Non-Archimedean Radial Calculus: Volterra Operator and Laplace Transform

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Abstract. In an earlier paper (A. N. Kochubei, Pacif. J. Math. 269 (2014), 355–369), the author considered a restriction of Vladimirov’s fractional differentiation operator $D^\alpha$, $\alpha > 0$, to radial functions on a non-Archimedean field. In particular, it was found to possess such a right inverse $I^\alpha$ that the appropriate change of variables reduces equations with $D^\alpha$ (for radial functions) to integral equations whose properties resemble those of classical Volterra equations. In other words, we found, in the framework of non-Archimedean pseudo-differential operators, a counterpart of ordinary differential equations. In the present paper, we begin an operator-theoretic investigation of the operator $I^\alpha$, and study a related analog of the Laplace transform.

Mathematics Subject Classification. Primary 47G10, Secondary 11S80, 35S10, 43A32.

Keywords. Fractional differentiation operator, Non-Archimedean local field, Radial functions, Volterra operator, Laplace transform.

1. Introduction

The basic linear operator defined on real- or complex-valued functions on a non-Archimedean local field $K$ (such as $K = \mathbb{Q}_p$, the field of $p$-adic numbers) is the Vladimirov pseudo-differential operator $D^\alpha$, $\alpha > 0$, of fractional differentiation [19]; for further development of this subject see [1, 3, 8, 11, 12, 22]. Note also the recent publications devoted to applications in geophysical models and to the study of related nonlinear equations [9, 10, 17, 18].

It was found in [13] that properties of $D^\alpha$ become much simpler on radial functions. Moreover, in this case it was found to possess a right inverse $I^\alpha$, which can be seen as a $p$-adic counterpart of the Riemann-Liouville fractional integral or, for $\alpha = 1$, the classical anti-derivative. The change of an unknown function $u = I^\alpha v$ reduces the Cauchy problem for an equation with the radial restriction of $D^\alpha$ to an integral equation with properties resembling those of classical Volterra equations. In other words, we found, in the
framework of non-Archimedean pseudo-differential operators, a counterpart of ordinary differential equations. In [13], we studied linear equations of this kind; nonlinear ones were investigated in [15]. Note that radial functions appear as exact solutions of the $p$-adic analog of the classical porous medium equation [9].

In this paper we study the operator $I^1$ on the ring of integers $O \subset K$ as an object of operator theory. The operator $I^1$ on $L^2(O)$ happens to be a sum of a bounded selfadjoint operator and a simple Volterra operator $I^1_0$ with a rank two imaginary part $J$, such that $\text{tr} \ J = 0$. The characteristic matrix-function $W(z)$ of $I^1_0$ is such that $W(z^{-1})$ is, in contrast to classical examples, an entire matrix function of zero order.

While the theory of Volterra operators and their characteristic functions is well-developed (see [4–6, 16, 21]), properties of the above operator are very different from those known for operators of classical analysis and their generalizations. Therefore, while $I^1$ and $I^1_0$ are just specific examples, they create a framework for future studies in this area.

Another subject touched in this paper is a version of the Laplace transform. The classical Laplace transform is based on the function $x \mapsto e^{-\lambda x}$ satisfying an obvious differential equation. A similar equation involving $D^\alpha$ has a unique radial solution [12]. This leads to a definition of the Laplace type transform in the above framework. We prove a uniqueness theorem and the inversion formula for this transform.

2. Preliminaries

2.1. Local Fields

Let $K$ be a non-Archimedean local field, that is a non-discrete totally disconnected locally compact topological field. It is well known that $K$ is isomorphic either to a finite extension of the field $\mathbb{Q}_p$ of $p$-adic numbers (if $K$ has characteristic 0), or to the field of formal Laurent series with coefficients from a finite field, if $K$ has a positive characteristic. For a summary of main notions and results regarding local fields see, for example, [11].

Any local field $K$ is endowed with an absolute value $| \cdot |_K$, such that $|x|_K = 0$ if and only if $x = 0$, $|xy|_K = |x|_K \cdot |y|_K$, $|x+y|_K \leq \max(|x|_K, |y|_K)$. Denote $O = \{x \in K : |x|_K \leq 1\}$, $P = \{x \in K : |x|_K < 1\}$. $O$ is a subring of $K$, and $P$ is an ideal in $O$ containing such an element $\beta$ that $P = \beta O$. The quotient ring $O/P$ is actually a finite field; denote by $q$ its cardinality. We will always assume that the absolute value is normalized, that is $|\beta|_K = q^{-1}$. The normalized absolute value takes the values $q^N$, $N \in \mathbb{Z}$. Note that for $K = \mathbb{Q}_p$ we have $\beta = p$ and $q = p$; the $p$-adic absolute value is normalized.

The additive group of any local field is self-dual, that is if $\chi$ is a fixed non-constant complex-valued additive character of $K$, then any other additive character can be written as $\chi_a(x) = \chi(ax)$, $x \in K$, for some $a \in K$. Below we assume that $\chi$ is a rank zero character, that is $\chi(x) \equiv 1$ for $x \in O$, while there exists such an element $x_0 \in K$ that $|x_0|_K = q$ and $\chi(x_0) \neq 1$. 
The above duality is used in the definition of the Fourier transform over \( K \). Denoting by \( dx \) the Haar measure on the additive group of \( K \) (normalized in such a way that the measure of \( O \) equals 1) we write

\[
\tilde{f}(\xi) = \int_K \chi(x\xi)f(x)\,dx, \quad \xi \in K,
\]

where \( f \) is a complex-valued function from \( L_1(K) \). As usual, the Fourier transform \( \mathcal{F} \) can be extended from \( L_1(K) \cap L_2(K) \) to a unitary operator on \( L_2(K) \). If \( \mathcal{F}f = \tilde{f} \in L_1(K) \), we have the inversion formula

\[
f(x) = \int_K \chi(-x\xi)\tilde{f}(\xi)\,d\xi.
\]

Working with functions on \( K \) and operators upon them we often use standard integration formulas; see [11,19]. The simplest of them are as follows:

\[
\int_{|x|_K \leq q^n} dx = q^n; \quad \int_{|x|_K = q^n} dx = \left(1 - \frac{1}{q}\right)q^n.
\]

\[
\int_{|x|_K \leq q^n} |x|_K^{\alpha-1} dx = \frac{1 - q^{-1}}{1 - q^{-\alpha}} q^{\alpha n}; \quad \text{here and above } n \in \mathbb{Z}, \alpha > 0.
\]

A function \( f : K \to \mathbb{C} \) is said to be locally constant, if there exists such an integer \( l \) that for any \( x \in K \)

\[
f(x + x') = f(x), \quad \text{whenever } |x'| \leq q^{-l}.
\]

The vector space \( \mathcal{D}(K) \) of all locally constant functions with compact supports is used as a space of test functions in analysis on \( K \). Note that the Fourier transform preserves \( \mathcal{D}(K) \). There exists a well-developed theory of distributions on local fields; see [1,11,19].

2.2. Vladimirov’s Operator

On a test function \( \varphi \in \mathcal{D}(K) \), the fractional differentiation operator \( D^\alpha \), \( \alpha > 0 \), is defined as

\[
(D^\alpha \varphi)(x) = \mathcal{F}^{-1} \left[ |\xi|_K^\alpha (\mathcal{F}(\varphi))(\xi) \right](x). \quad (2.1)
\]

Note that \( D^\alpha \) does not preserve \( \mathcal{D}(K) \); see [1] regarding the spaces of test functions and distributions preserved by this operator.

The operator \( D^\alpha \) can also be represented as a hypersingular integral operator:

\[
(D^\alpha \varphi)(x) = \frac{1 - q^\alpha}{1 - q^{-\alpha-1}} \int_K |y|_K^{-\alpha-1} \left[ \varphi(x - y) - \varphi(x) \right] dy. \quad (2.2)
\]
In contrast to (2.1), the expression in the right of (2.2) makes sense for wider classes of functions. In particular, \( D^\alpha \) is defined on constant functions and annihilates them. Denote for brevity \( \theta_\alpha = \frac{1 - q^\alpha}{1 - q^{-\alpha - 1}} \).

Below we consider the operator \( D^\alpha \) on a radial function \( u = u(|x|_K) \); here we identify the function \( x \mapsto u(|x|_K) \) on \( K \) with the function \( |x|_K \mapsto u(|x|_K) \) on \( q^Z \). This abuse of notation does not lead to confusion.

The explicit expression of \( D^\alpha u \) for a radial function \( u \) satisfying some growth restrictions near the origin and infinity was found in [13]. If \( u = u(|x|_K) \) is such that

\[
\sum_{k=-\infty}^{m} q^k |u(q^k)| < \infty, \quad \sum_{l=m}^{\infty} q^{-\alpha l} |u(q^l)| < \infty, \tag{2.3}
\]

for some \( m \in \mathbb{Z} \), then for each \( n \in \mathbb{Z} \) the expression in the right-hand side of (2.2) with \( \varphi(x) = u(|x|_K) \) exists for \( |x|_K = q^n \), depends only on \( |x|_K \), and

\[
(D^\alpha u)(q^n) = \theta_\alpha \left( 1 - \frac{1}{q} \right) q^{-(\alpha+1)n} \sum_{k=-\infty}^{n-1} q^k u(q^k) + q^{-\alpha n} \frac{q^{\alpha} + q - 2}{1 - q^{-\alpha - 1}} u(q^n) + \theta_\alpha \left( 1 - \frac{1}{q} \right) \sum_{l=n+1}^{\infty} q^{-\alpha l} u(q^l). \tag{2.4}
\]

Under the conditions (2.3), the expression (2.4) agrees also with the definition of \( D^\alpha \) in terms of Bruhat-Schwartz distributions (see Chapter 2 of [19]).

### 2.3. The Regularized Integral

The fractional integral mentioned in Introduction, was defined in [13] initially for \( \varphi \in D(K) \) as follows:

\[
(I^\alpha \varphi)(x) = (D^{\alpha} \varphi)(x) - (D^{-\alpha} \varphi)(0) \tag{\ast}
\]

where \( D^{-\alpha} \) is the right inverse of \( D^\alpha \) introduced by Vladimirov [19]:

\[
(D^{-\alpha} \varphi)(x) = (f_\alpha * \varphi)(x) = \frac{1 - q^{-\alpha}}{1 - q^{\alpha - 1}} \int_K |x - y|_{K}^{\alpha - 1} \varphi(y) \, dy, \quad \alpha \neq 1,
\]

\[
(D^{-1} \varphi)(x) = \frac{1 - q}{q \log q} \int_K \log |x - y|_K \varphi(y) \, dy.
\]

\( D^{-1} \) is a right inverse to \( D^1 \) only on such functions \( \varphi \) that

\[
\int_K \varphi(x) \, dx = 0.
\]

On such a function \( \varphi \) we have also \( D^{-1} D^1 \varphi = \varphi \).

The above definition (\ast) leads to explicit expressions

\[
(I^\alpha \varphi)(x) = \frac{1 - q^{-\alpha}}{1 - q^{\alpha - 1}} \int_{|y|_K \leq |x|_K} (|x - y|_K^{\alpha - 1} - |y|_K^{\alpha - 1}) \varphi(y) \, dy, \quad \alpha \neq 1,
\]
and
\[ (I^1 \varphi)(x) = \frac{1 - q}{q \log q} \int_{|y| K \leq |x| K} (\log |x - y| K - \log |y| K) \varphi(y) \, dy. \]

Note that the integrals are taken, for each fixed \( x \in K \), over bounded sets, and \((I^1 \varphi)(0) = 0\). These properties are different from those of the anti-derivatives \( D^{-\alpha} \) studied in [19].

Let \( u = u(|x| K) \) be a radial function, such that
\[
\sum_{k=-\infty}^{m} \max \left( q^k, q^{\alpha k} \right) |u(q^k)| < \infty, \quad \text{if } \alpha \neq 1,
\]
and
\[
\sum_{k=-\infty}^{m} |k| q^k |u(q^k)| < \infty, \quad \text{if } \alpha = 1,
\]
for some \( m \in \mathbb{Z} \). Then [13] \( I^\alpha u \) exists, it is a radial function, and for any \( x \neq 0 \),
\[
(I^\alpha u)(|x| K) = q^{-\alpha} |x|_K^\alpha u(|x| K) + \frac{1 - q^{-\alpha}}{1 - q^{\alpha - 1}} \int_{|y| K < |x| K} (|x|_K^{\alpha - 1} - |y|_K^{\alpha - 1}) u(|y| K) \, dy, \quad \alpha \neq 1,
\]
and
\[
(I^1 u)(|x| K) = q^{-1} |x|_K u(|x| K) + \frac{1 - q}{q \log q} \int_{|y| K < |x| K} (\log |x| K - \log |y| K) u(|y| K) \, dy. \quad (2.5)
\]

On an appropriate class of radial functions, \( I^\alpha \) is a right inverse to \( D^\alpha \) [13]. An important difference between \( D^{-\alpha} \) and \( I^\alpha \) is the bounded integration domain in the integral formulas for \( I^\alpha \).

### 2.4. Radial Eigenfunctions of \( D^\alpha \)

The operator \( D^\alpha \) defined initially on \( D(K) \) is, after its closure in \( L^2(K) \), a selfadjoint operator with a pure point spectrum \( \{ q^{\alpha N}, N \in \mathbb{Z} \} \) of infinite multiplicity and a single limit point zero.

It was shown in [12] that for each \( N \in \mathbb{Z} \), there exists a unique (up to the multiplication by a constant) radial eigenfunction
\[
v_N(|x| K) = \begin{cases} 
1, & \text{if } |x| K \leq q^{-N}, \\
-\frac{1}{q - 1}, & \text{if } |x| K = q^{-N + 1}, \\
0, & \text{if } |x| K \geq q^{-N + 2}, 
\end{cases} \quad (2.6)
\]
corresponding to the eigenvalue \( \lambda = q^{\alpha N} \). Below we interpret this function as an analog of the classical exponential function \( x \mapsto e^{-\lambda x} \). Note that \( v_N \in \mathcal{D}(K) \); this is a purely non-Archimedean phenomenon reflecting the unusual topological property of \( K \), its total disconnectedness.
The operator \( D_{O}^{\alpha} \) in the space \( L^2(O) \) on the ring of integers (unit ball) \( O \) is defined as follows. Extend a function \( \varphi \in D(O) \) (that is a function \( \varphi \in D(K) \) supported in \( O \)) onto \( K \) by zero. Apply \( D^{\alpha} \) and consider the resulting function on \( O \). After the closure in \( L^2(O) \) we obtain a selfadjoint operator \( D_{O}^{\alpha} \) with a discrete spectrum \([11,19]\) (here we do not touch different definitions from \([2], [14]\)).

Denote by \( \mathcal{H} \) the subspace in \( L^2(O) \) consisting of radial functions. The functions \( v_N, N = 1, 2, \ldots \) belong to \( \mathcal{H} \), as well as the function \( v_0(|x|_K) \equiv 1, \ |x|_K \leq 1. \)

By the definition of \( D_{O}^{\alpha} \), the functions \( v_N \) are its eigenfunctions corresponding to the eigenvalues \( q^{\alpha N} \). As for \( v_0 \), it is also an eigenfunction, with the eigenvalue \( \mu_0 = \frac{q - 1}{q^{\alpha + 1} - 1} q^{\alpha} \) \([11,19]\). Therefore \( \{v_N\}_{N \geq 0} \) is an orthonormal system in \( L^2(O) \), hence in \( \mathcal{H} \).

We have \( \|v_0\| = 1 (\| \cdot \| \) is the norm in \( \mathcal{H} \),
\[
\|v_N\|^2 = \int_{|x|_K \leq q^{-N}} dx + (q - 1)^{-2} \int_{|x|_K = q^{-N+1}} dx
= q^{-N} + (q - 1)^{-2} q^{-N+1} (1 - \frac{1}{q}) = (q - 1)^{-1} q^{-N},
\]

\[
\int_{|x|_K \leq 1} v_N(|x|_K) \, dx = 0, \ N \geq 1.
\]

Therefore the functions
\[
e_0(|x|_K) \equiv 1; \ e_N(|x|_K) = (q - 1)^{1/2} q^{N/2} v_N(|x|_K), \ N \geq 1, \quad (2.7)
\]
form an orthonormal system in \( \mathcal{H} \).

**Lemma 1.** The system \( \{e_N\}_{N \geq 0} \) is an orthonormal basis in \( \mathcal{H} \).

**Proof.** Let \( u \in \mathcal{H} \) be orthogonal to all the functions \( e_N \). Then
\[
\int_{|x|_K \leq 1} u(|x|_K) \, dx = 0,
\]
so that
\[
\sum_{j=-\infty}^{0} u(q^j)q^j = 0 \quad (2.8)
\]
and
\[
\int_{|x|_K \leq q^{-N}} u(|x|_K) \, dx - (q - 1)^{-1} \int_{|x|_K = q^{-N+1}} u(|x|_K) \, dx = 0,
\]
so that
\[
\sum_{j=-\infty}^{-N} u(q^j)q^j - (q - 1)^{-1} u(q^{-N+1}) = 0, \ N = 1, 2, \ldots. \quad (2.9)
\]
Subtracting from (2.8) the equality (2.9) with \( N = 1 \), we find that 
\[ u(1) = 0. \]
Now the equality (2.9) with \( N = 1 \) takes the form
\[ -\sum_{j=-\infty}^{-1} u(q^j)q^j = 0, \]
while (2.9) with \( N = 2 \) yields
\[ -2 \sum_{j=-\infty}^{-2} u(q^j)q^j - (q-1)^{-1}u(q^{-1}) = 0. \]
Subtracting we obtain that 
\[ u(q^{-1}) = 0. \]
Repeating the above reasoning we find that 
\[ u = 0. \]
\[ \square \]
Another (obvious) orthonormal basis in \( \mathcal{H} \) is 
\[ f_n(|x|_K) = \begin{cases} 
(1 - \frac{1}{q})^{-1/2}q^n/2, & \text{if } |x|_K = q^{-n}; \\
0, & \text{elsewhere}, 
\end{cases} \quad n = 0, 1, 2, \ldots. \quad (2.10) \]

The next result is of some independent interest.

**Proposition 1.** The set of “polynomials” 
\[ u(|x|_K) = \sum_{n=1}^{N} a_n |x|^n_K, \quad a_n \in \mathbb{C}, \quad N \geq 1, \quad (2.11) \]
is dense in \( \mathcal{H} \).

**Proof.** Suppose that a function \( F \in \mathcal{H} \) is orthogonal to all the functions 
\( X_l(|x|_K) = |x|^l_K, \ l \geq 1 \). Using the basis (2.10), write
\[ F = \sum_{n=0}^{\infty} c_n f_n, \quad \{c_n\} \in l^2. \]
We have
\[ \langle X_l, f_n \rangle = (1 - \frac{1}{q})^{-1/2}q^n/2 \int_{|x|_K = q^{-n}} |x|^l_K \ dx = (1 - \frac{1}{q})^{1/2}q^{-n/2-nl}, \]
so that
\[ \langle F, X_l \rangle = (1 - \frac{1}{q})^{1/2} \sum_{n=0}^{\infty} c_n q^{-n/2-nl} = 0, \quad l = 1, 2, \ldots. \]
Denoting \( \beta = q^{-1}, \ b_n = c_n q^{-n/2} \), we see that the vector \( (b_0, b_1, b_2, \ldots) \) \( \in l^2 \) is orthogonal in \( l^2 \) to each vector \( (1, \beta^1, \beta^2, \ldots) \), \( l \geq 1 \). It is known ([7], Problem 6) that the set of all these vectors is total in \( l^2 \), so that \( F = 0. \) \( \square \)

In fact, the above reasoning proves the density of polynomials (2.11) in a wider weighted space determined by the condition \( \{c_n q^{-n/2}\} \in l^2. \)
3. Integration Operators

3.1. The Operator $I^1$

Let us study $I^1$ as an operator in $\mathcal{H}$, find its matrix representation with respect to the basis $\{e_N\}$ and investigate the spectrum of $I^1$.

**Proposition 2.** The operator $I^1$ has the matrix representation

$$I^1 = \begin{pmatrix}
0 & -(q - 1)^{1/2}q^{-1/2} & -(q - 1)^{1/2}q^{-1} & \ldots & -(q - 1)^{1/2}q^{-n/2} & \ldots \\
0 & q^{-1} & 0 & \ldots & 0 & \ldots \\
0 & 0 & q^{-2} & \ldots & 0 & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & \ldots & q^{-n} & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots
\end{pmatrix}$$

(only the first row and the principal diagonal have nonzero elements). $I^1$ is a Hilbert-Schmidt operator. Apart from being a point of essential spectrum, $\lambda = 0$ is a simple eigenvalue. In addition, $I^1$ has simple eigenvalues $\lambda_m = q^{-m}$, $m = 1, 2, \ldots$.

**Proof.** Since the integral of each function $e_N$, $N \geq 1$, equals zero, we have $D^{-1}D^1e_N = e_N$. On the other hand, $D^1e_N = q^N e_N$, so that $D^{-1}e_N = q^{-N} e_N$, and by the definition (*) of $I^1$,

$$I^1e_N = q^{-N} e_N - (q - 1)^{1/2}q^{-N/2}e_0, \quad N = 1, 2, \ldots \quad (3.1)$$

Next, $(I^1e_0)(|x|\mathcal{K}), |x|\mathcal{K} \leq 1$, depends only on the values of $e_0$ for $|x|\mathcal{K} \leq 1$. Let $f(x) \equiv 1$, $x \in \mathcal{K}$. Then $I^1f = 0 \quad [13]$, so that

$$I^1e_0 = 0 \quad \text{in} \mathcal{H}. \quad (3.2)$$

The equalities (3.1) and (3.2) imply the required matrix representation, which implies the Hilbert-Schmidt property.

Let us find the eigenvalues of $I^1$. As we have seen, $I^1e_0 = 0$. Suppose that

$$u = \sum_{n=0}^{\infty} c_ne_n, \quad \{c_n\} \in l^2, \quad I^1u = \lambda u.$$ 

By (3.1) and (3.2),

$$I^1u = \sum_{n=1}^{\infty} q^{-n}c_ne_n - \left[ \sum_{n=1}^{\infty} (q - 1)^{1/2}q^{-n/2}c_n \right] e_0,$$

and we find that

$$\lambda c_0 = -(q - 1)^{1/2} \sum_{n=1}^{\infty} q^{-n/2}c_n;$$

$$\lambda c_n = q^{-n} c_n, \quad n \geq 1. \quad (3.3)$$

A nonzero value of $c_n (n \geq 1)$ is possible only for a single index $n = m$, and in this case $\lambda = q^{-m}$. Then the first equation in (3.3) gives $c_m = -(q - 1)^{-1/2}q^{-m/2}c_0$, so that

$$u = c_0e_0 - (q - 1)^{-1/2}q^{-m/2}c_0e_m.$$
is the unique (up to the multiplication by a constant) eigenfunction. □

3.2. A Local Representation

The definition (*) of the operator $I^\alpha$ involves operators in $L^2(K)$; then we make restrictions to $L^2(O)$ and $H$. In this section we show, for the case where $\alpha = 1$, that a similar representation containing only operators in $L^2(O)$ is also possible.

**Theorem 1.** If $u \in L^2(O)$, then

$$(I^1 u)(x) = \left( (D_O^1)^{-1} u \right) (x) - \left( (D_O^1)^{-1} u \right) (0).$$

**Proof.** In [14], we found the resolvent $(D_O^1 - \mu + \mu_0)^{-1}$ where $\mu_0 = \frac{q}{q+1}$ (the first eigenvalue of $D_O^1$), $\mu > 0$. In [14], in connection with nonlinear equations, we considered operators in $L^1(O)$, but the result is valid for $L^2(O)$ too. For $\mu = \mu_0$,

$$(D_O^1)^{-1} u(x) = \int_{|\xi|_{K} \leq 1} \mathcal{K}(x - \xi)u(\xi) d\xi + \mu_0^{-1} \int_{|\xi|_{K} \leq 1} u(\xi) d\xi,$$  

(3.5)

where for $|x|_K = q^m$, $m \leq 0$,

$$\mathcal{K}(x) = \int_{q \leq |\eta|_{K} \leq q^{-m+1}} |\eta|^{-1}_K \chi(\eta x) d\eta.$$

Using the well-known integration formula (see, for example, Sect. 1.5 in [11]), we get

$$\mathcal{K}(x) = \sum_{j=1}^{-m+1} q^{-j} \int_{|\eta|_{K} = q^j} \chi(\eta x) d\eta = (1 - \frac{1}{q}) \sum_{j=1}^{-m} 1 - q^{-1}$$

$$= -(1 - \frac{1}{q})^m q^{-1} = \frac{1 - q}{q \log q} \log |x|_K - q^{-1}.$$

By (3.5),

$$(D_O^1)^{-1} u(x) = \frac{1 - q}{q \log q} \int_{|\xi|_{K} \leq 1} \log |x - \xi|_K u(\xi) d\xi + \int_{|\xi|_{K} \leq 1} u(\xi) d\xi.$$

Comparing with the expression for $I^1$ and noticing that $|x - \xi|_K - |\xi|_K = 0$, if $|\xi|_K > |x|_K$, we obtain (3.4) □

3.3. The Volterra operator.

Let us consider the integral part of (2.5), the operator

$$(I^1_0 u)(x) = \frac{1 - q}{q \log q} \int_{|y|_K < |x|_K} (\log |x|_K - \log |y|_K) u(|y|_K) dy.$$

Recall [5] that a compact operator is called a Volterra operator, if its spectrum consists of the unique point $\lambda = 0$. An operator $A$ is called simple, if $A$ and $A^*$ have no common nontrivial invariant subspace, on which these operators
coincide. It is known [5] that a Volterra operator $A$ is simple, if and only if the equations $Af = 0$ and $A^*f = 0$ have no common nontrivial solutions.

The main technical tool in the study of $I_0^1$ is the identity [13]

$$\int_{|y| < |x|} (\log |x| - \log |y|) |y|^m_K \, dy = d_m |x|^{m+1}_K, \quad m = 0, 1, 2, \ldots, \quad (3.6)$$

where $0 < d_m \leq Aq^{-m}$, $A > 0$ does not depend on $m$.

**Theorem 2.** The operator $I_0^1$ in $\mathcal{H}$ is a simple Volterra operator with a rank 2 imaginary part $J = \frac{1}{2i} (A - A^*)$, such that $\text{tr} J = 0$.

**Proof.** 1) Suppose that $I_0^1 u = \lambda u$, $u \in \mathcal{H}$, $\lambda \in \mathbb{C}$, $\lambda \neq 0$. Then for $|x|_K \leq 1$,

$$|u(|x|_K)| \leq \frac{c}{|\lambda|} \|u\|_{L^2(O)} \left[ \int_{|y|_K < |x|_K} (\log |x| - \log |y|_K)^2 \, dy \right]^{1/2} \leq \frac{c}{|\lambda|} \|u\|_{L^2(O)} \left[ q^{-1}|x|_K (\log |x|_K)^2 \right]^{1/2} \leq H$$

where $c = \frac{q - 1}{q \log q}$, $H$ is a positive constant.

This implies the estimate

$$|u(|x|_K)| \leq \frac{cH}{|\lambda|} \int_{|y|_K < |x|_K} (\log |x| - \log |y|_K) \, dy,$$

and by the identity (3.6) with $m = 0$,

$$|u(|x|_K)| \leq \frac{cHA}{|\lambda|} |x|_K.$$

Similarly, the identity (3.6) with $m = 1$ gives

$$|u(|x|_K)| \leq \frac{c^2HA^2}{|\lambda|^2} q^{-1}|x|^2_K,$$

and we find by induction that

$$|u(|x|_K)| \leq \frac{c^{m+1}HA^{m+1}}{|\lambda|^{m+1}} q^{-1}q^{-2} \cdots q^{-m+1} |x|^{m+1}_K, \quad (3.7)$$

for an arbitrary natural number $m$.

Note that

$$q^{-1}q^{-2} \cdots q^{-m+1} = \left( \frac{1}{q} \right)^{m(m-1)/2}.$$

Together with (3.7), this shows that $u \equiv 0$.

2) It follows from the definition of $I_0^1$ that $\lambda = 0$ is an eigenvalue corresponding to the eigenfunction

$$u_0(|x|_K) = \begin{cases} 1, & \text{if } |x|_K = 1; \\ 0, & \text{if } |x|_K < 1. \end{cases} \quad (3.8)$$

Let us show that $\lambda = 0$ does not correspond to other eigenfunctions.
Suppose that $I_0^1 \varphi = 0$ for some $\varphi \in \mathcal{H}$, so that
\[ \sum_{j=-\infty}^{n-1} (n-j)q^j \varphi(q^j) = 0, \quad n = 0, -1, -2, \ldots \tag{3.9} \]
Together with (3.9), consider a similar equality with $n - 1$ substituted for $n$, that is,
\[ \sum_{j=-\infty}^{n-2} (n-1-j)q^j \varphi(q^j) = 0, \quad n = 0, -1, -2, \ldots \tag{3.10} \]
Subtracting (3.9) from (3.10) we find that
\[ \sum_{j=-\infty}^{n-1} q^j \varphi(q^j) = 0, \quad n = 0, -1, -2, \ldots \]
that is, in particular,
\[ q^{n-1} \varphi(q^{n-1}) + q^{n-2} \varphi(q^{n-2}) + \cdots = 0, \]
\[ q^{n-2} \varphi(q^{n-2}) + q^{n-3} \varphi(q^{n-3}) + \cdots = 0, \]
Subtracting the second equality from the first one, we find that $\varphi(q^{-1}) = \varphi(q^{-2}) = \ldots = 0$, so that $\varphi$ is proportional to the eigenfunction (3.8).

3) The imaginary part $J$ has the following matrix representation with respect to the basis $\{e_N\}$:
\[ J = \frac{(q-1)^{1/2}}{2i} \begin{pmatrix} 0 & -q^{-1/2} & -q^{-1} & \cdots & -q^{-N/2} & \cdots \\ -q^{-1/2} & 0 & 0 & \cdots & 0 & \cdots \\ -q^{-1} & 0 & 0 & \cdots & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -q^{-N/2} & 0 & 0 & \cdots & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}. \tag{3.11} \]

It is easy to write an integral representation
\[ (Ju)(|x|_K) = \frac{1 - q}{2iq \log q} \int_{|y|_K \leq 1} (\log |x|_K - \log |y|_K)u(|y|_K) dy, \tag{3.12} \]
that is
\[ (Ju)(|x|_K) = \frac{1 - q}{2iq \log q} \langle u, 1 \rangle \log |x|_K - \langle u, \log |x|_K \rangle 1 \tag{3.13} \]
hence $J$ is a rank 2 operator. We see from (3.11) that $\text{tr} \, J = 0$.

4) The only solution in $\mathcal{H}$ (up to the multiplication by a constant) of the equation $I_0^1 u = 0$ is the eigenfunction $u_0$ given by (3.8). Suppose that $(I_0^1)^* u_0 = 0$. Then $J u_0 = 0$. However by (3.12),
\[ (Ju_0)(|x|_K) = \frac{1 - q}{2iq \log q} \log |x|_K \int_{|y|_K = 1} dy = -\frac{(q - 1)^2}{2iq^2 \log q} \log |x|_K, \]
so that $Ju_0 \not\equiv 0$, and we have come to a contradiction. This proves that $I_0^1$ is a simple Volterra operator. \qed
Let us calculate the action of $I_0^1$ upon the basis $\{f_n\}$ defined in (2.10). We find for $|x|_K = q^{-j}, j \geq 0$, that

$$
(I_0^1 f_n) (|x|_K) = -\frac{(1 - q^{-1})^{1/2}}{\log q} q^{n/2} \int_{|y|_K < q^{-j}, |y|_K = q^{-n}} (\log |x|_K - \log |y|_K) \, dy
$$

$$
= (1 - q^{-1})^{1/2} q^{n/2} (j - n) \int_{|y|_K < q^{-j}, |y|_K = q^{-n}} dy
$$

$$
= \begin{cases} 
(1 - q^{-1})^{3/2} q^{-n/2} (j - n), & \text{if } n > j; \\
0, & \text{if } n \leq j.
\end{cases}
$$

This implies the equality

$$
\langle I_0^1 f_n, f_j \rangle = 0 \text{ for } n \leq j,
$$

meaning that $\{f_n\}$ is a basis of triangular representation for the operator $I_0^1$.

**Remark.** The operator $I_0^1$ is $S$-real with respect to the involution $S$ in $\mathcal{H}$ given by the complex conjugation. Therefore it is $S$-unicellular ([5], Appendix, Theorem 5.5). It is not clear whether it is unicellular in the usual (complex) sense. However it is unicellular in a smaller space $\mathcal{H}^p$ defined as a completion of the set of all “polynomials” $\varphi(|x|_K) = \sum_{j=0}^N c_j |x|^j_K$ with respect to the norm $\|\varphi\| = \{\sum |c_j|^p\}^{1/p}, 1 \leq p < \infty$. By virtue of (3.6), $I_0^1$ acts on the space $\mathcal{H}^p$ (isomorphic to $l^p$) as a weighted shift, for which the unicellularity was proved by Yakubovich [20].

### 3.4. Characteristic Function

Following the notation in [6], let us write (3.13) in the form

$$
\frac{1}{i} \left( I_0^1 - (I_0^1)^* \right) u = \sum_{\alpha, \beta = 1}^2 \langle u, h_\alpha \rangle j_{\alpha \beta} h_\beta
$$

where $h_1(|x|_K) = \frac{q - 1}{iq \log q} (= \text{const}), \ h_2(|x|_K) = -\log |x|_K, x \in O, j = (0 1 1)$.

For the operator $I_0^1$, we consider the $2 \times 2$ characteristic matrix-function of inverse argument

$$
W(z^{-1}) = E + izj \left[ \left( (E - zI_0^1)^{-1} h_\alpha, h_\beta \right) \right]_{\alpha, \beta = 1}^2
$$

where $E$ denotes both the unit operator in $\mathcal{H}$ and the unit matrix.

For the Volterra operator $I_1^1, W(z^{-1})$ is an entire matrix-function.

**Theorem 3.** Matrix elements of $W(z^{-1})$ are entire functions of zero order.
Proof. For small values of $|z|$, the Fredholm resolvent $(E - zI_0^1)^{-1}$ is given by the Neumann series

$$(E - zI_0^1)^{-1} f = \sum_{n=0}^{\infty} (zI_0^1)^n f, \quad f \in \mathcal{H}.$$  

In order to calculate the characteristic function, we have to compute the functions $(I_0^1)^n 1$ and $(I_0^1)^n \log |\cdot|_K$. The first of them is obtained easily from (3.6):

$$(I_0^1)^n 1 (|x|_K) = c^n \prod_{m=0}^{n-1} d_m \cdot |x|_K^n, \quad |x|_K \leq 1,$$

where $c = \frac{1 - q}{q \log q}$, $0 < d_m \leq Aq^{-m}$. Summing the progression we find that

$$(E - zI_0^1)^{-1} 1 (|x|_K) = \sum_{n=0}^{\infty} \rho_n z^n |x|_K^n, \quad |\rho_n| \leq C^n q^{-n^2/2}, \quad (3.14)$$

where $C > 0$ is a constant.

Let us consider $(I_0^1)^n \log |\cdot|_K$. We have

$$(I_0^1 \log |\cdot|_K) (|x|_K) = c \int_{|y|_K < |x|_K} (\log |x|_K - \log |y|_K) \log |y|_K dy.$$  

Setting $y = xt$, $|t|_K < 1$, we obtain

$$(I_0^1 \log |\cdot|_K) (|x|_K) = -c |x|_K \int_{|t|_K < 1} \log |t|_K (\log |x|_K + \log |t|_K) dt$$

$$= -ca_0 |x|_K \log |x|_k - cb_0 |x|_K \overset{\text{def}}{=} \sigma_1 |x|_K \log |x|_K - \eta_1 |x|_K$$

where

$$a_0 = \int_{|t|_K < 1} \log |t|_K dt, \quad b_0 = \int_{|t|_K < 1} \log^2 |t|_K dt.$$  

A similar calculation yields the expression

$$(I_0^1 (|\cdot|_K \log |\cdot|_K)) (|x|_K) = -ca_1 |x|_K^2 \log |x|_k - cb_1 |x|_K^2$$

where

$$a_1 = \int_{|t|_K < 1} |t|_K \log |t|_K dt, \quad b_1 = \int_{|t|_K < 1} |t|_K \log^2 |t|_K dt.$$  

Together with (3.6), this implies the formula

$$\left((I_0^1)^2 \log |\cdot|_K\right) (|x|_K) = c^2 a_0 a_1 |x|_K^2 \log |x|_k + c^2 a_0 b_1 |x|_K^2 - cb_0 d_1 |x|_K^2$$

$$\overset{\text{def}}{=} \sigma_2 |x|_K^2 \log |x|_K + \eta_2 |x|_K^2.$$  

$$\sigma_2 |x|_K^2 \log |x|_K + \eta_2 |x|_K^2.$$
Introducing similar constants for the next iterations,
\[ a_n = \int_{|t|<1} |t|^n_K \log |t|_K dt, \quad b_n = \int_{|t|<1} |t|^n_K \log^2 |t|_K dt, \]
and noticing that \(|a_n|, |b_n| \leq Mq^{-n}\), we prove by induction that
\[ (I_0^1)^n \log |\cdot|_K = \sigma_n |x|^n_K \log |x|_K + \eta_n |x|^n_K \tag{3.15} \]
where \(|\sigma_n|, |\eta_n| \leq C^n q^{-1} q^{-2} \cdots q^{-n+1} = C^n q^{-n(n-1)/2}.\]

It follows from (3.15) that
\[ (E - zI_0^1)^{-1} \log |\cdot|_K (|x|_K) = \sum_{n=0}^{\infty} \sigma_n z^n |x|^n_K \log |x|_K + \sum_{n=0}^{\infty} \eta_n z^n |x|^n_K \]
where \(|\sigma_n|, |\eta_n| \leq C^n q^{-n^2/2}.

Now we can compute the matrix-function \(W(z^{-1})\). By (3.14),
\[ \langle (E - zI_0^1)^{-1} h_1, h_1 \rangle = \text{const} \cdot \sum_{n=0}^{\infty} \rho_n z^n \int _{|x|_K \leq 1} dx = \sum_{n=0}^{\infty} \gamma_n z^n \]
where \(|\gamma_n| \leq C^n q^{-n^2/2},\) so that this matrix element is an entire function of zero order. Other matrix elements are estimated similarly on the basis of (3.15), by inserting 1 as an upper bound of \(|x|_K\) and taking into account the convergence of the integrals of \(\log |x|_K\) and \(\log^2 |x|_K\).

4. The Laplace Type Transform

4.1. Definition and Properties

Our definition of a Laplace type transform is based on the function \(v_N\) given by (2.6). It is essential that \(v_N \in \mathcal{D}(K)\). As we know, \(D^\alpha v_N = q^{\alpha N} v_N (\alpha > 0)\).

Let \(\xi \in K, \ |\xi|_K = q^N\). Then for any \(x \in K, v_N(|x|_K) = v_0(|x\xi|_K),\)
\[ D^\alpha v_0(|x\xi|_K) = D^\alpha v_N(|x|_K) = q^{\alpha N} v_N(|x|_K) = |\xi|^\alpha_K v_0(|x|_K).\]

We call the function
\[ \widehat{\varphi}(|\xi|_K) = \int_K v_0(|x\xi|_K) \varphi(|x|_K) \, dx \]
the Laplace type transform of a radial function \(\varphi \in L^1_{\text{loc}}(K)\). By the dominated convergence theorem, \(\widehat{\varphi}\) is continuous, bounded, and \(\widehat{\varphi}(|\xi|_K) \to 0, \ |\xi|_K \to \infty.\)

As a simple computation shows, if \(\varphi(|x|_K) \equiv \text{const},\) then \(\widehat{\varphi}(|\xi|_K) \equiv 0.\)

The above calculations, together with the selfadjointness of \(D^\alpha\) in \(L^2(K)\), show that
\[ \widehat{D^\alpha \varphi}(|\xi|_K) = \int_K (D^\alpha v_0(|x\xi|_K))(|x|_K) \varphi(|x|_K) \, dx = |\xi|^\alpha_K \widehat{\varphi}(|\xi|_K), \quad \xi \in K.\]

**Theorem 4.** (uniqueness) If \(\widehat{\varphi}(|\xi|_K) \equiv 0,\) then \(\varphi(|x|_K) \equiv \text{const}.\)
Proof. By the definition,

\[ \tilde{\varphi}(|\xi|_K) = \int_{|x|_K \leq |\xi|_K^{-1}} \varphi(|x|_K) \, dx - \frac{1}{q-1} \int_{|x|_K = q|\xi|_K^{-1}} \varphi(|x|_K) \, dx. \]

Let \( |\xi|_K = q^n, n \in \mathbb{Z} \). Then

\[ \tilde{\varphi}(q^n) = (1 - \frac{1}{q}) \sum_{j=-\infty}^{-n} \varphi(q^j)q^j - \varphi(q^{-n+1})q^{-n}. \]

If we denote \( \tilde{\varphi}(q^n) = f_n \), then

\[ f_{n+1} = (1 - \frac{1}{q}) \sum_{j=-\infty}^{-n-1} \varphi(q^j)q^j - \varphi(q^{-n})q^{-n-1}, \]

so that

\[ f_n - f_{n+1} = q^{-n} \left[ \varphi(q^{-n}) - \varphi(q^{-n+1}) \right]. \quad (4.1) \]

If \( \tilde{\varphi}(q^n) = 0 \) for all \( n \), then, by (4.1), \( \varphi(q^{-n}) = \varphi(q^{-n+1}) \) for all \( n \), so that \( \varphi(q^{-n}) \equiv \text{const.} \)

The identity (4.1) is of some independent interest, and we formulate it as a corollary.

Corollary 1. For all \( n \in \mathbb{Z} \),

\[ \tilde{\varphi}(q^n) - \tilde{\varphi}(q^{n+1}) = q^{-n} \left[ \varphi(q^{-n}) - \varphi(q^{-n+1}) \right]. \quad (4.2) \]

Corollary 2. A function \( \varphi \) is (strictly) monotone, if and only if \( \tilde{\varphi} \) is (strictly) monotone.

4.2. Inversion Formula

Theorem 5. For each \( m = 1, 2, \ldots, \)

\[ \varphi(q^m) = \varphi(1) + \sum_{j=0}^{m-1} q^{-j} \left[ \tilde{\varphi}(q^{-j+1}) - \tilde{\varphi}(q^{-j}) \right], \quad (4.3) \]

\[ \varphi(q^{-m}) = \varphi(1) + \sum_{j=1}^{m} q^j \left[ \tilde{\varphi}(q^j) - \tilde{\varphi}(q^{j+1}) \right], \quad (4.4) \]

Proof. According to (4.2),

\[ \varphi(1) - \varphi(q) = \tilde{\varphi}(1) - \tilde{\varphi}(q), \]
\[ \varphi(q) - \varphi(q^2) = q^{-1} \left[ \tilde{\varphi}(q^{-1}) - \tilde{\varphi}(1) \right], \]
\[ \varphi(q^2) - \varphi(q^3) = q^{-2} \left[ \tilde{\varphi}(q^{-2}) - \tilde{\varphi}(q^{-1}) \right], \]

etc. Summing up the first \( m \) equalities we obtain (4.3).

Similarly, by (4.2),

\[ \varphi(q^{-1}) - \varphi(1) = q \left[ \tilde{\varphi}(q) - \tilde{\varphi}(q^2) \right], \]
\[ \varphi(q^{-2}) - \varphi(q^{-1}) = q^2 \left[ \tilde{\varphi}(q^2) - \tilde{\varphi}(q^3) \right], \]

etc, and the summation yields (4.4).
Acknowledgements
This work was funded in part under the research project “Markov evolutions in real and p-adic spaces” of the Dragomanov National Pedagogic University of Ukraine.

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Received: May 21, 2020.
Revised: September 28, 2020.