UNIFORM ELLIPTICITY AND $p$-$q$ GROWTH

CRISTIANA DE FILIPPIS AND FRANCESCO LEONETTI

Abstract. Fix any two numbers $p$ and $q$, with $1 < p < q$; we give an example of an integral functional enjoying uniform ellipticity and $p$-$q$ growth.

Contents

1. Introduction 1
2. Example 3
3. Preliminary results 4
4. Proof of Theorem 2.1 5
5. Another example 8
References 9

1. Introduction

We consider integral functionals

\[
\int_{\Omega} f(Du(x))dx,
\]

where $u : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^N$, $\Omega$ is bounded and open and $f$ is continuous and nonnegative. About $f$ we assume $p$-$q$ growth

\[
c_1|z|^p - c_2 \leq f(z) \leq c_3|z|^q + c_4,
\]

where $c_1, c_2, c_3, c_4, p, q$ are constants with $c_1, c_3 \in (0, +\infty)$, $c_2, c_4 \in [0, +\infty)$ and $1 < p < q$. In this framework it is usual to assume that

\[
c_5(\mu + |z|)^{p-2} \leq \langle DDF(z) \frac{\lambda}{|\lambda|}, \frac{\lambda}{|\lambda|} \rangle,
\]

and

\[
|DDf(z)| \leq c_6(\mu + |z|)^{q-2},
\]

where $c_5, c_6, \mu$ are constants with $c_5, c_6 \in (0, +\infty)$ and $\mu \in [0, 1]$. Retaining only the informations about the growth in the large of the second derivative, as prescribed by (1.3)-(1.4), leads to the following bound on the ratio between the highest and the lower eigenvalue of $DDf$:

\[
\mathcal{R}(z) := \frac{\text{highest eigenvalue of } DDF(z)}{\text{lowest eigenvalue of } DDF(z)} \leq c_7(\mu + |z|)^{q-p},
\]

2010 Mathematics Subject Classification. 35J60, 35J70.
Key words and phrases. Regularity, uniform ellipticity, $p$-$q$-growth.

Acknowledgements. C. De Filippis is supported by the Engineering and Physical Sciences Research Council (EPSRC): CDT Grant Ref. EP/L015811/1. F. Leonetti is supported by MIUR, GNAMPA, INdAM, UNIVAQ.
for some positive constant $c_T$. The right hand side of (1.5), evidently blows up as $|z| \to \infty$, given that, in general, $q > p$. On the other hand, if by any chance the integrand $f$ features certain structural properties which make $R(z)$ bounded from above by a constant non-depending, in particular, from $z$, then we have uniform ellipticity. We are concerned with regularity of minimizers $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$ of (1.1); in this framework of $p$-$q$ growth, the following bound sometimes appears

\begin{equation}
q < p + c(n, p),
\end{equation}

where $c(n, p)$ is positive and tends to 0 when the dimension $n$ tends to $+\infty$; see [2, 6, 9–11, 13, 16, 18] and [17, Section 6]; see also [7, Section 6.2] where a simple argument is given. Now we assume the following structure condition

\begin{equation}
f(z) = g(|z|),
\end{equation}

with $g : [0, +\infty) \to [0, +\infty)$. Some papers require $g(0) = 0$, $g \in C^2([0, +\infty)) \cap C^1([0, +\infty))$ with $g'(t) > 0$ for $t > 0$; moreover, [3, 5, 8, 15] ask for

\begin{equation}
0 < m \leq \frac{g''(t)t}{g'(t)} \leq M < +\infty \quad \forall t > 0.
\end{equation}

Note that [1] requires (1.8) with $1 \leq m$; on the other hand, [14] asks for $M \leq 1$. In [4] they consider splitting densities $f(Du) = a(|(D_1u, ..., D_{n-1}u)|) + b(|D_nu|)$ and they require (1.8) for both $a$ and $b$. We remark that $g' > 0$ and (1.8) forces $g'' > 0$, so $g$ must be strictly convex; on the other hand, (1.8) allows $p$-$q$ growth whatever $p$ and $q$ are: in this paper we fix $p$ and $q$ with $1 < p < q$, no matter how far they are, and we show a convex function $g$ verifying (1.8), with $p$-$q$ growth. In [3] we find Theorem 1.15 that says

**Theorem 1.1.** We assume that $g(0) = 0$ and $g \in C^2([0, +\infty)) \cap C^1([0, +\infty))$; moreover, $g'(t) > 0$ for $t > 0$ and (1.8) holds true. If $u \in W^{1,1}_0(\Omega, \mathbb{R}^N)$ is a local minimizer of (1.1) under the structure condition (1.7) with $g$ as before, then $u$ is locally Lipschitz continuous in $\Omega$.

We are going to show an example for the previous Theorem 1.1: fix $p$ and $q$ with $1 < p < q$, then we give $g$ satisfying all the assumptions of Theorem 1.1 with the chosen $p$ and $q$: the restriction (1.6) does not apply! Moreover, such a $g$ gives an $f$ for which we have uniform ellipticity; indeed, let $g$ be any function in $C^2([0, +\infty))$ with $g'(t) > 0$ for $t > 0$, satisfying assumption (1.8); then, for the corresponding $f$ given by (1.7), we have

\[ \frac{\partial f}{\partial z_i}(z) = g'(|z|) \frac{z_i}{|z|} \]

and

\[ \frac{\partial^2 f}{\partial z_i \partial z_j}(z) = \left[ g''(|z|) - \frac{g'(|z|)}{|z|} \right] \frac{z_i z_j}{|z|^2} + \frac{g'(|z|)}{|z|} \delta_{ij}, \]

so that

\begin{equation}
\left\langle DDf(z), \frac{\lambda}{|\lambda|} \right\rangle = \left[ g''(|z|) - \frac{g'(|z|)}{|z|} \right] \frac{z}{|z|} \cdot \frac{\lambda}{|\lambda|} \left\langle \frac{z}{|z|} \cdot \frac{\lambda}{|\lambda|} \right\rangle^2 + \frac{g'(|z|)}{|z|},
\end{equation}

if we consider first the case $[\ldots] \geq 0$ and then the other case $[\ldots] < 0$, using (1.8), we get

\begin{equation}
\frac{\text{highest eigenvalue of } DDf(z)}{\text{lowest eigenvalue of } DDf(z)} \leq \max \left\{ M; \frac{1}{m} \right\};
\end{equation}

so, we are in the uniform ellipticity regime. So, after fixing $p$ and $q$ at will in $(1, +\infty)$, we are going to write an example of functional with $p$-$q$ growth and uniform ellipticity. For $1 < p < q$, set $a = \frac{2p}{q-p}$ and $b = \frac{p}{q-p}$. Then, we have $a, b > 0$, $1 < p = a - b < a + b = q$ and we can use the function $g$ defined in the next section 2.
2. Example

We fix $a, b \in (0, +\infty)$ with

$$1 < a - b.$$  \hfill (2.1)

We consider $g : [0, +\infty) \to [0, +\infty)$ such that

$$g(t) = t^{a + b \sin(\varphi(t))},$$  \hfill (2.2)

where $\varphi : \mathbb{R} \to \mathbb{R}$ is given by

$$\varphi(t) = \begin{cases} \frac{3}{2} \pi & \text{if } t \in (-\infty, 1], \\
\frac{3}{2} \pi + \varepsilon \ln(e + (t - 1)^4) & \text{if } t \in (1, +\infty), 
\end{cases}$$  \hfill (2.3)

with $\varepsilon > 0$. Note that

$$\varphi'(t) = \begin{cases} 0 & \text{if } t \in (-\infty, 1], \\
\frac{\varepsilon}{\ln(e + (t - 1)^4)} \frac{4(t-1)^2}{e^{(t-1)^4}} & \text{if } t \in (1, +\infty)
\end{cases}$$  \hfill (2.4)

and

$$\varphi''(t) = \begin{cases} 0 & \text{if } t \in (-\infty, 1], \\
\varepsilon \left\{ \frac{1}{\ln(e + (t - 1)^4)^2} \left[ \frac{4(t-1)^2}{e^{(t-1)^4}} \right]^2 \right. + \\
\left. \frac{12(t-1)^2 e^{4(t-1)^4}}{(e^{(t-1)^4})^2} \right\} & \text{if } t \in (1, +\infty)
\end{cases}$$  \hfill (2.5)

so that $\varphi \in C^2(\mathbb{R})$. Note that $\varphi'(t) > 0$ when $t > 1$; moreover, $\lim_{t \to +\infty} \varphi(t) = +\infty$. Then $\varphi(t)$ increases and takes all the values of the interval $[\frac{3}{2} \pi, +\infty)$. This means that, in (2.2), the exponent $a + b \sin(\varphi(t))$ oscillates between $a - b$ and $a + b$ infinitely many times as $t$ goes from 0 to $+\infty$; then $g(t)$ has $a - b$ growth from below and $a + b$ growth from above. As far as $\varepsilon$ is concerned, we require that

$$0 < \varepsilon < \min \left\{ 1, \frac{a - 1 - b}{224 b} \right\}.$$  \hfill (2.6)

We are going to prove the next

\textbf{Theorem 2.1.} Let us consider $a, b \in (0, +\infty)$ verifying (2.1); we take $g(t)$ given by (2.2) where $\varphi$ is defined in (2.3) and $\varepsilon$ satisfies (2.6). Then $g : [0, +\infty) \to [0, +\infty)$, $g(0) = 0$, $g(t) > 0$ for $t > 0$, $g \in C^1([0, +\infty)) \cap C^2((0, +\infty))$, $\lim_{t \to 0^+} g(t) = 0$, $\lim_{t \to +\infty} \frac{g(t)}{t} = +\infty$, $g'(0) = 0$ and

$$0 < \left\{ -b\varepsilon + a - b \right\} \frac{g(t)}{t} \leq g'(t) \leq \left\{ b\varepsilon + a + b \right\} \frac{g(t)}{t}$$  \hfill (2.7)

for every $t > 0$; moreover,

$$0 < \left\{ -b\varepsilon + a - b \right\} t^{a - b - 1} \leq g'(t) \leq \left\{ b\varepsilon + a + b \right\} \left[ t^{a - b - 1} + t^{a + b - 1} \right]$$  \hfill (2.8)

for every $t > 0$. As far as $g''$ is concerned, we get

$$0 < \left\{ -224 b\varepsilon + a - 1 - b \right\} \frac{g''(t)}{t} \leq g''(t) \leq \left\{ 224 b\varepsilon + a - 1 - b \right\} \frac{g''(t)}{t}$$  \hfill (2.9)

for every $t > 0$, thus $g$ is strictly convex in $[0, +\infty)$.

The present example is a modification of the one given in [12, 19]; in the present example the small new parameter $\varepsilon$ appears and it makes possible to get convexity and $p$-$q$ growth with any $p$ and $q$. 

3. Preliminary results

We need some preliminary estimates.

**Lemma 3.1.** For all $t \in (1, \infty)$ there holds that:

\begin{equation}
0 < \frac{(t-1)^3}{e + (t-1)^4} < 1.
\end{equation}

*Proof.* If $1 < t \leq 2$, then $0 < t - 1 \leq 1$ so that

\[ 0 < \frac{(t-1)^3}{e + (t-1)^4} \leq \frac{1}{e + (t-1)^4} \leq \frac{1}{e} < 1. \]

If $2 < t$, then $1 < t - 1$ so that

\[ 0 < \frac{(t-1)^3}{e + (t-1)^4} < \frac{(t-1)^4}{e + (t-1)^4} < 1. \]

The two cases give (3.1). □

**Lemma 3.2.** For all $t \in (1, \infty)$ there holds that:

\begin{equation}
0 < \frac{(t-1)^3 t}{e + (t-1)^4} < 2.
\end{equation}

*Proof.* We write $t = (t-1) + 1$ and we get

\[ 0 < \frac{(t-1)^3 t}{e + (t-1)^4} = \frac{(t-1)^3(t-1)}{e + (t-1)^4} + \frac{(t-1)^3}{e + (t-1)^4} < \frac{(t-1)^4}{e + (t-1)^4} + 1 < 1 + 1, \]

where we used (3.1). □

**Lemma 3.3.** For all $t \in (1, \infty)$ there holds that:

\begin{equation}
0 < \frac{(t-1)^2 t^2}{e + (t-1)^4} < 4.
\end{equation}

*Proof.* If $1 < t < 2$, then $0 < t - 1 < 1$ so that

\[ 0 < \frac{(t-1)^2 t^2}{e + (t-1)^4} < \frac{4}{e + (t-1)^4} < \frac{4}{e} < \frac{4}{2} = 2. \]

If $2 \leq t$, then $t \leq 2(t-1)$ so that

\[ 0 < \frac{(t-1)^2 t^2}{e + (t-1)^4} \leq \frac{(t-1)^2 4(t-1)^2}{e + (t-1)^4} = \frac{4(t-1)^4}{e + (t-1)^4} < 4. \]

The two cases give (3.3). □

**Lemma 3.4.** For all $t \in (1, \infty)$ there holds that:

\begin{equation}
0 < \frac{\ln t}{\ln(e + (t-1)^4)} < 1.
\end{equation}

*Proof.* If $1 < t \leq e$, then

\[ 0 < \frac{\ln t}{\ln(e + (t-1)^4)} \leq \frac{\ln e}{\ln(e + (t-1)^4)} < \frac{\ln e}{\ln e} = 1. \]

If $e < t$, then $t < (t-1)^2$: indeed, this last inequality is equivalent to $0 < t^2 - 3t + 1$: the two solutions of the equation $t^2 - 3t + 1 = 0$ are $\frac{3 \pm \sqrt{5}}{2}$ and $\frac{3 \pm \sqrt{5}}{2}$; note that $5 < 5, 29 = (2,3)^2$, so that $\frac{3 + \sqrt{5}}{2} < \frac{3 + 2}{2} = 2, 65 < e$; then $e < t$ implies $0 < t^2 - 3t + 1$ and $t < (t-1)^2$. This last inequality allows us to write

\[ 0 < \frac{\ln t}{\ln(e + (t-1)^4)} < \frac{\ln((t-1)^2)}{\ln(e + (t-1)^4)} < \frac{\ln((t-1)^2)}{\ln((t-1)^2)} = \frac{2}{4}. \]

The two cases give (3.4). □
In this section 3, \( \varphi \) is given by (2.3) with any \( \varepsilon > 0 \): in the forthcoming lemmas, no restriction from above on \( \varepsilon \) is required.

**Lemma 3.5.** Let \( \varepsilon > 0 \) be any number and \( \varphi \) be the function in (2.3). Then,
\[
0 < \varphi'(t) t \ln t \leq 8\varepsilon \quad \forall t \in (1, +\infty).
\]

*Proof.* We take into account formula (2.4) and estimates (3.4), (3.2):
\[
0 < \varphi'(t) t \ln t = \frac{\varepsilon}{\ln(e + (t - 1)^4)} \frac{4(t - 1)^3 t}{e + (t - 1)^4} \ln t = \frac{\varepsilon}{\ln(e + (t - 1)^4)} \frac{4(t - 1)^3 t}{e + (t - 1)^4} \leq 8\varepsilon.
\]
\[
\square
\]

**Lemma 3.6.** Let \( \varepsilon > 0 \) be any number and \( \varphi \) be the function in (2.3). Then,
\[
0 < \varphi'(t) t \leq 8\varepsilon \quad \forall t \in (1, +\infty).
\]

*Proof.* We take into account formula (2.4) and estimate (3.2):
\[
0 < \varphi'(t) t = \frac{\varepsilon}{\ln(e + (t - 1)^4)} \frac{4(t - 1)^3 t}{e + (t - 1)^4} = \frac{\varepsilon}{\ln(e + (t - 1)^4)} \frac{4(t - 1)^3 t}{e + (t - 1)^4} \leq 8\varepsilon.
\]
\[
\square
\]

**Lemma 3.7.** Let \( \varepsilon > 0 \) be any number and \( \varphi \) be the function in (2.3). Then,
\[
|\varphi''(t)| t^2 \ln t \leq 128\varepsilon \quad \forall t \in (1, +\infty).
\]

*Proof.* We take into account formula (2.5) and estimates (3.2), (3.3), (3.4):
\[
|\varphi''(t)| t^2 \ln t \leq \frac{\varepsilon \ln t}{\ln(e + (t - 1)^4)^2} \left[ \frac{4(t - 1)^3 t}{e + (t - 1)^4} \right]^2 + \frac{\varepsilon \ln t}{\ln(e + (t - 1)^4)^2} \frac{12\varepsilon(t - 1)^2 t^2 + 4(t - 1)^6 t^2}{|e + (t - 1)^4|^2} \leq \varepsilon(4^3 + 48 + 16) = 128\varepsilon.
\]
\[
\square
\]

4. **Proof of Theorem 2.1**

Definitions (2.2) and (2.3) say that, when \( t \in [0, 1] \), \( \varphi(t) = \frac{3}{2}\pi \) and \( g(t) = t^{a-b} \); condition (2.1) guarantees that \( 1 < a-b \) so that
\[
g(0) = 0,
\]
\[
\lim_{t \to 0^+} \frac{g(t)}{t} = 0,
\]
\[
g'(0) = 0;
\]
moreover, \( g(t) > 0 \) for \( t > 0 \). We recall that, for \( t > 1 \), \( t^{a-b} \leq g(t) \); again, condition (2.1) guarantees that \( 1 < a - b \) so that
\[
\lim_{t \to +\infty} \frac{g(t)}{t} = +\infty.
\]
Up to now, $g \in C^0([0, +\infty))$. For $t > 0$ we have

\begin{equation}
(4.5) \quad g(t) = t^{a+b\sin(\varphi(t))} = e^{[a+b\sin(\varphi(t))]\ln t},
\end{equation}

so that

\begin{equation}
(4.6) \quad g'(t) = e^{[a+b\sin(\varphi(t))]\ln t} \left\{ [b\cos(\varphi(t))]\varphi'(t) \ln t + [a + b\sin(\varphi(t))] \frac{1}{t} \right\} =
\end{equation}

\begin{equation}
\frac{g(t)}{t} \left\{ [b\cos(\varphi(t))]\varphi'(t) \ln t + [a + b\sin(\varphi(t))] \right\}.
\end{equation}

If $t \in (0, 1]$, then $\varphi(t) = \frac{3}{2} \pi$ and $\varphi'(t) = 0$, so that

\begin{equation}
(4.7) \quad g'(t) = \frac{g(t)}{t} [a - b] = [a - b] t^{a-b-1},
\end{equation}

again, condition (2.1) guarantees that $1 < a - b$ so that

\begin{equation}
(4.8) \quad \lim_{t \to 0^+} g'(t) = 0.
\end{equation}

Then, $g \in C^1([0, +\infty))$. Using formula (4.6), when $t > 0$, we have

\begin{equation}
\begin{aligned}
g''(t) &= \frac{g(t)}{t} \left\{ [b\cos(\varphi(t))]\varphi'(t) \ln t + [a + b\sin(\varphi(t))] \right\} +
\end{aligned}
\end{equation}

\begin{equation}
\begin{aligned}
&\frac{g(t)}{t} \left\{ [-b\sin(\varphi(t))]\varphi'(t) \ln t +
\end{aligned}
\end{equation}

\begin{equation}
\begin{aligned}
&\frac{g(t)}{t} \left\{ [b\cos(\varphi(t))]\varphi'(t) \ln t + \varphi'(t) \ln t + [b\cos(\varphi(t))]\varphi'(t) \right\}.
\end{aligned}
\end{equation}

Then $g \in C^2((0, +\infty))$. Now we are going to estimate $g'(t)$ by means of $\frac{g(t)}{t}$. First of all, we consider the case $t \in (0, 1]$: we can use formula (4.7) and we get $g'(t) = (a - b) \frac{g(t)}{t}$. After that, we deal with $t > 1$; we use formula (4.6) and estimate (3.5):

\begin{equation}
\begin{aligned}
\frac{g(t)}{t} \left\{ -6b \varepsilon + a - b \right\} \leq \frac{g(t)}{t} \left\{ [b\cos(\varphi(t))]\varphi'(t) \ln t + [a + b\sin(\varphi(t))] \right\} \leq \frac{g(t)}{t} \left\{ b\varepsilon + a + b \right\}.
\end{aligned}
\end{equation}

Note that $-6b \varepsilon + a - b < a - b < a + b < 6b \varepsilon + a + b$; then

\begin{equation}
\begin{aligned}
\frac{g(t)}{t} \left\{ -6b \varepsilon + a - b \right\} \leq g'(t) \leq \frac{g(t)}{t} \left\{ b\varepsilon + a + b \right\} \quad \forall t > 0.
\end{aligned}
\end{equation}

Up to now, we only used $a, b > 0$, $1 < a - b$ and $\varepsilon > 0$. Assumption (2.6) guarantees that $\varepsilon < \frac{a - b}{2ab}$; then $8b \varepsilon < 224b \varepsilon < a - 1 - b$, so that $1 < -8b \varepsilon + a - b$; this and positivity of $g$ give $g'(t) > 0$ when $t > 0$. Moreover, (4.11) can be written as follows

\begin{equation}
\begin{aligned}
\frac{1}{b\varepsilon + a + b} g'(t) \leq \frac{g(t)}{t} \leq \frac{1}{-b\varepsilon + a - b} g'(t) \quad \forall t > 0.
\end{aligned}
\end{equation}

We note that
We keep in mind that $t > 1$.

We divide by $t$ and we get

$$t^{a-b-1} \{ -b8\varepsilon + a - b \} \leq g'(t) \leq t^{a-b-1} \{ b8\varepsilon + a + b \} \quad \forall t > 0.$$ 

We need to estimate $g''(t)t$; to this aim, we use (4.9):

$$g''(t) = \begin{cases} g'(t) - \frac{g(t)}{t} & \{ b \cos(\varphi(t)) \} \varphi'(t) t \ln t + a + b \sin(\varphi(t)) \} + \\ \frac{g(t)}{t} \{ -b \sin(\varphi(t)) \} \varphi'(t) t \varphi'(t) t \ln t + \end{cases}$$

$$[b \cos(\varphi(t))] [\varphi''(t)t^2 \ln t + \varphi'(t)t(\ln t + 2)] \}.$$ 

We keep in mind (4.6) and we can write as follows

$$g''(t) = g'(t) \{ [b \cos(\varphi(t))] \varphi'(t) t \ln t + a - 1 + b \sin(\varphi(t)) \} +$$

$$\frac{g(t)}{t} \{ -b \sin(\varphi(t)) \} \varphi'(t) t \varphi'(t) t \ln t +$$

$$[b \cos(\varphi(t))] [\varphi''(t)t^2 \ln t + \varphi'(t)t(\ln t + 2)] \}.$$

For simplicity, define

$$\Phi_1(t) := [b \cos(\varphi(t))] \varphi'(t) t \ln t + a - 1 + b \sin(\varphi(t)) \}$$

$$\Phi_2(t) := -b \sin(\varphi(t)) \} \varphi'(t) t \varphi'(t) t \ln t + [b \cos(\varphi(t))] [\varphi''(t)t^2 \ln t + \varphi'(t)t(\ln t + 2)] \}.$$ 

in such a way that (4.18) reads as

$$g''(t) = g'(t) \Phi_1(t) + \frac{g(t)}{t} \Phi_2(t).$$

By (3.5), (3.6) and (3.7) we estimate for $t > 1$

$$-8\varepsilon b + a - 1 - b \leq \Phi_1(t) \leq 8\varepsilon b + a - 1 + b;$$

$$|\Phi_2(t)| \leq 8\varepsilon b + 128\varepsilon + 8\varepsilon + 16\varepsilon = 64\varepsilon + 152.$$ 

Now we estimate $g''(t)t$ from below; when $t > 1$ we keep in mind positivity of $g', g$ and estimates for $\Phi_1, \Phi_2$; we have

$$g''(t)t = g'(t)\Phi_1(t) + \frac{g(t)}{t} \Phi_2(t) \geq g'(t) \{ -8\varepsilon b + a - 1 - b \} + \frac{g(t)}{t} (-b)\varepsilon [64\varepsilon + 152] =: (I);$$

now we use the right hand side of (4.12) and we get

$$(I) \geq g'(t) \{ -8\varepsilon b + a - 1 - b + \frac{-b}[64\varepsilon + 152] \} \geq -8\varepsilon b + a - b \} =: (II);$$
now we use (2.6): \( \varepsilon < 1 \) gives \( 64 \varepsilon + 152 < 216 \) and \( \varepsilon < \frac{a - b}{224 \varepsilon} \) gives \( 1 < -b \varepsilon + a - b \), so that

\[
(4.20) \quad \frac{b \varepsilon (64 \varepsilon + 152)}{-b \varepsilon + a - b} < 216 b \varepsilon;
\]

then

\[ (\text{II}) \geq g'(t) \{-8 \varepsilon b + a - 1 - b - 216 \varepsilon b\}; \]

this means that, for \( t > 1 \) we have

\[ g''(t) t \geq g'(t) \{-224 \varepsilon b + a - 1 - b\}. \]

Note that we required \(-224 b \varepsilon + a - 1 - b > 0\) in our assumption (2.6).

When \( t \in (0, 1] \), we have \( \varphi(t) = \frac{1}{2} \pi \), \( \varphi'(t) = 0 = \varphi''(t) \); then \( g''(t) t = g'(t)(a - 1 - b) \). Moreover, \( g' \) is positive and

\[
(4.21) \quad a - 1 - b > a - 1 - b - 224 \varepsilon b > 0,
\]

then,

\[
(4.22) \quad g''(t) t \geq g'(t) \{-224 \varepsilon b + a - 1 - b\} \quad \forall t > 0.
\]

Since \( g'(t) > 0 \) when \( t > 0 \), this last inequality guarantees that \( g''(t) > 0 \) for all \( t > 0 \); then \( g' \) strictly increases in \((0, +\infty)\); since \( g' \) is continuous in \([0, +\infty)\), then \( g' \) strictly increases in \([0, +\infty)\); this guarantees that \( g \) is strictly convex in \([0, +\infty)\).

Now we estimate \( g''(t) t \) from above; when \( t > 1 \) we keep in mind positivity of \( g' \), \( g \) and estimates for \( \Phi_1, \Phi_2 \); we have

\[
g''(t) = g'(t) \Phi_1(t) + \frac{g(t)}{t} \Phi_2(t) \leq g'(t) \{8 \varepsilon b + a - 1 + b\} + \frac{g(t)}{t} \frac{b \varepsilon (64 \varepsilon + 152)}{-b \varepsilon + a - b} =: (\text{III});
\]

now we use the right hand side of (4.12) and we get

\[ (\text{III}) \leq g'(t) \left\{8 \varepsilon b + a - 1 + b + \frac{b \varepsilon (64 \varepsilon + 152)}{-8 \varepsilon b + a - b}\right\} =: (\text{IV}); \]

we use (4.20) and we get

\[ (\text{IV}) \leq g'(t) \{8 \varepsilon b + a - 1 + b + 216 \varepsilon b\}; \]

this means that, for \( t > 1 \) we have

\[ g''(t) t \leq g'(t) \{224 \varepsilon b + a - 1 + b\}. \]

When \( t \in (0, 1] \), we have \( \varphi(t) = \frac{1}{2} \pi \), \( \varphi'(t) = 0 = \varphi''(t) \); then \( g''(t) t = g'(t)(a - 1 - b) \). Moreover, \( g' \) is positive so that

\[
(4.23) \quad g''(t) t \leq g'(t) \{224 \varepsilon b + a - 1 + b\} \quad \forall t > 0.
\]

This ends the proof of Theorem 2.1. \( \square \)

5. ANOTHER EXAMPLE

Now we give an example in the subquadratic case by modifying a little bit the previous example of section 2: we introduce an additional restriction on \( a, b \) and we select a smaller \( \varepsilon \). More precisely, We fix \( a, b \in (0, +\infty) \) with (2.1) as in section 2; moreover, we require, in addition,

\[
(5.1) \quad a + b < 2.
\]

We consider \( g : [0, +\infty) \to [0, +\infty) \) given by (2.2) with \( \varphi \) as in (2.3) with \( \varepsilon > 0 \) satisfying (2.6) as in section 2; moreover, we require, in addition,
\[ \varepsilon < \frac{2 - a - b}{224 \cdot b}. \]

Please, note that (5.1) gives \(0 < 2 - a - b\), so the requirement (5.2) is in accordance with \(0 < \varepsilon\) and it implies
\[ 224b \varepsilon + a - 2 + b < 0. \]

This and the right hand side of (2.9) in Theorem 2.1 give

**Theorem 5.1.** Let us consider \(a, b \in (0, +\infty)\) verifying (2.1), (5.1); we consider \(g(t)\) given by (2.2) where \(\varphi\) is defined in (2.3) and \(\varepsilon\) satisfies (2.6), (5.2). Then
\[ g''(t) - \frac{g'(t)}{t} \leq \frac{224 \varepsilon + a - 2 + b}{t} g'(t) < 0 \quad \forall t > 0 \]
and we get \(M = 1\) in the right hand side of (1.8). Since
\[ \left( \frac{g'(t)}{t} \right)' = \frac{g''(t) t - g'(t)}{t^2} = \left( g''(t) - \frac{g'(t)}{t} \right) \frac{1}{t} < 0, \]
we get
\[ t \rightarrow \frac{g'(t)}{t} \quad \text{strictly decreases in } (0, +\infty). \]

**References**

[1] Baroni P. - *Riesz potential estimates for a general class of quasilinear equations* - Calc. Var. Partial Differential Equations, 53 (2015), 803-846.
[2] Baroni P., Colombo M., Mingione G. - *Harnack inequalities for double phase functionals* - Nonlinear Anal., 121 (2015), 206-222.
[3] Beck, L., Mingione, G. - *Lipschitz bounds and non-uniform ellipticity* - Comm. Pure Appl. Math. (2020). https://doi.org/10.1002/cpa.21880
[4] Bildhauer M., Fuchs M. - *Lipschitz bounds and non-uniform ellipticity* - Comm. Pure Appl. Math. (2020).
[5] Cianchi A., Maz'ya V. - *On the Regularity of Minima of Non-autonomous Functionals* - Arch. Ration. Mech. Anal., 188 (2009), 467-496.
[6] Cianchi A., Maz'ya V. - *Global boundedness of the gradient for a class of nonlinear elliptic systems* - Arch. Ration. Mech. Anal., 212 (2014), 129-177.
[7] Eleuteri M., Marcellini P., Mascolo E. - *Lipschitz estimates for systems with ellipticity conditions at infinity* - Ann. Mat. Pura Appl., 195 (2016), 1575-1603.
[8] Esposito L., Leonetti F., Mingione G. - *Higher integrability for minimizers of integral functionals with \(p, q\) growth* - J. Differential Equations, 195 (1999), 414-438.
[9] Esposito L., Leonetti F., Mingione G. - *Sharp regularity for functionals with \(p, q\) growth* - J. Differential Equations, 204 (2004), 5-55.
[10] Fusco N., Marcellini P. - *Regularity of quasiminimizers of functionals with non standard growth conditions* - Arch. Rational Mech. Anal., 105 (1989), 267-284.
[11] Lieberman G. - *Regularity of minima: an invitation to the dark side of the calculus of variations* - Appl. Math., 51 (2006), 355-425.
[12] Moscariello G., Nania L. - *Hölder continuity of minimizers of functionals with nonstandard growth conditions* - Ricerche Mat., 40 (1991), 259-273.
[19] Talenti G. - *Boundedness of minimizers* - Hokkaido Math. J., 19 (1990), 259-279.

Cristiana De Filippis, Mathematical Institute, University of Oxford, Andrew Wiles Building, Radcliffe Observatory Quarter, Woodstock Road, Oxford, OX26GG, Oxford, United Kingdom

E-mail address: Cristiana.DeFilippis@maths.ox.ac.uk

Francesco Leonetti, Università di L’Aquila, Dipartimento di Ingegneria e Scienze dell’Informazione e Matematica, Via Vetoio snr, 67100 L’Aquila, Italy

E-mail address: leonetti@uniqa.it