A DETAILED AND DIRECT PROOF OF SKOROHOD-WICHURA’S THEOREM

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ABSTRACT. The representation Skorohod theorem of weak convergence of random variables on a metric space goes back to Skorohod (1956) in the case where the metric space is the class of real-valued functions defined on [0,1] which are right-continuous and have left-hand limits when endowed with the Skorohod metric. Among the extensions of that to metric spaces, the version by Wichura (1970) seems to be the most fundamental. But the proof of Wichura seems to be destined to a very restricted public. We propose a more detailed proof to make it more accessible at the graduate level. However we do far more by simplifying it since important steps in the original proof are dropped, which leads to a direct proof that we hope to be more understandable to a larger spectrum of readers. The current version is more appropriate for different kinds of generalizations.

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1. INTRODUCTION

1.1. An easy entry. Let us begin by the simplest form of the Skorohod theorem which is the object of our study. By the Portmanteau Theorem (see Lo et al. (2016), Corollary 3, page 72), the weak convergence of probability measures on $\mathbb{R}$ is reduced to the convergence of cumulative distribution functions (cdf). Precisely, the sequence of probability measures $(P_n)_{n \geq 1}$ on the Borel space $(\mathbb{R}, B(\mathbb{R}))$ associated with the cdf’s
\[ F_n(x) = \mathbb{P}_n(x - \infty, x], \quad x \in \mathbb{R}, \quad n \geq 1, \]

weakly converges to the probability measure \( \mathbb{P}_\infty \) associated with the cdf \( F_\infty(x) = \mathbb{P}_\infty(x - \infty, x], \quad x \in \mathbb{R}, \)

denoted by \( \mathbb{P}_n \rightsquigarrow \mathbb{P}_\infty \), in the sense that for all \( f : \mathbb{R} \to \mathbb{R} \) continuous and bounded

\[
\int f \, d\mathbb{P}_n \to \int f \, d\mathbb{P}_\infty, \quad \text{as } n \to +\infty
\]

if and only if, for any continuity point \( x \) of \( F_\infty \) (denoted \( x \in C(F_\infty) \)),

\[
F_n(x) \to F_\infty(x), \quad \text{as } n \to +\infty.
\]

The convergence in Formula (1.2) is still denoted by \( F_n \rightsquigarrow F_\infty \). That formula itself becomes a definition of weak convergence and the notion is generalized to convergence of sequences of non-decreasing functions to another non-decreasing function. By using the generalized inverse (see Lo et al. (2016), page 112) of a non-decreasing function \( F \) defined by

\[
F^{-1}(y) = \inf \{ x \in \mathbb{R}, \quad F(x) \geq y \}, \quad y \in \mathbb{R},
\]

which is non-decreasing too (and is the inverse function of \( F \) if \( F \) is invertible), Billingsley (1968) proved that

\[
F_n \rightsquigarrow F_\infty \Rightarrow F_n^{-1} \rightsquigarrow F_\infty^{-1}.
\]

When applied to random variables \( (X_n)_{n \geq 1} \) and \( X_\infty \) taking values in \( \mathbb{R} \), we consider the probability laws and the cdf’s of those probability laws denoted by \( F_{X_\infty} \) and \( (F_{X_n})_{n \geq 1} \). Then Formula (1.2) becomes

\[
\forall x \in C(F_{X_\infty}), \quad F_{X_n}(x) \to F_{X_\infty}(x),
\]

and we still write \( X_n \rightsquigarrow X_\infty \).

The objective of the Skorohod problem is the following. In Formula (1.5), only the probability laws of \( X_\infty \) and the \( X_n \)’s matter. So the probability space on which \( X_\infty \) or each \( X_n, \ n \geq 1 \), is defined has no importance for the convergence. They may even be pair-wise distinct. As a result, we neither
have access to the paths of the random variables, nor can we make coherent numerical operations on them. On the other hand, it is also known that if \( X_{\infty} \) and all the \( X_n \)'s are on the same probability space, the almost-sure convergence and the convergence in probability imply the weak convergence. The following question arises:

(9) Given (1.5), can we construct a probability space \((\Omega, \mathcal{A}, \mathbb{P})\) holding a modification \({Y_{\infty}, Y_n, n \geq 1}\) of \({X_{\infty}, X_n, n \geq 1}\), in the sense that each \( Y_n \) has the same law as \( X_n \) for \( n \in \mathbb{N} \cup \{+\infty\} \), such that the weak convergence of \( Y_n \) to \( Y_{\infty} \) is an almost-sure convergence?

A successful answer to this question allows to treat problems of weak convergence as almost-sure convergences and to use analysis tools such as expansions, delta-methods, etc., in solving weak convergence problems.

The answer is relatively easy on \( \mathbb{R} \) due to Formula (1.4) based on Renyi’s representations of real valued random variables (see Lo et al. (2016), page 127). Indeed we may take \((\Omega, \mathcal{A}, \mathbb{P}) = ([0, 1], \mathcal{B}([0, 1]), \lambda)\), where \( \lambda \) is the Lebesgue measure which actually is a probability measure on \([0, 1]\). Let \( U : [0, 1] \to [0, 1] \) be the identity mapping. It is a real-valued random variable defined on \([0, 1]\) which is uniformly distributed on \((0, 1)\). By the Renyi representation, \({F_{X_{\infty}}^{-1}(U), F_{X_n}^{-1}(U), n \geq 1}\) is a modification of \({X_{\infty}, X_n, n \geq 1}\).

Using (1.4), we have

\[
\mathbb{P}\left(\{\omega \in \Omega, F_{X_n}^{-1}(U(\omega)) \nrightarrow F_{X_{\infty}}^{-1}(U(\omega))\}\right) \\
= \lambda(\{u \in [0, 1], F_{X_n}^{-1}(u) \nrightarrow F_{X_{\infty}}^{-1}(u)\}) \\
\leq \lambda(C(F_{X_{\infty}}^{-1})^c) = 0,
\]

since \( F_{X_{\infty}}^{-1} \) is non-decreasing and hence has at most a countable number of discontinuity points and countable sets are null sets for the Lebesgue measure.

This is the Skorohod theorem on \( \mathbb{R} \).

1.2. More general versions. The Skorohod theorem as well as the Skorohod topologies go back to the paper by Skorohod (1956) in which the theorem is stated on the class \( D([0, 1]) \) of real-valued functions defined on \([0, 1]\) which are right-continuous and have left-hand limits, endowed with
the topology he created and that is named after him. Since then, many authors have tried to extend it to general metric spaces, at least separable. (see Dudley (1966) for non-separable spaces).

The version given by Wichura (1970) can be considered as one of the most fundamental, from which further versions can be done. The proof of Wichura (1970) is indirect and certainly reserved to a very restricted public.

Given the importance of that theorem in weak convergence, we think that it deserves to have a more direct and pedagogical version. We want to make the proof available to readers coming first from courses of Topology, Measure and Integration, Mathematical Probability Theory but mastering the product of measurable spaces and product of measures, the beginners of graduate studies in summary. This what we intend to do here.

Before we proceed, we advise readers not familiar with arbitrary products of probability spaces, complete probability spaces and/or Caratheordory extensions of measures on semi-algebra to read about them in appropriate books, possibly in Chapter 9 in Lo (2018), Chapter 5 in Lo (2017b) (see part II on Outer measures and Doc. 04-02, Exercise 8).

Here, we mainly follow the ideas in the proofs of Wichura (1970), which is valid in any metric space, when the probability measure limit is tight. However, our modifications are significantly interesting in terms of new results concerning a number of assertions. We tried to clarify many details at a cost of a few more pages. The organization of the proofs is radically different. We will comment on the key points of our methods compared to the initial proof.

Here is the general statement of the main theorem.

**Theorem 1.** (Skorohod-Wichura) Let $D$ be an upward well-directed set, $(P_\alpha)_{\alpha \in D}$ be a family of probability laws on a metric space $(S, d)$ endowed with its Borel $\sigma$-algebra $S = B(S)$ and let $P_\infty$ be a tight probability law on $(S, d)$. Suppose that we have

\begin{equation}
(1.6) \quad P_\alpha \Rightarrow P_\infty \text{ on } D.
\end{equation}
Then, there exists a probability space \((\Omega, \mathcal{A}, \nu)\) holding a family of random variables \((X_\alpha)_{\alpha \in D}\) and a random variable \(X_\infty\) taking values in \(S\) such that:

(a) \(\nu_{X_\infty} = \nu_{X_\infty^{-1}} = \mathbb{P}_\infty\),

(b) \(\forall \alpha \in D, \nu_{X_\alpha} = \nu_{X_\alpha^{-1}} = \mathbb{P}_\alpha\)

and

(c) the field \((X_\alpha)_{\alpha \in D}\) almost surely converges to \(X_\infty\).

We will need some preliminary tools and remarks.

2. Preliminary tools

By assumption, \(\mathbb{P}_\infty\) is tight. So for any integer \(n \geq 1\), there is a compact subset \(K_n\) of \(S\) such that \(\mathbb{P}_\infty(K_n) \geq 1 - 1/n\). By taking \(S_\infty\) as the closure of \(\bigcup_{n \geq 1} K_n\), we have

\[ \mathbb{P}_\infty(S_\infty) = 1. \]

This means that \(S_\infty\) a \(\sigma\)-compact set (a countable union of compacts sets) and that \(S_\infty\) is a separable. We may replace the weak limit \(\mathbb{P}_\alpha \Rightarrow \mathbb{P}_\infty\) by \(\mathbb{P}_\alpha(\circ \cap S_\infty) \Rightarrow \mathbb{P}_\infty(\circ \cap S_\infty)\), meaning that the convergence holds on \(S_\infty\).

So we can do proceed to the proof when \(S\) is a separable and complete set (Polish space) on which each probability measure is tight. To begin, we give the following first tool.

**Proposition 1.** Let \(\mathbb{P}_\infty\) be a probability measure on the separable space \((S, d)\). Let \(\Delta > 0\) and \(\varepsilon > 0\). Then for any \(k \geq 1\), there exists a finite partition \(C_{j,k}, 0 \leq j \leq q(k)\) of \(\mathbb{P}_\infty\)-continuous borel set such that

\[ \forall (k \geq 1), \mathbb{P}_\infty(C_{0,k}) \leq 2^{-k}\varepsilon =: \varepsilon_k \]

and

\[ \forall (k \geq 1), (\forall 1 \leq j \leq q(k)), \text{diam}(C_{j,k}) \leq 2^{-k}\Delta. \]
Moreover, for all \( k \geq 1 \), any element of the partition \( C_{j,k+1} \), \( 0 \leq j \leq q(k+1) \) is the sum of \( \mathbb{P}_\infty \)-continuous sets, each of them being included in one element of the partition \( C_{j,k} \), \( 0 \leq j \leq q(k) \).

**Proof of Proposition 1.** Since \( S \) is separable, by the Lindelöf property, any open cover can be reduced to one of its countable sub-covers (See Choquet (1966)). Let \( D \) be a dense subset of \( S \). For a fixed real number \( \Delta > 0 \), we may cover \( S \) by a countable number of balls centered on points forming a subset \( D_0 \) of \( D \) with radius \( \Delta/2 \). For each \( s \in D_0 \), the borders of the balls \( B(s, \delta + \Delta/4) \), \( 0 \leq \delta \leq \Delta/4 \) are disjoint. Thus, we can find a value \( \delta \) such that \( \mathbb{P}_\infty(\partial B(s, \delta + \Delta/2)) = 0 \) (See Lo (2017b)). So \( S \) may be covered by a countable \( \mathbb{P}_\infty \)-continuous balls of diameters at most equal to \( \Delta_1 = \Delta \). If we denote these balls by \( D_{1,n}, n \geq 1 \), we have

\[
S = \bigcup_{n \geq 1} D_{1,n}.
\]

By the continuity of the probability measure, there exists \( n(1) \geq 1 \) such that

\[
\mathbb{P}_\infty\left( \bigcup_{1 \leq n \leq n(1)} D_{1,n} \right) \geq 1 - \varepsilon_1.
\]

Now we can use the classical method to transform the union of the \( D_{1,n}, 1 \leq n \leq n(1) \) into a sum of sets, that is

\[
S_1 = \bigcup_{1 \leq n \leq n(1)} D_{1,n} = \sum_{1 \leq n \leq n(1)} C_{n,1},
\]

with

\[
C_{1,1} = D_{1,1}, \quad C_{2,1} = D_{1,1}^c \cap D_{1,2} \subset D_{1,2}, \quad C_{j,1} = D_{1,1}^c \ldots D_{1,j-1}^c \cap D_{1,j} \subset D_{1,j}, \quad j \geq 3,
\]

(with \( q(1) = n(1) \)). By the following properties [where \( \text{int}(A) \) and \( \text{ext}(A) \) stand for the interior and of the exterior of a set \( A \), respectively],

\[
\partial(A) = (\text{int}(A) \cap \text{ext}(A))^c, \quad \text{int}(A^c) = \text{ext}(A) \quad \text{and} \quad \text{ext}(A^c) = \text{int}(A),
\]

we can see that a set and its complement has the same border and we already know that the border of a countable union of sets is included in the
union of the borders of the sets. When combined, the two points say that intersections and unions of sets are included in the union of the borders of those sets. So all the $C_{n,1}$, $1 \leq n \leq n(1)$, are disjoint and $\mathbb{P}_\infty$-continuous measurable sets of diameters at most equal to $\Delta_1 = \Delta$. By putting

$$C_{0,1} = S \setminus \bigcup_{1 \leq n \leq n(1)} D_{1,n},$$

we still get a $\mathbb{P}_\infty$-continuous measurable set. So we get that $S$ is partitioned into the disjoint measurable sets $C_{n,1}$, $0 \leq n \leq n(1)$, all $\mathbb{P}_\infty$-continuous and having diameters at most equal to $\Delta_1 = \Delta$ such that

$$\mathbb{P}_\infty(C_{0,1}) \leq \varepsilon_1.$$

In a second step, let us say that each $D_{1,j}$, $0 \leq j \leq n(1)$, may be covered (in $S$) by a countable of balls centered on elements of some $D_2(j) \subset D \cap D_{1,j}$ (we do not forget that $D_{1,j}$ is an open ball) with radius at most $\Delta/4$. We repeat the argument in the first step by using the balls $B(s, \delta + \Delta/8)$, $s \in D_2(j)$, $0 \leq \delta \leq \Delta/8$ and choose $\delta_s$ such that $B(s, \delta_s + \Delta/8)$ is $\mathbb{P}_\infty$-continuous and we get

$$D_{1,j} = D_{1,j} \cap \bigcup_{s \in D_2(j)} B(s, \delta_s + \Delta/4) \equiv \bigcup_{s \in D_2(j)} D_{s,j},$$

where the $D_{s,j}$’s are $\mathbb{P}_\infty$-continuous of diameter at most equal to $\Delta/2$. So we have

$$S_1 = \bigcup_{1 \leq j \leq n(1)} \bigcup_{s \in D_2(j)} D_{s,j} =: \bigcup_{n \geq 1} D_{2,n},$$

where the $D_{2,n}$ are open sets, all $\mathbb{P}_\infty$-continuous of diameter at most equal to $\Delta/2$. From there, we proceed as in the conclusion of the first step to get measurable sets $C_{n,2}$, $1 \leq n \leq n(2) = q(2)$, all $\mathbb{P}_\infty$-continuous of diameter at most equal to $\Delta/2$ such that for

$$C_{0,2} = S_1 \setminus \left( \bigcup_{1 \leq n \leq q(2)} D_{2,n} \right),$$

we have
\[ P_\infty(C_{0,2}) \leq \varepsilon_2, \]

and surely, for each \( j \in \{1, \cdots, q(2)\} \), \( C_{j,2} \) is in one of the \( C_{h,1}'s \).

From this the proof of the proposition is completed by induction. ■

We will also need the following result.

**Proposition 2.** Let \( \Omega_D = \prod_{\alpha \in D} S_\alpha \), \( S_D = \otimes_{\alpha \in D} S_\alpha, \nu \) be a non-countable product space holding a \( \sigma \)-finite measure which takes finite values on each element of a countable partition of \( \prod_{\alpha \in D} S_\alpha \) consisting of cylinders. Let \( A \in \otimes_{\alpha \in D} S_\alpha \) be a measurable set.

Then \( A \) is a subset of a measurable set \( B \) depending only on a countable number of factors \( \alpha \in D \).

**Proof of Proposition 2.** It is enough to do the proof with a finite measure \( \nu \) since the extension to a \( \sigma \)-finite measure is straightforward. Hence \( \nu \) is a measure on the algebra \( C_D \) of finite sums of cylinders. A measurable set \( A \) is of the form

\[ A = \prod_{\alpha \in V} A_\alpha \times \prod_{\alpha \not\in V} S_\alpha \equiv A_V \times S'_V, \quad (P1) \]

where \( V \) is a finite subset of \( D \) and \( A_\alpha \in S_\alpha \). We already suggested the reader to read Chapter 8 in Lo (2018) about product \( \sigma \)-algebras and relevant notation. Now, cylinders, as denoted in Formula (P1), depends only on a finite number of indices, since \( x = (x_\alpha)_{\alpha \in D} \) is in \( A \) if and only if \( x_V = (x_\alpha)_{\alpha \in V} \) is in \( A_V \).

Let us recall the projections on sub-products space of \( \Omega_D \). Define for any subset \( D_1 \subset D \), the projection \( \pi_{D_1} \) of \( \prod_{\alpha \in D} S_\alpha \) on \( \prod_{\alpha \in D_1} S_\alpha \), defined by

\[ \forall x = (x_\alpha)_{\alpha \in D} \in \prod_{\alpha \in D} S_\alpha, \quad \pi_{D_1}(x) = (x_\alpha)_{\alpha \in D_1}. \]

Let us proceed to the extension of the measure \( \nu \) from the algebra \( C_D \) to the measure \( \tilde{\nu} \) on \( \sigma \)-algebra \( S_D \) it generates by the outer-measure method.
But $\tilde{\nu}$ and $\nu$ are equal on $S_D$. So by the exterior measure definition (See Lo (2017b), Chapter 5, Doc 04-10), we have

$$\nu(A) = \inf \left\{ \nu(B), A \subset B = \bigcup_{n \geq 1} B_n, B_n \in C_D \right\}.$$ 

There is nothing to do when $A = \Omega_D$. If $\nu(A) < \nu(\Omega)$, for $\varepsilon < (\nu(\Omega) - \nu(A))/2$, there exists a countable number of finite sums of cylinders, denoted by $B_n$, $n \geq 1$ such that, for $B$ being the unions of the $B_n$’s,

$$A \subset B \text{ and } \nu(A) \leq \nu(B) < \nu(A) + \varepsilon.$$ 

To conclude, we take as $D_0$ the set of indices $\alpha$ involved in the cylinders forming the finite sums of cylinders $B_n, n \geq 1$. We have that $B$ is of the form

$$B = B_{D_0} \times S'_{D_0}. \quad \Box.$$ 

Now we give the complete proof of the main theorem.

**Proof of Theorem 1.** Now, let us describe the space on which holds our construction. We replicate the measurable space $(S, S)$ into $(S_\infty, S_\infty)$, $(S_\alpha, S_\alpha)$, $\alpha \in D$, in sense that each $(S_\alpha, S_\alpha)$ is identical to $(S, S)$ for $\alpha \in \{\infty\} \cup D$. We consider the measurable product space

$$\Omega = \prod_{\alpha \in \{\infty\} \cup D} S_\alpha = S_\infty \times \prod_{\alpha \in D} S_\alpha$$

and endow it with the product $\sigma$-algebra

$$\mathcal{A} = \otimes_{\alpha \in \{\infty\} \cup D} S_\alpha = S_\infty \otimes \otimes_{\alpha \in D} S_\alpha.$$ 

We take $(\beta_k)_{k \geq 1}$ an arbitrary sequence such that each $\beta_k > 0$,

$$\beta_1 > \beta_2 > \cdots > \beta_k \downarrow 0 \text{ as } k \uparrow +\infty \text{ and } \sum_{k \geq 1} \beta_k = 1,$$

(we may take $\beta_k = 2^{-k}, k \geq 1$, for example) and we denote $\beta^*_k = \beta_1 + \cdots + \beta_k, k \geq 1$. We denote elements of $\Omega$ as
\( \omega = (\omega_\infty, (\omega_\alpha)_{\alpha \in D}) \).

The simple projections \( \pi_\infty \) and \( \pi_\gamma \), for \( \gamma \in D \), are defined as

\[
\pi_\infty(\omega) = \pi_\infty(\omega_\infty, (\omega_\alpha)_{\alpha \in D}) = \omega_\infty,
\]

and

\[
\pi_\gamma(\omega) = \pi_\gamma(\omega_\infty, (\omega_\alpha)_{\alpha \in D}) = \omega_\gamma.
\]

We adopt the usual notation like in Lo (2018) (Chapter 9). For example, for a non-empty subset \( D_1 \subset \{\infty\} \cup D \), for

\[
A_{D_1} \subset \prod_{\alpha \in D_1} S_\alpha
\]

the cylinder of base \( A_{D_1} \) is denoted as

\[
A_{D_1} \times \prod_{\alpha \notin D_1} S_\alpha = \{(\omega_\infty, (\omega_\alpha)_{\alpha \in D}), (\omega_\alpha)_{\alpha \in D_1} \in A_{D_1}\},
\]

and we denote

\[
A_{D_1} \times \prod_{\alpha \notin D_1} S_\alpha = A_{D_1} \times S'_{D_1}.
\]

When we consider the measurable space \((\Omega, \mathcal{A})\), the identity mapping

\[
X : (\Omega, \mathcal{A}) \to \left( S_\infty \times \prod_{\alpha \in D} S_\alpha, S_\infty \otimes \otimes_{\alpha \in D} S_\alpha \right)
\]

is measurable and by the way, for any \( \omega = (\omega_\infty, (\omega_\alpha)_{\alpha \in D}) \in \Omega \),

\[
X(\omega) = (X_\alpha(\omega))_{\alpha \in \{\infty\} \cup D}.
\]

In the current case of the identity function, we have

\[
X_\infty(\omega) = \omega_\infty \text{ and } X_\alpha(\omega) = \omega_\alpha \text{ for } \alpha \in D
\]

and for \( A_\infty \subset S_\infty \) and \( A_\alpha \subset S_\alpha \) for \( \alpha \in D \), we have by the same notation,
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\[ X_{\infty}^{-1}(A_{\infty}) = A_{\infty} \times S'_{\{\infty\}} \] and \[ X_{\alpha}^{-1}(A_{\alpha}) = A_{\alpha} \times S'_{\{\alpha\}} \]

Later, we will create a probability measure \( \nu \) on \( (\Omega, \mathcal{A}) \) such that \( \nu X_{\alpha}^{-1} = P_{\alpha} \), \( \alpha \in \{\infty\} \cup S \).

After this important notation, we begin by exploiting the partitions and facts in Proposition 1 on each \( S_{\alpha} = S \) (for \( \alpha \in \{\infty\} \cup D \)) and we denote \( \Delta_k = \Delta/2^k \), \( \varepsilon_k = 2^{-k} \varepsilon \). For each \( k \geq 1 \), we denote

\[ I(k) = \{0 \leq j \leq q(k), \ P_{\infty}(C_{j,k}) \neq 0\} \]

By the weak convergence of \( P_{\alpha} \) to \( P_{\infty} \), and by the \( P_{\infty} \)-continuity of the \( C_{j,k} \)'s, we have for all \( j \in \{0, \ldots, q(k)\} \), that

\[ P_{\alpha}(C_{j,k}) \to P_{\infty}(C_{j,k}) \]

So, for a fixed \( c \in ]0,1[ \), for any \( k \geq 1 \), there exists \( \alpha_k \) such that for \( \alpha \geq \alpha_k \), for all \( j \in I(k) \),

\[ P_{\infty}(C_{j,k}) > 0 \] and \[ 1 - c \leq \frac{P_{\alpha}(C_{j,k})}{P_{\infty}(C_{j,k})} \leq 1 + c. \ (G1) \]

We may suppose \( (G1) \) holds for all \( \alpha \in D \) for \( k = 1 \). Now let us define

(2.1) \[ H_{\alpha,k}(\omega) = \frac{1}{1 - \beta_k^*}P_{\alpha}(\omega) - \sum_{j \in I(k)} \frac{\beta_k^*}{1 - \beta_k^*}P_{\alpha}(\omega/C_{j,k})P_{\infty}(C_{j,k}). \]

It is clear that each \( H_{\alpha,k} \) is \( \sigma \)-additive and \( H_{\alpha,k}(S_{\alpha}) = 1 \). Further, for any \( A_{\alpha} \in S_{\alpha} \), we have

\[ H_{\alpha,k}(A_{\alpha}) = \sum_{p \in I(k)} H_{\alpha,k}(A_{\alpha} \cap C_{p,k}), \]

and for all \( p \in I(k) \),
\[ H_{\alpha,k}(A_\alpha \cap C_{p,k}) = \frac{1}{1 - \beta_k} \mathbb{P}_\alpha(A_\alpha \cap C_{p,k}) - \frac{\beta_k}{1 - \beta_k} \mathbb{P}_\alpha(A_\alpha / C_{p,k}) \mathbb{P}_\infty(C_{p,k}) \]

Thus

\[ H_{\alpha,k}(A_\alpha \cap C_{p,k}) = \mathbb{P}_\alpha(A_\alpha \cap C_{p,k}) \left( \frac{1}{1 - \beta_k} - \frac{\beta_k}{1 - \beta_k} \frac{\mathbb{P}_\infty(C_{p,k})}{\mathbb{P}_\alpha(C_{p,k})} \right), \]

and the term between the parentheses is non-negative if and only if

\[ \beta_k \leq \frac{\mathbb{P}_\alpha(C_{p,k})}{\mathbb{P}_\infty(C_{p,k})} \equiv \eta_{\alpha,k,p}. \quad (G2) \]

Hence \( H_{\alpha,k} \) is non-negative whenever Formula (G2) holds for all \( p \in I(k) \).

From there, we re-use the method of the first part. We consider sequences

\[ 1 - c = \varepsilon_{1,0} < \varepsilon_{1,1} < \cdots < \varepsilon_{1,r} \uparrow 1, \text{ as } r \uparrow +\infty \]

and

\[ \varepsilon_{2,r} < \varepsilon_{2,r-1} < \cdots < \varepsilon_{2,1} < \varepsilon_{2,0} = 1 + c, \ \varepsilon_{2,r} \downarrow 1, \text{ as } r \uparrow +\infty. \]

For any fixed \( k \geq 1 \), we define a mapping \( \ell_1 : D \to \mathbb{N}^* \cup \{+\infty\} \) case by case.

Case 1 : Let us denote \( \eta_{\alpha,k} = \min_{p \in I(k)} \eta_{\alpha,k,p} \). For \( \alpha \in D \) fixed, if all the \( \eta_{\alpha,k} \)'s, are less that \( 1 - c \), we put \( \ell_1(\alpha) = +\infty \).

Case 2 : if all the \( \eta_{\alpha,k} \)'s are greater that \( 1 + c \), we take \( \ell_1(\alpha) = +\infty \).

Case 3 : if one of the two cases above fails, we may define

\[ J_1(\alpha) = \{ j \geq J, \ \exists (k \geq 1), \ \eta_{\alpha,k} \in [\varepsilon_{1,j}, \varepsilon_{1,j+1}] \} \]

and

\[ J_2(\alpha) = \{ j \geq 1, \ \exists (k \geq 1), \ \eta_{\alpha,k} \in [\varepsilon_{2,j+1}, \varepsilon_{2,j}] \}. \]
For $i \in \{1, 2\}$, if $J_i(\alpha)$ is empty or non-empty but unbounded, we put $\hat{\ell}(i, \alpha) = +\infty$ and if it is non-empty and bounded we set $\check{j}_i(\alpha) = \max J_i(\alpha)$ (for $\tau_i = 1_{(i=2)}$)

$$\hat{\ell}(i, \alpha) = \text{Argmax}_{\beta_k^{*} \leq \varepsilon_{i, j_i(\alpha) + \tau_i}} \beta_k^{*}.$$ 

In all cases, we take

$$\ell_1(\alpha) = \min(\hat{\ell}(1, \alpha), \hat{\ell}(2, \alpha)).$$

Conclusion, the function $\ell_1(\circ)$ is well-defined. Let us construct a second mapping $\ell_2$ as follows. We know that for any fixed $k \geq 1$, we have

$$\delta_{\alpha,k} = \max_{0 \leq h \leq q(k)} |\mathbb{P}_{\alpha}(C_{h,k}) - \mathbb{P}_{\infty}(C_{h,k})| \to 0 \text{ on } D.$$ 

From the the ranking $\beta_1 > \beta_2 > \cdots > \beta_j \searrow 0$, we define the non-empty set

$$J_3(\alpha) = \{ j \geq 1, (\exists k \geq 1), \delta_{\alpha,k} \in ]\beta_{j+1}, \beta_j]\}$$

and take $\ell_2(\alpha) = +\infty$ if $J_3(\alpha)$ is unbounded and $\ell_2(\alpha) = \max J_3(\alpha)$ otherwise. Finally, we take

$$\ell = \min(\ell_1, \ell_2).$$

Let us show that $\ell(\alpha) \to +\infty$ on $D$. Let $k_0 > 0$, and consider the unique value $j_1$ of $j$ such that $\beta_{k_0}^{*} \in ]\varepsilon_{1,j_1}, \varepsilon_{1,j_1+1}]$. All the $\eta_{\alpha,k_0,p}$’s converge to one and all the $\delta_{\alpha,k_0}$’s converge to zero. So, for some $\alpha_\infty$, for all $\alpha \geq \alpha_\infty$, we have that $\eta_{\alpha,k_0}$ is in some interval $]\varepsilon_{1,r}, \varepsilon_{2,r}]$, where $r \geq j_1$, and all the $\delta_{\alpha,k_0}$’s are in some interval $]\beta_{s+1}, \beta_s]$, $s \geq k_0$. By definition of $\ell_1$, in Case 3, and by definition of $\ell_2$, we get $\ell(\alpha) \geq \min(r, s) \geq k_0$. We have :

$$\exists \ell : D \to \mathbb{N}^* \cup \{+\infty\}, \lim_{D} \ell(\alpha) = +\infty.$$ 

Moreover, by construction, we have that, for all $\alpha \in D$ such that $\ell(\alpha) < +\infty$,

$$H_\alpha =: H_{\alpha,\ell(\alpha)}$$ 

is non-negative and hence is a probability measure such that (by inverting the formula)
\( P_\alpha(\omega) = (1 - \beta_{\ell(\alpha)}) \mathbb{H}_\alpha(\omega) + \beta_{\ell(\alpha)} \sum_{p \in I(\ell(\alpha))} \mathbb{P}_\infty(C_p,\ell(\alpha)) P_\alpha(\omega/C_p,\ell(\alpha)) \)

holds and we also have

\[ |P_\alpha(C_{h,\ell(\alpha)}) - P_\infty(C_{h,\ell(\alpha)})| \geq \beta_{\ell(\alpha)}. \quad (AP0) \]

Let us point out that \( \mathbb{H}_\alpha \) is used in all this chapter only if \( \ell(\alpha) < \infty \). Now we may construct the space on which the almost-sure limit will hold.

For any \( \alpha \in D \), for \( s \in S \), by the decomposition of \( S \) into the \( C_p,\ell(\alpha) \)'s, \( 0 \leq p \leq q(\ell(\alpha)) \), there exists a unique \( p(\alpha, s) \) such that \( s \in C_{p(\alpha, s),\ell(\alpha)} \). We define for \( \alpha \in D \), \( 1 \leq j < +\infty \), \( 0 \leq p \leq q(\ell(\alpha)) \),

\[ \nu_{j,s,\alpha} = \begin{cases} 
\mathbb{H}_\alpha & \text{if } j > \ell(\alpha) \\
\mathbb{P}_\alpha(\omega/C_{p(\alpha, s),\ell(\alpha)}) & \text{if } j \leq \ell(\alpha) 
\end{cases} \]

for \( \ell(\alpha) < +\infty \), and \( \nu_{j,s,\alpha} = \delta_s \) for \( \ell(\alpha) = +\infty \) and next, we define

\[ \nu_{j,s} = \delta_s \bigotimes_{\alpha \in D} \nu_{j,s,\alpha}. \]

We have to prove that for any \( j \geq 1 \), for \( A \in \mathcal{A} \), the mapping

\[ s \mapsto \nu_{j,s}(A), \quad (F1) \]

is measurable. Let us show this in three steps. (i) We first show that for \( A_\alpha \in S_\alpha \) fixed, for \( \alpha \in D \), for \( j \geq 1 \),

\[ s \mapsto \nu_{j,s,\alpha}(A_\alpha), \quad (F2) \]

is measurable. For \( \ell(\alpha) = +\infty \), this mapping is the indicator function of \( A_\alpha \).

For \( \ell(\alpha) < +\infty \), it is a constant function for \( j > \ell(\alpha) \) or, for \( 1 \leq j \leq \ell(\alpha) \), it is an elementary function associated to the subdivision of \( S \) into the \( C_p,\ell(\alpha) \)'s. So the function in (F2) est measurable (in \( s \in S \)). (ii) Next, we know that the product \( \sigma \)-algebra \( \mathcal{A} \) is generated by the class \( \mathcal{C} \) of all cylinders of the form
\[ A = A_\infty \times \prod_{r=1}^{p} A_{\alpha_r} \times S'_{(\infty, \alpha_1, \ldots, \alpha_p)}; \]

(Where \( p \) is finite, \( A_\infty \in S_\infty \), \( A_{\alpha_r} \in S_{\alpha_r} \), \( 1 \leq r \leq p \). For such cylinders, we have, for \( j \geq 1 \) fixed,

\[ \nu_{j,s}(A) = 1_{A_\infty}(s) \times \prod_{r=1}^{p} \nu_{j,s,\alpha}(A_{\alpha_r}), \]

and hence, the mapping in (F1) is measurable for all \( A \in C \). (iii) But \( C \) is a \( \pi \)-system containing the full space \( \Omega \). By Lo (2017b), Chapter 2, Exercise 4 in Doc 01-04), it is enough that show the class

\[ \{ A \in A, \ s \mapsto \nu_{j,s}(A) \text{ is measurable} \} \]

is a \( \lambda \)-system containing \( C \). But this is quite direct and left as an exercise. So we may integrate \( \nu_{j,s}(A) \) over \( s \in S \) and define

\[ A \ni A \mapsto \nu_j = \int_{S} \nu_{j,s}(A) \, dP_\infty(s), \]

so that, just by the monotone convergence theorem, \( \nu_j \) is a probability measure on \((\Omega, A)\) for each \( j \geq 1 \). Finally, we define

\[ \nu = \sum_{j \geq 1} \beta_j \nu_j. \]

We got the probability measure \( \nu \) and the random variables \( X_\infty \) and \( X_\alpha \) we were searching and we have :

**Lemma 1.** \( \nu \) is a probability measure and :

(i) \( \nu X^{-1}_\infty = P_\infty \).

(ii) For all \( \alpha \in D \) with \( \ell(\alpha) < +\infty \), \( \nu X^{-1}_\alpha = P_\alpha \).

(iii) For all \( \alpha \in D \) with \( \ell(\alpha) = +\infty \), \( \nu X^{-1}_\alpha = P_\infty \) (on \( S_\alpha \)). \( \Diamond \)
Proof of Lemma 1. The mapping \( \nu \) is a probability measure as a non-negative linear combination of probability measure associated to constants which add up to one. Further, we have:

(i) We have for all \( A_\infty \in \mathcal{S}_\infty \),

\[
\nu X^{-1}_\alpha (A_\infty) = \nu \left( A_\infty \times S'_\{\infty\} \right) \\
= \sum_{j \geq 1} \beta_j \int_S (1_{A_\infty}(s)) \, dP_\infty(s) \\
= \sum_{j \geq 1} \beta_j P_\infty(A_\infty) \\
= P_\infty(A_\infty).
\]

(ii) If \( \ell(\alpha) < +\infty \), we have for all \( A_\alpha \in \mathcal{S}_\alpha \),

\[
\nu X^{-1}_\alpha (A_\alpha) = \nu \left( A_\alpha \times S'_\{\alpha\} \right) \\
= \sum_{j \geq 1} \beta_j \int_S \nu_{j,s,\alpha}(A_\alpha) \, dP_\infty(s) \\
= \sum_{j > \ell(\alpha)} \beta_j \int_S \nu_{j,s,\alpha}(A_\alpha) \, dP_\infty(s) \\
+ \sum_{j \leq \ell(\alpha)} \beta_j \int_S \nu_{j,s,\alpha}(A_\alpha) \, dP_\infty(s),
\]

which, when combined with the definition of \( \nu_{j,s,\alpha} \) for \( \ell(\alpha) < +\infty \), \( \nu X^{-1}_\alpha (A_\alpha) \) leads to
\[
\nu X^{-1}_\alpha(A_\alpha) = \sum_{j > \ell(\alpha)} \beta_j \int_S \mathbb{H}_\alpha(A_\alpha) \ d\mathbb{P}_\infty(s) \\
+ \sum_{j \leq \ell(\alpha)} \beta_j \sum_{1 \leq p \leq q(\ell(\alpha))} \int_{C_p,\ell(\alpha)} \nu_{j,s,\alpha}(A_\alpha) \ d\mathbb{P}_\infty(s) \\
= (1 - \beta^*_{\ell(\alpha)})\mathbb{H}_\alpha(A_\alpha) \\
+ \sum_{j \leq \ell(\alpha)} \beta_j \sum_{1 \leq p \leq q(\ell(\alpha))} \int_{C_p,\ell(\alpha)} \mathbb{P}_\alpha(A_\alpha/C_{p,\ell(\alpha)}) \ d\mathbb{P}_\infty(s) \\
= (1 - \beta^*_{\ell(\alpha)})\mathbb{H}_\alpha(A_\alpha) + \beta^*_{\ell(\alpha)} \sum_{0 \leq p \leq q(\ell(\alpha))} \mathbb{P}_\alpha(A_\alpha/C_{p,\ell(\alpha)})\mathbb{P}_\infty(C_{p,\ell(\alpha)}).
\]

By comparing the later line and Formula (2.2), we get that \(\nu X^{-1}_\alpha = \mathbb{P}_\alpha\).

Finally for \(\ell(\alpha) = +\infty\), we have

\[
\nu X^{-1}_\alpha(A_\alpha) = \nu(A_\alpha \times S'_{\{\alpha\}}) \\
= \sum_{j \geq 1} \beta_j \int_S \nu_{j,s,\alpha}(A_\alpha) \ d\mathbb{P}_\infty(s) \\
= \sum_{j \geq 1} \beta_j \int_S 1_{A_\alpha}(s) \ d\mathbb{P}_\infty(s) \\
= \sum_{j \geq 1} \beta_j \mathbb{P}_\infty(A_\alpha) \\
= \mathbb{P}_\infty(A_\alpha).
\]

We are ready to conclude. Let us defined for \(k \geq 1\),

\[
A \equiv \bigcap_{k \geq 1} A_k \equiv \bigcap_{k \geq 1} \bigcup_{\ell(\alpha) \geq k} (d(X_\alpha, X_\infty) > \Delta_k).
\]

For a fixed \(\alpha \in D\), \(\ell(\alpha) \geq k\), we define \(G_{k,\alpha} = (d(X_\alpha, X_\infty) > \Delta_k)\). We do not know whether \(A\) or the \(A_k\)'s are measurable or not. But, by Proposition 2, there exists a measurable set \(A^*_k\) which includes \(A_k\) and depends only on a countable set \(D_{0,k}\) of indices, so that for \(D_0\) being the union of the \(D_{0,k}\)'s to which we add \(\{\infty\}\) if needed , we have
\[ A_k \subset A_k^* = A_{k,D_0}^* \times S'_{D_0}, \quad k \geq 1 \quad (CC1) \]

and by applying \( \Pi_{D_0} \) to that formula, we have

\[ \Pi_{D_0}(A_k) \subset A_{k,D_0}^*. \quad (CC2) \]

We also have by taking complements in \((CC1)\),

\[ \left( \bigcap_{\ell(\alpha) \geq k} (A_{k,D_0}^*)^c \right) \times S'_{D_0} \subset A_k^c. \quad (CC3) \]

But we have, as a general rule

\[ A_k \subset \pi_{D_0}(A_k) \times S'_{D_0}. \quad (CC4) \]

The combination of Formulas \((CC1)-(CC4)\) leads to

\[ (A_{k,D_0}^*)^c \times S'_{D_0} \subset \pi_{D_0}(A_k^c) \times S'_{D_0} \subset A_k^c. \]

Besides, it is clear that

\[ \pi_{D_0}(A_k^c) \times S'_{D_0} = \left( \bigcap_{\ell(\alpha) \geq k, \alpha \in D_0} G_{k,\alpha}^c \right) \times S'_{D_0}. \]

The combination of the two later facts yields

\[ A_k \subset \left( \bigcup_{\alpha \in D_0} G_{k,\alpha} \right) \times S'_{D_0} \subset A_k^*. \]

Let us study

\[ A_k^{**} \equiv \bigcup_{\ell(\alpha) \geq k, \alpha \in D_0} (d(X_\alpha, X_\infty) > \Delta_k). \quad (F1) \]

Let us fix \( k \geq 1 \) and \( \alpha \in D \) and set \( G_{k,\alpha} = (d(X_\alpha, X_\infty) > \Delta_k) \). We have, for any \( h \geq \ell(\alpha) \geq k \)
\[ \nu_j(G_{k,\alpha}) = \int_S \nu_{j,s}(G_{k,\alpha}) \, d\mathbb{P}_\infty(s) \]

\[ = \sum_{0 \leq p \leq q(\ell(\alpha))} \int_{C_{p,\ell(\alpha)}} \nu_{j,s}(G_{k,\alpha}) \, d\mathbb{P}_\infty(s) \]

\[ = \sum_{0 \leq p \leq q(\ell(\alpha))} \int_{C_{p,\ell(\alpha)}} \nu_{j,s}(X_\infty \in C_{0,h}) \cap G_{k,\alpha}) \, d\mathbb{P}_\infty(s) \]

\[ + \sum_{0 \leq p \leq q(\ell(\alpha))} \sum_{1 \leq u \leq q(h)} \int_{C_{p,\ell(\alpha)}} \nu_{j,s}(X_\infty \in C_{u,h}) \cap G_{k,\alpha}) \, d\mathbb{P}_\infty(s) \]

\[ =: T_1 + T_2. \]

We have

\[ T_1 \leq \sum_{0 \leq p \leq q(\ell(\alpha))} \int_{C_{p,\ell(\alpha)}} \nu_{j,s}(X_\infty \in C_{0,h}) \, d\mathbb{P}_\infty(s) \]

\[ = \sum_{0 \leq p \leq q(\ell(\alpha))} \int_{C_{p,\ell(\alpha)}} \delta_s(C_{0,h}) \, d\mathbb{P}_\infty(s) \]

\[ = \sum_{0 \leq p \leq q(\ell(\alpha))} \int_{C_{p,\ell(\alpha)}} 1_{C_{0,h}}(s) \, d\mathbb{P}_\infty(s) \]

\[ = \sum_{0 \leq p \leq q(\ell(\alpha))} \mathbb{P}_\infty(C_{p,\ell(\alpha)} \cap C_{0,h}) \]

\[ = \mathbb{P}_\infty(C_{0,h}) \]

\[ \leq \varepsilon_h. \]

Next, we can use Fubini’s theorem to see that

\[ T_2 = \sum_{0 \leq p \leq q(\ell(\alpha))} \sum_{1 \leq u \leq q(h)} \int_{C_{p,\ell(\alpha)}} \int_{S_\infty} 1_{C_{u,h}}(x_\infty) \left( \int_{S_\alpha} \nu_{j,s,\alpha}(B_c(x_\infty, \Delta_k)) \, d\delta_s(x_\infty) \right) \, d\mathbb{P}_\infty(s) \]

\[ = \sum_{0 \leq p \leq q(\ell(\alpha))} \sum_{1 \leq u \leq q(h)} \int_{C_{p,\ell(\alpha)}} 1_{C_{u,h}}(s) \nu_{j,s,\alpha}(B_c(s, \Delta_k)) \, d\mathbb{P}_\infty(s). \]
By construction, each $C_{u,h}$ is in some $C_{p,\ell(\alpha)}$. Hence, we may define for $0 \leq p \leq q(\ell(\alpha))$,

$$I(p, h) = \{u, \ u \in \{1, \ldots, q(h)\} \text{ and } C_{u,h} \subset C_{p,\ell(\alpha)}\}.$$ 

So for $u \notin I(p, h)$, $C_{u,h} \cap C_{p,\ell(\alpha)} = \emptyset$. Thus, for

$$B(p, h) = \sum_{u \in I(p, h)} C_{u,h},$$

the last equation reduces to

$$T_2 = \sum_{0 \leq p \leq q(\ell(\alpha))} \int_{B(p,h)} \nu_{j,s,\alpha}(B^c(s, \Delta_k)) d\mathbb{P}_\infty(s).$$

Let

$$T_2 = \sum_{0 \leq p \leq q(\ell(\alpha))} T(2, p) \text{ with } T(2, p) = \int_{B(p,h)} \nu_{j,s,\alpha}(B^c(s, \Delta_k)) d\mathbb{P}_\infty(s).$$

We have

$$T(2, p) = \int_{B(p,h)} \nu_{j,s,\alpha}(B^c(s, \Delta_k) \cap C_{p,\ell(\alpha)}) d\mathbb{P}_\infty(s)$$

$$+ \int_{C_{p,\ell(\alpha)}} \nu_{j,s,\alpha}(B^c(s, \Delta_k) \cap C_{p,\ell(\alpha)}^c) d\mathbb{P}_\infty(s)$$

$$= T(21, p) + T(22, p).$$

Now, in what follows, we may do the computations of the probability space $(S^2, \mathcal{A}^{\otimes 2}, \mathbb{P}_\infty \times \nu_{j,s,\alpha})$ since the measurable spaces $S_\infty$ and $S_\alpha$ are identical. So, by Tonelli’s theorem, we have

$$T(21, p) = \int_S 1_{B(p,h)}(s_1) \int_S 1_{C_{p,\ell(\alpha)}(s_2)} 1_{B^c(s_1, \Delta_k)}(s_2) \ d\mathbb{P}_\infty(s_1) \ d\nu_{j,s_1,\alpha}(s_2)$$

$$= \int_{S \times S} \left(1_{B(p,h)}(s_1) 1_{B^c(s_1, \Delta_k)}(s_2) \ d\mathbb{P}_\infty(s_1)\right) 1_{C_{p,\ell(\alpha)}(s_2)} \ d\nu_{j,s_1,\alpha}(s_2),$$
in which the expression between the big parentheses is zero since the
diameter of $C_{p, \ell(\alpha)}$ is less or equal to $\Delta_{\ell(\alpha)} \leq \Delta_k$ and for $(s_1, s_2) \in C_{p, \ell(\alpha)}^2$. So $T(21, p) = 0$.

Next, by Tonelli’s theorem again,

$$T(22, p) = \int_S 1_{B(p, h)}(s_1) \left( \int_S 1_{C^c_{p, \ell(\alpha)}}(s_2) \mu_{p, \ell(\alpha)}(s_1, s_2) ds_2 \right) d\mathbb{P}_\infty(s_1).$$

Either $\ell(\alpha) = +\infty$ and in that case, $\nu_{j, s_1, \alpha} = \delta_{s_1}$ and hence, the integral between the big parentheses, is

$$1_{C^c_{p, \ell(\alpha)}}(s_1) = 0,$$

and hence $T(22, p) = 0$. Or $\ell(\alpha) < \infty$ and in that case, $\nu_{j, s_1, \alpha}(\cdot) = \mathbb{P}_\alpha(\cdot/C_{p, \ell(\alpha)})$ (we recall that $B(p, h) \subset C_{p, \ell(\alpha)}$) and hence, the integral between the big parentheses, is

$$\mathbb{P}_\alpha(\{B^c(s_1, \Delta_k) \cap C^c_{p, \ell(\alpha)}\}/C_{p, \ell(\alpha)}) = 0.$$

We conclude that $T_2 = 0$ and thus $\nu_j(G_{k, \alpha}) \leq \varepsilon_h$ and finally, for any $\alpha \in D$, for any $k \geq 1$, for any $h \geq \ell(\alpha) \geq k$,

$$\nu(G_{k, \alpha}) \leq \varepsilon_h.(CC)$$

We need a little extra-work to do before concluding. For $k \geq 1$ fixed, we denote the countable set $\{\alpha \in D_0, \ell(\alpha) \geq k\}$ by $\{\alpha_1, \alpha_2, \ldots \}$.

We are going to use the construction that let Formula (CC) in an induction reasoning. For $r = 1$ we choose $h(1)$ such that

$$\nu(G_{\alpha_1, k}) \leq \varepsilon_h(1).$$

Next for $r = 2$, we choose $h(2) > \max(h(1), \ell(\alpha_2))$ to get

$$\nu(G_{\alpha_2, k}) \leq \varepsilon_h(2).$$

By induction, we get $k < h(1) < \cdots < h(2) < \cdots$ such that for any $r \geq 1$,

$$\nu(G_{\alpha_r, k}) \leq \varepsilon_h(r).$$
We conclude that
\[ \nu\left( \bigcup_{\ell(a) \geq k, \alpha \in D_0} (d(X_\alpha, X_\infty) > \Delta_k) \right) \leq \sum_{r \geq 1} \varepsilon_{k(r)}, \]
i.e.,
\[ \nu\left( A^{**}_k \right) \leq \sum_{j \geq k} \varepsilon_j. \]
Since the series \( \sum \varepsilon_j \) converges, we get that
\[ \nu(A^{**}) = 0, \]
where
\[ A^{**} = \bigcap_{k \geq 1} A^{**}_k. \]
By definition, the set \((X_\alpha \to X_\infty)\) includes \(A^c\), and we have
\[ A \subset A^{**}. \]
So the exterior measure of \((X_\alpha \to X_\infty)^c\) is given by
\[ \nu^*(X_\alpha \not\to X_\infty) = 0. \]
In a last move, we may extend \((\Omega_D, S_D, \nu)\) to a complete probability measure \((\hat{\Omega}_D, \hat{S}_D, \hat{\nu})\) (See Lo (2017b), Chapter 5, Doc 04-02, Exercise 8). We will have
\[ \hat{\nu}\left( \limsup_{k \to +\infty} \bigcup_{\ell(a) \geq k, \alpha \in D} (d(X_\alpha, X_\infty) < \Delta_k) \right) = 0. \]
Since the mappings \(X_\alpha, \alpha \in \{\infty\} \cup D\), do not change and all the previous laws concern \(\hat{S}_D\)-measurable mappings or sets (for the handling of which, \(\nu\) and \(\hat{\nu}\) are equivalent), we conclude that
\[ d(X_\alpha, X_\infty) \to_D 0, \ a.s.. \]
The proof is over. ■

Remarks.

(R1) For interested readers who want to compare with the original proof in Wichura (1970), the given proof is more direct in the sense that we did not use the step where \( D \) is countable and \( S \) is a finite metric space in Section 2. Neither we did require the existence of the mapping \( k(\gamma) \) of \( \gamma \in D \) in Formula (e) therein. In our approach, such a sequence came to birth itself.

(R2) We recall a very simple proof of the Skorohod theorem for \( \mathbb{R}^d \) with \( d = 1 \). It would be interesting to either find out if simple proofs are available or can be done for \( d \geq 2 \).

(R3) The present contribution is the result of a Msc Dissertation at University Gaston Berger (SENEGAL).

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