Structure in Dichotomous Preferences

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Abstract

Many hard computational social choice problems are known to become tractable when voters’ preferences belong to a restricted domain, such as those of single-peaked or single-crossing preferences. However, to date, all algorithmic results of this type have been obtained for the setting where each voter’s preference list is a total order of candidates. The goal of this paper is to extend this line of research to the setting where voters’ preferences are dichotomous, i.e., each voter approves a subset of candidates and disapproves the remaining candidates. We propose several analogues of the notions of single-peaked and single-crossing preferences for dichotomous profiles and investigate the relationships among them. We then demonstrate that for some of these notions the respective restricted domains admit efficient algorithms for computationally hard approval-based multi-winner rules.

1 Introduction

Preference aggregation is a fundamental problem in social choice, which has recently received a considerable amount of attention from the AI community. In particular, an important research question in computational social choice [Brandt et al., 2015] is the complexity of computing the output of various preference aggregation procedures. While for most common single-winner rules winner determination is easy, many attractive rules that output a committee (a fixed-size set of winners) or a ranking of the candidates are known to be computationally hard.

There are several ways to circumvent these hardness results, such as using approximate and parameterized algorithms. These standard algorithmic approaches are complemented by an active stream of research that analyzes the computational complexity of voting rules on restricted preference domains, such as the classic domains of single-peaked [Black, 1958] or single-crossing [Mirrlees, 1971] preferences. This research direction was popularized by Walsh [2007] and Faliszewski et al. [2011], and has lead to a number of efficient algorithms for winner determination under prominent voting rules as well as for manipulation and control, which can be used when voters’ preferences belong to one of these restricted domains [Walsh, 2007, Faliszewski et al., 2011, Brandt et al., 2010, Faliszewski et al., 2014, Betzler et al., 2013, Skowron et al., 2015b, Magiera and Faliszewski, 2014].

To the best of our knowledge, this line of work only considers settings where voters’ preferences are given by total orders over the set of candidates; indeed, this is perhaps the most widely studied setting in the area of computational social choice. However, computationally complex preference aggregation problems may also arise when voters’ preferences are dichotomous, i.e., each voter approves a subset of the candidates and disapproves the remaining candidates. Committee selection rules for voters with dichotomous preferences, or approval-based rules, have recently attracted some attention from the computational social choice community, and for two prominent such rules (specifically, Proportional Approval Voting (PAV) [Kilgour and Marshall, 2012] and Maximin Approval Voting (MAV) [Brams et al., 2007]) computing the winning committee is known to be NP-hard [Aziz et al., 2014, LeGrand et al., 2007]. It is therefore natural to ask if one could identify a suitable analogue of single-peaked/single-crossing preferences for the dichotomous setting, and design efficient algorithms for approval-based rules over such restricted dichotomous preference domains.

To address this challenge, in this paper we propose and explore a number of domain restrictions for dichotomous preferences that build on the same intuition as the concepts of single-peakedness and single-crossingness. Some of our restricted domains are defined by embedding voters or candidates into the real line, and requiring that the voters’ preferences over the candidates “respect” this embedding; others are obtained by viewing dichotomous preferences as weak orders and requiring them to admit a refinement that has a desirable structural property. Surprisingly, these approaches lead to a large number of concepts that are pairwise non-equivalent and capture different aspects of our intuition about what it means for preferences to be “one-dimensional”. We analyze the relationships among these restricted preference domains, (see Figure 5 for a summary), and discuss the complexity of detecting whether a given dichotomous profile belongs to one of these domains. We then demonstrate that considering these domains is useful from the perspective of algorithm design, by providing polynomial-time and FPT algorithms for PAV and MAV under some of these domain restrictions.
2 Basic Definitions

Let \( C = \{c_1, \ldots, c_m\} \) be a finite set of candidates. A partial order \( \succ \) over \( C \) is a reflexive, antisymmetric and transitive binary relation on \( C \); a partial order \( \succ \) is said to be total if for each \( c, d \in C \) we have \( c \succ d \) or \( d \succ c \). We say that a partial order \( \succ \) over \( C \) is a dichotomous weak order if \( C \) can be partitioned into two disjoint sets \( C^+ \) and \( C^- \) (one of which may be empty) so that \( c \succ d \) for each \( c \in C^+, d \in C^- \) and the candidates within \( C^+ \) and \( C^- \) are incomparable under \( \succ \).

An approval vote on \( C \) is an arbitrary subset of \( C \). We say that an approval vote \( v \) is trivial if \( v = \emptyset \) or \( v = C \). A dichotomous profile \( P = (v_1, \ldots, v_n) \) is a list of \( n \) approval votes; we will refer to \( v_i \) as the vote of voter \( i \). We write \( \pi_i = C \setminus v_i \). We associate an approval vote \( v_i \) with the dichotomous weak order \( \succ_i \) that satisfies \( c \succ_i d \) if and only if \( c \in v_i \) and \( d \in \pi_i \). Note that \( v_i = \emptyset \) and \( v_i = C \) correspond to the same dichotomous weak order, namely the empty one.

A partial order \( \succ' \) over \( C \) is a refinement of a partial order \( \succ \) over \( C \) if for every \( c, d \in C \) it holds that \( c \succ d \) implies \( c \succ' d \). A profile \( P' = (\succ_1, \ldots, \succ_n) \) of total orders is a refinement of a dichotomous profile \( P = (v_1, \ldots, v_n) \) if \( \succ_i \) is a refinement of \( \succ_{i'} \) for each \( i, i' \in \{1, \ldots, n\} \).

Let \( \prec \) be a total order over \( C \). A total order \( \succ \) over \( C \) is said to be single-peaked with respect to \( \prec \) if for any triple of candidates \( a, b, c \in C \) with \( a \prec b \prec c \) or \( c \prec b \prec a \) it holds that \( a \succ b \) implies \( b \succ c \). A profile \( P \) of total orders over \( C \) is said to be single-peaked if there exists a total order \( \prec \) over \( C \) such that all orders in \( P \) are single-peaked with respect to \( \prec \).

A profile \( P = (\succ_1, \ldots, \succ_n) \) of total orders over \( C \) is said to be single-crossing with respect to the given order of votes if for every pair of candidates \( a, b \in C \) such that \( a \succ b \) all votes where \( a \) is preferred to \( b \) precede all votes where \( b \) is preferred to \( a \); \( P \) is single-crossing if the votes in \( P \) can be permuted so that it becomes single-crossing with respect to the resulting order of votes.

A profile \( P = (\succ_1, \ldots, \succ_n) \) of total orders over \( C \) is said to be 1-Euclidean if there is a mapping \( \rho \) of voters and candidates into the real line such that \( c \succ_{i'} d \) if and only if \( |\rho(i) - \rho(c)| < |\rho(i) - \rho(d)| \). A 1-Euclidean profile is both single-peaked and single-crossing.

3 Preference Restrictions

We will now define a number of constraints that a dichotomous profile may satisfy. Most of these constraints can be divided into two basic groups: those that are based on ordering voters/candidates on the line and requiring the votes to respect this order (this includes VEI, VI, CEI, CI, DE, and DUE), and those that are based on viewing votes as weak orders and asking if there is a single-peaked/single-crossing/1-Euclidean profile of total orders that refines the given profile (this includes PSP, PSC, and PE); we remark that the study of the latter type of constraints was initiated by Lackner [2014]. We will also consider constraints that are based on partitioning voters/candidates (2PART and PART), as well as two constraints (WSC and SSC) that have been introduced in a recent paper of Elkind et al. [2015] in order to understand the best way of extending the single-crossing property to weak orders.

Fix a profile \( \mathcal{P} = (v_1, \ldots, v_n) \) over \( C \).
6. **Candidate Interval (CI):** We say that \( \mathcal{P} \) satisfies CI if candidates in \( C \) can be ordered so that each of the sets \( v_i \) forms an interval of that ordering. See Figure 4 for an example.

7. **Dichotomous Uniformly Euclidean (DUE):** We say that \( \mathcal{P} \) satisfies DUE if there is a mapping \( \rho \) of voters and candidates into the real line and a radius \( r \) such that for every voter \( i \) it holds that \( v_i = \{ c : |\rho(i) - \rho(c)| \leq r \} \).

8. **Dichotomous Euclidean (DE):** We say that \( \mathcal{P} \) satisfies DE if there is a mapping \( \rho \) of voters and candidates into the real line such that for every voter \( i \) there exists a radius \( r_i \) with \( v_i = \{ c : |\rho(i) - \rho(c)| \leq r_i \} \).

9. **Possibly single-peaked (PSP):** We say that \( \mathcal{P} \) satisfies PSP if there is a single-peaked profile of total orders \( \mathcal{P}' \) that is a refinement of \( \mathcal{P} \).

10. **Possibly single-crossing (PSC):** We say that \( \mathcal{P} \) satisfies PSC if there is a single-crossing profile of total orders \( \mathcal{P}' \) that is a refinement of \( \mathcal{P} \).

11. **Possibly Euclidean (PE):** We say that \( \mathcal{P} \) satisfies PE if there is a 1-Euclidean profile of total orders \( \mathcal{P}' \) that is a refinement of \( \mathcal{P} \).

12. **Seemingly single-crossing (SSC):** We say that \( \mathcal{P} \) satisfies SSC if the voters in \( \mathcal{P} \) can be reordered so that for each pair of candidates \( a, b \) \( \in C \) it holds that either all votes \( v_i \) with \( a \in v_i, b \not\in v_i \) precede all votes \( v_j \) with \( a \not\in v_j, b \in v_j \) or vice versa.

13. **Weakly single-crossing (WSC):** We say that \( \mathcal{P} \) satisfies WSC if the voters in \( \mathcal{P} \) can be reordered so that for each pair of candidates \( a, b \) \( \in C \) it holds that each of the vote sets \( V_1 = \{ v_i : a \in v_i, b \not\in v_i \} \), \( V_2 = \{ v_i : a \not\in v_i, b \in v_i \} \), \( V_3 = \{ v \in \mathcal{P} : v \not\in V_1 \cup V_2 \} \) forms an interval of this ordering, with \( V_3 \) appearing between \( V_1 \) and \( V_2 \).

### 3.1 Relations

The relationships among the properties defined above are depicted in Figure 5, where arrows indicate containment, i.e., more restrictive notions are at the top. All these containments are strict.

The four arrows at the top level of the diagram are immediate: any profile with at most two distinct votes where each candidate is approved in at least one of these votes satisfies VEI, CEI and WSC, and by definition 2PART is a special case of PART.

To understand the arrows in the next level, we first characterize the dichotomous profiles that are weakly single-crossing.

![Figure 4: Candidate Interval](image)

![Figure 5: Relations between notions of structure. Dashed lines indicate that the respective containment holds only subject to additional conditions.](image)

**Lemma 1.** A dichotomous profile \( \mathcal{P} \) satisfies WSC if and only if there exist three votes \( u, v, w \) such that

1. For every \( v_i \in \mathcal{P} \), it holds that \( v_i \in \{ v_i, v, w \} \), and
2. \( v_i \succ v \) is equal to \( v_i \succ w \) or \( v_i \succ u \).

**Proof sketch.** It is easy to check that every profile satisfying (1)–(2) satisfies WSC. For the converse direction, assume without loss of generality that the ordering of the votes \( v_1 \prec v_2 \prec \cdots \prec v_n \) witnesses that \( \mathcal{P} \) satisfies WSC. Let \( u = v_1, w = v_n, \) and set \( C_1 = u \cap w, C_2 = u \cap \pi, C_3 = \pi \cap w, C_4 = \pi \cap \pi \). The WSC property implies that for every \( \ell = 1, 2, 3, 4 \), every \( a, b \in C_\ell \), and every \( v_i \in \mathcal{P} \) we have \( a \in v_i \) if and only if \( b \in v_i \), i.e., candidates in each \( C_\ell \) occur as a block in all votes. Note that \( v_1 = u \cup C_2, v_n = w = C_1 \cup C_3 \).

Suppose that \( C_1, C_4 \neq \emptyset \). Then \( C_1 \subseteq v_i, C_4 \subseteq v_i \) for all \( v_i \in \mathcal{P} \). Indeed, fix a pair of candidates \( a \in C_1, b \in C_4 \). Both the first and the last voter strictly prefer \( a \) to \( b \), and therefore so do all other voters. Thus, if \( \mathcal{P} \) contains a vote \( v_i \neq u, w \), it has to be the case that \( v_i = C_1 \cup u \cap w \) or \( v_i = C_1 \cup C_2 \cup C_3 = u \cup w \); moreover, if both of these votes occur simultaneously and are distinct from each other and \( u, w \) (i.e., \( C_2, C_3 \neq \emptyset \)), the WSC property is violated. Indeed, suppose that \( v_1 = C_1, v_2 = C_1 \cup C_2 \cup C_3 \). Fix candidates \( a \in C_1, b \in C_4 \). If \( v_i \) appears before \( v_j \), consider a candidate \( c \in C_2 \); we get a contradiction as voters \( v_i \) and \( v_j \) are indifferent between \( a \) and \( b \), but \( v_j \) strictly prefers \( a \) to \( b \). When \( C_1 \) or \( C_4 \) is empty, the analysis is similar; note, however, that trivial votes \( v_i = C_1 \) and \( v_i = \emptyset \) may alternate arbitrarily without violating the WSC property (this is why the lemma is stated in terms of weak orders rather than approval votes).

We can now show that under mild additional conditions (no trivial voters/candidates) WSC implies VEI and CEI.

**Proposition 2.** Let \( \mathcal{P} \) be a dichotomous profile that either contains only two distinct votes or contains no vote \( v_i \) with \( v_i = \emptyset \). If \( \mathcal{P} \) satisfies WSC, then it satisfies VEI.
Proposition 4. A dichotomous profile is WSC, CEI and VEI if and only if it is a 2-partition profile.

Proof. It is immediate that a 2-partition profile is WSC, CEI, and VEI. For the converse direction, let \( \mathcal{P} \) be a CEI, VEI and WSC profile. By Lemma 1 \( \mathcal{P} \) contains at most three distinct votes \( u, v, w \) with \( v = u \cup w \) or \( v = u \cap w \). Since \( \mathcal{P} \) is CEI, we know from Lemma 3 that every candidate is approved at least once. Hence \( u \cup w = C \). Furthermore, every candidate is disapproved at least once. Thus, \( u \cap w = \emptyset \), since this intersection is also approved by \( v \). Thus, \( v \) is a trivial vote. This is possible because of Lemma 3 and hence \( v \) does not appear in \( \mathcal{P} \). We have shown that \( \mathcal{P} \) is a 2-partition profile.

Next, we will relate CEI and VEI to DUE.

Proposition 5. If a dichotomous profile \( \mathcal{P} \) satisfies CEI or VEI, then it satisfies DUE.

Proof. Suppose first that \( \mathcal{P} \) satisfies CEI with respect to the ordering \( c_1 < \cdots < c_m \) of candidates. Map the candidates into the real line by setting \( \rho(c_i) = i \), and let \( r = m \). We can now place each voter \( i \) to the left or to the right of all candidates at an appropriate distance so that the set of candidates within distance \( r \) from him coincides with \( v_i \). For VEI the argument is similar: if \( \mathcal{P} \) satisfies VEI with respect to the ordering \( v_1 \in \cdots \in v_n \) of voters, we place voters on the real line according to \( \rho(i) = i \), let \( r = n \), and place each candidate to the left or to the right of all voters at an appropriate distance.

The proof that WSC implies DUE is also based on our characterization of WSC preferences.

Proposition 6. If a dichotomous profile \( \mathcal{P} \) satisfies WSC, then it satisfies DUE.

Proof. Clearly empty votes can be ignored when checking whether a profile satisfies DUE, so assume \( \mathcal{P} \) contains empty votes. Then it contains at most three distinct votes \( u, v, w \) with \( v = u \cap w \) or \( v = u \cup w \). Let \( \rho(c) = 1 \) for \( c \in u \cap w \), \( \rho(c) = 2 \) for \( c \in u \cup w \), \( \rho(c) = 3 \) for \( c \in w \setminus u \), \( \rho(c) = 10 \) for \( c \notin u \cup w \). We set \( r = 1 \) if \( v = u \cap w \) and \( r = 2 \) if \( v = u \cup w \), and position the voters accordingly.

The last arrow on this level is from PART to DUE: here, the containment is straightforward, as the candidates approved by each voter can be placed as a block on the axis, with the respective voter(s) placed in the center of this block.

Proposition 7. If a dichotomous profile \( \mathcal{P} \) satisfies DUE then it satisfies both VI and CI.

Proof. Since \( \mathcal{P} \) satisfies DUE, we have an embedding \( \rho \) of votes and candidates into the real line. For VI, we order voters as induced by the \( \rho \) mapping; the voters approving some candidate form an interval on this induced order. For CI, we order candidates as induced by the \( \rho \) mapping; voters always approve a single interval on this ordering.

It is perhaps more surprising that the classes of CI, DE, PSP and PE preferences coincide.

Proposition 8. Let \( \mathcal{P} \) be a dichotomous profile. Then the following conditions are equivalent: (a) \( \mathcal{P} \) satisfies PE (b) \( \mathcal{P} \) satisfies PSP (c) \( \mathcal{P} \) satisfies CI (d) \( \mathcal{P} \) satisfies DE.

Proof sketch. Suppose \( \mathcal{P} \) satisfies PE, and let \( \mathcal{P}' \) be a refinement of \( \mathcal{P} \) that, together with a mapping \( \rho \), witnesses this. Then \( \mathcal{P}' \) is single-peaked and therefore \( \mathcal{P} \) satisfies PSP. If \( \mathcal{P} \) satisfies PSP, as witnessed by a refinement \( \mathcal{P}' \) and an axis \( < \), then \( \mathcal{P} \) satisfies CI with respect to \( < \). If \( \mathcal{P} \) satisfies CI with respect to an order \( < \) of candidates, we can map the candidates into the real axis in the order suggested by \( < \) so that the distance between every two adjacent candidates is 1. We can then choose an appropriate approval radius and position for each voter. Finally, if \( \mathcal{P} \) satisfies DE, as witnessed by a mapping \( \rho \), we can use this mapping to construct a refinement of \( \mathcal{P} \); by construction, this refinement is 1-Euclidean (we may have to modify \( \rho \) slightly to avoid ties).
Also, every PE profile is PSC since every 1-Euclidean refinement is also single-crossing. Interestingly, the converse is not true.

**Example 1.** Consider the profile $\mathcal{P} = \{(a, b), (a, c), (b, c)\}$ over $C = \{a, b, c\}$. It satisfies PSC, as witnessed by the single-crossing refinement $(a \succ b \succ c, c \succ a \succ b, b \succ c \succ a)$. However, in every refinement of $\mathcal{P}$ the first voter ranks $c$ last, the second voter ranks $b$ last, and the third voter ranks $a$ last. Thus, no such refinement can be single-peaked, and, consequently, no such refinement can be 1-Euclidean.

The equivalence between PSC and SSC is not entirely obvious; while it is clear that a profile that violates SSC also violates PSC, to prove the converse one needs to use an argument similar to the proof of Theorem 4 in Elkind et al., 2015. This has been shown in the extended version of Elkind et al., 2015.

**Proposition 9.** If a dichotomous profile $\mathcal{P}$ satisfies VI, it also satisfies SSC.

**Proof.** Assume that an VI profile is not SSC. Since it is not SSC, for every ordering of votes $\subseteq$ there are two candidates $a \succ b$ and votes $v_i \sqsubset v_j \sqsubset v_k$ such that $v_i : a \succ b$, $v_j : b \succ a$ and $v_k : a \succ b$. This implies, however, that for every $\subseteq$ there is a candidate $a$ and votes $v_i \sqsubset v_j \sqsubset v_k$ such that $v_i$ and $v_k$ approve of $a$ and $v_j$ disapproves $a$. This contradicts our assumption that the given profile is VI.

We are now going to list the remaining counter-examples for containment and thus show that the arrows in Figure 5 indeed indicate strict containment.

- CI $\not\supset$ VI: Consider $\{(a, b), (a, c), (b, c)\}$. This profile is CI with respect to $a \prec b \prec c$. It is not VI since the vote $\{a, b, c\}$ would have to be placed next to $\{a\}, \{b\}, \{c\}$.

- VI $\not\supset$ CI: Consider $\{(a, b), (a, c), (a, d)\}$. This profile is VI for the given order of voters. It is not CI since $a$ has to lie next to $b, c, d$.

- VEI $\not\supset$ CEI: Consider $\{(a, b), (a, d), (c, d)\}$. This profile is VEI for the given order of candidates. It is not CEI since $a$ has to lie next to $b$ and $c$ has to lie next to $d$. So $b \prec a \prec d \prec c$ is the only order witnessing CI, but the vote $\{a, d\}$ is not an extremal interval on this order.

- CEI $\not\supset$ VEI: Consider $\{(a, b), (a, c), (b, c)\}$. All votes are extremal intervals on the order $a \prec b \prec c$. The profile is however not VEI since $\{a, b\}$ has to lie next to $\{a\}$ and next to $\{b, c\}$ and $\{c\}$ next to $\{b, c\}$. So we obtain $\{c\} \sqsubseteq \{b, c\} \sqsubseteq \{a, b\} \sqsubseteq \{a\}$ as the only order witnessing VI, but this order does not satisfy VEI (consider candidate $b$).

- PART $\not\supset$ VEI, CEI, WSC: Consider the PART profile $\{a\}, \{b\}, \{c\}$.

All other counterexamples involving WSC immediately follow from Lemma 1 and Proposition 2 and all missing counterexamples involving PART can be obtained by picking intersecting votes.

### 3.2 Unique orders

If voter’s preferences are given by total orders, single-crossing profiles have a unique single-crossing order, i.e., only one specific order and its reverse witness the single-crossing property of the profile. For single-peaked profiles (of total orders) this is not the case. The question arises whether a similar phenomenon can be observed for dichotomous profiles. Clearly, this question only makes sense for profiles with distinct votes (for VI, VEI, WSC) and when all candidates are approved by some vote (for CI and CEI). Also, by unique we always mean that only one specific order and its reverse witness a certain restriction.

For dichotomous profiles satisfying SC, there is no unique order. The profile $\{(a), \{a, b\}, \{b, c\}\}$ is SC and all votes that put $\{b, c\}$ at an outermost position witness the SC property. Also profiles satisfying VI or CEI do not have unique orders witnessing these properties: e.g., consider $\\emptyset, \{a\}, \{b\}$ and $\{a\}, \{b\}, \{c\}$, respectively.

For profiles being WSC, VEI or CEI we can show that their corresponding orders are indeed unique. For profiles satisfying WSC, this follows from Lemma 1 for profiles satisfying either VEI or CEI the uniqueness can be shown as follows.

**Lemma 10.** For profiles containing distinct votes, VEI orders are unique.

**Proof.** Without loss of generality assume that $1 \sqsubset \cdots \sqsubset n$ be a VEI order. Assume towards a contradiction that $\sqsubset'$ is another VEI order that is neither $\sqsubset$ nor its reverse. Consequently, there exist three votes $v_i, v_j, v_k, i < j < k$ for which $\sqsubset$ and $\sqsubset'$ disagree on their order in the sense that $v_j$ is not in between $v_i$ and $v_k$ with respect to $\sqsubset'$. Without loss of generality let us assume $j \sqsubset' i \sqsubset' k$. Let us consider $C_X$ for every $X \subseteq \{i, j, k\}$ being defined as the set of all candidates approved by the votes corresponding to $X$ but not approved by those corresponding to $\{i, j, k\} \setminus X$. For example, $C_{ik}$ are those candidates approved by $c_i$ and $c_k$ but not by $c_j$. Since we have a CEI profile and $i \sqsubset j \sqsubset k$, we know that $C_{ik} \neq C_j = \emptyset$. Under our assumption that $\sqsubset'$ is also a VEI ordering with $j \sqsubset' i \sqsubset' k$, we know that $C_{jk} = C_i = \emptyset$. This implies that the candidate approved by $c_i$ are $C_{ijk} \cup C_{ij}$ and those approved by $c_j$ are also $C_{ijk} \cup C_{ij}$. This contradicts our assumption that all votes are distinct.

**Lemma 11.** If all candidates are approved by distinct sets of voters, CEI orders are unique.

**Proof.** First, let us observe that two candidates that are approved by the same voters certainly are indistinguishable; their positions on the CEI axis are interchangeable. Thus, our condition is necessary for the lemma to hold. The proof of this statement is similar to the previous proof. Without loss of generality assume that $c_1 \sqsubset \cdots \sqsubset c_m$ be a CEI order. Assume towards a contradiction that $\sqsubset'$ is another CEI order that is neither $\sqsubset$ nor its reverse. Consequently, there exist three votes $v_i, v_j, v_k, i < j < k$ for which $\sqsubset$ and $\sqsubset'$ disagree on their order in the sense that $c_j$ is not in between $c_i$ and $c_k$ with respect to $\sqsubset'$. Without loss of generality let us assume $c_j \sqsubset' c_i \sqsubset' c_k$. Let us consider $V_X$ for every $X \subseteq \{i, j, k\}$ being defined as the set of all votes that approve
the candidates in \( X \) and disapprove those in \( \{i, j\} \setminus X \). Since we have a VEI profile and \( c_i \sqsubseteq c_j \sqsubseteq c_k \), we know that \( V_{ik} = V_j = \emptyset \). Under our assumption that \( \sqsubseteq' \) is also a CEI ordering with \( c_j \sqsubseteq c_i \sqsubseteq c_k \), we know that \( V_{jk} = V_i = \emptyset \). This implies that the votes that approve \( c_i \) are \( V_{ijk} \cup V_{ij} \) and the votes approving \( c_j \) are \( V_{ijk} \cup V_{ij} \). This contradicts our assumption that all candidates are approved by a distinct set of voters.

### 3.3 Detection

To exploit the constraints defined in Section 3 we have developed algorithms that can decide whether a given profile belongs to one of the restricted domains defined by these constraints. Our results are summarized in Table 1.

| Constraint   | Complexity          |
|--------------|---------------------|
| 2PART        | poly (trivial)      |
| PART         | poly (trivial)      |
| VEI          | poly (CONSECUTIVE 1S) |
| CEI          | poly (trivial)      |
| WSC          | poly (CONSECUTIVE 1S) |
| DUE          | open                |
| VI           | poly (CONSECUTIVE 1S) |
| CI=DE=PSP=PE | poly (CONSECUTIVE 1S) |
| PSC=SSC      | open                |

Table 1: The complexity of detecting structure in dichotomous profiles

For WSC, Elkind et al. [2015] provide an algorithm that works for any weak orders (not just dichotomous ones). They leave the complexity of detecting PSC and SSC as an open problem, and we have not been able to resolve it for dichotomous weak orders. The problem of recognizing DUE preferences remains open as well, though it is plausible that a linear-programming based algorithms similar to those of [Doignon and Faliszewski, 1994, Knoblauch, 2010, Elkind and Faliszewski, 2014] exist.

### 4 Algorithms for Committee Selection

In this section, we consider two classic approval-based committee selection rules—Proportional Approval Voting (PAV) and Maximin Approval Voting (MAV)—and argue that we can design efficient algorithms for these rules when voters' preferences belong to some of the domains in our list (for some of the richer domains, we may need to place mild additional restrictions on voters' preferences).

We start by providing formal definitions of these rules.

**Definition 1.** Every non-increasing infinite sequence of non-negative reals \( w = (w_1, w_2, \ldots) \) that satisfies \( w_1 = 1 \) defines a committee selection rule \( w\text{-PAV} \). This rule takes a set of candidates \( C \), a dichotomous profile \( P = (v_1, \ldots, v_n) \) and a target committee size \( k \leq |C| \) as its input. For every size-\( k \) subset \( W \) of \( C \), it computes its \( w\text{-PAV} \) score as \( \sum_{v_i \in P} w_i (|W \cap v_i|) \), where \( w_i = \sum_{j=1}^{p} w_j \), and outputs a size-\( k \) subset with the highest \( w\text{-PAV} \) score, breaking ties arbitrarily. The \( w\text{-PAV} \) rule with \( w = (1, \frac{1}{2}, \frac{1}{3}, \ldots) \) is usually referred to simply as the PAV rule, and we write \( u(p) = 1 + \cdots + \frac{1}{p} \).

In what follows we assume that the entries of \( w \) are rational and \( w_i \) can be computed in time \( \text{poly}(i) \).

**Definition 2.** Given a set of candidates \( C \), a dichotomous profile \( P = (v_1, \ldots, v_n) \) and a target committee size \( k \leq |C| \), the MAV-score of a size-\( k \) subset \( W \) of \( C \) is computed as \( \max_{v_i \in P} (|W \cap v_i| + |v_i \setminus W|) \). MAV outputs a size-\( k \) subset with the lowest MAV score, breaking ties arbitrarily.

The \( w\text{-PAV} \) rule is defined by Kilgour and Marshall [2012], see also [Kilgour, 2010]. Intuitively, under this rule each voter is assumed to derive a utility of 1 from having exactly one of his approved candidates in the winning set: his marginal utility from having more of his approved candidates in the winning set is non-increasing. The goal of the rule is to maximize the sum of players’ utilities. In contrast, MAV [Brams et al., 2007] has an egalitarian objective: for each candidate committee, it computes the dissatisfaction of the least happy voter, and outputs a committee that minimizes the quantity.

Computing the winning committee under MAV and PAV is NP-hard, see, respectively, [LeGrand et al., 2007] and [Skowron et al., 2015a]. The hardness result for PAV extends to \( w\text{-PAV} \) as long as \( w \) satisfies \( w_1 > w_2 \); moreover, it holds even if each voter approves
of at most two candidates or if each candidate is approved by at most three voters.

We will now show that PAV admits an algorithm whose running time is polynomial in the number of voters and the number of candidates if the input profile satisfies CI or VI and, furthermore, each voter approves at most s candidates or each candidate is approved by at most d voters, where s and d are given constants. More specifically, we prove that PAV winner determination for CI and VI preferences is in FPT with respect to parameter s and in XP with respect to parameter d. For simplicity, we state our results for PAV; however, all of them can be extended to w-PAV.

In what follows, we write $[x : y]$ to denote the set $\{z \in \mathbb{Z} : x \leq z \leq y \}$.

**Theorem 13.** Given a dichotomous profile $P = (v_1, \ldots, v_n)$ over a candidate set $C = \{c_1, \ldots, c_m\}$ and a target committee size $k$, if $|v_i| \leq s$ for all $v_i \in P$ and $P$ satisfies VI, then we can find a winning committee under PAV in time $O(2^{2s} \cdot k \cdot n)$.

**Proof.** Assume that $P$ satisfies VI with respect to the order of voters $v_1 \preceq \cdots \preceq v_n$. For each triple $(i, A, \ell)$, where $i \in [1 : n]$, $A \subseteq v_i$, and $\ell \in [0 : k]$, let $r(i, A, \ell)$ be the maximum utility that the first $i$ voters can obtain from a committee $W$ such that $W \cap v_i = A$, $|W| = \ell$, and $W \subseteq v_1 \cup \cdots \cup v_i$.

We have $r(1, A, |A|) = u(|A|)$ for every $A \subseteq v_1$ and $r(1, A, \ell) = -\infty$ for every $A \subseteq v_1$, $\ell \in [0 : k] \setminus \{|A|\}$.

To compute $r(i + 1, A, \ell)$ for $i \in [1 : n - 1]$, $A \subseteq v_{i+1}$ and $\ell \in [0 : k]$, we let $p = |A| \setminus v_i$ and set

$$r(i + 1, A, \ell) = \max_{D \subseteq v_{i+1}} r(i, D \cup (A \cap v_i), \ell - p) + u(|A|).$$

Indeed, every committee $W$ with $|W| = \ell$, $W \cap v_{i+1} = A$, $W \subseteq v_1 \cup \cdots \cup v_i$ contains exactly $\ell - p$ candidates from $v_1 \cup \cdots \cup v_i$ and its intersection with $v_i$ is of the form $D \cup (A \cap v_i)$ and candidates in $D$ are approved by $v_i$, but not $v_{i+1}$. We output $\max_{A \subseteq v_n} r(n, A, k)$.

This dynamic program has $n \cdot 2^s \cdot (k + 1)$ states, and the value of each state is computed using $O(2^s)$ arithmetic operations. Assuming that basic calculations take constant time, we obtain a total runtime of $O(2^{2s} \cdot k \cdot n)$. □

A similar dynamic programming algorithm can be used if voters’ preferences satisfy CI.

**Theorem 14.** Given a dichotomous profile $P = (v_1, \ldots, v_n)$ over a candidate set $C = \{c_1, \ldots, c_m\}$ and a target committee size $k$, if $|v_i| \leq s$ for all $v_i \in P$ and $P$ satisfies CI, then we can find a winning committee under PAV in time $O(2^{s} \cdot n \cdot m)$.

**Proof.** Assume that $P$ satisfies CI with respect to the order of candidates $c_1 < \cdots < c_m$. For each triple $(j, A, \ell)$, where $j \in [1 : m]$, $A \subseteq \{c_{j+1}, \ldots, c_m\}$, and $\ell \in [0 : k]$, let $r(j, A, \ell)$ be the maximum utility that voters can obtain from a committee $W$ such that $W \subseteq \{c_1, \ldots, c_j\}$, $W \cap \{c_{j+1}, \ldots, c_m\} = A$, and $|W| = \ell$. Also, for each $j \in [1 : m - s + 1]$ and each $A \subseteq \{c_{j+s+1}, \ldots, c_m\}$ let $t(A, c_{j+s+1}) = \sum_{v_i \in P \setminus c_{j+s+1}} u(|A \cup v_i|)$. Note that all the quantities $t(\ldots)$ can be computed in time $O(2^s \cdot m \cdot n)$.

We have $r(1, 0, 0) = 0$, $r(1, \{c_1\}, 1) = \{(v_2 : c_2 \in v_1)\}$, and $r(1, A, \ell) = -\infty$ if $|A, \ell) \neq (\emptyset, 0)$, $(\{c_1\}, 1)$. The quantities $r(j + 1, A, \ell)$ for $j \in [1 : n - 1]$ can now be computed as follows. If $c_{j+1} \not\in A$, we set

$$r(j + 1, A, \ell) = \max \{r(j, A, \ell), r(j, A \cup \{c_{j+s}\}, \ell)\}.$$

Now, suppose that $c_{j+1} \in A$. Let $A' = A \setminus \{c_{j+1}\}$. Then $s(j + 1, A, \ell) = \max\{s_1, s_2\}$ where

$$s_1 = s(j, A' \cup \{c_{j+s}\}, \ell - 1) - t(A', c_{j+1}) + t(A, c_{j+1}),$$

$$s_2 = s(j, A', \ell - 1) - t(A', c_{j+1}) + t(A, c_{j+1}).$$

We output $\max_{A \subseteq \{c_{m-s+1}, \ldots, c_m\}} r(m, A, k)$. Our dynamic program has at most $2^s \cdot m \cdot (k + 1)$ states, and the utility of each state can be computed in time $O(1)$. Combining this with the time used to compute $t(\ldots)$, we obtain the desired bound on the running time. □

Our next two theorems also considers CI and VI preferences, and deal with the case where no candidate is approved by too many voters. Just as the algorithms in the proofs of Theorems 13 and 14 the algorithms for this case are based on dynamic programming.

**Theorem 15.** Given a dichotomous profile $P = (v_1, \ldots, v_n)$ over a candidate set $C = \{c_1, \ldots, c_m\}$ and a target committee size $k$, if $|\{i : c \in v_i\}| < d$ for all $c \in C$ and $P$ satisfies CI or VI, then we can find a winning committee under PAV in time poly(d, m, n, k^d).

**Proof.** Assume without loss of generality that the candidate order $c_1 < \cdots < c_m$ witnesses that $P$ is CI. For each voter $v_i \in P$, let $c_k$ and $c_v$ be, respectively, the first and the last candidate (with respect to $<$) approved by $v_i$, i.e., $v_i = \{c_j \mid b_i \leq j \leq e_i\}$. For $j \in [1 : m]$, we say that a voter $v_i$ is active at $j$ if $b_i \leq j \leq e_i$; we say that a voter $v_i$ is finished at $j$ if $e_i \leq j$. Let $B' = \{v_i \mid b_i = j\}$, $E' = \{v_i \mid e_i = j\}$. Given a set $W \subseteq C$, we will refer to the quantity $u(|W \cap v_i|)$ as the utility of voter $i$ from set $W$. Throughout the proof, we make the standard assumption that for any real-valued function $f$ we have $\max\{f(x) : x \in X\} = -\infty$ when $X = \emptyset$.

Let $R(j)$ be the set of all vectors $r \in [0 : k]^n$ such that for all $\ell \in [1 : n]$ it holds that $0 \leq r_\ell \leq \min\{|j - b_\ell + 1, k\}$ and, moreover, $r_\ell = 0$ whenever $v_i$ is not active at $c_j$. Vectors in $R(j)$ can be used to describe the impact of a set of candidates $C \in \{c_1, \ldots, c_m\}$ with $|C| \leq k$ on voters who are active at $c_j$: for each $v_i \in P$, $r_\ell$ indicates how many candidates in $C$ are approved by $v_i$. As there are at most $d$ voters who are active at $j$, we have $|R(j)| \leq (k + 1)^d$. For each $j \in [1 : m]$, $r \in [0 : \min\{j, k\}]$ and $r \in R(j)$, let $W_i(j, r)$ be the collection of all subsets of $C$ with the following properties: each $W \in W_i(j, r)$ satisfies $|W| = i$, $W \subseteq \{c_1, \ldots, c_j\}$, and, moreover, for each $\ell \in [1 : n]$ such that $v_i$ is active at $c_j$ it holds that $|v_i \cap W| = r_\ell$. Intuitively, $W_i(j, r)$ consists of all size-$i$ subsets of $\{c_1, \ldots, c_j\}$ whose impact on voters who are active at $c_j$ is described by $r$. Let $A(i, j, r)$ be the maximum total utility that voters who are finished at $j$ derive from a set in $W_i(j, r)$; note that $A(i, j, r) = -\infty$ if $W_i(j, r) = \emptyset$. Clearly, it is easy to compute $A(i, 1, r)$ for $i \in \{0, 1\}$ and all $r \in R(1)$.
We will now explain how to compute $A(i, j, r)$ given the values of $A(i', j - 1, r')$ for all $i' \in [0 : \min\{j - 1, k\}]$ and all $r \in R(j - 1)$.

Suppose first that $B_j \neq \emptyset$. By definition of $R(j)$ we have $r_x \in \{0, 1\}$ for each $v_x \in B_j$. Moreover, if we have $r_x \neq r_{x'}$ for some $v_x, v_{x'} \in B_j$, then $W(i, j, r) = \emptyset$ and consequently $A(i, j, r) = -\infty$: no subset of $\{c_1, \ldots, c_j\}$ can intersect $v_x$, but not $v_{x'}$ or vice versa.

Now, if $B_j \neq \emptyset$ and $r_x = 1$ for all $v_x \in B_j$, all sets in $W(i, j, r)$ contain $c_j$, and therefore
\[
A(i, j, r) = \max_{r' \in R'_1} A(i, j - 1, r') + \sum_{v_i \in E_j} u(r_i),
\]
where $R'_1$ is the set of all vectors $r' \in R(j - 1)$ with $r'_j = r'_j - 1$ for all voters $v_i$ that are active at both $c_j$ and $c_{j-1}$. Indeed, the second summand here is the total utility of voters in $E_j$; for every such voter $v_i$ we know that for any set of candidates $W \in W(i, j, r)$ he approves exactly $r_i$ candidates in $W$. The first summand is the maximum total utility of voters who are finished at $j - 1$ that can be achieved by picking a set $W'$ so that $W' \cup \{c_j\} \in W(i, j, r)$; every such set $W'$ is contained in $W(i, j - 1, r')$ for some vector $r'$ in $R(j - 1)$ that is consistent with $r$, i.e. satisfies $r'_j = r'_j - 1$ for all voters $v_i$ that are active at both $c_j$ and $c_{j-1}$.

By a similar argument, if $B_j \neq \emptyset$ and $r_x = 0$ for all $v_x \in B_j$, no set in $W(i, j, r)$ contains $c_j$, and therefore
\[
A(i, j, r) = \max_{r' \in R'_0} A(i, j - 1, r') + \sum_{v_i \in E_j} u(r_i),
\]
where $R'_0$ is the set of all vectors $r' \in R(j - 1)$ with $r'_j = r'_j$ for all voters $v_i$ that are active at both $c_j$ and $c_{j-1}$.

Finally, suppose that $B_j = \emptyset$. Then we have to consider both possibilities for $c_j$. To this end, define
\[
a_1 = \max_{r' \in R'_1} A(i - 1, j - 1, r') + \sum_{v_i \in E_j} u(r_i),
a_0 = \max_{r' \in R'_0} A(i - 1, j - 1, r') + \sum_{v_i \in E_j} u(r_i),
\]
where $R'_1$ and $R'_0$ are defined as above, and set
\[
A(i, j, r) = \max\{a_1, a_0\},
\]
again, the argument for correctness is the same as above.

To complete the proof, it remains to observe that the $PAV$-score of an optimal size-$k$ committee is given by $\max_{r \in R(m)} A(k, m, r)$. Once this score is computed, the respective committee can be found using standard dynamic programming techniques.

To bound the running time, note that our dynamic program has $O(km(k+1)^d)$ variables, and the argument above establishes that the value of $A(i, j, r)$ can be computed in time $O(d(k + 1)^d)$ given the values of $A(i', j - 1, r')$ for all $i' \in [0 : \min\{j, k\}]$, $r \in R(j - 1)$.

**Theorem 16.** Consider an election $(C, V)$ with $C = \{c_1, \ldots, c_m\}$, $V = \{v_1, \ldots, v_n\}$, and a target committee size $k$. If $|\{i \mid c \in v_i\}| \leq d$ for all $c \in C$ and $(C, V) \in \text{VI}$, then we can determine the winning committee under $PAV$ in time $poly(d, m, n, k^d)$.

**Proof.** Assume without loss of generality that the voter order $v_1 \subseteq \cdots \subseteq v_n$ witnesses that $(C, V)$ is in $\text{VI}$. For each candidate $c_j \in C$, let $v_{b_j}$ and $v_{e_j}$ be, respectively, the first and the last voter (with respect to $\subseteq$) who approve $c_j$, i.e., $\{v_i \in V \mid c_j \in v_i\} = \{v_i \mid b_j \leq i \leq e_j\}$.

Let $C_i^{\ell, r} = \{c_j \mid b_j = \ell, e_j = r\}$, $B_i^{\ell, r} = \{c_j \mid b_j \leq \ell\}$. Given a set $W \subseteq C$, we will refer to the quantity $u(W \cap v_i) = 1 + 1/2 + \cdots + 1/|W \cap v_i|$ as the utility of voter $i$ from set $W$. Throughout the proof, we make the standard assumption that for any real-valued function $f$ we have $\max\{f(x) \mid x \in X\} = -\infty$ when $X = \emptyset$.

Let $\mathcal{N}(i)$ be the set of all $m$-by-$m$ matrices over $[0 : k]$ that have the following property: for every matrix $N = (N_{\ell, r})_{\ell, r \in [1 : m]} \in \mathcal{N}(i)$, we have $0 \leq N_{\ell, r} \leq C_i^{\ell, r}$ if $\ell \leq i \leq r$ and $N_{\ell, r} = 0$ if $i < \ell$ or $i > r$. Matrices in $\mathcal{N}(i)$ can be used to describe the impact of a set of candidates $W$ on voter $v_i$; for each $\ell, r \in [1 : m]$, $N_{\ell, r}$ indicates how many candidates in $W$ are approved by $v_i$. Since $c_j \in v_i$ implies $i - d + 1 \leq b_j \leq i$, $i \leq e_j \leq i + d - 1$, we have $\mathcal{N}(i) \subseteq (k + 1)^d$.

For each $j \in [0 : k]$, $i \in [1 : n]$ and $N \in \mathcal{N}(i)$, let $W(i, j, N)$ be the collection of all size-$j$ subsets of $B$ such that $v_i \cap C_i^{\ell, r} = N_{\ell, r}$ for all $\ell, r \in [1 : m]$: we set $W(i, j, N) = \emptyset$ if $j \notin [0 : k]$, $i \notin [1 : n]$ or $N \notin \mathcal{N}(i)$. In words, $W(i, j, N)$ consists of all size-$j$ sets consisting of candidates that are approved by at least one voter in $v_1, \ldots, v_n$ whose impact on $v_i$ is described by $N$. Let $A(i, j, N)$ be the maximum total utility that voters in $\{v_1, \ldots, v_n\}$ derive from a set in $W(i, j, N)$; note that $A(i, j, N) = -\infty$ if $W(i, j, N) = \emptyset$. It is easy to compute $A(i, j, N)$ for all $j \in [0 : k]$ and all $N \in \mathcal{N}(1)$: we have $A(0, 0, N) = u(j)$ if $j = \sum_{r \in [1 : m]} N_{1, r}$ and $A(1, j, N) = -\infty$ otherwise. Also, for each $i \in [1 : n]$ we have $A(i, 0, N) = 0$ if $N_{i, 0} = 0$ for all $\ell \in [1 : m]$ and $A(i, 0, N) = -\infty$ otherwise.

We will now explain how to compute $A(i, j, N)$ given the values of $A(i - 1, j', N')$ for all $j' \in [1, j]$ and all $N' \in \mathcal{N}(i)$. Fix $i \in [2 : n], j \in [0 : k], N \in \mathcal{N}(i)$. Note first that for any set $W \in W(i, j, N)$ we have
\[
|v_i \cap W| = \sum_{r, \ell \in [1 : m]} n_{r, \ell};
\]
also, if $\sum_{r, \ell \in [1 : m]} n_{r, \ell} > j$, then $W(i, j, N) = \emptyset$.

Further, for every set $W \in W(i, j, N)$ the set $W \setminus \{c_1 \mid b_1 = i\}$ belongs to $W(i - 1, j', N')$ for $j' = j - 1 - |\{c_1 \mid b_1 = i\}|$ and for some matrix $N' \in \mathcal{N}(i)$ with $n'_{r, \ell} = n_{r, \ell}$ for $\ell \neq i$ and $r \neq i - 1$. Let $j' = j - 1 - |\{c_1 \mid b_1 = i\}|$, $N' \in \mathcal{N}(i - 1) \setminus n'_{i, \ell}$ for $\ell \neq i$, $r \neq i - 1$. Then we have
\[
A(i, j, N) = \max_{N' \in \mathcal{N}(i - 1)} A(i - 1, j', N') + u\left(\sum_{r, \ell \in [1 : m]} n_{r, \ell}\right)
\]
if $\sum_{r, \ell \in [1 : m]} n_{r, \ell} \leq j$ and $A(i, j, N) = -\infty$ otherwise.

To complete the proof, it remains to observe that the $PAV$-score of an optimal size-$k$ committee is given by $\max_{N \in \mathcal{N}(n)} A(n, k, N)$. Once this score is computed, the respective committee can be found using standard dynamic programming techniques, and the bound on running time follows immediately.
The reader may wonder if constraints on \(s\) and \(d\) in Theorems \([13][14]\) and \([15]\) are necessary. We conjecture that the answer is yes, i.e., winner determination under \(PAV\) remains hard under CI and VI preferences.

**Conjecture 17.** \(PAV\) is \(NP\)-hard even for CI and VI preferences.

However, for “truncated” weight vectors \(w\) we can find \(w\)-PAV winners in polynomial time. As the \((1, 0, \ldots )\)-\(PAV\) rule is essentially the classic Chamberlin–Courant rule [Chamberlin and Courant, 1983] for dichotomous preferences, our next result can be seen as an extension of the results of [Betzler et al., 2013] and [Skowron et al., 2015b] for the Chamberlin–Courant rule and single-peaked and single-crossing preferences: while we work on a less expressive domain (dichotomous preferences vs. total orders), we can handle a larger class of rules (all weight vectors with a constant number of non-zero entries rather than just \((1, 0, \ldots )\)).

**Theorem 18.** Consider a weight vector \(w\) where \(w_i = 0\) for \(i > i_0\) for some constant \(i_0\). Then given a dichotomous profile \(\mathcal{P} = (v_1, \ldots , v_n)\) over a candidate set \(C = \{c_1, \ldots , c_m\}\) and a target committee size \(k\), if \(\mathcal{P}\) satisfies VI, we can find a winning committee under \(w\)-PAV in polynomial time.

**Proof.** Assume that \(\mathcal{P}\) satisfies VI with respect to the order of voters \(v_1 \sqsubseteq \cdots \sqsubseteq v_n\).

The following algorithm is a refinement of Theorem \([13]\). For each triple \((i, A, \ell)\), where \(i \in [1 : n]\), \(A \subseteq v_i\), and \(\ell \in [0 : k]\), let \(r(i, A, \ell)\) be the maximum utility that the first \(i\) voters can obtain from a committee \(W\) such that \(|W| = \ell\), and \(W \subseteq v_1 \cup \cdots \cup v_i\) and \(A \subseteq W\).

We have \(r(1, A, \ell) = u(\ell)\) for every \(\ell \in [0 : |v_1|]\) and \(A \subseteq v_1\) with \(|A| = \min(i_0, \ell)\). In addition, we have \(r(1, A, \ell) = -\infty\) for every other \(A \subseteq v_1\) and \(\ell \in [0 : k]\). To compute \(r(i + 1, A, \ell)\) for \(i \in [1 : n - 1]\), \(A \subseteq v_{i+1}\) with \(|A| \leq i_0\) and \(\ell \in [|A| : k]\), we let \(s = |v_{i+1} \setminus (v_i \cup A)|\), i.e., the maximal number of candidates that might be have been added in the \(i + 1\) step to the committee but that do not show up in \(A\), and set

\[
r(i + 1, A, \ell) = \max\{r(i, D \cup (A \cap v_i), \ell - |A| - r) + u(|A|), 0\},
\]

where the maximum is taken over all \(D \subseteq v_i \setminus v_{i+1}\) with \(|D| \in [0 : i_0 - |A \cap v_i|]\) and all \(r \in [0 : s]\).

This dynamic program has \(n \cdot m^{i_0} \cdot (k + 1)\) states, and the value of each state is computed using \(O(m^{i_0} + 1)\) arithmetic operations. Assuming that basic calculations take constant time, we obtain a total runtime of \(O(n \cdot m^{i_0} + 1)\), which is polynomial for constant \(\xi\).

**Theorem 19.** Consider a weight vector \(w\) where \(w_i = 0\) for \(i > \xi\) for some constant \(\xi\). Then given a dichotomous profile \(\mathcal{P} = (v_1, \ldots , v_n)\) over a candidate set \(C = \{c_1, \ldots , c_m\}\) and a target committee size \(k\), if \(\mathcal{P}\) satisfies CI, we can find a winning committee under \(w\)-PAV in polynomial time.

**Proof.** Assume that \(\mathcal{P}\) satisfies CI with respect to the order of candidates \(c_1 \prec \cdots \prec c_m\). The following algorithm is a refinement of Theorem \([4]\). For two sets \(C_1, C_2 \subseteq C\) we write \(C_1 \prec C_2\) to denote that for all \(c \in C_1\) and \(d \in C_2\) it holds that \(c < d\). For each triple \((j, A, \ell)\), where \(j \in [1 : m]\), \(A \subseteq \{c_1, \ldots , c_j\}\), \(|A| \leq i_0\) and \(\ell \in [0 : k]\), let \(r(j, A, \ell)\) be the maximum utility that voters can obtain from a committee \(W\) such that \(A \subseteq W\), \(|W| = \ell\) and \(W \setminus A \subseteq A\). Also, for each \(j \in [1 : m - s + 1]\) and each \(A \subseteq \{c_{j+1}, \ldots , c_{j+1+s}\}\) let \(t(A, c) = \sum_{\ell \in [0 : k]} u(\ell | A \cup \{c\})\). It is essential that, given a committee \(W\) satisfying the conditions above, \(t(W, c) = t(A, c)\) assuming CI preferences and \(c' < c\) for all \(c' \in A \setminus \{c\}\). Furthermore, note that all the quantities \(t(\ldots)\) can be computed in time \(O(n \cdot m^{i_0} + 1)\) since we assume that \(u(\ldots)\) can be computed in constant time.

We have \(r(1, \emptyset, 0) = 0, r(1, \{c_1\}, 1) = |v_1 : c_1 \in v_1|\), and \(r(1, A, \ell) = -\infty\) if \(|A| \neq \emptyset\), \(\{c_1\}, 1\). The quantities \(r(j + 1, A, \ell)\) for \(j \in [1 : m - 1]\) and \(A \subseteq \{c_{j+1}, \ldots , c_{j+1+s}\}\) with \(|A| \leq i_0\) can now be computed as follows. If \(c_{j+1} \notin A\), we set

\[
r(j + 1, A, \ell) = r(j, A, \ell).
\]

Now, suppose that \(c_{j+1} \in A\). Let \(A' = A \setminus \{c_{j+1}\}\). Then

\[
s(j + 1, A, \ell) = \max\{s_1, s_2\} - t(A', c_{j+1}) + t(A, c_{j+1})\]

where

\[
s_1 = \max_{c \in A} s(j, A' \cup \{c\}, \ell - 1), \quad s_2 = s(j, A', \ell - 1).
\]

We output \(\max_{A \subseteq C, |A| \leq i_0} r(m, A, k)\). Our dynamic program has at most \(m^{i_0 + 1} \cdot (k + 1)\) states, and the utility of each state can be computed in time \(O(m)\). Combining this with the time used to compute \(t(\ldots)\), we obtain a total runtime of \(O(n \cdot m^{i_0 + 1} \cdot k)\), which is polynomial for fixed \(i_0\).

Moreover, for the more restricted domains, such as VEI, CEI, WSC and PART we can design polynomial-time algorithms for both \(MAV\) and \(PAV\), under no additional constraints on preferences (again, our results extend to \(w\)-PAV).

**Theorem 20.** Given a dichotomous profile \(\mathcal{P} = (v_1, \ldots , v_n)\) over a candidate set \(C = \{c_1, \ldots , c_m\}\) and a target committee size \(k\), if \(\mathcal{P}\) satisfies VEI, CEI, WSC or PART, we can find a winning committee under MAV and PAV in polynomial time.

**Proof sketch.** Consider first VEI. Assume without loss of generality that \(\mathcal{P}\) satisfies VEI for voter order \(v_1 \sqsubseteq \cdots \sqsubseteq v_n\). Each candidate in \(C\) belongs to one of the following four groups: \(C_1 = v_1 \setminus v_n\), \(C_2 = v_1 \setminus v_n\), \(C_3 = v_n \setminus v_1\), and \(C_4 = \emptyset\). Candidates in \(C_1\) are approved by all voters and candidates in \(C_4\) are not approved by any of the voters.

Suppose first that \(|C_1 \cup C_2 \cup C_3| < k\). Then there exists an optimal committee for both \(MAV\) and \(PAV\) that contains all candidates in \(C_1 \cup C_2 \cup C_3\) and exactly \(k - |C_1 \cup C_2 \cup C_3|\) candidates from \(C_4\). Hence, we can now assume that this is not the case. Then there exist an optimal committee that contains no candidates from \(C_4\).

Now, if \(|C_1| \geq k\), an optimal committee for both \(PAV\) and \(MAV\) consists of \(k\) candidates from \(C_1\), and if \(|C_1| < k\), there exists an optimal committee that contains all candidates in \(C_1\). It remains to decide how to allocate the remaining places among candidates in \(C_2\) and \(C_3\). To do so, we observe that there is a natural ordering over each of these sets: given a pair of candidates \((c, c')\) in \(C_2 \times C_2 \times C_3\), we write
\(c \leq c'\) if \(\{i : c \in v_i\} \subseteq \{i : c' \in v_i\}\). Note that every two candidates in \(C_2\) are comparable with respect to \(\preceq\), and so are every two candidates in \(C_3\). It is now easy to see that there exists an optimal committee (for \(PAV\) or \(MAV\)) that consists of candidates in \(C_1\), top \(p\) candidates in \(C_2\) with respect to \(\preceq\) and top \(r\) candidates in \(C_3\) with respect to \(\leq\) for some non-negative values of \(p, r\) with \(p + r + |C_1| = k\). Thus, by considering at most \(k^2\) possibilities for \(p\) and \(r\), we can find an optimal committee.

The argument for CEI is similar to the one for VEI: we have to decide how many candidates to select from each end of the candidate ordering witnessing that \(P\) satisfies CEI. For WSC, we can use the characterization in Lemma 1. The problem then boils down to deciding how many candidates to select from each of the sets \(u \setminus w, u \cap w\) and \(w \setminus u\). For PART and PAV, we can show that an optimal committee can be found by a natural greedy algorithm that at each point selects the candidate with the largest “marginal contribution” to the total utility. For PART and \(MAV\), we check, for each \(t = 0, \ldots, n\), whether there exists a committee whose \(MAV\)-score is at most \(t\). This is the case if for each voter \(v \in P\) we can select at least \((|v| + k - t)/2\) candidates from \(v\). Thus, if \(v_1, \ldots, v_k\) are the distinct votes in \(P\), we need to check that \(\sum_{i=1}^t |v_i| \leq \ell t - (\ell - 2)k\).

5 Conclusions and Open Problems

We have initiated research on analogues of the notions of single-peakedness and single-crossingness for dichotomous preference domains. We have proposed many constraints that capture some aspects of what it means for dichotomous preferences to be single-dimensional, explored the relationship among them, and showed that these constraints can be useful for identifying efficiently solvable special cases of hard voting problems on dichotomous domains. The algorithmic results in Section 4 can be seen as a proof that our approach has merit; however, there is certainly room for improvement there, both in terms of removing restrictions on the sizes of approval sets and number of voters that approve each candidate (for \(PAV\)) and in terms of considering larger domains, such as \(PSC\) for \(PAV\) and \(CI/VI\) for \(MAV\).

For many of our constraints, we have provided efficient algorithms for checking whether a given dichotomous profile satisfies that constraint; two notable open cases are DUE and \(PSC/SSC\). In particular, it would be interesting to understand if every profile that satisfies both VI and CI also satisfies DUE; this can be seen as an analogue of the question of whether every single-peaked single-crossing profile of total orders is \(1\)-Euclidean (see discussion in [Doignon and Falmagne, 1994; Elkind et al., 2014]). We can also ask if it is possible to detect if a given dichotomous profile is close to satisfying a structural constraint, and whether such “almost-structured” profiles have useful algorithmic properties; similar issues for profiles of total orders have recently received a lot of attention in the literature. [Cornaz et al., 2012; Cornaz et al., 2013; Bredereck et al., 2013; Erdélyi et al., 2013; Elkind and Lackner, 2014; Faliszewski et al., 2014].

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