A HYPERBOLIC METRIC AND STABILITY CONDITIONS ON K3 SURFACES WITH $\rho = 1$

KOTARO KAWATANI

1. Introduction

In this article we introduce a hyperbolic metric on the (normalized) space of stability conditions on projective K3 surfaces $X$ with Picard rank $\rho(X) = 1$. And we show that all walls are geodesic in the normalized space with respect to the hyperbolic metric. Furthermore we demonstrate how the hyperbolic metric is helpful for us by discussing mainly three topics. We first make a study of so called Bridgeland’s conjecture. In the second topic we prove a famous Orlov’s theorem without the global Torelli theorem. In the third topic we give an explicit example of stable complexes in large volume limits by using the hyperbolic metric. Though Bridgeland’s conjecture may be well-known for algebraic geometers, we would like to start from the review of it.

1.1. Bridgeland’s conjecture. In [4] Bridgeland introduced the notion of stability conditions on arbitrary triangulated categories $\mathcal{D}$. By virtue of this we could define the notion of “$\sigma$-stability” for objects $E \in \mathcal{D}$ with respect to a stability condition $\sigma$ on $\mathcal{D}$.

Bridgeland also showed that each connected component of the space $\text{Stab}(\mathcal{D})$ consisting of stability conditions on $\mathcal{D}$ is a complex manifold unless $\text{Stab}(\mathcal{D})$ is empty. Hence the non-emptiness of $\text{Stab}(\mathcal{D})$ is one of the biggest problem. Many researchers study this problem in various situations. For instance suppose $\mathcal{D}$ is the bounded derived category $\mathcal{D}(M)$ of coherent sheaves on a projective manifold $M$. In the case of $\dim M = 1$, the non-emptiness of $\text{Stab}(\mathcal{D}(M))$ was proven in the original article [3]. Furthermore the space $\text{Stab}(\mathcal{D}(M))$ was studied in detail by [17] (the genus is 0), [4] (the genus is 1) and [15] (the genus is greater than 1). In the case of $\dim M = 2$, the non-emptiness was proven by [5] (K3 or abelian surfaces) and [1] (other surfaces). In the case of $\dim M = 3$ it is discussed by [2]. These are just a handful of many studies.

As we stated before, the space $\text{Stab}(X)$ of stability conditions on the derived category $\mathcal{D}(X)$ of a projective K3 surface $X$ is not empty by [5]. This fact is proven by finding a distinguished connected component $\text{Stab}^\dagger(X)$. For $\text{Stab}^\dagger(X)$ Bridgeland conjectured the following:

*Date*: May 10, 2014, version 5.

2010 Mathematics Subject Classification. Primary 14F05; Secondly 14J28, 18E30, 32Q45.
Conjecture 1.1 (Bridgeland). The space \( \text{Stab}(X) \) is connected, that is, \( \text{Stab}(X) = \text{Stab}^\dagger(X) \). Furthermore the distinguished component \( \text{Stab}^\dagger(X) \) is simply connected.

As was proven by [5] and [10], if the conjecture holds then we can determine the group structure of \( \text{Aut}(D(X)) \) as follows: We have the covering map \( \pi: \text{Stab}^\dagger(X) \to \mathbb{P}^+_0(X) \) by [5, Theorem 1.1] (See also Theorem 2.5). Here \( \mathbb{P}^+_0(X) \) is a subset of \( H^*(X, \mathbb{C}) \) (See also Section 2.1). By virtue of [5] and [10], if Conjecture 1.1 holds we have the exact sequence of groups:

\[
1 \to \pi_1(\mathbb{P}^+_0(X)) \to \text{Aut}(D(X)) \xrightarrow{\kappa} O^+_{\text{Hodge}}(H^*(X, \mathbb{Z})) \to 1,
\]

where \( O^+_{\text{Hodge}}(H^*(X, \mathbb{Z})) \) is the Hodge isometry group of \( H^*(X, \mathbb{Z}) \) preserving the orientation of \( H^*(X, \mathbb{Z}) \). Hence Conjecture 1.1 predicts that the kernel \( \text{Ker}(\kappa) \) of the representation \( \kappa \) is given by the fundamental group \( \pi_1(\mathbb{P}^+_0(X)) \) and that \( \text{Aut}(D(X)) \) is given by an extension of \( \pi_1(\mathbb{P}^+_0(X)) \) and \( O^+_{\text{Hodge}}(H^*(X, \mathbb{Z})) \).

1.2. First theorem. Recall the right \( \widetilde{\text{GL}}^+(2, \mathbb{R}) \)-action on \( \text{Stab}(X) \) where \( \widetilde{\text{GL}}^+(2, \mathbb{R}) \) is the universal cover of \( \text{GL}^+(2, \mathbb{R}) \). We define \( \text{Stab}^n(X) \) by the quotient of \( \text{Stab}^\dagger(X) \) by the right \( \widetilde{\text{GL}}^+(2, \mathbb{R}) \) action. We call it a normalized stability manifold. For a projective K3 surface with \( \rho(X) = 1 \), we first introduce a hyperbolic metric on \( \text{Stab}^n(X) \). We also show that the hyperbolic metric is independent of the choice of Fourier-Mukai partners of \( X \):

**Theorem 1.2** (\( = \)Theorem 3.3). Assume that \( \rho(X) = 1 \).

1. \( \text{Stab}^n(X) \) is a hyperbolic 2 dimensional manifold.

2. Let \( Y \) be a Fourier-Mukai partner of \( X \) and \( \Phi: D(Y) \to D(X) \) an equivalence which preserves the distinguished component \( \text{Stab}^\dagger(X) \). Then the induced morphism \( \Phi^n: \text{Stab}^n(Y) \to \text{Stab}^n(X) \) is an isometry with respect to the hyperbolic metric.

Clearly if \( \text{Stab}(X) \) is connected it is unnecessary to assume that \( \Phi \) preserves the distinguished component.

We remark that there is another study by Woolf which focuses on the metric on \( \text{Stab}(D) \) (not normalized!). In [20], he showed that \( \text{Stab}(D) \) is complete with respect to the original metric introduced by Bridgeland. Our study is the first work which focuses on a different structure from Bridgeland’s original framework.

1.3. Second theorem. Next, by using the hyperbolic structure, we observe the simply connectedness of \( \text{Stab}^\dagger(X) \):

**Theorem 1.3** (\( = \)Theorem 4.1). Let \( X \) be a projective K3 surface with \( \rho(X) = 1 \). The following three conditions are equivalent.

1. \( \text{Stab}^\dagger(X) \) is simply connected.

2. \( \text{Stab}^n(X) \) is isomorphic to the upper half plane \( \mathbb{H} \).
Let $W(X)$ be the subgroup of $\text{Aut}(D(X))$ generated by two times compositions of the spherical twist $T_A$ by spherical locally free sheaves $A$. $W(X)$ is isomorphic to the free group generated by $T_A^2$:

$$W(X) = \bigstar_A^\infty (\mathbb{Z} \cdot T_A^2),$$

where $A$ runs through all spherical locally free sheaves and $\bigstar$ is the free product.

We give two remarks on Theorem 4.1. Firstly we could not prove the simply connectedness. However by using the hyperbolic structure on $\text{Stab}^\dagger(X)$, we can deduce the global geometry not only of $\text{Stab}^\dagger(X)$ but also of $\text{Stab}^\dagger(X)$ as follows. Since $\text{Stab}^\dagger(X)$ is a $\widetilde{\text{GL}}^+(2,\mathbb{R})$-bundle on $\text{Stab}^n(X)$, and we see $\text{Stab}^\dagger(X)$ is simply connected if and only if it is a $\widetilde{\text{GL}}^+(2,\mathbb{R})$-bundle over the upper half plane $\mathbb{H}$.

Secondly, if Conjecture 1.1 holds then we see the kernel $\text{Ker}(\kappa)$ is generated by $W(X)$ and the double shift $[2]$. Since the double shift $[2]$ commutes with any equivalence, the freeness of $W(X)$ implies $\text{Ker}(\kappa)/\mathbb{Z}[2]$ is free. However in higher Picard rank cases, it is thought that the generators of $\text{Ker}(\kappa)/\mathbb{Z}[2]$ have relations (See also Remark 4.3). Hence the freeness of $W(X)$ is a special phenomena.

1.4. Third theorem. In the third theorem, we study chamber structures on $\text{Stab}^\dagger(X)$ in terms of the hyperbolic structure on $\text{Stab}^n(X)$. Before we state the third theorem, let us recall chamber structures.

For a set $S \subset D(X)$ of objects which has bounded mass and an arbitrary compact subset $B \subset \text{Stab}^\dagger(X)$, we can define a finite collection of real codimension 1 submanifolds $\{W_\gamma\}_{\gamma \in \Gamma}$ satisfying the following property:

- Let $C \subset B \setminus \bigcup_{\gamma \in \Gamma} W_\gamma$ be an arbitrary connected component. If $E \in S$ is $\sigma$-semistable for some $\sigma \in C$ then $E$ is $\tau$-semistable for all $\tau \in C$.

Each $W_\gamma$ is said to be a wall and each connected component $C$ is said to be a chamber. In this paper we call all data of chambers and walls a chamber structure. We have to remark that chamber structures on $\text{Stab}^\dagger(X)$ descend to the normalized stability manifold $\text{Stab}^n(X)$. Namely $C/\widetilde{\text{GL}}^+(2,\mathbb{R})$ and $\{W_\gamma/\widetilde{\text{GL}}^+(2,\mathbb{R})\}$ also define a chamber structure on $\text{Stab}^n(X)$. Our third theorem is the following:

**Theorem 1.4 (Theorem 5.5).** All walls of chamber structures of $\text{Stab}^n(X)$ are geodesic.

1.5. Revisit of Orlov’s theorem. Generally speaking Fourier-Mukai transformations on $X$ may change chamber structures (This does not mean Fourier-Mukai transformations just permute chambers). By Theorems 3.3 and 5.3, we see that the image of walls by Fourier-Mukai transformations is also geodesic in $\text{Stab}^n(X)$. Applying this observation we show the following:
Proposition 1.5 (=Proposition 6.5). Let $X$ be a projective K3 surface with $\rho(X) = 1$ and $Y$ a Fourier-Mukai partner of $X$ with an equivalence $\Phi: D(Y) \to D(X)$.

If the induced morphism $\Phi_*: \text{Stab}(Y) \to \text{Stab}(X)$ preserves the distinguished component, then $Y$ is isomorphic to the fine moduli space of Gieseker stable torsion free sheaves.

We have to mention that a more stronger statement was already proven by Orlov in [18]: Any Fourier-Mukai partner of projective K3 surfaces is isomorphic to the fine moduli space of Gieseker stable sheaves. Our proof never needs the global Torelli theorem which was essential for Orlov’s proof. Hence our proof gives a new feature of stability condition; The theory of stability conditions substitutes for the global Torelli theorem. Since the strategy of Proposition 6.5 is technical, we will explain it in §6.1.

1.6. Stable complexes in the large volume limit. We also discuss the stability of complexes in large volume limits by using Lemma 3.2 which is crucial for Theorem 3.3 More precisely in Corollary 7.3 we prove that the complexes $T_A(O_x)$ are stable in the large volume limit where $T_A(O_x)$ is a spherical twist of $O_x$ by a spherical locally free sheaf. Originally it was expected that the $\sigma$-stability in the large volume limit is equivalent to Gieseker twisted stability (See also [5, §14]). However the possibility of stable complexes in the large volume limit is referred in [3]. We give an answer to this problem.

1.7. Contents. In Section 2 we prepare some basic terminologies. In Section 3 we prove the first main theorem. In Section 4 we prove the second main theorem. The third theorem will be proven in Section 5. The analysis of $\partial U(X)$, which is necessary for Theorem 4.1 will be also done in Section 5. In Section 6 we revisit Orlov’s theorem. In Section 7 we discuss the stability of $T_A^{-1}(O_x)$ in the large volume limit.

2. Preliminaries

In this section we prepare basic notations and lemmas. Let $(X, L)$ be a pari of a projective K3 surface with $\text{NS}(X) = \mathbb{Z}L$. Almost all notions are defined for general projective K3 surfaces. To simplify the explanations we focus on K3 surfaces with $\rho(X) = 1$.

2.1. Terminologies. The abelian category of coherent sheaves on $X$ is denoted by $\text{Coh}(X)$. Note that the numerical Grothendieck group $\mathcal{N}(X)$ is isomorphic to

$$H^0(X, \mathbb{Z}) \oplus \text{NS}(X) \oplus H^4(X, \mathbb{Z}).$$

We put $v(E) = ch(E)\sqrt{td_X}$ for $E \in D(X)$. Then we see

$$v(E) = r_E \oplus c_E \oplus s_E \in \mathcal{N}(X).$$
One can easily check that $r_E = \text{rank } E$, $c_E$ is the first Chern class $c_1(E)$ and $s_E = \chi(X, E) - \text{rank } E$. Hence for a vector $v = r \oplus c \oplus s \in \mathcal{N}(X)$, the component $r$ is called the \textit{rank} of $v$.

The Mukai pairing $\langle \cdot, \cdot \rangle$ on $H^*(X, \mathbb{Z})$ is given by

$$\langle r \oplus c \oplus s, r' \oplus c' \oplus s' \rangle = cc' - rs' - r's.$$ 

By Riemann-Roch theorem we see

$$\chi(E, F) = \sum (-1)^i \dim \text{Hom}^i_{D(X)}(E, F) = -\langle v(E), v(F) \rangle.$$ 

An object $A \in D(X)$ is said to be \textit{spherical} if $A$ satisfies

$$\text{Hom}^i_{D(X)}(A, A) = \begin{cases} \mathbb{C} & (i = 0, 2) \\ 0 & \text{(otherwise).} \end{cases}$$

We note that $v(A)^2 = -2$ if $A$ is spherical. By the effort of [19], for a spherical object $A$ we could define the autoequivalence $T_A$ called a \textit{spherical twist} (See also [7, Chapter 8]). By the definition of $T_A$ we have the following distinguished triangle for $E \in D(X)$:

$$(2.1) \quad \text{Hom}^i_{D(X)}(A, E) \otimes A \xrightarrow{\text{ev}} E \longrightarrow T_A(E),$$

where $\text{ev}$ is the evaluation map. We call the above triangle a \textit{spherical triangle}. We note that the vector of $T_A(E)$ can be calculated as follows

$$v(T_A(E)) = v(E) + \langle v(E), v(A) \rangle v(A).$$

Let $\Delta(X)$ be the set of $(-2)$-vectors:

$$\Delta(X) = \{ \delta \in \mathcal{N}(X) | \delta^2 = -2 \}$$

and let $\Delta^+(X)$ be the set $\{ \delta \in \Delta(X) | \delta = r \oplus c \oplus s, r > 0 \}$.

Following [5], we put

$$\mathcal{P}(X) = \{ v \in \mathcal{N}(X) \otimes \mathbb{C}[\Re(v) \text{ and } \Im(v) \text{ span a positive 2 plane}] \}$$

Since $\mathcal{P}(X)$ has two connected components, we define $\mathcal{P}^+(X)$ by the connected component containing $\exp(\sqrt{-1}\omega)$ where $\omega$ is an ample class. Then $\mathcal{P}^+(X)$ has the right $\text{GL}^+(2, \mathbb{R})$ action as the change of basis of the planes. This action is free. Hence there exists the quotient $\mathcal{P}^+(X) \rightarrow \mathcal{P}^+(X)/\text{GL}^+(2, \mathbb{R})$ which gives a principle $\text{GL}^+(2, \mathbb{R})$-bundle with a global section.

Under the assumption $\rho(X) = 1$, $\mathcal{P}^+(X)/\text{GL}^+(2, \mathbb{R})$ is isomorphic to the set $\mathcal{H}(X)$ where

$$\mathcal{H}(X) = \{ (\beta, \omega) = (xL, yL) | x + \sqrt{-1} \in \mathbb{H} \}.$$ 

Clearly $\mathcal{H}(X)$ is canonically isomorphic to $\mathbb{H}$. Then the global section $\mathcal{H}(X) \rightarrow \mathcal{P}^+(X)$ is given by

$$\mathcal{H}(X) \ni (x, y) \mapsto \exp(\beta + \sqrt{-1}\omega) \in \mathcal{P}^+(X).$$
In particular $P^+(X)$ is isomorphic to $\mathbb{H} \times GL^+(2, \mathbb{R})$. We put $P^+_0(X)$ by

$$P^+_0(X) = P^+(X) \setminus \bigcup_{\delta \in \Delta(X)} \langle \delta \rangle^\perp$$

where $\langle \delta \rangle^\perp$ is the orthogonal complement of $\delta$ with respect to the Mukai pairing on $H^*(X, \mathbb{H})$. Define

$$\mathcal{H}_0(X) = \{ v \in \mathcal{H}(X) \mid (\exp(v), \delta) \neq 0 (\forall \delta \in \Delta(X)) \}.$$ 

Then we see $P^+_0(X)$ is isomorphic to $\mathcal{H}_0(X) \times GL^+(2, \mathbb{R})$.

2.2. Stability conditions on K3 surfaces. Let $\text{Stab}(X)$ be the set of numerical locally finite stability conditions on $D(X)$. We put $\sigma = (A, Z) \in \text{Stab}(X)$ where $A$ is the heart of a bounded t-structure on $D$ and $Z$ is a central charge. Since the Mukai paring is non-degenerate on $N(X)$ we have the natural map:

$$\pi: \text{Stab}(X) \rightarrow N(X) \otimes \mathbb{C}, \quad \pi(\sigma) = Z^\vee$$

where $Z(E) = \langle Z^\vee, v(E) \rangle$.

In $\text{Stab}(X)$, there is a connected component $\text{Stab}^1(X)$ which contains the set $U(X)$:

$$U(X) = \{ \sigma = (A, Z) \in \text{Stab}(X) \mid Z^\vee \in P(X) \setminus \bigcup_{\delta \in \Delta(X)} \langle \delta \rangle^\perp, \quad O_x \text{ is } \sigma \text{-stable in the same phase for all } x \in X \}.$$ 

Let $\bar{U}(X)$ be the closure of $U(X)$ in $\text{Stab}(X)$. Then we see that $\bar{U}(X)$ be the set of stability conditions $\sigma$ such that $O_x$ ($\forall x \in X$) is $\sigma$-semistable in the same phase with $Z^\vee \in P(X) \setminus \bigcup_{\delta \in \Delta(X)} \langle \delta \rangle^\perp$. Define $\partial U(X)$ by $\bar{U}(X) \setminus U(X)$ and call it the boundary of $U(X)$.

We define the set $V(X)$ by

$$V(X) = \{ \sigma = (A, Z) \in U(X) \mid Z(O_x) = -1, \quad O_x \text{ is } \sigma \text{-stable with phase 1} \}.$$ 

One can see $U(X) = V(X) \cdot \overline{GL}^+(2, \mathbb{R}) \cong V(X) \times \overline{GL}^+(2, \mathbb{R})$ by [5, Proposition 10.3]. Furthermore the set $V(X)$ is parametrized by $(\beta, \omega) \in \mathcal{H}(X)$ in the following way:

For the pair $(\beta, \omega)$, put $A_{(\beta, \omega)}$ and $Z_{(\beta, \omega)}$ as follows:

$$A_{(\beta, \omega)} := \{ E^* \in D(X) \mid H^i(E^*) \begin{cases} \in \mathcal{T}_{(\beta, \omega)} & (i = 0) \\ \in \mathcal{F}_{(\beta, \omega)} & (i = -1) \\ = 0 & (\text{otherwise}) \end{cases} \}$$

$$Z_{(\beta, \omega)}(E) := \langle \exp(\beta + \sqrt{-1}\omega), v(E) \rangle,$$

where

$$\mathcal{T}_{(\beta, \omega)} := \{ E \in \text{Coh}(X) \mid E \text{ is a torsion sheaf or } \mu^-_{(E/\text{torsion})} > \beta \omega \}$$

and

$$\mathcal{F}_{(\beta, \omega)} := \{ E \in \text{Coh}(X) \mid E \text{ is torsion free and } \mu^+_{(E)} \leq \beta \omega \}.$$ 

\footnote{We remark that the definition of $P^+_0(X)$ is independent of the assumption $\rho(X) = 1$.}
Here $\mu_\omega^+(E)$ (respectively $\mu_\omega^-(E)$) is the maximal slope (respectively minimal slope) of semistable factors of a torsion free sheaf $E$ with respect to the slope stability. Since the pair $(\mathcal{T}_{(\beta, \omega)}, \mathcal{F}_{(\beta, \omega)})$ gives a torsion pair on $\text{Coh}(X)$, $\mathcal{A}_{(\beta, \omega)}$ is the heart of a bounded $t$-structure on $D(X)$. We denote the pair $(\mathcal{A}_{(\beta, \omega)}, Z_{(\beta, \omega)})$ by $\sigma_{(\beta, \omega)}$.

**Proposition 2.1** ([5, Proposition 10.3]). Assume that $(\beta, \omega)$ satisfies the condition

\begin{equation}
\langle \exp(\beta + \sqrt{-1} \omega), \delta \rangle \notin \mathbb{R}_{\leq 0}, \quad (\forall \delta \in \Delta^+(X))
\end{equation}

Then the pair $\sigma_{(\beta, \omega)}$ gives a numerical locally finite stability condition on $D(X)$. Furthermore we have

$$V(X) = \{ \sigma_{(\beta, \omega)} \in \text{Stab}^\dagger(X) \mid (\beta, \omega) \text{ satisfies the condition } (2.2) \}.$$  

**Remark 2.2.** We put $v(E) = r_E \oplus c_1(E) \oplus s_E$ for $E \in D(X)$. As the author remarked in [11, Section 4, (4.1)], for objects $E \in D(X)$ with rank $E \neq 0$, we can rewrite $Z_{(\beta, \omega)}(E)$ as follows,

\begin{equation}
Z_{(\beta, \omega)}(E) = \frac{v(E)^2}{2r_E} + \frac{r_E}{2} \left( \omega + \sqrt{-1} \left( \frac{c_1(E)}{r_E} - \beta \right) \right)^2.
\end{equation}

This equation (2.3) plays an important role in Lemma 3.2 which is crucial for Theorem 3.3.

**Definition 2.3.** For a projective K3 surface with $\rho(X) = 1$ we define the subgroup $W(X)$ of $\text{Aut}(D(X))$ generated by

$$W(X) = \langle T^2_A \mid A = \text{spherical locally free sheaf} \rangle.$$  

Then by using $U(X)$ and $W(X)$ we can describe $\text{Stab}^\dagger(X)$ in an explicit way:

**Proposition 2.4** ([5, Proposition 13.2]). Let $X$ be a projective K3 with $\rho(X) = 1$. The distinguished connected component $\text{Stab}^\dagger(X)$ is given by

$$\text{Stab}^\dagger(X) = \bigcup_{\Phi \in W(X)} \Phi_* (\text{Stab}^\dagger(U(X))).$$  

**Theorem 2.5** ([5]). The natural map $\pi : \text{Stab}^\dagger(X) \to N(X) \otimes \mathbb{C}$ has the image $\mathcal{P}_0^+(X)$. Furthermore $\pi$ is a Galois covering. The covering transformation group is the subgroup generated by equivalences in $\text{Ker}(\kappa)$ which preserve $\text{Stab}^\dagger(X)$.

**Corollary 2.6.** For a pair $(X, L)$, the induced map

$$\pi^n : \text{Stab}^n(X) \to \mathcal{H}_0^+(X)$$

is also a Galois covering map.
Proof. We have the following $\text{GL}^+(2, \mathbb{R})$-equivariant diagram:

\[
\begin{array}{ccc}
\text{Stab}^1(X)/\mathbb{Z}[2] & \xrightarrow{\pi'} & \mathcal{P}_0^+(X) \\
\downarrow & & \downarrow \\
\text{Stab}^n(X) & \xrightarrow{\pi^n} & \mathcal{H}_0^+_{\mathbb{Z}}(X).
\end{array}
\]

We note that both vertical maps are $\text{GL}^+(2, \mathbb{R})$-bundles and that $\pi'$ is also a Galois covering. By Theorem 2.5 the covering transformation group of $\pi'$ is a subgroup of $\text{Aut}(D(X))/\mathbb{Z}[2]$. Hence the right $\text{GL}^+(2, \mathbb{R})$-action on $\text{Stab}^1(X)/\mathbb{Z}[2]$ commutes with the covering transformations. Hence $\pi^n$ is also a Galois covering. □

2.3. On the fundamental group of $\mathcal{P}_0^+(X)$. We are interested in the fundamental group $\pi_1(\mathcal{P}_0^+(X))$. Generally speaking, it is highly difficult to describe the above condition (2.2) explicitly. Because of this difficulty, it becomes difficult to determine the relation between generators of $\pi_1(\mathcal{P}_0^+(X))$. Hence it seems impossible to determine the group structure of $\pi_1(\mathcal{P}_0^+(X))$. However, under the assumption $\rho(X) = 1$ it becomes easier.

Definition 2.7. Let $\delta = r \oplus c \oplus s \in \Delta(X)$. An associated point $p \in \mathfrak{H}(X)$ with $\delta \in \Delta(X)$ is the point $p \in \mathfrak{H}(X)$ such that $\langle \exp(p), \delta \rangle = 0$. We also denote the point by $p(\delta)$ and call it a spherical point. If $\delta$ is the Mukai vector of a spherical object $A$ we denote simply $p(v(A))$ by $p(A)$.

Remark 2.8. Let $\delta \in \Delta(X)$ and we put $\delta = r \oplus c \oplus s$. Since $c^2 \geq 0$ we see $r \neq 0$. Thus we have the disjoint sum $\Delta(X) = \Delta^+(X) \sqcup (-\Delta^+(X))$.

Now we have the explicit description of $p(\delta)$ as follows:

\[ p(\delta) = \left( \frac{c}{r}, \frac{1}{\sqrt{d|r|}}L \right) \in \mathfrak{H}(X), \]

where we put $L^2 = 2d$. Moreover one sees $p(\delta) = p(-\delta)$.

The key lemma of this subsection is that the set \{p(\delta) $\in \mathfrak{H}(X)$| $\delta \in \Delta(X)$\} is discreet in $\mathfrak{H}(X)$. To show this claim we introduce some notations.

Definition 2.9. Let $\delta = r \oplus c \oplus s \in \Delta^+(X)$.

1. We define the set $\Delta^{(i)}(X)$ by

$\Delta^{(i)}(X) = \{ r \oplus c \oplus s \in \Delta^+(X)| r \text{ is the } i\text{-th smallest in } \Delta^+(X) \}.$

We also define the rank associated to $\Delta^{(i)}(X)$ by $r$ for some $\delta = r \oplus c \oplus s \in \Delta^{(i)}(X)$.

2. We define the subset $\mathcal{V}(X)$ of $\mathfrak{H}(X)$ as follows.

$\mathcal{V}(X) = \{ (\beta, \omega) \in \mathfrak{H}(X)|(\beta, \omega) \text{ satisfies the condition (2.2) } \}.$

As we remarked in Proposition 2.1 this set is isomorphic to $V(X)$ consisting of stability conditions by the natural morphism $\pi$. 
3. Let $r_i$ be the rank associated to $\Delta^{(i)}(X)$. We define the subset $\mathcal{V}^{(i)}(X)$ of $\mathcal{V}(X)$ by

$$
\mathcal{V}^{(i)}(X) = \{ (\beta, \omega) \in \mathcal{V}(X) | \omega^2 > \frac{2}{r_i^2} \}.
$$

Remark 2.10. Let $X$ be a projective (not necessary Picard rank one) K3 surface. For any $\delta = r \oplus c \oplus s \in \Delta(X)$ with $r \geq 0$, there exists a spherical sheaf $A$ on $X$ such that $v(A) = \delta$ by [13]. In particular if $r > 0$ then we can take $A$ as a locally free sheaf. In addition if we assume $\text{NS}(X) = \mathbb{Z}L$ then we see $A$ is Gieseker-stable by [10] Proposition 3.14. Since we see $\gcd(r, n) = 1$ where $n$ satisfies $nL = c$, $A$ is $\mu$-stable by [9] Lemma 1.2.14.

Remark 2.11. For instance $\Delta^{(1)}(X)$ is the set of Mukai vectors of line bundles on $X$. Thus rank $\Delta^{(1)}(X) = 1$ for any $(X, L)$. However for $i > 1$, the rank of $\Delta^{(i)}(X)$ depends on the degree $L^2$.

Since rank $\Delta^{(1)}(X) = 1$, we see $(\beta, \omega)$ is in $\mathcal{V}^{(1)}(X)$ if and only if $\omega^2 > 2$. We have the following infinite filtration of $\mathcal{V}^{(i)}(X)$ ($i = 1, 2, 3 \cdots$)

$$
\mathcal{V}^{(1)}(X) \subset \mathcal{V}^{(2)}(X) \subset \cdots \subset \mathcal{V}^{(n)}(X) \subset \cdots \subset \mathcal{V}(X).
$$

Lemma 2.12. Notations being as above,

1. the set $S = \{ p(\delta) \in \mathcal{S}(X) | \delta \in \Delta(X) \}$ is a discreet set in $\mathcal{S}(X)$.

2. Furthermore the set $\mathcal{V}(X)$ is open in $\mathcal{S}(X)$.

Proof. Suppose that $\text{NS}(X) = \mathbb{Z}L$ with $L^2 = 2d$. Let $p(\delta)$ be the spherical point of $\delta \in \Delta^{+}(X)$. We put $\delta = r \oplus c \oplus s$ where $c = nL$ for some $n \in \mathbb{Z}$.

Recall that $p(\delta)$ is given by

$$
p(\delta) = (\frac{nL}{r}, \frac{1}{\sqrt{dr}}L).
$$

We also note that $\gcd(r, n) = 1$ since $\delta^2 = -2$ and $\text{NS}(X) = \mathbb{Z}L$. Let $B_\epsilon$ be the open ball whose center is $p(\delta)$ and the radius is $\epsilon$ (with respect to the usual metric). Since $r_{i+1} \geq r_i + 1$ (where $r_i$ is the rank of $\Delta^{(i)}(X)$) if $\epsilon$ is smaller than $\frac{1}{\sqrt{dr}}(\frac{1}{r_i} - \frac{1}{r_{i+1}})$ we see $B_\epsilon \cap S = \{ p(\delta) \}$.

We prove the second assertion. We define $S(\delta)$ for $\delta \in \Delta^{+}(X)$ as follows:

$$
S(\delta) = \{ (\beta, \omega) \in \mathcal{S}(X) | \beta = \frac{c}{r}, 0 < \omega^2 \leq \frac{2}{r_i^2} \}.
$$

Then one can check that

$$
\mathcal{V}(X) = \mathcal{S}(X) \setminus \bigcup_{\delta \in \Delta^{+}(X)} S(\delta).
$$

Hence we see

$$
\mathcal{V}^{(i)}(X) = \{ (\beta, \omega) \in \mathcal{S}(X) | \omega^2 > \frac{2}{r_i^2} \} \setminus \bigcup_{\delta \in \Delta^{(\leq i - 1)}} S(\delta),
$$

where $\Delta^{(\leq i)} = \bigcup_{j=1}^{i} \Delta^{(j)}(X)$. Since the set

$$
\{ \frac{c}{r} | \delta = r \oplus c \oplus s \in \Delta^{(\leq i)} \}
$$
is discrete in $\mathbb{R}L$, the set $\mathcal{V}^{(i)}(X)$ is open in $\mathcal{H}(X)$. Since we have

$$\mathcal{V}(X) = \bigcup_{i \in \mathbb{N}} \mathcal{V}^{(i)}(X),$$

the set $\mathcal{V}(X)$ is open in $\mathcal{H}(X)$. □

**Definition 2.13.** We set elements of the fundamental groups $\pi_1(\mathcal{H}_0(X))$ and of $\pi_1(GL^+(2, \mathbb{R}))$ as follows.

- We define $\ell_\delta$ by the loop which turns round only the spherical point $p(\delta) \in \mathcal{H}(X)$ counterclockwise;

\[
\begin{align*}
\ell_\delta & \colon [0, 1] \ni t \mapsto \left( \begin{array}{cc} \cos(2\pi t) & -\sin(2\pi t) \\ \sin(2\pi t) & \cos(2\pi t) \end{array} \right) \in GL^+(2, \mathbb{R}).
\end{align*}
\]

We define $g \in \pi_1(GL^+(2, \mathbb{R}))$ by

$$g \colon [0, 1] \ni t \mapsto \left( \begin{array}{cc} \cos(2\pi t) & -\sin(2\pi t) \\ \sin(2\pi t) & \cos(2\pi t) \end{array} \right) \in GL^+(2, \mathbb{R}).$$

We note that $g$ is a generator of $\pi_1(GL^+(2, \mathbb{R}))$ since $\pi_1(GL^+(2, \mathbb{R})) \cong \pi_1(SO(2)) \cong \mathbb{Z}$.

**Proposition 2.14.** The fundamental group $\pi_1(\mathcal{P}_0^+(X))$ is isomorphic to

$$\left( \prod_{\delta \in \Delta^+(X)} \mathbb{Z} \cdot \ell_\delta \right) \rtimes \mathbb{Z} \cdot g$$

where $\prod_{\delta \in \Delta^+(X)} \mathbb{Z} \cdot \ell_\delta$ is a free product of infinite cyclic groups $\mathbb{Z}$ generated by $\ell_\delta$.

**Proof.** Since $\mathcal{P}_0^+(X)$ is isomorphic to $\mathcal{D}_0^+(X) \times GL^+(2, \mathbb{R})$ we see $\pi_1(\mathcal{P}_0^+(X)) \cong \pi_1(\mathcal{D}_0^+(X)) \times \mathbb{Z} \cdot g$. As we remarked before we have $\Delta(X) = \Delta^+(X) \sqcup (-\Delta^+(X))$. Hence we see

$$\mathcal{D}_0^+(X) = \mathcal{D}^+(X) \setminus \bigcup_{\delta \in \Delta(X)} \langle \delta \rangle^\perp = \mathcal{D}^+(X) \setminus \bigcup_{\delta \in \Delta^+(X)} \langle \delta \rangle^\perp$$

Since $\mathcal{D}_0^+(X)$ is isomorphic to $\mathcal{H}_0(X)$ it is enough to show that

$$\pi_1(\mathcal{H}_0(X)) = \prod_{\delta \in \Delta^+} \mathbb{Z} \cdot \ell_\delta$$

We choose a base point $p$ of $\mathcal{H}_0(X)$ so that $p = \sqrt{-1}\omega$ with $\omega^2 \gg 2$. Let $\ell$ be the loop whose base point is $p$. Then there is a compact contractible subset $C$ whose interior $C^{\in \mathbb{R}}$ contains $\ell$. Then the following set is finite:

$$\{p(\delta) \in C^{\in \mathbb{R}} | \delta \in \Delta^+(X)\}.$$
Since the fundamental group of the complement of $n$-points in $C$ is the free group of rank $n$, we see the homotopy equivalence class of $\ell$ is uniquely given by

$$\ell_{\delta_1}^{k_1} \ell_{\delta_2}^{k_2} \ldots \ell_{\delta_m}^{k_m}$$

where each $k_i \in \mathbb{Z}$. In fact if another loop $m$ is homotopy equivalent to $\ell$ by $H: [0, 1] \times [0, 1] \to \mathcal{H}_0(X)$, then there is a contractible compact set $C'$ such that $(C')^m$ contains the image of $H$. Since there are at most finite spherical point in $(C')^m$, we see the above representation is unique. Thus we have finished the proof. □

To simplify the notations we denote $\ell_{v(A)}$ by $\ell_A$. By Remark 2.10 we see

$$\pi_1(\mathcal{H}_0(X)) = \langle \ell_A | A \text{ is spherical and locally free} \rangle = \ast \mathbb{Z}_{\ell_A}.$$ 

3. Hyperbolic structure on $\mathcal{H}_0(X)$

Let $\mathcal{H}_0^\dagger(X)$ be the connected components of $\mathcal{H}_0(X)$ introduced in §2. In this section we discuss a hyperbolic structure on the normalized stability manifold $\mathcal{H}_0^\dagger(X)$.

To simplify explanations of this section we always use the following notations. Let $(X_i, L_i)$ $(i = 1, 2)$ be projective K3 surfaces with $\text{NS}(X_i) = \mathbb{Z} L_i$ and let $\Phi: D(X_2) \to D(X_1)$ be an equivalence between them. The induced isometry $\mathcal{N}(X_2) \to \mathcal{N}(X_1)$ by $\Phi$ is denoted by $\Phi^N$.

For a closed point $p_i \in X_i$ we set

$$v(\Phi(O_{p_2})) = r_1 \oplus n_1 L_1 \oplus s_1 \quad \text{and} \quad v(\Phi^{-1}(O_{p_1})) = r_2 \oplus n_2 L_2 \oplus s_2.$$

Since $X_1$ and $X_2$ are Fourier-Mukai partners each other, we see $L_1^2 = L_2^2 = 2d$ for some $d \in \mathbb{N}$.

**Lemma 3.1.** Notations being as above,

(1) $r_1 = 0$ if and only if $r_2 = 0$. In particular if $r_2 = 0$ then $\Phi^N(O_{p_2}) = \pm v(O_{p_2}) = \pm (0 \oplus 0 \oplus 1)$.

(2) If $\Phi^N(O_{p_2}) = 0 \oplus 0 \oplus 1$ then $\Phi^N$ is numerically equivalent to $(M \otimes)^N$ where $M$ is in $\text{Pic}(X_1)$ under the canonical identification $\mathcal{N}(X_2) \cong \mathcal{N}(X_1)$.

**Proof.** By the symmetry it is enough to show that $r_2 = 0$ under the assumption $r_1 = 0$. If $r_1 = 0$, since $v(\Phi(O_{p_2}))$ is isotropic, we see $n_2^2 L_2^2 = 0$. Thus $n_1 = 0$. Moreover since $v(\Phi(O_{p_2}))$ is primitive, $s_1$ should be $\pm 1$. Hence $\Phi^N(0 \oplus 0 \oplus 1) = \pm (0 \oplus 0 \oplus 1)$. This gives the proof of the first assertion.

Second assertion essentially follows from the argument in the proof for [7], Corollary 10.12. Hence we recall his arguments.

Since $\rho(X_1) = 1$, there is the canonical isomorphism $f: \mathcal{N}(X_2) \to \mathcal{N}(X_1)$ where $f(0 \oplus 0 \oplus 1) = 0 \oplus 0 \oplus 1$, $f(0 \oplus L_2 \oplus 0) = 0 \oplus L_1 \oplus 0$ and $f(0 \oplus 0 \oplus 1) = 0 \oplus 0 \oplus 1$. We show that $\Phi^N = (\otimes M)^N$ ($\exists M \in \text{Pic}(X_1)$) under the canonical identification $f: \mathcal{N}(X_2) \to \mathcal{N}(X_1)$.
One can check easily

\[ v(\Phi^N(1\oplus 0\oplus 0)) = 1\oplus M \oplus \frac{M^2}{2} (\exists M \in \text{Pic}(X_1)), \]

by using the facts \( \langle 1\oplus 0\oplus 0, v(C_{p_2}) \rangle = -1 \) and \( \langle 1\oplus 0\oplus 0 \rangle^2 = 0 \). Now consider the functor

\[ \Psi = (\otimes M^{-1} \circ \Phi) : D(X_2) \rightarrow D(X_1) \rightarrow D(X_1). \]

Then we see \( \Psi^N(0\oplus 0\oplus 1) = 0\oplus 0\oplus 1 \) and \( \Psi^N(1\oplus 0\oplus 0) = 1\oplus 0\oplus 0 \). Thus \( \Psi^N \) induces the isomorphism

\[ \Psi^N : \text{NS}(X_2) \rightarrow \text{NS}(X_1). \]

Since \( \text{NS}(X_i) = \mathbb{Z}L_i \) we see \( \Psi^N(L_2) = \pm L_1 \). Since any equivalence preserves the orientations by \( \mathbb{10} \) we see \( \Psi^N(L_2) = L_1 \). This gives the proof of the second assertion. \( \square \)

**Lemma 3.2.** For \((\beta_i, \omega_i) \in \mathcal{H}(X_i) \ (i = 1, 2)\), we put \( \beta_i + \sqrt{-1}\omega_i = (x_i + \sqrt{-1}y_i)L_i \).

1. For any \( \beta_2 + \sqrt{-1}\omega_2 \in \mathcal{H}(X_2) \), there exist \( \beta_1 + \sqrt{-1}\omega_1 \in \mathcal{H}(X_1) \) and \( \lambda \in \mathbb{C}^* \) such that \( \Phi^N(\exp(\beta_2 + \sqrt{-1}\omega_2)) = \lambda \exp(\beta_1 + \sqrt{-1}\omega_1) \).
2. If \( r_1 \neq 0 \) then \( r_1r_2 > 0 \). Furthermore we have

\[ x_1 + \sqrt{-1}y_1 = \frac{1}{d\sqrt{r_1r_2}} \cdot \frac{-1}{(x_2 + \sqrt{-1}y_2) - \frac{w_2}{r_2}} + \frac{n_1}{r_1}. \]

In particular this gives a linear fractional transformation on \( \mathbb{H} \).

**Proof.** We put \( \mathcal{U}_2 = \exp(\beta_2 + \sqrt{-1}\omega_2) \) and \( \Phi^N(\mathcal{U}_2) = u \oplus v \oplus w \). Since we have \( \mathcal{U}_2^2 = 0 \) and \( \mathcal{U}_2 \mathcal{U}_2 > 0 \), we see the following:

(a) \( v^2 = 2uw \) and
(b) \( v\bar{w} - u\bar{w} - \bar{u}w > 0 \).

If \( u = 0 \) then \( v^2 \) should be 0. Since we have \( v^2 \geq 0 \) by the assumption, we see \( \Phi^N(\mathcal{U}_2) = 0 \oplus 0 \oplus w \). This contradicts the second inequality. Thus \( u \) should not be 0 and we see

\[
\Phi^N(\mathcal{U}_2) = u(1 \oplus \frac{v}{u} \oplus \frac{w}{u}) = u \left( 1 \oplus \frac{v}{u} \oplus \frac{1}{2} \left( \frac{v}{u} \right)^2 \right).
\]

Since \( \frac{v}{u} \) is in \( \text{NS}(X) \otimes \mathbb{C} \) we can put \( \frac{v}{u} = (x + \sqrt{-1}y)L_1 \) for some \( (x, y) \in \mathbb{R}^2 \).

By the inequality of (b), we see \( y \neq 0 \). Since \( \Phi \) preserves the orientation by \( \mathbb{10} \), we see \( y > 0 \). Thus we have proved the first assertion.

We prove the second assertion. By the first assertion we put

\[ \Phi^N(\exp(\beta_2 + \sqrt{-1}\omega_2)) = \lambda \exp(\beta_1 + \sqrt{-1}\omega_1). \]
Then we see

$$\lambda = -\langle \Phi^N(\exp(\beta_2 + \sqrt{-1}\omega_2)), v(\mathcal{O}_{p_1}) \rangle$$
$$= -\langle \exp(\beta_2 + \sqrt{-1}\omega_2), v(\Phi^{-1}(\mathcal{O}_{p_1})) \rangle$$
$$= -Z_{(\beta_2,\omega_2)}(\Phi^{-1}(\mathcal{O}_{p_1})), $$

and

$$-1 = \langle \exp(\beta_2 + \sqrt{-1}\omega_2), v(\mathcal{O}_{p_2}) \rangle$$
$$= \langle \Phi^N(\exp(\beta_2 + \sqrt{-1}\omega_2)), v(\mathcal{O}_{p_2}) \rangle$$
$$= \lambda \cdot Z_{(\beta_1,\omega_1)}(\Phi(\mathcal{O}_{p_2})).$$

Thus we have

$$1 = Z_{(\beta_2,\omega_2)}(\Phi^{-1}(\mathcal{O}_{p_1})) \cdot Z_{(\beta_1,\omega_1)}(\Phi(\mathcal{O}_{p_2}))$$

By Lemma 3.1 we see $r_1 \neq 0$ and $r_2 \neq 0$. Now recall Remark 2.2. Since $v(\Phi(\mathcal{O}_{p_2}))^2 = v(\Phi^{-1}(\mathcal{O}_{p_1}))^2 = 0$, we have

$$Z_{(\beta_2,\omega_2)}(\Phi^{-1}(\mathcal{O}_{p_1})) = \frac{r_2}{2} \left( y_2 + \sqrt{-1} \left( \frac{n_2}{r_2} - x_2 \right) \right)^2 L_2^2$$

and

$$Z_{(\beta_1,\omega_1)}(\Phi(\mathcal{O}_{p_2})) = \frac{r_1}{2} \left( y_1 + \sqrt{-1} \left( \frac{n_1}{r_1} - x_1 \right) \right)^2 L_1^2.$$  

Since $L_1^2 = L_2^2 = 2d$ we see

$$\langle x_1 - \frac{n_1}{r_1}, \sqrt{-1}y_1 \rangle = \frac{\pm 1}{d \sqrt{r_1 r_2}} \cdot \frac{1}{(x_2 - \frac{n_2}{r_2}) + \sqrt{-1}y_2}.$$  

Since the left hand side is in the upper half plane $\mathbb{H}$, $\sqrt{r_1 r_2}$ should be a real number. Thus we see $r_1 r_2 > 0$. Furthermore, since the imaginary part of the left hand side is positive we have

$$\langle x_1 - \frac{n_1}{r_1}, \sqrt{-1}y_1 \rangle = \frac{-1}{d \sqrt{r_1 r_2}} \cdot \frac{1}{(x_2 - \frac{n_2}{r_2}) + \sqrt{-1}y_2}.$$  

Thus we have finished the proof. \(\square\)

Recall that $\text{Stab}^n(X) = \text{Stab}^1(X)/\widetilde{\text{GL}}^+(2,\mathbb{R})$.

**Theorem 3.3.** Assume that $\rho(X) = 1$.

1. $\text{Stab}^n(X)$ is a hyperbolic 2 dimensional manifold.
2. Let $Y$ be a Fourier-Mukai partner of $X$ and $\Phi: D(Y) \to D(X)$ an equivalence. Suppose that $\Phi$ preserves the distinguished component. Then the induced morphism $\Phi^n: \text{Stab}^n(Y) \to \text{Stab}^n(X)$ is an isometry with respect to the hyperbolic metric.

**Proof.** By Corollary 2.8 we have the normalized covering map

$$\pi^n: \text{Stab}^n(X) \to \mathfrak{H}_0(X).$$
Since $\mathcal{H}_0(X)$ is isomorphic to the open subset of $\mathbb{H}$ by Lemma 2.12, we can define the hyperbolic metric on $\mathcal{H}_0(X)$ which is given by

$$ds^2 = \frac{dx^2 + dy^2}{y^2},$$

where $x + \sqrt{-1}y \in \mathbb{H}$. Since $\pi^n$ is a covering map, we can also define the hyperbolic metric on $\text{Stab}^n(X)$. Thus $\text{Stab}^n(X)$ is hyperbolic.

Now we prove the second assertion. If $v(\Phi(\mathcal{O}_y))$ is not $\pm(0 \oplus 0 \oplus 1)$ by Lemma 3.2, we see that the induced morphism between $\mathcal{H}_0(Y) \to \mathcal{H}_0(X)$ is given by the linearly fractional transformation. Since $\pi^n$ is an isometry, $\Phi^n$ is also an isometry. Suppose that $v(\Phi(\mathcal{O}_y)) = \pm(0 \oplus 0 \oplus 1)$. If necessary by taking a shift $[1]$ which gives the trivial action on $\mathcal{H}(X)$ we can assume that $v(\Phi(\mathcal{O}_y)) = 0 \oplus 0 \oplus 1$. Then, by Lemma 3.1, the induced action on $\mathcal{H}$ is given by a parallel transformation $z \mapsto z + n$ for some $n \in \mathbb{Z}$. Thus we have finished the proof. \(\square\)

4. Simply connectedness of $\text{Stab}^n(X)$

In this section we always assume $\rho(X) = 1$. Then, as was shown in the previous section, $\text{Stab}^n(X)$ is a hyperbolic manifold. By using the hyperbolic structure, we shall discuss the simply connectedness of $\text{Stab}^\dagger(X)$. Namely we show the following:

**Theorem 4.1.** The following conditions are equivalent.

1. $\text{Stab}^\dagger(X)$ is simply connected.
2. $\text{Stab}^n(X)$ is isomorphic to the upper half plane $\mathbb{H}$.
3. $W(X)$ is isomorphic to the free group generated by $T_A^n$:

   $$W(X) = \ast_A(\mathbb{Z} \cdot T_A^n),$$

   where $A$ runs through all spherical locally free sheaves.

**Proof.** We first show that $\text{Stab}^\dagger(X)$ is simply connected if and only if $\text{Stab}^n(X)$ is simply connected. Since the right action of $\widetilde{\text{GL}}^+(2, \mathbb{R})$ on $\text{Stab}^\dagger(X)$ is free, the natural map

$$\text{Stab}^\dagger(X) \to \text{Stab}^n(X)$$

gives the $\widetilde{\text{GL}}^+(2, \mathbb{R})$-bundle on $\text{Stab}^n(X)$. Thus there is an exact sequence of fundamental groups:

$$\pi_1(\widetilde{\text{GL}}^+(2, \mathbb{R})) \to \pi_1(\text{Stab}^\dagger(X)) \to \pi_1(\text{Stab}^n(X)) \to 1.$$

Since $\widetilde{\text{GL}}^+(2, \mathbb{R})$ is simply connected we see that $\pi_1(\text{Stab}^\dagger(X)) = \{1\}$ if and only if $\pi_1(\text{Stab}^n(X)) = \{1\}$.

Since $\text{Stab}^n(X)$ is a hyperbolic and complex manifold, $\text{Stab}^n(X)$ is isomorphic to $\mathbb{H}$ if and only if $\pi_1(\text{Stab}^n(X)) = \{1\}$ by Riemann’s mapping theorem. Thus we have proved that the first condition is equivalent to the second one.
We secondly show the first condition is equivalent to the third one. Let $\text{Cov}(\pi)$ be the covering transformation group of $\pi: \text{Stab}^\dagger(X) \to \mathcal{P}^+_0(X)$. We put $\tilde{W}(X)$ by the group generated by $W(X)$ and the double shift $[2]$. Note that $\tilde{W}(X)$ is isomorphic to $W(X) \times \mathbb{Z} \cdot [2]$.

We claim that $\tilde{W}(X)$ is isomorphic to $\text{Cov}(\pi)$. Recall that all spherical sheaf $A$ on $X$ with $\rho(X) = 1$ is $\mu$-stable by Remark 2.10. Hence any $\Phi \in \tilde{W}(X)$ gives a trivial action on $H^*(X, \mathbb{Z})$ and preserves the connected component $\text{Stab}^\dagger(X)$. Thus $\Phi$ gives the covering transformation by [5, Theorem 13.3]. Thus we have the group homomorphism $\tilde{W}(X) \to \text{Cov}(\pi)$. In particular by Proposition 2.4, we see this morphism is a surjection. Furthermore as is shown in [5, Theorem 13.3], this is injective. Thus we have proved our claim.

Since the covering $\pi: \text{Stab}^\dagger(X) \to \mathcal{P}^+_0(X)$ is a Galois covering, we have the exact sequence of groups:

$$1 \longrightarrow \pi_1(\text{Stab}^\dagger(X)) \longrightarrow \pi_1(\mathcal{P}^+_0(X)) \longrightarrow \text{Cov}(\pi) \longrightarrow 1.$$

As will be shown in Proposition 5.4 we see $\varphi(\ell_A) = T_A^2$ and $\varphi(g) = [2]$.

**Remark 4.2.** Since the quotient map $\text{Stab}^\dagger(X) \to \text{Stab}^n(X)$ is a $\widetilde{\text{GL}}^+(2, \mathbb{R})$-bundle, we see that $\text{Stab}^\dagger(X)$ is simply connected if and only if $\text{Stab}^\dagger(X)$ is a $\widetilde{\text{GL}}^+(2, \mathbb{R})$-bundle over $\mathbb{H}$. Thus we can deduce the global geometry of the stability manifold $\text{Stab}^\dagger(X)$.

**Remark 4.3.** We give some remarks for $W(X)$. Recall that any equivalence $\Phi \in \text{Aut}(D(X))$ induces the Hodge isometry $\Phi^H$ of $H^*(X, \mathbb{Z})$ in a canonical way. If Bridgeland’s conjecture holds, the group $W(X)$ is free group generated by $T_A^2$. Conversely if $W(X)$ is a free group generated by $T_A^2$, then $\varphi$ is an isomorphism. Hence $\text{Stab}^\dagger(X)$ is simply connected.

5. **Wall and the hyperbolic structure**

Let $X$ be a projective K3 surface with Picard rank one. We have two goals of this section. The first aim is to show Proposition 5.4 which is necessary for Theorem 4.1. The second aim is to show that any wall is geodesic.

Now we start this section from the following key lemma.
Lemma 5.1. Any $\sigma \in \partial U(X)$ is in a general position (See also [5, §12]). Namely the point $\sigma$ lies on only one irreducible component of $\partial U(X)$.

Before we start the proof, we remark that Maciocia proved a similar assertion in a slightly different situation in [13].

Proof. Suppose that there is an element $\sigma = (A, Z) \in \partial U(X)$ which is not general. Let $W_1$ and $W_2$ be two irreducible components of $\partial U(X)$ such that $\sigma \in W_1 \cap W_2$. By [5, Proposition 9.3] we may assume $\forall \tau_1 \in W_1 \setminus \{\sigma\}$ and $\forall \tau_2 \in W_2 \setminus \{\sigma\}$ are in general positions in a sufficiently small neighborhood of $\sigma$. Hence by [5, Theorem 12.1] there are two $(-2)$-vectors $\delta_i \in \Delta^+(X)$ ($i = 1, 2$) such that for any $\tau_i = (A_i, Z_i) \in W_i \setminus \{\sigma\}$ the imaginary part $\Im Z_i(O_x)Z_i(\delta_i)$ is 0 where $i \in \{1, 2\}$ and $x \in X$. Since these are closed conditions, the central charge $Z$ of $\sigma$ also satisfies the following condition:

\[
(5.1) \quad \Im Z(O_x)Z(\delta_1) = \Im Z(O_x)Z(\delta_2) = 0.
\]

By the assumption $\text{NS}(X) = ZL$, there exists $g \in GL^+(2, \mathbb{R})$ such that $Z'(E) := g^{-1} \circ Z(E) = \langle \exp(\beta + \sqrt{-1}\omega), v(E) \rangle$ where $(\beta, \omega) \in \text{Stab}(X)$.

Now we put $\delta_i = r_i + n_i L \oplus s_i$. Note that $r_i \neq 0$ since $n_i^2 L_i^2 \geq 0$. Since $Z'(O_x) = -1$ we see $\Im Z'(\delta_i)$ is zero by the condition (5.1). Thus we see

\[
\frac{n_1 L}{r_1} = \frac{n_2 L}{r_2} = \beta.
\]

Since $\delta_i = -2$ we see $\gcd(r_i, n_i) = 1$. Hence we have $\delta_1 = \delta_2$. This contradicts $W_1 \neq W_2$.

By Lemma 5.1 and [5, Theorem 12.1] we see $\partial U(X)$ is a disjoint union of real codimension 1 submanifolds:

\[
\partial U(X) = \bigsqcup_{A \text{ spherical locally free}} (W^+_A \sqcup W^-_A),
\]

where $W^+_A$ (respectively $W^-_A$) is the set of stability conditions whose type is $(A^+)$ (respectively $(A^-)$). In the following we give an explicit description of each component $W^+_A$.

Lemma 5.2. Let $X$ be a projective K3 surface with $\text{NS}(X) = ZL$ and let $A$ be a spherical locally free sheaf. We put $v(A) = r_A + n_A L \oplus s_A$ and define the set $S(v(A))$ by

\[
S(v(A)) = \{ (\beta, \omega) \in \mathcal{H}(X) | \beta = \frac{n_A L}{r_A}, 0 < \omega^2 < \frac{2}{r_A^2} \}.
\]

Then $W^+_A$ is isomorphic to $S(v(A)) \times \widetilde{GL}^+(2, \mathbb{R})$. In particular $W^+_A / \widetilde{GL}^+(2, \mathbb{R})$ is a hyperbolic segment spanned by two points in $\text{Stab}^n(X)$ which is isomorphic to $S(v(A))$.

Proof. We have to consider two cases: $\sigma \in W^+_A$ or $\sigma \in W^-_A$. Since the proof is similar, we give the proof only for the case $\sigma \in W^+_A$. 

Since \( \sigma \in W_+^A \), the Jordan-Hölder filtration of \( \mathcal{O}_x \) is given by the spherical triangle [2.1]
\[
A^{\oplus r_A} \longrightarrow \mathcal{O}_x \longrightarrow T_A(\mathcal{O}_x).
\]

By taking \( T_A^{-1} \) to the triangle (5.2) we have
\[
A^{\oplus r_A}[1] \longrightarrow T_A^{-1}(\mathcal{O}_x) \longrightarrow \mathcal{O}_x.
\]

Thus \( \mathcal{O}_x \) is \( T_A^{-1}\sigma \)-stable. Hence \( T_A^{-1}\sigma \) is in \( U(X) \).

Now we put \( T_A^{-1}\sigma = \tau = (A, Z) \). Since \( Z(A[1])/Z(\mathcal{O}_x) \in \mathbb{R}_{>0} \), we see that \( \tau \) is in the set
\[
W' = \{ \sigma(\beta, \omega) \in V(X) | \beta = \frac{n_{A[L]}}{r_A} < \omega^2 \} \cdot \widetilde{GL}(2, \mathbb{R}).
\]

Thus we see \( W_+^A \subset T_A W' \). To show the inverse inclusion, let \( \tau' = (A', Z') \) be in \( W' \). As we remarked in Remark 2.10, \( A \) is \( \mu \)-stable locally free sheaf. Then \( A[1] \) has no nontrivial subobject in \( A' \) by [8, Theorem 0.2]. Hence \( A[1] \) is \( \tau' \)-stable, in particular, with phase 1. Since \( T_A^{-1}(\mathcal{O}_x) \) is given by the extension (5.3) of \( \mathcal{O}_x \) and \( A^{\oplus r_A}[1] \), the object \( T_A^{-1}(\mathcal{O}_x) \) is strictly \( \tau' \)-semistable. Thus by taking \( T_A \) to the triangle (5.3), we obtain the Jordan-Hölder filtration (5.2). Hence we see \( W_+^A = T_A W' \).

Since the induced morphism between \( \mathcal{I}(X) \) by \( T_A \) is given by Lemma 3.2, we see
\[
W_+^A = T_A W' \cong S(v(A)) \times \widetilde{GL}(2, \mathbb{R}).
\]

For a spherical locally free sheaf \( A \) we define the point \( q = p(T_A(\mathcal{O}_x)) \in \mathcal{I}(X) \) by \( (\beta, \omega) = \left( \frac{c_1(A)}{r_A}, 0 \right) \). By the simple calculation we see that
\[
\langle \exp(q), v(T_A(\mathcal{O}_x)) \rangle = 0.
\]

Thus in the sense of Definition 2.7, \( p(T_A(\mathcal{O}_x)) \) could be regarded as the associated point of the isotropic vector \( v(T_A(\mathcal{O}_x)) \). In view of this we define the following notion:

**Definition 5.3.** An associated point \( p \in \mathcal{I}(X) \) with a primitive isotropic vector \( v \in \mathcal{N}(X) \) is the point which satisfies
\[
\langle \exp(p), v \rangle = 0.
\]

Clearly if \( v = r \oplus nL \oplus s \) then \( p \) is given by \( \frac{n}{r} \). In particular if \( v = 0 \oplus 0 \oplus 1 \) the associated point is \( \infty \in \mathcal{I}(X) \). We denote the point by \( p(v) \).

As an application of Lemma 5.2, we give the proof of a remained proposition:

**Proposition 5.4.** Let \( \varphi: \pi_1(\mathcal{P}_0^+(X)) \to \text{Cov}(\pi) \) be the morphism in the proof of Theorem 4.1. Then \( \varphi(\ell_A) = T_A^2 \) and \( \varphi(g) = [2] \).
Theorem 5.5. The set $\mathfrak{W}(S, B)$ is geodesic in $\text{Stab}^n(X)$.

Proof. Following [5] Proposition 9.3] let $\mathcal{T}$ be the set of objects
\[ \mathcal{T} = \{ A \in D(X) | \exists E \in S, \exists \sigma \in B \text{ such that } m_{\sigma}(A) \leq m_{\sigma}(E) \}. \]
We put the set of Mukai vectors in $\mathcal{T}$ by $I = \{v(A) | A \in \mathcal{T}\}$ and let $\gamma$ be the pair $\gamma = (v_i, v_j) \in I \times I$ which are not proportional. As was shown in [3, Proposition 9.3], each wall component $W_\gamma$ is given by

$$W_\gamma = \{ \sigma = (A, Z) \in \text{Stab}^1(X) | Z(v_i)/Z(v_j) \in \mathbb{R}_{>0}\}.$$ 

We put $W_\gamma/\text{Gl}^+(2, \mathbb{R})$ by $\mathfrak{W}_\gamma$. It is enough to prove that $\mathfrak{W}_\gamma$ is geodesic in $\text{Stab}^n(X)$.

Since $I$ is finite (Recall that $\mathcal{T}$ has bounded mass) we can take a sufficiently large $m \in \mathbb{Z}$ so that the rank of all vectors in $T^H_{mL}(I)$ are not $0$. For the set $T^H_{mL}(I)$ we define $\mathfrak{W}_\gamma^I$ by

$$\mathfrak{W}_\gamma^I = \{ [\sigma] = [(A, Z)] \in \text{Stab}^n(X) | Z(T^H_{mL}(v_i))/Z(T^H_{mL}(v_j)) \in \mathbb{R}_{>0} \}.$$ 

We may assume the central charge of $[\sigma] \in \mathfrak{W}_\gamma^I$ is given by

$$Z(E) = (\exp(\beta + \sqrt{-1}\omega), v(E))$$

where $(\beta, \omega) \in \delta(X)$.

We note that $\sigma \in \mathfrak{W}_\gamma^I$ satisfies the following equation

$$\text{Im}Z(T^H_{mL}(v_i))/Z(T^H_{mL}(v_j)) = 0.$$ 

Then one can easily check that the equation (5.4) defines hyperbolic line in $\delta(X)$. Since the hyperbolic structure is induced from $\delta(X)$ the set $\mathfrak{W}_\gamma^I$ is geodesic also in $\text{Stab}^n(X)$. Since we have $T^H_{mL}\mathfrak{W}_\gamma = \mathfrak{W}_\gamma$, the set $\mathfrak{W}_\gamma$ is also geodesic in $\text{Stab}^n(X)$ by Theorem 3.3. 

6. Revisit of Orlov’s theorem via hyperbolic structure

In this section we demonstrate applications of the hyperbolic structure on $\text{Stab}^n(X)$. Mainly we prove Orlov’s theorem without the global Torelli theorem but with assuming the connectedness of $\text{Stab}(X)$ in Proposition 6.5. Hence our application suggests that Bridgeland’s theory substitutes for the global Torelli theorem.

6.1. Strategy for Proposition 6.5. Since the proof of Proposition 6.5 is technical, we explain the strategy and the roles of some lemmas which we prepare in §6.2. Proposition 6.5 will be proved in §6.3.

If we have an equivalence $\Phi : D(Y) \to D(X)$ preserving the distinguished component then there exists $\Psi \in W(X)$ such that $(\Psi \circ \Phi)_*U(Y) \cap V(X) \neq \emptyset$ by Proposition 2.4. We want to take the large volume limit in the domain $(\Psi \circ \Phi)_*U(Y) \cap V(X)$. Because of the complicatedness of the set $V(X)$, we consider the subset $V(X)_{>2} = \{ \sigma(\beta, \omega) \in V(X) | \omega^2 > 2 \}$ and focus on the domain $D_{>2} = (\Psi \circ \Phi)_*U(Y) \cap V(X)_{>2}$.

To take the large volume limit, we have to know the shape of the domain $D_{>2}$. To know the shape of $D_{>2}$ we have to see where the boundary $(\Psi \circ \Phi)_*\partial U(Y)$ appears in $\text{Stab}^1(X)$. As we showed in Lemma 5.2, any connected component of $\partial U(Y)$ is the product of $\text{Gl}^+(2, \mathbb{R})$ and a hyperbolic segment spanned by two associated points. Since any equivalence $D(Y) \to D(X)$
induces an isometry between the normalized spaces $\text{Stab}^n(Y) \to \text{Stab}^n(X)$ by Theorem 5.3. We see that the image $(\Psi \circ \Phi)_* \partial U(Y)$ is also the products of $\widetilde{\text{GL}}^+(2, \mathbb{R})$ and hyperbolic segments spanned by two associated points (See also Lemma 6.1 below). This is the reason why the hyperbolic metric on $\text{Stab}^n(X)$ is important for us.

Here we have to recall that $\text{Stab}_\psi(X)$ is conjecturally $\widetilde{\text{GL}}^+(2, \mathbb{R})$-bundle over the upper half plane $\mathbb{H}$. Since we don't have the explicit isomorphism $\text{Stab}^n(X) \to \mathbb{H}$ yet, it is impossible to observe the place $(\Psi \circ \Phi)_* \partial U(Y)$ in $\text{Stab}_\psi(X)$. Instead of this observation, we study the numerical information of $(\Psi \circ \Phi)_* \partial U(Y)$, namely the image of $(\Psi \circ \Phi)_* \partial U(Y)$ by the quotient map $\pi_B: \text{Stab}_\psi(X) \to \mathbb{P}_0^+(X) \to \mathfrak{g}_0(X)$.

Set $\mathfrak{W} = \pi_B((\Psi \circ \Phi)_* \partial U(Y))$. As we showed in Lemma 6.2, $\mathfrak{W}$ is the disjoint sum of hyperbolic segments. As we show in Lemma 6.2 later, there are two types (I) and (II) of components of $\mathfrak{W}$. The type (I) is a hyperbolic segment which does not intersect the domain $\pi_B(V(X)_{>2})$ and the type (II) is a hyperbolic segment which does intersect $\pi_B(V(X)_{>2})$. Recall that our basic strategy is to take the limit in the domain $V(X)_{>2}$. If the family of type (II) components is unbounded in $\pi_B(V(X)_{>2})$, it may be impossible to take the large volume limit. Hence we have to show the boundedness of type (II) components (Proposition 6.3 and Corollary 6.4).

6.2. Technical lemmas. We prepare some technical lemmas. Throughout this section we use the following notations.

For a K3 surface $(X, L)$ we put $L^2 = 2d$. Suppose that $E \in D(X)$ satisfies $v(E)^2 = 0$ and $A \in D(X)$ is spherical. We put their Mukai vectors respectively

$$v(E) = r_E \oplus n_{\text{E}} L \oplus s_E \text{ and } v(A) = r_A \oplus n_A L \oplus s_A.$$ 

We denote $(\beta, \omega) \in \mathfrak{g}(X)$ by $(xL, yL)$.

The main object is the following set

$$\mathfrak{W}(A, E) = \{(\beta, \omega) \in \mathfrak{g}(X)| \Im Z_{(\beta, \omega)}(E) \bar{Z}_{(\beta, \omega)}(A) = 0\}.$$ 

One can easily check that the condition $\Im Z_{(\beta, \omega)}(E) \bar{Z}_{(\beta, \omega)}(A) = 0$ is equivalent to

$$N_{A,E}(x, y) = \lambda_E \left(\frac{-1}{r_A} + dr_A y^2 - \frac{d \lambda_A^2}{r_A} - \lambda_A(dr_E y^2 - \frac{\lambda_E^2}{r_E})\right) = 0,$$

where $\lambda_E = n_E - r_E x$ and $\lambda_A = n_A - r_A x$. We also have

$$N_{A,E}(x, y) = d(r_A n_E - r_E n_A) y^2 + d \lambda_E \lambda_A(\frac{n_E}{r_E} - \frac{n_A}{r_A}) - \frac{\lambda_E}{r_A}.$$ 

(6.1)

Lemma 6.1. Suppose that $0 < r_E$ and $\frac{n_E}{r_E} \neq \frac{n_A}{r_A}$. Then $\mathfrak{W}(A, E)$ is the half circle passing through the following 4 points:

$$(x, y) = (\alpha_E, 0), (\frac{n_E}{r_E}, 0), (\frac{n_A}{r_A}, \frac{1}{\sqrt{d|r_A|}}) \text{ and } (\alpha_A, \frac{1}{\sqrt{d|r_A|}}).$$
where \( \alpha_E = \frac{n_A}{r_A} - \frac{1}{\text{dr}_A(r_E - r_A)} \) and \( \alpha_A = \frac{n_E}{r_E} - \frac{1}{\text{dr}_A(r_E - r_A)} \). In particular the set \( \mathcal{W}(A, E) \) is a hyperbolic line passing through above 4 points.

**Proof.** We can prove Lemma 6.1 by the simple calculation of (6.1). □

In particular the first two points are associated points with respectively \( T_A(E) \) and \( E \). Hence we put them respectively

- \( p(T_A(E)) = (\alpha_E, 0) \),
- \( p(E) = (\frac{n_E}{r_E}, 0) \),
- \( p(A) = (\frac{n_A}{r_A}, \sqrt{d|r_A|}) \) and
- \( q = (\alpha_A, \frac{1}{\sqrt{d|r_A|}}) \).

We remark that if \( \frac{n_E}{r_E} = \frac{n_A}{r_A} \) then \( \mathcal{W}(A, E) \) is a hyperbolic line defined by \( x = \frac{n_E}{r_E} \).

**Lemma 6.2.** Suppose that \( 0 < r_E \) and \( 0 < \frac{n_E}{r_E} - \frac{n_A}{r_A} \). Then there two types of the configuration of the above four points on \( \mathcal{W}(A, E) \):

(I) If \( \frac{1}{d|r_A|} \leq \frac{n_E}{r_E} - \frac{n_A}{r_A} \) then we have \( \alpha_E < \frac{n_A}{r_A} \leq \alpha_A < \frac{n_E}{r_E} \). See also Figure 2 below.

(II) If \( 0 < \frac{n_E}{r_E} - \frac{n_A}{r_A} < \frac{1}{d|r_A|} \) then we have \( \alpha_E < \alpha_A < \frac{n_A}{r_A} < \frac{n_E}{r_E} \). See also Figure 3 below.

**Proof.** Similarly to Lemma 6.1 we could prove the assertion by simple calculations. □
Let \( \Phi : D(Y) \to D(X) \) be an equivalence preserving the distinguished component. Suppose \( E = \Phi(\mathcal{O}_y) \). By Lemma 5.2, \( \pi_S(\Phi_\ast \partial U(Y)) \) is the direct sum of hyperbolic segments \( p(A)p(T_A(E)) \) spanned by two points \( p(A) \) and \( p(T_A(E)) \). Clearly the segment \( p(A)p(T_A(E)) \) is a subset of \( \mathfrak{M}(A, E) \). Following Lemma 6.2 we have the disjoint sum:

\[
\pi_H(\Phi_\ast \partial U(Y)) = \biguplus_{\text{type(I)}} p(A')p(T_{A'}(E)) \sqcup \biguplus_{\text{type(II)}} p(A)p(T_A(E)).
\]

Since the type (II) segments become obstructions when we take the large volume limit in \( V(X) > 2 \). Hence we have to show the boundedness of type (II) segments. To show this, we give an upper bound of the diameter of the type (II) half circle \( W(A, E) \) in the following proposition. Clearly from Lemma 6.1 the diameter is given by \( n_E r_E - \alpha_E \).

**Proposition 6.3.** Suppose that \( r_E > 0 \) and \( 0 < \frac{n_E}{r_E} - \frac{n_A}{r_A} < \frac{1}{\sqrt{d|r_A|}} \). Then we have

\[
0 < \frac{n_E}{r_E} - \alpha_E \leq \frac{1}{r_E} + \frac{r_E}{d}.
\]

**Proof.** By the assumption one easily sees \( r_A \cdot (r_A n_E - r_E n_A) > 0 \). Hence we see

\[
\frac{n_E}{r_E} - \alpha_E = \left( \frac{n_E}{r_E} - \frac{n_A}{r_A} \right) + \frac{1}{d r_A^2 \left( \frac{n_E}{r_E} - \frac{n_A}{r_A} \right)}
\]

\[
= \frac{1}{r_A} \left( \frac{|r_A n_E - r_E n_A|}{r_E} + \frac{r_E}{d |r_A n_E - r_E n_A|} \right)
\]

\[
\leq \frac{|r_A n_E - r_E n_A|}{r_E} + \frac{r_E}{d |r_A n_E - r_E n_A|}.
\]

By the assumption we have

\[
\frac{|r_A n_E - r_E n_A|}{r_E} < \frac{r_E}{d |r_A n_E - r_E n_A|}.
\]

Since the continuous function \( f(t) = \frac{1}{t} + \frac{t}{d} \) on \( \mathbb{R}_{>0} \) is an increasing function for \( \frac{1}{t} < \frac{1}{d} \). Since we have \( \frac{r_E}{|r_A n_E - r_E n_A|} \leq r_E \) the following inequality holds:

\[
(6.3) \leq \frac{1}{r_E} + \frac{r_E}{d}.
\]

Thus we have proved the inequality. \( \square \)

The following corollary is a simple paraphrase of Proposition 6.3. However it is crucial for the proof of our main result, Proposition 6.5.

**Corollary 6.4.** Let \( \Phi : D(Y) \to D(X) \) be an equivalence which preserves the distinguished component. Set \( v(\Phi(\mathcal{O}_y)) = r \oplus n_LX \oplus s \) and \( L_X^2 = 2d \) and assume \( r > 0 \). Then the image \( \pi_S(\Phi_\ast \partial U(Y)) \) is in the following shaded
closed region \( R(Y, \Phi) \) where \( \pi_\beta: \text{Stab}^n(X) \to \mathcal{H}_0(X) \):
\[
R(Y, \Phi) = \{(x_L, y_L) \in \mathcal{H}(X) | \left( x - \frac{n}{r} + \frac{1}{2}\left( \frac{1}{d} + \frac{r}{d} \right) \right)^2 + y^2 \leq \frac{1}{4}\left( \frac{1}{d} + \frac{r}{d} \right)^2, \\
\left( x - \frac{n}{r} - \frac{1}{2}\left( \frac{1}{d} + \frac{r}{d} \right) \right)^2 + y^2 \leq \frac{1}{4}\left( \frac{1}{d} + \frac{r}{d} \right)^2 \text{ or } y^2 \leq \frac{1}{d} \}.
\]

*Figure 4. Figure for the region \( R(Y, \Phi) \).*

**Proof.** As we explained in (6.2), we see that
\[
\pi_\beta(\Phi_\ast \partial U(Y)) = \coprod_{\text{type(I)}} p(\mathcal{P}(\mathcal{A}')) p(\mathcal{T}_\mathcal{A}'(\Phi(\mathcal{O}_y))) \sqcup \coprod_{\text{type(II)}} p(A) p(\mathcal{T}_A(\Phi(\mathcal{O}_y)))
\]
where \( A \) and \( A' \) are spherical object of \( D(X) \). Clearly type (I) hyperbolic segments \( p(\mathcal{P}(\mathcal{A}')) p(\mathcal{T}_\mathcal{A}'(\Phi(\mathcal{O}_y))) \) are in the following region:
\[
\{(x_L, y_L)| y^2 \leq \frac{1}{d} \}.
\]
By Proposition 6.3 the type (II) hyperbolic segments are in the region
\[
\{(x_L, y_L) \in \mathcal{H}(X) | \left( x - \frac{n}{r} + \frac{1}{2}\left( \frac{1}{d} + \frac{r}{d} \right) \right)^2 + y^2 \leq \frac{1}{4}\left( \frac{1}{d} + \frac{r}{d} \right)^2 \text{ or } y^2 \leq \frac{1}{d} \}.
\]
This gives the proof. \( \square \)

### 6.3. Revisit of Orlov’s theorem

We prove the main result of this section.

**Proposition 6.5.** Let \((X, L_X)\) be a projective K3 surface with \( \rho(X) = 1 \) and \((Y, L_Y)\) a Fourier-Mukai partner of \((X, L_X)\). If an equivalence \( \Phi: D(Y) \to D(X) \) preserves the distinguished component, then \( Y \) is isomorphic to the fine moduli space of Gieseker stable torsion free sheaves.
Proof. We first put $L_X^2 = L_Y^2 = 2d$ and $v_0 = v(\Phi(\mathcal{O}_Y)) = r \oplus nL_X \oplus s$. If necessary by taking $T_{\mathcal{O}_X}$ and [1], we may assume $r > 0$. We denote the composition of two morphisms $\text{Stab}^1(X) \to \mathcal{P}_{\mathcal{H}_0}^X(X) \to \mathcal{H}_0(X)$ by $\pi_\mathcal{H}$. By the assumption we have $\Phi_* U(Y) \subset \text{Stab}^1(X)$.

We can take a stability condition $\tau \in U(Y)$ so that $\pi_\mathcal{H}(\Phi_\tau) = (\beta_0, \omega_0) = (aL_X, bL_X)$ with

(i) $a < \frac{n}{r} - \left(\frac{1}{r} + \frac{c}{d}\right)$ and

(ii) $2 < \omega_0^2$.

By the second condition (ii) and Lemma 5.2 we see $\pi_\mathcal{H} \circ \Phi_\tau$ does not lie on $\pi_\mathcal{H}(\partial U(X))$. Hence $\Phi_\tau$ is in a chamber of $\text{Stab}^1(X)$ by Proposition 2.4. Hence we see

$$\exists \Psi \in W(X) \times \mathbb{Z}[2] \text{ such that } (\Psi \circ \Phi_\tau) \in U(Y).$$

Now we put $\Phi' = \Psi \circ \Phi$ and take $\sigma_0 \in V(X)$ as $\sigma(\beta_0, \omega_0)$. Since $\Phi'_\tau$ and $\sigma_0$ belong to the same $\text{GL}^+(2, \mathbb{R})$-orbit, $\sigma_0$ is in $V(X) \cap \Phi'_\tau(U(Y))$. We define a family $\mathcal{F}$ of stability conditions as follows:

$$\mathcal{F} = \{ \sigma(\beta_0, t\omega_0) \in V(X) | 1 < t \in \mathbb{R} \}.$$  

Then we see $\pi_\mathcal{H}(\mathcal{F}) \cap \partial(Y, \Phi') = \emptyset$ by Corollary 6.4. Hence $\mathcal{F}$ does not meet $\Phi'_\tau(U(Y))$. Since $\sigma_0 \in \Phi'_\tau(U(Y))$ we see $\mathcal{F} \subset \Phi'_\tau(U(Y))$ and the object $\Phi'(\mathcal{O}_Y)$ is $\sigma$-stable for all $\sigma \in \mathcal{F}$. By Bridgeland’s large volume limit argument [5] Proposition 14.2 we see that $\Phi'(\mathcal{O}_Y)$ is a Gieseker semistable torsion free sheaf. Moreover by [10] Proposition 3.14 (or the argument of [11] Lemma 4.1) $\Phi'(\mathcal{O}_Y)$ is Gieseker stable. Since $v_0 = v(\Phi'(\mathcal{O}_Y))$ is isotropic and there is $u \in \mathcal{N}(X)$ such that $\langle v_0, u \rangle = 1$, there exists the fine moduli space $\mathcal{M}$ of Gieseker stable sheaves (See also [7] Lemma 10.22 and Proposition 10.20). Hence $Y$ is isomorphic to $\mathcal{M}$. □

Remark 6.6. Clearly the key ingredient of Proposition 6.5 is Corollary 6.4. The role of Corollary 6.4 is to detect the place of the numerical image of walls $\pi_\mathcal{H}(\Phi(\partial U(Y')))$. Without Theorems 3.3 and 5.5, it was difficult to detect the place of $\pi_\mathcal{H}(\Phi(\partial U(Y')))$. By virtue of these theorems, the problem is reduced to the problem with two associated points $p(A)$ and $p(T_A(\Phi(\mathcal{O}_Y)))$.

Remark 6.7. We explain the relation between author’s work and Huybrechts’s question in [8].

In [8] Proposition 4.1, it was proven that all non-trivial Fourier-Mukai partners of projective K3 surfaces are given by the fine moduli spaces of $\mu$-stable locally free sheaves (See also [8] Proposition 4.1]). We note that this proposition holds for all projective K3 surfaces. If the Picard rank is one, the proof of the proposition is based on the lattice argument. In the proof of [8] Proposition 4.1] Huybrechts asks whether there is a geometric proof.

---

\[\text{Since we are assuming } \rho(X) = 1, \text{ the Gieseker stability is equivalent to the twisted stability.}\]
In the previous work [12, Theorem 5.4], the author gave an answer of Huybrechts’s question, that is a geometric proof. However our proof is not completely independent of lattice theories, because it is based on Orlov’s theorem which strongly depends on the global Torelli theorem.

As a consequence of Proposition 6.5 and [12, Theorem 5.4], we could give the another proof of [8, Proposition 4.1] which is completely independent of the global Torelli theorem with assuming the connectedness of Stab($X$).

7. Stable complexes in large volume limits

Let $A$ be a spherical sheaf in $D(X)$. At the end of this paper we discuss the stability of the complex $T_{-1}A(O_X)$ in the large volume limit. More precisely we shall show that $T_{-1}A(O_X)$ is $\sigma_{(\beta, \omega)}$-stable if $\beta \omega < \mu \omega(A)$ and $\omega^2 > 2$. The possibility of stable complexes in the large volume limit is predicted in [3, Lemma 4.2 (c)].

For the vector $v(A) = r_A \oplus n_A L \oplus s_A$ we define the subset $D_A \subset H(X)$ as follows:

\[ D_A = \{(x_L, y_L) \in H(X) | (x - n_A r_A)^2 + (y - \frac{1}{2} \sqrt{d r_A})^2 < \frac{1}{4 d r_A^2}\} \]

Lemma 7.1. Notations being as above. In the domain $D_A$, there are no spherical point $p(\delta)$ with $(-2)$-vectors $\delta$. Moreover $D_A$ does not intersect $\pi H(\partial U(X))$.  

Proof. By the spherical twist $T_A$, we have the diagram:

\[
\begin{array}{ccc}
\text{Stab}^n(X) & \xrightarrow{T_A} & \text{Stab}^n(X) \\
\downarrow & & \downarrow \\
\mathcal{H}_0(X) & \xrightarrow{T_A^0} & \mathcal{H}_0(X).
\end{array}
\]

By Lemma 3.2 $T_A^0$ is given by the linear fractional transformation

\[ T_A^0(x + \sqrt{-1}y) = \frac{1}{d r_A} \cdot \frac{-1}{x + \sqrt{-1}y - \frac{n_A r_A}{r_A}} + \frac{n_A}{r_A} \]

We remark that $T_A^0$ is conjugate to the transformation $z \mapsto -1/d r_A z$.

Now we recall there are no spherical point $p(\delta)$ in the domain $\mathcal{H}(X)_{>2} = \{(\beta, \omega) \in H(X) | \omega^2 > 2\}$. One can easily check that $T_A^0(\mathcal{H}(X)_{>2}) = D_A$. Moreover it is clear that $\pi_{\mathcal{H}}(\partial U(X)) \cap \mathcal{H}(X)_{>2}$. This gives the proof. \Box

Define the subset $D_A^+ \subset V(X)$ by

\[ D_A^+ = \{\sigma(x_L, y_L) \in V(X) | x < \frac{n_A}{r_A}, (x_L, y_L) \in D_A\} \]

In the following proposition, we discuss the stability of sheaves $T_A(O_X)$ in the “small” volume limit $D_A^+$. 

---

We remark that $T_{-1}A(O_X)$ is a 2-terms complex such that $H^0(T_{-1}A(O_X)) = O_X$ and $H^{-1}(T_{-1}A(O_X)) = A^{r_A}$. 
Proposition 7.2. For any $\sigma \in D^+_A$, $T_A(O_x)$ is $\sigma$-stable. In particular $D^+_A \subset T_A U(X) \cap V(X)$.

Proof. To simplify the notation we set $A(x) = T_A(O_x)[-1]$. It is enough to show that $A(x)$ is $\sigma$-stable for all $\sigma \in D^+_A$.

One can see $A(x)$ is the kernel of the evaluation map $\text{Hom}(A, O_x) \otimes A \to O_x$ and is Gieseker stable. We note that there exists $\sigma \in D^+_A$ such that $A(x)$ is $\sigma$-stable by [12, Theorem 4.4 (2)]. In particular we see $D^+_A \cap T_A(U(X)) \neq \emptyset$. Hence it is enough to show $D^+_A \cap T_A(\partial U(X)) = \emptyset$. This is obvious by Lemma 7.1. $\square$

We set $(D^+_A)^{\vee} = \{\sigma_{(x_L,y_L)} \in V(X)| (yL)^2 > 2, x > \frac{n_A}{r_A}\}$.

Corollary 7.3. For any $\sigma \in (D^+_A)^{\vee}$, $T_A^{-1}(O_x)$ is $\sigma$-stable. In particular $(D^+_A)^{\vee} \subset T_A^{-1}(U(X))$.

Proof. Since $D^+_A \subset T_A(U(X)) \cap U(X)$ by Proposition 7.2 we see $T_A^{-1}(D^+_A) \subset U(X) \cap T_A^{-1}(U(X))$.

Since the $\sigma$-stability is equivalent to the $\sigma \cdot \tilde{g}$-stability for any $\tilde{g} \in \widetilde{GL}^+(2,\mathbb{R})$, it is enough to show that $T_A^{-1}(D^+_A)/\widetilde{GL}^+(2,\mathbb{R}) = (D^+_A)^{\vee}/\widetilde{GL}^+(2,\mathbb{R})$. This is obvious from Lemma 3.2. $\square$

Remark 7.4. In the article [3] Lemma 4.2 (c)], the possibility of the stable complexes in large volume limits is referred. Hence Corollary 7.3 gives the proof of this prediction.

Acknowledgement. The author is partially supported by Grant-in-Aid for Scientific Research (S), No 22224001.

References

[1] Arcara, D., Bertram, A. and Lieblich, M., Bridgeland-stable moduli spaces for K-trivial surfaces. preprint, J. of Eur. Math. Soc. 15 (2013), 1-38
[2] Bayer, A., Macrì, E. and Toda, Y., Bridgeland stability conditions on threefolds I: Bogomolov-Gieseker type inequalities. preprint, arXiv:1103.5010.
[3] Bayer, A., Polynomial Bridgeland stability conditions and the large volume limit. Geom. Topol. 13 (2009), 2389–2425.
[4] Bridgeland, T., Stability conditions on triangulated categories. Ann. of Math. 166 (2007), 317–345.
[5] Bridgeland, T., Stability conditions on K3 surfaces. Duke Math. Journal 141 (2008), 241–291.
[6] Hartmann, H., Cusps of the Kähler moduli space and stability conditions on K3 surfaces. Math. Ann. 354 (2012), 1–42.
[7] Huybrechts, D., Fourier-Mukai transformations in Algebraic Geometry. Oxford Mathematical Monographs, 2007.
[8] Huybrechts, D., Derived and abelian equivalence of K3 surfaces. Journal of Algebraic Geometry 17 (2008), 375–400.
[9] Huybrechts, D. and Lehn, M., The geometry of moduli spaces of sheaves. Aspects of Mathematics, 1997.
[10] Huybrechts, D., Macrì, E. and Stellari, P., Derived equivalences of K3 surfaces and orientation. Duke Math. Journal 149 (2009), 461–507.
[11] Kawatani, K., Stability conditions and μ-stable sheaves on K3 surfaces with Picard rank one. Osaka J. Math. 49 (2012), 1005–1034.
[12] Kawatani, K., Stability of Gieseker stable sheaves on K3 surfaces in the sense of Bridgeland and some applications. preprint, arXiv:1103.3921 (2011) to appear in Kyoto J. Math.
[13] Kuleshov, S. A., An existence theorem for exceptional bundles on K3 surfaces. Math. USSR Izv., 34 (1990) 373–388.
[14] Maciocia, A., Computing the walls associated to Bridgeland stability conditions on projective surfaces. preprint, arXiv:1202.4587 (2012).
[15] Macrì, E., Stability conditions on curves. Math. Res. Letters, 14 (2007) 657–672.
[16] Mukai, S., On the moduli spaces of bundles on K3 surfaces, I. In: Vector Bundles on Algebraic Varieties, Oxford Univ. Press, (1987) 341–413.
[17] Okada, S., Stability manifold of P1. Journal of Algebraic Geometry 15 (2006), 487–505.
[18] Orlov, D., Equivalences of derived categories and K3 surfaces. J. Math. Sci. 84 (1997), 1361–1381.
[19] Seidel, P. and Thomas, R., Braid group actions on derived categories of coherent sheaves, Duke Math. Journal, 108 (2001) 37–108.
[20] Woolf, J., Some metric properties of spaces of stability conditions. preprint, arXiv:1108.2668 (2011).