The $K$-way negativities as entanglement measures

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A classification of $N$-partite states, based on $K$-way ($2 \leq K \leq N$) negativities, is proposed. The $K$-way partial transpose with respect to a subsystem is defined so as to shift the focus to $K$-way coherences instead of $K$ subsystems of the composite system. For an $N$-partite system the fraction of $K$-way negativity ($2 \leq K \leq N$), contributing to global negativity, is obtained. After minimizing $K$-way negativities through local unitary qubit rotations, a combined analysis of $2$-way, $3$-way and global negativities is shown to provide distinct measures of genuine tripartite, W-state like and bipartite entanglement, for three qubit composite system. To illustrate the point, entanglement of three qubit GHZ class states, W-class states, three boson state and noisy states is analysed. In principle, a combined analysis of $K$-way negativities of all the subsystems should lead to the sufficient condition for $K$-way entanglement of the composite system. While pure $N$-partite entanglement of a composite system is generated by $N$-way coherences, $N$-partite entanglement in general can be present due to ($K < N$) $K$-way coherences as well.

PACS numbers: 03.67.Mn, 05.70.-a

I. INTRODUCTION

Entanglement of quantum systems plays a central role in quantum information processing. Quantum entanglement has made possible quantum state transport [1, 2], quantum communication over noisy channels [3], and quantum cryptography [4]. As such characterization of quantum entanglement is a fundamental issue. Several very interesting entanglement measures have been devised [5, 6, 7, 8, 9] for analyzing pure and mixed state entanglement of multipartite systems. Geometrical ideas have also been explored to understand entanglement in a number of papers [10, 11, 12]. Although bipartite entanglement is well understood, many interesting aspects of multipartite entanglement are still to be explored. Peres [6] and the Horodecki [7, 13, 14] have shown a positive partial transpose of a bipartite density operator to be a sufficient criterion for classifying bipartite entanglement. Positive partial transpose has been shown to be a necessary and sufficient condition [7] for separability of two-level bipartite systems. For higher dimensional systems positive partial transpose is a necessary condition [14]. Negativity [15, 16] based on Peres Horodecki PPT criterion has been shown to be an entanglement monotone [17, 18, 19]. Negativity is an interesting concept being related to the eigenvalues of partially transposed density matrix. For mixed states, use of negativity as an entanglement measure is restricted to positive states. For an $N$-partite composite system with three or more parties, it is found useful to split the system in $M$ parts with each party containing one or more sub-systems. A hierarchic classification of arbitrary $N$-qubit mixed states, based on separability and entanglement distillation properties of certain partitions, has been given by Diir and Cirac [20]. In this article, we propose a classification of $N$-partite states based on $K$-way negativities, where $2 \leq K \leq N$. The principle underlying the definition of $K$-way negativity for pure and mixed states of $N$-subsystems is NPT sufficient condition. However, $K$-way partial transpose with respect to a subsystem is defined so as to shift the focus to $K$-way coherences of the composite system instead of $K$ subsystems of the composite system. In principle, a combined analysis of $K$-way negativities for all the subsystems as well as the negativities as defined in [18], should provide a measure of $K$-partite entanglement of the composite system. The fraction of $K$-way negativity ($2 \leq K \leq N$), contributing to negativity of partial transpose of a given entangled state of an $N$-partite composite system, can be calculated. For a three qubit system, 2-way, 3-way and global negativities of all subsystems are shown to provide distinct measures of genuine tripartite, W-state like and bipartite entanglement of the composite system. Entanglement is invariant under local unitary rotations, whereas, coherences are not so.

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A given state operator $\hat{\rho}$ may be mapped to another state operator $\hat{\rho}_{\text{min}}$, having exactly the same entanglement as the state $\hat{\rho}$ but having $K$–way negativities that are either invariant or increase under local unitary rotations of qubits. Tripartite and bipartite entanglement measures for the states $\hat{\rho}$ and $\hat{\rho}_{\text{min}}$ may be written in terms of fractional negativities of $3$–way and $2$–way partial transpose of state $\hat{\rho}_{\text{min}}$. While pure N–partite entanglement of a composite system is generated by $N$–way coherences, $N$–partite entanglement in general can be present due to $K$–way ($2 \leq K < N$) coherences as well.

We explain the notation used for computational basis states in section II. The global negativity, which is twice the negativity as defined in ref. [18] is discussed in section III. The $K$–way partial transpose of an $N$–partite state is defined in section IV, followed by the definition of $K$–way negativity with respect to a given subsystem in section V. To clarify the definitions of section V, the construction of $2$–way and $3$–way partial transpose for three qubit states is given in section VI. Section VII deals with the contribution of $K$–way negativities for a particular subsystem to the global negativity of partially transposed state operator. Minimization of $2$–way and $3$–way negativities for a set of one parameter three qubit pure states is carried out in section VIII. Genuine three qubit entanglement induced by $3$–way coherence and three qubit entanglement induced by $2$–way coherence are also discussed, in the same section. Sections IX and X deal with entanglement of three qubit two parameter canonical states and three qubit noisy states, respectively. Conclusions are given in section XI.

**II. THE BASIS FOR N-PARTITE SYSTEM**

The Hilbert space, $C^d = C^{d_1} \otimes C^{d_2} \otimes \ldots \otimes C^{d_N}$, associated with a quantum system composed of $N$ sub-systems, is spanned by basis vectors of the form $|i_1i_2...i_N\rangle$, where $i_m = 0$ to $(d_m - 1)$, and $m = 1,...,N$. Here $d_m$ is the dimension of Hilbert space associated with $m^{th}$ sub-system. We define a raising operator for the $m^{th}$ sub-system as $\sigma_m^+ |i_m\rangle = |i_m + 1\rangle$ for $i_m < d_m - 1$, and $\sigma_m^+ |d_m - 1\rangle = 0$. Using this definition, we may rewrite

$$
|i_1i_2...i_N\rangle = (\sigma_1^+)^{i_1}(\sigma_2^+)^{i_2}... (\sigma_N^+)^{i_N}|0_10_2...0_N\rangle
$$

where $j = \sum_{m=1}^{N} i_m \left( \prod_{n=1}^{m} d_{n-1} \right)$ with $d_0 = 1$, and $\mu = \sum_{m=1}^{N} i_m$. The additional label $\mu$ counts the total number of local raising operations needed to get the basis state $|i_1i_2...i_N\rangle$ from the reference state $|0_10_2...0_N\rangle$. For an $N$ qubit quantum system, $\sigma_m^+$ is the usual spin raising operator for $m^{th}$ sub-system and the label $\mu$ counts the total number of spins that are flipped to get the basis vector $|i_1i_2...i_N\rangle$ from the reference state $|0_10_2...0_N\rangle$.

Using this notation the state operator for $N$–partite composite system operating on Hilbert space $C^d$ can be written as

$$
\hat{\rho} = \sum_{(j,\mu)} \langle j,\mu | \hat{\rho} | j,\mu \rangle | j,\mu \rangle \langle j,\mu |
+ \sum_{(j,\mu), (j',\mu')} \langle j,\mu | \hat{\rho} | j',\mu' \rangle | j,\mu \rangle \langle j',\mu' |. \tag{2}
$$

**III. GLOBAL NEGATIVITY**

Consider a pure state or a mixed state with a density matrix which is known to be positive. To measure overall entanglement of a subsystem $p$, we shall use global negativity $N_G$, which is twice the negativity as defined by Vidal and Werner [18]. It is an entanglement measure for classifying bipartite entanglement and is based on Peres-Horodecki negative partial transpose (NPT) sufficient condition. The global partial transpose of $\hat{\rho}$ (Eq.2) with respect
to sub-system \( p \) is defined as

\[
\hat{\rho}_G^p = \sum_{(j,\mu)} \langle j, \mu | \hat{\rho} | j, \mu \rangle | j, \mu \rangle \langle j, \mu |
\]

\[
+ \sum_{(j,\mu),(j',\mu')}^{(j,\mu),(j',\mu') \neq j} \langle j, \mu | \hat{\rho} | j', \mu' \rangle | j + (i_p' - i_p) \prod_{n=1}^{p} d_{n-1}, \mu + i_p' - i_p \rangle
\]

\[
\left< j' + (i_p - i_p') \prod_{n=1}^{p} d_{n-1}, \mu + i_p - i_p' \right| (3)
\]

The partially transposed matrix \( \hat{\rho}_G^p \) of an entangled state is not positive definite. Global Negativity is defined as

\[
N_G^p = \left\| \hat{\rho}_G^p \right\|_1 - 1 = 2 \sum_i |\lambda_i^-|
\]

where \( \left\| \hat{\rho}_G^p \right\|_1 \) and \( \lambda_i^- \) are the trace norm and negative eigen values of \( \hat{\rho}_G^p \), respectively. For a state having \( p \)th subsystem separable, the value of \( N_G^p \) is zero. Dimensionality of a composite entangled system determines the maximum possible value of \( N_G^p \). We recall here that the partial transposition with respect to subsystem \( p \) amounts to time reversal for the state of subsystem \( p \). For an entangled state this operation results in a state operator with negative eigenvalues.

In general, an \( N \)-partite system may not have pure \( N \)-partite entanglement because \( M \) (\( 1 \leq M \leq N/2 \) for \( N \) even and \( 1 \leq M \leq (N - 1)/2 \) for \( N \) odd) entangled subsystems are separable with respect to \( N - M \) subsystems. In other words, the state operator can be written as

\[
\hat{\rho} = \sum_i p_i \hat{\rho}_i (N - M) \otimes \hat{\rho}_i (M).
\]

For a bipartite system \( M = 1 \), as such, Peres Horodoski criteria separability condition turns out to be a necessary and sufficient condition for separability of the system. Since for a tripartite system \( M = 1 \) as well, the only condition for tripartite entanglement not to be present is that at least one of the three subsystems is separable (that is one out of \( N_G^1, N_G^2, \) and \( N_G^3 \), is zero). On the contrary, the necessary and sufficient condition for tripartite entanglement to be present is that \( \hat{\rho}_G^p \) with respect to none of the three subsystems is positive definite. That implies a non zero global negativity \( (N_G^p \neq 0) \) for transposition with respect to \( p \)th sub-system, for all possible values of \( p \). For a composite system with four subsystems \( M \) takes values 1 and 2. Hence there are two types of conditions that forbid the existence of four party entanglement, namely, the 4-party entanglement is not present if \( \hat{\rho}_G^p \) is positive for, at least, one of the four subsystems \( (M = 1) \) or a pair of subsystems \( (M = 2) \). Similar considerations apply to systems with \( N \) greater than four.

In the next two sections we fix the notation to write a \( K \)-way \( (2 \leq K \leq N) \) partial transpose with respect to a single subsystem and define \( K \)-way negativity. Once the presence of genuine \( K \)-way entanglement has been established, a measure of pure \( K \)-way entanglement can be obtained from the \( K \)-way negativities of the composite subsystem.

### IV. \( K \)-WAY PARTIAL TRANSPOSE

A typical off diagonal matrix element of the state operator \( \hat{\rho} \) that involves a change of state of \( K \) subsystems while leaving the state of \( N - K \) sub-systems unaltered looks like \( \langle i_1 i_2 ... i_K, i_{K+1}, ..., i_N | \hat{\rho} | i_1' i_2' ... i_K', i_{K+1}, ..., i_N \rangle \). The set of \( K \) distinguishable subsystems that change state while \( N - K \) sub-systems do not, can be chosen in \( D_K = \frac{N!}{(N-K)!K!} \) distinct ways. If we represent the basis vectors obtained from \( |i_1 i_2 ... i_N \rangle \) by changing the state of \( K \) subsystems by \( |i_1' i_2' ... i_K' \rangle \), where \( K = 0 \) to \( N \), the number of matrix elements of the type \( \langle i_1 i_2 ... i_K | \hat{\rho} | i_1' i_2' ... i_K' \rangle \) depends on \( D_K \) and the dimensions \( d_1, d_2 ... d_N \) of the subsystems. For now we stick to a simpler system of \( N \) qubits, that is \( d_1 = d_2 = ... = d_N = 2 \). Recalling that in the notation of section II the matrix element \( \langle i_1 i_2 ... i_N | \hat{\rho} | i_1' i_2' ... i_K' \rangle \) is written as \( \langle j, \mu | \hat{\rho} | j, \mu' \rangle \), where \( \mu = \sum_{m=1}^{N} i_m \) and \( \mu' = \sum_{m=1}^{N} i'_m \), we have \( K = \sum_{m=1}^{N} |(i'_m - i_m)| \). Alternatively, \( K = \mu' - \mu + 2 \sum_{m=1, i > i'}^{N} (i_m - i'_m) \) and \( \langle j, \mu | \hat{\rho} | j, \mu' \rangle = \langle j, \mu | \hat{\rho} | j, \mu' \rangle \left( \mu + K - 2 \sum_{m=1, i > i'}^{N} (i_m - i'_m) \right) \). Using the notation
\[ \mu'(K) = \mu + K - 2 \sum_{m=1, i > i'}^{N} (i_m - i'_m), \]

the operator \( \hat{\rho} \) can be split up into parts labelled by \( K \) with \((0 \leq K \leq N) \). By rewriting the state operator as

\[
\hat{\rho} = \sum_{(j,\mu), K=0} \langle j,\mu|\hat{\rho}|j,\mu\rangle |j,\mu\rangle + \sum_{(j,\mu),(j',\mu'(1))} \langle j,\mu|\hat{\rho}|jt,\mu'(1)\rangle |j,\mu\rangle \langle j',\mu'(1)| + \ldots \\
+ \sum_{(j,\mu),(jt,\mu'(2))} \langle j,\mu|\hat{\rho}|jt,\mu'(2)\rangle |j,\mu\rangle \langle j',\mu'(2)| + \ldots \\
+ \sum_{(j,\mu),(jt,\mu'(N))} \langle j,\mu|\hat{\rho}|jt,\mu'(N)\rangle |j,\mu\rangle \langle j',\mu'(N)| = \sum_{K=0}^{N} \hat{R}_K, \tag{6}
\]

where

\[
\hat{R}_K = \sum_{(j,\mu),(jt,\mu'(K))} \langle j,\mu|\hat{\rho}|jt,\mu'(K)\rangle |j,\mu\rangle \langle j',\mu'(K)|, \tag{7}
\]

the terms containing matrix elements \( \langle j,\mu|\hat{\rho}|jt,\mu'(K)\rangle \) that connect states with a fixed value of \( K \) are bunched together. We will refer to \( \hat{R}_K \) as the \( K \)-way coherence of \( N \)-qubit composite system. A three qubit system, for example, may have 1-way, 2-way, and 3-way coherence, \( \hat{R}_0 \) being the diagonal part of \( \hat{\rho} \). Genuine \( K \)-partite entanglement cannot be generated if no \( K \)-way coherence is present. But in general for \( 2 \leq K < N \), the \( K \)-way coherence can result in entanglement of more than \( K \) parties.

Of the \( N \) sub-systems comprising the composite system, the \( K \)-way partial transpose \( \hat{\rho}_K^{T_p} \) for the \( p^{th} \) system is constructed by partial transposition of \( \hat{R}_K \) with respect to subsystem \( p \) while leaving the rest of the state operator unchanged, that is

\[
\hat{\rho}_K^{T_p} = \sum_{K' \neq K} \hat{R}_{K'} + \hat{R}_0 + \sum_{(j,\mu),(jt,\mu'(K))} \langle j,\mu|\hat{\rho}|jt,\mu'(K)\rangle \left| j + (i_p - i'_p) \left( \prod_{n=1}^{p} d_{n-1} \right), \mu + (i'_p - i_p) \right\rangle \\
\left\langle j' + (i_p - i'_p) \left( \prod_{n=1}^{p} d_{n-1} \right), \mu'(K) + (i_p - i'_p) \right\rangle. \tag{8}
\]

We recall here that the global partial transposition with respect to subsystem \( p \) amounts to time reversal for the state of subsystem \( p \). For an entangled state this operation results in a state operator with negative eigenvalues. Taking \( K \)-way transpose with respect to \( p^{th} \) subsystem amounts to time reversal for the state of subsystem \( p \) in \( \hat{R}_K \). In case \( \sum_{K' \neq K} \hat{R}_{K'} = 0 \), the global and \( K \)-way partial transpose are the same. As such a \( K \)-way partial transpose on \( \hat{\rho} \) may result in an operator with negative eigenvalues iff \( \hat{R}_K \neq 0 \) and \( N^p_G \neq 0 \).

V. K-WAY NEGATIVITY

The \( K \)-way negativity of subsystem \( p \) to be calculated from \( K \)-way partial transpose of matrix \( \rho \) with respect to \( p \), is defined as

\[
N^p_K = \left\| \hat{\rho}_K^{T_p} \right\|_1 - 1, \tag{9}
\]

where \( \left\| \hat{\rho}_K^{T_p} \right\|_1 \) is the trace norm of \( \hat{\rho}_K^{T_p} \). The trace norm is calculated by using the relation \( \left\| \hat{\rho}_K^{T_p} \right\|_1 = 2 \sum |\lambda^-_i| + 1 \), \( \lambda^-_i \) being the negative eigenvalues of matrix \( \hat{\rho}_K^{T_p} \). The negativity \( N^p_K \) depends on the \( K \)-way coherence and is a measure of all possible types of entanglement attributed to \( K \)-way coherence. Intuitively, for a system to have pure \( N \)-partite entanglement, it is necessary that \( N \)-way coherences are non-zero for \( p = 1 \) to \( N \). On the other hand, \( N \)-partite entanglement can also be generated by \((N - 1)\)-way coherence. For a three qubit system, maximally entangled tripartite GHZ state is an example of pure tripartite entanglement involving 3-way coherence. On the
other hand, maximally entangled W-state is a manifestation of tripartite entanglement due to 2—way coherences. Entanglement of a subsystem is detected by global negativity $N^p_K$ (for all possible partitions of the system) and the hierarchy of negativities $N^p_K$ ($K = 2, \ldots N$), calculated from $\tilde{\rho}^T_p$ associated with sub-system $p = 1$ to $N$. However, $N^p_K$ is not independent of the global entanglement of the system. A necessary condition for pure $K$—partite entanglement to exist is that at least $K$ constituent systems have non zero global as well as $K$—way negativity.

VI. CONSTRUCTING $\tilde{\rho}^T_C$, $\tilde{\rho}^T_2$ AND $\tilde{\rho}^T_3$ FOR THREE QUBIT STATES

For a three qubit state, the global partial transpose with respect to first qubit is constructed by transposing the state of qubit one while keeping the composite state of qubits two and three unaltered. For the density operator

$$\tilde{\rho} = \sum_{(j,\mu)} \langle j, \mu| \tilde{\rho} |j, \mu\rangle \langle j, \mu| + \sum_{(j,\mu),(j',\mu')(K)} \langle j, \mu| \tilde{\rho} |j', \mu'(K)\rangle \langle j', \mu'(K)|$$

$$= \sum_{K=0}^{3} \tilde{R}_K.$$  \hspace{1cm} (10)

where $j = \sum_{m=1}^{3} i_m \left( \prod_{n=1}^{m} d_{n-1} \right)$ with $d_0 = 1$, and $\mu = \sum_{m=1}^{3} i_m$, the global partial transpose with respect to qubit $p$ reads as

$$\tilde{\rho}^T_C = \sum_{(j,\mu)} \langle j, \mu| \tilde{\rho} |j, \mu\rangle \langle j, \mu| + \sum_{(j,\mu),(j',\mu')(K)} \langle j, \mu| \tilde{\rho} |j', \mu'(K)\rangle \langle j', \mu'(K)|$$

$$|j + 2^{p-1}(i_p' - i_p), \mu + (i_p' - i_p)| \langle j', 2^{p-1}(i_p - i_p'), \mu'(K) + (i_p - i_p')|.$$  \hspace{1cm} (11)

We write the 2—way and 3—way partial transpose with respect to qubit $p$ as

$$\tilde{\rho}^T_2 = \sum_{(j,\mu)} \langle j, \mu| \tilde{\rho} |j, \mu\rangle \langle j, \mu| + \sum_{(j,\mu),(j',\mu')(K)} \langle j, \mu| \tilde{\rho} |j', \mu'(K)\rangle \langle j', \mu'(K)|$$

$$|j + 2^{p-1}(i_p' - i_p), \mu + (i_p' - i_p)| \langle j', 2^{p-1}(i_p - i_p'), \mu'(2) + (i_p - i_p')|.$$  \hspace{1cm} (12)

and

$$\tilde{\rho}^T_3 = \sum_{(j,\mu)} \langle j, \mu| \tilde{\rho} |j, \mu\rangle \langle j, \mu| + \sum_{(j,\mu),(j',\mu')(K)} \langle j, \mu| \tilde{\rho} |j', \mu'(K)\rangle \langle j', \mu'(K)|$$

$$|j + 2^{p-1}(i_p' - i_p), \mu + (i_p' - i_p)| \langle j', 2^{p-1}(i_p - i_p'), \mu'(3) + (i_p - i_p')|.$$  \hspace{1cm} (13)

Alternatively, the matrix representation may be used to show the action of 3—way map $T^1_3$ defined as $T^1_3 : \tilde{\rho} \rightarrow \tilde{\rho}^T_3$. The matrix $T^1_3$ acts only on those matrix elements for which $K = \mu' - \mu + 2 \sum_{m=1, i > i_m} (i_m - i_m') = 3$. We can arrange
these matrix elements in a single column with eight rows. Operation of the matrix $T^3_p$ on the column looks like

\[
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 
\end{bmatrix}
\begin{bmatrix}
\rho_{000,000} \\
\rho_{100,011} \\
\rho_{010,101} \\
\rho_{011,001} \\
\rho_{001,110} \\
\rho_{101,010} \\
\rho_{011,010} \\
\rho_{111,000}
\end{bmatrix} =
\begin{bmatrix}
\rho_{100,111} \\
\rho_{100,111} \\
\rho_{110,001} \\
\rho_{101,010} \\
\rho_{010,110} \\
\rho_{101,010} \\
\rho_{111,000} \\
\rho_{011,010}
\end{bmatrix}.
\]

(12)

The full $\rho$ matrix, written as a column, has 64 rows. The full map $T^3_p$ leaves the remaining 56 matrix elements without any change. Similarly, one can construct the maps $T^3_2$, $T^3_3$, and $T^3_3$, where $p = 1, 2, 3$. The eigenvalues of partially transposed matrices $T^3_p$, $T^3_2$, and $T^3_2$, are used to calculate the corresponding negativities.

\[\rho^G_G = \sum_\lambda \lambda^G+ |\Psi^G+\rangle \langle \Psi^G+| + \sum_\lambda \lambda^G- |\Psi^G-\rangle \langle \Psi^G-|,\]

where $\lambda^G+ (\lambda^G-)$ are the positive (negative) eigenvalues and eigenvectors, respectively. The trace norm of $\rho^G_G$ is

\[
\left\| \frac{\rho^T_T}{\rho^G_G} \right\|_1 = \sum_i \langle \Psi^G_\rightarrow | \frac{\rho^T_T}{\rho^G_G} | \Psi^G_i \rangle - \sum_i \langle \Psi^G_\rightarrow | \frac{\rho^T_T}{\rho^G_G} | \Psi^G_i \rangle = T \left( \frac{\rho^T_T}{\rho^G_G} \right) - 2 \sum_i \langle \Psi^G_\rightarrow | \frac{\rho^T_T}{\rho^G_G} | \Psi^G_i \rangle.
\]

Using $T \left( \frac{\rho^T_T}{\rho^G_G} \right) = 1$, the negativity of $\rho^T_T$ is given by

\[
N^P = -2 \sum_i \langle \Psi^G_\rightarrow | \frac{\rho^T_T}{\rho^G_G} | \Psi^G_i \rangle = -2 \sum_i \lambda^G_1.
\]

The global transpose with respect to subsystem $p$, may also be rewritten as

\[
\rho^T_T = \sum_{K=2}^N \rho^T_T - (N-2)\hat{\rho}.
\]

Substituting Eq. (16) in Eq. (15), and recalling that $\hat{\rho}$ is a positive operator with trace one, we get

\[
-2 \sum_i \lambda^G_1 = -2 \sum_{K=2}^N \sum_i \langle \Psi^G_\rightarrow | \rho^T_T | \Psi^G_i \rangle.
\]

Defining partial $K$—way negativity $E^p_K (K = 2 \text{ to } N)$ as

\[
E^p_K = -2 \sum_i \langle \Psi^G_\rightarrow | \rho^T_T | \Psi^G_i \rangle,
\]

we may split the global negativity for qubit $p$ as

\[
N^G = \sum_{K=2}^N E^p_K.
\]
The contribution $E^p_K$ depends on $\|\rho^p_K\|_1$, the ratio $E^p_K/N^p_K$ being the fraction of $K$-way negativity contributing to $N^p_G$.

The Peres-Horodecki criterion is the most useful sufficient condition for checking the separability of a subsystem in a bipartite split of composite quantum system. The sufficient condition for an $N$-partite pure state not to have genuine $N$-partite entanglement is that at least one of the global negativities is zero that is $N^p_G = 0$, where $p$ is one of the subsystems or one part of a bipartite split of the composite system. Consider an $N$-partite system in a pure state, $\hat{\rho} = \hat{\rho}_0 + \hat{\rho}_1 + \ldots + \hat{\rho}_N$. Recalling that $N^p_G = \sum_{K=2}^{N} E^p_K$, the separability of subsystem $p$ implies that $E^p_K = 0$, or $\rho^p_K \geq 0$, for $K = 2$ to $N$. In general, for a system having only genuine $K$-partite entanglement, $N^p_G = 0$ for $N - K$ of the subsystems. In addition, the lowest of the partial non zero $K$-way negativities measures $K$-partite entanglement, the same being a collective property of $K$-subsystems.

**VIII. LOCAL UNITARY OPERATIONS AND K-WAY COHERENCES**

An important point to note is that the trace norm satisfies

$$\left\| \frac{\mathcal{L}^p_T}{\rho^p_G} \right\|_1 \leq \sum_{K=2}^{N} \left\| \rho^p_K \right\|_1 - (N - 2) \left\| \hat{\rho} \right\|_1,$$

or in terms of negativities

$$N^p_G \leq \sum_{K=2}^{N} N^p_K.$$

The trace norm $\left\| \frac{\mathcal{L}^p_T}{\rho^p_K} \right\|_1$ is not invariant under local unitary rotations, where as $\left\| \frac{\mathcal{L}^p_T}{\rho^p_G} \right\|_1$ is invariant. For a given state $\hat{\rho}$, there exists a set of operators $\frac{\mathcal{L}^p_T}{\rho^p_K}$ for which $\sum_{p=1}^{N} N^p_K$ has the lowest value, for all $K$. The state $\hat{\rho}_{\text{min}}$
that yields transposed $K$-way operators with the lowest valued norm can be obtained from $\hat{\rho}$ by applying local unitary operations on constituent sub-systems. We conjecture that for the state having $\sum_{p=1}^{N} N_p^K$ at its minimum, $E_p^K$ measures the $K$-way entanglement of qubit $p$. Recalling that $N$-way entanglement is a collective property of $N$-qubits, $\min(E_1^N, E_2^N, ..., E_N^N)$ is a measure of genuine $N$-way entanglement of qubit $p$.

It is well known that the set of states that can be transformed into each other by local unitary operations lie on the same orbit and have the same entanglement as the canonical state expressed in terms of the minimum number of independent vectors [21]. Construction of pure three qubit canonical state has been given by Acin et al [22]. It is easily verified that for the states reducible to the canonical state by local unitary operations, although $N_p^G$ is invariant under local operations, $\sum_{p=1}^{N} N_p^G$ varies under local unitary operations.

Three qubit Greenberger-Horne-Zeilinger state

$$\Psi_{\text{GHZ}} = \frac{1}{\sqrt{2}} (|000\rangle + |111\rangle),$$

is a maximally entangled state having pure tripartite entanglement. Consider one parameter pure states

$$\Psi_1 = \sqrt{q} \Psi_{\text{GHZ}} + \sqrt{\left(\frac{1-q}{2}\right)} (|100\rangle + |011\rangle), \quad \hat{\rho}_1 = |\Psi_1\rangle \langle \Psi_1|,$$

where $0 \leq q \leq 1$. For these states, $N_2^1 = 0$ for all values of $q$ and $N_3^1 = N_3^3$. Fig. (1) displays total negativities $N_3^1 = \sum_{p=1}^{3} N_p^3$ and $N_2^1 = \sum_{p=1}^{3} N_p^2$ versus $q$ for the states $\hat{\rho}_1$. On applying a local rotation on qubit one, $N_3^1$ decreases, reaches a minimum and then increases. The unitary rotation

$$U_1(\theta_1) = \begin{bmatrix} \cos(\theta_1) & \sin(\theta_1) \\ -\sin(\theta_1) & \cos(\theta_1) \end{bmatrix}$$
that minimizes $N^t_3$ for a given value of parameter $q$ is found, numerically. The negativities $N^t_2(\theta_1)$ and $N^t_3(\theta_1)$ of partial 2–way and 3–way transpose of state

$$\hat{\rho}_1(\theta_1) = |\psi_1(\theta_1)\rangle \langle \psi_1(\theta_1)|,$$

where

$$\psi_1(\theta_1) = (U_1(\theta_1) \otimes I_2 \otimes I_3) \psi_1$$

are also displayed in Fig. (1).

The genuine tripartite entanglement, $E_3 = \min(E^t_1(\theta_1), E^t_2(\theta_1), E^t_3(\theta_1))$, is displayed as a solid line plot in Fig. (2). Fig. (2) also shows $E^t_2(\theta_1)$, which is a measure of bipartite entanglement of qubits 2 and 3. The state $\hat{\rho}_1(\theta_1)$ has the same degree of entanglement as the state $\hat{\rho}_1$ but lower values of 2–way and 3–way coherences. The canonical state corresponding to state $\psi_1$, found by the procedure given by Acin et al.\cite{22} is a GHZ like state

$$\psi'_1 = \sqrt{\frac{1 - 2q}{2}} |000\rangle + \sqrt{2q(1 - q)} |100\rangle + \sqrt{\frac{1}{2}} |111\rangle,$$

and is equal to state $\psi_1(\theta_1)$, obtained numerically.

In general, by successive unitary rotations of qubits one, two, and three, one may obtain a state operator with minimum 2–way and 3–way coherences. Minimization of $\sum_{p=1}^{N} N^K_p$, by successive unitary rotations of qubits, offers a method to reach the canonical state from a given state of the $N$ qubit composite system. In most cases, a simple computer program suffices to effect such a minimization, numerically. Analytical proof and further discussion on using minimization of $K$–way negativities to reach the canonical state, will be given elsewhere.

**IX. THREE QUBIT GHZ AND W-TYPE STATES**

There exists a class of tripartite states akin to maximally entangled W-state given by

$$\psi_W = \frac{1}{\sqrt{3}} (|100\rangle + |010\rangle + |001\rangle).$$
FIG. 4: Contour plots of genuine tripartite entanglement $E_3^2$, global Negativity $N_G^2$, and bipartite entanglement $E_{22}^2$, as a function of parameters $q$ and $a$ for two mode three boson state, $\hat{\rho}_3$. Solid line arrow and dashed line arrow indicate the states with maximal GHZ-type and maximal W-type entanglement, respectively.

To illustrate the ideas presented in previous sections, we analyze the entanglement of a set of single parameter pure states of the type

$$\Psi_2 = \sqrt{q}\Psi_{GHZ} + \sqrt{1-q}\Psi_W, \quad \hat{\rho}_2 = |\Psi_2\rangle \langle \Psi_2|,$$

for $0 \leq q \leq 1$.

Fig. (3) displays $E_1^2 = E_2^2 = E_3^2 = E_2$, $E_1^1 = E_3^3 = E_3$, and $N_G^1 = N_G^2 = N_G^3 = N_G$ as a function of parameter $q$ for a single qubit in state $\hat{\rho}_2$. Global negativity indicates that all three qubits are entangled as $q$ varies from zero to one. For $q = 0$, $E_2 = 0.94$, and $E_3 = 0$, that is the system does not have genuine tripartite entanglement. On the other hand for $q = 1$, the system has maximum genuine tripartite entanglement as is evidenced by $E_3 = 1.0$, and $E_2 = 0$. The measure of genuine tripartite entanglement is $E_3$ for $0 \leq q \leq 1$. The W-state like tripartite entanglement is generated by 2–way quantum correlations.

X. THREE QUBIT TWO PARAMETER CANONICAL STATE

To further illustrate the usage of $K$–way negativities, we consider a three qubit two parameter canonical state

$$\Psi_{q,a} = \sqrt{a}|000\rangle + \sqrt{(1-q)(1-a)}\Psi_W + \sqrt{q(1-a)}|111\rangle, \quad \hat{\rho}_3 = |\Psi_{q,a}\rangle \langle \Psi_{q,a}|.$$

Zheng et al. [23] have shown $\Psi_{q,a}$ to represent a two mode three boson state and calculated the concurrence and tangle for the state. Fig. (4) displays the contour plots of entanglement measures $E_3^2$, $N_G^2$, and $E_2^2$ for this state. For all values of parameters $a$ and $q$, genuine tripartite entanglement is given by $E_{33}^3 = E_{33}^3 = E_3$ and partial 2–way negativity resulting in bipartite and W-state like entanglement is given by $E_{22}^3 = E_{22}^3 = E_2$. The state with maximal genuine tripartite entanglement and the state with maximal W-type entanglement are indicated by solid line arrow and dashed line arrow, respectively.
X. THREE QUBIT NOISY STATE

Figs. (5) and (6) display entanglement measures for two parameter three qubit noisy states of the form

\[ \hat{\rho}_4 = a |\Psi_q\rangle \langle \Psi_q| + (1 - a) \frac{\hat{I}_8}{8}, \]  

(25)

where

\[ |\Psi_q\rangle = \frac{1}{\sqrt{2}} \left( |000\rangle + \sqrt{1 - q} |111\rangle + \sqrt{q} |110\rangle \right), \]

(26)

\[ 0 \leq q \leq 1 \text{ and } 0 \leq a \leq 1. \]  

Global negativity \( N^1_G = N^2_G \) for qubits one and two is independent of parameter \( q \) and varies linearly with \( a \) as seen in contour plots of Fig. (5). For \( a = 0 \) we have \( \hat{\rho}_4 = \frac{\hat{I}_8}{8} \) with \( N_G = E_3 = E_2 = 0 \) that is a separable mixed state. On the other hand for the case \( a = 1, q = 0 \) ( \( |\Psi_0\rangle = |\Psi_{GHZ}\rangle \) ), we have \( N_G = E_3 = 1.0, \) and \( E_2 = 0 \) for all qubits. Global negativity values indicate that all three sub-systems remain separable for \( a \leq 0.2. \) The contour plots of entanglement measures \( E^1_2 = E^2_3 \) and \( E_3 \) in Fig. (6) show the regions of maximum genuine tripartite entanglement, bipartite entanglement and separability. As qubit three has only tripartite entanglement measure \( E_3 \) being zero, the tripartite entanglement measure \( E_3 \) is equal to \( N^3_G. \) For the parameter values \( a = 1, q = 1, \) the system has maximum bipartite entanglement but no tripartite entanglement as the state looks like

\[ |\Psi_{q=1}\rangle = \left( |00\rangle + |11\rangle \right) \frac{1}{\sqrt{2}} |0\rangle. \]

XI. CONCLUSIONS

We have defined global and \( K \)-way negativities calculated from global and \( K \)-way partial transposes, respectively, of an \( N \)-partite state operator. For a given partition of an \( N \)-partite system, global negativity measures
FIG. 6: Contour plots of bipartite entanglement measure $E_{12}^1$ and genuine tripartite entanglement $E_3$ plotted as a function of parameters $q$ and $a$, for the state $\hat{\rho}_4$.

overall entanglement of parties. The $K$-way negativities for $2 \leq K \leq N$, on the other hand, provide a measure of $K$-way coherences of the system. Global negativity with respect to a subsystem can be written as a sum of partial $K$-way negativities. We conjecture that the partial $K$-way negativities provide an entanglement measure for $N$-partite canonical states. For canonical states, the coherences have their minimum value as such definite relations exist between the negativities and $K$-partite entanglement of these states. We have applied these ideas to one and two parameter three qubit states. For a three qubit system a combined analysis of 2-way, 3-way and global negativities with respect to all subsystems is shown to provide distinct measures of genuine tripartite, W-type, and bipartite entanglement of the composite system. Entanglement is invariant with respect to local unitary operations, whereas, coherences are not so. Extension to qutrits and application to systems with more than three parties, should be possible.

XIII. ACKNOWLEDGEMENTS

Financial support from National Council for Scientific and Technological Development (CNPq), Brazil and State University of Londrina, (Faep-UEL), Brazil is acknowledged.

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