Graph norms and Sidorenko’s conjecture

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Abstract

Let $H$ and $G$ be two finite graphs. Define $h_H(G)$ to be the number of homomorphisms from $H$ to $G$. The function $h_H(\cdot)$ extends in a natural way to a function from the set of symmetric matrices to $\mathbb{R}$ such that for $A_G$, the adjacency matrix of a graph $G$, we have $h_H(A_G) = h_H(G)$. Let $m$ be the number of edges of $H$. It is easy to see that when $H$ is the cycle of length $2n$, then $h_H(\cdot)^{1/m}$ is the $2n$-th Schatten-von Neumann norm. We investigate a question of Lovász that asks for a characterization of graphs $H$ for which the function $h_H(\cdot)^{1/m}$ is a norm.

We prove that $h_H(\cdot)^{1/m}$ is a norm if and only if a Hölder type inequality holds for $H$. We use this inequality to prove both positive and negative results, showing that $h_H(\cdot)^{1/m}$ is a norm for certain classes of graphs, and giving some necessary conditions on the structure of $H$ when $h_H(\cdot)^{1/m}$ is a norm. As an application we use the inequality to verify a conjecture of Sidorenko for certain graphs including hypercubes. In fact for such graphs we can prove statements that are much stronger than the assertion of Sidorenko’s conjecture.

We also investigate the $h_H(\cdot)^{1/m}$ norms from a Banach space theoretic point of view, determining their moduli of smoothness and convexity. This generalizes the previously known result for the $2n$-th Schatten-von Neumann norms.

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1 Introduction

Let $H$ and $G$ be graphs. A homomorphism from $H$ to $G$ is a mapping $h : V(H) \to V(G)$ such that for each edge $\{u, v\}$ of $H$, $\{h(u), h(v)\}$ is an edge of $G$. Let $h_H(G)$ denote the number of homomorphisms from $H$ to $G$. If $w$ is the adjacency matrix of the graph $G$, then

$$h_H(G) = \sum_{x_u \in V(G)} \prod_{(u,v) \in E(H)} w(x_u,x_v). \quad (1)$$

We might also divide $h_H(G)$ by the total number of mappings from $V(H)$ to $V(G)$ to obtain a normalized version:

$$t_H(G) := \frac{h_H(G)}{|V(G)|^{|V(H)|}}. \quad (2)$$

Thus $t_H(G)$ is the probability that a random mapping from $V(H)$ to $V(G)$ is a homomorphism.

The expression in the right hand side of (1) is quite common. Such sums appear as Mayer sums in classical statistical mechanics, Feynman sums in quantum field theory [32] and multicenter sums in quantum chemistry [41]. In the present article when we study $h_H(G)$, we usually think of $H$ as a fixed graph. In this case, as it has been stated formally in Lemma 2.1 in [21], when $G$ is a sufficiently dense graph, $h_H(G)$ is a good approximation for the number of copies of $H$ in $G$. This makes understanding the behavior of $h_H$ one of the main objectives of the extremal graph theory. Despite all the machinery that is developed in recent years [10, 2] and has been applied successfully to some important questions [23], still there are many questions regarding these functions that are remained unsolved. One of the important open questions in this area is the celebrated conjecture of Sidorenko [30]. The conjecture says that for every graph $G$, and every bipartite graph $H$ with $m$ edges, we have

$$t_H(G) \geq t_{K_2}(G)^m,$$

where $K_2$ is the graph comprising two vertices and a single edge between them. While the original motivation of this work was not to study this conjecture, during the research we realized that our results verify the conjecture for certain graphs including the hypercubes. In fact for such graphs we can prove statements that are surprisingly stronger than the assertion of Sidorenko’s conjecture. We discuss this more extensively in Section 2.5.

Let us explain our main motivation. First we need to define $h_H(\cdot)$ on a more general domain than graphs.

**Definition 1.1** Consider an index set $\mathcal{I}$. Let $WS(\mathcal{I})$ be the set of the symmetric real matrices indexed over $\mathcal{I}$, i.e.

$$WS(\mathcal{I}) = \{w : \mathcal{I} \times \mathcal{I} \to \mathbb{R} : w \text{ is symmetric}\}.$$

For a graph $H$ and $w \in WS(\mathcal{I})$, define

$$h_H(w) := \sum_{x_u \in \mathcal{I}} \prod_{(u,v) \in E(H)} w(x_u,x_v). \quad (3)$$

It turns out that for $C_4$, the cycle of size 4, the function $h_{C_4}(G)$ carries interesting information about $G$. For example if $t_{C_4}(G)^{1/4}$ is close to $t_{K_2}(G)$, then $G$ “looks random” in certain aspects [31]. Such graphs are usually referred to as quasi-random graphs. Another interesting fact about $h_{C_4}$ is that $h_{C_4}(\cdot)^{1/4}$ is a norm on $WS(\mathcal{I})$. These observations belong to the same circle of ideas employed by Szemerédi [33, 34] to prove his famous theorem on arithmetic progressions. In fact Szemerédi’s regularity lemma, the main tool in the proof of his theorem, roughly speaking says that every graph can be decomposed into a few number of subgraphs such that most of them are quasi-random (we refer the reader to Tao’s survey [35] for a precise formulation of the regularity lemma in terms of the
Recently Gowers \cite{13} \cite{14} defined a hypergraph version of this norm, and subsequently he \cite{12} and Nagle, Rödl, Schacht, and Skokan \cite{22} \cite{26} \cite{25} independently established a hypergraph regularity lemma which easily implies Szemerédi’s theorem in its full generality, and even stronger theorems such as Furstenberg-Katznelson’s multi-dimensional arithmetic progression theorem \cite{24} \cite{9}, a result that is the only known proof for it at the time was through ergodic theory \cite{11}. In fact arithmetic theorems such as Furstenberg-Katznelson’s multi-dimensional arithmetic progression theorem \cite{24} \cite{9}, a regularity lemma which easily implies Szemerédi’s theorem in its full generality, and even stronger Gowers norms play an essential role in the Green and Tao’s proof \cite{18} that the primes contain arbitrarily long arithmetic progressions and the current best bounds for the quantified version of the Szemerédi’s theorem is through the so called “inverse theorems” for these norms \cite{13} \cite{16} \cite{17} \cite{14}.

With all the known applications for the Gowers norms, it seems natural to believe that studying $h_H(\cdot)_{1/[E(H)]}$ for graphs other than $C_4$ might as well lead to some interesting applications. In fact the main goal of this article is to pursue a question of Lovász which asks for a characterization of graphs $H$ for which the function $h_H(\cdot)_{1/[E(H)]}$ is a norm. We prove both positive and negative results, showing that $h_H(\cdot)_{1/[E(H)]}$ is a norm for certain classes of graphs, and giving some necessary conditions on the structure of $H$ when $h_H(\cdot)_{1/[E(H)]}$ is a norm. We hope that the application to Sidorenko’s conjecture promises discoveries of more applications in the future.

We shall see in Section 2.3 that for $n > 1$, $h_{C_{2n}}(\cdot)_{1/2n}$ is the $2n$-th Schatten-von Neumann norm. Probably after the $\ell_p$ spaces and the Banach lattices, the Schatten-von Neumann spaces are the most well-studied normed spaces. Therefore it seems natural to study the $h_H(\cdot)_{1/[E(H)]}$ norms from a Banach space theoretic aspect too. In this direction we determine the moduli of convexity and smoothness of these spaces, the two dual parameters that play a fundamental role in Banach space theory. We discuss this further in Section 2.6.

\section{Definitions and main results}

\subsection{Notations and definitions}

For $n \in \mathbb{N}$, let $[n] := \{1, \ldots, n\}$. For two functions $f, g : \mathbb{R} \rightarrow \mathbb{R}^+$, we write $f = o(g)$, if and only if

$$\lim_{x \rightarrow \infty} f(x)/g(x) = 0.$$ 

A graph $G$ is a pair $(V, E)$ where $V$ is a finite set and $E$ is a multi-set (i.e. multiple copies of an element are allowed) of the edges, where every edge is an element of the form $\{u, v\}$ with $u, v$ distinct elements in $E$. So we allow our graphs to have multiple edges but no self-loops.

For a graph $G$, and an integer $k > 0$, we denote by $G^{(k)}$, the graph that is obtained by replacing every edge of $G$ by $k$ multiple edges.

For a graph $G = (V, E)$, a set $S \subseteq V$ is called an independent set if there is no edge with both endpoints in $S$. For the reasons that will be apparent soon we are mainly concerned about the bipartite graphs. In graph theory, $G = (V, E)$ is called a bipartite graph if $V$ can be partitioned into two disjoint independent sets $V_1$ and $V_2$. We call the partition of $V$ into $(V_1, V_2)$ a bipartition of $G$. Note that disconnected bipartite graphs have more than one bipartition. In this article we use a different definition that fixes one specific bipartition for $G$. So by a bipartite graph we mean a triple $G = (X, Y; E)$, where $X$ and $Y$ are two disjoint sets and $E$ is a multi-set of the elements of $X \times Y$. Note that here we fix the bipartition $(X, Y)$ as a part of the definition. Also note that contrary to our definition of graphs, here every edge is an ordered pair, and can be thought of as a directed edge from $X$ to $Y$.

Let $K_{m,n}$ be the complete bipartite graph, i.e. $K_{m,n} = (X,Y,X \times Y)$ where $|X| = m$ and $|Y| = n$, and note that with our definition unless $m = n$, $K_{m,n}$ is different from $K_{n,m}$. The $n$-dimensional hypercube $Q_n$ is the bipartite graph $(X,Y;E)$ where $X$ is the set of elements of $\{0,1\}^n$ with an even number of 1’s in their coordinates, and $Y = \{0,1\}^n \setminus X$. Moreover $(x, y) \in E$ if and only if $y$ differs only in one coordinate from $x$. 

\end{document}
For a bipartite graph $G$, the graph $G^{\oplus k}$ is defined in a similar way to the general graphs. We also define a product for bipartite graphs:

**Definition 2.1** Let $G = (X, Y; E)$ and $H = (X', Y'; E')$ be two bipartite graphs. Then define $G \times^b H$, the bi-product of $G$ and $H$, as the bipartite graph with bipartization $(X \times X', Y \times Y')$ where the multiplicity of $((x, x'), (y, y'))$ in $G \times^b H$ is equal to the product of the multiplicities of $(x, y) \in E$ and $(x', y') \in E'$.

By a normed space we mean a pair $(V, \| \cdot \|)$, where $V$ is a vector space over $\mathbb{R}$ and $\| \cdot \|$ is a function from $V$ to nonnegative reals satisfying

(i): $\|x\| = 0$ if and only if $x = 0$.

(ii): $\|\lambda x\| = |\lambda|\|x\|$ for all $x \in V$ and $\lambda \in \mathbb{R}$.

(iii): $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in V$.

We call $\|x\|$ the norm of $x$. A semi-norm is a function similar to a norm except that it might not satisfy (i). A quasi-norm is similar to a norm in that it satisfies the norm axioms, except that (iii) is replaced by $\|x + y\| \leq K(\|x\| + \|y\|)$ for some universal constant $K > 0$.

### 2.2 Graph norms

As we discussed in the introduction $h_{G,1}^{(1/4)}$ is a norm. Our main goal is to investigate a question of Lovász that asks for a characterization of graphs $H$ for which the function $h_H^{(1/m)}$ is a norm. Let $H$ be a nonbipartite graph and $w_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $w_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Then since $H$ is not bipartite we have $h_H(w_2) = 0$ and we get $h_H(w_1)^{1/|E(H)|} + h_H(w_2)^{1/|E(H)|} < h_H(w_1 + w_2)^{1/|E(H)|}$. This shows that for our purposes it is sufficient to restrict to the case where $H$ is bipartite. In this case we can use a more general setting than $W_S$ and remove the condition that $w$ is symmetric.

**Definition 2.2** Consider two index sets $\mathcal{I}$ and $\mathcal{J}$. Let

$$W(\mathcal{I} \times \mathcal{J}) = \{ w : \mathcal{I} \times \mathcal{J} \rightarrow \mathbb{R} \},$$

and

$$W^+(\mathcal{I} \times \mathcal{J}) = \{ w : \mathcal{I} \times \mathcal{J} \rightarrow \mathbb{R}^+ \}.$$  

For a bipartite graph $H = (X, Y; E)$ and $w \in W(\mathcal{I} \times \mathcal{J})$, define

$$h_H(w) := \sum_{x_\mathcal{I} \in \mathcal{I}} \sum_{y_\mathcal{J} \in \mathcal{J}} \prod_{(u,v) \in E} w(x_u, y_v);$$

(4)

$$\|w\|_H := |h_H(w)|^{1/|E|};$$

(5)

$$\|w\|_{r(H)} := h_H(|w|)^{1/|E|}.$$  

(6)

Furthermore let $W_H(\mathcal{I} \times \mathcal{J})$ and $W_{r(H)}(\mathcal{I} \times \mathcal{J})$ respectively denote the set of all $w \in W$ with $\|w\|_H < \infty$ and $\|w\|_{r(H)} < \infty$.

**Remark 2.3** Note that every bipartite graph $G = (X', Y'; E')$ can be represented by a zero-one matrix $w$ whose rows and columns are indexed respectively by elements of $X'$ and $Y'$, and every entry is equal to one, if and only if its corresponding row and column are adjacent in $G$. Then $h_H(w)$, as defined in (4), is the number of homomorphisms from $H$ to $G$ so that $X$ is mapped into $X'$, and $Y$ is mapped into $Y'$.
Remark 2.4 From now on, when there is no ambiguity we drop the variables from the subscript of sums. For example with this notation, we allow \( h_H(w) = \sum \prod_{(u,v) \in E(H)} w(u,v) \) or even \( h_H(w) = \sum \prod_{(u,v) \in E(H)} w \). We might also write \( \mathcal{W}_H \) instead of \( \mathcal{W}_H(I \times J) \).

Note that every \( w \in \mathcal{W}_H(I \times J) \) can be thought of as a matrix whose rows are indexed over \( I \) and whose columns are indexed over \( J \). Let \( w \in \mathcal{W}_H(I \times J) \) and \( w' \in \mathcal{W}_H(J \times K) \). Then the matrix multiplication of \( w \) to \( w' \) is defined. In order to distinguish between the matrix multiplication and the pointwise multiplication we denote the former by \( w \circ w' \), and the latter by \( w w' \). Moreover if \( w \in \mathcal{W}(I \times I) \), then \( w^n := w \circ \cdots \circ w \), where \( w \) appears \( n \) times in the right-hand side.

We shall see below that neither \( \| \cdot \|_H \) nor \( \| \cdot \|_{r(H)} \) is always a norm. We have the following observations:

Observation 2.5 Let \( H \) be a graph. Then

(i): If the function \( \| \cdot \|_H \) is a semi-norm on \( \mathcal{W}_H \), then \( \| \cdot \|_{r(H)} \) is a norm on \( \mathcal{W}_{r(H)} \).

(ii): If \( H \) has a vertex of odd degree, then \( \| \cdot \|_H \) is not always a norm.

Proof. Part (i) is trivial. To prove (ii), note that \( \| w \|_H = 0 \) for

\[
 w = \begin{bmatrix}
 1 & -1 \\
 -1 & 1 
\end{bmatrix}.
\]

Before continuing the discussion, let us first give a brief review on Schatten-von Neumann norms, and see why \( \| \cdot \|_{C_4} \) is a special case of those norms.

2.3 Schatten-von Neumann classes

Let \( A \) be a real matrix. The \( p \)-th Schatten norm of \( A \) is defined as

\[
\| A \|_{S_p} = \left( \text{tr}(A^t A)^{p/2} \right)^{1/p}.
\]

Note that when \( A \) is an \( n \times n \) matrix, \( \| A \|_{S_p} \) is just the \( \ell_p \) norm that is applied to the eigenvalues of \( |A| = (A^t A)^{1/2} \). This fact generalizes by the spectral theorem to the infinite case. It is well-known (but not trivial) that \( \| \cdot \|_{S_p} \) is a norm. This can be deduced from Theorem A below due to Schatten and von Neumann [27] [28] [29]. As it is mentioned in Remark 2.4 we can consider the elements of \( \mathcal{W}(I \times J) \) as matrices. We state the theorem of Schatten and von Neumann in this notation.

Theorem A. Suppose that \( 1 \leq p, q, r < \infty \) are such that \( \frac{1}{p} + \frac{1}{q} = \frac{1}{r} \). Let \( v \in \mathcal{W}(I_1, I_2) \) and \( w \in \mathcal{W}(I_2, I_3) \). Then

\[
\| v \circ w \|_{S_r} \leq \| v \|_{S_p} \| w \|_{S_q}.
\]

Consider \( C_{2n} \), a cycle of even length and a \( w \in \mathcal{W}(I \times J) \). Note that

\[
\| w \|_{S_{C_{2n}}}^2 = \text{tr}(w^t \circ w) = \sum_{i \in I} (w^t \circ w)_{ii} = \sum_{i_0, i_2, \ldots, i_{2n-2} \in I} \sum_{i_1, i_3, \ldots, i_{2n-1} \in J} w(i_0, i_1)w(i_1, i_2) \cdots w(i_{2n-2}, i_{2n-1})w(i_{2n-1}, i_0) = \| w \|_{S_{C_{2n}}}^2.
\]

For further reading about the Schatten-von Neumann norms we refer the reader to [5].
2.4 Hölder and weakly Hölder graphs

The following is a corollary of Theorem A.

**Corollary 2.6** Let \( k \geq 1 \) be an integer, \( V(C_{2k}) = X \cup Y \) be the bipartization of \( C_{2k} \), and \( w_e \in W(I \times J) \) for every \( e \in E(C_{2k}) \). Then

\[
\sum_{x_u \in I} \sum_{y_v \in J} \prod_{e=(u,v) \in E(C_{2k})} w_e(x_u, y_v) \leq \prod_{e \in E(C_{2k})} \|w_e\|_{C_{2k}}.
\]

**Proof.** Let us identify \( C_{2k} = (\{2i + 1 : 0 \leq i \leq k - 1\}, \{2i : 0 \leq i \leq k - 1\}; E) \), where

\[
E = \{(1,0),(3,2),\ldots,(2n-1,2n-2)\} \cup \{(1,2),(3,4),\ldots,(2n-1,0)\}.
\]

We have

\[
\sum_{e=(u,v)} \prod_{e \in (u,v)} w_e(x_u, y_v) = \text{tr}\left( w_t^{(1,0)} \circ w_t^{(1,2)} \circ w_t^{(2k-1,2k-2)} \circ w_t^{(2k-1,0)} \right) 
\leq \left\| w_t^{(1,0)} \circ w_t^{(1,2)} \circ w_t^{(2k-1,2k-2)} \circ w_t^{(2k-1,0)} \right\|_{S_1}.
\]

Since \( \frac{1}{2k} + \ldots + \frac{1}{2k} = \frac{1}{1} \), by repeatedly applying Theorem A and noting that always \( \|w\|_{S_p} = \|w^t\|_{S_p} \), we get

\[
\left\| w_t^{(1,0)} \circ w_t^{(1,2)} \circ w_t^{(2k-1,2k-2)} \circ w_t^{(2k-1,0)} \right\|_{S_1} \leq \prod_{e \in E} \|w_e\|_{C_{2k}} = \prod_{e \in E} \|w_e\|_{C_{2k}}.
\]

Corollary 2.6 inspires us to have the following definition.

**Definition 2.7** A bipartite graph \( H \) is called

- Hölder: If for every choice of \( \{w_e \in W(I \times J) : e \in E(H)\} \) we have
  \[
  \sum_{e \in E(H)} \prod_{e \in E(H)} w_e \leq \prod_{e \in E(H)} \|w_e\|_H. 
  \]

- Weakly Hölder: If for every choice of \( \{w_e \in W(I \times J) : e \in E(H)\} \) we have
  \[
  \sum_{e \in E(H)} \prod_{e \in E(H)} w_e \leq \prod_{e \in E(H)} \|w_e\|_{r(H)}. 
  \]

Note that Hölder implies weakly Hölder, and by Corollary 2.6, cycles of even length are Hölder. We prove the following theorem.

**Theorem 2.8** A graph \( H \) is Hölder if and only if \( \| \cdot \|_H \) is always a semi-norm. A graph \( H \) is weakly Hölder if and only if \( \| \cdot \|_{r(H)} \) is always a norm.

Let us state our positive results.

**Theorem 2.9** We have the following:

(i): If \( G \) and \( H \) are both Hölder (weakly Hölder) then so is \( G \times^b H \).

(ii): For every \( m, n \geq 1 \), the graph \( K_{m,n} \) is weakly Hölder. If both \( m \) and \( n \) are even then \( K_{m,n} \) is Hölder.

(iii): The hypercubes \( Q_n \) are weakly Hölder.
(iv): If $H$ is weakly Hölder, then $H^{2k}$ is Hölder for every integer $k > 0$.

We also prove the following necessary conditions for a graph to be weakly Hölder.

**Theorem 2.10** If $G$ is weakly Hölder, then

(i): For every subgraph $H \subseteq G$, we have $\frac{|E(H)|}{|V(H)|} \leq \frac{|E(G)|}{|V(G)|}$.

(ii): If $u$ and $v$ belong to the same part in the bipartization of $G$, then $\deg(u) = \deg(v)$.

**Remark 2.11** Theorem 2.10 implies that among trees only $K_{1,n}$ are weakly Hölder. As we shall see in Section 3.3, the proof of Theorem 2.10 shows that if a graph $G$ fails to satisfy at least one of Theorem 2.10 (i) or (ii), then the triangle inequality fails even if we restrict ourselves to the symmetric matrices.

### 2.5 Sidorenko’s conjecture

It is more natural to state Sidorenko’s conjecture in a continuous setting.

**Definition 2.12** Consider two probability spaces $(M, \mathcal{F}, \mu)$ and $(M', \mathcal{F}', \nu)$. Let

$$W(\mu \times \nu) = \{ w : \mu \times \nu \to \mathbb{R} : w \text{ is measurable} \},$$

and for a bipartite graph $H = (X, Y; E)$ and $w \in W(\mu \times \nu)$, define

$$h_H(w) := \int \prod_{(u,v) \in E} w(x_u, y_v) \prod_{x \in X} d\mu(x) \prod_{y \in Y} d\nu(y);$$

$$\|w\|_H := |h_H(w)|^{1/|E|};$$

$$\|w\|_{r(H)} := |h_H(|w|)|^{1/|E|}.$$  \hfill (9, 10, 11)

Furthermore let $W_H(\mu \times \nu)$ and $W_{r(H)}(\mu \times \nu)$ respectively denote the set of all $w \in W$ with $\|w\|_H < \infty$ and $\|w\|_{r(H)} < \infty$.

The $W_H(\mu \times \nu)$ spaces are related to $W_H(I \times J)$ spaces in the same way that $L_p$ spaces are related to $L_p$ spaces. So as one might expect it is easy to see that all the results that are mentioned in the previous sections hold for this setting as well. Now in this setting Sidorenko’s conjecture says that for every bipartite graph $H$ with $m$ edges and every $w \in W(\mu \times \nu)$, we have

$$h_H(|w|) \geq h_{K_2}(|w|)^m.$$  \hfill (12)

This simple-to-state conjecture is verified only for a handful of graphs [31] including trees, even cycles, and complete bipartite graphs. To see the importance of the conjecture note that the case where $H$ is a path is equivalent to the Blakley-Roy inequality [1] which has originally been proved by Blakley and Roy using spectral techniques.

Sidorenko’s conjecture has an interesting meaning: Fix a constant $0 \leq p \leq 1$. For an integer $n > 0$, let $G(n,p)$ be a random graph on $n$ vertices where each edge is present independently with probability $p$. Let $\mu$ be the uniform measure on the vertices of $G(n,p)$, and $w \in W(\mu \times \mu)$ be its adjacency matrix. Note that with high probability $h_{K_2}(w) = p \pm o(1)$, and $h_H(w) = p^m \pm o(1)$. So roughly speaking Sidorenko’s conjecture says that for every bipartite graph $H$, among all graphs with fixed number of vertices and edges, the random graphs asymptotically contain the least number of copies of $H$.

Balázs Szegedy [private communication] pointed out to the author that if $\| \cdot \|_{r(H)}$ is a norm, then Sidorenko’s conjecture holds for $H$. This can be easily seen from the convexity of norms. But now that we have Theorem 2.8 in fact we can say much more. Note that Sidorenko’s conjecture can be reformulated as the following.

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Conjecture 2.13 (Sidorenko’s Conjecture) Let $\mu$ be a probability measure. For every bipartite graph $H$ and every $w \in \mathcal{W}(\mu \times \mu)$, we have

$$\|w\|_{r(H)} \geq \|w\|_{r(K_2)}.$$ 

We have the following result as a corollary to Theorem 2.8 which implies a stronger statement than of Sidorenko’s conjecture’s when $\| \cdot \|_{r(H)}$ is a norm.

Theorem 2.14 Let $\mu$ and $\nu$ be two probability measures, and $H$ be a bipartite graph such that $\| \cdot \|_{r(H)}$ is a norm. Then for every subgraph $G \subseteq H$ and every $w \in \mathcal{W}(\mu \times \nu)$ we have

$$\|w\|_{r(H)} \geq \|w\|_{r(G)}.$$ 

Proof. For $e \in E(G)$, define $w_e = w$, and for $e \in E(H) \setminus E(G)$ define $w_e = 1$. Since $\| \cdot \|_{r(H)}$ is a norm, by Theorem 2.8 we have

$$\int \left| \prod_{e \in E(H)} w_e \right| \leq \prod_{e \in E(H)} \|w_e\|_{r(H)}.$$ 

By our choice of $w_e$ we get

$$h_G(|w|) \geq \int \left| \prod_{e \in E(H)} w_e \right| \leq \left( \prod_{e \in E(G)} \|w\|_{r(H)} \right) \left( \prod_{e \in E(H) \setminus E(G)} \|1\|_{r(H)} \right) \|E(G)\|,$$

or equivalently $\|w\|_{r(G)} \leq \|w\|_{r(H)}$.  

Remark 2.15 Now by combining Theorem 2.14 and Theorem 2.9 we see that $\| \cdot \|_{r(Q_n)}$ is an increasing sequence of norms on $\mathcal{W}(\mu \times \nu)$. Note that this is not true for $\mathcal{W}(\mathcal{I} \times \mathcal{J})$.

Consider a probability measure $\mu$ and a symmetric function $w \in \mathcal{W}(\mu \times \mu)$. In [6], Erdős and Simonovits proved that for positive integers $n \leq m$, we have $\|w\|_{r(P_{2n})} \leq \|w\|_{r(P_{2m})}$, where $P_k$ denotes the path of length $k$. They furthermore conjectured $\|w\|_{r(P_{2n-1})} \leq \|w\|_{r(P_{2m-1})}$. This conjecture would have been followed from Theorem 2.14 if $P_{2m-1}$ was weakly Hölder, but Theorem 2.11 shows that for $m > 2$, $P_{2m-1}$ is not weakly Hölder. 

2.6 Banach Space properties

In this section we study the Banach space properties of the graph norms.

The modulus of convexity of a Banach space $Y$ is a non-negative function $\delta_Y$, defined for $\epsilon > 0$ by

$$\delta_Y(\epsilon) = \inf \left\{ 1 - \frac{|x + y|}{2} : x, y \in Y, \|x\| = \|y\| = 1, \|x - y\| \geq \epsilon \right\}.$$ 

The modulus of smoothness of $Y$ is a function $\rho_Y$ defined for $\epsilon > 0$ by

$$\rho_Y(\epsilon) = \frac{1}{2} \sup \{|x + y| + |x - y| - 2 : x, y \in Y, \|x\| = 1, \|y\| = \epsilon \}.$$ 

Next theorem shows that if $H$ is a Hölder graph with $m$ edges, then $\| \cdot \|_H$ has the same moduli of smoothness and convexity as $\ell_m$. This was known [8] for $H = C_{2n}$ due to the relation to the Schatten-von Neumann norms.

Theorem 2.16 There exist constants $C_m > 0$ and $C'_m > 0$ such that the following holds. Let $H$ be a Hölder graph with $m$ edges, and $\mathcal{I}$ and $\mathcal{J}$ be two infinite sets. Then

$$C_m \delta_{\ell_m} \leq \delta_{\mathcal{W}_H} \leq \delta_{\ell_m}, \quad (12)$$

and

$$C'_m \rho_{\ell_m} \leq \rho_{\mathcal{W}_H} \leq \rho_{\ell_m}. \quad (13)$$

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For the definitions of type and cotype of a Banach space we refer the reader to [20]. It is known [7,8] that the modulus of convexity of power type \( q \geq 2 \) implies cotype \( q \), and the modulus of smoothness of power type \( 1 < p \leq 2 \) implies type \( p \). Thus Theorem 2.16 determines the type and cotype of \( \mathcal{W}_H \).

**Theorem 2.17** Let \( H \) be a graph with \( m \) edges such that \( \| \cdot \|_H \) is a norm on \( \mathcal{W}_H(I \times J) \). Then \( \mathcal{W}_H \) is of type 2 and cotype \( m \), and it is not of any cotype \( q < m \) if \( I \) and \( J \) are both infinite.

**Proof.** The type and cotype follows from the results of Figiel and Pisier [7,8]. It remains to show that if \( I \) and \( J \) are infinite, then \( \mathcal{W}_H \) is not of cotype \( q < m \). But if \( I \) and \( J \) are infinite, then \( \mathcal{W}_H \) contains all finite dimensional subspaces of \( \ell_m \) as subspace, and thus it cannot be of cotype \( q < m \).

3 **Proofs**

3.1 **Proof of Theorem 2.8**

Let us first develop some tools. Let \( w_1 \in \mathcal{W}(I \times J) \) and \( w_2 \in \mathcal{W}(I' \times J') \). Define \( w_1 \otimes w_2 \in \mathcal{W}((I \times I') \times (J \times J')) \), as \( [(x, x'), (y, y')] \mapsto w_1(x, y)w_2(x', y') \). We also define \( w^\otimes k = w \otimes \ldots \otimes w \), where \( w \) appears \( k \) times in the right-hand side. We have the following trivial observation.

**Lemma 3.1** Let \( H = (X, Y; E) \) be a bipartite graph, and \( w_e \in \mathcal{W}(I \times J) \) and \( w'_e \in \mathcal{W}(I' \times J') \) for \( e \in E \). Then

\[
\sum \prod_{e \in H} w_e \otimes w'_e = \left( \sum \prod_{e \in H} w_e \right) \left( \sum \prod_{e \in H} w'_e \right).
\]

Now with this lemma in hand we can prove Theorem 2.8 with the standard tensor power trick.

**Proof.**[Theorem 2.8] Let \( H \) be a Hölder graph with \( m \) edges and \( w_1, w_2 \in \mathcal{W}_H(I \times J) \). Then by expanding \( h_H(w_1 + w_2) \) and applying \([7] \) to each term, it is clear that

\[
h_H(w_1 + w_2) \leq (\|w_1\|_H + \|w_2\|_H)^m.
\]

This proves that \( \| \cdot \|_H \) is a semi-norm. Now suppose that \( H \) is not Hölder. Then there exists \( \{w_e \in \mathcal{W}(I \times J) : e \in E(H)\} \), such that

\[
\sum_{e \in E(H)} \prod_{e \in E(H)} w_e \geq \|w_e\|_H.
\]

After proper normalization we may assume that \( \|w_e\|_H \leq 1 \), for every \( e \in E(H) \), and \( \sum \prod_{e \in E(H)} w_e = c \), for some \( c > 1 \). Now by Lemma 3.1

\[
\left\| \sum_{e \in E(H)} w_e \otimes 2n \right\|_H^m = \sum_{x \in I, y \in J, e' \in E(H)} \prod_{e' \in E(H)} \left( \sum_{e \in E(H)} w_e \otimes 2n \right) = \sum_{f : E(H) \to E(H)} \left( \sum_{x \in I, y \in J} \prod_{e \in E(H)} w_e \otimes 2n \right) \geq \left( \sum_{x \in I, y \in J} \prod_{e \in E(H)} w_e \right)^{2n} = c^{2n},
\]

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while for every $e \in E(H)$, by Lemma 3.1 we have that 
\[
\|w_e^{2n}\|_H = \|w_e\|_H^{2n} \leq 1.
\]
Thus for large enough $n$, the triangle inequality fails:
\[
\sum_{e \in E(H)} \|w_e^{2n}\|_H \leq m < c^{2n/m} \leq \sum_{e \in E(H)} \|w_e^{2n}\|_H.
\]
The proof of the weakly Hölder case is similar.

It is easy to see that (14) implies the following corollary.

**Corollary 3.2** Let $H$ be a graph such that $\| \cdot \|_H$ is a quasi-norm (semi-quasi-norm, respectively) on $\mathcal{W}_H$. Then $\| \cdot \|_H$ is a norm (semi-norm respectively) on $\mathcal{W}_H$. The same statement holds for $\| \cdot \|_{r(H)}$.

### 3.2 Proof of Theorem 2.9

(i): Suppose that $G$ and $H$ are both Hölder with $m$ and $m'$ edges, respectively. Consider \( \{w_e \in \mathcal{W}(I \times J) : e \in \mathcal{E}(G \times H)\} \). We have
\[
\sum_{e \in E(G \times H)} \prod_{(a,b) \in E(G)} \prod_{(u,v) \in E(H)} w_{[(a,u),(b,v)]} \left( x_{[a,u]}, y_{[b,v]} \right) \]
\[
\leq \prod_{(a,b) \in E(G)} \left( \prod_{(u,v) \in E(H)} \left( \sum_{(a',b') \in E(G)} \prod_{(u',v') \in E(H)} w_{[(a',u'),(b',v')]} \left( x_{[a',u']}, y_{[b',v']} \right) \right) \right)^{1/m} \]
\[
= \prod_{(a,b) \in E(G)} \left( \prod_{(u,v) \in E(H)} \left( \sum_{(a',b') \in E(G)} \prod_{(u',v') \in E(H)} w_{[(a',u'),(b',v')]} \left( x_{[a',u']}, y_{[b',v']} \right) \right) \right)^{1/m'} \]
\[
\leq \prod_{(a,b) \in E(G)} \left( \prod_{(u,v) \in E(H)} \left( \sum_{(a',b') \in E(G)} \prod_{(u',v') \in E(H)} w_{[(a',u'),(b',v')]} \left( x_{[a',u']}, y_{[b',v']} \right) \right) \right)^{1/mm'} \]
\[
= \prod_{[(a,u),(b,v)] \in \mathcal{E}(G)} \|w_{[(a,u),(b,v)]}\|_{G \times H},
\]
where in (15) and (16) we applied (14). The case of weakly Hölder is similar.

(ii): The fact that $K_{1,2n}$ is Hölder follows from the classical Hölder inequality. Indeed let $X(K_{1,2n}) = \{u\}$ and $Y(K_{1,2n}) = \{v_1, \ldots, v_{2n}\}$. Then
\[
\sum_{i=1}^{2n} \prod_{i=1}^{2n} w_{(u,v_i)} = \sum_{x_i \in I} \prod_{i=1}^{2n} \left( \sum_{y_{v_i} \in J} w_{(u,v_i)} \right) \]
\[
\leq \prod_{i=1}^{2n} \left( \sum_{x_i \in I} \left( \sum_{y_{v_i} \in J} w_{(u,v_i)} \right)^{2n} \right)^{1/2n} = \prod_{i=1}^{2n} \|w_{(u,v_i)}\|_{K_{1,2n}}.
\]
Similarly one can show that $K_{1,n}$ is weakly Hölder. Now the assertion follows from (i) and the fact that $K_{m,n} = K_{1,n} \times K_{m,1}$. 

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Remark 3.5 Let \( \phi : V(Q_n) \to V(Q_n) \) be such that \( \phi(u) \in X \) and \( \phi(v) \in Y \) for every \( u \in X \) and \( v \in Y \), and furthermore \( (\phi(u), \phi(v)) \in E \) if \( (u, v) \in E \). Define
\[
R_\phi = \prod_{u \in X(Q_n), v \in Y(Q_n)} f_{\phi(u)}(x_u)g_{\phi(v)}(y_v) \prod_{(s, t) \in E(Q_n)} w_{\phi(s), \phi(t)}(x_s, y_t).
\]
For example, for \( e = (a, b) \), let \( \phi_e \) be defined as \( \phi_e(u) = a \) if \( u \in X(Q_n) \) and \( \phi_e(u) = b \), otherwise. Then \( R_e \) as it is defined in Claim 3.3 is in fact the same as \( R_{\phi_e} \), and if we denote by \( id \) the identity map from \( V(Q_n) \) to itself, then Claim 3.3 says that
\[
\sum |R_{id}| \leq \prod_{e \in E(Q_n)} \left( \sum |R_{\phi_e}| \right)^{1/|E(Q_n)|}.
\]
Proof. [Claim 3.3] We prove the claim by induction. Before engaging in the calculations, let us explain the intuition behind the proof. The variables $x_u, y_v$ assign some values to the vertices. The product in the left-hand side of (17) is the product of the functions $f_u, g_v$ and $w_e$ where $f_u$ and $g_v$ depend only on the values that are assigned to the vertices $u$ and $v$ respectively, and $w_e$ depends only on the values that are assigned to the endpoints of $e$. The first step in the proof is to group these functions together so that they can be interpreted as the same product but for $Q_{n-1}$ instead of $Q_n$. Then we can apply the induction hypothesis.

**Step 1:** We regroup the product in the left-hand side of (17) in the following way.

$$
\prod_{u \in X(Q_n), v \in Y(Q_n)} f_u g_v \prod_{e \in E(Q_n)} w_e = \prod_{u \in X(Q_{n-1}), v \in Y(Q_{n-1})} (f_{0u} w_{[0u,1u]} g_{1u})(f_{1v} w_{[1v,0v]} g_{0v}) \prod_{(s,t) \in E(Q_{n-1})} (w_{(0s,0t)} w_{(1t,1s)}).
$$

(19)

The left-hand side of (19) is the product in the left-hand side of (17), and the right-hand side of (19) can be interpreted as the same product but for $Q_{n-1}$ instead of $Q_n$. For $Q_{n-1}$ we use the induction hypothesis, for $Q_n$ we use the induction hypothesis for $Q_{n-1}$ but on different index sets in the following way: Let the value assigned to the endpoints of $e$ on the left-hand side of (17) is the product of the functions $f_u, g_v, w_e$. To use the induction hypothesis, for $Q_{n-1}$ we use the induction hypothesis on $Q_{n-1}$.

The right-hand side of (19) can be interpreted as the same product for $Q_{n-1}$ but on different index sets in the following way: Let the value assigned to the endpoints of $e$ on the left-hand side of (17) is the product of the functions $f_u, g_v, w_e$. To use the induction hypothesis, for $Q_{n-1}$ we use the induction hypothesis on $Q_{n-1}$.

More formally, to prove the claim for $n$ and $\mathcal{I} \times \mathcal{J}$, we use the induction hypothesis for $n-1$ with the index set $(\mathcal{I} \times \mathcal{J}) \times (\mathcal{I} \times \mathcal{J})$. Every vertex $v \in Q_{n-1}$ corresponds to two adjacent vertices 0v and 1v. To use the induction hypothesis, for $u \in X(Q_{n-1})$, define

$$
f'_u : \mathcal{I} \times \mathcal{J} \to \mathbb{R}
$$

$$
f'_u : [x, y] \mapsto f_{0u}(x)w_{[0u,1u]}(x, y)g_{1v}(y).
$$

For $v \in Y(Q_{n-1})$, define

$$
g'_v : \mathcal{I} \times \mathcal{J} \to \mathbb{R}
$$

$$
g'_v : [x, y] \mapsto f_{1v}(x)w_{[1v,0v]}(x, y)g_{0v}(y).
$$

and for $e = (u, v) \in E(Q_{n-1})$,

$$
w'_e : ([x, y], [x', y']) \mapsto w_{[0u,0v]}(x, y)w_{[1v,1u]}(x', y).
$$

Then

$$
\text{L.H.S. of (17)} = \text{R.H.S. of (19)} =
$$

$$
\sum_{u \in X(Q_{n-1}), v \in Y(Q_{n-1})} \prod_{e \in E(Q_{n-1})} f'_u([x_{0u}, y_{1u}])g'_v([x_{1v}, y_{0v}]) \prod_{e = (s,t) \in E(Q_{n-1})} w'_e([x_{0s}, y_{1s}], [x_{1t}, y_{0t}]).
$$

(20)

Then we apply the induction hypothesis to the right-hand side of (20) and obtain

$$
\text{R.H.S. of (20)} \leq \prod_{e = (a,b) \in E(Q_{n-1})} \left( \sum_{u \in X(Q_{n-1}), v \in Y(Q_{n-1})} \prod_{e \in E(Q_{n-1})} f'_u([x_{0u}, y_{1u}])g'_v([x_{1v}, y_{0v}]) \prod_{(s,t) \in E(Q_{n-1})} w'_{(a,b)}([x_{0s}, y_{1s}], [x_{1t}, y_{0t}]) \right)^{1/|E(Q_{n-1})|}
$$

$$
= \prod_{e = (a,b) \in E(Q_{n-1})} \left( \sum R_{\psi_e} \right)^{1/|E(Q_{n-1})|},
$$

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where for $e = (a, b)$

$$
\begin{align*}
\psi_e(0a) &= 0a & \forall u \in X(Q_{n-1}) \\
\psi_e(1a) &= 1a & \forall u \in X(Q_{n-1}) \\
\psi_e(0b) &= 0b & \forall v \in Y(Q_{n-1}) \\
\psi_e(1b) &= 1b & \forall v \in Y(Q_{n-1})
\end{align*}
$$

(21)

Combining this with (20) we obtain

$$
\text{L.H.S. of (17)} \leq \prod_{e = (a, b) \in E(Q_{n-1})} \left( \sum R_{\psi_e} \right)^{1/|E(Q_{n-1})|}.
$$

(22)

Step 2: In this step we obtain a different bound for the left-hand side of (17). In Step 1, for every $v \in Q_{n-1}$, we grouped the two vertices $0v$, $1v$ and the edge between them as one vertex (see (19)) and this reduced $Q_n$ to $Q_{n-1}$. In this step we reduce $Q_n$ to $Q_2$. For every vertex $s \in \{00, 11\} = X(Q_2)$, define

$$
f''_s = \prod_{u \in X(Q_{n-2}), v \in Y(Q_{n-2})} f_{su} g_{sv} \left( \prod_{(u, v) \in E(Q_{n-2})} w_{[su, sv]} \right),
$$

for every $t \in \{01, 10\} = Y(Q_2)$, define

$$
g''_t = \prod_{u \in X(Q_{n-2}), v \in Y(Q_{n-2})} f_{tv} g_{tu} \left( \prod_{(u, v) \in E(Q_{n-2})} w_{[tv, tu]} \right),
$$

and for every edge $e = (s, t) \in E(Q_2)$,

$$
w''_e = \prod_{u \in X(Q_{n-2}), v \in Y(Q_{n-2})} w_{[su, tu]} w_{[tv, sv]}.
$$

Note that the product in the left-hand side of (17) is equal to

$$
\left( \prod_{s \in X(Q_2), t \in Y(Q_2)} f''_s g''_t \right) \left( \prod_{(s, t) \in E(Q_2)} w''_{(s, t)} \right).
$$

We can apply Lemma 3.4 with proper index sets to these functions. We get

$$
\text{L.H.S of (17)} \leq \prod_{e = (s, t) \in E(Q_2)} \left( \sum R_{\rho_e} \right)^{1/4},
$$

(23)

where for $e = (s, t)$

$$
\begin{align*}
\rho_e(s') &= sv & \forall s' \in X(Q_2), v \in V(Q_{n-2}) \\
\rho_e(t') &= tv & \forall t' \in Y(Q_2), v \in V(Q_{n-2})
\end{align*}
$$

(24)

Step 3: In this step we combine Steps 1 and 2. Note that in (22), the product $R_{\psi_e}$ has the same form as the product in the left-hand side of (17). Thus we can apply Step 2 to $\sum R_{\psi_e}$. For $e \in E(Q_{n-1})$ we get

$$
\sum R_{\psi_e} \leq \prod_{e' = (s, t) \in E(Q_2)} \left( \sum R_{\rho_{e'} \circ \psi_e} \right)^{1/4}.
$$

(25)

Combining this with (22) we obtain

$$
\text{L.H.S. of (17)} \leq \prod_{e \in E(Q_{n-1})} \left( \prod_{e' \in E(Q_2)} \left( \sum R_{\rho_{e'} \circ \psi_e} \right)^{1/4|E(Q_{n-1})|} \right),
$$

(26)
Step 4: Now for some integer \(k > 0\) we repeatedly apply Step 3, and by (26) we get,
\[
\text{L.H.S. of (17)} \leq \prod_{e_1, \ldots, e_k \in E(Q_{n-1})} \left( \prod_{e'_1, \ldots, e'_k \in E(Q_2)} \left( \sum R_{\rho_{e'_1} \circ \psi_{e_1} \circ \ldots \circ \rho_{e'_k} \circ \psi_{e_k}} \right)^{4^{-k} |E(Q_{n-1})|^{-k}} \right). \tag{27}
\]

Let us first assume that \(\|w_e\|_\infty, \|f_u\|_\infty, \|g_v\|_\infty < C\) for some constant \(C > 0\). We shall deal with the general case later. Note first that for every arbitrary \(\phi : V(Q_n) \to V(Q_n)\), we have \(\sum R_{\phi} < L\) for some large number \(L\) which depends on \(C, f_u's, g_v's,\) and \(w_e's\) but does not depend on \(\phi\). Notice that for \(e = (a, b) \in E(Q_{n-1})\)
\[
\rho(00,01) \circ \psi_e = \phi(0a,0b), \tag{28}
\]
and
\[
\rho(11,10) \circ \psi_e = \phi(1a,1b), \tag{29}
\]
where \(\phi(0a,0b)\) and \(\phi(1a,1b)\) are defined as in Remark 3.5.

Next note that for every \(\tilde{e} \in E(Q_n), e \in E(Q_{n-1})\), and \(e' \in E(Q_2)\), we have \(\rho_{e'} \circ \psi_e \circ \phi_e = \tilde{\phi}_e\).

Then from (28) and (29) we can conclude that whenever there exists \(1 \leq i \leq k\) such that \(e'_i \in \{(00,01), (11,10)\}\), then \(\rho_{e'_i} \circ \psi_e \circ \ldots \circ \rho_{e'_k} \circ \psi_{e_k} = \phi_e\) for some \(e \in E(Q_n)\). Thus from (27), there exists numbers \(p_e \geq 0\) such that
\[
\sum_{e \in E(Q_n)} p_e = 1 - 2^{-k},
\]
and
\[
\text{L.H.S. of (17)} \leq \prod_{e \in E(Q_n)} \left( \sum R_{\phi_e} \right)^{p_e} L^{2^{-k}}. \tag{30}
\]

Since \(Q_n\) is edge transitive, by applying the bound (30) to different permutations of the edges and taking the geometric average, we finally conclude that
\[
\text{L.H.S. of (17)} \leq L^{2^{-k}} \prod_{e \in E(Q_n)} \left( \sum R_{\phi_e} \right)^{(1-2^{-k})/|E(Q_n)|}. \tag{31}
\]

By tending \(k\) to infinity, (31) reduces to (18).

Step 5: Now consider the general case where \(\|f_u\|_\infty, \|g_v\|_\infty, \|w_e\|_\infty\) need not be bounded. Fix \(C > 0\) and let \(f'_u := \max(f_u, C), g'_v := \max(g_v, C)\) and \(w'_e := \max(w_e, C)\). We know that Claim 3.3 holds for these functions. By tending \(C\) to infinity the dominated convergence theorem implies the claim for the general case as well.

3.3 Proof of Theorem 2.10

Let \(V(G) = X \cup Y\) be the bipartization of \(G\), and denote \(m = |E(G)|\), and \(n = |V(G)|\). Consider \(\mathcal{I} = \{1, \ldots, k\}\) for some \(k > 1\).

(i): Let \(\lambda \in \mathbb{R}^k\) be such that \(\lambda_i \geq 0\) for \(1 \leq i \leq k\). Define \(w \in \mathcal{W}^+(\mathcal{I} \times \mathcal{I})\) as
\[
w(x, y) = \begin{cases} 
\lambda_x & x = y \\
0 & \text{otherwise}
\end{cases}
\]

Note that \(\|1\|_{r(G)} = k^{n/m}\) and \(\|w\|_{r(G)} = \|\lambda\|_m\). Now let \(H\) be a subgraph of \(G\) with \(m'\) edges and \(n'\) vertices, and define \(w_e = w, \) if \(e \in E(H), \) and \(w_e = 1\) otherwise. Then by Theorem 2.8,
\[
k^{n-n'} \|\lambda\|_{m'}^{m'} = \sum_{e \in E(G)} w_e \leq \|1\|_{r(G)}^{m-m'} \|\lambda\|_m^{m'} = k^{n(m-m')/m} \|\lambda\|_m^{m'},
\]
and so
\[
\|\lambda\|_{m'} k^{\frac{n-n'}{m'}} \leq \|\lambda\|_m.
\]

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Since this holds for every \( \lambda \), we have \( \frac{m}{m'} - \frac{n'}{m'} \leq \frac{1}{m} - \frac{1}{m'} \), or \( \frac{m'}{m} \leq \frac{m}{m'} \).

(ii): Let \( w \in \mathcal{W}^+ \) be defined as

\[
w(x, y) = \begin{cases} 
1 & x = 1 \text{ or } y = 1 \\
0 & \text{otherwise}
\end{cases}
\]

Notice that if \( \prod_{(u, v) \in E(G)} w(x_u, y_v) \neq 0 \) then \( \{ u : x_u > 1 \} \cup \{ v : y_v > 1 \} \) is an independent set. Thus denoting by \( I(G) \) the set of all independent sets of \( G \), and by \( \alpha(G) \) the size of its largest independent set, we have

\[
h_G(w) = \sum_{S \in I(G)} (k-1)^{|S|} = Ck^{\alpha(G)} + o(k^{\alpha(G)}),
\]

where \( C \) is the number of the independent sets of \( G \) of size \( \alpha(G) \). Now let \( S \) be a largest independent set of \( G \), and let \( u \in V(G) \setminus S \). Without loss of generality assume that \( u \in X \). Let \( H = (X, Y, E \setminus E(u, S)) \), where \( E(u, S) = \{(u, v) \in E(G) : v \in S\} \). Define \( w_v = w \) if \( e \in E(H) \), and \( w_v = 1 \) otherwise. Note that

\[
\sum_{e=(u,v)\in E(G)} w_e \geq k(k-1)^{\alpha(G)},
\]

and from Theorem 22, we get

\[
k^{\alpha(G)} \leq k^{{n\text{deg}(u)}/{m}} \left( Ck^{\alpha(G)} + o(k^{\alpha(G)}) \right)^{{|E(H)|}/{m}}.
\]

Since this holds for every \( k > 1 \), we get

\[
k^{\alpha(G)+1} \leq k^{{n\text{deg}(u)}/{m}} k^{|E(H)|/{m}},
\]

and we get

\[
m \leq (n - \alpha(G))\text{deg}(u). \quad (32)
\]

Since this holds for all \( (n - \alpha(G)) \) vertices that are in \( V(G) \setminus S \), we conclude that \( V(G) \setminus S \) is an independent set and so without loss of generality we may assume that \( S = Y \) and \( |Y| \geq |X| \). Moreover (32) implies that all vertices in \( X = V(G) \setminus S \) have the same degree \( m/(n - |Y|) \).

Next let \( w_1 \in \mathcal{W}^+ \) be defined as

\[
w_1(x, y) = \begin{cases} 
1 & x, y = 1 \\
0 & \text{otherwise}
\end{cases}
\]

Then \( \|w_1\|_G = 1 \). We showed above that \( Y \) is the largest independent set of \( G \). Now consider \( v \in Y \) of degree \( d \). Let \( w_e = w_1 \) for every edge \( e \) incident to \( v \), and let \( w_e = w \) for the rest of the \( m - d \) edges. Then

\[
\sum_{e=(u,v)\in E(G)} w_e \geq (k-1)^{|Y|-1}.
\]

Hence for every \( k \),

\[
(k-1)^{|Y|-1} \leq 1^d \left( k^{\beta} + o(k^{\beta}) \right)^{(m-d)/m},
\]

which implies that \( d|Y| \leq m \). This shows that every vertex in \( Y \) is of degree \( d \).

### 3.4 Proof of Theorem 2.16

Before determining the moduli of smoothness and convexity of \( W_H \) we need the following well-known technical lemma (see [36]).

**Lemma 3.6** Let \( p \geq 2 \), and \( x, y \in \mathbb{R} \). Then
(i) We have
\[(x + y)^p + (x - y)^p \leq 2^{p-1}(x^p + y^p).\]

(ii) There exists a constant \(K_p\) such that if \(|y| \leq 1\), then
\[(1 + y)^p + (1 - y)^p - 2 \leq K_p|y|^2.\]

Now we can state the proof of Theorem 2.16. 

**Proof.**

The inequalities \(\delta_{\mathcal{W}_H} \leq \delta_{\ell_m}\) and \(\rho_{\ell_m} \leq \rho_{\mathcal{W}_H}\) follow from the fact that \(\ell_m\) is a subspace of \(\mathcal{W}_H\). The key observation to prove the theorem is that for \(w_1, w_2 \in \mathcal{W}_H\), we have
\[
\|w_1 + w_2\|^m_H + \|w_1 - w_2\|^m_H \leq (\|w_1\|_H + \|w_2\|_H)^m + (\|w_1\|_H - \|w_2\|_H)^m.
\]  \((33)\)

To prove \((33)\) expand the left-hand side. Some terms will be canceled, and then use the fact that \(H\) is Hölder to bound each of the remaining terms. From \((33)\) and Lemma 3.6 we get
\[
\|w_1 + w_2\|^m_H + \|w_1 - w_2\|^m_H \leq 2^{m-1}(\|w_1\|^m_H + \|w_2\|^m_H).
\]  \((34)\)

Suppose that \(\|w_1\|_H = \|w_2\|_H = 1\) and \(\|w_1 - w_2\|_H \leq \epsilon\). Then by \((34)\) we have
\[
\|w_1 + w_2\|^m_H \leq 2^m - \epsilon^m,
\]
or
\[
\left\| \frac{w_1 + w_2}{2} \right\|^m_H \leq \left(1 - \left(\frac{\epsilon}{2}\right)^m\right)^{1/m}.
\]

This shows that \(\delta_{\mathcal{W}_H}(\epsilon) \geq 1 - (1 - (\frac{\epsilon}{2})^m)^{1/m}\), and finishes the proof of \((12)\) because it is known (see \((20)\)) that \(\delta_{\ell_m}(\epsilon) \geq c_\epsilon/m^2m + o(\epsilon^m)\).

Combining \((33)\) and Lemma 3.6 \((\text{i})\) we get that for \(\|w_2\|_H \leq \|w_1\|_H = 1\),
\[
\|w_1 + w_2\|^m_H + \|w_1 - w_2\|^m_H \leq K_m\|w_2\|^2_H + 2.
\]  \((35)\)

Next note that for \(a \geq 0\) and \(p \geq 1\) we have \(a - 1 \leq (a^p - 1)/p\). From this we get
\[
\|w_1 + w_2\|_H + \|w_1 - w_2\|_H - 2 \leq m^{-1}(\|w_1 + w_2\|^m_H + \|w_1 - w_2\|^m_H - 2).
\]  \((36)\)

Combining \((35)\) and \((36)\) we have
\[
\|w_1 + w_2\|_H + \|w_1 - w_2\|_H - 2 \leq m^{-1}K_m\|w_2\|^2_H.
\]  \((37)\)

Thus for \(0 < \epsilon \leq 1\), we have \(\rho_{\mathcal{W}_H}(\epsilon) \leq m^{-1}K_m\epsilon^2\). This completes the proof of \((13)\) because it is known (see \((20)\)) that \(\rho_{\ell_m}(\epsilon) = (m - 1)\epsilon^2/8 + o(\epsilon^2)\).

**4 Concluding remarks and open questions**

- It is possible to generalize the framework of this article to hypergraphs and define \(\|\cdot\|_H\) norms when \(H\) is a \(k\)-partite \(k\)-uniform hypergraph. These norms will be defined on the space of the functions \(f : \mathcal{I}_1 \times \ldots \times \mathcal{I}_k \to \mathbb{R}\). When \(H\) is the complete \(k\)-partite \(k\)-uniform hypergraph and every part is of size exactly 2, we get the \(k\)-th Gowers norm. Again one can ask that for which hypergraphs the function is a norm.

- Is there any edge transitive bipartite graph that is not weakly Hölder?

- In Theorem \((2.9)\) \((\text{iii})\) we showed that hypercubes are weakly Hölder. We do not know the situation for any other graph that is of the form of the Cartesian products of even cycles and single edges. As the smallest case we suggest determining whether \(K_2 \times C_6\) is weakly Hölder or not.
• Prove or disprove that hypercubes are Hölder.

• Consider the graph $H$ that is obtained by removing the edges of a Hamiltonian cycle from $K_{5,5}$. This is the smallest graph for which Sidorenko’s conjecture is open [31]. Is this graph weakly Hölder? By Theorem 2.14, a positive answer verifies Sidorenko’s conjecture for this graph.

• Determine the moduli of smoothness and convexity of $r(H)$ when $r(H)$ is a norm.

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