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Uniqueness of static spacetimes with marginally outer trapped surfaces

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Abstract. Marginally outer trapped surfaces (MOTS) are routinely used as quasi-local replacements for black holes and it is expected that MOTS and event horizons should coincide in stationary and static spacetimes. In this short notice we present a uniqueness theorem of static spacetimes containing MOTS.

1. Introduction
The concept of black hole involves, by its very definition, global properties of a spacetime. In an evolutive approach (as in a numerical calculation, for instance) the complete knowledge of the evolved spacetime is not available a priori (and in many cases this remains true even after the evolution has been carried out). Thus, the concept of black hole cannot be immediately used and a suitable replacement thereof is required.

Trapped surfaces (and their various relatives) are widely believed to provide such natural replacements. In particular, marginally outer trapped surfaces (MOTS) are considered to be suitable quasi-local versions of the event horizon. It is clear however, that both concepts are a priori very different and that a careful study of the relationship between them is necessary. In this respect, Ben-Dov has proven \cite{Ben-Dov} that the boundary of the spacetime region containing weakly outer trapped surfaces (defined below) coincides with the event horizon in one of the simplest dynamical black hole configurations, namely the Vaidya spacetime. Nevertheless, from a 3+1 perspective the outermost MOTS in a given slice and the intersection of the event horizon with this slice do not coincide in general (consider for instance a spherical Cauchy foliation of the Vaidya spacetime, see e.g. \cite{Vaidya}).

In any case, it is expected that in a stationary situation, MOTS should always lie on the event horizon. There is no proof of this fact, however. A possible approach to this problem goes through studying black uniqueness theorems (in four dimensions) when the assumption of black hole is replaced by the existence of a MOTS. In this work, we present a uniqueness theorem for static spacetimes with MOTS provided some conditions are satisfied (Theorem 3 below).

2. Definitions
A static Killing initial data set (static KID) \((\Sigma, g, K; \xi)\) is a spacelike hypersurface (possibly with boundary \(\partial \Sigma\)) embedded in a static spacetime \((M, g^{(4)})\) with a static Killing field \(\xi\). Let us define \(\lambda = -g^{(4)}(\xi, \xi)\). Throughout this text a surface \(S\) is a 2-dimensional, orientable, embedded submanifold \(S\) in \(\Sigma\) which may have boundary \(\partial S\) provided \(\partial S \subset \partial \Sigma\).
We will assume that \((\Sigma, g, K; \vec{\xi})\) has a chosen asymptotically flat end \(\Sigma^\infty_0\). Then, a surface \(S \subset \Sigma\) is bounding if \(\Sigma \setminus S\) splits into two disjoint sets \(\Omega_1\) and \(\Omega_2\) such that

(i) \(\Omega_1\) is connected, \(\partial \Omega_1 = S\) and \(\Omega_1 = K \cup \Sigma^\infty_0\) with \(K\) compact and \(\Sigma^\infty_0\) is an asymptotically flat end of \(\Sigma^\infty_0\), and

(ii) \(\Omega_2 = \emptyset\) if and only if \(S = \partial \Sigma\).

The sets \(\Omega_1\) and \(\Omega_2\) will be called the exterior and the interior of \(S\) in \(\Sigma\), respectively.

Figure 1. In short, a bounding surface \(S\) separates the manifold \(\Sigma\) in two sets such that one of them, \(\Omega_1\) (the exterior region, in gray color), is connected and contains \(\Sigma^\infty_0\) and no other end or boundary. The surface \(S'\) is not bounding because the exterior region including \(\Sigma^\infty_0\) contains an additional boundary.

Using the fact that \(S\) is orientable, it can be easily proven that a surface \(S\) admits two vectors fields \(\{\vec{l}^+, \vec{l}^-\}\) which are null, future-directed and form a basis for the normal bundle of \(S\) in \(M\) at every point \(p \in S\). We will take \(\vec{l}^+\) and \(\vec{l}^-\) to be the direction whose orthogonal projection onto \(\Sigma\) points into the exterior of \(S\) in \(\Sigma\). Let us denote by \(\theta^+\) and \(\theta^-\) the null expansions of \(S\) along \(\vec{l}^+\) and \(\vec{l}^-\), respectively.

A closed (i.e. compact and without boundary) bounding surface \(S\) is a:

- Weakly outer trapped surface (WOTS) if \(\theta^+ \leq 0\).
- Marginally outer trapped surface (MOTS) if \(\theta^+ = 0\).
- Past weakly outer trapped surface (past WOTS) if \(\theta^- \geq 0\).

Finally, the trapped region \(T^+\) is the union of the interiors of all WOTS in \(\Sigma\) and, similarly, the past trapped region \(T^-\) is the union of all the interiors of all past WOTS in \(\Sigma\).

3. Motivation and statement of the result

Uniqueness theorems of black holes, which are cornerstones in General Relativity, have been proven in the stationary and the static cases for several matter models, such as e.g. vacuum and electrovacuum, under certain technical assumptions. To date, the most powerful method to prove uniqueness of static black holes is the so-called doubling method of Bunting and Masood-ul-Alam, which was used for the first time in [3] to show that the Schwarzschild family exhausts the set of static and vacuum non-degenerate black hole spacetimes, independently of whether the event horizon is a priori connected or not, thus improving the original uniqueness proof by Israel [4]. Although, as mentioned above, black holes require global hypotheses for their definition, in the static case these conditions can be translated into restrictions on the hypersurface \(\Sigma\) orthogonal to the static Killing vector. Thus, the theorem by Bunting and Masood-ul-Alam can be expressed as follows.
Theorem 1 (Bunting and Masood-ul-Alam, 1987, [3]) Let \((\Sigma, g, K; \vec{\xi})\) be a connected, asymptotically flat, static and vacuum KID. Assume time-symmetry (i.e. \(K = 0, \vec{\xi} \perp \Sigma\)), \(\partial \Sigma \neq \emptyset\) and \(\lambda_\Sigma \geq 0\) with \(\lambda = 0\) only on \(\partial \Sigma\).

Then \((\text{int } \Sigma, g)\) (where \(\text{int } \Sigma\) is the interior of \(\Sigma\)) is isometric to the \(\{t = 0, r > 2m\}\) slice of the Schwarzschild spacetime of mass \(m\), for some \(m > 0\).

The hypothesis of non-degeneracy of the event horizon, which in the theorem above appears implicitly in the fact that \(\Sigma\) has non-empty boundary and has no other ends besides the asymptotically flat one, was dropped by Chruściel in [5] after a suitable application of the doubling method. It is important to remark that the doubling method also works for other matter models. The key common feature that allows one to use the doubling method successfully in all these cases is the existence of a surface on \(\Sigma\) where \(\lambda = 0\) and in whose exterior \(\lambda > 0\). The existence of such a surface is guaranteed by the presence of an event horizon in the spacetime.

Since MOTS are expected to coincide with event horizons in the static case, it is natural to ask oneself whether the uniqueness theorems for static black holes can be extended to static spacetimes containing MOTS. A first answer to this question was given by Miao who proved the following theorem for the vacuum case.

Theorem 2 (Miao, 2005, [6]) Let \((\Sigma, g, K; \vec{\xi})\) be a time-symmetric, asymptotically flat, static, vacuum KID with compact minimal (i.e. with vanishing mean curvature) boundary.

Then \((\text{int } \Sigma, g)\) is isometric to the \(\{t = 0, r > 2m\}\) slice of the Schwarzschild spacetime of mass \(m\), for some \(m > 0\).

It is important to remark that in the time-symmetric setting all MOTS are bounding minimal surfaces and vice versa. In this work we try to generalize Miao’s theorem in two directions. First, we will allow any matter model for which the doubling method gives uniqueness and, second, we will relax the hypothesis of time-symmetry. The proof by Miao relies strongly on the vacuum field equations so the approach will require completely new methods. This will require in particular a proper understanding of MOTS in static spacetimes.

As a consequence of the vacuum field equations, the set \(\{\lambda = 0\}\) in a time-symmetric slice is a totally geodesic (in particular, minimal) boundary. Thus, recalling Theorem 1, Miao’s theorem can be rephrased as follows: No minimal surface can penetrate into the exterior region where the Killing vector is timelike in any time-symmetric, asymptotically flat, static and vacuum static initial data set. Thus, Miao’s result can be regarded as a confinement result for MOTS in time-symmetric and vacuum slices. Motivated by this fact, in [7] we proved that in a static KID, not necessarily time-symmetric, satisfying the null energy condition (which requires that the Einstein tensor contracted twice with any null vector gives a non-negative quantity), no MOTS can penetrate in the exterior region where the static Killing is timelike provided some technical hypotheses hold. The key tool which allowed us to achieve this result was a powerful theorem by Andersson and Metzger [8] which, roughly speaking, states that the boundary of the trapped region \(\partial T^+\) is a MOTS and, in fact, the outermost MOTS. Our result in [7] extends Theorem 2 as a confinement result for MOTS. However, Miao’s theorem also has a uniqueness part. This aspect can also be generalized as follows.

Theorem 3 Let \((\Sigma, g, K; \vec{\xi})\) be a static KID with a chosen asymptotically flat end \(\Sigma^\infty\) and satisfying the null energy condition. Suppose that \(\Sigma\) possesses a WOTS (in particular, MOTS) \(S\). Assume:

(i) \(T^- \subset T^+\);
(ii) \(H_1(\Sigma, \mathbb{Z}_2) = 0\) (i.e. the first homology group of \(\Sigma\) with coefficients in \(\mathbb{Z}_2\) is trivial);
(iii) The spacetime \((M, g^{(4)})\) containing \(\Sigma\) is such that all the Killing prehorizons are embedded (a Killing prehorizon is, by definition, a null and injectively immersed submanifold of \(M\) where the Killing vector \(\vec{\xi}\) is tangent and different from zero [9]).
(iv) The matter model is such that the Bunting and Masood-ul-Alam doubling method for time-symmetric slices gives uniqueness of black holes. Then \((\Sigma \setminus T^+, g, K)\) is a slice of such a unique spacetime.

Let us end this short communication by briefly commenting on the hypotheses of Theorem 3. To that aim, it will be necessary to give some hints on the proof. Let \(U\) be the connected component of \(\{\lambda > 0\}\) containing \(\Sigma^\infty_\infty\). Basically, the proof consists in showing that \(\partial U = \partial T^+\), which is equivalent to asserting that \(\lambda|_{\Sigma \setminus T^+} \geq 0\) and \(\lambda = 0\) only in \(\partial (\Sigma \setminus T^+)\). Once this is shown, hypothesis (iv) implies uniqueness. The proof of \(\partial U = \partial T^+\) is by contradiction: Assume \(\partial U \neq \partial T^+\) and then construct a WOTS outside \(\partial T^+\) (which is impossible by the theorem by Andersson and Metzger [8] mentioned above).

The construction of the WOTS outside \(\partial T^+\) is very specific and cannot cope with situations where hypothesis (i) does not hold. Nevertheless, we do not know whether condition (i) is really necessary for the validity of the theorem. In any case, in dynamical situations of strong gravitational collapse, it is to be expected that MOTS do form during the evolution, but no past WOTS are present at any time (since this would imply typically the presence of a singularity in the past). In this sense, hypothesis (i) seems reasonable. It is also important to mention that condition (i) is satisfied identically when the initial data set is time symmetric.

The WOTS outside \(\partial T^+\) is constructed by joining suitable pieces of surfaces with \(\lambda = 0\). The resulting set is proven to be injectively immersed, smooth, compact, without boundary and have vanishing null expansion. However, it may still be non-bounding. The topological condition (ii) is imposed to guarantee that this immersed submanifold is, in addition, bounding so that it splits the manifold into disjoint pieces. Hypothesis (iii) excludes the possibility that this immersed submanifold is non-embedded. It may well be the case that non-embedded prehorizons cannot exists under any circumstances but this is not known (see however [9] for a result in this direction under global spacetime assumptions). It should be remarked that the non-existence of non-embedded prehorizons is automatically satisfied if the spacetime is analytic.

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