DATA-DRIVEN BASIS FOR RECONSTRUCTING THE CONTRAST IN INVERSE SCATTERING: PICARD CRITERION, REGULARITY, REGULARIZATION, AND STABILITY

SHIXU MENG

Abstract. We consider the inverse medium scattering of reconstructing the medium contrast using Born data, including the full aperture, limited-aperture, and multi-frequency data. We propose data-driven basis functions for these inverse problems based on the generalized prolate spheroidal wave functions and related eigenfunctions. Such data-driven eigenfunctions are eigenfunctions of a Fourier integral operator; they remarkably extend analytically to the whole space, are doubly orthogonal, and are complete in the class of band-limited functions. We first establish a Picard criterion for reconstructing the contrast using the data-driven basis, where the reconstruction formula can also be understood from the viewpoint of data processing and analytic extrapolation. Another salient feature associated with the generalized prolate spheroidal wave functions is that the data-driven basis for a disk is also a basis for a Sturm-Liouville differential operator. With the help of Sturm-Liouville theory, we estimate the $L^2$ approximation error for a spectral cutoff approximation of $H^s$ functions. This yields a spectral cutoff regularization strategy for noisy data and an explicit stability estimate for contrast in $H^s (0 < s < 1/2)$ in the full aperture case. In the limited-aperture and multi-frequency cases, we also obtain spectral cutoff regularization strategies for noisy data and stability estimates for a class of contrast.

Key Words. inverse medium scattering, generalized prolate spheroidal wave function, Born approximation, Picard criterion, regularization, stability.

1. Introduction

Inverse scattering is of great importance in non-destructive testing, medical imaging, geophysical exploration, and numerous problems associated with target identification. Let us first introduce the inverse medium scattering problem in two dimensions. Let $k > 0$ be the wave number. A plane wave takes the following form:

$$e^{ikx \cdot \hat{\theta}}, \quad \hat{\theta} \in S := \{ x \in \mathbb{R}^2 : |x| = 1 \},$$

where $\hat{\theta}$ is the direction of propagation. Let $\Omega \subset \mathbb{R}^2$ be an open and bounded set with Lipschitz boundary $\partial \Omega$ such that $\mathbb{R}^2 \setminus \overline{\Omega}$ is connected. The set $\Omega$ is referred to as the medium. Let the real-valued function $q(x) \in L^\infty(\Omega)$ be the contrast of the medium supported in $\Omega$ (which gives rise to the refractive index $1 + q$) and $q \geq 0$ on $\Omega$. The medium scattering due to a plane wave $e^{ikx \cdot \hat{\theta}}$ is to find total wave field $e^{ikx \cdot \hat{\theta}} + u^s(x; \hat{\theta}; k)$ belonging to $H^1_{\text{loc}}(\mathbb{R}^2)$ such that

$$\Delta_x \left( u^s(x; \hat{\theta}; k) + e^{ikx \cdot \hat{\theta}} \right) + k^2 \left( 1 + q(x) \right) \left( u^s(x; \hat{\theta}; k) + e^{ikx \cdot \hat{\theta}} \right) = 0 \quad \text{in} \quad \mathbb{R}^2, \quad (1)$$

$$\lim_{r := |x| \to \infty} \sqrt{r} \left( \frac{\partial u^s(x; \hat{\theta}; k)}{\partial r} - iku^s(x; \hat{\theta}; k) \right) = 0,$$

1Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China. shixumeng@amss.ac.cn
where (2) holds uniformly in all directions. The scattered wave field is \( u^s(\cdot; \hat{\theta}; k) \). A solution is called radiating if it satisfies (2). This scattering problem is well-posed and there exists a unique radiating solution to (1)–(2); cf. [11, 21]. We refer to (1)–(2) as the full model.

Born approximation is a widely used method to treat inverse problems; cf. [11, 29, 32]. In the Born approximation region, one can approximate the solution \( u^s(\cdot; \hat{\theta}; k) \) by its Born approximation \( u^b_0(\cdot; \hat{\theta}; k) \) [11], which is the unique radiating solution to

\[
\Delta_x u^b_0(x; \hat{\theta}; k) + k^2 u^b_0(x; \hat{\theta}; k) = -k^2 q(x) e^{ikx \cdot \hat{\theta}} \quad \text{in} \quad \mathbb{R}^2.
\]

Every radiating solution of the Helmholtz equation has the following asymptotic behavior at infinity [7]:

\[
u^b_0(x; \hat{\theta}; k) = \frac{e^{ikr}}{\sqrt{8k\pi} \sqrt{r}} \left\{ \tilde{u}^\infty_0(\hat{x}; \hat{\theta}; k) + O\left(\frac{1}{r}\right) \right\} \quad \text{as} \quad r = |x| \to \infty,
\]

uniformly with respect to all directions \( \hat{x} := x/|x| \in \mathbb{S} \). The complex-valued function \( \tilde{u}^\infty_0(\hat{x}; \hat{\theta}; k) \) defined on \( \mathbb{S} \) is known as the scattering amplitude or far-field pattern with \( \hat{x} \in \mathbb{S} \) denoting the observation direction. It directly follows from [7] that

\[
\tilde{u}^\infty_0(\hat{x}; \hat{\theta}; k) = k^2 \int_{\Omega} e^{-ik\hat{x} \cdot p'} q(p') e^{ikp' \cdot \hat{\theta}} dp'.
\]  

(3)

The linear sampling method [10] and factorization method [18] developed in the 1990s apply to the full model of the inverse medium scattering and apply to the Born model (3) more explicitly (see [19] for a comparison between the full far-field operator and the Born far-field operator). Using a far-field operator \( L^2(\mathbb{S}) \to L^2(\mathbb{S}) \) whose kernel is the far-field pattern, these sampling methods are capable of reconstructing the support of the contrast \( q \), which requires little a priori information about the scattering objects, and provides both theoretical justifications and robust numerical algorithms. In particular, the linear sampling method [10] and the factorization method [18] as well as the convex scattering support [22, 36] use an infinite series and Picard criterion for the reconstruction. We refer the reader to [8, 21] for a more comprehensive introduction.

In this paper we follow the same idea and aim to reconstruct not only the shape but also the contrast by data-driven basis functions related to the generalized prolate spheroidal wave functions [33]. The idea begins with formulating the inverse problem as follows (see also [20, Section 7.4]).

**Inverse problem** with full aperture Born data: determine the contrast \( q \in L^2(\Omega) \) from

\[
\{ u^\infty_b(p; k) : p \in B(0, 2) \}
\]

where \( u^\infty_b(p; k) \) is given by (4)

\[
u^\infty_b(p; k) := \int_{\Omega} e^{ikp' \cdot \cdot \hat{\theta}} dp'.
\]  

(4)

This equivalent formulation is due to (3) and the fact that \( B(0, 2) \) is the interior of \( \{ \hat{\theta} - \hat{x} : \hat{x}, \hat{\theta} \in \mathbb{S} \} \). Similar to the linear sampling/factorization method where one investigates the far-field operator \( L^2(\mathbb{S}) \to L^2(\mathbb{S}) \), by processing the data as (4) we consider an operator \( L^2(\Omega) \to L^2(B(0, 2)) \) (which will be formulated shortly as an operator \( L^2(D_F) \to L^2(D_F) \) for some set \( D_F \) to treat the inverse problem. Note that by formulating the Born inverse scattering problem as (4), the comparison between the full far-field operator and the operator associated with (4) may be different from [19].
The inverse problems in the form of (4) is important in science, engineering, and technology. Its application is beyond Born inverse scattering: it also merits application in the inverse source problem (in which case $q$ stands for the intensity of the source, $p$ stands for the observation direction, and $k$ still stands for the frequency/wave number), computerized tomography [30], and Fourier analysis [33, 34, 35, 39]. We are most interested in inverse scattering and thereby have chosen to introduce the problem in the context of inverse scattering.

Our data-driven basis in this paper yields highly accurate and efficient reconstruction algorithms. Such a basis stems from an eigensystem \( \{ \psi_{m,n,\ell}(\cdot; c), \alpha_{m,n}(c) \}_{\ell \in \mathbb{N}} \) that satisfies

\[
\int_{B(0,1)} e^{i p' \cdot p} \psi_{m,n,\ell}(p'; c) \, dp' = \alpha_{m,n}(c) \psi_{m,n,\ell}(p; c), \quad p \in B(0,1),
\]

where $c > 0$ is a positive constant, $\mathbb{N} = \{0, 1, 2, \cdots\}$, $\mathbb{N}(m) = \{1\}$ if $m = 0$, and $\mathbb{N}(m) = \{1, 2\}$ if $m \geq 1$. Orthogonal basis functions serve as an important tool in analyzing inverse problems and in establishing regularization strategies [20, Chapters 2–3]. This data-driven basis originated in the work of Slepian [33] who gave a thorough theory on the generalized prolate spheroidal wave function (which is the radial part of $\psi_{m,n,\ell}(\cdot; c)$) and the related Fourier analysis in multiple dimensions. In one dimension, the generalized prolate spheroidal wave function corresponds to the prolate spheroidal wave function [35]. Remarkably, the (generalized) prolate spheroidal wave function is the eigenfunction of both a Fourier type integral operator and a Sturm-Liouville differential operator, which merits important applications in analytic extrapolation, approximation theory, uncertainty quantification, and Fourier analysis [33, 34, 35, 39]. Though such a data-driven basis applies naturally to (4), the application of generalized prolate spheroidal wave functions is rather limited in multi-dimensional inverse scattering and inverse source problems to the best of our knowledge. Indeed, Slepian’s discrete prolate spheroidal wave function (in one dimension) [34] applied naturally to the limited angle problem of computerized tomography [30, Chapter VI]. It amounts to extrapolating the limited angle data to obtain the full angle data. Recently, Slepian’s discrete prolate spheroidal wave function was applied in limited-aperture inverse scattering in $\mathbb{R}^2$ [12]. In particular, the discrete prolate spheroidal wave functions are doubly orthogonal (over the limited-aperture and the full aperture) and they serve as a data-driven basis for data completion or a Galerkin projection basis for the limited-aperture factorization method [21, Corollary 2.17]. It would be desirable to develop an explicit stability estimate for the data completion algorithm of [12] in the spirit of this work. Note that the discrete prolate spheroidal wave functions can be derived by a high-fidelity trigonometric approximation of the data followed by applying a reduced basis method with singular value decomposition [12]; this is in line with the reduced basis method in [31]. As such we are motivated to call the set of prolate related functions a data-driven basis. In a broader context, it is possible to apply a reduced basis method such as [31] or other physics-informed basis functions using machine learning such as [17] to study the inverse problems. The data-driven basis in this paper can also be learned via a Legendre-Galerkin neural network; in this paper we study this data-driven basis analytically by the generalized prolate spheroidal wave functions and remark that our result indeed serves as a mathematical foundation for relevant machine learning algorithms. We also note when our work is near completion that [16] applied the one dimensional prolate spheroidal wave functions together with the Radon transform to study reconstructions from the Fourier transform on the ball.
In this work we show how to use the generalized prolate spheroidal wave functions and their related eigenfunctions to solve the Born inverse scattering problem. We emphasize the following: (1) First, the generalized prolate spheroidal wave functions and their related eigenfunctions \cite{33} form a data-driven basis for a Fourier integral operator associated with (4). This allows us to establish a Picard criterion to reconstruct the contrast. This is in the spirit of the factorization/linear sampling method. (2) Second, note that if one has the knowledge of the data in the whole space $\mathbb{R}^2$ (which is guaranteed by unique continuation in theory), then one can solve for the contrast by classical inverse Fourier transform. The data-driven basis remarkably extends analytically to $\mathbb{R}^2$, is doubly orthogonal, and is complete in the class of band-limited functions \cite{33, 35}. We show that the reconstruction by Picard criterion can be understood from the viewpoint of data processing where one extrapolates the data to $\mathbb{R}^2$ first and then applies the inverse Fourier transform. The ill-posed nature of analytic extrapolation is revealed by the decay of the eigenvalues associated with the data-driven basis. (3) Third, the generalized prolate spheroidal wave functions (i.e., radial part of the data-driven basis) is also a basis for a positive definite Sturm-Liouville differential operator in the radial variable $s$ spheroidal wave functions (i.e., radial part of the basis) is indeed a basis for another self-adjoint, positive definite Sturm-Liouville differential operator in the variable $x$. Thereby one can study the regularity estimate using such a Sturm-Liouville operator. In particular, we show that the data-driven basis yields a natural definition of a suitable Sobolev space $H^s$ in connection with the classical Sobolev space $H^s$ and we estimate the $L^2$ approximation error for a spectral cutoff approximation of functions in $H^s$, $0 < s \leq 1$. This type of estimate is of independent interest in approximation theory using the data-driven basis. (4) Last but not least, as the data is “on the right hand side” of the operator equation, by the regularity estimate we are able to obtain an explicit regularization strategy for noisy data and establish a stability estimate for contrast (or more precisely, an extension of the contrast) belonging to the classical Sobolev space $H^s$, $0 < s < 1/2$. Such a stability estimate is expected to give a possible Hölder-logarithmic type stability estimate by choosing appropriate parameters in the regularization.

Limited-aperture data (cf. \cite{2, 4, 14, 15, 23}) and multi-frequency partial data (cf. \cite{1, 5, 13, 24, 28}) pose great challenges in inverse scattering. These partial data indicate that inverse problems are more ill-posed. In this work, in addition to the full aperture case, we moreover study the following limited-aperture and multi-frequency inverse problems.

**Inverse problem** with limited-aperture and multi-frequency data: determine the contrast $q \in L^2(\Omega)$ from

- limited-aperture Born data $\{\tilde{u}_b^\infty(\hat{x}; \hat{\theta}; k) : \hat{x}, \hat{\theta} \in S_L \}, S_L := \{x : x \in \mathbb{S}, \arg x \in [-\Theta, \Theta], 0 < \Theta < \pi \}$, where these data are equivalent to
  \[
  \{u_b^\infty(p; k) : p \in L \};
  \]
  here $L$ is the interior of $\{\hat{\theta} - \hat{x} : \hat{x}, \hat{\theta} \in S_L \}$ which is symmetric with respect to the origin, and $u_b^\infty$ is given by (4).

- multi-frequency Born data with two opposite observation directions $\{\tilde{u}_b^\infty(\pm \hat{x}^*; \hat{\theta}; k) : \hat{\theta} \in \mathbb{S}, k \in (0, K), K > 0 \}$ which are equivalent to
  \[
  \{u_b^\infty(p; K) : p \in M \};
  \]
  where $M$ is the interior of $\{a\hat{\theta} \pm ax^* : \hat{\theta} \in \mathbb{S}, a \in (0, 1) \}$ which is symmetric with respect to the origin, and $u_b^\infty$ is given by (4).

In each of the limited-aperture and multi-frequency cases, the inverse problem corresponds to inverting the integral operator with data in a symmetric set $L$ and $M$, respectively.
(as opposed to the full aperture case where we invert the integral operator with data in a disk $B(0, 2)$).

The application of the generalized prolate spheroidal wave functions (and their related functions) is far less developed for the limited-aperture and multi-frequency inverse scattering problems. Remarkably, there also exists a data-driven basis for a general symmetric set $\mathbb{A}$. The above Picard criterion and analytic extrapolation apply in exactly the same way to the limited-aperture and multi-frequency cases. Due to the fact that the area of the symmetric set is in general less than the area of $B(0, 2)$ in the full aperture case, the Picard criterion and analytic extrapolation expose more ill-posedness (see [12, 25, 37] for similar observations). Due to the lack of a Sturm-Liouville theory for the general symmetric set (to the best of our knowledge), the stability estimate differs from the one in full aperture case, where the stability estimate for partial data relies on less explicit a priori information on the contrast (for instance, we require that the contrast belongs to the range of some operator).

The remainder of the paper is organized as follows. In Section 2, we summarize the generalized prolate spheroidal wave functions and Fourier analysis [33] that are needed in our study. In Section 3 we generate the data-driven basis for the full aperture data and establish a Picard criterion to reconstruct the contrast in the spirit of the factorization/linear sampling method. We also show that the reconstruction by the Picard criterion can be understood from the viewpoint of data processing where one extrapolates the data to $\mathbb{R}^2$ first and then applies the inverse Fourier transform. Section 4 is devoted to the regularity estimate, regularization strategy, and stability estimate. In particular, with the help of the Sturm-Liouville theory developed in Section 4.1, we estimate in Section 4.2 the $L^2(B(0, 1))$ approximation error for a spectral cutoff approximation of functions in $H^s(B(0, 1))$, $0 < s \leq 1$. This allows us to develop the spectral cutoff regularization and its stability estimate in Section 4.3. Finally we develop in Section 5 the Picard criterion and stability estimate for the limited-aperture and multi-frequency cases.

Throughout the paper, we consistently use the superscript/subscript “F”, “L”, and “M” to represent that relevant functions, constants, and operators are for the full aperture data, the limited-aperture data, and the multi-frequency partial data, respectively.

2. Generalized prolate spheroidal wave functions and Fourier analysis

In this section, we summarize some of the main results [33] that are needed in our study. Let $c$ be a positive constant. We are interested in finding the eigensystem of the integral operator $\mathcal{F}_A^c : L^2(A) \rightarrow L^2(A)$

$$(\mathcal{F}_A^c \psi)(p) = \int_A e^{icp\cdot p'} \psi(p') dp', \quad \forall p \in A, \forall \psi \in L^2(A)$$

where $A$ is a symmetric ($x \in A$ if and only if $-x \in A$), bounded, open set. We summarize below a particular theory for a unit disk $B(0, 1)$ and a general theory for a general symmetric set $A$. All the following results in this section are from [33].

2.1. Generalized prolate spheroidal wave functions for unit disk.

When the set $A$ is a unit disk $B(0, 1)$, there exists an orthogonal eigensystem \{\psi_{m,n,\ell}(\cdot; c), \alpha_{m,n}(c)\}_{m,n,\ell}^{\ell \in \mathbb{N}} [33, pp. 3015–3018] that satisfies

$$(\mathcal{F}_{B(0,1)}^c \psi_{m,n,\ell}(\cdot; c))(p) = \int_{B(0,1)} e^{icp\cdot p'} \psi_{m,n,\ell}(p'; c) dp' = \alpha_{m,n}(c) \psi_{m,n,\ell}(p; c), \quad p \in B(0,1)$$ (5)
with \( N = \{0, 1, 2, \ldots \} \) and

\[
\mathbb{I}(m) = \begin{cases} 
\{1\} & m = 0 \\
\{1, 2\} & m \geq 1 
\end{cases}
\]

- Each eigenfunction \( \psi_{m,n,\ell}(\cdot; c) \) can be obtained via separation of variables

\[
\psi_{m,n,\ell}(\cdot; c)(x) = |x|^{-\frac{1}{2}} \varphi_{m,n}(|x|; c) Y_{m,\ell}(\theta_x), \quad x \in B(0, 1), \quad \theta_x = \arg x,
\]

where

\[
Y_{m,\ell}(\theta) = \begin{cases} 
1 & m = 0, \ell = 1 \\
\cos(m\theta) & m \geq 1, \ell = 1 \\
\sin(m\theta) & m \geq 1, \ell = 2
\end{cases}
\]

and \( \{\varphi_{m,n}(\cdot; c), \gamma_{m,n}(c)\}_{m,n \in \mathbb{N}} \) is the eigensystem that satisfies

\[
\int_0^1 J_m(crr')\sqrt{cr'} \varphi_{m,n}(r'; c) \, dr' = \gamma_{m,n}(c) \varphi_{m,n}(r; c), \quad 0 < r < 1,
\]

here the eigenvalue \( \gamma_{m,n}(c) \) is related to \( \alpha_{m,n}(c) \) by \( \gamma_{m,n}(c) = \frac{c^{1/2}}{2\pi r^{1/2}} \alpha_{m,n}(c) \). The function \( \varphi_{m,n}(\cdot; c) \) is chosen as real-valued and it is called a generalized prolate spheroidal wave function according to Slepian.

- Every eigenvalue \( \alpha_{m,n}(c) \) is non-zero and

\[ |\alpha_{m,n}(c)| \rightarrow 0, \quad m, n \rightarrow \infty. \]

\( \{\psi_{m,n,\ell}(\cdot; c)\}_{m,n \in \mathbb{N}}^{\ell \in \mathbb{I}(m)} \) is a complete and orthogonal set in \( L^2(B(0, 1)) \).

- By extending the domain of each \( \psi_{m,n,\ell}(\cdot; c) \) to \( \mathbb{R}^2 \) naturally (and without the danger of confusion) via

\[
\psi_{m,n,\ell}(p; c) := \frac{1}{\alpha_{m,n}(c)} \int_{B(0, 1)} e^{i p \cdot r'} \psi_{m,n,\ell}(p'; c) \, dp', \quad p \in \mathbb{R}^2,
\]

one can get the double orthogonality \([33, pp. 3013–3015]\), for any \( m, n \in \mathbb{N}, \ell \in \mathbb{I}(m) \),

\[
\int_{B(0, 1)} \psi_{m,n,\ell}(p; c) \psi_{m',n',\ell'}(p; c) \, dp = \left( \frac{c}{2\pi} \right)^2 |\alpha_{m,n}(c)|^2 \delta_{m,m'} \delta_{n,n'} \delta_{\ell,\ell'},
\]

and each \( \psi_{m,n,\ell}(\cdot; c) \) is normalized such that its energy in \( \mathbb{R}^2 \) is unit or equivalently

\[
\int_{B(0, 1)} \psi_{m,n,\ell}(p; c) \psi_{m',n',\ell'}(p; c) \, dp = \left( \frac{c}{2\pi} \right)^2 |\alpha_{m,n}(c)|^2 \delta_{m,m'} \delta_{n,n'} \delta_{\ell,\ell'},
\]

here \( \delta \) denotes the Kronecker delta. For the purpose of a clean presentation later on, we use \( \|\psi_{m,n,\ell}(\cdot; c)\| \) to represent the \( \|\psi_{m,n,\ell}(\cdot; c)\|_{L^2(B(0, 1))} \) norm of the eigenfunction \( \psi_{m,n,\ell}(\cdot; c) \) (and similar notation holds for other doubly orthogonal eigenfunctions); when we discuss the \( \|\psi_{m,n,\ell}(\cdot; c)\|_{L^2(\mathbb{R}^2)} \) norm, we shall state this explicitly.

- The remarkable property of the generalized prolate spheroidal wave function (eigenfunctions given by (7)) is that \( \varphi_{m,n}(\cdot; c) \) is also the eigenfunction of the (singular) Sturm–Liouville differential operator

\[
\mathcal{D}_{c,r} \varphi_{m,n}(\cdot; c) = \chi_{m,n}(c), \quad \text{where}
\]

\[
\mathcal{D}_{c,r} := -(1 - r^2) \frac{d^2}{dr^2} + 2r \frac{d}{dr} - \left( \frac{1/4 - m^2}{r^2} - c^2 r^2 \right),
\]

and
and \( \chi_{m,n}^\alpha(c) \) is the corresponding eigenvalue. When \( m = 1/2 \), this equation reduces to the equation for prolate spheroidal wave functions of order zero [35].

Remarkably, the (generalized) prolate spheroidal wave function is the eigenfunction of both a Fourier type integral operator and a Sturm-Liouville differential operator, which merits important applications in analytic extrapolation, approximation theory, uncertainty quantification, and Fourier analysis [33, 35].

2.2. Eigensystem for symmetric set.

Recall that \( A \subset \mathbb{R}^2 \) is a symmetric bounded open set. Consider the eigenvalue problem of finding eigenfunctions \( \psi_e(\cdot; c), \psi_o(\cdot; c) \in L^2(A) \) and eigenvalues \( \beta_e(c), \beta_o(c) \) such that

\[
\beta_e(c) \psi_e(p; c) = \int_A \cos(c \cdot p') \psi_e(p'; c) \, dp', \quad p \in A,
\]

\[
\beta_o(c) \psi_o(p; c) = \int_A \sin(c \cdot p') \psi_o(p'; c) \, dp', \quad p \in A.
\]

Since \( \cos \) is an even function and \( A \) is symmetric, it follows from [33, pp. 3014–3015] that there exists an eigensystem \( \{\psi_{e,n}(\cdot; c), \beta_{e,n}(\cdot; c)\}_{n=0}^\infty \) such that \( \{\psi_{e,n}(\cdot; c)\}_{n=0}^\infty \) is complete in the set of even functions in \( L^2(A) \), and \( \psi_{e,n}(\cdot; c) \) and \( \beta_{e,n}(\cdot; c) \) are real-valued. Similarly there exists an eigensystem \( \{\psi_{o,n}(\cdot; c), \beta_{o,n}(\cdot; c)\}_{n=0}^\infty \) such that \( \{\psi_{o,n}(\cdot; c)\}_{n=0}^\infty \) is complete in the set of odd functions in \( L^2(A) \), and \( \psi_{o,n}(\cdot; c) \) and \( \beta_{o,n}(\cdot; c) \) are real-valued. The set \( \{\psi_{o,n}(\cdot; c)\}_{n=0}^\infty \cup \{\psi_{e,n}(\cdot; c)\}_{n=0}^\infty \) is complete in \( L^2(A) \).

In this way one can obtain the eigensystem \( \{\psi_n(\cdot; c), \alpha_n(\cdot; c)\}_{n=0}^\infty : = \{\psi_{e,n}(\cdot; c), \beta_{e,n}(\cdot; c)\}_{n=0}^\infty \cup \{\psi_{o,n}(\cdot; c), i \beta_{o,n}(\cdot; c)\}_{n=0}^\infty \) that satisfies

\[
\alpha_n(c) \psi_n(p; c) = \int_A e^{icp\cdot p'} \psi_n(p'; c) \, dp', \quad p \in A.
\]

The eigenfunctions belong to \( L^2(A) \) and are real-valued, orthogonal, and either even (in which case the eigenvalue is real) or odd (in which case the eigenvalue is purely imaginary). Every eigenvalue \( \alpha_n(c) \) is non-zero due to an energy argument (using Fourier transform) [33, pp. 3012]. Here in this general symmetric case we have associated a single subscript \( n \) with the use of \( \psi_n \) to avoid any danger of confusion.

By extending the domain of \( \psi_n(\cdot; c) \) to \( \mathbb{R}^2 \) naturally via

\[
\psi_n(p; c) := \frac{1}{\alpha_n(c)} \int_A e^{icp\cdot p'} \psi_n(p'; c) \, dp', \quad p \in \mathbb{R}^2, \quad \forall n \in \mathbb{N},
\]

one can find the double orthogonality [33, p. 3013]

\[
\int_A \psi_m(p; c) \psi_n(p; c) \, dp = \left( \frac{c}{2\pi} \right)^2 |\alpha_n(c)|^2 \int_{\mathbb{R}^2} \psi_m(p; c) \psi_n(p; c) \, dp, \quad \forall m, n \in \mathbb{N}, \quad (13)
\]

and we normalize each eigenfunction such that its energy in \( \mathbb{R}^2 \) is unit or equivalently

\[
\int_A \psi_m(p; c) \psi_m'(p; c) \, dp = \left( \frac{c}{2\pi} \right)^2 |\alpha_m(c)|^2 \delta_{mm'}, \quad \forall m, m' \in \mathbb{N}. \quad (14)
\]

3. Picard Criterion and Analytic Extrapolation

In this section, we study the Picard criterion for reconstructing the contrast from the full aperture Born data and show that the reconstruction formula can be understood from the viewpoint of data processing and analytic extrapolation.

We begin with a formulation of the inverse problem. Recall in Section 1 that we aim to determine the contrast \( q \in L^2(\Omega) \) from

\[
\{u^\infty_k(p; k) : p \in B(0, 2)\}
\]
where \( u_\alpha^\infty(p; k) \) is given by (4), i.e.,
\[
u_\alpha^\infty(p; k) = \int_\Omega e^{ikp\cdot q(p')} \, dp', \quad \forall p \in B(0, 2).
\]
Now we choose a positive constant \( c_F := c_F(\Omega, k) \) depending on \( \Omega \) and \( k \) such that \( \Omega \subset D_F \) with \( D_F := \left\{ \frac{\alpha x}{k^n} \, : \, x \in B(0, 1) \right\} \). Let \( u_{b,F}^\infty \in L^2(D_F) \) be given by
\[
u_{b,F}^\infty(p) := \int_{D_F} e^{i \frac{\alpha x}{k^n} \cdot p'} \, q(p') \, dp', \quad \forall p \in D_F,
\]and let \( q \) be the extension of \( q \)
\[
q(x) := \left\{ \begin{array}{ll}
q(x) & x \in \Omega \\
0 & x \notin \Omega,
\end{array} \right. \quad a.e. \quad x \in \mathbb{R}^2.
\]
Then the knowledge of \( \{ u_\alpha^\infty(p; k) : p \in B(0, 2) \} \) amounts to the knowledge of \( \{ u_{b,F}^\infty(p) : p \in D_F \} \). Now the inverse problem is formulated as follows.

**Formulation of the inverse problem:** determine the contrast \( q \in L^2(D_F) \) from \( \{ u_{b,F}^\infty(p) : p \in D_F \} \).

We next introduce a suitable eigensystem for the study of (15). For the chosen positive constant \( c_F \), recalling in Section 2.1 that \( \{ \psi_{m,n,\ell}(; c_F), \alpha_{m,n}(c_F) \}_{m,n \in \mathbb{N}}^{\ell \in \mathbb{N}(m)} \) is an eigensystem in \( L^2(B(0, 1)) \), we introduce an orthogonal, complete set \( \{ \psi^F_{m,n,\ell}(; c_F) \}_{m,n \in \mathbb{N}}^{\ell \in \mathbb{N}(m)} \) in \( L^2(D_F) \) by
\[
\psi^F_{m,n,\ell}(x; c_F) := \frac{2k}{c_F} \psi_{m,n,\ell} \left( \frac{2k}{c_F} x; c_F \right), \quad x \in \mathbb{R}^2.
\]
It is then verified by a change of variable together with (8) that
\[
\int_{D_F} e^{i \frac{\alpha x}{k^n} \cdot p'} \psi^F_{m,n,\ell}(p'; c_F) \, dp' = \left( \frac{c_F}{2k} \right)^2 \alpha_{m,n}(c_F) \psi^F_{m,n,\ell}(p; c_F), \quad p \in \mathbb{R}^2,
\]
and by the double orthogonality from (9) that
\[
\int_{\mathbb{R}^2} \psi^F_{m,n,\ell}(p; c_F) \psi^{F,*}_{m',n',\ell'}(p; c_F) \, dp = \delta_{m,n'} \delta_{m',n} \delta_{\ell,\ell'},
\]
\[
\int_{D_F} \psi^F_{m,n,\ell}(p; c_F) \psi^{F,*}_{m',n',\ell'}(p; c_F) \, dp = \left( \frac{c_F}{2\pi} \right)^2 |\alpha_{m,n}(c_F)|^2 \delta_{m,n'} \delta_{m',n} \delta_{\ell,\ell'}.
\]

We are now ready to prove the following theorem. Let \( \langle \cdot, \cdot \rangle_D \) be the \( L^2(D) \)-inner product for a generic open bounded set \( D \) and \( \| \cdot \|_D \) be the induced norm. When there is no confusion, we drop the subscript for the best presentation.

**Theorem 1.** Let \( c_F > 0 \) be chosen such that \( \Omega \subset D_F = B(0, \frac{c_F}{k^n}) \). Let \( q \) be the extension (16) such that \( q = q \) in \( \Omega \) and \( q = 0 \) outside \( \Omega \) almost everywhere. Then \( q \) is solved by the Picard criterion
\[
q = \sum_{m,n \in \mathbb{N}} \left( \frac{2k}{c_F} \right)^2 \frac{1}{\alpha_{m,n}(c_F)} \left\| u_{b,F}^\infty \right\|_{D_F} \left\| \psi^F_{m,n,\ell}(; c_F) \right\|_{D_F} \psi^F_{m,n,\ell}(; c_F) \| \psi^F_{m,n,\ell}(; c_F) \|_{D_F}
\]
where the convergence is in \( L^2(D_F) \).

**Proof.** Note that \( q \in L^2(D_F) \) and \( \{ \psi^F_{m,n,\ell}(; c_F) \}_{m,n \in \mathbb{N}}^{\ell \in \mathbb{N}(m)} \) is complete in \( L^2(D_F) \); then \( q \) can be represented by a convergent series in \( L^2(D_F) \)
\[
q(p) = \sum_{m,n \in \mathbb{N}} \frac{\psi^F_{m,n,\ell}(p; c_F)}{\| \psi^F_{m,n,\ell}(; c_F) \|}, \quad p \in D_F.
\]
It suffices to determine \( q_{m,n,\ell} \). Note that \( \psi_{m,n,\ell}^F \) is an eigenfunction according to (18), we have from (15) that

\[
u_{b,F}^\infty(p) = \int_{D_F} e^{i \frac{\alpha_{m,n}^F}{c_F} q(p')} \omega(p') \, dp',
\]

where the series converges in \( L^2(D_F) \). From the full knowledge of \( \nu_{b,F}^\infty(p) \) in \( L^2(D_F) \), we have that

\[
q_{m,n,\ell} = \left( \frac{2k}{c_F} \right)^2 \frac{1}{\alpha_{m,n}(c_F)} \left\langle \nu_{b,F}^\infty, \frac{\psi_{m,n,\ell}^F(\cdot; c_F)}{\| \psi_{m,n,\ell}^F(\cdot; c_F) \|_{D_F}} \right\rangle_{D_F}.
\]

This completes the proof. \( \square \)

It is useful to make a remark on \( \Omega \) and \( D_F \). The parameter \( c_F \) is the product of the radius of the ball \( B(0,2k) \) that supports the restricted Fourier transform and the radius of the ball \( D_F \) that supports the contrast (i.e., \( c_F = 2k \frac{\alpha_{m,n}}{\alpha_{m,n}} \)). The performance of the reconstruction formula with noisy data may be observed by a stability estimate involving a delicate interplay of \( D_F, c_F, \) and several other parameters (cf. Theorem 3). At the current stage, we have not reached an explicit conclusion on how the choice of \( D_F \) explicitly determines the performance of reconstructing the original contrast \( q \). There is a similar piece of work on contrast reconstruction using a prolate-Galerkin linear sampling method, we refer the reader to [3] for related numerical experiments.

3.1. Data processing and analytic extrapolation.

The prolate spheroidal wave functions in one dimension extend analytically to \( \mathbb{R} \), are doubly orthogonal, and are complete in the class of band-limited functions \([35]\). Similarly the eigenfunctions \( \{ \psi_{m,n,\ell}^F(\cdot; c_F) \}_{m,n\in\mathbb{N}}^{\ell\in\mathbb{N}(m)} \) also extend analytically to \( \mathbb{R}^2 \) (18), are doubly orthogonal (19)–(20), and are complete in the class of band-limited functions in multiple dimensions (which is indicated by [33] with techniques in [35]; we include a proof for completeness). This is the key to extrapolating the data from \( D_F \) to the whole space \( \mathbb{R}^2 \). In this section we show how to interpret the reconstruction formula (21) of Theorem 1 from the viewpoint of data processing and analytic extrapolation.

The idea is to extrapolate \( \nu_{b,F}^\infty \) to the entire \( \mathbb{R}^2 \) so that \( q \) can then be solved by classical inverse Fourier transform. Let \( \hat{\nu}_{b,F}^\infty \) be an extrapolation of \( \nu_{b,F}^\infty \) such that \( \hat{\nu}_{b,F}^\infty = \nu_{b,F}^\infty \) in \( L^2(D_F) \). The natural analytic extrapolation is to require that \( \hat{\nu}_{b,F}^\infty \) has the following form:

\[
\hat{\nu}_{b,F}^\infty(p) = \int_{D_F} e^{i \frac{\alpha_{m,n}^F}{c_F} q(p')} \omega(p') \, dp', \quad \forall p \in \mathbb{R}^2,
\]

where we have extrapolated \( u_{b,F}^\infty \) analytically to \( \mathbb{R}^2 \) according to (15).

We introduce here the notation of band-limited function in the spirit of [33, 35]: a function \( u \in L^2(\mathbb{R}^2) \) is called a \( D_F \) band-limited function if and only if

\[
u(p) = \int_{D_F} e^{i \frac{\alpha_{m,n}^F}{c_F} q(p')} f(p') \, dp', \quad \forall p \in \mathbb{R}^2,
\]

for some \( f \in L^2(D_F) \). The family of band-limited functions enjoys the following property (which is indicated by [33] with techniques in [35]; we include a proof for completeness). Recall that \( \{ \psi_{m,n,\ell}^F(\cdot; c_F) \in L^2(\mathbb{R}^2) \}_{m,n\in\mathbb{N}}^{\ell\in\mathbb{N}(m)} \) is given by (17).

**Proposition 1.** \( \{ \psi_{m,n,\ell}^F(\cdot; c_F) \in L^2(\mathbb{R}^2) \}_{m,n\in\mathbb{N}}^{\ell\in\mathbb{N}(m)} \) is complete in the space of \( D_F \) band-limited functions in \( L^2(\mathbb{R}^2) \).
Proof. Suppose that \( u \in L^2(\mathbb{R}^2) \) is a \( D_F \) band-limited function given by (23) for some \( f \in L^2(D_F) \). Note that \( \{ \psi_{m,n,\ell}(\cdot; c_F) \in L^2(D_F) \}_{m,n\in\mathbb{N}} \) is complete in \( L^2(D_F) \); then \( f \) can be represented by a convergent series

\[
    f(p) = \sum_{m,n\in\mathbb{N}} f_{m,n,\ell} \frac{\psi_{m,n,\ell}(p; c_F)}{\|\psi_{m,n,\ell}(\cdot; c_F)\|}, \quad p \in D_F.
\]

This together with (23) gives that, for all \( p \in \mathbb{R}^2 \)

\[
    u(p) = \sum_{m,n\in\mathbb{N}} f_{m,n,\ell} \int_{D_F} e^{i\frac{4\pi}{k} p \varphi} \psi_{m,n,\ell}(p') \frac{1}{\|\psi_{m,n,\ell}(\cdot; c_F)\|} \, dp' = \sum_{m,n\in\mathbb{N}} f_{m,n,\ell} \left( \frac{c_F}{2k} \right)^2 \alpha_{m,n}(c_F) \frac{\psi_{m,n,\ell}(p; c_F)}{\|\psi_{m,n,\ell}(\cdot; c_F)\|}, \quad p \in \mathbb{R}^2.
\]

where in the last step we applied (18). This series is convergent in \( L^2(\mathbb{R}^2) \) due to (19)–(20) and the fact that the series of \( f \) converges. This proves the proposition.

Following this notation, the extrapolation \( \tilde{u}_{b,F}^{\infty} \) is the so-called \( D_F \) band-limited function. The following proposition states that such a band-limited function can be fully determined by its value restricted in \( D_F \).

**Proposition 2.** The band-limited function \( \tilde{u}_{b,F}^{\infty} \) (22) can be represented by

\[
    \tilde{u}_{b,F}^{\infty}(p) = \sum_{m,n\in\mathbb{N}} a_{m,n,\ell} \frac{1}{\left( \frac{2\pi}{k} \right)^2 |\alpha_{m,n}(c_F)|^2} \frac{1}{\|\psi_{m,n,\ell}(\cdot; c_F)\|^2} \left( \tilde{u}_{b,F}^{\infty}, \psi_{m,n,\ell}(\cdot; c_F) \right)_{D_F} \psi_{m,n,\ell}(p; c_F), \quad p \in \mathbb{R}^2. \tag{24}
\]

**Proof.** Let the band-limited function \( \tilde{u}_{b,F}^{\infty} \) be represented uniquely by an infinite series

\[
    \tilde{u}_{b,F}(p) = \sum_{m,n\in\mathbb{N}} a_{m,n,\ell} \psi_{m,n,\ell}(p; c_F), \quad p \in \mathbb{R}^2. \tag{25}
\]

Projecting \( \tilde{u}_{b,F}^{\infty} \) (25) onto \( \psi_{m,n,\ell} \) in the disk \( D_F \), one can solve for each coefficient \( a_{m,n,\ell} \) by the double orthogonality (19)–(20) to arrive at (24), where we have kept this particular form as each \( \psi_{m,n,\ell} \) is normalized to have unit norm in \( L^2(\mathbb{R}^2) \).

Now by applying inverse Fourier transform to (22), one can obtain from (24) (where in the second last step we shall apply (26) in the following remark) that

\[
    q(p) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{-ip \cdot p'} \tilde{u}_{b,F}^{\infty} \left( p' \frac{c_F}{4k^2} \right) \, dp' = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{-ip \cdot p'} \sum_{m,n\in\mathbb{N}} \frac{1}{\left( \frac{2\pi}{k} \right)^2 |\alpha_{m,n}(c_F)|^2} \left( \tilde{u}_{b,F}^{\infty}, \psi_{m,n,\ell}(\cdot; c_F) \right)_{D_F} \psi_{m,n,\ell}(p; c_F) \, dp' \\
    = \frac{1}{(2\pi)^2} \sum_{m,n\in\mathbb{N}} \frac{1}{\left( \frac{2\pi}{k} \right)^2 |\alpha_{m,n}(c_F)|^2} \left( \tilde{u}_{b,F}^{\infty}, \psi_{m,n,\ell}(\cdot; c_F) \right)_{D_F} \psi_{m,n,\ell}(p; c_F) \, dp' = \sum_{m,n\in\mathbb{N}} \frac{2k}{c_F} \left( \frac{c_F}{2k} \right)^2 \alpha_{m,n}(c_F) \frac{\psi_{m,n,\ell}(p; c_F)}{\|\psi_{m,n,\ell}(\cdot; c_F)\|}, \quad p \in D_F.
\]

This shows that we get the same result as in Theorem 1 by extrapolating the data first followed by applying the inverse Fourier transform.
Remark 1. In the last second step, we applied
\[
\psi_{m,n,\ell}(p';c_F) = \frac{1}{(2\pi)^2} \frac{(c_F/2k)^2 \alpha_{m,n}(c_F)}{e^{-ip\cdot p'}} \int_{\mathbb{R}^2} e^{-ip\cdot p'} \psi_{m,n,\ell} \left( p \frac{c_F}{4k^2}; c_F \right) \, dp, \quad p' \in D_F,
\]
for any \( m, n \in \mathbb{N}, \ell \in \mathbb{I}(m) \). This can be proved as follows: let \( \mathbb{I}_{D_F} \) be the characteristic function that \( \mathbb{I}_{D_F} = 1 \) in \( D_F \) and \( \mathbb{I}_{D_F} = 0 \) outside \( D_F \). Note that (18) gives
\[
\int_{\mathbb{R}^2} e^{i\frac{p^2}{2k}} \psi_{m,n,\ell}(p';c_F) \mathbb{I}_{D_F}(p') \, dp' = \frac{(c_F/2k)^2 \alpha_{m,n}(c_F)}{e^{-ip\cdot p'}} \psi_{m,n,\ell}(p; c_F), \quad p \in \mathbb{R}^2.
\]
Taking the inverse Fourier transform of the above equation yields
\[
\psi_{m,n,\ell}(p';c_F) \mathbb{I}_{D_F}(p') = \frac{1}{(2\pi)^2} \frac{(c_F/2k)^2 \alpha_{m,n}(c_F)}{e^{-ip\cdot p'}} \int_{\mathbb{R}^2} e^{-ip\cdot p'} \psi_{m,n,\ell} \left( p \frac{c_F}{4k^2}; c_F \right) \, dp,
\]
and this shows (26).

Remark 2. Note that analytic extrapolation is ill-posed in general (cf. [9, 26]), and (24) indeed performs an analytic extrapolation. Note that \( \psi_{m,n,\ell} \) is normalized to have unit norm in \( L^2(\mathbb{R}^2) \) (19), the ill-posedness of analytic extrapolation (24) is revealed by the fact that the eigenvalue \( \lambda_{m,n}(c_F) \) goes to zero as \( m, n \to \infty \).

4. Regularity, Approximation theory, Regularization and Stability

In addition to the salient feature that the data-driven basis \( \{ \varphi_{m,n,\ell}(:,c) \}_{m,n \in \mathbb{N}}^{\ell \in \mathbb{I}(m)} \) extends analytically to \( \mathbb{R}^2 \), is doubly orthogonal, and is complete in the class of band-limited functions, another feature is that \( \{ \varphi_{m,n,\ell}(:,c) \}_{m,n \in \mathbb{N}}^{\ell \in \mathbb{I}(m)} \) is also a basis for a Sturm-Liouville differential operator. This Sturm-Liouville differential operator brings additional regularity estimates that lead to an explicit stability estimate for a spectral cutoff regularization strategy. In this section, we study the relevant Sturm-Liouville theory in Section 4.1 and regularity estimates and approximation theory in Section 4.2, followed by a spectral cutoff regularization and its stability estimate in Section 4.3.

4.1. Sturm-Liouville theory.

Recall that the eigenfunction \( \psi_{m,n,\ell}(:,c) \), \( m, n \in \mathbb{N}, \ell \in \mathbb{I}(m) \), for the disk \( B(0,1) \) is given by (6), and Slepian [33] showed that the radial part \( \varphi_{m,n}(r; c) \) is also the eigenfunction of the following (singular) Sturm–Liouville differential operator (10)–(11), i.e.,
\[
\mathcal{D}_{c,rr} \varphi_{m,n}(c) = \chi_{m,n}(c), \quad \text{where}
\]
\[
\mathcal{D}_{c,rr} = -(1-r^2) \frac{d^2}{dr^2} + 2r \frac{d}{dr} - \left( \frac{1}{4} - \frac{m^2}{r^2} - c^2 r^2 \right),
\]
and \( \chi_{m,n}(c) \) is the corresponding eigenvalue. This is a Sturm-Liouville problem in the radial variable \( r \).

Now we have the following Sturm-Liouville problem in the variable \( x = (r \cos \theta, r \sin \theta)^T \) where \( T \) denotes the transpose. Let the Sturm-Liouville differential operator \( \mathcal{D}_{c,x} \) (see also [39]) be given by
\[
\mathcal{D}_{c,x} := -(1-r^2)\partial_r^2 + (3r-\frac{1}{r})\partial_r - \frac{1}{r^2}\partial_\theta^2 + c^2 r^2.
\]

Lemma 1. Let the eigenfunction \( \psi_{m,n,\ell}(:,c) \) be given by (6) for all \( m, n \in \mathbb{N}, \ell \in \mathbb{I}(m) \). Then each \( \psi_{m,n,\ell}(:,c) \) is also the eigenfunction to the following Sturm-Liouville problem:
\[
\mathcal{D}_{c,x} \psi_{m,n,\ell}(:,c) = \chi_{m,n}(c) \psi_{m,n,\ell}(:,c) \quad \text{in} \quad B(0,1),
\]
where we define the corresponding eigenvalue by \( \chi_{m,n}(c) \).
Proof. Note that \( \psi_{m,n,\ell}(\cdot; c) \) and \( \varphi_{m,n}(\cdot; c) \) are related by (6) for all \( m, n \in \mathbb{N}, \ell \in \mathbb{I}(m) \); then we can directly calculate

\[
\mathcal{D}_{c,x} \psi_{m,n,\ell}(\cdot; c) = \left[ -(1 - r^2) \frac{\partial^2}{\partial r^2} + (3r - \frac{1}{r}) \frac{\partial}{\partial r} - \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + c^2 r^2 \right] \left[ r^{-\frac{1}{2}} \varphi_{m,n}(\cdot; c)(r) Y_{m,\ell}(\theta) \right] = -r^{-\frac{1}{2}} \varphi_{m,n}(\cdot; c) \left[ \left( 1 - \frac{1}{r^2} \right) \frac{\partial^2}{\partial r^2} - 2r \frac{\partial}{\partial r} + \left( \frac{1}{4} - m^2 - c^2 r^2 + \frac{3}{4} \right) \right] \varphi_{m,n}(\cdot; c).
\]

Since \( \varphi_{m,n}(\cdot; c) \) is an eigenfunction of \( \mathcal{D}_{c,r} \), then \( \psi_{m,n,\ell}(\cdot; c) \) is an eigenfunction of \( \mathcal{D}_{c,x} \). This completes the proof. □

The operator \( \mathcal{D}_{c,x} \) is the natural multi-dimensional generalization of the Sturm-Liouville problem in one dimension considered by Slepian [33]. The operator \( \mathcal{D}_{c,x} \) is self-adjoint (see also [39]) in the sense that for any \( u, v \) in \( H^1(B(0,1)) \),

\[
(\mathcal{D}_{c,x} u, v)_{B(0,1)} = \langle \nabla u, \nabla v \rangle_{L^2(B(0,1);w)} + (x_2 \partial_{x_1} - x_1 \partial_{x_2}) u, (x_2 \partial_{x_1} - x_1 \partial_{x_2}) v \rangle_{L^2(B(0,1))} + c^2 \langle xu, xv \rangle_{L^2(B(0,1))},
\]

where \( (\cdot, \cdot)_{B(0,1)} \) denotes the duality paring, and \( L^2(B(0,1);w) \) is associated with the \( w(x) := 1 - |x|^2 \) weighted norm and inner product

\[
\|u\|_{L^2(B(0,1);w)} := \sqrt{\langle u, u \rangle_{L^2(B(0,1);w)}}, \quad \langle u, v \rangle_{L^2(B(0,1);w)} := \int_{B(0,1)} u(x)v(x)w(x) \, dx.
\]

The operator \( \mathcal{D}_{c,x} \) is strictly positive, since for any nonzero \( u \in H^1(B(0,1)) \),

\[
(\mathcal{D}_{c,x} u, u)_{B(0,1)} = \|\nabla u\|_{L^2(B(0,1);w)}^2 + \| (x_2 \partial_{x_1} - x_1 \partial_{x_2}) u \|_{L^2(B(0,1))}^2 + c^2 \| xu \|_{L^2(B(0,1))}^2 > 0.
\]

The eigensystem \( \{ \psi_{m,n,\ell}(\cdot; c), \chi_{m,n}(c) \}_{m,n,\ell \in \mathbb{N}} \) satisfies the following important property (cf. [39, Theorem 3.2]).

**Lemma 2.** For any \( c > 0 \), it holds that \( \{ \psi_{m,n,\ell}(\cdot; c) \}_{m,n,\ell \in \mathbb{N}} \) is a complete orthogonal basis in \( L^2(B(0,1)) \) and

\[
(m + 2n)(m + 2n + 2) < \chi_{m,n}(c) < (m + 2n)(m + 2n + 2) + c^2.
\]

Now we introduce the Sobolev space for any integer \( s > 0 \) (see [38] for the one-dimensional case)

\[
\widetilde{H}_s^c(B(0,1)) := \left\{ u \in L^2(B(0,1)) : \sum_{m,n \in \mathbb{N}} \chi_{m,n}(c) \left| \langle u, \frac{\psi_{m,n,\ell}(\cdot; c)}{\psi_{m,n,\ell}(\cdot; c)} \rangle \right|^2 < \infty \right\},
\]

equipped with the following energy norm in \( \widetilde{H}_s^c(B(0,1)) \), \( s = 1, 2, \cdots \) (cf. [27, Theorem 2.37 and Corollary 2.38]):

\[
\|u\|_{\widetilde{H}_s^c(B(0,1))}^2 := \sum_{m,n \in \mathbb{N}} \chi_{m,n}(c) \left| \langle u, \frac{\psi_{m,n,\ell}(\cdot; c)}{\psi_{m,n,\ell}(\cdot; c)} \rangle \right|^2 = \int_{B(0,1)} (D_{c,x} u) \overline{w} \, dx.
\]
Now we define the Sobolev space $\tilde{H}_c^{s}(B(0,1))$ for any real $s > 0$ by interpolation as in [27, Exercise B.8, p. 333], and the norm is characterized by

$$
\|u\|_{\tilde{H}_c^{s}(B(0,1))}^2 = \sum_{m,n \in \mathbb{N}, \ell \in I(m)} \chi_{m,n}^s(c) \left( \frac{\langle u, \psi_{m,n,\ell}(\cdot;c) \rangle}{\|\psi_{m,n,\ell}(\cdot;c)\|} \right)^2, \quad \forall s > 0.
$$

4.2. **Regularity estimate and approximation theory.**

We now estimate the $L^2(B(0,1))$ approximation error for a spectral cutoff approximation of functions in $H^s(B(0,1))$, $0 < s \leq 1$. By Lemma 2 the eigenvalue $\chi_{m,n}(c)$ approaches to infinity as $m, n \to \infty$; then for a given small value $\alpha > 0$, we can define a spectral cutoff approximation for any $u \in L^2(B(0,1))$:

$$
\pi_{\alpha,c} u := \sum_{m,n \in \mathbb{N}, \ell \in I(m)} \chi_{m,n}(c) \chi_{m,n}(c) \left( \frac{\langle u, \psi_{m,n,\ell}(\cdot;c) \rangle}{\|\psi_{m,n,\ell}(\cdot;c)\|} \right)^2.
$$

We begin with the result on approximations of any function $u \in \tilde{H}_c^{s}(B(0,1))$.

**Lemma 3.** Suppose that $u \in \tilde{H}_c^{s}(B(0,1))$ for $s \geq 0$. Then

$$
\|\pi_{\alpha,c} u - u\| \leq \alpha^{s/2} \|u\|_{\tilde{H}_c^{s}(B(0,1))}
$$

**Proof.** This proof follows directly from

$$
\|\pi_{\alpha,c} u - u\|^2 = \sum_{m,n \in \mathbb{N}, \ell \in I(m)} \chi_{m,n}(c) \chi_{m,n}(c) \left( \frac{\langle u, \psi_{m,n,\ell}(\cdot;c) \rangle}{\|\psi_{m,n,\ell}(\cdot;c)\|} \right)^2
$$

$$
\leq \alpha^s \sum_{m,n \in \mathbb{N}, \ell \in I(m)} \chi_{m,n}(c) \chi_{m,n}(c) \left( \frac{\langle u, \psi_{m,n,\ell}(\cdot;c) \rangle}{\|\psi_{m,n,\ell}(\cdot;c)\|} \right)^2
$$

This completes the proof. \boxed{}

The following lemma estimates the $\tilde{H}_c^{s}(B(0,1))$ norm by the $H^s(B(0,1))$ norm.

**Lemma 4.** Suppose that $u \in H^s(B(0,1))$ with $0 < s \leq 1$. Then $u \in \tilde{H}_c^{s}(B(0,1))$ and

$$
\|u\|_{\tilde{H}_c^{s}(B(0,1))} \leq C s^{1/2} (1 + c^2)^{1/2} \|u\|_{H^s(B(0,1))}
$$

for some positive constant $C \geq \sqrt{3}$ independent of $s$ and $c$.

**Proof.** We first prove the case when $s = 1$. This is a direct consequence of

$$
\|u\|_{\tilde{H}_c^{1}(B(0,1))}^2 = \int_{B(0,1)} \langle D_{c,x} u \rangle^2 dx
$$

$$
= \|\nabla u\|_{L^2(B(0,1))}^2 + \|(x_2 \partial_{x_1} - x_1 \partial_{x_2}) u\|_{L^2(B(0,1))}^2 + C^2 \|xu\|_{L^2(B(0,1))}^2
$$

$$
\leq \|\nabla u\|_{L^2(B(0,1))}^2 + 2 \|\nabla u\|_{L^2(B(0,1))}^2 + C^2 \|u\|_{L^2(B(0,1))}^2
$$

$$
\leq (3 + c^2) \|u\|_{H^1(B(0,1))}^2,
$$

which yields that

$$
\|u\|_{\tilde{H}_c^{1}(B(0,1))} \leq C (1 + c^2)^{1/2} \|u\|_{H^1(B(0,1))},
$$

(29)
for some positive constant $C \geq \sqrt{3}$ independent of $s$ and $c$. Note that $\tilde{H}^s(B(0,1))$ for any real $s > 0$ is given by interpolation as in [27, Exercise B.8, p. 333], and $H^s(B(0,1))$ for any real $s > 0$ is given by interpolation as in [27, Theorem B.8, p. 330]. Now an application of interpolation of Sobolev spaces [27, Theorem B.2, p. 320] together with estimate (29) yields (28). This completes the proof.

**Remark 3.** The regularity estimate in Lemma 4 is of independent interest in approximation theory using the generalized prolate spheroidal wave functions and the data-driven basis.

Now we are ready to prove the following theorem.

**Theorem 2.** Suppose that $u \in H^s(B(0,1))$ with $0 < s \leq 1$. Then

$$\|\pi_{\alpha,c} u - u\| \leq (\alpha C)^{s/2}(1 + c^2)^{s/2}\|u\|_{H^s(B(0,1))},$$

for some positive constant $C \geq \sqrt{3}$ independent of $\alpha$, $s$ and $c$.

**Proof.** This is a direct consequence of Lemma 3 and Lemma 4. This completes the proof.

Now we give a similar result for approximation of any function in $H^s(D_F)$, $0 < s \leq 1$. For any function $u^F \in L^2(D_F)$, we let

$$\pi_{\alpha}^F u^F := \sum_{m,n \in \mathbb{N}, \ell \in \mathbb{I}(m)} \frac{\chi_{m,n}(c_F) \alpha^{-1}}{\sqrt{\nu_{m,n,\ell}}} \left< u^F, \frac{\psi^F_{m,n,\ell}(\cdot; c_F)}{\|\psi^F_{m,n,\ell}(\cdot; c_F)\|_{D_F}} \right> \frac{\psi^F_{m,n,\ell}(\cdot; c_F)}{\|\psi^F_{m,n,\ell}(\cdot; c_F)\|_{D_F}}.$$

(30)

Now we can prove the following corollary.

**Corollary 1.** Suppose that $u^F \in H^s(D_F)$ with $0 < s \leq 1$. Then

$$\|\pi_{\alpha}^F u^F - u^F\|_{L^2(D_F)} \leq (\alpha C)^{s/2}(1 + c_F^2)^{s/2}\left(1 + \frac{c_F^2}{2k}\right)^{s/2} \|u^F\|_{H^s(D_F)},$$

for some positive constant $C \geq \sqrt{3}$ independent of $\alpha$, $s$, and $c_F$.

**Proof.** Let

$$u(p) := \frac{c_F^2}{2k} u^F \left( \frac{c_F^2}{2k} \right), \quad p \in B(0,1).$$

Then it follows from (30) and relation (17) between $\psi^F_{m,n,\ell}$ and $\psi^F_{m,n,\ell}(\cdot; c_F)$ that

This allows us to derive that

$$\|\pi_{\alpha}^F u^F - u^F\|^2_{L^2(D_F)} = \int_{D_F} \left| u^F(p) - (\pi_{\alpha}^F u^F)(p) \right|^2 dp

= \int_{B(0,1)} \left| \frac{c_F^2}{2k} u^F(p) - (\pi_{\alpha}^F u^F)(\frac{c_F^2}{2k}) \right|^2 \left( \frac{c_F^2}{2k} \right)^2 dp' = \|\pi_{\alpha,c_F} u - u\|_{L^2(B(0,1))}^2.$$
Together with Theorem 2, by setting \( c = c_F \), we have that
\[
\| \pi^F_\alpha u^F - u^F \|_{L^2(D_F)} \leq (\alpha C)^{s/2} (1 + c^2_F)^{s/2} \| u \|_{H^s(B(0,1))}
\] 
\[
\leq (\alpha C)^{s/2} (1 + c^2_F)^{s/2} \left( 1 + \frac{c_F}{2k} \right)^s \| u^F \|_{H^s(D_F)},
\]
where in the last step we have applied
\[
\| u \|_{H^s(B(0,1))} \leq \left( 1 + \frac{c_F}{2k} \right)^s \| u^F \|_{H^s(D_F)},
\]
which is due to the interpolation property of Sobolev spaces, \( \| u \|_{H^0(B(0,1))} = \| u^F \|_{H^0(D_F)} \) and
\[
\| u \|_{H^1(B(0,1))} = \sqrt{\| u^F \|_{L^2(D_F)}^2 + \frac{c_F}{2k} \| \nabla u^F \|_{L^2(D_F)}^2} \leq \left( 1 + \frac{c_F}{2k} \right)^s \| u^F \|_{H^1(D_F)}.
\]
This completes the proof.  

With the above regularity estimates, we are ready to study the following regularization strategy and stability estimate.

4.3. Regularization and stability.

In practice, the data is only known up to an error \( \delta \) with \( \| u^\infty_{b,F} - u^\infty_{b,F} \|_{L^2(D_F)} \leq \delta \). In the following we study a regularization strategy based on spectral cutoff. To begin with, for a small value \( \alpha > 0 \) let the operator \( \mathcal{R}^F_\alpha : L^2(D_F) \to L^2(D_F) \) be given by
\[
\mathcal{R}^F_\alpha u^F = \sum_{\mathcal{J}(\alpha)} \frac{1}{(2k)^2 \alpha_{m,n}(c_F)} \left\{ u^F_{m,n,\ell} \left( \frac{c_F}{2k} \right)^2 \right\}_{D_F} \left\| \psi_{m,n,\ell}^F (\cdot ; c_F) \right\|_{D_F} \psi_{m,n,\ell}^F (\cdot ; c_F),
\]
for all \( u^F \in L^2(D_F) \), where \( \mathcal{J}(\alpha) := \{ m, n \in \mathbb{N}, \ell \in \mathbb{I}(m) : \chi_{m,n}(c_F) < \alpha^{-1} \} \). We also introduce
\[
\beta(\alpha) := \min_{(m,n,\ell) \in \mathcal{J}(\alpha)} \left\{ \left( \frac{c_F}{2k} \right)^2 \left| \alpha_{m,n}(c_F) \right| \right\}.
\]
Let \( q^{\delta,\alpha} \) be a regularized solution given by
\[
q^{\delta,\alpha} := \mathcal{R}^F_\alpha u^\infty_{b,F} \]
\[
= \sum_{\mathcal{J}(\alpha)} \frac{1}{(2k)^2 \alpha_{m,n}(c_F)} \left\{ u^\infty_{b,F} \left( \frac{c_F}{2k} \right)^2 \right\}_{D_F} \left\| \psi_{m,n,\ell}^F (\cdot ; c_F) \right\|_{D_F} \psi_{m,n,\ell}^F (\cdot ; c_F).
\]
We first prove the following lemma.

Lemma 5. Let \( c_F \) be chosen such that \( \Omega \subset D_F = B(0, \frac{c_F}{2k}) \). Suppose that \( \| u^\infty_{b,F} - u^\infty_{b,F} \|_{L^2(D_F)} \leq \delta \). Let \( q^{\delta,\alpha} \) be a regularized solution given by (33) and \( \beta(\alpha) \) be given by (32). Then it holds that
\[
\| q^{\delta,\alpha} - q \|_{L^2(D_F)} \leq \frac{\delta}{\beta(\alpha)} + \| \pi^F_\alpha q - q \|_{L^2(D_F)}.
\]

Proof. First we observe that
\[
\| q^{\delta,\alpha} - q \|_{L^2(D_F)} = \| \mathcal{R}^F_\alpha u^\infty_{b,F} - \mathcal{R}^F_\alpha u^\infty_{b,F} + \mathcal{R}^F_\alpha u^\infty_{b,F} - q \|_{L^2(D_F)} \]
\[
\leq \| \mathcal{R}^F_\alpha u^\infty_{b,F} - \mathcal{R}^F_\alpha u^\infty_{b,F} \|_{L^2(D_F)} + \| \mathcal{R}^F_\alpha u^\infty_{b,F} - q \|_{L^2(D_F)}.
\]
It is directly seen that
\[
\|\mathcal{R}_\alpha^F u_{b,F}^{\infty,\delta} - \mathcal{R}_\alpha^F u_{b,F}^{\infty,\delta}\|_{L^2(D_F)} \leq \|\mathcal{R}_\alpha^F\| \|u_{b,F}^{\infty,\delta} - u_{b,F}^{\infty,\delta}\|_{L^2(D_F)} \leq \delta \|\mathcal{R}_\alpha^F\| \leq \frac{\delta}{\beta(\alpha)}
\]  
(36)
where in the last step we have applied \(\|\mathcal{R}_\alpha^F\| \leq 1/\beta\) due to (31) and that \((\frac{c_F}{2k})^2 |\alpha_{m,n}(c_F)| \geq \beta(\alpha)\) for \((m, n, \ell) \in \mathbb{J}(\alpha)\); here we note that \(\beta(\alpha)\) is given by (32).

For the second estimate, we observe from (21) that
\[
\|\mathcal{R}_\alpha^F u_{b,F}^{\infty,\delta} - \mathcal{Q}\|_{L^2(D_F)} = \sum_{\mathbb{J}(\alpha)} \left\| \psi_{m,n,\ell}^F \|\psi_{m,n,\ell}^F\| \right\|_{D_F} \psi_{m,n,\ell}^F \|\psi_{m,n,\ell}^F\| - \|\mathcal{Q}\|_{L^2(D_F)} \leq \|\pi_0^F \mathcal{Q} - \mathcal{Q}\|_{L^2(D_F)}.
\]
This completes the proof.

Now we are ready to prove the main theorem.

**Theorem 3.** Let \(c_F\) be chosen such that \(\Omega \subset D_F = B(0, \frac{sC}{D})\). Suppose that \(\|u_{b,F}^{\infty,\delta} - u_{b,F}^{\infty,\delta}\|_{L^2(D_F)} \leq \delta\). Let \(q^{s,\alpha}\) be a regularized solution given by (33) and \(\beta(\alpha)\) be given by (32). If \(q \in H^s(D_F)\) with \(0 < s < 1/2\), then it holds that
\[
\|q^{s,\alpha} - \mathcal{Q}\|_{L^2(D_F)} \leq \frac{\delta}{\beta(\alpha)} + (\alpha C)^{s/2}(1 + c_F^2)^{s/2} \left(1 + \frac{c_F}{2k}\right)^s \|q\|_{H^s(D_F)},
\]
(37)
where \(C \geq \sqrt{3}\) is a positive constant independent of \(\delta, \alpha, s\), and \(c_F\).

**Proof.** From Lemma 5, it is sufficient to estimate \(\|\pi_0^F \mathcal{Q} - \mathcal{Q}\|_{L^2(D_F)}\). Note that \(q \in H^s(D_F)\); then by Corollary 1,
\[
\|\pi_0^F \mathcal{Q} - \mathcal{Q}\|_{L^2(D_F)} \leq (\alpha C)^{s/2}(1 + c_F^2)^{s/2} \left(1 + \frac{c_F}{2k}\right)^s \|q\|_{H^s(D_F)}.
\]
This proves (37) and completes the proof.

**Remark 4.** The regularity assumption \(q \in H^s(D_F)\) with \(0 < s < 1/2\) is due to the fact that \(q\) is the extension of \(q\) by 0. In the case that \(\Omega\) coincides with \(D_F\), it is feasible to have the regularity assumption that \(q = q \in H^s(D_F)\) with any \(0 < s \leq 1\) where the main result still holds; it is also possible to generalize the result for any \(s > 0\).

**Remark 5.** Due to the asymptotic \((m + 2n)(m + 2n + 2) < \chi_{m,n}(c) < (m + 2n)(m + 2n + 2) + c^2\) in Lemma 2, one may obtain an explicit estimate on the set \(\mathbb{J}(\alpha) = \{m, n \in \mathbb{N}, \ell \in \mathbb{H}(m) : \chi_{m,n}(c_F) < \alpha^{-1}\}\) by
\[
\alpha < \chi_{m,n}(c_F) < \frac{1}{(m + 2n)^2}, \quad \forall (m, n, \ell) \in \mathbb{J}(\alpha).
\]
The parameter \(\beta(\alpha)\) depends on \(\alpha\) via (32) so that
\[
\beta(\alpha) \leq \left\{ \left(\frac{c_F}{2k}\right)^2 |\alpha_{m,n}(c_F)| \right\} , \quad \forall (m, n, \ell) \in \mathbb{J}(\alpha).
\]
Note that \(\alpha_{m,n}(c_F)\) is the eigenvalue of a Fourier type integral operator, so one expects that \(\alpha_{m,n}(c_F)\) decay exponentially to zero. Therefore for a given noise level \(\delta\), if we choose \(\beta(\alpha) = \delta^\gamma\) with \(0 < \gamma < 1\), \((\frac{\delta}{\beta(\alpha)}, \alpha^{s/2})\) is expected to appear in Hölder-logarithmic pair, and hence stability estimate (37) of Theorem 3 is expected in the Hölder-logarithmic type.
A further note on the computation. Noting that $\psi_{m,n,\ell}$ is the eigenfunction of the Sturm-Liouville operator, one can compute such an eigenfunction (as well as the corresponding Sturm-Liouville eigenvalue $\chi_{m,n}(c_F)$) in a very robust way using a Bouwkamp type algorithm [6] with Legendre polynomial expansion. After one computes the eigenfunction $\psi_{m,n,\ell}$, $\alpha_{m,n}(c_F)$ can be computed in a very stable manner using its relation to the generalized prolate spheroidal wave functions. We refer the reader to [38, 39] on these computational aspects. We further refer the reader to [3] for related numerical experiments on a linear sampling method using prolate spheroidal wave functions to achieve parameter identification.

5. LIMITED-APERTURE DATA AND MULTI-FREQUENCY DATA

The application of the generalized prolate spheroidal wave functions and their related functions is far less developed for the limited-aperture and multi-frequency inverse scattering problems. Remarkably, there also exists a data-driven basis for a general symmetric set [33]. In this section we first show that the Picard criterion also applies to the limited-aperture and multi-frequency cases, and then obtain a stability estimate for a spectral cutoff regularization.

5.1. Scaled eigensystem for symmetric set.

We first introduce a scaled eigensystem for a generic symmetric bounded open set $A_h := \{hx : x \in A\}$. Noting that $\{\psi_n(\cdot; c), \alpha_n(c)\}_{n=0}^\infty$ is an eigensystem given in Section 2.2, we introduce another orthogonal complete set $\{\tilde{\psi}_n(\cdot; c)\}_{n \in \mathbb{N}}$ in $L^2(A_h)$ by

$$\tilde{\psi}_n(x; c) := h^{-1}\psi_n(h^{-1}x; c), \quad x \in \mathbb{R}^2.$$ 

Then by a change of variable, it follows from (12) that

$$\int_{A_h} e^{i\pi \tau p \cdot p'} \tilde{\psi}_n(p'; c) \, dp' = h^2 \alpha_n(c) \tilde{\psi}_n(p; c), \quad p \in A_h,$n

and from (13)–(14) that

$$\int_{\mathbb{R}^2} \tilde{\psi}_n(p; c) \tilde{\psi}_{n'}(p; c) \, dp = \delta_{nn'},$$

$$\int_{A_h} \tilde{\psi}_n(p; c) \tilde{\psi}_{n'}(p; c) \, dp = \left(\frac{c}{2\pi}\right)^2 |\alpha_n(c)|^2 \delta_{nn'}.$$

5.2. Reconstruction formula.

In Section 5.2.1 and Section 5.2.2, we formulate the inverse problems for limited-aperture and multi-frequency data, respectively. In Section 5.2.3 we use the Picard criterion to obtain the reconstruction formula.

5.2.1. Limited-aperture data. We give a formulation of the Born inverse scattering with limited-aperture data. Recall in Section 1 that we aim to determine the contrast $q \in L^2(\Omega)$ from the limited-aperture Born data

$$\{\tilde{u}_b^\infty(\hat{x}; \hat{\theta}; k) : \hat{x}, \hat{\theta} \in S_L\}. \quad (38)$$

From the representation (3) of $\tilde{u}_b^\infty(\hat{x}; \hat{\theta}; k)$, the knowledge of $\{\tilde{u}_b^\infty(\hat{x}; \hat{\theta}; k) : \hat{x}, \hat{\theta} \in S_L\}$ is equivalent to the knowledge of $\{u_b^\infty(p; k) : p \in L\}$,

$$u_b^\infty(p; k) = \int_{\Omega} e^{ikp \cdot p'} q(p') \, dp', \quad \forall p \in L, \quad (39)$$
where $L$ is the interior of $\{\hat{\theta} - \hat{x} : \hat{x}, \hat{\theta} \in S_L\}$ which is symmetric with respect to the origin.

**Remark 6.** The set $L$ represents the size of the limited-aperture in a more explicit way. Note that as the aperture gets smaller, the area of the set $L$ becomes smaller, which indicates that the inverse problem gets more ill-posed (from the point of view of analytic unique continuation); see Figure 1 for an illustration.

Suppose that there exists a positive constant $c_L$ such that $\Omega \subset D_L := \{c_Lx/k : x \in L\}$. This is guaranteed for aperture size large than one half since the set $L$ contains a small neighborhood of the origin. For aperture size smaller than or equal to one half where the set $L$ does not contain a small neighborhood of the origin (see Figure 1), we have to impose a priori information that there exists $c_L$ such that $\Omega \subset D_L$ (this seems to be a restriction on $\Omega$, and it would be interesting to relax such a restriction; for instance, one possible way is to extrapolate the partial data in $L$ to obtain the full aperture data in $B(0, 2)$). By equation (39), we then have the knowledge of $\{u_{b,L}^\infty(p) : p \in D_L\}$ where

$$u_{b,L}^\infty(p) := \int_{D_L} e^{i\hat{x}^2p^2/2} \tilde{q}(p') dp' := (K^L \tilde{q})(p), \quad \forall p \in D_L,$$

and we denote the associated operator by $K^L : L^2(D_L) \to L^2(D_L)$. Now the limited-aperture inverse scattering is formulated as follows.

**Formulation of the inverse problem:** Assuming that there exists $c_L$ such that $\Omega \subset D_L$, determine the contrast $\tilde{q} \in L^2(D_L)$ from $\{u_{b,L}^\infty(p) : p \in D_L\}$.

![Figure 1](image_url)  
**Figure 1.** The domain $L$ with respect to different aperture sizes. From left to right: full, $\frac{3}{4}$, $\frac{1}{2}$, $\frac{1}{4}$.

Now we introduce a suitable eigensystem for the study of (40). Define the eigensystem $\{\psi_n^L(p; c_L), \alpha_n^L(c_L)\}_{n \in \mathbb{N}}$ in $L^2(D_L)$ (by setting $A = L$, $c = c_L$, and $h = c_L/k$ in Section 5.1) such that

$$\int_{D_L} e^{i\hat{x}^2p^2/2} \psi_n^L(p'; c_L) dp' = \left(\frac{c_L}{k}\right)^2 \alpha_n^L(c_L) \psi_n^L(p; c_L), \quad p \in D_L.$$

In this case the double orthogonality reads

$$\int_{\mathbb{R}^2} \psi_n^L(p; c_L) \psi_m^L(p; c_L) dp = \delta_{nn'},$$

$$\int_{D_L} \psi_n^L(p; c_L) \psi_m^L(p; c_L) dp = \left(\frac{c_L}{2\pi}\right)^2 |\alpha_n^L(c_L)|^2 \delta_{nn'}.$$

**5.2.2. Multi-frequency data.** In the multi-frequency case, recall in Section 1 that we aim to determine the contrast $\tilde{q} \in L^2(\Omega)$ from the following Born data with two opposite observation directions (which are $\pm \hat{x}^* \in \mathbb{S}$):

$$\{\tilde{u}_{b,\hat{k}}^\infty(\pm \hat{x}^*; \hat{\theta}; k) : \hat{\theta} \in \mathbb{S}, k \in (0, K), K > 0\}. \quad (41)$$
For this multi-frequency problem, we have that the knowledge of \( \{ \tilde{u}_h^\infty(\pm \hat{x}^*; \hat{\theta}; k) : \hat{\theta} \in \mathbb{S}, k \in (0, K) \} \) is equivalent to the knowledge of \( \{ u_h^\infty(p; K) : p \in M \} \) where \( M \) is the interior of \( \{ a \theta \pm a \hat{x}^* : \theta \in \mathbb{S}, a \in (0, 1) \} \) and

\[
u_h^\infty(p; K) = \int_\Omega e^{iKp_1 p'} q(p') \, dp', \quad \forall p \in M. \tag{42}
\]

Note that \( M \) is symmetric with respect to the origin. Suppose that there exists \( c_M \) such that \( \Omega \subset D_M = \{ y/K : y \in M \} \). By (42), we then have the knowledge of \( u_{h,M}(p) \) for \( p \in D_M \) where

\[
u_{h,M}(p) := \int_{D_M} e^{iK^2 p_1 p'} \psi_n^M(p; c_M) \, dp' := (K^M q)(p), \quad \forall p \in D_M,
\]

and we denote the associated operator by \( K^M : L^2(D_M) \to L^2(D_M) \). Now the multi-frequency inverse scattering is formulated as follows.

**Formulation of the inverse problem:** Assuming that there exists \( c_M \) such that \( \Omega \subset D_M \), determine the contrast \( q \in L^2(D_M) \) from \( \{ u_{h,M}(p) : p \in D_M \} \) .

Now we introduce a suitable eigensystem for the study of (43). Define the eigensystem \( \{ \psi_n^M(\cdot; c_M), \alpha_n^M(c_M) \}_{n \in \mathbb{N}} \) in \( D_M \) (by setting \( c = c_M, \, h = \frac{c}{K}, \) and \( A = M \) in Section 5.1) such that

\[
\int_{D_M} e^{iK^2 p_1 p'} \psi_n^M(p; c_M) \, dp' = \left( \frac{c_M}{K} \right)^2 \alpha_n^M(c_M) \psi_n^M(p; c_M), \quad p \in D_M.
\]

In this case the double orthogonality reads

\[
\int_{\mathbb{R}^2} \psi_n^M(p; c_M) \psi_n'(p; c_M) \, dp = \delta_{nn'},
\]

\[
\int_{D_M} \psi_n^M(p; c_M) \psi_n'(p; c_M) \, dp = \left( \frac{c_M}{2\pi} \right)^2 |\alpha_n^M(c_M)|^2 \delta_{nn'}.
\]

5.2.3. Picard criterion. Now we are ready to obtain the following reconstruction formulas by Picard criterion.

**Theorem 4.** Let \( q \) be the extension function that \( q = q \) in \( \Omega \) and \( q = 0 \) outside \( \Omega \) almost everywhere.

- Suppose that there exists \( c_L \) such that \( \Omega \subset D_L = \{ x/k : x \in L \} \), where \( L \) is the interior of \( \{ \hat{\theta} - \hat{x} : \hat{x} \in \mathbb{S}_L \} \) and \( \mathbb{S}_L = \{ x : x \in \mathbb{S}, \arg x \in [-\Theta, \Theta], 0 < \Theta < \pi \} \). Then \( q \) is uniquely solved by

\[
q = \sum_{n \in \mathbb{N}} \left( \frac{k}{c_L} \right)^2 \frac{1}{\alpha_n^L(c_L)} \left\langle u_{h,L}^\infty(\cdot; c_L), \psi_n^L(\cdot; c_L) \right\rangle_{D_L} \frac{\psi_n^L(\cdot; c_L)}{\| \psi_n^L(\cdot; c_L) \|},
\]

where the convergence is in \( L^2(D_L) \).

- Suppose that there exists \( c_M \) such that \( \Omega \subset D_M = \{ x/K : x \in M \} \) where \( M \) is the interior of \( \{ a \theta \pm a \hat{x}^* : \theta \in \mathbb{S}, a \in (0, 1) \} \). Then \( q \) is uniquely solved by

\[
q = \sum_{n \in \mathbb{N}} \left( \frac{K}{c_M} \right)^2 \frac{1}{\alpha_n^M(c_M)} \left\langle u_{h,M}^\infty(\cdot; c_M), \psi_n^M(\cdot; c_M) \right\rangle_{D_M} \frac{\psi_n^M(\cdot; c_M)}{\| \psi_n^M(\cdot; c_M) \|},
\]

where the convergence is in \( L^2(D_M) \).

**Proof.** The proof is exactly the same as the proof of Theorem 1. \( \square \)
Remark 7. Noting that the eigenfunctions \( \{ \psi_n^L(\cdot; c_L) \}_{n \in \mathbb{N}} \) and \( \{ \psi_n^M(\cdot; c_M) \}_{n \in \mathbb{N}} \) also extend analytically to \( \mathbb{R}^2 \), are doubly orthogonal, and are complete in a proper class of band-limited functions in multiple dimensions, we can also interpret the above reconstruction formulas from the viewpoint of data processing and analytic extrapolation (by following Section 3.1 in exactly the same way).

5.3. Regularization and stability.

In both the limited-aperture and the multi-frequency cases, we have considered an operator equation

\[
K\tilde{q} = \tilde{u}, \quad K : L^2(D) \to L^2(D)
\]

where

- for the limited-aperture case: \( K \) is given by (40), \( D = D_L \), and \( \tilde{u} = u_{b,L}^\infty \);
- for the multi-frequency case: \( K \) is given by (43), \( D = D_M \), and \( \tilde{u} = u_{b,M}^\infty \).

Therefore we first study the regularization and stability of this abstract equation (44), and then state the result for each of the above cases. In each case, we have found an orthogonal eigensystem \( \{ \phi_n, \mu_n \}_{n \in \mathbb{N}} \) in \( L^2(D) \) with real-valued \( \phi_n \) such that

\[
K\phi_n = \mu_n\phi_n, \quad \| \phi_n \| = \lambda_n > 0,
\]

\(|\mu_n| > 0, \lim_{n \to \infty} |\mu_n| = 0 \), and \( \tilde{q} \) is solved in the form of

\[
\tilde{q} = \sum_{n \in \mathbb{N}} \frac{1}{\mu_n} \langle \tilde{u}, \phi_n \rangle_{L^2(D)} \frac{\phi_n}{\lambda_n}.
\]

In practice, the data \( \tilde{u}^\delta \) is only known up to an error \( \delta \) with \( \| \tilde{u} - \tilde{u}^\delta \|_{L^2(D)} \leq \delta \) and this motivates us to consider a regularization strategy based on spectral cutoff where we define \( R_\alpha : L^2(D) \to L^2(D) \) by

\[
R_\alpha \tilde{u} = \sum_{|\mu_n| > \alpha} \frac{1}{\mu_n} \langle \tilde{u}, \phi_n \rangle_{L^2(D)} \frac{\phi_n}{\lambda_n}.
\]

Lemma 6. Suppose that \( \| \tilde{u}^\delta - \tilde{u} \|_{L^2(D)} \leq \delta \). Let \( \tilde{q} \in \text{Range}(K^*K)^{\sigma/2} \) for some \( \sigma > 0 \) and \( \| (K^*K)^{-\sigma/2} \tilde{q} \|_{L^2(D)} \leq E \) for some constant \( E \). Let \( \alpha(\delta) = c_0(\delta/E)^{1/(1+\sigma)} \) with some positive constant \( c_0 \); then it holds that

\[
\| R_\alpha(\delta) \tilde{u}^\delta - \tilde{q} \|_{L^2(D)} \leq \delta^{\sigma/(1+\sigma)}E^{1/(1+\sigma)}(1/c_0 + c_0^\sigma).
\]

Proof. Note that

\[
\| R_\alpha(\delta) \tilde{u}^\delta - \tilde{q} \|_{L^2(D)} = \| R_\alpha(\delta) \tilde{u}^\delta - R_\alpha(\delta) \tilde{u} + R_\alpha(\delta) \tilde{u} - \tilde{q} \|_{L^2(D)} \\
\leq \| R_\alpha(\delta) \tilde{u}^\delta - R_\alpha(\delta) \tilde{u} \|_{L^2(D)} + \| R_\alpha(\delta) \tilde{u} - \tilde{q} \|_{L^2(D)}.
\]

Now we estimate each of the terms in the right hand side of the above inequality.

For the first term, it is directly seen that

\[
\| R_\alpha(\delta) \tilde{u}^\delta - R_\alpha \tilde{u} \|_{L^2(D)} \leq \| R_\alpha \| \| \tilde{u}^\delta - \tilde{u} \|_{L^2(D)} \leq \delta \| R_\alpha \| \leq \frac{\delta}{\alpha}
\]

where in the last step we have applied \( \| R_\alpha \| \leq \frac{1}{\alpha} \) due to (47).

For the second term, we observe that \( \| (K^*K)^{-\sigma/2} \tilde{q} \|_{L^2(D)} \leq E \) implies that

\[
\| (K^*K)^{-\sigma/2} \tilde{q} \|_{L^2(D)}^2 = \sum_{n \in \mathbb{N}} |\mu_n|^{-2\sigma} |\langle \tilde{q}, \phi_n/\lambda_n \rangle_{L^2(D)}|^2 \leq E^2
\]
and thereby
\[
\|R_\alpha \tilde{u} - \tilde{q}\|_{L^2(D)}^2 = \sum_{|\mu_n| \leq \alpha} \left\| \frac{\phi_n}{\lambda_n^2 \mu_n} \phi_n \right\|_{L^2(D)}^2 = \sum_{|\mu_n| \leq \alpha} \frac{\|\tilde{q}, \phi_n/\lambda_n\|_{L^2(D)}^2}{\lambda_n^2 |\mu_n|^2}
\]
\[
\leq \alpha^{2\sigma} \| (K^* K)^{-\sigma/2} \tilde{q}\|_{L^2(D)}^2.
\]
Finally (49)–(51) yields that
\[
\|R_{\alpha(\delta)} u^\delta - \tilde{q}\|_{L^2(D)} \leq \frac{\delta}{\alpha(\delta)} + [\alpha(\delta)]^\sigma \| (K^* K)^{-\sigma/2} \tilde{q}\|_{L^2(D)} \leq \frac{\delta}{\alpha(\delta)} + [\alpha(\delta)]^\sigma E.
\]
Let \( \alpha(\delta) = c_0 (\delta/E)^{1/(1+\sigma)} \) with some positive constant \( c_0 \); then
\[
\|R_{\alpha(\delta)} u^\delta - \tilde{q}\|_{L^2(D)} \leq \frac{\delta}{c_0 (\delta/E)^{1/(1+\sigma)}} + (c_0 (\delta/E)^{1/(1+\sigma)})^\sigma E = \delta^{\sigma/(1+\sigma)} E^{1/(1+\sigma)} (1/c_0 + c_0^\sigma).
\]
This proves (48) and completes the proof. \( \Box \)

Note that the regularity assumption on \( \tilde{q} \) (i.e., \( \tilde{q} \in \text{Range}(K^* K)^{\sigma/2} \)) for some \( \sigma > 0 \) is less explicit compared to the one (i.e., \( q \in H^s(D_F) \) with \( 0 < s < 1/2 \)) in the full aperture case in Theorem 3. This is due to the lack of a Sturm-Liouville differential operator for the limited-aperture and multi-frequency cases (to the best of our knowledge). It may be possible to impose the same explicit assumption by combining the result of Theorem 3 and an extrapolation algorithm where the partial data is extrapolated to approximate the full aperture data with a stability estimate.

Now we apply Lemma 6 to the limited-aperture data and multi-frequency data cases.

**Theorem 5.** Let \( q \) be the extension function that \( q = q \) in \( \Omega \) and \( q = 0 \) outside \( \Omega \) almost everywhere.

- Suppose that there exists \( c_L \) such that \( \Omega \subset D_L = \{c_L x / k : x \in L\} \) where \( L \) is the interior of \( \{\hat{\theta} - \hat{x} : \hat{x} \in \mathbb{S}_L\} \) and \( \mathbb{S}_L = \{x : x \in \mathbb{S}, \arg x \in [-\Theta, \Theta], 0 < \Theta < \pi\} \). Let \( R_{\alpha}^L : L^2(D_L) \to L^2(D_L) \) be given by
  \[
  R_{\alpha}^L u = \sum_{(\mathbb{S}_L)^2 |\alpha(\mathbb{S}_L)| > \alpha} \left\langle u, \frac{\psi_n^L (\cdot; c_L)}{\|\psi_n^L (\cdot; c_L)\|_{D_L}}, \frac{\psi_n^L (\cdot; c_L)}{\|\psi_n^L (\cdot; c_L)\|_{D_L}} \right\rangle_{D_L}
  \]
Suppose that \( \|u_{k^L}^\infty - u_{k^L}^\infty\|_{L^2(D_L)} \leq \delta \). Let \( K^L \) be given by (40), and suppose that \( \tilde{q} \in \text{Range}(K^L K^L)^{\sigma/2} \) for some \( \sigma > 0 \) and \( \|(K^L K^L)^{-\sigma/2} \tilde{q}\|_{L^2(D_L)} \leq E_1 \) for some constant \( E_1 \). Let \( \alpha(\delta) = c_1 (\delta/E_1)^{1/(1+\sigma)} \) with some positive constant \( c_1 \); then it holds that
\[
\|R_{\alpha(\delta)} u_{k^L}^\infty - \tilde{q}\|_{L^2(D_L)} \leq \delta^{\sigma/(1+\sigma)} E_1^{1/(1+\sigma)} (1/c_1 + c_1^\sigma).
\]
- Suppose that there exists \( c_M \) such that \( \Omega \subset D_M = \{c_M x / K : x \in M\} \) where \( M \) is the interior of \( \{a \hat{\theta} \pm a \hat{x} : \hat{\theta} \in \mathbb{S}, a \in (0, 1)\} \). Let \( R_{\alpha}^M : L^2(D_M) \to L^2(D_M) \) be given by
  \[
  R_{\alpha}^M u = \sum_{(\mathbb{S}_M)^2 |\alpha(\mathbb{S}_M)| > \alpha} \left\langle u, \frac{\psi_n^M (\cdot; c_M)}{\|\psi_n^M (\cdot; c_M)\|_{D_M}}, \frac{\psi_n^M (\cdot; c_M)}{\|\psi_n^M (\cdot; c_M)\|_{D_M}} \right\rangle_{D_M}
  \]
Suppose that \( \|u_{b^M}^\infty - u_{b^M}^\infty\|_{L^2(D_M)} \leq \delta \). Let \( K^M \) be given by (43), and suppose that \( \tilde{q} \in \text{Range}(K^M K^M)^{\sigma/2} \) for some \( \sigma > 0 \) and \( \|(K^M K^M)^{-\sigma/2} \tilde{q}\|_{L^2(D_M)} \leq E_2 \)
for some constant $E_2$. Let $\alpha(\delta) = c_2(\delta/E_2)^{1/(1+\sigma)}$ with some positive constant $c_2$; then it holds that
\[
\|R_{\alpha(\delta)}^M u_{\infty, \delta} - q\|_{L^2(D_M)} \leq \delta^{\sigma/(1+\sigma)}E_2^{1/(1+\sigma)}(1/c_2 + c_2^2).
\]

Proof. These conclusions are the direct consequences of Lemma 6. \hfill $\square$

6. Conclusion

In this paper we propose data-driven basis functions for reconstructing the medium contrast using Born data, including the full aperture, limited-aperture, and multi-frequency partial data. The data-driven basis, which originated in the study of a Fourier integral operator, allows us to establish a Picard criterion for reconstructing the contrast. Another salient feature is that such a data-driven basis remarkably extends analytically to $\mathbb{R}^2$, is doubly orthogonal, and is complete in the class of band-limited functions. This yields that the reconstruction formula by Picard criterion can be understood from the viewpoint of data processing and analytic extrapolation. Another feature is that the data-driven basis for a disk is also a basis for a Sturm-Liouville differential operator. This Sturm-Liouville differential operator brings additional regularity estimates that lead to estimating the $L^2$ approximation error for a spectral cutoff approximation of functions in $H^s$. This approximation theory allows us to obtain a spectral cutoff regularization strategy with an explicit stability estimate for noisy data. In a broader context, the data-driven basis in this paper can also be learned via a Legendre-Galerkin neural network and our analysis indeed serves as a mathematical foundation towards relevant machine learning algorithms. The extension of our work to $\mathbb{R}^3$ necessarily requires the study of the generalization of the prolate spheroidal wave functions in $\mathbb{R}^3$. Perhaps the most exciting extension is to investigate a possible data-driven basis for the full model in which case some non-linear transformation is most likely needed.

References

[1] A Alzaalig, G Hu, X Liu and J Sun. Fast acoustic source imaging using multi-frequency sparse data, *Inverse Problems* **36**, 025009, 2020.
[2] L Audibert and H Haddar. The generalized linear sampling method for limited aperture measurements, *SIAM J. Imaging Sci.* **10**(2), pp. 845–870, 2017.
[3] L Audibert and S Meng. Shape and parameter identification by the linear sampling method for a restricted Fourier integral operator, arXiv preprint arXiv:2306.16199.
[4] G Bao and J Liu. Numerical solution of inverse scattering problems with multi-experimental limited aperture data, *SIAM J. Sci. Comput.* **25**, pp. 1102–1117, 2003.
[5] G Bao, P Li, J Lin and F Triki. Inverse scattering problems with multifrequencies, *Inverse Problems* **31**, 093001, 2015.
[6] C Bouwkamp. On the theory of spheroidal wave functions of order zero. *Nederl. Akad. Wetensch., Proc.* **53**, pp. 931–944, 1950.
[7] F Cakoni and D Colton. *Qualitative Approach to Inverse Scattering Theory*, Springer, 2016.
[8] F Cakoni, D Colton and H Haddar. *Inverse Scattering Theory and Transmission Eigenvalues*, 2nd ed., CBMS-NSF Regional Conf. Ser. in Appl. Math. 98, SIAM, Philadelphia, 2023.
[9] J Cheng, L Peng and M Yamamoto. The conditional stability in line unique continuation for a wave equation and an inverse wave source problem, *Inverse Problems* **21**, pp. 1993–2007, 2005.
[10] D Colton and A Kirsch. A simple method for solving inverse scattering problems in the resonance region. *Inverse Problems* **12**, pp. 383–393, 1996.
[11] D Colton and R Kress. *Inverse Acoustic and Electromagnetic Scattering Theory*, Springer Nature, New York, 2019.
[12] F Dou, X Liu, S Meng and B Zhang. Data completion algorithms and their applications in inverse acoustic scattering with limited-aperture backscattering data, *J. Comput. Phys.* **469**, 111550, 2022.
[13] R Griesmaier and C Schmiedecke. A factorization method for multi-frequency inverse source problems with sparse far field measurements, *SIAM J. Imag. Sci.* 10, pp. 2119–2139, 2017.

[14] I Harris and JD Rezac. A sparsity-constrained sampling method with applications to communications and inverse scattering, *J. Comput. Phys.* 451, 110890, 2022.

[15] M Ikehata, E Niemi and S Siltanen. Inverse obstacle scattering with limited-aperture data, *Inverse Probl. Imaging* 1, pp. 77–94, 2012.

[16] M. Isaev and R. G. Novikov, Reconstruction from the fourier transform on the ball via prolate spheroidal wave functions, Journal de Mathématiques Pures et Appliquées, 163 (2022), pp. 318–333.

[17] G Karniadakis, I Kevrekidis, L Lu, P Perdikaris, S Wang and L Yang. Physics-informed machine learning, *Nat. Rev. Phys.* 3, 422–440, 2021.

[19] A Kirsch. Remarks on the Born approximation and the Factorization Method, *Applicable Analysis* 96, no. 1, pp. 70–84, 2017.

[20] A Kirsch. *An Introduction to the Mathematical Theory of Inverse Problems*, Springer, Cham, 2021.

[21] A Kirsch and N Grinberg. *The Factorization Method for Inverse Problems*, Oxford University Press, Oxford, 2008.

[22] S Kusiak and J Sylvestre. The convex scattering support in a background medium. *SIAM J. Math. Anal.* 36, no. 4, pp. 1142–1158, 2005.

[23] J Li, H Liu and J Zou. Locating multiple multiscale acoustic scatterers. *Multiscale Model. Simul.* 12, no. 3, pp. 927–952, 2014.

[24] X Liu and S Meng. A multi-frequency sampling method for the inverse source problems with sparse measurements, arXiv preprint arXiv:2109.01434.

[25] X. Liu, S. Meng and B. Zhang. Modified sampling method with near field measurements, *SIAM J. Appl. Math.* 82(1), pp. 244–266, 2022.

[26] S Lu, B Xu and X Xu. Unique continuation on a line for the Helmholtz equation, *Appl. Anal.* 91(9), pp. 1761–1771, 2012.

[27] W McLean. *Strongly elliptic systems and boundary integral equations*, Cambridge university press, 2000.

[30] F Natterer. *The Mathematics of Computerized Tomography*, SIAM, Philadelphia, 2001.

[31] A Quarteroni, A Manzoni and F Negri. *Reduced basis methods for partial differential equations. An introduction*, Springer, Cham, 2016.

[32] V Serov. Inverse Born approximation for the nonlinear two-dimensional Schrödinger operator, *Inverse Problems* 23, pp. 1259–1270, 2007.

[33] D Slepian. Prolate spheroidal wave functions, Fourier analysis and uncertainty -IV; Extensions to many dimensions; generalized prolate spheroidal functions, *Bell System Tech. J.* 43, pp. 3009–3057, 1964.

[34] D Slepian. Prolate spheroidal wave functions, Fourier analysis, and uncertainty V: The discrete case, *Bell System Tech. J.* 57, pp. 1371–1430, 1978.

[35] D Slepian and H Pollak. Prolate Spheroidal Wave Functions, Fourier Analysis and Uncertainty –I, *Bell System Tech. J.* 40, pp. 43–64, 1961.

[36] J Sylvestre and J. Kelly. A scattering support for broadband sparse far field measurements, *Inverse Problems* 21, pp. 759–771, 2005.

[37] C Tsogka, D A. Mitsoudis and S Papadimitropoulos, Limited-aperture array imaging in acoustic waveguides, *Inverse Problems* 32, 125011, 2016.

[38] L Wang. Analysis of spectral approximations using prolate spheroidal wave functions, *Math. Comp.* 79(270):807–827, 2010.

[39] J Zhang, H Li, L Wang and Z Zhang. Ball prolate spheroidal wave functions in arbitrary dimensions. *Appl. Comput. Harmon. Anal.* 48, no. 2, pp. 539–569, 2020.