SIGN PATTERNS OF RATIONAL MATRICES WITH LARGE RANK

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Abstract. Let $A$ be a real matrix. The term rank of $A$ is the smallest number $t$ of lines (that is, rows or columns) needed to cover all the nonzero entries of $A$. We prove a conjecture of Li et al. stating that, if the rank of $A$ exceeds $t - 3$, there is a rational matrix with the same sign pattern and rank as those of $A$. We point out a connection of the problem discussed with the Kapranov rank function of tropical matrices, and we show that the statement fails to hold in general if the rank of $A$ does not exceed $t - 3$.

1. Introduction

The problem of constructing a matrix over a given ordered field with specified sign pattern and rank deserved a significant amount of attention in recent publications, see [3] and references therein. The present paper establishes a connection of this problem with that of computing certain rank functions arisen from tropical geometry. We prove the conjecture on sign patterns of rational matrices formulated in [3], and we present the examples showing the optimality of our result.

2. Preliminaries

The following notation is used throughout our paper. By $U^{m \times n}$ we denote the set of all $m$-by-$n$ matrices with entries from a set $U$, by $A_{ij} \in U$ we denote an entry of a matrix $A \in U^{m \times n}$. By $U_{(i)}$ we denote the $i$th row of $U$, and we call a line of a matrix any of its columns or rows.

A field $R$ is called ordered if, for some subset $P \subset R$ closed under addition and multiplication, the sets $P$, $-P$, and $\{0\}$ form a partition of $R$. The elements of $P$ are then called positive, and those from $-P$ negative. The sign pattern of a matrix $A \in R^{m \times n}$ is the matrix $S = S(A) \in \{+, -, 0\}^{m \times n}$ defined as $S_{ij} = +$ if $A_{ij}$ is positive, $S_{ij} = -$ if $A_{ij}$ is negative, and $S_{ij} = 0$ if $A_{ij} = 0$. The minimum rank of a sign pattern $S$ with respect to $R$ is the minimum of the ranks of matrices $B$ over $R$ satisfying $S(B) = S$.

There are a significant number of recent publications devoted to the study of the minimal ranks of sign patterns (see [3] and references therein), and our paper aims to prove a conjecture formulated in [3]. This conjecture relates the minimal rank of a pattern with a concept of the term rank of a matrix, which is defined as the smallest number of lines needed to include all the nonzero elements of that matrix. The classical König’s theorem states the the term rank of a matrix $A$ equals the maximum number of nonzero entries of $A$ no two of which belong to the same line, so the term rank of a sign pattern $S$ can be thought of as the maximum of...
the ranks of matrices $C$ over $R$ satisfying $S(C) = S$. Now we can formulate the conjecture by Li et al. relating the concepts of minimum and term ranks for sign pattern matrices.

**Conjecture 2.1.** [3, Conjecture 4.2] Assume $S$ is a sign pattern matrix with term rank equal to $t$, and let $r$ be the minimum rank of $S$ over the reals. If $r \geq t - 2$, then the minimum rank of $S$ over the rationals is $r$ as well.

In Section 3 we develop a combinatorial technique which allows to prove Conjecture 2.1. In Section 4 we establish the connection of the problem discussed with the Kapranov rank function of Boolean matrices introduced in [1]. We also make the use of matroid theory to prove the optimality of the bound in Conjecture 2.1 by showing that its statement fails to hold in general if $r$ is less than $t - 2$.

### 3. Proof of the result

We start with two easy observations helpful for further considerations.

**Observation 3.1.** Multiplying a row of a real matrix $A$ by a nonzero number will not change the minimal ranks of its sign pattern.

*Proof.* Trivial. □

**Observation 3.2.** Let $r$ and $t$ be, respectively, the minimum and maximum ranks of a sign pattern $S$ with respect to an ordered field $R$. Then, for any integer $h \in [r, t]$, there is a matrix over $R$ which has rank $h$ and sign pattern $S$.

*Proof.* Changing a single entry produces a matrix whose rank differs by at most 1 from that of the initial matrix. □

The following lemma gives a useful description of the rank of a block matrix. We say that a linear subspace $S \subset \mathbb{R}^d$ is *rational* if $S$ has a basis consisting of vectors that have rational coordinates only.

**Lemma 3.3.** Let $V_1 \in \mathbb{Q}^{p \times (p-1)}$ and $V_2 \in \mathbb{Q}^{(q-1) \times q}$ be rational matrices that have ranks $p - 1$ and $q - 1$, respectively. Then the set $W$ of all $W \in \mathbb{R}^{p \times q}$ for which the matrix $U = \begin{pmatrix} W & V_1 \\ V_2 & 0 \end{pmatrix}$ has rank $p + q - 2$ is a rational subspace.

*Proof.* Note that rational elementary transformations on the first $p$ rows or first $q$ columns of $U$ can not break the property of $W$ to be a rational subspace. So we can assume that $V_1$ and $V_2$ differ from the identity matrices by adding the zero column and row, which case is easy. □

Now we are ready to prove Conjecture 2.1 in a special case.

**Lemma 3.4.** For any real $m$-by-$n$ matrix $A$ of rank $n - 2$, there is a rational $m$-by-$n$ matrix which has rank $n - 2$ and sign pattern equal to that of $A$.

*Proof.* By the assumptions, there is a rank-two matrix $B \in \mathbb{R}^{n \times 2}$ for which the matrix $AB$ is zero. Observation 3.1 allows one to assume that the first column of $B$ consists of zeros and ones. Let $X$ be a matrix whose $(i, j)$th entry is a variable if $A_{ij} \neq 0$ and $X_{ij} = 0$ otherwise.

For a sufficiently large integer $N > 0$, we set $C_{jk} = [NB_{jk}]/N$. Note that, for every row index $i$, the matrix formed by the rows of $B$ with indexes $j$ satisfying $A_{ij} \neq 0$ has the same rank as the matrix formed by the rows of $C$ with the same
indexes. For every $i$, we assign to every free variable $X_{ig}$ of the linear system $X_{(i)}C = (00)$ the value $[NA_{ig}]/N$. Solving those systems, we get as a solution a rational matrix $X = X(N)$ which satisfies $XC = 0$. Since $X(N) → A$ as $N → ∞$, the matrices $X(N)$ and $A$ have the same sign pattern for sufficiently large $N$. □

Now let us prove the key result of the section.

**Theorem 3.5.** Let $A$ be a real matrix with term rank equal to $t$. If the rank of $A$ equals $t−2$, then there is a rational matrix which has rank $t−2$ and the same sign pattern as $A$.

**Proof.** 1. Up to row and column permutations, $A$ is an $n$-by-$m$ matrix of the form $(B C D O)$, where the matrix $B ∈ ℝ^{p×q}$ satisfies $p + q = t$, and $O$ is the zero matrix. If the rank of $D$ is less than $q − 1$, then by Lemma 3.4 we can construct a rational matrix $D'$ of rank $q−2$ with the same sign pattern as $D$. Choosing $B'$ and $C'$ as arbitrary matrices with sign patterns equal to those of $B$ and $C$, we get that the rank of $(B' C' D' O)$ is at most $t−2$, and we are done. We can assume in what follows that $D$ has rank at least $q−1$ and, similarly, that $C$ has rank at least $p−1$. Since the rank of $A$ is $t−2 = p + q − 2$, we conclude that the rank of $D$ exactly equals $q−1$ and the rank of $C$ is $p−1$. 2. By Step 1, the rows of $C$ are linearly dependent, and we can assume by Observation 3.1 that the coefficients of this linear dependence are rational. In other words, the columns of $C$ generate a rational subspace in $ℝ^p$. Since rational points are dense in rational subspaces, we can assume that the matrix $C$ (and the matrix $D$, similarly) consists of rational numbers, in which case the result follows from Lemma 3.3. □

Now we are ready to prove Conjecture 2.1.

**Theorem 3.6.** Let $A$ be a real matrix with term rank equal to $t$. If the rank of $A$ is at least $t−2$, then there is a rational matrix which has the same sign pattern and rank as those of $A$.

**Proof.** Note that adding a repeating row does not affect the rank of a matrix, and the term rank of the matrix obtained is either equal to or greater by one than that of the initial matrix. Therefore, adding a sufficient number of repeating rows to $A$, we get a matrix $A'$ satisfying the assumptions of Theorem 3.5. So we can find a rational matrix $B$ which has the same sign pattern as that of $A$ and rank not exceeding the rank of $A$. Now the result follows from Observation 3.2. □

4. Optimality of the result

To construct sign patterns of term rank $t$ realizable by real matrices of rank $t−3$ but not by rational matrices of that rank, we need to recall the definition of another rank concept. For $𝔽$ a field, define the Kapranov rank of a matrix $B ∈ {0,1}^{m×n}$ with respect to $𝔽$ as the smallest possible rank of a matrix $C ∈ ℝ^{m×n}$ satisfying $C_{ij} = 0$ if and only if $B_{ij} = 0$. The following lemma points out a connection between the quantity introduced (which we denote by $K_{𝔽}(B)$ in what follows) and the problem of pattern realisability.

**Lemma 4.1.** Assume $R_1$ is an ordered field, and a matrix $B ∈ {0,1}^{m×n}$ satisfies $r = K_{R_1}(B) < K_{R_2}(B)$ for any field $R_2$ strictly contained in $R_1$. Then there is a
sign pattern $S \in \{0, +, -\}^{m \times n}$ realizable by a matrix over $R_1$ of rank $r$ but not by a matrix over $R_2$ of that rank.

Proof. By definition of Kapranov rank, there is a matrix $A \in R_1^{m \times n}$ which has rank $r$ and satisfies $A_{ij} = 0$ if and only if $B_{ij} = 0$. Denoting the sign pattern of $A$ by $S$, one can see that $S$ is not realizable by a matrix over $R_2$ of rank $r$. $\square$

Now we see that a sign pattern realizable over $R_1$ but not over $R_2$ always exists if we have a zero-one matrix whose Kapranov rank over $R_2$ is greater than that over $R_1$. It turns out that producing zero-one matrices with this property can be performed by the use of matroid theory, and let us recall the basic definitions of this theory [2]. A matroid $M$ on a finite set $E$ is defined by the set $B \subset 2^E$ of its bases, which are supposed to satisfy the following conditions: (1) $B \neq \emptyset$; (2) if $A, B \in B$ and $a \in A \setminus B$, then there is a $b \in B \setminus A$ such that $A \setminus \{a\} \cup \{b\} \in I$. All the bases can easily be shown to have the same cardinality, and this cardinality is called the rank of a matroid $M$. A circuit in $M$ is a minimal set which is a subset of no $B \in B$. A dual matroid $M^*$ has as its bases the complements of the bases of $M$, and a circuit in $M^*$ is called a cocircuit for $M$. The matroid $M$ is representable over a field $F$ if we can assign vectors from $F^n$ to the elements of $E$ in such a way that a set $B$ is a basis of the linear span of $E$ if and only if $B \in B$. Finally, define a cocircuit matrix $C = C_M$ of $M$ as a matrix with rows indexed by elements of $E$ and columns indexed by cocircuits such that $C_{ij} = 1$ if $i$ belongs to the $j$th cocircuit and $C_{ij} = 0$ otherwise. The following theorem allows one to construct matrices whose Kapranov rank depends on a ground field.

Theorem 4.2. [1, Proposition 7.2 and Theorem 7.3] If $M$ is a matroid of rank $r$ and $C$ its cocircuit matrix, then $K_F(C) \geq r$ for any field $F$. For $F$ infinite, the condition $K_F(C) = r$ holds if and only if $M$ is representable over $F$.

The following well-known fact connects the notions of matroid duality and representability.

Theorem 4.3. [2] If a matroid $M$ is representable over a field $F$, then so is its dual $M^*$.

Note that the matroid duality is an involution, that is, the condition $(M^*)^* = M$ holds. This shows that Theorem [1,3] holds as well in the opposite direction. We also need the classical example of a non-representable matroid, which appeared in a foundational paper by Saunders MacLane [4].

Theorem 4.4. [4, Theorem 3] Let $K$ be a finite algebraic field over the field of rational numbers. Then there exists a matroid $M$ of rank 3 which is representable over $K$ but over no field strictly contained in $K$.

Now we are ready to prove the theorem stating that the bound of $t - 2$ is optimal in Theorem 3.6.

Theorem 4.5. Let $K$ be an ordered finite algebraic field over the field of rational numbers. Then there exists a matrix $A \in K^{n \times m}$ of rank $n - 3$ with the following property: the entries of any matrix $A' \in K^{n \times m}$ which has the same rank and sign pattern as those of $A$ generate the whole field $K$.

Proof. Using Theorem 4.4 we get a rank-three matroid $M$ representable over $K$ but not over any field strictly contained in $K$. Denoting the number of its vertices by $n$,
we see that by definitions the dual $M^*$ has rank $n - 3$. Then we use Theorem 4.3 to conclude that $M^*$ is representable over $K$ but not over any field strictly contained in $K$. From Theorem 4.2 it follows that the cocircuit matrix of $M^*$, which has $n$ rows, has also Kapranov rank $n - 3$ with respect to $K$ and greater Kapranov rank with respect to any field strictly contained in $K$. Application of Lemma 4.1 now completes the proof.

□

References

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