Positivity of Hadamard powers of tridiagonal matrices

Veer Singh Panwar ∗, A. Satyanarayana Reddy†
Department of Mathematics
Shiv Nadar University, Dadri
U.P. 201314, India.

Abstract

We characterize all tridiagonal infinitely divisible matrices. It is well known that if $A$ is an entrywise nonnegative, positive semidefinite matrix of order $n$, then $A^r = [a_{ij}^r]$ is positive semidefinite when $r \geq n - 2$ or $r$ is a positive integer. We prove that if $A$ is any positive definite (semidefinite) tridiagonal matrix, then $A^r$ is positive definite (semidefinite) for $r > 1$.

We give similar results for a special family of Pentadiagonal matrices.

AMS classification: 15B48, 47B36, 15B36

Keywords: Infinitely divisible matrices, tridiagonal matrices, Hadamard powers, pentadiagonal matrices, chain sequences, totally nonnegative matrices.

1 Introduction

Let $\mathbb{M}_n(\mathbb{R})$ denotes the set of all $n \times n$ matrices over $\mathbb{R}$. A matrix $A \in \mathbb{M}_n(\mathbb{R})$ is called nonnegative if all its entries are nonnegative. In this paper, every matrix belongs to $\mathbb{M}_n(\mathbb{R})$, and is nonnegative unless otherwise stated. A matrix is called totally nonnegative if all its minors are nonnegative. For more results on totally nonnegative matrices, see [4]. A matrix $A \in \mathbb{M}_n(\mathbb{R})$ is called positive semidefinite (PSD) (positive definite (PD)) if $A$ is symmetric and $\langle x, Ax \rangle \geq 0$ for all $x \in \mathbb{R}^n$ ($\langle x, Ax \rangle > 0$ for all $x \in \mathbb{R}^n$ and $x \neq 0$). The Hadamard product of $A = [a_{ij}]$ and $B = [b_{ij}]$ denoted by $A \circ B$ is defined as $A \circ B = [a_{ij}b_{ij}]$. If $r > 0$, then we denote $r^\text{th}$-Hadamard power of $A = [a_{ij}]$ by $A^r$ (or $(A)^r$), where $A^r = [a_{ij}^r]$.

A nonnegative, symmetric matrix $A$ is said to be infinitely divisible (ID) if $A^r$ is PSD for every $r > 0$. It is clear that every ID matrix is a PSD matrix, however, converse need not be true [1]. Some basic examples of ID matrices are nonnegative PSD matrices of order 2 and diagonal matrices with nonnegative diagonal entries. For more examples and results on ID matrices, see [2, 6]. A lot of work has been done on the real entrywise powers preserving the positive semidefiniteness of various families of matrices. For more information on this, see [3, 10, 11]. For $A \in \mathbb{M}_n(\mathbb{R})$, where $A$ is a PSD matrix, it is known that $A^r$ is PSD if $r$ is a positive integer or $r \geq n - 2$, see [3, 10], and this lower bound on the critical exponent $r$ is sharp [5, Theorem 2.2], i.e., for every non integer $r < n - 2$, there exists a positive semidefinite matrix $A \in \mathbb{M}_n(\mathbb{R})$ such that $A^r$ is not positive semidefinite.

A sequence $\{a_k\}_{k \geq 0}$ is called a chain sequence if there exist a parameter sequence $\{g_k\}_{k \geq 0}$ such that $0 \leq g_0 < 1$ and $0 < g_k < 1$ for $k \geq 1$ and $a_k = (1 - g_{k-1})g_k$ for $k \geq 1$, [3] page 91. A basic example of chain sequence is the constant sequence $\{\frac{1}{2}\}_{k \geq 1}$ with the parameter sequence $\{\frac{1}{2}\}_{k \geq 1}$. For more information and examples on chain sequences, see [11, 12, 9].

∗Email: vs728@snu.edu.in
†Email: satya.a@snu.edu.in
A matrix $T = [t_{ij}] \in \mathbb{M}_n(\mathbb{R})$, $1 \leq i, j \leq n$, where $t_{ij} = 0$ for $|i-j| > 1$ is called tridiagonal. A matrix $P = [p_{ij}] \in \mathbb{M}_n(\mathbb{R})$, $1 \leq i, j \leq n$, where $p_{ij} = 0$ for $|i-j| > 2$ is called pentadiagonal. Let $T, P \in \mathbb{M}_n(\mathbb{R})$ be symmetric tridiagonal and symmetric pentadiagonal matrices respectively, then

\[
T = \begin{bmatrix}
  a_1 & b_1 & 0 & \cdots & 0 \\
  b_1 & a_2 & b_2 & \cdots & 0 \\
  0 & b_2 & a_n & \cdots & 0 \\
  \vdots & \vdots & \ddots & \ddots & \vdots \\
  0 & 0 & \cdots & a_{n-1} & b_{n-1} \\
  0 & 0 & \cdots & b_{n-1} & a_n
\end{bmatrix},
\]

\[
P = \begin{bmatrix}
  a_1 & b_1 & c_1 & \cdots & 0 \\
  b_1 & a_2 & b_2 & \cdots & 0 \\
  c_1 & b_2 & a_3 & \cdots & 0 \\
  \vdots & \vdots & \ddots & \ddots & \vdots \\
  0 & 0 & \cdots & c_{n-2} & b_{n-1} \\
  0 & 0 & \cdots & b_{n-1} & a_n
\end{bmatrix}, \quad (1)
\]

where $a_i, b_j, c_k \geq 0$ for $1 \leq i \leq n, 1 \leq j \leq (n-1), 1 \leq k \leq (n-2)$. We discuss the infinite divisibility of tridiagonal and a special family of pentadiagonal matrices.

A matrix $A = [a_{ij}] \in \mathbb{M}_n(\mathbb{R})$ is called a band matrix of bandwidth $d$ if $a_{ij} = 0$ for $|i-j| > d$. Hence, the above matrices $T$ and $P$ are band matrices of bandwidth 1 and 2, respectively. A graph $G = (V, E)$, where $V = \{1, \ldots, n\}$, is called a band graph of bandwidth $d$ if $\{i, j\} \in E$ if and only if $i \neq j$ and $|i-j| \leq d$. Let $G = (V, E)$ be a band graph of bandwidth $d$, where $V = \{1, \ldots, n\}$, and $A = [a_{ij}] \in \mathbb{M}_n(\mathbb{R})$ be a PSD matrix such that $a_{ij} = 0$ for $i \neq j$ and $(i, j) \notin E$. Then $A^{sr}$ is positive semidefinite for $r \geq \min(d, n-2)$ \[8\] Cor 3.11. Hence, if $T$ and $P$ are PSD matrices, which are tridiagonal and pentadiagonal, respectively, then $T^{sr}$ and $P^{os}$ are PSD for $r \geq 1$ and $s \geq 2$, respectively. We show that if $T$ is a tridiagonal PD matrix, then $T^{sr}$ is PD for $r > 1$; and discuss some of its nice consequences. Then, we prove a similar result for a special family of pentadiagonal matrices.

In Section 2 we give a characterization for tridiagonal ID matrices. In Section 3 using chain sequences, we show that if $T$ is a tridiagonal PD matrix, then $T^{sr}$ is PD for $r > 1$. We show that if $P = [p_{ij}]$ is a PD (PSD) pentadiagonal matrix, where $p_{ij} = 0$ if $|i-j| = 1$, and $p_{ij} > 0$ if $|i-j| = 0$ or 2, then $P^{sr}$ is PD (PSD) for $r > 1$. We also give a characterization for such pentadiagonal ID matrices. We end the paper with some important remarks.

## 2 Tridiagonal infinitely divisible matrices

A symmetric block diagonal matrix is PSD (PD,ID) if and only if each block is PSD (PD, ID). Each principal submatrix of $A$ is ID if $A$ is ID. Every PSD matrix of order 2 is ID. The sum of two PSD matrices is PSD.

**Lemma 2.1.** Let $A = \begin{bmatrix} a_1 & b_1 & 0 \\ b_1 & a_2 & b_2 \\ 0 & b_2 & a_3 \end{bmatrix}$ be a PSD matrix of order 3. Then $A$ is ID if and only if $b_1b_2 = 0$.

**Proof.** Let $A$ be ID and $G = \lim_{r \to 0^+} A^{sr}$. Then $G$ is a PSD matrix, as it is the limit of a sequence of PSD matrices. Let $b_1$ and $b_2$ be positive. Since $A$ is PSD, $a_1, a_2$ and $a_3$ are positive. But then $G = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ is not a PSD, we get a contradiction. Hence $b_1b_2 = 0$. Conversely, if $b_1b_2 = 0$, then det$(A^{sr}) \geq 0$ for $r > 0$. Hence $A$ is ID.

\[\square\]

We now characterize all tridiagonal ID matrices of order $n \geq 4$.  

\[\]
Theorem 2.2. Let $T$ be a tridiagonal PSD matrix of order $n$ as given in equation (1). Then, for $n \geq 4$, $T$ is ID if and only if $b_ib_{i+1} = 0$ for every $i \in \{1, 2, \ldots, n-2\}$, i.e., the sequence $\{b_i\}_{i \geq 1}$ has no two consecutive positive entries.

Proof. Let $T$ be ID. Let $b_kb_{k+1} > 0$, for some $1 \leq k \leq (n-2)$, then by Lemma 2.1, the principal submatrix
\[
\begin{bmatrix}
    a_k & b_k & 0 \\
    b_k & a_{k+1} & b_{k+1} \\
    0 & b_{k+1} & a_{k+2}
\end{bmatrix}
\]
of $T$ is not ID. Thus $T$ is not ID, which contradicts the hypothesis. Hence, $b_ib_{i+1} = 0$ for every $i \in \{1, 2, \ldots, n-2\}$. Conversely, if $b_ib_{i+1} = 0$ for every $i \in \{1, 2, \ldots, n-2\}$, then $A$ becomes a block diagonal matrix, where each non-zero diagonal block is a PSD matrix of order 1 or 2, hence $T$ is ID.

Remark 2.3. From Theorem 2.2 we have, $T$ is an ID matrix if and only if $T$ is a block diagonal matrix, where each non-zero diagonal block is a PSD matrix of order 1 or 2.

In general, ID matrices are not closed under addition and multiplication. For example, let $X = [x_i x_j]$, where $x_1, \ldots, x_n$ are distinct positive real numbers and $J_n$ be the matrix of order $n$ with each of its entries equals to 1, then $X$ and $J_n$ are both ID, but their sum is not ID, see Theorem 1.1 of [10]. The Cauchy matrix $C = [c_{ij}] = \left[\frac{1}{i+j}\right], 1 \leq i, j \leq 3$ is ID [1] but its square $C^2$ is not ID because
\[
\det(C^2)^{\frac{1}{2}} = \det\left( (\frac{1}{i+j})^{\frac{1}{2}} \right) < 0.
\]
We say that two block diagonal matrices are of same structures if their corresponding blocks are square matrices of the same order. The set of block diagonal matrices of the same structure, is closed under addition, multiplication and multiplication by a nonnegative scalar. Hence, by Remark 2.3 we get the following:

Corollary 2.4. Let $T$ be a tridiagonal ID matrix of order $n \geq 2$ and $f(x) = \sum_{k=0}^{n} a_k x^k \in \mathbb{R}[x]$ be a polynomial, where $a_k \geq 0$ for all $0 \leq k \leq n$, then $f(T) = \sum_{k=0}^{n} a_k T^k$ is ID.

Observation 2.5. The results on tridiagonal ID matrices help us to provide an algorithm to check if a given $3 \times 3$ PSD matrix is ID or not. Consider the PSD matrix
\[
A = \begin{bmatrix}
    a_1 & b_1 & c_1 \\
    b_1 & a_2 & b_2 \\
    c_1 & b_2 & a_3
\end{bmatrix}.
\]

Let all entries of $A$ be positive. Then by Corollary 1.6 and Theorem 1.10 of [7], $A$ is ID if and only if
\[
\begin{bmatrix}
    \log(a_1 a_2) & \log(b_1 b_2) \\
    \log(b_1 b_2) & \log(a_2 a_3)
\end{bmatrix}
\]
is PSD. If $c_1 = 0$, then $A$ becomes tridiagonal and thus by Lemma 2.1, $A$ is ID if and only if $b_1 b_2 = 0$. In the remaining cases, one can easily observe that $A$ is ID if and only if $a_1 a_3 > 0$ and $b_1 = 0 = b_2$. Hence, this way we can characterize all ID matrices of order 3.

3 Hadamard powers preserving positivity

To prove our next result we need the following two results related to chain sequences:

Theorem 3.1. \cite{3} Theorem 5.7] If $\{a_k\}_{k=1}^n$ is a chain sequence and $0 < c_k \leq a_k$ for $k \geq 1$, then $\{c_k\}_{k=1}^n$ is also a chain sequence.

Let $T$ be a tridiagonal matrix of order $n$ as given in equation (1) with $a_i, b_j > 0$ for $1 \leq i \leq n, 1 \leq j \leq (n-1)$. Then we have,
Theorem 3.2. [8, Theorem 3.2] The matrix \( T \) defined as above is positive definite if and only if \( \left\{ \frac{b_i^2}{a_i a_{i+1}} \right\}_{i=1}^{n-1} \) is a chain sequence.

Let \( T \) be a tridiagonal matrix of order \( n \) as given in equation (1). If \( b_j = 0 \) for some \( 1 \leq j \leq (n-1) \), then \( T \) becomes a block diagonal matrix having two smaller diagonal blocks. Continuing this way with these smaller blocks and keep repeating the process, one can see that \( T \) is a block diagonal matrix, where each diagonal block is a tridiagonal matrix with positive entries on its upper and lower diagonals. Moreover, every such block of \( T \) is a PD matrix with positive entries on the main, upper and lower diagonals, if \( T \) is PD.

We are now ready to state and prove our first main result.

Theorem 3.3. Let \( T \) be a tridiagonal matrix of order \( n \). If \( T \) is PD, then \( T^{or} \) is PD for \( r > 1 \).

Proof. Let \( T \) be a tridiagonal PD matrix. Then \( T \) is a block diagonal matrix, where each diagonal block is a tridiagonal PD matrix with positive entries on its main, upper and lower diagonals. Hence, to show that \( T^{or} \) is PD for \( r > 1 \), it's sufficient to prove the our result for every symmetric tridiagonal matrix with positive entries on its main, upper and lower diagonals.

Let \( T_1 \) be a tridiagonal PD matrix of order \( n \) as given in equation (1), where \( a_i, b_i > 0 \) for \( 1 \leq i \leq n, 1 \leq j \leq (n-1) \). By Theorem 3.2, \( \left\{ \frac{b_i^2}{a_i a_{i+1}} \right\}_{i=1}^{n-1} \) is a chain sequence. Let \( 1 \leq i \leq n-1 \) and \( r > 1 \). Since \( T_1 \) is PD, \( a_i a_{i+1} - b_i^2 > 0 \), so \( \frac{b_i^2}{a_i a_{i+1}} < 1 \), which gives us \( 0 < \left( \frac{b_i^2}{a_i a_{i+1}} \right)^r < \frac{b_i^2}{a_i a_{i+1}} \). Thus by Theorem 3.1, \( \left\{ \frac{b_i^2}{a_i a_{i+1}} \right\}_{i=1}^{n-1} \) is also a chain sequence. Hence, by Theorem 3.2, the matrix \( T_1^{or} \) is PD. This completes the proof. \( \square \)

Corollary 3.4. Let \( T \) be a tridiagonal matrix of order \( n \). If \( T \) is PSD, then \( T^{or} \) is PSD for \( r > 1 \).

Proof. Let \( T \) be a tridiagonal PSD matrix of order \( n \). There exist a sequence \( \{T_k\}_{k \geq 1} \) of tridiagonal PD matrices, where \( T_k = T + \frac{1}{k}I \) for \( k \geq 1 \), such that \( T_k \) converges to \( T \) entrywise as \( k \to \infty \) [8, Page 432]. Let \( r > 1 \). By Theorem 3.3, we have, \( T_k^{or} \) is PD for \( k \geq 1 \). Since \( T^{or} \) is the limit of a sequence \( \{T_k^{or}\}_{k \geq 1} \) of PD matrices, \( T^{or} \) is PSD. Hence done. \( \square \)

The next result is an immediate consequence of Corollary 3.4 and the fact that every (nonnegative) tridiagonal PSD matrix is totally nonnegative [8, page 9].

Corollary 3.5. Let \( T \) be a tridiagonal matrix of order \( n \), which is symmetric and totally nonnegative. Then \( T^{or} \) is a totally nonnegative for \( r > 1 \).

Let \( A \) be a symmetric matrix of order \( n \). For \( \alpha = \{\alpha_1, \ldots, \alpha_k\} \subseteq \{1, 2, \ldots, n\} \), where \( \alpha_1 < \alpha_2 < \cdots < \alpha_k \), let \( A[\alpha] \) denotes the principal submatrix of \( A \) obtained by picking rows and columns indexed by \( \alpha \). \( A \) is PD if and only if all its leading principal minors are positive. If \( A \) is a PD matrix then all its principal submatrices are PD. Let \( \text{Perm}(i_1, \ldots, i_n), 1 \leq i_1, \ldots, i_n \leq n \) be the permutation matrix of order \( n \), whose \( k \)'th row is \( i_k \)'th row of the identity matrix of order \( n \). If \( A \) is a PD (PSD) matrix of order \( n \), then \( XAX^* \) is PD (PSD) for any square matrix \( X \) of order \( n \).

We give our second main result as follows:

Theorem 3.6. Let \( P = [p_{ij}] \) be a pentadiagonal matrix of order \( n \) as given in equation (1), where \( p_{ij} = 0 \) for \( |i-j| = 1 \), and \( p_{ij} > 0 \) for \( |i-j| = 0 \) and \( 2 \). We have,

1. If \( P \) is PD (PSD), then \( P^{or} \) is PD (PSD) for \( r > 1 \).

2. If \( P \) is ID if and only if the sequences \( \{c_{2i}\}_{i \geq 1} \) and \( \{c_{2i-1}\}_{i \geq 1} \) have no two consecutive positive entries.
Proof. Let \( P = [p_{ij}] \) be a pentadiagonal matrix of order \( n \) as given in equation (1), where \( p_{ij} = 0 \) for \( |i - j| = 1 \), and \( p_{ij} > 0 \) for \( |i - j| = 0, 2 \). Let \( A_i^* = P[\beta] \) and \( A_m^* = P[\gamma] \), where \( \beta = \{1, 3, \ldots, (2l - 1)\}, \gamma = \{2, 4, \ldots, 2m\} \) for \( 1 \leq l, m \leq k \) if \( n = 2k \) and \( 1 \leq l \leq (k + 1), 1 \leq m \leq k \) if \( n = 2k + 1 \). From the definition of \( P \), one can observe that the principal submatrices \( A_i^* \) and \( A_m^* \) of \( P \) are tridiagonal with the upper and lower diagonal entries from the set \( \{a_i\}_{i=1}^{n} \), and the main diagonal entries from the set \( \{a_i\}_{i=1}^{n} \), as given below:

\[
A_i^* = \begin{bmatrix}
    a_1 & c_1 & \cdots & 0 & 0 \\
    c_1 & a_2 & c_2 & 0 & 0 \\
    \vdots & \ddots & \ddots & \ddots & \vdots \\
    0 & \cdots & c_{2l-3} & a_{2l-3} & \cdots \\
    0 & \cdots & c_{2l-3} & a_{2l-1} & \cdots
\end{bmatrix}_{l \times l}, \quad A_m^* = \begin{bmatrix}
    a_2 & c_2 & \cdots & 0 & 0 \\
    c_2 & a_3 & c_3 & 0 & 0 \\
    \vdots & \ddots & \ddots & \ddots & \vdots \\
    0 & \cdots & c_{2m-2} & a_{2m-2} & \cdots \\
    0 & \cdots & c_{2m-2} & a_{2m} & \cdots
\end{bmatrix}_{m \times m}.
\]

Again, from the definition of \( P \), it can be observed that for every \( r > 0 \), \( P^{or} \) is congruent to the block matrix \( M^{or} \) via a permutation matrix \( X \) of order \( n \), i.e., \( M^{or} = XP^{or}X^* \) for \( r > 0 \), where

\[
X = \begin{cases}
    \text{Perm}(1, 3, \ldots, (2k - 1), 2, 4, \ldots, 2k), & \text{if } n = 2k \\
    \text{Perm}(1, 3, \ldots, (2k + 1), 2, 4, \ldots, 2k), & \text{if } n = 2k + 1
\end{cases}, \quad M = \begin{cases}
    \begin{bmatrix}
        A_k^* & 0 \\
        0 & A_k^*
    \end{bmatrix}, & \text{if } n = 2k, \\
    \begin{bmatrix}
        A_{k+1}^* & 0 \\
        0 & A_{k+1}^*
    \end{bmatrix}, & \text{if } n = 2k + 1.
\end{cases}
\]

We prove the required results for the case when \( n \) is even (the case when \( n \) is odd can be proved analogously). Let \( r > 1 \) and \( n = 2k \) for some positive integer \( k \geq 1 \). If \( P \) is PD (PSD), then by \( M = XPX^* \), \( M \) is PD (PSD), so \( A_k^* \) and \( A_k^{*^*} \) are PD (PSD) matrices. Hence, by Theorem 3.3 (Corollary 3.4), \( (A_k^*)^{or} \) and \( (A_k^{*^*})^{or} \) are PD (PSD), which gives \( M^{or} \) is PD (PSD), so \( P^{or} \) is PD (PSD), by \( P^{or} = X^{-1}M^{or}(X^{-1})^* \). This proves Part (1).

Since \( M^{or} = XP^{or}X^* \) for every \( r > 0 \), \( P \) is ID if and only if \( M \) is ID. Hence \( P \) is ID if and only if \( A_k^* \) and \( A_k^{*^*} \) are ID. Thus Part (2) follows by Theorem 2.2 (b).

\[\square\]

**Remark 3.7.** The results analogous to Theorem 3.3 and Corollary 3.4 are not true for all band matrices. For example, let

\[
K = \begin{bmatrix}
    2 & 3 & 4 & 5 & 0 \\
    3 & 5 & 7 & 9 & 0 \\
    4 & 7 & 10 & 13 & 0 \\
    5 & 9 & 13 & 17 & 0 \\
    0 & 0 & 0 & 0 & 1
\end{bmatrix},
\]

then \( K \) is a band matrix of bandwidth 3, which is PSD, but \( K^{or} \) is not PSD for \( 1 < r < 2 \). Again, since \( K \) is the limit of a sequence of PD band matrices of bandwidth 3 \([5, \text{Page 432}]\), the result analogous to Theorem 3.3 also doesn’t hold for all band matrices of bandwidth 3.

**Remark 3.8.** In Theorem 3.3, the lower bound 1 for the Hadamard power \( r \) is sharp. Let \( 0 < r_0 < 1 \).

Consider the tridiagonal PD matrix \( A(\epsilon) = \begin{bmatrix}
    1 & 1 & 0 \\
    1 & (2 + \epsilon) & 1 \\
    0 & 1 & 1
\end{bmatrix}, \text{ where } \epsilon \in \mathbb{R}^+. \) For every \( \epsilon < (2^{\frac{1}{r}} - 2) \),

\[
det(A(\epsilon)^{or}) = (2 + \epsilon)^{r_0} - 2 < 0, \text{ so } A(\epsilon)^{or} \text{ is not PD.}
\]
References

[1] Rajendra Bhatia. Infinitely divisible matrices. *The American Mathematical Monthly*, 113(3):221–235, 2006.
[2] Rajendra Bhatia and Hideki Kosaki. Mean matrices and infinite divisibility. *Linear algebra and its applications*, 424(1):36–54, 2007.
[3] Theodore S Chihara. *An introduction to orthogonal polynomials*. Courier Corporation, 2011.
[4] Shaun M Fallat and Charles R Johnson. *Totally nonnegative matrices*, volume 35. Princeton University Press, 2011.
[5] Carl H FitzGerald and Roger A Horn. On fractional hadamard powers of positive definite matrices. *Journal of Mathematical Analysis and Applications*, 61(3):633–642, 1977.
[6] Dominique Guillot, Apoorva Khare, and Bala Rajaratnam. Critical exponents of graphs. *Journal of Combinatorial Theory, Series A*, 139:30–58, 2016.
[7] Roger A Horn. The theory of infinitely divisible matrices and kernels. *Transactions of the American Mathematical Society*, 136:269–286, 1969.
[8] Roger A Horn and Charles R Johnson. *Matrix analysis*. Cambridge university press, 2012.
[9] Mourad EH Ismail and Martin E Muldoon. A discrete approach to monotonicity of zeros of orthogonal polynomials. *Transactions of the American Mathematical Society*, 323(1):65–78, 1991.
[10] Tanvi Jain. Hadamard powers of some positive matrices. *Linear Algebra and its Applications*, 528:147–158, 2017.
[11] Hubert Stanley Wall. *Analytic theory of continued fractions*. Courier Dover Publications, 2018.