WHEN STABLE COHEN-MACAULAY AUSLANDER ALGEBRA IS SEMISIMPLE

RASOOL HAFEZI

Dedicated to people in Afghanistan

Abstract. Let $\text{Gprj-Λ}$ denote the category of Gorenstein projective modules over an Artin algebra $\Lambda$ and the category $\text{mod-(Gprj-Λ)}$ of finitely presented functors over the stable category $\text{Gprj-Λ}$. In this paper, we study those algebras $\Lambda$ with $\text{mod-(Gprj-Λ)}$ to be a semisimple abelian category, and called $\Omega_{G}$-algebras. The class of $\Omega_{G}$-algebras contains important classes of algebras, including gentle algebras. Over an $\Omega_{G}$-algebra $\Lambda$, the structure of the almost split sequences in the morphism category $\text{H(Gprj-Λ)}$ and the monomorphism category $\text{S(Gprj-Λ)}$ of $\text{Gprj-Λ}$ is investigated. Among other applications, we provide some results for the Cohen-Macaulay Auslander algebras of $\Omega_{G}$-algebras.

1. Introduction and preliminaries

Let us begin with some notation.

Let $\mathcal{X}$ be an additive category. The Hom sets will be denoted either by $\text{Hom}_{\mathcal{X}}(\cdot, \cdot)$, $\mathcal{X}(\cdot, \cdot)$ or even just $(\cdot, \cdot)$, if there is no risk of ambiguity. By definition, a (right) $\mathcal{X}$-module is a contravariant additive functor $F : \mathcal{X} \rightarrow \text{Ab}$, where $\text{Ab}$ denotes the category of abelian groups. The $\mathcal{X}$-modules and natural transformations between them, called morphisms, form an abelian category denoted by $\text{Mod-}\mathcal{X}$. An $\mathcal{X}$-module $F$ is called finitely presented if there exists an exact sequence $\mathcal{X}(\cdot, X) \rightarrow \mathcal{X}(\cdot, X') \rightarrow F \rightarrow 0$, with $X$ and $X'$ in $\mathcal{X}$. All finitely presented $\mathcal{X}$-modules form a full subcategory of $\text{Mod-}\mathcal{X}$, denoted by $\text{mod-}\mathcal{X}$.

Throughout the paper $\Lambda$ denotes an Artin algebra and $\text{mod-\Lambda}$ the category of finitely generated right $\Lambda$-modules. From now on to the end of the introduction, let $\mathcal{X}$ be a full subcategory of $\text{mod-\Lambda}$. Assume $M \in \text{mod-\Lambda}$. A right $\mathcal{X}$-approximation of $M$ is a morphism $f : X \rightarrow M$ such that $X \in \mathcal{X}$ and any other morphism $X' \rightarrow M$ with $X' \in \mathcal{X}$ factors through $f$. $\mathcal{X}$ is called a contravariantly finite subcategory of $\text{mod-\Lambda}$ if every $M \in \text{mod-\Lambda}$ admits a right $\mathcal{X}$-approximation. Left $\mathcal{X}$-approximations and covariantly finite subcategories are defined dually. If $\mathcal{X}$ is both a contravariantly finite and a covariantly finite subcategory of $\text{mod-\Lambda}$, then it is called a functorially finite subcategory.

If $\mathcal{X}$ is contravariantly finite, then any morphism in $\mathcal{X}$ admits a weak kernel. Then it is known that $\text{mod-\mathcal{X}}$ is abelian, see [Au1, Chapter III, Section 2].

Let $\mathcal{X} \subseteq \text{mod-\Lambda}$ contain the subcategory $\text{prj-\Lambda}$ of projective modules in $\text{mod-\Lambda}$. We consider the stable category of $\mathcal{X}$, denoted by $\mathcal{X}^{\bullet}$. The objects of $\mathcal{X}^{\bullet}$ are the same as the objects of $\mathcal{X}$, which

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we usually denote by $\overline{X}$ when an object $X \in \mathcal{X}$ considered as an object in the stable category, and the morphisms are given by $\underline{\text{Hom}}_{\Lambda}(X, Y) = \text{Hom}_{\Lambda}(X, Y)/\mathcal{P}(X, Y)$, where $\mathcal{P}(X, Y)$ is the subgroup of $\text{Hom}_{\Lambda}(X, Y)$ consisting of those morphisms from $X$ to $Y$ which factor through a projective module in $\text{mod-}\Lambda$. We also denote by $f$ the residue class of $f : X \to Y$ in $\underline{\text{Hom}}_{\Lambda}(X, Y)$.

It is well-known that the canonical functor $\pi : \mathcal{X} \to \mathcal{X}$ induces a fully faithful functor $\pi^* : \text{mod-}\mathcal{X} \to \text{mod-}\mathcal{X}$. Hence due to this embedding we can identify the functors in $\text{mod-}\mathcal{X}$ as functors in $\text{mod-}\mathcal{X}$ vanish on the projective modules.

By our embedding, one can see that $\text{mod-}\mathcal{X}$ is a Serre subcategory, i.e., it is closed under taking subobjects, quotients and extensions. The Gabriel quotient category $\text{mod-}\mathcal{X}$ is by definition the localization of $\text{mod-}\mathcal{X}$ with respect to the collection of all morphisms whose kernels and cokernels are in $\text{mod-}\mathcal{X}$, see [G] for more details.

It is proved in [AHKe2, Theorem 3.5] that if $\text{prj-}\Lambda \subseteq \mathcal{X}$ is a contravariantly finite subcategory, then there exists an equivalence

$$\text{mod-}\mathcal{X} \cong \text{mod-}\Lambda.$$

The above equivalence is a relative version of an equivalence given by Auslander 50 years ago, and now is called the Auslander’s formula. A considerable part of Auslander’s work on the representation theory of finite dimensional, or more general Artin, algebras is motivated by this formula. Auslander’s formula suggests that for studying $\text{mod-}\Lambda$ one may study $\text{mod-}(\text{mod-}\Lambda)$ that has nicer homological properties than $\text{mod-}\Lambda$, and then translate the results back to $\text{mod-}\Lambda$.

Inspired by the influence of the Auslander’s formula on the modern representation theory, and having in hand a relative version of the formula, as presented in the above, we are interested in studying $\text{mod-}\mathcal{X}$ and $\text{mod-}\mathcal{X}$ for some certain subcategories. One of the important subcategories in the representation theory is the subcategory of Gorenstein projective modules, which plays a key role in Gorenstein homology algebra. In this paper we focus on this kind of subcategories.

Let us in below recall the notion of Gorenstein projective modules and relevant material.

A complex

$$P^\bullet : \ldots \to P^{-1} \xrightarrow{d^{-1}} P^0 \xrightarrow{d^0} P^1 \to \ldots$$

in $\text{prj-}\Lambda$ is called a totally acyclic complex if it is acyclic and the induced Hom complex $\text{Hom}_{\Lambda}(P^\bullet, \Lambda)$ is also acyclic. A module $G$ in $\text{mod-}\Lambda$ is called Gorenstein projective if it is isomorphic to a syzygy of a totally acyclic complex $P^\bullet$ [AB, EJ]. We denote by $\text{Gprj-}\Lambda$ the full subcategory of $\text{mod-}\mathcal{X}$ consisting of all Gorenstein projective modules. By definition is is plain $\text{Gprj-}\Lambda$ contains $\text{prj-}\Lambda$.

An Artin algebra $\Lambda$ is called of finite Cohen-Macaulay type, or CM-finite for short, if there are only finitely many isomorphism classes of indecomposable finitely generated Gorenstein projective modules [B].

Clearly, $\Lambda$ is a CM-finite algebra if and only if there is a finitely generated module $G$ such that $\text{Gprj-}\Lambda = \text{add-}(G)$, where $\text{add-}(G)$ is the additive subcategory of $\text{mod-}\Lambda$ consisting of all modules isomorphic to a direct summand of a finite direct sum of copies of $G$. In this case $G$ is called an additive generator of $\text{Gprj-}\Lambda$ and, in addition, if $G$ is basic, then the Artin algebra $\text{End}_{\Lambda}(G)$ is called the Cohen-Macaulay Auslander algebra of $\Lambda$. Moreover, the stable Cohen-Macaulay Auslander algebra of $\Lambda$ is defined as the endomorphism algebra $\text{End}_{\Lambda}(G)$ of $G$ in the stable category $\text{Gprj-}\Lambda$. If $\Lambda$ is a CM-finite algebra with $G$ as an additive generator, then the evaluation functor

$$\zeta_G : \text{mod-}(\text{Gprj-}\Lambda) \to \text{mod-}\text{End}_{\Lambda}(G)$$
defined by \( \zeta_G(F) = F(G) \) is an equivalence of categories [Au2, Proposition 2.7(c)]. Further, the restriction of \( \zeta_G \) to \( \text{mod-}(\text{Gprj}\Lambda) \) induces the equivalence \( \text{mod-}(\text{Gprj}\Lambda) \cong \text{mod-}\text{End}_\Lambda(G) \).

We say that \( \Lambda \) is an \( n \)-Iwanaga-Gorenstein, or simply \( n \)-Gorenstein, algebra if the injective dimension of \( \Lambda \) both as a left and a right \( \Lambda \)-module is at most \( n \). We often omit "\( n \)" when the value is not important to know. It is known that if \( \Lambda \) is Gorenstein, then \( \text{Gprj}\Lambda \) is contravariantly finite in \( \text{mod}\Lambda \), see e.g. [B, Proposition 4.7]. In particular, over Gorenstein algebras \( \text{mod-}(\text{Gprj}\Lambda) \) is abelian.

The simplest cases of studying \( \text{mod-}(\text{Gprj}\Lambda) \), at least in the homological dimensions sense, is when the global projective dimension of \( \text{mod-}(\text{Gprj}\Lambda) \) is zero, or a semisimple abelian category, i.e., any object is projective. We call an algebra with this property \( \Omega\text{G} \)-algebra; Some basic properties of them will be studied in Section 2. Especially we show in Proposition 2.4 any \( \Omega\text{G} \)-algebra is CM-finite. Hence, an \( \Omega\text{G} \)-algebra \( \Lambda \) is nothing else to say that the associated stable Cohen-Macaulay algebra of \( \Lambda \) is a semisimple algebra. Such an observation was investigated in the remarkable paper of Auslander and Reiten, where they showed that for a dualizing \( R \)-variety \( C \): the global dimension of \( \text{mod-}(\text{mod}\text{C}) \) is zero if and only if \( \text{mod}\text{C} \) is Nakayama and whose lowey length is at most two [AR1, Theorem 10.7]. The common case with our study is when \( C = \text{prj}\Lambda \) and \( \Lambda \) is a self-injective algebra.

The class of \( \Omega\text{G} \)-algebras contains important class of algebras in the representation theory such as the class of gentle algebras, and more general the class of quadratic monomial algebras (Remark 2.9). Gentle algebras can be found in many places of mathematics. The study of gentle algebras was initiated by Assem and Skowroński [AS] in order to study iterated tilted algebras of type \( A \). Viewing gentle algebras as \( \Omega\text{G} \)-algebras provide them a functorial approach. Even though the provided functorial approach is not completely determined gentle algebras, however it might be helpful to consider them in the larger class of \( \Omega\text{G} \)-algebras.

We are also interested in studying parallel of \( \text{mod-}(\text{Gprj}\Lambda) \). The \( \Omega\text{G} \)-algebras as the simplest case which we choose to study \( \text{mod-}(\text{Gprj}\Lambda) \) also affect nicely to the representation theory of the corresponding Cohen-Macaulay Auslander algebras as our results will indicate. To study \( \text{mod-}(\text{Gprj}\Lambda) \) and \( \text{mod-}(\text{Gprj}\Lambda) \), respectively, we apply the morphism category \( \text{H}(\text{Gprj}\Lambda) \) and the monomorphism category \( \text{S}(\text{Gprj}\Lambda) \) of \( \text{Gprj}\Lambda \).

Let us first recall the aforementioned categories as follows: the morphism category of \( \text{H}(\text{mod}\Lambda) \), for short \( \text{H}(\Lambda) \), has as objects the \( \Lambda \)-homomorphisms in \( \text{mod}\Lambda \), and morphisms are given by commutative diagrams. In fact, it is equivalent to the category of finitely generated modules over the lower triangular matrix ring \( T_2(\Lambda) = \begin{bmatrix} \Lambda & 0 \\ \Lambda & \Lambda \end{bmatrix} \). We denote an object in \( \text{H}(\Lambda) \) by \( (\begin{smallmatrix} A & B \\ \alpha & \beta \end{smallmatrix})_f \), where \( f : A \to B \) a map in \( \text{mod}\Lambda \), and a morphism in \( \text{H}(\Lambda) \) between \( (\begin{smallmatrix} A & B \\ \alpha & \beta \end{smallmatrix})_f \) and \( (\begin{smallmatrix} C & D \\ \gamma & \delta \end{smallmatrix})_g \) is denoted by \( (\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix})_g \), where \( \alpha : A \to C \) and \( \beta : B \to D \) in \( \text{mod}\Lambda \) such that \( \beta f = g \alpha \). Let \( \text{S}(\text{mod}\Lambda) \) (or for short \( \text{S}(\Lambda) \)) denote the subcategory of all monomorphisms in \( \text{mod}\Lambda \). For simplicity, especially in the diagrams or figures we write \( ABf \) in stead of \( (\begin{smallmatrix} A & B \\ \alpha & \beta \end{smallmatrix})_f \), moreover if the morphism \( f \) is clear from the context we only write \( AB \), especially for the case \( f \) is either the identity morphism or zero map. The category \( \text{S}(\Lambda) \) has been recently studied extensively by Ringel and Schmidmeier [RS]. Now the morphism (resp. monomorphism) category of \( \text{H}(\text{Gprj}\Lambda) \) (resp. \( \text{S}(\text{Gprj}\Lambda) \)) is the subcategory of \( \text{H}(\Lambda) \) (resp. \( \text{S}(\Lambda) \)) consisting of all objects \( (\begin{smallmatrix} A & B \\ \alpha & \beta \end{smallmatrix})_f \) such that \( A, B \in \text{Gprj}\Lambda \) (resp. \( A, B \) and \( \text{Cok}(f) \in \text{Gprj}\Lambda \)). Under the equivalence \( \text{H}(\Lambda) \cong \text{mod-}T_2(\Lambda) \), the subcategory \( \text{S}(\text{Gprj}\Lambda) \) is mapped into the subcategory \( \text{Gprj}\text{T}_2(\Lambda) \) of Gorenstein projective modules over \( T_2(\Lambda) \).
The reason of applying the categories $H(Gprj-\Lambda)$ and $S(Gprj-\Lambda)$ is based on the relationship given by the following functors:

$$\Phi : H(Gprj-\Lambda) \rightarrow \text{mod-}(Gprj-\Lambda), \quad (A) \mapsto \text{Cok}((-,-) \rightarrow (A) \rightarrow (-,B)),$$

where we apply the Yoneda functor on the morphism $f : A \rightarrow B$ and then take cokernel of the obtained morphism $(-,f) : (-,A) \rightarrow (-,B)$ in mod-(Gprj-\Lambda), and

$$\Psi : S(Gprj-\Lambda) \rightarrow \text{mod-}(Gprj-\Lambda), \quad (A) \mapsto \text{Cok}((-,-) \rightarrow (B \rightarrow (-,\text{Cok}(f)),$$

where $p : B \rightarrow \text{Cok}(f)$ is the canonical quotient map.

The functor $\Psi$ was studied in [E] and [RZ] for the case that mod-$\Lambda$ over certain algebras $\Lambda$ are involved in place of the subcategory $Gprj-\Lambda$, indeed, we have the functor $\Psi$ from $S(\Lambda)$ to mod-(mod-$\Lambda$). We also add here the origin of the functor $\Psi$ goes back to [AR2]. Respect to fairly large class of subcategories, including $Gprj-\Lambda$, the functor $\Psi$ is treated by the author in [H1]. It is proved that the functor $\Psi$ is dense, full and objective; hence, by [RZ, Appendix], the functor induces the equivalence

$$S(Gprj-\Lambda)/U \simeq \text{mod-}(Gprj-\Lambda),$$

where $U$ is the ideal of $S(Gprj-\Lambda)$ generated by objects of the forms $(\begin{array}{c} G \\ 0 \end{array})_1$ or $(\begin{array}{c} 0 \\ G \end{array})_0$, where $G$ runs through $Gprj-\Lambda$. The quotient category $S(Gprj-\Lambda)/U$ has the same objects as $S(Gprj-\Lambda)$. For objects $X$ and $Y$, let $I(X,Y)$ denotes the morphisms of $\text{Hom}_{H(\Lambda)}(X,Y)$ that factor through a finite direct sum of objects of the forms $(\begin{array}{c} G \\ 0 \end{array})_1$ or $(\begin{array}{c} 0 \\ G \end{array})_0$ where $G \in Gprj-\Lambda$. The morphism spaces of $S(Gprj-\Lambda)/U$ are defined as the quotients $S(Gprj-\Lambda)/U(X,Y) := \text{Hom}_{H(\Lambda)}(X,Y)/I(X,Y)$.

Similar observation also holds for the functor $\Phi$. This functor was already studied in [AR2], of course for the case that we are dealing with mod-$\Lambda$ in stead of $Gprj-\Lambda$. In the same manner of $\Psi$ one can prove that the functor $\Phi$ is full, dense and objective, see also [E, Proposition 2.5]. Hence, again by [RZ, Appendix], we have the equivalence

$$H(Gprj-\Lambda)/V \simeq \text{mod-}(Gprj-\Lambda)$$

where $V$ is the ideal of $S(Gprj-\Lambda)$ generated by objects of the forms $(\begin{array}{c} G \\ 0 \end{array})_1$ or $(\begin{array}{c} 0 \\ G \end{array})_0$, where $G$ runs through $Gprj-\Lambda$.

Therefore, the above equivalences make clear why we choose $H(Gprj-\Lambda)$ and $S(Gprj-\Lambda)$ to study mod-$(Gprj-\Lambda)$ and mod-$(Gprj-\Lambda)$, respectively. In spite of inducing the equivalence via the functors $\Psi$ and $\Phi$, they also behave well with respect to the Auslander-Reiten theory. More precisely, it is proved in [H1, Section 5] that apart from certain almost split sequences in $H(Gprj-\Lambda)$ the functor $\Psi$ carries all other almost split sequences in $H(Gprj-\Lambda)$ to the ones in mod-$(Gprj-\Lambda)$. An analogous result is valid for the functor $\Phi$. For this case, we refer to [HM, Section 5]. Although, in the latter paper the functor $\Phi$ is defined for the morphism category $H(\Lambda)$ of a sufficiently nice abelian category $\mathcal{A}$. But the proof still works in our setting. However, for our use in this paper we sketch a proof for the cases in place we are dealing with.

The paper is organized and illustrated in more details as follows. In Section 2, the notion of $\Omega_{\mathcal{Q}}$-algebras is defined. Some basic properties and examples of $\Omega_{\mathcal{Q}}$-algebras will be discussed. A classification of indecomposable non-projective Gorenstein projective modules over $\Omega_{\mathcal{Q}}$-algebras will be given that is a generalization of the ones in [CSZ] for quadratic monomial algebras and [Ka] for gentle algebras to $\Omega_{\mathcal{Q}}$-algebras (see Remark 2.7 for details). The classification is given by defining an equivalence relation on the indecomposable non-projective modules in $Gprj-\Lambda$. We shall also describe the singularity category over Gorenstein $\Omega_{\mathcal{Q}}$-algebras (Proposition
2.8). In Section 3, for an \( \Omega \)-algebra \( \Lambda \), the stable Auslander-Reiten quiver of \( \mathcal{S}(\text{Gprj-} \Lambda) \) is completely determined in the main theorem of this section (Theorem 3.10). In particular, we will establish a bijection between the components of the stable Auslander-Reiten quiver and the equivalence classes of the relation already defined. In addition, if \( \Lambda \) is assumed further to be a finite dimensional algebra over an algebraic closed field \( k \) of characteristic different from two, then we will show that the stable category of Gorenstein projective \( T_2(\Lambda) \)-modules is still described well. In Section 4, we will compute the almost split sequences in \( \text{H}(\text{Gprj-} \Lambda) \) over \( \Omega \)-algebras with certain ending terms. As an interesting application in Corollary 4.4 a connection between certain simple modules over the Cohen-Macaulay Auslander algebra of an \( \Omega \)-algebra is given. In the last section, we will study those \( \Omega \)-algebras that are 1-Gorenstein. In Proposition 5.1, we will an equivalent condition of such algebras via their radicals. We will show over a 1-Gorenstein \( \Omega \)-algebras \( \Lambda \): the Cohen-Macaulay Auslander algebra of \( \Lambda \) is representation-finite if and only if so is \( \Lambda \). Finally, in Proposition 5.5 we observe that \( \Omega \)-algebras (not necessary 1-Gorenstein) are a good source to produce CM-finite algebras.

2. \( \Omega \)-algebras

In this section, we will introduce the notion of \( \Omega \)-algebras. Some basics properties and examples of \( \Omega \)-algebras will be provided. In particular, we characterize the isoclasses of indecomposable non-projective Gorenstein-projective modules over an \( \Omega \)-algebra in terms of an equivalence relation defined on \( \text{Gprj-} \Lambda \). Next we will apply this characterization to describe the singularity categories over Gorenstein \( \Omega \)-algebras. Throughout this section we assume that \( \text{Gprj-} \Lambda \) is contravariantly finite in \( \text{mod-} \Lambda \).

Recall that a subcategory \( X \) of \( \text{mod-} \Lambda \) is resolving if it contains all projectives, and closed under extensions and the kernels of epimorphisms. On the other hand, since \( \text{Gprj-} \Lambda \) is resolving it also becomes a functorially finite subcategory of \( \text{mod-} \Lambda \) (due to [KS]). So, by [ASm], we can talk about the Auslander-Reiten theory in \( \text{Gprj-} \Lambda \).

By definition of the Gorenstein projective modules, we can see for each \( G \in \text{Gprj-} \Lambda \), \( G^* = \text{Hom}_\Lambda(G, \Lambda) \) is a Gorenstein projective module in \( \text{mod-} \Lambda^{\text{op}} \). Then the duality \( (-)^* : \text{prj-} \Lambda \rightarrow \text{prj-} \Lambda^{\text{op}} \) can be naturally generalized to the duality \( (-)^* : \text{Gprj-} \Lambda \rightarrow \text{Gprj-} \Lambda^{\text{op}} \).

Motivated by Section 5 of [H2] (an unpublished paper by the author) we have the following definition.

**Definition 2.1.** Let \( \Lambda \) be an Artin algebra. Then \( \Lambda \) is called \( \Omega \)-algebra if \( \text{mod-(Gprj-} \Lambda) \) is a semisimple abelian category, i.e. all whose objects are projective.

In the sequel, we give some basic properties of \( \Omega \)-algebras.

We start with following lemmas.

**Lemma 2.2.** Let \( \Lambda \) be an Artin algebra. Let \( P \) be an indecomposable projective in \( \text{mod-} \Lambda \).

(i) If \( f : G \rightarrow \text{rad}(P) \) is a minimal right \( \text{Gprj-} \Lambda \)-approximation. Then \( i \circ f \) is a minimal right almost split morphism in \( \text{Gprj-} \Lambda \). Here, \( i : \text{rad}(P) \hookrightarrow P \) denotes the canonical inclusion.

(ii) If \( g : G' \rightarrow \text{rad}(P^{*}) \) is a minimal right \( \text{Gprj-} \Lambda^{\text{op}} \)-approximation. Then \( (j \circ g)^* : P \rightarrow (G')^{*} \) is a minimal left almost split morphism in \( \text{Gprj-} \Lambda \). Here, \( j : \text{rad}(P^{*}) \hookrightarrow P^{*} \) denotes the canonical inclusion, and by abuse of notation we identify \( P^{**} \) and \( P \) because of the natural isomorphism \( P^{**} \cong P \).

**Proof.** (i) We only need to prove that for any non-split epimorphism \( h : G_0 \rightarrow P \), there exists morphism \( l : G_0 \rightarrow G \) so that \( i \circ f \circ l = h \). Since \( h \) is a non-split epimorphism, then \( \text{Im}(h) \) is a
proper submodule in \( P \), i.e. \( \text{Im}(h) \subseteq \text{rad}(P) \). So we can write \( h = i \circ s \) for some \( s : G_0 \to \text{rad}(P) \). Now since \( f \) is a right \( \text{Gprj-}\Lambda \)-approximation, then there is \( l : G_0 \to G \) such that \( f \circ l = s \) and clearly it works for what we need.

(ii) follows from (i) and the duality \((-)^* : \text{Gprj-}\Lambda \to \text{Gprj-}\Lambda^{\text{op}} \).

\[ \square \]

Lemma 2.3. Let \( \Lambda \) be an \( \Omega_G \)-algebra. Then any short exact sequence in \( \text{Gprj-}\Lambda \) is a direct sum of the short exact sequences of the form \( 0 \to \Omega_A(A) \to P \to A \to 0 \), \( 0 \to 0 \to B \xrightarrow{1} B \to 0 \) and \( 0 \to C \xrightarrow{1} C \to 0 \to 0 \) for some \( A, B \) and \( C \) in \( \text{Gprj-}\Lambda \).

Proof. Take an exact sequence \( 0 \to G_2 \to G_1 \to G_0 \to 0 \) in \( \text{Gprj-}\Lambda \). This short exact sequence induces the sequence

\[ 0 \to (-, G_2) \to (-, G_1) \to (-, G_0) \to F \to 0 \quad (1) \]

in \( \text{mod-}\text{Gprj-}\Lambda \). Since \( \text{mod-}(\text{Gprj-}\Lambda) \) is a semisimple abelian category then \( F \cong (\cdot, G) \) for some \( G \) in \( \text{Gprj-}\Lambda \). On the other hand, we know a minimal projective resolution of \( (\cdot, G) \) is as the following

\[ 0 \to (-, \Omega_A(G)) \to (\cdot, P) \to (\cdot, G) \to (\cdot, G) \to 0 \quad (2) \]

where \( P \to G \) is a projective cover of \( G \). By comparing (1) and (2) as two projective resolutions of \( F \) in \( \text{mod-}(\text{Gprj-}\Lambda) \), and using this fact that the latter is minimal then we get our result. Indeed, the fact we have used here is known in the homology algebra, that is, any projective resolution of an object in a perfect category is a direct sum of the minimal projective resolution of the given object and possibly some split exact complexes. \( \square \)

Proposition 2.4. Let \( \Lambda \) be an \( \Omega_G \)-algebra. Then \( \Lambda \) is \( \text{CM-finite} \).

Proof. Let \( G \) be an indecomposable non-projective Gorenstein projective module. According to Lemma 2.3, since an almost split sequence does not split, so the almost split sequence in \( \text{Gprj-}\Lambda \) obtained by getting the projective cover of \( G \). Thus it is of the form \( 0 \to \Omega_A(G) \to P \xrightarrow{i} G \to 0 \) in \( \text{Gprj-}\Lambda \). Let \( Q \) be an indecomposable direct summand of \( P \). The natural injection \( f : Q \to G \) is irreducible by using the general facts in the theory of almost split sequences for subcategories, e.g. [Kr, Theorem 2.2.2]. In view of Lemma 2.2 there is a minimal left almost split morphism \( g : Q \to Y \) in \( \text{Gprj-}\Lambda \). Hence \( G \) is a direct summand of \( Y \). We know that \( \text{mod-}\Lambda \) contains only finitely many indecomposable projective modules up to isomorphism. As we have seen any indecomposable non-projective Gorenstein projective module is related to an indecomposable projective module via a minimal left almost split morphism. Because of the uniqueness of the minimal left almost split morphisms, we get the result. We are done. \( \square \)

We recall from [C1, Lemma 3.4] that for a semisimple abelian category \( \mathcal{A} \) and an auto-equivalence \( \Sigma \) on \( \mathcal{A} \), there is an unique triangulated structure on \( \mathcal{A} \) with \( \Sigma \) the translation functor. Indeed, all the triangles are split. We denote the resulting triangulated category by \((\mathcal{A}, \Sigma)\). We call a triangulated category semisimple provided that it is triangle equivalent to \((\mathcal{A}, \Sigma)\) for some semisimple abelian category.

Recall from [RV] that a Serre functor for a \( R \)-linear triangulated category \((R \text{ is a commutative artinian ring})\) \( \mathcal{T} \) is an auto-equivalence \( S : \mathcal{T} \to \mathcal{T} \) together with an isomorphism \( D \text{Hom}_\mathcal{T}(X, -) \cong \text{Hom}_\mathcal{T}(-, S(X)) \) for each \( X \in \mathcal{T} \), here \( D \) denotes the ordinary duality.

Proposition 2.5. Let \( \Lambda \) be an \( \Omega_G \)-algebra. Then

(i) \( \text{Gprj-}\Lambda \) is a semisimple triangulated category;
(ii) For any non-projective Gorenstein projective indecomposable $G$, the (relative) Auslander-Reiten translation in $\text{Gprj-} \Lambda$ is the first syzygy $\Omega \Lambda(G)$:

(iii) Serre functor over $\text{Gprj-} \Lambda$ acts on objects by the identity, i.e. $S_{\Omega}(G) \simeq G$ for each $G$ in $\text{Gprj-} \Lambda$.

Proof. (i) Assume that $\Lambda$ is an $\Omega \Lambda$-algebra. Then by Lemma 2.3 we have: any short exact sequence with all terms in $\text{Gprj-} \Lambda$ can be written as a direct sum of the short exact sequences of the form $0 \to \Omega \Lambda(A) \to P \to A \to 0$, $0 \to 0 \to B \xrightarrow{1} B \to 0$ and $0 \to C \xrightarrow{1} C \to 0 \to 0$ for some $A, B$ and $C$ in $\text{Gprj-} \Lambda$. Hence by the structure of the triangles in $\text{Gprj-} \Lambda$, which is induced by the short exact sequences in $\text{Gprj-} \Lambda$, and in view of the shape of the short exact sequences in $\text{Gprj-} \Lambda$ as we mentioned in the above whenever $\Lambda$ is an $\Omega \Lambda$-Algebra, we get all possible triangles in $\text{Gprj-} \Lambda$ are a direct sums of the following trivial triangles $0 \to K[1] \xrightarrow{1} K[1]$, $K \xrightarrow{1} K \to 0 \to K[1]$ and $0 \to K \xrightarrow{1} K \to 0$. By the axioms of triangulated categories, every morphism $f : X \to Y$ in $\text{Gprj-} \Lambda$ is completed to a triangle, hence, by the above, we can write as $f = (f_1, f_0) : X_1 \oplus Y_1 \to X_2 \oplus Y_2$, where $f_1 : X_1 \to X_2$ is an isomorphism in $\text{Gprj-} \Lambda$. Define the natural injection $i : Y_1 \to X_1 \oplus Y_1$ and the natural projection $X_2 \oplus Y_2 \to Y_2$ as kernel and cokernel of $f$, respectively. Then this gives a semisimple abelian structure over $\text{Gprj-} \Lambda$. Now if we consider the equivalence $\Omega \Lambda : \text{Gprj-} \Lambda \to \text{Gprj-} \Lambda$, then the resulting triangulated category $(\text{Gprj-} \Lambda, \Omega)$ is triangle equivalent to $\text{Gprj-} \Lambda$, so we get (i).

(ii) follows from Lemma 2.3 since any almost split sequence does not split. We can deduce (iii) in view of (ii) and using this point that $\tau_{\Omega} = S_{\Omega} \circ \Omega \Lambda$.

The above proposition explains for a justification of the terminology of $\Omega \Lambda$-algebras. The notation “$\Omega \Lambda$” stand for the Auslander-Reiten translation $\tau_{\Omega}$ coincides with the first syzygy $\Omega \Lambda$.

Proposition 2.6. Let $\Lambda$ be an $\Omega \Lambda$-algebra. Then $\Lambda^{\text{op}}$ is also an $\Omega \Lambda$-algebra.

Proof. It is easily proved by using the duality $(-)^* : \text{Gprj-} \Lambda \to \text{Gprj-} \Lambda^{\text{op}}$. \hfill \Box

Let $\Lambda$ be an $\Omega \Lambda$-algebra. As there are only a finite number of non-isomorphic indecomposable Gorenstein projective modules (up to isomorphism) and the syzygy functor preserves the indecomposable Gorenstein projective modules ( [C2, Lemma 2.2]), so we get the non-projective Gorenstein projective modules are $\Omega \Lambda$-periodic modules, meaning that there exists $n > 0$, depending on a given non-projective Gorenstein projective module $G$, such that $\Omega_n \Lambda(G) \simeq G$.

On indecomposable non-projective modules in $\text{Gprj-} \Lambda$, we define an equivalence relation, that is, $G \sim G'$ if and only if there is some $n \in \mathbb{Z}$ such that $G \simeq \Omega_n \Lambda(G')$. Note that here $\Omega_n \Lambda(G)$ for $n < 0$ are defined by using right (minimal) projective resolution of $G$. In fact, since $\text{Gprj-} \Lambda$ is a Frobenius category with $\text{prj-} \Lambda$ as projective-injective objects, we can construct a right projective resolution by $\text{prj-} \Lambda$.

Let $\mathcal{C}(\Lambda)$ denote the set of the equivalence classes with respect to the relation $\sim$. Moreover, for an indecomposable non-projective Gorenstein projective module $G$ we write $l(G)$ for the minimum number $n > 0$ such that $\Omega_n \Lambda(G) \simeq G$. One can see the equivalence class $[G]$ has $l(G)$ elements up to isomorphism.

Let $G$ be an indecomposable non-projective module in $\text{Gprj-} \Lambda$. We point out if $\delta_1 : 0 \to \Omega \Lambda(G) \to P \to G \to 0$ is an almost split sequence in $\text{Gprj-} \Lambda$, then for every $i \geq 0$, the short exact sequence $0 \to \Omega \Lambda^{i+1}(G) \to P^i \to \Omega \Lambda(G) \to 0$, setting $P^0 = P$, is an almost split sequence in $\text{Gprj-} \Lambda$. Indeed, the middle term of the almost split sequence $0 \to \tau_{\Omega}(\Omega \Lambda(G)) \to B \to \Omega \Lambda(G) \to 0$ in $\text{Gprj-} \Lambda$ has no non-projective direct summand, otherwise the middle term of $\delta_1$ has an
indecomposable non-projective direct summand, which is a contradiction. Hence $B$ is projective so $\tau_G(\Omega_\Lambda(G)) \cong \Omega_\Lambda^2(A)$. Repeating the same argument we can prove for all $i \geq 0$. Dually, if the short exact sequence $\delta_2 : 0 \to G \to Q \to \Sigma_\Lambda(G) \to 0$ is an almost split sequence in $\text{Gprj-}\Lambda$, then for every $i \geq 0$, the short exact sequence $0 \to \Sigma_\Lambda(G) \to Q^i \to \Sigma_\Lambda^{i+1}(G) \to 0$, setting $Q^0 = Q$, is an almost split sequence in $\text{Gprj-}\Lambda$. Lastly, If both the short exact sequences $\delta_1$ and $\delta_2$ are almost split sequences in $\text{Gprj-}\Lambda$, then the $\tau_G$-orbit of $G$ consists entirely of $\Omega_\Lambda^i(G)$ and $\Sigma_\Lambda^i(G)$ for all $i \geq 0$. Therefore, in this case the $\tau_G$-orbit of $G$ is equal to the equivalence class $[G]$ in $\mathcal{C}(\Lambda)$ (see also Propositions 3.5 and 3.7 for an alternative way to get the above observation).

**Remark 2.7.** Let $\Lambda = kQ/I$ be a monomial algebra, that is, the quotient algebra of the path algebra $kQ$ of a finite quiver $Q$ modulo the ideal $I$ generated by the paths of length at least 2. Following [CSZ, Definition 3.7], we recall that a non-zero path $p$ in $\Lambda$ a perfect path, provided that there exists a sequence $p = p_1, p_2, \ldots, p_n, p_{n+1} = p$ of non-zero paths such that $(p_i, p_{i+1})$ are perfect pairs (see [CSZ, Definition 3.3]) for all $1 \leq i \leq n$. If the given non-zero paths $p_i$ are pairwise distinct, we refer to the sequence $p = p_1, p_2, \ldots, p_n$ as a relation cycle for $p$, which has length $n$. It is proved in [CSZ, Theorem 4.1] that there is a bijection between the set of perfect paths in $\Lambda$ and the isoclass set of indecomposable non-projective Gorenstein-projective $\Lambda$-modules, which is a generalization of the classification over gentle algebras given in [Ka]. The bijection is given by sending a perfect path $p$ to the $A$-module $pA$, where the right ideal $pA$ is generated by all paths $q \notin I$ such that $q = pf'$ for some path $f'$.

The situation is better when we concentrate on the quadratic monomial algebras. We recall that $\Lambda$ is quadratic monomial provided that the ideal $I$ generated by paths of length 2 in the rest of this remark assume $\Lambda$ is a quadratic monomial algebra. By [CSZ, lemma 3.4] for a perfect pair $(p, q)$ in $\Lambda$, both $p$ and $q$ are necessarily arrows. In particular, a perfect path is an arrow and its relation cycle consists entirely of arrows. Then by using [CSZ, Theorem 4.1] we get there is a bijection between perfect arrows in $\Lambda$ and the isoclass set of indecomposable non-projective Gorenstein-projective $\Lambda$-modules.

In [CSZ, Definition 5.1] the relation quiver $R_\Lambda$ of $\Lambda$ is defined as follows: the vertices are given by the arrows in $Q$, and there is an arrow $[\beta \alpha] : \alpha \to \beta$ if the head of $\beta$ is equal to the tail of $\alpha$ and $\beta \alpha$ lies in $I$. A component $D$ of $R_\Lambda$ is called perfect if it is a basic cycle [CSZ, definition 5.1]. From [CSZ, Lemma 5.3] we observe that an arrow $\alpha$ is perfect if and only if the corresponding vertex in $R_\Lambda$ belongs to a perfect component. As mentioned in the proof of [CSZ, Theorem 5.7], for a perfect arrow $\alpha$ lying in the perfect component $C$, its relation cycle is of the form $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_d$ with $\alpha_{d+1} = \alpha$. And we have $\Omega_\Lambda(\alpha_1 \Lambda) = \alpha_{i+1} \Lambda$ and $\Sigma(\alpha_i \Lambda) = \alpha_{i-1} \Lambda$. According to the above observation, we see that the set formed by all perfect components of the relation quiver $R_\Lambda$ are in one-to-one correspondence with the elements of $\mathcal{C}(\Lambda)$, by sending a given perfect component $D$ to the equivalence class $[\alpha \Lambda]$ in $\mathcal{C}(\Lambda)$, where $\alpha$ is a vertex of $D$.

Let $\mathcal{T}$ be an additive category and $F : \mathcal{T} \to \mathcal{T}$ an automorphism. Following [K], the orbit category $\mathcal{T}/F$ has the same objects as $\mathcal{T}$ and its morphisms from $X$ to $Y$ are $\oplus_{n \in \mathbb{Z}} \text{Hom}_\mathcal{T}(X, F^n(Y))$. Further, the singularity category $\mathbb{D}_{sg}(\Lambda)$ of an algebra $\Lambda$ is defined as the Verdier quotient of the bonded derived category $\mathbb{D}^b(\text{mod-}\Lambda)$ by the subcategory of bounded complexes of projective modules.

**Proposition 2.8.** Let $\Lambda$ be an $\Omega_\mathbb{Z}$-algebra over an algebraic closed filed $k$. If $\Lambda$ is a Gorenstein algebra, then there is an equivalence of triangulated categories

$$\mathbb{D}_{sg}(\Lambda) \simeq \prod_{[G] \in \mathcal{C}(\Lambda)} \mathbb{D}^b(\text{mod-}k)_{l(G)}$$
where \( D^b_{\text{mod-}}(\mod-k) \) denotes the triangulated orbit category of \( D^b(\mod-k) \) with respect to the functor \( F = [l(G)] \), the \( l(G) \)-the power of the shift functor.

**Proof.** By the Buchweitz’s equivalence over Gorenstein algebras, \( D_{sg}(\Lambda) \simeq \text{Gprj-}\Lambda \), so it is enough to describe \( \text{Gprj-}\Lambda \). Due to the lines before Remark 2.7 we see than any indecomposable non-projective in \( \text{Gprj-}\Lambda \) is isomorphic to an module in some equivalence class \([G]\). For any equivalence class \([G]\) \( \in \mathcal{C}(\Lambda) \), we have \( \sum l(G) \simeq G' \) in \( \text{Gprj-}\Lambda \) for any \( G' \in [G] \). Recall that the suspension functor \( \sum \) on \( \text{Gprj-}\Lambda \) is the quasi-inverse of the syzygy \( \Omega_{\Lambda} \). On the other hand, by the proof of Proposition 2.5, we deduce that any morphism between two objects in \( \text{Gprj-}\Lambda \) are either split monomorphism or split epimorphism. Hence for every indecomposable modules \( X \) and \( Y \) in \( \text{Gprj-}\Lambda \), \( \text{Hom}_{\Lambda}(X,Y) = 0 \) if \( X \not\cong Y \), and each non-zero morphism \( f \) in \( \text{Hom}_{\Lambda}(X,Y) \) is an isomorphism if \( X \cong Y \). In the latter case, \( \text{End}_{\Lambda}(X) = \text{End}_{\Lambda}(Y) \) (or \( \cong \text{End}_{\Lambda}(X,Y) \)) as \( k \)-vector spaces. But \( \text{End}_{\Lambda}(X) \) is a division algebra containing \( k \), as \( k \) is algebraic closed, this implies that \( \text{End}_{\Lambda}(G) \simeq k \) as \( k \)-vector space (or more as algebras). Thus, \( \text{Hom}_{\Lambda}(X,Y) \cong k \).

Now the rest of the proof goes through as for \([\text{Ka}, \text{Theorem 2.5(b)}] \). \( \Box \)

The above result is inspired by the similar equivalence in \([\text{Ka}] \) for gentle algebras and for quadratic monomial algebras in \([\text{CSZ}] \).

**Remark 2.9.**

1. The algebras of finite global dimension are triviality \( \Omega_{\mathcal{G}} \)-algebras, since \( \text{Gprj-}\Lambda = \text{proj-}\Lambda \). Although, our results in this paper are not interesting for them.
2. Let \( \Lambda = kQ/I \) be a quadratic monomial algebra. In \([\text{CSZ}, \text{Theorem 5.7}] \), it is proved that there is a triangle equivalence
   \[
   \text{Gprj-}\Lambda \simeq T_{d_1} \times T_{d_2} \times \cdots \times T_{d_m}
   \]
   where \( T_{d_n} = \text{mod-}(k^n, \sigma^n) \) for some natural number \( n \), and auto-equivalence \( \sigma_n : \text{mod-}k^n \to \text{mod-}k^n \). Thus by the above equivalence a quadratic monomial algebra is an \( \Omega_{\mathcal{G}} \)-algebra, as for each \( m \), \( \text{mod-}(\text{mod-}k^m) \) is semisimple. Especially, gentle algebras are \( \Omega_{\mathcal{G}} \)-algebras.
3. Let \( \Lambda \) be a simple gluing algebra of \( A \) and \( B \). See the papers \([\text{L1}] \) and \([\text{L2}] \) for two different ways of defining gluing algebras. In these two mentioned papers, for both types of gluing, is proved that \( \text{Gprj-}\Lambda \simeq \text{Gprj-}\Lambda \coprod \text{Gprj-}\Lambda \). Therefore, if \( \text{mod-}(\text{Gprj-}\Lambda) \) and \( \text{mod-}(\text{Gprj-}\Lambda) \) are semisimple, then \( \text{mod-}(\text{Gprj-}\Lambda) \), by definition, so is. So simple gluing operation preserve \( \Omega_{\mathcal{G}} \)-algebras. For example, cluster-tilted algebras of type \( \Lambda_n \) and endomorphism algebras of maximal rigid objects of cluster tube \( \mathcal{C}_n \) can be built as simple gluing \( \Omega_{\mathcal{G}} \)-algebras, see \([\text{L2}, \text{Corollaries 3.21 and 3.22}] \). Hence this kind of the cluster-tilted algebras are \( \Omega_{\mathcal{G}} \)-algebras.
4. By Theorem 3.24 in \([\text{AHV}] \), if \( \Lambda \) and \( \Lambda' \) are derived equivalent, then \( \text{Gprj-}\Lambda \simeq \text{Gprj-}\Lambda' \). Therefore, the property of being semisimple of \( \text{mod-}(\text{Gprj-}\Lambda) \) and \( \text{mod-}(\text{Gprj-}\Lambda') \) is preserved under the derived equivalence. Hence \( \Omega_{\mathcal{G}} \)-algebras are closed under derived equivalences.
5. There are many examples of CM-finite not-\( \Omega_{\mathcal{G}} \)-algebras. Let us first give a remark. It was shown in \([\text{AR1}, \text{Theorem 10.7}] \), for an arbitrary Artin algebra \( \Lambda \), \( \text{mod-}(\text{mod-}\Lambda) \) is a semisimple abelian category if and only if \( \Lambda \) is Nakayama algebra with loewy length at most 2. By using this fact, we see that the self-injective Nakayama algebras \( k[x]/(x^n), n > 2 \), are examples of CM-finite not-\( \Omega_{\mathcal{G}} \)-algebras.
3. The monomorphism category over $\Omega G$-algebras

In this section, all the almost split sequence in $S(\text{Gprj-}\Lambda)$ over an $\Omega G$-algebra will be computed. As an interesting application we will provide a nice description of the singularity category of $T_2(\Lambda)$.

The following result was first appeared in a primary version of [H1] available on arXiv. This result was omitted in the published version of it.

**Proposition 3.1.** Assume the subcategory of Gorenstein projective modules $\text{Gprj-}\Lambda$ is contravariantly finite in $\text{mod-}\Lambda$. Then $S(\text{Gprj-}\Lambda)$ is functorially finite in $H(\Lambda)$. In particular, $S(\text{Gprj-}\Lambda)$ has almost split sequences.

**Proof.** Since $S(\text{Gprj-}\Lambda)$ is indeed the subcategory of Gorenstein projective modules in $H(\Lambda)$ and so a resolving subcategory, then by [KS, Corollary 0.3] it is enough to show that $S(\text{Gprj-}\Lambda)$ is contravariantly finite in $H(\Lambda)$. We know from [RS, Theorem 2.5] that $S(\Lambda)$ is contravariantly finite in $H(\Lambda)$, hence by making use of this fact, it is enough to show that any object in $S(\Lambda)$ has a right $S(\text{Gprj-}\Lambda)$-approximation. Take an arbitrary object $(\frac{M_1}{M_2})_f$ in $S(\Lambda)$. Hence $f$ is a monomorphism. Let $G_1 \xrightarrow{g} \text{Cok}(f) \rightarrow 0$ be a minimal right $\text{Gprj-}\Lambda$-approximation of $\text{Cok}(f)$ in $\text{mod-}\Lambda$, which it exists by our assumption. Further, set $K_1 := \text{Ker}(g)$, since $g$ is minimal then by Wakamutsu’s Lemma, $K_1 \in \text{Gprj-}\Lambda^+$, i.e., $\text{Ext}^1(G, K_1) = 0$ for any $G$ in $\text{Gprj-}\Lambda$. Consider the following pull-back diagram.
Let \( G_2 \xrightarrow{t} U \to 0 \) be a minimal right Gprj-\( \Lambda \)-approximation. Again by applying Wakamutsu’s Lemma, \( K_2 := \text{Ker}(t) \) belongs to Gprj-\( \Lambda^\perp \). Now consider the following pull-back diagram

\[
\begin{array}{cccccccccccc}
& & & & & & 0 & & & & & & 0 \\
& & & & & & K_2 & \rightarrow & K_3 & \rightarrow & K_1 & \rightarrow & 0 \\
& & & & & 0 & \rightarrow & K_2 & \rightarrow & G_2 & \xrightarrow{t} & U & \rightarrow & 0 \\
& & & & & \downarrow{dt} & & \downarrow{d} & & \downarrow{d} & & \downarrow{d} & & \\
& & & & & M_2 & \rightarrow & M_2 & \rightarrow & M_2 & \rightarrow & M_2 & \rightarrow & 0 \\
& & & & & 0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 \\
\end{array}
\]

Since both \( K_1, K_2 \) are in Gprj-\( \Lambda^\perp \), so the first row in above digram follows \( K_3 \) so is. Finally, applying the Snake lemma for the following commutative diagram

\[
\begin{array}{cccccccccccc}
& & & & & & 0 & & & & & & 0 \\
& & & & & & G_3 & \rightarrow & G_2 & \rightarrow & 0 \\
& & & & & 0 & \rightarrow & M_1 & \xrightarrow{h} & U & \xrightarrow{i} & G_1 & \rightarrow & 0 \\
& & & & & \downarrow{ht} & & \downarrow{lt} & & \downarrow{lt} & & \downarrow{lt} & & \\
& & & & & 0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 \\
\end{array}
\]

yields the following short exact sequence

\[
0 \to K_2 \to G_3 \to M_1 \to 0,
\]

where \( G_3 := \text{Ker}(lt) \), that is a Gorenstein projective module since it is a kernel of an epimorphism in Gprj-\( \Lambda \). Putting together the maps obtained in the above, the following commutative diagram can be constructed

\[
\begin{array}{cccccccccccc}
& & & & & & 0 & & & & & & 0 \\
& & & & & & K_2 & \rightarrow & G_3 & \xrightarrow{p} & M_1 & \rightarrow & 0 \\
& & & & & 0 & \rightarrow & K_3 & \rightarrow & G_2 & \xrightarrow{dt} & M_2 & \rightarrow & 0 \\
& & & & & \downarrow{lt} & & \downarrow{lt} & & \downarrow{lt} & & \downarrow{lt} & & \\
& & & & & 0 & \rightarrow & K_1 & \rightarrow & G_1 & \xrightarrow{g} & \text{Cok}(f) & \rightarrow & 0 \\
& & & & & \downarrow{0} & & \downarrow{0} & & \downarrow{0} & & \downarrow{0} & & \\
& & & & & 0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 \\
\end{array}
\]

From the above digram, we obtain the following short exact sequence in \( H(\Lambda) \)
such that the middle term lies in \( S(Gprj-\Lambda) \) and \( K, K_3 \) are in \( Gprj-\Lambda^\perp \). Consider an arbitrary object \( \left( \frac{G}{G'} \right) _s \) in \( S(Gprj-\Lambda) \). In view of the following short exact sequence

\[
\begin{array}{ccccccccc}
\lambda : & 0 & \rightarrow & (\frac{K_3}{K_2})_v & \rightarrow & (\frac{G_3}{G_2})_w & \rightarrow & (\frac{P}{P_{dt}}) & \rightarrow & (\frac{M_1}{M_2})_f & \rightarrow & 0 \\
\end{array}
\]

where \( ij = s \) is an epi-mono factorization of \( s \) and \( \pi ' \) the canonical epimorphism, consequently \( j \) is an isomorphism. The sequence implies that \( \text{Ext}_1^{H(\Lambda)}((\frac{G}{G'})_s, (\frac{K_3}{K_2})_w) = 0 \) for any object \( (\frac{G}{G'})_s \) in \( S(Gprj-\Lambda) \). Indeed, by the known adjoint pairs between \( H(\Lambda) \) and \( \text{mod-}\Lambda \) along with the vanishing of \( \text{Ext} \) for \( K, K_3 \), we have \( \text{Ext}_1^{H(\Lambda)}((\frac{G}{G'})_s, (\frac{K_3}{K_2})_w) = \text{Ext}_1^A(G, K_2) = 0 \) and \( \text{Ext}_1^{H(\Lambda)}((\text{Cok}(s))_0, (\frac{K_3}{K_2})_w) = \text{Ext}_1^A(\text{Cok}(s), K_3) = 0 \), and consequently, the desired vanishing of \( \text{Ext}_3 \) in \( H(\Lambda) \). This follows the epimorphism included in the sequence \( (\lambda) \) is a right \( S(Gprj-\Lambda) \)-approximation of \( (\frac{M_1}{M_2})_f \) in \( H(\Lambda) \). Now the proof is complete. \( \square \)

We need the following lemmas.

**Lemma 3.2.** (1) Let \( P \) be an indecomposable projective module in \( \text{mod-}\Lambda \) with radical \( \text{rad}\,P \). The objects \( (\frac{0}{P})_0 \) and \( (\frac{0}{P})_1 \) are indecomposable projective objects in \( S(\Lambda) \) (and more in \( H(\Lambda) \)). Each indecomposable (relatively) projective object arises in this way. Further, the following inclusions play as sink maps in \( S(\Lambda) \) (and more in \( H(\Lambda) \))

\[
\begin{array}{ccccccccc}
(i) : & (\frac{0}{\text{rad}\,P})_0 & \rightarrow & (\frac{0}{P})_0 & \quad \text{and} \quad & (\frac{0}{P})_1 & \rightarrow & (\frac{0}{P})_1 , \\
\end{array}
\]

where \( i : \text{rad}\,P \rightarrow P \) the canonical inclusion, respectively.

(2) Let \( I \) be an indecomposable injective module in \( \text{mod-}\Lambda \) with socle \( \text{soc}\,I \). Then the objects \( (\frac{0}{I})_1 \) and \( (\frac{0}{I})_0 \) are indecomposable injective objects in \( S(\Lambda) \). Each indecomposable (relatively) injective object in \( S(\Lambda) \) arises in this way. Further, the following inclusions play as source maps in \( S(\Lambda) \)

\[
\begin{array}{ccccccccc}
(p) : & (\frac{0}{I})_1 & \rightarrow & (\frac{I/\text{soc}\,I}{I/\text{soc}\,I})_1 & \quad \text{and} \quad & (\frac{0}{I})_0 & \rightarrow & (\frac{\text{soc}\,I}{I})_0 , \\
\end{array}
\]

where \( p : I \rightarrow I/\text{soc}\,I \) the canonical quotient and \( j : \text{soc}\,I \hookrightarrow I \) the inclusion, respectively.

**Proof.** See [RS, Proposition 1.4]. \( \square \)

**Lemma 3.3.** (1) Let \( P \) be an indecomposable projective module in \( \text{mod-}\Lambda \) with radical \( \text{rad}\,P \). The objects \( (\frac{0}{P})_0 \) and \( (\frac{0}{P})_1 \) are indecomposable projective-injective objects. Each indecomposable injective object arises in this way.

(2) The following composition map is sink in \( S(Gprj-\Lambda) \)

\[
(\frac{0}{I}) \circ (\frac{0}{I}) : (\frac{0}{I})_0 \rightarrow (\frac{0}{\text{rad}\,P})_0 \rightarrow (\frac{0}{P})_0 
\]

where \( f : G \rightarrow \text{rad}\,P \) a minimal right \( Gprj-\Lambda \)-approximation and \( i : \text{rad}\,P \rightarrow P \) the canonical inclusion.

**Proof.** (1) See [HM, Proposition 1]. Set \( K := \text{Ker}(f) \). By Wakamutsu’s Lemma we have \( K \in Gprj-\Lambda^\perp \). For any \( (\frac{G}{G'})_e \) in \( S(Gprj-\Lambda) \), by applying \( \text{Ext}_1^{H(\Lambda)}(\cdot, (\frac{0}{K})_0) \) on the following short exact sequence
where, by construction, \( g \) is an isomorphism, we obtain \( \text{Ext}^1_{\mathcal{H}(\Lambda)}((C)_e, (0)_0) = 0 \). Hence 
\( (0)_K \in \mathcal{S}(\text{Gprj-}\Lambda) \). This implies that \( (\overline{0})_f : (C)_e \to (0)_{\text{rad}P} \) is a right \( \mathcal{S}(\text{Gprj-}\Lambda) \)-approximation, an moreover minimal in \( \mathcal{S}(\text{Gprj-}\Lambda) \) by using the (right) minimality of \( f \) in \( \text{mod-}\Lambda \). Now by using the fact we have already proved for the morphism \( (\overline{0})_f \) being a minimal right \( \mathcal{S}(\text{Gprj-}\Lambda) \)-approximation, we will show that \( (\overline{0})_f \) is right minimal almost split (sink). If \( (\overline{0})_f \circ (0)_g = (\overline{0})_h \), then \( h f = f \). Because of being right minimal of \( f \), we get \( h \) is an isomorphism, consequently so is \( (\overline{0})_h \). Thus \( (\overline{0})_h \) is right minimal. Finally, we will show that it is right almost split. Assume \( (\overline{0})_g : (\overline{X})_v \to (\overline{0})_{\mathcal{I}_v} \) is not a retraction. If \( (\overline{0})_g \) is not a retraction, then as the inclusion \( (\overline{0})_g \) is sink, there exists \( (\overline{0})_g = (\overline{0}) \circ (\overline{0})_v \). As we proved \( (\overline{0})_g \) is a right \( \mathcal{S}(\text{Gprj-}\Lambda) \)-approximation, so there is \( (\overline{0})_g : (\overline{X})_v \to (\overline{0})_{\mathcal{I}_v} \) such that \( (\overline{0})_g = (\overline{0}) \circ (\overline{0})_v \). Hence \( (\overline{0})_g \) factors through \( (\overline{0})_v \) via \( (\overline{0})_v \), as desired. We are done.

In the rest, we assume \( \mathcal{M} \) is either \( \text{mod-}\Lambda \) or \( \text{Gprj-}\Lambda \). According to which category we take then the relevant concept will be defined. For instance, when we say \( A \) is an injective object in \( \mathcal{M} \): if \( \mathcal{M} = \text{mod-}\Lambda \), then \( A \in \text{inj-}\Lambda \), and if \( \mathcal{M} = \text{Gprj-}\Lambda \), then \( A \in \text{prj-}\Lambda \).

**Lemma 3.4.** Assume \( \delta : 0 \to A \overset{f}{\to} B \overset{g}{\to} C \to 0 \) is an almost split sequence in \( \mathcal{M} \). Then

1. The almost split sequence in \( \mathcal{S}(\mathcal{M}) \) ending at \((\overline{0})_C\) has of the form

\[
\begin{array}{c}
0 \to (\overline{A})_1 \to (\overline{B})_f \to (\overline{C})_0 \to 0.
\end{array}
\]

2. Let \( e : A \to I \) be an injective envelop of \( A \). Then the almost split sequence in \( \mathcal{S}(\mathcal{M}) \)
ending at \((\overline{C})_1\) has of the form

\[
\begin{array}{c}
0 \to (\overline{A})_e \to (\overline{B})_h \to (\overline{C})_1 \to 0.
\end{array}
\]

where \( h \) is the map \((\overline{0})_I\) with \( e' : B \to I \) is an extension of \( e \). Indeed, it is induced from the following push-out diagram

\[
\begin{array}{ccc}
A & \overset{f}{\to} & B
\downarrow{e} & \downarrow{h}
\end{array}
\]

\[
\begin{array}{ccc}
I & \overset{(1)}{\to} & I \oplus C
\end{array}
\]

(3) Let \( p : P \to C \) be the projective cover of \( C \). Then the almost split sequence in \( \mathcal{S}(\mathcal{M}) \)
starting at \((\overline{0})_A\) has of the form

\[
\begin{array}{c}
0 \to (\overline{A})_0 \to (\overline{\Omega}(C))_h \to (\overline{0})_P \to 0.
\end{array}
\]

where \( h \) is the kernel of morphism \((f, p') : A \oplus P \to B \); here \( p' \) is a lifting of \( p \) to \( g \). Indeed, it is induced from the following pull-back diagram

\[
\begin{array}{ccc}
A & \to & B
\downarrow{g} & \downarrow{\Omega}(C)
\end{array}
\]

\[
\begin{array}{ccc}
I & \to & C
\downarrow{h} & \downarrow{p'}
\end{array}
\]

\[
\begin{array}{ccc}
P & \to & C
\end{array}
\]
Proof. We refer to [H1, Lemma 6.3]. □

Proposition 3.5. Assume that $N$ is an indecomposable non-projective module in $\mathcal{M}$. Suppose that there exists the almost split sequence $0 \to M \xrightarrow{i} P \xrightarrow{p} N \to 0$ in $\mathcal{M}$ with $P$ projective. If $M$ is non-projective and let $0 \to V \xrightarrow{i_1} Q \xrightarrow{p_1} M \to 0$ be the almost split sequence ending at $M$ in $\mathcal{M}$. Then, the following assertions hold.

1. $Q$ is injective in $\mathcal{M}$;
2. there are the following almost split sequences in $\mathcal{S}(\mathcal{M})$
   \[
   0 \to (\frac{V}{Q})_{i_1} \oplus (\frac{0}{M})_0 \to (\frac{Q}{Q})_1 \oplus (\frac{0}{M})_0 \to (\frac{M}{M})_1 \to 0;
   \]
3. in particular, there is the following full sub-quiver in $\Gamma_{\mathcal{M}}$

\[
\begin{array}{c}
0 \to (\frac{0}{M})_0 \\
\vdots \\
0 \to (\frac{V}{Q})_{i_1} \\
\vdots \\
0 \to (\frac{Q}{Q})_1 \\
\vdots \\
0 \to (\frac{N}{M})_0 \\
\end{array}
\]

where a decomposition $P = \oplus_i P_i$, resp. $Q = \oplus_i Q_i$ of $P$, resp. $Q$, into indecomposable modules.

Proof. Assume that $N$ satisfies the condition of the statement. According to Lemma 3.4(1) there is the almost split sequence

\[
\begin{array}{c}
0 \to (\frac{M}{M})_1 \\
\vdots \\
0 \to (\frac{M}{P})_{i_0} \\
\vdots \\
0 \to (\frac{0}{N})_0 \\
\end{array}
\]

in $\mathcal{S}(\mathcal{M})$. Hence $(\frac{M}{M})_1$ is a direct summand of the middle term of the almost split sequence $\eta$ in $\mathcal{S}(\mathcal{M})$ ending at $(\frac{M}{P})_{i_0}$. Again using Lemma 3.4(3) the middle term of $\eta$ is obtained by the
following pull-back diagram, that is $\left(\frac{M}{M \oplus P}\right)_h$:

```
\[
\begin{array}{c}
M \\
\downarrow h \\
M \\
\downarrow i_0 \\
P \\
\downarrow p_0 \\
N
\end{array}
\]
```

Since $\left(\frac{M}{M}\right)_1$ is a direct summand of $\left(\frac{M}{M \oplus P}\right)_h$ and $M$ being non-projective but $P$ a projective, we easily get the isomorphisms $\left(\frac{M}{M \oplus P}\right)_h \simeq \left(\frac{M}{M}\right)_1 \oplus \left(\frac{0}{P}\right)_0 = \left(\frac{M}{M}\right)_1 \oplus \left(\sum \frac{0}{P}\right)_0$, so the sequence $\eta$ gets the desired form.

Similarly, from the almost split sequence $\eta$ we obtain $\left(\frac{\eta}{\eta}\right)_0$ is a direct summand of the almost split sequence $\eta'$ ending at $\left(\frac{M}{M}\right)_1$ in $\mathcal{S}(\mathcal{M})$. We infer from Lemma 3.4(2):

$$\eta': 0 \rightarrow \left(\frac{M}{M}\right)_1 \rightarrow \left(\frac{\eta}{\eta}\right)_d \rightarrow \left(\frac{\mathcal{M}}{\mathcal{M}}\right)_1 \rightarrow 0.$$  

where $\epsilon$ is an injective envelop for when $\mathcal{M} = \text{mod-} \Lambda$ and a minimal left prj-$\Lambda$-approximation of $V$ for the case $\mathcal{M} = \text{Gprj-} \Lambda$. Because of this fact that $\left(\frac{M}{M}\right)_0$ is a direct summand of the middle term of $\eta'$ and $M$ being non-injective, we easily get the isomorphism $\left(\frac{\eta}{\eta}\right)_d \simeq \left(\frac{\eta}{\eta}\right)_0 \oplus \left(\frac{\eta}{\eta}\right)_\epsilon$.

As the middle term $\left(\frac{M}{M}\right)_i$ of the sequence $\lambda$ is indecomposable, the object $\left(\frac{\eta}{\eta}\right)_\epsilon$, must be injective. Thanks to the classification of the injective objects in $\mathcal{S}(\mathcal{M})$ (see Lemma 3.2 for when $\mathcal{M} = \text{mod-} \Lambda$ and Lemma 3.3 for the case $\mathcal{M} = \text{Gprj-} \Lambda$), $\epsilon$ is a split monomorphism and $Q$ an injective module, so the statement (1) follows. To complete the proof we have to show that $\epsilon'$ has no direct summand of the form $\left(\frac{0}{G}\right)_0$ for some injective module $J$. If this case holds, then the map in $\left(\frac{M}{M}\right)_1$ would not be the identity map, a contradiction. Indeed, for when $\mathcal{M} = \text{mod-} \Lambda$, if we assume that there is an indecomposable non-zero direct summand $\left(\frac{0}{G}\right)_0$ for some indecomposable injective module $J$, by Lemma 3.2 we have the source map $\left(\frac{\eta}{\eta}\right)_0 \rightarrow \left(\frac{\text{soc} J}{\text{soc} J}\right)_1$, where $i : \text{soc} J \rightarrow J$ the canonical inclusion, this means that $\left(\frac{M}{M}\right)_1 \simeq \left(\frac{\text{soc} J}{\text{soc} J}\right)_1$, and so $\text{soc} J \simeq J$, we reach a contradiction. If $\mathcal{M} = \text{Gprj-} \Lambda$, then by Lemma 3.3 the domain of the sink map ending at $\left(\frac{\eta}{\eta}\right)_0$ is $\left(\frac{0}{G}\right)_0$, where $G$ is a minimal right Gprj-$\Lambda$-approximation of radJ. Hence $V$ would be zero, which is a contradiction. We have proved (1) and (2) so far. The claim (3) follows from (1) and (2) along with Lemma 3.4. So we are done.

\[\square\]

Remark 3.6. (1) For the case $\mathcal{M} = \text{mod-} \Lambda$, there is a nice description of the almost split sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ with $B$ either a projective module or an injective module via whose ending terms as follows (see [ARS, §V Theorem 3.3]): (1) The module $B$ is injective if and only if $C$ is a non-projective simple module which is not a composition factor of rad$P$/soc$P$ for any projective module $P$. (2) The module $B$ is projective if and only if $C$ is a non-injective simple module which is not a composition factor of rad$I$/soc$I$ for any injective module $I$.

(2) For $\mathcal{M} = \text{Gprj-} \Lambda$ we do not know general information when the middle term of an almost split sequence in Gprj-$\Lambda$, as in the above proposition, is projective. But for the special case, when $\Lambda$ is a monomial algebra, according to [CSZ, Lemma 3.1] the projective module in the middle term must be indecomposable.

The following proposition is proved by the dual arguments to ones used in the above result.
Proposition 3.7. Assume $M$ is an indecomposable non-injective module in $\mathcal{M}$. Suppose that there exists the almost split sequence $0 \to M \xrightarrow{j_0} I \xrightarrow{q_0} N \to 0$ in $\mathcal{M}$ with $I$ injective in $\mathcal{M}$. If $N$ is non-injective and let $0 \to N \xrightarrow{j_1} I' \xrightarrow{q_1} W \to 0$ be the almost split sequence starting at $N$ in $\mathcal{M}$. Then, the following assertions hold.

1. $I'$ is projective in $\mathcal{M}$;
2. there are the following almost split sequences in $\mathcal{S}(\mathcal{M})$

\[
0 \to (\frac{M}{I})_0 \to (\frac{I}{0})_1 \oplus (\frac{N}{I})_0 \to (\frac{N}{N})_1 \to 0.
\]

\[
0 \to (\frac{0}{N})_0 \to (\frac{N}{N})_1 \oplus (\frac{0}{I'})_0 \to (\frac{N}{I'})_{j_1} \to 0;
\]

3. in particular, there is the following full sub-quiver in $\Gamma_{\mathcal{S}(\mathcal{M})}$

where a decomposition $I = \bigoplus^n I_i$, resp. $I' = \bigoplus^m I'_i$ of $I$, resp. $I'$, into indecomposable modules.

We establish the following construction by specializing the above last two propositions for when $\mathcal{M} = \text{Gprj-} \Lambda$.

Construction 3.8. Starting from an $N \in \text{Gprj-} \Lambda$ being indecomposable non-projective, so as stated in Proposition 3.5, and then using the proposition we get the following full sub-quiver of
But $m = r$, since $Q = \oplus_1^m Q_i = \oplus_1^r Q_i$. Iterating the same construction for $(\begin{smallmatrix} 0 \\ N \end{smallmatrix})_0$ and so on, we obtain the full sub-quiver of $\Gamma_{S(G_{proj})}$ containing all $\tau^i_0 \left( \begin{smallmatrix} 0 \\ N \end{smallmatrix} \right)_0$ for $i \geq 0$.

Following the inverse construction, but this turn by applying Proposition 3.7, we get the following sub-quiver of $\Gamma_{S(G_{proj})}$ for $M$.

But $u = d$, since $I' = \oplus_1^u I'_1 = \oplus_1^d I'_1$. Iterating the same construction for $(\begin{smallmatrix} 0 \\ N \end{smallmatrix})_1$ and so on, we find the full sub-quiver of $\Gamma_{S(G_{proj})}$ containing all $\tau^i_0 \left( \begin{smallmatrix} 0 \\ N \end{smallmatrix} \right)_1$ for $i \leq 0$. Having both above constructions together give us the part of the competent of $\Gamma_{S(G_{proj})}$ containing the $\tau_0$-orbit of $(\begin{smallmatrix} 0 \\ N \end{smallmatrix})_0$. By deleting projective-injective vertices we obtain that the following component of the stable Auslander-Reiten quiver $\Gamma_{S(G_{proj})}$:

which stops in both left and right sides as $\Lambda$ is CM-finite, by Proposition 2.4. Thus the underlying graph of such a component is a cycle. For instance, assume the module $N$, as stated in
Proposition 3.5, is of $\tau_G$-period 2, i.e., $\tau_G^2 N = N$, or equivalently of $\Omega_G$-period 2. Hence we have $V = N$. Based on our construction the corresponding components of $\Gamma^*_S(Gprj-\Lambda)$ containing $(\begin{smallmatrix} 0 \\ N \end{smallmatrix})_0$ is of the following form

$$
\begin{array}{c}
0 N \\
MP \\
NN \\
MM \\
0 M
\end{array}
$$

Let us formulate the above observation for the case that $\Lambda$ is an $\Omega_G$-algebra.

**Definition 3.9.** Denote $Z_n$ a basic $n$-cycle with the vertex set $\{1, 2, \ldots, n\}$ and the arrow set $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, where $s(\alpha_i) = i$ and $t(\alpha_i) = i + 1$ for each $1 \leq i \leq n$. Here, we identify $n + 1$ with 1. Let $I$ denote the arrow ideal of the path algebra $kZ_n$ over a field $k$. We recall that $I$ generated by all arrows in $Z_n$. Let $A_n = kZ_n/I^2$ be the corresponding quadratic monomial algebra. The symbol $kZ_n$ denotes also the path category of $Z_n$ and whose ideal arrow is defined similarly.

**Theorem 3.10.** Let $\Lambda$ be an $\Omega_G$-algebra. The following assertions hold.

1. Any indecomposable object in $S(Gprj-\Lambda)$ is isomorphic to either $(\begin{smallmatrix} 0 \\ G \end{smallmatrix})_0$ or $(\begin{smallmatrix} G \\ 0 \end{smallmatrix})_1$ or $(\begin{smallmatrix} \Omega_G(G) \\ P \end{smallmatrix})_1$, where $i$ the kernel of the projective cover $P \rightarrow G$, for some indecomposable non-projective Gorenstein projective module $G$. In particular, $S(Gprj-\Lambda)$ is of finite representation type, or equivalently $T_2(\Lambda)$ is CM-finite.

2. Let $G$ be an indecomposable non-projective module in $Gprj-\Lambda$. Assume $n$ is the least positive integer such that $\Omega^*_G(G) \cong Z_n$. Consider the following exact sequence obtained from the minimal projective resolution of $G$

$$
0 \rightarrow \Omega^*_G(G) \xrightarrow{p_0} P^{n-1} \xrightarrow{p_{n-1}} \cdots \xrightarrow{p_1} P_1 \xrightarrow{p_0} P_0 \xrightarrow{p_0} G \rightarrow 0,
$$

and for $0 \leq i \leq n-1$ denote $e_i : 0 \rightarrow K^i \xrightarrow{p_i} P_i \rightarrow K^{i-1} \rightarrow 0$ the induced short exact sequences by the above exact sequence, where $K^i$ is the kernel of $p_i$ (so by our convention $K^{n-1} = \Omega_G(G)$ and set $K^{-1} = G$). Then the vertices of the component $\Gamma^*_G$ containing the vertex $(\begin{smallmatrix} 0 \\ N \end{smallmatrix})_0$ are corresponds to the following indecomposable objects

$$
\left( \begin{smallmatrix} 0 \\ K^i \end{smallmatrix} \right)_0, \left( \begin{smallmatrix} K^i \\ 0 \end{smallmatrix} \right)_1, \left( \begin{smallmatrix} K^i \\ P^i \end{smallmatrix} \right)_j,
$$

where $i$ runs through $0 \leq i \leq n-1$. In particular, the component $\Gamma^*_G$ of $\Gamma^*_S(Gprj-\Lambda)$ has $3n$ vertices. Further, the stable Auslander-Reiten quiver $\Gamma^*_S(Gprj-\Lambda)$ is a disjoint union of the all components $\Gamma^*_G$, where $G$ belong to a fixed set of representatives of the all equivalence classes in $C(\Lambda)$. In particular, there is a bijection between the components of $\Gamma^*_S(Gprj-\Lambda)$ and the elements of $C(\Lambda)$.

3. Assume $\Lambda$ is a finite dimensional algebra over an algebraic closed filed $k$ of characteristic different from two. Let $\Gamma^*_G, \ldots, \Gamma^*_G$ denote all the components, and denote by $d_i$ the number of vertices in $\Gamma^*_G$, for each $1 \leq i \leq n$. Then there is an (additive) equivalence

$$
S(Gprj-\Lambda) \cong kZ_{d_1}/I_1^2 \times \cdots \times kZ_{d_n}/I_n^2,
$$

where $d_i$ is the number of vertices in $\Gamma^*_G$. 

\textbf{Proof.}
where \( KZ_d/I_1^2 \) is the quotient category of the path category \( kZ_n \) modulo the ideal \( I_1^2 \), the square of the arrow ideal \( I_1 \), indeed, generated by all paths of length \( \geq 2 \). Moreover, if \( \Lambda \) is Gorenstein, then \( \mathcal{D}_{sg}(T_2(\Lambda)) \cong kZ_d/I_1^2 \times \cdots \times kZ_n/I_n^2 \) (as additive categories).

**Proof.** (1) Assume \( \begin{pmatrix} G \\ G' \end{pmatrix} \) is an indecomposable object in \( \mathcal{S}(Gprj-\Lambda) \). It induces the short exact sequence \( 0 \rightarrow G \xrightarrow{f} G' \rightarrow \text{Cok}(f) \rightarrow 0 \) with all terms in \( Gprj-\Lambda \). But Lemma 2.3 implies that the sequence \( \lambda \) is isomorphic to the short exact sequences of the form either \( 0 \rightarrow \Omega_\Lambda(G) \rightarrow P \rightarrow G \rightarrow 0 \) or \( 0 \rightarrow G \xrightarrow{1} G \rightarrow 0 \) for some indecomposable module \( G \) in \( Gprj-\Lambda \). Hence we get the first part. The second part follows from the first part and using these two facts that \( \Omega_\Lambda \)-algebras are CM-finite (by Proposition 2.4) and the operation of taking projective cover gives us only a finite number indecomposable objects up to isomorphism in \( \mathcal{S}(Gprj-\Lambda) \).

(2) Applying Construction 3.8 for the two short exact sequences \( \epsilon_0, \epsilon_1 \) (if \( n = 1 \), applying for \( \epsilon_0, \epsilon_0 \)) we reach the following part of \( \Gamma_{\mathcal{S}(Gprj-\Lambda)} \)

Repeating the same construction for the pair of the short exact sequences \( (\epsilon_1, \epsilon_2) \) until the pair \( (\epsilon_{n-2}, \epsilon_{n-1}) \), we will obtain \( n - 1 \) full subquivers of \( \Gamma_{\mathcal{S}(Gprj-\Lambda)} \) as the above such the one corresponding to \( (\epsilon_{n-2}, \epsilon_{n-1}) \) has the object \( \begin{pmatrix} 0 \\ G \end{pmatrix} \) in the leftmost side. Hence the construction will stop at \( (n-1) \)-th step. By glowing the obtained full subquivers we obtain the full subquiver \( \tilde{\Gamma}_G \) of \( \Gamma_{\mathcal{S}(Gprj-\Lambda)} \) containing the \( \tau_G \)-orbit of \( \begin{pmatrix} 0 \\ N \end{pmatrix} \). Moreover, in view of the statement (1) we see that \( \Gamma_{\mathcal{S}(Gprj-\Lambda)} \) is a union of all full subquiver (not necessarily disjoint) \( \Gamma_{\mathcal{S}(Gprj-\Lambda)} \), where \( G \) runs through a fixed set of representatives of the all equivalence classes in \( \mathcal{C}(\Lambda) \). Be deleting the projective-injective vertices of the full subquiver \( \tilde{\Gamma}_G \) as we obtained in the above, then we obtain the component \( \Gamma_G \), as presented in the below

We have to identify the vertex corresponding to \( 0K^{n-1} \) with to \( 0G \) as by our assumption \( K^{n-1} = \Omega_\Lambda(G) \simeq G \). Now our construction reveals the all facts in the statement (2).

(3) Our assumption on \( \Lambda \) guarantees that the components of \( \Gamma_{\mathcal{S}(Gprj-\Lambda)} \) must be standard. Hence \( \mathcal{S}(Gprj-\Lambda) \) is equivalent to a product of the mesh categories of all the components. On the other hand,
the mesh category of the component $\Gamma_{G_i}$ is equivalent to the quotient category $kZ_{d_i}/I_{2_i}^2$, so the result. When $\Lambda$ is Gorenstein, so is $T_2(\Lambda)$ [A HKel, Corollary 4.3]. Then the last part follows from a known result of Buchweitz which implies $D_{sg}(T_2(\Lambda)) \cong S(Gprj-\Lambda)$. □

**Remark 3.11.** When $\Lambda$ is a Gorenstein finite dimensional algebra over an algebraic closed field $k$, as mentioned in the proof of the above theorem $T_2(\Lambda)$ so is. Hence by applying Happel’s work, $S(Gprj-\Lambda)$ gets a triangulated structure. Thus in the equivalence given in the third part of the theorem the left side is a triangulated category. We deduce by such an equivalence any path quotient category $kZ_{3n}/I_{2_i}^2$ can be enriched by a triangulated structure. It might be interesting to study the induced triangulated structure of $kZ_{3n}/I_{2_i}^2$. Viewing all the quotient categories $KZ_{d_i}/I_{2_i}^2$ with the induced triangulated structure, then the additive equivalence in theorem turns into a triangle equivalence.

4. **The morphism category over $\Omega_\mathbb{G}$-algebras**

This section is devoted to study of the morphism categories $H(Gprj-\Lambda)$ over $\Omega_\mathbb{G}$-algebras. We will compute some certain almost split sequences in $H(Gprj-\Lambda)$. It seems in contrast to $S(Gprj-\Lambda)$ it is difficult to compute all the almost split sequences. However, in the end of section for a very special case of $\Omega_\mathbb{G}$-algebras we will completely compute all the almost split sequences.

**Proposition 4.1.** Let $\Lambda$ be an Artin algebra such that the subcategory of Gorenstein projective modules $Gprj-\Lambda$ is contravariantly finite. Then $H(Gprj-\Lambda)$ is functorially finite in $H(\Lambda)$. In particular, $H(Gprj-\Lambda)$ has almost split sequences.

**Proof.** Thanks to [KS] it suffices to show that $H(Gprj-\Lambda)$ is contravariantly finite in $H(\Lambda)$. Assume an arbitrary object $(X, f)$ of $H(\Lambda)$ is given. Take a right $Gprj-\Lambda$-approximation $\alpha: G \to Y$ in $mod-\Lambda$. Let $(H, a, b)$ be a pull-back of $(\alpha, f)$ and $\beta: G' \to H$ a right $Gprj-\Lambda$-approximation.
of $H$ in mod-$\Lambda$. The below diagram contains all the data we have already defined.

$$
\begin{array}{c}
G' \xrightarrow{b\beta} X \\
\downarrow{\alpha\beta} \quad \downarrow{f} \quad \downarrow{H} \\
G \xrightarrow{a} Y
\end{array}
$$

By the above diagram we get the morphism $(b\beta) : (G'\alpha\beta) \rightarrow (X)f$ such that $(G'\alpha\beta)\in H(G\text{prj-}\Lambda)$. Using the universal property of pull-back, one can easily prove that the obtained morphism in $H(\Lambda)$ is a right $H(G\text{prj-}\Lambda)$-approximation of $(X)f$ in $H(\Lambda)$. \hfill \Box

We point out that as one can easily see that the proof of $H(G\text{prj-}\Lambda)$ being contravariantly finite works to show that $H(D)$ is contravariantly finite in $H(\Lambda)$, whenever $D$ is contravariantly finite in mod-$\Lambda$.

**Lemma 4.2.** Assume $\delta : 0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ is an almost split sequence in $\mathcal{M}$. Then

1. The almost split sequence in $H(\mathcal{M})$ ending at $(\frac{0}{C})_0$ has of the form

   $0 \rightarrow (A)_1 \xrightarrow{(f)} (A\beta)_f \xrightarrow{(g)} (\frac{0}{g})_0 \rightarrow 0.$

2. The almost split sequence in $H(\mathcal{M})$ ending at $(\frac{C}{C})_1$ has of the form

   $0 \rightarrow (\frac{0}{0})_0 \xrightarrow{(g)} (B\gamma)_g \xrightarrow{(f)} (\frac{C}{C})_1 \rightarrow 0.$

**Proof.** The proof is motivated for the case $\mathcal{C} = \text{mod-}\Lambda$. We refer to [MO, Proposition 3.1] or also [HMa, Lemma 5.3]. For convenience of the reader we provide a proof for the case $\mathcal{M} = \text{Gprj-}\Lambda$.

For (1), let $(\frac{0}{\delta}) : (\frac{X}{Y})_h \rightarrow (\frac{0}{C})_0$ be a non-retraction. Then the morphism $(\frac{0}{\delta}) : (\frac{0}{\delta})_0 \rightarrow (\frac{0}{C})_0$ in $S(\text{Gprj-}\Lambda)$ is a non-retraction as well. Because of Lemma 3.4(1) we know that the short exact sequence in the statement is almost split in $S(\text{Gprj-}\Lambda)$, hence there exists $(\frac{0}{\delta})_0 : (\frac{0}{\delta})_0 \rightarrow (A\beta)_f$ such that $(\frac{0}{\delta}) = (\frac{0}{\delta})_0 \circ (\frac{f}{g})$. Since $gh = 0$, there is a morphism $s : X \rightarrow A$ with $fs = th$. Now we see that the morphism $(\frac{0}{\delta})$ factors through $(\frac{0}{\delta})_0$ via $(\frac{f}{g})$. So we are done this case.

For (2), since $g : B \rightarrow C$ does not split, the map $(\frac{f}{g}) : (\frac{B}{C})_g \rightarrow (\frac{C}{C})_1$ so does not. Let $(\frac{0}{\delta}) : (\frac{X}{Y})_h \rightarrow (\frac{C}{C})_1$ be a map which is not a splittable epimorphism. Then $wh = v$. We claim $v$ is not a splittable epimorphism. Indeed, if $v$ is a splittable epimorphism, then there exists a morphism $s : C \rightarrow X$ such that $vs = 1$. Then we get the following commutative diagram

$$
\begin{array}{ccc}
C & \xrightarrow{s} & C \\
\downarrow{h} & \searrow{hs} & \downarrow{v} \\
X & \xrightarrow{h} & Y \\
\downarrow{v} & \swarrow{w} & \downarrow{w} \\
C & \xrightarrow{w} & C
\end{array}
$$
Hence \( (\psi) \) is a splittable epimorphism, a contradiction. As \( g : B \to C \) is a right almost split morphism, there exists a map \( r : X \to B \) such that \( gr = v \). Now we can make the following commutative diagram:

\[
\begin{array}{ccc}
X & \xrightarrow{h} & Y \\
\downarrow r & \quad & \downarrow w \\
B & \xrightarrow{g} & C \\
\downarrow g & \quad & \downarrow & \downarrow \\
C & \xrightarrow{g} & C
\end{array}
\]

Not that as we mention in the above diagram we can factor through \( (\psi) \). Hence \( (\psi) \) is a right almost split morphism in \( H(Gprj-\Lambda) \). This ends the proof. □

**Proposition 4.3.** Let \( \Lambda \) be an \( \Omega_G \)-algebra. Assume \( A \) is an indecomposable non-projective module in \( Gprj-\Lambda \). Consider the short exact sequences \( 0 \to \Omega_A^2(A) \xrightarrow{i_1} \Omega_A(A) \to 0 \) and \( 0 \to \Omega_A(A) \xrightarrow{i_0} \Omega_A(\Lambda(A)) \to 0 \) (induced by the minimal projective resolution of \( A \), and so almost split sequence in \( Gprj-\Lambda \)). The following short exact sequences in which the morphisms defined in an obvious way (see the proof below) is an almost split sequence in \( H(Gprj-\Lambda) \)

\[
0 \to \left( \frac{Q}{\Omega_A(A)} \right)_{p_1} \to \left( \frac{Q}{P} \right)_{i_0 p_1} \oplus \left( \frac{\Omega_A(\Lambda(A))}{\Omega_A(A)} \right)_{i_0} \to \left( \frac{\Omega_A(\Lambda(A))}{P} \right)_{i_0} \to 0.
\]

Further, the object \( \left( \frac{Q}{P} \right)_{i_0 p_1} \) is indecomposable.

**Proof.** According to Lemma 4.2 we have the following mesh diagrams in \( \Gamma_{H(Gprj-\Lambda)} \)

\[
\xymatrix{& Q \Omega_A(A) 
\ar[rr]
\ar[rd] & & \Omega_A(A) \oplus \Omega_A(A) 
\ar[rr]
\ar[rd] & & \Omega_A(\Lambda(A)) \Omega_A(A) 
\ar[rr]
\ar[rd] & & 0. 
\}
\]

By the above diagram we conclude \( \tau_{H(Gprj-\Lambda)} \left( \frac{\Omega_A(\Lambda(A))}{\Omega_A(A)} \right)_{i_0} = \left( \frac{Q}{\Omega_A(A)} \right)_{p_1} \). On the other hand, we have the following sequence in \( H(Gprj-\Lambda) \), the same as the one in the statement,

\[
0 \to \left( \frac{Q}{\Omega_A(\Lambda(A))} \right)_{p_1} \to \left( \frac{Q}{P} \right)_{i_0 p_1} \oplus \left( \frac{\Omega_A(\Lambda(A))}{\Omega_A(A)} \right)_{i_0} \to \left( \frac{\Omega_A(\Lambda(A))}{P} \right)_{i_0} \to 0,
\]

where \( \phi_1 = \left( \frac{1}{i_0} : \left( \frac{Q}{\Omega_A(A)} \right)_{p_1} \to \left( \frac{P}{\Omega_A(A)} \right)_{i_0 p_1} \right) \), \( \phi_2 = \left( \frac{p_1}{1} : \left( \frac{Q}{\Omega_A(A)} \right)_{p_1} \to \left( \frac{\Omega_A(\Lambda(A))}{\Omega_A(A)} \right)_{i_0} \right) \), \( \psi_1 = \left( \frac{p_1}{1} : \left( \frac{\Omega_A(\Lambda(A))}{\Omega_A(A)} \right)_{i_0} \to \left( \frac{\Omega_A(\Lambda(A))}{P} \right)_{i_0} \right) \), \( \psi_2 = \left( \frac{1}{i_0} : \left( \frac{\Omega_A(A)}{\Omega_A(A)} \right)_{i_0} \to \left( \frac{\Omega_A(\Lambda(A))}{P} \right)_{i_0} \right) \). To complete the proof in view of this fact that the left end term of the above non-split sequence is the Auslander-Reiten translation of the right end term, it suffices to prove that any non-isomorphism \( \left( \frac{i_0}{b} \right) : \left( \frac{Q}{\Omega_A(A)} \right)_{p_1} \to \left( \frac{Q}{\Omega_A(A)} \right)_{p_1} \) factors through \( \left( \frac{Q}{\Omega_A(\Lambda(A))} \right)_{p_1} \). Since \( p_1 \) is a projective cover, the morphism \( b \) is impossible to be an isomorphism. Otherwise, \( a \) is also an isomorphism, so this contradicts this fact that \( \left( \frac{i_0}{b} \right) \) is not an isomorphism. This implies that \( b \) factors through \( i_0 \) via some morphism, say \( d : P \to \Omega_A(A) \), as by our assumption \( i_0 \) is left minimal almost split. It is easy to see that
the morphism from the middle term of the above short exact sequence such whose restrictions to \( (\mathcal{Q})_{i_0 p_1} \) and \( (\Omega_\Lambda(A))_1 \) are \((\mathcal{Q})_{i_0 p_1} \) and the zero map, respectively. The following commutative diagram verifies the introduced morphism is well-defined in the morphism category.

![Diagram]

For the last statement, we can decompose as \((\mathcal{Q})_{i_0 p_1} = (\mathcal{X}_Y)_{i_0} \oplus (\mathcal{Z}_0)_{i_0} \oplus (\mathcal{W})_{i_0} \) with \( f \neq 0 \). Because of being projective of \( P \) and \( Q \) we get \( Z \) and \( W \) must be projective. Since \( \text{Im}(f) \cong \Omega_\Lambda(A) \) and \( \Omega_\Lambda(A) \) is indecomposable, we can deduce that \((\mathcal{X})_{i_0} \) is indecomposable. But \( Z = 0 \). Otherwise, it contradicts the fact that \( p_1 \) is a projective cover. We also have \( W = 0 \). Otherwise \( A \cong Z \), it means that \( A \) would be projective. It is absurd. Now the proof is completed.

The above result establishes some connection between certain simple modules over the Cohen-Macaulay Auslander algebras of \( \Omega_\Lambda \)-algebras.

**Corollary 4.4.** Let \( \Lambda \) be an \( \Omega_\Lambda \)-algebra. Denote by \( B \) the Cohen-Macaulay Auslander algebra of \( \Lambda \). If \( S \) a simple module in \( \text{mod}-B \) with the projective dimension 2, then \( \tau_B^{-1}S \cong \Omega_B(S') \), where \( S' \) is so a simple module in \( \text{mod}-A \) with the projective dimension 2. In particular, \( \text{Ext}_B^2(S,S') \cong D\text{Hom}_B(S',S) \).

**Proof.** Let \( G \) be a basic additive generator for \( \text{Gprj}_\Lambda \). As mentioned in the introduction the evaluation functor \( \zeta_G \) induces an equivalence \( \text{mod}-G \cong \text{mod}-B \). For a simple \( B \)-module \( S \) as stated in the corollary, there exists an indecomposable non-projective module \( A \) in \( \text{Gprj}_\Lambda \) such that the functor \((-A)/\text{rad}(-A)\) is mapped into \( S \) by \( \zeta_G \). Moreover, we have the following minimal projective resolution

\[
0 \to (-M) \overset{(-f)}{\to} (-N) \overset{(-g)}{\to} (-A) \to (-A)/\text{rad}(-A) \to 0,
\]

where \( 0 \to M \overset{f}{\to} N \overset{g}{\to} A \to 0 \) is an almost split sequence in \( \text{Gprj}_\Lambda \). See [AR3] for a proof of the facts. But, by Proposition 2.5, we may assume that the almost split sequence is equal to \( \delta_1 : 0 \to \Omega_\Lambda(A) \overset{i_0}{\to} P \overset{p_1}{\to} A \to 0 \).

Under the equivalence \( \text{mod}-(\text{Gprj}_\Lambda) \cong \text{mod}-B \) we show the statement for the functor \( S := (-,A)/\text{rad}(-,A) \). Applying Proposition 4.3 for the short exact sequences \( \delta_1 \) and \( \delta_2 : 0 \to A \overset{\delta_1}{\to} Q \overset{p_1}{\to} \Sigma_\Lambda(A) \to 0 \) (which exists as \( \text{Gprj}_\Lambda \) is Frobenius) follows the following

\[
\eta : 0 \to (P_A)_{p_0} \to (P_Q)_{i-1p_0} \oplus (A_A)_{i-1} \to (A_Q)_{i-1} \to 0.
\]

is an almost split sequence in \( H(\text{Gprj}_\Lambda) \). We know from the introduction (Section 1) the functor \( \Phi : H(\text{Gprj}_\Lambda) \to \text{mod}-(\text{Gprj}_\Lambda) \). We apply the functor \( \Phi \) on the above almost split sequence. It gives the following short exact sequence in \( \text{mod}-(\text{Gprj}_\Lambda) \)

\[
0 \to \Phi (P_A)_{p_0} \to \Phi (P_Q)_{i-1p_0} \to \Phi (A_A)_{i-1} \to 0.
\]
Note that $\Phi \left( \frac{A}{\mathcal{A}} \right)_1 = 0$. Using this fact that sequence $\eta$ is almost split in $\text{H}(\text{Gprj}-\Lambda)$, then it is straightforward to check that $\Psi(\eta)$ is an almost split sequence in $\text{mod-}(\text{Gprj}-\Lambda)$. See also [HMa, Proposition 5.7] for a proof. By definition, $S = \Phi \left( \frac{P}{A} \right)_{p_0}$, and by the almost split sequence $\Phi(\eta)$, $\tau_B^{-1}S = \Phi \left( \frac{P}{A} \right)_{\eta-1}$, remember here we applied the identification $\text{mod-}(\text{Gprj}-\Lambda)$ and $\text{mod-}B$. Take $S' = (-, \Sigma(A))/\text{rad}(-, A)$. Applying the Yoneda functor on the sequence $\delta_2$ yields

$(-, A) \rightarrow (-, Q) \rightarrow (-, \Sigma(A)) \rightarrow S' \rightarrow 0$

The above diagram completes the first part. For the second part, it is enough to use the Auslander-Reiten duality.

It might be a big and hard project to compute all the almost split sequences in $\text{H}(\text{Gprj}-\Lambda)$. However, we will do it for a very special case when $\Lambda = A_n = kZ_n/I^2$. As $A_n$ is a Nakayama algebra with loewy length two, by it [AR1, Theorem 10.7], it is an $\Omega^2$-algebra. Also, it is self-injective, hence mod-$A_n = \text{Gprj}-A_n$.

For our computation we need the following general result.

**Proposition 4.5.** Let $\Lambda$ be a self-injective algebra. Assume $0 \rightarrow \Omega_\Lambda(M) \overset{i_0}{\rightarrow} P^0 \overset{p_0}{\rightarrow} M \rightarrow 0$ and $0 \rightarrow \Omega_\Lambda^1(M) \overset{i_1}{\rightarrow} P^1 \overset{p_1}{\rightarrow} \Omega_\Lambda(M) \rightarrow 0$ are almost split sequences in $\text{mod-}\Lambda$. The following short exact sequences in which the morphisms defined in an obvious way are almost split sequences in $\text{H}(\Lambda)$

(a) $0 \rightarrow \left( \Omega_\Lambda(M) \overset{i_0}{\rightarrow} P^0 \right)_0 \rightarrow \left( \frac{0}{M} \right)_0 \oplus \left( \frac{P^0}{M} \right)_{p_0} \oplus \left( \Omega_\Lambda(M) \overset{i_0}{\rightarrow} P^0 \right)_1 \rightarrow \left( \frac{P^0}{M} \right)_{p_0} \rightarrow 0.$

(b) $0 \rightarrow \left( \Omega_\Lambda(M) \overset{i_0}{\rightarrow} P^0 \right)_{i_0 p_1} \rightarrow \left( \frac{P^0}{M} \right) \oplus \left( \Omega_\Lambda(M) \overset{i_0}{\rightarrow} P^0 \right)_{i_0} \rightarrow \left( \frac{P^0}{M} \right)_{i_0 p_1} \rightarrow 0.$

(c) $0 \rightarrow \left( \Omega_\Lambda(M) \overset{i_0}{\rightarrow} P^0 \right)_0 \rightarrow \left( \frac{0}{M} \right)_0 \oplus \left( \frac{P^1}{M} \right)_{i_0 p_1} \rightarrow \left( \frac{P^1}{M} \right)_{i_0 p_1} \rightarrow 0.$

(d) $0 \rightarrow \left( \Omega_\Lambda(M) \overset{i_0}{\rightarrow} P^0 \right)_0 \rightarrow \left( \frac{0}{M} \right)_0 \rightarrow \left( \frac{P^1}{M} \right)_{i_0 p_1} \rightarrow \left( \frac{P^1}{M} \right)_{i_0 p_1} \rightarrow 0.$

Proof. Set first throughout the proof $\tau_H := \tau_{\Omega(\Lambda)}$.

(a) We mention the assumption of $P^0$ being injective implies that $i_0$ is an injective envelop of $\Omega_\Lambda(A)$. Then by [H3, Proposition 4.4] we obtain $\tau_H \left( \frac{P^0}{M} \right)_{p_0} = \left( \frac{\Omega_\Lambda(M)}{P^0} \right)_{i_0}$. Hence to show that the sequence in (a) is almost split it suffices to prove that any non-isomorphism $\left( \frac{\alpha_1}{\alpha_2} \right) : \left( \frac{\Omega_\Lambda(M)}{P^0} \right)_{i_0} \rightarrow \left( \frac{\Omega_\Lambda(M)}{P^0} \right)_{i_0}$ factors through the monomorphism lying in the sequence. It is not true $\alpha_1$ to be an isomorphism. Otherwise, $\alpha_1$ would be an isomorphism, as $i_0$ is an injective envelop, this means that $\left( \frac{\alpha_1}{\alpha_2} \right)$ is an isomorphism, a contradiction. Because of being $i_0$ a left minimal almost split, there is a morphism $\beta : P^0 \rightarrow \Omega_\Lambda(M)$ such that $\beta i_0 = \alpha_1$. As $(\alpha_2 - i_0 \beta)i_0 = 0$,
so there is \( \eta : M \to P^0 \) such that \( \eta p_0 = \alpha_2 - i_0 \beta \). Define the map from the middle term of the sequence in (a) to \( (\Omega_\Lambda(M)_{p^0})_{i_0} \) such its restriction on the direct summands \( (\frac{p_0}{\eta})_0, (\frac{p^0}{1})_1 \) and \( (\Omega_\Lambda(M)_{0})_0 \) are \( (\begin{smallmatrix} 0 & \eta \\ \beta & i_0 \beta \end{smallmatrix}) \) and \( (\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}) \), respectively. One can easily check that the defined map gives the desired factorization.

We have by Proposition 4.3 the following almost split sequence in \( H(\Lambda) \)

\[
\begin{array}{c}
(f) & 0 & \longrightarrow & \left( \frac{P_1}{\Omega_\Lambda(M)} \right)_{p_1} & \oplus & \left( \frac{\Omega_\Lambda(M)}{\Omega_\Lambda(M)} \right)_1 & \longrightarrow & \left( \frac{\Omega_\Lambda(M)}{p^0} \right)_{i_0} & \longrightarrow & 0.
\end{array}
\]

The almost split sequences in (a) and (f) imply that \( \tau_H \left( \frac{\Omega_\Lambda(M)}{p^0} \right)_{i_0} = \left( \frac{p_1}{p^0} \right)_{i_0 p_1} \) and the indecomposable object \( \left( \frac{\Omega_\Lambda(M)}{p^0} \right)_{i_0} \) is a direct summand of the middle term of the almost split sequence \( \hat{b} \) ending at \( \left( \frac{\Omega_\Lambda(M)}{p^0} \right)_{i_0} \). By comparing the domains and codomains of the ending terms of the almost split sequence \( \hat{b} \) (or the same dimension shifting), we see that the middle term of \( \hat{b} \) has exactly direct summands (up to isomorphism) \( \left( \frac{\Omega_\Lambda(M)}{p^0} \right)_{i_0} \) and \( \left( \frac{p_1}{p^0} \right)_{i_0} \), so we have \( \hat{b} \) is the same the sequence in (b). This ends the proof of the sequence in (b) to be an almost split sequence in \( H(\Lambda) \).

(c) Applying [H3, Proposition 3.4], we get \( \tau_H \left( \frac{p^0}{p_1} \right)_{i_0 p_1} = \left( \frac{\Omega_\Lambda(M)}{0} \right)_0 \). Indeed, \( \text{Cok}(i_0 p_1) = M \) and \( \tau_\Lambda M = \Omega_\Lambda(M) \), and then we can apply the result to get the aforementioned claim. Also, by the almost split sequence (f) we see that \( \left( \frac{p_1}{\Omega_\Lambda(M)} \right)_{i_0} \) is a direct summand of the almost split sequence \( \hat{c} \) ending at \( \left( \frac{p^0}{p^0} \right)_{i_0 p_1} \). Next by comparing the domain and codomain, we see that the middle term of the \( \hat{c} \) has exactly direct summands \( \left( \frac{p_1}{\Omega_\Lambda(M)} \right)_{p_1} \) and \( \left( \frac{p^0}{0} \right)_{0} \), so \( \hat{c} \) is the same as the sequence in (c). (e) is a direct consequence of (b) and (c).

In the sequel, we apply the above proposition several times for the case \( \Lambda = A_n = kZ_n/I^2 \). Note that all indecomposable \( A_n \)-modules are either projective or simple. Take an arbitrary simple module \( S \). On can see that \( \Omega_A^0(S) = S \) and \( n \) is the least number with such a property. Consider the short exact sequences \( \epsilon_i : 0 \to K_i \to P^i \to K^{i-1} \to 0 \), for \( 0 \leq i \leq n-1 \), the induced short exact sequences by the minimal projective resolution of \( S \), here \( K^{-1} = S \) and \( K^{n-1} = \Omega_A^0(S) \).

Apply first Proposition 4.5 for the pair \( (\epsilon_0, \epsilon_1) \) and of course in view of Lemma 4.2 we get the following part of \( \Gamma_{H(\Lambda)} \)

\[
\begin{array}{c}
0 \Omega_{A_1}(S) & \overset{0}{\longrightarrow} & P^0 \Omega_{A_1}(S) & \overset{0}{\longrightarrow} & 0 \\
\Omega_{A_1}(S) & \overset{P^1}{\longrightarrow} & P^1 \Omega_{A_1}(S) & \overset{P^1}{\longrightarrow} & \Omega_{A_1}(S) P^0 \\
\Omega_{A_1}(S) & \overset{P^0}{\longrightarrow} & P^0 \Omega_{A_1}(S) & \overset{P^0}{\longrightarrow} & P^0 S \\
\Omega_2(S) & \overset{1}{\longrightarrow} & \Omega_2(S) \Omega_{A_1}(S) & \overset{1}{\longrightarrow} & 0 S \\
\end{array}
\]

Continuing the above procedure for the rest of the short exact sequences, then we reach to a part (as above) of \( \Gamma_{H(\Lambda)} \) such that the vertices in the leftmost side are \( 0 \Omega_{A_1}(S) = 0S \) and
\(\Omega_{n}^{2}(S)0 = \Omega_{n}A_{s}(S)0\). Therefore, we obtain entirely \(\Gamma_{H(A_{n})}\) by gluing the \(n\) parts. The immediate consequence of the above construction is that \(H(A_{n})\) is of finite representation type, and consequently by [AR2, Theorem 1.1] the Auslander algebra of \(A_{n}\) so is.

For instance, for the case \(n = 2\), the \(\Gamma_{H(A_{2})}\) is of the form:

\[
\begin{array}{cccccc}
\text{SS} & \text{p0} & \text{p10} & \text{p1P0} & \text{p0p0} & \text{p0S} \\
p0S & \text{p0p1} & \text{p1P0} & \text{p1A0} & \text{p0p0} & \text{p0S} \\
oS & \text{p0p1} & \text{p0p0} & \text{p0} & \text{S0} & \text{A0} \\
oP1 & \text{p0p0} & \text{p0S} & \text{S0} & \text{A0} & \text{A0}
\end{array}
\]

Firstly, if we delete the projective vertices and injective vertices (those vertices are corresponded to \((P_{1})_{0}\) or \((P_{0})_{0}\) or \((S_{0})_{0}\) for some indecomposable projective \(A_{2}\)-module \(P\) of \(\Gamma_{H(A_{2})}\), denote by \(\Gamma_{H(A_{2})}^{ss}\) the resulting quiver, according to our above computation we see that \(\Gamma_{H(A_{2})}^{ss} = Z\mathbb{A}_{3}/(\tau^4)\).

Secondly, first consider these two facts as follows: (1) in a straightforward way as we did in the proof of Proposition 4.3, all the almost split sequences in \(H(A_{2})\) have no ending terms of the form \((M)_{1}\) or \((M)_{0}\), for some indecomposable module \(M\), their image under the functor \(\Phi : H(A_{2}) \rightarrow \text{mod-(mod-}A_{2})\) are again almost split in \(\text{mod-(mod-}A_{2})\), or in \(\text{mod-B}\), where \(B\) is the Auslander algebra of \(A_{2}\). See also [HMa, Proposition 5.7] for a proof of the mentioned fact in a general situation. (2) projective-injective functors in \(\text{mod-(mod-}A_{2})\) are those representable functors of the form \(\text{Hom}_{A_{2}}(\cdot, P)\), where \(P\) is a projective module in \(\text{mod-}A_{2}\). Following these two aforementioned facts and our computation for \(\Gamma_{H(A_{2})}\) we get the stable Auslander-Reiten quiver \(\Gamma_{B}^{ss}\), obtained by deleting projective-injective vertices, is equal to the extended Dynkin quiver of type \(\hat{A}_{8}\) with clockwise orientation, as presented in below,

\[
\begin{array}{cc}
P^{1}P^{0} & \\
P^{1}\Omega_{A_{2}}(S) & \Omega_{A_{2}}(S)P^{0} \\
0\Omega_{A_{2}}(S) & 0S \\
SP^{1} & P^{0}S \\
P^{0}P^{1} & 
\end{array}
\]

5. 1-Gorenstein \(\Omega_{G}\)-algebras

In this section we mainly study \(\Omega_{G}\)-algebras which are also 1-Gorenstein. Recall that an algebra \(\Lambda\) is 1-Gorenstein if the injective dimension of \(\Lambda\) as left or right module (in this case one side is enough) is at most one. In the last result, we show the \(\Omega_{\Lambda}\)-algebras (not necessarily 1-Gorenstein) are a good source to produce CM-finite algebras.

In the following we shall give a characterization of 1-Gorenstein \(\Omega_{G}\)-algebras in terms of radical of projective modules.
Proposition 5.1. The following conditions are equivalent:

(i) $\Lambda$ is a 1-Gorenstein $\Omega_\Lambda$-algebra.
(ii) $\text{rad}\Lambda \oplus \Lambda$ generates $\text{Gprj}\Lambda$, i.e., $\text{Gprj}\Lambda = \text{add}(\text{rad}\Lambda \oplus \Lambda)$.

Proof. (i) $\Rightarrow$ (ii). 1-Gorensteinness of $\Lambda$ implies $\Omega_\Lambda^1(\text{mod}\Lambda) = \text{Gprj}\Lambda$, in particular, $\text{rad}\Lambda$ is a Gorenstein projective module. We observe by Lemma 2.2 that for each indecomposable projective $Q$, $\text{rad}(Q) \hookrightarrow Q$ is a minimal right almost split morphism in $\text{Gprj}\Lambda$. Let $G$ be an indecomposable non-projective Gorenstein projective module. There exists an indecomposable non-projective Gorenstein projective module $G'$ such that $\Omega_\Lambda(G') = G$. Then we have the almost split sequence $0 \rightarrow G \rightarrow P \rightarrow G' \rightarrow 0$ in $\text{Gprj}\Lambda$, by using the assumption of being $\Omega_\Lambda$-algebra of $\Lambda$.

Assume $Q$ is an indecomposable direct summand of $P$. From the left minimal almost split morphism $f$ we get an irreducible morphism from $G$ to $Q$. As we mentioned in the above there exits the right minimal almost split morphism $\text{rad}(Q) \hookrightarrow Q$. Hence $G$ must be a direct summand of $\text{rad}(Q)$, so a direct summand of $\text{rad}\Lambda$, as required.

(ii) $\Rightarrow$ (i). Since $\text{rad}\Lambda$ is a Gorenstein projective module, we obtain this fact that the simple modules are Gorenstein projective. We can prove that the global Gorenstein projective dimension of $\text{mod}\Lambda$ is at most one, equivalently, $\Lambda$ is a Gorenstein projective module of $\Lambda$. It follows the first case.

The established claim gives us that for any indecomposable non-projective Gorenstein projective indecomposable $X$, if $g : Y \rightarrow X$ is a right minimal almost split morphism in $\text{Gprj}\Lambda$, then $Y$ must be a projective module. Note that by (ii) the subcategory $\text{Gprj}\Lambda$ is of finite representation type, and so it has almost split sequences. Let an indecomposable non-projective representable functor $(-, \underline{G})$ is given. We may assume that $G$ is indecomposable non-projective in $\text{Gprj}\Lambda$. As we said there is an almost split sequence $\lambda : 0 \rightarrow \tau_G G \rightarrow E \rightarrow G \rightarrow 0$ in $\text{Gprj}\Lambda$. We know from our claim that $E$ is projective. Since $E$ is projective after applying the Yoneda functor on $\lambda$ we get the following sequence in $\text{mod}(\text{Gprj}\Lambda)$

$$0 \rightarrow (-, \tau_G G) \rightarrow (-, E) \rightarrow (-, G) \rightarrow (-, \underline{G}) \rightarrow 0.$$  

As $\lambda$ is almost split in $\text{Gprj}\Lambda$, the above sequence implies that $(-, \underline{G})$ is simple in $\text{mod}(\text{Gprj}\Lambda)$. Consequently, as any simple functor in $\text{mod}(\text{Gprj}\Lambda)$ arising in this way, induced by an almost
split sequence, see e.g. [Au3, Chapter 2], we get it a semisimple abelian category. So Λ is an $\Omega_G$-algebra. We are done.

In the following, some examples of 1-Gorenstein $\Omega_G$-algebras are given.

**Example 5.2.**  
(i) Clearly, hereditary algebras are 1-Gorenstein $\Omega_G$-algebras. So 1-Gorenstein $\Omega_G$-algebras can be considered as a generalization of hereditary algebras.

(ii) Recently due to Ming Lu and Bin Zhu in [LZ] a criteria was given for which monomial algebras are 1-Gorenstein algebras. Hence by having in hand such criteria we can search among quadratic monomial algebras to find 1-Gorenstein $\Omega_G$-algebras. In particular, we can specialize on gentle algebras to find which of them are 1-Gorenstein, as studied in [CL1].

(iii) The cluster-tilted algebras are defined in [BMR1] and [BMR2] are an important class of 1-Gorenstein algebras. So among cluster-tilted algebras we can find some examples of 1-Gorenstein $\Omega_G$-algebras (see below). For example, the cluster-tilted algebras of type $A$ since are gentle algebra, so in this case we are dealing with 1-Gorenstein $\Omega_G$-algebras. For the other types $D$ and $E$, in [CGL] the singularity categories of cluster-tilted algebras are described by the stable categories of some self-injective algebras. In particular, those cluster-tilted algebras of type $D$ and $E$ which are singularity equivalent to the self-injective Nakayama algebra $\Lambda(3, 2)$, the Nakayama algebra with cycle quiver with 3 vertices modulo the ideal generated by the paths of length 2, are other examples of $\Omega_G$-algebras. The following cluster-tilted algebra given by the quiver

```
1: \lambda \mu \delta \gamma
2: \beta
3: \gamma
4: \alpha
```

bound by the quadratic monomial relations $\alpha \beta = 0$, $\gamma \delta = 0$, $\delta \lambda = 0$, $\lambda \gamma = 0$, $\beta \mu = 0$, and $\mu \alpha = 0$ is also a 1-Gorenstein $\Omega_G$-algebra.

There is a similar result of the following theorem in [CL2, Theorem 4.4], where the authors proved that over a gentle algebra $\Lambda$: the Cohen-Macaulay Auslander algebra $\text{Aus}(\text{Gprj-}\Lambda)$ of $\Lambda$ is representation-finite if and only if so is $\Lambda$.

To prove we need some preparation from [AHKa] as follows. Let $(\Lambda_3, J)$ be the quiver $\Lambda_3 : v_2 \xrightarrow{a} v_1 \xrightarrow{b} v_0$ with relation $J$ generated by $ab$. By definition, a representation $M$ of $(\Lambda_3, J)$ over $\Lambda$ is a diagram

$$M : (M_2 \xrightarrow{f_2} M_1 \xrightarrow{f_1} M_0)$$

of $\Lambda$-modules and $\Lambda$-homomorphisms such that $f_1 f_2 = 0$, i.e. a sequence $M_2 \xrightarrow{f_2} M_1 \xrightarrow{f_1} M_0$ in mod-$\Lambda$. A morphism between the representations $M$ and $N$ is a triplet $\alpha = (\alpha_2, \alpha_1, \alpha_0)$ of $\Lambda$-homomorphisms such that the diagram

```
M_2 \xrightarrow{f_2} M_1 \xrightarrow{f_1} M_0
\downarrow{\alpha_2} \quad \downarrow{\alpha_1} \quad \downarrow{\alpha_0}
N_2 \xrightarrow{g_2} N_1 \xrightarrow{g_1} N_0
```
is commutative. Denote by \( \text{rep}(\mathbb{A}, \mathbb{J}, \Lambda) \) the category of all representations of \((\mathbb{A}, \mathbb{J})\) over \( \text{mod-}\Lambda \). Consider the following subcategory of \( \text{rep}(\mathbb{A}, \mathbb{J}, \Lambda) \):

\[
\mathcal{L}(\text{Gprj}\Lambda) = \{(G_2 \xrightarrow{f_2} G_1 \xrightarrow{f_1} G_0) \mid G_0, G_1, G_2 \in \text{Gprj}\Lambda \text{ and } 0 \to G_2 \xrightarrow{f_2} G_1 \xrightarrow{f_1} G_0 \text{ is exact}\}.
\]

Following [AHKa, Section 2], we define the functor \( \mathcal{F} : \mathcal{L}(\text{Gprj}\Lambda) \to \text{mod-}\text{(Gprj}\Lambda) \) by sending a representation \( G : (G_2 \xrightarrow{f_2} G_1 \xrightarrow{f_1} G_0) \) in \( \mathcal{L}(\text{Gprj}\Lambda) \) to the functor \( F_G \) in \( \text{mod-}(\text{Gprj}\Lambda) \) lying in the following exact sequence in \( \text{mod-}(\text{Gprj}\Lambda) \) (obtained by applying the Yoneda functor over \( G \), if we consider it as a left exact sequence in \( \text{mod-}\Lambda \))

\[
0 \to (-, G_2) \to (-, G_1) \to (-, G_0) \to F_G \to 0.
\]

It is proved in [AHKa, Proposition 2.2] over 1-Gorenstein algebra \( \Lambda \), the functor \( \mathcal{F} \) is full, faithful and objective. Hence, by [RZ, Appendix], we have the equivalence \( \mathcal{L}(\text{Gprj}\Lambda)/<\mathcal{K}> \cong \text{mod-}\text{(Gprj}\Lambda) \), where \( <\mathcal{K}> \) is the ideal of \( \mathcal{L}(\text{Gprj}\Lambda) \) generated by all the kernel objects of the functor \( \mathcal{F} \). Note that by the definition one can see an indecomposable kernel object is isomorphic to either representation \( (0 \to X \xrightarrow{1} X) \) or \( (X \xrightarrow{1} X \to 0) \) for some indecomposable module \( X \) in \( \text{Gprj}\Lambda \).

An important application from the equivalence which we will use in the next theorem is: \( \mathcal{L}(\text{Gprj}\Lambda) \) is of finite representation type if and only if so is \( \text{mod-}(\text{Gprj}\Lambda) \), whenever \( \Lambda \) is assumed to be 1-Gorenstein.

**Theorem 5.3.** Assume that \( \Lambda \) is a 1-Gorenstein \( \Omega_3\)-algebra. Then the Cohen-Macaulay Auslander algebra of \( \Lambda \) is representation-finite if and only if so is \( \Lambda \).

**Proof.** Specializing [AHKe2, Lemma 3.3] to the subcategory \( \text{Gprj}\Lambda \) provides us the fully faithful functor \( \vartheta : \text{mod-}\Lambda \to \text{mod-}(\text{Gprj}\Lambda) \). The embedding follows the “only if” part.

For the “if” part assume that \( \Lambda \) is representation-finite. Based on the discussion given in the lines before the theorem, we observe that the subcategory \( \mathcal{L}(\text{Gprj}\Lambda) \) of \( \text{rep}(\mathbb{A}, \mathbb{J}, \Lambda) \) is of finite representation type if and only if \( \text{mod-}(\text{Gprj}\Lambda) \) so is. Thus it is enough to show that \( \mathcal{L}(\text{Gprj}\Lambda) \) is of finite representation type. To do this, we divide the proof into three cases as follows.

Denote first by \( \mathcal{D} \) the full additive subcategory of \( \mathcal{L}(\text{Gprj}\Lambda) \) consisting of all indecomposable objects are isomorphic to either \( (G \xrightarrow{1} G \to 0) \) or \( (0 \to G \xrightarrow{1} G) \) with indecomposable object \( G \) in \( \text{Gprj}\Lambda \). Since \( \mathcal{D} \) has only finitely many indecomposable representations up to isomorphism, as \( \Lambda \) is CM-finite, hence we may only consider the indecomposable representations in \( \mathcal{L}(\text{Gprj}\Lambda) \) not in \( \mathcal{D} \) throughout the proof.

**Case 1:** We will show in this step that there is only a finite number of indecomposable representations, up to isomorphism, of \( \mathcal{L}(\text{Gprj}\Lambda) \) of the form \( (0 \to G \xrightarrow{1} P) \) with \( P \) projective module and \( f \) an injection. Let \( \mathcal{D}_1 \) denote the subcategory of \( \mathcal{L}(\text{Gprj}\Lambda) \) which consists of all representations \( (0 \to X \xrightarrow{g} Q) \) with \( X \in \text{Gprj}\Lambda, Q \in \text{prj}\Lambda \) and \( g \) an injection. Indeed, \( \mathcal{D}_1 \) is generated by the indecomposable representations as we want to show in this step that there are finitely many isomorphism classes of them. We define a functor \( \mathcal{F} : \mathcal{D}_1 \to \text{mod-}\Lambda \) by sending \( (0 \to X \xrightarrow{g} Q) \) to the \( \text{Cok}(f) \), which is a full and dense functor. Note that since \( \Lambda \) is 1-Gorenstein then for each module \( M \) in \( \text{mod-}\Lambda \) we have a short exact sequence \( 0 \to G \to P \to M \to 0 \) with \( P \) projective and \( G \) Gorenstein projective. This shows that the functor \( \mathcal{F} \) is dense. Also by the projectivity of the last terms of the representations in \( \mathcal{D}_1 \) we can prove that the functor \( \mathcal{F} \) is full. Finally, it is not difficult to see that the kernel of \( \mathcal{F} \) consists of all morphisms in \( \mathcal{D}_1 \) factor through representations of the form \( (0 \to Q \xrightarrow{1} Q) \) with \( Q \) a projective module; denote
by \( \mathcal{K}_1 \) the subcategory of \( \mathcal{L}(\text{Gprj-} \Lambda) \) consisting of all such forms. So the functor is objective as well. Thus, by [RZ, Appendix], the functor \( \mathcal{F} \) induces an equivalence between the quotient (additive) category \( \mathcal{D}_1 \) and \( \text{mod-} \Lambda \). Therefore, by the induced equivalence and using our assumption of \( \Lambda \) being representation-finite, we deduce that \( \mathcal{D}_1 \) so is, as desired.

**Case 2:** In this step we will prove that there is a finite number of indecomposable representations up to isomorphism in \( \mathcal{L}(\text{Gprj-} \Lambda) \) of the form \( (0 \to G \overset{f}{\to} G') \) with only \( f \) to be an injection and not any restriction on \( G' \). Take an indecomposable object \( (0 \to G \overset{f}{\to} G') \) of such a stated form. Since \( G' \) is a Gorenstein projective module then it can be embedded into a projective module, namely \( i : G' \to P \). Now we have the representation \( (0 \to G \overset{i}{\to} P) \) which lies in \( \mathcal{D}_1 \).

Considering \( (0 \to G \overset{i}{\to} P) \) as an object in the Krull-Schmidt category \( \text{rep}(A_3, J, \Lambda) \), then we can decompose it as

\[
(0 \to G \overset{i}{\to} P) = \oplus (0 \to G_i \overset{f_i}{\to} P_i),
\]

where the \( P_i \) must be projective modules. Since \( \mathcal{D}_1 \) is clearly closed under direct summands, then the \( (0 \to G_i \overset{f_i}{\to} P_i) \) are indecomposable objects in \( \mathcal{D}_1 \). By the decomposition (\( \dagger \)), we also obtain the following decomposition

\[
(0 \to G \overset{f}{\to} G') = \oplus (0 \to G_i \overset{f_i}{\to} \text{Im}(f_i)).
\]

Clearly \( \text{Im}(f_i) \) are in \( \text{Gprj-} \Lambda \) as \( \Lambda \) is a 1-Gorenstein. On the other hand, since \( (0 \to G \overset{f}{\to} G') \) is indecomposable then there is some \( j \) such that \( (0 \to G \overset{f}{\to} G') = (0 \to G_j \overset{f_j}{\to} \text{Im}(f_j)) \). As we have seen any indecomposable representation in Case 2 can be uniquely determined by an indecomposable object in \( \mathcal{D}_1 \). But by Case 1, \( \mathcal{D}_1 \) is of finite representation type, so this completes the proof of this step.

**Case 3:** Let \( A = (G_1 \overset{f}{\to} G_2 \overset{g}{\to} G_3) \) be an indecomposable representation in \( \mathcal{L}(\text{Gprj-} \Lambda) \). Without of loss generality we may assume that \( A \) is the form neither \( (0 \to G \overset{f}{\to} G) \) nor \( (G \overset{1}{\to} G \to 0) \). If we consider the representation \( A \) as a left exact sequence in \( \text{mod-} \Lambda \), then we have the following diagram

\[
\begin{array}{ccc}
0 & \rightarrow & G_1 \\
\downarrow & & \downarrow f \\
G_2 & \rightarrow & G_3 \\
\downarrow e & & \downarrow g \\
G & \rightarrow & \\
\end{array}
\]

in which \( g = me \) is an epi-mono factorization of \( g \). Since \( \Lambda \) is 1-Gorenstein then \( G \) belongs to \( \text{Gprj-} \Lambda \). In view of Lemma 2.3, concerning the structure of the short exact sequences in \( \text{Gprj-} \Lambda \) over an \( \Omega_2 \)-algebra, and \( X \) being indecomposable, we obtain \( G_2 \) a projective module and \( e \) a projective cover. In fact, if the short exact sequence \( 0 \to G_1 \overset{f}{\to} G_2 \overset{g}{\to} G \to 0 \) has a direct summand of the form \( (M \overset{1}{\to} M \to 0) \) or \( (0 \to M \overset{1}{\to} M) \), then it contradicts the indecomposability of \( A \).

Since \( (0 \to G \overset{m}{\to} G_3) \) is a monomorphism then we can consider it as an object appearing in Case 2. We decompose the representation into indecomposable representations as the following

\[
(0 \to G \overset{m}{\to} G_3) = \oplus (0 \to H_i \overset{m_i}{\to} M_i).
\]

Let \( p_i : P_i \rightarrow H_i \) be a projective cover of \( H_i \) for each \( i \). Because \( e : G_2 \rightarrow G \) is a projective cover of \( G = \oplus H_i \), by the above decomposition we have the equality, so we may assume \( G_2 = \oplus P_i \),
due to uniqueness of projective covers, and may identify $e$ with a morphism such that whose matrix presentation is a diagonal matrix with $p_i$ on the $(i, i)$-th entry (and so $f$ with a matrix presentation with $h_i = \text{Ker}(p_i) \to P_i$ on the $(i, i)$-entry and $G_1 = \oplus \text{ker}(p_i)$, i.e., $\text{ker}(p_i) = \Omega_\Lambda(H_i)$). All together, we have that the following decomposition $(G_1 \xrightarrow{f} G_2 \xrightarrow{g} G_3) = \oplus (\Omega(H_i) \xrightarrow{h_i} P_i \xrightarrow{m_i} M_i)$.

As $X$ is an indecomposable representation then it is isomorphic to $(\Omega(H_j) \xrightarrow{h_j} P_j \xrightarrow{m_j} M_j)$ for some $j$. As we have observed the representations $(\Omega(H_i) \xrightarrow{h_i} P_i \xrightarrow{m_i} M_i)$ are constructed uniquely by the indecomposable representations $(0 \to H_i \xrightarrow{m_i} M_i)$. But, from Case 2 we deduce that there is a finite number of choices of the representations $(0 \to H_i \xrightarrow{m_i} M_i)$ up to isomorphism. So we get we have finitely many choices up to isomorphism for the given indecomposable $A$. We are done.

An immediate consequence of the above theorem and Proposition 5.1 is the following:

**Corollary 5.4.** Let $\Lambda$ be a 1-Gorenstein $\Omega_G$-algebra. If $\Lambda$ is representation-finite, then $\text{End}(\Lambda \oplus \text{rad}\Lambda)$ so is.

It is interesting to see that whether Theorem 5.3 holds for a more general algebras such as 1-Gorenstein CM-finite algebra.

We end up this paper by the following result which shows that the $\Omega_G$-algebras are good source to produce CM-finite algebra via path algebras.

We first briefly outline some background about the path algebras. Given a finite-dimensional algebra $\Lambda$ over a field $k$, and a finite acyclic quiver $Q$. Let $\Lambda Q = kQ \otimes_k \Lambda$, where $kQ$ is the path algebra of $Q$ over $k$. We call $\Lambda Q$ the path algebra of a finite acyclic quiver $Q$ over $\Lambda$. As in the case of $\Lambda = k$, mod-$\Lambda Q$ is equivalent to the category rep($Q, \Lambda$) of representations of $Q$ over $\Lambda$. The notion of Gorenstein projective representations are defined analog with Gorenstein projective modules. We refer to [LuZ1] for more details. For instance if $Q = v_1 \to \cdots \to v_n$, the path algebra $\Lambda Q$ is given by the lower triangular matrix algebra of $\Lambda$:

\[
T_n(\Lambda) = \begin{bmatrix}
\Lambda & 0 & \cdots & 0 \\
\Lambda & \Lambda & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\Lambda & \Lambda & \cdots & \Lambda
\end{bmatrix}
\]

Moreover, a representation $X$ of $Q$ over $\Lambda$ is a datum as described in below:

\[
X = (X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_n)
\]

where $X_i$ is a $\Lambda$-module and $f_i : X_i \to X_{i+1}$ is a $\Lambda$-homomorphism. A morphism from representation $X$ to representation $Y$ is a datum $(\alpha_i : X_i \to X_{i+1})_{1 \leq i \leq n-1}$ such that for each $1 \leq i \leq n - 1$

\[
\begin{array}{c}
X_i \\
\downarrow \quad \alpha_i \\
Y_i
\end{array} \xrightarrow{\quad \alpha_{i+1} \quad} \begin{array}{c}
X_{i+1} \\
\downarrow \\
Y_{i+1}
\end{array}
\]

commutes.
**Proposition 5.5.** Let $\Lambda$ be an $\Omega_G$-algebra. Let $Q$ be the following linear quiver with $n \geq 1$ vertices

$$v_1 \to \cdots \to v_n.$$ 

Then the path algebra $\Lambda Q$ (or $T_n(\Lambda)$) is CM-finite.

**Proof.** Since the case $n = 1$ is clear, we assume $n \geq 2$. By the local characterization given in [LuZI, Theorem 5.1] or [EHS, the dual of Theorem 3.5.1] for the Gorenstein projective representations, we obtain for a given representation $X = (X_1 \overset{f_1}{\rightarrow} \cdots \rightarrow X_{n-1} \overset{f_{n-1}}{\rightarrow} X_n)$ in $\text{rep}(Q, \Lambda)$: $X$ is a Gorenstein projective representation if and only if it satisfies the following conditions

1. For $1 \leq i \leq n$, $X_i$ are Gorenstein projective modules.
2. For $1 \leq i \leq n-1$, $\text{Cok}(f_i)$ are Gorenstein projective modules and $f_i$ are monomorphisms.

Assume $X$ is an indecomposable Gorenstein projective representation in $\text{rep}(Q, \Lambda)$ then we claim that it is isomorphism to the following indecomposable representation (which is obvious to be Gorenstein projective from the above characterization and also in the end of the proof we will show that it is indecomposable): Let $G$ be an indecomposable Gorenstein projective module and $1 \leq i \leq j \leq n$, consider

$$(\dagger) \quad Y_{[i,j,G]} = (0 \rightarrow \cdots \rightarrow \Omega_{\Lambda}(G) \overset{1}{\rightarrow} \Omega_{\Lambda}(G) \cdots \rightarrow \Omega_{\Lambda}(G) \overset{l}{\rightarrow} P \overset{1}{\rightarrow} \cdots \overset{l}{\rightarrow} P)$$

where the first $G$ is settled in the $i$-th vertex and the last one in the $j$-th vertex, the map $l$ attached to the arrow $v_j \rightarrow v_{j+1}$ is the inclusion $\Omega_{\Lambda}(G) \hookrightarrow P$. Set $\text{Cok}(l) := G'$. Because of being indecomposable of $G$ and $G'$, we also deduce that the map $l : G \rightarrow P$ is a minimal left prj-$\Lambda$-approximation and the induced map $P \rightarrow G'$ is a projective cover.

Let $m$ be the least integer such that $X_{m} \neq 0$. We prove the claim by the inverse induction on $m$. If $m = n$, then the case is clear. Assume $m < n$. Denote by $X'$ the sub-representation of $X$ such that $X'_m = 0$ and $X'_j = X_d$ for all $m < d \leq n$. In view of Lemma 2.3 and being indecomposable of $X$, the monomorphism $f_m : X_m \rightarrow X_{m+1}$ is isomorphic to one of the following cases: (1) : $0 \rightarrow G$, (2) : $G \overset{1}{\rightarrow} G$, (3) : $\Omega_{\Lambda}(G) \overset{l}{\rightarrow} P$. The first case is impossible as $X_m$ would be 0, a contradiction. If the second case holds, then one can see easily $\text{End}_{\Lambda Q}(X) \simeq \text{End}_{\Lambda Q}(X')$.

Applying the above characterization follows that $X'$ remains to be Gorenstein projective. Hence $X'$ is an indecomposable Gorenstein projective representation with $X'_m = 0$. Our inductive assumption implies that $X' \simeq Y_{[m-1,j,G]}$ and $1 \leq j \leq n$. Thus, $X \simeq Y_{[m,j,G]}$. If the case (3) holds, then we show that $X \simeq Y_{[m,m+1,G]}$. If $m+1 = n$, then there is nothing to prove. Assume $m + 1 < n$. Analogous to the above for the monomorphism $f_{m+1} : P \rightarrow X_{m+2}$, we have three cases. The first and third cases are impossible. Just it remains the second case. This implies we may identify $f_{m+1}$ by the identity map of $P$. Continuing this inductive procedure leads to the desired form. Lastly, we complete the induction proof.

Now we prove $Y_{[i,j,G]}$ defined in the above is indecomposable. Assume an endomorphism $\alpha = (\alpha_i)_i^n$ of $Y := Y_{[i,j,G]}$. As it satisfies the commutativity conditions, we obtain $\alpha_i = \alpha_d = \alpha_j$, for $i \leq d \leq j$ and $\alpha_{j+1} = \alpha_d = \alpha_n$ for $j+1 \leq d \leq n$. If $\alpha_i = \alpha_j$ is an automorphism, then, by applying this fact that $l$ is a minimal left prj-$\Lambda$-approximation, $\alpha_{j+1}$ is an automorphism. Hence $\alpha$ is an automorphism. If $\alpha_i = \alpha_j$ is not an automorphism, then it is nilpotent as $G$ is indecomposable. Due to the commutative diagram assigned to the arrow $v_j \rightarrow v_{j+1}$ we have the
following commutative diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & G & \xrightarrow{I} & P & \xrightarrow{G'} & 0 \\
\downarrow{\alpha_j} & & \downarrow{\alpha_{j+1}} & & \downarrow{\beta} & & \\
0 & \rightarrow & G & \xrightarrow{I} & P & \xrightarrow{G'} & 0.
\end{array}
\]

But the induced map \(\beta\) is not an automorphism. Otherwise, as \(P \rightarrow \text{Cok}(l) = G'\) is a projective cover, we get \(\alpha_{j+1}\) so is. Then by the above diagram we obtain \(\alpha_j\) is an automorphism, a contradiction. Hence, \(\beta\) is nilpotent, as \(G'\) is indecomposable. \(\Box\)

**Remark 5.6.** By use of [ABHV, Corollary 7.6] we observe that over an Gorenstein algebra \(\Lambda\) Proposition 5.5 is independent of the orientation of the quiver \(Q\). Hence in this way we can produce more CM-finite algebras. In general, we believe directly, as we did in Proposition 5.5, by using local characterization of Gorenstein projective representations given in [LuZ1, Theorem 5.1] for a finite acyclic quiver and even more by the similar one given in [LuZ2] for acyclic quivers with monomial relations, one can generate some more CM-finite algebras by taking (quotient algebra of) path algebras of \(\Omega_G\)-algebras.

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**References**

[ABHV] J. Asadollahi, P. Bahiraei, R. Hafezi and R. Vahed, *On relative derived categories*, Comm. Algebra 44 (2016), no. 12, 5454-5477.

[AHKa] J. Asadollahi, R. Hafezi and Z. Karimi, *I-Gorenstein algebras of finite Cohen-Macaulay type*, Accepted for publication in the Michigan Mathematical Journal.

[AHKel] J. Asadollahi, R. Hafezi and M.H. Keshavarz, *Injective representations of bound quiver algebras*, J. Algebra Appl. 17 (2018), no. 1, 1850001, 13 pp

[AHKKe2] J. Asadollahi, R. Hafezi and M.H. Keshavarz, *Auslander’s formula for contravariantly finite subcategories*, J. Math. Soc. Japan 73 (2021), no. 2, 329-349.

[AHV] J. Asadollahi, R. Hafezi and R. Vahed, *Derived equivalences of functor categories*, J. Pure Appl. Algebra 223 (2019), no. 3, 1073-1096.

[A] H. Asashiba, *Gluing derived equivalences together*, Adv. Math. 235 (2013), 134-160.

[AS] I. Assem and A. Skowroński, *Iterated tilted algebras of type An*, Math. Z. 195 (1987), 269-290.

[Au1] M. Auslander, *Representation Dimension of Artin Algebras*, Queen Mary College Notes, 1971.

[Au2] M. Auslander, *Representation theory of artin algebras I*, Comm. Algebra 1 (1974), 177-268.

[Au3] M. Auslander, *Functors and morphisms determined by objects*, in *Representation theory of algebras*. (Proc. Conf., Temple Univ., Philadelphia, Pa., 1976) , 1244. Lecture Notes in Pure Appl. Math., 37, Dekker, New York, 1978.

[AB] M. Auslander and M. Bridger, *Stable module theory*, Mem. Amer. Math. Soc. 94 (1969).

[AR1] M. Auslander and I. Reiten, *Stable equivalence of dualizing R-varieties*, Adv. Math. 12(3) (1974), 306-366.

[AR2] M. Auslander and I. Reiten, *On the representation type of triangular matrix rings*, J. London Math. Soc. 12(2) (1975/76), no. 3, 371-382.

[AR3] M. Auslander and I. Reiten, *Representation theory of Artin algebras. VI. A functorial approach to almost split sequences*, Comm. Algebra 6 (1978), no. 3, 257-300.

[ARS] M. Auslander, I. Reiten and S.O. Smalø, *Representation theory of Artin algebras*, Cambridge Studies in Advanced Mathematics, 36. Cambridge University Press, Cambridge, 1995. xiv+423 pp. ISBN: 0-521-41134-3.
M. Auslander and S.O. Smalø, Almost split sequences in subcategories, J. Algebra 69(2) (1981), 426-454.

A. Beligiannis, On algebras of finite Cohen-Macaulay type, Adv. Math. 226 (2011), 1973-2019.

A. Buan, R. Marsh and I. Reiten, Cluster-tilted algebras, Trans. Amer. Math. Soc. 359(1) (2007), 323-332.

A. Buan, R. Marsh and I. Reiten, Cluster-tilted algebras of finite representation type, J. Algebra 306(2) (2006), 412-431.

R.O. Buchweitz, Maximal cohen-macaulay modules and tate-cohomology over gorenstein rings, unpublished (1986), available at http://hdl.handle.net/1807/16682. (1986).

X.W. Chen, The singularity category of an algebra with radical square zero, Doc. Math. 16 (2011), 921-936.

X.W. Chen, Algebras with radical square zero are either self-injective or CM-free, Proc. Amer. Math. Soc. 140 (2012), no. 1, 93-98.

X.W. Chen, D. Shen and G. Zhou, The Gorenstein-projective modules over a monomial algebra, Proceedings of the Royal Society of Edinburgh Section A: Mathematics, 148A (2018), 1115-1134.

X. Chen, S. Geng and M. Lu, The singularity categories of the cluster-tilted algebras of Dynkin type, Algebr. Represent. Theory 18 (2015), no. 2, 531-554.

X. Chen and M. Lu, Desingularization of quiver Grassmannians for gentle algebras, Algebr. Represent. Theory 19 (2016), no. 6, 1321-1345.

X. Chen and M. Lu, Cohen-Macaulay Auslander algebras of gentle algebras, Comm. Algebra 47 (2019), no. 9, 3597-3613.

E. Enochs and O. Jenda, Gorenstein injective and projective modules, Math. Z. 220 (1995), no. 4, 611-633.

H. Eshraghi, R. Hafezi and Sh. Salarian, Total acyclicity for complexes of representations of quivers, Comm. Algebra 41 (2013), no. 12, 4425-4441.

R. Hafezi, From subcategories to the entire module categories, Forum Math. 33 (2021), no. 1, 245-270.

R. Hafezi, Auslander-Reiten duality for subcategories, available on arXiv:1705.06684.

R. Hafezi, Almost split sequences in morphism categories, available on arXiv:2103.08883.

R. Hafezi, On Cohen-Macaulay Auslander algebras, available on arXiv:1802.05156.

R. Hafezi and E. Mahdavi, Covering theory, (mono)morphism categories and stable Auslander algebras, available on arXiv:2011.08646.

R. Hafezi and I. Muchtadi, Different exact structures on the monomorphism categories, Appl. Categ. Structures 29 (2021), no. 1, 31-68.

D. Happel, Triangulated Categories in the Representation Theory of Finite Dimensional Algebras, London Mathematical Society, Lecture Notes Series 119. Cambridge: Cambridge University Press, 1988.

P. Gabriel, Des catégories abéliennes, Bull. Soc. Math. France 90 (1962), 323-448.

M. Lu, Gorenstein defect categories of triangular matrix algebras, J. Algebra 480 (2017), 346-367.

M. Lu, Gorenstein properties of simple gluing algebras, Algebr. Represent. Theory 22 (2019), no. 3, 517-543.

Z.W. Li and P. Zhang, A construction of Gorenstein-projective modules, J. Algebra 323 (2010), no. 6, 1802-1812.

M. Lu and B. Zhu, Singularity categories of Gorenstein monomial algebras, J. Pure Appl. Algebra 225 (2021), no. 8, Paper No. 106551, 39 pp.

X.H. Luo and P. Zhang, Monic representations and Gorenstein-projective modules, Pacific J. Math. 264 (2013), no. 1, 163-193.

X.H. Luo and P. Zhang, Separated monic representations I: Gorenstein-projective modules, J. Algebra 479 (2017), 1-34.

M. Kalck, Singularity categories of gentle algebras, Bull. Lond. Math. Soc. 47 (2015), no. 1, 65-74.

B. Keller, On triangulated orbit categories, Doc. Math. 10 (2005), 551-581.

H. Krause and ø. Solberg, Applications of cotorsion pairs, J. London Math. Soc. (2) 68 (2003), 631-650.

M. Kreiss, Auslander-Reiten theory in functorially finite resolving subcategories, PhD thesis, 2013, available on arXiv:1501.01238.

R. Martínez-Villa and M. Ortiz-Morales, Tilting theory and functor categories III. The maps category, Int. J. Algebra 5 (2011), no. 9-12, 529-561.
[RV] I. Reiten and M. Van den Bergh, Noetherian hereditary abelian categories satisfying Serre duality, J. Amer. Math. Soc. 15 (2002) 295-366.

[RS] C.M. Ringel and M. Schmidmeier, The Auslander-Reiten translation in submodule categories, Trans. Amer. Math. Soc. 360 (2008), no. 2, 691-716.

[RZ] C.M. Ringel and P. Zhang, From submodule categories to preprojective algebras, Math. Z. 278 (2014), no. 1-2, 55-73.

Department of Mathematics, University of Isfahan, P.O.Box: 81746-73441, Isfahan, Iran

Email address: hafezi@ipm.ir, hafezira@gmail.com