A UNIVERSAL COREGULAR COUNTABLE SECOND-COUNTABLE SPACE

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Abstract. A Hausdorff topological space \( X \) is called superconnected (resp. coregular) if for any nonempty open sets \( U_1, \ldots, U_n \subseteq X \), the intersection of their closures \( \overline{U_1} \cap \cdots \cap \overline{U_n} \) is not empty (resp. the complement \( X \setminus (\overline{U_1} \cap \cdots \cap \overline{U_n}) \) is a regular topological space). A canonical example of a coregular superconnected space is the projective space \( \mathbb{P}^{\infty} \) of the topological vector space \( \mathbb{Q}^{<\omega} = \{ (x_n)_{n \in \omega} : \{ n \in \omega : x_n \neq 0 \} < \omega \} \) over the field of rationals \( \mathbb{Q} \). The space \( \mathbb{P}^{\infty} \) is the quotient space of \( \mathbb{Q}^{<\omega} \setminus \{ 0 \} \) by the equivalence relation \( x \sim y \) iff \( \mathbb{Q} \cdot x = \mathbb{Q} \cdot y \).

We prove that every countable second-countable coregular space is homeomorphic to a subspace of \( \mathbb{P}^{\infty} \), and a topological space \( X \) is homeomorphic to \( \mathbb{P}^{\infty} \) if and only if \( X \) is countable, second-countable, and admits a decreasing sequence of closed sets \( (X_n)_{n \in \omega} \) such that (i) \( X_0 = X, \bigcap_{n \in \omega} X_n = \emptyset \), (ii) for every \( n \in \omega \) and a nonempty open set \( U \subseteq X_n \) the closure \( \overline{U} \) contains some set \( X_m \), and (iii) for every \( n \in \omega \) the complement \( X \setminus X_n \) is a regular topological space. Using this topological characterization of \( \mathbb{P}^{\infty} \) we find topological copies of the space \( \mathbb{P}^{\infty} \) among quotient spaces, orbit spaces of group actions, and projective spaces of topological vector spaces over countable topological fields.

1. Introduction

A topological space \( X \) is called functionally Hausdorff if for any distinct points \( x, y \in X \) there exists a continuous function \( f : X \to \mathbb{R} \) such that \( f(x) \neq f(y) \). It is easy to see that each countable functionally Hausdorff space is (totally) disconnected. On the other hand, there are many examples of countable Hausdorff spaces which are connected and even superconnected. A topological space \( X \) is defined to be superconnected if for any nonempty open sets \( U_1, \ldots, U_n \) in \( X \) the intersection of their closures \( \overline{U_1} \cap \cdots \cap \overline{U_n} \) is not empty. It is easy to see that continuous images of superconnected spaces remain superconnected.

One of standard examples of a superconnected Hausdorff space is the famous Golomb space, introduced by Brown [6] and popularized by Golomb [11], [12]. The Golomb space is the space \( \mathbb{N} \) of positive integer numbers, endowed with the topology generated by the base consisting of the arithmetic progressions \( \mathbb{N} \cap (a + b \mathbb{Z}) \) with relatively prime numbers \( a, b \in \mathbb{N} \). Using the Chinese remainder theorem, it can be shown (see [11]) that the closure of the arithmetic progression \( \mathbb{N} \cap (a + b \mathbb{Z}) \) in the Golomb space contains the arithmetic progression \( p_1 \cdots p_n \mathbb{N} \), where \( p_1, \ldots, p_n \) are prime divisors of \( b \). This implies that the Golomb space is superconnected. Then any continuous image of the Golomb space is superconnected as well. One of such images is the Kirch space [13], which is the space \( \mathbb{N} \) endowed with the topology generated by the base consisting of the arithmetic progressions \( \mathbb{N} \cap (a + b \mathbb{Z}) \) where the numbers...
$a, b$ are relatively prime and $b$ is not divided by a square of a prime number. The Kirch space is known to be superconnected and locally connected. In [2] and [3] it was shown that the Golomb and Kirch spaces are topologically rigid, i.e., have trivial homeomorphism group.

Another natural example of a superconnected countable Hausdorff space is the infinite rational projective space $\mathbb{Q}P^\infty$, which is the projective space of the topological vector space $\mathbb{Q}^\omega = \{ (x_n)_{n \in \omega} : |\{ n \in \omega : x_n \neq 0 \}| < \omega \}$ over the field $\mathbb{Q}$ of rational numbers. Here the countable power $\mathbb{Q}^\omega$ is endowed with the Tychonoff product topology. The space $\mathbb{Q}P^\infty$ is defined as the quotient space of the space $\mathbb{Q}_0^\omega = \mathbb{Q}^\omega \setminus \{0\}^\omega$ by the equivalence relation $x \sim y$ iff $x = \lambda y$ for some nonzero rational number $\lambda$. The superconnectedness of the rational projective space $\mathbb{Q}P^\infty$ was first noticed by Gelfand and Fuks in their paper [10]. It is easy to show (see also Theorem 6) that the infinite rational projective space $\mathbb{Q}P^\infty$ is topologically homogeneous (i.e., for any points $x, y \in \mathbb{Q}P^\infty$ there exists a homeomorphism $h$ of $\mathbb{Q}P^\infty$ such that $h(x) = y$). So, $\mathbb{Q}P^\infty$ is not homeomorphic to the Golomb or Kirch space. Another important property that distinguished the space $\mathbb{Q}P^\infty$ from the Golomb and Kirch spaces is the coregularity of $\mathbb{Q}P^\infty$.

A topological space $X$ is called coregular if $X$ is Hausdorff and for any nonempty open sets $U_1, \ldots, U_n \subset X$ the complement $X \setminus (U_1 \cap \cdots \cap U_n)$ is a regular topological space. We recall that a topological space $X$ is regular if it is Hausdorff and for every closed set $F \subset X$ and point $x \in X \setminus F$ there are disjoint open sets $U, V$ in $X$ such that $x \in U$ and $F \subseteq V$. It is easy to see that a topological space containing more than one point is regular if and only if it is coregular and not superconnected. In Theorems 2 and 5 we shall prove that the space $\mathbb{Q}P^\infty$ is coregular and moreover, it is contains a topological copy of each coregular countable second-countable space. Let us recall that a topological space $X$ is second-countable if it has a countable base of the topology.

Observe that for a coregular topological space $X$ with a countable base of the topology $\{ U_n \}_{n \in \omega}$, the sequence $\{ X_n \}_{n \in \omega}$ of the sets $X_n = \overline{U_0 \cap \cdots \cap U_n}$ has the following two properties: (i) for any nonempty open set $U \subset X$ the closure $\overline{U}$ contains some set $X_n$, and (ii) for every $n \in \omega$ the complement $X \setminus X_n$ is a regular topological space. This property of the sequence $\{ X_n \}_{n \in \omega}$ motivates the following definition.

**Definition 1.** A sequence $\{ X_n \}_{n \in \omega}$ of closed subsets of a topological space $X$ is called
- **vanishing** if $X_0 = X$, $\bigcap_{n \in \omega} X_n = \emptyset$ and $X_{n+1} \subseteq X_n$ for every $n \in \omega$;
- a **coregular skeleton** for $X$ if $\{ X_n \}_{n \in \omega}$ is vanishing and has two properties: (i) for every nonempty open set $U \subset X$ the closure $\overline{U}$ contains some set $X_n$ and (ii) for every $n \in \omega$ the complement $X \setminus X_n$ is a regular topological space;
- a **superconnecting skeleton** for $X$ if $\{ X_n \}_{n \in \omega}$ is vanishing and for every nonempty open set $U \subset X$ there exists $n \in \omega$ such that $\emptyset \neq X_n \subseteq \overline{U}$;
- an **inductively superconnecting skeleton** for $X$ if $\{ X_n \}_{n \in \omega}$ is vanishing and for every $n \in \omega$ and nonempty open set $U \subset X_n$ there exists $m \in \omega$ such that $\emptyset \neq X_m \subseteq \overline{U}$;
- a **superskeleton** if it is both a coregular skeleton and an inductively superconnecting skeleton;
- a **canonical superskeleton** if $\{ X_n \}_{n \in \omega}$ is a superskeleton and for every $n \in \omega$ the set $X_{n+1}$ is nowhere dense in $X_n$.

It is clear that a topological space $X$ is coregular if it has a coregular skeleton, and $X$ is superconnected of it has a superconnecting skeleton.
The principal result of this paper is the following topological characterization of the infinite rational projective space $\mathbb{Q}P^\infty$.

**Theorem 1.** A topological space is homeomorphic to the space $\mathbb{Q}P^\infty$ if and only if it is countable, second-countable and possesses a superskeleton.

The proof of Theorem 1 will be presented in Section 4. Since the proof is long and technical, we postpone it till the end of the paper, and first we apply Theorem 1 to finding topological copies of the space $\mathbb{Q}P^\infty$ among quotient spaces of topological spaces by equivalence relations and orbit spaces of group actions.

## 2. Topological copies of the space $\mathbb{Q}P^\infty$ in “nature”

Let $E$ be an equivalence relation on a topological space $X$. A subset $A \subseteq X$ is called $E$-saturated if $A$ coincides with its $E$-saturation $EA = \bigcup_{x \in A} Ex$ where $Ex = \{y \in X : (x, y) \in E\}$ is the equivalence class of a point $x \in X$. Let $X/E$ be the space of $E$-equivalence classes $\{Ex : x \in X\}$ and $q : X \to X/E$ be the map assigning to each point $x \in X$ its equivalence class $Ex \in X/E$. The space $X/E$ carries the quotient topology, consisting of all subsets $\overline{U}$ is open in $X$.

**Proposition 1.** Let $E \subseteq X \times X$ and equivalence relation on a topological space $X$ satisfying the following conditions:

1. the set $E$ is closed in $X \times X$;
2. for any open set $U \subseteq X$ its $E$-saturation $EU$ is open in $X$;
3. $X$ admits a vanishing sequence $(X_n)_{n \in \omega}$ of non-empty $E$-saturated closed sets such that:
   a. for any $n \in \omega$ and nonempty $E$-saturated open set $U \subseteq X_n$, the closure $\overline{U}$ contains some set $X_m$;
   b. for any $n \in \omega$, $E$-saturated open set $U \subseteq X$ and $x \in U$, there exists an open $E$-saturated neighborhood $V \subseteq X$ of $x$ such that $\overline{V} \subseteq U \cap X_n$.

Then the quotient space $X/E$ possesses a superskeleton. If the space $X$ is first-countable and $X/E$ is countable, then $X/E$ is homeomorphic to $\mathbb{Q}P^\infty$.

**Proof.** Let $Y$ be the quotient space $X/E$. The condition (2) implies that the quotient map $q : X \to Y$ is open. Then for any $E$-saturated closed set $A \subseteq X$ its image $q(A) = Y \setminus q(X \setminus A)$ is closed in $Y$. In particular, for every $n \in \omega$ the image $Y_n = q(X_n)$ is a closed subset of $Y$ and hence $(Y_n)_{n \in \omega}$ is a vanishing sequence of nonempty closed sets in $Y$.

For any $n \in \omega$ and nonempty open set $U \subseteq Y_n$ the preimage $q^{-1}(U)$ is an open $E$-saturated set in $X_n$ and by condition (3a), the closure $\overline{q^{-1}(U)}$ contains some set $X_m$. Then $Y_m = q(X_m) \subseteq q(\overline{q^{-1}(U)}) \subseteq q(\overline{q^{-1}(U)}) = \overline{U}$. This means that $(Y_n)_{n \in \omega}$ is an inductively superconnecting skeleton for the space $Y$. By [8, 2.4.C], the condition (1,2) imply that the quotient space $Y = X/E$ is Hausdorff. Now the condition (3a),(3b) ensure that the skeleton $(Y_n)_{n \in \omega}$ is coregular and hence $(Y_n)_{n \in \omega}$ is a superskeleton for $X/E$.

If the space $X$ is first-countable and $X/E$ is countable, then the openness of the quotient map $q : X \to X/E$ implies the first-countability of the quotient space $X/E$. Being countable and first-countable, the space $X/E$ is second-countable. By Theorem 1 the space $X/E$ is homeomorphic to $\mathbb{Q}P^\infty$. □
Next, we consider orbit spaces of group actions, which are special examples of quotient spaces. By a group act we understand a topological space $X$ endowed with an action $\alpha : G \times X \to X$ a group $G$. The action $\alpha$ satisfies the following axioms:

- for every $g \in G$ the map $\alpha(g, \cdot) : X \to X$, $\alpha(g, \cdot) : x \mapsto gx := \alpha(g, x)$, is a homeomorphism of $X$;
- for the identity $1_G$ of the group $G$ and every $x \in X$ we have $1_Gx = x$;
- $(gh)x = g(hx)$ for all $g, h \in G$ and $x \in X$.

In this case we also say that $X$ is a $G$-space. We say that a $G$-space $X$ has closed orbits if for any point $x \in X$ its orbit $Gx = \{gx : g \in G\}$ is a closed subset of $X$. A subset $A \subseteq X$ is called $G$-invariant if it coincides with its $G$-saturation $GA = \bigcup_{x \in A} Gx$. The action of $G$ on $X$ induces the equivalence relation $E = \{(x, gx) : x, g \in G\}$. The quotient space $X/E$ by this equivalence relation is called the orbit space of the $G$-space and is denoted by $X/G$.

**Proposition 2.** Let $X$ be a $G$-space with closed $G$-orbits, possessing a vanishing sequence $(X_n)_{n \in \omega}$ of nonempty $G$-invariant closed subsets such that

1. for any $n \in \omega$ and nonempty open $G$-invariant set $U \subseteq X_n$, the closure $\overline{U}$ contains some set $X_m$;
2. for any $n \in \omega$, point $x \in X \setminus X_n$, and open $G$-invariant neighborhood $U \subseteq X$ of $x \in U$, there exists an open $G$-invariant neighborhood $V \subseteq X$ of $x$ such that $V \subseteq U \cup X_n$.

Then the orbit space $X/G$ has a superskeleton. If $X$ is first-countable and $X/G$ is countable, then the space $X/G$ is homeomorphic to $\mathbb{QP}^\infty$.

**Proof.** The closedness of orbits implies that the orbit space $Y = X/G$ is a $T_1$-space. Since each open set $U \subseteq X$ has open $G$-saturation $GU = \bigcup_{g \in G} gU$, the quotient map $q : X \to X/G$ is open. Then for every $n \in \omega$, the image $Y_n = q(X_n)$ of the closed $G$-invariant set $X_n$ is a closed subset of $Y$ and hence $(Y_n)_{n \in \omega}$ is a vanishing sequence of nonempty closed sets in $Y$. The condition (2) implies that for every $n \in \omega$, the open subspace $Y \setminus Y_n$ of the $T_1$-space $Y$ is regular and hence Hausdorff. Since $\bigcap_{n \in \omega} Y_n = \emptyset$, the space $Y$ is Hausdorff and the equivalence relation

$$E = \{(x, gx) : x \in X, g \in G\} = \{(x, y) \in X \times X : q(x) = q(y)\}$$

is closed in $X \times X$. Now we can apply Proposition 1 and conclude that $(Y_n)_{n \in \omega}$ is a superskeleton in the space $X/G$ (and the space $X/G$ is homeomorphic to $\mathbb{QP}^\infty$ if $X$ is first-countable and $X/G$ is countable).

Now we find topological copies of the space $\mathbb{QP}^\infty$ among infinite projective spaces of singular $G$-spaces.

**Definition 2.** A topological space $X$ endowed with a continuous action $\alpha : G \times X \to X$ of a Hausdorff topological group $G$ is called singular if it has the following properties:

- the topological space $X$ is regular and infinite;
- the set $\text{Fix}_G(X) = \{x \in X : Gx = \{x\}\}$ is a singleton;
- for every $x \notin \text{Fix}_G(X)$ the map $\alpha_x : G \to X$, $\alpha_x : g \mapsto gx = \alpha(g, x)$, is injective and open;
- the orbit $Gx$ of every point $x \in X \setminus \text{Fix}_G(X)$ contains the singleton $\text{Fix}_G(X)$ in its closure $\overline{Gx}$. 

(v) for any points \( x \in X \setminus \text{Fix}_G(X) \) and \( y \in X \), there exists a neighborhood \( U \subseteq X \) of \( y \) such that for any neighborhood \( W \subseteq X \) of the singleton \( \text{Fix}_G(X) \), there exists a neighborhood \( V \subseteq X \) of \( \text{Fix}_G(X) \) such that \( \alpha_u(\alpha_x^{-1}(V)) \subseteq W \) for every \( u \in U \).

**Example 1.** There are many natural examples of singular \( G \)-spaces:

1. The complex plane \( \mathbb{C} \) endowed with the action of the multiplicative group \( \mathbb{C}^* \) of non-zero complex numbers.
2. Any subfield \( \mathbb{F} \subseteq \mathbb{C} \) endowed with the action of the multiplicative group \( \mathbb{F}^* = \mathbb{F} \setminus \{0\} \).
3. The real line \( \mathbb{R} \) endowed with the action of the multiplicative group \( \mathbb{R}^* \) of non-zero real numbers.
4. The real line \( \mathbb{R} \) endowed with the action of the multiplicative group \( \mathbb{R}_+ \) of positive real numbers.
5. The closed half-line \( \overline{\mathbb{R}} = [0, \infty) \) endowed with the action of the multiplicative group \( \mathbb{R}_+ \).
6. The one-point compactification \( \overline{\mathbb{R}} \) of the space \( \mathbb{R} \) of real numbers endowed with the natural action of the additive group \( \mathbb{R} \).
7. The space \( \mathbb{Q} \) of rationals, endowed with the action of the multiplicative group \( \mathbb{Q}^* \) of non-zero rational numbers.
8. The space \( \mathbb{Q} \) of rationals, endowed with the action of the multiplicative group \( \mathbb{Q}_+ \) of positive rational numbers.
9. The one-point compactification \( \overline{\mathbb{Z}} = \mathbb{Z} \cup \{+\infty\} \) of the discrete space \( \mathbb{Z} \) endowed with the natural action of the additive group \( \mathbb{Z} \) of integer numbers.
10. The one-point compactification of any non-compact locally compact topological group \( G \), endowed with the natural action of the topological group \( G \).

Given a singular \( G \)-space \( X \), consider the \( G \)-space \( X^\omega \) endowed with the Tychonoff product topology and the coordinatewise action of the group \( G \). Let \( s \) be the unique point of the singleton \( \text{Fix}(X; G) \). We shall be interested in two special subspaces of \( X^\omega \):

\[
X^<_\omega := \{ x \in X^\omega : |\{ n \in \omega : x(n) \neq s \}| < \omega \} \quad \text{and} \quad X^<\omega := X^\omega \setminus \{s\}^\omega.
\]

The orbit space \( X^<_\omega / G \) is called the *infinite projective space* of the singular \( G \)-space \( X \) and is denoted by \( X\mathbb{P}^\infty \).

If \( X = \mathbb{F} \) is a non-discrete topological field endowed with the action of its multiplicative group \( \mathbb{F}^* \), then \( \mathbb{F}^<_\omega \) is a topological vector space over the field \( \mathbb{F} \) and \( \mathbb{F}\mathbb{P}^\infty \) is the projective space of the topological vector space \( \mathbb{F}^<_\omega \) in the standard sense. In particular, \( \mathbb{Q}\mathbb{P}^\infty \) is the projective space of the topological vector space \( \mathbb{Q}^<_\omega \) over the topological field \( \mathbb{Q} \) of rational numbers.

**Theorem 2.** The infinite projective space \( X\mathbb{P}^\infty \) of any singular \( G \)-space \( X \) possesses a canonical superskeleton. If the singular \( G \) space \( X \) is countable and metrizable, then its infinite projective space \( X\mathbb{P}^\infty \) is homeomorphic to the space \( \mathbb{Q}\mathbb{P}^\infty \).

**Proof.** Let \( s \) be the unique point of the singleton \( \text{Fix}_G(X) \). It will be convenient to identify elements \( x \in X^<_\omega \) with sequences \( (x_n)_{n \in \omega} \).

Since the group \( G \) acts by homeomorphisms on the space \( X^<_\omega \), for every open set \( U \subseteq X^<_\omega \) the set \( GU = \bigcup_{g \in G} gU \) is open. This implies that the quotient map \( q : X^<_\omega \to X\mathbb{P}^\infty \) is open.

Now using Proposition [2], we shall prove that \( X\mathbb{P}^\infty \) possesses a canonical superskeleton. For every \( n \in \omega \) consider the \( G \)-invariant subspace

\[
X_n = \{ x \in X^<_\omega : \forall i \in n \ x_i = s \}
\]
Claim 2. \( \alpha \) which contradicts the choice of the neighborhood \( \mathcal{F} \cap \mathcal{T} \) of the Tychonoff product topology on \( X \). Consequently, \( \{s\} = X \setminus \bigcup_{x \in X \setminus \{s\}} Gx \) is closed in \( X \) and \( \{s\} \) is closed in \( X^n \), which implies that \( X_n \) is closed in \( X \).

Taking into account the openness of the quotient map \( q : X^n \to Y \) and the \( G \)-invariantness of the closed sets \( X_n \), we conclude that for every \( n \in \omega \) the set \( Y_n = q(X_n) \) is closed in \( X^\omega \). Therefore, \( (Y_n)_{n \in \omega} \) is a vanishing sequence of nonempty closed sets in the space \( X^\omega \). Since the singleton \( \{s\} \) is nowhere dense in \( X \) (by Definition 2(iv)), for every \( n \in \omega \) the set \( X_{n+1} \) is nowhere dense in \( X_n \) and then the set \( Y_{n+1} \) is nowhere dense in \( Y_n \).

Claim 1. For any point \( x \in X^\omega \) its orbit \( Gx \) is closed in \( X^\omega \).

Proof. Given any point \( y \in X^\omega \setminus Gx \), we should find an open neighborhood \( V \) of \( y \) in \( X^\omega \) such that \( V \cap Gx = \emptyset \). Since \( y \in X^\omega \), the set \( \Omega = \{ n \in \omega : y_n \neq s \} \) is finite and nonempty. If \( x_n \notin G y_n \) for some \( n \in \Omega \), then \( G x_n \cap G y_n = \emptyset \) and by Definition 2(iii), \( G y_n \) is an open neighborhood of \( y_n \) in \( X \) and hence \( V = \{ v \in X^\omega : v_n \in G y_n \} \) is an open neighborhood of \( y \) that is disjoint with the orbit \( Gx \) of \( x \). So, we assume that for every \( n \in \Omega \), the point \( x_n \) belongs to the orbit \( G y_n \) and hence \( x_n = g_n y_n \) for some \( g_n \in G \).

If \( g_n \neq g_k \) for some numbers \( n, k \in \Omega \), then we can choose an open neighborhood \( W \subseteq G \) of the identity in the Hausdorff topological group \( G \) such that \( W g_n^{-1} \cap W k^{-1} = \emptyset \). By Definition 2(iii), the sets \( W y_n \) and \( W y_k \) are open neighborhoods of \( y_n \) and \( y_k \), respectively. We claim that the open neighborhood \( V = \{ v \in X^\omega : v_n \in W y_n, v_k \in W y_k \} \) of \( y \) does not intersect the orbit \( Gx \) of \( x \). In the opposite case we can find an element \( g \in G \) such that \( g x_n \in W y_n \) and \( g x_k \in W y_k \). Then \( g y_n y_n = g x_n \in W y_n \) and \( g y_k y_k = g x_k \in W y_k \). The injectivity of the maps \( \alpha_{y_n} \) and \( \alpha y_k \) guarantees that \( g y_n \in W \) and \( g y_k \in W \). Then \( g_n \in g^{-1} W \subseteq g_k W^{-1} W \) and finally, \( g_n W^{-1} \cap g_k W^{-1} = \emptyset \), which contradicts the choice of the neighborhood \( W \). This contradiction shows that \( V \cap Gx = \emptyset \).

Finally, assume that \( g_n = g_k \) for all \( n, k \in \Omega \). Fix any number \( n \in \Omega \). Since \( y \notin G x \), there exists \( m \in \omega \setminus \Omega \) such that \( y_m = s \neq g x_m \). Then \( g_n^{-1} x_m \neq y_m \) and by the Hausdorff property of \( X \), we can find an open neighborhood \( W_m \subseteq X \) of \( y_m = s \) such that \( g_n^{-1} x_m \notin \overline{W_m} \). By the continuity of the map \( \alpha_{x_m} : G \to X, \alpha_{x_m} : g \mapsto g x_m \), the set \( F = \alpha_{x_m}(\overline{W_m}) \) is closed in \( G \) and does not contain \( g_n^{-1} \). Find an open neighborhood \( U \subseteq G \) of the identity such that \( F \cap U g_n^{-1} = \emptyset \). We claim that the open neighborhood \( V = \{ v \in X^\omega : v_n \in U y_n, v_m \in W_m \} \) of \( y \) does not intersect the orbit \( Gx \). In the opposite case we can find an element \( g \in G \) such that \( g x_n \in U y_n \) and \( g x_m \in W_m \). Then \( g y_n y_n = g x_n \in U y_n \) and the injectivity of the map \( \alpha_{y_n} \) implies that \( g y_n \in U \). Also the inclusion \( g x_m \in W_m \) implies \( g \in F \). Then \( g y_n \in U \cap F g y_n \), which contradicts the choice of the neighborhood \( U \). \( \square \)

Claim 2. For every \( n \in \omega \), the closure \( \overline{U} \) of any nonempty \( G \)-invariant open set \( U \subseteq X_n \) contains some space \( X_m \).

Proof. Fix any point \( x \in U \). For every \( m > n \), identify the ordinal \( m \) with the set \( \{0, \ldots, m - 1\} \) and consider the projection \( \pi_m : X_m \to X^{m \setminus n}, \pi_m : x \mapsto (x_n, \ldots, x_{m-1}) \). By the definition of the Tychonoff product topology on \( X_n \), there exists \( m \geq n \) and an open neighborhood \( V \subseteq X^{m \setminus n} \) of \( \pi_m(x) \) such that \( \pi_m^{-1}(V) \subseteq U \). Then \( U = GU \supseteq G \cdot \pi_m^{-1}(V) = \pi_m^{-1}(GV) \). We claim that \( \{s\}^{m \setminus n} \subseteq \overline{GV} \). Given any open set \( W \subseteq X^{m \setminus n} \) that contains the singleton \( \{s\}^{m \setminus n} \), find an open neighborhood \( W_s \subseteq X \) of \( s \) such that \( W_s^{m \setminus n} \subseteq W \). Take any point
By the choice of the set \( \mathcal{A} \), for every \( i \in \mathbb{N} \), there exists a neighborhood \( W_i \subset X \) of \( s \) such that \( \alpha_{x_i}(\alpha_z^{-1}(W_i)) \subset W_z \). Definition \( 2(iv) \) ensures that the intersection \( G_z \cap \bigcap_{i \in \mathbb{N}} W_i \) contains some point \( w \neq s \). Then the element \( g = \alpha_z^{-1}(w) \) is well-defined and \( gx_i = \alpha_{x_i}(g) \in \alpha_{x_i}(\alpha_z^{-1}(W_i)) \subset W_z \). The point \( (gx_i)_{i \in \mathbb{N}} \) belongs to the intersection \( W_m \cap \bigcap_{i \in \mathbb{N}} gV_i \subset W \cap gV \subset W \cap GV \), witnessing that \( \{s\}_m^{\mathbb{N}} \subset \overline{GV} \).

Since the projection \( \pi_m : X_n \to X_m^{\mathbb{N}} \) is an open \( G \)-equivariant map, \[
X_m = \pi_m^{-1}(\{s\}_m^{\mathbb{N}}) \subset \overline{\pi_m^{-1}(GV)} \subset \overline{\pi_m^{-1}(GV)} \subset U.
\]

\( \square \)

**Claim 3.** For any \( n \in \mathbb{N} \), point \( x \in X \setminus X_n \), and \( G \)-invariant open neighborhood \( U \subset X^{<\omega} \) of \( x \), there exists a \( G \)-invariant open neighborhood \( V \subset U \) of \( x \) such that \( V \subset U \cup X_n \).

**Proof.** By the definition of the space \( X^{<\omega} \), \( \exists x \) there exists an index \( i \in \omega \) such that \( x_i \neq s \). By the definition of the Tychonoff product topology on the regular topological space \( X^{<\omega} \), there exist \( m > \max\{n, i\} \) and an open set \( W \subset X^m \) such that \( \pi_m^{-1}(W) \subset \overline{\pi_m^{-1}(W)} \subset U \) where \( \pi_m : X^{<\omega} \to X^m \), \( \pi_m : y \mapsto (y_0, \ldots, y_{m-1}) \), is the projection onto the first \( m \) coordinates. By Definition \( 2(v) \), for every \( j \in m \) the point \( x_j \) has a neighborhood \( W_j \subset X \) such that for any neighborhood \( O \subset X \) of \( s \) there exists a neighborhood \( O' \) of \( s \) such that \( \alpha_u(O') \subset O \) for every \( u \in W_j \). Replacing \( W_j \) by smaller neighborhoods, we can assume that \( \bigcap_{j \in m} W_j \subset W \).

Now consider the “hyperplane” \( H = \{ y \in X^m : y_i = x_i \} \subset X^m \), the open set \( V_i = \{(y_j)_{j \in \mathbb{N}} \in X^m : y_i \in Gx_i \} \subset X^m \), and the continuous map \( r : V_i \to H, \ r : (y_j)_{j \in \mathbb{N}} \mapsto ((\alpha_{x_i}(y_i))^{-1} \cdot y_j)_{j \in \mathbb{N}} \).

The map \( r \) assigns to each \( y \in V_i \) the unique point of the intersection \( Gy \cap H \). The continuity of the map \( r \) follows from the continuity of the action \( \alpha \), the openness and injectivity of the map \( \alpha_{x_i} \), and the continuity of the inversion in the topological groups \( G \). The preimage \( V_m := r^{-1}(\prod_{j \in \mathbb{N}} W_j) \) is an open \( G \)-invariant set in \( X^m \setminus \{s\}_m \) and the preimage \( V := \pi_m^{-1}(V_m) \) is an open \( G \)-invariant neighborhood of \( x \) in \( X^{<\omega} \). We claim that \( V \subset U \cup X_m \subset U \cup X_n \).

Observe that \( \pi_m = r^{-1}(W_m) = GW_m \), where \( W_m := H \cap \prod_{j \in \mathbb{N}} W_j \) is an open neighborhood of \( \pi_m(x) \) in \( H \) and \( \overline{W_m} \subset \overline{W} \). The continuity of the map \( r \) ensures that the set \( G \cdot \overline{W_m} = r^{-1}(\overline{W_m}) \) is closed in \( V_i \) and hence \( \overline{V_m} \subset (X^m \setminus V_i) \cup GV_m \).

We claim that no point of the set \( (X^m \setminus V_i) \setminus \{s\}_m \) belongs to the closure \( \overline{V_m} \). Fix any point \( y = (y_j)_{j \in \mathbb{N}} \in X^m \setminus V_i \) with \( y \notin \{s\}_m \). By the definition of the set \( V_i \), we have \( y_i \notin Gx_i \). If \( y_i \neq s \), then \( U_y = \{ z \in X^m : z_i \in GY_i \} \) is an open neighborhood of \( y \), which is disjoint with the set \( V_i \supset V_m \).

So, assume that \( y_i = s \). Since \( y \notin \{s\}_m \), there exists \( k \in m \) such that \( y_k \neq s \). By the Hausdorff property of \( X \), there exists an open neighborhood \( O \subset X \) of \( s \) such that \( y_k \notin \overline{O} \).

By the choice of the set \( W_k \), there exists an open neighborhood \( O' \subset X \) of \( s \) such that \( \alpha_u(O') \subset O \) for every \( u \in W_k \). Now consider the open neighborhood \( U_y = \{ z \in X^m : y_i \in O', \ y_k \notin \overline{O} \} \) of \( y \) in \( X^m \). We claim that \( U_y \cap V_m = \emptyset \). To derive a contradiction, assume that \( U_y \cap V_m \) contains some point \( z \). Since \( z \in V_m = GW_m \), the point \( z \) can be written as \( z = gh \) for some \( g \in G \) and \( h \in W_m = H \cap \prod_{j \in \mathbb{N}} W_j \). Write \( h \) as \( (h_j)_{j \in \mathbb{N}} \).

It follows from \( h \in H \) that \( h_i = x_i \) and \( h_k \in W_k \). On the other hand, \( z \in U_y \) implies \( gx_i = gh_i \in O' \) and \( gh_k \notin \overline{O} \). The inclusions \( gx_i \subset O' \) and \( h_k \subset W_k \) imply \( g \in \alpha_{x_i}(O') \) and
Claim 1 implies that the G-space $X_0^{<\omega}$ satisfies the conditions of Proposition 2. The proof of this proposition implies that the sequence $(Y_n)_{n\in\omega}$ is a superskeleton form the orbit space $X^\infty = X_0^{<\omega}/G$. Since each space $Y_{n+1}$ is nowhere dense in $Y_n$, the superskeleton $(Y_n)_{n\in\omega}$ is canical. If the space $X$ is countable and first-countable, then $X^\infty$ is homeomorphic to $\mathbb{Q}^\infty$ by Theorem 1.

**Remark 1.** The skeleton $(Y_n)_{n\in\omega}$ contructed in the proof of Theorem 2 will be called the canonical superskeleton of the space $X^\infty$.

Let $F$ be a topological field. Three elements $F^*x, F^*y, F^*z$ of the projective space $FP^\infty$ are called collinear if the union $F^*x \cup F^*y \cup F^*z$ is contained in some 2-dimensional vector subspace of $F^{<\omega}$.

For two topological fields $F_1, F_2$ a map $f : F_1^\infty \to F_2^\infty$ is called affine if for any collinear elements $F_1^*x, F_1^*y, F_1^*z \in F_1^\infty$, the elements $f(F_1^*x), f(F_1^*y), f(F_1^*z)$ are collinear in the projective space $F_2^2P^\infty$. A bijective map $f : F_1^\infty \to F_2^\infty$ is called an affine isomorphism if both maps $f$ and $f^{-1}$ are affine. If an affine isomorphism $f : F_1^\infty \to F_2^\infty$ is also a homeomorphism, then $f$ is called an affine topological isomorphism. The projective spaces $F_1^\infty, F_2^\infty$ are called affinely isomorphic (resp. affinely homeomorphic) if there exists an affine topological isomorphism $f : F_1^\infty \to F_2^\infty$.

In spite of the fact that for any countable subfields $F_1, F_2 \subseteq \mathbb{C}$, the infinite projective spaces $F_1P^\infty$ and $F_2P^\infty$ are homeomorphic (by Theorem 2), we have the following rigidity result for affine isomorphisms between infinite projective spaces.

**Proposition 3.** Two (topological) fields $F_1, F_2$ are (topologically) isomorphic if and only if their infinite projective spaces $F_1^\infty, F_2^\infty$ are affinely isomorphic (affinely homeomorphic).

**Proof.** If $\sigma : F_1 \to F_2$ is a (topological) isomorphism of the (topological) fields $F_1, F_2$, then the map $f$ induces the affine (topological) isomorphism

$$\tilde{\sigma} : F_1^\infty \to F_2^\infty, \quad \tilde{\sigma} : F_1^* \cdot (x)_{n\in\omega} \mapsto F_2^* \cdot (\sigma(x))_{n\in\omega},$$

of the projective spaces $F_1^\infty$ and $F_2^\infty$. This proves the “only if” part of the proposition.

To prove the “if” part, assume that $g : F_1^\infty \to F_2^\infty$ is an affine (topological) isomorphism between the projective spaces $F_1^\infty$ and $F_2^\infty$. Identify the space $F_1^3$ with the 3-dimensional vector subspace $\{(x_n)_{n\in\omega} \in F_1^{<\omega} : \forall n \geq 3 \ (x_n = 0)\}$ of the topological vector space $F_1^{<\omega}$. Consider the vectors $e_1 = (1, 0, 0), e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$ in $F_1^3$, and observe that the elements $F_1^*e_1, F_1^*e_2, F_1^*e_3$ are not collinear in the projective space $F_1^3P^\infty$. Then their images $g(F_1^*e_1), g(F_1^*e_2), g(F_1^*e_3)$ are not collinear in the projective space $F_2P^\infty$ and hence the union $g(F_1^*e_1) \cup g(F_1^*e_2) \cup g(F_1^*e_3)$ is contained in a unique 3-dimensional vector subspace $L$ of the topological vector space $F_2^{<\omega}$. We can choose a basis $e_1 \in g(F_1^*e_1), e_2 \in g(F_1^*e_2), e_3 \in g(F_1^*e_3)$ for the space $L$ such that $F_2^2(e_1 + e_2 + e_3) = g(F_1^*(e_1 + e_2 + e_3))$. Then the affine (topological) isomorphism $g$ induces an affine (topological) isomorphism $h = g|F_1^2P^2$ of the projective planes $F_1P^2 = (F_1^3 \setminus \{0\}^3)/F_1^*$ and $F_2P^2 = (F_2^3 \setminus \{0\}^3)/F_2^*$ such that $h(F_1^*e_i) = F_2^*e_i$ for $i \in \{1, 2, 3\}$.
Lemma 1. A (Hausdorff second-countable) topological space is coregular if (and only if) it has a coregular skeleton.

Proof. To prove the “if” part, assume that a topological space $X$ has a coregular skeleton $(X_n)_{n \in \omega}$. First we show that the space $X$ is Hausdorff. Fix any distinct points $x, y \in X$. By

$$h(F_1(x_1e_1 + x_2e_2 + x_3e_3)) = F_2(\sigma(x_1)e_1' + \sigma(x_2)e_2' + \sigma(x_3)e_3').$$

Therefore, the fields $F_1, F_2$ are isomorphic.

If the affine isomorphism $g$ is a homeomorphism, then so is the map $h$. In this case we shall prove that the field isomorphism $\sigma$ is a homeomorphism. For this observe that for every $i \in \{1, 2\}$, the map $f_i : F_i \to F_i^P$, $f : x \mapsto F_i^*(e_1 + x e_2)$ is a topological embedding. Since $\sigma = f_1^{-1} \circ h \circ f_1$ and $\sigma^{-1} = f_1^{-1} \circ h^{-1} \circ f_2$, then maps $\sigma, \sigma^{-1}$ are continuous and $\sigma : F_1 \to F_2$ is a topological isomorphism of the topological fields $F_1, F_2$.

By Example 12.5, the spaces $\mathbb{C}, \mathbb{R}, \mathbb{R}^+$ endowed with suitable group actions are singular $G$-spaces. By Theorem 2 the infinite projective spaces $\mathbb{C}P^\infty, \mathbb{R}P^\infty, \mathbb{R}_+P^\infty$ possess (canonical) superskeleta. It can be shown that each of these spaces has a countable base of the topology consisting of sets, homeomorphic to the space $\mathbb{R}^{<\omega}$, so is a (non-metrizable) $\mathbb{R}^{<\omega}$-manifold.

The distinguishing topological property of the space $\mathbb{R}_+P^\infty$ is possessing a superskeleton $(Y_n)_{n\in\omega}$ such that for every $n < m$ in $\omega$ the complement $Y_n \setminus Y_m$ is contractible.

This observation and the topological characterization of the space $\mathbb{Q}P^\infty$ suggests the following topological characterization of the space $\mathbb{R}_+P^\infty$.

Conjecture 1. A Hausdorff topological space $X$ is homeomorphic to $\mathbb{R}_+P^\infty$ if and only if $X$ possesses a superskeleton $(X_n)_{n\in\omega}$ such that for every $n < m$ in $\omega$ the set $X_n$ is a $Z$-set in $X_n$ and the space $X_n \setminus X_m$ is homeomorphic to $\mathbb{R}^{<\omega}$.

A closed subset $A$ of a topological space $X$ is called a $Z$-set in $X$ if the set $C(\mathbb{R}^\omega, X \setminus A)$ is dense in the space $C(\mathbb{R}^\omega, X)$ of continuous functions from the Hilbert cube $\mathbb{R}^\omega = [0, 1]^\omega$ to $X$, endowed with the compact-open topology. For more information on Infinite-Dimensional Topology, see the monographs [4], [15], [5], [17]. For the topological characterization of the space $\mathbb{R}^{<\omega}$, see [16], [7], [5], §1.6., [17] §4.3.

It can be shown that the spaces $\mathbb{R}P^\infty, \mathbb{C}P^\infty, \mathbb{R}_+P^\infty$ contain dense subspaces, homeomorphic to $\mathbb{Q}P^\infty$.

Problem 1. Does the Golomb (or Kirch) space contain a subspace homeomorphic to $\mathbb{Q}P^\infty$?

3. SOME PROPERTIES OF SKELETA IN TOPOLOGICAL SPACES

In this section we establish some properties of various skeleta in topological spaces. First we fix some standard notations.

For a subset $A$ of a topological space $X$ by $\overline{A}$ and $\partial A$ we denote the closure and boundary of $A$ in $X$. By $\omega$ we denote the smallest infinite ordinal, and by $\mathbb{N}$ the set $\omega \setminus \{0\}$ of positive integer ordinals. Ordinals are identified with the sets of smaller ordinals. So, $n = \{0, \ldots, n - 1\}$ for any natural number $n \in \omega$.

Lemma 1. A (Hausdorff second-countable) topological space is coregular if (and only if) it has a coregular skeleton.

Proof. To prove the “if” part, assume that a topological space $X$ has a coregular skeleton $(X_n)_{n\in\omega}$. First we show that the space $X$ is Hausdorff. Fix any distinct points $x, y \in X$. By
Definition [1] \((X_n)_{n \in \omega}\) is a vanishing sequence of closed sets in \(X\). Consequently, \(\bigcap_{n \in \omega} X_n\) and we can find a number \(m \in \omega\) such that \(x, y \in X \setminus X_m\). Since \((X_n)_{n \in \omega}\) is a coregular skeleton, the space \(X \setminus X_m\) is regular and hence Hausdorff. Then the points \(x, y\) have disjoint open neighborhoods \(O_x, O_y\) in the space \(X \setminus X_m\). Since \(X \setminus X_m\) is open in \(X\), the sets \(O_x, O_y\) remain open in \(X\), witnessing that \(X\) is Hausdorff.

Now fix any non-empty open sets \(U_1, \ldots, U_k \subseteq X\). By Definition [1] for every \(i \in \{1, \ldots, k\}\) there exists a number \(n_i \in \omega\) such that \(X_{n_i} \subseteq \overline{U}_i\). Then for the number \(n = \max_{i \leq k} n_i\) we have

\[
X_n \subseteq \bigcap_{i=1}^n X_{n_i} \subseteq \bigcap_{i=1}^k \overline{U}_i.
\]

Since the skeleton \((X_i)_{i \in \omega}\) is coregular, the space \(X \setminus X_n\) is regular and so is its subspace \(X \setminus (\overline{U}_1 \cap \cdots \cap \overline{U}_k)\). This completes the proof of the coregularity of \(X\).

To prove the “only if” part, assume that the space \(X\) is Hausdorff, second-countable, and coregular. If \(X\) is regular, then put \(X_0 = X, X_n = \emptyset\) for all \(n \in \mathbb{N}\), and observe that \((X_n)_{n \in \omega}\) is a coregular skeleton for \(X\). So, we assume that \(X\) is not regular and hence infinite.

Fix a countable base \(\{U_n\}_{n \in \mathbb{N}}\) of the topology of \(X\) such that \(U_n \neq \emptyset\) for all \(n \in \mathbb{N}\). Let \(X_0 = X\) and \(X_n = \overline{U}_1 \cap \cdots \cap \overline{U}_n\) for every \(n \in \mathbb{N}\).

To show that \(\bigcap_{n \in \omega} X_n = \emptyset\), fix any point \(x \in X\). Since the space \(X\) is infinite and Hausdorff, there exists \(n \in \omega\) such that \(x \notin \overline{U}_n\) and hence \(x \notin X_n\). This shows that the sequence \((X_n)_{n \in \omega}\) is vanishing. By the coregularity of \(X\), for every \(n \in \omega\) the space \(X \setminus X_n\) is regular. Also observe that every nonempty open set \(U \subseteq X\) contains some set \(U_n\) and then \(X_n \subseteq \overline{U}_n \subseteq \overline{U}\). Therefore, the vanishing sequence \((X_n)_{n \in \omega}\) is a coregular skeleton for the space \(X\). \(\Box\)

**Lemma 2.** A (Hausdorff infinite second-countable) topological space is superconnected if (and only if) it possesses a superconnecting skeleton.

**Proof.** To prove the “if” part, assume that a topological space \(X\) has a superconnecting skeleton \((X_n)_{n \in \omega}\). By Definition [1] \((X_n)_{n \in \omega}\) is a vanishing sequence in \(X\). To see that \(X\) is superconnected, fix any non-empty open sets \(U_1, \ldots, U_k \subseteq X\). By Definition [1] for every \(i \in \{1, \ldots, k\}\) there exists a number \(n_i \in \omega\) such that \(\emptyset \neq X_{n_i} \subseteq \overline{U}_i\). Then for the number \(n = \max_{i \leq k} n_i\) we have

\[
\emptyset \neq X_n \subseteq \bigcap_{i=1}^n X_{n_i} \subseteq \bigcap_{i=1}^k \overline{U}_i,
\]

witnessing that \(\overline{U}_1 \cap \cdots \cap \overline{U}_k \neq \emptyset\) and \(X\) is superconnected.

To prove the “only if” part, assume that an infinite Hausdorff second-countable space \(X\) is superconnected and fix a countable base \(\{U_n\}_{n \in \mathbb{N}}\) of the topology of \(X\) such that \(U_n \neq \emptyset\) for all \(n \in \mathbb{N}\). Let \(X_0 = X\). By the superconnectedness of \(X\), for every \(n \in \mathbb{N}\) the closed set \(X_n = \overline{U}_1 \cap \cdots \cap \overline{U}_n\) is not empty.

To show that \(\bigcap_{n \in \omega} X_n = \emptyset\), fix any point \(x \in X\). Since the space \(X\) is infinite and Hausdorff, there exists \(n \in \omega\) such that \(x \notin \overline{U}_n\) and hence \(x \notin X_n\). This shows that the sequence \((X_n)_{n \in \omega}\) is vanishing.

To see that \((X_n)_{n \in \omega}\) is a superconnecting skeleton for \(X\), take any nonempty open set \(U \subseteq X\) and find \(n \in \omega\) such that \(U_n \subseteq U\). Then

\[
\emptyset \neq X_n = \overline{U}_1 \cap \cdots \cap \overline{U}_n \subseteq \overline{U}_n \subseteq \overline{U}.
\]
Lemma 3. Let $X$ be an infinite Hausdorff topological space and $(X_n)_{n \in \omega}$ be its superconnecting skeleton. Then the space $X$ is crowded and for every $n \in \omega$, the space $X_n$ is infinite.

Proof. First we show that for every $n \in \omega$ the set $X_n$ is infinite. To derive a contradiction, assume that for some $n \in \omega$ the space $X_n$ is finite. We can assume that $n$ is the smallest number with this property. Then for every $i \in n$ the space $X_i$ is infinite and hence contains some point $x_i$. Since $X$ is infinite and Hausdorff, there exists a nonempty set $U \subseteq X$ whose closure $\overline{U}$ does not intersect the finite set $X_n \cup \{x_i\}_{i \in n}$. Since the skeleton $(X_k)_{k \in \omega}$ is superconnecting, the closure $\overline{U}$ contains some set $X_i \neq \emptyset$. Assuming that $i \geq n$ we conclude $\emptyset \neq X_i = X_i \cap \overline{U} \subseteq X_n \cap \overline{U} = \emptyset$, which is a contradiction. So, $i < n$ and then $x_i \in X_i \subseteq \overline{U}$, which contradicts the choice of $U$. This contradiction witnesses that the spaces $X_n$ are infinite.

Assuming that the space $X$ is not crowded, we can find an isolated point $x$ in $X$. Then $U = \{x\}$ is an open subspace of $X$ such that $\overline{U} = \{x\}$ by the Hausdorff property of $X$. Since $(X_n)_{n \in \omega}$ is a superconnecting skeleton of $X$, there exist $n \in \omega$ such that $X_n \subseteq \overline{U} = \{x\}$ and hence $X_n$ is finite, which is a desired contradiction. □

Lemma 4. Let $X$ be an infinite topological space and $(X_n)_{n \in \omega}$ be its superskeleton. Then for every $n \in \omega$ the space $X_n$ is crowded.

Proof. By Definition 1 for every $n \in \omega$ the sequence $(X_m)_{m=n}^{\infty}$ is a superconnecting coregular skeleton for the space $X_n$. Then the space $X = X_0$ is coregular and hence Hausdorff. By Lemma 3, for every $n \in \omega$ the set $X_n$ is infinite. Applying Lemma 3 to the superconnecting skeleton $(X_m)_{m=n}^{\infty}$ for the infinite Hausdorff space $X_n$, we conclude that $X_n$ crowded. □

Lemma 5. Let $X$ be an infinite Hausdorff space and $(X_n)_{n \in \omega}$ be its superconnecting or coregular skeleton. Then for some $n \in \omega$ the set $X_n$ is nowhere dense in $X = X_0$.

Proof. Being infinite and Hausdorff, the space $X$ contains two disjoint nonempty open sets $U, V$. By Definition 1 the closures $\overline{U}, \overline{V}$ contain some set $X_n$. Then the set $X_n \subseteq \overline{U} \cap \overline{V} \subseteq \overline{U} \cap \overline{X \setminus U} = \partial U$ is nowhere dense in $X$. □

Lemma 6. If $(X_n)_{n \in \omega}$ is a superskeleton for an infinite topological space $X$, then for some increasing number sequence $(n_k)_{k \in \omega}$ the sequence $(X_{n_k})_{k \in \omega}$ is a canonical superskeleton for $X$.

Proof. Applying Lemma 5 construct inductively an increasing number sequence $(n_k)_{k \in \omega}$ such that $n_0 = 0$ and for every $k \in \omega$ the set $X_{n_k+1}$ is nowhere dense in $X_{n_k}$.

A subset of a topological space is called regular open if it coincides with the interior of its closure. A topological space $X$ is called semiregular if it is Hausdorff and has a base consisting of regular open sets.

Lemma 7. Each coregular topological space $X$ is semiregular.

Proof. If the space $X$ is finite, then it is discrete (being Hausdorff) and hence regular and semiregular. So, we assume that $X$ is infinite.

To show that $X$ is semiregular, fix any point $x \in X$ and an open neighborhood $U$ of $x$ in $X$. Taking into account that $X$ is infinite and Hausdorff, we can replace $U$ by a smaller neighborhood of $x$ and assume that $X$ contains a non-empty open set $W$, which is disjoint with $U$. Then $U \cap \overline{W} = \emptyset$. Since $X$ is coregular, the space $X \setminus \overline{W}$ is regular. Then the point
Lemma 8. A topological space $X$ is regular if and only if its square $X \times X$ is coregular.

Proof. The “only if” part is trivial. To prove the “if” part, assume that the space $X \times X$ is coregular. Then $X \times X$ is Hausdorff and so is the space $X$. If $X$ is finite, then $X$ is discrete and hence regular. So, assume that $X$ is infinite. Then we can fix any point $x \in X$ and find a non-empty open set $U \subseteq X$ such that $x \notin U$. By the coregularity of $X \times X$ the complement $X \times X \setminus U \times U$ is a regular space and so is its subspace $\{x\} \times X$ and the space $X$. □

4. Main Results

In this section we prove a difficult Theorem 3 implying Theorem 4 and many other important properties of the space $\mathbb{Q}P^\infty$. Let us recall that a function $f : X \to Y$ between topological spaces $X, Y$ is called a topological embedding if $f$ is a homeomorphism between $X$ and the subspace $f(X)$ of $Y$.

Theorem 3. Let $X$ be a countable second-countable space, $(X_n)_{n \in \omega}$ be a coregular skeleton in $X$, and $A$ be a nowhere dense closed set in $X$. Let $Y$ be a countable second-countable space, $(Y_n)_{n \in \omega}$ be a canonical superskeleton in $Y$, and $B$ be a subset of $Y$ such that for every $n \in \omega$ the intersection $B \cap Y_n$ is nowhere dense in $Y_n$. Let $f : A \to B$ be a homeomorphism such that $f^{-1}(Y_n) = A \cap X_n$ for all $n \in \omega$. Then there exist a topological embedding $\tilde{f} : X \to Y$ such that $f|A = \tilde{f}$ and $\tilde{f}^{-1}(Y_n) = X_n$ for all $n \in \omega$. If the sequence $(X_n)_{n \in \omega}$ is a canonical superskeleton in $X$, the set $B$ is closed in $Y$, and for every $n \in \omega$ the set $A \cap X_n$ is nowhere dense in $X_n$, then $\tilde{f}(X) = Y$ and $\tilde{f}$ is a homeomorphism.

Proof. For constructing the topological embedding $\tilde{f} : X \to Y$ we should make some preliminary work with the spaces $X$ and $Y$.

We start with the space $X$ endowed with a coregular skeleton $(X_n)_{n \in \omega}$. Let $\ell_X : X \to \omega$ be the function assigning to each $x \in X$ the largest number $n$ such that $x \in X_n$ (such the number $n$ exists because $\bigcap_{i \in \omega} X_i = \emptyset$). The function $\ell_X$ will be called the level map of $X$.

Denote by $\tau_X$ the topology of $X$. Using the first-countability of $X$, for each point $x \in X$, fix a neighborhood base $\{O_n^X(x)\}_{n \in \omega}$ at $x$ such that $O_{n+1}^X(x) \subseteq O_n^X(x) \subseteq X \setminus X_{1+\ell_X(x)}$ for every $n \in \omega$. Fix a well-order $\preceq_X$ on the set $X' := X \setminus A$ such that for any $x \in X'$ the set $\downarrow x := \{z \in X' : z \preceq_X x\}$ is finite. For a nonempty subset $S \subseteq X'$ by $\min S$ we denote the smallest element of the set $S$ with respect to the well-order $\preceq_X$.

Next, do the same for the space $Y$ endowed with a canonical superskeleton $(Y_n)_{n \in \omega}$. Denote by $\tau_Y$ the topology of $Y$. Let $\ell_Y : Y \to \omega$ be the function assigning to each $y \in Y$ the largest number $n$ such that $x \in Y_n$. Using the first-countability of $Y$, for every point $y \in Y$ choose a neighborhood base $\{O_n^Y(y)\}_{n \in \omega}$ at $y$ such that $O_{n+1}^Y(y) \subseteq O_n^Y(y) \subseteq Y \setminus Y_{1+\ell_Y(y)}$ for all $n \in \omega$. Fix a well-order $\preceq_Y$ on the set $Y' := Y \setminus B$ such that for any $y \in Y$ the set $\downarrow y := \{z \in Y : z \preceq_Y y\}$ is finite.

If the set $X' = X \setminus A$ is finite, then let $\tilde{f} : X \to Y$ be any injective function such that $\tilde{f}|A = f$ and $f(x) \in \ell_Y^{-1}(\ell_X(x)) \setminus B$ for any $x \in X'$. The choice of $f$ is possible since for every $n \in \omega$ the sets $B \cap Y_n$ and $Y_{n+1}$ are nowhere dense in $Y_n$. Then $\tilde{f}$ is a required extension of $f$. 

$x$ has an open neighborhood $V \subseteq X \setminus W$ such that $V \subseteq U$ and $V \cap (X \setminus W) \subseteq U$. Let $O$ be the interior of the set $\overline{V}$ in $X$. Observe that $O \cap W \subseteq \overline{V} \cap W \subseteq \overline{U} \cap W = \emptyset$ and hence $O \cap W = \emptyset$. Then $O \subseteq \overline{V} \setminus W \subseteq U$. Taking into account that the set $O$ is regular open, we conclude that the space $X$ is semiregular. □
So, we assume that the open subspace $X' = X \setminus A$ of $X$ is infinite. In this case we shall construct the topological embedding $\bar{f}$ by induction over the index set $\Gamma = \omega \cup (\omega \times \omega)$ endowed with the strict well-order $\prec$ uniquely defined by the following conditions:

a) for numbers $n, m \in \omega$ we have $n < m$ iff $n < m$;

b) for a number $n \in \omega$ and a pair $(i, j) \in \omega \times \omega$ we have $(i, j) \prec n$ iff $i + j < n$;

c) for a pair $(i, j) \in \omega \times \omega$ and a number $n \in \omega$ we have $n \prec (i, j)$ iff $n \leq i + j$;

d) for two pairs $(i, j), (n, m) \in \omega \times \omega$ we have $(i, j) \prec (n, m)$ iff either $i + j < n + m$ or $i + j = n + m$ and $i < n$.

The initial elements of the well-ordered set $\Gamma$ are:

$$0, (0, 0), 1, (1, 0), (0, 1), 2, (2, 0), (1, 1), (0, 2), 3, (3, 0), (2, 1), (1, 2), (0, 3), 4, \ldots$$

For every element $\gamma \in \Gamma$ let $\downarrow \gamma = \{ \alpha \in \Gamma : \alpha \prec \gamma \}$. Writing $k \in \downarrow \gamma$ (resp. $(i, j) \in \downarrow \gamma$) we shall understand that $k \in \omega \cap \downarrow \gamma$ (resp. $(i, j) \in (\omega \times \omega) \cap \downarrow \gamma$).

Write the set $\omega$ as the union $\Omega \cup \Omega \cup \Omega$ of pairwise disjoint sets $\Omega, \Omega, \Omega$ such that $|\Omega| = |A|$, $|\Omega| = |X'| = \omega$, $0 \in \Omega$, and $|\Omega| \in \{0, \omega\}$. We choose the set $\Omega$ to be infinite iff the skeleton $(X_n)_{n \in \omega}$ is a canonical superskeleton for $X$, the set $B$ is closed in $Y$, and for every $n \in \omega$ the set $A \setminus X_n$ is nowhere dense in $X_n$. Let $\xi : \Omega \to A$ be a bijective function.

Now we are ready to start the inductive construction of the topological embedding $\bar{f} : X \to Y$ extending the homeomorphism $f$.

Inductively we shall construct sequences of points $\{x_n\}_{n \in \omega} \subseteq X$, $\{y_n\}_{n \in \omega} \subseteq Y$, a double sequences of open sets $\{U_{n,k}\}_{n,k \in \omega} \subseteq \tau_X$, $\{V_{n,k}\}_{n,k \in \omega} \subseteq \tau_Y$, and a function $\ell : \Gamma \to \omega$ such that for any $\gamma \in \Gamma$ the following conditions are satisfied:

1. If $\gamma = n$ for some number $n \in \omega$, then
   
   a) $\ell(\gamma) = \ell_X(x_n) = \ell_Y(y_n)$;
   
   b) $x_n \notin \{x_k\}_{k \in \uparrow \gamma}$ and $y_n \notin \{y_k\}_{k \in \downarrow \gamma}$;
   
   c) $\{(i, j) \in \downarrow \gamma : x_n \in U_{i,j}\} = \{(i, j) \in \downarrow \gamma : y_n \in V_{i,j}\}$;
   
   d) If $n \in \Omega$, then $x_n = \xi(n)$ and $y_n = f(x_n)$;
   
   e) If $n \in \Omega$, then $x_n = \min(X' \setminus \{x_k\}_{k \in \uparrow \gamma})$ and $y_n \notin B$;
   
   f) If $n \in \Omega$, then $y_n = \min(Y' \setminus \{y_k\}_{k \in \downarrow \gamma})$ and $x_n \notin A$.

2. If $\gamma = (n, k)$ for some $n, k \in \omega$, then
   
   a) $\ell(\gamma) = 2 + \max\{\ell(\alpha) : \alpha \in \downarrow \gamma\}$;
   
   b) for any $m \in \omega \cap \downarrow \gamma$ with $m \neq n$, we have $x_m \notin U_{n,k}$ and $y_m \notin V_{n,k}$;
   
   c) $x_n \in U_{n,k} \subseteq O_k^X(x_n) \subseteq X \setminus X_{1+\ell(n)}$ and $y_n \in V_{n,k} \subseteq O_k^Y(x_n) \subseteq Y \setminus Y_{1+\ell(n)}$;
   
   d) $\{(i, j) \in \downarrow \gamma : U_{n,k} \subseteq U_{i,j}\} = \{(i, j) \in \downarrow \gamma : x_n \in U_{i,j}\}$ and $\{(i, j) \in \downarrow \gamma : V_{n,k} \subseteq V_{i,j}\} = \{(i, j) \in \downarrow \gamma : y_n \in V_{i,j}\}$;
   
   e) $\{(i, j) \in \downarrow \gamma : U_{n,k} \cap U_{i,j} = \emptyset\} = \{(i, j) \in \downarrow \gamma : x_n \notin U_{i,j}\}$ and $\{(i, j) \in \downarrow \gamma : V_{n,k} \cap V_{i,j} = \emptyset\} = \{(i, j) \in \downarrow \gamma : y_n \notin V_{i,j}\}$;
   
   f) $X_{\ell(\gamma)} = \partial U_{n,k}$ and $Y_{\ell(\gamma)} = \partial V_{n,k} \subseteq U_{n,k} \cap Y_{\ell(n)}$;
   
   g) if $n \in \Omega$, then $f(U_{n,k} \cap A) = V_{n,k} \cap B$;
   
   h) if $n \notin \Omega$, then $U_{n,k} \cap A = \emptyset = V_{n,k} \cap B$.

0. We start the inductive construction letting $x_0$ be the smallest point of the well-ordered set $(X', \preceq_X)$. Since the set $Y_{1+\ell_X(x_0)}$ is nowhere dense in $Y_{\ell_X(x_0)}$, the set $Y_{\ell_X(x_0)} \setminus Y_{1+\ell_X(x_0)}$ is
not empty and hence contains some point \( y_0 \). Such choice of \( y_0 \) guarantees that the condition (1) is satisfied for \( \gamma = 0 \in \overline{\Omega} \).

Now assume that for some \( \gamma \in \Gamma \), we have defined the function \( \ell \) on the set \( \downarrow \gamma \) and constructed points \( x_n, y_n \) and open sets \( U_{i,j}, V_{i,j} \) for all \( n \in \omega \cap \downarrow \gamma \) and \( (i, j) \in (\omega \times \omega) \cap \downarrow \gamma \) so that the inductive conditions (1)–(2) are satisfied.

To fulfill the inductive step, consider two possible cases.

1. First assume that \( \gamma = n \) for some number \( n \in \omega \). This case has three subcases.

1’. If \( n \in \Omega \), then put \( x_n = \xi(n) \in A \) and \( y_n = f(x_n) \in B \). Our assumption on the map \( f \) ensures that \( \ell_X(x_n) = \ell_Y(y_n) \). So we can put \( \ell(\gamma) := \ell_X(x_n) = \ell_Y(y_n) \) and see that the inductive conditions (1a), (1b) are satisfied. To see that (1c) is satisfied, take any pair \((i, j) \prec \gamma = n\).

   First we assume that \( x_n \in U_{i,j} \). If \( x_i \notin A \), then \( i \notin \Omega \) and we obtain a contradiction \( x_n \in A \cap U_{i,j} = \emptyset \) applying the inductive condition (2h). This contradiction shows that \( x_i \notin A \). Then \( y_n = f(x_n) \in f(A \cap U_{i,j}) = B \cap V_{i,j} \subseteq V_{i,j} \) by the inductive condition (2g). By analogy we can show that \( x_n \in V_{i,j} \) implies \( x_n \in U_{i,j} \). This means that the condition (1c) is satisfied.

   Now assume that \( x_n \in \overline{U_{i,j}} \). If \( x_n \in U_{i,j} \), then \( y_n \in V_{i,j} \subseteq \overline{V_{i,j}} \) by the (already proved) inductive condition (1c). So, we assume that \( x_n \in \overline{U_{i,j}} \setminus U_{i,j} = \partial U_{i,j} = X_{\ell(i,j)} \) (for the last equality, see the inductive condition (2f)). Then \( \ell_Y(y_n) = \ell_X(x_n) \geq \ell(i, j) \) and hence \( y_n \in Y_{\ell(i,j)} \subseteq \overline{V_{i,j}} \) by the condition (2f). By analogy we can prove that \( y_n \in \overline{V_{i,j}} \) implies \( x \in \overline{U_{i,j}} \). This means that the condition (1d) is satisfied. It is clear that the conditions (1e)–(1g) are satisfied, too.

1”. Next, assume that \( n \in \overline{\Omega} \). This case requires much more work. Define the point \( x_n \) by the formula (1f) and put \( \ell(n) = \ell_X(x_n) \). It remains to find a point \( y_n \in Y \) satisfying the conditions (1a)–(1d).

**Lemma 9.** For any nonempty set \( J \subseteq (\omega \times \omega) \cap \downarrow \gamma \), integer number \( l < \min \{ \ell(i, j) : (i, j) \in J \} \), and nonempty open set \( W \subseteq Y \) such that \( W \cap \bigcap_{(i, j) \in J} Y_{\ell(i, j) - 1} \neq \emptyset \), there exists a point \( y \in W \cap Y_1 \setminus Y_{l+1} \) such that \( y \notin \bigcup_{(i, j) \in J} \overline{V_{i,j}} \).

**Proof.** Write the set \( J \) as \( \{(i_1, j_1), \ldots, (i_m, j_m)\} \) for some pairs \( (i_m, j_m) \cdots < (i_1, j_1) \). If \( J \) is empty, then \( m = 0 \). The inductive condition (2a) ensures that \( \ell(i_k, j_k) + 2 \leq \ell(i_{k-1}, j_{k-1}) \) for any \( k \in \{2, \ldots, m\} \).

Let \( v_1 \) be any point in the set \( W \cap Y_{\ell(i_1, j_1) - 1} \setminus Y_{\ell(i_1, j_1)} \). Such point exists since the intersection \( W \cap Y_{\ell(i_1, j_1) - 1} \) is nonempty and \( Y_{\ell(i_1, j_1)} \) is nowhere dense in \( Y_{\ell(i_1, j_1) - 1} \).

By the inductive conditions (2c),(2f),(2a) we have
\[
\overline{V_{i_1,j_1}} = V_{i_1,j_1} \cup \partial V_{i_1,j_1} \subseteq (Y \setminus Y_{l+1}) \cup Y_{\ell(i_1,j_1)}
\]
and \( \ell(i_1) + 2 \leq \ell(i_1, j_1) \). Hence, the set \( Y_{\ell(i_1,j_1) - 1} \setminus Y_{\ell(i_1,j_1)} \) is disjoint with \( \overline{V_{i_1,j_1}} \). Then \( v_1 \notin \overline{V_{i_1,j_1}} \) and we can choose an open neighborhood \( W_1 \subseteq W \) of \( v_1 \) such that \( W_1 \cap \overline{V_{i_1,j_1}} = \emptyset \).

Inductively we shall construct a sequence of points \( v_2, \ldots, v_m \in Y \) and a sequence of open sets \( W_2, \ldots, W_m \) in \( Y \) such that for every \( k \in \{2, \ldots, m\} \) the following conditions are satisfied:

(i) \( v_k \in W_{k-1} \cap Y_{\ell(i_k,j_k) - 1} \setminus Y_{\ell(i_k,j_k)} \);

(ii) \( v_k \in W_k \subseteq W_{k-1} \setminus \overline{V_{i_k,j_k}} \).
Assume that for some \( k \in \{2, \ldots, m\} \) we have constructed points \( v_1, \ldots, v_{k-1} \) and an open set \( W_1, \ldots, W_{k-1} \) satisfying the conditions (i), (ii). Since \( v_{k-1} \in Y_{\ell(i_k,j_k)} \) and the set \( Y_{\ell(i_k,j_k)} \) is nowhere dense in \( Y_{\ell(i_k,j_k)-1} \), we can choose a point \( v_k \in W_{k-1} \cap Y_{\ell(i_k,j_k)-1} \). By the inductive conditions (2c) and (2f), \( V_{i_k,j_k} = V_{i_k,j_k} \cup \partial V_{i_k,j_k} \subseteq (Y \setminus Y_{1+\ell(i_k)}) \cup Y_{\ell(i_k,j_k)} \) and \( \ell(i_k) + 1 \leq \ell(i_k,j_k) - 1 \). Consequently, the set \( Y_{\ell(i_k,j_k)-1} \setminus Y_{\ell(i_k,j_k)} \) is disjoint with \( V_{i_k,j_k} \). So, \( v_k \notin V_{i_k,j_k} \) and we can choose an open neighborhood \( W_k \subseteq W_{k-1} \) of \( v_k \) such that \( W_k \cap V_{i_k,j_k} = \emptyset \). This completes the inductive step.

After completing the inductive construction, consider the point \( v_m \in Y_{\ell(i_m,j_m)-1} \) and its neighborhood \( W_m \subseteq W \). The inductive condition (ii) guarantees that

\[
W_m \cap \bigcup_{k=1}^{m} V_{i_k,j_k} = \bigcup_{k=1}^{m} (W_m \cap V_{i_k,j_k}) \subseteq \bigcup_{k=1}^{m} (W_k \cap V_{i_k,j_k}) = \emptyset.
\]

Taking into account that \( l < \min \{\ell(i,j) : (i,j) \in J\} \leq \ell(i_m,j_m) \), we conclude that \( v_m \in Y_{\ell(i_m,j_m)-1} \subseteq Y_1 \). Since the set \( Y_{l+1} \) is nowhere dense in \( Y_l \), there exists a point \( y \in W_m \cap Y_{1} \setminus Y_{l+1} \). Since \( W_m \) is disjoint with \( \bigcup_{(i,j) \in J} V_{i,j} \), the point \( y_n \) does not belong to \( \bigcup_{(i,j) \in J} V_{i,j} \).

Now we are able to find a \( y_n \) satisfying the conditions (1a)–(1d).

Consider the sets \( I(x_n) = \{(i,j) \in 1 : x_n \in U_{i,j}\} \) and \( J(x_n) = \{(i,j) \in 1 : x_n \notin U_{i,j}\} \).

**Claim 4.**

1. For any \( (i,j) \in I(x_n) \cup J(x_n) \) we have \( \ell_X(x_n) < \ell(i,j) \).
2. For any \( (i,j) \in I(x_n) \) we have \( \ell_X(x_n) \leq \ell(i) \).

**Proof.**

1. If \( (i,j) \in I(x_n) \cup J(x_n) \), then \( x_n \notin \partial U_{i,j} = X_{i,j} \) and hence \( \ell_X(x_n) < \ell(i,j) \).
2. Now assume that \( (i,j) \in I(x_n) \). Then \( x_n \in U_{i,j} \subseteq X \setminus X_{1+\ell_X(i)} \) and hence \( \ell_X(x_n) < 1 + \ell_X(i) = 1 + \ell(i) \) according to (1a).

Choose a minimal subset \( I \subseteq I(x_n) \) such that for every \( (i,j) \in I(x_n) \) there exists \( (p,q) \in I \) such that \( U_{p,q} \subseteq U_{i,j} \). It is clear that \( \bigcap_{(i,j) \in I(x_n)} U_{i,j} = \bigcap_{(i,j) \in I} U_{i,j} \).

**Claim 5.**

\( \bigcap_{(i,j) \in I(x_n)} V_{i,j} = \bigcap_{(i,j) \in I} V_{i,j} \).

**Proof.**

It suffices to show that for any \( (i,j) \in I(x_n) \) there exists \( (p,q) \in I \) such that \( V_{p,q} \subseteq V_{i,j} \). Given any pair \( (i,j) \in I(x_n) \), find a \( (p,q) \in I \) such that \( x_p \in U_{p,q} \subseteq U_{i,j} \) (such a pair exists by the choice of the set \( I \)). If \( (i,j) \prec (p,q) \), then \( y_p \in V_{i,j} \) by the condition (1c) and \( x_p \in U_{p,q} \) by the condition (2d).

If \( p \prec (i,j) \), the the condition (2b) implies that \( p = i \) and condition (2d) ensures that \( V_{p,q} \subseteq V_{i,j} \).

**Claim 6.**

For any pairs \( (i,j) \prec (p,q) \) in \( I \) we have \( \ell(p) \geq \ell(i,j) > \ell(i) \).

**Proof.**

First we show that \( x_p \in \partial U_{i,j} \). Assuming that \( x_p \notin \partial U_{i,j} \), we conclude that \( x_p \in U_{i,j} \) or \( x_p \notin U_{i,j} \). If \( x_p \in U_{i,j} \), then the inductive condition (2d) guarantees that \( x_p \in U_{p,q} \subseteq U_{i,j} \) and the minimality of \( I \) ensures that \( (i,j) \notin I \), which contradicts our assumption. So, \( x_p \notin U_{i,j} \).

In this case, \( U_{p,q} \cap U_{i,j} = \emptyset \) by the condition (2e), but this contradicts \( x_n \in U_{i,j} \cap U_{p,q} \). Therefore, \( x_p \in \partial U_{i,j} = X_{i,j} \) and \( \ell(p) = \ell_X(x_p) \geq \ell(i,j) > \ell_X(x_i) = \ell(i) \).

Write the set \( I \) as \( \{(i_1,j_1), \ldots, (i_m,j_m)\} \) for some pairs \( (i_m,j_m) \prec \cdots \prec (i_1,j_1) \). If \( I \) is empty, then \( m = 0 \). Claims 4 and 5 imply that

\[
\ell(i_1,j_1) > \ell(i_1,j_2) \geq \ell(i_2,j_2) \geq \cdots \geq \ell(i_m,j_m) > \ell(i_m) \geq \ell_X(x_n).
\]
This chain of inequalities allows us to write the set $J(x_n)$ as the union

$$J(x_n) = \left( \bigcup_{k=1}^{m} J_k \right) \cup \left( \bigcup_{k=0}^{m} J_k' \right)$$

of the sets

- $J_k = \{(i, j) \in J(x_n) : \ell(i_k, j_k) > \ell(i, j) > \ell(i_k)\}$ for $k \in \{1, \ldots, m\}$,
- $J_0 = \{(i, j) \in J(x_n) : \ell(i, j) > \ell(i_1, j_1)\}$,
- $J_k' = \{(i, j) \in J(x_n) : \ell(i_k) > \ell(i, j) > \ell(i_{k+1}, j_{k+1})\}$ for $k \in \{1, \ldots, m - 1\}$,
- $J_m' = \{(i, j) \in J(x_n) : \ell(i, j) > \ell(i, j) > \ell_Y(x_n)\}$.

Since the sets $\{(i, j) : (i, j) \in J(x_n)\}$ and $\{(i_k), \ell(i_k, j_k)\}_{k=1}^{m}$ are disjoint, the union $\bigcup_{k=1}^{m} J_k \cup \bigcup_{k=0}^{m} J_k'$ is indeed equal to $J(x_n)$.

By Lemma 9, there exists a point $v_0' \in Y_{\ell(i_1, j_1)}$ such that $v_0' \not\in \bigcup_{(i,j) \in J'} \nabla_{i,j}$. Then $W_0' := Y \setminus \bigcup_{(i,j) \in J'} \nabla_{i,j}$ is an open neighborhood of $v_0'$.

Inductively we shall construct a sequence of points $v_1, v_1', \ldots, v_n, v_n'$ and a sequence of open sets $W_1 \supseteq W_1' \supseteq \cdots \supseteq W_m \supseteq W_m'$ in $Y$ such that for every $k \in \{1, \ldots, m\}$ the following conditions are satisfied:

1. $v_k \in W_{k-1}' \cap V_{i_k,j_k} \cap Y_{\ell(i_k)}$;
2. $v_k \in W_k = W_{k-1}' \cap V_{i_k,j_k}$;
3. $W_k \cap \bigcup_{(i,j) \in J_k} \nabla_{i,j} = \emptyset$;
4. $v_k' \in W_k' \subseteq W_k \setminus \bigcup_{(i,j) \in J_k'} \nabla_{i,j}$;
5. if $k < m$, then $v_k' \in Y_{\ell(i_{k+1}, j_{k+1})}$;
6. $v_m' \in Y_{\ell_Y(x_n)} \setminus Y_{1+\ell_Y(x_n)}$.

To make an inductive step, assume that for some $k \in \{1, \ldots, m\}$ a point $v'_{k-1}$ and an open set $W'_k$ with $v'_{k-1} \in W_{k-1} \cap Y_{\ell(i_k,j_k)}$ have been constructed. By the inductive condition (2f), $Y_{\ell(i_k,j_k)} = \partial V_{i_k,j_k} \cap Y_{\ell(i_k)}$. Consequently, there exists a point $v_k \in W_{k-1} \cap V_{i_k,j_k} \cap Y_{\ell(i_k)}$. Put $W_k := W_{k-1} \cap V_{i_k,j_k}$. It is clear that the inductive conditions (a), (b) are satisfied.

**Claim 7.** $V_{i_k,j_k} \cap \bigcup_{(i,j) \in J_k} \nabla_{i,j} = \emptyset$.

**Proof.** To derive a contradiction, assume that $V_{i_k,j_k} \cap \nabla_{i,j} \neq \emptyset$ for some $(i, j) \in J_k$. The definition of the set $J_k \ni (i, j)$ yields $\ell(i_k, j_k) > \ell(i, j) > \ell(i_k)$ and hence $(i, j) \prec (i_k, j_k)$ by the condition (2a). It follows from (2c) and $V_{i_k,j_k} \cap \nabla_{i,j} \neq \emptyset$ that $y_{i_k} \in \nabla_{i,j}$ and hence $x_{i_k} \in \overline{U}_{i,j}$ according to the condition (1d). Assuming that $x_{i_k} \in U_{i,j}$, we obtain $U_{i_k,j_k} \subseteq U_{i,j}$ by the inductive condition (2d). Then $x_n \in U_{i_k,j_k} \subseteq U_{i,j}$, which contradicts the inclusion $(i, j) \in J_k$. Therefore, $x_{i_k} \not\in U_{i,j}$ and hence $y_{i_k} \not\in V_{i,j}$ by condition (1c). Then $y_{i_k} \in \nabla_{i,j} \setminus V_{i,j} = \partial V_{i,j}$ and hence $\ell(i_k) = \ell_Y(y_{i_k}) \geq \ell(i, j)$, which contradicts the inclusion $(i, j) \in J_k$. \qed

Claim 7 and the condition (b) imply the condition (c).

Since $v_k \in W_k \cap Y_{\ell(i_k)} \subseteq W_k \cap \bigcup_{(i,j) \in J_{k-1}} Y_{\ell(i,j)}$, we can apply Lemma 9 and find a point $v'_{k-1} \in W_k$ and a neighborhood $W'_k$ of $v'_{k-1}$ satisfying the inductive conditions (d),(e),(f).

After completing the inductive construction, we conclude that the open subset $W_m' \cap Y_{\ell(n)} \setminus Y_{1+\ell(n)}$ of the crowded space $Y_{\ell(n)}$ contains the point $v'_m$ and hence is not empty. Since the
space $Y_{\ell(n)}$ is crowded (see Lemma 4) and the set $\tilde{B} \cap Y_{\ell(n)}$ is nowhere dense in $Y_{\ell(n)}$, there exists a point

$$y_n \in W'_n \cap (Y_{\ell(n)} \setminus Y_{1+\ell(n)}) \setminus (\tilde{B} \cup \{y_k\}_{k \in \gamma}).$$

The inductive conditions (a),(c),(d) and Claim 5 imply that $I(x_n) \subseteq I(y_n)$ and $J(x_n) \subseteq J(y_n)$, where

$$I(y_n) = \{(i, j) \in \downarrow \gamma : y_n \in V_{i,j}\} \quad \text{and} \quad J(y_n) = \{(i, j) \in \downarrow \gamma : y_n \notin \overline{V}_{i,j}\}.$$

The condition (1c) will follow as soon as we show that $\partial U \cap y_n = \partial V$ and hence $y_n \in Y_{\ell(i,j)} = \partial V_{i,j}$ and hence $y_n \notin V_{i,j}$ and $(i, j) \notin I(y_n)$, which contradicts the choice of $(i, j)$. This completes the proof of condition (1c).

To prove the condition (1d), assume that $J(x_n) \neq J(y_n)$ and find a pair $(i, j) \in J(y_n) \setminus J(x_n)$. Then $x_n \in \overline{U}_{i,j}$. Assuming that $(i, j) \in I(x_n) = I(y_n)$ and hence $y_n \in V_{i,j} \subseteq \overline{V}_{i,j}$, which contradicts $(i, j) \in J(y_n)$. Therefore, $x_n \in \overline{U}_{i,j} \setminus U_{i,j} = \partial U_{i,j} = X_{\ell(i,j)}$ and $\ell_Y(y_n) = \ell_X(x_n) \geq \ell(i, j)$ and then the condition (2f) ensures that $y_n \in Y_{\ell(i,j)} = \partial V_{i,j}$ and hence $(i, j) \notin J(y_n)$, which contradicts the choice of the pair $(i, j)$. This contradiction completes the proof of the condition (1d). It is clear that the conditions (1e)–(1g) holds.

1′′. $n \in \widehat{\Omega}$. In this case the set $\overline{\Omega}$ is not empty and hence $(X_n)_{n \in \omega}$ is a canonical superskeleton for $X$, the set $B$ is closed in $X$ and for every $n \in \omega$ the set $A \cap X_n$ is nowhere dense in $X_n$. In this case we put $y_n = \min(Y'' \setminus \{y_k\}_{k \in \gamma})$ and repeating the argument from the case 1′, can find a point $x_n \in X$ satisfying the conditions (1a)–(1g).

2. Now consider the second case: $\gamma = (n, k)$ for some $(n, k) \in \omega \times \omega$. Since $n \prec (n, k)$, the points $x_n, y_n$ have been already defined. So, we can choose open sets $U \subseteq X$ and $V \subseteq Y$ such that

- $x_k \notin U \neq X$ and $y_k \notin V \neq Y$ for every $k \in \downarrow \gamma \setminus \{x_n\}$,
- $x_n \in U \subseteq O_X^\gamma(x_n) \cap \bigcap_{(i, j) \in I(x_n)} U_{i,j} \setminus \bigcup_{(i, j) \in J(x_n)} \overline{U}_{i,j}$, and
- $y_n \in V \subseteq O_Y^\gamma(y_n) \cap \bigcap_{(i, j) \in I(y_n)} V_{i,j} \setminus \bigcup_{(i, j) \in J(y_n)} \overline{V}_{i,j}$,

where

$$I(x_n) = \{(i, j) \in \downarrow \gamma : x_n \in U_{i,j}\}, \quad I(y_n) = \{(i, j) \in \downarrow \gamma : y_n \in V_{i,j}\},$$

$$J(x_n) = \{(i, j) \in \downarrow \gamma : x_n \notin \overline{U}_{i,j}\}, \quad J(y_n) = \{(i, j) \in \downarrow \gamma : y_n \notin \overline{V}_{i,j}\}.$$

If $n \notin \Omega$, then $x_n \notin A, y_n \notin \overline{B}$ (by the inductive conditions (1f), (1g)) and we can (and will) additionally assume that

$$U \cap A = \emptyset = V \cap \tilde{B}.$$

Since $(X_i)_{i \in \omega}$ is a coregular skeleton for the space $X$ and $(Y_i)_{i \in \omega}$ is a superskeleton for the space $Y$, there exists a number $l \geq 2 + \max\{\ell(\alpha) : \alpha \in \downarrow \gamma\}$ such that

$$X_i \subseteq \overline{U} \cap X \setminus \overline{U} \subseteq \partial U \quad \text{and} \quad Y_i \subseteq \overline{V} \cap Y_{\ell(n)} \setminus \overline{Y} \subseteq \partial V.$$

Since the skeleton $(Y_i)_{i \in \omega}$ is coregular, the complement $Y \setminus Y_i$ is a regular topological space. Being second-countable, the regular space $Y \setminus Y_i$ is metrizable (by the Urysohn Metrization Theorem [2, 4.2.9]). Being countable, the metrizable space $Y \setminus Y_i$ is zero-dimensional. Then we can find a closed-and-open neighborhood $V' \subseteq Y \setminus Y_i$ of the point $y_n$ such that $V' \subseteq V$. Then $\partial V' \subseteq Y_i$. 


By analogy we prove that the space $X \setminus X_I$ is metrizable and zero-dimensional. By Theorem \([8, 7.1.11]\), the countable zero-dimensional space $X \setminus X_I$ is strongly zero-dimensional (which means that any disjoint closed sets in $X \setminus X_I$ can be separated by closed-and-open neighborhoods). Observe that the sets $f^{-1}(V')$ and $f^{-1}((Y \setminus Y_I) \setminus V')$ are two closed disjoint sets in $A \setminus X_I$ and $X \setminus X_I$. By the strong zero-dimensionality of $X \setminus X_I$, there exists a closed-and-open set $U'$ in $X \setminus X_I$ such that

$$
\{x_n\} \cup f^{-1}(V') \subseteq U' \subseteq U \setminus f^{-1}((Y \setminus Y_I) \setminus V').
$$

Then $\partial U' \subseteq X_I$ and $f(A \cap U') = B \cap V'$.

Since $(X_i)_{i \in \omega}$ is a coregular skeleton for $X$ and $(Y_i)_{i \in \omega}$ is a superconnecting skeleton for $Y$, there exists a number $p > l$ such that $X_p \subseteq \partial U'$ and $Y_p \subseteq \partial V'$.

**Lemma 10.** There exists closed-and open subset $V_{n,k} \subseteq Y \setminus Y_p$ such that $V' \subseteq V_{n,k} \subseteq V$, $(V_{n,k} \setminus V') \cap B = \emptyset$ and $\partial V_{n,k} = Y_I \subseteq V_{n,k} \cap Y_{\ell(n)}$.

**Proof.** By the Urysohn Metrization Theorem \([8, 4.2.9]\), the second-countable regular space $Y \setminus Y_p$ is metrizable. So, we can find a metric $d$ generating the topology of $Y$. Since the set $Y_I \setminus Y_p \subseteq Y$ is countable and nonempty, there exists a function $h : \omega \to Y_I \setminus Y_p$ such that for every $y \in Y_I \setminus Y_p$ the preimage $h^{-1}(y)$ is infinite. Since $Y_I \setminus Y_p \subseteq Y \setminus Y_{\ell(n)}$ and the set $B \cap Y_{\ell(n)}$ is nowhere dense in $Y_{\ell(n)}$, for every $m \in \omega$ we can find a point $v_m \in V \cap Y_{\ell(n)} \setminus B$ such that $d(v_m, h(m)) < 2^{-m}$. Since the space $Y \setminus Y_p$ is zero-dimensional, the point $v_m$ has a closed-and-open neighborhood $W_m$ in $Y \setminus (Y_p \cup B)$ such that $W_m \subseteq V \cap \{y \in Y \setminus Y_p : d(y, v_m) < 2^{-m}\}$. Then the boundary $\partial W_m$ of $W_m$ in $Y$ is contained in $Y_p$. We claim that the open neighborhood

$$
V_{n,k} = V' \cup \bigcup_{m \in \omega} W_m \subseteq V
$$

of $y_n$ has the required property: $X_I = \partial V_{n,k} \subseteq V_{n,k} \cap Y_{\ell(n)}$.

First we show that $Y_I \subseteq V_{n,k} \cap Y_{\ell(n)}$. Given any point $a \in Y_I$ and open neighborhood $O_a \subseteq Y$ of $a$, use the nowhere dense of $Y_p$ in $Y_I$ and find a point $b \in O_a \cap (Y_I \setminus Y_p)$. Since the metric $d$ generates the topology of the space $Y \setminus Y_p$, there exists a number $q \in \omega$ such that the ball $B(b; 2^{-q}) = \{y \in Y \setminus Y_I : d(y, b) < 2^{-q}\}$ is contained in $O_a$. Since the set $h^{-1}(b)$ is infinite, there exists $m > q$ such that $d(y, h(m)) = b$. Then $d(b, v_m) = d(h(m), v_m) < 2^{-m} < 2^{-q}$ and hence $v_m \in O_a$. On the other hand, $v_m \in V_{n,k} \cap Y_{\ell(n)}$, witnessing that $O_a \cap \{y \in Y \setminus Y_{\ell(n)}\} \neq \emptyset$ and hence $a \in V_{n,k} \cap Y_{\ell(n)} \subseteq V_{n,k}$.

On the other hand, $a \in Y_I \subseteq Y_p \subseteq X \setminus V \subseteq Y \setminus V_{n,k}$ and hence $a \in V_{n,k} \cap Y \setminus V_{n,k} = \partial V_{n,k}$. Therefore, $Y_I \subseteq \partial V_{n,k}$. Assuming that $Y_I \neq \partial V_{n,k}$, we can find a point $z \in \partial V_{n,k} \setminus Y_I$. Choose $s \in \omega$ such that the ball $B(z; 2^{-s}) = \{y \in Y : d(z, y) < 2^{-s}\}$ does not intersect the closed subset $Y_I \setminus Y_p$ of $Y \setminus Y_p$. We claim that for every $m \geq s + 2$, the ball $B(z; 2^{-s-1})$ does not intersect the set $W_m$. Assuming that $B(z; 2^{-s-1}) \cap W_m$ contains some point $w$, we conclude that

$$
d(z, h(m)) \leq d(z, w) + d(w, v_m) + d(v_m, h(m)) < 2^{-s-1} + 2^{-m} + 2^{-m} = 2^{-s-1} + 2^{-m+1} \leq 2^{-s-1} + 2^{-s-1} = 2^{-s},
$$

which contradicts the choice of $s$. Since $z \in \partial V_{n,k}$, $z \notin V_{n,k}$ and hence $z \notin V' \cup \bigcup_{m < 2+s} W_m$. It follows from $\partial V' \cup \bigcup_{m < 2+s} \partial W_m \subseteq Y_I \neq z$ that $z \notin \bigcup V' \cup \bigcup_{m < 2+s} W_m$. Then we can find a
neighborhood $O_z \subseteq B(z; 2^{-s-1})$ such that $O_z \cap \overline{V} \cup \bigcup_{m<2+s} W_m = \emptyset$ and hence
\[ O_z \cap V_{n,k} \subseteq (O_z \cap V') \cup \left( \bigcup_{m<2+s} O_z \cap W_m \right) \cup \left( \bigcup_{m \geq 2+s} B(s; 2^{-s-1}) \cap W_m \right) = \emptyset, \]
which contradicts $z \in \partial V_{n,k}$. This contradiction shows that $\partial V_{n,k} = Y_I \subseteq \overline{V}_{n,k} \cap Y_{\ell(n)}$. The choice of the sets $W_m \subseteq Y \setminus B$ ensures that
\[ (V_{n,k} \setminus V) \cap B \subseteq \bigcup_{m \in \omega} (W_m \cap B) = \emptyset. \]

By analogy we can prove the following lemma.

**Lemma 11.** There exists closed-and-open subset $U_{n,k} \subseteq X \setminus X_p$ such that $U' \subseteq U_{n,k} \subseteq U$, $(U_{n,k} \setminus U') \cap A = \emptyset$ and $\partial U_{n,k} = X_I$. Moreover, if $\ov{\Omega} \neq \emptyset$, then $X_I \subseteq U_{n,k} \cap X_{\ell(n)}$.

Finally, observe that the number $\ell(n,k) := l$ and the sets $U_{n,k}$ and $V_{n,k}$ constructed in Lemmas 10 and 11 satisfy the conditions (2a)–(2i). This completes the inductive step.

After completing the inductive construction, observe that the inductive condition (1f) implies that $X = \{ x_n \}_{n \in \omega}$. So, we can consider the map $f : X \rightarrow Y$ such that $f(x_n) = y_n$ for every $n \in \omega$. The inductive condition (1e) ensures that $\ov{f[A]} = f$. We claim that the map $f$ is a topological embedding.

To see that $f$ is continuous, take any $n \in \omega$ and any neighborhood $O(y_n)$ of the point $y_n = f(x_n)$. Find $k \in \omega$ such that $O_k^Y(y_n) \subseteq O(y_n)$. The inductive condition (2b) guarantees that $V_{n,k} \subseteq O_k^Y(y_n)$. We claim that $f(U_{n,k}) \subseteq V_{n,k}$. Indeed, for any $x_m \in U_{n,k} \setminus \{ x_n \}$, the inductive condition (2b) ensures that $(n,k) < m$. Then $y_m \in V_{n,k}$ by the condition (1c). Therefore, $f(U_{n,k}) \subseteq V_{n,k} \subseteq O_k^Y(y_n) \subseteq O(y_n)$, witnessing that the map $f$ is continuous. By analogy we can prove the continuity of the map $f^{-1} : f(X) \rightarrow X$.

If $(X_n)_{n \in \omega}$ is a canonical superskeleton for $X_n$, the set $B$ is closed in $Y$, and for every $n \in \omega$ the set $A \cap X_n$ is nowhere dense in $X_n$, then the set $\ov{\Omega}$ is infinite and the inductive condition (1g) implies that $\ov{f(X)} = \{ y_n \}_{n \in \omega} = Y$ and hence the topological embedding $f$ is a homeomorphism. \qed

Now we deduce some corollaries of Theorem 1.

**Corollary 1.** Let $X,Y$ be two countable second-countable topological spaces, $(X_n)_{n \in \omega}$ and $(Y_n)_{n \in \omega}$ be canonical superskeleta in the spaces $X,Y$, and $A,B$ be closed nowhere dense sets in the spaces $X,Y$, respectively. Let $h : A \rightarrow B$ be a homeomorphism such that $h(A \cap X_n) = (B \cap Y_n)$ for every $n \in \omega$. Then there exists a homeomorphism $h : X \rightarrow Y$ such that $h|A = h$ and $h(X_n) = Y_n$ for all $n \in \omega$.

**Corollary 2.** Let $X,Y$ be two countable second-countable topological spaces and $(X_n)_{n \in \omega}$ and $(Y_n)_{n \in \omega}$ be canonical superskeleta in the spaces $X,Y$, respectively. Then there exists a homeomorphism $h : X \rightarrow Y$ such that $h(X_n) = Y_n$ for all $n \in \omega$.

Lemma 5 and Corollary 2 imply another corollary.

**Corollary 3.** Let $X,Y$ be two countable second-countable topological spaces and $(X_n)_{n \in \omega}$ and $(Y_n)_{n \in \omega}$ be superskeleta in the spaces $X,Y$, respectively. Then there exists an increasing number sequence $(n_k)_{k \in \omega}$ and a homeomorphism $h : X \rightarrow Y$ such that $h(X_{n_k}) = Y_{n_k}$ for all $k \in \omega$. 
Now we are able to prove Theorem 4 reformulating it as follows.

**Theorem 4** (Characterization of \(\mathbb{Q}P^\infty\)). A topological space \(X\) is homeomorphic to the space \(\mathbb{Q}P^\infty\) if and only if \(X\) is countable second-countable and admits a vanishing sequence \((X_n)_{n \in \omega}\) of nonempty closed sets that has two properties:

1. for every \(n \in \omega\) and a nonempty open set \(U \subseteq X_n\) the closure \(\overline{U}\) contains some set \(X_m\);
2. for every \(n \in \omega\) the complement \(X \setminus X_n\) is a regular topological space.

**Proof.** The “only if” part follows from Theorem 2. To prove the “if” part, assume that the space \(X\) is countable, second-countable and \(X\) has a vanishing sequence of nonempty closed sets \((X_n)_{n \in \omega}\) satisfying the conditions (1),(2). By Definition 3 \((X_n)_{n \in \omega}\) is a superskeleton for \(X\). By Theorem 2 the space \(\mathbb{Q}P^\infty\) also is countable, second-countable and has a superskeleton. By Corollary 3 the spaces \(X\) and \(\mathbb{Q}P^\infty\) are homeomorphic. \(\square\)

Now we prove a universality property of the space \(\mathbb{Q}P^\infty\).

**Theorem 5** (Universality of \(\mathbb{Q}P^\infty\)). Each countable second-countable coregular space \(X\) is homeomorphic to a subspace of \(\mathbb{Q}P^\infty\).

**Proof.** By Lemma 1 the space \(X\) admits a coregular skeleton \((X_n)_{n \in \omega}\). By Theorem 2 the space \(\mathbb{Q}P^\infty\) has a canonical superskeleton \((Y_n)_{n \in \omega}\). Applying Theorem 1 with \(A = B = \emptyset\), we obtain a topological embedding \(f : X \to Y\) such that \(f^{-1}(Y_n) = X_n\) for all \(n \in \omega\). \(\square\)

Now we prove a (rather strong) homogeneity property of the space \(\mathbb{Q}P^\infty\).

A subset \(A\) of a topological space \(X\) is called

- deep if for any non-empty open sets \(U_1, \ldots, U_n \subseteq X\) the set \(A \setminus (\overline{U_1} \cap \cdots \cap \overline{U_n})\) is finite.
- shallow if there exist non-empty open sets \(U_1, \ldots, U_n \subseteq X\) such that \(A \cap (\overline{U_1} \cap \cdots \cap \overline{U_n}) = \emptyset\).

This definition implies that for any deep (resp. shallow) set \(A\) in a topological space \(X\) and any homeomorphism \(h : X \to X\) the set \(h(A)\) is deep (resp. shallow). Observe also that any infinite set in a second-countable space contains an infinite subset which is either deep or shallow. The definition implies that any finite set in a Hausdorff space is shallow.

**Theorem 6** (Dichotomic Homogeneity of \(\mathbb{Q}P^\infty\)). Let \(A, B\) be two closed discrete subsets of \(\mathbb{Q}P^\infty\). If the sets \(A, B\) are either both deep or both shallow, then any bijection \(f : A \to B\) extends to a homeomorphism \(h\) of \(\mathbb{Q}P^\infty\) such that \(h(A) = B\).

**Proof.** By Theorem 2 the space \(\mathbb{Q}P^\infty\) has a canonical superskeleton \((X_n)_{n \in \omega}\). If both sets \(A, B\) are shallow, then we can find a number \(m \in \omega\) such that \(X_m\) is disjoint with the set \(A \cup B\). Let \(Y_0 = X_0\) and \(Y_n = X_{m+n}\) for \(n \in \mathbb{N}\). Observe that \((Y_n)_{n \in \omega}\) is a canonical superskeleton for the space \(\mathbb{Q}P^\infty\) such that \(A \cup B \subseteq Y_0 \setminus Y_1\). Since the space \(\mathbb{Q}P^\infty\) is crowded, the sets \(A, B\) are nowhere dense in \(X\). Applying Corollary 1 we can find a homeomorphism \(h : X \to X\) such that \(h|A = f\) and \(h(Y_n) = Y_n\) for all \(n \in \omega\).

The case of deep sets \(A, B\) is more tricky. Let \(\ell : X \to \omega\) be the function assigning to each point \(x \in X\) the unique number \(n \in \omega\) such that \(x \in X_n \setminus X_{n+1}\). The deepness of \(A, B\) implies that for every \(n \in \omega\) the set \((A \cup B) \setminus X_n\) is finite. Choose an increasing sequence \((n_k)_{k \in \omega}\) such that \(n_0 = 0\) and for every \(k \in \omega\) the following conditions hold:

- \(n_{k+1} > \ell(f(a))\) for any \(a \in A \setminus X_{n_{k+1}}\);
- \(n_{k+1} > \ell(f^{-1}(b))\) for any \(b \in B \setminus X_{n_{k+1}}\).
For every $k \in \omega$ consider the finite sets
\[ A_k = A \cap X_{nk} \setminus X_{nk+1}, \quad B_k = B \cap X_{nk} \setminus X_{nk+1}, \]
\[ A_k^+ = \{a \in A : f(a) \in X_{nk} \setminus X_{nk+1}\}, \quad B_k^+ = \{b \in B : f^{-1}(b) \in X_{nk} \setminus X_{nk+1}\}, \]
\[ A_k^- = \{a \in A : f(a) \notin X_{nk}\}, \quad B_k^- = \{b \in B : f^{-1}(b) \notin X_{nk}\}. \]

**Claim 8.** For any $k \in \omega$ we have
\[ f(A_k^-) = B_k^-, \quad f(A_k^+) = B_k^+, \quad f^{-1}(B_k^+) = A_k^-+1, \quad \text{and} \quad f(A_k^-+1) = B_k^+. \]

**Proof.** The equality $f(A_k^-) = B_k^-$ follows from the definition of the sets $A_k^-$ and $B_k^-$. To show that $f(A_k^-) = B_k^+$, take any $a \in A_k^+$. Then $f(a) \in B \cap X_{nk+1}$ by the definition of $A_k^+$. The definition of the number $n_{k+2}$ guarantees that $\ell(f(a)) < n_{k+2}$ and hence for every $a \in A_k^+$,
\[ f(a) \in B \cap X_{nk+1} \setminus X_{nk+2} = B_{k+1}. \]

Since $a = f^{-1}(f(a)) \notin X_{nk+1}$, the point $f(a)$ belongs to $B_{k+1}^-$. Therefore, $f(A_k^+) \subseteq B_{k+1}^-$. Now take any point $b \in B_{k+1}^-$ and observe that the point $a = f^{-1}(b)$ does not belong to $X_{nk+1}$ by the definition of the set $B_{k+1}^-$. Assuming that $a \notin X_{nk}$, we conclude that $\ell(b) = \ell(f(a)) < n_{k+1}$ and hence $b \notin X_{nk+1}$, which contradicts the choice of $b$. This contradiction shows that $a \in A \cap X_{nk} \setminus X_{nk+1}$. Since $b = f(a) \in X_{nk+1}$, the point $a$ belongs to the set $A_k^+$ and hence $b \in f(A_k^+)$. Therefore, $f(A_k^+) = B_{k+1}^+$. By analogy we can prove that $f^{-1}(B_k^+) = A_{k+1}^-$ and hence $f(A_{k+1}^-) = B_k^+$. \qed

**Claim 9.** For every $k \in \omega$ we have $A_k^+ \cup B_k^+ \subseteq X_{nk+1} \setminus X_{nk+1}$.

**Proof.** Assuming that $A_k^+ \notin X_{nk+1}$, we can find a point $a \in A_k^+ \setminus X_{nk+1}$. The choice of $n_{k+1}$ ensures that $n_{k+1} > \ell(f(a))$ and hence $f(a) \notin X_{nk+1}$, which contradicts the inclusion $a \in A_k^+$. This contradiction shows that $A_k^+ \subseteq X_{nk+1} \setminus X_{nk+1}$. By analogy we can prove that $B_k^+ \subseteq X_{nk+1} \setminus X_{nk+1}$. \qed

The coregularity of the skeleton $(X_n)_{n \in \omega}$ guarantees that for every $k \in \omega$ the countable space $X_{nk+1} \setminus X_{nk+1}$ is regular and hence metrizable and zero-dimensional. Then we can find a closed-and-open sets $U_k$ and $V_k$ in $X_{nk+1} \setminus X_{nk+1}$ such that $A_k \cap U_k = A_k^+$ and $B_k \cap V_k = B_k^+$. Let $Y_0 = Z_0 = X_0$ and for every $k \in \mathbb{N}$ let
\[ Y_k := U_k \cup X_{nk+1} \quad \text{and} \quad Z_k := V_k \cup X_{nk+1}, \]
and observe that $Y_k = U_k \cup X_{nk+1} \subseteq X_{nk+1} \subseteq X_{nk+1} \subseteq Y_{k-1}$. The nowhere density of the set $X_{nk+1}$ in $X_{nk}$ implies the nowhere density of $Y_k$ in $X_{nk}$ and also in $Y_{k-1}$. By analogy we can show that $Z_k$ is nowhere dense in $Z_{k-1}$. It is easy to check that $(Y_k)_{k \in \omega}$ and $(Z_k)_{k \in \omega}$ are canonical superskeleta for $X$ such that
\[ A \cap Y_k \setminus Y_{k+1} = A_{k+1}^- \cup A_{k+1}^- \cup A_{k+1}^+ \quad \text{and} \quad B \cap Z_k \setminus Z_{k+1} = B_{k+1}^+ \cup B_{k+1}^+ \cup B_{k+1}^- \]
for every $k \in \omega$. This implies that $f(A \cap Y_k) = B \cap Z_k$ for every $k \in \omega$. Since the spaces $A, B$ are discrete, the intersections $A \cap Y_k$ and $B \cap Z_k$ are nowhere dense in the crowded spaces $Y_k, Z_k$, respectively. Applying Theorem 8 we can find a homeomorphism $h : X \to Y$ such that $h|A = f$ and $h(Y_k) = Z_k$ for all $k \in \omega$. \qed

Since any finite set in a Hausdorff topological space is shallow, Theorem 8 implies that the following finite homogeneity property of the space $QP^\infty$. 


Corollary 4 (Finite homogeneity of $\mathbb{P}_\infty^\infty$). Any bijective function $f : A \to B$ between two finite subsets $A, B$ of the space $\mathbb{P}_\infty^\infty$ can be extended to a homeomorphism of $\mathbb{P}_\infty^\infty$.

Remark 2. By Lemma 8, the product $\mathbb{P}_\infty^\infty \times \mathbb{P}_\infty^\infty$ is not coregular and hence cannot be homeomorphic to $\mathbb{P}_\infty^\infty$. Nonetheless, $\mathbb{P}_\infty^\infty \times \mathbb{P}_\infty^\infty$ contains a dense subspace homeomorphic to $\mathbb{P}_\infty^\infty$, see [18].

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References

[1] T. Banakh, J. Mioduszewski, S. Turek, On continuous self-maps and homeomorphisms of the Golomb space, Comment. Math. Univ. Carolin. 59:4 (2018) 423–442.
[2] T. Banakh, D. Spirito, S. Turek, The Golomb space is topologically rigid, preprint (https://arxiv.org/abs/1912.01994).
[3] T. Banakh, Y. Stelmakh, S. Turek, The Kirch space is topologically rigid, in preparation.
[4] C. Bessaga, A. Pe\l czynski, Selected Topics in Infinite-Dimensional Topology, MM 58, Polish Sci. Publ., Warsaw, 1975.
[5] T. Banakh, T. Radul, M. Zarichnyi, Absorbing sets in infinite-dimensional manifolds, Math. Studies, Monog. Ser. 1, VNTL Publ., Lviv, 1996.
[6] M. Brown, A countable connected Hausdorff space, Bull. Amer. Math. Soc. 59 (1953), 367. Abstract #423.
[7] D. Curtis, T. Dobrowolski, J. Mogilski, Some applications of the topological characterizations of the sigma-compact spaces $l_1^\infty$ and $\Sigma$, Trans. Amer. Math. Soc. 284:2 (1984) 837–846.
[8] R. Engelking, General Topology, Heldermann Verlag, Berlin, 1989.
[9] R. Hartshorne, Foundations of projective geometry, Harvard University, W. A. Benjamin, Inc., New York, 1967.
[10] I.M. Gelfand, D.B. Fuks, The topology of noncompact Lie groups, Funct. Analysis and Its Appl. 1:4 (1967) 285–295.
[11] S. Golomb, A connected topology for the integers, Amer. Math. Monthly, 66 (1959), 663–665.
[12] S. Golomb, Arithmetica topologica, in: General Topology and its Relations to Modern Analysis and Algebra (Proc. Sympos., Prague, 1961), Academic Press, New York; Publ. House Czech. Acad. Sci., Prague (1962) 179–186; available at https://dml.cz/bitstream/handle/10338.dmlcz/700933/Toposym_01-1961-1_41.pdf.
[13] A.M. Kirch, A countable, connected, locally connected Hausdorff space, Amer. Math. Monthly 76 (1969), 169–171.
[14] A. Kryftis, A constructive approach to affine and projective planes, Ph.D. Dissertation, Univ. of Cambridge, 2015 (https://arxiv.org/pdf/1601.04998.pdf).
[15] J. van Mill, Infinite-Dimensional Topology, Prerequisites and Introduction, North-Holland Math. Library 43. Elsevier Sci. Publ. B.V., Amsterdam, 1989.
[16] J. Mogilski, Characterizing the topology of infinite-dimensional $\sigma$-compact manifolds, Proc. Amer. Math. Soc. 92:1 (1984) 111–118.
[17] K. Sakai, Topology of infinite-dimensional manifolds, Springer (to appear).
[18] Ya. Stelmakh, Almost coregular topological spaces, in preparation.

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