Position-dependent mass charged particles in magnetic and Aharonov-Bohm flux fields: separability, exact and conditionally exact solvability

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Abstract: Using cylindrical coordinates, we consider position-dependent mass (PDM) charged particles moving under the influence of magnetic, Aharonov-Bohm flux, and a pseudoharmonic or a generalized Killingbeck-type potential fields. We implement the PDM-minimal-coupling recipe [26], along with the PDM-momentum operator [27], and report separability under radial cylindrical and azimuthal symmetrization settings. For the radial Schrödinger part, we transform it into a radial one-dimensional Schrödinger-type and use two PDM settings, $g (\rho) = \eta \rho^2$ and $g (\rho) = \eta / \rho^2$, to report on the exact solvability of PDM charged particles moving in three fields: magnetic, Aharonov-Bohm flux, and pseudoharmonic potential fields. Next, we consider the radial Schrödinger part as is and use the biconfluent Heun differential forms for two PDM settings, $g (\rho) = \lambda \rho$ and $g (\rho) = \lambda / \rho^2$, to report on the conditionally exact solvability of our PDM charged particles moving in three fields: magnetic, Aharonov-Bohm flux, and generalized Killingbeck potential fields. Yet, we report the spectral signatures of the one-dimensional $z$-dependent Schrödinger part on the overall eigenvalues and eigenfunctions, for all examples, using two $z$-dependent potential models (infinite potential well and Morse-type potentials).

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I. INTRODUCTION

In the non-relativistic Schrödinger equation, a position-dependent deformation of the mass of the particle, or a position-dependent deformation of the coordinates involved, are two parallel assumptions that yield, in short, to position-dependent mass (PDM) concept. This concept have attracted much attention in the literature over the years, for both classical and quantum mechanical systems (a sample of references can be found in e.g., [1–27]). Hereby, the most prominent PDM non-relativistic Hamiltonian is known as the von Roos Hamiltonian [1] (in $\hbar = 2m_0 = 1$ units)

$$\hat{H} = -\frac{1}{4} \left[ M (\vec{r})^a \vec{\nabla} M (\vec{r})^b \cdot \vec{\nabla} M (\vec{r})^c + M (\vec{r})^c \vec{\nabla} M (\vec{r})^b \cdot \vec{\nabla} M (\vec{r})^a \right] + V (\vec{r}). \quad (1)$$

Where $M (\vec{r}) = m_0 m (\vec{r})$, $m_0$ is the rest mass, $m (\vec{r})$ is a position-dependent dimensionless scalar multiplier that forms the position-dependent mass $M (\vec{r})$, $V (\vec{r})$ is the potential field, and $a, b, c$ are called the ambiguity parameters that satisfy von Roos constraint $a + b + c = -1$. Yet, this Hamiltonian is known to be associated with an ordering ambiguity problem as a result of the non-unique representation of the kinetic energy operator. An obvious radical change in the profile of the kinetic energy term occurs when the values of the ambiguity parameters are changed (consequently, the profile of the effective potential will radically change). There exist an infinite number of ambiguity parametric settings that satisfy the von Roos constraint above. In the literature, however, one may find many suggestions on the ambiguity parametric values [2–14]. Yet, the only physically acceptable condition (along with the von Roos constraint) on the ambiguity parameters is that $a = c$ to ensures continuity at the abrupt heterojunction (e.g., Refs. [12, 16]). The rest are based on different eligibility proposals which are, at least, mathematically challenging and useful models that enrich the class of exactly solvable or conditionally exactly solvable quantum mechanical systems [17, 20]. Nevertheless, it was only very recently that a PDM momentum operator is constructed by Mustafa and Algadhi [27] and resulted in fixing the ordering ambiguity parameters at $a = c = -1/4$ and $b = -1/2$ (known in the literature as Mustafa and Mazharimousavi’s parametric settings [13]).

On the other hand, quantum mechanical charged particles of constant mass moving in a uniform magnetic field (with some occasional inclusion of the Aharonov-Bohm flux field) have been a subject of research interest over the years...
(e.g., see the sample of references [28–34] and related references cited therein). Only a handful number of attempts were made to treat PDM charged particles in uniform magnetic field [24–26, 35]. Nevertheless, to facilitate exact solvability for PDM charged particles in electromagnetic fields, Mustafa and Algadhi [27] have used some canonical point transformation that maps the PDM Hamiltonian into conventional constant mass setting. In so doing, the exact solutions for PDM systems are inferred from those for conventional constant mass systems. Moreover, in their very recent study, Eshghi et al. [35] have used Ben Danial and Duke’s parametric settings $a = c = 0$ and $b = -1$ (c.f., e.g., [2–4, 22–24]) and considered PDM-charged particles moving in both magnetic and Aharonov-Bohm flux fields.

In the current methodical proposal, however, we use the freshly constructed PDM momentum operator of Mustafa and Algadhi [27] and study, within cylindrical and azimuthal symmetrization settings, the PDM-charged particles moving under the influence of magnetic and Aharonov-Bohm flux fields. The PDM minimal coupling [26] is used in the process. We also explore the separability of the corresponding PDM-Schrödinger equation, along with its feasible exact and conditionally exact solvability through two sets of illustrative examples. To the best of our knowledge, and within the current methodical proposal setting, no such studies have been carried out and/or are available in the literature so far.

The organization of this paper is in order. In section 2, we use the PDM-minimal-coupling recipe [26], along with the PDM-momentum operator [27], and discuss the separability of the problem within the cylindrical coordinates $(\rho, \varphi, z)$, indulging azimuthal symmetry. A purely radial $\rho$-coordinate dependent, (12) below, and a simplistic one-dimensional $z$-dependent, (11) below, Schrödinger equations resulted in the process. In connection with the radial $\rho$-dependent part, moreover, we choose to treat it in two different ways. The first of which is to transform it into a radial one-dimensional Schrödinger form and discuss, in section 3, its exact solvability using a pseudoharmonic potential (which is usually used for quantum dots and antidotes, e.g., [31–33]). In the same section, we report exact eigenvalues and eigenfunctions for two PDM models, $g(\rho) = \eta \rho^2$ and $g(\rho) = \eta/\rho^2$. For the second treatment of the radial $\rho$-dependent part (12), nevertheless, we choose to use, in section 4, the biconfluent Heun differential form (c.f., e.g., [33–35]) and discuss the separability of the problem within the cylindrical coordinates $\rho, \varphi, z$-dependent part, (12) below, and a simplistic one-dimensional $z$-dependent Schrödinger part, (11) below, on the overall spectra are reported for each of the four models used. We conclude in section 5.

II. PDM-MOMENTUM OPERATOR AND MINIMAL-COUPLING: CYLINDRICAL COORDINATES AND SEPARABILITY

In this section, we start with the PDM momentum operator

\[ \hat{P}(\vec{r}) = -i \left[ \vec{\nabla} - \frac{1}{4 \left( \frac{\vec{m}(\vec{r})}{m(\vec{r})} \right)} \right], \]

suggested by Mustafa and Algadhi [27] (the readers are advised to refer to [27] for more details on the issue) which obviously collapses into $\hat{P} = -i \vec{\nabla}$ for constant mass settings. The PDM momentum operator is to be substituted in the PDM-Schrödinger equation

\[ \left[ \left( \frac{\hat{P}(\vec{r}) - e \vec{A}(\vec{r})}{\sqrt{m(\vec{r})}} \right)^2 + W(\vec{r}) \right] \psi(\vec{r}) = E \psi(\vec{r}) ; \quad W(\vec{r}) = e \varphi(\vec{r}) + V(\vec{r}), \]

with electromagnetic interaction. Where $\vec{A}(\vec{r})$ is the vector potential, $e \varphi(\vec{r})$ is a scalar potential and $V(\vec{r})$ is any other potential energy than the electric and magnetic ones. The PDM minimal coupling [26, 27] is used in the process. Consequently, in a straightforward manner, equation (3) would read

\[ \left[ -\frac{1}{m(\vec{r})} \vec{\nabla}^2 + \left( \frac{\vec{m}(\vec{r})}{m(\vec{r})^2} \right) \cdot \vec{\nabla} + \frac{1}{4 \left( \frac{\vec{m}(\vec{r})}{m(\vec{r})^2} \right)} - \frac{7}{16} \left( \frac{\vec{m}(\vec{r})}{m(\vec{r})^3} \right)^2 + \frac{2}{m(\vec{r})} \vec{A}(\vec{r}) \cdot \vec{\nabla} + \frac{ie}{m(\vec{r})} \left( \vec{\nabla} \cdot \vec{A}(\vec{r}) \right) - i e \vec{A}(\vec{r}) \cdot \left( \frac{\vec{m}(\vec{r})}{m(\vec{r})^2} \right) + \frac{e^2 \vec{A}(\vec{r})^2}{m(\vec{r})} + W(\vec{r}) \right] \psi(\vec{r}) = E \psi(\vec{r}). \]
Here, we consider the interaction of a PDM particle of charge $e$ moving in the vector potential

$$
\vec{A}(\vec{r}) = \vec{A}_1(\vec{r}) + \vec{A}_2(\vec{r}); \quad \begin{cases} \nabla \times \vec{A}_1(\vec{r}) = \vec{B} = B_0 \hat{z} \\ \nabla \times \vec{A}_2(\vec{r}) = 0 \end{cases},
$$

(5)

where a constant, uniform magnetic field $\vec{B} = B_0 \hat{z}$ is applied in the $z$-direction, $\vec{A}_1(\vec{r}) = (0, B_0 \rho/2, 0)$ and $\vec{A}_2(\vec{r}) = (0, \Phi_{AB}/2\pi \rho, 0)$ are given in the cylindrical coordinates, with $\vec{A}_2(\vec{r})$ describing the so called Aharonov-Bohm flux field $\Phi_{AB}$ effect (c.f., e.g., [24, 31–33] and related references cited therein). At this point, one should be aware that our charge $e = \pm |e|$, and we put no restriction on its positivity or negativity as yet. Consequently, our PDM charged particle interacts with the total vector potential

$$
\vec{A}(\vec{r}) = \left(0, \frac{B_0}{2} \rho + \frac{\Phi_{AB}}{2\pi \rho}, 0\right),
$$

(6)

that satisfies the Coulomb gauge $\nabla \cdot \vec{A} = 0$. Moreover, we shall use the assumptions that

$$
m(\vec{r}) = m(\rho, \varphi, z) = g(\rho) f(\varphi) k(z) = g(\rho); f(\varphi) = k(z) = 1,
$$

(7)

and

$$
g(\rho) W(\rho, \varphi, z) = V(\rho) + V(\varphi) + V(z) = V(\rho) + V(z); V(\varphi) = 0,
$$

(8)

where, $V(\varphi) = 0$ assumes azimuthal symmetrization and our PDM scalar multiplier $m(\vec{r}) = g(\rho)$ is only radially cylindrically symmetric. This would secure separability of the problem at hand.

Under such assumptions construction, we may now follow the conventional textbook separation of variables and use the substitution

$$
\psi(\vec{r}) = \psi(\rho, \varphi, z) = R(\rho) Z(z) e^{i m \varphi},
$$

(9)

(where $m = 0, \pm 1, \pm 2, ..., \pm \ell$ is the magnetic quantum number, and $\ell$ is angular momentum quantum number) in (4) to obtain.

$$
\begin{align*}
\left[ \frac{R''(\rho)}{R(\rho)} - \frac{g'(\rho)}{g(\rho)} - \frac{1}{\rho} \right] R'(\rho) &+ \frac{1}{4} \left( g''(\rho) + \frac{g'(\rho)}{\rho g(\rho)} \right) + \frac{7}{16} \left( \frac{g'(\rho)}{g(\rho)} \right)^2 \\
&- \frac{\tilde{m}^2}{\rho^2} - \frac{e^2 \Phi_{AB}^2}{4\pi^2 \rho^2} + \frac{e \Phi_{AB}}{2\pi} - e B_\rho m - \frac{e^2 B_\rho^2 \rho^2}{4} + g(\rho) E - V(\rho) \\
&\quad + \left[ \frac{Z''(z)}{Z(z)} - V(z) \right] = 0.
\end{align*}
$$

(10)

It is obvious that this equation decouples into two parts, a purely $z$-dependent part

$$
[-\partial_z^2 + V(z)] Z(z) = k_z^2 Z(z),
$$

(11)

and a radial-dependent cylindrically-azimuthal part

$$
\begin{align*}
\left[ \frac{R''(\rho)}{R(\rho)} - \frac{g'(\rho)}{g(\rho)} - \frac{1}{\rho} \right] R'(\rho) &+ \frac{1}{4} \left( g''(\rho) + \frac{g'(\rho)}{\rho g(\rho)} \right) + \frac{7}{16} \left( \frac{g'(\rho)}{g(\rho)} \right)^2 \\
&- \frac{\tilde{m}^2}{\rho^2} + e B_\rho m - k_z^2 - \frac{e^2 B_\rho^2 \rho^2}{4} + g(\rho) E - V(\rho) = 0,
\end{align*}
$$

(12)

where $\alpha = \Phi_{AB}/\Phi_0$, $\Phi_0 = 2\pi/e$ is the Aharonov-Bohm flux quantum (within the current units settings, of course), and $\tilde{m} = m - \alpha$ is a new irrational magnetic quantum number that indulges within the Aharonov-Bohm quantum
number \( \alpha \). At this point, nevertheless, one may need to get rid of the first-order derivative and bring the radial part into the one-dimensional Schrödinger form. In so doing, one may use the substitution

\[
R(\rho) = \sqrt{\frac{g(\rho)}{\rho}} U(\rho),
\]

(13)
to obtain

\[
\left\{ -\frac{d^2}{d\rho^2} + \frac{\dot{m}^2 - 1/4}{\rho^2} + V_{eff}(\rho) \right\} U(\rho) = \tilde{E} U(\rho).
\]

(14)

Where,

\[
V_{eff}(\rho) = V(\rho) + \frac{e^2 B_0^2 \rho^2}{4} - g(\rho) E + \left[ \frac{5}{16} \left( \frac{g'(\rho)}{g(\rho)} \right)^2 \right. - \left. \frac{1}{4} \left( \frac{g''(\rho)}{g(\rho)} \right) - \frac{1}{4} \left( \frac{g'(\rho)}{g(\rho)} \right) \right],
\]

(15)
and

\[
\tilde{E} = eB_0\tilde{m} - k_z^2
\]

(16)
represents the eigenvalues of (14) which are to be used to find the eigenvalues of the radial PDM problem at hand, i.e., to find \( E_{n_{\rho,m,\alpha}} \) in (15). Obviously, result (14) retrieves the constant mass textbook settings for \( g(\rho) = 1 \). However, we are interested in the case where \( g(\rho) \neq const \).

On the technical mathematical side of the current methodical proposal, we have, at our disposal, three types of Schrödinger differential equations to deal with. The \( z \)-dependent part of (11), the \( \rho \)-dependent part of (12) and the one-dimensional \( \rho \)-dependent part of (14). The \( \rho \)-dependent parts (12) and (14) are to be shown useful in their own skin and serve different PDM and/or interaction potential settings. This is clarified in the forthcoming illustrative examples. The first batch of which consists of exactly solvable models and the second consists of conditionally exactly solvable models. Whereas, the \( z \)-dependent part of (11) will have its own spectral signatures on the overall spectra of the decoupled problem in (3). The strategy of our methodological proposal in handling (3) is clear, therefore.

III. RADIAL CYLINDRICAL ONE-DIMENSIONAL PDM-SCHRÖDINGER FORM: A PSEUDOHARMONIC POTENTIAL AND EXACT SOLVABILITY

In this section, consider our charged PDM particle moving in the so called pseudoharmonic potential \[32\]

\[
V(\rho) = V_1\rho^2 + \frac{V_2}{\rho^2} - 2V_0 ; \quad V_1 = \frac{V_0}{\rho_0^2}, \quad V_2 = V_0\rho_0^2
\]

(17)
in the presence of a uniform magnetic field and an Aharonov-Bohm flux field of (6). Where, \( V_0 \) is the chemical potential and \( \rho_0 \) is the zero point of the pseudoharmonic potential. This potential includes both a harmonic quantum dot potential \( V_1\rho^2 \) and antidote potential \( V_2\rho^2 \) \[31, 32\]. The details of quantum dots and antidotes lie far beyond the scope of the current study and can be traced from \[31, 32\]. Such a pseudoharmonic potential is most suited for the one-dimensional PDM-Schrödinger form (14) and anticipated to be exactly solvable for a sample of PDM settings. Therefore, we treat, in what follows, some special PDM settings so that their exact solutions are inferred from some models that are known to be exactly solvable.

A. Model-I: a radial cylindrical PDM \( g(\rho) = \eta\rho^2 \)

Consider a charged particle with radial cylindrical PDM \( g(\rho) = \eta\rho^2 \) moving in the pseudoharmonic potential (17), under the influence of a uniform magnetic and an Aharonov-Bohm flux fields of (6). Then the effective potential \( V_{eff}(\rho) \) of (15) would read

\[
V_{eff}(\rho) = V_1\rho^2 + \frac{V_2}{\rho^2} - 2V_0 + \frac{e^2 B_0^2 \rho^2}{4} - \eta E\rho^2 + \frac{1}{4\rho^2}.
\]

(18)
Hence, equation (14) collapse into

$$\left\{ -\frac{d^2}{d\rho^2} + \frac{\tilde{l}^2 - 1/4}{\rho^2} + \frac{(4V_1 - 4\eta E + e^2B_0^2)}{4} \right\} U(\rho) = E_{eff} U(\rho),$$

(19)

where

$$\tilde{l}^2 - 1/4 = \tilde{m}^2 + V_2 \iff \tilde{l} = \sqrt{(m - \alpha)^2 + V_2 + 1/4}. \quad (20)$$

Equation (19) is, in fact, the well known two-dimensional radial cylindrical harmonic oscillator problem (c.f., e.g., [34]) that admits exact solution in the form of

$$E_{eff} = \sqrt{4V_1 - 4\eta E + e^2B_0^2} \left(2n_\rho + \left|\tilde{l}\right| + 1\right) = 2V_o + eB_0 (m - \alpha) - k_z^2. \quad (21)$$

Which would, in turn, imply that the eigenvalues are given by

$$E_{n_\rho,m,\alpha} = \frac{1}{4\eta} \left[ 4V_1 + e^2B_0^2 - \left( \frac{2V_o + eB_0 (m - \alpha) - k_z^2}{2n_\rho + 1 + \sqrt{(m - \alpha)^2 + V_2 + 1/4}} \right)^2 \right] \quad (22)$$

and radial wavefunctions are obtained in a similar manner to read

$$R_{n_\rho,m,\alpha}(\rho) \sim \rho^{-\left|\tilde{l}\right| - 1} \exp\left( -\frac{\sqrt{e^2B_0^2 + 4V_1 - 4\eta E_{n_\rho,m,\alpha}}}{4} \rho^2 \right) \ _1F_1\left( -n_\rho, \left|\tilde{l}\right| + 1; \frac{\sqrt{e^2B_0^2 + 4V_1 - 4\eta E_{n_\rho,m,\alpha}}}{2} \rho^2 \right) \quad (23)$$

**B. Model-II: a radial cylindrical PDM g(\rho) = \eta/\rho^2**

A charged particle with radial cylindrical PDM g(\rho) = \eta/\rho^2 moving in the pseudopotential field (17), along with a uniform magnetic and an Aharonov-Bohm flux fields of (6), would imply that equation (14) be rewritten as

$$\left\{ -\frac{d^2}{d\rho^2} + \frac{\tilde{l}^2 - 1/4}{\rho^2} + \frac{(4V_1 + e^2B_0^2)}{4} \right\} U(\rho) = E_{eff} U(\rho), \quad (24)$$

where

$$\tilde{l}^2 - 1/4 = \tilde{m}^2 + V_2 - \eta E \iff \tilde{l} = \sqrt{(m - \alpha)^2 + V_2 - \eta E + 1/4}. \quad (25)$$

Equation (24) is, again, in the form of the well known two-dimensional radial cylindrical harmonic oscillator and admits the exact solution

$$E_{eff} = \sqrt{4V_1 + e^2B_0^2} \left(2n_\rho + \sqrt{(m - \alpha)^2 + V_2 - \eta E + 1/4 + 1}\right) = 2V_o + eB_0 (m - \alpha) - k_z^2, \quad (26)$$

to yield the eigenvalues

$$E_{n_\rho,m,\alpha} = \frac{1}{\eta} \left\{ (m - \alpha)^2 + V_2 + 1/4 - \left[ \frac{2V_o + eB_0 (m - \alpha) - k_z^2}{\sqrt{4V_1 + e^2B_0^2}} \right] - (2n_\rho + 1) \right\}^2 \quad (27)$$

and the corresponding radial eigenfunctions

$$R_{n_\rho,m,\alpha}(\rho) \sim \rho^{-\left|\tilde{l}\right| - 1} \exp\left( -\frac{\sqrt{e^2B_0^2 + 4V_1}}{4} \rho^2 \right) \ _1F_1\left( -n_\rho, \left|\tilde{l}\right| + 1; \frac{\sqrt{e^2B_0^2 + 4V_1}}{2} \rho^2 \right) \quad (28)$$
In this section, we use the radial cylindrical PDM-Schrödinger form (12) and consider a generalized Killingbeck potential field (e.g., [40]) of the form

\[ V(\rho) = V_0 + V_1 \rho + V_2 \rho^2 + V_3 \rho + V_4 \rho^2. \]  

(29)

When such potential field is substituted in (12), one obtains

\[
\begin{aligned}
&\frac{R''(\rho)}{R(\rho)} - \frac{\beta^2}{\rho^2} + \tilde{k}^2 - \gamma^2 \rho^2 + g(\rho) E - V_i \rho - \frac{V_0}{\rho} = 0, \\
&\frac{\beta^2}{\rho^2} = \tilde{m}^2 + V_4, \quad \tilde{k}^2 = eB_z \tilde{m} - k_z^2 - V_0, \\
&\gamma^2 = \frac{e^2 B^2}{4} + V_2, \quad \alpha = \Phi_{AB}/\Phi_o, \quad \Phi_o = 2\pi/e.
\end{aligned}
\]  

(30)

In the sample illustrative of examples below, we wish to benefit from the known solutions of the biconfluent Heun equation using two different PDM settings.

A. Model-III: a radial cylindrical PDM \( g(\rho) = \lambda \rho \)

A charged PDM particle with radial cylindrical PDM \( g(\rho) = \lambda \rho \) moving in the potential field (29), under the influence of both a uniform magnetic and an Aharonov-Bohm flux fields of (6), would be described by the radial Schrödinger equation (30) as

\[
\frac{R''(\rho)}{R(\rho)} - \frac{\beta^2}{\rho^2} = \frac{3}{16} - \gamma^2 \rho^2 - \frac{V_3}{\rho} + \frac{V_1}{\rho} \rho + \tilde{k}^2 = 0.
\]  

(32)

Which, in a straightforward manner, collapses into the standard one-dimensional Schrödinger form of the biconfluent Heun equation (c.f., e.g., [35] and related references cited therein)

\[
R''(\rho) + \left[ \frac{1 - \tilde{\alpha}^2}{4 \rho^2} - \frac{\tilde{\beta}}{2\rho} - \tilde{\beta} \rho - \tilde{\gamma}^2 - \tilde{\beta}^2 \right] R(\rho) = 0,
\]  

(33)

where

\[
\begin{aligned}
&\tilde{\alpha}^2 = 1 - \frac{3}{16} - \beta^2, \quad -\tilde{\delta}/2 = -V_4, \quad -\tilde{\beta} = \lambda E - V_i, \\
&\gamma^2 = 1 = \frac{e^2 B^2}{4} + V_2, \quad \tilde{\gamma} - \tilde{\beta}^2/4 = \tilde{k}^2.
\end{aligned}
\]  

(34)

We now use the transformation recipe

\[
R(\rho) = \rho^{(1+\tilde{\alpha}^2)/2} \exp \left[ -\frac{\tilde{\beta} \rho + \rho^2}{2} \right] U(\rho)
\]  

(35)

in (33) to obtain the biconfluent Heun-type equation

\[
\rho U''(\rho) + \left[ 1 + \tilde{\alpha} - \tilde{\beta} \rho - 2 \rho^2 \right] U'(\rho) + \left\{ (\tilde{\gamma} - 2 - \tilde{\alpha}) \rho - \frac{1}{2} \left[ \tilde{\delta} + [1 + \tilde{\alpha}] \tilde{\beta} \right] \right\} U(\rho) = 0.
\]  

(36)
Which is known to admit solutions in the form of biconfluent Heun functions

\[ U (\rho) = H_B \left( \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}; \rho \right), \tag{37} \]

where,

\[ \tilde{\gamma} - 2 - \tilde{\alpha} = 2n_\rho ; \quad n_\rho = 0, 1, 2 \cdots, \tag{38} \]

provides the essential quantization and

\[
\begin{align*}
\tilde{\gamma} &= \frac{\tilde{\beta}^2}{4} + \tilde{k}^2 = \frac{(\lambda E - V_i)^2}{4} + eB_\phi (m - \alpha) - k_z^2 - V_0, \\
\tilde{\alpha} &= 2\sqrt{(m - \alpha)^2 + V_4 + \frac{1}{16}}. 
\end{align*} \tag{39} \]

This would, in turn, imply that the eigenvalues are given as

\[ E_{n_\rho,m_\alpha} = \frac{1}{\lambda} \left[ V_i + 2 \left( 2 \left[ n_\rho + 1 + \sqrt{(m - \alpha)^2 + V_4 + \frac{1}{16}} \right] - eB_\phi (m - \alpha) + k_z^2 + V_0 \right) \right]^{1/2} \tag{41} \]

and the radial eigenfunctions are

\[ R_{n_\rho,m_\alpha} (\rho) \sim \rho^{(1+\tilde{\alpha})/2} \exp \left( -\frac{\tilde{\beta} \rho + \rho^2}{2} \right) H_B \left( \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}; \rho \right), \tag{42} \]

Where \( \tilde{\alpha} \) and \( \tilde{\beta} \) are defined, respectively, in (40) and (34). However, for more details on the biconfluent Heun the readers are advised to refer to the sample of references \[35–40\].

### B. Model-IV: a radial cylindrical PDM \( g (\rho) = \lambda/\rho^2 \)

For a charged PMD particle with \( g (\rho) = \lambda/\rho^2 \) moving in the vicinity of the three fields above (i.e., the potential of (29), the uniform magnetic and the Aharonov-Bohm flux fields of (6)), the radial Schrödinger equation (30) along with the substitution (13) would collapse into

\[
U'' (\rho) - \frac{\xi^2}{\rho^2} - \gamma^2 \rho^2 - V_i \rho - \frac{V_4}{\rho} + \tilde{k}^2 = 0 ; \quad \xi^2 = \beta^2 - \lambda E. \tag{43} \]

Which, in a straight forward manner, reduces to

\[
U'' (\rho) + \left[ \frac{1 - \tilde{\alpha}^2}{4\rho^2} - \frac{\tilde{\delta}}{2\rho} - \tilde{\beta} \rho - \rho^2 + \tilde{\gamma} - \frac{\tilde{\beta}^2}{4} \right] U (\rho) = 0, \tag{44} \]

where

\[
\begin{align*}
(1 \cdash \tilde{\alpha}^2) / 4 &= -\xi^2 = \lambda E - (m - \alpha)^2 - V_i, \quad \tilde{\delta} / 2 = V_4, \quad \tilde{\beta} = V_i, \\
\tilde{\gamma} - \tilde{\beta}^2 / 4 &= \tilde{k}^2 = eB_\phi (m - \alpha) - k_z^2 - V_0 \quad \gamma^2 = 1 = e^2B_\phi^2 / 4 + V_2, 
\end{align*} \tag{45} \]

Next, we use a transformation recipe similar to (35) and substitute

\[
U (\rho) = \rho^{(1+\tilde{\alpha})/2} \exp \left[ -\left( \tilde{\beta} \rho + \rho^2 \right) / 2 \right] Y (\rho) \tag{46} \]

in (44) to obtain a biconfluent Heun-type equation

\[
\rho Y'' (\rho) + \left[ 1 + \tilde{\alpha} - \tilde{\beta} \rho - 2\rho^2 \right] Y' (\rho) + \left\{ \tilde{\gamma} - 2 - \tilde{\alpha} \right\} \rho - \frac{1}{2} \left( \tilde{\delta} + [1 + \tilde{\alpha}] \tilde{\beta} \right) \} Y (\rho) = 0. \tag{47} \]
Which admits solutions in the form of biconfluent Heun functions

\[ Y(\rho) = H_B(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}; \rho) . \]  

(48)

provided that

\[ \tilde{\gamma} - 2 - \tilde{\alpha} = 2n_\rho ; \; n_\rho = 0, 1, 2 \cdots , \]  

(49)

gives again the essential quantization. Where, in this case,

\[ \tilde{\gamma} = \frac{\tilde{\beta}^2}{4} + \tilde{k}^2 = eB_z (m - \alpha) - k^2 - V_0 + \frac{V^2}{4} , \]  

(50)

\[ \tilde{\alpha} = \sqrt{1 + 4 \left[ (m - \alpha)^2 + V_4 - \lambda E \right]} . \]  

(51)

This would, in turn, imply that the eigenvalues are given by

\[ E_{n_\rho, m, \alpha} = \frac{1}{\lambda} \left\{ (m - \alpha)^2 + V_4 + \frac{1}{4} - \left[ 2(n_\rho + 1) + k^2 + V_0 - \frac{V^2}{4} - eB_o (m - \alpha) \right]^2 \right\} , \]  

(52)

and the radial eigenfunctions are

\[ R_{n_\rho, m, \alpha}(\rho) \sim \rho^{(\tilde{\gamma}^2 - 2)/2} \exp \left( -\frac{\tilde{\beta} \rho + \rho^2}{2} \right) H_B(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}; \rho) . \]  

(53)

Where \( \tilde{\alpha} \) and \( \tilde{\beta} \) are defined, respectively, in (51) and (45).

In the two examples reported above, III and IV, it is obvious that the exact analytical solutions offered by the biconfluent Heun-type equations belong to the set of PDM-Schrödinger equations that are conditionally exactly solvable. This is mandated by the condition \( \gamma^2 = 1 = e^2 B_2^2/4 + V_2^2 \) in (34) and again in (45). This would, effectively, imply that \( V_2 = 1 - e^2 B_2^2/4 \) is a condition imposed by the exact solvability of the biconfluent Heun-type equation that renders our radial PDM-Schrödinger equation (12) conditionally exactly solvable. Whereas, in Model-II of the preceding section, we have used the same mass setting but not the same condition imposed upon Model-IV above. That is why the results for the two models are not the same as should be expected.

V. SPECTRAL SIGNATURES OF THE ONE-DIMENSIONAL \( z \)-DEPENDENT SCHröDINGER PART ON THE OVERALL SPECTRA

In this section, we shall include the \( z \)-dependent part (11) of the PDM Schrödinger equation in (10)

\[ [-\partial_z^2 + V(z)] Z(z) = k_z^2 Z(z) , \]  

and explore its contribution to the spectra of the four examples discussed above. On the mathematical theoretical side, nevertheless, we may consider any of the conventional textbook exactly solvable one-dimensional Schrödinger equations and indulge their signatures on the overall spectra. Therefore, there exist a large number of feasible one-dimensional potentials that may contribute to the problem at hand. However, for the sake of clarification and illustration of the current methodical proposal, we only choose two one-dimensional potentials, an infinite potential well and a Morse-type oscillator potential.

A. Case 1: Infinite potential well

Let us assume that our charged PDM particle is also bound to move within an impenetrable potential well of width \( L \) on the \( z \)-axis, i.e.,

\[ V(z) = \begin{cases} 
0 & ; \; 0 < z < L \\
\infty & ; \; \text{elsewhere} . \end{cases} \]  

(54)
This would by, the textbook boundary conditions, manifest an exact solution in the form of

\[ Z(z) \sim \sin(k_z z) \implies k_z L = (n_z + 1) \pi \implies k_z^2 = \frac{(n_z + 1)^2 \pi^2}{L^2}; \quad n_z = 0, 1, 2, \ldots \] (55)

Under such settings, the total eigenenergies and eigenfunctions of the four examples above are, respectively, in order. For the two exactly solvable models, I and II, we get

\[ E_{n_\rho, m, \alpha, n_z} = \frac{1}{4\eta} \left[ 4V_1 + e^2B_o^2 - \frac{2V_o + eB_o (m - \alpha) - (n_z + 1)^2 \pi^2/L^2}{2n_\rho + 1 + \sqrt{(m - \alpha)^2 + V_z + 1/4}} \right] \] (56)

\[ \psi_{n_\rho, m, \alpha, n_z} (\rho, \varphi, z) = \mathcal{N} \sin \left( \frac{(n_z + 1) \pi}{L} z \right) \rho^{1+|\ell|} \exp \left( -\frac{\sqrt{e^2B_o^2 + 4V_z - 4\eta E_{n_\rho, m, \alpha, n_z} \rho^2}}{2} \right) \times \mathbf{1}_1 \mathbf{F}_1 \left( \begin{array}{c} -n_\rho; |\ell| + 1; \sqrt{e^2B_o^2 + 4V_z - 4\eta E_{n_\rho, m, \alpha, n_z} \rho^2} \end{array} \right) e^{im\varphi}. \] (57)

for Model-I, and

\[ E_{n_\rho, m, \alpha, n_z} = \frac{1}{\eta} \left\{ (m - \alpha)^2 + V_z + 1/4 - \left[ \frac{2V_o + eB_o (m - \alpha) - (n_z + 1)^2 \pi^2/L^2}{\sqrt{4V_1 + e^2B_o^2}} \right] (2n_\rho + 1) \right\}^2, \] (58)

\[ \psi_{n_\rho, m, \alpha, n_z} (\rho, \varphi, z) = \mathcal{N} \sin \left( \frac{(n_z + 1) \pi}{L} z \right) \rho^{-1+|\ell|} \exp \left( -\frac{\sqrt{e^2B_o^2 + 4V_z - 4\eta E_{n_\rho, m, \alpha, n_z} \rho^2}}{2} \right) \times \mathbf{1}_1 \mathbf{F}_1 \left( \begin{array}{c} n_\rho; |\ell| + 1; \sqrt{e^2B_o^2 + 4V_z - 4\eta E_{n_\rho, m, \alpha, n_z} \rho^2} \end{array} \right) e^{im\varphi}. \] (59)

for Model-II. Moreover, for the two conditionally exactly solvable models, III and IV, we obtain

\[ E_{n_\rho, m, \alpha, n_z} = \frac{1}{\lambda} \left[ V_1 + 2 \left( 2n_\rho + 1 + \sqrt{(m - \alpha)^2 + V_4 + 1/16} - eB_o (m - \alpha) + \frac{(n_z + 1)^2 \pi^2}{L^2} + V_o \right) \right], \] (60)

\[ \psi_{n_\rho, m, \alpha, n_z} (\rho, \varphi, z) = \mathcal{N} \sin \left( \frac{(n_z + 1) \pi}{L} z \right) \rho^{(1+\tilde{\alpha}^2)/2} \exp \left( -\frac{\tilde{\beta} \rho + \rho^2}{2} \right) H_B \left( \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}; \rho \right) e^{im\varphi}. \] (61)

for Model-III, and

\[ E_{n_\rho, m, \alpha, n_z} = \frac{1}{\lambda} \left\{ (m - \alpha)^2 + V_4 + 1/4 - \left[ 2(n_\rho + 1) + \frac{(n_z + 1)^2 \pi^2}{L^2} + V_o - \frac{V_z^2}{4} - eB_o (m - \alpha) \right] \right\}^2, \] (62)

\[ \psi_{n_\rho, m, \alpha, n_z} (\rho, \varphi, z) = \mathcal{N} \sin \left( \frac{(n_z + 1) \pi}{L} z \right) \rho^{(\tilde{\alpha}^2 - 2)/2} \exp \left( -\frac{\tilde{\beta} \rho + \rho^2}{2} \right) H_B \left( \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}; \rho \right) e^{im\varphi}. \] (63)

for Model-IV.
B. Case 2: A Morse-type potential

If our charged PDM-particle is also influenced by a Morse-type potential (c.f., e.g., [41, 42])

\[ V(z) = D \left[ \exp(-2\sigma z) - 2 \exp(-\sigma z) \right] \]

in the z-direction, would result in the exact eigenvalues and eigenfunctions given, respectively, as

\[ k_z^2 = \left( \frac{\sqrt{D}}{\sigma} - n_z - \frac{1}{2} \right)^2, \]

\[ Z(z) \sim z^{k_z} e^{-z^2/2} L_{n_z}^{2k_z}(z), \]

where \( L_{n_z}^{2k_z}(z) \) are the Laguerre polynomials. In this case, the total eigenenergies and eigenfunctions of the four examples at hand are in order.

For the two exactly solvable models, I and II, we get

\[
E_{n_p,m,\alpha,n_z} = \frac{1}{4\eta} \left[ 4V_1 + e^2B_0^2 - \left( \frac{2V_0 + eB_0 (m - \alpha) - \left( \frac{\sqrt{D}}{\sigma} - n_z - \frac{1}{2} \right)^2}{2n_p + 1 + \sqrt{(m - \alpha)^2 + V_z + 1/4}} \right)^2 \right],
\]

\[
\psi_{n_p,m,\alpha,n_z}(\rho,\varphi,z) = \mathcal{N} z^{k_z} e^{-z^2/2} L_{n_z}^{2k_z}(z) \rho^{1+|\ell|} \exp \left( -\frac{\sqrt{e^2B_0^2 + 4V_2 - 4\eta E_{n_p,m,\alpha,n_z}^2 \rho^2}}{2} \right) e^{im\varphi}.
\]

for Model-I, and

\[
E_{n_p,m,\alpha,n_z} = \frac{1}{\eta} \left\{ (m - \alpha)^2 + V_z + 1/4 - \left[ \frac{2V_0 + eB_0 (m - \alpha) - \left( \frac{\sqrt{D}}{\sigma} - n_z - \frac{1}{2} \right)^2}{\sqrt{4V_1 + e^2B_0^2}} \right] - (2n_p + 1) \right\}^2,
\]

\[
\psi_{n_p,m,\alpha,n_z}(\rho,\varphi,z) = \mathcal{N} z^{k_z} e^{-z^2/2} L_{n_z}^{2k_z}(z) \rho^{-1+|\ell|} \exp \left( -\frac{\sqrt{e^2B_0^2 + 4V_2 - 4\eta E_{n_p,m,\alpha,n_z}^2 \rho^2}}{2} \right) e^{im\varphi}.
\]

for Model-II. Likewise, for the two conditionally exactly solvable models, III and IV, we obtain

\[
E_{n_p,m,\alpha,n_z} = \frac{1}{\lambda} \left[ V_1 + 2 \left\{ n_p + 1 + \sqrt{(m - \alpha)^2 + V_4 + \frac{1}{16} - eB_0 (m - \alpha) + \left( \frac{\sqrt{D}}{\sigma} - n_z - \frac{1}{2} \right)^2 + V_0} \right\} \right],
\]

\[
\psi_{n_p,m,\alpha,n_z}(\rho,\varphi,z) = \mathcal{N} z^{k_z} e^{-z^2/2} L_{n_z}^{2k_z}(z) \rho^{(1+\alpha^2)/2} \exp \left( -\frac{\tilde{\beta} \rho + \rho^2}{2} \right) H_B \left( \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}; \rho \right) e^{im\varphi}.
\]

for Model-III, and

\[
E_{n_p,m,\alpha,n_z} = \frac{1}{\lambda} \left\{ (m - \alpha)^2 + V_4 + \frac{1}{4} - \left[ 2(n_p + 1) + \left( \frac{\sqrt{D}}{\sigma} - n_z - \frac{1}{2} \right)^2 + V_0 - \frac{V_2^2}{4} - eB_0 (m - \alpha) \right] \right\}^2,
\]

\[
\psi_{n_p,m,\alpha,n_z}(\rho,\varphi,z) = \mathcal{N} z^{k_z} e^{-z^2/2} L_{n_z}^{2k_z}(z) \rho^{-1+|\ell|} \exp \left( -\frac{\tilde{\beta} \rho + \rho^2}{2} \right) H_B \left( \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}; \rho \right) e^{im\varphi}.
\]
ψ_{n\rho,m,\alpha,nz} (\rho, \varphi, z) = N z^{kz} e^{-z^2/2} I_{n\rho}^2 (\rho(\alpha^2 - 2)/2) \exp \left( -\frac{\tilde{\beta} \rho + \rho^2}{2} \right) H_B \left( \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}; \rho \right) e^{im\varphi}. \quad (74)

for Model-IV.

VI. CONCLUDING REMARKS

Using cylindrical coordinates (\rho, \varphi, z), we have considered PDM charged particles moving under not only the influence of magnetic and Aharonov-Bohm flux fields, but also other interaction potential fields (the reader is advised to refer to Wang [43] for some comprehensive background on the Aharonov-Bohm effect). We have implemented the PDM-minimal-coupling recipe [26], along with the PDM-momentum operator [27], and explored the separability of the problem under radial cylindrical and azimuthal symmetrization settings. A simple one-dimensional textbook purely z-dependent (11) and a purely radial \rho-dependent (12) Schrödinger equations are obtained. For the radial \rho-dependent (12) Schrödinger equation, we have transformed it into a radial one-dimensional Schrödinger form (14) and used two PDM settings, \( g(\rho) = \eta \rho^2 \) and \( g(\rho) = \eta/\rho^2 \), to report on the exact solvability (both eigenvalues and eigenfunctions) of our PDM charged particles moving in three fields: the magnetic, the Aharonov-Bohm flux, and the pseudoharmonic potential (i.e., usual settings for charged particles in quantum dots and antidotes, e.g., [31–33], but here we have PDM charged particles). This is documented in section 3. Moreover, we have used the radial \rho-dependent (12) as is and used the biconfluent Heun differential forms for two PDM settings, \( g(\rho) = \lambda \rho \) (36) and \( g(\rho) = \lambda/\rho^2 \) (47), to report on the conditionally exact solvability (both eigenvalues and eigenfunctions) of our PDM charged particles moving in three fields: the magnetic, the Aharonov-Bohm flux, and the generalized Killingbeck potential (reported in section 4). Yet, the spectral signatures of the one-dimensional z-dependent Schrödinger part (11) on the overall eigenvalues and eigenfunctions reported, in section 5, using two z-dependent potential models (infinite potential well (54) and Morse type potentials (64)) for each of the four examples used in section 3 and 4. To the best of our knowledge, the current study has never been reported elsewhere.

In the light of our experience above, our observations are in order.

We have considered two PDM-radial Schrödinger-like equations, (12) and (14), as a result of textbook separation of variables procedure. Our illustrative examples were chosen in the most simplistic format so that our approach can be clearly followed and implemented (documented in sections 3 and 4). However, the discussion above should not only be restricted to the analytically exact (pseudoharmonic potential of (17)) or analytically conditionally exact (generalized Killingbeck-type potential of (29) via the biconfluent Heun equation) solvabilities reported, but also it should, technically and/or in principle, be applicable to Schrödinger-like models that admit numerically exact or numerically conditionally exact solvabilities (c.f. e.g., [14] and related references cited therein). It may very well be applied to quasi-exactly solvable models (c.f. e.g., Quesne [45, 46] and related references cited therein), or even to non-Hermitian and pseudo-Hermitian Hamiltonian settings (c.f. e.g., [47–50] and related references cited therein). Likewise, this would hold true for the z-dependent Schrödinger-like equation (11), where we have only used two analytically exactly solvable models (infinite potential well (54) and Morse type potentials (64)).

On the vector potential side, we have used a vector potential (5) that leads to a uniform constant magnetic field. However, in the construction of vector potential, the magnetic field may turn out to be position-dependent. Therefore, it would be more appropriate and/or more general approach to work with a vector potential

\[ \vec{A}_1(\vec{r}) = S(\rho) \vec{A}_1(\vec{r}) = \left( 0, \frac{B_z}{2\rho} S(\rho), 0 \right), \quad (75) \]

where \( S(\rho) \) is a scalar multiplier that may absorb any position-dependent terms that may emerge in the construction process of the vector potential \( \vec{A}_1(\vec{r}) \) (c.f. e.g., [27]). It would be interesting to explore this problem in the near future.
