Quantum Gravity Microstates from Fredholm Determinants

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A large class of two dimensional quantum gravity theories of Jackiw-Teitelboim (JT) gravity \cite{1} is a two dimensional model of gravity coupled to a scalar $\phi$, with Euclidean action:

$$ I = -\frac{1}{2} \int_M \sqrt{g} \phi (R + 2) - \int_{\partial M} \sqrt{h} \phi_b (K - 1) - \frac{S_0}{2\pi} \left( \frac{1}{2} \int_M \sqrt{g} R + \int_{\partial M} \sqrt{h} K \right) , \quad (1) $$

where $R$ is the Ricci scalar and in the boundary terms, $K$ is the trace of the extrinsic curvature for induced metric $h_{ij}$ and $\phi_b$ is the boundary value of $\phi$. The constant $S_0$ multiplies the Einstein-Hilbert action, which yields the Euler characteristic $\chi(M) = 2 - 2g - b$ of the spacetime manifold $M$ (with boundary $\partial M$), with $g$ handles and $b$ boundaries. The partition function of the full quantum gravity theory $Z(\beta)$ at some inverse temperature $\beta = 1/T$ is given by the path integral over $M$ with a boundary of length $\beta$. It has a topological expansion $Z(\beta) = \sum_{g=0}^{\infty} Z_g(\beta)$, where $Z_g(\beta)$ has a factor $e^{-\chi(M) S_0 \beta}$.

JT gravity is of interest not just as a solvable toy model of gravity, but also because it is a universal sector of the non-trivial dynamics being at the (length $\beta$) boundary $\partial M$. It has a Schwarzian action, and $\text{(2)}$

$$ Z_0(\beta) = \frac{e^{S_0} e^{\pi^2 / 4\sqrt{\beta}}} {4\sqrt{\beta}} , \quad \rho_0(E) = e^{S_0 \sinh(2\pi \sqrt{E}) / 4\pi^2} , \quad (2) $$

where the spectral density $\rho_0(E)$ comes from the Laplace transform: $Z_0(\beta) = \int_0^{\infty} \rho_0(E) e^{-\beta E} dE$. This class of models can be explicitly solved to any order to yield $Z_0(\beta)$ (e.g., yielding corrections to $\rho_0(E)$), and correlations thereof. This was shown in Refs. $[\text{4}, \text{5}]$, along with a striking equivalence (to be recalled below) to models of random $N \times N$ matrices at large $N$, where the topological expansion parameter $e^{-S_0} \sim 1/N$.

This Letter’s results go well beyond the topological perturbative expansion to uncover non-perturbative physics, computing the details of individual underlying energy levels of the spectrum. This reveals the physics of the microscopic degrees of freedom of the quantum gravity theory, or, from the higher dimensional perspective, the underlying black hole microstates. The entropy counts microstates via $S_0 = \log N$. Expanding in $1/N$ around large $N$ will not resolve their individual character. For this, a non-perturbative formulation is needed that allows $O(e^{-N})$ physics to be extracted, a program begun in earnest (in this context) in Ref. $[\text{7}]$. The key new tool, introduced in this Letter, will be a Fredholm determinant of an operator that naturally arises from the underlying matrix model description. It yields exquisite details of individual energy levels.

The results for the first six levels for JT gravity are shown in Fig. $[\text{1}]$ For illustration, the plots (and all others herein) are for $e^{-S_0} = 1$. For a fixed reference value of $E$, smaller $e^{-S_0}$ (larger extremal entropy $S_0$) results in an increase in the number of levels found to the left, which is nicely consistent with their microstate interpretation. A key feature is that the knowledge of the spectrum is fundamentally statistical, increasingly so at lower energies. $[\text{8}]$ At higher $E$, the energy levels become more sharply defined (their variance decreases), and also form a continuum. In this regime the spacetime language (and the perturbation theory described above) is a good ap-
Non-Perturbative Matrix Models.—The direct non-perturbative completion of Ref. [5]’s Hermitian matrix model definition of JT gravity has instabilities. One way to see this is to note that perturbatively it is a combination of the family of \((2k-1, 2)\) minimal models (coupled to gravity), obtained as Hermitian matrix models: It was known long ago that the even models are non-perturbatively unstable due to eigenvalue tunneling to infinite negative values. However Ref. [7] supplied a non-perturbative completion of the JT gravity matrix...
model that preserves the perturbative expansion to all orders. It is built by double-scaling complex matrices $M$ for which the potential is built from $MM$ [21]. It can be thought of as a model of an Hermitian matrix with a manifestly positive eigenvalue spectrum. The model is unaffected by non-perturbative instabilities. [22]

A useful way of seeing the difference between the two types of model is through their orthogonal polynomial formulation. The Hermitian matrix model integral can be written entirely in terms of a system of $N$ polynomials $P_i(\lambda)$, orthogonal with respect to the measure $d\lambda e^{-V(\lambda)}$. The polynomials themselves are related according to $\lambda P_i(\lambda)=P_{i+1}(\lambda)+R_iP_{i-1}(\lambda)$ and the $R_i$ satisfy a recursion relation determined by $V(\lambda)$. Knowing the coefficients $R_i$ turns out to be equivalent to solving the partition function integral or any insertion into the integral of traces of powers of $N$-large $P$. According to the integral of traces of powers of $R$ (acting as an operator inside the integral): $\int \rho(E)\lambda^k d\lambda$, $k$ runs over the whole real line, via $X=0$ perturbation theory) yields the identity that Ref. [5] constructed. Meanwhile, the non-perturbative states at $E<0$ are not present, while the perturbative physics remains the same. There is an analogous model for any $k$ with this kind of behaviour, and Ref. [7] showed how to build a non-perturbative completion of JT gravity using them. It is a matter of finding the correct equation for $u(x)$. It is constructed as follows: The general double scaled matrix model yields the form (7) but with $R=\sum_{k=1}^{\infty} t_k R_k[u] + x$, where $R_k[u]$ are the “Gelfand-Dikii” [25] polynomials in $u(x)$ and its derivatives, beginning as $R_k[u]=u(x)^k + \cdots$. (Their details are not needed here.) The $t_k$ are couplings. The specific values $t_k=\pi^{2k-2}/k!(k-1)!$ yield an equation that to leading order gives the JT spectral density [2], with $h=e^{-S_0}$.

Solving the equation for $u(x)$ order by order (expanding about large $x<0$ perturbation theory) yields the identical physics that Ref. [7] constructed. Meanwhile, the full solution for $u(x)$ (constructed numerically to good accuracy in Ref. [20]) determines $\mathcal{H}$. The spectral problem can be solved numerically to yield wavefunctions $\psi(E,x)$, and the integral [6] yields the non-perturbative $\rho(E)$. It is plotted in Fig. 1 as the solid black curve. At large $E$ it asymptotes to the classical Schwarzian result [2] (dashed line), but at small $E$ there are undulations that are invisible in perturbation theory. [27] They are the precursors of the underlying microphysics.
The properties of det$(I_F)$ is a cumulative probability density function (CDF). The zero) is $E$ interval $(0, s)$, the probability of finding no eigenvalues on the $E$ can be used to focus on one energy/eigenvalue at a time $K$. wavefunctions are a hybrid of Airy and Bessel functions, defined kernels to the menagerie. They may be of wider Letter’s studies of JT gravity now add some new well- $n$ of interest, (Airy, Bessel, sine, Laguerre, th generalizations thereof, giving a portmanteau of $K$ kernels are a powerful tool that have been used in the statistical physics literature for computing random matrix model properties such as correlation functions [23]. The determinantal structure arises because of the van der Monde determinant $\prod_{i \neq j} (\lambda_i - \lambda_j)^2$ that comes as the Jacobian for going from $M$ variables to eigenvalues $\lambda$. The orthogonal polynomial basis inherits all this structure, and can be used as the building blocks, in their guise as the $\psi(E, x)$. For example the core object is the “kernel”:

$$K(E, E') = \int_{-\infty}^{0} \psi(E, x)\psi(E', x) dx . \quad (8)$$

In fact, it has already played a role, as its diagonal is the non-perturbative density, Eq. (6). Clearly there is a lot more information to be extracted from its off-diagonal terms. Various combinations of the kernel, Laplace transformed, can be used to write multi-point correlation functions [23] for $Z(\beta)$, but the raw form will be a much more direct tool for studying the spectrum.

It is the context of the Fredholm determinant construction that gives $K$ its kernel moniker. In solving problems of the form [29]: $f(E) = \int_{a}^{b} K(E, E')f(E')dE' = g(E)$, on some interval $(a, b)$ (or union of intervals) in the $E-$plane, the properties of det$(I - K)$ are important. $K$ is the integral operator with kernel $K(E, E')$. The random matrix model literature is filled with various kinds of kernel of interest, (Airy, Bessel, sine, Laguerre, etc.). This Letter’s studies of JT gravity now add some new well-defined kernels to the menagerie. They may be of wider mathematical interest since (e.g., for the $k=1$ model) the wavefunctions are a hybrid of Airy and Bessel functions, and 4th generalizations thereof, giving a portmanteau of Wigner and Wishart behaviour, scaled.

Returning to the physics, the Fredholm determinant can be used to focus on one energy/eigenvalue at a time as follows. For the 1st energy level (labelled henceforth as the 0th, for the ground state, $E_0$, of JT gravity system), the probability of finding no eigenvalues on the interval $(0, s)$ (chosen since the lowest possible energy is zero) is $E(0; s) = \det(I - K(0, s))$. Crucially (for later) this is a cumulative probability density function (CDF). The probability density function (PDF) for finding an energy is $F(0; s) = -dE(0; s)/ds$. For orientation, in the case of the Airy model the interval would be $(-\infty, s)$, and using the Airy kernel yields the famous Tracy-Widom distribution [30] of the smallest eigenvalue. Here, the JT gravity system will reveal a new kind of smallest eigenvalue distribution.

More generally, for the $n$th energy level, the probability (CDF) for the interval $(0, s)$ is written as:

$$E(n; s) = \sum_{j=0}^{n} \frac{(-1)^j}{j!} \frac{d^j}{ds^j} \det(I - zK(0, s)) \bigg|_{z=1} . \quad (9)$$

Correspondingly the PDF is: $F(n; s) = -dE(n; s)/ds$.

The major challenge now is to compute the determinant of the infinite dimensional operator in Eq. (9). This requires much more care, and is prone to severe numerical difficulties even though the problem is effectively discrete ($\sim 700$ energies were used). Useful at this point is the impressive work of Bornemann [31] that shows how to use quadrature on a relatively small number of points in the energy interval $(0, s)$ to compute Fredholm determinants: Using Clenshaw-Curtis quadrature to break up the interval into $m$ points $e_i$ and compute weights $w_i \ (i = 1 \cdots m)$, an integral $\int_{0}^{s} f(E)dE$ on the interval would be computed as $\sum_{i} w_i f(e_i)$. So similarly the determinant becomes:

$$\det(I - zK(0, s)) \rightarrow \det(\delta_{ij} - zw_j K(e_i, e_j) w_i z_j) . \quad (10)$$

If the $\psi(E, x)$ are known accurately (such as for Airy and Bessel kernels) the method gives impressive results with modest values of $m$ such as 8, or 16. In the case in hand, the $\psi(E, x)$ were found as approximate numerical solutions to an eigenstate problem for which the potential $u(x)$ was itself a solution to a difficult (15th order [26]) non-linear equation, so some challenging numerical difficulties due to errors can be expected. However, they can be surmounted well enough to get very good results for the definition of JT gravity discussed here [32].

The result for the zeroth level (the ground state) is the focus of the inset of Fig. 1. A small amount of smoothing has been applied to remove numerical noise. To the left, at zero, the CDF $E(0; s)$ shows that there is a non-zero (but small) chance of finding the ground state there, falling steadily to zero to the far right with the increasing unlikelihood of finding the ground state at very high energies. Its derivative, the PDF $F(0; s)$ is also shown in the inset. It peaks at around 0.75. The mean of the distribution gives the average ground state of the ensemble $\langle E_0 \rangle \approx 0.66$. (For illustration purposes all computations were done using $h=1$). Computing the results for higher levels is straightforward, although numerical inaccuracies in finding the first level get successively amplified with each level. The first six levels are shown in the main part of Fig. 1. What has been uncovered here with the Fredholm determinant technique are the explicit probability peaks for individual energies/microstates that, when added together, produce the previously found non-perturbative undulations in the spectral density (black line). In principle any quantity can now be computed using this information, as will be demonstrated next.

Quenched Free Energy.—It is important to compute the quantity $F_0(T) = -\beta^{-1}\langle \log Z(\beta) \rangle$ for JT gravity, but it is difficult. Ref. [9] pointed out that connected diagrams with multiple boundaries (“replica wormholes”, as they were implementing a gravitational replica trick) should play a crucial role. Ref. [10] observed that in addition a non-perturbative formulation was needed, such as a matrix model. There have been various useful partial results from toy models [11] [12], at low T. Ref. [11]
noted that a better tool was needed, in order to incorporate the properties of individual peaks. The Fredholm determinant has performed handily in that regard, and so computation of $F_Q(T)$ is now rather straightforward.

A numerical approach is natural (since all the explicit data about the levels are numerical), simply directly sampling ensembles. In fact, this was done recently in Ref. [11] for the comparatively simple cases of the Airy model and a variety of Bessel models. There, all that was needed was Gaussian random matrices (readily numerically generated) plus scaling to an endpoint. A matrix model for JT gravity has much more exotic probability distributions however, so direct sampling seems doomed. However, the individual probability distributions for each level can now be generated by reverse-engineering the above results for the $E(n; s)$, which (recall) are CDFs.

A key result from the theory of statistics is that any probability distribution function can be generated from a uniform probability distribution by mapping from the associated CDF. So, ensembles can be generated as follows: For a particular sample, generate the $n$th energy level $E_n$ with the appropriate probability (using uniform PDFs on a computer and converting using the CDF $E(n; s)$ to sample the correct PDF $F(n; s)$). Then compute $\log (Z(\beta)) = \log (\sum_n e^{-\beta E_n})$. This was done for an ensemble of just 5000 samples, and then averaged. For contrast, the annealed quantity $F_A(T) = -\beta^{-1} \log (Z(\beta))$ can be computed too (averaging the partition function over the ensemble and taking the log at the end).

Using just the first six levels yields the content of the inset of Fig. 2. As higher levels are added, the details of the curves settle swiftly. Only successively higher $T$ details are affected by adding successively higher levels, the process asymptoting to the classical result at large $T$. In the main plot of Fig. 2 a result with 150 levels is given. It is constructed by approximating the $E(n; s)$ for levels above about $n=10$. In this regime the approximation is already very much under control. The point is that even by 6 levels (see Fig. 1), the peaks have narrowed and overlap significantly, as the system returns to the continuum (classical) regime. The $n>10$ peaks are well approximated by narrow Gaussians, with their location, mean, and standard deviation determined by the classical density curve [2]. So CDFs (error functions for Gaussians) were used for high levels. As a test of this, the red dashed line is the annealed free energy computed using the $Z(\beta)$ obtained by simply Laplace transforming $\rho(E)$ (integrated to the appropriate cutoff energy corresponding to the 150th level). The agreement with the result computed by direct ensemble averaging (red crosses) is remarkably good. This shows that the result for $F_Q(T)$ is accurate. Overall, rather nicely, $F_Q(T)$ is monotonically decreasing (i.e., the entropy $S(T)$ is manifestly positive), with zero slope at $T=0$, corresponding to entropy $S_0$ at extremality. Additionally, this result confirms an earlier suggestion [10] that there is no sign of the replica symmetry breaking transition conjectured in Ref. [9].

**Final Remarks.**—The studies of this Letter (with several JT supergravity examples to appear) show that matrix models of JT gravity are completely tractable models of a key quantum gravity phenomenon: A cross-over from smooth, classical spacetime geometry (large $E$ (or $T$) in Fig. 1 (or 2)) to a regime where spacetime is not enough, and a description in terms of discrete microstates takes over (small $E$ (or $T$)). The microstates are always there, of course, but the regime where their individuality is inevitable is when the variance or randomness becomes pronounced, while inter-spacing is large. This is a regime beyond smooth spacetimes, even with many handles and wormholes. It is no longer geometrical, but the matrix model computes the physics just as readily. The key tool used here was the Fredholm determinant, whose kernel is built from sewing together two copies of the wavefunctions $\psi(E, x)$. Interestingly, the $\psi(E, x)$ have an interpretation [33] as a type of D-brane probe, in the language of minimal string theory. In a sense then, the Fredholm tool is a D-brane probe that detects an increased spreading out or variance (or itself spreads out) as it moves to lower energies. This is reminiscent of aspects of the D-brane probes involved in the “enhançon mechanism” [34]. This might be relevant when considering what lessons these phenomena might teach about higher dimensional quantum gravity. Finally, as mentioned in the introduction, the microstates uncovered in detail here also model the microstates of the higher dimensional black holes whose near-horizon low $T$ dynamics is controlled by a JT gravity. It will be interesting to see what other features of JT gravity and black hole physics (in various dimensions) might be accessible using matrix model technology.

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This includes variants with non-orientable $M$. This Letter will not dwell on the (logically distinct) issue of interpreting holographic correspondences when gravity seems (discussed in Ref. [35], and now widely debated) of interest to ensemble average over non-gravitational duals. Later work [36] extended the connection to models with more general potentials for supersymmetric extensions. This includes variants with non-orientable $M$, and also supersymmetric extensions. Later work [30] extended the connection to models with more general potentials for $\phi$.}

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