DETERMINANT BUNDLES AND GEOMETRIC QUANTIZATION
OF VORTEX MODULI SPACES ON COMPACT KÄHLER SURFACES

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Abstract. In this paper we first show that on projective manifolds \((M, \omega)\), there are holomorphic determinant bundles (in the sense of Knusen-Mumford used by Bismut, Gillet, Soulé) which play the role of the geometric quantum bundle, namely one for each input data of a Hermitian holomorphic line bundle \(L\) of non-trivial Chern class on a compact Kähler manifold \(Z\) (with Todd genus non-zero) and a choice of a geometric quantization of \((M, \omega)\). Next we further study the generalization of the vortex equations on Kähler 4-manifold which has been studied earlier by Bradlow. We show that when the Kähler 4-manifold avoids some obstructions then the regular part of the moduli space is a Kähler manifold and admit a pull back of a Quillen determinant bundle as the quantum line bundle, i.e. the curvature is proportional to the Kähler form. Thus they can be quantized geometrically. In fact we show that the moduli space of the usual vortex equations on a projective Kähler 4-manifold is projective when the moduli space is smooth. Since in Kähler 4-manifold the vortex moduli and the Seiberg Witten moduli coincide our effort gives a quantization of Seiberg Witten moduli by determinant bundles.

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1. Introduction

Given a symplectic manifold \((M, \omega)\) with \(\omega\) integral (i.e. its cohomology class is in the image of \(H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{R})\)), geometric prequantization is the construction of a Hermitian line bundle with a connection (called the prequantum line bundle) whose curvature \(\rho\) is proportional to the symplectic form \(\omega\). This is always possible as long as \(\omega\) is integral. This method of quantization, developed by Kostant and Souriau, assigns to functions \(f \in C^\infty(M)\), an operator, \(\hat{f} = -i\nabla_{X_f} + f\) acting on the Hilbert space of square integrable sections of \(L\) (the wave functions). Here \(\nabla = d - i\theta\) where locally \(\omega = d\theta\) and \(X_f\) is defined by \(\omega(X_f, \cdot) = -df(\cdot)\). We have taken \(\hbar = 1\). The general reference for this is Woodhouse [36].

This assignment has the property that that the Poisson bracket (induced by the symplectic form), namely, \(\{f_1, f_2\}_PB\) corresponds to an operator proportional to the commutator \([\hat{f}_1, \hat{f}_2]\) for any two functions \(f_1, f_2\).

The Hilbert space of prequantization is usually too huge for most purposes. Geometric quantization involves construction of a polarization of the symplectic manifold such that we now take polarized sections of the line bundle, yielding a finite dimensional Hilbert space in most cases. However, \(\hat{f}\) does not map the
polarized Hilbert space to the polarized Hilbert space in general. Thus only a few observables from the set of all $f \in C^\infty(M)$ are quantizable.

When $M$ is a compact Kähler manifold with $\omega$ an integral Kähler form and $L$ the prequantum line bundle, one can take as the Hilbert space of quantization the space of holomorphic sections of $L^{\otimes \mu}$ for $\mu \in \mathbb{Z}$ large enough. See [35] for example for an explanation. We call such a $L^{\otimes \mu}$ a quantum bundle.

The determinant line bundle was originally constructed by Knusden and Mumford and later on generalized by Quillen as the determinant line bundle of a family of Cauchy Riemann operators on the space of connections on certain vector bundles on a compact Riemann surface [30]. There are many generalizations, see Bismut and Freed, [5] and in Bismut, Gillet, Soulé [2].

In the notation of Bismut, Gillet, Soulé [2], one can take $B$ to be the moduli space of connections of a vector bundle on a compact Riemann surface $\Sigma$, $Z$ to be $\Sigma$ and $M$ to be the trivial product $M = Z \times B$. Define the determinant of a family of Cauchy Riemann operators on $\Sigma$ parametrized by $B$, the space of connections. This yields a Quillen bundle in the definition according to [30].

We will denote a bundle as a determinant bundle if the bundle or its rational power is a bundle in the sense of Quillen [30] or Bismut and Freed [5] or Bismut, Gillet, Soulé [2].

Let $(X, \omega)$ be a compact integral Kähler manifold such that $L$ is a holomorphic positive line bundle whose curvature is proportional to the Kähler form $\omega$. Since for $\mu > \mu_0 > 0$ for some $\mu_0$ the holomorphic bundle $L^{\otimes \mu}$ has enough holomorphic sections for an embedding in projective space it can be considered as a quantum bundle. The Hilbert space of quantization will be the space of holomorphic sections of $L^{\otimes \mu}$. It should be noted that since $X$ is compact square integrability of holomorphic sections follows automatically.

An interesting question is whether this bundle $L$ or $L^{\otimes \mu}$ can be thought of as a determinant bundle. This is the main thrust of this paper.

In [13], Dey and Mathai had shown that one can realize a certain tensor product of $L$ as a Quillen bundle, i.e. determinant of a family of Cauchy Riemann operators on $\mathbb{C}P^1$, parametrized by $X$. Even when $(X, \omega)$ is just compact and symplectic, with integral symplectic form, Gromov embedding theorem enables us to embed $X$ into $\mathbb{C}P^N$ and they had shown in [13] that the quantum bundle is again of Quillen type, i.e. determinant of a family of Cauchy Riemann operators on $\mathbb{C}P^1$, parametrized by $X$.

In [7], Biswas proved an equivariant version of the results in [13] where the Cauchy Riemann operators can be on any compact Riemann surface (not necessarily $\mathbb{C}P^1$).

In [9], [10], [14] Dey and in [18] Romao and Eriksson have constructed Quillen bundles on the moduli space of vortices on a Riemann surface.

In [11], [12] Dey has constructed Quillen bundles on the moduli space of Higgs bundles for the three Kähler forms mentioned in the paper by Hitchin [24] which constitute a hyperKähler structure.

Donaldson and Kronheimer have given an exposition of the Quillen construction over the moduli space of ASD connections on a Riemann surface, see for instance (6.5.4) in [16].
Motivated by these examples, in this paper we have shown that on projective manifolds \( M \), the geometric quantum bundle can be realized in many ways as a Quillen bundle in the sense of Bismut, Gillet, Soulé, namely one for each input data of a Hermitian holomorphic line bundle \( L \) on a compact Kähler manifold \( Z \) and a geometric quantization of \((M, \omega)\).

The vortex equations when defined on a Kähler surface \((X, \omega)\) yields a non-empty moduli space under certain conditions, see Bradlow [3]. In fact in this case, the moduli space is compact, as it can be described as a Seiberg-Witten moduli space (see Appendix). and the regular part of the moduli space is Kähler. Denoting the Kähler form to be \( \Omega \) and if the Poincaré dual to \( \omega \) (the Kähler form of the Kähler surface) has a representative which is a Riemann surface (one dimensional complex submanifold), then there is a pullback of a Quillen bundle on the regular part of the moduli space whose curvature is proportional to \( \Omega \) (provided a mild obstruction is bypassed). As an application we show that for projective 4-manifolds, the vortex moduli space is projective (if the moduli space is smooth or regular).

In Appendix we recall that there is a connection with Seiberg-Witten moduli space and the vortex moduli space for compact Kähler 4-manifolds, as showed by Bradlow and Garcia-Prada [6].

2. Quantum line bundle as a determinant bundle

We begin by mentioning a well known fact namely, for \( Z \) a Riemann surface and \( M = A \), the space of unitary connections on a vector bundle \( E \) on the Riemann surface, the three constructions, namely the one of Bismut and Freed [5] and Bismut, Gillet and Soulé, [2] and the one of Quillen [30], matches. The equivalence of the first case and third case has been in discussed in the introduction of [2] and proved in details from section \( f \) to section \( k \). The equivalence of all three has also been discussed in the introduction of [2] and the proof is discussed in an series of related papers. The interested reader may look into these as it is beyond our scope of our paper to describe them fully.

In Quillen’s case, the family of \( \bar{\partial} \) operators on a Riemann surface \( Z \) is parametrized by \( A \) where \( A \) is the space of unitary connections of a vector bundle \( E \) on \( Z \). The \( \bar{\partial}_A \) acts on sections of the vector bundle \( E \) for \( A \in A \).

We have a following:

**Nomenclature:** For purposes of this paper, we shall use the term determinant line bundle if the line bundle in question is a rational power of a Quillen bundle or a bundle in the sense of Knusden-Mumford as in Bismut and Freed [5] and Bismut, Gillet and Soulé, [2]. We will still use the term Quillen bundle if the bundle is as in [30] with a modified Quillen metric.

We recall

**Definition 2.1.** A holomorphic line bundle equipped with a connection and Hermitian metric on a compact Kähler manifold with integral Kähler form is called quantum line bundle if the curvature of the line bundle (w.r.t. the connection) is proportional to the Kähler form with the appropriate proportionality constant (which can be positive or negative). In this case the Hilbert space consists of square integrable polarized sections with polarization by holomorphic tangent space for positive constant or antiholomorphic for negative constant.
We have the following theorem:

**Theorem 2.1.** If $M$ is a compact Kähler manifold with integral Kähler form. Then the quantum bundle, or its non-zero power, positive or negative, is holomorphically equivalent to a determinant line bundle (modulo tensoring with a flat bundle), thus making the determinant bundle also a quantum line bundle. There are choices i.e. one for each choice of a compact Kähler manifold $Z$ with non-zero Todd genus and having a holomorphic Hermitian bundle $L$ with non-trivial Chern class. (We may have to replace $L$ with $L^\otimes n$ for some $n$).

**Proof.** Let $L$ be a quantum line bundle on $M$.

Let $Z$ be a compact Kähler manifold with Todd genus $\neq 0$ with a holomorphic line bundle $L$ on it with non-trivial Chern class.

Let $Y = Z \times M$. Let $L$ be a quantum line bundle on $M$ and $L$ a holomorphic line bundle on $Z$. Then define $\zeta = p_1^*(L) \otimes p_2^*(L)$. Let $Q$ be the determinant line bundle according to Bismut, Gillet, Soulé [2] corresponding to $\zeta$.

According to Bismut, Gillet and Soulé, the curvature $Q_\zeta$ of $Q$ is given by the degree two term of $2i\pi \int_Z \text{Tr}[\exp(\Omega_\zeta)]$ where $R_Z$ is the curvature of $T^0_1Z$ and $\Omega_\zeta$ is the curvature of the holomorphic Hermitian connection on $\zeta$.

Note that $\Omega_\zeta = \mathcal{K}_L + \mathcal{K}_L$ where $\mathcal{K}_L$ and $\mathcal{K}_L$ are the curvatures of $L$ and $L$. In our case $R_Z$ and $\mathcal{K}_L$ are absorbed in the integral leaving only an constant multiple of $\mathcal{K}_L$ as the degree two part of the above expression. Thus curvature $Q_\zeta$ is thus proportional to $\mathcal{K}_L$ with a rational proportionality constant. Suppose the constant is strictly positive. In this case the determinant bundle is a positive rational tensor power of $L$. If the constant is strictly negative, the determinant bundle is a positive rational tensor power of $L^{-1}$ modulo tensoring with a flat bundle.

This proportionality constant may be zero. Then $Q_\zeta$ cannot be the curvature to a quantum bundle or its tensor product since symplectic forms on compact manifolds have non trivial cohomology. In this case since the Chern class of our line bundle $L$ on $Z$ is non trivial and $Z$ has non-zero Todd genus, replacing $L$ with $L = L^\otimes n$ for an appropriate $n$ in the above expression, we will get a non-zero constant. This can be seen as follows. The constant of proportionality looks like $\int_Z (Td) + nd_1 + n^2d_2 + ...$ (a terminating polynomial) where $d_i$ are constants. Here $Td$ denotes $Td(\frac{\mathcal{K}_L}{2\pi i}) \neq 0$. We get a non-identically zero polynomial in $n$. Since a polynomial has finite number of zeros there will be some $n$ for which the constant is non zero.

This implies that the determinant bundle and rational powers (positive or negative) of the quantum ample bundle differ by atmost by a holomorphic bundle which is flat. So we can quantize by a determinant line bundle. Since the determinant bundle and the quantum bundle we started with and its rational powers may differ by flat bundles they may not be holomorphically equivalent, so we may hope to get new quantization.

\[ \square \]

The above theorem says that under the conditions above, we can quantize $M$ by the determinant bundle.

A similar result for the abelian vortex moduli space on a Riemann surface, Dey, [9], [10].

One can define the vortex equations on compact Kähler 4-manifolds. The moduli space was studied by Bradlow [3]. We geometrically quantize the moduli space where the equations are now defined on a compact Kähler surfaces.
3. Vortex equations on a Riemann surface, Quillen bundle and curvature

We review some material from [9], [10], [14]. On a Riemann surface $M$ the abelian vortex equation are given by

$$\bar{\partial}_A \phi = 0,$$  \hspace{1cm} (1)

$$F_A = \frac{1}{2} (\tau - |\phi|^2) \omega.$$  \hspace{1cm} (2)

where $\phi$ is defined as a smooth section of an Hermitian holomorphic line bundle $L$ and $A$ is the unitary connection of the principal bundle $P$ associated to $L$. The form $\omega$ is an imaginary valued Kähler form on $M$ and $\tau$ a real constant.

3.1. The symplectic form. Let $A$ be the space of all unitary connections on $P$ the associated principal bundle of the vortex bundle $L$ and $\Gamma(M, L)$ be sections of $L$. We define the configuration space as $C = A \times \Gamma(M, L)$. The space $C$ is an infinite dimensional affine space with differential structure determined by tangent spaces whose tangent vectors are of the form $(\alpha, \phi)$ where $\alpha$ is a $u(1)$ valued one form and $\phi$ a section. Let $p = (A, \Psi) \in C$, $X = (\alpha_1, \beta)$, $Y = (\alpha_2, \eta) \in T_pC$. We define the following $L^2$-metric on $C$.

$$G(X, Y) = \int_M \alpha_1 \wedge \alpha_2 + i \int_M (\bar{\beta} \eta + \bar{\eta} \beta) \omega$$  \hspace{1cm} (3)

and an almost complex structure $I = (\ast, i)$ on $T_pC$ where $\ast dz_1 = -i dz_1$ and $\ast d\bar{z}_1 = id\bar{z}_1$.

We define

$$\Omega(X, Y) = -\int_M \alpha_1 \wedge \alpha_2 - \frac{1}{2} \int_M (\beta \bar{\eta} - \bar{\beta} \eta) \omega$$  \hspace{1cm} (4)

such that $G(IX, Y) = \Omega(X, Y)$.

Let $\zeta \in Maps(M, u(1))$ be an element of the Lie algebra of the gauge group. Note that $\bar{\zeta} = -\zeta$. It generates a vector field $X_\zeta$ on $C$ as follows:

$$X_\zeta(A, \phi) = (d\zeta, -\zeta \phi) \in T_pC,$$  \hspace{1cm} (5)

where $p = (A, \phi) \in C$. We show next that $X_\zeta$ is Hamiltonian. Namely, define $H_\zeta : C \to \mathbb{C}$ as follows:

$$H_\zeta(p) = \int_M \zeta(F_A - \frac{1}{2} (\tau - |\phi|^2) \omega)$$  \hspace{1cm} (6)

Then for $X = (\alpha, \beta) \in T_pC$,

$$dH_\zeta(X) = -\int_M d\zeta \wedge \alpha - \frac{1}{2} \int_M \bar{\beta}(-\zeta) \phi - \beta(\zeta) \bar{\phi} \omega$$  \hspace{1cm} (7)

$$dH_\zeta(X) = \Omega(X_\zeta, X)$$  \hspace{1cm} (8)

Thus we can define the moment map $\mu : C \to \Omega^2(M, u(n)) = g^*$ (the dual of the Lie algebra of the gauge group) to be

$$\mu(A, \phi) = F_A - \frac{1}{2} (\tau - |\phi|^2) \omega$$  \hspace{1cm} (9)
The moduli space of the vortex equations (1) and (2) is defined as the quotient of the space of solutions by the gauge group. It inherits its topology by the quotient topology and the differential structure by taking quotient of the space of tangent vectors of the solution space by the gauge group. By Kähler reduction (see 2.3, [18]) this form descends to the moduli space, giving it a Kähler structure. In fact it is related to the Manton-Nasir form [29].

Note: In this section we had taken $\omega = h^2 dz \wedge d\bar{z}$, an imaginary valued symplectic form on the Riemann surface which is also Kähler. If instead we take it to be the symplectic form on the real tangent space, namely, $\tilde{\omega} = i \omega$, then the second vortex equation looks like:

$$F_A = \frac{i}{2}(|\phi|^2 - \tau)\tilde{\omega}. \quad (10)$$

This is the form in which the equation been written in [3].

3.2. The modified Quillen metric and curvature. In [29], it was discussed that the Manton-Nasir-Samols form on the vortex moduli space is integral when the Riemann surface has volume an integral multiple of $4\pi\tau$ (see 11).

Let $\text{det}(\bar{\partial})$ denote the Quillen bundle defined on $A$ as in [30]. Let $pr : C = A \times \Gamma(L) \to A$. We denote the Quillen bundle $P = pr^*(\text{det}(\bar{\partial}))$ which is well defined on $C = A \times \Gamma(L)$ which is an affine space. We can equip $P$ a modified Quillen metric, namely, we multiply the Quillen metric [30] by the factor $e^{-\frac{i}{4\pi} \int_M |\psi|^2 H \omega}$.

The Quillen metric contributes $-\frac{1}{2\pi} \int_S \alpha_1 \wedge \alpha_2$ to the curvature [30], and the factor $e^{-\frac{i}{4\pi} \int_M |\psi|^2 H \omega}$ contributes $\frac{1}{2\pi} \int_S (\beta \bar{\eta} - \bar{\beta} \eta) \omega$ to the curvature.

Thus we have the following:

The curvature of $P$ with the modified Quillen metric is indeed $\frac{1}{2\pi} \Omega$ on the affine space $C$.

$\Omega$ descends to the moduli space as a Kähler form by Kähler reduction.

If descent of $\Omega$ is integral on the moduli space then the bundle descends to the moduli space. This follows from considerations in [30] (chapter on prequantization) and [9], [10], [14] etc.

Let $\Omega_{MN}$ be the Manton-Nasir form as defined in [29]. Then $\Omega_{Samols} = \Omega_{MN}$. (see [29] equation 2.16, arxiv version and discussion above the equation).

It can be shown that $\left[\frac{\Omega}{2\pi}\right] = [\Omega_{Samols}] = [\Omega_{MN}]$, see for instance [13].

The condition that the bundle descends (i.e. $\Omega$ is integral on the moduli space) is that the Riemann surface has volume $A$ an integral multiple of $A$. This is because

$$\left[\frac{\Omega}{2\pi}\right] = [\Omega_{MN}] = [(\frac{\tau A}{2} - 2\pi N)\eta + 2\pi(\sigma_1 + \ldots + \sigma_n)] \quad (11)$$

where $A$ is the volume of the Riemann surface $\Sigma$ and $\sigma_i$'s and $\eta$ are integral cohomological classes of the moduli space (defined in [29]).

4. Generalizations of above theory to the moduli space of vortices on a Kähler surface

For Kähler surfaces $X$ (i.e. compact Kähler 4 real dimensional manifolds) vortex equations are as follows [3].
with \( \omega \) complex structure

\[ \Lambda F_A = \frac{i}{2} (|\phi|^2 - \tau) \]  

(13)

\[ F_A^{0,2} = 0 \]  

(14)

where \( \Lambda F_A \) is the contraction of \( F_A \) with a suitable Kähler form \( \omega \) (i.e. the symplectic form on the real tangent space) where \( F_A \) is the curvature of a line bundle \( L \) on \( X \) with connection \( A \), \( \phi \) is a section.

Let the configuration space be \( \mathcal{C} = \mathcal{A} \times \Gamma(L) \), where \( \mathcal{A} \) is the affine space of unitary connections on \( L \) and \( \Gamma(L) \) is the space of sections of \( L \). The space \( \mathcal{C} \) is an infinite dimensional affine space and its tangent space is similar to that discussed in the Riemann surface case discussed in previous section.

### 4.1. The moduli space as a Kähler manifold.

In the first part of this section we closely follow Riera \[31]\.

Let \( \mathcal{A} \) be the space of \( U(1) \)-connections on \( P \) the associated bundle of a line bundle \( L \). This is an affine space modelled on \( \Omega^1(P \times \text{Ad} u(1)) \). We define a complex structure \( I_A \) on \( \mathcal{A} \) as follows. Given any \( A \in \mathcal{A} \), the tangent space \( T_A \mathcal{A} \) can be canonically identified with \( \Omega^1(P \times \text{Ad} u(1)) = \Omega^0(T^*X) \otimes P \times \text{Ad} u(1)) \). Then we set \( I_A = -I^* \otimes 1 \), where \( I \) is the complex structure of the tangent bundle which it inherits since the manifold is complex. The complex structure \( I_A \) is integrable. We also define on \( \mathcal{A} \) a symplectic form \( \omega_A \). Let \( \Lambda : \Omega^{p,q}(X) \to \Omega^{p-1,q-1}(X) \) be the adjoint of the map given by wedging with \( \omega \) which is also equal to the contraction with \( \omega \) with respect to Kähler metric \( \Omega \). Then, if \( A \in \mathcal{A} \) and \( \alpha_1, \alpha_2 \in T_A \mathcal{A} = \Omega^1(P \times \text{Ad} u(1)) \), we set

\[ \omega_A(\alpha_1, \alpha_2) = -\int_X \Lambda(B(\alpha_1, \alpha_2)) \frac{\omega \wedge \omega}{2} \]  

(15)

Here \( B : \Omega^1(P \times \text{Ad} u(1)) \otimes \Omega^1(P \times \text{Ad} u(1)) \to \Omega^2(X) \) is the combination of the usual wedge product with a bi-invariant nondegenerate pairing \( \langle , \rangle \), on \( u(1) \). Since \( u(1) \) is one dimensional we can consider \( B \) as wedge of imaginary one forms.

It turns out that \( \omega_A \) is a symplectic form on \( \mathcal{A} \), and it is compatible with the complex structure \( I_A \). Hence \( \mathcal{A} \) is a Kähler manifold. Let \( X_1 = (\alpha_1, \beta) \) and \( X_2 = (\alpha_2, \eta) \) are in \( T_{(A, \phi)} \mathcal{C} \).

On \( \mathcal{C} \) we define

\[ \Omega_X(X_1, X_2) = -\int_X \Lambda(B(\alpha_1, \alpha_2)) \frac{\omega \wedge \omega}{2} + \frac{i}{2} \int_X (\beta \eta - \bar{\beta} \bar{\eta}) \frac{\omega \wedge \omega}{2} \]  

(16)

where \( \omega \) is now a Kähler form (a symplectic form compatible with complex structure and Kähler metric). The form \( \Omega_X \) is the Kähler form with respect to the following Kähler metric.

\[ g(X_1, X_2) = \int_M *\alpha_1 \wedge \alpha_2 \wedge \omega + \int_M (\frac{\beta \eta + \bar{\beta} \bar{\eta}}{2}) \frac{\omega \wedge \omega}{2} \]  

(17)

where \( * \) defined in preceding section. There exists a moment map for the action of \( \mathcal{G}_{U(1)} \) on \( \mathcal{A} \), which takes the following form (see for example \[10, 26\]):

\[ \mu : A \to \text{Lie} \mathcal{G}_{U(1)}^* \]

\[ \mathcal{A} \to \Lambda(F_A) \]
Here $F_A$ denotes the curvature of $A$. It lies in $\Omega^2(P \times_{\text{Ad}} \text{u}(1))$, so $\Lambda(F_A) \in \Omega^0(P \times_{\text{Ad}} \text{u}(1)) \subset \Omega^0(P \times_{\text{Ad}} \text{u}(1))^*$, the last inclusion being given by the integral on $X$ of the pairing $<<$. Let $\mu(A, \phi) = \Lambda F_A - i(\phi^2 - \tau)$.

Following the same arguments from previous section we have $\mu = \mu(A, \phi)$ above is a moment map for the action of the gauge group on $C$. The moduli space inherits the quotient topology by the gauge group action and the differential structure is similar to the one discussed in section 3 in the Riemann surface case.

Since the moduli space is same as the Seiberg-Witten moduli by Appendix, the regular part of the moduli space is Kähler with Kähler form $\Omega_X$ by corollary 4.2 Becker [1] mainly by infinite dimensional Kähler reduction technique.

The above argument works for Kähler surfaces for more general Kähler manifolds it has been shown in [31].

4.2. Determinant bundle construction on the moduli space. The construction is similar to Donaldson’s construction of the Quillen bundle on the moduli space of ASD connections on a Kähler surface [16], section (6.5.4).

Let $A \in \mathcal{A}$. Let us restrict the connection $A$ to the Poincaré dual $S$ of the Kähler form $\omega$ on $X$. Let the restricted connection be denoted by $A^R$ and the space of restricted connection be denoted by $\mathcal{A}^R$.

Let us consider the first term of the symplectic form. For the Kähler surface the symplectic form $\Omega_A$ is from (14)

$$\Omega_A(\alpha_1, \alpha_2) = -\int_X (\Lambda(B(\alpha_1, \alpha_2)) \frac{\omega \wedge \omega}{2}. \quad (18)$$

$\Lambda$ is the contraction with respect to the Kähler form $\omega$ and $B$ is wedging of $\alpha_1$ and $\alpha_2$. So we get

$$\Omega_A(\alpha_1, \alpha_2) = -\int_X \alpha_1 \wedge \alpha_2 \wedge \omega. \quad (19)$$

Recall the closed Kähler form $\omega$ belongs to a cohomology class $[\omega]$ and thus there exists a homology class $D$ which is the Poincaré dual of the cohomology class $[\omega]$. If we take submanifold representative of $D$, say $S$, we have

$$\int_X (\alpha_1 \wedge \alpha_2) \wedge \omega = \int_S \alpha_1 \wedge \alpha_2 = -\Omega_R(\alpha_1, \alpha_2) \quad (20)$$

for $\alpha_1 \wedge \alpha_2$ closed, where we denote $\Omega_R(\alpha_1, \alpha_2) = -\int_S \alpha_1 \wedge \alpha_2$.

Let the Poincaré dual of $\omega$ be $S$.

**Theorem 4.1.** If $S$ is a Riemann surface i.e. it is a complex one dimensional submanifold of the Kähler surface $X$, then we have a pullback of a determinant line bundle in the sense of Quillen with a metric on the vortex configuration space. Its curvature form is cohomologous to the standard Kähler form on the configuration space (with the appropriate proportionality factor) and their difference is given by the differential of a gauge invariant one form. Moreover if this bundle descends to the vortex moduli space, its curvature with respect to a certain connection will be proportional to the Kähler form obtained by the moment map reduction of the Kähler form to the regular part of the vortex moduli space.

**Proof.** We assume $S$ is connected, though the proof goes through otherwise as well. We observe the restriction of forms on $X$ will give a map from $\mathcal{A}$ to $\mathcal{A}^R$ the connection space of the Riemann surface $S$. We can pullback the Quillen bundle
over $A^R$ with curvature proportional to $\Omega_R(\alpha_1, \alpha_1) = -\int_X \alpha_1 \wedge \alpha_2$ to a holomorphic bundle (since $S$ is a Riemann surface by the above assumption). We claim that if the pullback of this bundle descends on $A/\mathcal{G}$ its curvature proportional to

$$-\int_X (\alpha_1 \wedge \alpha_2) \wedge \omega = \Omega_A(\alpha_1, \alpha_2)$$

(21)

The above statement follows verbatim from section (6.5.4), [16].

On $C = A \times \Gamma(L)$ we defined the Kähler form which can now be written as $\Omega_X(X_1, X_2) = -\int_X \alpha_1 \wedge \alpha_2 \wedge \omega + \frac{i}{2} \int_X (\beta \bar{\eta} - \bar{\beta} \eta) \frac{\omega}{2}$, where $X_1 = (\alpha_1, \beta)$ and $X_2 = (\alpha_2, \eta)$ as before. By discussion in the previous subsection the regular part of the moduli space is Kähler with Kähler form the descendant of the above form $\Omega_X$.

The Quillen bundle is the standard Quillen bundle on the Riemann surface, namely $\text{det}(\bar{\partial}_A^R)$ on the Riemann surface configuration space. The pullback bundle induced by the restriction map with metric [30] is modified with the factor $e^{\frac{i}{2\pi} \int_X |\phi|^2 \omega}$ as in the Riemann surface case [10].

The conditions for descent of the line bundle will be discussed in the next subsection. If this line bundle descends as in the Riemann surface case, the cohomology of the curvature is proportional to the cohomology of the descendant of $\Omega_X$ by [22] since it can be checked as in [16]

$$\Omega_A(\alpha_1, \alpha_2) = i^*(\Omega_B)(\alpha_1, \alpha_2) + d(\Phi)(\alpha_1, \alpha_2)$$

(22)

where notation is as in [16] and exactness of the second integrands of the two forms $\Omega_X$ and $2\pi$ times curvature of the pullback of modified Quillen metric on the Riemann surface moduli. From the above equation since $\Phi$ and $d(\Phi)$ is gauge invariant we get the statement above [21]. (Here $i$ is the restriction map).

4.3. Integrality. Let $X_1 = (\alpha_1, \beta)$ and $X_2 = (\alpha_2, \eta)$ be tangent to the configuration space. The form $\Omega_X$ given by

$$\Omega_X(X_1, X_2) = -\int_X \alpha_1 \wedge \alpha_2 \wedge \omega + \frac{i}{2} \int_X (\beta \bar{\eta} - \bar{\beta} \eta) \frac{\omega}{2}$$

(23)

on the Kähler surface configuration and the solution subspaces. It may not descend to an integral form on the moduli space. Same holds for corresponding form $\Omega_S$ for the moduli space in the the Riemann surface case. Deriving their result from the Samol’s metric, Manton and Nasir gave a cohomological description of $\frac{1}{2\pi}$ times the form on the vortex moduli space for a Riemann surface. Now since $\frac{A}{2\pi} = \Omega_{Samol}$, $\Omega_{MN}$, from the discussion in subsection 3.2 the cohomology class of $\frac{1}{2\pi}$ times the form as in [18] is

$$\left[\frac{\Omega_S(A)}{2\pi}\right] = \left[\left(\frac{\tau A}{2} - 2\pi N\right) \eta + 2\pi (\sigma_1 + \ldots + \sigma_n)\right]$$

(24)

where $A$ is the area of the Riemann surface $S$ and $\sigma_i$s and $\eta$ are integral cohomological classes of the moduli space (defined in [20]). From equation (23) it is clear $\frac{\Omega_S(A)}{2\pi}$ is integral if $\frac{\tau A}{2\pi} = n$ where $n$ is an integer.

Taking the form $\omega_1 = k\omega$ where $k = n\frac{A}{2\pi}$ we get an integral Kähler form on the moduli space for the Riemann surface. Using $\omega_1$ instead of $\omega$ we get an ample line bundle on moduli space of vortices on the Riemann surface $S$.  

We first define $\Psi$, a holomorphic map between the vortex moduli space $\mathcal{M}_X$ for the Kähler surface $X$ and the vortex moduli space $\mathcal{M}_S$ for the Riemann surface $S$ (with vortex equation with $\tau$ large enough such the moduli space is non-empty [3]).

The pullback of the holomorphic line bundle by $\Psi$ will be a holomorphic line bundle on the moduli space of vortices for the Kähler surface and we will show the pullback form defining the first Chern class is cohomologous to $-\frac{\Omega_X(\omega_1)}{4\pi^2}$ where $\Omega_X(\omega_1)$ is the form $\Omega_X$ with $\omega$ replaced by $\omega_1$.

**Obstructions $O(1)$ and $O(2)$:** Let the vortex line bundle and the Poincaré dual $S$ to $\omega$ be such that a non-identically-zero holomorphic section does not vanish entirely on $S$ or a component of $S$ when $S$ is disconnected, i.e. the Chern class of the vortex line bundle is such that the Poincaré dual does not contain a component of $S$. If this condition is not satisfied it is called obstruction $O(1)$.

When every representative of the Poincaré dual to $\omega$ is not a complex submanifold of $X$, we call it obstruction $O(2)$.

**Lemma 4.2.** When obstructions $O_1$ and $O(2)$ is not satisfied, there is a holomorphic map $\Psi$ between the vortex moduli space $\mathcal{M}_X$ for the Kähler surface $X$ and the vortex moduli space $\mathcal{M}_S$ for the Riemann surface $S$.

**Proof.** Here we again assume $S$ is connected. Let us define $\Psi : \mathcal{M}_X \rightarrow \mathcal{M}_S$ by $\Psi([A, \phi]) = [A_R, \phi_R]$ where $\phi_R$ denotes restriction of $\phi$ to $S$ and $A_R$ is the only connection (see [3] sections 3 and 4) which is a solution to the vortex equation on the Riemann surface $S$ with section $\phi_R$ and holomorphic structure the restriction of the holomorphic structure due to $A$. Since the first vortex equation of $X$ restricts to the first vortex equation of $S$ as the later being a complex one dimensional submanifold of $X$ the restriction of section and holomorphic structure works out.

Orbits map to orbits: If two elements $(A, \phi)$ and $(A_1, \phi_1)$ are related by gauge transformations in the Kähler surface configuration space then the section part restrictions are related by restriction of gauge transformations, and the connection parts will be related by the same gauge transforms by uniqueness and since the equations are gauge invariant. Thus the map $\Psi$ is well-defined.

Map between the moduli spaces: From a close inspection of [3] one can see the moduli of solutions $\mathcal{M}_X$ is given by effective divisors. For an effective divisor in the Kähler surface $X$ we can take a holomorphic section and holomorphic structure as a representative and its restriction on the Riemann surface $S$ (which is a one dimensional complex submanifold) will define a divisor on the Riemann surface provided the restriction is not identically zero which is avoided since obstruction $O_1$ is not satisfied. This coincides with the map $\Psi$ between between moduli spaces which we defined in the first line. Though our connection part may not agree with the restriction of a connection of a Kähler surface moduli but the holomorphic structure do match because of the first vortex equation and $S$ being a complex one dimensional submanifold.

Holomorphicity of the map $\Psi$: The map between the moduli is holomorphic as the Riemann surface $S$ is a complex one dimensional submanifold of $X$ and the section part is just the restriction. Since acting the section part of a tangent vector of the space of solutions by the complex structure of the section part changes the connection part of the vector by its complex structure (since the moduli of solutions are complex analytic) the map is holomorphic. □
Since we chose \( k = n \frac{i \pi}{4} \) (where \( n \) an integer and \( A \) area of \( S \) induced by \( \omega \)), the negative of the Kähler form \( \frac{\Omega_{S(A_{\alpha_1})}}{4\pi^2} \) on the moduli space is integral (\( A_{\alpha_1} \) is the area of \( S \) with volume form \( \omega_1 = k \omega \)) and hence its cohomology class will be a Chern class of a holomorphic line bundle \( L \) and so its pullback by \( \Psi \) to the Kähler surface moduli space \( \mathcal{M}_X \) will be a holomorphic line bundle \( \Psi^*(L) \).

**Lemma 4.3.** The cohomology class of the form \( -\frac{\Omega_X(\omega_1)}{4\pi^2} \) is the Chern class of \( \Psi^*(L) \). Thus from this we get a quantization of vortex moduli by determinant bundle in the sense mentioned in Section 2

**Proof.** The Chern class of the bundle \( \Psi^*(L) \) is the cohomology class of the pull back of \( -\frac{\Omega_{S(A_{\alpha_1})}}{4\pi^2} \). The form in the configuration space level is given by

\[
\frac{\Omega_{S(A_{\alpha_1})}}{4\pi^2} = -\int_S \alpha_1 \wedge \alpha_2 + i \int_S (\beta \bar{\eta} - \bar{\beta} \eta) \frac{\omega}{2} \tag{25}
\]

Though our map is not exactly the restriction the holomorphic structure part of \( A_R \) and \( A^R \) (the actual restriction of \( A \) where \( [(A, \phi)] \) is a solution in \( X \)) are same(\( \partial_A_R = \bar{\partial}_A^R \)). The reason behind this is that the Riemann surface \( S \) is a one dimensional complex submanifold of \( X \) and so the first vortex equation of \( X \) restricts to the first vortex equation of \( S \). Now since the fibre of the Quillen determinant bundle as defined in [30] and the metric depends on the holomorphic structure (the delbar part) the pullback of \( \Psi \) and descent of the pull back bundle in moduli level described in 4.1 yield holomorphically equivalent isometric bundles. Thus we get a determinant bundle since pullback of a determinant bundle is a determinant bundle (see introduction [2]) with the required curvature from below.

The following argument has been made in theorem 4.1 but we repeat for the reader’s convenience. From (22) for the form \( \omega_1 \) we get the first integrand of the numerator of the right hand side of (25) is cohomologous to first integrand of \( \Omega_X(\omega_1) \) by the correspondence in the above paragraph since \( \Psi \) and the restriction map pullbacks produce isometric bundles. The second term of (25) is \( \int_S (i \beta \bar{\eta} - i \bar{\beta} \eta) \frac{\omega}{2} \). Similarly it can be shown the form representing the second term of the \( \frac{\Omega_X(\omega_1)}{4\pi^2} \) is exact. Both cases the one forms are gauge invariant. So the difference of the two forms \( \Psi^*(\frac{\Omega_{S(A_{\alpha_1})}}{4\pi^2}) \) and \( \frac{\Omega_X(\omega_1)}{4\pi^2} \) descend to exact forms making them cohomologous. Since moduli for the form \( \omega_1 \) and the form \( \omega \) is biholomorphic and the pull back of the Kähler form of \( \omega_1 \) moduli is cohomologus to th \( k \) times a Kähler form in the \( \omega \) moduli by arguments as above, we get a quantization of the original \( \omega \) moduli by a determinant bundle. (the bundle \( L^{-1} \) is a Quillen bundle as \( - \) times its curvature is the integral descent of the Quillen curvature and since positive or negative rational powers of Quillen bundle are determinant bundles by our convention (see section 2) and since determinant bundles are closed under the pullbacks our claim follows). \( \square \)

**4.4. Projectivity of moduli space.** The above theory can be generalized to \( S \) having more than one connected components. In that case the obstruction to getting a holomorphic bundle on the regular part the whole moduli space is existence of sections whose zero sets contain a component of \( S \).

The moduli space \( \mathcal{M}_X \) of vortex equations on a Kähler surface \( X \) is compact since it is equivalent to the Seiberg Witten moduli.
On the other hand, we proved that the regular moduli space $\mathcal{M}_X$ has a Kähler form $\Omega_X$, which is integral under the condition that obstructions $O(1)$ and $O(2)$ can be avoided.

Under this condition we have shown, there is a determinat line bundle whose curvature is proportional to $\Omega_X$. Thus from results proved in previous subsection we have a quantization by a determinant bundle of the vortex Kähler surface moduli space.

Below we give a large class of manifolds for which the obstructions can be avoided. We mention the following proposition which follows also from Bradlow’s work \[3\]. The main result in Bradlow’s paper implies that the moduli space can be interpreted as a Hilbert scheme of hypersurfaces in the base (Kähler, projective) manifold of a fixed degree (the degree of the line bundle). That such a Hilbert scheme is projective is a well-known fact (Grothendieck’s EGA or FGA). This observation is due to N. Romao.

There may be other proofs of the following proposition probably one due to J.M. Baptista whose reference we are unable to provide.

We mention this proposition as it follows easily from the results of this section without going into the theory of Hilbert schemes. It also provides an example of the ample or the quantum bundle being a determinant bundle and it may differ from the one shown in \[15\] as in our case the input bundle is on a Kähler surface while there’s was a bundle on $\mathbb{C}P^1$.

**Proposition 4.4.** Let $X$ be a projective Kähler surface with integral Kähler form $\omega$. Then the moduli space $\mathcal{M}$ of vortex equations on $X$ is projective if the moduli space is smooth.

**Proof.** Since the moduli space is smooth and compact, we have to show that obstructions $O_1$ and $O_2$ are avoided in order to get an ample line bundle. Since the manifold is projective we can have a very ample bundle whose zero of a generic section will make us avoid obstruction $O_2$. To avoid obstruction $O_1$, suppose a bundle $L$ has an holomorphic section which is zero on divisor $D$ and if the divisor is smooth and irreducible.

$$deg(L) - deg(D) \geq 0 \quad (26)$$

Let $\mathcal{L}$ be the ample bundle then $\mathcal{L}^p$ for large $p$ be a very ample bundle then by increasing $p$ we can have

$$deg(L) - deg(\mathcal{L}^p) < 0 \quad (27)$$

So the obstruction $O_1$ can be avoided for the vortex moduli space for $X$ with Kähler form $p\omega$ if there is smooth and irreducible divisor of $\mathcal{L}^p$. But this is guaranteed by Bertini’s theorem since the bundle is very ample. For the vortex moduli space corresponding to $X$ with Kähler form $\omega$ this holds too (as the corresponding Kähler manifolds $(X, \omega)$ and $(X, p\omega)$ are biholomorphc and the pull back of the Kähler form of $p\omega$ moduli is cohomologus to $p$ times a Kähler ähler form in the $\omega$ moduli).

\[\square\]

**Corollary 4.5.** The vortex moduli space on the Kähler surface is projective if the moduli space is smooth and there exists a closed surface surface $S$ which is a complex one dimensional submanifold whose homology class is the Poincaré dual of the cohomology class of the Kähler form and none of the representatives of the
Poincaré dual of the Chern class of the vortex line bundle contain a component of $S$.

5. Appendix 1: The Vortex and the Seiberg-Witten correspondence

In this section we briefly review the Seiberg-Witten equations for the Kähler surface and the analysis of these equations in this case. This section closely follows Bradlow and Garcia-Prada, [6].

In [6], Bradlow and Garcia-Prada wrote the Seiberg-Witten equations as

$$\bar{\partial}_A \phi + \bar{\partial}_A^* \beta = 0$$

(28)

$$\Lambda F_A = i(|\phi|^2 - |\beta|^2)$$

(29)

$$F_A^{2,0} = -\bar{\phi} \beta$$

(30)

$$F_A^{0,2} = \beta \phi$$

(31)

where notation is as in [6], i.e. $(\phi, \beta)$ is a section of the $S_L^+$ bundle, $A$ is connection on $L$ and $\hat{A}$ is the induced connection on $\hat{L}$ and $\Lambda F_A$ is the contraction of the curvature with the Kähler form $\omega$. It is not difficult to see (37) that the solutions to these equations are such that either $\beta = 0$ or $\phi = 0$, and it is not possible to have irreducible solutions of both types simultaneously for a fixed Spin$^c$-structure.

We thus have one of the following two situations:

(i) $\beta = 0$ and the equations reduce to

$$F_A^{0,2} = 0$$

$$\bar{\partial}_A \phi = 0$$

$$\Lambda F_A = i|\phi|^2$$

and similar equations for

(ii) $\phi = 0$, i.e.

$$F_A^{0,2} = 0$$

$$\bar{\partial}_A \beta = 0$$

$$\Lambda F_A = i|\beta|^2$$

Remark 5.1. We have omitted the equation $F_A^{2,0} = 0$, since by unitarity of the connection this is equivalent to $F_A^{0,2} = 0$.

The Hodge star operator interchanges these two cases, and we can thus concentrate on case $\beta = 0$. Equations are essentially the equations known as the vortex equations. These have been extensively studied (e.g. in [6], [6], [20], [21]) for compact Kähler manifolds of arbitrary dimension. The equations are the following: Let $(X, \omega)$ be a compact Kähler manifold of arbitrary dimension, and let $(L, h)$ be a Hermitian $C^\infty$ line bundle over $X$. Let $\tau \in R$. The $\tau$ -vortex equations

$$F_A^{0,2} = 0$$

$$\bar{\partial}_A \phi = 0$$

$$\Lambda F_A = \frac{1}{2}(|\phi|^2 - \tau)$$

are equations for a pair $(A, \phi)$ consisting of a connection on $(L, h)$ and a smooth section of $L$. The first equation means that $A$ defines a holomorphic structure on $L$, while the second says that $\phi$ must be holomorphic with respect to this holomorphic structure.

Let $s$ be the scalar curvature of $X$, the Bradlow and Garcia-Prada [6] obtain that the Sieberg written equations are equivalent to

$$F_A^{0,2} = 0$$
\[ \overline{\partial}_A \phi = 0 \]
\[ \Lambda F_A = \frac{i}{2}(|\phi|^2 + s) \]

These are the vortex equations on \( \hat{L} \), but with the parameter \( \tau \) replaced by minus the scalar curvature. One can perturb the above equations by \(-s + f\), when \( \beta = 0 \), equations reduce to the constant function vortex equations (see e.g. [22]).

6. Conclusion:

1. We have shown that for various vortex moduli spaces for compact Kähler surfaces the quantum bundle is a holomorphic determinant line bundle. If the compact Kähler surface is projective, we have that the vortex moduli spaces are projective, if they are smooth. There are obstructions to this result which we showed can be surmounted. Though the moduli space has been proven to be a Hilbert scheme in certain cases by Bradlow, we donot use the theory of Hilbert schemes.

2. We also showed that on projective manifolds \((M, \omega)\), there are holomorphic determinant bundles (in the sense of Knusden-Mumford used by Bismut, Gillet, Soulé) which play the role of the geometric quantum bundle, namely one for each input data of a Hermitian holomorphic line bundle \( L \) of non-trivial Chern class on a compact Kähler manifold \( Z \) (with Todd genus non-zero) and a choice of a geometric quantization of \((M, \omega)\).

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