ORTHOGONAL DIRICHLET POLYNOMIALS WITH CONSTANT WEIGHT

Dedicated to Academician Professor Gradimir Milovanović on the occasion of his 70th birthday.

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Let \( \{\lambda_j\}_{j=1}^{\infty} \) be a sequence of distinct positive numbers. We analyze the orthogonal Dirichlet polynomials \( \{\psi_{n,T}\} \) formed from linear combinations of \( \{\lambda_j^{-it}\}_{j=1}^{n} \), associated with constant (or Legendre) weight on \([-T,T]\). Thus

\[
\frac{1}{2T} \int_{-T}^{T} \psi_{n,T}(t) \overline{\psi_{m,T}(t)} dt = \delta_{mn}.
\]

Moreover, we analyze how these polynomials behave as \( T \) varies.

1. FILL SECTION TITLE HERE

Throughout, let

\[
\{\lambda_j\}_{j=1}^{\infty}
\]

be a sequence of distinct positive numbers.

Given \( m \geq 1 \), a Dirichlet polynomial of degree \( \leq n \) [16], [17] associated with this sequence of exponents has the form

\[
\sum_{n=1}^{m} a_n \lambda_n^{-it} = \sum_{n=1}^{m} a_n e^{-i(\log \lambda_n)t},
\]

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where \( \{a_n\} \subset \mathbb{C} \). We denote the set of all such polynomials by \( \mathcal{L}_n \).

The traditional orthogonal Dirichlet polynomials are just the “monomials” \( \{\lambda_n^{-it}\} \) themselves. Indeed, in the theory of almost-periodic functions [1], [2], heavy use is made of orthogonality in the mean:

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T \lambda_j^{-it} \overline{\lambda_k^{-it}} dt = \delta_{jk}.
\]

In the hope that a more standard orthogonality relation might have some advantages, the author [6], investigated Dirichlet orthogonal polynomials associated with the arctangent density. Thus \( \phi_n \in \mathcal{L}_n \) has positive leading coefficient, and

\[
\int_{-\infty}^{\infty} \phi_n(t) \overline{\phi_m(t)} \frac{dt}{\pi(1+t^2)} = \delta_{mn}, \quad m,n \geq 1.
\]

These Dirichlet orthogonal polynomials admit a very simple explicit expression, at least when \( 0 < \lambda_1 < \lambda_2 < \cdots \) : for \( n \geq 2 \),

\[
\phi_n(t) = \frac{\lambda_n^{1-it} - \lambda_{n-1}^{1-it}}{\sqrt{\lambda_n^2 - \lambda_{n-1}^2}}.
\]

These orthonormal polynomials have been applied in several questions by Weber and Dimitrov as well as the author [4], [7], [15], [17], [18], [19]. In a subsequent paper [8], the author considered orthogonal Dirichlet polynomials for the Laguerre weight, though it turned out that much of the material there was already subsumed by Müntz orthogonal polynomials [3]. Müntz orthogonal polynomials have also been a topic investigated by Gradimir Milovanovic [10], [11], [12], to whom this paper is dedicated.

Very recently [9], we investigated Dirichlet orthogonal polynomials for rational weights

\[
w(t) = \sum_{m=1}^{L} \frac{c_m}{\pi \left(1 + (b_m t)^2\right)}
\]

and appropriately chosen \( \{c_j\} \). Here \( L \geq 1 \), and \( 1 = b_1 < b_2 < \cdots < b_L \). We obtained a simple explicit determinantal expression for the orthonormal polynomials, but could only resolve positivity of the weight for the case \( L = 2 \).

In this paper, we let \( T > 0 \), and consider \( \psi_{n,T} \in \mathcal{L}_n \), with positive leading coefficient \( \gamma_{n,T} \), such that

\[
(\psi_{n,T}, \psi_{m,T})_T = \frac{1}{2T} \int_{-T}^{T} \psi_{n,T}(t) \overline{\psi_{m,T}(t)} dt = \delta_{mn}.
\]

We are especially interested in how \( \psi_{n,T} \) behaves as \( T \) varies, and in particular how it behaves as \( T \to \infty \). Next, define the \( n \)th reproducing kernel

\[
K_{n,T}(u,v) = \sum_{j=1}^{n} \psi_{j,T}(u) \overline{\psi_{j,T}(v)}.
\]
In the sequel, we also use
\[ S(u) = \frac{\sin u}{u}. \]

From the simple relation
\[
\left( \lambda_j^{-it}, \lambda_k^{-it} \right)_T = \frac{1}{2T} \int_{-T}^{T} (\lambda_j/\lambda_k)^{-it} dt = S(T \log (\lambda_j/\lambda_k)),
\]
and standard determinantal representations for orthonormal functions with respect to a given inner product, we see that
\[
\psi_{n,T}(x) = \frac{(-1)^{n+1}}{\sqrt{A_{n-1,T}A_{n,T}}} \times \det \begin{bmatrix}
\lambda_1^{-ix} & \lambda_2^{-ix} & \lambda_3^{-ix} & \cdots & \lambda_n^{-ix} \\
1 & S(T \log \lambda_1/\lambda_2) & S(T \log \lambda_1/\lambda_3) & \cdots & S(T \log \lambda_1/\lambda_n) \\
S(T \log \lambda_2/\lambda_1) & 1 & S(T \log \lambda_2/\lambda_3) & \cdots & S(T \log \lambda_2/\lambda_n) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
S(T \log \lambda_{n-1}/\lambda_1) & S(T \log \lambda_{n-1}/\lambda_2) & S(T \log \lambda_{n-1}/\lambda_3) & \cdots & S(T \log \lambda_{n-1}/\lambda_n)
\end{bmatrix},
\]
so the leading coefficient of \( \psi_{n,T}(x) \) is
\[
\gamma_{n,T} = \sqrt{\frac{A_{n-1,T}}{A_{n,T}}},
\]
where
\[
A_{n,T} = \det [S(T \log \lambda_j/\lambda_k)]_{1 \leq j,k \leq n}.
\]

It follows easily from the determinantal expression and the fact that
\[
\lim_{x \to \infty} S(x) = 0,
\]
that
\[
\lim_{T \to \infty} \psi_{n,T}(x) = \lambda_n^{-ix}
\]
and that \( \psi_{n,T} \) is an infinitely differentiable function of \( T \).

One of the motivations for our study is the celebrated Montgomery-Vaughan inequality and its ramifications. In one form its asserts that [14, p. 74, Corollary 2], [13, p. 128, Thm. 1]
\[
\left| \sum_{j=1}^{n} a_j \lambda_j^{-it} \right|^2 dt = (T + 2\pi \delta^{-1}) \sum_{j=1}^{n} |a_j|^2,
\]
where
\[
\delta = \min \{|\log \lambda_j - \log \lambda_k| : 1 \leq j, k \leq n \text{ and } j \neq k\},
\]
while $|\varepsilon| \leq 1$. We hope that a theory of orthogonal Dirichlet polynomials might contribute to this circle of ideas and to estimates involving Dirichlet polynomials. We begin with a simple result related to the Montgomery-Vaughan inequality: write for $j \geq 1$, $T > 0$,

$$\lambda_j^{-it} = \sum_{k=1}^{j} c_{T,j,k} \psi_{k,T}(t).$$

Also write

$$\psi_{n,T}(t) = \sum_{j=1}^{n} d_{T,n,j} \lambda_j^{-it}.$$ 

Let

$$C_{T,n} = \begin{bmatrix} c_{T,1,1} & c_{T,2,1} & c_{T,3,1} & \cdots & c_{T,n,1} \\ 0 & c_{T,2,2} & c_{T,3,2} & \cdots & c_{T,n,2} \\ 0 & 0 & c_{T,3,3} & \cdots & c_{T,n,3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & c_{T,n,n} \end{bmatrix}.$$ 

Theorem 1. (a) For any complex numbers $\{a_j\}_{j=1}^{n}$,

$$\frac{1}{2T} \int_{-T}^{T} \left| \sum_{j=1}^{n} a_j \lambda_j^{-it} \right|^2 dt = \|C_{T,n}a\|^2$$

where $a = [a_1 \ a_2 \ \ldots \ a_n]^T$ and the norm is the usual Euclidean norm. In particular,

$$\sup_{\{a_j\}} \frac{1}{2T} \int_{-T}^{T} \left| \sum_{j=1}^{n} a_j \lambda_j^{-it} \right|^2 dt / \sum_{j=1}^{n} |a_j|^2 = \|C_{T,n}\|^2,$$

where the norm is the usual matrix norm induced by the Euclidean norm.

(b) The coefficients $\{c_{T,j,k}\}$ and $\{d_{T,n,k}\}$ are real.

(c) For $j, k \geq 1$,

$$\min_{\{j,k\}} \sum_{\ell=1}^{\min(j,k)} c_{T,k,\ell} c_{T,j,\ell} = S(T \log \lambda_j / \lambda_k).$$

Next, we consider $\psi'_{n,T}$:
Theorem 2.
(a) \[
\psi'_{n,T}(t) = (-i \log \lambda_n) \psi_{n,T}(t) \\
+ \frac{1}{2T} \left( \psi_{n,T}(t) K_{n-1,T}(t,T) - \overline{\psi_{n,T}(t)} K_{n-1,T}(t,-T) \right) \\
= (-i \log \lambda_n) \psi_{n,T}(t) \\
+ \frac{i}{T} \sum_{j=1}^{n-1} \psi_{j,T}(t) \text{Im} \left( \psi_{n,T}(T) \overline{\psi_{j,T}(T)} \right).
\] (15)
(b) \[
\frac{1}{2T} \int_{-T}^{T} |\psi'_{n,T}|^2 = (\log \lambda_n)^2 + \frac{1}{T^2} \sum_{j=1}^{n-1} \left| \text{Im} \left( \psi_{n,T}(T) \overline{\psi_{j,T}(T)} \right) \right|^2 \\
= (\log \lambda_n)^2 + \frac{1}{T} \text{Re} \left( \psi_{n,T} \overline{\psi_{n,T}} \right)(T).
\] (16)

Next, we compare the orthonormal polynomials \(\psi_{n,T}\) and \(\psi_{n,S}\) for different \(S,T\):

Theorem 3. Let \(S > T\).
(a) \[
\Delta_{n,T} = \frac{1}{2T} \int_{-T}^{T} \left| \psi_{n,S}(t) - \frac{\gamma_{n,S}}{\gamma_{n,T}} \psi_{n,T}(t) \right|^2 dt \\
\leq \frac{S}{T} - \left( \frac{\gamma_{n,S}}{\gamma_{n,T}} \right)^2.
\] (17)
(b) \[
\frac{\gamma_{n,S}}{\gamma_{n,T}} \leq \left( \frac{S}{T} \right)^{1/2}.
\] (18)
(c) \[
K_{n,T}(x,x) + \left( \frac{S}{T} - 2 \right) K_{n,S}(x,x) \geq 0.
\] (19)

Finally, we consider the rate of change of several quantities w.r.t. \(T\):

Theorem 4.
(a) \[
\frac{\partial}{\partial T} K_{n,T}(x,x) = \frac{1}{T} K_{n,T}(x,x) - \frac{1}{2T} \left( |K_n(x,T)|^2 + |K_n(x,-T)|^2 \right).
\] (20)
(b) \[
\frac{\partial}{\partial T} (\ln \gamma_{n,T}) = \frac{1}{2T} (1 - |\psi_{n,T}(T)|^2).
\] (21)
(c)

\[
\frac{\partial}{\partial T} \ln A_{n,T} = -\frac{1}{T} (n - K_{n,T}(T,T)).
\]

(d) Let \( c_{T,j,k} \) be the connection coefficient as in (1.9). Then

\[
\frac{\partial}{\partial T} c_{T,j,k} + \frac{1}{2T} c_{T,j,k} = \frac{1}{2T} \left[ \lambda_j^{-it} \overline{\psi_{k,T}(T)} + \lambda_j^T \psi_{k,T}(T) \right] + \frac{1}{2T} \int_{-T}^{T} \lambda_j^{-it} \frac{\partial}{\partial T} \psi_{k,T}(t) dt.
\]

We prove Theorems 1 and 2 in Section 2, and Theorems 3 and 4 in Section 3.

1. PROOF OF THEOREMS 1 AND 2

Proof of Theorem 1

(a)

\[
\frac{1}{2T} \int_{-T}^{T} \left| \sum_{k=1}^{n} a_k \lambda_k^{-it} \right|^2 dt = \frac{1}{2T} \int_{-T}^{T} \left| \sum_{k=1}^{n} a_k \left( \sum_{j=1}^{k} c_{T,k,j} \psi_{j,T}(t) \right) \right|^2 dt
\]

\[
= \frac{1}{2T} \int_{-T}^{T} \left| \sum_{j=1}^{n} \left( \sum_{k=j}^{n} a_k c_{T,k,j} \right) \psi_{j,T} \right|^2 dt
\]

\[
= \sum_{j=1}^{n} \left| \sum_{k=j}^{n} a_k c_{T,k,j} \right|^2
\]

\[
= \sum_{j=1}^{n} \left| (C_{T,n}a)_j \right|^2 = \|C_{T,n}a\|^2.
\]

(b) First, if \( j < k \),

\[
\frac{1}{2T} \int_{-T}^{T} \lambda_j^{-it} \overline{\psi_{k,T}(-t)} dt
\]

\[
= \frac{1}{2T} \int_{-T}^{T} \lambda_j^{-is} \psi_{k,T}(s) ds
\]

\[
= \frac{1}{2T} \int_{-T}^{T} \lambda_j^{-is} \psi_{k,T}(s) ds = 0.
\]

Also,

\[
\psi_{k,T}(-t) = \sum_{j=1}^{k} \frac{1}{d_{T,k,j}} \lambda_j^{-it}
\]
so is also an orthonormal polynomial (recall the leading coefficient is positive). By
uniqueness,

\begin{equation}
\psi_{k,T}(-t) = \psi_{k,T}(t),
\end{equation}

and hence the \( \{d_{T,k,j}\} \) are real. Next, by orthogonality,

\begin{align*}
c_{T,j,k} &= \frac{1}{2T} \int_{-T}^{T} \lambda_j^{-it} \psi_{k,T}(t) dt \\
&= \frac{1}{2T} \int_{-T}^{T} \lambda_j^{i t} \psi_{k,T}(t) dt \\
&= \frac{1}{2T} \int_{-T}^{T} \lambda_j^{-it} \psi_{k,T}(-t) dt \\
&= \frac{1}{2T} \int_{-T}^{T} \lambda_j^{-it} \psi_{k,T}(t) dt = c_{T,j,k}.
\end{align*}

Thus the \( \{c_{T,k,j}\} \) are real.

(c) From (1.4), (2.1), and (a),

\begin{align*}
\sum_{1 \leq j,k \leq n} a_j a_k S(T \log \lambda_j / \lambda_k) \\
&= \sum_{\ell=1}^{n} \left| \sum_{k=\ell}^{n} a_k c_{T,k,\ell} \right|^2 \\
&= \sum_{1 \leq k,j \leq n} a_j a_k \sum_{\ell=1}^{\min\{j,k\}} c_{T,k,\ell}^2 c_{T,j,\ell}.
\end{align*}

Choosing some \( a_j = 1 \) and all remaining \( a' \)'s \( = 0 \) gives

\[ \sum_{\ell=1}^{j} c_{T,j,\ell}^2 = 1. \]

Next choose distinct \( j,k \) and \( a_j = a_k = 1 \) with all remaining \( a' \)'s \( = 0 \). Then we obtain

\[ 2S(T \log \lambda_j / \lambda_k) + 1 = 2 \sum_{\ell=1}^{\min\{j,k\}} c_{T,k,\ell} c_{T,j,\ell} + \sum_{\ell=1}^{j} c_{T,j,\ell}^2. \]

Thus we obtain (1.14) in full generality. \( \square \)

**Proof of Theorem 2**

(a) Write

\[ \psi_{n,T}'(t) = (-i \log \lambda_n) \psi_{n,T}(t) + \sum_{j=1}^{n-1} \beta_j \psi_{j,T}(t). \]
Here, integrating by parts, for $j \leq n - 1$,
\[
\beta_j = \frac{1}{2T} \int_{-T}^{T} \psi'_{n,T} (t) \overline{\psi_{j,T} (t)} dt
\]
\[
= \frac{1}{2T} \left\{ \psi_{n,T} (T) \overline{\psi_{j,T} (T)} - \psi_{n,T} (-T) \overline{\psi_{j,T} (-T)} \right\}
\]
\[
- \frac{1}{2T} \int_{-T}^{T} \psi_{n,T} (t) \overline{\psi_{j,T} (t)} dt
\]
\[
= \frac{1}{2T} \left\{ \psi_{n,T} (T) \overline{\psi_{j,T} (T)} - \psi_{n,T} (-T) \overline{\psi_{j,T} (-T)} \right\}
\]
\[
= \frac{i}{T} \text{Im} \left( \psi_{n,T} (T) \overline{\psi_{j,T} (T)} \right).
\]

So
\[
\psi'_{n,T} (t) = (-i \log \lambda_n) \psi_{n,T} (t)
\]
\[
+ \frac{1}{2T} \left\{ \psi_{n,T} (T) \sum_{j=1}^{n-1} \overline{\psi_{j,T} (T) \psi_{j,T} (t)} - \psi_{n,T} (-T) \sum_{j=1}^{n-1} \overline{\psi_{j,T} (-T) \psi_{j,T} (t)} \right\}
\]
\[
= (-i \log \lambda_n) \psi_{n,T} (t) + \frac{1}{2T} \left\{ \psi_{n,T} (T) K_{n-1} (t, T) - \psi_{n,T} (-T) \overline{K_{n-1} (t, -T)} \right\}.
\]

Also,
\[
\psi'_{n,T} (t) = (-i \log \lambda_n) \psi_{n,T} (t) + \frac{i}{T} \sum_{j=1}^{n-1} \text{Im} \left( \psi_{n,T} (T) \overline{\psi_{j,T} (T)} \right) \psi_{j,T} (t).
\]

(b) The first identity in (1.16) follows from the second identity in (1.15). Next, integrating by parts gives
\[
\frac{1}{2T} \int_{-T}^{T} |\psi'_{n,T}|^2
\]
\[
= \frac{1}{2T} \int_{-T}^{T} \overline{\psi'_{n,T}} \psi'_{n,T}
\]
\[
= \frac{1}{2T} \left\{ \left( \psi_{n,T} \overline{\psi'_{n,T}} \right) (T) - \left( \psi_{n,T} \overline{\psi'_{n,T}} \right) (-T) \right\} - \frac{1}{2T} \int_{-T}^{T} \psi_{n,T} \psi''_{n,T}.
\]

Here using (a) twice,
\[
\psi''_{n,T} = - (\log \lambda_n)^2 \psi_{n,T} + P,
\]
where $P \in L_{n-1}$, so
\[
\frac{1}{2T} \int_{-T}^{T} \psi_{n,T} \overline{\psi''_{n,T}} = - (\log \lambda_n)^2.
\]
Next, \( \psi'_{n,T}(t) = -i \sum_{j=1}^{n} f_j (\log \lambda_j) \lambda_j^{-it} \),

where all \( f_j \) are real, so

\[
\psi'_{n,T}(-T) = -i \sum_{j=1}^{n} f_j (\log \lambda_j) \lambda_j^{iT} = -\psi'_{n,T}(T).
\]

Substituting this and (2.2) into (2.3), gives

\[
\frac{1}{2T} \int_{-T}^{T} |\psi'_{n,T}|^2 \leq \frac{1}{2T} \left\{ (\psi'_{n,T}\psi'_{n,T})(T) + (\psi_{n,T}\psi'_{n,T})(T) \right\} + (\log \lambda_n)^2
\]

\[
= \frac{1}{T} \text{Re} \left( \psi_{n,T}\psi'_{n,T}(T) + (\log \lambda_n)^2 \right).
\]

\[\square\]

2. PROOF OF THEOREMS 3 AND 4

Proof of Theorem 3
(a)

\[
\Delta_{n,T} := \frac{1}{2T} \int_{-T}^{T} \left| \psi_{n,S}(t) - \frac{\gamma_{n,S}}{\gamma_{n,T}} \psi_{n,T}(t) \right|^2 dt
\]

\[
= \frac{1}{2T} \int_{-T}^{T} \left( \psi_{n,S}(t) - \frac{\gamma_{n,S}}{\gamma_{n,T}} \psi_{n,T}(t) \right) \left( \psi_{n,S}(t) - \frac{\gamma_{n,S}}{\gamma_{n,T}} \psi_{n,T}(t) \right) dt
\]

\[
= \frac{1}{2T} \int_{-T}^{T} \psi_{n,S}(t) - \frac{\gamma_{n,S}}{\gamma_{n,T}} \psi_{n,T}(t) \left( \psi_{n,S}(t) \right) dt
\]

\[
= \frac{1}{2T} \int_{-T}^{T} |\psi_{n,S}(t)|^2 dt - \left( \frac{\gamma_{n,S}}{\gamma_{n,T}} \right)^2 \leq \frac{S}{T} - \left( \frac{\gamma_{n,S}}{\gamma_{n,T}} \right)^2.
\]

(b) Then also,

\[
\frac{\gamma_{n,S}}{\gamma_{n,T}} \leq \left( \frac{S}{T} \right)^{1/2}.
\]
(28) \[
\frac{1}{2T} \int_{-T}^{T} |K_{n,S}(x,t) - K_{n,T}(x,t)|^2 dt = \frac{1}{2T} \int_{-T}^{T} |K_{n,S}(x,t)|^2 dt - 2K_{n,S}(x,x) + K_{n,T}(x,x).
\]

Also,

\[
\frac{1}{2T} \int_{-T}^{T} |K_{n,S}(x,t)|^2 dt = \frac{S}{T} \left[ \frac{1}{2S} \int_{-S}^{S} |K_{n,S}(x,t)|^2 dt - \frac{1}{2S} \int_{T \leq |t| \leq S} |K_{n,S}(x,t)|^2 dt \right] = \frac{S}{T} \left[ K_{n,S}(x,x) - \frac{1}{2S} \int_{T \leq |t| \leq S} |K_{n,S}(x,t)|^2 dt \right].
\]

So substituting in (3.1) above,

\[
0 \leq \frac{S}{T} \left[ K_{n,S}(x,x) - \frac{1}{2S} \int_{T \leq |t| \leq S} |K_{n,S}(x,t)|^2 dt \right] - 2K_{n,S}(x,x) + K_{n,T}(x,x)
= K_{n,T}(x,x) + \left( \frac{S}{T} - 2 \right) K_{n,S}(x,x)
\]

\[
- \frac{1}{2T} \int_{T \leq |t| \leq S} |K_{n,S}(x,t)|^2 dt.
\]

In particular, (1.19) follows. \(\square\)

**Proof of Theorem 4**

(a) Let \(S > T\). Now from (3.2) above,

\[
K_{n,T}(x,x) - K_{n,S}(x,x) \geq \left( 1 - \frac{S}{T} \right) K_{n,S}(x,x)
+ \frac{1}{2T} \int_{T \leq |t| \leq S} |K_{n,S}(x,t)|^2 dt
\]

so

\[
\frac{K_{n,T}(x,x) - K_{n,S}(x,x)}{T - S} \leq \frac{1}{T} K_{n,S}(x,x)
+ \frac{1}{T - S} \frac{1}{2T} \int_{T \leq |t| \leq S} |K_{n,S}(x,t)|^2 dt.
\]

(30)
Next,

\[
0 \leq \frac{1}{2S} \int_{-S}^{S} |K_{n,S}(x,t) - K_{n,T}(x,t)|^2 \, dt \\
= K_{n,S}(x,x) - 2K_{n,T}(x,x) + \frac{1}{2S} \int_{-S}^{S} |K_{n,T}(x,t)|^2 \, dt \\
= K_{n,S}(x,x) - 2K_{n,T}(x,x) \\
+ \frac{T}{S} \left[ K_{n,T}(x,x) + \frac{1}{2T} \int_{T \leq |t| \leq S} |K_{n,T}(x,t)|^2 \, dt \right] \\
= K_{n,S}(x,x) + \left( \frac{T}{S} - 2 \right) K_{n,T}(x,x) + \frac{1}{2S} \int_{T \leq |t| \leq S} |K_{n,T}(x,t)|^2 \, dt,
\]

so

\[
K_{n,S}(x,x) - K_{n,T}(x,x) \geq \left( 1 - \frac{T}{S} \right) K_{n,T}(x,x) \\
- \frac{1}{2S} \int_{T \leq |t| \leq S} |K_{n,T}(x,t)|^2 \, dt.
\]

Then

\[
\frac{K_{n,S}(x,x) - K_{n,T}(x,x)}{S - T} \geq \frac{1}{S} K_{n,T}(x,x) \\
- \frac{1}{S - T} \frac{1}{2S} \int_{T \leq |t| \leq S} |K_{n,T}(x,t)|^2 \, dt.
\]

Together with (3.3), this establishes

\[
\frac{1}{S} K_{n,T}(x,x) - \frac{1}{S - T} \frac{1}{2S} \int_{T \leq |t| \leq S} |K_{n,T}(x,t)|^2 \, dt \\
\leq \frac{K_{n,S}(x,x) - K_{n,T}(x,x)}{S - T} \\
\leq \frac{1}{T} K_{n,S}(x,x) + \frac{1}{T - S} \frac{1}{2T} \int_{T \leq |t| \leq S} |K_{n,S}(x,t)|^2 \, dt.
\]

Inasmuch as \(K_{n,T}(x,x) - K_{n,S}(x,x) \to 0\) as \(|S - T| \to 0\), (indeed, the representation (1.5) shows that \(\psi_{n,T}\) and hence \(K_{n,T}\) are infinitely differentiable in \(T\)), this last inequality yields that

\[
\frac{\partial}{\partial T} K_{n,T}(x,x) = \frac{1}{T} K_{n,T}(x,x) - \frac{1}{2T} \left( |K_{n}(x,T)|^2 + |K_{n}(x,-T)|^2 \right).
\]
(b) \[
\frac{1}{2T} \int_{-T}^{T} |\psi_{n,S}(t) - \psi_{n,T}(t)|^2 dt \\
= \frac{1}{2T} \int_{-T}^{T} |\psi_{n,S}(t)|^2 dt - \frac{2\gamma_{n,S}}{\gamma_{n,T}} + 1 \\
= \frac{S}{T} \left( 1 - \frac{1}{2S} \int_{-T}^{T} |\psi_{n,S}(t)|^2 dt \right) - \frac{2\gamma_{n,S}}{\gamma_{n,T}} + 1,
\]

So \[
2\frac{\gamma_{n,T} - \gamma_{n,S}}{\gamma_{n,T}} \geq 1 - \frac{S}{T} + \frac{1}{2T} \int_{-T}^{T} |\psi_{n,S}(t)|^2 dt.
\]

Then recalling that \(\psi_{n,S}(-t) = \overline{\psi_{n,S}(t)}\),

\[
2\frac{\gamma_{n,T} - \gamma_{n,S}}{\gamma_{n,T}} \leq \frac{1}{2T} \gamma_{n,T} + \frac{1}{T - S} \gamma_{n,T} \frac{1}{2T} \int_{-T}^{T} |\psi_{n,S}(t)|^2 dt.
\]

In the other direction,

\[
\frac{1}{2S} \int_{-S}^{S} |\psi_{n,S}(t) - \psi_{n,T}(t)|^2 dt \\
= 1 - \frac{2\gamma_{n,T}}{\gamma_{n,S}} + \frac{1}{2S} \int_{-S}^{S} |\psi_{n,T}(t)|^2 dt \\
= 1 - \frac{2\gamma_{n,T}}{\gamma_{n,S}} + \frac{T}{S} \frac{1}{2S} \int_{-T}^{T} |\psi_{n,T}(t)|^2 dt,
\]

so \[
2\frac{\gamma_{n,S} - \gamma_{n,T}}{\gamma_{n,S}} \geq 1 - \frac{T}{S} - \frac{1}{2S} \int_{-T}^{T} |\psi_{n,T}(t)|^2 dt,
\]

and hence \[
\frac{\gamma_{n,S} - \gamma_{n,T}}{S - T} \geq \frac{\gamma_{n,S}}{2S} - \frac{\gamma_{n,S}}{2S} \frac{1}{S - T} \int_{-T}^{T} |\psi_{n,T}(t)|^2 dt.
\]

Combined with (3.5), this gives

\[
\frac{\gamma_{n,S}}{2S} - \frac{\gamma_{n,S}}{2S} \frac{1}{S - T} \int_{-T}^{T} |\psi_{n,T}(t)|^2 dt \leq \frac{\gamma_{n,S} - \gamma_{n,T}}{S - T} \\
\leq \frac{\gamma_{n,T}}{2T} - \frac{1}{S - T} \frac{\gamma_{n,T}}{2T} \int_{-T}^{T} |\psi_{n,S}(t)|^2 dt.
\]

This gives \[
\frac{\partial \gamma_{n,T}}{\partial T} = \frac{\gamma_{n,T}}{2T} - \frac{\gamma_{n,T}}{2T} |\psi_{n,T}(T)|^2,
\]
and hence the result.

(c) Recall that
\[ \gamma_{n,T} = \sqrt{\frac{A_{n-1,T}}{A_{n,T}}}, \]
so
\[ \gamma_{2,T} \cdots \gamma_{n,T} = \sqrt{\frac{A_{1,T}}{A_{n,T}}} = \sqrt{\frac{1}{A_{n,T}}}. \]
Thus
\[ \frac{\partial}{\partial T} \ln \sqrt{\frac{1}{A_{n,T}}} = \sum_{j=2}^{n} \frac{\partial (\ln \gamma_{j,T})}{\partial T} = \sum_{j=2}^{n} \frac{1}{2T} (1 - |\psi_{j,T}(T)|^2) \]
\[ \Rightarrow \frac{\partial}{\partial T} \ln A_{n,T} = -\frac{1}{T} \sum_{j=2}^{n} (1 - |\psi_{j,T}(T)|^2) \]
\[ = -\frac{1}{T} (n - K_{n,T}(T,T)), \]
recall that \( \psi_{1,T}(x) = \lambda_1^{-ix} \) so \( |\psi_{1,T}(x)| = 1. \)

(d) \( c_{S,j,k} - c_{T,j,k} \)
\[ = \frac{1}{2S} \int_{-S}^{S} \lambda_j^{-iu} \psi_{k,S}(t) dt - \frac{1}{2T} \int_{-T}^{T} \lambda_j^{-iu} \overline{\psi}_{k,T}(t) dt \]
\[ = \frac{1}{2S} \int_{T \leq |t| \leq S} \lambda_j^{-iu} \psi_{k,S}(t) dt + \frac{1}{2} \left( \frac{1}{S} - \frac{1}{T} \right) \int_{-T}^{T} \lambda_j^{-iu} \overline{\psi}_{k,S}(t) dt \]
\[ + \frac{1}{2T} \int_{-T}^{T} \lambda_j^{-iu} \left[ \frac{\psi_{k,S}(t)}{S} - \frac{\overline{\psi}_{k,T}(t)}{T} \right] dt \]
so
\[ \frac{c_{S,j,k} - c_{T,j,k}}{S - T} \]
\[ = \frac{1}{S - T} \left( \frac{1}{2S} \int_{T \leq |t| \leq S} \lambda_j^{-iu} \psi_{k,S}(t) dt - \frac{1}{2} \frac{1}{ST} \int_{-T}^{T} \lambda_j^{-iu} \overline{\psi}_{k,S}(t) dt \right) \]
\[ + \frac{1}{2T} \int_{-T}^{T} \lambda_j^{-iu} \left[ \psi_{k,S}(t) - \frac{\psi_{k,T}(t)}{S} \right] dt \]
Now let \( S \rightarrow T. \)
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