TORIC POISSON STRUCTURES

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Abstract. Let $T_C$ be a complex algebraic torus and let $X(\Sigma)$ be a smooth projective $T_C$-variety. In this paper, a real $T_C$-invariant Poisson structure $\Pi_\Sigma$ is constructed on the complex manifold $X(\Sigma)$, the symplectic leaves of which are the $T_C$-orbits in $X(\Sigma)$. It is shown that each leaf admits a Hamiltonian action by a sub-torus of the compact torus $T \subset T_C$. However, the global action of $T_C$ on $(X(\Sigma), \Pi_\Sigma)$ is Poisson but not Hamiltonian. The main result of the paper is a lower bound for the first Poisson cohomology of these structures. For the simplest case, $X(\Sigma) = \mathbb{C}P^1$, the Poisson cohomology is computed using a Mayer-Vietoris argument and known results on planar quadratic Poisson structures and in the example the bound is optimal. The paper concludes with the example of $\mathbb{C}P^n$, where the modular vector field with respect to a particular Delzant Liouville form admits a curious formula in terms of Delzant moment data. This formula enables one to compute the zero locus of this modular vector field and relate it to the Euclidean geometry of the moment simplex.

1. Introduction

Symplectic structures on compact toric manifolds $X(\Sigma)$ have been extensively studied (cf. [3], [10], [12]) providing a wealth of examples of compact symplectic manifolds with symmetry. The purpose of this paper is to construct and study a real Poisson structure $\Pi_\Sigma$ on $X(\Sigma)$ whose symplectic leaves are the complex torus orbits, so that $\Pi_\Sigma$ is not regular and yet has a dense open leaf. Since degenerate Poisson structures play an important role in study of homogeneous spaces (e.g.: groups [6], [14]; flag varieties [6], [7], [19]; symmetric spaces [2], [8], [21]), it is interesting to consider such Poisson structures in the almost homogeneous setting of toric varieties. In the natural system of holomorphic coordinates, $\Pi_\Sigma$ turns out to be a real quadratic Poisson structure. Structures of these types have found application in combinatorics ([9], [22]) and have been the focus of first attempts at developing a theory of local invariants for Poisson structures ([4], [13]). In addition to the construction of $\Pi_\Sigma$ and its local form, global properties are addressed, such as the symplectic geometry of its leaves and the modular class, the main result being the following lower bound for the first Poisson cohomology (cf. section 5).

Theorem 1.1. $\dim_{\mathbb{R}} H^1(X(\Sigma), \Pi_\Sigma) \geq 2n + 1$ where $n$ is the complex dimension of the maximal torus acting effectively on $X(\Sigma)$.

A standard example used for illustrating differences between symplectic and Poisson geometry is the quadratic Poisson structure in the plane

$$\Pi = \frac{1}{2}(x^2 + y^2)\partial_x \wedge \partial_y.$$ 

It is degenerate only at the origin and the open symplectic leaf has interesting topology. Motivated by questions in [18], Nakanishi in [15] computed the Poisson
cohomology of this structure showing that it was finite dimensional and that, in particular, \( \dim H^1(\mathbb{R}^2, \Pi) = 2 \). It is invariant under the action of the circle \( T \) by rotations of the plane about the origin and on the open symplectic leaf \( \mathbb{R}^2 \setminus \{0\} \), this action is Hamiltonian. However, the global action of \( T \) on \( (\mathbb{R}^2, \Pi) \) is Poisson but not Hamiltonian as the momentum map on \( \mathbb{R}^2 \setminus \{0\} \) fails to extend to the origin, having a logarithmic singularity there. This generates one dimension in the Mayer-Vietoris argument and the theorems of Nakanishi.

The source of these infinitesimal outer automorphisms can be seen by rewriting \( \Pi \) in complex coordinates \( z = x + iy \). Then \( \Pi = i|z|^2 \partial_x \wedge \partial_z \) and a short calculation shows that \( \Pi \) is invariant under the action of the complex torus \( T_C \). Indeed, if \( \zeta \in T_C \), then \( z \mapsto z \zeta = z' \) sends \( z \partial_z \mapsto z' \partial_{z'} \) and thus preserves \( \Pi \). The local Hamiltonian functions for the action of \( T_C/T \) fail to extend to the entirety of the open symplectic leaf for topological reasons, let alone to all of \( C \). So, the real Lie algebra \( \mathbb{R} + i\mathbb{R} \) of the complex torus \( T_C \) is included in \( H^1(C, \Pi) \).

In this paper, this example is generalized to Poisson structures on smooth compact toric varieties. Let \( T \) be a compact abelian Lie group. Let \( \Lambda^* \subset t^* \) denote the dual group of \( T \) viewed as lattice in the dual of its Lie algebra \( t \). Given a Delzant polytope \( \Delta \) in \( t^* \), one can construct a smooth algebraic manifold \( X(\Sigma) \) from the data of its dual fan \( \Sigma \). The complexified torus \( T_C \) acts on \( X(\Sigma) \) with an open dense orbit. A version of this construction is presented in section 2 in order to establish some notation to be used in later sections. The exposition there is a blend of similar discussions in [17], and chapter VII of [1], in the smooth projective case. Although standard, it is included here for completeness as results in subsequent sections depend on the details of this construction.

Roughly, \( X(\Sigma) \) arises as the quotient of \( \mathbb{C}^d \), where \( d \) is the number of facets of \( \Delta \), by the action of a complex algebraic torus. The Poisson structure \( \Pi_{\Sigma} \) is that co-induced by the quotient map from the Poisson structure \( \Pi \oplus \Pi \oplus \cdots \oplus \Pi \) on \( \mathbb{C}^d \). On \( X(\Sigma) \) there is a distinguished system of complex algebraic coordinate charts associated to the vertices of \( \Delta \) and in these coordinates \( \Pi_{\Sigma} \) is a homogeneous quadratic Poisson structure. The symplectic leaves of the prototype on \( \mathbb{C}^d \) are precisely the orbits of \( T_C^\times \) on \( \mathbb{C}^d \). Perhaps not surprising then is the result that the symplectic leaves of \( \Pi_{\Sigma} \) on \( X(\Sigma) \) turn out to be the \( T_C \)-orbits in \( X(\Sigma) \). By construction, the action of \( T_C \) on \( (X(\Sigma), \Pi_{\Sigma}) \) is Poisson, but no subgroup of \( T_C \) acts in a globally Hamiltonian way. However, each symplectic leaf admits a Hamiltonian action by a sub-torus of the compact torus \( T \). This is the content of section 4.

Considering the results of Nakanishi in the planar case, a natural question to ask is whether the image of \( t + it \) in \( H^1(X(\Sigma), \Pi_{\Sigma}) \) generates it. In section 5 it is shown that this is not the case. The modular class of \( (X(\Sigma), \Pi_{\Sigma}) \) is always non-trivial and and independent of the image of \( t + it \). While this does not pin down \( H^1(X(\Sigma), \Pi_{\Sigma}) \) in general, it does exhaust the classes for the simplest toric variety, \( \mathbb{C}P^1 \). In section 6.2, the Poisson cohomology of \((\mathbb{C}P^1, \Pi_{\Sigma})\) is computed using a Mayer-Vietoris argument and the theorems of Nakanishi.

Of course, much has been written about the connection between smooth projective \( T_C \)-varieties and smooth compact connected symplectic manifolds of dimension \( 2 \dim T \) admitting a Hamiltonian action by \( T \). Through the fundamental work of Delzant [3] such symplectic manifolds are characterized by their momentum polytope \( \Delta \subset t^* \) in that each is isomorphic to \( X(\Sigma) \) (where \( \Sigma \) is the dual fan of \( \Delta \))
equipped with Kähler metric with respect to which the action of $T$ is Hamiltonian with momentum map image $\Delta$. The family of Delzant polytopes having a given dual fan thus produces a family of Kähler metrics on $X(\Sigma)$. It has thus far proved difficult to assess the compatibility of $\Pi_{\Sigma}$ with a Poisson structure $\pi_{\Delta}$ determined by Delzant’s construction. However, the final section of this paper contains a curious formula for the modular vector field of $\Pi_{\Sigma}$ on $\mathbb{CP}^n$ with respect to a particular Delzant Liouville volume $\frac{1}{n!}\omega^\Delta_{\Sigma}$ in terms of Delzant momentum map data. A consequence of this formula is a characterization of the zero set of the modular vector field in terms of the centroids of the faces of the momentum simplex.

Before continuing to the next section, some notation and conventions to be used in the paper are fixed. Throughout the paper $T$ denotes the group of complex numbers of modulus one and $h$ will denote a fixed, but arbitrary, positive real parameter. The Lie algebra of $T$ will be identified with $\mathbb{R}$ by the map $\mathbb{R} \rightarrow T : s \mapsto \exp(s \pi) h$ where $h$ is $h$ divided by the circumference of the unit circle. The kernel of this map is the lattice $h\mathbb{Z} \subset \mathbb{R}$. By taking products, this induces an identification of the Lie algebra of $T^d$ with $\mathbb{R}^d$ and the kernel of the product map $\mathbb{R}^d \rightarrow T^d$ is the lattice $(h\mathbb{Z})^d \subset \mathbb{R}^d$. For a convex set $F$ in a real vector space, $F^\circ$ will denote the relative interior, i.e., its interior as a topological subspace of its affine hull.

If $\omega$ is a 2-form on a smooth manifold $M$, we denote by $\omega^\#: TM \rightarrow T^*M$ the bundle map defined by contraction in the second argument, $v \mapsto \omega^\#(v) := \omega(\cdot, v)$. If $\pi$ is a bi-vector field on $M$ then $\pi^\#: T^*M \rightarrow T**M$ is defined by contraction in the first argument, $\nu \mapsto \pi^\#(\nu) := \pi(\nu, \cdot)$. If $M$ is finite dimensional, as are all the spaces considered in this article, then there is a canonical bundle isomorphism $T^**M \simeq TM$. In that case, if $\omega$ is a symplectic form then $\omega^\#$ is an isomorphism at each point and the image of $\omega$ under the map $\bigwedge^2 T^*M \rightarrow \bigwedge^2 T**M$ induced by $(\omega^\#)^{-1}$ can be canonically identified with a non-degenerate Poisson structure $\pi$ on $M$. With these conventions, $\pi^\# \circ \omega^\#: TM \rightarrow TM$ is the identity and the Hamiltonian function for the standard Poisson structure generating rigid counter-clockwise rotation of the plane is positive, i.e., if $\pi = \partial_x \wedge \partial_y$, then $\pi^\#(d\left(\frac{1}{2}(x^2 + y^2)\right)) = -y\partial_x + x\partial_y$.

We will regard $T\mathbb{C}$ as acting on $\mathbb{C}$ by right multiplication, $\mathbb{C} \times T\mathbb{C} \rightarrow \mathbb{C} : (z, \zeta) \mapsto z\zeta$. As a real vector space, $\mathbb{C}$ will be identified with $\mathbb{R}^2$ via the map $z \mapsto \text{Re}(z)e_1 + \text{Im}(z)e_2$. The conventions adopted here for the relation between tensors in this complex coordinate and in the real coordinates $x = \text{Re}(z)$ and $y = \text{Im}(z)$ are the following. Set $dz = dx +idy$ and $d\overline{z} = dx -idy$, and $\partial_x = \frac{1}{2}(\partial_x - i\partial_y)$ and $\partial_y = \frac{1}{2}(\partial_x + i\partial_y)$ so that $\partial_x = \partial_x + i\partial_y = 2\text{Re}(\partial_x)$ and $\partial_y = (i\partial_x - i\partial_y) = 2\text{Re}(i\partial_x)$ and $\partial_x$, $\partial_y$ is dual as a complex basis to $dz, d\overline{z}$. In short, the holomorphic cotangent bundle has been identified with the real cotangent bundle using the real projection, whereas the holomorphic tangent bundle has been identified with the real tangent bundle using twice the real projection. Under the identifications, the field $z\partial_x + \text{c.c.}$ is radial vector field $x\partial_x + y\partial_y$ and $iz\partial_x + \text{c.c.}$ is the vector field generating rigid counter-clockwise rotation about the origin.

The organization of the paper is as follows. Section 2 reviews the construction of smooth compact $T\mathbb{C}$-manifolds. The Delzant construction is reviewed in section 3 for later use in section 4 where a formula involving Delzant moment data is derived.

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1. In dealing with analysis on the unit circle, factors of its circumference show up everywhere in computations and different authors have used different conventions in attempts to deposit these factors in one place or another. The reason for inserting the parameter $h$ and the symbol $h$ here is simply to keep track of those factors while freeing up the symbol $\pi$ to represent other things.
for the modular vector field of $\Pi_{\mathbb{C}}$ on $\mathbb{CP}^n$ relative to the Delzant Liouville volume form. The Poisson structure $\Pi_{\mathbb{C}}$ is constructed in section 4 and its local geometry and the symplectic geometry of its leaves are analyzed. The main result on the first Poisson cohomology of $(X(\Sigma), \Pi_{\mathbb{C}})$ is proved in section 5 and in section 6.2 the Poisson cohomology of $(\mathbb{CP}^1, \Pi_{\mathbb{C}})$ is computed by a Mayer-Vietoris argument.

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2. COMPACT SMOOTH ALGEBRAIC $T_{\mathbb{C}}$-MANIFOLDS.

Let $T$ be a compact abelian Lie group of dimension $n$. Write $\Lambda^*$ for the dual group of $T$ viewed as a lattice in $t^*$, the dual of the Lie algebra $t$ of $T$, and let $\Lambda$ denote the dual lattice in $t$. Let $t_{\mathbb{C}}$ and $T_{\mathbb{C}}$ denote the complexifications of $t$ of $T$, respectively.

Let $\Delta$ be a convex polytope in $t^*$. An edge emanating from a vertex $\nu \in \Delta$ is said to be generated by $\eta \in t^*$ if and only if the edge is contained in the ray $\{\nu + t\eta : t > 0\}$. A convex polytope $\Delta$ in $t^*$ is Delzant if and only if for each vertex of $\Delta$, the edges emanating from $\nu$ are generated by a basis of $\Lambda^*$. Given a Delzant polytope $\Delta$ in $t^*$, let $\Sigma$ denote the fan in $\Lambda$ over its dual polytope (cf. [12] §1.5). To $\Sigma$ one can associate a smooth complex projective variety $X(\Sigma)$ on which $T_{\mathbb{C}}$ acts with an open orbit. As a complex manifold, $X(\Sigma)$ may be constructed as the quotient an open subset of $\mathbb{C}^d$ by the free action of a complex torus.

The combinatorial properties of the polytope $\Delta$, and hence of the dual fan $\Sigma$, are the essential data used in the construction. So, there are two dual points of view one can take, using either then polytope or the fan to index the relevant quantities. From the point of view of algebraic geometry, it is natural to use fan as the constructions can be generalized to singular toric varieties whose fans are not the dual fan of a Delzant polytope. From the point of view of symplectic geometry, however, the polytope is natural choice. The subject of Poisson geometry overlaps both of these worlds, so this article will adopt the parallel use of both the polytope and its dual fan.

Let $\Delta$ be a Delzant polytope in $t^*$ and enumerate the facets $1, 2, \ldots, d$ of $\Delta$. Correspondingly, this enumerates the elements of the 1-skeleton of $\Sigma$, denoted $\Sigma^{(1)}$. This enumeration sets up an inclusion reversing correspondence between the set of faces of $\Delta$ and a subset of the power set of $\{1, 2, \ldots, d\}$, partially ordered by inclusion. Namely, each face $F$ of codimension $f$ corresponds to the subset $\{s_1, \ldots, s_f\}$ of labels of the facets whose intersection is $F$. For the $\ell^{th}$ ray in $\Sigma^{(1)}$, let $u_{\ell}$ be the unique primitive element of $\Lambda$ which generates it. Then the cone in $\Lambda$ generated by $\{u_{s_1}, u_{s_2}, \ldots, u_{s_f}\}$, denoted cone $\{u_{s_1}, \ldots, u_{s_f}\}$, is the strongly convex rational polyhedral cone in $\Sigma$ determined by $F$. As $\Delta$ necessarily has dimension $n$, each vertex $V$ corresponds to a subset $\{s_1, \ldots, s_n\}$ of cardinality $n = \dim t$. Order $u_{s_1}, \ldots, u_{s_n}$ by requiring that $s_1 < s_2 < \cdots < s_n$, then $u_{s_1}, \ldots, u_{s_n}$ is a basis of $t$ and the dual basis of $t^*$ generates the edges of $\Delta$ emanating from the vertex $V$.

**Notation 2.1.** Define a $\mathbb{Z}$-linear map $p: (h\mathbb{Z})^d \rightarrow \Lambda$ by the assignments $h\epsilon_\ell \mapsto u_\ell$ for each $\ell = 1, 2, \ldots, d$ and also write $p: \mathbb{R}^d \rightarrow t$ and $p: \mathbb{C}^d \rightarrow t_{\mathbb{C}}$ for the extensions to real and complex scalars, respectively. Up to the enumeration of the facets of $\Delta$ (or equivalently the rays in $\Sigma^{(1)}$) the map $p$ is uniquely associated with $\Delta$ (and thus $\Sigma$). Let $n \subset \mathbb{R}^d$ denote the kernel of $p: \mathbb{R}^d \rightarrow t$ so that $n \cap (h\mathbb{Z})^d$ is the kernel of $p: (h\mathbb{Z})^d \rightarrow \Lambda$ and $n_{\mathbb{C}} = \ker(p: \mathbb{C}^d \rightarrow t_{\mathbb{C}})$ is the complexification of $n$. Let
\[\iota: n \cap (h\mathbb{Z})^d \to (h\mathbb{Z})^d\] denote the inclusion and, as before, also write \(\iota: n \to \mathbb{R}^d\) and \(\iota: n_C \to \mathbb{C}^d\) for the extensions to real and complex scalars, respectively.

Let \(N_C \subset \mathbb{T}_C\) denote the complex algebraic subgroup of \(\mathbb{T}_C\) corresponding to \(n_C\). Essentially, \(X(\Sigma)\) is the quotient of \(\mathbb{C}^d\) by the action of \(N_C\). More precisely, it is the quotient by \(N_C\) of a dense open subset \(U_\Sigma \subset \mathbb{C}^d\) on which \(N_C\) acts freely. As \(U_\Sigma\) is defined as a union of \(\mathbb{T}_C^d\) orbits on \(\mathbb{C}^d\), notation for describing these must be introduced.

**Notation 2.2.** Given \(S \subset \{1, 2, \ldots, d\}\), let

\[(C_0)^S = \{z = (z_1, \ldots, z_d) \in \mathbb{C}^d: z_\ell = 0 \Leftrightarrow \ell \in S^c\}\]

where \(S^c\) denotes the complement of \(S\).

For \(d = 1\), the orbits of \(\mathbb{T}_C\) are the origin \((C_0)^0\) and its complement \((C_0)^1\). By taking products it is clear that the assignment \(S \mapsto (C_0)^S\) gives a bijection between the orbits of \(\mathbb{T}_C^d\) on \(\mathbb{C}^d\) and the powerset of \(\{1, 2, \ldots, d\}\). It is important to note that the closure of \((C_0)^S\) is the union of the orbits \((C_0)^{S'}\) indexed by \(S' \subset S\), and thus equals the linear subspace \(\mathbb{C}^S \subset \mathbb{C}^d\) spanned by \(\{e_s: s \in S\}\). The isotropy subgroup of \((C_0)^S\) is \(\mathbb{T}_C^S := \exp(C^{S'})\).

**Notation 2.3.** Let

\[U_\Sigma := \bigcup_S (C_0)^S \subset \mathbb{C}^d\]

where the union is taken over \(S \subset \{1, 2, \ldots, d\}\) such that \(S^c\) labels a face of \(\Delta\) (equivalently, the subsets \(S\) such that cone \(\{u_s: s \in S^c\} \in \Sigma\)).

**Proposition 2.4.** \(U_\Sigma\) is an open subset of \(\mathbb{C}^d\) on which \(N_C\) acts freely.

**Proof.** The union in (1) is disjoint since \((C_0)^{S_1}\) and \((C_0)^{S_2}\) are different \(\mathbb{T}_C^d\)-orbits if \(S_1 \neq S_2\). In fact, as \(\mathbb{C}^d\) is a disjoint union of \(\mathbb{T}_C^d\)-orbits,

\[U_\Sigma = \mathbb{C}^d \setminus \bigcup_{S'} (C_0)^{S'}\]

where the union is over the subsets \(S'\) such that \((S')^c\) does not label a face of \(\Delta\). Note that if \(S \subset S'\) and \((S')^c\) does not label a face of \(\Delta\) then neither does \(S\) because \(S^c \supset (S')^c\). Hence, if \((S')^c\) does not label a face of \(\Delta\), then the closure of \((C_0)^{S'}\), i.e.,

\[\mathbb{C}^{S'} := \bigcup_{S \subset S'} (C_0)^S\]

is contained in the union \(\bigcup_S (C_0)^S\) over sets \(S\) for which \(S^c\) does not label a face of \(\Delta\). It follows that the set being deleted from \(\mathbb{C}^d\) in (2) is a closed set, and hence \(U_\Sigma\) is open.

To prove that the \(N_C\)-action is free, one needs to show that the isotropy subgroup of \(N_C\) at each point of \(U_\Sigma\) is trivial. Consider one \(\mathbb{T}_C^d\)-orbit \((C_0)^S\) in \(U_\Sigma\). The isotropy subgroup of each \(z \in (C_0)^S\) is \(N_C \cap \mathbb{T}_C^S\). If \(S^c\) labels a face of \(\Delta\) then the set \(\{u_s: s \in S^c\}\) is a subset of an integral basis of \(\Lambda\). Thus \(N_C \cap \mathbb{T}_C^S = 1\). \(\square\)

**Notation 2.5.** The quotient space \(U_\Sigma/N_C\) is a smooth complex manifold with a right action of the complex torus \(\mathbb{T}_C \simeq N_C/\mathbb{T}_C^d\). Denote this quotient by \(X(\Sigma)\).

The following proposition is obvious.
Proposition 2.6. The $T_C$-orbits in $X(\Sigma)$ of complex codimension $\ell$ are in bijection with the faces of the polytope $\Delta$ of real codimension $\ell$ (equivalently, the cones in $\Sigma$ of dimension $\ell$) for each $\ell = 0, 1, \ldots, n$. In particular, the interior of $\Delta$ (the zero cone in $\Sigma$) corresponds to an open dense orbit of $T_C$ and the vertices of $\Delta$ (the $n$-dimensional cones in $\Sigma$) correspond to the fixed points of $T_C$.

The remainder of this section is devoted to the construction of a distinguished atlas of local holomorphic coordinates for $X(\Sigma)$. Given $z = (z_1, \ldots, z_d) \in U_\Sigma \subset \mathbb{C}^d$, write $[z_1 : z_2 : \ldots : z_d]$ for the class $zN_C$ in $X(\Sigma)$.

Proposition 2.7. Given a vertex $V$ of $\Delta$ (or an $n$-dimensional cone of $\Sigma$), let $U_V = \cup_{S(C_0)^S}$ where the union is taken over all subsets $S \subset \{1, 2, \ldots, d\}$ such that $S^c \subset S$, i.e., $S^c$ labels a face containing $V$ ($S^c$ labels a cone in $\Sigma$ contained in the cone corresponding to $V$). Then $\{U_V/N_C : V$ is a vertex of $\Delta\}$ is an open cover of $X(\Sigma)$.

Proof. Let $V$ be a vertex of $\Delta$. The faces of $\Delta$ containing $V$ are in bijection with the subsets $S^c \subset S_V$. Hence $U_V \subset U_\Sigma$. The complement is the union

$$U_\Sigma \setminus U_V = \bigcup_{S} (C_0)^S$$

where the union is taken over the subsets $S$ such that $S^c$ labels a face of $\Delta$, but $S^c \nsubseteq S_V$. If $S^c \nsubseteq S_V$, then every set containing $S^c$ does not contain $S_V$. Thus, the closure of $(C_0)^S$ belongs to the union in (3). Being equal to a finite union of closed sets, $U_\Sigma \setminus U_V$ is therefore closed. Hence $U_V$ is open in $U_\Sigma$ and $U_V/N_C$ is open in $X(\Sigma)$.

For each vertex, there is at least one facet which does not meet it. Hence, the sets $U_V$ such that $V$ is a vertex of $\Delta$ cover $U_\Sigma$. The result follows. $\square$

Proposition 2.8. For each vertex $V$ of $\Delta$ (or an $n$-dimensional cone of $\Sigma$), the principal $N_C$-bundle $U_\Sigma \to X(\Sigma)$ is trivial over the open set $U_V/N_C$.

Proof. Let $V$ be a vertex of $\Delta$. No generality is lost in assuming that $S_V = \{1, 2, \ldots, n\}$ as the facets of $\Delta$ may be relabeled if necessary. Then $U_V = \{z \in \mathbb{C}^d : z_\ell \neq 0, \ell = n + 1, n + 2, \ldots, d\}$.

As $u_1, u_2, \ldots, u_n$ form a basis for $\Lambda$, there exists a unique integral $n \times (d - n)$ matrix $A$ such that $u_{n+\ell} + \sum_{k=1}^{n} A_{k\ell} u_k = 0$ for each $\ell = 1, 2, \ldots, d - n$. The vectors $h c_1, \ldots, h c_{d-n}$ where $c_\ell = e_{n+\ell} + \sum_{k=1}^{n} A_{k\ell} u_k$ then form a basis for the kernel of $p : (h\mathbb{Z})^d \to \Lambda$. The induced isomorphism $T^d_{C} \simeq N_C$ then parameterizes the action of $N_C$ on $\mathbb{C}^d$ so that $\zeta = (\zeta_1, \ldots, \zeta_{d-n}) \in T^d_{C} \simeq N_C$ acts on $z = (z_1, \ldots, z_d) \in \mathbb{C}^d$ by

$$z\zeta = (z_1 \zeta_1^{a_1}, \ldots, z_n \zeta_1^{a_n}, z_{n+1} \zeta_1, \ldots, z_d \zeta_{d-n})$$

where $\zeta^{a_\ell}$ denotes the monomial $\zeta_1^{a_1} \cdots \zeta_{d-n}^{a_{d-n}}$. Thus, the quotient map $U_V \to U_V/N_C$ has the form

$$(z_1, \ldots, z_d) \mapsto [z_1 \tilde{z}^{-a_1} : \cdots : z_n \tilde{z}^{-a_n} : 1 : \cdots : 1].$$

where $\tilde{z}^{-a_\ell}$ denotes the monomial $z_{n+1}^{-a_1} \cdots z_{n+d-1}^{-a_{d-n}}$. The map $U_V/N_C \to U_\Sigma$ defined by $[w_1 : \cdots : w_n : 1 : \cdots : 1] \mapsto (w_1, \ldots, w_n, 1, \ldots, 1)$ clearly gives a section, trivializing the $N_C$-bundle. $\square$

Notation 2.9. The previous two propositions show that there is a distinguished coordinate chart $\psi : U_V/N_C \to \mathbb{C}^n$ associated to every vertex of $\Delta$ (or $n$-dimensional
cone in $\Sigma$). Given a vertex $V$ labeled by $S_V = \{s_1, \ldots, s_n\}$, $s_1 < s_2 < \ldots < s_d$
let $\sigma$ denote the permutation sending $k \mapsto s_k$ for $k = 1, 2, \ldots, n$ and $n + \ell \mapsto s_{n+\ell}$
for $\ell = 1, 2, \ldots, d - n$ where $s_{n+1} < \ldots < s_d$. Find the unique set of integers
$a_{k\ell}$ such that $u_{s_{n+\ell}} + \sum_{k=1}^n a_{k\ell} u_{s_k} = 0$ for each $\ell = 1, 2, \ldots, d - n$. Then define
$w_V : U_V / N_C \to \mathbb{C}^n$ by $z N_C \mapsto (w_1, \ldots, w_n)$ where $w_\ell = z_{s_{\ell}} z^{-a_\ell}$ and $z^{-a_\ell}$ is the
monomial $z_{s_{n+1}} \ldots z_{s_d}$.
In shorthand notation, $w = z^{[1|-A]_\sigma}$.

It should be noted that in the coordinates associated to vertex $V$ labeled by $S_V = \{s_1, \ldots, s_n\}$, the elements $u_{s_1}, \ldots, u_{s_n} \in \mathfrak{t}$ act by the vector fields
$(w/h) i w_1 \partial_{w_1} + c.c., \ldots, (w/h) i w_n \partial_{w_n} + c.c.,$
respectively. Thus, the open $T_C$ orbit is characterized by the algebraic inequality
$w_1 \ldots w_n \neq 0$.

3. The Delzant Construction

The original symplectic convexity theorem of Atiyah and Guillemin-Sternberg
asserts that the image of the momentum map $\Phi : X \to \mathfrak{t}^*$ of a Hamiltonian torus ac-
ction by $T$ on a compact connected symplectic manifold $(X, \omega)$ is a convex polytope
$\Delta \subset \mathfrak{t}^*$. In [3], Delzant determined the geometry of $\Delta$ under the additional assumptions
that the $T$-action is effective and $\dim X = 2 \dim T$. What is more, given $\Delta$ he
produced a construction of a compact connected Hamiltonian $T$-manifold (which is
also Kähler) $(X_\Delta, \omega_\Delta, \Phi_\Delta)$ of dimension $2 \dim T$ on which $T$ acted effectively with
$\Phi_\Delta(X_\Delta) = \Delta$. Further, he showed that any other such Hamiltonian $T$-manifold
$(X, \omega, \Phi)$ with $\Phi(X) = \Delta$ was necessarily $T$-equivariantly symplectomorphic to
$(X_\Delta, \omega_\Delta, \Phi_\Delta)$. A consequence of a theorem of Frances Kirwan (cf. Theorem 7.4
in [11]) is that $X_{\Delta}$ is homeomorphic to the compact smooth
$T_C$-manifold $X(\Sigma)$ constructed from the dual fan $\Sigma$ of $\Delta$. For torus actions, every range translate of
a momentum map is again a momentum map. Thus, Delzant’s construction yields a family of Kähler metrics on $X(\Sigma)$ parameterized by the translation classes of Delzant polytopes in $\mathfrak{t}^*$.

In the briefest possible terms, here is the construction. Let $\Delta \subset \mathfrak{t}^*$ be a Delzant
polytope with $d$ facets. When $\mathbb{C}^d$ is equipped with the non-degenerate Poisson structure

$$\pi = \sum_{\ell=1}^d i \partial_{z_\ell} \wedge \partial_{\bar{z}_\ell},$$

the standard action of the real torus $T^d$ on $\mathbb{C}^d$ is Hamiltonian and $J : \mathbb{C}^d \to (\mathbb{R}^d)^*$
given by

$$J(z) = \sum_{\ell=1}^d \frac{1}{2} |z_\ell|^2 \epsilon_\ell$$

is a momentum map. Let $\omega$ denote the symplectic form induced by $\pi$. Fixing an
enumeration of the facets of $\Delta$, define $p$, $t$, and $n$, as in Notation 2.1. Taking duals, the sequence

$$0 \leftarrow n^* \leftarrow \mathbb{R}^d)^* \leftarrow (\mathbb{R}^d)^* \leftarrow (\mathbb{R}^d)^* \leftarrow t^* \leftarrow 0$$

is exact. The composition $\mu = t^* \circ J : (\mathbb{R}^d)^* \to n^*$ is a momentum map for the
action of $N = \exp(n) \subset T^d$ on $({\mathbb{C}^d}, \pi)$. 

For $\lambda \in (\mathbb{R}^d)^*$, write $\lambda \geq 0$ if $\lambda$ takes non-negative values on the cone spanned by $\{e_1, \ldots, e_d\}$ in $\mathbb{R}^d$ and write $P^* = \{\lambda \in (\mathbb{R}^d)^*: \lambda \geq 0\}$. Assume that $0 \in \Delta^o$, then there exists a unique $\lambda \in (P^*)^o$ such that

$$\Delta = \{\nu \in \mathfrak{t}^*: hp^*(\nu) + \lambda \geq 0\}. \tag{7}$$

The method of Delzant is to construct $(X_\Delta, \omega_\Delta)$ by symplectic reduction of the Hamiltonian $N$-manifold $(\mathbb{C}^d, \omega, \mu)$ at the $\mu$-regular value $\kappa = t^*(\lambda)$.

The image under $J$ of the level set $\mu^{-1}(\kappa)$ is precisely the image of $\Delta \subset \mathfrak{t}^*$ under the affine map $\nu \mapsto hp^*(\nu) + \lambda$. As $\mu$ is $\mathbb{T}^d$-invariant, there exists a smooth $T \simeq N/\mathbb{T}^d$-invariant map $\Phi_\Delta: \mu^{-1}(\kappa)/N \to \mathfrak{t}^*$ making the diagram

$$\begin{array}{ccc}
\mu^{-1}(\kappa)/N & \xrightarrow{h^p + \lambda} & \mathfrak{t}^* \\
\downarrow & & \downarrow \\
\mu^{-1}(\kappa) & \xrightarrow{J} & (\mathbb{R}^d)^* \\
\end{array} \tag{8}$$

commute. By symplectic reduction, the quotient $X_\Delta = \mu^{-1}(\kappa)/N$ obtains a symplectic structure from $\omega$. Multiplying the induced Poisson structure by the parameter $h$, one obtains a non-degenerate Poisson structure $\pi_\Delta$ on $X_\Delta$ with respect to which the $T$-action is Hamiltonian with momentum map $\Phi_\Delta$. By construction, $\Phi_\Delta(X_\Delta) = \Delta$.

The level set $\mu^{-1}(\kappa)$ is the boundary of a convex set of dimension greater than 1, and thus is connected. Moreover, the map $J$ is proper and carries $\mu^{-1}(\kappa)$ onto the convex polytope $hp^*(\Delta) + \lambda \subset (\mathbb{R}^d)^*$ and is thus compact. Thus $X_\Delta$ is compact and connected and hence so is $X(\Sigma)$ by Kirwan’s theorem.

Of importance for this paper is the fact that $\Phi_\Delta$ provides a correspondence between the $T_\Sigma$-orbits in $X(\Sigma)$ and the faces of $\Delta$. In particular, every $T_\Sigma$-orbit is of the form $\Phi_\Delta^{-1}(F^\circ)$ for a unique face $F$ of $\Delta$, and $T_\Sigma$ orbits of dimension $\ell$ correspond to faces of dimension $\ell$.

**Remark 3.1.** The assumption that $0 \in \Delta^o$ can be dropped if one first translates $\Delta$ to $\Delta' = \Delta - \nu$ for some $\nu \in \Delta^\circ$, so that $0 \in (\Delta')^o$, then applies the construction above to get $\omega_{\Delta'}$ and $\Phi_{\Delta'}$ and sets $\omega_\Delta = \omega_{\Delta'}$ and $\Phi_\Delta = \Phi_{\Delta'} + \nu$.

## 4. Toric Poisson Structures

Let $\Pi$ denote bi-vector field

$$\Pi = \sum_{\ell=1}^d i|z|z_\ell^2 \partial_{z_\ell} \wedge \partial_z \tag{9}$$

on $\mathbb{C}^d$. In the case that $d = 1$, $\Pi$ defines a Poisson structure in the plane.

**Theorem 4.1.** _The bi-vector field $\Pi$ defines a Poisson structure on $\mathbb{C}^d$ which is invariant under the action of the complex torus $T_\Sigma^d$. _

**Proof.** For $d = 1$, $\Pi = |z|^2 \partial_{\bar{z}} \wedge \partial_z$ is automatically Poisson being a bi-vector field on a space of real dimension 2. The bi-vector field in (9) is the product Poisson structure induced on $\mathbb{C}^d$ and is therefore Poisson. Alternatively, one could note that the map $\exp: \mathbb{C}^d \to \mathbb{C}^d$, defined by $(z_1', \ldots, z_d') \mapsto (e^{z_1'}, \ldots, e^{z_d'})$ has open dense image and carries the Poisson structure $\pi$ to $\Pi$. As $\Pi$ is smooth $[\Pi, \Pi] = \exp_*[\pi, \pi] = 0$ on an open dense set, $\Pi$ is Poisson. With this point of view, the invariance of $\Pi$
under the action of $T^d_C$ is an immediate consequence of the translation invariance of $\pi$ on $C^d$. \hfill \Box

It is interesting that this quadratic Poisson structure arises naturally as the image of a constant Poisson structure under the exponential map. In dimension $d = 1$, $\Pi = i|z|^2 \partial_{\bar{z}} \wedge \partial_z$ has precisely two symplectic leaves, the origin in $\mathbb{C}$ and its complement $\mathbb{C}_0$. Given the arguments in the previous proof, the following theorem is immediate.

**Theorem 4.3.** The symplectic leaves of $(\mathbb{C}^d, \Pi)$ are precisely the orbits of $T^d_C$ on $\mathbb{C}^d$.

In particular, although this Poisson structure is not regular, the symplectic leaves are finite in number and can be enumerated by the $2^d$ subsets of the set of indices $\{1, 2, \ldots, d\}$, $S \mapsto (\mathbb{C}_0)^S$ as in the previous section.

**Theorem 4.2.** The symplectic leaves of $(\mathbb{C}^d, \Pi)$ are precisely the orbits of $T^d_C$ on $\mathbb{C}^d$.

**Proof.** Due to the product structure of $\Pi$ it suffices to establish the first claim in dimension $d = 1$. The infinitesimal action of $T$ on $\mathbb{C}$ is generated by the vector field $\frac{i}{\hbar}z \partial_z + c.c.$ The function $\frac{1}{\hbar} \log |z|^2$, defined for $z \neq 0$, satisfies $\Pi^T(\frac{1}{\hbar}d \log |z|^2) = \frac{i}{\hbar}z \partial_z + c.c.$ there. So, on the open symplectic leaf $\mathbb{C}_0$, the action of $T$ is Hamiltonian. Since $\Pi$ is non-degenerate on $\mathbb{C}_0$, any other Hamiltonian function must differ from $\frac{1}{\hbar} \log |z|^2$ by a constant there and since none of these functions extend to the origin, it follows that the action of $T$ on $(\mathbb{C}, \Pi)$ is Poisson but not Hamiltonian. However, its restriction to $\mathbb{C}_0$ is Hamiltonian.

The action of $T_C/T$ is generated infinitesimally by the radial field $z \partial_z + c.c.$ A Hamiltonian function with respect to $\Pi$ for this field is $\frac{1}{\hbar} \text{Arg}(z)$ which can be defined on $\mathbb{C} \setminus (-\infty, 0]$. Of course, this function can not even be extended to $\mathbb{C}_0$, let alone to all of $\mathbb{C}$. So the action of $T_C/T$ is not Hamiltonian on $(\mathbb{C}_0, \Pi)$ or $(\mathbb{C}, \Pi)$.

For general $d$ consider the leaf $(\mathbb{C}_0)^S$ indexed by an arbitrary $S \subset \{1, 2, \ldots, d\}$. By the arguments above, it is clear that the action of $T^d_C/T^S_C$ on $(\mathbb{C}_0)^S$ is Poisson but not globally Hamiltonian. However, one can define

$$\Phi_S(z) = \sum_{s \in S} \frac{1}{\hbar} \log |z_s|^2 \epsilon_s,$$

on $(\mathbb{C}_0)^S$. Then $\Phi_S: (\mathbb{C}_0)^S \to (\mathbb{R}^d)^*$ is a momentum mapping for the action of the compact sub-torus $T^S$ on $(\mathbb{C}_0)^S$. \hfill \Box

Let $T$ be a compact abelian Lie group, let $\Delta$ be a Delzant polytope in $\mathfrak{t}^*$, and let $\Sigma$ be its dual fan in $\mathfrak{t}$. Since $\Pi$ on $\mathbb{C}^d$ is invariant under the action of $T^d_C$, so is its restriction to $U_C$ and thus $\Pi$ descends to the toric variety $X(\Sigma) = U_C/N_C$.

**Notation 4.4.** Let $\Pi_\Sigma$ denote the Poisson structure on $X(\Sigma)$ coinduced by the quotient map $q: (U_C, \Pi) \to U_C/N_C = X(\Sigma)$.

**Theorem 4.5.** The symplectic leaves of $(X(\Sigma), \Pi_\Sigma)$ are in bijection with the set of cones in $\Sigma$, or equivalently, with the set of faces of the polytope $\Delta$. 


The next theorem shows how to construct \( \Pi_\Sigma \) locally and the following notation will arise and be used throughout the rest of the paper.

**Notation 4.6.** Given a vertex \( V \) of \( \Delta \) labeled by the set \( S_V = \{s_1, \ldots, s_n\} \) where \( s_1 < s_2 < \cdots < s_n \). Let \( s_{n+1} < \cdots < s_d \) denote the elements of \( S_V^c \). Let \( A_V \) denote the \( n \times (d - n) \) integral matrix such that \( u_{s_{n+\ell}} + \sum_{k=1}^n u_{s_k} A_{k\ell} = 0 \). Then \( B_V = 1 + A_V (A_V)^t \).

**Theorem 4.7.** Let \( V \) be a vertex of \( \Delta \). This determines a \( n \times (d - n) \) integral matrix \( A_V \) and an associated coordinate chart \( w_V : z N_C \mapsto (w_1, \ldots, w_n) \). In terms of these coordinates the Poisson structure \( \Pi_\Sigma \) has the form

\[
(11) \quad \Pi_\Sigma = \sum_{p,q=1}^n i B_{pq} \bar{w}_p \partial \bar{w}_q \wedge \partial w_q
\]

where \( B_{pq} \) are the components of the symmetric positive definite integral matrix \( B_V = 1 + A_V (A_V)^t \). In particular, \( \Pi_\Sigma \) is a homogeneous quadratic Poisson structure in each vertex chart.

**Proof.** Again, it suffices to consider the case where \( S_V = \{1, 2, \ldots, n\} \). Now for each \( s = 1, 2, \ldots, n \),

\[
\frac{\partial w_s}{\partial z_r} = z_r z_s^{-a_s r} \text{ if } r = 1, 2, \ldots, n
\]

whereas

\[
\frac{\partial w_s}{\partial z_r} = -a_{sr} z_s z_r^{-a_s r} = -a_{sm} w_s
\]

for each \( r = n + m \), where \( m = 1, 2, \ldots, d - n \). Thus, under the quotient map \( z_r \partial z_r \mapsto w_r \partial w_r \) for each \( r = 1, 2, \ldots, n \) whereas \( z_r \partial z_r \mapsto - \sum_{s=1}^n a_{sm} w_s \partial w_s \) for each \( r = n + m \) where \( m = 1, 2, \ldots, d - n \). Hence, \( \Pi \) maps to

\[
\Pi_\Sigma = \sum_{r=1}^n i |w_r|^2 \partial \bar{w}_r \wedge \partial w_r + \sum_{m=1}^{d-n} \left( \sum_{s=1}^n a_{sm} \partial \bar{w}_s \partial w_s \right) \wedge \left( \sum_{t=1}^n a_{tm} w_t \partial w_t \right)
\]

\[
= \sum_{\ell=1}^n \left( 1 + \sum_{m=1}^{d-n} a_{tm} a_{\ell m} \right) i |w_\ell|^2 \partial \bar{w}_s \wedge \partial w_\ell
\]

\[
+ \sum_{k<\ell} \left( \sum_{m=1}^{d-n} a_{km} a_{\ell m} \right) i w_k w_\ell \partial \bar{w}_m \wedge \partial w_\ell + c.c.
\]

The coefficients are clearly the components of the matrix \( B_V = 1 + A_V (A_V)^t \) as claimed. \( \square \)

**Corollary 4.8.** Let \( V \) be a vertex of \( \Delta \). In terms of the vertex coordinates \( w_1, \ldots, w_n \),

\[
(12) \quad \frac{1}{n!} \Pi_\Sigma = i^n \det(B_V) |w_1|^2 \cdots |w_n|^2 \partial \bar{w}_1 \wedge \partial w_1 \wedge \cdots \wedge \partial \bar{w}_n \wedge \partial w_n
\]

**Proof.** Let \( \tilde{\pi} = \sum_{p,q=1}^n B_{pq} \bar{w}_p \partial \bar{z}_q \wedge \partial z_q \) on \( \mathbb{C}^n \). Then \( \tilde{\pi} \mapsto \Pi_\Sigma \) under the map \( z \mapsto \exp(z) = w \). Furthermore,

\[
\frac{1}{n!} \tilde{\pi}^n = i^n \det(B_V) \partial \bar{z}_1 \wedge \partial z_1 \wedge \cdots \wedge \partial \bar{z}_n \wedge \partial z_n
\]
and therefore
\[ \frac{1}{\pi^n} \Pi^\alpha = \frac{1}{\pi^n} (\exp_\pi \pi^n) = \exp_\pi (\frac{1}{\pi^n} \pi^n) = i^n \det (B_V) |w_1|^2 \ldots |w_n|^2 \partial_{w_1} \wedge \ldots \wedge \partial_{w_n} \wedge \partial_{w_n} \]
as was to be shown. \[\square\]

**Theorem 4.9.** The action of $T_C$ on $(X(\Sigma), \Pi^\alpha)$ is Poisson but not Hamiltonian. However, each symplectic leaf of $(X(\Sigma), \Pi^\alpha)$ has a Hamiltonian action by a subtorus of $T$.

**Proof.** Consider the $T_C$-orbit in $X(\Sigma)$ corresponding to a face $F$ of $\Delta$, i.e., $\Phi^{-1}_\Delta(F^\circ)$. In assuming that $F$ is labeled by $\{1, 2, \ldots, s\}$ where $s < n$, no generality is lost. Then this orbit is contained in the open set $U_V/N_C$ where $V$ is the vertex labeled by $\{1, 2, \ldots, n\}$ and is characterized in the coordinates by the inequality $w_1 \ldots w_s \neq 0$ together with the equations $w_{s+1} = 0, \ldots, w_n = 0$. It thus suffices to define the momentum map for the action of $T_F := \exp(\text{span}(u_{s+1}, \ldots, u_n))$ on this leaf in local coordinates.

Let $\eta_1, \ldots, \eta_n$ be the frame for $t^*$ dual to the frame $u_1, \ldots, u_n$ for $t$. Let $B_V$ denote the principal $s \times s$ block of the matrix $B_V$ and let $b_V$ denote its inverse, this exists because $B_V$ is positive definite and therefore each of its principal minors is non-zero. Define $\psi_F : \Phi^{-1}_\Delta(F^\circ) \rightarrow t^*$ by

\[ \psi_F(w_1, \ldots, w_s) = (h/\hbar) \sum_{k, \ell=1}^s b_{k\ell} \log |w_\ell|^2 \eta_k. \]

Then $d(\psi_F, u_m) = \sum_{\ell=1}^s b_{m\ell} d \log |w_\ell|^2$ and thus

\[ \Pi^\alpha_{\Sigma}(d(\psi_F, u_m)) = (h/\hbar) \sum_{p, q=1}^s \sum_{\ell=1}^n iB_{pq} w_p \partial_{\eta_p} \wedge \partial^\#_q \left( \sum_{\ell=1}^n b_{m\ell} \log |w_\ell|^2 \right) \]

\[ = (h/\hbar) \sum_{p, q=1}^s iB_{pq} b_{m\ell} w_q \partial_{\eta_q} + c.c. \]

\[ = (h/\hbar) \sum_{p, q=1}^s iB_{pq} b_{m\ell} w_q \partial_{w_q} + c.c. \]

This shows that the action of $T_F$ on $\Phi^{-1}_\Delta(F^\circ)$ is Hamiltonian. In particular, the action of $T$ on the open symplectic leaf is Hamiltonian, but clearly it does not extend to all of $X(\Sigma)$. This completes the proof. \[\square\]

5. **The Modular Class and $H^1(X(\Sigma), \Pi^\alpha)$**

If $(M, \pi_M)$ is a Poisson manifold, then $d_\pi : \mathcal{V}^k(M) \rightarrow \mathcal{V}^{k+1}(M)$ defined by $d_{\pi_M}(W) = [W, \pi_M]$ for each $W \in \mathcal{V}^k(M)$ is a differential, where $[\cdot, \cdot]$ is the Schouten bracket of multi-vector fields. The algebra of multi-vector fields $\mathcal{V}(M)$ together with $d_{\pi_M}$ forms a complex. The homology $H^\bullet(M, \pi_M)$ of this complex is called the Poisson cohomology of $M$. The anchor map $\pi^\# : T^*M \rightarrow TM$ induces a chain map from the de Rham complex to the Poisson complex and therefore induces a map $\pi^\# : H^\bullet_{DR}(M) \rightarrow H^\bullet(M, \pi_M)$. When $\pi_M$ is non-degenerate on $M$, the induced map is an isomorphism. So for symplectic manifolds, the Poisson cohomology is finite dimensional. At the other extreme, every manifold admits the zero poisson structure $\pi_M = 0$ and in this case $H^\bullet(M, \pi_M) = \mathcal{V}^\bullet(M)$ which is infinite dimensional.
For structures in between these two extremes, the Poisson cohomology can be quite difficult to compute because of the lack of a powerful method for computation.

Every Poisson manifold \((M, \pi_M)\) has a distinguished class in the first Poisson cohomology called the modular class (cf. [20]). A vector field representing this class may be computed as follows. Choose a smooth positive density \(\mu\) on \(M\). For each smooth function \(f\), the Lie derivative of \(\mu\) with respect to the Hamiltonian vector field \(\pi^{\#}(df)\) is again a density and thus can be expressed uniquely as \(\theta_\mu(f)\mu\) where \(\theta_\mu(f)\) is a smooth function. Due to the skew-symmetry of \(\pi_M\), the assignment \(f \mapsto \theta_\mu(f)\) turns out to be a derivation on smooth functions and hence defines a vector field \(\theta_\mu\) called the modular vector field of \(\pi_M\) associated with the density \(\mu\). If \(a\) is a positive function on \(M\), then \(a\theta_\mu\) is a positive density and \(\theta_{a\mu} = \theta_\mu - \pi^{\#}(d \log a)\).

In the case that \((M, \pi_M)\) is a non-degenerate Poisson manifold and \(\mu = \frac{1}{\pi} \pi_M^n\) is the Liouville volume form of the symplectic structure, \(\theta_\mu = 0\).

**Theorem 5.1.** Let \((M, \pi_M)\) be an orientable Poisson manifold with the property that \(\pi_M\) is non-degenerate on a open dense subset of \(M\) but is not regular. Then the modular class of \((M, \pi_M)\) is non-trivial.

**Proof.** Suppose \(\dim M = 2n\) and let \(\mu\) be a nowhere vanishing 2\(n\)-form on \(M\). Let \(\tilde{\mu}\) be the nowhere vanishing 2\(n\)-vector field dual to \(\mu\) in the sense that \((\mu, \tilde{\mu}) = 1\) at each point of \(M\). Let \(U\) be the union of the open symplectic leaves of \(\pi_M\). Then \(\frac{1}{\pi} \pi_M^n = a\tilde{\mu}\) where \(a \in C^\infty(M)\) is a function vanishing on the complement of \(U\). On the open symplectic leaves, \(a\tilde{\mu}\) is dual to the 2\(n\)-form \(\frac{1}{\pi} \mu\) which is the Liouville volume form of \(\pi_M\) on the open symplectic leaves. Thus

\[
\theta_{\frac{1}{\pi} \mu} = \theta_\mu - \pi_M^n(df \log(1/a))
\]

on \(U\) which implies that \(\theta_\mu = -\pi_M^n(df \log(a))\) on \(U\) as \(\theta_{\frac{1}{\pi} \mu} = 0\). Any other primitive of \(\theta_\mu\) on \(U\) must differ from \(d \log(a)\) by a locally constant function and none of these extend to the boundary of \(U\) as \(a = 0\) there. Thus, \(\theta_\mu \neq 0\) in \(H^1(M, \pi_M)\). □

**Corollary 5.2.** The modular class of \((X(\Sigma), \Pi_\Sigma)\) is non-zero.

**Proof.** The toric variety \(X(\Sigma)\) is a complex manifold. Thus it is oriented. As the symplectic leaves of \(\Pi_\Sigma\) are the \(T_C\)-orbits in \(X(\Sigma)\), \(\Pi_\Sigma\) is not regular and yet is non-degenerate on an open dense set. □

A consequence of the proof of Theorem 5.1 is that there is a linear injection \(t + it \to H^1(X(\Sigma), \Pi_\Sigma)\) because the action of \(T_C\) is Poisson and yet fails to be Hamiltonian because the momentum map for the action on the open symplectic leaf fails to extend. It is natural to ask whether the modular class is disjoint from this image and this question is answered by the next result. For brevity, the following notation is introduced.

**Notation 5.3.** If \(u_1, \ldots, u_n\) is the basis of \(\Lambda\) associated to given vertex of \(\Delta\), write \(R_1, \ldots, R_n\) and \(D_1, \ldots, D_n\) for the vector fields on \(X(\Sigma)\) corresponding to \(u_1, \ldots, u_n\) and \(iu_1, \ldots, iu_n\), respectively. In terms of the local coordinates \(w_1, \ldots, w_n\) associated to the same vertex, \(R_k = (h/h)iw_k \partial_{w_k} + c.c.\) and \(D_k = (h/h)w_k \partial_{w_k} + c.c.\) for each \(k = 1, 2, \ldots, n\).

As the action of \(T_C\) is Poisson, the vector fields \(R_k\) and \(D_k\) determine classes in \(H^1(X(\Sigma), \Pi_\Sigma)\) for each \(k\) and a \(\mathbb{R}\)-linear map \(t + it \to H^1(X(\Sigma), \Pi_\Sigma)\).
Theorem 5.4. For each vertex of $\Delta$, the associated classes

$$[R_1, \ldots, [R_n], [D_1], \ldots, [D_n], [\theta_\mu] \in H^1(X(\Sigma), \Pi_\Sigma)$$

are linearly independent. Hence $\dim H^1(X(\Sigma), \Pi_\Sigma) \geq 2n + 1$.

Proof. Let $w_1, \ldots, w_n$ be the associated vertex coordinates. Let

$$\lambda = i^n dw_1 \wedge d\overline{w_1} \wedge \cdots \wedge dw_n \wedge d\overline{w_n}$$

and let $\frac{1}{m!} \Omega^n_\Sigma$ denote the Liouville volume form for $\Pi_\Sigma$ on the symplectic leaf $\{w_1 \ldots w_n \neq 0\}$. Then $\lambda = a \frac{1}{m!} \Omega^n_\Sigma$ where $a = \det(B_V)|w_1|^2 \ldots |w_n|^2$.

Now $d \log a = \sum_{\ell=1}^n d \log |w_\ell|^2$ and therefore

$$\theta_\lambda = -\Pi_\Sigma^d(d \log a)$$

$$= - \sum_{\ell,p,q=1}^n i B_{pq} \overline{w_p} w_q \frac{\delta_{\ell,q}}{w_\ell} + c.c.$$
Theorem 6.2. The Poisson cohomology of $(\mathbb{CP}^1, \Pi_\Sigma)$ is finite dimensional. As real vector spaces

\begin{itemize}
  \item[i)] $H^0(\mathbb{CP}^1, \Pi_\Sigma) = \mathbb{R}$,
  \item[ii)] $H^1(\mathbb{CP}^1, \Pi_\Sigma) = \mathbb{R}^3$, and
  \item[iii)] $H^2(\mathbb{CP}^1, \Pi_\Sigma) = \mathbb{R}^4$.
\end{itemize}

Proof. The proof applies the Mayer-Vietoris sequence for Poisson cohomology to the affine open cover of $\mathbb{CP}^1$ by the open sets $U = U(1)/N_\mathbb{C}$ and $V = U(2)/N_\mathbb{C}$ associated to the vertices of $\Delta$. Let $w$ denote the coordinate in the first chart and $w'$ the coordinate in the second chart. In terms of $w$, $\Pi_\Sigma = 2(h/\hbar) i|w|^2 \partial_{\overline{w}} \wedge \partial_w$ and the expression is identical in terms of $w'$ where $w' = 1/w$ on the overlap. The intersection $U \cap V$ is the open symplectic leaf of $\Pi_\Sigma$, thus $H^*(U \cap V, \Pi_\Sigma) \simeq H^*_D(\mathbb{CP}^1 \cap \mathbb{CP}^1)$. As $\Pi_\Sigma$ is non-degenerate on an open dense set the anchor map induces an isomorphism $H^0(\mathbb{CP}^1, \Pi_\Sigma) \simeq H^0_D(S^1)$. Thus, the initial row of the Mayer-Vietoris sequence is exact and the interesting part begins with $H^1(\mathbb{CP}^1, \Pi_\Sigma)$.

By the theorems of Nakanishi the terms $H^*(U, \Pi_\Sigma)$ are known. The isomorphism $H^*(U \cap V, \Pi_\Sigma) \simeq H^*_{DR}(\mathbb{CP}^1 \cap \mathbb{CP}^1) \simeq H^*_{DR}(S^1)$ gives that $H^1(U \cap V, \Pi_\Sigma) = \mathbb{R}$ and $H^2(U \cap V, \Pi_\Sigma) = 0$. Hence, the interesting part of the Mayer-Vietoris sequence takes the form

$$0 \to H^1(\mathbb{CP}^1, \Pi_\Sigma) \to \mathbb{R}^2 \oplus \mathbb{R}^2 \xrightarrow{\delta} H^2(\mathbb{CP}^1, \Pi_\Sigma) \to \mathbb{R}^2 \oplus \mathbb{R}^2 \to 0$$

where $\delta : H^1(U \cap V, \Pi_\Sigma) \to H^2(\mathbb{CP}^1, \Pi_\Sigma)$ is the connecting homomorphism. From this exact sequence, it is clear that the Poisson cohomology of $(\mathbb{CP}^1, \Pi_\Sigma)$ is finite dimensional as $\dim H^1 = 1 = \dim H^2$ and $\dim H^1 \leq 4$. A generator of $H^1_{DR}(U \cap V)$ is $d\text{Arg}(w)$, and under the anchor map $\Pi_\Sigma^\#(d\text{Arg}(z)) = \frac{2}{(\hbar/\delta)D} \delta$ which extends globally to all of $\mathbb{CP}^1$. Therefore, the connecting homomorphism $\delta : H^1(U \cap V, \Pi_\Sigma) \to H^2(\mathbb{CP}^1, \Pi_\Sigma)$ is the zero map, so $H^1(\mathbb{CP}^1, \Pi_\Sigma) \simeq \mathbb{R}^3$ and $H^2(\mathbb{CP}^1, \Pi_\Sigma) = \mathbb{R}^4$. The proof is complete. \hfill $\square$

It is interesting to note that the $T$-invariant generators of $H^2(\mathbb{CP}^1, \Pi_\Sigma)$ are not all algebraic even though $\Pi_\Sigma$ is real algebraic. The space of $T$-invariant smooth algebraic bi-vector fields on $\mathbb{CP}^1$ is only 3-dimensional as each is of the form

$$i(a + b|w|^2 + c|w|^4) \partial_{\overline{w}} \wedge \partial_w$$

expressed in the coordinate associated to vertex 1 where $a, b, c \in \mathbb{R}$. The bi-vectors $i\partial_{\overline{w}} \wedge \partial_w$, $i|w|^2 \partial_{\overline{w}} \wedge \partial_w$, and $i|w|^4 \partial_{\overline{w}} \wedge \partial_w$, which vanish at fixed point 2 but not 1, vanish at 1 and 2, and vanish 1 but not 2, respectively, form a basis for this space. In their span, with $a = 1$, $b = 2$, and $c = 1$ is the non-degenerate Poisson structure $\pi_\Delta = i(1 + |w|^2)^2 \partial_{\overline{w}} \wedge \partial_w$ arising the Delzant construction. A fourth independent generator is the real analytic bi-vector $-|w|^2 \partial_{\overline{w}} \wedge \partial_w$ which vanishes only at vertex 2 but with infinite order.
7. The modular vector field on \( \mathbb{CP}^n, \Pi_\Sigma \).

Let \( T \) be a Lie group isomorphic to \( \mathbb{T}^n \) and let \( u_1, \ldots, u_n \) be a basis \( \Lambda \subset \mathfrak{t} \) and set \( u_{n+1} = -\sum_{\ell=1}^n u_\ell \). Let \( \eta_1, \ldots, \eta_n \) be the dual basis of \( \Lambda^* \subset \mathfrak{t}^* \). Let \( \Delta' \) be the convex hull of the points \( \{0, \eta_1, \ldots, \eta_n\} \) in \( \mathfrak{t}' \). Then \( \Delta' \) is a standard \( n \)-simplex and a Delzant polytope and its dual fan \( \Sigma \) is the data needed to construct \( \mathbb{CP}^n \) as a toric variety. Every other polytope whose dual fan in \( \Sigma \) differs from \( \Delta' \) only by a translation or overall change of scale. The translate \( \Delta = \Delta' - \nu \) where \( \nu = \frac{1}{n+1}(\eta_1 + \ldots + \eta_n) \) is a Delzant polytope containing the origin as an interior point. The Kähler form \( \omega_{\Delta} \) it determines on \( X(\Sigma) = \mathbb{CP}^n \) is the one relevant for the discussion in this section. Define the map \( A_\lambda \) with \( (14) \)

\[
\frac{1}{n+1}(\epsilon_1 + \ldots + \epsilon_{n+1}), \quad \lambda = \frac{1}{n+1}(\epsilon_1 + \ldots + \epsilon_{n+1}),
\]

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\[
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\]

the associated affine coordinates are

\[
(14) \quad w_1 = z_1 z_{n+1}^{-1}, \ldots, w_n = z_n z_{n+1}^{-1}
\]

as the matrix \( A_V \) is the column vector whose transpose is \((1, \ldots, 1)\). Let \( \phi \) be the generator of \( n^* \) dual to the generator \( c \) of \( n \) and linearly identify \( n^* \) with \( \mathbb{R} \) via the assignment \( \phi \mapsto 1 \). Then \( t \phi \mapsto t \) gives a global coordinate \( \mu(\mathcal{U}_\Sigma) \to \mathbb{R}_{>0} \) and the diffeomorphism \( \mathcal{U}_\Sigma / N \to X(\Sigma) \times \mu(\mathcal{U}_\Sigma) \) is expressed in terms of the coordinates \((w_1, \ldots, w_n, t)\) for \( \mathcal{U}_{S_V} / N_C \times \mu(\mathcal{U}_\Sigma) \) by \( w \) as above and

\[
t = \langle \mu, c \rangle = \frac{1}{\hbar}|z_1|^2 + \ldots + \frac{1}{\hbar}|z_{n+1}|^2.
\]

In these coordinates, the image of \( \pi t \) under the quotient map lands in the subbundle \( \Lambda^2 T \mathbb{C}^n \) of \( \Lambda^2 T(\mathbb{C}^n \times \mathbb{R}_+) \), so in doing computations it suffices to calculate only these components. Under the quotient map

\[
\partial z_k \mapsto z_{n+1}^{-1}\partial w_k
\]

plus terms in \( T\mathbb{R}_+ \) for \( k = 1, 2, \ldots, n \) and

\[
\partial z_{n+1} \mapsto \sum_{p=1}^n (-z_p) z_{n+1}^{-2} \partial w_p
\]

plus terms in \( T\mathbb{R}_+ \). Therefore,

\[
\hbar \pi \mapsto \pi(t, A) = \hbar i \sum_{k=1}^n |z_{n+1}|^{-1} (1 + |z_k|^2 |z_{n+1}|^2)^{-2} \partial \pi_k \land \partial w_k
\]

\[
+ \hbar i \sum_{k < \ell} z_k z_{\ell} (|z_{n+1}|^2)^{-2} \partial \pi_k \land \partial w_\ell + c.c.
\]

It remains to determine the functions \( z_1, \ldots, z_{n+1} \) in terms of \( w = (w_1, \ldots, w_n) \) and \( t \). Write \( ||w||^2 \) for the sum \( |w_1|^2 + \ldots + |w_n|^2 \). The point \((w, t) \in \mathbb{C}^n \times \mathbb{R}_+ \) corresponds to the unique \( N \)-orbit in \( \mathcal{U}_\Sigma \) which passes through the point \((w_1 F, \ldots, w_n F, F) \)
where \( F = F(w) \) is a positive real number such that \( t = \mu(w_1F, \ldots, w_nF, F) \). In other words
\[
t = \frac{1}{h}|w_1|^2F^2 + \ldots + \frac{1}{h}|w_n|^2F^2 + \frac{1}{h}F^2,
\]
or equivalently, \( F = \sqrt{t/h/(1 + \|w\|^2)} \). Substituting \( z_\ell = w_\ell F \) for \( \ell = 1, 2, \ldots, n \) and \( z_{n+1} = F \), gives
\[
\pi t = \frac{i}{(h/h)t} \sum_{k=1}^{n} (1 + |w|^2)(1 + |w_k|^2) \partial_{\bar{w}_k} \wedge \partial_{w_k} + \frac{i}{(h/h)t} \sum_{k<\ell} w_k w_\ell (1 + \|w\|^2) \partial_{\bar{w}_k} \wedge \partial_{w_\ell} + c.c.
\]
(15)

Note that it is independent of the parameter \( h \) as \( h/h \) is the circumference of the unit circle. The expression for \( \pi \Delta \) is then obtained by setting \( t = 1 \) in (15).

The components of the momentum map in terms of the basis \( u_1, \ldots, u_n \) are easily found to be
\[
\langle \Phi_\Delta(w), u_\ell \rangle = \frac{(n+1)|w_\ell|^2 - 1 - \|w\|^2}{(n+1)(1 + \|w\|^2)}
\]
(16)

To compute this, observe that the map \( hp^* \) sends \( u_\ell \mapsto \epsilon_\ell - \epsilon_{n+1} \) for each \( \ell = 1, \ldots, n \). For each \( \ell = 1, 2, \ldots, n \) one can solve for the components of the momentum map \( \Phi_\Delta \). Applying both sides to \( e_\ell \) gives
\[
\langle \Phi_\Delta(w), u_\ell \rangle + \frac{1}{n+1} = \frac{|w_\ell|^2}{1 + \|w\|^2} \text{ for } \ell = 1, 2, \ldots, n
\]
(17)

from which the equations above follow.

The local formulas for \( \Pi_\Sigma \) can be obtained using Theorem 4.7. Here the matrix \( B_V = 1 + A_V A^*_V \) is
\[
B_V = \begin{pmatrix}
2 & 1 & \cdots & 1 \\
1 & 2 & & \\
& 1 & \ddots & \\
& & & 2
\end{pmatrix}
\]
whence
\[
\Pi_\Sigma = \sum_{k=1}^{n} 2 |w_k|^2 \partial_{\bar{w}_k} \wedge \partial_{w_k} + \sum_{k<\ell} i w_k w_\ell \partial_{\bar{w}_k} \wedge \partial_{w_\ell} + c.c.
\]
(18)

Moreover, by Corollary 4.8
\[
\frac{1}{i^n} \Pi_\Sigma^n = i^n(n+1)|w_1|^2 \ldots |w_n|^2 \partial_{\bar{w}_1} \wedge \partial_{w_1} \wedge \cdots \wedge \partial_{\bar{w}_n} \wedge \partial_{w_n}
\]
(19)

as the determinant of the matrix \( B_V \) is \( (n+1) \) (it is of the form identity plus a rank one matrix whose non-zero eigenvalue is \( n \)).

**Proposition 7.1.** In terms of the local coordinates \( w_1, \ldots, w_n \),
\[
\frac{1}{i^n} \pi_\Delta^n = \frac{i^n}{(h/h)^n} (1 + \|w\|^2)^{n+1} \partial_{\bar{w}_1} \wedge \partial_{w_1} \wedge \cdots \wedge \partial_{\bar{w}_n} \wedge \partial_{w_n}.
\]
(20)
Proof. Regard \( w = (w_1, \ldots, w_n) \) as a column vector. From \([15]\) we see that \( \pi_\Delta \) has a local expression of the form \( \frac{1}{(\pi/h)} (1 + \| w \|^2) \xi \) where \( \xi \) is a bi-vector field on \( \mathbb{C}^n \) of pure bi-degree \((1, 1)\). Thus, one can use the identity \( \pi_\Delta^n = \frac{n}{(\pi/h)} (1 + \| w \|^2)^n \xi^n \) to simplify the calculation. In terms of the basis \( dw_1, \ldots, dw_n, dw_1, \ldots, dw_n \), \( \xi \) is represented by a matrix \( X \) of the form

\[
X = \begin{pmatrix} 0 & X' \\ -X' & 0 \end{pmatrix}
\]

relative to the splitting of the cotangent space into \((0, 1)\) and \((1, 0)\) parts where \( X' = 1 + wu^* \). Thus,

\[
\xi^n = n! \text{Pfaff}(X) \partial_{\omega_1} \wedge \partial_{\omega_1} \wedge \ldots \wedge \partial_{\omega_n} \wedge \partial_{\omega_n}.
\]

From the block form of \( X \), it follows that \( \text{Pfaff}(X) = \det(X') \).

The operator \( wu^* \) has rank one when \( w \neq 0 \) and a generator of the non-zero eigenspace is the vector \( w \) with associated eigenvalue \( \| w \|^2 \). Thus, \( \det(X') = \det(1 + wu^*) = (1 + \| w \|^2) \) and this relationship holds as \( w \to 0 \). The result follows immediately. \( \square \)

**Theorem 7.2.** Let \( V \) be a vertex of \( \Delta \). For convenience, assume that it is labeled by \( \{1, 2, \ldots, n\} \). Let \( R_1, \ldots, R_n \) denote the vector fields on \( \mathbb{C}P^n \) corresponding to the vectors \( u_1, \ldots, u_n \in t \) and write \( u \) for the frame \( u = (u_1, \ldots, u_n) \). Then the modular vector field \( \theta_\mu \) of \( \Pi_\Sigma \) associated to the Delzant volume form \( \mu = \frac{1}{n!} \omega^n_\Delta \) has the form

\[
\theta_\mu = \frac{(n + 1)}{h/h} \sum_{\ell=1}^{n} (\Phi_\Delta, (uB_V)_\ell) R_\ell
\]

where \( B_V \) is the integral matrix attached to \( V \).

Proof. As \( \theta_\mu \) is smooth, it suffices to establish (21) in an affine coordinate chart. Let \( V \) be the vertex of \( \Delta \) labeled by \( S_V = \{1, 2, \ldots, n\} \) and let \( w_1, \ldots, w_n \) be the associated holomorphic coordinates. Let \( \Omega_\Sigma \) be the symplectic form of \( \Pi_\Sigma \) on the open leaf \( \{w_1 \neq \ldots \neq w_n \neq 0\} \) and set \( a = \omega^n_\Delta / \Omega_\Sigma^n = \Pi_\Sigma^n / \pi^n_\Delta \) so that \( \theta_\mu = \Pi_\Sigma^n (-d \log a) \) there. Then

\[
a = (n + 1)(\hbar/h)^n \frac{|w_1|^2 \ldots |w_n|^2}{(1 + \| w \|^2)^{n+1}}
\]

by (19) and (20) and therefore

\[
d \log a = \sum_{\ell=1}^{n} \frac{d|w_\ell|^2}{|w_\ell|^2} - (n + 1) \sum_{\ell=1}^{n} \frac{d|w_\ell|^2}{(1 + \| w \|^2)}
\]

\[
= \sum_{\ell=1}^{n} \left( 1 - \frac{(n + 1)|w_\ell|^2}{(1 + \| w \|^2)} \right) \frac{d|w_\ell|^2}{|w_\ell|^2}.
\]

By factoring out \(-(n + 1)\) from each coefficient of \( d|w_\ell|^2 / |w_\ell|^2 = d \log |w_\ell|^2 \), one obtains a familiar term. Indeed

\[
d \log a = -(n + 1) \sum_{\ell=1}^{n} \left( \frac{|w_\ell|^2}{(1 + \| w \|^2)} - \frac{1}{n+1} \right) d \log |w_\ell|^2
\]

\[
= -(n + 1) \sum_{\ell=1}^{n} \langle \Phi_\Delta, u_\ell \rangle d \log |w_\ell|^2
\]
Figure 1. Delzant polytopes for $\mathbb{C}P^n$ for $n = 1$ and $n = 2$. The points marked show the momentum map image of the zero set of the modular vector field $\theta_\mu$ of $\Pi_\Sigma$, computed with respect to $\mu = \frac{1}{n!} \omega^n$, the Delzant Liouville volume form. For $n = 2$, the annihilator subspaces of $(uB_V)_1$, $(uB_V)_2$, and $(uB_V)_1 - (uB_V)_2$ are indicated by the dashed lines $L_1$, $L_2$, and $L_3$, respectively.
q = 1, 2, . . . , n. As $B_V$ is invertible, $uB_V$ is also a frame, so this condition stipulates that $\Phi_\Delta = 0$, i.e., the image under $\Phi_\Delta$ of the zero locus of $\theta_\mu$ on $\Phi_\Delta^{-1}(\Delta^c)$ is $0$ in $t^*$ which is the centroid of $\Delta$ by construction.

Now suppose $F$ is labeled by $\{1, 2, . . . , s\}$. Then $R_1, . . . , R_s$ vanish on $\Phi_\Delta^{-1}(\Delta^c)$. Given the form of $B_V$, the equations $(\eta, u_p) = \frac{1}{n+1}$ for $p = 1, . . . , s$ and $(\eta, (uB_V)_q) = 0$ for $q = s + 1, s + 2, . . . , n$ are independent. Thus, it suffices then to show that $\eta = c_F$ is a solution as this solution will be unique. Let $V_\ell$ denote the vertex opposite to the facet of the simplex $\Delta$ labeled by $\ell$. Then the vertices of $F$ are $\{V_{s+1}, V_{s+2}, . . . , V_{n+1}\}$. Now $V_\ell = \eta_k - \frac{1}{n+1} \sum_{r=1}^n \eta_r$ for each $k = 1, 2, . . . , n$ while $V_{n+1} = -\frac{1}{n+1} \sum_{r=1}^n \eta_r$. Therefore, the centroid of $F$ is

$$c_F = \frac{1}{n-s+1} (V_{s+1} + V_{s+2} + . . . + V_{n+1})$$

By construction $(c_F, u_p) = \frac{1}{n+1}$ for each $p = 1, . . . , n$. If $q \in \{s + 1, s + 2, . . . , n\}$ then

$$\langle c_F, (uB_V)_q \rangle = \frac{1}{n-s+1} \sum_{p=1}^n B_{pq} \left( \sum_{t=1}^{n-s} \eta_{s+t} - \frac{n-s+1}{n+1} \sum_{r=1}^n \eta_r, u_p \right)$$

$$= \frac{1}{n-s+1} \left( \sum_{r=1}^s B_{rq} \left( -\frac{n-s+1}{n+1} \right) + \sum_{t=1}^{n-s} B_{(s+t),q} \left( 1 - \frac{n-s+1}{n+1} \right) \right).$$

Given that $B_{pq} = 1$ if $q \neq p$ and $B_{pp} = 2$, it follows that

$$\langle c_F, (uB_V)_q \rangle = \frac{1}{n-s+1} \left( s \left( -\frac{n-s+1}{n+1} \right) + (n-s+1) \left( 1 - \frac{n-s+1}{n+1} \right) \right)$$

$$= 0$$

for each $q \in \{s + 1, s + 2, . . . , n\}$.

Therefore, the image under $\Phi_\Delta$ of the zero locus of $\theta_\mu$ on $\Phi_\Delta^{-1}(\Delta^c)$ is precisely the centroid $c_F$ of $F$. This completes the proof. \qed

**Corollary 7.4.** The zero locus of $\theta_\mu$ on $\mathbb{C}P^n$ is a disjoint union of compact tori, one of dimension $\ell$ for each face $F$ of dimension $\ell$ of $\Delta$.

**Proof.** By the symplectic convexity theorem, the pre-image under $\Phi_\Delta$ of a point interior to a face $F$ of $\Delta$ of dimension $\ell$ is a real torus of dimension $\ell$. Thus, the claim follows immediately from the previous theorem. \qed

What is intriguing about the preceding computations is that the Poisson structures $\pi_\Delta$ and $\Pi_\Sigma$ are not compatible when $n > 1$ (for $n = 1$ the Schouten bracket on $\mathbb{C}P^2$ is trivial) and yet the modular vector field for $\Pi_\Sigma$ relative to the Delzant Liouville volume $\frac{1}{\omega_\Delta} \omega_\Delta^{-n}$ can be computed in terms of moment data. Although the modular class has been studied and used as a tool in applications of Poisson geometry, good examples are lacking, and this formula contributes a natural family.
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