OVERTWISTED DISCS IN PLANAR OPEN BOOKS

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Abstract. Using open book foliations we show that an overtwisted disc in a planar open book can be put in a topologically nice position. As a corollary, we prove that a planar open book whose fractional Dehn twist coefficients greater than one for all the boundary components supports a tight contact structure.

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1. Introduction

There is a rigid dichotomy between tight and overtwisted contact structures on 3-manifolds. All the contact structures are locally identical, so tightness and overtwistedness are global properties.

Eliashberg’s classification of overtwisted contact structures [14] states that overtwisted contact structures are classified by the homotopy types of 2-plane fields. On the other hand, tight contact structures are more subtle, and classification of tight contact structures is still open except for several simple cases including some Seifert fibered spaces.

It is often hard to determine whether a given contact structure is tight or overtwisted. Some types of fillability, such as (weak and strong) symplectic or Stein, imply tightness of the contact structures. Non-vanishing of Ozsváth and Szabó’s contact invariant shows tightness [26]. In convex surface theory, Giroux’s criterion [17] is useful to find overtwisted discs, and Honda’s state traversal method [19] provides a way to prove tightness. In [16] Eliashberg and Thurston use confoliations to prove that a contact structure obtained by $C^0$-small perturbation of a taut foliation is universally tight (cf. [20]).
In this paper we give a new tightness criteria (Corollaries 1.2 and 1.3) using the strong topological and combinatorial aspects of open book foliations [22, 23, 24, 25]. Here is our main theorem:

**Theorem 1.1.** Let \((S, \phi)\) be a planar open book which supports an overtwisted contact structure. Then there exists a transverse overtwisted disc \(D\) such that:

\[(SE1): \text{All the valence } \leq 1 \text{ vertices of the graph } G_\sim(D) \text{ are strongly essential.}\]

The graph \(G_\sim(D)\) and strongly essential vertices are defined in Section 2. Roughly speaking, Theorem 1.1 shows that we can put a transverse overtwisted disc so that it intersects each page of the open book in some nice way. In the theory of Haken 3-manifolds, one uses essential surfaces to cut 3-manifolds and study the structure and properties of the manifolds. We apply this classical scheme to contact 3-manifolds and surfaces admitting open book foliations and analyze topological features of the open books.

For a boundary component \(C \subset \partial S\) let \(c(\phi, C)\) denote the fractional Dehn twist coefficient (FDTC) of \(\phi\) with respect to \(C\). See [21] for a more formal definition. In [21, Theorem 1.1] Honda, Kazez and Matić prove that an open book \((S, \phi)\) supporting a tight contact structure implies that \(\phi\) is right-veering, in particular, \(c(\phi, C) \geq 0\) for all the boundary components \(C\) of \(S\). The next corollaries assert the converse direction of the theorem under some assumptions on FDTC.

**Corollary 1.2.** Let \((S, \phi)\) be a planar open book. If \(c(\phi, C) > 1\) for all the boundary components \(C \subset \partial S\) then \((S, \phi)\) supports a tight contact structure.

The next is a slightly stronger statement:

**Corollary 1.3.** Let \((S, \phi)\) be a planar open book. If \(c(\phi, C) \geq 1\) for all the boundary components \(C \subset \partial S\) and the equality holds for at most one boundary component then \((S, \phi)\) supports a tight contact structure.

It is interesting to compare these corollaries with the result of Colin and Honda in [13]. They show that for a (not necessarily planar) open book \((S, \phi)\) with pseudo-Anosov monodromy, if \(c(\phi, C_i) \geq \frac{k}{n_i} (k \geq 2)\) for every boundary component \(C_i\) of \(S\), where \(n_i\) is the number of prongs around \(C_i\) of the transverse measured (stable) foliation for \(\phi\), then \((S, \phi)\) supports a universally tight contact structure ([13] treats the connected binding case, but by Baldwin and Etnyre [11, Theorem 4.5] the same result holds for the general case). They show tightness by proving the non-vanishing of the contact homology. It is important to note that the foundation of contact homology requires hard analysis, and is geometric in the sense that its definition uses Reeb vector fields and contact forms. On the other hand, our argument using open book foliations is topological in nature.

It is also worth mentioning that Corollary 1.3 may show tightness for cases not covered by Colin and Honda’s result: Consider a planar open book \((S, \phi)\) with pseudo-Anosov
monodromy such that \( n_{i_0} = 1 \) and \( c(\phi, C_{i_0}) = 1 \) for some boundary component \( C_{i_0} \), and \( c(\phi, C) > 1 \) for the remaining boundary components.

**Remark.** Let \( S = S_{0,4} \) be a sphere with four discs removed. Call the boundary circles \( A, B, C \) and \( D \). Let \( E \) be a simple closed curve in \( S \) that separates \( A, B \) from \( C, D \). For \( h, i, k > 0 \), let \( \Phi_{h,i,k} = T_A^h T_B^i T_C^k T_D^{k-1} \), where \( T_X \) denotes the right-handed Dehn twist along \( X \in \{ A, B, C, D, E \} \). In [23, Theorem 4.1] we show that the open book \(( S, \Phi_{h,i,k})\) is non-destabilizable and supports an overtwisted contact structure. The FDTCs of \( \Phi = \Phi_{h,i,k} \) are 

\[
(c(\Phi, A), c(\Phi, B), c(\Phi, C), c(\Phi, D)) = (h, i, 1, 1).
\]

Thus, the conditions in Corollaries 1.2 and 1.3 are best possible even if we add a reasonable assumption that \(( S, \phi)\) is non-destabilizable.

## 2. Review of open book foliations

In this section we summarize definitions and properties of open book foliations used in this paper. For details, see [22, 24, 25]. The idea of open book foliations originally came from Bennequin’s work [2] and Birman-Manasco’s braid foliations [4, 5, 6, 7, 8, 9, 11, 12].

Let \( S = S_{g,r} \) be a genus \( g \) surface with \( r (> 0) \) boundary components, and \( \phi \in \text{Diff}^+ (S, \partial S) \) an orientation preserving diffeomorphism of \( S \) fixing the boundary \( \partial S \) pointwise. Suppose that the open book \(( S, \phi)\) supports the closed oriented contact 3-manifold \(( M, \xi)\) via the Giroux correspondence [18]. The manifold \( M \) is often denoted by \( M_{(S,\phi)} \). Let \( B \) denote the binding of the open book and \( \pi : M \setminus B \to S^1 \) the fibration whose fiber \( S_t := \pi^{-1}(t) \) is called a page.

Let \( F \subset M_{(S,\phi)} \) be an embedded, oriented surface possibly with boundary. If \( F \) has boundary we require that \( \partial F \) is a closed braid with respect to \(( S, \phi)\), that is, \( \partial F \) is positively transverse to every page. Up to isotopy of \( F \) fixing \( \partial F \) we may put \( F \) so that the singular foliation given by the intersection with the pages

\[
\mathcal{F}_{ob}(F) = \{ F \cap S_t \mid t \in [0,1] \}
\]

admits the following conditions \(( F \ i)–( F \ iv)\), see [22, Theorem 2.5]. We call \( \mathcal{F}_{ob}(F) \) an open book foliation on \( F \).

- \(( F \ i)\): The binding \( B \) pierces the surface \( F \) transversely in finitely many points. Moreover, \( v \in B \cap F \) if and only if there exists a disc neighborhood \( N_v \subset \text{Int}(F) \) of \( v \) on which the foliation \( \mathcal{F}_{ob}(N_v) \) is radial with the node \( v \), see Figure 1(1, 2). We call \( v \) an elliptic point.

- \(( F \ ii)\): The leaves of \( \mathcal{F}_{ob}(F) \) along \( \partial F \) are transverse to \( \partial F \).

- \(( F \ iii)\): All but finitely many fibers \( S_t \) intersect \( F \) transversely. Each exceptional fiber is tangent to \( F \) at a single point \( \in \text{Int}(F) \). In particular, \( \mathcal{F}_{ob}(F) \) has no saddle-saddle connections.
\((\mathcal{F} \text{ iv}):\) All the tangencies of \(F\) and fibers are of saddle type, see Figure \(\text{I}(3, 4)\). We call them hyperbolic points.

A leaf \(l\) of \(\mathcal{F}_{ob}(F)\), a connected component of \(F \cap S_t\), is called regular if \(l\) does not contain a tangency point, and singular otherwise. The regular leaves are classified into the following three types:

- \(a\)-arc : An arc where one of its endpoints lies on \(B\) and the other lies on \(\partial F\).
- \(b\)-arc : An arc whose endpoints both lie on \(B\).
- \(c\)-circle : A simple closed curve.

We say that an elliptic point \(v\) is positive (resp. negative) if the binding \(B\) is positively (resp. negatively) transverse to \(F\) at \(v\). The sign of the hyperbolic point \(h\) is positive (resp. negative) if the positive normal direction of \(F\) at \(h\) agrees (resp. disagrees) with the direction of \(t\). We denote the sign of a singular point \(x\) by \(\text{sgn}(x)\). See Figure \(\text{I}\) where we describe an elliptic point by a hollowed circle with its sign inside, a hyperbolic point by a black dot with the sign indicated nearby, and positive normals \(\vec{n}_F\) to \(F\) by dashed arrows.

![Figure 1. Signs of singularities and normal vectors \(\vec{n}_F\).](image)

Hyperbolic points in \(\mathcal{F}_{ob}(F)\) are classified into six types according to the types of nearby regular leaves: Type \(aa\), \(ab\), \(bb\), \(ac\), \(bc\), and \(cc\) as depicted in Figure \(\text{II}\). Such a model neighborhood is called a region. We denote by \(\text{sgn}(R)\) the sign of the hyperbolic point contained in the region \(R\). The surface \(F\) is decomposed into a union of regions so that whose interiors are disjoint [22, Proposition 2.15].

We will often take the following homotopical properties of leaves into account.

**Definition 2.1.** Let \(b\) be a \(b\)-arc in \(S_t\). We say that:
(1) $b$ is essential if $b$ is not boundary-parallel in $S_t \setminus (S_t \cap \partial F)$,
(2) $b$ is strongly essential if $b$ is not boundary-parallel in $S_t$,
(3) $b$ is separating if $b$ separates the page $S_t$ into two connected components.

For a $b$-arc the conditions ‘boundary parallel in $S_t$’ and ‘non-strongly essential’ are equivalent.

**Definition 2.2.** An elliptic point $v$ is called strongly essential if every $b$-arc that ends at $v$ is strongly essential. An open book foliation $\mathcal{F}_{ob}(F)$ is called (strongly) essential if all the $b$-arcs are (strongly) essential.

The next lemma may be one of the most useful results in open book foliation theory, which claims that existence of strongly essential vertices provides us an estimate of the FDTC. See [24, Section 5] for further relationships between open book foliations and the FDTC.

**Lemma 2.3.** [24, Lemma 5.1] Let $v$ be an elliptic point of $\mathcal{F}_{ob}(F)$ lying on a binding component $C \subset \partial S$. Assume that $v$ is strongly essential and there are no $a$-arcs around $v$. Let $p$ (resp. $n$) be the number of positive (resp. negative) hyperbolic points that are joined with $v$ by a singular leaf.

(1) If $\text{sgn}(v) = +1$ then $-n \leq c(\phi, C) \leq p$.

(2) If $\text{sgn}(v) = -1$ then $-p \leq c(\phi, C) \leq n$.

The embedding of $F$ near a hyperbolic point is described as follows: Recall that a hyperbolic point is nothing but a saddle tangency of a page and $F$. Consider a saddle-shaped
subsurface of $F$ whose leaves $l_1$ and $l_2$ (possibly $l_1 = l_2$) as in Figure 3 are sitting on a page $S_t$. As $t$ increases (the page moves up) the leaves approach along a properly embedded arc $\gamma \subset S_t$ (dashed in Figure 3) joining $l_1$ and $l_2$ and switch configuration. See the passage in Figure 3. We call $\gamma$ the describing arc of the hyperbolic singularity. Up to isotopy, $\gamma$ is uniquely determined and conversely $\gamma$ uniquely determines embedding of the saddle. We often put the sign of a hyperbolic point near its describing arc.

The movie presentation of $F$ determines how $F$ is embedded up to isotopy: Take $0 = s_0 < s_1 < \cdots < s_k = 1$ so that $S_{s_i}$ is a regular page and there exists exactly one hyperbolic point $h_i$ in each interval $(s_i, s_{i+1})$. The sequence of the slices $(S_{s_i}, S_{s_i} \cap F)$ with the describing arc of $h_i$ is called a movie presentation.

**Example 2.4.** Let $(D^2, id)$ be an open book decomposition of $S^3$ and consider a 2-sphere $F$ embedded in $S^3$ as shown in Figure 4-(a). Figure 4-(b) depicts the whole picture of $\mathcal{F}_{ob}(F)$ and Figure 4-(c) is the movie presentation. See [22, 23] for more examples of movie presentations.

The graph $G_- = G_-(F)$ of $\mathcal{F}_{ob}(F)$ is a graph consists of negative elliptic points and unstable separatrices for negative hyperbolic points in $aa$-$ab$- and $bb$-tiles. The vertices of $G_-$ are the negative elliptic points in $ab$- and $bb$-tiles and the end points of the edges of $G_-$ that lie on $\partial F$, called fake vertices. Similarly, $G_+$ the graph consists of positive elliptic points and stable separatrices of positive hyperbolic points is defined.

**Definition 2.5.** [22, Definition 4.1] An embedded disc $D \subset M(S, \phi)$ whose boundary is a positively braided unknot is called a transverse overtwisted disc if

1. $G_-$ is a connected tree with no fake vertices.
2. $G_+$ is homeomorphic to $S^1$.
3. $\mathcal{F}_{ob}(D)$ contains no c-circles.
As proved in [22, Proposition 4.2, Corollary 4.6], an open book \((S, \phi)\) supports an overtwisted contact structure if and only if \(\mathcal{M}_{(S, \phi)}\) contains a transverse overtwisted disc.

In [25], we study operations on open book foliations including a b-arc foliation change:

**Theorem 2.6 (b-arc foliation change).** [25, Theorem 3.1, Proposition 3.2] Assume that the open book foliation \(\mathcal{F}_{\text{ob}}(F)\) contains two tiles \(R_1, R_2\) satisfying the following conditions (i)–(iii), see Figure 5-(a):

(i): \(R_i\) \((i = 1, 2)\) is either an ab-tile or a bb-tile.

(ii): \(\text{sgn}(R_1) = \text{sgn}(R_2) = \varepsilon \in \{+1, -1\}\).

(iii): \(R_1\) and \(R_2\) are adjacent at a separating b-arc, \(b\).

Then there is an ambient isotopy \(\Phi_\tau : M \rightarrow M\) supported on \(M \setminus B\) such that:

(1) \(F' = \Phi_1(F)\) admits an open book foliation \(\mathcal{F}_{\text{ob}}(F')\). If \(\mathcal{F}_{\text{ob}}(F)\) is essential then so is \(\mathcal{F}_{\text{ob}}(F')\).

(2) The region decomposition of \(\mathcal{F}_{\text{ob}}(F')\) contains regions \(R'_1, R'_2\), see Figure 5-(b,c) such that:

(a) \(R'_1\) and \(R'_2\) are either \(aa\), \(ab\), or \(bb\)-tile,

(b) \(\text{sgn}(R'_1) = \text{sgn}(R'_2) = \varepsilon\) as in (ii),

(c) \(\Phi_1(R_1 \cup R_2) = R'_1 \cup R'_2\)
(d) $R'_1 \cap R'_2$ is exactly one leaf $l$ of type $a$ or $b$.

(e) The numbers of the hyperbolic points connected to $v$ and $A$ by a singular leaf decrease both by one, though the total number of hyperbolic points remains the same.

(3) $\Phi_t$ preserves the region decomposition of $F \setminus (R_1 \cup R_2)$.

(4) If $\partial F$ is non-empty then $\Phi_t(\partial F)$ is a closed braid w.r.t. $(S, \phi)$ for all $t \in [0, 1]$, i.e., $L = \partial F$ and $L' = \partial F'$ are braid isotopic.

Figure 5. (a)$\rightarrow$(b) and (a)$\rightarrow$(c) are b-arc foliation changes.

3. Outline of the proof of Theorem 1.1

The rest of the paper is devoted to proving Theorem 1.1. In this section we overview the proof. Assume that a planar open book $(S, \phi)$ supports an overtwisted contact structure. We start with an arbitrary transverse overtwisted disc $D$ in $M(S, \phi)$ and introduce a complexity of $D$ which measures how far $D$ is from having the property (SE1). We construct a new transverse overtwisted disc $D'$ whose complexity is less than that of $D$. By standard induction on the complexity we finish the proof.
In Section 4 as an intermediate step we construct an embedded disc $D_0$ by replacing a boundary parallel b-arc of $D$ with an inessential b-arc. The disc $D_0$ may not be a transverse overtwisted disc.

In Section 5 we study the open book foliation of $D_0$ and show that $\mathcal{F}_{ob}(D_0)$ is obtained by “blowing up” $\mathcal{F}_{ob}(D)$ along several leaves (cf. Figure 19).

In Section 6 we construct a transverse overtwisted disc $D'$ from $D_0$. After studying basic properties of $D'$ we define a complexity of a transverse overtwisted disc and prove that $D'$ has less complexity than the original $D$.

In Section 7 we complete the proof of Theorem 1.1 and give related results, comments and questions.

4. Movie presentation of the intermediate disc $D_0$

In this section we construct the intermediate disc $D_0$ from a given transverse overtwisted disc $D \subset M(S, \phi)$ with (SE1). Without loss of generality, we may assume that $\mathcal{F}_{ob}(D)$ is essential, for otherwise we can remove inessential b-arcs to make $\mathcal{F}_{ob}(D)$ essential (see [24, Theorem 3.2]).

Let $v \in G_{-}(D)$ be a valence one, non-strongly essential vertex. That is, $v$ is a negative elliptic point of $\mathcal{F}_{ob}(D)$ and $m$ positive ab-tiles $R_1, \ldots, R_m$ and one negative bb-tile $R_-$ meet at $v$. Let $\Omega_0, \ldots, \Omega_m$ denote the positive elliptic points which are connected to $v$ by a b-arc, $\Omega_i \in \partial R_i \cap \partial R_{i+1}$ and $\Omega_0, \Omega_m \in \partial R_-$. For $t \in [0,1]$, we denote the b-arc in $S_t$ that ends at $v$ by $b_t$ and the hyperbolic point in $R_i$ (resp. $R_-$) by $h_i$ (resp. $h_-$), see Figure 6(a).

![Figure 6](image-url)

**Figure 6.** (a) $\mathcal{F}_{ob}(D)$ near the valence one vertex $v$. (b) The leaf box $A$ in the page $S_0$ bounded by the b-arc $b_0$. 
We may assume that every singular fiber contains exactly one hyperbolic point and the page $S_0 = \phi(S_1)$ is a regular fiber. Denote the singular fiber that contains $h_i$ by $S_{t_i}$ where $t_i \in (0, 1)$. There exists a small $\varepsilon > 0$ such that:

- $h_- \in S_{1-\varepsilon}$.
- For any distinct singular fibers $S_t$ and $S_{t'}$ we have $|t - t'| > 2\varepsilon$.
- The family $\{S_t \mid t_i - \varepsilon \leq t \leq t_i + \varepsilon\}$ contains no hyperbolic points other than $h_i$.
- $0 < t_1 < t_2 < \cdots < t_m < 1$.

**Lemma 4.1.** With some perturbation of $D$ we may assume that:

- **(P1):** The $b$-arc $b_t$ is non-separating for all $t \in (t_1, t_m)$ with $t \neq t_1, \ldots, t_m$.

**Proof.** Assume that $b_t$ is separating for some $t \in (t_i, t_{i+1})$. Then the ab-tiles $R_t$ and $R_{t+1}$ meet along a separating b-arc. Since $\text{sgn}(R_t) = \text{sgn}(R_{t+1})$ applying b-arc foliation change (Theorem 2.6) the region $R_t \cup R_{t+1}$ is replaced by the union of one positive ab-tile and one positive aa-tile. See the passage (a) in Figure 7. The new aa-tile can be eliminated by destabilizing the closed braid $\partial D$, see Figure 7(b). The resulting disc is a transverse overtwisted disc.

As a consequence, the family of separating b-arcs $\{b_t \mid t \in (t_i, t_{i+1})\}$ disappear and the number of positive elliptic points connected to $v$ decreases by one. \hfill \Box

Suppose that there exists a valence one, non-strongly essential vertex $v \in G_- (D)$ satisfying (P1). This implies that either $b_0$ or $b_{1-2\varepsilon}$ is boundary parallel. In the rest of the paper we may assume that $b_0$ is boundary parallel. (The other case can be treated similarly.) Thus, $b_0$ cobounds a disc $\Delta \subset S_0$ with the binding. Let $\partial' \Delta := \partial \Delta \setminus b_0$. Since $\mathcal{F}_{ob}(D)$ is essential $\text{Int}(\Delta)$ intersects $D$. Hence $\partial' \Delta$ contains elliptic points, $x_1, \ldots, x_n$. Let $A := \text{Int}(\Delta) \cap D \subset S_0$ a set of leaves. See Figure 8(b) where $A$ is represented by a box, called the leaf box $A$.

The following is a key observation.

**Lemma 4.2.** If $\mathcal{F}_{ob}(D)$ satisfies (P1) then $D$ satisfies the property

- **(P2):** $\Omega_i \not\subset \partial' \Delta$ for all $i = 1, \ldots, m$. That is, $\Omega_i \neq x_1, \ldots, x_n$ for all $i = 1, \ldots, m$.

**Proof.** We first show that $\Omega_m \not\subset \partial' \Delta$. If $\Omega_m \subset \partial' \Delta$ then the $b$-arc $b_{1-2\varepsilon}$ connecting $v$ and $\Omega_m$ is included in $\Delta$ since $b_{1-2\varepsilon}$ and $b_0$ (projected on $S$) have zero geometric intersection and the page surface $S$ is planar. That is, $b_{1-2\varepsilon}$ lies on the left of $b_0 \subset \partial \Delta$ near $v$. However, since $h_-$ is a negative hyperbolic point $b_0$ should lie on the left of $b_{1-2\varepsilon}$ from $v$, which is a contradiction.

Next we assume that $\Omega_i \subset \partial' \Delta$ for some $i = 1, \ldots, m - 1$. Again by the planar assumption on $S$ the $b$-arc $b_{t_i+\varepsilon}$ connecting $v$ and $\Omega_i$ is separating, which contradicts (P1). \hfill \Box
Figure 7. (Lemma 4.1): If the bb-tiles $R_i$ and $R_{i+1}$ are adjacent at a separating b-arc then we apply:
(a) b-arc foliation change.
(b) destabilization of the closed braid $\partial D$.
(c) rescaling the open book foliation.

Now by modifying $D$ we construct a new embedded disc $D_0$ whose b-arc $b_0$ is inessential. We do this by moving the leaf box $A$ out of $\Delta$ at the cost of introducing new singular points. The disc $D_0$ may not be a transverse overtwisted disc, but is similar to a transverse overtwisted disc in the sense that $\partial D_0$ violates the Bennequin-Eliashberg inequality. The definition of $D_0$ is given by a movie presentation:

**Step 1:** Movie for $t \in [0, t_1 + \varepsilon]$.

See Figure 8 where the left three sketches depict the movie presentation of $D$ for $t \in [0, t_1 + \varepsilon]$ near $\partial^c \Delta \cup b_{t_1 + \varepsilon} \cup \Omega_1$. Since $D \cap (\Delta \times [0, t_1]) \cong A \times [0, t_1]$ the region $\Delta \times [0, t_1]$ contains no hyperbolic points.

The movie presentation of $D_0$ for $t \in [0, t_1 + \varepsilon]$ near $\partial^c \Delta \cup b_{t_1 + \varepsilon} \cup \Omega_1$ is shown in the right three sketches of Figure 8. There are no leaves in $\Delta \times [0, t_1]$, i.e., no elliptic points on $\partial^c \Delta$. Instead, on the positive side of $\Omega_1$ the new elliptic points, $x_1^{(1)}, \ldots, x_n^{(1)}$, and the leaf box $A$ are placed.
Away from the neighborhood of $\partial \Delta \cup b_{t_1 + \varepsilon} \cup \Omega_1$ the movie presentations of $D$ and $D_0$ are the same, except that on the positive side of $\Omega_i$ (for $i = 2, \ldots, m$) we put elliptic points $x_1^{(i)}, \ldots, x_n^{(i)}$ and the leaf box $A$ as in Figure 9. The property (P2) guarantees that $x_1^{(i)}, \ldots, x_n^{(i)} \notin \partial \Delta$ for all $i = 1, \ldots, m$.

**Figure 8. Step 1:** The movie presentations of $D$ and $D_0$ for $t \in [0, t_1 + \varepsilon]$.

**Figure 9. Step 1:** For $t \in [0, t_1 + \varepsilon]$ put $A$ and $x_1^{(i)}, \ldots, x_n^{(i)}$ near $\Omega_i$. 
Step 2: Movie for $t \in [t_i + \varepsilon, t_{i+1} - \varepsilon]$ ($i = 1, 2, \ldots, m - 1$) and $[t_m + \varepsilon, 1 - 2\varepsilon)$.

The left sketch of Figure 10 depicts the slice $D \cap S_t$ for $t \in [t_i + \varepsilon, t_{i+1} - \varepsilon)$ near $\partial' \Delta \cup b_t \cup \Omega_i$. Let $X_t \subset S_t$ be the set of leaves connected to $\partial' \Delta$ (i.e., end at $x_1, \ldots, x_n$). By legs of $X$ we mean subset of $X$ near $x_1, \ldots, x_n$. We construct $D_0$ for the interval $[t_i + \varepsilon, t_{i+1} - \varepsilon)$ by sliding the legs of $X_t \subset S_t$ along the $b$-arc $b_t$, see the right sketch of Figure 10. This is made possible due to (P2).

Figure 10. Step 2: The slices $D_0 \cap S_t$ and $D_0 \cap S_t$ for $t \in [t_i + \varepsilon, t_{i+1} - \varepsilon)$.

If for some $t^* \in [t_i + \varepsilon, t_{i+1} - \varepsilon)$ the leaf box $X_{t^*}$ contains a hyperbolic point $h^*$ then we can use the same describing arc for $h^*$ (see Figure 11) since it does not intersect the $b$-arc $b_{t^* - \varepsilon}$.

Away from the neighborhood of $\partial' \Delta \cup b_t \cup \Omega_i$ the movie presentations of $D$ and $D_0$ are identical, except that on the positive side of each of $\Omega_j$ for $j \neq i$ the elliptic points $x_1^{(j)}, \ldots, x_n^{(j)}$ and the leaf box $A$ are placed (see Figure 9).

Step 3: Movie for $t \in [t_i - \varepsilon, t_i + \varepsilon]$ where $i = 2, \ldots, m$.

This step is the core of the construction of $D_0$ and divided into two sub-steps. Figure 12 shows the slices $D_0 \cap S_{t_i - \varepsilon}$ and $D_0 \cap S_{t_i + \varepsilon}$. Since $h_{t_i}$ is the only hyperbolic point of $\mathcal{F}_{ob}(D)$ in the interval $[t_i - \varepsilon, t_i + \varepsilon]$ we have $X_t \cong X_{t_i - \varepsilon}$ for all $t \in [t_i - \varepsilon, t_i + \varepsilon]$, hence we may denote $X_t$ simply by $X$.

Step 3-1: Movie for $t \in [t_i - \varepsilon, t_i + \frac{1}{2}\varepsilon]$. 
Figure 11. **Step 2**: Modification of a hyperbolic point.

Figure 12. **Step 3**: Before and after Step 3.

See Figure 13. For $t \in [t_i - \varepsilon, t_i + \frac{\varepsilon}{2}]$ the slice $D_0 \cap S_t$ is defined by

- adding the leaf box $A$ on the positive side of each $\Omega_k$ for $k \neq i - 1$ and
- sliding the legs of $X$ along $b_t$ to the positive side of $\Omega_{i-1}$. 

Figure 13. **Step 3-1:** In the interval \([t_i - \varepsilon, t_i + \frac{\varepsilon}{2}]\), add a corresponding hyperbolic point \(h_i\).

- on the page \(S_{t_i - \varepsilon}\) we copy the describing arc of the hyperbolic point \(h_i \in S_{t_i}\).

**Step 3-2:** Movie for \(t \in (t_i + \frac{\varepsilon}{2}, t_i + \varepsilon]\).

By introducing hyperbolic points we move the legs of the leaf box \(X\) from the positive side of \(\Omega_{i-1}\) to the positive side of \(\Omega_i\).

**3-2-1:** First we consider a simple case where the leaf in the original leaf box \(A\) for \(D\) (Figure 6) from \(x_1\) is an a-arc. Then on the page \(S_{t_i + \varepsilon/2}\) the leaf from \(x_1^{(i)}\) is also an a-arc. Join this a-arc and the leg of the leaf box \(X\) landing on \(x_1^{(i-1)}\) by a (+) describing arc that lies near \(b_{t_i + \varepsilon/2}\) as in Figure 14-(1). As a consequence, the leaf from \(x_1^{(i-1)}\) becomes an a-arc and \(x_1^{(i)}\) is connected to \(X\) as in Figure 14-(2).

**3-2-2:** Next assume that the leaf in the original leaf box \(A\) for \(D\) from \(x_1\) is a b-arc, \(b\), joining \(x_1\) and \(x_j\) for some \(j \neq 1\). Let \(A'\) denote the sub-leaf box of \(A\) enclosed by \(b\) and
$A'': = A \setminus (A' \cup b)$. By Step 3-1, in the slice $D_0 \cap S_{t_{i+\varepsilon}/2}$ the elliptic points $x_1^{(i)}$, $x_j^{(i)}$ are joined by a b-arc, $b^{(i)}$, as depicted in Sketch (1) of Figure 15.

(a) Join the b-arc $b^{(i)}$ and the leg of the leaf box $X$ landing on $x_1^{(i-1)}$ by a describing arc that lies near $b_{t_i+\varepsilon}$ as in Figure 15 (1). The sign of the new hyperbolic point, $h$, is equal to the sign of the elliptic point $x_1$. The configuration changes as in the passage (a) of Figure 15. Namely, the leaf box $X$ and $x_1^{(i)}$ are connected, and $x_j^{(i)}$ and $x_1^{(i-1)}$ are joined by a b-arc.

(b) Apply to the leaf box $A'$ the operation in Step 3-2-1 or Step 3-2-2 (a) depending on the type of the leaf from $x_2^{(i)}$. See the passage (b) of Figure 15. The leaf box $A'$ is moved from near $\Omega_i$ to near $\Omega_{i-1}$.

(c) Introduce a hyperbolic point, $\overline{h}$, between the leaves from $x_1^{(i-1)}$ and $x_j^{(i-1)}$ to enclose $A'$ by a b-arc, $b^{(i-1)}$. See the passage (c) of Figure 15. The sign of the hyperbolic point $\overline{h}$ is opposite to that of $h$;

\begin{equation}
\text{sgn}(h) = -\text{sgn}(\overline{h}) = \text{sgn}(x_1).
\end{equation}

(d) Apply the above (a, b, c) to the leaf box $A''$.

As a consequence, the entire leaf box $A$ is moved to the positive side of $\Omega_{i-1}$ from $\Omega_i$ and the leaves from $x_1^{(i-1)}, \ldots, x_n^{(i-1)}$ are connected to $A$. See Figure 12.

Step 4: Movie for $t \in [1 - 2\varepsilon, 1]$.

In the interval $[1 - 2\varepsilon, 1]$ there is only one hyperbolic point $h_- \in S_{1-\varepsilon}$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure14.png}
\caption{Step 3-2-1: Movie of $D_0$.}
\end{figure}
By (Step 2) for \([t_m + \varepsilon, 1 - 2\varepsilon]\), the slice \(D_0 \cap S_{1-2\varepsilon}\) is obtained from the slice \(D \cap S_{1-2\varepsilon}\) by sliding the legs of \(A\) along \(b_{1-2\varepsilon}\). On the page \(S_{1-\varepsilon}\) we introduce a negative hyperbolic point corresponding to \(h_\ast\) (see the upper right sketch of Figure 16). Note that the slices \(D \cap S_1\) and \(D_0 \cap S_1\) are identical except for the added \(m\) copies of \(A\). Since these copies exist near the binding, the copies of \(A\) on \(S_1\) and the copies of \(A\) on \(S_0\) are identified under the monodromy \(\phi\). This completes the construction of \(D_0\).
In this section we describe the open book foliation of $D_0$ and explain how $\mathcal{F}_{ob}(D_0)$ is related to $\mathcal{F}_{ob}(D)$.

Assume that the leaf from $x_j$ in the original leaf box $A$ is an a-arc. In Step 3-2-1 (Figure 15) in order to switch the leg of $X$ from $x_j^{(i-1)}$ to $x_j^{(i)}$ we have introduced a positive hyperbolic point in the interval $[t_i + \frac{\varepsilon}{2}, t_i + \varepsilon]$. Figure 17 depicts the open book foliation $\mathcal{F}_{ob}(D_0)$ near $x_j^{(i-1)}$ and $x_j^{(i)}$.

Suppose that the leaf from $x_j$ in $A$ is a b-arc connecting $x_j$ and $x_k$ where $j < k$ (cf. Step 3-2-2 and Figure 15). If $\text{sgn}(x_j) = -\text{sgn}(x_k) = +1$ then the open book foliation $\mathcal{F}_{ob}(D_0)$ in the neighborhood of $x_j^{(i-1)}, x_j^{(i)}, x_k^{(i-1)}, x_k^{(i)}$ is as in Figure 18-(1). Note by the sign condition (4.1) we have $\text{sgn}(h) = -\text{sgn}(\overline{h}) = +1$. Similarly, Figure 18-(2) depicts the case when $\text{sgn}(x_j) = -\text{sgn}(x_k) = -1$.

In summary, the open book foliation $\mathcal{F}_{ob}(D_0)$ is obtained from $\mathcal{F}_{ob}(D)$ by “blowing-up” along the leaf on the page $S_0$ from $x_j$ ($j = 1, \ldots, n$). That is, if the leaf is an a-arc then its neighborhood is replaced by region consisting of $m$ positive elliptic and $(m - 1)$ positive hyperbolic points. See Figure 19-(a). If the leaf is a b-arc, then its neighborhood is replaced by region consisting of $m$ positive and $m$ negative elliptic points and $(m - 1)$ positive and $(m - 1)$ negative hyperbolic points. The signs are determined by the condition (4.1). See Figure 19-(b).
The following proposition highlights features of the disc $D_0$:

**Proposition 5.1.** Let $D_0$ be the disc constructed in Section 4.

(i) The $b$-arc $b_0 \subset S_0$ of $\mathcal{F}_{ob}(D_0)$ connecting $v$ and $\Omega_0$ is inessential.

(ii) $sl(\partial D_0, D_0) = 1$.

(iii) $G_{- -}(D_0)$ is a tree and the number of the valence 1 vertices of $G_{- -}(D_0)$ and that of $G_{- -}(D)$ are the same.

**Remark 5.2.** If $S$ is a non-planar surface, it may be possible that $\Omega_i \in \partial' \Delta$ for some $i$ and exist a non-separating $b$-arc connecting $v$ and $\Omega_i$, i.e., the property (P2) may not hold. Then the elliptic points $x^{(i)}_1, \ldots, x^{(i)}_n$ of $D_0$ are placed on $\partial \Delta$ and Proposition 5.1(i) may not hold.
By Proposition 5.1(i), the b-arc $b_0$ is inessential. We remove $v$ by compressing the disc $\Delta$ co-bounded by $b_0$ and $\partial'\Delta$ as shown in Figure 20 and flatten the local extrema. Call the resulting disc $D_1$. The open book foliation changes as in the passage $(1) \rightarrow (2) \rightarrow (3)$ of Figure 21.

This does not affect the self-linking number and we have $sl(\partial D_1, D_1) = sl(\partial D_0, D_0) = 1$ by Proposition 5.1(ii). The Bennequin-Eliashberg inequality [15] does not hold for $D_1$. Thus,
we can apply the construction discussed in [22, Section 4] to $D_1$ and obtain a transverse overtwisted disc, $D'$. We call the whole construction

$$D \rightarrow D_0 \rightarrow D_1 \rightarrow D'$$

deforming $D$ at $v$.

**Proposition 6.1.** The number of the valence 1 vertices of the graph $G_{-}(D')$ is smaller than or equal to that of $G_{-}(D)$.

**Proof.** Note that $G_{-}(D') = G_{-}(D_1)$. By the construction of $D_1$ the graph $G_{-}(D_1)$ is obtained from $G_{-}(D_0)$ by removing the vertex $v$ and the edge from $v$. The assertion follows from Proposition 5.1-(iii). \hfill $\square$

**Remark 6.2.** Here are remarks on $D_0$ and $D_1$.

1. If $m = 1$ then $D_1$ is already a transverse overtwisted disc so $D' = D_1$. In this case, the operation $D \rightarrow D'$ is nothing but an exchange move studied in [25].

2. The passage $D \rightarrow D_0 \rightarrow D_1$ does not require that $D$ is a disc. We only need the assumption that $v$ is a non-strongly essential, valence one vertex of $G_{-}(D)$. Similar construction may apply to general surfaces embedded in $M_{(S, \phi)}$.

At a first glance, $F_{ob}(D')$ looks more complicated than $F_{ob}(D)$, since we have introduced many singularities, including negative elliptic points in order to remove the vertex $v \in G_{-}(D)$. Our next task is to define a complexity of a transverse overtwisted disc and show that the complexity of $D'$ is smaller than that of $D$.

**Definition 6.3.** Let $D$ be a transverse overtwisted disc. Let

$$V_D := \text{the set of valence one, non-strongly essential vertices of } G_{-}(D).$$

For $v \in V_D$, the branch $B(v)$ is the maximal connected subgraph of $G_{-}(D)$ containing $v$ and valence $\leq 2$, non-strongly essential vertices of $G_{-}(D)$. See Figure 22.
Definition 6.4. Let \( v \in \mathcal{V}_D \) and \( w \in B(v) \). Let \( b \subset S_t \) be a boundary-parallel b-arc starting from \( w \). Then \( b \) cobounds a disc \( \Delta \subset S_t \) with a subarc, \( \alpha \), of the binding. We define the nesting level of \( b \) as follows (see also Figure 23):

1. If \( B(v) \cap \alpha \) is empty then we define the nesting level of \( b \) to be zero.
2. If \( B(v) \cap \alpha \) is non-empty then let \( k \) be the maximal nesting level of the b-arcs that end at \( B(v) \cap \alpha \). The nesting level of \( b \) is defined to be \( k + 1 \).

Definition 6.5. For \( v \in \mathcal{V}_D \) and \( w \in B(v) \) we define \( NL(w) \in \mathbb{Z} \) the nesting level of \( w \) (with respect to \( D \)) by

\[
NL_D(w) = \max\{\text{nesting level of } b \mid b \text{ is a boundary parallel b-arc from } w\}.
\]
Definition 6.6. The complexity of $v \in V_D$ is a sequence of non-negative integers
\[ C_D(v) = (\ldots, C_k, C_{k-1}, \ldots, C_1, C_0) \]
where $C_k$ represents the number of the vertices of $B(v)$ of the nesting level $k$. By Definition 6.5 we have $C_k \leq C_{k-1}$ for all $k$. We compare sequences by the natural lexicographical order (from the left). For example $(\ldots, 0, 0, 1, 1, 4) > (\ldots, 0, 0, 9, 42)$.

Definition 6.7. Let
\[ C_D := \min\{C_D(v) \mid v \in V_D\} \]
We have $C_D = (\ldots, 0, 0)$ if and only if all the valence one vertices of $G_-(D)$ are strongly essential. Define the complexity of $D$ to be the pair
\[ \mathcal{C}(D) := (|V_D|, C_D) \]
and compare $\mathcal{C}(D)$ by the lexicographical order.

The following is a key to the proof of Theorem 1.1.

Proposition 6.8. Let $D$ be a transverse overtwisted disc with $|V_D| \geq 1$. Let $v \in V_D$ such that $C_D = C_D(v)$, and $D'$ the transverse overtwisted disc obtained by deforming $D$ at $v$. Then $\mathcal{C}(D') < \mathcal{C}(D)$.

Proof. By the proof of Proposition 6.1 we have $|V_{D'}| \leq |V_D|$. If $|V_{D'}| < |V_D|$ then $\mathcal{C}(D') < \mathcal{C}(D)$.

Assume that $|V_{D'}| = |V_D|$, that is $|B(v)| \geq 2$. Let $v' \in V_D$ be the vertex which is adjacent to $v \in V_D$. By Definition 6.3 and the construction of $D'$ we have $v' \in V_{D'}$.

To compare $C_D$ and $C_{D'}$ we examine the nesting levels of the vertices in $B(v')$. Recall that in the construction of $D_0$ (hence $D'$) we have added copies of the leaf box $A$. There are two types of vertices $x \in B(v')$:

**Type A**: $x$ is a newly introduced negative elliptic point on the foot of $A$. (i.e., $x$ is a negative elliptic point of the form $x_i^{(j)}$ in Section 5)

**Type B**: $x$ is a vertex that comes from a vertex, $x^* \in B(v)$.

The original leaf box $A$ for $D$ is contained in the half-disc $\Delta$ bounded by the b-arc $b$ from $v$. So for a vertex $x \in B(v')$ of Type A, we have
\[ NL_{D'}(x) \leq NL_D(v) - 1. \]

For a vertex $x \in B(v')$ of Type B with the corresponding vertex $x^* \in B(v)$, the added copies of the leaf box $A$ do not affect the nesting level, i.e.,
\[ NL_{D'}(x) = NL_D(x^*). \]

To get the new transverse overtwisted disc $D'$ we have removed the vertex $v$ of the nesting level $NL_D(v)$, so these observations improve:
\[ C_{D'} \leq C_{D'}(v') < C_D(v) = C_D. \]
7. Proofs of Theorem 1.1 and Corollaries 1.2 and 1.3

Proof of Theorem 1.1. Let \( (S, \phi) \) be a planar open book supporting an overtwisted contact structure. Take a transverse overtwisted disc \( D \subset (S, \phi) \). By [24, Theorem 3.2] we may assume that \( \mathcal{F}_{ob}(D) \) is an essential open book foliation.

If \( \mathcal{F}_{ob}(D) \) contains only one negative elliptic point \( v \) we have shown in [24, Theorem 6.2, Claim 6.3] that \( v \) is strongly essential and the condition \((SE1)\) is satisfied.

If \( \mathcal{F}_{ob}(D) \) contains more than one negative elliptic points then by Proposition 6.8 we can find a transverse overtwisted disc \( D' \) with \( c(D') = (0, \ldots, 0) \), i.e., \((SE1)\) is satisfied. \(\square\)

Proof of Corollaries 1.2 and 1.3. Suppose that a planar open book \( (S, \phi) \) supports an overtwisted contact structure. By Theorem 1.1 there exists a transverse overtwisted disc \( D \) with the property \((SE1)\).

To prove Corollary 1.2 let \( v \in G_{--}(D) \) be a valence \( \leq 1 \) vertex and assume that \( v \) lies on the binding component \( C \). Lemma 2.3 and \((SE1)\) imply that \( c(\phi, C) \leq 1 \), which is a contradiction.

To prove Corollary 1.3 let \( \partial S = C_1 \cup \cdots \cup C_r \) and assume that \( c(\phi, C_i) > 1 \) for \( i \neq 1 \). If \( G_{--}(D) \) consists of one vertex \( v \in C_j \) then \( \phi \) is not right-veering w.r.t. \( C_j \) so \( c(\phi, C_j) \leq 0 \), which is a contradiction. Hence \( G_{--}(D) \) contains more than one vertices. The property \((SE1)\) imposes that if a valence 1 vertex \( v \in G_{--}(D) \) lies on \( C_j \) then \( c(\phi, C_j) \leq 1 \). Thus \( C_j = C_1 \). In other words, all the valence one vertices of \( G_{--}(D) \) must lie on the same binding component \( C_1 \). Then the estimate of FDTC in [24, Theorem 5.4], which is a refined version of Lemma 2.3, implies the strict inequality \( c(\phi, C_1) < 1 \). \(\square\)

To close the paper, we give several related questions and comments.

First, as noted in Remark 6.2 our construction of \( D \rightarrow D_0 \rightarrow D_1 \) discussed in [11–14] is valid for general surfaces \( F \) in planar open books. We call the operation removing a non-strongly essential valence one vertex \( v \) deforming \( F \) at \( v \).

Additional argument similar to the one for the exchange move in [25] shows that if \( F' \) is a surface obtained by deforming \( F \) at \( v \), the braids \( \partial F \) and \( \partial F' \) are transversely isotopic (if \( F \) has boundary) and \( F \) and \( F' \) are isotopic in \( M_{(S, \phi)} \).

This observation, combined with the complexity defined in [6] we get the following result concerning a “nice” position of general Seifert surfaces in planar open books.

**Theorem 7.1.** Let \( F \) be a Seifert surface of a closed braid \( L \) w.r.t. a planar open book \( (S, \phi) \). Then there exists a surface \( F' \) isotopic to \( F \) such that \( \partial F' \) is transversely isotopic to \( L \) and
(SE1'): all the valence \( \leq 1 \) vertices of \( G_{-} (F') \) are strongly essential.

The condition (SE1') only concerns the vertices of valence \( \leq 1 \). One may ask whether one can further modify and put the surface \( F \) so that:

(SE): all the vertices of \( G_{-} (F) \) are strongly essential,

while preserving the transverse knot type of its boundary.

It is also interesting to ask whether Theorem 1.1 holds for non-planar open books. In our whole arguments, we use the planar assumption only to guarantee the property (P2) of Lemma 4.2 and we deform \( D \) at \( v \) (cf. Remark 5.2).

ACKNOWLEDGEMENT

The authors would like to thank Joan Birman, Bill Menasco and John Etnyre. T.I. was supported by JSPS Research Grant-in-Aid for Research Activity Start-up. K.K. was partially supported by NSF grants DMS-0806492 and DMS-1206770.

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