Central Limit Theory for Linear Spectral Statistics of Normalized Separable Sample Covariance Matrix

Yu Long, Xie Jiahui and Zhou Wang*
Department of Statistics and Applied Probability,
National University of Singapore

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Abstract

This paper focuses on the separable covariance matrix when the dimension $p$ and the sample size $n$ grow to infinity but the ratio $p/n$ tends to zero. The separable sample covariance matrix can be written as $n^{-1}A^{1/2}XBX^\top A^{1/2}$, where $A$ and $B$ correspond to the cross-row and cross-column correlations, respectively. We normalize the separable sample covariance matrix and prove the central limit theorem for corresponding linear spectral statistics, with explicit formulas for the mean and covariance function. We apply the results to testing the correlations of a large number of variables with two common examples, related to spatial-temporal model and matrix-variate model, which are beyond the scenarios considered in existing studies. The asymptotic sizes and powers are studied under mild conditions. The computations of the asymptotic mean and variance are involved under the null hypothesis where $A$ is the identity matrix, with simplified expressions which facilitate to practical usefulness. Extensive numerical studies show that the proposed testing statistic performs sufficiently reliably under both the null and alternative hypothesis, while a conventional competitor fails to control empirical sizes.

Keywords: random matrix theory, spatial-temporal model, in-balanced dataset, high dimension, hypothesis testing.

*Zhou Wang, Professor, National University of Singapore, Singapore, 117546. wangzhou@nus.edu.sg.
1 Introduction

The recent decades have seen impressive development of random matrix theory with important applications in statistics, physics, econometrics and engineering. Related research involves but not limited to empirical spectral distribution (ESD), linear spectral statistics (LSS), the largest or smallest eigenvalues for Wigner matrix, sample covariance matrix, sample correlation matrix or more complex structures such as F matrix and higher-order tensors. See Bai & Silverstein (2010), Cai et al. (2020) and references therein for details. To this end, for any Hermitian matrix $A$ of size $n \times n$, its ESD is defined by

$$F_A(x) = \frac{1}{n} \sum_{j=1}^{n} I(\lambda_j^A \leq x),$$

where $\lambda_j^A$ is the $j$-th largest eigenvalue of $A$. Moreover, the LSS corresponding to $A$ are quantities of the form

$$\frac{1}{n} \sum_{j=1}^{n} f(\lambda_j^A) = \int f(x) dF_A(x),$$

where $f$ is some continuous and bounded real function on $(-\infty, \infty)$.

The sample covariance matrix is one of the most widely studied random matrices in the literature. On one hand, the leading eigenvalues of sample covariance matrix draw a lot of attention in statistical applications such as principal component analysis and factor analysis. Under the high dimensional setting, both the Gaussian and non-Gaussian cases have been considered in the literature, see Bao et al. (2015), El Karoui (2007), Knowles & Yin (2017). On the other hand, many statistics in traditional multivariate analysis can be written as functions of eigenvalues of sample covariance matrices. Central limit theorem (CLT) for LSS of sample covariance matrix then finds a lot of applications especially in multivariate hypothesis testing of a specific covariance structure. One can refer to Bai & Silverstein (2008) and Najim & Yao (2016) for example. Recently, Bai & Yao (2008) and Banna et al. (2020) extend the results to spiked covariance model while Hu et al. (2019) generalizes the results by considering elliptical distributions.

However, the mentioned work usually focuses on sample covariance matrix with the form

$$S_n = \frac{1}{n} A_p^{1/2} X_n X_n^* A_p^{1/2},$$

where $X_n$ contains independent samples and $A_p$ is a nonnegative Hermitian matrix representing the cross-row covariance structure. Hence, it excludes the case where both cross-row
and cross-column correlations exist. For instances, in spatial-temporal models, it’s necessary to consider both of the correlations across space and time. Under such cases, we may write the observed data matrix as $A_{p}^{1/2}X_{n}B_{n}^{1/2}$, and write the sample covariance matrix as

$$S_{n} = \frac{1}{n}A_{p}^{1/2}X_{n}B_{n}X_{n}^{*}A_{p}^{1/2},$$

(1.1)

named as separable sample covariance matrix.

Random matrix theory on separable sample covariance matrix has grown more popular in recent years. The limits of its empirical spectral distribution and the corresponding Stieltjes transforms have been shown in Zhang (2007) and Couillet & Hachem (2014). The fluctuation of its leading eigenvalues has been studied in Yang (2019) and Ding & Yang (2019). The former proves the universality for the distribution of the largest sample eigenvalues while the latter considers spiked separable covariance model and shows the phase transition phenomenon for the leading eigenvalues. As for the LSS, Bai et al. (2019) gives the central limit theorem (CLT) while Li et al. (2021) explores its application in testing white noise for time series data.

Up to our knowledge, for the separable covariance matrix, almost all the existing literature only consider the case where the dimension $p$ and the sample size $n$ are comparable, i.e., $p/n \to c \in (0, \infty)$. Motivated by the following two examples, in this paper we are more concerned with the case where $p$ and $n$ both tend to infinity with the ratio $p/n \to 0$. For the first example, we consider spatial-temporal data matrix consisting of series of pollutant concentration such as PM2.5 collected at several environmental monitoring stations. Due to the high cost of building a monitoring station, the number of spatial locations is usually much smaller than the length of series. In the second example, we consider a series of matrix-variate observations $\{Y_{t}\}_{1 \leq t \leq n}$ such as the pixel matrices of images, where $Y_{t}$ are high-dimensional ($p \times q$) and can be decomposed by $Y_{t} = A_{p}^{1/2}X_{t}B_{q}^{1/2}$. Define the column-column covariance matrix as

$$M_{n} = \frac{1}{nq} \sum_{t=1}^{n} Y_{t}Y_{t}^{\top} = \frac{1}{nq}A_{p}^{1/2}(X_{1}, \ldots, X_{n})(I_{n} \otimes B_{q})(X_{1}, \ldots, X_{n})^{\top}A_{p}^{1/2},$$

which can be regarded as a separable sample covariance matrix. The matrix $M_{n}$ has found important applications in data reduction technique such as image representation in Yang et al. (2004) and the matrix factor model in Chen et al. (2020) and Yu et al. (2020). It can also be applied to the estimation of Kronecker product covariance matrix structure for matrix-variate data, see Leng & Pan (2018). We clarify here that although $A_{p}^{1/2}$ in Bai
et al. (2019) is $p \times m$ where $m$ can be much larger than $p$, Bai et al. (2019) does not cover the above two common examples.

The existing results with $p/n \to c$ can be invalid if $p/n \to 0$ since the spectral distribution of $S_n$ is degenerate. The latter case is more challenging and relatively less studied in the literature even for simpler structures. The first breakthrough dates back to Bai & Yin (1988). They prove that the ESD of the normalized sample covariance matrix $\bar{S}_n$ tends to the semicircle law when $p/n \to 0$, where $\bar{S}_n$ is defined by

$$\bar{S}_n = \frac{1}{\sqrt{np}}(X_nX_n^T - nI_p),$$

$I_p$ is $p \times p$ identity matrix, and $X_p$ is $p \times n$ random matrix with independent and identically distributed (i.i.d.) entries. That is, both $A_p$ and $B_n$ are assumed to be identity matrices. Following this line, Chen & Pan (2012) proves the convergence of the largest eigenvalue of $\bar{S}_n$ while Chen & Pan (2015) shows the central limit theorem for the LSS of $\bar{S}_n$. We remark that the latter actually considers the case $n/p \to 0$, which is equivalent to $p/n \to 0$ by exchanging $X_n$ and $X_n^T$. The CLT is then applied to the testing of identity for large dimension covariance matrix in Li & Yao (2016).

The problem becomes more challenging when $A_p$ and (or) $B_n$ are no more identity matrices. Under such scenarios, Wang & Paul (2014) shows the most general result for the limiting spectral density (LSD) of $\bar{S}_n$. More recently, Bhattacharjee & Bose (2016) studies the LSD for polynomial-forms for a series of separable sample covariance matrix, which covers the setting in Wang & Paul (2014). There are also some results in time series models, which contain both cross-sectional and temporal correlations, hence closely related to the separable covariance model. The LSD of large-dimensional autocovariance matrices are studied in Wang et al. (2017), while Bhattacharjee & Bose (2019) further generalizes the results and establishes the asymptotic normality of trace polynomials for the autocovariance matrix. However, they basically focus on the sample autocovariance matrix and only a special case of the linear spectral statistics. Up to our knowledge, no results are found on the CLT for LSS of separable sample covariance matrix when $p/n \to 0$.

In this paper, we study the LSS of normalized separable sample covariance matrix defined in section 2 when $p \to \infty$ and $p/n \to 0$. The contributions of this paper are summarized as follows. First, we prove that the LSS of normalized separable sample covariance matrix converges weakly to a Gaussian limit under mild conditions, with explicit formulas for the asymptotic mean and covariance function. It turns out that they are dependent on the spectrums of both $A_p$ and $B_n$. The calculation of the asymptotic covariance function follows a similar strategy as in Chen & Pan (2015), but the calculation of the asymptotic
mean is much more challenging because the explicit equation therein does not hold anymore for general \( \mathbf{A}_p \) and \( \mathbf{B}_n \). Our results rely on a new method involving recursive expansion of the Stieltjes transform and rigorous control of all the error terms, which can be our second contribution. Third, we apply the CLT to testing the correlations of a large number of variables in spatial-temporal models and matrix-variate models, which are beyond the scenarios considered in existing literature. This can be supplement to the covariance matrix tests ever studied in Chen et al. (2010). We propose new testing statistic which seems to be the only reliable approach in the two examples.

The remaining of this paper is organized as follows. Section 2 presents the main theoretical results on the CLT of the LSS. We first introduce some general assumptions, then move to the Stieltjes transform under a simple case where \( \mathbf{A}_p \) is diagonal, and eventually generalize the result for LSS with general \( \mathbf{A}_p \). In Section 3, we apply the theorems to testing the correlations of a large number of random variables with the two preceding motivation examples. Section 4 is devoted to extensive numerical studies, which verify the reliable performance of the proposed testing statistics. In Section 5, we apply the approach to testing correlations of the Fama-French 10 \( \times \) 10 portfolio series. Section 6 concludes the paper and discusses future work. All the technical proofs of the theorems, corollaries and lemmas appeared in the paper are put into our supplementary material.

We introduce some notation here. Let \( \| \cdot \| \) denote the spectral norm of a matrix or Euclidean norm of a vector. \( \mathbf{A}^\top \) (or \( \mathbf{A}^* \)) denotes the transpose (or conjugate transpose) of a real (complex) matrix \( \mathbf{A} \). \( A_{ij} \) is the \((i,j)\)-th entry of \( \mathbf{A} \). For complex number \( z \), \( \Im z \) and \( \Re z \) denote its imaginary and real parts, respectively. \( i = \sqrt{-1} \). \( \bar{\lambda}_{\mathbf{B}^h} = n^{-1} \text{tr}(\mathbf{B}^h) \) for symmetric matrix \( \mathbf{B} \) and any integer \( h > 0 \), where \( \text{tr}(\cdot) \) denotes the trace.

## 2 Main results

### 2.1 Model setup

Note that the fundamental large number in this paper is \( p \) rather than \( n \). We only consider the case where the entries in \( \mathbf{A}_p \), \( \mathbf{B}_n \) and \( \mathbf{X}_p \) are real numbers. Motivated by Bai & Yin (1988), Wang & Paul (2014) and Chen & Pan (2015), we normalize \( \mathbf{S}_p \) by

\[
\bar{\mathbf{S}}_p = \frac{1}{\sqrt{np}} (\mathbf{A}_p^{1/2} \mathbf{X}_p \mathbf{B}_n \mathbf{X}_p^\top \mathbf{A}_p^{1/2} - n \bar{\lambda}_{\mathbf{B}_n} \mathbf{A}_p),
\]
where $\mathbf{A}_p^{1/2}$ and $\mathbf{B}_n^{1/2}$ are $p \times p$ and $n \times n$ deterministic matrices, respectively, and $\mathbf{X}_p$ is $p \times n$ consisting of independent real random variables with mean 0 and variance 1. $\bar{\lambda}_n = n^{-1} \sum_{j=1}^n \lambda_j^{B_n}$ and $n \bar{\lambda}_p \mathbf{A}_p$ is the expectation of $\mathbf{A}_p^{1/2} \mathbf{X}_p \mathbf{X}_p^\top \mathbf{A}_p^{1/2}$. For simplicity, we may suppress the subscript $p$ and $n$, and write $\bar{\mathbf{S}}, \mathbf{A}, \mathbf{X}$ and $\mathbf{B}$ directly without invoking any confusion. We formalize the basic assumptions of this paper in the below.

**Condition 2.1.** Suppose that

1. $\mathbf{X} = (x_{ij})_{p \times n}$ where $\{x_{ij} : i = 1, \ldots, p, j = 1, \ldots, n\}$ are i.i.d. real random variables with $\mathbb{E}(x_{11}) = 0$, $\mathbb{E}(x_{11}^2) = 1$, $\mathbb{E}(x_{11}^2) = \nu_4$, and $\mathbb{E}|x_{11}|^{4+\delta_0} < \infty$ for some $\delta_0 > 0$.

2. $p/n \to 0$ as $p \to \infty$.

3. $\mathbf{A}_p$ and $\mathbf{B}_n$ are non-negative deterministic real symmetric matrices with bounded eigenvalues $(a_1, \ldots, a_p)$ and $(b_1, \ldots, b_n)$, respectively in decreasing order. Moreover, the ESDs of $\mathbf{A}_p$ and $\mathbf{B}_n$ converge to some probability functions $F^A$ and $F^B$ which are not degenerate at 0 as $p \to \infty$, respectively. The eigenvalues of $\mathbf{A}_p$ and $\mathbf{B}_n$ are in the support of $F^A$ and $F^B$ for large $p$, respectively.

The above conditions are standard and common in related literature. The condition of finite $(4+\delta_0)$-th moment is to derive almost surely upper and lower bounds for the eigenvalues of $\bar{\mathbf{S}}_p$, which is not stringent in real applications. In the below, we introduce some preliminary results which are the foundations of the central limit theory.

### 2.2 Preliminary results

We first introduce the results on the limiting spectral distribution of $\bar{\mathbf{S}}_p$ from Wang & Paul (2014). To characterize the limit of $F^{\bar{\mathbf{S}}_p}(x)$, let’s define the Stieltjes transform of any distribution function $F(x)$ to be

$$m_F(z) = \int \frac{1}{x - z} dF(x), \quad z \in \mathbb{C}^+.$$ 

Then, for the empirical spectral distribution $F^{\bar{\mathbf{S}}_p}(x)$, its Stieltjes transform can be written as

$$m_{F^{\bar{\mathbf{S}}_p}}(z) = \int \frac{1}{x - z} dF^{\bar{\mathbf{S}}_p}(x) = \frac{1}{p} \text{tr}(\bar{\mathbf{S}}_p - z \mathbf{I})^{-1} = \frac{1}{p} \sum_{j=1}^p \frac{1}{\lambda_j^{\bar{\mathbf{S}}_p} - z}, \quad z \in \mathbb{C}^+.$$ 

The following lemma is adapted from Wang & Paul (2014).
Lemma 2.2 (Wang & Paul (2014)). Suppose that Condition 2.1 holds. Then the ESD of $\bar{S}_p$ almost surely converges to a non-random probability function $F$ whose Stieltjes transform $m(z)$ is the unique solution satisfying $\Im m(z) > 0$ and

$$m(z) = -\int \frac{1}{z + x\bar{\lambda}_{B^2} s(z)} dF^A(x), \quad s(z) = -\int \frac{x}{z + x\bar{\lambda}_{B^2} s(z)} dF^A(x), \quad z \in \mathbb{C}^+. \quad (2.2)$$

where $\bar{\lambda}_{B^2} = \lim_{n \to \infty} n^{-1} \text{tr}(B^2_n) = \int x^2 dF_B(x)$ is a positive constant.

Before moving to the central limit theorem for LSS of $\bar{S}_p$, we first give almost surely upper and lower bounds for its largest and smallest eigenvalues in the next Lemma 2.3.

Lemma 2.3 (Bound on $\|\bar{S}_p\|$). Suppose that Condition 2.1 holds. Then the spectral norm of $\bar{S}_p$ satisfies

$$\|\bar{S}_p\| \leq 2 \limsup_p a_1 b_1, \quad \text{almost surely.}$$

In practice, since the convergence rates of $F^A_p$ to $F^A$ and and $F^{B_o}_n$ to $F^B$ are unknown, we need their finite sample counterparts and define

$$m_p(z) = -\int \frac{1}{z + x\bar{\lambda}_{B^2_n} s_p(z)} dF^A_p(x), \quad s_p(z) = -\int \frac{x}{z + x\bar{\lambda}_{B^2_n} s_p(z)} dF^A_p(x),$$

where $\bar{\lambda}_{B^2_n} = n^{-1} \text{tr}(B^2)$ Further let $F_p(x)$ be the probability function associated with $m_p(z)$. With the bound on the support of $\bar{S}_p$ in Lemma 2.3, define

$$\tilde{G}_p(f) := p \int_{-\infty}^{+\infty} f(x)d(F^S_p(x) - F_p(x)),$$

where $f \in \mathcal{M} := \{\text{functions which are analytic in an open domain containing} [-2c, 2c] \}$ with $c = \limsup_p a_1 b_1$. Define an event $U_p := \{\|\bar{S}_p\| \leq 2 \limsup a_1 b_1\}$. Then by Lemma 2.3 and the Cauchy integral formula, with probability tending to one $U_p$ happens and

$$\tilde{G}_p(f) = -\frac{1}{2\pi i} \oint_{\mathcal{C}} f(z)p|m_{F} s_p(z) - m_p(z)|dz,$$

where $\mathcal{C}$ is the contour formed by the boundary of the rectangle with four vertices $(\pm u_0, \pm iv_0)$ where $u_0 = 2 \limsup_p a_1 b_1 + \epsilon_0$ with sufficiently small $\epsilon_0$, and $v_0$ is any positive number so that $f$ is analytic in a neighborhood of $\mathcal{C}$. Hence, it’s sufficient to study the Stieltjes transform on the contour $\mathcal{C}$. Our target is then to find some mean correction term $X_p(z)$ such that the process

$$M_p(z) = p[m_{F} s_p(z) - m_p(z) - X_p(z)] \quad (2.3)$$

converges weakly to some limit which needs to be specified.
2.3 Special case with diagonal $A_p$

In this subsection, we focus on a simple case where $A_p$ is diagonal. Note that the scenario considered in Chen & Pan (2015) is contained in this special case. To avoid dealing with small imaginary part of $z$, we denote

$$\mathcal{C}_u = \{u + iv : u \in [-u_0, u_0]\}, \quad \mathcal{C}_l = \{-u_0 + iv : v \in [p^{-1} \eta_p, v_0]\},$$

$$\mathcal{C}_r = \{u_0 + iv : v \in [p^{-1} \eta_p, v_0]\}, \quad \mathcal{C}_0 = \{-u_0 + iv : v \in [0, p^{-1} \eta_p]\} \cup \{u_0 + iv : v \in [0, p^{-1} \eta_p]\},$$

where $\{\eta_p\}$ is a sequence decreasing to 0 satisfying $\eta_p \geq p^{-\alpha}$ for some $\alpha \in (0, 1)$, and $v_0$ is a constant. The following theorem specifies the limit of the process $M_p(z)$ for $z \in \mathcal{C}_u$.

**Theorem 2.4** (Stieltjes transform). With Condition 2.1, if $A_p$ is diagonal and let the mean correction term in (2.3) be

$$X_p(z) = -\frac{1}{p} A_p(z) \times \sum_{k=1}^{p} \frac{1}{a_k^2} \tilde{\epsilon}_k(z)^3 - Y_p(z) \times \frac{1}{p} \sum_{k=1}^{p} a_k \tilde{\epsilon}_k(z)^2 \left(1 - \frac{1}{p} \sum_{k=1}^{p} a_k \tilde{\epsilon}_k(z)^2 \right)^{-1},$$

where

$$A_p(z) = (\nu_4 - 3) p^{-1} \sum_{j=1}^{n} B_{jj}^2 + \bar{\lambda}_{B_n^2} + B_p(z),$$

$$B_p(z) = \frac{1}{p} \sum_{k=1}^{p} a_k^2 \bar{\lambda}_{B_n^2} \tilde{\epsilon}_k(z)^2 \left(1 - \frac{1}{p} \sum_{k=1}^{p} a_k^2 \bar{\lambda}_{B_n^2} \tilde{\epsilon}_k(z)^2 \right)^{-1},$$

$$\tilde{\epsilon}_k(z) = \left[z + a_k \bar{\lambda}_{B_n^2} s_p(z) \right]^{-1},$$

and $Y_p(z)$ is the solution satisfying $Y_p(z) = o(1)$ as $p \to \infty$ to the equation

$$x = \bar{\lambda}_{B_n^2} \left( -\frac{1}{p} A_p(z) \times \sum_{k=1}^{p} \frac{1}{a_k^2} \tilde{\epsilon}_k(z)^3 + s_p(z) \right) + \left( \frac{1}{n} \sum_{j=1}^{n} \frac{(\lambda_{jB_n}^B)^2}{1 - \lambda_{jB_n}^B \sqrt{\frac{p}{n}} D_{p2}(x, z)} \right) D_{p2}(x, z)$$

with

$$D_{p2}(x, z) := \frac{1}{p} \sum_{k=1}^{p} \frac{a_k \tilde{\epsilon}_k}{1 - x a_k \tilde{\epsilon}_k},$$

then the process $\{M_p(z) : z \in \mathcal{C}_u\}$ converges weakly to a two-dimensional Gaussian process $M(\cdot)$ with mean zero and the covariance function

$$\Lambda(z_1, z_2) := \frac{\partial^2}{\partial z_1 \partial z_2} \lim_{p \to \infty} \frac{1}{p} \sum_{k=1}^{p} \tilde{\epsilon}_k(z_1) \tilde{\epsilon}_k(z_2) \left( \frac{2}{p} a_k^2 \bar{\lambda}_{B_n^2} \sum_{i < k} a_i^2 \tilde{\epsilon}_i(z_1) \tilde{\epsilon}_i(z_2) \right) \frac{2}{1 - \frac{1}{p} \bar{\lambda}_{B_n^2} \sum_{i < k} a_i^2 \tilde{\epsilon}_i(z_1) \tilde{\epsilon}_i(z_2)} + a_k^2 \left[ \frac{\nu_4 - 3}{n} \sum_{j=1}^{n} B_{jj}^2 + 2 \bar{\lambda}_{B_n^2} \right].$$
As in Chen & Pan (2015), the process of \( \{M(z), z \in \mathcal{C}_u\} \) can be extended to \( \{M(z), z \in \mathcal{C}_u \cup \bar{\mathcal{C}}_u\} \), where \( \bar{\mathcal{C}}_u = \{z : \bar{z} \in \mathcal{C}_u\} \) by noting the facts (i) \( M(z) \) is symmetric, i.e., \( M(\bar{z}) = M(z) \) and (ii) the mean and covariance functions are continuous and independent of \( v_0 \).

It seems that the mean and covariance functions in Theorem 2.4 are very complicated. In practice, the central limit theorem is usually applied to testing the correlations of a large number of random variables. Hence, we are more concerned with the null hypothesis that these variables are uncorrelated, which reduces to the case \( A_p = I_p \) with normalized data. On the other hand, testing a general covariance matrix, i.e., \( A_p = A^0 \), is equivalent to testing \( A_p = I_p \) after left-multiplying \((A^0)^{-1/2}\) to the data. Based on the above arguments, in the below we focus on this special case and aim to simplify the formulas for the mean correction term and the covariance function in Theorem 2.4. When \( A_p = I_p \), it’s straightforward that

\[
m_p(z) = s_p(z) = -\frac{1}{z + \lambda_{B^2} s_p(z)} = -\bar{\epsilon}_k(z), \quad B_p(z) = \frac{\lambda_{B^2} m_p^2(z)}{1 - \lambda_{B^2} m_p^2(z)} = \lambda_{B^2}^2 m'_p(z),
\]

The next corollary then follows directly.

**Corollary 2.5 (Simplified results for \( A_p = I_p \)).** Under the same conditions as in Theorem 2.4, further if \( A_p = I_p \), the results then hold with mean correction term being

\[
\begin{align*}
\lambda'_p(z) &= \frac{1}{p} A_p(z) \times m_p^3(z) - \frac{\mathcal{Y}_p(z) \times m_p^2(z)}{1 + \mathcal{Y}_p(z) m_p(z)},
\end{align*}
\]

where

\[
A_p(z) = (\nu_4 - 3)n^{-1} \sum_{j=1}^{n} B_{jj}^2 + \lambda_{B^2} + \lambda_{B^2}^2 m'_p(z),
\]

and \( \mathcal{Y}_p(z) \) is the solution satisfying \( \mathcal{Y}_p(z) = o(1) \) as \( p \to \infty \) to equation

\[
x = \left( \frac{1}{p} A_p(z) \times m_p^3(z) + m_p(z) \right) \times \bar{\lambda}_{B^2} + \left( \frac{1}{n} \sum_{j=1}^{n} \frac{\lambda_j^{B^n}}{1 - \lambda_j^{B^n} \sqrt{\frac{p}{n} D_{p2}(x, z)}} \right) D_{p2}(x, z)
\]

with

\[
D_{p2}(x, z) = \frac{-m_p(z)}{1 + x m_p(z)},
\]

while the covariance function being

\[
\Lambda(z_1, z_2) = m'(z_1) m'(z_2) \left( \nu_4 - 3 \right) \lim_{n \to \infty} n^{-1} \sum_{j=1}^{n} B_{jj}^2 + \frac{2 \bar{\lambda}_{B^2}}{(1 - \lambda_{B^2} m(z_1) m(z_2))^2}.
\]
It’s also worth mentioning that if further $B_n = I_n$, our results are consistent with those in Chen & Pan (2015). The covariance function is exactly the same as that in Proposition 3.1 of Chen & Pan (2015), while for the mean correction term we claim the following corollary. Note that $p/n \to 0$ in this paper while they assume $n/p \to 0$.

**Corollary 2.6** (Comparison with Chen & Pan (2015)). Under the conditions in Corollary 2.5, if further $B_n = I_n$, the mean correction term will satisfy

$$
\tilde{A}X_p^2 + \tilde{B}X_p + \tilde{C} = o(p^{-1}),
$$

where $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ are from Chen & Pan (2015) and defined by

$$
\tilde{A} = m - \sqrt{\frac{p}{n}(1 + m^2)}, \quad \tilde{B} = m^2 - 1 - \sqrt{\frac{p}{n}m(1 + 2m^2)}, \quad \tilde{C} = \frac{m^3}{p}A - \sqrt{\frac{p}{n}m^4}.
$$

$m$ and $A$ are the abbreviations for $m(z)$ and $A_p(z)$, respectively.

By the proof of Proposition 6.1 in Chen & Pan (2015), the above corollary indicates that the mean correction term $\mathcal{X}_p(z)$ is asymptotically equivalent to that in Chen & Pan (2015) when $B_n = I_n$.

### 2.4 For general $A_p$

Now we are ready to extend the CLT in Theorem 2.4 to the original linear spectral statistics. The general case where $A_p$ is not diagonal will be considered in this part. Define

$$
G_p(f) = p \int_{-\infty}^{+\infty} f(x)d(F_S^p(x) - F_p(x)) + \frac{p}{2\pi i} \oint_{C} f(z)\mathcal{X}_p(z)dz.
$$

The results are summarized in the following theorem.

**Theorem 2.7** (Linear spectral statistics). Under Condition 2.1, further assume either of the following two assumptions holds,

1. $A_p$ is diagonal,
2. $\mathbb{E}x_{11}^4 = 3$,
then for any \( f_1, \ldots, f_l \in \mathcal{M} \), the finite dimensional random vector \((G_p(f_1), \ldots, G_p(f_l))\) converges weakly to a Gaussian vector \((Y(f_1), \ldots, Y(f_l))\) with mean function \( \mathbb{E}Y(f) = 0 \) and covariance function

\[
cov(Y(f_j), Y(f_k)) = -\frac{1}{4\pi^2} \oint_{C_1} \oint_{C_2} f_j(z_1)f_k(z_2)\Lambda(z_1, z_2)dz_1dz_2,
\]

where \( \Lambda(z_1, z_2) \) is defined in Theorem 2.4. The contours \( C_1 \) and \( C_2 \) are non-overlapping, counterclockwise, and enclosing the interval \([-2 \lim_{a_1, b_1} a_1 b_1, 2 \lim_{a_1, b_1} a_1 b_1]\).

### 3 Applications to hypothesis testing

In this section, we investigate the statistical applications of the above theorems to testing the correlations of a large number of variables with a separable structure. We first introduce two motivation examples.

#### 3.1 Two examples

##### 3.1.1 Example 1: testing covariance matrix in spatial-temporal models

The testing of an identity covariance matrix has been studied extensively in the literature, see Ledoit & Wolf (2002), Bai et al. (2009), Chen et al. (2010) to name a few. Usually the testing statistics are based on the sample covariance matrix \( n^{-1} \sum_j Y_j Y_j^\top \) since its expectation is exactly equal to the population covariance matrix when \( \{Y_j\} \) are i.i.d. samples. The ultra-high dimensional scenarios where \( p \) is much larger than \( n \) are growing popular for real applications. Under such cases, most of the existing testing statistics can be written as linear spectral statistics of the normalized sample covariance matrix. However, as we claimed in the introduction, the existing literature only considers the special case where \( A \) and \( B \) are both identity matrices. Such an assumption is not realistic, which excludes many common scenarios such as spatial-temporal models.

Unlike the existing studies, we allow \( Y_j \) to be serially correlated and model the spatial and temporal correlations by two matrices \( \Sigma_1 \) and \( \Sigma_2 \). More precisely, we assume \( Y = (Y_1, \ldots, Y_n) \) can be written as

\[
Y = A_1 X A_2, \quad \Sigma_1 = A_1 A_1^\top, \quad \Sigma_2 = A_2 A_2^\top.
\]

(3.1)

Here we write \( A_1 \) and \( A_2 \) rather than \( A^{1/2} \) and \( B^{1/2} \) due to the identification issues with orthogonal rotations. The expectation of the sample covariance matrix is then \( n^{-1} \text{tr}(\Sigma_2) \Sigma_1 \).
Therefore, we can apply the preceding CLT to testing the hypothesis $\Sigma_1 = I$ when $p/n \to 0$. Detailed testing statistics are given in the next subsection. Note that testing a specific covariance matrix, i.e., $\Sigma_1 = \Sigma_0$, is equivalent to testing $\Sigma_1 = I$ if we left-multiply each $Y_j$ with $\Sigma_0^{-1/2}$. The ultra-high dimensional case where $p \gg n$ will be discussed in the simulation studies.

### 3.1.2 Example 2: testing correlations for matrix-variate data

Recently, higher-order structures such as matrix-variate or tensor-variate data are drawing more and more attention. Typical examples for matrix variate observations are the pixel matrices of images. To ease the analysis, existing studies usually propose different assumptions on the cross-row or/and cross-column correlations of the matrices. However, a fundamental question is whether the rows or columns of matrix observations are uncorrelated. In the below, we explain how to test the cross-row/column correlations based on the CLT in the above theorems.

Specifically, let $\{Y_j\}_{1 \leq j \leq n}$ be a collection of $p \times q$ large matrices, where each $Y_j$ can be decomposed as $Y_j = A_1X_jA_2$. Here $A_1$ and $A_2$ are $p \times p$ and $q \times q$ deterministic matrices respectively, while $X_j$ is $p \times q$ random matrix with i.i.d. entries. If the entries of $X_j$ are from standard normal distributions, we say $Y_j$ follows the matrix normal distribution which means

$$
\text{Vec}(Y_j) \sim \mathcal{N}(0, \Sigma_2 \otimes \Sigma_1), \quad \Sigma_2 = A_2A_2^\top, \Sigma_1 = A_1A_1^\top.
$$

$\Sigma_1$ and $\Sigma_2$ correspond to the cross-row and cross-column correlations, respectively. The target is to test $\Sigma_1 = I$ or $\Sigma_2 = I$.

We first consider the special case where $A_2 = I$. We can put $\{Y_j\}$ together by

$$
Z = (Y_1, \ldots, Y_n) = A_1(X_1, \ldots, X_n) = A_1(x_1, \ldots, x_{nq}).
$$

Then, it reduces to testing the covariance matrix of $x_j$. In practice, usually $n \asymp p \asymp q$, hence $p/(nq) \to 0$. Therefore, the existing statistics in the literature will work, such as that in Chen & Pan (2015) which relies on the linear spectral statistics of $S_p := (nq)^{-1}(ZZ^\top - nI)$. However, for general $A_2$, no existing statistics are available.

### 3.2 Testing statistic

Indeed, for general $A_2$ in the above example two, $Z$ is equal to $A_1(X_1, \ldots, X_n)(I_n \otimes A_2)$, which has a similar separable structure to that in example one. Therefore, we focus on the
first example. We formalize the testing as

\[ H_0 : \Sigma_1 = A_1 A_1^\top = I, \quad \text{v.s.} \quad H_1 : \Sigma_1 = A_1 A_1^\top \neq I. \]

The testing statistic is constructed based on the mean-corrected linear spectral statistics \( G_p(f) \) defined in (2.5), by replacing \( A_p \) and \( B_n \) there with \( I \) and \( \Sigma_2 \). Specifically, let

\[ T = \frac{G_p(f)}{\sigma}, \]

where \( \sigma^2 \) is the corresponding limiting variance in Theorem 2.7. Although many functions \( f \) are available in real practice, we take \( f(x) = x^2 \) in this paper to ease the calculations. We first claim the following theorem which provides a useful result for the asymptotic variance.

**Theorem 3.1.** Under Condition 2.1 and \( A_p = I_p \), the covariance function in Theorem 2.7 can be simplified as

\[
\text{cov}(Y(f_j), Y(f_k)) = \frac{1}{4\pi^2} \int_{-\sqrt{4\lambda B^2}}^{\sqrt{4\lambda B^2}} \int_{-\sqrt{4\lambda B^2}}^{\sqrt{4\lambda B^2}} f'_j(t_1) f'_k(t_2) H(t_1, t_2) dt_1 dt_2,
\]

where

\[
H(t_1, t_2) = \frac{1}{\lambda^2 B^2} \left( (\nu_4 - 3) \frac{1}{n} \sum_j B_j^2 \right) \sqrt{4\lambda B^2 - t_1^2} \sqrt{4\lambda B^2 - t_2^2} + 2 \log \left( \frac{4\lambda B^2 - t_1 t_2 + \sqrt{(4\lambda B^2 - t_1^2)(4\lambda B^2 - t_2^2)}}{4\lambda B^2 - t_1 t_2 - \sqrt{(4\lambda B^2 - t_1^2)(4\lambda B^2 - t_2^2)}} \right).
\]

Moreover, if \( f_j(x) = f_k(x) = x^2 \), we have \( \text{cov}(Y(f_j), Y(f_k)) = 4\lambda^2 B^2 \).

Therefore, for the testing problem we have the following corollary directly.

**Corollary 3.2.** Under Condition 2.1 (replace \( B_n \) with \( \Sigma_2 \)) and the null hypothesis \( H_0 \), let \( f(x) = x^2 \), then

\[
T = \frac{G_p(f)}{2\lambda \Sigma_2} \xrightarrow{d} \mathcal{N}(0, 1), \quad p \to \infty.
\]

As a result, we reject the null hypothesis as long as \( |T| > \Psi(\alpha/2) \), where \( \Psi(\alpha/2) \) is the \((1 - \alpha/2)\) quantile of the standard normal distribution and \( \alpha \) is the significant level. Indeed, due to the identification issue, under the null hypothesis where \( \Sigma_1 = I, A_1 \) can
be any orthogonal matrix. To be concrete, Theorem 3.1 indicates $T \overset{d}{\to} \mathcal{N}(0,1)$ if $T$ is calculated with $A_1^T Y$, not exactly $Y$. However, the eigenvalues of $YY^T$ and $A_1^T YY^T A_1$ are totally the same under this case, then the result in (3.2) still holds.

For the alternative hypothesis, we consider two common choices. In the first case, denoted as $H_{1,1}$, let $A_1$ be diagonal with non-identical entries and spectral distribution

$$(1 - \rho_1)\delta_1 + \rho_1\delta_{1 + c},$$

(3.3)

where $\rho_1 \in (0, 1)$, $\delta$ is the Dirac measure and $|c| > 0$ is constant. In the second case, denoted as $H_{1,2}$, we consider cross-row correlations and let

$$(A_1)_{ij} = \begin{cases} 1, & i = j, \\ \rho_2, & i \neq j, \end{cases}$$

(3.4)

where $\rho_2 \in (0, 1)$ is a constant. The next theorem shows the asymptotic powers under $H_{1,1}$ and $H_{1,2}$.

**Theorem 3.3.** Assume $A_1$ satisfies the alternative hypothesis $H_{1,1}$ or $H_{1,2}$, while the other assumptions in Condition 2.1 holds. Then as $p \to \infty$,

$$\mathbb{P}[|T| > \Psi(\alpha/2)] \to 1.$$  

### 3.3 Computations

The testing statistic $T$ relies on unknown parameters of the matrix $\Sigma_2$. In practice, to avoid estimating such a large-dimensional matrix, we shall propose additional assumptions on the spectrum of $\Sigma_2$. In many real examples, the spectral distribution of $\Sigma_2$ is usually associated with finite number of discrete points. In the below, we assume $\Sigma_2$ has empirical spectral distribution

$$\alpha\delta_{c_1} + (1 - \alpha)\delta_{c_2},$$

(3.5)

where $\alpha \in [0, 1]$ and $c_1 > c_2 > 0$. For example, if $\Sigma_2$ is structured as

$$(\Sigma_2)_{ij} = \begin{cases} 1, & i = j, \\ \rho, & (i, j) \in \{(2k, 2k - 1), (2k - 1, 2k)\}_{k=1,\ldots,n/2}, \\ 0, & \text{otherwise}, \end{cases}$$

(3.6)

then $\alpha = 0.5$, $c_1 = 1 + \rho$ and $c_2 = 1 - \rho$. The special structure in (3.6) happens in real world if the monitoring stations evaluate the environmental quality twice a day and only
the measurements in the same day are dependent. It will be more realistic to estimate $\rho$ rather than the whole matrix $\Sigma_2$ or its eigenvalues.

Given $\Sigma_1 = I$ and the spectrum of $\Sigma_2$ as in (3.5), we are ready to show the numerical calculation of the mean correction for $G_p(f)$. The following process can also be easily extended to more general $\Sigma_2$ with more than two mass points. When $\Sigma_1 = I$, the spectral distribution corresponding to $m_p(z)$ is a rescaled semicircle law with density

$$\frac{1}{2\pi\lambda_{\Sigma_2}^2} \sqrt{4\lambda_{\Sigma_2}^2 - x^2}.$$ 

Therefore, it’s easy to calculate $\int f(x)dF_p(x) = \lambda_{\Sigma_2}^2$. What remains is the calculation of integral corresponding to the mean correction term $X_p$. Generally, there is no closed-form formula for this term and we need numerical approximations. For the special $\Sigma_2$ in (3.6), we can simplify the equation for $Y_p$ to

$$x = \left(\frac{1}{p} A_p(z) \times m_p^3(z) + m_p(z)\right) \times (\alpha c_1^2 + (1 - \alpha)c_2^2) + \left(\frac{\alpha c_1^2}{1 - \sqrt{\frac{p}{n}}c_1 D_{p2}(x, z)} + \frac{(1 - \alpha)c_2^2}{1 - \sqrt{\frac{p}{n}}c_2 D_{p2}(x, z)}\right) D_{p2}(x, z).$$

Further, with the definition of $D_{p2}(x, z)$, we can reorganize the above equation in polynomial form as

$$A_1 x^3 + A_2 x^2 + A_3 x + A_4 = 0,$$  \hspace{1cm} (3.7)

where

$$A_1 = m^2,$$

$$A_2 = m \left(2 + \sqrt{\frac{p}{n}}m(c_1 + c_2) - \left(\frac{1}{p} A_p(z) \times m_p^3(z) + m_p(z)\right) \times (\alpha c_1^2 + (1 - \alpha)c_2^2) m^2,\right)$$

$$A_3 = (1 + \sqrt{\frac{p}{n}}c_1 m)(1 + \sqrt{\frac{p}{n}}c_2 m) + m^2(\alpha c_1^2 + (1 - \alpha)c_2^2)$$

$$- m \left(2 + \sqrt{\frac{p}{n}}m(c_1 + c_2)\right)(\alpha c_1^2 + (1 - \alpha)c_2^2)\left(\frac{1}{p} A_p(z) \times m_p^3(z) + m_p(z)\right),$$

$$A_4 = -(1 + \sqrt{\frac{p}{n}}c_1 m)(1 + \sqrt{\frac{p}{n}}c_2 m)(\alpha c_1^2 + (1 - \alpha)c_2^2)\left(\frac{1}{p} A_p(z) \times m_p^3(z) + m_p(z)\right)$$

$$+ m \left(\alpha c_1^2(1 + \sqrt{\frac{p}{n}}c_2 m) + (1 - \alpha)c_2^2(1 + \sqrt{\frac{p}{n}}c_1 m)\right),$$

$$15$$
and we write \( m \) for \( m_p(z) \) for simplicity. On the other hand, by the relationship \( z = -(m^{-1} + \bar{\lambda}\Sigma_3^2 m) \), we have

\[
-\frac{p}{2\pi i} \oint_C f(z) \chi_p(z) dz = \frac{p}{2\pi i} \oint_{|m|=d} f(-m^{-1} - \bar{\lambda}\Sigma_2^2 m) \chi_p(m)(m^{-2} - \bar{\lambda}\Sigma_2^2) dm, \tag{3.8}
\]

where \( d < (\bar{\lambda}\Sigma_2^2)^{-1/2} \) by the limiting spectral density and the contour \(|m|=d\) is counterclockwise. We simplify the numerical approximation with (3.7) and (3.8).

4 Simulation studies

In this section, we do numerical experiments to investigate the central limit theory and investigate the performance of the proposed testing statistic. As we have claimed, the testing problems in the two preceding examples are essentially equivalent. Hence, we only consider example one in the simulation. It contains three parts. Part one investigates the empirical sizes under the null hypothesis, which also verifies the central limit theory when \( \Sigma_1 = I \). Part two checks the empirical powers of the testing statistic under the two alternative hypothesis. Part three further verifies the central limit theory when \( \Sigma_1 \) is not identity matrix. All the simulation results are based on 5000 replications.

Figure 1: Empirical densities for the proposed testing statistics under the null hypothesis of example one with \( p = 100 \) and \( n = 10000 \). The red real lines correspond to the standard normal density. Left: Gaussian distribution; Right: Bernoulli distribution.
4.1 Under the null hypothesis

We generate data based on (3.1) in the above example one. Choose \( p \) from \{20, 50, 80, 100\} and \( n \) from \{2000, 5000, 8000, 10000\}. Let \( A_1 = I \), draw the entries of \( x_{ij} \) from i.i.d. standard normal distribution \( \mathcal{N}(0, 1) \) or Bernoulli distribution with \( \mathbb{P}(x_{ij} = \pm 1) = 0.5 \), and set \( \Sigma_2 \) as in (3.6) with \( \rho = 0.5 \). Numerical calculation with Matlab shows that under the null hypothesis, the mean correction term in (3.8) is independent of \( n \) and \( p \), equal to 1.25 for normal distribution and −0.75 for Bernoulli distribution. With the simulated data, we plot the empirical densities of the proposed testing statistics \( T \) in Figure 1 under the two distributions with 5000 replications when \( p = 100 \) and \( n = 10000 \). The empirical cumulative functions corresponding to the quantiles of standard normal distribution are displayed in Table 1. The empirical densities and cumulative functions imply the asymptotic normality of \( T \) under the null hypothesis.

Table 1: Standard normal quantiles (\( \Psi(\alpha) \)) and the corresponding empirical cumulative functions of the testing statistic. \( p = 100, n = 10000 \).

| \( \Psi(\alpha) \) | -1.6449 | -1.2816 | -1.0364 | -0.8416 | -0.6745 | -0.5244 | -0.3853 | -0.2533 | -0.1257 | 0.0000 |
|-------------------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| \( \alpha \)      | 0.0500  | 0.1000  | 0.1500  | 0.2000  | 0.2500  | 0.3000  | 0.3500  | 0.4000  | 0.4500  | 0.5000  |
| Normal            | 0.0520  | 0.1042  | 0.1580  | 0.2084  | 0.2542  | 0.3006  | 0.3508  | 0.4000  | 0.4496  | 0.5024  |
| Bernoulli         | 0.0526  | 0.1012  | 0.1542  | 0.2020  | 0.2538  | 0.3038  | 0.3596  | 0.4080  | 0.4630  | 0.5096  |
| \( \Psi(\alpha) \) | 0.1257  | 0.2533  | 0.3853  | 0.5244  | 0.6745  | 0.8416  | 1.0364  | 1.2816  | 1.6449  |         |
| \( \alpha \)      | 0.5500  | 0.6000  | 0.6500  | 0.7000  | 0.7500  | 0.8000  | 0.8500  | 0.9000  | 0.9500  |         |
| Normal            | 0.5504  | 0.6010  | 0.6436  | 0.6888  | 0.7384  | 0.7880  | 0.8438  | 0.8958  | 0.9458  |         |
| Bernoulli         | 0.5606  | 0.6108  | 0.6600  | 0.7094  | 0.7584  | 0.8052  | 0.8506  | 0.8964  | 0.9444  |         |

Next, we aim to investigate the empirical sizes of the proposed testing statistic (\( T_1 \)) when \( p \) and \( n \) grow. To show the bias by neglecting the temporal correlations, the testing statistic in Chen & Pan (2015) (\( T_2 \)) is regarded as a competitor, which takes \( \Sigma_2 \) as identity matrix. The empirical sizes with \( \alpha = 0.05 \) and \( \alpha = 0.10 \) of the two approaches under two distributions are shown in Tables 2 and 3. It’s seen that the proposed testing statistic performs very reliably under both distributions no matter \( \alpha = 0.05 \) or \( \alpha = 0.10 \), while the conventional one in Chen & Pan (2015) fails to control the Type-one errors.
Table 2: Empirical sizes of two statistics in example one. $\alpha = 0.05$.

| $x_{ij}$ | $\mathcal{N}(0,1)$ | $\mathcal{B}$ | $\mathcal{T}_1$ (proposed) | $\mathcal{T}_2$ (Chen & Pan (2015)) |
|---------|--------------------|---------------|-----------------------------|-------------------------------------|
|         | $(n/p)$ | 20 | 50 | 80 | 100 | 20 | 50 | 80 | 100 |
| 2000    |        | 0.0590 | 0.0620 | 0.0660 | 0.0732 | 0.6828 | 1 | 1 | 1 |
| 5000    |        | 0.0562 | 0.0554 | 0.0614 | 0.0630 | 0.6916 | 0.9998 | 1 | 1 |
| 8000    |        | 0.0548 | 0.0544 | 0.0512 | 0.0616 | 0.6810 | 0.9996 | 1 | 1 |
| 10000   |        | 0.0580 | 0.0522 | 0.0546 | 0.0566 | 0.6836 | 1 | 1 | 1 |

Table 3: Empirical sizes of two statistics in example one. $\alpha = 0.10$.

| $x_{ij}$ | $\mathcal{N}(0,1)$ | $\mathcal{B}$ | $\mathcal{T}_1$ (proposed) | $\mathcal{T}_2$ (Chen & Pan (2015)) |
|---------|--------------------|---------------|-----------------------------|-------------------------------------|
|         | $(n/p)$ | 20 | 50 | 80 | 100 | 20 | 50 | 80 | 100 |
| 2000    |        | 0.1148 | 0.1154 | 0.1268 | 0.1342 | 0.7634 | 1 | 1 | 1 |
| 5000    |        | 0.1092 | 0.1126 | 0.1162 | 0.1180 | 0.7770 | 1 | 1 | 1 |
| 8000    |        | 0.1026 | 0.1112 | 0.1010 | 0.1082 | 0.7692 | 0.9998 | 1 | 1 |
| 10000   |        | 0.1092 | 0.1096 | 0.1058 | 0.1094 | 0.7684 | 1 | 1 | 1 |

| $x_{ij}$ | $\mathcal{B}$ | $\mathcal{T}_1$ (proposed) | $\mathcal{T}_2$ (Chen & Pan (2015)) |
|---------|---------------|-----------------------------|-------------------------------------|
|         | $(n/p)$ | 20 | 50 | 80 | 100 | 20 | 50 | 80 | 100 |
| 2000    |        | 0.0940 | 0.0970 | 0.1088 | 0.1080 | 0.7828 | 1 | 1 | 1 |
| 5000    |        | 0.0946 | 0.0978 | 0.1000 | 0.1028 | 0.7864 | 0.9998 | 1 | 1 |
| 8000    |        | 0.0920 | 0.1022 | 0.1004 | 0.0994 | 0.7912 | 1 | 1 | 1 |
| 10000   |        | 0.0856 | 0.0954 | 0.1036 | 0.1044 | 0.7808 | 1 | 1 | 1 |
4.2 Under the alternative hypothesis

For the alternative hypothesis, we consider two cases as in Theorem 3.3. For $H_{1,1}$ in (3.3), we set $\rho_1 = 0.5$ and $c = 0.02$, while for $H_{1,2}$ in (3.4) we set $\rho_2 = 0.003$. The choices of parameters $c$ and $\rho_2$ are for better presentation. The empirical powers of the proposed statistic under these two hypothesis and two distributions are reported in Tables 4 and 5. With $n$ and $p$ growing, the powers tend to 1 as expected.

Table 4: Empirical powers of the proposed testing statistic under $H_{1,1}$.

| $x_{ij}$ | $(n\backslash p)$ | $\alpha = 0.05$ | $\alpha = 0.10$ |
| --- | --- | --- | --- |
| | | 20 | 50 | 80 | 100 | 20 | 50 | 80 | 100 |
| $N(0,1)$ | 2000 | 0.2258 | 0.4072 | 0.6084 | 0.7310 | 0.3182 | 0.5114 | 0.7036 | 0.8118 |
| | 5000 | 0.5306 | 0.7180 | 0.8706 | 0.9314 | 0.6292 | 0.8032 | 0.9218 | 0.9640 |
| | 8000 | 0.7854 | 0.9158 | 0.9798 | 0.9914 | 0.8524 | 0.9538 | 0.9916 | 0.9956 |
| | 10000 | 0.8966 | 0.9730 | 0.9944 | 0.9986 | 0.9388 | 0.9874 | 0.9976 | 0.9998 |
| Bernoulli | 2000 | 0.1906 | 0.3782 | 0.6062 | 0.7498 | 0.2764 | 0.4990 | 0.7162 | 0.8306 |
| | 5000 | 0.5162 | 0.7316 | 0.8892 | 0.9416 | 0.6282 | 0.8202 | 0.9336 | 0.9698 |
| | 8000 | 0.8374 | 0.9420 | 0.9864 | 0.9940 | 0.8988 | 0.9672 | 0.9936 | 0.9982 |
| | 10000 | 0.9444 | 0.9862 | 0.9968 | 0.9996 | 0.9708 | 0.9942 | 0.9986 | 0.9998 |

Table 5: Empirical powers of the proposed testing statistic under $H_{1,2}$.

| $x_{ij}$ | $(n\backslash p)$ | $\alpha = 0.05$ | $\alpha = 0.10$ |
| --- | --- | --- | --- |
| | | 20 | 50 | 80 | 100 | 20 | 50 | 80 | 100 |
| $N(0,1)$ | 2000 | 0.1138 | 0.3844 | 0.7646 | 0.9302 | 0.1858 | 0.4924 | 0.8346 | 0.9598 |
| | 5000 | 0.3248 | 0.9566 | 1 | 1 | 0.4162 | 0.9784 | 1 | 1 |
| | 8000 | 0.5960 | 0.9996 | 1 | 1 | 0.6870 | 1 | 1 | 1 |
| | 10000 | 0.7394 | 1 | 1 | 1 | 0.8182 | 1 | 1 | 1 |
| Bernoulli | 2000 | 0.1056 | 0.3890 | 0.7858 | 0.9364 | 0.1668 | 0.5052 | 0.8594 | 0.9658 |
| | 5000 | 0.3200 | 0.9586 | 1 | 1 | 0.4284 | 0.9788 | 1 | 1 |
| | 8000 | 0.5994 | 0.9996 | 1 | 1 | 0.6980 | 1 | 1 | 1 |
| | 10000 | 0.7568 | 1 | 1 | 1 | 0.8300 | 1 | 1 | 1 |
4.3 Non-identity $\Sigma_1$

The ultra-high dimensional scenarios where $p \gg n$ are also of interest in some real applications. We may exchange $A_1$ and $A_2$ under such cases. That is, under the null hypothesis, it’s equivalent to considering $\Sigma_2 = I$ while $\Sigma_1$ is general with $p/n \to 0$. Theorem 2.7 indicates that the CLT still holds at least for Gaussian distributions. In the below, we verify this result with simulated data.

Now we let $A_2 = I$ and $x_{ij} \overset{i.i.d.}{\sim} \mathcal{N}(0,1)$, $p = 100$, $n = p^2$. $A_1$ is generated by $A_1 = UDU^\top$, where $U$ is any real orthogonal matrix and the spectrum of $D$ satisfies

$$
\beta c_1 + (1 - \beta) c_2, \quad c_1 > c_2 > 0.
$$

We try two combinations of $\beta, c_1, c_2$ and calculate the asymptotic means and variances by numerical approximation with Matlab, based on the results in Theorems 2.4 and 2.7. We list the results in the next Table 6. The empirical densities of the testing statistic and the empirical cumulative functions corresponding to standard norm quantiles are presented in Figure 2 and Table 7. These results indicate the central limit theory for non-identity $\Sigma_1$.

| Setting | $\beta$ | $c_1$ | $c_2$ | $p \int f(x) dF_p(x)$ | $\frac{1}{2\pi} \int f(z) \chi_p(z) dz$ | variance |
|---------|--------|------|------|------------------|---------------------------------|---------|
| 1       | 0.5    | 1.2  | 0.8  | 108.16           | -1.2416                         | 6.1670  |
| 2       | 0.3    | 1.2  | 1.0  | 128.14           | -1.3221                         | 7.0634  |

5 Real data analysis: Fama-French $10 \times 10$ portfolios

In this section, we use the proposed CLT and the testing procedure to analyze a real data example. The data set consists of monthly returns of 100 portfolios, which can be freely downloaded from [http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html](http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html). The portfolios can be further categorized into 10 levels of capital sizes and 10 levels of book-to-equity ratios. Hence, it’s naturally structured as $10 \times 10$ matrix-variate series, also known as the Fama-French $10 \times 10$ portfolios. Considering the missing rate, we only use the data from January-1964 to December-2020, with totally 684 months. We impute the missing values (missing rate is 0.25%) by linear interpolation. The data set was
Figure 2: Empirical densities for the proposed testing statistics for non-identity $\Sigma_1$ under two parameter settings. $p = 100$ and $n = p^2$. The red real lines correspond to the standard normal density. Left: Setting 1; Right: Setting 2.

Table 7: Standard normal quantiles ($\Psi(\alpha)$) and the corresponding empirical cumulative functions of the testing statistic for non-identity $\Sigma_1$. $p = 100$, $n = p^2$.

| $\Psi(\alpha)$ | -1.6449 | -1.2816 | -1.0364 | -0.8416 | -0.6745 | -0.5244 | -0.3853 | -0.2533 | -0.1257 | 0.0000 |
|-----------------|--------|--------|--------|--------|--------|--------|--------|--------|--------|-------|
| $\alpha$       | 0.0500 | 0.1000 | 0.1500 | 0.2000 | 0.2500 | 0.3000 | 0.3500 | 0.4000 | 0.4500 | 0.5000 |
| Setting 1       | 0.0532 | 0.1002 | 0.1526 | 0.2034 | 0.2570 | 0.3084 | 0.3600 | 0.4060 | 0.4564 | 0.5052 |
| Setting 2       | 0.0498 | 0.1010 | 0.1500 | 0.2054 | 0.2606 | 0.3088 | 0.3612 | 0.4074 | 0.4604 | 0.5102 |
| $\Psi(\alpha)$ | 0.1257 | 0.2533 | 0.3853 | 0.5244 | 0.6745 | 0.8416 | 1.0364 | 1.2816 | 1.6449 |
| $\alpha$       | 0.5500 | 0.6000 | 0.6500 | 0.7000 | 0.7500 | 0.8000 | 0.8500 | 0.9000 | 0.9500 |
| Setting 1       | 0.5582 | 0.6086 | 0.6610 | 0.7112 | 0.7534 | 0.8034 | 0.8526 | 0.9018 | 0.9476 |
| Setting 2       | 0.5618 | 0.6064 | 0.6544 | 0.6988 | 0.7474 | 0.7928 | 0.8430 | 0.8928 | 0.9484 |
ever analyzed in Wang et al. (2019) and Yu et al. (2020). Following their preprocessing
procedures, we first subtract the monthly market excess returns and then standardize the
series one by one.

As in the preceding example two, we assume a separable structure for the $10 \times 10$
matrix-variate series that

$$Y_t = A_1 X_t A_2, \quad 1 \leq t \leq n, n = 684,$$

where $A_1$ and $A_2$ are $10 \times 10$ matrices representing the row-row and column-column
correlations, respectively. We are interested in testing whether the rows or columns are un-
correlated, i.e., whether $\Sigma_1 = A_1 A_1^\top$ or $\Sigma_2 = A_2 A_2^\top$ is identity matrix.

We illustrate the procedure for testing $\Sigma_1 = I$. To calculate the testing statistics, the
spectrum of $\Sigma_2$ and the fourth moments $\nu_4$ of $X_{t,ij}$ are required. Under the null hypothesis,
since $nq \gg p$, we can roughly estimate $\Sigma_2$ by

$$\hat{\Sigma}_2 \approx \frac{1}{10 \times 684} \sum_{t=1}^{684} Y_t^\top Y_t.$$

Figure 3 plots the estimated $\hat{\Sigma}_2$ and its eigenvalues. On one hand, the off-diagonal entries
are far away from 0, which indicates that $\Sigma_2$ is not identity. On the other hand, the first
eigenvalue of $\hat{\Sigma}_2$ is spiked while the others are almost the same. Hence, we assume the
spectral distribution of $\Sigma_2$ to be

$$0.1 \delta_{\hat{\lambda}_1} + 0.9 \delta_{\bar{\lambda} - 1},$$

where $\hat{\lambda}_1 = 5.4$ is the largest eigenvalue of $\hat{\Sigma}_2$ and $\bar{\lambda} - 1 = 0.51$ is the mean of the left
eigenvalues.

Similarly, we can also estimate $\Sigma_1$ by a parallel procedure, which is plotted in Figure
4. Indeed, Figure 4 implies clearly $\Sigma_1$ shall not be identity. However, a statistically formal
test is still necessary in order to know how much we can believe in this conclusion. For $\nu_4$, a
natural estimator is

$$\hat{\nu}_4 = \frac{1}{10 \times 10 \times 684} \sum_{t=1}^{684} \sum_{i=1}^{10} \sum_{j=1}^{10} W_{t,ij}^4 = 11.2, \quad \text{where } W_t := \hat{\Sigma}_1^{-1/2} Y_t \hat{\Sigma}_2^{-1/2}.$$

It’s seen that $\hat{\nu}_4$ is much larger than 3, which is due to the heavy-tailed characteristic of
real financial data set.
(a) heatmap of $\hat{\Sigma}_2$

(b) Eigenvalues of $\hat{\Sigma}_2$

Figure 3: Estimated $\Sigma_2$ (left) for the Fama-French data set and its eigenvalues (right).

(a) heatmap of $\hat{\Sigma}_1$

(b) Eigenvalues of $\hat{\Sigma}_1$

Figure 4: Estimated $\Sigma_1$ (left) for the Fama-French data set and its eigenvalues (right).
Based on the argument in Section 3 and numerical approximation, the statistic for testing $\Sigma_1 = \mathbf{I}$ is equal to 7373.692. Hence, we reject the null hypothesis and believe $\Sigma_1 \neq \mathbf{I}$ confidently, which is consistent with the estimator in Figure 4. On the other hand, we can also test $\Sigma_2 = \mathbf{I}$ with a similar procedure. The testing statistics is equal to 7817.638, which suggests significant rejection with a very small P-value.

6 Conclusions and discussions

This paper investigates limiting distributions of linear spectral statistics for normalized separable sample covariance matrix. The central limit theory is provided when the dimension tends to infinity with a slower rate than the sample size, with explicit formulas for the mean and covariance functions. We apply the results to testing the correlations of a large number of variables in spatial-temporal model and matrix-variate model. We propose new testing statistics, and study the asymptotic powers under common alternative hypothesis. The numerical studies verify the usefulness of our approach.

We are interested in extending the results of the current paper in the following two directions for future work. First, the current settings exclude spiked models where some eigenvalues of $\mathbf{A}$ or $\mathbf{B}$ can be well separated from the others. As shown by our real data analysis, the spiked case is very common in real world. In spiked models, the spiked sample eigenvalues and non-spiked ones usually show different convergence rates which should be treated separately. We leave this as one of our future work, especially for the matrix-variate models, which are closely-related to the matrix-valued factor models with applications in econometrics and finance. Second, for non-diagonal $\mathbf{A}$, the current theorem requires that the fourth moments of $x_{ij}$ are equal to 3. This condition is from a comparison of characteristic functions between Gaussian case and general scenario in the technical proof. The generalization from Gaussian to general distributions is an essential challenge in random matrix theory, and deserves independent study.

7 Supplementary material

All the technical proofs of the mentioned theorems, corollaries and lemmas are organized in our supplementary material.

Supplement to “Central Limit Theory for Linear Spectral Statistics of Normalized Separable Sample Covariance Matrix”
This supplementary material provides technical proofs of all the theorems, corollaries and lemmas appeared in the main paper. There are six sections. Section A provides an outline of the proof. Section B truncates the entries $x_{ij}$ and proves Lemma 2.3, corresponding to the upper bound for the spectral norm of $\overline{S}_p$. Sections C and D are devoted to the proof of Theorem 2.4, that is, the convergence of the Stieltjes transform process $M_p(z)$, which is the most important result in the paper. We decompose $M_p(z)$ into random part and non-random part, which are studied in the two sections respectively. Section E completes the proof of CLT for the LSS in Theorem 2.7 and all the remaining corollaries in Section 2. Section F proves the results in Section 3.

A Outline of the proof

The major target of this supplementary material is to prove the CLT of LSS in Theorem 2.7 of the main paper. As claimed in the main paper, based on the Cauchy integral formula, with probability tending to 1,

$$G_p(f) = -\frac{1}{2\pi i} \oint_C M_p(z)dz.$$ 

Hence, we are more concerned with the Stieltjes transform process $M_p(z)$. Theorem 2.4 shows the convergence of $M_p(z)$ on the contour $C_u$ with diagonal $A_p$. Therefore, the proof is divided into two parts. The first part proves Theorem 2.4, organized in Sections B to D. The second part is in Section E, which considers contours $C_{l,r,0}$ and general $A_p$.

To prove Theorem 2.4, we decompose $M_p(z)$ into random part and non-random part as follows,

$$M_p(z) = p[m_Fs_p(z) - E m_Fs_p(z)] + p[E m_Fs_p(z) - m_p(z) - X_p(z)] = M_{p1}(z) + M_{p2}(z).$$

The random part $M_{p1}(z)$ mainly contributes to the covariance function while the non-random part $M_{p2}(z)$ mainly contributes to the mean correction term $X_p(z)$.

To deal with $M_{p1}(z)$, we expand the Stieltjes transform and observe that the randomness mainly comes from a sum of martingale as in equation (C.8). With the decomposition in this equation, we easily verify the finite-dimensional distribution of $M_{p1}(z)$ and specify the covariance function, following a similar expansion strategy as in Chen & Pan (2015). Finally, we finish the proof for $M_{p1}(z)$ by verifying the tightness. The details are put in Section C.
For $M_{p,q}(z)$, we first expand the diagonal entries of $m_{F,S_p}(z)$ with Schur’s complement formula. It’s seen that the diagonal entries will converge to the terms $\tilde{e}_k(z)$ defined in Theorem 2.4. However, the convergence rates are slow, so we need more detailed calculations of the errors to specify those which are not smaller than $O(p^{-1})$. The proof procedure is more complicated than that in Chen & Pan (2015), where $E_{m_{F,S_p}(z)}$ can be explicitly solved by a quadratic equation of one variable. With general $A_p$ and $B_p$, we find that the equation therein is not correct anymore. Instead, we expand the diagonal entries of the Stieltjes transform using the inverse matrix formula recursively. After each expansion step, we remove some ”big” errors by rigorous approximation. Finally, we sum up all these non-negligible errors, which lead to the rmean correction term $X_p(z)$ in Theorem 2.4.

To complete the proof of Theorem 2.7, we first prove that the integrals corresponding to $C_{l,r,0}$ are negligible. Since the event $U_p(\epsilon_0) := \{ \| S_p \| \leq 2 \limsup a_1 b_1 + \epsilon_0/2 \}$ happens with probability tending to 1 for large $p$, we aim to prove that under this event

$$\lim_{v_0 \downarrow 0} \lim_{p \to \infty} E \left| \int_{C_{l,r}} f(z) M_p(z) dz \right| \to 0,$$

which is verified in Section E. Moreover, for non-diagonal $A_p$, note that the theorem still holds if $x_{ij}$ are i.i.d. standard normal variables, due to the orthogonal invariance property. For non-Gaussian variables, we use the Lindeberg replacement technique and compare the characteristic functions of LSS with Gaussian and non-Gaussian distributions. Under the additional constraint $\nu_4 = 3$, we can replace standard normal variables with general $x_{ij}$, which has negligible effects on the limiting distribution of the LSS. Then, Theorem 2.7 holds.

What remains is to verify the results in Section 3, i.e., the asymptotic distribution under the null and the asymptotic powers under the two alternative hypothesis. The results under the null hypothesis follow from Theorem 2.7 if the asymptotic variance in Theorem 3.1 holds. Theorem 3.1 is actually an extension of the results in Bai & Yin (1988). The only difference is that the ESD in our model converges to a rescaled semicircle law, which is dependent on the spectrum of $B_n$. For the alternative hypothesis, we actually prove a more general result, i.e., the power converges to 1 as long as $p^{-1} \text{tr}(A_p - I_p)^2 \geq \epsilon_0$ for some $\epsilon_0 > 0$ and sufficiently large $p$. Then, the theorem holds because the two alternative hypothesis satisfy this condition.
B Truncation and the proof of Lemma 2.3

In this section, we first truncate the random variables $x_{ij}$ by $\delta_p \sqrt{np}$, which helps in controlling some higher-order moments of $x_{ij}$ in the proof. Then, we prove Lemma 2.3, which provides preliminary upper and lower bounds for the eigenvalues of $\bar{S}_p$. In the proof, we may suppress the dependence on $p$ and $z$ if it doesn’t cause any confusion.

B.1 Truncation

We follow the truncation technique in Chen & Pan (2012) and Chen & Pan (2015). By Condition 2.1, there is a series of $\delta_p$s satisfying

$$\lim_{p \to \infty} \delta_p^{-4} \mathbb{E}|x_{11}|^4 I(|x_{11}| > \delta_p \sqrt{np}) = 0,$$

$$\delta_p \downarrow 0, \delta_p \sqrt{np} \uparrow \infty.$$  

Define the truncated variables as $\hat{x}_{ij} = x_{ij} I(x_{ij} \leq \delta_p \sqrt{np})$, and the standardized truncated variables as $\tilde{x}_{ij} = (\hat{x}_{ij} - \mathbb{E}\hat{x}_{ij})/\sigma$, where $\sigma^2$ is the variance of $\hat{x}_{ij}$. Let $\hat{X} = (\hat{x}_{ij})$, $\tilde{X} = (\tilde{x}_{ij})$, $\hat{S}_p$ and $\tilde{S}_p$ be the normalized separable sample covariance matrix by replacing $X$ with $\hat{X}$ and $\tilde{X}$, respectively. Then, similarly to the proof of Theorem 1 in Chen & Pan (2012), $\mathbb{P}(\bar{S}_p \neq \hat{S}_p, i.o.) = 0$ by choosing a proper series of $\delta_p$. Hence, below we aim to control the effects by replacing $\hat{X}$ with $\tilde{X}$.

Some elementary calculations yield that

$$|\mathbb{E}\tilde{x}_{ij}| = |\mathbb{E}x_{ij} - \mathbb{E}x_{ij} I(|x_{ij}| > \delta_p \sqrt{np})| \leq \mathbb{E}|x_{ij}| I(|x_{ij}| > \delta_p \sqrt{np})$$  

$$\leq (\delta_p \sqrt{np})^{-3} \mathbb{E}|x_{11}|^4 I(|x_{11}| > \delta_p \sqrt{np}) = o((np)^{-3/4}),$$  

$$|1 - \sigma^2| \leq \mathbb{E}|x_{11}^2| I(|x_{11}| > \delta_p \sqrt{np}) \leq o((np)^{-1/2}).$$  

Then, by the Wely’s theorem,

$$|G_p(f, \tilde{X}) - G_p(f, \hat{X})| \leq \sum_{j=1}^p |f(\lambda_j(\hat{S}_p)) - f(\lambda_j(\tilde{S}_p))| \leq C_f \sum_{j=1}^p |\lambda_j(\hat{S}_p) - \lambda_j(\tilde{S}_p)|$$

$$\leq C_f \times p \|\hat{S}_p - \tilde{S}_p\| \leq C_f \times \sqrt{\frac{p}{n}} \|A\| \left(\|\tilde{X} - \hat{X}\|B\hat{X}^\top + \|(\tilde{X} - \hat{X})B\hat{X}^\top\|\right).$$  

Note that by the i.i.d. assumption,

$$\hat{X} - \tilde{X} = \frac{\sigma - 1}{\sigma} \hat{X} + \frac{1}{\sigma} \mathbb{E}\hat{x}_{11} 1_p 1_n^\top.$$

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Therefore,
\[
\sqrt{\frac{p}{n}} \|(\hat{X} - \tilde{X})B\hat{X}^\top\| \leq \frac{\sigma - 1}{\sigma} \|B\| \times \sqrt{mp} \times \left\| \frac{1}{n} \hat{X} \hat{X}^\top \right\| + \sqrt{\frac{p}{n}} \times \frac{1}{\sigma} \|E\hat{x}_{11}\| \times \sqrt{p} \times O_p(\sqrt{mp}) \\
\leq o_p(1),
\]
where the \(o_p(1)\) is by (B.1), (B.3), \(p/n \rightarrow 0\) and the fact \(n^{-1} \hat{X}\hat{X}^\top = O_{a.s.}(1)\). Similarly, we can prove
\[
\sqrt{\frac{p}{n}} \|(\hat{X} - \tilde{X})B\tilde{X}^\top\| = o_p(1).
\]
Consequently, the asymptotic distributions of \(G_p(f, \tilde{X})\) and \(G_p(f, \hat{X})\) are the same. Similarly, the asymptotic distributions of \(M_p(z, \tilde{X})\) and \(M_p(z, \hat{X})\) are also the same. On the other hand, by (B.4)
\[
\frac{1}{\sqrt{np}} \|(\hat{X} - \tilde{X})B\hat{X}^\top\| \leq \frac{C}{\sqrt{mp}} \|A\| \left( \left\| (\hat{X} - \tilde{X})B\hat{X}^\top \right\| + \left\| (\hat{X} - \tilde{X})B\tilde{X}^\top \right\| \right).
\]
By (B.5),
\[
\frac{1}{\sqrt{np}} \|(\hat{X} - \tilde{X})B\hat{X}^\top\| \leq \frac{\sigma - 1}{\sigma} \|B\| \times \sqrt{\frac{n}{p}} \times \left\| \frac{1}{n} \hat{X} \hat{X}^\top \right\| + \frac{1}{\sqrt{mp}} \times \frac{1}{\sigma} \|E\hat{x}_{11}\| \times \sqrt{mp} \times O_{a.s.}\left( (np)^{3/4} \right) \\
= o_{a.s.}(1),
\]
where we use \(|\hat{x}_{ij}| \leq \delta_p \sqrt{np}\) for the term \(1_n^\top \hat{X}^\top\). Similarly,
\[
\frac{1}{\sqrt{np}} \|(\hat{X} - \tilde{X})B\tilde{X}^\top\| = o_{a.s.}(1).
\]
In conclusion, replacing \(X\) with \(\tilde{X}\) will have negligible effects on the LSS and eigenvalues. Hence, we will focus on \(\tilde{X}\) rather than \(X\) in the proof. For simplicity, we still write \(X\) but assume that
\[
|x_{ij}| \leq \delta_p \sqrt{np}, \quad \mathbb{E}x_{ij} = 0, \quad \mathbb{E}x_{ij}^2 = 1, \quad \mathbb{E}x_{ij}^4 = \nu_4 + o(1). \tag{B.6}
\]
B.2 The proof of Lemma 2.3

We prove Lemma 2.3 here, which gives a rough bound for the support of $F\bar{S}_p$. By definition of $\bar{S}_p$,

$$\|\bar{S}_p\| \leq \left\| \frac{1}{\sqrt{np}}(A^{1/2}XBX^\top A^{1/2} - n\bar{\lambda}_B A) \right\| \leq \left\| A \right\| \left\| \frac{1}{\sqrt{np}}(XBX^\top - n\bar{\lambda}_B) \right\|$$

$$\leq a_1 \left( \max_i \frac{1}{\sqrt{np}}|x_i^\top Bx_i - n\bar{\lambda}_B| + \left\| \frac{1}{\sqrt{np}}B_x^0 \right\| \right),$$

where $B_x^0$ is defined by

$$(B_x^0)_{ij} = \begin{cases} 0, & i = j, \\ x_i^\top Bx_j, & i \neq j. \end{cases}$$

We first deal with $B_x^0$. For simplicity, we suppress the index $x$. Let $e_i$ be the $p$-dimensional vector with the $i$-th entry being 1 and the others being 0. Then,

$$\left\| \frac{1}{\sqrt{np}}B_0^0 \right\|^2 = \sup_{\|\xi\| = 1} \frac{1}{np}\|B_0^0\xi\| = \sup_{\|\xi\| = 1} \sum_{i=1}^p \frac{1}{np} \left( x_i^\top (X^\top - x_i e_i^\top)\xi \right)^2$$

$$= \sup_{\|\xi\| = 1} \sum_{i=1}^p \frac{1}{np} \left\| B(X^\top - x_i e_i^\top)\xi x_i^\top \right\|^2 \leq \sup_{\|\xi\| = 1} \sum_{i=1}^p \frac{1}{np} \|B\|^2 \left\| (X^\top - x_i e_i^\top)\xi x_i^\top \right\|^2$$

$$\leq \sup_{\|\xi\| = 1} \|B\|^2 \sum_{i=1}^p \frac{1}{np} \left( x_i^\top (X^\top - x_i e_i^\top)\xi \right)^2 = \|B\|^2 \times \left\| \frac{1}{\sqrt{np}}\tilde{B}_x^0 \right\|^2$$

where $\tilde{B}_x^0$ is defined by

$$(\tilde{B}_x^0)_{ij} = \begin{cases} 0, & i = j, \\ x_i^\top x_j, & i \neq j. \end{cases}$$

By the proof of Theorem 2 in Chen & Pan (2012), we conclude that almost surely

$$\|(np)^{-1/2}\tilde{B}_x^0\| \leq 2. \quad \text{(B.8)}$$

Now we come back to the first term in (B.7) and show that it’s $o_{a.s.}(1)$. Write

$$\frac{1}{\sqrt{np}}(x_i^\top Bx_i - n\bar{\lambda}_B) = \frac{1}{\sqrt{np}} \sum_{j=1}^n B_{jj}(x_{ij}^2 - 1) + \frac{1}{\sqrt{np}} \sum_{j \neq l} B_{jl}x_{ij}x_{il}. \quad \text{(B.9)}$$
Since $\|B\|$ is bounded and $x_{ij}$ are i.i.d., a similar technique to proving equation (9) in Chen & Pan (2012) leads to
\[
\max_i \frac{1}{\sqrt{np}} \left| \sum_{j=1}^{n} B_{jj} (x_{ij}^2 - 1) \right| = o_{a.s.}(1). \tag{B.10}
\]
Hence, we only focus on the second term of (B.9). Using Lemma 5 in Pan & Zhou (2011), for any $j \leq n$ and $k \geq 2$,
\[
\mathbb{P} \left( \max_i \frac{1}{\sqrt{np}} \left| \sum_{l \neq j}^{n} B_{jl} x_{ij} x_{il} \right| > \epsilon \right) \leq p \mathbb{E} \left[ \sum_{l \neq j}^{n} B_{jl} x_{ij} x_{il} \right] \leq C_{k,\epsilon} p \|B\|^k / (np)^{k/2}.
\]
Take $k = 4 + \delta_0$ to get
\[
\mathbb{P} \left( \max_i \frac{1}{\sqrt{np}} \left| \sum_{l \neq j}^{n} B_{jl} x_{ij} x_{il} \right| > \epsilon \right) \leq \frac{C_{\epsilon}}{p^{1+\delta_0/2}}.
\]
Since $\delta_0 > 0$, by the Borel-Cantelli theorem and the fact $\sum_{p=1}^{\infty} p^{-1-\delta_0/2} < \infty$, we conclude that
\[
\max_i \frac{1}{\sqrt{np}} \left| \sum_{l \neq j}^{n} B_{jl} x_{ij} x_{il} \right| = o_{a.s.}(1). \tag{B.11}
\]
The lemma then follows from (B.8), (B.9), (B.10) and (B.11). The next two sections are devoted to the proof of Theorem 2.4.

C \quad The random part with diagonal $A_p$

We rewrite $M_p(z)$ as the sum of random part $M_{p1}(z)$ and non-random part $M_{p2}(z)$ by
\[
M_p(z) = p[m_F s_p(z) - \mathbb{E} m_F s_p(z)] + p[m_F s_p(z) - m_p(z) - \mathcal{X}_p(z)] := M_{p1}(z) + M_{p2}(z).
\]
Then, Theorem 2.4 can be concluded with detailed analysis of the random part and non-random part. In this section, we focus on the random part and assume $A$ is diagonal. The proof in this part is adapted from Chen & Pan (2015) where both $A$ and $B$ are identity matrices. The differences lie in the following two aspects. First, after expanding the Stieltjes transform by deleting each row of $X$, there will be a coefficient corresponding to $a_k$ since the diagonal entries of $A$ are no more identical. Second, we will have more error terms at some steps which are from the off-diagonal entries of $B$. 

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C.1 Expansion of the Stieltjes transform

We first introduce some notations. Write \( Y = A^{1/2}XB^{1/2} \). Let \( X_k \) be the \((p - 1) \times n\) matrix after removing \( x_k^\top \) from \( X \), where \( x_k^\top \) is the \( k \)-th row vector of \( X \). Similarly, let \( y_k^\top \) be the \( k \)-th row of \( Y \), \( Y_k \) be the \((p - 1) \times n\) matrix after removing the \( k \)-th row from \( Y \), and \( S_k = \frac{1}{\sqrt{np}}(Y_kY_k^\top - n\bar{\lambda}_BA_k) \), where \( A_k \) is obtained by deleting the \( k \)-th row and column in \( A \). Let \( D := S - zI_p \) and \( D_k = \tilde{S}_k - zI_{p-1} \), where \( z = u + iv \in \mathbb{C}_u \). Note that the \( k \)-th diagonal entry of \( D \) is \( d_k := \frac{1}{\sqrt{np}}(y_k^\top y_k - n\bar{\lambda}_A k) - z \) and the \( k \)-th row of \( D \) with the \( k \)-th element deleted is \( q_k^\top := \frac{1}{\sqrt{np}}y_k^\top Y_k^\top \).

Define the \( \sigma \)-field generated by \((x_1, \ldots, x_k)\) as \( \mathcal{F}_k \) and the conditional expectation \( \mathbb{E}_k(\cdot) := \mathbb{E}(\cdot | \mathcal{F}_k) \). By Shur’s complement formula and the inverse formula, \( \frac{1}{\sqrt{np}}y_k^\top Y_k^\top = \frac{1}{\sqrt{np}}(y_k^\top y_k - n\bar{\lambda}_A k) - z \),

\[
\begin{align*}
\text{tr}(D^{-1}) &= \frac{1}{d_k - q_k^\top D_k^{-1}q_k} + \text{tr}(D_k - d_k^{-1}q_kq_k^\top)^{-1}, \\
\text{tr}(D_k - d_k^{-1}q_kq_k^\top)^{-1} &= \text{tr}(D_k^{-1}) + d_k^{-1}q_k^\top D_k^{-1}(D_k - d_k^{-1}q_kq_k^\top)^{-1}q_k, \\
d_k^{-1}q_k^\top D_k^{-1}(D_k - d_k^{-1}q_kq_k^\top)^{-1}q_k &= \frac{q_k^\top D_k^{-2}q_k}{1 - d_k^{-1}q_k^\top D_k^{-1}q_k}.
\end{align*}
\]

Therefore, we have

\[
\text{tr}(D^{-1} - D_k^{-1}) = -\frac{1 + q_k^\top D_k^{-2}q_k}{-d_k + q_k^\top D_k^{-1}q_k}, \quad \text{(C.4)}
\]

As a result,

\[
p[m_{\mathcal{F}_k} - \mathbb{E}(m_{\mathcal{F}_k})] = \sum_{k=1}^p (\mathbb{E}_k - \mathbb{E}_{k-1})\text{tr}(D^{-1} - D_k^{-1}) = \sum_{k=1}^p \rho_k \quad \text{(C.5)}
\]

where

\[
\rho_k := -(\mathbb{E}_k - \mathbb{E}_{k-1})\beta_k(1 + q_k^\top D_k^{-2}q_k), \quad \beta_k := \frac{1}{-d_k + q_k^\top D_k^{-1}q_k},
\]

\[
\zeta_k := -\tilde{\beta}_k\beta_k\eta_k(1 + q_k^\top D_k^{-2}q_k),
\]

\[
\tilde{\beta}_k := \frac{1}{z + \frac{1}{np}\text{tr}M_k^{(s)}}, \quad M_k^{(s)} := a_kB^{1/2}Y_k^\top D_k^{-s}Y_kB^{1/2}, s = 1, 2,
\]

\[
\eta_k := \frac{1}{\sqrt{np}}(y_k^\top y_k - n\bar{\lambda}_A k) - \gamma_{k1},
\]

\[
\gamma_{ks} := q_k^\top D_k^{-s}q_k - (np)^{-1}\text{tr}M_k^{(s)}, s = 1, 2, \quad \kappa_k := \tilde{\beta}_k\gamma_{k2}.
\]
Note that in the above derivation we use
\[ \beta_k = \tilde{\beta}_k + \beta_k \eta_k \tilde{\beta}_k. \] (C.6)

We now provide some useful bounds for further use. First, since the eigenvalues of \( D \) have the form of \( 1/(\lambda_j(\bar{S}) - z) \), considering the imaginary part we have \( \|D^{-1}\| \leq v^{-1} \). Similarly \( \|D_k^{-1}\| \leq v^{-1} \). Besides, note that \( \beta_k \) is the \( k \)-th diagonal entry of \( D \), hence \( |\beta_k| \leq v^{-1} \). By symmetry and the eigenvalue decomposition of \( D_k^{-1} \), it’s also easy to see that all the diagonal entries of \( M_k(1) \) have positive imaginary part, then \( |\tilde{\beta}_k| \leq v^{-1} \). Moreover, let \( \Gamma = \{\xi_1, \ldots, \xi_{p-1}\} \) and \( \Theta = \text{diag}(\theta_1, \ldots, \theta_{p-1}) \) be the eigenvectors and eigenvalues of \( \bar{S}_k \). Then,

\[
\begin{align*}
\frac{1}{np} \text{tr} M_k(1) &= \left| \frac{1}{np} \text{tr} B^{1/2} Y_k^\top \Gamma (\Theta - z I_{p-1})^{-1} \Gamma^\top Y_k B^{1/2} \right| \\
&= \left| \frac{1}{np} \text{tr} \left( B^{1/2} Y_k^\top \Gamma (\Theta - z)^{-1} (\Theta - \bar{z})^{-1} (\Theta - u) \Gamma^\top Y_k B^{1/2} \right) + iv \times B^{1/2} Y_k^\top \Gamma (\Theta - z)^{-1} (\Theta - \bar{z})^{-1} \Gamma^\top Y_k B^{1/2} \right| \\
&\leq \frac{1}{np} \text{tr} \left( B^{1/2} Y_k^\top (D_k^{-1}(z)D_k^{-1}(\bar{z}))^{1/2} Y_k B^{1/2} \right) \\
&\leq \frac{b_1}{np} \text{tr} \left( Y_k^\top (D_k^{-1}(z)D_k^{-1}(\bar{z}))^{1/2} Y_k \right). \\
\end{align*}
\] (C.7)

Note that for sufficiently large \( p \),

\[
\begin{align*}
\frac{1}{np} \text{tr} \left( Y_k Y_k^\top (D_k^{-1}(z)D_k^{-1}(\bar{z}))^{1/2} \right) &= \frac{1}{np} \text{tr} \left( \sqrt{np}(D_k(z) + z I_{p-1}) + n \bar{\lambda}_B A_k \right) (D_k^{-1}(z)D_k^{-1}(\bar{z}))^{1/2} \\
&= \frac{1}{np} \text{tr} \left( \sqrt{np}(D_k(z)D_k^{-1}(\bar{z}))^{1/2} + z \sqrt{np}(D_k^{-1}(z)D_k^{-1}(\bar{z}))^{1/2} + n \bar{\lambda}_B A_k (D_k^{-1}(z)D_k^{-1}(\bar{z}))^{1/2} \right) \\
&\leq C \left( \sqrt{\frac{p}{n}} + 1 \right) \leq C.
\end{align*}
\]

With a similar technique applied to \( M_k(2) \), we conclude

\[
\left| \frac{1}{np} \text{tr} M_k(1) \right| \leq C, \quad \left| \frac{1}{np} \text{tr} M_k(2) \right| \leq C.
\]
On the other hand,
\[
|\mathbf{1} + \mathbf{q}_k^\top \mathbf{D}_k^{-2} \mathbf{q}_k| \beta_k \leq (1 + |\mathbf{q}_k^\top \mathbf{D}_k^{-2} \mathbf{q}_k|) \times |\beta_k|
\]
\[
\leq \left(1 + \sum_{j} \frac{|q_k^\top \xi_j|}{|\theta_j - z|^2}\right) \times \frac{1}{3(-d_k + \mathbf{q}_k^\top \mathbf{D}_k^{-1} \mathbf{q}_k)}
\]
\[
= \left(1 + \sum_{j} \frac{|q_k^\top \xi_j|}{|\theta_j - z|^2}\right) \times \frac{1}{v\left(1 + \sum_{j} \frac{|q_k^\top \xi_j|^2}{|\theta_j - z|^2}\right)}
\]
\[
\leq v^{-1}.
\]

Using (C.6), we further decompose \(\zeta_k\) into
\[
\zeta_k = -\tilde{\beta}^2 \eta_k [1 + (np)^{-1} \text{tr} \mathbf{M}_k^{(2)}] - \tilde{\beta}^2 \eta_k \gamma_k - \tilde{\beta}^2 \eta_k^2 (1 + \mathbf{q}_k^\top \mathbf{D}_k^{-2} \mathbf{q}_k)
\]
\[
:= \zeta_{k1} + \zeta_{k2} + \zeta_{k3}.
\]

To this end, we should first show the following lemma on large deviation bounds.

**Lemma C.1.** For \(z = u + iv\) with \(v > 0\), under Condition 2.1 and (B.6), we have
\[
\mathbb{E} |\gamma_{ks}|^2 \leq Cp^{-1}, \quad \mathbb{E} |\gamma_{ks}|^4 \leq C\left(\frac{1}{p^2} + \frac{p}{n^2} + \frac{1}{np}\right),
\]
\[
\mathbb{E} |\eta_k|^2 \leq Cp^{-1}, \quad \mathbb{E} |\eta_k|^4 \leq C\left(\frac{\delta_p^4}{p} + \frac{1}{p^2} + \frac{p}{n^2} + \frac{1}{np}\right),
\]
where \(\delta_p \to 0\) slowly is the truncation parameter.

The proof of Lemma C.1 is given in the next subsection. With this lemma, the Burkholder inequality and the Cauchy-Schwartz inequality, we have
\[
\mathbb{E} \left|\sum_{k=1}^{p} (\mathbb{E}_{k} - \mathbb{E}_{k-1}) \zeta_{k3}\right|^2 \leq \sum_{k=1}^{p} \mathbb{E} \left|\tilde{\beta}_k^2 \beta_k \eta_k^2 (1 + \mathbf{q}_k^\top \mathbf{D}_k^{-2} \mathbf{q}_k)\right|^2 \leq C\delta_p^4.
\]

Similarly,
\[
\mathbb{E} \left|\sum_{k=1}^{p} (\mathbb{E}_{k} - \mathbb{E}_{k-1}) \zeta_{k2}\right|^2 \leq \sum_{k=1}^{p} \mathbb{E} \left|\tilde{\beta}_k^2 \beta_k \gamma_k\right|^2 \leq C\left(\delta_p^2 + \frac{p}{n}\right).
\]

Consequently, we can write
\[
p[m_{Fk}(z) - \mathbb{E}(m_{Fk}(z))] = \sum_{k=1}^{p} \mathbb{E}_k \left[\mathbf{1} + \frac{1}{np} \text{tr} \mathbf{M}_k^{(2)}\right] \tilde{\beta}_k^2 \beta_k \gamma_k + o_{L^2}(1)
\]
\[
:= \sum_{k} \mathbb{E}_k (\alpha_k(z)) + o_{L^2}(1).
\]

(C.8)
It’s easy to see that $\mathbb{E}_{k-1} \alpha_k(z) = 0$. Then, for the random part, it’s sufficient to prove the finite-dimensional convergence and the tightness of $M_{p1}(z)$. For the finite-dimensional convergence, we only need to consider the sum

$$
\sum_{j=1}^l \xi_j \sum_{k=1}^p \mathbb{E}_k(\alpha_k(z_j)) = \sum_{k=1}^p \left( \sum_{j=1}^l \xi_j \mathbb{E}_k(\alpha_k(z_j)) \right),
$$

where $\xi_1, \ldots, \xi_j$ are complex numbers and $l$ is a positive integer. Before moving forward, we first give the proof of Lemma C.1 in the next subsection.

C.2 Proof of Lemma C.1

Proof. Note that $y_k^\top y_k - n \bar{\lambda} B a_k = a_k \left[ \sum_{j=1}^n B_{jj} (x_{kj}^2 - 1) + \sum_{j \neq l} B_{jl} x_{kj} x_{kl} \right]$, hence by independence,

$$
\mathbb{E} \left| \frac{1}{\sqrt{np}} \left( y_k^\top y_k - n \bar{\lambda} B a_k \right) \right|^2 \leq \frac{C}{p},
$$

$$
\mathbb{E} \left| \frac{1}{\sqrt{np}} \left( y_k^\top y_k - n \bar{\lambda} B a_k \right) \right|^4 \leq \mathbb{E} \left| \frac{C}{\sqrt{np}} \left[ \sum_{j=1}^n b_j (x_{kj}^2 - 1) \right] \right|^4 + C \mathbb{E} \left| \frac{1}{\sqrt{np}} \sum_{i \neq j} B_{ij} x_{ki} x_{kj} \right|^4 \leq \frac{C}{n^2 p^2} \left( \sum_{j=1}^n (x_{kj}^2 - 1)^2 \right)^2 + \frac{C}{n^2 p^2} (\mathbb{E} x_{11}^4)^2 \|B\|_F^4 \leq \frac{C}{n^2 p^2} \left( \sum_{j=1}^n \mathbb{E} x_{kj}^4 \right)^2 + \frac{C}{n^2 p^2} \sum_{j=1}^n \mathbb{E} x_{jk}^8 + \frac{C}{p^2} \leq C \left( \frac{1}{p^2} + \frac{\delta^4}{p} \right),
$$

where for the third inequality we use Burkholder’s inequality and Lemma 5 in Pan & Zhou (2011). On the other hand, note that

$$
q_k^\top D_k^{-s} q_k - \frac{1}{np} \text{tr}(M_k^{(s)}) = \frac{1}{np} \left( x_k^\top M_k^{(s)} x_k - \text{tr}(M_k^{(s)}) \right)
$$

$$
:= \frac{1}{np} x_k^\top H x_k + \frac{1}{np} \sum_{j=1}^n M_{jj}^{(s)} (x_{kj}^2 - 1),
$$

where $H = M_k^{(s)} - \text{diag}(M_{11}^{(s)}, \ldots, M_{nn}^{(s)})$ and $M_{jj}^{(s)}$ is the $j$-th diagonal entry of $M_k^{(s)}$. Let $B_j^\top$ be the $j$-th row of $B$, then it’s not hard to verify

$$
\mathbb{E} |M_{jj}^{(s)}|^4 \leq C \mathbb{E} \|B_j^\top X_k A_k D_k^{-s} A_k X_k B_j\|^4 \leq C \mathbb{E} \left( \sum_l \|B_{jl} x_{(k),l}\|^2 \right)^4 = C \mathbb{E} \left( \sum_l B_{jl}^2 \|x_{(k),l}\|^2 \right)^4,
$$

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where \( x_{(k),l} \) is the \( l \)-th column of \( X_k \). Then, by Burkholder’s inequality,

\[
\mathbb{E}|M_{jj}^{(s)}| \leq C \mathbb{E}
\left( \sum_l (B_{jl}^2 \|x_{(k),l}\|)^2 \right)^2 = C \left( \sum_l \mathbb{E}(B_{jl}^2 \|x_{(k),l}\|)^2 \right)^2 + C \sum_l \mathbb{E}(B_{jl}^2 \|x_{(k),l}\|)^4
\]

\[
\leq C p^4 \left( \sum_l B_{jl}^4 \right)^2 + C \left( \sum_l B_{jl}^8 \mathbb{E}\|x_{(k),1}\|^8 \right) \leq C (p^4 + np^2),
\]

where we use the facts that

\[
\sum_l B_{jl}^{2h} \leq \sum_l B_{jl}^2 \times \|B\|^{2h-2} \leq \|B\|^{2h},
\]

and

\[
\mathbb{E}\|x_{(k),1}\|^8 \leq \left( \sum_h \mathbb{E}|x_{(k),th}|^4 \right)^2 + \sum_h \mathbb{E}|x_{(k),th}|^8 \leq C (p^2 + \delta^4 np^2).
\]

Let \( E_{jk}(\cdot) = \mathbb{E}(\cdot | x_{k1}, \ldots, x_{kj}) \), then by Burkholder’s inequality and the independence between \( M_k^{(s)} \) and \( x_k \), we have

\[
\mathbb{E} \left| \frac{1}{np} \sum_{j=1}^n M_{jj}^{(s)}(x_{kj}^2 - 1) \right|^2 \leq \frac{C}{n^2 p^2} \sum_{j=1}^n \mathbb{E} \left| M_{jj}^{(s)}(x_{kj}^2 - 1) \right|^2 \leq \frac{C}{p},
\]

\[
\mathbb{E} \left| \frac{1}{np} \sum_{j=1}^n M_{jj}^{(s)}(x_{kj}^2 - 1) \right|^4 \leq \frac{C}{n^4 p^4} \left( \sum_{j=1}^n \mathbb{E}|M_{jj}^{(s)}(x_{kj}^2 - 1)|^2 \right)^2 + \frac{C}{n^4 p^4} \sum_{j=1}^n \mathbb{E}|M_{jj}^{(s)}|^4 (x_{kj}^2 - 1)^4 \leq C \left( \frac{1}{np} + \frac{p}{n^2} \right).
\]

It remains to deal with \( x_k^\top H x_k \). By Lemma 5 in Pan & Zhou (2011), for any \( h \geq 2 \),

\[
\mathbb{E}|x_k^\top H x_k|^h \leq C (\mathbb{E}|x_{11}|^h)^2 \mathbb{E}(\text{tr} HH^\top)^{h/2}.
\]

Let \( h = 2 \), then

\[
\mathbb{E}|x_k^\top H x_k|^2 \leq \mathbb{E}\|H\|^2 \leq \mathbb{E}\|M_k^{(s)}\|^2.
\]

By the definition of \( M_k^{(s)} \),

\[
\|M_k^{(s)}\|^2 \leq \|D^{1-s} Y_k Y_k^\top\|^2 = \left\| D^{1-s} \left( \sqrt{np} (D_k + z I_{p-1}) + n \bar{\lambda}_p A_k \right) \right\|^2 \leq C \left( np \|D_k^{1-s}\|^2 + \|D_k^{1-s}\|^2 + n^2 \|D_k^{1-s}\|^2 \right) \leq C n^2 p.
\]
Therefore,
\[
\mathbb{E} \left| \frac{1}{np} x_k^\top H x_k \right|^2 \leq C', \quad \mathbb{E} \left| \frac{1}{np} x_k^\top H x_k \right|^4 \leq C'.
\]
Combining the above results, we then conclude the lemma.

\[ \square \]

### C.3 Finite-dimensional distribution

Recall the expansion in (C.8). By the central limit theorem for martingale, for the random part it’s sufficient to verify the two conditions in Lemma 9.12 in Bai & Silverstein (2010). The condition (9.9.2) therein is easily verified by
\[
\sum_{k=1}^p \mathbb{E} \left| \sum_{j=1}^l \xi_j \mathbb{E}_k(\alpha_k(z_j)) \right|^4 \leq C \sum_{k=1}^p \mathbb{E} \left( |\eta_k|^4 + |\gamma_k|^4 \right) \to 0.
\]

Hence, in the following we aim to check another condition, which is equivalent to finding the limit in probability of the covariance
\[
\Lambda_p(z_1, z_2) := \sum_{k=1}^p \mathbb{E}_{k-1} [\mathbb{E}_k(\alpha_k(z_1)) \cdot \mathbb{E}_k(\alpha_k(z_2))].
\]

Recall the expression of \(\alpha_k(z)\) in (C.8),
\[
\alpha_k(z) = -\left(1 + \frac{1}{np} \text{tr} M^{(2)}_k \right) \tilde{\beta}_k^2 \eta_k - \gamma_k \tilde{\beta}_k = \frac{\partial}{\partial z} (\tilde{\beta}_k \eta_k).
\]

By the dominated convergence theorem, we then focus on
\[
\frac{\partial^2}{\partial z_2 \partial z_1} \sum_{k=1}^p \mathbb{E}_{k-1} [\mathbb{E}_k(\tilde{\beta}_k(z_1) \eta_k(z_1)) \cdot \mathbb{E}_k(\tilde{\beta}_k(z_2) \eta_k(z_2))].
\]

We first aim to find the limit of \(\tilde{\beta}_k\). It’s easy to get
\[
\tilde{\beta}_k - \epsilon_k = -\epsilon_k \tilde{\beta}_k \frac{1}{np} \left( \text{tr} M^{(1)}_k - \mathbb{E} \text{tr} M^{(1)}_k \right), \quad \epsilon_k = \frac{1}{z + \frac{1}{np} \mathbb{E} \text{tr} M^{(1)}_k}.
\]

To this end, we need more notations as follows. We remove \(a_k\) from \(M^{(1)}_k\) and write \(M_k\). Abuse the notation a little to let \(\{e_i, i = 1, \ldots, k-1, k+1, \ldots, p\}\) be the \((p-1)\)-dimensional
vector with the $i$-th (or $(i-1)$-th if $k < i$) entry being 1 and the others being 0. Then, we have $Y_k = Y_{kl} + e_l y_l^\top$, where $Y_{kl}$ is obtained by replacing the $l$-th (or $(l-1)$-th) row of $Y_k$ with 0. Further write

$$h_l^\top = \frac{1}{\sqrt{np}} y_l^\top Y_{kl}^\top + \frac{1}{\sqrt{np}} (y_l^\top y_l - na_l \bar{\lambda}_B) e_l^\top, \quad r_l = \frac{1}{\sqrt{np}} Y_{kl} y_l,$$

$$D_{kl,r} = D_k - e_l h_l^\top = \frac{1}{\sqrt{np}} (Y_{kl} Y_{kl}^\top - n \bar{\lambda}_B A(l)) - z I_{p-1}, \quad \zeta_l = \frac{1}{1 + \vartheta_l}, \quad \vartheta_l = h_l^\top D_{kl,r}^{-1} e_l,$$

$$D_{kl} = D_{kl,r} - r_l e_l^\top = \frac{1}{\sqrt{np}} (Y_{kl} Y_{kl}^\top - n \bar{\lambda}_B A(l)) - z I_{p-1}, \quad M_{kl} = B^{1/2} Y_{kl} D_{kl}^{-1} Y_{kl} B^{1/2},$$

where $A(l)$ is obtained by replacing the $l$-th diagonal entry of $A$ with 0. Then, by the fact $|\beta| \leq C$, $|\epsilon_k| \leq C$,

$$E|\beta_k - \epsilon_k|^2 \leq CE \left[ \frac{1}{np} \left( \text{tr} M_k - \text{Etr} M_k \right) \right]^2 \leq CE \left[ \frac{1}{np} \sum_{l=1}^p (E_l - E_{l-1})(\text{tr} M_k - \text{tr} M_{kl}) \right]^2.$$  

By some elementary but tedious calculations (see section 5.2 in Chen & Pan (2015)), we have

$$M_k = M_{kl} - \frac{a_l \zeta_l}{znp} M_{kl} x_l x_l^\top M_{kl} + \frac{a_l \zeta_l}{z \sqrt{np}} M_{kl} x_l x_l^\top B + B x_l x_l^\top \frac{a_l \zeta_l}{z \sqrt{np}} M_{kl} - \frac{a_l \zeta_l}{z} B x_l x_l^\top B$$

$$= B_1(z) + B_2(z) + B_3(z) + B_4(z) + B_5(z).$$

(C.9)

Moreover, $\vartheta_l$ can be further simplified as

$$\vartheta_l = \frac{a_l}{znp} x_l^\top M_{kl} x_l - \frac{a_l}{z \sqrt{np}} (x_l^\top B x_l - n \bar{\lambda}_B).$$

Similarly to the proof of Lemma C.1, we claim

$$\vartheta_l - \frac{a_l}{znp} \text{tr} M_{kl} \xrightarrow{L_4} 0, \quad \zeta_l - \frac{1}{1 + \frac{a_l}{znp} \text{tr} M_{kl}} \xrightarrow{L_4} 0.$$  

(C.10)

The result on $\zeta_l$ holds because the imaginary part of $z \zeta_l^{-1}$ is $\Im z + \Im a_l/(np)x_l^\top M_{kl} x_l$, which is always larger than $v_0$. Hence,

$$|\zeta_l| \leq |z| v_0^{-1},$$

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and similarly
\[ \left| \frac{1}{1 + \frac{M_{kl}}{z}} \right| \leq |z|v_0^{-1}. \]

By the above argument, with Burkholder’s inequality it’s easy to get
\[ \mathbb{E} |\tilde{\beta}_k - \epsilon_k|^2 \leq \frac{C}{p}. \]  
(C.11)

That is, \( \tilde{\beta}_k - \epsilon_k = o_p(1) \). Note that \( |\beta_k| \leq C \) and \( |\epsilon_k| \leq C \). Then by dominated convergence theorem, for any integer \( t \),
\[ \mathbb{E} |\tilde{\beta}_k - \epsilon_k|^t \to 0. \]

Therefore,
\[
\mathbb{E} \left| \sum_{k=1}^{p} \mathbb{E}_{k-1} \left[ \mathbb{E}_k (\tilde{\beta}_k(z_1) \eta_k(z_1)) \cdot \mathbb{E}_k (\tilde{\beta}_k(z_2) \eta_k(z_2)) - \mathbb{E}_k (\epsilon_k(z_1) \eta_k(z_1)) \cdot \mathbb{E}_k (\epsilon_k(z_2) \eta_k(z_2)) \right] \right|
\leq \sum_{k=1}^{p} \mathbb{E} \left[ \left| (\tilde{\beta}_k(z_1) - \epsilon_k(z_1)) \eta_k(z_1) \right| \cdot \mathbb{E}_k (\tilde{\beta}_k(z_2) \eta_k(z_2)) \right]
+ \sum_{k=1}^{p} \mathbb{E} \left[ \left| \mathbb{E}_k (\tilde{\beta}_k(z_1) \eta_k(z_1)) \cdot \mathbb{E}_k ((\tilde{\beta}_k(z_2) - \epsilon_k(z_2)) \eta_k(z_2)) \right| \right]
\to 0.
\]

As a result, it suffices to consider
\[
\frac{\partial^2}{\partial z_2 \partial z_1} \sum_{k=1}^{p} \epsilon_k(z_1) \epsilon_k(z_2) \mathbb{E}_{k-1} \left[ \mathbb{E}_k \eta_k(z_1) \cdot \mathbb{E}_k \eta_k(z_2) \right].
\]

Actually, in the next section, we prove the convergence of the ESD of \( \bar{S}_p \), and the results (D.3) to (D.6) therein show that
\[ \epsilon_k(z) \to \left[ z + a_k \lambda \mathbf{B}^2 s_p(z) \right]^{-1} := \bar{\epsilon}_k \to \left[ z + a_k \lambda \mathbf{B}^2 s(z) \right]^{-1}, \]
which implies that it suffices to consider
\[
\frac{\partial^2}{\partial z_2 \partial z_1} \sum_{k=1}^{p} \bar{\epsilon}_k(z_1) \bar{\epsilon}_k(z_2) \mathbb{E}_{k-1} \left[ \mathbb{E}_k \eta_k(z_1) \cdot \mathbb{E}_k \eta_k(z_2) \right]. \tag{C.12}
\]

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In the following, we further simplify the expression (C.12). We write

\[ E_k \eta_k(z) = \frac{a_k}{\sqrt{np}} \sum_{j=1}^{n} B_{jj} (x_{kj}^2 - 1) + \frac{a_k}{\sqrt{np}} \sum_{i \neq j} B_{ij} x_{ki} x_{kj} - \frac{1}{np} \left( \sum_{i \neq j} x_{ki} x_{kj} E_k M_{k,ij}^{(1)}(z) + \sum_{i} (x_{ki}^2 - 1) E_k M_{k,ii}^{(1)}(z) \right). \]

Hence, after some calculations

\[ E_{k-1}[E_k \eta_k(z_1) \cdot E_k \eta_k(z_2)] = \frac{a_k}{np} \left( \mathbb{E}(x_{11}^2 - 1) \sum_j B_{jj}^2 + 2 \sum_{i \neq j} B_{ij}^2 \right) + A_1 + A_2 + A_3 + A_4 + A_5 + A_6, \]

where

\[ A_1 = -\frac{a_k}{np \sqrt{np}} \mathbb{E}(x_{11}^2 - 1)^2 \sum_j B_{jj} E_k M_{k,jj}^{(1)}(z_1), \quad A_2 = -\frac{a_k}{np \sqrt{np}} \mathbb{E}(x_{11}^2 - 1)^2 \sum_j B_{jj} E_k M_{k,jj}^{(1)}(z_2), \]

\[ A_3 = \frac{2}{n^2 p^2} \sum_{i \neq j} E_k M_{k,ij}^{(1)}(z_1) E_k M_{k,ij}^{(1)}(z_2), \quad A_4 = \frac{1}{n^2 p^2} \mathbb{E}(x_{11}^2 - 1)^2 \sum_j E_k M_{k,jj}^{(1)}(z_1) E_k M_{k,jj}^{(1)}(z_2), \]

\[ A_5 = -\frac{2a_k}{np \sqrt{np}} \sum_{i \neq j} B_{ij} E_k M_{k,ij}^{(1)}(z_1), \quad A_6 = -\frac{2a_k}{np \sqrt{np}} \sum_{i \neq j} B_{ij} E_k M_{k,ij}^{(1)}(z_2). \]

By the proof of Lemma C.1, we already know that \( \mathbb{E}|M_{k,jj}^{(1)}|^4 \leq C(p^4 + np^2) \), then it’s easy to conclude

\[ \mathbb{E} \left| \sum_{k=1}^{p} A_j \right| \to 0, \text{ for } j = 1, 2, 4. \]

Moreover,

\[ \mathbb{E} \left| \sum_{k=1}^{p} A_5 \right| = \mathbb{E} \left| \sum_{k=1}^{p} \frac{2a_k}{np \sqrt{np}} \text{tr}(BM_k^{(1)}) \right| - o(1) \to 0. \]

Similar results hold for \( A_6 \). On the other hand,

\[ \sum_{k=1}^{p} A_3 = \frac{2}{p} \sum_{k=1}^{p} Z_k - \frac{2}{n^2 p^2} \sum_{k=1}^{p} \sum_{j=1}^{n} E_k M_{k,jj}^{(1)}(z_1) E_k M_{k,jj}^{(1)}(z_2) = \frac{2}{p} \sum_{k=1}^{p} Z_k + o_{L_1}(1), \]

where \( Z_k \) is defined by

\[ Z_k = \frac{1}{n^2 p} \text{tr} \left( E_k M_k^{(1)}(z_1) \cdot E_k M_k^{(1)}(z_2) \right). \]
That is, we only need to find the limit in probability of
\[ \tilde{\Lambda}_p(z_1, z_2) := \frac{1}{p} \sum_{k=1}^{p} \tilde{\epsilon}_k(z_1) \tilde{\epsilon}_k(z_2) \left( 2Z_k + a_k^2 \left( \nu_4 - 3 \right) \frac{1}{n} \sum_j B_{jj}^2 + 2\tilde{\lambda} B^2 \right). \]

### C.4 Decomposition for \( Z_k \)

Note that
\[ D_k = \sum_{i \neq k} e_i h_i^\top - zI_{p-1}. \]

Multiplying \( D_k^{-1} \) to both sides leads to
\[ zD_k^{-1} = -I_{p-1} + \sum_{i \neq k} e_i h_i^\top D_k^{-1}. \]

Therefore,
\[
zM_k = -B^{1/2}Y_k^\top Y_k B^{1/2} + \sum_{i \neq k} \left( B^{1/2}Y_k^\top e_i h_i^\top D_k^{-1} Y_k B^{1/2} \right)
= -B^{1/2}Y_k^\top Y_k B^{1/2} + \sum_{i \neq k} \left( \zeta_i B^{1/2} y_i h_i^\top D_k^{-1} (Y_k + e_i y_i^\top) B^{1/2} \right)
= -B^{1/2}Y_k^\top Y_k B^{1/2} + \sum_{i \neq k} \left( \zeta_i B^{1/2} y_i \frac{1}{\sqrt{np}} y_i^\top Y_k D_k^{-1} Y_k B^{1/2} \right)
+ \sum_{i \neq k} \left( \zeta_i \tilde{\vartheta}_i B^{1/2} y_i y_i^\top B^{1/2} \right)
= -\sum_{i \neq k} \left( \zeta_i B^{1/2} y_i y_i^\top B^{1/2} \right) + \sum_{i \neq k} \left( \zeta_i B^{1/2} y_i \frac{1}{\sqrt{np}} y_i^\top Y_k D_k^{-1} Y_k B^{1/2} \right)
= -\sum_{i \neq k} \left( \zeta_i B^{1/2} y_i y_i^\top B^{1/2} \right) + \sum_{i \neq k} \left( \zeta_i B x_i \frac{a_i}{\sqrt{np}} x_i^\top M_k \right). \]

On the other hand, by taking \( l = i \) in (C.9), we write
\[
z_1 Z_k = \frac{a_k^2}{n^2 p} \text{tr} \left( E_k z_1 M_k(z_1) \cdot E_k M_k(z_2) \right) := C_1(z_1, z_2) + C_2(z_1, z_2), \]
with

\[ C_1(z_1, z_2) = -\frac{a_i^2}{n^2p} \sum_{i<k} E_k(\zeta_i(z_1) y_i^\top B^{1/2} E_k[\sum_{j=1}^5 B_j(z_2)] B^{1/2} y_i) \]

\[ -\frac{a_i^2}{n^2p} \sum_{i>k} E_k(\zeta_i(z_1) y_i^\top B^{1/2} E_k M_k(z_2) B^{1/2} y_i) := \sum_{j=1}^6 C_{1j}, \]

\[ C_2(z_1, z_2) = \frac{a_i^2}{n^2p} \sum_{i<k} E_k \left( \frac{a_i \zeta_i(z_1)}{\sqrt{np}} x_i^\top M_{ki}(z_1) E_k[\sum_{j=1}^5 B_j(z_2)] B x_i \right) \]

\[ + \frac{a_i^2}{n^2p} \sum_{i>k} E_k \left( \frac{a_i \zeta_i(z_1)}{\sqrt{np}} x_i^\top M_{ki}(z_1) E_k M_k(z_2) B x_i \right) := \sum_{j=1}^6 C_{2j}, \]

where \( C_{1j}, C_{2j} \) correspond to \( B_j \) for \( j = 1, \ldots, 5 \), while \( C_{16} \) and \( C_{26} \) correspond to \( i > k \). Now we aim to control these terms one by one.

We first introduce some useful bounds. For any \( n \times n \) matrix \( Q \) independent of \( x_i \), we claim

\[ E(x_i^\top Q x_i - \text{tr} Q)^2 \leq C E\| Q \|_F^2, \] \hspace{1cm} (C.14)

\[ E(x_i^\top Q x_i)^2 \leq C E(\| Q \|_F^2 + \| Q \|_F^2) \leq C n E\| Q \|_F^2, \] \hspace{1cm} (C.15)

\[ E(x_i^\top M_{ki} Q x_i - \text{tr} M_{ki} Q)^2 \leq C E\| M_{ki} Q \|_F^2 \leq C n^2p E\| Q \|_F^2, \] \hspace{1cm} (C.16)

\[ E(x_i^\top M_{ki} Q x_i)^2 \leq C E(\| M_{ki} Q \|_F^2 + \| M_{ki} Q \|_F^2) \leq C n^2p^2 E\| Q \|_F^2. \] \hspace{1cm} (C.17)

The proof of (C.14) to (C.16) is similar to Lemma C.1, while (C.17) is concluded from \( [(np)^{-1} \text{tr} M_{ki} Q] \leq C\| Q \| \) whose proof is similar to (C.7). Note that \( E_k M_k(z_2) \) is also independent of \( x_i \) for \( i > k \). Therefore, by the Cauchy Schwartz inequality,

\[ E|C_{1j}| \leq C \sqrt{\frac{p}{n}}, \quad j = 1, 2, 3, 4, 6, \]

\[ E|C_{2j}| \leq C \sqrt{\frac{p}{n}}, \quad j = 1, 2, 4, 5, 6. \]

Hence, we only need to consider \( C_{15} \) and \( C_{23} \). Specifically,

\[ C_{15} = \frac{a_k^2}{n^2p} \sum_{i<k} E_k \left( \zeta_i(z_1) y_i^\top B^{1/2} \frac{a_i E_k z_i}{z_2} B x_i x_i^\top B B^{1/2} y_i \right) \]

\[ = \frac{i.p.}{p} \sum_{i<k} E_k \left( \zeta_i(z_1) y_i^\top B^{1/2} \frac{a_i \zeta_i(z_1) \zeta_i(z_2)}{z_2} B x_i x_i^\top B B^{1/2} y_i \right). \]
On the other hand,
\[
C_{23} = \frac{a_k^2}{n^2p} \sum_{i<k} \mathbb{E}_k \left( \frac{a_i \zeta_i(z_1)}{\sqrt{np}} x_i^\top M_{ki}(z_1) \mathbb{E}_k \left[ \frac{a_i \zeta_i(z_2)}{z_2 \sqrt{np}} M_{ki} x_i x_i^\top B \right] B x_i \right)
\]

\[
\xrightarrow{i.p.} \sum_{k=1}^p \mathbb{E}_k \left( \frac{a_i \zeta_i(z_1)}{\sqrt{np}} x_i \right) \mathbb{E}_k \left( \frac{a_i \zeta_i(z_2)}{z_2 \sqrt{np}} \frac{1}{n} \sum_{i<k} a_i^2 \overline{\epsilon}_i(z_1) \overline{\epsilon}_i(z_2) \right).
\]

That is,
\[
\mathbb{Z}_k \xrightarrow{i.p.} \frac{a_k^2 \lambda_k^2}{\sum_{i<k} a_i^2 \overline{\epsilon}_i(z_1) \overline{\epsilon}_i(z_2)} + \frac{1}{\lambda_k^2 \sum_{i<k} a_i^2 \overline{\epsilon}_i(z_1) \overline{\epsilon}_i(z_2)},
\]

which implies
\[
\mathbb{Z}_k \xrightarrow{i.p.} \frac{2a_k^2 \lambda_k^2}{1 - \lambda_k^2} \frac{1}{\sum_{i<k} a_i^2 \overline{\epsilon}_i(z_1) \overline{\epsilon}_i(z_2)} + a_k^2 \left( \nu_4 - 3 \right) \frac{1}{n} \sum_{i<j} B_{ij}^2 + 2 \lambda_k^2 \right).$$

\[C.5 \quad \text{Tightness of } M_{p1}(z)\]

We end this section with a proof of the tightness of $M_{p1}(z)$. By Burkholder’s inequality,
\[
\mathbb{E} \left( \sum_{k=1}^p \sum_{j=1}^t \xi_j \mathbb{E}_{k-1} \left( \alpha_k(z_j) \right) \right)^2 \leq C,
\]

which ensures the first condition in Theorem 12.3 of Billingsley (2013). For the second condition, similarly to Bai & Silverstein (2010) and Chen & Pan (2015), we aim to verify
\[
\frac{\mathbb{E} |M_{p1}(z_1) - M_{p1}(z_2)|^2}{|z_1 - z_2|^2} \leq C, \quad z_1, z_2 \in \mathcal{C}_u.
\]

By (C.5),
\[
M_{p1}(z_1) - M_{p1}(z_2) = \sum_k \left( \rho_k(z_1) - \rho_k(z_2) \right)
\]
\[
= \sum_k \left( \mathbb{E}_k - \mathbb{E}_{k-1} \right) \left( \beta_k(z_1)(1 + q_k^\top D_k^{-2}(z_1)q_k) - \beta_k(z_2)(1 + q_k^\top D_k^{-2}(z_2)q_k) \right).
\]

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Note that
\[ \beta_k(z_1) - \beta_k(z_2) = \beta_k(z_1) \beta_k(z_2) \left( z_2 - z_1 + q_k D_k(z_2)^{-1} q_k - q_k D_k(z_1)^{-1} q_k \right), \]
\[ q_k D_k(z_2)^{-1} q_k - q_k D_k(z_1)^{-1} q_k = q_k^T D_k(z_2)^{-1} (z_2 - z_1) D_k(z_1)^{-1} q_k. \]
Moreover,
\[ \beta_k(z_1) \beta_k(z_2) = \beta_k(z_1) \beta_k(z_2) + \beta_k(z_1) \beta_k(z_1) \eta_k(z_1) \beta_k(z_2) \]
\[ = \beta_k(z_1) \beta_k(z_2) + \beta_k(z_1) \beta_k(z_2) \eta_k(z_2) + \beta_k(z_1) \beta_k(z_1) \eta_k(z_1) \beta_k(z_2). \]
Then, by Burkholder’s inequality and the facts that \( (E_k - E_{k-1}) \beta_k(z_1) \beta_k(z_2) = 0 \), \( |\beta_k| \leq C \), \( E|\eta_k|^2 \leq Cp^{-1} \), we conclude
\[ E \left| \sum_k (E_k - E_{k-1}) \beta_k(z_1) \beta_k(z_2) (z_2 - z_1) \right|^2 \leq C|z_1 - z_2|^2. \]

For the other terms, actually it’s very similar though with more tedious calculations. For example, we write
\[ (z_2 - z_1) \beta_k(z_1) \beta_k(z_2) q_k^T D_k(z_2)^{-1} D_k(z_1)^{-1} q_k \]
\[ = (z_2 - z_1) \beta_k(z_1) \beta_k(z_2) \left( q_k^T D_k(z_2)^{-1} D_k(z_1)^{-1} q_k - \frac{a_k}{np} \text{tr} B^{1/2} Y_k^T D_k(z_2)^{-1} D_k(z_1)^{-1} Y_k B^{1/2} \right) \]
\[ + (z_2 - z_1) \beta_k(z_1) \beta_k(z_2) \frac{a_k}{np} \text{tr} B^{1/2} Y_k^T D_k(z_2)^{-1} D_k(z_1)^{-1} Y_k B^{1/2}. \]

The first term can be bounded with Burkholder’s inequality. For the second term, use the expansion for \( \beta_k(z_1) \beta_k(z_2) \) and note that
\[ (E_k - E_{k-1}) \beta_k(z_1) \beta_k(z_2) \frac{a_k}{np} \text{tr} B^{1/2} Y_k^T D_k(z_2)^{-1} D_k(z_1)^{-1} Y_k B^{1/2} = 0. \]

For the remaining terms, we omit the details.

**D  The non-random part with diagonal \( A_p \)**

In this section, we focus on the non-random part \( M_{p2}(z) \) under the special case where \( A \) is diagonal. This is more challenging than the trivial case in Chen & Pan (2015) where \( A_p \) and \( B_n \) are both identity matrices. First, we show how to find the limit of \( \text{E}(m_{p \cdot p}(z)) \) using a method which is different from Wang & Paul (2014).
D.1 The limit of $E m_{Fsp}(z)$

We already know that

$$E \text{tr}(D^{-1}) = - \sum_{k=1}^{p} E(\beta_k) = - \sum_{k=1}^{p} E \left( \frac{1}{z + \frac{1}{np} y_k^\top Y_k^\top D_k^{-1} Y_k y_k - \frac{1}{\sqrt{np}} (y_k^\top y_k - n \lambda_B a_k)} \right).$$

Let

$$\epsilon_k = \frac{1}{z + \frac{1}{np} \text{tr} M_k^{(1)}},$$

$$\mu_k = \frac{1}{\sqrt{np}} (y_k^\top y_k - n \lambda_B a_k) - \frac{1}{np} y_k^\top Y_k^\top D_k^{-1} Y_k y_k + \frac{1}{np} \text{tr} M_k^{(1)}. \quad (D.1)$$

Then, by the identity $\beta_k = \epsilon_k + \epsilon_k \beta_k \mu_k$ and the fact that

$$\frac{1}{p} \sum_{k=1}^{p} E|\epsilon_k \beta_k \mu_k| \leq C \sum_{k=1}^{p} \left( E|\mu_k|^2 \right)^{1/2} \to 0,$$

we have

$$E(m_{Fsp}(z)) = - \frac{1}{p} \sum_{k=1}^{p} \epsilon_k + o(1). \quad (D.2)$$

Moreover, by (C.13), we have

$$\frac{z}{np} \text{tr} M_k^{(1)} = \frac{a_k}{np} \text{tr} \left[ - \sum_{i \neq k} \left( \zeta_i B^{1/2} y_i y_i^\top B^{1/2} \right) + \sum_{i \neq k} \left( \zeta_i B x_i \frac{a_i}{\sqrt{np}} x_i^\top M_{ki} \right) \right].$$

Recall that

$$\zeta_i - \frac{1}{1 + \frac{a_i}{zp} \text{tr} M_{ki}} \overset{L_i}{\to} 0.$$

Hence, after some calculations, we have

$$\frac{z}{np} \text{tr} M_k^{(1)} = - \frac{a_k}{np} \sum_{i \neq k} \mathbb{E} \frac{a_i \text{tr} B^2}{1 + \frac{a_i}{zp} \text{tr} M_{ki}} + o(1) = za_k \lambda_B^2 \frac{1}{p} \text{tr} D_k^{-1} A_k + o(1). \quad (D.3)$$

On the other hand, by Burkholder’s inequality and the expansion in (C.9), it’s not difficult to verify

$$E \left| \frac{1}{np} \text{tr} M_{ki} - \frac{1}{np} \text{tr} M_{ki} \right|^2 \to 0, \quad E \left| \frac{1}{np} \text{tr} M_k - \frac{1}{np} \text{tr} M_k \right|^2 \to 0.$$
Therefore, (D.3) also implies

\[
\frac{1}{p} \text{Etr} D_k^{-1} A_k = - \frac{1}{p} \sum_{j \neq k} a_j \lambda_{B_2} \text{Etr} D_k^{-1} A_k + o(1). \tag{D.4}
\]

Let \( s_p(z) \) and \( m_p(z) \) be the respective solutions in \( \mathbb{C}^+ \) to

\[
s_p(z) = - \int \frac{x}{z + x \lambda_{B_2} s_p(z)} dF_{A_p}, \quad m_p(z) = - \int \frac{1}{z + x \lambda_{B_2} s_p(z)} dF_{A_p} \quad z \in \mathbb{C}^+. \tag{D.5}
\]

Then, as \( p \to \infty \), for \( s(z) \) defined in Lemma 2.2,

\[
s_p(z) - s(z) = \int \left( \frac{x}{z + x \lambda_{B_2} s(z)} - \frac{x}{z + x \lambda_{B_2} s_p(z)} \right) dF_{A_p} + \int \frac{x}{z + x \lambda_{B_2} s(z)} d(F^A - F_{A_p}) \] 

\[
= \int \left( \frac{x^2 \lambda_{B^2} [s_p(z) - s(z)]}{[z + x \lambda_{B_2} s(z)][z + x \lambda_{B_2} s_p(z)]} \right) dF_{A_p} + o(1),
\]

where we use the fact that \( F_{A_p} \to F^A \) and for \( x \in \text{supp}(F^A) \cup \text{supp}(F_{A_p}) \),

\[
\left| \frac{x}{z + x \lambda_{B_2} s(z)} \right| \leq \frac{x}{3z} \leq C.
\]

Moreover, by the definition in (D.5), there exists some positive constant \( C_z \) satisfying \( |s_p(z)| \leq C_z \). Similar conclusion holds for \( s(z) \). Then, considering the imaginary part, for sufficiently large \( p \) we have

\[
\left| 1 - \int \left( \frac{x^2 \lambda_{B^2}}{[z + x \lambda_{B_2} s(z)][z + x \lambda_{B_2} s_p(z)]} \right) dF_{A_p} \right| \geq \int \frac{\lambda_{B^2} x^2}{4 [||z||^2 + (x \lambda_{B_2} C_z)^2]} dF_{A_p} > 0.
\]

Then, as \( p \to \infty \)

\[
s_p(z) - s(z) \to 0.
\]

Similarly, we have

\[
s(z) - \frac{1}{p} \text{Etr} D_k^{-1} A_k \to 0, \quad m_p(z) \to m(z), \quad \frac{1}{p} \text{Etr} D^{-1} - m(z) \to 0. \tag{D.6}
\]

where \( m(z) \) is defined in Lemma 2.2.
D.2 Convergence of $E M_p^2(z)$

Now we aim to find the limit of $E M_p^2(z)$, which is more challenging due to the multiplication with $p$. That is, we need to study the $o(1)$ terms in (D.2), (D.3) and (D.4). For simplicity, we write $m\bar{F}_p$ for $m_F s_p(z)$ and define

$$
\bar{\epsilon}_k = \frac{1}{n_p} \bar{E} \text{tr} \bar{M} \rightarrow \frac{1}{z + a_k \lambda_{B^2 s_p}(z)} := \tilde{\epsilon}_k, \quad M := B^{1/2} Y^\top D^{-1} Y B^{1/2},
$$

$$
\bar{\mu}_k = \frac{1}{\sqrt{n_p}} (y_k^\top y_k - n \lambda_B a_k) - \frac{1}{n_p} y_k^\top Y_k D_k^{-1} Y_k y_k + \frac{a_k}{n_p} \bar{E} \text{tr} \bar{M}.
$$

Note the difference between $\epsilon_k, \bar{\epsilon}_k$ and $\tilde{\epsilon}_k$. Then,

$$
W_1 := E m\bar{F}_p - m_p = -\frac{1}{p} \sum_{k=1}^p E \beta_k + \frac{1}{p} \sum_{k=1}^p \frac{1}{z + a_k \lambda_{B^2 s_p}(z)}
$$

$$
= -\frac{1}{p} \sum_{k=1}^p \beta_k \tilde{\epsilon}_k \bigg[ \bar{\mu}_k + a_k \bar{E} \bar{\mu}_k - \frac{1}{n_p} \bar{E} \text{tr} \bar{M} \bigg]. \quad \text{(D.7)}
$$

Before moving forward, we first introduce several bounds in the next lemma which are useful in later proofs. The proof of this lemma is postponed to the next subsection.

**Lemma D.1.** Under condition 2.1 and (B.6), for $z \in C_u$,

$$
\frac{1}{n_p} |\text{tr} MB^h| \leq C, \text{ for any fixed } h \geq 0,
$$

$$
\left| \frac{1}{n_p} (E \text{tr} M - E \text{tr} M_k) \right| \leq \frac{C}{p}, \text{ for any } k,
$$

$$
E |\bar{\mu}_k|^3 = o(p^{-1}), \quad E |\bar{\mu}_k|^4 = o(p^{-1}).
$$

The results also hold if we replace $M$ and $M_k$ with $M_k$ and $M_{ki}$, respectively.

We first focus on $p^{-1} \sum_k \beta_k \bar{\epsilon}_k \bar{\mu}_k$. By the decomposition $\beta_k = \bar{\epsilon}_k + \bar{\epsilon}_k^2 \bar{\mu}_k + \bar{\epsilon}_k \bar{\mu}_k + \beta_k \bar{\epsilon}_k \bar{\mu}_k^3$, we have

$$
E \sum_{k=1}^p \beta_k \bar{\epsilon}_k \bar{\mu}_k = \sum_{k=1}^p \bar{\epsilon}_k \left( \bar{\epsilon}_k \bar{\mu}_k + \bar{\epsilon}_k^2 \bar{\mu}_k^2 + \bar{\epsilon}_k \bar{\mu}_k^3 + \bar{\epsilon}_k \beta_k \bar{\mu}_k^4 \right) := H_1 + H_2 + H_3 + H_4.
$$

For $H_4$, Lemma D.1 implies $E |\beta_k||\bar{\mu}_k|^4 = o(p^{-1})$, where the $o(p^{-1})$ is actually uniformly for $k$. A similar conclusion holds for $H_3$. For $H_1$,

$$
E \bar{\mu}_k = \frac{a_k}{n_p} \left( E \text{tr} M - E \text{tr} M_k \right).
$$
By a similar expansion in (C.9), we have
\[
\mathbb{E}\tilde{\mu}_k = \frac{a_k}{np} \mathbb{E} \left( - \frac{a_k \zeta_k}{znp} x_k^\top M_k^2 x_k + \frac{a_k \zeta_k}{z \sqrt{np}} x_k^\top B M_k x_k + \frac{a_k \zeta_k}{z} x_k^\top M_k B x_k - \frac{a_k \zeta_k}{z} x_k^\top B^2 x_k \right).
\]

Therefore, after some calculations
\[
\sum_k \tilde{\zeta}_k \tilde{\epsilon}_k \mathbb{E}\tilde{\mu}_k = \sum_k \frac{a_k \tilde{\zeta}_k \tilde{\epsilon}_k}{np} \mathbb{E} \left( - \frac{a_k \zeta_k}{znp} \text{tr} M_k^2 - \frac{a_k \zeta_k}{z} \text{tr} B^2 \right) + O \left( \sqrt{\frac{p}{n}} \right)
\]
\[
\quad \rightarrow \sum_k \frac{a_k^2 \tilde{\epsilon}_k^2}{np} \left( - \frac{1}{np} \mathbb{E} \text{tr} M_k^2 - \text{tr} B^2 \right) + o(1).
\]

Moreover, by (C.9) and (C.13),
\[
\frac{z}{n^2p} \mathbb{E} \text{tr} M_k^2 = \frac{z}{n^2p} \mathbb{E} \left( \sum_i a_i^2 \epsilon_i^2 (x_i^\top B^2 x_i)^2 \right) + \frac{z}{n^2p} \mathbb{E} \left( \sum_i \frac{a_i^2 \epsilon_i^2}{np} x_i^\top M_{ki}^2 x_i x_i^\top B^2 x_i \right) + o(1).
\]

Then, we conclude that
\[
\frac{1}{n^2p} \mathbb{E} \text{tr} M_k^2 = \frac{1}{p} \sum_i \frac{a_i^2 \tilde{\lambda}_{B_{ki}}^2}{(z + a_i \lambda_{B_{ki}} s_p(z))^2} + \frac{1}{p} \sum_i \frac{a_i^2 \tilde{\lambda}_{B_{ki}}^2}{(z + a_i \lambda_{B_{ki}} s_p(z))^2} \times \frac{1}{n^2p} \mathbb{E} \text{tr} M_k^2 + o(1).
\]

On the other hand, it’s easy that \((n^2 p)^{-1} \mathbb{E} \text{tr} M_k^2 = (n^2 p)^{-1} \mathbb{E} \text{tr} M^2 + o(1).

Now we calculate \(H_2\). By the definition of \(\tilde{\mu}_k\),
\[
\mathbb{E} (\tilde{\mu}_k)^2 = \mathbb{E} (\tilde{\mu}_k - \mathbb{E} \tilde{\mu}_k)^2 + O(p^{-2}), \quad \mathbb{E} (\tilde{\mu}_k - \mathbb{E} \tilde{\mu}_k)^2 = S_1 + S_2,
\]
where
\[
S_1 = \frac{1}{np} \mathbb{E} (y_k^\top y_k - n \tilde{\lambda}_{B} a_k)^2 + \mathbb{E} \gamma_{k1}^2, \quad S_2 = S_{21} + S_{22},
\]
\[
S_{21} = \frac{a_k^2}{n^2p^2} \mathbb{E} (\text{tr} M_k - \mathbb{E} \text{tr} M_k)^2,
\]
\[
S_{22} = - \frac{2a_k}{np \sqrt{np}} \mathbb{E} (y_k^\top y_k - n \tilde{\lambda}_{B} a_k) (x_k^\top M_k x_k - \mathbb{E} \text{tr} M_k).
\]

We start with \(S_{21}\). By Burkholder’s inequality and (C.9),
\[
|S_{21}| \leq \frac{1}{n^2p^2} \sum_{i \neq k} \mathbb{E} \left( \mathbb{E}_i - \mathbb{E}_{i-1} \right) (\text{tr} M_k - \mathbb{E} \text{tr} M_k)^2 = o(p^{-1}).
\]

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Similarly, on the other hand,
\[
|S_{22}| \asymp \frac{1}{np\sqrt{np}} \mathbb{E}\left( \sum_i B_{ii}(x^2_{ki} - 1) + \sum_{i \neq j} B_{ij}x_{ki}x_{kj} \right) \left( \sum_i M_{k,ii}(x^2_{ki} - 1) + \sum_{i \neq j} M_{k,ij}x_{ki}x_{kj} \right) \\
\leq \frac{1}{np\sqrt{np}} \mathbb{E}\left( (\nu_4 - 2) \sum_i B_{ii}M_{k,ii} + 2\text{tr}M_kB \right) \leq O\left( \frac{1}{\sqrt{np}} \right) = o(p^{-1}).
\]
Hence, we only need to consider \(S_1\). Note that
\[
\frac{1}{np} \mathbb{E}(y_k\top y_k - n\tilde{\lambda}_B a_k)^2 = \frac{a^2_k}{np} \mathbb{E}\left( \sum_i B_{ii}(x^2_{ki} - 1) + \sum_{i \neq j} B_{ij}x_{ki}x_{kj} \right)^2 = \frac{a^2_k}{p} \left( (\nu_4 - 3) \frac{1}{n} \sum_j B_{jj}^2 + 2\tilde{\lambda}_B^2 \right).
\]
Similarly,
\[
\sum_k \mathbb{E} \gamma_k^2 = \frac{1}{p} \sum_k \frac{2a^2_k}{n^2p} \text{tr}M_k^2 + o(1).
\]
As a conclusion,
\[
\mathbb{E} \sum_{k=1}^p \beta_k \tilde{\epsilon}_k \tilde{\mu}_k = \frac{1}{p} \sum_k a_k^2 \tilde{\epsilon}_k \left( \frac{\nu_4 - 3}{n} \sum_j B_{jj}^2 + \tilde{\lambda}_B^2 + \frac{1}{n^2p} \mathbb{E}\text{tr}M^2 \right) + O(1) = A \times \frac{1}{p} \sum_k a_k^2 \epsilon_k + o(1).
\]
Hence,
\[
\mathcal{W}_1 = -\frac{1}{p} A \times \frac{1}{p} \sum_k a_k^2 \epsilon_k - \mathcal{W}_2 \times \frac{1}{p} \sum_{k=1}^p \mathbb{E} a_k \beta_k \tilde{\epsilon}_k + o(p^{-1}), \quad \mathcal{W}_2 := (\tilde{\lambda}_B^2 s_p(z) - \frac{1}{np} \mathbb{E}\text{tr}M).
\]
(D.8)
Now we aim to find the limit of \(p \mathcal{W}_2\). By a similar expansion in (C.13),
\[
z p \mathcal{W}_2 = \frac{1}{n} \sum_i \mathbb{E}\text{tr}\left( \zeta_i B^{1/2} y_i B^{1/2} \right) - \frac{1}{n} \mathbb{E} \sum_i \text{tr}\left( \zeta_i B x_i \frac{a_i}{\sqrt{np}} x_i \top M_i \right) + p\tilde{\lambda}_B^2 s_p(z).
\]
Actually, here \(z^{-1}\zeta_i\) is exactly \(\beta_i\), then
\[
p \mathcal{W}_2 = \sum_i a_i \mathbb{E}(\beta_i - \bar{\epsilon}_i) \frac{1}{n} x_i \top B^2 x_i - \frac{1}{n} \mathbb{E} \sum_i \text{tr}\left( \beta_i B x_i \frac{a_i}{\sqrt{np}} x_i \top M_i \right) + o(1).
\]
Note that
\[
\sum_i a_i \mathbb{E}(\beta_i - \bar{\epsilon}_i) \left( \frac{1}{n} x_i \top B^2 x_i - \tilde{\lambda}_B^2 \right) = \sum_i a_i \mathbb{E}(\beta_i - \bar{\epsilon}_i) \left( \frac{1}{n} x_i \top B^2 x_i - \tilde{\lambda}_B^2 \right) \\
= \sum_i a_i \mathbb{E} \beta_i \bar{\epsilon}_i \tilde{\mu}_i \left( \frac{1}{n} x_i \top B^2 x_i - \tilde{\lambda}_B^2 \right) \leq O(\sqrt{p/n}) \to 0.
\]
Therefore,

\[ pW_2 = \lambda B^2 \sum_i a_i \mathbb{E}(\beta_i - \bar{\epsilon}_i) - \frac{1}{n} \mathbb{E} \sum_i \text{tr} \left( \beta_i B x_i \frac{a_i}{\sqrt{np}} x_i^\top M_i \right) + o(1). \] (D.9)

We start with the first term. Similarly to the previous proof, we have

\[ \sum_i a_i \mathbb{E}(\beta_i - \bar{\epsilon}_i) = A \times \frac{1}{p} \sum_i a_i^3 \bar{\epsilon}_i + W_2 \times \sum_i a_i^2 \bar{\epsilon}_i \mathbb{E} \beta_i + o(1). \]

On the other hand, by (D.3) to (D.6), \( W_2 = o(1) \). Then, we can further expand \( \sum_i a_i^2 \bar{\epsilon}_i \mathbb{E} \beta_i \) and repeat the procedure iteratively. In this process, the \( o(1) \) terms are summable and will still be \( o(1) \). Then,

\[ pW_2 = A \times \frac{1}{p} \sum_i a_i^3 \bar{\epsilon}_i + \sum_{j=1}^{\infty} B_j - \frac{1}{n} \mathbb{E} \sum_i \text{tr} \left( \beta_i B x_i \frac{a_i}{\sqrt{np}} x_i^\top M_i \right) + o(1), \]

where

\[ B_j = W_2^j \sum_i (a_i \bar{\epsilon}_i)^{j+1}. \]

That is, in equation (D.9),

\[ \sum_i a_i \mathbb{E}(\beta_i - \bar{\epsilon}_i) = A \times \frac{1}{p} \sum_i a_i^3 \bar{\epsilon}_i + ps_p(z) + \sum_i \frac{a_i}{1 - W_2 a_i \bar{\epsilon}_i} + o(1) := D(W_2) + o(1). \] (D.10)

Now we move to the second term in (D.9),

\[ \frac{1}{n} \sum_{i=1}^{p} \mathbb{E} \left( \beta_i \frac{a_i}{\sqrt{np}} x_i^\top M_i B x_i \right) = \frac{1}{n} \sum_{i=1}^{p} \mathbb{E} \left( (\bar{\epsilon}_i + \beta_i \bar{\epsilon}_i (\bar{\mu}_i + a_i W_2) \frac{a_i}{\sqrt{np}} x_i^\top M_i B x_i \right) := J_1 + J_2 + J_3. \]

For \( J_2 \), with the bounds on \( |\beta_i| \) and \( |\bar{\epsilon}_i| \) we observe that

\[ |J_2 - \frac{1}{n} \sum_{i=1}^{p} \mathbb{E} \beta_i \bar{\epsilon}_i \bar{\mu}_i \frac{a_i}{\sqrt{np}} \text{tr} M_i B| \leq \frac{1}{n} \sum_{i=1}^{p} \mathbb{E} |\beta_i| |\bar{\epsilon}_i||\bar{\mu}_i| \frac{a_i}{\sqrt{np}} |x_i^\top M_i B x_i - \text{tr} M_i B| = o(1). \]

Moreover,

\[ \frac{1}{n} \sum_{i=1}^{p} \mathbb{E} \beta_i \bar{\epsilon}_i \bar{\mu}_i \frac{a_i}{\sqrt{np}} \text{tr} M_i B = \frac{1}{n} \sum_{i=1}^{p} \mathbb{E} (\bar{\epsilon}_i + \beta_i \bar{\epsilon}_i \bar{\mu}_i) \bar{\epsilon}_i \bar{\mu}_i \frac{a_i}{\sqrt{np}} \text{tr} M_i B := J_{21} + J_{22}. \]
For $J_{22}$,
\[ |J_{22}| \leq \frac{1}{n} \sum_{i=1}^{p} \mathbb{E}[\tilde{e}_i|\beta_i||\bar{e}_i|\bar{\mu}_i] \left| \frac{a_i}{\sqrt{np}} \text{tr} M_i B \right| = o(1). \]

On the other hand, for $J_{21}$, note that
\[ \frac{1}{n\sqrt{np}} \mathbb{E}\tilde{\mu}_i \text{tr} M_i B = \frac{a_i}{n\sqrt{np}} \mathbb{E}\left( \frac{1}{np} \text{tr} M_i - \frac{1}{np} \mathbb{E}\text{tr} M_i \right) \text{tr} M_i B \]
\[ = \frac{a_i}{n\sqrt{np}} \mathbb{E}\left( \frac{1}{np} \text{tr} M_i - \frac{1}{np} \mathbb{E}\text{tr} M_i \right) \text{tr} M_i B + \frac{a_i}{n\sqrt{np}} \mathbb{E}\left( \frac{1}{np} \mathbb{E}\text{tr} M_i - \frac{1}{np} \mathbb{E}\text{tr} M_i \right) \text{tr} M_i B \]
\[ := J_{211} + J_{212}. \]

For $J_{211}$, note that
\[ J_{211} = \frac{a_i}{n\sqrt{np}} \mathbb{E}\left( \frac{1}{np} \text{tr} M_i - \frac{1}{np} \mathbb{E}\text{tr} M_i \right) \left( \text{tr} M_i B - \mathbb{E}\text{tr} M_i B \right) \]
\[ \leq C \sqrt{\frac{p}{n}} \mathbb{E}\left( \frac{1}{np} \text{tr} M_i - \frac{1}{np} \mathbb{E}\text{tr} M_i \right)^2 \times \mathbb{E}\left( \frac{1}{np} \text{tr} M_i B - \frac{1}{np} \mathbb{E}\text{tr} M_i B \right)^2 = o(p^{-1}), \]
where we use the expansion in (C.9), Burkholde’s inequality and the martingale decomposition
\[ \frac{1}{np} \text{tr} M_i - \frac{1}{np} \mathbb{E}\text{tr} M_i = \sum_{j} (\mathbb{E}j - \mathbb{E}j-1) \left( \frac{1}{np} \text{tr} M_i - \frac{1}{np} \text{tr} M_{ij} \right). \]

For $J_{212}$, similarly by (C.9),
\[ |J_{212}| \leq C \sqrt{\frac{p}{n}} \mathbb{E}\left( \frac{1}{np} \mathbb{E}\text{tr} M_i - \frac{1}{np} \mathbb{E}\text{tr} M \right) \frac{1}{np} \mathbb{E}\text{tr} M_i B = o(p^{-1}). \]

As a result, we conclude that $|J_2| = o(1)$. For $J_3$, we can expand $\beta_i$ once more and write
\[ J_3 = W_2 \frac{1}{n} \sum_{i=1}^{p} \mathbb{E}a_i \tilde{e}_i \left( \tilde{e}_i + \tilde{\epsilon}_i \beta_i (\tilde{\mu}_i + W_2) \right) \frac{a_i}{\sqrt{np}} x_i^\top M_i B x_i := J_{31} + J_{32} + J_{33}. \]

Similarly to $J_2$, we can show that $J_{32} = o(W_2)$. Furthermore, we can keep expanding $\beta_i$ in $J_{33}$ and repeat the procedure iteratively. At each step $k \geq 1$, we get a new error term
\[ W_2^{k-1} \frac{1}{n} \sum_{i=1}^{p} (a_i \tilde{e}_i)^k \frac{1}{\sqrt{np}} \mathbb{E}\text{tr} M_i B, \]

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and a negligible term $o(W_k^{-1})$. Since $W_2 \to 0$, such negligible terms are summable and we conclude

$$
\frac{1}{n} \sum_{i=1}^{p} \mathbb{E} \left( \beta_i \frac{a_i}{\sqrt{np}} x_i^\top M_i B x_i \right) = \frac{1}{\sqrt{np}} \mathbb{E} \text{tr} M_i B \times \left( \sum_{k=1}^{\infty} W_k^{-1} \frac{1}{n} \sum_{i=1}^{p} (a_i \tilde{\epsilon}_i)^k \right) + o(1)
$$

$$
= \frac{1}{\sqrt{np}} \mathbb{E} \text{tr} M_i B \times \frac{1}{n} \sum_{i=1}^{p} \frac{a_i \tilde{\epsilon}_i}{1 - W_2 a_i \tilde{\epsilon}_i} + o(1)
$$

$$
= \sqrt{\frac{p}{n}} \mathbb{E} \text{tr} MB \times \frac{1}{n} \sum_{i=1}^{p} \frac{a_i \tilde{\epsilon}_i}{1 - W_2 a_i \tilde{\epsilon}_i} + o(1). \quad \text{(D.11)}
$$

Now we focus on the term $\mathbb{E} \text{tr} MB$. Based on (C.13),

$$
\sqrt{\frac{p}{n}} \mathbb{E} \text{tr} MB = - \sqrt{\frac{p}{n}} \tilde{\lambda}_B^3 \mathbb{E} \sum_i a_i \beta_i + \sqrt{\frac{p}{n}} \frac{1}{nn} \sum_i \mathbb{E} \left( \beta_i \frac{a_i}{\sqrt{np}} x_i^\top M_i B^2 x_i \right) + o(1)
$$

$$
:= P_1 + P_2 + o(1).
$$

For $P_1$, writing $\beta_i = \beta_i - \tilde{\epsilon}_i + \tilde{\epsilon}_i$ and using (D.10),

$$
P_1 = - \sqrt{\frac{p}{n}} \tilde{\lambda}_B^3 D(W_2) + \sqrt{\frac{p}{n}} \tilde{\lambda}_B^3 ps_p(z) + o(1) = - \sqrt{\frac{p}{n}} \tilde{\lambda}_B^3 \times pD_2(W_2) + o(1),
$$

where

$$
D_2(W_2) := \frac{1}{p} \sum_{i=1}^{p} \frac{a_i \tilde{\epsilon}_i}{1 - W_2 a_i \tilde{\epsilon}_i}.
$$

For $P_2$, similarly to (D.11), we conclude

$$
P_2 = \frac{p}{nn} \mathbb{E} \text{tr} MB^2 \times \frac{1}{p} \sum_{i=1}^{p} \frac{a_i \tilde{\epsilon}_i}{1 - W_2 a_i \tilde{\epsilon}_i} + o(1).
$$

Hence, we can keep expanding $\text{tr} MB^2$ iteratively, and at each step $k \geq 1$ we get a term

$$
-p \times \left( \sqrt{\frac{p}{n}} \tilde{\lambda}_B^{k+2} \times \left( \frac{1}{p} \sum_{i=1}^{p} \frac{a_i \tilde{\epsilon}_i}{1 - W_2 a_i \tilde{\epsilon}_i} \right)^k \right)^{k+1},
$$

while the other error terms are summable to $o(1)$. Hence, we conclude

$$
- \frac{1}{n} \sum_{i=1}^{p} \mathbb{E} \left( \beta_i \frac{a_i}{\sqrt{np}} x_i^\top M_i B x_i \right) = \left( \frac{1}{n} \sum_{j=1}^{n} \frac{(\lambda_j^B)^2}{1 - \lambda_j^B \sqrt{\frac{p}{n}} D_2(W_2) - \tilde{\lambda}_B^2} \right) \times pD_2(W_2) + o(1).
$$

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Now let $Y_p$ be the solution to
\[ x = \lambda B^2 \times \frac{1}{p} D(x) + \left( \frac{1}{n} \sum_{j=1}^{n} \frac{(\lambda_j^B)^2}{1 - \lambda_j} \right) \sqrt{\frac{\lambda_j^B}{p}} D_2(x), \]
and satisfy $Y_p = o(1)$. Then,
\[ p(W_2 - Y_p) = o(1), \]
which further implies
\[ p(W_1 + \frac{1}{p} A \times \frac{1}{p} \sum_k a_k^2 \sum_k \lambda_k a_k \beta_k \times Y_p) = o(1). \]
Moreover, note that $Y_p = o(1)$, then similarly to (D.10), we can prove that
\[ Y_p \sum_{k=1}^{p} a_k \tilde{\epsilon}_k \beta_k = Y_p \left( \sum_{j=1}^{\infty} W_2^{j-1} \sum_{k=1}^{p} \tilde{\epsilon}_k^{j+1} \right) + o(1) = Y_p \sum_k \frac{a_k \tilde{\epsilon}_k^2}{1 - W_2 a_k \tilde{\epsilon}_k} + o(1) \]
\[ = Y_p \sum_k \frac{a_k \tilde{\epsilon}_k^2}{1 - Y_p a_k \tilde{\epsilon}_k} + o(1). \]
Therefore,
\[ p \left( W_1 + \frac{1}{p} A \times \frac{1}{p} \sum_k a_k^2 + Y_p \times \sum_k \frac{a_k \tilde{\epsilon}_k^2}{1 - Y_p a_k \tilde{\epsilon}_k} \right) = o(1). \]
We then find the limit of the non-random part and complete the proof of Theorem 2.4. It remains to prove the bounds in Lemma D.1.

### D.3 Proof of Lemma D.1

**Proof.** The result for $(np)^{-1} |\text{tr} MB^h|$ is easy by using the similar technique in deriving (C.7). For the second result, it follows directly from the expansion in (C.9), and the bounds for $\zeta_k$ and $(np)^{-1} \text{tr} MB^h$. Hence, we only prove the bounds for the moments of $\bar{\mu}_k$.

By definition,
\[ \bar{\mu}_k = \frac{1}{\sqrt{np}} (y_k y_k - n \lambda_B a_k) - \left( \frac{1}{np} y_k y_k D_k^{-1} y_k y_k - \frac{a_k \text{E} \text{tr} M}{np} \right) \]
\[ + \left( \frac{a_k \text{E} \text{tr} M}{np} - \frac{a_k \text{tr} M}{np} \right) + \left( \frac{a_k \text{E} \text{tr} M}{np} - \frac{a_k \text{tr} M}{np} \right) \]
\[ := L_1 + L_2 + L_3 + L_4. \]
For $L_1$ and $L_2$, by the proof of Lemma C.1,
$$E|L_1|^4 = o(p^{-1}), \quad E|L_2|^4 = o(p^{-1}).$$

For $L_3$, we write
$$\frac{1}{np} \text{tr} M_k - \frac{1}{np} \text{tr} M_k = -\frac{1}{np} \sum_{i=1}^{p} (E_i - E_{i-1}) \text{tr}(M_k - M_{ki}).$$

Therefore, by Burkholder's inequality and the expansion in (C.9), together with the bounds in the proof of Lemma C.1, we conclude
$$E|L_3|^4 = o(p^{-1}).$$

The last term $L_4$ can be handled similarly by Burkholder's inequality, the expansion in (C.9) and the bounds in the proof of Lemma C.1. Consequently, we have
$$E|\mu_k|^4 = o(p^{-1}).$$

The result for $E|\bar{\mu}_k|^3$ then follows directly from the Cauchy-Schwartz inequality and the fact that $E|\bar{\mu}_k|^2 = O(p^{-1})$.

\section{Proof of Corollaries 2.5, 2.6 and Theorem 2.7}

\subsection{Proof of Corollaries 2.5 and 2.6}

We first prove Corollary 2.5. The mean correction term is straightforward from the expressions in Lemma 2.4. For the covariance function, note that
$$\Lambda(z_1, z_2) = \frac{\partial^2}{\partial z_1 \partial z_2} m(z_1)m(z_2) \left( \frac{2}{p} \lim_{p \to \infty} \sum_{k=1}^{p} \frac{k \lambda_B^2 m(z_1)m(z_2)}{1 - \frac{k}{p} \lambda_B^2 m(z_1)m(z_2)} + \frac{\nu_4 - 3}{n} \sum_{j=1}^{n} B_{jj}^2 + 2\bar{\lambda}_B^2 \right)$$
$$= \frac{\partial^2}{\partial z_1 \partial z_2} m(z_1)m(z_2) \left( - \frac{2 \log(1 - \lambda_B^2 m(z_1)m(z_2))}{m(z_1)m(z_2)} + (\nu_4 - 3)n^{-1} \sum_{j=1}^{n} B_{jj}^2 \right)$$
$$= m'(z_1)m'(z_2) \left( (\nu_4 - 3) \lim_{n \to \infty} n^{-1} \sum_{j=1}^{n} B_{jj}^2 + \frac{2\lambda_B^2}{(1 - \lambda_B^2 m(z_1)m(z_2))^2} \right).$$
where in the second step we use the fact
\[
\lim_{p \to \infty} \frac{1}{p} \sum_{k=1}^{p} \frac{\bar{\lambda}^2 B^2 m(z_1) m(z_2)}{1 - \frac{k}{p} \bar{\lambda} B^2 m(z_1) m(z_2)} = \int_0^1 \frac{t \bar{\lambda}^2 B^2 m(z_1) m(z_2)}{1 - t \bar{\lambda} B^2 m(z_1) m(z_2)} dt
\]
\[
= \bar{\lambda} B^2 \left( -1 - (\bar{\lambda} B^2 m(z_1) m(z_2))^{-1} \log \left(1 - \bar{\lambda} B^2 m(z_1) m(z_2)\right) \right).
\]

Corollary 2.5 is then verified. Now we move to the proof of Corollary 2.6. When \( B_n = I_n \),
\[
m_p(z) = s_p(z) = s(z) = m(z) = -\frac{1}{z + m(z)},
\]
then the mean correction term can be further simplified as
\[
\mathcal{X}_p = \frac{1}{p} \mathcal{A} \times m^3 - \frac{\mathcal{Y} \times m^2}{1 + m \mathcal{Y}},
\]
where we suppress the dependence on \( z \) and \( p \) for simplicity and
\[
\mathcal{A} = \nu_4 - 2 + m'.
\]
Moreover, \( \mathcal{D}(x) \) and \( \mathcal{D}_2(x) \) are simplified to
\[
\mathcal{D}(x) = -\mathcal{A} m^3 + pm - p \frac{m}{1 + mx}, \quad \mathcal{D}_2(x) = - \frac{m}{1 + mx}.
\]
Therefore, \( \mathcal{Y} \) is the solution to
\[
x = -\frac{1}{p} \mathcal{A} m^3 + m - \frac{m}{1 + mx} - \left( \frac{1}{1 + \sqrt{\frac{m}{n} 1 + mx}} - 1 \right) \frac{m}{1 + mx}
\]
\[
= -\frac{1}{p} \mathcal{A} m^3 + m - \frac{m}{1 + \sqrt{\frac{m}{n} 1 + mx}},
\]
and satisfy \( \mathcal{Y} = o(1) \). The above equation is equivalent to
\[
(x - m + \frac{1}{p} \mathcal{A} m^3) \left( 1 + \sqrt{\frac{p}{n} \frac{m}{1 + mx}} \right) + \frac{m}{1 + mx} = 0.
\]
\[
\Leftrightarrow (x - m + \frac{1}{p} \mathcal{A} m^3) \left( 1 + m x + \sqrt{\frac{p}{n} m} \right) + m = 0.
\]
We remove the $o(p^{-1})$ terms and let $\tilde{Y}$ be the solution to
\begin{equation}
mx^2 + \left(1 + \sqrt{\frac{p}{n}} m - m^2\right)x + \frac{1}{p} Am^3 - \sqrt{\frac{p}{n}} m^2 = 0, \tag{E.1}
\end{equation}
and define $\tilde{X}_p$ by replacing $Y$ with $\tilde{Y}$. Then, it’s sufficient to verify
\[ H := \tilde{A} \tilde{X}_p^2 + \tilde{B} \tilde{X}_p + \tilde{C} = o(p^{-1}). \]

Note that
\[ H = \left( m - \sqrt{\frac{p}{n}} (1 + m^2) \right) \left( \frac{1}{p} Am^3 - \frac{\tilde{Y} m^2}{1 + m \tilde{Y}} \right)^2 
   + \left( m^2 - 1 - \sqrt{\frac{p}{n}} m (1 + 2m^2) \right) \left( \frac{1}{p} Am^3 - \frac{\tilde{Y} m^2}{1 + m \tilde{Y}} \right) + \frac{m^3}{p} A - \frac{\sqrt{C}}{n} m^4. \]

We first remove all the $o(p^{-1})$ terms and write
\[ H = \left( m - \sqrt{\frac{p}{n}} (1 + m^2) \right) \left( \frac{\tilde{Y} m^2}{1 + m \tilde{Y}} \right)^2 
   - \left( m^2 - 1 - \sqrt{\frac{p}{n}} m (1 + 2m^2) \right) \frac{\tilde{Y} m^2}{1 + m \tilde{Y}} + \frac{1}{p} Am^5 - \sqrt{\frac{C}{n}} m^4 + o(p^{-1}). \]

Therefore, we only need to prove
\[ H_1 := m^2 \left( m - \sqrt{\frac{p}{n}} (1 + m^2) \right) \tilde{Y}^2 - \left( m^2 - 1 - \sqrt{\frac{p}{n}} m (1 + 2m^2) \right) \tilde{Y} (1 + m \tilde{Y}) 
   + \left( \frac{1}{p} Am^3 - \sqrt{\frac{C}{n}} m^2 \right) (1 + m \tilde{Y})^2 = o(p^{-1}). \]

Once again, we remove all the $o(p^{-1})$ terms. After some calculations,
\[ H_1 := m \tilde{Y}^2 - \left( m^2 - 1 - \sqrt{\frac{p}{n}} m \right) \tilde{Y} + \frac{1}{p} Am^3 - \sqrt{\frac{p}{n}} m^2 + o(p^{-1}), \tag{E.2} \]
which is exactly consistent with (E.1). The corollary is then verified.
E.2 Proof of Theorem 2.7 for diagonal $A_p$

In the last two sections, we have proved the weak convergence of the process $M_p(z)$ on $C_u$. By the Cauchy integral formula, to complete the proof for diagonal $A_p$, we still need to show that the integral on $C_{0,t,r}$ is negligible. Let $u_0 \geq 2 \limsup a_1b_1 + \epsilon_0$ for some small positive constant $\epsilon_0$. Since the event $U_p(\epsilon_0) := \{||S_p|| \leq 2 \limsup a_1b_1 + \epsilon_0/2\}$ happens with probability 1 for large $p$, below we aim to prove that under this event

$$\lim \lim_{v_0,0,p \to \infty} \mathbb{E} \left| \int_{C_{0,t,r}} f(z)M_p(z)dz \right| = 0.$$ 

We start with $C_0$. Since $||S_p|| \leq 2 \limsup a_1b_1 + \epsilon_0/2$, we have $|m_{F_Sp}(z)| \leq 2/\epsilon_0$. Therefore,

$$\mathbb{E}|M_{p1}(z)| \leq p \times \sqrt{\mathbb{E}|m_{F_Sp}(z)|^2} \leq pC.$$ 

On the other hand, $|M_{p2}(z)| \leq pC$. Hence,

$$\lim \lim_{v_0,0,p \to \infty} \mathbb{E} \left| \int_{C_0} f(z)M_p(z)dz \right| \leq \lim \lim_{v_0,0,p \to \infty} \int_{C_0} f(z)\mathbb{E}|M_{p1}(z)|^2 + |M_{p2}(z)|^2dz \leq C \times p\epsilon_0 \to 0.$$ 

Next, for $z \in C_{t,r}$, it’s sufficient to prove that under $U_p(\epsilon_0)$, for sufficiently large $p$,

$$\mathbb{E}|M_{p1}(z)| \leq C, \quad |M_{p2}(z)| \leq C, \quad \text{for } z \in C_{t,r}.$$ 

We start with $M_{p2}(z)$. Actually, by careful investigation of the previous proof in Section D, we observe that the condition $\exists z \geq v_0$ ($v_0$ is some constant) only contributes to the bounds like

$$|\beta_k| \leq C, \quad \left| \frac{1}{np} \text{tr}M \right| \leq C, \quad |\bar{\epsilon}(z)| \leq C, \quad |\bar{\epsilon}_k| \leq C, \quad \mathcal{W}_2 \to 0. \quad (E.3)$$

Hence, we only need to check these bounds for $z \in C_{t,r}$ and verify the results in Lemma D.1. For $\beta_k$, note that $-\beta_k$ is still the $k$-th diagonal element of $D^{-1}$. Under the event $U_p(\epsilon_0)$, we have $||S_p|| \leq 2 \limsup a_1b_1 + \epsilon_0/2$, which implies $|\lambda_j^{S_p} - z| \geq \epsilon_0/2$ and $||D||^{-1} \leq 2/\epsilon_0$. Therefore, $|\beta_k|$ is still bounded. Moreover, it still holds that $(np)^{-1}|\text{tr}M| \leq C$ by similar technique as in (C.7).

Next, we focus on $\bar{\epsilon}_k$. Since $\exists s(z) > 0$, the support of $F^A$ is bounded and the function $\frac{x}{z + x\lambda_B^2 s(z)}$ is continuous, from the equations satisfied by $s(z)$, we conclude that there exists a positive constant $\delta_1$ such that for any $t$ in the support of $F^A$,

$$\inf_{z \in C_{t,r}} |z + t\lambda_B^2 s(z)| \geq \delta_1.$$ 

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On the other hand, \( s_p(z) \to s(z) \) and the eigenvalues of \( A_p \) are all in the support of \( F^A \), then \( |\bar{\epsilon}_k| \leq C \).

Lastly, we check \( W_2 = o(1) \). By the expansion in (C.13),
\[
W_2 = \lambda_B^2 \frac{1}{p} \sum_i a_i (E_i \beta_i - \bar{\epsilon}_i) + o(1) = -\lambda_B^2 \left[ \frac{1}{p} \text{Etr}D^{-1}A - s_p(z) \right] + o(1).
\]
Actually, by the definition of \( \beta_i \), (C.13) and the continuous mapping theorem, we know that
\[
\beta_i \xrightarrow{\text{i.p.}} \frac{1}{z + \frac{1}{np} \text{tr}M^{(1)}} z + \frac{a_i \text{Etr}M}{z + a_i \lambda_B^2 \frac{1}{p} \text{Etr}D^{-1}A}.
\]
Then, \( |\epsilon_k(z)| \leq C \) for large \( p \). Note that \( \beta_i \) and \( p^{-1} \text{tr}D^{-1}A \) are bounded and \( \text{tr}D^{-1}A = -\sum_i a_i \beta_i \). Hence, by the dominated convergence theorem,
\[
\frac{1}{p} \text{Etr}D^{-1}A = \frac{1}{p} \sum_i \frac{1}{z + a_i \lambda_B^2 \frac{1}{p} \text{Etr}D^{-1}A} + o(1),
\]
which further implies \( p^{-1} \text{Etr}D^{-1}A - s_p(z) = o(1) \) and \( W_2 = o(1) \). Furthermore, with these preliminary bounds, we observe that the results in Lemma D.1 still hold. Then, \( |M_{p2}(z)| \leq C \) for \( z \in \mathcal{C}_{l,r} \).

Now we move to the calculation of \( \mathbb{E}[M_{p1}(z)] \) for \( z \in \mathcal{C}_{l,r} \). We can not use the decomposition in (C.8) because the bound for \( \bar{\beta}_k \) is not guaranteed to hold anymore. The strategy is to replace \( \bar{\beta}_k \) with \( \epsilon_k \). Recall the definition of \( \epsilon_k \) and \( \mu_k \) in (D.1). Based on (C.5) and the relationship \( \beta_k = \epsilon_k + \beta_k \epsilon_k \mu_k \),
\[
M_{p1}(z) = -\sum_{k=1}^p (\mathbb{E}_k - \mathbb{E}_{k-1}) \beta_k (1 + q_k^\top D^{-2} q_k)
\]
\[
= -\sum_{k=1}^p (\mathbb{E}_k - \mathbb{E}_{k-1}) \beta_k (1 + \frac{1}{np} \text{tr}M_k^{(2)} + \gamma_k^2)
\]
\[
= -\sum_{k=1}^p (\mathbb{E}_k - \mathbb{E}_{k-1}) (\beta_k \gamma_k^2) - \sum_{k=1}^p (\mathbb{E}_k - \mathbb{E}_{k-1}) \beta_k \epsilon_k \mu_k \left( 1 + \frac{1}{np} \text{tr}M_k^{(2)} \right).
\]
Under the event \( U_p(\epsilon_0) \), with the bounds in (E.3), it’s easy to see the second moments of \( \gamma_k^2 \) and \( \mu_k \) are still \( O(p^{-1}) \). Hence, by Burkholder’s inequality, it reduces to proving \( |\epsilon_k| \leq C \). Using the expansions in (C.9) and (C.13), we conclude that
\[
\epsilon_k \to \frac{1}{z + a_k \lambda_B^2 \frac{1}{p} \text{Etr}D^{-1}A} \to \mathbb{E} \beta_k \leq C.
\]
As a result, \( |M_{p1}(z)| \leq C \) for large \( p \) and the proof for diagonal \( A_p \) has been completed.
E.3 Proof of Theorem 2.7 for general $A_p$

Up to now we only proved the results for diagonal $A_p$. In this subsection, we extend the results to general non-negative definite matrix $A_p$. Note that if the entries of $X_p$ are i.i.d. standard Gaussian variables, we can regard $A_p$ as diagonal because standard Gaussian vectors are orthogonally invariant. Therefore, the results hold for Gaussian case. We then follow the interpolation strategy in Bai et al. (2019) to compare the characteristic functions of linear spectral statistics under the Gaussian case and general case.

The proof is essentially adapted from Bai et al. (2019) and we use similar notation therein. Let $X_p$ be the $p \times n$ random matrix whose entries are i.i.d. from some general distributions. Let $Z_p$ be $p \times n$ random matrix whose entries are i.i.d. standard Gaussian variables. Define

$$W_p(\theta) = (w_{jk}) = X_p \sin \theta + Z_p \cos \theta, \quad Y_p(\theta) = A_p^{1/2}W_p(\theta)B_p^{1/2},$$

$$G_p(\theta) = \frac{1}{\sqrt{np}}Y_p(\theta)Y_p(\theta)^\top, \quad S_p(\theta) = \frac{1}{\sqrt{np}}(Y_p(\theta)Y_p(\theta)^\top - n\bar{\lambda}_B A_p).$$

Then, $S_p(0)$ is the matrix of interest. The proofs are very similar to those in Bai et al. (2019) by the observation that $\partial G / \partial \theta = \partial S / \partial \theta$. Hence, we only show the necessary steps for self-completeness.

Furthermore, let

$$H_p(t, \theta) = e^{itS_p(\theta)}, \quad S(\theta) = \text{tr}(S_p(\theta)), \quad S^0(\theta) = S(\theta) - p \int f(x)dF_p(x), \quad Z_p(x, \theta) = \mathbb{E}e^{ixS^0(\theta)}.$$

We may suppress the dependence on $p$ and $\theta$ for simplicity. Therefore, it’s sufficient to prove that

$$\frac{\partial Z_p(x, \theta)}{\partial \theta} \to 0$$

uniformly in $\theta$ over the interval $[0, \pi/2]$, because

$$Z_p(x, \pi/2) - Z_p(x, 0) = \int_0^{\pi/2} \frac{\partial Z_p(x, \theta)}{\partial \theta} d\theta.$$

Let $f(\lambda)$ be a smooth function with the Fourier transform

$$\hat{f}(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(\lambda)e^{-it\lambda}d\lambda.$$
Firstly, we calculate the derivative of \( S(\theta) \). By the inverse Fourier transform formula,

\[
\frac{\partial S(\theta)}{\partial w_{jk}} = \int_{-\infty}^{+\infty} \hat{f}(u) \text{tr} \frac{\partial H_p(u)}{\partial w_{jk}} du,
\]

while

\[
\frac{\partial [H_p(u)]_{dl}}{\partial w_{jk}} = \sum_{a,b=1}^{p} \frac{\partial [H_p(u)]_{dl}}{\partial \bar{s}_{ab}} \times \frac{\partial \bar{s}_{ab}}{\partial w_{jk}},
\]

where \( \bar{s}_{ab} \) is the \((a, b)\)-th entry of \( \bar{S}_p \). By Lemma 0.16 in Bai et al. (2019),

\[
\frac{\partial H_p(t)}{\partial \bar{s}_{ab}} = i \int_{0}^{t} e^{is_p} e_a e_b^T e^{i(t-s)} s_p ds = i \int_{0}^{t} H_p(s) e_a e_b^T H_p(t-s) ds.
\]

On the other hand,

\[
\frac{\partial s_{ab}}{\partial w_{jk}} = \frac{\partial g_{ab}}{\partial w_{jk}} = \left[ \frac{G_p}{\partial w_{jk}} \right]_{ab} = \frac{1}{\sqrt{np}} [A^{1/2}]_{aj} [BW^T A^{1/2}]_{kb} + \frac{1}{\sqrt{np}} [A^{1/2}WB]_{ak} [A^{1/2}]_{jb}.
\]

Let \( h_{ij} \) be the \((i, j)\)-th entry of \( H \), and \( f * g(t) = \int_{0}^{t} f(s) g(t-s) ds \). Then

\[
\frac{\partial h_{dl}(t)}{\partial w_{jk}} = i \sqrt{n_p} \sum_{a,b} h_{da} * h_{bl}(t) \left\{ [A^{1/2}]_{aj} [BW^T A^{1/2}]_{kb} + \frac{1}{\sqrt{np}} [A^{1/2}WB]_{ak} [A^{1/2}]_{jb} \right\}
\]

\[
= i \sqrt{n_p} [HA^{1/2}]_{dj} * [BW^T A^{1/2} H]_{kl}(t) + i \sqrt{n_p} [HA^{1/2}WB]_{dk} * [A^{1/2}H]_{ji}(t).
\]

Therefore,

\[
\text{tr} \frac{\partial H_p(u)}{\partial w_{jk}} = 2i \sqrt{n_p} \sum_{d=1}^{p} [HA^{1/2}]_{dj} * [BW^T A^{1/2} H]_{kd}(u) = 2i u \sqrt{n_p} [A^{1/2} H(u) A^{1/2} WB]_{jk}.
\]

We then conclude

\[
\frac{\partial S(\theta)}{\partial w_{jk}} = 2i \sqrt{n_p} \int_{-\infty}^{+\infty} \hat{f}(u) [A^{1/2} H(u) A^{1/2} WB]_{jk} du = \frac{2}{\sqrt{n_p}} [A^{1/2} \tilde{f}(S_p) A^{1/2} WB]_{jk},
\]

where

\[
\tilde{f}(S_p) = i \int_{-\infty}^{+\infty} \hat{f}(u) H(u) du.
\]
Furthermore, 
\[
\frac{\partial Z_p(x, \theta)}{\partial \theta} = \frac{2 xi}{\sqrt{n p}} \sum_{j=1}^{p} \sum_{k=1}^{n} \mathbb{E} w'_{jk}[A^{1/2} \tilde{f}(\tilde{S}_p)A^{1/2}WB]_{jk}e^{ixS_0(\theta)},
\]
where 
\[
w'_{jk} = \frac{dw_{jk}}{d\theta} = x_{jk} \cos \theta - y_{jk} \sin \theta.
\]
Let \( W_{p,jk}(w, \theta) \) denote the corresponding matrix \( W_p \) by replacing the \((j, k)\)-th entry \((w_{jk})\) with \( w \). Let 
\[
\psi_{jk}(w) = \left[ A^{1/2} \tilde{f}(\tilde{S}_{p,jk}(w, \theta))A^{1/2}W_{p,jk}(w, \theta)B \right]_{jk}e^{ixS_{0,jk}(w, \theta)}.
\]
Then, by Taylor’s expansion, 
\[
\psi_{jk}(w) = \sum_{l=0}^{3} \frac{1}{l!} w'_{jk} \psi_{jk}(l) + \frac{1}{4!} w^4_{jk} \psi_{jk}^{(4)}(\rho w_{jk}), \quad \rho \in (0, 1),
\]
which implies 
\[
\frac{\partial Z_p(x, \theta)}{\partial \theta} = \frac{2 xi}{\sqrt{n p}} \sum_{j=1}^{p} \sum_{k=1}^{n} \mathbb{E} w'_{jk} \sum_{l=0}^{3} \frac{1}{l!} w'_{jk} \psi_{jk}(l) + \frac{1}{4!} w^4_{jk} \psi_{jk}^{(4)}(\rho w_{jk}).
\]
It’s easy to see 
\[
\mathbb{E} w'_{jk} w^0_{jk} = 0, \quad \mathbb{E} w'_{jk} w^1_{jk} = 0, \quad \mathbb{E} w'_{jk} w^2_{jk} = \mathbb{E} w^3_{jk} \sin^2 \theta \cos \theta, \quad \mathbb{E} w'_{jk} w^3_{jk} = o(1),
\]
under the condition \( \nu_4 = 3 \). Therefore, it’s sufficient to prove that 
\[
\left| \frac{1}{\sqrt{n p}} \sum_{j} \sum_{k} \mathbb{E} w'_{jk} w^l_{jk} \psi_{jk}(0) \right| \rightarrow 0, \quad l = 2, 4.
\]
To this end, we claim that the expansions for \( \psi_{jk}(0) \) are almost the same as those in Bai et al. (2019), except that the scaling coefficient in the denominator is \( \sqrt{n p} \) here rather than \( n \). This has no effects on the results because we can slightly modify Lemma 0.8 in Bai et al. (2019) to a rate of \( n^{1/2}p^{3/4} \) when \( p/n \rightarrow 0 \). Below we write down the modified lemma and its proof, to conclude this subsection. For further detailed proofs, one can refer to Bai et al. (2019).
Lemma E.1. Suppose $A$, $B$ and $C$ are $p \times p$, $p \times n$ and $n \times n$ random matrices respectively, $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ are $p \times n$ random matrices. Moreover, the moments of their spectral norms are bounded and $p/n \to 0$. Then we get as $p \to \infty$,

$$\left| \sum_{jk} \mathbb{E} A_{jj} B_{jk} C_{kk} \right| \leq C n^{1/2} p^{-3/4}, \quad \left| \sum_{jk} \mathbb{E} A_{jk} B_{jk} C_{jk} \right| \leq C n^{1/2} p^{-3/4}.$$ 

Proof. The results follow from the Cauchy Schwartz inequality directly. Specifically,

$$\left| \sum_{jk} \mathbb{E} A_{jj} B_{jk} C_{kk} \right| \leq \left( \sum_k \mathbb{E} \left| C_{kk} \right|^2 \right)^{1/2} \times \left( \sum_j \mathbb{E} \left| \sum_k A_{jj} B_{jk} \right|^2 \right)^{1/2} \leq C \sqrt{n} \times \left( \mathbb{E} \sum_{j_1, j_2} A_{j_1 j_1} A_{j_2 j_2} \sum_k B_{j_1 k} B_{j_2 k} \right)^{1/2} \leq C \sqrt{n} \times \left( \sum_{j_1, j_2} \mathbb{E} A_{j_1 j_1}^2 A_{j_2 j_2}^2 \right)^{1/4} \left( \sum_{j_1, j_2} \mathbb{E} (BB^\top)^2_{j_1 j_2} \right)^{1/4} \leq C n^{1/2} p^{-3/4}.$$ 

On the other hand,

$$\left| \sum_{jk} \mathbb{E} A_{jk} B_{jk} C_{jk} \right| \leq \left( \sum_{j,k} \mathbb{E} \left| A_{jk} \right| \left| B_{jk} \right| \right)^{1/2} \times \left( \sum_{j,k} \mathbb{E} \left| A_{jk} \right| \left| B_{jk} \right| \left| C_{jk} \right| \right)^{1/2} \leq \left( \sum_{j,k} \mathbb{E} \left| A_{jk} \right|^2 \right)^{3/4} \times \left( \sum_{j,k} \mathbb{E} \left| B_{jk} \right|^4 \right)^{1/4} \leq C \left( \sum_{j,k} \mathbb{E} \left| A_{jk} \right|^2 \right)^{3/8} \times \left( \sum_{j,k} \mathbb{E} \left| B_{jk} \right|^2 \right)^{3/8} \times (np)^{1/4} \leq C n^{1/2} p^{3/4},$$

which concludes the lemma. □
F Proof of the results in Section 3

F.1 Proof of Theorem 3.1

Based on Corollary 2.5, denote
\[
\tilde{\Lambda}(z_1, z_2) := -2 \log(1 - \bar{\lambda}_B^2 m(z_1) m(z_2)) + (\nu_4 - 3)n^{-1} \sum_{j=1}^{n} B_{jj}^2 m(z_1) m(z_2),
\]
and consider the covariance function in Theorem 2.7 with two functions \(f\) and \(g\). It can be written as
\[
\Gamma(z_1, z_2) = -\frac{1}{4\pi^2} \oint_{C_1} \oint_{C_2} f(z_1) g(z_2) \frac{\partial^2}{\partial z_1 \partial z_2} \tilde{\Lambda}(z_1, z_2) dz_1 dz_2,
\]
where \(C_i\) are disjoint the contours formed by vertex \((\pm(2 \limsup b_1 + \epsilon_i) \pm iv_i)\) with some small \(\epsilon_i\) and \(v_i\). Integrating by parts, we have
\[
\Gamma(z_1, z_2) = -\frac{1}{4\pi^2} \oint_{C_1} \oint_{C_2} f'(z_1) g'(z_2) \tilde{\Lambda}(z_1, z_2) dz_1 dz_2.
\]

Denote \(A(z_1, z_2) := f'(z_1) g'(z_2) \tilde{\Lambda}(z_1, z_2)\) and \(c = 2 \limsup b_1\). Letting \(v_j \to 0\) and \(\epsilon_j \to 0\), we have
\[
\Gamma(z_1, z_2) = -\frac{1}{4\pi^2} \int_{-c}^{c} \int_{-c}^{c} [A(t_1^-, t_2^-) - A(t_1^-, t_2^+) - A(t_1^+, t_2^-) + A(t_1^+, t_2^+)] dt_1 dt_2,
\]
where \(t_j^\pm := t_j \pm i0\). We first consider \(f\) and \(g\) to be real-valued functions. Note that
\[
m(t \pm i0) = \begin{cases} 
- t + \text{sign}(t) \sqrt{t^2 - 4\lambda_B^2}, & t^2 \geq 4\lambda_B^2, \\
\frac{-t \pm i \sqrt{4\lambda_B^2 - t^2}}{2\lambda_B^2}, & t^2 < 4\lambda_B^2.
\end{cases}
\]
Let $\bar{c} = \sqrt{4\lambda_B^2}$. Then we have

\[
\begin{align*}
\int_{-\bar{c}}^{\bar{c}} \int_{-\bar{c}}^{\bar{c}} f'(t_1)g'(t_2) \left[ m(t_1^+)m(t_2^-) - m(t_1^-)m(t_2^+) - m(t_1^+)m(t_2^-) + m(t_1^-)m(t_2^+) \right] dt_1 dt_2 \\
= - \int_{-\bar{c}}^{\bar{c}} \int_{-\bar{c}}^{\bar{c}} f'(t_1)g'(t_2) \frac{1}{\lambda_B^2} \sqrt{4\lambda_B^2 - t_1^2} \sqrt{4\lambda_B^2 - t_2^2} dt_1 dt_2, \\
\int_{-\bar{c}}^{\bar{c}} \int_{-\bar{c}}^{\bar{c}} f'(t_1)g'(t_2) \left[ \log(1 - \bar{\lambda}_B^2 m(t_1^-)m(t_2^-)) - \log(1 - \bar{\lambda}_B^2 m(t_1^+)m(t_2^-)) \\
- \log(1 - \bar{\lambda}_B^2 m(t_1^-)m(t_2^+)) + \log(1 - \bar{\lambda}_B^2 m(t_1^+)m(t_2^+)) \right] dt_1 dt_2 \\
= \int_{-\bar{c}}^{\bar{c}} \int_{-\bar{c}}^{\bar{c}} f'(t_1)g'(t_2) \log \left| \frac{1 - \bar{\lambda}_B^2 m(t_1^-)m(t_2^-)}{1 - \bar{\lambda}_B^2 m(t_1^+)m(t_2^-)} \right|^2 dt_1 dt_2 \\
= \int_{-\bar{c}}^{\bar{c}} \int_{-\bar{c}}^{\bar{c}} f'(t_1)g'(t_2) \log \left( \frac{4\lambda_B^2 - t_1 t_2 + \sqrt{(4\lambda_B^2 - t_1^2)(4\lambda_B^2 - t_2^2)}}{4\lambda_B^2 - t_1 t_2 - \sqrt{(4\lambda_B^2 - t_1^2)(4\lambda_B^2 - t_2^2)}} \right) dt_1 dt_2.
\end{align*}
\]

Hence, the expression in Theorem 3.1 follows. Moreover, the above argument still holds for complex-valued functions $f$ and $g$.

When $f(x) = g(x) = x^2$, it’s easy to see the first term of $H(t_1, t_2)$ is an odd function on $t_1$ or $t_2$, then the corresponding integral is zero. For the second term, take $t_1 = \sqrt{4\lambda_B^2 \cos \theta}$ and $t_2 = \sqrt{4\lambda_B^2 \sin \theta}$. Then, the result follows from Remark 1.3 in Chen & Pan (2015).

### F.2 Proof of Theorem 3.3

Indeed, we prove a more general result that the power tends to 1 as long as $p^{-1}(\text{tr}(A_p - I_p))^2 \geq c_0$ for some $c_0 > 0$ and sufficiently large $p$. Let $\tilde{S}_p^0$ be the normalized separable sample covariance matrix corresponding to $A_p = I_p$, i.e.,

\[
\tilde{S}_p^0 = \frac{1}{\sqrt{np}} \left[ XB_n X^\top - (\text{tr}B_n)I_p \right],
\]

and $G_p^0(f)$ defined by (2.5) using $\tilde{S}_p^0$. Therefore, under the alternative, with $f(x) = x^2$,

\[
G_p(f) = \frac{1}{2\lambda_B^2} \left( \text{tr}S_p^2 - \text{tr}(\tilde{S}_p^0)^2 \right) + \frac{G^0_p(f)}{2\lambda_B^2},
\]

where

\[
S_p = \frac{1}{\sqrt{np}} \left[ A_p^{1/2} XB_n X^\top A_p^{1/2} - (\text{tr}B_n)I_p \right].
\]
By Theorem 3.1, $G_p^0(f) = O_p(1)$. Therefore, it’s sufficient to prove that
\[ \text{tr} \bar{S}_p^2 - \text{tr}(\bar{S}_p^0)^2 \overset{i.p.}{\rightarrow} \infty, \text{ as } p \rightarrow \infty. \]

By definition,
\[ \frac{1}{n} \left( \text{tr} \bar{S}_p^2 - \text{tr}(\bar{S}_p^0)^2 \right) \]
\[ = \frac{1}{p} \left( \frac{1}{n^2} [A_p^{1/2}XB_nX^T A_p^{1/2} - (\text{tr}B_n)I_p]^2 - \frac{1}{n^2} [XB_nX^T - (\text{tr}B_n)I_p]^2 \right) \]
\[ = \frac{1}{p} \left( \frac{1}{n^2} \left[ (A_p^{1/2} - I)XB_nX^T (A_p^{1/2} - I) + (A_p^{1/2} - I)XB_nX^T + XB_nX^T (A_p^{1/2} - I) \right]^2 \right) \]
\[ + \frac{2}{p} \text{tr} \left( \frac{1}{n^2} \left[ (A_p^{1/2} - I)XB_nX^T (A_p^{1/2} - I) + (A_p^{1/2} - I)XB_nX^T + XB_nX^T (A_p^{1/2} - I) \right] \times [XB_nX^T - (\text{tr}B_n)I_p] \right) \]
\[ := L_1 + L_2. \]

By the inequality $|p^{-1} \text{tr} A_1A_2| \leq \|A_1\|\|A_2\|$ for any $p \times p$ matrices $A_1$ and $A_2$, we have
\[ |L_2| \leq C_0 \left| \frac{1}{n} \left[ (A_p^{1/2} - I)XB_nX^T (A_p^{1/2} - I) + (A_p^{1/2} - I)XB_nX^T + XB_nX^T (A_p^{1/2} - I) \right] \right| \]
\[ \times \left| \frac{1}{n} [XB_nX^T - (\text{tr}B_n)I_p] \right| \]
\[ \leq C_0 \times \left| \frac{1}{n} \|XB_nX^T\| \times \frac{1}{n} [XB_nX^T - (\text{tr}B_n)I_p] \right| \]
\[ = o_p(1), \]
where we use the facts $\|n^{-1}XB_nX^T\| \leq C_0$ for some constant $C_0$ and $\|n^{-1}[XB_nX^T - (\text{tr}B_n)I_p]\| \rightarrow 0$ by Lemma 2.3. On the other hand, for $L_1$, denote
\[ J_1 := \frac{1}{n} \left[ (A_p^{1/2} - I)XB_nX^T (A_p^{1/2} - I) + (A_p^{1/2} - I)XB_nX^T + XB_nX^T (A_p^{1/2} - I) \right]. \]

Then, by $\|n^{-1}[XB_nX^T - (\text{tr}B_n)I_p]\| \rightarrow 0$, it’s easy to see
\[ \left\| J_1 - \frac{1}{n} \left[ (\text{tr}B_n)(A_p^{1/2} - I)^2 + 2(\text{tr}B_n)(A_p^{1/2} - I) \right] \right\| = o_p(1), \]

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which further implies

\[\left| L_1 - \frac{(\text{tr}B_n)^2}{n^2 p} \text{tr} \left[ (A_p^{1/2} - I)^2 + 2(A_p^{1/2} - I) \right]^2 \right| = o_p(1).\]

Note that

\[\frac{1}{p} \text{tr} \left[ (A_p^{1/2} - I)^2 + 2(A_p^{1/2} - I) \right]^2 = \frac{1}{p} \text{tr} [A_p - I]^2 \geq c_0,
\]

for sufficiently large \( p \). Therefore, with probability approaching 1,

\[
\frac{1}{n} \left| \text{tr} \bar{S}_p^2 - \text{tr} (\bar{S}_p^0)^2 \right| \geq c_0 \Rightarrow \left| \text{tr} \bar{S}_p^2 - \text{tr} (\bar{S}_p^0)^2 \right| \to \infty.
\]

Under \( H_{1,1} \) or \( H_{1,2} \), we have \( p^{-1} \text{tr} (A_p - I)^2 \geq c_0 \) for sufficiently large \( p \), which concludes the theorem.

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