1 Introduction

Spin has been shown to be amenable to a new treatment which provides new insight into the origin of its matrix description[1-6]. This treatment is based on the interpretation of quantum mechanics due to Landé [7-10]. The ideas on which the treatment is based are general, and though the treatment has initially been applied to spin systems, it should be of general applicability. In this paper, we prove this by successfully applying this approach to the description of polarization. We obtain new generalized formulas for the polarization states and for the operators that enter into the calculation of polarization expectation values.

This paper is organized as follows. In Section 2, we start off by giving the basic theory underlying the new approach. In Section 3, we review the connection between the differential and the matrix eigenvalue equation. Section 4 is devoted to the application of the theory to the case of polarization measurements. Thus, we derive the generalized probability amplitudes for polarization measurements and calculate the probabilities corresponding to them in Section 4.1. In Section 4.2 we give the general formulas for the matrix operators of any observable whose eigenvalues depend on the polarization direction. These formulas are straightaway used to obtain the generalized polarization operator in Section 4.3. We consider expectation values for polarization measurements in Section 4.4. In Section 4.5, we connect the present treatment with the standard approach by showing how the generalized formulas we present here reduce to the standard ones in the appropriate limit. We wrap up the paper in Section 5, with a Discussion and then a Conclusion.

2 Basic Theory

The basic theory to be employed is a development of the Landé interpretation of quantum mechanics[7-10]. According to this approach, nature is inherently indeterministic, and quantum mechanics must necessarily reflect this. Let a quantum system have the observables $A$, $B$ and $C$ which have the respective eigenvalue spectra $A_1$, $A_2$... $B_1$, $B_2$... and $C_1$, $C_2$... If the system is initially in a state corresponding to the eigenvalue $A_i$, a measurement of $B$ yields any of the eigenvalues $B_j$ with probabilities determined by the probability amplitudes $\chi(A_i, B_j)$. A measurement of $C$ results in one of the eigenvalues $C_j$ with probabilities determined by the probability amplitudes $\psi(A_i, C_j)$. If the system is initially in a state corresponding to the eigenvalue $B_i$, a measurement of $C$ gives any of the eigenvalues $C_j$ with probabilities determined by the probability amplitudes $\phi(B_i, C_j)$. The probability amplitudes display a two-way symmetry contained in the Hermiticity condition

$$\psi(C_j, A_i) = \psi^*(A_i, C_j).$$

(1)

These probability amplitudes are orthogonal:
\[ \sum_j \psi^*(A_i, C_j) \psi(A_k, C_j) = \delta_{ik}. \] \hspace{1cm} (2)

As is to be expected of probability amplitudes pertaining to one system, they are not independent and are connected to one another by the relation

\[ \psi(A_i, C_n) = \sum_j \chi(A_i, B_j) \phi(B_i, C_n). \] \hspace{1cm} (3)

Each of the other two types of probability amplitudes can similarly be expressed in terms of the other two sets. Relation (3) is our fundamental expression and most of what follows is based upon it.

### 3 Differential and Matrix Eigenvalue Equations

In order to better understand the present approach, we review some results regarding the connection between the differential and the matrix eigenvalue equations for the same observable[1]. In the Landé formalism, the eigenfunction is a probability amplitude connecting two states connected together by a measurement. For this reason it is characterized by two labels. One defines the state that obtains before measurement while the other refers to the state that comes about as a result of the measurement. The eigenvalue refers to the initial state, while the continuous variable in terms of which the corresponding differential operator is framed describes the final state. Thus, the eigenvalue equation for the operator \( A(x) \) is written out as

\[ A(x)\psi(a_k, x) = a_k \psi(a_k, x). \] \hspace{1cm} (4)

where \( \psi(a_k, x) \) is an eigenfunction of \( A \) with eigenvalue \( a_k \).

We now transform this differential eigenvalue equation to a matrix eigenvalue equation in the usual way. We use Eq. (3) to express \( \psi(a_k, x) \) as an expansion:

\[ \psi(a_k, x) = \sum_{j=1}^{N} \chi(a_k, B_j) \phi(B_j, x). \] \hspace{1cm} (5)

Going through the usual steps, we find the matrix eigenvalue equation

\[
\begin{pmatrix}
A_{11} - a_k & A_{12} & \cdots & A_{1N} \\
A_{21} & A_{22} - a_k & \cdots & A_{2N} \\
\vdots & \vdots & \ddots & \cdots \\
A_{N1} & A_{N2} & \cdots & A_{NN} - a_k
\end{pmatrix}
\begin{pmatrix}
\chi(a_k, B_1) \\
\chi(a_k, B_2) \\
\vdots \\
\chi(a_k, B_N)
\end{pmatrix} = 0,
\] \hspace{1cm} (6)

where

\[ A_{mj} = \langle \phi(B_m, x) | A(x) | \phi(B_j, x) \rangle. \] \hspace{1cm} (7)
The expectation value of the arbitrary quantity $R(x)$ is given by

$$\langle R \rangle = \sum_{i=1}^{N} \sum_{j=1}^{N} \chi^*(a_k, B_i) R_{ij} \chi(a_k, B_j) = [\chi(a_k)]^\dagger [R] [\chi(a_k)]$$

(8)

where

$$R_{ij} = \langle \phi(B_i, x) | R(x) | \phi(B_j, x) \rangle$$

(9)

$$[R] = \begin{pmatrix}
  R_{11} & R_{12} & \ldots & R_{1N} \\
  R_{21} & R_{22} & \ldots & R_{2N} \\
  \vdots & \vdots & \ddots & \vdots \\
  R_{N1} & R_{N2} & \ldots & R_{NN}
\end{pmatrix}$$

(10)

and

$$[\chi(a_k)] = \begin{pmatrix}
  \chi(a_k, B_1) \\
  \chi(a_k, B_2) \\
  \vdots \\
  \chi(a_k, B_N)
\end{pmatrix}$$

(11)

A very important point to note from Eq. (6) is that the elements of the eigenvectors $[\chi(a_k)]$ of the operator $[A]$ are the probability amplitudes $\chi(a_k, B_j)$. This is true even in the cases where the matrix eigenvalue equation does not directly follow from a differential eigenvalue equation. This is a very important result because it makes almost trivial the task of obtaining the eigenvectors of the operator $[A]$. Basically, as soon as the probability amplitudes for the measurement under consideration are known, the eigenvectors are also known. We have extensively used this result in our work on spin[1-5].

4 Application to Polarization

4.1 Probability Amplitudes and Probabilities

As we have done for spin, we shall use Eq. (3) to obtain generalized expressions for the probability amplitudes describing polarization measurements. We shall then use these to obtain the matrix treatment of polarization.

We consider a photon travelling in the $z$ direction. Its polarization is measured with respect to the $x$ direction by means of the angle $\theta$. If the photon is polarized at the angle $\theta$ with respect to the $x$ direction, then according to standard theory, its state is

$$[\chi(\theta^+)] = \begin{pmatrix}
  \cos \theta \\
  \sin \theta e^{i\alpha}
\end{pmatrix}$$

(12)

where $\alpha$ is the relative phase between the $x$ and $y$ components of the electric field vector. If the polarization is at right angles to $\theta$, the state is
\[ \chi(\theta^-) = \begin{pmatrix} -\sin \theta \\ \cos \theta e^{i\alpha} \end{pmatrix}. \] (13)

When \( \theta = 0 \) and \( \alpha = 0 \), the state Eq. (12) gives \( x \) polarization, while the state Eq. (13) describes \( y \) polarization. On the other hand, if \( \theta = \pi/4 \) and \( \alpha = \pi/2 \), Eq. (12) describes right circular polarization while Eq. (13) describes left circular polarization.

In the spirit of the Landé approach, we should begin our treatment of polarization by talking about probability amplitudes. Suppose that the photon is initially polarized in the direction \( \theta_a \) and its polarization in the direction \( \theta_b \) is then measured. We denote the probability amplitude for this measurement by \( \chi(\theta_a^+, \theta_b^+) \). Thus, the probability amplitude for finding the polarization to be along the direction defined by \( \theta_b \) is \( \chi(\theta_a^+, \theta_b^+) \), while that for finding it to be perpendicular to that direction is \( \chi(\theta_a^+, \theta_b^-) \). By the same token, if the polarization is known to be initially perpendicular to the direction defined by \( \theta_a \), the probability amplitudes for finding it, upon measurement, to be parallel and perpendicular to the direction defined by \( \theta_b \) are \( \chi(\theta_a^-, \theta_b^+) \) and \( \chi(\theta_a^-, \theta_b^-) \) respectively. The polarizations parallel and perpendicular to a given direction define two orthogonal states of an observable which is characterized by the angle defining that direction. We may expand the probability amplitude \( \chi \) using the fundamental relation Eq. (3): thus, for example,

\[ \chi(\theta_a^+, \theta_b^+) = \psi(\theta_a^+, \theta_c^+ \phi(\theta_c^+, \theta_b^+) + \psi(\theta_a^+, \theta_c^- \phi(\theta_c^-, \theta_b^-) \] (14)

where \( \theta_c \) is an arbitrary third direction.

Owing to the fact that all the probability amplitudes occurring refer to polarization-to-polarization measurements, we can use the symbol \( \chi \) for \( \psi \) and \( \phi \). Thus, we have

\[ \chi(\theta_a^+, \theta_b^-) = \chi(\theta_a^+, \theta_c^+) \chi(\theta_c^+, \theta_b^-) + \chi(\theta_a^+, \theta_c^-) \chi(\theta_c^-, \theta_b^-). \] (15)

The other expansions are

\[ \chi(\theta_a^+, \theta_b^-) = \chi(\theta_a^+, \theta_c^+) \chi(\theta_c^+, \theta_b^-) + \chi(\theta_a^+, \theta_c^-) \chi(\theta_c^-, \theta_b^-), \] (16)

\[ \chi(\theta_a^-, \theta_b^+) = \chi(\theta_a^-, \theta_c^+) \chi(\theta_c^+, \theta_b^-) + \chi(\theta_a^-, \theta_c^-) \chi(\theta_c^-, \theta_b^-) \] (17)

and

\[ \chi(\theta_a^- \theta_b^-) = \chi(\theta_a^-, \theta_c^+) \chi(\theta_c^+, \theta_b^-) + \chi(\theta_a^-, \theta_c^-) \chi(\theta_c^-, \theta_b^-). \] (18)

We now need to obtain the explicit forms of these probability amplitudes. We shall deduce these generalized probability amplitudes by considering Eqs. (11) and (13).

According to the theory in Section 3, the elements of a matrix state are probability amplitudes in their own right. This holds true for all polarization state vectors. This means that from the standard state
\[ \chi(\theta_a^+, \theta_f^+) = \cos \theta_a \]

we can make the deductions

\[ \chi(\theta_a^+, \theta_f^+) = \cos \theta_a \] (20)

and

\[ \chi(\theta_a^+, \theta_f^-) = \sin \theta_a e^{i \alpha}, \] (21)

where \( \theta_f \) is an unknown direction.

On the other hand, using

\[ \chi(\theta_a^-) = \left( \begin{array}{c} -\sin \theta_a \\ \cos \theta_a e^{i \alpha} \end{array} \right), \] (22)

we deduce that

\[ \chi(\theta_a^-, \theta_f^+) = -\sin \theta_a \] (23)

and

\[ \chi(\theta_a^-, \theta_f^-) = \cos \theta_a e^{i \alpha}. \] (24)

But suppose that the initial direction is \( \theta_b \) instead of \( \theta_a \). Then, from the standard states

\[ \chi(\theta_b^+) = \left( \begin{array}{c} \cos \theta_b \\ \sin \theta_b e^{i \alpha} \end{array} \right), \] (25)

and

\[ \chi(\theta_b^-) = \left( \begin{array}{c} -\sin \theta_b \\ \cos \theta_b e^{i \alpha} \end{array} \right), \] (26)

we deduce that

\[ \chi(\theta_b^+, \theta_f^+) = \cos \theta_b, \] (27)

\[ \chi(\theta_b^+, \theta_f^-) = \sin \theta_b e^{i \alpha}, \] (28)

\[ \chi(\theta_b^-, \theta_f^+) = -\sin \theta_b \] (29)

and

\[ \chi(\theta_b^-, \theta_f^-) = \cos \theta_b e^{i \alpha}. \] (30)

Using Eq. (3), together with Eq. (11), we can now eliminate the unknown direction \( \theta_f \) by expanding the probability amplitudes \( \chi(\theta_a^+, \theta_b^+) \) and \( \chi(\theta_a^-, \theta_b^-) \) over the states corresponding to it. Thus, we obtain the generalized probability amplitudes

\[ \chi(\theta_a^+, \theta_b^+) = \cos \theta_a \cos \theta_b + \sin \theta_a \sin \theta_b e^{i(\alpha_a - \alpha_b)}, \] (31)

\[ \chi(\theta_a^+, \theta_b^-) = -\cos \theta_a \sin \theta_b + \sin \theta_a \cos \theta_b e^{i(\alpha_a - \alpha_b)}, \] (32)

\[ \chi(\theta_a^-, \theta_b^+) = -\sin \theta_a \cos \theta_b + \cos \theta_a \sin \theta_b e^{i(\alpha_a - \alpha_b)} \] (33)
\[ \chi(\theta_a, \theta_b) = \sin \theta_a \sin \theta_b + \cos \theta_a \cos \theta_b e^{i(\alpha_a - \alpha_b)} . \] (34)

In labelling the probability amplitudes, we have assumed that \( \theta_a \) refers to the initial state. This is for two reasons. First, we used a similar approach when we treated spin by this approach. On that occasion, we found that the angle which explicitly appeared in the formulas for the standard generalized results indeed referred to the initial state. Second, the standard treatment defines only the initial state, and so the angle which appears in the formulas must refer to that state.

We observe that by setting \( \theta_b = \theta_f = 0 \), we obtain the standard generalized results for the probability amplitudes. However, we now recognise that these are not so generalized after all, and are only a special case of the more generalized results Eqs. (31) - (34).

The probabilities corresponding to these probability amplitudes are

\[ P(\theta_a^+, \theta_b^+) = |\chi(\theta_a^+, \theta_b^+)|^2 \]
\[ = \cos^2 \theta_a \cos^2 \theta_b + \sin^2 \theta_a \sin^2 \theta_b + \frac{1}{2} \sin 2\theta_a \sin 2\theta_b \cos(\alpha_a - \alpha_b) \] (35)

and

\[ P(\theta_a^+, \theta_b^-) = P(\theta_a^+, \theta_b^-) \]
\[ = \cos^2 \theta_a \sin^2 \theta_b + \sin^2 \theta_a \cos^2 \theta_b - \frac{1}{2} \sin 2\theta_a \sin 2\theta_b \cos(\alpha_a - \alpha_b) \] (36)

4.2 Matrix Operators and States

Consider the quantity \( R(\theta_b) \), which is measured at the same time as the polarization direction and whose values depend on the direction of polarization. The value of \( R(\theta_b) \) when the polarization is found to be parallel to \( \theta_b \) is denoted by \( R_+ \), while the value when the polarization is found perpendicular to \( \theta_b \) is called \( R_- \). Suppose further that the polarization is initially known to be along \( \theta_a \). Then the expectation value of \( R(\theta_b) \) is

\[ \langle R(\theta_b) \rangle = |\chi(\theta_a^+, \theta_b^+)|^2 R_+ + |\chi(\theta_a^+, \theta_b^-)|^2 R_- \] (38)

We expand \( \chi(\theta_a^+, \theta_b^+) \) and \( \chi(\theta_a^+, \theta_b^-) \) thus:

\[ \chi^*(\theta_a^+, \theta_b^+) = \chi^*(\theta_a^+, \theta_b^+) \chi^*(\theta_a^+, \theta_b^+) + \chi^*(\theta_a^+, \theta_b^-) \chi^*(\theta_a^+, \theta_b^-) \] (39)

and

\[ \chi(\theta_a^+, \theta_b^+) = \chi(\theta_a^+, \theta_b^+) \chi(\theta_a^+, \theta_b^+) + \chi(\theta_a^+, \theta_b^-) \chi(\theta_a^+, \theta_b^-) . \] (40)

Hence,
\[ \langle R(\theta_b) \rangle = \left( \chi_{11} \, \chi_{12} \right) \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} \left( \chi_{11} \, \chi_{12} \right) \]

\[ = [\chi_+]^\dagger [R] [\chi_+], \quad (41) \]

where

\[ [\chi_+] = \begin{pmatrix} \chi_{11} \\ \chi_{12} \end{pmatrix}, \quad (42) \]

and

\[ [R] = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}. \quad (43) \]

Here,

\[ \chi_{11} = \chi(\theta_a^+, \theta_c^+) \quad \text{and} \quad \chi_{12} = \chi(\theta_a^+, \theta_c^-), \quad (44) \]

while

\[ R_{11} = \chi^*(\theta_a^+, \theta_b^+)\chi(\theta_a^+, \theta_b^+)R_+ + \chi^*(\theta_a^+, \theta_b^-)\chi(\theta_a^+, \theta_b^-)R_-, \quad (45) \]

\[ R_{12} = \chi^*(\theta_a^-, \theta_b^+)\chi(\theta_a^-, \theta_b^-)R_+ + \chi^*(\theta_a^-, \theta_b^-)\chi(\theta_a^-, \theta_b^-)R_-, \quad (46) \]

\[ R_{21} = \chi^*(\theta_a^+, \theta_b^+)\chi(\theta_a^-, \theta_b^+)R_+ + \chi^*(\theta_a^-, \theta_b^-)\chi(\theta_a^-, \theta_b^-)R_-, \quad (47) \]

and

\[ R_{22} = \chi^*(\theta_a^-, \theta_b^+)\chi(\theta_a^-, \theta_b^-)R_+ + \chi^*(\theta_a^-, \theta_b^-)\chi(\theta_a^-, \theta_b^-)R_. \quad (48) \]

If on the other hand the initial state is such that the polarization is at right angles to \( \theta_a \), the state that appears in Eq. (41) becomes

\[ [\chi_-] = \begin{pmatrix} \chi_{21} \\ \chi_{22} \end{pmatrix} = \begin{pmatrix} \chi(\theta_a^-, \theta_c^+) \\ \chi(\theta_a^-, \theta_c^-) \end{pmatrix}. \quad (49) \]

Using the generalized probability amplitudes Eqs. (31) - (34), we find that the generalized polarization states, denoted by \([\chi_{\alpha_a}^\pm]\), are

\[ [\chi_{\alpha_a}^+] = \begin{pmatrix} \cos \theta_a \cos \theta_c + \sin \theta_a \sin \theta_c e^{i(\alpha_a-\alpha_c)} \\ -\cos \theta_a \sin \theta_c + \sin \theta_a \cos \theta_c e^{i(\alpha_a-\alpha_c)} \end{pmatrix}, \quad (50) \]

and

\[ [\chi_{\alpha_a}^-] = \begin{pmatrix} -\sin \theta_a \cos \theta_c + \cos \theta_a \sin \theta_c e^{i(\alpha_a-\alpha_c)} \\ \sin \theta_a \sin \theta_c + \cos \theta_a \cos \theta_c e^{i(\alpha_a-\alpha_c)} \end{pmatrix}. \quad (51) \]

In order to obtain Eqs. (50) and (51) \( b \) by \( c \) in Eqs. (31) - (34). The states are orthogonal and are each normalized to unity.
Using the expressions Eqs. (31) - (34) for the generalized probability amplitudes with the arguments changed appropriately, so that \( \theta_a \rightarrow \theta_c \) and \( \theta_b \) remains unchanged, we obtain

\[
R_{11} = \left[ \cos^2 \theta_c \cos^2 \theta_b + \sin^2 \theta_c \sin^2 \theta_b + \frac{1}{2} \sin 2\theta_c \sin 2\theta_b \cos (\alpha_c - \alpha_b) \right] R_+ + \left[ \sin^2 \theta_c \cos^2 \theta_b - \frac{1}{2} \sin 2\theta_c \sin 2\theta_b \cos (\alpha_c - \alpha_b) \right] R_-,
\]

\( \text{(52)} \)

\[
R_{12} = \left[ -\frac{1}{2} \sin 2\theta_c \cos 2\theta_b + \frac{1}{2} \sin 2\theta_b \cos 2\theta_c \cos (\alpha_c - \alpha_b) \right] + \frac{1}{2} \sin 2\theta_c \cos 2\theta_b \cos (\alpha_c - \alpha_b) + \frac{i}{2} \sin 2\theta_b \sin (\alpha_c - \alpha_b) \]

\( \text{and} \)

\[
R_{21} = \left[ -\frac{1}{2} \sin 2\theta_c \cos 2\theta_b + \frac{1}{2} \sin 2\theta_b \cos 2\theta_c \cos (\alpha_c - \alpha_b) \right] + \frac{i}{2} \sin 2\theta_b \sin (\alpha_c - \alpha_b) \]

\( \text{(53)} \)

\[
R_{22} = \left[ \sin^2 \theta_c \cos^2 \theta_b + \cos^2 \theta_c \sin^2 \theta_b - \frac{1}{2} \sin 2\theta_c \sin 2\theta_b \cos (\alpha_c - \alpha_b) \right] R_+ + \left[ \sin^2 \theta_c \sin^2 \theta_b + \cos^2 \theta_c \cos^2 \theta_b + \frac{1}{2} \sin 2\theta_c \sin 2\theta_b \cos (\alpha_c - \alpha_b) \right] R_-.
\]

\( \text{(54)} \)

4.3 Polarization Operator and its Eigenvectors

Suppose that the quantity \( R \) is the polarization itself, which we denote by \( p \). We agree that if measurement finds the polarization parallel to \( \theta_b \), we assign the value +1 to this observable. If the polarization is found perpendicular to \( \theta_b \), we assign the value -1. Then the operator Eq. (31) is found have the elements

\[
p_{11} = \cos 2\theta_c \cos 2\theta_b + \sin 2\theta_c \sin 2\theta_b \cos (\alpha_c - \alpha_b),
\]

\( \text{(56)} \)

\[
p_{12} = -\sin \theta_c \cos \theta_b + \cos 2\theta_c \sin 2\theta_b \cos (\alpha_c - \alpha_b) + i \sin 2\theta_b \sin (\alpha_c - \alpha_b),
\]

\( \text{(57)} \)
\[ p_{21} = -\sin \theta_c \cos \theta_b + \cos 2\theta_c \sin 2\theta_b \cos (\alpha_c - \alpha_b) - i \sin 2\theta_b \sin (\alpha_c - \alpha_b) \] (58)

and

\[ p_{22} = -\cos 2\theta_c \cos 2\theta_b - \sin 2\theta_c \sin 2\theta_b \cos (\alpha_c - \alpha_b). \] (59)

According to the reasoning in Section 3, the eigenvectors of this operator are

\[ [\xi_{\pm}] = \begin{pmatrix} 
\chi(\theta^+_b, \theta^+_c) \\
\chi(\theta^-_b, \theta^-_c)
\end{pmatrix} = \begin{pmatrix} 
cos \theta_b \cos \theta_c + \sin \theta_b \sin \theta_c e^{i(\alpha_b - \alpha_c)} \\
-\cos \theta_b \sin \theta_c + \sin \theta_b \cos \theta_c e^{i(\alpha_b - \alpha_c)}
\end{pmatrix} \] (60)

for eigenvalue \( \lambda = 1 \), and

\[ [\xi_{-}] = \begin{pmatrix} 
\chi(\theta^-_b, \theta^+_c) \\
\chi(\theta^-_b, \theta^-_c)
\end{pmatrix} = \begin{pmatrix} 
-\sin \theta_b \cos \theta_c + \cos \theta_b \sin \theta_c e^{i(\alpha_b - \alpha_c)} \\
\sin \theta_b \sin \theta_c + \cos \theta_b \cos \theta_c e^{i(\alpha_b - \alpha_c)}
\end{pmatrix} \] (61)

for \( \lambda = -1 \). Direct calculation shows that indeed, the eigenvalue equations

\[ [p][\xi_{\pm}] = \pm [\xi_{\pm}] \] (62)

are satisfied.

### 4.4 Expectation Values

The expectation value of the polarization operator itself is now easily obtained. Suppose that the polarization state that obtains before measurement corresponds to polarization in the direction \( \theta_a \), and the polarization is measured in the direction \( \theta_b \). Then this expectation value is

\[
\langle p \rangle_+ = P(\theta^+_a, \theta^+_b)(+1) + P(\theta^-_a, \theta^-_b)(-1) = \cos 2\theta_a \cos 2\theta_b + \sin 2\theta_a \sin 2\theta_b \cos (\alpha_a - \alpha_b)
\] (63)

Similarly,

\[
\langle p \rangle_- = P(\theta^-_a, \theta^+_b)(+1) + P(\theta^-_a, \theta^-_b)(-1) = -\cos 2\theta_a \cos 2\theta_b - \sin 2\theta_a \sin 2\theta_b \cos (\alpha_a - \alpha_b)
\] (64)

for the cases when the polarization is initially parallel and perpendicular to the angle \( \theta_a \) respectively.

### 4.5 Connection With Standard Formulas

The standard results are obtained from the current generalized results by setting the final angle equal to zero. Thus if we set \( \theta_b = 0, \alpha_b = 0 \) in Eqs. (59) - (60), we obtain the standard formulas

\[ \chi^+_a = \cos \theta_a \] (65)
\[ \chi_a^- = \sin \theta_a e^{i\alpha_a}, \quad (66) \]

The other two formulas which result from Eqs. (33) - (34) can be obtained from Eqs. (65) - (66) by having \( \theta_a \to \theta_a + \pi/2 \), because this is the condition that defines a state orthogonal to the one corresponding to \( \theta_a \). Thus, we get

\[ \chi_{a\perp}^+ = -\sin \theta_a \]

and

\[ \chi_{a\perp}^- = \cos \theta_a e^{i\alpha_a} \]

where we have used the notation \( a \perp \) to indicate the quantities corresponding to the direction perpendicular to \( \theta_a \).

The polarization states in this limit are

\[ [\chi_{\theta_a}^+] = \begin{pmatrix} \cos \theta_a \\ -\sin \theta_a e^{i\alpha_a} \end{pmatrix} \quad (69) \]

and

\[ [\chi_{\theta_a}^-] = \begin{pmatrix} -\sin \theta_a \\ \cos \theta_a e^{i\alpha_a} \end{pmatrix} \quad (70) \]

In order to obtain the standard polarization operator and its eigenvectors, we have to set \( \theta_c = 0 \), \( \alpha_c = 0 \) in Eqs. (56) - (61) because this is the angle that now corresponds to the final direction. The polarization operator then becomes

\[ [p] = \begin{pmatrix} \cos 2\theta_b & \sin \theta_b e^{i(\alpha_a - \alpha_b)} \\ \sin \theta_b e^{-i(\alpha_a - \alpha_b)} & -\cos 2\theta_b \end{pmatrix} \quad (71) \]

while its eigenvectors become

\[ [\xi_+] = \begin{pmatrix} \cos \theta_b \\ \sin \theta_b e^{i(\alpha_b - \alpha_c)} \end{pmatrix} \quad (72) \]

for eigenvalue +1 and

\[ [\xi_-] = \begin{pmatrix} -\sin \theta_b \\ \cos \theta_b e^{i(\alpha_b - \alpha_c)} \end{pmatrix} \quad (73) \]

for eigenvalue -1.

### 5 Conclusion

In this paper, we have presented generalized formulas for the treatment of polarization. These should prove useful in the description of phenomena in which polarization plays a part by giving us increased freedom of description. Particularly noteworthy in this regard are the formulas for the elements of the
matrix form of the operator of any observable whose eigenvalues depend on the polarization.

The results in this paper demonstrate once again the utility of the approach to quantum mechanics due to Landé. The debate on the interpretation of quantum mechanics continues unabated, and sometimes ascends to the metaphysical. At other times it seems to be redundant, as when variants of the theory are produced which yield only the standard results while allowing a different interpretation of the foundations. The present approach at least produces generalizations and new results which other approaches appear unable to do. To this extent, it justifies itself. We have no doubt that these generalizations will in future lead to important advances in both the understanding and the application of quantum mechanics.

6 References

1. Mweene H. V., "Derivation of Spin Vectors and Operators From First Principles", quant-ph/9905012
2. Mweene H. V., "Generalized Spin-1/2 Operators and Their Eigenvectors", quant-ph/9906002
3. Mweene H. V., "Vectors and Operators for Spin 1 Derived From First Principles", quant-ph/9906043
4. Mweene H. V., "Alternative Forms of Generalized Vectors and Operators for Spin 1/2", quant-ph/9907031
5. Mweene H. V., "Spin Description and Calculations in the Landé Interpretation of Quantum Mechanics", quant-ph/9907033
6. Mweene H. V., "A New Approach to the Treatment of Systems of Compounded Angular Momentum", quant-ph/9907082
7. Landé A., "From Dualism To Unity in Quantum Physics", Cambridge University Press, 1960.
8. Landé A., "New Foundations of Quantum Mechanics", Cambridge University Press, 1965.
9. Landé A., "Foundations of Quantum Theory," Yale University Press, 1955.
10. Landé A., "Quantum Mechanics in a New Key," Exposition Press, 1973.