On the Domain of Applicability of General Relativity

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Abstract

We consider the domain of applicability of general relativity (GR), as a classical theory of gravity, by considering its applications to a variety of settings of physical interest as well as its relationship with real observations. We argue that, as it stands, GR is deficient whether it is treated as a microscopic or a macroscopic theory of gravity. We briefly discuss some recent attempts at removing this shortcoming through the construction of a macroscopic theory of gravity. We point out that such macroscopic extensions of GR are likely to be non-unique and involve non-Riemannian geometrical frameworks.

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1 Introduction

Ever since the inception of general relativity (GR), many attempts have been made to extend or generalise it. The motivation for these attempts has mainly come from outside, relating essentially to the questions of its quantisation and its unification with other forces of nature. They therefore amount to an external critique of GR. In its applications to classical gravitational phenomena, however, GR has been generally assumed to be satisfactory.

Our aim here is to take GR as a classical theory of gravity and ask:

(i) what is its precise domain of applicability?

(ii) can it unambiguously deal with real settings and observations?

In this sense we are concerned with an internal critique of the theory.

We start by recalling that both in testing and applying GR as a theory of gravity, it is usually assumed\(^\text{1}\) that both the gravitational phenomena under study and the observations involved in deducing information about them are ideal, in the sense that particles are taken to be test particles and the observations are assumed to have infinite resolution. Clearly, in real applications, neither of these assumptions could be justified. The question then arises as to the precise status of GR when applied to real phenomena and put into correspondence with real observations.

To answer this, we discuss some of the major difficulties that arise when GR is applied in real settings. It turns out that the central problem is naturally tied up with the fact that both real observations and real phenomena are extended in nature; the former due to the fact that all real observations unavoidably involve finite resolutions and the latter because, strictly speaking, there are no point particles in reality. To demonstrate the difficulty, we consider two problems: the motion of particles and the question of cosmology. These considerations naturally lead us to the question of whether GR can be consistently treated either as a microscopic or macroscopic theory of gravity. We shall discuss each of these scenarios in turn and argue that, as it stands, GR is deficient in both cases. We shall then briefly consider the question of whether the theory can be successfully made compatible with real observations and phenomena internally (i.e. by only employing concepts internal to GR) and uniquely. This leads us to the question of averaging and macroscopic considerations of GR which we shall briefly discuss and conclude that the answer is again likely to be negative on both accounts.

We should also add here that despite our employment of GR as the classical theory of gravity, the main points raised here are also of relevance for other (alternative) theories of gravity, and in their comparison with observations. As a result what we have to say is of potential importance in making a comparative study of such theories and therefore in determining the ”correct” classical theory of gravity.

\(^{1}\)There are, however, exceptions. See, for example, Ellis \(^{2}\), Sciama \(^{3}\), Ehlers \(^{4}\).
The organisation of the paper is as follows. In Section 2 we consider the application of GR to cosmology and the problem of motion of bodies. In Section 3 we discuss the correspondence of GR with observations and discuss some of the problems that arise when comparison is made between predictions of GR and observations. A statement of the problem of the domain of applicability of GR is discussed in Section 4. Sections 5 and 6 contain the treatment of GR respectively as microscopic and macroscopic theories of gravity, together with some of the features and shortcomings of each scenario. In Section 7 we briefly discuss some of the attempts that have recently been made to develop a macroscopic theory of gravity, together with their corresponding problems and finally Section 8 contains our conclusions.

2 Applications of GR in some physical settings

The usual starting point in the classical studies of gravitational phenomena, including the universe itself, are Einstein's field equations (EFE)

\[ r_{ab} - \frac{1}{2}g_{ab}g^{cd}r_{cd} = -\kappa t_{ab}, \]  

(1)

together with the equations of motion

\[ t^b_{ab} = 0, \]  

(2)

which are known to follow from (1). It is, therefore, these equations which are employed in order to interpret local (e.g. solar system) and large scale (e.g. cosmological) observations, on the one hand, and to construct mathematical models in order to predict the evolution of gravitational phenomena, on the other.

From a theoretical point of view, the main idea is to treat these equations as a correspondence rule\(^3\), whereby given the form of the stress-energy tensor \( t_{ab} \), the geometry can be specified by solving equations (1)\(^4\). The resulting predictions of the theory are then to be compared with real observations, in order to ascertain its viability as a theory of gravity on all relevant scales, including the cosmological ones.

Now it is well known that general relativity has been extremely successful in accounting for local observations, including the usual classical tests in the solar system and the observations of binaries \(^9\). We shall come back to the question of motion of bodies in GR in the following sections. Here, however, it is instructive to contrast this with the status of GR as a theory of gravity on cosmological scales, for which there is little detailed direct evidence. The main reason for this is not just the usual “uncertainty” brought about by the error bars

\(^3\)We shall not dwell on the exact nature of this correspondence, specially its (i) uniqueness, i.e whether given \( t_{ab} \), the metric \( g_{ab} \) is specified uniquely and (ii) stability, i.e. whether small errors in the specification of \( t_{ab} \) or the simplifying assumptions usually employed in the construction of models can give rise to qualitative changes in the corresponding geometry \(^6, 7\). These questions are, however, of potential importance in determining the overall status of the theory in practice.

\(^4\)In reality one assumes, in addition to \( t_{ab} \), a number of simplifying assumptions, such as symmetry, which partially nail down the geometry as well, as is the case, for example, in cosmology, with the assumptions of isotropy and homogeneity which lead to Friedmann–Lemaître–Robertson–Walker (FLRW) geometry. Einstein’s field equations will then specify the unknown function(s) in the metric and hence the dynamical
which are bound to be present in all cosmological observations. It relates also to the very nature of real cosmological observations and the difficulties that arise when attempts are made to construct an appropriate theoretical framework for their interpretations within GR.

To discuss the domain of applicability of GR as a classical theory of gravity we shall focus on a number of settings of physical importance - namely the cases of cosmology and the motion of particles - where applications of GR involve fundamental difficulties.

2.1 GR and cosmology

A fundamental problem arising in the applications of GR to cosmology concerns the question of scales over which the theory is supposed to hold. This is due to the presence of a hierarchy of the cosmological scales of observational interest in the universe, on the one hand, and the absence of an intrinsic scale in the theory, on the other.

In the usual practice of standard cosmology, this problem is circumvented by making the following assumptions, which are usually made implicitly:

1. in the real complicated "lumpy" universe with a discrete matter distribution (of stars, galaxies, clusters of galaxies, etc.), the stress-energy tensor \( t_{ab} = t_{ab}^{\text{discrete}} \) can be adequately approximated by a "smoothed", or hydrodynamic, stress-energy tensor \( T_{ab}^{\text{(hydro)}} \), usually taken to be representable by a simple perfect fluid.

2. as \( t_{ab}^{\text{(discrete)}} \rightarrow T_{ab}^{\text{(hydro)}} \) on the right hand side of the EFE, the left hand side remains unchanged under such a change and therefore the appropriate field equations for describing the matter distribution \( T_{ab}^{\text{(hydro)}} \) still take the form

\[
R_{ab} - \frac{1}{2} G_{ab} R = -\kappa T_{ab}^{\text{(hydro)}},
\]

where capital letters denote quantities which correspond to the smoothed matter distribution \( T_{ab}^{\text{(hydro)}} \), and \( G_{ab} \) and \( R_{ab} \) are the metric and the Ricci tensors describing the corresponding geometry which is taken to be the same pseudo-Riemannian spacetime geometry as for (1).

3. the corresponding equations of motion follow from (3) and are given in the form

\[
T_{a;b}^{\text{(hydro)}} = 0,
\]

with \( T_{a;b}^{\text{(hydro)}} = T_{ac}^{\text{(hydro)}} G^{cb} \).

Given the intricacy and the detail involved in the real stress-energy tensor \( t_{ab} = t_{ab}^{\text{discrete}} \), these assumptions allow a large number of astrophysical and cosmological problems to be treated, which would otherwise have remained impossibly difficult to tackle. Implicit in these assumptions is that the solutions of (3) approximate well the solutions of (1) with the corresponding \( t_{ab}^{\text{discrete}} \), at least in the regions between the discrete matter constituents. This is a very strong theoretical assumption which is usually made without justification, in

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5As well as all other alternative theories of gravity proposed so far.
6For example, Ellis singles out five such scales, down to the scale of relevance for stars.
7There are authors, however, who mention this assumption explicitly. See, for example, [2, 10, 11, 12].
The main point here is that it is not a priori clear how good these assumptions are. What is therefore required, is a consistent application of GR to cosmology, where both sides of the EFE (1) are averaged out simultaneously, in order to find out the extent to which the above assumptions are in fact justified (see [2, 11, 12, 13] and references therein).

2.2 Motion of bodies in GR

Another important problem of real significance, from a physical point of view, is the description of motion of particles. There are two ways in which particles are treated within the context of GR: either as ideal particles or as what we shall refer to as real particles. We shall briefly discuss each of these in turn.

2.2.1 Ideal particles

The usual notion of particle motion in GR concerns the motion of ideal (test) particles, which are postulated to move along timelike geodesics of the (pseudo)-Riemannian spacetime manifold. The problem, however, is that so far no unambiguous and problem-free model has been put forward in the context of GR, for the description of the motion of such particles. The main difficulty stems from the problematic relation between the field equations and the equations of motion. It is in fact important to distinguish between the “laws of motion” and the “equations of motion” (see [14, 15, 16]). The main issue is that even though equations (2) follow directly from (1), only in certain special circumstances, such as the case of monopole singularities, where the influence of a moving monopole particle on the background metric is negligible, does the geodesic law of motion follow from (2). The difficulty arises from the fact that even though one can construct model candidates for point-like particles which satisfy equations (2), no such examples exist as yet which also satisfy the field equations. For example, it is known [17, 18] that given a smooth spacetime, a distributional model of point-particle energy-momentum tensor $t^{ab(\text{particle})}$ supported by a timelike worldline $L = \zeta(s)$, which obeys the dominant energy condition and satisfies (2), is necessarily monopolar

$$t^{ab(\text{particle})}(x) = \text{const} \int \dot{\zeta}^a \dot{\zeta}^b \delta(x - \zeta(s)) ds,$$

where $\text{const} > 0$ and $L$ is a geodesic. The problem is that this is not compatible with the EFE and worse still even the field equations themselves lose their mathematical validity, since the metric diverges at the particle location.

On the other hand, modelling of ideal particles moving along geodesics as an exact solution to EFE representing a test particle moving along a geodesic has not been successful as
yet (see [19, 20] and references therein).

An alternative approach to the problem of ideal particles is to treat them as extended bodies\(^{11}\), in the limit of all moments characterising them vanishing. Such point-like, single-pole, extended bodies appear to avoid the difficulties with the infinite matter density and metric divergences. The problem, however, is that, similar to the previous case, there is no way of combining the equations of motion (2) with the field equations (1) (with \(t_{ab} = T^{(\text{interior})}_{ab}\)).

### 2.2.2 Real particles

Turning now to the motion of extended particles, we recall that despite a great deal of efforts, as yet no non-approximate full treatment of motion of real particles exists [21, 22, 24]. The difficulties are similar to those encountered in a realistic treatment of cosmology, including the requirement for coordinate free definitions of integrals of tensor fields over the spacelike surfaces of the particles’ world-tubes, in order to determine moments in terms of which such particles could be characterised. A great deal of effort has gone into deriving approximate equations of motion of such particles, up to orders where gravitational backreaction also appears [24], but it still remains unclear whether such formal approximations converge or provide metrics that approximate actual solutions [17].

Another important feature of the motion of real particles is their non(Riemannian)-geodesic nature. To get a partial feel for this, let us consider the motion of a small spinning sphere of radius \(R\), which is assumed to have zero quadruple and higher multipoles [21, 22]. The motion of the particle can then be represented by a line \(L\) inside its world tube, the points of which are denoted by \(X^i\). Assuming the particle to be small, letting \(R \to 0\) and \(\delta x^i = x^i - X^i\), the equation of motion of such a particle may be written as

\[
\frac{D}{D\gamma} \left( m u^i \right) + \frac{D}{D\gamma} \left( u_j \frac{DS^j}{D\gamma} \right) + \frac{1}{2} S^{kl} u^m R^i_{mkl} = 0, \tag{6}
\]

where the \(S^{kl}\) take account of the particle’s spin and are defined as

\[
S^{ij} = \int \delta x^i T^{j4(\text{interior})} dv - \int \delta x^j T^{i4(\text{interior})} dv. \tag{7}
\]

This shows clearly that the first term in equation (7), \(D (mu^i) / D\gamma = 0\), corresponds to the usual Riemannian geodesic equation in the \(g_{ij}\) space, with the other terms appearing as soon as the \(S^{kl}\)'s are non-zero, i.e. the internal degrees of freedom of particles - here the spin - are taken into account. These extra spin-dependent terms clearly demonstrate that the motion of such particles cannot be geometrised within the Riemannian geometry and therefore a more general geometrical framework is required for their geometrisation.

Another important difficulty arises when one considers real observations, which we shall turn to now.

\(^{11}\) Represented as timelike world-tubes, with finite spacelike 3-volume sections, which are the supports of the corresponding interior stress-energy tensors \(T^{(\text{interior})}_{ab}\), describing the interior of the bodies and vanishing outside [21, 22].

\(^{12}\) We should also add here that the derivation of the geodesic equations for a point particle, based on the variational principle given by Fock [23], uses essentially a similar representation of a point particle as a
3 The correspondence between theory and observations in GR

In order to test the viability of physical theories (including GR), especially in their role as theoretical frameworks within which observations are analysed, it is necessary to devise ways whereby they can be put into one-to-one correspondence with observations and measurements. The setting up of such a correspondence would involve two steps: (i) to locate quantities within each theory which have observational counterparts or can be expressed in terms of such quantities and (ii) to ensure that such quantities possess identical spacetime domains of definition as those employed in observations. We shall refer to theories characterisable in terms of such operationally defined quantities as complete. An important feature of completeness is that it makes the question of viability of the theories decidable.

Now the first of these steps is not in principle difficult, even though there are quantities in physical theories which are not operationally defined, or inversely, there are physically motivated concepts which cannot be unambiguously defined in certain theories, such as, for example, the notion of energy and mass in GR. Regarding the second step, the main difficulty in most (dynamical) physical theories arises from the fact that they are formulated as differential equations, whereas all observational devices have a finite resolution. This is of vital importance, especially in view of the fact that, as was emphasized long ago by Bohr and Rosenfeld [25], only spacetime averages of field quantities have physical meaning. Consequently, point-like quantities are not operationally definable. This dichotomy is crucial in the case of GR, where the mathematical quantities of the theory are essentially point-like whereas the observations (especially in the case of cosmology) are invariably extended, in the sense of covering a (large) neighbourhood in the spacetime. For example, the theoretical procedures [4] of measuring the curvature tensor by means of the equations of geodesic deviation or through the employment of geodesic triangles are infinitesimal in nature, as they are based on standard calculus and therefore employ infinitesimal distances and times. On the other hand, real measurements (and observations) are extended by nature, as they involve averages over spacetime regions of finite characteristic length (see [25, 26]).

As an example of the fundamental difficulties of this type that one encounters in GR, let us recall the constructive-axiomatic approach to GR developed by Ehlers, Pirani and Schild (EPS) [4, 5]. The main aim of this work is to make transparent, in an axiomatic way, the relationship between the geometrical structures, including the Riemannian nature of spacetime, on the one hand, and the observable phenomena, on the other. More precisely, starting from the paths of light rays and the trajectories of idealised particles and relating them to the conformal and projective structures of spacetime respectively, they claim that these substructures, together with an additional assumption regarding the constancy of norm of vectors under parallel transport, would uniquely fix the underlying geometry of spacetime to be Riemannian.

There are two important points to note about this scheme: one theoretical and one observational. Firstly, the EPS assumptions do not necessarily reduce the geometry of spacetime to Riemannian.

\[ \text{We should emphasise the distinction between the viability of a theory and its completeness. Completeness is a necessary but not a sufficient condition for the viability of a theory.} \]
to be Riemannian; this is only true if the starting geometrical framework chosen is Weylian. Therefore, starting from more general geometrical frameworks, such as for example the Finslerian one, the imposition of EPS conditions does not necessarily reduce the underlying geometry of the spacetime to be Riemannian \[27, 28\]. In this sense then the whole scheme is dependent upon the theoretical framework chosen, i.e. the starting Weylian geometrical framework. Secondly, the type of observations implied by the EPS scheme are ideal, since in this scheme it is the ideal particles which are used in order to determine the projective structure of the spacetime. Now given that all real particles are extended (and usually have spin), an important question, from the point of view of our discussion here, is what happens if the idealised test particles in the EPS scheme are replaced by real physical extended (spinning) particles? In other words what is the resulting geometry for which the trajectories of such particles are geodesics? This is a very difficult problem to treat in its fullness, as even a satisfactory description of motion of extended bodies does not as yet exist (see Dixon \[22\] and Ehlers \[14, 29\] for a detailed discussion). Nevertheless, the equation of motion of the spinning particle discussed in the previous section indicates that GR (or at least its Riemannian geometrical component) can not be made constructively (-axiometrically) compatible with the motion of real (spinning) particles, treated as primary objects of the theory. Therefore, the consideration of real particles is likely to lead to geometrical settings that are more general than Riemannian. We shall see this thread appearing again when we consider the possibility of a macroscopic theory of gravity in Section 7.

4 Domain of applicability of GR

Our considerations so far seem to indicate that GR has fundamental difficulties in successfully treating both ideal and real particles, as well as dealing with the cosmological phenomena. The main sources of these difficulties (apart from the fact that no successful treatment of point particles exists) are related to the facts that (i) real phenomena are extended and (ii) real observations have finite resolutions and therefore always involve some form of spacetime “smoothing” or “averaging” in practice.

To see whether GR is complete, we start by asking whether there exist scales over which Einstein’s equations hold exactly\(^{14}\). There are in principle two different possible answers to this question: either GR, as it stands, is a microscopic theory or it is macroscopic, in the sense of being adequate to describe gravitation on a specified range of scales, with given matter models specifying these scales. We should contrast this with the usual practice which employs GR in order to describe classical gravitational phenomena on all scales, encountered both in theory and practice.

In the following Sections we consider each of these scenarios in turn and ask whether GR can be successfully treated in either way.

\(^{14}\)It is worthwhile to recall that this question has an old history going back to Einstein himself who realised the potential mathematical and physical problems that arise when attempts are made to construct point-like matter models within GR. This led Einstein to conclude that GR is adequate to describe macroscopic processes with continuously distributed matter model (see \[30\] for a discussion).
5 GR as a microscopic theory

In this section we start by assuming that GR, as it stands, is a classical microscopic theory of gravity and ask whether we could do so consistently. The usual motivation for treating GR as a microscopic theory of gravity seems to be based on its success in accounting for local phenomena, including the solar system tests and the observations of binaries. From a more formal point of view, the approach of EPS, which sets up a correspondence between the motion of ideal particles and the projective geometry of spacetime, could also be counted as a plus for this interpretation. There are, however, many problems with a microscopic interpretation of GR. From both a physical and mathematical point of view, the problems which arise when GR is taken as a classical microscopic theory of gravity have similarities with those which arise when electrodynamics and Newtonian gravity are considered as microscopic theories. There are important differences, however, due to the specific nature of general relativistic gravitation. The following are some of the defining features and difficulties of such a microscopic interpretation.

(I) The energy-momentum tensor: It assumed to be expressible as a discrete distribution of point-like matter constituents ("particles") localized at points \( x_A \), with the energy-momentum tensor supported by timelike world lines and given by

\[
t_{ab}^{(\text{micro})}(x_1, x_2, \ldots, x_N) = \sum_{A=1}^{N} t_{ab}^{(\text{particle})}(x_A),
\]

where \( N \) is the number of the constituents.\(^{15}\) The corresponding field dynamics and the laws of motion are then given by (1) and (2) respectively, where in both cases the stress-energy tensor is replaced by \( t_{ab}^{(\text{micro})} \).

(II) Vacuum nature: outside point sources, the general relativistic microscopic gravitation is an inherently vacuum phenomenon, satisfying the vacuum field equations

\[
r_{ab} - \frac{1}{2} g_{ab} g^{cd} r_{cd} = 0,
\]

with the discrete sources (8) acting as some sort of boundary conditions. The totality of such sources can be defined as the set of all possible singularities of the solutions of the vacuum field equations (9), satisfying all possible symmetries, asymptotic conditions\(^{16}\), etc. So, for example, an isolated point mass can be defined as the singularity of the corresponding spherically symmetric static solution of (9) with the integration constant identified with its mass. The task of compiling a complete list of all such microscopic sources allowed by the equations (1) with (8), or (9), is extremely difficult and is equivalent to finding the set of all vacuum solutions to (9). As examples of candidates for such sources we may consider, ideal particles, isolated point masses (i.e. sources for Schwarzschild black holes), point masses with spin

\(^{15}\)It is worth pointing out that \( N \) is not sufficiently large here to make it physically adequate to apply statistical or kinetic methods for the description of the matter distribution (see [31] and references therein).

\(^{16}\)This is physically similar to the case of classical microscopic electrodynamics, as developed in the Lorentz theory of electrons [32], which describes the dynamics of point-like charges in vacuum by means of microscopic field equations, the Lorentz-Maxwell equations.

\(^{17}\)The same approach may be formulated also for classical microscopic electrodynamics.
(i.e intrinsic angular momentum), like the source for the Kerr solution\(^{18}\) and rotating (test or interacting) particles moving freely or in the field of the other masses. There are also other examples of solutions to the vacuum EFE \(^{33}\), which are regular everywhere except at a number of singularities and which could be thought of as special mass distributions, such as the sources of Weyl’s and Curzon’s solutions, and solutions involving more than one Schwarzschild or Kerr black holes. Such sources, though “extended” in the sense of being non-local (line-like, etc.), can still be considered as microscopic.

(III) Modelling point particles: As was pointed out in Subsection 2.2.1, there are fundamental difficulties in successfully modelling point particles within GR. One could, however, assume that such particles are external to the theory, as is the case with point-like charges in the classical theory of electromagnetism \(^{32}\), and point masses in Newtonian gravity. Regarding the former, we note that all attempts to overcome the difficulties involved in modelling electrons, including the infinities that arise due to the self-energy, failed until it was realised that point charges must be treated as the singularities of the electromagnetic field and they are therefore incompatible with the field equations. In this case there are essentially two ways out, both external to the original theory. The first involves quantum electrodynamics, where the interaction of electrons are viewed quantum mechanically and where the problem of self-energy of electron is dealt with by employing renormalisations, and the second involves the employment of a continuous model of charged matter, resulting in Maxwell’s macroscopic electrodynamics, which has been shown \(^{32}\) to result from a space-time smoothing (or averaging) of the equations and relations of the microscopic theory. A similar situation also arises if Newtonian gravity \(^{34}\) is considered as a microscopic theory. The field theoretic structures of both microscopic (taken as standard Newtonian gravity) and macroscopic (upon a space averaging of Newtonian gravity) theories can be shown to be the same, apart from the discrete matter distribution being replaced by a continuous one \(^{35}\), which constitutes the problem of construction of continuous matter models in Newtonian gravity.

(IV) Equations of motion: As was discussed in Section 2, no satisfactory treatment of ideal particles exists which is at the same time compatible with the EFE.

(V) Newtonian limit: One may expect that the correspondence principle between GR, as a microscopic theory, and Newtonian gravity as a non-relativistic microscopic theory of gravity, should in principle provide a limiting procedure for the field equations (1) (together with (8)). For the left hand side of (1), there is a well-known procedure which reduces it to the Poisson equation. For the right hand side, the problem is how to define the limiting case for the sources. This is not clear, but the Israel-Carter-Robinson theorems \(^{36}\) may be considered to have established, at least for the case of one isolated mass (Schwarzschild solution), that the simplest general relativistic analogues of point-like sources in Newtonian gravity are black holes. As far as the geodesics of the test particles are concerned, the correspondence holds trivially. What has not been shown, however, is that the solution of the field equations for a test particle \(^{19,20}\) (see Section 2) possesses a Newtonian limit, nor has it been proved that the Newtonian limit for the case of \(N\) point-like sources (black holes,
test particles, etc.) exists (see [15, 17] for references). It therefore follows that there is as yet no rigorous formulation of the correspondence principle for GR, treated as a microscopic theory

(VI) Relation to observations: Because of the finiteness of their resolutions, all observations (and measurements) involve finite regions of spacetime and are therefore extended in the sense of involving averages (time, space, spacetime, statistical, etc.) of measured quantities. On the other hand, GR as a microscopic theory cannot internally produce quantities which are similarly extended and are therefore operationally definable. This is of relevance both for the case of extended particles as well as for cosmology.

These considerations indicate that microscopic GR cannot be complete. We should mention here that there are real settings, however, where it is possible (and physically adequate) to approximate real matter sources by point-like models. In such cases, we may treat GR as an adequate microscopic theory in practice. We should, however, bear in mind that this is an approximation and the important question in this respect is the estimation of the errors that this approximation involves in each particular setting.

6 GR as a macroscopic theory

Next we consider whether GR, as it stands, can be considered as a consistent macroscopic theory of gravity. If so, a great deal of objections raised above against the microscopic theory, such as compatibility with extended phenomena would be removed. Further, since macroscopic theory presupposes its objects to be averages, it would therefore be compatible with non-local measurements and observations.

The following are some of the defining features and difficulties that arise when GR is treated as a macroscopic theory:

(I) The energy-momentum tensor: The main assumptions of such a theory are that (a) there exist macroscopic (continuous) matter distributions with hydrodynamic stress-energy tensors $T^{(\text{macro})}_{ab}$, $T^{(\text{macro})}_{ab}(x) = T^{(\text{hydro})}_{ab}(x)$, (10) which are supported by world-tubes $\Sigma$ (a region of spacetime filled with matter), $x \in \Sigma$, and (b) macroscopic gravitational field dynamics due to macroscopic matter distributions (10) are governed by Einstein’s field equations (13), which in turn define an averaged metric $G_{ab}$. The corresponding law of motion is then given by (4) and, unlike microscopic theory which essentially has a vacuum character, the macroscopic theory is supposed to describe the field

\footnote{For example not all vacuum solutions of Einstein’s equations are known to have a Newtonian limit \[24, 17\].}

\footnote{The macroscopic model of matter in a classical macroscopic theory of gravity (assumed here to be GR) is to be postulated analogously to that of the macroscopic model of charge and current distributions in Maxwell’s classical theory of electromagnetism \[38\], where charge $\rho^{(\text{macro})}(x)$ and current distributions $j^{(\text{macro})}(x)$ are given as continuous (hydrodynamic) distributions $\rho^{(\text{hydro})}(x)$ and $j^{(\text{hydro})}(x)$ together with the definition of a charge configuration, its boundary and its exterior.}
both inside and outside extended bodies.

Mathematically, the continuous matter model may take the form of any stress-energy tensor supported by a world-tube and satisfying the appropriate differentiability and energy conditions. From a physical point of view, however, in addition to details of hydrodynamics and thermodynamics, one also requires information regarding the scales over which physical quantities are defined.

(II) The nature of macroscopic gravity: The macroscopic gravitational field due to a matter configuration satisfies the field equations inside of that configuration and the macroscopic vacuum equations

\[ R_{ab} - \frac{1}{2} G_{ab} G^{cd} R_{cd} = 0 \]  

in the exterior region. Also due to the matter characteristics undergoing a discontinuous jump at the boundary of the body, the field characteristics must be matched on the boundary in an appropriate way (see for example, and the references therein).

(III) Motion of real particles: As was mentioned in Section 2, no consistent description exists of motions of extended sources within GR, which is at the same time compatible with the field equations. Of potential interest here is the result by Lichnerowicz which shows that EFE together with a suitable source model (e.g., perfect fluid) determines the motion of the source as well as the evolution of the gravitational field. Interestingly, however, there is no analogous result for a single extended body in vacuum, or a system of such bodies.

(IV) Correspondence with microscopic theory: Starting from GR as a macroscopic theory, a fundamental question is the nature of the corresponding underlying microscopic theory from which the macroscopic theory may be derived upon some assumptions.

(V) Newtonian limit: It is not clear what the correspondence principle for such a theory is and what should be taken as the macroscopic analogue of the Newtonian theory in this case.

(VI) Intrinsic scales: An important distinguishing feature of GR (relative to electrodynamics and Newtonian theory) is the nonlinearity of its field equations. An immediate consequence of this is that by assuming the left hand side of the EFE to remain of the same form as (3), one is ignoring any reference to intrinsic scales implied in the definition of
The above discussions indicate that the treatments of GR (as it stands) as either microscopic or macroscopic are problematic. The question is how to remedy this fundamental shortcoming? One line research has been to attempt to construct a macroscopic theory, starting with GR as a microscopic theory of gravity. This is in fact the reverse of the situation that arose in electromagnetism, where it was proposed by Lorentz that, in addition to the usual macroscopic theory of Maxwell, there exists a microscopic level of description of electromagnetic phenomena \[12\], from which the macroscopic theory can be derived by an appropriate spacetime averaging procedure.

The formulation of a macroscopic theory of gravity, in the sense of Lorentz, is, however, much more complicated and requires the following three problems to be tackled: (i) how to define spacetime averages on a curved spacetime manifold, (ii) how to average the left hand side of the EFE to establish the form of the macroscopic field operator and (iii) how to average the microscopic matter distribution on the right hand side of the EFE in order to construct a macroscopic model of gravitating matter.

To accomplish (i), one requires a generalisation of the spacetime averaging procedure for flat space, defined in Cartesian coordinates (see, for example, \[38, 41\]). An important feature of such averages is that they keep the volume of the spacetime regions constant so as to ensure their applicability to all scales of interest. In this way the volume may be taken as a free parameter of the averaged theory, with the corresponding equations valid on any scale. The particular value of this volume (scale) would then need to be fixed by a model of the matter. In this connection, it has recently been shown (see \[12, 42, 43\]) that there exist a class of volume-preserving idempotent averaging operators, which allows the definition of a covariant spacetime averaging procedure, by generalising the flat spacetime case and keeping the averaging volume as a free parameter.

The resolution of the question (ii), however, turns out to be much more difficult than the associated question in electrodynamics. The main reason is that, as distinct from Maxwell-Lorentz equations, the EFE are nonlinear with the important consequence that the field correlation functions arising in the process of averaging cannot be defined in terms of the EFE themselves. This is a direct consequence of the fact that in such settings the averages

\[T_{\alpha \beta}^{(\text{macro})}\]

and, therefore, the corresponding correlations. It should be emphasised that in such a theory it is \[T_{\alpha \beta}^{(\text{macro})}\] which carries all the information about scales and since its definition relies totally on the model of matter, the effect of changes in scales only concerns the right hand side and not the fundamental structure of the field equations themselves. A related problem is whether there exists a built-in scale which may serve as a correlation length in GR. This is unlikely, in view of the fact that GR as a classical theory of gravity has only two constants, namely the universal constant of gravitation \(G\) and the speed of light \(c\), neither of which (nor any combination of them) possess such a scale\[^23\].

7 Towards a theory of macroscopic gravity

The above discussions indicate that the treatments of GR (as it stands) as either microscopic or macroscopic are problematic. The question is how to remedy this fundamental shortcoming? One line research has been to attempt to construct a macroscopic theory, starting with GR as a microscopic theory of gravity. This is in fact the reverse of the situation that arose in electromagnetism, where it was proposed by Lorentz that, in addition to the usual macroscopic theory of Maxwell, there exists a microscopic level of description of electromagnetic phenomena \[12\], from which the macroscopic theory can be derived by an appropriate spacetime averaging procedure.

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\[^23\]Here we are confining ourselves to the case where the cosmological constant \(\Lambda = 0\).
of the products of a quantity $Q$ are not equal to the products of its averages, i.e.

$$\langle Q^m \rangle \langle Q^n \rangle \neq \langle Q^{m+n} \rangle,$$

where $m$ and $n$ are positive integers and $\langle .. \rangle$ denotes an average. As a result, averaging in general requires correlation terms which need to be specified externally to the theory. In this way the averaging procedures are non-unique in at least two ways: firstly due to the freedom that exists in the choice of the procedure itself and secondly due to the assumptions necessary to estimate the correlation terms within each procedure. A number of such procedures have been proposed in the literature (for a review see [13]), but these are mostly perturbative in nature and only go as far as the second order in perturbations. Nevertheless, all these schemes have already demonstrated that any attempt to average out the terms of the second order in the metric perturbations\(^{24}\) does require the evaluation of correlation terms which again can only be defined externally to the theory. Consequently, it seems that the elements of externality and non-uniqueness are generic properties of all such averaging procedures\(^{25}\).

A non-perturbative approach recently put forward by Zalaletdinov [12, 42] (see reviews on the approach in [13, 43, 46]) consists of a spacetime averaging of Cartan’s structure equations for the (pseudo)-Riemannian geometry of spacetime and the EFE. The outcome of this procedure is a set of averaged equations in the form

$$M_{ab} - \frac{1}{2} G_{ab} G^{cd} M_{cd} = -\kappa T_{ab}^{(macro)},$$

where $G_{ab}$ is the macroscopic metric, $M_{ab}$ is the Ricci tensor corresponding to the Riemannian curvature tensor $M_{abcd}$ and the macroscopic stress-energy tensor $T_{ab}^{(macro)} = T^{(macro)}_{a} G_{cb}$ is of the form

$$T_{a}^{(macro)} = \langle t_{a}^{(micro)} \rangle + C_{a}^{b},$$

where $\langle t_{a}^{(micro)} \rangle$ is the averaged energy-momentum tensor and $C_{a}^{b}$ embodies the field correlation terms. These equations have been shown to possess the following properties: (i) in the high frequency limit (up to second order perturbations) they reduce [42] to Isaacson’s equations [47] and (ii) in the case where all correlation functions vanish they reduce [43] to the usual EFE

$$M_{ab} - \frac{1}{2} G_{ab} G^{cd} M_{cd} = -\kappa \langle t_{ab}^{(micro)} \rangle,$$

where in cosmological applications, the tensor on the right hand side is usually taken as a perfect fluid stress-energy tensor. Equations (13) are of the same form as (8), together with $\langle t_{ab}^{(micro)} \rangle = T_{ab}^{(macro)} = T_{ab}^{(hydro)}$ and the identification $M_{ab} = R_{ab}$. An interesting outcome of this that it makes transparent and precise the implicit assumptions that are usually made in general relativistic cosmological modelling, namely, that all correlation functions due to the macroscopic nature of gravity vanish.

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\(^{24}\)We should point out that in perturbative approaches $Q$ is usually taken to be a metric perturbation with the property that $\langle Q \rangle = 0$. But the second order term $\langle Q^2 \rangle$ is already non-trivial, $\langle Q^2 \rangle \neq 0$.

\(^{25}\)There are similarities here with the so called closure problem [44], which arises in the formulation of the problem of turbulence, where the moment equations need to be truncated (closed), by hand, i.e. externally, which in turn leads to the non-uniqueness of the procedure.
Finally, point (iii) concerns the question of construction of a model of the macroscopic gravitating matter in the form

$$\langle t^{(\text{micro})}_{ab} \rangle = T^{(\text{hydro})}_{ab}.$$  \hspace{1cm} (16)

In analogy to electrodynamics, this would require a model of microscopic matter to be specified, which is a very difficult task with no clear indications so far as to how to proceed (see an approach in [18]). The usual practice in cosmology, however, is to assume a phenomenological model for $T^{(\text{hydro})}_{ab}$, such as a perfect fluid, the validity of which remains unclear.

To conclude, despite important steps that have been taken towards a macroscopic extension of GR, major problems remain including the questions of externality and non-uniqueness that seem to accompany the processes of averaging, as well as the question of construction of appropriate matter sources. The task of completing this programme remains paramount, especially in view of recent cosmological observations which seem to indicate that the dynamical effects of such correlation functions may be of relevance on the dynamics of large enough scales.

Finally, we should mention that the spacetime averaging of Cartan’s structure equations for the (pseudo)-Riemannian geometry of spacetime seems to lead naturally to non-Riemannian features [12, 42]. This is interesting, especially in view of the fact that, as was discussed in Section 2, attempts at describing the motion of extended bodies also lead to non-Riemannian frameworks. This seems to indicate that a successful description of extended gravitational phenomena is likely to involve non-Riemannian considerations.

## 8 Conclusions

By considering a number of applications of GR in real settings, as well its relation to real observations, we argue that GR, as it stands, is deficient as a classical theory of gravity. This seems to be true whether GR is treated as a microscopic or macroscopic theory. In this way, therefore, there seems to be no scales over which the theory, as it stands, holds precisely. There is a sense, however, in which the treatment of GR as a microscopic theory is more consistent, for in such a setting it possesses all the problems “typical” for known classical microscopic theories.

We briefly discuss some of the recent attempts to construct a macroscopic theory of gravity, starting from GR as a microscopic theory, as an attempt to remedy this fundamental shortcoming. We conclude that such constructions are likely to include external features, be non-unique and involve non-Riemannian geometrical frameworks. In this way a full consistent theory of classical gravity with a build-in length remains to be developed. This is important, not only as a matter of principle, but especially in view of recent work which indicate that averaging could have important consequences for the dynamics of the universe and might, for example, be capable of resolving the so-called age problem in cosmology (see an approach in [13]).

From the perspective of our discussion here, it is worthwhile to recall that the text book successes of GR as a theory of gravity rely on the fact that the setting usually chosen for
the classical tests (the one body problem) is vacuum, and effectively microscopic in nature. As a result, their good agreement with observations is no surprise as the errors (due to the deviations of the Sun from an effective point particle) are likely to be small. What needs to be done is to extend GR in order to provide a physically and mathematically adequate framework in order to estimate the errors involved in settings which are far from microscopcity, as in the case of the motion of extended bodies and cosmology. The non-uniqueness of the macroscopic extensions of GR are only likely to be removable ultimately in reference with observations in such settings.

Finally, since the main points raised here are also applicable to alternative theories of gravity, our discussion is also of potential relevance to testing and assessing the viability of such theories in real settings.

Acknowledgments

RT was supported by PPARC UK Grant number H09454. RZ was supported by a Royal Society fellowship and would like to thank the School of Mathematical Sciences for hospitality. We also would like to thank George Ellis for helpful comments on the manuscript and discussions, and Henk van Elst for careful reading of the manuscript and comments.
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