MSO Undecidability for some Hereditary Classes of Unbounded Clique-Width

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Abstract

Seese's conjecture for finite graphs states that monadic second-order logic (MSO) is undecidable on all graph classes of unbounded clique-width. We show that to establish this it would suffice to show that grids of unbounded size can be interpreted in two families of graph classes: minimal hereditary classes of unbounded clique-width; and antichains of unbounded clique-width under the induced subgraph relation. We explore a number of known examples of the former category and establish that grids of unbounded size can indeed be interpreted in them.

1. Introduction

The monadic second-order logic (MSO) of graphs has been an object of intensive research for many years now. It is a logic that is highly expressive and yet very well behaved on many interesting classes of graphs. It has enabled the extension of many automata-theoretic and algebraic techniques to the construction of algorithms on graphs. It has become a reference logic against which many others are compared. A key area of investigation is determining on which classes of graphs is MSO algorithmically well-behaved.

The good algorithmic behaviour of MSO on a class $C$ of graphs is usually taken to mean one of two things: the evaluation (or model-checking) problem for MSO sentences on $C$ is tractable; or the satisfiability problem of MSO sentences on $C$ is decidable. Usually, these two are linked. Broadly speaking, the only way we know to show that the MSO theory of a class $C$ is decidable is to show that $C$ can be obtained by means of an MSO transduction from a class of trees which itself has a decidable theory and this also yields efficient evaluation algorithms for MSO sentences on $C$. And, the only way we know to show that the MSO theory of $C$ is undecidable is to show that there is an MSO transduction that yields arbitrarily large grids on $C$ and this also yields an obstacle to the tractability of MSO evaluation on $C$.

Seese [18] formalizes the first of these observations into a conjecture: if the MSO theory of a class $C$ is decidable, there is an MSO transduction $\Psi$ and a class $T$ of trees such that $\Psi$ maps $T$ to $C$. This remains an open question nearly three decades after it was first posed despite considerable research effort around it. By a theorem of Courcelle and Engelfriet [5],
it is known that the classes of graphs obtained by MSO transductions from trees are exactly those of bounded clique-width. Thus, Seese’s conjecture can be understood as saying that any class of graphs of unbounded clique-width has an undecidable MSO theory. If we similarly formalize the second observation above about grids and combine it with this, we can formulate the following stronger conjecture: every class \( \mathcal{C} \) of graphs of unbounded clique-width admits an MSO transduction that defines arbitrarily large grids. Seese’s conjecture is often formulated in this stronger form as it seems the only reasonable route to proving it. This can be seen as an interesting analogue of the Robertson-Seymour grid minor theorem to the effect that any class of graphs of unbounded treewidth admits arbitrarily large grids as minors.

In recent years there has been growing interest in clique-width as a measure of the complexity of graphs from a structural and algorithmic point of view, quite separate from questions of logic \([8, 3, 16, 9]\). In particular, it provides a route for extending algorithmic methods that have had great success on sparse graph classes \([15]\) to more general classes of graphs. A class of graphs may be of bounded clique-width while containing dense graphs—the classic example being the class of cliques.

In the context of the structural study of classes of bounded clique-width, there is particular interest in hereditary classes, that is classes of graphs closed under the operation of taking induced subgraphs. This is because the induced subgraph relation behaves well with respect to clique-width. If a graph \( H \) is a subgraph or a minor of a graph \( G \), the clique-width of \( H \) can be greater than that of \( G \) but if \( H \) is an induced subgraph of \( G \), then the clique-width of \( H \) is no more than that of \( G \). Hence, the hereditary closure of a class \( \mathcal{C} \) of bounded clique-width still has bounded clique-width.

The induced subgraph relation is not as well-behaved as the graph minor relation. By the Robertson-Seymour graph minor theorem \([17]\), the graph minor relation is a well-quasi-order. This is not true of the induced subgraph relation. It is also possible to construct infinite descending chains, under inclusion, of classes of graphs, each of unbounded clique-width. Indeed, Lozin \([13]\) identified the first example of a hereditary class \( \mathcal{C} \) of graphs of unbounded clique-width that are minimal with this property—that is, no hereditary proper subclass of \( \mathcal{C} \) has unbounded clique-width. Since then, many other such classes have been constructed. Collins et al. \([2]\) show how to obtain an infinite family of such classes. Atminas et al. \([1]\) construct examples of such classes which are characterized by a finite collection of forbidden induced subgraphs. Lozin et al. \([14]\) construct an example of a minimal hereditary class of unbounded clique-width that is well-quasi-ordered under the induced substructure relation.

This exploration of novel examples of classes of unbounded clique-width also suggests an approach to establishing Seese’s conjecture for finite graphs. We establish in Section 3 that Seese’s conjecture follows from the conjunction of the following two statements: (1) every collection of graphs of unbounded clique-width that forms an infinite anti-chain under the induced subgraph relation interprets arbitrarily large grids; and (2) every minimal hereditary class of unbounded clique-width interprets arbitrarily large grids. This suggests a programme to establish Seese’s conjecture by systematically studying antichains and minimal hereditary classes of unbounded clique-width. While we do not yet know of a complete classification of minimal hereditary classes of unbounded clique-width, we make progress in this programme by considering all known classes as of now and showing that in all cases we can indeed interpret grids of unbounded size. We systematically investigate
these classes in Sections 4 - 6.

It is worth mentioning some significant lines of investigation related to Seese’s conjecture. Courcelle [4] shows that proving Seese’s conjecture for finite graphs is equivalent to proving the relativized version of the conjecture for particular classes of graphs, two examples being bipartite graphs and split graphs. He further shows the conjecture to be true when relativized to uniformly k-sparse graphs and interval graphs. Another line of work addresses variants of Seese’s conjecture obtained by considering logics other than MSO. One such result by Seese [18] shows that guarded second-order logic (GSO) is undecidable on classes of unbounded clique-width. Similarly, Courcelle and Oum [7] show that the extension C2MSO of MSO obtained by considering modulo 2 counting quantifiers is also undecidable on classes of unbounded clique-width. In all of these cases, the proof goes via interpreting grids in unbounded clique-width classes. There has also been interesting progress looking at Seese’s conjecture for structures other than graphs. A significant paper here is by Hliněný and Seese [10] who show the conjecture to be true for matroids representable over any finite field.

2. Preliminaries

For a simple, undirected loop-free graph $G$, we write $V(G)$ for the vertices of $G$ and $E(G)$ for the edges. We consider monadic second-order logic (MSO) over vocabularies $\tau$ containing the binary relation $E$ and finitely many unary relation symbols. An MSO formula over the vocabulary $\tau$ is an expression that is inductively constructed from atomic MSO formulae using the Boolean connectives $\land, \lor$, and $\neg$, and existential quantification over vertex variables and set variables. Here an atomic MSO formula is an expression of the form $E(x,y)$ or $Q(x)$ or $X(y)$ or $x=y$ where $x,y$ are vertex variables, the predicates $E,Q$ belong to $\tau$ and $X$ is a set variable. A first order, or FO, formula is an MSO formula that does not contain any set variable. We often write $\varphi(\bar{x},\bar{X})$ to denote a formula whose free variables are among $\bar{x}$ and $\bar{X}$, the former being a tuple of vertex variables and the latter a tuple of set variables.

A $\tau$-labeled graph is a $\tau$-structure that interprets $E$ as an irreflexive and symmetric binary relation. Given a $\tau$-labeled graph $G$ and an MSO formula $\varphi(\bar{x},\bar{X})$ where the length of $\bar{x}$ is $k$, we can think of $\varphi$ as defining a $k$-ary relation on an expansion of $G$ with an interpretation $\bar{A}$ of $\bar{X}$. Specifically this relation, denoted $\varphi(G,\bar{A})$, is given by $\varphi(G,\bar{A}) = \{\bar{a} \mid G \models \varphi(\bar{a},\bar{A})\}$ where $\models$ denotes the “models” relation [12 Chap. 7]. For vocabularies $\tau$ and $\sigma$ of labeled graphs and a sequence $\bar{Z}$ of set variables, an MSO $\tau$-$\sigma$ interpretation with parameters $\bar{Z}$ is a sequence $\Psi(\bar{Z}) = (\psi(x,\bar{Z}), (\psi_R(\bar{y}_R,\bar{Z}))_{R\in\sigma})$ of MSO($\tau$) formulas, such that $x$ is a single vertex variable and $\bar{y}_R$ is a sequence of vertex variables whose length equals the arity of $R$ for each $R \in \sigma$. Given a $\tau$-labeled graph $G$ and an interpretation $\bar{A}$ of $\bar{Z}$ in $G$, the $\tau$-$\sigma$ interpretation $\Psi(\bar{Z})$ defines a $\sigma$-labeled graph $H = \Psi((G,\bar{A}))$ such that (i) the vertex set of $H$ is $\psi(G,\bar{A})$, and (ii) the relation $R \in \sigma$ is interpreted in $H$ as the set $R_H = \psi_R(G,\bar{A})$. Thus if $\bar{Z} = (Z_1,\ldots,Z_l)$, then $\Psi(\bar{Z})$ defines a function from the class of $(\tau \cup \{Z_1,\ldots,Z_l\})$-labeled graphs to the class of $\sigma$-labeled graphs. We call the function too an MSO interpretation. If $l = 0$, we call the interpretation $\Psi$ parameterless, and such a $\Psi$ defines a function from $\tau$-labeled graphs to $\sigma$-labeled graphs. An example of a parameterless interpretation is $\Theta = (\theta,\theta_E)$ where $\theta(x) := (x = x)$ and $\theta_E(x,y) := \neg E(x,y)$.
the function it defines is graph complementation mapping a graph to its complement. An example of an interpretation with parameters is \( \Psi(Z) = (\psi(x, Z), \psi_E(x, y, Z)) \) where \( \psi(x, Z) := Z(x) \) and \( \psi_E(x, y, Z) := E(x, y) \); the function that it defines produces on an input \((G, A)\), the subgraph of \(G\) induced by \(A\). Given a class \(\mathcal{C}\) of \(\tau\)-labeled graphs and an interpretation \(\Psi\) with parameters \(\bar{Z}\), we denote by \(\Psi(\mathcal{C})\) the class of \(\sigma\)-labeled graphs given by \(\Psi(\mathcal{C}) = \{\Psi((G, A)) \mid G \in \mathcal{C}\}\), and \(A\) is an interpretation of \(\bar{Z}\) in \(G\). Since they are functions, one can compose interpretations and it is known that the class of MSO interpretations is closed under function composition [11]. We call MSO interpretations with parameters as simply MSO interpretations for ease of readability, and denote them with the uppercase letters \(\Phi, \Gamma, \Psi, \Theta, \) etc.

The notion of clique-width is a structural parameter of graphs that was introduced by Courcelle, Engelfriet and Rozenberg in [3] as a generalization of the well-known notion of treewidth. Clique-width handles dense graphs as well in contrast to treewidth that deals with only sparse graphs, and yet enjoys many of nice algorithmic and logical properties that tree-width does. We do not give the definitions of the clique-width and tree-width here as we need only specific properties of these for our results that we state below; we point the reader to [5] [15] for more about the notions and results concerning them. We denote the clique-width and tree-width of a graph \(G\) as \(\text{cwd}(G)\) and \(\text{twd}(G)\) respectively. As examples, a clique has clique-width 1, and a cograph has clique-width 2. It is known for any graph \(G\), that \(\text{cwd}(G) \leq 4 \cdot 2^{\text{twd}(G)} - 1 + 1 [8]\). A class of graphs is said to have bounded clique-width if for some number \(k \geq 1\), every graph in the class has clique-width at most \(k\). As seen above, cliques, cographs and bounded treewidth graphs have bounded clique-width. A graph class has unbounded clique-width if it does not have bounded clique-width. Examples of graph classes of unbounded clique-width include grids, interval graphs, and line graphs [4].

A graph \(H\) is an induced subgraph of \(G\) if \(V(H) \subseteq V(G)\) and for any \(x, y \in V(H), (x, y) \in E(H)\) if, and only if, \((x, y) \in E(G)\). We write \(H \subseteq G\) to denote that \(H\) is an induced subgraph of \(G\). A graph class is said to be hereditary if it is closed under induced subgraphs. For any class \(\mathcal{C}\), we write \(\mathcal{C}^\downarrow\) to denote the hereditary closure of \(\mathcal{C}\)—i.e. class of graphs \(H\) that are induced subgraphs of some graph in \(\mathcal{C}\). The class of all graphs of clique-width at most \(k\) is hereditary since the clique-width of an induced subgraph of \(G\) is never more than the clique-width of \(G\). An antichain under the induced subgraph relation is a set \(\mathcal{A}\) of graphs such that if \(G\) and \(H\) are distinct graphs in \(\mathcal{A}\), then neither of \(G \subseteq H\) or \(H \subseteq G\) holds. Usually when we say “antichain” without further qualification, we mean an antichain under the induced subgraph relation. A graph class \(\mathcal{C}\) is said to be well-quasi-ordered (WQO) under induced subgraphs if it does not contain any infinite antichains. For example, the class of all cliques is WQO under induced subgraphs.

The MSO theory of a graph class \(\mathcal{C}\) is the class of all MSO sentences that are true in all graphs of \(\mathcal{C}\). Seese’s conjecture states any class whose MSO theory is decidable has bounded clique-width. An \(m \times n\) grid is a graph on \(m \cdot n\) vertices whose vertex set \(V = \{(i, j) \mid 1 \leq i \leq m, 1 \leq j \leq n\}\) and whose edge set \(E = \{((i, j), (i, j + 1)) \mid 1 \leq i \leq m, 1 \leq j < n\} \cup \{((i, j), (i + 1, j)) \mid 1 \leq i < m, 1 \leq j \leq n\}\). The grid is square if \(m = n\). We say a class \(\mathcal{C}\) of graphs interprets grids via an MSO interpretation \(\Phi\), if \(\Phi(\mathcal{C})\) contains isomorphic copies of arbitrarily large square grids. Any class of graphs that interprets grids via an MSO interpretation has an undecidable MSO theory. It is known that if the clique-width of a class \(\mathcal{C}\) is bounded and \(\Phi\) is an MSO interpretation, then the clique-width of \(\Phi(\mathcal{C})\)
is also bounded [5 Cor. 7.38].

We say that a class of graphs \( \mathcal{C} \) is HUCW if it is hereditary and has unbounded clique-width. An HUCW graph class is said to be minimal if it does not contain a proper subclass that is HUCW. For example, bipartite permutation graphs and unit interval graphs are two minimal HUCW graph classes [13]. The existence of infinitely many minimal HUCW classes is established in [2].

### 3. Minimal Classes and Well-Quasi-Ordering

In this section we lay out an approach to studying Seese’s conjecture which motivates our study of MSO decidability for minimal HUCW classes. The first observation is that, if \( \mathcal{C} \) is a counter-example to Seese’s conjecture, then so is \( \mathcal{C} \downarrow \). Recall that a counter-example to Seese’s conjecture would be a class \( \mathcal{C} \) that has unbounded clique-width and a decidable MSO theory. Clearly if \( \mathcal{C} \) has unbounded clique-width, then so does \( \mathcal{C} \downarrow \). The following proposition shows that MSO decidability is also inherited by the hereditary closure.

**Proposition 3.1.** If the MSO theory of \( \mathcal{C} \) is decidable, then so is the MSO theory of \( \mathcal{C} \downarrow \).

**Proof.** For any MSO sentence \( \varphi \) and a set variable \( X \) not appearing in \( \varphi \), the relativization of \( \varphi \) to \( X \) is an MSO formula \( \varphi^X(X) \) that relativizes all the quantifiers appearing in \( \varphi \) to \( X \). That is, \( \varphi^X(X) \) is obtained from \( \varphi \) by replacing every sub-formula in \( \varphi \) of the form (i) \( \exists \alpha \) with \( \exists z(\forall z(\varphi^X(z) \land \alpha^X)) \); (ii) \( \forall \alpha \) with \( \forall z(\varphi^X(z) \rightarrow \alpha^X) \); (iii) \( \exists Z \alpha \) with \( \exists Z((\forall z(z \in X) \land \alpha^X)) \); and (iv) \( \forall V \alpha \) with \( \forall Z((\forall z(z \in X) \rightarrow \alpha^X)) \). Here \( z \) is a first-order variable and \( Z \) a set variable, and \( Z \in X \) is shorthand for \( \forall w(Z(w) \rightarrow X(w)) \). The key property of relativization is that for any given graph \( G \), if \( G^A \) denotes the subgraph of \( G \) induced by a subset \( A \subseteq V(G) \), then

\[
G \models \varphi^X(A) \text{ if, and only if, } G^A \models \varphi.
\]

It immediately follows that an MSO sentence \( \varphi \) is true in all graphs in \( \mathcal{C} \downarrow \) if, and only if, \( \forall X \varphi^X(X) \) is true in all graphs in \( \mathcal{C} \).

Thus, if the MSO theory of \( \mathcal{C} \) is decidable, we can decide if a given MSO sentence \( \varphi \) is true in all graphs in \( \mathcal{C} \downarrow \), by deciding if \( \varphi':= \forall X \varphi^X(X) \) is in this theory.

Hence, if there is a counter-example to Seese’s conjecture, we have one that is a hereditary class of unbounded clique-width, i.e. an HUCW class as introduced in Section 2. In the present section, we establish some basic facts about the HUCW classes that allow us to structure the search for such a counter-example, or indeed the attempt to show that there isn’t one.

The relation of being an induced subgraph is not a well-quasi-order as it admits infinite anti-chains. As an example, let \( I_n \) be the graph on \( n + 4 \) vertices \( e_0, e_1, e_2, e_3, c_1, \ldots, c_n \) where for each \( i < n \) there is an edge between \( c_i \) and \( c_{i+1} \), and in addition we have edges \( e_0 - c_1, e_1 - c_1, e_2 - c_2, \) and \( e_3 - c_3 \). In short, there is a path of length \( n \) with two additional vertices at each end to mark the ends. Then, it is clear the the collection \( (I_n)_{n \in \mathbb{N}} \) is an antichain in the induced subgraph order. This particular antichain has bounded clique-width. It is also possible to construct infinite antichains of unbounded clique-width. An example is obtained by taking the collection of \( n \times n \) grids and adding an additional two
vertices at each corner to form a triangle. In what follows, whenever we refer to an antichain we mean one under the induced subgraph relation.

From an infinite antichain of unbounded clique-width, it is easy to construct an infinite descending chain of classes of graphs (under the inclusion relation) all of which are HUCW. Thus, it was a significant discovery to find that there are actually HUCW classes $\mathcal{C}$ which are minimal: no proper hereditary subclass of $\mathcal{C}$ has unbounded clique-width. The first such example is due to Lozin [13]. Collins et al. [2] constructed an infinite family of such classes and Lozin et al. [14] give an example which is itself well-quasi-ordered under the induced substructure relation. We examine these in some detail in subsequent sections.

If it could be shown that every class that is HUCW contains as a subclass a minimal HUCW class, then showing that every minimal HUCW class interprets arbitrarily large grids would suffice to prove Seese’s conjecture. Indeed, if $\mathcal{C}$ interprets grids of unbounded size, so does every class that contains $\mathcal{C}$. However, as we show in Section 3.2 below, there are indeed HUCW classes that contain no minimal HUCW subclass. This is linked to the existence of anti-chains of unbounded clique-width. Specifically, we establish the following three facts.

1. If $\mathcal{C}$ is a minimal HUCW class, then it cannot contain an infinite antichain of unbounded clique-width.
2. There exist HUCW classes which contain no minimal such class.
3. If $\mathcal{C}$ is a HUCW class that contains no minimal class, it must contain an antichain of unbounded clique width.

From these, it follows that one could prove Seese’s conjecture for finite graphs by establishing two things: (1) every antichain of unbounded clique-width interprets arbitrarily large grids; and (2) every minimal HUCW class interprets arbitrarily large grids.

Before looking at these, we make a further observation that is useful in establishing interpretability of grids in the classes we consider. It shows that to prove that we can interpret grids, it is sufficient to interpret a class $\mathcal{C}$ such that we can interpret grids in $\mathcal{C} \downarrow$. 

**Lemma 3.2.** Suppose $\mathcal{C}$ is a class that interprets grids. Let $\mathcal{D}$ be a class for which there exists an MSO interpretation $\Xi$ such that the hereditary closure of $\Xi(\mathcal{D})$ contains $\mathcal{C}$. Then $\mathcal{D}$ interprets grids as well.

**Proof.** Let $\Theta$ be an MSO interpretation such that $\Theta(\mathcal{C})$ contains all square grids. Let $\Gamma(Z) = (\gamma(x,Z), \gamma_E(x,y,Z))$ be the MSO interpretation such that $\gamma(x,Z) := Z(x)$ and $\gamma_E(x,y,Z) := E(x,y)$. Then for any class $\mathcal{Y}$ of graphs, $\Gamma(\mathcal{Y})$ is indeed the hereditary closure of $\mathcal{Y}$. Consider now the composition $\Omega = \Theta \circ \Gamma \circ \Xi$ (viewing $\Theta$, $\Gamma$ and $\Xi$ as functions) – this is also an MSO interpretation (cf. Section 2). We see that $\Gamma(\Xi(\mathcal{D}))$ contains the class $\mathcal{C}$ by the premise of the lemma, and hence $\Omega(\mathcal{D})$ contains $\Theta(\mathcal{C})$ which in turn contains all square grids.

### 3.1. Antichains and Minimal Classes

We first establish the relationship between the existence of antichains of unbounded clique-width and the minimality of HUCW classes. These are established in Theorems 3.5 and 3.6.
We say that a sequence \((\mathcal{C}_i)_{i \in \omega}\) is an infinite strictly descending HUCW-chain if for each \(i\), \(\mathcal{C}_i\) is an HUCW class and \(\mathcal{C}_{i+1}\) is a proper subset of \(\mathcal{C}_i\). We say that \(\mathcal{C}\) contains an infinite strictly descending HUCW-chain if there is such a chain with \(\mathcal{C}_i \subseteq \mathcal{C}\) for all \(i\).

**Lemma 3.3.** The following are equivalent:

1. \(\mathcal{C}\) contains an infinite strictly descending HUCW-chain whose intersection is a class of bounded clique width.
2. \(\mathcal{C}\) contains an antichain of unbounded clique width.

**Proof.** \(\mathbb{1} \rightarrow \mathbb{2}\): If \(\{G_1, G_2, \ldots\}\) is such an antichain, then let \(\mathcal{C}_i\) be the hereditary closure of \(\{G_i, G_{i+1}, \ldots\}\) for \(i \geq 1\). Then \(\mathcal{C}_1 \supsetneq \mathcal{C}_2 \supsetneq \ldots\) is an infinite strictly descending HUCW-chain whose intersection is empty and hence of bounded clique width.

\(\mathbb{1} \rightarrow \mathbb{2}\): Let \(\mathcal{C}_1 \supsetneq \mathcal{C}_2 \supsetneq \ldots\) be such a descending HUCW-chain and \(\mathcal{C}_\omega = \bigcap_{i \geq 1} \mathcal{C}_i\). Let \(\mathcal{D}_i = \mathcal{C}_i \setminus \mathcal{C}_{i+1}\) for \(i \geq 1\). Then for \(1 \leq i < j\), we have \(\mathcal{D}_i \cap \mathcal{D}_j = \emptyset\); hence \(\mathcal{D}_i \cap \mathcal{D}_j = \mathcal{D}_i \cap \mathcal{C}_\omega = \emptyset\). Further, \(\mathcal{C}_i = (\bigcup_{k \leq \omega} \mathcal{D}_k) \cup \mathcal{C}_\omega\).

**Claim 3.4.** The following are true:

1. For \(1 \leq i < j\), no graph in \(\mathcal{D}_i\) is an induced subgraph of a graph in \(\mathcal{D}_j\).
2. For \(i \geq 1\), for every graph \(G \in \mathcal{D}_i\), there exists a number \(f(G) > i\) such that for all \(j \geq f(G)\), no graph in \(\mathcal{C}_j \setminus \mathcal{C}_\omega\) is an induced subgraph of \(G\).

**Proof.** \(\mathbb{1}\): If \(G \subseteq H\) for some \(G \in \mathcal{D}_i\) and \(H \in \mathcal{D}_j\), then since \(\mathcal{D}_j \subseteq \mathcal{C}_j\) and \(\mathcal{C}_j\) is hereditary, we would have \(G \in \mathcal{C}_j\); but that contradicts the fact that \(\mathcal{D}_i \cap \mathcal{C}_j = \emptyset\).

\(\mathbb{2}\): Let \(H_1, \ldots, H_r\) be an enumeration of the induced subgraphs of \(G\) that are not in \(\mathcal{C}_\omega\). Clearly \(r\) is finite since \(G\) is finite. Since \(\mathcal{C}_i = (\bigcup_{k \leq \omega} \mathcal{D}_k) \cup \mathcal{C}_\omega\), there exist numbers \(j_1, \ldots, j_r \in [i, \omega)\) such that \(H_i \in \mathcal{D}_{j_i}\) for \(i \in \{1, \ldots, r\}\). It then follows by the properties of the \(\mathcal{D}_i\)'s above that \(f(G) = \max\{j_i \mid 1 \leq i \leq r\} + 1\) is indeed as desired. \(\Box\)

We now use the above claim to inductively construct an antichain of \(\mathcal{C}\) of unbounded clique width. Let \(G_0\) be a graph in \(\mathcal{D}_0\). Assume that we have constructed graphs \(G_0, \ldots, G_i\) for \(i \geq 0\) such that (i) \(G_j \in \mathcal{D}_i\) and \(l_j > l_{j-1}\) for \(1 \leq j \leq i\); (ii) \(\{G_0, \ldots, G_i\}\) is an antichain; and (iii) the clique-width of \(G_j\) is strictly greater than that of \(G_{j-1}\) for \(1 \leq j \leq i\). Let \(k = \max\{f(G_j) \mid 1 \leq j \leq i\} > l_i\) where \(f\) is as in Claim 3.3. Consider the class \(\mathcal{C}_k \setminus \mathcal{C}_\omega\). By Lemma 3.3, all graphs in this class are incomparable with each of \(G_0, \ldots, G_i\) in the induced subgraph order. Further, since \(\mathcal{C}_k\) has unbounded clique width while \(\mathcal{C}_\omega\) has bounded clique width, we have \(\mathcal{C}_k \setminus \mathcal{C}_\omega\) has unbounded clique width, whereby there exists \(G_{i+1} \in \mathcal{C}_k \setminus \mathcal{C}_\omega\) such that \(G_{i+1}\) has clique width greater than that of \(G_i\). Let \(l_{i+1} > k > l_i\) be such that \(G_{i+1} \in \mathcal{D}_{l_{i+1}}\). Then we see that \(G_{i+1}\) is indeed as desired to complete the induction. \(\Box\)

We are now ready to prove the two results linking minimality of HUCW classes and the existence of infinite antichains.

**Theorem 3.5.** If \(\mathcal{C}\) is minimal HUCW, then \(\mathcal{C}\) does not contain an antichain of unbounded clique width.

**Proof.** If \(\mathcal{C}\) contains an antichain of unbounded clique width, then by Lemma 3.3, \(\mathcal{C}\) contains an infinite strictly descending HUCW-chain, and hence in particular a proper subclass that is HUCW. Hence \(\mathcal{C}\) is not minimal. \(\Box\)
Theorem 3.6. If $\mathcal{C}$ is HUCW and does not contain a minimal HUCW class, then there exists in $\mathcal{C}$ an antichain of unbounded clique-width.

Proof. We assume without loss of generality that the vertices of the graphs of $\mathcal{C}$ belong to the set $\mathbb{N}$ of natural numbers, so that $\mathcal{C}$ is countable. Suppose that $\mathcal{C}$ does not contain a minimal class. Consider the sequence $\langle \mathcal{C}_\lambda \rangle_{\lambda \geq 0}$ of classes of structures, for ordinals $\lambda$, defined inductively as follows. Firstly $\mathcal{C}_0 = \mathcal{C}$. Inductively, assume that for all $\nu < \lambda$, the class $\mathcal{C}_\nu$ has been defined and that $\mathcal{C}_\nu \subseteq \mathcal{C}$ for all $\nu < \lambda$. If $\lambda$ is a limit ordinal, define $\mathcal{C}_\lambda = \bigcap_{\nu < \lambda} \mathcal{C}_\nu$. If $\lambda$ is a successor ordinal of say $\lambda^-$, then define $\mathcal{C}_\lambda$ as follows. If $\mathcal{C}_{\lambda^-}$ is not HUCW, then $\mathcal{C}_\lambda = \mathcal{C}_{\lambda^-}$. Otherwise $\mathcal{C}_{\lambda^-}$ is HUCW and $\mathcal{C}_{\lambda^-} \subseteq \mathcal{C}$; then $\mathcal{C}_{\lambda^-}$ cannot be minimal since by our premise, $\mathcal{C}$ does not contain any minimal HUCW class. Define $\mathcal{C}_\lambda$ then as some proper subclass of $\mathcal{C}_{\lambda^-}$ that is HUCW (that is identified by say an oracle). This completes the construction of the sequence $\langle \mathcal{C}_\lambda \rangle_{\lambda \geq 0}$.

Consider now the set $\mathcal{P}$ of ordinals defined as $\mathcal{P} = \{ \lambda \mid \mathcal{C}_\lambda$ is not HUCW$\}$. By the definition above, if $\lambda \in \mathcal{P}$, then all ordinals greater than $\lambda$ are in $\mathcal{P}$ as well. Now since the ordinals are well ordered, $\mathcal{P}$ has a minimum, call it $\mu^*$. We make the following observations about $\mu^*$:

1. $\mu^*$ must be a limit ordinal. If it is a successor ordinal of say $\lambda$, then $\mathcal{C}_\lambda$ must be HUCW since $\mu^*$ is the minimum ordinal in $\mathcal{P}$. But if $\mathcal{C}_\lambda$ is HUCW, then $\mathcal{C}_{\mu^*}$ must be a HUCW class by the inductive definitions above. Therefore, $\mathcal{C}_{\mu^*} = \bigcap_{\nu < \mu^*} \mathcal{C}_\nu$ where $\mathcal{C}_\nu$ is HUCW for all $\nu < \mu^*$.

2. $\mu^*$ is countable – this is because $\mathcal{C}$ is countable.

3. $\mathcal{C}_{\mu^*}$ is a hereditary class of bounded clique width. Let $G \in \mathcal{C}_{\mu^*}$ and $H \subseteq G$. Then by (1) above, $G \in \mathcal{C}_\nu$ for all $\nu < \mu^*$. Since each $\mathcal{C}_\nu$ is hereditary, we have $H \in \mathcal{C}_\nu$ for all $\nu < \mu^*$. Then $H \in \mathcal{C}_{\mu^*}$. So $\mathcal{C}_{\mu^*}$ is hereditary. That $\mathcal{C}_{\mu^*}$ has bounded clique width now follows from the fact that $\mathcal{C}_{\mu^*}$ is not HUCW.

Now since $\mu^*$ is countable, it has cofinality $\omega$ so that there exists an increasing function $f: \mathbb{N} \to \mu^*$ (where $\mu^*$ is seen as the set of ordinals less than $\mu^*$) such that if $\mathcal{F}_i = \mathcal{C}_{f(i)}$ for $i \in \mathbb{N}$, then $\bigcap_{i \in \mathbb{N}} \mathcal{F}_i = \mathcal{C}_{\mu^*}$. We observe that $\mathcal{F}_1 \supsetneq \mathcal{F}_2 \supsetneq \ldots$ is an infinite strictly descending HUCW-chain in $\mathcal{C}$, whose intersection $\mathcal{C}_{\mu^*}$ is a class of bounded clique-width. It now follows by Lemma 3.3 that $\mathcal{C}$ contains an antichain of unbounded clique-width.

The converse of Theorem 3.6 does not hold. That is to say, we can construct an HUCW class that both contains a minimal HUCW class and contains an infinite antichain of unbounded clique-width. Indeed, if $\mathcal{C}_1$ is a minimal HUCW class and $\mathcal{C}_2$ the hereditary closure of an infinite antichain of unbounded clique-width then clearly $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$ verifies this property.

3.2. HUCW Classes which Contain No Minimal Class

We give a construction of a class of graphs $\Gamma$ which is HUCW but does not contain a minimal HUCW class. For this, it suffices to show that if $\mathcal{C}$ is any hereditary subclass of $\Gamma$ of unbounded clique-width, it contains an infinite anti-chain under the induced subgraph relation, of unbounded clique-width.
Towards this, let \( G_{n,n} \) denote the \( n \times n \) grid. Note that, in \( G_{n,n} \), every vertex has degree 2, 3 or 4, and there are exactly four vertices (at the corners) of degree 2. For \( n \geq 3 \), we define \( T_n \) as the graph obtained from \( G_{n,n} \) by:

1. removing every vertex \( v \) of degree 2 and inserting an edge between the two neighbours of \( v \); and

2. replacing every vertex \( v \) of degree 4 by four new vertices \( v_1, v_2, v_3, v_4 \) which are connected in a 4-cycle so that the four edges incident on \( v \) are now each incident on one of the four new vertices.

It is easily seen that \( T_n \) is 3-regular, and it is more convenient to work with then grids. The number of vertices in \( T_n \) is less than \( 4n^2 \).

Recall that a graph \( H \) is a subdivision of a graph \( G \) if it is obtained from \( G \) by replacing every edge by a simple path. For a positive integer \( t \), we write \( G^t \) for the \( t \)-subdivision of \( G \): the graph obtained from \( G \) by replacing each edge of \( G \) by a path of length \( t \). We make the following simple observation for later use:

**Lemma 3.7.** If \( H \) is a subdivision of \( G \) and \( \text{twd}(G) = k \), then \( \text{twd}(H) \leq \max(k, 3) \).

**Proof.** Suppose \( (T, \beta) \) is a tree decomposition of \( G \) of width \( k \). To obtain a tree decomposition of \( H \), consider an edge \( \{u, v\} \) of \( G \) which is subdivided into a path \( u = p_0, \ldots, p_t = v \) in \( H \). As \( \{u, v\} \) is an edge of \( G \), there must be a node \( a \) of \( T \) such that \( \{u, v\} \in \beta(t) \). We attach a path \( a_1, \ldots, a_t \) of length \( t \) to \( a \) and let \( \beta(a_t) = \{u, v, p_t, p_{t+1}\} \). Doing this for each edge gives us a tree decomposition of \( H \) whose width is \( \max(k, 3) \).

Define the class \( \Gamma = \{ H \mid H \subseteq T^n \text{ for some } n > 2 \} \), i.e. the collection of graphs that are induced subgraphs of the \( n \)-subdivision of \( T_n \) for some \( n \). Note that in any graph \( H \in \Gamma \) every vertex has degree 2 or 3. We call the vertices of degree 3 the branch vertices. We can now establish some useful properties of the graphs in \( \Gamma \).

**Lemma 3.8.** If \( H \in \Gamma \) has at most \( m \) branch vertices, then \( \text{cwd}(H) \leq 3 \cdot 2^{m-1} \).

**Proof.** Since \( H \) has at most \( m \) branch vertices, it is the subdivision of some graph \( G \) with \( m \) vertices. Hence, by Lemma 5.7, the treewidth of \( H \) is at most \( m \). Now, for any graph \( G \), \( \text{cwd}(G) \leq 3 \cdot 2^{\text{twd}(G)-1} \) [5, Prop. 2.114], and the result follows.

**Lemma 3.9.** There is a computable function \( f \) such that for every positive integer \( n \), if \( H \) is a subdivision of \( T_{f(n)} \) then the clique-width of \( H \) is greater than \( n \).

**Proof.** It is easily seen that there is an MSO interpretation mapping any subdivision \( H \) of \( T_n \) to \( T_n \). This is because we can define the branch vertices in \( H \) (as the vertices of degree 3) and the edge relation relates two branch vertices if there is a path between them that does not pass through any other branch vertex. Similarly, there is an MSO transduction (with a 1-parameter expansion) that maps \( T_n \) to \( G_{n,n} \). Then by [5, Cor. 7.38], there is a computable function \( g : \mathbb{N} \rightarrow \mathbb{N} \) such that \( \text{cwd}(G_{n,n}) \leq f(\text{cwd}(H)) \). And, by [5, Prop. 2.106], \( \text{cwd}(G_{n,n}) = n + 1 \), and the result follows.

For a graph \( H \in \Gamma \), write \( \text{mn}(H) \) for the length of the shortest path between two branch vertices of \( H \). We are now ready to establish Theorem 3.10.
Theorem 3.10. There is a hereditary class of unbounded clique-width that does not contain any minimal hereditary class of unbounded clique-width.

Proof. The class is $\Gamma$. It is hereditary by definition and has unbounded clique-width by Lemma 3.9. Thus, it remains to show that for every class $\mathcal{C} \subseteq \Gamma$, if $\mathcal{C}$ has unbounded clique-width, then $\mathcal{C}$ contains an infinite anti-chain of unbounded clique-width.

So, suppose $\mathcal{C} \subseteq \Gamma$ has unbounded clique-width. We define the following sequence of graphs. First, let $G_0$ be any graph in $\mathcal{C}$ containing at least two branch vertices. Suppose we have defined $G_i$ for $i \geq 0$, and let $t = \max(\text{cwd}(G_i), \text{mn}(G_i))$. We then choose $G_{i+1}$ to be any graph in $\mathcal{C}$ with $\text{cwd}(G_{i+1}) > 3 \cdot 2^{t^2} - 1$.

It is clear that the sequence of graphs $(G_i : i \in \omega)$ is of unbounded clique-width, since $\text{cwd}(G_i) < \text{cwd}(G_{i+1})$ for all $i$. We now argue that this is also an anti-chain. For any $i < j$, clearly $G_j$ cannot be an induced subgraph of $G_i$ since $\text{cwd}(G_i) < \text{cwd}(G_j)$, so it remains to show that $G_i$ is not an induced subgraph of $G_j$. Since $\text{cwd}(G_j) > 3 \cdot 2^{t^2} - 1$, where $t = \max(\text{cwd}(G_i), \text{mn}(G_i))$, it follows by Lemma 3.8 that $G_j$ has more than $4t^2$ branch vertices. Since $T_n^m$ contains fewer than $4n^2$ branch vertices, it follows that $G_j$ is not an induced subgraph of $T_n^m$ for any $n \leq t$. Hence, $\text{mn}(G_j)$ is at least $t + 1$. However, by choice of $t$, $\text{mn}(G_i) \leq t$ and so $G_i$ contains two branch vertices at distance at most $t$. We conclude that $G_i$ is not an induced subgraph of $G_j$. □

4. Word-defined minimal classes

As the starting point of our exploration of minimal hereditary classes of unbounded clique-width, we consider the construction given by Collins et al. [2] to demonstrate that there are infinitely many such classes. To be precise, they construct a hereditary class $\mathcal{J}_\alpha$ of graphs for each $\omega$-word $\alpha \in \{0, 1, 2\}^\omega$. They show that as long as $\alpha$ contains either infinitely many 1s or infinitely many 2s, the class $\mathcal{J}_\alpha$ has unbounded clique-width. Moreover, they show that for infinitely many distinct such $\alpha$, $\mathcal{J}_\alpha$ is also minimal. The conditions under which $\mathcal{J}_\alpha$ is minimal need not concern us here as we are able to show that whenever $\alpha$ contains either infinitely many 1s or infinitely many 2s, $\mathcal{J}_\alpha$ interprets all square grids via MSO transductions. In particular, this covers all minimal classes $\mathcal{J}_\alpha$ of unbounded clique-width. Before we proceed to a proof, we give a precise definition of the classes $\mathcal{J}_\alpha$.

The class $\mathcal{J}_\alpha$ is defined as the class of all finite induced subgraphs of a single countably infinite graph $\mathcal{P}_\alpha$. The set of vertices of $\mathcal{P}_\alpha$ is $\{v_{i,j} \mid i, j \in \mathbb{N}\}$. We think of the set as an infinite collection of columns $V_j = \{v_{i,j} \mid i \in \mathbb{N}\}$. All edges are between vertices in adjacent columns, i.e. there is no edge between $v_{i,j}$ and $v_{i',j'}$ unless $j' = j + 1$ or $j' = j - 1$. The edges between successive columns are defined by the word $\alpha$ according to the following rules.

1. If $\alpha_i = 0$, then $\{v_{i,j}, v_{k,j+1}\} \in E(\mathcal{P}_\alpha)$ if, and only if, $i = k$.
2. If $\alpha_i = 1$, then $\{v_{i,j}, v_{k,j+1}\} \in E(\mathcal{P}_\alpha)$ if, and only if, $i \neq k$ for $i, k \in \mathbb{N}$.
3. If $\alpha_i = 2$, then $\{v_{i,j}, v_{k,j+1}\} \in E(\mathcal{P}_\alpha)$ if, and only if, $i \leq k$ for $i, k \in \mathbb{N}$.

The class $\mathcal{J}_\alpha$ is now given by $\mathcal{J}_\alpha = \{G \mid G$ is a finite induced subgraph of $\mathcal{P}_\alpha\}$. We show the following theorem in this section.
Theorem 4.1. Let \( \alpha \in \{0, 1, 2\}^\omega \) be such that \( \alpha \) contains infinitely many 1s or infinitely many 2s. Then there exists an MSO interpretation \( \Theta \) such that \( \Theta(S_\alpha) \) contains the class of all square grids.

To prove Theorem 4.1, we show the existence of an MSO interpretation \( \Psi \) such that the hereditary closure of \( \Psi(S_\alpha) \) contains the class of all square grids. Lemma 3.2 ensures that this indeed suffices.

4.1. Interpreting Grids

We now describe the construction of the interpretation. What we show is that we can find in \( S_\alpha \) a sequence of graphs \( G_n \) for \( n \in \mathbb{N} \) within which we can interpret upper triangular grids. One can think of an upper triangular grid \( U_t \) as the subgraph of the \( t \times t \) grid induced by the vertices above the main diagonal. That is to say those vertices in the vertex set \( \{(i, j) \mid 1 \leq i, j \leq t \} \) with \( i \leq j \). It is clear that \( U_t \) has as an induced subgraph a \( r \times r \) grid, where \( r = \lfloor \frac{t}{2} \rfloor \).

Let \( \alpha \in \{0, 1, 2\}^\omega \) be an \( \omega \)-word containing infinitely many 1s or infinitely many 2s. We write \( \alpha_i \) for the \( i \)th element of the word. Let \( p \in \omega \) be the least value such that \( \alpha_p \neq 0 \). Fix \( n \geq 1 \) and let \( l \) be the length of the shortest subsequence of \( \alpha \) starting at \( \alpha_p \) which contains exactly \( 2n + 2 \) elements which are not 0. We write \( \beta_0 : \beta_{l-1} \) for this sequence, so \( \beta_0 = \alpha_p \).

Recall that the vertices of \( P_\alpha \) are \( \{v_{i, j} \mid i, j \in \mathbb{N} \} \), and we write \( V_j \) for the set \( \{v_{i, j} \mid i \in \mathbb{N} \} \).

We define the graph \( G_n \) to be the subgraph of \( P_\alpha \) induced by the set \( X = \bigcup_{i=0}^{i=l-1} X_i \) where \( X_i \subseteq V_{p+i} \) is defined as follows for \( 0 \leq i < l \).

1. \( X_0 = \{v_{0, p}, v_{1, p}, v_{2, p}\} \); and
2. \( X_{i+1} = \{v_{0, p+i+1}, v_{1, p+i+1}, \ldots, v_{l-1, p+i+1}\} \) where \( t = |X_i| \) if \( \beta_i = 0 \) and \( t = |X_i| + 1 \) otherwise.

It is clear that \( G_n \in S_\alpha \). We call the sets \( X_i \) the columns of \( G_n \) and the sets \( Y_i = \{v_{i, j} \mid p \leq j < p + l - 1 \} \) the rows of \( G_n \). In particular, we refer to \( Y_0 \) as the top row of \( G_n \).

Consider the expansion \( H_n \) of \( G_n \) with unary predicates \( \text{Colour}_1, \text{Colour}_2, \text{top}, \text{bottom}, \text{penult}, \text{prepenult}, \text{first} \) and \( \text{last} \) which are interpreted as follows. For \( i \in \{1, 2\} \), the predicate \( \text{Colour}_i \) is interpreted as the set \( \bigcup_{j=1}^{\beta(j) = i} X_j \); \( \text{top} \) is interpreted as the top row of \( G_n \); \( \text{bottom} \) is the set of all vertices \( v_{i, j} \) such that \( i' \leq i \) for all \( v_{i', j} \in X_{j-p} \); \( \text{prepenult} \) is the set of all vertices \( v_{i, j} \) such that \( v_{i+1, j} \) is in \( \text{bottom} \); \( \text{penult} \) is the set of all vertices \( v_{i, j} \) such that \( v_{i+2, j} \) is in \( \text{bottom} \); and finally, \( \text{first} \) and \( \text{last} \) are interpreted as the sets \( X_0 \) and \( X_{l-1} \) respectively.

We construct below an MSO interpretation \( \Psi \) with parameters \( \text{Colour}_1, \text{Colour}_2, \text{top}, \text{bottom}, \text{penult}, \text{prepenult}, \text{first} \) and \( \text{last} \) such that \( \Psi(H_n) \) contains the \( n \times n \) grid as an induced subgraph. Then the hereditary closure of \( \Psi(S_\alpha) \) contains the class of all square grids, establishing Theorem 4.1 via Lemma 3.2.

Towards defining \( \Psi \) we need a number of auxiliary predicates which lead up to the definition of binary predicates \( H\text{-edge} \) and \( V\text{-edge} \) which define the horizontal and vertical edges of a grid-like structure. We give these details below.
1. **samecolumn**($x,y$): This predicate is true of $x,y$ in $H_n$ if $x,y \in X_i$ for some $i \in \{0,\ldots,l-2\}$ and $\beta_i \in \{1,2\}$ as long as neither of $x,y$ is in bottom or penult. Towards defining **samecolumn**($x,y$), we define the predicate **Colour**$_0(x)$ which is true of $x$ in $G_n$ if $x \in X_i$ for some $i$ with $\beta_i = 0$.

\[
\begin{align*}
\text{Colour}_0(x) & := \neg(\text{Colour}_1(x) \lor \text{Colour}_2(x)) \\
\text{rightlower}(x) & := \text{bottom}(x) \lor \text{penult}(x) \lor \text{last}(x) \\
\text{samecolumn}(x,y) & := \neg(\text{rightlower}(x) \lor \text{rightlower}(y)) \land \\
& \quad \neg \text{Colour}_0(x) \land \bigwedge_{i=1}^{\beta_i} \text{Colour}_i(x) \leftrightarrow \text{Colour}_i(y) \land \\
& \quad \forall z(\text{bottom}(z) \to (E(x,z) \leftrightarrow E(y,z)))
\end{align*}
\]

To understand the last condition, note that if $x$ and $j$ are in the same column $X_i$ with $\beta_i \in \{1,2\}$ and neither is in the bottom or penultimate row in $X_i$, then the bottom elements of $X_{i-1}$ and $X_{i+1}$ are neighbours of either both $x$ and $y$ or neither. On the other hand, suppose $x$ and $y$ are in different columns, say $X_i$ and $X_j$ respectively with $i < j$. Since $\beta_j$ is 1 or 2, we know that $|X_{j+1}| = |X_j| + 1$, and hence every element of $X_j$, in particular $y$, is adjacent to the bottom element $z$ of $X_{j+1}$. Since $x$ is not in a column adjacent to $X_{j+1}$, it cannot have an edge to $z$, and hence $x$ and $y$ do not satisfy the predicate **samecolumn**.

2. **adjcolumn**($x,y$): This predicate is true of $x,y$ in $H_n$ if for some $i,j$ with $|i-j| = 1$ and $\beta_i \in \{1,2\}$, it is the case that $x \in X_i$ but not in bottom or penult and $y \in X_j$.

\[
\text{adjcolumn}(x,y) := \neg \text{rightlower}(x) \land \neg \text{Colour}_0(x) \land \exists u (\text{samecolumn}(x,u) \land E(u,y))
\]

3. **domain**($x$): This predicate is true of $x$ in $H_n$ if it is not one of the “boundary” vertices of $G_n$.

\[
\text{domain}(x) := \neg(\text{top}(x) \lor \text{first}(x) \lor \text{rightlower}(x))
\]

4. **rhscolumn**[$i;S$]($x$): For $i \in \{0,1,2\}$ and $S \subseteq \{0,1,2\}$, this predicate is true of $x,y$ in $H_n$ if **domain**($x$) and **domain**($y$) both hold and if $x \in X_j$ and $y \in X_{j+1}$ for some $j$ with $\beta_j = i$ and $\beta_{j+1} \in S$. We need this predicate only for the following specific values of $[i;S]$:
(i) $[0;\{1,2\}]$, (ii) $[1;\{0,2\}]$ and (iii) $[2;\{0,1,2\}]$. We provide these
5. **H-definition** below.

\[
\begin{align*}
\text{rhscolumn}[2; \{0, 1, 2\}](x, y) & := \text{domain}(x) \land \text{Colour}_2(x) \land \text{domain}(y) \land \\
& \quad \forall i \in [0, 2] \; \text{Colour}_i(y) \rightarrow \gamma_i(x, y) \\
\eta_0(x, y) & := \text{adjcolumn}(x, y) \land \\
& \quad \exists v \; \text{samecolumn}(x, v) \land \text{top}(v) \land E(v, y) \\
\eta_1(x, y) & := \text{adjcolumn}(x, y) \land \\
& \quad \exists v \; \text{samecolumn}(x, v) \land \text{prepenult}(v) \land \\
& \quad \exists w \; \text{samecolumn}(y, w) \land \neg E(w, v) \\
\eta_2(x, y) & := \eta_1(x, y) \\
\text{rhscolumn}[1; \{0, 2\}](x, y) & := \text{domain}(x) \land \text{Colour}_1(x) \land \text{domain}(y) \land \\
& \quad \neg \text{Colour}_1(y) \land \bigwedge_i \text{Colour}_i(y) \rightarrow \gamma_i(x, y) \\
\text{rhscolumn}[0; \{1, 2\}](x, y) & := \text{domain}(x) \land \text{Colour}_0(x) \land \text{domain}(y) \land \\
& \quad \neg \text{Colour}_0(y) \land \text{adjcolumn}(y, x) \land \\
& \quad \neg \exists v \; \text{samecolumn}(y, v) \land \text{top}(v) \land E(v, x)
\end{align*}
\]

These predicates are meant to give an orientation to some edges in the symmetric relation \(\text{adjcolumn}(x, y)\). Thus, it is sufficient to argue that if \(x\) is in \(X_i\), then \(y\) cannot be in \(X_{i-1}\). We present the argument for the case when \(\beta_i = 2\). Other cases can be argued similarly.

Suppose \(\beta_i = 2\). There are three subcases depending on the value of \(\beta_{i-1}\). If \(\beta_{i-1} = 0\), then the only element \(z\) of \(X_{i-1}\) that is adjacent to the top element of \(X_i\) is the top element of \(X_{i-1}\). But then \(\text{domain}(z)\) does not hold. Thus no \(y \in X_{i-1}\) satisfies the formula \(\alpha_0(x, y)\). If \(\beta_{i-1} = 1\) or \(\beta_{i-1} = 2\), then the only element of \(X_{i-1}\) that is not adjacent to the element of \(X_i\) that satisfies \(\text{prepenult}\) is the penultimate element of \(X_{i-1}\) if \(\beta_{i-1} = 1\) or the bottom element of \(X_{i-1}\) if \(\beta_{i-1} = 2\). But neither of these elements is in \(\text{domain}\).

5. **H-edge** \((x, y)\): This predicate is true of \(x, y\) in \(H_n\) if both \(\text{domain}(x)\) and \(\text{domain}(y)\) hold, \(x\) and \(y\) are in the same row and adjacent columns of \(H_n\) and either (i) \(x \in X_i\) and \(y \in X_{i+1}\) for some \(i\); or (ii) \(y \in X_i\) and \(x \in X_{i+1}\) with \(\beta_i = \beta_{i+1} \neq 2\).

\[
\begin{align*}
\text{H-edge}(x, y) & := \text{domain}(x) \land \text{domain}(y) \land \\
& \quad \forall i \in [2] \; \text{Colour}_i(x) \rightarrow \gamma_i(x, y) \\
\gamma_2(x, y) & := \text{rhscolumn}[2; \{0, 1, 2\}](x, y) \land E(x, y) \land \\
& \quad \forall z \; \text{rhscolumn}[2; \{0, 1, 2\}](x, z) \land \text{lessthan}(z, y) \land z \neq y \rightarrow \neg E(x, z) \\
\text{lessthan}(z, y) & := \forall v \; \text{samecolumn}(x, v) \land E(z, v) \rightarrow E(y, v) \\
\gamma_1(x, y) & := (\text{Colour}_1(y) \land (3z \land \text{samecolumn}(y, z) \land E(z, y))) \land \neg E(x, y) \lor \\
& \quad \text{rhscolumn}[1; \{0, 2\}](x, y) \land \neg E(x, y) \\
\gamma_0(x, y) & := (\text{Colour}_0(y) \land E(x, y)) \lor (\text{rhscolumn}[0; \{1, 2\}](x, y) \land E(x, y))
\end{align*}
\]

Suppose \(x \in X_i\) and \(y \in X_j\). In all cases in the definition above except when \(\beta_i = \beta_j \neq 2\), it is the case that \(\text{rhscolumn}[\cdot](x, y)\) is true, which means \(j = i + 1\). In the case when \(\beta_i = \beta_j = 0\), we see that \(x\) and \(y\) are required to be adjacent, and when \(\beta_i = \beta_j = 1\), \(x\) and \(y\) are required to be non-adjacent with the additional condition that there is some element \(z\) in the same column as \(y\) that is adjacent to \(x\) – both of
these cases can happen only when \( x \) and \( y \) are in adjacent columns and in the same row. We therefore are left with arguing that when \( j = i + 1 \), then \( x \) and \( y \) satisfy \( H\)-edge\((x, y)\) if, and only if, they are in the same row.

If \( x \in X_i \) and \( \beta_i = \{0, 1\} \), the element \( y \) of \( X_{i+1} \) that is in the same row as \( x \) is easily distinguished. If \( \beta_i = 0 \), \( y \) is the only element of \( X_{i+1} \) that is adjacent to \( x \) and if \( \beta_i = 0 \) it is the only element of \( X_{i+1} \) not adjacent to \( x \). When \( \beta_i = 2 \), we see that for elements \( z = v_{j,i+p+1} \) and \( y = v_{j',i+p+1} \) of \( X_{i+1} \), we have \( j \leq j' \) if, and only if, every element of \( X_i \) that is adjacent to \( z \) is also adjacent to \( y \). This is expressed by the predicate \text{lessThan}(z, y)\). With this linear order on \( X_{i+1} \) defined, we see that an element \( y \) of \( X_{i+1} \) is in the same row as \( x \) if, and only if, \( x \) and \( y \) are adjacent, and no element of \( X_{i+1} \) that is less than \( y \) is adjacent to \( x \).

6. V-edge\((x, y)\)\): This predicate is true of \( x, y \) in \( H_n \) if \( \text{domain}(x) \) and \( \text{domain}(y) \) both hold and \( x = v_{j,i+p} \) and \( y = v_{j+1,i+p} \) for some \( i \) with \( \beta_i \neq 0 \). In the following definition, \( \text{TCH-edge}(x, y) \) denotes that the pair \((x, y)\) is in the reflexive and transitive closure of \( H\)-edge\). The reflexive and transitive closure of any binary relation is easily defined in MSO.

\[
\text{V-edge}(x, y) := \neg \text{Colour}_0(x) \land \text{sameColumn}(x, y) \land \\
\exists u \exists v (\text{prepenultEdge}(u, v) \land \\
(\text{TCH-edge}(u, x) \land \text{TCH-edge}(v, y)))
\]

\[
\text{prepenultEdge}(u, v) := \neg \text{Colour}_0(u) \land \text{sameColumn}(u, v) \land \text{prepenult}(v) \land \\
\exists z (\text{prepenult}(z) \land \text{H-edge}(z, u))
\]

The formula \text{prepenultEdge}(u, v)\) defines those pairs \( u, v \) in the domain where \( u = v_{j,i+p} \)
and \( v = v_{j+1,i+p} \) for some \( i \) where the bottom element in the column \( X_i \) is \( v_{j+3,i+p} \). To see why this definition is correct, note that \text{prepenult}(v)\) is true precisely when \( v = v_{j+1,i+p} \) in this column. To identify \( u = v_{j,i+p} \), we exploit crucially the special way in which the columns were chosen in \( G_n \); if \( \beta_i \in \{1, 2\} \) then \( |X_i| = |X_{i-1}| + 1 \). This ensures that \( H\)-edge\((z, u)\) holds for the element \( z \in X_{i-1} \) for which \text{prepenult}(z)\) holds. Given the definition of \text{prepenultEdge}(u, v)\) we see that every pair of elements \( x, y \) in the domain that are in the same column and in consecutive rows, is just a “horizontal translate” of a pair \((u, v)\) satisfying \text{prepenultEdge}(u, v)\). That is, \( x \) and \( y \) are reachable from \( u \) and \( v \) respectively by \( H\)-edge\)-paths.

We are now ready to define the MSO interpretation \( \Psi \). Define an “upper triangular” \( r \times r \) grid as the graph \( U_r \) whose vertex set is \( \{ u_{i,j} \mid 1 \leq j \leq r, i \leq j \} \) and whose edge set is \( \{ \{ u_{i,j}, u_{i,j+1} \mid 1 \leq j < r, i \leq j \} \cup \{ u_{i,j}, u_{i+1,j} \mid 1 < j \leq r, i < j \} \). A uniform subdivision of \( U_r \) is the graph obtained by choosing a subset \( S \subseteq \{1, \ldots , r-1\} \) and for each \( j \in S \) and each \( i \leq j \), replacing the edge \( \{ u_{i,j}, u_{i,j+1} \} \) with a path on \( k_j \) vertices for some \( k_j \geq 2 \). It is easy to show that there exists a parameterless MSO interpretation \( \Gamma \) from graphs to graphs such that if \( Z \) is a uniform subdivision of \( U_r \), then \( \Gamma(Z) \) is \( U_r \). Observe that \( U_{2r} \) contains the \( r \times r \) grid as an induced subgraph.

We now define \( \Psi \) as the composition given by \( \Psi = \Gamma \circ \Delta \) where \( \Delta = (\Delta_V(x), \Delta_E(x, y)) \) is as below. The formulae below contain the predicates \( \text{Colour}_1, \text{Colour}_2, \text{top}, \text{bottom}, \text{penult}, \text{prepenult}, \text{first} \) and \( \text{last} \) which constitute the parameters of \( \Psi \).

\[
\Delta_V(x) := \text{domain}(x)
\]
\[
\Delta_E(x, y) := H\text{-edge}(x, y) \lor H\text{-edge}(y, x) \lor V\text{-edge}(x, y) \lor V\text{-edge}(y, x)
\]
We observe that for the graph $H_n$ defined above, $\Delta(H_n)$ is indeed isomorphic to a uniform subdivision of $U_{2n}$. Then $\Psi(H_n)$ is isomorphic to $U_{2n}$ and hence contains the $n \times n$ grid as an induced subgraph.

**Proof of Theorem 4.1.** Given $\alpha \in \{0,1,2\}^\omega$ containing infinitely many 1’s or infinitely many 2’s, consider the MSO interpretation $\Psi$ as described above. The hereditary closure of $\Psi(\mathcal{S}_\alpha)$ contains the class of all square grids. Taking $\mathcal{C}$ in Lemma 3.2 to be the class of all square grids, $\mathcal{D}$ to be $\mathcal{S}_\alpha$ and $\mathcal{E}$ to be $\Psi$, we are indeed done. \hfill\box

## 5. Transferring Grid Interpretability via MSO Transductions

The main result of this section is as below.

**Theorem 5.1.** The following minimal hereditary classes of graphs of unbounded clique-width interpret grids:

1. Bichain graphs
2. Split permutation graphs
3. Bipartite permutation graphs
4. Unit interval graphs

We first show the result for bichain graphs. The proof of the result for the subsequent graph classes can be shown using interpretations in bichain graphs, or using results established earlier.

**Bichain graphs.** We need some terminology to talk about these graphs. Given a graph $G$, a sequence $v_1, \ldots, v_k$ of vertices of $G$ is said to be a **chain** if $N(v_i) \subseteq N(v_j)$ whenever $i \leq j$, where $N(v) := \{u \mid E(u,v)\}$ denotes the neighbourhood of $v$. A bipartite graph $(A \cup B, E)$ is called a $k$-chain graph if each of the two parts $A$ and $B$ can be further partitioned into at most $k$ chains. A bichain graph is a 2-chain graph.

We now describe the bichain graph $Z_n$ as defined in [1]. This graph is $n$-universal in that, all bichain graphs on at most $n$ vertices are induced subgraphs of $Z_n$. The graph has vertex set $\{z_{i,j} \mid 1 \leq i \leq n, 1 \leq j \leq n\}$ (which can thus be seen as an $n \times n$ grid of points), and $\{z_{i,j}, z_{i',j'}\}$ is an edge if, and only if, one of the following holds: (i) $j$ is odd, $j' = j + 1$ and $i < i'$; (ii) $j$ is even, $j' = j + 1$ and $i' \leq i$; or (iii) $j$ is even, $j'$ is odd and $j' \geq j + 3$. Since bichain graphs are hereditary, it follows that the class Bichain of all bichain graphs is exactly the hereditary closure of the class $\{Z_n \mid n \geq 1\}$.

For any $n$ we define $H_n$ to be the expansion of $Z_n$ with the unary predicates $\text{top}$, $\text{bottom}$, $\text{even}$, $\text{first}$ and $\text{last}$ that are interpreted as follows: $\text{top}$ is the set $\{z_{1,j} \mid 1 \leq j \leq n\}$; $\text{bottom}$ is the set $\{z_{n,j} \mid 1 \leq j \leq n\}$; $\text{even}$ is the set $\{z_{i,j} \mid j \text{ is even}\}$; $\text{first}$ is the set $\{z_{i,1} \mid 1 \leq i \leq n\}$; and $\text{last}$ is the set $\{z_{i,n} \mid 1 \leq i \leq n\}$. We construct an FO interpretation $\Psi$ with parameters $\text{top}$, $\text{bottom}$, $\text{even}$, $\text{first}$ and $\text{last}$ such that $\Psi(H_{n+2})$ is an $n \times n$ grid. Then $\Psi(\text{Bichain})$ contains the class of all square grids, establishing Theorem 5.1 via Lemma 3.2.
To help us define Ψ, we need some auxiliary predicates that we define next. The first of these is the predicate \( \text{samecolumn}(x, y) \) which is true of \( x, y \) in \( H_n \) if, and only if \( x \) and \( y \) appear in the same column and neither is the bottom element of that column.

\[
\text{samecolumn}(x, y) := \neg (\text{bottom}(x) \lor \text{bottom}(y)) \land \forall z (\text{bottom}(z) \rightarrow (E(x, z) \leftrightarrow E(y, z)))
\]

It is clear that this formula is true for any \( x \) and \( y \) in the same column of \( H_n \) as long as they are not bottom elements. To see that no other pair satisfies the formula, let \( x = z_{i,j} \) and \( y = z_{i',j'} \) with \( j < j' \). We argue by cases. If \( j' \) is odd, then \( y \) is adjacent to the bottom element \( u \) of column \( j'+1 \). Moreover, since \( j'+1 \) is then even, \( u \) is not adjacent to any \( z_{i,j} \) with \( j < j' \). On the other hand, if \( j' \) is even, then we consider whether \( j \) is odd or even. If \( j \) is odd, \( x \) is adjacent to the bottom element of column \( j+1 \) and \( y \) is not while of \( j \) is odd, \( x \) is adjacent to the bottom element of column \( j'+1 \) and \( y \) is not.

We now define the predicates \( \text{adjcolumn}(x, y) \), \( \text{domain}(x) \), \( \text{rightcol}(x, y) \), \( \text{linord}(x, y) \), \( \text{H-edge}(x, y) \) and \( \text{V-edge}(x, y) \). For \( x, y \) in \( H_n \), the predicate \( \text{adjcolumn}(x, y) \) is true if \( x \) and \( y \) are in adjacent columns and neither is the bottom element of its column. The predicate \( \text{domain}(x) \) is true if \( x \) is not a “boundary” vertex of \( H_n \). For \( x = z_{i,j} \) and \( y = z_{i',j'} \) which are not boundary vertices, the predicate \( \text{rightcol}(x, y) \) is true if \( j = j' + 1 \); the predicate \( \text{linord}(x, y) \) is true if \( j = j' \) and \( i \leq i' \); the predicate \( \text{H-edge}(x, y) \) is true if \( j = j' + 1 \) and \( i = i' \); and the predicate \( \text{V-edge}(x, y) \) is true if \( j = j' \) and \( i' = i + 1 \). The definitions of the predicates below are easy to verify.

\[
\text{adjcolumn}(x, y) := \neg (\text{bottom}(x) \lor \text{bottom}(y)) \land
\exists u \exists v (\text{samecolumn}(u, x) \land \text{samecolumn}(v, y) \land E(u, v)) \land
\exists u \exists v (\text{samecolumn}(u, x) \land \text{samecolumn}(v, y) \land \neg E(u, v))
\]

\[
\text{domain}(x) := \neg (\text{top}(x) \lor \text{bottom}(x) \lor \text{first}(x) \lor \text{last}(x))
\]

\[
\text{rightcol}(x, y) := \text{domain}(x) \land \text{domain}(y) \land \text{adjcolumn}(x, y) \land
(\text{even}(x) \leftrightarrow \exists u, v (\text{samecolumn}(u, x) \land \text{top}(u) \land \text{samecolumn}(v, y) \land \text{top}(v) \land E(u, v)))
\]

\[
\text{linord}(x, y) := \text{domain}(x) \land \text{domain}(y) \land \text{samecolumn}(x, y) \land
\exists z \left( \text{rhscolumn}(x, z) \land
( \neg \text{even}(x) \land \forall u ((\text{samecolumn}(z, u) \land E(y, u)) \rightarrow E(x, u))) \lor
(\text{even}(x) \land \forall u ((\text{samecolumn}(z, u) \land E(x, u)) \rightarrow E(y, u))) \right)
\]

\[
\text{H-edge}(x, y) := \text{domain}(x) \land \text{domain}(y) \land \text{rhscolumn}(x, y) \land
(\neg \text{even}(x) \land \neg E(x, y) \land
\forall z (\text{samecolumn}(z, y) \land z \neq y \land \text{linord}(y, z)) \rightarrow E(x, z)) \lor
(\text{even}(x) \land E(x, y) \land
\forall z (\text{samecolumn}(z, y) \land z \neq y \land \text{linord}(y, z)) \rightarrow \neg E(x, z)))
\]

\[
\text{V-edge}(x, y) := \text{domain}(x) \land \text{domain}(y) \land
\text{linord}(x, y) \land \forall z (\text{samecolumn}(z, x) \land \text{linord}(z, y) \rightarrow \text{linord}(z, x))
\]

Consider now the FO interpretation \( \Psi = (\Psi_V, \Psi_E) \) given as below.

\[
\Psi_V(x) := \text{domain}(x)
\]

\[
\Psi_E(x, y) := \text{H-edge}(x, y) \lor \text{H-edge}(y, x) \lor \text{V-edge}(x, y) \lor \text{V-edge}(y, x)
\]
It is clear that $\Psi(H_{n+2})$ is the $n \times n$ grid.

Proof of Theorem 5.1(1). Let $\Psi$ be the FO interpretation as described above; then $\Psi(\text{Bichain})$ contains the class of all square grids.

Split permutation graphs. Recall that a split graph is a graph $G$ whose vertex set can be partitioned into two sets $C$ and $I$ such that $C$ induces a clique in $G$ and $I$ is an independent set in $G$. Following [1], we use the following characterization of split permutation graphs.

Proposition 5.2 (Proposition 2.3, [1]). Let $G$ be a split graph given together with a partition of its vertex set into a clique $C$ and an independent set $I$. Let $H$ be the bipartite graph obtained from $G$ by deleting the edges of $C$. Then $G$ is a split permutation graph if, and only if, $H$ is a bichain graph.

Proof of Theorem 5.1(2). Let $\Psi$ be the FO interpretation as described above. Then $\Psi(\text{SP})$, and hence its hereditary closure, contains the class Bichain. We are then done by Theorem 5.1(1) and Lemma 3.2.

Bipartite permutation graphs. These graphs, as the name suggests, are graphs that are bipartite as well as being permutation graphs. For our purposes, the following characterization is useful. Consider the graph $P_n$ on vertex set $\{v_{i,j} \mid 1 \leq i, j \leq n\}$ where the only edges are between $v_{i,j}$ and $v_{i+1,j'}$ for $j' \leq j$. Then, the class of bipartite permutation graphs is exactly the hereditary closure of the class $\{P_n \mid n \geq 1\}$. Now, it is easily seen that this class is exactly the class $\mathcal{S}_\alpha$ as described in Section 4 for $\alpha = 2^\omega$, and this has been observed in [2]. Thus, Theorem 5.1(3) follows from Theorem 4.1.

Unit interval graphs. A unit interval (UI) graph is an interval graph which has an interval representation in which every interval is of unit length. Courcelle [4] has shown that Seese’s conjecture holds for the class of interval graphs in the sense that any such class of unbounded clique-width has an undecidable MSO theory. It follows, in particular, that this is true of the unit interval graphs, establishing Theorem 5.1(4).

6. Power graphs

In this section, we consider the class of power graphs as defined in [14] in the context of well-quasi-ordering (WQO) and clique-width. Most of the classes that we have seen so far can be shown to not be WQO under the induced subgraph relation; in particular, all word defined classes, unit interval graphs and bipartite permutation graphs can be seen to contain the antichain $\{I_n \mid n \geq 1\}$ described following Prop. 3.1. We do not know whether bichain graphs and split permutation graphs are WQO under induced subgraphs, though they have been shown to not be labeled WQO [1]. In contrast, the power graphs constitute
a class of graphs that is HUCW, that is WQO under induced subgraph \cite{14} and, as we show, is a minimal HUCW class. The minimality follows from arguments contained in \cite{13}, but was not observed there. We now define the class of power graphs, show that they are minimal and then, in the remainder of the section that they admit interpretability of grids.

For \( n \geq 1 \), we define the graph \( D_n \) as follows. The vertex set of \( D_n \) is \([n] = \{1, \ldots, n\}\). For each \( i < n \), there is an edge between \( i \) and \( i+1 \)—we call these path edges. Furthermore, there is an edge between \( i \) and \( j \) if the largest power of 2 that divides \( i \) is the same as the largest power of 2 that divides \( j \)—we call these clique edges. To understand this terminology, note that we can see \( D_n \) as consisting of a simple path of length \( n \), along with, for each \( k \) such that \( 2^k \leq n \), a clique on all vertices \( j = 2^k \cdot (2r+1) \) for some \( r \geq 0 \)—we call this the power clique corresponding to \( k \). In particular, taking \( k = 0 \), there is a clique formed by all the odd elements, which we call the odd clique. Observe that the path edges, which are the only edges with endpoints in different power cliques always have one end point in the odd clique and one outside it. The class of power graphs, denoted \textbf{Power-graphs}, is now defined as the hereditary closure of the class \( \{D_n \mid n \geq 1\} \).

6.1. Minimality of \textbf{Power-graphs}

**Proposition 6.1.** The class \textbf{Power-graphs} is a minimal hereditary class of unbounded clique-width.

Given a graph \( G \in \textbf{Power-graphs} \), define an interval in \( G \) to be a set of consecutive numbers in (the vertex set of) \( G \), and a factor of \( G \) to be the subgraph of \( G \) induced by an interval. We now recall the following two results proved in \cite{14}.

**Lemma 6.2** (Lemma 11, \cite{14}). Let \( G \) be a graph in \textbf{Power-graphs}. Then there exists an integer \( t = t(G) \) such that for any \( n \geq t \), every factor of \( D_n \) of length at least \( t \) contains \( G \) as an induced subgraph.

**Theorem 6.3** (Theorem 2, \cite{14}). Let \( G \) be a graph in \textbf{Power-graphs} such that the length of the longest factor in \( G \) is \( t \). Then the clique-width of \( G \) is at most \( 2(\log t + 4) \).

**Proof of Proposition 6.1** That \textbf{Power-graphs} is a hereditary class of unbounded clique-width has already been shown in \cite{14}. Towards showing minimality, consider a proper hereditary subclass \( \mathcal{I} \) of \textbf{Power-graphs}; then \( \mathcal{I} \) excludes a graph \( G \in \textbf{Power-graphs} \). Let \( t = t(G) \) be as given by Lemma 6.2. Let \( \mathcal{I} = \mathcal{I}_1 \cup \mathcal{I}_2 \) where \( \mathcal{I}_1 = \mathcal{I} \cap \{D_n \mid n < t\} \) and \( \mathcal{I}_2 = \mathcal{I} \cap \{D_n \mid n \geq t\} \). Observe that \( \mathcal{I}_1 \) is a finite class.

We now show that for each graph \( X \in \mathcal{I}_2 \), every factor of \( X \) has length \( < t \). For suppose \( X \) has a factor \( Y \) of length \( \geq t \). Then since \( X \in \textbf{Power-graphs} \), there exists \( p \geq 1 \) such that \( X \subseteq D_p \) and so \( Y \) is also a factor of \( D_p \). Hence by Lemma 6.2 we have \( G \in \textbf{Power-graphs} \) is an induced subgraph of \( Y \), whereby it is also an induced subgraph of \( X \). Since \( \mathcal{I} \) is hereditary, \( G \in \mathcal{I} \) which is a contradiction.

So every factor of \( X \) has length \( < t \). By \cite{63} every \( X \in \mathcal{I}_2 \) has clique-width \( \leq k = 2(\log t + 4) \). Then \( \mathcal{I}_2 \) has bounded clique-width, and hence so does \( \mathcal{I} \) since \( \mathcal{I}_1 \) is finite. \( \square \)

6.2. Interpreting grids in \textbf{Power-graphs}

**Theorem 6.4.** There exists an MSO interpretation \( \Theta \) such that \( \Theta(\textbf{Power-graphs}) \) contains all square grids.
We show Theorem 6.4 by showing that there exists an MSO interpretation Φ such that the hereditary closure of Φ(Power-graphs) contains all bipartite permutation graphs. We are then done by Theorem 5.1 and Lemma 3.2. Indeed, it suffices to show that we can interpret grids in the set \( \{ D_n \mid n \in \mathbb{N} \} \) as this is contained in Power-graphs.

We construct a number of auxiliary predicates along the way.

We first show that there exists an FO formula odd(x) such that for \( n \geq 9 \), if \( x \) is a number in \( D_n \), then \( \text{odd}(x) \) is true if, and only if, \( x \) is an odd number.

\[
\text{odd}(x) := \exists y \exists z \exists w \left( \text{“} x, y, z, w \text{ form a } 4\text{-clique except for the } z \rightarrow w \text{ edge”} \right)
\]

It is easy to see that for \( n \geq 9 \), all odd numbers in \( D_n \) satisfy odd(x): if \( x \) is an odd number with \( x < n-3 \), then choose \( y = x+2 \), \( w = x+4 \) and \( z = x+1 \); else, choose \( y = x-2 \), \( w = x-4 \) and \( z = x-1 \).

To show that the even numbers of \( D_n \) do not satisfy odd(x), we first observe that in any power clique other than the odd clique, since the numbers in the clique are of the form \( 2^k \cdot (2r+1) \) for fixed \( k \), the difference between any two numbers in the clique is at least \( 2^{k+1} \), which is at least 4 since \( k \geq 1 \). Suppose now that odd(x) is true for an even \( x \), witnessed by \( y, z, w \). We have the following two cases.

- The edge between \( x \) and \( y \) is a clique edge. Then \( |x-y| \geq 4 \). If \( z \) is in a different power clique, then \( |x-z| = 1 \) and \( |z-y| = 1 \), whereby \( |x-y| \leq 2 \) — a contradiction. Thus \( z \) is in the same power clique as \( x \) and \( y \). By the same argument, \( w \) is in the same power clique as \( x \) and \( y \), so there is a clique edge \( z \rightarrow w \), giving a contradiction.

- The edge between \( x \) and \( y \) is a path edge. Then \( |x-y| = 1 \). If \( z \) is in the same power clique as \( x \), then \( |x-z| \geq 4 \) and \( |y-z| = 1 \), which is a contradiction. Thus \( z \) is in the same power clique as \( y \), since otherwise \( |y-z| = 1 \) and \( |x-z| = 1 \) which is inconsistent with \( |x-y| = 1 \). By the same argument, \( w \) is in the same clique as \( y \), so there is a clique edge \( z \rightarrow w \), giving a contradiction.

**Remark** 6.5. The formula odd(x) is central to our construction below and we assume henceforth that \( n \geq 9 \).

clique(x,y) and pathedge(x,y): The formula clique(x,y) is true of the pair \( (x,y) \) in \( D_n \) if, and only if, \( x \) and \( y \) are in the same power clique. The formula pathedge(x,y) is true if, and only if, \( |x-y| = 1 \).

\[
\text{clique}(x,y) := E(x,y) \land (\text{odd}(x) \oplus \text{odd}(y))
\]

\[
\text{pathedge}(x,y) := E(x,y) \land \neg \text{pathedge}(x,y).
\]

path(P,x,y): This predicate is true of all triples \( (P,x,y) \) for an MSO variable \( P \) and \( x,y \in D_n \) if \( P \) is the (unique) path between \( x \) and \( y \), whose edges are all path edges. Below \( \exists!w \) denotes “there is a unique \( w \) such that...”.

\[
\text{path}(P,x,y) := (P(x) \land P(y) \land \\
\exists!w(P(w) \land \text{pathedge}(x,w)) \land \\
\exists!w(P(w) \land \text{pathedge}(y,w)) \land \\
\forall w ((P(w) \land w \neq x \land w \neq y) \rightarrow \\
\exists u \exists v (P(u) \land P(v) \land \text{pathedge}(u,w) \land \text{pathedge}(v,w) \land u \neq v))
\]

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between\( (x, y, z) \): This predicate is true of all triples \((x, y, z)\) in \(D_n\) such that \(y\) appears somewhere along the (unique) path between \(x\) and \(z\) \((y\) could be one of \(x\) or \(z\)).

\[
\text{between}(x, y, z) := \exists P(\text{path}(P, x, z) \land P(y))
\]

We now make a few observations about \(D_n\). Note that since the path edges are definable, and they form a simple path from 1 to \(n\), the only possible automorphisms are the trivial one and the map that reverses the order, in particular mapping \(n\) to 1. Moreover, since the odd numbers are definable, for the order reversing map to be an automorphism, \(n\) must be odd. We can say more: a more careful analysis shows that the order reversal preserves all power cliques if, and only if, \(n = 2^k - 1\) for some \(k\). However, for our purposes it suffices to note that whenever \(n\) is even \(D_n\) has no non-trivial automorphisms. The predicates we define next are for even \(n > 9\).

one\((x)\): This predicate is satisfied by \(x\) in \(D_n\) if, and only if, \(x = 1\). It defines the unique (when \(n\) is even) odd element that has only one path edge incident on it.

\[
\text{one}(x) := \text{odd}(x) \land \neg \exists z_1 \exists z_2 (\text{pathedge}(x, z_1) \land \text{pathedge}(x, z_2) \land z_1 \neq z_2)
\]

This now allows us to orient the path edges to obtain the natural successor relation on \(D_n\).

\[
\text{succ}(x, y) := \text{pathedge}(x, y) \land \exists z (\text{one}(z) \land \text{between}(z, x, y)).
\]

As usual, we can then define in MSO a formula \(\text{linord}(x, y)\) which defines the reflexive and transitive closure of \(\text{succ}\).

cliquemin\((x)\): This predicate is true of \(x\) in \(D_n\) if, and only if, \(x\) is the minimum element of its power clique \((i.e. \ x = 2^k \ for \ some \ k \geq 0)\).

\[
\text{cliquemin}(x) := \forall y (\text{clique}(x, y) \rightarrow \text{linord}(x, y)).
\]

The linear order defined by \(\text{linord}\) then allows us to linearly order the power cliques.

cliqueord\((x, y)\): This predicate is true of the pair \((x, y)\) in \(D_n\) if, and only if, the minimum element in the power clique of \(x\) is less than the minimum element in the power clique of \(y\).

\[
\text{cliqueord}(x, y) := \exists z_1 z_2 (\text{cliquemin}(z_1) \land \text{cliquemin}(z_2) \land \text{clique}(x, z_1) \land \text{clique}(y, z_2) \land \text{linord}(z_1, z_2))
\]

This ordering of the power cliques and the fact that \(\text{linord}\) linearly orders each clique gives us sufficient structure to define arbitrarily large grids. To see this concretely, consider the following relation.

cliquemin-succ\((x, y)\): This predicate is true if \(x\) is in the power clique corresponding to \(k\) and \(y\) in the power clique corresponding to \(k + 1\) for some \(k\).

\[
\text{cliquemin-succ}(x, y) := \neg \text{clique}(x, y) \land \text{cliqueord}(x, y) \land \\
\forall z (\text{cliqueord}(x, z) \rightarrow (\text{clique}(z, y) \lor \text{cliqueord}(y, z))
\]

Consider now the relation \(\text{forward}(x, y)\) defined by

\[
\text{forward}(x, y) := \text{cliquemin-succ}(x, y) \land \text{linord}(x, y).
\]

This relates an element \(x\) in the power clique corresponding to \(k\) to all elements of the power clique corresponding to \(k + 1\) that are greater than \(x\).
Then, the interpretation

\[
\Phi_V(x) := \text{True} \\
\Phi_E(x, y) := \text{forward}(x, y) \lor \text{forward}(y, x)
\]

maps \(D_n\) to a graph whose edge relation is the symmetric closure of \(\text{forward}\). We claim that the graph \(\Phi(D_n)\) contains a large bipartite permutation graph as an induced subgraph. To see this, choose the largest value \(k\) such that the power clique corresponding to \(k\) contains at least \(k\) elements in \(D_n\) (in other words \(2^k(2k - 1) \leq n\)). Consider the subgraph of \(D_n\) induced by the set of vertices \(\{v_{i,j} \mid 0 \leq i, j \leq k - 1\}\) where \(v_{i,j} = j \cdot 2^{k+1} + 2^i + 1\). Each \(v_{i,j}\) is then in the power clique corresponding to \(i + 1\) and it is easily checked that there is an edge between \(v_{i,j}\) and \(v'_{i',j'}\) in \(\Phi(D_n)\) precisely when \(i' = i + 1\) and \(j \leq j'\).

**Proof of Theorem 6.4.** As established above, the graph \(\Phi(D_n)\) contains an induced subgraph isomorphic to the bipartite permutation graph \(P_k\) as long as \(2^k(2k - 1) \leq n\) and \(n \geq 9\). Then the hereditary closure of \(\Phi(\text{Power-graphs})\) contains all bipartite permutation graphs, whereby, by Theorem 5.1 and Lemma 3.2, we are done.

7. Conclusion

The study of monadic second-order logic on graphs has attracted great attention in recent years. An important aspect of work on this logic is to classify classes of graphs into those on which MSO is well behaved and those on which it is not. Seese’s conjecture is an important focus of this classification effort. In its stronger form it offers a dichotomy: any class of graphs is either interpretable in trees and therefore has bounded clique-width and is well-behaved or it interprets arbitrarily large grids and its MSO theory is then undecidable.

We show that Seese’s conjecture could be established by considering two kinds of graph classes: the minimal hereditary classes of unbounded clique-width and the antichains of unbounded clique-width. Showing that all such classes interpret unbounded grids would suffice. While we do not have a complete taxonomy of such classes, we investigated all the ones we know and showed that none of them provide a counter-example.

It is also worth pointing out that for many of the classes we consider, the original proofs that they have unbounded clique-width are non-trivial. The interpretation of grids in the classes also provides a uniform method of proving that they have unbounded clique-width.

As a final remark, it is worth noting that there are standard graph operations which allow us to construct new minimal HUCW graph classes from the ones we have. For example, taking the graph complement of all graphs in a class \(\mathcal{C}\) yields a class that is also minimal HUCW if \(\mathcal{C}\) is. Since this operation is itself an MSO interpretation, the results about interpreting arbitrarily large grids apply to the resulting classes as well.

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