Global well-posedness and scattering for the
defocusing cubic Schrödinger equation on
waveguide $\mathbb{R}^2 \times T^2$

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Abstract

We consider the problem of large data scattering for the defocusing
cubic nonlinear Schrödinger equation on $\mathbb{R}^2 \times T^2$. This equation is crit-
ical both at the level of energy and mass. The key ingredients contain
a global-in-time Strichartz estimate, resonant system approximation and
profile decomposition. Assuming the large data scattering for the 2d cubic
resonant system, we can prove the large data scattering for this problem.

1 Introduction

Let us consider the defocusing cubic nonlinear Schrödinger equation on $\mathbb{R}^2 \times T^2$,

$$(i\partial_t + \Delta_{\mathbb{R}^2 \times T^2})u = F(u) = |u|^2u,$$

$$u(0, x) = u_0 \in H^1(\mathbb{R}^2 \times T^2)$$

where $\Delta_{\mathbb{R}^2 \times T^2}$ is the Laplace-Beltrami operator on $\mathbb{R}^2 \times T^2$ and $u : \mathbb{R} \times \mathbb{R}^2 \times T^2 \to \mathbb{C}$ is a complex-valued function. A natural question is that given a finite (in the
sense of $H^1$ norm) initial data, can we obtain a global well-posed solution that
scatters?

The equation (1.1) has the following conserved quantities:

1. Energy $E$: $E(u) = \frac{1}{2}||\nabla u||^2_{L^2(\mathbb{R}^2 \times T^2)} + \frac{1}{4}||u||^4_{L^4(\mathbb{R}^2 \times T^2)}$.
2. Mass $M$: $M(u) = ||u||^2_{L^2(\mathbb{R}^2 \times T^2)}$.
3. Full energy $L$: $L(u) = \frac{1}{2}M(u) + E(u)$.
4. Momentum: $P(u) = Im \int \bar{u}(x, y, t) \nabla u(x, y, t) dx dy$. 

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Actually the equation (1.1) is a special case of the general nonlinear equation on the waveguide $\mathbb{R}^m \times \mathbb{T}^n$:
\[
(i\partial_t + \Delta_{\mathbb{R}^m \times \mathbb{T}^n})u = F(u) = |u|^{p-1}u,
\]
\[
u(0, x) = u_0 \in H^1(\mathbb{R}^m \times \mathbb{T}^n).
\] (1.2)

As for the background, we know that there are many existing results regarding NLS problems on Euclidean space. In this paper, the nonlinear Schrödinger equation is discussed on a semiperiodic space, i.e. $\mathbb{R}^2 \times \mathbb{T}^2$. The motivation is to better understand the broad question of the effect of the geometry of the domain on the asymptotic behavior of large solutions to nonlinear dispersive equations.

The study of solutions of the nonlinear Schrödinger equation on compact or partially compact domains has been the subject of many works. Such equations have also been studied in applied sciences and various background. Those spaces are also called waveguide manifolds. In paper [23, 24], there is a good overview of the main results for waveguide manifolds.

The studies on global well-posedness for energy critical and subcritical equations seem to point to the absence of any geometric obstruction to global existence. Moreover, it is clear that the geometry influences the asymptotic dynamics of solutions. Thus, it is meaningful to explore when one can obtain the simplest asymptotic behavior, i.e. scattering, which means that all nonlinear solutions asymptotically resemble linear solutions. Based on the theory of NLS on Euclidean space, i.e. $\mathbb{R}^d$, the equation
\[
(i\partial_t + \Delta_{\mathbb{R}^d})u = F(u) = |u|^{p-1}u,
\]
\[
u(0, x) = u_0 \in H^1(\mathbb{R}^d)
\] (1.3)

would scatter in the range $1 + \frac{4}{d} \leq p \leq 1 + \frac{4}{d-2}$. When $p = 1 + \frac{4}{d}$, the equation (1.3) is mass critical; when $p = 1 + \frac{4}{d-2}$, the equation (1.3) is energy critical; when $p < 1 + \frac{4}{d}$, the equation is mass subcritical; when $p > 1 + \frac{4}{d-2}$, the equation (1.3) is energy supercritical; when $1 + \frac{4}{d} < p < 1 + \frac{4}{d-2}$, the equation is mass supercritical and energy subcritical.

Naturally, we are also interested in the range of $p$ for well-posedness and scattering of the NLS on $\mathbb{R}^m \times \mathbb{T}^n$. Based on the existing results and theories, we expect that the solution of (1.2) globally exists and scatters in the range $1 + \frac{4}{m} \leq p \leq 1 + \frac{4}{m+n-2}$. And fortunately the index ($p = 2$) in equation (1.1) lies in the range (exactly at the endpoints of the interval), it is reasonable for us to consider this problem.

Also, we are mainly inspired by a similar result [13] which studies the defocusing quintic NLS on space $\mathbb{R} \times \mathbb{T}^2$ by Zaher Hani and Benoît Pausader. For that
problem, the defocusing NLS equation is also critical both at the level of energy and mass. [15, 16, 17, 18, 19] are also some of the important related results. Also, the introduction of [8] has a summary of the main known results about NLS problems on waveguides, i.e. \( \mathbb{R}^m \times \mathbb{T}^n \).

As in [13], we also need to assume the large data scattering of a cubic resonant system (1.5). The next task for us to do is to deal with the regarding resonant system, i.e. to prove this conjecture 1.2. An inspiring thing is that [33] (by K. Yang and L. Zhao) has proved the large data scattering for a similar resonant system recently. Conjecture 1.1 seems to be a reachable problem to work on, and we leave it for a later work.

The first result asserts that the small data leads to solutions that are global and scatter.

**Theorem 1.1** There exists \( \delta > 0 \) such that any initial data \( u_0 \in H^1(\mathbb{R}^2 \times \mathbb{T}^2) \) satisfying

\[
\|u_0\|_{H^1(\mathbb{R}^2 \times \mathbb{T}^2)} \leq \delta
\]

leads to a unique global solution \( u \in X_1^1(\mathbb{R}) \) which scatters in the sense that there exists \( v^{\pm \infty} \in H^1(\mathbb{R}^2 \times \mathbb{T}^2) \) such that

\[
\|u(t) - e^{it\Delta} v^{\pm \infty}\|_{H^1(\mathbb{R}^2 \times \mathbb{T}^2)} \to 0 \quad \text{as} \quad t \to \pm \infty. \tag{1.4}
\]

The uniqueness space \( X_1^1 \subset C_t(\mathbb{R} : H^1(\mathbb{R}^2 \times \mathbb{T}^2)) \) was essentially introduced by Herr-Tataru-Tzvetkov [15]. In order to extend our analysis to large data, we use a method formalized in [20, 21]. One key ingredient is a linear and nonlinear profile decomposition for solutions with bounded energy. The so-called profiles correspond to sequences of solutions exhibiting an extreme behavior. It is there that the “energy critical” and “mass critical” nature of our equation become manifest.

For this problem, in view of the scaling-invariant of the IVP (1.1) under

\[
\mathbb{R}^2 \times \mathbb{T}^2 \to M_\lambda := \mathbb{R}^2 \times (\lambda^{-1}\mathbb{T}^2)_y, \quad u \to \tilde{u}(x,y,t) = \lambda u(\lambda x, \lambda y, \lambda^2 t).
\]

There are two situations:

When \( \lambda \to 0 \), the manifolds \( M_\lambda \) will be similar to \( \mathbb{R}^4 \) and we can use the 4D energy critical result [23] by (E. Ryckman and M. Visan). The appearance is a manifestation of the energy-critical nature of the nonlinearity.

When \( \lambda \to \infty \), the manifolds \( M_\lambda \) becomes thinner and thinner and resembles \( \mathbb{R}^2 \). The problem will become similar to the cubic mass critical NLS problem on \( \mathbb{R}^2 \).
\[(i\partial_t + \Delta_x)u = |u|^2u, \quad u(0) \in H^1(\mathbb{R}^2). \quad (1.5)\]

The cubic resonant system:

We consider the cubic resonant system,
\[(i\partial_t + \Delta_x)u = \sum_{(p_1,p_2,p_3) \in R(j)} u_{p_1} \bar{u}_{p_2} u_{p_3},\]
\[R(j) = \{(j_1,j_2,j_3) \in (\mathbb{Z}^2)^3 : j_1 - j_2 + j_3 = j \quad \text{and} \quad |j_1|^2 - |j_2|^2 + |j_3|^2 = |j|^2\} \quad (1.6)\]
with unknown \(\bar{u} = \{u_j\}_{j \in \mathbb{Z}^2}\), where \(u_j : \mathbb{R}_x^2 \times \mathbb{R}_t \rightarrow \mathbb{C}\).

In the special case when \(u_j = 0\) for \(j \neq 0\), it is exactly the equation (1.5). Similar systems of nonlinear Schrödinger equations arise in the study of nonlinear optics in waveguides.

As we show in Section 8, the system (1.6) is Hamiltonian and it has a nice local theory and retains many properties of (1.5). In view of this and of the result of Dodson [9], it seems reasonable to assume the following conjecture. Another reason for us to assume this is that K. Yang and L. Zhao [33] have proved the large data scattering for a similar resonant system when index \(j \in \mathbb{Z}\).

**Conjecture 1.1** Let \(E \in (0, \infty)\). For any smooth initial data \(\bar{u}_0\) satisfying:
\[E_{ls}(\bar{u}_0) = \frac{1}{2} \sum_{j \in \mathbb{Z}^2} \langle j \rangle^2 ||u_{0,j}||^2_{L^2_x(\mathbb{R}^2)} \leq E.\]
There exists a global solution \(\bar{u}(t)\), \(\bar{u}(t = 0) = \bar{u}_0\) with conserved \(E_{ls}(\bar{u}(t))\) satisfying:
\[||\bar{u}||^2_{\bar{W}} := \sum_{j \in \mathbb{Z}^2} \langle j \rangle^2 ||u_j||^2_{L^4_t(\mathbb{R}_t \times \mathbb{R}^2_x)} \leq \Lambda_{ls}(E_{ls}(\bar{u}_0)) \quad (1.7)\]
for some finite non-decreasing function \(\Lambda_{ls}(E)\).

**Remark.** As for a more general case, when \(n = 2\) and \(p = 1 + \frac{4}{m}\), the Initial Value Problem (1.2) is also both critical at the level of mass and energy. If \(m > 2\), \(p\) would no longer be an integer, which may cause some trouble for the resonant system approximation.

We now give the main result of this paper which asserts the large data scattering for (1.1) conditioned on Conjecture 1.1.

**Theorem 1.2** Assume that Conjecture 1.1 holds for all \(E \leq E_{ls, max}\), then any initial data \(u_0 \in H^1(\mathbb{R}^2 \times \mathbb{T}^2)\) satisfying
\[L(u_0) = \int_{\mathbb{R}_x^2 \times \mathbb{T}^2} \left(\frac{1}{2} ||\nabla u_0||^2 + \frac{1}{2} |u_0|^2 + \frac{1}{4} |u_0|^4\right) dx \leq E_{ls, max}\]
leads to a solution $u \in X_1^1(\mathbb{R})$ which is global, and scatters (in the sense of (1.3)). In particular if $E_{\text{max}}^{ls} = +\infty$, then all solutions of (1.1) with finite energy and mass scatter.

As a consequence of the local theory for the system (1.6), Conjecture 1.1 holds below a nonzero threshold $E_{\text{max}}^{ls} > 0$, so Theorem 1.2 is non-empty and indeed strengthens Theorem 1.1. Another point worth mentioning is that while Theorem 1.2 is stated as an implication, it is actually an equivalence as it is easy to see that one can reverse the analysis needed to understand the behavior of large-scale profile initial data for (1.1) in order to control general solutions of (1.6) and prove Conjecture 1.1 assuming that Theorem 1.2 holds (cf. Section 8). The “scattering threshold” for (1.1) and the resonant system (1.6) are same.

The proof of the Theorem 1.2 follows from a standard skeleton based on the Kenig-Merle machinery [20, 21] which illustrates a classical and vivid road map for global well-posedness (and scattering) problem. Mainly there are three important points: global Strichartz estimates, large-scale profile and the resonant cubic system and profile decomposition.

The paper is organized as follows: in Section 2, we introduce some notations and function spaces; in Section 3, we prove the global Strichartz estimates that will be used later; in Section 4, we will prove the local well-posedness and small data scattering of (1.1); in Section 6, we obtain a good linear profile decomposition that leads us to analyze, which is what we do in Section 5 for the Euclidean and large-scale profiles. In Section 7, we prove the contradiction argument leading to Theorem 1.2 (Main Theorem). In Section 8, we prove the local theory for the cubic resonant system (1.6) and also give a proof of a lemma (local-in-time $L^p$ estimate) in Section 3.

## 2 Notations and function spaces

About the notation, we write $A \lesssim B$ to say that there is a constant $C$ such that $A \leq CB$. We use $A \simeq B$ when $A \lesssim B \lesssim A$. Particularly, we write $A \lesssim_u B$ to express that $A \leq C(u)B$ for some constant $C(u)$ depending on $u$.

In addition to the usual isotropic Sobolev spaces $H^s(\mathbb{R}^2 \times \mathbb{T}^2)$, we have non-isotropic versions. For $s_1, s_2 \in \mathbb{R}$ we define:

$$H^{s_1, s_2}(\mathbb{R}^2 \times \mathbb{T}^2) = \{ u : \mathbb{R}^2 \times \mathbb{T}^2 \to \mathbb{C} : (\xi)^{s_1} (n)^{s_2} \hat{u}(\xi, n) \in L^2_{\xi, n}(\mathbb{R}^2 \times \mathbb{Z}^2) \}. \quad (2.1)$$

Particularly $H^{0,1}(\mathbb{R}^2 \times \mathbb{T}^2)$ is a Hilbert space with inner product:

$$\langle u, v \rangle_{H^{0,1}} = \langle u, v \rangle_{L^2} + \langle \nabla_y u, \nabla_y v \rangle_{L^2}.$$
We can also define a discrete analogue. For \( \vec{\phi} = \{ \phi_p \}_{p \in \mathbb{Z}^2} \) a sequence of real-variable functions, we let

\[
h^{s_1} H^{s_2} := \{ \vec{\phi} = \{ \phi_p \} : \| \vec{\phi} \|_{H^{s_1} H^{s_2}}^2 = \sum_{p \in \mathbb{Z}^2} (p)^{2s_1} \| \phi_p \|_{H^{s_2}}^2 < +\infty \}.
\] (2.2)

We can naturally identify \( H^{0,1}(\mathbb{R}^2 \times \mathbb{T}^2) \) and \( h^1 L^2 \) by via the Fourier transform in the periodic variable \( y \).

**Function spaces.** In this paper, we will use some function spaces. For example, the \( X^1 \) space was essentially introduced by Herr-Tataru-Tzvetkov [15].

For \( C = [-\frac{1}{2}, \frac{1}{2}]^4 \in \mathbb{R}^4 \) and \( z \in \mathbb{R}^4 \), we denote by \( C_z = z + C \) the translate by \( z \) and define the sharp projection operator \( P_{C_z} \) as follows (\( F \) is the Fourier transform):

\[
F(P_{C_z} f) = \chi_{C_z}(\xi) F(f)(\xi).
\]

We use the same modifications of the atomic and variation space norms that were employed in some other papers [15, 16]. Namely, for \( s \in \mathbb{R} \), we define:

\[
\| u \|_{X^s(\mathbb{R})}^2 = \sum_{z \in \mathbb{Z}^4} \langle z \rangle^{2s} \| P_{C_z} u \|_{U^s_\Delta(\mathbb{R}; L^2)}^2
\]

and similarly we have,

\[
\| u \|_{Y^s(\mathbb{R})}^2 = \sum_{z \in \mathbb{Z}^4} \langle z \rangle^{2s} \| P_{C_z} u \|_{V^s_\Delta(\mathbb{R}; L^2)}^2
\]

where the \( U^s_\Delta \) and \( V^s_\Delta \) are the atomic and variation spaces respectively of functions on \( \mathbb{R} \) taking values in \( L^2(\mathbb{R}^2 \times \mathbb{T}^2) \). There are some nice properties of those spaces. We refer to [15, 16] for the description and properties. For convenience, we also give the some definitions here.

**Definition 2.1** Let \( 1 \leq p < \infty \), and \( H \) be a complex Hilbert space. A \( U^p \)-atom is a piecewise defined function, \( a : \mathbb{R} \to H \)

\[
a = \sum_{k=1}^{K} \chi_{[t_{k-1}, t_k)} \phi_{k-1}
\]

where \( \{ t_k \}_{k=0}^{K} \in \mathbb{Z} \) and \( \{ \phi_{k} \}_{k=0}^{K-1} \subset H \) with \( \sum_{k=0}^{K} \| \phi_k \|_H^p = 1 \). Here we let \( \mathbb{Z} \) be the set of finite partitions \( -\infty < t_0 < t_1 < ... < t_K \leq \infty \) of the real line.

The atomic space \( U^p(\mathbb{R}; H) \) consists of all functions \( u : \mathbb{R} \to H \) such that \( u = \sum_{j=1}^{\infty} \lambda_j a_j \) for \( U^p \)-atoms \( a_j \), \( \{ \lambda_j \} \in l^1 \), with norm

\[
\| u \|_{U^p} := \inf \{ u = \sum_{j=1}^{\infty} \lambda_j a_j, \lambda_j \in \mathbb{C}, a_j \ U^p \text{-atom} \}.\]
Definition 2.2 Let $1 \leq p < \infty$, and $H$ be a complex Hilbert space. We define $V^p(\mathbb{R}, H)$ as the space of all functions $v : \mathbb{R} \to H$ such that

$$||u||_{V^p} := \sup_{\{t_k\}_{k=1}^K} \left( \sum_{k=1}^K ||v(t_k) - v(t_{k-1})||_H^p \right)^{\frac{1}{p}} \leq +\infty,$$

where we use the convention $v(\infty) = 0$. Also, we denote the closed subspace of all right-continuous functions $v : \mathbb{R} \to H$ such that $\lim_{t \to -\infty} v(t) = 0$ by $V^p_c(\mathbb{R}, H)$.

Definition 2.3 For $s \in \mathbb{R}$, we let $U^p H^s$ resp. $V^p \Delta H^s$ be the spaces of all functions such that $e^{-it\Delta}u(t)$ is in $U^p(\mathbb{R}, H^s)$ resp. $V^p(\mathbb{R}, H)$, with norms

$$||u||_{U^p H^s} = ||e^{-it\Delta}u||_{U^p(\mathbb{R}, H^s)}, \quad ||u||_{V^p \Delta H^s} = ||e^{-it\Delta}u||_{V^p(\mathbb{R}, H^s)}.$$

For this problem, we choose $H$ to be $L^2(\mathbb{R}^2 \times \mathbb{T}^2)$. Norms $X^s_0$ and $Y^s$ are both stronger than the $L^\infty(\mathbb{R}; H^s)$ norm and weaker than the norm $U^2_\Delta(\mathbb{R}; H^s)$. Moreover, they satisfy the following property (for $p > 2$):

$$U^2_\Delta(\mathbb{R}; H^s) \hookrightarrow X^s_0 \hookrightarrow Y^s \hookrightarrow V^2(\mathbb{R}; H^s) \hookrightarrow U^p(\mathbb{R}; H^s) \hookrightarrow L^\infty(\mathbb{R}; H^s).$$

For an interval $I \subset \mathbb{R}$, we can also define the restriction norms $X^s(I)$ and $Y^s(I)$ in the natural way: $||u||_{X^s(I)} = \inf \{||v||_{X^s_0(\mathbb{R})} : v \in X^s_0(\mathbb{R}) \text{ satisfying } v|_I = u|_I\}$. And similarly for $Y^s(I)$.

A modification for to $X^s_0(\mathbb{R})$:

$$X^s(\mathbb{R}) := \{u : \phi_{-\infty} = \lim_{t \to -\infty} e^{-it\Delta}u(t) \text{ exists in } H^s, u(t) - e^{it\Delta}\phi_{-\infty} \in X^s_0(\mathbb{R})\}$$

equipped with the norm:

$$||u||^2_{X^s(\mathbb{R})} = ||\phi_{-\infty}||^2_{H^s(\mathbb{R}^2 \times \mathbb{T}^2)} + ||u - e^{it\Delta}\phi_{-\infty}||^2_{X^s_0(\mathbb{R})}. \quad (2.3)$$

Our basic space to control solutions is $X^1(I) = X^1(I) \cap C(I; H^1)$. Also we use $X^1_{c, loc}(I)$ to express the set of all solutions in $C_{loc}(I; H^1)$ whose $X^1(J)$-norm is finite for any compact subset $J \subset I$.

In order to control the nonlinearity on interval $I$, we need to define ‘$N$-Norm’ as follows, on an interval $I = (a, b)$ we have:

$$\|h\|_{N^s(I)} = \left\| \int_a^t e^{i(t-s)\Delta} h(s) ds \right\|_{X^s(I)}. \quad (2.4)$$

And then we can define the following spacetime norm, i.e. ‘$Z$-norm’ by

$$\|u\|_{Z(I)} = \left( \sum_{N \geq 1} N^2 \|1_I(t)P_N u\|_{L^4_{t,x,y}([0,T] \times \mathbb{T}^2 \times \mathbb{R})}^4 \right)^{\frac{1}{4}}.$$
where $I_\gamma = [2\pi \gamma, 2\pi (\gamma + 1)]$. Here we decompose $\mathbb{R}$ into $\mathbb{R} = \bigcup_{\gamma \in \mathbb{Z}} 2\pi[\gamma, \gamma + 1)$. $Z$ is a weaker norm than $X^1$, in fact:

$$||u||_{Z(I)} \lesssim ||u||_{X^1(I)}.$$ It follows from Strichartz estimate.

We also need the following theorem which has analogues in [13, 15, 16].

**Theorem 2.1** [[13, 15, 16]] If $f \in L^1_t(I, H^1(\mathbb{R}^2 \times \mathbb{T}^2))$, then

$$||f||_{N(I)} \lesssim \sup_{v \in \mathcal{Y}^{-1}(I), ||v||_{\mathcal{Y}^{-1}(I)} \leq 1} \int_{I \times (\mathbb{R}^2 \times \mathbb{T}^2)} f(x,t) v(x,t) dx dt.$$ Also, we have the following estimate holds for any smooth function $g$ on an interval $I = [a,b]$:

$$||g||_{X^1(I)} \lesssim ||g(0)||_{H^1(\mathbb{R}^2 \times \mathbb{T}^2)} + (\sum_N ||P_N (i\partial_t + \Delta) g||^2_{L^1_t(I,H^1(\mathbb{R}^2 \times \mathbb{T}^2)))}^{1/2}. $$

**Proof:** The proof follows from [15, Proposition 2.11] and [16, Proposition 2.10].

## 3 Global Strichartz estimate

**Theorem 3.1** Then we can prove the following Strichartz Estimate:

$$\|e^{it\Delta_{\mathbb{R}^2 \times \mathbb{T}^2}} P \leq N u_0 \|_{L^p_{x,y,t}(\mathbb{R}^2 \times \mathbb{T}^2 \times [2\pi \gamma, 2\pi (\gamma + 1)])} \lesssim N^{2 - \frac{8}{p}} \|u_0\|_{L^2(\mathbb{R}^2 \times \mathbb{T}^2)}$$

whenever

$$p > \frac{10}{3} \quad \text{and} \quad \frac{1}{q} + \frac{1}{p} = \frac{1}{2}. $$

**Proof:** The main idea of the proof is similar to [13, Theorem 3.1], i.e. use $T - T^*$ argument, a partition of unity and then estimate the diagonal part and non-diagonal part separately. One remarkable difference is that in the diagonal estimate part, we can not use Bourgain’s $L^p$ estimate directly on $\mathbb{T}^2([2])$ as in [13] since we need a Strichartz estimate with a threshold less than 4 (precisely it is $\frac{10}{3}$) to do the interpolation later. And we use Hardy-Littlewood circle method as in [19, Proposition 2.1] to obtain the local-in-time $L^p$ estimate.

First, let us prove a more precise conclusion and we can get the estimate by duality:

**Lemma 3.2** For any $h \in C^\infty_c(\mathbb{R}_x^2 \times \mathbb{T}_y^2 \times \mathbb{R}_t)$, then there holds

$$\left\| \int_{s \in \mathbb{R}} e^{-is\Delta_{\mathbb{R}_x^2 \times \mathbb{T}_y^2}} P \leq N h(x, y, s) ds \right\|_{L^p_{x,y,t}(\mathbb{R}_x^2 \times \mathbb{T}_y^2 \times \mathbb{R}_t)} \lesssim \left( N^{2 - \frac{8}{p}} ||h||_{L^p_{x,y,t}(\mathbb{R}^2 \times \mathbb{T}^2 \times [2\pi \gamma, 2\pi (\gamma + 1)])} + N^{1 - \frac{5}{3}} ||h||_{L^{p'}_{x,y,t}(\mathbb{R}^2 \times \mathbb{T}^2 \times [2\pi \gamma, 2\pi (\gamma + 1)])} \right).$$

for any $(p,q)$ satisfies (3.2).
Lemma 3.3
Let
For the diagonal part:
methods.
We will estimate the diagonal part and the non-diagonal part by using different
Here we have,
Proof: In order to distinguish between the large and small time scales, we choose
a smooth partition of unity $1 = \sum_{\gamma \in \mathbb{Z}} \chi(t - 2\pi\gamma)$ with $\chi$ supported in $[-2\pi, 2\pi]$. We also denote by $h_\alpha(t) = \chi(t)h(2\pi\alpha + t)$. Using the semigroup property and the unitarity of $e^{it\Delta_{\mathbb{R}^2 \times \mathbb{T}^2}}$ we can get:

$$\left\| \int_{s \in \mathbb{R}} e^{-is\Delta_{\mathbb{R}^2 \times \mathbb{T}^2}} P_{\leq N} h(x, y, s) ds \right\|_{L^2_x, y(\mathbb{R}^2 \times \mathbb{T}^2)}$$
$$= \int_{s, t \in \mathbb{R}} \langle e^{-is\Delta_{\mathbb{R}^2 \times \mathbb{T}^2}} P_{\leq N} h(s), e^{-it\Delta_{\mathbb{R}^2 \times \mathbb{T}^2}} P_{\leq N} h(t) \rangle_{L^2_x, y(\mathbb{R}^2 \times \mathbb{T}^2)} e^{it\Delta_{\mathbb{R}^2 \times \mathbb{T}^2}} ds dt$$
$$= \sum_{\alpha, \beta} \int_{s, t \in \mathbb{R}} \langle e^{-is\Delta_{\mathbb{R}^2 \times \mathbb{T}^2}} P_{\leq N} h_\alpha(s), e^{-it\Delta_{\mathbb{R}^2 \times \mathbb{T}^2}} P_{\leq N} h_\beta(t) \rangle_{L^2_x, y(\mathbb{R}^2 \times \mathbb{T}^2)} ds dt$$
$$= \sigma_d + \sigma_{nd}.$$ 

Here we have,

$$\sigma_d = \sum_{\alpha \in \mathbb{Z}, |\gamma| \leq 9} \int_{s, t \in \mathbb{R}} \langle e^{-i(s-2\pi\gamma)\Delta_{\mathbb{R}^2 \times \mathbb{T}^2}} P_{\leq N} h_\alpha(s), e^{-it\Delta_{\mathbb{R}^2 \times \mathbb{T}^2}} P_{\leq N} h_{\alpha+\gamma}(t) \rangle_{L^2_x, y} ds dt.$$ 

$$\sigma_{nd} = \sum_{\alpha, \gamma \in \mathbb{Z}, |\gamma| > 9} \int_{s, t \in \mathbb{R}} \langle e^{-i(s-2\pi\gamma)\Delta_{\mathbb{R}^2 \times \mathbb{T}^2}} P_{\leq N} h_\alpha(s), e^{-it\Delta_{\mathbb{R}^2 \times \mathbb{T}^2}} P_{\leq N} h_{\alpha+\gamma}(t) \rangle_{L^2_x, y} ds dt.$$ 

Here ‘d’ is short for ‘diagonal’ and ‘nd’ is short for ‘non-diagonal’.

We will estimate the diagonal part and the non-diagonal part by using different methods.

For the diagonal part: First we need a local-in-time $L^p$ estimate as follows:

**Lemma 3.3** Let $p_1 = \frac{10}{3}$, then for any $p > p_1$, $N \geq 1$, and $f \in L^2(\mathbb{R}^2 \times \mathbb{T}^2)$,

$$\|e^{it\Delta} P_N f\|_{L^p(\mathbb{R}^2 \times \mathbb{T}^2 \times [0, 2\pi])} \lesssim_p N^{2-\frac{2}{p}} \|f\|_{L^2(\mathbb{R}^2 \times \mathbb{T}^2)} \quad (3.4)$$

We will give the proof of Lemma 3.3 in the Appendix (Section 8).

According to the estimate (3.4) above, by duality we have

$$\left\| \int_{s \in \mathbb{R}} e^{-is\Delta_{\mathbb{R}^2 \times \mathbb{T}^2}} P_{\leq N} h(s) ds \right\|_{L^2_x, y(\mathbb{R}^2 \times \mathbb{T}^2)} \lesssim N^{2-\frac{4}{p}} \|h\|_{L^p_x, y((\mathbb{R}^2 \times \mathbb{T}^2) \times [-2\pi, 2\pi])} \quad (3.5)$$

where $h$ is supported in $[-2\pi, 2\pi]$. And consequently,
inequality and the discrete Hardy-Sobolev inequality as below:

and we can apply it to estimate the non-diagonal part by using H"older’s inequality and the discrete Hardy-Sobolev inequality as below:

\[
\sigma_d = \sum_{\alpha \in \mathbb{Z}, |\alpha| \leq 9} \int_{s,t \in \mathbb{R}} e^{-i(s-2\pi \gamma)\Delta_{k^2 \times y^2}} P_{\leq N} h_\alpha(s), e^{-i\Delta_{k^2 \times y^2}} P_{\leq N} h_{\alpha+\gamma}(t) \| \lambda_{x,y} \times \lambda_{x,y} \| ds dt
\]

\[
\leq \sum_{\alpha \in \mathbb{Z}, |\alpha| \leq 9} \| \int_{s \in \mathbb{R}} e^{-i(s-2\pi \gamma)\Delta_{k^2 \times y^2}} P_{\leq N} h_\alpha(2\pi \gamma + s) ds \| \| L_{x,y} \| \| e^{-i\Delta_{k^2 \times y^2}} P_{\leq N} h_{\alpha+\gamma}(t) ds \| L_{x,y}^2,
\]

\[
\lesssim N^{2(2-\frac{6}{p})} \sum_{\alpha} \| h_\alpha \|_{L_{x,y,t}^p(\mathbb{R}^2 \times \mathbb{T}^2 \times [-2\pi, 2\pi])}^2.
\]

This finished the estimate for the diagonal part.

For the non-diagonal part: We need a lemma (Lemma 3.4) that we will introduce later and we can apply it to estimate the non-diagonal part by using H"older’s inequality and the discrete Hardy-Sobolev inequality as below:

\[
\sigma_{nd} = \sum_{\alpha, \gamma \in \mathbb{Z}, |\gamma| > 10} \int_{t \in \mathbb{R}} \int_{s \in \mathbb{R}} e^{-i(s-2\pi \gamma)\Delta_{k^2 \times y^2}} P_{\leq N} h_\alpha(s), e^{-i\Delta_{k^2 \times y^2}} P_{\leq N} h_{\alpha+\gamma}(t) \| \lambda_{x,y} \times \lambda_{x,y} \| dt
\]

\[
\lesssim N^{2-\frac{6}{p}} \sum_{\alpha, \gamma \in \mathbb{Z}, |\gamma| > 10} |\gamma|^{\frac{2}{p}-1} \| h_\alpha \|_{L_{x,y,t}^p} \| h_{\alpha+\gamma} \|_{L_{x,y,t}^p}
\]

\[
\lesssim N^{2-\frac{6}{p}} \| h_\alpha \|_{L_{x,y,t}^p(\mathbb{R}^2 \times \mathbb{T}^2 \times [-2\pi, 2\pi])}^2.
\]

\[
\text{Lemma 3.4} \quad \text{Suppose } \gamma \in \mathbb{Z} \text{ satisfies } |\gamma| \geq 3 \text{ and that } p > \frac{10}{3}. \text{ For any function } h \in L_{x,y,t}^p(\mathbb{R}^2 \times \mathbb{T}^2 \times [-2\pi, 2\pi]), \text{ there holds that:}
\]

\[
\| \int_{s \in \mathbb{R}} \chi(s) e^{-i(t-s+2\pi \gamma)\Delta_{k^2 \times y^2}} P_{\leq N} h(s) ds \|_{L_{x,y,t}^p(\mathbb{R}^2 \times \mathbb{T}^2 \times [-2\pi, 2\pi])} \lesssim |\gamma|^{\frac{2}{p}-1} N^{2-\frac{6}{p}} \| h \|_{L_{x,y,t}^p(\mathbb{R}^2 \times \mathbb{T}^2 \times [-2\pi, 2\pi])}.
\]

\textbf{Proof:} The proof of this lemma is similar as in [13]. The main idea of the proof is to study the Kernel } K_{N,\gamma} \text{ and decompose the corresponding index set into three parts and estimate over the three parts separately. Notice that a significant difference is the non-stationary phase estimate because of the dimension, we will have:

\[
\| K_{N,\gamma} \|_{L_{x,y,t}^p} \lesssim |\gamma|^{-1} N^2
\]

instead of

\[
\| K_{N,\gamma} \|_{L_{x,y,t}^p} \lesssim |\gamma|^{-\frac{2}{3}} N^2.
\]

And

\[
\| F_{x,y,t} K_{N,\gamma} \|_{L_{x,y,t}^p} \lesssim 1
\]

still holds.
For difference as follows:

\[ K = \text{following decomposition:} \]

where

\[ a + b \]

is a large constant to be decided later. For fixed \( \alpha, \beta \), we will decompose \( K_{\alpha \beta} \) and estimate them as follows.

1. \( S_1 = (\alpha, \beta) : |C_1| \leq |C_2| \leq (1 + N)^2 \).
2. \( S_2 = (\alpha, \beta) : (|C_1| - 1) \leq |C_2| \leq |C_2| \).
3. \( S_3 = (\alpha, \beta) : |C_1| - 1 \leq |C_2| \leq |C_2| \).

Then we can estimate as in [13]. Eventually for the conclusion, there is one difference as follows:

\[ \|g\|_{L^p} \leq C \|g\|_{L^p} \leq \max(2N^{-1}, N^{-1}) \leq \|g\|_{L^p} \leq 2N^{-1} \]

Also the Kernel is defined as:

\[ K_{\alpha \beta} = \int k_{\alpha \beta}(x, y, t) e^{-i(x+y+t)} \eta_{\alpha \beta}(x, y, t) \]

We define \( K_{\alpha \beta}(x, y, t) \) similarly for \( \beta \in \mathbb{Z} \). And we have the following decomposition:

\[ K_{\alpha \beta}(x, y, t) = \sum_{k \in \mathbb{Z}} K_{\alpha \beta}^k(x, y, t) \]

where \( S_1, S_2, S_3 \) are three index sets. And similarly in this case we have the following decomposition:

\[ K_{\alpha \beta} = K_{\alpha \beta}^1 + K_{\alpha \beta}^2 + K_{\alpha \beta}^3 \]

For a dyadic number, we define \( g^\delta(x, y, t) = -\sqrt{\Delta g(x, y, t)} \) which has modulus in \( L^1 \). We define \( h^\delta \) similarly for \( \beta \in \mathbb{Z} \). And we have the following decomposition:

\[ g^\delta(x, y, t) = \sum_{k \in \mathbb{Z}} g^\delta_k(x, y, t) \]

(3.8)
if $p > \frac{10}{3}$. The rest follows as in [13] so we omitted.

This finished the estimate for the non-diagonal part.

4 Local well-posedness and small-data scattering

Recall “Z-norm” is

$$
\|u\|_{Z(I)} = \left( \sum_{N \geq 1} N^2 \|1_I(t)P_N u\|_{L^4_{x,y,t}(\mathbb{R}^2\times\mathbb{T}^2\times\mathbb{R})}^4 \right)^{\frac{1}{4}}.
$$

Now for convenience, we define “Z’-norm” which is a mixture of Z-norm and $X^1$-norm as follows

$$
\|u\|_{Z’(I)} = \|u\|_{Z(I)}^{\frac{1}{4}}\|u\|_{X^1(I)}^{\frac{3}{4}}.
$$

(4.1)

**Lemma 4.1 (Bilinear Estimate)** Suppose that $u_i = P_{N_i} u$, for $i = 1, 2$ satisfying $N_1 \geq N_2$. There exists $\delta$ such that the following estimate holds for any interval $I \in \mathbb{R}$:

$$
\|u_1 u_2\|_{L^2_{x,y,t}(\mathbb{R}^2\times\mathbb{T}^2\times I)} \lesssim \left( \frac{N_2}{N_1} + \frac{1}{N_2} \right)^{\delta} \|u_1\|_{Y^0(I)}\|u_2\|_{Z’(I)}.
$$

(4.2)

**Proof:** Without loss of generality, we can assume that $I = \mathbb{R}$. On one hand, we need the following estimate which follows from [16, Proposition 2.8],

$$
\|u_1 u_2\|_{L^2_{x,y,t}(\mathbb{R}^2\times\mathbb{T}^2\times \mathbb{R})} \lesssim N_2 \left( \frac{N_2}{N_1} + \frac{1}{N_2} \right)^{\delta}\|u_1\|_{Y^0(\mathbb{R})}\|u_2\|_{Y^0(\mathbb{R})}.
$$

(4.3)

And it suffices to prove the following estimate, if it is hold then we can just combine the two inequalities (noticing the definition of Z’-norm) and we will get the lemma finished.

$$
\|u_1 u_2\|_{L^2_{x,y,t}(\mathbb{R}^2\times\mathbb{T}^2\times \mathbb{R})} \lesssim \|u_1\|_{Y^0(\mathbb{R})}\|u_2\|_{Z(\mathbb{R})}.
$$

(4.4)

We first notice that, by orthogonality considerations, we may replace $u_1$ by $P_C u_2$ where $C$ is a cube of dimension $N_2$. By using Hölder’s inequality, we have,

$$
\|(P_C u_1) u_2\|_{L^2_{x,y,t}} \lesssim \|P_C u_1\|_{L^4_{x,y,t}(\mathbb{R}^2\times\mathbb{T}^2\times \mathbb{R})}\|u_2\|_{L^4_{x,y,t}(\mathbb{R}^2\times\mathbb{T}^2\times \mathbb{R})} \lesssim \frac{N_2^\frac{3}{4}}{N_1^\frac{1}{4}} \|P_C u_1\|_{Y^0}\|u_2\|_{Z(\mathbb{R})}.
$$

Here we have used some properties of the function spaces and another form of Strichartz inequality.
The Strichartz inequality in another form: for $p > \frac{4}{13}$ and $q$ as in Theorem 3.1, the following estimate holds for any time interval $I \subset \mathbb{R}$ and every cube $Q \subset \mathbb{R}^4$ of size $N$:

$$
\|1_{I}(t)P_{Q}u\|_{L^{p}_{x,y,1}L^{q}_{t}} \lesssim N^{2-\frac{4}{p}}\|u\|_{L^{\infty}_{x}(I;L^{2}(\mathbb{R}^2 \times \mathbb{T}^2))}.
$$

(4.5)

By using the atomic properties, it follows from the Strichartz inequality straightly.

**Lemma 4.2 (Nonlinear Estimate)** For $u_i \in X^1(I)$, $i = 1, 2, 3$. There holds that

$$
\|\tilde{u}_1 \tilde{u}_2 \tilde{u}_3\|_{H(I)} \leq \sum_{(i,j,k) = (1,2,3)} \|u_i\|_{X^1(I)}\|u_j\|_{Z'(I)}\|u_k\|_{Z'(I)}
$$

(4.6)

where $\tilde{u}_i$ is either $u_i$ or $\tilde{u}_i$.

**Proof:** It suffices to prove the following estimate: (Without loss of generality, let $I = \mathbb{R}$)

$$
\|\sum_{K \geq 1} P_{K}u_1 \prod_{i=2}^{3} P_{\leq C \cdot K} \tilde{u}_i\|_{H(\mathbb{R})} \lesssim C \|u_1\|_{X^1(\mathbb{R})}\|u_2\|_{Z'(\mathbb{R})}\|u_3\|_{Z'(\mathbb{R})}.
$$

(4.7)

It suffices to prove for any $u_0 \in Y^{-1}$ and $\|u_0\|_{Y^{-1}} \leq 1$ (By using Theorem 2.1)

$$
\sum_{N_1} \int_{\mathbb{R}^2 \times T^2 \times \mathbb{R}} \tilde{u}_0 P_{N_1}u_1 \prod_{i=2}^{3} (P_{\leq C \cdot N_1} \tilde{u}_i) dx dy dt \leq \|u_0\|_{Y^{-1}}\|u_1\|_{X^1(\mathbb{R})}\|u_2\|_{Z'(\mathbb{R})}\|u_3\|_{Z'(\mathbb{R})}.
$$

(4.8)

Now we split them as follows, let $u_i = \sum_{N_i \geq 1} P_{N_i}u_i$, $i = 0, 1, 2, 3$, denoting $u_{N_j} = P_{N_j}u_j$ and then the estimate would follow from the following bound:

$$
\sum_{s(N_0, N_1, N_2, N_3)} \left| \int u_0^{N_0}u_1^{N_1}u_2^{N_2}u_3^{N_3} dx dy dt \right| \lesssim \|u_0\|_{Y^{-1}}\|u_1\|_{X^1(\mathbb{R})}\|u_2\|_{Z'(\mathbb{R})}\|u_3\|_{Z'(\mathbb{R})}.
$$

(4.9)

Here we have set of index $S$ to be $\{(N_0, N_1, N_2, N_3) : N_1 \sim \max(N_2, N_0) \geq N_2 \geq N_3\}$ and we split $S$ into the disjoint union of $S_1$ and $S_2$ and $S_1$ is for the elements in $S$ that satisfy $N_1 \sim N_2$ and $S_2$ if for the elements in $S$ that satisfy $N_1 \sim N_0$. And we will estimate $S_1$ and $S_2$ separately. We omit the $S_2$ part since the estimate is similar.

By using bilinear estimate (3.2) and some basic inequalities and the properties of function spaces, we have, a term in $S_1$:

$$
\left| \int u_0^{N_0}u_1^{N_1}u_2^{N_2}u_3^{N_3} dx dy dt \right| \leq \|u_0^{N_0}u_2^{N_2}\|_{L^2}\|u_1^{N_1}u_3^{N_3}\|_{L^2}
$$

$$
\leq \left( \frac{N_2}{N_0} + \frac{1}{N_2} \right)^\delta \left( \frac{N_3}{N_1} + \frac{1}{N_3} \right)^\delta \|u_0^{N_0}\|_{Y^0(\mathbb{R})}\|u_1^{N_1}\|_{Y^0(\mathbb{R})}\|u_2^{N_2}\|_{Z'(\mathbb{R})}\|u_3^{N_3}\|_{Z'(\mathbb{R})}.
$$

(4.9)
By using Cauchy-Schwarz, the sum of the terms in $S_1$:

$$S_1 \lesssim \sum_{N_1 \sim N_0} \left( \frac{N_2}{N_0} + \frac{1}{N_2} \right)^\delta \left( \frac{N_3}{N_1} + \frac{1}{N_3} \right)^\delta \parallel u_0^N \parallel_{Y^\sigma(\mathbb{R})} \parallel u_1^N \parallel_{Y^\sigma(\mathbb{R})} \parallel u_2^N \parallel_{Z'(\mathbb{R})} \parallel u_3^N \parallel_{Z'(\mathbb{R})}$$

$$\lesssim \sum_{N_1 \sim N_0} \frac{N_0}{N_1} \parallel u_0^N \parallel_{Y^{-1}(\mathbb{R})} \parallel u_1^N \parallel_{Y^1(\mathbb{R})} \parallel u_2^N \parallel_{Z'(\mathbb{R})} \parallel u_3^N \parallel_{Z'(\mathbb{R})}$$

$$\lesssim \parallel u_0^N \parallel_{Y^{-1}(\mathbb{R})} \parallel u_1^N \parallel_{X^1(\mathbb{R})} \parallel u_2^N \parallel_{Z'(\mathbb{R})} \parallel u_3^N \parallel_{Z'(\mathbb{R})}.$$

This finished Lemma 4.2.

**Theorem 4.3 [Local Well-posedness]** Let $E > 0$ and $\parallel u_0 \parallel_{H^1(\mathbb{R}^2 \times \mathbb{T})} < E$, then there exists $\delta_0 = \delta_0(E) > 0$ such that if

$$\parallel e^{it\Delta} u_0 \parallel_{Z(I)} < \delta$$

for some $\delta \leq \delta_0$, $0 \in I$. Then there exists a unique strong solution $u \in X^1(I)$ satisfying $u(0) = u_0$ and we can get an estimate,

$$\parallel u(t) - e^{it\Delta} u_0 \parallel_{X^1(I)} \leq (E\delta)^{\frac{2}{3}}. \quad (4.10)$$

Observe that if $u \in X^1(\mathbb{R})$, then $u$ scatters as $t \to \pm \infty$ as in (1.3). Also, if $E$ is small enough, $I$ can be taken to $\mathbb{R}$ which proves Theorem 1.1.

**Proof:** First, we consider a mapping defined as follows,

$$\Phi(u) = e^{it\Delta} u_0 - \int_0^t e^{i(t-s)\Delta} |u(s)|^2 u(s) ds.$$

And we define a set $B = \{ u \in X^1_c(I) : \parallel u \parallel_{X^1(I)} \leq 2E \text{ and } \parallel u \parallel_{Z(I)} \leq 2\delta \}$. Now we will check two properties of $\Phi$: 1. $\Phi$ maps $B$ to $B$.  2. $\Phi$ is a contraction mapping.

1. For $u \in B$, we can use the nonlinear estimate in Lemma 3.2 and let $\delta \leq 1$ and small enough to make $E^3\delta$ small enough, we have:

$$\parallel \Phi(u) \parallel_{X^1(I)} \leq \parallel e^{it\Delta} u_0 \parallel_{X^1(I)} + \parallel |u|^2 u \parallel_{N(I)} \leq E + CE^{\frac{2}{3}} \delta^{\frac{2}{3}} \leq 2E,$$

$$\parallel \Phi(u) \parallel_{Z(I)} \leq \parallel e^{it\Delta} u_0 \parallel_{Z(I)} + \parallel |u|^2 u \parallel_{N(I)} \leq \delta + CE^{\frac{2}{3}} \delta^{\frac{2}{3}} \leq 2\delta.$$

2. $\parallel \Phi(u) - \Phi(v) \parallel_{X^1(I)} \lesssim \parallel u - v \parallel_{X^1(I)} (\parallel u \parallel_{X^1(I)} + \parallel v \parallel_{X^1(I)}) (\parallel u \parallel_{Z'(I)} + \parallel v \parallel_{Z'(I)})$

$$\leq C \parallel u - v \parallel_{X^1(I)} E^{\frac{2}{3}} \delta^{\frac{2}{3}},$$

$$\leq C \parallel u - v \parallel_{X^1(I)} [E^{\frac{3}{2}} \delta^{\frac{3}{2}}],$$

$$\leq C \frac{1}{2} \parallel u - v \parallel_{X^1(I)}.$$

Thus the result now follows from the Picard’s fixed point argument.
Theorem 4.4 [Controlling Norm] Let $u \in X^1_{c,loc}(I)$ be a strong solution on $I \in \mathbb{R}$ satisfying

$$\| u \|_{Z(I)} < \infty. \quad (4.11)$$

Then we have two conclusions,

1. If $I$ is finite, then $u$ can be extended as a strong solution in $X^1_{c,loc}(I')$ on a strictly larger interval $I' \subseteq I' \subset \mathbb{R}$. In particular, if $u$ blows up in finite time, then the $Z$ norm of $u$ has to blow up.

2. If $I$ is infinite, then $u \in X^1_{c}(I)$.

Proof: Without loss of generality, for the finite case we can assume $I = [0, T)$ and we want to extend it to $[0, T + v)$ for some $v > 0$. Denoting $E = \sup_I \| u(t) \|_{H^1(\mathbb{R}^2 \times T^2)}$ and using the time-divisibility of ‘$Z$-norm’, there exists $T_1$ such that $T - 1 < T_1 < T$ such that

$$\| u \|_{Z([T_1, T])} \leq \epsilon$$

where $\epsilon$ is to be decided. This allows to conclude:

$$\| u(t) - e^{i(t-T_1)\Delta} u(T_1) \|_{X^1([T_1, T])} \lesssim \| u \|_{X^1([T_1, T])}^3 \| u \|_{Z([T_1, T])}^{\frac{3}{2}} \leq C \epsilon^{\frac{3}{2}} \| u \|_{X^1([T_1, T])}^2 \leq \frac{3}{4} \epsilon_0(E).$$

By bootstrap argument, we get,

$$\| u \|_{X^1([T_1, T])} \lesssim E.$$

If $\epsilon$ is small enough and, making $\epsilon$ possibly smaller, we have,

$$\| e^{i(t-T_1)\Delta} u(T_1) \|_{Z([T_1, T])} \leq \| u \|_{Z([T_1, T])} + \| e^{i(t-T_1)\Delta} u(T_1) - u(t) \|_{Z([T_1, T])} \leq \epsilon + C' \epsilon^{\frac{3}{2}} E^{\frac{3}{2}} \leq \frac{3}{4} \epsilon_0(E).$$

Notice that we can let $\epsilon$ small enough s.t. $\epsilon < \frac{1}{4}$ and $\epsilon E < (\frac{1}{2} \epsilon_0(E))^{\frac{3}{2}}$.

This allows to find an interval $[T_1, T + v]$ for which:

$$\| e^{i(t-T_1)\Delta} u(T_1) \|_{Z([T_1, T+v])} < \delta_0.$$

That finishes the proof by using the Theorem 4.3 (Local Well-posedness).

Using the symmetries of the equation, the above argument also covers the case when $I$ is an arbitrary bounded interval.
For the infinite case, without loss of generality, it is enough to consider the case \( I = (a, \infty) \). Choosing \( T \) to be large enough so that
\[
\| u \|_{Z([T, \infty))} \leq \epsilon
\]
we get that for any \( T' > T \):
\[
\| u(t) - e^{i(t-T)\Delta} u(T) \|_{X^1([T,T'])} \lesssim \| u \|_{Z([T,T'])}^3 \| u \|_{Z([T,T])}^3 \leq C \epsilon^2 \| u \|_{Z([T,T'])}^3
\]
which gives that \( \| u \|_{X^1([T,T'])} \lesssim E \) for any \( T' > T \) and we have
\[
\| e^{i(t-T)\Delta} u(T) \|_{Z([T,\infty))} \leq 2\epsilon \leq \delta_0(E)
\]
if \( \epsilon \) small enough. Now the result follows from by using Theorem 4.3.

**Theorem 4.5** [Stability Theory] Let \( I \in \mathbb{R} \) be an interval, and let \( \tilde{u} \in X^1(I) \) solve the approximate solution,
\[
(i\partial_t + \Delta_{\mathbb{R}^2 \times \Sigma^2})\tilde{u} = \rho|\tilde{u}|^2\tilde{u} + \epsilon \quad \text{and} \quad \rho \in [0,1]. \tag{4.12}
\]
Assume that:
\[
\| \tilde{u} \|_{Z(I)} + \| \tilde{u} \|_{L_{\rho}^p(I, H^1(\mathbb{R}^2 \times \Sigma^2))} \leq M. \tag{4.13}
\]
There exists \( \epsilon_0 = \epsilon_0(M) \in (0,1] \) such that if for some \( t_0 \in I \):
\[
\| \tilde{u}(t_0) - u_0 \|_{H^1(\mathbb{R}^2 \times \Sigma^2)} + \| \epsilon \|_{N(I)} \leq \epsilon < \epsilon_0, \tag{4.14}
\]
then there exists a solution \( u(t) \) to the exact equation:
\[
(i\partial_t + \Delta_{\mathbb{R}^2 \times \Sigma^2}) u = |u|^2 u \tag{4.15}
\]
with initial data \( u_0 \) satisfies
\[
\| u \|_{X^1(I)} + \| \tilde{u} \|_{X^1(I)} \leq C(M), \quad \| u - \tilde{u} \|_{X^1(I)} \leq C(M)\epsilon. \tag{4.16}
\]

**Proof:** The proof is very similar to the proof of [13, Proposition 4.7]. The proof relies tightly on the estimate of Lemma 4.2 (nonlinear estimate) and the division of the intervals.

First, we consider for an interval \( J \in I \) s.t. \( \| \tilde{u} \|_{Z(J)} \leq \epsilon \) (That is the additional smallness assumption, \( \epsilon \) is to be decided). We will prove the theorem under the assumption first.

Then by local existing argument for the approximate equation, there exists \( \delta_1(M) \) that if
\[
\| e^{i(t-t_*)\Delta} \tilde{u}(t_*) \|_{Z(J)} + \| \epsilon \|_{N(J)} \leq \delta_1
\]
for some \( t_* \in J \), then \( \tilde{u} \in X^1(J) \) is unique and satisfies:
\[
\| \tilde{u} - e^{i(t-t_*)\Delta} \tilde{u}(t_*) \|_{X^1(J)} \leq C \| \tilde{u} \|_{X^1(J)}^2 \| \tilde{u} \|_{Z(J)}^2 + \| \epsilon \|_{N(J)}.
\]

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We can conclude
\[ \|u\|_{X^1(J)} \lesssim M + 1 \quad \text{and} \quad \|e^{i(t-t_*)\Delta}u(t_*)\|_{Z(J)} \lesssim \epsilon. \]
if \( \epsilon < \epsilon_1(M) \) is small enough.

Second, let us estimate the difference of the solutions. Consider solution \( u \) with initial data \( u_* \) satisfying \( \|u_* - \tilde{u}(t_*)\|_{H^1} \leq \epsilon \) and living on an interval \( J_u \subset J \) containing \( t_* \).

We want to prove the following estimate for some constant \( C \) independent of \( J_u \) to be specified later:
\[ \|u - \tilde{u}\|_{X^1(J_u)} \leq C\epsilon. \]  
(4.17)

Let \( w = u - \tilde{u} \), then we know that \( w \) satisfies:
\[ (i\partial_t + \Delta)w = \rho(|\tilde{u} + w|^2(\tilde{u} + w) - |\tilde{u}|^2\tilde{u}) - \epsilon. \]

Adopting the bootstrap hypothesis:
\[ \|w\|_{X^1(J_u \cap [t_*, t_* + t])} \leq 2C\epsilon. \]

For convenience, we denote \( J_u \cap [t_* - t, t_* + t] \) by \( J_t \), by using nonlinear estimate, we compute:
\[ \|w\|_{X^1(J_t)} \lesssim \|u(t_*) - \tilde{u}(t_*)\|_{H^1(\mathbb{R}^2 \times \mathbb{T}^2)} + \|w\|_{X^1(J_t)}\|\tilde{u}\|_{X^1(J_t)}\|\tilde{u}\|_{Z(J_t)} + \|e\|_{N(J_t)} \]
\[ \lesssim \epsilon + \|w\|_{X^1(J_t)}\|\tilde{u}\|_{X^1(J_t)}\|\tilde{u}\|_{Z(J_t)} \]
\[ \lesssim C_1\epsilon + C_1 M^2\epsilon^2\|w\|_{X^1(J_t)}. \]

As a result, if \( \epsilon < \epsilon_1(M) \) with \( \epsilon_1(M) \) small enough in terms of \( M \), we conclude that \( \|u - \tilde{u}\|_{X^1(J_t)} \leq 2C_1\epsilon \), which close the the bootstrap argument with \( C = 2C_1 \). This finishes the proof under the additional assumption \( (\|\tilde{u}\|_{Z(J)} \leq \epsilon) \).

Now, to generalize the argument to the whole interval \( I \), we split \( I \) into \( N = C(M, \epsilon_1(M)) \) intervals \( I_k = [T_k, T_{k+1}) \) such that:
\[ \|u\|_{Z(I_k)} \leq \frac{\epsilon_1(M)}{100} \quad \text{and} \quad \|e\|_{N(I_k)} \leq \frac{\epsilon_1(M)}{100}. \]

If \( \epsilon_0(M) \) is chosen sufficiently small in terms of \( N, M \) and \( \epsilon_1(M) \), we can iterate the first part of the proof on each interval \( I_k \) while keeping the condition
\[ \|u(T_k) - \tilde{u}(T_k)\|_{H^1(\mathbb{R}^2 \times \mathbb{T}^2)} + \|e\|_{N(I_k)} + \|u\|_{Z(I_k)} < \epsilon_1(M) \]
always satisfied for each \( k \). This finishes the proof.
5 Nonlinear Analysis of the Profiles

In this section, we describe and analyze the main profiles that appear in our linear and nonlinear profile decomposition.

5.1 Euclidean Profiles. The Euclidean profiles define a regime where we can compare solutions of cubic NLS on \( \mathbb{R}^4 \) with those on \( \mathbb{R}^2 \times \mathbb{T}^2 \).

We fix a spherically symmetric function \( \eta \in C_0^\infty(\mathbb{R}^4) \) supported in the ball of radius 2 and equal to 1 in the ball of radius 1. Given \( \phi \in H^1(\mathbb{R}^4) \) and a real number \( N \geq 1 \), we define:

\[
Q_N \phi \in H^1(\mathbb{R}^4) \quad (Q_N \phi)(x) = \eta(\frac{x}{N^2})\phi(x)
\]

\[
\phi_N \in H^1(\mathbb{R}^4) \quad \phi_N(x) = N(Q_N \phi)(Nx)
\]

\[
f_N \in H^1(\mathbb{R}^2 \times \mathbb{T}^2) \quad f_N(y) = \phi_N(\Psi^{-1}(y))
\]

where \( \Psi \) is the identity map from the unit ball of \( \mathbb{R}^4 \) to \( \mathbb{R}^2 \times \mathbb{T}^2 \). Thus \( Q_N \phi \) is a compactly supported modification of the profile \( \phi \), \( \phi_N \) is an \( \dot{H}^1 \)-invariant rescaling of \( Q_N \phi \), and \( f_N \) is the function obtained by transferring \( \phi_N \) to a neighborhood of 0 in \( \mathbb{R}^2 \times \mathbb{T}^2 \). Notice that

\[
\|f_N\|_{H^1(\mathbb{R}^2 \times \mathbb{T}^2)} \lesssim \|\phi\|_{\dot{H}^1(\mathbb{R}^4)}.
\]

Then we use the main theorem of [22] (by E. Ryckman and M. Visan) in the following form:

**Theorem 5.1** Assume \( \psi \in \dot{H}^1(\mathbb{R}^4) \), then there is a unique global solution \( v \in C(\mathbb{R} : \dot{H}^1(\mathbb{R}^4)) \) of the initial-value problem

\[
(i\partial_t + \Delta_{\mathbb{R}^4})v = v|v|^2, \quad v(0) = \psi,
\]

and

\[
\|\nabla v\|_{(L^\infty_t \cap L^2_t \cap L^4_t)(\mathbb{R}^4 \times \mathbb{R})} \leq \tilde{C}(E_{\mathbb{R}^4}(\psi)).
\]

Moreover this solution scatters in the sense that there exists \( \psi^{\pm \infty} \in \dot{H}^1(\mathbb{R}^4) \) such that

\[
\|v(t) - e^{it\Delta} \psi^{\pm \infty}\|_{\dot{H}^1(\mathbb{R}^4)} \to 0
\]

as \( t \to \pm \infty \). Besides if \( \psi \in H^5(\mathbb{R}^4) \), then \( v \in C(\mathbb{R} : H^5(\mathbb{R}^4)) \) and

\[
\sup_{t \in \mathbb{R}} \|v(t)\|_{H^5(\mathbb{R}^4)} \lesssim \|\psi\|_{H^5(\mathbb{R}^4)}.
\]

Based on the existing result, we have:
Theorem 5.2 Assume \( \phi \in \dot{H}^1(\mathbb{R}^4) \), \( T_0 \in (0, \infty) \), and \( \rho \in \{0, 1\} \) are given, and define \( f_N \) as before. Then the following conclusions hold:

1. There is \( N_0 = N_0(\phi, T_0) \) sufficiently large such that for any \( N \geq N_0 \), there is a unique solution \( U_N \in C((-T_0 N^{-2}, T_0 N^{-2}); H^1(\mathbb{R}^2 \times T^2)) \) of the initial-value problem

\[
(i \partial_t + \Delta) U_N = \rho U_N |U_N|^2, \quad \text{and} \quad U_N(0) = f_N. \tag{5.5}
\]

Moreover, for any \( N \geq N_0 \),

\[
\|U_N\|_{X^1(-T_0 N^{-2}, T_0 N^{-2})} \lesssim E_{\rho \phi}(\phi). \tag{5.6}
\]

2. Assume \( \epsilon_1 \in (0, 1] \) is sufficiently small (depending only on \( E_{\rho \phi}(\phi) \)), \( \phi' \in H^5(\mathbb{R}^4) \), and \( \|\phi - \phi'\|_{H^1(\mathbb{R}^4)} \leq \epsilon_1 \). Let \( v' \in C(\mathbb{R}: H^5) \) denote the solution of the initial-value problem

\[
(i \partial_t + \Delta_{\mathbb{R}^4}) v' = \rho v'|v'|^2, \quad v'(0) = \phi'.
\]

For \( R \geq 1 \) and \( N \geq 10R \), we define

\[
v'_R(x, t) = \eta\left(\frac{x}{R}\right) v'(x, t) \quad (x, t) \in \mathbb{R}^4 \times (-T_0, T_0)
\]

\[
v'_{R, N}(x, t) = N v'_R(N x, N^2 t) \quad (x, t) \in \mathbb{R}^4 \times (-T_0 N^{-2}, T_0 N^{-2}) \tag{5.7}
\]

\[
V_{R, N}(y, t) = v'_{R, N}(\Psi^{-1}(y), t) \quad (y, t) \in \mathbb{R}^2 \times T^2 \times (-T_0 N^{-2}, T_0 N^{-2})
\]

Then there is \( R_0 \geq 1 \) (depending on \( T_0 \) and \( \phi' \) and \( \epsilon_1 \)) such that, for any \( R \geq R_0 \) and \( N \geq 10R \),

\[
\lim_{N \to \infty} \|U_N - V_{R, N}\|_{X^1(-T_0 N^{-2}, T_0 N^{-2})} \lesssim E_{\rho \phi}(\phi) \epsilon_1. \tag{5.8}
\]

Proof: It suffices to prove part (2). All implicit constants are allowed to depend on \( \|\phi\|_{H^1(\mathbb{R}^4)} \). The idea of the proof is to show that with \( R_0 \) chosen large enough, \( V_{R, N} \) is an approximate solution.

First, we define:

\[
e_R(t, x) := (i \partial_t + \Delta_{\mathbb{R}^4}) v'_R - \rho |v'_R|^2 v'_R.
\]

Using the fact that \( \sup_t \|v'(t)\|_{H^5} \lesssim \|\phi'\|_{H^5} 1 \), we get that:

\[
|e_R(t, x)| + |\nabla_{\mathbb{R}^4} e_R(t, x)| \lesssim 1_{[R/2, R]}(|v'(t, x)| + |\nabla_{\mathbb{R}^4} v'(t, x)| + |\Delta_{\mathbb{R}^4} v'(t, x)|)
\]

which directly gives that there exists \( R_0 \geq 1 \) such that for all \( R > R_0 \)

\[
\lim_{R \to \infty} \|e_R| + |\nabla_{\mathbb{R}^4} e_R\|_{L^1 L^1_{t, x \mathbb{R}^4}(-T, T)} = 0.
\]

Letting

\[
e_{R, N}(x, t) := (i \partial_t + \Delta_{\mathbb{R}^4}) v'_{R, N} - \rho |v'_{R, N}|^2 v'_{R, N},
\]

\[
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\]
we have that for any $R > R_0$ and $N \geq 1$:

$$\|e_{R,N} + |\nabla_{\mathbb{R}^2} e_{R,N}|\|_{L^1_t L^2_y(\mathbb{R}^2 \times (-TN^{-2},TN^{-2}))} \leq 2\epsilon_1$$  \hfill (5.9)

with $V_{R,N}$ defined on $\mathbb{R}^2 \times \mathbb{T}^2 \times (-TN^{-2},TN^{-2})$, we let

$$E_{R,N}(y,t) = (i\partial_t + \Delta_{R^2})V_{R,N} - \rho\partial V_{R,N}^2 V_{R,N} = e_{R,N}(\Psi^{-1}(y),t).$$

For $R > R_0$ and $N \geq 10R$:

$$\|E_{R,N} + |\nabla_{\mathbb{R}^2} E_{R,N}|\|_{L^1_t L^2_y(\mathbb{R}^2 \times (-TN^{-2},TN^{-2}))} \lesssim \epsilon_1$$

from which it follows (using Theorem 2.1) that:

$$\|E_{R,N}\|_{L^1_t H^1_y(\mathbb{R}^2 \times \mathbb{T}^2 \times (-TN^{-2},TN^{-2}))} \lesssim \epsilon_1.$$

To verify the requirements of Theorem 4.5, we use (5.3) to conclude that:

$$\|V_{R,N}\|_{L^2_y H^2_y(\mathbb{R}^2 \times \mathbb{T}^2 \times (-TN^{-2},TN^{-2}))} \lesssim 1.$$

As for the $Z$-norm control, we choose $N$ to be big enough so that $TN^{-2} \leq \frac{1}{2}$

which makes all summations in the $Z$-norm consist of at most two terms, after which we estimate the $Z$-norm by using Littlewood-Paley theory and Sobolev embedding theorem as follows:

$$\|K_+^i \|_{P_y V_{R,N}}\|_{L^1_t L^2_y(\mathbb{R}^2 \times \mathbb{T}^2 \times (-TN^{-2},TN^{-2}))} \|_{L^4_y} \lesssim \|(1 - \Delta)^{1/2} V_{R,N}\|_{L^4_y} \lesssim \|(1 - \Delta)^{1/2} \varphi_{R,N}\|_{L^4_y} \lesssim_E (\varphi) \epsilon_1.$$

At last, we know for $R_0$ big enough and $R > R_0$, $N \geq 10R$,

$$\|f_N - V_{R,N}(0)\|_{H^1(\mathbb{R}^2 \times \mathbb{T}^2)} \lesssim \|Q_N \phi - \phi\|_{H^1(\mathbb{R}^4)} + \|\phi' - \phi\|_{H^1(\mathbb{R}^4)} + \|\phi' - V_R'(0)\|_{H^1(\mathbb{R}^4)} \lesssim \epsilon_1.$$

This completes the verification of the requirements of Theorem 4.5 which concludes the proof.

**Lemma 5.3** Suppose that $\phi \in \dot{H}^1(\mathbb{R}^4)$, $\epsilon > 0$, and $I \subset \mathbb{R}$ is an interval. Assume that

$$\|\phi\|_{\dot{H}^1(\mathbb{R}^4)} \leq 1, \quad \|\nabla_x e^{it\Delta} \phi\|_{L^2_t L^4_y(\mathbb{R}^4 \times I)} \leq \epsilon.$$ \hfill (5.10)

For $N \geq 1$, we define as before:

$$Q_N \phi = \eta(N^{-1/2}x)\phi(x), \quad \phi_{N} = N(Q_N \phi)(Nx), \quad f_N(y) = \phi_N(\Psi^{-1}(y)).$$

Then there exists $N_0 = N_0(\phi, \epsilon)$ such that for any $N \geq N_0$,

$$\|e^{it\Delta} f_N\|_{L^2_y(\mathbb{R} \times (-N^{-2}I, N^{-2}I))} \lesssim \epsilon.$$

**Proof:** It suffices to prove that there exists $T_0$ such that for any $N > 1$:

$$\|e^{it\Delta} f_N\|_{Z(\mathbb{R} \times (-N^{-2}T_0, N^{-2}T_0))} \lesssim \epsilon.$$ \hfill (5.11)
as the rest follows from Lemma 5.2 (with $\rho = 0$). Without loss of generality, by limiting arguments, we may assume that \( \phi \in C_0^\infty(\mathbb{R}^4) \). We have, for any \( p \),

\[
\begin{aligned}
f_{N,p}(x) &= \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} \phi_N(x,y)e^{-i(y,p)} \, dy = \frac{N}{(2\pi)^2} \int_{\mathbb{R}^2} e^{-i(y,p)} \phi(Nx,Ny) \, dy.
\end{aligned}
\]

And using dispersive estimate and unitarity, we have

\[
\sup_{x \in \mathbb{R}^2} \sum_{|p| \leq M} |e^{it\Delta} f_{N,p}(x)| \lesssim \frac{M^2}{|t|} ||f_N||_{L_{x,y}^4} \lesssim \frac{M^2 N^{-3}}{|t|} \tag{5.12}
\]

and

\[
||e^{it\Delta} P_M f_N(t)||_{L^2_{x,y}([\mathbb{R}^2 \times \mathbb{T}^2])} \lesssim \min \{ (\frac{M}{N})^{2 - \frac{4}{p}}, (\frac{N}{M})^{100} \} \tag{5.13}
\]

Then by interpolation we have (choose \( l = 0,10000 \)):

\[
\begin{aligned}
||e^{it\Delta} P_M f_N(t)||_{L^2_{x,y}([\mathbb{R}^2 \times \mathbb{T}^2])} &\lesssim \frac{N^{-1}}{|t|^{1 - \frac{2}{p}}} [(\frac{M}{N})^{2 - \frac{4}{p}}] \\
&\lesssim \frac{N^{-1}}{|t|^{1 - \frac{2}{p}}} \min \{ (\frac{M}{N})^{2 - \frac{4}{p}}, (\frac{N}{M})^{100} \}. \tag{5.14}
\end{aligned}
\]

As a result,

\[
\begin{aligned}
\sum_M M^2 ||e^{it\Delta} P_M f_N(t)||_{L^2_{x,y}([\mathbb{R}^2 \times \mathbb{T}^2 : |t| \geq T^2])} \lesssim T^{-\frac{4}{p}}. \tag{5.15}
\end{aligned}
\]

According to the definition of \( Z \)-norm, we finished the proof.

We can now describe the nonlinear solutions of (1.1) corresponding to data concentrating a point. Let \( \mathcal{F}_c \) denote the set of renormalized Euclidean frames

\[
\mathcal{F}_c := \{(N_k,t_k,x_k)_{k \geq 1} : N_k \in [1,\infty), x_k \in \mathbb{R}^2 \times \mathbb{T}^2, N_k \to \infty, \text{ and either } t_k = 0 \text{ for any } k \geq 1 \text{ or } \lim_{k \to \infty} N_k^2|t_k| = \infty \}.
\]

Given \( f \in L^2(\mathbb{R}^2 \times \mathbb{T}^2), \ t_0 \in \mathbb{R}, \) and \( x_0 \in \mathbb{R}^2 \times \mathbb{T}^2, \) we define:

\[
\begin{aligned}
\pi_{x_0} f &= f(x-x_0), & \Pi_{(t_0,x_0)} f &= (e^{-i t_0 \Delta_{x_0}^{\mathbb{T}^2}} f)(x-x_0) = \pi_{x_0} e^{-i t_0 \Delta_{x_0}^{\mathbb{T}^2}} f. \tag{5.16}
\end{aligned}
\]

Also for \( \phi \in \dot{H}^1(\mathbb{R}^4) \) and \( N \geq 1 \), we denote the function obtained in by:

\[
T_N \phi := N \hat{\phi}(N \Psi^{-1}(x)) \quad \text{where} \quad \hat{\phi}(y) := \eta(\frac{y}{N^\frac{3}{2}}) \phi(y) \tag{5.17}
\]

and as before observe that \( T_N : \dot{H}^1(\mathbb{R}^4) \to H^1(\mathbb{R}^2 \times \mathbb{T}^2) \) with \( ||T_N \phi||_{H^1(\mathbb{R}^2 \times \mathbb{T}^2)} \lesssim ||\phi||_{\dot{H}^1(\mathbb{R}^4)}. \)
Theorem 5.4 Assume that \( \mathcal{O} = (N_k, t_k, x_k)_k \in \tilde{F}_{e}, \phi \in \dot{H}^1(\mathbb{R}^4) \), and let \( U_k(0) = \Pi_{t_k, x_k}(T^e_{N_k} \phi) \):

1. For \( k \) large enough, there is a nonlinear solution \( U_k \in X^1(\mathbb{R}) \) of the equation (1.1) satisfying:

\[
||U_k||_{X^1(\mathbb{R})} \lesssim E_{\mathbb{R}^4}(\phi) \quad (5.18)
\]

2. There exists a Euclidean solution \( u \in C(\mathbb{R} : \dot{H}^1(\mathbb{R}^4)) \) of

\[
(i\partial_t + \Delta_{\mathbb{R}^4}) u = |u|^2 u \quad (5.19)
\]

with scattering data \( \phi^{\pm \infty} \) defined as in (5.4) such that for all \( T \geq T(\phi, \epsilon) \) there exists \( R(\phi, \epsilon, T) \) such that for all \( R \geq R(\phi, \epsilon, T) \), there holds that

\[
||U_k(\pm \infty) - u||_{X^1(\mathbb{R})} \leq \epsilon \quad (5.20)
\]

for \( k \) large enough, where

\[
(\pi - x_k \tilde{u})(x, t) = N_k \eta(N_k \Psi^{-1}(x)/R) u(N_k \Psi^{-1}(x), N_k^2(t - t_k)) \quad (5.21)
\]

In addition, up to a subsequence,

\[
||U_k(t) - \Pi_{(t_k - t, x_k)} T_{N_k}^e \phi^{\pm \infty}||_{X^1(\{ |t - t_k| \leq T N_k^{-2} \})} \leq \epsilon \quad (5.22)
\]

for \( k \) large enough (depending on \( (\phi, \epsilon, T, R) \)).

Proof: Without loss of generality, we may assume \( x_k = 0 \) for all \( k \). We first consider the case when \( t_k = 0 \) for all \( k \).

If \( u \) is the solution with initial data \( \phi \), there exists a time \( T_0 = T_0(\phi, \epsilon) \) such that:

\[
||\nabla u||_{L^2 L^4(\mathbb{R}^4 \times \{|t| \geq T \})} \ll E(\phi) \quad (5.23)
\]

Theorem 5.2 tells us that for any \( T \geq T_0 \), there is \( R_0 = R_0(T, \phi, \epsilon) \) such that for all \( R \geq R_0 \) and \( N_k \geq 10 R \), it holds that:

\[
||U_k - \tilde{u}_k||_{X^1(\{|t| \leq T N_k^{-2} \})} \leq \epsilon
\]

Equation (5.23) along with Lemma 5.3 imply that \( e^{it\Delta} U_k(\pm T N_k^{-2}) \) is sufficiently small in \( Z(\{|t| \geq T N_k^{-2} \}) \) thus guaranteeing that \( U_k \) extends to a global solution in \( X^1(\mathbb{R}) \) that satisfies (5.22).

We now consider the other case when \( N_k^2 |t_k| \to \infty \). For definiteness, we assume that \( N_k^2 t_k \to \infty \) and \( u \) be the solution to (5.20) satisfying:

\[
||\nabla (u(t) - e^{it\Delta} \phi)||_{L^2(\mathbb{R}^4)} \to 0
\]
as $t \to -\infty$. Let $\tilde{\phi} = u(0)$ and let $V_k$ be the solution of (1.1) with initial data $T_{N_k} \tilde{\phi}$.

Applying the first case of the proof to the frame $(N_k, 0, 0)$ and the family $V_k$ we conclude that:

$$||V_k(-t_k) - \Pi_{t_k,0} \phi||_{H^1(\mathbb{R}^2 \times \mathbb{T}^2)} \to 0,$$

as $k \to \infty$.

The conclusion of the proof now follows from Theorem 4.5 and by noticing the behavior of $V_k$.

**Large Scale Profiles.** Also we will analyze large-scale profiles that appear in the profile decomposition in the next section. We need some notation: given $\psi \in H^{0,1}(\mathbb{R}^2 \times \mathbb{T}^2)$ and $M \leq 1$, we define the large-scale rescaling

$$T^{ls}_M \psi(x,y) = M \tilde{\psi}^\ast(Mx,y)$$

where $\tilde{\psi}^\ast(x,y) = P_{x \leq M^{-1/100}} \psi(x,y)$.

(5.24)

It is not hard to see that, $T^{ls}_M : H^{0,1}(\mathbb{R}^2 \times \mathbb{T}^2) \to H^1(\mathbb{R}^2 \times \mathbb{T}^2)$ is a linear bounded operator.

It is crucial to study the behavior of nonlinear solutions $u_k$ of (1.1) with initial data as above. And these solutions are tightly related to the solutions of the following cubic resonant systems.

**The cubic resonant system.** We consider the cubic resonant system:

$$(i \partial_t + \Delta_x) u_j = \sum_{(p_1, p_2, p_3) \in R(j)} u_{p_1} \bar{u}_{p_2} u_{p_3},$$

$R(j) = \{(j_1, j_2, j_3) \in (\mathbb{Z}^2)^3 : j_1 - j_2 + j_3 = j \text{ and } |j_1|^2 - |j_2|^2 + |j_3|^2 = |j|^2\}$

with initial data $\vec{u}(0) = \{u_j(0)\} \in h^1 L^2$, where is defined in (2.2). The following energy

$$E_{ls}(\vec{u}) = \sum_{p \in \mathbb{Z}^2} (1 + |p|^2) ||u_p||_{L^2_\mathcal{H}(\mathbb{R}^2)}$$

is conserved and so is the $h^1 L^2$ norm of any solution of (1.6).

**Conjecture 5.1** In addition to Conjecture 1.1, any initial data $\vec{u}_0$ of finite $E_{ls}$ energy leads to a global solution of (1.6) satisfying

$$||\vec{u}||^2_{W} := \sum_{p \in \mathbb{Z}^2} [1 + |p|^2] ||u_p||^2_{L^4_{x,t}(\mathbb{R}^2 \times \mathbb{T}^2)} \leq S(E_{ls}(\vec{u}))$$

as $t \to -\infty$. Let $\tilde{\phi} = u(0)$ and let $V_k$ be the solution of (1.1) with initial data $T_{N_k} \tilde{\phi}$.

Applying the first case of the proof to the frame $(N_k, 0, 0)$ and the family $V_k$ we conclude that:

$$||V_k(-t_k) - \Pi_{t_k,0} \phi||_{H^1(\mathbb{R}^2 \times \mathbb{T}^2)} \to 0,$$

as $k \to \infty$.

The conclusion of the proof now follows from Theorem 4.5 and by noticing the behavior of $V_k$.

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where $\tilde{\psi}^\ast(x,y) = P_{x \leq M^{-1/100}} \psi(x,y)$.

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It is not hard to see that, $T^{ls}_M : H^{0,1}(\mathbb{R}^2 \times \mathbb{T}^2) \to H^1(\mathbb{R}^2 \times \mathbb{T}^2)$ is a linear bounded operator.

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$R(j) = \{(j_1, j_2, j_3) \in (\mathbb{Z}^2)^3 : j_1 - j_2 + j_3 = j \text{ and } |j_1|^2 - |j_2|^2 + |j_3|^2 = |j|^2\}$

with initial data $\vec{u}(0) = \{u_j(0)\} \in h^1 L^2$, where is defined in (2.2). The following energy

$$E_{ls}(\vec{u}) = \sum_{p \in \mathbb{Z}^2} (1 + |p|^2) ||u_p||_{L^2_\mathcal{H}(\mathbb{R}^2)}$$

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$$||\vec{u}||^2_{W} := \sum_{p \in \mathbb{Z}^2} [1 + |p|^2] ||u_p||^2_{L^4_{x,t}(\mathbb{R}^2 \times \mathbb{T}^2)} \leq S(E_{ls}(\vec{u}))$$

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where $S$ is some nondecreasing finite function. Also, this solution scatters in the sense that there exists $\vec{v}^{\pm \infty} \in h^1 L^2$ such that
\[
\sum_{p \in \mathbb{Z}^2} [1 + |p|^2] ||u_p(t) - e^{it\Delta_x} \vec{v}^{\pm \infty}_p||_{L^2_p(\mathbb{R}^2)} \to 0 \quad \text{as} \quad t \to \pm \infty.
\]
(5.26)

As we show in the Section 8, this conjecture is also implied by the Theorem 1.2. In addition, by using the local well-posedness theory for (1.6), this conjecture is true under the smallness hypothesis $E_{ls}(\vec{u}) < \delta$ for some $\delta > 0$. Finally the result of [9] implies that the conjecture is true if one adds the additional condition that $\vec{u}(0)$ is a scalar.

By using the conjecture above and the persistence of regularity part of Theorem 8.1, we have:

**Theorem 5.5** Assume that Conjecture 5.1 hold true. Suppose that $\vec{u}_0 \in h^1 L^2$ and that $\vec{u} \in C(\mathbb{R} : h^1 L^2)$ is the solution of (1.6) with initial data $\vec{u}_0$ given by Conjecture 5.1. Suppose also that $\vec{v}_0 \in h^3 H^2$ satisfies
\[
||\vec{u}_0 - \vec{v}_0||_{h^1 L^2} \lesssim \epsilon,
\]
and that $\vec{v}(t)$ is the solution to (1.6) with initial data $\vec{v}(0) = \vec{v}_0$. Then, it holds that:
\[
||(1 - \Delta_x)\{(1 + |p|^2)\vec{v}\}_p||_{L^\infty_t(h^1 L^2) \cap \vec{W}(\mathbb{R})} \lesssim ||u_0||_{h^1 L^2} ||v_0||_{h^3 H^2},
\]
\[
||\vec{u} - \vec{v}||_{L^\infty_t(h^1 L^2) \cap \vec{W}(\mathbb{R})} \lesssim ||u_0||_{h^1 L^2} \epsilon
\]
and there exists $\vec{\omega}^{\pm} \in h^3 H^2$ such that
\[
\sum_{p \in \mathbb{Z}^2} [1 + |p|^2] ||v_p(t) - e^{it\Delta_x} \vec{\omega}^{\pm}_p||_{L^2_p(\mathbb{R}^2)} \to 0 \quad \text{as} \quad t \to \pm \infty.
\]

**Lemma 5.6** Assume that Conjecture 5.1 holds true. Let $\psi \in H^{0,1}(\mathbb{R}^2 \times \mathbb{T}^2)$, $T_0 \in (0, \infty)$, and $\rho \in \{0, 1\}$ be given, and define $f_M = T^\rho_M \psi(x,y)$. The following conclusions hold:

1. There is $M_0 = M_0(\phi, T_0)$ sufficiently small such that for all $M \leq M_0$, there is a unique solution $U_M \in C((-T_0 M^2, T_0 M^2); H^1(\mathbb{R}^2 \times \mathbb{T}^2))$ of the initial-value problem
   \[
   (i\partial_t + \Delta_{\mathbb{R}^2 \times \mathbb{T}^2})U_M = \rho|U_M|^2 U_M, \quad U_M(0) = f_M.
   \]
   (5.27)

Moreover, for any $M \leq M_0$,
\[
||U_M||_{X^{1,-T_0 M^{-2},T_0 M^{-2}}(\mathbb{R}^2 \times \mathbb{T}^2)} \lesssim E_{ls}(\psi) 1.
\]
(2) Assume $\varepsilon_1 \in (0, 1]$ is sufficiently small (depending only on $E_{1s}(\psi)$), $v_0 \in h^3 H^2$, and $\|\psi - \vec{v}_0\|_{H^1 L^2} \leq \varepsilon_1$. Let $\vec{v} \in C(\mathbb{R} : h^3 H^2)$ denote the solution of the initial-value problem

$$(i \partial_t + \Delta_x) v_j = \rho \sum_{(p_1, p_2, p_3) \in R(j)} v_{p_1} v_{p_2} v_{p_3}, \quad v_j(0) = v_{0,j}, j \in \mathbb{Z}^2.$$ 

For $M \geq 1$ we define

$$v_{j,M}(x, t) = M v_j(Mx, M^2 t), \quad (x, t) \in \mathbb{R}^2 \times (-T_0 M^{-2}, T_0 M^{-2}),$$

$$V_M(x, y, t) = \sum_{q \in \mathbb{Z}^2} e^{-it|q|^2} e^{i(y, q)} v_{q,M}(x, t), \quad (x, y, t) \in \mathbb{R}^2 \times \mathbb{T}^2 \times (-T_0 M^{-2}, T_0 M^{-2}),$$

then

$$\limsup_{M \to 0} \|U_M - V_M\|_{L^1(-T_0 M^{-2}, T_0 M^{-2})} \lesssim E_{1s}(\psi) \varepsilon_1. \quad (5.28)$$

Proof: When $\rho = 0$, it is trivial. It suffices to prove (2).

First, by using Stricharz estimate on $\mathbb{R}^2$, we have that

$$\sum_{q \in \mathbb{Z}^2} \langle q \rangle^2 \|v_q\|_{L^2_t L^6_x}^2 \lesssim E_{1s}(\vec{v}), \quad \sum_{q \in \mathbb{Z}^2} \langle q \rangle^6 \|v_q\|_{L^6_t W^{2,6}_x}^2 \lesssim E_{1s}(\vec{v}) \|\vec{v}\|_{h^3 H^2}^2. \quad (5.30)$$

We now want to show that $V_M$ is an approximate solution to (5.28) in the sense of Theorem 4.5.

$$(i \partial_t + \Delta_{\mathbb{R}^2 \times \mathbb{T}^2}) V_M - |V_M|^2 V_M = - \sum_{q \in \mathbb{Z}^2} e^{-it|q|^2} e^{i(y, q)} \sum_{\vec{p} \in NR(q)} v_{p_1, M} v_{p_2, M} v_{p_3, M} = RHS$$

$$\Phi_{q, \vec{p}} = |p_1|^2 - |p_2|^2 + |p_3|^2 - |q|^2$$

where

$$NR(q) = \{\vec{p} = (p_1, p_2, p_3) : p_1 - p_2 + p_3 - q = 0; \Phi_{q, \vec{p}} \neq 0\}.$$ 

Naturally, $NR$ is short for non-resonant. In addition, we now claim that

$$\|RHS\|_{N^1(1)} \lesssim \|\vec{v}_0\|_{h^3 H^2} M. \quad (5.31)$$

As in [13], We will estimate the high frequency part and low frequency part separately.

$$RHS = P_{>2-10}^x RHS + P_{<2-10}^x RHS = P_{\text{high}} RHS + P_{\text{low}} RHS.$$ 

For the high frequency part, we obtain

$$\|P_{>2-10}^x RHS\|_{N^1(0, S)}^2 \lesssim \|P_{>2-10}^x \partial_x RHS\|_{N^1(0, S)}^2 = \sum_{q \in \mathbb{Z}^2} \langle q \rangle^{-2} \langle \langle q \rangle \rangle^2 \sum_{\vec{p} \in NR(q)} \|\partial_x v_{p_1, M} v_{p_2, M} v_{p_3, M}\|_{L^2_t H^2_x}^2.$$

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Since $q \leq \max\{p_1, p_2, p_3\}$, we see that, for any $q$,

$$\langle q \rangle^2 \sum_{\vec{p} \in \mathbb{N}^\mathbb{R}(q)} \partial_x \left\{ v_{p_1, M \overrightarrow{v}_{p_2, M} v_{p_3, M}} \right\} \| L^1_t H^1_x \| \lesssim \sum_{\vec{p} \in \mathbb{N}^\mathbb{R}(q)} \langle p_1 \rangle^2 \| \partial_x (v_{p_1, M}) \| L^4_{t} W^2_x \| \prod_{j=2}^{3} \{ (p_j)^2 \| v_{p_j, M} \| L^4_{t} W^2_x \}.$$

Thus the high frequency part can be handled.

For the low frequency part, we can use integration by parts,

$$\int_{0}^{S} e^{i(S-\sigma)\Delta_{x^2} + \varphi} P_{\text{low}} \text{RHS}(\sigma) d\sigma$$

$$= - \sum_{\vec{q} \in \mathbb{Z}^2} \sum_{\vec{p} \in \mathbb{N}^\mathbb{R}(q)} e^{-iS\|q^2 + \Phi_{q, \vec{p}}\|} e^{i(y, \vec{q})} \int_{0}^{S} e^{i\sigma \Delta_x + \Phi_{q, \vec{p}}} P_{\text{low}} (v_{p_1, M \overrightarrow{v}_{p_2, M} v_{p_3, M}}) d\sigma$$

$$= \sum_{\vec{q} \in \mathbb{Z}^2} \sum_{\vec{p} \in \mathbb{N}^\mathbb{R}(q)} e^{-iS\|q^2 + \Phi_{q, \vec{p}}\|} e^{i(y, \vec{q})} \times$$

$$\left\{ |e^{i(S-\sigma)\Delta_x + \Phi_{q, \vec{p}}}| (\Delta_x + \Phi_{q, \vec{p}})^{-1} P_{\text{low}} (v_{p_1, M \overrightarrow{v}_{p_2, M} v_{p_3, M}}) \right\}_{0}^{S}$$

$$- i \int_{0}^{S} e^{i(\sigma - \sigma)\Delta_x + \Phi_{q, \vec{p}}} (\Delta_x + \Phi_{q, \vec{p}})^{-1} P_{\text{low}} \partial_\sigma (v_{p_1, M \overrightarrow{v}_{p_2, M} v_{p_3, M}}) d\sigma \right\}.$$

Now we need to estimate three terms: ‘$\Delta_x$-boundary’, ‘0-boundary’, and the other term. We will estimate them separately.

One important thing we should be clear is that $(\Delta_x + \Phi_{q, \vec{p}})^{-1}$ is bounded on $L^2_x (\mathbb{R}^2)$ noticing the non-resonant condition and the low frequency cutoff as in [7, 13].

For the ‘$\Delta_x$-boundary’:

$$\| \sum_{\vec{q} \in \mathbb{Z}^2} \sum_{\vec{p} \in \mathbb{N}^\mathbb{R}(q)} e^{-iS\|q^2 + \Phi_{q, \vec{p}}\|} e^{i(y, \vec{q})} (\Delta_x + \Phi_{q, \vec{p}})^{-1} P_{\text{low}} (v_{p_1, M \overrightarrow{v}_{p_2, M} v_{p_3, M}}) \| H^1_t (I)$$

$$\lesssim \sum_{\vec{q} \in \mathbb{Z}^2} \sum_{\vec{p} \in \mathbb{N}^\mathbb{R}(q)} \| v_{p_1, M \overrightarrow{v}_{p_2, M} v_{p_3, M}} (0) \| H^2_x (\mathbb{R}^2)$$

$$+ \sum_{\vec{q} \in \mathbb{Z}^2} \langle q \rangle^2 \| (i \partial_t + \Delta_x) \sum_{\vec{p} \in \mathbb{N}^\mathbb{R}(q)} v_{p_1, M \overrightarrow{v}_{p_2, M} v_{p_3, M}} \| L^2_t H^1_x$$

$$\lesssim \| \vec{v}_0 \|_{L^3_t H^2_x} M^4.$$
For the '0-boundary':

\[
\| \sum_{q \in \mathbb{Z}^2} \sum_{\vec{p} \in \mathcal{N}(R(q))} e^{-iS\Delta_{\mathbb{R}^2 \times \mathbb{T}^2}} e^{i(y,q)} (\Delta_x + \Phi_{\vec{q},\vec{p}})^{-1} P_{low} [v_{p_1, M} \overline{v_{p_2, M}} v_{p_3, M}](0) \|_{L^2_x(L^2_t)}^2 \lesssim \|v_0\|_{L^{3}_{x,t}}^2 M^4.
\]

For the other term:

\[
\| \sum_{q \in \mathbb{Z}^2} \sum_{\vec{p} \in \mathcal{N}(R(q))} e^{-iS[q^2 + \Phi_{\vec{q},\vec{p}}]} e^{i(y,q)} \int_0^S e^{i(S-\sigma)\Delta_x + \Phi_{\vec{q},\vec{p}}}(\Delta_x + \Phi_{\vec{q},\vec{p}})^{-1} P_{low} \partial_x \{v_{p_1, M} \overline{v_{p_2, M}} v_{p_3, M}\} d\sigma \|_{L^2_x(L^2_t)}^2 \lesssim \|v_0\|_{L^{3}_{x,t}}^2 M^4.
\]

This finishes the proof of (5.23).

We also have that

\[
\|V_M\|_{L^\infty_t H^1_x(\mathbb{R}^2 \times \mathbb{T}^2 \times I)} \leq \sum_{q \in \mathbb{R}^2} \langle q \rangle^2 \|v_{q,M}\|_{L^\infty_t H^1_x} \leq C\|\bar{u}(0)\|_{L^1_{t,x}}^2 + C(M)\|\bar{v}\|_{L^1_{t,x}}^2
\]

and that

\[
\|V_M\|_{X^1(I)} \lesssim \|\bar{v}(0)\|_{L^1_{t,x}}^2 1 + C(\|\bar{v}\|_{H^2}) M. \tag{5.32}
\]

Moreover, we have (Using Lemma 8.2)

\[
\| (i\partial_t + \Delta_{\mathbb{R}^2 \times \mathbb{T}^2}) V_M \|_{L^2_x(L^2_t)}^2 \lesssim \sum_{q \in \mathbb{Z}^2} \langle q \rangle^2 \|v_{q,M}\|_{L^2_t W^1_{x,s}}^2 \lesssim \left( \sum_{q \in \mathbb{Z}^2} \Pi_{k=1}^2 \langle q \rangle^2 \|v_{q,M}\|_{L^2_t W^1_{x,s}}^2 \right)^2 \lesssim \left( \sum_{\vec{p} \in \mathcal{N}(R(q))} \langle \vec{p} \rangle^2 \|v_{p,M}\|_{L^2_t W^1_{x,s}}^2 \right)^3.
\]

which justifies (5.33).

By using Theorem 4.5, we conclude that, for M small enough (depending on \( \bar{v}_0 \)), the solution \( U_M \) of (1.1) with initial data \( V_M(0) \) exists on \( I \) and that

\[
\|U_M - V_M\| \lesssim \epsilon_1 + C(\|\bar{v}_0\|_{H^3}) M,
\]

which ends the proof.

**Lemma 5.7** For any \( \psi \in H^{0,1}(\mathbb{R}^2 \times \mathbb{T}^2) \) and any \( \epsilon > 0 \), there exists \( T_0 = T(\psi, \epsilon) \) and \( M_0 = M(\psi, \epsilon) \) such that for any \( T \geq T_0 \) and any \( M \leq M_0 \),

\[
\|e^{i\Delta_{\mathbb{R}^2 \times \mathbb{T}^2}} T_M^2 \psi\|_{Z(M^2 | t \geq T_0)} \lesssim \epsilon.
\]
**Proof:** By Stricharz estimate on $\mathbb{R}^2$, and dominated convergence theorem, there exists $T_0$ such that

$$\sum_{p \in \mathbb{Z}^2} \langle p \rangle^2 \langle e^{it\Delta} \psi \rangle^2_{L^2(\mathbb{R}^2 \times [t \geq T_0])} \leq \epsilon^{1000}.$$  \hspace{1cm} (5.33)

Let $I = \{ |t| \geq T_0 \}$ and $I_M = \{ M^2 |t| \geq T_0 \}$. We have that

$$e^{it\Delta_{k^2 \times 2^2}}T^{ls}_M \psi = \sum_{q \in \mathbb{Z}^2} e^{i\langle q,y \rangle |q|^2 t} (M e^{it\Delta} \tilde{\psi}_p^M (Mx))$$

$$= \sum_{q \in \mathbb{Z}^2} e^{i\langle q,y \rangle |q|^2 t} v_{q,M}(t, x)$$

where we denoted by:

$$v_{p,M}(x, t) = M e^{it\Delta} \tilde{\psi}_p^M (Mx).$$

Noticing that $e^{i\langle q,y \rangle} v_{q,M}(x, t)$ is supported in Fourier space in the box centered at $q$ of radius 2 and Bernstein’s inequality in $y$, we can estimate:

$$||P_N e^{it\Delta_{k^2 \times 2^2}}T^{ls}_M \psi||_{L^4_x,y_t} \lesssim N^{\frac{1}{4}} \left( \sum_{|q| \sim N} ||v_{q,M}(x, t)||^2_{L^4_x} \right)^\frac{1}{2} \lesssim N^{\frac{1}{4}} \left( \sum_{|q| \sim N} ||v_{q,M}||^2_{L^4_x} \right)^\frac{1}{2}.$$

Thus we know:

$$\sum_{N \geq 1} N^2 ||P_N e^{it\Delta_{k^2 \times 2^2}}T^{ls}_M \psi||_{L^4_x,y_t, (\mathbb{R}^2 \times \mathbb{T}^2 \times I_{M})}^2 \lesssim \sum_{N \geq 1} N^2 \left( \sum_{|q| \sim N} ||v_{q,M}||^2_{L^4_x, \mathbb{R}^2} \right)^2 \lesssim \sum_{N \geq 1} N^2 \left( \sum_{|q| \sim N} ||v_{q,M}||^2_{L^4_y, (\mathbb{R}^2 \times I_M)} \right)^2 \lesssim \sum_{q \in \mathbb{Z}^2} ||e^{it\Delta} \tilde{\psi}_q^M||_{L^4_t, (\mathbb{R}^2 \times I)}^2 \lesssim \epsilon^{2000}.$$  \hspace{1cm} (5.34)

According to the definition of $Z$-norm, we finish the proof of this lemma.

Now we can describe the nonlinear solutions of the Initial Value Problem (1.1) corresponding to large-scale profile. In view of the profile analysis in the next section, we need to consider the renormalized large-scale frames by:

$$\tilde{F}_{ls} := \{(M_k, t_k, p_k, \xi_k) : M_k \leq 1, M_k \to 0, p_k = (x_0, 0) \in \mathbb{R}^2 \times \mathbb{T}^2, \text{ and } \xi_k \in \mathbb{R}$$

,with $\xi_k \rightarrow \xi_\infty \in \mathbb{R}$ and either $t_k = 0$ or $M_k^{-1} t_k \to \pm \infty$ and either $\xi_k = 0$ or $M_k^{-1} \xi_k \to \pm \infty \}.$

**Theorem 5.8 Assume Conjecture 5.1 holds true. Fix a renormalized large-scale frame $(M_k, t_k, (x_k, 0), \xi_k)_k \in \tilde{F}_{ls}$ and $\psi \in H^{0,1}(\mathbb{R}^2 \times \mathbb{T}^2)$ let**

$$U_k(0) = \Pi_{t_k, x_k} e^{i\xi_k x} T^{ls}_{M_k} \psi.$$
(1) For $k$ large enough (depending on $\psi$, $S$), there is a nonlinear solution $U_k \in X_1^c(\mathbb{R})$ of the solution (1.1) satisfying:

$$\|U_k\|_{X_1^c(\mathbb{R})} \lesssim E_{\psi}(\psi).$$  \hspace{1cm} (5.35)

(2) There exists a solution $\bar{v} \in C(\mathbb{R} : h^1L^2)$ of (1.6) with scattering data $v_0^{\pm \infty}$ such that the following holds, up to a subsequence: for $\epsilon > 0$, there exists $T(\psi, \epsilon)$ such that for all $T \geq T(\psi, \epsilon)$, there holds that

$$\|U_k - W_k\|_{X_1^c(\{|t-t_k| \leq TM_k^{-2}\})} \leq \epsilon,$$  \hspace{1cm} (5.36)

for $k$ large enough, where

$$W_k(x,t) = e^{-i|\xi_k|^2} e^{ix\xi_k} V_{M_k}(x-x_k-2\xi_k \eta, y, \eta), \quad \eta = t - t_k$$

with $V_k$ defined as before. Moreover,

$$\|U_k(t) - \Pi_{k-t_k} e^{ix\xi_k T_{M_k}^{\frac{t_k}{TM_k}}} \|_{X_1^c(\{|t-t_k| \geq TM_k^{-2}\})} \leq \epsilon,$$  \hspace{1cm} (5.37)

for $k$ large enough (depending on $\psi, \epsilon, T$).

**Proof:** Without loss of generality, we may assume that $x_k = 0$. Using a Galilean transform and the fact that $\xi_k$ is bounded, we may assume that $\xi_k = 0$ for all $k$.

First we can consider the case when $t_k = 0$ for all $k$ and we let $\bar{v}$ be the solution of (1.6) with initial data $\bar{\psi}$. Then by using Theorem 5.5, we see that there exists $T_0 = T_0(\psi, \epsilon)$ such that

$$\sup_{t \geq T_0} \|\bar{u}(t) - e^{it\Delta_x} \bar{v}_{0}^{\pm \infty}\|_{h^1L^2} + \|e^{it\Delta_x} \bar{v}_{0}^{\pm \infty}\|_{\bar{W}(\{t \geq T_0\})} \leq \epsilon,$$  \hspace{1cm} (5.38)

fix $T \geq T_0$. Applying Lemma 5.6, we see that, if $k$ is large enough,

$$\|U_k - V_{M_k}\|_{X_1^c(\{|t| \leq TM_k^{-2}\})} \leq \epsilon.$$

By using Stricharz estimates, (5.39) and Lemma 5.8, we see

$$\|e^{it\Delta_x v_{0}^{\pm \infty}} u_k(\pm TM_k^{-2})\|_{Z(\{t \geq TM_k^{-2}\})} \leq \epsilon.$$  \hspace{1cm} (5.39)

Now, Theorem 4.3 implies $U_k$ extends to a global solution $U_k \in X_1^c(\mathbb{R})$ satisfying (5.38).

Now we consider the other case when $M_k^{-2} |t_k| \to \infty$. For definiteness, we assume that $M_k^{-2} |t_k| \to +\infty$ and let $\bar{v}$ be the solutions to (1.6) satisfying

$$\|\bar{v}(t) - e^{it\Delta_x} \bar{v}_{0}^{\pm \infty}\|_{h^1L^2} \to 0$$
as $t \to -\infty$. Let $\psi' = \tilde{v}(0) \in H^{0,1}(\mathbb{R}^2 \times \mathbb{T}^2)$ and let $V_k$ be the solution of (1.1) with initial data $T^{ls}_{M_k} \psi$. Applying the first case of the proof to the frame $(N_k,0,0,0)$ and the family $V_k$ we conclude that:

$$||V_k(-t_k) - \Pi_{t_k,0} T^{ls}_{M_k} \psi||_{H^1(\mathbb{R}^2 \times \mathbb{T}^2)} \to 0$$

as $k \to \infty$. The conclusion of the proof now follows from Theorem 4.5 and the behavior of $V_k$.

6 Profile Decomposition

Then we can define three different kinds of profiles corresponding to different frames. We use frames to make the different profiles written in the same form as in [13, 17, 18, 19].

**Definition 6.1 (Frames and Profiles)** (1) We define a frame to be sequence $(N_k,t_k,p_k,\xi_k) \in 2^2 \times \mathbb{R} \times (\mathbb{R}^2 \times \mathbb{T}^2) \times \mathbb{R}$ which contains $4$ elements of information. And we can distinguish three types of profiles as follows.

a) A Euclidean frame is a sequence $F_e = (N_k,t_k,p_k,0)$ with $N_k \geq 1, N_k \to \infty, t_k \in \mathbb{R}, p_k \in \mathbb{R}^2 \times \mathbb{T}^2$.

b) A Large-scale frame is a sequence $F_{ls} = (M_k,t_k,p_k,\xi_k)$ with $M_k \leq 1, M_k \to 0, t_k \in \mathbb{R}, p_k \in \mathbb{R}^2 \times \mathbb{T}^2$.

c) A Scale-1 frame is a sequence $F_1 = (1,t_k,p_k,0)$ with $t_k \in \mathbb{R}, p_k \in \mathbb{R}^2 \times \mathbb{T}^2$.

(2) We say that two frames $(N_k,t_k,p_k,\xi_k)$ and $(M_k,s_k,q_k,\eta_k)$ are orthogonal if

$$\lim_{k \to +\infty} (\frac{N_k^2}{M_k^2} [t_k-s_k] + 2N_k^{-1}|\xi_k-\eta_k| + N_k|p_k-q_k| - 2(t_k-s_k)\xi_k|\eta_k|) = +\infty.$$

(3) We associate a profile defined as:

a) If $\mathcal{O} = (N_k,t_k,p_k,0)\in$ a Euclidean frame and for $\phi \in H^1(\mathbb{R}^4)$ we define the Euclidean profile associated to $(\phi, \mathcal{O})$ as the sequence $\tilde{\phi}_{\mathcal{O},k}$ with

$$\tilde{\phi}_{\mathcal{O},k} = \Pi_{t_k,p_k}(T^{ec}_{N_k}\phi)(x,y).$$

b) If $\mathcal{O} = (M_k,t_k,p_k,\xi_k)\in$ a large scale frame, if $p_k = (x_k,0)$ and if $\psi \in H^{0,1}(\mathbb{R}^2 \times \mathbb{T}^2)$, we define the large scale profile associated to $(\psi, \mathcal{O})$ as the sequence $\tilde{\psi}_{\mathcal{O},k}$ with

$$\tilde{\psi}_{\mathcal{O},k} = \Pi_{t_k,p_k}[e^{i\xi_k x} T_{M_k}^{ls} \psi(x,y)].$$

c) If $\mathcal{O} = (1,t_k,p_k,0)\in$ a scale one frame, if $W \in H^1(\mathbb{R}^2 \times \mathbb{T}^2)$, we define the scale one profile associated to $(W, \mathcal{O})$ as $\tilde{W}_{\mathcal{O},k}$ with

$$\tilde{W}_{\mathcal{O},k} = \Pi_{t_k,p_k} W.$$  

(4) Finally, we say that a sequence of functions $\{f_k\} \subset H^1(\mathbb{R}^2 \times \mathbb{T}^2)$ is absent from a frame $\mathcal{O}$ if, up to a subsequence:

$$\langle f_k, \tilde{\psi}_{\mathcal{O},k} \rangle_{H^1 \times H^1} \to 0$$

as $k \to \infty$ for any profile $\tilde{\psi}_{\mathcal{O},k}$ associated with $\mathcal{O}$. 

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Lemma 6.1 (Frame equivalence and orthogonality) (1) Suppose that \( \mathcal{O} \) and \( \mathcal{O}' \) are equivalent Euclidean (respectively large-scale or scale-one) frames, then there exists an isometry \( L \) of \( H^1(\mathbb{R}^4) \) (resp. \( H^{0,1}(\mathbb{R}^2 \times \mathbb{T}^2), H^1(\mathbb{R}^2 \times \mathbb{T}^2) \)) such that, for any profile generator \( \psi \in H^1(\mathbb{R}^4) \) (resp. \( H^{0,1}(\mathbb{R}^2 \times \mathbb{T}^2), H^1(\mathbb{R}^2 \times \mathbb{T}^2) \)), it holds that, up to a subsequence:

\[
\limsup_{k \to +\infty} \|L\psi_{\mathcal{O},k} - \tilde{\psi}_{\mathcal{O}',k}\|_{H^1(\mathbb{R}^2 \times \mathbb{T}^2)} = 0. \tag{6.1}
\]

(2) Suppose that \( \mathcal{O} \) and \( \mathcal{O}' \) are orthogonal frames and \( \tilde{\psi}_{\mathcal{O},k} \) and \( \tilde{\phi}_{\mathcal{O}',k} \) are two profiles associated with \( \mathcal{O} \) and \( \mathcal{O}' \) respectively. Then

\[
\lim_{k \to +\infty} \langle \tilde{\psi}_{\mathcal{O},k}, \tilde{\phi}_{\mathcal{O}',k} \rangle_{H^1(\mathbb{R}^2 \times \mathbb{T}^2)} = 0,
\]

\[
\lim_{k \to +\infty} \|\tilde{\psi}_{\mathcal{O},k}\|_{L^2} + \|\tilde{\phi}_{\mathcal{O}',k}\|_{L^2} = 0.
\]

(3) If \( \mathcal{O} \) is a Euclidean frame and \( \tilde{\psi}_{\mathcal{O},k} \), and \( \tilde{\phi}_{\mathcal{O}',k} \) are two profiles associated to \( \mathcal{O} \), then:

\[
\lim_{k \to +\infty} \langle \tilde{\psi}_{\mathcal{O},k}, \tilde{\phi}_{\mathcal{O}',k} \rangle_{H^1(\mathbb{R}^2 \times \mathbb{T}^2)} = \langle \psi, \phi \rangle_{H^1(\mathbb{R}^4)},
\]

\[
\lim_{k \to +\infty} \|\tilde{\psi}_{\mathcal{O},k}\|_{L^2} + \|\tilde{\phi}_{\mathcal{O}',k}\|_{L^2} = 0.
\]

(4) If \( \mathcal{O} \) is a scale-1 frame and \( \tilde{\psi}_{\mathcal{O},k} \), \( \tilde{\phi}_{\mathcal{O},k} \) are two profiles associated to \( \mathcal{O} \), then:

\[
\lim_{k \to +\infty} \langle \tilde{\psi}_{\mathcal{O},k}, \tilde{\phi}_{\mathcal{O},k} \rangle_{H^1(\mathbb{R}^2 \times \mathbb{T}^2)} = \langle \psi, \phi \rangle_{H^1(\mathbb{R}^2 \times \mathbb{T}^2)}.
\]

(5) If \( \mathcal{O} \) is a large-scale frame and \( \tilde{\psi}_{\mathcal{O},k} \), \( \tilde{\phi}_{\mathcal{O},k} \) are two profiles associated to \( \mathcal{O} \), then:

\[
\lim_{k \to +\infty} \|\tilde{\psi}_{\mathcal{O},k}\|_{L^4_{\mathcal{O},k}}(\mathbb{R}^2 \times \mathbb{T}^2) = 0,
\]

\[
\lim_{k \to +\infty} \langle \tilde{\psi}_{\mathcal{O},k}, \tilde{\phi}_{\mathcal{O},k} \rangle_{H^1(\mathbb{R}^2 \times \mathbb{T}^2)} = \langle \psi, \phi \rangle_{H^{0,1}(\mathbb{R}^2 \times \mathbb{T}^2)} + |\xi||\langle \psi, \phi \rangle_{L^2(\mathbb{R}^2 \times \mathbb{T}^2)}| \approx \langle \psi, \phi \rangle_{H^{0,1}(\mathbb{R}^2 \times \mathbb{T}^2)}.
\]

The proof is straightforward. (cf. [17, 18, 19])

The following theorem is the main theorem, i.e. profile decomposition.

**Theorem 6.2 (Profile Decomposition)** Assume \( \{\phi_k\}_k \) is a sequence of functions satisfying

\[
\sup_{k \geq 0} \|\phi_k\|_{L^2(\mathbb{R}^2 \times \mathbb{T}^2)} + \|\nabla_{x,y} \phi_k\|_{L^2(\mathbb{R}^2 \times \mathbb{T}^2)} < E \leq \infty. \tag{6.2}
\]

Then there exists a subsequence (for convenience which we also denote by \( \phi_k \)) , a family of Euclidean profiles \( \hat{\psi}_{\mathcal{O},k} \), a family of large scale profiles \( \hat{\psi}_{\mathcal{O}',k} \), a
family of scale-1 profiles $\tilde{W}_{\gamma,k}$ such that, for any $A \geq 1$ and any $k \geq 0$ in the subsequence

$$\phi_k(x,y) = \sum_{1 \leq \alpha \leq A} \phi_{\alpha,k}^\alpha + \sum_{1 \leq \beta \leq A} \psi_{\beta,k}^\beta + \sum_{1 \leq \gamma \leq A} \tilde{W}_{\gamma,k}^\gamma + R_k^A(x,y),$$  \hspace{1cm} (6.3)$$

with

$$\lim_{A \to \infty} \limsup_{k \to \infty} \| e^{it\Delta_{\gamma^2}} R_k^A \|_{L^2(\mathbb{R})} = 0.$$  \hspace{1cm} (6.4)$$

In addition, all the frames are pairwise orthogonal and we have the following orthogonality property:

$$M(\phi_k) = \sum_{1 \leq \beta \leq A} M(\psi^\beta) + \sum_{1 \leq \gamma \leq A} M(W^\gamma) + M(R_k^A) + o_{A,k \to +\infty}(1),$$

$$\| \nabla_{x,y} \phi_k \|_{L^2(\mathbb{R}^2 \times T^2)}^2 = \sum_{1 \leq \alpha \leq A} \| \varphi^\alpha \|_{H^1(\mathbb{R}^4)}^2 + \sum_{1 \leq \beta \leq A} \| \xi^\beta \|_{L^2(\mathbb{R}^2)}^2 M(\psi^\beta) + \| \nabla_y \psi^\beta \|_{L^2}^2 + \sum_{1 \leq \gamma \leq A} \| \nabla_{x,y} W^\gamma \|_{L^2(\mathbb{R}^2 \times T^2)}^2 + \| \nabla_{x,y} R_k^A \|_{L^2(\mathbb{R}^2 \times T^2)}^2 + o_{A,k \to +\infty}(1),$$

$$\| \phi_k \|_{L^4(\mathbb{R}^2 \times T^2)}^4 = \sum_{1 \leq \alpha \leq A} \| \varphi^\alpha \|_{L^4}^4 + \sum_{1 \leq \gamma \leq A} \| W^\gamma \|_{L^4}^4 + o_{A \to +\infty, k \to +\infty}(1),$$  \hspace{1cm} (6.5)$$

where $\xi^\beta_k = \lim_{k \to +\infty} \xi^\beta_k$, $o_{A,k \to +\infty}(1) \to 0$ as $k \to +\infty$ for each fixed $A$, and $o_{A \to +\infty, k \to +\infty}(1) \to 0$ in the ordered limit $\lim_{A \to +\infty} \lim_{k \to +\infty}$. 

As in [13, 17, 18], this follows from iteration of the following statement,

**Lemma 6.3** Let $\delta > 0$. Assume that $\phi_k$ is a sequence satisfying:

$$\sup_{k \geq 0} \| \phi_k \|_{L^2(\mathbb{R}^2 \times T^2)} + \| \nabla_{x,y} \phi_k \|_{L^2(\mathbb{R}^2 \times T^2)} < E \leq \infty.$$  \hspace{1cm} (6.6)$$

then there exists a subsequence (for convenience, which we also denote by $\phi_k$), $A = A(E, \delta)$ Euclidean profiles $\phi_{\alpha,k}^\alpha$, $A$ large scale profiles $\psi_{\beta,k}^\beta$, and $A$ scale 1 profiles $\tilde{W}_{\gamma,k}^\gamma$ such that, for any $k \geq 0$ in the subsequence

$$\phi_k(x,y) = \sum_{1 \leq \alpha \leq A} \phi_{\alpha,k}^\alpha + \sum_{1 \leq \beta \leq A} \psi_{\beta,k}^\beta + \sum_{1 \leq \gamma \leq A} \tilde{W}_{\gamma,k}^\gamma + R_k^A(x,y)$$  \hspace{1cm} (6.7)$$

with

$$\limsup_{k \to \infty} \sup_t \| e^{it\Delta_{\gamma^2}} R_k^A \|_{L^2(\mathbb{R})} \leq \delta.$$ 

Also, the frames are pairwise orthogonal and we have the following orthogonality property:

$$M(\phi_k) = \sum_{1 \leq \beta \leq A} M(\psi^\beta) + \sum_{1 \leq \gamma \leq A} M(W^\gamma) + M(R_k^A) + o_{k \to +\infty}(1),$$
\[ ||\nabla_{x,y}\phi_k||_{L^2(\mathbb{R}^2 \times \mathbb{T}^2)}^2 = \sum_{1 \leq \alpha \leq A} ||\varphi^\alpha||_{H^1(\mathbb{R}^4)}^2 + \sum_{1 \leq \beta \leq A} ||\xi^\beta||_{H^2(\mathbb{R}^4)}^2 M(\psi^\beta) + ||\nabla_y\psi^\beta||_{L^2}^2 \]
\[ + \sum_{1 \leq \gamma \leq A} ||\nabla_{x,y}W^\gamma||_{L^2(\mathbb{R}^2 \times \mathbb{T}^2)}^2 + ||\nabla_{x,y}R_k^A||_{L^2(\mathbb{R}^2 \times \mathbb{T}^2)}^2 + o_k \to +\infty(1), \]

\[ ||\phi_k||_{L^4(\mathbb{R}^2 \times \mathbb{T}^2)}^4 = \sum_{1 \leq \alpha \leq A} ||\varphi^\alpha||_{L^4(\mathbb{R}^4)}^4 + \sum_{1 \leq \gamma \leq A} ||W^\gamma||_{L^4(\mathbb{R}^2 \times \mathbb{T}^2)}^4 \]
\[ + ||R_k^A||_{L^4(\mathbb{R}^2 \times \mathbb{T}^2)}^4 + o_k \to +\infty(1), \]

where \(o_k \to +\infty(1) \to 0\) as \(k \to +\infty\).

The proof will be completed in two steps: first, we extract the Euclidean and scale-1 profiles by studying the defects of compactness of the Strichartz estimate. This extraction leaves only sequences whose linear flow has small critical Besov scale-1 profiles by studying the defects of compactness of the Strichartz estimate. Proof: We first claim that if \(\Lambda_\infty(\{f_k\}) \geq v\), then there exists a frame \(O\) and an associated profile \(\psi_{O,k}\) satisfying

\[ \limsup_{k \to \infty} ||\tilde{\psi}_{O,k}||_{H^1} \lesssim 1, \]  

(6.11)

and

\[ \limsup_{k \to \infty} |\langle f_k, \tilde{\psi}_{O,k} \rangle|_{H^1 \times H^1} \gtrsim v. \]  

(6.12)

In addition, if \(f_k\) was absent from a family of frames \(O\), then \(O\) is orthogonal to all the previous frames.
Let us prove the claim above first. By assumption, up to extracting a subsequence, there exists a sequence \( \{N_k, t_k, (x_k, y_k)\} \) such that, for all \( k \)
\[
\frac{1}{2} v \leq N_k^{-1} |(e^{it_k \Delta} P_{N_k} f_k)(x_k, y_k)| \leq |(f_k, N_k^{-1} (e^{it_k \Delta} P_{N_k}) \delta_{(x_k, y_k)})|_{H^1 \times H^{-1}}.
\]

Let us consider two situations.

First, assume that \( N_k \) remains bounded, then up to a subsequence, we can assume that \( N_k \to N_\infty \) and since \( N_k \) is dyadic, we may assume that \( N_k = N_\infty \) for all \( k \). In this case, we define the scale-1 profile \( \mathcal{O} = (1, t_k, (x_k, y_k), 0) \) and
\[
\psi = (1 - \Delta)^{-1} N_k^{-1} P_{N_k} \delta_{(0,0)}.
\]

Now assume that \( N_k \to +\infty \) and we define the Euclidean frame \( \mathcal{O} = (N_k, t_k, (x_k, y_k), 0) \) and the function:
\[
\psi = \mathcal{F}_{\mathbb{R}^4}^{-1} (|\xi|^2 |\eta^4(\xi) - \eta^4(2\xi)|) \in H^1(\mathbb{R}^4).
\]

By using Poisson Summation Formula, we can prove that
\[
\lim_{k \to +\infty} ||(1 - \Delta) T_{N_k} \psi - N_k^{-1} P_{N_k} \delta_0||_{L^4} = 0.
\]

Thus, by definition, we have \( ||(1 - \Delta) T_{N_k} \psi - N_k^{-1} P_{N_k} \delta_0||_{H^{-1}} \to 0 \) and then we conclude,
\[
\frac{1}{2} v \lesssim |(f_k, N_k^{-1} (e^{it_k \Delta} P_{N_k}) \delta_{(x_k, y_k)})| \lesssim |(f_k, (1 - \Delta) \tilde{\psi}_{\mathcal{O}, k})|_{H^1 \times H^{-1}}.
\]

The last claim about orthogonality \( \mathcal{O} \) with \( \mathcal{O}^\alpha \) follows from Lemma 6.2 and the existence of a nonzero scalar product in (6.12).

Now continuing with the sequence \( \{f_k\} \) as above, if the frame selected was a Scale-1 frame, we consider
\[
g_k(x, y) := e^{it_k \Delta} f_k((x, y) + p_k) = \Pi_{-(t_k, p_k)} f_k.
\]

This is a bounded sequence in \( H^1 \), up to a subsequence, we assume it converges weakly to \( \varphi \in H^1 \). We then define the profile corresponding to \( \mathcal{O} \) as \( \tilde{\psi}_{\mathcal{O}, k} \). By its definition and (6.6), \( \varphi \) has norm smaller than \( E \). Also, we have,
\[
\frac{v}{2} \lesssim \lim_{k \to +\infty} \langle f_k, \psi_{\mathcal{O}, k} \rangle_{H^1 \times H^1} \lesssim \lim_{k \to +\infty} \langle g_k, \psi \rangle_{H^1 \times H^1} = \langle \varphi, \psi \rangle_{H^1 \times H^1}.
\]

Consequently, we get that
\[
||\varphi||_{H^1} \gtrsim v. \tag{6.13}
\]

We also observe that since \( g_k - \psi \) weakly converges to 0 in \( H^1 \), there holds that
\[
||A f_k||_{L^2}^2 = ||A g_k||_{L^2}^2 = ||A g_k - \varphi||_{L^2}^2 + ||A \varphi||_{L^2}^2 + o_k(1) = ||A (f_k - \varphi_{\mathcal{O}, k})||_{L^2}^2 + ||A \varphi||_{L^2}^2 + o_k(1) \tag{6.14}
\]
for $A = 1$ or $A = \nabla_{x,y}$. The situation for Euclidean frame is similar.

For the Euclidean case, for $k$ large enough, we consider

$$\varphi_k(y) = N_k^{-1} \eta^4 (y/N_k)(\Pi_{-t_k,x_k}f_k)(\Psi(y/N_k)), \quad y \in \mathbb{R}^4.$$  

This is a bounded sequence in $\dot{H}^1(\mathbb{R}^4)$. We can extract a subsequence that converges weakly to a function $\varphi \in \dot{H}^1(\mathbb{R}^4)$ satisfying

$$||\varphi||_{\dot{H}^1(\mathbb{R}^4)} \lesssim 1.$$  

Now, let $\gamma \in C_0^\infty(\mathbb{R}^4)$; for $k$ large enough,

$$\langle f_k, \tilde{\gamma}O,k \rangle_{H^1 \times H^1(\mathbb{R}^2 \times T^2)} = \langle \varphi, \gamma \rangle_{H^1 \times H^1(\mathbb{R}^4)} + o_k(1).$$

We conclude that

$$||\varphi||_{\dot{H}^1(\mathbb{R}^4)} \gtrsim v \quad (6.15)$$

and that

$$h_k = f_k - \tilde{\varphi}O,k$$

is absent from the frame $\mathcal{O}$. Using Lemma 6.2, we see that

$$||h_k||_{L^2}^2 = ||f_k||_{L^2}^2 + o_k(1),$$

$$||\nabla_{x,y}h_k||_{L^2}^2 = ||\nabla_{x,y}f_k||_{L^2}^2 + ||\nabla\varphi||_{L^2}^2 - 2(\nabla f_k, \nabla \tilde{\varphi}O,k)_{L^2 \times L^2}$$

$$= ||\nabla_{x,y}f_k||_{L^2}^2 - ||\nabla\varphi||_{L^2}^2 + o_k(1).$$

Defining $f_0^0 = \phi_k$ and for each $\alpha$, $f_{k+1}^\alpha = f_k^\alpha - \tilde{\phi}O,\alpha,k$ where is the profile given based on the considerations above; iterating this claim at most $O(v^{-2})$ times and replacing $\phi_k$ by

$$\phi_k = \phi_k - \sum_{\alpha} \tilde{\phi}O,\alpha,k.$$  

We obtain that $\{\phi_k\}_k$ satisfies

$$\limsup_{k \to +\infty} ||\phi_k||_{H^1} \leq E < +\infty \quad (6.16)$$

and

$$\limsup_{k \to +\infty} \sup_{N \geq 1, t, x, y} N^{-1} |(e^{it\Delta}P_N\phi_k')(x, y)| < v. \quad (6.17)$$

This proves the Lemma 6.4.

**Proof of Lemma 6.3:** First for $v = v(\delta, E)$ to be decided later we use Lemma 6.5 and extract some profiles. Then, we replace $\phi_k$ by $\phi_k'$, thus ensuring that (6.10) holds for the sequence $\{\phi_k\}_k$. We now consider

$$\Lambda_0(\{\phi_k'\}) = \limsup_{k \to +\infty} ||e^{it\Delta} \phi_k'||_{Z(\mathbb{R})}.$$  

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If $\Lambda_0(\{\phi_k^\prime\}) < \delta$, we may set $R_k^A = \phi_k'$ for all $k$ and we get Lemma 6.4.

Now we claim that if $\{\phi_k'^\prime\}_k$ satisfies $\Lambda_0(\{\phi_k'^\prime\}_k) \geq \delta$ and $\{\phi_k'^\prime\}_k$ is orthogonal to a family of frames $O^\alpha$, $1 \leq \alpha \leq A$, then there exists a frame $O$ orthogonal to $O^\alpha$ and an associated profile $\tilde{\varphi}_{O,k}$ such that, after passing to a subsequence, we have that

$$\limsup_{k \to +\infty} \|\tilde{\varphi}_{O,k}\|_{H^1(\mathbb{R}^2 \times \mathbb{T}^2)} \gtrsim \delta \quad 1 \phi_k'-\tilde{\varphi}_{O,k} \text{ is absent from } O. \quad (6.18)$$

Once the claim is established then the end of the proof follows by iterating the extraction process as in Lemma 6.4. Thus, now we will focus on the claim.

Since $\Lambda_0(\{\phi_k'^\prime\}) \geq \delta$, by Hölder’s inequality and Strichartz estimates (3.2), we have that,

$$c_k^N = N^{\frac{1}{2}}\|e^{it\Delta}P_N\phi_k'^\prime\|_{L^4(\mathbb{R}^2 \times \mathbb{T}^2 \times \mathbb{R})},$$

$$\|c_k^N\|_{L^2} \leq \|c_k^N\|_{L^4} \|c_k^N\|_{L^\infty} \leq (\sum_{N \geq 1} N^2\|P_N\phi_k'^\prime\|_{L^2}^2)^{\frac{1}{2}} (\sup_N c_k^N)^{\frac{1}{2}}.$$

Using (6.16), we obtain that there exists a sequence of scales $N_k \geq 1$ such that,

$$\left(\frac{\delta}{2}\right)^2 < \Lambda_0(\{\phi_k'^\prime\})^2 \leq E_1^\frac{1}{2} N_k^\frac{1}{2} \|e^{it\Delta}P_N\phi_k'^\prime\|_{L^4(\mathbb{R}^2 \times \mathbb{T}^2 \times \mathbb{R})}.$$

We conclude that there exists a sequence $h_k \in C_\infty^\infty(\mathbb{R}^2 \times \mathbb{T}^2 \times \mathbb{R})$ such that

$$1 \leq \|h_k\|_{L^\infty} \leq 2,$$

$$\left(\frac{\delta}{2}\right)^2 E_1^{-\frac{1}{2}} N_k^{-\frac{1}{2}} \leq \langle h_k, e^{it\Delta}P_N\phi_k'^\prime \rangle_{L^2 \times L^2}.$$

Now for a given threshold $B$, we introduce the partition function $\chi_B(\gamma)$ satisfies

$$\chi_B(\gamma) = 1 \text{ if } \|h_k\|_{L^4(\mathbb{R}^2 \times \mathbb{T}^2 \times I_\gamma)} \geq B \quad \chi_B(\gamma) = 0 \text{ otherwise}$$

And we decompose as follows,

$$h_k(x,y,t) = h_k^B + h_k^<B = h_k(x,y,t)\chi_B(\left[\frac{t}{2\pi}\right]) + h_k(x,y,t)(1-\chi_B(\left[\frac{t}{2\pi}\right]))$$

so we have

$$\|h_k\|_{L^4(\mathbb{R}^2 \times \mathbb{T}^2 \times \mathbb{R})} \leq \|h_k^B\|_{L^4(\mathbb{R}^2 \times \mathbb{T}^2 \times \mathbb{R})} + \|h_k^<B\|_{L^4(\mathbb{R}^2 \times \mathbb{T}^2 \times \mathbb{R})},$$

$$\sup_{\gamma} \|h_k^<B\|_{L^4(\mathbb{R}^2 \times \mathbb{T}^2 \times I_\gamma)} \leq B.$$
Using Strichartz estimates, we have that for \( \frac{10}{3} < p_1 < 4 \):

\[
\limsup_{k \to +\infty} N_k^{\frac{1}{p_1} - 1} \left\| e^{it\Delta} P_{N_k} \phi_k' \right\|_{L^\infty_t L^{p_1}(\mathbb{R}^3)} \lesssim \limsup_{k \to +\infty} \| \phi_k' \|_{H^1} \lesssim E.
\]

Interpolating with (6.17), we obtain that

\[
\limsup_{k \to +\infty} N_k^{\frac{1}{p_1}} \left\| e^{it\Delta} P_{N_k} \phi_k' \right\|_{L^\infty_t L^4(\mathbb{R}^3)} \lesssim E^{\frac{2}{p_1}} v^{\frac{4-p_1}{4}}.
\]

An observation is that:

\[
|\text{supp}_\gamma(h_k^B)| \leq \left( \frac{2}{B} \right)^\frac{4}{3}.
\]

Thus by Hölder’s inequality in \( \gamma \),

\[
\langle e^{it\Delta} P_{N_k} \phi_k', h_k^B \rangle \leq \| e^{it\Delta} P_{N_k} \phi_k' \|_{L^\infty_t L^4(\mathbb{R}^3)} \| h_k^B \|_{L^4(\mathbb{R}^3)} \left( \frac{2}{B} \right)^{\frac{4-p_1}{4}}
\]

\[
\lesssim \left( \frac{1}{B} \right)^{\frac{4-p_1}{8}} N_k^{\frac{1}{4}} E^{\frac{2}{p_1}} v^{\frac{4-p_1}{4}}.
\]

Eventually, for any fixed \( B > 0 \), we can choose \( v = v(B, \delta) \) such that

\[
v^{\frac{4-p_1}{4}} = c \delta^2 E^{-\frac{1}{2}} B^{\frac{4-p_1}{8}},
\]

for \( c > 0 \) a constant sufficiently small so that,

\[
\left( \frac{\delta}{2} \right)^2 E^{-\frac{1}{2}} \leq 2 \langle h_k^B, e^{it\Delta} P_{N_k} \phi_k' \rangle_{L^2 \times L^2} \leq 2E \| h_k^B \|_{L^2(\mathbb{R}^3)} \left( \frac{2}{B} \right)^{\frac{4-p_1}{4}}. \tag{6.19}
\]

Using Strichartz estimate (3.3), we obtain that

\[
\| \int_{\mathbb{R}} e^{-is\Delta} P_{N_k} h_k^B(x, y, s) ds \|_{L^2_{x,y} L^\infty_{s}([0, T])} \lesssim N_k^\frac{7}{8} \left\| h_k^B \right\|_{L^4_{x,y} L^\infty_{s}([0, T])} + N_k^{\frac{1}{2}} \left\| h_k^B \right\|_{L^8_{x,y} L^\infty_{s}([0, T])}
\]

\[
\lesssim N_k^\frac{7}{8} \left\| h_k^B \right\|_{L^4_{x,y} L^\infty_{s}([0, T])} B^\frac{1}{2} + N_k^{\frac{1}{2}} \left\| h_k^B \right\|_{L^4_{x,y} L^\infty_{s}([0, T])} B^\frac{1}{2}
\]

Choosing \( B \) such that

\[
B^\frac{1}{2} = \epsilon E^{-\frac{1}{2}} \delta^2.
\]

for some constant \( \epsilon > 0 \) small enough and plugging into (6.19), we obtain that

\[
\delta^2 \lesssim \epsilon \delta^2 + E^{\frac{3}{2}} N_k^{-\frac{1}{3}}.
\]

If \( \epsilon > 0 \) is small enough, from the estimate above we can obtain a uniform bound for \( N_k \) and the bound only relies on \( E \) and \( \delta \)

\[
N_k \lesssim_{E, \delta} 1. \tag{6.20}
\]
To sum up, we have showed that if $\Lambda_0(\{\phi'_k\}) > \delta$, then there exists a sequence of scales $N_k$ satisfying (6.20) and such that
\[ \| P_{N_k} e^{it\Delta} \phi_k' \|_{L^4_x,\gamma_t(R^2 \times T^2 \times \mathbb{R})} > c(\delta, E) \]
for some $c(\delta, E) > 0$ which denotes a small positive constant depending only on $\delta$ and $E$.

Now, we write
\[ P_{N_k} e^{it\Delta} \phi_k'(x, y) = \sum_{z=(z_1, z_2) \in \mathbb{Z}^2: |z_1| \leq N_k} e^{-it|z|^2} e^{i(z,y)} \eta_{N_k}^2(z) [e^{it\Delta} \phi''_{z,k}(x, y)] \]
where
\[ \phi''_{z,k}(x) = \frac{1}{(2\pi)^2} \int_{T^2} P_{N_k} \phi_k'(x, y) e^{-i(z,y)} dy. \]
Extracting a subsequence, we conclude that there exists $z$ such that, for all $k$,
\[ \text{supp}(F_x \phi''_{z,k}) \subset [-3N_k, 3N_k]^2, \]
\[ \| e^{it\Delta} \phi''_{z,k} \|_{L^4_x,\gamma_t(R^2 \times \mathbb{R})} > c(\delta, E), \quad (6.21) \]
\[ \| \phi''_{z,k} \|_{H^1(R^2)} \leq M. \]

Now we need to apply the following result from [1, 5, 28]. (We use the version as [28, Corollary A.3]):

**Theorem 6.5** For any $M$, $c(\delta, E) > 0$, there exists a finite set $C \subset L^2(\mathbb{R}^2)$ of functions satisfying,
\[ \| v \|_{L^1(\mathbb{R}^2)} = 1, \quad \| v \|_{L^2(\mathbb{R}^2)} \leq S(E, M, \delta), \quad \forall v \in C, \quad (6.22) \]
and $\kappa(M, c(\delta, E)) > 0$ such that whenever $u \in L^2(\mathbb{R}^2)$ obeys the bounds
\[ \| u \|_{L^2(\mathbb{R}^2)} \leq M, \quad \| e^{it\Delta} u \|_{L^4_x,\gamma_t(R^2 \times \mathbb{R})} \geq c(\delta, E). \]
Then there exists $v \in C$ and $(\lambda, \xi_0, t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2$ such that
\[ \langle u, v \rangle_{L^2 \times L^2(\mathbb{R}^2)} \geq \kappa, \quad v(x) = \lambda e^{ix\xi_0} e^{it_0 \Delta x} v(\lambda(x - x_0)). \]

Using this Theorem, after extraction, there exists a function $v \in L^2(\mathbb{R}^2)$ satisfying (6.22) and a sequence $(\lambda_k, \xi_k, t_k, x_k) \in \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2$ such that
\[ \langle \phi''_k, v_k \rangle \geq \kappa, \quad v_k(x) = \lambda_k e^{ix\xi_k} e^{it_k \Delta x} v(\lambda_k(x - x_k)). \quad (6.23) \]

We may assume $v$ has a compactly supported Fourier transform. Also we claim that $\lambda_k$ and $|\xi_k|$ remain bounded. By computation,
\[ F_{\mathbb{R}^2} v_k(\xi) = \lambda_k^{-1} e^{ix\xi_k} e^{-ix\xi_k} [F_{\mathbb{R}^2} e^{it_k \Delta x} v](\frac{\xi - \xi_k}{\lambda_k}), \]

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and consider the sequence

Assume now that \( \lambda_k \) remains bounded below. Then, up to extracting a subsequence, we may assume that \( \lambda_k \to \lambda_\infty \in (0, \infty) \). Similarly, we may assume that \( \xi_k \to \xi_\infty \). Then, setting

\[
\tilde{\psi}(x, y) = e^{i(x,y)} e^{ix\xi_\infty \lambda_\infty v(\lambda_\infty x)}, \quad t'_k = -\lambda_k^{-2} t_k, \quad x'_k = x_k + 2t'_k \xi_k
\]

and defining the frame \( \mathcal{O} = \{(1, t'_k, (x'_k, 0), 0)_k\} \), we see from (6.23) that

\[
\kappa \leq \langle P_{N_k} \tilde{\Phi}_k, e^{i(x,y)} e^{ix\xi_\infty \lambda_\infty v(\lambda_\infty x-k)} \rangle
\]

\[
\leq \langle \tilde{\Phi}_k, e^{i(x,M \xi_k - t'_k (|z|^2 - |\xi_\infty|^2))} P_{N_k} e^{-it'k \Delta_{x_2}^2} \tilde{\psi}(x-x'_k) \rangle + \alpha_k(1).
\]

Since \( 1 \leq N_k \leq 1 \), up to a subsequence, we may assume that \( N_k = N_\infty \). As a result, setting \( \psi = P_{N_\infty} \psi \), we see the scale-one profile \( \tilde{\psi}_{\mathcal{O}, k} \) satisfies (6.11) and (6.12). We also conclude that \( \mathcal{O} \) is orthogonal to \( \mathcal{O}^\alpha \), \( 1 \leq \alpha \leq A \). As in the proof of Lemma 6.4, we find \( \varphi \) satisfying (6.18).

Assume now that \( \lambda_k \to 0 \). Let \( M_k \) be a dyadic number such that \( 1 \leq \lambda_k^{-1} M_k \leq 2 \) and consider the sequence

\[
\Phi_k(x,y) = M_k^{-1} e^{-itk\Delta_x} [e^{i(a_k/x)} \Phi_k(x + \frac{x}{M_k}, y)].
\]

We have that

\[
\|\Phi_k\|_{L^2(\mathbb{R}^2 \times T^2)} + \|\nabla_y \Phi_k\|_{L^2(\mathbb{R}^2 \times T^2)} \leq M^2 + E^2.
\]

Up to a subsequence, we may assume that \( \Phi_k \to \Phi \) in \( H^{0,1}(\mathbb{R}^2 \times T^2) \). We define

\[
t'_k = -M_k^{-2} t_k, \quad \xi'_k = -\xi_k, \quad x'_k = x_k + 2t'_k \xi_k,
\]

and \( \mathcal{O} = (M_k, t'_k, (x'_k, 0), \xi'_k) \). Then we obtain that \( \mathcal{O} \) is orthogonal to \( \mathcal{O}^\alpha \) from (6.23) and we see from the definition of \( \Phi_k \) that (6.18) holds with \( \varphi = e^{i\theta_k} \Phi \) for some \( \theta_k \in \mathbb{R}/\mathbb{Z} \).

This finishes the proof of Lemma 6.3.

### 7 Induction on Energy

Then we are now ready to prove the main theorem. We follow an induction on energy method formalized in [20, 21]. Define

\[
\Lambda(L) = \sup\{\|u\|_{Z^{1}(I)}^2 : u \in X_{loc}^{1}(I), E(u) + \frac{1}{2} M(u) \leq L\}
\]

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where the supremum is taken over all strong solutions of full energy less than $L$. By the local theory, this is sublinear in $L$ and finite for $L$ sufficiently small. We also define

$$L_{\text{max}} = \sup\{ L : \Lambda(L) < +\infty \}.$$  

We want to show that $L_{\text{max}} = +\infty$. That is our goal. If $L_{\text{max}} = +\infty$ holds, we can extend the small data result to our main theorem, i.e. Theorem 1.2 by using the Extension Theorem (Theorem 4.4).

The key proposition is:

**Theorem 7.1** Assume that $L_{\text{max}} < +\infty$ and the Conjecture 1.2 holds for $L_{\text{max}}$. Let $\{t_k\}_k, \{a_k\}_k, \{b_k\}_k$ be arbitrary sequences of real numbers and $\{u_k\}$ be a sequence of solutions to (1.1) such that $u_k \in X^1_{c,\text{loc}}(t_k - a_k, t_k + b_k)$ and satisfying

$$L(u_k) \to L_{\text{max}}, \quad ||u_k||_{Z(t_k-a_k,t_k)} \to +\infty, \quad ||u_k||_{Z(t_k,t_k+b_k)} \to +\infty. \quad (7.1)$$

Then passing to a subsequence, there exists a sequence $x_k \in \mathbb{R}^2$ and $\omega \in H^1(\mathbb{R}^2 \times \mathbb{T}^2)$ such that

$$\omega_k(x,y) = u_k(x - x_k, y, t_k) \to \omega \quad (7.2)$$

strongly in $H^1(\mathbb{R}^2 \times \mathbb{T}^2)$.

We will give the proof later by using Theorem 6.2 (profile decomposition). Based on Theorem 7.1, we can prove:

**Corollary 7.2** Assume that $L_{\text{max}} < +\infty$ and the Conjecture 1.2 holds for some $E_{\text{max}}^s \geq L_{\text{max}}$. Then there exists $u \in X^1_{c,\text{loc}}(\mathbb{R})$ solving (1.1) and a Lipschitz function $\underline{x} : \mathbb{R} \to \mathbb{R}^2$ such that $L(u) = L_{\text{max}}$ and

$$\sup_{t \in \mathbb{R}} |\underline{x}'(t)| \lesssim 1 \quad (7.3)$$

$$(u(x - \underline{x}(t), y, t) : t \in \mathbb{R}) \text{ is precompact in } H^1(\mathbb{R}^2 \times \mathbb{T}^2).$$

**Proof:** Assuming that $L_{\text{max}} < +\infty$, we can find a sequence of solutions of (1.1) $u_k$ satisfying (7.1). Apply Theorem 7.1 we can extract a subsequence and obtain a sequence $x_k$ such that Corollary 7.2 holds for some $\omega \in H^1(\mathbb{R}^2 \times \mathbb{T}^2)$. Thus $L(\omega) = L_{\text{max}} < +\infty$. Let $U \in C(I : H^1)$ be the maximal strong solution of (1.1) with initial data $\omega$, defined on $I = (-a_{\infty}, b_{\infty})$. According to local theory and (7.1),

$$||U||_{Z(-a_{\infty},0)} = ||U||_{Z(0,b_{\infty})} = +\infty. \quad (7.4)$$

We claim that there exists $\kappa > 0$ such that for all $t \in I$,

$$||U||_{Z(t-2\kappa, t+2\kappa)} \leq 2. \quad (7.5)$$

If $U$ is global, $a_{\infty} = b_{\infty} = +\infty$. 40
Assume if (7.5) is not true. Then there exists a sequence \( t_k \in I \) such that
\[
||U||_{(t_k-\frac{1}{t_k+1}, t_k+\frac{1}{t_k+1})} \geq 2.
\]
We can apply Theorem 7.1 to the sequence \( U(t_k) \) and obtain that, up to a subsequence, there exists \( x_k \) such that \( U_k(x, y) = \omega(x, y) \) in \( H^1 \).

Let \( W \) be the nonlinear solution of (1.1) with initial data \( \omega' \). By the local theory, we know there exists \( \kappa^* > 0 \) such that
\[
||W||_{Z(-\kappa^*, \kappa^*)} \leq 1
\]
and by the stability theory, we obtain that, for \( k \) large enough
\[
||U||_{Z(t_k-\kappa^*, t_k+\kappa^*)} \leq 2
\]
which gives a contradiction for \( k \) large enough.

Now we prove (7.3). We define the sequence of times \( t_k = k\kappa \) and for each \( t_k \), we define \( x_k \) and \( R_k \) such that
\[
\frac{1}{2} \int_{|x-x_k| \leq R_k} \int_{\mathbb{T}^2} |u(t_k, x, y)|^2 + |\nabla u(t_k, x, y)|^2 + \frac{1}{2} |u(t_k, x, y)|^4 \, dx \, dy = \frac{99}{100} L_{\max} \tag{7.6}
\]
and \( R_k \) is the minimal with this property. While \( x_k \) is not necessarily unique, we claim that there exists \( D \) such that for all \( k \)
\[
R_k \leq D, \quad |x_k - x_{k+1}| \leq D \tag{7.7}
\]
and that
\[
\{ u(t_k + s, x - x_k) \}, k \in \mathbb{Z}, s \in (-\kappa, \kappa) \text{ is precompact in } H^1(\mathbb{R}^2 \times \mathbb{T}^2). \tag{7.8}
\]
The fact that the \( R_k \) are uniformly bounded comes from the compactness up to translations of \( \{ u(t_k) \}_k \). Assume that \( \{ v_k(x, y) = u(x - x_k, y, t_k) \} \) was not precompact in \( H^1(\mathbb{R}^2 \times \mathbb{T}^2) \). In that case, there exists \( \epsilon > 0 \) and a subsequence \( k' \) such that for all \( k_1', k_2' \),
\[
||v_{k_1'} - v_{k_2'}||_{H^1(\mathbb{R}^2 \times \mathbb{T}^2)} > \epsilon. \tag{7.9}
\]
Now apply Theorem 7.1, we see that there exists a sequence \( x_k' \) and a subsequence of \( k' \) such that
\[
v_{k_1'}(x - x_k', y) \to \omega(x, y) \text{ strongly in } H^1(\mathbb{R}^2 \times \mathbb{T}^2).
\]
From (7.6), \( \{ x_k' \}_k \) remains bounded, so that the convergence of \( v_{k_1'} \) contradicts (7.9). Using (7.5) and the precompactness of \( \{ v_k \}_k \), we obtain (7.8). Similarly, this implies the second statement in (7.7). Choose \( x(t) \) to be a Lipschitz function satisfying \( x(t_k) = x_k \), we obtain (7.3). This ends the proof.

Now we can finish the proof of our main theorem.
Theorem 7.3 Assume that $u$ satisfies the conclusion of Corollary 7.2, then $u = 0$. In particular, $L^* \geq E^{*\alpha}_{\text{max}}$.

Proof: Assume $u \neq 0$. Then, from the compactness property, we see that there exists $\rho > 0$ such that
\[
\inf_{t \in T} \min(||u(t)||_{L^4_x}, ||u(t)||_{L^2_x}) \geq \rho. \tag{7.10}
\]
Now let us consider the conserved momentum
\[
P(u) = Im \int_{\mathbb{R}^2 \times T^2} \bar{u}(x, y, t) \partial_x u(x, y, t) dxdy.
\]
Considering the Galilean transform
\[
v(z, t) = e^{-i|\xi_0|^2 t + i \langle z, \xi_0 \rangle} u(z - 2\xi_0 t, t),
\]
let
\[
\xi_0 = -\frac{P(u)}{M(u)}
\]
without loss of generality, we can assume that
\[
P(u) = 0. \tag{7.11}
\]
Then we define the Virial action by
\[
A_R(t) = \int_{\mathbb{R}^2 \times T^2} \chi_R(x_1 - \bar{z}_1(t))(x_1 - \bar{z}_1(t)) Im[\bar{u}(x, y, t) \partial_x u(x, y, t)] dxdy
\]
for $\chi_R(x) = \chi(x/R)$ and $\chi$ satisfies $\chi(x) = 1$ when $|x| \leq 1$ and $\chi(x) = 0$ when $|x| \geq 2 \ (x \in \mathbb{R})$.

On one hand, clearly
\[
\sup_t |A_R(t)| \lesssim R. \tag{7.12}
\]
On the other hand, we compute that
\[
\frac{d}{dt} A_R = -\bar{z}'_1(t) Im \int_{\mathbb{R}^2 \times T^2} \bar{u}(x, y, t) \partial_x u(x, y, t) dxdy
- \bar{z}'_1(t) \int_{\mathbb{R}^2 \times T^2} \{(\chi')_R(x - \bar{z}_1(t))(x - \bar{z}_1(t)) \frac{x - \bar{z}_1(t)}{R} - (1 - \chi_R(x - \bar{z}_1(t)))\} Im[\bar{u}(x, y, t) \partial_x u(x, y, t)] dxdy
+ \int_{\mathbb{R}^2 \times T^2} \chi_R(x_1 - \bar{z}_1(t))(x_1 - \bar{z}_1(t)) Im[\bar{u}(x, y, t) \partial_x u(x, y, t)] dxdy.
\]
The first term will vanish automatically based on the assumption and the second term can be bounded by
\[
\int_{\{|x - \bar{z}(t)| \geq R\}} \int_{T^2} \left|u(x, y, t)^2 + |\nabla u(x, y, t)|^2\right| dxdy = O_R(t),
\]
\[ \sup_t O_R(t) \to 0 \quad \text{as} \quad R \to +\infty. \]

Notice that
\[ \partial_t \text{Im}[\bar{u}(x,y,t)\partial_x u(x,y,t)] = \partial_x \Delta \frac{|u|^2}{2} - 2 \text{div} \{ \text{Re} \partial_x \bar{u} \nabla u \} - \frac{1}{4} \partial_{x_1} |u|^4. \]

For the last term, we have
\[ \frac{d}{dt} A_R = \int_{\mathbb{R}^2 \times T^2} \chi_R(x_1 - \bar{x}_1(t)) \left[ \frac{1}{4} |u(x,y,t)|^4 + \frac{1}{2} |\partial_x u(x,y,t)|^2 \right] dx \, dy \\
+ \int_{\mathbb{R}^2 \times T^2} \chi_R'(x_1 - \bar{x}_1(t)) \frac{x_1 - \bar{x}_1(t)}{R} \left[ \frac{1}{4} |u(x,y,t)|^4 + \frac{1}{2} |\partial_x u(x,y,t)|^2 \right] dx \, dy \\
- \int_{\mathbb{R}^2 \times T^2} \frac{|u(x,y,t)|^2}{2} \partial_{x_1}^2 \chi_R(x_1 - \bar{x}_1(t))(x_1 - \bar{x}_1(t)) dx \, dy + O_R(t) \\
= \int_{\mathbb{R}^2 \times T^2} \left[ \frac{1}{4} |u(x,y,t)|^4 + \frac{1}{2} |\partial_x u(x,y,t)|^2 \right] dx \, dy + O_R(t). \]

Integrating this equality, we obtain
\[ |A_R(t) - A_R(0)| \geq C t \rho - t \sup_t O_R(t). \]

Taking \( R \) sufficiently large enough, we obtain, when \( t \) is sufficiently large, there is a contradiction. This finishes the proof of Theorem 7.3.

**Proof of Theorem 7.1:** Without loss of generality, we may assume that \( t_k = 0 \). We apply the profile decomposition, i.e. Theorem 6.3 to the sequence \( \{u_k(0)\} \) which is bounded in \( H^1(\mathbb{R}^2 \times T^2) \). Then we get:
\[ u_k(0) = \sum_{1 \leq \alpha \leq J} \tilde{\varphi}_{\alpha,k} + \sum_{1 \leq \beta \leq J} \tilde{\psi}_{\beta,k} + \sum_{1 \leq \gamma \leq J} \tilde{W}_{\gamma,k} + R_k^J(x,y). \]

There are three cases to be discussed: no profile, one profile and multiple profiles.

**Case 1:** There is no profiles. So if we take \( J \) sufficiently large, we will have:
\[ ||e^{it\Delta} u_k(0)||_{Z(\mathbb{R})} = ||e^{it\Delta} R_k^J||_{Z(\mathbb{R})} \leq \delta_0/2 \]
for \( k \) sufficiently large, where \( \delta_0 \) is given in Theorem 4.3. Then we know that \( u_k \) can be extended on \( \mathbb{R} \) and that
\[ \lim_{k \to +\infty} ||u_k||_{Z(\mathbb{R})} \leq \delta_0. \]

It is a contradiction. Hence, we consider the situation when there are at least one profile.
Now for every linear profile, we define the associated nonlinear profile as the maximal solution of (1.1) with the same initial data as in [13].

From Section 5 and Section 6, we can define:

\[ L_E(\alpha) := \lim_{k \to +\infty} (E(\tilde{\varphi}_\alpha O, k) + \frac{1}{2} M(\tilde{\varphi}_\alpha O, k)) = E_{\mathbb{R}^4}(\phi^\alpha) \in (0, L_{\text{max}}], \]

\[ L_{ls}(\beta) := \lim_{k \to +\infty} (E(\tilde{\psi}_\beta S, k) + \frac{1}{2} M(\tilde{\psi}_\beta S, k)) = ||\psi^\beta||^2_{H^{0.1}(\mathbb{R}^2 \times T^2)} \in (0, L_{\text{max}}], \]

\[ L_1(\gamma) := \lim_{k \to +\infty} (E(\tilde{\psi}_\gamma O, k) + \frac{1}{2} M(\tilde{\psi}_\gamma O, k)) = E(W) + \frac{1}{2} M(W^\gamma) \in (0, L_{\text{max}}]. \]

Noticing that:

\[ \lim_{J \to +\infty} \left[ \sum_{1 \leq \alpha, \beta, \gamma \leq J} [L_E(\alpha) + L_{ls}(\beta) + L_1(\gamma)] + \lim_{k \to +\infty} L(R_k^1) \right] \leq L_{\text{max}}. \quad (7.13) \]

**Case 2a:** \( L_E(1) = L_{\text{max}} \), there is only one Euclidean profile, that is

\[ u_k(0) = \tilde{\varphi}_{\varepsilon,k} + o_k(1) \]

in \( H^1(\mathbb{R}^2 \times T^2) \), where \( \varepsilon \) is a Euclidean frame. In this case, since the corresponding nonlinear profile \( U_k \) satisfies \( ||U_k||_{Z(\mathbb{R})} \lesssim E_{\mathbb{R}^4}(\phi) \) and \( \lim_{k \to +\infty} ||U_k(0) - u_k(0)||_{H^1(\mathbb{R}^2 \times T^2)} \to 0 \).

We may use Theorem 4.5 to deduce that

\[ ||u_k||_{Z(\mathbb{R})} \lesssim ||u_k||_{X^1(\mathbb{R})} \lesssim L_{\text{max}} \]

which contradicts (7.1).

**Case 2b:** \( L_{ls}(1) = L_{\text{max}} \), there is only one large scale profile, that is

\[ u_k(0) = \tilde{\psi}_{S,k} + o_k(1) \]

in \( H^1 \), where \( S \) is a large-scale frame. In this case, the corresponding nonlinear profile \( U_k \) satisfies \( ||U_k||_{Z(\mathbb{R})} \lesssim ||\psi||_{H^{0.1}} \) and \( \lim_{k \to +\infty} ||U_k(0) - u_k(0)||_{H^1(\mathbb{R}^2 \times T^2)} \to 0 \).

We may use Theorem 4.5 to deduce that

\[ ||u_k||_{Z(\mathbb{R})} \lesssim ||u_k||_{X^1(\mathbb{R})} \lesssim L_{\text{max}} \]

which contradicts (7.1).

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Case 2c: \( L_1(1) = L_{\text{max}} \), there is only one scale-one profile, we have that
\[
 u_k(0) = \tilde{\omega}_{O, k} + o_k(1)
\]
in \( H^1(\mathbb{R}^2 \times \mathbb{T}^2) \), where \( O = \{1, t_k, x_k, 0\} \) is a scale-one frame. If \( t_k \equiv 0 \), this is precisely the conclusion (7.2).

If \( t_k \to +\infty \), then
\[
 || e^{it\Delta_{R^2 \times T^2} \tilde{\omega}_{O, k}} ||_{Z(\alpha_k, 0)} \leq || e^{it\Delta_{R^2 \times T^2} \omega} ||_{Z(-\infty, -t_k)}
\]
which goes to 0 as \( t_k \to +\infty \). Using Theorem 4.3, we see that, for \( k \) large enough,
\[
 || u_k ||_{Z(-\infty, 0)} \leq \delta_0.
\]
It contradicts (7.1). The case \( t_k \to -\infty \) is similar.

Case 3: \( L_\mu(1) < L_{\text{max}} \) for all \( \mu \in \{E, ls, 1\} \). In this case, we construct an approximate solution of (1.1) with initial data \( u_k(0) \) whose \( Z \)-norm is finite and derive a contradiction by using Theorem 4.5. For this case, first, there exists \( \eta > 0 \) such that for all \( \alpha \geq 1 \), \( \mu \in \{E, ls, 1\} \) and \( L_\mu(\alpha) < L_{\text{max}} - \eta \), we have that all nonlinear profiles are global and satisfy, for any \( k, \alpha \geq 1 \) and \( \mu \in \{E, ls, 1\} \) (after extracting a subsequence)
\[
 || U_{\mu, \alpha} ||_{X^1(\mathbb{R})} \lesssim 1.
\]
(7.14)

One thing we need to mention is that from now on all implicit constants are allowed to depend on \( \Lambda(L_{\text{max}} - \eta/2) \). Using Theorem 4.5, it follows that
\[
 || U_{\mu, \alpha} ||_{X^1(\mathbb{R})} \lesssim 1.
\]
(7.14)

For \( J, k \geq 1 \), we define
\[
 U_{\text{prof}, k}^J = \sum_{1 \leq \alpha \leq J} \sum_{\mu \in \{E, ls, 1\}} U_{\mu, \alpha}^k.
\]
(7.15)

We can prove that \( || U_{\text{prof}, k}^J ||_{X^1(\mathbb{R})} \lesssim 1. \)

More precisely, we can show that there exists a constant \( Q \lesssim 1 \) such that
\[
 || U_{\text{prof}, k}^J ||_{X^1(\mathbb{R})}^2 + \sum_{1 \leq \alpha \leq J, \mu \in \{E, ls, 1\}} || U_{\mu, \alpha}^k ||_{X^1(\mathbb{R})}^2 \leq Q^2
\]
(7.16)

uniformly in \( J \), for all \( k \) sufficiently large. Let \( \delta_0(2L_{\text{max}}) \) defined in Theorem 4.3. We know that there are only finitely many profiles such that \( L(\alpha) \geq \frac{\delta_0}{3} \).

Without loss of generality, we may assume that for all \( \alpha \geq A, L(\alpha) \leq \delta_0 \). (Notice
Now we can define the approximate solution. We let $(\text{7.13})$

$$||U_{\text{prof},k}^J||_{X^1(\mathbb{R})} = || \sum_{1 \leq \alpha \leq J} \sum_{\mu \in \{E,ls,1\}} U_k^{\mu,\alpha}||_{X^1(\mathbb{R})}$$

$$\leq || \sum_{1 \leq \alpha \leq A} \sum_{\mu \in \{E,ls,1\}} U_k^{\mu,\alpha}||_{X^1(\mathbb{R})} + || \sum_{A \leq \alpha \leq J} \sum_{\mu \in \{E,ls,1\}} (U_k^{\mu,\alpha} - e^{it\Delta}U_k^{\mu,\alpha}(0))||_{X^1(\mathbb{R})}$$

$$+ || e^{it\Delta} \sum_{A \leq \alpha \leq J} \sum_{\mu \in \{E,ls,1\}} U_k^{\mu,\alpha}(0)||_{X^1(\mathbb{R})}$$

$$\leq || \sum_{1 \leq \alpha \leq A} \sum_{\mu \in \{E,ls,1\}} U_k^{\mu,\alpha}||_{X^1(\mathbb{R})} + \sum_{A \leq \alpha \leq J} \sum_{\mu \in \{E,ls,1\}} L_\mu(\alpha) \frac{\beta}{2}$$

$$+ || \sum_{A \leq \alpha \leq J} \sum_{\mu \in \{E,ls,1\}} U_k^{\mu,\alpha}(0)||_{H^2(\mathbb{R}^2 \times T^2)}$$

$$\lesssim A + \sum_{A \leq \alpha \leq J} \sum_{\mu \in \{E,ls,1\}} L_\mu(\alpha) \frac{\beta}{2} + || \sum_{A \leq \alpha \leq J} \sum_{\mu \in \{E,ls,1\}} U_k^{\mu,\alpha}(0)||_{H^1(\mathbb{R}^2 \times T^2)} \lesssim 1.$$

The bound on $\sum_{1 \leq \alpha \leq J} \sum_{\mu \in \{E,ls,1\}} ||U_k^{\mu,\alpha}||^2_{X^1}$ is similar.

Then we are now ready to construct the approximate solution. Let $F(z) = z|z|^2$ and also we have

$$F'(G)u = 2G\bar{G}u + G^2\bar{u}. \quad (7.17)$$

For each $B$ and $J$, we define $g_k^{B,J}$ to be the solution of the initial value problem:

$$i\partial_t g + \Delta g - F'(U_{\text{prof},k}^B)g = 0, \quad g(0) = R_k^J. \quad (7.18)$$

The solution $g_k^{B,J}$ is well defined on $\mathbb{R}$ for $k > k_0(B,J)$ and satisfies:

$$||g_k^{B,J}||_{X^1(\mathbb{R})} \leq Q'. \quad (7.19)$$

For some $Q'$ independent of $J$ and $B$. This follows by splitting $\mathbb{R}$ into $O(Q)$ intervals $I_j$ over which $||U_{\text{prof},k}^B||_{Z(I_j)}$ is small and applying the local theory on each subinterval.

Now we can define the approximate solution. We let $(A$ will be chosen shortly$)$

$$U_{\text{app},k}^J = U_{\text{prof},k}^A + g_k^{A,J} + U_{\text{prof},k}^J \quad \text{where} \quad U_{\text{prof},k}^A = \sum_{A \leq \alpha \leq J} \sum_{\mu} U_k^{\mu,\alpha}$$

which has $u_k(0)$ as its initial data and satisfies, for any $1 \leq A \leq J$, the bound:

$$||U_{\text{app},k}^J||_{X^1(\mathbb{R})} \leq 3(Q + Q')$$

for all $k \geq k_0(J)$. According to Theorem 4.5 with $M = 6(1 + Q + Q')$ gives us an $\epsilon_1 = \epsilon_1(M) \leq \frac{1}{K(1+Q+Q')}$ for some $K$ sufficiently large, such that if the error
By Lemma 4.2, (7.16) and (7.20), we estimate:

\[ \|U_{\text{prof},k}^A\|_{X^1(\mathbb{R})}^2 + \sum_{A < \alpha \leq J} \sum_{\mu} \|U_{k}^{\mu,\alpha}\|_{X^1(\mathbb{R})}^2 \leq \epsilon_1^{10}. \]  
(7.20)

for any \( J \geq A \) and \( k \) sufficiently large.

After fixing \( A \) we can bound the error term:

\[ \epsilon_k^J = (i \partial_t + \Delta)U^A_{\text{prof},k} - F(U^A_{\text{prof},k}) \]  
(7.21)
\[ = -F(U^A_{\text{prof},k} + g^A_{k,J} + U_{\text{prof},k}^{A,J}) + \sum_{1 \leq \alpha \leq J, \mu} F(U_{k}^{\mu,\alpha}) + F'(U_{\text{prof},k}^A) g^A_{k,J} \]  
(7.22)
\[ = -F(U_{\text{prof},k}^A + g^A_{k,J} + U_{\text{prof},k}^{A,J}) + F(U_{\text{prof},k}^A + g_{k,J}^A) + F(U_{\text{prof},k}^A) \]  
(7.23)
\[ - F(U_{\text{prof},k}^A + g_{k,J}^A) + F(U_{\text{prof},k}^A) + F'(U_{\text{prof},k}^A) g_{k,J}^A \]  
(7.24)
\[ - F(U_{\text{prof},k}^A) + \sum_{A \leq \alpha \leq A} F(U_{k}^{\mu,\alpha}) \]  
(7.25)
\[ - F(U_{\text{prof},k}^{A,J}) + \sum_{A+1 \leq \alpha \leq J, \mu} F(U_{k}^{\mu,\alpha}). \]  
(7.26)

We will estimate the four terms separately.

By Lemma 4.2, (7.16) and (7.20), we estimate:

\[ \|(7.23)|N(\mathbb{R})| \lesssim (\|U_{\text{prof},k}^A + g_{k,J}^A\|_{X^1(\mathbb{R})} + \|U_{\text{prof},k}^{A,J}\|_{X^1(\mathbb{R})})^2 \|U_{\text{prof},k}^{A,J}\|_{X^1(\mathbb{R})} \leq \epsilon_1^{1/4} \]  
for \( k \) large enough. By Lemma 4.2 and Lemma 7.5, we estimate:

\[ \|(7.24)|N(\mathbb{R})| \lesssim (\|U_{\text{prof},k}^A\|_{X^1(\mathbb{R})} + \|g_{k,J}^A\|_{X^1(\mathbb{R})})^2 \|g_{k,J}^A\|_{Z'(\mathbb{R})} \lesssim (Q + Q')^2 \|g_{k,J}^A\|_{Z'(\mathbb{R})} \leq \epsilon_1^{1/4} \]  
if \( J \) is big enough and for \( k > k_0(J) \). By Lemma 7.4, we estimate:

\[ \|(7.25)|N(\mathbb{R})| \lesssim A \sum_{(\alpha_1, \alpha_2) \neq (\alpha_2, \alpha_1)} \|U_{k}^{\mu_1,\alpha_1} U_{k}^{\mu_2,\alpha_2} U_{k}^{\mu_3,\alpha_3}\|_{N(\mathbb{R})} \leq \epsilon_1^{1/4} \]  
if \( k \) is big enough. By (7.20), we estimate:

\[ \|(7.26)|N(\mathbb{R})| \lesssim \|U_{\text{prof},k}^{A,J}\|_{X^1(\mathbb{R})}^3 + \sum_{A < \alpha \leq J} \|U_{\text{prof},k}^{\mu,\alpha}\|_{X^1(\mathbb{R})} \leq \epsilon_1^{1/4}. \]

By using Theorem 4.5, we get that \( u_k \) extends as a solution in \( X^1(\mathbb{R}) \) satisfying:

\[ \|u_k\|_{Z(\mathbb{R})} < +\infty \]
which contradicts (7.1).

There are two more lemmas which are used in the estimates above.

**Lemma 7.4** Assume that $U^\alpha_k, U^\beta_k, U^\gamma_k$ are three nonlinear profiles from the set 
\[ \{U^{\mu,\alpha}_k : 1 \leq \alpha \leq A, \mu \in \{E, ls, 1\}\} \] such that $U^\alpha_k$ and $U^\beta_k$ correspond to orthogonal frames. Then for these nonlinear profiles:
\[
\limsup_{k \to +\infty} ||\hat{U}^\alpha_k \hat{U}^\beta_k \hat{U}^\gamma_k||_{N(\mathbb{R})} = 0
\]
where for $\delta \in \{\alpha, \beta, \gamma\}$, $\hat{U}^\delta_k \in \{U^\delta_k, \bar{U}^\delta_k\}$.

**Lemma 7.5** For any fixed $A$, it holds that:
\[
\limsup_{J \to \infty} \limsup_{k \to \infty} ||g^A_J||_{Z(\mathbb{R})} = 0.
\]

The proofs of Lemma 7.4 and Lemma 7.5 are similar to the proofs in [13, 19].

# 8 Local Theory of the Resonant System

**Theorem 8.1** (Local well-posedness and small-data scattering for (1.6))
Let $\bar{u}(0) = \{u_p(0)\}_p \in h^1L^2$ satisfies $||\bar{u}_0||_{h^1L^2} \leq E$, then:

1. There exists an open interval $I$ which contains 0 and a unique solution $\bar{u}(t)$ of (1.6) in $C^0_t(I : h^1L^2) \cap \tilde{W}(I)$.

2. There exists $E_0$ such that $E \leq E_0$, $\bar{u}(t)$ is global and scatters.

3. Persistence of regularity: if $\bar{u}(0) \in h^\eta H^k$ for some $\eta \geq 1$ and $k \geq 0$, then $\bar{u} \in C^0(I : h^\eta H^k)$.

**Proof:** The proof follows from a simple fixed point theorem (and classical arguments), once we have established the nonlinear estimate. By Strichartz estimate, we have
\[
||u_j||_{L^4_{x,t}(\mathbb{R}^2 \times I)} \lesssim ||u_j(0)||_{L^2} + \sum_{R(j)} ||u_{p_1} \bar{u}_{p_2} u_{p_3}||_{L^\frac{4}{3}_{x,t}(\mathbb{R}^2 \times I)}
\]
where $R(j)$ was defined in (1.6). Multiplying by $\langle j \rangle$ and square-summing, the first term on the right-handed side is bounded by the square of the $h^1L^2_x$-norm.
For the second term, we compute as follows
\[
\sum_{j \in \mathbb{Z}^2} \langle j \rangle^2 \left[ \sum_{R(j)} \| u_{p_1} \bar{u}_{p_2} u_{p_3} \|_{L^2_{7/6}(\mathbb{R}^2 \times I)}^2 \right]
\lesssim \sum_{j \in \mathbb{Z}^2} \langle j \rangle^2 \left[ \sum_{R(j)} \Pi_k^3 \| u_{p_k} \|_{L^1_{1/3}(\mathbb{R}^2 \times I)}^2 \right]
\lesssim \sum_{j \in \mathbb{Z}^2} \{ \sum_{R(j)} \Pi_k^2 \langle p_k \rangle^2 \| u_{p_k} \|_{L^1_{1/3}(\mathbb{R}^2 \times I)}^2 \} \times \langle j \rangle^2 \sum_{R(j)} \langle p_1 \rangle^{-2} \langle p_2 \rangle^{-2} \langle p_3 \rangle^{-2}
\lesssim \sum_{j \in \mathbb{Z}^2} \sum_{R(j)} \Pi_k^3 \langle p_k \rangle^2 \| u_{p_k} \|_{L^1_{1/3}(\mathbb{R}^2 \times I)} \lesssim \| \bar{\nu} \|_{\tilde{W}(I)}^6.
\]
We obtain
\[
\| \bar{\nu} \|_{\tilde{W}(I)} \lesssim \| e^{it\Delta_x} \bar{u}_0 \|_{\tilde{W}(I)} + \| \bar{\nu} \|_{\tilde{W}(I)}^3.
\]
Also we know
\[
\| e^{it\Delta_x} \bar{u}_0 \|_{\tilde{W}(I)} \lesssim \| \bar{u}_0 \|_{h^1 L^2} \lesssim E.
\]
We can now run a classical fixed-point argument in $W(I) \cap C_t(I : h^1 L^2)$ provided $I$ or $E$ is small enough. The rest of the theorem follows from standard arguments.

**Lemma 8.2** There holds that
\[
\sup_{j \in \mathbb{Z}^2} \{ \langle j \rangle^2 \sum_{R(j)} \langle p_1 \rangle^{-2} \langle p_2 \rangle^{-2} \langle p_3 \rangle^{-2} \} \lesssim 1.
\]

**Proof:** Without loss of generality, we may assume that
\[
|p_1| \leq |p_3|, \quad \max(|j|, |p_2|) \sim |p_3|.
\]
Also we can see that $p_1$ is on a specific circle $C$,
\[
|p_1 - \frac{p_2 - \bar{j}}{2}|^2 = \left( \frac{p_2 - \bar{j}}{2} \right)^2.
\]
\[
S_1 = \sum_{(p_1, p_2, p_3) \in R(j); |p_1| \leq |p_3|; |p_2| \leq |p_1|} \langle p_1 \rangle^{-2} \langle p_2 \rangle^{-2} \frac{\langle j \rangle^2}{\langle p_3 \rangle^2}
\lesssim \sum_{(p_1, p_2, j + p_2 - p_1) \in R(j); |p_2| \leq |p_1|} \langle p_1 \rangle^{-2} \langle p_2 \rangle^{-2} \left[ \frac{\langle j \rangle}{\max(|j|, |p_2|)} \right]^2
\lesssim \sum_{p_2} \langle p_2 \rangle^{-2} \sum_{p_1} \langle p_1 \rangle^{-2}
\lesssim \sum_{p_2} \langle p_2 \rangle^{-2} (|p_2|)^{-1} \lesssim 1.
\]

The sum when $|p_1| \leq |p_2|$ is bounded similarly, using the following lemma to bound the sum over $p_2$ instead of the bound over $p_1$. 49
Lemma 8.3  For any $P \in \mathbb{R}^2$, $R > 0$ and $A > 1$ there hold that:

$$\sum_{|p| \geq A} \frac{1}{(p)^2} \lesssim A^{-1}$$

where $C(P, R)$ denotes the circle of radius $R$ centered at $P$.

Proof: It is exactly as same as [13, Lemma 8.3].

Lemma 8.4  Assume the conclusion of Theorem 1.2 holds for all initial data $u_0 \in H^1(\mathbb{R}^2 \times \mathbb{T}^2)$ with full energy $L(u_0) < E_{\text{max}}$. Then Conjecture 1.2 holds true for all initial data $\tilde{u}_0 \in h^1 L^2_z$ satisfying $E_{ls}(\tilde{u}_0) < E_{\text{max}}$. Furthermore, if all finite full-energy solutions scatter for (1.1), then the same thing holds for finite $E_{ls}$-energy solutions of (1.6).

Proof: It follows as in [13, Lemma 8.4].

Now let us focus on the proof of Lemma 3.3 (Local-in-time $L^p$ estimate). The idea of the proof is similar to [19, Proposition 2.1]. The main ingredient is the following distributional inequality:

Lemma 8.5  Assume $p_0 > \frac{10}{3}$, $N \geq 1$, $\lambda \in [N^\frac{2p_0-6}{p_0-3}, 2^{10} N^2]$, $|m|_\mathcal{L}^2(\mathbb{R}^2 \times \mathbb{Z}^2) \leq 1$, and $m(\xi) = 0$ for $|\xi| > N$, then

$$\left| \left\{ (x, t) \in \mathbb{R}^2 \times \mathbb{T}^2 \times [-2^{-10}, 2^{-10}] : \left| \int_{\mathbb{R}^2 \times \mathbb{Z}^2} m(\xi) e^{-it|\xi|^2} e^{ix \cdot \xi} d\xi \right| \geq \lambda \right\} \right| \lesssim N^{2p_0-6} \lambda^{-p_0}.$$  

(8.1)

First of all, Lemma 3.3 follows from the lemma above (Lemma 8.5).

Proof of Lemma 3.3: We let

$$F(x, t) = \int_{\mathbb{R}^2 \times \mathbb{Z}^2} m(\xi) e^{-it|\xi|^2} e^{ix \cdot \xi} d\xi,$$

where $m$ is as in Lemma 8.5, it suffices to prove that if $p > \frac{10}{3}$ and $N \geq 1$, then

$$\left\| 1_{[-2^{-10}, 2^{-10}]}(t) F \right\|_{\mathcal{L}^p(\mathbb{R}^2 \times \mathbb{T}^2 \times \mathbb{R})} \lesssim_p N^{2-\frac{4}{p}}.$$  

(8.2)

We may assume $p \in (\frac{10}{3}, 4]$ and $N \gg 1$. Then

$$\left\| 1_{[-2^{-10}, 2^{-10}]}(t) F \right\|_{\mathcal{L}^p(\mathbb{R}^2 \times \mathbb{T}^2 \times \mathbb{R})}^p \leq \sum_{2^l \leq 2^{10} N^2} 2^{pl} \left| \left\{ (x, t) \in \mathbb{R}^2 \times \mathbb{T}^2 \times [-2^{-10}, 2^{-10}] : |F(x, t)| \geq 2^l \right\} \right|.$$

If $2^l \geq N^\frac{2p_0-6}{p_0-3}$, $p_0 \in (\frac{10}{3}, p)$, we use the distributional inequality (8.1). If $2^l \leq N^\frac{2p_0-6}{p_0-3}$, we use the following bound:

$$2^{2l} \left| \left\{ (x, t) \in \mathbb{R}^2 \times \mathbb{T}^2 \times [-2^{-10}, 2^{-10}] : |F(x, t)| \geq 2^l \right\} \right| \leq \left\| F \right\|_{\mathcal{L}^2(\mathbb{R}^2 \times \mathbb{T}^2 \times \mathbb{R})}^2 \lesssim 1.$$
Therefore
\[
\|1_{[-2^{-10},2^{-10}]}(t)F\|_{L^p(P(R^2 \times T^2 \times \mathbb{R})} \lesssim \sum_{2^l \leq N \frac{2p-6}{p0}} 2^{(p-2)l} + \sum_{N \frac{2p0-6}{p0} \leq 2^l \leq 2^{10} N^2} 2^{pl} \cdot N^{2p0-6} 2^{-p0l}
\]
\[
\lesssim N^{2p-6},
\]
(8.3)

which gives (8.2). It suffices to prove Lemma 8.5. Now let us focus on Lemma 8.5.

Proof of Lemma 8.5: We may assume that $N \gg 1$ and consider the kernel $K_N : \mathbb{R}^2 \times T^2 \times \mathbb{R} \to \mathbb{C},$
\[
K_N(x, t) = \eta^1(2^l t/(2\pi)) \int_{\mathbb{R}^2 \times T^2} e^{-it\xi^2} e^{ix \cdot \xi} (\xi/N) d\xi
\]
(8.4)

Let
\[
S_\lambda := \{(x, t) \in \mathbb{R}^2 \times T^2 \times [-2^{-10}, 2^{-10}] : |\int_{\mathbb{R}^2 \times T^2} m(\xi)e^{-it\xi^2} e^{ix \cdot \xi} d\xi| \geq \lambda\}
\]

and fix a function $f : \mathbb{R}^2 \times T^2 \times [-2^{-10}, 2^{-10}] \to \mathbb{C}$ such that
\[
|f| \leq 1_{S_\lambda}
\]
(8.5)

and
\[
\lambda |S_\lambda| \leq \left| \int_{\mathbb{R}^2 \times T^2} f(x, t) \cdot |\int_{\mathbb{R}^2 \times T^2} m(\xi)e^{-it\xi^2} e^{ix \cdot \xi} d\xi| dx dt\right|.
\]
(8.6)

Using the assumption on $m$ we estimate the right-hand side of the inequality (8.6) above by
\[
\left| \prod_{j=1}^4 \eta^1(\xi_j/N) \cdot \int_{\mathbb{R}^2 \times T^2} f(x, t)e^{-it\xi^2} e^{ix \cdot \xi} d\xi dtdx dt\right|_{L^2_x}.
\]

Thus
\[
\lambda^2 |S_\lambda|^2 \leq \int_{\mathbb{R}^2 \times T^2} \int_{\mathbb{R}^2 \times T^2} f(x, t) \overline{f(y, s)} K_N(t-s, x-y)dtdx dsdy.
\]
(8.7)

Using Lemma 8.6 below, we estimate the right-hand-side in (8.7) as follows
\[
\int_{\mathbb{R}^2 \times T^2} \int_{\mathbb{R}^2 \times T^2} f(x, t) \overline{f(y, s)} K_N(t-s, x-y)dtdx dsdy
\]
\[
\leq \sum_{\mu \in \{1, 2, 3\}} \left| \int_{\mathbb{R}^2 \times T^2} \int_{\mathbb{R}^2 \times T^2} f(x, t) \overline{f(y, s)} K^\mu_N(t-s, x-y)dtdx dsdy\right|
\]
\[
\leq (\lambda^2/2) ||f||_{L^1_x}^2 + C\lambda^2 (N^{2p0-6} \lambda^{-p0}) ||f||_{L^2_x}^2 + C\lambda^2 (N^{2p0-6} \lambda^{-p0}) \frac{r-1}{r} ||f||_{L^{p0+1}}^2
\]
\[
\leq (\lambda^2/2)|S_\lambda|^2 + C\lambda^2 (N^{2p0-6} \lambda^{-p0}) |S_\lambda| + C\lambda^2 (N^{2p0-6} \lambda^{-p0}) \frac{r-1}{r} |S_\lambda|^{\frac{r+1}{r}}.
\]
(8.8)
Using (8.7), it follows that
\[ |S_{\lambda}| \lesssim N^{2p_0-6}\lambda^{-p_0} + (N^{2p_0-6}\lambda^{-p_0})^{(r-1)/r} |S_{\lambda}|^{1/r}. \]
which gives (8.1) and finishes the proof of Lemma 8.5. Now, we only need to prove the following lemma, which is obviously a crucial step in the proof of Lemma 8.5.

**Lemma 8.6** Assume \( \lambda \in [N^{2p_0-6}/2^10N^2] \) as in Lemma 8.5 and \( r \in [2,4] \), there is a decomposition
\[ K_N = K_{1,\lambda}^N + K_{2,\lambda}^N + K_{3,\lambda}^N \]
such that
\[ \|K_{1,\lambda}^N\|_{L^\infty(\mathbb{R}^2 \times \mathbb{T}^2 \times \mathbb{R})} \leq \frac{\lambda^2}{2}, \]
\[ \|K_{2,\lambda}^N\|_{L^\infty(\mathbb{R}^2 \times \mathbb{T}^2 \times \mathbb{R})} \lesssim \lambda^2 (N^{2p_0-6}\lambda^{-p_0}), \]
\[ \|K_{3,\lambda}^N\|_{L^r(\mathbb{R}^2 \times \mathbb{T}^2 \times \mathbb{R})} \lesssim \lambda^2 (N^{2p_0-6}\lambda^{-p_0})^{(r-1)/r}. \]

**Proof of Lemma 8.6:** For a continuous function \( h : \mathbb{R} \to \mathbb{C} \) and any \( (\xi,\tau) \in \mathbb{R}^2 \times \mathbb{Z}^2 \times \mathbb{R} \)
\[ F[K_N(x,t) \cdot h(t)](\xi,\tau) = C\eta^4(\xi/N) \int_{\mathbb{R}} h(t)\eta^1(2^j t/(2\pi))e^{-it(\tau+i|\xi|^2)}dt. \] (8.10)

It is shown in \([2, \text{Lemma 3.18}]\) that
\[ \left| \sum_{n \in \mathbb{Z}} e^{-it|n|^2} e^{ixn} \eta^1(\xi/N)^2 \right| \lesssim \frac{N}{\sqrt{q(1 + N|t/(2\pi)| - a/q^1/2)}} \] (8.11)
if
\[ t/(2\pi) = a/q + \beta, \quad q \in \{1, \ldots, N\}, a \in \mathbb{Z}, (a,q) = 1, |\beta| \leq (Nq)^{-1}. \] (8.12)

Also for \( t \) as in (8.12), we have,
\[ |K_N(x,t)| \lesssim \frac{N^2}{q(1 + N|t/(2\pi)| - a/q^{1/2})^2} \left( \frac{N}{1 + N|t/(2\pi)|^2} \right)^2. \] (8.13)

For \( j \in \mathbb{Z} \) we define \( \eta_j, \eta \geq 0 : \mathbb{R} \to [0,1], \)
\[ \eta_j(s) := \eta^1(2^j s) - \eta^1(2^{j+1} s), \quad \eta_{\geq j}(s) := \sum_{k \geq j} \eta_k(s). \]

Fix integers \( K, L, \) satisfying
\[ K \in \mathbb{Z}_+, \quad N \in [2^{K+4}, 2^{K+5}), \quad L \in \mathbb{Z} \cap [0, 2K+20], \quad \lambda^{p_0-2} N^{6-2p_0} \in [2^L, 2^{L+1}). \] (8.14)
Now we start with the important decomposition as follows:

\[ 1 = \left[ \sum_{k=0}^{K-1} \sum_{j=0}^{K-k-1} p_{k,j}(s) \right] + e(s) \]

\[ p_{k,j}(s) := \sum_{q=2^k}^{2^{k+1}-1} \sum_{a \in \mathbb{Z}, (a,q)=1} \eta_{j+k+10}(s/(2\pi) - a/q) \quad \text{if } j \leq K - k - 1, \quad (8.15) \]

\[ p_{k,K-k}(s) := \sum_{q=2^k}^{2^{k+1}-1} \sum_{a \in \mathbb{Z}, (a,q)=1} \eta_{j+k+10}(s/(2\pi) - a/q). \]

Let \( T_K = \{(k,j) \in \{0,\ldots,K-1\} \times \{0,\ldots,K\} : k+j \leq K \} \). In view of Dirichlet’s lemma, we observe that

\[ \text{if } t \in \text{supp}(e) \text{ satisfies } (8.12), \text{ then either } N \lesssim q \text{ or } (Nq)^{-1} \approx |t/(2\pi) - a/q|. \quad (8.16) \]

We define the first component of the kernel \( K_{N,1}^{2,\lambda} \),

\[ K_{N,1}^{2,\lambda}(x,t) = K_N(x,t) \cdot \eta^1(2^{L-40}t/(2\pi)). \quad (8.17) \]

It follows from (8.10) that

\[ ||\hat{K}_{N,1}^{2,\lambda}||_{L^\infty(\mathbb{R}^2 \times \mathbb{T}^2 \times \mathbb{R})} \lesssim 2^{-L} \lesssim N^{2p_0-6}\lambda^{2-p_0}, \quad (8.18) \]

which agrees with (8.9).

Therefore we may assume that \( L \geq 45 \) and write

\[ K_N(x,t) - K_{N,1}^{2,\lambda}(x,t) = \sum_{l=4}^{L-41} K_N(x,t) \cdot \eta_l(t/(2\pi)). \quad (8.19) \]

Using (8.13) and (8.16), for any \((k,j) \in T_K\) and \( l \in [4, L-41] \cap \mathbb{Z} \),

\[ \sup_{x,t} |K_N(x,t) \cdot \eta_l(t/(2\pi))p_{k,j}(t)| \lesssim 2^{l2^{K+j}} \]

\[ \sup_{x,t} |K_N(x,t) \cdot \eta_l(t/(2\pi))e(t)| \lesssim 2^{l2^K} \quad (8.20) \]

We analyze two cases.

**Case 1:** \( L \leq 2K - \delta K, \delta = \frac{1}{100} \). In this case we set

\[ K_N^{1,\lambda}(x,t) := K_N(x,t) \cdot \left[ \sum_{l=4}^{L-41} \eta_l(t/(2\pi))[e(t)+ \sum_{k,j \in T_K, 2j \leq L} p_{k,j}(t)+ \sum_{k,j \in T_K, 2j > L} \rho_{k,j}p_{k,0}(t)] \right], \quad (8.22) \]
\[ K^2_N(x,t) := K^2_{N,1}(x,t) + K_N(x,t) \left( \sum_{l=4}^{L-41} \eta_l(t/(2\pi)) \left( \sum_{k,j \in T_K, 2j > L} (p_{k,j}(t) - \rho_{k,j,pk,0}(t)) \right) \right), \]

(8.23)

\[ K^3_N(x,t) := 0, \]

(8.24)

where \( K^2_{N,1} \) is defined as in (8.17) and

\[ \rho_{k,j} := 2^{-j} \text{ if } j \leq K - k - 1 \text{ and } \rho_{k,N-k} := 2^{-K+k+1}. \]

(8.25)

The bound on \( K^1_N \) is trivial. Using (8.20) and (8.21), for any \((x,t) \in \mathbb{R}^2 \times \mathbb{T}^2 \times \mathbb{R}\), we have

\[
|K^1_N(x,t)| \leq \sum_{l=4}^{L-41} |K_N(x,t) \cdot \eta_l(t/(2\pi))e(t)| \\
+ \sum_{l=4}^{L-41} \sum_{k,j \in T_K, 2j \leq L} |K_N(x,t) \cdot \eta_l(t/(2\pi))p_{k,j}(t)| \\
+ \sum_{l=4}^{L-41} \sum_{k,j \in T_K, 2j > L} |K_N(x,t) \cdot \eta_l(t/(2\pi))p_{k,j}p_{k,0}(t)| \\
\leq 2^LqK + K2^LqK^{L/2} + K2^L2K \\
\leq K_2(L-2K)(3p_0-10)/(2(p_0-2)) \cdot 2^{2L/(p_0-2)}2K(4p_0-12)/(p_0-2).
\]

(8.26)

Recall that \((2K-L) \geq \delta K, N \gg 1\), and \(p_0 > \frac{40}{3}\). Notice that \(\lambda^2 \approx 2^{2L/(p_0-2)}2K(4p_0-12)/(p_0-2)\) the desired bound \(|K^1_N(x,t)| \leq \lambda^2/2\) follows.

It remains to bound (8.9) on the kernel \(K^2_N\) which follows as in [19].

**Case 2:** \(L \geq 2K - \delta K, \delta = \frac{1}{100}\). For \(b \in \mathbb{Z}_+ \) sufficiently large, in this case we set

\[
K^1_N(x,t) := K_N(x,t) \cdot \left( \sum_{l=4}^{L-41} \eta_l(t/(2\pi))e(t) + \sum_{k,j \in T_K, 2j \leq L-b} p_{k,j}(t) \right). \]

(8.27)

Using the bound (8.20) and (8.21), for \((x,t) \in \mathbb{R}^2 \times \mathbb{T}^2 \times \mathbb{R}\)

\[
|K^1_N(x,t)| \leq 2^LqK + 2^LqK^{L/2}b/2 \\
\leq 2^{-b/2}(L-2K)(3p_0-10)/(2(p_0-2)) \cdot 2^{2L/(p_0-2)}2K(4p_0-12)/(p_0-2).
\]

(8.28)

Since \(\lambda^2 \approx 2^{2L/(p_0-2)}2K(4p_0-12)/(p_0-2)\), it follows that \(|K^1_N(x,t)| \leq \lambda^2/2\) if \(b\) is fixed sufficiently large.
Now, let
\[ L_N(x, t) := K_N(x, t) - K_{N,1}^2(x, t) - K_{N,1}^1(x, t) = \sum_{l=4}^{L-41} \sum_{k,j \in T_k, 2j > L-b} K_N(x, t) \cdot \eta_l(t) p_{k,j}(t). \]

It remains to prove that one can decompose \( L_N = L_{N}^{2^\lambda} + L_{N}^{3^\lambda} \) satisfying
\[
\| \widehat{L_{N}^{2^\lambda}} \| \lesssim 2^{-L}, \quad \| \widehat{L_{N}^{3^\lambda}} \| \lesssim \lambda^{2/2} \tau^{-L(r-1)/r}. \tag{8.29}
\]

We let \( \tilde{\eta}_L(s) := \sum_{l=4}^{L-41} \eta_l(s) = \eta_1(2^4 s) - \eta_1(2^{L-40} s). \) And we have
\[
\widehat{L_N}(\xi, \tau) = C \sum_{k,j \in T_k, 2j > L-b} \eta^4(\xi/N) \int_{\mathbb{R}} \tilde{\eta}_L(t) \eta_1^1(2^5 t) p_{k,j}(2\pi t) e^{-2\pi it(\tau+|\xi|^2)} dt. \tag{8.30}
\]

The cardinality of the set \( T_{K,L} := \{ k, j \in T_K : 2j > L-b \} \) is bounded by \( C(1 + |2K-L|)^2 \). Let \( f_{k,j} : \mathbb{R} \to \mathbb{C}, \)
\[
f_{k,j}(\mu) := \int_{\mathbb{R}} \tilde{\eta}_L(t) \eta_1^1(2^5 t) p_{k,j}(2\pi t) e^{-2\pi it\mu} dt. \tag{8.31}
\]

It suffices to prove that for \( (k, j) \in T_{K,L}, \) one can decompose
\[
f_{k,j} = f_{k,j}^2 + f_{k,j}^3, \tag{8.32}
\]
satisfying
\[
\| f_{k,j}^2 \|_{L^\infty(\mathbb{R})} \lesssim 2^{-L}(1 + |2K-L|)^{-2} \tag{8.33}
\]
and
\[
\| f_{k,j}^3 \|_{L^r(\mathbb{R})} \lesssim (\lambda N^2)^{2/\tau} 2^{-L(r-1)/r}(1 + |2K-L|)^{-2} \tag{8.34}
\]
The rest of the proof follows as in [19, Proposition 2.1].

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