Statistics of Holons and Spinons in the t-J Model

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Abstract

We study the statistics of holons and spinons in the 2-dimensional t-J model of high-Tc superconductors by applying the gauge theory of separation phenomena that we have developed recently. This study is motivated by the observation that, near the half filling, the spin degrees of freedom of the t-J model are correctly described by bosonic variables as in the slave-fermion representation, while, at intermediate hole concentrations, spinons behave as fermions as in the slave boson representation, supporting a large Fermi surface. In the previous papers, we studied the charge-spin separation (CSS) in high-Tc superconductors and the particle-flux separation (PFS) in the fractional quantum Hall effect (FQHE), and showed that both of them are understood as phase transitions of certain gauge theories into their deconfinement phases, where the gauge fields are nothing but the phases of mean fields sitting on links. For example, the CSS occurs when the dynamics of the gauge field that glues a holon and a spinon is relaized in the deconfinement phase; holons and spinons appear as quasixcitations. In the present paper, we start with the slave-boson representation of the t-J model, in which an electron is a composite of a fermionic spinon

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and a bosonic holon. Then we represent each fermionic spinon as a composite of a new boson and one (or odd number of) Chern-Simons (CS) flux. A new PFS, similar to the PFS in FQHE, occurs when the dynamics of gauge field that glues each new bosonic spinon and a CS flux lies in the deconfinement phase. The following four phases are possible: (1) electron phase (no separations), (2) anomalous metallic phase with fermionic spinons and bosonic holons (CSS), (3) double-boson phase (bosonic holons and spinons) with antiferromagnetic long-range order (CSS and PFS), and (4) bosonic electron phase (PFS). We discuss the interplay of CSS and PFS. Especially, it is concluded that the spinons behave as bosons rather than fermions near the half filling due to the PFS.
1 Introduction

Since the discovery of high-Tc superconductivity, strongly-correlated electron systems are one of the most interesting topics in condensed matter physics. Especially, the metallic phase of high-Tc superconductors has various anomalous properties. The anomalous experimental observations can be consistently explained by assuming that charge and spin degrees of freedom of electrons (holons and spinons, respectively) move independently, i.e., the charge-spin separation (CSS) takes place [1]. In the mean-field (MF) theory of the t-J model in the slave-boson or slave-fermion representation [2, 3], the CSS appears quite naturally. However, the MF calculations themselves are not sufficient, because the phase degrees of freedom of the MF’s sitting on links behave like gauge fields that glue holons and spinons, and careful study of the dynamics of these effective gauge fields is required to justify the results of MF theory.

In the previous papers [4], two of the present authors examined the phase structure of the above effective gauge fields. As we showed there, the CSS can be understood as a deconfinement phenomenon, i.e., gauge dynamics operating between holons and spinons is so weak that holons and spinons are deconfined (liberated). There exists a confinement-deconfinement (CD) phase transition at certain critical temperature, $T_{CD}$, and the CSS occurs at low temperatures ($T$) below $T_{CD}$ (deconfinement phase), while at $T > T_{CD}$, holons and spinons are confined and appear solely as electrons (confinement phase).

In the present paper, we shall address yet another interesting problem of the strongly-correlated electron systems, i.e., statistics transmutation of the quasiexcitations in the CSS phase. Actually, in high-Tc superconductors, it is generally expected that, at intermediate hole concentrations, spinons behave as fermions, while, near the half filling, spin dynamics is well described by bosonic variables as in the $O(3)$ nonlinear $\sigma$-model. In two-dimensional systems, a statistics transmutation of particles is properly described by the Chern-Simons (CS) gauge theory. For example, let us recall
the fractional quantum Hall effect (FQHE) at filling fractions $\nu = 1/(2n+1)$, which is another interesting two-dimensional strongly-correlated electron system. The FQHE is quite successfully described by the CS Ginzburg-Landau theory, which takes a form of the CS gauge theory coupled with a bosonized electron field. A FQH state is characterized here as a Bose condensation of these bosonized electrons.

Another example is quasiexcitations in the half-filled Landau level. Jain proposed the idea of composite fermions (CF); an electron is a composite of a CF and two units of solenoidal CS fluxes. By assuming that CF’s and fluxes move independently, one may develop a MF theory, in which fluxes are globally cancelled by the external magnetic field, and the resulting system is a collection of quasifree CF’s weakly interacting with a gauge field. The perturbative analyses of such a system give rise to interesting results. In a previous paper, we called this possible separation phenomena, “particle-flux separation” (PFS), due to its close resemblance to the CSS, and studied the mechanism of PFS by applying the gauge-theoretical method similar to the case of CSS. We found that there is a CD phase transition at some critical temperature $T_{\text{PFS}}$; At $T < T_{\text{PFS}}$, the system is in the deconfinement phase and electrons are desassociated into CF’s and fluxes, i.e., the PFS takes place, while at $T > T_{\text{PFS}}$, the system is in the confinement phase and CF’s and fluxes are confined into electrons.

Being motivated by these analyses, we shall study statistics transmutation of the quasiexcitations of the t-J model by employing the CS gauge-field formalism together with the above gauge theory of separation phenomena.

In Sect.2, we start with the slave-boson representation of the t-J model, in which an electron $C_{x\sigma}(x; \text{site}, \sigma; \text{spin})$ is expressed as a composite of a fermionic spinon $f_{x\sigma}$ and a bosonic holon $b_x$. As in the usual MF theory, we decouple the model by introducing auxiliary MF’s sitting on links. Since their phases are gauge fields that glue holons and spinons, analysis of the phase dynamics is crucial to study CSS. By using the CS gauge theory, we furthermore express a fermionic spinon as a
composite of a new boson (bosonic spinon) $\phi_{x\sigma}$ and odd number of CS flux quanta $W_x$ to rewrite the Hamiltonian in the double-boson representation. Next, we introduce another auxiliary link field, the phase of which is a new gauge field that glues a bosonic spinon and a CS flux. We study its dynamics, under the effects of the other gauge fields introduced to study CSS, following our general method to the separation phenomena. Phase structure of this gauge dynamics is closely related to the statistics of spinons. If this gluing gauge field is in a deconfinement phase, bosonic spinons and CS fluxes are deconfined and move independently; spinons behave as bosons rather than fermions; statistics of the spinon is transmuted. On the other hand, if it is in a confinement phase, composites do not break; quasiexcitations are original fermionic spinons themselves in the slave-boson formalism from which we have started. In the rest of the present paper, we shall call the above separation of a bosonic spinon and a CS flux in the t-J model also PFS due to its common nature to the PFS at FQHE. Schematically, one may summerize the relation as

\begin{align}
C_{x\sigma} & = b_x^\dagger \times f_{x\sigma} \rightarrow b_x^\dagger + f_{x\sigma}, \quad \text{(CSS)} \\
 f_{x\sigma} & = \phi_{x\sigma} \times W_x^\dagger \rightarrow \phi_{x\sigma} + W_x^\dagger, \quad \text{(PFS)}
\end{align}

(1.1)

where the symbol “$\times$” means confinement and “$+$” means deconfinement (separation).

In Sect.2.2, we list up the possible four phases which the system may exhibit, using a general argument based on the gauge symmetries of the double-boson t-J model. According to whether the CSS and/or the PFS take place, one expects the following four phases:

1. the electron phase (no separation),
2. the abnormal metallic phase with fermionic spinons and bosonic holons (CSS),
3. the double-boson phase (bosonic holons and spinons) with antiferromagnetic long-range order (CSS and PFS),
4. the bosonic electron phase (PFS).
In Sect.3.1, we formulate a MF theory by integrating out bosonic holons and spinons using the hopping expansion. This determines the magnitudes of link fields; they develop nonvanishing values at sufficiently low $T$. Thus the phases of link variables can be introduced. They are gauge fields and play an important role at low energies. In Sect.3.2, we derive an effective gauge theory of the above gauge fields by the hopping expansion. Its action has a form similar to the well-known lattice gauge theory developed for strong interactions.

In Sect.4, the phase structure of the effective gauge theory is studied explicitly. In Sect.4.1, we first investigate a canonical gauge model from a general point of view. According to the Polyakov-Susskind theory of CD transition, we map the gauge model to the (an)isotropic classical XY spin model. In Sect.4.2 we apply the results obtained there to the effective gauge theory of Sect.3.2. We show that, both for CSS and PFS, there exist CD phase transitions and derive the equations that determine the critical temperatures $T_{CSS}$, $T_{PFS}$; the CSS and PFS take place at $T < T_{CSS}$ and $T < T_{PFS}$, respectively. Furthermore, we give a plausible argument that the spinons behave as bosons rather than fermions near the half filling, in accordance with the slave-fermion representation.

In Sect.5, we present conclusions and point out the remaining problems. Generally, one may conceive the following two representations of fermionic electron operator;

Double Boson :

$$C_{x\sigma} = b^\dagger_x \times f_{x\sigma} = b^\dagger_x \times \phi_{x\sigma} \times W^\dagger_x, \quad (1.2)$$

Double Fermion :

$$C_{x\sigma} = \psi^\dagger_x \times a_{x\sigma} = \psi^\dagger_x \times \eta_{x\sigma} \times W^\dagger_x, \quad (1.3)$$

where $\psi_x$ is the fermionic holon operator and $a_{x\sigma}$ is the bosonic spinon operator in the slave-fermion representation and $\eta_{x\sigma}$ is the fermionic spinon operator. In the present paper, we mainly focus on the statistics transmutation of spinons. As explained above, the bosonic spinons are expected to be superior assignment than the fermionic
spinons at small hole concentrations. We start with the double boson representation (1.2) rather than the double fermion representation (1.3), and demonstrate this superiority via the PFS. It is obvious that the statistics of holons is also an important problem, and one should study this problem in order to obtain a coherent picture of quasiexcitations in the t-J model. For example, if holons behave as fermions at low \( T \) and intermediate hole concentrations, a hole pair with electric charge +2e, not a single hole, is a natural candidate for the condensed objects that develop the superconductivity \([9]\). All these problems are under study, and results will be reported in future publications.

2 The t-J model in double-boson representation

In Sect.2.1, we rewrite the t-J model in several steps to arrive at the double boson representation, (2.11) below, which is useful to discuss the CSS and PFS. In Sect.2.2, we present a general discussion on what kinds of phases this model may realize, being based on the two local \( U(1) \) gauge symmetries (2.15) and (2.16) of the model.

2.1 Double-boson representation

The t-J model on a two-dimensional lattice is defined by the following Hamiltonian and the local constraint,

\[
H_{tJ} = -t \sum_{x,i,\sigma} C_{x+i,\sigma}^\dagger C_{x,\sigma} + J \sum_{x,i} \left( S_{x+i} \cdot S_x - \frac{1}{4} n_{x+i} n_x \right),
\]

\[
n_x = \sum_{\sigma} C_{x,\sigma}^\dagger C_{x,\sigma} \leq 1,
\]

where \( C_{x,\sigma} \) is the electron annihilation operator at site \( x \) and spin \( \sigma (= 1, 2) \), and the spin operator is given by \( S_x = C_{x,\sigma}^\dagger \sigma C_{x,\sigma} / 2 \) with the Pauli matrices \( \sigma \). The suffix \( i (= 1, 2) \) denotes the lattice directions and also the unit lattice vectors. In the slave-boson representation the electron operator is written as

\[
C_{x,\sigma} = b_{x,\sigma}^\dagger f_{x,\sigma},
\]
where $b_x$ is the bosonic holon operator and $f_{x\sigma}$ is the fermionic spinon operator. The local constraint in (2.1) is expressed as

$$ (f_{x_1}^\dagger f_{x_1} + f_{x_2}^\dagger f_{x_2} + b_x^\dagger b_x - 1)|\text{phys}\rangle = 0, \tag{2.3} $$

which excludes double occupancy of electrons at each $x$. By substituting (2.2) into (2.1), the Hamiltonian is rewritten as

$$ H = -t \sum (b_{x+i}^\dagger f_{x+i,\sigma}^\dagger f_{x+1,\sigma} b_x + \text{H.c.}) - \frac{J}{2} \left( \sum f_{x\sigma}^\dagger \tilde{f}_{x+i,\sigma} \tilde{f}_{x+i,\sigma}' f_{x\sigma}' \right) - \mu_f \sum f_{x\sigma}^\dagger f_{x\sigma} - \mu_b \sum b_x^\dagger b_x - \sum \lambda_x \left( \sum f_{x\sigma}^\dagger f_{x\sigma} + b_x^\dagger b_x - 1 \right), \tag{2.4} $$

where $\epsilon_{\sigma\sigma'}$ is the antisymmetric tensor. We added the chemical potential terms and the Lagrange multiplier $\lambda_x$ for the local constraint for later convenience. $\mu_b$ and $\mu_f$ are chosen so that the hole concentration is $\delta$, i.e., $\langle b_x^\dagger b_x \rangle = \delta$, and so $\langle \sum_{\sigma} f_{x\sigma}^\dagger f_{x\sigma} \rangle = \rho = 1 - \delta$.

By using the Hubbard-Stratonovich transformation in path-integral formalism, let us rewrite (2.4) in the form used in a MF theory [2]. Explicitly, by introducing auxiliary complex link variables, $\chi_{xi}$ and $D_{xi}$, that are fluctuating "MF’s", we get [10]

$$ H = \sum \left[ \frac{3J}{8} |\chi_{xi}|^2 + \frac{1}{2J} |D_{xi}|^2 \right] - \sum \left[ \chi_{xi} \left( \frac{3J}{8} \sum f_{x+i,\sigma}^\dagger f_{x\sigma} + tb_{x+i}^\dagger b_x \right) + \text{H.c.} \right] - \frac{1}{2} \sum [D_{xi} f_{x\sigma}^\dagger \tilde{f}_{x+i,\sigma} + \text{H.c.}] + \frac{8t^2}{3J} \sum b_{x+i}^\dagger b_{x+i} b_x^\dagger b_x - \frac{3J}{8} \sum f_{x+i,\sigma}^\dagger f_{x+i,\sigma} f_{x\sigma'}^\dagger f_{x\sigma} - \mu_f \sum f_{x\sigma}^\dagger f_{x\sigma} - \mu_b \sum b_x^\dagger b_x - \sum \lambda_x \left( \sum f_{x\sigma}^\dagger f_{x\sigma} + b_x^\dagger b_x - 1 \right) \tag{2.5} $$

By differentiating $H$, we get the relations,

$$ \frac{3J}{8} \langle \chi_{xi} \rangle = \langle \frac{3J}{8} \sum f_{x+i,\sigma}^\dagger f_{x\sigma} + tb_{x+i}^\dagger b_x \rangle, $$

$$ \frac{1}{2J} \langle D_{xi} \rangle = \frac{1}{2} \langle \sum f_{x\sigma}^\dagger \tilde{f}_{x+i,\sigma} \rangle. \tag{2.6} $$

So $\chi_{xi}$ is the hopping amplitude of holons and spinons, while $D_{xi}$ is the amplitude of resonating valence bonds of antiferromagnetism. In the usual MF approximation,
one simply determines the values of MF’s, $\langle \chi_{xi} \rangle$ and $\langle D_{xi} \rangle$ as functions of $\delta$ and $T$, assuming some spatial periodicity. The effects of fluctuations around these MF’s, particularly the CD transition and the CSS, are studied in Ref. [4] based on this Hamiltonian [11].

According to the lattice CS gauge theory [12], let us introduce a bosonic spinon field $\phi_{x\sigma}$ [13],

$$\phi_{x\sigma} = e^{-iq \sum_{y} \theta(x-y)(\hat{\rho}(y) - \rho)} f_{x\sigma},$$
\[
\hat{\rho}(x) = \sum_{\sigma} f_{x\sigma}^\dagger f_{x\sigma} = \sum_{\sigma} \phi_{x\sigma}^\dagger \phi_{x\sigma},
\]

(2.7)

where $q$ is an odd integer and $\theta(x)$ is the multi-valued lattice angle function with $\theta(0) = 0$. One can verify that $\phi_{x\sigma}$, $\phi_{y\sigma}'$ and their Hermitian conjugate operators satisfy the bosonic commutation relations for $x \neq y$ and the fermionic anticommutation relations for $x = y$. These relations imply that $\phi_{x}$ describes hard-core bosons. By substituting (2.7), $H$ becomes

$$H = \sum \left[ \frac{3J}{8}|\chi_{xi}|^2 + \frac{1}{2J}|D_{xi}|^2 \right] - \sum \left[ \chi_{xi} \left( \frac{3J}{8} \sum \phi_{x+i,\sigma}^\dagger e^{-iq \sum \theta_{i}(x-y)(\hat{\rho}(y) - \rho)} \phi_{x\sigma} \right) \right.$$

$\left. + \left( t b_{x+i}^\dagger b_{x} + \text{H.c.} \right) \right] - \frac{1}{2} \sum [D_{xi} \phi_{x\sigma}^\dagger \phi_{x+i,\sigma} + \text{H.c.}] + \frac{8t^2}{3J} \sum b_{x+i}^\dagger b_{x} + \mu \sum \phi_{x\sigma}^\dagger \phi_{x\sigma} - \mu b \sum b_{x}^\dagger b_{x} - \sum \lambda_{x} \left( \sum \phi_{x\sigma}^\dagger \phi_{x\sigma} + b_{x}^\dagger b_{x} - 1 \right),$

(2.8)

where $\nabla_{i}$ is the difference operator on the lattice, and we put $\mu_{\phi} = \mu_{f}$.

From the $\phi_{x\sigma}$-hopping term in (2.8), it is obvious that the bosonic spinons move in the combined field of $\chi_{xi}$ and the CS magnetic field $B^{CS}_{x}$,

$$B^{CS}_{x} \equiv \epsilon_{ij} \nabla_{i} A^{CS}_{xj} = 2\pi q \hat{\rho}(x)$$

$$A^{CS}_{xi} = q \sum_{y} \nabla_{i} \theta_{i}(x-y) \hat{\rho}(y),$$

(2.9)

where we have used the fact

$$\nabla_{i} \theta(x) = 2\pi \epsilon_{ij} \nabla_{j} G(x),$$

(2.10)
and $G(x)$ is the two-dimensional lattice Green function.

As mentioned in the Introduction, two of the present authors discussed quasiexcitations in the half-filled Landau level by using the gauge theory similar to that for CSS. There we introduced a gauge field which glues a CF and CS fluxes and studied its dynamics. We found that the PFS occurs at $T < T_{\text{PFS}}$, where these CF’s are stable quasiexcitations.

Here, we apply the same method and introduce auxiliary complex variables $V_{x'i}$ on links by the Hubbard-Stratonovich transformation. Then $H$ is rewritten as follows;

$$
H = \sum \left[ \frac{3J}{8} |\chi_{x'i}|^2 + \frac{1}{2J} |D_{x'i}|^2 + |V_{x'i}|^2 \right] - \left[ \sum \chi_{x'i} \left( \frac{3J}{8\gamma} \sum \phi_{x+i,\sigma} V_{x'i} \phi_{x,\sigma} \right.ight.
$$

$$
+ \left. \frac{1}{2} \sum D_{x'i} \phi_{x+i,\sigma} \phi_{x,\sigma} + \gamma \sum V_{x'i} W_{x+i}^\dagger W_x + \text{H.c.} \right] - \sum g_\phi^2 \phi_{x+i,\sigma} \phi_{x+i,\sigma} - \mu_b \sum b_{x'}^\dagger b_{x'} - \mu \sum \left( \sum \phi_{x+i,\sigma} \phi_{x,\sigma} + b_{x'}^\dagger b_{x'} - 1 \right),
$$

where

$$
W_x = \exp[-i \int_0^\beta \sum_{x,i} \phi_{x,i,\sigma} \partial_t \phi_{x,i,\sigma} + i \omega x],
$$

$$
g_\phi^2 = \frac{3J}{8} + \left( \frac{3J}{8\gamma} |\chi_{x'i}| \right)^2,
$$

and we have introduced the parameter $\gamma$ for dimensional reason. In (2.12), $\omega_x$ is an arbitrary function, which corresponds to the longitudinal part of the “CS gauge field” (see Ref.8).

The partition function $Z \equiv \text{Tr} \exp(-\beta H)$ of the system is expressed in the path-integral formalism as

$$
Z = \int [d\phi][db][d\chi][dD][dV][d\omega] \exp A,
$$

$$
A = \int_0^\beta d\tau \left[ - \sum \phi_{x+i,\sigma} \partial_\tau \phi_{x,\sigma} - \sum b_{x'}^\dagger \partial_\tau b_{x'} - H \right],
$$

where the imaginary time $\tau$ runs from 0 to $\beta \equiv 1/T$. Here $[db][d\chi][dD][dV]$ are usual path integrals of complex variables, and $[d\omega]$ is the path integral for $0 < \omega_x(\tau) < 2\pi$.
which respects the gauge symmetry $[U(1)_{\text{PFS}}$ explained below]. The integral $[d\phi]$ is ambiguous since no simple coherent states of hard-core bosons are known. We take a possible option that $\phi_x(\tau)$ is a complex number and the measure includes a factor $\lim_{\lambda_\phi \to \infty} \exp[-\int d\tau \lambda_\phi \sum f(\bar{\phi}_x \phi_x)]$, where $f(x)$ is some smooth function which is zero for $x = 0, 1$ and positive otherwise. This option is too formal generally, but is sufficient for the hopping expansion that we shall use; We can calculate each terms of hopping expansions systematically respecting the hard-core nature (See Sect.3 for details).

### 2.2 Gauge symmetries $U(1)_{\text{CSS}}, U(1)_{\text{PFS}}$ and possible phases

The Hamiltonian (2.11) is invariant under the following two kinds of time-independent local $U(1)$ gauge transformations;

\begin{align}
U(1)_{\text{CSS}}: \\
(\phi_{x\sigma}, b_x) &\rightarrow e^{i\alpha_x}(\phi_{x\sigma}, b_x), \\
\chi_{xi} &\rightarrow e^{i\alpha_{x+1}}\chi_{xi}e^{-i\alpha_x} \\
D_{xi} &\rightarrow e^{i\alpha_{x+1}}D_{xi}e^{i\alpha_x}, \tag{2.15}
\end{align}

and

\begin{align}
U(1)_{\text{PFS}}: \\
(\phi_{x\sigma}, W_x) &\rightarrow e^{i\beta_x}(\phi_{x\sigma}, W_x) \\
\omega_x &\rightarrow \omega_x + \beta_x \\
V_{xi} &\rightarrow e^{i\beta_{x+1}}V_{xi}e^{-i\beta_x}. \tag{2.16}
\end{align}

We shall derive an effective gauge theory of (2.11) in Sect.3 and discuss its dynamics in Sect.4. Since the basic characteristics of general gauge dynamics are determined sorely from the knowledge of gauge symmetry, it will be helpful to discuss the possible phases of the t-J model, based on the gauge symmetries $U(1)_{\text{CSS}}$ and $U(1)_{\text{PFS}}$, before going into details.
Generally speaking, dynamics of gauge theory is categorized into two phases; confinement phase and deconfinement phase. In the confinement phase, fluctuations of gauge field are very large, i.e., it is a disordered phase of the gauge symmetry, and only charge-neutral bound objects can appear as physical excitations. Contents of quasiparticles are quite different from the original “elementary” charged particles, which appear in the Hamiltonian. On the other hand, in the deconfinement phase, fluctuations of gauge field is not so strong, and charged particles appear as physical excitations.

Since our system possesses two independent gauge symmetries, $U(1)_{\text{CSS}}$ and $U(1)_{\text{PFS}}$, there are totally four possible phases. They are listed up with their physical picture as follows;

1. $(U(1)_{\text{CSS}}, U(1)_{\text{PFS}}) = (\text{confinement, confinement})$
   
   Only charge-neutral bound states of $\phi_{x\sigma}$ and $b_x$ as well as $\phi_{x\sigma}$ and $W_x$ are physical quasiexcitations. They are composites of $\phi_{x\sigma}$, $b_x$ and $W_x$, or equivalently $f_{x\sigma} = W_x \phi_{x\sigma}$ and $b_x$, which are nothing but the electrons $C_{x\sigma} = b_x^\dagger f_{x\sigma}$. Therefore, this phase is the electron phase. In the t-J model, it is expected to appear at sufficiently high $T$.

2. $(U(1)_{\text{CSS}}, U(1)_{\text{PFS}}) = (\text{deconfinement, deconfinement})$
   
   In this phase, bosonic excitations $\phi_{x\sigma}$ and $b_x$ appear as quasiexcitations. Since fluctuations of link fields $\chi_{xi}, D_{xi}, V_{xi}$ are small, a simple MF theory is applicable starting from the Hamiltonian (2.11). It is expected that one obtains the results similar as in the MF theory of slave-fermion formalism of the t-J model for spinons in which bose statistics is assigned to spinons. The MF’s $\chi_{xi}$ etc. may not have a simple uniform configuration but $\sqrt{2}a$ or $2a$ periodicity as in the spiral state in the slave-fermion MF theory, where $a$ is the lattice spacing. The bose condensation of spinons $\phi_{x\sigma}$ at sufficiently low $T$ corresponds to a long-range order of antiferromagnetism. Both the spiral state and the state with antiferromagnetic long-range order belong to this phase.
3. \((U(1)_{CSS}, U(1)_{PFS})=(\text{deconfinement, confinement})\)  
\(\phi_{x\sigma}\) and \(b_x\) are not bound and so the CSS takes place, while \(\phi_{x\sigma}\) and \(W_x\) are bound to form \(f_{x\sigma}\). Thus the quasiexcitations are fermionic spinons \(f_{x\sigma}\) and bosonic holons \(b_x\). The usual slave-boson representation is useful for this phase, and there appears a large Fermi surface consistent with the Luttinger theorem. This phase corresponds to the anomalous metallic state in high-\(T_c\) superconductors.

4. \((U(1)_{CSS}, U(1)_{PFS})=(\text{confinement, deconfinement})\)  
\(\phi_{x\sigma}\) and \(b_x\) are bound, while \(\phi_{x\sigma}\) and \(W_x\) are not bound. So the quasiexcitations are “bosonic electrons”, \(b_x^\dagger \phi_{x,\sigma}\); which are bosons, each having spin 1/2 and charge \(-e\). In high-\(T_c\) superconductors, no states corresponding to this phase seem to be observed. However, if holons cannot move freely by some effects like localization, only relevant quasiexcitations are the composite spin excitations localized at site \(x\) and described by the operator \(n_x = \frac{1}{2} \phi_x^\dagger \sigma \phi_x\). The spinons \(\phi_{x\sigma}\) themselves cannot condense because \(U(1)_{CSS}\) is in the confinement phase. Therefore, this phase corresponds to the insulator of cuprates with no long-range magnetic order. The dynamics of these massive spin excitations is well-described by the \(CP^1\) or \(O(3)\) nonlinear-\(\sigma\) model (see detailed discussion in Ref.\[13\]).

In the rest of the present paper, we shall study which phase of these four will be realized at each values of \(T\) and \(\delta\).

### 3 Hopping expansion and effective gauge theory

#### 3.1 Amplitudes of link fields

We shall first study the amplitudes of link fields. From the Hamiltonian \((2.11)\), one may obtain a MF-like Hamiltonian. However, from the experience of MF studies of the t-J model, it is expected that nonuniform configurations of the phases of link
fields may appear [14]. This problem will be studied in some detail in the following subsection. In this section we shall focus on the single-link potential, which essentially determines the amplitudes of link fields. For this purpose we simple set in (2.11),

$$\chi_{xi} = \chi_0, \quad D_{xi} = D_0, \quad V_{xi} = V_0,$$  \hspace{1cm} (3.1)

where \(\chi_0\), \(D_0\), and \(V_0\) are assumed to be real constants (this assumption is not essential in the following calculation). The MF Hamiltonian is given by

$$H_{MF} = \sum \left[ \frac{3J}{8} |\chi_0|^2 + \frac{1}{2} J |D_0|^2 + |V_0|^2 \right] - \left[ \sum \chi_0 \left( \frac{3J}{8} \sum \phi_{x+i,\sigma}^\dagger V_0 \phi_{x,\sigma} \right) + \frac{1}{2} \sum D_0 \phi_{x,\sigma}^\dagger \phi_{x+i,\sigma} + \gamma \sum V_0 \tilde{W}_{x+i}^\dagger W_x + \text{H.c.} \right]$$

$$- \frac{8J}{3} \sum b_{x+i}^\dagger b_x + \mu_\phi \sum \phi_{x,\sigma}^\dagger \phi_{x,\sigma}$$

$$- \mu_b \sum \delta^\dagger b_x - \lambda_x \left( \sum \phi_{x,\sigma}^\dagger \phi_{x,\sigma} + b_x^\dagger b_x - 1 \right).$$  \hspace{1cm} (3.2)

The propagators of the fields \(\phi_{x,\sigma}\) and \(b_x\) in the hopping expansion are obtained as follows;

$$\langle b_x(\tau_1)b_y(\tau_2) \rangle = \delta_{xy} G_b(\tau_1 - \tau_2),$$

$$G_b(\tau) = \frac{e^{\mu_b \tau}}{1 - e^{\beta \mu_b}} \left[ \theta(\tau) + e^{\beta \mu_b} \theta(-\tau) \right],$$

$$\delta = \frac{e^{\beta \mu_b}}{1 - e^{\beta \mu_b}},$$

$$\langle \phi_{x,\sigma}(\tau_1)\phi_{y,\sigma'}(\tau_2) \rangle = \delta_{\sigma\sigma'} \delta_{xy} G_\phi(\tau_1 - \tau_2),$$

$$G_\phi(\tau) = \frac{e^{\mu_\phi \tau}}{1 + e^{\beta \mu_\phi}} \left[ \theta(\tau) - e^{\beta \mu_\phi} \theta(-\tau) \right],$$

$$1 - \delta = 2 \cdot \frac{e^{\beta \mu_\phi}}{1 + e^{\beta \mu_\phi}},$$  \hspace{1cm} (3.3)

where we have treated \(\phi_{x,\sigma}\) as hard-core bosons as mentioned.

Here we briefly comment on the treatment of the local-constraint Lagrange multiplier \(\lambda_x\) in \(H_{MF}\) (3.2). If the CSS takes place, the spinons and holons appear as quasiexcitations. A straightforward loop calculation shows that, due to the contribution from the loop diagrams of spinons and holons, fluctuations of \(\lambda_x\) become massive,
and so the local constraint becomes irrelevant at low energies \[4\]. This is consistent with the CSS. On the other hand, if the CSS does not occur, the t-J model must be studied in term of the original electrons, and the local constraint must be treated faithfully. One well-known example of such treatments is the high-$T$ expansion. As we are interested in the state of CSS in the present paper, we shall ignore the local constraint in most of the later discussion.

Let us consider the effective potential $P$ of the link fields. From (3.2), on the tree level, we have

$$P_{\text{tree}} = \frac{3J}{8} \gamma \phi^2 + \frac{1}{2J} (\phi \chi_0^2 + V_0^2).$$  \hspace{1cm} (3.4)

The $\phi_x$-hopping terms in (3.2) give the following terms of $\chi_0$ and $D_0$ in the second-order of the hopping expansion;

$$P_{\phi}^{(2)} = -\frac{1}{2} \left( \frac{3J}{8} \gamma \right)^2 (1 - \delta^2) \beta \chi_0^2 V_0^2 - L(\delta) \beta D_0^2,$$  \hspace{1cm} (3.5)

where $L(\delta) = \frac{\delta}{4} \left( \ln \frac{1 + \delta}{1 - \delta} \right)^{-1}$, and we have used the propagator (3.3).

Similarly, the $b_x$-hopping term gives rise to

$$P_{b}^{(2)} = -t^2 \delta (1 + \delta) \beta \chi_0^2.$$  \hspace{1cm} (3.6)

Contribution from the flux-hopping term $V_0 W_{x+i}^\dagger W_x$ is also calculated in a straightforward manner. To this end, it is useful to use the following expression of $W_{x+i}^\dagger W_x$,

$$W_{x+i} W_x^\dagger = \exp[ -2\pi qi \sum_y \epsilon \nabla_j G(x - y) (\hat{\rho}(y) - \rho) + i \nabla_i \omega_x ].$$  \hspace{1cm} (3.7)

From (3.7), we get

$$P_{W}^{(2)} = -V_0^2 \frac{\gamma^2}{\beta} \int_0^\beta d\tau_1 d\tau_2 \exp[ -2\pi qi \sum \epsilon \nabla_j G(x - y) (\phi_{y,\sigma}(\tau_1) - \phi_{y,\sigma}^\dagger(\tau_2))]].$$  \hspace{1cm} (3.8)

The above expectation value $\langle \exp[ \nabla G \phi^\dagger \phi] \rangle$ is easily evaluated by the hopping expansion. As $\phi_{x\sigma}$ describes hard-core bosons, the following identity is satisfied for an arbitrary c-number $\alpha$,

$$e^{\alpha \phi^\dagger \phi} = 1 + (e^\alpha - 1) \phi^\dagger \phi.$$  \hspace{1cm} (3.9)
In the leading order of the hopping expansion, we can use this identity to evaluate the following averages;

\[ \langle \exp[\alpha(\phi_{x,\sigma}^{\dagger}(\tau_1) - \phi_{x,\sigma}^{\dagger}(\tau_2))] \rangle = \langle [1 + (e^{\alpha} - 1)\phi_{x,\sigma}^{\dagger}(\tau_1)][1 + (e^{-\alpha} - 1)\phi_{x,\sigma}^{\dagger}(\tau_2)] \rangle = 1. \]  

(3.10)

As a result, we get

\[ P^{(2)}_W = -\gamma^2 \beta V_0^2. \]  

(3.11)

From (3.4), (3.3), (3.6) and (3.11), it is obvious that \( \chi_0, D_0, \) and \( V_0 \) should develop nonvanishing expectation values at sufficiently low \( T \), since the coefficients in \( P^{(2)} = P_{\text{tree}} + P^{(2)}_{\phi} + P^{(2)}_b + P^{(2)}_W \) become negative. This result is supported by the higher-order terms which may be calculated systematically (see Ref.\[4, 8\] for similar calculations).

### 3.2 Effective Gauge Theory

As shown in the previous subsection, the amplitudes of the link fields develop nonzero expectation values at low \( T \). Therefore one can define the phases of link fields. From the local gauge symmetries (2.15) and (2.16), the most important degrees of freedom at low \( T \) are these phases. Actually as we explained before, they can be regarded as gauge fields. Let us introduce \( U(1) \) variables, \( U_{xi} \)'s, corresponding to them through

\[ \chi_{xi} = \chi_0 U_{xi}^\chi, \hspace{1em} D_{xi} = D_0 U_{xi}^D, \hspace{1em} V_{xi} = V_0 U_{xi}^V, \]

\[ U_{xi}^\chi, U_{xi}^D, U_{xi}^V \in U(1). \]  

(3.12)

We substitute (3.12) into the Hamiltonian of (2.11), and integrate out the “elementary” fields \( \phi_{x,\sigma} \) and \( b_x \) by the hopping expansion as in Sect.3.1 in order to obtain an effective Lagrangian of the composite gauge fields, \( U_{xi} \)'s. This effective Lagrangian must possess the local gauge symmetries corresponding to \( U(1)_{\text{CSS}} \) and \( U(1)_{\text{PFS}} \) of (2.13) and (2.16).
The action $A_{\text{EGT}}$ of the effective gauge theory of $U_{\mathbf{x}}$'s is defined as

$$\exp(A_{\text{EGT}}[U]) = \int [d\phi][db][d\omega] \exp A.$$  \hspace{1cm} (3.13)

The hopping expansion below gives (approximately) the following canonical form with the electric term $A^e$ and the magnetic term $A^m$;

$$A_{\text{EGT}}[U] = A^e + A^m,$$  \hspace{1cm} (3.14)

$$A^e = -\int d\tau \sum_{x,i} \left[ \partial_\tau U^\dagger_{\mathbf{x}i} \partial_\tau U_{\mathbf{x}i} + \cdots \right],$$

$$A^m = \int d\tau \sum_x \left[ U_{\mathbf{x},2} U^\dagger_{\mathbf{x}+1,2} U^\dagger_{\mathbf{x}+1,1} U_{\mathbf{x},1} + \cdots \right].$$ \hspace{1cm} (3.15)

Each hopping term and four-point interaction in $A$ gives rise to each contribution to the effective action:

$$A^e = A^e_\phi + A^e_b + A^e_W + A^e_{\phi,4} + A^e_{b,4}$$

$$A^m = A^m_\phi + A^m_b + A^m_W + A^m_{\phi,4} + A^m_{b,4}. \hspace{1cm} (3.16)$$

From the $\phi$-hopping term, we have

$$A^e_\phi = A^e_{\phi,\chi V} + A^e_{\phi,D}. \hspace{1cm} (3.17)$$

The first term is

$$A^e_{\phi,\chi V} = \left(\frac{3J|\chi_0 V_0|}{8\gamma}\right)^2 \sum_{x,i} \int_0^\beta d\tau_1 d\tau_2 \bar{U}^\chi_{\mathbf{x}i}(\tau_1) \bar{U}^V_{\mathbf{x}i}(\tau_1) U^\chi_{\mathbf{x}i}(\tau_2) U^V_{\mathbf{x}i}(\tau_2) \times \sum_\sigma \langle \phi^\dagger_{\mathbf{x},\sigma}(\tau_1) \phi_{\mathbf{x}+i,\sigma}(\tau_1) \phi^\dagger_{\mathbf{x}+i,\sigma}(\tau_2) \phi_{\mathbf{x},\sigma}(\tau_2) \rangle$$

$$= \left(\frac{3J|\chi_0 V_0|}{8\gamma}\right)^2 \sum_{x,i} \frac{1}{2} (1 - \delta^2) \beta^2 \sum_n \bar{U}^\chi_{\mathbf{x}i,n} \bar{U}^V_{\mathbf{x}i,\tau_1} \sum_l U^\chi_{\mathbf{x}i,l} U^V_{\mathbf{x}i,\tau_1, l}, \hspace{1cm} (3.18)$$

where we have introduced Fourier expansions for $U_{\mathbf{x}i}(\tau)$'s;

$$U_{\mathbf{x}i}(\tau) = \sum_n e^{i\omega_n \tau} U_{\mathbf{x}i,n},$$

$$\sum_n U^\dagger_{\mathbf{x}i,n} U_{\mathbf{x}i,n+m} = \delta_{m,0}, \hspace{1cm} (3.19)$$
where \( \omega_n = 2\pi n / \beta \), \( n = 0, \pm 1, \pm 2, \cdots \). It is obvious from its definition that \( U_{xi,n} \) is the amplitude for those configurations with the winding number \( n \) in the \( \tau \)-direction. Therefore, if the static mode \( U_{xi,0} \) dominates over all the other oscillating modes \( U_{xi,n \neq 0} \), the gauge dynamics is in the deconfinement phase. On the other hand, if the amplitudes of oscillating modes are excited well and independently, then the gauge dynamics is in the confinement phase. More systematic investigation on this gauge dynamics can be given by the Polyakov-Susskind theory [4, 10], as we shall see in Sect.4.1.

Similarly, \( A_{\phi,D}^e \) is evaluated as

\[
A_{\phi,D}^e = \frac{|D_0|^2}{4} \sum_{x,i} \int_0^\beta d\tau_1 d\tau_2 \bar{U}_{xi}^D(\tau_1)U_{xi}^D(\tau_2) \\
\times \sum_{\sigma,\sigma'} \langle \phi_{x+i,\sigma}(\tau_1)\phi_{x\sigma}(\tau_1)\phi_{x+i,\sigma'}^\dagger(\tau_2)\phi_{x\sigma'}^\dagger(\tau_2) \rangle \\
= \frac{|D_0|^2}{4} \sum_{x,i} \int_0^\beta d\tau_1 d\tau_2 \bar{U}_{xi}^D(\tau_1)U_{xi}^D(\tau_2) \\
\times 2(1 + e^{\beta\mu\phi})^{-2}e^{2\mu\phi(\tau_1 - \tau_2)}[\theta(\tau_1 - \tau_2) + e^{2\beta\mu\phi}\theta(\tau_2 - \tau_1)]. \tag{3.20}
\]

The above integral is approximately evaluated as

\[
\int_0^\beta d\tau_1 \int_0^\beta d\tau_2 e^{2\mu\phi(\tau_1 - \tau_2)}\theta(\tau_1 - \tau_2)\bar{U}_{xi}^D(\tau_1)U_{xi}^D(\tau_2) \\
\simeq \int_0^\beta d\tau_1 \int_0^\beta d\epsilon e^{2\mu\phi\epsilon}\bar{U}_{xi}^D(\tau_1)[U_{xi}^D(\tau_1) - \epsilon \partial_{\tau_1} U_{xi}^D(\tau_1) + \frac{1}{2}\epsilon^2 \partial_{\tau_1}^2 U_{xi}^D(\tau_1)]. \tag{3.21}
\]

The second integral in (3.20) is evaluated in a similar way;

\[
\int_0^\beta d\tau_1 \int_0^\beta d\tau_2 e^{2\mu\phi(\tau_1 - \tau_2)}\theta(\tau_2 - \tau_1)\bar{U}_{xi}^D(\tau_1)U_{xi}^D(\tau_2) \\
\simeq e^{2\beta\mu\phi} \int_0^\beta d\tau_1 \int_0^\beta d\epsilon e^{-2\mu\phi\epsilon}\bar{U}_{xi}^D(\tau_1) \\
\times [U_{xi}^D(\tau_1) + \epsilon \partial_{\tau_1} U_{xi}^D(\tau_1) + \frac{1}{2}\epsilon^2 \partial_{\tau_1}^2 U_{xi}^D(\tau_1)], \tag{3.22}
\]

where we have used the periodicity \( U_{xi}^D(\tau \pm \beta) = U_{xi}^D(\tau) \). Collecting (3.21) and (3.22), we have

\[
A_{\phi,D}^e \simeq \frac{|D_0|^2}{2(1 + e^{\beta\mu\phi})^2} \sum_{x,i} \int_0^\beta d\tau \int_0^\beta d\epsilon (e^{2\mu\phi\epsilon} + e^{2\mu\phi(\beta - \epsilon)})
\]
perturbative calculation in powers of $\phi$. Then we obtain in the leading order,

$$
\phi \text{in the action as } \phi^\dagger A \chi V
$$

For $A^e_4$, we have

$$
A^e_4 = t^2 |x_0|^2 \sum_{x,i} \int_0^\beta d\tau_1 d\tau_2 \bar{U}^x_{x\dagger}(\tau_1) U^x_{x\dagger}(\tau_2) \times (b^\dagger_{x+i}(\tau_1)b_x(\tau_1)b_{x+i}(\tau_2))
$$

$$
= t^2 |x_0|^2 (1 + \delta) \beta^2 \sum_{x,i} \bar{U}^x_{x\dagger,0} U^x_{x\dagger,0}.
$$

For $A^e_W$, the identity (3.10) is very useful, giving rise to

$$
A^e_W = \gamma^2 |V_0|^2 \sum_{x,i} \int_0^\beta d\tau_1 d\tau_2 \bar{U}^V_{x\dagger}(\tau_1) U^V_{x\dagger}(\tau_2) \times (W^x_{x\dagger}(\tau_1)W_{x+i}(\tau_1)W^\dagger_{x+i}(\tau_2)W_x(\tau_2))
$$

$$
= \gamma^2 |V_0|^2 \beta^2 \sum_{x,i} \bar{U}^V_{x\dagger,0} U^V_{x\dagger,0}.
$$

It is not straightforward to evaluate the full effect of $A^e_{\phi,4}$ and $A^e_{b,4}$. We shall use perturbative calculation in powers of $\phi^4$ and $b^4$. To do this, let us rewrite the $\phi^4$-term in the action as $\phi^4_{x+i,\sigma}=N[\phi^\dagger_{x+i,\sigma}\phi^\dagger_{x+i,\sigma'}\phi_{x\sigma}+\phi^\dagger_{x\sigma}\phi_{x\sigma}]+(\rho/2)\phi^\dagger_{x+i,\sigma}\phi_{x+i,\sigma}+\phi^\dagger_{x\sigma}\phi_{x\sigma}−\rho\delta_{\sigma\sigma'}/2$, where $N[\phi^\dagger_{x\sigma}\phi_{x\sigma}]=\phi^\dagger_{x\sigma}\phi_{x\sigma}−\rho\delta_{\sigma\sigma'}/2$, and similar one for the $b^4$-term. The second term above is absorbed into the chemical potential term and the third term is an irrelevant constant. Then we obtain in the leading order,

$$
A^e_{\phi,4} = A^e_{4,\chi V} + A^e_{4,D},
$$

$$
A^e_{4,\chi V} = \frac{g^2}{8}\left(\frac{3J|\chi_0|}{8\gamma}\right)^2 |V_0|^2 \sum_{x,i} \int_0^\beta d\tau_1 d\tau_2 d\tau_3 \bar{U}^x_{x\dagger}(\tau_3) \bar{U}^V_{x\dagger}(\tau_3) \bar{U}^x_{x\dagger}(\tau_2) U^V_{x\dagger}(\tau_2)
\times \sum_{\sigma} \langle N[\phi^\dagger_{x+i,\sigma}\phi^\dagger_{x+i,\sigma'}\phi_{x\sigma}]N[\phi^\dagger_{x\sigma}\phi_{x\sigma}]\rangle \langle \phi^\dagger_{x\sigma}(\tau_3)\phi_{x+i,\sigma}(\tau_3)\phi^\dagger_{x+i,\sigma}(\tau_2)\phi_{x\sigma}(\tau_2)\rangle
$$

$$
= -\frac{g^2}{8}\left(\frac{3J|\chi_0|}{8\gamma}\right)^2 |V_0|^2 \sum_{x,i} \frac{1}{8} (1 - \delta^2)^2 \beta^3 \sum_n \bar{U}^x_{x,n} \bar{U}^V_{x,n} \sum_l U^x_{x,l} U^V_{x,l}.
$$

$$
A^e_{4,D} = \frac{g^2 |D_0|^2}{4} \sum_{x,i} \int_0^\beta d\tau_1 d\tau_2 d\tau_3 \bar{U}^D_{x\dagger}(\tau_2) U^D_{x\dagger}(\tau_2)
\times \langle N[\phi^\dagger_{x+i,\sigma}\phi_{x+i,\sigma'}(\tau_1)]N[\phi^\dagger_{x\sigma}\phi_{x\sigma}(\tau_1)]\rangle \langle \phi^\dagger_{x+i,\gamma}(\tau_2)\phi_{x\gamma}(\tau_2)\phi^\dagger_{x\gamma}(\tau_3)\phi_{x+i,\gamma'}(\tau_3)\rangle
$$

$$
\sim -\frac{g^2 |D_0|^2}{4} \beta^4 F(\delta) \sum_{x,i} \int_0^\beta d\tau [\partial_\tau \bar{U}^D_{x\dagger}(\tau)\partial_\tau U^D_{x\dagger}(\tau)],
$$

19
where $F(\delta)$ is some positive-definite function of $\delta$ and $g^2_\delta = \frac{2t^4}{3} + \left(\frac{3t^4}{8}|\chi_0|\right)^2$. Similarly from the $b^4$-term, we have

$$A^c_{b,4} = -\frac{8t^4}{3J}|\chi_0|^2 \sum_{x,i} \beta \int_0^\beta d\tau_1 d\tau_2 d\tau_3 \bar{U}_x^\chi(\tau_3) U_{xi}^\chi(\tau_2) \left(N[b^1_{x+i} b_{x+i}(\tau_1)] N[b^1_x b_x(\tau_1)] b^1_x(\tau_3) b^1_{x+i}(\tau_2) b_{x+i}(\tau_2)\right)$$

$$\times \langle N[b^1_{x+i} b_{x+i}(\tau_1)] N[b^1_x b_x(\tau_1)] b^1_x(\tau_3) b^1_{x+i}(\tau_2) b_{x+i}(\tau_2)\rangle$$

$$= -\frac{8t^4}{3J}|\chi_0|^2 \delta^2 (1 + \delta)^2 \beta^3 \sum_{x,i} \bar{U}_{x,i,0} U_{x,i,0}^\chi. \quad (3.28)$$

From (3.23), (3.24) and (3.25), it is obvious that, at high $T$, $\beta \to 0$, the coefficients of oscillating modes $U_{x,n} \neq 0$ etc., as well as the static modes, become small, so that these modes are excited randomly, so both $U_{CSS}^1$ and $U_{PFS}^1$ are in disordered-confinement phase. On the other hand, at low $T$, $\beta \to \infty$, excitations of all the oscillating modes are suppressed, so both $U_{CSS}^1$ and $U_{PFS}^1$ are in the deconfinement phase.

We shall see how the other terms would affect the above result. It is obvious that $A^c_{b,4}$ (3.28) disfavors the deconfinement of $U_{CSS}^1$. However its effect is small for small $\delta$. $A_{\phi,\chi V}^c$ (3.18) and $A_{4,\chi V}^c$ (3.26) give the couplings between $U_{x,i}^\chi$ and $U_{x,i}^V$. The sign of the coefficient of this mixing term, $\sum_n \bar{U}_{x,i,n} \bar{U}_{x,i,-n} U_{x,i,l} U_{x,i,-l}$, depends on the values of $T$, $\chi_0$, etc. Let us see the effect of this term. For example, let us assume that as $T$ goes down, a phase transition to the deconfinement phase of $U_{CSS}^1$ takes place first before the deconfinement transition of $U_{PFS}^1$ Then the static modes $U_{x,i,n=0}^\chi$ and/or $U_{x,i,n=0}^D$ develop nonvanishing values. Therefore, one may use a decoupling procedure to evaluate the mixing term as

$$\sum_n \bar{U}_{x,i,n} \bar{U}_{x,i,-n} U_{x,i,l} U_{x,i,-l}$$

$$\to \sum_n \bar{U}_{x,i,n} U_{x,i,n} \langle \bar{U}_{x,i,-n} U_{x,i,-n} \rangle + \sum_n \bar{U}_{x,i,n} U_{x,i,n} \langle \bar{U}_{x,i,-n} U_{x,i,-n} \rangle$$

$$\sim \langle \bar{U}_{x,i,0} U_{x,i,0} \rangle \cdot \bar{U}_{x,i,0} U_{x,i,0}. \quad (3.29)$$

This implies that, if the coefficient is positive, the deconfinement phase of $U_{PFS}^1$ is favored. On the other hand, if it is negative, the confinement phase is favored.
The magnetic terms of the effective gauge theory are obtained in a similar way. They play a minor role in the discussion of the CD phase transition \cite{4, 16}. Their explicit form is

\[ A_{m}^{m} = |\chi_{0}V_{0}|^{4} \left( \frac{3J_{s}}{8\gamma} \right)^{4} \sum_{x} \prod_{i} d\tau_{i} \bar{U}_{x_{i}}^{x}(\tau_{i}) \bar{U}_{x_{i+1}}^{x}(\tau_{i+1}) \bar{U}_{x_{i+2}}^{y}(\tau_{i+2}) \times U_{x_{i+2},1}^{x}U_{x_{i+2},1}^{y}(\tau_{3})U_{x_{i+2},2}^{x}U_{x_{i+2},2}^{y}(\tau_{4}) \prod_{i=1}^{4} G_{\phi}(\tau_{i} - \tau_{i+1}) + \text{H.c.}, \] (3.30)

where \( \tau_{i+4} = \tau_{i} \).

Similarly,

\[ A_{m}^{m} = t^{4}|\chi_{0}|^{4} \sum_{x} \prod_{i} d\tau_{i} \bar{U}_{x_{i}}^{x}(\tau_{i}) \bar{U}_{x_{i+1}}^{x}(\tau_{i+1}) \times \prod_{i=1}^{4} G_{b}(\tau_{i} - \tau_{i+1}) + \text{H.c.}, \] (3.31)

\[ A_{m}^{m} = \gamma^{4}|V_{0}|^{4} \sum_{x} \prod_{i} d\tau_{i} \bar{U}_{x_{i}}^{y}(\tau_{i}) \bar{U}_{x_{i+1}}^{y}(\tau_{i+1}) \times \langle W_{x_{i}}^{x}W_{x_{i+1}}^{y}W_{x_{i+2}}^{x}W_{x_{i+2}}^{y}W_{x_{i+2}}^{y} \rangle + \text{H.c.} \]

\[ = \gamma^{4}|V_{0}|^{4} \sum_{x} \beta^{4}(\bar{U}_{x_{i}}^{y} \bar{U}_{x_{i+1}}^{y} \bar{U}_{x_{i+2}}^{y} \bar{U}_{x_{i+2}}^{y}) \times e^{-2\pi i q \rho} [1 - \rho + e^{2\pi i q \rho}] + \text{H.c.} \] (3.32)

From (3.30), (3.31) and (3.32), it is obvious that the magnetic terms determine the spatial configuration of \( U_{x_{i}} \)'s and the CD picture obtained from the electric terms are usually not affected qualitatively by the presence of these magnetic terms.

In the following section, we shall give a somewhat detailed study of the phase structure and phase transitions of the effective gauge theory.

### 4 Phase structure of the effective gauge theory

In Sect.3.2, we have obtained the explicit form of the effective gauge theory. The electric terms \( A_{e}^{e} \) and \( A_{W}^{e} \) have the following form;

\[ \sum_{x,i} \beta^{2} \bar{U}_{x_{i},0}U_{x_{i},0}. \] (4.1)
From the unitarity condition (3.19), the above term is rewritten as follows;

\[ \beta^2 \bar{U}_{xi,0} U_{xi,0} = \beta^2 \left( 1 - \sum_{n \neq 0} \bar{U}_{xi,n} U_{xi,n} \right) \]

\[ \sim \beta^2 - \frac{2\beta^3}{(2\pi)^2} \int_0^{\beta} d\tau \partial_\tau \bar{U}_{xi} \partial_\tau U_{xi}. \] (4.2)

The mixing terms \( A_{\phi,\chi}^e \) and \( A_{4,\chi}^e \) reduce to the form of (4.1) in the vicinity of CD phase transition or in the deconfinement phase of one of the gauge symmetries, because \( U^x_{xi,0} \) and/or \( U^y_{xi,0} \) dominate over nonstatic modes (see (3.29)). Therefore, essential structure of the CD phase transition is well approximated by the canonical form of the electric term (4.2).

In the next subsection, we shall study the CD phase transition of this canonical gauge theory by mapping it to a classical spin model. The results obtained there will help us to follow the detailed study of our effective gauge theory of the t-J model in Sect4.2.

### 4.1 Finite-temperature properties of the canonical gauge theory by a map to the XY model

Let us consider the canonical gauge model, the action \( A(U, V) \) of which is given by

\[ A(U, V) = - \int d\tau \sum_{x,i} [a_U \partial_\tau \bar{U}_{xi} \partial_\tau U_{xi} + a_V \partial_\tau \bar{V}_{xi} \partial_\tau V_{xi}], \] (4.3)

where we consider two kinds of \( U(1) \) gauge fields \( U_{xi} \) and \( V_{xi} \) and single gauge symmetry. For the second field \( V_{xi} \), we use the same notation as the link field \( V_{xi} \) in Sect.3, but no confusions should arise since the former appear only in this subsection. Let \( U_{xi} \) and \( V_{xi} \) transform under a gauge transformation under consideration as follows;

\[ U_{xi} \rightarrow e^{i\alpha_x+i} U_{xi} e^{-i\alpha_x}, \]
\[ V_{xi} \rightarrow e^{i\alpha_x+i} V_{xi} e^{i\alpha_x}. \] (4.4)
It is useful to introduce canonical angles and their conjugate electric operators;

\[ U_{xi} = \exp(i\theta_{U,xi}), \quad V_{xi} = \exp(i\theta_{V,xi}), \]

\[ A(U,V) = -\sum_{x,i} [a_U \theta_{U,xi}^2 + a_V \theta_{V,xi}^2], \]

\[ E_{U,xi} = -2a_U \dot{\theta}_{U,xi} \leftrightarrow -i \frac{\partial}{\partial \theta_{U,xi}}, \]

\[ E_{V,xi} = -2a_V \dot{\theta}_{V,xi} \leftrightarrow -i \frac{\partial}{\partial \theta_{V,xi}}. \]  

(4.5)

Using the electric field operators (4.5), the Hamiltonian \( H \) and the generator \( Q_x \) of the local gauge transformation (4.4) are given as

\[ H = \sum_{x,i} \left[ \frac{1}{4a_U} E_{U,xi}^2 + \frac{1}{4a_V} E_{V,xi}^2 \right], \]

\[ Q_x = \sum_i [\nabla_i E_{U,xi} + E_{V,x+i,i} + E_{V,xi}]. \]  

(4.6)

The partition function \( Z \) is the trace over the physical states that are gauge singlet, i.e., satisfy \( Q_x = 0 \). So it is written as

\[ Z = \text{Tr} \prod_x \delta_{Q_x,0} \exp(-\beta H). \]  

(4.7)

The trace symbol above implies the sum over the eigenvalues of \( E_{U,xi} \) and \( E_{V,xi} \) which are integers since \( E_{xi} \)'s are conjugate to compact angle variables. We introduce Lagrange multiplier angle field \( \gamma_x[\in (0, 2\pi)] \) to enforce the gauge constraint \( Q_x = 0 \). Then \( Z \) of (4.7) is given as

\[ Z = \prod_{x,i} \sum_{E_{U,xi}=-\infty}^{\infty} \sum_{E_{V,xi}=-\infty}^{\infty} \int \prod_x \frac{d\gamma_x}{2\pi} \exp \left[ -\beta H + i \sum_x \gamma_x Q_x \right] \]

\[ \propto \int \prod_x \frac{d\gamma_x}{2\pi} \exp \left[ -\frac{1}{2\beta} \sum_{x,i} [a_U (\nabla_i \gamma_x)^2 + a_V (\gamma_{x+i} + \gamma_x)^2] \right], \]  

(4.8)

where we have introduced the periodic Gaussian function,

\[ \sum_{n=-\infty}^{\infty} \exp(-cn^2 + i\gamma n) = \left( \frac{\pi}{c} \right)^{1/2} \sum_{m=-\infty}^{\infty} \exp \left\{ -\frac{1}{4c} (\gamma - 2\pi m)^2 \right\} \]

\[ \equiv \left( \frac{\pi}{c} \right)^{1/2} \tilde{\exp} \left( -\frac{1}{4c} \gamma^2 \right). \]  

(4.9)
The last expression of (4.8) is nothing but the partition function of a classical XY spin model with a “double”-Villain action. This spin model has a global $Z_2$ symmetry under $\gamma_x \rightarrow \gamma_x + \pi$, and belongs to the same universality class as the following XY model with “double”-cosine action, or “anisotropic” XY model,

$$Z_{XY} = \int \prod_x \frac{d\gamma_x}{2\pi} \exp \left[ J_1 \sum_{x,i} \cos(\gamma_{x+i} - \gamma_x) + J_2 \sum_{x,i} \cos(\gamma_{x+i} + \gamma_x) \right]$$

$$= \int \prod_x \frac{d\gamma_x}{2\pi} \exp \sum_{x,i} \left[ (J_1 + J_2) \cos \gamma_x \cos \gamma_{x+i} + (J_1 - J_2) \sin \gamma_x \sin \gamma_{x+i} \right],$$

$$J_1 \equiv \beta^{-1} a_U, \quad J_2 \equiv \beta^{-1} a_V.$$

By the MF theory [1] and also by the numerical calculations [18], it is shown that the model has three phases;

1. Disordered phase for $J_1 + J_2 \ll 1$,

   The spin-spin correlation functions behave as

   $$\langle e^{i\gamma_x e^{-i\gamma_0}} \rangle \sim e^{-c|x|}, \quad |x| \rightarrow \infty$$

   (4.11)

2. Quasi-long-range-ordered phase for $J_1 \gg 1, J_2 = 0$ or $J_1 = 0, J_2 \gg 1$,

   This is the well-known Kosterlitz-Thouless (KT) phase of the $O(2)$-invariant classical XY spin model in two-dimensions;

   $$\langle e^{i\gamma_x e^{-i\gamma_0}} \rangle \sim |x|^{-c}, \quad |x| \rightarrow \infty$$

   (4.12)

   or

   $$\langle e^{(-)^x i\gamma_x e^{-i\gamma_0}} \rangle \sim |x|^{-c}, \quad |x| \rightarrow \infty$$

   (4.13)
3. Ordered phase for $J_1 + J_2 \gg 1, J_1 \neq 0, J_2 \neq 0$,

The $Z_2$ symmetry is spontaneously broken in this phase, and the ground state is the configuration with $\gamma_x = 0 \pmod{\pi}$.

$$\langle e^{i\gamma_x} e^{-i\gamma_0} \rangle \sim \text{const.} + e^{-c|x|}. \quad (4.14)$$

$$|x| \to \infty$$

Let us put a pair of oppositely-charged static sources at $x = 0$ and $x = R$ in the original gauge theory. In this case, the constraint changes as $Q_x = \delta_{x,0} - \delta_{x,R}$. It is easily seen [16] that the potential energy $W(R)$ of this state is related with the above spin-spin correlation functions as follows;

$$\langle e^{i\gamma_R} e^{-i\gamma_0} \rangle = \exp[-\beta W(R)]. \quad (4.15)$$

Therefore,

1. Disordered phase: $W(R) \propto R$.
   This phase corresponds to the confinement phase. quasiexcitations are only charge-neutral compounds.

2. Quasi-long-range ordered phase: $W(R) \propto \ln R$.
   This phase corresponds to the massless Coulomb phase. Because of the long-range interaction by the massless gauge boson, some nontrivial infrared behaviors are expected. Resummation method of the perturbative expansion or renormalization-group study are useful to investigate this phase [19].

3. Ordered phase: $W(R) \propto e^{-cR}$.
   This phase corresponds to the Higgs phase with a massive gauge boson. There is only short-range interaction between charged particles.

The above result is qualitatively understood from the fact that $\gamma_x$ is the Lagrange multiplier for the constraint of the local-gauge invariance. In the disordered phase of
the spin system, $\gamma_x$ fluctuates strongly. This means that the local constraint in the original gauge theory operates quite strictly. On the other hand, in the ordered phase of the spin system, $\gamma_x$ has a nontrivial expectation value and fluctuations around it are very small. In the original gauge theory, this means that the local-gauge invariance is not faithfully respected by quasiexcitations. Charged particles can appear as the physical excitations. The phase of the quasi-long-range order is in between.

One might conceive the possibility of mixed dynamics; i.e., the $U$ field (say, for the $b - \bar{b}$ channel) is in confinement phase whereas the $V$ (for the $b - \bar{b}$ channel) field is in deconfinement (or the other way around). To study the dynamics of $V$ field, one needs to calculate the potential energy of a pair of sources of same charges, hence the spin correlation functions, $\langle \exp(i(-1)^{x_1+x_2}\alpha_x)\exp(i\alpha_0)\rangle$. By using a MF theory and the high-temperature expansion for the $U$-dynamics and $\langle \exp(i(-1)^{x_1+x_2}\alpha_x)\exp(i\alpha_0)\rangle$ for the $V$-dynamics, behave similarly at large $|x|$. Therefore two gauge dynamics are always in the same phase and such mixed phases do not exist.

4.2 CSS and PFS in the double-boson t-J model

In the previous subsection, we examined the phase structure of the canonical gauge theory. It is rather straightforward to apply these results to the effective gauge theory of the t-J mode in the double-boson representation, which we derived in Sect.3. By using (4.2), $A_b^e$ of (3.24) and $A_W^e$ of (3.25) can be rewritten as follows;

$$A_b^e \simeq -t^2|\chi_0|^2 \frac{2}{(2\pi)^2} \delta(1 + \delta) \beta^3 \int d\tau \sum_{x,i} \partial_\tau \bar{U}_x^i \partial_\tau U_x^i,$$

$$A_W^e \simeq -\gamma^2|\xi_0|^2 \frac{2}{(2\pi)^2} \beta^3 \int d\tau \sum_{x,i} \partial_\tau \bar{U}_x^i \partial_\tau U_x^i. \quad (4.16)$$

Similarly, the kinetic term (3.23) of $U^D$ becomes

$$A^{e}_{\phi,D} \simeq -|D_0|^2 g(\delta) \beta^3 \int d\tau \sum_{x,i} \partial_\tau \bar{U}_x^D \partial_\tau U_x^D, \quad (4.17)$$
where
\[ g(\delta) = \frac{1}{4(1 + e^{\beta \mu e})(\beta^3)} \int_0^\beta d\epsilon \left( e^{2 \mu \epsilon} + e^{2 \mu e (\beta - \epsilon)} \varepsilon^2 \right), \] (4.18)
and a similar expression for \( A_{4,D} \) of (3.27).

Let us assume that, as lowering \( T \), the CD phase transition of \( U(1)_{CSS} \) occurs first before the CD transition of \( U(1)_{PFS} \), i.e., \( T_{CSS} > T_{PFS} \). Then we determine the transition temperature \( T_{CSS} \) as follows. In this case, the mixing terms (3.18) and (3.26) give only minor effects, because \( U(1)_{PFS} \) is still in a disordered-confinement phase at \( T = T_{CSS} \), and the associated gauge field \( U_{xi} \) fluctuates rather randomly. Explicitly, the decoupled term (3.29) is negligible since \( \langle \bar{U}_{xi,0} U_{xi,0} \rangle \) is small. Therefore, from (4.16) and (4.17), we estimate
\[ a_U = 2 t^2 |\chi_0|^2 \delta (1 + \delta) \frac{(2\pi)^2}{\beta^3}, \]
\[ a_V = |D_0|^2 g(\delta) \beta^3, \]
in (4.13). From the mapping (4.10) considered in Sect.4.1, the couplings \( J_1 \) and \( J_2 \) in the mapped XY spin model are given by
\[ J_1 = 2 t^2 |\chi_0|^2 \delta (1 + \delta) \frac{(2\pi)^2}{\beta^2}, \]
\[ J_2 = |D_0|^2 g(\delta) \beta^2. \] (4.19)

The phase transition line is approximately given by \( J_1 + J_2 = 1 \). At high \( T \) we have \( J_1 + J_2 < 1 \), since the magnitudes \( \chi_0, D_0, V_0 \) are certainly decreasing functions of \( T \), so the \( U(1)_{CSS} \) is in the confinement phase. At low \( T \), \( J_1 + J_2 > 1 \) and it is in the deconfinement phase, as we assumed. More precisely, there are two kinds of deconfinement phase. For \( \chi_0 \neq 0 \) and \( D_0 \neq 0 \), the Higgs phase appears. If one of \( \chi_0 \) and \( D_0 \) vanishes, then the Coulomb phase appears. As we explained in Sect.3.2, this latter transition (\( D_0 = 0 \)) corresponds to the CSS (see Ref.4 for more detailed discussion), and \( T_{CSS} \) is estimated as [20]
\[ T_{CSS} \simeq \left( 2 t^2 |\chi_0|^2 \frac{(1 + \delta)}{(2\pi)^2} + |D_0|^2 g(\delta) \right)^{1/2}. \] (4.20)
At $T < T_{\text{CSS}}$, the static component of $U_{xi}^\chi$ dominates, $U_{xi}^\chi(\tau) \sim U_{xi,0}^\chi$. From (3.29), the mixing terms $A_4^{c_e}V$ and $A_4^{c_e}V$ give similar effects upon the dynamics of $U_{xi}^V$ as for the previous case (4.16). In order to estimate them, we have to know the quantitative behavior of $\chi_0, V_0, \text{etc.}$ Such a quantitative investigation of them is under study and will appear elsewhere. It is however expected that, at some parameter region of the $T - \delta$ plane, the deconfinement phase of $U(1)_{\text{PFS}}$ is realized.

On the contrary, if we assume that the CD transition of $U(1)_{\text{PFS}}$ takes place at higher $T$ than that of $U(1)_{\text{CSS}}$, one obtains similar results, but the roles of $U(1)_{\text{CSS}}$ and $U(1)_{\text{PFS}}$ are interchanged.

Let us consider the phase structure near the half filling. Some qualitative but still interesting results are obtained there without knowledge of detailed behavior of the amplitudes. Let us consider the region $T < T_{\text{CSS}}$. Near the half filling $\delta \sim 0$, the coefficient of $A_6^c$ of (3.24) is very small and so the zero mode $U_{xi,0}^\chi$ has a finite but very small expectation value $\langle \bar{U}_{xi,0}^\chi U_{xi,0}^\chi \rangle$ even if the gauge dynamics of $U(1)_{\text{CSS}}$ is in the deconfinement phase as we assumed. This leads that the mixing terms $A_4^{c_e}V$ and $A_4^{c_e}V$ play no important role as the decoupling (3.29) shows. On the other hand, the term $A_4^{c,D}D$ of (4.17) has a finite coefficient as $\delta \to 0(\mu_0 \to 0)$,

$$g(\delta = 0) = \frac{1}{48}.$$  

The coefficient of the term $A_4^{c,D}D$ of (3.27) has a similar behavior. Thus, for $D_0 \neq 0$, the zero modes $D_{xi,0}$ develop at low $T$. $T_{\text{CSS}}$ is estimated by (4.20). Concerning to the PFS, we apply the result of Sect.4.1 to the $U_{xi}^V$-dynamics of the term $A_4^cW$ (4.16). The condition $J_1 + J_2 = 1(J_2 = 0)$ reads $2\gamma^2|V_0|^2(2\pi)^{-2}\beta^2 = 1$, so it predicts the CD transition at

$$T_{\text{PFS}} \simeq \frac{\gamma}{\sqrt{2\pi}}|V_0|.$$  

The PFS takes place at $T < T_{\text{PFS}}$. Therefore, at sufficiently low $T$, the effective gauge theory should be in the phase of $(U(1)_{\text{CSS}}, U(1)_{\text{PFS}}) = (\text{deconfinement, deconfinement})$, whose physical picture is discussed in Sect.2.2. In particular, the spinons behave as
bosons. A simple MF study is applicable, and existence of a long-range antiferromagnetic order is expected at low \( T \) (at \( T = 0 \) in the exactly two-dimensional case). Here we point out that, if the condition \( 2(\gamma |V_0|/2\pi)^2 > |D_0|^2 g(0) \) is satisfied, the expressions (4.20) and (4.21) show that the phase like \( (U(1)_{CSS}, U(1)_{PFS}) = \text{(confinement, deconfinement)} \) is realized at \( T_{CSS} < T < T_{PFS} \). As explained in Sect.2.2, if holons cannot move by some localization effect, this phase corresponds to an insulator with massive spin excitations.

Now let us consider the other case of intermediate hole concentrations. Here \( A^c_b \) is dominant due to the large coefficient and the associated gauge field \( U_{xi}^X \) becomes static at low \( T \). In this case, besides \( A^c_b \) and \( A^c_W \), the \( \phi^4 \)-term and \( A^c_{4,V} \) play an important role. From (3.18) and (3.26), we see that the term \( A^c_{4,V} \) dominates over \( A^c_{\phi,4V} \) at low \( T, T < \frac{1}{4} g^2 \delta^2 (1 - \delta^2) \). As one can see from (3.29) and the fact that the coefficient in \( A^c_{4,V} \) of (3.26) is negative, the term \( A^c_{4,V} \) disfavors co-existence of two sets of the static modes, \( U_{xi,0}^X \) and \( U_{xi,0}^V \). This indicates that if \( T_{CSS} > T_{PFS} \), the CSS takes place at low \( T \), but the PFS is suppressed by the \( \phi^4 \)-interaction. Therefore, the phase \( (U(1)_{CSS}, U(1)_{PFS}) = \text{(deconfinement, confinement)} \) is expected. This concludes that the spinons are bosonic at intermediate \( \delta \)'s. The importance of 4-point interactions in separation phenomena was recognized in the PFS of FQHE. Actually, we remember that, for quasiexcitations in the half-filled Landau level, the Coulomb repulsion is essential for the stability of composite fermions \( \square \).

Let us discuss the effect of magnetic terms at intermediate \( \delta \)'s in some detail. It is well known that fermions under a constant magnetic field \( B \) lower their energy drastically when \( B \) is just a unit flux per fermion, since each fermion may absorb a flux quantum to convert itself to a single boson and these bosons may condense in the lowest energy state. Statistical transmutation of electrons in FQHE at \( \nu = 1/(2n + 1) \) is an example. However, in our double-boson representation, we face bosons in a magnetic field and no statistical transmutation occurs in contrast with the case of fermions. Instead we can argue the relative stability of CSS and PFS. The magnetic
term $A^m_\phi$ (3.31) favors the uniform fluxless configuration of $U^X_{xi}$, whereas $A^m_W$ (3.32) favors spatial-flux configuration of $U^V_{xi}$ with $2q$ flux quanta for spinon. From this difference in fluxes which a $b$-particle and a $\phi$-particle feel, one may argue that the CSS is more stable than the PFS, i.e., $T_{CSS} > T_{PFS}$. For bosons in a constant magnetic field $B$, there appear the same one-particle Landau levels as in the case of fermions. However, all the bosons can occupy the lowest state, the energy of which is the zero-point oscillation $\propto B^{1/2}$. Let us assume that only the CSS takes place. Then neither of the quasiexcitations, the $b$-particles nor the composites $f_{x\sigma} = W^\dagger_x \phi_x$ feel any finite magnetic field, since the CS fluxes, $W^\dagger_x$, feel the field $\bar{U}_{xi}$ which cancels $U^V_{xi}$ for $\phi_x$’s. There are no zero-point energy. On the contrary, let us assume only the PFS takes place. Then each quasiexcitation, $b^\dagger_x \phi_{x\sigma}$, feels $2q$ flux and so stores zero-point energy. This leads us to the above conclusion of relative stability.

Anyway, as we explained in Sect.2.2, the region in question corresponds to the anormalous metallic phase of high-Tc cuprates. On the other hand, at sufficiently high $T$, it is obvious that both $U(1)_{CSS}$ and $U(1)_{PFS}$ are in the confinement phase, and the quasiexcitations are the original electrons. This result is also in agreement with experiments.

In Fig.1 we present a possible phase diagram in the $T-\delta$ plane. The line of $T_{CSS}$ is based on the numerical calculations in Ref.[17]. The line of $T_{PFS}$ is drawn assuming that it is a decreasing function of $\delta$. This is expected from $A^e_{4,\phi V}$ (3.26) and the fact that the amplitude $\chi_0$ increases very rapidly as a function of $\delta$ [17]. Also in most of the interesting region of $\delta$, the estimated $T_{CSS}$ is smaller than $J$ or $g_{\phi}^2$. Therefore the region below $T_{CSS}$ is partitioned into two region, the low-$\delta$ region named the region (I), and the high-$\delta$ region, the region (II). These regions correspond to $(U(1)_{CSS}, U(1)_{PFS})=(deconfienement, deconfinement)$, and $(deconfienement, confinement)$, respectively. The spinons are bosonic in the former region, while they are fermionic in the latter region. As explained above, in the region (I) near the half filling, the terms in the effective gauge theory, $A^e_{\phi,D}, A^e_{4,D}$ and $A^e_W$, are dominant. In the region (II) of
the intermediate hole concentration, the terms $A_b^e$, $A_w^e$ and $A_{1,1}^e$ are effective.

5 Conclusions

In this paper, we have studied nature of the quasiexcitations of the two-dimensional t-J model, starting from the slave-boson representation. Especially we are interested in when the statistics of quasiexcitations transmute as $T$ and $\delta$ change and how they are related to the CSS. To this end, we used a general gauge-theoretical approach to separation phenomena developed in the previous papers [4, 8] for two independent separation phenomena, CSS and PFS.

The CSS is understood as a CD phase transition of the gauge fields, the phase degrees of freedom of the link “mean fields” expressing the correlations among nearest-neighbor holons and spinons. In Ref. [4, 17], the critical temperature $T_{CSS}$ of the CD transition is calculated.

To study statistics of quasiexcitations, the CS gauge theory is a suitable apparatus. This problem is closely related to the idea of CF in the half-filled Landau level. In that case, however, an electron transmutes to another fermion, a CF, by attaching two (or even number of) CS flux quanta. In the present t-J model a fermionic spinon in the slave-boson formalism is viewed as a composite of a bosonic spinon and odd number of CS flux quanta. The possibility of dissociation of a spinon into a bosonic spinon and CS flux, i.e., PFS, implies the change of statistics of spinons. As in the case of CSS, we calculated the transition temperature $T_{PFS}$.

Explicitly, we first categorized the possible phases of the t-J model based on the two local gauge symmetries of the system, $U(1)_{CSS}$ and $U(1)_{PFS}$. By using the hopping expansion, we obtained the explicit form of the effective gauge theory. Then, by mapping this effective theory to the classical (an)isototropic XY spin model in two dimensions and using the known results on this model, we identified the physical nature of each phases. The result is summerized in Fig.1. Especially, our investigation
indicates that, near the half filling and at low \( T \), the PFS takes place and spinons behave like bosons rather than fermions.

For more quantitative study, detailed MF calculations are required. In the framework of slave-boson formalism, this is carried out and a line of the CSS transition in the \( T - \delta \) plane is obtained \cite{17}. What is interesting is that \( T_{\text{CSS}} \) is only \( 10 \sim 20 \% \) of the MF critical temperature. This exhibits that the effects of gauge-field fluctuations are very important not only quantitatively but also qualitatively.

It is also interesting and complementary to the present study to start with the slave-fermion representation and then move to the double-boson (or double-fermion) formalism using the CS gauge theory. Parallel discussion is possible in this formalism and we expect nontrivial and interesting result which will be reported elsewhere. In particular it will shed some light on the statistics of holons; a topic that we do not discuss sufficiently in the present paper. One may conceive yet other representations to start with, although they might look excentric, including anyonic excitations. We plan to pursue the problem of statistics in the t-J model by using techniques based on gauge theory similar to the present ones, but from a more general point of view in scope of these different starting representations.

Concerning to the statistics of quasiexcitations in the t-J model, there are some recent works \cite{21}. However, the present approach is distinguished from them by applying the knowledge of strong-coupling gauge theory. We stress that the notion of confinement and deconfinement is powerful to understand separation phenomena of degrees of freedom in condensed-matter physics in an intuitive and coherent manner.

In this paper, we showed explicitly how the gauge-theoretical approach is used to the combined situation of two typical separation phenomena, CSS and PFS. According to the gauge theory of CD transitions, we expect two genuine phase transitions of KT type; the CSS transition at \( T_{\text{CSS}} \) where the CSS takes place, and the PFS transition at \( T_{\text{PFS}} \) where the statistics of spinons changes. These should be taken seriously as well as the experimentally established transitions; the antiferromagnetic transition.
at $T_{\text{Neel}}$ \cite{15} and the superconducting transition at $T_c$ \cite{9}. Sufficient experimental indications seem to have appeared for the CSS \cite{17}. It seems remaining to find such indications for PFS.
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[11] The present Hamiltonian includes the ”counter terms”, the $\phi^4$-term and the $b^4$-term. They are necessary to maintain the equivalence to the original t-J Hamiltonian since they work as the counter terms to cancel unwelcome terms produced by the Hubbard-Storatonovich transformation. In the MF analysis of Ref.[2], they are ignored, but they play an important role in the CSS and PFS as shown in Sect.4.2.

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FIGURE CAPTION

FIG. 1. A possible phase diagram in the $T - \delta$ plane. The region below $T_{\text{CSS}}$ is partitioned into two regions, (I) the low-$\delta$ region where the PFS occurs, and (II) the high-$\delta$ region without PFS. The spinons are bosonic in (I), and fermionic in (II). In Ref. [17], $T_{\text{CSS}}$ is explicitly evaluated as a function of $\delta$ by numerical calculations. These two curves $T_{\text{CSS}}$ and $T_{\text{PFS}}$ should be taken seriously as well as the experimentally established curves; the antiferromagnetic transition line $T_{\text{Neel}}$ [15] and the superconducting transition line $T_c$ [9].