theta)_{\text{even}} e^{-t\tilde{B}(\theta)^2_{\text{even}}(t, (w, x), (w', y))} \psi_2(y),
\]

\[
\mathcal{R}_{even}(t, (w, x), (w', y)) \in \mathcal{R}_{\text{even}}(t, (w, x), (w', y))
\]
where \( \tilde{B}(\theta) := \frac{d}{d\theta} \tilde{\theta}(\theta) \) and \( \tilde{\theta}_{\text{doub}}(t, (w, x), (w', y)) \) is the heat kernel of \( e^{-t\tilde{B}(\theta)^2} \) on \( M_{\text{doub}} := M \cup Y M \), the closed double of \( M \). Then \( \tilde{R}_{\text{even}}(t, (w, x), (w', y))(\theta) \) is a parametrix for \( \tilde{B}(\theta)_{\text{even}} \tilde{\theta}_{\text{even}}(t, (w, x), (w', y))(\theta) \), the kernel of \( \tilde{B}(\theta)_{\text{even}} e^{-t\tilde{B}(\theta)^2}_{\text{even, } \tilde{\theta}(\theta)} \) on \( M \) and the standard computation shows that for \( 0 < t \leq 1 \),

\[
|\tilde{B}(\theta)_{\text{even}} \tilde{\theta}_{\text{even}}(t, (w, x), (w, x))(\theta) - \tilde{R}_{\text{even}}(t, (w, x), (w, x))(\theta)| \leq c_3 e^{-\frac{c_4 t}{d}}
\]

(6.7)

for some positive constants \( c_3 \) and \( c_4 \), which implies that

\[
\text{Tr} \left( \tilde{B}(\theta) e^{-t\tilde{B}(\theta)^2}_{\tilde{\theta}(\theta)} \right) \sim \text{Tr} \left( \tilde{R}_{\text{even}}(t, (w, x), (w, x))(\theta) \right) = \text{Tr} \left( \tilde{B}(\theta) e^{-t\tilde{B}(\theta)^2}_{\tilde{\theta}(\theta)} \psi_1(x) \right) + \text{Tr} \left( \tilde{B}(\theta) e^{-t\tilde{B}(\theta)^2}_{\tilde{\theta}(\theta)} \psi_2(x) \right).
\]

(6.8)

We note that

\[
e^{-t(\tilde{B}(\theta))^2} = \begin{pmatrix} e^{-tB^2} & 0 \\ 0 & e^{-tB^2} \end{pmatrix}, \quad \tilde{B}(\theta) = B \begin{pmatrix} \cos \theta \text{Id} & -\sin \theta \text{Id} \\ -\sin \theta \text{Id} & \cos \theta \text{Id} \end{pmatrix},
\]

which shows that

\[
\text{Tr} \left( \tilde{B}(\theta) e^{-t(\tilde{B}(\theta))^2} \psi_2(u) \right) = 0.
\]

(6.9)

Using (6.3), and the decomposition (6.8), we have

\[
\text{Tr} \left( \tilde{B}(\theta) e^{-t(\tilde{B}(\theta))^2}_{\tilde{\theta}(\theta)} \psi_1(x) \right) = \text{Tr} \left\{ \begin{pmatrix} \cos \theta & -\sin \theta \\ -\sin \theta & -\cos \theta \end{pmatrix} \gamma (\nabla\partial_x + A) \left\{ \frac{1}{\sqrt{4\pi t}} \left( e^{-\frac{(x-y)^2}{4t}} + \left( I - 2\tilde{P}(\theta) \right) e^{-\frac{(x+y)^2}{4t}} \right) e^{-t\tilde{A}^2} \right\} \psi_1(x) \right\}
\]

\[
= \left( \frac{1}{\sqrt{4\pi t}} \int_0^\infty x e^{-\frac{x^2}{4t}} \psi_1(x) dx \right) \text{Tr} \left\{ \begin{pmatrix} \cos \theta & -\sin \theta \\ -\sin \theta & -\cos \theta \end{pmatrix} \gamma \left( I - 2\tilde{P}(\theta) \right) e^{-t\tilde{A}^2} \right\} 
\]

\[
+ \left( \frac{1}{\sqrt{4\pi t}} \int_0^\infty \psi_1(x) dx \right) \text{Tr} \left\{ \begin{pmatrix} \cos \theta & -\sin \theta \\ -\sin \theta & -\cos \theta \end{pmatrix} \gamma A e^{-t\tilde{A}^2} \right\} 
\]

\[
+ \left( \frac{1}{\sqrt{4\pi t}} \int_0^\infty \psi_1(x) e^{-\frac{x^2}{4t}} dx \right) \text{Tr} \left\{ \begin{pmatrix} \cos \theta & -\sin \theta \\ -\sin \theta & -\cos \theta \end{pmatrix} \gamma A \left( I - 2\tilde{P}(\theta) \right) e^{-t\tilde{A}^2} \right\}
\]

\[
=: (I) + (II) + (III).
\]

(6.10)

Since \( \gamma A = -A\gamma \), we have

\[
(II) = 0.
\]

(6.11)

We note that
\[
\text{Tr} \left\{ \left[ \begin{array}{cc}
\cos \theta & -\sin \theta \\
-\sin \theta & -\cos \theta
\end{array} \right] \gamma A \left( I - 2 \tilde{P}(\theta) \right) e^{-t \tilde{A}^2} \right\} = \text{Tr} \left\{ \left[ \begin{array}{cc}
\cos \theta & -\sin \theta \\
-\sin \theta & -\cos \theta
\end{array} \right] \left[ \begin{array}{cc}
\sin \theta & \cos \theta \\
\cos \theta & -\sin \theta
\end{array} \right] \tilde{\gamma}(\theta) A \left( I - 2 \tilde{P}(\theta) \right) e^{-t \tilde{A}^2} \right\}. \tag{6.12}
\]

Since \( \left[ \begin{array}{cc}
\cos \theta & -\sin \theta \\
-\sin \theta & -\cos \theta
\end{array} \right] \) and \( \left[ \begin{array}{cc}
\sin \theta & \cos \theta \\
\cos \theta & -\sin \theta
\end{array} \right] \), \( \tilde{\gamma}(\theta) \) and \( A \), \( \tilde{\gamma}(\theta) \) and \( \left( I - 2 \tilde{P}(\theta) \right) \) anticommute with each other (Lemma 4.5), we have

\[
\text{Tr} \left\{ \tilde{\gamma}(\theta) \left[ \begin{array}{cc}
\sin \theta & \cos \theta \\
\cos \theta & -\sin \theta
\end{array} \right] \left\{ \begin{array}{cc}
\cos \theta & -\sin \theta \\
-\sin \theta & -\cos \theta
\end{array} \right\} A \left( I - 2 \tilde{P}(\theta) \right) e^{-t \tilde{A}^2} \right\} = -\text{Tr} \left\{ \gamma \left[ \begin{array}{cc}
\cos \theta & -\sin \theta \\
-\sin \theta & -\cos \theta
\end{array} \right] A \left( I - 2 \tilde{P}(\theta) \right) e^{-t \tilde{A}^2} \right\} = 0,
\tag{6.13}
\]

which shows that

\[
(\text{III}) = 0. \tag{6.14}
\]

For (I) we note that

\[
I - 2 \tilde{P}(\theta) = \left[ \begin{array}{cc}
0 & -P(\theta)^* \\
-P(\theta) & 0
\end{array} \right] = -P(\theta)^* \frac{I + i \tilde{\gamma}(\theta)}{2} - P(\theta) \frac{I - i \tilde{\gamma}(\theta)}{2} = -A(\theta) \sin \theta + i B \tilde{\gamma}(\theta) \cos \theta,
\]

which leads to

\[
\text{Tr} \left\{ \left[ \begin{array}{cc}
\cos \theta & -\sin \theta \\
-\sin \theta & -\cos \theta
\end{array} \right] \gamma \left( I - 2 \tilde{P}(\theta) \right) e^{-t \tilde{A}^2} \right\} = \text{Tr} \left\{ \left[ \begin{array}{cc}
\cos \theta & -\sin \theta \\
-\sin \theta & -\cos \theta
\end{array} \right] \gamma \left( -A(\theta) \sin \theta + i B \tilde{\gamma}(\theta) \cos \theta \right) e^{-t \tilde{A}^2} \right\} = -\sin \theta \text{Tr} \left\{ \left[ \begin{array}{cc}
\cos \theta & -\sin \theta \\
-\sin \theta & -\cos \theta
\end{array} \right] A(\theta) e^{-t \tilde{A}^2} \right\} + i \cos \theta \text{Tr} \left\{ \left[ \begin{array}{cc}
\cos \theta & -\sin \theta \\
-\sin \theta & -\cos \theta
\end{array} \right] \gamma B \tilde{\gamma}(\theta) e^{-t \tilde{A}^2} \right\}.
\]

Let \( K = \left( \begin{array}{cc}
-1 & 0 \\
0 & 1
\end{array} \right) \). Simple computation shows that
The following lemma is straightforward by using (4.35), Lemma 4.4, Lemma 4.7 and (6.4).

\[ \text{Tr} \left\{ \begin{array}{cc} \cos \theta & -\sin \theta \\ -\sin \theta & -\cos \theta \end{array} \right\} \gamma A(\theta) e^{-tA^2} = \text{Tr} \left\{ \gamma (B_Y^2)^{-1} (B_Y^2)^+ e^{-tB_Y^2} \begin{pmatrix} 0 & \text{Id} \\ -\text{Id} & 0 \end{pmatrix} \right\} = 0, \]  
\[ \text{Tr} \left\{ \begin{array}{cc} \cos \theta & -\sin \theta \\ -\sin \theta & -\cos \theta \end{array} \right\} \gamma B \tilde{\gamma}(\theta) e^{-tA^2} = \text{Tr} \left( \begin{pmatrix} 0 & \text{Id} \\ -\text{Id} & 0 \end{pmatrix} \gamma B e^{-tA^2} \right) \]  
\[ = 2 \text{Tr} \left( \beta \Gamma^Y \left( \begin{array}{cc} -K & 0 \\ 0 & -K \end{array} \right) e^{-tB_Y^2} \right) = -2 \text{Tr} \left( \beta \Gamma^Y \left( \begin{array}{cc} -1 & 0 \\ 0 & 1 \end{array} \right) e^{-tB_Y^2} \right) = 0. \]  
\[ (6.15) \]

In the last equality we used the fact that \( \text{Tr} \left( *Y e^{-tB_Y^2} \right) = 0 \) since \( H^*(Y, E|_Y) = 0 \). Hence (I) = 0. Equations from (6.9) to (6.16) show that

\[ \text{Tr} \left( \left( \frac{d}{d\theta} \tilde{B}(\theta) \right) e^{-t\tilde{B}(\theta)^2 P_{\tilde{\gamma}}(\theta)} \right) = 0. \]  
\[ (6.17) \]

6.2. Asymptotic expansion of \( \text{Tr} \left\{ \left( \tilde{B}(\theta) Q(x) - Q(x) \tilde{B}(\theta) \right) e^{-t\tilde{B}(\theta)^2 P_{\tilde{\gamma}}(\theta)} \right\} \). Since \( Q(x) \) is supported in \([0,1] \times Y\), the standard theory for heat kernel ([1], [3]) implies that the asymptotic expansions of

\[ \text{Tr} \left( \left( \tilde{B}(\theta) Q(x) - Q(x) \tilde{B}(\theta) \right) e^{-t\tilde{B}(\theta)^2 P_{\tilde{\gamma}}(\theta)} \right) \text{ and } \text{Tr} \left( \left( \tilde{B}(\theta)^{\text{cyl}} Q(x) - Q(x) \tilde{B}(\theta)^{\text{cyl}} \right) e^{-t(\tilde{B}(\theta)^{\text{cyl}})^2} \right) \]

are equal up to \( e^{-c} \) for some \( c > 0 \). With a little abuse of notation we write \( \tilde{B}(\theta)^{\text{cyl}} \) by \( \tilde{B}(\theta) \) again. Simple computation using (4.22) shows that

\[ \text{Tr} \left\{ \left( \tilde{B}(\theta) Q(x) - Q(x) \tilde{B}(\theta) \right) e^{-t\tilde{B}(\theta)^2 P_{\tilde{\gamma}}(\theta)} \right\} = \text{Tr} \left\{ \tilde{\gamma}(\theta) e^{i\phi(x) T(\theta)} (i\phi'(x) T'(\theta)) e^{-i\phi(x) T(\theta)} e^{-t\tilde{B}(\theta)^2 P_{\tilde{\gamma}}(\theta)} \right\} \]
\[ + \text{Tr} \left\{ \tilde{\gamma}(\theta), Q(x) \right\} \nabla_{\theta_x} e^{-t\tilde{B}(\theta)^2 P_{\tilde{\gamma}}(\theta)} \right) + \text{Tr} \left\{ [\tilde{\gamma}(\theta), \tilde{A}], Q(x) \right\} e^{-t\tilde{B}(\theta)^2 P_{\tilde{\gamma}}(\theta)} \right) \}
\[ =: (I) + (II) + (III). \]  
\[ (6.18) \]

The following lemma is straightforward by using (4.35), Lemma 4.4, Lemma 4.7 and (6.4).

**Lemma 6.2.**

1. \( T'(\theta) = i \mathfrak{B}^* \mathfrak{A}(\theta) \left( \frac{I - \tilde{i\gamma}(\theta)}{2} \right) + \frac{i\theta}{2} \mathfrak{B}^* \mathfrak{A}(\theta). \)

2. \( Q(x) = \phi'(x) \mathfrak{B}^* \mathfrak{A}(\theta) \left( \frac{I - \tilde{i\gamma}(\theta)}{2} \right) \eta(x), \quad \tilde{Q}(x) := \int_1^u \phi'(u) e^{i\phi(u) T(\theta)} \mathfrak{B}^* \mathfrak{A}(\theta) e^{-i\phi(u) T(\theta)} du. \)

3. \( \tilde{\gamma}(\theta) \tilde{Q}(x) = -\tilde{Q}(x) \tilde{\gamma}(\theta), \quad \tilde{\gamma}(\theta) \left( \mathfrak{B}^* \mathfrak{A}(\theta) \frac{I - \tilde{i\gamma}(\theta)}{2} \right) = \left( \mathfrak{B}^* \mathfrak{A}(\theta) \frac{I - \tilde{i\gamma}(\theta)}{2} \right) \tilde{\gamma}(\theta). \)

Using (6.5) and Lemma 6.2 we have
Using the decomposition \((6.20)\), \((6.32)\) and Lemma 4.4 we have

\[
\tilde{\gamma}(\theta) \tilde{Q}(x) \tilde{P}(\theta) = \int_{-1}^{1} \phi'(u) e^{i\phi(u)T(\theta)} \tilde{\gamma}(\theta) \mathcal{B}^* \mathfrak{A}'(\theta) e^{-i\phi(u)T(\theta)} \tilde{P}(\theta) du,
\]

\[
e^{\pm i\phi(u)T(\theta)} = \begin{bmatrix} \cos(\phi(u)\theta) \mp \sin(\phi(u)\theta) & 0 \\ \mathcal{B}^* \mathfrak{A}(\theta) & 0 \end{bmatrix} \begin{bmatrix} 1 & \text{Id} \\ -\text{Id} & -\text{Id} \end{bmatrix}. \quad (6.20)
\]

Since \(\tilde{\gamma}(\theta) = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}\) and \(\mathcal{B}^* \mathfrak{A}'(\theta) = \begin{bmatrix} 0 & \mathcal{B}^* \mathfrak{A}'(\theta) \\ \mathfrak{A}'(\theta) & 0 \end{bmatrix}\) with respect to the decomposition \((6.20)\), simple computation shows that

\[
e^{i\phi(u)T(\theta)} \tilde{\gamma}(\theta) \mathcal{B}^* \mathfrak{A}'(\theta) e^{-i\phi(u)T(\theta)} \tilde{P}(\theta)
\]

\[
= \frac{i}{2} \left\{ \cos(\phi(u)\theta) \mathcal{B}^* \mathfrak{A}'(\theta) + \sin(\phi(u)\theta) \mathfrak{A}(\theta) \mathfrak{A}'(\theta) \right\} \begin{bmatrix} P(\theta) & \text{Id} \\ -\text{Id} & -P(\theta)^* \end{bmatrix}\n\]

\[
= \frac{i}{2} \left\{ \cos(\phi(u)\theta) \mathcal{B}^* \mathfrak{A}'(\theta) + \sin(\phi(u)\theta) \mathfrak{A}(\theta) \mathfrak{A}'(\theta) \right\} \begin{bmatrix} P(\theta) & I - i\tilde{\gamma}(\theta) \\ I + i\tilde{\gamma}(\theta) & P(\theta)^* \end{bmatrix}\n\]

\[
= \frac{i}{2} \left\{ \cos(\phi(u)\theta) \mathcal{B}^* \mathfrak{A}'(\theta) + \sin(\phi(u)\theta) \mathfrak{A}(\theta) \mathfrak{A}'(\theta) \right\} \left\{ \mathcal{B} \cos \theta - i\mathfrak{A}(\theta) \tilde{\gamma}(\theta) \sin \theta + i\tilde{\gamma}(\theta) \right\}. \quad (6.21)
\]

Since \(\tilde{\gamma}(\theta)\) anticommutes with \(\mathcal{B}^* \mathfrak{A}'(\theta)\) and \(\mathfrak{A}(\theta) \mathfrak{A}'(\theta)\) (cf. Lemma 4.4), we have

\[
\text{Tr} \left\{ \left( \cos(\phi(u)\theta) \mathcal{B}^* \mathfrak{A}'(\theta) + \sin(\phi(u)\theta) \mathfrak{A}(\theta) \mathfrak{A}'(\theta) \right) \left( i\tilde{\gamma}(\theta) e^{-t\Lambda^2} \right) \right\} = 0. \quad (6.22)
\]

Using Lemma 4.4 again, we have

\[
\text{Tr} \left\{ \left( \cos(\phi(u)\theta) \mathcal{B}^* \mathfrak{A}'(\theta) + \sin(\phi(u)\theta) \mathfrak{A}(\theta) \mathfrak{A}'(\theta) \right) \left( \mathcal{B} \cos \theta - i\mathfrak{A}(\theta) \tilde{\gamma}(\theta) \sin \theta e^{-t\Lambda^2} \right) \right\}
\]

\[
= \cos(\phi(u)\theta) \cos \theta \left\{ \mathfrak{A}'(\theta) e^{-t\Lambda^2} \right\} - \sin(\phi(u)\theta) \cos \theta \left\{ \mathcal{B} e^{-t\Lambda^2} \right\}
\]

\[
- i \cos(\phi(u)\theta) \sin \theta \left\{ \mathcal{B}^* \left( \begin{bmatrix} 0 & \text{Id} \\ -\text{Id} & 0 \end{bmatrix} \tilde{\gamma}(\theta) e^{-t\Lambda^2} \right) + i \sin(\phi(u)\theta) \sin \theta \left\{ \mathfrak{A}'(\theta) \tilde{\gamma}(\theta) e^{-t\Lambda^2} \right\}. \quad (6.23)
\]
Simple computation shows that for $K = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\text{Tr} \left( \mathfrak{A}'(\theta)e^{-t\mathfrak{A}^2} \right) = \text{Tr} \left( \mathfrak{B}^* \left( \begin{array}{cc} 0 & \text{Id} \\ -\text{Id} & 0 \end{array} \right)e^{-t\mathfrak{A}^2} \right) = \text{Tr} \left( \mathfrak{A}'(\theta)e^{-t\mathfrak{A}^2} \right) = 0,$$ \hspace{1cm} (6.24)

$$\text{Tr} \left( \begin{array}{cc} 0 & \text{Id} \\ -\text{Id} & 0 \end{array} \right)e^{-t\mathfrak{A}^2} = \text{Tr} \left( \beta Y e^{-t\mathfrak{A}^2} \left( \begin{array}{cc} -K & 0 \\ 0 & -K \end{array} \right) \right) = -2 \text{Tr} \left( \beta Y e^{-t\mathfrak{B}_2^0} K \right)$$

$$= 2 \text{Tr} \left( \Gamma Y e^{-t\mathfrak{B}_2^0} |_{\Omega_{\text{even}}(Y,E|Y)} + \Gamma Y e^{-t\mathfrak{B}_2^0} |_{\Omega_{\text{odd}}(Y,E|Y)} \right) = 0.$$ \hspace{1cm} (6.25)

Hence, equations from (6.19) to (6.24) show that

$$(\text{II}) = \text{Tr} \left\{ \tilde{\gamma}(\theta) , \mathcal{Q}(x) | \nabla_{\partial_x} e^{-t\mathfrak{B}(\theta)} \tilde{\gamma}(\theta) \right\} = 0.$$ \hspace{1cm} (6.26)

Using (6.25), we have

$$(\text{I}) = \text{Tr} \left\{ \tilde{\gamma}(\theta) e^{i\phi(x)T'(\theta)} (i\phi'(x)T'(\theta)) e^{-i\phi(x)T'(\theta)} e^{-t\mathfrak{B}(\theta)^2} \right\}$$

$$= \int_0^\infty \text{Tr} \left\{ \tilde{\gamma}(\theta) e^{i\phi(x)T'(\theta)} (i\phi'(x)T'(\theta)) e^{-i\phi(x)T'(\theta)} \left\{ \frac{1}{\sqrt{4\pi t}} \left( I + \left( I - 2\tilde{P}(\theta) \right) e^{-\frac{t^2}{4}} \right) e^{-t\mathfrak{A}^2} \right\} \right\} dx$$

$$= \frac{i}{\sqrt{4\pi t}} \left( \int_0^\infty \phi'(x) dx \right) \text{Tr} \left( \tilde{\gamma}(\theta) T'(\theta) e^{-t\mathfrak{A}^2} \right)$$

$$+ \frac{i}{\sqrt{4\pi t}} \left( \int_0^\infty \phi'(x) e^{-\frac{t^2}{4}} \text{Tr} \left( \tilde{\gamma}(\theta) e^{i\phi(x)T'(\theta)} T'(\theta) e^{-i\phi(x)T'(\theta)} \left( I - 2\tilde{P}(\theta) \right) e^{-t\mathfrak{A}^2} \right) dx \right)$$

$$=: (I_1) + (I_2).$$ \hspace{1cm} (6.27)

Since $\phi'(x) = 0$ near $x = 0$ and has a compact support, we have

$$(I_2) = O(e^{-\frac{t}{8}}).$$ \hspace{1cm} (6.28)

Using the assertion (1) in Lemma 6.2 and Lemma 4.4, we have

$$\text{Tr} \left( \tilde{\gamma}(\theta) T'(\theta) e^{-t\mathfrak{A}^2} \right) = i \text{Tr} \left( \tilde{\gamma}(\theta) \mathfrak{B}^* \mathfrak{A}(\theta) \frac{I - i\tilde{\gamma}(\theta)}{2} e^{-t\mathfrak{A}^2} \right) + \frac{i\theta}{2} \text{Tr} \left( \tilde{\gamma}(\theta) \mathfrak{B}^* \mathfrak{A}'(\theta) e^{-t\mathfrak{A}^2} \right)$$

$$= i \text{Tr} \left( \tilde{\gamma}(\theta) \mathfrak{B}^* \mathfrak{A}(\theta) \frac{I - i\tilde{\gamma}(\theta)}{2} e^{-t\mathfrak{A}^2} \right)$$

$$= \frac{i}{2} \text{Tr} \left( \tilde{\gamma}(\theta) \mathfrak{B}^* \mathfrak{A}(\theta) e^{-t\mathfrak{A}^2} \right) - \frac{1}{2} \text{Tr} \left( \mathfrak{B}^* \mathfrak{A}(\theta) e^{-t\mathfrak{A}^2} \right).$$ \hspace{1cm} (6.29)

For $K = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and the decomposition (2.3), we have

$$\mathfrak{B}^* \mathfrak{A}(\theta) = -\beta Y \left( \mathfrak{B}_Y^2 \right)^{-1} \left( \left( \mathfrak{B}_Y^2 \right)^{-} - \left( \mathfrak{B}_Y^2 \right)^{+} \right) \begin{pmatrix} \cos \theta & K \\ -\sin \theta & K \end{pmatrix},$$

$$\tilde{\gamma}(\theta) \mathfrak{B}^* \mathfrak{A}(\theta) = -i \left( \mathfrak{B}_Y^2 \right)^{-1} \left( \left( \mathfrak{B}_Y^2 \right)^{-} - \left( \mathfrak{B}_Y^2 \right)^{+} \right) \begin{pmatrix} 0 & K \\ -K & 0 \end{pmatrix}.$$ \hspace{1cm} (6.30)
which imply that

$$\text{Tr} \left( B^* A(\theta) e^{-t\hat{A}^2} \right) = \text{Tr} \left( \tilde{\gamma}(\theta) B^* A(\theta) e^{-t\hat{A}^2} \right) = 0$$

(6.31)

and hence

$$(I) = \text{Tr} \left\{ \tilde{\gamma}(\theta) e^{i\phi(x)T(\theta)} (i\phi'(x)T'(\theta)) e^{-i\phi(x)T(\theta)} e^{-t\hat{B}(\theta)^2} \right\} = O(e^{-\tau}) \text{ for some } c > 0.$$  \hspace{1cm} (6.32)

Finally, we are going to analyze (III) := \text{Tr} \left\{ \tilde{\gamma}(\theta) \bar{A}, Q(x) \right\} e^{-t\hat{B}(\theta)^2} \text{ in } (6.18). Using (6.3) and Lemma 6.2 we have

$$(III) = \text{Tr} \left\{ \left[ \tilde{\gamma}(\theta) \bar{A}, Q(x) \right] e^{-t\hat{B}(\theta)^2} \right\}
= \text{Tr} \left\{ \left[ \tilde{\gamma}(\theta) \bar{A}, Q(x) \right] \left( \frac{1}{\sqrt{4\pi t}} \left( I + (I - 2\bar{P}(\theta)) e^{-\frac{x^2}{4t}} \right) \right) e^{-t\hat{A}^2} \right\}
= \frac{1}{\sqrt{4\pi t}} \int_0^\infty \left( 1 + e^{-\frac{x^2}{4t}} \right) \text{Tr} \left\{ \left[ \tilde{\gamma}(\theta) \bar{A}, Q(x) \right] e^{-t\hat{A}^2} \right\} dx
- \frac{2}{\sqrt{4\pi t}} \int_0^\infty \left( e^{-\frac{x^2}{4t}} \right) \text{Tr} \left\{ \left[ \tilde{\gamma}(\theta) \bar{A}, Q(x) \right] \bar{P}(\theta) e^{-t\hat{A}^2} \right\} dx
= \frac{\theta}{\sqrt{4\pi t}} \int_0^\infty \left( e^{-\frac{x^2}{4t}} \right) \text{Tr} \left\{ \left[ \tilde{\gamma}(\theta) \bar{A}, \tilde{Q}(x) \right] \bar{P}(\theta) e^{-t\hat{A}^2} \right\} dx
+ \frac{2}{\sqrt{4\pi t}} \int_0^\infty \left( \phi(x) e^{-\frac{x^2}{4t}} \right) \text{Tr} \left\{ \left[ \tilde{\gamma}(\theta) \bar{A}, \left[ B^* A(\theta) \right] e^{-t\hat{A}^2} \right\} dx \right\}(6.33)
$$

where in the last expression we used the decomposition (4.26). Using Lemma 6.2 we have

$$\text{Tr} \left\{ \left[ \tilde{\gamma}(\theta) \bar{A}, \tilde{Q}(x) \right] \bar{P}(\theta) e^{-t\hat{A}^2} \right\} = \text{Tr} \left\{ \left( \tilde{\gamma}(\theta) \bar{A} \tilde{Q}(x) - \tilde{Q}(x) \tilde{\gamma}(\theta) \bar{A} \right) \bar{P}(\theta) e^{-t\hat{A}^2} \right\}
= \text{Tr} \left\{ \tilde{\gamma}(\theta) \left( \bar{A} \tilde{Q}(x) + \tilde{Q}(x) \bar{A} \right) \bar{P}(\theta) e^{-t\hat{A}^2} \right\}
= \text{Tr} \left\{ \tilde{\gamma}(\theta) \left( \bar{A} \tilde{Q}(x) + \tilde{Q}(x) \bar{A} \right) \left( I - \bar{P}(\theta) \right) e^{-t\hat{A}^2} \right\}
= \frac{1}{2} \text{Tr} \left\{ \tilde{\gamma}(\theta) \left( \bar{A} \tilde{Q}(x) + \tilde{Q}(x) \bar{A} \right) e^{-t\hat{A}^2} \right\} = 0,$$  \hspace{1cm} (6.34)

since $\bar{A} e^{-t\hat{A}^2} = e^{-t\hat{A}^2} \bar{A}$ and $\bar{A} \tilde{\gamma}(\theta) = -\tilde{\gamma}(\theta) \bar{A}$. Using Lemma 4.1 and 4.32, by simple computation we have
\[
\begin{align*}
\text{Tr} \left( \left[ \tilde{\gamma}(\theta) \tilde{A}, \left[ \begin{array}{cc}
\mathcal{B}^* & \mathcal{A}(\theta) \\
0 & 0
\end{array} \right] \tilde{P}(\theta) e^{-t\tilde{\Delta}^2} \right] \right) \\
= - \frac{i}{2} \text{Tr} \left\{ \left[ \begin{array}{cc}
\mathcal{B}^* & \mathcal{A}(\theta) \\
\tilde{A} & \mathcal{B}^* \mathcal{A}(\theta) \\
\mathcal{B}^* & \mathcal{A}(\theta) \\
\tilde{A} & \mathcal{B}^* \mathcal{A}(\theta) cos \theta
\end{array} \right] e^{-t\tilde{\Delta}^2} \right\} \\
= - \frac{i}{2} \text{Tr} \left\{ \left[ \begin{array}{cc}
\mathcal{B}^* & \mathcal{A}(\theta) \\
\tilde{A} & \mathcal{B}^* \mathcal{A}(\theta) \\
\mathcal{B}^* & \mathcal{A}(\theta) \\
\tilde{A} & \mathcal{B}^* \mathcal{A}(\theta) cos \theta
\end{array} \right] \frac{I - i\tilde{\gamma}(\theta)}{2} e^{-t\tilde{\Delta}^2} \right\} \\
= - \frac{i}{2} \text{Tr} \left\{ \left[ \begin{array}{cc}
\tilde{A} & \mathcal{B}^* \\
\mathcal{B}^* & \tilde{A}
\end{array} \right] e^{-t\tilde{\Delta}^2} \right\} - \frac{1}{2} \cos \theta \text{Tr} \left( \tilde{A} \mathcal{A}(\theta) \tilde{\gamma}(\theta) e^{-t\tilde{\Delta}^2} \right) \tag{6.35}
\end{align*}
\]

Setting \( K = \left( \begin{array}{cc}
-1 & 0 \\
0 & 1
\end{array} \right) \), \( L = \left( \begin{array}{cc}
0 & -1 \\
-1 & 0
\end{array} \right) \) and \( J = \left( \begin{array}{cc}
0 & 1 \\
-1 & 0
\end{array} \right) \), we have

\[
\begin{align*}
\tilde{A} \mathcal{B}^* &= \beta (\Gamma^Y \nabla^Y + \nabla^Y \Gamma^Y) \left( \begin{array}{cc}
0 & -J \\
J & 0
\end{array} \right) \\
\tilde{A} \mathcal{A}(\theta) \tilde{\gamma}(\theta) &= -i\beta (\Gamma^Y \nabla^Y + \nabla^Y \Gamma^Y) \left( \begin{array}{cc}
(B_\gamma^Y)^{-1} & (B_\gamma^Y)^- \\
(B_\gamma^Y) & -(B_\gamma^Y)^+
\end{array} \right) \left( \begin{array}{cc}
L & 0 \\
0 & L
\end{array} \right), \tag{6.36}
\end{align*}
\]

which shows that

\[
\text{Tr} \left( \tilde{A} \mathcal{B}^* e^{-t\tilde{\Delta}^2} \right) = \text{Tr} \left( \tilde{A} \mathcal{A}(\theta) \tilde{\gamma}(\theta) e^{-t\tilde{\Delta}^2} \right) = 0. \tag{6.37}
\]

By (6.36) and (6.37), we have

\[
\text{Tr} \left( \left[ \begin{array}{cc}
\tilde{\gamma}(\theta) \tilde{A}, & \mathcal{B}^* \mathcal{A}(\theta) \\
0 & 0
\end{array} \right] \tilde{P}(\theta) e^{-t\tilde{\Delta}^2} \right) = 0 \quad \text{and hence (III) = 0.} \tag{6.38}
\]

Adding up the above arguments, we have

\[
\text{Tr} \left\{ \left[ \begin{array}{cc}
\mathcal{B}(\theta) Q(x) - Q(x) \mathcal{B}(\theta) \\
0 & 0
\end{array} \right] e^{-t\tilde{\Delta}^2} \tilde{P}(\theta) e^{-t\tilde{\Delta}^2} \right\} \sim O(e^{-\frac{t}{2}}) \quad \text{for } t \to 0^+ \tag{6.39}
\]

and this completes the proof of Theorem 5.2.

References

[1] M. F. Atiyah, V. K. Patodi and I. M. Singer, Spectral asymmetry and Riemannian geometry, I, Math. Proc. Cambridge Philos. Soc. 77 (1975), 43-69.
[2] C. B"ar Zero sets of solutions to semilinear elliptic systems of first order Invent. Math. 138 (1999), 183-202.
[3] B. Booss-Bavnbek and K. Wojciechowski, Elliptic Boundary Value Problems for Dirac Operators, Birkh"auser, Boston, 1993.
[4] M. Braverman and T. Kappeler, Refined analytic torsion, J. Diff. Geom. 78 (2008), no. 2, 193-267.
[5] M. Braverman and T. Kappeler, Refined Analytic Torsion as an Element of the Determinant Line, Geom. Topol. 11 (2007), 139-213.
[6] J. Brüning and M. Lesch, *On the η-invariant of certain nonlocal boundary value problems*, Duke Math. J. 96 (1999), no. 2, 425-468.

[7] D. Burghelea, L. Friedlander and T. Kappeler, *Mayer-Vietoris type formula for determinants of elliptic differential operators*, J. of Funct. Anal. 107 (1992), 34-66.

[8] M. Farber and V. Turaev, *Absolute torsion*, Tel Aviv Topology Conference: Rothenberg Festschrift (1998), Contemp. Math., vol. 231, Amer. Math. Soc., Providence, RI, 1999, pp. 73–85.

[9] M. Farber and V. Turaev, *Poincaré-Reidemeister metric, Euler structures, and torsion*, J. Reine Angew. Math. 520 (2000), 195–225.

[10] P. B. Gilkey, *Invariance Theory, the Heat Equation, and the Atiyah-Singer Index Theorem*, 2nd Edition, CRC Press, Inc., 1994.

[11] R.-T. Huang and Y. Lee *The refined analytic torsion and a well-posed boundary condition for the odd signature operator*, arXiv:1004.1753.

[12] R.-T. Huang and Y. Lee *The gluing formula of the refined analytic torsion for an acyclic Hermitian connection*, arXiv:1103.3571, To appear in Manuscripta Mathematica.

[13] J. Kazdan, *Unique continuation in geometry* Comm. Pure Appl. Math. 41 (1988), 667-681.

[14] P. Kirk, M. Lesch, *The η-invariant, Maslov index and spectral flow for Dirac-type operators on manifolds with boundary*, Forum Math., 16 (2004), no. 4, 553-629.

[15] Y. Lee, *Burghelea-Friedlander-Kappeler’s gluing formula for the zeta determinant and its applications to the adiabatic decompositions of the zeta-determinant and the analytic torsion* Trans. Amer. Math. Soc. 355 no. 10 (2003), 4093-4110.

[16] Y. Lee *The zeta-determinants of Dirac Laplacians with boundary conditions on the smooth self-adjoint Grassmanian* J. Geom. Phys. 57 (2007), 1951-1976.

[17] W. Müller, *Analytic torsion and R-torsion of Riemannian manifolds*, Adv. in Math. 28 (1978), 233-305.

[18] L. Nicolaescu *The Maslov index, the spectral flow, and decomposition of manifolds*, Duke Math. J. 80 (1995), 485-533.

[19] D. B. Ray and I. M. Singer, *R-torsion and the Laplacian on Riemannian manifolds* Adv. in Math. 7 (1971), 145-210.

[20] V. G. Turaev, *Reidemeister torsion in knot theory*, Russian Math. Survey 41 (1986), 119-182.

[21] V. G. Turaev, *Euler structures, nonsingular vector fields, and Reidemeister-type torsions*, Math. USSR Izvestia 34 (1990), 627-682.

[22] B. Vertman, *Refined analytic torsion on manifolds with boundary*, Geom. Topol. 13 (2009), 1989-2027.

[23] B. Vertman, *Gluing formula for refined analytic torsion*, arXiv:0808.0451

**Department of Mathematics, National Central University, Chung-Li 320, Taiwan**

*E-mail address: rthuang@math.ncu.edu.tw*

**Department of Mathematics, Inha University, Incheon, 402-751, Korea**

*E-mail address: yoonweon@inha.ac.kr*