Mass loading induced dephasing in nanomechanical resonators

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Abstract
This paper presents a study of dephasing of an underdamped nanomechanical resonator subject to random mass loading of small particles. A frequency noise model is presented which describes dephasing due to the attachment and detachment of particles at random points and particle diffusion along the resonator. This situation is commonly encountered in current mass measurement experiments using nanoelectromechanical (NEM) resonators. The conditions which can lead to inhomogeneous broadening and fine structure in the modes’ absorption spectra are discussed. It is also shown that the spectra of the higher-order cumulants of the (complex) vibrational mode amplitude are sensitive to the parameters characterizing the frequency noise process. Hence, measurement of these cumulants can provide information not only about the mass but also about other parameters of the particles (diffusion coefficient and attachment–detachment rates).

The effect of dephasing can also be observed in the power spectrum or absorption spectrum of the underdamped oscillator as broadening and even departure from the Lorentzian absorption line [12–14]. This broadening is not a consequence of an increase of damping. Studying frequency noise in nanomechanical resonators is important not only to find cures for the spectral broadening in high quality factor nanoresonators [15] but also in the application of the latter as sensors (e.g. NEM-based mass measurement applications) [16].

The eigenfrequency noise may have different origins. For instance, in an electrical read-out method of the nanobeam displacement, charge fluctuations in the nanoresonator cause frequency fluctuations due to capacitive coupling with gate electrodes [7, 8]. Also, random frequency modulation of a certain vibrational mode can result from parametric coupling with other modes that are, for example, thermally driven [17, 18].

In the context of NEM-based mass measurement experiments, particles enter and leave the resonator at random points and they also diffuse along the resonator [16]. Since the eigenfrequency shift of a vibrational mode depends on the particle position, random mass loading of particles has been usually perceived as a limiting factor to the precision...
of NEM-based mass sensors. However, we can show that
higher-order cumulants of the (complex) vibrational mode
amplitude can be used to characterize the frequency noise
process [14] and determine the particle parameters (mass,
diffusion coefficient, attachment and detachment rates).

In order to quantify the effect of random mass loading in
the average response of nanomechanical resonators,
a frequency noise model is presented which describes
the quasicontinuous type of frequency noise present in
NEM-based mass sensors. This model contains, previously
studied, frequency noise models for particle diffusion [13]
and discrete frequency jumps [12] as limiting cases.

Bachtold et al have recently realized CNT-based mass
sensors where Xe atoms and naphthalene molecules attach,
detach and diffuse along a CNT nanoresonator [16]. In typical
NEM-based mass measurement experiments the incoming

effect of the driven mode, beyond experimental control.

It has been demonstrated that eigenmodes of nanomechanical
resonators can exhibit low intrinsic losses with quality factors
\( Q \sim 10^5 \) [19, 9]. The vibrational amplitude, \( q(t) \), of a certain
eigenmode, which is weakly driven near resonance by a force
\( F \cos(\omega_0 t) \), satisfies the following equation,

\[
\ddot{q} + 2\Gamma \dot{q} + [\omega_0^2 + 2\omega_0 \xi(t)]q = \frac{F}{M} \cos(\omega_0 t) + \frac{f_\text{tr}(t)}{M},
\]

where \( \omega_0, M \) and \( 2\Gamma \) are the bare resonance frequency,
effective mass and energy relaxation rate of the driven mode,
respectively. Equation (1) also includes the eigenfrequency noise,
\( \xi(t) \), and the zero-mean additive thermal noise, \( f_\text{tr}(t) \).
It is assumed that \( \omega_0 \gg \Gamma, \Delta \equiv (\langle \xi(t)^2 \rangle - \langle \xi(t) \rangle^2)^{1/2}, \langle \delta \omega \rangle, \)
\( \xi \equiv 1 \); where \( \delta \omega = \omega - \omega_0 \) is the frequency detuning and \( \xi \) is
the frequency noise correlation time. It is also assumed that
the Fourier components of \( \xi(t) \) at frequencies \( \omega \approx 2\omega_0 \)
are negligibly small (no parametric resonance). The relative size
between the quantities \( \Delta, \Gamma, \xi \equiv 1 \) and \( \langle \delta \omega \rangle \) is arbitrary.

If the above restrictions hold then we can use the
rotating-wave-approximation (RWA) to study the dynamics of
\( q(t) \) at frequencies \( \omega \approx \omega_0 \). The slow (complex) dynamical
variables \( u(t) \) and \( u^*(t) \) can be introduced and are defined by

\[
u(t) = \exp(-i\omega_0 t)[i\omega_0 q(t) + \dot{q}(t)]/2i\omega_0,
\]
and
\[
q(t) = u(t) \exp(i\omega_0 t) + u^*(t) \exp(-i\omega_0 t).
\]
After averaging out fast oscillating terms, within the oscillation period \( 2\pi/\omega_0 \), the equation of motion for \( u(t) \) is [20]

\[
\dot{u}(t) = -[\Gamma + i(\delta \omega - \xi(t))]u - i\frac{F}{4M\omega_0} + \tilde{f}_\text{tr}(t),
\]

where \( \tilde{f}_\text{tr}(t) = f_\text{tr}(t) \exp(-i\omega_0 t)/2i\omega_0 M \). The value of the
moments of the complex vibrational amplitude, \( \langle u^n(t) \rangle_{\text{st}} \), do
not depend on the additive thermal noise, \( f_\text{tr}(t) \), if the latter is
uncorrelated with the frequency noise process, \( \xi(t) \), and the
RWA holds (equation (3)).

The frequency noise becomes correlated with the additive
thermal noise process in the case where the former depends
on the oscillator state (back-action). For instance, back-action
occurs in the diffusion-induced bistability mechanism of
driven nanomechanical resonators, discussed in [21], and in
nonlinear oscillators, where the vibrational frequency depends
on the vibrational amplitude. These extra mechanisms of
dephasing can be neglected by weakly driving the vibrational
mode. Thus, \( f_\text{tr}(t) \) is dropped from equation (3).

The frequency noise process, \( \xi(t) = \xi(\xi,t) \), is assumed to
be defined in terms of a Markovian stochastic process \( \xi(t) \),
which is modeled by the master equation

\[
\partial_t p(\xi,t) = \sum_{\xi'} \tilde{W}(\xi,\xi') p(\xi', t).
\]

Above, \( p(\xi,t) \) is the probability distribution of \( \xi \) at time \( t \) and
\( \tilde{W}(\xi,\xi') \) is the matrix of transition probabilities per unit
of time between different \( \xi \)-states. It is assumed that \( \tilde{W} \) does not
depend on the oscillator state, \( u(t) \). Thus, \( f_\text{tr}(t) \) and \( \xi(t) \)
are independent processes.

The evolution of the joint probability distribution of the
oscillator state and the eigenfrequency noise process, \( P \), is
described by the Einstein–Fokker–Planck equation [22]

\[
\partial_t P(u, u^*, \xi, t) = \tilde{W} P + \partial_u [\Gamma + i(\delta \omega - \xi(t))] u P
\]

\[
+ \frac{F}{4M\omega_0} \partial_u P + \text{c.c.},
\]

where \( \tilde{W}(\xi) \) is a scalar function of the noise state \( \xi \).

The stationary value of the \( n \)th moment of the complex
vibrational amplitude \( u(t) \) is given by

\[
\langle u^n(t) \rangle_{\text{st}} = \sum_{\xi} \int \int du \, du^* \, u^n P_{\text{st}}(u, u^*, \xi),
\]

\[
= \left( \frac{F}{4M\omega_0} \right)^n \chi^{(n)}(\delta \omega),
\]

where \( P_{\text{st}}(u, u^*, \xi) \) is the stationary solution of equation (5)
and \( \chi^{(n)}(\delta \omega) \) is the scaled \( n \)th moment of \( u(t) \).

The scaled \( n \)th moment, \( \chi^{(n)}(\delta \omega) \), is written as a sum of
scaled partial moments, \( \tilde{\chi}^{(n)}(\xi; \delta \omega) \), defined by

\[
\tilde{\chi}^{(n)}(\xi; \delta \omega) = \left( \frac{4M\omega_0}{F} \right)^n \int \int du \, du^* \, u^n P_{\text{st}}(u, u^*, \xi),
\]

\[
\chi^{(n)}(\delta \omega) = \sum_{\xi} \tilde{\chi}^{(n)}(\xi; \delta \omega).
\]

The scaled partial moments are complex quantities which are
coupled by the equations [14]

\[
\tilde{W}(\chi^{(n)}) - n(\Lambda - i \xi(\xi)) \chi^{(n)} = ni\tilde{\chi}^{(n-1)},
\]

where \( \Lambda = \Gamma + 4i\delta \omega \), \( \tilde{\chi}^{(0)}(\delta \omega, \xi) = p_{\text{st}}(\xi) \) and \( p_{\text{st}}(\xi) \) is the
stationary solution of equation (4).
Similarly, the (scaled) oscillator susceptibility, $\chi(\delta \omega)$, is defined by

$$\chi(\delta \omega) = (4Mm_o/F)(u^*(t))_{at}.$$  

The scaled oscillator susceptibility can also be written in terms of scaled partial susceptibilities $\chi(\xi; \delta \omega)$ $\chi(\delta \omega) = \sum \chi(\xi; \delta \omega)$, which are coupled by the equation that results from complex conjugation of equation (8) with $n = 1$.

The limit cases of slow ($\tau_c \gg \Gamma^{-1}, \Delta^{-1}$) and fast ($\tau_c \ll \Gamma^{-1}, \Lambda^{-1}$) frequency noise have been discussed previously [12, 13]. Both limit cases are described by equation (8) and they correspond to slow (fast) diffusion and rare (frequent) adsorption and desorption events. In what follows, the experimentally relevant case of rare adsorption and desorption events, and slow and fast diffusion are discussed [16].

The eigenfrequency noise due to random mass loading in NEM resonators is quasicontinuous. It evolves in time either continuously, when particles diffuse along the nanoresonator, or discontinuously, when particles enter and leave the nanoresonator. Now, let us describe a model for the frequency noise when a single particle acts on the resonator. If many particles act independently on the resonator then it is only necessary to model the frequency noise process due to a single particle (see appendix B). This frequency noise model includes the frequency noise models for particle diffusion [13] and discrete frequency jumps [12] as limit cases.

The state of the particle is described by two variables $x(t) \in [0, 1]$ and $\tau(t) \in [-L/2, \Lambda/2]$, where $L$ is the resonator length. The former variable has the value of one (zero) when the particle is (is not) on the nanoresonator. The position of the particle on the nanoresonator at time $t$ is $x(t)$. The frequency noise $\Xi(t)$ in equation (1) is defined in terms of the variables $x(t)$ and $\tau(t)$ by the relation

$$\Xi(x, \tau) = -v \nu \phi^2(x),$$

where $\nu(x)$ is the spatial profile of the driven vibrational mode, $v = m_o \omega_0 / 2M$, $m_o$ is the analyte mass, $M = \int dx m \rho(x) \phi^2(x)$ is the vibrational mode effective mass and $\rho_0$ is the resonator bare mass density.

The transition probability matrix $W(x, \eta; x', \eta')$ for the frequency noise process of a single particle is defined by

$$\partial p_{\eta=1}(x, t) = -\Gamma_0(x)p_1(x, t) + \Gamma_1(x)f(x)p_0(t) + L_D(x)p_0(x, t),$$

$$\partial p_{\eta=0}(t) = \int_{-L/2}^{L/2} dx \left[ \Gamma_0(x)p_1(x, t) - \Gamma_1(x)f(x)p_0(t) \right],$$

where $p_1(x, t)$ is the probability density that the particle is on the nanoresonator at position $x$ and time $t$. Similarly, $p_0(t)$ is the probability that the particle is not on the nanoresonator at instant $t$. The incoming flux distribution per particle is $f(x)$, which is normalized as $\int dx f(x) = 1$, and $\Gamma_{i0}(x)$ is the probability per unit of time that the particle is adsorbed (desorbed) at the point $x$. The detachment rate follows the Arrhenius law, $\Gamma_{0i}(x) \sim \exp(-E_a/k_BT)$, where $k_BT$ is the thermal energy and $E_a$ is the activation energy [16]. The attachment rate is proportional to the incoming flux of particles and to the cross-section of the nanoresonator. For small vibration amplitudes, the attachment and detachment rates do not depend on the oscillator state, $u(t)$.

The operator $L_D(x)$ describes the diffusion of an overdamped particle,

$$L_Dp_1(x, t) = \tilde{u}(U'(x)p_1(x, t))/k + D\delta^2 p_1,$$

where $\nu$ and $D$ are the particle friction and diffusion coefficients, respectively, and $U(x)$ is a confining potential acting on the particle. The following reflecting boundary conditions are also assumed: $U'(x)p_1(x, t)/k + D\delta^2 p_1(x, t) = 0$ at $x = \pm L/2$. In thermal equilibrium, the diffusion coefficient is determined by the coupling between the particle and the nanobeam vibrational modes (phonon bath) and by temperature. If the excited vibrational mode is of low order and weakly driven, the nanobeam temperature and the other (not directly driven) modes are not significantly affected by the external drive. Hence, for underdamped vibrational modes, it is a good approximation to consider that the diffusion coefficient is independent of the vibrational mode state, $u(t)$.

In order that the RWA is valid, the noise correlation time, $\tau_c \sim \min[L^2 / D, \Gamma_0^{-1}, \Gamma_1^{-1}]$, should be much larger than the oscillation period, $2\pi / \nu_0$.

Note that equation (11) can be extended to include other subsystems. For instance, the substrate may be considered as a new subsystem which exchanges particles with the resonator subsystem through the clamping regions. In this case additional exchange rates $\Gamma_{i0}(x)$ need to be provided.

3. Dephasing from mass loading of a single particle

The model introduced in section 2 for the frequency noise is used to calculate the oscillator susceptibility and also the nth moment of the complex vibrational amplitude, $u(t)$. Let us consider the fundamental flexural mode of a one-dimensional resonator with mode shape $\phi(x) = \sqrt{2}\cos(\pi x / L)$. The shift of the vibrational frequency due to a particle at position $x$ is $-v\phi^2(x)$. It is assumed that the particle may be adsorbed at any point on the resonator with equal probability; i.e. $f(x) = 1 / L$ in equation (11). Also, the adsorbed particle freely diffuses along the resonator ($U = 0$ in equation (12)) with reflecting boundary conditions, $\partial p_1(x = \pm L/2, t) = 0$. Below, the case where the maximum frequency shift per particle, $2\nu$, is larger than the oscillator damping rate, $\Gamma_1$, and the adsorption rate, $\Gamma_0$, and desorption rate $\Gamma_0$ [16] is discussed. First let us discuss the results for the oscillator susceptibility.

As discussed in section 2, the scaled oscillator susceptibility, $\chi(\delta \omega)$, is given by

$$\chi(\delta \omega) = \tilde{\chi}_0(\delta \omega) + \int_{-L/2}^{L/2} dx \tilde{\chi}_1(\delta \omega, x),$$

where $\tilde{\chi}_0(\delta \omega)$ and $\tilde{\chi}_1(\delta \omega, x)$ are the scaled partial susceptibilities associated with no particle and one particle at position $x$ on the resonator. The equation for the scaled partial
susceptibilities (see equation (18)) can be solved analytically in terms of continued fractions (see appendix A).

Figure 1 depicts the scaled absorption spectrum, $\text{Im}(\chi(\delta\omega))$, for the limit cases of a slow and fast diffusing particle and $\nu \gg \Gamma_i, \Gamma_o, \Gamma$. In the slow diffusion limit, the absorption spectrum exhibits inhomogeneous broadening. For frequency detuning $\delta\omega \approx -2\nu$, the scaled partial susceptibility of a particle at position $x$ is $\tilde{\chi}_i(\delta\omega, x) \approx \langle \chi_i(\delta\omega, x) \rangle + i(\delta\omega + \nu \phi^2(x))$ and the scaled oscillator susceptibility is [13]

$$\chi(\delta\omega) \approx \frac{Lp_{1,\text{st}}}{\sqrt{(\Gamma + \Gamma_o - i\delta\omega)(\Gamma + \Gamma_o - i(\delta\omega + 2\nu))}}. \quad (14)$$

The scaled partial susceptibility $\tilde{\chi}_0(\delta\omega)$ contributes to the total oscillator susceptibility mainly for frequency detuning $\delta\omega \approx 0$. This is reasonable because $\tilde{\chi}_0(\delta\omega)$ corresponds to no adsorbed particle (zero frequency shift). As depicted in figure 1, $\text{Im}(\chi_0(\delta\omega))$ is approximately Lorentzian with a half-width $\approx \Gamma_i + \Gamma_o$ and height proportional to the stationary probability of no particle on the resonator: $p_{0,\text{st}} = \Gamma_o/(\Gamma_i + \Gamma_o)$. The asymmetry observed in $\text{Im}(\chi_0(\delta\omega))$ at $\delta\omega = 0$ is the result of coupling with the scaled partial susceptibilities $\tilde{\chi}_i(x, \delta\omega \approx 0)$.

The limit case of a fast diffusing particle is depicted in the inset of figure 1, assuming that $\nu \gg \Gamma_i, \Gamma_o, \Gamma$. In this limit, the oscillator absorption spectrum can be approximated by two Lorentzians centered at $\delta\omega = 0$ and $\delta\omega = -\nu \phi(x)^2 \Gamma_i$. These Lorentzians have half-widths approximately equal to $\Gamma_i + \Gamma_o + \nu^2 L^2/8\pi^2 D$ and heights approximately equal to $p_{0,\text{st}}/(\Gamma_i + \Gamma_o)$ and $(Lp_{1,\text{st}})/(\Gamma_i + \Gamma_o + \nu^2 L^2/8\pi^2 D)$, respectively. These results agree with dephasing due to telegraph frequency noise, which is obtained in the limit $D \rightarrow \infty$ [14]. Thus, the effect of a finite but large diffusion coefficient ($D \gg L^2\Gamma_i$) is to increase the half-width of the Lorentzian centered at $\delta\omega = -\nu$ by an amount $\approx \nu^2 L^2/8\pi^2 D$.

As the particle diffusion gets slower, this Lorentzian gets broader and it eventually loses its Lorentzian shape and acquires the shape given by equation (14).

Higher-order moments of $u(t)$ are calculated by solving numerically the coupled system (8). In the characterization of the dephasing process, cumulants are more important than the moments of $u(t)$ because the former vanish in the absence of frequency noise [14]. Figure 2 shows the spectrum of the imaginary part of the third cumulant, $\chi_3 = (u^3) - 3(u) \cdot (u^2) + 2(u)^3$, for both limit cases of a fast and a slow diffusing particle. The magnitude and shape of the higher-order cumulants’ spectra depend on the dephasing process parameters (i.e. diffusion coefficient and attachment and detachment rates); moreover, this dependency is more significant for higher-order cumulants of $u(t)$.

Using a frequency noise model such as the one introduced in section 2, measurement of the cumulants of $u(t)$ can provide enough information to characterize the dephasing process and determine the particle parameters. This is demonstrated in figure 3 where the imaginary parts of the second and third cumulant are shown to be more sensitive to a small change of the diffusion coefficient (solid and dashed lines correspond to $D = 3.33L^2\nu$ and $D = 3.50L^2\nu$, respectively). Note that the other particle parameters and oscillator damping rate have been chosen such that the first cumulant is minimally affected (i.e. $\Gamma_i + \Gamma_o, \Gamma_i + \Gamma_o + \nu^2 L^2/8\pi^2 D$, and $p_{0,\text{st}}$ are fixed in the solid and dashed curves of figure 3).

In order to measure the moments and cumulants of the slowly varying complex amplitude $u(t) \equiv X(t) - iY(t)$, one has to measure the in-phase, $X(t)$, and quadrature, $Y(t)$, envelopes of the oscillations: $q(t) = 2X(t)\cos(\omega t) + 2Y(t)\sin(\omega t)$. In carbon-based nanoresonators, these measurements can be performed using the FM mixing technique [23]. Notice that the recording time of $X(t)$ and $Y(t)$ has to be much larger than the correlation time of the frequency noise and the oscillator energy relaxation time.
4. Dephasing from mass loading of many particles

In section 1 we considered dephasing due to particles acting independently on the nanoresonator. Particles act independently on the resonator if the average particle density on the resonator is small such that they do not interact with each other. For simplicity, it is assumed that the particles have identical mass, adsorption and desorption rates and diffusion coefficient. Hence, the total frequency noise $\xi(t)$ is equal to the sum of independent and identical processes $\xi_i(t)$, which are realizations of the process (10),

$$\xi(t) = \sum_{i=1}^{N} \xi_i(t),$$

where $N$ is the total number of particles in the system. Particles can be either in the gas subsystem or in the resonator subsystem. The number of particles on the resonator $n(t)$ evolves according to a Bernoulli process with master equation

$$p_n = -\left[\Gamma_o(N - n) + \Gamma_i(N - n + 1)\right]p_n + \left[\Gamma_i(N - n - 1)\right]p_{n-1},$$

where $N_0 = N\Gamma/(\Gamma_i + \Gamma_o)$ is the average number of particles on the resonator. The stationary distribution of equation (16) for $N_0 \ll N$ and $\Gamma_i = (N_0/N)\Gamma_o \ll \Gamma_o$ is the Poisson distribution: $p_n^0 = \exp(-N_0)N_0^n/n!$.

Figure 4 depicts the oscillator absorption spectrum for $N_0 = 3$ and $\nu \gg \Gamma_o, \Gamma_i$. The limit of fast diffusing particles agrees with the results obtained previously for discrete frequency jumps with a stationary Poisson distribution. The absorption spectrum displays a fine structure (dotted curve) and it is formed by Lorentzians centered at frequencies $\delta \omega = -j\nu$ ($j = 0, 1, \ldots$) with half-widths $\Gamma_j = \Gamma + j\nu$ and $(j + N_0)\Gamma_o$ is equal to the reciprocal mean time of having $j$ particles on the resonator), and the heights are proportional to the stationary distribution $p_n^0$ [12].

The effect of a finite diffusion coefficient, $D$, is that the half-width of the above Lorentzians is increased by an amount $\sim j\nu^2 \Gamma/(8\pi^2 D)$, see figure 4 (solid curve). In the limit of slow diffusion, we can again observe inhomogeneous broadening due to the randomness associated with the particle insertion point. This is depicted in the inset of figure 4. Here, the absorption spectrum, $\Im \chi(\delta \omega)$, also exhibits sharp peaks. However, these peaks are not Lorentzians and they get sharper as $\Gamma_o$ decreases. They also appear in the case of dephasing due to a single particle acting on the resonator, see equation (14). The above results agree with Monte Carlo simulations [24, 25].

5. Conclusions

The effect of random mass loading of small particles on the resonant response of the fundamental flexural mode of a nanobeam has been discussed. A model for the frequency noise has been presented which accounts for particle attachment and detachment at random points on the resonator and particle diffusion along the resonator. Analytical and numerical results have been presented for the mode susceptibility and for some higher-order cumulants of the complex vibrational mode amplitude. In the case of rare adsorption and desorption events and fast diffusion, the oscillator absorption spectrum exhibits fine structure. The diffusion contribution to the linewidths of the (approximately) Lorentzian lines, centered at $-j\nu$, is $\sim j\nu^2 \Gamma/(8\pi^2 D)$, where $D$ is the diffusion coefficient, $j = (0, 1, \ldots)$, $\nu$ is the frequency shift due to a particle located at the vibrational mode antinode and $L$ is the nanobeam length. It has also been demonstrated that higher-order cumulants are more sensitive to the parameters of the frequency noise and their measurement can be used to develop a mass sensing scheme, which is able not only to determine the mass of the particle but also its diffusion coefficient, attachment and detachment rates to the nanoresonator.
time of frequency noise due to only particle diffusion is \( \tau \) independently on the resonator, see section 4. The scaled susceptibility \( \chi(\delta \omega) \) is given by the complex equation for the scaled susceptibility is given by the complex conjugate of equation (8) with \( n = 1 \) and \( \tilde{W} \) defined by the master equation (11), whose stationary solution is \( p_{0, st} = \Gamma_0/\Gamma_1 + \Gamma_0 \) and \( p_{1, st}(x) = (1/L) \cdot \Gamma_1/(\Gamma_1 + \Gamma_0) \). The total scaled susceptibility is equal to the sum of the scaled partial susceptibilities \( \chi_{0}(\delta \omega) \) (no particle on resonator) and \( \chi_1(x; \delta \omega) \) (particle at position \( x \))

\[
\chi(\delta \omega) = \chi_{0}(\delta \omega) + \int_{-L/2}^{L/2} dx \chi_1(x; \delta \omega),
\]

and the scaled partial susceptibilities satisfy the equations

\[
\begin{align*}
[A^* - i \nu \phi^2(x) + \Gamma_0 - D \delta^2(x)] \chi_0 &= \chi_{1, st}(x), \\
(A^* + \Gamma_1) \chi_0 &= \Gamma_0 \int dx \chi_1(\delta \omega, x) = i \psi_{0, st},
\end{align*}
\]

where \( A^* = \Gamma - i \delta \omega \). The solution for \( \chi_1(x; \delta \omega) \) is sought in the form

\[
\chi_1(x; \delta \omega) = \sum_{k \geq 0} A_k \cos(2\pi kx/L).
\]

The coefficients \( A_k \) satisfy a difference equation system which can be solved in terms of continued fractions

\[
\int dx \chi_1(x; \delta \omega) = \frac{L \cdot i \psi_{1, st} + i \Gamma_1 \psi_{0, st}}{R(D; \delta \omega) - \Gamma_0/\Gamma_1} + \chi_{0, st},
\]

\[
\chi_{0}(\delta \omega) = \frac{\Gamma_0 \int dx \chi_1(\delta \omega, x) + i \psi_{0, st}}{\Gamma + \Gamma_1 - i \delta \omega},
\]

where \( R \) is given by

\[
R(D; \delta \omega) = \Theta(0) + \frac{\nu^2/2}{\Theta(1) + \frac{\nu^2/2}{\Theta(2) + \frac{\nu^2/2}{\cdots}}},
\]

and \( \Theta(n) = \Gamma + \Gamma_0 - i(\delta \omega + \nu) + n^2 \delta^2_D \). The correlation time of frequency noise due to only particle diffusion is \( \tau_D = (L/2\pi)^2/D \) [13].

Appendix B

Here we consider the calculation of the (scaled) susceptibility function, \( \chi(\delta \omega) \), for the case of many particles acting independently on the resonator, see section 4. The scaled oscillator susceptibility can be expressed as

\[
\chi(\delta \omega) = i \int_{-\infty}^{0} dt e^{\lambda^* t} \cdot g^N(t),
\]

where \( N \) is the total number of particles in the system (in and out of the resonator), \( \Lambda^* = \Gamma - i \delta \omega \) and the dephasing-dependent term \( g(t) \) is equal to

\[
g(t) = \int dx \tilde{\chi}_1(x, t) + \tilde{\chi}_0(t).
\]

The time-dependent partial susceptibilities \( \tilde{\chi}_1(x, \delta \omega) \) and \( \tilde{\chi}_0(\delta \omega) \) are related to \( \tilde{\chi}_1(x, \delta \omega) \) and \( \tilde{\chi}_0(\delta \omega) \) by the Fourier transform

\[
\tilde{\chi}_1(x; \delta \omega) = \int_{-\infty}^{0} dt e^{\lambda^* t} \tilde{\chi}_1(x, t),
\]

and a similar relation between \( \tilde{\chi}_0(t) \) and \( \tilde{\chi}_0(\delta \omega) \) holds. These new (scaled) partial susceptibilities satisfy the equations

\[
\partial_t \tilde{\chi}_1 = [\Gamma_0 - i \nu \phi^2(x)] \tilde{\chi}_1 - \Gamma_0 f(x) \tilde{\chi}_0,
\]

\[
\partial_t \tilde{\chi}_0 = \Gamma_0 \tilde{\chi}_0 - \Gamma_1 \tilde{\chi}_1 - \Gamma_0 \int dx \tilde{\chi}_1(x, t),
\]

with initial conditions \( \tilde{\chi}_1(x, 0) = i \psi_{1, st}(x) = (i/L) \cdot \Gamma_1/(\Gamma_1 + \Gamma_0) \) and \( \tilde{\chi}_0(0) = i \psi_{0, st} = i \Gamma_0/\Gamma_1 \) with reflecting boundary conditions: \( \partial_t \tilde{\chi}_1(x = \pm L/2, t) = 0 \). Note that \( g(t) \) can also be obtained by taking the inverse Fourier transform of the result equation (19) with \( \Gamma = 0 \).

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