The free derivative is a pervasive tool in free analysis that provides deep insight into many interesting problems; recent papers discussing the free derivative provide stronger results than their classical counterparts [Pas14, Aug19]. In particular, the free inverse function theorem has been studied in a variety of settings [Pas14, AKV15, AM16, Man20].

This paper answers the question when is a free map the derivative of a free analytic function? While a resolution to the question can be found in [KVV14], or more recently, [KVSV20], the techniques used therein will be less familiar with a non-specialist of free analysis. This article, on the other hand, is set up to be an analogue of two very well-known theorems in analysis: derivatives are curl-free and a curl-free vector field (on a simply connected domain) is a derivative. The statements of our main theorems, their proofs, and even the definitions used within are natural generalizations of their classical counterparts. By following a well-known proof arc, this article provides a satisfying proof that is accessible to both veterans and neophytes of free analysis.

1.1. Preliminaries. Throughout the paper we fix \( g, h \in \mathbb{Z}^+ \) and let \( M(\mathbb{C})^g = (M_n(\mathbb{C})^g)_{n=1}^\infty \). It serves as our universal free set. A subset \( \Omega \subseteq M(\mathbb{C})^g \) is sequence \( \Omega = (\Omega[n])_{n=1}^\infty \), where \( \Omega[n] \subseteq M_n(\mathbb{C})^g \). We say \( \Omega \) is a free set if

1. \( X \oplus Y = (X_1 \oplus Y_1, \ldots, X_g \oplus Y_g) = (\begin{pmatrix} X_1 & 0 \\ 0 & Y_1 \end{pmatrix}, \ldots, \begin{pmatrix} X_g & 0 \\ 0 & Y_g \end{pmatrix}) \in \Omega[n + m] \);
2. \( S^{-1}XS = (S^{-1}X_1S, \ldots, S^{-1}X_gS) \in \Omega[n] \);

for all \( m, n \in \mathbb{Z}^+ \) and \( X = (X_1, \ldots, X_g) \in \Omega[n], Y = (Y_1, \ldots, Y_g) \in \Omega[m] \) and \( S \in \text{GL}_n(\mathbb{C}) \).

If \( \Omega \) is a free set then \( \Omega \times M(\mathbb{C})^g \) is defined to be the free set \( (\Omega[n] \times M_n(\mathbb{C})^g)_{n=1}^\infty \).

A free set \( \Omega \) is said to be a free domain if each \( \Omega[n] \) is open. We note that while our definition requires \( \Omega \) to be closed under simultaneous conjugation by similarities, in certain settings it is desirable to assume the weaker condition of \( \Omega \) being closed under simultaneous conjugation by unitaries, see [JKM+19, HKMV19].

If \( f = (f[n])_{n=1}^\infty \) where \( f[n] : \Omega[n] \to M_n(\mathbb{C})^h \), then we write \( f : \Omega \to M(\mathbb{C})^h \). If, in addition,

1. \( f(X \oplus Y) = \begin{pmatrix} f(X) & 0 \\ 0 & f(Y) \end{pmatrix} \)
(2) $f(S^{-1}XS) = S^{-1}f(X)S$

whenever $X = (X_1, \ldots, X_g) \in \Omega[n]$, $Y = (Y_1, \ldots, Y_g) \in \Omega[m]$ and $S \in \text{GL}_n(\mathbb{C})$ then $f$ is a **free map**. A free map is **continuous** if each $f[n]$ is continuous and **analytic** if each $f[n]$ is analytic. When $f$ is analytic, then its **nc directional derivative** at $X \in \Omega[n]$ in the direction of $H \in M_n(\mathbb{C})$ is

$$Df(X)[H] = \lim_{z \to 0} \frac{f(X + zH) - f(X)}{z}.$$ 

A basic result of free analysis [HKM11, KVV14] says a continuous free map on a free domain is analytic and

$$f \left( \begin{array}{cc} X & H \\ 0 & X \end{array} \right) = \left( \begin{array}{cc} f(X) & Df(X)[H] \\ 0 & f(X) \end{array} \right).$$

The nc directional derivative is a Fréchet derivative, hence it is linear in the direction of the derivative. By virtue of a few simple block matrix computations, the derivative map $Df : \Omega \times M(\mathbb{C})^g \rightarrow M(\mathbb{C})^h$ is easily seen to be free analytic and is linear in its second set of coordinates (such a map will be called a **free vector field**). On the other hand, if we are given an analytic free vector field then we can ask whether it is the derivative of a free map. This question is akin to asking when a given smooth vector field is the gradient of a smooth function.

Our two main results provide necessary and sufficient conditions for an analytic free vector field to be the derivative of an analytic free map. The first is a free analog of the Clairaut-Schwarz Theorem on the equality of mixed partial derivatives:

**Theorem 2.4** (Free Clairaut-Schwarz Theorem). Suppose $\Omega$ is a free domain and $f : \Omega \rightarrow M(\mathbb{C})^h$ is free analytic. If $F : \Omega \times M(\mathbb{C})^g \rightarrow M(\mathbb{C})^h$ is defined by $F(X, H) = Df(X)[H]$, then $F$ is an analytic free vector field and free curl-free: $DF(X, H)[K, L] = DF(X, K)[H, L]$ for all $X \in \Omega$ and $H, K, L \in M(\mathbb{C})^g$.

The second theorem guarantees the existence of a potential function when the analytic free vector field is free-curl free:

**Theorem 3.6.** Suppose $\Omega$ is a connected free domain and $T$ is an analytic free vector field on $\Omega \times M(\mathbb{C})^g$. Then $T$ is free-curl free if and only if there exists a free analytic function $f$ on $\Omega$ such that $T(X, H) = Df(X)[H]$ for all $X \in \Omega$ and $H \in M(\mathbb{C})^g$.

See [Ste18, KVSV20] for related results for difference-differential operators from a different perspective.

Finally, the proof of Theorem 3.6 requires several pieces, of which the following is of independent interest.

**Proposition 3.5.** Suppose $\Gamma \subseteq M_n(\mathbb{C})^g$ is nonempty, open and $\Gamma$ is similarity invariant. If $\beta : \Gamma \rightarrow M_n(\mathbb{C})^h$ is analytic and respects conjugation by unitaries then $\beta$ respects conjugation by similarities.

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2. NECESSITY

The first classical result investigated is the Clairaut-Schwarz Theorem on the equality of mixed partial derivatives: if $\phi$ is any scalar-valued $C^2$ function, then $\text{curl}(\nabla \phi) = 0$. Our proof, below, of the free analog of this classical result relies only on our domain being closed under direct sums and unitary conjugation, rather than direct sums and conjugation by similarities.

**Definition 2.1.** Suppose $T : \Omega \times M(\mathbb{C})^{g} \to M(\mathbb{C})^{h}$. The map $T$ is said to be a **free vector field** if $T$ is free as a function from $\Omega \times M(\mathbb{C})^{g}$ to $M(\mathbb{C})^{h}$ and $T$ is linear in its second coordinates: $T(X, cH + K) = cT(X, H) + T(X, K)$ for all $X \in \Omega$, $H, K \in M(\mathbb{C})^{g}$ and $c \in \mathbb{C}$.

Suppose that in addition to being a free vector field, $T$ is also analytic. Then the **free-curl** of $T$ is the difference $DT(X, H)[K, 0] - DT(X, K)[H, 0]$ (for each $X$ we get a map $(H, K) \mapsto DT(X, H)[K, 0] - DT(X, K)[H, 0]$). If the free-curl is zero for each $X \in \Omega$, then $T$ is said to be **free-curl free**.

**Example 2.2.** Let $T : \Omega \times M(\mathbb{C})^{g} \to M(\mathbb{C})^{h}$ with $T(X, H) = XH - HX$ and note that $T$ is a free vector field. In order for $T$ to be the derivative of a free map, we need $XH - HX = 0$ for all $X \in \Omega$ and $H \in M(\mathbb{C})^{g}$. Hence our domain must only contain points $X$ such that $XH = HX$ for all $H$, that is $X$ is always a scalar matrix. However, the set of scalar matrices is not open and on this set, $XH - HX$ is identically zero.

**Lemma 2.3.** Suppose $\Omega$ is a free domain. If $T$ is an analytic free vector field on $\Omega \times M(\mathbb{C})^{g}$, then $DT(X, H)[K, 0] = DT(X, K)[H, 0]$ if and only if $DT(X, H)[K, L] = DT(X, K)[H, L]$ for all $L \in M(\mathbb{C})^{g}$.

**Proof.** Suppose $DT(X, H)[K, 0] = DT(X, K)[H, 0]$ and let $L$ be given. Since $T$ is a free vector field we immediately observe that $DT(X, H)[0, L] = T(X, L)$. Thus,

$$DT(X, H)[K, L] = DT(X, H)[K, 0] + DT(X, H)[0, L] = DT(X, K)[H, 0] + T(X, L) = DT(X, K)[H, 0] + DT(X, K)[0, L] = DT(X, K)[H, L].$$

The other direction is obtained immediately by choosing $L = 0$. \hfill \blacksquare

**Theorem 2.4** (Free Clairaut-Schwarz Theorem). Suppose $\Omega$ is a free domain and $f : \Omega \to M(\mathbb{C})^{h}$ is free analytic. If $F : \Omega \times M(\mathbb{C})^{g} \to M(\mathbb{C})^{h}$ is defined by $F(X, H) = DF(X, H)[H]$, then $F$ is an analytic free vector field and free curl-free: $DF(X, H)[K, L] = DF(X, K)[H, L]$ for all $X \in \Omega$ and $H, K, L \in M(\mathbb{C})^{g}$.

**Proof.** It is sufficient to show $DF(X, H)[K, 0] = DF(X, K)[H, 0]$ by Lemma 2.3. Recall that for any free analytic map we have

$$f \begin{pmatrix} X & H \\ 0 & X \end{pmatrix} = \begin{pmatrix} f(X) & F(X, H) \\ 0 & f(X) \end{pmatrix}.$$ 

Since $F$ is free analytic,

$$F \begin{pmatrix} (X, H) & (K, 0) \\ (0, 0) & (X, H) \end{pmatrix} = \begin{pmatrix} F(X, H) & DF(X, H)[K, 0] \\ 0 & F(X, H) \end{pmatrix}.$$
In particular,
\[
\begin{pmatrix}
F(X, H) & DF(X, H)[K, 0] \\
0 & F(X, H)
\end{pmatrix}
= F
\begin{pmatrix}
(X, H) & (K, 0) \\
(0, 0) & (X, H)
\end{pmatrix}
= F
\begin{pmatrix}
X & K \\
0 & X
\end{pmatrix}
\begin{pmatrix}
H & 0 \\
0 & H
\end{pmatrix}
= Df
\begin{pmatrix}
X & K \\
0 & X
\end{pmatrix}
\begin{pmatrix}
H & 0 \\
0 & H
\end{pmatrix}.
\]

Thus,
\[
f
\begin{pmatrix}
X & K & H & 0 \\
0 & X & 0 & H \\
0 & 0 & X & K \\
0 & 0 & 0 & X
\end{pmatrix}
= \begin{pmatrix}
 f(X) & F(X, K) & Df
\begin{pmatrix}
X & K \\
0 & X
\end{pmatrix}
\begin{pmatrix}
H & 0 \\
0 & H
\end{pmatrix}
\end{pmatrix}
= \begin{pmatrix}
 f(X) & F(X, K) & F(X, H) & DF(X, H)[K, 0] \\
0 & f(X) & 0 & F(X, H) \\
0 & 0 & f(X) & F(X, K) \\
0 & 0 & 0 & f(X)
\end{pmatrix}.
\]

Moreover, letting
\[
U = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]
we see that
\[
\begin{pmatrix}
X & K & H & 0 \\
0 & X & 0 & H \\
0 & 0 & X & K \\
0 & 0 & 0 & X
\end{pmatrix}
= U \begin{pmatrix}
X & H & K & 0 \\
0 & X & 0 & K \\
0 & 0 & X & H \\
0 & 0 & 0 & X
\end{pmatrix} U^{-1}
\]
and the free structure of \( f \) implies
\[
f
\begin{pmatrix}
X & K & H & 0 \\
0 & X & 0 & H \\
0 & 0 & X & K \\
0 & 0 & 0 & X
\end{pmatrix}
= Uf
\begin{pmatrix}
X & H & K & 0 \\
0 & X & 0 & K \\
0 & 0 & X & H \\
0 & 0 & 0 & X
\end{pmatrix} U^{-1}.
\]

Since
\[
f
\begin{pmatrix}
X & K & H & 0 \\
0 & X & 0 & H \\
0 & 0 & X & K \\
0 & 0 & 0 & X
\end{pmatrix}
= \begin{pmatrix}
 f(X) & F(X, K) & F(X, H) & DF(X, H)[K, 0] \\
0 & f(X) & 0 & F(X, H) \\
0 & 0 & f(X) & F(X, K) \\
0 & 0 & 0 & f(X)
\end{pmatrix} \]
and

\[
Uf \left( \begin{array}{cccc}
X & H & K & 0 \\
0 & X & 0 & K \\
0 & 0 & X & H \\
0 & 0 & 0 & X \\
\end{array} \right) U^{-1} = U \left( \begin{array}{cccc}
f(X) & F(X, H) & F(X, K) & DF(X, K)[H, 0] \\
0 & f(X) & 0 & F(X, K) \\
0 & 0 & f(X) & F(X, H) \\
0 & 0 & 0 & f(X) \\
\end{array} \right) U^{-1}
\]

we conclude \( DF(X, H)[K, 0] = DF(X, K)[H, 0] \). Therefore, for all \( X \in \Omega \) and \( H, K, L \in M(\mathbb{C})^g \) we have \( DF(X, H)[K, L] = DF(X, K)[H, L] \), as desired.

**Corollary 2.5.** Suppose \( T \) is a free vector field on \( \Omega \times M(\mathbb{C})^g \). If there exist \( X \in \Omega \) and \( H, K \in M(\mathbb{C})^g \) such that \( DT(X, H)[K, 0] \neq DT(X, K)[H, 0] \), then \( T \) is not the derivative of an analytic free map on \( \Omega \).

**Proof.** This is the contrapositive of Theorem 2.4.

Theorem 2.4 gives us a clean necessary condition for an analytic free vector field to be the derivative of a free analytic map.

**Example 2.6.** Once more, let us set \( T(X, H) = XH - HX \). Since

\[
DT(X, H)[K, 0] = KH - HK \neq HK - KH = DT(X, K)[H, 0]
\]
on any open free set, we see that \( T \) is not the derivative of an analytic free map.

**Example 2.7.** Similarly, in the classical setting \((-y, x)\) is a standard example of a non-conservative vector field. In our case choosing the free vector field \( T(X_1, X_2, H_1, H_2) = X_1H_2 - H_1X_2 \), we see that

\[
DT(X, H)[K, 0] = K_1H_2 - H_1K_2 \neq H_1K_2 - K_1H_2 = DT(X, K)[H, 0]
\]
unless \( K_1H_2 = H_1K_2 \) for all \( K, H \in M(\mathbb{C})^2 \) – a preposterous idea! Thus, \( T \) is not the derivative of an analytic free map.

**Remark 2.8.** As mentioned above, the fact that the proof of Theorem 2.4 relies only on \( \Omega \) being open and invariant under direct sums and unitary conjugation implies that it can be extended to the operator NC setting. However, the sufficiency proofs rely on our domain being closed under conjugation by similarities as well as evaluations on finite operators.

**Remark 2.9.** Many of the results found in this section ostensibly could be obtained by restricting a free function to a fixed level \( n \) and identifying it as an \( n^2g \) tuple of analytic maps in \( n^2g \) commuting variables. The classical theory then applies and gives us necessary and sufficient condition for the existence of a scalar potential function. However, this misses the free forest for the classical trees.

3. Sufficiency

In the classical setting, if \( \text{curl}(\Phi) = 0 \) on a simply connected domain, then there exists a scalar potential function for \( \Phi \). In outline, the proof of this fact proceeds by showing that curl free implies path independent (on a simply connected domain) to guarantee that a potential
function constructed via line integrals from an anchor point in the domain is well defined (independent of the choice of path).

In the free setting free-curl free implies path independent regardless of the geometry of the domain $\Omega$. Proving that the natural candidate for a potential function is a free function requires some care.

**Definition 3.1.** Suppose $\Omega$ is a free domain and suppose $T$ is an analytic free vector field on $\Omega \times M(\mathbb{C})^g$. For any smooth path $\gamma : [0, 1] \to \Omega[n]$ the entries of $T(\gamma(t), \gamma'(t))$ are analytic functions of $t$, and we define

$$I(T, \gamma) := \int_0^1 T(\gamma(t), \gamma'(t)) \, dt$$

to be the result of applying the integral entry-wise.\(^1\)

We say $T$ is **path independent** if for all $n \in \mathbb{Z}^+$, whenever $\gamma_1, \gamma_2 : [0, 1] \to \Omega[n] \times M_n(\mathbb{C})^g$ are smooth, $\gamma_1(0) = \gamma_2(0)$ and $\gamma_1(1) = \gamma_2(1)$, then

$$I(T, \gamma_1) = \int_0^1 T(\gamma_1(t), \gamma_1'(t)) \, dt = \int_0^1 T(\gamma_2(t), \gamma_2'(t)) \, dt = I(T, \gamma_2).$$

**Lemma 3.2.** Suppose $\Omega$ is a free domain and $T$ is an analytic free vector field. If $\gamma : [0, 1] \to \Omega[n]$ and $\eta : [0, 1] \to \Omega[m]$ are smooth paths and $S \in GL_n(\mathbb{C})$ then

$$I(T, \gamma + \eta) = I(T, \gamma) \oplus I(T, \eta).$$

and

$$I(T, S^{-1}\gamma S) = S^{-1}I(T, \gamma)S.$$

**Proof.** Note $(S^{-1}\gamma S)'(t) = S^{-1}\gamma'(t)S$ and $(\gamma + \eta)'(t) = \gamma'(t) \oplus \eta'(t)$. Hence, the linearity of integration and the free nature of $T$ yield

$$I(T, \gamma + \eta) = \int_0^1 T(\gamma(t), \gamma'(t) + \eta'(t)) \, dt = \int_0^1 T(\gamma(t), \gamma'(t)) \oplus T(\eta(t), \eta'(t)) \, dt = I(T, \gamma) \oplus I(T, \eta)$$

and

$$I(T, S^{-1}\gamma S) = \int_0^1 T(S^{-1}\gamma S, S^{-1}\gamma S) \, dt = \int_0^1 S^{-1}T(\gamma(t), \gamma'(t)) \, dt = S^{-1}I(T, \gamma)S.$$

**Proposition 3.3.** Suppose $\Omega$ is a free domain and $T$ is an analytic free vector field. If $T$ is free-curl free on $\Omega$ (that is, $DT(X, H)[K, 0] = DT(X, K)[H, 0]$), then $T$ is path independent.

**Proof.** Our first observation is that since the derivative of a free function is a Fréchet derivative, it respects the chain rule and the Fundamental Theorem of Calculus:

$$\int_0^1 D\alpha(\gamma(t))[\gamma'(t)] \, dt = \int_0^1 (\alpha \circ \gamma)'(t) \, dt = \alpha(\gamma(1)) - \alpha(\gamma(0)). \quad (3.1)$$

\(^1\)Any smooth path will be bounded away from the boundary of $\Omega[n]$, hence the integrals defined above will have no convergence issues.
Fix $n \in \mathbb{Z}^+$ such that $\Omega[n] \neq \emptyset$ and suppose $\gamma_1$ and $\gamma_2$ are smooth paths in $\Omega[n]$ with the same endpoints. Let

$$S = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \dot{\gamma}(t) = \begin{pmatrix} \gamma_1(t) & 0 & 0 & 0 \\ 0 & \gamma_2(t) & 0 & 0 \\ 0 & 0 & \gamma_1(t) & 0 \\ 0 & 0 & 0 & \gamma_2(t) \end{pmatrix}$$

and observe that

$$S\dot{\gamma}(t)S^{-1} = \begin{pmatrix} \gamma_1(t) & 0 & 0 & \gamma_1(t) - \gamma_2(t) \\ 0 & \gamma_2(t) & 0 & 0 \\ 0 & 0 & \gamma_1(t) & 0 \\ 0 & 0 & 0 & \gamma_2(t) \end{pmatrix}.$$

In particular, $S\dot{\gamma}(t)S^{-1}$ when viewed as a $2 \times 2$ block matrix is precisely in the form $\begin{pmatrix} X & \mathbf{H} \\ 0 & X \end{pmatrix}$; precisely the form needed for realizing the derivative via point evaluation. Hence with a choice of $\gamma_{1,1'} = (\gamma_1, \gamma_1')$ and $\gamma_{2,2'} = (\gamma_2, \gamma_2')$ and an application of Lemma 3.2 we see

$$T(S\dot{\gamma}(t)S^{-1}, S\dot{\gamma}'(t)S^{-1}) =$$

$$S \begin{pmatrix} T(\gamma_{1,1'}) & 0 & 0 & 0 \\ 0 & T(\gamma_{2,2'}) & 0 & 0 \\ 0 & 0 & T(\gamma_{1,1'}) & 0 \\ 0 & 0 & 0 & T(\gamma_{2,2'}) \end{pmatrix} S^{-1} =$$

$$= \begin{pmatrix} T(\gamma_{1,1'}) & 0 & 0 & T(\gamma_{1,1'}) - T(\gamma_{2,2'}) \\ 0 & T(\gamma_{2,2'}) & 0 & 0 \\ 0 & 0 & T(\gamma_{1,1'}) & 0 \\ 0 & 0 & 0 & T(\gamma_{2,2'}) \end{pmatrix} DT\begin{pmatrix} \gamma_{1,1'} & 0 \\ 0 & \gamma_{2,2'} \end{pmatrix} \begin{pmatrix} 0 & \gamma_{1,1'} - \gamma_{2,2'} \\ 0 & 0 \end{pmatrix}.$$ (3.2)

Expanding out the derivative in the upper right hand corner and applying the fact that we assumed $T$ is free-curl free:

$$DT\begin{pmatrix} \gamma_{1,1'} & 0 \\ 0 & \gamma_{2,2'} \end{pmatrix} \begin{pmatrix} 0 & \gamma_{1,1'} - \gamma_{2,2'} \\ 0 & 0 \end{pmatrix} =$$

$$= DT\begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix} \begin{pmatrix} \gamma_1' & 0 \\ 0 & \gamma_2' \end{pmatrix} \begin{pmatrix} 0 & \gamma_1 - \gamma_2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & \gamma_1' - \gamma_2' \\ 0 & 0 \end{pmatrix}.$$ (3.3)

This is precisely the derivative of the composition of $T$ with a smooth path. Choosing

$$\dot{\gamma}(t) = \begin{pmatrix} \gamma_1(t) & 0 \\ 0 & \gamma_2(t) \end{pmatrix}$$

and applying equations (3.1) and (3.3) we see

$$\mathcal{I}(DT, \dot{\gamma}) = \int_0^1 DT(\dot{\gamma}(t))|\dot{\gamma}'(t)| \, dt = T(\dot{\gamma}(1)) - T(\dot{\gamma}(0)).$$
However,
\[ \tilde{\gamma}(1) = \left( \begin{pmatrix} \gamma_1(1) & 0 \\ 0 & \gamma_2(1) \end{pmatrix}, \begin{pmatrix} 0 & \gamma_1(1) - \gamma_2(1) \end{pmatrix} \right) = \left( \begin{pmatrix} \gamma_1(1) & 0 \\ 0 & \gamma_2(1) \end{pmatrix}, \begin{pmatrix} 0 & 0 \end{pmatrix} \right) \]
and similarly
\[ \tilde{\gamma}(0) = \left( \begin{pmatrix} \gamma_1(0) & 0 \\ 0 & \gamma_2(0) \end{pmatrix}, \begin{pmatrix} 0 & \gamma_1(0) - \gamma_2(0) \end{pmatrix} \right) = \left( \begin{pmatrix} \gamma_1(0) & 0 \\ 0 & \gamma_2(0) \end{pmatrix}, \begin{pmatrix} 0 & 0 \end{pmatrix} \right). \]

Thus, linearity of \( T \) in its second coordinate implies \( T(\tilde{\gamma}(1)) = 0 = T(\tilde{\gamma}(0)) \), hence \( I(DT, \tilde{\gamma}) = 0 \). In light of equation (3.2) it follows that
\[ I(T, S^{\hat{\gamma}} S^{-1}) = \begin{pmatrix} I(T, \gamma_1) & 0 & 0 & 0 \\ 0 & I(T, \gamma_2) & 0 & 0 \\ 0 & 0 & I(T, \gamma_1) & 0 \\ 0 & 0 & 0 & I(T, \gamma_2) \end{pmatrix}. \]

Finally, another application of Lemma 3.2 yields
\[ I(T, S^{\tilde{\gamma}} S^{-1}) = S I(T, \tilde{\gamma}) S^{-1} = \begin{pmatrix} I(T, \gamma_1) & 0 & 0 & I(T, \gamma_1) - I(T, \gamma_2) \\ 0 & I(T, \gamma_2) & 0 & 0 \\ 0 & 0 & I(T, \gamma_1) & 0 \\ 0 & 0 & 0 & I(T, \gamma_2) \end{pmatrix}. \]

Therefore
\[ I(T, \gamma_1) = \int_0^1 T(\gamma_1(t), \gamma_1'(t)) \, dt = \int_0^1 T(\gamma_2(t), \gamma_2'(t)) \, dt = I(T, \gamma_2) \]
and we conclude \( T \) is path independent.

**Remark 3.4.** Peculiarly enough, there was no mention of the geometry of \( \Omega \) as one expects in the classical setting. This is a common phenomenon in Free Analysis and for some insight into this curiosity, we use an argument from [Pas20]. If \( \gamma \) and \( \eta \) are two distinct non-intersecting paths in \( \Omega \) with the same endpoints then setting
\[ \Gamma = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} \gamma(t) \\ \eta(t) \end{pmatrix} : a, b \in \mathbb{R}, a^2 + b^2 = 1, t \in [0, 1] \right\} \subseteq \Omega \]
we see that \( \Gamma \cong S^2 \), which is simply connected.

We know from Theorem 2.4 that derivatives of free maps are free-curl free. If \( \Omega \) is connected, then Theorem 3.6 provides the converse. However, for the first time in this article, the proof of Theorem 3.6 requires a delicate touch. Specifically, by defining our potential level-wise, we open up the possibility that the different levels do not align.

In the classical setting, if \( F \) is a path-independent vector field, then fix an “anchor” point \( x_0 \) in the domain and define a potential function \( \varphi \) with \( \varphi(x) \) equal to the result of integrating \( F \) from \( x_0 \) to \( x \) along any path. We still employ this exact idea in the free setting: for each level \( n \), we choose an “anchor” point \( Z_n \in \Omega \) and define \( \alpha_n(X) \) to be the integral of \( T \) from \( Z_n \) to \( X \), along any path. However, there is no guarantee that the \( Z_n \) are related in any obvious fashion (beware: there exist free domains that contain no scalar matrices), nor do we have any guarantee that \( \alpha_n \) respects similarities (or even unitaries). Thus, the proof of Theorem 3.6 is an imitation of the classical proof with three additional steps to fix any misalignments in the level-wise potentials:
(i) At each level $n$, fix an “anchor” point $Z \in \Omega[n]$ and define $\alpha_n(X) = \int_{z_0}^{X} T$, integrated along any path.

(ii) Define $\beta_n(X)$ as the Haar integral of $U^* \alpha_n(UXU^*)U$ over the unitary group – this results in a function that respects conjugation by unitaries.

(iii) Use the entry-wise analyticity of $\beta_n$ to show that $\beta_n$ respects similarities.

(iv) Use level-wise direct sums to find constants $b_n$ such that $\Phi_n = \beta_n + b_n$ defines an analytic free map.

Step (iii) is interesting in its own right and may have application outside of this paper. Thus, we present Step (iii) as a self-contained Proposition.

**Proposition 3.5.** Suppose $\Gamma \subseteq M_n(\mathbb{C})^\Phi$ is nonempty, open and $\Gamma$ is similarity invariant. If $\beta : \Gamma \to M_n(\mathbb{C})^h$ is analytic and respects conjugation by unitaries then $\beta$ respects conjugation by similarities.

**Proof.** Let $S_n$ denote the set of $n \times n$ self-adjoint matrices. We claim that if $\eta : M_n(\mathbb{C}) \to \mathbb{C}^k$ is analytic and vanishes on $S_n$, then $\eta$ vanishes on $M_n(\mathbb{C})$.

Accordingly, suppose $\eta$ is an analytic map vanishing on $S_n$. Let $A,B \in S_n$ and define the map $\zeta : \mathbb{C} \to \mathbb{C}^k$ by $\zeta(z) = \eta(A + izB)$. Note that if $w$ is pure imaginary then $A + iwB \in S_n$ and we must have $\zeta(w) = 0$. Hence, $\zeta$ is an analytic map that vanishes on the imaginary axis, thus $\zeta$ is identically zero. In particular, $0 = \zeta(1) = \zeta(A + iB)$. Every $X \in M_n(\mathbb{C})$ can be decomposed as $A + iB$ for some $A,B \in S_n$, therefore $\eta$ vanishes on $M_n(\mathbb{C})$ and our claim is proved.

Let $U_n$ denote the group of $n \times n$ unitary matrices. Define $\psi : GL_n(\mathbb{C}) \to M_n(\mathbb{C})^h$ by

$$\psi(S) = S\beta(S^{-1}XS)S^{-1} - \beta(X).$$

We see immediately that $\psi$ is analytic and vanishes on $U_n$. Next, let $\varepsilon : M_n(\mathbb{C}) \to GL_n(\mathbb{C})$ be defined by $\varepsilon(X) = e^{iX}$. Note that $\varepsilon$ is surjective and $\varepsilon$ maps $S_n$ onto $U_n$. Hence, the composition $\psi \circ \varepsilon : M_n(\mathbb{C}) \to M_n(\mathbb{C})^h$ is analytic and vanishes on $S_n$. By our claim, $\psi \circ \varepsilon = 0$, hence the surjectivity of $\varepsilon$ implies $\psi$ vanishes on $GL_n(\mathbb{C})$. Therefore $\beta$ respects conjugation by similarities.

**Theorem 3.6.** Suppose $\Omega$ is a connected free domain and $T$ is an analytic free vector field on $\Omega \times M(\mathbb{C})^\Phi$. Then $T$ is free-curl free if and only if there exists a free analytic function $f$ on $\Omega$ such that $T(X,H) = Df(X)[H]$ for all $X \in \Omega$ and $H \in M(\mathbb{C})^\Phi$.

**Proof.** The first direction is handled by Theorem 2.4.

Conversely, suppose $T$ is an analytic free vector field on $\Omega \times M(\mathbb{C})^\Phi$. Let $\mathcal{N}$ be the set of all positive integers $n$ such that $\Omega[n] \neq \emptyset$. We first construct an analytic free map $\Phi$ on $\Omega \times \Omega$ by

$$\Phi(X,Y) = \int_0^1 T(\gamma(t),\gamma'(t)) \, dt$$

where $\gamma$ is any smooth path in $\Omega$ such that $\gamma(0) = Y$ and $\gamma(1) = X$. This map is well-defined since Proposition 3.3 tells us that $T$ is path independent. Moreover, $\Phi(X,Y) + \Phi(Y,Z) = \Phi(X,Z)$ for all $X,Y,Z \in \Omega[n]$ and all $n \in \mathcal{N}$.

Let $\gamma$ and $\eta$ be smooth paths from $Y$ to $X$ and $Z$ to $W$, respectively. Hence $S^{-1}\gamma S$ is a path from $S^{-1}YS$ to $S^{-1}XS$ while $\gamma \oplus \eta$ is a path from $Y \oplus Z$ to $X \oplus W$. Thus, Lemma 3.2
shows us that $\Phi$ is free. Moreover, for any smooth path $\gamma$ from $Y$ to $X$ and any smooth path $\eta$ from $0$ to $H$,

$$\Phi\left(\begin{pmatrix} X & H \\ 0 & X \end{pmatrix}, \begin{pmatrix} Y & 0 \\ 0 & Y \end{pmatrix}\right) = \int_0^1 T\left(\begin{pmatrix} \gamma(t) & \eta(t) \\ 0 & \gamma(t) \end{pmatrix}, \begin{pmatrix} \gamma'(t) & \eta'(t) \\ 0 & \gamma'(t) \end{pmatrix}\right) dt$$

$$= \int_0^1 \begin{pmatrix} T(\gamma(t), \gamma'(t)) & DT(\gamma(t), \gamma'(t))[\eta(t), \eta'(t)] \\ 0 & T(\gamma(t), \gamma'(t)) \end{pmatrix} dt$$

$$= \left(\begin{array}{cc} \Phi(X, Y) & T(\gamma(1), \eta(1)) - T(\gamma(0), \eta(0)) \\ 0 & \Phi(X, Y) \end{array}\right)$$

$$= \left(\begin{array}{cc} \Phi(X, Y) & T(X, H) - T(Y, 0) \\ 0 & \Phi(X, Y) \end{array}\right)$$

$$= \left(\begin{array}{cc} \Phi(X, Y) & T(X, H) \\ 0 & \Phi(X, Y) \end{array}\right).$$

Thus,

$$D\Phi(X, Y)[H, 0] = T(X, H) \quad \text{and} \quad D\Phi(X, Y)[0, K] = -T(Y, K). \quad (3.4)$$

Next, for each $n \in \mathcal{N}$, choose $Z_n \in \Omega[n]$ and define $\alpha_n : \Omega[n] \to M_n(\mathbb{C})^h$ by

$$\alpha_n(X) = \Phi(X, Z_n).$$

Using the fact that $\Phi(X, Y) = -\Phi(Y, X)$ we see

$$\alpha_n(X) - \alpha_n(Y) = \Phi(X, Z_n) - \Phi(Y, Z_n) = \Phi(X, Y)$$

for all $X, Y \in \Omega[n]$.

For each $n \in \mathcal{N}$ let $U_n$ denote the group of $n \times n$ unitary matrices and define $\beta_n : \Omega[n] \to M(\mathbb{C})^h$ via Haar integration:

$$\beta_n(X) = \int_{U_n} U^*\alpha_n(UXU^*)U \, dU.$$

Let $V \in U_n$ and note that

$$\beta_n(V^*XV) = \int_{U_n} U^*\alpha_n(UV^*XVU^*)U \, dU$$

$$= \int_{U_n} V^*W^*\alpha_n(W^*XW^*)WV \, d(WV)$$

$$= V^*\left[\int_{U_n} W^*\alpha_n(W^*XW^*)W \, dW\right]V$$

$$= V^*\beta_n(X)V,$$
where the invariance of the Haar measure is used. Moreover, for any \(X, Y \in \Omega[n]\) we have

\[
\beta_n(X) - \beta_n(Y) = \int_{H_n} U^* (\alpha_n(U^*XU) - \alpha_n(U^*YU)) U \, dU
\]

\[
= \int_{H_n} U^* (\Phi(U^*XU, U^*YU)) U \, dU
\]

\[
= \int_{H_n} \Phi(X, Y) \, dU
\]

\[
= \Phi(X, Y).
\]

Hence, Equation 3.4 implies \(D\beta_n(X)[H] = D\Phi(X, Y)[H, 0] = T(X, H)\). Since \(\beta_n\) is analytic and respects conjugation by unitaries, Proposition 3.5 implies \(\beta_n\) also respects conjugation by similarities.

Our last step is to construct \(f\) from \(\beta\) by adding appropriate scalars. Suppose \(X \in \Omega[m]\), \(Y \in \Omega[n]\) and let

\[
\beta_{m+n} \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix},
\]

where \(A, B, C, D\) are tuples of size \(m \times m, m \times n, n \times n\) and \(n \times n\), respectively. Take any nonzero \(\mu, \nu \in \mathbb{C}\) and note \(S = \mu I_m \oplus \nu I_n\) is invertible and \(S^{-1}(X \oplus Y)S = X \oplus Y\). Hence

\[
\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \beta_{m+n} (S^{-1}(X \oplus Y)S) = S^{-1}\beta_{m+n} (X \oplus Y)S = \begin{pmatrix} A & \frac{\mu}{\nu}B \\ C & D \end{pmatrix}
\]

and we conclude that \(B\) and \(C\) are zero.

Next, for any \(X_1, X_2 \in \Omega[m]\) and \(Y_1, Y_2 \in \Omega[n]\) we see that

\[
\beta_{m+n} \begin{pmatrix} X_1 & 0 \\ 0 & Y_1 \end{pmatrix} - \beta_{m+n} \begin{pmatrix} X_2 & 0 \\ 0 & Y_2 \end{pmatrix} = \Phi \left( \begin{pmatrix} X_1 & 0 \\ 0 & Y_1 \end{pmatrix}, \begin{pmatrix} X_2 & 0 \\ 0 & Y_2 \end{pmatrix} \right) = \Phi(X_1, X_2) \begin{pmatrix} 0 & 0 \\ 0 & \Phi(Y_1, Y_2) \end{pmatrix}.
\]

Hence, \(A\) is independent of our choice of \(Y\) and by a similar argument, \(D\) is independent of our choice of \(X\). Treating \(A\) and \(D\) as functions we see from Equation (3.5) that

\[
A(X_1) - A(X_2) = \Phi(X_1, X_2) = \beta_m(X_1) - \beta_m(X_2)
\]

and

\[
D(Y_1) - D(Y_2) = \Phi(Y_1, Y_2) = \beta_n(Y_1) - \beta_n(Y_2).
\]

Rearranging these equations shows

\[
A(X_1) - \beta_m(X_1) = A(X_2) - \beta_m(X_2)
\]

for all \(X_1, X_2 \in \Omega[m]\). Hence \(A - \beta_m\) is constant as must be \(D - \beta_n\). We let \(A - \beta_m = C_m\) and \(D - \beta_n = C_n\).
Now take $S_m$ and $S_n$ to be similarities of size $m \times m$ and $n \times n$, respectively. Observe

$$\begin{pmatrix}
A(S_m^{-1}XS_m) & 0 \\
0 & D(S_n^{-1}YS_n)
\end{pmatrix} = \beta_{m+n} \begin{pmatrix}
S_m^{-1}XS_m & 0 \\
0 & S_n^{-1}YS_n
\end{pmatrix} = \begin{pmatrix}
S_m^{-1} & 0 \\
0 & S_m^{-1}
\end{pmatrix} \beta_{m+n} \begin{pmatrix}
X & 0 \\
0 & Y
\end{pmatrix} \begin{pmatrix}
S_m & 0 \\
0 & S_n
\end{pmatrix} = \begin{pmatrix}
S_m^{-1}A(X)S_m & 0 \\
0 & S_n^{-1}D(Y)S_n
\end{pmatrix}.$$

Hence, $A$ and $D$ respect similarities. Moreover,

$$C_m = A(S_m^{-1}XS_m) - \beta_m(S_m^{-1}XS_m) = S_m^{-1}(A(X) - \beta_m(X))S_m = S_m^{-1}C_mC_m.$$

Thus, $C_m = c_mI_m$ for some scalar tuple $c_m \in \mathbb{C}^h$. A similar argument shows that $C_n = c_nI_n$ for some $c_n \in \mathbb{C}^h$. Therefore, $\beta_{m+n}$ “nearly” respects direct sums.

For any $m, n \in \mathcal{N}$ we let $c_{m+n}^m$ and $c_{m+n}^n$ be the scalars in $\mathbb{C}^h$ such that

$$\beta_{m+n} \begin{pmatrix}
X & 0 \\
0 & Y
\end{pmatrix} - \begin{pmatrix}
\beta_m(X) & 0 \\
0 & \beta_n(Y)
\end{pmatrix} = \begin{pmatrix}
c_{m+n}^mI_m & 0 \\
0 & c_{m+n}^nI_n
\end{pmatrix}$$

for any $X \in \Omega_m$ and $Y \in \Omega_n$. We note that these constants are well-defined since $\beta_{m+n}$ respects similarities:

$$\begin{aligned}
\beta_{m+n} \begin{pmatrix}
Y & 0 \\
0 & X
\end{pmatrix} - \begin{pmatrix}
\beta_n(Y) & 0 \\
0 & \beta_m(X)
\end{pmatrix} &= \begin{pmatrix}
0 & I_n \\
I_m & 0
\end{pmatrix} \left[ \begin{pmatrix}
\beta_{m+n}(X) & 0 \\
0 & \beta_{m+n}(Y)
\end{pmatrix} - \begin{pmatrix}
\beta_m(X) & 0 \\
0 & \beta_n(Y)
\end{pmatrix} \right] \begin{pmatrix}
0 & I_n \\
I_m & 0
\end{pmatrix} \\
&= \begin{pmatrix}
0 & I_n \\
I_m & 0
\end{pmatrix} \begin{pmatrix}
c_{m+n}^mI_m & 0 \\
0 & c_{m+n}^nI_n
\end{pmatrix} \begin{pmatrix}
0 & I_n \\
I_m & 0
\end{pmatrix} = \begin{pmatrix}
c_{m+n}^mI_m & 0 \\
0 & c_{m+n}^nI_n
\end{pmatrix}.
\end{aligned}$$

Suppose now that $k, m, n \in \mathcal{N}$ and $X \in \Omega[k]$, $Y \in \Omega[m]$ and $Z \in \Omega[n]$. It follows that

$$\begin{aligned}
\beta_{k+m+n} \begin{pmatrix}
X & 0 & 0 \\
0 & Y & 0 \\
0 & 0 & Z
\end{pmatrix} - \begin{pmatrix}
\beta_k(X) & 0 & 0 \\
0 & \beta_m(Y) & 0 \\
0 & 0 & \beta_n(Z)
\end{pmatrix} &= \beta_{k+m+n} \begin{pmatrix}
X & 0 & 0 \\
0 & Y & 0 \\
0 & 0 & Z
\end{pmatrix} - \begin{pmatrix}
\beta_k(X) & 0 & 0 \\
0 & \beta_m(Y) & 0 \\
0 & 0 & \beta_n(Z)
\end{pmatrix} \\
&= \begin{pmatrix}
\beta_{k+m+n}(X) & 0 & 0 \\
0 & \beta_{m+n}(Y) & 0 \\
0 & 0 & \beta_{n+m}(Z)
\end{pmatrix} - \begin{pmatrix}
\beta_k(X) & 0 & 0 \\
0 & \beta_m(Y) & 0 \\
0 & 0 & \beta_n(Z)
\end{pmatrix} \\
&= \begin{pmatrix}
\beta_{k+m+n}(X) - \beta_k(X) & 0 & 0 \\
0 & \beta_{m+n}(Y) - \beta_m(Y) & 0 \\
0 & 0 & \beta_{n+m}(Z) - \beta_n(Z)
\end{pmatrix} \\
&= \begin{pmatrix}
\beta_{k+m+n}(X) & 0 & 0 \\
0 & \beta_{m+n}(Y) & 0 \\
0 & 0 & \beta_{n+m}(Z)
\end{pmatrix} - \begin{pmatrix}
\beta_k(X) & 0 & 0 \\
0 & \beta_m(Y) & 0 \\
0 & 0 & \beta_n(Z)
\end{pmatrix} + \begin{pmatrix}
(\beta_{k+m+n}(X) - \beta_k(X))I_k & 0 & 0 \\
0 & (\beta_{m+n}(Y) - \beta_m(Y))I_m & 0 \\
0 & 0 & (\beta_{n+m}(Z) - \beta_n(Z))I_n
\end{pmatrix} \\
&= \begin{pmatrix}
(\beta_{k+m+n}(X) - \beta_k(X))I_k & 0 & 0 \\
0 & (\beta_{m+n}(Y) - \beta_m(Y))I_m & 0 \\
0 & 0 & (\beta_{n+m}(Z) - \beta_n(Z))I_n
\end{pmatrix}.
\end{aligned}$$

Interchanging the roles of $X, Y$ and $Z$, we have the (redundant) equations

$$\begin{gathered}
c_{k+m+n}^k = \frac{\beta_{k+m+n}(X) - \beta_k(X)}{\beta_{k+m+n}(X) - \beta_k(X)} \\
c_{k+m+n}^m = \frac{\beta_{k+m+n}(Y) - \beta_m(Y)}{\beta_{k+m+n}(Y) - \beta_m(Y)} \\
c_{k+m+n}^n = \frac{\beta_{k+m+n}(Z) - \beta_n(Z)}{\beta_{k+m+n}(Z) - \beta_n(Z)}.
\end{gathered} \tag{3.6}$$

Now, let $n_0 = \min(\mathcal{N})$ and for each $k \in \mathcal{N}$, let $b_k = c_{k+n_0}^k - c_{k+n_0}^k$ and define

$$f_k(X) = \beta_k(X) + b_kI_k$$
for all $X \in \Omega[k]$. Since $f_k$ differs from $\beta_k$ by a scalar matrix, $f_k$ respects conjugation by similarities. Setting $f = (f_k)_{k \in \mathcal{N}}$, we claim $f : \Omega \to M(\mathbb{C})^h$ also respects direct sums. Accordingly, suppose $k, m \in \mathcal{N}$, $X \in \Omega[k]$ and $Y \in \Omega[m]$ and consider

$$f_{k+m} \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} = (f_k(X) & 0 \\ 0 & f_m(Y)) .$$

We show that this difference is zero. By their respective definitions,

$$f_{k+m} \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} - (f_k(X) & 0 \\ 0 & f_m(Y)) = \beta_{k+m} \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} - (\beta_k(X) & 0 \\ 0 & \beta_m(Y)) + b_{k+m} I_{k+m} - (d_k I_k & 0 \\ 0 & b_m I_m)$$

$$= (c_{k+m}^k I_k & 0 \\ 0 & c_{k+m}^m I_m) + b_{k+m} I_{k+m} - (b_k I_k & 0 \\ 0 & b_m I_m).$$

The first $k \times k$ block is $(c_{k+m}^k + b_{k+m} - b_k) I_k$. By our definition of $b_{k+m}$ and $b_k$ and a few applications of Equation (3.6) we see

$$c_{k+m}^k + b_{k+m} - b_k = c_{k+m}^k + [c_{k+m}^k - c_{k+m+n_0}^k] - [c_{k+n_0}^k - c_{k+n_0}^n]$$

$$= (c_{k+m+n_0}^k - c_{k+n_0}^k) - (c_{k+m+n_0}^n - c_{k+n_0}^n)$$

$$= (c_{k+m+n_0}^k - c_{k+n_0}^k) = 0 .$$

Hence, for all $k, m \in \mathcal{N}$, $f_{k+m}(X \oplus Y) = f_k(X) \oplus f_m(Y)$ for all $X \in \Omega[k]$ and $Y \in \Omega[m]$. Thus, $f$ respects direct sums and conjugation by similarities. Moreover, $Df(X)[H] = D\beta(X)[H] = T(X, H)$ since at each level, $f$ and $\beta$ differ by a scalar matrix. Therefore there exists an analytic free function $f$ such that $Df(X)[H] = T(X, H)$.

**Corollary 3.7.** Suppose $\Omega$ is a free domain that is connected. If $(\phi_n)_{n=1}^\infty$ is a sequence of level-wise analytic functions on $\Omega$ such that $(D\phi_n)_{n=1}^\infty$ is an analytic free vector field, then there exists an analytic free map $f$ on $\Omega$ such that $\phi$ and $f$ differ by a constant at each level if and only if $(D\phi_n)_{n=1}^\infty$ is free-curl free.

**Proof.** Suppose $T = (D\phi_n)_{n=1}^\infty$. If $T$ is free-curl free then Theorem 3.6 tells us there exists a free analytic map $f$ on $\Omega$ such that $Df(X)[H] = T(X, H)$. At level $n$, $Df(X)[H] = T(X, H) = D\phi_n(X)[H]$, hence $f$ and $\phi$ must differ by a constant.

On the other hand, if $f$ and $\phi$ differ by a constant at each level, then their derivatives are the same, hence $Df(X)[H] = T(X, H) = D\phi(X)[H]$. Since $f$ is free analytic, Theorem 2.4 implies $T$ is free-curl free.

4. Applications

In this section we present two applications of the ideas of the main article. The first result proves the existence of free pluriharmonic conjugates while the second shows that if a derivative is an nc rational, then it must be the derivative of an nc rational.
4.1. Conjugates of Free Pluriharmonic Functions. In this subsection we present an elementary proof of the existence of conjugates of free pluriharmonic functions. While this fact has been previously established, see [Pas20, Corollary 2.2], we provide an alternate proof taking advantage of Theorem 3.6.

The ideas and proofs in this subsection are due to Robert Martin who kindly gave permission for the author to reproduce his work.

Definition 4.1. Recall that $S_n$ denotes the set of $n \times n$ self-adjoint matrices. Let $S^g = (S_n^g)_{n=1}^{\infty}$ denote our NC self-adjoint universe. For each $n$, let $U_n$ denote set of unitary matrices in $M_n(\mathbb{C})$. A subset $\Gamma \subseteq S^d$ is a real free set if it closed under direct sums and conjugation by joint unitary similarities.

Suppose $\Gamma \subseteq S^d$ is a real free set. If $u = (u[n])_{n=1}^{\infty}$ where $u[n] : \Gamma[n] \to S_n$, then we write $u : \Gamma \to S$. If, in addition,

1. $u(X \oplus Y) = \begin{pmatrix} u(X) & 0 \\ 0 & u(Y) \end{pmatrix}$
2. $u(V^{-1}XV) = V^{-1}u(X)V$

whenever $X = (X_1, \ldots, X_g) \in \Gamma[n]$, $Y = (Y_1, \ldots, Y_g) \in \Gamma[m]$ and $V \in U_n$, then $u$ is a real free map or real free function.

If $\Gamma$ is a real free set that is closed under conjugation by similarities, then Proposition 3.5 implies that any real free map on $\Gamma$ is an honest-to-goodness free map.

Definition 4.2. If $X = \text{Re}X + i\text{Im}X \in M_n(\mathbb{C})^g$, we write $\overrightarrow{X} := (\text{Re}X, \text{Im}X) \in S^2$, and $\overline{X} := (-\text{Im}X, \text{Re}X) = i\overrightarrow{X}$. We say $\Gamma \subseteq S^g$ is a real free domain if $\Gamma = \overrightarrow{\Omega}$ for some free domain $\Omega$ and if $\Omega$ is connected, then we say $\Gamma$ is connected.

Let $f$ be an analytic free map on a free domain $\Omega \subseteq M(\mathbb{C})^g$. If $u = \text{Re}f$, then we view $u$ as a real free function defined on the real free domain

$$
\text{Dom } u := \left\{ \overrightarrow{X} : X \in \Omega \right\}
$$

The following results are found in [tHK20, Theorem 4.1] and [Kle20, Theorem 3.5.2], respectively.

Theorem 4.3 (NC Cauchy-Riemann equations). Suppose $f$ is an analytic free map on a free domain $\Omega$. If $u = \text{Re}f$ and $v = \text{Im}f$, then

$$
Du(\overrightarrow{X})[\overrightarrow{H}] = Dv(\overrightarrow{X})[\overrightarrow{H}].
$$

Theorem 4.4 (NC Laplace equations). Suppose $f$ is an analytic free map on a free domain $\Omega$. If $u = \text{Re}f$ and $v = \text{Im}f$, then

$$
D^2u(\overrightarrow{X})[\overrightarrow{H}, \overrightarrow{K}] + D^2u(\overrightarrow{X})[\overrightarrow{H}, \overrightarrow{K}] = 0
$$

and

$$
D^2v(\overrightarrow{X})[\overrightarrow{H}, \overrightarrow{K}] + D^2v(\overrightarrow{X})[\overrightarrow{H}, \overrightarrow{K}] = 0.
$$

Definition 4.5. ([Kle20, Definition 3.5.6]). A real free function, $u$, on a real free domain, $\Gamma$, is free pluriharmonic if it obeys the NC Laplace equations.

We need one last technical result before proceeding.
Theorem 4.6. ([Ham82, Theorems 3.5.2 & 3.5.3, Corollary 3.5.4]). Suppose $\mathcal{F}$ and $\mathcal{G}$ are Fréchet spaces, $\mathcal{U} \subseteq \mathcal{F}$ is an open domain, and $T : \mathcal{U} \to \mathcal{G}$ is a continuous map. If $D^2 T$ is jointly continuous as a map on $\mathcal{U} \times \mathcal{F} \times \mathcal{F}$, then $D^2 T(X)[Y, Z]$ is linear in both $Y$ and $Z$ and $D^2 T(X)[Y, Z] = D^2 T(X)[Z, Y]$.  

We now present the main theorem of this subsection and its proof.

Theorem 4.7. Suppose $u : \Gamma \to S$ is a free pluriharmonic function on the connected real free domain $\Gamma \subseteq S^{2g}$, and assume that $u$ is jointly level-wise continuous on $\Gamma \times S^{2g} \times S^{2g}$. Then $u$ has a free pluriharmonic conjugate $v : \Gamma \to S$ so that $f = u + iv : \Omega \to M(\mathbb{C})$ is an analytic free map on a connected free domain $\Omega$ with $\overline{\Omega} = \Gamma$.

Proof. Define $\Omega = \{ Z = X + iY : \nabla = (X, Y) \in \Gamma \}$. By the assumptions on $\Gamma$, $\Omega$ is a connected free domain. For each $n \in Z^+$, $X \in \Omega[n]$ and $H \in M_n(\mathbb{C})^g$, we define $T(X, H) := Du(X)[H] - iDu(X)[\overline{H}]$. Note that if $u$ were the real part of an analytic free map, $f$, then by the NC Cauchy-Riemann equations, $Df(X)[H]$ must have this form. Next, we observe that $T$ is graded, preserves direct sums and conjugation by unitaries. Since $T$ is linear in $H$ it follows that it is a free vector field, moreover Proposition 3.5 implies $T$ is an analytic free vector field. Thus, Theorem 3.6 implies that $T(X, H) = Df(X)[H]$ for some analytic free map $f$ if and only if $T$ is free-curl free.

With the free-curl of $T$ in mind, we note $DT(X, H)[K, 0] = D^2 u(X)[\overline{H}, K] - iD^2 u(X)[H, K]$ and $DT(X, K)[H, 0] = D^2 u(X)[K, \overline{H}] - iD^2 u(X)[K, H]$. Thus, if $D^2 u(X)[\overline{H}, K] = D^2 u(X)[K, \overline{H}]$ (4.1) and $D^2 u(X)[H, K] = D^2 u(X)[K, H]$ (4.2) are both true, then it follows that $T$ is free-curl free.

Our hypothesis on $u$ and Hamilton’s Theorem 4.6 immediately imply Equation (4.1). Next we observe $D^2 u(X)[\overline{H}, K] = D^2 u(X)[K, \overline{H}]$ by Equation (4.1) $\Rightarrow -D^2 u(X)[\overline{K}, \overline{H}]$ by NC Laplace equations $\Rightarrow D^2 u(X)[K, H]$.

Hence, $T$ is free-curl free and Theorem 3.6 implies $Du(X)[H] - iDu(X)[\overline{H}] = T(X, H) = Df(X)[H]$ for an analytic free map $f$ with free domain $\Omega$. By virtue of our construction, $Du(X)[H] = \text{Re} Df(X)[H]$ so that we can choose $f$ with $u = \text{Re} f$. Therefore $v = \text{Im} f$ is a free pluriharmonic conjugate of $u$. $\blacksquare$
4.2. **NC rational derivatives are derivatives of nc rationals.** The titular proposition of this subsection is proved using rational formal power series, so we set about defining the appropriate objects. Let \( \mathbf{x} = \{x_1, \ldots, x_g\} \) and \( \mathbf{h} = \{h_1, \ldots, h_g\} \) be sets of freely noncommuting indeterminates. The set \( \langle \mathbf{x} \rangle \) is the free monoid, i.e. the set of all words formed from the letters \( x_1, \ldots, x_g \). The free algebra \( \mathbb{C}\langle \mathbf{x} \rangle \) is the set of all finite \( \mathbb{C} \)-linear combinations of elements of \( \langle \mathbf{x} \rangle \).

A **formal power series** \( S \) is a function \( S : \langle \mathbf{x} \rangle \to \mathbb{C} \) and the image of \( w \) under \( S \) is the coefficient of \( w \), denoted \([S,w]\). The set of all formal power series (in \( \mathbf{x} \) over \( \mathbb{C} \)) is denoted \( \mathbb{C}\langle \mathbf{x} \rangle \). For any word \( w \in \langle \mathbf{x} \rangle \) and series \( S \in \mathbb{C}\langle \mathbf{x} \rangle \), we let \( w^{-1}S = \sum_{w \in \langle \mathbf{x} \rangle} [S, wv]v \) (effectively the backwards shift by \( w \)).

A subset \( M \) of \( \mathbb{C}\langle \mathbf{x} \rangle \) is called **stable** if, for all \( S \in M \) and \( w \in \langle \mathbf{x} \rangle \), the series \( w^{-1}S \in M \). A series \( S \) is **rational** if and only if it is contained in a stable finitely generated left submodule of \( \mathbb{C}\langle \mathbf{x} \rangle \). The set of all rational formal power series in \( \mathbb{C}\langle \mathbf{x} \rangle \) forms an algebra and is denoted by \( \mathbb{C}\langle \mathbf{x} \rangle_0 \) (this notation is due to the fact that these series can exactly be realized as noncommutative rational functions with 0 in their domain). If \( S \) is rational, then \( w^{-1}S \) is rational for any \( w \in \mathbf{x} \).

**Proposition 4.8.** Suppose \( f \in \mathbb{C}\langle \mathbf{x} \rangle \). If \( Df(x)[\mathbf{h}] \) is a rational formal power series then \( f(x) \) is as well.

**Proof.** For each \( 1 \leq i \leq g \) we let \( \partial_i f := Df(x)[0, \ldots, 0, h_i, 0, \ldots, 0] \). The map \( \zeta_i(F(x)[\mathbf{h}]) = F(x)[0, \ldots, h_i, \ldots, 0] \) is an algebra homomorphism and since \( Df(x)[\mathbf{h}] \) is a rational series by hypothesis, it follows that \( \partial_i f = \zeta_i(Df(x)[\mathbf{h}]) \) is a rational series as well. Consequently, \( h_i^{-1}\partial_i f \) is rational. Next, we write

\[
f = c_0 + \sum_{j=1}^{g} x_j f_j
\]

for some \( f_i \in \mathbb{C}\langle \mathbf{x} \rangle \). Hence,

\[
\partial_i f = h_i f_i + \sum_{j=1}^{g} x_j \partial_i f_j \in \mathbb{C}\langle \mathbf{x}, \mathbf{h} \rangle_0.
\]

As noted above, the rationality of \( \partial_i f \) implies that \( h_i^{-1}\partial_i f \) is rational as well. Since \( h_i^{-1}\partial_i f_i = f_i \), we have that each \( f_i \) is rational. Therefore, \( f = c_0 + \sum_{i=1}^{g} x_i f_i \) is a rational formal power series.

**Remark 4.9.** Not every nc rational can be represented as a rational formal power series, however, every nc rational can be represented as a rational generalized series over \( M_n(\mathbb{C})\langle \mathbf{x} \rangle \), for some \( n \), see [Vol18]. Moreover, the proof of Proposition 4.8 works in exactly the same way for generalized series, allowing us to conclude that every nc rational derivative is indeed the derivative of an nc rational.

**References**

[AKV15] Gulnara Abduvalieva and Dmitry S. Kaluzhnyi-Verbovetskyi. Implicit/inverse function theorems for free noncommutative functions. J. Funct. Anal., 269(9):2813--2844, 2015. https://doi-org.lp.hscl.ufl.edu/10.1016/j.jfa.2015.07.011.

[AM16] Jim Agler and John E. McCarthy. The implicit function theorem and free algebraic sets. Trans. Amer. Math. Soc., 368(5):3157--3175, 2016. https://doi-org.lp.hscl.ufl.edu/10.1090/tran/6546.
[Aug19] Meric L. Augat. The free grothendieck theorem. Proceedings of the London Mathematical Society, 118(4):787–825, 2019. https://londmathsoc.onlinelibrary.wiley.com/doi/abs/10.1112/plms.12200.

[Ham82] Richard S. Hamilton. The inverse function theorem of Nash and Moser. Bulletin (New Series) of the American Mathematical Society, 7(1):65 – 222, 1982.

[HKM11] J. William Helton, Igor Klep, and Scott McCullough. Proper analytic free maps. Journal of Functional Analysis, 260(5):1476 – 1490, 2011. https://doi.org/10.1016/j.jfa.2010.11.007.

[HKMV19] J. William Helton, Igor Klep, Scott McCullough, and Jurij Volčič. Biaalytic free maps between spectrahedra and spectraballs. Preprint, 2019. https://arxiv.org/abs/1804.09743.

[JKM+19] Michael Jury, Igor Klep, Mark E. Mancuso, Scott McCullough, and James E. Pascoe. Noncommutative partial convexity via Γ-convexity. Preprint, 2019. https://arxiv.org/abs/1908.05949.

[Kle20] E. M. Klem. Complex analysis and realization theory for classes of free noncommutative functions. PhD thesis, Vrije Universiteit Amsterdam, 2020.

[KSV20] Dmitry Kaliuzhnyi-Verbovetskyi, Leonard Stevenson, and Victor Vinnikov. Integrability of free noncommutative functions, 2020.

[KV14] D.S. Kaliuzhnyi-Verbovetskyi and V. Vinnikov. Foundations of Free Noncommutative Function Theory. Mathematical Surveys and Monographs. American Mathematical Society, 2014.

[Man20] Mark E. Mancuso. Inverse and implicit function theorems for noncommutative functions on operator domains. Journal of Operator Theory, 83(2):447–473, 2020. http://dx.doi.org/10.7900/jot.2018oct21.2237.

[Pas14] J. E. Pascoe. The inverse function theorem and the jacobian conjecture for free analysis. Mathematische Zeitschrift, 278(3):987–994, 2014. http://dx.doi.org/10.1007/s00209-014-1342-2.

[Pas20] J.E. Pascoe. Noncommutative free universal monodromy, pluriharmonic conjugates, and plurisubharmonicity. Preprint, 2020. https://arxiv.org/abs/2002.07801.

[Ste18] Leonard C. Stevenson. Calculus of higher order noncommutative functions, June 2018. PhD Thesis, Drexel University.

[tHK20] S. ter Horst and E. M. Klem. Cauchy-riemann equations for free noncommutative functions. In Raúl E. Curto, William Helton, Huaxin Lin, Xiang Tang, Rongwei Yang, and Guoliang Yu, editors, Operator Theory, Operator Algebras and Their Interactions with Geometry and Topology, pages 333–357, Cham, 2020. Springer International Publishing.

[Vol18] Jurij Volčič. Matrix coefficient realization theory of noncommutative rational functions. Journal of Algebra, 499:397 – 437, 2018.