Spectral properties of the Neumann–Poincaré operator on rotationally symmetric domains

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Abstract
This paper concerns the spectral properties of the Neumann–Poincaré operator on two- and three-dimensional bounded domains which are invariant under either rotation or reflection. We prove that if the domain has such symmetry, then the domain of definition of the Neumann–Poincaré operator is decomposed into invariant subspaces defined as eigenspaces of the unitary transformation corresponding to rotation or reflection. Thus, the spectrum of the Neumann–Poincaré operator is the union of those on invariant subspaces. In two dimensions, an $m$-fold rotationally symmetric simply connected domain $D$ is realized as the $m$th-root transform of a domain, say $\Omega$. We prove that the spectrum on one of invariant subspaces is the exact copy of the spectrum on $\Omega$. It implies in particular that the spectrum on the transformed domain $D$ contains the spectrum on the original domain $\Omega$ counting multiplicities. We present a matrix representation of the Neumann–Poincaré operator on the $m$-fold rotationally symmetric domain using the Grunsky coefficients. We also discuss some examples including lemniscates, $m$-star shaped domains and the Cassini oval.

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1 Introduction

The purpose of the present paper is to investigate the spectral properties of the NP operator (NP is the acronym of Neumann–Poincaré) on the boundary of the domain in
Let $D$ be a bounded domain with the Lipschitz continuous boundary. The NP operator, denoted by $K_{\partial D}$, is the boundary integral operator on $\partial D$ defined by

$$K_{\partial D}[\varphi](x) := \int_{\partial D} \partial_{v_y} \Gamma(x - y) \varphi(y) \, d\sigma(y), \quad x \in \partial D, \quad (1.1)$$

where $\partial_{v_y}$ denotes the outward normal derivative at the point $y \in \partial D$ and $\Gamma(x - y)$ is the fundamental solution to the Laplacian, i.e., $\Gamma(x) = \frac{1}{2\pi} \ln |x|$ in two dimensions and $\Gamma(x) = -\frac{1}{4\pi |x|^2}$ in three dimensions. The integral above is understood as the Cauchy principal value if $\partial D$ is merely Lipschitz continuous.

The NP operator appears naturally when solving the classical boundary value problem on $D$ in terms of layer potential, which was initiated by C. Neumann and Poincaré as the name of the operator suggests. Recently there is rapidly growing interest in the spectral properties of the NP operator in relation to the plasmonic resonance on meta material and significant new results are being produced. We refer to recent surveys [3, 12] for historical account and recent development on the operator. In relation to the subject of this paper, we mention that $K_{\partial D}$ can be realized as a self-adjoint operator on $\mathcal{H}^{1/2}(\partial D)$ ($\mathcal{H}^{1/2}(\partial D)$ is the Sobolev space of order 1/2 on $\partial D$) by introducing a new inner product [13]. If $\partial D$ is $C^{1,\alpha}$-smooth for some $\alpha > 0$, then $K_{\partial D}$ is compact on $\mathcal{H}^{1/2}(\partial D)$ and hence its spectrum consists of eigenvalues and their limit point 0; If it has a corner, then $K_{\partial D}$ admits nontrivial essential spectrum (see, for example, [15]).

Let $D \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded domain with the Lipschitz boundary. Let $\mathcal{R}$ be either a rotation by the angle $2\pi/m$ for some positive integer $m$ (about a line if $d = 3$ and a point if $d = 2$) or a reflection (with respect to a plane if $d = 3$ and a line if $d = 2$). Suppose that $\partial D$ is invariant under $\mathcal{R}$, namely, $\mathcal{R}(\partial D) = \partial D$. Then, the operator $U$ on $\mathcal{H}^{1/2}(\partial D)$, defined by

$$(U\varphi)(x) := \varphi(\mathcal{R}x), \quad x \in \partial D, \quad \varphi \in \mathcal{H}^{1/2}(\partial D), \quad (1.2)$$

is unitary on $\mathcal{H}^{1/2}(\partial D)$.

If $\partial D$ has the $m$-fold rotational symmetry, namely, if $\mathcal{R}$ is rotation, then $U^m = I$, $I$ being the identity operator. Therefore, if $\lambda$ is an eigenvalue of $U$, then $\lambda = \zeta^l_m$ for some $l = 0, 1, \ldots, m - 1$, where $\zeta_m = e^{2\pi i/m}$. Let $X_l$, $l = 0, 1, \ldots, m - 1$, be the eigenspace of the $U$ corresponding to the eigenvalue $\zeta^l_m$. If $\mathcal{R}$ is a reflection, then eigenvalues of $U$ are 1 and $-1$. Let $X_0$ and $X_1$ be the corresponding eigenspaces.

We obtain the following theorem.

**Theorem 1.1** Let $D$ be a bounded domain in $\mathbb{R}^d$, $d = 2, 3$, with the Lipschitz boundary. Let $\mathcal{R}$ be either a rotation by $2\pi/m$ or a reflection. Suppose that $D$ is invariant under $\mathcal{R}$ and let $X_0, \ldots, X_{m-1}$ be eigenspaces of $U$ as defined above ($m = 2$ if $\mathcal{R}$ is a reflection). Then, $U$ is diagonalizable and

$$\mathcal{H}^{1/2}(\partial D) = \bigoplus_{l=0}^{m-1} X_l, \quad (1.3)$$

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where the sum is direct.

Note that $X_l$ has a geometrical meaning. If $\varphi \in X_l$, then $\varphi(\mathcal{R}x) = \zeta_l^m \varphi(x)$. Thus, values of $\varphi$ on $\partial D$ are determined by its values on $\{ x \in \partial D \mid 0 < \text{angle}(x) \leq \frac{2\pi}{m} \}$, where $\text{angle}(x)$ is the angle of $x$ with respect to the rotation axis. For a reflection case, $X_0$ is the set of all functions even with respect to the reflection, and $X_1$ that of odd functions.

If $D$ is invariant under $\mathcal{R}$, a rotation or a reflection, one can see easily from the definition (1.1) of $\mathcal{K}_{\partial D}$ that $U$ commutes with $\mathcal{K}_{\partial D}$, namely,

$$U \mathcal{K}_{\partial D} = \mathcal{K}_{\partial D} U.$$  \hfill (1.4)

Therefore, the eigenspace $X_l$ is invariant under $\mathcal{K}_{\partial D}$ for all $l$, i.e.,

$$\mathcal{K}_{\partial D}(X_l) \subset X_l.$$  \hfill (1.5)

In fact, if $\varphi \in X_l$, then $U \varphi = \zeta_l^m \varphi$. Thus, we have from (1.4) that

$$U \left( \mathcal{K}_{\partial D} \varphi \right) = \mathcal{K}_{\partial D} \left( U \varphi \right) = \zeta_l^m \mathcal{K}_{\partial D} \varphi,$$

which implies $\mathcal{K}_{\partial D} \varphi \in X_l$. So we obtain the following result as an immediate corollary of Theorem 1.1. Here and throughout this paper, $\sigma(\mathcal{K}_{\partial D}, \mathcal{H})$ denotes the spectrum of $\mathcal{K}_{\partial D}$ on $\mathcal{H}$ for a subspace $\mathcal{H}$ of $\mathcal{H}^{1/2}(\partial D)$ invariant under $\mathcal{K}_{\partial D}$. When $\mathcal{H} = \mathcal{H}^{1/2}(\partial D)$, we denote it by $\sigma(\mathcal{K}_{\partial D})$.

**Corollary 1.2** Under the same assumption, we have

$$\sigma(\mathcal{K}_{\partial D}) = \bigcup_{l=0}^{m-1} \sigma(\mathcal{K}_{\partial D}, X_l).$$  \hfill (1.6)

If $D \subset \mathbb{C}$ is a simply connected bounded domain with $C^{1,\alpha}$-smooth boundary, then there exist a unique positive constant $R$ and a unique univalent function $\Psi$ from $\{|z| > R\}$ onto $\mathbb{C} \setminus \overline{D}$ which admits a Laurent series expansion of the form

$$\Psi(z) = z + a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots.$$  \hfill (1.7)

Here and afterwards, such a (exterior) Riemann mapping for $D$ is denoted by $\Psi_D$. For two dimensional case, $\partial D$ is assumed to be $C^{1,\alpha}$-smooth to guarantee that the Riemann mapping $\Psi_D$ has enough regularity up to $\partial D$. In fact, the Riemann mapping $\Psi_D$ is $C^{1,\alpha}$ up to the boundary if $\partial D$ is $C^{1,\alpha}$ [16, Theorem 3.6]. In particular, $(\Psi_D^{-1})^n$, $n \in \mathbb{Z}$, forms a basis for $\mathcal{H}^{1/2}(\partial \Omega)$.

If $D$ is $m$-fold rotationally symmetric, then the eigenspace $X_l$ can be characterized by the functions $(\Psi_D^{-1})^n$. In fact, by uniqueness of the Riemann mapping, we have
\[ \Psi_D^{-1}(\zeta_m w) = \zeta_m \Psi_D^{-1}(w). \]

It then follows that \((\Psi_D^{-1}(\zeta_m w))^{jm+l} = \zeta_m^{l}(\Psi_D^{-1}(w))^{jm+l}, \)
\(i.e., \ U[f^{jm+l}] = \zeta_m^{l} f^{jm+l} \) where \( f = \Psi_D^{-1}. \) So, we infer
\[ X_l = \text{span}\{(\Psi_D^{-1})^n : n = jm + l, \ j \in \mathbb{Z}\}, \ l = 0, 1, \ldots, m - 1. \quad (1.8) \]

The NP operator on a planar \( m \)-fold rotationally symmetric simply connected domain has particularly interesting spectral property which results from the fact that any such a domain is realized as the \( m \)th-root transform of a certain domain. The \( m \)th-root transform is defined as follows. Let \( \Omega \) be a simply connected bounded domain containing 0 in its interior. If \( \Psi_\Omega \) is the Riemann mapping for \( \Omega \) defined in \(|z| > R\}, it is known (see [6, p. 28]) that \( \Psi_\Omega(z^m)^{1/m} \) is univalent on \(|z| > R^{1/m} \) for a positive integer \( m \). The \( m \)th-root transform of \( \Omega \), denoted by \( \Omega_m \), is defined to be the image of \(|z| > R^{1/m} \) under the map \( \Psi_\Omega(z^m)^{1/m}. \) In other words, \( \Omega_m \) is defined by the relation
\[ \Psi_{\Omega_m}(z) := \Psi_\Omega(z^m)^{1/m} \quad (1.9) \]
for \(|z| > R^{1/m}. \) It is easy to see that \( \Omega_m \) is \( m \)-fold rotationally symmetric. It is also easy to prove (actually it is left as an exercise in [6]) that any simply connected \( m \)-fold rotationally symmetric domain is an \( m \)th-root transform of a certain domain.

We prove that if \( D \) is the \( m \)th-root transform of \( \Omega \), then \( \sigma(K_{\partial D}) \) contains an exact copy of \( \sigma(K_{\partial \Omega}) \) as stated in the following theorem.

**Theorem 1.3** Let \( D \subset \mathbb{C} \) be a simply connected \( m \)-fold rotationally symmetric domain with \( C^{1,\alpha} \) boundary for some \( \alpha > 0 \). Suppose that \( D \) is the \( m \)th-root transform of \( \Omega \). It holds that
\[ \sigma(K_{\partial D}, X_0) = \sigma(K_{\partial \Omega}), \quad (1.10) \]
counting multiplicities.

It is worth mentioning that the eigenspace \( X_0 \) can be written as
\[ X_0 = \text{span}\{((\Psi_\Omega^{-1}(w^m))^j : j \in \mathbb{Z}\}. \]
as one can see from (1.8) and (1.9).

The relation (1.10) shows in particular that
\[ \sigma(K_{\partial \Omega}) \subset \sigma(K_{\partial D}). \]

If \( m \geq 2 \), then the inclusion is proper counting multiplicities since \( X_l (l \geq 1) \) is not empty. There are domains for which the inclusion is proper as sets, namely, there is an eigenvalue on \( \partial \Omega_m \) which is not an eigenvalue on \( \partial \Omega \). For example, let \( \Omega \) be a disk centered at a point other than the origin. Then spectrum \( \sigma(K_{\partial \Omega}) \) consists of 0 and 1/2, where 0 is an eigenvalue of infinite multiplicities and 1/2 is a simple eigenvalue. The \( m \)th-root transform \( \Omega_m \) of \( \Omega \) is the \( m \)-leaf symmetric lemniscate and \( \sigma(K_{\partial \Omega_m}) \) contains infinitely many nonzero eigenvalues in addition to the eigenvalue 0 of infinite
multiplicities since the a bounded planar domain must be a disk if the NP operator on the domain is of finite rank (see [18, Theorem 7.6] for a proof).

There are not many classes of domains where NP spectra (spectra of the NP operators) are known. Ellipses are among them (see, for example, [13]). Since the NP spectrum is invariant under Möbius transform [17] (see also [10]), the NP spectrum on the limaçon of Pascal can be computed. The limaçon of Pascal is the image of the unit disc under the map \( w = z + A z^2 \) for some constant \( A \) and can be realized as the Möbius transform of an ellipse (see [2]). On the other hand, it is proved in [13] that lemniscates has 0 as an NP eigenvalue and its multiplicity is infinite. Lemniscates have NP eigenvalues other than 0 even though we don’t know what they are. Theorem 1.3 yields a way to construct domains, via \( m \)-th-root transforms, whose partial NP eigenvalues can be computed.

Theorem 1.3 is proved using the representation of the NP operator in terms of the Grunsky coefficients which is obtained in [11]. In subsection 3.2, we briefly review the Grunsky coefficients and present a simple alternative proof of the representation based on Cauchy’s theorem. Using the representation we also give an alternative proof of the fact that the NP operator on lemniscates have the infinite dimensional kernel which was proved in [13, Theorem 9] as mentioned before. The representation enables us to prove that if \( D \) is the \( m \)-th-root transform of \( \Omega \), then there is a unitary transform \( U \) from \( X_0 \) onto \( \mathcal{H}^{1/2}(\partial \Omega) \) such that

\[
K_{\partial D}|_{X_0} = U^{-1} K_{\partial \Omega} U,
\]

where \( K_{\partial D}|_{X_0} \) denotes the \( K_{\partial D} \) restricted to \( X_0 \), and hence (1.10) follows.

This paper is organized as follows. Theorem 1.1 is proved in the next section. Section 3 is to discuss the representation of the NP operator in terms of the Grunsky coefficients and to prove Theorem 1.3. In Sect. 4, we present a matrix representation of the NP operator on \( m \)-fold rotationally symmetric planar domains. In the last section, we discuss two examples: \( m \)-star shaped domains and the Cassini oval. The Cassini oval is the 2nd-root transform of a disk.

2 Proof of Theorem 1.1

Lemma 2.1 For each \( l = 0, 1, \ldots, m - 1 \), let

\[
p_l(z) = \prod_{k=0}^{m-1} (z - \zeta_m^k) = \frac{z^m - 1}{z - \zeta_m^l}.
\]

Then, the following identity holds:

\[
z \sum_{l=0}^{m-1} p_l(z) = mz^m.
\]
Proof Suppose that \( |z| > 1 \). We have
\[
z \sum_{l=0}^{m-1} p_l(z) = (z^m - 1) \sum_{l=0}^{m-1} \frac{1}{z^{m_l}} = (z^m - 1) \sum_{k=0}^{\infty} \left( \sum_{l=0}^{m-1} \frac{\zeta_m^{lk}}{z} \right) \frac{1}{z^k}.
\]
Since \( \sum_{l=0}^{m-1} \zeta_m^{lk} = m \) if \( k = jm \) for some \( j \) and \( \sum_{l=0}^{m-1} \zeta_m^{lk} = 0 \) otherwise, we have
\[
z \sum_{l=0}^{m-1} p_l(z) = m(z^m - 1) \sum_{j=0}^{\infty} \frac{1}{z^{jm}}.
\]
So, (2.2) holds for \( |z| > 1 \). Since both sides of (2.2) are polynomials, it holds for all \( z \).

Proof of Theorem 1.1. To prove Theorem 1.1, it suffices to prove
\[
\mathcal{H}^{1/2}(\partial D) = X_0 + X_1 + \cdots + X_{m-1}.
\]
The identity (2.2) reads
\[
\frac{1}{m} U \sum_{l=0}^{m-1} p_l(U) = U^m = I.
\]
For each \( \varphi \in \mathcal{H}^{1/2}(\partial D) \), let for \( l = 0, \ldots, m - 1 \),
\[
\varphi_l = \frac{1}{m} U p_l(U)[\varphi].
\]
Then, we have
\[
\varphi = \sum_{l=0}^{m-1} \varphi_l.
\]
Moreover, since \( (U - \zeta_m^l I)p_l(U) = U^m - I = 0 \) which can be read from (2.1), we have
\[
(U - \zeta_m^l I)[\varphi_l] = \frac{1}{m} U(U^m - I)[\varphi] = 0,
\]
and hence \( \varphi_l \in X_l \) for each \( l \). This proves (1.3) and the proof for the rotationally symmetric case is complete.

If \( D \) is symmetric with respect to the reflection \( \mathcal{R} \), it can be treated as the 2-fold symmetric case since \( \mathcal{R}^2 = I \).
3 Grunsky coefficients and the proof of Theorem 1.3

3.1 Double layer potential and Cauchy transform

Note that the NP operator $K_{\partial\Omega}[\varphi](z)$ is defined only for $z \in \partial\Omega$, in other words, $K_{\partial\Omega}$ is an operator on $\partial\Omega$. If we define the integral for $z$ outside $\partial\Omega$, it is called the double layer potential, that is,

$$D_{\partial\Omega}[\varphi](z) := \int_{\partial\Omega} \frac{\Gamma(w - z) \varphi(w)}{w - z} \, d\sigma(w), \quad z \in \mathbb{C} \setminus \partial\Omega. \quad (3.1)$$

The NP operator and the double layer potential enjoy the following jump relation: for $z \in \partial\Omega$

$$D_{\partial\Omega}[\varphi]|_{\pm}(z) := \lim_{t \to +0} D_{\partial\Omega}[\varphi](z \pm t \nu_z) = \left( \mp \frac{1}{2} + K_{\partial\Omega} \right) [\varphi](z), \quad (3.2)$$

where $\nu_z$ denotes the (complexified) outward unit normal vector at $z$ (see, for example, [1]).

Let $C_{\partial\Omega}$ be the Cauchy transform, that is,

$$C_{\partial\Omega}[\varphi](z) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{\varphi(w)}{w - z} \, dw, \quad z \in \mathbb{C} \setminus \partial\Omega. \quad (3.3)$$

The following relation holds (see [18, p. 67]):

$$D_{\partial\Omega}[\varphi](z) = \frac{C_{\partial\Omega}[\varphi](z) + \overline{C_{\partial\Omega}[\overline{\varphi}](z)}}{2}. \quad (3.4)$$

In fact, it can be proved using the relations

$$\partial_{\nu_z} = \Re \left[ (v_1 \partial_x + v_2 \partial_y) + i(-v_1 \partial_y + v_2 \partial_x) \right] = 2\Re(v_z \partial_z)$$

and $\tau_w d\sigma(w) = dw$, where $\tau_w$ is the unit tangential vector (with the positive orientation) at $w \in \partial\Omega$. Here $z = x + iy$, $\partial_z = \frac{1}{2}(\partial_x - i \partial_y)$, and $\Re$ denotes the real part.

3.2 The NP operator and Grunsky coefficients

From now on we denote the set $\{|z| < R\}$ by $B_R$. Let $\mathcal{H}^{1/2}(\partial B_R)$ be the Sobolev space of order 1/2 and let $\mathcal{H}_0^{1/2}(\partial B_R)$ be the subspace of functions of mean value zero, i.e., the collection of $f \in \mathcal{H}^{1/2}(\partial B_R)$ of the form $f(z) = \sum_{n \neq 0} a_n (z/R)^n$. For such a function, the homogeneous $\mathcal{H}^{1/2}$ norm is defined by

$$\|f\|^2 = \sum_{n \neq 0} |n| |a_n|^2. \quad (3.5)$$
and it is equivalent to the usual $H^{1/2}$ norm. For an integer $n \neq 0$, let

$$f_n(z) = \frac{1}{\sqrt{|n|}} \left( \frac{z}{R} \right)^n, \quad z \in \partial B_R.$$  

(3.6)

The functions $f_n$ form an orthonormal basis for $H^{1/2}_0(\partial B_R)$.

Let $\Omega$ be a simply connected bounded planar domain with $C^{1,\alpha}$ boundary. Let $\Psi = \Psi_\Omega$, i.e., the Riemann mapping from $\mathbb{C} \setminus \overline{B_R}$ onto $\mathbb{C} \setminus \overline{\Omega}$ of the form (1.7). In this subsection, we omit $\Omega$ for ease of notation. Let $H^{1/2}_0(\partial \Omega)$ be the collection of functions $g$ such that $g \circ \Psi \in H^{1/2}_0(\partial B_R)$, and define the norm on $H^{1/2}_0(\partial \Omega)$ by

$$\|g\|_* := \|g \circ \Psi\|.$$  

(3.7)

Since $\partial \Omega$ is assumed to be $C^{1,\alpha}$, $\Psi$ is $C^{1,\alpha}$ up to $\partial \Omega$. Thus, the norm $\|\|_*$ is equivalent to the usual $H^{1/2}$ norm on $H^{1/2}_0(\partial \Omega)$.

For each integer $n \neq 0$, define $g_n$ by

$$g_n(w) := (f_n \circ \Psi^{-1})(w), \quad w \in \partial \Omega.$$  

(3.8)

Then, $g_n, n \neq 0$, form an orthonormal basis for $H^{1/2}_0(\partial \Omega)$ with respect to the inner product induced by the norm $\|\|_*$. We note that since $f_n = f_{-n}$,

$$\overline{g_n} = g_{-n}.$$  

(3.9)

One can see from the form (1.7) of $\Psi$ that if $n > 0$, $(\Psi^{-1}(w))^n$ can be uniquely decomposed as

$$\left( \Psi^{-1}(w) \right)^n = F_n(w) + \hat{F}_n(w),$$  

(3.10)

where $F_n(w)$ is a polynomial and $\hat{F}_n(w)$ is an analytic function in $\mathbb{C} \setminus \overline{\Omega}$ such that $\hat{F}_n(w) \to 0$ as $|w| \to \infty$. The function $F_n(w)$ is called the Faber polynomial of degree $n$ generated by $\Psi^{-1}$. Let the Laurent series expansion of $\hat{F}_n(\Psi(z))$ be given by

$$\hat{F}_n(\Psi(z)) = -\sum_{k=1}^{\infty} c_{n,k} \frac{z^k}{\Psi^{-1}(w)^k},$$  

(3.11)

so that

$$\left( \Psi^{-1}(w) \right)^n = F_n(w) - \sum_{k=1}^{\infty} \frac{c_{n,k}}{\Psi^{-1}(w)^k}.$$  

(3.12)

The coefficients $c_{n,k}$ are called Grunsky coefficients. See [5] for the Faber polynomials and the Grunsky coefficients.
The following theorem is obtained in [11]. We include an alternative proof based on the Cauchy integral formula.

**Theorem 3.1** [11] Let

\[ \mu_{n,k} = \frac{\sqrt{k}}{2 \sqrt{n}} c_{n,k} R^{n+k}, \quad n, k = 1, 2, \ldots \]  

(3.13)

It holds that

\[ \mathcal{K}_{\partial \Omega}[g_n] = \sum_{k=1}^{\infty} \mu_{n,k} g_{-k}, \]  

(3.14)

\[ \mathcal{K}_{\partial \Omega}[g_{-n}] = \sum_{k=1}^{\infty} \mu_{n,k} g_k, \]  

(3.15)

for all \( n = 1, 2, \ldots \).

**Proof** We assume \( R = 1 \) for simplicity. Since \( F_n \) is analytic in \( \Omega \), \( \hat{F}_n(w) \to 0 \) as \( |w| \to \infty \), and both of them are continuous up to the boundaries, the Cauchy integral formula and Cauchy’s theorem yield that

\[ \mathcal{K}_{\partial \Omega}[g_n](w) = \frac{F_n(w)}{|n|^{1/2}} \]  

if \( w \in \Omega \). Since \( \Psi^{-1}(w)^{-n} \to 0 \) as \( |w| \to \infty \), we see that \( \mathcal{K}_{\partial \Omega}[g_{-n}](w) = 0 \) if \( w \in \Omega \). It then follows from (3.4) and (3.9) that

\[ \mathcal{D}_{\partial \Omega}[g_n](w) = F_n(w)/(2|n|^{1/2}) \]  

if \( w \in \Omega \). We then infer from the jump relation (3.2) that

\[ \mathcal{K}_{\partial \Omega}[g_n] = \mathcal{D}_{\partial \Omega}[g_n] - \frac{1}{2} g_n = - \frac{\hat{F}_n}{2 \sqrt{|n|}}. \]

Now, (3.14) follows from (3.11) and the definition (3.8) of \( g_n \). (3.15) follows from (3.9) and (3.14).

\( \square \)

Theorem 3.1 shows that if \( g = \sum_{n=1}^{\infty} (a_n g_n + b_n g_{-n}) \in \mathcal{H}^{1/2}_0(\partial \Omega) \), then

\[ \mathcal{K}_{\partial \Omega}[g] = \sum_{k=1}^{\infty} \left( \sum_{n=1}^{\infty} \mu_{n,k} a_n \right) g_{-k} + \sum_{k=1}^{\infty} \left( \sum_{n=1}^{\infty} \mu_{n,k} b_n \right) g_k. \]

Thus, if we identify \( \mathcal{H}^{1/2}_0(\partial \Omega) \) by \( \ell^2 \times \ell^2 \) via \( g \mapsto (a, b) \) where \( a = (a_1, a_2, \ldots) \) and \( b = (b_1, b_2, \ldots) \), then the NP operator \( \mathcal{K}_{\partial \Omega} \) is identified with the operator, denoted by \([\mathcal{K}_{\partial \Omega}]\), on \( \ell^2 \times \ell^2 \) defined by

\[ [\mathcal{K}_{\partial \Omega}](a, b) = (\mathcal{M} b, \mathcal{M} a), \]  

(3.16)

where \( \mathcal{M} = \mathcal{M}_{\partial \Omega} \) is the one-sided infinite matrix defined by

\[ \mathcal{M} = (\mu_{n,k})_{n,k \geq 1}. \]  

(3.17)

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Here the superscript $t$ denotes the transpose. In [11], $K_{\partial \Omega}$ is represented by the two-sided infinite matrix.

Here comes a fact of crucial importance: Grunsky theorem says that

$$kc_{n,k} = nc_{k,n}, \quad k, n = 1, 2, \ldots$$

(3.18)

(see [9] for a proof), which implies that $\mathcal{M}$ is symmetric, namely,

$$\mathcal{M}^t = \mathcal{M}.$$  (3.19)

It implies as was shown in [11] that $[K_{\partial \Omega}]$ is self-adjoint on $\ell^2 \times \ell^2$, and equivalently, $K_{\partial \Omega}$ is self-adjoint on $\mathcal{H}^{1/2}_{0}(\partial \Omega)$ equipped with the norm $\| \|_e$ defined by (3.7).

We call the constants $\mu_{n,k}$ defined by (3.13) the modified Grunsky coefficients on $\Omega$. Sometimes we write $\mu_{n,k} = \mu_{n,k}^{\Omega}$ to specify the domain where they are defined.

### 3.3 Proof of Theorem 1.3

Let $D$ be a simply connected $m$-fold rotationally symmetric (with respect to the point 0) domain. Let $\Psi_D$ be the Riemann mapping for $D$ given in (1.7). Let $g_n^D \in \mathcal{H}^{1/2}(\partial D)$ be a function defined in (3.8), namely,

$$g_n^D(w) := \frac{1}{\sqrt{|n|}} \left( \frac{\Psi_D^{-1}(w)}{R} \right)^n, \quad w \in \partial D,$$

(3.20)

and $g_0^D(w) = 1$ for convenience. Then, (1.8) can be rewritten as

$$X_l = \text{span}\{g_n^D : n = jm + l, \ j \in \mathbb{Z}\}, \ l = 0, 1, \ldots, m - 1.$$  (3.21)

Let $F_n^D(w)$ and $c_{n,k}^D$ be the $n$th Faber polynomial and Grunsky coefficients of $D$, defined in (3.10) and (3.11), respectively, so that the following relation holds

$$\left( \Psi_D^{-1}(w) \right)^n = F_n^D(w) - \sum_{k=1}^{\infty} \frac{c_{n,k}^D}{\Psi_D^{-1}(w)^k}.$$  (3.22)

Let $\mu_{n,k}$ be the modified Grunsky coefficients defined by (3.13).

**Lemma 3.2** Let $D$ be a simply connected $m$-fold rotationally symmetric (with respect to the point 0) domain. Then, for $n, k = 1, 2, \ldots$,

$$\mu_{n,k}^D = 0 \text{ if } n + k \not\equiv 0 \ (\text{mod } m).$$  (3.23)

**Proof** Suppose that $g_n^D \in X_l$ for some $l$. Then, we see from (3.21) that

$$n \equiv l \ (\text{mod } m).$$  (3.24)
We have from Theorem 3.1
\[ K_{\partial D}[g_n^D] = \sum_{k=1}^{\infty} \mu_{n,k}^D g_{-k}^D \]

Note that \( K_{\partial D}[g_n^D] \in X_l \) by (1.5). Since \( g_n^D \) is orthogonal to each other, we infer that
\[ \mu_{n,k}^D = 0 \text{ if } -k \not\equiv l \pmod{m}. \]
This together with (3.24) leads us to (3.23). \( \square \)

Let \( \Omega \) be the simply connected domain whose \( m \)th-root transform is \( D \), namely,
\[ \Psi_D(z) = \Psi_\Omega(z^m)^{1/m}. \] (3.25)

Let \( F_n^\Omega(w) \) and \( c_{n,k}^\Omega \) be the \( n \)th Faber polynomial and Grunsky coefficients of \( \Omega \) so that the relation (3.22) holds with \( D \) replaced with \( \Omega \). Let \( \mu_{n,k}^\Omega \) be the modified Grunsky coefficients.

**Lemma 3.3** If \( D \) is the \( m \)th-root transform of \( \Omega \), then
\[ \mu_{mn,km}^D = \mu_{n,k}^\Omega \] (3.26)
for all \( k, n = 1, 2, \ldots \).

**Proof** One can see from (3.25) that the following holds:
\[ \Psi_D^{-1}(w)^m = \Psi_\Omega^{-1}(w^m). \]

We thus have
\[ \left( \Psi_D^{-1}(w) \right)^{mn} = \left( \Psi_\Omega^{-1}(w^m) \right)^n = F_n^\Omega(w^m) - \sum_{k=1}^{\infty} \frac{c_{n,k}^\Omega}{\left( \Psi_\Omega^{-1}(w^m) \right)^k}. \]

On the other hands, we have from (3.23) that
\[ \left( \Psi_D^{-1}(w) \right)^{mn} = F_{mn}^D(w) - \sum_{k=1}^{\infty} \frac{c_{mn,km}^D}{\left( \Psi_D^{-1}(w) \right)^k} = F_{mn}^D(w) - \sum_{k=1}^{\infty} \frac{c_{mn,km}^D}{\left( \Psi_D^{-1}(w) \right)^{mk}} = F_{mn}^D(w) - \sum_{k=1}^{\infty} \frac{c_{mn,km}^D}{\left( \Psi_\Omega^{-1}(w^m) \right)^k}. \]

Therefore we have \( c_{mn,km}^D = c_{n,k}^\Omega \) and (3.26) follows. \( \square \)
Proof of Theorem 1.3. Let $\tilde{X}_0 := X_0 \cap \mathcal{H}_0^{1/2}(\partial \Omega)$. We prove
\[
\sigma(\mathcal{K}_{\partial D}, \tilde{X}_0) = \sigma(\mathcal{K}_{\partial \Omega}, \mathcal{H}_0^{1/2}(\partial \Omega)).
\] (3.27)

Since $\mathcal{K}_{\partial D}[1] = 1/2$ for any domain $D$, $\sigma(\mathcal{K}_{\partial D}, X_0) = \sigma(\mathcal{K}_{\partial D}, \tilde{X}_0) \cup \{1/2\}$ and $\sigma(\mathcal{K}_{\partial \Omega}) = \sigma(\mathcal{K}_{\partial \Omega}, \mathcal{H}_0^{1/2}(\partial \Omega)) \cup \{1/2\}$. Thus (1.10) follows from (3.27).

To prove (3.27), define the linear transform $U : \tilde{X}_0 \to \mathcal{H}_0^{1/2}(\partial \Omega)$ by $U(g_{mn}) = g_n^\Omega$ for all $n \neq 0$. Since both $g_{mn}^D$ and $g_n^\Omega$ are orthonormal basis of each space, $U$ is a unitary transform. If $n > 0$, then by (3.14), we have
\[
\mathcal{K}_{\partial \Omega}[U(g_{mn})] = \mathcal{K}_{\partial \Omega}[g_n^\Omega] = \sum_{k=1}^\infty \mu_{n,k}^\Omega g_{-k}.
\]
By (3.26), we have
\[
\mathcal{K}_{\partial \Omega}[U(g_{mn})] = \sum_{k=1}^\infty \mu_{mn, mk}^D g_{-k} = \sum_{k=1}^\infty \mu_{mn, mk}^D U(g_{-mk}).
\]
It then follows from (3.23) that
\[
\mathcal{K}_{\partial \Omega}[U(g_{mn})] = U\mathcal{K}_{\partial D}[g_{mn}].
\] (3.28)
The case for $n < 0$ can be dealt with in the same way to prove (3.28). Thus, we have
\[
\mathcal{K}_{\partial \Omega} = U\mathcal{K}_{\partial D}|_{\tilde{X}_0},
\]
from which (3.27) follows. ☐

3.4 Lemniscates

Let $\Omega$ be the lemniscate whose boundary is defined by
\[
\partial \Omega = \{z : |P(z)| = R^n\},
\] (3.29)
where $P$ is a polynomial of degree $n$ and $R$ is a number sufficiently large so that all the zeros of $P$ lie inside $\Omega$.

Here we give an alternative proof, using the representation of $\mathcal{K}_{\partial \Omega}$ in terms of the Grunsky coefficients, of the following theorem proved in [13].

Theorem 3.4 [13] If $\Omega$ is a lemniscate whose boundary is defined by (3.29), then $\mathcal{K}_{\partial \Omega}$ has an infinite dimensional kernel.
Proof If \( P(z) \) is given by 
\[ P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0, \]
then the Riemann mapping \( \Psi = \Psi_\Omega \) is given by,
\[ \Psi^{-1}(w) = w \left( 1 + \frac{a_{n-1}}{w} + \frac{a_{n-2}}{w^2} + \cdots + \frac{a_0}{w^n} \right)^{1/n} \]  
(3.30)

(see [14]). Since \( (\Psi^{-1}(w))^{nk} = P(w)^k, \) \( F_{nk}(w) = 0 \) and hence the Grunsky coefficients \( c_{nk,l} \) is 0 for all \( k \) and \( l \). Thus \( K_{\partial \Omega \{ g_{nk} \} = 0 \) for all \( k \neq 0 \) by Theorem 3.1.

\( \square \)

Remark 1 An \( m \)th-root transform of a lemniscate is also a lemniscate. In fact, if the lemniscate \( \Omega \) is given as before. Then its \( m \)th-root transform \( \Omega_m \) is given by
\[ \Psi^{-1}(w^m)^{1/m} = w \left( 1 + \frac{a_{n-1}}{w^m} + \frac{a_{n-2}}{w^{2m}} + \cdots + \frac{a_0}{w^{nm}} \right)^{1/m}, \]

namely, it is the lemniscate defined by the polynomial \( P(z) = z^{nm} + a_{n-1}z^{(n-1)m} + \cdots + a_0. \)

4 Matrix representations

Let \( D \) be a simply connected \( m \)-fold rotationally symmetric domain and let \( \Omega \) be the domain such that \( D = \Omega_m \), the \( m \)th-root transform of \( \Omega \). The matrix representations of \( K_{\partial D} \) would yield a clear picture on the spectral properties.

As in (3.16), \( K_{\partial D} \) is represented by the operator \( [K_{\partial D}] \) on \( \ell^2 \times \ell^2 \) defined by
\[ [K_{\partial D}] (a, b) = (\mathcal{M}_{\partial D}b, \mathcal{M}_{\partial D}a), \]
where \( \mathcal{M}_{\partial D} = (\mu_{n,k})_{n,k \geq 1} \). According to (3.23), \( \mathcal{M}_{\partial D} \) takes the form

\[
\begin{bmatrix}
0 & \mu_{1,m-1}^D & 0 & \mu_{1,2m-1}^D & 0 & \cdots \\
0 & 0 & \mu_{2,m-1}^D & 0 & \mu_{2,2m+1}^D & 0 & \cdots \\
\mu_{m-1,1}^D & 0 & 0 & \mu_{m-1,m+1}^D & 0 & \cdots \\
0 & \mu_{m,m-1}^D & 0 & 0 & \mu_{m,m+1}^D & \cdots \\
0 & 0 & \mu_{m+1,m-1}^D & 0 & \mu_{m+1,m+1}^D & 0 & \cdots \\
\mu_{2m-1,1}^D & 0 & 0 & \mu_{2m-1,m+1}^D & 0 & \cdots \\
0 & \mu_{2m,m}^D & 0 & 0 & \mu_{2m,2m+1}^D & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots 
\end{bmatrix}
\]
For $j = 0, 1, \cdots, m_*$, where $m_* = m/2$ if $m$ is even and $m_* = (m - 1)/2$ if $m$ is odd, we define the subspace $\mathcal{H}_j$ by

$$
\mathcal{H}_j = \begin{cases} 
X_j \cap \mathcal{H}_0^{1/2}(\partial D) & \text{if } j = 0, \\
X_j & \text{if } m \text{ is even and } j = m/2, \\
X_j \oplus X_{m-j} & \text{otherwise.}
\end{cases}
$$

(4.1)

Then, $\mathcal{H}_0^{1/2}(\partial D)$ is the direct sum of $\mathcal{H}_0, \mathcal{H}_1, \ldots, \mathcal{H}_{m_*}$, namely,

$$
\mathcal{H}_0^{1/2}(\partial D) = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_{m_*}.
$$

(4.2)

The NP operator $\mathcal{K}_{\partial D}$ on $\mathcal{H}_j$ is represented by the operator $[\mathcal{K}_{\partial D}|_{\mathcal{H}_j}]$ on $\ell^2 \times \ell^2$ defined by

$$
[\mathcal{K}_{\partial D}|_{\mathcal{H}_j}](a, b) = (\mathcal{M}_j b, \mathcal{M}_j a),
$$

(4.3)

where $\mathcal{M}_j$ is given as follow:

(i) If $j = 0$ or $j = m/2$ ($m$ is even), then

$$
\mathcal{M}_j = \begin{bmatrix}
\mu^D_{m-j,m-j} & \mu^D_{m-j,2m-j} & \mu^D_{m-j,3m-j} & \cdots \\
\mu^D_{2m-j,m-j} & \mu^D_{2m-j,2m-j} & \mu^D_{2m-j,3m-j} & \cdots \\
\mu^D_{3m-j,m-j} & \mu^D_{3m-j,2m-j} & \mu^D_{3m-j,3m-j} & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{bmatrix}.
$$

(4.4)

(ii) Otherwise,

$$
\mathcal{M}_j = \begin{bmatrix}
0 & \mu^D_{j,m-j} & 0 & \mu^D_{j,2m-j} & \cdots \\
\mu^D_{m-j,j} & 0 & \mu^D_{m-j,m+j} & 0 & \cdots \\
0 & \mu^D_{m+j,m-j} & 0 & \mu^D_{m+j,2m-j} & \cdots \\
\mu^D_{2m-j,j} & 0 & \mu^D_{2m-j,m+j} & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}.
$$

(4.5)

Note that

$$
\mathcal{M}_0 = \mathcal{M}_{\partial \Omega},
$$

(4.6)

which is due to (3.26).

We note that $\mathcal{M}_j = \mathcal{M}_j$. So, if $\mathcal{M}_j$ is real, then it admits the diagonalization, and its eigenvalues completely determines those of $[\mathcal{K}_{\partial D}|_{\mathcal{H}_j}]$. In fact, we have the following proposition.

**Proposition 4.1** Suppose that $\mathcal{M}_j$ is real. If $\lambda$ is an eigenvalue of $[\mathcal{K}_{\partial D}|_{\mathcal{H}_j}]$, then either $\lambda$ or $-\lambda$ is an eigenvalue of $\mathcal{M}_j$. Conversely, if $\lambda$ is an eigenvalue of $\mathcal{M}_j$, then both $\lambda$ and $-\lambda$ are eigenvalues of $[\mathcal{K}_{\partial D}|_{\mathcal{H}_j}]$. 

\(\square\) Springer
Proof Suppose that \( \lambda \) is an eigenvalue of \( K_{\partial D}|_{\mathcal{H}_j} \). According to (4.3), there is \( a, b \in \ell^2 \) (not both zero) such that

\[
M_j b = \lambda a, \quad M_j a = \lambda b.
\]

Thus we have

\[
M_j (a + b) = \lambda (a + b), \quad M_j (a - b) = -\lambda (a - b).
\]

Since either \( a + b \) or \( a - b \) is nonzero, we infer that \( \lambda \) or \(-\lambda\) is an eigenvalue of \( M_j \).

If \( \lambda \) is an eigenvalue of \( M_j \), then there is nonzero \( a \in \ell^2 \) such that

\[
M_j a = \lambda a.
\]

Therefore, we have

\[
\begin{align*}
\left[ K_{\partial D}|_{\mathcal{H}_j} \right] (a, a) &= (M_j a, M_j a) = \lambda (a, a), \\
\left[ K_{\partial D}|_{\mathcal{H}_j} \right] (a, -a) &= (-M_j a, M_j a) = -\lambda (a, -a).
\end{align*}
\]

We conclude that both \( \lambda \) and \(-\lambda\) are eigenvalues of \( K_{\partial D}|_{\mathcal{H}_j} \).

\[\Box\]

5 Examples

5.1 \( m \)-star-shaped domains

For a positive integer \( m \), let \( S_m \) be the regular \( m \)-star, namely,

\[
S_m = \{ x \xi^k_m : 0 \leq x \leq 4^{1/m}, \ k = 0, 1, \ldots, m - 1 \}, \quad (5.1)
\]

where \( \xi_m = e^{2\pi i / m} \). Let

\[
\Psi_m(z) = z \left( 1 + \frac{1}{z^m} \right)^{1/m}, \quad m = 1, 2, \ldots.
\]

It is known (see [4, 8]) that \( \Psi_m \) maps \( |z| > 1 \) conformally onto \( \mathbb{C} \setminus S_m \). The mapping \( \Psi_m \) is the \( m \)th-root transform of \( \Psi_1 \), that is, \( \Psi_m(z) = \Psi_1(z^m)^{1/m} \).

Fix \( R > 1 \) and let \( \Omega_m \) be bounded domains such that \( \partial \Omega_m = \{ \Psi_m(z) : |z| = R^{1/m} \} \). See Fig. 1 for shapes of \( \partial \Omega_m \). It is known that the eigenvalues of the NP operator on the ellipse \( x^2/a^2 + y^2/b^2 = 1 \) \((a \geq b)\) are \( \pm(a - b)^n/2(a + b)^n, \ n = 1, 2, \ldots, \) and \( 1/2 \) (see, for example, [13, Proposition 8]). Thus one can see easily that \( \sigma(K_{\partial \Omega_1}) = \{ \pm R^{-2n}/2 : n = 1, 2, \ldots \} \cup \{ 1/2 \} \). Since \( \Omega_m \) is the \( m \)th-root transform of \( \Omega_1 \), we infer from the Theorem 1.3 that \( \sigma(K_{\partial \Omega_m}) \) contains this set. Note that \( \partial \Omega_1 \) is an ellipse and \( \sigma(K_{\partial \Omega_1}) = \{ \pm R^{-n}/2 : n = 1, 2, \ldots \} \cup \{ 1/2 \} \). Thus \( \sigma(K_{\partial \Omega_2}) = \sigma(K_{\partial \Omega_1}) \cup \sigma(K_{\partial \Omega_2}, \mathcal{H}_1) \) and \( \sigma(K_{\partial \Omega_2}, \mathcal{H}_1) = \{ \pm R^{-2n+1}/2 : n = 0, 1, \ldots \} \).
Fig. 1  $m$-star-shaped domains $\partial \Omega_m$, $m = 3, 4, 5$ (from left to right), which are images of $\{|z| = R^{1/m}\}$ under $\Psi_m$. Here, $R = 1.1$. All of them have $\pm R^{-2n}/2$ ($n = 1, 2, \ldots$) as their NP eigenvalues.

5.2 Cassini oval

A Cassini oval $D$ is a lemniscate defined by (3.29) with the polynomial $P(z) = z^2 - 1$. See Fig. 2. The Riemann mappings $\Psi_D$ for $D$ is given by

$$\Psi_D^{-1}(w) = w \left(1 - \frac{1}{w^2}\right)^{\frac{1}{2}}. \quad (5.2)$$

Note that $\Psi_D$ is the 2nd-root transform of $\Psi(z) = z + 1$ and hence $D$ is the 2nd-root transform of $\Omega = \{|w - 1| = R^2\}$. So $\sigma(K_{\partial \Omega})$ contains of 0 as an eigenvalue of infinite multiplicity since $\sigma(K_{\partial D})$ does. This fact is already known by Theorem 3.4. Here, we look into $\sigma(K_{\partial D}, \mathcal{H}_1)$, spectrum of $K_{\partial D}$ on the invariant subspace $\mathcal{H}_1$.

If $n > 0$, then

$$\left(\Psi_D^{-1}(w)\right)^{2n+1} = w^{2n+1} \left(1 - \frac{1}{w^2}\right)^{\frac{2n+1}{2}} = w^{2n+1} \sum_{j=0}^{\infty} (-1)^j \left(\frac{2n+1}{2}\right)_{j} \frac{1}{w^{2j}}. $$

Therefore, the Faber polynomial $F_{2n+1}(w)$ is given by

$$F_{2n+1}(w) = \sum_{j=0}^{n} (-1)^{n-j} \left(\frac{2n+1}{2}\right)_{j} w^{2j+1}. \quad (5.3)$$

Fig. 2 Cassini oval when $R = 1.1$
Table 1  Eigenvalues of $[M_1]_n$, $n = 10, 25, 50, 100$, when $R = 1.1$

|       | $[M_1]_{10}$ | $[M_1]_{25}$ | $[M_1]_{50}$ | $[M_1]_{100}$ |
|-------|--------------|--------------|--------------|----------------|
| $\lambda_1$ | 0.249194    | 0.249279    | 0.249280     | 0.249280       |
| $\lambda_2$ | 0.0188675    | 0.019039    | 0.0190397    | 0.0190397      |
| $\lambda_3$ | 0.00126322   | 0.00135824  | 0.00135840   | 0.00135840     |
| $\lambda_4$ | 0.0000716940 | 0.0000967768| 0.0000968816 | 0.0000968816   |
| $\lambda_5$ | $3.10066 \times 10^{-6}$ | $6.86233 \times 10^{-6}$ | $6.90960 \times 10^{-6}$ | $6.90960 \times 10^{-6}$ |
| $\lambda_6$ | $9.74582 \times 10^{-8}$ | $4.77676 \times 10^{-7}$ | $4.92791 \times 10^{-7}$ | $4.92793 \times 10^{-7}$ |
| $\lambda_7$ | $2.16813 \times 10^{-9}$ | $3.16948 \times 10^{-8}$ | $3.51450 \times 10^{-8}$ | $3.51461 \times 10^{-8}$ |
| $\lambda_8$ | $3.24992 \times 10^{-11}$ | $1.93082 \times 10^{-9}$ | $2.50618 \times 10^{-9}$ | $2.50662 \times 10^{-9}$ |
| $\lambda_9$ | $2.94949 \times 10^{-13}$ | $1.05077 \times 10^{-10}$ | $1.78615 \times 10^{-10}$ | $1.78772 \times 10^{-10}$ |
| $\lambda_{10}$ | $1.22538 \times 10^{-15}$ | $5.05405 \times 10^{-12}$ | $1.27027 \times 10^{-11}$ | $1.27501 \times 10^{-11}$ |

It then follows that

\[
F_{2n+1}(\Psi_D(z)) = \sum_{j=0}^{n} (-1)^{n-j} \left( \frac{2n+1}{2} \right) \left( \frac{2j+1}{2} \right) \left( 1 + \frac{1}{z^2} \right)^{\frac{2j+1}{2}} \sum_{k=0}^{\infty} \frac{1}{z^{2k}}.
\]

In particular, they are real. Thus, eigenvalues of $\mathcal{M}_1$ (and numbers of the opposite sign) are eigenvalues of $\mathcal{K}_{\partial D}$ on $\mathcal{H}_1$ by Proposition 4.1.
By the definition (3.13) of the modified Grunsky coefficients, we see that the matrix $\mathcal{M}_1$ is of the form

$$\mathcal{M}_1 = \begin{bmatrix}
\frac{1}{2} R^2 & -\frac{\sqrt{3}}{8} R^4 & \frac{\sqrt{5}}{16} R^6 & \cdots \\
-\frac{\sqrt{3}}{8} R^4 & \frac{1}{8} R^6 & -\frac{3\sqrt{15}}{128} R^8 & \cdots \\
\frac{\sqrt{5}}{16} R^6 & -\frac{3\sqrt{15}}{128} R^8 & \frac{9}{128} R^{10} & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{bmatrix}.$$ (5.5)

It is not clear whether the eigenvalues of $\mathcal{M}_1$ can be computed explicitly. But, eigenvalues can be computed numerically. Moreover, since $R > 1$, finite submatrices yield good approximations of eigenvalues as Table 1 shows.

Let $[\mathcal{M}_1]_n$ be the submatrix of $\mathcal{M}_1$ of size $n \times n$ obtained by taking the first $n$ rows and columns. Let $\lambda_j$ be eigenvalues enumerated according to the rule $|\lambda_1| \geq |\lambda_2| \geq \cdots$. Table 1 exhibits the first 10 eigenvalues when $n = 10, 25, 50, 100$. There eigenvalues are computed by using Python built-in function eigvalsh for numerical computation of eigenvalues of Hermitian matrix.

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**Conflict of interest** On behalf of all authors, the corresponding author states that there is no conflict of interest.

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