Hydrogen atom in fuzzy spaces - Exact solution

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Abstract. We investigate consequences of space non-commutativity in quantum mechanics of the hydrogen atom. We introduce rotationally invariant noncommutative space \( \hat{R}_3 \) - an analog of the hydrogen atom (H-atom) configuration space which is generated by noncommutative coordinates realized as operators in an auxiliary (Fock) space \( \mathcal{F} \). The Hilbert space \( \hat{H} \) of wave functions \( \hat{\psi} \) is formed by properly weighted Hilbert-Schmidt operators in \( \mathcal{F} \). We define an analog of the H-atom Hamiltonian in \( \hat{R}_3 \) and explicitly determine the bound state energies \( E^{\lambda}_{n} \) and the corresponding eigenstates \( \hat{\psi}^{\lambda}_{njm} \).

1. Introduction

Basic ideas of non-commutative geometry have been developed in [1] and, in a form of matrix geometry, in [2]. The main applications have been considered in the area of quantum field theory in order to understand, or even to remove, UV singularities, and/or eventually to construct a proper base for the quantum gravity.

The analysis performed in [3] led to the conclusion that quantum vacuum fluctuations could create (micro)black holes which prevent localization of space-time points. Mathematically this requires non-commutative coordinates \( x^\mu \) in space-time satisfying specific uncertainty relations. The simplest set of operators \( \hat{x}^\mu \) representing \( x^\mu \) should satisfy Heisenberg-Moyal commutation relations

\[
[\hat{x}^\mu, \hat{x}^\nu] = i \theta^{\mu\nu}, \quad \mu, \nu = 0, 1, 2, 3,
\]

where \( \theta^{\mu\nu} \) are given numerical constants that specify the non-commutativity of the space-time in question.

Later in [4] it was shown that field theories in NC spaces with (1) can emerge as effective low energy limits of string theories. These results supported a vivid development of the non-commutative QFT. However, such models contain various unpleasant and unwanted features. The divergences are not removed, on the contrary, UV-IR mixing appears, [5]. The Lorentz invariance is broken down to \( SO(2) \times SO(1, 1) \), but even this is sufficient to prove the classical CPT and Spin-statistics theorems, [6]. This was not accidental and led to the twisted Poincaré reinterpretation of NC space-time symmetries, [7].

However, it could be interesting to reverse the approach. Not to use the NC geometry to improve the foundation of QFT, what is a very complicated task, but to test the effect of non-commutativity of the space on the deformation of the well-defined quantum mechanics (QM):

- Various QM systems have been investigated in 3D space with Heisenberg-Moyal commutation relations \( [\hat{x}_i, \hat{x}_j] = i\theta^{ij}, \quad i, j = 1, 2, 3 \), e.g. harmonic oscillator, Aharonov-Bohm
effect, Coulomb problem, see [8], [9]. However, in such 3D NC space the rotational symmetry is violated and there are systems, such as $H$-atom, that are tightly related to the rotational symmetry.

- The rotational symmetry survives in 2D Heisenberg-Moyal space with NC coordinates $\hat{x}_1, \hat{x}_2$ satisfying the $\mathcal{F}$ commutation relations $[\hat{x}_1, \hat{x}_2] = i\theta$ in an auxiliary Hilbert space. In [10] a planar spherical well was described in detail, remarkable to see how the persisted rotational symmetry helps to solve exactly the problem in question.

Our aim is to extend such scheme to the QM problems with rotationally symmetric potentials $V(r)$ in the configuration space $R^3_0 \equiv R^3 \setminus \{0\}$. We restrict ourselves to the Coulomb potential which, in the usual (commutative) setting, is a solution of the Poisson equation vanishing at infinity:

$$\Delta V(r) = 0 \Rightarrow V(r) = -\frac{q}{r}. \quad (2)$$

For $H$-atom, in a Gaussian system of units, is a square of electric charge $e^2$. In this case we are dealing with Schrödinger equation

$$-\frac{\hbar^2}{2m} \Delta \psi(x) - \frac{e^2}{r} \psi(x) = E \psi(x), \quad r = |x| > 0 \quad (3)$$

in the Hilbert space $\mathcal{H}_0 = \mathcal{L}^2(R^3, d^3x)$. Expressing the wave function as

$$\psi(x) = R_j(r) H_{jm}(x), \quad H_{jm}(x) \sim r^j Y_{jm}(\theta, \varphi), \quad (4)$$

and putting $\alpha = 2m e^2/\hbar^2$ and $\kappa = \sqrt{-2mE/\hbar}$, we obtain the radial Schrödinger equation:

$$r R_j''(r) + 2(j + 1)R_j'(r) + \alpha R_j(r) = \kappa^2 r R_j(r). \quad (5)$$

Its solution is given in terms of confluent hypergeometric function:

$$R_j(r) = e^{-\kappa r} F\left(j + 1 - \frac{\alpha}{2\kappa}, 2j + 2; 2\kappa r\right). \quad (6)$$

For bound states $E < 0$, i.e. real-valued $\kappa$, the solution should have a finite norm in $\mathcal{H}_0$. This is the case when the first argument of the degenerated hypergeometric function is zero or a negative integer, and this determines the discrete energy eigenvalues:

$$\frac{\alpha}{2\kappa_n} = n = j + 1, j + 2, \ldots \Rightarrow E_n = -\frac{\hbar^2}{2m} \kappa_n^2 = -\frac{me^4}{2\hbar^2 n^2}. \quad (7)$$

In this paper we extend the QM solution of the Coulomb problem to the non-commutative rotationally invariant space. In Section 2 we define $\hat{R}^3_0$, a rotationally invariant NC generalization of the configuration space $R^3_0$, we introduce the generators of rotations and the NC analog of their eigenfunctions. In Section 3 we define Hilbert space $\hat{\mathcal{H}}$ - the NC analog of $\mathcal{H}$, and we introduce the NC analog of the Coulomb problem Hamiltonian acting in $\hat{\mathcal{H}}$. In Section 4 we exactly solve the corresponding NC analog of the Schrödinger equation. Last Section 5 contains conclusions.

2. The noncommutative space $\hat{R}^3_0$

In this section we define the noncommutative space $\hat{R}^3_0$, possessing full rotational invariance, as a sequence of fuzzy spheres introduced, in various contexts, in [11]. A similar construction of a 3D noncommutative space, as a sequence of fuzzy spheres, was proposed in [12]. However, various fuzzy spheres are related to each other differently not leading to $R^3$ at large distances.
We realize the noncommutative coordinates in $\hat{R}_0^3$ in terms of 2 pairs of boson annihilation and creation operators $\hat{a}_\alpha, \hat{a}^\dagger_\alpha, \alpha = 1, 2$, satisfying usual commutation relations, see [13]:

$$[\hat{a}_\alpha, \hat{a}^\dagger_\beta] = \delta_{\alpha\beta}, \quad [\hat{a}_\alpha, \hat{a}_\beta] = [\hat{a}^\dagger_\alpha, \hat{a}^\dagger_\beta] = 0. \quad (8)$$

They act in an auxiliary Fock space $\mathcal{F}$ spanned by normalized vectors

$$|n_1, n_2\rangle = \frac{(\hat{a}^\dagger_1)^{n_1}(\hat{a}^\dagger_2)^{n_2}}{\sqrt{n_1!n_2!}} |0\rangle. \quad (9)$$

Here, $|0\rangle \equiv |0, 0\rangle$ denotes the normalized vacuum state: $\hat{a}_1 |0\rangle = \hat{a}_2 |0\rangle = 0$.

The noncommutative coordinates $\hat{x}_j$, $j = 1, 2, 3$, in the space $\hat{R}_0^3$ are given as

$$\hat{x}_j = \lambda \hat{a}^\dagger_j \sigma_j \hat{a} = \lambda \sigma_{ij} \hat{a}^\dagger_i \hat{a}_j, \quad j = 1, 2, 3, \quad (10)$$

where $\lambda$ is a universal length parameter. The coordinates $\hat{x}_j$ satisfy rotationally invariant commutation rules:

$$[\hat{x}_i, \hat{x}_j] = 2i \lambda \varepsilon_{ijk} \hat{x}_k, \quad [\hat{x}_i, \hat{x}_j] = 0, \quad (11)$$

where $\hat{\sigma} = \lambda \hat{N}$, and $\hat{N} = \hat{a}^\dagger \hat{a} = \hat{a}_\alpha^\dagger \hat{a}_\alpha$.

Let us consider the linear space of normal ordered polynomials:

$$\hat{\Psi} = \sum C_{m_1m_2n_1n_2} (\hat{a}^\dagger_1)^{m_1} (\hat{a}^\dagger_2)^{m_2} (\hat{a}_1)^{n_1} (\hat{a}_2)^{n_2}, \quad (12)$$

where the summation is finite over nonnegative integers satisfying $m_1 + m_2 = n_1 + n_2$. In this space we define generators of rotations $L_j$, $j = 1, 2, 3$, as follows

$$L_j \hat{\Psi} = \frac{i}{2} [\hat{a}^\dagger_j \sigma_j \hat{a}, \hat{\Psi}], \quad j = 1, 2, 3, \quad (13)$$

obeying proper commutation relations

$$[L_i, L_j] \hat{\Psi} = (L_i L_j - L_j L_i) \hat{\Psi} = i \varepsilon_{ijk} L_k \hat{\Psi}. \quad (14)$$

With respect to the rotations (13) the doublet of annihilation (creation) operators transforms as spinor (conjugated spinor), whereas the triplet of NC coordinates as vector. The standard eigenfuctions $\hat{\psi}_{jm}$, $j = 0, 1, 2, \ldots$, $m = -j, \ldots, +j$, satisfying

$$L_j^2 \hat{\psi}_{jm} = j(j + 1) \hat{\psi}_{jm}, \quad L_3 \hat{\psi}_{jm} = m \hat{\psi}_{jm}, \quad (15)$$

are given by the formula

$$\hat{\psi}_{jm} = \lambda^j \sum_{jm} (\hat{a}^\dagger_1)^{m_1} (\hat{a}^\dagger_2)^{m_2} m_1!m_2! : R_j(\hat{\sigma}) : \frac{(\hat{a}^\dagger_1)^{n_1} (\hat{a}^\dagger_2)^{n_2} n_1!n_2!}{N!} \quad (16)$$

with the summation over all nonnegative integers satisfying $m_1 + m_2 = n_1 + n_2 = j$, $m_1 - m_2 - n_1 + n_2 = 2m$. Thus $\hat{\psi}_{jm} = 0$ when restricted to the subspaces $\mathcal{F}_N = \{|n_1, n_2| n_1 + n_2 = N\}$ with $N < j$. The symbol : $R_j(\hat{\sigma}) :$ represents a normal ordered analytic function in the operator $\hat{\sigma}$:

$$: R_j(\hat{\sigma}) : = \sum_k c_k^j : \hat{\sigma}^k : = \sum_k c_k^j \lambda^k \frac{\hat{\sigma}!}{(N - k)!}. \quad (17)$$

For any fixed : $R_j(\hat{\sigma}) :$ equation (16) defines a representation space for a unitary irreducible representation with spin $j$. 

3
3. Quantum mechanics in space $\hat{\mathbb{R}}^3_\lambda$

Let $\mathcal{H}$ denote the Hilbert space generated by functions (16) with weighted Hilbert-Schmidt norm

$$
\|\hat{\Psi}\|^2 = 4\pi \lambda^3 \text{Tr}[(\hat{N} + 1) \hat{\Psi} \dagger \hat{\Psi}] = 4\pi \lambda^2 \text{Tr}[\hat{r} \hat{\Psi} \dagger \hat{\Psi}], \quad \hat{r} = \lambda (\hat{N} + 1).
$$

Here, $\hat{r} = \lambda (\hat{N} + 1) = \hat{\varrho} + \lambda$ is the operator that approximates the Euclidean distance from the origin in an optimal way: $r^2 - \hat{r}_j^2 = \lambda^2$. In next Section we give a strong argument supporting the exceptional role of $\hat{r}$.

The rotationally invariant weight $w(\hat{r}) = 4\pi \lambda^2 \hat{r}$ is determined by the requirement that a ball in $\hat{\mathbb{R}}^3_\lambda$ with radius $r = \lambda(N+1)$ should possess a standard volume in the limit $r \to \infty$. The weighted trace $4\pi \lambda^2 \text{Tr}[\hat{r} \ldots]$ at large distances goes over to the usual volume integral $\int d^3\vec{x} \ldots$.

The 3D noncommutative space proposed in [12] corresponds to the choice $w(\hat{r}) = \text{const}$ and at large distances does not correspond to the flat space $\mathbb{R}^3_0$.

The generators of rotations $L_j$, $j = 1, 2, 3$ are hermitian (self-adjoint) operators in $\mathcal{H}$. The norm (18) of the corresponding operator eigenfunctions $\hat{\Psi}_{jm}$ reads:

$$
\|\hat{\Psi}_{jm}\|^2 = \frac{4\pi \lambda^3 + 2j}{(2j)!^2} \sum_{N=0}^\infty (N + j + 1) \left(\frac{N + j + 1}{2j + 1}\right) |\mathcal{R}_j(N)|^2,
$$

$$
\mathcal{R}_j(N) = \langle n, N - n | R_j(\hat{\varrho}) | n, N - n \rangle
$$

Now we are ready to define the kinetic term of a Hamiltonian in the noncommutative case. First we postulate the NC analog of the Laplacian in $\hat{\mathbb{R}}^3_\lambda$ as follows:

$$
\Delta_\lambda \hat{\Psi} = - \frac{1}{\lambda^2} \left[\hat{a}_\alpha, [\hat{a}_\alpha, \hat{\Psi}]\right] = - \frac{1}{\lambda^2(N+1)} \left[\hat{a}_\alpha, [\hat{a}_\alpha, \hat{\Psi}]\right].
$$

This choice is motivated by the following facts:

- A double commutator is an analog of a second order differential operator;
- The factor $\hat{r}^{-1}$ guarantees that the operator $\Delta_\lambda$ is hermitian (self-adjoint) in $\mathcal{H}$, and finally,
- The factor $\lambda^{-1}$, or $\lambda^{-2}$ respectively, guarantees the correct physical dimension of $\Delta_\lambda$ and its non-trivial commutative limit.

Calculating the action of (20) on $\hat{\Psi}_{jm}$ given in (16) we can check whether the postulate (20) is a reasonable choice. The corresponding formula is derived in Appendix:

$$
\Delta_\lambda \hat{\Psi}_{jm} = - \frac{1}{\lambda^r} \left[\hat{a}_\alpha, [\hat{a}_\alpha, \hat{\Psi}_{jm}]\right]
$$

$$
= \frac{\lambda^j}{\lambda^r} \sum_{(jm)} \frac{(\hat{a}_1^{m_1})(\hat{a}_2^{m_2})}{m_1! m_2!} \cdot :\hat{\varrho} R'(\hat{\varrho}) + 2(j+1)R'(\hat{\varrho}) + \hat{a}_1^{n_1} (-\hat{a}_2^{n_2}) : \frac{n_1! n_2!}{n_1! n_2!}.
$$

Here the symbols $R'(\hat{\varrho})$ is defined as a formal derivative $\partial_{\hat{\varrho}}$:

$$
R(\hat{\varrho}) = \sum_{k=0}^\infty c_k \hat{\varrho}^k \Rightarrow \quad R'(\hat{\varrho}) = \sum_{k=1}^\infty k c_k \hat{\varrho}^{k-1}. \quad (22)
$$

The higher derivatives as given by repeated application of this rule. In the commutative limit $\lambda \to 0$ formally $\hat{\varrho} \to r$, and we see that (21) guarantees that $\Delta_\lambda$ reduces just to the standard Laplacian. Based on that, we postulate the kinetic term of the Hamiltonian as follows

$$
H_0 \hat{\Psi} = - \frac{\hbar^2}{2m} \Delta_\lambda \hat{\Psi} = \frac{\hbar^2}{2m\lambda^r} \left[\hat{a}_\alpha, [\hat{a}_\alpha, \hat{\Psi}]\right].
$$

(23)
4. The Coulomb problem in $\hat{R}^3_0$

In the commutative case the Coulomb potential is a radial solution of the equation (2) finite at infinity. Due to our choice of the noncommutative Laplacian $\Delta_\lambda$ the equivalent equation is

$$[\hat{a}_\alpha^\dagger, [\hat{a}_\alpha, V(\hat{N})]] = 0.$$  

It can be rewritten as a simple recurrent relation

$$(\hat{N} + 2) V(\hat{N} + 1) - (\hat{N} + 1) V(\hat{N}) = (\hat{N} + 1) V(\hat{N}) - \hat{N} V(\hat{N} - 1),$$

which possesses just the Coulomb type solution vanishing at large distances:

$$V(\hat{N}) = -\frac{q}{\lambda(N+1)} = -\frac{q}{\hat{r}},$$

where $q$ is an arbitrary constant. We see that the dependence $\hat{r}^{-1}$ of the NC Coulomb potential is inevitable.

Thus, the noncommutative analog of the Schrödinger equation with the Coulomb potential in $\hat{R}^3_0$ is

$$\frac{\hbar^2}{2m\lambda}\hat{a}_\alpha^\dagger, [\hat{a}_\alpha, \hat{\Psi}] - \frac{q}{\hat{r}} \hat{\Psi} = E \hat{\Psi} \leftrightarrow \frac{1}{\lambda} [\hat{a}_\alpha^\dagger, [\hat{a}_\alpha, \hat{\Psi}]] - \alpha \hat{\Psi} = -\kappa^2 \hat{r} \hat{\Psi},$$

where $\alpha = 2me^2/\hbar^2$ and $\kappa = \sqrt{-2mE}/\hbar$.

Inserting $\hat{\Psi}_{jm}$ into (26) we obtain the NC analog of radial Schrödinger equation:

$$:\hat{\rho} \hat{R}_j'' + 2(j+1) \hat{R}_j' + \alpha \hat{R}_j : = \kappa^2 : [\hat{\rho} \hat{R}_j + \lambda((j+1) \hat{R}_j + \hat{\rho} \hat{R}_j')] : .$$

The term proportional to $\lambda$ represents the NC correction (their calculation is given in Appendix). For $\lambda \to 0$ this NC radial Schrödinger equation reduces to the standard radial Schrödinger equation. We associate the following ordinary differential equation to the mentioned operator radial Schrödinger equation (27):

$$\hat{\rho} \mathcal{R}_j'' + 2(j+1) \mathcal{R}_j' + \alpha \mathcal{R}_j = \kappa^2 [\hat{\rho} \mathcal{R}_j + \lambda((j+1) \hat{R}_j + \hat{\rho} \hat{R}_j')].$$

If the function $\mathcal{R}_j = \mathcal{R}_j(\hat{\rho}) = \sum_k c_k^j \hat{\rho}^k$ solves the associated ordinary differential equation (28), then

$$\hat{\rho} \mathcal{R}_j'' = \mathcal{R}_j(\hat{\rho}) : = \sum_k c_k^j : \hat{\rho}^k : = \sum_k c_k^j \lambda^k \frac{\hat{N}^k}{(N-k)!} : .$$

solves the operator radial equation (27). Moreover, the operator function $\hat{\mathcal{R}}_j = : \mathcal{R}_j(\hat{\rho}) :$ possesses a finite norm in $\mathcal{H}$ provided the function $\mathcal{R}_j = \mathcal{R}_j(\rho)$ has finite norm in $\mathcal{H}$ (since the norm (18) asymptotically reduces to the usual QM norm).

The solution $\mathcal{R}_j$ of the associated radial Schrödinger equation (28) is given similarly as in the standard Coulomb problem in terms of the confluent hypergeometric function:

$$\mathcal{R}_j(\hat{\rho}) = e^{-b\hat{\rho}} F\left(j + 1 - \frac{\alpha}{2d\kappa}, 2j + 2; 2\hat{\rho}\kappa d\right).$$

The dimensionless quantities $b$ and $d$ given as

$$b = \sqrt{1 + \eta^2} - \eta, \quad d = \sqrt{1 + \eta^2}, \quad \eta = \frac{1}{2} \lambda\kappa$$
specify the NC corrections. They enter $b$ and $d$ via the parameter $\eta = \lambda \kappa / 2$. So they vanish not only in the commutative limit $\lambda \to 0$, but also for $\kappa \to 0$.

We shall restrict ourselves to the determination of the bound states spectra with $E < 0$ and $\kappa > 0$. In this case $R_{\nu}(\hat{\varrho})$ should be normalizable. This is ensured if the first argument of the confluent hypergeometric function is zero or negative integer, what determines the discrete energy eigenvalues (remember that $d$ is $\kappa$-dependent, see (31)):

$$E_n^\lambda = -\frac{\hbar^2 \kappa_n^2}{2m} = -\frac{\hbar^2}{2m} \frac{2}{1 + \sqrt{1 + \lambda^2/a_0^2 n^2}},$$  

(32)

where $a_0 = 2/\alpha = \hbar^2 / me^2 = 5.29 \times 10^{-11} m$ is the Bohr radius. The first factor in $E_n^\lambda$ is just the standard bound state energy of the Coulomb problem, whereas the second one represents the noncommutative correction. The NC corrections vanish in the limit $\lambda/n \to 0$, i.e., in the commutative limit $\lambda \to 0$, or for fixed $\lambda$, in the quasi-classical limit $n \to \infty$ for highly excited states.

The solution $\hat{\Psi}^\lambda_{\nu jm}$ of the operator equation (27) corresponding to the energy $E_n^\lambda$ is

$$\hat{\Psi}^\lambda_{\nu jm} = N^\lambda_{\nu jm} \sum_{(jm)} \frac{\hat{a}_1^{m_1} \hat{a}_2^{m_2}}{m_1! m_2!} : R_{\nu j}(\hat{\varrho}) : \frac{\hat{a}_1^{n_1} (-\hat{a}_2)^{n_2}}{n_1! n_2!},$$  

(33)

where $N^\lambda_{\nu jm}$ denotes the normalization factor and

$$R_{\nu j}(\hat{\varrho}) = e^{-b_\nu \kappa_n \hat{\varrho}} F(j + 1 - n, 2j + 2, 2\kappa_n d_n) \kappa_n = \sqrt{-2mE_n^\lambda / \hbar}.$$  

(34)

The normalization constant $N^\lambda_{\nu jm}$ can be determined by (19) (we skip the calculation).

5. Conclusions

We carefully defined the NC rotationally invariant analog of the QM configuration space and the Hilbert space of operator wave functions in NC configuration space. The central point of our construction was the definition of $\Delta^{\lambda}$ the NC analog of Laplacian, supplemented by a consequent definition of the weighted Hilbert-Schmidt norm and a definition of the Coulomb potential satisfying NC Poisson equation.

With this input this Hilbert space we introduced the NC analog of $H$-atom Hamiltonian and explicitly determined the bound-state energies $E_n^\lambda$ and corresponding eigenstates $\hat{\psi}^\lambda_{\nu jm}$ (see equations (32), (33) and (34)).

We found that the discrete parameters $n, j, m$ have the same meaning and range as in the standard (commutative) Coulomb problem, and moreover, the bound-state energies and eigenstates possess a smooth commutative limit $\lambda \to 0$. This paper does not deal with the case of the scattering in the NC configuration space - this will be discussed elsewhere.

The noncommutativity parameter $\lambda$ is not fixed within our model. However, it can be estimated by some other physical requirement. For example, one can postulate, that the rest energy $mc^2$ of electron is equal to the electrostatic energy of its Coulomb field. We stress that in the NC case the electrostatic energy of electron is finite (no cut-off at short distance is needed). A straightforward calculation gives the relation (the details will be published, see [14]):

$$mc^2 = \frac{3 e^2}{8} \lambda \Rightarrow \lambda = \frac{3 e^2}{8 mc^2} \equiv \lambda_0.$$  

(35)
This $\lambda_0$ is fraction of the classical radius of electron $r_0 = e^2/mc^2$: $\lambda_0 = 1.06 \times 10^{-15} \, m = 1.06 \, fm$ (the similarity to proton radius is purely accidental). The NC corrections to the $H$-atom energy levels given in (32) are of order $(\lambda_0/a_0)^2 = (9/64) \alpha_0^2 \approx 4 \times 10^{-11}$ (here $\alpha_0 \approx 1/137$ is fine structure constant). Such tiny corrections to energy levels are beyond any experimental evidence. Moreover, at $\lambda_0 \approx 1 \, fm$ relativistic and QFT effects become essential.

Our investigation indicates that the noncommutativity of the configuration space is fully consistent with the general QM axioms. However, a more detailed analysis of the Coulomb problem in \( \hat{R}_\lambda^3 \) would be a desirable dealing, e.g., with the following aspects:

- Coulomb scattering problem, dyon problem (electron in the electric point charge and magnetic monopole field);
- Coulomb problem in \( \hat{R}_0^3 \) and its dynamical symmetry, hidden supersymmetry and integrability of the Coulomb system.

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### Appendix

**Proof of the NC Laplacian action (21).** Let us begin with the calculation of the double commutator in question (we skip indices $j$ and $m$):

\[
[\hat{a}_\alpha^\dagger, [\hat{a}_\alpha, \hat{\Psi}_{jm}]] = \lambda^j [\hat{a}_\alpha^\dagger, [\hat{a}_\alpha, \sum_{(jm)} (\hat{a}_1^\dagger)^{m_1} (\hat{a}_2^\dagger)^{m_2} / m_1! m_2! ] : \hat{R} : \hat{a}_1^{n_1} (\hat{a}_2)^{n_2} / n_1! n_2! ]
\]

\[
= \lambda^j \sum_{(jm)} [\hat{a}_\alpha, \frac{(\hat{a}_1^\dagger)^{m_1} (\hat{a}_2^\dagger)^{m_2}}{m_1! m_2! }] [\hat{a}_\alpha^\dagger, : \hat{R} : \hat{a}_1^{n_1} (\hat{a}_2)^{n_2} / n_1! n_2! ]
\]

\[
+ \lambda^j \sum_{(jm)} [\hat{a}_\alpha, \frac{(\hat{a}_1^\dagger)^{m_1} (\hat{a}_2^\dagger)^{m_2}}{m_1! m_2! }] : \hat{R} : [\hat{a}_\alpha^\dagger, \frac{(\hat{a}_1^\dagger)^{n_1} (\hat{a}_2)^{n_2}}{n_1! n_2! } ]
\]

\[
+ \lambda^j \sum_{(jm)} \frac{(\hat{a}_1^\dagger)^{m_1} (\hat{a}_2^\dagger)^{m_2}}{m_1! m_2! } [\hat{a}_\alpha^\dagger, : \hat{R} : \hat{a}_\alpha, : \hat{R} : ] \frac{(\hat{a}_1^\dagger)^{n_1} (\hat{a}_2)^{n_2}}{n_1! n_2! },
\]

(1)

where \( \hat{R} = \sum_{k=0}^{\infty} c_k^j \hat{N}^k \), i.e., \( c_k^j = \lambda^j c_k \). Now we shall use the following commutation relations

\[
[\hat{a}_\alpha^\dagger, \hat{N}^k ] = - k \hat{a}_\alpha^\dagger : \hat{N}^{k-1} : \Rightarrow [\hat{a}_\alpha^\dagger, : \hat{R} : ] = - \hat{a}_\alpha^\dagger : \partial_N \hat{R} :,
\]

\[
[\hat{a}_\alpha, \hat{N}^k ] = k : \hat{N}^{k-1} : \hat{a}_\alpha \Rightarrow [\hat{a}_\alpha, : \hat{R} : ] = : \partial_N \hat{R} : \hat{a}_\alpha,
\]

(2)

where \( \partial_N \) denotes the derivatives with respect to \( \hat{N} \): \( \partial_N \hat{R} = \sum_{k=1}^{\infty} k c_k^j \hat{N}^{k-1} \).

It is easy to see that the second line in (1) vanish, and the the first and third line give the same contribution

\[
\sum_{(jm)} \frac{(\hat{a}_1^\dagger)^{m_1} (\hat{a}_2^\dagger)^{m_2}}{m_1! m_2! } (- j : \partial_N \hat{R} : \frac{\hat{a}_1^{n_1} (\hat{a}_2)^{n_2}}{n_1! n_2! }).
\]

(3)
From (2) the double commutator $[\hat{a}_\alpha^\dagger, [\hat{a}_\beta, : \hat{R} :]]$ follows directly, and this gives the value of the third line in (1)

$$
\sum_{(jm)} \frac{(\hat{a}_1^\dagger)^{m_1} (\hat{a}_2^\dagger)^{m_2}}{m_1! m_2!} \left( - : \hat{N} \partial_{\hat{N}}^2 \hat{R} : + 2 : \partial_{\hat{N}} \hat{R} : \right) \frac{\hat{a}_1^{n_1} (-\hat{a}_2)^{n_2}}{n_1! n_2!}.
$$

(4)

Introducing parameter $\lambda$ and switching to the derivatives with respect to $\hat{\varrho}$ the last two equations yields (21).

**Calculation of the NC corrections coming from r.h.s. in (27).** From equation (which can be proved by induction in $k$)

$$
:\hat{N}^k : |n_1, n_2\rangle = \frac{N!}{(N-k)!} |n_1, n_2\rangle, \quad N = n_1 + n_2
$$

(5)

it follows easily

$$
\hat{N} : \hat{N}^k : = : \hat{N}^{k+1} : + k : \hat{N}^k : \quad \Rightarrow \quad \hat{N} : \hat{R} : = : \hat{N} \hat{R} : + : \hat{N} \partial_{\hat{N}} \hat{R} : .
$$

(6)

This relation gives directly

$$
(\hat{N} + 1) \sum_{(jm)} \frac{(\hat{a}_1^\dagger)^{m_1} (\hat{a}_2^\dagger)^{m_2}}{m_1! m_2!} : \hat{R} : \frac{\hat{a}_1^{n_1} (-\hat{a}_2)^{n_2}}{n_1! n_2!} = \sum_{(jm)} \ldots [(\hat{N} + j + 1) : \hat{R} :] \ldots
$$

$$
= \sum_{(jm)} \ldots : [(\hat{N} + j + 1) \hat{R} + \hat{N} \partial_{\hat{N}} \hat{R}] : \ldots ,
$$

(7)

where we have replaced both untouched factors containing annihilation and creation operators by dots. Introducing parameter $\lambda$ again and switching to the derivatives with respect to $\hat{\varrho}$ we obtain the desired NC corrections in (27).

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