Quantum Caustics for Systems with Quadratic Lagrangians in Multi-Dimensions

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Abstract. We study quantum caustics (i.e., the quantum analogue of the classical singularity in the Dirichlet boundary problem) in $d$-dimensional systems with quadratic Lagrangians of the form $L = \frac{1}{2}P_{ij}(t) \dot{x}^i \dot{x}^j + Q_{ij}(t) x^i \dot{x}^j + \frac{1}{2}R_{ij}(t) x^i x^j + S_i(t) x^i$. Based on Schulman's procedure in the path-integral we derive the transition amplitude on caustics in a closed form for generic multiplicity $f$, and thereby complete the previous analysis carried out for the maximal multiplicity case ($f = d$). The unitarity relation, together with the initial condition, fulfilled by the amplitude is found to be a key ingredient for determining the amplitude, which reduces to the well-known expression with Van-Vleck determinant for the non-caustics case ($f = 0$). Multiplicity dependence of the caustics phenomena is illustrated by examples of a particle interacting with external electromagnetic fields.

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1. Introduction

The semiclassical approximation of quantum mechanics [1] is a powerful means to investigate various — often non-perturbative — aspects of quantization in reference to classical mechanics. In the path-integral framework, it amounts to keeping only up to quadratic terms in the Lagrangian, and as such the core of the semiclassical approximation resides in the evaluation of the integral for quadratic systems. The path-integral then becomes Gaussian allowing for a closed form for the transition amplitude in terms of the familiar Van Vleck determinant. That is, in \(d\)-dimensions, the amplitude for the transition between \(a, b \in \mathbb{R}^d\) during the time interval \([0, T]\) reads

\[
K(b, T; a, 0) = \sqrt{\det \left( \frac{i}{2\pi\hbar} \frac{\partial^2 I[\bar{x}]}{\partial a^i \partial b^j} \right)} e^{\frac{\bar{x}}{\hbar} I[\bar{x}]}.
\]

(1.1)

Here \(I[\bar{x}] = \int_0^T dt L\) is the action for the quadratic part of the Lagrangian along the classical path \(\bar{x}(t)\) which satisfies the Dirichlet boundary conditions corresponding to the transition. The point to be noted is that the transition kernel \(K(b, T; a, 0)\) in (1.1) develops singularities if there exists no (or more than one) classical paths which meet the Dirichlet boundary conditions, and accordingly there arise physical phenomena characteristic to the singularities known as caustics in geometrical optics [2].

Historically, the semiclassical analysis of caustics phenomena was conducted intensively in late seventies in association with the catastrophe theory (see, e.g., [3, 4] and references therein). There, one takes into account cubic terms to avoid the singularities (and hence the caustics no longer exist in the strict sense), which is in fact an appropriate procedure for most realistic physical systems. The path-integral is then approximated by the ‘generalized Airy integral’ governed by catastrophe polynomials [5, 6, 7]. On the other hand, for endpoints beyond caustics, namely when the endpoints go beyond singular (conjugate) points, the analysis for pure quadratic systems was carried out in evaluating the correction in phase factor [8] as well as in providing a more general basis for the semiclassical expansion in the path-integral [9]. The latter considered also endpoints on caustics, but only for the extreme case when the caustics occur maximally, that is, when the multiplicity \(f\) of the caustics coincides with the dimension \(d\). Specific examples on caustics have been studied independently in [10, 11] (see also [12, 13]).
The aim of this paper is to present a complete analysis of quantum caustics for quadratic systems by deriving the transition amplitude in a closed form under generic multiplicity $f$ on caustics in $d$-dimensions, extending that of [9] and of our own [14]. We shall find that, in both classical and quantum regimes, caustics have a rich structure characterized by the multiplicity, and this will be demonstrated by two examples furnished later. Schulman’s procedure [5, 4] will be adopted for evaluating the kernel on caustics, but the key ingredient to get the closed form turns out to be the unitarity relation and the initial condition satisfied by the kernel. This we find is amusing, as the set of these requirements alone is sufficient to get the kernel formula (1.1) for regular (non-caustics) cases without recourse to any involved measures.

This paper is organized as follows. After the Introduction, in Section 2 we provide a general argument for caustics both in classical and quantum mechanics. Using the unitarity relation as a key ingredient, the transition amplitude is derived explicitly for the generic case of multiplicity. For illustration, in Section 3 we consider a particle moving under external electric and magnetic fields exhibiting caustics phenomena with different multiplicities. Section 4 is devoted to our conclusion.
2. Caustics in Classical and Quantum Mechanics

In this section we present a general theory of caustics for quadratic systems in $d$-dimensions in both classical mechanics and quantum mechanics. We first provide a framework for classical caustics which are classified by the multiplicity $f$ given by the co-dimension of the surface formed by the set of focal points. We then derive the transition kernel for generic $f$ based on the unitarity relation and the initial condition satisfied by the kernel. The result reduces to the standard formula (1.1) for the nonsingular ($f = 0$) case and to the one previously obtained for the maximally singular ($f = d$) case.

2.1. Classical caustics

The systems we are interested in are those described by Lagrangians in $d$-dimensions which are at most quadratic:

$$L = \frac{1}{2} P_{ij}(t) \dot{x}^i \dot{x}^j + Q_{ij}(t) x^i \dot{x}^j + \frac{1}{2} R_{ij}(t) x^i x^j + S_i(t) x^i.$$  \hspace{1cm} (2.1)

The coefficient functions $P_{ij}(t), Q_{ij}(t), R_{ij}(t), S_i(t)$ with $i, j = 1, 2, \ldots, d$ are smooth functions of time $t$, and it is understood that repeated indices are summed over unless otherwise stated. We take $P_{ij}(t)$ and $R_{ij}(t)$ symmetric as a matrix, $P^T(t) = P(t), R^T(t) = R(t)$, and assume that $P(t)$ is positive-definite, i.e., all of its minors are positive for any $t \in [0, T]$. The equations of motion derived from the action read

$$\Lambda x + S = 0,$$  \hspace{1cm} (2.2)

where we have used the matrix-valued operator,

$$\Lambda := -\frac{d}{dt} \left( P(t) \frac{d}{dt} + Q^T(t) \right) + \left( Q(t) \frac{d}{dt} + R(t) \right).$$  \hspace{1cm} (2.3)

We note that the operator $\Lambda$ is self-adjoint in the space of functions which vanish at the time boundary $t = 0$ and $T$.

We shall consider the Dirichlet problem associated with the equation (2.2), that is, we look for the solution $\bar{x}(t)$ of (2.2) satisfying the boundary conditions,

$$\bar{x}(0) = a, \quad \bar{x}(T) = b.$$  \hspace{1cm} (2.4)
to some given vectors $a, b \in \mathbb{R}^d$. The usual procedure for this is to choose first two independent sets of solutions, \{v_k(t)\} and \{u_k(t)\} for $k = 1, 2, \ldots, d$, which obey the homogeneous (Jacobi) equation,

$$\Lambda v_k = 0, \quad \Lambda u_k = 0, \quad k = 1, 2, \ldots, d . \quad (2.5)$$

The set \{v_k\} is specified by requiring that

$$v^i_k(0) = \delta_{ik}, \quad \dot{v}^i_k(0) = 0, \quad i, k = 1, 2, \ldots, d . \quad (2.6)$$

On the other hand, for \{u_k\} we impose only the conditions,

$$u^i_k(0) = 0, \quad i, k = 1, 2, \ldots, d , \quad (2.7)$$

for the moment and leave $\dot{u}^i_k(0)$ undetermined. These solutions form a complete set of solutions for the homogeneous equation.\footnote{Note that our boundary conditions, (2.6) and (2.7), differ slightly from those used in Ref.[9].} Later, we shall impose other conditions to specify the set uniquely.

We also choose a special solution $s(t)$ of the full equations of motion (2.2), $\Lambda s + S = 0$, obeying the initial condition $s^i(0) = 0$. (For our present purpose we do not need to specify the initial velocities $\dot{s}^i(0)$.) Then, in terms of $2d$ constants, $A^k, B^k$ for $k = 1, 2, \ldots, d$, the general solution of the equations of motion (2.2) is given by

$$\bar{x}(t) = A^k u_k(t) + B^k v_k(t) + s(t) . \quad (2.8)$$

For convenience, we introduce the matrices $U(t)$ and $V(t)$ from the solutions by $U_{ik}(t) := u^i_k(t)$ and $V_{ik}(t) := v^i_k(t)$, respectively, and thereby rewrite the general solution (2.8) as

$$\bar{x}(t) = U(t) A + V(t) B + s(t) , \quad (2.9)$$

with $A = (A_1, \ldots, A_d)^T$ and $B = (B_1, \ldots, B_d)^T$. Then the initial condition in (2.4) implies $B = a$ whereas the final condition in (2.4) is met by choosing

$$A = U^{-1}(T) (b - V(T) a - s(T)) . \quad (2.10)$$

It is thus clear that the solution to the Dirichlet problem (2.4) does not exist if $\det U(T) = 0$, and this is the cause of caustics.
To formulate the caustics more precisely, let us define \( f \) such that \( d - f \) becomes the rank of the matrix \( U(T) \), and choose the (new) set of solutions \( \{u_k\} \) satisfying

\[
\mathbf{u}_k(T) = \mathbf{0} \quad \text{for} \quad k = 1, 2, \ldots, f. \tag{2.11}
\]

We also demand, for simplicity, that the non-vanishing part of the solutions in the set be normalized as

\[
u_k^i(T) = \delta_{i k} \quad \text{for} \quad i, k = f + 1, f + 2, \ldots, d. \tag{2.12}
\]

Hence, at \( t = T \) the matrix \( U(T) \) takes the form,

\[
U(T) = \begin{pmatrix} 0 & * \\ 0 & 1 \end{pmatrix}. \tag{2.13}
\]

Here the upper left ‘0’ is a \( f \times f \) null matrix, the lower left ‘0’ is a \((d - f) \times f \) null matrix, ‘1’ is a \((d - f) \times (d - f) \) identity matrix, and ‘*’ represents a \( f \times (d - f) \) matrix consisting of undetermined elements. As we shall see shortly, the number \( f \) gives the co-dimension of the surface where caustics occur, with \( f = d \) and \( f = 0 \) being the two extremes. We call these extreme cases ‘full caustics’ and ‘non-caustics’, respectively, and use ‘partial caustics’ for other intermediate cases.

Note that the choice, (2.11) and (2.12), can always be realized by linear transformations in the solution space formed by \( \{u_k\} \) after some exchanges of the coordinates, if necessary. It then follows from the solution (2.8) that the surface of caustics (i.e., the set of all endpoints) conjugate to the initial point \( a \) is given by

\[
x = \mathbf{u}_k(T)A^k + h(a), \tag{2.14}
\]

where \( A^k, k = f + 1, \ldots, d \), parameterize the \((d - f)\)-dimensional caustic surface, and we have used

\[
h(a) := \mathbf{v}_k(T)a^k + s(T). \tag{2.15}
\]

Thus, if \( f \neq 0 \), then for our Dirichlet problem the best we can do (in our coordinate frame) is to find a solution that meets the conditions \( \bar{x}^i(T) = b^i \) for \( i = f + 1, \ldots, d \) by choosing

\[
A^k = A^k(b) := b^k - h^k(a) \quad \text{for} \quad k = f + 1, f + 2, \ldots, d. \tag{2.16}
\]
Figure 1. A schematic picture of a caustics plane and the decomposition of the endpoint vector $\bar{x}(T)$ by the projection operator. The vectors $U^\perp(T) h(a)$ and $b - \bar{x}(T)$ become orthogonal to $U(T) b$ when $U(T)$ is symmetric.

To specify the solution uniquely, for the remaining components $k = 1, \ldots, f$ we set $A^k = 0$ for simplicity. With this choice the classical solution turns out to be

$$\bar{x}^i(t) = \left\{ v^i_l(t) - \sum_{k=f+1}^{d} v^i_k(T) u^i_k(t) \right\} a^l + \sum_{k=f+1}^{d} u^i_k(t) b^k + s^i(t) - \sum_{k=f+1}^{d} u^i_k(t) s^k(T), \quad (2.17)$$

whose endpoint,

$$\bar{x}^i(T) = u^i_k(T) b^k + \{ \delta^i_k - u^i_k(T) \} h^k(a), \quad i = 1, 2, \ldots, d, \quad (2.18)$$

reduces to $b^i$ for $i = f + 1, f + 2, \ldots, d$ as required.

An important point to note is that the matrix $U(T)$, normalized as (2.13), fulfills $(U(T))^2 = U(T)$ and hence it acts as a projection operator onto the caustic surface given by (2.14). Accordingly, the matrix $U^\perp(T) := 1 - U(T)$ projects vectors down to the complementary space 'orthogonal' to the surface. (Strictly speaking, the space is not quite orthogonal to the surface because $U(T)$ may not be symmetric.) In terms of these projection operators the endpoint of our classical solution (2.18) is

$$\bar{x}(T) = U(T) b + U^\perp(T) h(a), \quad (2.19)$$

where now its geometrical meaning is evident (Fig.1).
We here derive some useful identities which will be important later. Notice first that, for arbitrary vector-valued functions $f$ and $g$, we have

$$\int_0^T dt \left( f \cdot \Lambda g - g \cdot \Lambda f \right) = \left[ \left( \dot{f} \cdot P g - \dot{g} \cdot P f \right) + f \cdot (Q - Q^T) g \right]_0^T , \quad (2.20)$$

where we have used the inner product $a \cdot b = a^i b^i$. If we choose for the functions the solutions $f = u_k$, $g = v_l$ of the Jacobi equation (2.5), then from the boundary conditions, (2.6), (2.7), (2.11) and (2.12), we find that in matrix form the relations (2.20) become

$$V^T(T)P(T)\dot{U}(T) - \dot{V}^T(T)P(T)U(T) + V^T(T)\{Q^T(T) - Q(T)\}U(T) = P(0)\dot{U}(0) . \quad (2.21)$$

We may also choose $f = u_k$ and $g = u_l$ to get

$$U^T(T)P(T)\dot{U}(T) - \dot{U}^T(T)P(T)U(T) + U^T(T)\{Q^T(T) - Q(T)\}U(T) = 0 . \quad (2.22)$$

In particular, for $l > f$ and $k \leq f$ the $(lk)$-component of the identities (2.22) reads

$$u_l(T) \cdot P(T) \dot{u}_k(T) = 0 . \quad (2.23)$$

In the non-caustics case where one has $U(T) = 1$, one can eliminate $Q(T)$ from the identity (2.21) by using (2.22) to find

$$\left\{ -\dot{V}^T(T) + V^T(T)\dot{U}^T(T) \right\} P(T) = P(0)\dot{U}(0) . \quad (2.24)$$

Let us recall the general definition of the Jacobi fields. Denote by $\bar{x}(p, t)$ the classical solution with

$$\bar{x}(p, 0) = a, \quad \frac{\partial L[\bar{x}]}{\partial \bar{x}} \bigg|_{t=0} = p , \quad (2.25)$$

for some given $a, p \in \mathbb{R}^d$. Then the Jacobi fields are given by

$$J_{ik}(t) := \frac{\partial \bar{x}^i(p, t)}{\partial p_k} . \quad (2.26)$$

For our quadratic system (2.1) we have the classical solution in the form (2.8) and $p_k$ is given by $P_{ki}(0) \dot{x}^i(0) + a^i Q_{ik}(0)$. From these the Jacobi fields are found to be

$$J(t) = U(t)\dot{U}^{-1}(0)P^{-1}(0) . \quad (2.27)$$
This shows manifestly that $J(t)$ fulfills the Jacobi equation (2.5) with the initial condition $J(0) = 0$.

At this point we remark that, as can be seen from the explicit form of the classical solution (2.8) the classical action for the solution is at most quadratic in the boundary values,

$$I[\bar{x}] = W_{ij}a^i a^j + X_{ij}a^i b^j + Y_{ij}b^i b^j + E_i a^i + F_i b^i + G,$$

(2.28)

where $W_{ij}, X_{ij}, \ldots, G$ are some functions of $T$. (Note that in (2.28) the summation for $a^i$ over $i$ runs from 1 to $d$ whereas for $b^i$ it runs only from $f + 1$ to $d$ due to the choice in (2.16).) For instance, in terms of our Jacobi fields, $X_{ij}, i = 1, \ldots, d, j = f + 1, \ldots, d$, are given by

$$X_{ij} = \frac{1}{2} \left[ -P_{ik}(0)\dot{u}_j^k(0) + \left\{ \dot{v}_i^k(T) - \sum_{l=f+1}^{d} v_l^i(T)\dot{u}_j^k(T) \right\} P_{kn}(T)u_j^n(T) \right]$$

$$+ \frac{1}{2} \left\{ v_i^k(T) - \sum_{l=f+1}^{d} v_l^i(T)u_l^k(T) \right\} \left[ P_{kn}(T)\dot{u}_j^n(T) + \left\{ Q_{kn}(T) + Q_{nk}(T) \right\} u_j^n(T) \right].$$

(2.29)

In particular, for non-caustics case the identity (2.24) can be used to simplify (2.29) into

$$X = -\frac{1}{2} \left\{ P(0)\dot{U}(0) + \left[ -\dot{V}^T(T) + V^T(T)\dot{U}^T(T) \right] P(T) \right\} = -P(0)\dot{U}(0).$$

(2.30)

Using (2.30) and (2.27), one can immediately confirm the relation,

$$-J(T) X = -X J(T) = 1,$$

(2.31)

which follows from the general definition (2.25).

2.2. Quantum caustics

We now move on to quantum mechanics and consider the problem corresponding to the classical Dirichlet problem (2.4), that is, we wish to find the transition kernel $K(b, T; a, 0)$ whose path-integral expression is

$$K(b, T; a, 0) = \int_{x(0) = a}^{x(T) = b} Dx \ e^{\frac{i}{\hbar}I[x]}.$$

(2.32)
To this end, as we did for the one-dimensional case, we first decompose any path \( x(t) \) with the boundary values (2.4) as

\[
x(t) = \bar{x}(t) + \rho(t) + \eta(t) .
\]

(2.33)

Here \( \bar{x}(t) \) is a classical path starting from \( \bar{x}(0) = a \) and ending at a point in the caustic surface (2.14), and for definiteness we choose to be the one (2.17) mentioned earlier. The function \( \rho(t) \) is a compensating function satisfying \( \rho(0) = 0 \) and

\[
\rho(T) = b - \bar{x}(T) = U^\perp(T)(b - h(a)) ,
\]

(2.34)

which is designed to fill the gap between the given endpoint \( b \) for the transition and the actual endpoint \( \bar{x}(T) \) of the classical solution. Note that \( U(T)\rho(T) = 0 \) implies \( \rho^i(T) = 0 \) for \( i = f + 1, \ldots, d \).

The final piece \( \eta(t) \) in (2.33) representing fluctuations fulfills \( \eta(0) = \eta(T) = 0 \) and may be expanded as \( \eta(t) = \sum_n a_n \chi_n(t) \) in terms of the orthonormal vector-valued eigenfunctions \( \{\chi_n\} \) associated with the self-adjoint operator \( \Lambda \):

\[
\Lambda(t) \chi_n(t) = \lambda_n \chi_n(t) ,
\]

(2.35)

with

\[
\chi_n(0) = \chi_n(T) = 0 ; \quad \int_0^T dt \chi_n(t) \cdot \chi_m(t) = \delta_{nm} .
\]

(2.36)

The action for an arbitrary path \( x(t) \) then becomes

\[
I[x] = I[\bar{x} + \rho + \eta]
= I[\bar{x} + \rho] + \frac{1}{2} \sum_n \lambda_n a_n^2 + \sum_n \lambda_n a_n \int_0^T dt \rho(t) \cdot \chi_n(t) + \rho(T) \cdot P(T) \sum_n a_n \dot{\chi}_n(T) .
\]

(2.37)

Notice that, when caustics occur, the solutions \( \{u_k\} \) for \( k = 1, 2, \ldots, f \) may be chosen such that they are orthonormal each other, \( \int_0^T dt \, u_k(t) \cdot u_l(t) = \delta_{kl} \), by performing linear transformations among themselves. This implies that \( \{u_k\} \) is just the set of \( f \) zero modes \( \{\varphi_k\} \) with \( \lambda_k = 0 \) in the orthonormal set of eigenfunctions. Keeping this in mind, by
change of the measure $\mathcal{D}x = \mathcal{D}\eta \propto \prod_n da_n$, we carry out the path-integration (2.32) to obtain

$$K(b, T; a, 0) = \left(\frac{2\pi}{i}\right)^{f/2} \mathcal{N} \left[\prod_n' \lambda_n\right]^{-1/2} \prod_{k=1}^f \delta(\rho(T) \cdot P(T) \dot{u}_k(T)) e^{i I[\dot{x} + \rho]}, \quad (2.38)$$

where $\mathcal{N}$ is a normalization constant and the prime in $\prod_n'$ indicates that the zero modes are omitted in the product. If we denote by $\det' M$ the minor corresponding to the first $f \times f$ part of a $d \times d$ matrix $M$, then from the property (which we shall show later),

$$\det'(P(T)\dot{U}(T)) \neq 0, \quad (2.39)$$

we find

$$\prod_{k=1}^f \delta(\rho(T) \cdot P(T) \dot{u}_k(T)) = |\det'(P(T)\dot{U}(T))|^{-1} \prod_{i=1}^f \delta(\rho^i(T)). \quad (2.40)$$

Thus, if we let $m(T)$ be the Morse index of the operator $\Lambda$ (associated with the period $[0, T]$) which gives the number of non-positive modes $\lambda_n \leq 0$ in (2.35), and combine (2.40) with (2.34), we get the kernel in the polar form,

$$K(b, T; a, 0) = R(T) \prod_{i=1}^f \delta \left([U^\perp(T)(b - h(a))]^i\right) e^{i \Theta(b, T; a, 0)}. \quad (2.41)$$

The phase part is given by

$$\Theta(b, T; a, 0) := \frac{1}{\hbar} I[\dot{x}] - \frac{\pi}{2} m(T) + \gamma, \quad (2.42)$$

where $\gamma$ is a constant independent of $T$, and we have replaced $I[\dot{x} + \rho]$ with $I[\dot{x}]$ under the presence of the delta-functions in (2.41). The result (2.41) shows that, when caustics occur, allowed transitions are those satisfying $\rho(T) = 0$, that is, those whose boundaries admit classical solutions. In other words, classically forbidden processes remain to be forbidden even quantum mechanically.

The key ingredient for determining the modulus part $R(T)$ of the kernel (2.41) is the unitarity relation,

$$\prod_{i=1}^d \delta(a^i - c^i) = \int \prod_{i=1}^d db^i K^*(b, T; c, 0) K(b, T; a, 0). \quad (2.43)$$
Plugging (2.41) into (2.43) and noticing
\[ \Theta(b, T; a, 0) - \Theta(b, T; c, 0) = \frac{1}{\hbar} \sum_{j=f+1}^{d} X_{ij}(a^i - c^i)b^j + \Phi(a - c) , \]  

(2.44)

where \( \Phi(a - c) \) stands for the terms which are independent of \( b \) and vanish at \( a = c \), we observe that the r.h.s. of (2.43) becomes
\[
R^2(T) \int \prod_{j=1}^{d} db^j \prod_{l=1}^{f} \delta \left( \left[ U^\perp(T)(b - h(a)) \right]^l \right) \\
\times \prod_{m=1}^{f} \delta \left( \left[ U^\perp(T)(b - h(a)) \right]^m \right) e^{i\Theta(b,T;a,0) - i\Theta(b,T;c,0)} \\
= R^2(T) \int \prod_{j=f+1}^{d} db^j \prod_{m=1}^{f} \delta \left( \left[ U^\perp(T)(h(a) - h(c)) \right]^m \right) e^{i\sum_{j=f+1}^{d} X_{ij}(a^i - c^i)b^j + i\Phi(a - c)} \\
= R^2(T) \prod_{m=1}^{f} \delta \left( \left[ U^\perp(T)(h(a) - h(c)) \right]^m \right) (2\pi\hbar)^{d-f} \prod_{j=f+1}^{d} \delta \left( X_{ij}(a^i - c^i) \right) e^{i\Phi(a - c)} .
\]

(2.45)

We then use \( h(a) - h(c) = V(T)(a - c) \) together with the matrix \( Z \) defined by
\[
Z_{ij} = \begin{cases} 
\{ \delta_n^i - u_n^i(T) \} v_n^j(T), & \text{for } j = 1, 2, \ldots, f; \\
X_{ji}, & \text{for } j = f + 1, f + 2, \ldots, d,
\end{cases}
\]

(2.46)

to rewrite the unitarity relation (2.43) as
\[
\prod_{i=1}^{d} \delta(a^i - c^i) = (2\pi\hbar)^{d-f} R^2(T) \prod_{i=1}^{d} \delta \left( Z_{ij}(a^j - c^j) \right) e^{i\Phi(a - c)} \\
= (2\pi\hbar)^{d-f} R^2(T) |\det Z|^{-1} \prod_{i=1}^{d} \delta(a^i - c^i) .
\]

(2.47)

From this the modulus part is found to be
\[
R(T) = (2\pi\hbar)^{-\frac{d-f}{2}} \sqrt{|\det Z|} .
\]

(2.48)

We here point out that the fact \( \det Z \neq 0 \) can be shown directly, but it is also obvious from the observation that otherwise the r.h.s. of (2.47) does not match the l.h.s. due to the difference in the structure of delta-functions.
It remains to determine the constant $\gamma$ in the phase factor (2.42). We do this by looking at the initial condition for the kernel,

$$\lim_{T \to 0^+} K(b, T; a, 0) = \prod_{i=1}^{d} \delta(b^i - a^i) .$$

(2.49)

Note that the limit $T \to 0$ cannot be taken for our kernel (2.41), since caustics take place only at finite $T$. However, the formal limit still makes sense if we put $U(T) \to U(0) = 0$ and $V(T) \to V(0) = 1$ in (2.41) and thereby isolate the delta-functions $\prod_{i=1}^{f} \delta([U^\perp(T)(b - h(a))]^i) \to \prod_{i=1}^{f} \delta(b^i - a^i)$ so that the rest of the kernel describes the transition on the caustics plane. To use (2.49), we also have to evaluate the classical action in the phase factor for the solution (2.8), but for our purpose we only need the asymptotic form of the solution,

$$\bar{x}(t) = a + (b - a) \frac{t}{T} + \mathcal{O}(T) ,$$

(2.50)

which leads to the classical action,

$$I[\bar{x}] = \frac{1}{2T} (b - a) \cdot P(0)(b - a) + \frac{1}{4} (b - a) \cdot \dot{P}(0)(b - a) + \frac{1}{2} (b + a) \cdot Q(0)(b - a) + \mathcal{O}(T) .$$

(2.51)

From (2.46) we observe that $\det Z$ reduces in the limit to $\det'' X$ which is given by the minor corresponding to the last $(d - f) \times (d - f)$ part of the matrix $X$. Thus from the classical action (2.51) we find

$$| \det Z | = T^{-(d-f)} \det'' P(0) + \mathcal{O}(T^{-(d-f)+1}) .$$

(2.52)

With the help of the identity $\lim_{\epsilon \to 0} (2\pi i \epsilon)^{-n/2} e^{i\epsilon \cdot A x/2\epsilon} = (\det A)^{-1/2} \delta^{(n)}(x)$ valid for an $n$-dimensional vector $x$ and an $n \times n$ matrix $A$, together with the property $m(T) = 0$ for $T \to 0$, we can readily evaluate the (formal) limit of the kernel (2.41) to get

$$\lim_{T \to 0^+} K(b, T; a, 0) = i^{d-f} \ e^{i\gamma} \prod_{i=1}^{d} \delta(b^i - a^i) .$$

(2.53)

Comparing with (2.49) we find that the constant $\gamma$ is determined by $e^{i\gamma} = i^{-(d-f)/2}$.

Having found both the modulus and the phase part, we obtain the closed form of the transition kernel on caustics:

$$K(b, T; a, 0) = (2\pi i \hbar)^{-\frac{d-f}{2}} \sqrt{|\det Z|} \prod_{i=1}^{f} \delta \left([U^\perp(T)(b - h(a))]^i\right) e^{\frac{i}{\hbar} I[x] - i\frac{2\pi}{\hbar} m(T)} .$$

(2.54)
In particular, for the full caustics case \( f = d \) the kernel reduces to
\[
K(b, T; a, 0) = \sqrt{|\det V(T)|} \prod_{i=1}^{d} \delta (b^i - h^i(a)) \ e^{\frac{i}{\hbar} I[x]-\frac{i}{\hbar} m(T)} .
\] (2.55)

On the other hand, in the non-caustics case \( f = 0 \) the matrix \( Z \) becomes the Van Vleck matrix \( X \) with \( X_{ij} = \partial^2 I[\bar{x}]/\partial a^i \partial b^j \), and hence the kernel,
\[
K(b, T; a, 0) = (2\pi i \hbar)^{-\frac{d}{2}} \sqrt{|\det X|} e^{\frac{i}{\hbar} I[\bar{x}]-\frac{i}{\hbar} m(T)} ,
\] (2.56)
is indeed the standard one (1.1) for the quadratic system (with the phase correction by the Morse index included) — here we derived it directly from the unitarity relation (2.43) and the initial condition (2.49) without relying on involved procedures, such as one using discretized multiple integrations and recursion relations employed in the literature. We also mention that, with the generic Jacobi fields \( J(t) \) satisfying (2.31), the transition kernel (2.56) becomes the generalized Gel’fand-Yaglom formula obtained previously [8].

Before closing this section, we wish to prove the property (2.39) used to obtain the kernel (2.41). To this end, first we assume that (2.39) does not hold. Then we can find a linear combination \( u(t) := \sum_{k=1}^{f} c_k u_k(t) \) out of the solutions \( \{u_k; k = 1, \ldots, f\} \) such that \( P(T) \dot{u}(T) = (0, *)^T \), where ‘0’ is an \( f \)-dimensional null vector. But since (2.23) is equivalent to \( U^T(T) P(T) \dot{u}_k(T) = 0 \) for \( k = 1, \ldots, f \), we deduce \( U^T(T) P(T) \dot{u}(T) = 0 \). This suggests that the last \( d-f \) components of the vector \( P(T) \dot{u}(T) \) also vanish identically, i.e., \( P(T) \dot{u}(T) \) is actually a null vector. Then from the positive definiteness of \( P(T) \) we find \( \dot{u}(T) = 0 \). Combined with \( u(T) = 0 \) which follows from the caustics conditions, we come to the conclusion that \( (u(T), \dot{u}(T)) = 0 \) as a \( 2d \)-dimensional vector. This however contradicts with the non-triviality of the solutions \( \{u_k\}, \{v_k\} \) at \( t = T \),
\[
\det \begin{pmatrix} U(T) & V(T) \\ \dot{U}(T) & \dot{V}(T) \end{pmatrix} \neq 0 ,
\] (2.57)
which ensures that, as a set of \( 2d \)-dimensional vectors, \( (u_k(T), \dot{u}_k(T)) \) (together with \( (v_k(T), \dot{v}_k(T)) \)) for \( k = 1, \ldots, d \) form a complete basis. We therefore see that our assumption is wrong, proving (2.39) as claimed.
3. Caustics under External Electromagnetic Fields

In this section we provide two examples to illustrate caustic phenomena for a particle in three dimensions. The first example discusses partial caustics \( f = 2 \), which however can be more naturally viewed as two-dimensional full caustics plus a decoupled motion in the third dimension. The second example shows non-trivial partial caustics \( f = 1 \). Electromagnetic fields are used as driving forces acting on the test particle, and the two examples show that caustics with different focal dimensions are possible for different field configurations.

3.1. Full caustics

Let us consider a point particle with mass \( m \) and charge \( q \) subject to a non-static but uniform electric field \( \mathbf{E}_{\text{ext}} = (E_x(t), E_y(t), E_z(t))^T \) and, assuming an appropriate background current, a static uniform magnetic field \( \mathbf{B}_{\text{ext}} = (0, 0, B)^T \). These fields can be derived from a vector potential \( \mathbf{A} \) in the symmetric gauge, \( \mathbf{A} = \frac{B}{2}(-y, x, 0)^T \), and a scalar potential \( \phi \) is given by \( \phi = x \cdot \mathbf{E}_{\text{ext}} \). Noting the position of the particle by \( \mathbf{x} = (x, y, z)^T \) its Lagrangian reads

\[
L_0 := \frac{m}{2} \dot{x}^2 + q \dot{x} \cdot \mathbf{A} - q \phi. \tag{3.1}
\]

Although essentially the same model has been considered in [13], we re-examine it in detail to illuminate the phenomenon of caustics from our point of view.

Due to the special field configuration the Lagrangian (3.1) splits into two parts,

\[
L_0 = L + L_z, \tag{3.2}
\]

of which \( L_z = \frac{m}{2} \dot{z}^2 - q E_z(t) z \) describes an accelerated motion in the \( z \)-direction whereas

\[
L = \frac{m}{2} (\dot{\mathbf{r}}^2 + \omega \mathbf{r} \cdot \Omega \dot{\mathbf{r}}) - q \mathbf{E} \cdot \mathbf{r}, \tag{3.3}
\]

governs the motions in the \( xy \)-plane where we use \( \mathbf{r} = (x, y)^T \) and \( \mathbf{E} = (E_x, E_y)^T \). We have also introduced the cyclotron angular velocity \( \omega := q B/m \) and the skew matrix

\[
\Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]

By splitting off the \( z \)-component in (3.2) we obtain the system (3.3) on which we shall concentrate in the following.

As is evident from the discussion in the previous Section, the intrinsic behavior of the system on caustics does not depend on terms linear in the coordinates, \(-q \mathbf{E} \cdot \mathbf{x}\).
Without this term the system exhibits the discrete symmetry under the ‘parity’ conjugation \((x(t), \omega) \leftrightarrow (y(t), -\omega)\), which has important consequences on the caustics.

The equation of motion for a classical path \(\vec{r}\) reads

\[
\Lambda \vec{r} + S = 0, 
\]

in which the differential operator \(\Lambda\) and the nonlinear term \(S\) are given by

\[
\Lambda = -m \frac{d^2}{dt^2} + m\omega\Omega \frac{d}{dt}, \quad S(t) = -qE(t),
\]

where \(1\) denotes a two by two unit matrix. To provide a special solution of (3.4) we focus on the first time derivative \(y(t) := \dot{\vec{r}}(t)\), which fulfills

\[
m(\ddot{y} + \omega^2 y) + q(\omega\Omega E + \dot{E}) = 0. \tag{3.6}
\]

This equation is solved by

\[
y(t) = -\frac{q}{m\omega} \int_0^t dt' \sin \omega(t - t') \{\omega\Omega E(t') + \dot{E}(t')\} + g(t), \tag{3.7}
\]

wherein \(g(t)\) stands for solutions to the homogeneous part of (3.6) given by a linear combination of \(\sin \omega t\) and \(\cos \omega t\). By integrating the solution (3.7) and thereby imposing the boundary condition \(\vec{s}(0) = 0\) we obtain the special solution,

\[
\vec{s}(t) = \frac{q}{m\omega} \Omega \int_0^t dt' \left(e^{\omega(t-t')} - 1\right) E(t'). \tag{3.8}
\]

On the other hand, the solution space of the homogeneous part of (3.4) is formed by the four independent functions,

\[
\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \sin \omega t \\ \cos \omega t \end{pmatrix}, \begin{pmatrix} \cos \omega t \\ -\sin \omega t \end{pmatrix}. \tag{3.9}
\]

In particular, the Jacobi fields \(v_1\) and \(v_2\) which satisfy (2.6) are given by the first two constant vectors, respectively, while \(u_1\) and \(u_2\) are given by linear combinations of the four. The general solution for the equation of motion (3.4) is then obtained by (2.8) (with \(\vec{x}\) replaced by \(\vec{r}\)).

In order to establish the conditions for caustics, let us consider the eigenvalue equation (2.35) with (2.36) for the present case. Although the eigenfunctions of this equation can
be found in the usual manner, for completeness let us briefly comment on their derivation.

First, for the eigenfunctions we make the ansatz $\chi_n(t) = c e^{i\xi t}$, where $c$ is a constant complex two-vector and $\xi$ a complex number. From (2.35) one deduces the coefficient equation $[(m\xi^2 - \lambda_n)1 + im\Omega]c = 0$, which in turn implies that $\xi$ can take one of the following four values,

$$\begin{align*}
\xi_1 &= -\omega + \frac{\sqrt{\omega^2 + (4\lambda_n/m)}}{2}, & \xi_2 &= -\omega - \frac{\sqrt{\omega^2 + (4\lambda_n/m)}}{2}, \\
\xi_3 &= \omega + \frac{\sqrt{\omega^2 + (4\lambda_n/m)}}{2}, & \xi_4 &= \omega - \frac{\sqrt{\omega^2 + (4\lambda_n/m)}}{2},
\end{align*}$$

(3.10)

to each of which there corresponds a fixed vector $c = c_i$ given by $c_1 = c_2 = (i, 1)^T$ and $c_3 = c_4 = (-i, 1)^T$, respectively. The general solution to (2.35) reads (a_i complex)

$$\left( \begin{array}{c} i \\ 1 \end{array} \right) (a_1 e^{i\xi_1 t} + a_2 e^{i\xi_2 t}) + \left( \begin{array}{c} -i \\ 1 \end{array} \right) (a_3 e^{i\xi_3 t} + a_4 e^{i\xi_4 t}),$$

(3.11)

upon which the boundary condition in (2.35) is to be imposed. This leads to $a_1 + a_2 = 0$, $a_3 + a_4 = 0$, $e^{i\xi_1 T} = e^{i\xi_2 T}$, and $e^{i\xi_3 T} = e^{i\xi_4 T}$, from which we infer $m\omega^2 + 4\lambda_n > 0$ and deduce the eigenvalues

$$\lambda_n = -\frac{m\omega^2}{4} + m \left( \frac{n\pi}{T} \right)^2$$

(3.12)

for positive integer $n$. There exist two real eigenfunctions to each eigenvalue, namely,

$$\chi_n(+) = \sqrt{\frac{T}{2}} \sin \left( \frac{n\pi}{T} t \right) \left( \sin \left( \frac{\omega t}{2} \right) \right)$$

(3.13)

and

$$\chi_n(-) = \sqrt{\frac{T}{2}} \sin \left( \frac{n\pi}{T} t \right) \left( -\cos \left( \frac{\omega t}{2} \right) \right).$$

(3.14)

One can check that these two series of eigenfunctions (3.13) and (3.14) are indeed orthonormal, $\int_0^T dt \chi_n^{(l)} \cdot \chi_n^{(l')} = \delta_{nm} \delta_{ll'}$ and complete. The twofold degeneracy comes from the discrete symmetry of the Lagrangian (3.3). Clearly, zero modes $\lambda_n = 0$ appear if

$$\omega = \omega_n \quad \text{where} \quad \omega_n := \frac{2n\pi}{T},$$

(3.15)

in which case caustics occur. Note that at any of the above $\omega_n$ for some positive integer $n$, there appear two zero modes due to the degeneracy. We thus find that, as far as the the motions on the $xy$-plane are concerned, we have the full caustics, $f = 2$. 

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Let us turn our attention to quantum mechanics and obtain the transition amplitude on the caustics at $\omega = \omega_n$. Being the full caustics, the amplitude can be obtained immediately by using the general formula for the kernel (2.55), where now we have $|\det V(T)| = 1$ and $h(a) = a + s(T)$ with $s(T)$ being the endpoint value of the special solution (3.8). We also notice from (3.12) that the Morse index is given by $m(T) = 2n$ (where the factor 2 comes from the twofold degeneracy) and the classical action reads

$$I[\bar{r}] = -\frac{q}{2} a \cdot \int_0^T dt \left( e^{\Omega(T-t)} + 1 \right) E(t) + s(T) \cdot \frac{m}{2} \dot{s}(T) - \frac{q}{2} \int_0^T dt s(t) \cdot E(t). \quad (3.16)$$

Combining these, for $\omega = \omega_n$ we obtain

$$K(b, T; a, 0) = (-1)^n \delta^{(2)}(b - a - s(T)) e^{\pi I[\bar{r}]/2}. \quad (3.17)$$

So far we have discussed the two dimensional subsystem governed by the Lagrangian $L$, which reveals full caustics at $\omega = \omega_n$. When the total system $L_0$ in (3.2) is considered, these caustics are only partial, since the motion in the z-direction is free of caustics. This fact is also expressed in the total transition amplitude from the initial position $(a_x, a_y, a_z)^T$ to the final one $(b_x, b_y, b_z)^T$, which is the product of the kernel (3.17) and the contribution coming from the z-component

$$K(b, T; a, 0) \times K_z(b_z, T; a_z, 0), \quad (3.18)$$

where $K_z(b_z, T; a_z, 0)$ can be calculated as

$$K_z(b_z, T; a_z, 0) = \sqrt{\frac{m}{2 \pi i h T}} e^{\pi I_z}, \quad (3.19)$$

with

$$I_z = \frac{m}{2T} \left[ (b_z - a_z)^2 + \frac{2}{m} \int_0^T dt E_z(t) \{ a_z T + t(b_z - a_z) \} \right]$$

$$- \frac{2}{m^2} \int_0^T dt \int_0^t dt' E_z(t) E_z(t')(T - t)t' \right]. \quad (3.20)$$

Since the z-component $b_z$ of the endpoint can take an arbitrary value, depending on the initial momentum in this direction, the set of image points of an initial point $a$ makes up a straight line in the z-direction. The transition kernel obtained here, (3.18) with (3.17), (3.19) and (3.20), agrees with the one [13] obtained previously by a different method.
3.2. Partial caustics

The above example, when viewed as that of partial caustics, is rather trivial due to the decoupling of the $z$-component. However, partial caustics do not always have such a trivial situation. To examine a non-trivial case of partial caustics let us consider the following Lagrangian in three dimensions,

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{qB}{2}(x\dot{y} - y\dot{x}) - \alpha qyz.$$  \hfill (3.21)

It describes a point particle of mass $m$ and charge $q$ moving in a static uniform magnetic field in the $z$-direction $B_{\text{ext}} = (0, 0, B)^T$ and a static but non-uniform electric field $E_{\text{ext}} = -\alpha(0, z, y)^T$, $\alpha \neq 0$. Introducing $\omega = qB/m$ and $\gamma = q\alpha/m$ the equation of motion reads

$$\ddot{x}(t) + \begin{pmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \dot{x}(t) + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \gamma \\ 0 & \gamma & 0 \end{pmatrix} x(t) = 0.$$  \hfill (3.22)

We note that, like the previous example, the Lagrangian and the equation of motion are invariant under the ‘parity’ transformation,

$$(x, y, z, \omega, \gamma) \rightarrow (x, -y, z, -\omega, -\gamma).$$  \hfill (3.23)

In order to construct classical solutions we make the ansatz $x(t) = he^{i\lambda t}$ ($h$ a complex constant vector and $\lambda$ a complex number) and substitute it into (3.22), yielding

$$\left[ \lambda^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - i\lambda \begin{pmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & \gamma \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \gamma \end{pmatrix} \right] h = 0.$$  \hfill (3.24)

These equations possess the pair of solutions $(\lambda, h)$ given by

$$\lambda_1 = 0; \quad h_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad h'_1 = \begin{pmatrix} \gamma t \\ 0 \\ -\omega \end{pmatrix}, \quad \text{(double root)}$$

$$\lambda_2 = \xi; \quad h_2 = \begin{pmatrix} -i\omega \\ \xi \\ \gamma/\xi \end{pmatrix}, \quad \lambda_3 = -\xi; \quad h_3 = \begin{pmatrix} +i\omega \\ \xi \\ \gamma/\xi \end{pmatrix} = h_2^*, \quad (3.25)$$

$$\lambda_4 = i\eta; \quad h_4 = \begin{pmatrix} +\omega \\ -\eta \\ \gamma/\eta \end{pmatrix}, \quad \lambda_5 = -i\eta; \quad h_5 = \begin{pmatrix} -\omega \\ -\eta \\ \gamma/\eta \end{pmatrix},$$

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where we have used

$$\xi = \sqrt{\frac{\beta + 1}{2}} \omega, \quad \eta = \sqrt{\frac{\beta - 1}{2}} \omega, \quad \beta = \sqrt{1 + 4 \frac{\gamma^2}{\omega^4}} > 1. \quad (3.26)$$

By taking linear combinations of the complex-valued solutions $h_j e^{i\lambda_j t}$ for $j = 2$ and $3$ and likewise for $j = 4$ and $5$ we obtain the following real solutions besides $h_1$ and $h'_1$,

$$f_2(t) = \begin{pmatrix} \omega \sin \xi t \\ \xi \cos \xi t \\ (\gamma/\xi) \cos \xi t \end{pmatrix}, \quad f_3(t) = \begin{pmatrix} \omega \cos \xi t \\ -\xi \sin \xi t \\ -(\gamma/\xi) \sin \xi t \end{pmatrix},$$

$$f_4(t) = \begin{pmatrix} -\omega \sinh \eta t \\ -\eta \cosh \eta t \\ (\gamma/\eta) \cosh \eta t \end{pmatrix}, \quad f_5(t) = \begin{pmatrix} -\omega \cosh \eta t \\ -\eta \sinh \eta t \\ (\gamma/\eta) \sinh \eta t \end{pmatrix}. \quad (3.27)$$

The general real solution then reads

$$x(t) = Ah_1 + Bh'_1 + Cf_2(t) + Df_3(t) + Ef_4(t) + Ff_5(t), \quad (3.28)$$

with $A$, $B$, $C$, $D$, $E$, and $F$ some real constants.

To judge whether or not caustics occur in this system we have to examine zero-modes, i.e., classical solutions with vanishing boundary conditions $x(0) = x(T) = 0$. The initial condition put into (3.28) gives

$$A = \omega(F - D), \quad C = \frac{\eta}{\xi} E, \quad B = \frac{\eta \omega \beta}{\gamma} E, \quad (3.29)$$

and so the Jacobi field becomes $Dw_1(t) + Ew_2(t) + Fw_3(t)$ with

$$w_1(t) = \begin{pmatrix} \omega (\cos \xi t - 1) \\ -\xi \sin \xi t \\ -(\gamma/\xi) \sin \xi t \end{pmatrix},$$

$$w_2(t) = \begin{pmatrix} -\frac{\omega}{\eta} (\sinh \eta t - \gamma^2 t / (\omega^2 \eta)) + (\omega/\xi)(\sin \xi t + \gamma^2 t / (\omega^2 \xi)) \\ -\frac{\omega}{\xi} (\cos \eta t) + \frac{\omega}{\xi} (\cos \xi t) \\ (\gamma/\eta^2)(\cosh \eta t - 1) + (\gamma/\xi^2)(\cosh \xi t - 1) \end{pmatrix},$$

$$w_3(t) = \begin{pmatrix} -\omega (\cosh \eta t - 1) \\ -\eta \sinh \eta t \\ (\gamma/\eta) \sinh \eta t \end{pmatrix}. \quad (3.30)$$

The final condition $x(T) = 0$ for a non-trivial choice of the three remaining parameters $D$, $E$, and $F$ is fulfilled whenever the matrix formed by the Jacobi fields becomes singular at $t = T$,

$$0 = \det (w_1(T), w_2(T), w_3(T))$$

$$= \omega^3 \beta \left[ -\frac{2}{\xi} (\cos \xi T - 1) \sinh \eta T - \frac{2}{\eta} \sin \xi T (\cosh \eta T - 1) + \beta T \sin \xi T \sinh \eta T \right]. \quad (3.31)$$
One sufficient condition for caustics is clearly seen to be given by

\[ \xi = \xi(\omega) = \omega_n , \quad (3.32) \]

with \( \omega_n \) appearing in (3.15), which is indirectly a condition on the cyclotron frequency \( \omega \).

In this case the coefficients in (3.28) for the zero-mode solution read

\[ A = -\omega D, \quad D = \text{arbitrary}, \quad B = C = E = F = 0 . \quad (3.33) \]

Since there is just one zero-mode, we have the partial caustics situation \( f = 1, d = 3 \).

The vectors \( w_i(t), i = 1, 2, 3 \), are related to the special choice of the Jacobi fields \( u_i(t) \) in (2.11) to (2.12) in the following way

\[ u_1(t) = M w_1(t), \]
\[
\begin{align*}
    u_2(t) &= \frac{1}{\sigma} M \left[ \frac{\eta}{\omega} \cdot \frac{\sinh \eta T}{1 - \cosh \eta T} w_2(t) + \frac{1}{\omega} w_3(t) \right], \\
    u_3(t) &= \frac{1}{\sigma} M \left[ w_2(t) + \frac{1}{\eta} \cdot \frac{\sinh \eta T - \eta \beta T}{1 - \cosh \eta T} w_3(t) \right],
\end{align*} \quad (3.34)\]

with

\[ \sigma := 2 + \frac{\eta \beta T \sinh \eta T}{1 - \cosh \eta T}, \quad M := \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \quad (3.35) \]

The function \( u_1(t) \) is the zero-mode solution, and the caustics surface is spanned by the two independent vectors \( u_2(T) \) and \( u_3(T) \), see Fig. 2. On the other hand, the set of Jacobi fields \( v_i(t), i = 1, 2, 3 \), with the boundary conditions \( v_i^j(0) = \delta_i^j \) and \( \dot{v}_i^j(0) = 0 \) introduced in (2.6) can be expressed as follows

\[ v_1(t) = h_1, \]
\[ v_2(t) = -\frac{\omega}{\gamma} h'_1 + \frac{\xi}{\omega^2 \beta} f_2(t) - \frac{\eta}{\omega^2 \beta} f_4(t), \]
\[ v_3(t) = -\frac{\gamma}{\omega^2 \beta} f_2(t) + \frac{\gamma}{\omega^2 \beta} f_4(t). \quad (3.36) \]

At this point it is worth mentioning the behavior of the Jacobi fields on caustics in the limit \( \alpha \to 0 \), since in this limit the Lagrangian (3.21) reduces to the Lagrangian (3.1)\(^3\) Besides this, there can be other caustics, i.e., those which occur at \( \xi \neq \omega_n \) for which the determinant (3.31) vanishes.
Figure 2. The behavior of the Jacobi fields and the caustics plane for the parameter choice $\omega = 1$, $\gamma = 0.2$. For each of the three end points on the caustics plane we can see two different paths starting at the origin, which reflects the existence of the zero-mode $u_1(t)$.

of the previous subsection up to the term $-q\phi$ linear in the coordinates, which does not affect the behavior of the Jacobi fields. One expects that the zero-mode solutions obtained in the previous subsection for caustics $\omega = \omega_n$,

$$
\chi_n(+)\bigg|_{\omega=\omega_n} = \frac{1}{2} \sqrt{\frac{T}{2}} \begin{pmatrix} 1 - \cos \omega_n t \\ \sin \omega_n t \end{pmatrix}, \\
\chi_n(-)\bigg|_{\omega=\omega_n} = \frac{1}{2} \sqrt{\frac{T}{2}} \begin{pmatrix} -\sin \omega_n t \\ 1 - \cos \omega_n t \end{pmatrix},
$$

are recovered as the $xy$-components of the first two of the Jacobi fields in (3.30), and, furthermore, that the twofold degeneracy appears. Indeed, since $\xi = \omega_n$ and

$$
\begin{pmatrix} \gamma \\ \eta \\ \gamma/\eta \\ \omega \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ 0 \\ \omega \\ \omega_n \end{pmatrix} \quad \text{for} \quad \alpha \rightarrow 0,
$$

we see that $-w_1(t)/\omega$ goes over into the upper vector and $-w_2(t)$ into the lower vector of (3.37). Whereas $w_1(t)$ is already a zero-mode of the original system $w_2(t)$ is not, which
nonetheless turns into a zero-mode in the limit. On the other hand, \( w_3(t)/\eta \) goes over into the function \( (0, 0, t)^T \), which describes free motion of the \( z \)-component in the direction of the caustics line.

When passed over to quantum mechanics, the transition amplitude for the present partial caustics can also be obtained from the general formula (2.54). However, we shall not record it here as it becomes rather cumbersome due to the structure of the matrix \( Z(T) \) in (2.46).
4. Conclusion

In this paper we studied caustics in quantum mechanics in multi-dimensions for Lagrangians which are at most quadratic by extending our previous study in one-dimension [14, 16]. Our main result is the kernel formula (2.54) which gives the transition amplitude in $d$-dimensions. As in one-dimension, we found that transitions prohibited classically remain to be so even quantum mechanically. The complication that arises in the multi-dimensional case is that we now have partial caustics, that is, the dimension of the caustics surface where the classical trajectories (and hence the quantum amplitudes) concentrate is not zero but smaller than $d$. More precisely, one looks at the number $f$ of zero modes of the operator defined by the quadratic part of the Lagrangian, and if $f = d$ caustics take place maximally in all directions (full caustics), if $f = 0$ there arises none (non-caustics), and otherwise we have partial caustics. For the full caustics case, the transition kernel has been previously obtained in [9] based on a different formalism of the path-integral (‘sum over all continuous vector fields along a classical path vanishing at boundaries’). Here we have adopted a more intuitive and seemingly easier method presented in [4], and thereby obtained the kernel formula explicitly for generic caustics. In the full caustics case the kernel formula we obtained reduces to the one given in [9] as required, whereas in the non-caustics case it recovers the one (Gel’fand-Yaglom formula) which is familiar in semi-classical approximations. The crucial ingredient of our derivation is the unitarity relation combined with the initial condition satisfied by the kernel, and to our amusement our method turns out to be much simpler than the previous ones [15, 8] to get the well-known Van-Vleck formula for semiclassical approximations.

The two examples we presented illustrate how partial caustics occur both classically and quantum mechanically. The first is the system of a charged particle under constant magnetic and electric fields perpendicular to each other, which is found to be the case $d = 3$ and $f = 2$. The caustics observed there is, however, not quite partial intrinsically, because one can find an appropriate frame of coordinates in which the partial caustics can be regarded as being a sum of $d = 1$ non-caustics and $d = 2$ full caustics. The second example is given by the system of a charged particle under a certain (rather eccentric) electric field and provides the case $d = 3$ and $f = 1$. In contrast to the first example, this does not seem to admit a trivial decoupling into full and non-caustics beforehand and hence may be regarded as one which exhibits intrinsic partial caustics.
For further extension of the present work, one obvious direction is to study quantum caustics in field theory, especially in the context where the semiclassical approximation becomes important. This includes analyses of solitons/instantons in, e.g., the sine-Gordon theory, nonlinear sigma models, and the Yang-Mills theory possibly coupled to various matter fields. (In fact it was the question of the role of instantons in the nonlinear sigma model [17, 18] which led us to the study of caustics originally [14].) Another direction is to find applications in condensed matter physics, where often situations like the first example of Section 3 are considered. To this end we feel that the quantum feature of caustics should be addressed in various physical aspects, such as in the spread of wave packets in the Gaussian slit experiment [16]. We hope that our present study serves as a basis for future investigations associated with caustics phenomena including those mentioned here.

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