Non-Degeneracy of Kobayashi Volume Measures for Singular Directed Varieties

YA DENG

Abstract

In this note, we prove the generic Kobayashi volume measure hyperbolicity of singular directed varieties \((X, V)\), as soon as the canonical sheaf \(K_V\) of \(V\) is big in the sense of Demailly.

1 Introduction

Let \((X, V)\) be a complex directed manifold, i.e X is a complex manifold equipped with a holomorphic subbundle \(V \subset T_X\). Demailly’s philosophy in introducing directed manifolds is that, there are certain fonctorial constructions which work better in the category of directed manifolds (ref. [Dem12]), even in the “absolute case”, i.e. the case \(V = T_X\). Therefore, it is usually inevitable to allow singularities of \(V\), and \(V\) can be seen as a coherent subsheaf of \(T_X\) such that \(T_X/V\) is torsion free. In this case \(V\) is a subbundle of \(T_X\) outside an analytic subset of codimension at least 2, which is denoted by \(\text{Sing}(V)\). The Kobayashi volume measure can also be defined for (singular) directed manifolds.

Definition 1.1. Let \((X, V)\) be a directed manifold with \(\dim(X) = n\) and \(\text{rank}(V) = r\). Then the Kobayashi volume measure of \((X, V)\) is the pseudometric defined on any \(\xi \in \wedge^r V_x\) for \(x \notin \text{Sing}(V)\), by

\[
e_{X,V}(\xi) := \inf \{ \lambda > 0; \exists f: B_r \to X, f(0) = x, \lambda f_*(\tau_0) = \xi, f_*(T_{B_r}) \subset V\},
\]

where \(B_r\) is the unit ball in \(\mathbb{C}^r\) and \(\tau_0 = \frac{\partial}{\partial t_1} \wedge \cdots \wedge \frac{\partial}{\partial t_r}\) is the unit \(r\)-vector of \(\mathbb{C}^r\) at the origin. We say that \((X, V)\) is generic volume measure hyperbolic if \(e_{X,V}\) is generically positive definite.

In [Dem12] the author also introduced the definition of canonical sheaf \(K_V\) for any singular directed variety \((X, V)\), and he showed that the “bigness” of \(K_V\) will imply that, any non-constant entire curve \(f: \mathbb{C} \to (X, V)\) must satisfy certain global algebraic differential equations. In this note, we will study the Kobayashi volume measure of the singular directed variety \((X, V)\), and give another intrinsic explanation for the bigness of \(K_V\). Our main theorem is as follows:

Theorem 1.1. Let \((X, V)\) be a compact complex directed variety (\(V\) is singular) with \(\text{rank}(V) = r\) and \(\dim(X) = n\). If \(V\) is of general type (see Definition 2.1 below), with the base locus \(\text{Bs}(V) \subset X\) (see also Definition 2.1), then Kobayashi volume measure of \((X, V)\) is non-degenerate outside \(\text{Bs}(V)\). In particular \((X, V)\) is generic volume measure hyperbolic.

Remark 1.1. In the absolute case, Theorem 1.1 is proved in [Gri71] and [KO71]; for the smooth directed variety it is proved in [Dem12].
2 Proof of main theorem

Proof. Since the singular set \( \text{Sing}(V) \) of \( V \) is an analytic set of codimension \( \geq 2 \), the top exterior power \( \wedge^r V \) of \( V \) is a coherent sheaf of rank 1, and is included in its bidual \( \wedge^r V^{**} \), which is an invertible sheaf (of course, a line bundle). We will give an explicit construction of the multiplicative cocycle which represent the line bundle \( \wedge^r V^{**} \).

Since \( V \subset T_X \) is a coherent sheaf, we can take an open covering \( \{U_\alpha\} \) satisfying that on each \( U_\alpha \) one can find sections \( e_1^{(\alpha)}, \ldots, e_k^{(\alpha)} \in \Gamma(U_\alpha, T_X|_{U_\alpha}) \) which generate the coherent sheaf \( V \) on \( U_\alpha \). Thus the sections \( e_{i_1}^{(\alpha)} \wedge \cdots \wedge e_{i_r}^{(\alpha)} \in \Gamma(U_\alpha, \wedge^r T_X|_{U_\alpha}) \) with \( (i_1, \ldots, i_r) \) varying among all \( r \)-tuples of \( (1, \ldots, k_\alpha) \) generate the coherent sheaf \( \wedge^r V|_{U_\alpha} \), which is a subsheaf of \( \wedge^r T_X|_{U_\alpha} \). Let \( \gamma_I^{(\alpha)} := e_{i_1}^{(\alpha)} \wedge \cdots \wedge e_{i_r}^{(\alpha)} \), then by \( \text{Cod}(\text{Sing}(V)) \geq 2 \) we know that the common zero of all \( \gamma_I^{(\alpha)} \) is contained in \( \text{Sing}(V) \), and thus all \( \gamma_I^{(\alpha)} \) are proportional outside \( \text{Sing}(V) \). Therefore there exists a unique \( v_\alpha \in \Gamma(U_\alpha, \wedge^r T_X|_{U_\alpha}) \), and holomorphic functions \( \{\lambda_I\} \) which do not have common factors, such that \( \gamma_I^{(\alpha)} = \lambda_I v_\alpha \) for all \( I \). By this construction we can show that on \( U_\alpha \cap U_\beta, v_\alpha \) and \( v_\beta \) coincide up to multiplication by a nowhere vanishing holomorphic function, i.e.

\[
 v_\alpha = g_{\alpha \beta} v_\beta
\]
on \( U_\alpha \cap U_\beta \neq \emptyset \), where \( g_{\alpha \beta} \in \mathcal{O}_X(U_\alpha \cap U_\beta) \). This multiplicative cocycle \( \{g_{\alpha \beta}\} \) corresponds to the line bundle \( \wedge^r V^{**} \). Then fix a Kähler metric \( \omega \) on \( X \), it will induce a metric \( h_\omega \) on \( \wedge^r T_X \) and thus also induce a singular hermitian metric \( h_\omega \) of \( \wedge^r V^{***} \) whose local weight \( \varphi_\alpha \) is equal to \( \log |v_\alpha|^2_{h_\omega} \). It is easy to show that \( h_\omega \) has analytic singularities, and the set of singularities \( \text{Sing}(h_\omega) = \text{Sing}(V) \). Indeed, we have \( \text{Sing}(h_\omega) = \cup_\alpha \{p \in U_\alpha | v_\alpha(p) = 0\} \). Now we make the following definition.

Definition 2.1. With the notions above, \( V \) is called to be of general type if there exists a singular metric \( h \) on \( \wedge^r V^{***} \) with analytic singularities satisfying the following two conditions:

(1) The curvature current \( \Theta_h \geq \epsilon \omega \), i.e., it is a Kähler current.

(2) \( h \) is more singular than \( h_\omega \), i.e., there exists a globally defined quasi-psh function \( \chi \) which is bounded from above such that

\[
 e^\chi \cdot h = h_\omega.
\]

Since \( h \) and \( h_\omega \) has both analytic singularities, \( \chi \) also has analytic singularities, and thus \( e^\chi \) is a continuous function. Moreover, \( e^{\chi(p)} > 0 \) if \( p \notin \text{Sing}(h) \). We define the base locus of \( V \) to be

\[
 \text{Bs}(V) := \cap h(\text{Sing}(h)),
\]

where \( h \) varies among all singular metric on \( \wedge^r V^{***} \) satisfying the Properties (1) and (2) above.

Now fix a point \( p \notin \text{Bs}(V) \), then by Definition 2.1 we can find a singular metric \( h \) with analytic singularities satisfying the Property (1) and (2) above, and \( p \notin \text{Sing}(h) \).

Let \( f \) be any holomorphic map from the unit ball \( \mathbb{B}_r \subset C^r \) to \( (X, V) \) such that \( f(0) = p \), then locally we have

\[
 f_*(\frac{\partial}{\partial t_1} \wedge \cdots \wedge \frac{\partial}{\partial t_r}) = a^{(\alpha)}(t) \cdot v_\alpha|_f,
\]

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where \( a^{(a)}(t) \) is meromorphic functions, with poles contained in \( f^{-1}(\text{Sing}(V)) \), and satisfies

\[
|f_s(\frac{\partial}{\partial t_1} \wedge \cdots \wedge \frac{\partial}{\partial t_r})|^2 = |a^{(a)}(t)|^2 \cdot |v_a|^2_{\tilde{H}} \leq C.
\]

Since \( f_s(\frac{\partial}{\partial t_1} \wedge \cdots \wedge \frac{\partial}{\partial t_r}) \) can be seen as a (meromorphic!) section of \( f^* \wedge^r V^{**} \), then we define

\[
\delta(t) := |f_s(\frac{\partial}{\partial t_1} \wedge \cdots \wedge \frac{\partial}{\partial t_r})|^2_{h^{-1},r} = |a^{(a)}(t)|^2 \cdot e^{\phi_a},
\]

where \( \phi_a \) is the local weight of \( h \). By Property (2) above, we have a globally defined quasi-psh function \( \chi \) on \( X \) which is bounded from above such that

\[
\delta(t) = e^{\chi} \cdot |f_s(\frac{\partial}{\partial t_1} \wedge \cdots \wedge \frac{\partial}{\partial t_r})|^2_{\omega} \leq C_1. \tag{2.1}
\]

Now we define a semi-positive metric \( \tilde{\gamma} \) on \( \mathbb{B}_r \) by putting \( \tilde{\gamma} := f^* \omega \), then we have

\[
\frac{|f_s(\frac{\partial}{\partial t_1} \wedge \cdots \wedge \frac{\partial}{\partial t_r})|_{\omega}}{\det \tilde{\gamma}} \leq C_0(f(t)), \tag{2.2}
\]

where \( C_0(z) \) is a bounded function on \( X \) which does not depend on \( f \), and we take \( C_2 \) to be its upper bound. One can find a conformal \( \lambda(t) \) to define \( \gamma = \lambda \tilde{\gamma} \) satisfying

\[
\det \gamma = \delta(t)^{\frac{1}{2}}.
\]

Combined (2.1) and (2.2) together we obtain

\[
\lambda \leq C_2 \cdot e^{\frac{\chi}{2}}.
\]

Since \( \Theta_h \geq e\omega \), by (2.1) we have

\[
\ddc \log \det \gamma \geq \frac{e}{2} f^* \omega = \frac{e}{2\lambda} \gamma \geq \frac{e}{2C_2^2} e^{-\frac{\chi}{2} \gamma}. \tag{2.3}
\]

By Property (2) in Definition 2.1 of \( h \), there exists a constant \( C_3 > 0 \) such that

\[
e^{-\frac{\chi}{2} \gamma} \geq C_3.
\]

Let \( A := \frac{eC_3}{2C_2^2} \), and we know that it is a constant which does not depend on \( f \). Then by Ahlfors-Schwarz Lemma (see Lemma 2.1 below) we have

\[
\delta(0) \leq \left( \frac{r+1}{A} \right)^{2r}.
\]

Since \( p \notin \text{Sing}(h) \), then we have \( e^{\chi(p)} > 0 \), and thus

\[
|f_s(\frac{\partial}{\partial t_1} \wedge \cdots \wedge \frac{\partial}{\partial t_r})|_{\omega}(0) \leq e^{-\chi(p)} \delta(0) = e^{-\chi(p)} \cdot \left( \frac{r+1}{A} \right)^{2r}.
\]

By the arbitrariness of \( f \), and Definition 1.1, we conclude that \((X, V)\) is generic volume measure hyperbolic and \( e_{X,V}^r \) is non-degenerate outside \( \text{Bs}(V) \). \( \square \)
Lemma 2.1 (Ahlfors-Schwarz). Let $\gamma = \sqrt{-1} \sum \gamma_{jk}(t) dt_j \wedge dt_k$ be an almost everywhere positive hermitian form on the ball $B(0, R) \subset \mathbb{C}^n$ of radius $R$, such that

$$-\text{Ricci}(\gamma) := \sqrt{-1} \partial \bar{\partial} \log \det \gamma \geq A \gamma$$

in the sense of currents for some constant $A > 0$. Then

$$\det(\gamma)(t) \leq \left( \frac{R^2 + 1}{AR^2} \right)^{r-1} \left( 1 - \frac{t^2}{R^2} \right)^{r+1}.$$ 

Remark 2.1. If $V$ is regular, then $V$ is of general type if and only if $\Lambda^r V$ is a big line bundle. In this situation, the base locus $\text{Bs}(V) = B_+(\Lambda^r V)$, where $B_+(\Lambda^r V)$ is the augmented base locus for the big line bundle $\Lambda^r V$ (ref. [Laz04]).

With the notions above, we define the coherent ideal sheaf $\mathcal{I}(V)$ to be germ of holomorphic functions which is locally bounded with respect to $h_s$, i.e., $\mathcal{I}(V)$ is the integral closure of the ideal generated by the coefficients of $v_n$ in some local trivialization in $\Lambda^r T_X$. Let $\Lambda^r V^{***}$ be denoted by $K_V$, and $K_V := K_V \otimes \mathcal{I}(V)$, then $K_V$ is the canonical sheaf of $(X, V)$ defined in [Dem12]. It is easy to show that the zero scheme of $\mathcal{I}(V)$ is equal to $\text{Sing}(h_s) = \text{Sing}(V)$. $K_V$ is called to be a big sheaf iff for some log resolution $\mu : \widetilde{X} \to X$ of $\mathcal{I}(V)$ with $\mu^* \mathcal{I}(V) = \mathcal{O}_{\widetilde{X}}(-D)$, $\mu^* K_V - D$ is big in the usual sense. Now we have the following statement:

**Proposition 2.1.** $V$ is of general type if and only if $K_V$ is big. Moreover, we have

$$\text{Bs}(V) \subset \mu(B_+(\mu^* K_V - D)) \cup \text{Sing}(h_s) = \mu(B_+(\mu^* K_V - D)) \cup \text{Sing}(V).$$

*Proof.* From Definition 2.1, the condition that $K_V$ is a big sheaf implies that $K_V$ and $\mu^* K_V - D$ are both big line bundles. For $m > 0$, we have the following isomorphism

$$\mu^* : H^0(X, (mK_V - A) \otimes \mathcal{I}(V)^m) \xrightarrow{\sim} H^0(\widetilde{X}, m\mu^* K_V - \mu^* A - mD). \tag{2.3}$$

Fix an ample divisor $A$, then for $m \gg 0$, the base locus (in the usual sense) $B(m\mu^* K_V - mD - \mu^* A)$ is stably contained in $B_+(\mu^* K_V - D)$ (ref. [Laz04]). Thus we can take a $m > 0$ to choose a basis $s_1, \ldots, s_k \in H^0(\widetilde{X}, m\mu^* K_V - mD - \mu^* A)$, whose common zero is contained in $B_+(\mu^* K_V - D)$. Then by (2.3) there exists $\{e_i\}_{1 \leq i \leq k} \subset H^0(X, (mK_V - A) \otimes \mathcal{I}(V)^m)$ such that

$$\mu^*(e_i) = s_i.$$

Then we can define a singular metric $h_m$ on $mK_V - A$ by putting $||\xi||_{h_m}^2 := \frac{||\xi||^2}{\sum_{i=1}^k |e_i|^2}$ for $\xi \in (mK_V - A)$. We take a smooth metric $h_A$ on $A$ such that the curvature $\Theta_A \geq \epsilon \omega$ is a smooth Kähler form. Then $h := (h_m h_A)^{1/m}$ defines a singular metric on $K_V$ with analytic singularities, such that its curvature current $\Theta_h \geq \frac{1}{m} \Theta_A$. From the construction we know that $h$ is more singular than $h_s$, and $\text{Sing}(h) \subset \mu(B_+(\mu^* K_V - D)) \cup \text{Sing}(h_s)$. \hfill \Box

**Remark 2.2.** From Proposition 2.1 we can take Definition 2.1 as another equivalent definition of the bigness of $K_V$, and it is more analytic. From Theorem 1.1 we can replace the condition that $V$ is of general type by the bigness of $K_V$, and it means that the definition of canonical sheaf of singular directed varieties is very natural.
A direct consequence of Theorem 1.1 is the following corollary, which was suggested in [GPR13]:

**Corollary 2.1.** Let \((X, V)\) be directed varieties with \(\text{rank}(V) = r\), and \(f\) be a holomorphic map from \(\mathbb{C}^r\) to \((X, V)\) with generic maximal rank. Then if \(K_V\) is big, the image of \(f\) is contained in \(\text{Bs}(V) \subseteq X\).

The famous conjecture by Green-Griffiths stated that in the absolute case the converse of Theorem 1.1 should be true. It is natural to ask whether we have similar results for any directed varieties. A result by Marco Brunella (ref. [Bru10]) gives a weak converse of Theorem 1.1 for 1-directed variety:

**Theorem 2.1.** Let \(X\) be a compact Kähler manifold equipped with a singular holomorphic foliation \(\mathcal{F}\) by curves. Suppose that \(\mathcal{F}\) contains at least one leaf which is hyperbolic, then the canonical bundle \(K_{\mathcal{F}}\) is pseudoeffective.

Indeed, Brunella proved more than the results stated in the theorem above. By putting on \(K_{\mathcal{F}}\) precisely the Poincaré metric on the hyperbolic leaves, he construct a singular hermitian metric \(h\) (maybe not with analytic singularities) on \(K_{\mathcal{F}}\), such that the set of points where \(h\) is unbounded locally are polar set \(\text{Sing}(\mathcal{F}) \cup \text{Parab}(\mathcal{F})\), where \(\text{Parab}(\mathcal{F})\) are the union of parabolic leaves, and the curvature \(\Theta_h\) of the metric \(h\) is a positive current. Therefore, it seems that Brunella’s theorem can be strengthened, i.e., not only \(K_{\mathcal{F}}\) is pseudo-effective, but also the canonical sheaf \(K_{\mathcal{F}}\) is pseudoeffective. However, as is shown in the following example, even if all the leaves of \(\mathcal{F}\) are hyperbolic, the canonical sheaf can not be pseudoeffective.

**Example 2.1.** A foliation by curves of degree \(d\) on the complex projective space \(\mathbb{C}P^n\) is generated by a global section
\[
s \in H^0(\mathbb{C}P^n, T_{\mathbb{C}P^n} \otimes \mathcal{O}(d - 1)).
\]

From the results by Lins Neto and Soares [LS96] and Brunella [Bru06], we know that a generic one-dimensional foliation \(\mathcal{F}\) of degree \(d\) satisfies the following property:

(a) the set of the singularities \(\text{Sing}(\mathcal{F})\) of \(\mathcal{F}\) is discrete;

(b) each singularity \(p \in \text{Sing}(\mathcal{F})\) is non-degenerate, i.e. the Milnor number of \(\mathcal{F}\) at \(p\) is 1;

(c) no \(d + 1\) points in \(\text{Sing}(\mathcal{F})\) lie on a projective line;

(d) all the leaves of \(\mathcal{F}\) are hyperbolic.

Hence the Baum-Bott formula states that
\[
\# \text{Sing}(\mathcal{F}) = c_n(T_{\mathbb{C}P^n} + \mathcal{O}(d - 1))
= \sum_{i=0}^{n} c_{n-i}(\mathbb{C}P^n)c_1(\mathcal{O}(d - 1))^i
= \sum_{i=0}^{n} \binom{n+1}{i+1}(d - 1)^i
= \frac{d^{n+1} - 1}{d - 1},
\]
and thus the canonical sheaf $K_F = O(d - 1) \otimes I_{\text{Sing}(F)}$, where $I_{\text{Sing}(F)}$ is the maximal ideal of $\text{Sing}(F)$. By property (c) above it is easy to prove that for $d \gg 0$ $K_F$ is not pseudo-effective.

**Remark 2.3.** In [McQ08] the author introduces the definition *canonical singularities* for foliations, in dimension 2 this definition is equivalent to *reduced singularities* in the sense of Seidenberg. The generic foliation by curves of degree $d$ in $\mathbb{C}P^n$ is another example of canonical singularities. In this situation, one can not expect to improve the “bigness” of canonical sheaf $K_F$ by blowing-up. Indeed, this birational model is “stable” in the sense that, $\pi_*K_{\tilde{F}} = K_F$ for any birational model $\pi : (\tilde{X}, \tilde{F}) \to (X, F)$. However, on the complex surface endowed with a foliation $F$ with reduced singularities, if $f$ is an entire curve tangent to the foliation, and $T[f]$ is the Ahlfors current associated with $f$, then in [McQ98] it is shown that the positivity of $T[f] \cdot T_F$ can be improved by an infinite sequence of blowing-ups, due to the fact that certain singularities of $F$ appearing in the blowing-ups are “small”, i.e. the lifted entire curve will not pass to these singularities. Since $T[f] \cdot T_F$ is related to value distribution, and thus these small singularities are negligible. In [Den16] this “Diophantine approximation” idea has been generalized to higher dimensions.

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**References**

[Bru06] M. Brunella, Inexistence of invariant measures for generic rational differential equations in the complex domain. Bol. Soc. Mat. Mexicana (3), 2006, 12(1): 43-49.

[Bru10] M. Brunella, Uniformisation of foliations by curves. Holomorphic dynamical systems. Springer Berlin Heidelberg, 2010: 105-163.

[Dem12] J.P. Demailly, Kobayashi pseudo-metrics, entire curves and hyperbolicity of algebraic varieties, Lecture notes in Summer School in Mathematics 2012.

[Dem14] J.P. Demailly, Towards the Green-Griffiths-Lang conjecture, arXiv: 1412.2986v2 [math.AG]

[Den16] Y. Deng, Degeneracy of entire curves into higher dimensional complex manifolds, in preparation.

[GPR13] C. Gasbarri, G. Pacienza, E. Rousseau, Higher dimensional tautological inequalities and applications. Mathematische Annalen, 2013, 356(2): 703-735.

[Gri71] P.A. Griffiths, Holomorphic mapping into canonical algebraic varieties. Annals of Mathematics, 1971: 439-458.

[KO71] S. Kobayashi, T. Ochiai, Mappings into compact complex manifolds with negative first Chern class. Journal of the Mathematical Society of Japan, 1971, 23(1): 137-148.
[Laz04] R. Lazarsfeld, Positivity in algebraic geometry II. Springer-Verlag, 2004.

[LS96] A. Lins Neto, M.G. Soares, Algebraic solutions of one-dimensional foliations. Journal of Differential Geometry, 1996, 43(3): 652-673.

[McQ98] M. McQuillan, Diophantine approximations and foliations. Publications Mathmatiques de l’Institut des Hautes tudes Scientifiques, 1998, 87(1): 121-174.

[McQ08] M. McQuillan, Canonical models of foliations. Pure and applied mathematics quarterly, 2008, 4(3).

INSTITUT FOURIER & UNIVERSITY OF SCIENCE AND TECHNOLOGY OF CHINA
YA DENG
Email: Ya.Deng@fourier-grenoble.fr