EXTENDED CANONICAL DUALITY AND CONIC PROGRAMMING FOR SOLVING 0-1 QUADRATIC PROGRAMMING PROBLEMS

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ABSTRACT. An extended canonical dual approach for solving 0-1 quadratic programming problems is introduced. We derive the relationship between the optimal solutions to the extended canonical dual problem and the original problem and prove that there exists no duality gap in-between. The extended canonical dual approach leads to a sufficient condition for global optimality, which is more general than known results of this kind. To solve the extended canonical dual problem, we construct corresponding conic programming problems and study their relationship to the extended canonical dual problem. Using this relationship, we design an algorithm for solving the extended canonical dual problem. Our work extends the known solvable sub-class of 0-1 quadratic programming problems.

1. Introduction. In this paper, we study the following 0-1 quadratic integer programming problem:

\[
\begin{align*}
\min & \quad F(x) = \frac{1}{2} x^T Q x + c^T x \\
\text{s.t.} & \quad x \in \{0, 1\}^n,
\end{align*}
\]

where \( Q \) is an \( n \times n \) real symmetric matrix and \( c \) is a vector in \( \mathbb{R}^n \).

The 0-1 quadratic integer programming problem (QIP in short) appears in many applications [19]. For example, the well studied max-cut problem in combinatorial optimization is a special sub-class of QIP. It is known that QIP is NP-Hard [7]. In other words, it is impossible to be solved generally in polynomial time, unless \( P=NP \).

To find an exact global optimal solution to QIP, we usually count on some well performed enumeration strategies. The branch-and-bound algorithm with a lower
bound estimation \[14\] applies such a strategy. However, in the worst case, the branch-and-bound algorithm may still need to enumerate through all the feasible points. Taking the computation time into consideration, researchers study approximation algorithms for finding an approximate solution to some classes of quadratic integer problems \[8, 12\]. For example, Goemans et al. proposed an efficient approximation algorithm to achieve an approximation solution to the max-cut problem with a guaranteed approximation ratio of 0.878 \[8\].

Another effort on solving QIP is to identify some sub-classes that can be solved in polynomial time. Identifying polynomially solvable subclasses of binary quadratic programming problems not only offers theoretical insights into the complex nature of the problem, but also provides platforms for designing relaxation schemes to obtain exact solutions. For example, Allemand et al. discussed a polynomial time solvable case in \[1\]. With the assumption that the quadratic objective function has a low rank spectral decomposition, they adopted a well-designed enumeration algorithm to achieve a global optimal solution in polynomial time. Recently, Fang et al. used a canonical duality approach to provide a polynomial-time solvable sub-class of QIP \[3\]. Wang et al. further extended the result and proposed an approximation algorithm for solving the max-cut problem \[18\]. Canonical duality theory was firstly proposed by Gao \[5\]. It has been developed for solving global optimization problems \[3, 4, 10, 17, 20, 6\]. Our work, motivated by these works, extends the canonical duality theory to solve the 0-1 quadratic programming problem, discovers a new global optimality condition for solving QIP and identifies a new solvable sub-class of QIP.

The rest of this paper is arranged as follows. We first briefly review the main results of the canonical duality theory and define an extended canonical dual problem. Then we present a perfect dual relationship between QIP and its extended canonical dual problem in Section 2. Followed by constructing a conic relaxation problem for QIP in Section 3, we discuss the relationship between the dual of this conic programming problem and the extended canonical dual problem. In Section 4, we propose an efficient algorithm for solving the extended canonical dual problem to generate an exact global optimal solution to QIP under appropriate assumptions. Some concluding remarks are given in the last section.

2. Canonical duality theory and extended canonical dual problem. In this paper, we adopt the following notations: \(M_n\) denotes the set of all \(n \times n\) real symmetric matrices, \(S_n\) the set of all \(n \times n\) symmetric positive semidefinite matrices, and \(N_n\) the set of all \(n \times n\) matrices with non-negative elements. Given a vector \(x \in \mathbb{R}^n\), \([x]_i\) or \(x_i\) denotes the \(i\)-th component of \(x\), \(Diag(x)\) denotes the \(n \times n\) diagonal matrix with \(x_i\) being the \(i\)-th diagonal element. Especially, for a vector \(\lambda \in \mathbb{R}^n\), \(\Lambda\) denotes the diagonal matrix \(Diag(\lambda)\). For two vectors \(x, y \in \mathbb{R}^n\), \(x \odot y\) is a vector in \(\mathbb{R}^n\) with \([x]_i\, [y]_i\) being its \(i\)-th component. For a real symmetric matrix \(U, U \succeq 0\) means \(U\) is positive semidefinite and \(U \succ 0\) means \(U\) is positive definite. For two matrices \(A\) and \(B\), denote \(A \cdot B = \text{trace}(A^T B)\). For a given optimization problem \((\ast)\), its optimal objective value is denoted by \(V(\ast)\).

Now we focus on the problem QIP. Since its 0-1 constraints can be equivalently written as \(x_i^2 - x_i = 0\) for \(i = 1, 2, \cdots, n\), the Lagrangian function of QIP is defined
by
\[ L(x, \lambda) = \frac{1}{2} x^T Q x + c^T x + \sum_{i=1}^{n} \lambda_i (x_i^2 - x_i). \]

Let \( \nabla_x L(x, \lambda) = (Q + 2\Lambda)x + c - \lambda = 0 \). Assuming that \( Q + 2\Lambda \) is invertible, we have a unique solution \( x_\lambda = -(Q + 2\Lambda)^{-1}(c - \lambda) \). The original canonical dual function \( P^d(\lambda) \) is defined as
\[ P^d(\lambda) = L(x_\lambda, \lambda) = -\frac{1}{2}(c - \lambda)^T (Q + 2\Lambda)^{-1}(c - \lambda), \]
for \( \lambda \in \mathcal{F} = \{ \lambda \in \mathbb{R}^n \mid \det(Q + 2\Lambda) \neq 0 \} \).

From Theorems 1 and 2 in [3], we know the following results:

**Lemma 2.1.** The canonical dual function \( P^d(\lambda) \) is differentiable at any \( \lambda \in \mathcal{F} \) with\[
\nabla P^d(\lambda) = x_\lambda \circ x_\lambda - x_\lambda,
\]
where \( x_\lambda = -(Q + 2\Lambda)^{-1}(c - \lambda) \).

**Theorem 2.2.** If \( \lambda^* \in \mathcal{F} \) is a critical point of \( P^d(\lambda) \), then \( x_{\lambda^*} = -(Q + 2\Lambda^*)^{-1}(c - \lambda^*) \in \{0, 1\}^n \) is a KKT point of the problem QIP and \( P^d(\lambda^*) = F(x_{\lambda^*}) \).

The above theorem says that every critical point of the canonical dual function corresponds to a KKT point of the primal problem. The next theorem indicates that if a critical point \( \lambda^* \) satisfies the “positive definite condition” of \( Q + 2\Lambda^* > 0 \), then its corresponding primal solution \( x_{\lambda^*} \) is indeed a global optimal solution to the problem QIP.

**Theorem 2.3.** If \( P^d(\lambda) \) has a critical point \( \lambda^* \) such that \( Q + 2\Lambda^* > 0 \), then \( x_{\lambda^*} = -(Q + 2\Lambda^*)^{-1}(c - \lambda^*) \) is a global optimal solution of the problem QIP.

**Proof.** Since \( \lambda^* \) is a critical point of \( P^d(\lambda) \), we have
\[ \nabla P^d(\lambda)|_{\lambda=\lambda^*} = x_{\lambda^*} \circ x_{\lambda^*} - x_{\lambda^*} = 0. \]
Consequently, \( x_{\lambda^*} \in \{0, 1\}^n \). Notice that \( x_{\lambda^*} = -(Q + 2\Lambda^*)^{-1}(c - \lambda^*) \) is a solution for \( \nabla_x L(x, \lambda^*) = 0 \). When \( Q + 2\Lambda^* > 0 \), \( L(x, \lambda^*) \) is strictly convex in \( x \) and \( x_{\lambda^*} \) must be a global minimizer of \( L(x, \lambda^*) \) over \( \mathbb{R}^n \). Now, for any \( x \in \{0, 1\}^n \subset \mathbb{R}^n \), we have \( F(x) = F(x) + \sum_{i=1}^{n} \lambda_{i}^*(x_i^2 - x_i) = L(x, \lambda^*) \geq L(x_{\lambda^*}, \lambda^*) = F(x_{\lambda^*}) \). Hence \( x_{\lambda^*} \) is a global optimal solution of QIP.

Theorem 2.3 is a classical global optimality condition in literature. Similar conditions can be found in several papers, e.g., Proposition 3.2 of [9] and Theorem 2 of [6].

With the definition of the original canonical dual function and its properties in mind, we now consider an extended canonical dual problem. We first extend the feasible domain from \( \mathcal{F} = \{ \lambda \in \mathbb{R}^n \mid \det(Q + 2\Lambda) \neq 0 \} \) to \( \mathcal{G} = \{ \lambda \in \mathbb{R}^n \mid (Q + 2\Lambda)x + (c - \lambda) = 0 \ has \ a \ solution \ x \in \mathbb{R}^n \} \). Evidently, we know \( \mathcal{F} \subseteq \mathcal{G} \). Hence the set \( \mathcal{G} \) is an extension of \( \mathcal{F} \). Next we want to define a function \( \mathcal{P}^e(\lambda) \) on \( \mathcal{G} \) such that \( \mathcal{P}^e(\lambda) = P^d(\lambda) \) for any \( \lambda \in \mathcal{F} \). For this purpose, we introduce the following lemma:

**Lemma 2.4.** Let \( A \) be a real symmetric matrix in \( \mathcal{M}_n \) and \( b \in \mathbb{R}^n \). If \( A \) is not invertible and the system of linear equations \( Ax + b = 0 \) is solvable, then
\[ \frac{1}{2} x_a^T Ax_a + b^T x_a = \frac{1}{2} x_b^T Ax_b + b^T x_b \]
holds for any two solutions \( x_a \) and \( x_b \) of \( Ax + b = 0 \).
Lemma 2.7. For function $P$, we need to introduce the extended gradient as well. Moreover, we say $\lambda$ as below.

For an extended critical point $\bar{x}$ of $P$ of the problem QIP and the extended critical points of $\bar{P}$, we see a conventional critical point of $P$ if $\bar{P}$ is any solution of the system of linear equations $(Q+2\Lambda)x + (c - \lambda) = 0$. Therefore, $\hat{x} = 0$, $\tilde{x} = \bar{x} - \hat{x} \in \partial P^c(\lambda)$. Thus $\bar{x}$ is an extended critical point of $P^c(\lambda)$ if $0 \in \partial P^c(\lambda^*)$.

By this definition, we see a conventional critical point of $P^d(\lambda)$ is also an extended critical point of $P^c(\lambda)$. Meanwhile, the extended critical point has the following property:

Lemma 2.7. $\bar{x} \in \bar{G}$ is an extended critical point of $P^c(\lambda)$ if and only if $\{x \in \mathbb{R}^n \mid (Q + 2\Lambda)x + (c - \lambda) = 0\} \cap \{0, 1\}^n \neq \emptyset$.

Proof. If $\tilde{x} \in \{x \in \mathbb{R}^n \mid (Q + 2\Lambda)x + c - \bar{\lambda} = 0\} \cap \{0, 1\}^n \neq \emptyset$, then $0 = \tilde{x} \circ \tilde{x} - \tilde{x} \in \partial P^c(\lambda)$. Thus $\lambda$ is an extended critical point of $P^c(\lambda)$. On the other hand, if $\lambda$ is an extended critical point of $P^c(\lambda)$, then there exists an $\tilde{x}$ such that $(Q + 2\Lambda)\tilde{x} + c - \lambda = 0$ and $\tilde{x} \circ \tilde{x} - \tilde{x} = 0$. Therefore $\tilde{x} \in \{0, 1\}^n$ and $\{x \in \mathbb{R}^n \mid (Q + 2\Lambda)x + c - \bar{\lambda} = 0\} \cap \{0, 1\}^n \neq \emptyset$.

Definition 2.8. For an extended critical point $\bar{x}$ of $P^c(\lambda)$, define any $x_{\bar{x}} \in \{x \in \mathbb{R}^n \mid (Q + 2\Lambda)x + c - \lambda = 0\} \cap \{0, 1\}^n$ as a primal solution of QIP corresponding to $\lambda$ (note any such $x_{\bar{x}}$ has the same primal objective value by Lemma 2.4).

From Lemma 2.7 and Definition 2.8, we see a perfect dual relationship in the next theorem.

Theorem 2.9. If $\lambda^* \in \bar{G}$ is an extended critical point of $P^c(\lambda)$, then $x_{\lambda^*} \in \{x \in \mathbb{R}^n \mid (Q + 2\Lambda^*)x + c - \lambda^* = 0\} \cap \{0, 1\}^n$ is a KKT solution of the problem QIP. Moreover, $P^c(\lambda^*) = F(\lambda^*)$.

Proof. Lemma 2.7 implies that, for an extended critical point $\lambda^*$, there exists a corresponding primal solution $x_{\lambda^*}$ such that $(Q + 2\Lambda^*)x_{\lambda^*} + c - \lambda^* = 0$ and $x_{\lambda^*} \in \{0, 1\}^n$. Hence $x_{\lambda^*}$ is a feasible solution of the problem QIP and it satisfies the KKT condition. Meanwhile, we have $F(x_{\lambda^*}) = \frac{1}{2}x_{\lambda^*}^TQx_{\lambda^*} + c^Tx_{\lambda^*} + \sum_{i=1}^{n} \lambda^*_i([x_{\lambda^*}]_i^2 - |x_{\lambda^*}|_i) = L(x_{\lambda^*}, \lambda^*) = P^c(\lambda^*)$.

The next lemma further specifies the relationship between the feasible solutions of the problem QIP and the extended critical points of $P^c(\lambda)$.
Lemma 2.10. For any $\bar{x} \in \{0,1\}^n$, there exists an extended critical point $\lambda_{\bar{x}} \in \mathcal{G}$ of $P^c(\lambda)$ such that $\bar{x} \in \{ x \in \mathbb{R}^n \mid (Q + 2\Lambda) x + c - \lambda_{\bar{x}} = 0 \} \cap \{0,1\}^n$.

Proof. For any $\bar{x} \in \{0,1\}^n$, notice that the system of linear equations $(Q + 2\Lambda)\bar{x} + c - \lambda = 0$ is equivalent to $[2\text{Diag}(\bar{x}) - \text{Diag}(e)]\lambda + Q\bar{x} + c = 0$, where $e = [1,1,\ldots,1]^T$. Moreover, since the determinant of the matrix $[2\text{Diag}(\bar{x}) - \text{Diag}(e)]$ is not zero, we know $(Q + 2\Lambda)\bar{x} + c - \lambda = 0$ for $\lambda$ always has a unique solution $\lambda_{\bar{x}} \in \mathcal{G}$, which is exactly the extended critical point. \hfill $\Box$

Remark 1. Theorem 2.9 and Lemma 2.10 indicate that for any extended critical point $\lambda^* \in \mathcal{G}$, there always exists at least one primal solution corresponding to $\lambda^*$ that satisfies $x_{\lambda^*} \in \{ x \in \mathbb{R}^n \mid (Q + 2\Lambda^*) x + c - \lambda^* = 0 \} \cap \{0,1\}^n$. On the other hand, for any primal solution $\bar{x} \in \{0,1\}^n$, there always exists a unique dual extended critical point $\lambda_{\bar{x}}$ corresponding to it. This correspondence leads to the perfect dual relation in the sense of $P^c(\lambda^*) = F(x_{\lambda^*})$.

Now, we define the extended canonical dual problem (ECD in short) of the problem QIP as below.

$$\begin{array}{ll}
\min & P^c(\lambda) \\
\text{s.t.} & \lambda \in \mathcal{G} \\
& 0 \in \partial P^c(\lambda).
\end{array}$$

(ECD)

ECD is to find an extended critical point of the extended canonical dual function with a minimum objective value. Combining Theorem 2.9 and Lemma 2.10, we have the next theorem.

Theorem 2.11 (perfect dual relation). There is a perfect dual relation between problems QIP and ECD in the sense that $V(QIP) = V(ECD)$. Moreover, for any dual optimal solution $\lambda^*$ to problem ECD, there exists a corresponding primal optimal solution $x_{\lambda^*} \in \{ x \in \mathbb{R}^n \mid (Q + 2\Lambda^*) x + c - \lambda^* = 0 \} \cap \{0,1\}^n$ to problem QIP.

Proof. From Theorem 2.9 and Lemma 2.10, we know that each primal feasible solution has a corresponding dual extended critical solution, and each dual extended critical solution has at least one corresponding primal solution with the same objective value. Thus, the optimal solution of problem QIP must correspond to an optimal extended critical solution of problem ECD and $V(ECD) = V(QIP)$. \hfill $\Box$

Note that the extended canonical dual problem ECD is well defined with a perfect dual relation in the extended definition. This new setting extends the previous results shown in [3], in which the canonical dual function is defined only when $Q + 2\Lambda$ is invertible and the perfect dual relationship is not always satisfied.

3. Extended canonical duality and conic programming. Our goal is to solve the dual problem ECD for obtaining a global optimal solution to the primal problem QIP. In general, this is an NP-hard problem, because solving QIP is NP-hard. What we are interested in doing is to identify a solvable sub-class of QIP such that its dual problem ECD can be solved efficiently.

Our approach is based on the concept of linear conic relaxation. We start from reformulating the primal problem QIP as a linear conic programming problem.
First, let us define a cone
\[ D_{n+1} = \{ U \in \mathcal{M}_{n+1} \mid \begin{bmatrix} 1 \\ x \end{bmatrix}^T U \begin{bmatrix} 1 \\ x \end{bmatrix} \geq 0 \text{ for every } x \in [0,1]^n \}, \]
which is convex, and a set of matrices
\[ Z = \{ X \in \mathcal{M}_{n+1} \mid X = \begin{bmatrix} 1 \\ x \end{bmatrix} \begin{bmatrix} 1 \\ x \end{bmatrix}^T \text{ for some } x \in [0,1]^n \}. \]
Let \( \text{Cone}(Z) \) be the convex cone generated by \( Z \) and denote \( D^*_{n+1} = \text{Cone}(Z) \).

Then we see the relation of \( D_{n+1} \) and \( D^*_{n+1} \) in the next result.

**Lemma 3.1.** \( D_{n+1} \) is the dual cone of \( D^*_{n+1} \).

**Proof.** For any \( X \in D^*_{n+1} \), there exists a positive \( r \) with \( x^1, x^2, \ldots, x^r \in [0,1]^n \) and \( \lambda_1, \lambda_2, \ldots, \lambda_r \geq 0 \) such that
\[ X = \lambda_1 \begin{bmatrix} 1 \\ x^1 \end{bmatrix} \begin{bmatrix} 1 \\ x^1 \end{bmatrix}^T + \lambda_2 \begin{bmatrix} 1 \\ x^2 \end{bmatrix} \begin{bmatrix} 1 \\ x^2 \end{bmatrix}^T + \cdots + \lambda_r \begin{bmatrix} 1 \\ x^r \end{bmatrix} \begin{bmatrix} 1 \\ x^r \end{bmatrix}^T. \]

If \( U \in D_{n+1} \), then
\[ U \cdot X = \lambda_1 \begin{bmatrix} 1 \\ x^1 \end{bmatrix}^T U \begin{bmatrix} 1 \\ x^1 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ x^2 \end{bmatrix}^T U \begin{bmatrix} 1 \\ x^2 \end{bmatrix} + \cdots + \lambda_r \begin{bmatrix} 1 \\ x^r \end{bmatrix}^T U \begin{bmatrix} 1 \\ x^r \end{bmatrix} \geq 0. \]

This means \( U \in (D^*_{n+1})^* \) and hence \( D_{n+1} \subseteq (D^*_{n+1})^* \).

On the other hand, for any \( x \in [0,1]^n \),
\[ X = \begin{bmatrix} 1 \\ x \end{bmatrix} \begin{bmatrix} 1 \\ x \end{bmatrix}^T \in D^*_{n+1}. \]

If \( U \in (D^*_{n+1})^* \), then \( U \cdot X \geq 0 \) and
\[ \begin{bmatrix} 1 \\ x \end{bmatrix}^T U \begin{bmatrix} 1 \\ x \end{bmatrix} \geq 0. \]

This means that \( U \in D_{n+1} \) and, consequently, \( (D^*_{n+1})^* \subseteq D_{n+1} \). Therefore, \( D_{n+1} = (D^*_{n+1})^* \). \( \square \)

Note that Lemma 3.1 also implies that \( D^*_{n+1} \) is the dual cone of \( D_{n+1} \).

**Remark 2.** Similar definitions for \( D_{n+1} \) and \( D^*_{n+1} \) can also be found in Strum and Zhang [15].

Now, we consider the following conic optimization problem with the rank one constraint:
\[
\begin{align*}
\min & \quad \frac{1}{2} Q \cdot X + c^T x \\
\text{s.t.} & \quad X_{ii} = x_i \text{ for } i = 1, 2, \ldots, n \\
& \quad \begin{bmatrix} 1 \\ x \end{bmatrix} \begin{bmatrix} x^T \\ X \end{bmatrix} = Y \\
& \quad Y \in D^*_{n+1} \\
& \quad \text{rank}(Y) = 1.
\end{align*}
\]

An immediate result can stated as below.
Lemma 3.2. \((x, X)\) is a feasible solution of the problem QIP2 if and only if \(x \in \{0, 1\}^n\) with \(X = xx^T\).

Proof. If \(x \in \{0, 1\}^n\) and \(X = xx^T\), then it is easy to verify that \((x, X)\) satisfies all constraints of the problem QIP2. Hence \((x, X)\) is a feasible solution to QIP2.

On the other hand, if \((x, X)\) is a feasible solution of the problem QIP2, then the constraint of \(\text{rank}(Y) = 1\) leads to \(X = xx^T\) and \(X_{ii} = x_i^2\) for \(i = 1, 2, \ldots, n\). From the constraint of \(X_{ii} = x_i\), we can get \(x_i^2 = x_i\). Consequently, \(x \in \{0, 1\}^n\).

The above lemma shows that problems QIP2 and QIP have an equivalent feasible domain. Noting that they have the same objective function, we have the following result.

Theorem 3.3. Problems QIP2 and QIP are equivalent.

Note that the rank one constraint of the problem QIP2 is not a convex constraint. By dropping this constraint, we may relax QIP2 to the following linear conic programming problem (COP in short):

\[
\begin{align*}
\min & \quad \frac{1}{2} \mathbf{Q} : \mathbf{X} + \mathbf{c}^T \mathbf{x} \\
\text{s.t.} & \quad X_{ii} = x_i \text{ for } i = 1, 2, \ldots, n \\
& \quad \begin{bmatrix} 1 & \mathbf{x}^T \\ \mathbf{x} & \mathbf{X} \end{bmatrix} = \mathbf{Y} \\
& \quad \mathbf{Y} \in D_{n+1}^+.
\end{align*}
\]

(COP)

Using the conic duality theory, we define the dual problem of COP as

\[
\begin{align*}
\max & \quad \frac{1}{2} \sigma \\
\text{s.t.} & \quad \begin{bmatrix} -\sigma & (\mathbf{c} - \mathbf{\lambda})^T \\ \mathbf{c} - \mathbf{\lambda} & \mathbf{Q} + 2\mathbf{\Lambda} \end{bmatrix} \in D_{n+1}.
\end{align*}
\]

(COD)

Since the dual conic programming problem COD itself is hard to solve, we do not tackle it directly. Instead, we study its relation to the canonical dual problem ECD first. This may provide hints for us to solve the canonical dual problem using a conic programming method.

To do so, we first define a dual matrix corresponding to \(\mathbf{\lambda}^*\), for any feasible solution \(\mathbf{\lambda}^* \in \mathcal{G}\) of ECD, as

\[
\mathcal{D}(\mathbf{\lambda}^*) = \begin{bmatrix} -2P^c(\mathbf{\lambda}^*) & (\mathbf{c} - \mathbf{\lambda}^*)^T \\ \mathbf{c} - \mathbf{\lambda}^* & \mathbf{Q} + 2\mathbf{\Lambda} \end{bmatrix}.
\]

Then we have an immediate result.

Theorem 3.4. If the canonical dual problem ECD has an extended critical point \(\mathbf{\lambda}^* \in \mathcal{G}\) such that \(\mathcal{D}(\mathbf{\lambda}^*) \in D_{n+1}\), then \((\sigma^*, \mathbf{\lambda}^*)\) with \(\sigma^* = 2P^c(\mathbf{\lambda}^*)\) is an optimal solution of the conic dual problem COD. Moreover, the corresponding primal solution \(\mathbf{x}_{\mathbf{\lambda}^*}\) is a global optimal solution of the original problem QIP.

Proof. Since the corresponding primal solution \(\mathbf{x}_{\mathbf{\lambda}^*} \in \{0, 1\}^n\), \((\mathbf{x}_{\mathbf{\lambda}^*}, X^*)\) with \(X^* = x_{\mathbf{\lambda}^*}x_{\mathbf{\lambda}^*}^T\) must be feasible to both of QIP2 and COP. Noticing that

\[
\mathcal{D}(\mathbf{\lambda}^*) = \begin{bmatrix} -\sigma^* & (\mathbf{c} - \mathbf{\lambda}^*)^T \\ \mathbf{c} - \mathbf{\lambda}^* & \mathbf{Q} + 2\mathbf{\Lambda} \end{bmatrix} \in D_{n+1},
\]
we know \((\sigma^*, \lambda^*)\) is a feasible solution of the problem COD. Using the perfect dual relation 
\(P^c(\lambda^*) = F(x_{\lambda^*})\) of Theorem 2.11, we can verify that 
\[
\begin{bmatrix}
1 & x_{\lambda^*}^T \\
x_{\lambda^*} & x_{\lambda^*} x_{\lambda^*}^T
\end{bmatrix}
\begin{bmatrix}
-\sigma^* (c - \lambda^*)^T \\
c - \lambda^* Q + 2\Lambda^*
\end{bmatrix} = 0.
\]
Therefore, the complementarity condition is satisfied. From the optimality condition for linear conic programming, we know that \((x_{\lambda^*}, X^*)\) is optimal to the problem COP and, consequently, optimal to the problem QIP2. Since problems QIP2 and QIP are equivalent, we know that \(x_{\lambda^*}\) is an optimal solution of the problem QIP. \(\square\)

The above theorem provides a new sufficient condition of global optimality for solving problem QIP. This condition actually extends the global optimality condition stated in Theorem 2.3 (or Theorem 3 in [3]), since each case that satisfies the condition in Theorem 2.3 also satisfies the condition in Theorem 3.4. The next example illustrates the difference of the global optimality conditions stated in Theorems 2.3 and 3.4.

**Example 1.** Consider the following quadratic programming problem:

\[
\begin{align*}
\min & \quad x^T Q x = x^T \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x \\
\text{s.t.} & \quad x \in \{0, 1\}^2.
\end{align*}
\]

The canonical dual function is defined as 
\[
P^d(\lambda) = -\frac{1}{4} \lambda^T \begin{bmatrix} \lambda_1 & 1 \\ 1 & \lambda_2 \end{bmatrix}^{-1} \lambda.
\]
It is easy to verify that \(\lambda^* = 0\) is a critical point of \(P^d(\lambda)\) with \(x_{\lambda^*} = 0\). Then we have 
\[
Q + 2\Lambda^* = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},
\]
which is invertible but not positive semidefinite. Hence the condition in Theorem 2.3 is not satisfied. However, we see that 
\[
\mathcal{D}(\lambda^*) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \in D_3,
\]
since every term of \(\mathcal{D}(\lambda^*)\) is non-negative. Applying Theorem 3.4, we know \(x_{\lambda^*}\) is globally optimal.

Since the rest of this section is centering around identifying a sub-class of QIP that can be globally solved, we state the following extended global optimality condition as an assumption:

**Extended Global Optimality Condition.** *The extended canonical dual function has an extended critical point \(\lambda^* \in \mathcal{G}\) with \(\sigma^* = 2P^c(\lambda^*)\) such that \(\mathcal{D}(\lambda^*) = \begin{bmatrix} -\sigma^* (c - \lambda^*)^T \\ c - \lambda^* Q + 2\Lambda^* \end{bmatrix} \in D_{n+1}\). The corresponding primal solution of \(\lambda^*\) is denoted by \(x_{\lambda^*}\).*
It is interesting to note that, from Theorem 3.4, the $\lambda^*$ in the Extended Global Optimality Condition must be an optimal solution of the problem COD. But, solving COD directly may not provide $\lambda^*$, because COD may have multiple optimal solutions in general. This situation is illustrated by the next example.

**Example 2.** Consider the following quadratic programming problem:

$$\min x^T Q x + c^T x = x^T \begin{bmatrix} 0 & 4 \\ 4 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix}^T x$$

Subject to $x \in \{0, 1\}^2$.

The canonical dual function is defined as

$$P^c(\lambda) = -\frac{1}{4}(e - \lambda)^T \begin{bmatrix} 4 & \lambda_1 \\ \lambda_1 & 4 \end{bmatrix}^{-1} (e - \lambda),$$

where $e = [1, 1]^T$. It is easy to verify that $\lambda^* = [1, 1]^T$ is an extended critical point of $P^c(\lambda)$ with $x_{\lambda^*} = 0$.

The corresponding conic dual problem becomes

$$\max \frac{1}{2}\sigma$$

Subject to

$$\begin{bmatrix} -\sigma & 1 - \lambda_1 & 1 - \lambda_2 \\ 1 - \lambda_1 & 2\lambda_1 & 8 \\ 1 - \lambda_2 & 8 & 2\lambda_2 \end{bmatrix} \in D_3.$$

It is not difficult to see that any $(\sigma, \lambda_1, \lambda_2)$ with $\sigma = 0$ and $0 \leq \lambda_1, \lambda_2 \leq 1$ is an optimal solution of this conic relaxation problem. In particular, by Theorem 3.4, $(\sigma^*, \lambda_1^*, \lambda_2^*) = (0, 1, 1)$ corresponding to the extended critical point of $P^c(\lambda)$ is an optimal solution of the conic relaxation problem. But solving this problem may or may not lead to $\lambda^*$ directly.

In order to find the $\lambda^*$ as stated in the Extended Global Optimality Condition, we need to conduct further analysis.

**Lemma 3.5.** Under the Extended Global Optimality Condition with $\lambda^*$ and $x_{\lambda^*}$ being defined, if $(\sigma_D, \lambda_D)$ is an optimal solution of the problem COD, then $x_{\lambda^*}$ is a global optimal solution of the following mathematical optimization problem:

$$\min x^T (Q + 2\Lambda_D) x + 2(c^T - \lambda_D^T) x$$

Subject to $x \in [0, 1]^n$. (MOP)

**Proof.** From the given condition, we know that

$$\begin{bmatrix} -\sigma_D & (c - \lambda_D)^T \\ c - \lambda_D & Q + 2\Lambda_D \end{bmatrix} \in D_{n+1}$$

and

$$\begin{bmatrix} 1^T \\ x \end{bmatrix}^T \begin{bmatrix} -\sigma_D & (c - \lambda_D)^T \\ c - \lambda_D & Q + 2\Lambda_D \end{bmatrix} \begin{bmatrix} 1 \\ x \end{bmatrix} \geq 0$$

for any $x \in [0, 1]^n$. Consequently, $x^T (Q + 2\Lambda_D) x + 2(c^T - \lambda_D^T) x \geq \sigma_D$ for any $x \in [0, 1]^n$. Moreover, since

$$\begin{bmatrix} 1 \\ x_{\lambda^*} \end{bmatrix}^T \begin{bmatrix} 1 \\ x_{\lambda^*} \end{bmatrix} \geq 0$$

for any $x_{\lambda^*} \in [0, 1]^n$. Consequently, $x^T (Q + 2\Lambda_D) x + 2(c^T - \lambda_D^T) x \geq \sigma_D$ for any $x \in [0, 1]^n$. Moreover, since
is an optimal solution of COP, by the complementarity condition of conic programming, we have
\[
\begin{bmatrix}
1 & x_{\lambda^*}^T \\
x_{\lambda^*} & x_{\lambda^*}x_{\lambda^*}^T
\end{bmatrix} \cdot \begin{bmatrix}
-\sigma_D & (c - \lambda_D)^T \\
(c - \lambda_D) & Q + 2\Lambda_D
\end{bmatrix} = 0.
\]
Therefore, \(x_{\lambda^*}^T(Q + 2\Lambda_D)x_{\lambda^*} + 2(c^T - \lambda_D^T)x_{\lambda^*} = \sigma_D\). This completes the proof. \(\square\)

The optimality condition of the problem MOP results in the next Lemma.

**Lemma 3.6.** Under the Extended Global Optimality Condition with \(\lambda^*\) and \(x_{\lambda^*}\) being defined, let \((\sigma_D, \lambda_D)\) be an optimal solution of the problem COD. If \([x_{\lambda^*}]_i = 1\), then \([(2Q + 4\Lambda_D)x_{\lambda^*} + 2(c - \lambda_D)]_i \leq 0\). If \([x_{\lambda^*}]_i = 0\), then \([(2Q + 4\Lambda_D)x_{\lambda^*} + 2(c - \lambda_D)]_i \geq 0\).

**Proof.** Notice that the gradient of the objective function of the problem MOP at \(x_{\lambda^*}\) is \((2Q + 4\Lambda_D)x_{\lambda^*} + 2(c - \lambda_D)\). If \([x_{\lambda^*}]_i = 1\), then \(-e_i\) is a feasible direction for MOP at \(x_{\lambda^*}\). Since \(x_{\lambda^*}\) is a global optimal solution of MOP, its optimality condition results in \(-e_i^T[(2Q + 4\Lambda_D)x_{\lambda^*} + 2(c - \lambda_D)] \geq 0\), i.e., \([(2Q + 4\Lambda_D)x_{\lambda^*} + 2(c - \lambda_D)]_i \leq 0\). Similarly, if \([x_{\lambda^*}]_i = 0\), then \(e_i\) is a feasible direction for MOP at \(x_{\lambda^*}\), thus \([(2Q + 4\Lambda_D)x_{\lambda^*} + 2(c - \lambda_D)]_i \geq 0\). \(\square\)

Combining Lemmas 3.5 and 3.6, we have the next result.

**Lemma 3.7.** Under the Extended Global Optimality Condition with \(\lambda^*\) and \(x_{\lambda^*}\) being defined, if \((\sigma_D, \lambda_D)\) is an optimal solution of the problem COD, then \(\lambda_D \leq \lambda^*\).

**Proof.** Let \(r = [(2Q + 4\Lambda_D)x_{\lambda^*} + 2(c - \lambda_D)] - [(2Q + 4\lambda^*_D)x_{\lambda^*} + 2(c - \lambda^*)].\) Knowing \((2Q + 4\lambda^*_D)x_{\lambda^*} + 2(c - \lambda^*) = 0\) and Lemma 3.6, we see that (i) if \([x_{\lambda^*}]_i = 0\), then \([r]_i \geq 0\); (ii) if \([x_{\lambda^*}]_i = 1\), then \([r]_i \leq 0\). Noticing that \(r_i = (4[x_{\lambda^*}]_i - 2)[\lambda_D - \lambda^*]_i\), and \([x_{\lambda^*}]_i = 0\) or \(1\), we further have \([\lambda_D]_i \leq [\lambda^*]_i\). \(\square\)

Now we can establish the relationship between the optimal solutions of problems COD and ECD as follows.

**Theorem 3.8.** Under the Extended Global Optimality Condition with \(\lambda^*\) and \(x_{\lambda^*}\) being defined, if \((\sigma_D, \lambda_D)\) is an optimal solution of the problem COD, then \(\lambda^*\) is the unique optimal solution of the following conic program:

\[
\begin{align*}
\max_{\lambda} & \quad e^T\lambda \\
\text{s.t.} & \quad \begin{bmatrix}
-\sigma_D & (c - \lambda)^T \\
(c - \lambda) & Q + 2\Lambda
\end{bmatrix} \in D_{n+1}.
\end{align*}
\]

(COP2)

**Proof.** Notice that, for any feasible solution \(\lambda\) of the problem COP2, \((\sigma_D, \lambda)\) is an optimal solution of the problem COD. From Lemma 3.7, we have \(\lambda \leq \lambda^*\). Since \(\lambda^*\) is a feasible solution of COP2, it must be the unique optimal solution of COP2. \(\square\)

In summary, under the Extended Global Optimality Condition, the extended critical point \(\lambda^*\) corresponds to one of the optimal solutions of the problem COD and \(\lambda^*\) is actually the unique optimal solution of the conic program COP2.
4. Practical algorithm and numerical experiments. Conic programming problems with different underlying structures have been studied recently, for example, Strum and Zhang treated cones of non-negative quadratic functions in [15] and Burer handled copositive cones in [2]. However, to our knowledge, there is no known algorithm that can efficiently solve the conic programming problem COP and its dual problem. In order to develop a practical computational algorithm, we will restrict this conic programming problem to a better structured cone. The idea is to substitute the cone $D_{n+1}$ by some computable cone $C_{n+1}$ such that $C_{n+1} \subseteq D_{n+1}$ and $D^*_{n+1} \subseteq C^*_{n+1}$ for the dual cones. Given such a cone $C_{n+1}$, we define the following conic programming problem:

\[
\max \frac{1}{2} \sigma \\
\text{s.t.} \begin{bmatrix} -\sigma & (c - \lambda)^T \\ c - \lambda & \lambda Q + 2\Lambda \end{bmatrix} \in C_{n+1}.
\] (P1)

Then we have the next result.

**Theorem 4.1.** The optimal value of the problem P1 is a lower bound for the problem QIP. Moreover, if there exists an extended critical point $\lambda^* \in G$ such that $D(\lambda^*) \in C_{n+1}$, then this lower bound coincides with the optimal value of the problem QIP.

**Proof.** Since $C_{n+1} \subseteq D_{n+1}$, we have $V(P1) \leq V(COD) \leq V(QIP)$. If there exists an extended critical point $\lambda^*$ such that $D(\lambda^*) \in C_{n+1} \subseteq D_{n+1}$, then, by Theorem 3.4, its corresponding primal solution $x_{\lambda^*}$ is a global optimal solution of QIP. Hence $V(P1) = V(QIP)$ and the lower bound becomes the optimal value of the problem QIP. \(\square\)

Assuming that $\sigma_D$ is optimal to the problem P1, we define another conic programming problem:

\[
\max e^T \lambda \\
\text{s.t.} \begin{bmatrix} -\sigma_D & (c - \lambda)^T \\ c - \lambda & \lambda Q + 2\Lambda \end{bmatrix} \in C_{n+1}.
\] (P2)

Then we have the next theorem.

**Theorem 4.2.** If $P^c(\lambda)$ has an extended critical point $\lambda^* \in G$ such that $D(\lambda^*) \in C_{n+1}$ then $\lambda^*$ is the unique optimal solution to the problem P2.

**Proof.** From Theorem 4.1, we know the value $\sigma_D$ is equal to $2V(QIP)$, which also equals to $\sigma_D$ as defined in the problem COP2. Since $D(\lambda^*) \in C_{n+1} \subseteq D_{n+1}$, from Theorem 3.8, we know $\lambda^*$ is optimal to COP2 and any feasible solution of the problem P2 is also feasible for COP2. Hence $\lambda^*$ is the unique optimal solution of P2. \(\square\)

Now, let $(Q + 2\Lambda^*)^+$ denote the Moore-Penrose pseudoinverse of the matrix $(Q + 2\Lambda^*)$ [13]. With problems P1 and P2, we have the following algorithm which provides either a global optimal solution to the problem QIP or a lower bound for QIP:
Algorithm 1. (QIP Algorithm):
Step 1: For a given problem QIP, construct the conic programming problem P1.
Step 2: Solve the problem P1 for an optimal solution σD.
Step 3: Construct the conic programming problem P2 using Algorithm 1.
Step 4: Solve the problem P2 to obtain an optimal solution λ*.
Step 5: If xλ* = (Q + 2Λ*)+(c − λ*) ∈ {0, 1}n, then return xλ* as a global optimal solution of the problem QIP. Otherwise, Algorithm 1 will return a lower bound 1/2σD for the problem QIP.

Remark 3. For those cases satisfying the condition of Theorem 4.2, if Q + 2Λ* is invertible, then (Q + 2Λ*)+ becomes the conventional inverse matrix, and xλ* = (Q + 2Λ*)−1(c − λ*) satisfies that xλ* ∈ {0, 1}n automatically.

The next theorem validates Algorithm 1.

Theorem 4.3. If Algorithm 1 returns a solution xλ* successfully, then xλ* is a global optimal solution of the problem QIP. Otherwise, Algorithm 1 will return a lower bound.

Proof. If xλ* is returned successfully in Step 5 of Algorithm 1, then D(λ*) ∈ Cn+1 ⊆ Dn+1. Meanwhile, since xλ* ∈ {0, 1}n ∩ \{x ∈ \mathbb{R}^n| (Q + 2Λ*)x + c − λ* = 0\}, we know λ* is an extended critical point of Pc(λ). By Theorem 3.4, xλ* must be a global optimal solution of the problem QIP. Otherwise, from Theorem 4.1, a lower bound 1/2σD for QIP is returned successfully.

It is important to note that, for practical computations, there are many choices for Cn+1. For example, the positive semidefinite cones Sn+1 are computable cones for consideration. We may also consider some cones with better approximation effects (than semidefinite cones), for example, the cones of Sn + \mathcal{N}_n = \{A + B | A ∈ \mathcal{S}_n, B ∈ \mathcal{N}_n\}. Moreover, the sequence of cones in [11] used by Klerk et al. to approximate the copositive cone from inside are also potential candidates.

The following three examples are used to illustrate how Algorithm 1 works.

Example 3. First, we consider the three-dimensional example used in [3] with

\[ Q = \begin{bmatrix} -22 & 9 & 1 \\ 9 & -140 & 6 \\ 1 & 6 & -80 \end{bmatrix} \]

and c = [2, 6, 1]^T.

By choosing C_4 = \mathcal{S}_4 to run Algorithm 1, we obtain the Lagrangian vector λ* = [12, 128, 73]^T in Step 4. Then by computing xλ* = (Q + 2Λ*)+(c − λ*) = [0, 1, 1]^T ∈ {0, 1}^3, we find the global optimal solution as reported before.

Example 4. Now, consider an eight-dimensional example with

\[ Q = \begin{bmatrix} 131 & 28 & -12 & 4 & -72 & 29 & 136 & 140 \\ 28 & 124 & 41 & -72 & -80 & -117 & 118 & -4 \\ -12 & 41 & 39 & -49 & -26 & -49 & -10 & 50 \\ 4 & -72 & -49 & 192 & 142 & 27 & -64 & -40 \\ -72 & -80 & -26 & 142 & 143 & 59 & -113 & -53 \\ 29 & -117 & -49 & 27 & 59 & 372 & -107 & -81 \\ 136 & 118 & -10 & -64 & -113 & -107 & 111 & 109 \\ 140 & -4 & 50 & -40 & -53 & -81 & 109 & 411 \end{bmatrix} \]
and $c = [-197, -166, -81, -93, -36, 219, -182, -403]^T$.

If we choose $C_9 = S_9$ to run Algorithm 1, then we find

$$\lambda_*^* = [38.82, 50.99, 35.52, -10.55, 30.44, -17.48, 35.67, -73.41]^T$$

in Step 4. But

$$x_{\lambda_*^*} = -(Q + 2\lambda_*^*)^T(c - \lambda_*^*)$$

is not feasible. Hence we only obtain a lower bound of $\sigma_D = -504.18$. However, if we choose $C_9 = S_9 + \mathcal{N}_9$ to run Algorithm 1 again, we obtain

$$\lambda^* = [27, 39, 36, -16, 23, -49, 31, -70]^T$$

with its corresponding primal solution

$$x_{\lambda^*} = [0, 1, 1, 1, 1, 0, 1, 1]^T \in \{0, 1\}^8,$$

which is a global optimal solution with the objective value of $-502$.

**Example 5.** Lastly, we consider a ten-dimensional example with

$$Q = \begin{bmatrix}
95 & 14 & -38 & 25 & 51 & 5 & 128 & -55 & 85 & 11 \\
14 & 185 & -37 & -12 & 12 & 2 & -20 & 6 & -21 & 61 \\
-38 & -37 & 128 & -32 & 31 & -10 & 63 & 28 & 121 & -11 \\
25 & -12 & -32 & 99 & 34 & 38 & -6 & -69 & -15 & -51 \\
51 & 12 & 31 & 34 & 122 & 12 & 53 & -36 & 42 & 5 \\
5 & 2 & -10 & 38 & 12 & 59 & 7 & -68 & 16 & -2 \\
128 & -20 & 63 & -6 & 53 & 7 & 118 & 23 & 107 & 26 \\
-55 & 6 & 28 & -69 & -36 & -68 & 23 & 213 & 4 & 50 \\
85 & -21 & 121 & -15 & 42 & 16 & 107 & 4 & 247 & -18 \\
11 & 61 & -11 & -51 & 5 & -2 & 26 & 50 & -18 & 183
\end{bmatrix}$$

and $c = [2, -203, 51, 18, -44, -46, -47, -105, 22, -236]^T$.

If we choose $C_{11} = S_{11}$ to run Algorithm 1, then we find

$$\lambda_*^* = [47.82, -34.17, 13.49, -21.83, 3.19, 16.98, 36.89, -19.60, 36.85, -8.50]^T$$

in Step 4. But

$$x_{\lambda_*^*} = -(Q + 2\lambda_*^*)^T(c - \lambda_*^*)$$

is not feasible. Hence a lower bound of $\sigma_D = -256.95$ is obtained. However, if we choose $C_{11} = S_{11} + \mathcal{N}_{11}$ to apply Algorithm 1, then we find the Lagrangian vector

$$\lambda^* = [2, -39, -11, -23, -17, 17, -17, -27, -12, -5]^T$$

with its corresponding primal solution

$$x_{\lambda^*} = -(Q + 2\lambda^*)^{-1}(c - \lambda^*) = [0, 1, 0, 1, 0, 1, 0, 1, 0, 1]^T \in \{0, 1\}^{10},$$

which is a global optimal solution with the objective value of $-247.5$.

It is interesting to mention that the matrices $Q + 2\Lambda^*$ are not positive semidefinite for Examples 4 and 5. In this situation, the canonical dual method proposed in [3] fails to find the global optimal solution, but Algorithm 1 still works.
5. **Concluding remarks.** In this paper, we have presented an extended canonical dual approach for solving 0-1 quadratic integer programming problems. By studying the relationship between the extended canonical dual problem and corresponding conic programming problems, we have discovered a new global optimality condition for solving QIP and identified a new solvable sub-class of QIP with a practical algorithm.

Our new global optimality condition is more general than those reported in literature, including Theorem 3 of [3] for the \{0, 1\}-constrained QIP, Lemma 2 of [16] for the \{-1, 1\}-constrained QIP, Proposition 3.2 of [9] for the quadratically constrained quadratic programs, and other articles of this kind.

Notice that the known results mentioned above actually are different forms of the following optimality condition:

**Positive Semidefinite Condition.** Let \( x^* \in \{0, 1\}^n \) and \( \lambda^* \in \mathbb{R}^n \), if \( Q + 2\lambda^* \succeq 0 \) and \( (Q + 2\lambda^*)x^* + (c - \lambda^*) = 0 \), then \( x^* \) is an optimal solution of QIP.

Now, if \( Q + 2\lambda^* \succeq 0 \) and \( (Q + 2\lambda^*)x^* + (c - \lambda^*) = 0 \), then \( L(x, \lambda^*) \) is convex for \( x \) and \( \nabla_x L(x, \lambda^*)|_{x=x^*} = 0 \). Hence \( x^* \) is a minimizer of \( L(x, \lambda^*) \). If we let

\[
\sigma^* = 2P^T(\lambda^*) = 2L(x^*, \lambda^*) \quad \text{and} \quad D(\lambda^*) = \begin{bmatrix} -\sigma^* & (c - \lambda^*)^T \\ c - \lambda^* & Q + 2\lambda^* \end{bmatrix},
\]

then, for any \( x \in \mathbb{R}^n \), we have

\[
\begin{bmatrix} 1 \\ x \end{bmatrix}^T \begin{bmatrix} -\sigma^* & (c - \lambda^*)^T \\ c - \lambda^* & Q + 2\lambda^* \end{bmatrix} \begin{bmatrix} 1 \\ x \end{bmatrix} = 2L(x, \lambda^*) - 2L(x^*, \lambda^*) \geq 0.
\]

From this inequality, it is easy to verify that

\[
\begin{bmatrix} -\sigma^* & (c - \lambda^*)^T \\ c - \lambda^* & Q + 2\lambda^* \end{bmatrix} \in S_{n+1} \subseteq D_{n+1},
\]

which also implies that \( Q + 2\lambda^* \succeq 0 \). In this sense, our Extended Global Optimality Condition is more general than the Positive Semidefinite Condition, since problems satisfy the Positive Semidefinite Condition always satisfy the Extended Global Optimality Condition.

To the best of our knowledge, a known solvable subclass of QIP is the set of those satisfying the Positive Definite Condition of \( Q + 2\lambda^* > 0 \) (See [3]). In contrast, our proposed Algorithm 1 extends this solvable sub-classes to those QIP satisfying the condition in Theorem 4.2 with \( x_{\lambda^*} = -(Q + 2\lambda^*)^+(c - \lambda^*) \in \{0, 1\}^n \). This is certainly more general than the Positive Definite Condition. The performance of Algorithm 1 depends on the choice of the cone \( C_{n+1} \). The positive semidefinite cone \( S_{n+1} \) is only one acceptable choice. As Examples 4 and 5 show, we may choose a cone tighter than \( S_{n+1} \) for better performance than previously known algorithms.

To continue this research, we are currently extending our work to handle the quadratically constrained quadratic programming problems. We are also investigating better computable approximation of the cone \( D_{n+1} \) to further improve the performance of Algorithm 1.

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