Empirical description of transverse momentum spectra of identified particles produced in proton-proton collisions at $\sqrt{s} = 200$ GeV

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Abstract: Transverse momentum spectra of identified particles produced in high energy proton-proton ($p+p$) collisions are empirically described in the framework of participant quark model or the multisource model at the quark level, in which the source itself is exactly the participant quark. Each participant (constituent) quark contributes to the transverse momentum spectrum, which are described by a revised Tsallis–Pareto-type (TP-like) function. The transverse momentum spectrum of the hadron is the fold of two or more TP-like functions. For a lepton, the transverse momentum spectrum is the fold of two TP-like functions due to two participant quarks, e.g. projectile and target quarks, taking part in the collisions. A discussed theoretical approach seems to describe the $p+p$ collisions data at center-of-mass energy $\sqrt{s} = 200$ GeV very well.

Keywords: Transverse momentum spectra, identified particles, empirical description, TP-like function

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I. INTRODUCTION

As one of the “first day” measurable quantities, the transverse momentum ($p_T$) spectra of various particles produced in high energy proton-proton (hadron-hadron), proton-nucleus (hadron-nucleus), and nucleus-nucleus collisions are of special importance because, it reveals about the temperature and collectivity in the produced systems. The distribution range of $p_T$ is generally very wide, from 0 to more than 100 GeV/c, which is collision energy dependent. In very low-, low-, high-, and very high-$p_T$ regions [1], the shapes of $p_T$ spectrum for given particles are possibly different from each other. In some cases, the differences are very large and the spectra show different empirical laws.

Generally, the spectrum in (very) low-$p_T$ region is contributed by (resonance decays or other) soft excitation process. The spectrum in (very) high-$p_T$ region is naturally contributed by (very) hard scattering process. There is no such boundary in $p_T$ to separate soft and hard processes. At a given collision energy, for different collision species, looking into the spectral shape, a theoretical function that best fits to the $p_T$-spectra is usually chosen to extract information like rapidity density, $dN/dy$, kinetic freeze-out temperature, $T_{\text{kin}}$ or $T_0$ and average radial flow velocity, $\langle \beta_T \rangle$ or $\beta_T$. The low-$p_T$ region up to $\sim 2-3$ GeV/c is well described by a Boltzmann–Gibbs function, whereas the high-$p_T$ part is dominated by a power-law tail. It is interesting to note that there are many different functions, sometimes motivated by experimental trend of the data or sometimes theoretically, to have a proper spectral description thereby leading to a physical picture. The widely used functions are:

1. An exponential function in $p_T$ or $m_T$ [2]:

$$f(p_T) = p_T \times A \times \left( \frac{e^{m_0/T}}{T^2 + Tm_0} \right), \tag{1}$$

$$f(m_T) = m_T \times A \times \left( \frac{e^{m_0/T}}{T^2 + Tm_0} \right). \tag{2}$$

Here, $A$ is the normalization constant, $T$ is the effective temperature (thermal temperature and
collective radial flow) and \( m_T = \sqrt{p_T^2 + m_0^2} \) is the transverse mass, with \( m_0 \) being the identified particle rest mass.

2. A Boltzmann distribution:

\[
f(p_T) = p_T \times A \times m_T \times \left( e^{-m_0/T} \right) \frac{e^{m_0/T}}{2T^3 + 2T^2 m_0 + T m_0^2}.
\]

3. Bose–Einstein/Fermi–Dirac distribution:

\[
f(p_T) = p_T \times A \times m_T \times \frac{1}{e^{m_0/T} + 1} \times \left( e^{m_0/T} + 1 \right),
\]

4. Power-law or Hagedorn function [3]:

\[
f(p_T) = p_T \times A \times \left( 1 + \frac{p_T}{p_0} \right)^{-n} \rightarrow \begin{cases} \exp\left(-\frac{n p_T}{p_0}\right), & \text{for } p_T \rightarrow 0, \\ \left(\frac{p_0}{p_T}\right)^n, & \text{for } p_T \rightarrow \infty, \end{cases}
\]

where \( p_0 \) and \( n \) are fitting parameters. This becomes a purely exponential function for small \( p_T \) and a purely power-law function for large \( p_T \) values.

5. Tsallis–Lévy [4, 5] or Tsallis–Pareto-type function [4, 6]:

\[
f(p_T) = p_T \times A \times \frac{A(n-1)(n-2)}{nT[nT+m_0(n-2)]} \times \left( 1 + \frac{m_T - m_0}{nT} \right)^{-n}.
\]

Note here that a multiplicative pre-factor of \( p_T \) in the above functions are used assuming that the \( p_T \)-spectra do not have a \( p_T \) factor in the denominator (see the expression for the invariant yield) and all the functions are normalized so that the integral of the functions provides the value of “\( A \)”. When the first three functions describe the \( p_T \)-spectra up to a low \( p_T \) around 2–3 GeV/c, the fourth function i.e. the power-law describes the high-\( p_T \) part of the spectrum. The last two functions (power-law or Hagedorn function and Tsallis–Lévy or Tsallis–Pareto-type function), which are more empirical in nature, lack a microscopic picture, however, describe the full spectra. The Tsallis distribution function, while describing the spectra in \( p+p \) collisions, has brought up the concept of non-extensive entropy, contrary to the low-\( p_T \) domain pointing to an equilibrated system usually described by Boltzmann-Gibbs extensive entropy.

The two behaviors in (very) low- and (very) high-\( p_T \) regions are difficult to coordinate simultaneously by a simple probability density function. Instead, one can use a two-component function [7], the first component \( f_1(p_T) \) is for the (very) low-\( p_T \) region and the second component \( f_2(p_T) \) is for the (very) high-\( p_T \) region, to superpose a new function \( f(p_T) \) to fit the \( p_T \) spectra. There are two forms of superpositions, \( f(p_T) = k f_1(p_T) + (1-k) f_2(p_T) \) or \( f(p_T) = A_1 \theta(p_T - p_{\text{low}}) f_1(p_T) + A_2 \theta(p_T - p_{\text{high}}) f_2(p_T) \) [3, 8, 9], where \( k \) denotes the contribution fraction of the first component, \( A_1 \) and \( A_2 \) are constants which make the two components are equal to each other at \( p_T = p_1 \), and \( \theta(x) \) is the usual step function which satisfies \( \theta(x) = 0 \) if \( x < 0 \) and \( \theta(x) = 1 \) if \( x \geq 0 \).

It is known that there are entanglements in determining parameters in the two components in the first superposition [8]. There is possibly a non-smooth interlinkage at \( p_T = p_1 \) between the two components in the second superposition [9]. These two issues are not our expectation. To avoid the entanglements and non-smooth interlinkage, we hope to use a new function to fit simultaneously the spectra in whole \( p_T \) region for various particles. After sounding many functions out, a Tsallis–Pareto-type function [4, 6] which empirically describes both the low-\( p_T \) exponential and the high-\( p_T \) power-law [10–13] is the closest to our target, though the Tsallis–Pareto-type function is needed to revise in some cases.

In this work, to describe the spectra in whole \( p_T \) range which includes (very) low and (very) high \( p_T \) regions, the Tsallis–Pareto-type function is empirically revised by a simple method. To describe the spectra in whole \( p_T \) range as accurately as possible, the contribution of participant quark to the spectrum is also empirically taken to be the revised Tsallis–Pareto-type (TP-like) function with another set of parameters. Then, the \( p_T \) distribution of given particles is the fold of a few TP-like functions. To describe the spectra of identified particles in whole \( p_T \) range, both the TP-like function and the fold of a few TP-like functions are used to fit the data measured in proton-proton (\( p+p \)) collisions at center-of-mass energy \( \sqrt{s} = 200 \text{ GeV} \) by the PHENIX Collaboration [14–18].
The remainder of this paper is structured as follows. The formalism and method are described in Section 2. Results and discussion are given in Section 3. In Section 4, we summarize our main observations and conclusions.

II. FORMALISM AND METHOD

According to refs. [4, 6], the Tsallis–Pareto-type function which empirically describes both the low-\( p_T \) exponential and the high-\( p_T \) power-law can be simplified presented as [10–13],

\[
 f(p_T) = C \times p_T^{a_0} \times \left( 1 + \frac{\sqrt{p_T^2 + m_0^2} - m_0}{nT} \right)^{-n} \tag{7}
\]

in terms of \( p_T \) probability density function, where the parameter \( T \) describes the excitation degree of the considered source, the parameter \( n \) describes the degree of non-equilibrium of the considered source, and \( C \) is the normalization constant which depends on \( T, n, \) and \( m_0 \). Equation (7) is in fact a rewrite of Eq. (6).

As an empirical formula, the Tsallis–Pareto-type function is successful in the description of \( p_T \) spectra in many cases. However, our exploratory analysis shows that Eq. (7) is not accurate in describing the spectra in whole \( p_T \) range in some cases. In particular, Eq. (7) cannot describe flexibility the spectra in very low-\( p_T \) region, which is contributed by the resonance decays. We would like to revise empirically Eq. (7) by adding a power index \( a_0 \) on \( p_T \). After the revision, we have

\[
 f(p_T) = C \times p_T^{a_0} \times \left( 1 + \frac{\sqrt{p_T^2 + m_0^2} - m_0}{nT} \right)^{-n} \tag{8}
\]

where \( C \) is the normalization constant which is different from that in Eq. (7). To be convenient, the two normalization constants in Eqs. (7) and (8) are denoted by the same symbol \( C \). Eq. (8) can be used to fit the spectra in whole \( p_T \) range. The revised Tsallis–Pareto-type function [Eq. (8)] is called the TP-like function by us.

Our exploratory analysis shows that Eq. (8) is not accurate in describing the spectra in whole \( p_T \) range, too, though it is more accurate than Eq. (7). To obtain accurate results, the contribution \( (p_{ti}) \) of the \( i \)-th participant quark to \( p_T \) is assumed to obey

\[
 f_i(p_{ti}) = C_i \times p_{ti}^{a_{i0}} \times \left( 1 + \frac{\sqrt{p_{ti}^2 + m_{0i}^2} - m_0}{nT} \right)^{-n} \tag{9}
\]

where the subscript \( i \) is used for the quantities related to the participant quark \( i \), and \( m_{0i} \) is empirically the constituent mass of the considered quark \( i \). The value of \( i \) can be 2 or 3 even 4 or 5 due to the number of participant (or constituent) quarks. Eq. (9) is also the TP-like function with different mass from Eq. (8).

It should be noted that \( m_0 \) in Eq. (8) is for a particle, and \( m_{0i} \) in Eq. (9) is for the quark \( i \). For example, if we study the \( p_T \) spectrum of protons, we have \( m_0 = 0.938 \) GeV/\( c^2 \) and \( m_{02} = m_{03} = 0.31 \) GeV/\( c^2 \). In the case of studying the \( p_T \) spectrum of photons, we have \( m_0 = 0 \) and \( m_{01} = m_{02} = 0.31 \) GeV/\( c^2 \) if we assume that two lightest quarks taking part in the collision on photon production.

There are two participant quarks to constitute usually mesons, namely the quarks 1 and 2. The \( p_T \) spectra of mesons are the fold of two TP-like functions. We have

\[
 f(p_T) = \int_0^{p_T} f_1(p_{t1}) f_2(p_T - p_{t1}) dp_{t1} = \int_0^{p_T} f_2(p_{t2}) f_1(p_T - p_{t2}) dp_{t2}. \tag{10}
\]

At the level of current knowledge, leptons have no further structures. However, to produce a lepton in a common process, two participant quarks, a projectile quark and a target quark, are assumed to take part in the interactions. The \( p_T \) spectra of leptons are in fact the fold of two TP-like functions, that is Eq. (10) in which \( m_{01} \) and \( m_{02} \) are empirically the constituent mass of the lightest quark. To produce leptons in a special process such as \( c \bar{c} \rightarrow \mu^+ \mu^- \), \( m_{01} \) and \( m_{02} \) are the constituent mass of \( c \) quark.

There are three participant quarks to constitute usually baryons, namely the quarks 1, 2 and 3. The \( p_T \) spectra of baryons are the fold of three TP-like functions. We have the fold of the first two TP-like functions to be

\[
 f_{12}(p_{t12}) = \int_0^{p_{t12}} f_1(p_{t1}) f_2(p_{t12} - p_{t1}) dp_{t1} = \int_0^{p_{t12}} f_2(p_{t2}) f_1(p_{t12} - p_{t2}) dp_{t2}. \tag{11}
\]

The fold of the first two TP-like functions and the third
TP-like function is

\[ f(p_T) = \int_0^{p_T} f_{12}(p_{t12}) f_3(p_T - p_{t12}) dp_{t12} = \int_0^{p_T} f_3(p_3) f_{12}(p_T - p_3) dp_3. \] (12)

Equation (8) can fit approximately the spectra in whole \( p_T \) range for various particles at the particle level, in which \( m_0 \) is the rest mass of the considered particle. In principle, Eqs. (10) and (12) can fit the spectra in whole \( p_T \) range for various particles at the quark level, in which \( m_{0i} \) is the constituent mass of the quark \( i \). If Eq. (8) is a revision of Eq. (7), Eqs. (10) and (12) are the results of the multiset model [19, 20] at the quark level. In the multiset model, one, two, or more sources are assumed to emit particles due to different production mechanisms, source temperatures and event samples. In a given event sample, the particles with the same source temperature are assumed to emit from the same source by the same production mechanism. We can also call Eqs. (10) and (12) the results of participant quark model due to they being the contributions of participant quarks.

We would like to explain the normalization constant in detail. As a probability density function, \( f(p_T) \) is not compared directly with the experimental data presented in literature in some cases, where \( N \) denotes the number of considered particles. Generally, the experimental data are presented in forms of i) \( dN/dp_T \), ii) \( d^2N/dydp_T \), and iii) \( (1/2\pi p_T)d^2N/dydp_T = Ed^2N/dp^3 \), where \( E \) and \( p \) denote the energy and momentum of the considered particle, respectively. One can use \( N_0 f(p_T) = dN_0 f(p_T)/dy \), and \( (1/2\pi p_T)N_0 f(p_T)/dy / \) to fit them accordingly, where \( N_0 \) denotes the normalization constant.

Let \( \sigma \) denote the cross-section, the forms of data have usually i) \( d\sigma/dp_T \), ii) \( d^2\sigma/dydp_T \), and iii) \( (1/2\pi p_T)d^2\sigma/dydp_T = Ed^2\sigma/dp^3 \). One can use \( \sigma_0 f(p_T) = \sigma_0 (p_T)/dy \), and \( (1/2\pi p_T)\sigma_0 f(p_T)/dy \) to fit them accordingly, where \( \sigma_0 \) denotes the normalization constant. The data presented in terms of \( m_T \) can also be studied due to the conserved probability density and the relation between \( m_T \) and \( p_T \). In particular, \( (1/2\pi p_T)d^2\sigma/dydp_T = (1/2\pi m_T)d^2\sigma/dydm_T \), where \( \sigma \) can be replaced by \( N \).

### III. RESULTS AND DISCUSSION

Figure 1(a) shows the \( p_T \) spectra (the invariant cross-section), \( Ed^2\sigma/dp^3 \), of some hadrons with given combinations and decay channels including \( (\pi^+ + \pi^-)/2 \) plus \( \pi^0 \rightarrow \gamma \gamma \), \( (K^+ + K^-)/2 \) plus \( K_S^0 \rightarrow \pi^0 \pi^0 \), \( \eta \rightarrow \gamma \gamma \) plus \( \eta \rightarrow \pi^0 \pi^+ \pi^- \), \( \omega \rightarrow e^+e^- \) plus \( \omega \rightarrow \pi^0 \pi^+ \pi^- \) plus \( \omega \rightarrow \pi^0 \gamma \), \( (p + \bar{p})/2 \), \( \eta' \rightarrow \eta \pi^+ \pi^- \), \( \phi \rightarrow e^+e^- \) plus \( \phi \rightarrow K^+K^- \), \( J/\psi \rightarrow e^+e^- \), and \( \psi' \rightarrow e^+e^- \) produced in \( p + p \) collisions at 200 GeV. Different symbols represent different particles and their different decay channels measured by the PHENIX Collaboration [14]. The results corresponding to \( \pi, K, \eta, \omega, p, \) and \( \eta' \) are scaled by multiplying 10, 10^2, 10^3, 10^4, 10^5, and 10, respectively. The results corresponding to \( \phi, J/\psi, \) and \( \psi' \) are not re-scaled.

In Fig. 1(a), the dotted and dashed curves are our fitted results by using Eqs. (8) (for mesons and baryons) and (10) (for mesons) or (12) (for baryons) respectively. The values of free parameters \( (T, n, a_0) \), normalization constant \( (\sigma_0) \), \( \chi^2 \), and degree of freedom (dof) obtained from Eq. (8) are listed in Table I, while the values of parameters and \( \chi^2/dof \) obtained from Eqs. (10) or (12) are listed in Table II. In Eq. (8), \( m_0 \) is taken to be order the rest mass of \( \pi, K, \eta, \omega, p, \eta', \phi, J/\psi, \) and \( \psi' \) for the cases from \( (\pi^+ + \pi^-)/2 \) to \( \psi' \rightarrow e^+e^- \) sequenced due to the order shown in Fig. 1(a). In the fit process at the quark level, the quark structure of \( \pi^0 \) results in its \( f(p_T) \) to be the half of the sum of \( u\bar{u} \)’s \( f(p_T) \) and \( d\bar{d} \)’s \( f(p_T) \). Because the constituent masses of \( u \) and \( d \) are the same [21], \( \pi^0 \)’s \( f(p_T) \) is equal to \( u\bar{u} \)’s \( f(p_T) \) or \( d\bar{d} \)’s \( f(p_T) \). The quark structure of \( \eta \) results in its \( f(p_T) \) to be \( \cos^2 \phi \times u\bar{u} \)’s \( f(p_T) \) plus \( \sin^2 \phi \times s\bar{s} \)’s \( f(p_T) \) due to the quark structures of \( \eta_q \) and \( \eta_s \), where \( \phi = 39.3^\circ \pm 1.0^\circ \) is the mixing angle [22]. The quark structure of \( \eta' \) results in its \( f(p_T) \) to be \( \sin^2 \phi \times u\bar{u} \)’s \( f(p_T) \) plus \( \cos^2 \phi \times s\bar{s} \)’s \( f(p_T) \).

To show the departures of the fit from the data, following Fig. 1(a), Figs. 1(b) and 1(c) show the ratios of data to fit obtained from Eqs. (8) and (10) or (12) respectively. One can see that the fits are around the data in whole \( p_T \) range, except for a few large departures. The experimental data on the mentioned hadrons measured in \( p + p \) collisions at 200 GeV by the PHENIX Collaboration [14] can be fitted by Eqs. (8) (for mesons and baryons) and (10) (for mesons) or (12) (for baryons).

From the values of \( \chi^2 \) and data over fit, one can see that Eq. (10) or (12) seems to be better than Eq. (8).
FIG. 1: (a) The invariant cross-section of some hadrons with given combinations and decay channels produced in p + p collisions at 200 GeV. Different symbols represent different particles and their different decay channels measured by the PHENIX Collaboration [14], some of them are scaled by different amounts marked in the panel. The dotted and dashed curves are our fitted results by using Eqs. (8) and (10) or (12) respectively. (b) The ratio of data to fit obtained from Eq. (8). (c) The ratio of data to fit obtained from Eq. (10) or (12).
FIG. 2: (a) The invariant cross-section of photons and some leptons with given combinations and producing channels produced in $p+p$ collisions at 200 GeV. Different symbols represent different particles and their producing channels measured by the PHENIX Collaboration [15–18]. The dotted and dashed curves are our fitted results by using Eqs. (8) and (10) respectively. (b) The ratio of data to fit obtained from Eq. (8). (c) The ratio of data to fit obtained from Eq. (10).
TABLE I: Values of $T$, $n$, $a_0$, $\sigma_0$, $\chi^2$, and dof corresponding to the dotted curves in Fig. 1, which are fitted by the TP-like function [Eq. (8)]. The last dof is 0 which appears as “–” in the table.

| Particle | $T$ (GeV) | $n$ | $a_0$ | $\sigma_0$ (mb) | $\chi^2$/dof |
|----------|-----------|-----|-------|-----------------|-------------|
| $(\pi^+ + \pi^-)/2$ | 0.128 ± 0.002 | 9.409 ± 0.200 | 0.890 ± 0.030 | 37.042 ± 1.307 | 5/39 |
| $\pi^0$ | 0.177 ± 0.002 | 9.500 ± 0.030 | 0.887 ± 0.010 | 3.197 ± 0.044 | 7/27 |
| $(K^+ + K^-)/2$ | | | | | |
| $K_S^0$ | | | | | |
| $\eta$ | 0.195 ± 0.002 | 9.889 ± 0.040 | 1.000 ± 0.010 | 1.755 ± 0.088 | 6/32 |
| $\omega$ | 0.193 ± 0.001 | 9.460 ± 0.100 | 0.900 ± 0.020 | 3.073 ± 0.065 | 23/34 |
| $(p + \bar{p})/2$ | 0.149 ± 0.002 | 9.100 ± 0.700 | 1.040 ± 0.030 | 1.291 ± 0.044 | 11/13 |
| $\eta'$ | 0.210 ± 0.002 | 10.001 ± 0.245 | 0.980 ± 0.002 | 0.584 ± 0.018 | 4/8 |
| $\phi$ | 0.204 ± 0.003 | 9.424 ± 0.185 | 1.000 ± 0.035 | 0.307 ± 0.013 | 11/15 |
| $J/\psi$ | 0.416 ± 0.008 | 11.004 ± 0.450 | 0.997 ± 0.015 | $(5.365 ± 0.132) \times 10^{-4}$ | 4/22 |
| $\psi'$ | 0.452 ± 0.003 | 8.349 ± 0.052 | 0.959 ± 0.010 | $(9.234 ± 0.044) \times 10^{-5}$ | 1/– |

Figure 2(a) shows the invariant cross-section of photons and some leptons with given combinations and producing channels including $(e^+ + e^-)/2$, $(\mu^+ + \mu^-)/2$ (open heavy-flavor decays), Drell–Yan $\rightarrow \mu^+\mu^-$, $c\bar{c} \rightarrow \mu^+\mu^-$, and $b\bar{b} \rightarrow \mu^+\mu^-$ produced in $p+p$ collisions at 200 GeV. Different symbols represent different particles and their producing channels measured by the PHENIX Collaboration [15–18]. The dotted and dashed curves are our fitted results by using Eqs. (8) and (10) respectively, where two participant quarks are considered in the formation of mentioned particles. The values of parameters and $\chi^2$/dof obtained from Eqs. (8) and (10) are listed in Tables III and IV respectively. In Eq. (8), $m_0$ is taken to be the rest mass of $\gamma$, $e$, $\mu$, $2\mu_1$, $2\mu_2$, and $4\mu$ for the cases from $\gamma$ to $b\bar{b} \rightarrow \mu^+\mu^-$ sequenced due to the order shown in Fig. 2(a). In Eq. (10), $m_{01} + m_{02}$ are taken to be the constituent masses of $u + u$, $u + u$, $u + c$, $u + u$, $c + c$, and $b + b$ sequenced due to the same order as particles.

Following Fig. 2(a), Figs. 2(b) and 2(c) show the ratios of data to fit obtained from Eqs. (8) and (10) respectively. One can see that the fits are around the data in whole $p_T$ range, except for a few large departures. The experimental data on the mentioned photons and leptons measured in $p+p$ collisions at 200 GeV by the PHENIX Collaboration [15–18] can be fitted by Eqs. (8) and (10). From the values of $\chi^2$ and data over fit, one can see that Eq. (10) seems to be better than Eq. (8).

The values of $a_0$ in Table 1 show that maybe Eq. (8) is not necessary due to $a_0 \approx 1$. However, the values of $a_0$ in Table 3 show that Eq. (8) is indeed necessary due to $a_0 \neq 1$. In general, Eq. (8) is necessary in the data-driven analysis due to the fact that $a_0 \neq 1$ in some cases.

To see the dependences of the spectra on free parameters, Figure 3 presents variant pion spectra with different parameters in Eqs. (8) and (10). From the upper panel [Figs. 3(a), 3(b), and 3(c)] to middle panel [Figs. 3(d), 3(e), and 3(f)] then to lower panel [Figs. 3(g), 3(h), and 3(i)], $T$ changes from 0.1 GeV to 0.15 GeV then to 0.2 GeV. From the left panel to middle panel then to right panel, $n$ changes from 5 to 10 then to 15. In each panel, the solid, dotted, dashed, and dot-dashed curves without (with) open circles correspond to the spectra with $a_0 = -1$, 0, 1, and 2, respectively, from Eq. (8) [Eq. (10)]. One can see that the probability in high $p_T$ region increases with the increase of $T$, decreases with the increase of $n$, and increase with the increase of $a_0$. From negative to positive, $a_0$ determines obviously the shape in low $p_T$ region.

From the shapes of curves in Fig. 3, one can see that the parameter $a_0$ introduced in the TP-like function [Eq. (8)] by us determines mainly the trend of curve in low-$p_T$ region. If the contribution of resonance decays affect obviously the shape of spectrum in low-$p_T$ region, one may use a more negative $a_0$ in the fit process. Due to the introduction of $a_0$, the TP-like function is more flexible than the Tsallis-Pareto-type function. In fact, $a_0$ is a sensitive quantity to describe the contribution of resonance decays.
TABLE II: Values of $T$, $n$, $a_0$, $\sigma_0$, $\chi^2$, and dof corresponding to the dashed curves in Fig. 1, which are fitted by the fold [Eq. (10) or (12)] of two or three TP-like functions. The last dof is 0 which appears as “-” in the table.

| Particle | Quark structure | $T$ (GeV) | $n$ | $a_0$ | $\sigma_0$ (mb) | $\chi^2$/dof |
|----------|----------------|-----------|-----|-------|----------------|-------------|
| $(\pi^+ + \pi^-)/2$ | $ud$, $d\bar{u}$ | $0.205 \pm 0.004$ | 7.629 $\pm 0.025$ | $-0.540 \pm 0.020$ | 37.600 $\pm 4.011$ | 5/39 |
| $\pi_0$ | $(u\bar{d} - d\bar{u})/\sqrt{2}$ | 0.196 $\pm 0.001$ | 7.816 $\pm 0.030$ | $-0.091 \pm 0.007$ | 2.913 $\pm 0.044$ | 4/27 |
| $(K^+ + K^-)/2$ | $u\bar{s}$, $s\bar{u}$ | 0.212 $\pm 0.001$ | 8.109 $\pm 0.030$ | 0.000 $\pm 0.011$ | 1.838 $\pm 0.066$ | 4/32 |
| $K_S^0$ | $d\bar{s}$ | 0.222 $\pm 0.001$ | 8.394 $\pm 0.089$ | 0.000 $\pm 0.010$ | 2.787 $\pm 0.124$ | 21/34 |
| $\eta$: $\eta_0$, $\eta_8$ | $(u\bar{d} + d\bar{u})/\sqrt{2}$, $s\bar{s}$ | 0.233 $\pm 0.002$ | 8.315 $\pm 0.405$ | 0.000 $\pm 0.010$ | 0.593 $\pm 0.022$ | 3/8 |
| $\omega$ | $(u\bar{u} + d\bar{d})/\sqrt{2}$ | 0.236 $\pm 0.003$ | 8.232 $\pm 0.200$ | 0.000 $\pm 0.020$ | 0.307 $\pm 0.018$ | 10/15 |
| $(p + \bar{p})/2$ | $u\bar{u}$, $d\bar{d}$ | 0.162 $\pm 0.002$ | 7.600 $\pm 0.080$ | $-0.130 \pm 0.030$ | 1.288 $\pm 0.029$ | 3/13 |
| $\gamma$: $\eta_0$, $\eta_8$ | $(u\bar{d} + d\bar{u})/\sqrt{2}$, $s\bar{s}$ | 0.233 $\pm 0.002$ | 8.315 $\pm 0.405$ | 0.000 $\pm 0.010$ | 0.593 $\pm 0.022$ | 3/8 |
| $J/\psi$ | $c\bar{c}$ | 0.439 $\pm 0.008$ | 8.545 $\pm 0.035$ | 0.000 $\pm 0.015$ | $(5.275 \pm 0.132) \times 10^{-4}$ | 4/22 |
| $\psi'$ | $c\bar{c}$ | 0.503 $\pm 0.005$ | 7.025 $\pm 0.030$ | 0.055 $\pm 0.004$ | $(9.232 \pm 0.044) \times 10^{-5}$ | 1/− |

TABLE III: Values of $T$, $n$, $a_0$, $\sigma_0$, $\chi^2$, and dof corresponding to the dotted curves in Fig. 2, which are fitted by the TP-like function [Eq. (8)].

| Particle | $T$ (GeV) | $n$ | $a_0$ | $\sigma_0$ (mb) | $\chi^2$/dof |
|----------|-----------|-----|-------|----------------|-------------|
| $\gamma$ | 0.258 $\pm 0.001$ | 9.413 $\pm 0.020$ | 1.750 $\pm 0.010$ | 4.836 $\pm 0.044$ | 2/14 |
| $(e^+ + e^-)/2$ | 0.203 $\pm 0.002$ | 7.840 $\pm 0.040$ | 0.002 $\pm 0.011$ | 14.114 $\pm 0.371$ | 17/24 |
| $(\mu^+ + \mu^-)/2$ | 0.125 $\pm 0.001$ | 9.208 $\pm 0.050$ | 0.799 $\pm 0.003$ | 13.486 $\pm 0.131$ | 5/9 |
| (open heavy-flavor decays) | | | | | |
| Drell–Yan $\rightarrow \mu^+\mu^-$ | 0.349 $\pm 0.003$ | 8.849 $\pm 0.100$ | 2.200 $\pm 0.020$ | $(1.556 \pm 0.125) \times 10^{-4}$ | 8/8 |
| $c\bar{c} \rightarrow \mu^+\mu^-$ | 0.385 $\pm 0.005$ | 17.200 $\pm 1.000$ | 1.590 $\pm 0.008$ | $(4.197 \pm 0.251) \times 10^{-6}$ | 10/11 |
| $b\bar{b} \rightarrow \mu^+\mu^\pm$ | 0.489 $\pm 0.004$ | 30.211 $\pm 1.000$ | 2.113 $\pm 0.030$ | $(7.162 \pm 0.314) \times 10^{-9}$ | 8/6 |

IV. SUMMARY AND CONCLUSIONS

We summarize here our main observations and conclusions.

(a) The transverse momentum spectra in terms of the invariant cross-section of various particles (some hadrons with given combinations and decay channels, photons, and some leptons with given combinations and producing channels) produced in high energy proton-proton collisions have been studied by a TP-like function (a revised Tsallis–Pareto-type function). Meanwhile, the transverse momentum spectra have also been studied in the framework of participant quark model or the multisource model at the quark level. In the model, the source itself is exactly the participant quark. Each participant quark contributes the transverse momentum spectrum to be the TP-like function.

(b) For a hadron, the participant quarks are in fact its constituent quarks. The transverse momentum spectrum of the hadron is the fold of two or more TP-like functions. For a lepton, the transverse momentum spectrum is the fold of two TP-like functions due to two participant quarks, e.g. projectile and target quarks, taking part in the collisions. The TP-like function and the fold of a few TP-like functions can fit the experimental data of various particles produced in proton-proton collisions at 200 GeV measured by the PHENIX Collaboration.

(c) In the TP-like function and the fold of a few TP-like functions, the main parameters $T$, $n$, and $a_0$ are sensitive to the spectra. In variant pion spectra from the TP-like function and the fold of two TP-like functions, the probability in high transverse momentum region increases with the increase of $T$, decreases with the increase of $n$, and increase with the increase of $a_0$. From
negative to positive, $a_0$ determines obviously the shape in low transverse momentum region, which is sensitive in describing the contribution of resonance decays.

**Data Availability**

The data used to support the findings of this study are included within the article and are cited at relevant places within the text as references.

**Compliance with Ethical Standards**

The authors declare that they are in compliance with ethical standards regarding the content of this paper.

**Conflicts of Interest**

The authors declare that there are no conflicts of interest regarding the publication of this paper. The funders had no role in the design of the study; in the collection, analyses, or interpretation of the data; in the writing of the manuscript, or in the decision to publish the results.

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**TABLE IV:** Values of $T$, $n$, $a_0$, $\sigma_0$, $\chi^2$, and dof corresponding to the dashed curves in Fig. 2, which are fitted by the fold [Eq. (10)] of two TP-like functions.

| Particle | Quark-like | $T$ (GeV) | $n$ | $a_0$ | $\sigma_0$ ($\mu$b) | $\chi^2$/dof |
|----------|------------|-----------|-----|-------|---------------------|--------------|
| $\gamma$ | uu         | 0.383 ± 0.001 | 6.793 ± 0.040 | 0.060 ± 0.002 | 4.967 ± 0.044 | 2/14         |
| $(e^+ + e^-)/2$ | uu | 0.255 ± 0.002 | 6.378 ± 0.030 | $-0.696$ ± 0.003 | 13.946 ± 0.176 | 5/24         |
| $(\mu^+ + \mu^-)/2$ | uc | 0.167 ± 0.001 | 5.935 ± 0.055 | $-0.802$ ± 0.003 | 12.439 ± 0.270 | 4/9          |
| (open heavy-flavor decays) | | | | | | |
| Drell–Yan $\rightarrow \mu^+ \mu^-$ | uū | 0.418 ± 0.010 | 5.616 ± 0.040 | 0.398 ± 0.008 | $(1.560 ± 0.125) \times 10^{-4}$ | 8/8          |
| $c\bar{c} \rightarrow \mu^+ \mu^-$ | c\bar{c} | 0.221 ± 0.005 | 6.402 ± 0.500 | 0.066 ± 0.010 | $(4.051 ± 0.188) \times 10^{-6}$ | 7/11         |
| $\bar{b}b \rightarrow \mu^\pm \mu^\mp$ | \bar{b}b | 0.235 ± 0.007 | 11.253 ± 2.000 | 0.049 ± 0.020 | $(7.162 ± 0.251) \times 10^{-9}$ | 3/6          |

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FIG. 3: Variant pion spectra with different parameters in Eqs. (8) and (10). From the upper panel [Figs. 3(a), 3(b), and 3(c)] to middle panel [Figs. 3(d), 3(e), and 3(f)] then to lower panel [Figs. 3(g), 3(h), and 3(i)], $T$ changes from 0.1 GeV to 0.15 GeV then to 0.2 GeV. From the left panel to middle panel then to right panel, $n$ changes from 5 to 10 then to 15. In each panel, the solid, dotted, dashed, and dot-dashed curves without (with) open circles are obtained by $a_0 = -0.1, 0, 1,$ and 2, respectively, from Eq. (8) [Eq. (10)].

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