Quantized Fields à la Clifford and Unification

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It is shown that the generators of Clifford algebras behave as creation and annihilation operators for fermions and bosons. They can create extended objects, such as strings and branes, and can induce curved metric of our spacetime. At a fixed point, we consider the Clifford algebra \( Cl(8) \) of the 8-dimensional phase space, and show that one quarter of the basis elements of \( Cl(8) \) can represent all known particles of the first generation of the Standard model, whereas the other three quarters are invisible to us and can thus correspond to dark matter.

1 Introduction

Quantization of a classical theory is a procedure that appears somewhat enigmatic. It is not a derivation in a mathematical sense. It is a recipe of how to replace, e.g., the classical phase space variables, satisfying the Poisson bracket relations, with the operators satisfying the corresponding commutation relations \([1]\). What is a deeper meaning for replacement is usually not explained, only that it works. A quantized theory so obtained does work and successfully describes the experimental observations of quantum phenomena.

On the other hand, there exists a very useful tool for description of geometry of a space of arbitrary dimension and signature \([2]-[7]\). This is Clifford algebra. Its generators are the elements that satisfy the well-known relations, namely that the anticommutators of two generators are proportional to the components of a symmetric metric tensor. The space spanned by those generators is a vector space. It can correspond to a physical space, for instance to our usual three dimensional space, or to the four dimensional spacetime. The generators of a Clifford algebra are thus basis vectors of a physical space. We will interpret this as a space of all possible positions that the center of mass of a physical object can posses. A physical object has an extension that can be described by an effective oriented area, volume, etc. While the center of mass position is described by a vector, the oriented area is described by a bivector, the oriented volume by a trivector, etc. In general, an extended object is described \([8, 9, 10, 11, 12]\) by a superposition of scalars, vectors, bivectors, trivectors,

\(^1\)A chapter in the book Beyond Peaceful Coexistence; The Emergence of Space, Time and Quantum (Edited by: Ignazio Licata, Foreword: G. ’t Hooft, World Scientific, 2016)
etc., i.e., by an element of the Clifford algebra. The Clifford algebra associated with an extended object is a space, called \textit{Clifford space}\footnote{Here we did not go into the mathematical subtleties that become acute when the Clifford space is not flat but curved. Then, strictly speaking, the Clifford space is a \textit{manifold}, such that the tangent space in any of its points is a Clifford algebra. If Clifford space is \textit{flat}, then it is isomorphic to a Clifford algebra.}

Besides the Clifford algebras whose generators satisfy the anticommutation relations, there are also the algebras whose generators satisfy commutation relations, such that the commutators of two generators are equal to the components of a metric, which is now \textit{antisymmetric}. The Clifford algebras with a symmetric metric are called \textit{orthogonal Clifford algebras}, whereas the Clifford algebras with an antisymmetric metric are called \textit{symplectic Clifford algebras}\footnote{Here we did not go into the mathematical subtleties that become acute when the Clifford space is not flat but curved. Then, strictly speaking, the Clifford space is a \textit{manifold}, such that the tangent space in any of its points is a Clifford algebra. If Clifford space is \textit{flat}, then it is isomorphic to a Clifford algebra.}.

We will see that symplectic basis vectors are in fact quantum mechanical operators of bosons\cite{14,15}. The Poisson brackets of two classical phase space coordinates are equal to the commutators of two operators. This is so, because the Poisson bracket consists of the derivative and the symplectic metric which is equal to the commutator of two symplectic basis vectors. The derivative acting on phase space coordinates yields the Kronecker delta and thus eliminates them from the expression. What remains is the commutator of the basis vectors.

Similarly, the basis vectors of an orthogonal Clifford algebra are quantum mechanical operators for fermions. This becomes evident in the new basis, the so called Witt basis. By using the latter basis vectors and their products, one can construct spinors.

Orthogonal and symplectic Clifford algebras can be extended to infinite dimensional spaces\cite{14,15}. The generators of those infinite dimensional Clifford algebras are fermionic and bosonic field operators. In the case of fermions, a possible vacuum state can be the product of an infinite sequence of the operators\cite{14,15}. If we act on such a vacuum with an operator that does not belong to the set of operators forming that vacuum, we obtain a “hole” in the vacuum. This hole behaves as a particle. The concept of the Dirac sea, which is nowadays considered as obsolete, is revived within the field theories based on Clifford algebras. But in the latter theories we do not have only one vacuum, but many possible vacuums. This brings new possibilities for further development of quantum field theories and grand unification. Because the generators of Clifford algebras are basis vectors on the one hand, and field operators on the other hand, this opens a bridge towards quantum gravity. Namely, the expectation values of the “flat space” operators with respect to suitable quantum states composed of many fermions or bosons, can give “curved space” vectors, tangent to a manifold with non vanishing curvature. This observation paves the road to quantum gravity.
2 Clifford space as an extension of spacetime

Let us consider a flat space $M$ whose points are possible positions of the center of mass $P$ of a physical object $\mathcal{O}$. If the object’s size is small in comparison to the distances to surrounding objects, then we can approximate the object with a point particle. The squared distance between two possible positions, with coordinates $x^\mu$ and $x^\mu + \Delta x^\mu$, is

$$\Delta s^2 = \Delta x^\mu g_{\mu \nu} \Delta x^\nu.$$  \hfill (1)

Here index $\mu$ runs over dimensions of the space $M$, and $g_{\mu \nu}$ is the metric tensor. For instance, in the case in which $M$ is spacetime, $\mu = 0, 1, 2, 3$, and $g_{\mu \nu} = \eta_{\mu \nu} = \text{diag}(1, -1, -1, -1)$ is the Minkowski metric. The object $\mathcal{O}$ is then assumed to be extended in spacetime, i.e., to have an extension in a 3D space and in the direction $x^0$ that we call “time”.

There are two possible ways of taking the square root of $\Delta s^2$.

Case I.

$$\Delta s = \sqrt{\Delta x^\mu g_{\mu \nu} \Delta x^\nu}$$  \hfill (2)

Case II.

$$\Delta x = \Delta x^\mu \gamma_\mu$$  \hfill (3)

In Case I, the square root is a scalar, i.e., the distance $\Delta s$.

In Case II, the square root is a vector $\Delta x$, expanded in term of the basis vectors $\gamma_\mu$, satisfying the relations

$$\gamma_\mu \cdot \gamma_\nu \equiv \frac{1}{2} (\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu) = g_{\mu \nu}.$$  \hfill (4)

If we write $\Delta x = \Delta x^\mu \gamma_\mu = (x^\mu - x_0^\mu) \gamma_\mu$ and take $x_0^\mu = 0$, we obtain $x = x^\mu \gamma_\mu$, which is the position vector of the object’s $\mathcal{O}$ center of mass point $P$ (Fig. 1), with $x^\mu$ being the coordinates of the point $P$.

![Figure 1: The center of mass point $P$ of an extended object $\mathcal{O}$ is described by a vector $x^\mu \gamma_\mu$.](image-url)
In spite of being extended in spacetime and having many (practically infinitely many) degrees of freedom, we can describe our object $O$ by only four coordinates $x^\mu$, the components of a vector $x = x^\mu \gamma_\mu$.

The $\gamma_\mu$ satisfying the anticommutation relations (4) are generators of the Clifford algebra $Cl(1, 3)$. A generic element of $Cl(1, 3)$ is a superposition

$$X = \sigma_1 \gamma_1 + \frac{1}{2!} x^{\mu \nu} \gamma_\mu \wedge \gamma_\nu + \frac{1}{3!} x^{\mu \nu \rho} \gamma_\mu \wedge \gamma_\nu \wedge \gamma_\rho + \frac{1}{4!} x^{\mu \nu \rho \sigma} \gamma_\mu \wedge \gamma_\nu \wedge \gamma_\rho \wedge \gamma_\sigma,$$

(5)

where $\gamma_\mu \wedge \gamma_\nu$, $\gamma_\mu \wedge \gamma_\nu \wedge \gamma_\rho$, and $\gamma_\mu \wedge \gamma_\nu \wedge \gamma_\rho \wedge \gamma_\sigma$ are the antisymmetrized products $\gamma_\mu \gamma_\nu$, $\gamma_\mu \gamma_\nu \gamma_\rho$, and $\gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma$, respectively. They represent basis bivectors, 3-vectors and 4-vectors, respectively. The terms in Eq. (5) describe a scalar, an oriented line, area, 3-volume and 4-volume. The antisymmetrized product of five gammas vanishes identically in four dimensions.

A question now arises as to whether the object $X$ of Eq. (5) can describe an extended object in spacetime $M_4$. We have seen that $x = x^\mu \gamma_\mu$ describes the centre of mass position. We anticipate that $\frac{1}{2!} x^{\mu \nu} \gamma_\mu \wedge \gamma_\nu$ describes an oriented area associated with the extended object. Suppose that our object $O$ is a closed string. At first approximation its is described just by its center of mass coordinates (Fig. 2a). At a better approximation it is described by the quantities $x^{\mu \nu}$, which are the projections of the oriented area, enclosed by the string, onto the coordinate planes (Fig. 2b). If we probe the string at a better resolution, we might find that it is not exactly a string, but a closed membrane (Fig. 3). The oriented volume, enclosed by this 2-dimensional membrane is described by the quantities $X^{\mu \nu \rho}$. At even better resolution we could eventually see that our object $O$ is in fact a closed 3-dimensional membrane, enclosing a 4-volume, described by $x^{\mu \nu \rho \sigma}$. Our object $O$ has finite extension in the 4-dimensional spacetime. It is like an instanton.

Figure 2: With a closed string one can associate the center of mass coordinates (a), and the area coordinates (b).

Let us now introduce a more compact notation by writing

$$X = \sum_{r=0}^{4} x^{\mu_1 \mu_2 \ldots \mu_r} \gamma_{\mu_1 \mu_2 \ldots \mu_r} \equiv x^M \gamma_M,$$

(6)
where $\gamma_{\mu_1 \mu_2 ... \mu_r} \equiv \gamma_{\mu_1} \wedge \gamma_{\mu_2} \wedge ... \wedge \gamma_{\mu_r}$, and where we now assume $\mu_1 < \mu_2 < ... < \mu_r$, so that we do not need a factor $1/r!$. Here $x^M$ are interpreted as quantities that describe an extended instantonic object in $M_4$. On the other hand, $x^M$ are coordinates of a point in the 16-dimensional space, called Clifford space $C$. In other words, from the point of view of $C$, $x^M$ describe a point in $C$.

Figure 3: Looking with a sufficient resolution one can detect eventual presence of volume degrees of freedom.

The coordinates $x^M$ of Clifford space can describe not only closed, but also open branes. For instance, a vector $x^\mu \gamma_\mu$ can denote position of a point event with respect to the origin (Fig. 1), or it can describe a string-like extended object (an instantonic string in spacetime). Similarly, a bivector $x^{\mu \nu} \gamma_\mu \wedge \gamma_\nu$ can describe a closed string (2a), or it can describe an open membrane. Whether the coordinates $x^M \equiv x^{\mu_1 \mu_2 ... \mu_r}$ describe a closed $r$-brane or an open $(r + 1)$-brane is determined by the value of the scalar and pseudoscalar coordinates, i.e., by $\sigma$ and $\tilde{\sigma}$ (for more details see Ref.[16]).

A continuous 1-dimensional set of points in $C$ is a curve, a worldline, described by the mapping

$$x^M = X^M(\tau),$$

where $\tau$ is a monotonically increasing parameter and $X^M$ embedding functions of the worldline in $C$. We assume that it satisfies the action principle

$$I[X^M] = \mathcal{M} \int d\tau \left( G_{MN} \dot{X}^M \dot{X}^N \right)^{1/2},$$

where $G_{MN}$ is the metric in $C$, and $\mathcal{M}$ a constant, analogous to mass. From the point of view of spacetime, the functions $X^M(\tau) \equiv X^{\mu_1 \mu_2 ... \mu_r}(\tau), \ r = 0, 1, 2, 3, 4$, describe evolution of an extended instantonic object in spacetime. Some examples are in Fig. 4 (see also [10]).

In this setup, there is no “block universe” in spacetime. There do not exist infinitely long worldlines or worldtubes in spacetime. Infinitely long worldlines exist in $C$-space, and in this sense a block universe exists in $C$-space.

The action (8) is invariant under reparametrizations of $\tau$. A consequence is the constraint among the canonical momenta $P_M = \partial L/\partial \dot{X}^M = \mathcal{M} \dot{X}_M/\sqrt{g_{JK} \dot{X}^J \dot{X}^K}$:

$$P^M P_M - \mathcal{M}^2 = 0.$$
The metric of Clifford space is given by the scalar product of two basis vectors,

$$\eta_{MN} = \gamma^\dagger_M \gamma = < \gamma^\dagger_M \gamma_N >_0,$$

where “$\dagger$” is the operation that reverses the order of vectors in the product $\gamma_M = \gamma_{\mu_1} \gamma_{\mu_2} \cdots \gamma_{\mu_r}$, so that $\gamma^\dagger_M = \gamma_{\mu_r} \cdots \gamma_{\mu_2} \gamma_{\mu_1}$. The superscript “0” denotes the scalar part of an expression. For instance,

$$< \gamma_{\mu} \gamma_{\nu} >_0 = \eta_{\mu \nu}, \quad < \gamma_{\mu} \gamma_{\nu} \gamma_{\alpha} >_0 = 0, \quad < \gamma_{\mu} \gamma_{\nu} \gamma_{\alpha} \gamma_{\beta} >_0 = \eta_{\mu \beta} \eta_{\nu \alpha} - \eta_{\mu \alpha} \eta_{\nu \beta}. \quad (11)$$

So we obtain

$$\eta_{MN} = \text{diag}(1, 1, 1, 1, 1, 1, -1, -1, -1, -1, -1, -1, -1, -1, -1), \quad (12)$$

which means that the signature of $C$-space is $(+ + + + + + + + - - - - - - - - -)$, or shortly, $(8, 8)$.

The quadratic form reads

$$X^\dagger X = \eta_{MN} x^M x^N = \sigma^2 + \eta_{\mu \nu} x^\mu x^\nu + (\eta_{\mu \beta} \eta_{\nu \alpha} - \eta_{\mu \alpha} \eta_{\nu \beta}) x^{\mu \alpha} x^{\nu \beta} + \eta_{\mu \nu} \tilde{x}^\mu \tilde{x}^\nu - \tilde{\sigma}^2,$$

$$= \eta_{\bar{\mu} \bar{\nu}} \tilde{x}^{\bar{\mu}} \tilde{x}^{\bar{\nu}} + \sigma^2 - \tilde{\sigma}^2, \quad (13)$$

where $x^{\hat{\mu}} = (x^\mu, x^{\mu \nu}, \tilde{x}^\mu)$, with $\tilde{x}^\mu \equiv \frac{1}{3!} \epsilon^{\mu \nu \rho \sigma} x^{\nu \rho \sigma}$ being the pseudoscalar coordinates, whereas $\sigma$ is the scalar and $\tilde{\sigma} \equiv \frac{1}{4!} \epsilon_{\mu \nu \rho \sigma} x^{\mu \nu \rho \sigma}$ the pseudoscalar coordinate in $C$-space.

Upon quantization, $P_M$ become operators $P_M = -i \partial/\partial x^M$, and the constraint (9) becomes the Klein-Gordon equation in $C$-space:

$$\left( \partial_M \partial^M + M^2 \right) \Psi(x^M) = 0. \quad (14)$$
In the new coordinates,
\[ s = \frac{1}{2}(\sigma + \tilde{\sigma}) \quad \lambda = \frac{1}{2}(\sigma - \tilde{\sigma}), \]  \hspace{1cm} (15)
in which the quadratic form is
\[ X^\dagger X = \eta_{\hat{\mu}\hat{\nu}}x^{\hat{\mu}}x^{\hat{\nu}} - 2s\lambda, \]  \hspace{1cm} (16)
the Klein-Gordon equation reads
\[ \eta^{\hat{\mu}\hat{\nu}}\partial_{\hat{\mu}}\partial_{\hat{\nu}}\phi - 2\partial_s\partial_\lambda\phi = 0. \]  \hspace{1cm} (17)
If we take the ansatz
\[ \phi(x^{\hat{\mu}}, s, \lambda) = e^{i\Lambda\lambda}\psi(s, x^{\hat{\mu}}), \]  \hspace{1cm} (18)
then Eq. (17) becomes
\[ \eta^{\hat{\mu}\hat{\nu}}\partial_{\hat{\mu}}\partial_{\hat{\nu}}\phi - 2i\Lambda\partial_s\phi = 0, \]  \hspace{1cm} (19)
i.e.,
\[ i\frac{\partial\psi}{\partial s} = \frac{1}{2\Lambda}\eta^{\hat{\mu}\hat{\nu}}\partial_{\hat{\mu}}\partial_{\hat{\nu}}\psi. \]  \hspace{1cm} (20)
This is the generalized Stueckelberg equation. It is like the Schrödinger equation, but it describes the evolution of the wave function \( \psi(s, x^{\hat{\mu}}) \) in the 14-dimensional space whose points are described by coordinates \( x^{\hat{\mu}} \). The evolution parameter is \( s \).

A remarkable feature of this setup is that the evolution parameter has a clear physical meaning: it is given in terms of the scalar, \( \sigma \), and the pseudoscalar, \( \tilde{\sigma} \), coordinate according to Eq. (15). The latter quantities, as shown before, are given by a configuration of the object, sampled in terms of the coordinates \( X^M \) of the Clifford space \( C \).

The wave function \( \psi(s, x^{\hat{\mu}}) \) is the probability amplitude that at a given value of the evolution parameter \( s \) we will find an instantonic extended object with coordinates \( x^{\hat{\mu}} \).

This is illustrated in Fig. 5. In principle all points of \( C \)-space are possible in the sense that we can find there an instantonic extended object. A wave packet determines a subset of point of \( C \) that are more probable to “host” the occurrence of an instantonic object (an event in \( C \)). The wave function determines the probability amplitude over the points of \( C \). Its square determines the probability density. From the point of view of spacetime, wave function determines which instantonic extended objects are more likely to occur. It determines the probability amplitude, and its square the probability density of occurrence of a given instantonic extended object. The probability amplitude \( \psi \) is different at different values of the evolution parameter \( s \). In other words, \( \psi \) changes (evolves) with \( s \).
Figure 5: Extended instantonic object in spacetime (a) is represented by a point in $C$-space (b). Quantum mechanically, the extended object is blurred (c). In $C$-space, we have a blurred point, i.e., a “cloud” of points occurring with probability density $|\psi(s, x^\mu)|^2$.

Instead of one extended object, described by $x^M$, we can consider several or many extended objects, described by $x^{iM}$, $i = 1, 2, ..., n$. They form an instantonic configuration $\{\mathcal{O}^i\} = \{\mathcal{O}^i\}, i = 1, 2, ..., n$. The space of all possible instantonic configurations will be called configuration space $C$. The infinitesimal distance between two configurations, i.e., between two points in $C$, is

$$dS^2 = \eta_{iM(jN)}dx^{iM}dx^{jN}, \quad (21)$$

where $\eta_{iM(jN)} = \delta_{ij}\eta_{MN}$ is the metric of a flat configuration space.

We will assume that the Klein-Gordon equation (14) can be generalized so to hold for the wave function $\psi(x^{iM})$ in the space of instantonic configurations $\{\mathcal{O}^i\}$:

$$\left(\eta^{iM(jN)}\partial_{iM}\partial_{jN} + K^2\right)\phi(x^{iM}) = 0, \quad \partial_{iM} \equiv \frac{\partial}{\partial X^{iM}} \quad (22)$$

Let us choose a particular extended object, $\mathcal{O}^1$, with coordinates $x^{1M} \equiv x^M = (\sigma, x^\mu, x^{\mu\nu}, \tilde{x}^\mu, \tilde{\sigma})$. The coordinates of the remaining extended objects within the configuration are $x^{2M}, x^{3M}, ...$. Let us denote them $x^{iM}$, $\tilde{i} = 2, 3, ..., N$. Following the same procedure as in Eqs. (15)–(20), we define $s$ and $\lambda$ according to (15) to the
first object. We have thus split the coordinates $x^i M$ of the configuration according to

$$x^i M = (s, \lambda, x^\mu, x^\bar{i} M) = (s, \lambda, x^\bar{M}), \quad (23)$$

where $x^\bar{M} = (x^\mu, x^\bar{i} M)$. By taking the ansatz

$$\phi(s, \lambda, x^\bar{M}) = e^{i\Lambda \lambda} \psi(s, x^\bar{M}), \quad (24)$$

Eq. (22) becomes

$$\eta^\bar{M}\bar{N} \partial_{\bar{M}} \partial_{\bar{N}} \psi - 2i \Lambda \partial_s \psi = 0, \quad (25)$$

i.e.,

$$i \partial_s \psi = \frac{1}{2\Lambda} \eta^\bar{M}\bar{N} \partial_{\bar{M}} \partial_{\bar{N}} \psi. \quad (26)$$

Eq. (26) describes evolution of a configuration composed of a system of instantonic extended objects.

The evolution parameter $s$ is given by the configuration itself (in the above example by one of its parts), and it distinguishes one instantonic configuration from another instantonic configuration. So we have a continuous family of instantonic configurations, evolving with $s$. Here, “instantonic configuration” or “instantonic extended object” is a generalization of the concept of “event”, associated with a point in spacetime. An event, by definition is “instantonic” as well, because it occurs at one particular point in spacetime.

A configuration can be very complicated and self-referential, and thus being a record of the configurations at earlier values of $s$. In this respect this approach resembles that by Barbour\[18\], who considered “time capsules” with memory of the past. As a model, he considered a triangleland, whose configurations are triangles. Instead of triangleland, we consider here the Clifford space, in which configurations are modeled by oriented $r$-volumes ($r = 0, 1, 2, 3, 4$) in spacetime. In this respect our model differs from Barbour’s model, in which the triangles are in 3-dimensional space. Instead of 3-dimensional space, I consider a 4-dimensional space with signature $(+−−−)$. During the development of physics it was recognized that a 3-dimensional space is not suitable for formulation of the theory describing the physical phenomena, such as electromagnetism and moving objects. In other words, the theory of relativity requires 4-dimensional space, with an extra dimension $x^0$, whose signature is opposite to the signature of three spatial dimensions. The fourth dimension was identified with time, $x^0 \equiv t$. Such identification, though historically very useful, has turned out to be misleading \[19\], \[20\], \[21\], \[9\]. In fact, $x^0$ is not the true time, it is just a coordinate of the fourth dimension. The evolution time is something else. In the Stueckelberg theory \[19\], \[20\], \[21\], \[9\] its origin remains unexplained. In the approach with Clifford space, the evolution time (evolution parameter) is $s = (\sigma + \tilde{\sigma})/2$, i.e., a superposition of the scalar coordinate, $\sigma$, and the pseudoscalar coordinate, $\tilde{\sigma}$. This
is the parameter that distinguishes configurations within a 1-dimensional family. In principle, the configurations can be very complicated and self-referential, including conscious experiences of an observer. Thus $s$ distinguishes different conscious experiences of an observer \[20, 9\]; it is the time experienced by a conscious observer. A wave function $\phi(x^M) = e^{iA} \psi(s, x^M)$ “selects” in the vast space $\mathcal{C}$ of all possible configurations $x^M$ a subspace $\mathcal{S} \in \mathcal{C}$ of configurations. More precisely, $\phi$ assigns a probability density over the points of $\mathcal{C}$, so that some points are more likely to be experienced by an observer than the other points. In particular, $\phi(x^M)$ can be a localized wave packet evolving along $s$. For instance, such a wave packet can be localized around a worldline $x^M = X^M_0(s)$ in $\mathcal{C}$, which from the point of view of $M_4$, is a succession (evolution) of configurations $X^M_0$ at different values of the parameter $s$. If configurations are complicated and include the external world and an observer’s brain, such wave packet $\psi(s, x^M)$ determines the evolution of conscious experiences of an observer coupled by his sense organs to the external world.

The distinction between the evolution time $\tau$ and the coordinates $x^0$ in the wave function $\psi(s, x^M)$, can help in clarifying the well known Libet experiment \[22\]. The latter experiment seemingly demonstrates that we have no “free will”, because shortly before we are conscious of a decision, our brain already made the decision. This experiment is not in conflict with free will, if besides the theory described above, we as well invoke the Everett many worlds interpretation of quantum mechanics \[23\], and the considerations exposed in Ref. \[9\]. Further elaboration of this important implication of the Stueckelberg and Everett theory is beyond the scope of this chapter. But anyone with a background in those theories can do it after some thinking. An interested reader can do it as an exercise.

\section{Generators of Clifford algebras as quantum mechanical operators}

\subsection{Orthogonal and symplectic Clifford algebras}

After having exposed a broader context of the role of Clifford algebras in physics, let me now turn to a specific case and consider the role of Clifford algebras in quantization. The inner product of generators of Clifford algebra gives the metric. We distinguish two cases:

(i) If metric is \textit{symmetric}, then the inner product is given by the \textit{anticommutator} of generators; this is the case of an \textit{orthogonal Clifford algebra}:

$$\frac{1}{2} \{ \gamma_a, \gamma_b \} \equiv \gamma_a \cdot \gamma_b = g_{ab}.$$  \hspace{1cm} (27)

(ii) If metric is \textit{antisymmetric}, then the inner product is given by the commutator
of generators; this is the case of a symplectic Clifford algebra:

\[ \frac{1}{2}[q_a, q_b] \equiv q_a \wedge q_b = J_{ab}. \]  \hfill (28)

Here \( q_a \) are the symplectic basis vectors that span a symplectic space, whose points are associated with symplectic vectors \[ z = z^a q_a. \]  \hfill (29)

Here \( z^a \) are commuting phase space coordinates,

\[ z^a z^b - z^b z^a = 0. \]  \hfill (30)

An example of symplectic space in physics is phase space, whose points are coordinates and momenta of a particle:

\[ z^a = (x^\mu, p^\mu) \equiv (x^\mu, \bar{x}^\mu) \equiv (x^\mu, x^\mu). \]  \hfill (31)

The corresponding basis vectors then split according to

\[ q_a = (q_\mu^{(x)}, q_\mu^{(p)}) \equiv (q_\mu, \bar{q}_\mu) \equiv (q_\mu, q_\mu), \quad \mu = 1, 2, ..., n, \]  \hfill (32)

and the relation (28) becomes

\[ \frac{1}{2}[q_\mu^{(x)}, q_\nu^{(p)}] \equiv \frac{1}{2}[q_\mu, q_\nu] = J_{\mu\nu} = g_{\mu\nu}, \]

\[ [q_\mu^{(x)}, q_\nu^{(x)}] = 0, \quad [q_\mu^{(p)}, q_\nu^{(p)}] = 0, \]  \hfill (33)

where we have set

\[ J_{ab} = \begin{pmatrix} 0 & g_{\mu\nu} \\ -g_{\mu\nu} & 0 \end{pmatrix}. \]  \hfill (34)

Here, depending on the case considered, \( g_{\mu\nu} \) is the euclidean, \( g_{\mu\nu} = \delta_{\mu\nu}, \mu, \nu = 1, 2, ..., n \), or the Minkowski metric, \( g_{\mu\nu} = \eta_{\mu\nu} \). In the latter case we have \( \mu, \nu = 0, 1, 2, ..., n - 1 \).

We see that (33) are just the Heisenberg commutation relations for coordinate and momentum operators, identified as\(^3\)

\[ \hat{x}_\mu = \frac{1}{\sqrt{2}} q_\mu^{(x)}; \quad \hat{p}_\mu = \frac{i}{\sqrt{2}} q_\mu^{(p)}. \]  \hfill (35)

Then we have

\[ [\hat{x}_\mu, \hat{p}_\nu] = ig_{\mu\nu}, \quad [\hat{x}_\mu, \hat{x}_\nu] = 0, \quad [\hat{p}_\mu, \hat{p}_\nu] = 0. \]  \hfill (36)

\(^3\) We insert factor \( i \) in order to make the operator \( \hat{p}_\mu \) hermitian.
Instead of a symplectic vector \( z^a q_a \), let us now consider another symplectic vector, namely

\[
F = \frac{\partial f}{\partial z^a} q^a,
\]

where \( f = f(z) \) is a function of position in phase space. The wedge product of two such vectors is

\[
F \wedge G = \frac{\partial f}{\partial z^a} q^a \wedge \frac{\partial g}{\partial z^b} = \frac{\partial f}{\partial z^a} J^{ab} \frac{\partial g}{\partial z^b},
\]

where in the last step we used the analog of Eq. (28) for the reciprocal quantities \( q^a = J^{ab} q_b \), where \( J^{ab} \) is the inverse of \( J_{ab} \).

Eq. (38) is equal to the Poisson bracket of two phase space functions. namely, using (31) and (34), we have

\[
\frac{\partial f}{\partial x^\mu} J^{ab} \frac{\partial g}{\partial p^\nu} \eta_{\mu \nu} = \frac{\partial f}{\partial x^\mu} \frac{\partial g}{\partial x^\nu} \eta_{\mu \nu} \equiv \{ f, g \}_{PB}.
\]

In particular, if

\[
f = z^c, \quad g = z^d,
\]

Eqs. (38), (39) give

\[
q^a \wedge q^b = J^{ab} = \{ z^a, z^b \}_{PB}.
\]

We see that the Heisenberg commutation relations for operators \( \hat{x}^\mu, \hat{p}^\mu \) are obtained automatically, if we express the Poisson bracket relations in terms of the wedge product of the symplectic vectors

\[
F = \frac{\partial f}{\partial z^a} q^a = \frac{\partial f}{\partial x^\mu} q^a_{(x)} + \frac{\partial f}{\partial p^\mu} q^a_{(p)} \quad \text{and} \quad G = \frac{\partial g}{\partial z^a} = \frac{\partial g}{\partial x^\mu} q^a_{(x)} + \frac{\partial g}{\partial p^\mu} q^a_{(p)}
\]

By having taken into account not only the coordinates and functions in a symplectic space, but also corresponding basis vectors, we have found that basis vectors are in fact quantum mechanical operators [14]. Moreover, the Poisson bracket between classical phase space variable, \( \{ z^a, z^b \}_{PB} \), is equal to the commutator, \( \frac{1}{2}[q^a, q^b] = q^a \wedge q^b \), of vectors (i.e., of operators) \( q^a \) and \( q^b \) [14]. According to this picture, quantum operators are already present in the classical symplectic form, if we write the symplectic metric as the inner product of symplectic basis vectors. The latter vectors are just the quantum mechanical operators.

Analogous procedure can be performed with orthogonal Clifford algebras. Then a point in phase space can be described as a vector

\[
\lambda = \lambda^a \gamma_a,
\]

where \( \lambda^a \) are anticommuting phase space coordinates,

\[
\lambda^a \lambda^b + \lambda^b \lambda^a = 0,
\]

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and $\gamma_a$ basis vectors, satisfying Eq. (27). If we split the vectors $\gamma_a$ and the metric $\gamma_{ab}$ according to
\begin{equation}
\gamma_a = (\gamma_{\mu}, \bar{\gamma}_{\mu}), \quad \mu = 0, 1, 2, ..., n - 1,
\end{equation}
\begin{equation}
g_{ab} = \begin{pmatrix} g_{\mu\nu} & 0 \\ 0 & g_{\mu\nu} \end{pmatrix}
\end{equation}
and introduce a new basis, the so called Witt basis,
\begin{equation}
\theta_{\mu} = \frac{1}{\sqrt{2}} (\gamma_{\mu} + i \bar{\gamma}_{\mu}),
\end{equation}
\begin{equation}
\bar{\theta}_{\mu} = \frac{1}{\sqrt{2}} (\gamma_{\mu} - i \bar{\gamma}_{\mu}),
\end{equation}
then the Clifford algebra relations (27) become
\begin{align}
\theta_{\mu} \cdot \bar{\theta}_{\nu} &\equiv \frac{1}{2} (\theta_{\mu} \bar{\theta}_{\nu} + \bar{\theta}_{\nu} \theta_{\mu}) = \eta_{\mu\nu}, \\
\theta_{\mu} \cdot \theta_{\nu} &= 0, \\
\bar{\theta}_{\mu} \cdot \bar{\theta}_{\nu} &= 0.
\end{align}
These are the anticommutation relations for fermionic creation and annihilation operators.

Let us now introduce functions $\tilde{f}(\lambda)$ and $\tilde{g}(\lambda)$, and consider the vectors
\begin{equation}
\tilde{F} = \frac{\partial \tilde{f}}{\partial \lambda^a} \gamma^a, \quad \tilde{g} = \frac{\partial \tilde{g}}{\partial \lambda^a} \gamma^a
\end{equation}
The dot product of those vectors is
\begin{equation}
\tilde{F} \cdot \tilde{G} = \frac{\partial \tilde{f}}{\partial \lambda^a} \gamma^a \cdot \frac{\partial \tilde{g}}{\partial \lambda^b} \gamma^b = \frac{\partial \tilde{f}}{\partial \lambda^a} g^{ab} \frac{\partial \tilde{g}}{\partial \lambda^b} = \{\tilde{f}, \tilde{g}\}_{PB},
\end{equation}
where $g^{ab} = \gamma^a \cdot \gamma^b$ is the inverse of $g_{ab}$.

Eq. (50) shows that the dot product, which in the orthogonal case corresponds to the inner product, is equal to the Poisson bracket of two phase space functions, now composed with the symmetric metric $g_{ab}$.

If
\begin{equation}
\tilde{f} = \lambda^c, \quad \tilde{g} = \lambda^d,
\end{equation}
Eq. (50) gives
\begin{equation}
\tilde{F} \cdot \tilde{G} = \gamma^c \cdot \gamma^d = g^{cd},
\end{equation}
which in the Witt basis read as the fermionic anticommutation relations (48). This means that the Poisson bracket between the (classical) phase space variables $\lambda^a, \lambda^b$ is equal to the anticommutator of the "operators" $\gamma^a$ and $\gamma^b$:
\begin{equation}
\{\lambda^a, \lambda^b\}_{PB} = \frac{1}{2} \{\gamma^a, \gamma^b\} = g^{ab}.
\end{equation}
Again we have that the basis vectors behave as quantum mechanical operators.
3.2 Equations of motion for a particle’s coordinates and the corresponding basis vectors

We will now consider a point particle, described by the phase space action

$$ I = \frac{1}{2} \int d\tau \left( \dot{z}^a J_{ab} \dot{z}^b + z^a K_{ab} \dot{z}^b \right), \quad (54) $$

where

$$ \frac{1}{2} z^a K_{ab} \dot{z}^b = H \quad (55) $$

is the Hamiltonian, the quantity $K_{ab}$ being a symmetric $2n \times 2n$ matrix.

Variation of the action (54) with respect to $z^a$ gives

$$ \dot{z}^a = J^{ab} \frac{\partial H}{\partial z^b}, \quad (56) $$

which are the Hamilton equations of motion.

A solution of equation (56) is a trajectory $z$ in phase space. We can consider a trajectory as an infinite dimensional vector with components $z^a(\tau) \equiv z^a(\tau)$. Here $a(\tau)$ is the index that denotes components; it is a double index, with $a$ being a discrete index, and $(\tau)$ a continuous one. Corresponding basis vectors are $q_a(\tau) \equiv q_a(\tau)$, and they satisfy the relations

$$ q_a(\tau) \wedge q_b(\tau) = J_{a(\tau)b(\tau)} = J_{ab}(\tau - \tau'), \quad (57) $$

which are an extension of the relations (28) to our infinite dimensional case.

A trajectory is thus described by the vector

$$ z = z^a(\tau) q_a(\tau) \equiv \int d\tau z^a(\tau) q_a(\tau). \quad (58) $$

The phase space velocity vector is

$$ v = \dot{z}^a(\tau) q_a(\tau) = -z^a(\tau) \dot{q}_a(\tau), \quad (59) $$

where we have assumed that the “surface” term vanishes:

$$ v = \int d\tau \dot{z}^a(\tau) q_a(\tau) = -\int d\tau z^a(\tau) \dot{q}_a(\tau) + z^a(\tau) q_a(\tau) \bigg|_{\tau_2}^{\tau_1}. \quad (60) $$

The last term vanishes if $z^a(\tau_2) q_a(\tau_2) = z^a(\tau_1) q_a(\tau_1)$.

The action (54) can be written as

$$ I = \frac{1}{2} \left( \dot{z}^a J_{a(\tau)b(\tau)} \dot{z}^b(\tau') + z^a K_{a(\tau)b(\tau)} \dot{z}^b(\tau') \right), \quad (61) $$
where \( J_{a(\tau)b(\tau')} \) is given in Eq. (57), and
\[
K_{a(\tau)b(\tau')} = K_{ab} \delta(\tau - \tau').
\]
(62)
The corresponding equations of motion are
\[
\dot{z}^{a(\tau)} = J^{a(\tau)c(\tau'')} K_{c(\tau'')b(\tau')} z^{b(\tau')}.
\]
(63)
Multiplying both sides of the latter equation by \( q^{a(\tau)} \), we obtain
\[
\dot{z}^{a(\tau)} q^{a(\tau)} = -q^{a(\tau)} K_{a(\tau)b(\tau')} z^{b(\tau')}.
\]
(64)
We have raised the index by \( J^{a(\tau)c(\tau'')} \) and taken into account that \( J^{a(\tau)c(\tau'')} = -J^{c(\tau'')a(\tau)} \). Eq. (64) is just Eq. (63), expressed in terms of the basis vectors. Both equations are equivalent.

Using the relation (59) in Eq. (64), we obtain
\[
\dot{z}^{b(\tau')} q^{b(\tau')} = q^{a(\tau)} K_{a(\tau)b(\tau')} z^{b(\tau')}.
\]
(65)
Apart from the surface term that we have neglected in Eq. (60), the last equation, (65), is equivalent to the classical equation of motion (56), only the \( \tau \)-dependence has been switched from the components to the basis vectors.

A curious thing happens if we assume that Eq. (65) holds for an arbitrary trajectory (Fig. 6).

Figure 6: If the operator equations of motion (65) hold for any path \( z^{a(\tau)} \) this means that coordinates and momenta are undetermined.

Then, instead of (65), we can write
\[
\dot{q}^{b(\tau')} = q^{a(\tau)} K_{a(\tau)b(\tau')}
\]
(66)
Inserting into the latter equation the explicit expression (62) for \( K_{a(\tau)b(\tau')} \) and writing \( \dot{q}^{b(\tau)} = \dot{q}^{b(\tau)}, \ q^{a(\tau)} = q^{a(\tau)} \), we obtain
\[
\dot{q}^{a(\tau)} = K_{ab} q^{b(\tau)}.
\]
(67)
This can be written as
\[ \dot{q}_a = [q_a, \hat{H}], \]  
(68)
where
\[ \hat{H} = \frac{1}{2} q^a K_{ab} q^b, \]  
(69)
is the Hamilton operator, satisfying
\[ [q_a, \hat{H}] = K_{ab} q^b. \]  
(70)

Starting from the classical action (54), we have arrived at the Heisenberg equations of motion (68) for the basis vectors \( q_a \). On the way we have made a crucial assumption that the particle does not follow a trajectory \( z^a(\tau) \) determined by the classical equations of motion, but that it can follow any trajectory. By the latter assumption we have passed from the classical to the quantized theory. We have thus found yet another way of performing quantization of a classical theory, Our assumption that a trajectory (a path) can be arbitrary, corresponds to that by Feynman path integrals. In our procedure we have shown how such an assumption of arbitrary path leads to the Heisenberg equations of motion for operators.

### 3.3 Supersymmetrization of the action

The action (54) can be generalized \[14\] so to contain not only a symplectic, but also an orthogonal part. For this purpose, we introduce the generalized vector space whose elements are
\[ z = z^A q_A, \]  
(71)
where
\[ z^A = (z^a, \lambda^a), \quad z^a = (x^\mu, \bar{x}^\mu), \quad \lambda^a = (\lambda^\mu, \bar{\lambda}^\mu) \]  
(72)
are coordinates, and
\[ q_A = (q_a, \gamma_a), \quad q_a = (q_\mu, \bar{q}_\mu), \quad \gamma_a = (\gamma_\mu, \bar{\gamma}_\mu) \]  
(73)
are basis vectors. The metric is
\[ \langle q_A q_B \rangle_0 = G_{AB} = \begin{pmatrix} J_{ab} & 0 \\ 0 & g_{ab} \end{pmatrix}, \]  
(74)
where \( J_{ab} = -J_{ba} \) and \( g_{ab} = g_{ba} \).
Let us consider a particle moving in such space. Its worldline is
\[ z^A = Z^A(\tau). \] (75)

An example of a possible action is
\[ I = \frac{1}{2} \int d\tau \dot{Z}^A G_{AB} Z^B + \text{interaction terms}. \] (76)

Using (72)–(74), the latter action can be split as
\[ I = \frac{1}{2} \int d\tau \left( \dot{x}^\mu \eta_{\mu\nu} \dot{x}^\nu - \dot{\bar{x}}^\mu \eta_{\mu\nu} \dot{\bar{x}}^\nu + \dot{\lambda}^\mu \eta_{\mu\nu} \lambda^\nu + \dot{\bar{\lambda}}^\mu \eta_{\mu\nu} \bar{\lambda}^\nu \right) + \text{interaction terms} \] (77)

Here \( z^a \) are commuting, and \( \lambda^a \) anticommuting (Grassmann) coordinates. The canonical momenta are
\[ p^{(x)}_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = \frac{1}{2} \eta_{\mu\nu} \dot{x}^\nu, \quad p^{(x)}_\bar{\mu} = \frac{\partial L}{\partial \dot{\bar{x}}^\mu} = -\frac{1}{2} \eta_{\mu\nu} \dot{\bar{x}}^\nu, \]
\[ p^{(\lambda)}_\mu = \frac{\partial L}{\partial \dot{\lambda}^\mu} = \frac{1}{2} \eta_{\mu\nu} \lambda^\nu, \quad p^{(\lambda)}_\bar{\mu} = \frac{\partial L}{\partial \dot{\bar{\lambda}}^\mu} = \frac{1}{2} \eta_{\mu\nu} \bar{\lambda}^\nu. \] (78)

Instead of the coordinates \( \lambda^a = (\lambda^\mu, \bar{\lambda}^\mu) \), we can introduce the new coordinates
\[ \lambda^a = (\lambda^\mu, \bar{\lambda}^\mu), \quad \lambda^\mu \equiv \xi^\mu = \frac{1}{\sqrt{2}}(\lambda^\mu - i\bar{\lambda}^\mu), \]
\[ \bar{\lambda}^\mu \equiv \bar{\xi}^\mu = \frac{1}{\sqrt{2}}(\lambda^\mu + i\bar{\lambda}^\mu), \] (79)
in which the metric is
\[ g'_{ab} = \gamma'_a \cdot \gamma'_b = \begin{pmatrix} 0 & \eta_{\mu\nu} \\ \eta_{\mu\nu} & 0 \end{pmatrix}. \] (80)

In the new coordinates we have
\[ \dot{\lambda}^a g_{ab} \lambda^b = \dot{\lambda}^a g_{ab} \lambda^b = \dot{\xi}^\mu \eta_{\mu\nu} \xi^\nu + \dot{\bar{\xi}}^\mu \eta_{\mu\nu} \bar{\xi}^\nu. \] (81)

Now the pairs of canonically conjugate variables are \((\xi^\mu, \frac{1}{2} \bar{\xi}^\mu)\) and \((\bar{\xi}^\mu, \frac{1}{2} \xi^\mu)\), whereas in the old coordinates the pairs were \((\lambda^\mu, \frac{1}{2} \lambda^\mu)\) and \((\bar{\lambda}^\mu, \frac{1}{2} \bar{\lambda}^\mu)\), which was somewhat unfortunate, because the variables in the pair were essentially the same.

The interaction term can be included by replacing the \(\tau\)-derivative in the action (76) with the covariant derivative:
\[ \dot{Z}^A \rightarrow \dot{Z}^A + A^A C Z^C. \] (82)
So we obtain\[14\]
\[ I = \frac{1}{2} \int \mathrm{d} \tau \left( \dot{Z}^A + A^A_C Z^C \right) G_{AB} Z^B. \] (83)

This is a generalized Bars action\[24\], invariant under $\tau$-dependent (local) rotations of $Z^A$. As discussed in\[14\], the gauge fields $A^A_C(\tau)$ are not dynamical; they have the role of Lagrange multipliers, whose choice determines a gauge, related to the way of how the canonically conjugated variables can be locally rotated into each other.

For a particular choice of $A^A_C$, we obtain
\[ A^A_C Z^C G_{AB} Z^B = \alpha \ p^\mu p_\mu + \beta \lambda^\mu p_\mu + \gamma \bar{\lambda}^\mu p_\mu. \] (84)

Here $\alpha, \beta, \gamma$ are Lagrange multipliers contained in $A^A_C$. Other choices of $A^A_C$ are possible, and they give expressions that are different from (84). A nice theory of how its works in the bosonic subspace, was elaborated by Bars (see, e.g., refs.\[24\]).

The action (83), for the case (84), gives the constraints
\[ p^\mu p_\mu = 0, \quad \lambda^\mu p_\mu = 0, \quad \bar{\lambda}^\mu p_\mu = 0, \] (85)
or equivalently
\[ p^\mu p_\mu = 0, \quad \xi^\mu p_\mu = 0, \quad \bar{\xi}^\mu p_\mu = 0, \] (86)
if we use coordinates $\xi^a = (\xi^\mu, \bar{\xi})$, defined in Eq. (79).

Upon quantization we have
\[ \hat{p}^\mu \hat{p}_\mu \Psi = 0, \quad \hat{\lambda}^\mu \hat{p}_\mu \Psi = 0, \quad \hat{\bar{\lambda}}^\mu \hat{p}_\mu \Psi = 0, \] (87)
or equivalently
\[ \hat{p}^\mu \hat{p}_\mu \Psi = 0, \quad \hat{\xi}^\mu \hat{p}_\mu \Psi = 0, \quad \hat{\bar{\xi}}^\mu \hat{p}_\mu \Psi = 0, \] (88)
where the quantities with hat are operators, satisfying
\[ [\hat{x}^\mu, \hat{p}^\nu] = i\eta^{\mu\nu}, \quad [\hat{x}^\mu, \hat{x}^\nu] = 0, \quad [\hat{p}^\mu, \hat{p}^\nu] = 0, \] (89)
\[ \{\hat{\lambda}^\mu, \hat{\lambda}^\nu\} = 2i\eta^{\mu\nu}, \quad \{\hat{\bar{\lambda}}^\mu, \hat{\bar{\lambda}}^\nu\} = 2i\eta^{\mu\nu}, \quad \{\hat{x}^\mu, \hat{\bar{\lambda}}^\nu\} = 0. \] (90)
\[ \{\hat{\xi}^\mu, \hat{\bar{\xi}}^\nu\} = \eta^{\mu\nu}, \quad \{\hat{\xi}^\mu, \hat{\xi}^\nu\} = 0, \quad \{\hat{\bar{\xi}}^\mu, \hat{\bar{\xi}}^\nu\} = 0. \] (91)

The operators can be represented as
\[ \hat{x}^\mu \rightarrow x^\mu, \quad \hat{\lambda}^\mu \rightarrow -i \frac{\partial}{\partial x^\mu}, \quad \hat{\bar{\lambda}}^\mu \rightarrow \xi^\mu, \quad \hat{\bar{\xi}}^\mu \rightarrow \frac{\partial}{\partial \xi^\mu}, \] (92)
where
\[ x^\mu x^\nu - x^\nu x^\mu = 0, \quad \xi^\mu \xi^\nu + \bar{\xi}^\mu \bar{\xi}^\nu = 0. \] (93)

\[ 18 \]
A state $\Psi$ can be represented as a wave function $\psi(x^\mu, \xi^\mu)$ of commuting coordinates $x^\mu$ and anticommuting (Grassmann) coordinates $\xi^\mu$.

In Eq. (87) we have two copies of the Dirac equation, where $\hat{\lambda}^\mu$ and $\tilde{\lambda}^\mu$ satisfy the Clifford algebra anticommutation relations (90), and are related to $\gamma^\mu$, $\bar{\gamma}^\mu$ according to

$$\hat{\lambda}^\mu = \gamma^\mu, \quad \tilde{\lambda}^\mu = i\bar{\gamma}^\mu.$$  \hspace{1cm} (94)

Using (92), we find that the quantities $\gamma_\mu$, $\bar{\gamma}_\mu$, satisfying

$$\gamma_\mu \cdot \gamma_\nu = \eta_{\mu\nu}, \quad \bar{\gamma}_\mu \cdot \bar{\gamma}_\nu = \eta_{\mu\nu},$$  \hspace{1cm} (95)

can be represented according to

$$\gamma_\mu = \frac{1}{\sqrt{2}} \left( \xi_\mu + \frac{\partial}{\partial \xi^\mu} \right), \quad \bar{\gamma}_\mu = \frac{1}{\sqrt{2}} \left( \xi_\mu - \frac{\partial}{\partial \xi^\mu} \right).$$  \hspace{1cm} (96)

If we expand $\psi(x^\mu, \xi^\mu)$ in terms of the Grassmann variables $\xi^\mu$, we obtain a finite number (i.e., $2^n$) of terms:

$$\psi(x^\mu, \xi^\mu) = \sum_{r=0}^{n} \psi_{\mu_1\mu_2...\mu_r} \xi^{\mu_1} \xi^{\mu_2} ... \xi^{\mu_r}.$$  \hspace{1cm} (97)

In the case of 4D spacetime, $n = 4$, the wave function has $2^4 = 16$ components. The state $\Psi$ can then be represented as a column $\psi^\alpha(x)$, $\alpha = 1, 2, .., 16$, and the operators $\gamma^\mu$, $\bar{\gamma}^\mu$ as $16 \times 16$ matrices. Because we have built our theory over the $8D$ phase space, our spinor has not only four, but sixteen components. This gives a lot of room for unified theories of particles and fields [25, 26, 27, 28, 29].

4 Basis vectors, Clifford algebras, spinors and quantized fields

4.1 Spinors as particular Clifford numbers

We have seen that the generators of Clifford algebras have the properties of quantum mechanical operators. Depending on the kind of Clifford algebra, they satisfy the commutation or anti commutation relations for bosonic or fermionic creation and annihilation operators.

From the operators $\theta_\mu$ and $\bar{\theta}_\mu$, defined in Eq. (45), we can build up spinors by taking a “vacuum”

$$\Omega = \prod_\mu \bar{\theta}_\mu,$$  \hspace{1cm} (98)

which satisfies $\bar{\theta}_\mu \Omega = 0$.
and acting on it by “creation” operators $\theta_{\mu}$. So we obtain a “Fock space” basis for spinors
\[ s_\alpha = (1\Omega, \theta_{\mu}\Omega, \theta_{\mu}\theta_{\nu}\Omega, \theta_{\mu}\theta_{\nu}\theta_{\rho}\Omega, \theta_{\mu}\theta_{\nu}\theta_{\rho}\theta_{\sigma}\Omega), \quad (99) \]
in terms of which any state can be expanded as
\[ \Psi_\Omega = \sum \psi^\alpha s_\alpha, \quad \alpha = 1, 2, ..., 2^n. \quad (100) \]
Components $\psi^\alpha$ can be spacetime dependent fields. With the operators $\theta_{\mu}, \bar{\theta}_{\mu}$ we can construct spinors as the elements of a minimal left ideal of a Clifford algebra $Cl(2n)$. We will take the dimension of spacetime $n = 4$, so that our phase space will have dimension 8, and the Clifford algebra, built over it, will be $Cl(2, 6)$ which we will simply denote $Cl(8)$ or, in general, $Cl(2n)$.

Besides (98), there are other possible vacuums, e.g.,
\[ \Omega = \prod_{\mu} \theta_{\mu}, \quad \theta_{\mu}\Omega = 0, \quad (101) \]
\[ \Omega = \left( \prod_{\mu \in R_1} \theta_{\mu} \right) \left( \prod_{\mu \in R_2} \bar{\theta}_{\mu} \right), \quad \begin{align*} 
\theta_{\mu}\Omega & = 0, \quad \text{if} \ \mu \in R_2 \\
\bar{\theta}_{\mu}\Omega & = 0, \quad \text{if} \ \mu \in R_2. \quad (102) 
\end{align*} \]
where
\[ R_1 = \{ \mu_1, \mu_2, ..., \mu_r \}, \quad R_2 = \{ \mu_{r+1}, \mu_{r+2}, ..., \mu_n \} \quad (103) \]
There are $2^n$ vacuums of such a kind. By taking all those vacuums, we obtain the Fock space basis for the whole $Cl(2n)$. If $n = 4$, the latter algebra consists of 16 independent minimal left ideals, each belonging to a different vacuum (102) and containing 16-component spinors ($2^n = 16$ if $n = 4$), such as (100). A generic element of $Cl(8)$ is the sum of the spinors $\Psi_{\Omega, i}, i = 1, 2, 3, 4, ..., 16$ belonging to the ideal associated with a vacuum $\Psi_{\Omega, i}$:
\[ \Psi = \sum \Psi_{\Omega, i} = \psi^{\alpha i} s_{\alpha i} \equiv \psi^{\tilde{\alpha}} s_{\tilde{\alpha}}, \quad \tilde{\alpha} = 1, 2, 3, 4, ..., 256, \quad (104) \]
where $s_{\tilde{\alpha}} \equiv s_{\alpha i}, \alpha, i = 1, 2, ..., 16$, is the Fock space basis for $Cl(8)$, and $\psi^{\tilde{\alpha}} \equiv \psi^{\alpha i}$ are spacetime dependent fields. The same element $\Psi \in Cl(8)$ can be as well expanded in terms of the multivector basis,
\[ \Psi = \psi^{\tilde{\alpha}} \gamma_{\tilde{\alpha}}, \quad \tilde{\alpha} = 1, 2, 3, 4, ..., 256, \quad (105) \]
where
\[ \gamma_{\tilde{\alpha}} = 1, \gamma_{a_1}, \gamma_{a_1} \wedge \gamma_{a_2}, ..., \gamma_{a_1} \wedge \gamma_{a_2} \wedge ... \wedge \gamma_{a_{2n}}, \quad (106) \]
which can be written compactly as

\[ \gamma_A = \gamma_{a_1} \wedge \gamma_{a_2} \wedge ... \wedge \gamma_{a_r}, \quad r = 0, 1, 2, ..., 2n. \] (107)

We see that if we construct the Clifford algebra of the 8-dimensional phase space, then we have much more room for unification of elementary particles and fields than in the case of \( Cl(1, 3) \), constructed over 4D spacetime. We have a state \( \Psi \) that can be represented by a 16 × 16 matrix, whose elements can represent all known particles of the 1st generation of the Standard model. Thus, 64 elements of this 16 × 16 matrix include the left and right handed (L,R) versions of the states \((e, \nu_e), (u, d)_r, (u, d)_b,(u, d)_g\), and their antiparticles, times factor two, because all those states, satisfying the generalized Dirac equation\(^{27, 28}\) (see also Sec.5.2), can in principle be superposed with complex amplitudes.

If we take \textit{space inversion} (P) of those 64 states by using the same procedure as in Ref. \[32\], we obtain another 64 states of the 16 × 16 matrix representing \( \Psi \), namely the states of mirror particles (P-particles). Under time reversal (T) (see Ref. \[32\]), we obtain yet another 64 states corresponding to time reversed particles (T-particles). And finally, under PT, we obtain 64 states of time reversed mirror particles (PT-particles). Altogether, we have \( 4 \times 64 = 256 \) states:

\[ s_{\alpha_i} = \left( \begin{array}{cccc}
(e \, \nu) & (e \, \nu) & (e \, \nu) & (e \, \nu) \\
(\bar{e} \, \bar{\nu}) & (\bar{e} \, \bar{\nu})_p & (\bar{e} \, \bar{\nu})_T & (\bar{e} \, \bar{\nu})_PT \\
u \, d & (u \, d) & (u \, d)_r & (u \, d)_r,P \\
(\bar{u} \, \bar{d}) & (\bar{u} \, \bar{d})_r & (\bar{u} \, \bar{d})_{rT} & (\bar{u} \, \bar{d})_{r,PT} \\
u \, d & (u \, d) & (u \, d)_r & (u \, d)_r,P \\
(\bar{u} \, \bar{d}) & (\bar{u} \, \bar{d})_r & (\bar{u} \, \bar{d})_{rT} & (\bar{u} \, \bar{d})_{r,PT} \\
u \, d & (u \, d) & (u \, d)_r & (u \, d)_r,P \\
(\bar{u} \, \bar{d}) & (\bar{u} \, \bar{d})_r & (\bar{u} \, \bar{d})_{rT} & (\bar{u} \, \bar{d})_{r,PT}
\end{array} \right), \quad (108)\]

where

\[ \begin{pmatrix} e & \nu \\ \bar{e} & \bar{\nu} \end{pmatrix} \equiv \begin{pmatrix} e_L & ie_L & \nu_L & i\nu_L \\ e_R & ie_R & \nu_R & i\nu_R \\ \bar{e}_L & i\bar{e}_L & \bar{\nu}_L & i\bar{\nu}_L \\ \bar{e}_R & i\bar{e}_R & \bar{\nu}_R & i\bar{\nu}_R \end{pmatrix}, \quad (109)\]

and similarly for \( u, \, d \).

Those states interact with the corresponding gauge fields\(^{5}\) which include the gauge fields of the Standard model, such as the photon, weak bosons and gluons. There exist also mirror versions, as well as T and PT versions of the standard gauge

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\(^4\) Unification based on Clifford algebras in phase space was considered by Zenczykowski \[25\].

\(^5\) How this works in the case of \( Cl(1, 3) \) is shown in Ref. \[32\] (see also \[27\] \[28\]).
bosons.

Of the 256 particle states in Eq. (108), only 1/4 interact with our usual photons, whereas the remaining 3/4 do not interact with our photons, but they may interact with mirror photons, T-photons or PT-photons. This scheme thus predicts the existence of dark matter. If the matter in the universe were evenly distributed over the ordinary particles, P-particles, T-particles and PT-particles, then 1/4 of the matter would be visible, and 3/4 dark. In reality, the distribution of matter in the universe need not be even over the four different version of the particles. It can deviate from even distribution, but we expect that the deviation is not very big. According to the current astronomical observations about 81.7% of matter in the universe is dark, and only 18.3% is visible. This roughly corresponds to the ratio 1/4 of the “visible states” in matrix (108).

4.2 Quantized fields as generalized Clifford numbers

We can consider a field as an infinite dimensional vector. As an example, let us take

$$\Psi = \psi^i(x)h_i(x) \equiv \int d^n x \psi^i(x)h_i(x),$$

where \(i = 1, 2\), \(x \in \mathbb{R}^3\) or \(x \in \mathbb{R}^{1,3}\) are, respectively, a discrete index, and \(x\) a continuous index, denoting, e.g., a point in 3D space, or an event in 4D spacetime. The infinite dimensional vector \(\Psi\) is decomposed with respect to an infinite dimensional basis, consisting of vectors \(h_i(x) \equiv h_i(x)\), satisfying

$$h_i(x) \cdot h_j(x') \equiv \frac{1}{2}(h_i(x)h_j(x') + h_j(x')h_i(x)) = \rho_{ij}(x)(x'),$$

where \(\rho_{ij}(x)(x')\) is the metric of the infinite dimensional space \(S\). The latter space may in general have non-vanishing curvature [9]. If, in particular, the curvature of \(S\) is “flat”, then we may consider a parametrization of \(S\) such that

$$\rho_{ij}(x)(x') = \delta_{ij}\delta(x - x').$$

---

6 Here we extend the concept of mirror particles and mirror gauge fields. The idea of mirror particles was first put forward by Lee and Yang [35] who realized that “…there could exist corresponding elementary particles exhibiting opposite asymmetry such that in the broader sense there will still be over-all right-left symmetry.” Further they wrote: “If this is the case, it should be pointed out that there must exist two kinds of protons \(p_R\) and \(p_L\), the right-handed one and the left-handed one.” Lee and Yang thus considered the possibility of mirror particles, though they did not name them so, and as an example they considered ordinary and mirror protons. Later, Kobzarev et al. [36], instead of P-partners, considered CP-partners of ordinary particles and called them “mirror particles”. They argued that a complete doubling of the known particles and forces, except gravity, was necessary. Subsequently, the idea of mirror particles has been pursued in refs [37–42]. The connection between mirror particles and dark matter was suggested in Ref. [43], and later explored in many works, e.g., in [44–50]. An explanation of mirror particles in terms of algebraic spinors (elements of Clifford algebras) was exposed in Refs. [82, 83]. For a recent review see [51].
In Eq. (111), we have a generalization of the Clifford algebra relations \[ \text{(4)} \] to infinite dimensions.

Instead of the basis in which the basis vectors satisfy Eq. (111), we can introduce the Witt basis

\[ h_i(x) = \frac{1}{\sqrt{2}} (h_1(x) + i h_2(x)), \]

\[ \bar{h}_i(x) = \frac{1}{\sqrt{2}} (h_1(x) - i h_2(x)), \]

in which we have

\[ h_i(x) \cdot \bar{h}_{i'}(x') = \delta_{i(x)}(x'), \]

\[ h_i(x) \cdot h_{i'}(x') = 0, \quad \bar{h}_i(x) \cdot \bar{h}_{i'}(x') = 0. \]

The vector \( h_{i(x)} \) and the corresponding components \( \psi_{i(x)} \) may contain an implicit discrete index \( \mu = 0, 1, 2, ..., n \), so that Eq. (110) explicitly reads

\[ \Psi = \psi_{i\mu}(x) h_{i\mu}(x) = \psi_{\mu}(x) h_\mu(x) + \bar{\psi}_{\mu}(x) \bar{h}_\mu(x). \]

Then, Eqs. (115), (116) become the anticommuting relations for fermion fields:

\[ h_{\mu}(x) \cdot \bar{h}_{\nu}(x') = \eta_{\mu\nu} \delta_{(x)(x')}, \]

\[ h_{\mu}(x) \cdot h_{\nu}(x') = 0, \quad \bar{h}_{\mu}(x) \cdot \bar{h}_{\nu}(x') = 0. \]

The quantities \( h_{\mu}(x), \bar{h}_{\mu}(x) \) are a generalization to infinite dimensions of the Witt basis vectors \( \theta_\mu, \bar{\theta}_\mu \), defined in Eq. (45).

Using \( \bar{h}_\mu(x) \), we can define a vacuum state as the product \[ \Omega = \prod_{\mu,x} \bar{h}_\mu(x), \quad \bar{h}_\mu(x) \Omega = 0. \]

Then, using the definition (117) of a vector \( \Psi \), we have

\[ \Psi \Omega = \psi_{\mu}(x) h_\mu(x) \Omega. \]

Because \( \bar{h}_\mu(x) \Omega = 0 \), the second part of \( \Psi \) disappears in the above equation.

The infinite dimensional vector \( \Psi \), defined in Eq. (117), consists of two parts, \( \psi_{\mu}(x) h_\mu(x) \) and \( \bar{\psi}_{\mu}(x) \bar{h}_\mu(x) \), which both together span the phase space of a field theory.

The vector \( \psi_{\mu}(x) h_\mu(x) \) can be generalized to an element of an infinite dimensional Clifford algebra:

\[ \psi_0 \mathbb{1} + \psi_{\mu}(x) h_\mu(x) + \psi_{\mu}(x) \nu(x') h_\mu(x) h_\nu(x') + ... \]

Acting with the latter object on the vacuum \[ \Omega \] we obtain

\[ \Psi \Omega = (\psi_0 \mathbb{1} + \psi_{\mu}(x) h_\mu(x) + \psi_{\mu}(x) \nu(x') h_\mu(x) h_\nu(x') + ...) \Omega. \]
This state is the infinite dimensional space analog of the spinor as an element of a left ideal of a Clifford algebra. At a fixed point \( x \equiv x^\mu \) there is no “sum” (i.e., integral) over \( x \) in expression (122), and we obtain a spinor with \( 2^n \) components. It is an element of a minimal left ideal of \( Cl(2n) \). In 4D spacetime, \( n = 4 \), and we have \( Cl(8) \) at fixed \( x \).

Besides the vacuum (120) there are other vacuums, such as

\[
\Omega = \prod_{\mu,x} h_{\mu(x)} , \quad h_{\mu(x)} \Omega = 0, \tag{124}
\]

and, in general,

\[
\Omega = \left( \prod_{\mu \in R_1, x} \bar{h}_{\mu(x)} \right) \left( \prod_{\mu \in R_2, x} h_{\mu(x)} \right). \tag{125}
\]

Here \( R = R_1 \cup R_2 \) is the set of indices \( \mu = 0, 1, 2, ..., n \), and \( R_1, R_2 \) are subsets of indices, e.g., \( R_1 = \{1, 3, 5, ..., n\} \), \( R_2 = \{2, 4, ..., n-1\} \).

Expression (125) can be written as

\[
\Omega = \prod_x \left( \prod_{\mu \in R_1} \bar{h}_{\mu(x)} \right) \left( \prod_{\mu \in R_2} h_{\mu(x)} \right) = \prod_x \Omega_{(x)}, \tag{126}
\]

where

\[
\Omega_{(x)} = \left( \prod_{\mu \in R_1} \bar{h}_{\mu(x)} \right) \left( \prod_{\mu \in R_2} h_{\mu(x)} \right), \tag{127}
\]

is a vacuum at a fixed point \( x \). At a fixed \( x \), we have \( 2^n \) different vacuums, and thus \( 2^n \) different spinors, defined analogously to the spinor (123), belonging to different minimal ideal of \( Cl(2n) \).

The vacuum (125) can be even further generalized by taking different domains \( R_1, R_2 \) of spacetime positions \( x \):

\[
\Omega = \left( \prod_{\mu \in R_1, x \in R_1} \bar{h}_{\mu(x)} \right) \left( \prod_{\mu \in R_2, x \in R_2} h_{\mu(x)} \right), \tag{128}
\]

In such a way we obtain many other vacuums, depending on a partition of \( \mathbb{R}^n \) into two domains \( R_1 \) and \( R_2 \) so that \( \mathbb{R}^n = R_1 \cup R_2 \).

Instead of the configuration space, we can take the momentum space, and consider, e.g., positive and negative momenta. In Minkowski spacetime we can have a vacuum of the form

\[
\Omega = \left( \prod_{\mu, p^0 > 0, \mathbf{p}} \bar{h}_{\mu(p^0, \mathbf{p})} \right) \left( \prod_{\mu, p^0 < 0, \mathbf{p}} h_{\mu(p^0, \mathbf{p})} \right), \tag{129}
\]

24
which is annihilated according to
\[ \bar{h}_\mu(p^0 > 0, \mathbf{p}) \Omega = 0, \quad h_\mu(p^0 < 0, \mathbf{p}) \Omega = 0. \] (130)

For the vacuum (129), \( \bar{h}_\mu(p^0 > 0, \mathbf{p}) \) and \( h_\mu(p^0 < 0, \mathbf{p}) \) are annihilation operators, whereas \( \bar{h}_\mu(p^0 < 0, \mathbf{p}) \) and \( h_\mu(p^0 > 0, \mathbf{p}) \) are creation operators from which one can compose the states such as
\[ \left( \psi_0 \mathbb{1} + \psi_\mu(p^0 > 0, \mathbf{p}) \bar{h}_\mu(p^0 > 0, \mathbf{p}) + \psi_\mu(p^0 > 0, \mathbf{p}) \psi_\nu(p^0 > 0, \mathbf{p}') h_\mu(p^0 > 0, \mathbf{p}) h_\nu(p^0 > 0, \mathbf{p}') + \ldots \right) \Omega. \] (131)

The vacuum, satisfying (130), has the property of the bare Dirac vacuum. This can be seen if one changes the notation according to
\[ \bar{h}_\mu(p^0 > 0, \mathbf{p}) \equiv b_\mu^\dagger(\mathbf{p}), \quad h_\mu(p^0 > 0, \mathbf{p}) \equiv b_\mu(\mathbf{p}) \] (132)
\[ \bar{h}_\mu(p^0 < 0, \mathbf{p}) \equiv d_\mu(\mathbf{p}), \quad h_\mu(p^0 < 0, \mathbf{p}) \equiv d_\mu^\dagger(\mathbf{p}) \] (133)
\[ \Omega \equiv |0\rangle_{\text{bare}}. \] (134)

A difference with the usual Dirac theory is that our operators have index \( \mu \) which takes four values, and not only two values, but otherwise the principle is the same.

The operators \( b_\mu^\dagger \) and \( b_\mu \), respectively, create and annihilate a positive energy fermion, whereas the operators \( d_\mu, d_\mu^\dagger \) create and annihilate a negative energy fermion. This is precisely a property of the bare Dirac vacuum. Instead of the bare vacuum, in quantum field theories we consider the the physical vacuum
\[ |0\rangle = \prod_{\mu, \mathbf{p}} d_\mu(\mathbf{p}) |0\rangle_{\text{bare}}, \] (135)
in which the negative energy states are filled, and which in our notation reads
\[ \Omega_{\text{phys}} = \prod_{\mu, p^0 < 0, \mathbf{p}} \bar{h}_\mu(p^0, \mathbf{p}) \Omega. \] (136)

We see that in a field theory à la Clifford, a vacuum is defined as the product of fermionic operators (generators in the Witt basis). The Dirac (physical) vacuum is defined as a sea of negative energy states according to (135) or (136). Today it is often stated that the Dirac vacuum as the sea of negative energy states is an obsolete concept. But within a field theory based infinite dimensional Clifford algebras, a vacuum is in fact a “sea” of states defined by infinite (uncountable) product of operators.

With respect to the vacuum (129), one kind of particles are created by the positive energy operators \( h_\mu(p^0 > 0, \mathbf{p}) \), whilst the other kind of particles are created by the
negative energy operator, $\bar{h}_\mu(p^0 < 0, p)$. The vacuum with reversed properties can also be defined, besides many other possible vacuums. All those vacuums participate in a description of the interactive processes of elementary particles. What we take into account in our current quantum field theory calculations seem to be only a part of a larger theory that has been neglected. It could be that some of the difficulties (e.g., infinities) that we have encountered in QFTs so far, are partly due to neglect of such a larger theory.

In an analogous way we can also construct [14] bosonic states as elements of an infinite dimensional symplectic Clifford algebra. The generators of the latter algebra are bosonic field operators. We will use them in the next subsection when constructing the action and field equations.

### 4.3 The action and field equations

A sympletic vector is [14]

$$\Phi = \phi^i(x) k_i(x) = \phi^3(x) k_1(x) + \phi^2(x) k_2(x)$$

$$\equiv \phi^r(x) k_\rho(x) + \Pi(x) k_\Pi(x),$$

$$x \in \mathbb{R}^3 \text{ or } x \in \mathbb{R}^{1,3}. \quad (137)$$

Here $\phi^i(x) = (\phi^r(x), \Pi(x))$ are components and $k_i(x)$, $i = 1, 2$, basis vectors, satisfying

$$k_i(x) \wedge k_j(x') = \frac{1}{2} [k_i(x), k_j(x')] = J_{i(x)j(x')}, \quad (138)$$

where

$$J_{i(x)j(x')} = \begin{pmatrix} 0 & \delta_{i(x)}(x') \\ -\delta_{j(x')}(x) & 0 \end{pmatrix} \quad (139)$$

The action is

$$I = \int d\tau \left[ \frac{1}{2} \dot{\phi}^i(x) J_{i(x)j(x')} \phi^j(x') - H \right], \quad (140)$$

where

$$H = \frac{1}{2} \phi^i(x) K_{i(x)j(x')} \phi^j(x') \quad (141)$$

is the Hamiltonian, and

$$\frac{1}{2} \phi^i(x) J_{i(x)j(x')} \phi^j(x') = \frac{1}{2} (\Pi \dot{\phi} - \phi \Pi) \quad (142)$$

the symplectic form.

In particular [14], if $x \equiv x' \in \mathbb{R}^3$, $r = 1, 2, 3$, and

$$K_{i(x)j(x')} = \begin{pmatrix} (m^2 + \partial^r \partial_r) \delta(x - x') & 0 \\ 0 & \delta(x - x') \end{pmatrix}, \quad (143)$$

In particular [14], if $x \equiv x' \in \mathbb{R}^3$, $r = 1, 2, 3$, and

$$K_{i(x)j(x')} = \begin{pmatrix} (m^2 + \partial^r \partial_r) \delta(x - x') & 0 \\ 0 & \delta(x - x') \end{pmatrix}, \quad (143)$$
then we obtain the phase space action for a classical scalar field.

If \( x \equiv x^r, \ r = 1, 2, 3, \) and

\[
K_{i(x)j(x')} = \left( -\frac{1}{2m}\partial^r\partial_r + V(x) \right) \delta(x - x') g_{ij}, \quad g_{ij} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]

(144)

then the action \( (140) \) describes the classical Schrödinger field.

If \( x \equiv x^\mu \in \mathbb{R}^{1,3}, \ \mu = 0, 1, 2, 3, \) and

\[
K_{i(x)j(x')} = \left( -\frac{1}{2\Lambda}\partial^\mu\partial_\mu \right) \delta(x - x') g_{ij}, \quad g_{ij} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]

(145)

then from \( (140) \) we obtain the action for the classical Stueckelberg field.

From the action \( (140) \) we obtain the following equations of motion

\[
\dot{\phi}^{i(x)} = J^{i(x)j(x')} \frac{\partial H}{\partial \phi^{j(x')}},
\]

(146)

where \( \partial/\partial \phi^{i(x')} \equiv \delta/\delta \phi^{i(x')} \) is the functional derivative. By following the analogous procedure as in Sec. 3.2, we obtain \( (147) \) the equations of motion for the operators:

\[
\dot{k}_{i(x')} = k^{i(x)} K_{i(x)j(x')} = [k_{j(x')}, \hat{H}],
\]

(147)

where

\[
\hat{H} = \frac{1}{2} k^{i(x)} K_{i(x)j(x')} k^{j(x')}.
\]

(148)

The Heisenberg equations of motion \( (147) \) can be derived from the action

\[
I = \frac{1}{2} \int \text{d}\tau \left( \dot{k}^{i(x)} J^{i(x)j(x')} k^{j(x')} + k^{i(x)} K_{i(x)j(x')} k^{j(x')} \right).
\]

(149)

The Poisson bracket between two functionals of the classical phase space fields is

\[
\{ f(\phi^{i(x')}), g(\phi^{j(x')}) \}_PB = \frac{\partial f}{\partial \phi^{i(x')}} J^{i(x)j(x')} \frac{\partial g}{\partial \phi^{j(x')}}.
\]

(150)

In particular, if \( f = \phi^{k(x')}, \ g = \phi^{\ell(x')}, \) Eq. \( (150) \) gives \( (14) \)

\[
\{ \phi^{k(x')}, \phi^{\ell(x')} \}_PB = J^{k(x')\ell(x')} = k^{k(x')} \wedge k^{\ell(x')} \equiv \frac{1}{2} [k^{k(x')}, k^{\ell(x')}].
\]

(151)

On the one hand, the Poisson bracket of two classical fields is equal to the symplectic metric. On the other hand, the symplectic metric is equal to the wedge product of basis vectors. In fact, the basis vectors are quantum mechanical operators, and they satisfy the quantum mechanical commutation relations

\[
\frac{1}{2} [k_{\phi}(x), k_{\Pi}(x')] = \delta(x - x'),
\]

(152)

or

\[
[\hat{\phi}(x), \hat{\Pi}(x')] = i\delta(x - x'),
\]

(153)

if we identify \( \frac{1}{\sqrt{2}} k_{\phi}(x) \equiv \hat{\phi}(x), \ \frac{1}{\sqrt{2}} k_{\Pi}(x') \equiv \hat{\Pi}(x') \).

A similar procedure can be repeated for fermionic vectors \( (14) \).
5 Towards quantum gravity

5.1 Gravitational field from Clifford algebra

The generators of a Clifford algebra, $\gamma_\mu$, $\bar{\gamma}_\mu$, are (i) tangent vectors to a manifold which, in particular, can be spacetime. On the other hand, (ii) the $\gamma_\mu$, $\bar{\gamma}_\mu$ are superpositions of fermionic creation and annihilation operators, as shown in Eqs. (47), (48). This two facts, (i) and (ii), must have profound and far reaching consequences for quantum gravity. Here I am going to expose some further ingredients that in the future, after having been fully investigated, will illuminate the relation between quantum theory and gravity.

As a first step let us consider a generalized spinor field, defined in Sec. 4.1:

$$\Psi = \psi^A \gamma_A. \quad (154)$$

We are interested in the expectation value of a a vector $\gamma_\mu$ with respect to the state $\Psi$:

$$\langle \gamma_\mu \rangle_1 \equiv \langle \Psi^\dagger \gamma_\mu \Psi \rangle_1 = \langle \psi^A \gamma_A \gamma_\mu \psi^B \rangle_1 \quad (155)$$

The subscript 1 means vector part of the expression. Recall from Sec. 2 that $\dagger$ means reversion. Taking

$$\langle s^A_\gamma \gamma_\mu s^B_\gamma \rangle_1 = C^{cA}_{AB} \gamma_c, \quad (156)$$

we have

$$\langle \gamma_\mu \rangle_1 = e_\mu^c \gamma_c, \quad (157)$$

where

$$e_\mu^c = \psi^A C^{cA}_{AB} \psi^B \quad (158)$$

is the fierbein. The vector $\gamma_\mu$ gives the flat spacetime metric

$$\gamma_\mu \cdot \gamma_\nu = \eta_{\mu\nu}. \quad (159)$$

The expectation value vector $\langle \gamma_\mu \rangle_1$ gives a curved spacetime metric

$$g_{\mu\nu} = \langle \gamma_\mu \rangle_1 \cdot \langle \gamma_\nu \rangle_1 = e_\mu^c e_\nu^d \eta_{cd} \quad (160)$$

which, in general, differs from $\eta_{\mu\nu}$. If $\psi^A$ depends on position $x \equiv x^\mu$ in spacetime, then also $e_\mu^c$ depends on $x$, and so does $g_{\mu\nu}$.

From Eq. (156) we obtain

$$e_\mu^a = \langle \gamma_\mu \rangle \cdot \gamma^a. \quad (161)$$

From the fierbein we can calculate the spin connection,

$$\omega^a_{\mu\nu} = \frac{1}{2} (e^p_\rho e_{[\mu,\rho]}^a - e^p_\mu e_{[\nu,\rho]}^a + e^p_\rho e_{[\sigma,\rho]}^a e_{\mu,\sigma}). \quad (162)$$
The curvature is
\[ R_{\mu \nu}^{ab} = \partial_\mu \omega_\nu^{ab} - \partial_\nu \omega_\mu^{ab} + \omega_\mu^{ac} \omega_\nu^{cb} - \omega_\nu^{ac} \omega_\mu^{cb} \] (163)

In order to see whether the curvature vanishes or not, let us calculate \( \omega^{[\mu, \nu]}_{ab} \) by using (158) in which we write \( \tilde{\psi} \psi^* \equiv \tilde{\psi} A \tilde{\psi} B \equiv \psi A \psi B \).

We obtain
\[ \omega^{[\mu, \nu]}_{ab} = \frac{1}{2} \left[ C_{\tilde{A}B}^{b \tilde{C}D \mu} (\psi^{\tilde{A}B} \psi^\tilde{C}D, \rho)_{\nu} - C_{\tilde{A}B}^{b \tilde{C}D \nu} (\psi^{\tilde{A}B} \psi^\tilde{C}D, \rho)_{\mu} \right. \\
- C_{\tilde{A}B}^{a \tilde{C}D \mu} (\psi^{\tilde{A}B} \psi^\tilde{C}D, \rho)_{\nu} + C_{\tilde{A}B}^{a \tilde{C}D \nu} (\psi^{\tilde{A}B} \psi^\tilde{C}D, \rho)_{\mu} \\
+ \text{more terms} - (\mu \rightarrow \nu, \nu \rightarrow \mu) \]. (165)

The latter expression does not vanish identically. In general it could be different from zero, which would mean that also the curvature (163) is different from zero, and that the generalized spinor field \( \psi^{\tilde{A}}(x) \) induces gravitation. This assertion should be checked by explicit calculations with explicit structure constants \( C_{\tilde{A}B}^{b \tilde{C}D \mu} \) and/or their symmetry relations.

If \( \psi^{\tilde{A}}(x) \) indeed induces gravitation, then we have essentially arrived at the basis of quantum gravity. At the basic level, gravity is thus caused by a spacetime dependent (generalized) spinor field \( \psi^{\tilde{A}}(x) \) entering the expression (158) for vierbein. If \( \psi^{\tilde{A}}(x) \) is constant, or proportional to \( e^{ip_\mu x^\mu} \) which, roughly speaking, means that there is no non trivial matter, then \( R_{\mu \nu}^{ab} = 0 \). This has its counterpart in the (classical) Einstein’s equation which say that matter curves spacetime.

### 5.2 Action principle for the Clifford algebra valued field

Let us assume that the field (154) satisfies the action principle\(^7\)

\[ I = \frac{1}{2} \int d^4 x \partial_\mu \phi^A \partial_\nu \phi^B \eta_{AB} \eta^{\mu \nu} \] (166)

for a system of scalar fields \( \phi^A \) that may contain an implicit index \( i = 1, 2 \), denoting real and imaginary components. Here \( \eta_{AB} \) is the metric of the 16D Clifford space, whereas \( \eta_{\mu \nu} \) is the metric of the 4D Minkowski space. The action (166) is not invariant under reparametrizations of coordinates \( x^\mu \) (i.e., of general coordinate transformations). A possible way to make the action invariant is to replace \( \eta^{\mu \nu} \) with \( g^{\mu \nu} \), and include a kinetic term for \( g^{\mu \nu} \). Another possible way is to consider the action

\[ I = \int d^4 x \det(\partial_\mu \phi^A \partial_\nu \phi^B \eta_{AB})^{1/2}. \] (167)

\(^7\)If reduced to a subspace of the Clifford space, this action contains a mass term.
This is an action for a 4D surface $V_4$, embedded in the 16D space, the embedding functions being $\phi^A(x^\mu)$. The induced metric on $V_4$ is

$$g_{\mu\nu} = \partial_\mu \phi^A \partial_\nu \phi^B \eta_{AB} \quad (168)$$

The theory based on the nonlinear action (167) is complicated and difficult to quantize. Therefore we will return to the action (166) and try to explore how far can we arrive in inducing non trivial spacetime metric according to the lines indicated in Sec. 5.1. The equations of motion derived from (166) are

$$\eta^{\mu\nu} \partial_\mu \partial_\nu \phi^A = 0. \quad (169)$$

A field $\phi^A$ that satisfied the latter equation satisfies also the Dirac like equation

$$\gamma^\mu \partial_\mu \phi^A = 0, \quad \gamma^\mu \cdot \gamma^\nu = \eta^{\mu\nu}. \quad (170)$$

This is so because of the relation (154) and the fact that $\psi^{\tilde{A}}$ are spinor components belonging to all left minimal ideals of the considered Clifford algebra. Eq. (170) can be contracted by $\gamma_A$, and we obtain the Dirac-Kähler equation

$$\gamma^\mu \partial_\mu \phi^A \gamma_A = 0, \quad (171)$$

where

$$\gamma_A = (1, \gamma_{a_1}, \gamma_{a_1} \wedge \gamma_{a_2}, \ldots, \gamma_{a_1} \wedge \ldots \wedge \gamma_{a_4}). \quad (172)$$

In (171) we have a geometric form of the equation. We can put it in a sandwich between $\gamma^B$ and $\gamma_A$, or equivalently, between $s^B$ and $s^\tilde{A}$, according to

$$\langle \gamma^B \gamma^\mu \gamma_A \rangle_s \partial_\mu \phi^A = 0, \quad \text{or} \quad \langle s^B \gamma^\mu s^\tilde{A} \rangle_s \partial_\mu \phi^{\tilde{A}} = 0. \quad (173)$$

Here "$S$" denotes scalar part, $\langle \rangle_0$ multiplied by the dimension of the spinor space. Here $\langle \gamma^B \gamma^\mu \gamma_A \rangle_s \equiv (\gamma_\mu)^B_A$ and $\langle s^B \gamma^\mu s^\tilde{A} \rangle_s \equiv (\gamma_\mu)^{\tilde{B}}_{\tilde{A}}$ are $16 \times 16$ matrices, representing the vectors $\gamma_\mu$. Those matrices are reducible to four $4 \times 4$ blocks

$$\langle s^B \gamma^\mu s^\tilde{A} \rangle_s \equiv (\gamma_\mu)^\beta_\alpha, \quad \alpha, \beta = 1, 2, 3, 4 \quad \text{spinor index}, \quad (174)$$

which are just the (usual) Dirac matrices.

Eq. (173) can be derived from the action

$$I = \int d^4x \langle \phi^A \gamma_A \gamma^\mu \gamma_B \partial_\mu \phi^B \rangle_s, \quad (175)$$

which can also be written in terms of the generalized spinors $\psi^{\tilde{A}} s^\tilde{A}$:

$$I = \int d^4x \langle \psi^{\tilde{A}} s^\tilde{A} \gamma^\mu \gamma_B \partial_\mu \psi^B \rangle_s. \quad (176)$$
The action \((175)\) or \((176)\) is not invariant under general coordinate transformations of \(x^\mu\). For this aim one has to consider position dependent Clifford numbers, giving the connection according to \([28]\)

\[
\partial_\mu \gamma_A = \Gamma_\mu^B A \gamma_B, \quad \partial_\mu s_A = \Gamma_\mu^B A s_B
\] (177)

from which we find\([28]\) that \(\partial_\mu \psi^B\) and \(\partial_\mu \tilde{\psi}^\tilde{B}\) must be replaced with the covariant derivatives

\[
D_\mu \phi^B = \partial_\mu \phi^B + \Gamma_\mu^{BC} \phi_C \quad \text{and} \quad D_\mu \phi^{\tilde{B}} = \partial_\mu \phi^{\tilde{B}} + \Gamma_\mu^{\tilde{B}\tilde{C}} \phi^{\tilde{C}}.
\] (178)

Then, in particular, the position dependent \(\gamma^\mu\) gives curved metric according to

\[
\gamma^\mu(x) \cdot \gamma^\nu(x) = g_{\mu\nu}.
\]

In addition, one also needs to include a kinetic term for \(g_{\mu\nu}\) or the connection \(\Gamma_\mu^{BC}\) (or for \(\Gamma_\mu^{\tilde{B}\tilde{C}}\)).

Alternatively, one can find a solution \(\phi^A\) (or, equivalently \(\tilde{\psi}^{\tilde{A}}\)) of the flat space equation \((171)\), with \(\gamma^\mu \cdot \gamma^\nu = \eta_{\mu\nu}\), and calculate the expectation value \(\langle \gamma^\mu \rangle\) according to Eq. \((155)\), and then obtain the metric

\[
g_{\mu\nu} = \langle \gamma_\mu \rangle \cdot \langle \gamma_\nu \rangle
\] (179)

of a curved spacetime, induced by the fields \(\phi^A\). No kinetic term for the field \(g_{\mu\nu}(x)\) or the corresponding connection is necessary in such a procedure. A curved spacetime metric comes directly from the fields \(\phi^A\) (or \(\tilde{\psi}^{\tilde{A}}\)) which are solutions of the flat space equation \((171)\).

In both procedures the metric is given in terms of the fields \(\phi^A\) (or \(\tilde{\psi}^{\tilde{A}}\)). Equating the metrics \((168)\) and \((179)\), we have

\[
\partial_\mu \phi^A \partial_\nu \phi^B \eta_{AB} = \phi^A C_{AB\mu} \phi^B \phi^C C_{CD\nu} \phi^D \gamma_a \gamma_b.
\] (180)

Here we have used Eqs. \((155)-(158)\) in which we replaced \(\tilde{\psi}^{\tilde{A}}\) with \(\phi^A\), as suggested by \((154)\). Eq. \((180)\) is a condition that the fields \(\phi^A\) must satisfy. Such a condition can be satisfied if we start from the action

\[
I = \int d^4x \left(\frac{1}{2} \partial_\mu \phi^A \partial_\nu \phi^B \eta_{AB} \eta^{\mu\nu} - \frac{1}{3!} \lambda_{ABCD} \phi^A \phi^B \phi^C \phi^D\right)
\] (181)

with a quartic self-interaction term. The equations of motion are then

\[
\partial_\mu \partial^\mu \phi_A + \frac{1}{3!} \lambda_{ABCD} \phi^B \phi^C \phi^D = 0,
\] (182)

from which we obtain

\[
\int d^4x \left(\partial_\mu \phi^A \partial^\mu \phi_A - \frac{1}{3!} \lambda_{ABCD} \phi^B \phi^C \phi^D\right).
\]
\[
= - \int d^4x \left( \phi^A \partial_\mu \partial^\mu \phi_A + \frac{1}{3!} \lambda_{ABCD} \phi^B \phi^C \phi^D \right) = 0. \tag{183}
\]

The latter equation also comes from (180) after contracting with \( \eta^{\mu\nu} \) and integrating over \( x \), provided that we identify
\[
\frac{1}{3!} \lambda_{ABCD} = C^a_{AB\mu} C^b_{CD\nu} \eta^{\mu\nu} \eta_{ab}, \tag{184}
\]
where \( \eta_{ab} = \gamma_a \cdot \gamma_b \).

In the action (181) we have yet another possible generalization of the non-interacting action (166) (the other generalization was the “minimal surface” action (167)). We have thus arrived at a fascinating result that the spacetime metric \( g_{\mu\nu} \) can be induced by Clifford algebra valued field \( \phi^A \gamma_A \) that satisfies the quartic action principle (181).

5.3 Fermion creation operators, branes as vacuums, branes with holes, and induced gravity

The procedure described in Sec. 5.1 can be considered as a special case of quantized fields (123) at a fixed spacetime point \( x \). We will now start from a generic object of the form (123). It consists of the terms such as
\[
\psi^{\mu_1(x_1)\mu_2(x_2)\ldots\mu_r(x_r)} h_{\mu_1(x_1)} h_{\mu_2(x_2)} \ldots h_{\mu_r(x_r)} \Omega, \tag{185}
\]
where we assume that \( \Omega \) is the vacuum given by Eq. (120). The operator \( h_{\mu_i(x_i)} \) creates a fermion at a point \( x_i \). The product of operators \( h_{\mu_i(x_1)} h_{\mu_j(x_j)} \) creates a fermion at \( x_i \) and another fermion at \( x_j \). By a generic expression (185) we can form any structure of fermions, e.g., a spin network. In the limit in which there are infinitely many densely packed fermions, we obtain arbitrary extended objects, such as strings, membranes, \( p \)-branes, or even more general objects, including instantonic branes, considered in Sec. 2.

Let us use the following compact notation for a state of many fermions forming an extended object in spacetime:
\[
\left( \prod_{\mu, x \in \mathcal{R}} h_{\mu(x)} \right) \Omega. \tag{186}
\]

Here the product runs over spacetime points \( x \in \mathcal{R} \) of a region \( \mathcal{R} \) of spacetime \( M_D \). In particular, \( \mathcal{R} \) can be a \( p \)-brane’s world sheet \( V_{p+1} \), whose parametric equation is \( x^\mu = X^\mu(\sigma^a) \), \( \mu = 0, 1, 2, \ldots, D - 1 \), \( a = 1, 2, \ldots, p + 1 \), or it can be a brane-like instantonic object, also described by some functions \( X^\mu(\sigma) \). Then the product of operators in Eq. (186) can be written in the form
\[
\prod_{\mu, x = X(\sigma)} h_{\mu(x)} \equiv h[X^\mu(\sigma)], \tag{187}
\]
where \( h[X^\mu(\sigma)] \) is the operator that creates a brane or an instantonic brane (that we will also call “brane”). Here a brane is an extended objects consisting of infinitely many fermions, created according to

\[
\psi_{\text{brane}} = h[X^\mu(\sigma)]\Omega = \left( \prod_{\mu,x=X(\sigma)} h_\mu(x) \right) \Omega \tag{188}
\]

To make contact with the usual notation, we identify

\[
\Omega \equiv |0\rangle, \quad h[X^\mu(\sigma)] \equiv b^\dagger[X^\mu(\sigma)], \quad \psi_{\text{brane}} \equiv |X^\mu(\sigma)\rangle, \tag{189}
\]

and write

\[
|X^\mu(\sigma)\rangle = b^\dagger[X^\mu(\sigma)]|0\rangle. \tag{190}
\]

A generic single brane state is a superposition of the brane states:

\[
|\Psi\rangle = \int |X^\mu(\sigma)\rangle \mathcal{D}X(\sigma) \langle X^\mu(\sigma)|\Psi\rangle. \tag{191}
\]

In the notation of Eqs. (185)–(188), the latter expression reads

\[
\Psi = \int \mathcal{D}X(\sigma) \psi[X^\mu(\sigma)] h[X^\mu(\sigma)]\Omega, \tag{192}
\]

where

\[
\psi[X^\mu(\sigma)] = \lim_{r \to \infty, \Delta x_i \to 0} \psi^{\mu_1(x_1) \ldots \mu_r(x_r)}. \tag{193}
\]

However, besides single brane states, there are also two-brane, three-brane, and in general, many-brane states. The brane Fock-space states are thus

\[
b^\dagger[X_1^\mu(\sigma)]|0\rangle, \quad b^\dagger[X_1^\mu(\sigma)]b^\dagger[X_2^\mu(\sigma)]|0\rangle, \quad b^\dagger[X_1^\mu(\sigma)] \ldots b^\dagger[X_r^\mu(\sigma)]|0\rangle, \ldots \tag{194}
\]

A generic brane state is a superposition of those states.

If we act on the brane state (188) with the operator \( \bar{h}_{\mu'(x')} \), we have

\[
\bar{h}_{\mu'(x')} \Psi_{\text{brane}} = \bar{h}_{\mu'(x')} \left( \prod_{\mu,x=X(\sigma)} h_\mu(x) \right) \Omega. \tag{195}
\]

If \( x' \) is outside the brane, then nothing happens. But is \( x' \) is a position on the brane, then (195) is a state in which the particle at \( x' \) with the spin orientation \( \mu \) is missing. In other words, (195) is a brane state with a hole at \( x' \).

We may also form two hole state, many-hole states, and the states with a continuous set of holes,

\[
\left( \prod_{\mu,x \in \mathcal{R}_1} \bar{h}_\mu(x) \right) \Psi_{\text{brane}} = \left( \prod_{\mu,x \in \mathcal{R}_1} \bar{h}_\mu(x) \right) \left( \prod_{\mu,x=X(\sigma)} h_\mu(x) \right) \Omega, \tag{196}
\]
where $\mathcal{R}_1 \subset \mathcal{R} = \{X^\mu(\sigma)\}$. For instance, $\mathcal{R}_1$ can be a string or a brane of a lower dimensionality than the brane $X^\mu(\sigma^a)$.

If the space into which the brane is embedded has many dimensions, e.g., $D = 10 > p + 1$, then the brane’s worldsheet $V_{p+1}$ can represent our spacetime,\footnote{For more details on how an instantonic brane is related to our evolving spacetime, see Refs. [9, 52].} which, if $p + 1 > 4$, has extra dimensions. The induced metric on $V_{p+1}$ can be curved, and so we have curved spacetime. We have thus arrived at the brane world scenario. Holes in the brane are particles. More precisely, the point like holes in the worldsheet $V_{p+1}$ are instantonic point particles, whereas the string like holes are instantonic strings, which can be either space like or time like (see Refs. [9, 52]).

Let me now outline how the induced metric on a brane $V_{p+1}$ could be formally derived in terms of the operators $h_{\mu(x)}, \bar{h}_{\mu(x)}$. The corresponding operators in orthogonal basis are (see (113), (114)),

\[
\begin{align*}
h_{1\mu(x)} &= \frac{1}{\sqrt{2}}(h_{\mu(x)} + \bar{h}_{\mu(x)}), \\
h_{2\mu(x)} &= \frac{1}{i\sqrt{2}}(h_{\mu(x)} - \bar{h}_{\mu(x)}),
\end{align*}
\]

satisfy the Clifford algebra relations

\[
h_{i\mu(x)} \cdot h_{j\nu(x')} = \delta_{ij} \eta_{\mu\nu} \delta(x - x').
\]

In particular,

\[
h_{1\mu(x)} \cdot h_{1\nu(x)} = \eta_{\mu\nu} \delta(0).
\]

Comparing the latter result with

\[
\gamma_\mu \cdot \gamma_\nu = \eta_{\mu\nu},
\]

we find that\footnote{Such notation could be set into a rigorous form if, e.g., in Eq. (199) we replace $\delta(x - x')$ with $\frac{1}{\sqrt{\pi}} \exp[-(x - x')^2]$ and $\delta(0)$ with “$\delta(0)$”. Then Eq. (200) is replaced by $h_{1\mu(x)} \cdot h_{1\nu(x)} = \eta_{\mu\nu}$ “$\delta(0)$”. By inserting into the latter equation the relation $h_{1\mu(x)} = \gamma_\mu \sqrt{\delta(0)}$, we obtain $\gamma_\mu \cdot \gamma_\nu = \eta_{\mu\nu}$, which also holds in the limit $a \to 0$, because “$\delta(0)$” has disappeared from the equation.}

\[
h_{1\mu(x)} = \gamma_\mu \sqrt{\delta(0)}.
\]

This means that up to an infinite constant, $h_{1\mu(x)}$ is proportional to $\gamma_\mu$, a basis vector of Minkowski spacetime. Thus, a proper renormalization of $h_{1\mu(x)}$ gives $\gamma_\mu$.

In a given quantum state $\Psi$ we can calculate the expectation value of $h_{i\mu(x)}$ according to

\[
\langle h_{i\mu(x)} \rangle = \langle \Psi | h_{i\mu(x)} | \Psi \rangle_1,
\]

\[\text{For more details on how an instantonic brane is related to our evolving spacetime, see Refs. [9, 52].}\]
where the subscript 1 means vector part of the expression in the bracket. The inner product gives the expectation value of the metric:

$$\langle \rho_{ij(x) j'j'(x')} \rangle = \langle h_{i}(x) \rangle \cdot \langle h_{j'}(x') \rangle. \quad (204)$$

This is the metric of an infinite dimensional manifold that, in general, is curved. In Refs. [9], a special case of such a manifold, for $i = j = 1$, called membrane space $\mathcal{M}$, was considered. It was shown how to define connection and curvature of $\mathcal{M}$.

Taking $i = j = 1$ and $x = x'$ in Eq. (204), we have

$$\langle \rho_{1i}(x) \rangle \cdot \langle \rho_{1j}(x) \rangle. \quad (205)$$

Upon renormalization according to (202) (see Footnote 8), we obtain

$$\langle g_{\mu\nu}(x) \rangle = \frac{1}{\sqrt{\delta(0)}} \langle \gamma_{\mu}(x) \cdot \gamma_{\nu}(x) \rangle. \quad (206)$$

where

$$\langle g_{\mu\nu}(x) \rangle = \langle \rho_{1i}(x) \rangle \cdot \langle \rho_{1j}(x) \rangle \cdot \frac{1}{\sqrt{\delta(0)}} \quad (207)$$

is a position dependent metric of spacetime. We expect that the corresponding Riemann tensor is in general different from zero.

As an example let us consider the expectation value of a basis vector $h_{1\mu}(x)$ in the brane state (188):

$$\langle h_{1\mu}(x) \rangle = \langle \Psi_{\text{brane}}^\dagger h_{1\mu}(x) \Psi_{\text{brane}} \rangle_1 = \langle \Psi_{\text{brane}}^\dagger \frac{1}{\sqrt{2}}(h_{\mu}(x) + \bar{h}_{\mu}(x)) \Psi_{\text{brane}} \rangle_1. \quad (208)$$

From Eq. (195) in which the vacuum $\Omega$ is defined according to (120), we have

$$\bar{h}_{\mu}(x) \Psi_{\text{brane}} = \begin{cases} 
\Psi_{\text{brane}}(\tilde{x}), & x \in \text{brane}; \\
0, & x \notin \text{brane}.
\end{cases} \quad (209)$$

Here $\Psi_{\text{brane}}(\tilde{x})$, with the accent “〜” on $x$, denotes the brane with a hole at $x$. The notation $x \in \text{brane}$ means that $x$ is on the brane, whereas $x \notin \text{brane}$ means that $x$ is outside the brane created according to (188).

Because $(\bar{h}_{\mu}(x) \Psi_{\text{brane}})^\dagger = \Psi_{\text{brane}}^\dagger h_{\mu}(x)$, we also have

$$\Psi_{\text{brane}}^\dagger h_{\mu}(x) = \begin{cases} 
\Psi_{\text{brane}}^\dagger(\tilde{x}), & x \in \text{brane}; \\
0, & x \notin \text{brane}.
\end{cases} \quad (210)$$

For the expectation value of $h_{1\mu}(x)$ we then obtain

$$\langle h_{1\mu}(x) \rangle = \begin{cases} 
\frac{1}{\sqrt{2}} \langle \Psi_{\text{brane}}^\dagger(\tilde{x}) \Psi_{\text{brane}} \rangle_1 + \frac{1}{\sqrt{2}} \langle \Psi_{\text{brane}}^\dagger \Psi_{\text{brane}}(\tilde{x}) \rangle_1, & x \in \text{brane}; \\
0, & x \notin \text{brane}.
\end{cases} \quad (211)$$
A similar expression we obtain for \( \langle h_{2\mu(x)} \rangle \). The expectation value of the metric \( \langle h_{2\mu(x)} \rangle \) is
\[
\langle h_{2\mu(x)} \rangle = \left\{ \begin{array}{ll}
\langle \rho_{ij}(x) \rangle |_{\text{brane}}, & \text{on the brane;} \\
0, & \text{outside the brane.}
\end{array} \right.
\]
(212)

An interesting result is that outside the brane the expectation value of the metric is zero. Outside the brane, there is just the vacuum \( \Omega \). The expectation value of a vector \( h_{ij}(x) \) in the vacuum, given by (120), is zero, and so is the expectation value \( \langle \rho_{ij}(x) \rangle \). This makes sense, because the vacuum \( \Omega \) has no orientation that could be associated with a non vanishing effective vector. In \( \Omega \) there also are no special points that could determine distances, and thus a metric. This is in agreement with the concept of configuration space, developed in Refs. [9], (see also Sec. 2), according to which outside a configuration there is no space and thus no metric: a physical space is associated with configurations, e.g., a system of particles, branes, etc.; without a configuration there is no physical space. In other words, a concept of a physical space unrelated to a configuration of physical objects has no meaning. Our intuitive believing that there exists a three (four) dimensional space(time) in which objects live is deceiving us. The three (four) dimensional space(time) is merely a subspace of the multidimensional configuration space of our universe, in which only position of a single particle is allowed to vary, while positions of all remaining objects are considered as fixed. Of course this is only an idealization. In reality, other objects are not fixed, and we have to take into account, when describing the universe, their configuration subspaces as well. Special and general relativity in 4-dimensional spacetime is thus a special case of a more general relativity in configuration space. Quantization of general relativity has failed, because it has not taken into account the concept of configuration space, and has not recognized that 4D spacetime is a subspace of the huge configuration space associated with our universe. The approach with quantized fields presented in this work has straightforwardly led us to the concept of many particle configurations and effective curved spaces associated with them.

If in Eq. (212) we take \( i = j, \ x = x' \), and use Eqs. (205–207), then we obtain
\[
\langle g_{\mu\nu}(x) \rangle = \left\{ \begin{array}{ll}
g_{\mu\nu}(x) |_{\text{brane}} \neq 0, & \text{on the brane;} \\
0, & \text{outside the brane.}
\end{array} \right.
\]
(213)

It is reasonable to expect that detailed calculations will give the result that \( g_{\mu\nu}(x) |_{\text{brane}} \) is the induced metric on the brane, i.e.,
\[
g_{\mu\nu}(x) |_{\text{brane}} = \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} \equiv f_{ab}.
\]
(214)

Notice that the expectation value of the metric is not defined as \( \langle \rho_{ij}(x) \rangle = \langle \Psi^\dagger \rho_{ij}(x) \Psi \rangle \), but as \( \langle \rho_{ij}(x) \rangle = \langle h_{ij}(x) \rangle \cdot \langle h_{ij}(x) \rangle \).
Recall that the brane can be our spacetime. We have thus pointed to a possible derivation of a curved spacetime metric from quantized fields in higher dimensions.

6 Quantized fields and Clifford space

In the previous section we considered fermion states that are generated by the action of creation operators on the vacuum $\Omega$ according to Eq. (185). In particular, a many fermion state can be a brane, formed according to eq. (186). In Sec. 2 we showed that a brane can be approximately described by a polyvector (5) (see also (6)), which is a superposition of the Clifford algebra basis elements

$$\gamma_\mu \wedge \gamma_\nu, \gamma_\mu \wedge \gamma_\nu \wedge \gamma_\rho, \gamma_\mu \wedge \gamma_\nu \wedge \gamma_\rho \wedge \gamma_\sigma.$$  

(215)

This means that a Fock space element of the form (186) can be mapped into a polyvector:

$$\left( \prod_{\mu, x \in \mathcal{R}} h_\mu(x) \right) \Omega \rightarrow x^M \gamma_M.$$  

(216)

As an example let us consider the case in which the region $\mathcal{R}$ of spacetime is a closed line, i.e., a loop. The holographic projections of the area enclosed by the loop are given in terms of the bivector coordinates $X_{\mu \nu}$. The loop itself is described by a bivectors $X_{\mu \nu} \gamma_\mu \wedge \gamma_\nu$. So we have the mapping

$$\left( \prod_{\mu, x \in \text{loop}} h_\mu(x) \right) \Omega \rightarrow x^{\mu \nu} \gamma_{\mu \nu}.$$  

(217)

With the definite quantum states, described by Eq. (186) or (188) (see also (190)), which are the brane basis states, analogous to position states in the usual quantum mechanics, we can form a superposition (192) (see also (191)). To such an indefinite brane state there corresponds a state with indefinite polyvector coordinate $X^M$:

$$\int \mathcal{D}X(\sigma) \Psi[X(\sigma)] h[X(\sigma)] \Omega \rightarrow \phi(x^M).$$  

(218)

In particular, if $h[X(\sigma)] \Omega$ is a loop, then we have the mapping

$$\int \mathcal{D}X(\sigma) \Psi[X(\sigma)] h[X(\sigma)] \Omega \rightarrow \phi(x^{\mu \nu}).$$  

(219)

The circle is thus closed. With the mapping (216) we have again arrived at the polyvector $x^M \gamma_M$ introduced in Sec. 2. The polyvector coordinates $x^M$ of a classical

$^{11}$ Of course, there is a class of loops, all having the same $X^{\mu \nu}$. 

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system satisfy the dynamics as formulated in Refs. [28, 10, 11]. That dynamics can be generalized to super phase space as discussed in Sec. 3, where besides the commuting coordinates $x^\mu$, $\mu = 0, 1, 2, 3$, we introduced the Grassmann coordinates $\xi^\mu$. In the quantized theory, the wave function $\psi(x^\mu, \xi^\mu)$ represents a 16-component field, $\phi^A$, $A = 1, 2, ..., 16$, that depends on position $x^\mu$ in spacetime, and satisfies the Dirac equation (170) and the multicomponent Klein-Gordon equation (169). In analogous way, besides commuting polyvector coordinates $x^M$, $M = 1, 2, ..., 16$, we have the corresponding Grassmann coordinates $\xi^M$, and the wave function $\phi(x^M)$ is generalized to $\phi(x^M, \xi^M)$. The expansion of $\phi(x^M, \xi^M)$ in terms of $\xi^M$ gives a $2^{16}$-component field, $\phi^A$, $A = 1, 2, ..., 2^{16}$, that depends on position $x^M$ in Clifford space, and satisfies the generalized Dirac equation, $\gamma^M \partial_M \phi^A(x^M)$.

As the evolution parameter, i.e., the time along which the wave function evolves, we can take the time like coordinate $x^0$, or the time-like coordinate $\sigma$. Alternatively, we can take the light-like coordinate $s$, defined in Eq. (15), as the evolution parameter. Then, as shown in Ref. (16), the Cauchy problem can be well posed, in spite of the fact that in Clifford space there are eight time-like dimensions, besides eight space-like dimensions. Moreover, according to Refs. [53, 54, 55], there are no ghosts in such spaces, if the theory is properly quantized, and in Refs. [56, 59, 60] it was shown that the stability of solutions can be achieved even in the presence of interactions.

We can now develop a theory of such quantized fields in Clifford space along similar lines as we did in Secs. 4 and 5 for the quantized fields in the ordinary spacetime. So we can consider the analog of Eqs. (186)–(214) and arrive at the induced metric on a 4-dimensional surface $V_4$ embedded in the 16-dimensional Clifford space. Whereas in Eqs. (186)–(214) we had hoc postulated the existence of extra dimensions, we now see that extra dimensions are incorporated in the configuration space of brane like objects created by the fermionic field operators $h_{\mu(x)}$. Our spacetime can thus be a curved surface embedded in such a configuration space.

7 Conclusion

Clifford algebras are very useful to describe extended objects as points in Clifford spaces, which are subspaces of configuration spaces. The Stueckelberg evolution parameter can be associated with the scalar and the pseudoscalar coordinate of the Clifford space.

The generators of orthogonal and symplectic Clifford algebras, i.e., the orthogonal and symplectic basis vectors, behave, respectively, as fermions and bosons. Quantization of a classical theory is the shift of description from components to the (orthogonal or symplectic) basis vectors.

We have found that a natural space to start from is a phase space, which can be
either orthogonal or symplectic. We united both those phase spaces into a super phase space, whose points are described by anticommuting (Grassmann) and commuting coordinates, the basis vectors being the generators of orthogonal and symplectic Clifford algebras. We have considered the Clifford algebra $Cl(8)$ constructed over the 8-dimensional orthogonal part of the super phase space. Remarkably, the 256 spinor states of $Cl(8)$ can be associated with all the particles of the Standard Model, as well as with additional particles that do not interact with our photons and are therefore invisible to us. This model thus predicts dark matter. Moreover, it appears to be a promising step towards the unification of elementary particles and interactions (see also [27, 28, 11].

Both, orthogonal and symplectic Clifford algebras can be generalized to infinite dimensions, in which case their generators (basis vectors) are bosonic and fermionic field creation and annihilation operators. In the Clifford algebra approach to field theories, a vacuum is the product of infinite, uncountable number of Fermionic field creation operators. They can form many sorts of possible vacuums as the seas composed of those field operators. In particular, strings and branes can be envisaged as being such seas. The field operators, acting on such brane states, can create holes in the branes, that behave as particles. From the expectation values of vector operators in such a one, two, or many holes brane state, we can calculate the metric on the brane. According to the brane world scenario, a brane can be our world. We have found that holes in a fermionic brane behave as particles, i.e., matter, in our world, and that the metric on the brane can be quantum mechanically induced by means of the fermionic creation and annihilation operators. We have thus found a road to quantum gravity that seems to avoid the usual obstacles.

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