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LITTLEWOOD-PALEY DECOMPOSITION OF OPERATOR DENSITIES
AND APPLICATION TO A NEW PROOF OF THE LIEB-THIRRING
INEQUALITY

JULIEN SABIN

Abstract. The goal of this note is to prove an analogue of the Littlewood-Paley decomposition for densities of operators and to use it in the context of Lieb-Thirring inequalities.

Introduction

Let $d \geq 1$ and $\psi$ a smooth function on $\mathbb{R}^d$, supported in $\mathbb{R}^d \setminus \{0\}$, satisfying
\[ 1 = \sum_{j \in \mathbb{Z}} \psi(2^{-j}\xi), \quad \forall \xi \in \mathbb{R}^d \setminus \{0\}. \tag{1} \]

An example of such a function is given in [11, Lemma 8.1]. In particular, the function $\psi$ can be chosen to be radial and non-negative. We define the Littlewood-Paley multiplier localizing on frequencies $|\xi| \sim 2^j$ by
\[ P_j u := F^{-1}(\xi \mapsto \psi_j(\xi)F u(\xi)), \quad \psi_j := \psi(2^{-j} \cdot), \quad j \in \mathbb{Z}, u \in S'(\mathbb{R}^d), \]
where $F$ denotes the Fourier transform. The Littlewood-Paley theorem [11, Thm. 8.3] states that for any $1 < p < \infty$, there exists $C > 0$ such that for any $u \in L^p(\mathbb{R}^d)$ one has
\[ \frac{1}{C} \| u \|_{L^p} \leq \left\| \left( \sum_{j \in \mathbb{Z}} |P_j u|^2 \right)^{1/2} \right\|_{L^p} \leq C \| u \|_{L^p}. \tag{2} \]

This harmonic analysis result has countless applications, from functional inequalities to nonlinear PDEs. It allows to obtain information about $L^p$-properties of a function $u$ from the frequency-localized pieces $P_j u$. For instance, it leads to a very short proof of the Sobolev embedding $H^s(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d)$ for $p = 2d/(d - 2s)$, $0 < s < d/2$, as we recall in Section 2.1. It was also used, for instance, to prove Strichartz-type inequalities [9, 3]. We refer to [8] for more general applications of Littlewood-Paley theory.

This note is devoted to a generalization of (2) to densities of operators. When $\gamma \geq 0$ is a finite-rank operator on $L^2(\mathbb{R}^d)$, its density is defined as
\[ \rho_{\gamma}(x) := \gamma(x, x), \quad \forall x \in \mathbb{R}^d, \]
where $\gamma(\cdot, \cdot)$ denotes the integral kernel of $\gamma$. We prove that for any $1/2 < p < \infty$, there exists $C > 0$ such that for any finite-rank $\gamma \geq 0$ with $\rho_{\gamma} \in L^p(\mathbb{R}^d)$ we have
\[ \frac{1}{C} \| \rho_{\gamma} \|_{L^p(\mathbb{R}^d)} \leq \sum_{j \in \mathbb{Z}} \rho_{P_j \gamma P_j} \|_{L^p(\mathbb{R}^d)} \leq C \| \rho_{\gamma} \|_{L^p(\mathbb{R}^d)}. \tag{3} \]
When $\gamma$ is a rank-one operator, this last inequality is equivalent to the usual Littlewood-Paley estimates (2). Indeed, if $u$ with $\|u\|_{L^2} = 1$ belongs to the range of $\gamma$, then $\rho_\gamma = |u|^2$.

The motivation to generalize the Littlewood-Paley decomposition to operator densities comes from many-body quantum mechanics. Indeed, a simple way to describe a system of $N$ fermions in $\mathbb{R}^d$ is via an orthogonal projection $\gamma$ on $L^2(\mathbb{R}^d)$ of rank $N$. The quantity $\rho_\gamma$ then describes the spatial density of the system. Variational or time-dependent models depending on $\gamma$ then typically include interactions between the particles via non-linear functionals of $\rho_\gamma$, like in Hartree-Fock models [10, 1, 2, 4]. As a consequence, $L^p$-properties of $\rho_\gamma$ are often needed to control these interactions. When $\gamma$ is a rank-one operator, these properties can be derived via Littlewood-Paley estimates (we typically think of Sobolev-type or Strichartz-type estimates). The estimate (3) allows to treat the rank $N$ case, and we illustrate this on the concrete example of the Lieb-Thirring inequality, which is a rank $N$ generalization of the Sobolev inequality.

In Section 1 we prove the inequality (3). In Section 2 we apply it to give a new proof of the Lieb-Thirring inequality.

1. Littlewood-Paley for densities

In this section we prove the generalization of the Littlewood-Paley theorem to densities of operators. We will see that the proof is a simple adaptation of the proof of the usual Littlewood-Paley theorem. Thus, let us first recall briefly the proof of (2). It is usually done via Khinchine’s inequality [11, Lemma 5.5], see the proof of Theorem 8.3 in [11]: if one denotes by $(r_j)$ a sequence of independent random variables taking values in $\{-1, 1\}$ and satisfying $\mathbb{P}(r_j = \pm 1) = 1/2$, one has

$$\frac{1}{C} \left( \sum_j |a_j|^2 \right)^{p/2} \leq \mathbb{E} \left| \sum_j a_j r_j \right|^p \leq C \left( \sum_j |a_j|^2 \right)^{p/2},$$

for any set of coefficients $(a_j) \subset \mathbb{C}$, for some $C > 0$, and for any $1 \leq p < \infty$. From this one deduces that

$$\left\| \left( \sum_j |P_j u|^2 \right)^{1/2} \right\|_{L^p} \leq \mathbb{E} \int_{\mathbb{R}^d} \left| \sum_j r_j P_j u(x) \right|^p dx.$$

The Fourier multiplier by the function $\xi \mapsto \sum_j r_j \psi_j(\xi)$ is bounded from $L^p(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$ for any $1 < p < \infty$, with a bound independent of the realization of the $(r_j)$. Indeed, one has to notice that for any given $\xi \in \mathbb{R}^d$, there are only a finite number of non-zero terms in the sum $\sum_j r_j \psi_j(\xi)$ (and this number only depends on $\psi$). The Mikhlin multiplier theorem [11, Thm. 8.2] shows the boundedness of the Fourier multiplier. We deduce from all this the inequality

$$\mathbb{E} \int_{\mathbb{R}^d} \left| \sum_j r_j P_j u(x) \right|^p dx \lesssim \mathbb{E} \int_{\mathbb{R}^d} |u(x)|^p dx = \|u\|_{L^p}^p.$$
The reverse inequality is done by a duality argument where the condition \((1)\) appears: we use the identity
\[
\int_{\mathbb{R}^d} f \overline{g} \, dx = \sum_j \int_{\mathbb{R}^d} P_j f \overline{P_j g} \, dx,
\]
where \(\overline{P_j}\) is another sequence of Littlewood-Paley multipliers such that \(\overline{P_j} P_j = P_j\) (which may be built from a \(\overline{\psi}\) which is identically 1 on the support of \(\psi\)). The fact that we cannot take \(\overline{P_j} = P_j\) is related to the deep fact that we cannot choose \(P_j\) to be a projection (that is, we cannot take \(\psi_j = 1(2^j \leq \cdot < 2^{j+1})\)); indeed such a \(P_j\) is not bounded on \(L^p(\mathbb{R}^d)\) (except for \(d = 1\) or \(p = 2\)) by Fefferman’s famous result \([5]\).

The main result of this section is the following lemma.

**Lemma 1.** For any \(1/2 < p < \infty\), there exists \(C > 0\) such that for any \(N \geq 1\), for any \((\lambda_k)_{k=1}^N \subset \mathbb{R}_+\) and any functions \((u_k)_{k=1}^N\) in \(L^{2p}(\mathbb{R}^d)\) we have
\[
\frac{1}{C} \left\| \sum_k \lambda_k |u_k|^2 \right\|_{L^p} \leq \left\| \sum_{j,k} \lambda_k |P_j u_k|^2 \right\|_{L^p} \leq C \left\| \sum_k \lambda_k |u_k|^2 \right\|_{L^p}. \tag{4}
\]

Lemma 1 implies the Littlewood-Paley decomposition \((3)\) for densities using the spectral decomposition of \(\gamma\). We first need a version of Khinchine’s inequality for tensor products, which is proved for instance in \([13]\ \text{Appendix D}]\). We however include a proof here for completeness.

**Lemma 2.** Let \((a_{j,k}) \subset \mathbb{C}\) a sequence of coefficients and \((r_j)\) a sequence of independent random variables such that \(\mathbb{P}(r_j = \pm 1) = 1/2\). Then, we have
\[
\left( \sum_{j,k} |a_{j,k}|^2 \right)^{p/2} \leq \mathbb{E} \left[ \sum_{j,k} a_{j,k} r_j r_k \right]^{p/2},
\]
for all \(1 \leq p < \infty\), where the implicit constant is independent of \((a_{j,k})\).

**Remark 3.** The reverse inequality also holds; we however do not need it here.

**Remark 4.** This inequality does not follow from the Khinchine inequality from abstract arguments because the sequence \((r_j r_k)\) is not independent anymore: knowing \(r_1 r_2\) and \(r_1 r_3\) implies that we know \(r_2 r_3\) as well.

**Proof of Lemma 2.** We only prove it for \(1 \leq p \leq 2\), which is sufficient since \(\mathbb{E}|g|^p \geq (\mathbb{E}|g|^2)^{p/2}\) for \(p \geq 2\). We first apply Khinchine’s inequality with respect to the random parameter associated to \((r_k)\):
\[
\mathbb{E} \left[ \sum_{j,k} a_{j,k} r_j r_k \right]^{p/2} \geq \mathbb{E} \left( \sum_k \left| \sum_j a_{j,k} r_j \right|^2 \right)^{p/2},
\]
where $\mathbb{E}_1$ denotes the expectation with respect to the random parameter associated to $(r_j)$. Since $p/2 \leq 1$, we may apply the reverse Minkowski inequality to infer that
\[
\mathbb{E}_1 \left( \sum_k \left( \sum_j a_{j,k} r_j \right)^2 \right)^{p/2} \geq \left( \sum_k \left( \mathbb{E}_1 \left( \sum_j a_{j,k} r_j \right)^p \right)^{2/p} \right)^{p/2}.
\]
Using a second time Khinchine’s inequality leads to
\[
\left( \sum_k \left( \mathbb{E}_1 \left( \sum_j a_{j,k} r_j \right)^p \right)^{2/p} \right)^{p/2} \geq \left( \sum_{j,k} |a_{j,k}|^2 \right)^{p/2}.
\]
From this tensorized Khinchine inequality, we deduce one side of the desired inequality.

**Lemma 5.** Let $(\lambda_k) \subset \mathbb{R}_+$ a finite sequence of coefficients and $(u_k)$ a finite sequence in $L^{2p}(\mathbb{R}^d)$. Then, we have
\[
\left\| \sum_{j,k} \lambda_k |P_j u_k|^2 \right\|_{L^p} \leq \left\| \sum_k |u_k|^2 \right\|_{L^p},
\]
for all $1/2 < p < \infty$, where the implicit constant is independent of $(\lambda_k), (u_k)$.

**Proof.** By Lemma 2
\[
\left\| \sum_{j,k} \lambda_k |P_j u_k|^2 \right\|_{L^p} \leq \mathbb{E} \int_{\mathbb{R}^d} \left| \sum_{j,k} \lambda_k^{1/2} r_j r_k P_j u_k(x) \right|^{2p} dx.
\]
By the boundedness of the Fourier multiplier by $\xi \mapsto \sum_j r_j \hat{\psi}_j(\xi)$ on $L^{2p}$, we have
\[
\mathbb{E} \int_{\mathbb{R}^d} \left| \sum_{j,k} \lambda_k^{1/2} r_j r_k P_j u_k(x) \right|^{2p} dx \leq \mathbb{E} \int_{\mathbb{R}^d} \left| \sum_k \lambda_k^{1/2} r_k u_k(x) \right|^{2p} dx.
\]
Applying again Khinchine’s inequality, we have
\[
\int_{\mathbb{R}^d} \mathbb{E} \left| \sum_k \lambda_k^{1/2} r_k u_k(x) \right|^{2p} dx \leq \int_{\mathbb{R}^d} \left( \sum_k \lambda_k |u_k(x)|^2 \right)^p dx.
\]

The other side of the inequality uses Lemma 5.

**Lemma 6.** Let $(\lambda_k) \subset \mathbb{R}_+$ a finite sequence of coefficients and $(u_k)$ a finite set of functions in $L^{2p}(\mathbb{R}^d)$. Then, we have
\[
\left\| \sum_k \lambda_k |u_k|^2 \right\|_{L^p} \leq \left\| \sum_{j,k} \lambda_k |P_j u_k|^2 \right\|_{L^p},
\]
for all $1/2 < p < \infty$, where the implicit constant is independent of $(\lambda_k), (u_k)$.

---

\footnote{Stating that $\| \sum f_k \|_{L^{p/2}} \geq \sum_k \| f_k \|_{L^{p/2}}$ for any $f_k \geq 0$.}
Remark 7. The right side of (6) is well-defined due to Lemma 5.

Proof. For any $V \geq 0$, we have

$$\int_{\mathbb{R}^d} \left( \sum_k \lambda_k |u_k(x)|^2 \right) V(x) \, dx = \sum_k \lambda_k \int_{\mathbb{R}^d} u_k(x) V(x) u_k(x) \, dx$$

$$= \sum_{j,k} \lambda_k \int_{\mathbb{R}^d} \tilde{P}_j u_k(x) \tilde{P}_j V u_k(x) \, dx,$$

where the sequence $\tilde{P}_j$ was defined earlier. By H"older’s inequality,

$$\sum_{j,k} \lambda_k \int_{\mathbb{R}^d} \tilde{P}_j u_k(x) \tilde{P}_j V u_k(x) \, dx \leq \int_{\mathbb{R}^d} \left( \sum_{j,k} \lambda_k |P_j u_k(x)|^2 \right)^{1/2} \left( \sum_{j,k} \lambda_k |\tilde{P}_j V u_k(x)|^2 \right)^{1/2}$$

$$\leq \left\| \sum_{j,k} \lambda_k |P_j u_k(x)|^2 \right\|_{L^p}^{1/2} \left\| \sum_{j,k} \lambda_k |\tilde{P}_j V u_k(x)|^2 \right\|_{L^p/(2p-1)}^{1/2}.$$ 

By Lemma 5 using that $p/(2p-1) > 1/2$, we have

$$\left\| \sum_{j,k} \lambda_k |\tilde{P}_j V u_k(x)|^2 \right\|_{L^p/(2p-1)} \lesssim \left\| V^2 \sum_k \lambda_k |u_k|^2 \right\|_{L^p/(2p-1)},$$

which leads to the desired result by choosing $V = (\sum_k \lambda_k |u_k|^2)^{p-1}$. \qed

2. Application: Lieb-Thirring inequalities

In this section, we explain how to use the Littlewood-Paley decomposition (3) to provide a simple proof of the Lieb-Thirring inequality. We first compare the Littlewood-Paley decompositions (2) and (3), and argue why they cannot be used in the same way.

2.1. Comparison of the two Littlewood-Paley decompositions. The Lieb-Thirring inequality generalizes to densities of operators the Gagliardo-Nirenberg-Sobolev inequality

$$\| u \|_{L^{2+4/d}} \lesssim \| u \|_{L^2}^{2/d} \| \nabla u \|_{L^2}^{d/(d-2)} , \ \forall u \in H^1(\mathbb{R}^d). \quad (7)$$

This last inequality can be proved very easily using the usual Littlewood-Paley decomposition (2). Indeed, by H"older’s inequality we have

$$\| P_j u \|_{L^{2+4/d}} \leq \| P_j u \|_{L^2}^{2/d} \| P_j u \|_{L^\infty}^{2/d}$$

$$\lesssim \| P_j u \|_{L^2}^{2/d} \| \mathcal{F}(P_j u) \|_{L^1}^{2/d}$$

$$\lesssim 2^{j(2/d)} \| P_j u \|_{L^2}^{2/d}$$

$$\lesssim \| P_j u \|_{L^2}^{2/d} \| \nabla P_j u \|_{L^2}^{d/(d-2)}.$$
meaning that the Gagliardo-Nirenberg-Sobolev inequality is immediate for frequency-localized functions. To get it for any function, we use the Littlewood-Paley decomposition \((2)\) and obtain

\[
\|u\|_{L^{2+4/d}}^2 \lesssim \sum_j \|P_j u\|_{L^{2+4/d}}^2 \
\lesssim \sum_j \|P_j u\|_{L^2}^{4/2 + d} \|\nabla P_j u\|_{L^2}^{2d/2} 
\lesssim \left( \sum_j \|P_j u\|_{L^2}^2 \right)^{2/d + 2} \left( \sum_j \|\nabla P_j u\|_{L^2}^2 \right)^{d/d + 2} 
\lesssim \|u\|_{L^2}^{4/d + 2} \|\nabla u\|_{L^2}^{2d/d + 2}. 
\]

We see here the power of the Littlewood-Paley decomposition: it allows to deduce functional inequalities from their version for frequency-localized functions. This has been used in several contexts, for instance concerning Strichartz inequalities \([9, 3]\). In particular, notice that we have used something much weaker than the Littlewood-Paley decomposition, namely the inequality

\[
\|u\|_{L^p}^2 \lesssim \sum_j \|P_j u\|_{L^p}^2, 
\]

which follows from \((2)\) by a triangle inequality. We now explain why the same strategy does not work in the context of the Lieb-Thirring inequality. This inequality reads

\[
\text{Tr}(-\Delta) \gamma \gtrsim \int_{\mathbb{R}^d} \rho_\gamma(x)^{1+\frac{d}{4}} \, dx, 
\]

for any finite-rank \(0 \leq \gamma \leq 1\). To see that it is indeed a generalization of the Gagliardo-Nirenberg-Sobolev inequality, notice that it is equivalent to the inequality

\[
\int_{\mathbb{R}^d} \left( \sum_{k=1}^N \lambda_k |u_k(x)|^2 \right)^{1+\frac{d}{4}} \, dx \lesssim \sum_{k=1}^N \lambda_k \int_{\mathbb{R}^d} |\nabla u_k(x)|^2 \, dx, 
\]

for any \((\lambda_k) \subset \mathbb{R}_+, (u_k) \subset H^1(\mathbb{R}^d)\) orthonormal in \(L^2(\mathbb{R}^d)\), and any \(N \geq 1\). The usual Gagliardo-Nirenberg-Sobolev inequality thus corresponds to the particular case \(N = 1\) of the Lieb-Thirring inequality. However, the Lieb-Thirring inequality does not follow from the Gagliardo-Nirenberg-Sobolev and the triangle inequalities, they only imply that

\[
\int_{\mathbb{R}^d} \left( \sum_{k=1}^N \lambda_k |u_k(x)|^2 \right)^{1+\frac{d}{4}} \, dx \lesssim \left( \sum_{k=1}^N \lambda_k \right)^{\frac{d}{4}} \sum_{k=1}^N \lambda_k \int_{\mathbb{R}^d} |\nabla u_k(x)|^2 \, dx, 
\]

which is weaker than the Lieb-Thirring inequality, especially for large \(N\). Let us notice that Frank, Lieb, and Seiringer have proved in \([7]\) an equivalence between the Gagliardo-Nirenberg-Sobolev and (the dual version of) the Lieb-Thirring inequality.
Again, for frequency-localized $\gamma$, this inequality is elementary: the constraint $0 \leq \gamma \leq 1$ implies that $0 \leq P_\gamma P_j \leq P_j^2$ and hence $0 \leq \rho_{P_j \gamma} P_j(x) \lesssim 2^{d j}$ for all $x \in \mathbb{R}^d$. As a consequence,

$$\left\| \rho_{P_j \gamma} P_j \right\|_{L^{1+2/d}} \leq \left\| \rho_{P_j \gamma} P_j \right\|_{L^{1+2/d}}^{\frac{d}{2}} \left\| \rho_{P_j \gamma} P_j \right\|_{L^{1+2/d}} \lesssim \left( \text{Tr} (\Delta) P_j \gamma P_j \right)^{\frac{d}{2}} \lesssim \left( \text{Tr} (\Delta) P_j \gamma P_j \right)^{\frac{d}{4+2}},$$

which is exactly the Lieb-Thirring inequality. Here, we used the fact that $\int \rho_\gamma = \text{Tr} \gamma$. Using the same idea as in the proof of the Gagliardo-Nirenberg-Sobolev inequality, we find that for any $\gamma$,

$$\left\| \rho_\gamma \right\|_{L^{1+2/d}} \lesssim \sum_j \left\| \rho_{P_j \gamma} P_j \right\|_{L^{1+2/d}} \lesssim \sum_j \left( \text{Tr} (\Delta) P_j \gamma P_j \right)^{\frac{d}{4+2}},$$

which we cannot sum. Indeed, the inequality

$$\sum_j \left( \text{Tr} (\Delta) P_j \gamma P_j \right)^{\frac{d}{4+2}} \leq \left( \sum_j \left( \text{Tr} (\Delta) P_j \gamma P_j \right) \right)^{\frac{d}{4+2}} \sim \left( \text{Tr} (\Delta) \gamma \right)^{\frac{d}{4+2}}$$

is of course wrong because $d/(d+2) < 1$. We thus see the difference between the applications of the Littlewood-Paley decompositions for functions or for densities of operators: one cannot directly resum the frequency-localized inequalities in the context of operators. Of course, the reason behind it is the use of the rough triangle inequality $\|\rho_\gamma\|_{L^p} \lesssim \sum_j \|\rho_{P_j \gamma} P_j\|_{L^p}$, which one should not do for operators. We now explain how to go beyond this difficulty.

2.2. Proof of the Lieb-Thirring inequality. Let us prove the Lieb-Thirring inequality using the Littlewood-Paley decomposition for densities. Hence, let $0 \leq \gamma \leq 1$ an operator on $L^2(\mathbb{R}^d)$, which we may assume to be of finite rank. Since $1 = \sum_j P_j$ with $P_j \geq 0$, we deduce that $1 \geq \sum_j P_j^2$.

We thus have

$$\text{Tr} (\Delta) \gamma \geq \sum_j \text{Tr} \sqrt[\gamma]{P_j (\Delta) P_j \sqrt[\gamma]{\gamma}} \geq \sum_j 2^{2j} \text{Tr} \sqrt[\gamma]{P_j^2} \sqrt[\gamma]{\gamma} = \int_{\mathbb{R}^d} \sum_j 2^{2j} \rho_{P_j \gamma} P_j(x) \, dx. \quad (8)$$

Lemma 8. Let $(\alpha_j)_{j \in \mathbb{Z}}$ a sequence of real numbers satisfying $0 \leq \alpha_j \leq 2^{jd}$ for all $j$. Then, we have the inequality

$$\left( \sum_j \alpha_j \right)^{\frac{1}{1+\frac{d}{4}}} \lesssim \sum j 2^{2j} \alpha_j.$$  

Let us first notice that the lemma implies the Lieb-Thirring inequality: indeed, since $0 \leq \gamma \leq 1$ we deduce that $0 \leq P_j \gamma P_j \leq P_j^2$ and hence $0 \leq \rho_{P_j \gamma} P_j(x) \lesssim 2^{jd}$ for all $x \in \mathbb{R}^d$. Hence, from the Lemma and (8) we deduce that

$$\text{Tr} (\Delta) \gamma \gtrsim \int_{\mathbb{R}^d} \left( \sum_j \rho_{P_j \gamma} P_j(x) \right)^{1+\frac{d}{4}} \, dx \gtrsim \int_{\mathbb{R}^d} \rho_\gamma(x)^{1+\frac{2}{d}} \, dx,$$

where in the last inequality we used the Littlewood-Paley theorem for densities. Let us now prove the lemma.
Proof of Lemma 8. We split the following sum as
\[
\sum_j \alpha_j = \sum_{j \leq J} \alpha_j + \sum_{j > J} \alpha_j.
\]
We estimate the first sum using that \(0 \leq \alpha_j \leq 2^j d\):
\[
\sum_{j \leq J} \alpha_j \lesssim 2^d J,
\]
and the second sum is estimated in the following way:
\[
\sum_{j > J} \alpha_j \lesssim 2^{-2J} \sum_j 2^{2j} \alpha_j.
\]
We thus find that for all \(J\),
\[
\sum_j \alpha_j \lesssim 2^d J + 2^{-2J} \sum_j 2^{2j} \alpha_j.
\]
Optimizing over \(J\) leads to the result. \(\square\)

Of course, the same strategy of proof allows to obtain more general inequalities of the type
\[
\text{Tr}(-\Delta)^{b} \gamma \gtrsim \int_{\mathbb{R}^d} \rho_\gamma(x)^{1 + \frac{2b}{d+2}} dx,
\]
for all \(0 \leq \gamma \leq (-\Delta)^a\), with \(b \geq 0\) and \(a > -d/2\). In particular, the case \(d \geq 3\), \(a = -1\), \(b = 1\) is due to Rumin [12] and was shown to be equivalent to the CLR inequality by Frank [6]. Our method is similar to the one used by Rumin, except that he uses a continuous decomposition
\[
-\Delta = \int_0^{\infty} 1(-\Delta > \tau) d\tau
\]
instead of a dyadic decomposition coming from Littlewood-Paley. Rumin’s method is actually far more powerful when dealing with these kind of inequalities, and was shown to work when replacing \(-\Delta\) by general \(a(-i\nabla)\) by Frank [6]. The dyadic decomposition seems useless in these more general cases since it does not distinguish the high/low values of \(a\). We expect that the Littlewood-Paley decomposition might be useful when one wants to exploit the “almost orthogonality” between the blocks \((P_j)\): we have \(P_j P_k = 0\), except for finite number of blocks, a phenomenon which does not appear in Rumin’s decomposition. This orthogonality might be useful when dealing with higher Schatten spaces \(\mathcal{S}^\alpha\) compared to the trace-class \(\mathcal{S}^1\) which appears for instance in the Lieb-Thirring inequality. We hope to find such applications in the future.

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References

[1] A. Bove, G. Da Prato, and G. Fano, An existence proof for the Hartree-Fock time-dependent problem with bounded two-body interaction, Commun. Math. Phys., 37 (1974), pp. 183–191.

[2] ———, On the Hartree-Fock time-dependent problem, Commun. Math. Phys., 49 (1976), pp. 25–33.

[3] N. Burq, P. Gérard, and N. Tzvetkov, Strichartz inequalities and the nonlinear Schrödinger equation on compact manifolds, Amer. J. Math., 126 (2004), pp. 569–605.

[4] J. Chadam, The time-dependent Hartree-Fock equations with Coulomb two-body interaction, Commun. Math. Phys., 46 (1976), pp. 99–104.

[5] C. Fefferman, The multiplier problem for the ball, Ann. of Math. (2), 94 (1971), pp. 330–336.

[6] R. L. Frank, Cwikel’s theorem and the CLR inequality, J. Spectr. Theory, 4 (2014), pp. 1–21.

[7] R. L. Frank, E. H. Lieb, and R. Seiringer, Equivalence of Sobolev inequalities and Lieb-Thirring inequalities, in XVIth International Congress on Mathematical Physics, World Sci. Publ., Hackensack, NJ, 2010, pp. 523–535.

[8] M. Frazier, B. Jawerth, and G. Weiss, Littlewood-Paley theory and the study of function spaces, vol. 79 of CBMS Regional Conference Series in Mathematics, Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1991.

[9] M. Keel and T. Tao, Endpoint Strichartz estimates, Amer. J. Math., 120 (1998), pp. 955–980.

[10] E. H. Lieb and B. Simon, The Hartree-Fock theory for Coulomb systems, Commun. Math. Phys., 53 (1977), pp. 185–194.

[11] C. Muscalu and W. Schlag, Classical and multilinear harmonic analysis. Vol. I, vol. 137 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 2013.

[12] M. Rumin, Spectral density and Sobolev inequalities for pure and mixed states, Geom. Funct. Anal., 20 (2010), pp. 817–844.

[13] E. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton Mathematical Series, 30, Princeton University Press, 1970.

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