Non-extensive thermodynamics and stationary processes of localization

Leone Fronzoni\textsuperscript{1,2}, Paolo Grigolini\textsuperscript{3,4,5}, Simone Montangero\textsuperscript{1}

\textsuperscript{1}Dipartimento di Fisica dell’Università di Pisa and INFN, Via Buonarroti 2, 56127 Pisa, Italy
\textsuperscript{2}Centro interdisciplinare per lo studio dei Sistemi Complessi Via Bonanno 25B, Pisa, Italy
\textsuperscript{3}Dipartimento di Fisica dell’Università di Pisa, Piazza Torricelli 2, 56127 Pisa, Italy
\textsuperscript{4}Center for Nonlinear Science, University of North Texas, P.O. Box 305370, Denton, Texas 76203
\textsuperscript{5}Istituto di Biofisica del CNR, Via San Lorenzo 26, 56127 Pisa, Italy

(Received January 10, 2022)

We focus our attention on dynamical processes characterized by an entropic index $Q < 1$. According to the probabilistic arguments of Tsallis and Bukman \cite{Tsallis1996, Bukman1996} these processes are subdiffusional in nature. The non-extensive generalization of the Kolmogorov-Sinai entropy yielding the same entropic index implies the stationary condition. We note, on the other hand, that enforcing the stationary property on subdiffusion has the effect of producing a localization process occurring within a finite time scale. We thus conclude that the stationary dynamic processes with $Q < 1$ must undergo a localization process occurring at a finite time. We check the validity of this conclusion by means of a numerical treatment of the dynamics of the logistic map at the critical point.

05.45.+b,03.65.Sq,05.20.-y

I. INTRODUCTION

There is a growing interest on the subject of non-extensive thermodynamics introduced ten years ago by Tsallis \cite{Tsallis1988} by means of the entropy

$$S_q = \frac{1 - \sum_{i=1}^{W} p_i^q}{q - 1}. \quad (1)$$

Note that this entropy is characterized by the index $q$ whose departure from the conventional value $q = 1$ signals the thermodynamic effects of either long-range correlations in fractal dynamics \cite{Tsallis1988, Grigolini2017} or the non-local character of quantum mechanics \cite{Zurek1982}. The growing interest for Tsallis’ non-extensive entropy is testified by the exponentially growing list of publications on this hot issue \cite{Miyata2019}. Of remarkable interest for the subject of fractal dynamics is the discovery recently made in Refs. \cite{Fronzoni2019, Fronzoni2020} that the entropic index $q$ also determines the specific analytical form, more general than the exponential form, adopted by two trajectories moving from infinitely close initial conditions, to depart from one another.

This important result is based on the generalization of the Kolmogorov-Sinai (KS) entropy \cite{Kolmogorov1958, Sinai1970} and, consequently, of the important theorem of Pesin \cite{Pesin1976}. In Refs. \cite{Fronzoni2019, Fronzoni2020} it was shown that the adoption of the Tsallis entropy yields a natural generalization of the Kolmogorov-Sinai (KS) entropy \cite{Kolmogorov1958}. For the sake of simplicity, henceforth we shall be referring ourselves to this form of entropy as Kolmogorov-Sinai-Tsallis (KST) entropy. The adoption of the KST entropy yields a generalization of the Pesin theorem \cite{Kolmogorov1958, Sinai1970} and, with it, the important result:

$$\delta(t) = \lim_{\Delta y(0) \to 0} \frac{\Delta y(t)}{\Delta y(0)} = [1 + \lambda_q t(1 - q)]^{1/(1-q)}. \quad (2)$$

The symbol $\Delta y(t)$ denotes the distance at time $t$ between two trajectories departing from two close initial conditions. Note that this prescription for $q > 1$ implies a divergence at finite time, and, consequently, a faster than exponential departure of two initially very close trajectories from one to the other. The case $q < 1$ yields a power law dependence on time, which can be interpreted as being slower than the exponential departure. It is well known that the case $q = 1$ results in ordinary Brownian diffusion. In Ref. \cite{Fronzoni2019} it was shown that the case $q > 1$ leads to superdiffusion. Consequently, the case $q < 1$, with a departure of the trajectories slower than exponential, ought to result, in general, in subdiffusion. It has to be pointed out that this dynamical argument fits the theoretical prediction of Ref. \cite{Fronzoni2019}. The authors of this interesting paper assign an entropic index $q$ to a non-linear Fokker-Planck equation, and prove that the condition $q < 1$ generates a subdiffusion process. We are convinced that there is an intimate relation between dynamic and probabilistic approach \cite{Fronzoni2019, Fronzoni2020}: as a consequence of this conviction we are led to believe that also a dynamic process characterized by $q < 1$ must produce a form of subdiffusion.

As well known \cite{Miyata2019, Fronzoni2019}, the concept of KS entropy implies the assumption of an invariant distribution. This condition, which must be extended to the KST entropy, is, in turn, closely related to the stationary assumption recently adopted \cite{Fronzoni2019} to establish a dynamic derivation of the diffusion processes. We want to prove that this assumption is the key physical ingredient necessary to make the dynamic treatment of anomalous diffusion compatible with a treatment based on non-extensive thermodynamics. This has also the effect of making subdiffusion as well as superdiffusion \cite{Fronzoni2019} compatible with the extended thermodynamics of Tsallis.

On the other hand, in Section II we will show that the only form of subdiffusion compatible with the stationary condition is a localization process taking place with a finite time scale. For all these reasons we are led to make...
the conjecture that a generator of diffusion characterized by the entropic index \( q < 1 \) must result in a localization process, whose time scale is finite, and it is expected to tend to infinity with \( q \to 1 \).

We shall not produce a general proof of this conjecture. We shall limit ourselves to a numerical investigation of a dynamical generator with \( q < 1 \) and we shall show, using in fact a numerical approach, that it results in a localization process. The dynamic generator of diffusion studied in this paper is the logistic map. The rationale for this choice is that the logistic map is one of the most used prototypes to illustrate the transition to chaos [10]. On top of that, there are already some papers [3–5] devoted to studying the sensitivity to initial condition in the case of the logistic map, and claiming that the proper entropic index in this case is \( q < 1 \). Thus, to prove our conjecture that the stationary dynamics with \( q < 1 \) yield localization, we are only left with the problem of proving, with numerical calculations, that the diffusion process generated by the logistic map is in fact characterized by localization.

The paper is organized as follows. In Section II we review the dynamic approach to diffusion, a picture based on very simple mathematical arguments. We show that the condition that anomalous diffusion is slower than normal diffusion, and at the same time compatible with the stationary assumption, yields localization occurring within a finite time scale. Section III is devoted to discussing the time evolution of the KST entropy of the logistic map at the onset of chaos, and to the detection of the entropic index \( q < 1 \). In Section IV we check the theoretical conjecture that \( q < 1 \) yields localization with a finite time scale. We study also the localization breakdown caused by the control parameter approaching the condition of full chaos. Finally, Section V is devoted to concluding remarks.

## II. A DYNAMIC APPROACH TO DIFFUSION

The simplest dynamical approach to diffusion is established by the equation of motion

\[
\frac{dx}{dt} = \xi(t),
\]

where \( D(t) \) can be interpreted as being a time dependent diffusion coefficient defined by

\[
D(t) \equiv \langle \xi(t)^2 \rangle_{eq} \int_0^t \Phi_\xi(t')dt'
\]

and the symbol \( \Phi_\xi(t) \) denotes the autocorrelation function of the fluctuation \( \xi(t) \) defined by

\[
\Phi_\xi(t) \equiv \frac{\langle \xi(0)\xi(t) \rangle_{eq}}{\langle \xi^2 \rangle_{eq}}.
\]

Note that the stationary property implied by the one-time correlation function of Eq. (7) is generated by averaging over an equilibrium distribution, or invariant measure, of initial condition. Ordinary statistical mechanics is ensured by the condition that the correlation time \( \tau \) defined by

\[
\tau \equiv \int_0^t \Phi_\xi(t')dt'
\]

exits and is finite, namely, that the important condition

\[
0 < \tau < \infty
\]

is fulfilled.

The important condition of Eq. (10) can be violated in different ways. A first example is given by a physical condition described by the autocorrelation function \( \Phi_\xi(t) \) with the time asymptotic property

\[
\lim_{t \to \infty} \Phi_\xi(t) = \frac{\text{const}}{t^\beta}
\]

and

\[
0 < \beta < 1.
\]

This corresponds to the time dependent diffusion coefficient of Eq. (10) becoming infinite in the asymptotic time limit. Another way of violating this condition, on which we focus our attention in this paper, is when the time \( \tau \) of Eq. (8) vanishes. In this case, in the asymptotic time limit the time dependent diffusion coefficient of Eq. (10) vanishes, thereby implying that also the localization process might take place asymptotically in time.

The methods of nonequilibrium statistical mechanics can be extended to deal with processes violating the important condition of Eq. (10). Examples can be found in the recent literature, see for instance [18], and in interesting review papers [19,20]. Furthermore, as an effect of the increasing popularity of the non-extensive entropy proposed ten year ago by Tsallis [2], it is becoming clear that also the regime of anomalous diffusion can be given a thermodynamical as well as dynamical significance [13].

We aim at establishing a unified mechanical and thermodynamical picture ranging from normal to anomalous...
diffusion, namely, including both super and subdiffusion. For this ambitious task to be successfully completed it is necessary to turn our attention to the stationary property. First of all, we have to point out that the analysis made in Ref. [13] was limited to the case where the fluctuating variable $\xi(t)$ is dichotomous, and the map of Geisel and Thomae [22] is used to establish the distribution of sojourn times in one or the other of the velocity states [23]. This waiting time distribution, called $\psi(t)$, has the inverse power law structure:

$$\lim_{t \to \infty} \psi(t) = \frac{\text{const}}{t^\mu} \tag{12}$$

with

$$2 < \mu < 3. \tag{13}$$

The restriction to values of $\mu > 2$ is enforced by the need of establishing a contact with the stationary theory earlier developed. In fact, as recently pointed by Zaslavsky [24], in the specific case when the waiting time distribution has an inverse power behavior, the asymptotic time regime of the distribution of Poincaré recurrence time is dominated by the distribution of times of sojourn at the border between the chaotic sea and the stability islands.

On the other hand the Kac theorem [25] establishes that ergodicity implies that the distribution of Poincaré recurrence times is characterized by a finite first moment, thereby yielding $\mu > 2$. According to Ref. [23], $\psi(t)$ is proportional to the second time derivative of the correlation function $\Phi_\xi(t)$ of Eq.(14). Thus, $\mu = \beta + 2$ and the condition $\mu > 3$ implies $\beta > 1$. The values of $\mu > 3$, corresponding to a condition where also the second moment of $\psi(t)$ is finite, means that the ensuing diffusion process collapses into the attraction basin of the central limit theorem, and consequently into the domain of ordinary statistical mechanics. In conclusion, the stationary condition and our wish to study anomalous condition force us to consider the interval defined by Eq.(11).

The stationary condition involved in the dynamical theory of diffusion behind Eqs.(5) and (6) is shared by the mathematical foundation of the KS entropy, resting on the assumption that the dynamical system under study has an invariant measure [12]. This makes it possible to settle the apparent paradox associated with the use of thermodynamics along the lines established by Tsallis [18], we now establish an important property of subdiffusion: A dynamic process fulfilling the stationary condition yields localization within a finite time scale. We have seen that, in addition to superdiffusion, another way of breaking the condition for normal diffusion is given by the property

$$\lim_{t \to \infty} D(t) = 0. \tag{17}$$

Note that at $t = 0$ the diffusion coefficient $D(t)$ vanishes, due to its own definition. Thus we expect $D(t)$ to reach the maximum value $D_{\text{max}}$ at a finite time $\tau$, corresponding to the duration of the transition from microscopic dynamics to the “macroscopic” diffusion regime. Let us define $(t > \tau)$

$$R(t) \equiv \frac{D(t)}{D_{\text{max}}} \tag{18}$$

and the corresponding localization time:

$$T \equiv \int_0^\infty dt R(t). \tag{19}$$

The case $T = \infty$ corresponds to the asymptotic property of Eq.(14). This is a case of no interest for the discussion of this paper, focusing on the stationary condition. Note, however, that the stationary condition implies that use is made of the correlation function $\Phi_\xi(t)$. The asymptotic behavior of this correlation function can be derived from Eq.(4) by time differentiation. Let us also adopt the following subdiffusion condition

$$\lim_{t \to \infty} \langle x^2(t) \rangle = \text{const} \cdot t^{2-\beta}, \tag{20}$$

with

$$1 < \beta < 2. \tag{21}$$

This is the same subdiffusion regime as that of Eq.(14). The change of notation with the adoption of the new symbol $\beta$ is due to the fact that we want now to establish a connection with a stationary condition and with
the correlation function $\Phi_\xi(t)$. From Eqs. (4) and (3) we see that the asymptotic behavior of the correlation function $\Phi_\xi(t)$ is proportional to the second time derivative of $x^2(t)$ of Eq. (22). Thus we see that asymptotic limit of this correlation function is characterized by negative tail:

$$\lim_{t \to \infty} \Phi_\xi(t) = -\frac{\text{const}}{t^\beta}. \quad (22)$$

This negative tail would yield a negative contribution to the diffusion coefficient balancing, in part or totally, the positive contribution stemming from the short times, thus leading to either normal diffusion or localization at finite times. If we want to set the condition of anomalous diffusion with a vanishing diffusion coefficient, we are left with a localization process taking place with the finite time scale $T$.

In conclusion, the main result of this Section is that a dynamic process with $q < 1$ must involve localization with finite time scale. The demonstration proceeds as follows. First we assume that the property found by the authors of Ref. Ref. [21] that $q < 1$ implies subdiffusion is of general validity. The authors base their arguments on Fokker-Planck-like equations, and consequently on a probabilistic view, which is historically dictated by the need itself of understanding thermodynamics. We are convinced that a unifying perspective directly relating thermodynamics to dynamics is possible. On the basis of this conviction we make the conjecture that $q < 1$ implies subdiffusion even when a totally dynamics perspective is adopted. We note that the extension of KS entropy rests on the existence of an invariant distribution and consequently is compatible with a dynamic approach to diffusion resting on the stationary assumption. Finally, the adoption of the dynamic approach to diffusion shows that stationary subdiffusion means localization occurring within a finite time scale.

III. THERMODYNAMICS OF FRACTAL DYNAMICS

The authors of Ref. [27] expressed the Kolmogorov-Sinai-Tsallis (KST) entropy in terms of an average over the invariant distribution $p(x)$ by means of the following expression:

$$H_q(t) = [1 - k(q)] \int dx p(x)^q \delta(t,x)^{1-q} / (q - 1). \quad (23)$$

Note that the departure point of the theory leading to this interesting result is given by a repartition of the phase space of the system under study into cells of small size. This is mirrored by the fact that $k(q) = (2l)^{q-1}$, where $2l$ denotes the size of these cells. The function $\delta(t,x)$ is defined by Eq. (2) with the dependence on the initial condition $x$ made explicit.

It is worth noticing that the interesting result of ref. [27], as expressed by Eq. (23), fits the predictions of the more heuristic treatment of Refs. [8,9]. Let us illustrate that this aspect in detail. Let us assume that the function $\delta(t)$ of Eq. (2) is expressed by

$$\delta(t) = [1 + (1 - q)\lambda Q t^{1-(1-q)}]. \quad (24)$$

Note that we are using the capital letter $Q$ rather than $q$ on purpose. This is because we are assuming that that only one power law exists, that this is characterized by the power index $\beta = \frac{1}{1-(1-q)}$ and that the result $Q$ is the magic value that the analysis made in terms of the KST ought to discover. Plugging Eq. (24) into the KST entropy of Eq. (23) we immediately assess that the KST entropy become a linear function of time only when we assign to the mobile entropic index $q$ the magic value $q = Q$.

Note that the threshold of chaos is characterized by fractal dynamics. This means that system’s dynamics significantly depart from the condition of full chaos, where there is a nice connection between thermodynamics and mechanics can be established using the KS entropy. The non-extensive thermodynamics of Tsallis makes it possible to extend a thermodynamic perspective to the case of fractal dynamics. We want to point out again that the onset of thermodynamics is associated, in a full accordance with the traditional wisdom [24], to the presence of a time region where the generalized form of Kolmogorov entropy grows linearly in time.

Actually, as earlier pointed out by the authors of Refs. [29,30], in the real dynamic cases, the sensitivity to initial condition does not result in a simple expression as that of Eq. (24). Rather, a distribution of power indexes is found. Work in progress of our group aims at assessing which value of $Q$ will emerge from the search for linear increase of $H_q(t)$ of Eq. (23), if a proper average over this distribution of power indexes is made. Although a final conclusion has not yet been reached, we are convinced that the prediction $Q < 1$ of Refs. [8,9] is correct. Thus we can restate our conviction that the family of logistic-like maps studied in Section IV has to produce subdiffusion under the specific form of localization with a finite localization time.

IV. THE LOGISTIC MAP AND THE PROCESS OF DYNAMICAL LOCALIZATION WITH A FINITE TIME SCALE

We devote this Section to checking the conjecture that $Q < 1$ yields a localization process with a finite localization time. For the reasons explained in Section I, we use the logistic map as a diffusion generator. More precisely, we follow [8] and study the logistic-like family of maps:

$$y_{n+1} = 1 - \mu |y_n|^z, \quad (25)$$

with

$$z > 1; 0 < \mu \leq 2; n = 0, 1, 2; \ldots; y \in [-1, 1]. \quad (26)$$
First of all, we evaluate numerically the invariant distribution, a sample of which is illustrated in Fig. 1 for the case $z = 2, \mu = \mu_c(2) = 1.4011551 \ldots$. Then we define the fluctuation

$$\xi_n \equiv y_n - < y_n >_{eq}, \quad (27)$$

where $< y_n >_{eq}$ denotes an average over the invariant distribution. This makes it possible for us to identify the fluctuation of Eq.(27) with the diffusion generator of Eq.(8). In fact, it is evident that with $n \to \infty$ the discrete variable of Eq.(9) becomes virtually continuous as the variable $\xi$ of Eq.(8). Then we evaluate $x(t)$ using the discrete counterpart of Eq.(8) with different values of $z$, we determine $x(t)^2$ and we average it over the set of initial conditions that in the case $z = 2$ corresponds to the invariant distribution of Fig.1.

Actually, we do our calculations for different values of $z$, and for each $z$ we assign to $\mu$ the corresponding critical value $\mu_c(z)$. According to the authors of Ref [3] the best fitting to their numerical result is given by:

$$Q(z) = 1 - a_0/(z - 1)^{a_1}, \quad (28)$$

with $a_0 = 0.75$ and $a_1 = 0.60$. This means that with increasing $z$ the value of $Q$ tends to the value $Q = 1$, signalling the condition of standard statistical mechanics [3]. This implies that for $z \to \infty$, the diffusion process must tend to become ordinary Brownian diffusion again. Let us see how the time evolution of the second moment $M_2(t) = < x^2(t) >$ mirrors this earlier entropic analysis [3]. The time evolution of $M_2(t)$ is illustrated by Fig.2. We see that the time evolution of $M_2(t)$ shows a scenery distinctly different from that of ordinary statistical mechanics, corresponding to a function $M_2(t)$ increasing linearly in time. At $z = 2$ the function $M_2(t)$ immediately settles in a state characterized by wild fluctuations but not exceeding a finite upper value. We also see that increasing $z$ from $z = 2$ to $z = 100$ has the effect of resulting in a visible localization onset time. In fact in the case $z = 100$ it takes the system about 25 time steps to settle in another state characterized by wild fluctuations around a time independent mean value. Although the intensity of the fluctuations is larger than in the case $z = 2$, the transient nature of this transition to the localized state is clear. Exploring the time evolution of the logisticlike map of Eq.(23) for larger values of $z$ so as to make more extended the time necessary to produce localization involves some technical problems [32]. However, on the basis of the results illustrated in Fig.2 we make the conjecture that the effect of increasing $z$ must be that of making increasingly larger the localization time so that the condition of ordinary statistical mechanics $Q(\infty) = 1$ is recovered when the localization time becomes infinite.

We explored another path to full chaos, namely to the entropic condition characterized by $Q = 1$. This is obtained by keeping $z = 2$ and moving $\mu(2)$ from the critical value $\mu = \mu_c(2) = 1.4011551 \ldots$ to the value $\mu = 2$, in correspondence of which the logistic map is characterized by full chaos [16]. First of all we adopt the same numerical method as that used in Refs. [3] to establish the entropic index $Q$ corresponding to a given dynamical condition. We use this same method with a changing value of the control parameter $\mu$: we select some values of $\mu$ in the interval $[1.4011551 \ldots, 2]$ that do not correspond to the periodic windows. The behavior of $\delta(t)$ in these cases is illustrated by Fig.3 and the second moment of the diffusing distribution is illustrated by Fig.4.

These numerical results are somewhat unexpected. In fact, we would have expected the localization time to become increasingly larger till to diverge at the condition of full chaos. On the contrary, we see that there exists a localization state regime. This is of infinite time duration at the critical threshold $\mu = \mu_c$. As the control parameter $\mu$ exceeds this critical value the localization state become unstable. The closer and the closer the control parameter $\mu$ to the full chaos condition, the shorter and the shorter the lifetime of this localized state. In the literature of localization processes it is possible to find a discussion of the effect produced by the environmental fluctuations on the phenomenon of localization exhibited by the kicked rotator (see, for instance, Ref. [33]). This is a kind of localization breakdown taking place as soon as the interaction with environment is switched on. The diffusion coefficient of the ensuing process of ordinary diffusion is proportional to the square of the intensity of the coupling between system and environment. Here, on the contrary, we see that the localization breakdown occurs in time, at times of increasingly larger duration as the control parameter approaches the threshold value $\mu_c$.

In conclusion, it seems that the effect revealed by the numerical calculations of this paper is related to the conjecture of Ref. [27] that the entropic index $Q$ is time dependent. As a result of a proper average, made on the the crowding index distribution of Ref. [34], rather than on the coordinate $x$, we expect that Eq.(23) results in a linear in time entropy increase at a "magic" entropic value $Q$. This value, in turn, is not the same at all times. This is equivalent to saying that, in a sort of effective sense, the form of Eq.(24) is replaced by

$$\delta(t) = [1 + (1 - Q(t))\lambda_Q(t)]^{1/(1 - Q(t))}. \quad (29)$$

In accordance with the main result of this paper, let us assume that $Q(t) < 1$ means localization occurring within a finite time scale, and, as shown by Fig.2, almost instantaneously at $z = 2$. Then the results of Fig.4 can be immediately accounted for if we assign to $Q(t)$ the following time behavior: $Q(t)$ keeps a value smaller than the ordinary prescription $Q = 1$ for an extended time interval corresponding to the time duration of the localized states of Fig.4, then $Q(t)$ makes an abrupt jump to the traditional value $Q = 1$. Note that Fig.3, and other figures as well, concerning different values of system’s parameters and not shown here, exhibit an interesting property worth of a comment. The overall time behavior
of $\delta(t)$ looks like exponential in accordance with the fact that the localization breakdown implies that the standard form of sensitivity to the initial conditions is recovered. However, a more careful analysis of the numerical results shows that the exponential increase of $\delta(t)$ and the multifractal oscillations of Refs. [3–5] coexist in the same curve. The multifractal oscillations last for a finite amount of time at a given level, with a behavior very similar to that illustrated in Refs. [3–5]. At a given time a sort of abrupt transition to another level, with the same kind of multifractal oscillations, take place. The overall behavior looks like that exponential-like behavior of Fig.3. We think that the abrupt transition from a localized state to the regime of Brownian diffusion, shown in Fig.4, is a consequence of the coexistence of these two properties. Thus we make the following attractive conjecture: The occurrence of a localization process with a finite time scale, as a result of the stationary condition to the regime of Brownian diffusion, shown in Fig.4, is a consequence of the coexistence of these two properties. Thus we make the following attractive conjecture: The occurrence of a localization process with a finite time scale, as a result of the stationary condition to the regime of Brownian diffusion, shown in Fig.4, is a consequence of the coexistence of these two properties. Thus we make the following attractive conjec-
ture: The occurrence of a process of subdiffusion characterized by $Q < 1$ is expected to yield subdiffusion. On the other hand, according to the compelling results of Section III a process of subdiffusion generated by stationary fluctuations must lead to localization within a finite time scale. All these arguments lead us to make the plausible conjecture that dynamics with $Q < 1$ generate localization, and that the localization time is finite.

This is not a rigorous demonstration. However, the numerical results illustrated in Section IV fully support this conjecture and, in our opinion, should serve the useful purpose of stimulating a search for a more rigorous demonstration.

V. CONCLUDING REMARKS

The so called KST entropy shares with the conventional KS entropy the stationary condition, which is made evident by Eq.(23), resting in fact on an average over the invariant distribution. The findings of Ref. [21] refer to a probabilistic level of description, preceding the first attempts at establishing a dynamic foundation of non-extensive thermodynamics [24,25]. We make the plausible assumption that the unification of dynamics and thermodynamics [24,25] can be extended. As ambitious as this task is, this assumption is plausible, and the first results obtained [13] are very encouraging. Consequently, it is reasonable to make the conjecture that the results of Ref. [21] have a general validity and apply also to merely dynamic processes, described with no use of probabilistic ingredients. This means that a dynamic process characterized by $Q < 1$ is expected to yield subdiffusion. On the other hand, according to the compelling results of Section III a process of subdiffusion generated by stationary fluctuations must lead to localization within a finite time scale. All these arguments lead us to make the plausible conjecture that dynamics with $Q < 1$ generate localization, and that the localization time is finite.

This is not a rigorous demonstration. However, the numerical results illustrated in Section IV fully support this conjecture and, in our opinion, should serve the useful purpose of stimulating a search for a more rigorous demonstration.

[1] [http://tsallis.cat.cbpf.br/biblio.htm]
[2] C. Tsallis, J. Stat. Phys. 52, 479 (1988).
[3] M.L. Lyra and C. Tsallis, Phys. Rev. Lett. 80, 53 (1998).
[4] C. Tsallis, A.R. Plastino and W.-M. Zheng, Chaos, Solitons and Fractals 8, 885 (1997).
[5] U.M.S. Costa, M.L. Lyra, A.R. Plastino, C. Tsallis, Phys. Rev. E 56, 245 (1997).
[6] L. Palatella, P. Grigolini, quant-ph/9810083.
[7] S. Abe, A.K. Rajagopal, quant-ph/9904088.
[8] A.N. Kolmogorov, Dokl. Acad. Sci. USSR 119(5), 861 (1958).
[9] Ya.G. Sinai, Dokl. Acad. Sci. USSR 124(4), 768 (1959).
[10] Ya.B. Pesin, Russian Mathematical Surveys (4)32,55(1977).
[11] J.P. Eckmann and D. Ruelle, Rev. Mod. Phys. 57, 617 (1985).
[12] S.M. Pincus, Proc. Natl. Acad. Sci. USA 88, 2297 (1991).
[13] M. Buiatti, P. Grigolini, A. Montagnini, Phys. Rev. Lett. 82, 3383 (1999).
[14] C. Jarzynski, Phys. Rev. Lett. 74, 2937 (1995).
[15] M. Bianucci, R. Mannella, B.J. West, and P. Grigolini, Phys. Rev. E 51, 3002 (1995).
[16] B.-L. Hao, *Elementary Symbolic Dynamics* (World Scientific, Singapore, 1999) p. 361.
[17] R. Mannella, B.J. West, and P. Grigolini, Fractals 2, 81 (1994).
[18] G. Trefán, E. Floriani, B.J. West, and P. Grigolini, Phys. Rev. E 50, 2564 (1994).
[19] T. Geisel, Lecture Notes in Physics (Springer) 450, 153 (1995).
[20] J. Klafter, G. Zumofen and M.F. Shlesinger, Lecture Notes in Physics (Springer) 450, 183 (1995).
[21] C. Tsallis, D.J. Bukman, Phys. Rev. E 54, R2197 (1996).
[22] T. Geisel and S. Thomae, Phys. Rev. Lett. 52, 1936 (1984).
[23] P. Allegrini, P. Grigolini, and B.J. West, Phys. Rev. E 54, 4760 (1996).
[24] G. M. Zaslavsky, *Physics of Chaos in Hamiltonian Systems on the Foundations of Statistical Physics*, Imperial College Press, London (1988).
[25] Kac, *Probability and Related Topics in Physical Sciences* (Interscience, New York, 1958).
[26] R. Bettin, R. Mannella, B.J. West, P. Grigolini, Phys. Rev. E 51, 212 (1995).
[27] J. Yang, P. Grigolini, [cond-mat]/996291.
[28] C. Tsallis, Braz. J. Phys. 29, 1 (1999).
[29] G. Anania and A. Politi, Europhys. Lett. 7 (1988) 119.
[30] H. Hata, T. Horita and H. Mori, Prog. Theor. Phys. 82, 897 (1989).
[31] P.M. Izrailev, Phys. Rep. 196, 299 (1990).
[32] J.P. Van der Werle, H.W. Capel and R. Kluiving, Physica A 145, 425 (1987).
[33] T. Dittrich and R. Graham, Annals of Phys. 200, 363
[34] C. Beck and F. Schlögl, *Thermodynamics of chaotic systems* (Cambridge University Press, Cambridge, 1993)

FIG. 1. The invariant distribution of the logistic map of Eq. (25), with $z = 2$, at the critical value $\mu = 1.4011551$.

FIG. 2. The second moment $M_2 \equiv < x^2 >$ of the logistic map of Eq. (25) as a function of time. We show the case $z = 2$ (bottom curve) and the case $z = 100$ (upper curve). The bottom dashed line and the upper dotted line are eye guides illustrating that in both cases the second moment fluctuations take place around time independent mean values. The dot-dash line is an eye guide illustrating the existence of a finite time of transition to the localized state in the case $z = 100$.

FIG. 3. The function $\delta(t)$ of Eq. (2) of the logistic map of Eq. (25) as a function of time. We show the case $z = 2$ with $\mu = 1.45$ (bottom), $\mu = 1.55$ (middle) and the case $\mu = 1.95$ (upper curve).

FIG. 4. The second moment $< x^2 >$ of the logistic map of Eq. (25) as a function of time. We show the case $z = 2$ with $\mu = 1.45$ (bottom), $\mu = 1.55$ (middle) and the case $\mu = 1.95$ (upper curve).