Quasisymmetric functions for nestohedra

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Abstract

To a generalized permutohedron is associated a quasisymmetric function which is the generating function of all positive integer points in interiors of maximal cones of the normal fan. We describe this function for the class of nestohedra as a Hopf algebra morphism from a combinatorial Hopf algebra on building sets. For the class of graph-associahedra the corresponding quasisymmetric function is a new isomorphism invariant of graphs.

Keywords: Hopf algebra, nestohedron, quasisymmetric function, P-partition

1 Introduction

Let $Q$ be a convex polytope. The normal fan $\Sigma_Q$ is the set of cones over the faces of the polar polytope $Q^*$. The polytope $Q$ is simple if and only if the normal fan $\Sigma_Q$ is simplicial. The polytope $Q$ is a Delzant polytope if its normal fan $\Sigma_Q$ is regular, i.e. the generators of the normal cone $\sigma_v$ at any vertex $v \in Q$ can be chosen to form an integer basis of $\mathbb{Z}^n$.

The permutohedron $P_n$ is an $(n-1)$-dimensional polytope which is the convex hall $P_n = \text{Conv}\{x_\omega \mid \omega \in S_n\}$, where $x \in \mathbb{R}^n$ is a point with strictly increasing coordinates $x_1 < \cdots < x_n$ and $x_\omega = (x_{\omega(1)}, \ldots, x_{\omega(n)})$ for a permutation $\omega \in S_n$.

The normal fan $\Sigma_{P_n}$ of the permutohedron $P_n$ is the braid arrangement fan. A generalized permutohedron $Q$ is a polytope whose normal fan $\Sigma_Q$ is refined by the braid arrangement fan $\Sigma_{P_n}$. The generalized permutohedra, studied by Postnikov in [6], include some interesting classes of polytopes, such as matroid polytopes, graphic zonotopes, graph-associahedra and nestohedra.

Let $Q$ be a generalized permutohedron in $\mathbb{R}^n$. A function $f : [n] \to \mathbb{N}$ is $Q$-generic if it lies in the interior of the normal cone $\sigma_v$ for some vertex $v \in Q$. Thus a $Q$-generic function $f$, as an element of $(\mathbb{R}^n)^*$, uniquely maximizes over $Q$ at a vertex. Define the power series
\[ F(Q) = \sum_{f} \prod_{i=1}^{n} x_{f(i)}, \]

where the sum is over all \( Q \)-generic functions. The generating function \( F(Q) \) is a homogeneous quasisymmetric function of degree \( n \). Therefore it can be expanded in monomial basis of quasisymmetric functions

\[ F(Q) = \sum_{\alpha \vdash n} \zeta_{\alpha}(Q) M_{\alpha}, \tag{1} \]

where \( M_{\alpha} = \sum_{a_{1} < \ldots < a_{k}} x_{a_{1}}^{a_{1}} \cdots x_{a_{k}}^{a_{k}} \) for a composition \( \alpha = (a_1, \ldots, a_k) \vdash n \) of the integer \( n \).

This power series is introduced and its main properties are derived by Billera, Jia and Reiner in ([2], Section 9). If \( Q = Z_{\Gamma} \) is a graphic zonotope the function \( F(Z_{\Gamma}) \) is easily seen to be Stanley’s chromatic symmetric function of the graph \( \Gamma \). It is known that chromatic symmetric function of a graph is obtained as the image of the canonical morphism from the chromatic Hopf algebra of graphs to symmetric functions, see ([1], Example 4.5). For the matroid base polytope \( Q = P_{M} \) the quasisymmetric function \( F(P_{M}) \) is an isomorphism invariant of a matroid \( M \). Billera, Jia and Reiner in [2] constructed the combinatorial Hopf algebra of matroids and proved that the assignment \( M \mapsto F(P_{M}) \) is a Hopf algebra morphism to quasisymmetric functions. We particularly respond to [2, Problem 9.3] and study the quasisymmetric functions \( F(Q) \) for the class of nestohedra.

The nestohedron \( Q = P_{B} \) is a simple polytope which can be obtained from a simplex by a sequence of face truncations. The family of faces by which we perform truncations is encoded by a building set \( B \), which is a subset of the face lattice of the simplex. For the class of nestohedra we describe coefficients \( \zeta_{\alpha}(P_{B}) \) in the expansion (1) in terms of underlying building sets. We construct a certain combinatorial Hopf algebra of building sets \( B \) and show that the canonical morphism of \( B \) sends a building set \( B \) precisely to the generating function \( F(P_{B}) \) of the corresponding nestohedron \( P_{B} \). This Hopf algebra on building set is of different nature than the Hopf algebra \( BSet \) studied in [4], [5], which is a subalgebra of the chromatic Hopf algebra of hypergraphs.

A graph \( \Gamma \) canonically produces the graphical building set \( B(\Gamma) \). This assignment determines the Hopf subalgebra of graphs \( G \subset B \). The polytopes \( P_{B(\Gamma)} \) are called graph-associahedra. The canonical morphism of \( G \), which assigns the quasisymmetric function \( F(P_{B(\Gamma)}) \) to a graph \( \Gamma \), defines a new isomorphism invariant of graphs different in the nature from the Stanley’s chromatic symmetricfunction.

In section 2 we review the necessary facts about nestohedra. In section 3 we give an expansion of \( F(P_{B}) \) as a sum of certain \( P_{v} \)-partition enumerators over all vertices \( v \in P_{B} \). In section 4 we prove the main contribution of the paper that the assignment \( B \mapsto F(P_{B}) \) is a Hopf algebra morphism.
2 Nestohedra

In this section we review the necessary definitions and facts about nestohedra. This class of polytopes is introduced and studied in [3], [6], [7], [8].

A hypergraph $B$ on the finite set $V = V(B)$ is a collection of nonempty subsets of $V$. For convenience we suppose that $\{i\} \in B, i \in V$. For a subset $I \subset V$, let $B |_I = \{ J \subset I \mid J \in B \}$ be the induced subhypergraphs. The contraction of $I \subset V$ from $B$ is the hypergraph $B/I = \{ J \subset V \setminus I \mid J \in B \text{ or } I' \cup J \in B \text{ for some } I' \subset I \}$.

Let $\Delta[n] = \text{Conv}\{e_1, \ldots, e_n\}$ be the standard coordinate simplex in $\mathbb{R}^n$. To a subset $I \subset [n]$ corresponds the face $\Delta_I = \text{Conv}\{e_i \mid i \in I\} \subset \Delta[n]$. For a hypergraph $B$ on $V = [n]$ define the polytope $P_B$ as the Minkowski sum of simplices

$$P_B = \sum_{I \in B} \Delta_I.$$

By properties of the Minkowski sum we have

$$P_B = \sum_{I \in B} \text{Conv}\{e_i \mid i \in I\} = \text{Conv}\sum_{I \in B} \{e_i \mid i \in I\}.$$

The polytope $P_B$ is simple if additionally the hypergraph $B$ satisfies the following condition:

- If $I, J \in B$ and $I \cap J \neq \emptyset$ then $I \cup J \in B$.

In that case $B$ is called a building set and the polytope $P_B$ is called a nestohedron.

**Example 2.1.** Given a simple graph $\Gamma$ on the vertex set $V$, the graphical building set $B(\Gamma)$ is defined as the collection of all $I \subset V$ such that induced graphs $\Gamma |_I$ are connected. For the graph $\Gamma$ and a subset $I \subset V$, the contraction $\Gamma/I$ is a graph on the vertex set $V \setminus I$ with two vertices $u$ and $v$ connected by the edge if either $\{u, v\}$ is an edge of $\Gamma$ or there are a path $u, w_1, \ldots, w_k, v$ in $\Gamma$ with $w_1, \ldots, w_k \in I$. Then it is immediate that $B(\Gamma |_I) = B(\Gamma) |_I$ and $B(\Gamma/I) = B(\Gamma)/I$. The polytope $P_{B(\Gamma)}$ is called a graph-associahedron. For instance the series $P_n, A_n, C_n, n > 2$ of permutohedra, associahedra and ciclohedra correspond to complete graphs, paths and cycles on $n$ vertices, respectively.

Let $B_{\text{max}}$ be the collection of maximal by inclusion elements of a building set $B$. We say that a building set $B$ is connected if $V \in B$. Since the Minkowski sum is the product for polytopes which are contained in the complementary subspaces, we have

$$P_B = \sum_{I \in B_{\text{max}}} \sum_{J \in B|_I} \Delta_J = \prod_{I \in B_{\text{max}}} P_{B|_I}.$$

Thus we may restrict ourselves to connected building sets. The realization of nestohedra is given by the following proposition.
**Proposition 2.2** ([3], Proposition 3.12). Let $B$ be a connected building set on the finite set $V = [n]$ and $\mu(B)$ be the number of elements of $B$. The nestohedron $P_B$ can be described as the intersection of the hyperplane $H_{[n]}$ with the halfspaces $H_{I, \geq}$ corresponding to all $I \in B \setminus \{[n]\}$, where

$$H_{[n]} = \{ x \in \mathbb{R}^n \mid \sum_{i \in [n]} x_i = \mu(B) \},$$

$$H_{I, \geq} = \{ x \in \mathbb{R}^n \mid \sum_{i \in I} x_i \geq \mu(B \mid I) \}.$$  

As a consequence we obtain that the nestohedron $P_B$ can be obtained by a sequence of face truncations from the simplex $\Delta = H_{[n]} \cap \cap_{i=1}^n H_{[i], \geq}$. Let $H_I = \partial H_{I, \geq}$ be the hyperplane corresponding to $I \subset [n]$. We index the face lattice of $\Delta$ by $\Delta \cap \cap_{i \in I} H_{[i]}$, $I \subset [n]$. Then perform the face truncations $\Delta \cap H_{I, \geq}$ prescribed by non-singleton sets $I \in B$ in any reverse order. It follows that facets of the nestohedron $P_B$ are indexed by the elements $I \in B \setminus \{[n]\}$. A facet $F_I \subset P_B$ is isomorphic to the product $P_{B/I} \times P_{B/I}$. 

**Example 2.3.** The permutohedron $P_n$ is obtained by truncations along all faces of the simplex $\Delta$ in reverse order and each facet of $P_n$ is of the form $P_k \times P_{n-k}$, for some $1 \leq k \leq n - 1$. 

The face lattice of $P_B$ is described by the following proposition.

**Proposition 2.4** ([3], Theorem 3.14; [6], Theorem 7.4). Given a connected building set $B$ on $V = [n]$, let $\{F_I \mid I \in B \setminus \{[n]\}\}$ be the set of facets of the nestohedron $P_B$. The intersection $F_{I_1} \cap \ldots \cap F_{I_k}, k \geq 2$ is a nonempty face of $P_B$ if and only if

(N1) $I_i \subset I_j$ or $I_j \subset I_i$ or $I_i \cap I_j = \emptyset$ for any $1 \leq i < j \leq k$.

(N2) $I_{j_1} \cup \ldots \cup I_{j_p} \notin B$ for any pairwise disjoint sets $I_{j_1}, \ldots, I_{j_p}$.

A subcollection $\{I_1, \ldots, I_k\} \subset B$ that satisfies the conditions (N1) and (N2) is called a nested set. The collection $N_B$ of all nested sets form a simplicial complex called the nested set complex. The nested set complex $N_B$ is a simplicial polytope which is realized as the polar polytope $P^*_B$.

### 3 Expansion in P-partition enumerators

The proposition 2.4 implies that vertices of $P_B$ correspond to maximal nested sets. We denote this correspondence by $v \mapsto N_v$. To a vertex $v \in P_B$ associate the poset $(N_v \cup \{[n]\}, \subseteq)$. For $I \in N_v \cup \{[n]\}$ let $i_I \in [n]$ be the element such that $\{i_I\} = I \setminus \{J \in N_v \mid J \subseteq I\}$. The correspondence $I \mapsto i_I$ is a well defined bijection by the characterization of maximal nested sets ([6], Proposition 7.6). It defines the partial order $\leq_v$ on $[n]$ by $i_I \leq_v i_J$ if and only if $I \subset J$. Denote this poset on $[n]$ by $P_v$. 

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The edges which are incident to a vertex \( v \in P_B \) correspond to the nested sets \( N_v \setminus \{ I \}, I \in N_v \). The edge \( vv' \in P_B \) corresponds to the nested set \( N_v \cap N_{v'} \in N_B \). The following proposition, which is a consequence of proposition 2.2, describes the coordinates and normal cones at vertices of \( P_B \).

**Proposition 3.1.** Let \( v \in P_B \) be a vertex of the nestohedron \( P_B \) and \( N_v \in N_B \) be the corresponding maximal nested set.

(i) The coordinates of the vertex \( v \) are given by

\[
 x_{ij} = \mu(B \mid I) - \sum_{J \subset I} \mu(B \mid J), I \in N_v \cup \{ [n] \},
\]

where the sum is over all \( J \in N_v \) which are covered by \( I \).

(ii) The normal cone \( \sigma_v \) at the vertex \( v \) is determined by the inequalities

\[
 x_{ij} < x_{iI}, \text{ for all } J \supset I.
\]

Recall the basic facts about \( P \)-partitions. Given a finite poset \( P \) of cardinality \( n \) and a labelling \( \omega : P \to [n] \), a \((P, \omega)\)-partition is an order-preserving map \( f : P \to \mathbb{N} \) such that \( f(p) < f(q) \) for all \( p <_P q \) with \( \omega(p) > \omega(q) \). For any labelled poset \((P, \omega)\) it is defined the generating function

\[
 F(P, \omega) = \sum_f \prod_{p \in P} x_{f(p)},
\]

where the sum is over all \((P, \omega)\)-partitions. The function \( F(P, \omega) \) is quasisymmetric. A labelling \( \omega : P \to [n] \) is called strict if \( \omega(p) > \omega(q) \) whenever \( p <_P q \), or natural if \( \omega(p) < \omega(q) \) for \( p <_P q \). The following theorem is a specialization of the general statement for generalized permutohedron to nestohedra.

**Theorem 3.2** ([2], Theorem 9.2). For a connected building set \( B \),

\[
 F(P_B) = \sum_{v \in P_B} F(P_v, \omega_v),
\]

where \( \omega_v \) is any strict labelling of \( P_v \).

**Proof.** Any \( P_B \)-generic function \( f : [n] \to \mathbb{N} \) which is maximized at a vertex \( v \) lies at the normal cone \( f \in \sigma_v \). By proposition 3.1 (ii), these functions are precisely \((P_v, \omega_v)\)-partitions. \(\)

Let \( P_{opp} \) be the opposite poset of \( P \) with the reverse order \( p \leq_{P_{opp}} q \) if and only if \( p \leq_P q \). The labelling \( \omega \) is strict on \( P \) if and only if it is natural on \( P_{opp} \). Define \( F^*(P, \omega) = F(P_{opp}, \omega) \). The antipode \( S \) acts on the monomial basis of quasisymmetric functions by

\[
 S(M_\alpha) = (-1)^{k(\alpha)} \sum_{\beta \leq_\alpha^*} M_\beta,
\]
where $\alpha^*$ is the reverse composition of $\alpha$ and the sum is over all refinement of $\alpha^*$. The functions $F(P, \omega)$ and $F^*(P, \omega)$ are related with

$$S(F(P, \omega)) = (-1)^{|P|} F^*(P, \omega).$$

For a connected building set $B$ on the finite set $V = [n]$ and the nestohedron $P_B$ theorem 3.2 allows to define a quasisymmetric function

$$F^*(P_B) = \sum_{v \in P_B} F^*(P_v, \omega_v),$$

which satisfies the identity $S(F(P_B)) = (-1)^n F^*(P_B)$. A function $f : [n] \to \mathbb{N}$ is minimized at a vertex $v \in P_B$ if and only if $f$ is a $(P_B^{opp}, \omega_v)$-partition. Let $c(f)$ be the number of $f$-minimizing vertices of $P_B$. Hence we have the following identity

$$F^*(P_B) = \sum_f c(f) \prod_{i=1}^n x_{f(i)}. \quad (2)$$

Let $\chi(P_B, m) = F(P_B)(1^m)$ be the principal specialization of $F(P_B)$ which is a polynomial in $m$ that counts the number of $P_B$-generic function $f : [n] \to [m]$. It is related with the principal specialization $\chi^*(P_B, m) = F^*(P_B)(1^m) = \sum_{f : [n] \to [m]} c(f)$ by

$$\chi(P_B, -m) = (-1)^n \chi^*(P_B, m).$$

Specially, for $m = 1$, we obtain that $\chi(P_B, -1) = (-1)^n |\text{Vert}(P_B)|$ counts the number of vertices of $P_B$.

4 Hopf algebra morphism

The goal of this section is to show that the assignment of quasisymmetric function $F(P_B)$ to a building set $B$ is a Hopf algebra morphism. Let $\mathcal{B}$ be the graded vector space generated by the set of all isomorphism classes of building sets. The grading is defined by the number of vertices. Define the multiplication and comultiplication by

$$B_1 \cdot B_2 = B_1 \sqcup B_2 \quad \text{and} \quad \Delta(B) = \sum_{I \subset V} B |_I \otimes B/I.$$ 

The unit is the building set $B_\emptyset$ on the empty set and the counit is defined by $\epsilon(B_\emptyset) = 1$ and zero otherwise.

**Proposition 4.1.** The vector space $\mathcal{B}$ with the above defined operations is a graded commutative and non-cocommutative connected bialgebra.

**Proof.** The only nontrivial parts of the statement are the coassociativity and the compatibility of operations, which follows from the properties $(B/I |_J = (B |_{I \cup J})/I, (B/I)/J = B/(I \cup J)$ for any disjoint $I, J \subset V$ and $(B_1 \cdot B_2) |_{I_1 \cup I_2} = B_1 |_{I_1 \cdot B_2} |_{I_2}, (B_1 \cdot B_2)/(I_1 \cup I_2) = B_1/I_1 \cdot B_2/I_2$ for all $I_1 \subset V_1, I_2 \subset V_2$. \[\square\]
The antipode of $B$ is determined by Takeuchi’s formula for the antipode of a graded connected bialgebra

$$S(B) = \sum_{k \geq 1} (-1)^k \sum_{\mathcal{L}_k} \prod_{j=1}^{k} (B |_{I_j})/I_{j-1},$$

(3)

where the inner sum goes over all chains of subsets $\mathcal{L}_k : \emptyset = I_0 \subset I_1 \subset \cdots \subset I_{k-1} \subset I_k = V$.

**Remark 4.2.** The vector space $G$ spanned by all isomorphism classes of simple graphs is endowed with the Hopf algebra structure by operations

$$\Gamma_1 \cdot \Gamma_2 = \Gamma_1 \cup \Gamma_2 \text{ and } \Delta(\Gamma) = \sum_{I \subseteq V} \Gamma |_I \otimes \Gamma/I.$$

The map that associates the graphical building set $B(\Gamma)$ to a graph $\Gamma$ is extended to a Hopf algebra monomorphism $i : G \to B$.

Given a decomposition $\alpha = (a_1, \ldots, a_k) = n$, we say that the chain $\mathcal{L} : \emptyset = I_0 \subset I_1 \subset \cdots \subset I_{k-1} \subset I_k = V$ is a splitting chain of the type type($\mathcal{L}$) = $\alpha$ if $|I_j \setminus I_{j-1}| = a_j$ and $(B |_{I_j})/I_{j-1}$ is discrete for all $1 \leq j \leq k$.

**Theorem 4.3.** For a connected building set $B$ the generating function $F(P_B)$ is given by

$$F(P_B) = \sum_{\alpha = n} \zeta_{\alpha}(B) M_{\alpha},$$

where $\zeta_{\alpha}(B)$ is the number of all splitting chains of the type $\alpha$.

**Proof.** Let $\mathcal{L}$ be a splitting chain of the length $k$. The sets $I_j \setminus I_{j-1}, 1 \leq j \leq k$ decompose the set of vertices $V = [n]$. Define the level of a vertex $i \in V$ by $l(i) = j$ if $i \in I_j \setminus I_{j-1}$. Let $S_i = \{i\} \cup \max\{J \subset I_{l(i)}-1 | \{i\} \cup J \in B\}$ for $i \in V$. Since $B$ is connected and $B/I_{k-1}$ is discrete it follows that $|I_k \setminus I_{k-1}| = 1$, i.e. $S_i = V$ for the unique $i \in V$. Let $N(\mathcal{L}) = \{S_i | i \in V\} \setminus \{V\}$.

**Claim:** The collection $N(\mathcal{L})$ is a maximal nested set.

(N1) Suppose that $S_i \cap S_j \neq \emptyset$ for some $i, j \in V$. If $l = l(i) = l(j)$ then $S_i \cup S_j \in B$ and $\{i, j\} \in (B |_{I_l})/I_{l-1}$. If $l(j) < l(i)$ then $i \in S_i \cup S_j \in B$ which implies $S_j \subset S_i$.

(N2) If $S = S_{i_1} \cup \ldots \cup S_{i_p} \in B$ then $S = S_{i_j}$ for a vertex $i_j \in V$ with the maximal level $l = \max\{l(i_1), \ldots, l(i_p)\}$. Therefore $S_{i_1}, \ldots, S_{i_p}$ is not a disjoint collection.

Denote by $v(\mathcal{L})$ the vertex of $P_B$ which corresponds to $N(\mathcal{L})$. It defines the map $g : \mathcal{L} \mapsto v(\mathcal{L}) \in P_B$. We show the following identity

$$F(P_v, \omega_v) = \sum_{\mathcal{L} \in g^{-1}(v)} M_{\text{type}(\mathcal{L})},$$
for any strict labelling $\omega_v$.

Let $L \in g^{-1}(v)$ be a splitting chain. The associated level function $i \mapsto l(i)$ is obviously a $(P_v, \omega_v)$-partition. It shows that the monomial quasisymmetric function $M_{type(L)}$ is a summand of $F(P_v, \omega_v)$. On the other hand, for a $(P_v, \omega_v)$-partition with the set of values $i_1 < \cdots < i_k$, define the decomposition of the set $V$ by $I_j = f^{-1}(\{i_j\}), 1 \leq j \leq k$. Then $L : I_1 \subset I_1 \cup I_2 \subset \cdots \subset I_1 \cup \cdots \cup I_k = V$ is a splitting chain of $B$ and $N(L) = N_v$. The statement of theorem follows from theorem 3.2.

**Corollary 4.4.** The map $F : B \to QSym$, defined by $F(B) = F(P_B)$, is a morphism of combinatorial Hopf algebras.

**Proof.** The character $\zeta : B \to k$, defined by $\zeta(B) = 1$ if $B$ is discrete and zero otherwise, turns $B$ into the combinatorial Hopf algebra (see [1] for a general background on combinatorial Hopf algebras). For the combinatorial Hopf algebra $(B, \zeta)$ there is a unique morphism $\Psi : B \to QSym$ to the combinatorial Hopf algebra of quasisymmetric functions ([1], Theorem 4.1). By theorem 4.3, this morphism is given exactly by the map $F$.

As a consequence we obtain the following identities for the function $F$:

\[
F(P_{B_1} \times P_{B_2}) = F(P_{B_1})F(P_{B_2}),
\]

\[
\Delta(F(B)) = \sum_{I \subset V} F(P_{B|I}) \otimes F(P_{B/I}).
\]

In addition, the mapping $F^* : B \to QSym$ defined by (2) is also a morphism of combinatorial Hopf algebras. The formula for antipode (3) implies

\[
(-1)^n F^*(P_B) = \sum_{k \geq 1} (-1)^k \sum_{\mathcal{L}_k} \prod_{j=1}^{k} F(P_{(B|I_j)/I_{j-1}}).
\]

**Remark 4.5.** The function $F(P_B)$ is not a combinatorial invariant of nestohedra. For example, the building sets $B_1 = \{1, 2, 3, 4, 12, 13\}$ and $B_2 = \{1, 2, 3, 4, 12, 13, 34\}$ on the four element set $V = [4]$ have $P_{B_1}$ and $P_{B_2}$ combinatorially equivalent to the 3-cube, but $F(B_1) \neq F(B_2)$.

**Problem 4.6.** In what extent the function $F(P_{B(\Gamma)})$ differs simple graphs? Find two non-isomorphic graphs $\Gamma_1$ and $\Gamma_2$ such that $F(P_{B(\Gamma_1)}) = F(P_{B(\Gamma_2)})$.

**Problem 4.7.** In any combinatorial Hopf algebra are defined the generalized Dehn-Sommerville relations which characterize the odd subalgebra (see [1], Section 5). Find a building set that satisfy the generalized Dehn-Sommerville relations for $B$. In [4] is defined the class of eulerian building sets that satisfies the generalized Dehn-Sommerville relations for the chromatic Hopf algebra of building set $BSet$. Similarly defined eulerian subalgebra of $B$ is trivial.

8
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