SCORZA QUARTICS OF TRIGONAL SPIN CURVES AND THEIR VARIETIES OF POWER SUMS

HIROMICHI TAKAGI AND FRANCESCO ZUCCONI

Abstract. Our fundamental result is the construction of new subvarieties in the varieties of power sums for the Scorza quartic of any general pairs of trigonal curves and non-effective theta characteristics. This is a generalization of Mukai’s description of smooth prime Fano threefolds of genus twelve as the varieties of power sums for plane quartics. Among other applications, we give an affirmative answer to the conjecture of Dolgachev and Kanev on the existence of the Scorza quartic for any general pairs of curves and non-effective theta characteristics.

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1. Introduction

1.1. Varieties of power sums and the Waring problem.

Throughout the paper, we work over $\mathbb{C}$, the complex number field.

The problem of representing a homogeneous form as a sum of powers of linear forms has been studied since the last decades of the 19th century. This is called the Waring problem for a homogeneous form. We are interested in the study of the global structure of a suitable compactification of the variety parameterizing all such representations of a homogeneous form. To give a precise definition of such a compactification consider a $(v+1)$-dimensional vector space $V$. Let $F \in S^m V$ be a homogeneous forms of degree $m$ on $V$, where $V$ is the dual vector space of $V$. Let $\mathbb{P}_v V$ be the projective space parameterizing one-dimensional vector subspaces in $V$, which is sometime denoted by $\mathbb{P}^v$.

Definition 1.1.1. The varieties of power sums of $F$ is the following set with reduced structure:

$$\text{VSP} (F,n) := \{ ([H_1], \ldots, [H_n]) | H_1^m + \cdots + H_n^m = F \} \subset \text{Hilb}^n (\mathbb{P}_v V).$$

We call the Waring rank of $F$ the minimum of $n$ such that $\text{VSP} (F,n) \neq \emptyset$.

There are other compactifications, for example, the one in the $n$-th symmetric product of $\mathbb{P}_v V$, but for our treatment we need the one in the Hilbert scheme.

As far as we know, the first global descriptions of positive dimensional varieties of power sums for some homogeneous forms were given by S. Mukai.

The most intensively studied cases of varieties of power sums, including Mukai’s case, are where $F$ is a general $(v+1)$-nary homogeneous form of degree $m$ for some $m, v \in \mathbb{N}$, and $n$ is the Waring rank of $F$, which we denote by $n(m,v)$.

By a standard parameter count, we can easily compute the expected dimension of $\text{VSP} (F,n)$ for a general homogeneous form $F$. Since the dimension of the vector space of $(v+1)$-nary homogeneous forms of degree $m$ is $\binom{m+v}{m}$, the expected dimension is

$$\text{expdim} \text{VSP} (F,n) := (v+1)n - \binom{m+v}{m}.$$

Thus it is expected that

$$n(m,v) = \left\lfloor \frac{1}{v+1} \binom{m+v}{m} \right\rfloor.$$

It is known, however, that there are exceptions to $n(m,v)$ by the following result of J. Alexander and A. Hirschowitz [AH95]:

| $m$ | $v$ | $n(m,v)$ |
|-----|-----|----------|
| 2   | arbitrary | $v+1$ |
| 3   | 4   | 8        |
| 4   | 2   | 6        |
| 4   | 3   | 10       |
| 4   | 4   | 15       |

Here is the table of the known descriptions of $\text{VSP} (F, n(m,v))$. 
Varieties of power sums

In the table,

- \( W \) is a fivefold and is the variety of lines in the fivefold linear complete intersection \( \mathbb{P}^{10} \cap \text{OG}(5,10) \subset \mathbb{P}^{15} \) of the ten-dimensional orthogonal Grassmannian \( \text{OG}(5,10) \),
- \( S \) is a smooth symplectic fourfold obtained as a deformation of the Hilbert square of a polarized \( K3 \) surface of genus eight, and
- see the introduction of [RS00] or [Dol04] for the references of the results in the 19th and early 20th centuries.

As we can see in the table, the study before Mukai’s one were devoted only to the cases where \( \dim \text{VSP}(F,n(m,v)) = 0 \) and mostly the cases where \( F \) has a unique representation. Recently, using the technique of birational geometry, M. Mella proved in [Mel06] that, if \( m > v > 1 \), then the uniqueness holds only in the case where \( (m,v) = (5,2) \).

In [IR01a], Iliev and Ranestad treat some special \( (v+1) \)-nary cubics \( F \) and prove that, if \( v \geq 8 \), then the Waring rank of \( F \) is less than that of a general cubic.

In [IK99] Iliev and Kanev study varieties of power sums more systematically.

### 1.2. Mukai’s contribution.

Let \( V_{22} \) be a smooth prime Fano threefold of genus twelve, namely, a smooth projective threefold such that \( -K_{V_{22}} \) is ample, the class of \( -K_{V_{22}} \) generates \( \text{Pic} \ V_{22} \), and the genus \( g(V_{22}) := \left( -\frac{K_{V_{22}}}{2} \right)^3 + 1 \) is equal to twelve. \( V_{22} \) can be embedded into \( \mathbb{P}^{13} \) by the linear system \( |-K_{V_{22}}| \). Mukai discovered the following remarkable result [Muk92] §6, Theorem 11] (see also [DK93], [Sch01], and [Dol04] Theorem 3.12 for some details):

**Theorem 1.2.1.** For a general ternary quartic form \( F_4 \), \( \text{VPS}(F_4,6) \subset \text{Hilb}^6 \mathbb{P}^2 \) is a smooth prime Fano threefold of genus twelve, where we use the dual notation for later convenience. Moreover every general \( V_{22} \) is of this form.

To characterize a general \( V_{22} \) he studied the Hilbert scheme of lines on a general \( V_{22} \subset \mathbb{P}^{13} \) showing that it is isomorphic to a smooth plane quartic curve \( \mathcal{H}_1 \subset \mathbb{P}^2 \). He thought how to recover \( V_{22} \) by \( \mathcal{H}_1 \). For this, one more data was necessary. Using the incidence relation on \( \mathcal{H}_1 \times \mathcal{H}_1 \) defined by intersections of lines on \( V_{22} \), he found a non-effective theta characteristic \( \theta \) on \( \mathcal{H}_1 \). As explained in [DK93] §6,7], there is a beautiful result of G. Scorza which asserts that, associated to the pair \( (\mathcal{H}_1, \theta) \),...
there exists another plane quartic curve \( \{ F_4 = 0 \} \) in the same ambient plane as \( \mathcal{H}_1 \). (By saluting Scorza, \( \{ F_4 = 0 \} \) is called the Scorza quartic.) Then, finally, Mukai proved that \( V_{22} \) is recovered as VSP \((F_4, 6)\). Mukai observed that conics on \( V_{22} \) are parameterized by the plane \( \mathcal{H}_2 \) and \( \mathcal{H}_2 \) is naturally considered as the plane \( \overline{\mathbb{P}}^2 \) dual to \( \mathbb{P}^2 \). Moreover, he showed, for one representation of \( F_4 \) as a power sum of linear forms \( H_1, \ldots, H_6 \), the six points \([H_1], \ldots, [H_6] \in \mathbb{P}^2\) correspond to six conics through one point of \( V_{22} \).

Even if \( F_4 \) is taken as a special ternary quartic, VSP \((F_4, 6)\) may be still a smooth prime Fano threefold of genus twelve. Mukai [Muk92, §7] shows that, if \( F_4 \) is the square of a non-degenerate quadratic form, then VSP \((F_4, 6)\) is so called the Mukai-Umemura threefold discovered in [MU83] as a smooth SO(3, \( \mathbb{C} \))-equivariant compactification of SO(3, \( \mathbb{C} \))/Icosa. N. Manolache and F.-O. Schreyer [MS01] and F. Melliez and K. Ranestad [MR05] show that, if \( F_4 \) is the Klein quartic, then VSP \((F_4, 6)\) is a smooth compactification of the moduli space of \((1, 7)\)-polarized abelian surfaces.

1.3. Geometry of conics and lines and the main result.

Our main result, given in the end of the section 2, is a generalization of Mukai’s result Theorem 1.2.1; we describe certain subvarieties of the varieties of power sum of special quartic forms in any number \( v + 1 \) of variables. The quartics correspond to the ones of Theorem 1.2.1 if \( v = 2 \).

For this we generalize Mukai’s study of the geometries of lines and conics on \( V_{22} \). We recall Iskovskih’s description of the so-called double projection of a \( V_{22} \) from a general line as follows:

![Diagram]

where

- \( f' \) is the blow-up along a general line,
- \( B \) is the smooth quintic del Pezzo threefold, namely, a smooth projective threefold such that \(-K_B = 2H\), where \( H \) is the ample generator of Pic \( B \) and \( H^3 = 5 \), and
- \( f \) is the blow-up along a smooth rational curve of degree five (with respect to \( H \)).

Generalizing this situation we consider a general smooth rational curve of degree \( d \) on \( B \), where \( d \) is an arbitrary integer greater than or equal to 5. In 2.2 we establish the existence of such a \( C \) and we study some of its properties, especially, the relations to lines and conics on \( B \) intersecting it. Let \( f : A \to B \) be the blow up of \( B \) along \( C \). In 2.3.2 and 2.4.2 we define lines and conics on \( A \), which are appropriate generalizations of lines and conics on \( V_{22} \). We say \( l \) is a line on \( A \) if \( l \) is a reduced connected curve with \(-K_A \cdot l = 1, E_C \cdot l = 1 \) and \( p_a(l) = 0 \), where \( E_C := f^{-1}(C) \) is the exceptional divisor of \( f : A \to B \). We say \( q \) is a conic on \( A \) if \( q \) is a reduced connected curve with \(-K_A \cdot q = 2, E_C \cdot q = 2 \) and \( p_a(q) = 0 \).

We see that lines on \( A \) are parameterized by a smooth trigonal canonical curve \( \mathcal{H}_1 \) of genus \( d - 2 \) (Corollary 2.3.1). Conics on \( A \) turn out to be parameterized by a smooth surface \( \mathcal{H}_2 \). The study of \( \mathcal{H}_2 \) is quite delicate. For this purpose, we
Moreover the image is an irreducible component of $F$ quartic and $\text{Im } \Phi = \text{VSP } (\text{Theorem 1.3.3}) = \text{Theorem 2.5.12}$ mapping a point $a$ to the intersection of the bisecants $\beta_i$ and $C$, $i = 1, \ldots, \frac{(d-2)(d-3)}{2}$.

Moreover, we show that if $d \geq 6$, then $|D_i|$ is very ample and embeds $\mathcal{H}_2$ in $\mathbb{P}^{d-3}$, and if $d = 5$, $|D_i|$ defines a birational morphism $\mathcal{H}_2 \to \mathbb{P}^2$. Here we use the dual notation for later convenience. If $d \geq 6$, then $\mathcal{H}_2$ is so called the White surface (see [Whi24] and [Gim89]). It is interesting for us that the classical White surface naturally appears in this set up.

A deeper understanding of the geometry of conics requires the notion of intersection of two conics and, more precisely, the divisor $D_q \subset \mathcal{H}_2$ parameterizing conics which intersect a fixed conic $q$. It is easy to see that $D_q \sim 2D_i$.

Now assuming $d \geq 6$ we consider $\mathcal{H}_2 \subset \mathbb{P}^{d-3} = \mathbb{P}_* \tilde{V}$. By the double projection of $B$ from a general point $b$, we see that there are $n := \frac{(d-1)(d-2)}{2}$ conics (counted with multiplicities) through $b$. It is crucial that the number $n$ is equal to the dimension of the quadratic forms on $\mathbb{P}^{d-3}$. Nevertheless, infinitely many conics on $A$ pass through a point on the strict transform of a bi-secant line of $C$. Hence to have a finiteness result we have to consider the blow-up $\rho: \tilde{A} \to A$ along the strict transforms of bi-secant lines of $C$ on $B$. Then by a careful analysis on mutually intersecting conics on $A$ we construct a morphism $\Phi: \tilde{A} \to \text{Hilb}^n \mathbb{P}^{d-3}$ obtained by attaching the process which associates $n$ conics on $A$ to each point $\tilde{a}$ of $\tilde{A}$; see Definition 1.3.2 for the precise definition of attached conics. To produce the quartic we are looking for, we show that the proper locus $\{ [q] \in \mathcal{H}_2 \mid [q] \in D_q \}$ on $\mathcal{H}_2$ is cut out by a quartic, whose equation is denoted by $\tilde{F}_4$. Moreover we show that $\tilde{F}_4$ is non-degenerate, this means that the polar map induced by $\tilde{F}_4$ from $S^2V$ to $S^2V$ is an isomorphism. Then the required quartic $F_4$ is the dual quartic to $\tilde{F}_4$, namely, the quartic form in $S^4V$ such that its induced polar map from $S^2V$ to $S^2V$ is the inverse of that of $\tilde{F}_4$.

For the precise statement of our main result, we need the following definition:

**Definition 1.3.2.** For a subvariety $S$ of $\mathbb{P}_* \tilde{V}$, we set

$$\text{VSP } (F, n; S) := \{(H_1, \ldots, H_n) \mid [H_i] \in S, H_1^n + \cdots + H_n^n = F\} \subset \text{VSP } (F, n)$$

and we call it the varieties of power sums of $F$ confined in $S$.

As far as we know, $\text{VSP } (F, n; S)$ is essentially a new object to study.

Our main theoretical result is the following:

**Theorem 1.3.3** (=Theorem 2.5.12). There is an injection $\Phi: \tilde{A} \to \text{Hilb}^n \mathbb{P}^{d-3}$ mapping a point $a$ of $\tilde{A}$ to the point representing the $n$ conics on $A$ attached to $a$. Moreover the image is an irreducible component of $\text{VSP } (F_4, n; \mathcal{H}_2)$.

In the sequel 1.4 we explain a more significant geometrical meaning of the special quartic $F_4$.

Based on Mukai’s result we can state the following conjecture: $\Phi$ is an embedding and $\text{Im } \Phi = \text{VSP } (F_4, n; \mathcal{H}_2)$.
We remark that, for \( d \leq 8 \), the number \( n \) is equal to the Waring rank of a general \((d - 2)\)-nary quartic, and especially, the cases where \( d = 5, 6, 7 \) cover exceptional cases of Alexander and Hirschowitz.

Even if \( d = 5 \), we have a similar result, which is an elaboration on Theorem 1.2.1. The explanation is technical: see 2.5.3.

1.4. Applications.

In the section 3, we give some applications of our study of \( A \) for a pair of a canonical curve of any genus and a non-effective theta characteristic, a spin curve for short.

Dolgachev and Kanev [DK93, §9] give a modern account of Scorza’s beautiful construction of a certain quartic hypersurface, so called the Scorza quartic, associated to every spin curve. It is expected that the Scorza quartic is useful for the study of a spin curve but no deeper properties of the Scorza quartic were unknown. Firstly, its construction is not so explicit. Secondly, Scorza’s construction itself depends on three assumptions on spin curves (see [DK93, (9.1) (A1)–(A3)]) and it were unknown whether these conditions are fulfilled for a general spin curve of genus > 3. Thus the existence of the Scorza quartic was conditional except for the genus 3 case, where Scorza himself solved the problem. We give contributions for these two subjects.

In 3.1, using the incidence correspondence on \( \mathcal{H}_1 \times \mathcal{H}_1 \) defined by intersections of lines on \( A \), we define a non-effective theta characteristic \( \theta \) on the trigonal curve \( \mathcal{H}_1 \). This is a generalization of Mukai’s result explained as in 1.2.

In 3.2 we observe that there is a natural duality between \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \), which induces the natural identification \( \mathbb{P}^{d-3} = \mathbb{P}^* H^0(\mathcal{H}_1, K_{\mathcal{H}_1}) \), where for clarity reasons we denote by \( \mathbb{P}^{d-3} \) the projective space dual to the ambient projective space \( \mathbb{P}^{d-3} \) of \( \mathcal{H}_2 \), and by \( \mathbb{P}^* H^0(\mathcal{H}_1, K_{\mathcal{H}_1}) \) the ambient projective space of the canonical embedding of \( \mathcal{H}_1 \).

In 3.3 we recall the definition of the discriminant loci and we compute it explicitly for \( (\mathcal{H}_1, \theta) \). In 3.4 we recall the precise definition of the Scorza quartic for a spin curve.

By virtue of our explicit computation of the discriminant, we prove in 3.5 that the pair \( (\mathcal{H}_1, \theta) \) satisfies the conditions [DK93, (9.1) (A1)–(A3)], which guarantee the existence of the Scorza quartic for the pair \( (\mathcal{H}_1, \theta) \). Then, by a standard deformation theoretic argument, we can then verify that the conditions (A1)–(A3) hold also for a general spin curve, hence we answer affirmatively to the Dolgachev-Kanev Conjecture:

**Theorem 1.4.1** (=Theorem 3.5.3). *The Scorza quartic exists for a general spin curve.*

Moreover we can find explicitly the Scorza quartic for \( (\mathcal{H}_1, \theta) \). In fact, by definition, the Scorza quartic \( \{ F'_4 = 0 \} \) for \( (\mathcal{H}_1, \theta) \) lives in \( \mathbb{P}^* H^0(\mathcal{H}_1, K_{\mathcal{H}_1}) \), but, as we remark above, we can consider \( \{ F'_4 = 0 \} \subseteq \mathbb{P}^{d-3} \). In 3.6 we prove that the special quartic \( \{ F_4 = 0 \} \subseteq \mathbb{P}^{d-3} \) in Theorem 1.3.3 coincides with the Scorza quartic \( \{ F'_4 = 0 \} \).

We recommend the readers who are interested only in the subsections 3.1–3.5 to skip the subsection 2.5.
Finally, in [3.7] we show that $A$ is recovered from the pair $(\mathcal{H}_1, \theta)$. This implies that $(\mathcal{H}_1, \theta)$’s fill up an open subset of the moduli of trigonal spin curves. In particular, $\mathcal{H}_1$ is a general trigonal curve for a general $C$.

1.5. Final remarks.

In this paper, we only consider a general rational curve on $B$ but there are interesting special cases. In the forthcoming paper, applying the method of this paper, we will study the blow-ups of $B$ along special smooth rational curves of degree six and pairs of canonical curves of genus four and even theta characteristics.

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2. Blowing up the quintic del Pezzo threefold $B$ along a $\mathbb{P}^1$ of degree $d$

In this section, we study the geometries of the blow-up $A$ of the quintic del Pezzo threefold $B$ along a smooth rational curve of degree $d$, which is nothing but the special threefold we mention in the abstract. In 2.1 we review the description of lines and conics on $B$, and 2-ray games originating from $B$. Based on this, we construct in 2.2 smooth rational curves of degree $d$, where $d$ is an arbitrary positive integer, having nice intersection properties with respect to lines and conics. The results in 2.2 are delicate but their proof is more or less based on standard parameter count. In 2.3 and 2.4 we study the families of curves on $A$ of degree one or two with respect to the anti-canonical sheaf of $A$ (we call them lines and conics on $A$ respectively). The curve $\mathcal{H}_1$ parameterizing lines on $A$ and the surface $\mathcal{H}_2$ parameterizing conics on $A$ are two of the main characters in this paper. See Corollary 2.3.1 and Theorem 2.3.18 for a quick view of their properties. Finally in 2.5 we prove the main theorem (Theorem 2.5.12). See 2.5.3 for the relationship of our result with Mukai’s one we mentioned in the introduction.

2.1. Review on geometries of $B$.

Let $V$ be a vector space with $\dim_{\mathbb{C}} V = 5$. The Grassmannian $G(2, V)$ embeds into $\mathbb{P}^9$ and we denote the image by $G \subset \mathbb{P}^9$. It is well-known that the quintic del
Pezzo 3-fold, i.e., the Fano 3-fold $B$ of index 2 and of degree 5 can be realized as $B = G \cap \mathbb{P}^6$, where $\mathbb{P}^6 \subset \mathbb{P}^9$ is transversal to $G$ (see [Fuj81, Ilk77 Thm 4.2 (iii), the proof p.511-p.514]).

Let $H_B^1$ and $H_B^2$ be the Hilbert scheme, respectively, of lines and of conics on $B$. We collect basic known facts on lines and conics on $B$ almost without proof.

2.1.1. Lines on $B$. Let $\pi : \mathbb{P} \to H_B^1$ be the universal family of lines on $B$ and $\varphi : \mathbb{P} \to B$ the natural projection. By [FN89a, Lemma 2.3 and Theorem I], $H_B^1$ is isomorphic to $\mathbb{P}^2$ and $\varphi$ is a finite morphism of degree three. In particular the number of lines passing through a point is three counted with multiplicities. We recall some basic facts about $\pi$ and $\varphi$ which we use in the sequel.

Before that, we fix some notation.

**Notation 2.1.1.** For an irreducible curve $C$ on $B$, denote by $M(C)$ the locus $\subset \mathbb{P}^2$ of lines intersecting $C$, namely, $M(C) := \pi(\varphi^{-1}(C))$ with reduced structure. Since $\varphi$ is flat, $\varphi^{-1}(C)$ is purely one-dimensional. If $\deg C \geq 2$, then $\varphi^{-1}(C)$ does not contain a fiber of $\pi$, thus $M(C)$ is a curve. See Proposition 2.1.3 for the description of $M(C)$ in case $C$ is a line.

**Definition 2.1.2.** A line $l$ on $B$ is called a special line if $\mathcal{N}_{l/B} \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$.

**Proposition 2.1.3.** It holds:

1. for the branched locus $B_\varphi$ of $\varphi : \mathbb{P} \to B$ we have:
   1.1) $B_\varphi \subset |-K_B|$,  
   1.2) $\varphi^*B_\varphi = R_1 + 2R_2$,  
   1.3) $R_1 \cong R_2 \cong \mathbb{P}^1 \times \mathbb{P}^1$, and  
   1.4) $\varphi : R_1 \to B_\varphi$ and $\varphi : R_2 \to B_\varphi$ are injective,
2. $R_2$ is contracted to a conic $Q_2$ by $\pi : \mathbb{P} \to H_B^1$. Moreover $Q_2$ is the branched locus of $\pi_{|R_2} : R_2 \to H_B^1$.
3. $Q_2$ parameterizes special lines. If $l$ is not a special line on $B$, then $\mathcal{N}_{l/B} = \mathcal{O}_l \oplus \mathcal{O}_l$.
4. if $l$ is a special line, then $M(l)$ is a line, and $M(l)$ is tangent to $Q_2$ at $|l|$. If $l$ is not a special line, then $M(l)$ is the disjoint union of a line and the point $|l|$. By abuse of notation, we denote by $M(l)$ the one-dimensional part of $M(l)$ for any line $l$. Vice-versa, any line in $H_B^1$ is of the form $M(l)$ for some line $l$,
5. the locus swept by lines intersecting $l$ is a hyperplane section $T_l$ of $B$ whose singular locus is $l$. For every point $b$ of $T_l \setminus l$, there exists exactly one line which belongs to $M(l)$ and passes through $b$. Moreover, if $l$ is not special, then the normalization of $T_l$ is $\mathbb{F}_1$ and the inverse image of the singular locus is the negative section of $\mathbb{F}_1$, or, if $l$ is special, then the normalization of $T_l$ is $\mathbb{F}_3$ and the inverse image of the singular locus is the union of the negative section and a fiber, and
6. if $l$ is not a special line, then $\varphi^{-1}(l)$ is the disjoint union of the fiber of $\pi$ corresponding to $l$, and the smooth rational curve dominating $M(l)$.

**Proof.** See [FN89a §2] and [Ili94 §1].

By the proof of [FN89a] we see that $B$ is stratified according to the ramification of $\varphi : \mathbb{P} \to B$ as follows:

$$B = (B \setminus B_\varphi) \cup (B_\varphi \setminus C_\varphi) \cup C_\varphi,$$
where \( C_\phi \) is a smooth rational normal sextic and if \( b \in B \setminus B_\phi \) exactly three distinct lines pass through it, if \( b \in (B_\phi \setminus C_\phi) \) exactly two distinct lines pass through it, one of them is special, and finally \( C_\phi \) is the loci of \( b \in B \) through which it passes only one line.

### 2.1.2. Conics on \( B \)

**Proposition 2.1.4.** The Hilbert scheme of conics on \( B \) is isomorphic to \( \mathbb{P}^4 = \mathbb{P}_* \tilde{V} \). The support of a double line is a special line and the double lines are parameterized by a rational normal quartic curve \( \Gamma \subset \mathbb{P}_* \tilde{V} \) and the secant variety of \( \Gamma \) is a singular cubic hypersurface which is the closure of the loci parameterizing reducible conics.

**Proof.** See [Ili94, Proposition 1.2.2].

The identification is given by the map \( sp: \mathcal{H}_2^B \to \mathbb{P}_* \tilde{V} \) with \([c] \mapsto \langle Gr(c) \rangle = \mathbb{P}_c^3 \subset \mathbb{P}_* V\), where for a general conic \( c \subset B \) we set \( Gr(c) := \cup \{ r \in \mathbb{P}_* V \mid [r] \in c \} \simeq \mathbb{P}^1 \times \mathbb{P}^1 \).

### 2.1.3. Two-ray games based on \( B \)

We are interested in the geometry of the 3-fold \( A \) obtained by blowing-up \( B \) along a curve \( C := C_\phi \) as constructed in Proposition 2.2.4. To understand this geometry we need to describe some two-ray games originating from \( B \).

**Definition 2.1.5.** Let \( b \) be a point of \( B \). We call the rational map from \( B \) defined by the linear system of hyperplane sections singular at \( b \) the **double projection from \( b \)**.

**Proposition 2.1.6.** (1) Let \( b \) be a point of \( B \). Then the target of the double projection from \( b \) is \( \mathbb{P}^2 \), and the double projection from \( b \) and the projection \( B \longrightarrow \overline{B}_b \) from \( b \) fit into the following diagram:

\[
\begin{array}{ccc}
B & \xrightarrow{\pi_{1b}} & \overline{B}_b \\
\downarrow \pi_{1b} & & \downarrow \pi_{2b} \\
B_b & \\ \end{array}
\]

where \( \pi_{1b} \) is the blow-up of \( B \) at \( b \), \( B_0 \longrightarrow B_1' \) is the flop of the strict transforms of lines through \( b \), and \( \pi_{2b}: B'_0 \to \mathbb{P}^2 \) is a (unique) \( \mathbb{P}^1 \)-bundle structure. We denote by \( E_b \) the \( \pi_{1b} \)-exceptional divisor and by \( E'_b \) the strict transform of \( E_b \) on \( B'_0 \). Moreover we have the following descriptions:

(1-1) \[-K_{B'_0} = H + L,\]

where \( H \) is the strict transform of a general hyperplane section of \( B \), and \( L \) is the pull back of a line on \( \mathbb{P}^2 \).

(1-2) if \( b \notin B_\phi \), then the strict transforms \( l'_i \) of three lines \( l_i \) through \( b \) on \( B_b \) have the normal bundle \( O_{\mathbb{P}^2}(-1) \oplus O_{\mathbb{P}^2}(-1) \). The flop \( B_0 \longrightarrow B'_0 \) is the Atiyah flop. In particular, \( E'_b \to E_b \) is the blow-up at the three points \( E_b \cap l'_i \).

If \( b \in B_\phi \setminus C_\phi \), then \( E_\phi \longrightarrow E'_b \) can be described as follows: let \( l \) and \( m \) be two lines through \( b \), where \( l \) is special, and \( m \) is not special. Let \( l' \) and \( m' \) be the strict transforms of \( l \) and \( m \) on \( B_0 \). First blow up \( E_0 \) at two
points \( t_1 := E_b \cap l' \) and \( t_2 := E_b \cap m' \) and then blow up at a point \( t_3 \) on the exceptional curve \( e \) over \( t_1 \). Finally, contract the strict transform of \( e \) to a point. Then we obtain \( E'_b \) (this is a degeneration of the case (a)). See [FN89b] in case of \( b \in C_\\phi \), and

\[(1-3) \quad \text{a fiber of } \pi_{2b} \text{ not contained in } E'_b \text{ is the strict transform of a conic through } b, \text{ or the strict transform of a line } \not\ni b \text{ intersecting a line through } b. \]

The description of the fibers of \( \pi_{2b} \) contained in \( E'_b \) is as follows: if \( b \not\in B_\\phi \), then \( \pi_{2b}|_{E'_b}: E'_b \rightarrow \mathbb{P}^2 \) is the blow-down of the strict transforms of three lines connecting two of \( E_b \cap l'_i \), namely, \( E_b \rightarrow \mathbb{P}^2 \) is the Cremona transformation.

Assume that \( b \in B_\\phi \backslash C_\\phi \). Then \( \pi_{2b}|_{E'_b}: E'_b \rightarrow \mathbb{P}^2 \) is the blow-down of the strict transforms of two lines, one is the line connecting \( t_1 \) and \( t_2 \), the other is the line whose strict transform passes through \( t_3 \). \( E_b \rightarrow \mathbb{P}^2 \) is a degenerate Cremona transformation. See [FN89b] in case of \( b \in C_\\phi \).

(2) Let \( l \) be a line on \( B \). Then the projection of \( B \) from \( l \) is decomposed as follow:

\[(2.2) \quad \pi_{1l} \quad \pi_{2l} \]

\[\pi \]

\[B \rightarrow B \rightarrow Q, \]

where \( \pi_{1l} \) is the blow-up along \( l \) and \( B \rightarrow Q \) is the projection from \( l \) and \( \pi_{2l} \) contracts onto a rational normal curve of degree 3 the strict transform of the loci swept by the lines of \( B \) touching \( l \). Moreover

\[(2.3) \quad -K_{B_l} = H + H_Q, \]

where \( H \) and \( H_Q \) are the pull backs of general hyperplane sections of \( B \) and \( Q \) respectively. We denote by \( E_l \) the \( \pi_{1l} \)-exceptional divisor.

(3) Let \( q \) be a smooth conic on \( B \). Then the projection of \( B \) from \( q \) behaves as follow:

\[(2.4) \quad \pi_{1q} \quad \pi_{2q} \]

\[\pi \]

\[B \rightarrow B \rightarrow \mathbb{P}^3, \]

where \( \pi_{1q} \) is the blow-up of \( B \) along \( q \) and \( \pi_{2q}: B_q \rightarrow \mathbb{P}^3 \) is the divisorial contraction of the strict transform \( T_q \) of the loci swept by the lines touching \( q \). Moreover

\[(2.5) \quad -K_{B_q} = H + H_P, \]

where \( H \) and \( H_P \) are the pull backs of general hyperplane sections of \( B \) and \( \mathbb{P}^3 \) respectively.

**Proof.** These results come from explicit computations and are more or less known. Especially, for (2), refer [Fuj81], and for (3) (and (2)), refer [MM81], No. 22 for (3) (No. 26 for (2)). See also [MM85], p.533 (7.7) for a discussion.

(1) is less known. We have only found the paper [FN89b], in which they deal with the most difficult case (c). Here we only sketch the construction of the flop in the middle case (b) to intend the reader to get a feeling of birational maps from \( B \).
Let $b$ be a point of $B_{\varphi} \setminus C_{\varphi}$. We use the notation of the statement of (1-2). The flop of $m'$ is the Atiyah flop. We describe the flop of $l'$. By $\mathcal{N}_{l'/B} \simeq O_{\mathbb{P}^1}(1) \oplus O_{\mathbb{P}^1}(-1)$, it holds that $\mathcal{N}_{l'/B_1} \cong O_{\mathbb{P}^1} \oplus O_{\mathbb{P}^1}(-2)$. Hence the flop of $l'$ is a special case of Reid’s one [Rei83, Part II]. We show that the width is two in Reid’s sense. Let $T_1$ be the normalization of $T_l$. By Proposition 2.1.3 (5), $T_1 \simeq \mathbb{P}^3$ and the inverse image of the singular locus of $T_l$ is the union of the negative section $C_0$ and a fiber $r$. Let $\mu: B_b \to B_b$ be the blow-up along $l'$ and $F$ the exceptional divisor. Let $T_2$ be the strict transform of $T_l$ on $B_b$. Then $T_2$ is the blow-up of $T_1$ at two points $s_1 \in C_0$ and $s_2 \in r$. Denote by $C_0'$ and $r'$ the strict transforms of $C_0$ and $r$. We prove that $\mathcal{N}_{r'/B_b} \simeq O_{\mathbb{P}^1}(-1)^{\oplus 2}$. Note that $F \cap T_2 = C_0' \cup r'$.

The curves $C_0'$ and $r'$ are two sections on $F$. Let $T_1'$ be the image of $T_2$ on $B_b$. By $\mathcal{N}_{T_1'/B_b} \cong O_{\mathbb{P}^1} \oplus O_{\mathbb{P}^1}(-2)$ and $T_2 = \mu^* T_1 - 2F$, it holds $F \cong \mathbb{F}_2$, and $T_2|_F \cong 2G_0 + 3\gamma$, where $G_0$ is the negative section of $F$ and $\gamma$ is a fiber of $F \to l'$. Note that $F \cdot C_0' = (F|_{T_2} \cdot C_0')_{T_2} = -3$ and $F \cdot r' = (F|_{T_2} \cdot r')_{T_2} = 0$, and $F \cdot G_0 = 0$ and $F \cdot (G_0 + 3\gamma) = -3$. Thus we have $C_0' \sim G_0 + 3\gamma$ and $r' = G_0$ on $F$. Now we see that $-K_{B_b} \cdot r' = (\mu^* (-K_{B_b}) - F) \cdot r' = 0$. Therefore, by $(r')^2 = -1$ on $T_2$, it holds that $\mathcal{N}_{r'/B_b} \simeq O_{\mathbb{P}^1}(-1)^{\oplus 2}$.

It is easy to see that we can flop $r'$. Let $\tilde{B}_b \to \tilde{B}_b'$ be the flop of $r'$ (now we consider locally around $r'$). Let $F'$ be the strict transform of $F$ on $\tilde{B}_b'$. By [Rei83], $F' \simeq F$ and there is a blow-down $\tilde{B}_b' \to \tilde{B}_b''$ of $F'$ such that $\tilde{B}_b''$ is smooth. $\tilde{B}_b \to \tilde{B}_b''$ is the flop of $l'$.

By this description of the flop, we can easily obtain (1-2).}

As a first application of the above operations, we have the following result, which we often use:

**Corollary 2.1.7.** Let $b_1$ and $b_2$ be two (possibly infinitely near) points on $B$ such that there exists no line on $B$ through them. Then there exists a unique conic on $B$ through $b_1$ and $b_2$.

**Proof.** We project $B$ from $b_1$ as in (2.1). Then the assertion follows by the description of fibers of $\pi_{2b_1}$ as in Proposition 2.1.6 (1-3). \qed

2.2. **Construction of smooth rational curves $C_d$ of degree $d$ on $B$.**

We construct smooth rational curves of degree $d$ on $B$ with certain properties.

**Proposition 2.2.1.** There exists a smooth rational curve $C_d$ of degree $d$ on $B$ such that

(a) a general line on $B$ intersecting $C_d$ is uni-secant,
(b) $C_d$ is obtained as a smoothing of the union of a smooth rational curve $C_{d-1}$ of degree $d - 1$ on $B$ and a general uni-secant line of it on $B$,
(c) $\mathcal{N}_{C_d/B} \cong O_{\mathbb{P}^1}(d - 1) \oplus O_{\mathbb{P}^1}(d - 1)$. In particular $h^1(\mathcal{N}_{C_d/B}) = 0$ and $h^0(\mathcal{N}_{C_d/B}) = 2d$, and
(d) if $d = 5$, then $C_5$ is a normal rational curve and is contained in a unique hyperplane section $S$, which is smooth. If $d \geq 6$, then $C_d$ is not contained in a hyperplane section.

**Proof.** We argue by induction on $d$.

If $d = 1$, we have the assertion since $\mathcal{N}_{C_1/B} \cong O_{\mathbb{P}^1} \oplus O_{\mathbb{P}^1}$ for a general line $C_1$. 

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Now assume that \(C_{d-1}\) is a general smooth rational curve of degree \(d - 1\) on \(B\). By induction, a general secant line \(l\) of \(C\) on \(Q\) is uni-secant. Set \(Z := C_{d-1} \cup l\) and \(N_{Z/Q} := \text{Hom}_{\mathcal{O}_Q}(\mathcal{I}_Z, \mathcal{O}_Q)\). By induction, the normal bundle of \(C_{d-1}\) satisfies (c).

Thus, by \(N_{l/Q} \cong \mathcal{O}_{P_1} \oplus \mathcal{O}_{P_2}\) and [HI85] Theorem 4.1 and its proof, it holds \(h^1(N_{Z/Q}) = 0\), and moreover \(Z := C_{d-1} \cup l\) is strongly smoothable, namely, we can find a smoothing \(\mathcal{C}\) of \(Z\) with the smooth total space. By the upper semi-continuity theorem, we have \(h^1(N_{C_{d}/Q}) = 0\).

Thus the Hilbert scheme of \(Q\) is smooth at \([C_d]\) and is of dimension \(2d\), which is also the dimension of the component of the Hilbert scheme containing \([C_d]\) on \(B\). On the other hand, varying \(C_{d-1}\) and \(l\), we have a family of reducible curves of dimension \(2(d - 1) + 1 = 2d - 1\). Thus the smoothing constructed as above is general in the component of the Hilbert scheme whose generic point parameterizes smooth rational curves of degree \(d\).

It is easy to see that a general line \(m\) intersecting \(C_{d-1}\) does not intersect \(l\), thus \(m\) is a uni-secant line of \(C_{d-1} \cup l\). This implies (a) for \(C_d\) by a deformation theoretic argument.

To check the form of the normal bundle, simply assume by induction that \(N_{C_{d-1}/Q} := \mathcal{O}_{P_1}(d - 2) \oplus \mathcal{O}_{P_2}(d - 2)\). Set \(N_{C_d/Q} := \mathcal{O}_{P_1}(a_d) \oplus \mathcal{O}_{P_2}(b_d)\) (\(a_d \geq b_d\)) for the smoothing \(C_d\) of \(Z\). We show that \(a_d = b_d = d - 1\).

It suffices to prove \(h^0(N_{Z/Q}(-d)) = 0\). In fact, then, by the upper semi-continuous theorem, we have \(h^0(N_{C_d/Q}(-d)) = 0\) and \(a_d, b_d \leq d - 1\). Thus, by \(a_d + b_d = 2d - 2\), it holds \(a_d = b_d = d - 1\).

The equality \(h^0(N_{Z/Q}(-d)) = 0\) easily follows from the following three exact sequences, where \(t := C_{d-1} \cap l\):

\[
0 \to N_{Z/Q} \to N_{Z/Q} \otimes \mathcal{O}_{C_{d-1}} \to N_{Z/Q} \otimes \mathcal{O}_l \to 0.
\]

\[
0 \to N_{C_{d-1}/Q} \to N_{Z/Q} \otimes \mathcal{O}_{C_{d-1}} \to T^1(t) \to 0.
\]

\[
0 \to N_{l/Q} \to N_{Z/Q} \otimes \mathcal{O}_l \to T^1(t) \to 0.
\]

Finally we prove (d). In case \(d = 5\), we have only to notice that a general hyperplane section of \(B_5\) is a del Pezzo surface of degree 5 which contains a smooth \(C_5\). For \(d \geq 6\), the assertion follows by induction. \(\square\)

We denote by \(H^B_d\) the union of components of the Hilbert scheme of \(B\) whose general points parameterize smooth rational curves of degree \(d\) obtained inductively as in Proposition 2.2.1.

The following proposition describe relations between \(C_d\) and lines and conics on \(B\).

**Proposition 2.2.2.** A general \(C_d\) as in Proposition 2.2.1 satisfies the following conditions:

1. There exist no \(k\)-secant lines of \(C_d\) on \(B\) with \(k \geq 3\),
2. There exist at most finitely many bi-secant lines of \(C_d\) on \(B\), and any of them intersects \(C_d\) simply,
3. Bi-secant lines of \(C_d\) on \(B\) are mutually disjoint,
4. Neither a bi-secant line nor a line through the intersection point between a bi-secant line and \(C_d\) is a special line,
(5) there exist at most finitely many points $b$ outside $C_d$ such that all the lines through $b$ intersect $C_d$, and such points exist outside bi-secant lines of $C_d$.

(6) there exist no $k$-secant conics of $C_d$ with $k \geq 5$.

(7) there exist at most finitely many quadri-secant conics of $C_d$ on $B$, and no quadri-secant conic is tangent to $C_d$, and

(8) $q_{C_d}$ has no point of multiplicity greater than two for any multi-secant conic $q$.

**Proof.** We can prove the assertions by simple dimension count based upon Proposition 2.2.1. We assume that $d \geq 4$ since otherwise we can verify the assertion easily.

(1). Let $\mathcal{D}$ be the closure of the set

$$\{([C_d],[l]) \mid C_d \cap l \text{ consists of 3 points}\} \subset \mathcal{H}_d^B \times \mathcal{H}_1^B.$$ 

Let $\pi_d: \mathcal{D} \to \mathcal{H}_d^B$ and $\pi_1: \mathcal{D} \to \mathcal{H}_1^B$ be the natural morphisms induced by the projections. The claim follows if we show that $\dim_{\mathbb{C}} \mathcal{D} \leq 2d - 1$ since $\dim \mathcal{H}_d^B = 2d$.

Thus we estimate $\dim_{\mathbb{C}} \text{Hom}^2 (\mathbb{P}^1, B; p_i \mapsto s_i, i = 1, 2, 3)$. If $p_i, i = 1, 2, 3$ are fixed points of $\mathbb{P}^1, [\pi]$ is a general point and the degree is measured by $-K_B$. By $d \geq 4$ and Proposition 2.2.1 (c), it holds that $h^0(\mathbb{P}^1, \pi^* T_B(-p_1 - p_2 - p_3)) = 2d - 6$ and $h^1(\mathbb{P}^1, \pi^* T_B(-p_1 - p_2 - p_3)) = 0$. Then

$$\dim_{\mathbb{C}} \text{Hom}^2 (\mathbb{P}^1, B; p_i \mapsto s_i, i = 1, 2, 3)_{[\pi]} = h^0(\pi^* T_B(-p_1 - p_2 - p_3)) = 2d - 6.$$ 

This implies that $\dim_{\mathbb{C}} \pi^{-1}_1(\{l\}) \leq 2d - 6 - 3 = 2d - 3$ since the three points can be chosen arbitrarily. Then $\dim_{\mathbb{C}} \mathcal{D} \leq 2d - 1$ since $\dim_{\mathbb{C}} \mathcal{H}_1^B = 2$.

(2). Now let $\mathcal{D}$ be the closure of the set

$$\{([C_d],[l]) \mid C_d \cap l \text{ consists of 2 points}\} \subset \mathcal{H}_d^B \times \mathcal{H}_1^B.$$ 

As before, let $\pi_d: \mathcal{D} \to \mathcal{H}_d^B$ and $\pi_1: \mathcal{D} \to \mathcal{H}_1^B$ be the natural morphisms induced by the projections. By $d \geq 4$ and Proposition 2.2.1 (c), it holds that $h^0(\mathbb{P}^1, \pi^* T_B(-p_1 - p_2)) = 2d - 3$ and $h^1(\mathbb{P}^1, \pi^* T_B(-p_1 - p_2)) = 0$. Then

$$\dim_{\mathbb{C}} \text{Hom}^2 (\mathbb{P}^1, B; p_i \mapsto s_i, i = 1, 2)_{[\pi]} = h^0(\pi^* T_B(-p_1 - p_2)) = 2d - 3.$$ 

Since $\dim_{\mathbb{C}} \text{Aut} (\mathbb{P}^1, p_1, p_2) = 1$ it holds that $\dim_{\mathbb{C}} \pi^{-1}_1(\{l\}) \leq 2d - 3 + 2 - 1 = 2d - 2$. Hence $\dim_{\mathbb{C}} \mathcal{D} = 2d$. Hence $C_d$ has only a finite number of bisecant lines.

We now show that the loci where $C_d$ has a tangent bisecant is a codimension one loci inside $\mathcal{H}_d^B$. Let $B_t$ be the blow-up of $B$ in a point $t \in C_d$ and let $l$ be a bi-secant which is tangent to $C_d$ at $t$ (if it exists). Let $E$ be the exceptional divisor, and $C'$ and $l'$ the strict transforms of $C$ and $l$ respectively. By hypothesis there exists a unique point $s \in E \cap C' \cap l'$. We estimate $\dim_{\mathbb{C}} \text{Hom}^{d-2} (\mathbb{P}^1, B_t, p \mapsto s)_{[\pi]}$, where $p$ is a fixed point of $\mathbb{P}^1$, $[\pi]$ is a general point, and the degree is measured by $-K_{B_t}$. In this case $h^0(\pi^* T_{B_t}(-p)) = 2d - 2$ hence $\dim_{\mathbb{C}} \pi^{-1}_1(\{l\}) \leq 2d - 2 + 1 - 2 = 2d - 3$. This implies the claim.

The cases (3), (4) and (5) are similar. Thus we only give few comments for (5). Set $\mathcal{D}$ be the closure of the set

$$\{([C_d],[l_1],[l_2],[l_3]) \mid C_d \cap l_1 \neq \emptyset (i = 1, 2, 3),\] $$

$$l_1 \cap l_2 \cap l_3 \neq \emptyset, l_1 \cap l_2 \cap l_3 \notin C_d, l_i \text{ are distinct}\}$$

$$\subset \mathcal{H}_d^B \times \mathcal{H}_1^B \times \mathcal{H}_{11}^B.$$ 

For the former half of (5), we have only to prove that $\dim \mathcal{D} \leq 2d$. This can be carried out by a similar dimension count as above. For the latter half of (5), we use
the inductive construction of $C_d$ besides dimension count. We can omit the proof of (6)–(8) since are definitely similar to those of (1)–(3).

\[ \square \]

**Notation 2.2.3.** Denote by $\beta_i$ ($i = 1, \cdots, s$) bi-secant lines of $C_d$.

In the following proposition, we describe some more relations between $C_d$ and lines on $B$ by using $M(C_d) \subset H^B_2$.

**Proposition 2.2.4.** A general $C_d$ as in Proposition 2.2.1 satisfies the following conditions:

1. $C_d$ intersects $B_2$ simply,
2. $M_d := M(C_d)$ intersects $Q_2$ simply,
3. $M_d$ is an irreducible curve of degree $d$ with only simple nodes (recall that in Proposition 2.1.3 we abuse the notation by denoting the one-dimensional part of $\pi(\varphi^{-1}(C_1))$ by $M(C_1)$),
4. for a general line $l$ intersecting $C_d$, $M_d \cup M(l)$ has only simple nodes as its singularities, and
5. $M_d \cup M(\beta_i)$ has only simple nodes as its singularities.

**Proof.** We show the assertion inductively using the smoothing construction of $M_d$ from the union of $C_{d-1}$ and a general uni-secant line $l$ of $C_{d-1}$.

In case of $d = 1$, by letting $C_1$ be a general line, the assertion follows from Proposition 2.1.3. By induction on $d$ assume that we have a smooth $C_{d-1}$ ($d \geq 2$) satisfying (1)–(5). We verify $C_{d-1} \cup l$ satisfies the following (1)–(5)', which are suitable modifications of (1)–(5):

1. $C_{d-1} \cup l$ intersects $B_2$ simply by (1) for $C_{d-1}$ and generality of $l$.
2. $M_{d-1} \cup M(l)$ intersects $Q_2$ simply by (2) for $C_{d-1}$ and generality of $l$.
3. $M_{d-1} \cup M(l)$ is not irreducible but is of degree $d$ and has only simple nodes by (4) for $C_{d-1}$.
4. $M_{d-1} \cup M(l) \cup M(m)$ has only simple nodes as its singularities for a general line $m$ intersecting $C_{d-1}$.

Indeed, since $m$ is also general, $M_{d-1} \cup M(m)$ has only simple nodes by (4) for $C_{d-1}$. Thus we have only to prove that $M_{d-1} \cap M(l) \cap M(m) = \emptyset$, namely, there is no secant line of $C_{d-1}$ intersecting both $l$ and $m$. Fix a general $l$ and move $m$. If there are secant lines $r_m$ of $C_{d-1}$ intersecting both $l$ and $m$ for general $m$'s, then $r_m$ moves whence we have $M(l) \subset M_{d-1}$, a contradiction.

5. For a bi-secant line $\beta$ of $C_{d-1} \cup l$ except the lines through $C_{d-1} \cap l$, the curve $M_{d-1} \cup M(l) \cup M(\beta)$ has only simple nodes as its singularities.

Indeed, if $\beta$ is a bi-secant line of $C_{d-1}$, then the assertion follows from (5) for $C_{d-1}$ by a similar way to the proof of (4)'. Suppose that $\beta$ is a uni-secant line of $C_{d-1}$ intersecting $l$. We have only to prove that there is no secant line of $C$ intersecting both $l$ and $\beta$. If there is such a line $r$, then $l$, $\beta$ and $r$ pass through one point. This does not occur for general $l$ and $\beta$ by Proposition 2.2.2 (5).

Thus, by a deformation theoretic argument, we see that $C_d$ satisfies (1)–(5). \[ \square \]

**Notation 2.2.5.** Consider the double projection from $b$, see proposition 2.2.6 (1)]. Throughout the paper, we denote by $C_b'$, $C_b''$ and $C_b$ the strict transforms of $C := C_d$ on $B_b$, $B_b'$ and $\mathbb{P}^2$ respectively.

The following result is one of the key properties of the component $H^B_2$. Its importance and difficulty lies in the actual fact that it holds for every $b \in B$. 

\[ \square \]
Proposition 2.2.6. Let $C_d$ be a general smooth rational curve of degree $d$ on $B$ constructed as in Proposition [2.2.1]. Then, for any point $b \in B$, the restriction of $\pi_b$ to $C_d$ is birational if $d \geq 5$.

Proof. We prove the assertion by induction based on the construction of $C_d$ from $C_{d-1} \cup l$, where $l$ is a general uni-secant line of $C_{d-1}$ on $B$.

Assume that $d = 5$ and $\pi_{b|C_b}$ is not birational for $b$. Then $C_b$ is a line or conic in $\mathbb{P}^2$. Let $S$ be the pull-back of $C_b$ by $\pi_{2b}$. If $C_b$ is a line, then $C_5$ is contained in a singular hyperplane section, which is the strict transform of $S$ on $B$ (recall that $B \dashrightarrow \mathbb{P}^2$ is the double projection from $b$). This contradicts Proposition [2.2.1] (d).

Assume that $C_b$ is a conic.

The only possibility is that $L \cdot C''_b = 4$ and $C''_b \rightarrow C_b$ is a double cover since $\deg C_b \cdot \deg(C''_b \rightarrow C_b) \leq 5$. By Proposition [2.1.6] (1-1), it holds $H \cdot C''_b = 6$. Then by $L = H - 2E'_b$, we have $E'_b \cdot C''_b = 1$. Note that $E'_b S = 2$. On $S$, we can write $E'_b S \sim C_0 + pl$ and $C''_b \sim 2C_0 + ql (p, q \geq 0)$, where $C_0$ is the negative section of $S$ and $l$ is a fiber of $S \rightarrow C''_b$. Set $e := -C_0^2$. By $E'_b \cdot C''_b = 1$ and $E'_b S = 2$, we have $q + 2p = 2e = 1$ and $2p - e = 2$. Thus $e = 2p - 2$ and $q = 2p - 3$. Since $C''_b$ is irreducible, $q \geq 2e$, whence $2p - 3 \geq 2(2p - 2)$, i.e., $p = 0$ and $q = -3$, a contradiction.

Assume that $d \geq 6$. Let $\mathcal{C} \rightarrow \Delta$ be the one-parameter smoothing of $C_{d-1} \cup l$ such that $\mathcal{C}$ is smooth. We consider the trivial family of the double projections $B \times \Delta \dashrightarrow \mathbb{P}^2 \times \Delta$ from $b \times \Delta$. Denote by $\mathcal{C}'$, $\mathcal{C}''_b$ and $\mathcal{C}_b$ the strict transforms of $\mathcal{C}$ on $B' \times \Delta$, $B''_b \times \Delta$ and $\mathbb{P}^2 \times \Delta$ respectively. We also denote by $\mathcal{C}'_{d-1,b}$, $\mathcal{C}''_{d-1,b}$, and $\mathcal{C}_{d-1,b}$ the strict transforms of $C_{d-1}$ on $B'_b$, $B''_b$ and $\mathbb{P}^2$ respectively. It suffices to prove that there exists at least one point on $C_{d-1}$ where $\mathcal{C} \dashrightarrow \mathcal{C}_b$ is birational.

Indeed, set

$$
\mathcal{N} := \{(b, t) \in B \times \Delta \mid \mathcal{C} \dashrightarrow \mathcal{C}_b \text{ is not birational at any point of } \mathcal{C}_t \}
$$

and let $\Delta' \subset \Delta$ be the image of $\mathcal{N}$ by the projection to $\Delta$. $\mathcal{N}$ is a closed subset, and so is $\Delta'$ since $B \times \Delta \rightarrow \Delta$ is proper. Thus $\Delta'$ consists of finitely many points since the origin is not contained in $\Delta'$. For a point $t \in \Delta$ sufficiently near the origin, $\mathcal{C}_t \dashrightarrow \mathcal{C}_{t,b}$ is birational for any $b$.

By induction, we may assume that $\mathcal{C}_{d-1} \dashrightarrow \mathcal{C}_{d-1,b}$ is birational. Note that $\mathcal{C}_{d-1,b}$ is not a line since otherwise $\mathcal{C}_{d-1}$ is contained in a singular hyperplane section as we see above in the case of $\mathcal{C}_0$, a contradiction. As for $l$, if $b \notin l$, then the image of $l$ is a line or a point on $\mathbb{P}^2$. If $b \in l$, then the strict transform of $l$ on $B_b$ is a flopping curve. Thus $\mathcal{C}_b$ contains the line corresponding to $l$. We investigate the other possible reducible components of the central fiber $\mathcal{C}_{b,0}$ of $\mathcal{C}_b \rightarrow \Delta$. If $b \notin C_{d-1} \cup l$, then the only possibility is that $\mathcal{C}_{b,0}$ contains the image of a flopped curve, which is a line on $\mathbb{P}^2$. Thus $\mathcal{C} \dashrightarrow \mathcal{C}_b$ is birational at a point of $C_{d-1}$.

If $b \in C_{d-1} \cup l$, then $\mathcal{C}_{b,0}$ contains the image $m_b$ of the strict transform $m''_b$ of a line $m'_b$ in $E_b$ through $E_b \cap (C_{d-1,b} \cup l'_b)$, where $l'_b$ is the strict transform of $l$ on $B_b$. The line $m'_b$ is nothing but the exceptional curve for $\mathcal{C}_{b} \rightarrow \mathcal{C}$ (recall that $\mathcal{C}$ is a smooth surface). Moreover, if $b \in l$, then by the description of $E_b \dashrightarrow \mathbb{P}^2$, $m_b$ is a line since $l_b$ is a flopping curve. Thus $\mathcal{C} \dashrightarrow \mathcal{C}_b$ is birational at a point of $C_{d-1}$. Suppose that $b \in C_{d-1} \setminus l$. If $m_b$ intersects a flopping curve, $m_b$ is a line or a point, thus $\mathcal{C} \dashrightarrow \mathcal{C}_b$ is birational at a point of $C_{d-1}$. In the other case, $m_b$ is a conic. If $b \notin \cup_i \beta_i$, then $\deg C_{d-1,b} = d - 3$ by Proposition [2.1.6] (1-1). By $d \geq 6$, $C_{d-1,b}$ is not a conic, thus $\mathcal{C} \dashrightarrow \mathcal{C}_b$ is birational at a point of $C_{d-1}$. Assume $b \in \beta_i$. Then
deg \( C_{d-1,b} = d - 4 \). We have only to show that if \( d = 6 \), then \( C_{d-1,b} \neq m_b \). By Proposition 2.2.2 (4), the flop is of type (a) in Proposition 2.1.6 (1-2). The line \( m'_b \) intersects three lines which are the strict transforms of three fibers of \( \pi_b \) contained in \( E'_b \). On the other hand, by \( E'_b \cdot C''_{d-1,b} = 2 \), the curve \( C''_{d-1,b} \) intersects at most two fibers of \( \pi \) contained in \( E'_b \). Thus \( C_{d-1,b} \neq m_b \), and \( C \rightarrow C_b \) is birational at a point of \( C_{d-1} \).

We restate the proposition in terms of the relation between \( C_d \) and multi-secant conics of \( C_d \) on \( B \) as follows:

**Corollary 2.2.7.** Let \( b \) be a point of \( B \) not in any bi-secant line of \( C_d \) on \( B \). If \( d \geq 5 \), then there exist finitely many \( k \)-secant conics of \( C_d \) on \( B \) through \( b \) with \( k \geq 2 \) if \( b \notin C_d \) (resp. with \( k \geq 3 \) if \( b \in C_d \)).

**Proof.** For a point \( b \in B \) outside bi-secant lines of \( C_d \) on \( B \), there exist a finite number of singular multi-secant conics of \( C_d \) through \( b \) since the number of lines through \( b \) is finite, and the number of lines intersecting both a line through \( b \) and \( C_d \) is also finite by Proposition 2.2.1 (3). Therefore we have only to consider smooth multi-secant conics \( q \) of \( C_d \) through \( b \). By Proposition 2.1.6 (1-3), the strict transform \( q' \) of such a conic \( q \) on \( B'_b \) is a fiber of \( \pi_{2b} \). If \( b \notin C_d \), then \( q' \) intersects \( C'_b \) twice or more counted with multiplicities, thus by Proposition 2.2.6, the finiteness of such a \( q \) follows. We can prove the assertion in case of \( b \in C_d \) similarly, thus we omit the proof.

**Remark.** We refine this statement in Lemmas 2.4.13 and 2.5.7.

### 2.3. Curve \( \mathcal{H}_1 \) parameterizing marked lines.

We fix a general \( C := C_d \) as in 2.2. Let \( f : A \rightarrow B \) be the blow-up along \( C \). We start the study of the geometry of \( A \). The first step consists of finding the curves, if any, which replace the lines of ordinary geometry.

#### 2.3.1. Construction of \( \mathcal{H}_1 \) and marked lines. Set \( \mathcal{H}_1 := \varphi^{-1}C \subset \mathbb{P} \) and \( M := M_d \). We begin with a few corollaries of Proposition 2.2.3.

**Corollary 2.3.1.** If \( d \geq 2 \), then \( \mathcal{H}_1 \) is a smooth curve of genus \( d - 2 \) with the triple cover \( \mathcal{H}_1 \rightarrow C \). In particular, if \( d \geq 3 \), then \( \mathcal{H}_1 \) is a smooth non-hyperelliptic trigonal curve of genus \( d - 2 \).

**Proof.** By Propositions 2.1.3 and 2.2.4 (1), it holds that \( \mathcal{H}_1 \) is smooth and the ramification for \( \mathcal{H}_1 \rightarrow C \) is simple by Proposition 2.2.4 (1). Since \( B_x \subset -K_B \) and \( d = \deg C \), we can compute \( g(\mathcal{H}_1) \) by the Hurwitz formula:

\[
g(\mathcal{H}_1) - 2 = 3 \times (-2) + d \times 2, \text{ equivalently, } g(\mathcal{H}_1) = d - 2.
\]

**Corollary 2.3.2.** The number \( s \) of nodes of \( M \) is \( \frac{(d-2)(d-3)}{2} \), whence \( C \) has \( \frac{(d-2)(d-3)}{2} \) bi-secant lines on \( B \).

**Proof.** By the inductive construction of \( C \) we see that \( \pi_{\mathcal{H}_1} : \mathcal{H}_1 \rightarrow M \) is birational. By 2.2.4 (3) \( p_a(M) = \frac{(d-1)(d-2)}{2} \). Then by 2.2.4 (3) we know the number of nodes of \( M \) since \( g(\mathcal{H}_1) = d - 2 \). The latter half follows since a bi-secant line of \( C \) corresponds to a node of \( M \).
Now we select some lines on $B$ which we use in the sequel. Note that

$$\mathcal{H}_1 = \{ ([l], t) \mid [l] \in M, t \in C \cap l \} \subset M \times C,$$

and the elements of $\mathcal{H}_1$ deserve a name:

**Definition 2.3.3.** The pair of a secant line $l$ of $C$ on $B$ and a point $t \in C \cap l$ is called a marked line.

Let $(l, t)$ be a marked line. If $C \cap l$ is one point, then $\{t\} = C \cap l$ is uniquely determined. For a bi-secant line $\beta_i$ of $C$, there are two choices of $t$. Thus $\mathcal{H}_1$ parameterizes marked lines.

**2.3.2. Lines on the blow-up $A$ of $B$ along $C_d$.**

We prove that each marked line corresponds to a curve of anticanonical degree 1 on the blow-up $A$ of $B$ along $C$. This gives us a suitable notion of line on $A$.

**Notation 2.3.4.**

1. Let $f: A \to B$ be the blowing up along $C$ and $E_C$ the $f$-exceptional divisor.
2. $\{p_{ij}, p_i\} = C \cap \beta_i \subset B$.
3. $\zeta_{ij} = f^{-1}(p_{ij}) \subset E_C \subset A$, and
4. $\beta_i \cap \zeta_{ij} = p_{ij} \in E_C \subset A$,

where $i = 1, \ldots, s$ and $j = 1, 2$.

**Definition 2.3.5.** We say that a connected curve $l \subset A$ is a line on $A$ if

1. $-K_A \cdot l = 1$, and
2. $E_C \cdot l = 1$.

We point out that since $-K_A = f^*(-K_B) - E_C$ and $E_C \cdot l = 1$ then $f(l)$ is a line on $B$ intersecting $C$. More precisely:

**Proposition 2.3.6.** A line $l$ on $A$ is one of the following curves on $A$:

1. the strict transform of a uni-secant line of $C$ on $B$, or
2. the union $l_{ij} = \beta_i \cup \zeta_{ij}$, where $i = 1, \ldots, s$ and $j = 1, 2$.

In particular $l$ is reduced and $p_a(l) = 0$.

**Notation 2.3.7.** For a line $l$ on $A$, we usually denote by $\overline{l}$ its image on $B$.

**Corollary 2.3.8.** The curve $\mathcal{H}_1 \subset \mathbb{P}$ is the Hilbert scheme of the lines of $A$.

**Proof.** Let $\mathcal{H}_1'$ be the Hilbert scheme of lines on $A$, which is a locally closed subset of the Hilbert scheme of $A$. By the obstruction calculation of the normal bundles of the components of lines on $A$, it is easy to see that $\mathcal{H}_1'$ is a smooth curve. Denote by $\mathcal{U}_1 \to \mathcal{H}_1'$ the universal family of the lines on $A$ and let $\overline{\mathcal{U}_1}$ be the image of $\mathcal{U}_1$ on $B \times \mathcal{H}_1'$ (with induced reduced structure).

**Claim 2.3.9.** $\overline{\mathcal{U}} \to \mathcal{H}_1'$ is a $\mathbb{P}^1$-bundle.

**Proof of the claim.** Let $\mathcal{L}$ be the pull-back of the ample generator of Pic $B$ by

$$\mathcal{U}_1 \to A \times \mathcal{H}_1' \to B \times \mathcal{H}_1' \to B.$$

Since $g: \mathcal{U}_1 \to \mathcal{H}_1'$ is flat and $h^0(l, \mathcal{L}_l) = 2$ for a line $l$ on $B$, $\mathcal{E} := g_* \mathcal{L}$ is a locally free sheaf of rank two. $\mathbb{P}(\mathcal{E})$ is nothing but the $\mathbb{P}^1$-bundle contained in $B \times \mathcal{H}_1'$ whose fiber is the image of a line on $A$. This implies that $\mathbb{P}(\mathcal{E}) = \overline{\mathcal{U}}$ as schemes and $\overline{\mathcal{U}}$ is a $\mathbb{P}^1$-bundle. \qed
By the claim same we have a natural morphism $\mathcal{H}'_1 \to \mathbb{P}^2$, whose image is $M$. By Proposition 2.3.6 $\mathcal{H}'_1 \to M$ is birational and surjective. Since $\mathcal{H}'_1$ and $\mathcal{H}_1$ are smooth, they are both normalizations of $M$, then $\mathcal{H}'_1 \simeq \mathcal{H}_1$. □

Remark. For a bi-secant line $\beta_i$, we have two choices of marking, $p_{ij}$ or $p_{ij}$. We describe which line on $A$ corresponds to $(\beta_i, p_{ij})$. Denote by $\mathcal{U}_1 \to \mathcal{H}_1$ the universal family of the lines on $A$ and consider the following diagram:

$$
\begin{array}{ccc}
\mathcal{U}_1 & \subset & A \times \mathcal{H}_1 \\
\downarrow & & \downarrow \\
\overline{\mathcal{U}}_1 & \subset & B \times \mathcal{H}_1.
\end{array}
$$

Then $\mathcal{U}_1 \to \overline{\mathcal{U}}_1$ is the blow-up along $(C \times \mathcal{H}_1) \cap \overline{\mathcal{U}}_1$, which is the union of a section of $\overline{\mathcal{U}}_1 \to \mathcal{H}_1$ consisting markings and finite set of points $(p_{i,3-j}, [\beta_i, p_{ij}])$. Thus the marked line $(\beta_i, p_{ij})$ corresponds to the line $l_{i,3-j}$.

2.4. Surface $\mathcal{H}_2$ parameterizing marked conics.

Now we define a notion of conic on $A$. We proceed as in the case of lines, first defining the notion of marked conic.

2.4.1. Construction of $\mathcal{H}_2$ and marked conics.

Definition 2.4.1. The pair of a $k$-secant conic $q$ on $B$ with $k \geq 2$ and a zero-dimensional subscheme $\eta \subset C$ of length two contained in $q|_C$ is called a marked conic.

From now on, we assume that $d \geq 3$.

Marked conics are parameterized by

$$
\mathcal{H}'_2 := \{([q], [\eta]) \mid [q] \in \overline{\mathcal{H}}_2, \eta \subset C \cap q \subset \overline{\mathcal{H}}_2 \times S^2C
$$

with reduced structure, where $\overline{\mathcal{H}}_2 \subset \mathbb{P}^4$ is the locus of multi-secant conics of $C$ on $B$.

By Corollary 2.7.1 and $d \neq 1$, the natural projection of $\mathcal{H}'_2 \to S^2C$ is one to one outside $[\beta_i|_C]$ and the diagonal of $S^2C$, thus by the Zariski main theorem, it is an isomorphism outside $[\beta_i|_C]$ and the diagonal of $S^2C$.

We denote by $e'_i$ the fiber of $\mathcal{H}'_2 \to S^2C$ over a $[\beta_i|_C]$. Since $B$ is the intersection of quadrics, any conic cannot intersect a line twice properly. Thus any conic $\supset \beta_i|_C$ contains $\beta_i$. This implies that $e'_i \simeq \mathbb{P}^1$, and $e'_i$ parameterizes marked conics of the form

$$
\{([\beta_i \cup \alpha], [\beta_i|_C]) \mid \alpha \text{ is a line such that } \alpha \cap \beta_i \neq \emptyset\}.
$$

Over the diagonal of $S^2C$, $\mathcal{H}'_2 \to S^2C$ is finite since for $t \in C$, there exist a finite number of reducible conics with $t$ as a singular point or conics tangent to $C$ at $t$.

Hence $\mathcal{H}'_2$ is the union of the unique two-dimensional component, which dominates $S^2C$, and possibly lower dimensional components mapped into the diagonal of $S^2C$ or $e'_i$. Note that $\mathcal{H}'_2 \to \overline{\mathcal{H}}_2$ is finite since choices of markings of a multi-secant conic of $C$ is finitely many by $d \geq 3$.

Claim 2.4.2. $e'_i$ is contained in the unique two-dimensional component of $\mathcal{H}'_2$. 

Proof. We have only to prove that $\overline{\mathcal{H}}_2$ is two-dimensional near the generic point of the image of $e'_i$ since $\mathcal{H}_2' \rightarrow \overline{\mathcal{H}}_2$ is one to one near the generic point of the image of $e'_i$. Let $\mathcal{V}_2 \rightarrow \mathcal{H}_2'^B \cong \mathbb{P}^4$ be the universal family of conics on $B$ and $\overline{\mathcal{H}}'_2$ the inverse image of $C \times C$ by $\mathcal{V}_2 \times \mathcal{V}_2 \rightarrow B \times B$. Since the morphism $\mathcal{V}_2 \times \mathcal{V}_2 \rightarrow \mathcal{V}_2 \rightarrow \mathbb{P}^4$ is flat, $\mathcal{V}_2 \times \mathcal{V}_2$ is purely six-dimensional. Thus any component of $\overline{\mathcal{H}}_2$ has dimension greater than or equal to two. Though the inverse image of the diagonal of $C \times C$ is three-dimensional, any other component of $\overline{\mathcal{H}}'_2$ is at most two-dimensional by a simple dimension count similar to the proof of Proposition 2.2.2. We can prove that a general $C$ does not satisfy this condition.

Notation 2.4.3. Let $\mathcal{H}_2$ be the normalization of the unique two-dimensional component of $\mathcal{H}_2'$ and $\overline{\mathcal{H}}_2 \subset \overline{\mathcal{H}}_2$ the image of $\mathcal{H}_2$. Denote by $\eta$ the natural morphism $\mathcal{H}_2 \rightarrow S^2C$. Set

$$c_i := [\beta_i(c)] \in S^2C \cong \mathbb{P}^2,$$

and

$$e_i := \eta^{-1}(c_i),$$

where $i = 1, \ldots, s$.

By the above consideration, $\eta: \mathcal{H}_2 \rightarrow S^2C$ is isomorphic outside $[\beta_i(c)]$ and $\mathcal{H}_2 \rightarrow \overline{\mathcal{H}}_2$ is the normalization. Thus we see that $\mathcal{H}_2$ parameterizes marked conics outside the inverse image of $c_i$. We need to understand the inverse image by $\eta$ of the diagonal.

Claim 2.4.4. Assume that $([q], [2b]) \in \mathcal{H}_2$ for $b \in C$ and a conic $q$. Then

(1) $q$ is reduced,

(2) if $q$ is smooth at $b$, then $q$ is tangent to $C$ at $b$, and

(3) if $q$ is singular at $b$, then the strict transform of $q$ is connected on $A$. Moreover, $b \notin \beta_i$ nor $B_\varphi$.

Proof. By Proposition 2.1.6 (1-3) and a degeneration argument, $q$ corresponds to the fiber of $\pi_{2b}$ through the point $t'$ in $C_b' \cap E_b'$ coming from $t := C_b \cap E_b$.

(1) Assume by contradiction that $q$ is non-reduced. By Proposition 2.1.4, $q$ is a multiple of a special line $l$. By Proposition 2.2.2 (4), $l$ is a uni-secant line of $C$. Let $m$ be the other line through $b$ (by generality of $C$, we have $l \neq m$). Let $l'$ and $m'$ be the strict transforms of $l$ and $m$ on $B_b$ respectively. By Proposition 2.1.6 (1-3), the fiber of $\pi_{2b}$ through $t'$ is the strict transform of the line in $E_b$ joining $l' \cap E_b$ and $m' \cap E_b$. Hence by the assumption, $l' \cap E_b$, $m' \cap E_b$ and $C_b' \cap E_b$ are collinear. By dimension count similar to the proof of Proposition 2.2.2, we can prove that a general $C$ does not satisfy this condition.

(2) This follows from the previous discussion.

(3) Set $q = l_1 \cup l_2$, where $l_1$ and $l_2$ are the irreducible components of $q$, and let $l'_1$ be the strict transform of $l_1$ on $B_b$. By (1), it holds $l_1 \neq l_2$. Then the fiber of $\pi_{2b}$ corresponding to $q$ is the strict transform of the line on $E_b$ through $E_b \cap l'_1$ and $E_b \cap l'_2$. Note that $A$ is obtained from $B_b$ by blow-up $B_b$ along $C_b'$ and then contracting the strict transform of $E_b$. Thus the former half of the assertion follows. The latter half follows again by simple dimension count.
2.4.2. Conics on $A$.

**Definition 2.4.5.** We say that a curve $q \subset A$ is a **conic** on $A$ if

(i) $q$ is connected and reduced,  
(ii) $-K_A \cdot q = 2$,  
(iii) $E_C \cdot q = 2$, and  
(iv) $p_a(q) = 0$.

Using this definition, we can classify conics on $A$ similarly to Proposition 2.3.6.

**Proposition 2.4.6.** Let $q$ be a conic on $A$. Then $\overline{\mathcal{Q}} := f(q) \subset B$ is a $k$-secant conic of $C$ with $k \geq 2$. Moreover one of the following holds:

(a) $\overline{\mathcal{Q}}$ is smooth at $\overline{\mathcal{Q}} \cap C$. $q$ is the union of the strict transform $q'$ of $\overline{\mathcal{Q}}$ and $k-2$ distinct fibers $\zeta_1, \ldots, \zeta_{k-2}$ of $E_C$ such that $\zeta_i \cap q' \neq \emptyset$,

(b) $\overline{\mathcal{Q}}$ is the union of two uni-secant lines $l$ and $m$ such that $C \cap l \cap m \neq \emptyset$. $q$ is the union of the strict transforms $l$ and $m$ of $\overline{l}$ and $\overline{m}$ respectively (we assume that $l \cap m \neq \emptyset$), or

(c) $\overline{\mathcal{Q}}$ is the union of $\beta_i$ and a line $\mathcal{T}$ through a $p_{ij}$. $q$ is the union of the fiber $\zeta_{ij}$ over $p_{ij}$ and the strict transforms $\beta_i'$ and $\tau'$ of $\beta_i$ and $\mathcal{T}$ respectively.

**Notation 2.4.7.** We usually denote by $\overline{\mathcal{Q}} \subset B$ the image of a conic $q$ on $A$.

Let $\mathcal{H}_2^A$ be the normalization of the two-dimensional part of the Hilbert scheme of conics on $A$, which is a locally closed subset of the Hilbert scheme of $A$. Let $\mu: \mathcal{U}_2 \to \mathcal{H}_2^A$ be the pull-back of the universal family of conics on $A$.

**Lemma 2.4.8.** Let $\overline{\mathcal{U}}_2$ be the image of $\mathcal{U}_2$ on $B \times \mathcal{H}_2^A$ (with induced reduced structure) then $\overline{\mathcal{U}}_2 \to \mathcal{H}_2^A$ is a conic bundle.

**Proof.** The proof is similar to that of Claim 2.3.9.

Let $\mathcal{L}$ be the pull-back of the ample generator of Pic $B$ by

$$\mathcal{U}_2 \to A \times \mathcal{H}_2^A \to B \times \mathcal{H}_2^A \to B.$$  

Since $\mu: \mathcal{U}_2 \to \mathcal{H}_2^A$ is flat and $h^0(q, \mathcal{L}, q) = 3$ for a conic $q$ on $A$ (recall that $q$ is reduced), then $E := \mu_* \mathcal{L}$ is a locally free sheaf of rank 3. Letting $\mathbb{P}^6 = (B)$, $\mathbb{P}(E)$ is the $\mathbb{P}^2$-bundle contained in $\mathbb{P}^6 \times \mathcal{H}_2^A$ whose fiber is the plane spanned by the image of a conic on $A$. Let $Q := (B \times \mathcal{H}_2^A) \cap \mathbb{P}(E)$, where the intersection is taken in $\mathbb{P}^6 \times \mathcal{H}_2^A$. A scheme theoretic fiber of $Q \to \mathcal{H}_2^A$ is the image of a conic of $A$ since $B$ is the intersection of quadrics. Then $Q = \overline{\mathcal{U}}_2$ as schemes and $\overline{\mathcal{U}}_2$ is a conic bundle. □

**Proposition 2.4.9.** The two surfaces $\mathcal{H}_2^A$ and $\mathcal{H}_2$ are isomorphic.

**Proof.** By Lemma 2.4.8 there exists a natural morphism $\overline{\nu}: \mathcal{H}_2^A \to \mathcal{H}_2$. By Proposition 2.4.6 $\overline{\nu}$ is finite and birational, hence $\overline{\nu}$ lifts to the morphism $\nu: \mathcal{H}_2^A \to \mathcal{H}_2$ since $\mathcal{H}_2 \to \mathcal{H}_2$ is the normalization. By the Zariski main theorem, $\nu$ is an inclusion. By Claim 2.4.4 (1) and (3), and Proposition 2.4.6 $\nu$ is also surjective. □

By Proposition 2.4.9 we can pass freely from conics on $A$, that is elements of $\mathcal{H}_2^A$ to marked conics and vice-versa according to the kind of argument we will need. In particular we can speak of the universal family $\mu: \mathcal{U}_2 \to \mathcal{H}_2$ of marked conics meaning $\mathcal{U}_2 := \mathcal{U}_2^A$ and $\mathcal{H}_2^A$ identified to $\mathcal{H}_2$ via $\nu$.  

Query: What is the meaning of a conic on $A$ according to Definition 2.4.5?  

Response: A conic on $A$ is defined as a curve $q \subset A$ that is connected and reduced, satisfies $-K_A \cdot q = 2$, $E_C \cdot q = 2$, and $p_a(q) = 0$. This definition allows for the classification of conics on $A$ similarly to Proposition 2.3.6.
Corollary 2.4.10. The Hilbert scheme of conics on $A$ is an irreducible surface (and $H_2$ is the normalization). The normalization is injective, namely, $H_2$ parameterizes conics on $A$ in one to one way.

Proof. By Proposition 2.4.6 the image of $H_2$ in the Hilbert scheme parameterizes all the conics, thus the first part follows.

For the second part, we have already seen that $H_2$ parameterizes conics on $A$ in one to one way outside $\cup_i e_i$. Let $\alpha$ be a general line intersecting $\beta_i$, and $\alpha'$ the strict transform of $\alpha$ on $A$. By easy obstruction calculation, we see that the Hilbert scheme of conics on $A$ is smooth at $[\beta_i' \cup \alpha']$. Thus general points of $e_i$ also parameterizes conics on $A$. Then, however, since $e_i' \simeq \mathbb{P}^1$, where $e_i'$ is the inverse image of $[\beta_i|C]$ by $H_2' \rightarrow S^2C$, it holds that $e_i' \simeq e_i' \simeq \mathbb{P}^1$ ($H_2 \rightarrow S^2C$ has only connected fibers). This implies the assertion.

2.4.3. Description of $H_2$.
We want to investigate further the morphism $\eta: H_2 \rightarrow S^2C \simeq \mathbb{P}^2$.

Notation 2.4.11. For a point $b \in C$, set

$$L_b := \{[q] \in H_2 \mid \exists, b' \neq b, f(q) \cap C = \{b, b'\}\}.$$ 

By Corollary 2.1.7 $\eta(L_b)$ is a line in $S^2C \simeq \mathbb{P}^2$.

To understand better $\eta: H_2 \rightarrow \mathbb{P}^2$ we need to find special loci inside $H_2$. A natural step is to study the locus of conics which intersect a fixed line.

Let $U_1 \subset U_2 \times H_1$ be the pull-back of $U_1$ via the following diagram:

$$U_1' \subset U_2 \times H_1 \xrightarrow{A \times H_1} A \times H_1 \xrightarrow{U_1} H_1,$$

where $\tilde{D}_1$ is the image of $U_1'$ on $H_2 \times H_1$.

By definition

$$\tilde{D}_1 = \{([q], [l]) \mid q \cap l \neq \emptyset\} \subset H_2 \times H_1.$$ 

First we need to know which component of $\tilde{D}_1$ is divisorial or dominates $H_1$.

Let $\psi: U_2 \rightarrow A$ be the morphism obtained via the universal family $\mu: U_2 \rightarrow H_2$.

Next lemma is necessary to prove the finitess of $\psi$ outside $\cup_{i=1}^n \beta_i' \subset A$.

Lemma 2.4.12. Let $\overline{t}$ be a general uni-secant line of $C$ and $l_b \subset \mathbb{P}^2$ the image of $\overline{t}$ by the double projection from a point $b$. For a general point $b \notin C$, deg $C_b = d$ and $C_b \cup l_b$ has only simple nodes. For a general point $b$ of $C$, deg $C_b = d - 2$ and $C_b \cup l_b$ has only simple nodes.

Proof. The claims for deg $C_b$ follows from Propositions 2.1.6 (1-1) and 2.2.6. As for the singularity of $C_b \cup l_b$, the claim follows from simple dimension count. For simplicity, we only prove that for a general point $b$, $C_b$ has only simple nodes. By Proposition 2.2.2 we may assume that any multi-secant conic through $b$ is smooth, bi-secant and intersects $C$ simply. Let $\overline{t}$ be a smooth bi-secant conic through $b$. We may assume that $N_{\overline{t}|C_b} \simeq \mathcal{O}_{\mathbb{P}^1}(1)^{\otimes 2}$. Let $q'$ be the strict transform of $\overline{t}$ on $B'_b$. Let $B'' \rightarrow B'_b$ be the blow-up along $q'$, $E_{q'}$ the exceptional divisor and $C''$ the strict transform of $C''$. Note that $E_{q'} \simeq \mathbb{P}^1 \times \mathbb{P}^1$ since $N_{q'/B'_b} \simeq \mathcal{O}_{\mathbb{P}^2}^{\otimes 2}$. Then $C_b$ has simple

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nodes at the image of \( q' \) if and only if the two points in \( E_{q'} \cap \tilde{C}' \) does not belong to the same ruling with the opposite direction to a fiber of \( E_{q'} \rightarrow q' \). Let \( \tilde{D}_q \rightarrow B \) be the blow-up along \( q \), \( E_{\tilde{q}} \) the exceptional divisor and \( \tilde{C} \) the strict transform of \( C \). It is easy to see that a ruling of \( E_{\tilde{q}} \) with the opposite direction to a fiber of \( E_{\tilde{q}} \rightarrow \tilde{q} \) corresponds to that of \( E_{q'} \) with the opposite direction to a fiber of \( E_{q'} \rightarrow q' \). Thus \( C_b \) has simple nodes at the image of \( q' \) if and only if the two points in \( E_{\tilde{q}} \cap \tilde{C} \) does not belong to the same ruling with the opposite direction to a fiber of \( E_{\tilde{q}} \rightarrow \tilde{q} \). We can show that this is the case for a general \( b \) by simple dimension count. \( \square \)

From now on, we assume \( d \geq 5 \) throughout the paper since we need Proposition 2.2.6.

We do not have the finiteness of \( \psi \) all over \( A \). To obtain a finite morphism, we blow-up \( A \) more in 2.5.1. Till now we can prove:

**Proposition 2.4.13.** \( \psi \) is finite of degree \( n := \frac{(d-1)(d-2)}{2} \) and flat outside \( \cup_{i=1}^s \beta_i' \).

**Proof.** Let \( a \in A \setminus \cup_{i=1}^s \beta_i' \) and set \( b := f(a) \). If \( b \not\in C \), then the finiteness of \( \psi \) over \( a \) follows from Corollary 2.2.7. Moreover, by Lemma 2.4.12, the number of conics through a general \( a \) is \( n \). Thus \( \deg \psi = n \). We will prove that \( \psi \) is finite over \( a \in E_C \setminus \cup_{i=1}^s \beta_i' \). Once we prove this, the assertion follows. Indeed, \( \mathcal{U}_2 \) is Cohen-Macaulay since \( \mathcal{H}_2 \) is smooth and any fiber of \( \mathcal{U}_2 \rightarrow \mathcal{H}_2 \) is reduced, thus \( \psi \) is flat.

Let \( a \in E_C \setminus \cup_{i=1}^s \beta_i' \). The assertion is equivalent to that only finitely many conics belonging to \( L_b \) pass through \( a \). If \( b \notin \cup_{i=1}^s \beta_i \), then \( L_b \) is irreducible. If \( b \in \cup_{i=1}^s \beta_i \), then \( L_b = L_b' + e_i \), where \( L_b' \) is the strict transform of \( \eta(L_b) \) whence is irreducible. Note that almost all the conics belonging to \( e_i \) does not pass through \( a \notin \cup_{i=1}^s \beta_i' \). Let \( S_b \subset A \) be the locus swept by the conics of the family \( L_b \) if \( b \notin \cup_{i=1}^s \beta_i \), or the locus swept by the conics of the family \( L_b' \) if \( b \in \cup_{i=1}^s \beta_i \). \( S_b \) is irreducible. Let \( S_b := f(S_b), \mathcal{S}_b \) and \( \mathcal{S}'_b \) the strict transforms of \( S_b \) on \( B_b \) and \( B'_b \) respectively. Then \( \mathcal{S}'_b = \pi_{2b} C_b \). Let \( d_b := \deg C_b \). By Proposition 2.1.1 (1-1), \( d_b = d-2 \) if \( b \notin \cup_{i=1}^s \beta_i \), or \( d-3 \) if \( b \in \cup_{i=1}^s \beta_i \). Since \( \mathcal{S}'_b \sim d_b L \) and \( L = H - 2E_b \), we have \( \mathcal{S}'_{b|E_b} \) is a curve of degree \( 2d_b \) in \( E_b \cong \mathbb{P}^2 \).

Since \( A \) is obtained from \( B_b \) by blowing up \( C'_b \) and then contracting the strict transform of \( E_b \), a point \( a \) over \( b \) corresponds to a line \( l_a \) in \( E_b \) through \( t := E_b \cap C'_b \). If \( C''_b \) does not intersect fibers of \( \pi_{2b} \) contained in \( E'_b \), then \( \mathcal{S}'_{b|E_b} \) is irreducible. Thus no \( l_a \) is contained in \( \mathcal{S}'_{b|E_b} \) and we are done. Assume that \( C''_b \) intersects a fiber \( l' \) of \( \pi_{2b} \) contained in \( E'_b \). By Claim 2.4.14 (3), \( b \not\in B_\varphi \) nor \( b \notin \cup_{i=1}^s \beta_i \) for a general \( C \). Since \( L_b \) is irreducible, it suffices to prove the finiteness and nonemptiness of the set of conics through a general point \( b \) over \( A \). Equivalently, we only have to show that a general \( l_a \) intersects \( \mathcal{S}'_{b|E_b} \) outside \( t \). Since \( l' \) intersects \( C''_b \) simply at one point, \( C_b \) is smooth at the image \( l' \) of \( l' \) on \( \mathbb{P}^2 \). Thus \( \mathcal{S}'_{b|E_b} = C''_b + l, \) where \( C''_b \) and \( l \) are the strict transforms of \( C_b \) and \( l' \). Note that \( C''_b \) is smooth at \( t \) and \( \deg C''_b = 2d_b - 1 = 2d - 5 \geq 5 \) by \( d \geq 5 \). Thus a general \( l_a \) intersects \( C''_b \) outside \( t \).

**Remark.** Though we do not need it later, we describe the fiber of \( \psi \) over a general point \( a \in E_C \setminus \cup_{i=1}^s \beta_i' \) for reader’s convenience.

Set \( b := f(a) \). As in the proof of Proposition 2.4.13 a point \( a \) over \( b \) corresponds to a line \( l_a \) in \( E_b \) passing through \( E_b \cap C'_b \). By Lemma 2.4.12 it holds that \( \deg C_b =
By definition of $d$ and $l$, there exist $2(d-3)$ conics through $b$. By Proposition 2.4.10 corresponding to a tri-secant conic $\gamma$, there is a unique conic $q$ on $A$ containing the fiber of $E_C$ over $b$ and such a conic on $A$ contains $a$. Thus we obtain $\frac{(d-3)(d-4)}{2}$ conics through $a$. By definition of $L_b$, these conics do not belong to $L_b$.

We need more $n - \frac{(d-3)(d-4)}{2} = 2d - 5$ conics through $a$. We show that there exist $2(d-2) - 1$ conics through $a$ on $A$ coming from the family parameterized by $L_b$. We use the notation of the proof of Proposition 2.4.13. For a general $b \in C$, $C''_b$ does not intersect fibers of $\pi_{2b}$ contained in $E'_b$. Thus $\hat{S}_{b|E_b}$ is an irreducible curve of degree $2(d-2)$ on $E_b$. Thus there are $2(d-2)$ intersection points of $\hat{S}'_{b|E_b}$ and $l_a$. Among those, the intersection point $C''_b \cap E_b$ does not correspond to a conic on $A$ through $a$ since it comes from the tangent of $C$. Thus we have $2(d-2) - 1$ conics as desired.

We need to study mutual intersection of a conic and a line in special cases. Let $F \subset \mathcal{H}_2 \times \mathcal{H}_1$ be the image in $\mathcal{H}_2 \times \mathcal{H}_1$ of the inverse image of $((\cup \beta_i^j) \times \mathcal{H}_1) \cap \mathcal{U}_1$; that is

$$F := \{([q],[l]) \mid q \cap \beta_i^j \cap l \neq \emptyset\}.$$  

A point $([q],[l]) \in F$ iff i) $l = l_{ij}$ and $q \cap \beta_i^j \neq \emptyset$ or ii) $l \neq l_{ij}$, $l \cap \beta_i^j \neq \emptyset$ and $q \cap \beta_i^j \cap l \neq \emptyset$. For every $i = 1, \ldots, s$, $j = 1, 2$ the family of those $([q],[l])$ which satisfies i) or ii) has dimension one and clearly does not dominate $\mathcal{H}_1$.

**Corollary 2.4.14.** Any component of $\hat{D}_1$ which is not contained in $F$ dominates $\mathcal{H}_1$. Moreover, any non-divisorial component of $\hat{D}_1$ outside $F$ (if it exists) is a one-dimensional component whose generic point parameterizes reducible conics, namely, a one-dimensional component of

$$\{([q],[l]) \mid l \subset q\}.$$  

**Remark.** Here we leave the possibility that a one-dimensional component whose generic point parameterizes reducible conics is contained in a divisorial component of $\hat{D}_1$. We, however, prove that this is not the case in Corollary 2.4.19. Hence, finally, the fiber of $\hat{D}_1 \to \mathcal{H}_1$ over a general $[l] \in \mathcal{H}_1$ parameterizes conics which properly intersect $l$.

**Proof.** By Proposition 2.4.13 $\mathcal{U}_1 \to A$ is finite and flat outside $\cup \beta_i^j$. Then $\mathcal{U}_2 \times \mathcal{H}_1 \to A \times \mathcal{H}_1$ is flat outside $(\cup \beta_i^j) \times \mathcal{H}_1$. By base change, $\mathcal{U}_1' \to \mathcal{U}_1$ is flat and finite outside $((\cup \beta_i^j) \times \mathcal{H}_1) \cap \mathcal{U}_1$. Then every irreducible component of $\mathcal{U}_1'$ which is not mapped to $(\cup \beta_i^j) \times \mathcal{H}_1 \cap \mathcal{U}_1$ is two-dimensional, and dominates $\mathcal{U}_1$, hence dominates $\mathcal{H}_1$. Therefore any component of $\hat{D}_1$ which is not contained in $F$ dominates $\mathcal{H}_1$.

We find a possible non-divisorial component of $\hat{D}_1$ outside $F$. Let $q \subset U'_1$ be a curve mapped to a point, say, $([q],[l])$ on $\mathcal{H}_2 \times \mathcal{H}_1$. The image of $\gamma$ on $A$ is an irreducible component of $q$, say, $q_1$. The image of $\gamma$ on $\mathcal{U}_1$ is $q_1 \times [l]$, thus $q_1$ is also an irreducible component of $l$. We have the following three possibilities:

1. $l$ is irreducible, hence $q_1 = l$ and $q = l \cup m$, where $m$ is another line. Such $([q],[l])$ form the one-dimensional family of reducible conics,
2. $l = l_{ij}$ and $\beta_i^j \subset q$. Namely $[q] \in \epsilon_i$, or $q = \beta_i^j \cup \alpha \cup \zeta_{ik}$, where $\alpha$ is the strict transform of a line on $B$ intersecting $\beta_i$ and $C$ outside $\beta_i \cap C$, or
3. $l = l_{ij}$ and $\zeta_{ij} \subset q$ and $f(q)$ is a tri- or quadri-secant conic of $C$ such that $p_{ij} \in f(q)$. 


Thus we have the second assertion. \qed

**Notation 2.4.15.** Let \( D_1 \subset \mathcal{H}_2 \times \mathcal{H}_1 \) be the divisorial part of \( \hat{D}_1 \). Since \( \mathcal{H}_1 \) is a smooth curve \( D_1 \to \mathcal{H}_1 \) is flat. Let \( D_1 \) be the fiber of \( D_1 \to \mathcal{H}_1 \) over \(|l| \in \mathcal{H}_1 \). Clearly we can write \( D_1 \leftarrow \mathcal{H}_2 \).

Next two lemmas are basic to understand the geometry of \( \mathcal{H}_2 \).

**Lemma 2.4.16.** Let \( \mathcal{T}_1 \) and \( \mathcal{T}_2 \) be two general secant lines of \( C \) such that \( \mathcal{T}_1 \cap \mathcal{T}_2 = \emptyset \). Let \( B \longrightarrow Q \longrightarrow \mathbb{P}^2 \) be the successive linear projections from \( \mathcal{T}_1 \) and then the strict transform of \( \mathcal{T}_2 \) on \( Q \). Let \( \mathcal{T} \) be another general secant line of \( C \), and \( C', \mathcal{T} \subset \mathbb{P}^2 \) be the images of \( C \) and \( \mathcal{T} \) respectively. Then \( C \cup \mathcal{T} \longrightarrow C' \cup \mathcal{T} \) is birational and \( C' \cup \mathcal{T} \) has only simple nodes as its singularities. In particular (since \( \deg C' = d - 2 \) and \( C' \) is rational) \( C' \) has simple nodes, equivalently, there exist \( \frac{(d-3)(d-4)}{2} \) bi-secant conics of \( C \) intersecting both \( \mathcal{T}_1 \) and \( \mathcal{T}_2 \).

\[ \text{Proof.} \] We show the assertion using the inductive construction of \( C = C_d \). The assertion follows for \( d = 3 \) directly. Consider a smoothing from \( C_{d-1} \cup \overline{m} \) to \( C_{d} \). Let \( \overline{m}_1 \) and \( \overline{m}_2 \) be two general secant lines of \( C_{d-1} \) such that \( \overline{m}_1 \cap \overline{m}_2 = \emptyset \). Let \( B \longrightarrow Q \longrightarrow \mathbb{P}^2 \) be the successive linear projections from \( \overline{m}_1 \) and then the strict transform of \( \overline{m}_2 \) on \( Q \). Let \( \overline{m} \) be another general secant line of \( C_{d-1} \), and \( C'_{d-1}, \overline{m} \subset \mathbb{P}^2 \) be the images of \( C_{d-1} \) and \( \overline{m} \) respectively. Then we have only to show that \( C_{d-1} \cup \overline{m} \cup \overline{r} \longrightarrow C'_{d-1} \cup \overline{m}' \cup \overline{r}' \) is birational and \( C'_{d-1} \cup \overline{m}' \cup \overline{r}' \) has only simple nodes as its singularities assuming \( C_{d-1} \cup \overline{m} \cup \overline{r} \) is birational and \( C'_{d-1} \cup \overline{m}' \cup \overline{r}' \) has only simple nodes as its singularities. Since \( \overline{m}' \) is also general, \( C_{d-1} \cup \overline{m} \longrightarrow C'_{d-1} \cup \overline{m}' \) is birational and \( C'_{d-1} \cup \overline{m}' \) has only simple nodes as its singularities. Thus \( C_{d-1} \cup \overline{m} \cup \overline{r} \longrightarrow C'_{d-1} \cup \overline{m}' \cup \overline{r}' \) is clearly birational. To show \( C'_{d-1} \cup \overline{m}' \cup \overline{r}' \) has only simple nodes as its singularities, it suffices to prove that there are no secant conics of \( C_{d-1} \) intersecting all the \( \overline{m}_1 \), \( \overline{m}_2 \), \( \overline{m} \) and \( \overline{r} \). This follows from the fact that a secant conic \( \overline{q} \) of \( C_{d-1} \) intersects finitely many secant lines of \( C_{d-1} \) by \( M(\overline{r}) \not\subset M(C_{d-1}) \). \qed

**Lemma 2.4.17.** Let \( \mathcal{T}_0 \) be a general secant line of \( C \). Let \( B \longrightarrow Q \longrightarrow \mathbb{P}^2 \) be the successive linear projections from \( \mathcal{T}_0 \) and then the strict transform of a \( \beta_i \) on \( Q \). Let \( \mathcal{T} \) be another general secant line of \( C \), and \( C' \) and \( \mathcal{T} \subset \mathbb{P}^2 \) be the images of \( C \) and \( \mathcal{T} \) respectively. Then \( C \cup \mathcal{T} \longrightarrow C' \cup \mathcal{T} \) is birational and \( C' \cup \mathcal{T} \) has only simple nodes as its singularities. In particular (since \( \deg C' = d - 3 \) and \( C' \) is rational) \( C' \) has simple nodes, equivalently, there exist \( \frac{(d-3)(d-4)}{2} \) bi-secant conics of \( C \) intersecting \( \beta_i \) and \( \mathcal{T}_0 \) except conics containing \( \beta_i \).

\[ \text{Proof.} \] Similarly to the previous lemma, we show the assertion using the inductive construction of \( C = C_d \). The assertion follows for \( d = 4 \) directly. Consider a smoothing from \( C_{d-1} \cup \overline{m} \) to \( C_d \). Let \( \overline{m}_0 \) be a general secant line of \( C_{d-1} \), and \( \beta \) a bi-secant line of \( C_{d-1} \cup \overline{m} \) different from two lines through \( C_{d-1} \cap \overline{m} \). Let \( B \longrightarrow Q \longrightarrow \mathbb{P}^2 \) be the successive linear projections from \( \overline{m}_0 \) and then the strict transform of \( \beta \) on \( Q \). Let \( \overline{r} \) be another general secant line of \( C_{d-1} \), and \( C'_{d-1}, \overline{m}' \) and \( \overline{r}' \subset \mathbb{P}^2 \) be the images of \( C_{d-1} \), \( \overline{m} \) and \( \overline{r} \) respectively.

First we suppose that \( \beta \) is a bi-secant line of \( C_{d-1} \). Then we have only to show that \( C_{d-1} \cup \overline{m} \cup \overline{r} \longrightarrow C'_{d-1} \cup \overline{m}' \cup \overline{r}' \) is birational and \( C'_{d-1} \cup \overline{m}' \cup \overline{r}' \) has only
simple nodes as its singularities assuming \( C_{d-1} \cup \mathcal{P} \to C'_{d-1} \cup \mathcal{P} \) is birational and \( C'_{d-1} \cup \mathcal{P} \) has only simple nodes as its singularities. The proof is the same as that of Lemma 2.4.16, so we omit it.

Next suppose that \( \beta \) is a uni-section line of \( C_{d-1} \) intersecting \( \mathcal{P} \) outside \( C_{d-1} \cap \mathcal{P} \). Note that, by the projection \( B \to \mathbb{P}^2 \), \( \mathcal{P} \) is contracted to a point. Moreover, \( \beta \) is a general uni-section line since so is \( \mathcal{P} \). Thus, by Lemma 2.4.16 \( C_{d-1} \cup \mathcal{P} \cup \mathcal{P} \to C'_d \cup \mathcal{P} \cup \mathcal{P} \) is birational and \( C'_{d-1} \cup \mathcal{P} \cup \mathcal{P} \) has only simple nodes as its singularities.

Now we reach the precise description of \( \mathcal{H}_2 \).

**Theorem 2.4.18.** \( \eta \colon \mathcal{H}_2 \to \mathbb{P}^2 \) is the blow-up at \( c_1, \ldots, c_s \) and \( e_i \) are \( \eta \)-exceptional curves. Moreover \( \mathcal{H}_2 \) has the following properties:

1. \( D_l \sim (d-3)h - \sum_{i=1}^{s} e_i \),

where \( h \) is the strict transform of a general line on \( \mathbb{P}^2 \).

2. \( h^1(\mathcal{H}_2, \mathcal{O}_{\mathcal{H}_2}((d-4)h - \sum_{i=1}^{s} e_i)) = 0 \).

3. \( |D_l| \) is base point free. In case of \( d = 5 \), the image of \( \Phi|_{D_l} \) is \( \mathbb{P}^2 \). In case of \( d \geq 6 \), \( D_l \) is very ample and \( |D_l| \) embeds \( \mathcal{H}_2 \) into \( \mathbb{P}^{d-3} \).

4. If \( d \geq 6 \), then \( \mathcal{H}_2 \subset \mathbb{P}^{d-3} \) is projectively Cohen-Macaulay and is the intersection of cubics.

**Remark.** (1) If \( d \geq 6 \), then \( \mathcal{H}_2 \subset \mathbb{P}^{d-3} \) is so called the **white surface** (see [Whi24] and [Gim89]). In [Man01], the white surface attains the maximal degree among projectively Cohen Macaulay rational surfaces in a fixed projective space.

(2) We use the dual notation \( \mathbb{P}^{d-3} \) for later convenience.

**Proof.** (1) Let \( \pi_C \colon C \times C \to S^2 C \) be the natural projection and \( L_b \) a ruling of \( C \times C \) in one fixed direction such that \( \pi_C(L_b) = \eta(L_b) \). By applying the Bertini theorem to \( L'_b \), we see that \( \pi^* \eta(D_l) \) and \( L'_b \) intersect simply for a general \( b \in C \) whence \( \eta(D_l) \) intersects \( \eta(L_b) \) simply since \( \pi_C \) is étale at \( \pi_C^* \eta(D_l) \cap L_b \). Then \( D_l \) intersects \( L_b \) simply since \( \eta \) is isomorphic at \( D_l \cap L_b \). For a general \( b \in C \), we consider the double projection \( \pi_b \colon B \to \mathbb{P}^2 \) from \( b \) as in Proposition 2.1.6(1). Let \( \tilde{l} \) be a general line on \( B \) intersecting \( C \) and \( l_b \) the image of \( \tilde{l} \) by \( \pi_b \). We can assume that \( \tilde{l} \neq c := C \cap \mathcal{P} \). Obviously \( l_b \) is a line. By Lemma 2.4.12 \( \deg C_b = d - 2 \) and the curve \( C_b \cup l_b \) has only simple nodes. Hence the number of points in \( C_b \cap l_b \) is \( d - 2 \), one of them counts for the unique conic through \( b \) and \( c \). This last conic gives a conic on \( A \) which does not intersects \( l \). The other \( d - 3 \) points count for elements of \( D_l \). Then \( \eta(D_l) \) is a curve of degree \( d - 3 \).

Let \( \tilde{l}_1 \) and \( \tilde{l}_2 \) be two general secant lines of \( C \) such that \( \tilde{l}_1 \cap \tilde{l}_2 = \emptyset \). By Lemma 2.4.10 \( \#(D_{l_1} \cup D_{l_2}) = \frac{(d-3)(d-4)}{2} \). This immediately gives for the intersection product \( D_{l_1} \cdot D_{l_2} \geq \frac{(d-3)(d-4)}{2} \). On the other hand, \( D_l \cap e_i \neq \emptyset \) for a general \( l \) since \( D_l \cap e_i \) contains the point corresponding to a marked conic \( (\beta_l \cup \alpha, \beta_{iC}) \), where \( \alpha \) is the unique line intersecting \( \beta_l \) and \( l \). Moreover, for two general \( l_1 \) and \( l_2 \), \( D_{l_1} \cap D_{l_2} \cap e_i = \emptyset \). Now the curves \( e_i \) have negative self intersection then
By Lemma 2.4.17, we only prove (2.7). Since
\( (2.8) \)

It is easy to see that
\( D \)
conics intersecting
\( e \)

Consequently, \( D_1 \sim (d - 3)h - \sum_{i=1}^{s} e_i \) for a general \([l] \in \mathcal{H}_1\), and, by the flatness of \( D_1 \to \mathcal{H}_1\), that holds for any \([l] \in \mathcal{H}_1\).

(2) Let \( L'_p_{ij} = L_{p_{ij}} - e_i \) (note that \( e_i \subset L_{p_{ij}} \)). We see that \( L'_p_{ij} \subset D_{l_{ij}} \) and \( D_{l_{i1}} - L'_p_{i1} = D_{l_{i2}} - L'_p_{i2} \), which we denote by \( D_{\beta_i} \). Note that
\[
D_{\beta_i} \sim (d - 4)h - \sum_{k \neq i} e_k.
\]

It is easy to see that \( D_{\beta_i} \) have the following properties:

(2.7)
\[
D_{\beta_i} \cap e_i = \emptyset.
\]

(2.8)
\[
D_{\beta_i} \cap D_{\beta_j} \cap D_{\beta_k} = \emptyset.
\]

We only prove (2.7). Since \( D_{\beta_i} \cap e_i \neq \emptyset \) would imply that \( e_i \) is a component of \( D_{\beta_i} \), it suffices to prove that, for a general \( l \), \( D_{\beta_i} \cap D_l \) does not contain a point of \( e_i \).

By Lemma 2.4.17 \( D_{\beta_i} \cap D_l \) contains \( \frac{(d-4)(d-5)}{2} \) points corresponding to bi-secant conics intersecting \( \beta_i \) and \( l \) except conics containing \( \beta_i \). On the other hand, we have \( D_l \cdot D_{\beta_i} = \frac{(d-4)(d-5)}{2} \), thus the conics we count in Lemma 2.4.17 correspond to all the intersection of \( D_{\beta_i} \cap D_l \). Consequently, \( D_{\beta_i} \cap D_l \) does not contain a point of \( e_i \).

By (2.7) and the trivial equality
\[
(d - 4)h - \sum_{i \geq k+1} e_i = D_{\beta_k} + e_1 + \cdots + e_{k-1},
\]
we obtain \( e_k \not\in B_s |(d - 4)h - \sum_{i \geq k+1} e_i| \).

Since \( \mathcal{O}_{\mathcal{H}_2}((d - 4)h - \sum_{i \geq k+1} e_i) \otimes \mathcal{O}_{\mathcal{H}_2} \mathcal{O}_{e_k} \simeq \mathcal{O}_{e_k} \) we have that
\[
H^0(\mathcal{H}_2, \mathcal{O}_{\mathcal{H}_2}((d - 4)h - \sum_{i \geq k+1} e_i)) \to H^0(\mathcal{H}_2, \mathcal{O}_{e_k})
\]
is surjective. Hence by the exact sequence
\[
0 \to \mathcal{O}_{\mathcal{H}_2}((d - 4)h - \sum_{i \geq k} e_i) \to \mathcal{O}_{\mathcal{H}_2}((d - 4)h - \sum_{i \geq k} e_i) \to \mathcal{O}_{e_k} \to 0,
\]
we have \( H^1(\mathcal{H}_2, \mathcal{O}_{\mathcal{H}_2}((d - 4)h - \sum_{i=1}^{s} e_i)) \simeq H^1(\mathcal{H}_2, \mathcal{O}_{\mathcal{H}_2}(d - 4)h) \). Since it is easy to see that \( h^1(\mathcal{H}_2, \mathcal{O}_{\mathcal{H}_2}(d - 4)h) = 0 \), we have (2).

(3) Since no conic on \( A \) intersects all the line on \( A \), \( |D_l| \) has no base point. In case \( d = 5 \), the image of \( \Phi_{|\mathcal{H}_1|} \) is \( \mathbb{P}^2 \) by \( (D_l)^2 = 1 \).

Assuming \( d \geq 6 \), we prove that \( D_l \) is very ample. By (2) and [DG88, Theorem 3.1], it suffices to prove that
\[
h^0(\mathcal{H}_2, \mathcal{O}_{\mathcal{H}_2}(h - \sum_{j=1}^{d-3} e_{i_j})) = 0
\]
for any set of \( d - 3 \) exceptional curves \( e_{i_1}, \ldots, e_{i_{d-3}} \). Assume by contradiction that there exists an effective divisor \( L \in |h - \sum_{j=1}^{d-3} e_{i_j}| \) for a set of \( d - 3 \) exceptional curves \( e_{i_1}, \ldots, e_{i_{d-3}} \). By \( \frac{(d-2)(d-3)}{2} - (d - 3) \geq 3 \), we find at least three \( e_i \) such that \( i \not\in \{j_1, \ldots, j_{d-3}\} \). For an \( i \not\in \{j_1, \ldots, j_{d-3}\} \), noting \( D_l \sim D_{\beta_i} + h - e_i \), \( D_l \cdot L = 0 \), and \( L \cdot (h - e_i) > 0 \), we have \( L \subset D_{\beta_i} \). This contradicts (2) since the number of \( i \) such that \( i \not\in \{j_1, \ldots, j_{d-3}\} \) is at least 3.
We show that \( h^0(\mathcal{H}_2, \mathcal{O}_{\mathcal{H}_2}(D_1)) = d - 2 \). By the Riemann-Roch theorem, 
\( \chi(\mathcal{O}_{\mathcal{H}_2}(D_1)) = d - 2 \). Since \( h^2(\mathcal{H}_2, \mathcal{O}_{\mathcal{H}_2}(D_1)) = h^0(\mathcal{H}_2, \mathcal{O}_{\mathcal{H}_2}(-D_1 + K_{\mathcal{H}_2})) = 0 \), 
we see that \( h^0(\mathcal{H}_2, \mathcal{O}_{\mathcal{H}_2}(D_1)) = d - 2 \) is equivalent to \( h^1(\mathcal{H}_2, \mathcal{O}_{\mathcal{H}_2}(D_1)) = 0 \).
Since \( |D_1| \) has no base point, so is \( |(d - 3)h - \sum e_i| \). Thus the proof that 
\( h^1(\mathcal{H}_2, \mathcal{O}_{\mathcal{H}_2}(D_1)) = 0 \) is almost the same as the above one showing (2) and we omit it.

(4) follows from [Gim89, Proposition 1.1]. \( \square \)

**Remark.** In case of \( d = 5 \), the morphism defined by \( |D_1| \) contracts three curves 
\( D_{e_i} \) (\( i = 1, 2, 3 \)), which are nothing but the strict transforms of three lines passing 
through two of \( c_j \). Namely, the composite \( S^2C \twoheadrightarrow \mathcal{H}_2 \twoheadrightarrow \mathbb{P}^2 \) is the Cremona transformation.

The following corollary contains the nontrivial result that for a general \([l] \in \mathcal{H}_1 \), 
\( D_l \) parameterizes conics which properly intersect \( l \).

**Corollary 2.4.19.** For a general \([l] \in \mathcal{H}_1 \), \( D_l \) does not contain any point correspon-
ding to the line pairs \( l \cup m \) with \([m] \in \mathcal{H}_1 \).

**Proof.** Fix \([m] \in \mathcal{H}_1 \) such that \( l \cup m \) is a line pair. If \((m, b)\) is the marked line given 
by \( m \) then we have \( d - 2 \) line pairs \( l \cup m, l_1 \cup m, \ldots, l_{d-3} \cup m \). Since \( L_b \sim h \) then 
\( h \cdot D_l = d - 3 \) and definitely \([l_1 \cup m], \ldots, [l_{d-3} \cup m] \in D_l \). Thus \([l \cup m] \notin D_l \). \( \square \)

### 2.5. Varieties of power sums for special non-degenerate quartics \( F_4 \)

In Proposition 2.4.13 we have seen that \( \psi: U_2 \to A \) is finite and flat outside 
\( \cup_{i=1}^n \beta'_i \). We can modify the morphism \( \psi: U_2 \to A \) to obtain a finite one. See 
Proposition 2.5.7 which is the goal of 2.5.1. This and our understanding of the geometry of \( \mathcal{H}_2 \) 
give an important morphism whose target is \( \text{VSP}(F_4, n; \mathcal{H}_2) \): see Theorem 2.5.12

**2.5.1. Special blow-up \( \tilde{A} \) of \( A \).** Similarly to (2.10), we consider the following diagram:

\[
\begin{array}{ccc}
\mathcal{U}_2 & \xrightarrow{(\psi, \text{id})} & A \times \mathcal{H}_2 \\
\downarrow & & \downarrow \\
\tilde{D}_2 & \hookrightarrow & \mathcal{H}_2 \times \mathcal{H}_2
\end{array}
\]

Let \( \mathcal{U}_2 \subset \mathcal{U}_2 \times \mathcal{H}_2 \) be the pull-back of \( U_2 \) and \( \tilde{D}_2 \) the image of \( \mathcal{U}_2 \) on \( \mathcal{H}_2 \times \mathcal{H}_2 \).
Similarly to the investigation of the diagram \( (2.4) \), we see that the image \( \mathcal{F}' \) in 
\( \mathcal{H}_2 \times \mathcal{H}_2 \) of the inverse image of \( \cup_{i=1}^n \beta'_i \times \mathcal{H}_2 \) is not divisorial nor does not dominate \( \mathcal{H}_2 \). 
Moreover, any component of \( \tilde{D}_2 \) outside \( \mathcal{F}' \) dominates \( \mathcal{H}_2 \), and is divisorial 
or possibly the diagonal of \( \mathcal{H}_2 \times \mathcal{H}_2 \). Note that unlike the diagram \( (2.4) \), there 
is no other non-divisorial component in this case. Compare the proof of Corollary 2.4.14. 
Here we leave the possibility that the diagonal of \( \mathcal{H}_2 \times \mathcal{H}_2 \) is contained in 
the divisorial component of \( \tilde{D}_2 \). We, however, prove this is not the case in Lemma 2.5.12.

Let \( D_2 \subset \mathcal{H}_2 \times \mathcal{H}_2 \) be the union of the divisorial components of \( \tilde{D}_2 \) 
with reduced structure. \( D_2 \) is Cartier since \( \mathcal{H}_2 \times \mathcal{H}_2 \) is smooth. \( D_2 \to \mathcal{H}_2 \) is flat since \( D_2 \) is 
Cohen-Macaulay, \( \mathcal{H}_2 \) is smooth and \( D_2 \to \mathcal{H}_2 \) is equi-dimensional. Let \( D_q \) be the 
fiber of \( D_2 \to \mathcal{H}_2 \) over \([q] \in \mathcal{H}_2 \) via the projection to the second factor.
We are almost ready to define the modification of $\psi: U_2 \to A$ we are looking for. To find the range we consider the blow-up of $A$ along $\bigcup_{i=1}^n \beta_i'$ and we denote it by $\rho: \tilde{A} \to A$.

**Lemma 2.5.1.**

$$\mathcal{N}_{\beta_i'/A} = \mathcal{O}_{\beta_i'}(-1) \oplus \mathcal{O}_{\beta_i'}(-1).$$

**Proof.** We prove the assertion by using the inductive construction of $C_d$. The assertion is clear for $d = 1$ since $C_1$ has no bi-secant line. Suppose the assertion holds for $C_{d-1}$. Choose a general uni-secant line $\mathcal{I} \subset B$ of $C_{d-1}$. Let $\mathcal{m}_1, \ldots, \mathcal{m}_{d-2}$ be the lines on $B$ intersecting both $C_{d-1}$ and $\mathcal{I}$ outside $C_{d-1} \cap \mathcal{I}$. Let $A' \to B$ be the blow-up along $C_{d-1} \cup \mathcal{I}$. Note that the smoothing $C_{d-1} \cup \mathcal{I}$ to $C_d$ induces that of $A'$ to $A$. Let $\tilde{\mathcal{m}}_i$ be the strict transform of $\mathcal{m}_i$ on $A'$. By the smoothing construction of $C_d$ from $C_{d-1} \cup \mathcal{I}$ and the assumption on induction, we have only to prove $\mathcal{N}_{\tilde{\mathcal{m}}_i/A'} = \mathcal{O}_{\tilde{\mathcal{m}}_i}(-1) \oplus \mathcal{O}_{\tilde{\mathcal{m}}_i}(-1)$. Let $A'_1 \to B$ be the blow-up along $\mathcal{I}$ and $A'_2 \to A'_1$ the blow-up along the strict transform of $C_{d-1}$. Denote by $m'_i$ and $m''_i$ the strict transform of $\tilde{\mathcal{m}}_i$ on $A'_1$ and $A'_2$ respectively. Then $\mathcal{N}_{\tilde{\mathcal{m}}_i/A'} = \mathcal{N}_{m'_i/A'_1} = \mathcal{O}_{m'_i}(-1)$, since $m'_i$ is a fiber of $A'_1 \to Q$ (cf. Proposition 2.1.6 (2)). We have $\mathcal{N}_{m''_i/A'_2} = \mathcal{O}_{m''_i}(-1)$. Let $F$ be the exceptional divisor of $A'_1 \to Q$ and $F'$ the strict transform of $F$ on $A'_2$. We may suppose $F$ and $C'_{d-1}$ intersect transversely, thus $F' \to F$ is the blow-up at $d-2$ points $m'_i \cap C'_{d-1}$ $(i = 1, \ldots, d - 2)$. Thus $F' \cdot m''_i = -1$ and $\mathcal{N}_{m''_i/F'} = \mathcal{O}_{m''_i}(-1)$, and this implies the assertion. We add the following piece of notation:

**Notation 2.5.2.** (1) $\xi_{ij} := \rho^{-1}(\beta_i')$. By Lemma 2.5.1 $E_i \simeq \mathbb{P}^1 \times \mathbb{P}^1$.

(2) $f_i :=$ a general fiber of $p_i|E_i: E_i \to \beta_i'$.

(3) $\gamma_i :=$ a general fiber of the other projection $E_i \to \mathbb{P}^1$.

(4) $\tilde{E}_C :=$ the strict transform of $E_C$, and

(5) $\tilde{\xi}_{ij} :=$ the strict transform of the fiber $\xi_{ij}$ of $E_C$ over $p_{ij} \in C \cap \beta_i$, where $i = 1, \ldots, s$ and $j = 1, 2$.

The domain of the finite morphism is $\tilde{U}_2 := U_2 \times_A \tilde{A}$; in other words, $\tilde{U}_2$ is the blow-up of $U_2$ along $U_2 \cap (\bigcup_{i=1}^n \beta_i' \times \gamma_2)$. We obtain that the natural morphism $\tilde{U}_2 \to \tilde{A}$ is finite after an analysis of the morphism $U_2 \to A$ in the neighborhood of some special conics and via the suitable notion of conic on $\tilde{A}$.

Note that, by Proposition 2.2.4 (5), there are $d-4$ lines $\alpha_1, \ldots, \alpha_{d-4}$ distinct from $\beta_i$ and intersecting both $C$ and $\beta_i$ outside $C \cap \beta_i$. Set $t_k := \alpha_k \cap C$. Corresponding to $\alpha_k$, there are two marked conics $(\alpha_k \cup \beta_i; p_{i1}, t_k)$ and $(\alpha_k \cup \beta_i; p_{i2}, t_k)$, which does not belong to $e_i$ (by the choice of marking). We denote by $\xi_{ijk}$ the conics on $A$ corresponding to $(\alpha_k \cup \beta_i; p_{ij}, t_k)$, where $i = 1, \ldots, s$, $j = 1, 2$, and $k = 1, \ldots, d - 4$.

**Lemma 2.5.3.** $\xi_{ijk}$ does not belong to $D_{\beta_i}$.

**Proof.** By the projection from $\beta_i$, the image $\tilde{q}$ of a general conic $q$ belonging to $D_{\beta_i}$ maps to a bi-secant line of the image $C' \subset Q$ of $C$, and $\alpha_k$ maps to a point $p_{ijk}$. Let $p'_{ij}$ be the point of $C'$ corresponding to $p_{ij}$. Let $F$ be the exceptional divisor over $\beta_i$, and $F'$ the image of $F$ on $Q$. We say a ruling of $F' \simeq \mathbb{P}^1 \times \mathbb{P}^1$ is horizontal if it does not come from a fiber of $F \to \beta_i$. If $[\xi_{ijk}] \in D_{\beta_i}$, then $\xi_{ijk}$ corresponds to a bi-secant line of $C'$, which must be the horizontal ruling of $F'$ through $p'_{ij}$ and $p_{ijk}$. By inductive construction of $C$, we can prove that $p'_{ij}$ and $p_{ijk}$ do not lie on a horizontal ruling. Thus we have the claim. □
**Definition 2.5.4.** We say that a curve $\tilde{q} \subset \tilde{A}$ is a conic on $\tilde{A}$ if

1. $\tilde{q}$ is connected and reduced,
2. $-K_{\tilde{A}} \cdot \tilde{q} = 2$,
3. $E_C \cdot \tilde{q} = 2$,
4. $E_i \cdot \tilde{q} = 0$, and
5. $p_a(\tilde{q}) = 0$.

Similarly to the case of conics on $A$, we know there exists a unique two-dimensional component of the Hilbert scheme of $\tilde{A}$ parameterizing conics on $\tilde{A}$. Let $H^{\tilde{A}}_2$ be the normalization of the two-dimensional component. Similarly to the proof of Proposition 2.4.9, we can show $H^{\tilde{A}}_2 \to \overline{H}_2$. Since $H_2 \to \overline{H}_2$ is the normalization, we have also a natural morphism $H^{\tilde{A}}_2 \to H_2$. We do not need a full classification of conics on $\tilde{A}$ but only the following:

**Lemma 2.5.5.** (1) There is a unique conic $\tilde{q}$ on $\tilde{A}$ corresponding to a conic $q$ on $A$ belonging to $e_i$, and moreover, $\tilde{q}$ is isomorphic to $q$ over the component $\beta_i'$.

In particular, $H^{\tilde{A}}_2 \to H_2$ is isomorphic near $e_i$.

(2) A conic belonging to $D_{\beta_i}$ is smooth near $\beta_i'$. There is a unique conic $\tilde{q}$ on $\tilde{A}$ corresponding to a conic $q$ on $A$ belonging to $D_{\beta_i}$, and, over $\beta_i'$, $\tilde{q}$ is isomorphic to the union of $q$ and the fiber of $E_i$ over $q \cap \beta_i'$. In particular, $H^{\tilde{A}}_2 \to H_2$ is isomorphic near $D_{\beta_i}$.

**Proof.** This follows from an explicit calculation as in the proof of Proposition 2.3.6. For the first statement of (2), we use Proposition 2.2.2 (5) and Lemma 2.5.3.

Let $\Gamma := U_2 \cap (\cup_{i=1}^s \beta_i' \times H_2)$. Outside $\cup_i \beta_i' \times e_i$, $\Gamma$ is set-theoretically the disjoint union

$$\Gamma_i := \{ (x, [q]) \mid [q] \in D_{\beta_i}, x \in q \cap \beta_i' \} (i = 1, \ldots, s),$$

which is a section of $\mu$ over $D_{\beta_i}$, and

$$\Gamma_{ijk} := \{ (x, [\xi_{ijk}]) \mid x \in \beta_i' \} (k = 1, \ldots, d - 4, j = 1, 2).$$

**Lemma 2.5.6.** Along $\Gamma_{ijk}$, $U_2$ is smooth and $\Gamma$ is reduced.

**Proof.** To show that $U_2$ is smooth near $\Gamma_{ijk}$, we have only to see that the conic $\xi_{ijk}$ is strongly smoothable. Note that $N_{\beta_i'/A} \cong O_{P_i}(-1)^{d/2}$, $N_{\alpha_k'/A} \cong O_{P_1} \oplus O_{P_1}(-1)$ and $N_{\alpha_{3-j}} \cong O_{P_1} \oplus O_{P_1}(-1)$. We apply [HHS] Theorem 4.1 by setting $C = \beta_i'$, $D = \alpha_k' \cup \xi_{i,j,3-j}$ and $S = (\alpha_k' \cap \beta_i') \cup (\xi_{i,j,3-j} \cap \beta_i')$. We check the conditions a) and b) of [ibid.]. The condition a) clearly holds. The condition b) follows from the following two facts:

1. let $F$ be the exceptional divisor of the blow up of $B$ along $\alpha_k$. Note that $F \cong P^1 \times P^1$. We say a fiber of $F \to P^1$ in the other direction to $F \to \alpha_k$ a horizontal fiber. Then the intersection points of the strict transform of $C$ and $F$, and the strict transform of $\beta_i$ and $F$ do not lie on a common horizontal fiber.

This can be proved by the inductive construction of $C = C_d$ in a similar fashion to the proof of Lemma 2.5.1 and

2. let $G$ be the exceptional divisor of the blow up of $A$ along $\xi_{i,j,3-j}$. Note that $G \cong F_3$. Then the intersection points of the strict transform of $\beta_i'$ and $G$ does not lie on the negative section of $G$.

This can be easily proved by noting $\xi_{i,j,3-j}$ is a fiber of $E$. 
Thus, by [HH85] Theorem 4.1, $\xi_{ijk}$ is strongly smoothable.

Second, we prove that $\Gamma$ is reduced along $\Gamma_{ijk}$. We have only to prove that $\mathcal{U}_2 \to A$ is unramified along $\Gamma_{ijk}$ since then $\Gamma$ is the étale pull-back of $\beta'_i$ near $\Gamma_{ijk}$, hence is reduced.

By the inductive construction of $C = C_d$ and the following exact sequence:

$$0 \to \mathcal{N}_{\beta'_i/A} \to \mathcal{N}_{\xi_{ijk}/A}|\beta'_i' \to T_{\mathcal{S}}^1 \to 0,$$

we can prove that $\mathcal{N}_{\xi_{ijk}/A}|\beta'_i'$ is isomorphic to the universal family of conics on $A$ over $e_i$ and $D_{\beta_i}$. Thus $\mathcal{U}_2$ is Cohen-Macaulay over $e_i$ and $D_{\beta_i}$ since so are the fibers. Note that $\mathcal{U}_2 \to \mathcal{U}_2$ is the blow-up along $\Gamma_i$ near $\Gamma_i$ and is an isomorphism near $\beta'_i \times e_i$.

Lemma 2.5.5 shows that $\mathcal{U}_2 \to \mathcal{U}_2$ is the blow-up along $\Gamma_{ijk}$ near $\xi_{ijk} \times [\xi_{ijk}]$, and $\mathcal{U}_2$ is smooth over $\Gamma_{ijk}$. Thus $\mathcal{U}_2$ is Cohen-Macaulay. To see $\tilde{\psi}$ is finite, we have only to note that the inverse images of $\beta'_i \times e_i$ on $\mathcal{U}_2$ and the exceptional divisor of $\mathcal{U}_2 \to \mathcal{U}_2$ are not contracted by $\tilde{\psi}$.

From now on in the section 3, we assume that $d \geq 6$ and we consider $\mathcal{H}_2 \subset \tilde{\mathbb{P}}^{d-3}$.

Consider the following diagram:

\[
\begin{array}{ccc}
\mathcal{H}_2 & \xrightarrow{\tilde{\psi}} & \tilde{\mathcal{U}}_2 \\
\mu \searrow & \nearrow \tilde{\psi} & \\
& \tilde{\mathbb{A}} & \\
\end{array}
\]

Definition 2.5.8. Let $\tilde{a}$ be a point of $\tilde{\mathbb{A}}$. We say that $[\tilde{\psi}^{-1}(\tilde{a})] \in \text{Hilb}^n\tilde{\mathbb{P}}^{d-3}$ is the cluster of conics attached to $\tilde{a}$ and denote it by $[Z_{\tilde{a}}]$. A conic $q$ such that $[q] \in \text{Supp} Z_{\tilde{a}}$ is called a conic attached to $\tilde{a}$.

Remark. Though we do not need it later, we describe the fiber of $\tilde{\psi}$ over a general point $\tilde{a} \in E_i$ for some $i$ for reader’s convenience. In other words, we exhibit $n$ conics attached to $\tilde{a}$.

Set $a := \rho(\tilde{a}) \in \mathbb{A}$ and $b := f(a) \in \beta_i$. We use notations of Proposition 2.4.13. Since $\deg C_b = d - 2$, the number of bi-secant conics through $b$ not belonging to the family $e_i$ is given by the number of double points of $C_b$, which is $\frac{(d-3)(d-4)}{2}$. Moreover $2(d-4)$ conics $\xi_{ijk}$ through $a$.

The number of remaining conics is $3 = n - \frac{(d-3)(d-4)}{2} - 2(d-4)$. Such conics will belong to $e_i$. By Lemma 2.5.5 $\mathcal{U}_2 \to \mathcal{H}_2$ is isomorphic to the universal family of conics on $A$ over $e_i$. Thus a desired conic on $A$ is the image of a conic $\tilde{q}$ on $A$ such that $\tilde{a} \in \tilde{q}$ and $\rho(\tilde{q})$ belongs to $e_i$. We show there are three such conics. Let $S_i$ be the strict transform on $\tilde{A}$ of the locus of lines intersecting $\beta_i$. Then it is easy to
see that $S_{i,E_i}$ does not contain any fiber $\gamma_i$ of the second projection $\sigma_i : E_i \to \mathbb{P}^1$. Moreover $S_{i,E_i} \sim 2\gamma_i + 3f_i$. Let $\gamma_i'$ be the fiber of $\sigma_i$ through $\tilde{a}$. Then $\gamma_i'$ intersect $S_i$ at three points. Corresponding to these three points, there are three lines on $B$ intersecting $\beta_i$. Denote by $l_1, l_2$ and $l_3 \subset A$ the strict transforms of these three lines. Then $\beta_i' \cup l_j$ ($j = 1, 2, 3$) are the conics on $A$ what we want.

By Proposition 2.5.7 and the universal property of Hilbert schemes, we obtain a naturally defined map $\Psi : \tilde{A} \to \text{Hilb}^n \tilde{P}^{d-3}$. This is clearly injective because $n$ conics attached to a point $\tilde{a} \in \tilde{A}$ uniquely determines $\tilde{a}$.

The task is to understand the image of $\Psi$.

2.5.2. Morphism from $\tilde{A}$ to VSP.

To understand the image of $\Psi : \tilde{A} \to \text{Hilb}^n \tilde{P}^{d-3}$ we construct explicitly a quartic polynomial which plays the role of the plane quartic in the Mukai’s interpretation of $V_{22}$.

**Lemma 2.5.9.** $D_2$ does not contain the diagonal of $\mathcal{H}_2 \times \mathcal{H}_2$. In particular we have the following:

let $\tilde{a}$ be a general point of $\tilde{A}$ and $q_1, q_2, \ldots, q_n \in \mathcal{H}_2$ the conics attached to $\tilde{a}$. Then

$$D_{q_i}([q]) \neq 0$$

for $1 \leq i \leq n$.

**Proof.** Here we assume $d \geq 3$. It suffices to prove that $D_{q}([q]) \neq 0$ for a general $[q] \in \mathcal{H}_2$. This is equivalent to that the image $D_{q}^p$ on $\mathcal{H}_2$ of $D_{q}$ does not contain $[\mathfrak{q}]$. Note that $D_{q}^p$ is the closure of the locus of multi-secant conics of $C$ intersecting properly $\mathfrak{q}$. Now the assertion follows from the inductive construction of $C_d$ from $C_{d-1} \cup \mathcal{T}$. From now on, we denote $D_{q}^p$ for $C_d$ by $D_{q,d}^p$. If $d = 3$, then $D_{q} \sim 0$, thus the assertion trivially true. If $D_{q,d-1}^p([\mathfrak{q}]) \neq 0$ for a general multi-secant conic $\mathfrak{q}$ of $C_{d-1}$, then $D_{q,d}^p([\mathfrak{q}]) \neq 0$ for a general multi-secant conic $\mathfrak{q}$ of $C_d$. \hfill $\square$

The proof of the following lemma is almost identical to the one of Theorem 2.4.18 then we omit it:

**Lemma 2.5.10.** $D_{q} \sim 2(d-3)h - 2\sum e_i$ for a conic $q$, namely, $D_{q}$ is a quadric section of $\mathcal{H}_2 \subset \tilde{P}^{d-3}$.

We proceed to construct the quartic polynomial. By the seesaw theorem, it holds that $D_2 \sim p_1^* D_q + p_2^* D_q$. Consider the morphism $\mathcal{H}_2 \times \mathcal{H}_2$ into $\tilde{P}^{d-2} \times \tilde{P}^{d-3}$ defined by $[p_1^* D_l + p_2^* D_l]$, which is an embedding since $d \geq 6$. Since it is easy to see that

$$H^0(\mathcal{H}_2 \times \mathcal{H}_2, D_2) \simeq H^0(\tilde{P}^{d-3} \times \tilde{P}^{d-3}, \mathcal{O}(2, 2)),$$

it holds that $D_2$ is the restriction of a unique $(2, 2)$-divisor on $\tilde{P}^{d-3} \times \tilde{P}^{d-3}$, which we denote by $\{D_2 = 0\}$. Since $\{D_2 = 0\}$ is also symmetric, we may take the equation $\tilde{D}_2$ so that it is the bi-homogenization of an equation $\tilde{F}_4$ of a quartic in $\tilde{P}^{d-3}$ (cf. [DK93 §1]). Moreover the fiber of $\{\tilde{D}_2 = 0\}$ over a point $p \in \tilde{P}^{d-3}$ is defined by the polar $P_p(\tilde{F}_4)$, which we denote by $\tilde{D}_p$. For $[q] \in \mathcal{H}_2$, we denote $\tilde{D}_{[q]}$ simply by $D_q$.

By construction, $D_q = \{\tilde{D}_q = 0\} \cap (\mathcal{H}_2 \times \mathcal{H}_2)$. We may choose the defining equation $H_q$ of the hyperplane of $\tilde{P}^{d-3}$ corresponding to $[q]$ such that $P_{H_q}(\tilde{F}_4) = \tilde{D}_q$. 
From now on, we write $\mathbb{P}^{d-3} = \mathbb{P}_a V$, where $V$ is the $d - 2$-dimensional vector space. The crucial point in the following assertions is that the number of the conics attached to a point of $\widetilde{A}$ coincides with $\text{dim}_C S^2 V$.

Let $\bar{a}$ be a general point of $\widetilde{A}$ and $q_1, \ldots, q_n$ are the conics attached to $\bar{a}$. By the definition of $\widetilde{D}_{q_i}$ and generality of $\bar{a}$, we have the following (we use Lemma 2.6.9):

$$\widetilde{D}_{q_i}([q_i]) = 0 \ (j \neq i) \text{ and } \widetilde{D}_{q_i}([q_i]) \neq 0. \quad (2.11)$$

(2.11) implies $\widetilde{D}_{q_1}, \ldots, \widetilde{D}_{q_n}$ are linearly independent. Thus by $P_{H_{q_i}^2}(\widetilde{F}_4) = \widetilde{D}_{q_i}$, it holds that the apolarity map

$$\text{ap}_{\widetilde{F}_4} : S^2 \bar{V} \rightarrow S^2 V$$

$$G \mapsto P_G(\widetilde{F}_4)$$

is an isomorphism. Moreover, $H_{q_1}^2, \ldots, H_{q_n}^2$ are linearly independent. Thus $\widetilde{F}_4$ is non-degenerate in the sense of Dolgachev. By [Dol04 §2.3], there exists a unique quartic form $F_4$ such that $\text{ap}_{\widetilde{F}_4} = \text{ap}_{F_4}^{-1}$. In particular, it holds

$$P_{D_{q_i}}(F_4) = H_{q_i}^2,$$

$F_4$ is called the quartic form dual to $\widetilde{F}_4$.

To see the relation between the set of conics attached to a general point of $\widetilde{A}$ and the representation of $F_4$ as a sum of powers of linear forms we need to find conditions which force $n$ conics to be attached to $\bar{a} \in \widetilde{A}$. Next lemma is sufficient for our purposes.

**Lemma 2.5.11.** Let $q_1, \ldots, q_n$ be $n$ distinct conics such that

1. $\widetilde{D}_{q_i}([q_j]) = 0$ for all $i \neq j$,
2. all the $\overline{q}_i$ are smooth,
3. if three of $\overline{q}_i$ pass through a point $b$, then any other $\overline{q}_i$ does not intersect a line through $b$ outside $b$, and
4. no two of $\overline{q}_i$ intersect at a point of $C \cup \bigcup_i \beta_i$.

Then the $q_i$’s are attached to a point of $\widetilde{A}$.

**Proof.** By the assumption (1), $\overline{q}_1, \ldots, \overline{q}_n$ are mutually intersecting multi-secant conics of $C$. By the assumption (4), it suffices to prove $\overline{q}_1, \ldots, \overline{q}_n$ pass through one point of $B$.

**Step 1.** Let $b \in B$ be a point such that five of $\overline{q}_i$, say, $\overline{q}_1, \ldots, \overline{q}_5$ pass through $b$. Then all the $\overline{q}_i$ pass through $b$.

By the double projection from $b$, $\overline{q}_1, \ldots, \overline{q}_5$ are mapped to points $p_1, \ldots, p_5$ on $\mathbb{P}^2$. Suppose by contradiction that a smooth conic $\overline{q}_j$ does not pass through $b$. Let $q'_j$, $q''_j$ and $\overline{q}_j$ be the strict transforms of $\overline{q}_j$ on $B_b$, $B'_b$ and $\mathbb{P}^2$, and set $S := \pi_{2b}^* q_j$. By the assumption (3), $\overline{q}_j$ does not intersect a line through $b$. Thus $\overline{q}_j$ is a smooth conic through $p_1, \ldots, p_5$. The conic $\overline{q}_j$ is unique since a conic through five points is unique. It holds that $-K_{B'_b} \cdot q''_j = 4$ and $S \cdot q''_j = 4$, thus $S \simeq \mathbb{P}^2$ and $q''_j$ is the negative section. This implies that $q_j$ is also unique. By reordering, we may assume that $j = n$. We have the configuration such that all the conics pass through $b$ except $q_n$. Denote by $p_i$ the image of $q_i$ ($i \neq n$). Then $\overline{q}_n$ and $C_b$ intersect at $p_i$. By $d \geq 6$, it holds $\text{deg} C_b \geq 3$, thus $\overline{q}_n \neq C_b$. By the assumption (4), $b \notin C$. Therefore $\overline{q}_n$ and $C_b$ intersect at $n - 1$ singular points of $C_b$. Since $\text{deg} C_b \leq d$, it
holds \(2(n - 1) \leq 2d\), a contradiction.

**Step 2.** If four conics \(\mathcal{F}_1, \ldots, \mathcal{F}_4\) pass through one point \(b\), then all the conics pass through \(b\).

By contradiction and Step 1, we may assume that all the conics except \(\mathcal{F}_1, \ldots, \mathcal{F}_4\) do not pass through \(b\). Pick up two any conics, say, \(\mathcal{F}_5\) and \(\mathcal{F}_6\), not passing through \(b\). Considering the double projection from \(b\) as in Step 1. Denote by \(\mathcal{F}_i\) (\(j \geq 5\)) the image of \(\mathcal{F}_j\) on \(\mathbb{P}^2\). By the assumption (3), \(\mathcal{F}_5\) and \(\mathcal{F}_6\) do not intersect a line through \(b\), thus \(\mathcal{F}_5\) and \(\mathcal{F}_6\) are conics on \(\mathbb{P}^2\). Therefore \(\mathcal{F}_5 \cap \mathcal{F}_6\) lies on one of \(\mathcal{F}_1, \ldots, \mathcal{F}_4\), since otherwise \(\mathcal{F}_5\) and \(\mathcal{F}_6\) would intersect at five points and this is a contradiction as in Step 1. Thus any two conics intersect on \(\mathcal{F}_1, \ldots, \mathcal{F}_4\). Let \(p_i\) be the intersection of \(\mathcal{F}_i \cap \mathcal{F}_5\) for \(i = 1, \ldots, 4\). Then \(\mathcal{F}_j\) \((j \geq 5)\) pass through one of \(p_i\). Thus one of \(p_i\), say, \(p_1\), there pass through at least \(\lceil \frac{(n-5)}{4}\rceil\) conics. By Step 1, \(\lceil \frac{(n-5)}{4}\rceil \leq 2\) (already \(\mathcal{F}_1\) and \(\mathcal{F}_5\) pass through \(p_1\)). This implies \(d = 6\). We exclude this case in Step 3. Note that if \(d = 6\), then the four conics \(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_5,\) and \(\mathcal{F}_6\) mutually intersect and the all the intersection points are different. By reordering conics, we assume that \(\mathcal{F}_i\) \((1 \leq i \leq 4)\) satisfy this property.

**Step 3.** We complete the proof.

Assume by contradiction that \(\mathcal{F}_1, \ldots, \mathcal{F}_n\) do not pass through one point on \(B\). If \(d \geq 7\), then, by Steps 1 and 2,

\[
(2.12) \quad \text{at most three of } \mathcal{F}_i\text{'s pass through any intersection point.}
\]

Let \(m\) be the number of conics in a maximal tree \(T\) of \(\mathcal{F}_i\)'s such that two conics in \(T\) pass through any intersection point. Note that \(T\) is connected since \(\mathcal{F}_i\)'s mutually intersect. The number of the intersection points of \(\mathcal{F}_i\)'s contained in \(T\) is \(\frac{m(m-1)}{2}\).

By the maximality of \(T\), a conic not belonging to \(T\) passes through one of the intersection points of conics in \(T\). By \((2.12)\), no two conics not belonging to \(T\) pass through one of the intersection point of conics in \(T\). Hence it holds \(\frac{m(m-1)}{2} + m \geq n\).

This implies that \(m \geq d - 2\) by \(n = \frac{(d-1)(d-2)}{2}\). By reordering, we assume that \(\mathcal{F}_1, \ldots, \mathcal{F}_m\) belong to \(T\). If \(d = 6\), then we take \(\mathcal{F}_1, \ldots, \mathcal{F}_4\) as in the last part of Step 2. Consider the projection \(\mathbb{P}^3 \rightarrow \mathbb{P}^2\) from \(\mathcal{F}_1\). Then \(\mathcal{F}_2, \ldots, \mathcal{F}_m\) are mapped to lines \(l_2, \ldots, l_m\) intersecting mutually on \(\mathbb{P}^3\) and the intersection points are different. Thus \(l_2, \ldots, l_m\) span a plane, which in turn shows that \(\mathcal{F}_1, \ldots, \mathcal{F}_m\) span a hyperplane section \(H\) on \(B\). Since \(C\) intersects \(\mathcal{F}_i\) at two point or more, \(C\) intersects \(H\) at \(2m\) points or more by the assumption (4). But \(2m \geq 2(d-2) > d\), \(C\) must be contained in \(H\), a contradiction to Proposition 2.2.1 (d).

We think the next theorem to be of theoretical relevance in itself and as a first result to understand varieties of sum of powers confined in a subvariety.

**Theorem 2.5.12.** \(\text{Im } \Phi\) is an irreducible component of \(\text{VSP} (F_4, n; \mathcal{H}_2)\).

**Proof.** Set

\[
Z := \{( [H_1], \ldots, [H_n] ) \in \text{Hilb}^n \mathcal{H}_d | H_1^4 + \cdots + H_n^4 = F_4, [H_i] \in \mathcal{H}_2 \}.
\]

For a general point \(\mathcal{A}\) and conics \(q_1, \ldots, q_n\) attached to \(\mathcal{A}\), we have \((2.11)\). Conversely, \(n\) conics \(q_i\) satisfying \((2.11)\) and the assumptions (2)–(4) of Lemma 2.5.11 determine a point of \(\mathcal{A}\). Note that the assumptions (2)–(4) of Lemma 2.5.11 are open conditions. Thus we have only to prove that \((2.11)\) is equivalent to

\[
(2.13) \quad \alpha_1 H_{q_1}^4 + \cdots + \alpha_n H_{q_n}^4 = F_4\quad \text{with some nonzero } \alpha_i \in \mathbb{C}.
\]
We see that (2.13) is equivalent to
\[(2.14) \text{ If } \{G = 0\} \subset \mathbb{P}^{d-3} \text{ is any quartic through } [q_1], \ldots, [q_n], \text{ then } P_{F_4}(G) = 0.\]

Indeed, by the apolarity pairing, \((G, H^4_{q_1}) = 0 \Leftrightarrow G([q_i]) = 0, \text{ thus, the assumption on } G \text{ is equivalent to } G \subset \langle H^4_{q_1}, \ldots, H^4_{q_n} \rangle^\perp. \text{ Therefore (2.13) is equivalent to } \langle H^4_{q_1}, \ldots, H^4_{q_n} \rangle^\perp \subset \langle F_4 \rangle^\perp. \text{ Since } F_4 \text{ is non-degenerate, this is equivalent to (2.13).}\n
We show (2.11) implies (2.14). If (2.11) holds, then \(\tilde{D}_{q_i} (i \neq 1) \text{ generate the space of linear forms passing through } [q_1], \text{ we may write } G = Q_2 \tilde{D}_{q_2} + \cdots + Q_n \tilde{D}_{q_n}, \text{ where } Q_i \text{ are quadratic forms on } \mathbb{P}^{d-3}. \text{ By } G([q_i]) = 0 \text{ for } i \neq 1, \text{ we have } Q_i([q_i]) \tilde{D}_{q_i}([q_i]) = 0. \text{ Thus } Q_i \text{ is a linear combination of } \tilde{D}_{q_j} (j \neq i). \text{ Consequently, } G \text{ is a linear combination of } \tilde{D}_{q_1}, \tilde{D}_{q_2}, \ldots, \tilde{D}_{q_n} (1 \leq i < j \leq n). \text{ Thus } P_{F_4}(G) = 0 \text{ follows from that } \]
\[P_{F_4}(\tilde{D}_{q_1}, \tilde{D}_{q_2}) = P_{H_{q_1}}(\tilde{D}_{q_2}) = \tilde{D}_{q_j}([q_i]) = 0.\]

Finally we show (2.13) implies (2.11). By (2.13), it holds
\[H^2_{q_i} = P_{D_{q_i}}(F_4) = \sum j (\tilde{D}_{q_j}, H^4_{q_i}) H^2_{q_j}.\]

Since \(H^2_{q_j}\) are linearly independent, (2.11) holds. \(\square\)

**Definition 2.5.13.** We say \(\text{Im } \Phi\) is the main component of \(\text{VSP } (n, F_4; \mathcal{H}_2)\).

The following lemma characterizes the main component of \(\text{VSP } (n, F_4; \mathcal{H}_2)\), which will play a crucial role in Lemma 2.5.14.

**Lemma 2.5.14.** Let \((\mathcal{H}^o_2)^o\) and \((\text{Hilb}^k \mathbb{P}^{d-3})^o (k \in \mathbb{N})\) be the complements of all the small diagonals of \(\mathcal{H}^o_2\) (\(k\) times product of \(\mathcal{H}_2\)) and \(\text{Hilb}^k \mathbb{P}^{d-3}\) respectively. Set
\[\text{VSP}^o(F_4, n; \mathcal{H}_2) := \{(H_1), \ldots, (H_n) \mid (H_i) \in \mathcal{H}_2, H^1_{i1} + \cdots + H^m_{in} = F_4\}.\]

Let \(V^o\) be the inverse image of \(\text{VSP}^o(F_4, n; \mathcal{H}_2)\) by the natural projection \((\mathcal{H}^o_2)^o \to (\text{Hilb}^k \mathbb{P}^{d-3})^o\). Let \((\mathcal{H}^o_2)^o \to (\mathcal{H}^o_2)^o\) be the projection to any of two factors. Then a component of \(V^o\) dominating \(D_2\) dominates the main component of \(\text{VSP} (F_4, n; \mathcal{H}_2)\).

**Proof.** Let \(\{[q_1], [q_2]\} \in D_2 \cap (\mathcal{H}^o_2)^o\) be a general point and \(\{q_i\} (i = 1, \ldots, n)\) any set of mutually conjugate \(n\) conics including \(q_1\) and \(q_2\). Since \(q_1\) and \(q_2\) are general, we may assume that all the \(q_i\) are general. By Lemma 2.5.11 and Theorem 2.5.12 it suffices to prove that \(q_1, \ldots, q_n\) satisfies the conditions (2)-(4) of Lemma 2.5.11.

(2). Let \(\mathfrak{r}_1\) and \(\mathfrak{r}_2\) are mutually intersecting smooth conics on \(B\) and \(\mathfrak{r}_3\) a line pair on \(B\) intersecting both \(\mathfrak{r}_1\) and \(\mathfrak{r}_2\). Since the Hilbert scheme of conics on \(B\) is 4-dimensional, the pair of \(\mathfrak{r}_1\) and \(\mathfrak{r}_2\) depends on 7 parameters. If we fix \(\mathfrak{r}_1\) and \(\mathfrak{r}_2\), then \(\mathfrak{r}_3\) depends on 1 parameter. Thus the configuration \(\mathfrak{r}_1, \mathfrak{r}_2, \mathfrak{r}_3\) depends on 8 parameters. Fix \(\mathfrak{r}_1, \mathfrak{r}_2\) and \(\mathfrak{r}_3\). We count the number of parameters of \(C_d\) such that \(C_d\) intersects each of \(\mathfrak{r}_i (i = 1, 2, 3)\) twice. The number of parameters is \(h^0((O_{P_1}(d-1) \oplus O_{P_2}(d-1)) \oplus O_{P_1}(-6)) + 6 = 2d - 12 + 6 = 2d - 6\), where +6 means the sum of the numbers of parameters of two points on \(\mathfrak{r}_i (i = 1, 2, 3)\). By \(2d - 6 + 8 = 2d + 2\), a general \(C_d\) has 2-dimensional pairs of mutually intersecting bi-secant conics which intersect at least one bi-secant line pair of \(C_d\). Thus general pairs of mutually intersecting bi-secant conics of \(C_d\), which form a 3-dimensional family, do not intersect a bi-secant line pair of \(C_d\).

(3). Assume by contradiction that \(\overline{\mathfrak{r}_i}, \overline{\mathfrak{r}_j}\) and \(\overline{\mathfrak{r}_k}\) pass through a point \(b\), then \(\overline{\mathfrak{r}_l}\) does not pass through \(b\) but intersects a line through \(b\). Then by the double
projection from $b$, $\overline{q}_j$ is mapped to a line through the three singular points of the image of $C_b$ corresponding to $\overline{q}_i$, $\overline{q}_j$ and $\overline{q}_k$. Thus we have only to prove that for a general point of $b$ on $B$, three double points of the image of $C_b$ do not lie on a line.

Fix a general point $b \in B$. Let $\overline{q}_1, \overline{q}_2, \overline{q}_3$ be three conics on $B$ through $b$ such that by the double projection from $b$, they are mapped to three colinear points on $\mathbb{P}^2$.

The number of parameters of $C_d$’s intersecting each of $\overline{q}_i$ twice is $h^0((\mathcal{O}_{\mathbb{P}^1}(d-1) \oplus \mathcal{O}_{\mathbb{P}^1}(d-1)) \otimes \mathcal{O}_{\mathbb{P}^1}(-6)) = 2d - 12$ since $h^1((\mathcal{O}_{\mathbb{P}^1}(d-1) \oplus \mathcal{O}_{\mathbb{P}^1}(d-1)) \otimes \mathcal{O}_{\mathbb{P}^1}(-6)) = 0$

Note that the number of parameters of $\overline{q}_1, \overline{q}_2, \overline{q}_3$ is 5 since that of lines in $\mathbb{P}^2$ is 2, and that of three points on a line is 3. Thus the number of parameters of $C_d$’s such that its image of the double projection from $b$ has three colinear double points is at most $2d - 1$. Hence a general $C_d$ does not satisfy this property.

(4). Let $r_1$ and $r_2$ be a general pair of mutually conjugate conics on $A$ such that $\overline{q}_1$ and $\overline{q}_2$ are smooth, and $\overline{q}_1$ and $\overline{q}_2$ intersect at a point on $C \cup \cup \beta_i$. Such general pairs of conics $r_1$ and $r_2$ form a two-dimensional family since dim $C \cup \cup \beta_i = 1$ and if one point $t$ of $C \cup \cup \beta_i$ is fixed, then such pairs of conics such that $t \in \overline{q}_1 \cap \overline{q}_2$ form a one-dimensional family. For a general pair of $r_1$ and $r_2$, the number of the sets of $n$ mutually conjugate conics including $r_1$ and $r_2$ is finite since $D_{r_1}$ and $D_{r_2}$ has no common component. Thus $\{q_i\}$ does not contain such a pair by generality whence $\{q_i\}$ satisfies (4).

2.5.3. Relation with Mukai’s result. Here we sketch how the argument goes on if $d = 5$ and explain a relation of our result with Theorem 2.4.18

Assume that $d = 5$. Associated to the birational morphism $\mathcal{H}_2 \to \mathbb{P}^2$, there exists a non finite birational morphism

$$\Phi: \tilde{A} \to V_{22} := \text{VSP}(F_4, 6) \subset \text{Hilb}^6\mathbb{P}^2,$$

which fits into the following diagram:

\begin{center}
\begin{tikzpicture}[node distance=2cm, auto]
  \node (A) {$A$};
  \node (A') {$A'$};
  \node (V) {$V_{22}$};
  \node (B) [below of=A] {$B$};
  \draw[->] (A) to node[above] {$\rho'$} (A');
  \draw[->] (A) to node[below] {$\rho$} (V);
  \draw[->] (B) to node[below] {$f$} (A);
  \draw[->] (V) to node[below] {$\Phi$} (A');
  \draw[->] (B) to node[below] {$f'$} (V);
\end{tikzpicture}
\end{center}

where

- $V_{22}$ is a smooth prime Fano threefold of genus twelve,
- $\rho'$ is the blow-down of the three $\rho$-exceptional divisors $E_i$ ($i = 1, 2, 3$) over the strict transform $\beta_i'$ in the other direction. In other words, $A \longrightarrow A'$ is the flops of $\beta_1', \beta_2'$ and $\beta_3'$ (cf. Lemma 2.5.1), and
- the morphism $f'$ contracts the strict transform of the unique hyperplane section $S$ containing $C$ (see Proposition 2.2.1 (d)) to a general line on $V_{22}$.

The rational map $V_{22} \longrightarrow B$ is the famous double projection of $V_{22}$ from a general line $m$ first discovered by Iskovskih (see [Isk78]).

We explain how $f'$ and $\rho'$ are interpreted in our context. As we remarked after the proof of Theorem 2.4.18, the morphism $\mathcal{H}_2 \to \mathbb{P}^2$ defined by $|D_i|$ contracts three curves $D_{e_i}$ which parameterize conics intersecting $\beta_i'$. By noting $S$ is covered
by the images of such conics, this corresponds to that the morphism $f'$ contracts the strict transform of $S$

We can see that any conic on $A$ except one belonging to $D_0$, corresponds to that on $V_{22}$ in the usual sense, and the component of Hilbert scheme of $V_{22}$ parameterizing conics is naturally isomorphic to $\mathbb{P}^2$. The three conics on $V_{22}$ corresponding to the images of $D_0$ are $\beta_3^3 \cup m$, where $\beta_3^3$ are the images of the flopped curve corresponding to $\beta_3^3$.

Let $a \in E_i$. Then six conics on $A$ attached to $a$ are $\xi_{ij}$ ($j = 1, 2$), a conic $q_0$ from $D_0$, and three conics from $e_i$ (see the remark at the end of 2.5.1). Moreover, if $a$ moves in a fiber $\gamma$ of the other projection $E_i \to \mathbb{P}^1$, then only the conic $q_0$ from $D_0$ varies. By the contraction $H_2 \to \mathbb{P}^2$, there is no difference among points on $\gamma$. This is the meaning of the contraction $\rho'$ of $E_i$ in the other direction.

Finally we remark that $H_1$ is also naturally isomorphic to the component of Hilbert scheme of $V_{22}$ parameterizing lines.

3. THE EXISTENCE OF THE SCORZA QUARTIC

In this section we will use the geometries of $H_1$ and $H_2$ to give an affirmative answer to the conjecture of Dolgachev and Kanev [DK93, Introduction p. 218] (see Theorem 3.5.3).

3.1. Theta-correspondence on $H_1 \times H_1$.

In this subsection, we regard $H_1$ as the component of the Hilbert scheme of $A$ parameterizing lines on $A$.

We will define a non-effective theta characteristic on $H_1$ by investigating the following set:

$$I := \{(l_1, l_2) \in H_1 \times H_1 \mid l_1 \text{ and } l_2 \text{ intersect}\}.$$

We need a more precise and technical definition of $I$. First we reconsider the desingularization morphism $\pi_{H_1} : H_1 \to M \subset \mathbb{P}^2$; see Corollary 2.3.2.

Lemma 3.1.1. $h^0(H_1, (\pi_{H_1})^*O_M(1)) = 3$.

Proof. Let $h : S \to H^P_1 \simeq \mathbb{P}^2$ be the blow-up of $H^P_1$ at the $s = \frac{(d-2)(d-3)}{2}$ nodes of $M$. Then $H_1 \sim d\lambda - 2 \sum_{i=1}^s \varepsilon_i$, where $\lambda$ is the pull-back of a general line and $\varepsilon_i$ are exceptional curves. By the exact sequence

$$0 \to O_S(\lambda - H_1) \to O_S(\lambda - H_1) \to \mathcal{O}_{H_1}(\pi_{H_1})^*O_M(1) \to 0$$

together with $h^0(O_S(\lambda)) = 3$ and $h^0(O_S(\lambda - H_1)) = h^1(O_S(\lambda)) = 0$, we see that $h^0(H_1, (\pi_{H_1})^*O_M(1)) = 3$ is equivalent to $h^1(O_S(\lambda - H_1)) = 0$. By the Riemann-Roch theorem, we have $\chi(O_S(\lambda - H_1)) = 0$. Thus by $h^0(O_S(\lambda - H_1)) = 0$, $h^1(O_S(\lambda - H_1)) = 0$ is equivalent to $h^2(O_S(\lambda - H_1)) = 0$. By the Serre duality, $h^2(O_S(\lambda - H_1)) = h^0(O_S((d-4)\lambda - \sum_{i=1}^s \varepsilon_i))$. Thus we have only to prove that there exists no plane curve of degree $d - 4$ through $s$ nodes of $M$. We prove this fact by using the inductive construction of $C = C_d$. In case $d = 2$, the assertion is obvious. From now on in the proof, we put the suffix $d$ to the object depending on $d$. For example, $s_d := \frac{(d-2)(d-3)}{2}$. Assuming $h^0(O_{S_d}((d-4)\lambda_d - \sum_{i=1}^{s_d} \varepsilon_{i,d})) = 0$, we prove $h^0(O_{S_{d+1}}((d-3)\lambda_{d+1} - \sum_{i=1}^{s_d} \varepsilon_{i,d+1})) = 0$.

Recall that we constructed $C_{d+1}$ by the smoothing of the union of $C_d$ and a general uni-secant line $\mathcal{T}$ of $C_d$. By a standard degeneration argument, we have only to prove that there exists no plane curve of degree $d - 3$ through $s_{d+1}$ nodes of
$M_d \cup M(\overline{l})$, where $s_d$ of $s_{d+1}$ nodes are those of $C_d$ and the remaining $s_{d+1} - s_d = d - 2$ nodes are $M_d \cap M(\overline{l})$ except the two points corresponding to the two other lines $\overline{t}'$ through $C_d \cap \overline{t}$. Assume that there exists a plane curve $G$ of degree $d - 3$ through $s_{d+1}$ nodes of $M_d \cup M(\overline{l})$. Then $G \cap M(\overline{l})$ contains at least $d - 2$ points. Since $\deg G = d - 3$, this implies $M(\overline{l}) \subset G$. Thus there exists a plane curve of degree $d - 4$ through $s_d$ nodes of $M_d$, a contradiction.

We denote by $\delta$ the $g^3_3$ on $H_1$ which defines $\varphi_{|H_1}: H_1 \to C$. Let $l, l'$ and $l''$ be three lines on $A$ such that $[l] + [l'] + [l''] \sim \delta$. Then $\overline{t}, \overline{t}'$ and $\overline{t}''$ are lines through one point of $C$. Set

$$\theta := (\pi_{|H_1})^* O_M(1) - \delta.$$ 

Let $l$ be any line on $A$ and $l', l''$ lines such that $[l] + [l'] + [l''] \sim \delta$. By $\theta + [l] = \pi_{|H_1}^* O_M(1) - [l'] - [l'']$ and Lemma 3.1.1, we have $h^0(H_1, O_{H_1}(\theta + [l])) = 1$. Let $p_i: H_1 \times H_1 \to H_1$ ($i = 1, 2$) be the two projections and $\Delta$ the diagonal of $H_1 \times H_1$. Set $L := O_{H_1 \times H_1}(p_2^* \theta + \Delta)$. By $h^0(H_1, O_{H_1}(\theta + [l])) = 1$ for any $[l] \in H_1$, we see that $p_1^* L$ is an invertible sheaf. Define an ideal sheaf $I$ by $p_1^* p_1^* L = L \otimes I$. $I$ is an invertible sheaf and let $I$ be the divisor defined by $I$. Then we can extract the following definition:

**Definition 3.1.2.** $I$ is called the theta-correspondence. We will denote by $I([l])$ the fiber of $I \to H_1$ over $[l]$.

The following result is a generalization of Mukai’s result [Muk04 §4, Theorem] in our setting:

**Proposition 3.1.3.** $\theta$ is a non-effective theta characteristic.

**Proof.** By invoking [DK93 Lemma 7.2.1] and the definition of $I$, it suffices to prove the following:

(a) $h^0(H_1, O_{H_1}(\theta + [l])) = 1$ for any $[l] \in H_1$.
(b) $I$ is reduced.
(c) $I$ is disjoint from the diagonal.
(d) $I$ is symmetric, and
(e) $I$ is a $(g(H_1), g(H_1))$-correspondence.

Let $l$ be any line on $A$ and $l', l''$ lines such that $[l] + [l'] + [l''] \sim \delta$.

We have proved (a) already.

Noting that the line in $\mathbb{P}^2$ joining $[\overline{l}]$ and $[\overline{t}']$ parameterizes the lines on $B$ intersecting $\overline{t}$, we see that the fiber of $I \to H_1$ over a general $[l]$ is reduced. Hence $I$ is reduced.

We prove (c). It is equivalent to show that the support of $I([l])$ does not contain $[l]$. By definition $\theta + [l] = \pi_{|H_1}^* O_M(1) - [l'] - [l'']$. If $\overline{t}$ is a uni-secant and is not special, then $M(\overline{l})$ does not contain $[\overline{l}]$, thus we are done. If $\overline{t}$ is special, then, by Propositions 2.1.3 (4) and 2.2.4 (2), we are done. If $\overline{t}$ is a bi-secant then by Proposition 2.2.4 (4), we are done.

We prove (d). Let $m$ be a line on $A$ such that $[m]$ is contained in the support of $I([l])$. It suffices to prove that for a general $l$, $[l]$ is contained in the support of $I([m])$. For a general $l$, we may assume that $m \neq l'$ or $l''$. Then it is easy to verify this fact.

Finally we prove (e). Since $I$ is symmetric and $\deg(\theta + [l]) = d - 2 = g(H_1)$, the divisor is a $(g(H_1), g(H_1))$-correspondence.
3.2. **Duality between $H_1$ and $H_2$.**

Denote by $\mathbb{P}^{d-3}$ the projective space dual to $\mathbb{P}^{d-3}$. The family

\[
D_1 \longrightarrow H_2 \times H_1 \\
\downarrow \\
H_1
\]

induces the morphism

\[
H_1 \rightarrow \mathbb{P}^{d-3} \\
[l] \mapsto [D_l].
\]

by the universal property of the Hilbert scheme. Since $D_l \neq D_{l'}$ for $l \neq l'$, $H_1 \rightarrow \mathbb{P}^{d-3}$ is injective.

Consider the projection $D_1 \rightarrow H_2$ and denote by $H_q$ the fiber over $[q]$. Since $D_1$ is a Cartier divisor in a smooth 3-fold $H_1 \times H_2$ then $D_1$ is Cohen-Macaulay. Since no conic on $A$ intersects infinitely many lines on $A$, $D_1 \rightarrow H_2$ is finite. Then $D_1 \rightarrow H_2$ is flat since $H_2$ is smooth. Note that for a general conic $q, H_q$ parameterizes all the lines intersecting $q$. By considering the morphism $\pi_{|H_1|} : H_1 \rightarrow M \subset \mathbb{P}^2$, it is easy to see that for a general conic $q, H_q \in |\pi^*O_M(2) - 2\delta|$, namely, $H_q \sim 2\theta \sim K_{H_1}$. By the flatness of $D_1 \rightarrow H_2$, it holds $H_q$ for any $q$.

Recall that we denote by $\{H_q = 0\}$ the hyperplane in $\mathbb{P}^{d-3}$ corresponding to $[q] \in \mathbb{P}^{d-3}$. Note that, for $[l] \in H_1$ and $[q] \in H_2$, $[l] \in \{H_q = 0\}$ if and only if $D_l([q]) = 0$. Thus $H_q = \{H_q = 0\}$. Consequently, the injection $H_1 \rightarrow \mathbb{P}^{d-3}$ is the canonical embedding $\Phi_{|K_{H_1}|} : H_1 \rightarrow \mathbb{P}^{d-3}$ by $H_q \sim K_{H_1}$.

In case $d = 5$, a similar construction gives the duality of the canonical embedding $H_1 \subset \mathbb{P}^2$ and the birational morphism $H_2 \rightarrow \mathbb{P}^2$.

3.3. **Discriminant locus.**

We follow [DK93, 7.1.4 p.279]. Let $\Gamma \subset \mathbb{P}^{g-1}$ be a canonical curve of genus $g$ and $\theta'$ a non-effective even theta characteristic on $\Gamma$. By the Riemann-Roch theorem, it holds that $h^0(\theta' + x) = 1$ for a point $x \in \Gamma$. Let

\[
I := \{(x, y) \mid y \text{ is in the support of the unique member of } |\theta' + x| \} \subset \Gamma \times \Gamma.
\]

We call this the *theta-correspondence*, which is consistent with Definition 3.1.2. We denote by $I(x)$ the fiber of $I \rightarrow \Gamma$ over $x$ and call it the *theta-polyhedron* attached to $x$. In other words, $I(x)$ is the unique member of $|\theta' + x|$ as a divisor.

Since the linear hull $\langle I(x) - y \rangle$ is a hyperplane of $\mathbb{P}^{g-1}$, then we can define a morphism $\pi_{\theta'} : I \rightarrow |K_{\Gamma}| = \mathbb{P}^{g-1}$ as a composition of the embedding $I \hookrightarrow \Theta_{\Gamma}$ and the Gauss map $\gamma : \Theta_{\Gamma}^{ns} \rightarrow \mathbb{P}^{g-1}$,

where $\Theta_{\Gamma} \subset J(\Gamma)$ is the theta divisor and $\Theta_{\Gamma}^{ns}$ is the nonsingular locus of $\Theta_{\Gamma}$.

**Definition 3.3.1.** The image $\Gamma(\theta')$ of the above morphism $\pi_{\theta'} : I \rightarrow \mathbb{P}^{g-1}$ is called the *discriminant locus* of $(\Gamma, \theta')$.

Set-theoretically $\pi_{\theta'}$ is the map $(x, y) \mapsto \langle I(x) - y \rangle$. The hyperplane $\langle I(x) - y \rangle$ is called the *face* of $I(x)$ opposed to $y$.

From now on in the section 4, we assume that $d \geq 6$ for the pair $(H_1, \theta)$ and we consider $\mathcal{H}_2 \subset \mathbb{P}^{d-3}$.

For the pair $(H_1, \theta)$, we can interpret $\Gamma(\theta)$ by the geometry of lines and conics on $A$ as follows:
Proposition 3.3.2. For the pair \((\mathcal{H}_1, \theta)\), the discriminant locus \(\Gamma(\theta)\) is contained in \(\mathcal{H}_2\), and the generic point of the curve \(\Gamma(\theta)\) parameterizes line pairs.

Proof. Take a general point \(([l_1], [l_2]) \in I\), equivalently, take two general intersecting lines \(l_1\) and \(l_2\). \(l_1 \cup l_2\) is a line pair and the lines corresponding to the points of \(I([l_1]) - [l_2]\) are lines intersecting \(l_1\) except \(l_2\). Thus by 3.2 the point in \(\mathbb{P}^{d-3}\) corresponding to the hyperplane \(I([l_1]) - [l_2]\) is nothing but \([l_1 \cup l_2] \in \mathcal{H}_2\). This implies the assertion. \(\square\)

Proposition 3.3.3. The curve \(\Gamma(\theta)\) belongs to the linear system \(|3(d-2)h - 4\sum_{i=1}^s e_i|\) on \(\mathcal{H}_2\).

Proof. We can write:

\[
\Gamma(\theta) \sim ah - \sum m_i e_i,
\]

where \(a \in \mathbb{Z}\) and \(m_i \in \mathbb{Z}\). For a general \(b \in C\), \(L_b\) intersects \(\Gamma(\theta)\) simply. Thus \(a\) is the number of line pairs whose images on \(B\) pass through \(b\). There exists three lines \(l_1, l_2\) and \(l_3\) through \(b\). It suffices to count the number of reducible conics on \(B\) having one of \(l_i\) as a component except \(l_1 \cup l_2, l_2 \cup l_3\) and \(l_3 \cup l_1\). Thus \(a = 3(d-2)\).

We will count the number of line pairs belonging to \(e_i\). Each of such line pairs is of the form \(l_{ij;k} \cup l_{ij}\), where \(l_{ij;k}\) \((k = 1, 2)\) is the strict transform of the line through \(p_{ij}\) distinct from \(\beta_i\). Thus the number of such pairs is four and \(m_i \geq 4\).

Finally we will count the number of line pairs intersecting a general line \(l\). By Corollary 2.4.19, \(D_l\) does not contain any line pair \(l \cup l'\). Since the number of lines on \(A\) intersecting a fix line on \(A\) is \(d - 2\), we see that \(D_l \cdot \Gamma(\theta) \geq (d - 2)(d - 3)\). Then

\[
(d - 2)(d - 3) \leq \Gamma(\theta) \cdot D_l = (d - 3)a - \sum_{i=1}^s m_i.
\]

where \(s = \frac{(d-2)(d-3)}{2}\). This implies that \(m_i = 4\). \(\square\)

Corollary 3.3.4. For \((\mathcal{H}_1, \theta)\), it holds that \(\deg \Gamma(\theta) = g(g - 1)\) and \(p_a(\Gamma(\theta)) = \frac{3}{2}g(g - 1) + 1\).

Proof. The invariants of \(\Gamma(\theta)\) are easily calculated. \(\square\)

3.4. Definition of the Scorza quartic.

By Definition 3.3.1 we have the following diagram:

\[
\begin{array}{ccc}
I \subset \Gamma \times \Gamma & \xleftarrow{\pi_\theta} & \Gamma(\theta') \subset \mathbb{P}^{g-1} \\
& \quad \quad \downarrow p & \\
& \Gamma \subset \mathbb{P}^{g-1}.
\end{array}
\]

We can define:

\[
\overline{D}_H := \pi_\theta^* p^*(H \cap \Gamma),
\]

where \(H\) is an hyperplane of \(\mathbb{P}^{g-1}\). It is easy to see:

\[
\deg \overline{D}_H = 2g(g - 1).
\]

Let \(S^m V\) the space of \(m\)-th symmetric forms on the vector space \(V\). Note that an element of \(S^m V\) defines a hypersurface of degree \(m\) in \(\mathbb{P}_V V\). Let \(F \in S^{2k} V\) be a non-degenerate homogeneous form of degree \(2k\) and \(\hat{F} \in S^{2k} V\) the dual homogeneous
form to $F$ defined as in [Dol04] §2.3. Following [Dol04] 4.1, we define the variety of the conjugate pairs
\[ \text{CP}(F) := \{ ([H_1], [H_2]) \in \mathbb{P}_s \hat{V} \times \mathbb{P}_s \hat{V} \mid \langle H^k_1, P_{H^k_2}(F) \rangle = 0 \}, \]
where $\langle \cdot, \cdot \rangle$ is the polarity pairing. Let
\[ \Delta := \text{CP}(F) \cap (\text{the diagonal of } \mathbb{P}_s \hat{V} \times \mathbb{P}_s \hat{V}). \]
Since the diagonal of $\mathbb{P}_s \hat{V} \times \mathbb{P}_s \hat{V}$ is isomorphic to $\mathbb{P}_s \hat{V}$ then $\Delta \simeq \{ \hat{F} = 0 \}$.

Set $D'_H := P_{H^k}(\hat{F})$ for a hyperplane $H \subset \mathbb{P}_s \hat{V}$. Then we can write:
\[ \text{CP}(F) = \{ ([H_1], [H_2]) \in \mathbb{P}_s \hat{V} \times \mathbb{P}_s \hat{V} \mid D'_{H_2}([H_1]) = 0 \}. \]

**Definition 3.4.1.** A non-degenerate quartic $\{ F'_i = 0 \}$ is called the Scorza quartic for $(\Gamma, \theta')$ if $\{ D'_H = 0 \} \cap \Gamma(\theta') = \overline{D}_H$ for a hyperplane $\{ H = 0 \}$ such that $\Gamma \cap \{ H = 0 \}$ is reduced, where $D'_H$ is defined as above for $F'_4$.

3.5. **Dolgachev-Kanev’s conjecture on the existence of the Scorza quartic.**

We show that the following properties hold for general pairs of canonical curves $\Gamma$ and even theta characteristics $\theta'$ as Dolgachev and Kanev conjectured.

(A1) The number of theta-polyhedrons having a general face in common is two.

Equivalently, the degree of the map $I \to \Gamma(\theta')$ is two,

(A2) $\Gamma(\theta')$ is not contained in a quadric, and

(A3) $I$ is reduced.

By [DK93] Theorem 9.3.1, these three conditions are sufficient for the existence of the Scorza quartic for the pair $(\Gamma, \theta')$.

First we show that for our trigonal curve $\mathcal{H}_1$ and the even theta characteristic $\theta$ defined by intersecting lines the above conditions hold.

**Lemma 3.5.1.** $(\mathcal{H}_1, \theta)$ satisfies (A1)–(A3).

**Proof.** (A1) This condition means that for general lines $l$ and $l'$ on $A$ such that $(\{l\}, \{l'\}) \in I$ the face $I(\{l\}) - \{l'\})$ belongs only to $I(\{l\})$ and to $I(\{l'\})$.

By contradiction assume that there exists a line $m$ on $A$ such that $m \neq l, m \neq l'$ and $\langle I(\{l\}) - \{l'\} \rangle$ is a face of $I(\{m\})$. Then some $d - 3$ points of $I(\{m\})$ lie on the hyperplane $\langle I(\{l\}) - \{l'\} \rangle$, equivalently, $m$ intersects $d - 3$ lines on $A$ corresponding to the points of $I(\{l\}) \cup I(\{l'\})$ except $l$ and $l'$. By $d \geq 6$, it holds that, for $l$ or $l'$, say, $l$, there exist two lines intersecting both $l$ and $m$. Consider the projection $B \to Q$ from $f(l) = \overline{l}$:

\[
\begin{array}{ccc}
\pi_1 & & \pi_2 \\
B & & Q \\
\end{array}
\]

We use the notation of Proposition 2.1.6 (2). Now notice that by generality of $l$, $\overline{l} \neq \overline{m} := f(m)$ is equivalent to have $l \neq m$. Since there exist two lines intersecting both $\overline{l}$ and $\overline{m}$, we have $\overline{l} \cap \overline{m} = \emptyset$. Thus the strict transform $\overline{m}'$ of $\overline{m}$ on $Q$ is a line.

Since there exist two lines intersecting both $\overline{l}$ and $\overline{m}$, $\overline{m}'$ intersects the image $E'_\overline{l}$ of $E'_\overline{l}$ at two points. Since $E'_\overline{l}$ is a hyperplane section on $Q$, this implies that $\overline{m}' \subset E'_\overline{l}$, a contradiction.

(A2) This condition is satisfied by Theorem 2.3.18 (4) and Proposition 3.3.3.

(A3) We prove this in the proof of Proposition 3.1.3. $\square$
The set \( \{ \theta \} \).

**Proposition 3.6.1.**

**Proof.** The assertion is equivalent to the following equality:

\[ I'(\theta) := I(\tau). \]

where \( \tau \) is the involution on \( I \) induced by that of \( \Gamma \times \Gamma \) permuting the factors. Note that \( I \to \Gamma(\theta) \) factor through \( I'(\theta) \).

**Corollary 3.5.2.** For \((H_1, \theta)\), it holds \( \Gamma(\theta)' \simeq \Gamma(\theta) \).

**Proof.** By Lemma 3.5.1, (A1) holds for \((H_1, \theta)\). Thus, by [DK93, Corollary 7.1.7], we have \( p_a(\Gamma(\theta)') = \frac{3}{2}g(g-1) + 1 \). Thus, by Corollary 3.3.4, \( p_a(\Gamma(\theta)') = p_a(\Gamma(\theta)) \).

By (A1) again, the natural morphism \( \Gamma(\theta)' \to \Gamma(\theta) \) is birational. Therefore it holds \( \Gamma(\theta)' \simeq \Gamma(\theta) \).

By a moduli argument we prove the conjecture for a general pair \((\Gamma, \theta')\).

**Theorem 3.5.3.** A general spin curve satisfies the conditions (A1)–(A3). In particular, the Scorza quartic exists for a general spin curve.

**Proof.** Let \( M \) be the moduli space of couples \((\Gamma, \theta')\), where \( \Gamma \) is a curve of genus \( g \) and \( \theta' \) is a theta characteristic such that \( h^0(\Gamma, \theta') = 0 \). Classically, \( M \) is known to be irreducible (see [Co]). Let \( U \) be a suitable finite cover of an open neighborhood of a couple \((H_1, \theta)\) such that there exists the family \( \mathcal{G} \to U \) of pairs of canonical curves and non-effective theta characteristics. Denote by \((\Gamma_u, \theta_u)\) the fiber of \( \mathcal{G} \to U \) over \( u \in U \). By Lemma 3.5.1, \((H_1, \theta)\) satisfies (A1)–(A3). Since the conditions (A1) and (A3) are open conditions, these are true on \( U \). Thus we have only to prove that the condition (A2) is still true on \( U \). Let \( J \to U \) be the family of Jacobians and \( \Theta \to U \) the corresponding family of theta divisors. By [DK93, p.279-282], the family \( I \) of theta-correspondences embeds into \( \Theta \), and by the family of Gauss maps \( \Theta \to \mathbb{P}^{g-1} \times U \), we can construct the family \( \mathcal{G} \to U \) whose fiber \( \mathcal{G}_u \subset \mathbb{P}^{g-1} \) is the discriminant \( \Gamma(\theta_u) \). By Corollary 3.5.2, it holds \( \Gamma(\theta)' \simeq \Gamma(\theta) \) for \((H_1, \theta)\). Thus we have also \( \Gamma(\theta_u)' \simeq \Gamma(\theta_u) \) for \( u \in U \). By [DK93, Corollary 7.1.7], we see that \( p_a(\Gamma(\theta_u)) \) and \( \deg \Gamma(\theta_u) \) are constant for \( u \in U \). Thus \( \mathcal{G} \to U \) is a flat family since the Hilbert polynomials are constant. Since no quadric contains \( \Gamma(\theta) \) for \((H_1, \theta)\), neither does \( \Gamma(\theta_u) \) for \( u \in U \) by the upper semi-continuity theorem.

**Remark.** Let \((\Gamma, \theta')\) be a general pair of a canonical curve \( \Gamma \) and a non-effective theta characteristic \( \theta' \). In the proof of Theorem 3.5.3, we prove that \( \Gamma(\theta)' \simeq \Gamma(\theta) \).

### 3.6. \( F_4 \) is the Scorza quartic for \((H_1, \theta)\).

Note that, for \( F_4 \), it holds \( D_2 = \text{CP}(F_4)|_{H_2 \times H_2} \) and \( \bar{D}_q = P_{H_2}(\bar{F}_4) \).

Let \( \{ F'_4 = 0 \} \) be the Scorza quartic for \((H_1, \theta)\). Let \( D'_H \) be defined as in 3.4 for \( F'_4 \). We simply denote \( D'_{H_q} \) by \( D'_q \).

**Proposition 3.6.1.** \( F_4 \) is the Scorza quartic for \((H_1, \theta)\).

**Proof.** The assertion is equivalent to the following equality:

\[ \text{CP}(F'_4)|_{H_2 \times H_2} = \text{CP}(F_4)|_{H_2 \times H_2}. \]

Since both sides have the structures of quadric section bundles over \( H_2 \), it suffices to prove \( \{ D'_q = 0 \} = \{ \bar{D}_q = 0 \} \) for a generic \( q \). Since there is no quadric containing \( \Gamma(\theta) \) by Lemma 3.5.1, it is sufficient to show that \( \{ D'_q = 0 \} \cap \Gamma(\theta) = \{ \bar{D}_q = 0 \} \cap \Gamma(\theta) \). The set \( \{ \bar{D}_q = 0 \} \cap \Gamma(\theta) \) consists of points corresponding to the line pairs intersecting \( q \). On the other hand, \( \{ D'_q = 0 \} \cap \Gamma(\theta) = \bar{D}_{H_q} \) by the definition of the Scorza quartic.
since $H_{q}$ is reduced for a general $q$. By the definition of $D_{H_{q}}$, we can easily show the set $D_{H_{q}}$ also consists of points corresponding to the line pairs intersecting $q$. Thus $\{D_{q}' = 0\} \cap \Gamma(\theta) = \{D_{q} = 0\} \cap \Gamma(\theta)$ as desired.

3.7. Moduli space of trigonal spin curves.

As in Mukai’s case we can reconstruct the threefold $\tilde{A}$, that is the couple $(B,C)$, via the curve $H_{1}$ and a non-effective theta characteristic $\theta$ on it.

**Proposition 3.7.1.** $\tilde{A}$ is recovered from $(H_{1}, \theta)$.

**Proof.** From $(H_{1}, \theta)$, we can define $\Gamma(\theta)$ as in Definition 3.3.1 and $F_{4}$ by Proposition 3.6.1. By Theorem 2.4.1 and Proposition 3.3.3, $H_{2}$ is recovered from $\Gamma(\theta)$ as the intersection of cubics containing $\Gamma(\theta)$. By Theorem 2.5.12 and Lemma 2.5.14, $\tilde{A}$ is recovered from $F_{4}$ and $H_{2}$. □

For the next result, we denote $\tilde{A}$ by $\tilde{A}_{d}$.

Recall that we denote by $H_{B}^d$ the union of components of the Hilbert scheme of $B$ whose general points parameterize smooth rational curves of degree $d$ obtained inductively as in Proposition 2.2.1. By the remark after the proof of Proposition 3.7.5, $H_{B}^d$ is irreducible.

We identify $H_{B}^d$ with the moduli space of $\tilde{A}_{d}$, which we denote by $M_{d}$. Let $M_{g}'$ and $\tilde{M}_{g}'$ be the moduli space of trigonal curves of genus $g$ and the moduli space of pairs of trigonal curves of genus $g$ and even theta characteristics, respectively. We can define the rational map $\pi_{M}: M_{d} \dashrightarrow \tilde{M}_{d-2}'$ by setting $\tilde{A}_{d} \mapsto (H_{1}, \theta)$.

**Corollary 3.7.2.** $\pi_{M}$ is birational. Moreover, $\text{Im} \pi_{M}$ is an irreducible component of $\tilde{M}_{d-2}'$ dominating $\tilde{M}_{d-2}'$. In particular a general $H_{1}$ is a general trigonal curve of genus $d - 2$.

**Proof.** The first assertion follows from Proposition 3.7.1.

Since $\dim H_{B}^d = 2d$ and $\dim \text{Aut} (B, C_{d}) \leq 3$, we see that $\dim M_{d} \geq 2d - 3$. On the other hand, $\dim M_{d-2}' = 2d - 3$ and a smooth curve has only a finite number of theta-characteristics. Thus the latter part follows from the first. □

Combining Theorem 2.5.12, Proposition 3.6.1 and Corollary 3.7.2, we obtain:

**Corollary 3.7.3.** Let $F_{4}$ be the Scorza quartic for a general trigonal spin curve of genus $d - 2$ ($d \geq 6$) and $H_{2}$ the intersection of cubics containing the discriminant locus of the trigonal spin curve. Set $n := \frac{(d-1)(d-2)}{2}$. Then the normalization of the main component of $\text{VSP}(F_{4}, n; H_{2})$ is isomorphic to the blow-up of quintic del Pezzo threefold $B$ along a general smooth rational curve of degree $d$ and then the strict transforms of its bi-secant lines on $B$.

By our study we see some other problems which are of a certain interest:

**Problem 3.7.4.** (1) Is the Hilbert scheme of curves of $B$ whose generic point corresponds to a smooth rational curve of degree $d$ irreducible, namely, is $H_{B}^d$ the unique irreducible component ?

(2) Is $M_{g}'$ irreducible ?

If $d = 5$, (2) is true by [DK93, Lemma 7.7.1]. We show that (1) is true also for $d \leq 6$. (Probably if $d \leq 5$, then it is known. Our contribution is for $d = 6$).

**Proposition 3.7.5.** If $d \leq 6$, then the answer to Problem 3.7.4(1) is affirmative.
proof. For a smooth projective variety $X$ in some projective space, let $C_d^0(X)$ be the components of the Hilbert scheme of $X$ whose general points parameterize smooth rational curves of degree $d$. By [Per02, C$^0_d(G(a, b))$] is irreducible, where $G(a, b)$ is the Grassmannian parameterizing $a$-dimensional sub-vector spaces in a fixed $b$-dimensional vector space. The claim is that $C_d^0(\mathbb{P}^6 \cap G(2, 5))$ is irreducible, where $\mathbb{P}^6 \subset \mathbb{P}^9$ is transversal to $G(2, 5)$. The claim is true for $d = 1$ since $H^B_1 \simeq \mathbb{P}^2$.

Let $B$ be the irreducible family of del Pezzo 3-folds $B = G(2, 5) \cap \mathbb{P}^6$, where $\mathbb{P}^6 \subset \mathbb{P}^9$ is transversal to $G(2, 5)$. Let $$J = \{[(C_d^0, [B]) \in C_d^0(G(2, 5)) \times B \mid C_d^0 \subset B]\}.$$ The claim is equivalent to show that a general fiber $J \rightarrow B$ is irreducible. Since $d \leq 6$, a smooth rational curve of degree $d$ is contained in at least a six-dimensional projective space. Thus a general fiber of $J \rightarrow C_d^0(G(2, 5))$ is non-empty and irreducible. Since $C_d^0(G(2, 5))$ is irreducible, it holds $J$ is irreducible. By the argument of [MT01] Proof of Theorem 3.1 p.17, we have only to show that there is one particular component $C_d^0(B)$ of a general fiber $J \rightarrow B$ invariant under monodromy.

By induction let us assume that $C_{d-1}^0(B)$ is irreducible. Let $[C_{d-1}^0 \in C_{d-1}^0(B)]$ be a generic element. The family of lines $[l] \in H^B_1$ which intersect a general element of $C_{d-1}^0(B)$ is irreducible by Proposition 22.2.4 (3). This implies that the family $C_{d-1,1}^0(B)$ of reducible curves $C_d^0 = C_{d-1}^0 \cup l$ such that $[C_{d-1}^0 \in C_{d-1}^0(B)]$, $[l] \in H^B_1$ and length $C_{d-1}^0 \cap l = 1$ is irreducible. Similarly to the proof of Proposition 22.2.4, we see that the locus containing the points corresponding to the smoothings of curves from $C_{d-1,1}^0(B)$ is an irreducible component of $J$.

Remark. The proof of the proposition shows that $H^B_d$ is irreducible for any $d$.

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Graduate School of Mathematical Sciences, the University of Tokyo, Tokyo, 153-8914, Japan, takagi@ms.u-tokyo.ac.jp

D.I.M.I., the University of Udine, Udine, 33100 Italy, francesco.zucconi@dimi.univud.it