Abstract

Modular transformations of string theory (including the well-known stringy dualities) play a crucial role in the discussion of discrete flavor symmetries in the Standard Model. They are at the origin of $CP$-transformations and provide a unification of $CP$ with traditional flavor symmetries. Here, we present a novel, fully systematic method to reliably compute the unified flavor symmetry of the low-energy effective theory, including enhancements from the modular transformations of string theory. The unified flavor group is non-universal in moduli space and exhibits the phenomenon of “Local Flavor Unification” where different sectors of the theory can be subject to different flavor structures.
1 Introduction

The origin of flavor remains one of the most challenging questions in the Standard Model (SM) of particle physics. String theory, as a consistent ultra-violet completion of the SM, can provide some useful ideas to attack this puzzle. Previous discussions of the origin of flavor symmetry in string theory [1–3] relied on some “guesswork” based on the geometry of compactified space and properties of string selection rules. While this led to models with appealing discrete flavor symmetries, it typically did not address the origin of CP. A first step to include CP was made in ref. [4], where a CP candidate was identified as an outer automorphism of the traditional flavor group. This provided a string theory origin of the general mechanism of group theoretical CP violation discussed earlier in [3–10]. Still, a comprehensive picture of the origin of flavor and CP remained illusive: A priori, it is not clear whether the interpretation of the geometry of compact dimensions and string theory (space group) selection rules [11–13] gives the complete set of symmetries. A more general mechanism is needed to clarify the situation.

In the present paper we shall present such a general mechanism. It is based on the consideration of outer automorphisms of the Narain lattice and the Narain space group [14–16]. The full set of symmetries is determined by the properties of the Narain space group [17]. In this way, the Narain space group encodes all the information from the string theory models under consideration, providing a unified description of the traditional flavor symmetry with CP (or CP-like) transformations as its outer automorphisms [4]. Apart from the traditional flavor symmetries discussed so far, the full flavor symmetry uncovered in our approach also includes duality (modular) transformations that exchange winding and momentum states [18] and act nontrivially on the twisted states of string theory [19–24]. This provides a new perspective on the theory of flavor and CP which was already outlined in our earlier paper [17]. The main results of the new scheme include:

- The traditional flavor symmetries are only one part of this picture (these are the symmetries that are universal in the moduli space of string theory).

- Modular (including duality) transformations of string theory are new ingredients of the full flavor structure. At some specific lower-dimensional regions (e.g. points or lines) in moduli space, the modular transformations become symmetries and lead to an enhancement of the flavor group, as illustrated in figure 1.

- CP (or CP-like) transformations are shown to be part of these modular transformations [17]. CP is an exact symmetry only within the self-dual regions and spontaneously broken otherwise [4]. The modular enhanced flavor structure, therefore, leads to a unification of traditional flavor symmetries and CP.

- The full unified flavor group is non-universal in moduli space, while the traditional flavor group is its subgroup that is preserved universally in moduli-space.

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1This observation was reported in our earlier paper [17] and was subsequently incorporated in an explicit bottom-up flavor construction in [25].
The moduli space of the Kähler modulus $T$ for the two-dimensional $\mathbb{Z}_3$ orbifold. The traditional flavor symmetry $\Delta(54)$ is conserved at all points universally. If the Kähler modulus $T$ is stabilized on the indicated lines or points, the traditional flavor symmetry is enhanced by those modular transformations that leave the corresponding lines or points invariant. Importantly, these enhancements correspond to outer automorphisms of the traditional flavor symmetry.

- The non-universality of the flavor symmetry allows for different flavor structures in different sectors of the theory. This allows the implementation of the significant difference of flavor structure in the quark and lepton sector of the Standard Model.

Figure 1: The moduli space of the Kähler modulus $T$ for the two-dimensional $\mathbb{Z}_3$ orbifold. The traditional flavor symmetry $\Delta(54)$ is conserved at all points universally. If the Kähler modulus $T$ is stabilized on the indicated lines or points, the traditional flavor symmetry is enhanced by those modular transformations that leave the corresponding lines or points invariant. Importantly, these enhancements correspond to outer automorphisms of the traditional flavor symmetry.

The paper is organized as follows. In section 2 we shall introduce the Narain lattice, the Narain space group, and its outer automorphisms. In section 3 we present technicalities of modular (and duality) transformations in two compact dimensions. The connections between modular transformations and flavor symmetries are subject of section 4. We shall discuss the enhancement of the traditional flavor symmetry and classify the possible enhanced symmetry groups in the two-dimensional case. In section 5 we consider the modular transformations of the two-dimensional $\mathbb{Z}_3$ orbifold and classify all outer automorphisms of the corresponding $\mathbb{Z}_3$ Narain space group. In section 6 we present the landscape of the enhanced flavor symmetries of the $\mathbb{Z}_3$ orbifold. The traditional flavor symmetry is $\Delta(54)$ and it is universal in moduli space. On fixed lines and circles (of modular transformations) we find an enhancement of $\Delta(54)$ to $\text{SG}(108,17)$ which includes a $\mathcal{CP}$-like transformation as an exact symmetry. There are several different possibilities for such an enhancement, corresponding to several different ways of enhancing $\Delta(54)$ by different $\mathbb{Z}_2$ subgroups of $S_4$ – the group of outer automorphisms of $\Delta(54)$. The location of the fixed straight lines and circles is shown in figure 1. The enhancements
combine at points where two lines meet and the flavor group is enhanced to $SG(216,87)$. The maximal enhancement to $SG(324,39)$ is obtained at points where three lines meet. In section 7 we present the lessons from string theory for flavor model building. We shall compare the string theory point of view with previous bottom-up attempts in flavor model building that implement modular symmetries and $C\mathbb{P}$. More technical details of the construction are relegated to the appendices. Sections 2-5 are rather technical and may be skipped in a first reading. The main results of the paper can be appreciated by looking at figures 1 and 2 and reading sections 6 and 7.

2 Outer automorphisms of the Narain space group

2.1 Narain lattice

A toroidal compactification of $D$ bosonic string coordinates $y$ can be described most conveniently in the Narain formulation, where $y$ is separated into $D$ right- and $D$ left-moving degrees of freedom $y_R$ and $y_L$. Combined into a $2D$-dimensional Narain coordinate $Y$ this reads

$$
\begin{pmatrix}
y \\
y^\ast
\end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix}1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix}y_R \\
y_L
\end{pmatrix} \text{ and } Y := \begin{pmatrix}y_R \\
y_L
\end{pmatrix},
$$

(1)

where $\tilde{y}$ denotes the T-dual coordinate of $y$. Then, $Y$ is compactified on a $2D$-dimensional torus that is parameterized by a Narain vielbein matrix $E$, i.e.

$$
Y \sim Y + E\hat{N}, \text{ where } \hat{N} = \begin{pmatrix}n \\
m
\end{pmatrix} \in \mathbb{Z}^{2D}.
$$

(2)

Here, $\hat{N} \in \mathbb{Z}^{2D}$ contains the string’s winding and Kaluza-Klein quantum numbers $n$ and $m$, respectively. To render the world-sheet theory modular invariant the Narain vielbein $E$ has to span an even self-dual lattice $\Gamma = \{E\hat{N} \mid \hat{N} \in \mathbb{Z}^{2D}\}$ with metric $\eta$ of signature $(D,D)$. Consequently, one can always choose $E$ such that

$$
E^T \eta E = \hat{\eta}, \text{ where } \eta := \begin{pmatrix}-1 & 0 \\
0 & 1
\end{pmatrix} \text{ and } \hat{\eta} := \begin{pmatrix}0 & 1 \\
1 & 0
\end{pmatrix},
$$

(3)

and the Narain vielbein can be parameterized as

$$
E := \frac{1}{\sqrt{2}} \begin{pmatrix}e^{-T} & -\sqrt{\alpha'} e^{-T} \\ \sqrt{\alpha'} & e^{-T}
\end{pmatrix} \begin{pmatrix}G - B \\
G + B
\end{pmatrix}.
$$

(4)

In this definition of the Narain vielbein, $e$ denotes the vielbein of the $D$-dimensional geometrical torus $\mathbb{T}^D$ with metric $G := e^T e$, $e^{-T}$ corresponds to the inverse transposed matrix of $e$, $B$ is the anti-symmetric background $B$-field ($B = -B^T$), and $\alpha'$ is called the Regge slope. Then,
the scalar product of two Narain lattice vectors \( \lambda_i = E \hat{N}_i \in \Gamma \) with \( \hat{N}_i \in \mathbb{Z}^{2D} \) for \( i \in \{1, 2\} \) can be evaluated using eq. (3),

\[
\lambda_1^T \eta \lambda_2 = \hat{N}_1^T E^T \eta E \hat{N}_2 = \hat{N}_1^T \eta \hat{N}_2 = n_1^T m_2 + m_1^T n_2 \in \mathbb{Z} .
\] (5)

As a remark, the Narain vielbein \( E \) can be extended to the heterotic string by including Wilson line background fields that act on the 16 extra left-moving bosonic gauge degrees of freedom of the heterotic string, see e.g. [26].

### 2.2 Narain space group

Next, we generalize the Narain lattice construction to \( \mathbb{Z}_K \) orbifolds in the Narain formulation. In this case, the toroidal compactification from eq. (2) is extended by the action of a \( \mathbb{Z}_K \) Narain twist \( \Theta \), i.e.

\[
Y \sim \Theta^k Y + E \hat{N}, \quad \text{where} \quad \Theta = \begin{pmatrix} \theta_R & 0 \\ 0 & \theta_L \end{pmatrix} \quad \text{and} \quad \Theta^K = 1 .
\] (6)

The integer \( k \in \{0, \ldots, K - 1\} \) enumerates the twisted sectors. We have to demand the block-structure of the Narain twist \( \Theta \) in eq. (6) such that \( \Theta \) cannot mix left- and right-moving modes of the string. In addition, the matrices \( \theta_R \) and \( \theta_L \) must be orthogonal in order to leave the string’s mass invariant (c.f. eqs. (94) and (97) in appendix A.3). For supersymmetric orbifolds it is moreover necessary that \( \theta_R \in \text{SO}(D) \). We define a \( \mathbb{Z}_K \) Narain space group \( S_{\text{Narain}} \) as the multiplicative closure of a finite list of generators:

\[
S_{\text{Narain}} := \langle (\Theta, 0), (1, E_i) \text{ for } i \in \{1, \ldots, 2D\} \rangle ,
\] (7)

where the vector \( E_i \) corresponds to the \( i \)-th column of the vielbein matrix \( E \), and we restrict ourselves to \( \mathbb{Z}_K \) Narain space groups without roto-translations\(^3\). Then, a general \( \mathbb{Z}_K \) Narain space group element \( g \) reads

\[
g = (\Theta^k, E \hat{N}) \in S_{\text{Narain}} \quad \text{where} \quad k \in \{0, \ldots, K - 1\} \quad \text{and} \quad \hat{N} \in \mathbb{Z}^{2D} ,
\] (8)

and acts on the Narain coordinates \( Y \) as \( Y \mapsto gY = \Theta^k Y + E \hat{N} \), see eq. (6). The Narain space group has to close under multiplication, especially \( (\Theta, 0)(1, E \hat{N}) (\Theta^{-1}, 0) = (1, \Theta E \hat{N}) \) has to be an element of the Narain space group for all \( \hat{N} \in \mathbb{Z}^{2D} \). Hence, \( \Theta \) has to be an outer automorphism of the Narain lattice, \( \Theta \Gamma = \Gamma \). Importantly, the Narain space group gives a natural framework to discuss the various classes of closed strings on Narain orbifolds, as briefly reviewed in appendix A. Simultaneously it yields the discrete symmetries of the string setting via its outer automorphisms, as discussed in sections 2.3 and 2.4.

\(^2\)The name “\( \mathbb{Z}_K \) Narain space group” refers to the \( K \)-fold twist \( \Theta \) in \( S_{\text{Narain}} \) and should not indicate that the group is Abelian. By definition of \( S_{\text{Narain}} \), it is clear that it is actually non-Abelian, and, moreover, non-compact.

\(^3\)A \( \mathbb{Z}_K \) roto-translation would be generated by \( (\Theta, V) \), where \( V \not\in \Gamma \) but \( (\Theta, V)^K = (1, \lambda) \) with \( \lambda \in \Gamma \).
Finally, by conjugation with the Narain vielbein \((E, 0)\) we change the basis to the so-called lattice basis. To highlight this, all quantities in the lattice basis are written with a hat. For example, for each \(g \in \hat{S}_{\text{Narain}}\) we define\(^4\)

\[
\hat{g} := (E^{-1}, 0) (\Theta^k, E N) (E, 0) = (\hat{\Theta}^k, \hat{N}) \in \hat{S}_{\text{Narain}},
\]
and the Narain twist in the lattice basis is defined as \(\hat{\Theta} := E^{-1} \Theta E\), where \(\hat{\Theta} \in \text{GL}(2D, \mathbb{Z})\) follows automatically from the fact that \(\hat{\Theta}\) is an outer automorphism of the Narain lattice. Depending on the choice of the Narain twist \(\Theta\), the condition \(\hat{\Theta} \in \text{GL}(2D, \mathbb{Z})\) can freeze some of the free parameters (i.e. moduli) of the Narain vielbein \(E\) to some special values.

Due to its general form in eq. (6), the twist in the lattice basis has to satisfy

\[
\hat{\Theta}^T \hat{\eta} \hat{\Theta} = \hat{\eta} \quad \text{and} \quad \hat{\Theta}^T \mathcal{H} \hat{\Theta} = \mathcal{H},
\]
where we have introduced the so-called generalized metric \(\mathcal{H} := E^T E\). Hence, the Narain twist \(\hat{\Theta}\) leaves the Narain scalar product eq. (5) as well as the generalized metric invariant.

In this work, we concentrate on so-called symmetric \(Z_K\) orbifolds, meaning that we assume \(\theta := \theta_R = \theta_L\), i.e. the Narain twist \(\Theta\) acts left-right-symmetric.

### 2.3 Outer automorphisms of the Narain lattice

A natural framework to understand the origin of modular transformations in string theory is the Narain lattice. In general, two lattices \(\Gamma\) and \(\Gamma'\) are identical if their vielbeins \(E\) and \(E'\) are related by a transformation \(\Sigma \in \text{GL}(2D, \mathbb{Z})\), such that\(^5\)

\[
E \mapsto E' = E \Sigma^{-1} \quad \text{for} \quad \Sigma \in \text{GL}(2D, \mathbb{Z}).
\]

If \(\Gamma = \Gamma'\) is a Narain lattice we have to demand in addition that \(\hat{\Sigma}\) leaves the Narain metric \(\hat{\eta}\) invariant, \(\hat{\Sigma}^T \hat{\eta} \hat{\Sigma} = \hat{\eta}\). This follows by using that both, \(E\) and \(E'\), satisfy eq. (3). We use these conditions to define the group \(O_\hat{\eta}(D, D, \mathbb{Z})\) of outer automorphisms of the Narain lattice,\(^6\)

\[
O_\hat{\eta}(D, D, \mathbb{Z}) := \{ \hat{\Sigma} \mid \hat{\Sigma} \in \text{GL}(2D, \mathbb{Z}) \quad \text{with} \quad \hat{\Sigma}^T \hat{\eta} \hat{\Sigma} = \hat{\eta} \}.
\]

In section 3 below, we will analyze in more detail the group \(O_\hat{\eta}(D, D, \mathbb{Z})\) and its action on the moduli of the theory. We will then confirm that the outer automorphisms of the Narain lattice \(\Gamma\) give rise to the modular transformations of this string setting.\(^\square\) Observe that, due to eq. (10), also the Narain twist \(\hat{\Theta}\) must be an element of \(O_\hat{\eta}(D, D, \mathbb{Z})\), with the additional constraint \(\hat{\Theta}^T \mathcal{H} \hat{\Theta} = \mathcal{H}\).

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\(^4\)To highlight whenever we work in the lattice basis, we also write \(\hat{S}_{\text{Narain}}\) for the Narain space group, even though \(\hat{S}_{\text{Narain}} \cong S_{\text{Narain}}\) are, of course, one and the same group.

\(^5\)We use the inverse matrix \(\Sigma^{-1}\) here for later convenience.

\(^6\)Here, we ignore the continuous translational outer automorphisms of the Narain lattice as they correspond to additional \(U(1)\) symmetries, which will be broken to discrete subgroups in section 2.4 after orbifolding.
2.4 Outer automorphisms of the Narain space group

After having discussed the outer automorphisms of the Narain lattice \( \Gamma \), we continue with the outer automorphisms of the Narain space group \( \hat{S}_{\text{Narain}} \). They are of special interest as they yield the modular transformations as well as the flavor symmetries of the effective four-dimensional theory after orbifolding \[17\]. An outer automorphism of the Narain space group can be written as a transformation with

\[
\hat{h} := (\hat{\Sigma}, \hat{T}) \notin \hat{S}_{\text{Narain}},
\]

that acts as conjugation on \( \hat{S}_{\text{Narain}} \) and, by the definition of an automorphism, maps the Narain space group to itself. In detail, for each element \( (\hat{\Theta}^k, \hat{N}) \in \hat{S}_{\text{Narain}} \) we have to ensure that

\[
(\hat{\Theta}^k, \hat{N}) \xrightarrow{\hat{h}} (\hat{\Sigma}, \hat{T}) (\hat{\Theta}^k, \hat{N}) (\hat{\Sigma}, \hat{T})^{-1} \in \hat{S}_{\text{Narain}}.
\]

The action of such an outer automorphism on strings is discussed in appendix A.2. This Narain approach can be viewed as a stringy completion of a purely geometrical approach to identify flavor symmetries, for example in complete intersection Calabi-Yau manifolds \[27, 28\]. As discussed in ref. \[17\] one can find a set of generators of the group of outer automorphisms that is of the form

\[
\left\{(\hat{\Sigma}_1, 0), (\hat{\Sigma}_2, 0), \ldots, (1, \hat{T}_1), (1, \hat{T}_2), \ldots\right\}.
\]

In other words, the group of outer automorphisms can be generated by pure twists \((\hat{\Sigma}_i, 0) \notin \hat{S}_{\text{Narain}}\) and pure translations \((1, \hat{T}_j) \notin \hat{S}_{\text{Narain}}\). Roto-translations are not needed to generate the outer automorphism group. Consequently, the translational part \(\hat{T}_j\) of \((1, \hat{T}_j) \notin \hat{S}_{\text{Narain}}\) must be fractional, i.e. \(\hat{T}_j \not\in \mathbb{Z}^{2D}\) and \(0 \leq \hat{T}_j < 1\). Moreover, the twist \(\hat{\Sigma}_i\) of \((\hat{\Sigma}_i, 0) \notin \hat{S}_{\text{Narain}}\) cannot be a rotation of the type \(\hat{\Theta}^\ell\) that has been used to construct the orbifold, i.e. \(\hat{\Sigma}_i \neq \hat{\Theta}^\ell\) for \(\ell \in \{0, \ldots, K - 1\}\). Otherwise, \((1, \hat{T}_j)\) and \((\hat{\Sigma}_i, 0)\) would be inner automorphisms of the Narain space group, which is excluded by assumption.

The condition \[14\] has important consequences for the special case of pure lattice translations, \((1, \hat{N}) \in \hat{S}_{\text{Narain}}\), and purely rotational outer automorphisms, \(\hat{h} = (\hat{\Sigma}, 0) \notin \hat{S}_{\text{Narain}}\). Namely, taking

\[
(1, \hat{N}) \xrightarrow{\hat{h}} (\hat{\Sigma}, 0) (1, \hat{N}) (\hat{\Sigma}^{-1}, 0) = (1, \hat{\Sigma} \hat{N}) \xrightarrow{\hat{h}} \hat{S}_{\text{Narain}},
\]

we find that \(\hat{N} \xrightarrow{\hat{h}} \hat{\Sigma} \hat{N} \in \mathbb{Z}^{2D}\), for any \(\hat{N} \in \mathbb{Z}^{2D}\). Hence, a necessary condition for any twist \(\hat{\Sigma} \in \text{GL}(2D, \mathbb{Z})\) to be an outer automorphism \(\hat{h} = (\hat{\Sigma}, 0)\) of the Narain space group is that \(\hat{\Sigma}\) itself has to be an outer automorphism of the Narain lattice, that is

\[
\hat{\Sigma} \in O_{\eta}(D, D, \mathbb{Z}).
\]

\[7\] However, this statement in general does not hold in the case when the Narain space group \(\hat{S}_{\text{Narain}}\) itself has roto-translations as generators.
3 Modular transformations

Let us now specialize to $D = 2$ dimensions and analyze how transformations $E \mapsto E' = E \hat{\Sigma}^{-1}$ with $\hat{\Sigma} \in \text{O}_\eta(2,2,\mathbb{Z})$ act on the Kähler ($T$) and complex structure modulus ($U$). Specifically, we will recapitulate how the modular transformations of the theory originate from the symmetries of the Narain lattice.

In $D = 2$ dimensions a general two-torus $\mathbb{T}^2$ is parameterized by three real numbers (the lengths of the basis vectors $e_1$ and $e_2$ of the geometrical vielbein $e$ and their relative angle $\phi$). The strength of the anti-symmetric $B$-field is another free parameter. These four numbers can be combined into the so-called Kähler modulus $T$ and complex structure modulus $U$, which are given by

$$T = T_1 + i T_2 := \frac{1}{\alpha'} \left( B_{12} + i \sqrt{\det G} \right),$$

$$U = U_1 + i U_2 := \frac{1}{G_{11}} \left( G_{12} + i \sqrt{\det G} \right) = \frac{|e_2|}{|e_1|} e^{i\phi}.$$  

(18a)

(18b)

$T$ and $U$ describe all deformations of the $(2,2)$ Narain lattice. Consider now the matrices

$$\hat{K}_S := \begin{pmatrix} 0 & \epsilon \\ \epsilon & 0 \end{pmatrix} \quad \text{and} \quad \hat{K}_T := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\hat{C}_S := \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix} \quad \text{and} \quad \hat{C}_T := \begin{pmatrix} \gamma & 0 \\ 0 & \gamma^{-1} \end{pmatrix},$$

(19a)

(19b)

It is easily confirmed that $\hat{K}_{S,T}$ and $\hat{C}_{S,T}$ are elements of $O_\eta(2,2,\mathbb{Z})$, as defined in eq. (12). Furthermore, using the presentation

$$\text{SL}(2,\mathbb{Z}) = \langle S, T \mid S^4 = 1, S^2 = (ST)^3 \rangle,$$

(20)

and noting that $\hat{K}_S^2 = \hat{C}_S^2 = -1$, one confirms that $\hat{K}_{S,T}$ and $\hat{C}_{S,T}$ generate the modular group $[\text{SL}(2,\mathbb{Z})_T \times \text{SL}(2,\mathbb{Z})_U]/\mathbb{Z}_2 \subset O_\eta(2,2,\mathbb{Z})$. The $\mathbb{Z}_2$ quotient here is generated by the abstract element $(S_T)^2(S_U)^2$ with $S_T \in \text{SL}(2,\mathbb{Z})_T$ and $S_U \in \text{SL}(2,\mathbb{Z})_U$, corresponding to the elements $\hat{K}_S$ and $\hat{C}_S$ of the representation (19). Finally, the set of generators of $O_\eta(2,2,\mathbb{Z})$ is completed by the mutually commuting $\mathbb{Z}_2$ matrices

$$\hat{\Sigma}_s := \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \hat{M} := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$  

(21)

Altogether this establishes the structure of $O_\eta(2,2,\mathbb{Z})$ as

$$O_\eta(2,2,\mathbb{Z}) \cong [(\text{SL}(2,\mathbb{Z})_T \times \text{SL}(2,\mathbb{Z})_U) \times (\mathbb{Z}_2 \times \mathbb{Z}_2)]/\mathbb{Z}_2.$$  

(22)

*The Kähler modulus $T$ here should not be confused with the outer automorphism translations $\hat{T}$ above or the abstract $\text{SL}(2,\mathbb{Z})$ group element $T$ below. We think it is always self-explanatory and clear from the context when we refer to which object.

*Note that the other factorized duality $\hat{M}'$ is also part of $O_\eta(2,2,\mathbb{Z})$ through the following relation: $\hat{M}' = K_S^3 \hat{C}_S \hat{M}$. 

7
Next, we compute the transformation properties of the moduli \( T \) and \( U \) under \( \hat{K}_S, \hat{C}_S, \hat{\Sigma}_s, \) and \( \hat{M} \). It is convenient to use the generalized metric \( \mathcal{H} = E^T E \) for this. As the Narain vielbein depends on the moduli \( E = E(T, U) \), cf. eq. (4), so does the generalized metric \( \mathcal{H} = \mathcal{H}(T, U) \). Under a transformation \( (11) \) \( \mathcal{H} \) transforms as

\[
\mathcal{H}(T, U) \overset{\hat{\Sigma}}{\longrightarrow} \mathcal{H}(T', U') = \hat{\Sigma}^{-T} \mathcal{H}(T, U) \hat{\Sigma}^{-1}.
\]  

(23)

This equation can be used to read off the transformations of the moduli\(^{10}\)

\[
T \overset{\hat{\Sigma}}{\longrightarrow} T' = T'(T, U) \quad \text{and} \quad U \overset{\hat{\Sigma}}{\longrightarrow} U' = U'(T, U).
\]  

(24)

Since \( \pm \hat{\Sigma} \) both yield exactly the same transformation of the moduli in eq. (23), each factor of \( \text{SL}(2, \mathbb{Z}) \) acts only as \( \text{PSL}(2, \mathbb{Z}) \) on the moduli. We find that \( \hat{K}_S \) and \( \hat{K}_T \) induce the transformations

\[
\hat{K}_S : T \mapsto -\frac{1}{T}, \quad U \mapsto U, \quad \tag{25a}
\]

\[
\hat{K}_T : T \mapsto T + 1, \quad U \mapsto U, \quad \tag{25b}
\]

as expected for the modular group \( \text{PSL}(2, \mathbb{Z})_T \) of the Kähler modulus \( T \). Furthermore, \( \hat{C}_S \) and \( \hat{C}_T \) generate the transformations

\[
\hat{C}_S : T \mapsto T, \quad U \mapsto -\frac{1}{U}, \quad \tag{26a}
\]

\[
\hat{C}_T : T \mapsto T, \quad U \mapsto U + 1, \quad \tag{26b}
\]

giving rise to the modular group \( \text{PSL}(2, \mathbb{Z})_U \) of the complex structure modulus \( U \). Finally, \( \hat{\Sigma}_s \) reflects the real parts of both, \( T \) and \( U \),

\[
\hat{\Sigma}_s : T \mapsto -\bar{T}, \quad U \mapsto -\bar{U}, \quad \tag{27}
\]

while \( \hat{M} \) interchanges the Kähler and complex structure modulus

\[
\hat{M} : T \mapsto U, \quad U \mapsto T, \quad \tag{28}
\]

see e.g. ref. [21]. We will see later that \( \hat{\Sigma}_s \) is related to \( \mathcal{CP} \) or \( \mathcal{CP} \)-like transformations \([17]\), while \( \hat{M} \) induces the so-called mirror symmetry.

\[^{10}\text{As an alternative, one may use section 3.3 of ref. [26] with } \tilde{M} \in \{ (\hat{K}_S)^{-1}, (\hat{K}_T)^{-1}, (\hat{C}_S)^{-1}, (\hat{C}_T)^{-1}, \hat{\Sigma}, \hat{M} \}, \text{where we use the inverse matrices due to our altering definition of the action in eq. (11).}\]
4 Connection between modular transformations and flavor symmetries

Modular transformations and flavor symmetries are closely connected in string theory [17]. We distinguish them here by their action on the moduli $T$ and $U$ and their breaking-behavior under non-vanishing vacuum expectation values (VEVs) of $\langle T \rangle$ and $\langle U \rangle$.

As discussed in section 2.2 orbifolding can freeze some of the moduli that parameterize the Narain vielbein $E$ to special values. Let us denote these frozen moduli collectively by $M_{\text{fix}}$ and the unfrozen ones by $M$. For example, as we will see in detail in section 5 for a symmetric $\mathbb{Z}_3$ orbifold in $D = 2$ dimensions the complex structure modulus is frozen, $M_{\text{fix}} = U = \exp (2\pi i / 3)$, while the Kähler modulus remains unconstrained, $M = T$.

Now, let us consider an outer automorphism of the Narain space group, eq. (14) and denote it by $\hat{h} = (\hat{\Sigma}, \hat{T}) \notin \hat{S}_{\text{Narain}}$, where $\hat{\Sigma} \in O_{\hat{\eta}}(D,D,\mathbb{Z})$. Then, we can distinguish three cases of transformations:

1. The traditional flavor symmetry. It is defined as the subgroup of the outer automorphisms of the Narain space group that remains unbroken at every point in moduli space $\langle M \rangle$. All transformations $\hat{h}$ which trivially leave the moduli invariant,

$$ M \overset{\hat{h}}{\mapsto} M'(M) = M , $$

belong to this class. In general, these transformations include all translational outer automorphisms $\hat{h} = (1, \hat{T}) \notin \hat{S}_{\text{Narain}}$ and the trivially acting element $\hat{h} = (-1, 0)$. However, note that the latter element can also be an inner automorphism if it is taken to be part of the orbifold twist, in which case it does not appear as a flavor symmetry. Together, the translations $(1, \hat{T}_j) \notin \hat{S}_{\text{Narain}}$ and possibly the inversion $(-1, 0) \notin \hat{S}_{\text{Narain}}$ generate what we call the traditional flavor symmetry.

2.a) The modular transformations after orbifolding. These are given by those outer automorphisms $\hat{h} = (\hat{\Sigma}, \hat{T}) \notin \hat{S}_{\text{Narain}}$ that give rise to nontrivial modular transformations,

$$ M \overset{\hat{h}}{\mapsto} M'(M) \neq M , $$

for generic values of the moduli $M$. Consequently, modular transformations are generally spontaneously broken by the VEVs of the moduli $\langle M \rangle$, i.e. at a generic point in moduli space. This may also include transformations involving complex conjugation and the mirror symmetry.

2.b) The unified flavor symmetry. This symmetry is a combination of the traditional flavor symmetry (1.) together with specific enhancements from the modular transformations (2.a) that depend on the location in moduli space. This happens due to the fact that some transformations from case (2.a) have fixed points, i.e.

$$ M \overset{\hat{h}}{\mapsto} M'(M) \overset{!}{=} M , $$

9
for some special values of the moduli $M$, even though in general $M'(M) \neq M$. If the VEVs of the moduli are stabilized precisely at these fixed points, then the corresponding outer automorphism $\hat{h}$ enhances the traditional flavor symmetry to build up the unified flavor symmetry. For example, $M \mapsto M'(M) = -1/M$ is a nontrivial modular transformation which has a fixed point at $M = i$.

This enhancement may include modular transformations which involve complex conjugation or permutation of the moduli, which are generically related to $CP$ and $CP$-like transformations or to so-called mirror symmetry, respectively. Hence, not all of these transformations are flavor symmetries in the traditional sense, for what reason we decided to call the resulting group the “unified flavor symmetry”. Depending on the localization in moduli space there can be various different “unified flavor symmetries”. These unified flavor symmetries share the property that they are broken spontaneously to the traditional flavor symmetry, case (1.), once the moduli are deflected from any of their fixed points, eq. (31), to a generic point in moduli space.

4.1 Enhancements of the flavor symmetry

Let us discuss cases (1.) and (2.b) in more detail. As already remarked above, we exclude here transformations which are part of the orbifold twist $\hat{\Theta}$ and focus on the true outer automorphisms.

Focusing on case (2.b) we are looking for outer automorphisms of the Narain space group $\hat{h} = (\hat{\Sigma}, \hat{T})$ with $\hat{\Sigma} \in O_{\hat{\eta}}(D, D, \mathbb{Z})$ that leave the moduli invariant only at some special points but not at a generic point in moduli space. Hence, using eq. (23) we find the condition

$$\mathcal{H}(M) \xrightarrow{\hat{\Sigma}} \mathcal{H}(M') = \hat{\Sigma}^{-T} \mathcal{H}(M) \hat{\Sigma}^{-1} = \mathcal{H}(M),$$

which has to be solved for $M$ in order to identify values of the moduli with potentially enhanced symmetry. We can define $\Sigma := E \hat{\Sigma} E^{-1}$ and rewrite eq. (32) as

$$\Sigma^T \Sigma \xrightarrow{\hat{\Sigma}} \mathcal{H}(M).$$

Moreover, combining this condition with $\Sigma^T \eta \Sigma = \eta$ from eq. (12), we have to demand

$$\Sigma^T \Sigma \xrightarrow{\hat{\Sigma}} 1,$$

in analogy to the Narain twist $\Theta$ in eq. (6). We remark that this block-diagonal structure of orthogonal matrices in $\Sigma$ automatically ensures that the left- and right-moving masses of a general untwisted string are invariant under a transformation with $\Sigma$. All potential enhancements of the traditional flavor symmetry originate from outer automorphisms $(\hat{\Sigma}, 0) \notin \hat{S}_{\text{Narain}}$, where $\Sigma$ is of the form stated in eq. (34).

---

11The transformations that leave the moduli invariant at every point in moduli space are classified as class (1.) above. Therefore, if they are not part of the orbifold twist, they are already included as part of the traditional flavor symmetry.
In the next section, we will make use of this block-diagonal structure of \( \Sigma \) in \( D = 2 \) dimensions to classify — independent of the choice of orbifold twist — all rotational outer automorphisms of the Narain space group that leave invariant the moduli at some special regions in moduli space. In contrast, the translational outer automorphisms depend on the chosen orbifold and, hence, they must be discussed case by case, see section 5.4.1 for an example.

4.2 Classifying the possible enhancements of flavor symmetries in \( D = 2 \)

We now specialize to the case of two extra dimensions compactified on a symmetric \( \mathbb{Z}_K \) orbifold with \( K \neq 2 \) and classify all possible enhancements of the traditional flavor symmetry. All of them can be described by different \( \Sigma \)'s of the form stated in eq. (34). As discussed in ref. [17] the automorphism condition (14) in \( D = 2 \) restricts the determinants of \( \sigma_R \) and \( \sigma_L \). In more detail, eq. (14) implies that for all \( k \) there is a \( k' \) such that

\[
\sigma_R \theta^k = \theta^{k'} \sigma_R \quad \text{and} \quad \sigma_L \theta^k = \theta^{k'} \sigma_L.
\]

(35)

The determinant of \( \sigma_R \in \text{O}(2) \) is constrained to be \( \pm 1 \). Let us assume first that \( \sigma_R \) is a rotation and not a reflection, i.e. \( \det(\sigma_R) = 1 \). Then, we use that in \( D = 2 \) dimensions all rotations necessarily commute. Consequently, \( k = k' \) in eq. (35) and we find

\[
\sigma_L \theta = \theta \sigma_L.
\]

(36)

Moreover, by assumption we are considering the case \( \theta \neq -1 \) since \( K \neq 2 \). Hence, \( \sigma_L \) must be a rotation, too. Thus, in this case we get \( \det(\sigma_R) = \det(\sigma_L) = +1 \). Repeating these arguments for \( \sigma_R \) being a reflection, i.e. \( \det(\sigma_R) = -1 \), we obtain in general

\[
\det(\sigma_R) = \det(\sigma_L).
\]

(37)

This restricts \( \Sigma \) to four cases (now given in the lattice basis \( \hat{\Sigma} = E^{-1} \Sigma E \)):

1. Symmetric rotations \( \hat{S}_{\text{rot}}(\alpha) \) with \( \sigma_R = \sigma_L \) and \( \det(\sigma_R) = +1 \),

2. Symmetric reflections \( \hat{S}_{\text{refl}}(\alpha) \) with \( \sigma_R = \sigma_L \) and \( \det(\sigma_R) = -1 \),

3. Asymmetric rotations \( \hat{A}_{\text{rot}}(\alpha_R, \alpha_L) \) with \( \sigma_R \neq \sigma_L \) and \( \det(\sigma_R) = \det(\sigma_L) = +1 \),

4. Asymmetric reflections \( \hat{A}_{\text{refl}}(\alpha_R, \alpha_L) \) with \( \sigma_R \neq \sigma_L \) and \( \det(\sigma_R) = \det(\sigma_L) = -1 \).

Here, the symmetric transformations are parameterized by one angle \( \alpha \) (being either the rotation angle or the angle of the reflection axis) and asymmetric transformations are parameterized by two angles \( \alpha_R \) and \( \alpha_L \) (being either the rotation angles or the angles of the reflection axes).

\(^{12}\)We exclude the case \( K = 2 \) of \( \mathbb{Z}_2 \) orbifolds here, because for this case there are, in general, outer automorphisms which do not obey eq. (37). For example, for \( K = 2 \) there can be outer automorphisms that act as a rotation on the right-mover and as a reflection on the left-mover.
By the definition of outer automorphisms, the rotations must map the four-dimensional Narain lattice to itself. Thus, the order of a four-dimensional rotation (associated to \( \hat{S}_{\text{rot}}(\alpha) \) or \( \hat{A}_{\text{rot}}(\alpha_R, \alpha_L) \)) is restricted to the orders of the possibly allowed crystallographic rotations in four dimensions. These orders can easily be found from the Euler-\( \phi \) function \([29]\), and they are given by

\[
\{1, 2, 3, 4, 5, 6, 8, 10, 12\}.
\] (38)

Another condition is that the \( 4 \times 4 \) matrix \( \hat{\Sigma} \) must be integral in the lattice basis, c.f. eq. (12). As this matrix is obtained from \( \Sigma \) (given by the four cases above) via \( \hat{\Sigma} = E^{-1}\Sigma E \), this integral condition is in general only fulfilled for special values of the moduli that parameterize the vielbein \( E \) of the Narain lattice. This explains why the unified flavor symmetry – originating from the outer automorphism group of the Narain space group – can depend on the value of the moduli.

After having classified all outer automorphisms of the Narain space group, we also want to analyze how they act on both, untwisted and twisted string states. The transformation of untwisted strings can be computed easily as we show in appendix A.3. However, in order to identify the actual flavor symmetry generated by the outer automorphisms, one also has to identify the transformation properties of twisted strings. We will proceed to do this for the example of a \( \mathbb{Z}_3 \) orbifold in the next section.

One of the main explicit results of the present work then is a complete classification of all outer automorphisms \( \hat{h} = (\hat{\Sigma}, \hat{T}) \notin \hat{S}_{\text{Narain}} \) for the two-dimensional \( \mathbb{Z}_3 \) orbifold presented in the next section.

5 Outer automorphisms of the \( \mathbb{Z}_3 \) Narain space group

To be specific, we analyze the symmetric \( \mathbb{Z}_3 \) orbifold in \( D = 2 \) as our main example. We begin in section 5.1 by defining the Narain space group of the symmetric \( \mathbb{Z}_3 \) orbifold in \( D = 2 \) dimensions. Then, in section 5.2, we identify those modular transformations that remain unbroken after orbifolding and analyze the transformation properties of untwisted and twisted strings in section 5.3. Finally, in section 5.4, we classify the outer automorphisms into the above types (1.) and (2.b), i.e. into traditional and unified flavor symmetries, respectively.

5.1 \( \mathbb{Z}_3 \) Narain space group

In the lattice basis of the \( (2, 2) \) Narain formulation, the symmetric \( \mathbb{Z}_3 \) twist \( \hat{\Theta} \) reads (cf. section 2.2)

\[
\hat{\Theta} = \begin{pmatrix}
0 & -1 & 0 & 0 \\
1 & -1 & 0 & 0 \\
0 & 0 & -1 & -1 \\
0 & 0 & 1 & 0
\end{pmatrix} \in O_\eta(2, 2, \mathbb{Z}) \Leftrightarrow \theta = \theta_R = \theta_L = \begin{pmatrix}
-\frac{1}{2} & -\frac{\sqrt{3}}{2} \\
\frac{\sqrt{3}}{2} & -\frac{1}{2}
\end{pmatrix} \right) .
\] (39)
It can be decomposed into the generators $\hat{C}_S$ and $\hat{C}_T$ of the modular group $\text{SL}(2,\mathbb{Z})_U$ as

$$\hat{\Theta} = \left(\hat{C}_S\right)^3 \hat{C}_T \in \text{SL}(2,\mathbb{Z})_U.$$  \hspace{1cm} (40)

Consequently, we can use eq. (26) to show that the moduli $T$ and $U$ transform under the twist action (40) as

$$\hat{\Theta} : T \mapsto T, \quad U \mapsto -\frac{1}{U+1}.$$  \hspace{1cm} (41)

Therefore, $\hat{\Theta}$ is a symmetry of the Narain lattice $\Gamma$ spanned by the vielbein $E$ for an arbitrary value of the Kähler modulus $T$ but fixed complex structure modulus,

$$U \equiv -\frac{1}{U+1} \iff U = \exp(2\pi i/3) \Rightarrow E = E(T).$$  \hspace{1cm} (42)

Thus, the complex structure modulus $U$ is frozen at $\exp(2\pi i/3)$ which corresponds to the case $R := |e_1| = |e_2|$, enclosing an angle of $120^\circ$. This might have been expected from geometrical considerations of the symmetric $\mathbb{Z}_3$ orbifold, see eq. (18b). Hence, the metric $G$ and the $B$-field $B$ are fixed up to two free parameters: the overall radius $R$ and the parameter $b$ of the anti-symmetric $B$-field, i.e.

$$e = R \begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} \end{pmatrix}, \quad G = \frac{R^2}{2} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \quad \text{and} \quad B = b\alpha' \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$  \hspace{1cm} (43)

In this case, the Kähler modulus, eq. (18a), reads

$$T = b + i\frac{\sqrt{3}}{2}r,$$  \hspace{1cm} (44)

where we have defined $r := R^2/\alpha'$.

### 5.2 Modular transformations after orbifolding

Modular transformations were introduced in section 3 as outer automorphisms of the Narain lattice. Now, in order to remain unbroken after orbifolding, a modular transformation has to be an (outer) automorphism of the Narain space group $\hat{S}_{\text{Narain}}$ as well, see case (2.a) in section 4. In the following we analyze which elements $\hat{\Sigma}$ of the modular group eq. (22) of the general $(2,2)$ Narain lattice satisfy the condition that for each $k \in \{0,1,2\}$ there is a $k' \in \{0,1,2\}$ such that the $\mathbb{Z}_3$ orbifold twist $\hat{\Theta}$ from eq. (39) satisfies the condition

$$\hat{\Sigma} \hat{\Theta}^k \hat{\Sigma}^{-1} = \hat{\Theta}^{k'},$$  \hspace{1cm} (45)

which originates from eq. (14). In this case, $\hat{\Sigma}$ is an unbroken modular transformation after orbifolding.

We note that $\hat{\Theta} \in \text{SL}(2,\mathbb{Z})_U$, see eq. (40). Moreover, elements from $\text{SL}(2,\mathbb{Z})_T$ commute with those from $\text{SL}(2,\mathbb{Z})_U$. It is therefore obvious that the generators $\hat{K}_S$ and $\hat{K}_T$ of $\text{SL}(2,\mathbb{Z})_T$ commute with $\hat{\Theta}$,

$$\hat{K}_S \hat{\Theta} \hat{K}_S^{-1} = \hat{\Theta} \quad \text{and} \quad \hat{K}_T \hat{\Theta} \hat{K}_T^{-1} = \hat{\Theta}.$$  \hspace{1cm} (46)
Hence, modular transformations from $\text{SL}(2, \mathbb{Z})_T$ are automorphisms of the $\mathbb{Z}_3$ Narain space group and, therefore, remain unbroken after orbifolding. Note that these transformations do not interchange the twisted sectors, i.e. a string with constructing element $\hat{g} = (\hat{\Theta}^k, \hat{N}) \in \hat{S}_{\text{Narain}}$ from the $k$-th twisted sector is mapped by a modular transformation $\text{SL}(2, \mathbb{Z})_T$ to a string from the same twisted sector, see eq. (90) in appendix A.

On the other hand, $\hat{C}_S \hat{\Theta} \hat{C}_S^{-1} \neq \hat{\Theta}^{k'}$ and $\hat{C}_T \hat{\Theta} \hat{C}_T^{-1} \neq \hat{\Theta}^{k'}$, for any $k' \in \{0, 1, 2\}$. Consequently, the generators of the modular group $\text{SL}(2, \mathbb{Z})_U$ are not automorphisms of the $\mathbb{Z}_3$ Narain space group – in other words, $\text{SL}(2, \mathbb{Z})_U$ is broken by the orbifold. This can also be understood in the following way: As we have seen in eq. (42), the complex structure modulus $U$ has to be frozen at $U = \exp(2\pi i/3)$ for the symmetric $\mathbb{Z}_3$ orbifold. Hence, any modular transformation that does not leave $U = \exp(2\pi i/3)$ invariant must be broken. Indeed, there is a $\mathbb{Z}_6$ subgroup of $\text{SL}(2, \mathbb{Z})_U$, generated by $\hat{C}_S \hat{C}_T$, which leaves $U = \exp(2\pi i/3)$ invariant. This $\mathbb{Z}_6$ can be written as $\mathbb{Z}_3 \times \mathbb{Z}_2$, where the $\mathbb{Z}_3$ is generated by the orbifold twist $\hat{\Theta} = (\hat{C}_S)^3 \hat{C}_T$ while the $\mathbb{Z}_2$ is generated by $\hat{C}_S^2$. Thus, the $\mathbb{Z}_3$ factor is an inner, not an outer, automorphism of the Narain space group, while the $\mathbb{Z}_2$ factor can be written as $\hat{C}_S^2 = \hat{K}_*^2$ and, hence, also appears from the unbroken modular group $\text{SL}(2, \mathbb{Z})_T$. Consequently, the modular group $\text{SL}(2, \mathbb{Z})_U$ does not contain any independent outer automorphisms of the $\mathbb{Z}_3$ Narain space group and, therefore, does not contribute to the flavor symmetry.

In addition to elements of $\text{SL}(2, \mathbb{Z})_U$ or $\text{SL}(2, \mathbb{Z})_T$, we consider the $\mathcal{CP}$-like transformation $\hat{K}_*$ defined as

$$
\hat{K}_* := \hat{C}_S \hat{C}_T \hat{C}_S \hat{\Sigma}_* = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \text{with} \quad (\hat{K}_*)^2 = 1 .
$$

This transformation is an outer automorphism of the $\mathbb{Z}_3$ Narain space group

$$
\hat{K}_* \hat{\Theta} \hat{K}_*^{-1} = \hat{\Theta}^2 ,
$$

and hence remains unbroken after orbifolding. Very importantly, $\hat{K}_*$ interchanges strings from the first and second twisted sector, see eq. (90) in appendix A. Consequently, eq. (49) has significant consequences for the heterotic string: the heterotic string has 16 extra left-moving bosonic degrees of freedom $X^I$ for $I = 1, \ldots, 16$, which give rise to the 10D gauge symmetry, for example $E_8 \times E_8$, and four right-moving complex world-sheet fermions $\Psi^a$ (in light-cone gauge). In order to promote the interchange of twisted sectors induced by $\hat{K}_*$ to an automorphism of the heterotic string, the action of $\hat{K}_*$ has to be extended to

$$
X^I \mapsto -X^I \quad \text{and} \quad \Psi^a \mapsto \bar{\Psi}^a .
$$

Hence, the transformation $\hat{K}_*$ maps all gauge representations to their complex conjugates and interchanges left-chiral and right-chiral target-space fermions. Thus, it corresponds to a
Finally, under a transformation $\hat{K}_*$ the moduli transform as

$$
\hat{K}_*: T \mapsto -\bar{T}, \quad U \mapsto -\frac{\bar{U}}{1+\bar{U}}.
$$

(51)

This leaves the special choice of moduli

$$
T = i T_2 \quad \text{and} \quad U = \exp(2\pi i/3)
$$

(52)

invariant for all values of $T_2 \in \mathbb{R}$. Since the complex structure modulus $U$ is frozen by the $\mathbb{Z}_3$ orbifold to precisely this value according to eq. (42), we confirm that the transformation $\hat{K}_*$ is unbroken by the $\mathbb{Z}_3$ orbifold as long as $T$ takes purely imaginary values.

In conclusion, we find that the maximal modular group after orbifolding is generated by $(\hat{K}_S, \hat{K}_T, \hat{K}_*)$ and has the structure

$$
\text{SL}(2,\mathbb{Z})_T \ltimes \mathbb{Z}_2,
$$

(53)

where the $\mathbb{Z}_2$ factor acts as physical $\mathbb{CP}$ transformation, at least for the gauge sector. In fact, using the presentation [31]

$$
\text{GL}(2,\mathbb{Z}) = \langle S, T, K | S^4 = 1, S^2 = (ST)^3, K^2 = 1, (SK)^2 = 1, (TK)^2 = 1 \rangle,
$$

(54)

we find that our generators actually form a representation of the extended modular group $\text{GL}(2,\mathbb{Z}) \cong \text{SL}(2,\mathbb{Z}) \times \mathbb{Z}_2$. Very importantly, we will see next that all massless string states transform trivial under a certain subgroup of $\text{GL}(2,\mathbb{Z})$ (i.e. under a congruence subgroup, where elements of $\text{GL}(2,\mathbb{Z})$ are only defined mod 3). Hence, massless strings form representations of the finite modular group $\text{GL}(2,3)$. Furthermore, it is clear that the $\mathbb{CP}$-like transformation $\hat{K}_*$ can be spontaneously broken, if $\langle T \rangle$ is deflected away from a purely imaginary value.

### 5.3 Modular transformations of strings

Next, we discuss strings on the symmetric $\mathbb{Z}_3$ orbifold in $D = 2$ and their transformation properties under the modular group $\text{GL}(2,\mathbb{Z})$ generated by $\hat{K}_S$, $\hat{K}_T$, and $\hat{K}_*$. A detailed description of strings on orbifolds is reviewed in appendix A. In summary, there are two classes of strings on orbifolds, invariant under the $\mathbb{Z}_3$ twist: First of all, there are untwisted strings $V(\hat{N})^{\text{orb}}$ with constructing elements $[g] = \{(1, E \hat{N}), (1, E \Theta \hat{N}), (1, E \Theta^2 \hat{N})\} \subset S_{\text{Narain}}$. Here, $\hat{N} = (n_1, n_2, m_1, m_2)^T \in \mathbb{Z}^4$ parameterizes the Kaluza-Klein (KK) momentum $(m_1, m_2)$ and winding numbers $(n_1, n_2)$ of the untwisted string on the orbifold. At a generic point in $T$-moduli space, only the untwisted string with $\hat{N} = (0, 0, 0, 0)^T$ is massless. Yet, we are interested in the full tower of massive strings. It is convenient to group these untwisted strings into nine classes

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[13] The corresponding (outer) automorphism of the gauged semi-simple Lie groups is unique and corresponds to the usual, most general physical $\mathbb{CP}$ transformation. However, for the (discrete, possibly enhanced) flavor symmetry, it is possible that the corresponding transformation “only” acts as a $\mathbb{CP}$-like symmetry, which is exactly the mechanism of physical $\mathbb{CP}$ violation which we had already discussed in [4].
| generator $\hat{\Sigma}$ | transformation of $T$-modulus | transformation of charges $(M,N)$ for untwisted strings | six-dimensional representation $\hat{\Sigma}_6$ for twisted strings $(X, Y, Z, \bar{X}, \bar{Y}, \bar{Z})$ |
|---|---|---|---|
| $\hat{K}_S$ | $T \mapsto -\frac{1}{T}$ | $(M, N) \mapsto (N, -M)$ | $\begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 \\
1 & \omega & \omega^2 & 0 & 0 & 0 \\
1 & \omega^2 & \omega & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & -1 & -1 \\
0 & 0 & 0 & -1 & -\omega^2 & -\omega \\
0 & 0 & 0 & -1 & -\omega & -\omega^2
\end{pmatrix}$ |
| $\hat{K}_T$ | $T \mapsto T + 1$ | $(M, N) \mapsto (M - N, N)$ | $\begin{pmatrix}
\omega^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & \omega & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$ |
| $\hat{K}_s$ | $T \mapsto -\bar{T}$ | $(M, N) \mapsto (M, -N)$ | $\begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}$ |

Table 1: Modular transformations after $\mathbb{Z}_3$ orbifolding: SL(2,$\mathbb{Z}$)$_T$ modular transformations of the $T$-modulus are generated by $\hat{K}_S$ and $\hat{K}_T$ and extended by the $\mathbb{CP}$-like transformation $\hat{K}_s$, see section 5.2. Here and in the following, $\omega := e^{2\pi i/3}$.

of strings $V^{(M,N)}$ for $M, N \in \{0, 1, 2\}$ according to their discrete KK and winding charges, which are defined modulo 3 as

$$ (M, N) = (-m_1 + m_2, n_1 + n_2). \quad (55) $$

Second, there are twisted strings: We denote them by $(X, Y, Z)$ and $(\bar{X}, \bar{Y}, \bar{Z})$ for the three twisted strings localized at the three fixed points in the first and second twisted sector, respectively. Note that $(\bar{X}, \bar{Y}, \bar{Z})$ give rise to the right-chiral CPT-conjugates of $(X, Y, Z)$ needed to promote $(X, Y, Z)$ to complete left-chiral superfields.

Now, we can use the results of appendix A.3 to compute the transformation of orbifold-invariant untwisted strings $V(\hat{N})^{\text{orb}}$ under the generators of unbroken modular transformations $\hat{\Sigma} \in \{\hat{K}_S, \hat{K}_T, \hat{K}_s\}$.

$$ \hat{\Sigma} \in \{\hat{K}_S, \hat{K}_T, \hat{K}_s\}. \quad (56) $$
These modular transformations act naturally on untwisted strings $V(\hat{N})^{\text{orb}}$, i.e.
\[ V(\hat{N})^{\text{orb}} \rightarrow V(\hat{\Sigma}^{-1}\hat{N})^{\text{orb}}. \] (57)

Hence, modular transformations $\hat{\Sigma}$ in general permute untwisted strings, as indicated by the transformations of the classes $V^{(M,N)}$ of untwisted strings given in the third column of table I.

Finally, we compute the transformation properties of twisted strings $(X, Y, Z)$ and $(\bar{X}, \bar{Y}, \bar{Z})$. To do so, we examine the OPEs [19, 20] between twisted strings $(X, Y, Z)$ and $(\bar{X}, \bar{Y}, \bar{Z})$ which yield the classes $V^{(M,N)}$ of untwisted strings, see eq. (103) in appendix A.4. Since we know the transformations of $V^{(M,N)}$, we can infer the transformations of the twisted strings, where possible phases have been fixed using three approaches that all lead to the same result: i) By requiring minimality, such that the flavor groups computed in section 6 are as small as possible; ii) by requiring that the resulting flavor groups are identical for equivalent regions in moduli space; and iii) by using ref. [21] (ignoring those contributions that originate from other parts of the full string vertex operator and the contribution that depends on the $T$-modulus). Doing so, we obtain the six-dimensional transformation matrices $\hat{\Sigma}_6$ of the six twisted strings $(X, Y, Z, \bar{X}, \bar{Y}, \bar{Z})$ under $(\hat{\Sigma}, 0) \notin \hat{S}_{\text{Narain}}$, see also [20, 21, 24]. The results are listed in the last column of table I.

At low energies one can integrate out all massive strings and the effective low-energy theory depends only on the massless strings given by the untwisted string $V(\hat{N} = 0)^{\text{orb}}$ and the twisted strings. Under modular transformations with $\hat{K}_S$ and $\hat{K}_T$ the untwisted string $V(0)^{\text{orb}}$ is invariant, while the twisted strings transform with the six-dimensional matrices $\hat{K}_{S,6}$ and $\hat{K}_{T,6}$ as given in table I. It turns out that these matrices generate the finite modular group $T'$, which is the double covering group of $A_4 \cong \Gamma_3$. Including the $\mathbb{CP}$-like transformation $\hat{K}_s$, the finite modular group $T' \cong \text{SL}(2,3)$ is enhanced to $\text{GL}(2,3)$, see appendix A.5 for further details. In summary, the modular group of massless strings on the two-dimensional $\mathbb{Z}_3$ orbifold is $\text{GL}(2,3)$, a group of order 48, and it includes the $\mathbb{CP}$-like transformation $\hat{K}_s$.

5.4 Classification of flavor symmetries

Let us now give a complete classification of all outer automorphisms of the $\mathbb{Z}_3$ Narain space group of type (1.) and (2.b), i.e. that leave the moduli invariant, at least at some points in moduli space. By doing so, we obtain the unified flavor symmetry of the symmetric $\mathbb{Z}_3$ orbifold in two dimensions. As stated in section 2.4 the group of outer automorphisms can be generated in our case by pure twists $\hat{h} = (\hat{\Sigma}, 0) \notin \hat{S}_{\text{Narain}}$ and pure translations $\hat{h} = (1, \hat{T}) \notin \hat{S}_{\text{Narain}}$. Moreover, according to section 4.2 there are four classes of twist outer automorphisms: symmetric rotations, symmetric reflections, asymmetric rotations, and asymmetric reflections. In the following, we will discuss the pure translations and these four classes of twists individually.

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Figure 2: Fixed points and fixed curves in the moduli space of the Kähler modulus \( T = b + i \sqrt{3/2} r \) under modular transformations composed out of the generators \( \hat{K}_S, \hat{K}_T, \) and \( \hat{K}_s \).

5.4.1 Pure translations

The pure translational outer automorphisms \( \hat{h} = (1, \hat{T}) \) with \( \hat{T} \notin \mathbb{Z}^4 \) can be determined by

\[
\left(1 - \hat{\Theta}\right) \hat{T} \in \mathbb{Z}^4,
\]

using eq. (14) with \( k = 1 \). It turns out that there are two translations, denoted by A and B,

\[
A = (1, \hat{T}_1), \quad B = (1, \hat{T}_2) \quad \text{with} \quad \hat{T}_1 = \begin{pmatrix} 1/3 \\ 2/3 \\ 0 \\ 0 \end{pmatrix}, \quad \hat{T}_2 = \begin{pmatrix} 0 \\ 0 \\ 1/3 \\ 1/3 \end{pmatrix}, \quad (59)
\]

which generate all translational outer automorphisms — at any point in \( T \)-moduli space. Hence, A and B are generators of the traditional flavor symmetry as defined in section 6.

The translation A shifts the winding number, while B shifts the KK number. Using eq. (1) these shifts can be translated to shifts of the geometrical coordinates \( y \) and their T-duals \( \tilde{y} \): the translation A shifts \( y \) (and simultaneously \( \tilde{y} \) if the B-field is nontrivial), while B shifts only the T-dual coordinate \( \tilde{y} \) but leaves the geometrical coordinate \( y \) inert. Consequently, the translation B can not be obtained in a purely geometrical approach but only in the Narain construction.
5.4.2 Symmetric rotations

We start with the most general left-right-symmetric rotation in \((2,2)\) dimensions \(S_{\text{rot.}}(\alpha)\), where the rotation angle \(\alpha\) is constrained crystallographically according to eq. (38). By changing the basis to the Narain lattice basis, \(\hat{S}_{\text{rot.}}(\alpha) \in \text{GL}(4,\mathbb{Z})\), one finds that left-right-symmetric rotations in \((2,2)\) dimensions are automorphisms of the \(\mathbb{Z}_3\) Narain space group only for \(\alpha = 2\pi \ell/6\) and \(\ell \in \{0,\ldots,5\}\), independent of the value of the \(T\)-modulus. Hence, one confirms that the order 6 transformation \(\hat{S}_{\text{rot.}}(2\pi/6) = \left(\hat{C}_S\hat{C}_T\right)^5\) with \(\left(\hat{S}_{\text{rot.}}(2\pi/6)\right)^6 = \mathbb{1}\), (60) generates all symmetric rotations. Since \(\left(\hat{S}_{\text{rot.}}(2\pi/6)\right)^2 = \hat{\Theta}\) is the orbifold twist \(\hat{\Theta}\), see eq. (39), this \(\mathbb{Z}_6\) can be written as \(\mathbb{Z}_3 \times \mathbb{Z}_2\) generated by the inner automorphism \(\hat{\Theta}\) and the outer automorphism

\[
C := (\hat{S}_{\text{rot.}}(\pi), 0) \quad \text{where} \quad \hat{S}_{\text{rot.}}(\pi) = \left(\hat{S}_{\text{rot.}}(2\pi/6)\right)^3 = -\mathbb{1} = \left(\hat{K}_S\right)^2.
\]

We denote this rotational outer automorphism \((\hat{S}_{\text{rot.}}(\pi), 0)\) of the \(\mathbb{Z}_3\) Narain space group \(\hat{S}_{\text{Narain}}\) by \(C\).

In summary, the symmetric rotation \(C = (\hat{S}_{\text{rot.}}(\pi), 0)\) is unbroken at any point in \(T\)-moduli space. Hence, \(C\) is an outer automorphism of type (1.) and belongs, together with \(A\) and \(B\), to the traditional flavor symmetry as defined in section 4.

5.4.3 Symmetric reflections

Next, we discuss left-right-symmetric reflections \(S_{\text{refl.}}(\alpha)\) in \((2,2)\) dimensions. Again, we change the basis to the Narain lattice basis, \(\hat{S}_{\text{refl.}}(\alpha) = E^{-1}S_{\text{refl.}}(\alpha)E \in \text{GL}(4,\mathbb{Z})\), (63) then, one finds that the general left-right-symmetric reflection \(\hat{S}_{\text{refl.}}(\alpha)\) is an integer matrix only for \(\alpha = 2\pi \ell/6\) with \(\ell \in \{0,\ldots,5\}\), for all choices of the radius \(r\) but only for quantized values of the \(B\)-field \(b = n_B/2\) with \(n_B \in \mathbb{Z}\). Thus, the symmetric reflection \(\hat{S}_{\text{refl.}}(\alpha)\) depends additionally on \(n_B \in \mathbb{Z}\) and we write \(\hat{S}_{\text{refl.}}(n_B)(\alpha)\). One confirms that the transformations

\[
\hat{S}_{\text{refl.}}(n_B)(\alpha) = \left(\hat{S}_{\text{rot.}}(\pi)\right)^v \left(\hat{\Theta}\right)^w \hat{S}_{\text{refl.}}(2\pi/6)^{(n_B)}(2\pi/6)
\]

describe all symmetric reflections for \(v \in \{0,1\}\) and \(w \in \{0,1,2\}\) such that the reflection axis has an angle \(\alpha = 2\pi/6(1 + 3v + 2w)\). Consequently, we can choose \(\hat{S}_{\text{refl.}}(2\pi/6)\) as the sole
generator of the symmetric reflections, and define
\[ \hat{S}_{\text{refl}}(n_B) := \hat{S}_{\text{refl}}(2\pi/6) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ -n_B & n_B & 1 & 1 \\ n_B & 0 & 0 & -1 \end{pmatrix} . \] (65)

Using section 3, this outer automorphism can be decomposed into the generators of the modular group as
\[ \hat{S}_{\text{refl}}(n_B) = (\hat{K}_T)^{n_B} \hat{K}_* \text{ for } n_B \in \mathbb{Z} . \] (66)

This reflective outer automorphism \( \hat{S}_{\text{refl}}(n_B), 0 \) of \( \hat{S}_{\text{Narain}} \) is denoted by \( D(n_B) \) for \( n_B \in \mathbb{Z} \).

In summary, the symmetric reflection \( D(n_B) = (\hat{S}_{\text{refl}}(n_B), 0) \) is unbroken for an arbitrary radius \( r \) in \( T \)-moduli space, but only for \( b = n_B/2 \) with \( n_B \in \mathbb{Z} \), see figure 2. Hence, \( D(n_B) \) is an outer automorphism of type (2.b) and belongs to the unified flavor symmetry as defined in section 4.

5.4.4 Asymmetric rotations

Left-right asymmetric rotations
\[ \hat{A}_{\text{rot}}(\alpha_R, \alpha_L) = E^{-1} A_{\text{rot}}(\alpha_R, \alpha_L) E \in \text{GL}(4, \mathbb{Z}) \] (67)
are automorphisms of the \( \mathbb{Z}_3 \) Narain space group for two classes of values of the \( T \)-modulus. Firstly, at all points \( T_1 = b, T_2 = 1/c \) that fulfill
\[ c, b \cdot c \in \mathbb{Z} \text{ and } b^2 c + \frac{1}{c} \in \mathbb{Z} , \] (68)
the left-right asymmetric rotations are generated by the transformation
\[ \hat{A}_{\text{rot}}(2\pi/12, 2\pi/12) = c \begin{pmatrix} 0 & b & 1 & 0 \\ -b & b & 1 & 1 \\ b^2 + \frac{1}{c^2} & -b^2 - \frac{1}{c^2} & -b & -b \\ 0 & b^2 + \frac{1}{c^2} & b & 0 \end{pmatrix} . \] (69)

Secondly, at all points \( T_1 = b, T_2 = \sqrt{3} r/2 \) that fulfill
\[ \frac{1}{r}, \frac{|T|^2}{r} \in \mathbb{Z} \text{ and } \frac{2b}{r} \in \mathbb{Z}^{\text{odd}} , \] (70)
the left-right asymmetric rotations are generated by the symmetric \( \mathbb{Z}_6 \) rotation \( \hat{S}_{\text{rot}}(2\pi/6) \) and the transformation
\[ \hat{A}_{\text{rot}}(2\pi/3, 0) = \begin{pmatrix} -\frac{b}{r} + \frac{1}{2} & \frac{b}{r} - \frac{1}{2} & \frac{1}{r} & \frac{1}{r} \\ -\frac{b}{r} + \frac{1}{2} & 0 & 0 & \frac{1}{r} \\ \frac{b^2}{r} + \frac{3r}{4} & 0 & 0 & -\frac{b}{r} - \frac{1}{2} \\ \frac{b^2}{r} - \frac{3r}{4} & \frac{b^2}{r} + \frac{3r}{4} & \frac{b}{r} + \frac{1}{2} & \frac{b}{r} + \frac{1}{2} \end{pmatrix} . \] (71)
Together with the already discussed outer automorphisms, the two asymmetric rotations above generate all possible asymmetric rotations. For example, at first sight $\hat{A}_{\text{rot}}(0, 2\pi/3)$ is an additional left-right asymmetric transformation. However, this transformation is not independent since

$$\hat{A}_{\text{rot}}(0, 2\pi/3) = \left( \hat{A}_{\text{rot}}(2\pi/3, 0) \right)^2 \left( \hat{S}_{\text{rot}}(2\pi/6) \right)^2.$$  \hfill (72)

The points of the first class eq. (68) correspond to the blue squares in figure 2, while the second class eq. (70) is located at the green curls.

Let us give one example per class: one solution of eq. (68) is given by $b = 0$ and $r = \frac{2}{\sqrt{3}}$ (i.e. $c = 1$) yielding $\hat{A}_{\text{rot}}(2\pi/12, 2\pi/12) = \hat{K}_S \hat{\Theta}$, while one solution of eq. (70) is given by $b = b = 1/2$ and $r = 1$ resulting in $\hat{A}_{\text{rot}}(2\pi/3, 0) = \hat{K}_T \hat{K}_S \hat{S}_{\text{rot}}(2\pi/6)$. Since $\hat{\Theta}$ in the first class is an inner automorphism and $\hat{S}_{\text{rot}}(2\pi/6)$ in the second class is part of the traditional flavor symmetry, see section 5.4.2, the corresponding enhancements are entirely generated by $\hat{K}_S$ and $\hat{K}_T \hat{K}_S$, respectively.

In summary, there are two independent asymmetric rotations $(\hat{A}_{\text{rot}}(2\pi/12, 2\pi/12), 0)$ and $(\hat{A}_{\text{rot}}(0, 2\pi/3), 0)$. Both are unbroken only at special points in $T$-moduli space, given by eq. (68) and eq. (70). Hence, these asymmetric rotations are outer automorphisms of type (2.b) and, therefore, contribute to the unified flavor symmetries as defined in section 4.

5.4.5 Asymmetric reflections

Finally, left-right-asymmetric reflections

$$\hat{A}_{\text{refl}}(\alpha_R, \alpha_L) = E^{-1} A_{\text{refl}}(\alpha_R, \alpha_L) E \in \text{GL}(4, \mathbb{Z})$$  \hfill (73)

are automorphisms of the $\mathbb{Z}_3$ Narain space group only for special values of the $T$-modulus and special angles $\alpha_R \neq \alpha_L$ of the reflection axis. One can confirm that the transformations

$$\hat{A}_{\text{refl}}(\alpha_R, \alpha_L) = \begin{pmatrix} -w & 0 & 0 & v \\ -w & w & v & v \\ \frac{1-w^2}{v} & \frac{1-w^2}{v} & -w & -w \\ \frac{1-w^2}{v} & 0 & 0 & w \end{pmatrix}$$  \hfill (74)

describe all asymmetric reflections up to $\mathbb{Z}_6$ rotations given in eq. (61) for

$$v, w \in \mathbb{Z} \quad \text{and} \quad \frac{1-w^2}{v} \in \mathbb{Z}.$$  \hfill (75)

Furthermore, the $T$-modulus is constrained to live on a circle of radius $1/|v|$ with center at $(w/v, 0)$,

$$\left( T_1 - \frac{w}{v} \right)^2 + (T_2)^2 = \frac{1}{v^2}.$$  \hfill (76)

For example, a solution of eq. (75) is given by $v = \pm 1$ and $w = 0$ yielding a circle of radius 1, centered at $(0, 0)$. In this case eq. (74) is given by $\hat{A}_{\text{refl},1} := (\hat{K}_S)^3 \hat{K}_s$ for $v = 1$ and $\hat{A}_{\text{refl},1'} := \hat{K}_S \hat{K}_s$ for $v = -1$. 

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In summary, the asymmetric reflection $\hat{A}_{\text{refl}}(\alpha_R, \alpha_L, 0)$ is unbroken only on the circles eq. (76) in $T$-moduli space, see figure 2. Hence, $(\hat{A}_{\text{refl}}(\alpha_R, \alpha_L, 0))$ belongs to the unified flavor symmetry as defined in section 4.

6 Unified flavor symmetries of the $\mathbb{Z}_3$ orbifold

After having classified the generators of the (traditional and unified) flavor symmetries for the $\mathbb{Z}_3$ orbifold on the level of outer automorphisms of the corresponding Narain space group in section 5.4, we determine the resulting, moduli-dependent flavor symmetries in this section. To do so, it is necessary to identify the transformation properties of twisted strings in order to get faithful representations of the resulting flavor groups.

6.1 Traditional flavor symmetry at a generic point in $\langle T \rangle$: $\Delta(54)$

The outer automorphisms $\hat{h}$ of the $(2, 2)$-dimensional $\mathbb{Z}_3$ Narain space group of type (1.), i.e. the ones that are unbroken at a generic point $\langle T \rangle$ in $T$-moduli space, can be generated by two translations $A$ and $B$, defined in section 5.4.1, and one symmetric $\mathbb{Z}_2$ rotation $C$, defined in section 5.4.2. In order to identify the actual symmetry group of the traditional flavor symmetry, we determine the transformation properties of untwisted and twisted strings under these actions in the following.

First, we consider an orbifold-invariant untwisted string $V(\hat{N})_{\text{orb}}$ with winding and KK charges $N, M \in \{0, 1, 2\}$ such that $\hat{N} \in \Gamma_{MN} \subset \mathbb{Z}^4$. Then, using eq. (95) from appendix A.3 we obtain the following transformation properties with respect to the translational generators of the outer automorphisms $A$ and $B$ and with respect to the rotation $C$:

\begin{align*}
V(\hat{N})_{\text{orb}} & \xrightarrow{A} \omega^{2M} V(\hat{N})_{\text{orb}}, \\
V(\hat{N})_{\text{orb}} & \xrightarrow{B} \omega^N V(\hat{N})_{\text{orb}}, \\
V(\hat{N})_{\text{orb}} & \xrightarrow{C} V(-\hat{N})_{\text{orb}},
\end{align*}

(77a, 77b, 77c)

where $\omega = \exp^{2\pi i/3}$. Consequently, an untwisted string $V(\hat{N})_{\text{orb}}$ with $\hat{N} \in \Gamma_{MN}$ transforms with phases $\omega^{2M}$ and $\omega^N$ under the translations $A$ and $B$, respectively, and (for $\hat{N} \neq 0$) gets interchanged with $V(-\hat{N})_{\text{orb}}$ under $C$.

Next, we compute the transformation properties of twisted strings $(X, Y, Z, \bar{X}, \bar{Y}, \bar{Z})$ under $A$, $B$, and $C$. For the symmetric rotation $C$, we can translate eq. (62) into the six-dimensional representation $\hat{S}_{\text{rot}}(\pi) = (\hat{K}_S)^2 \mapsto (\hat{K}_{S,6})^2$ using table 1 and obtain

\[
\begin{pmatrix}
X \\
Y \\
Z \\
\bar{X} \\
\bar{Y} \\
\bar{Z}
\end{pmatrix} \xrightarrow{C} \begin{pmatrix}
-1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix}
\begin{pmatrix}
X \\
Y \\
Z \\
\bar{X} \\
\bar{Y} \\
\bar{Z}
\end{pmatrix}.
\]

(78)
irrep | untw. vertex operator | \((M, N)\) | A | B | C
--- | --- | --- | --- | --- | ---
\(1_0\) | \(V(\hat{N})_{\text{orb}} + V(\hat{-N})_{\text{orb}}\) | (0, 0) | 1 | 1 | 1
\(1'\) | \(V(\hat{N})_{\text{orb}} - V(\hat{-N})_{\text{orb}}\) | (0, 0) | 1 | 1 | −1

| irrep | twisted vertex operator | \(N\) | A | B | C
--- | --- | --- | --- | --- | ---
\(2_1\) | \(\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}\) | 0 | \(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\) | \(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\) | \(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\)
\(2_2\) | \(\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}\) | 1 | \(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\) | \(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\) | \(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\)
\(2_3\) | \(\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}\) | 2 | \(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\) | \(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\) | \(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\)

Table 2: \(\Delta(54)\) transformation properties of \(\mathbb{Z}_3\) orbifold-invariant untwisted strings \(V(\hat{N})_{\text{orb}}\) and twisted strings \((X, Y, Z)\) and \((\bar{X}, \bar{Y}, \bar{Z})\) from the first and second twisted sector, respectively. Note that \(\hat{N} \in \Gamma^{(M,N)}\) and the case \(\hat{N} = 0\) is excluded for \(1'\).

On the other hand, for the translations \(A\) and \(B\) we use the OPEs (103) from appendix A.4 to identify the transformation properties of twisted strings from the corresponding ones of untwisted strings. Hence, the strategy is exactly complementary to the one used in ref. [4] where the transformation of doublets was extracted from the assumed transformation properties of the triplets. As a result, the six-dimensional representation of the outer automorphisms \(A\), \(B\), and \(C\) generate the traditional flavor symmetry \(\Delta(54)\). The results are summarized in table 2.

Note that \(C\) corresponds to a rotation in extra dimensions, hence, one is tempted to interpret \(C\) as an \(R\)-symmetry of \(\mathcal{N} = 1\) supersymmetry. Consequently, \(\Delta(54)\) gets promoted to the first example of a non-Abelian discrete \(R\)-symmetry [32] from strings (where the Grassmann variable of \(\mathcal{N} = 1\) superspace transforms in a nontrivial one-dimensional representation). A detailed discussion of the transformation of world-sheet fermions under the rotation \(C\) is needed to settle this question.

Moreover, since \(A\) and \(B\) are defined as translational outer automorphisms they seem to commute and generate \(\mathbb{Z}_3 \times \mathbb{Z}_3\) on first sight. This would have been true if they had affected
only the untwisted strings, cf. the simultaneously diagonal operators for doublets in table 2. However, by analyzing the action of A and B on twisted strings, c.f. table 2, we realize that A and B do not commute. Interestingly, \( A^2B^2 \) and B do commute and it is precisely them which give rise to the well-known \( \mathbb{Z}_3 \times \mathbb{Z}_3 \) point and space group selection rules [11,13].

6.2 Unified flavor symmetry at \( b = \frac{\text{integer}}{2} \): SG(108, 17)

Let us now discuss the possible enhancements of the traditional flavor symmetry by modular transformations, which together give rise to the unified flavor symmetry. For quantized values of the \( B \)-field \( b = \frac{nB}{2} \) with \( n_B \in \mathbb{Z} \) and generic radii \( r \) the traditional flavor symmetry \( \Delta(54) \) gets enhanced by a left-right-symmetric reflection \( D(n_B) = (\hat{K}_T)^{n_B} \hat{K}_s, \) as described in section 5.4.3.

In order to determine the resulting unified flavor symmetry, we use table 1 to construct the six-dimensional (faithful) representation \((\hat{K}_T, 6) (n_B) (\hat{K}_s, 6)\) of \( \hat{S}_{\text{refl}}(n_B) \) that acts on the six twisted strings, i.e.

\[
\begin{pmatrix}
X \\
Y \\
Z \\
\bar{X} \\
\bar{Y} \\
\bar{Z}
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 & \omega^{2n_B} & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\omega^{n_B} & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
X \\
Y \\
Z \\
\bar{X} \\
\bar{Y} \\
\bar{Z}
\end{pmatrix}.
\]

Note that this six-dimensional representation of \( \hat{S}_{\text{refl}}(n_B) \) has a mod-3-periodicity in \( n_B \), i.e. \( \hat{S}_{\text{refl}}(n_B + 3) = \hat{S}_{\text{refl}}(n_B) \). Hence, there are three different \( \mathbb{Z}_2 \) enhancements of \( \Delta(54) \). In all three cases, one finds that the enhanced symmetry is SG(108, 17), see figure 1. For example, this can done by using GAP [33]. We note that SG(108, 17) is a group of \( \mathbb{CP} \)-type IIA — in contrast to \( \Delta(54) \) which is a \( \mathbb{CP} \)-type I group [8,9].

Consequently, we obtain a coherent picture of spontaneous \( \mathbb{CP} \)-breaking: using the results from section 3 the \( T \)-modulus transforms under \( \hat{S}_{\text{refl}}(n_B) \) as \( T \mapsto n_B - T \). Hence, the VEV \( \langle T \rangle \) is invariant for quantized \( B \)-field \( b = \frac{nB}{2} \), i.e.

\[
\langle T \rangle \mapsto n_B - \langle T \rangle = \langle T \rangle \quad \text{for} \quad \langle T \rangle = \frac{n_B}{2} + i \frac{\sqrt{3}}{2} \langle r \rangle.
\]

Here, \( n_B \in \mathbb{Z} \) specifies lines in \( T \)-moduli space, see figure 2 where the traditional flavor symmetry \( \Delta(54) \) is enhanced to SG(108, 17) and the \( \mathbb{CP} \)-like transformation \( \hat{S}_{\text{refl}}(n_B) \) is unbroken. Furthermore, moving \( \langle T \rangle \) away from the symmetry-enhanced lines,

\[
\langle T \rangle = \frac{n_B}{2} + i \frac{\sqrt{3}}{2} \langle r \rangle \quad \text{to} \quad \langle T' \rangle = \langle T \rangle + \delta T \quad \text{where} \quad \text{Re}(\delta T) \neq 0,
\]

leads to spontaneous symmetry breaking of the flavor group SG(108, 17) to \( \Delta(54) \), i.e. the \( \mathbb{CP} \)-like transformation \( \hat{S}_{\text{refl}}(n_B) \) gets broken spontaneously [17]. This is another example of spontaneous breaking of \( \mathbb{CP} \) where a \( \mathbb{CP} \)-type IIA group gets broken to a group of \( \mathbb{CP} \)-type I, implying that \( \mathbb{CP} \) will be violated by quantized geometrical phases [4].
6.3 Unified flavor symmetry at $|T|^2 = 1$: SG(108, 17)

A similar picture emerges on special circles in $T$-moduli space. For example, let us discuss the semi-circle $|T|^2 = 1$ with $T_2 > 0$ in figure 2. There, the traditional flavor symmetry $\Delta(54)$ gets enhanced by two left-right-asymmetric reflections $\hat{A}_{\text{refl},1} := (\hat{K}_S)^3 \hat{K}_s$ and $\hat{A}_{\text{refl},1'} := \hat{K}_S \hat{K}_s$ corresponding to $v = \pm 1$ and $w = 0$, respectively, see section 5.4.5. Since $\hat{A}_{\text{refl},1'} = \hat{S}_{\text{rot}}(\pi) \hat{A}_{\text{refl},1}$ and $\hat{S}_{\text{rot}}(\pi)$ is contained in $\Delta(54)$, these reflections are not independent.

In order to identify the unified flavor symmetry, we use table 1 to construct the six-dimensional (faithful) representation of

$$\hat{A}_{\text{refl},1} = \left(\hat{K}_S\right)^3 \hat{K}_s \mapsto \left(\hat{K}_S, 6\right)^3 \hat{K}_s, 6.$$ (82)

Again using GAP [33], we find that this $\mathbb{Z}_2$ outer automorphism enhances the traditional flavor group $\Delta(54)$ to SG(108, 17), see figure 1. Hence, similar to the $\mathbb{C}P$-like transformations that are conserved on straight lines in $T$-moduli space, for example $\hat{S}_{\text{refl}}(0)$ at $b = 0$, there are three $\mathbb{C}P$-like transformations that are conserved on the respective circles. These three transformations correspond to three more $\mathbb{Z}_2$ outer automorphisms of $\Delta(54)$ which are conjugate to the three previously identified $\mathbb{Z}_2$ in full $S_4$ group of outer automorphisms of $\Delta(54)$. Consequently, the unified flavor symmetry is SG(108, 17) on all of these lines and circles. However, the respective groups SG(108, 17) are not identical, but conjugate to each other.

6.4 Unified flavor symmetry at $b = 0$ and $r = 2/\sqrt{3}$: SG(216, 87)

Let us now consider the point $(b, r) = (0, 2/\sqrt{3})$ in figure 2. At this point, we identify the following type (1.) and type (2.b) outer automorphisms of the Narain space group:

1. $\Delta(54)$ is the traditional flavor symmetry at a generic point $(b, r)$ in $T$-moduli space, see section 6.1.

2. On the line at $b = 0$ we find a left-right-symmetric reflection $\hat{S}_{\text{refl}}(0) = \hat{K}_s$, see section 5.4.3 for $n_B = 0$.

3. From the point $(b, r) = (0, 2/\sqrt{3})$ we get a $\mathbb{Z}_4$ left-right-asymmetric rotation $\hat{K}_S$ (ignoring the inner automorphism $\hat{\Theta}$), see section 5.4.4.

4. On the circle of radius 1, centered at the origin we obtain two left-right-asymmetric reflections $\hat{A}_{\text{refl},1} := (\hat{K}_S)^3 \hat{K}_s$ and $\hat{A}_{\text{refl},1'} := \hat{K}_S \hat{K}_s$ corresponding to $v = \pm 1$ and $w = 0$, respectively, see section 5.4.5.

Then, using the six-dimensional representations of $\Delta(54)$, $\hat{K}_s$, and $\hat{K}_S$ from tables 1 and 2 one finds the unified flavor symmetry at $b = 0$ and $r = 2/\sqrt{3}$ as the closure of all above transformations 1.-4., resulting in SG(216, 87). This is the unified flavor symmetry at all blue squares in figure 1. We stress that the specific left-right asymmetric rotations, 3., are already contained in the closure of 1., 2., and 4. That is, the symmetry at the intersecting points of
lines and circles is already fully described by the intersecting symmetries and not enhanced beyond that.

6.5 Unified flavor symmetry at \( b = \frac{1}{2} \) and \( r = 1 \): SG(324, 39)

Next, we analyze the point \((b, r) = (\frac{1}{2}, 1)\) in \( T \)-moduli space, figure 2. There, we identify the following type (1.) and type (2.b) outer automorphisms of the Narain space group:

1. At a generic point \((b, r)\) we found \( \Delta(54) \) generated by A, B, and C as the traditional flavor symmetry, see section 6.1.
2. On the red line at \( b = \frac{1}{2} \) we get additionally a left-right-symmetric reflection \( D(1) = (\hat{S}_{\text{refl}}(1), 0) \), where \( \hat{S}_{\text{refl}}(1) = \hat{K}_T \hat{K}_s \), see section 5.4.3 for \( n_B = 1 \).
3. Moreover, at \((b, r) = (\frac{1}{2}, 1)\) there is a left-right-asymmetric \( Z_3 \) rotation \( \hat{A}_{\text{rot}}((\frac{2\pi}{3}), 0) = \hat{K}_T \hat{K}_s \hat{S}_{\text{rot}}((\frac{2\pi}{6})) \), see section 5.4.4.
4. On the circle of radius 1, centered at the origin we obtain two left-right-asymmetric reflections \( \hat{A}_{\text{refl}} := (\hat{K}_S)^3\hat{K}_s \) and \( \hat{A}_{\text{refl}'} := \hat{K}_S\hat{K}_s \) corresponding to \( v = \pm 1 \) and \( w = 0 \), respectively, see section 5.4.5.
5. Finally, on the circle of radius 1, centered at \((1, 0)\) there are two left-right-asymmetric reflections \( \hat{A}_{\text{refl}2} := \hat{K}_S\hat{K}_s\hat{K}_T\hat{K}_S \) and \( \hat{A}_{\text{refl}2'} := (\hat{K}_S)^3\hat{K}_s\hat{K}_T\hat{K}_S \) corresponding to \( v = w = \mp 1 \), respectively, see section 5.4.5.

These automorphisms are not independent. Indeed, we identify the following relations

\[
\hat{A}_{\text{refl}1'} = \hat{S}_{\text{rot}}(\pi) \hat{A}_{\text{refl}1}, \\
\hat{A}_{\text{rot}((\frac{2\pi}{3}), 0)} = \hat{S}_{\text{refl}}(1) \hat{A}_{\text{refl}1'} \hat{\Theta}^2, \\
\hat{A}_{\text{refl}2} = \hat{A}_{\text{refl}1} \hat{A}_{\text{rot}((\frac{2\pi}{3}), 0)} \hat{\Theta}, \\
\hat{A}_{\text{refl}2'} = \hat{S}_{\text{rot}((\pi))} \hat{A}_{\text{refl}2},
\]

where \( \hat{\Theta} \) is an inner automorphism of the Narain space group. Consequently, the unified flavor symmetry at \((b, r) = (\frac{1}{2}, 1)\) can be generated by the transformations A, B, and C from \( \Delta(54) \) and by \( \hat{S}_{\text{refl}}(1) \) and \( \hat{A}_{\text{refl}1} \). Then, one can use tables 1 and 2 to obtain the six-dimensional representations of \( \Delta(54) \) and of the generators

\[
\hat{S}_{\text{refl}}(1) = \hat{K}_T \hat{K}_s \text{ and } \hat{A}_{\text{refl}1} = (\hat{K}_S)^3\hat{K}_s.
\]

Again using GAP to compute the closure, we find that this six-dimensional representation generates SG(324, 39). This is the unified flavor symmetry at \((b, r) = (\frac{1}{2}, 1)\) and at all green curls in figure 1. Again, no extra generators besides those already conserved on the lines and semi-circles are needed. Finally, let us remark that at the green curls in figure 1 also the gauge symmetry gets enhanced by a \( U(1) \times U(1) \) factor, see e.g. ref. [3].
7 Conclusions and outlook

In the present paper we have presented a general method to deduce the flavor symmetries of string models. This led to a hybrid system of a unified flavor group composed of two distinct components. There is on one hand the traditional flavor group that is universal in moduli space. At some specific regions in moduli space, on the other hand, it is enhanced via duality symmetries that also include $C\bar{P}$-like transformations. The full flavor group is thus non-universal in moduli space and it allows different flavor- and $C\bar{P}$-structures for different sectors of the theory (dependent of the location of fields in the compact extra dimensions). At a generic point in moduli space the enhanced symmetries are broken spontaneously. For values of the moduli close to the self-dual points a hierarchy of flavor parameters can emerge. String theory thus provides us with some specific rules or lessons for flavor model building.

Up to now there has been substantial work on "bottom-up" model constructions of flavor that consider the concept of modular symmetries \([34-59]\). The main focus there was on the description of the flavor structure of the lepton sector of the Standard Model (based on the finite modular groups $\Gamma_N$ for $N = 2, 3, 4, 5$), where modular transformations seem to be particularly successful. There has been less work on the quark sector [46,47,52,55] and the question of $C\bar{P}$ symmetries has usually not been discussed (with the recent exception of [25]). Time has come to analyze possible connections of the bottom-up constructions with the rules and lessons from string theory presented in this paper. In order to compare the bottom-up constructions with the top-down picture, however, we first have to clarify some apparent differences between the two approaches:

- In string theory the modular transformations act nontrivially on the super- and Kähler-potential of the low-energy effective field theory [23,24], while the phenomenological bottom-up models assume an invariant superpotential. It remains to be seen how this property of string theory can be accommodated in the bottom-up approach and whether (and how) this might affect the phenomenological predictions of the models.

- String theory provides a hybrid flavor picture including the traditional flavor symmetries and parts of the finite modular group $\Gamma_N$ (here $\Gamma_3$, and actually, its $C\bar{P}$-enhanced double covering group GL(2,3)). That is, not the full modular group may be realized in the low-energy effective theory: the stabilization of the $T$-modulus necessarily leads to a spontaneous breaking of parts of the modular group. In our example, the maximal enhancement of the non-modular flavor symmetry by modular transformations is from $\Delta(54)$ to SG(324,39). This group only includes some generators of GL(2,3). At this point we need more general string theory constructions to see whether present bottom-up constructions can be embedded in a string theory framework. In any case, we would expect that both, the traditional flavor symmetries and modular symmetries should be part of the fully unified picture of flavor and $C\bar{P}$.

More work is needed to answer these questions. This would require more explicit model
building in string theory along the lines discussed in \cite{60,64}. One should keep in mind that our present discussion has concentrated on general aspects of the flavor structure, illustrated on a toy model in $D = 2$ compact extra dimensions. Even in this simplified case we were told a first lesson: string theory gives rise to potentially large flavor groups. In the simple $D = 2$ example we already obtained a group as large as $\text{SG}(324, 39)$. This has to be generalized to $D = 6$ within models that accommodate the spectrum of the standard model \cite{65}, where even larger groups are likely to emerge \cite{66,67}. With the tools described in the present paper we could then explore the full “landscape” of flavor symmetries in $D = 6$, try to make connections to the existing bottom-up constructions, and extend the existing constructions of the lepton sector to a fully unified picture. A second generic lesson from string theory concerns $\mathcal{C}\mathcal{P}$. It naturally appears as part of the modular symmetries at some specific regions in moduli space and is spontaneously broken if one moves away from these self-dual points and lines. The phenomenological properties will coincide with those discussed in ref. \cite{4}, where $\mathcal{C}\mathcal{P}$-violation is connected to the heavy winding modes of string theory. A third lesson from string theory is the appearance of a unified symmetry of flavor and $\mathcal{C}\mathcal{P}$ that is non-universal in moduli space. It includes the traditional flavor symmetry and modular symmetries at some specific regions on moduli space. Different sectors of a theory might have different flavor and $\mathcal{C}\mathcal{P}$ symmetries, and this might explain the different flavor and $\mathcal{C}\mathcal{P}$ structure of the quark and lepton sectors of the standard model. Reminiscent of the concept of local grand unification \cite{68,69} one might call this “Local Flavor Unification”, as the flavor properties are connected to the location of fields in the compact extra dimensions. This provides a new perspective for flavor model building inspired by string theory.

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\section{String states in Narain orbifolds}

This appendix gives a brief review on string states on Narain orbifolds and their properties. We start by defining closed strings on Narain orbifolds via orbifold boundary conditions in section A.1. Next, we analyze their transformation properties in section A.2 resulting in the observation that the outer automorphisms of the Narain space group $S_{\text{Narain}}$ give rise to (flavor) symmetries of the string setup. Afterwards, section A.3 introduces vertex operators of untwisted strings, while section A.4 specializes to the example of the symmetric $\mathbb{Z}_3$ orbifold in $D = 2$ dimensions. Finally, in section A.5 some details on the irreducible representations of the finite modular group $T'$ are presented with a focus on the transformation of the $\Delta(54)$
triplet of twisted strings under $T'$ as $2' \oplus 1$.

### A.1 Boundary conditions of closed strings

The orbifold boundary condition of a closed bosonic string $Y(\tau, \sigma)$ in the Narain formulation is given by

$$Y(\tau, \sigma + 1) = g Y(\tau, \sigma) = \Theta^k Y(\tau, \sigma) + E \hat{N}, \quad (85)$$

where $g = (\Theta^k, E \hat{N}) \in S_{\text{Narain}}$ is called the constructing element, see eq. (6). Since $Y \sim \tilde{g} Y$ are identified on the orbifold for all $\tilde{g} \in S_{\text{Narain}}$ the boundary conditions with constructing elements $g$ and $\tilde{g}^{-1} g \tilde{g}$ describe the same closed string. Hence, a closed string on the orbifold is associated to a conjugacy class of constructing elements

$$[g] = \{ \tilde{g}^{-1} g \tilde{g} \mid \tilde{g} \in S_{\text{Narain}} \}. \quad (86)$$

The resulting string state is denoted by $| [g] \rangle$.

Note that there is a transformation $y_R \mapsto y_R + \xi$ and $y_L \mapsto y_L - \xi$ such that $y \sim y_R + y_L$ is invariant [26]. This left-right asymmetric translation can be used match the Narain conjugacy class $[g]$ from eq. (86) to the corresponding conjugacy class of the geometrical space group.

### A.2 Transformations of closed strings under outer automorphisms

A transformation with $h = (\Sigma, E \hat{T}) \notin S_{\text{Narain}}$ acts as

$$Y \mapsto h Y = \Sigma Y + E \hat{T}. \quad (87)$$

Consequently, $h$ transforms a boundary condition (85) with constructing element $g \in S_{\text{Narain}}$ according to

$$h Y(\tau, \sigma + 1) = g h Y(\tau, \sigma) \quad \Leftrightarrow \quad Y(\tau, \sigma + 1) = (h^{-1} g h) Y(\tau, \sigma). \quad (88)$$

To ensure that $h$ is a consistent transformation, this boundary condition must belong to some, maybe different, constructing element of the orbifold theory. Hence,

$$h^{-1} g h \in S_{\text{Narain}} \text{ even though } h \notin S_{\text{Narain}}, \quad (89)$$

for all $g \in S_{\text{Narain}}$. Consequently, a transformation $h \notin S_{\text{Narain}}$ must be an outer automorphism of the Narain space group $S_{\text{Narain}}$ and, in general, it acts nontrivially on string states, i.e.

$$| [g] \rangle \xrightarrow{h} \varphi_{g,h} | [h^{-1} g h] \rangle, \quad (90)$$

up to a possible phase $\varphi_{g,h}$.
Examples: Let us discuss two examples. First we take an untwisted string with constructing element $g = (1, E \hat{N}) \in S_{\text{Narain}}$, i.e. a string with winding and KK numbers given by $\hat{N} \in \mathbb{Z}^{2D}$ that lives in the bulk of the orbifold. Then, we analyze the action of a purely rotational outer automorphism $h = (\Sigma, 0) \notin S_{\text{Narain}}$. Using $\hat{\Sigma} := E^{-1} \Sigma E$ in eq. (90) yields

$$\left|\left(1, E \hat{N}\right)\right| \xrightarrow{h} \left|\left(1, \Sigma^{-1} E \hat{N}\right)\right| = \left|\left(1, E \hat{\Sigma}^{-1} \hat{N}\right)\right|,$$

(91)

where we have already used that the phase $\varphi_{g,h}$ of an untwisted string $g$ is trivial for a purely rotational transformation $h$, as we will see explicitly in eq. (95) in appendix A.3.

As a second example, take a twisted string with constructing element $g = (\Theta, E \hat{N}) \in S_{\text{Narain}}$ which is localized at the fixed point of $g$. In this case, we take an outer automorphism $h = (1, E \hat{T}) \notin S_{\text{Narain}}$ and eq. (90) yields

$$\left|\left(\Theta, E \hat{N}\right)\right| \xrightarrow{h} \varphi_{g,h} \left|\left(\Theta, E \hat{N} - (1 - \Theta) E \hat{T}\right)\right|.$$

(92)

Hence, translations $h = (1, E \hat{T}) \notin S_{\text{Narain}}$ can permute twisted string states localized at different fixed points. The determination of the phase $\varphi_{g,h}$ of a twisted string is more involved. An example is given in section 6.1.

A.3 Untwisted vertex operators

A closed bosonic string compactified on a $D$-dimensional torus with winding numbers $n \in \mathbb{Z}^D$ and KK numbers $m \in \mathbb{Z}^D$ corresponds to a string eq. (85) with constructing element $g = (1, E \hat{N}) \in S_{\text{Narain}}$ and the associated vertex operator reads

$$V(\hat{N}) = \exp\left(2\pi i P^T \eta Y\right) \quad \text{where} \quad P = \begin{pmatrix} p_R \\ p_L \end{pmatrix} = E \hat{N} \quad \text{and} \quad \hat{N} = \begin{pmatrix} n \\ m \end{pmatrix} \in \mathbb{Z}^{2D},$$

(93)

ignoring the co-cycle and normal-ordering. The Narain momentum $P = E \hat{N}$ contains right- and left-moving momenta that enter the string’s total mass, i.e.

$$M^2 \propto P^2 = (p_R)^2 + (p_L)^2 = \hat{N}^T \mathcal{H} \hat{N},$$

(94)

plus further contributions and using $\mathcal{H} = E^T E$. Compared to refs. [4,20] we have set $V(\hat{N}) \equiv V^{p,w}$, where the momentum vector $p$ and the winding vector $w$ are given by the KK numbers $m$ and winding numbers $n$, respectively.

Let us analyze the transformation of a vertex operator $V(\hat{N})$ under a general transformation $h = (\Sigma, E \hat{T})$ of the right- and left-moving bosonic string coordinates $Y \xrightarrow{h} \Sigma Y + E \hat{T}$ from eq. (87). This yields

$$V(\hat{N}) \xrightarrow{h} \exp\left(2\pi i \hat{N}^T \eta \hat{T}\right) V(\Sigma^{-1} \hat{N}),$$

(95)

under the assumption $\hat{\Sigma} = E^{-1} \Sigma E \in O_\eta(D,D,\mathbb{Z})$, see eq. (12). We will use this result frequently, when we discuss the transformation of bosonic strings under outer automorphisms of
the \( \mathbb{Z}_3 \) Narain space group, see for example section 5.3. Furthermore, note that this transformation eq. (95) is in agreement with \( \hat{N} \to \hat{N}' = \hat{\Sigma}^{-1} \hat{N} \) from eq. (11).

Using eq. (95), one easily verifies that a vertex operator \( V(\hat{N}) \) is invariant under a shift by a Narain lattice vector \( Y \to Y + E \hat{T} \) with \( \hat{T} \in \mathbb{Z}^{2D} \), i.e.

\[
V(\hat{N}) \mapsto \exp \left( \frac{2\pi i \hat{T}^T \hat{\eta}}{1} \right) V(\hat{N}) = V(\hat{N}).
\]

(96)

On the other hand, for a fractional shift \( Y \to Y + E \hat{T} \) with \( \hat{T} \not\in \mathbb{Z}^{2D} \) a vertex operator \( V(\hat{N}) \) obtains in general a nontrivial phase.

Under the \( \mathbb{Z}_K \) orbifold action \( Y \to \Theta Y \) with \( \Theta E = E \hat{\Theta} \) a bosonic string vertex operator eq. (93) transforms as \( V(\hat{N}) \to V(\hat{\Theta}^{-1} \hat{N}) \), using eq. (95). Consequently, assuming orbifold-invariance of the other string degrees of freedom, the \( \mathbb{Z}_K \) orbifold-invariant combination for \( \hat{N} \neq 0 \) reads

\[
V(\hat{N})^{\text{orb}} = \frac{1}{\sqrt{K}} \sum_{k=0}^{K-1} V(\hat{\Theta}^k \hat{N}),
\]

(97)

and the orthogonal linear combinations are removed from the orbifold spectrum. This vertex operator corresponds to an orbifold-invariant string state \( | [g] \rangle \) with constructing element \( g = (1, E \hat{N}) \in S_{\text{Narain}} \).

**A.4 Vertex operators of the \( \mathbb{Z}_3 \) Narain orbifold**

Let us now specialize to the symmetric \( \mathbb{Z}_3 \) orbifold in two dimensions, see section 5.1. The full particle spectrum of the \( \mathbb{Z}_3 \) Narain orbifold contains untwisted strings with constructing elements \( g = (1, E \hat{N}) \in S_{\text{Narain}} \), where \( \hat{N} \in \mathbb{Z}^4 \) gives the winding numbers \( n \) and KK numbers \( m \). Then, the orbifold-invariant untwisted vertex operators read

\[
V(\hat{N})^{\text{orb}} = \frac{1}{\sqrt{3}} \left( V(\hat{N}) + V(\hat{\Theta} \hat{N}) + V(\hat{\Theta}^2 \hat{N}) \right) \quad \text{for} \quad \hat{N} = \begin{pmatrix} n \\ m \end{pmatrix} \in \mathbb{Z}^4,
\]

(98)

for \( \hat{N} \neq 0 \) and \( V(\hat{N})^{\text{orb}} = V(\hat{N}) \) for \( \hat{N} = 0 \). In addition, there are twisted strings \( (X, Y, Z) \) with constructing elements \( (\Theta, E \hat{N}) \) from the first twisted sector and \( (\hat{X}, \hat{Y}, \hat{Z}) \) with constructing elements \( (\hat{\Theta}^2, E \hat{N}) \) from the second twisted sector, where we focus on the respective twist fields. Furthermore, we note that in the following we ignore other contributions to the full string vertex operators like co-cycles, world-sheet fermions, oscillator excitations, and the 16 gauge degrees of freedom. Their inclusion can only yield additional transformation phases but cannot change the non-Abelian structure of the flavor groups which is the main concern of this work.

For an untwisted string \( V(\hat{N})^{\text{orb}} \) one can define two discrete \( \mathbb{Z}_3 \) charges: KK charge \( M \in \{0, 1, 2\} \) and winding charge \( N \in \{0, 1, 2\} \), i.e.

\[
(M, N) = (-m_1 + m_2, n_1 + n_2),
\]

(99)
where both charges are defined mod 3, see e.g. ref. [4]. Note that each term \( V(\hat{\Theta}^k \tilde{N}) \) with 
\( k = 0, 1, 2 \) in the orbifold-invariant vertex operator \( \hat{\Theta} \) carries the same \( \mathbb{Z}_3 \) charges, i.e.

\[
(M, N) \rightarrow (2m_1 + m_2, n_1 - 2n_2) = (M, N) ,
\]

(100)

using that \( M \) and \( N \) are defined modulo 3. Then, we can arrange all orbifold-invariant untwisted strings \( V(\tilde{N})^{\text{orb}} \) into nine classes \( V^{(M,N)} \) depending on their \( \mathbb{Z}_3 \times \mathbb{Z}_3 \) charges \( (M, N) \) using \( \tilde{N} \in \Gamma^{(M,N)} \subset \mathbb{Z}^3 \) for \( M, N \in \{0, 1, 2\} \), where we define the sublattices

\[ \Gamma^{(M,N)} = \{ \tilde{N} \in \mathbb{Z}^4 \mid \tilde{N} \sim \hat{\Theta} \tilde{N} \text{ and } M = -m_1 + m_2 \text{ mod } 3 \text{ and } N = n_1 + n_2 \text{ mod } 3 \} . \]

(101)

Explicitly, the nine classes \( V^{(M,N)} \) of orbifold-invariant untwisted strings \( V(\tilde{N})^{\text{orb}} \) are defined as

\[
V^{(M,N)} = \sum_{\tilde{N} \in \Gamma^{(M,N)}} C(\tilde{N}) V(\tilde{N})^{\text{orb}} \quad \text{for} \quad M, N = 0, 1, 2 ,
\]

(102)

where the coefficients \( C(\tilde{N}) \) are given in ref. [20].

The classes \( V^{(M,N)} \) of untwisted strings appear in the operator product expansions (OPEs) between twisted strings \( (X, Y, Z) \) from the first twisted sector and twisted strings \( (\bar{X}, \bar{Y}, \bar{Z}) \) from the second twisted sector [20], i.e.

\[
V^{(0,0)} = \frac{1}{3} \left( X \bar{X} + Y \bar{Y} + Z \bar{Z} \right) ,
\]

(103a)

\[
V^{(0,2)} = \frac{1}{3} \left( Y \bar{Z} + Z \bar{X} + X \bar{Y} \right) ,
\]

(103b)

\[
V^{(1,0)} = \frac{1}{3} \left( Z \bar{Y} + X \bar{Z} + Y \bar{X} \right) ,
\]

(103c)

\[
V^{(1,2)} = \frac{1}{3} \left( X \bar{X} + \omega Y \bar{Y} + \omega^2 Z \bar{Z} \right) ,
\]

(103d)

\[
V^{(2,0)} = \frac{1}{3} \left( Z \bar{X} + \omega^2 X \bar{X} + \omega Y \bar{Y} \right) ,
\]

(103e)

where \( \omega = \exp(2\pi i/3) \). These OPEs will turn out to be crucial in order to translate the transformation properties eq. (98) of untwisted strings to the twisted strings.

A.5 Irreducible representations of the finite modular group \( T' \)

In addition to their transformation under the traditional flavor symmetry \( \Delta(54) \), where they transform as \( 6 = 3 \oplus 3 \), the twisted string states \( (X, Y, Z, \bar{X}, \bar{Y}, \bar{Z}) \) of the \( \mathbb{Z}_3 \) orbifold also transform under modular transformations \( S, T \in \text{SL}(2, \mathbb{Z})_T \) with the six-dimensional matrices \( \hat{K}_{S,6} \) and \( \hat{K}_{T,6} \) given in table [1]. However, these two matrices do not correspond to a faithful representation of \( \text{SL}(2, \mathbb{Z})_T \), (for example, \( (\hat{K}_{T,6})^3 = 1 \) even though \( T \in \text{SL}(2, \mathbb{Z})_T \) has infinite order) but they generate the finite modular group \( T' \equiv \text{SL}(2, 3) \) (the double covering group of
Table 3: Character table of the finite modular group $T' \cong \text{SL}(2, 3)$. Here, $\omega := e^{\frac{2 \pi i}{3}}$.

$$
\begin{array}{c|cccccccc}
T' & [1] & [s^2] & [t] & [t^2] & [s] & [s^2t] & [s^2t^2] \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1' & 1 & 1 & \omega & \omega^2 & 1 & \omega & \omega^2 \\
1'' & 1 & 1 & \omega^2 & \omega & 1 & \omega^2 & \omega \\
2 & 2 & -2 & -1 & -1 & 0 & 1 & 1 \\
2' & 2 & -2 & -\omega & -\omega^2 & 0 & \omega & \omega^2 \\
2'' & 2 & -2 & -\omega^2 & -\omega & 0 & \omega^2 & \omega \\
3 & 3 & 3 & 0 & 0 & -1 & 0 & 0 \\
\end{array}
$$

$A_4 \cong \Gamma_3$ of order 24), which can be defined by the presentation$^{14}$

$$
T' = \left\langle s, t \mid s^4 = t^3 = (st)^3 = 1, s^2 t = t^2 s^2 \right\rangle.
$$

(104)

Indeed, the matrices $\hat{K}_S,6$ and $\hat{K}_T,6$ generate a reducible, six-dimensional representation of $T'$. They are block-diagonal with $3 \times 3$ blocks corresponding to the strings from first and second twisted sector, respectively (these blocks are exchanged by the action of the $\mathcal{CP}$-like transformation $\hat{K}_i$, as expected). This six-dimensional representation decomposes into irreducible representations of $T'$ as (see also refs. $^{21,38}$

$$
6 = (2' \oplus 1) \oplus (2'' \oplus 1).
$$

(105)

This decomposition can be made explicit by the following basis change: Focusing on the upper three-dimensional block only, $(X, Y, Z)$ are rotated into $X_0 = -X$ and $X_\pm = (Y \pm Z)/\sqrt{2}$ by the orthogonal transformation

$$
\begin{pmatrix}
X_+ \\
X_0 \\
X_-
\end{pmatrix}
= \begin{pmatrix}
0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\
-1 & 0 & 0 \\
0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}
\end{pmatrix}
\begin{pmatrix}
X \\
Y \\
Z
\end{pmatrix}.
$$

(106)

Then, for the first twisted sector, $\hat{K}_S$ and $\hat{K}_T$ take the form

$$
\hat{K}_{S,3}^{33} = \begin{pmatrix}
\frac{1}{\sqrt{3}} & \sqrt{2/3}i & 0 \\
\sqrt{2/3}i & -\frac{1}{\sqrt{3}} & 0 \\
0 & 0 & 1
\end{pmatrix}
$$

and

$$
\hat{K}_{T,3}^{33} = \begin{pmatrix}
1 & 0 & 0 \\
0 & \omega^2 & 0 \\
0 & 0 & 1
\end{pmatrix},
$$

(107)

proving the $2 \oplus 1$ block-structure. The basis change for the lower $3 \times 3$ blocks of $\hat{K}_{S,6}$ and $\hat{K}_{T,6}$ works completely analogous, and the second twisted sector states $(\bar{X}, \bar{Y}, \bar{Z})$ transform with the complex conjugate of the above matrices. Using the character table of $T'$, as given in

$^{14}$An in depth discussion of the group $T'$ is given in chapter 5 of ref. $^{70}$. The generators used there are related to our generators as $s \doteq (\hat{K}_S)^3$ and $t \doteq \hat{K}_T$.
table 3, it is straightforward to verify the decomposition (105). In summary, the three twisted strings \((X, Y, Z)\) corresponding to the three fixed points of the two-dimensional \(\mathbb{Z}_3\) orbifold do not transform as an irreducible 3 of \(T'\) but as a doublet 2' and a trivial singlet 1.

If one additionally takes \(\hat{K}_g\) into account, \(T' \cong SL(2,3)\) gets enlarged to \(GL(2,3)\) and the six-dimensional representation eq. (105) decomposes into irreducible representations of \(GL(2,3)\) as \(4 \oplus 2\).

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