Ground state and orbital stability for the NLS equation on a general starlike graph with potentials

Claudio Cacciapuoti\textsuperscript{1}, Domenico Finco\textsuperscript{2} and Diego Noja\textsuperscript{3}

\textsuperscript{1} Dipartimento di Scienza e Alta Tecnologia, Università dell’Insubria, Via Valleggio 11, 22100 Como, Italy
\textsuperscript{2} Facoltà di Ingegneria, Università Telematica Internazionale Uninettuno, Corso Vittorio Emanuele II 39, 00186 Roma, Italy
\textsuperscript{3} Dipartimento di Matematica e Applicazioni, Università di Milano Bicocca, via R. Cozzi, 55, 20125 Milano, Italy

E-mail: claudio.cacciapuoti@uninsubria.it, d.finco@uninettunouniversity.net and diego.noja@unimib.it

Received 22 September 2016, revised 19 June 2017
Accepted for publication 30 June 2017
Published 21 July 2017

Recommended by Dr Jean-Claude Saut

Abstract

We consider a nonlinear Schrödinger equation (NLS) posed on a graph (or network) composed of a generic compact part to which a finite number of half-lines are attached. We call this structure a starlike graph. At the vertices of the graph interactions of $δ$-type can be present and an overall external potential is admitted. Under general assumptions on the potential, we prove that the NLS is globally well-posed in the energy domain. We are interested in minimizing the energy of the system on the manifold of constant mass ($L^2$-norm). When existing, the minimizer is called ground state and it is the profile of an orbitally stable standing wave for the NLS evolution. We prove that a ground state exists for sufficiently small masses whenever the quadratic part of the energy admits a simple isolated eigenvalue at the bottom of the spectrum (the linear ground state). This is a wide generalization of a result previously obtained for a star-graph with a single vertex. The main part of the proof is devoted to prove the concentration compactness principle for starlike structures; this is non trivial due to the lack of translation invariance of the domain. Then we show that a minimizing, bounded, $H^1$ sequence for the constrained NLS energy with external linear potentials is in fact convergent if its mass is small enough. Moreover we show that the ground state bifurcates from the vanishing solution at the bottom of the linear spectrum. Examples are provided with a discussion of the hypotheses on the linear part.
1. Introduction

Analysis on metric graphs and networks is a growing subject with many potential applications of a physical and technological character. The interest in these structures, also from a mathematical point of view, lies in the fact that while they are relatively simple analytically, being essentially one dimensional, they can have in a sense arbitrary complexity due to nontrivial connectivity and topology.

A large part of the literature is devoted to linear equations on graphs (see [16, 30] for an overview of the theory and the many applications), with special emphasis on Schrödinger equation describing the so called quantum graphs. Recently nonlinear equations have attracted attention, and a certain amount of mathematical work has been done on nonlinear Schrödinger equation (NLS) on quantum graphs, at least in some special situations (see for example [1–4, 9–11, 18, 29, 31, 32, 37, 38]). In this paper we settle some issues about the nonlinear Schrödinger equation on a quantum graph \( \mathcal{G} \), composed by a compact core to which a finite number of half-lines are attached (and at least one). We refer to this structure as a starlike graph (see figure 1).

Our main interest is in showing that the NLS dynamics admits on a starlike graph a ground state under mild and natural hypotheses. As ground state we mean a standing solution of the NLS on the graph which minimizes the system energy at a fixed constant mass, i.e. \( L^2 \)-norm. A previous result in a very special case was given in the paper [5], where a single vertex with \( N \) half lines—a so called star graph—with a delta interaction was considered. Here we extend that result widely generalizing the topology of the compact core, and admitting the (possible) presence of external potentials on the graph. We however retain the power nonlinearity to avoid wordy statements but this limitation is not really necessary. The NLS on the graph is an equation of the form

\[
\frac{d}{dt}\Psi = H\Psi - |\Psi|^{2\mu}\Psi
\]

where:

(i) \( \Psi \) is a multicomponent function where every component is a complex function on a single edge of the graph;

(ii) The operator \( H \) is a Schrödinger operator on the graph acting on every edge as \( -\frac{d^2}{dx^2} + W \) and complemented with suitable boundary condition to make it selfadjoint on its domain;

(iii) The nonlinearity, of power type and again defined edge by edge, is focusing (the minus sign).

For further details and complete hypotheses and definitions, see the following section. The previous equation is globally well-posed in energy or form domain \( H^1(\mathcal{G}) \), which is the usual Sobolev space including continuity at vertices, for every \( \mu \in [0, 2) \), the subcritical range, see section 2.6 below for a proof. In the critical case \( \mu = 2 \) the solution is only defined for small initial data, as in the case of the NLS equation on the line. In any case the mass of the solution, i.e. its \( L^2 \)-norm \( \|\Psi\|^2 \), and the energy
E[Ψ] = E^{\text{lin}}[Ψ] - \frac{1}{\mu + 1} ||Ψ||^2_{2\mu+2} = ||Ψ'||^2 + (Ψ, WΨ) + \sum_{v \in V} \alpha(v) |Ψ(v)|^2 = \frac{1}{\mu + 1} ||Ψ||^2_{2\mu+2}

are conserved quantities. Of special importance is the quadratic contribution to the energy
E^{\text{lin}}[Ψ] = ||Ψ'||^2 + (Ψ, WΨ) + \sum_{v \in V} \alpha(v) |Ψ(v)|^2.

It contains three terms. The kinetic energy, a potential term defined by W and the last term which is the energy associated to delta interactions concentrated at vertices v of the graph; we do not assume definite sign on the strengths \alpha(v) of the interaction at vertices.

Our hypotheses are rather simple and they regard only the topology of the graph and the quadratic part of the energy.

Assumption 1. \(G\) is a connected graph with a finite number of edges and vertices, and it is composed by a compact core and at least one infinite edge (one half-line).

Assumption 2. \(W = W_+ - W_-\) with \(W_+ \geq 0\), \(W_+ \in L^1(G) + L^\infty(G)\), and \(W_- \in L^r(G)\) for some \(r \in [1, 1 + 1/\mu]\).

Assumption 3. \(\inf \sigma(H) := -E_0\), \(E_0 > 0\) and it is an isolated eigenvalue.

We remark that, our assumptions imply that \(-E_0\) is a simple eigenvalue and the corresponding eigenfunction, denoted by \(\Phi_0\), is strictly positive (see section 2.4).

The following two theorems contain the main results of this paper: under the above assumptions we prove the existence of a nonlinear ground state and we identify it as an element of a branch of stationary states bifurcating from the linear ground state.

Theorem 1. Let \(0 < \mu < 2\) and consider on a starlike graph \(G\) the following minimization problem:
\[-\nu = \inf \{E[Ψ] \text{ s.t. } Ψ \in H^1(G), M[Ψ] = m\}. \tag{1.2}\]

If assumptions 1–3 hold true, then \(mE_0 < \nu < +\infty\) for any \(m > 0\). Moreover, there exists \(m^* > 0\) such that for \(0 < m < m^*\) there exists \(\tilde{Ψ}_m \in H^1(G)\), with \(M[\tilde{Ψ}_m] = m\), such that \(E[\tilde{Ψ}_m] = -\nu\).

In order to characterize the minimizer \(\tilde{Ψ}_m\) of theorem 1 we study the stationary equation
\[HΦ - |Φ|^2 Φ = -ωΦ \quad Φ \in D(H), \quad ω > 0, \tag{1.3}\]
and we prove in proposition 5.1 in section 5 the existence of a branch \(Φ(ω)\) of stationary states bifurcating from the bottom of the linear spectrum \(-E_0\) in the direction of the linear
The eigenvector $\Phi_0$. The bifurcation branch can be equivalently parametrized by the mass $m$, so that one has $\Phi(\omega) = \Phi(\omega(m))$.

**Theorem 2.** Let $0 < \mu < 2$ and assume that assumptions 1–3 hold true, then there exists $m^* > 0$ such that for $0 < m < m^*$ the minimizer $\Psi_m$ belongs to the branch defined in proposition 5.1, that is $\Psi_m = \Phi(\omega(m))$.

We briefly comment on the assumptions.

Assumption 1 is a topological one. We remark that if $G$ is a compact connected graph without infinite edges, the minimization problem 1.2 admits a solution whenever the energy functional $E[\Psi]$ is bounded from below.

Assumption 2 is a rather weak hypothesis which is sufficient to guarantee that $E^{\text{lin}}$ is the quadratic form of a self-adjoint operator bounded from below, see also remark 2.1. We stress that the stronger assumption $W \in L^r(G)$ is needed only in the final part of the proof of theorem 1, to guarantee that the $W$-terms in the energy functional $E[\Psi]$ are negligible whenever the energy functional is evaluated on sequences that escape at infinity on one of the half-lines (runaway sequences), see equation (4.10) below. All the results before the limit (4.10) hold true under the weaker assumption $W \in L^1(G) + L^\infty(G)$.

Assumption 3 is used to apply bifurcation theory. We will prove in section 2.4 that in the present setting the bottom of the spectrum $-E_0$ is a simple eigenvalue. We stress that in particular this property is satisfied in many relevant examples, such as the following:

(a) No delta terms, i.e. $\alpha(v) = 0$ for all $v$ (also called Kirchhoff boundary conditions at vertices, see, e.g. [27]) and a sufficiently well behaved and decaying external potential attractive in the mean, i.e. such that $\int_G W < 0$. In the pure Kirchhoff case (with no potentials) an extensive analysis of NLS with power nonlinearity has been given in the recent papers [9, 10], where in particular it is shown that existence of a ground state for subcritical nonlinearity holds true only in some exceptional cases, the simplest one being the tadpole graph [18, 31]. Here we show that summing a small negative potential restores the ground state generically for small mass.

(b) Absence of potential term and delta interactions negative in the mean: $\sum_{v \in V} \alpha(v) < 0$ (See also [22] for an explicit example in this case).

(c) A mixing of the two: delta interaction at the vertices and well behaved potentials with negative potential energy: $\sum_{v \in V} \alpha(v) + \int_G W < 0$; see section 2.4 for further precise information.

Notice that at the level of quadratic form and in this one dimensional problem, strictly speaking, one could consider on the same footing both the delta terms and the regular potential term. We have a preference to keep separate the two contributions because this is the usual way they are treated in quantum graph literature.

We comment now briefly on the proof strategy. As in [5] we want to make use of concentration compactness techniques, but we have to cope with the lack of translational invariance of the graph. We show that for starlike graphs the concentration compactness lemma 3.7 is valid. We note that with respect to the standard concentration compactness result in $\mathbb{R}^n$ see, e.g. [19, 20], we have to split the compact case in two sub-cases, named runaway and convergent. In the runaway case a minimizing, bounded in $H^1(G)$, sequence $\Psi_n$ eventually escapes on a single distinguished external edge, in the sense that any of its $L^p$-norms with $p \geq 2$ on the other edges vanishes and the same occurs for the $L^p$-norm on any bounded part of the distinguished edge. In the convergence case, which is the one we are interested in, an $H^1(G)$-bounded sequence admits a converging subsequence in $L^p(G)$, $p \geq 2$. So that, to get
convergence, we have to exclude vanishing, dichotomy and runaway case. In particular, to exclude the runaway case we derive a lower bound for the energy and we show that such a lower bound is not compatible with a previous estimate of the energy of the ground state. Here is the only point where we use the hypothesis of small mass in theorem 1. With the present technique it is not possible to exclude that for big masses the minimizing sequence is runaway. Indeed, in [6], for the simpler case of a star graph, it was shown that for large mass minimizing sequences are runaway, and the symmetric stationary state is only a local minimizer of the constrained energy functional. We remark that in the case of the line with a delta interaction, the existence of the ground state for every value of the mass was given in [8], which covers also other examples of point interactions.

A further analysis of the stationary equation via bifurcation techniques allows to give further information on the ground state. When global well-posedness of the model holds true, the ground state, being a constrained minimum of the energy, is orbitally stable. We provide a global well-posedness result in $H^1(G)$ in theorem 3, filling a gap in the literature. We also show well-posedness for strong solutions (domain operator) as an intermediate step to prove energy conservation.

We end the introduction with an outline of the paper. In section 2 we give preliminary definitions and results on quantum graphs (sections 2.1 and 2.2). We precise the hypotheses on the quadratic part of the energy and comment about the validity of assumption 3 (sections 2.3 and 2.4); in particular we give a result about positivity improving of the linear semigroup which implies simplicity of the ground state. Finally we give well-posedness in energy and operator domain and prove mass and energy conservation for the time dependent NLS equation on a starlike graph (sections 2.5 and 2.6). In section 3 the concentration compactness lemma is extended to the case of starlike networks. All statements are given explicitly, but only the steps which need essential modification of the original result valid on $\mathbb{R}^n$ are proved, while references are provided for the missing but straightforward steps. In section 4 the variational analysis needed to prove theorem 1 is given. Finally in section 5 the bifurcation analysis showing the existence of a branch of standing waves emanating from the vanishing solution under the validity of assumptions 2 and 3 is proved, see proposition 5.1. Estimates on the size of the branch element in terms of relevant parameters are given as well. The identification of the states on the bifurcation branch with the solutions of the variational problem is shown at the end of the section completing the proof of theorem 2. Throughout the paper $c$ and $C$ denote generic positive constants whose value may change from line to line.

2. Preliminaries

2.1. Quantum graphs

We consider a connected metric graph $G = (V, E)$ where $V$ is the set of vertices and $E$ is the set of edges. We assume that the cardinalities $|V|$ and $|E|$ of $V$ and $E$ are finite. We identify each edge $e \in E$ with length $L_e \in (0, \infty]$ with the interval $I_e = [0, L_e]$, if $L_e$ is finite, or $[0, \infty)$, if $L_e$ is infinite. The set of edges with finite length is denoted by $E^f$ while the set of edges with infinite length is denoted by $E^i$. Moreover we associate each finite length edge with two vertices, and each infinite length edges with one vertex. The notation $\vec{v} \in e$ with $\vec{v} \in V$ and $e \in E$, denotes that $\vec{v}$ is a vertex of the edge $e$. Two vertices $\vec{v}_1$ and $\vec{v}_2$ are adjacent, $\vec{v}_1 \sim \vec{v}_2$, if they are vertices of a common edge which connects them. The degree of a vertex is the number of edges emanating from it. We denote by $\{e \prec \vec{v}\}$ the set of edges connected to the vertex $\vec{v}$. We fix a coordinate $x$ on each interval $I_e$ such that $x = 0$ and $x = L_e$ correspond to vertices if $L_e < \infty$ while if $L_e = \infty$ the vertex attached to the rest of the graph corresponds to $x = 0$. 
Any choice of the orientation of finite length edges is equivalent for our purposes. To avoid ambiguities, from now on we will denote points on the graph with \( x = (e, y) \), where \( e \in E \) identifies the edge and \( y \in I_e \) is the coordinate on the corresponding edge. The length of a path is well defined due to the coordinates on the edges and therefore there is a natural distance on \( \mathcal{G} \). Given \( x \) and \( y \) on \( \mathcal{G} \), the distance \( d(x, y) \) is defined as the infimum of the length of the paths connecting the two points. Then \((\mathcal{G}, d)\) is a locally compact metric space and it is compact if and only if \( L_e < \infty \) for all \( e \in E \). In this paper we will assume that there is at least one edge with infinite length, so that the considered graph is non-compact. A function \( \Psi : \mathcal{G} \to \mathbb{C}[E] \) is equivalent to a family of functions \( \{\psi_e\}_{e \in E} \) with \( \psi_e : I_e \to \mathbb{C} \). In our notation, if \( x = (e, y) \)

\[
\Psi(x) = \psi_e(y).
\]

The spaces \( L^p(\mathcal{G}) \), \( 1 \leq p \leq \infty \), are made of functions \( \Psi \) such that \( \psi_e \in L^p(I_e) \) for all \( e \in E \) and

\[
\|\Psi\|_p^p = \sum_{e \in E} \|\psi_e\|_{L^p(I_e)}^p, \quad 1 \leq p < \infty \quad \|\Psi\|_\infty = \max_{e \in E} \|\psi_e\|_{L^\infty(I_e)}.
\]

We denote by \( (\cdot, \cdot) \) the inner product associated with \( L^2(\mathcal{G}) \). When \( p = 2 \), the index will be omitted. We denote by \( C(\mathcal{G}) \) the set of continuous functions on \( \mathcal{G} \) and introduce the spaces

\[
H^1(\mathcal{G}) := \{ \Psi \in C(\mathcal{G}) \text{ s.t. } \psi_e \in H^1(I_e) \forall e \in E \}
\]
equipped with the norm

\[
\|\Psi\|_{H^1(\mathcal{G})}^2 = \sum_{e \in E} \|\psi_e\|_{H^1(I_e)}^2,
\]

and

\[
H^2(\mathcal{G}) := \{ \Psi \in H^1(\mathcal{G}) \text{ s.t. } \psi_e \in H^2(I_e) \forall e \in E \}
\]
equipped with the norm

\[
\|\Psi\|_{H^2(\mathcal{G})}^2 = \sum_{e \in E} \|\psi_e\|_{H^2(I_e)}^2.
\]

We note that \( H^1(\mathcal{G}) \) is a Hilbert space. In the following, whenever a functional norm refers to a function defined on the graph, we omit the symbol \( \mathcal{G} \).

### 2.2. Gagliardo–Nirenberg inequalities on graphs

Let \( \mathcal{G} \) be any non-compact graph, then if \( p, q \in [2, +\infty) \), with \( p \geq q \), and \( \alpha = \frac{2}{q - p} (1 - q/p) \), there exists \( C \) such that

\[
\|\Psi\|_p \leq C \|\Psi\|^{\alpha}_q \|\Psi\|^{-\alpha}_p,
\]

for all \( \Psi \in H^1(\mathcal{G}) \).

A proof of inequality (2.1) for \( q = 2 \), which is easily generalized to any \( q \geq 2 \), is in [10]. If the graph is compact inequality (2.1) does not hold true (it is clearly violated by constant functions), but it can be replaced by the weaker inequality

\[
\|\Psi\|_p \leq C \|\Psi\|^{\alpha}_{H^1} \|\Psi\|^{-\alpha}_q,
\]

which holds true on any graph if \( p, q \in [2, +\infty) \), with \( p \geq q \), and \( \alpha = \frac{2}{q - p} (1 - q/p) \), for all \( \Psi \in H^1(\mathcal{G}) \).
A proof of inequality (2.2) for compact graphs is in [30], for non compact graphs it is a trivial consequence of (2.1).

See also [24] for a collection of useful inequalities on graphs. In what follows we shall always use the weaker inequality (2.2).

2.3. Linear Hamiltonian and quadratic form

We denote by $H$ the Hamiltonian with a $\delta$ coupling of strength $\alpha(\varphi) \in \mathbb{R}$ at each vertex and a potential term $W$ on each edge. It is defined as the operator in $L^{2}(\tilde{G})$ with domain

$$D(H) := \left\{ \psi \in H^{2} \text{ s.t. } \sum_{e \in \mathbb{E}} \partial_{e} \psi_{e}(v) = \alpha(\varphi)\psi_{v}(v) \quad \forall v \in V \right\},$$

where we have denoted by $\partial_{e}$ the outward derivative from the vertex, it coincides with $\frac{d}{dt}$ or $-\frac{d}{dt}$ according the orientation on the edge. The action of $H$ is defined by

$$(H\psi)_{e} = -\psi''_{e} + W_{e}\psi_{e},$$

where $W_{e}$ is the component of the potential $W$ on the edge $e$.

In the following we will write $V = V_{-} \cup V_{0} \cup V_{+}$ where $V_{-}$, respectively $V_{0}$, $V_{+}$, is the set of vertices such that $\alpha(\varphi)$ is negative, respectively null, positive. As recalled in the Introduction, assumption 2 implies in particular that operator $H$ is a self-adjoint operator on $L^{2}(G)$. The quadratic form of this operator is defined on the energy space given by $\mathcal{H}^{1}(G)$ and it is explicitly given by

$$E^{\text{lin}}[\psi] = \|\psi'\|^{2} + (\psi, W\psi) + \sum_{v \in V} \alpha(\varphi)|\psi_{v}(v)|^{2}.$$

Notice that $\psi(\varphi)$ is well defined due to the continuity condition in $\mathcal{H}^{1}(G)$.

**Remark 2.1.** Indeed one can prove that under assumption 2 one has

$$\left| (\psi, W\psi) + \sum_{v \in V} \alpha(\varphi)|\psi_{v}(v)|^{2} \right| \leq a\|\psi'\|^{2} + b\|\psi\|^{2}, \quad \text{with } 0 < a < 1, \quad b > 0,$$

which, by KLMN theorem, implies that the form $E^{\text{lin}}$ is closed and hence defines a selfadjoint operator. It is easy to prove that the corresponding operator coincides with $H$. To prove that the bound (2.3) holds true, first note that by assumption 2 we have that $W \in L^{1}(G) + L^{\infty}(G)$. Moreover, by Gagliardo–Nirenberg inequalities, setting $W = W_{1} + W_{\infty}$

$$\|(\psi, W\psi)\| \leq \|W_{1}\| \|\psi\|_{L^{\infty}}^{2} + \|W_{\infty}\| \|\psi\|^{2},$$

where we used the trivial inequality $\|\psi\|_{L^{\infty}} \leq \|\psi\|^{2}_{L^{2}} + \|\psi\|^{2}/(2\varepsilon)$ for all $\varepsilon > 0$. Similarly,

$$\sum_{v \in V} \alpha(\varphi)|\psi_{v}(v)|^{2} \leq C_{\varepsilon}\|\psi\|^{2}_{L^{2}} \leq C_{\varepsilon}\|\psi\|_{L^{2}}^{2} \leq \varepsilon\|\psi'\|^{2} + b_{\varepsilon}\|\psi\|^{2}.$$

We note that, as a direct consequence of equation (2.3), there exist $c_{1}, c_{2}, \Lambda > 0$ such that
In what follows we will denote by $\|\Psi\|_H$ the graph norm, defined as $\|\Psi\|_H = \|(H + E_0 + 1)\Psi\|$. 

2.4. Linear ground state

One has that the bottom of the spectrum of $H$ is given by 

$$-E_0 = \inf \left\{ E_{\text{lin}}[\Psi], \Psi \in H^1(\mathcal{G}), \|\Psi\| = 1 \right\},$$

and it is a negative, isolated eigenvalue by assumption 3. We will show that our assumptions imply that $-E_0$ is a simple eigenvalue, corresponding to a strictly positive eigenfunction denoted by $\Phi_0$. This allows to apply bifurcation theory from a simple eigenvalue and to construct the nonlinear ground state.

Assumption 2 with the additional request that the potential $W$ is relatively compact with respect to the laplacian on the graph (Kirchhoff or delta boundary conditions or a mixing of the two) assures that the Hamiltonian $H$ admits an essential spectrum $\sigma_e(H) = [0, +\infty)$. So that, with this additional condition, a necessary hypothesis for assumption 3 to be satisfied is that at least a negative eigenvalue exists. It is straightforward to prove, considering a trial function constant on the compact part of the graph and smoothly vanishing at infinity that if $\sum_{x \in \mathcal{V}} \alpha(x) + \int_{\mathcal{G}} W$ is negative the quadratic form is negative on this trial function and so a negative eigenvalue exists. Moreover the delta interactions contribute at most with a finite number of eigenvalues and the same holds true if $W_-$ is vanishing sufficiently fast at infinity.

The additional request $\int_{\mathcal{G}} |W(x)|(1 + |x|)dx < \infty$, as in the line or half line cases is sufficient to guarantee that the discrete spectrum is finite. In particular $-E_0 < 0$ is an isolated eigenvalue.

The non degeneracy of the smallest eigenvalue (and the fact that the corresponding eigenfunction is strictly positive) is a subtler problem. When a ground state exists this property is assured by and is equivalent to the fact that the resolvent $(H + \lambda)^{-1}$ associated to $H$ is positivity improving for all $\lambda > E_0$ (see [34], theorem XIII.44).

**Proposition 2.2.** Let assumptions 1 and 2 hold true. Then the resolvent $(H + \lambda)^{-1}$ is positivity improving for all $\lambda > E_0$ where $-E_0 = \inf \sigma(H)$.

**Proof.** We focus on the case $W_- \in L^\infty(\mathcal{G})$ and we show that the resolvent $(H + \lambda)^{-1}$ is positivity improving for some $\lambda$ large enough. By Trotter–Kato formula and a uniform approximation argument (see e.g. [34, theorem XIII.45], [33, theorem VIII.25c] and an analogous argument in [25]), it is possible to prove that the same statement holds true also for $W_\in L^r(\mathcal{G})$ for any $r \geq 1$.

We notice preliminarily that by the Beurling–Deny theorem, the resolvent $(H + \lambda)^{-1}$ as well as the semigroup $e^{-tH}$ are positivity preserving respectively for all $t > 0$ and for all $\lambda > E_0$.

By [35, theorem XIII.44], it is enough to prove that the resolvent $(H + \lambda)^{-1}$ is positivity improving for some $\lambda$ large enough. We take $\lambda > \max\{E_0, \|W_-\|_\infty\}$. Let $\Phi \in L^2(\mathcal{G})$ be such that $\Phi \geq 0$ a.e. in $\mathcal{G}$, $\Phi \neq 0$, and set $\Psi = (H + \lambda)^{-1}\Phi$. Since $(H + \lambda)^{-1}$ is positivity preserving we have that $\Psi \geq 0$. We are left to prove that $\Psi > 0$. We shall proceed by contradiction, by proving that if $\Psi = 0$ for some $x \in \mathcal{G}$, then it must be $\Psi = 0$ in $\mathcal{G}$.

Let us consider first the internal edges. By the definition of $\Psi$ we have that $\psi_\nu \in H^2\left(\{0, L_\nu\}\right)$,

$$\psi''_\nu + (W_{\nu,-} - W_{\nu,+} - \lambda)\psi_\nu = -\phi_\nu \leq 0, \quad \forall x \in \{0, L_\nu\}$$

$$c_1 \|\Psi\|^2_{\mu_0} \leq E_{\text{lin}}[\Psi] + \Lambda M[\Psi] \leq c_2 \|\Psi\|^2_{\mu_0}.$$
and \( W_{e,-} - W_{e,+} - \lambda \leq 0 \) for all \( e \in E^m \). Then, by the maximum principle, see, e.g. [23, theorem 9.6], we have that \( \psi_e \) cannot achieve a nonpositive minimum in \((0, L_e)\) unless it is a constant. Hence, since \( \psi_e \geq 0 \), if \( \psi_e(x) = 0 \) for some \( x \in (0, L_e) \), or \( x = 0 \) or \( x = L_e \) or \( \psi_e(x) = 0 \) for all \( x \in [0, L_e] \). Since we are reasoning on the assumption that \( \Psi = 0 \) for some \( x \in \mathcal{G} \), in the case of external edges we can repeat the same argument by choosing a large enough open subset of \((0, +\infty)\). We conclude that if \( \Psi = 0 \) for some \( x \in \mathcal{G} \) or \( x \) coincides with a vertex of the graph or \( \Psi = 0 \) on the edge containing \( x \), in both cases there will be at least one vertex in which \( \Psi = 0 \). To conclude the proof it is enough to show that if \( \Psi = 0 \) in one vertex of the graph, then it must be equal to zero everywhere. To start with, we note that, by the same maximum principle used above, if \( \psi_e(0) = \psi_e'(0) = 0 \) then it must be \( \psi_e(x) = 0 \) for all \( x \in [0, L_e] \) (the same holds true if \( \psi_e(L_e) = \psi_e'(L_e) = 0 \)). To see that this is indeed the case, it is enough to recall that by Sobolev extension theorem one can define \( \tilde{\psi}_e \in H^1((-\delta, L_e)) \), for some \( \delta > 0 \), by extending \( \psi_e \) to zero. Applying the maximum principle to \( \tilde{\psi}_e \) (after extending to zero also \( W_e \) and \( \phi_e \)) one obtains that it must be \( \tilde{\psi}_e = 0 \).

Next, assume that \( \Psi_0(x) = 0 \) for some \( x \in V \). Then, by the boundary condition encoded in \( \mathcal{D}(H) \), one has \( \sum_{e \ni x} \delta_e \psi_e(x) = 0 \). Just to fix the ideas, assume that for a \( \psi_e \) in the previous sum the vertex \( x \) coincides with \( x = 0 \); it turns out that it must be \( \psi_e'(0) \geq 0 \), because \( \psi_e \) is a nonnegative, \( C^1([0, L_e]) \) function, such that \( \psi_e(0) = 0 \). By taking into account the boundary condition, we conclude that \( \psi_e'(0) = 0 \) for all \( e \in \mathcal{E} \). Hence, if \( \psi_e = 0 \) in \( I_e \) for all \( e \in \mathcal{E} \) (by the extension argument described above) and by continuity \( \Psi = 0 \) in \( \mathcal{G} \).

We add, by way of information, that simplicity of all eigenvalues of a quantum graph, in absence of tadpoles, and with delta interactions at the vertices can be shown to be a generic property up to changing edge lengths and intensity of delta interactions (see [17] for details).

2.5. Energy of the nonlinear problem

The nonlinear energy reads

\[
E[\Psi] = E^\text{lin}[\Psi] - \frac{1}{\mu + 1} \|\Psi\|_{2\mu + 2}^{2\mu + 2}
\]

\[
= \|\Psi'\|^2 + (\Psi, W\Psi) + \sum_{x \in V} \alpha(x) |\Psi(x)|^2 - \frac{1}{\mu + 1} \|\Psi\|_{2\mu + 2}^{2\mu + 2}
\]

and it is defined on \( H^1(\mathcal{G}) \). The mass functional is given by

\[
M[\Psi] = |\Psi|^2.
\]

Restricted on the mass constraint the nonlinear energy is bounded from below, as a consequence of the Gagliardo–Nirenberg inequalities on graphs and of the hypotheses on the external potentials, in particular on \( W_- \). This is shown in section 4, at the beginning of the proof of theorem 1.

2.6. Well-posedness

The local well-posedness for equation (1.1) in \( H^1(\mathcal{G}) \) proceeds along well known lines as an application of Banach fixed point theorem, see proposition 2.3. Global well-posedness (proved in theorem 3 below) then follows by conservation laws (see proposition 2.8). In order to prove conservation laws in \( H^1(\mathcal{G}) \), as an intermediate step, we study the dynamics in \( \mathcal{D}(H) \), propositions 2.5–2.7.
We will give only a representative result; a more general or optimal result could be obtained by making use of local in time Strichartz estimates, but we avoid this way for two reasons. The first one is that our interest in this paper is to establish theorem 1, which is a variational property of the NLS on the graph in $H^1(\mathcal{G})$. In the presence of global well-posedness in the same space the existence of the ground state implies by well known arguments (see, e.g. [21]) its orbital stability. We do not need deeper or finer results at this level and in any case the picture is clear.

We stress however that, to the best of our knowledge, the result given in theorem 3 below is not present in the literature.

The second reason is that Strichartz estimates should be preliminarily proven for starlike graphs, and this would bring us too far apart. In fact dispersive estimates are known for graphs, but only in some special examples, in particular for trees with Kirchhoff or delta vertices [14, 15] and on the tadpole graph [13].

To proceed we introduce the following integral form of equation (1.1)

$$\Psi(t) = e^{-iHt}\Psi_0 + i \int_0^t e^{-iH(t-s)}|\Psi(s)|^{2\mu}\Psi(s) \, ds \equiv T(\Psi)(t). \tag{2.4}$$

**Proposition 2.3 (Local well-posedness in $H^1(\mathcal{G})$).** Let $\mu > 0$ and assumptions 1 and 2 hold true. For any $\Psi_0 \in H^1(\mathcal{G})$, there exists $T > 0$ such that the equation (2.4) has a unique solution $\Psi \in C([0,T),H^1(\mathcal{G})) \cap C^1([0,T),H^1(\mathcal{G}))$. Moreover, equation (2.4) has a maximal solution defined on an interval of the form $[0,T^*)$, and the following ‘blow-up alternative’ holds: either $T^* = \infty$ or

$$\lim_{t \to T^*} \|\Psi(t)\|_{H^1} = +\infty.$$  

**Proof.** Consider the space $C([0,T],H^1(\mathcal{G})) := C_T H^1$ with the norm $\|\Psi\|_{C_T H^1} = \sup_{t \in [0,T]} \|\Psi(t)\|_{H^1}$ and a closed ball $\overline{B}_R \subset C_T H^1$. It is well known that $\overline{B}_R$ is a complete metric space. We prove that $T : \overline{B}_R \to \overline{B}_R$ and moreover $T$ is a contraction on $\overline{B}_R$ if $R$ and $T$ are suitably chosen.

We start by noting that for any $\Psi \in H^1$ one has that

$$\|e^{-iHt}\Psi\|_{H^1} \leq C\|\Psi\|_{H^1}. \tag{2.5}$$

This inequality follows from the conservation of the $L^2$-norm $\|e^{-iHt}\Psi\| = \|\Psi\|$, and from

$$(1 - a)\|e^{-iHt}\Psi\|^2 - b\|e^{-iHt}\Psi\|^2 \leq \mathcal{E}^{\text{lin}}[e^{-iHt}\Psi] = \mathcal{E}^{\text{lin}}[\Psi] \leq (1 + a)\|\Psi\|^2 + b\|\Psi\|^2$$

where we used the conservation of the linear energy $\mathcal{E}^{\text{lin}}[e^{-iHt}\Psi] = \mathcal{E}^{\text{lin}}[\Psi]$, and the bound (2.3).

By the bound (2.5), Schwarz inequality, and the property of $H^1(\mathcal{G})$ of being a Banach algebra one has

$$\|T(\Psi)(t)\|_{H^1} = \left\|e^{-iHt}\Psi_0 + i \int_0^t e^{-iH(t-s)}|\Psi(s)|^{2\mu}\Psi(s) \, ds \right\|_{H^1} \leq C\|\Psi_0\|_{H^1} + C \int_0^t \|\Psi(s)\|_{H^1}^{2\mu} ds \leq C\|\Psi_0\|_{H^1} + C(\mu) \int_0^t \|\Psi(s)\|_{H^1}^{2\mu+1} ds.$$
Now, taking the supremum in time
\[ \| T(\Psi) \|_{C_tH^1} \leq C \| \Psi_0 \|_{H^1} + TC(\mu) \| \Psi \|_{C_tH^1}^{2\mu + 1}. \]

We take \( R \) such that \( C \| \Psi_0 \|_{H^1} \leq R/2 \), and in the last inequality we want
\[ TC(\mu) \| \Psi \|_{C_tH^1}^{2\mu + 1} \leq \frac{R}{2}. \]

The latter inequality holds true up to taking \( T \) small enough, indeed for \( \Psi \in \overline{B}_R \) one has
\[ C(\mu)T \| \Psi \|_{C_tH^1}^{2\mu + 1} \leq C(\mu)TR^{2\mu + 1} \leq \frac{R}{2} \]
if \( T \leq \frac{C(\mu)}{2R^{2\mu}} \). And this shows that \( T : \overline{B}_R \rightarrow \overline{B}_R \).

Now we show that we can achieve contractivity of \( T \), possibly choosing a smaller \( T \) if needed.

We have to bound in \( C_tH^1 \)
\[ T(\Psi_1) - T(\Psi_2) = i \int_0^t e^{-iH(t-s)} (|\Psi_1|^{2\mu} \Psi_1 - |\Psi_2|^{2\mu} \Psi_2) \, ds. \]
By use of mean value theorem one has
\[ \| |\Psi_1|^{2\mu} \Psi_1 - |\Psi_2|^{2\mu} \Psi_2 \|_{H^1} \leq C(\mu)(|\Psi_1|^{2\mu} + |\Psi_2|^{2\mu}) \| \Psi_1 - \Psi_2 \|_{H^1} \]
and from this, using again Sobolev immersions in one dimension,
\[ \| |\Psi_1|^{2\mu} \Psi_1 - |\Psi_2|^{2\mu} \Psi_2 \|_{H^1} \leq C(\mu)(|\Psi_1|^{2\mu} + |\Psi_2|^{2\mu}) \| \Psi_1 - \Psi_2 \|_{H^1} \]
As before,
\[
\| T(\Psi_1) - T(\Psi_2) \|_{C_tH^1} \leq \sup_{t \in [0,T]} \left\| \int_0^t e^{-iH(t-s)} (|\Psi_1|^{2\mu} \Psi_1 - |\Psi_2|^{2\mu} \Psi_2) \, ds \right\|_{H^1}
\leq TC(\mu)(\| \Psi_1 \|_{C_tH^1}^{2\mu} + \| \Psi_2 \|_{C_tH^1}^{2\mu}) \| \Psi_1 - \Psi_2 \|_{C_tH^1}, \tag{2.6}
\]
and now it is enough to choose \( T \) so small to have
\[ TC(\mu)(\| \Psi_1 \|_{C_tH^1}^{2\mu} + \| \Psi_2 \|_{C_tH^1}^{2\mu}) < 1 \]
for \( \Psi_1, \Psi_2 \in \overline{B}_R \), which is always possible.

The blow-up alternative is shown by bootstrap.

For the extension of the solution to \( C^1([0,T], H^1(\mathcal{G})^*) \) the procedure is similar to the standard case of the equation on \( \mathbb{R} \). Some caution is only needed because of the meaning to give to the equation. One extends first the operator \( H \) to \( H^1(\mathcal{G}) \) with values in \( H^1(\mathcal{G})^* \) by means of the sesquilinear form \( B \) associated to \( E \), the (bounded from below) quadratic form of the operator \( H \):
\[ (\Psi_1, H\Psi_2) = B(\Psi_1, \Psi_2) \]

3281
as in the standard definition of the weak laplacian. This allows to show by direct calculation that one has in $H^1(G)^*$
\[
\frac{d}{dt} e^{-it\partial} \varphi = -iH e^{-it\partial} \varphi
\]
and that a $C^0([0,T), H^1(G))$ solution of equation (2.4) is a $C^1([0,T], H^1(G)^*)$ solution of equation (1.1) and vice versa.

**Remark 2.4.** Continuous dependence of the solution on initial data is a consequence of Gronwall inequality applied to the bound
\[
\|\varphi(t) - \Phi(t)\|_{H^1} \leq \|e^{-it\partial} (\varphi_0 - \Phi_0)\|_{H^1} + C(\|\varphi\|_{C^0_t H^1}^2 + \|\Phi\|_{C^0_t H^1}^2) \int_0^t \|\varphi(s) - \Phi(s)\|_{H^1} ds
\]
which can be proved by an argument similar to the one used for the bound (2.6), see, e.g. [35].

In order to prove the global well-posedness in $H^1(G)$ we will need to prove energy conservation. To this aim, as an intermediate step, we have to analyze the dynamics in $\mathcal{D}(H)$. Proofs are only sketched for sake of brevity.

**Proposition 2.5 (Local well-posedness in $\mathcal{D}(H)$).** Let $\mu > 0$ and assumptions 1 and 2 hold true. For any $\varphi_0, \Phi_0 \in \mathcal{D}(H)$, there exists $T > 0$ such that the equation (2.4) has a unique solution $\varphi \in C([0,T), \mathcal{D}(H)) \cap C^1([0,T), L^2(G))$. Moreover, equation (2.4) has a maximal solution defined on an interval of the form $[0,T^*)$, and the following ‘blow-up alternative’ holds: either $T^* = \infty$ or
\[
\lim_{t \to T^*} \|\varphi(t)\|_{H^1} = +\infty.
\]

**Proof.** The proof is another application of Banach fixed point theorem for the map $\mathcal{T}$ defined above in the space
\[
\mathcal{X} = C^0([0,T), \mathcal{D}(H)) \cap C^1([0,T), L^2(G))
\]
equipped with the norm
\[
\|\varphi\|_{\mathcal{X}} := \sup_{t \in [0,T]} \|\varphi(t)\|_{H^1} + \sup_{t \in [0,T]} \|\partial_t \varphi(t)\|.
\]
The main point is that we need a different representation of this map obtained by integrating by parts w.r.t. to $s$ (see for technical details [7, theorem 3.1]) in order to prove the required mapping properties in higher regularity setting. After integration by parts, and by using Gagliardo–Nirenberg inequality to bound nonlinear terms (such as $\|\varphi(t)\|^{2\mu} \partial_t \varphi(t)\| \leq \|\varphi(t)\|^{2\mu} \|\partial_t \varphi(t)\| \leq \|\varphi(t)^{2\mu+1}\|$ for all $t \in [0,T]$), one ends up with the bounds
\[
\|\mathcal{T}(\varphi)\|_{\mathcal{X}} \leq C\|\varphi_0\|_{H^1} + T C \|\varphi\|_{\mathcal{X}}^{2\mu+1}
\]
and
\[
\|\mathcal{T}(\varphi_1) - \mathcal{T}(\varphi_2)\|_{\mathcal{X}} \leq T C (\|\varphi_1\|_{\mathcal{X}}^{2\mu} + \|\varphi_2\|_{\mathcal{X}}^{2\mu}) \|\varphi_1 - \varphi_2\|_{\mathcal{X}}.
\]
The proof is concluded by applying Banach fixed point theorem. □
Notice that such solutions are strong solutions, i.e. they satisfy (1.1).

**Proposition 2.6 (Conservation laws for initial data in $D(H)$).** Let $\mu > 0$. For any solution $\Psi \in C^0([0, T), D(H)) \cap C^1([0, T), L^2(G))$ to the problem (2.4), the following conservation laws hold at any time $t$:

$$M[\Psi(t)] = M[\Psi(0)], \quad E[\Psi(t)] = E[\Psi(0)].$$

**Proof.** Conservation of the mass follows immediately from the identity

$$\frac{d}{dt} M[\Psi(t)] = 2\Re \left( \Psi(t), \frac{d\Psi(t)}{dt} \right)$$

and equation (1.1). In order to prove conservation of the energy, after noting that, by proposition 2.5, $(\Psi(t), H\Psi(t))$ is differentiable, it is sufficient to prove that:

$$\frac{d}{dt}(\Psi(t), H\Psi(t)) = 2\Re \left( \frac{d\Psi(t)}{dt}, H\Psi(t) \right)$$

$$\frac{d}{dt}(\Psi(t), |\Psi(t)|^{2\mu}\Psi(t)) = (2\mu + 2)\Im |\Psi(t)|^{2\mu}\Psi(t), H\Psi(t))$$

and use equation (1.1). □

**Proposition 2.7 (Global well-posedness in $D(H)$).** Let $0 < \mu < 2$ and assumptions 1–3 hold true. Then, any solution of equation (2.4) with $\Psi_0 \in D(H)$ is global in time.

**Proof.** By Gagliardo–Nirenberg estimates (2.2), conservation of the $L^2$-norm and energy, and hypotheses on the potential, one obtains a uniform bound on the $H^1(G)$-norm of the solution (see estimate (4.4) proven in section 4). Hence, again by Gagliardo–Nirenberg inequality and conservation laws, one has the bound $||\Psi(t)||_H \leq C(||\Psi_0||, E[\Psi_0])$. Then, similarly to the bound (2.7), one can prove

$$||\Psi(t)||_H \leq C||\Psi_0||_H + C_0 \int_0^t ||\Psi(s)||_H ds$$

(where $C_0$ depends only on $||\Psi_0||$ and $E[\Psi_0]$). The latter bound implies

$$||\Psi(t)||_H \leq a(\Psi_0) e^{b(t)}$$

by a Gronwall-type argument. Globality of solutions follows from the blow-up alternative. □

**Proposition 2.8 (Conservation laws for initial data in $H^1$).** Let $\mu > 0$. For any solution $\Psi \in C^0([0, T), H^1(G)) \cap C^1([0, T), H^1(G^*)^*)$ to the problem (2.4), the following conservation laws hold at any time $t$:

$$M[\Psi(t)] = M[\Psi(0)], \quad E[\Psi(t)] = E[\Psi(0)].$$

**Proof.** First one proves that $e^{iHt}\Psi(t)$ is differentiable and

$$\frac{d}{dt} e^{iHt}\Psi(t) = ie^{iHt} |\Psi(t)|^{2\mu}\Psi(t)$$

3283
Then
\[
\frac{d}{dt}M[\Psi(t)] = \frac{d}{dt}(e^{iHt}\Psi(t), e^{iHt}\Psi(t))
\]
\[
= 2\Re(e^{iHt}\Psi(t), \frac{d}{dt}e^{iHt}\Psi(t)) = 2\Im(e^{iHt}\Psi(t), e^{iHt}|\Psi(t)|^2\mu(t)) = 0
\]
and conservation of mass is proved. In order to prove conservation of energy we work by approximation. Since \(\mathcal{D}(H)\) is a core of \(\mathcal{D}(E)\) we can approximate the initial datum \(\Psi_0\) with a sequence \(\Psi_{0,n} \in \mathcal{D}(H)\) converging in the uniform topology of \(H^1(\mathcal{G})\). Let \(\Psi_{0,n}(t)\) be the solutions of (2.4). By a Gronwall argument one proves that \(\Psi_{0,n}(t)\) converges to \(\Psi(t)\) in \(H^1(\mathcal{G})\) then
\[
E[\Psi(t)] = \lim_{n \to \infty} E[\Psi_{0,n}(t)] = \lim_{n \to \infty} E[\Psi_{0,n}] = E[\Psi_0].
\]

**Theorem 3 (Global well-posedness in \(H^1\)).** Let \(0 < \mu < 2\). For any \(\Psi_0 \in H^1(\mathcal{G})\), the equation (2.4) has a unique solution \(\Psi \in C^0([0, \infty), H^1(\mathcal{G})) \cap C^1([0, \infty), H^1(\mathcal{G})^*)\).

**Proof.** As in the proof of proposition 2.7, by Gagliardo–Nirenberg inequalities and conservation laws, one obtains a uniform bound on the \(H^1(\mathcal{G})\)-norm of the solution, see equation (4.4). So, no blow-up in finite time of the \(H^1\)-norm of the solution can occur, and by the blow-up alternative, the solution is global in time.

## 3. Concentration compactness lemma

As noted in [5] where the special case of star graphs was treated, concentration compactness techniques on the real line (or more generally in \(\mathbb{R}^n\)) can be adapted to certain domains where translation invariance is absent. With respect to the classical result (see, e.g. [19, 20] for expositions and references) the main point is a finer analysis of the compact case, which is split into two sub-cases: convergent and runaway (see lemma 3.7 below). In this section we extend the concentration compactness lemma to a generic connected noncompact graph with a finite number of internal and external edges. In the course of the analysis, where the proofs of single steps require only minor modifications with respect to the standard case, we omit the details and we refer to the already cited texts [19, 20].

We need preliminary an information about the metric structure of the graph. We denote by \(d(\mathbf{x}, \mathbf{y})\) the distance between two points of the graph, defined as the infimum of the length of the paths connecting \(\mathbf{x}\) to \(\mathbf{y}\).

**Proposition 3.1.** Let \(\mathbf{x} = (e, x) \in \mathcal{G}\), fix the edge \(e \in E\) and let \(I_e\) be the associated (open) interval, moreover fix a point \(y \in \mathcal{G}\). The function
\[
d_{e, y}(x) : I_e \to \mathbb{R}
\]
\[
d_{e, y}(x) := d(\mathbf{x}, \mathbf{y})
\]
is continuous and piecewise linear. In particular, \(d'_{e, y}\) is a piecewise constant function with at most one discontinuity point \(x^* \in I_e\), and \(d'_{e, y}(x) = 1\) or \(d'_{e, y}(x) = -1\) for all \(x \in I_e \setminus \{x^*\}\).

**Proof.** Assume first that \(y \notin e\). If \(e\) is an internal edge (with length \(L_e < \infty\)), let \(a\) and \(b\) be the vertices that identify the endpoints of the edge \(e\), note that if \(e\) is a loop \(a\) and \(b\) coincide. Without loss of generality, set \(a = (e, 0)\) and \(b = (e, L_e)\). Then
\[ d_{e,Y}(x) = \min \{ d(g, y) + x, d(b, y) + L_e - x \}. \]

If \( e \) is an external edge, let \( g \equiv (e, 0) \) be its endpoint, then one has
\[ d_{e,Y}(x) = d(g, y) + x. \]

On the other hand, if \( y \in e \), one has
\[ d_{e,Y}(x) = |x - y|. \]

The properties of \( d_{e,Y} \) follow from its explicit form.

We denote by \( B(y, t) \) the open ball of radius \( t \) and center \( y \)
\[ B(y, t) := \{ x \in \mathcal{G} \text{ s.t. } d(x, y) < t \}. \]

We denote by \( \| \cdot \|_{L^2(\mathcal{G})} \) the \( L^2(\mathcal{G}) \) norm restricted to the ball \( B(y, t) \).

We define the volume of the set \( B(y, t) \) by
\[ \text{Vol}(B(y, t)) = \sum_{e} \int_{B_e} (1_{B(y, t)})_e(x)dx \]
where \( 1_{B(y, t)} \) is the characteristic function of the set \( B(y, t) \).

We have the following bounds on the volume of the sets \( B(y, t) \) and \( B(y, t) \setminus B(y, s) \):

**Proposition 3.2.** Let \( 0 < s < t < \infty \), then
\[ \text{Vol}(B(y, t)) \leq 2Nt \quad \text{and} \quad \text{Vol}(B(y, t) \setminus B(y, s)) \leq 2N(t-s). \]

**Proof.** We prove only the second bound, the proof of the first one is similar. By definition one has
\[ B(y, t) \setminus B(y, s) = \{ x \in \mathcal{G} \text{ s.t. } s \leq d(x, y) < t \}, \]
and
\[ \text{Vol}(B(y, t) \setminus B(y, s)) = \sum_{e} \int_{B_e \setminus B(y, s)} (1_{B(y, t) \setminus B(y, s)})_e(x)dx. \]
We have that, for each \( e \in E \),
\[ (1_{B(y, t) \setminus B(y, s)})_e(x) = \begin{cases} 1 & \text{if } s \leq d_{e,Y}(x) < t \\ 0 & \text{otherwise} \end{cases} \]
By proposition 3.1, it is easy to convince oneself that for any edge \( e \)
\[ \int_{B_e} (1_{B(y, t) \setminus B(y, s)})_e(x)dx \leq 2(t-s). \]
From which the bound on the volume immediately follows.

Next we prove a result on the convergence of bounded sequences in \( H^1(\mathcal{G}) \).

**Proposition 3.3.** Let \( \{ \Psi_n \}_{n \in \mathbb{N}} \) be such that \( \Psi_n \in H^1(\mathcal{G}) \) and \( \| \Psi_n \|_{H^1} \leq c \). Then there exists a subsequence \( \{ \Psi_{n_k} \}_{k \in \mathbb{N}} \) and a function \( \Psi \in H^1(\mathcal{G}) \) such that \( \Psi_{n_k} \rightharpoonup \Psi \) weakly in \( H^1(\mathcal{G}) \).
and $\Psi_{n_k} \to \Psi$ in $L^\infty(B(y,t))$, for any fixed $y$ and $t$.

**Proof.** Since $\Psi_{n_k}$ is bounded in $H^1(G)$, there exists a subsequence $\Psi_{n_{k_l}}$ and a function $\Psi \in H^1(G)$, such that $\Psi_{n_{k_l}}$ converges to $\Psi$ weakly in $H^1(G)$, see, e.g., theorem 2.18 in [28].

By Gagliardo–Nirenberg inequality the sequence $\Psi_{n_{k_l}}$ is uniformly bounded in $L^\infty(G)$. Then, by Rellich-Kondrashov theorem, there exists a subsequence, still denoted by $\Psi_{n_{k_l}}$, such that $(\Psi_{n_{k_l}})_e \to (\Psi)_e$ in $L^\infty(I_e)$ for all the internal edges $e \in E^n$, and $(\Psi_{n_{k_l}})_e \to (\Psi)_e$ in $L^\infty(I)$ for all the external edges $e \in E^e$ and for any bounded subinterval $I$ of $\mathbb{R}$.

Moreover, since the functions $\Psi_{n_{k_l}}$ are continuous in the vertices, so is $\Psi$ and this concludes the proof of the proposition. \hfill \Box

**Remark 3.4.** As a trivial consequence of proposition 2.3, one has that the subsequence $\Psi_{n_{k_l}}$ convergence to $\Psi$ also in $L^p(B(y,t))$, for all $p \geq 1$ and any fixed $y$ and $t$.

For any function $\Psi \in L^2$ and $T > 0$ we define the concentration function $\rho(\Psi,t)$ as

$$\rho(\Psi,t) = \sup_{y \in G} \|\Psi\|_{L^2(B(y,t))}^2.$$  \hfill (3.1)

In the following proposition we prove two important properties of the concentration function: that the sup at the r.h.s. of equation (3.1) is indeed attained at some point of $G$ and the Hölder continuity of $\rho(\Psi, \cdot)$.

**Proposition 3.5.** Let $\Psi \in L^2$ be such that $\|\Psi\| > 0$, then

(i) $\rho(\Psi, \cdot)$ is non-decreasing, $\rho(\Psi, 0) = 0$, $0 < \rho(\Psi, t) \leq M[\Psi]$ for $t > 0$ and $\lim_{t \to \infty} \rho(\Psi, t) = M[\Psi]$.

(ii) There exists $\Psi(t, \cdot) \in G$ such that

$$\rho(\Psi, t) = \|\Psi\|^2_{L^2(B(\mathbf{y},\varepsilon))}.$$  \hfill (3.2)

(iii) If $\Psi \in L^p$ for some $2 \leq p \leq \infty$, then

$$|\rho(\Psi, t) - \rho(\Psi, s)| \leq c \|\Psi\|^2_\infty |t - s|^{\frac{p-2}{p}}$$  \hfill (3.3)

for all $s, t > 0$ and where $c$ is independent of $\Psi$, $s$ and $t$.

**Proof.** The proofs of (i) and (ii) follow directly from the proof of lemma 1.7.4 in [19]. To prove (iii) one uses the inequality

$$|\rho(\Psi, t) - \rho(\Psi, s)| \leq \|\Psi\|^2_{L^2(B(\mathbf{y},\varepsilon))},$$

see lemma 1.7.4 in [19], and the inequalities:

$$\|\Psi\|^2_{L^2(B(\mathbf{y},\varepsilon))} \leq \|\Psi\|^2,$$

for $p = 2$:

$$\|\Psi\|^2_{L^2(B(\mathbf{y},\varepsilon))} \leq \left[\text{Vol}(B(\mathbf{y},\varepsilon) \setminus B(\mathbf{y},s))\right]^{\frac{p-2}{p}} \|\Psi\|^2 \leq (2N|t - s|) \frac{p-2}{p} \|\Psi\|^2_p,$$

for $2 < p < \infty$; and

3286
\[ \|\Psi\|_{L^2(B(y,s))}^2 \leq 2N|t-s|\|\Psi\|_\infty \]

for \( p = \infty \).

For any sequence \( \Psi_n \in L^2 \) we define the concentrated mass parameter \( \tau \) as
\[
\tau = \lim_{t \to \infty} \liminf_{n \to \infty} \rho(\Psi_n, t).
\]

Te parameter \( \tau \) plays a key role in the concentration compactness lemma because it distinguishes the occurrence of vanishing, dichotomy or compactness in \( H^1(\mathcal{G}) \)-bounded sequences.

The following lemma (see for the standard case lemma 1.7.5 in [19]), proves that \( \tau \) can be computed as the limit of \( \rho \) on a suitable subsequence.

**Lemma 3.6.** Let \( m > 0 \) and \( \{\Psi_n\}_{n \in \mathbb{N}} \) be such that: \( \Psi_n \in H^1(\mathcal{G}) \),
\[
M[\Psi_n] \to m \quad \text{as} \quad n \to \infty, \quad (3.4)
\]
and
\[
\sup_{n \in \mathbb{N}} \|\Psi_n\| < \infty. \quad (3.5)
\]

Then there exist a subsequence \( \{\Psi_{n_k}\}_{k \in \mathbb{N}} \), a nondecreasing function \( \gamma(t) \), and a sequence \( t_k \to \infty \) with the following properties:

(i) \( \rho(\Psi_{n_k}, \cdot) \to \gamma(\cdot) \in [0, m] \) as \( k \to \infty \) uniformly on bounded sets of \([0, \infty)\).
(ii) \( \tau = \lim_{t \to \infty} \gamma(t) = \lim_{k \to \infty} \rho(\Psi_{n_k}, t_k) = \lim_{k \to \infty} \rho(\Psi_{n_k}, t_k/2) \).

**Proof.** We refer to [19, lemma 1.7.5] for the details of the proof. Here we just remark that the equicontinuity of the sequence \( \rho(\Psi_{n_k}, \cdot) \), needed to apply Arzelà–Ascoli theorem, follows from (3.3), and from the fact that, by Gagliardo–Nirenberg inequality and assumptions (3.4) and (3.5), \( \|\Psi_n\|_\infty \) is uniformly bounded in \( n \).

We are now ready to prove the concentration compactness lemma. Although the statement of the lemma is similar both to the standard case (see [19, proposition 1.7.6]) and to lemma 3.3 in [5] where the case of star graph is treated, its proof requires several adjustments and changes and for this reason we provide all the details. We also remark that the argument used here to prove the existence of runaway sequences is simpler than the one used in [5].

**Lemma 3.7 (concentration compactness).** Let \( m > 0 \) and \( \{\Psi_n\}_{n \in \mathbb{N}} \) be such that:
\[
\Psi_n \in H^1(\mathcal{G}), \quad (3.6)
\]

\[
\sup_{n \in \mathbb{N}} \|\Psi_n\| < \infty. \quad (3.7)
\]

Then there exists a subsequence \( \{\Psi_{n_k}\}_{k \in \mathbb{N}} \) such that:

(i) (Convergence) If \( \tau = m \), at least one of the two following cases occurs:
   
   (i1) (Convergence) There exists a function \( \Psi \in H^1(\mathcal{G}) \) such that \( \Psi_{n_k} \to \Psi \) in \( L^p \) as \( k \to \infty \) for all \( 2 \leq p \leq \infty \).
   
   (i2) (Runaway) There exists \( \varepsilon^* \in \mathbb{E}^* \), such that for any \( t > 0 \), and \( 2 \leq p \leq \infty \)
\[
\lim_{k \to \infty} \left( \sum_{e \notin e^*} \| (\Psi_{n_k})_e \|^p_{L^p_G} + \| (\Psi_{n_k})_e^* \|^p_{L^p((0,1))} \right) = 0. \tag{3.8}
\]

(ii) (Vanishing) If \( \tau = 0 \), then \( \Psi_{n_k} \to 0 \) in \( L^p \) as \( k \to \infty \) for all \( 2 < p \leq \infty \).

(iii) (Dichotomy) If \( 0 < \tau < m \), then there exist two sequences \( \{ R_k \}_{k \in \mathbb{N}} \) and \( \{ S_k \}_{k \in \mathbb{N}} \) in \( H^1(G) \) such that

\[
\text{supp } R_k \cap \text{supp } S_k = \emptyset \tag{3.9}
\]

\[
| R_k(\lambda) | + | S_k(\lambda) | \leq | \Psi_{n_k}(\lambda) | \quad \forall \lambda \in G \tag{3.10}
\]

\[
\lim_{k \to \infty} M[R_k] = \tau \quad \lim_{k \to \infty} M[S_k] = m - \tau \tag{3.11}
\]

\[
\liminf_{k \to \infty} \left( \| \Psi_{n_k} \| - \| R_k \| - \| S_k \| \right) > 0 \tag{3.12}
\]

\[
\lim_{k \to \infty} \left( \| \Psi_{n_k} \|^p - \| R_k \|^p - \| S_k \|^p \right) = 0 \quad 2 \leq p < \infty \tag{3.13}
\]

\[
\lim_{k \to \infty} \| \Psi_{n_k} \|^2 - \| R_k \|^2 - \| S_k \|^2 \|_{\infty} = 0. \tag{3.14}
\]

Proof. Let \( \{ \Psi_{n_k} \}_{k \in \mathbb{N}} \), \( \gamma(\cdot) \) and \( t_k \) be the subsequence, the function and the sequence defined in lemma 3.6.

Proof of (i). Suppose \( \tau = m \). By lemma 3.6 (ii), for any \( m/2 < \lambda < m \) there exists \( t_k \) large enough such that \( \gamma(t_k) > \lambda \). Then by lemma 3.6 (i), for \( k \) large enough \( \rho(\Psi_{n_k}, t_k) > \lambda \).

Set \( y_k(t) \equiv \gamma(\Psi_{n_k}, t) \), where \( \gamma(\Psi_{n_k}, t) \) was defined in proposition 3.5 (ii). For \( k \) large enough, we have that

\[
d(\bar{y}_k(t_{m/2}), y_k(t_{\lambda})) \leq t_{m/2} + t_{\lambda}. \tag{3.16}
\]

To prove (3.16), assume that \( d(\bar{y}_k(t_{m/2}), y_k(t_{\lambda})) > t_{m/2} + t_{\lambda} \), then the balls \( B(\bar{y}_k(t_{m/2}), t_{m/2}) \) and \( B(y_k(t_{\lambda}), t_{\lambda}) \) would be disjoint, thus implying

\[
M[\Psi_{n_k}] > \| \Psi_{n_k} \|^2_{B(y_k(t_{m/2}), t_{m/2})} + \| \Psi_{n_k} \|^2_{B(\Psi_{n_k}, t_{\lambda})} > \frac{m}{2} + \lambda \geq m
\]

which is impossible because \( M[\Psi_{n_k}] \to m \). Next we distinguish two cases: \( \{ y_k(t_{m/2}) \}_{k \in \mathbb{N}} \) bounded (it belongs to a finite ball on the graph) and \( \{ y_k(t_{m/2}) \}_{k \in \mathbb{N}} \) unbounded (there is no finite ball on the graph containing the sequence).

Case \( y_k(t_{m/2}) \) bounded. By proposition 3.3 and remark 3.4, we have that there exists a subsequence \( \Psi_{n_k} \) and a function \( \Psi \in H^1(G) \) such that \( \Psi_{n_k} \to \Psi \) weakly in \( H^1(G) \) and \( \Psi_{n_k} \to \Psi \) in \( L^2(B(\Psi, t)) \) for any fixed \( y \) and \( t \).

The function \( \Psi \) might be \( \Psi \) the null function, next we show that for \( y_k \) bounded this is not the case. We prove indeed that \( M[\Psi] = m \) which, together with the weak convergence in \( H^1(G) \), implies that \( \Psi_{n_k} \to \Psi \) in \( L^2 \), then the convergence in \( L^p \) for \( 2 < p \leq \infty \) follows from
Gagliardo–Nirenberg inequality.

Fix $\lambda \in (m/2, m)$, and let $t_\lambda$ be such that $\rho(\Psi_{m_\lambda}, t_\lambda) > \lambda$ for $k$ large enough. Since, by (3.16), $y_k(t_\lambda)$ is bounded as well, up to choosing a subsequence which we still denote by $y_k$, we can assume that $y_k(t_\lambda) \to y^*(t_\lambda)$ and $y_k(t_{m/2}) \to y^*(t_{m/2})$. Then, for any fixed $\varepsilon > 0$ and $k$ large enough we have $d(y^*(t_{m/2}), y_k(t_{m/2})) \leq \varepsilon$, so that, by (3.16) and the triangle inequality, it follows that $d(y^*(t_{m/2}), y_k(t_\lambda)) \leq \varepsilon + t_{m/2} + t_\lambda$. Setting $T = 2(\varepsilon + t_{m/2} + t_\lambda)$ we certainly have that $B(y_k(t_\lambda), T) \subseteq B(y^*(t_{m/2}), T)$ so that

$$
\|\Psi_{m_\lambda}\|^2_{H^2(t_{m/2}, T)} \geq \|\Psi_{m_\lambda}\|^2_{H^2(y_k(t_\lambda), T)} = \rho(\Psi_{m_\lambda}, t_\lambda) > \lambda.
$$

Since

$$
M[\Psi] \geq \|\Psi\|^2_{H^2(t_{m/2}, T)} = \lim_{k \to \infty} \|\Psi_{m_\lambda}\|^2_{H^2(y_k(t_\lambda), T)},
$$

we have that $M[\Psi] \geq \lambda$. As we can choose $\lambda$ arbitrarily close to $m$, we get $M[\Psi] \geq m$. On the other hand, by weak convergence, we have that

$$
M[\Psi] \leq \liminf_{k \to \infty} M[\Psi_{m_\lambda}] = m,
$$

so that $M[\Psi] = m$.

Assume now that $y_k(t_{m/2})$ is unbounded. Then, up to choosing a subsequence, which we still denote by $y_k$, we can assume that there exists $e^* \in E^\infty$ such that $\{y_k(t_{m/2})\}_{k \in \mathbb{N}}$ belongs to the the edge $e^*$ and $y_k(t_{m/2}) \to \infty$.

Fix $\varepsilon$ and $t$. Set $\lambda = m - \varepsilon$ and $t_\lambda$ such that for $k$ large enough $\rho(\Psi_{m_\lambda}, t_\lambda) > \lambda$. By (3.16) we have that $y_k(t_\lambda) \to \infty$, so that, for large enough, $y_k(t_\lambda) - t_\lambda > t$ and

$$
\int_{t}^{\infty} |(\Psi_{m_\lambda})_{e^*}(x)|^2 \, dx \geq \|\Psi_{m_\lambda}\|^2_{L^2(t_\lambda, \infty)} \geq \rho(\Psi_{m_\lambda}, t_\lambda) > \lambda = m - \varepsilon.
$$

On the other hand, by (3.6) and for $k$ large enough, one has that

$$
M[\Psi_{m_\lambda}] = \sum_{e \neq e^*} \|\Psi_{m_\lambda}\|^2_{L^2(t_\lambda)} + \int_{t}^{\infty} |(\Psi_{m_\lambda})_{e^*}(x)|^2 \, dx + \int_{t}^{\infty} |(\Psi_{m_\lambda})_{e^*}(x)|^2 \, dx < m + \varepsilon,
$$

so that

$$
\sum_{e \neq e^*} \|\Psi_{m_\lambda}\|^2_{L^2(t_\lambda)} + \int_{0}^{t} |(\Psi_{m_\lambda})_{e^*}(x)|^2 \, dx < 2\varepsilon.
$$

The limit (3.8) for $p > 2$ follows by Gagliardo–Nirenberg inequalities applied to the graph $\mathcal{G}_t$ obtained from $\mathcal{G}$ by cutting the edge $e^*$ at length $t$. We remark that the graph $\mathcal{G}_t$ might be compact.

Proof of (ii). We start with the proof of a useful inequality, see equation (3.17) below. Let $L_{\max}$ be the maximal length of the internal edges. For any internal edge $e \in E^m$, by Gagliardo–Nirenberg inequality applied to the interval $I_e$, by equation (3.1), and since $\rho(\Psi, \cdot)$
is non-decreasing, one has that
\[
\|\psi\|_{L^6(\mathbb{R}_+)}^6 \leq c_c \|\phi\|_{L^6(\mathbb{R}_+)}^6 \|\psi\|_{H^1(\mathbb{R}_+)}^2
\]
\[
\leq c_c \rho(\psi, L_{\text{max}}/2)^2 \|\psi\|_{H^1(\mathbb{R}_+)}^2 \leq c \rho(\Psi, L_{\text{max}}/2)^2 \|\psi\|_{H^1(\mathbb{R}_+)}^2
\]
where \(c_c\) is a constant that depends on the edge \(e \in E^n\) (on the length of the interval \(I_e\)) and we set \(c = \max_{e \in E^n} c_c\). On the other hand, for any external edge \(e \in E^{\text{ext}}\), one has
\[
\|\psi\|_{L^6(\mathbb{R}_+)}^6 = \sum_{n=0}^{\infty} \|\psi\|_{L^6((nL_{\text{max}}(n+1)L_{\text{max}}))}^6
\]
\[
\leq c \sum_{n=0}^{\infty} \|\psi\|_{L^6((nL_{\text{max}}(n+1)L_{\text{max}}))}^6 \|\psi\|_{H^1((nL_{\text{max}}(n+1)L_{\text{max}}))}^2
\]
\[
\leq c \rho(\psi, L_{\text{max}}/2)^2 \sum_{n=0}^{\infty} \|\psi\|_{H^1((nL_{\text{max}}(n+1)L_{\text{max}}))}^2 = c \rho(\Psi, L_{\text{max}}/2)^2 \|\psi\|_{H^1(\mathbb{R}_+)}^2,
\]
where \(c\) is a constant that depends on \(L_{\text{max}}\). Summing up on internal and external edges we get
\[
\|\Psi\|_{L^6}^6 \leq c \rho(\Psi, L_{\text{max}}/2)^2 \|\Psi\|_{H^1}^2.
\] (3.17)

Suppose now that \(r = 0\). By lemma 3.6, \(\tau = \lim_{k \to \infty} \rho(\psi, t_k) = 0\). Then since \(\rho(\Psi, \cdot)\) is non-decreasing and \(t_k \to \infty\), \(\lim_{k \to \infty} \rho(\psi, L_{\text{max}}/2) = 0\), and \(\lim_{k \to \infty} \|\psi\|_{H^1} = 0\) by (3.17).

The statement for \(2 < p < 6\) follows from the Hölder inequality \(\|\Psi\|_{L^p} \leq \|\Psi\|_{L^6}^{\frac{6}{p}} \|\Psi\|_{L^{\infty}}^{\frac{6-p}{p}}\), while for \(6 < p \leq \infty\) one uses inequality (2.2) with \(q = 6\).

Proof of (iii). Suppose that \(0 < \tau < m\) and let \(\theta\) and \(\varphi\) be two cut-off functions such that \(\theta, \varphi \in C(\mathbb{R}_+)\), \(0 \leq \theta, \varphi \leq 1\) and
\[
\begin{align*}
\theta(t) &= \begin{cases} 
1 & 0 \leq t \leq 1/2 \\
0 & t \geq 3/4
\end{cases} \\
\varphi(t) &= \begin{cases} 
0 & 0 \leq t \leq 3/4 \\
1 & t \geq 1
\end{cases}
\end{align*}
\]

Set \(y(t_k) \equiv y(\psi, t_k)\), where \(y(\psi, t)\) was defined in proposition 3.5 (ii). Define the following cut off functions
\[
\Theta_k(\chi) = \theta \left( \frac{d(\chi, y(t_k/2))}{t_k} \right) \quad \Phi_k(\chi) = \varphi \left( \frac{d(\chi, y(t_k/2))}{t_k} \right).
\] (3.18)

We remark that \((\Theta_k)_e(x) = \theta(d(\chi, y(t_k/2)))/t_k)\) with \(d\chi\) given as in proposition 3.1, and similarly for \(\Phi_k\).

Let \(R_k\) be defined by
\[
R_k(\chi) = \Theta_k(\chi)\psi_k(\chi),
\]
and let \(S_k\) be defined by
\[
S_k(\chi) = \Phi_k(\chi)\psi_k(\chi),
\]
products to be understood pointwise. We remark that \(R_k\) (\(S_k\) resp.) coincides with \(\psi_k\) in the ball \(B(y(t_k/2), t_k/2)\) (in the set \(G_\theta(B(y(t_k/2), t_k)\) resp.) and \(R_k = 0\) (\(S_k = 0\) resp.) in the
set $G(B(y(t_k/2), 3t_k/4)$ (in the ball $B(y(t_k/2), 3t_k/4)$ resp.). Properties (3.9) and (3.10) are immediate. Property (3.11) also immediately follows from the definitions of $R_k$ and $S_k$ and from proposition 3.1. Next we notice that by proposition 3.5, (ii),
\[
\rho(\Psi_{n_k}, t_k/2) = \|\Psi_{n_k}\|^2_{B(y(t_k/2), 3t_k/4)} \leq M[R_k].
\]
Moreover, since $\theta(t) \leq 1$,
\[
M[R_k] = \|\Psi_{n_k}\|^2_{B(y(t_k/2), 3t_k/4)} \leq \|\Psi_{n_k}\|^2_{B(y(t_k/2), 3t_k/4)} = \rho(\Psi_{n_k}, t_k),
\]
where we have taken into account the optimality of $y(t_k)$ according to proposition 3.5, (ii) and the definition of $\rho(\Psi, t)$. Therefore
\[
\lim_{k \to \infty} M[R_k] = \tau
\]
by lemma 3.6, (ii). Define $Z_k := \Psi_{n_k} - R_k - S_k$ and notice that
\[
\text{supp}(Z_k) \subseteq B(y(t_k/2), t_k) \setminus B(y(t_k/2), t_k/2)
\]
and $|Z_k| \leq |\Psi_{n_k}|$, to be understood pointwise. Then one has
\[
M[Z_k] \leq \|\Psi_{n_k}\|^2_{B(y(t_k/2), 3t_k/4)} \leq \rho(\Psi_{n_k}, t_k) - \rho(\Psi_{n_k}, t_k/2)
\]
again by the optimality properties of $y(t_k)$. It follows from (3.19) and lemma 3.6, (ii) that
\[
M[Z_k] \to 0 \quad \text{as} \quad k \to \infty,
\]
and therefore $M[S_k] \to m - \tau$ which concludes the proof of (3.12).

To prove (3.14) and (3.15) we use
\[
\|\Psi_{n_k}(\xi)^p - |R_k(\xi)|^p - |S_k(\xi)|^p\| \leq c_p |\Psi_{n_k}(\xi)|^{p-1}|Z_k(\xi)| \quad p \geq 1,
\]
(3.21)
to be understood pointwise, which in turn implies
\[
\|\Psi_{n_k}\|_p - \|R_k\|_p - \|S_k\|_p \leq c \|\Psi_{n_k}\|^{p-1}_2 \|Z_k\|_p \leq c \|Z_k\|_p \quad p \geq 2
\]
where we used (3.6), (3.7), and Gagliardo–Nirenberg inequality (2.2). The limit (3.14) then follows from $\|Z_k\| \to 0$. To prove (3.15) we use (3.21) with $p = 1$, and the fact that, by $\|Z_k\|_W \leq c, \|Z_k\| \to 0$, and Gagliardo–Nirenberg inequality, one has $\|Z_k\|_\infty \to 0$.

Concerning the inequality (3.13), first notice that
\[ |(\Psi_m)_x| \leq |(R_k)_x| - |(S_k)_x| \]
\[ = |(\Psi_m)_x|^2 [1 - (\Theta_k)^2 - (\Phi_k)^2] \]
\[ - |(\Psi_m)_e| [(|\Theta_k|^2 + (\Phi_k)^2)] - \text{Re}(\overline{\Psi_m} \cdot (\Psi_m)_x) [(|\Theta_k|^2 + (\Phi_k)^2)]' \]
\[ \geq - \frac{c}{t_k^2} |(\Psi_m)_e|^2 - \frac{c}{t_k} |(\Psi_m)_e| |(\Psi_m)_e| \]

for almost all \( x \in I_e \), where we used \( 1 - (\Theta_k)^2 - (\Phi_k)^2 \geq 0 \) and the fact that \( |(\Theta_k)_e(x)| \leq c/t_k \), \( |(\Phi_k)_e(x)| \leq c/t_k \) for almost all \( x \in I_e \) (see the remark below equation (3.18) and proposition 3.1). The inequality (3.13) follows by integrating on \( I_e \), and summing up on \( e \), and by recalling that \( t_k \rightarrow \infty \).

**Remark 3.8.** We note that equation (3.8) in lemma 3.7-(ii) implies that in the runaway case
\[ \lim_{{k \rightarrow \infty}} ||\Psi_m||_{{L^p(B(x,t))}} = 0 \]
for any \( 2 \leq p \leq \infty \), \( \gamma \in \mathcal{G} \), and \( t > 0 \).

4. Variational analysis

**Proof of theorem 1.** We prove first that \( mE_0 < \nu < \infty \). The lower bound \( \nu > mE_0 \) is a direct consequence of the fact that \( E[\Psi] < E^\text{lin}[\Psi] \), for all \( \Psi \in H^1(\mathcal{G}) \), and that, by the definition of \( E_0 \) and \( E^\text{lin} \), one has
\[ \inf \{E^\text{lin}[\Psi] \text{ s.t. } \Psi \in H^1(\mathcal{G}), \ M[\Psi] = m \} = -mE_0. \]

To prove that \( \nu < +\infty \) we first note that, by using Hölder and Gagliardo–Nirenberg inequalities, one can prove the bounds:
\[ \|\Psi\|_{{L^{2q+2}}^2} \leq c \||\Psi\|_{{L^q}}^\mu \||\Psi\|_{{L^{2q}}}^{2+\mu}; \]  \hspace{1cm} (4.1)
\[ (\Psi, W_- \Psi) \leq \|W_-\|_r \||\Psi\|_{{L^{2q/r}}}^2 \|\Psi\|_{{L^{2q}}}^{2(1-\alpha)} \]  \hspace{1cm} (4.2)

for all \( q \in [2, 2r/(r-1)] \) and with \( \alpha = \frac{2}{2q} \left( 1 - \frac{2(r-1)}{2r} \right) \); and
\[ |\Psi(x)|^2 \leq \|\Psi\|_{{L^2}}^2 \leq c \||\Psi\|_{{H^r}} \||\Psi\| \quad \forall x \in V. \]  \hspace{1cm} (4.3)

We remark that the inequalities (4.1)–(4.3) hold true for any connected finite graph. If \( M[\Psi] = m \), by (4.1)–(4.3) we have
\[ E[\Psi] + m \geq \|\Psi\|_{{H^r}}^2 - Cm^{\frac{2+\mu}{\mu+1}} \|\Psi\|_{{H^r}}^\mu - C\sqrt{m} \sum_{g \in \mathcal{G}} |\alpha(g)| \|\Psi\|_{{H^r}} - Cm^{1-1/(2r)} \|W_-\|_r \||\Psi\|_{{H^r}}^{1/r}. \]

We notice that for any \( a, b, c, d > 0 \), \( r \geq 1 \), and \( 0 < \mu < 2 \) there exist \( \delta, \beta > 0 \) such that \( ax^2 - bx^\mu - cx - dx^{1/r} > \delta x^2 - \beta \), for any \( x \geq 0 \), then
\[ E[\Psi] + m \geq \delta \|\Psi\|_{{H^r}}^2 - \beta, \]  \hspace{1cm} (4.4)

which implies \( \nu \leq \beta + m \).
In the remaining part of the proof we shall prove that we can choose \( m^* \) such that for \( m < m^* \) minimizing sequences have a convergent subsequence.

Let \( \{ \Psi_n \}_{n \in \mathbb{N}} \) be a minimizing sequence, i.e. \( \Psi_n \in H^1(G) \), \( M[\Psi_n] = m \), and \( \lim_{n \to \infty} E[\Psi_n] = -\nu \). Concerning the mass constraint, we remark that it is enough to assume \( M[\Psi_n] \to m \) as \( n \to \infty \), in such a case one can define \( \tilde{\Psi}_n = \sqrt{m} \Psi_n / \| \Psi_n \| \) and note that \( \lim_{n \to \infty} E[\tilde{\Psi}_n] = \lim_{n \to \infty} E[\Psi_n] \).

We shall prove that there exists \( \hat{\Psi} \in H^1(G) \) such that \( M[\hat{\Psi}] = m \), \( E[\hat{\Psi}] = -\nu \) and \( \Psi_n \to \hat{\Psi} \) in \( H^1(G) \).

We can assume that \( E[\Psi_n] \leq -\nu/2 \) then by inequality (4.4), up to taking a subsequence, we can assume that
\[
\sup_{n \in \mathbb{N}} \| \Psi_n \|_{H^1} \leq \infty,
\]
moreover the following lower bound holds true
\[
\frac{1}{\mu + 1} \| \Psi_n \|_{2 \mu + 2}^2 + (\Psi, W_\mu \Psi) + \sum_{v \in V^-} |\alpha(v)||\Psi_n(v)|^2 \geq \frac{\nu}{2}.
\]

Next we use lemma 3.7 and prove that vanishing and dichotomy cannot occur for \( \{ \Psi_n \}_{n \in \mathbb{N}} \). Set \( \tau = \lim_{t \to \infty} \liminf_{n \to \infty} \rho(\Psi_n, t) \). First we prove that vanishing cannot occur. If \( \tau = 0 \), then by lemma 3.7 there would exist a subsequence \( \Psi_{n_k} \) such that \( \| \Psi_{n_k} \|_p \to 0 \) for all \( 2 < p \leq \infty \) but this, together with equations (4.2) and (4.3), would contradict (4.5).

To prove that dichotomy cannot occur, suppose \( 0 < \tau < m \), then there would exist \( R_k \) and \( S_k \) satisfying (3.9)–(3.15). In particular we know that
\[
\liminf_{k \to \infty} (\| \Psi_{n_k} \|_p^2 - \| R_k \|_p^2 - \| S_k \|_p^2) \geq 0
\]
and
\[
\liminf_{k \to \infty} (\| \Psi_{n_k} \|_p^2 - \| R_k \|_p^2 - \| S_k \|_p^2) = 0 \quad 2 \leq p < \infty
\]

Moreover we claim that
\[
\lim_{k \to \infty} (\Psi_{n_k}, W\Psi_{n_k}) - (R_k, WR_k) - (S_k, WS_k) \geq 0,
\]
we postpone the proof of this claim to the end of the discussion. Summing up, we arrive at
\[
\liminf_{k \to \infty} (E[\Psi_{n_k}] - E[R_k] - E[S_k]) \geq 0,
\]
which implies
\[
\limsup_{k \to \infty} (E[R_k] + E[S_k]) \leq -\nu.
\]
Notice that, given \( \Psi \in H^1(G) \) and \( \delta > 0 \), then
We remark that \( R_k, S_k \in H^1(\mathcal{G}) \), since \( \Psi_{n_k} \) satisfies the continuity condition at the vertices and the multiplication with the cut-off functions preserves that. Let \( \delta_k = \sqrt{m/M[R_k]} \) and \( \gamma_k = \sqrt{m/M[S_k]} \) such that \( M[\delta_k R_k], M[\gamma_k S_k] = m \). Then, using the above equality and the fact that \( E[\delta_k R_k], E[\gamma_k S_k] \geq -\nu \), one has

\[
E[R_k] \geq -\nu \left( \frac{1}{\delta_k^2} + \frac{1}{\gamma_k^2} \right) + \frac{\delta_k^{2\mu} - 1}{\mu + 1} \|R_k\|_{2\mu+2}^2 + \frac{\gamma_k^{2\mu} - 1}{\mu + 1} \|S_k\|_{2\mu+2}^2.
\]

from which

\[
E[R_k] + E[S_k] \geq -\nu \left( \frac{1}{\delta_k^2} + \frac{1}{\gamma_k^2} \right) + \frac{\delta_k^{2\mu} - 1}{\mu + 1} \|R_k\|_{2\mu+2}^2 + \frac{\gamma_k^{2\mu} - 1}{\mu + 1} \|S_k\|_{2\mu+2}^2.
\]

Notice that by (3.12)

\[
\frac{1}{\delta_k^2} \to \frac{\tau}{m} \quad \text{and} \quad \frac{1}{\gamma_k^2} \to 1 - \frac{\tau}{m}.
\]

Let \( \theta = \min\{\left(\tau/m\right)^{-\nu}, (1 - \tau/m)^{-\nu}\} \) and notice that \( \theta > 1 \) since \( 0 < \tau/m < 1 \). Therefore

\[
\liminf_{k \to \infty} (E[R_k] + E[S_k]) \geq -\nu + \frac{\theta - 1}{\mu + 1} \liminf_{k \to \infty} \|\Psi_{n_k}\|_{2\mu+2}^2 \geq -\nu,
\]

where we used the fact that \( \liminf_{k \to \infty} \|\Psi_{n_k}\|_{2\mu+2}^2 \neq 0 \). The latter claim is proved by noticing that \( \liminf_{k \to \infty} \|\Psi_{n_k}\|_{2\mu+2}^2 = 0 \), together with \( \|\Psi_{n_k}\|_{\mu} \) bounded and equations (4.2) and (4.3), would imply \( \liminf_{k \to \infty} \|\Psi_{n_k} W_{-\Psi_{n_k}}\| = 0 \) and \( \liminf_{k \to \infty} \|\Psi_{n_k}\|_{\infty} = 0 \). Hence, there would be a contradiction with inequality (4.5). We conclude that if \( 0 < \tau < m \) we get a contradiction, cfr. inequalities (4.7) and (4.8). To end the analysis of the case \( 0 < \tau < m \) we are left to prove the claim (4.6). We rewrite \( W = W_+ - W_- \) and consider first the term with \( W_+ \). We have that

\[
(W_+ \Psi_{n_k} - R_k W_+ R_k - S_k W_+ S_k) = \sum_{e} \int_{\mathcal{L}_e} (W_+)_{\Psi_{n_k}} \left[ 1 - (\Theta_k)^2_{\Psi_{n_k}} - (\Phi_k)^2_{\Psi_{n_k}} \right] (|\Psi_{n_k}|)^2 \, dx \geq 0.
\]

Since \( R_k \) and \( S_k \) have disjoint supports, we have that

\[
|\Psi_{n_k} W_- \Psi_{n_k}| - (R_k, W_- R_k) - (S_k, W_- S_k) \leq |(Z_k, W_- Z_k)| + 2 |(R_k, W_- Z_k)| + 2 |(S_k, W_- Z_k)|
\]

\[
\leq |(Z_k, W_- Z_k)| + 2 (R_k, W_- R_k)^{1/2} (Z_k, W_- Z_k)^{1/2} + 2 (S_k, W_- Z_k)^{1/2} (Z_k, W_- Z_k)^{1/2}.
\]

The terms containing \( R_k \) and \( S_k \) are bounded by lemma 3.7 and inequality (4.2). The terms
containing $Z_k$, go to zero by inequality (4.2) and because $\|Z_k\| \to 0$ by equation (3.20). From which the claim (4.6) follows.

Since $0 \leq \tau < m$ leads us to a contradiction, it must be $\tau = m$.

Now we prove that for $m < m^*$ the minimizing sequence is not runaway. Here the limitation on the mass plays a role for the first time. By absurd suppose that there exists $|L_{\tau 1}| \Psi = n (\Psi, \| = |W_n| \neq 0$. $\varepsilon_n \| \| (\| 2 \leq m$.

By equation (3.8) (see also remark 3.8), from which the second limit in (4.9).

We start by noticing that this is a direct consequence of lemma 3.7 and inequality (4.2) applied to the edge $I_{e}$. We are left to prove that

$$\lim_{n \to \infty} \int_{I_{e}} (W_{-})_{\varepsilon} (\Psi_n)_{\varepsilon}^2 \, dx = 0. \quad (4.10)$$

We start by noticing that $\|\Psi_n\|_{L^p}$ is uniformly bounded, hence, so is $\|\Psi_n\|_p$ for all $p \in [2, +\infty]$, by (2.2) (with $q = 2$). As a consequence, we have that for any $\varepsilon > 0$ there exists $R > 0$ (independent of $n$) such that

$$\int_{R}^{+\infty} (W_{-})_{\varepsilon} (\Psi_n)_{\varepsilon}^2 \, dx \leq \| (W_{-})_{\varepsilon} \|_{L^1(R, \infty)} \|\Psi_n\|_{L^2(R, \infty)}^2 \leq \varepsilon,$$

with $r'$ such that $r'^{-1} + r'^{-1} = 1$. For such $R$, there exists $n_0$ such that for all $n > n_0$ one has

$$\int_{0}^{R} (W_{-})_{\varepsilon} (\Psi_n)_{\varepsilon}^2 \, dx \leq \| W_{-} \|_{L^p} \| (\Psi_n)_{\varepsilon} \|_{L^2(R, \infty)}^2 \leq \varepsilon$$

by (3.8) (see also remark 3.8), from which the second limit in (4.9).

Recalling that, by lemma 3.7—equation (3.8), one has $\lim_{n \to \infty} \| (\Psi_n)_{\varepsilon} \|_{L^2(R, \infty)} = 0$ for all $\varepsilon \neq e^*$, and by equation (4.9), we infer

$$\lim_{n \to \infty} E[\Psi_n] \geq \lim_{n \to \infty} \int_{0}^{\infty} (\Psi_n)_{\varepsilon}^2 \, dx - \frac{1}{\mu + 1} \int_{0}^{\infty} (\Psi_n)_{\varepsilon}^2 \, dx. \quad (4.11)$$

Let $\chi : R_+ \to [0, 1]$ be a function such that $\chi \in C^\infty(R_+)$, $\chi(0) = 0$ and $\chi(x) = 1$ for all $x \geq 1$. Define

$$\psi_n^\ast(x) := \chi(x)(\Psi_n)_{\varepsilon}(x),$$

so that $\psi_n^\ast(0) = 0$, and $\| \psi_n^\ast \|_{L^2(R_+)} \leq \varepsilon$. By lemma 3.7—equation (3.8), for all $p \geq 2$,

$$\lim_{n \to \infty} \| \Psi_n \|_p = \lim_{n \to \infty} \| (\Psi_n)_{\varepsilon} \|_{L^p(0, \infty)} = \lim_{n \to \infty} \| \psi_n^\ast \|_{L^p(0, \infty)}, \quad (4.12)$$

where we used the fact that $\lim_{n \to \infty} \| (\Psi_n)_{\varepsilon} \|_{L^1(0, 1)} = 0$, and the trivial bound $\| \psi_n^\ast \|_{L^1(0, 1)} \leq \| \chi \|_{L^1(0, 1)} \| (\Psi_n)_{\varepsilon} \|_{L^1(0, 1)}$. In particular, $\lim_{n \to \infty} \| (\Psi_n)_{\varepsilon} \|_{L^2(0, \infty)}^2 = \lim_{n \to \infty} \| \psi_n^\ast \|_{L^2(0, \infty)}^2 = m$. Moreover we have that
To prove the latter inequality, we note that
\[
\lim_{n \to \infty} \int_{0}^{\infty} |(\Psi_{n})_{e}^{t}|^2 \, dx = \lim_{n \to \infty} \int_{0}^{\infty} |\Psi_{n}^{*} e^{t}|^2 (1 - \chi^2) \, dx \\
+ \lim_{n \to \infty} \int_{0}^{1} |(\Psi_{n})_{e}^{t}|^2 \chi^2 + 2 \chi \Re(\Psi_{n})_{e}^{t} \Psi_{n}^{*} \, dx \\
= \lim_{n \to \infty} \int_{0}^{1} |(\Psi_{n})_{e}^{t}|^2 (1 - \chi^2) \, dx \geq 0,
\]
where we used again lemma 3.7—equation (3.8) and the bounds \( \| \chi \|_{\infty}, \| \chi' \|_{\infty} \leq c \).

We have the following chain of inequalities/identities
\[
\lim_{n \to \infty} E[\Psi_{n}] \\
\geq \lim_{n \to \infty} \int_{0}^{\infty} |\psi_{n}^{*} e^{t}|^2 \, dx - \frac{1}{\mu + 1} \int_{0}^{\infty} |\psi_{n}^{*} e^{t}|^{2 \mu + 2} \, dx \\
(\text{we used equations (4.11) - (4.13)}) \\
\geq \inf \left\{ \int_{0}^{\infty} |\psi^{*}(x)|^2 \, dx - \frac{1}{\mu + 1} \int_{0}^{\infty} |\psi(x)|^{2 \mu + 2} \, dx \, \text{s.t.} \, \psi \in H^{1}(\mathbb{R}^{+}), \, \psi(0) = 0, \, \| \psi \|_{L^{2}(\mathbb{R}^{+})} = m \right\} \\
(\text{we used the fact that } \psi_{n}^{*} \in H^{1}(\mathbb{R}^{+}), \, \psi_{n}^{*}(0) = 0, \, \text{and } \| \psi_{n}^{*} \|_{L^{2}(\mathbb{R}^{+})} \to m \text{ as } n \to \infty) \\
= \inf \left\{ \int_{\mathbb{R}} |\psi^{*}(x)|^2 \, dx - \frac{1}{\mu + 1} \int_{\mathbb{R}} |\psi(x)|^{2 \mu + 2} \, dx \, \text{s.t.} \, \psi \in H^{1}(\mathbb{R}), \, \psi(0) = 0 \, \forall x \leq 0, \, \| \psi \|_{L^{2}(\mathbb{R})} = m \right\} \\
(\text{where we used the fact that } \psi \in H^{1}(\mathbb{R}_{+}) \text{ and } \psi(0) = 0 \text{ if and only if its zero extension belongs to } H^{1}(\mathbb{R}), \text{ see, e.g.}[12, \text{theorem 5.29}]) \\
\geq \inf \left\{ \int_{\mathbb{R}} |\psi^{*}(x)|^2 \, dx - \frac{1}{\mu + 1} \int_{\mathbb{R}} |\psi(x)|^{2 \mu + 2} \, dx \, \text{s.t.} \, \psi \in H^{1}(\mathbb{R}), \, \| \psi \|_{L^{2}(\mathbb{R})} = m \right\} \\
(\text{we enlarged the set on which the } \inf \text{ is taken). (4.14)}
\]

It is well known that the infimum in the latter minimization problem is indeed attained and that the minimizing function (up to translations and phase multiplications) is given by the soliton profile
\[
\phi(x) = [(\mu + 1) \omega_{R}]^{\frac{1}{2}} \operatorname{sech} \left( \frac{x}{\sqrt{2} \mu \omega_{R}} \right).
\]
The frequency \( \omega_{R} \) is fixed by the mass constraint through the relation
\[
m = \| \phi \|_{L^{2}(\mathbb{R})}^2 = 2 \left( \frac{\mu + 1}{\mu} \right)^{\frac{1}{2}} \omega_{R}^{\frac{1}{2} - \frac{1}{\mu}} \int_{0}^{1} (1 - t^2)^{\frac{1}{2} - 1} \, dt,
\]
which gives
\[
\omega_{R} = \left( \frac{2 (\mu + 1)}{\mu} \right)^{\frac{1}{2}} \int_{0}^{1} (1 - t^2)^{\frac{1}{2} - 1} \, dt \right)^{-\frac{2 \mu}{2 \mu - 1}} m^{\frac{2 \mu - 1}{2 \mu}}.
\]
The infimum in the minimization problem (4.14) is given by the nonlinear energy of the soliton

$$\int_{\mathbb{R}} |\phi'(x)|^2 \, dx - \frac{1}{\mu + 1} \int_{\mathbb{R}} |\phi(x)|^{2\mu+2} \, dx = -\frac{2}{2 + \mu} \omega R \, m = -\gamma_\mu m^{1+\frac{2\mu}{\mu+2}},$$

with $\gamma_\mu = \frac{2 - \mu}{2 + \mu} \left( \frac{\mu+1}{\mu} \int_{0}^{1} (1 - t^2)^{\frac{1}{\mu}} \, dt \right) - \frac{2\mu}{\mu+2}$. So that by the inequality (4.14), we conclude that if $\Psi_n$ is a runaway sequence it must be

$$\lim_{n \to \infty} E[\Psi_n] \geq -\gamma_\mu m^{1+\frac{2\mu}{\mu+2}}.$$

To show that for $m$ small enough a minimizing sequence cannot be runaway it is enough to notice that

$$E[\Phi_0] < E^{\text{lin}}[\Phi_0] = -m E_0$$

For sufficiently small $m$ we have $-m E_0 < -\gamma_\mu m^{1+\frac{2\mu}{\mu+2}}$ and hence, a minimizing sequence cannot be runaway.

By lemma 3.7 we conclude that for all $0 < m < m^*$ there exists a state $\hat{\Psi}_m \in H^1(G)$ such that minimizing sequences converge, up to taking subsequences, to $\hat{\Psi}_m$ in $L^p$ for $p \geq 2$. In particular, $M[\hat{\Psi}_m] = m$, and the potential, vertices, and nonlinear terms in $E[\Psi_n]$ converge to the corresponding ones in $E[\hat{\Psi}_m]$. Taking into account also the weak lower continuity of the $H^1$ norm we have

$$E[\hat{\Psi}_m] \leq \lim_{n \to \infty} E[\Psi_n] = -\nu$$

which implies that $E[\hat{\Psi}_m] = -\nu$. Since $E[\hat{\Psi}_m] = \lim_{n \to \infty} E[\Psi_n]$ then $||\hat{\Psi}_m|| = \lim_{n \to \infty} ||\Psi_n||$ and we have proved that $\Phi_n \to \hat{\Psi}_m$ in $H^1$.

5. Bifurcation analysis

In this section we prove theorem 2. Preliminarily we describe the bifurcation of stationary solutions from the linear ground state. A reference on the standard one dimensional case is [36].

**Proposition 5.1 (Bifurcation from the linear ground state).** If assumptions 1–3 hold true, then there exists $\delta > 0$ such that, for any $\omega \in (E_0, E_0 + \delta)$, equation (1.3) admits a unique (up to phase multiplication) solution $\Phi(\omega)$. Moreover, the function $m(\omega) := ||\Phi(\omega)||^2$ belongs to $C^1(E_0, E_0 + \delta)$, is such that

$$m(\omega) = \left( \frac{\omega - E_0}{||\Phi_0||^{2\mu+2}} \right)^{\frac{1}{2 \mu+2}} + o \left( (\omega - E_0)^{\frac{1}{2 \mu+2}} \right),$$

and it is invertible. Denoting its inverse by $\omega(m)$, one has that the function $E(m) := E[\Phi(\omega(m))]$ is continuous for $m > 0$ small enough, and

$$E(m) = -E_0 m + o(m).$$

**Proof of proposition 5.1.** We follow the approach used in [26]. Without loss of generality...
ity, we can take $\Phi(\omega)$ real valued. We start by noting that $D(H)$ with the graph norm $\|\Phi\|_H$ is a Banach space.

We note that the following inequality holds true:

$$\|\psi\|^{2+\mu} \leq C\|\psi\|_H^{2+\mu+1}. $$

To prove it we use first Hölder and Gagliardo–Nirenberg inequalities to obtain

$$\|\psi\|^{2+\mu} \leq C\|\psi\|^\mu \|\psi\|_H^{\mu+1}. $$

Then we prove that $\|\psi\| \leq C\|\psi\|_H$. To this aim, we use the fact that $E[\psi] = (\psi, H\psi)$, which in turn implies

$$\|\psi\|^2 \leq (\psi, H\psi) + (\psi, W_-\psi) + \sum_{\mathcal{V} \in \mathcal{V}_-} |\alpha(\mathcal{V})||\psi(\mathcal{V})|^2$$

$$\leq \|\psi\|_H^2 + C_0 \left(\|\psi\|_H^{2+\mu} + \|\psi\|^\mu \|\psi\|_H\right). \quad (5.3)$$

for some large constant $C_0$. Here we used again Gagliardo–Nirenberg inequality and

$$(\psi, W_-\psi) \leq \|W_-\|_{L^1} \|\psi\|_H^{2+\mu} \leq C\|\psi\|_H^{2+\mu+1},$$

see also equation (4.2) below. By the bound (5.3) we infer that if $\|\psi\| > 4C_0\|\psi\|_H$ it must be $\|\psi\|^2 \leq 2\|\psi\|_H^2$, hence $\|\psi\| \leq C\|\psi\|_H$.

Let us introduce the map $F : D(H) \times \mathbb{R}_+ \to L^2$

$$F(\Phi, \omega) = (H + \omega)\Phi - |\Phi|^{2+\mu}\Phi. \quad (5.4)$$

It is clear that $F \in C^1(D(H) \times \mathbb{R}_+, L^2)$. Notice that

$$D_\Phi F(0, \omega)\Phi = (H + \omega)\Phi.$$

We use the Lyapunov–Schmidt method to study the existence of solutions of

$$F(\Phi, \omega) = 0, \quad (5.5)$$

which is equivalent to equation (1.3). Let us introduce two orthogonal projectors in $L^2$

$$P = \Phi_0 (\Phi_0, \cdot) \quad Q = I - P$$

considered as operators on $D(H)$. We decompose accordingly

$$\Phi = a\Phi_0 + \Theta,$$

where $a = (\Phi_0, \Phi)$ and $Q\Theta = 0$. This decomposition is well defined on $D(H)$, i.e. $\Theta \in D(H)$. Moreover, since if $\Phi$ is a solution of equation (5.4) so is $-\Phi$ we can assume $a \geq 0$. Then equation (5.5) is equivalent to the system

$$\begin{cases}
QF(a\Phi_0 + \Theta, \omega) = 0 \\
P\Phi(a\Phi_0 + \Theta, \omega) = 0
\end{cases} \quad (5.6)$$

The first equation in (5.6) is called the auxiliary equation, the second one is the bifurcation equation. We introduce the map $G : \mathbb{R}_+ \times D(H) \times \mathbb{R}_+ \to L^2$.
Nonlinearity 30 (2017) 3271

C Cacciapuoti et al

\[ G(a, \Theta, \omega) := QF(a\Phi_0 + \Theta, \omega) = Q(H + \omega)Q\Theta - Q[a\Phi_0 + \Theta]^2(2\mu)(a\Phi_0 + \Theta), \]

hence, the auxiliary equation is equivalently written as \( G(a, \Theta, \omega) = 0 \). Since

\[ D_\Theta G(0, 0, E_0) = Q(H + E_0)Q \]

is invertible then, by the implicit function theorem in Banach spaces, the auxiliary equation defines locally in a neighborhood \( I = (0, \varepsilon) \times (E_0 - \delta, E_0 + \delta) \) a unique function \( \Theta_+(a, \omega) \) in \( C(I, D(H)) \) such that \( Q\Theta_+ = \Theta_+ \),

\[ QF(a\Phi_0 + \Theta_+(a, \omega), \omega) = 0, \]

and \( \lim_{(a, \omega) \to (0^+, E_0)} \| \Theta_+(a, \omega) \|_H = 0 \). Indeed from the equation \( G(a, \Theta_+(0, \omega), \omega) = 0 \), and since \( Q(H + \omega)Q \) is invertible with bounded inverse in a neighborhood of \( E_0 \), one has that

\[ \| \Theta_+(a, \omega) \|_H \leq C \left( \| a\Phi_0 \|^2 \| a\Phi_0 \| + \| \Theta_+(a, \omega) \|^{2\mu} \| \Theta_+(a, \omega) \| \right) \leq C \left( \| a\Phi_0 \|^{2\mu+1} + \| \Theta_+(a, \omega) \|^{2\mu+1} \right). \]

Since taking \( \varepsilon \) and \( \delta \) small enough we can make \( \| \Theta_+ \|_H \) arbitrarily small we have

\[ \| \Theta_+(a, \omega) \|_H \leq Ca^{2\mu+1}. \]  

(5.7)

Now we turn our attention to the bifurcation equation. First we write it explicitly using the definition of \( P \).

\[ a(\omega - E_0) - \left( \Phi_0, |a\Phi_0 + \Theta_+(a, \omega)|^{2\mu}(a\Phi_0 + \Theta_+(a, \omega)) \right) = 0. \]

This is an implicit equation w.r.t. two real parameters \( a \) and \( \omega \), we assume \( a \neq 0 \) and recast it in the following form

\[ f(a, \omega) \equiv (\omega - E_0) - a^{2\mu} \left( \Phi_0, \left| \Phi_0 + \frac{\Theta_+(a, \omega)}{a} \right|^{2\mu} \left( \Phi_0 + \frac{\Theta_+(a, \omega)}{a} \right) \right) = 0. \]  

(5.8)

We want to use the implicit function theorem in (5.8) to make explicit \( \omega(a) \). By the bound (5.7) it is immediate that \( f(a, \omega) \) is continuous, moreover

\[ \partial_a f(a, \omega) = 1 - a^{2\mu} \left( \Phi_0, (2\mu + 1) \left| \Phi_0 + \frac{\Theta_+(a, \omega)}{a} \right|^{2\mu} \frac{\partial_a \Theta_+(a, \omega)}{a} \right). \]

Which shows that \( \partial_a f(a, \omega) \) is also continuous. Notice that

\[ \partial_a \Theta_+(a, \omega) = -D_\Theta G^{-1}(a, \Theta_+(a, \omega), \omega) D_a G(a, \Theta_+(a, \omega), \omega) = -D_\Theta G^{-1}(a, \Theta_+(a, \omega), \omega) Q\Theta_+(a, \omega). \]

Hence, by (5.7), we have

\[ \| \partial_a \Theta_+(a, \omega) \|_H \leq C a^{2\mu+1}, \]
which implies that \( \partial_\omega f(0,E_0) \neq 0 \). We conclude that (5.8) defines uniquely a continuous function \( \omega_*(a) \) in a neighborhood of the origin such that \( \omega_*(0) = E_0 \). Moreover, it is clear that for small \( a \)

\[
a^{2\mu} \left( \phi_0, \left( \phi_0 + \frac{\Theta_*(a,\omega)}{a} \right) \right) \geq 0
\]

and then \( \omega_* - E_0 \geq 0 \) that is \( \omega_* \geq E_0 \). We can give a more precise asymptotic behavior, that is

\[
\omega_*(a) = E_0 + a^{2\mu}||\phi_0||_{2\mu+2}^{2\mu+2} + O(a^{4\mu}).
\]

Concerning the regularity properties of \( \omega_* \), exploiting the identity \( \partial_\omega f(a,\omega_*(a)) = 0 \) we conclude that \( \omega_* \in C^1(0,\varepsilon) \). Indeed, by L’Hôpital’s rule, we infer

\[
||\phi_0||_{2\mu+2}^{2\mu+2} = \lim_{a \to 0^+} \frac{\omega_*(a) - E_0}{a^{2\mu}} = \lim_{a \to 0^+} \frac{\omega_*(a)'}{2\mu a^{2\mu - 1}},
\]

hence

\[
\omega_*(a)' = 2\mu ||\phi_0||_{2\mu+2}^{2\mu+2} a^{2\mu - 1} + o(a^{2\mu - 1}),
\]

which guarantees that \( \omega_* \) is strictly increasing, hence invertible, in \((0,\varepsilon)\). We denote its inverse by \( a_*(\omega) \). Obviously \( a_* \in C^1(E_0,E_0 + \delta) \) and

\[
a_*(\omega) = \left( \frac{\omega - E_0}{||\phi_0||_{2\mu+2}^{2\mu+2}} \right)^{\frac{1}{\mu}} + O \left( (\omega - E_0)^{\frac{1}{\mu} + 1} \right), \tag{5.9}
\]

by the inequality \( (A + B)^{\frac{1}{\mu}} - B^{\frac{1}{\mu}} \leq C A^{\frac{1}{\mu} - 1} B \), which holds true for all \( 0 < B < A/2 \). The sought solution is given by

\[
\Phi(\omega) = a_*(\omega)\phi_0 + \Theta_*(a_*(\omega),\omega).
\]

We are left to prove properties (5.1) and (5.2).

As \( \omega \to E_0 \to 0 \), due to (5.7) and (5.9) we have

\[
m(\omega) = ||\Phi(\omega)||^2 = a_*(\omega)^2 + ||\Theta_*(a_*(\omega),\omega)||^2 = \left( \frac{\omega - E_0}{||\phi_0||_{2\mu+2}^{2\mu+2}} \right)^{\frac{1}{\mu}} + o \left( (\omega - E_0)^{\frac{1}{\mu}} \right),
\]

which proves equation (5.1). By the regularity of \( a_* \) and \( \Theta_*(a,\omega) \) it follows that \( m(\omega) \) is in \( C^1(E_0,E_0 + \delta) \) and it is invertible because

\[
\left( \frac{||\phi_0||_{2\mu+2}^{2\mu+2}}{||\phi_0||_{2\mu+2}^{2\mu+2}} \right)^{\frac{1}{\mu}} = \lim_{\omega \to E_0^+} \frac{m(\omega)}{(\omega - E_0)^{\frac{1}{\mu}}} = \lim_{\omega \to E_0^+} \mu \frac{m(\omega)'}{(\omega - E_0)^{1 - \frac{1}{\mu}}}. \]

Denoting its inverse by \( \omega(m) \) one has

\[
\left( \frac{\omega(m) - E_0}{||\phi_0||_{2\mu+2}^{2\mu+2}} \right)^{\frac{1}{\mu}} = m + o(m). \tag{5.10}
\]

Computing the energy we have that
E[Φ(ω)] = E[Φ(ω)] = \frac{||Φ(ω)||^{2\mu + 2}_{2\mu + 2}}{\mu + 1} - E_0a_0(\omega)^2 + (\Theta_0(a_0(\omega), \omega), H\Theta_0(a_0(\omega), \omega)) = -E_0a_0(\omega)^2 + o(a_0(\omega)^2).

\[ E(m) = E[Φ(ω(m))] \] is continuous as a function of \( m \) (indeed it is \( C^1(0, \bar{m}) \) for \( \bar{m} \) small enough), and recalling (5.9) and (5.10) we get (5.2).

**Proof of theorem 2.** Let us consider a sequence \( \{m_n\}_{n \in \mathbb{N}} \) in \( (0, m^*) \) and such that \( m_n \to 0 \). Correspondingly we have a sequence \( \{\nu_n\}_{n \in \mathbb{N}} \) in \( (0, \nu^*) \) and such that \( \nu_n \to 0 \). The authors are grateful to Gregory Berkolaiko, Rupert Frank, Pavel Exner, and Delio Mugnolo for useful discussions. DF and DN acknowledge the support of FIRB 2012 project.
‘Dispersive dynamics: Fourier Analysis and Variational Methods’, Ministry of University and Research of Italian Republic (code RBFR12MXPO). CC acknowledges the support of the FIR 2013 project ‘Condensed Matter in Mathematical Physics’, Ministry of University and Research of Italian Republic (code RBFR13WAET)

References

[1] Adami R, Cacciapuoti C, Finco D and Noja D 2011 Fast solitons on star graphs Rev. Math. Phys. 23 409–51
[2] Adami R, Cacciapuoti C, Finco D and Noja D 2012 On the structure of critical energy levels for the cubic focusing NLS on star graphs J. Phys. A: Math. Theor. 45 192001
[3] Adami R, Cacciapuoti C, Finco D and Noja D 2012 Stationary states of NLS on star graphs Europhys. Lett. 100 10003
[4] Adami R, Cacciapuoti C, Finco D and Noja D 2014 Variational properties and orbital stability of standing waves for NLS equation on a star graph J. Differ. Equ. 257 3738–77
[5] Adami R, Cacciapuoti C, Finco D and Noja D 2014 Constrained energy minimization and orbital stability for the NLS equation on a star graph Ann. Inst. Henri Poincaré, Anal. Non Linear 31 1289–310
[6] Adami R, Cacciapuoti C, Finco D and Noja D 2016 Stable standing waves for a NLS on star graphs as local minimizers of the constrained energy J. Differ. Equ. 260 7397–415
[7] Adami R and Noja D 2009 Existence of dynamics for a 1D NLS equation perturbed with a generalized point defect J. Phys. A: Math. Theor. 42 495302
[8] Adami R, Noja D and Visciglia N 2013 Constrained energy minimization and ground states for NLS with point defects Discrete Comput. Geom. B 18 1155–88
[9] Adami R, Serra E and Tilli P 2015 NLS ground states on graphs Calc. Var. PDEs 54 743–61
[10] Adami R, Serra E and Tilli P 2016 Threshold phenomena and existence results for NLS ground states on metric graphs J. Func. Anal. 271 201–23
[11] Adami R, Serra E and Tilli P 2017 Negative energy ground states for the L2-critical NLSE on metric graphs Commun. Math. Phys. 352 387–406
[12] Adams R A and Fournier J J F 2003 Sobolev Spaces (Pure and Applied Mathematics Series vol 140) (Boston, MA: Academic)
[13] Mehmeti F A, Ammari K and Nicaise S 2017 Dispersive effects for the Schrödinger equation on a tadpole graph J. Math. Anal. Appl. 448 262–80
[14] Banica V and Ignat L I 2011 Dispersion for the Schrödinger equation on networks J. Math. Phys. 52 083703
[15] Banica V and Ignat L I 2014 Dispersion for the Schrödinger equation on the line with multiple Dirac delta potentials and on delta trees Anal. PDE 7 903–27
[16] Berkolaiko G and Kuchment P 2013 Introduction to Quantum Graphs (Mathematical Surveys and Monographs vol 186) (Providence, RI: American Mathematical Society)
[17] Berkolaiko G and Liu W 2017 Simplicity of eigenvalues and non-vanishing of eigenfunctions of a quantum graph J. Math. Anal. Appl. 455 803–818
[18] Cacciapuoti C, Finco D and Noja D 2015 Topology induced bifurcations for the NLS on the tadpole graph Phys. Rev. E 91 013206
[19] Cazenave T 2003 Semilinear Schrödinger Equations (Courant Lecture Notes in Mathematics vol 10) (Providence, RI: American Mathematical Society)
[20] Cazenave T 2006 An Introduction to Semilinear Elliptic Equations (Rio de Janeiro: IM-UFRJ)
[21] Cazenave T and Lions P-L 1982 Orbital stability of standing waves for some nonlinear Schrödinger equations Commun. Math. Phys. 85 549–61
[22] Exner P and Jex M 2012 On the ground state of quantum graphs with attractive δ-coupling Phys. Lett. A. 376 713–7
[23] Gilbarg D and Trudinger N 2001 Elliptic Partial Differential Equations of Second Order (Berlin: Springer)
[24] Haeseler S 2011 Heat kernel estimates and related inequalities on metric graphs (arXiv:1101.3010v1)
[25] Keller M, Lenz D, Vogt H and Wojciechowski R 2015 Note on basic features of large time behaviour of heat kernels J. Reine Angew. Math. 708 73–95

3302
[26] Kirr E, Kevrekidis P G and Pelinovsky D E 2011 Symmetry-breaking bifurcation in the nonlinear Schrödinger equation with symmetric potentials Commun. Math. Phys. 308 795–844
[27] Kostrykin V and Schrader R 1999 Kirchhoff’s rule for quantum wires J. Phys. A: Math. Gen. 32 595–630
[28] Lieb E H and Loss M 2001 Analysis (Graduate Studies in Mathematics vol 14) 2nd edn (Providence, RI: American Mathematical Society)
[29] Marzuola J and Pelinovsky D E 2016 Ground states on the dumbbell graph Appl. Math. Res. Express 98–145
[30] Mugnolo D 2014 Semigroup Methods for Evolution Equations on Networks (Berlin: Springer)
[31] Noja D, Pelinovsky D and Shaikhova G 2015 Bifurcation and stability of standing waves in the nonlinear Schrödinger equation on the tadpole graph Nonlinearity 28 2343–78
[32] Pelinovsky D E and Schneider G 2017 Bifurcations of standing localized waves on periodic graphs Ann. Henri Poincaré 18 1185–211
[33] Reed M and Simon B 1978 Methods of Modern Mathematical Physics I (Analysis of Operators) (London: Academic)
[34] Reed M and Simon B 1978 Methods of Modern Mathematical Physics IV (Analysis of Operators) (London: Academic)
[35] Segal I 1963 Non-linear semi-groups Ann. Math. 78 339–64
[36] Stuart C A 2006 Uniqueness and stability of ground states for some nonlinear Schrödinger equations J. Eur. Math. Soc. 8 399–414
[37] Sobirov Z, Matrasulov D, Sabirov K, Sawada S and Nakamura k 2010 Integrable nonlinear Schrödinger equation on simple networks: connection formula at vertices Phys. Rev. E 81 066602
[38] Sabirov K K, Sobirov Z A, Babajanov D and Matrasulov D U 2013 Stationary nonlinear Schrödinger equation on simplest graphs Phys. Lett. A 377 860–5