Some Considerations on Universality

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The paper puts into discussion the concept of universality, in particular for structures not of the power of Turing computability. The question arises if for such structures a universal structure of the same kind exists or not. For that the construction of universal Turing machines and those with some constraints are presented in some detail.

1 Introduction

Without the investigation of the concept of universality it is quite possible that our modern computers wouldn’t exist. Although they are finite automata, at least theoretically, practically they can be treated as Turing machines processing any special program which can be seen as a special Turing machine. This paper presents some general considerations on that concept, presenting in some more detail universality for general Turing machines, deterministic as well as non-deterministic ones. Universal devices not only have been considered for Turing machines (TM’s), but also for cyclic Post machines (CPM’s), cellular automata (CA’s), random access machines (RAM’s), and others.

Very small universal devices mainly have been constructed for DTM’s, CPM’s, and RAM’s. Until recently, the encoding of special deterministic Turing machines (DTM’s) for small universal (deterministic) Turing machines (U(D)TM’s) was achieved via universal tag systems, needing exponential space and time for the simulation. A polynomial amount of space and time was shown in [7][8].

Universality has mainly been investigated intensively for the class of Turing computability whereas there exist only few publications on universality for restricted systems, having the same restriction. Such a restriction is e.g. space or time complexity, or working conditions. Furthermore, the concept of universality also should be investigated for other underlying structures than free monoids, e.g. commutative monoids or more general structures. Actually the ideas presented here arose from the question if there exists a universal (higher order) Petri nets capable of simulating all special Petri nets but not having power of Turing computability.

The concept of universality discussed here is somehow restricted, since there should not just be a black box with input special machine (e.g. a DTM) together with its input, and output of the somehow simulated result, without considering also e.g. space and time amount of the simulation [4][1]. It should also somehow simulate the (local) behaviour (e.g. in space and time).

To clarify the problem universality for Turing machines will be considered in more detail.

2 General Turing Machines

For general Turing machines (TM’s), in particular for the deterministic version (DTM), there exists the concept of universality. This means that a single DTM can be constructed which can simulate any special DTM in a certain way. For that purpose it is necessary to encode a DTM as well as its input.
Since the encodings have to be invertible, the corresponding encoding functions have to be injective, and the encoding as well as decoding functions have to be computable by DTM’s.

Formally, assume an enumerable universal alphabet $\Sigma$, such that that any special DTM $M$ has a finite input and work alphabet $\Sigma_M \subset \Sigma$. Similarly, assume an enumerable universal set of states $Q$ with $Q \cap \Sigma = \emptyset$, such that any special DTM $M$ has a finite set of states $Q_M \subset Q$. In both cases, $\Sigma$ and $Q$ are equivalent to $\mathbb{N}$. It can be also assumed that $Q = \{q_1, \ldots, q_{|Q_M|}\}$, and that the initial state of $M$ is $q_1$ and the final state $q_{|Q_M|}$. Furthermore, that by renaming the symbols of $M$ are $\Sigma_M = \{s_1, \ldots, s_{|\Sigma_M|}\}$.

Any DTM $M$ defines a local function $f_M$ given by the program of $M$, affecting the configurations of $M$. Those can be written as $w_Lq_xw_R$ where $w_L, w_R \in \Sigma^+_M$, and $q \in Q_M$. An initial configuration has the form $\xi = w_Lq_1w_R$, and a final configuration the form $w_Lq_{|Q_M|}w_R$, where $w_L \in \Sigma^+_M$ and $w_R \in \Sigma^+_M$. Quite often also a normal form $\xi = q_1w$, and $c_F = q_{|Q_M|}w$, respectively, is used where $w \in \Sigma^+_M$. A step of $M$ can then be expressed by $f_M(w_Lq_xw_R) = w_L'q_x'w_R'$, with the usual interpretation of new symbol, new state and movement. Thus $f_M : \Sigma^+_M \Sigma^+_M \to \Sigma^+_M \Sigma^+_M$. $f_M$ can be extended to a, in general partial, function $f_M^* : \Sigma^+_M \Sigma^+_M \to \Sigma^+_M \Sigma^+_M$ by the definition $f_M^*(w_Lq_1w_R) = w_Lq_{|Q_M|}w_R'$. This also defines a function $\tilde{f}_M : \Sigma^+_M \to \Sigma^+_M$ by ($w = w_Lw_R$ and $w' = w_L'w_R'$), $f_M(w) = w' \iff f_M^*(w_Lq_1w_R) = w_Lq_{|Q_M|}w_R'$. For the above mentioned normal form this gives $f_M(q_1w) = q_{|Q_M|}w'$ and

For a universal DTM $U$ the program as well as the configurations of special DTM $M$ have to be encoded. Simple encoding and decoding functions should be used for that purpose. They have to be injective and computable by DTM’s $\Psi_M$ and $\Phi_M$, as well as the inverses by some DTM’s $\Psi_U$ and $\Phi_U$. Actually, these $DTM$’s are related to $M$ since some information on $Q_M$ and $\Sigma_M$ is necessary. Let $\Sigma_U \subset \Sigma$ be the alphabet, and $Q_U \subset Q$ the set of states of $U$. One possibility of encoding are functions $f_{\Psi_M} : (\Sigma_M \cup Q_M) \to (\Sigma_U \setminus \{\#\})^+$ and $f_{\Phi_M} : (\Sigma_M \cup Q_M) \to (\Sigma_U \setminus \{\#\})^+$ where $\# \in \Sigma_U$ is a special separator.

The initial and final configurations of $U$ for simulating $M$ can be chosen in a normal form as $q_1x y f_{\Psi_M}(P_M)\#y f_{\Phi_M}(c_1)$ and $q_{|Q_M|}x y f_{\Psi_M}(P_M)\#y f_{\Phi_M}(c_F)$ where $x, y \in \Sigma_U$ are special symbols indicating the current position in the program $P_M$ and configuration $c_M$, $P_M$ is the program of $M$, and $\# \in \Sigma_U$ is a separator. Both functions, $f_{\Psi_M}$ and $f_{\Phi_M}$, are injective. The configurations of $U$ simulating $M$ have one of the forms

For the simulation the following condition for one step $c_M \to c'_M$ of $M$, or reaching a final configuration of the initial configuration of $M$ holds (here for normal forms):

\[
\forall M \forall c_M \exists n \in \mathbb{N} : f_M^n(q_x y f_{\Psi_M}(P_M)\#y f_{\Phi_M}(c_M)) = q_x y f_{\Psi_M}(P_M)\#y f_{\Phi_M}(c'_M) \\
\text{or} \\
\forall M \exists n \in \mathbb{N} : f_M^n(q_1x y f_{\Psi_M}(P_M)\#y f_{\Phi_M}(c_1)) = q_{|Q_M|}x y f_{\Psi_M}(P_M)\#y f_{\Phi_M}(c_F).
\]

Often $\{0, 1\} \subset \Sigma_U$ is chosen for the encoding alphabet. $\Sigma_M$, $Q_M$, $P_M$, and configurations $c_M$ are encoded either in unary or binary. Then $s_i$ can be encoded by $1^i$ in unary or by $\text{bin}(i)$ in binary. Analogously for $q_j$. A configuration $s_{i_1} \cdots s_{i_\ell} q j s_{i_{\ell+1}} \cdots s_{i_v}$ can be encoded e.g. by $1_{\ell+1} 0 \cdots 1_{v+1} 0 \cdots 1_{v'}$ in unary or by $\text{bin}(i_1) a \cdots \text{bin}(i_v) \text{bin}(j) \text{bin}(i_{\ell+1}) a \cdots \text{bin}(i_v) a$ in binary where $a, b$ are separating symbols. Encoding $0, 1, a, c$ e.g. by $00, 01, 10, 11$ then gives a binary encoding using only 0, 1.

The unary encoding needs $|w|([|\Sigma_M| + 1] + |Q_M| + 4)$ space, whereas the binary encoding needs $2(|w|([\log_2|\Sigma_M| + 1] + [\log_2|Q_M|] + 2)$ space.
Encoding and also decoding can be achieved by very simple DTM’s, actually by deterministic finite state transducers (DFST’s) consuming linear time. For binary encoding the transducer also needs information on the sizes $|\Sigma_M|$ and $|Q_M|$.

A special class of TM is the class of TM with alphabet $\{0, 1\}$. In that case only states have to be encoded. Thus the bounds are $3|w| + |Q_M| + 4$ or $4|w| + 2\lceil \log_2 |Q_M| \rceil + 4$.

The considerations so far also hold for non-deterministic TM’s (NTM’s). Encodings again can be achieved by DTM’s. Instead of functions one has to consider binary relations between configurations. The universal NTM works like the deterministic counterpart, with the difference that because of the encoding of the non-deterministic program $P_M$ the universal NTM has choices for its steps.

It is also an interesting aspect to look for small universal NTM.

### 3 Turing Machines with Constraint

In the case of TM’s with some constraint the DTM for encoding and decoding should be restricted to the same constraint. Such a constraint is e.g. space or time complexity. But it could be also reversibility or some property of the function or relation defined by the TM. In any case, besides alphabet and set of states of the special TM also the complexity function has to be encoded.

In [2] it has been shown that for the class of DTM’s with a binary alphabet $\{0, 1\}$ and a space complexity function belonging to a subclass of primitive recursive functions over 1 variable, there exists a universal DTM with the same complexity constraint. Note that only one variable is needed for the computation of the complexity function. Examples of such primitive recursive complexity functions are $g(x) = c \cdot x$, $g(x) = c \cdot x^k$, $g(x) = c \cdot 2^x$, $g(x) = c \cdot 2^{2^x}$, polynomials with non-negative coefficients

$$p(x) = \sum_{i=1}^{k} c_i x^{k-i} \in \mathbb{N},$$

g(x) = 2^{p(x)} \text{ where } p(x) \text{ is a polynomial as just defined.}

This subclass of primitive recursive functions $g$ does not use projections and has the property:

$$\exists c_g \in \mathbb{N} \forall \bar{x} \in \mathbb{N}^k : c_g + g(\bar{x}) \geq \sum_{i=1}^{k} x_i,$$

which in the case of 1 variable results in $c_g + g(x) \geq x$.

Thus $g(x) = \lceil \log_2(x) \rceil$ does not belong to this subclass.

If $r_g$ is the length of the representation of the space complexity function $g$ then the universal DTM needs only $s_g(x) = g(x) + r_g$ space.

For the analogous time complexity constraint there is a price of polynomial cost. In this case there is a polynomial $p_g$ with non-negative coefficients for every primitive recursive function from the subclass defined above such that the time complexity of the universal DTM fulfills $t_g(x) \leq p_g(g(x))$ for all $x \in \mathbb{N}$.

Since the space and time complexity functions are computed by DTM’s it follows by straight forward arguments that the results from above also hold for non-deterministic TM’s (NTM’s), i.e. there exists a universal NTM for the subclass of primitive recursive functions as complexity constraints.

Thus there exist universal DTM’s and NTM’s with complexity constraints for each complexity function from the special class of primitive recursive functions (for time complexity only with additional polynomial). In particular, there exist universal (N)LBA’s and DLBA’s.
4 Finite Automata and Finite State Transducers

In the case of deterministic and non-deterministic finite automata (DFA’s, NFA’s), as well as deterministic and non-deterministic finite state transducers (DFST’s, NFST’s) the encoding and decoding of special FA and FST should be achieved by DFST’s. Otherwise, the encoding might contain too much computation power, going even beyond the power of FA’s or FST’s.

In [3] it was shown that under the above condition that encoding of special FA’s and FST’s has to be achieved by a DSFT, there exists no universal DFA, NFA, DFST and NFST. The proof uses a contradiction on the number of states of such a universal automaton.

Thus there are no universal FA’s in this strong sense.

In the same paper ([3]) it has also been shown that there exists a universal FA for the class of all FA’s with a bounded number \( k \) of states. However, that universal FA does not belong to the same class, needing strictly more states.

5 Pushdown Automata and Pushdown Transducers

For universal DPDA’s and NPDA’s, as well as for corresponding transducers (DPDT’s and NPDT’s), the encoding of special DPDA and NPDA should be done by a DPDT. From the considerations above for Turing machines it follows that deterministic finite state transducers suffice for encoding and decoding. It is unknown that under such conditions there exist universal automata of that kind. But it is my conjecture that such do not exist, since there is a problem in accessing the program of a special PDA. Possibly this can be shown using Kolmogorov complexity arguments.

6 Rewriting Systems

In the case of word grammars or other word rewriting systems like L-systems, it is necessary to use a deterministic or monogeneous grammar or rewriting system of the same type for encoding the special ones.

For general semi Thue systems (STS) or type-0 grammars this is straightforward since any such system can be simulated by a TM, and vice versa. In a similar way, type-1 or monotone grammars can be simulated by NLBA’s and vice versa.

For type-2 grammars, however, my conjecture is that there does not exist a universal grammar of that type. Here the encoding should be achieved by some kind of deterministic or monogeneous type-2 grammar used as transducer. The same seems to hold for ETOL systems where encoding should be done by an EDTOL system used as transducer. An ETOL system [9] consists of a finite number of not necessarily disjoint finite sets of context-independent rules which in one derivation step have to be applied in parallel to all symbols in a word using only, not necessarily identical rules, of one set. In an EDTOL system there is only one rule for each symbol.

7 Multiset Grammars

Whereas for word grammars the encoding of special grammars can be done by a FST it is a problem for multiset grammars, at least for simple multisets as elements from \( \mathbb{N}^k \). This arises when looking for a universal vector addition system (VAS) or a universal Petri net (PN). Most probably, such don’t exist,
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at least for simple multisets, i.e. elements from $\mathbb{N}^k$. The reason for that possibly is the commutativity of the basic underlying operation $\oplus$, whereas for word systems catenation is not commutative.

Possibly, higher order multisets, used as tokens in higher order Petri nets, might be a solution for the construction of universal higher order Petri nets. However, they should not be of the power of Turing machines.

8 General Universality

A possible definition of general universality to cover e.g. all the cases considered so far, may be the following construction.

Let $\mathcal{M}$ be a universe of elements, equipped with a binary operation $\circ$ and $\mathcal{R}$ a collection (class) of binary relations $r \subseteq \mathcal{M} \times \mathcal{M}$ with the property $\circ: \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{M}$ for $r \in \mathcal{R}$.

To define a universal relation $R$ for $\mathcal{R}$ encoding of elements $x \in \mathcal{M}$ and of $r$ itself is necessary. This might be achieved by injective functions $f_r: \mathcal{M} \rightarrow \mathcal{M}'$ and $g: \mathcal{R} \rightarrow \mathcal{M}''$ from the same class $\mathcal{R}$, such that

$$\forall r \in \mathcal{R} \forall x, y \in \mathcal{M} \exists n \in \mathbb{N}: (g(r) \circ f_r(x), g(r) \circ f_r(y)) \in \mathcal{R}^n \Leftrightarrow (x, y) \in r,$$

where $\mathcal{M}_R = \mathcal{M}' \cup \mathcal{M}''$ and $\mathcal{M}' \cap \mathcal{M}'' = \emptyset$, and $R \subseteq \mathcal{M} \times \mathcal{M}$.

But there is a problem about the operation $\circ$ since actually it is a ternary relation on $\mathcal{M}$.

More general, with a single pairing function $\phi: \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{M}$, the condition is

$$\forall r \in \mathcal{R} \forall x, y \in \mathcal{M} \exists n \in \mathbb{N}: (g(r), \phi(f_r(x), f_r(y))) \in \mathcal{R}^n \Leftrightarrow (x, y) \in r.$$

Another possibility may be to choose as basic universe a Scott domain as for the construction of a model for the $\lambda$-calculus.

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