Abstract. We introduce an additive basis of the integral cohomology ring of the Peterson variety which reflects the geometry of certain subvarieties of the Peterson variety. We explain the positivity of the structure constants from a geometric viewpoint, and provide a manifestly positive combinatorial formula for them. We also prove that our basis coincides with the additive basis introduced by Harada-Tymoczko.

1. Introduction

Let \( n \) be a positive integer and \( Fl_n \) the full-flag variety of \( \mathbb{C}^n \). Namely, \( Fl_n \) is the collection of nested sequences of linear subspaces of \( \mathbb{C}^n \) given as follows:

\[
Fl_n = \{ V_\bullet = (V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}^n) \mid \dim_\mathbb{C} V_i = i \ (1 \leq i \leq n) \}.
\]

Let \( N \) be an \( n \times n \) regular nilpotent matrix viewed as a linear map \( N: \mathbb{C}^n \to \mathbb{C}^n \). The Peterson variety \( Pet_n \subseteq Fl_n \) is defined by

\[
Pet_n := \{ V_\bullet \in Fl_n \mid NV_i \subseteq V_{i+1} \ (1 \leq i \leq n-1) \},
\]

where \( NV_i \) denotes the image of \( V_i \) under the map \( N: \mathbb{C}^n \to \mathbb{C}^n \). It was introduced by Dale Peterson to study the quantum cohomology ring of \( Fl_n \), and it has been appeared in several contexts (e.g. [5, 8, 19, 25, 31]).

For a permutation \( w \in S_n \), let \( X_w \subseteq Fl_n \) be the Schubert variety associated with \( w \), and \( \Omega_w \subseteq Fl_n \) the dual Schubert variety associated with \( w \). We denote by \([n-1]\) the set of integers \( 1, 2, \ldots, n-1 \). For a subset \( J \subseteq [n-1] \), let \( w_J \) be the longest element of the Young subgroup \( \mathfrak{S}_J \) of the permutation group \( \mathfrak{S}_n \) associated with \( J \) (see Section 2.1 for details), and set

\[
X_J := X_{w_J} \cap Pet_n \quad \text{and} \quad \Omega_J := \Omega_{w_J} \cap Pet_n.
\]

Then \( X_J \) and \( \Omega_J \) in \( Pet_n \) play similar roles to that of Schubert varieties and dual Schubert varieties in \( Fl_n \), and provide important information on the topology of \( Pet_n \).

In this paper, we construct an additive basis \( \{ \varpi_J \mid J \subseteq [n-1] \} \) of the integral cohomology ring \( H^*(Pet_n; \mathbb{Z}) \) which reflects the geometry of \( X_J \) and \( \Omega_J \) (Theorem 4.14). As a consequence, we may consider the structure constants for the multiplication rule:

\[
\varpi_J \cdot \varpi_K = \sum_{L \subseteq [n-1]} d_{JK}^L \varpi_L, \quad d_{JK}^L \in \mathbb{Z}.
\]

(1.1)

It turns out that all \( d_{JK}^L \) are non-negative integers, and we give a geometric proof of this positivity (Proposition 4.16). We also provide a manifestly positive combinatorial formula for \( d_{JK}^L \) (Theorem 5.6) in terms of left-right diagrams, which we introduce in this paper.
To find our formula for $d_{JK}^L$, we prove several properties of the cohomology classes $\varpi_J$ which are inherited from the geometry of $X_J$ and $\Omega_J$. In particular, writing $\Omega_J$ as an intersection of divisors on $\text{Pet}_n$ provides the geometric idea behind our formula for $d_{JK}^L$ in terms of left-right diagrams.

We also show that our basis $\{\varpi_J \mid J \subseteq [n-1]\}$ coincides with the additive basis of the cohomology ring $H^*(\text{Pet}_n; \mathbb{C})$ with $\mathbb{C}$-coefficients introduced by Harada-Tymoczko ([20]). Their basis is obtained by taking restriction of certain Schubert classes to $\text{Pet}_n$, and it is called the Peterson Schubert basis. It has been studied by Bayegan-Harada ([9]), Drellich ([14]) and Goldin-Gorbutt ([18]). In [18], Goldin and Gorbutt gave combinatorial formulas for the structure constants of Harada-Tymoczko’s basis (in a certain equivariant setting) which are manifestly positive and integral. Thus, after taking the non-equivariant limit, their formulas and ours both describe the same structure constants, but these formulas have different perspectives; their approach is mostly combinatorial whereas our approach is based on the geometry of $X_J$ and $\Omega_J$. We include a short comparison of their formulas and ours in Section 6.

Interestingly, our computations match with those of Berget-Spink-Tseng ([10, Sect. 7]) on a certain subring of the cohomology ring of a toric variety which is called the permutohedral variety. One of their results can be interpreted as a formula describing the structure constants $d_{JK}^L$ as products of mixed Eulerian numbers which were introduced and studied by Postnikov ([30]). With this connection in mind, our formula (Theorem 5.6) for $d_{JK}^L$ can also be thought as computing some products of mixed Eulerian numbers by using the geometry of $\text{Pet}_n$. We explain this connection in Section 6 including the relations with the works of Nadeau-Tewari ([28]) and the second author ([21]).

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2. Basic notations

In this section, we recall some terminologies which will be used in this paper.

2.1. Combinatorics on the Dynkin diagram of type A. Let $n(\geq 2)$ be a positive integer. We use the notation $[n-1] := \{1, 2, \ldots, n-1\}$, and we regard it as the set of vertices of the Dynkin diagram of type $A_{n-1}$ for the rest of the paper. Namely, two vertices $i, j \in [n-1]$ are connected by an edge if and only if $|i-j| = 1$. See Figure 1.

![Figure 1. The Dynkin diagram of type A_{n-1}.](image-url)
The flag variety \( F \subseteq [n-1] \) as a full-subgraph of the Dynkin diagram appeared above. We may decompose it into the connected components:

\[
(2.1) \quad J = J_1 \sqcup J_2 \sqcup \cdots \sqcup J_m,
\]

where each \( J_k \) (\( 1 \leq k \leq m \)) is the set of vertices of a maximal connected subgraph of \( J \). To determine each \( J_k \) uniquely, we require that elements of \( J_k \) are less than elements of \( J_{k'} \) when \( k < k' \).

**Example 2.1.** Let \( n = 10 \) and \( J = \{1, 2, 4, 5, 6, 9\} \). Then we have

\[
J = \{1, 2\} \sqcup \{4, 5, 6\} \sqcup \{9\} = J_1 \sqcup J_2 \sqcup J_3.
\]

For \( J \subseteq [n-1] \), let us introduce a Young subgroup given by

\[
\mathfrak{S}_J := \mathfrak{S}_{J_1} \times \mathfrak{S}_{J_2} \times \cdots \mathfrak{S}_{J_m} \subseteq \mathfrak{S}_n,
\]

where each \( \mathfrak{S}_{J_k} \) is the subgroup of \( \mathfrak{S}_n \) generated by the simple reflections \( s_i \) for all \( i \in J_k \). Let \( w_J \) be the longest element of \( \mathfrak{S}_J \), i.e.,

\[
(2.2) \quad w_J := w_0^{(J_1)}w_0^{(J_2)}\cdots w_0^{(J_m)} \in \mathfrak{S}_J,
\]

where each \( w_0^{(J_k)} \) is the longest element of the permutation group \( \mathfrak{S}_{J_k} \) (\( 1 \leq k \leq m \)).

**Example 2.2.** Let \( n = 10 \) and \( J = \{1, 2\} \sqcup \{4, 5, 6\} \sqcup \{9\} \) as above. By identifying the permutation \( w_J \) with its permutation matrix, we have

\[
w_J = w_0^{(J_1)}w_0^{(J_2)}w_0^{(J_3)} = (s_1s_2s_1)(s_4s_5s_6s_4s_5s_4)(s_9) = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}.
\]

We can identify each permutation \( w \in \mathfrak{S}_n \) with its permutation flag \( V_* \in F_{Fl_n} \) defined by \( V_i = \langle e_{w(1)}, e_{w(2)}, \ldots, e_{w(i)} \rangle \) for \( 1 \leq i \leq n \), where \( e_1, e_2, \ldots, e_n \) denotes the standard basis of \( \mathbb{C}^n \), and the right hand side is the linear subspace of \( \mathbb{C}^n \) spanned by \( e_{w(1)}, e_{w(2)}, \ldots, e_{w(i)} \). Using this identification, we explain how the permutations \( w_J \) are related to the Peterson variety. Let \( GL_n(\mathbb{C}) \) be the general linear group of invertible \( n \times n \) complex matrices. Let \( T \subseteq GL_n(\mathbb{C}) \) be the maximal torus consisting of diagonal matrices. Let us identify \( \mathbb{C}^\times \) with a subgroup of \( T \) as follows:

\[
(2.3) \quad \mathbb{C}^\times = \left\{ \begin{pmatrix} g & g^2 & \cdots & g^n \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \end{pmatrix} \in T \mid g \in \mathbb{C}^\times \right\}.
\]

The flag variety \( Fl_n \) admits a natural \( GL_n(\mathbb{C}) \)-action by regarding each element \( g \in GL_n(\mathbb{C}) \) as an automorphism \( g: \mathbb{C}^n \to \mathbb{C}^n \). Restricting this \( GL_n(\mathbb{C}) \)-action on \( Fl_n \),
to the above subgroup \( \mathbb{C}^\times \), it is well-known that the fixed point set \((\mathcal{F}_n)^{\mathbb{C}^\times}\) is the set of the permutation flags, i.e. \((\mathcal{F}_n)^{\mathbb{C}^\times} = \mathfrak{S}_n\) (e.g. [16, The proof of Lemma 2 in Sect. 10.1]). It is straightforward to see that this \( \mathbb{C}^\times \)-action on \( \mathcal{F}_n \) preserves \( \text{Pet}_n \), and it was shown in [20] that the fixed point set \((\text{Pet}_n)^{\mathbb{C}^\times}\) is given by
\[
(\text{Pet}_n)^{\mathbb{C}^\times} = \{w_J \in \mathfrak{S}_n \mid J \subseteq [n-1]\}.
\]
Because of this relation, the combinatorics of \( w_J \) will be important to understand the structure of \( \text{Pet}_n \).

For \( 1 \leq i \leq n-1 \), let \( s_i \in \mathfrak{S}_n \) be the simple reflection which interchanges \( i \) and \( i+1 \). We denote by \( \leq \) the Bruhat order on \( \mathfrak{S}_n \), that is, we have \( u \leq v \) \((u,v \in \mathfrak{S}_n)\) if and only if a reduced expression of \( u \) is a subword of a reduced expression of \( v \).

**Lemma 2.3.** For \( J \subseteq [n-1] \) and \( 1 \leq i \leq n-1 \), we have \( s_i \leq w_J \) if and only if \( i \in J \).

**Proof.** Recall that \( w_J \) is the product of longest elements in \( \mathfrak{S}_{J_k} \) for \( 1 \leq k \leq m \):
\[
w_J = w_0^{(J_1)}w_0^{(J_2)}\cdots w_0^{(J_m)}.
\]
Since each \( w_0^{(J_k)} \) \((1 \leq k \leq m)\) preserves the decomposition (2.1), it follows that the length of \( w_J \) is the same as the sum of the length of \( w_0^{(J_k)} \) for \( 1 \leq k \leq m \). Thus, the products of reduced expressions of \( w_0^{(J_k)} \) for \( 1 \leq k \leq m \) give a reduced expression of \( w_J \). Here, an arbitrary reduced expression of \( w_0^{(J_k)} \) contains a simple reflection \( s_i \) if and only if \( i \in J_k \). Therefore, it follows that a reduced expression of \( w_J \) contains \( s_i \) if and only if \( i \in J \) (see Example 2.2). This implies the desired claim. \( \square \)

The following claim appears in [24, Lemma 6], but we give a proof for the reader.

**Lemma 2.4.** For \( J, J' \subseteq [n-1] \), we have
\[
w_{J'} \leq w_J \quad \text{if and only if} \quad J' \subseteq J.
\]

**Proof.** We first prove that \( J' \subseteq J \) under the assumption \( w_{J'} \leq w_J \). For this, take an arbitrary element \( i \in J' \). By the previous lemma, we have \( s_i \leq w_{J'} \). Combining this with the assumption \( w_{J'} \leq w_J \), we obtain that \( s_i \leq w_J \). Thus, it follows that \( i \in J \) by the previous lemma again.

We next prove that \( w_{J'} \leq w_J \) under the assumption \( J' \subseteq J \). Take the decomposition
\[
J' = J'_1 \sqcup J'_2 \sqcup \cdots \sqcup J'_{m'},
\]
into the connected components as in (2.1). Since each \( J'_\ell \) \((1 \leq \ell \leq m')\) is connected, it is contained in some connected component \( J_k \) of \( J \). This leads us to consider a map
\[
\varphi : \{1, 2, \ldots, m'\} \to \{1, 2, \ldots, m\}
\]
which we define by the conditions \( J'_i \subseteq J_{\varphi(i)} \) for \( 1 \leq i \leq m' \). Then we have that
\[
\bigsqcup_{\varphi(i) = k} J'_i \subseteq J_k
\]
by the definition of the map \( \varphi \). This implies that
\[
\prod_{\varphi(i) = k} w_0^{(J'_i)} \leq w_0^{(J_k)} \quad \text{in} \ \mathfrak{S}_{J_k}
\]
since $w^{(J_k)}_0$ is the longest permutation in $S_{J_k}$. Recalling that each $J_k$ is a connected component of $J$, these inequalities for $1 \leq k \leq m$ imply that $w_{J'} \leq w_J$. □

**Example 2.5.** Let $n = 8$, $J' = \{1, 4, 5, 7\}$ and $J = \{1, 2, 4, 5, 6, 7\}$ so that we have $J' \subseteq J$. In this case, we have

$$J' = \{1\} \sqcup \{4, 5\} \sqcup \{7\} = J'_1 \sqcup J'_2 \sqcup J'_3$$

and hence

$$w_{J'} = (s_1)(s_4s_5s_4)(s_7) \leq (s_1s_2s_1)(s_4s_5s_6s_7s_4s_5s_6s_4s_5s_4) = w_J.$$ 

**2.2. Hessenberg varieties.** In this subsection, we briefly recall the notion of Hessenberg varieties. They are (possibly reducible) subvarieties of the flag variety $Fl_n$, and will appear in the next section. A function $h: [n] \to [n]$ is a Hessenberg function if it satisfies the following two conditions:

(i) $h(1) \leq h(2) \leq \cdots \leq h(n)$,

(ii) $h(j) \geq j$ for all $j \in [n].$

Note that $h(n) = n$ by definition. We may identify a Hessenberg function $h$ with a configuration of (shaded) boxes on a square grid of size $n \times n$ which consists of boxes in the $i$-th row and the $j$-th column satisfying $i \leq h(j)$ for $i, j \in [n]$, as we illustrate in the following example.

**Example 2.6.** Let $n = 5$. The Hessenberg function $h: [5] \to [5]$ given by

$$(h(1), h(2), h(3), h(4), h(5)) = (2, 3, 3, 5, 5)$$

corresponds to the configuration of the shaded boxes drawn in Figure 2.

![Figure 2](image)

**Figure 2.** The configuration of the shaded boxes corresponding to $h$.

For an $n \times n$ matrix $X$ considered as a linear map $X: \mathbb{C}^n \to \mathbb{C}^n$ and a Hessenberg function $h: [n] \to [n]$, the Hessenberg variety associated with $X$ and $h ([12, 33])$ is defined as

$$\text{Hess}(X,h) = \{V \in Fl_n \mid XV_j \subseteq V_{h(j)} \text{ for all } j \in [n]\}.$$ 

Let $N$ be the $n \times n$ regular nilpotent matrix in Jordan canonical form, i.e.,

$$N = \begin{pmatrix}
0 & 1 & 0 & \cdots & \cdots \\
0 & 1 & \cdots & \cdots \\
& & \ddots & \ddots \\
\cdots & \cdots & \ddots & 0 & 1 \\
& & & & 0 & 1
\end{pmatrix}.$$ 

Then $\text{Hess}(N,h)$ is called a regular nilpotent Hessenberg variety. It is known that $\text{Hess}(N,h)$ is an irreducible projective variety of (complex) dimension $\sum_{i=1}^n (h(i) - i)$.
implies that we see that there exists a $C$-fixed point $x_{w_j} \in \text{Pet}_n$ by definition. In particular, we obtain the well-known formula
\[ \dim_{\mathbb{C}} \text{Pet}_n = n - 1. \]
For Hessenberg functions $h, h': [n] \to [n]$, it is clear that if $h(i) \leq h'(i)$ for $1 \leq i \leq n$ then $\text{Hess}(N, h) \subseteq \text{Hess}(N, h')$. For example, the Hessenberg function $h: [5] \to [5]$ given in Example 2.6 defines a 3-dimensional regular nilpotent Hessenberg variety $\text{Hess}(N, h)$ which is contained in $\text{Pet}_5(\subseteq \text{Fl}_5)$.

3. Geometric constructions

In this section, we introduce two kinds of subvarieties $X_J$ and $\Omega_J$ in $\text{Pet}_n$ for each $J \subseteq [n-1]$, and we establish geometric properties of them. They will play important roles to construct an additive basis of the integral cohomology ring $H^*(\text{Pet}_n; \mathbb{Z})$ in the next section.

3.1. Analogue of Schubert varieties in the Peterson variety. For $w \in \mathfrak{S}_n$, let $X_w \subseteq \text{Fl}_n$ be the Schubert variety associated with $w$ and $\Omega_w \subseteq \text{Fl}_n$ the dual Schubert variety associated with $w$ ([16, Sect. 10]). We note that $\dim_{\mathbb{C}} X_w = \text{codim}_{\mathbb{C}}(\Omega_w, \text{Fl}_n) = \ell(w)$, where $\ell(w)$ is the length of $w$.

**Definition 3.1.** For $J \subseteq [n-1]$, we define
\[ (3.1) \quad X_J := X_{w_J} \cap \text{Pet}_n \quad \text{and} \quad \Omega_J := \Omega_{w_J} \cap \text{Pet}_n, \]
where $w_J \in \mathfrak{S}_n$ is the permutation defined in (2.2).

Peterson ([29]) studied a particular open affine subset of $\Omega_J$ to construct the quantum cohomology ring of $\text{Fl}_n$ (cf. [25, 31]). Also, Insko ([23]) and Insko-Tymoczko ([24]) studied $X_J$ to show the injectivity of the homomorphism $H_n(\text{Pet}_n; \mathbb{Z}) \to H_n(\text{Fl}_n; \mathbb{Z})$.

It turns out that $X_J$ and $\Omega_J$ in $\text{Pet}_n$ play similar roles to that of Schubert varieties and dual Schubert varieties in $\text{Fl}_n$. As an illustrating property, we begin with the following claim. Recall that we have $X_w \cap \Omega_v = \emptyset$ in $\text{Fl}_n$ if and only if $w \geq v$.

**Proposition 3.2.** For $J, J' \subseteq [n-1]$, we have
\[ X_J \cap \Omega_{J'} \neq \emptyset \quad \text{if and only if} \quad J \supseteq J'. \]

Moreover, when $J = J'$, we have $X_J \cap \Omega_J = \{w_J\}$.

**Proof.** If $X_J \cap \Omega_{J'} \neq \emptyset$, then we have $(X_{w_J} \cap \Omega_{w_{J'}}) \cap \text{Pet}_n \neq \emptyset$ by definition. Note that $(X_{w_J} \cap \Omega_{w_{J'}}) \cap \text{Pet}_n$ is complete, and it is preserved by the $\mathbb{C}^*$-action on $\text{Pet}_n$ described in Section 2. Thus, it follows that it contains a $\mathbb{C}^*$-fixed point (e.g. [22, Chap. VIII, Sect. 21.2]). Since we have
\[ ((X_{w_J} \cap \Omega_{w_{J'}}) \cap \text{Pet}_n)^{\mathbb{C}^*} = (X_{w_J} \cap \Omega_{w_{J'}})^{\mathbb{C}^*} \cap (\text{Pet}_n)^{\mathbb{C}^*}, \]
we see that there exists a $\mathbb{C}^*$-fixed point $w_K \in \text{Pet}_n$ (see (2.4)) such that $w_K \in X_{w_J}$ and $w_K \in \Omega_{w_{J'}}$. The former condition implies that $w_J \geq w_K$, and the latter condition implies that $w_K \geq w_{J'}$ ([16, Sect. 10.2 and 10.5]). Thus, we obtain $w_J \geq w_{J'}$, and it follows that $J \supseteq J'$ from Lemma 2.4.

If $J \supseteq J'$, then we have $w_J \geq w_{J'}$ by Lemma 2.4. This implies that $w_J \in X_{w_J} \cap \Omega_{w_{J'}}$, and hence $X_J \cap \Omega_{J'} \neq \emptyset$ follows. \qed
A distinguished property of $X_J$ is that it is a regular nilpotent Hessenberg variety for a certain Hessenberg function. Let us explain this in the following. For each $J \subseteq [n-1]$, there is a natural Hessenberg function which is determined by $J$ as follows. Let $h_J : [n] \to [n]$ be a function given by

$$h_J(i) = \begin{cases} i + 1 & \text{if } i \in J \\ i & \text{if } i \notin J \end{cases} \tag{3.2}$$

for $1 \leq i \leq n$. Then $h_J$ is a Hessenberg function, and we have $\text{Hess}(N, h_J) \subseteq \text{Pet}_n$ since the Hessenberg function for $\text{Pet}_n$ is given by $h(i) = i + 1$ for $1 \leq i \leq n - 1$ as we saw in Section 2.2.

**Example 3.3.** Let $n = 10$ and $J = \{1, 2\} \cup \{4, 5, 6\} \cup \{9\} = J_1 \cup J_2 \cup J_3$. Then the configuration of boxes of $h_J$ is given in Figure 3. Compare the figure with the permutation matrix of $w_J$ in Example 2.2.

![Figure 3. The Hessenberg function $h_J$.](image)

As we mentioned above, Insko-Tymoczko ([24]) studied $X_J$, and they proved the most part of the following claim. Recall that $X_J$ is defined to be the intersection $X_{w_J} \cap \text{Pet}_n$ where $X_{w_J}$ is the Schubert variety associated with $w_J$.

**Proposition 3.4.** For $J \subseteq [n-1]$, we have

$$X_J = X_{w_J} \cap \text{Pet}_n = \text{Hess}(N, h_J) \tag{3.3}$$

where $N$ is the regular nilpotent matrix given in (2.7), and $X_{w,J}$ is the Schubert cell associated with $w_J$. In particular, we have $\dim \mathbb{C} X_J = |J|$.

To prove this, we need the following lemma. Let $X_w^o \subseteq Fl_n$ be the Schubert cell associated with $w$ and and $\Omega_w^o \subseteq Fl_n$ the dual Schubert cell associated with $w$.

**Lemma 3.5.** The following are equivalent.

1. $X_w^o \cap \text{Pet}_n \neq \emptyset$
2. $\Omega_w^o \cap \text{Pet}_n \neq \emptyset$
3. $w \in \text{Pet}_n$ (i.e. $w = w_J$ for some $J \subseteq [n-1]$)

**Proof.** It is clear that (3) implies (1). To see that (1) implies (3), take an element $z \in X_w^o \cap \text{Pet}_n \neq \emptyset$. Since $X_w^o \cap \text{Pet}_n \subseteq X_w^o$ is preserved under the $\mathbb{C}^\times$-action on $X_w^o$,
it follows that \( t \cdot z \in X^o_w \cap \text{Pet}_n \) for all \( t \in \mathbb{C}^\times \). Noticing that \( X^o_w \cap \text{Pet}_n \subseteq X^o_w \) is a closed subset, we have
\[
(3.4) \quad \lim_{t \to 0} t \cdot z \in X^o_w \cap \text{Pet}_n.
\]
Under the standard identification \( X^o_w = \mathbb{C}^{l(w)} \) (cf. [16, Sect. 10.2]), the \( \mathbb{C}^\times \)-action on \( X^o_w \) is identified with a linear action with positive weights. Thus it follows that \( \lim_{t \to 0} t \cdot z = 0 \) which corresponds to \( w \in X^o_w \) (cf. the proof of Lemma 5 in [24]). Thus it follows that \( w \in \text{Pet}_n \) by (3.4). The equivalence of (2) and (3) follows by an argument similar to that for the equivalence of (1) and (3). \( \square \)

**Proof of Proposition 3.4.** Let us first prove that
\[
(3.5) \quad X^o_{w_j} \cap \text{Pet}_n \subseteq \text{Hess}(N, h_j) \quad \text{for each } J \subseteq [n-1].
\]
For this, take an arbitrary element \( V_\bullet \in X^o_{w_j} \cap \text{Pet}_n \). Then we have
\[
NV_i \subseteq V_{i+1} \quad (1 \leq i \leq n-1).
\]
To see that \( V_\bullet \in \text{Hess}(N, h_j) \), we need to show that
\[
(3.6) \quad NV_p \subseteq V_p \quad \text{for } p \notin J.
\]
Since we are assuming \( V_\bullet \in X^o_{w_j} \cap \text{Pet}_n \), the flag \( V_p \) lies in the Schubert cell \( X^o_{w_j} \). Here, the permutation \( w_j \) is a product of longest permutations of the symmetric group of smaller ranks as given in (2.2). Thus it follows (from e.g. [16, Sect. 10.2]) that
\[
V_p = \langle e_1, e_2, \ldots, e_p \rangle \quad \text{for } p \notin J,
\]
where \( e_1, e_2, \ldots, e_n \) are the standard basis of \( \mathbb{C}^n \). Since \( Ne_1 = 0 \) and \( Ne_i = e_{i-1} \) for \( 2 \leq i \leq n \), it is clear that (3.6) follows. Thus we obtain (3.5).

Now, let us prove the claim (3.3) of this proposition. Since we have \( X_{w_j} = \bigcup_{v \leq w_j} X^o_v \), it follows that
\[
X_J = X_{w_j} \cap \text{Pet}_n = \bigcup_{v \leq w_j} (X^o_v \cap \text{Pet}_n) = \bigcup_{J' \subseteq J} (X^o_{w_{j'}} \cap \text{Pet}_n),
\]
where the last equality follows from Lemmas 2.4 and 3.5. For each intersection \( X^o_{w_{j'}} \cap \text{Pet}_n \) in the right-most side, we have that \( X^o_{w_{j'}} \cap \text{Pet}_n \subseteq \text{Hess}(N, h_{j'}) \) by (3.5). The condition \( J' \subseteq J \) implies that \( h_{j'}(i) \leq h_j(i) \) for \( 1 \leq i \leq n \), and hence we have \( \text{Hess}(N, h_{j'}) \subseteq \text{Hess}(N, h_j) \). Combining this with the previous inclusion, we see that
\[
X^o_{w_{j'}} \cap \text{Pet}_n \subseteq \text{Hess}(N, h_j)
\]
in (3.7). Thus it follows that
\[
(3.8) \quad X_J \subseteq \text{Hess}(N, h_j).
\]
Note that \( X^o_{w_j} \cap \text{Pet}_n \subseteq X_J (= X_{w_j} \cap \text{Pet}_n) \) by definition, and hence we have that \( X^o_{w_j} \cap \text{Pet}_n \subseteq X_J \) by taking the closure. Combining this with (3.8), we obtain that
\[
(3.9) \quad \overline{X^o_{w_j} \cap \text{Pet}_n} \subseteq X_J \subseteq \text{Hess}(N, h_j).
\]
In this sequence, the both sides have the same dimension. This is because we have \( \dim_{\mathbb{C}} X^o_{w_j} \cap \text{Pet}_n = |J| \) from [24, Lemma 9] and \( \dim_{\mathbb{C}} \text{Hess}(N, h_j) = |J| \) from [7, Lemma 7.1]. Since \( \text{Hess}(N, h_j) \) is irreducible, the two inclusions in (3.9) are equalities. This completes the proof. \( \square \)
Combining Proposition 3.4 and a result of Drellich [13, Theorem 4.5], we may express $X_J$ as a product of Peterson varieties of smaller ranks as follows. Let $J = J_1 \sqcup J_2 \sqcup \cdots \sqcup J_m$ be the decomposition into the connected components. Then, by definition, we have $w_J = w_0^{(J_1)} w_0^{(J_2)} \cdots w_0^{(J_m)}$, where each $w_0^{(J_k)}$ is a product of longest elements in $\mathfrak{S}_{J_k}$ for $1 \leq k \leq m$. Hence it follows that the Schubert variety $X_{w_J} \subseteq Fl_n$ associated with $w_J$ is isomorphic to the product of the flag varieties of smaller ranks:

$$X_{w_J} = \prod_{k=1}^m X_{w_0^{(J_k)}} \cong \prod_{k=1}^m Fl_{n_k},$$

where we set

$$n_k := |J_k| + 1 \quad \text{for } 1 \leq k \leq m.$$

By restricting this isomorphism to $X_J = X_{w_J} \cap Pet_n$, it follows from Proposition 3.4 and [13, Theorem 4.5] that $X_J$ is isomorphic to a product of Peterson varieties of smaller ranks.

**Corollary 3.6.** For $J \subseteq [n-1]$, we have

$$X_J = \prod_{k=1}^m X_{J_k} \cong \prod_{k=1}^m Pet_{n_k},$$

where $J = J_1 \sqcup J_2 \sqcup \cdots \sqcup J_m$ is the decomposition into the connected components and $n_k = |J_k| + 1$ ($1 \leq k \leq m$).

**Example 3.7.** Let $n = 10$ and $J = \{1, 2\} \sqcup \{4, 5, 6\} \sqcup \{9\} = J_1 \sqcup J_2 \sqcup J_3$. The representation matrix of $w_J$ is given in Example 2.2, and we have

$$X_{w_J} \cong Fl_3 \times Fl_4 \times Fl_2,$$

$$X_J \cong Pet_3 \times Pet_4 \times Pet_2.$$

Compared to Schubert varieties and dual Schubert varieties in $Fl_n$, the structures of $X_J$ and $\Omega_J$ in $Pet_n$ are rather simple as we explain below. To begin with, we make the following definition.

**Definition 3.8.** For $1 \leq i \leq n - 1$, let

$$D_i := X_{[n-1]\setminus\{i\}} \quad \text{and} \quad E_i := \Omega_{\{i\}}.$$ (3.10)

where $X_{[n-1]\setminus\{i\}}$ and $\Omega_{\{i\}}$ are defined in (3.1).

**Lemma 3.9.** For $1 \leq i \leq n - 1$, the following hold.

1. $D_i$ and $E_i$ have codimension 1 in $Pet_n$.
2. $D_i \cap E_i = \emptyset$.

**Proof.** For (1), we have $\dim \mathbb{C} D_i = \dim \mathbb{C} X_{[n-1]\setminus\{i\}} = n - 2$ by Proposition 3.4. We also have

$$E_i = \Omega_{\{i\}} = \Omega_{w_{\{i\}}} \cap Pet_n = \Omega_{s_i} \cap Pet_n.$$

It is well-known that $\Omega_{s_i}$ is irreducible and it has complex codimension 1 in $Fl_n$ ([16, Sect. 10.2]). Hence $\Omega_{s_i}$ in $Fl_n$ is locally cut out by a single function. We also know that $\Omega_{s_i} \cap Pet_n$ is a non-empty proper subset of $Pet_n$ since we have $s_i \in \Omega_{s_i} \cap Pet_n$ and $id = w_0 \in Pet_n \setminus \Omega_{s_i}$. Thus, it follows that $\dim \mathbb{C} E_i = n - 2$. For (2), the claim follows from Proposition 3.2. \qed
In the next subsection, we will see that $D_i$ and $E_i$ are divisors\footnote{In this paper, a divisor on a variety $Y$ always means the support of an effective Cartier divisor on $Y$, i.e., the zero locus of a section of a line bundle over $Y$.} on $Pet_n$. The following claim means that $X_J$ and $\Omega_J$ can be described as intersections of divisors on $Pet_n$.

**Proposition 3.10.** For $J \subseteq [n-1]$, we have

\[(3.11) \quad X_J = \bigcap_{i \notin J} D_i \quad \text{and} \quad \Omega_J = \bigcap_{i \in J} E_i.\]

**Proof.** By (3.7), we have

\[
X_{w_{[n-1]\setminus\{i\}}} \cap Pet_n = \bigcup_{J' \subseteq [n-1]\setminus\{i\}} (X_{w_{J'}}^\circ \cap Pet_n).
\]

This implies from the definition of $D_i$ that

\[
\bigcap_{i \notin J} D_i = \bigcap_{i \notin J} (X_{w_{[n-1]\setminus\{i\}}} \cap Pet_n) = \bigcup_{J' \subseteq J} (X_{w_{J'}}^\circ \cap Pet_n).
\]

Combining this with (3.7), we obtain the desired claim for $X_J$. An argument similar to this proves that

\[
\bigcap_{i \in J} E_i = \bigcap_{i \in J} (\Omega_{w_{\{i\}}} \cap Pet_n) = \bigcup_{J'' \supseteq J} (\Omega_{w_{J'' \setminus \{i\}}} \cap Pet_n) = \Omega_{w_J \cap Pet_n} = \Omega_J
\]

by Lemmas 2.4 and 3.5. \hfill \Box

**Example 3.11.** Let $n = 9$ and $J = \{2, 3, 4\} \sqcup \{7, 8\}$ so that $[n-1] \setminus J = \{1\} \sqcup \{5, 6\}$. Then we have

\[
X_J = D_1 \cap D_5 \cap D_6 \quad \text{and} \quad \Omega_J = E_2 \cap E_3 \cap E_4 \cap E_7 \cap E_8.
\]

### 3.2. Defining equations of $X_J$ and $\Omega_J$

Let $B \subseteq GL_n(\mathbb{C})$ be the Borel subgroup of upper-triangular matrices in $GL_n(\mathbb{C})$. Then we have the standard identification $Fl_n = GL_n(\mathbb{C})/B$ as is well known. Recall that $GL_n(\mathbb{C}) \to GL_n(\mathbb{C})/B (= Fl_n)$ is a principal $B$-bundle. Thus, for a complex $B$-representation space $V$, we have the associated complex vector bundle\footnote{We take the $B$-action on the product $GL_n(\mathbb{C}) \times V$ so that $[g, v] = [gb, b^{-1} \cdot v]$ in the quotient.} over $GL_n(\mathbb{C})/B$:

\[
GL_n(\mathbb{C}) \times^B V \to GL_n(\mathbb{C})/B \quad ; \quad [g, v] \mapsto gB.
\]

For a weight $\mu: T \to \mathbb{C}^\times$, we obtain $\mu: B \to \mathbb{C}^\times$ by composing that with the canonical projection $B \to T$, which we also denote by the same symbol. We denote by $\mathbb{C}_\mu = \mathbb{C}$ the corresponding 1-dimensional representation space of $B$. Set

\[(3.12) \quad L_\mu = GL_n(\mathbb{C}) \times^B \mathbb{C}_\mu^* = GL_n(\mathbb{C}) \times^B \mathbb{C}_{-\mu},\]

where $\mathbb{C}_\mu^*$ is the dual representation space of $\mathbb{C}_\mu$. We also denote the restriction $L_\mu|_{Pet_n}$ by the same symbol $L_\mu$ when there are no confusion.

Let us introduce two representations of $B$ associated with each $J \subseteq [n-1]$ as follows. For $1 \leq i \leq n-1$, let $\omega_i: T \to \mathbb{C}^\times$ be the $i$-th fundamental weight of $T$ given by
diag(t₁, t₂, ..., tₙ) \mapsto t₁t₂ \cdots tᵢ. For J \subseteq [n − 1], we obtain a representation space of T given by a direct sum

$$\bigoplus_{i \in J} \mathbb{C}^{α_i}.$$  

Through the canonical projection B \to T, we regard this as a representation of B. To introduce the other representation of B associated with J, let αᵢ (1 ≤ i ≤ n − 1) be the i-th simple root defined as a weight αᵢ: T \to \mathbb{C}^× given by diag(t₁, t₂, ..., tₙ) \mapsto tᵢtᵢ⁻¹. Let H_J \subseteq \mathfrak{g}l_n(\mathbb{C}) be the Hessenberg subspace (cf. [33, Sect. 2]) corresponding to the Hessenberg function h_J defined in (3.2), that is,

$$H_J := b \oplus \bigoplus_{i \in J} \mathfrak{g}_{-α_i} \subseteq \mathfrak{g}l_n(\mathbb{C}),$$  

where b = Lie(B) is the Lie algebra of B and each \mathfrak{g}_{-α_i} is the standard root space of \mathfrak{g}l_n(\mathbb{C}) associated with the i-th negative simple root −αᵢ (1 ≤ i ≤ n − 1). Since H_J is preserved by the adjoint action of B on \mathfrak{g}l_n(\mathbb{C}), the quotient space

$$H_{[n-1]}/H_J$$

is a representation space of B. Now, these two representations of B induce the following vector bundles over Flₙ:

$$U_J := GL_n(\mathbb{C}) \times^B \left( H_{[n-1]}/H_J \right),$$

$$V_J := GL_n(\mathbb{C}) \times^B \left( \bigoplus_{i \in J} \mathbb{C}^{*}_{-α_i} \right).$$

If there are no confusion, we denote the restrictions of U_J and V_J on Petₙ by the same symbol. Note that we have

$$\text{rank } U_J = (n − 1) − |J|,$$
$$\text{rank } V_J = |J|. \tag{3.14}$$

Recall that

$$Petₙ = \{ gB \in GL_n(\mathbb{C})/B \mid g⁻¹Ng ∈ H_{[n-1]} \}$$

(cf. [24, 31] or [33, Sect. 2]). Thus, the following map gives a section of U_J over Petₙ:

$$\phi_J: Petₙ \to U_J \quad ; \quad \phi_J(gB) = [g, [g⁻¹Ng]],$$

where [g⁻¹Ng] ∈ H_{[n-1]}/H_J is the class represented by g⁻¹Ng ∈ H_{[n-1]}. For 1 ≤ i ≤ n − 1, let

$$\det_i: GL_n(\mathbb{C}) \to \mathbb{C}_{ωᵢ}(= \mathbb{C})$$

be the function which takes the leading principal minor of order i. This is a B-equivariant function with respect to the multiplication of B on GLₙ(\mathbb{C}) from the right. Thus, we have a well-defined section

$$\psi_J: Petₙ \to V_J \quad ; \quad gB \mapsto \left[ g, \sum_{i \in J} \det_i(g) \right].$$

The following claim means that X_J and Ω_J in Petₙ are defined by the equation \phi_J = 0 and \psi_J = 0, respectively.
Proposition 3.12. For $J \subseteq [n-1]$, we have
\[ X_J = Z(\psi_J) \quad \text{and} \quad \Omega_J = Z(\phi_J), \]
where $Z(\phi_J)$ and $Z(\psi_J)$ denote the zero loci of the sections $\phi_J$ and $\psi_J$, respectively.

Proof. Since the defining condition of $\text{Hess}(N, h_J)$ is precisely that $g^{-1}Ng \in H_J$ (e.g. [33, Sect. 2]), it is clear that we have $X_J = Z(\phi_J)$. It is known that
\[ \Omega_{s_1} = \{ gB \in \text{GL}_n(\mathbb{C})/B \mid \det_i(g) = 0 \} \]
as subsets of $\text{Fl}_n$ (cf. [16, Proposition 9 in Sect. 10.6]). Thus, it follows that
\[ Z(\psi_{\{i\}}) = \Omega_{s_1} \cap \text{Pet}_n = E_i. \]

Now, we obtain that
\[ Z(\psi_J) = \bigcap_{i \in J} Z(\psi_{\{i\}}) = \bigcap_{i \in J} E_i = \Omega_J \]
by Proposition 3.10.

Corollary 3.13. For $1 \leq i \leq n - 1$, $D_i$ and $E_i$ are divisors on $\text{Pet}_n$.

Proof. By (3.14), it follows that $U_{[n-1]\setminus\{i\}}$ is a line bundle, and we have $D_i = X_{[n-1]\setminus\{i\}} = Z(\phi_{[n-1]\setminus\{i\}})$ by the previous proposition. This means that $D_i$ is a divisor on $\text{Pet}_n$. Similarly, $E_i = \Omega_{\{i\}} = Z(\psi_{\{i\}})$ is a divisor on $\text{Pet}_n$ since $V_{\{i\}}$ is a line bundle by (3.14).

Corollary 3.14. For $J \subseteq [n-1]$, we have $\dim_{\mathbb{C}} \Omega_J = |J|$ in $\text{Pet}_n$.

Proof. Recall that $\Omega_J = Z(\psi_J)$ and $\text{rank} V_J = |J|$. This means that $\Omega_J$ in $\text{Pet}_n$ is locally cut out by $|J|$ functions, and it follows that each irreducible component of $\Omega_J$ has codimension at most $|J|$ in $\text{Pet}_n$ ([17, Proposition 14.1]). This implies that
\[ \dim_{\mathbb{C}} \Omega_J \geq (n - 1) - |J| \]
(3.15)
since $\dim_{\mathbb{C}} \text{Pet}_n = n - 1$ and $\Omega_J = \Omega_{w_J} \cap \text{Pet}_n \neq \emptyset$. We show that the equality holds by induction on $(n - 1) - |J|$. When $J = [n-1]$, we have $\Omega_{[n-1]} = \Omega_{w_0} \cap \text{Pet}_n = \{ w_0 \}$ so that the claim is obvious. Let $J \subsetneq [n-1]$, and assume by induction that $\dim_{\mathbb{C}} \Omega_K = (n - 1) - |K|$ for $J \subsetneq K \subsetneq [n-1]$. We prove that $\dim_{\mathbb{C}} \Omega_J = (n - 1) - |J|$. Take an element $i \in [n-1] \setminus J$, and set $K = J \cup \{i\}$. Then we have
\[ \Omega_K = \Omega_J \cap E_i \subseteq \Omega_J \]
by Proposition 3.10. This means that $\Omega_K$ is the zero locus of the section $\psi_{\{i\}}$ of the line bundle $V_{\{i\}}$ restricted over $\Omega_J$. Thus we see that
\[ \dim_{\mathbb{C}} \Omega_K \geq \dim_{\mathbb{C}} \Omega_J - 1, \]
(3.17)
which follows by applying [17, Proposition 14.1] to each irreducible component of $\Omega_J$. Namely, the dimension decreases at most by 1 in (3.16). Since we have $\dim_{\mathbb{C}} \Omega_K = (n - 1) - |K|$ by the inductive hypothesis, we can rewrite (3.17) as
\[ \dim_{\mathbb{C}} \Omega_J \leq \dim_{\mathbb{C}} \Omega_K + 1 = (n - 1) - |K| + 1 = (n - 1) - |J|. \]
Combining this with (3.15), we obtain the desired equality. \qed
Remark 3.15. The vector bundles $U_J$ and $V_J$ are decomposed into line bundles as

$$U_J = \bigoplus_{i \in J} L_{\alpha_i} \quad \text{and} \quad V_J = \bigoplus_{i \in J} L_{\beta_i}.$$ 

The first equality follows since we have $H_{[n-1]}/H_J \cong \oplus_{i \notin J} C_{-\alpha_i}$ as representations of $B$, where the right hand side is a direct sum representation. These decomposition can be thought as the analogue of Proposition 3.10 in the language of vector bundles over $\text{Pet}_n$.

4. The cohomology ring of $\text{Pet}_n$

In this section, we construct an additive basis of the integral cohomology ring $H^*(\text{Pet}_n; \mathbb{Z})$ by incorporating the geometry established in the previous section. We also introduce the structure constants of the basis, and provide a geometric proof for their positivity.

For an algebraic variety $X$ which admits a cellular decomposition by complex affines spaces (which is also called a paving by affines), an irreducible Zariski closed subset $Y$ of $X$ has its fundamental cycle (as a reduced scheme) in $H_{2d}(Y; \mathbb{Z})$, where $d = \dim_{\mathbb{C}} Y$. By abusing notations, we use the same symbol for its image in $H_{2d}(X; \mathbb{Z})$ under the induced map $i^* : H_{2d}(Y; \mathbb{Z}) \to H_{2d}(X; \mathbb{Z})$, where $i$ is the inclusion map $i : Z \hookrightarrow X$. See [16, Appendix B] or [17, Chap. 19] for the details.

4.1. The homology group of $\text{Pet}_n$. Recall that we have a decomposition of $\text{Fl}_n$ by the Schubert cells:

$$\text{Fl}_n = \bigsqcup_{w \in S_n} X_w^\circ.$$ 

This induces a decomposition

$$\text{Pet}_n = \bigsqcup_{J \subseteq [n-1]} (X_J^\circ \cap \text{Pet}_n),$$

by Lemmas 2.4 and 3.5. It is known from [29, 33] that each $X_J^\circ \cap \text{Pet}_n$ is isomorphic to an affine cell $\mathbb{C}^{[J]}$ and that they form a paving by affines. Recall also from Proposition 3.4 that we have

$$X_J = X_J^\circ \cap \text{Pet}_n$$

and that $\dim_{\mathbb{C}} X_J = |J|$ for $J \subseteq [n-1]$. This implies that the cycles represented by $X_J$ for $J \subseteq [n - 1]$ form a $\mathbb{Z}$-basis of the homology group $H_*(\text{Pet}_n; \mathbb{Z})$.

Proposition 4.1. ([29, 33]) For each $J \subseteq [n - 1]$, we have $[X_J] \in H_{2d}(\text{Pet}_n; \mathbb{Z})$, and the set $\{[X_J] \mid J \subseteq [n - 1]\}$ is a $\mathbb{Z}$-basis of $H_*(\text{Pet}_n; \mathbb{Z})$;

$$H_*(\text{Pet}_n; \mathbb{Z}) = \bigoplus_{J \subseteq [n-1]} \mathbb{Z}[X_J].$$

Example 4.2. Let $n = 4$ so that $[n - 1] = \{1, 2, 3\}$. Then we have

$$H_*(\text{Pet}_n; \mathbb{Z}) = \mathbb{Z}[X_0] \oplus (\mathbb{Z}[X_{11}] \oplus \mathbb{Z}[X_{22}] \oplus \mathbb{Z}[X_{33}])$$

$$\oplus (\mathbb{Z}[X_{12}] \oplus \mathbb{Z}[X_{13}] \oplus \mathbb{Z}[X_{23}] \oplus \mathbb{Z}[X_{1,2,3}]).$$
Remark 4.3. Compared to $X_{m,j}^\ast \cap \text{Pet}_n (\cong \mathbb{C}^{[J]})$, the geometry of $\Omega_{m,j} \cap \text{Pet}_n$ is known to encode the quantum cohomology ring of a partial flag variety specified by $J$ ([29, 31]).

4.2. The cohomology group of $\text{Pet}_n$. For each weight $\mu: T \to \mathbb{C}^\ast$, we constructed a line bundle $L_\mu$ over $\text{Fl}_n$ in Section 3.2. Recall also that $\alpha_i$ and $\varpi_i$ are the $i$-th simple root and the $i$-th fundamental weight of $T$ $(1 \leq i \leq n - 1)$, respectively. It is well-known that we have an isomorphism

$$\bigoplus_{i=1}^{n-1} \mathbb{Z}\varpi_i \cong H^2(\text{Fl}_n; \mathbb{Z}); \quad \mu \mapsto e(L_\mu),$$

where we regard each $\mu = a_1\varpi_1 + \cdots + a_{n-1}\varpi_{n-1}$ ($a_1, \ldots, a_{n-1} \in \mathbb{Z}$) as the weight $\mu: T \to \mathbb{C}^\ast$ given by $\text{diag}(t_1, \ldots, t_n) \mapsto (t_1^{a_1}t_2^{a_2} \cdots (t_1 \cdots t_{n-1})^{a_{n-1}}$. Let $i: \text{Pet}_n \hookrightarrow \text{Fl}_n$ be the inclusion map. Insko ([23]) proved that the induced homomorphism $i^*: H_\ast(\text{Pet}_n; \mathbb{Z}) \to H_\ast(\text{Fl}_n; \mathbb{Z})$ is an injection whose image is a direct summand of $H_\ast(\text{Fl}_n; \mathbb{Z})$. This means that the map $i^*: H_2(\text{Pet}_n; \mathbb{Z}) \to H_2(\text{Fl}_n; \mathbb{Z})$ on degree 2 is an isomorphism since rank $H_2(\text{Fl}_n; \mathbb{Z}) = \text{rank } H_2(\text{Pet}_n; \mathbb{Z}) = n - 1$ (see Proposition 4.1), and hence the restriction map

$$i^*: H^2(\text{Fl}_n; \mathbb{Z}) \cong H^2(\text{Pet}_n; \mathbb{Z})$$

on degree 2 cohomology group is an isomorphism. By combining these isomorphisms, we obtain that $\bigoplus_{i=1}^{n-1} \mathbb{Z}\varpi_i \cong H^2(\text{Pet}_n; \mathbb{Z})$. In the rest of the paper, we identify $\bigoplus_{i=1}^{n-1} \mathbb{Z}\varpi_i$ with $H^2(\text{Pet}_n; \mathbb{Z})$ through this isomorphism, and we use the same symbol $\mu \in H^2(\text{Pet}_n; \mathbb{Z})$ for the element $e(L_\mu)$ by abusing notation. For example, we write

$$\alpha_i = e(L_{\alpha_i}) \quad \text{and} \quad \varpi_i = e(L_{\varpi_i})$$

as elements in $H^2(\text{Pet}_n; \mathbb{Z})$.

In Section 3.2, we constructed vector bundles $U_J$ and $V_J$ over $\text{Pet}_n$. Adopting the above notation, we may express the Euler class $e(V_J)$ as a monomial of $\varpi_i$ for each $i \in J$. Namely, for $J \subseteq [n - 1]$, we have

$$e(V_J) = \prod_{i \in J} \varpi_i$$

since the vector bundle $V_J$ decomposes into line bundles as follows:

$$V_J = \text{GL}_n(\mathbb{C}) \times^B \left( \bigoplus_{i \in J} \mathbb{C}^* \right) = \bigoplus_{i \in J} L_{\varpi_i}.$$ 

For $J \subseteq [n - 1]$, take the decomposition $J = J_1 \sqcup J_2 \sqcup \cdots \sqcup J_m$ into the connected components. We set

$$m_J := |J_1|!|J_2|! \cdots |J_m|!.$$ 

(4.2)

Definition 4.4. For $J \subseteq [n - 1]$, let

$$\varpi_J := \frac{1}{m_J} e(V_J) = \frac{1}{m_J} \prod_{i \in J} \varpi_i,$$

where $m_J$ is defined in (4.2).
The cohomology class $\varpi_J$ is defined to be an element of $H^{2|J|}(\text{Pet}_n; \mathbb{Q})$, but we will show that it belongs to the integral cohomology group $H^{2|J|}(\text{Pet}_n; \mathbb{Z})$.

**Example 4.5.** Let $n = 9$ and $J = \{2, 3, 4, 7, 8\}$ so that $J = \{2, 3, 4\} \sqcup \{7, 8\}$ is the decomposition into the connected components. Then we have

$$
\varpi_J = \frac{1}{3!2!}(\varpi_2 \varpi_3 \varpi_4)(\varpi_7 \varpi_8) = \frac{1}{12} \varpi_2 \varpi_3 \varpi_4 \varpi_7 \varpi_8.
$$

Compare this with Example 3.11.

**Remark 4.6.** We also have

$$
e(U_J) = \prod_{i \notin J} \alpha_i
$$

(cf. Remark 3.15). These decompositions of $e(U_J)$ and $e(V_J)$ can be thought as the cohomological analogue of Proposition 3.10.

The main purpose of this subsection is to prove that the set of cohomology classes $\{\varpi_J \mid J \subseteq [n-1]\}$ forms a module basis of the integral cohomology group $H^*(\text{Pet}_n; \mathbb{Z})$. We will state this in Theorem 4.14, and we devote the rest of this subsection for its proof.

**Lemma 4.7.** For $1 \leq i \leq n - 1$, we have

$$
\alpha_i \varpi_i = 0 \quad \text{in} \quad H^4(\text{Pet}_n, \mathbb{Z}).
$$

**Proof.** Notice that $\alpha_i \varpi_i$ is the Euler class of the rank 2 vector bundle $U_{[n-1]\setminus\{i\}} \oplus V_{\{i\}} = L_{\alpha_i} \oplus L_{\varpi_i}$ (cf. Remark 3.15). From Section 3.2, we have the section $\phi_{[n-1]\setminus\{i\}} + \psi_{\{i\}}$ of this bundle whose zero locus is $Z(\phi_{[n-1]\setminus\{i\}}) \cap Z(\psi_{\{i\}}) = D_i \cap E_i$ as we saw in the proof of Corollary 3.13. Now, by Lemma 3.9 (2), this is the empty set. Thus, $\phi_{[n-1]\setminus\{i\}} + \psi_{\{i\}}$ on $\text{Pet}_n$ is a nowhere-zero section, and hence the Euler class $\alpha_i \varpi_i$ vanishes ([27, Property 9.7]).

**Remark 4.8.** In [15, Corollary 3.4] and [19, Theorem 4.1], the equations $\alpha_i \varpi_i = 0$ for $1 \leq i \leq n - 1$ appeared as the fundamental relations in the presentation of the cohomology ring $H^*(\text{Pet}_n; \mathbb{C})$.

For $1 \leq i \leq n$, let $F_i$ be the $i$-th tautological vector bundle over $\text{Fl}_n$ whose fiber at $V_i \in \text{Fl}_n$ is $V_i$. As a convention, we denote by $F_0$ the trivial sub-bundle of $F_1$ of rank 0. The quotient line bundle $L_i := F_i/F_{i-1}$ is called the $i$-th tautological line bundle, and we set

$$
x_i := c_1(L_i^*) \in H^2(\text{Fl}_n; \mathbb{Z}) \quad (1 \leq i \leq n),
$$

where we note that $x_1 + \cdots + x_n = 0$. We will also denote by the same symbol the restriction of $x_i$ to $H^2(\text{Pet}_n; \mathbb{Z})$. It is well-known that for $1 \leq i \leq n - 1$, we have

$$
\alpha_i = x_i - x_{i+1},
$$

$$
\varpi_i = x_1 + x_2 + \cdots + x_i.
$$

For $1 \leq i < j \leq n$, let

$$
\alpha_{i,j} := \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1} = x_i - x_j.
$$

For a homology cycle $Z \in H_k(\text{Fl}_n; \mathbb{Z})$ of degree $k$, the Poincaré dual of $Z$ is the (unique) cohomology class $\gamma \in H^{2d-k}(\text{Fl}_n; \mathbb{Z})$ ($d = \dim \mathbb{C} \text{Fl}_n$) satisfying $\gamma \cap [\text{Fl}_n] = Z$. In the
following lemma, we regard $\text{Fl}_{n-1}$ as a subvariety of $\text{Fl}_n$ whose flags are contained in the linear subspace of $\mathbb{C}^n$ generated by $e_1, e_2, \ldots, e_{n-1}$. The claim (ii) of the following lemma seems to be well-known, but we provide a proof using Hessenberg varieties for the completeness of the paper.

**Lemma 4.9.** The following hold:

(i) The Poincaré dual of $[\text{Pet}_n] \in H_*(\text{Fl}_n; \mathbb{Z})$ is $\prod_{i \geq 2} \alpha_{i,j} \in H^*(\text{Fl}_n; \mathbb{Z})$.

(ii) The Poincaré dual of $[\text{Fl}_{n-1}] \in H_*(\text{Fl}_n; \mathbb{Z})$ is $\frac{1}{n} \alpha_{1,n} \alpha_{2,n} \cdots \alpha_{n-1,n} \in H^*(\text{Fl}_n; \mathbb{Z})$.

**Proof.** We first prove the claim (i). Recall from (3.2) and (3.13) that $H_J \subseteq \mathfrak{gl}_n(\mathbb{C})$ for $J \subseteq [n-1]$ is the Hessenberg space corresponding to the Hessenberg function $h_J$. Consider the associated vector bundle

$$\mathcal{N} := \text{GL}_n(\mathbb{C}) \times^B (\mathfrak{gl}_n(\mathbb{C})/H_{[n-1]})$$

over $\text{Fl}_n = \text{GL}_n(\mathbb{C})/B$. By an argument similar to that used in Section 3.2, $\text{Pet}_n$ can be written as the zero locus of a section of the vector bundle $\mathcal{N}$, and it is shown in [2, Corollary 3.9] that the Poincaré dual of $[\text{Pet}_n]$ is the Euler class $e(\mathcal{N}) \in H^*(\text{Fl}_n; \mathbb{Z})$. It is straightforward to verify that $e(\mathcal{N}) = \prod_{i \geq 2} \alpha_{i,j}$ by the same inductive argument as that in [2, Sect. 4] using short exact sequences of vector bundles.

Next we prove the claim (ii). Let $S$ be an $n \times n$ regular semisimple matrix (i.e. a diagonal matrix with distinct eigenvalues) and $\text{Hess}(S, h_0)$ a regular semisimple Hessenberg variety, where $h_0$ is a Hessenberg function $h_0 : [n] \to [n]$ given by

$$h_0(i) := \begin{cases} n-1 & (1 \leq i \leq n-1) \\ n & (i = n). \end{cases}$$

It is shown in [7, Sect. 3 and Sect. 4] that the Poincaré dual of $[\text{Hess}(S, h_0)]$ is

$$(x_1 - x_n)(x_2 - x_n) \cdots (x_{n-1} - x_n) \in H^*(\text{Fl}_n; \mathbb{Z}),$$

where the left hand side is equal to $\alpha_{1,n} \alpha_{2,n} \cdots \alpha_{n-1,n}$ by the definition of $\alpha_{i,j}$. It is also known that $\text{Hess}(S, h_0)$ has $n$ connected components and that all the connected components give the same cycle $[\text{Fl}_{n-1}]$ ([32, Sect. 3]). Thus the Poincaré dual of $n[\text{Fl}_{n-1}]$ is $\alpha_{1,n} \alpha_{2,n} \cdots \alpha_{n-1,n}$, which implies the claim (ii). \hfill $\Box$

**Remark 4.10.** [7, Corollary 7.2] with the formula for the double Schubert polynomial in [7, p.2613] gave a more general formula than that of Lemma 4.9 (i) for regular nilpotent Hessenberg schemes which were not known to be reduced when it was published. After that, [1] proved that they are in fact reduced when they contain $\text{Pet}_n$ (cf. [1, Remark 3.8]), and the formula are now generalized in [2] for an arbitrary Lie type.

For an (irreducible) projective variety $Y$, we denote the fundamental cycle of $Y$ as $[Y] \in H_{2d}(Y; \mathbb{Z})$, where $d = \dim_\mathbb{C} Y$. For a cohomology class $\beta \in H^{2d}(Y; \mathbb{Z})$, we write

$$\int_Y \beta := \langle [Y], \beta \rangle_Y \quad (\in \mathbb{Z}),$$

where the right hand side is the value of the standard paring

$$\langle , \rangle_Y : H_{2d}(\text{Pet}_n; \mathbb{Z}) \times H^{2d}(\text{Pet}_n; \mathbb{Z}) \to \mathbb{Z}.$$
Proposition 4.11. We have
\[ \int_{\text{Pet}_n} \omega_1 \omega_2 \cdots \omega_{n-1} = (n-1)!. \]

Proof. Let us first prove that
\[ (4.5) \quad \int_{\text{Pet}_n} \omega_1 \omega_2 \cdots \omega_{n-1} = (n-1) \int_{\text{Pet}_{n-1}} \omega_1 \omega_2 \cdots \omega_{n-2}. \]

For the \( \omega_{n-1} \) in the left hand side of (4.5), notice that
\[ \omega_{n-1} = \frac{1}{n} \sum_{i=1}^{n-1} i \alpha_i, \]
which follows from (4.4). Since we have \( \omega_i \alpha_i = 0 \) for \( 1 \leq i \leq n-1 \) from Lemma 4.7, we see that
\[ (4.6) \quad \int_{\text{Pet}_n} \omega_1 \omega_2 \cdots \omega_{n-1} = \frac{n-1}{n} \int_{\text{Pet}_n} \omega_1 \omega_2 \cdots \omega_{n-2} \alpha_{n-1}. \]
By Lemma 4.9 (i), the right hand side of (4.6) can be computed as the following integral over \( Fl_n \):
\[ \frac{n-1}{n} \int_{Fl_n} \omega_1 \omega_2 \cdots \omega_{n-2} \alpha_{n-1} \prod_{j-i \geq 2} \alpha_{i,j}. \]
Since we have \( \alpha_{n-1} = \alpha_{n-1,n} \), the last expression can be written as
\[ \frac{n-1}{n} \int_{Fl_n} \omega_1 \omega_2 \cdots \omega_{n-2} (\alpha_{1,n} \alpha_{2,n} \cdots \alpha_{n-1,n}) \prod_{j-i \geq 2, j \neq n} \alpha_{i,j}. \]
By Lemma 4.9 (ii), we can rewrite this as the following integral over \( Fl_{n-1} \):
\[ (n-1) \int_{Fl_{n-1}} \omega_1 \omega_2 \cdots \omega_{n-2} \prod_{j-i \geq 2, j \neq n} \alpha_{i,j}. \]
Applying Lemma 4.9 (i) to \( Pet_{n-1} \subseteq Fl_{n-1} \), this can be written as the following integral over \( Pet_{n-1} \):
\[ (n-1) \int_{Pet_{n-1}} \omega_1 \omega_2 \cdots \omega_{n-2}. \]
Hence we proved (4.5).

Now using (4.5) repeatedly, we obtain that
\[ \int_{Pet_n} \omega_1 \omega_2 \cdots \omega_{n-1} = (n-1)! \int_{Pet_2} \omega_1. \]
Noticing that \( Pet_2 = Fl_2 = \mathbb{P}^1 \), we see that \( \omega_1 (= x_1) \) is the first Chern class of the dual of the standard tautological line bundle over \( \mathbb{P}^1 \) by (4.3). Thus the integral in the right hand side is equal to 1, which completes the proof. \( \square \)
Lemma 4.12. For $J \subseteq [n-1]$, we have
\[
\int_{X_J} e(V_J) = m_J,
\]
where $m_J = |J_1||J_2|\cdots|J_m|!$ is defined in (4.2).

Proof. The isomorphism given in Corollary 3.6 induces an isomorphism
\[
H^*(X_J; \mathbb{Z}) \cong \bigotimes_{k=1}^{m} H^*(\text{Pet}_{n_k}; \mathbb{Z})
\]
which sends $e(V_J)(= \prod_{i \in J} \varpi_i) \in H^{2|J|}(X_J; \mathbb{Z})$ to $\bigotimes_{k=1}^{m} \varpi_1 \varpi_2 \cdots \varpi_{|J_k|}$. It also induces an isomorphism
\[
H_*(X_J; \mathbb{Z}) \cong \bigotimes_{k=1}^{m} H_*(\text{Pet}_{n_k}; \mathbb{Z})
\]
which sends $[X_J] \in H_2|J|(X_J; \mathbb{Z})$ to $\bigotimes_{k=1}^{m} [\text{Pet}_{n_k}]$. Now the claim follows from Proposition 4.11. □

Proposition 4.13. For $J, K \subseteq [n-1]$ such that $|J| = |K|$, the degree of the homology class $[X_J]$ is the same as the degree of the Euler class $e(V_K)$, and we have
\[
\langle [X_J], e(V_K) \rangle_{\text{Pet}_n} = \begin{cases} 
  m_J & \text{if } J = K, \\
  0 & \text{if } J \neq K.
\end{cases}
\]

Proof. Note that we have
\[
\langle [X_J], e(V_K) \rangle_{\text{Pet}_n} = \int_{X_J} e(V_K).
\]
(4.7)

For the case $J = K$, the claim follows from the previous lemma. Let us consider the case $J \neq K$. This condition and $|J| = |K|$ imply that $J \nsubseteq K$. Recall from Proposition 3.12 that we have the section $\psi_K: \text{Pet}_n \to V_K$ such that $Z(\psi_K) = \Omega_K$. Thus, Proposition 3.2 implies that the vector bundle $V_K$ restricted on $X_K$ admits a nowhere-zero section (given by $\psi_K$). Thus the Euler class $e(V_K)$ vanishes on $X_J$, and hence the right hand side of (4.7) is equal to 0 in this case. □

For $J \subseteq [n-1]$, recall from Definition 4.4 that
\[
\varpi_J = \frac{1}{m_J} e(V_J) = \frac{1}{m_J} \prod_{i \in J} \varpi_i.
\]

Theorem 4.14. For each $J \subseteq [n-1]$, the cohomology class $\varpi_J$ is an element of the integral cohomology $H^{2|J|}(\text{Pet}_n; \mathbb{Z})$, and the set
\[
\{ \varpi_J \in H^*(\text{Pet}_n; \mathbb{Z}) \mid J \subseteq [n-1] \}
\]
is a $\mathbb{Z}$-basis of $H^*(\text{Pet}_n; \mathbb{Z})$.

Proof. Recall from Proposition 4.1 that $\{[X_J] \mid J \subseteq [n-1]\}$ forms a $\mathbb{Z}$-basis of $H_*(\text{Pet}_n; \mathbb{Z})$. Since the paring between $H_*(\text{Pet}_n; \mathbb{Z})$ and $H^*(\text{Pet}_n; \mathbb{Z})$ is perfect, the previous proposition implies the desired claim. □
Example 4.15. Let $n = 4$ so that $[n - 1] = \{1, 2, 3\}$. The additive basis given in Theorem 4.14 is

$$H^*(\text{Pet}_n; \mathbb{Z}) = \mathbb{Z}\mathcal{w}_0 \oplus (\mathbb{Z}\mathcal{w}_{(1)} \oplus \mathbb{Z}\mathcal{w}_{(2)} \oplus \mathbb{Z}\mathcal{w}_{(3)}) \oplus (\mathbb{Z}\mathcal{w}_{(1,2)} \oplus \mathbb{Z}\mathcal{w}_{(1,3)} \oplus \mathbb{Z}\mathcal{w}_{(2,3)}) \oplus \mathbb{Z}\mathcal{w}_{(1,2,3)}.$$  

As we saw in Proposition 4.13, this is the dual basis of the basis of the homology group $H_*(\text{Pet}_n; \mathbb{Z})$ given in Example 4.2.

4.3. Structure constants and their positivity. By Theorem 4.14, we can study the cohomology ring $H^*(\text{Pet}_n; \mathbb{Z})$ in terms of the basis $\{\mathcal{w}_j\}_{J \subseteq [n-1]}$. Specifically, we expand the product of two classes $\mathcal{w}_J$ and $\mathcal{w}_K$ as a linear combination of the basis:

$$\mathcal{w}_J \cdot \mathcal{w}_K = \sum_{L \subseteq [n-1]} d^L_{JK} \mathcal{w}_L, \quad d^L_{JK} \in \mathbb{Z}. \tag{4.8}$$

The coefficients $d^L_{JK}$ are called the structure constant for the basis $\{\mathcal{w}_j\}_{J \subseteq [n-1]}$. In the following, we explain a geometric interpretation of $d^L_{JK}$, and deduce their positivity. Note that the degree of $\mathcal{w}_L$ in $H^*(\text{Pet}_n; \mathbb{Z})$ is $2|L|$ and that the degree of $\mathcal{w}_J \cdot \mathcal{w}_K$ in $H^*(\text{Pet}_n; \mathbb{Z})$ is $2(|J| + |K|)$. Thus we may assume that

$$|L| = |J| + |K| \tag{4.9}$$

for each summand of (4.8) since we have $d^L_{JK} = 0$ otherwise. Then by Proposition 4.13, we have

$$d^L_{JK} = \langle [X_L], \mathcal{w}_J \cdot \mathcal{w}_K \rangle_{\text{Pet}_n} = \frac{1}{m_J m_K} \int_{X_L} \left( \prod_{j \in J} \mathcal{w}_j \right) \left( \prod_{k \in K} \mathcal{w}_k \right), \tag{4.10}$$

where $m_J$ and $m_K$ are the positive integers defined in (4.2). Now recall that each $\mathcal{w}_i \in H^2(X_L; \mathbb{Z})$ is the Euler class of the line bundle $V_{i(i)}$ corresponding to the divisor $E_i := Z(\psi_{i(i)})$ on $\text{Pet}_n$. Hence it follows from (4.10) that the structure constant $d^L_{JK}$ computes an intersection number of (possibly duplicate) divisors $E_i \cap X_L$’s on $X_L$ up to a constant multiple given by $\frac{1}{m_J m_K}$ (cf. [26, Sect. 1.1.C]). This provides a geometric interpretation of $d^L_{JK}$ in (4.8), and it leads us to the following positivity.

Proposition 4.16. We have $d^L_{JK} \geq 0$ for all $J, K, L \subseteq [n-1]$.

Proof. Recall that each line bundle $L_{i\omega}$ over $F_{i\omega}$ is nef for $1 \leq i \leq n - 1$ (e.g. [11, the proof of Proposition 1.4.1] or [3, Lemma 3.5]). Hence the restriction of $L_{i\omega}$ over $X_L$ is nef as well for $1 \leq i \leq n - 1$. Thus the claim $d^L_{JK} \geq 0$ follows from (4.10) and the positivity of intersection numbers of nef divisors [26, Example 1.4.16]. □

5. Structure constants and Left-right diagrams

Recall from the previous section that the structure constants $d^L_{JK}$ are defined to be the coefficients of the expansion formula (4.8) for the product $\mathcal{w}_J \cdot \mathcal{w}_K$ for $J, K \subseteq [n-1]:$

$$\mathcal{w}_J \cdot \mathcal{w}_K = \sum_{L \subseteq [n-1]} d^L_{JK} \mathcal{w}_L, \quad d^L_{JK} \in \mathbb{Z}.$$
In this section, we provide a manifestly positive combinatorial formula which computes the structure constant $d_{JK}^L$ for all $J, K, L \subseteq \{a - 1\}$. We start with the following lemma which tells us how to expand a monomial of $\omega_1, \ldots, \omega_{n-1}$ containing a square in the simplest case.

**Lemma 5.1.** For $1 \leq a \leq i \leq b \leq n - 1$, we have

$$\omega_i \cdot (\omega_a \omega_{a+1} \cdots \omega_b) = \frac{b - i + 1}{b - a + 2} \omega_{a-1} \omega_a \cdots \omega_b + \frac{i - a + 1}{b - a + 2} \omega_a \cdots \omega_b \omega_{b+1}$$

in $H^*(\text{Pet}_n; \mathbb{Z})$, where we take the convention $\omega_0 = \omega_n = 0$.

**Proof.** We prove the claim by induction on $b - a (\geq 0)$. When $b - a = 0$, we have $a = i = b$ so that the left hand side is $\omega_a^2$. Noticing that $\alpha_a = -\omega_{a-1} + 2\omega_a - \omega_{a+1}$ (with the above convention), we have that

$$\omega_a(-\omega_{a-1} + 2\omega_a - \omega_{a+1}) = 0$$

by Lemma 4.7. Thus the claim in this case follows since this equality can be expressed as

$$\omega_a^2 = \frac{1}{2} \omega_{a-1} \omega_a + \frac{1}{2} \omega_a \omega_{a+1}.$$  

We now prove the claim for the case $a < b$. Assume by induction that the claim holds for any $a' \leq i' \leq b'$ with $b' - a' < b - a$. When $i = a$, we have

$$\omega_a(\omega_a \omega_{a+1} \cdots \omega_b) = \left(\omega_a(\omega_a \omega_{a+1} \cdots \omega_{b-1})\right) \omega_b$$

$$= \left(\frac{b - a}{b - a + 1} \omega_{a-1} \omega_a \cdots \omega_{b-1} + \frac{1}{b - a + 1} \omega_a \omega_{a+1} \cdots \omega_b\right) \omega_b$$

(by the inductive hypothesis)

$$= \frac{b - a}{b - a + 1} \omega_{a-1} \omega_a \cdots \omega_b + \frac{1}{b - a + 1} \omega_a \left(\omega_{a+1} \cdots \omega_{b-1} \omega_b^2\right)$$

$$= \frac{b - a}{b - a + 1} \omega_{a-1} \omega_a \cdots \omega_b$$

$$+ \frac{1}{(b - a + 1)^2} \omega_a \left(\omega_a \omega_{a+1} \cdots \omega_b + (b - a) \omega_{a+1} \omega_{a+2} \cdots \omega_{b+1}\right)$$

(by the inductive hypothesis).

Since the left hand side and the second summand of the right hand side are proportional, this equation can be written as

$$\frac{(b - a + 1)^2 - 1}{(b - a + 1)^2} \omega_a(\omega_{a+1} \cdots \omega_b) = \frac{b - a}{b - a + 1} \omega_{a-1} \omega_a \cdots \omega_b$$

$$+ \frac{b - a}{(b - a + 1)^2} \omega_a \omega_{a+1} \cdots \omega_{b+1}.$$  

Noticing that $(b - a + 1)^2 - 1 = (b - a)(b - a + 2)$ for the numerator of the coefficient of the left hand side, we obtain that

$$\omega_a^2(\omega_{a+1} \cdots \omega_b) = \frac{b - a + 1}{b - a + 2} \omega_{a-1} \omega_a \cdots \omega_b + \frac{1}{b - a + 2} \omega_a \omega_{a+1} \cdots \omega_{b+1}$$

(5.1)
which verifies the claim for the case \( i = a \). Now suppose that \( a < i (\leq b) \). We then have that

\[
\varpi_i(\varpi_a \varpi_{a+1} \cdots \varpi_b) = \varpi_a(\varpi_i(\varpi_{a+1} \cdots \varpi_b))
\]

\[
= \varpi_a(\varpi_{a-1} \varpi_{a+1} \cdots \varpi_b + \frac{i-a}{b-a+1} \varpi_{a+1} \varpi_{a+2} \cdots \varpi_{b+1})
\]

(by the induction hypothesis)

\[
= \frac{b-i+1}{b-a+1} \varpi_a(\varpi_{a+1} \cdots \varpi_b) + \frac{i-a}{b-a+1} \varpi_{a+1} \varpi_{a+2} \cdots \varpi_{b+1}
\]

\[
= \frac{b-i+1}{b-a+1} \left( \frac{b-a+1}{b-a+2} \varpi_{a-1} \varpi_a \cdots \varpi_b + \frac{1}{b-a+2} \varpi_a \varpi_{a+1} \cdots \varpi_{b+1} \right)
\]

\[
+ \frac{i-a}{b-a+1} \varpi_{a+1} \varpi_{a+2} \cdots \varpi_{b+1} \quad \text{(by (5.1))}
\]

Thus we complete the proof by induction. \( \square \)

Lemma 5.1 is the simplest case of expansions, but it turns out that it provides an effective way for computing the expansion of \( \varpi_J \cdot \varpi_K \) for \( J, K \subseteq [n-1] \) as we see in the following example.

**Example 5.2.** Let \( n = 10 \), and take \( J = \{1, 3, 5, 6, 7\} \) and \( K = \{3, 6, 8\} \). The product \( \varpi_J \cdot \varpi_K \) can be computed by using Lemma 5.1 repeatedly as follows. We first extract \( \varpi_i \)'s which produce squares:

\[
\varpi_J \cdot \varpi_K = \left( \frac{1}{1!1!3!} \varpi_1 \varpi_3 \varpi_5 \varpi_6 \varpi_7 \right) \cdot \left( \frac{1}{1!1!1!1!} \varpi_3 \varpi_6 \varpi_8 \right)
\]

\[
= \frac{1}{3!} \varpi_6 \cdot \varpi_3 \cdot (\varpi_1 \varpi_3 \varpi_5 \varpi_6 \varpi_7 \varpi_8).
\]

By applying Lemma 5.1 to \( \varpi_2^2 \), this can be computed as

\[
\frac{1}{3!} \varpi_6 \cdot \varpi_3 \cdot (\varpi_1 \varpi_3 \varpi_5 \varpi_6 \varpi_7 \varpi_8)
\]

\[
= \frac{1}{3!} \varpi_6 \cdot \left( \varpi_1 \left( \frac{1}{2} \varpi_2 \varpi_3 + \frac{1}{2} \varpi_3 \varpi_4 \right) \varpi_5 \varpi_6 \varpi_7 \varpi_8 \right)
\]

\[
= \frac{1}{3!} \cdot \frac{1}{2} \varpi_6 \cdot (\varpi_1 \varpi_2 \varpi_3 \varpi_5 \varpi_6 \varpi_7 \varpi_8) + \frac{1}{3!} \cdot \frac{1}{2} \varpi_6 \cdot (\varpi_1 \varpi_3 \varpi_4 \varpi_5 \varpi_6 \varpi_7 \varpi_8).
\]

Now by applying Lemma 5.1 to \( \varpi_5 \varpi_2^2 \varpi_7 \varpi_8 \) in the first summand and \( \varpi_3 \varpi_4 \varpi_5 \varpi_6^2 \varpi_7 \varpi_8 \) in the second summand, we can continue our computation as

\[
\frac{1}{3!} \cdot \frac{1}{2} \varpi_6 \cdot (\varpi_1 \varpi_2 \varpi_3 \varpi_5 \varpi_6 \varpi_7 \varpi_8) + \frac{1}{3!} \cdot \frac{1}{2} \varpi_6 \cdot (\varpi_1 \varpi_3 \varpi_4 \varpi_5 \varpi_6 \varpi_7 \varpi_8)
\]

\[
= \frac{1}{3!} \cdot \frac{1}{2} \left( \varpi_1 \varpi_2 \varpi_3 \left( \frac{3}{5} \varpi_4 \varpi_5 \varpi_6 \varpi_7 \varpi_8 + \frac{2}{5} \varpi_5 \varpi_6 \varpi_7 \varpi_8 \varpi_9 \right) \right).
\]
For each box in the grid, we define the row number as the number which is written below the column containing the box. Let Example 5.3. We now introduce a combinatorial object which effectively computes the structure (5.2)
\[
\left( \frac{3}{7} \varpi_1 \varpi_2 \varpi_3 \varpi_4 \varpi_5 \varpi_6 \varpi_7 \varpi_8 + \frac{4}{7} \varpi_3 \varpi_4 \varpi_5 \varpi_6 \varpi_7 \varpi_8 \varpi_9 \right) 
\]
\[
= \frac{1}{3!} \left( 3 \frac{1}{2} \frac{3}{5} + \frac{1}{2} \frac{3}{7} \right) \cdot 8! \varpi_{\{1,2,3,4,5,6,7,8\}} 
\]
\[
+ \frac{1}{3!} \cdot \frac{2}{5} \cdot 3! \cdot 5! \varpi_{\{1,2,3,5,6,7,8,9\}} + \frac{1}{3!} \cdot \frac{4}{7} \cdot 7! \varpi_{\{1,3,4,5,6,7,8,9\}} 
\]
\[
= 3456 \varpi_{\{1,2,3,4,5,6,7,8\}} + 24 \varpi_{\{1,2,3,5,6,7,8,9\}} + 240 \varpi_{\{1,3,4,5,6,7,8,9\}}. 
\]
Thus we conclude that
\[
\varpi_J \cdot \varpi_K = 3456 \varpi_{\{1,2,3,4,5,6,7,8\}} + 24 \varpi_{\{1,2,3,5,6,7,8,9\}} + 240 \varpi_{\{1,3,4,5,6,7,8,9\}} 
\]
which gives a particular case of the expansion (4.8). As one can see, the geometric idea behind this computation is the realization of \( \Omega_J \) by intersecting the divisors \( E_i \); see (3.11) and (4.1).

Let \( J, K, \) and \( L \) be subsets of \([n-1]\). By tracking the computations in the above example, it is straightforward to see that if \( \varpi_L \) appears in the expansion of the product \( \varpi_J \cdot \varpi_K \), then \( L \) must contain \( J \cup K \). Combining this with (4.9), we see that
\[
(5.2) \quad d^L_{JK} = 0 \quad \text{unless} \quad L \supseteq J \cup K \quad \text{and} \quad |L| = |J| + |K|. 
\]
We now introduce a combinatorial object which effectively computes the structure constants \( d^L_{JK} \). Because of (5.2), we always assume that \( L \supseteq J \cup K \) and \( |L| = |J| + |K| \) in what follows. We first prepare the following two steps.

1. Write the elements of \([n-1]\) in increasing order, and draw a square grid of size \((1 + |J \cap K|) \times |L|\) over the subset \( L \subseteq [n-1] \). On the left side of the grid, write the elements of \( J \cap K \) in increasing order from the second row to the bottom row.

For each box in the grid, we define the row number of the box as the number which is written beside the row containing the box, and define the column number of the box as the number which is written below the column containing the box.

2. Shade the boxes in the first row whose column numbers belong to \( J \cup K(\subseteq L) \). Mark each box with a cross \( \times \) whose row number is the same as the column number.

**Example 5.3.** Let \( n = 10 \) and take \( J = \{1,3,5,6,7\} \) and \( K = \{3,6,8\} \) as in the previous example. We depict the resulting grids after the steps (1) and (2) for the following two choices of \( L \).

(i) If \( L = \{1,2,3,4,5,6,7,8\} \), then the resulting grid is as follows.

\[
\begin{array}{cccccccc}
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
3 & & & & & & & \\
& & & & & & & \\
6 & & & & & & & \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 9 \\
\end{array} 
\]

For example, the row number of the marked box in the second row is 3.
(ii) If \( L' = \{1, 2, 3, 5, 6, 7, 8, 9\} \), then the resulting grid is as follows.

\[
\begin{array}{cccccccc}
3 & 4 & 5 & 6 & 7 & 8 & 9 \\
6 & & & & & & \\
\end{array}
\]

We now play a combinatorial game on the grid prepared above. Let us explain the rule of the game inductively.

(The game):

Assume that some boxes in the \( i \)-th row are shaded \((1 \leq i < |J \cap K| + 1)\). Then shade the boxes in the \((i + 1)\)-th row whose column numbers are the same as those of the shaded boxes in the \(i\)-th row. If there is a non-shaded box adjacent to the left (L) or the right (R) of the consecutive string of the shaded boxes in the \((i + 1)\)-th row containing the marked box, then shade one of them darkly. In this case, continue to the next row. If there are no such boxes, then we stop the game.

We say that the combinatorial game explained above is successful if we can continue the game to the bottom row. We define a left-right diagram associated with \((J, K, L)\) as a configuration of boxes on a square grid of size \((1 + |J \cap K|) \times |L|\) over \(L(\subseteq [n - 1])\) which obtained as the resulting configuration of the shaded boxes of a successful game. We denote by \(\Delta^L_{JK} \) the set of left-right diagrams associated with \((J, K, L)\).

**Example 5.4.** We take the triples \((J, K, L)\) and \((J, K, L')\) given in Example 5.3.

(i) The left-right diagrams associated with \((J, K, L)\) are \(P_1\) and \(P_2\) in Figure 4.
(ii) The left-right diagram associated with \((J, K, L')\) is the \(P'\) in Figure 5.

\[
\begin{array}{c|c}
3 & x \\
6 & \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\hline
(L) & \\
3 & x \\
6 & \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\hline
(R) & \\
3 & x \\
6 & \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\end{array}
\]

\begin{center}
\textbf{Figure 5.} The (unique) game for the \((J, K, L')\).
\end{center}

Next, we define the weight of a left-right diagram \(P \in \Delta^L_{JK}\) as follows. For each row of \(P\) (except for the first row), we consider the consecutive string of the shaded boxes which contains the marked box in the focused row. Then the set of the column numbers for these boxes must be of the form \(\{a, a+1, \ldots, b\}\) for some \(a, b \in L\), and the column number \(i\) of the marked box satisfies \(a \leq i \leq b\). Motivated by Lemma 5.1, we assign to this row a positive rational number given by

\[
\begin{align*}
\frac{b - i + 1}{b - a + 2} & \quad \text{if the additional box \(\square\) is to the left of the marked box \(\times\)}, \\
\frac{i - a + 1}{b - a + 2} & \quad \text{if the additional box \(\square\) is to the right of the marked box \(\times\)}.
\end{align*}
\]

Note that the column number of the additional box is \(a - 1\) in the former case and \(b + 1\) in the latter case (cf. Lemma 5.1). We may pictorially interpret this rational number as follows.

- The denominator is the number of the shaded boxes counted from the additional box \(\square\) to the terminal box lying on the opposite side of the string of shaded boxes across the marked box \(\times\).
- The numerator is the number of the shaded boxes counted from the marked box \(\times\) to the same terminal box as above.

We define the weight of \(P\) as the product of these positive rational numbers assigned to the rows of \(P\) (except for the first row), and denote it by \(\text{wt}(P)\).

**Example 5.5.** Continuing with Example 5.4, the weights of the left-right diagrams \(P_1, P_2, P'\) can be computed as follows.
(i) The weights of the left-right diagrams $P_1$ and $P_2$ associated with the $(J, K, L)$ are
\[
\text{wt}(P_1) = \frac{1}{2} \cdot \frac{3}{5} \quad \text{and} \quad \text{wt}(P_2) = \frac{1}{2} \cdot \frac{3}{7}.
\]
(See Figure 6). By construction, these weights appear in the computation of the coefficient of the $\varpi_L = \varpi_{\{1,2,3,4,5,6,7,8\}}$ in Example 5.2.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure6}
\caption{The computations of the weights of $P_1$ and $P_2$.}
\end{figure}

(ii) The weight of the left-right diagram $P' \in \Delta_{JK}'$ can be computed as
\[
\text{wt}(P') = \frac{1}{2} \cdot \frac{2}{5}.
\]
(See Figure 7). This weight appears in the computation of the coefficient of the $\varpi_{L'} = \varpi_{\{1,2,3,5,6,7,8,9\}}$ in Example 5.2 as well.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure7}
\caption{The computation of the weight of $P'$.}
\end{figure}

We now summarize our computation of the structure constants. For $J \subseteq [n-1]$, recall from Definition 4.4 that
\[
\varpi_J = \frac{1}{m_J} e(V_J) = \frac{1}{m_J} \prod_{i \in J} \varpi_i.
\]

**Theorem 5.6.** Let $J, K$ be subsets of $[n-1]$. In $H^*(\text{Pet}_n; \mathbb{Z})$, we have
\[
\varpi_J \cdot \varpi_K = \sum_{K \cup J \subseteq L \subseteq [n-1]} d_{JK}^{L} \varpi_L, \quad d_{JK}^{L} \in \mathbb{Z},
\]
and the structure constant $d_{JK}^{L}$ in this equality is given by
\[
d_{JK}^{L} = \frac{m_L}{m_J m_K} \sum_{P \in \Delta_{JK}^{L}} \text{wt}(P),
\]
where $\Delta_{JK}^{L}$ is the set of left-right diagrams and $\text{wt}(P)$ is the weight of $P$ defined above. In particular, we have $d_{JK}^{L} = 0$ in (5.3) if and only if $\Delta_{JK}^{L} = \emptyset$. 

Proof. Recall from Definition 4.4 that

\begin{equation}
\varpi_J \cdot \varpi_K = \frac{1}{m_J m_K} \left( \prod_{j \in J} \varpi_j \right) \cdot \left( \prod_{k \in K} \varpi_k \right).
\end{equation}

If \( J \cap K = \emptyset \), then this does not contain a square of \( \varpi_1, \ldots, \varpi_{n-1} \), and it is clearly equal to \( \frac{m_L}{m_J m_K} \varpi_L \).

Thus we may assume that \( J \cap K \neq \emptyset \) which implies that the right hand side of (5.4) contains some squares. By extracting the terms which produce the squares, we can express the product in the right hand side of (5.4) as

\begin{equation}
\left( \prod_{j \in J} \varpi_j \right) \cdot \left( \prod_{k \in K} \varpi_k \right) = \left( \prod_{i \in J \cap K} \varpi_i \right) \cdot \left( \prod_{q \in J \cup K} \varpi_q \right).
\end{equation}

We compute the product in the right hand side of this equality. For this purpose, take the decomposition \( J \cup K = M_1 \sqcup \cdots \sqcup M_s \) into the connected components. Let \( i \) be the smallest element of \( J \cap K \). Then we have \( i \in M_r \) for some \( 1 \leq r \leq s \). Since \( M_r \) is connected, we can express it as \( M_r = \{ a, a+1, \ldots, b \} \) for some \( a, b \in J \cup K \) with \( a \leq i \leq b \). Then, by Lemma 5.1 the product \( \varpi_i \cdot \left( \prod_{q \in J \cup K} \varpi_q \right) \) can be expanded as

\[
\varpi_i \cdot \left( \prod_{q \in J \cup K} \varpi_q \right) = \frac{b - i + 1}{b - a + 2} \prod_{q \in J \cup K \setminus \{ a - 1 \}} \varpi_q + \frac{i - a + 1}{b - a + 2} \prod_{q \in J \cup K \cup \{ b + 1 \}} \varpi_q,
\]

where we have no squares of \( \varpi_q \)'s in the right hand side since \( M_r \) is a connected component of \( J \cup K \). If \( |J \cap K| \geq 2 \), then let \( i' \) be the smallest element of \( J \cap K \setminus \{ i \} \). Multiplying \( \varpi_{i'} \) to the right hand side of this equality, we can expand it by square-free monomials in \( \varpi_1, \ldots, \varpi_{n-1} \) by Lemma 5.1 again (cf. Example 5.2). Repeating this procedure for each element of \( J \cap K \) in increasing order, we obtain that

\[
\left( \prod_{i \in J \cap K} \varpi_i \right) \cdot \left( \prod_{q \in J \cup K} \varpi_q \right) = \sum_{L \supseteq J \cup K \atop |L| = |J| + |K|} \left( \sum_{P \in \Delta^L_{JK}} \text{wt}(P) \right) \prod_{q \in L} \varpi_q,
\]

by the construction of the left-right diagrams and their weights. Combining this with (5.4) and (5.5), we obtain that

\[
\varpi_J \cdot \varpi_K = \sum_{L \supseteq J \cup K \atop |L| = |J| + |K|} \left( \frac{m_L}{m_J m_K} \sum_{P \in \Delta^L_{JK}} \text{wt}(P) \right) \varpi_L,
\]

which implies the desired claim. \( \square \)

Example 5.7. Let \( n = 10 \) and take \( J = \{ 1, 3, 5, 7 \} \), \( K = \{ 3, 6, 8 \} \) as in Example 5.2. We compute the coefficients in (5.3) for the following two choices of \( L \). Note that we have \( m_J = 3! \) and \( m_K = 1 \).
(i) For $L = \{1, 2, 3, 4, 5, 6, 7, 8\}$, we have $m_L = 8!$, and the weights of left-right diagrams associated with the $(J, K, L)$ are computed in Example 5.5. Hence we obtain that

$$d_{JK}^L = \frac{m_L}{m_J m_K} \sum_{P \in \Delta_{JK}^L} \text{wt}(P) = \frac{8!}{3!} \left( \frac{1}{2} \cdot \frac{3}{5} + \frac{1}{2} \cdot \frac{3}{7} \right) = 3456$$

which coincides with the coefficient of $\omega_L = \omega_{\{1, 2, 3, 4, 5, 6, 7, 8\}}$ in Example 5.2.

(ii) For $L' = \{1, 2, 3, 5, 6, 7, 8, 9\}$, we have $m_{L'} = 3! \cdot 5!$, and the weight of the left-right diagram associated with the $(J, K, L')$ are computed in Example 5.5. Hence we obtain that

$$d_{JK}^{L'} = \frac{m_{L'}}{m_J m_K} \sum_{P \in \Delta_{JK}^{L'}} \text{wt}(P) = \frac{3! \cdot 5!}{3!} \left( \frac{1}{2} \cdot \frac{2}{5} \right) = 24$$

which coincides with the coefficient of $\omega_{L'} = \omega_{\{1, 2, 3, 5, 6, 7, 8, 9\}}$ in Example 5.2.

**Remark 5.8.** Theorem 5.6 provides the combinatorial description of the computation demonstrated in Example 5.2. As we observed there, the geometric idea behind our computation is the realization of $\Omega_J$ by intersecting the divisors $E_i$.

### 6. Relations to other works

In this section, we clarify how the results in this paper are related to other works in existing literatures. Especially, we explain the relations to the work of Goldin-Gorbutt ([18]) on Peterson Schubert calculus and to the works of Berget-Spink-Tseng ([10]), Nadeau-Tewari ([28]), and the second author ([21]) on mixed Eulerian numbers. We emphasize that [10, 18, 28] are announced earlier than this paper.

#### 6.1. Relations to Peterson Schubert calculus

We begin with reviewing the motivation of Peterson Schubert calculus from [9, 14, 18, 20]. We first note that these papers studied the equivariant cohomology ring of $\text{Pet}_n$ with respect to the $\mathbb{C}^*$-action explained in Section 2.1, but we focus on the ordinary cohomology ring to compare with our computation (see [9, 14, 18, 20] for the results in the equivariant cohomology). Recall that the dual Schubert variety $\Omega_w$ associated with $w \in \mathfrak{S}_n$ determines the homology cycle $[\Omega_w]$ in $H_*(Fl_n; \mathbb{Z})$. We denote by $\sigma_w \in H^{2t(w)}(Fl_n; \mathbb{Z})$ the Poincaré dual of $[\Omega_w]$, which is called the Schubert class associated with $w$. It is well-known that the set of Schubert classes $\{\sigma_w \mid w \in \mathfrak{S}_n\}$ forms an additive basis of $H^*(Fl_n; \mathbb{Z})$. Thus we may express the product $\sigma_u \cdot \sigma_v$ as a linear combination of the Schubert classes:

$$\sigma_u \cdot \sigma_v = \sum_{w \in \mathfrak{S}_n} c_{uw}^w \sigma_w, \quad c_{uw}^w \in \mathbb{Z}.$$

Computations of the structure constants $c_{uw}^w$ is called Schubert calculus on the flag variety $Fl_n$. Geometrically, $c_{uw}^w$ is the intersection number $\int_{Fl_n} (\sigma_u \cdot \sigma_v \cdot \sigma_{w_0 w})$, and this implies the positivity for the structure constants, i.e., $c_{uv}^w \geq 0$ by Kleiman’s transversality theorem (see e.g. [11, Sect. 1.3]).

Motivated by this, Harada and Tymoczko considered the following problem in [20]. Let $p_w \in H^*(\text{Pet}_n; \mathbb{C})$ denote the image of the Schubert class $\sigma_w \in H^*(Fl_n; \mathbb{C})$ under
the restriction map $H^*(Fl_n; \mathbb{C}) \to H^*(\text{Pet}_n; \mathbb{C})$. They called $p_w$ the Peterson Schubert class corresponding to $w$. Since the restriction map $H^*(Fl_n; \mathbb{C}) \to H^*(\text{Pet}_n; \mathbb{C})$ is surjective ([20, 23]), it is natural to ask whether there exists a natural subset of Peterson Schubert classes $p_w$ which forms an additive basis of $H^*(\text{Pet}_n; \mathbb{C})$. They gave an answer to this question as follows. Let $J = \{j_1 < j_2 < \cdots < j_m\}$ be a subset of $[n-1]$. They defined the element $v_J \in \mathfrak{S}_n$ to be the product of simple transpositions whose indices are in $J$, in increasing order, that is,

$$v_J := s_{j_1}s_{j_2}\cdots s_{j_m}.$$  

\textbf{Theorem 6.1.} ([20, Theorem 4.12]) The set $\{p_{v_J} \mid J \subseteq [n-1]\}$ forms a $\mathbb{C}$-basis of $H^*(\text{Pet}_n; \mathbb{C})$.

By this theorem, we may expand the product $p_{v_J} \cdot p_{v_K}$ in terms of the Peterson Schubert classes $p_{v_L}$:

$$p_{v_J} \cdot p_{v_K} = \sum_{L \subseteq [n-1]} c^L_{JK} p_{v_L}, \quad c^L_{JK} \in \mathbb{C}. \quad (6.2)$$

Computing the structure constants $c^L_{JK}$ is called Peterson Schubert calculus in [18]. Harada and Tymoczko also gave Monk’s formula for $c^L_{JK}$ in [20, Theorem 6.12], which is the case for $|J| = 1$. Recently, Goldin and Gorbutt gave combinatorial formulas for the structure constants $c^L_{JK}$ in [18, Theorems 1,4,6,7] which are manifestly positive and integral. In particular, their formulas imply the positivity for the structure constants.

\textbf{Theorem 6.2.} ([18, Corollary 8]) The structure constants $c^L_{JK}$ in (6.2) are non-negative integers for all $J, K, L \subseteq [n-1]$.

This theorem ensures that all the coefficients $c^L_{JK}$ in (6.2) are (non-negative) integers, but it is not obvious whether $\{p_{v_J} \in H^{2|J|}(\text{Pet}_n; \mathbb{Z}) \mid J \subseteq [n-1]\}$ forms a $\mathbb{Z}$-basis of $H^*(\text{Pet}_n; \mathbb{Z})$. Moreover, it is natural to ask a geometric reason of this positivity for the structure constants $c^L_{JK}$ (cf. [9, Remark 3.4] and [20, p.43, question (2)]). In what follows, we give an answer to this question. Recall from [9] that we have Giambelli’s formula for the Peterson Schubert classes.

\textbf{Theorem 6.3.} (Giambelli’s formula for the Peterson variety, [9, Theorem 3.2]) For $J \subseteq [n-1]$, we have

$$p_{v_J} = \frac{1}{|J_1|!|J_2|!\cdots|J_m|!} \prod_{i \in J} p_{s_i}, \quad (6.3)$$

where $J_k (1 \leq k \leq m)$ are the the connected components of $J$.

\textbf{Remark 6.4.} Drellich gave Giambelli’s formula for arbitrary Lie types in [14].

As is well-known, the Schubert class $\sigma_{s_i}$ can be written as $\sigma_{s_i} = x_1 + \cdots + x_i = \varpi_i$ in $H^2(Fl_n; \mathbb{Z})$, where $x_1, \ldots, x_n$ are defined in (4.3). This implies that

$$p_{s_i} = \varpi_i \quad \text{in} \quad H^2(\text{Pet}_n; \mathbb{Z}),$$

for $1 \leq i \leq n-1$ by taking the restriction. Thus, the right hand side of (6.3) is nothing but $\varpi_J$ in Definition 4.4. As a consequence of Theorems 4.14 and 6.3, we obtain the following result which explains the geometric background of the Peterson Schubert calculus.
Corollary 6.5. For $J \subseteq [n-1]$, we have $p_{w_J} = \omega_J$. In particular, the set
$$\{p_{w_J} \in H^{2|J|}(\text{Pet}_n; \mathbb{Z}) \mid J \subseteq [n-1]\}$$
forms a $\mathbb{Z}$-basis of $H^*(\text{Pet}_n; \mathbb{Z})$. Moreover, the structure constant $c^L_{JK}$ in (6.2) is equal to the structure constant $d^L_{JK}$ in Theorem 5.6.

Remark 6.6. This implies that Lemma 5.1 is essentially a special case of Monk’s formula [20, Theorem 6.12].

By Corollary 6.5, the structure constants $d^L_{JK}$ can also be computed by the formulas for $c^L_{JK}$ proved earlier by Goldin-Gorubtt [18] in the $\mathbb{C}^*$-equivariant setting (see Section 2.1). Their approach to the structure constants is mostly combinatorial whereas our approach is geometric based on the properties of $X_J$ and $\Omega_J$. We end this subsection by giving a short observation on the difference of their formulas and ours.

Suppose that $J, K, L \subseteq [n-1]$ are all connected subsets such that $J \cup K \subseteq L$, $|L| = |J| + |K|$. Then, we may write $J = [a_1, a_2]$, $K = [b_1, b_2]$, $L = [c_1, c_2]$, and we may assume that $a_1 \leq b_1$ by interchanging the roles of $J$ and $K$ if necessary. In this case, their formula ([18, Corollary 2]) for $c^L_{JK}$ is quite simple:
$$c^L_{JK} = \frac{(a_2 - b_1 + 1)(b_2 - a_1 + 1)}{(a_1 - c_1)(b_1 - c_1)}.$$

For general $J, K, L \subseteq [n-1]$, their computation of $c^L_{JK}$ consists of three (ordered) formulas ([18, Theorems 3, 5, 6]) each of which successively makes a reduction to the computations in the former case.

In contrast, our formula has several terms even when $J, K, L \subseteq [n-1]$ are all connected, however it provides a single formula which covers all the cases of general $J, K, L \subseteq [n-1]$.

6.2. Relations to mixed Eulerian numbers. We next explain the relations of the results in this paper to the works on mixed Eulerian numbers introduced and studied by Postnikov ([30]).

We briefly recall the definition of mixed Eulerian numbers. For $a_1, \ldots, a_n \in \mathbb{R}^n$, the permutahedron $P_n(a_1, \ldots, a_n)$ is defined to be the convex hull of the $S_n$-orbits of $(a_1, \ldots, a_n)$ in $\mathbb{R}^n$:
$$P_n(a_1, \ldots, a_n) = \text{ConvexHull}\{(a_{w(1)}, \ldots, a_{w(n)}) \in \mathbb{R}^n \mid w \in S_n\}.$$  

This is at most $(n-1)$-dimensional, and it sits inside of an affine hyperplane in $\mathbb{R}^n$. The $(n-1)$-dimensional volume (computed by projecting down to $\mathbb{R}^{n-1}$) of $P_n(a_1, \ldots, a_n)$ in terms of $u_i = a_i - a_{i+1}$ for $1 \leq i \leq n-1$ can be written as
$$\text{Vol } P_n(a_1, \ldots, a_n) = \sum_{c_1, \ldots, c_{n-1}} A_{c_1, \ldots, c_{n-1}} \frac{u_1^{c_1}}{c_1!} \cdots \frac{u_{n-1}^{c_{n-1}}}{c_{n-1}!},$$

where the sum is taken over all non-negative integers $c_1, \ldots, c_{n-1}$ with $c_1 + \cdots + c_{n-1} = n-1$. The coefficients $A_{c_1, \ldots, c_{n-1}}$ are called mixed Eulerian numbers which are known to be non-negative integers (see [30] for details).
In [10], Berget-Spink-Tseng studied log-concavity of matroid $h$-vectors in relation to mixed Eulerian numbers. For that purpose, they considered the invariant subring of the Chow ring of the permutohedral variety with respect to the action of the symmetric group. They introduced a basis $\delta_S$ of this invariant subring, and they proved that the structure constants of this basis can be written as products of mixed Eulerian numbers ([10, Proposition 7.7 and Corollary 7.9]). This invariant subring is known to be isomorphic to $H^*(\text{Pet}_n; \mathbb{Z})$ by [6, Theorem 1.1] (cf. [4, Theorem B] for $\mathbb{Q}$-coefficients), and one can see that their basis corresponds to $\sigma J$ in $H^*(\text{Pet}_n; \mathbb{Z})$ (compare [10, Corollary 7.9] and Lemma 5.1 in this paper). Therefore, our formula (Theorem 5.6) can also be regarded as computing some products of mixed Eulerian numbers by using the geometry of $\text{Pet}_n$.

Nadeau-Tewari ([28]) also found a relation between mixed Eulerian numbers and intersection numbers of Schubert varieties and the permutohedral variety for an arbitrary Lie type. After [10] and [28], the second author of this paper investigated in [21] a connection between Peterson Schubert calculus and mixed Eulerian numbers. More precisely, it was shown that the mixed Eulerian numbers can be written as intersection numbers of Schubert divisors in Peterson variety for an arbitrary Lie type ([21, Theorem 1.1]). We remark that, for type A, this formula was proved in [10] and [28] independently. Including this paper, all of these works are done independently, and these established connections between Peterson Schubert calculus and mixed Eulerian numbers.

To end this paper, let us lastly deduce the formula for $d_{JK}^L$ in terms of mixed Eulerian numbers in the context of Peterson Schubert calculus. For $J, K, L \subseteq [n-1]$, recall from (4.10) that we have

$$d_{JK}^L = \langle [X_L], \sigma J \cdot \sigma K \rangle_{\text{Pet}_n} = \frac{1}{m_J m_K} \int_{X_L} \left( \prod_{j \in J} \sigma_j \right) \left( \prod_{k \in K} \sigma_k \right)$$

if $|J| + |K| = |L|$ and that we have $d_{JK}^L = 0$ if $|J| + |K| \neq |L|$. Taking the decomposition $L = L_1 \sqcup \cdots \sqcup L_q$ into the connected components of $L$, we have $X_L = \prod_{i=1}^q X_{L_i}$ by Corollary 3.6. Hence, the integration over $X_L$ above can be written as a product of integrations over $X_{L_i}$ for $1 \leq i \leq q$:

$$\int_{X_L} \left( \prod_{j \in J} \sigma_j \right) \left( \prod_{k \in K} \sigma_k \right) = \prod_{i=1}^q \int_{X_{L_i}} \left( \prod_{j \in J \cap L_i} \sigma_j \right) \left( \prod_{k \in K \cap L_i} \sigma_k \right).$$

Denoting $\ell_i := |L_i| + 1$, we have $X_{L_i} \cong \text{Pet}_{\ell_i}$ by Corollary 3.6 again. Namely, each integration in the last equality is an intersection number of divisors on $\text{Pet}_{\ell_i}$. We note that under this isomorphism $\sigma_r \in H^*(\text{Pet}_{\ell_i}; \mathbb{Q})$ $(1 \leq r \leq |L_i|)$ corresponds to $\sigma_{r + \min \{L_i-1\}} \in H^*(X_{L_i}; \mathbb{Q})$ since we have $\text{Pet}_{\ell_i} \subseteq \text{Fl}(\mathbb{C}^{\ell_i})$ and $X_{L_i} \subseteq \text{Pet}_n \subseteq \text{Fl}(\mathbb{C}^n)$.

As explained above, the second author gave a formula which computes those intersection numbers as mixed Eulerian numbers ([21, Theorem 1.1]). By applying it to the integrations above, we obtain the following formula for which we take the convention that $A_{c_1, \ldots, c_p} = 0$ unless $c_1 + \cdots + c_p = p$ for positive integers $p$. 


Theorem 6.7. For $J, K, L \subseteq [n-1]$, we have
\[ d_{JK}^L = \frac{1}{m_J m_K} \prod_{i=1}^{q} A^{c_1^{(i)} \cdots c_{i-1}^{(i)}}_{L_{i}} \] where $L = L_1 \sqcup \cdots \sqcup L_q$ is the decomposition into the connected components of $L$ and $c_1^{(i)}, \ldots, c_{i-1}^{(i)}$ are the multiplicities of the product $(\prod_{j \in J \cap L_i} \varpi_j)(\prod_{k \in K \cap L_i} \varpi_k)$ given by
\[ c_r^{(i)} := \begin{cases} 2 & \text{if } r + \min L_i - 1 \in J \cap K, \\ 1 & \text{if } r + \min L_i - 1 \in (J \cup K) - (J \cap K), \\ 0 & \text{otherwise} \end{cases} \] for $1 \leq i \leq q$ and $1 \leq r \leq |L_i|$ (which means $r + \min L_i - 1 \in L_i$).

Remark 6.8. As we noted above, this formula can also be deduced from [10, Proposition 7.7].

Remark 6.9. The indexes of the mixed Eulerian numbers appearing in Theorem 6.7 are always less than or equal to 2. In [10] and [21], mixed Eulerian numbers with arbitrary indexes are considered.
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