NOTES ON FORMAL SMOOTHNESS

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Dedicated to Robert Wisbauer on the occasion of his 65th birthday

Abstract. The definition of an S-category is proposed by weakening the axioms of a Q-category introduced by Kontsevich and Rosenberg. Examples of Q- and S-categories and (co)smooth objects in such categories are given.

1. Introduction

In [12] Kontsevich and Rosenberg introduced the notion of a Q-category as a framework for developing non-commutative algebraic geometry. Relative to such a Q-category they introduced and studied the notion of a formally smooth object. Depending on the choice of Q-category this notion captures e.g. that of a smooth algebra of [16], which arose a considerable interest since its role in non-commutative geometry was revealed in [8].

The aim of these notes is to give a number of examples of Q-categories, and their weaker version which we term S-categories, of interest in module, coring and comodule theories, and to give examples of smooth objects in these Q-categories. Crucial to the definition of an S-category is the notion of a separable functor introduced in [14]. In these notes we consider only the separability of functors with adjoints. This case is fully described by the Rafael Theorem [15]: A functor which has a right (resp. left) adjoint is separable if and only if the unit (resp. counit) of adjunction is a natural section (resp. retraction). For a detailed discussion of separable functors we refer to [7].

Throughout these notes, by a category we mean a set-category (i.e. in which morphisms form sets), by functors we mean covariant functors. All rings are unital and associative.

Definition 2.1. An S-category is a pair of functors \( \mathcal{X} = ( \mathcal{X} \xrightarrow{u^*} \mathcal{X} ) \) such that \( u^* \) is separable and left adjoint of \( u_* \).

This means that in an S-category \( \mathcal{X} = ( \mathcal{X} \xrightarrow{u^*} \mathcal{X} ) \) the unit of adjunction \( \eta : \mathcal{X} \rightarrow u_*u^* \) has a natural retraction \( \nu : u_*u^* \rightarrow \mathcal{X} \). Therefore, for all objects \( x \) of \( \mathcal{X} \) and \( y \) of \( \mathcal{X} \), there exist morphisms

\[ \mathcal{X}(y, u^*(x)) \rightarrow \mathcal{X}(u_*(y), x), \quad g \mapsto \nu_x \circ u_*(g). \]
The notion of an S-category is a straightforward generalisation of that of a Q-category, introduced in [12]. The latter is defined as a pair of functors $X = (\bar{X} \xrightarrow{u_*} X)$ such that $u^*$ is full and faithful and left adjoint of $u_*$. In a Q-category the unit of adjunction $\eta$ is a natural isomorphism, hence, in particular, a section. Thus any Q-category is also an S-category. Following the Kontsevich-Rosenberg terminology (prompted by algebraic geometry) the functors $u_*$ and $u^*$ constituting an S-category are termed the direct image and inverse image functors, respectively.

**Definition 2.2.** We say that an S-category $X = (\bar{X} \xrightarrow{u_*} X)$ is supplemented if there exists a functor $u !_!: \bar{X} \to X$ and a natural transformation $\bar{\eta} : \bar{X} \to u^*u !$. In particular, an S-category $X = (\bar{X} \xrightarrow{u_*} X)$ is supplemented if $u^*$ has a left adjoint. Furthermore, $X$ is supplemented if the functor $u_*$ is separable, since, in this case, the counit of adjunction has a section which we can take for $\bar{\eta}$ (and $u !_! = u_*$). This supplemented S-category is termed a self-dual supplemented S-category.

In a supplemented S-category, for any $y \in \bar{X}$, there is a canonical morphism in $X$, natural in $y$,$$
 r_y : u_*(y) \to u ! (y),
$$
defined as a composition
$$r_y : u_*(y) \xrightarrow{u_*(\bar{\eta}_y)} u_*u^*u !_!(y) \xrightarrow{\nu ! (y)} u ! (y).$$
The existence of canonical morphisms $r_y$ allows us to make the following

**Definition 2.3.** Given a supplemented S-category $X = (\bar{X} \xrightarrow{u_*} X)$, with the natural map $r : u_* \to u !$, an object $x$ of $X$ is said to be:

(a) **formally $X$-smooth** if, for any $y \in \bar{X}$, the mapping $X(x, r_y)$ is surjective;

(b) **formally $X$-cosmooth** if, for any $y \in \bar{X}$, the mapping $X(r_y, x)$ is surjective.

**Remark 2.4.** We would like to stress that the notion of formal $X$-(co)smoothness is relative to the choice of the retraction of the unit of adjunction, and the choice of $u !_!$ and $\bar{\eta}$, since the definition of $r$ depends on all these data.

Dually to S- and Q-categories one defines $S^o$-categories and $Q^o$-categories.

**Definition 2.5.** An $S^o$-category (respectively $Q^o$-category) is a pair of functors $X = (\bar{X} \xrightarrow{u_*} X)$ such that $u^*$ is separable (resp. fully faithful) and right adjoint of $u_*$. Thus an adjoint pair of separable functors gives rise to a supplemented S- and $S^o$-category. In these notes (with a minor exception) we concentrate on S-categories.

### 3. Examples of Q- and S-categories

The following generic example of a Q-category was constructed by Kontsevich and Rosenberg in [12].
Example 3.1 (The Q-category of morphisms). Let $\mathcal{X}$ be any category, and let $\mathcal{X}^2$ be the category of morphisms in $\mathcal{X}$ defined as follows. The objects of $\mathcal{X}^2$ are morphisms $f$, $g$ in $\mathcal{X}$. Morphisms in $\mathcal{X}^2$ are commutative squares

\[
\begin{array}{c}
x \xrightarrow{f} y \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
x' \xrightarrow{g} y'
\end{array}
\]

where the vertical arrows are in $\mathcal{X}$. Now, set $\bar{\mathcal{X}} = \mathcal{X}^2$. The inverse image functor $u^*$ is

\[
u^* : x \mapsto \left( \begin{array}{ccc} x & \to & x \\ \downarrow & \downarrow & \downarrow \\ f & \downarrow & y \\
\end{array} \right), \quad \left( \begin{array}{ccc} x & \to & y \\ \downarrow & \downarrow & \downarrow \\ f & \downarrow & y \\
\end{array} \right) \mapsto \left( \begin{array}{ccc} x & \to & x \\ \downarrow & \downarrow & \downarrow \\ f & \downarrow & y \\
\end{array} \right).
\]

The direct image functor $u_*$ is defined by

\[
u_* : \left( \begin{array}{ccc} x & \to & y \\ \downarrow & \downarrow & \downarrow \\ f & \downarrow & y \\
\end{array} \right) \mapsto x, \quad \left( \begin{array}{ccc} x & \to & y \\ \downarrow & \downarrow & \downarrow \\ f & \downarrow & y \\
\end{array} \right) \mapsto \left( \begin{array}{ccc} x & \to & x \\ \downarrow & \downarrow & \downarrow \\ f & \downarrow & y \\
\end{array} \right).
\]

Note that, for all objects $x$ and morphisms $f$ in $\mathcal{X}$,

\[
u_* \nu^*(x) = \nu_* ( \begin{array}{ccc} x & \to & x \\ \downarrow & \downarrow & \downarrow \\ f & \downarrow & y \\
\end{array} ) = x, \quad \nu_* \nu^*(f) = f.
\]

Hence, for all objects $x$ in $\mathcal{X}$, there is an isomorphism (natural in $x$), $\eta_x : x \to \nu_* \nu^*(x)$, $\eta_x = x$.

Note further that for all objects $x \xrightarrow{f} y$ in $\mathcal{X}^2$, $\nu_* \nu^*(f) = x$, and we can define a morphism $\varepsilon_f : \nu^* \nu_*(f) \to f$ by

\[
\varepsilon_f = \left( \begin{array}{ccc} x & \to & x \\ \downarrow & \downarrow & \downarrow \\ \nu^* \nu_*(f) & \downarrow & \nu^* \nu_*(f) \\
\end{array} \right).
\]

In this way, $\nu_*$ is the right adjoint of $\nu^*$ with counit $\varepsilon$ and unit $\eta$. The unit is obviously a natural isomorphism, hence $\nu^*$ is full and faithful and, thus, a Q-category $\mathcal{X} = (\bar{\mathcal{X}} \xrightarrow{\nu_*} \mathcal{X})$ is constructed. $\mathcal{X}$ is supplemented, since $\nu^*$ has a left adjoint

\[
u_1 : \left( \begin{array}{ccc} x & \to & y \\ \downarrow & \downarrow & \downarrow \\ \nu_* \nu^*(f) & \downarrow & \nu_* \nu^*(f) \\
\end{array} \right) \mapsto y, \quad \left( \begin{array}{ccc} x & \to & y \\ \downarrow & \downarrow & \downarrow \\ \nu_* \nu^*(f) & \downarrow & \nu_* \nu^*(f) \\
\end{array} \right) \mapsto \left( \begin{array}{ccc} y & \to & y \\ \downarrow & \downarrow & \downarrow \\ \nu_* \nu^*(f) & \downarrow & \nu_* \nu^*(f) \\
\end{array} \right).
\]

The unit of the adjunction $\nu_1 \dashv \nu^*$ is, for all $f : x \to y$,

\[
\bar{\eta}_f = \left( \begin{array}{ccc} x & \to & y \\ \downarrow & \downarrow & \downarrow \\ \nu^* \nu_*(f) & \downarrow & \nu^* \nu_*(f) \\
\end{array} \right),
\]
and thus the corresponding maps \( r \) come out as

\[ r_f = f. \]

Consequently, an object \( x \in \mathcal{X} \) is formally \( \mathcal{X} \)-smooth (when \( \mathcal{X} \) is supplemented by \( u! \) and \( \bar{\eta} \)) provided, for all \( y \xrightarrow{f} z \in \mathcal{X} \), the mapping

\[ \mathcal{X}(x, y) \to \mathcal{X}(x, z), \quad g \mapsto f \circ g, \]

is surjective. Similarly, \( x \) is formally \( \mathcal{X} \)-cosmooth if and only if the mappings

\[ \mathcal{X}(z, x) \to \mathcal{X}(y, x), \quad g \mapsto g \circ f, \]

are surjective.

This generic example has a useful modification whereby one takes for \( \bar{\mathcal{X}} \) any full subcategory of \( \mathcal{X}^2 \) which contains all the identity morphisms in \( \mathcal{X} \).

**Example 3.2** (The Wisbauer Q-category). Let \( R \) by a ring and \( M \) be a left \( R \)-module. Following [17, Section 15] \( \sigma[M] \) denotes a full subcategory of the category \( R\mathcal{M} \) of left \( R \)-modules, consisting of objects subgenerated by \( M \). Since \( \sigma[M] \) is a full subcategory of \( R\mathcal{M} \), the inclusion functor

\[ u^* : \sigma[M] \to R\mathcal{M}, \]

is full and faithful. It also has the right adjoint, the trace functor (see [17, 45.11] or [5, 41.1]),

\[ u_* = T^M : R\mathcal{M} \to \sigma[M], \quad T^M(L) = \sum \{ f(N) \mid N \in \sigma[M], f \in \text{Hom}_R(M, L) \}. \]

Hence there is a Q-category \( \mathcal{X} = ( \bar{\mathcal{X}} \xrightarrow{u^*} \mathcal{X} ) \) with \( \mathcal{X} = \sigma[M] \) and \( \bar{\mathcal{X}} = R\mathcal{M} \).

All the remaining examples come from the theory of corings.

**Example 3.3** (Comodules of a locally projective coring). This is a special case of Example 3.2. Let \( (C, \Delta_C, \varepsilon_C) \) be an \( A \)-coring which is locally projective as a left \( A \)-module. Let \( \hat{R} = \mathcal{C} = \text{Hom}_{A-}(\mathcal{C}, A) \) be a left dual ring of \( C \) with the unit \( \varepsilon_C \) and product, for all \( r, s \in \hat{R} \),

\[ rs : C \xrightarrow{\Delta_C} C \otimes_A C \xrightarrow{C \otimes_A s} C \xrightarrow{r} A. \]

Take \( \mathcal{X} = \mathcal{M}^C \), the category of right \( C \)-comodules, and \( \bar{\mathcal{X}} = R\mathcal{M} \). Define a functor

\[ u^* : \mathcal{M}^C \to R\mathcal{M}, \quad M \mapsto M, \]

where right \( C \)-comodule \( M \) is given a left \( R \)-module structure by \( rm = \sum m_i r(m_i) \). Since \( \mathcal{C} \) is a locally projective left \( A \)-module, the functor \( u^* \) has a right adjoint, the rational functor (see [5, 20.1]),

\[ u_* = \text{Rat}_C : R\mathcal{M} \to \mathcal{M}^C, \quad \text{Rat}_C(M) = \{ n \in M \mid n \text{ is rational} \}, \]

where an element \( n \in M \) is said to be rational provided there exists \( \sum_i m_i \otimes_A c_i \in M \otimes_A C \) such that, for all \( r \in \hat{R} \), \( rm = \sum m_i r(c_i) \). Here, the left \( R \)-module \( M \) is seen as a right \( A \)-module via the anti-algebra map \( A \to \hat{R}, a \mapsto \varepsilon_C(-a) \).
Example 3.4 (Coseparable corings). Recall that an $A$-coring $(C, \Delta_C, \varepsilon_C)$ is said to be coseparable \cite{10} if there exists a $(C, \mathcal{C})$-bicomodule retraction of the coproduct $\Delta_C$. This is equivalent to the existence of a cointegral defined as an $(A, A)$-bimodule map $\delta : C \otimes A C \to A$ such that $\delta \circ \Delta_C = \varepsilon_C$, and

$$(C \otimes_{A} \delta) \circ (\Delta_C \otimes_{A} C) = (\delta \otimes_{A} C) \circ (C \otimes_{A} \Delta_C).$$

Furthermore, this is equivalent to the separability of the forgetful functor $(-)_A : \mathcal{M}_C \to \mathcal{M}_A$ \cite[Theorem 3.5]{11}. Since this forgetful functor is a left adjoint to $- \otimes_{A} C : \mathcal{M}_A \to \mathcal{M}_C$, a coseparable coring $C$ gives rise to an $S$-category $X$ with

$$X = \mathcal{M}_C, \quad \bar{X} = \mathcal{M}_A, \quad u^* = (-)_A, \quad u_* = -\otimes_{A} C.$$

This $S$-category is denoted by $X^C_\delta$. By \cite[Theorem 3.5]{11}, the retraction $\nu$ of the unit of the adjunction is given explicitly, for all $M \in \mathcal{M}_C$,

$$\nu_M : M \otimes_{A} C \to M, \quad m \otimes_{A} C \mapsto \sum m_{(0)} \delta(m_{(1)} \otimes_{A} C).$$

In general, $X^C_\delta$ need not to be supplemented. However, if there exists $e \in C^A := \{ c \in C \mid \forall a \in A, ac = ca \}$, then $X^C_\delta$ can be supplemented with

$$u_1 = -\otimes_{A} C, \quad \bar{\eta}_M : M \to M \otimes_{A} C, \quad m \mapsto m \otimes_{A} e.$$

This supplemented $S$-category is denoted by $X^C_{\delta, e}$.

Recall that an $A$-coring $C$ is said to be cosplit if there exists an $A$-central element $e \in C^A$ such that $\varepsilon_C(e) = 1$. By \cite[Theorem 3.3]{11} this is equivalent to the separability of the functor $- \otimes_{A} C$, and thus a cosplit corings gives rise to an $S^\circ$-category. Therefore, a coring which is both cosplit and coseparable induces a self-dual, supplemented $S$-category.

In addition to the defining adjunction of an $A$-coring, $(-)_A \dashv - \otimes_{A} C$, for any right $C$-comodule $P$, there is a pair of adjoint functors

$$- \otimes_{B} P : \mathcal{M}_B \to \mathcal{M}_C, \quad \text{Hom}_C(P, -) : \mathcal{M}_C \to \mathcal{M}_B,$$

where $B$ is any subring of the endomorphism ring $S = \text{End}_C(P)$ (cf. \cite[18.21]{9}). Depending on the choice of $C$, $P$ and $B$ this adjunction provides a number of examples of Q-categories.

Example 3.5 (Comatrix corings). Take a $(B, A)$-bimodule $P$ that is finitely generated and projective as a right $A$-module. Let $e \in P \otimes_{A} P^*$ be the dual basis (where $P^* = \text{Hom}_A(P, A)$), and let $C = P^* \otimes_{B} P$ be the comatrix coring associated to $P$ \cite{9}. The coproduct and counit in $C$ are given by

$$\Delta_C(\xi \otimes_{B} P) = \xi \otimes_{B} e \otimes_{B} P, \quad \varepsilon_C(\xi \otimes_{B} P) = \xi(P),$$

for all $p \in P$ and $\xi \in P^*$. $P$ is a right $C$-comodule with the coaction $\delta^P : p \mapsto e \otimes_{B} P$. Let

$$X = \mathcal{M}_B, \quad \bar{X} = \mathcal{M}_C, \quad u^* = -\otimes_{B} P, \quad u_* = \text{Hom}_C(P, -).$$

In view of \cite[Proposition 2.3]{6}, $X = (\bar{X} \xrightarrow{u_*} \bar{X})$ is a Q-category if and only if the map

$$B \to P \otimes_{A} P^*, \quad b \mapsto be,$$

is pure as a morphism of left $B$-modules (equivalently, $P$ is a totally faithful left $B$-module).
Example 3.6 (Strongly \((\mathcal{C}, A)\)-injective comodules). Let \(\mathcal{C}\) be an \(A\)-coring, let \(P\) be a right \(\mathcal{C}\)-comodule and \(S = \text{End}^\mathcal{C}(P)\). Following [18, 2.9], \(P\) is said to be strongly \((\mathcal{C}, A)\)-injective if the coaction \(\varrho^P : P \to P \otimes_A \mathcal{C}\) has a left \(S\)-module right \(\mathcal{C}\)-comodule retraction. For such a comodule, define

\[
\mathcal{X} = M_S, \quad \tilde{\mathcal{X}} = \mathcal{M}^\mathcal{C}, \quad u^* = - \otimes_S P, \quad u_* = \text{Hom}^\mathcal{C}(P, -).
\]

In view of [18, 3.2], if \(P\) is a finitely generated and projective as a right \(A\)-module, then \(\mathcal{X} = (\tilde{\mathcal{X}} \xrightarrow{u_*} \mathcal{X})\) is a \(Q\)-category.

Example 3.7 ((\(\mathcal{C}, A\))-injective Galois comodules). Recall that a right \(\mathcal{C}\)-comodule is said to be \((\mathcal{C}, A)\)-injective, provided there is a right \(\mathcal{C}\)-colinear retraction of the coaction. The full subcategory of \(\mathcal{M}^\mathcal{C}\) consisting of all \((\mathcal{C}, A)\)-injective comodules is denoted by \(\mathcal{IC}\).

Let \(P\) be a right comodule of an \(A\)-coring \(\mathcal{C}\), and let \(S = \text{End}^\mathcal{C}(P)\) and \(T = \text{End}_A(P)\). Following [18, 4.1], \(P\) is said to be a Galois comodule if, for all \(N \in \mathcal{IC}\), the evaluation map

\[
\text{Hom}^\mathcal{C}(P, N) \otimes_S P \to N, \quad f \otimes p \mapsto f(p),
\]

is an isomorphism of right \(\mathcal{C}\)-comodules.

Let \(P\) be a Galois comodule, and assume that the inclusion \(S \to T\) has a right \(S\)-module retraction. By [18, 4.3] this is equivalent to say that \(P\) is a \((\mathcal{C}, A)\)-injective comodule, and hence one can consider the following pair of categories and adjoint functors:

\[
\tilde{\mathcal{X}} = M_S, \quad \mathcal{X} = \mathcal{IC}, \quad u_* = - \otimes_S P : \tilde{\mathcal{X}} \to \mathcal{X}, \quad u^* = \text{Hom}^\mathcal{C}(P, -) : \mathcal{X} \to \tilde{\mathcal{X}}.
\]

Since the evaluation map is the counit of the adjunction \(u_* \dashv u^*\), the Galois property of \(P\) means that the functor \(u^*\) is fully faithful. Thus \(\mathcal{X} = (\tilde{\mathcal{X}} \xrightarrow{u_*} \mathcal{X})\) is a \(Q^0\)-category.

4. Examples of smooth and cosmooth objects

Let \(\mathcal{C}\) be an \(A\)-coring, set \(\mathcal{X} = \mathcal{M}^\mathcal{C}\), and consider the full subcategory of \(\mathcal{X}^2\) consisting of all monomorphisms in \(\mathcal{M}^\mathcal{C}\) with an \(A\)-module retraction. With these data one constructs a \(Q\)-category as in Example 3.1. This \(Q\)-category is denoted by \(\mathcal{XC}\).

Theorem 4.1. A right \(\mathcal{C}\)-comodule \(M\) is \((\mathcal{C}, A)\)-injective if and only if \(M\) is a formally \(\mathcal{XC}\)-cosmooth object.

Proof. In view of the discussion at the end of Example 3.1 an object \(M \in \mathcal{X} = \mathcal{M}^\mathcal{C}\) if formally \(\mathcal{XC}\)-cosmooth if and only if, for all morphisms \(f : N \to N'\) in \(\mathcal{M}^\mathcal{C}\) with right \(A\)-module retraction, the maps

\[
\vartheta_f : \text{Hom}^\mathcal{C}(N', M) \to \text{Hom}^\mathcal{C}(N, M), \quad g \mapsto g \circ f,
\]

are surjective. This means that, for all \(h \in \text{Hom}^\mathcal{C}(N, M)\), there is \(g \in \text{Hom}^\mathcal{C}(N', M)\) completing the following diagram

\[
\begin{array}{ccc}
M & \xrightarrow{f} & N' \\
\downarrow{h} & & \downarrow{f} \\
0 & \xrightarrow{0} & N
\end{array}
\]
where the arrow $N' \to N$ is in $\mathcal{M}_A$, and thus is equivalent to $M$ being $(C, A)$-injective, see [3, 18.18]. □

The arguments used in the proof of Theorem 4.1, in particular, the identification of (co)smooth objects as object with a (co)splitting property, apply to all $Q$-categories of the type described in Example 3.1. This leads to reinterpretation of smooth algebras and coalgebras in abelian monoidal categories studied in [3].

Example 4.2. Let $(V, \otimes)$ be an abelian monoidal category, i.e. a monoidal category which is abelian and such that the tensor functors $- \otimes v$, $v \otimes -$ are additive and right exact, for all objects $v$ of $V$. Let $\mathfrak{X}$ be the category of algebras in $V$, and let $\mathfrak{X}$ be a full subcategory of $\mathfrak{X}^2$, consisting of Hochschild algebra extensions, i.e. of all surjective algebra morphisms split as morphisms in $V$ and with a square-zero kernel. Denote the resulting $Q$-category by $\mathbb{HAE}$. In view of [3, Theorem 3.8], an algebra in $V$ is formally smooth in the sense of [3, Definition 3.9], i.e. it has the Hochschild dimension at most 1, if and only if it is a formally $\mathbb{HAE}$-smooth object.

In particular if $(V, \otimes)$ is the category of vector spaces (with the usual tensor product), we obtain the characterisation of smooth algebras [16] (or semi-free algebras in the sense of [8]), described in [12, Proposition 4.3].

Example 4.3. Let $(V, \otimes)$ be an abelian monoidal category. Let $\mathfrak{X}$ be the category of coalgebras in $V$, and let $\mathfrak{X}$ be a full subcategory of $\mathfrak{X}^2$, consisting of Hochschild coalgebra extensions, i.e. of all injective coalgebra morphisms $\sigma : C \to E$ split as morphisms in $V$ and with the property $(p \otimes p) \circ \Delta_E = 0$, where $p : E \to \text{coker} \sigma$ is the cokernel of $\sigma$. Denote the resulting $Q$-category by $\mathbb{HCE}$. In view of [3, Theorem 4.16], a coalgebra in $V$ is formally smooth in the sense of [3, Definition 4.17] if and only if it is a formally $\mathbb{HCE}$-cosmooth object.

The following example is taken from [2].

Example 4.4. Let $A$ and $B$ be rings, and let $M$ be a $(B, A)$-bimodule. Denote by $\mathcal{E}_M$ the class of all $(B, B)$-bilinear maps $f$ such that $\text{Hom}_B(M, f)$ splits as an $(A, B)$-bimodule map. A $B$-bimodule $P$ is said to be $\mathcal{E}_M$-projective, provided every morphism $N \to P$ in $\mathcal{E}_M$ has a section. By the argument dual to that in the proof of Theorem 4.1 one can reinterpret $\mathcal{E}_M$-projectivity as formal smoothness as follows.

Take $\mathfrak{X}$ to be the category of $B$-bimodules and $\mathfrak{X} = \mathcal{E}_M$, a full subcategory of $\mathfrak{X}^2$. Denote the resulting $Q$-category by $\mathbb{E}$. A $B$-bimodule $P$ is formally $\mathbb{E}$-smooth if and only if, for all $f : N \to N' \in \mathcal{E}_M$, the function

$$\Theta(f) : \text{Hom}_{B, B}(P, N) \to \text{Hom}_{B, B}(P, N'), \quad g \mapsto f \circ g,$$

is surjective. In terminology of [11, Chapter X], $\mathbb{E}$-smoothness of $P$ is equivalent to the $\mathcal{E}_M$-projectivity of $P$.

Thus $M$ is formally smooth if and only if $\text{ker } ev_M$ is formally $\mathcal{E}_M$-smooth.
Proposition 4.5. Let $C$ be a coseparable $A$-coring with a cointegral $\delta$ and an $A$-central element $e$, and let $\mathcal{K}_{b,e}^C$ be the associated supplemented $S$-category. A right $C$-comodule $M$ is formally $\mathcal{K}_{b,e}^C$-smooth if and only if the map 

$$\kappa_M : M \to M, \quad m \mapsto \sum m_{(0)} \delta(e \otimes_A m_{(1)}),$$

is a right $A$-linear section (i.e. $\kappa_M$ has a left inverse in End$_A(M)$).

Proof. In this case, for all $N \in \mathcal{M}_A$, the canonical morphisms $r_N$ read

$$r_N : N \otimes_A C \to N \otimes_A C, \quad n \otimes_A c \mapsto \sum n \otimes_A e(1) \delta(e(2) \otimes_A c).$$

Using the (defining adjunction) isomorphisms Hom$^C(M, N \otimes_A C) \cong $ Hom$_A(M, N)$, the maps

$$\text{Hom}^C(M, r_N) : \text{Hom}^C(M, N \otimes_A C) \to \text{Hom}^C(M, N \otimes_A C),$$

can be identified with

$$\vartheta_{M,N} : \text{Hom}_A(M, N) \to \text{Hom}_A(M, N), \quad f \mapsto (N \otimes_A e \circ r_N \circ (f \otimes_A C) \circ \varrho^M,$$

where $\varrho^M : M \to M \otimes_A C$ is the coaction. Hence Hom$^C(M, r_N)$ are surjective for all $N$ if and only if $\vartheta_{M,N}$ are surjective for all $N$. These can be computed further, for all $m \in M$, $f \in \text{Hom}_A(M, N)$,

$$\vartheta_{M,N}(f)(m) = (N \otimes_A e \circ r_N((f(m_{(0)})) \otimes_A m_{(1)}))$$

$$= (N \otimes_A e \circ (f(m_{(0)}) \otimes_A e(1) \delta(e(2) \otimes_A m_{(1)}))) = \sum f(m_{(0)}) \delta(e \otimes_A m_{(1)}))$$

$$= \sum f(m_{(0)}) \delta(e \otimes_A m_{(1)})) = f(\kappa_M(m)), $$

by the right $A$-linearity of $f$. Hence

$$\vartheta_{M,N}(f) = f \circ \kappa_M.$$ 

If $\kappa_M$ has a retraction $\lambda_M \in \text{End}_A(M)$, then for all $f \in \text{Hom}_A(M, N)$,

$$\vartheta_{M,N}(f \circ \lambda_M) = f \circ \lambda_M \circ \kappa_M = f,$$

i.e., the $\vartheta_{M,N}$ are surjective. If, on the other hand, all the $\vartheta_{M,N}$ are surjective, choose $N = M$ and take any $\lambda_M \in \vartheta_{M,M}^{-1}(M)$. Then

$$M = \vartheta_{M,M}(\lambda_M) = \lambda_M \circ \kappa_M,$$

so $\lambda_M$ is a retraction of $\kappa_M$ as required. □

Example 4.6 (Modules graded by $G$-sets). Let $G$ be a group, $X$ be a (right) $G$-set and let $A = \oplus_{\sigma \in G}$ be a $G$-graded $k$-algebra. Following [13], a $kX$-graded right $A$-module $M = \oplus_{x \in X} M_x$ is said to be graded by $G$-set $X$ provided, for all $x \in X, \sigma \in G$,

$$M_x A_\sigma \subseteq M_{x\sigma}.$$ 

A morphism of such modules is an $A$-linear map which preserves the $X$-grading. The resulting category is denoted by gr-$(G, A, X)$. It is shown in [7, Section 4.6] that gr-$(G, A, X)$ is isomorphic to the category of right comodules of the following coring $C$. As a left $A$-module $C = A \otimes kX$. The right $A$-multiplication is given by

$$(a \otimes x)a_\sigma = aa_\sigma \otimes x\sigma, \quad \forall a \in A, x \in X, a_\sigma \in A_\sigma.$$
The coproduct and counit are defined by
\[ \Delta_C(a \otimes x) = (a \otimes x) \otimes_A (1_A \otimes x), \quad \varepsilon_C(a \otimes x) = a. \]

An object \( M = \bigoplus_{x \in X} M_x \) in \( \text{gr}-(G, A, X) \) is a right \( C \)-comodule with the coaction \( q^M : M \to M \otimes_A C, m_x \mapsto m_x \otimes_A 1_A \otimes x \), where \( m_x \in M_x \). Also in [7, Section 4.6] it is shown that \( C \) is a coseparable coring with a cointegral (cf. [19, Proposition 2.5.3])
\[ \delta : C \otimes A C \simeq A \otimes kX \otimes kX \to A, \quad a \otimes x \otimes y \mapsto a \delta_{x,y}. \]

Thus \( \text{gr}-(G, A, X) \) gives rise to an \( S \)-category as in Example 3.4.

Let \( X^G := \{ x \in X \mid \forall \sigma \in G, x \sigma = x \} \) be the set of one-point orbits of \( G \) in \( X \). If \( X^G \neq \emptyset \), the above \( S \)-category can be supplemented as in Example 3.4 by
\[ e := 1_A \otimes z, \quad z \in X^G. \]

In this case, for any \( M \in \text{gr}-(G, A, X) \), the map \( \kappa_M \) in Proposition 4.5 comes out as
\[ \kappa_M(m_x) = m_x \delta_{x,z}, \quad \forall m_x \in M_x. \]

Thus a graded module \( M \in \text{gr}-(G, A, X) \) is formally \( C_{A,kX} \)-smooth if and only if it is concentrated in degree \( z \), i.e., \( M = M_z \).

Given an \( A \)-coring \( C \), the set of right \( A \)-module maps \( C \to A, C^* \), is a ring with the unit \( \varepsilon_C \) and the product, for all \( \xi, \xi' \in C^* \),
\[ \xi \xi' : C \xrightarrow{\Delta_C} C \otimes_A C \xrightarrow{\varepsilon_C \otimes A^C} C \xrightarrow{\xi} A. \]

**Proposition 4.7.** Let \( C \) be a coseparable \( A \)-coring with a cointegral \( \delta \) and an \( A \)-central element \( e \), and let \( C_{A,kX} \) be the associated supplemented \( S \)-category. Then the following statements are equivalent:

1. All right \( C \)-comodules are formally \( C_{A,kX} \)-cosmooth.
2. The right \( A \)-linear map
   \[ \lambda : C \to A, \quad \xi \mapsto \delta(e \otimes_A \xi), \]
   has a left inverse in the dual ring \( C^* \).
3. The regular right \( C \)-comodule \( C \) is formally \( C_{A,kX} \)-cosmooth.

**Proof.** Note that \( C^* \) can be identified with \( \text{End}^C(C) \) via the map \( \xi \mapsto (\xi \otimes_A C) \circ \Delta_C \) (with the inverse \( f \mapsto \varepsilon_C \circ f \)). Under this identification the product in \( C^* \) coincides with the composition in \( \text{End}^C(C) \). Hence (2) is equivalent to saying that the map \( r_A = (\lambda \otimes A^C) \circ \Delta_C \) has a retraction in \( \mathfrak{M}^C \). Denote this retraction by \( s_A \). Note further that, since \( \delta \) is a cointegral, the \( r_N \) defined in the proof of Proposition 4.5 can be written as \( r_N = N \otimes_AR_A \).

This implies that \( s_A \) is a section of \( r_A \) if and only if \( s_N = N \otimes A_{SA} \) is a retraction of \( r_N = N \otimes A_{RA} \), for all right \( A \)-modules \( N \). Finally observe that for all \( M \in \mathfrak{M}^C \) and \( N \in \mathfrak{M}_A \), the maps \( \varphi_{M,N} := \text{Hom}^C(r_N, M) \) come out explicitly as
\[ \varphi_{M,N} : \text{Hom}^C(N \otimes_A C, M) \ni f \mapsto f \circ r_N \in \text{Hom}^C(N \otimes_A C, M). \]

(2) \( \Rightarrow \) (1) The property \( s_N \circ r_N = N \otimes_A C \), implies that, for all right \( C \)-comodules \( M \) and right \( A \)-modules \( N \), the maps \( \varphi_{M,N} \) are surjective. Hence all right \( C \)-comodules are formally \( C_{A,kX} \)-cosmooth.

The implication (1) \( \Rightarrow \) (3) is obvious.
(3) ⇒ (2) If \( \mathcal{C} \) is formally \( X_{\delta,e}^C \)-cosmooth, then \( \varphi_{\mathcal{C},A} : \text{End}^C(\mathcal{C}) \to \text{End}^C(\mathcal{C}) \) is surjective. Hence there exists \( s_A \in \text{End}^C(\mathcal{C}) \) such that
\[
\mathcal{C} = \varphi_{\mathcal{C},A}(s_A) = s_A \circ r_A.
\]
This completes the proof. \( \square \)

A coseparable \( A \)-coring \( \mathcal{C} \) with a cointegral \( \delta \) is said to be Frobenius-coseparable if there exists \( e \in \mathcal{C}^A \) such that, for all \( c \in \mathcal{C} \), \( \delta(c \otimes_A e) = \delta(e \otimes_A c) = \varepsilon_C(c) \). The element \( e \) is called a Frobenius element. In particular a Frobenius-coseparable coring is a Frobenius coring, see [5, 27.5].

**Corollary 4.8.** Let \( \mathcal{C} \) be a Frobenius-coseparable \( A \)-coring with cointegral \( \delta \) and Frobenius element \( e \). Then any right \( \mathcal{C} \)-comodule is formally \( X_{\delta,e}^C \)-cosmooth and \( X_{\delta,e}^C \)-smooth.

**Proof.** The maps \( \kappa_M \) in Proposition 4.5 are all identity morphisms, hence they are sections and thus every right \( \mathcal{C} \)-comodule is formally \( X_{\delta,e}^C \)-smooth. The map \( \lambda \) in Proposition 4.7 coincides with the counit \( \varepsilon_C \). Since \( \varepsilon_C \) is a unit in \( \mathcal{C}^* \), it has a left inverse, and thus every right \( \mathcal{C} \)-comodule is formally \( X_{\delta,e}^C \)-cosmooth. \( \square \)

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NOTES ON FORMAL SMOOTHNESS

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