Fair Division through Information Withholding

Hadi Hosseini\textsuperscript{1}, Sujoy Sikdar\textsuperscript{2}, Rohit Vaish\textsuperscript{3}, Jun Wang\textsuperscript{3}, and Lirong Xia\textsuperscript{3}

\textsuperscript{1}Rochester Institute of Technology, hhvcs@rit.edu
\textsuperscript{2}Washington University in St. Louis, sujoy@wustl.edu
\textsuperscript{3}Rensselaer Polytechnic Institute, \{vaishr2, wangj38, xial\}@rpi.edu

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Abstract

Envy-freeness up to one good (EF\textsubscript{1}) is a well-studied fairness notion for indivisible goods that addresses pairwise envy by the removal of at most one good. In the worst case, each pair of agents might require the (hypothetical) removal of a different good, resulting in a weak aggregate guarantee. We study allocations that are nearly envy-free in aggregate, and define a novel fairness notion based on information withholding. Under this notion, an agent can withhold (or hide) some of the goods in its bundle and reveal the remaining goods to the other agents. We observe that in practice, envy-freeness can be achieved by withholding only a small number of goods overall. We show that finding allocations that withhold an optimal number of goods is computationally hard even for highly restricted classes of valuations. In contrast to the worst-case results, our experiments on synthetic and real-world preference data show that existing algorithms for finding EF\textsubscript{1} allocations withhold close-to-optimal number of goods.

1 Introduction

When dividing discrete objects, one often strives for a fairness notion called envy-freeness (Foley, 1967), under which no agent prefers the allocation of another agent to its own. Envy-free outcomes might not exist in general (even with only two agents and a single indivisible good), motivating the need for approximations. Among the many approximations of envy-freeness proposed in the literature (Lipton et al., 2004; Budish, 2011; Nguyen and Rothe, 2014; Caragiannis et al., 2016), the notion called envy-freeness up to one good (EF\textsubscript{1}) has received significant attention recently. EF\textsubscript{1} requires that pairwise envy can be eliminated by the removal of some good in the envied bundle. It is known that an EF\textsubscript{1} allocation always exists and can be computed in polynomial time (Lipton et al., 2004).

On closer scrutiny, however, we find that EF\textsubscript{1} is not as strong as one might think. Indeed, an EF\textsubscript{1} allocation could entail the (hypothetical) removal of many goods, because the elimination of envy for different pairs of agents may require the removal of distinct goods. To see this, consider an instance with six goods $g_1, \ldots, g_6$ and three agents $a_1, a_2, a_3$ whose (additive) valuations are as shown below:

|   | $g_1$ | $g_2$ | $g_3$ | $g_4$ | $g_5$ | $g_6$ |
|---|------|------|------|------|------|------|
| $a_1$ | 1    | 1    | 4    | 1    | 1    | 4    |
| $a_2$ | 1    | 4    | 1    | 1    | 4    | 1    |
| $a_3$ | 1    | 4    | 1    | 1    | 4    | 1    |

Observe that the allocation shown via circled goods is EF\textsubscript{1}, since any pairwise envy can be addressed by removing an underlined good. However, each pair of agents requires the removal of a different good (e.g., $a_1$'s envy towards $a_2$ is addressed by removing $g_3$ whereas $a_3$'s envy towards $a_2$ is addressed by removing $g_4$, and so on), resulting in a weak approximation overall (since all goods need to be removed over all pairs of agents).
The above example shows that EF1, on its own, is too coarse to distinguish between allocations that remove a large number of goods (such as the one with circled entries) and those that remove only a few (such as the one with underlined entries, which, in fact, is envy-free). This limitation highlights the need for a fairness notion that (a) can distinguish between allocations in terms of their aggregate approximation, and (b) retains the “up to one good” style approximation of EF1 that has proven to be practically useful (Goldman and Procaccia, 2014). Our work aims to fill this important gap.

We propose a new fairness notion called envy-freeness up to $k$ hidden goods (HEF-$k$) defined as follows: Say there are $n$ agents, $m$ goods, and an allocation $A = (A_1, \ldots, A_n)$. Suppose there is a set $S$ of $k$ goods (called the hidden set) such that each agent $i$ withholds the goods in $A_i \cap S$ (i.e., the hidden goods owned by $i$) and only discloses the goods in $A_i \setminus S$ to the other agents. Any other agent $h \neq i$ only observes the goods disclosed by $i$ (i.e., those in $A_i \setminus S$), and its valuation for $i$’s bundle is therefore $v_h(A_i \setminus S)$ instead of $v_h(A_i)$. Additionally, agent $h$’s valuation for its own bundle is $v_h(A_h)$ (and not $v_h(A_h \setminus S)$) because it can observe its own hidden goods. If, under the disclosed allocation, no agent prefers the bundle of any other agent (i.e., if $v_h(A_i) \geq v_h(A_i \setminus S)$ for every pair of agents $i, h$), then we say that $A$ is envy-free up to $k$ hidden goods (HEF-$k$). In other words, by withholding the information about $S$, allocation $A$ can be made free of envy.

Notice how HEF-$k$ addresses the limitations associated with EF1: Like EF1, HEF-$k$ is a relaxation of envy-freeness that is defined in terms of the number of goods. However, unlike EF1, HEF-$k$ offers a precise quantification of the extent of information that must be withheld in order to achieve envy-freeness.

Clearly, any allocation can be made envy-free by hiding all the goods (i.e., if $k = m$). The real strength of HEF-$k$ lies in $k$ being small; indeed, an HEF-0 allocation is envy-free. As we will demonstrate below, there are natural settings that admit HEF-$k$ allocations with a small $k$ (i.e., hide only a small number of goods) even when (exact) envy-freeness is unlikely.

**Information Withholding is Meaningful in Practice.**

To understand the usefulness of HEF-$k$, we generated a synthetic dataset where we varied the number of agents $n$ from 5 to 10, and the number of goods $m$ from 5 to 20 (we ignore the cases where $m < n$). For every fixed $n$ and $m$, we generated 100 instances with binary valuations. Specifically, for every agent $i$ and every good $j$, the valuation $v_{i,j}$ is drawn i.i.d. from Bernoulli(0.7). Figure 1a shows the heatmap of the number of instances out of 100 that do not admit envy-free outcomes. (Thus, a ‘hot’ cell indicated by red color is one where none of the 100 instances admits an envy-free allocation.) Figure 1b shows the heatmap of the number of goods that must be hidden in the worst-case. That is, the color of each cell denotes the smallest $k$ such that each of the corresponding 100 instances admits some HEF-$k$ allocation.

![Heatmap of the fraction of instances that are not envy-free.](image1)

![Heatmap of the number of goods that must be hidden.](image2)

**Figure 1:** In both figures, each cell corresponds to 100 instances with binary valuations for a fixed number of goods $m$ (on X-axis) and a fixed number of agents $n$ (on Y-axis).

It is evident from Figure 1 that even in the regime where envy-free outcomes are unlikely (in
particular, the red-colored cells in Figure 1a), there exist HEF-\( k \) allocations with \( k \leq 3 \) (the light blue-colored cells in Figure 1b). This observation, along with the foregoing discussion, suggests that fairness through information withholding is a well-motivated approach towards approximate envy-freeness that could provide promising results in practice.

Our Contributions  We make contributions on three fronts.

- On the conceptual side, we propose a novel fairness notion called envy-freeness up to \( k \) hidden goods (HEF-\( k \)) as a fine-grained generalization of envy-freeness in terms of aggregate approximation.

- Our theoretical results (Section 4) show that computing HEF-\( k \) allocations is computationally hard even for highly restricted classes of valuations (Theorem 1 and Corollary 1). We show a similar result when HEF-\( k \) is required alongside Pareto optimality (Theorem 2). A related technical contribution is an alternative proof of NP-completeness of determining the existence of an envy-free allocation for binary valuations (Proposition 3).

- Our experiments show that HEF-\( k \) allocations with a small \( k \) often exist, even when (exact) envy-free allocations do not (Figure 1). We also compare several known algorithms for computing EF1 allocations on synthetic and real-world preference data, and find that the round-robin algorithm and an algorithm of Barman et al. (2018) withhold close-to-optimal number of goods, often hiding no more than three items (Section 5).

2 Related Work

An emerging line of work in the fair division literature considers relaxations of envy-freeness by limiting the information available to the agents. Notably, Aziz et al. (2018) consider a setting where each agent is aware only of its own bundle and has no knowledge about the allocations of the other agents. They propose the notion of epistemic envy-freeness (EEF) under which each agent believes that an envy-free allocation of the remaining goods among the other agents is possible. Note that in EEF, each agent might consider a different hypothetical assignment of the remaining goods, and each of these could be significantly different from the actual underlying allocation. By contrast, under HEF-\( k \), each agent evaluates its valuation with respect to the same (underlying) allocation. Chen and Shah (2017) study a related model where agents have probabilistic beliefs about the allocations of the other agents, and envy is defined in expectation. Chan et al. (2019) study a setting similar to Aziz et al. (2018) wherein each agent is unaware of the allocations of the other agents, with the guarantee that it does not get the worst bundle.

Another related line of work considers settings where the agents constitute a social network and can only observe the allocations of their neighbors (Abebe et al., 2017; Bei et al., 2017; Chevaleyre et al., 2017; Aziz et al., 2018; Beynier et al., 2018; Bredereck et al., 2018). These works place an informational constraint on the set of agents, whereas our model restricts the set of revealed goods per agent.

Several other forms of fairness approximations have been proposed recently, such as introducing side payments (Halpern and Shah, 2019), permitting sharing of some goods (Sandomirskiy and Segal-Halevi, 2019), or donating a small fraction of goods (Caragiannis et al., 2019; Chaudhury et al., 2020).

3 Preliminaries

Problem instance  An instance \( \mathcal{I} = ([n], [m], \mathcal{V}) \) of the fair division problem is defined by a set of \( n \in \mathbb{N} \) agents \( [n] = \{1, 2, \ldots, n\} \), a set of \( m \in \mathbb{N} \) goods \( [m] = \{1, 2, \ldots, m\} \), and a valuation profile \( \mathcal{V} = \{v_1, v_2, \ldots, v_n\} \) that specifies the preferences of every agent \( i \in [n] \) over each subset of the goods in \([m] \) via a valuation function \( v_i : 2^{[m]} \to \mathbb{N} \cup \{0\} \). Notice that each agent’s valuation for any subset of goods is assumed to be a non-negative integer. We will assume that the valuation functions are
an EF local search and price-rise subroutines in a Fisher market associated with the fair division instance, and i.e., has a polynomial dependence on $v$. We say that an instance has binary valuations if for every $i \in [n]$ and every $j \in [m]$, $v_{i,j} \in \{0, 1\}$.

**Allocation** An allocation $A := (A_1, \ldots, A_n)$ refers to an $n$-partition of the set of goods $[m]$, where $A_i \subseteq [m]$ is the bundle allocated to agent $i$. Given an allocation $A$, the utility of agent $i \in [n]$ for the bundle $A_i$ is $v_i(A_i) = \sum_{j \in A_i} v_{i,j}$.

**Definition 1 (Envy-freeness).** An allocation $A$ is envy-free (EF) if for every pair of agents $i, h \in [n]$, $v_i(A_i) \geq v_i(A_h)$. An allocation $A$ is envy-free up to one good (EF1) if for every pair of agents $i, h \in [n]$ such that $A_i \neq \emptyset$, there exists some good $j \in A_h$ such that $v_i(A_i) \geq v_i(A_h \setminus \{j\})$. An allocation $A$ is strongly envy-free up to one good (sEF) if for every agent $h \in [n]$ such that $A_h \neq \emptyset$, there exists a good $g_h \in A_h$ such that for all $i \in [n]$, $v_i(A_i) \geq v_i(A_h \setminus \{g_h\})$. The notions of EF, EF1, and sEF are due to Foley (1967), Budish (2011), and Conitzer et al. (2019), respectively.

**Definition 2 (Envy-freeness with hidden goods).** An allocation $A$ is said to be envy-free up to $k$ hidden goods (HEF-k) if there exists a set $S \subseteq [m]$ of at most $k$ goods such that for every pair of agents $i, h \in [n]$, we have $v_i(A_i) \geq v_i(A_h \setminus S)$. An allocation $A$ is envy-free up to $k$ uniformly hidden goods (uHEF-k) if there exists a set $S \subseteq [m]$ of at most $k$ goods satisfying $|S \cap A_i| \leq 1$ for every $i \in [n]$ such that for every pair of agents $i, h \in [n]$, we have $v_i(A_i) \geq v_i(A_h \setminus S)$. We say that allocation $A$ hides the goods in $S$ and reveals the remaining goods. Notice that a uHEF-k allocation is also HEF-k but the converse is not necessarily true. Indeed, in Proposition 2, we will present an instance that, for some $k \in \mathbb{N}$, admits an HEF-k allocation but no uHEF-k allocation.

**Remark 1.** It follows from the definitions that HEF-0 $\Rightarrow$ HEF-1 $\Rightarrow$ HEF-2 $\ldots$, and that an allocation satisfies HEF-0 if and only if it satisfies EF. It is also easy to verify that an allocation is sEF if and only if it is uHEF-n. This is because the unique hidden good for every agent is also the one that is (hypothetically) removed under sEF1. Additionally, as discussed in Section 1, an EF1 allocation might not be uHEF-k for any $k \leq n$.

We say that allocation $A$ is HEF with respect to set $S$ if $A$ becomes envy-free after hiding the goods in $S$, i.e., for every pair of agents $i, h \in [n]$, we have $v_i(A_i) \geq v_i(A_h \setminus S)$. We say that $k$ goods must be hidden under $A$ if $A$ is HEF with respect to some set $S$ such that $|S| = k$, and there is no set $S'$ with $|S'| < k$ such that $A$ is HEF with respect to $S'$.

**Definition 3 (Pareto optimality).** An allocation $A$ is Pareto dominated by another allocation $B$ if $v_i(B_i) \geq v_i(A_i)$ for every agent $i \in [n]$ with at least one of the inequalities being strict. A Pareto optimal (PO) allocation is one that is not Pareto dominated by any other allocation.

**Definition 4 (EF1 algorithms).** We will now describe four known algorithms for finding EF1 allocations that are relevant to our work.

**Round-robin algorithm (RoundRobin):** Fix a permutation $\sigma$ of the agents. The RoundRobin algorithm cycles through the agents according to $\sigma$. In each round, an agent gets its favorite good from the pool of remaining goods.

**Envy-graph algorithm (EnvyGraph):** This algorithm, proposed by Lipton et al. (2004), works as follows: In each step, one of the remaining goods is assigned to an agent that is not envied by any other agent. The existence of such an agent is guaranteed by resolving cyclic envy relations (if any exists) in a combinatorial structure called the envy-graph of an allocation.

**Fisher market-based algorithm (Alg-EF1+PO):** This algorithm, due to Barman et al. (2018), uses local search and price-rise subroutines in a Fisher market associated with the fair division instance, and returns an EF1 and PO allocation. The bound on running time of this algorithm is pseudopolynomial, i.e., has a polynomial dependence on $v_{i,j}$ instead of $\log v_{i,j}$.

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1A slightly weaker notion than EF1 was previously studied by Lipton et al. (2004). However, their algorithm can be shown to compute an EF1 allocation.
Maximum Nash Welfare solution (MNW): The Nash social welfare of an allocation \( A \) is defined as
\[
\text{NSW}(A) := \left( \prod_{i \in [n]} v_i(A_i) \right)^{1/n}.
\]
The MNW algorithm computes an allocation with the highest Nash social welfare (called a Nash optimal allocation). It is known that a Nash optimal allocation is both EF1 and PO (Caragiannis et al., 2016).

Remark 2. Conitzer et al. (2019) observed that RoundRobin, Alg-EF1+PO, and MNW algorithms all satisfy sEF1. It is easy to see that EnvyGraph algorithm is also sEF1. Among these four algorithms, only MNW and Alg-EF1+PO are provably also PO.\(^2\) The allocations computed by all four algorithms have the property that there exists some agent that is not envied by any other agent. Indeed, MNW and Alg-EF1+PO are both PO and therefore cannot have cyclic envy relations, and RoundRobin and EnvyGraph algorithms have this property by design. For such an agent (not necessarily the same agent for all four algorithms), no good needs to be removed under sEF1. Therefore, from Remark 1, all these algorithms are also envy-free up to \( n - 1 \) uniformly hidden goods, or uHEF-(\( n - 1 \)).

Proposition 1. Given an instance with additive valuations, a uHEF-(\( n - 1 \)) allocation always exists and can be computed in polynomial time, and a uHEF-(\( n - 1 \)) + PO allocation always exists and can be computed in pseudopolynomial time.

Remark 3. Note that for any \( k < n - 1 \), an HEF-k allocation might fail to exist. Indeed, with \( n \) agents that have identical and positive valuations for \( m = n - 1 \) goods, some agent will surely miss out and force the allocation to hide all \( n - 1 \) (i.e., \( k + 1 \) or more) goods. Therefore, the bound in Proposition 1 for uHEF-k (and hence, for HEF-k) is tight in terms of \( k \).

3.1 Relevant Computational Problems

Definition 5 formalizes the decision problem of checking whether a given instance admits a fair (i.e., HEF-k) allocation.

Definition 5 (HEF-k-EXISTENCE). Given an instance \( I \), does there exist an allocation \( A \) and a set \( S \subseteq [m] \) of at most \( k \) goods such that \( A \) is HEF with respect to \( S \)?

Notice that a certificate for HEF-k-EXISTENCE consists of an allocation \( A \) as well as a set \( S \) of at most \( k \) hidden goods.

Another relevant computational question involves checking whether a given allocation \( A \) is HEF with respect to some set \( S \subseteq [m] \) of at most \( k \) goods.

Definition 6 (HEF-k-VERIFICATION). Given an instance \( I \) and an allocation \( A \), does there exist a set \( S \subseteq [m] \) of \( k \) goods such that \( A \) is HEF with respect to \( S \)?

For additive valuations, both HEF-k-EXISTENCE and HEF-k-VERIFICATION are in NP. The next problem pertains to the existence of envy-free allocations.

Definition 7 (EF-EXISTENCE). Given an instance \( I \), does there exist an envy-free allocation for \( I \)?

EF-EXISTENCE is known to be NP-complete (Lipton et al., 2004). From Remark 1, it follows that HEF-k-EXISTENCE is NP-complete when \( k = 0 \) for additive valuations.

4 Theoretical Results

This section presents our theoretical results concerning the existence and computation of HEF-k and uHEF-k allocations. We will first show that uHEF-k is a strictly more demanding notion than HEF-k (Proposition 2).

\(^2\)It is also known that RoundRobin and EnvyGraph fail to satisfy PO; see, e.g., (Conitzer et al., 2017).
Proposition 2. There exists an instance \( I \) that, for some fixed \( k \in \mathbb{N} \), admits an HEF-\( k \) allocation but no uHEF-\( k \) allocation.

Proof. Consider the fair division instance \( I \) with five agents \( a_1, \ldots, a_5 \) and six goods \( g_1, \ldots, g_6 \) shown in Table 1. Observe that the allocation \( A = (A_1, \ldots, A_5) \) with \( A_1 = \{g_1, g_2\}, A_2 = \{g_3\}, A_3 = \{g_4\}, A_4 = \{g_5\}, A_5 = \{g_6\} \) satisfies HEF-2 with respect to the set \( S = \{g_1, g_2\} \).

| \( g_1 \) | \( g_2 \) | \( g_3 \) | \( g_4 \) | \( g_5 \) | \( g_6 \) |
|-----|-----|-----|-----|-----|-----|
| \( a_1 \) | 1 | 1 | 2 | 0 | 0 | 0 |
| \( a_2 \) | 1 | 1 | 2 | 0 | 0 | 0 |
| \( a_3 \) | 10 | 10 | 1 | 1 | 1 | 1 |
| \( a_4 \) | 10 | 10 | 1 | 1 | 1 | 1 |
| \( a_5 \) | 10 | 10 | 1 | 1 | 1 | 1 |

Table 1: The instance used in the proof of Proposition 2.

We will show that \( I \) does not admit a uHEF-2 allocation. Suppose, for contradiction, that there exists an allocation \( B \) satisfying uHEF-2. Then, \( B \) must hide \( g_1 \) and \( g_2 \) (otherwise, at least one of \( a_3, a_4 \) or \( a_5 \) will envy the owner(s) of these goods). Thus, in particular, the good \( g_3 \) must be revealed by \( B \). Assume, without loss of generality, that \( g_3 \) is not assigned to \( a_1 \) in \( B \) (otherwise, a similar argument can be carried out for \( a_2 \)). Then, \( B \) must assign both \( g_1 \) and \( g_2 \) to \( a_1 \) (so that \( a_1 \) does not envy the owner of \( g_3 \)). However, this violates the one-hidden-good-per-agent property of uHEF-\( k \), which is a contradiction.

Recall from Section 3.1 that HEF-\( k \)-Existence is NP-complete when \( k = 0 \). This still leaves open the question whether HEF-\( k \)-Existence is NP-complete for any fixed \( k \in \mathbb{N} \). Our next result (Theorem 1) shows that this is indeed the case, even under the restricted setting of identical valuations (i.e., for every \( j \in [m] \), \( v_{i,j} = v_{h,j} \) for every \( i, h \in [n] \)).

Theorem 1 (Hardness of HEF-\( k \)-Existence). For any fixed \( k \in \mathbb{N} \), HEF-\( k \)-Existence is NP-complete even for identical valuations.

Proof. We will show a reduction from Partition, which is known to be NP-complete (Garey and Johnson, 1979). An instance of Partition consists of a multiset \( X = \{x_1, x_2, \ldots, x_n\} \) with \( x_i \in \mathbb{N} \) for all \( i \in [n] \). The goal is to determine whether there exists \( Y \subseteq X \) such that \( \sum_{x_i \in Y} x_i = \sum_{x_i \in X \setminus Y} x_i = T \), where \( T := \frac{1}{2} \sum_{x_i \in X} x_i \).

We will construct a fair division instance with \( k + 3 \) agents \( a_1, \ldots, a_{k+3} \) and \( n + k + 1 \) goods. The goods are classified into \( n + 1 \) main goods \( g_1, \ldots, g_{n+1} \) and \( k \) dummy goods \( d_1, \ldots, d_k \). The (identical) valuations are defined as follows: Every agent values the goods \( g_1, \ldots, g_n \) at \( x_1, \ldots, x_n \) respectively; the good \( g_{n+1} \) at \( T \), and each dummy good at \( 4T \).

(\( \Rightarrow \)) Suppose \( Y \) is a solution of Partition. Then, an HEF-\( k \)-allocation can be constructed as follows: Assign the main goods corresponding to the set \( Y \) to agent \( a_1 \) and those corresponding to \( X \setminus Y \) to agent \( a_2 \). The good \( g_{n+1} \) is assigned to agent \( a_3 \). Each of the remaining \( k \) agents is assigned a unique dummy good. Note that every agent in the set \{\( a_1, a_2, a_3 \)\} envies every agent in the set \{\( a_4, \ldots, a_{k+3} \)\}, and these are the only pairs of agents with non-zero envy. Therefore, the allocation can be made envy-free by hiding the \( k \) dummy goods, i.e., the allocation is HEF with respect to the set \{\( d_1, \ldots, d_k \)\}.

(\( \Leftarrow \)) Now suppose there exists an HEF-\( k \) allocation \( A \). Since there are \( k \) dummy goods and \( k + 3 \) agents, there must exist at least three agents that do not receive any dummy good in \( A \). Without loss of generality, let these agents be \( a_1, a_2 \) and \( a_3 \) (otherwise, we can reindex). We claim that all dummy goods must be hidden under \( A \). Indeed, agent \( a_1 \) does not receive any dummy good, and therefore its maximum possible valuation can be \( v(g_1 \cup \cdots \cup g_{n+1}) = 3T < v(d_j) \) for any dummy good \( d_j \). If some dummy good \( d_j \) is not hidden, then \( a_1 \) will envy the owner of \( d_j \), contradicting HEF-\( k \). Therefore, all dummy goods must be hidden, and since there are \( k \) such goods, these are the only ones that can be hidden.
The above observation implies that the good \( g_{n+1} \) must be revealed by \( A \). Furthermore, \( g_{n+1} \) must be assigned to one of \( a_1, a_2 \) or \( a_3 \) (otherwise, by pigeonhole principle, one of these agents will have valuation at most \( \frac{2T}{3} \) and will envy the owner of \( g_{n+1} \)). If \( g_{n+1} \) is assigned to \( a_3 \), then the remaining main goods \( g_1, \ldots, g_n \) must be divided between \( a_1 \) and \( a_2 \) such that \( v(A_1) \geq T \) and \( v(A_2) \geq T \). This gives a partition of the set \( X \).

Another commonly used preference restriction is that of binary valuations (i.e., for every \( i \in [n] \) and \( j \in [m] \), \( v_{i,j} \in \{0, 1\} \)). We note that even under this restriction, HEF-\( k \)-Existence remains NP-complete when \( k = 0 \) (Corollary 1). This observation follows from a result of Aziz et al. (2015), who showed that determining the existence of an envy-free allocation is NP-complete even for binary valuations (Proposition 3). We provide an alternative proof of this statement in Section 7.1 in the appendix.

**Proposition 3** (Aziz et al., 2015; Theorem 11). EF-Existence is NP-complete even for binary valuations.

**Corollary 1.** For \( k = 0 \), HEF-\( k \)-Existence is NP-complete even for binary valuations.

Proposition 3 is also useful in establishing the computational hardness of finding an HEF-\( k \)+PO allocation. Note that unlike Corollary 1, Theorem 2 holds for every fixed \( k \in \mathbb{N} \).

**Theorem 2 (Hardness of HEF-\( k \)+PO).** Given any instance \( \mathcal{I} \) with binary valuations and any fixed \( k \in \mathbb{N} \cup \{0\} \), it is NP-hard to determine if \( \mathcal{I} \) admits an allocation that is envy-free up to \( k \) hidden goods (HEF-\( k \)) and Pareto optimal (PO).

**Proof.** (Sketch) Starting from any instance of EF-Existence with binary valuations (Proposition 3), we add to it \( k \) new goods and \( k + 1 \) new agents such that all new goods are approved by all new agents (and no one else). Also, the new agents have zero value for the existing goods. In the forward direction, an arbitrary allocation of new goods among the new agents works. In the reverse direction, PO forces each new (respectively, existing) good to be assigned among new (respectively, existing) agents only. The imbalance between new agents and new goods means that all (and only) the new goods must be hidden. Then, the restriction of the HEF-\( k \) allocation to the existing agents/goods gives the desired EF allocation.

We will now proceed to analyzing the computational complexity of HEF-\( k \)-Verification. Here, we show a hardness-of-approximation result (Theorem 3). The inapproximability factor is stated in terms of the aggregate envy, defined as follows: Given any allocation \( A \), the aggregate envy in \( A \) is the sum of all pairwise envy values, i.e.,

\[
E := \sum_{h \in [n]} \sum_{i \neq h} \max\{0, v_i(A_h) - v_i(A_i)\}.
\]

Note that HEF-\( k \)-Verification is stated as a decision problem (Definition 6). However, one can consider an approximation version of this problem as follows: A \( c \)-approximation algorithm for HEF-\( k \)-Verification takes as input a fair division instance and an allocation, and computes a set of goods of size at most \( c \cdot k^{\text{opt}} \), where \( k^{\text{opt}} \) is the size of the smallest hidden set for the given allocation. Under this definition, Theorem 3 can be interpreted as follows: Given any \( \varepsilon > 0 \), there is no polynomial-time \((1 - \varepsilon) \cdot \ln E\)-approximation algorithm for HEF-\( k \)-Verification, unless \( \text{P} = \text{NP} \).

**Theorem 3 (HEF-\( k \)-Verification inapproximability).** Given any \( \varepsilon > 0 \), it is NP-hard to approximate HEF-\( k \)-Verification to within \((1 - \varepsilon) \cdot \ln E\) even for binary valuations, where \( E \) is the aggregate envy in the given allocation.

**Proof.** We will show a reduction from Hitting Set. An instance of Hitting Set consists of a finite set \( X = \{x_1, \ldots, x_p\} \), a collection \( \mathcal{F} = \{F_1, \ldots, F_q\} \) of subsets of \( X \), and some \( k \in \mathbb{N} \). The goal is to determine whether there exists \( Y \subseteq X \), \(|Y| \leq k\) that intersects every member of \( \mathcal{F} \) (i.e., for every
We will construct a fair division instance with \( n = q + 1 \) agents and \( m = p + \sum_{i=1}^{q}(|F_i| - 1) \) goods. The agents are classified into \( q \) dummy agents \( a_1, \ldots, a_q \) and one main agent \( a_{q+1} \). The goods are classified into \( p \) main goods \( g_1, \ldots, g_p \) and \( q \) distinct sets of dummy goods, where the \( i \)-th set consists of the goods \( f^i_1, \ldots, f^i_{|F_i|-1} \).

The valuations are as follows: The main agent approves all the main goods, i.e., for all \( j \in [p] \), \( v_{q+1}(\{g_j\}) = 1 \). Each dummy agent \( a_i \) approves the dummy goods in the \( i \)-th set as well as those main goods that intersect with \( F_i \), i.e., for every \( i \in [q] \), \( v_i(\{f_i^j\}) = 1 \) for all \( j \in [|F_i| - 1] \), and \( v_i(\{g_j\}) = 1 \) whenever \( x_j \in F_i \). All other valuations are set to 0.

The input allocation \( A = (A_1, \ldots, A_{q+1}) \) is defined as follows: The main agent \( a_{q+1} \) is assigned all the main goods, i.e., \( A_{q+1} := \{g_1, \ldots, g_p\} \). For every \( i \in [q] \), the dummy agent \( a_i \) is assigned the \( |F_i| - 1 \) dummy goods in the \( i \)-th set, i.e., \( A_i := \{f^i_1, \ldots, f^i_{|F_i|-1}\} \). Note that in the allocation \( A \), each dummy agent envies the main agent by one approved good, and these are the only pairs of agents with envy.

\((\Rightarrow)\) Suppose \( Y \subseteq X \), \( |Y| \leq k \) is solution of the Hitting Set instance. We claim that the allocation \( A \) is HEF with respect to the set \( S := \{g_j : x_j \in Y\} \) with \( |S| \leq k \). Indeed, since \( S \) is induced by a hitting set, each dummy agent approves at least one good in \( S \). Therefore, by hiding the goods in \( S \), the envy from the dummy agents can be eliminated.

\((\Leftarrow)\) Now suppose there exists \( S \subseteq [m] \), \( |S| \leq k \) such that \( A \) is HEF with respect to \( S \). Then, for every \( i \in [q] \), the set \( S \) must contain at least one good that is approved by the dummy agent \( a_i \) (otherwise \( A \) will not be envy-free after hiding the goods in \( S \)). It is easy to see that the set \( Y := \{x_j : g_j \in S\} \) constitutes the desired hitting set of cardinality at most \( k \).

Finally, to show the hardness-of-approximation, notice that the aggregate envy in \( A \) is \( q \) because each dummy agent envies the main agent by one unit of utility. The claim now follows by substituting \( |F| = q = E \) in the inapproximability result of Hitting Set stated above.

Our next result (Theorem 4) provides an approximation algorithm that (nearly) matches the hardness-of-approximation result in Theorem 3. We remark that the algorithm in Theorem 4 applies to any instance with additive and possibly non-binary valuations.

**Theorem 4 (Approximation algorithm).** There is a polynomial-time algorithm that, given as input any instance of HEF-\( k \)-Verification, finds a set \( S \subseteq [m] \) with \( |S| \leq k^{opt} \cdot \ln E + 1 \) such that the given allocation is HEF with respect to \( S \). Here, \( E \) and \( k^{opt} \) denote the aggregate envy and the number of goods that must be hidden under the given allocation, respectively.

The proof of Theorem 4 is deferred to Section 7.2 in the appendix but a brief idea is as follows: For any set \( S \subseteq [m] \), define the residual envy function \( f : 2^{[m]} \to \mathbb{R} \) so that \( f(S) \) is the aggregate envy in allocation \( A \) after hiding the goods in \( S \). That is,

\[
f(S) := \sum_{h \in [n]} \sum_{i \neq h} \max\{0, v_i(A_h \setminus S) - v_i(A_i)\}.
\]

The relevant observation is that \( f \) is supermodular. Given this observation, the approximation guarantee in Theorem 4 can be obtained by the standard greedy algorithm for submodular maximization, or, equivalently, supermodular minimization (Nemhauser et al., 1978); see Algorithm 1 in Section 7.2.

## 5 Experimental Results

We have seen that the worst-case computational results for HEF-\( k \), even in highly restricted settings, are largely negative (Section 4). In this section, we will examine whether the known algorithms for computing approximately envy-free allocations—in particular, the four E1 algorithms described in Definition 4 in Section 3—can provide meaningful approximations to HEF-\( k \) in practice. Recall from
Normalized average-case regret

| Alg-EF1+PO | RoundRobin | MNW | EnvyGraph |
|------------|------------|-----|-----------|
| ![Heatmap](image1.png) | ![Heatmap](image2.png) | ![Heatmap](image3.png) | ![Heatmap](image4.png) |

Number of goods that must be hidden on average (averaged over non-EF instances only)

| Alg-EF1+PO | RoundRobin | MNW | EnvyGraph |
|------------|------------|-----|-----------|
| ![Heatmap](image1.png) | ![Heatmap](image2.png) | ![Heatmap](image3.png) | ![Heatmap](image4.png) |

Table 2: Results for synthetic data.

Remark 2 that all four discussed algorithms—RoundRobin, MNW, Alg-EF1+PO, and EnvyGraph—satisfy uHEF-\((n - 1)\).

We evaluate each algorithm in terms of (a) its regret (defined below), and (b) the number of goods that the algorithm must hide. Given an instance \(I\) and an allocation \(A\), let \(\kappa(A, I)\) denote the number of goods that must be hidden under \(A\). The regret of allocation \(A\) is the number of extra goods that must be hidden under \(A\) compared to the optimal. That is, \(\text{reg}(A, I) := \kappa(A, I) - \min_{\kappa} \kappa(B, I)\). Similarly, given an algorithm \(\text{Alg}\), the regret of \(\text{Alg}\) is given by \(\text{reg}(\text{Alg}(I), I)\), where \(\text{Alg}(I)\) is the allocation returned by \(\text{Alg}\) for the input instance \(I\). Note that the regret can be large due to the suboptimality of an algorithm, but also due to the size of the instance. To negate the effect of the latter, we normalize the regret value by \(n - 1\), which is the worst-case upper bound on the number of hidden goods for all four algorithms of interest.

5.1 Experiments on Synthetic Data

The setup for synthetic experiments is similar to that used in Figure 1. Specifically, the number of agents, \(n\), is varied from 5 to 10, and the number of goods, \(m\), is varied from 5 to 20 (we ignore the cases where \(m < n\)). For every fixed \(n\) and \(m\), we generated 100 instances with binary valuations drawn i.i.d. from Bernoulli distribution with parameter 0.7 (i.e., \(v_{i,j} \sim \text{Ber}(0.7)\)). Table 2 shows the heatmaps of the normalized regret (averaged over 100 instances) and the number of goods that must be hidden (averaged over non-EF instances, i.e., whenever \(k \geq 1\)) for all four algorithms.\(^3\)

It is clear that Alg-EF1+PO and RoundRobin algorithms have a superior performance than MNW and EnvyGraph. In particular, both Alg-EF1+PO and RoundRobin have small normalized regret, suggesting that they hide close-to-optimal number of goods. Additionally, the number of hidden goods itself is small for these algorithms (in most cases, no more than three goods need to be hidden), suggesting that the worst-case bound of \(n - 1\) is unlikely to arise in practice. Overall, our experiments suggest that Alg-EF1+PO and RoundRobin can achieve useful approximations to HEF-\(k\) in practice, especially in comparison to MNW and EnvyGraph.\(^4\)

5.2 Experiments on Real-World Data

For experiments with real-world data, we use the data from the popular fair division website Spliddit (Goldman and Procaccia, 2014). The Spliddit data has 2212 instances in total, where the number of

\(^3\)Additional results for \(v_{i,j} \sim \text{Ber}(0.7)\), and \(v_{i,j} \sim \text{Ber}(0.5)\) can be found in Section 7.4 in the appendix.

\(^4\)In Section 7.3 in the appendix, we provide two families of instances where the normalized worst-case regret of MNW is large.
agents $n$ varies between 3 and 10, and the number of goods $m \geq n$ varies between 3 and 93. Unlike the synthetic data, the distribution of instances here is rather uneven (see Figure 3 in Section 7.4 in the appendix); in fact, 1821 of the 2212 instances have $n = 3$ agents and $m = 6$ goods. Therefore, instead of using heatmaps, we compare the algorithms in terms of their normalized regret (averaged over the entire dataset) and the cumulative distribution function of the hidden goods (see Figure 2).

Figure 2 presents an interesting twist: MNW is now the best performing algorithm, closely followed by RoundRobin and Alg-EF1+PO. For any fixed $k$, the fraction of instances for which these three algorithms compute an HEF-$k$ allocation is also nearly identical. As can be observed, these algorithms almost never need to hide more than three goods. By contrast, EnvyGraph has the largest regret and significantly worse cumulative performance. Therefore, once again, Alg-EF1+PO and RoundRobin algorithms perform competitively with the optimal solution, making them attractive options for achieving fair outcomes without withholding too much information.

6 Future Work

Analyzing the asymptotic behavior of HEF-$k$ allocations, as has been done for envy-free allocations (Dickerson et al., 2014; Manurangsi and Saksompong, 2019), is an interesting direction for future work. It would also be interesting to explore the connections with other recently proposed relaxations that involve discarding goods (Caragiannis et al., 2019; Chaudhury et al., 2020) or sharing a small subset of goods (Sandomirskiy and Segal-Halevi, 2019).

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7 Appendix

7.1 Proof of Proposition 3

Recall the statement of Proposition 3.

**Proposition 3** (Aziz et al., 2015; Theorem 11). **EF-Existence is NP-complete even for binary valuations.**

Our proof of Proposition 3 uses a reduction from Equitable Coloring, which is defined below.

**Definition 8** (**Equitable Coloring**). Given a graph $G$ and a number $\ell \in \mathbb{N}$, does there exist a proper $\ell$-coloring of $G$ such that all color classes are of equal size?

The standard definition of **Equitable Coloring** requires the color classes to differ in size by at most one. We overload the term to refer to the version where all color classes are of the same size. **Equitable Coloring** can be shown to be NP-complete by a straightforward reduction from **Graph $k$-Colorability** (Garey and Johnson, 1979). In addition, we can assume that $\ell \geq 3$ without loss of generality.

**Proof.** (of Proposition 3) We will show a reduction from **Equitable Coloring**. Recall from Definition 8 that an instance of **Equitable Coloring** consists of a graph $G = (V, E)$ and a number $\ell \in \mathbb{N}$. The goal is to determine if $G$ admits a proper $\ell$-coloring wherein the color classes are of the same size. For simplicity, we will write $n := |V|$ and $m := |E|$. Note that we can assume, without loss of generality, that $G$ is connected.\(^5\) Since a connected graph with $n$ vertices has at least $n - 1$ edges, we have that

$$m \geq n - 1. \hspace{1cm} (1)$$

In addition, we will also assume that each vertex in $G$ has degree at least two. Indeed, for any vertex $v$ with degree at most one, we can add $\ell$ new vertices $v_1, v_2, \ldots, v_\ell$ that are connected as follows: The vertices $v_1, \ldots, v_\ell$ constitute an $\ell$-clique (that is, for every pair of distinct $i, j \in [\ell]$, $v_i$ is connected to $v_j$), and $v$ is connected to each vertex in $\{v_2, \ldots, v_{\ell - 1}\}$ but not $v_1$. Call the new graph $G'$. It is easy to see that $G'$ has an equitable $\ell$-coloring if and only if $G$ does.

We will construct a fair division instance with $m + n$ goods and $m + \ell$ agents. The agents are classified into $m$ edge agents $a_1, \ldots, a_m$ and $\ell$ dummy agents $d_1, \ldots, d_\ell$. The goods are classified into $n$ vertex goods $v_1, \ldots, v_n$ and $m$ edge goods $e_1, \ldots, e_m$. Note that we use the same notation for the vertices (edges) and the corresponding vertex (edge) goods.

The preferences of the agents are defined as follows: For every edge $e = (v_i, v_j)$, an edge agent $a_e$ approves all the edge goods and exactly two vertex goods $v_i$ and $v_j$. Each dummy agent approves all the vertex goods and has zero value for the edge goods.

($\Rightarrow$) Suppose $G$ admits an equitable coloring with each color class of size $\frac{n}{\ell}$. Then, an envy-free allocation $A$ can be constructed as follows: Assign each edge good $e$ to the edge agent $a_e$ and each vertex good $v$ to the dummy agent $d_i$ if vertex $v$ has color $i$. Notice that all goods are allocated under $A$. Also note that no two edge agents envy each other since each of them gets exactly one edge good. Furthermore, due to the proper coloring condition, for any edge $e = (v_i, v_j)$ in $G$, the corresponding vertex goods $v_i$ and $v_j$ are assigned to distinct dummy agents. Hence, no edge agent envies a dummy agent. The dummy agents have zero value for the edge goods and therefore do not envy the edge agents. Finally, since all color classes are of the same size, each dummy agent gets exactly $\frac{n}{\ell}$ approved goods, and therefore does not envy any other dummy agent. Overall, the allocation is envy-free.

($\Leftarrow$) Now suppose there exists an envy-free allocation $A$. We will show that $A$ satisfies Properties 1 to 6 that will help us infer an equitable coloring of $G$.

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\(^5\)Not be confused with the number of agents, $n$, and the number of goods, $m$, as defined in Section 3.

\(^6\)Given any graph $G$, we can construct another connected graph $G' = (V', E')$ as follows: Let $V' := V \cup \{x_1, \ldots, x_\ell\} \cup \{y_1, \ldots, y_{\ell + 1}\}$. There is an edge between every pair of vertices in $\{x_1, \ldots, x_\ell\}$ so as to induce an $\ell$-clique. In addition, each $y_i$ is connected to every vertex in $\{x_1, \ldots, x_\ell\}$ as well as to every vertex in $V$. It is easy to see that $G'$ admits an equitable $\ell$-coloring if and only if $G'$ admits a proper $(\ell + 1)$-coloring. Indeed, the $x_i$’s consume $\ell$ colors, and, as a result, all $y_i$’s must have the same color. This, in turn, leaves exactly $\ell$ colors for the vertices in $V$. Furthermore, there are $\frac{n}{\ell} + 1$ vertices in each color class, implying that the coloring is equitable.
Property 1. No edge agent can get two or more edge goods under A.

Proof. (of Property 1) Suppose, for contradiction, that some edge agent $a_e$ gets two or more edge goods. Then, any other edge agent $a_e'$ has a utility of at least 2 for the bundle of $a_e$. For A to be envy-free, $a_e'$ must have a utility of at least 2 for its own bundle. For binary valuations, this means that $a_e'$ must be assigned two or more goods that it approves. Therefore, we need at least $2m$ goods to satisfy the edge agents. The total number of available goods is $m + n$, which, using Equation (1), evaluates to at most $2m + 1$. This leaves at most one good to be allocated among $\ell$ dummy agents. Since $\ell \geq 3$, some dummy agent is bound to be envious, contradicting the envy-freeness of A.

Property 2. Every dummy agent gets at least one vertex good under A.

Proof. (of Property 2) Fix a vertex good $v$ and a dummy agent $d$. Then, either $v$ is assigned to $d$, or $d$ gets some other (approved) good to prevent it from envying the owner of $v$. Since the only goods approved by the dummy agents are the vertex goods, the claim follows.

Property 3. No dummy agent can get an edge good under A.

Proof. (of Property 3) Suppose, for contradiction, that a dummy agent $d$ gets an edge good $e$ under A. From Property 2, we know that $d$ also gets some vertex good, say $v_0$. By assumption, the graph $G$ has minimum degree two, so there must exist some edge $e_1 = (v_0, v_1)$ adjacent to the vertex $v_0$. Notice that the edge agent $a_{e_1}$ has a utility of (at least) 2 for the bundle of $d$. Therefore, for A to be envy-free, $a_{e_1}$ must get at least two goods that it approves. Property 1 limits the number of edge goods assigned to any edge agent to at most one. Therefore, in addition to some edge good, $a_{e_1}$ must also get the vertex good $v_1$. Once again using the bound on minimum degree of $G$, we get that there must exist some edge $e_2 = (v_1, v_2)$ adjacent to the vertex $v_1$. A similar argument shows that the vertex good $v_2$ must be assigned to the edge agent $a_{e_2}$. Continuing in this manner, we will eventually encounter an edge $e_i = (v_{i-1}, v_i)$ such that $v_{i-1}$ is already assigned to $a_{e_{i-1}}$ and $v_i$ is already assigned to either $d$ or some other edge agent. This would imply that $a_{e_i}$ is envious of some other agent under A—a contradiction.

Property 4. Every edge agent gets exactly one edge good under A.

Proof. (of Property 4) Follows from Properties 1 and 3.

Property 5. No edge agent can get a vertex good under A.

Proof. (of Property 5) Suppose, for contradiction, that some edge agent $a_{e_0}$ is assigned a vertex good $v_0$. Let $e_1 = (v_0, v_1)$ be an edge incident to the vertex $v_0$ in $G$ (such an edge must exist due to the bound on minimum degree). From Property 4, we know that each edge agent gets exactly one edge good. Thus, the edge agent $a_{e_1}$ has a utility of (at least) 2 for the bundle of the agent $a_{e_0}$. For A to be envy-free, $a_{e_1}$ must receive two or more goods that it approves, only one of which can be an edge good. Therefore, agent $a_{e_1}$ must also receive the vertex good $v_1$. Now let $e_2 = (v_1, v_2)$ be another edge incident to the vertex $v_1$ in $G$ (again, such an edge exists because $v_1$ has degree at least two). A similar argument implies that the vertex good $v_2$ must be assigned to the edge agent $a_{e_2}$. Continuing in this manner, let $e_i = (v_{i-1}, v_i)$ denote the first edge in the sequence for which one of the following mutually exclusive conditions is true:

1. Either, the vertex good $v_i$ is assigned to an agent different from $a_{e_i}$, or
2. the vertex good $v_i$ is assigned to $a_{e_i}$ and $v_i = v_0$ (thus $a_{e_i} = a_{e_0}$).

Notice that due to the finiteness of the graph $G$, there must exist an edge $e_i$ satisfying one of the aforementioned conditions. We will now argue that each of these conditions leads to a contradiction.
First, suppose that the vertex good \( v_i \) is assigned to an agent different from \( a_e \). Then, the edge agent \( a_e \) has a utility of (at least) 2 for the bundle of edge agent \( a_e-1 \) and a utility of 1 for its own bundle, contradicting the envy-freeness of \( A \).

Next, suppose that \( v_i = v_j \). That is, there exists a cycle \( C = \{(v_0, v_1), (v_1, v_2), \ldots, (v_{i-1}, v_0)\} \) in the graph \( G \) such that for every \( j \in \{0, 1, \ldots, i-1\} \), the vertex good \( v_j \) is assigned to the edge agent \( a_e \).

Recall from Property 2 that each dummy agent gets at least one vertex good. Since all vertex goods corresponding to the vertices in \( C \) are assigned to the edge agents, there must exist at least one vertex outside the cycle \( C \). Furthermore, since \( G \) is connected, there must exist a path from this vertex to a vertex in \( C \), say \( v_1 \). Thus, there must exist a vertex \( v'_1 \in V \) such that \( (v_1, v'_1) \in E \) and \( v'_1 \notin C \). Let \( e'_1 := (v_1, v'_1) \). Then, the edge agent \( a_e \) has a utility of 2 for the bundle of agent \( a_e \) (recall that \( a_e \) receives an edge good and the vertex good \( v_1 \)), and must therefore be assigned the vertex good \( v'_1 \). Since the vertex \( v'_1 \) has degree at least two, there must exist another edge \( e'_2 = (v'_1, v'_2) \) in \( G \). By a similar argument as before, the edge agent \( a_e \) must be assigned the vertex good \( v'_2 \). Continuing in this manner, we will encounter an edge, say \( e'_i = (v'_{i-1}, v'_i) \) such that the vertex good \( v'_i \) is assigned to an agent different from \( a_e \). This means that the edge agent \( a_e \) has a utility of (at least) 2 for the bundle of the edge agent \( a_e-1 \) and a utility of 1 for its own bundle, contradicting the envy-freeness of \( A \).

This completes the proof of Property 5.

**Property 6.** For any edge \( e = (v_i, v_j) \), no dummy agent is assigned both vertex goods \( v_i \) and \( v_j \) under \( A \).

**Proof.** (of Property 6) Suppose, for contradiction, that for some edge \( e = (v_i, v_j) \), a dummy agent \( d \) is assigned both \( v_i \) and \( v_j \). Property 4 implies that the utility of \( d \) for its own bundle is exactly 1. However, the utility of \( d \) for the bundle of \( d \) is 2, contradicting the envy-freeness of \( A \).

It follows from Property 5 that all vertex goods must be allocated among the dummy agents. Now consider the following coloring of the graph \( G \): For each vertex \( v \), the color of \( v \) is the index of the dummy agent that gets the vertex good \( v \). Property 6 implies that the coloring is proper. Furthermore, due to envy-freeness of \( A \), agents with identical valuations must have equal utilities. Therefore, each dummy agent gets the same number of vertex goods, implying that the coloring is equitable. This completes the proof of Proposition 3.

### 7.2 Proof of Theorem 4

Recall the statement of Theorem 4.

**Theorem 4 (Approximation algorithm).** There is a polynomial-time algorithm that, given as input any instance of HEF-k-Verification, finds a set \( S \subseteq [m] \) with \( |S| \leq k^{\text{opt}} \cdot \ln E + 1 \) such that the given allocation is HEF with respect to \( S \). Here, \( E \) and \( k^{\text{opt}} \) denote the aggregate envy and the number of goods that must be hidden under the given allocation, respectively.

Recall from Section 4 that given any allocation \( A \), the residual envy function \( f : 2^{[m]} \to \mathbb{R} \) is defined as follows:

\[
f(S) := \sum_{h \in [n]} \sum_{i \neq h} \max\{0, v_i(A_h \setminus S) - v_i(A_h)\}.
\]

Here, \( f(S) \) is the aggregate envy in \( A \) after hiding the goods in \( S \subseteq [m] \). We will show in Lemma 1 that \( f \) is supermodular, i.e., for any pair of sets \( S, T \subseteq [m] \) such that \( S \subseteq T \) and any good \( j \notin T \), \( f(S) - f(S \cup \{j\}) \geq f(T) - f(T \cup \{j\}) \). The proof of Theorem 4 will then follow from the standard greedy algorithm for submodular maximization, or, equivalently, supermodular minimization (Nemhauser et al., 1978).
Lemma 1. The residual envy function \( f \) is supermodular.

Proof. We will start with the necessary notation. For any agent \( h \in [n] \) and any other agent \( i \in [n] \setminus \{h\} \), define \( f_{h,i}(S) := \max\{0, v_i(A_h \setminus S) - v_i(A_i)\} \) as the envy of \( i \) towards \( h \) after hiding the goods in \( S \). Also, let \( f_h(S) := \sum_{i \neq h} f_{h,i}(S) \) denote the total (aggregate) envy towards \( h \). We therefore have \( f(S) = \sum_{h \in [n]} f_h(S) = \sum_{h \in [n]} \sum_{i \neq h} f_{h,i}(S) \).

Notice that \( f_{h,i}(S) \) is a monotone non-increasing set function, i.e., for any \( r \), \( f(r) \geq f(r \setminus \{j\}) \). By a similar reasoning for the set \( r \), we have that \( i \notin E_h(S) \), then additivity of valuations implies \( v_i(A_r \setminus S) = v_i(A_r \setminus S \cup \{j\}) \). Thus,

\[
    f(S) = f(S \cup \{j\}) = f_r(S) - f_r(S \cup \{j\}) = \sum_{i \neq r} f_{r,i}(S) - f_{r,i}(S \cup \{j\}) = \sum_{i \in E_r(S)} f_{r,i}(S) - f_{r,i}(S \cup \{j\}) = \sum_{i \in E_r(S) \cap N_j} f_{r,i}(S) - f_{r,i}(S \cup \{j\}),
\]

where the first equality uses the fact that for any \( h \neq r \), we have \( f_h(S) = f_h(S \cup \{j\}) \), the third equality uses the fact that \( i \notin E_r(S) \), then \( f_{r,i}(S) = f_{r,i}(S \cup \{j\}) = 0 \), and the fourth equality uses the fact that \( v_i(A_r \setminus S) = v_i(A_r \setminus S \cup \{j\}) \) whenever \( i \notin N_j \). By a similar reasoning for the set \( T \), we get that

\[
    f(T) - f(T \cup \{j\}) = \sum_{i \in E_r(T) \cap N_j} f_{r,i}(T) - f_{r,i}(T \cup \{j\}).
\]

Recall that \( E_r(T) \subseteq E_r(S) \). Therefore, Equation (2) can be rewritten as

\[
    f(S) - f(S \cup \{j\}) = \sum_{i \in E_r(T) \cap N_j} f_{r,i}(S) - f_{r,i}(S \cup \{j\}) + \sum_{i \in E_r(S) \setminus E_r(T) \cap N_j} f_{r,i}(S) - f_{r,i}(S \cup \{j\}) \geq \sum_{i \in E_r(T) \cap N_j} f_{r,i}(S) - f_{r,i}(S \cup \{j\}),
\]

where the inequality follows from the use of the monotonicity of \( f_{r,i} \) for all \( i \in E_r(S) \setminus E_r(T) \cap N_j \).

Therefore, from Equations (3) and (4), it suffices to show that for every \( i \in E_r(T) \cap N_j \), \( f_{r,i}(S) - f_{r,i}(S \cup \{j\}) \geq 0 \). We will prove this by contradiction.

Suppose, for contradiction, that for some \( i \in E_r(T) \cap N_j \), we have \( f_{r,i}(S) - f_{r,i}(S \cup \{j\}) < 0 \). Then, we must have \( i \in E_r(S \cup \{j\}) \), since otherwise we get \( i \notin E_r(T \cup \{j\}) \) and therefore \( f_{r,i}(S \cup \{j\}) = f_{r,i}(T \cup \{j\}) = 0 \). This would imply that \( f_{r,i}(S) < f_{r,i}(T) \), which contradicts the monotonicity of \( f_{r,i} \). Hence, for any \( i \in E_r(T) \cap N_j \), we also have that \( i \in E_r(S \cup \{j\}) \).

Notice that for any \( i \in E_r(S \cup \{j\}) \cap N_j \), we have \( f_{r,i}(S) - f_{r,i}(S \cup \{j\}) = v_{i,j} \) by the additivity of valuations. However, this would require that \( f_{r,i}(T) - f_{r,i}(T \cup \{j\}) > v_{i,j} \), which is a contradiction. Therefore, the function \( f \) must be supermodular.

We are now ready to prove Theorem 4.

Proof. (of Theorem 4) Note that allocation \( A \) is HEF with respect to a set \( S \) if and only if \( f(S) \leq 0 \). For integral valuations, \( f(S) \leq 0 \) if and only if \( f(S) < 0 \). Therefore, it suffices to compute a set \( S \) in polynomial time such that \( |S| \leq k_{\text{opt}} \cdot \ln E + 1 \) and \( f(S) < 0 \).
**Algorithm 1**: Greedy Approximation Algorithm for HEF-$k$-Verification

**Input**: An instance $\langle [n], [m], V \rangle$ and an allocation $A$.

**Output**: A set $S \subseteq [m]$.

1. Initialize $S = \emptyset$.
2. while $f(S) \geq 1$ do
   3. Set $j' \leftarrow \arg \max_{j \in [m] \setminus S} f(S) - f(S \cup \{j\})$  \hspace{1cm} \text{tiebreak lexicographically}
   4. Update $S \leftarrow S \cup \{j'\}$
3. return $S$

Consider the greedy algorithm described in Algorithm 1. At each step, the algorithm adds to the current set the good that provides the largest reduction in the residual envy. This process is continued as long as $f(S) \geq 1$. Since there are $m$ goods, it is clear that the algorithm terminates in at most $m$ steps. Furthermore, from the above observation, it follows that the allocation $A$ is HEF with respect to the set $S$ returned by the algorithm. Therefore, all that remains to be shown is a bound on $|S|$.

Observe that $f(\emptyset) = E$. Recall from the proof of Lemma 1 that $f$ is a sum of monotone non-increasing set functions, and is therefore itself monotone non-increasing. Define another set function $g : 2^{[m]} \to \mathbb{R}$ as follows:

$$g(S) := E - f(S).$$

Notice that $g$ is a non-negative, monotone non-decreasing, and integer-valued submodular function with $g(\emptyset) = 0$. Therefore, our goal is to find a set $S$ such that $g(S) > E - 1$.

We will now use the result of Nemhauser et al. (1978) for submodular maximization stated below as Proposition 4. In particular, let $p := k_{\text{opt}}$ be the size of the optimal hidden set (i.e., the number of goods that must be hidden under $A$). Then,

$$\max_{S: |S| \leq p} g(S) = E - \min_{S: |S| \leq k_{\text{opt}}} f(S) = E.$$

From the bound in Proposition 4, we have that

$$(1 - e^{-q/p}) \max_{S: |S| \leq p} g(S) > E - 1$$
$$\iff (1 - e^{-q/k_{\text{opt}}}) \cdot E > E - 1$$
$$\iff 1 - e^{-q/k_{\text{opt}}} > 1 - 1/E$$
$$\iff \ln \frac{1}{E} > -q/k_{\text{opt}}$$
$$\iff q > k_{\text{opt}} \ln E.$$

Thus, after $q > k_{\text{opt}} \ln E$ steps, any set $S$ constructed by the algorithm satisfies $g(S) > E - 1$, or, equivalently, $f(S) < 1$, giving us the desired bound $|S| \leq k_{\text{opt}} \ln E + 1$. This completes the proof of Theorem 4. □

**Proposition 4** (Nemhauser et al. (1978), Krause and Golovin (2014)). Let $g : 2^{[m]} \to \mathbb{R}_{\geq 0}$ be a monotone non-decreasing submodular function, and let $\{S_i\}_{i \geq 0}$ be the sequence of sets constructed in Algorithm 1. Then, for any positive integers $p$ and $q$, we have that

$$g(S_q) \geq (1 - e^{-q/p}) \max_{S: |S| \leq p} g(S).$$
|     | $g_1$ | $g_2$ | $g_3$ | $g_4$ | $g_5$ |
|-----|-------|-------|-------|-------|-------|
| $a_1$ | 1     | 0     | 0     | 0     | 0     |
| $a_2$ | 10    | 1     | 0     | 0     | 0     |
| $a_3$ | 0     | 10    | 1     | 0     | 0     |
| $a_4$ | 0     | 0     | 10    | 1     | 0     |
| $a_5$ | 0     | 0     | 0     | 10    | 1     |

Table 3: The instance used in proof of Proposition 5.

### 7.3 MNW can have large regret in the worst-case

This section presents two results concerning the worst-case regret of MNW solution. In Proposition 5, we will provide a family of instances for which the normalized regret of MNW approaches 1 (i.e., the maximum possible value). In Proposition 6, we will show a slightly weaker limit ($1/2$ instead of 1) that holds even for the restricted domain of binary valuations. We will use $\kappa^{\text{opt}}(I) := \min_A \kappa(A, I)$ to denote the smallest number of goods that must be hidden under any allocation in the instance $I$.

**Proposition 5.** There exists a family of instances for which the normalized regret of any Nash optimal allocation approaches 1 in the limit.

**Proof.** Consider the fair division instance $I$ with five agents $a_1, \ldots, a_5$ and five goods $g_1, \ldots, g_5$ shown in Table 3. The unique Nash optimal allocation (say $A$) for this instance assigns $g_i$ to $a_i$ for every $i \in [5]$. Thus, the goods $g_1, g_2, g_3, g_4$ must be hidden under $A$, i.e., $\kappa(A, I) = 4$. On the other hand, an allocation (say $B$) that assigns $g_5$ to $a_1$, and $g_{i-1}$ to $a_i$ for every $i \in \{2, \ldots, 5\}$ only needs to hide the good $g_1$. Indeed, $\kappa^{\text{opt}}(I) = 1$ since any allocation must hide $g_1$ to avoid envy from $a_1$ or $a_2$. The desired family of instances is the natural extension of the above example to $n$ agents and $n$ goods. In the limit, the normalized regret of the Nash optimal allocation is $\lim_{n \to \infty} \frac{(n-1)\cdot 1 - 1}{n-1} = 1$. \(\square\)

**Proposition 6.** There exists a family of instances with binary valuations for which the normalized regret of any Nash optimal allocation approaches $1/2$ in the limit.

**Proof.** Fix some $t \in \mathbb{N}$. Consider an instance $I_n$ with $2t + 1$ agents, consisting of $t$ groups of ordinary agents $\{a_i, b_i\}_{i \in [t]}$ and one special agent $s$. The goods are also classified into $t$ groups, with group $i$ comprising of five goods $g_{i,1}, \ldots, g_{i,5}$. For each $i \in [t]$, both $a_i$ and $b_i$ approve all five goods in group $i$ and have zero value for all the other goods. The special agent $s$ approves all the goods.

The above instance admits an envy-free allocation $A$ in which $s$ gets one good from each group, and the other goods are allocated evenly among the group members. That is, for each $i \in [t]$, $a_i$ gets $\{g_{i,1}, g_{i,2}\}$, $b_i$ gets $\{g_{i,3}, g_{i,4}\}$, and $s$ gets $g_{i,5}$. Thus, $\kappa^{\text{opt}}(I_n) = 0$.

Let $B$ denote any Nash optimal allocation. It is easy to see that $B$ is one of the following two canonical forms:

- Either $s$ gets two goods from two different groups and the rest of the goods are assigned ‘evenly,’ i.e., for each $i \in [t - 2]$, $a_i$ gets $\{g_{i,1}, g_{i,2}, g_{i,3}\}$ and $b_i$ gets $\{g_{i,4}, g_{i,5}\}$, and for $i \in \{t - 1, t\}$, $a_i$ gets $\{g_{i,1}, g_{i,2}\}$, $b_i$ gets $\{g_{i,3}, g_{i,4}\}$ and $s$ gets $g_{i,5}$,

- or, $s$ gets three goods from three different groups and the other goods are assigned ‘evenly,’ i.e., for each $i \in [t - 3]$, $a_i$ gets $\{g_{i,1}, g_{i,2}, g_{i,3}\}$ and $b_i$ gets $\{g_{i,4}, g_{i,5}\}$, and for $i \in \{t - 2, t - 1, t\}$, $a_i$ gets $\{g_{i,1}, g_{i,2}\}$, $b_i$ gets $\{g_{i,3}, g_{i,4}\}$ and $s$ gets $g_{i,5}$.

Either way, $B$ must hide at least $t - 3$ goods (one good in each of the groups $1, \ldots, t - 3$ to avoid envy from $b_i$). Thus, $\reg(B, I_n) = \kappa(B, I_n) = t - 3$.

The desired family of instances can be obtained by choosing an arbitrarily large $t$. In the limit, the normalized regret of $B$ is $\lim_{t \to \infty} \frac{t-3}{2t} = \frac{1}{2}$. \(\square\)
### Normalized worst-case regret

| Algorithm          | Regret  |
|--------------------|---------|
| Alg-EF1+PO         | 0.6     |
| RoundRobin         | 0.2     |
| MNW                | 0.8     |
| EnvyGraph          | 0.9     |

### Frequency of envy-freeness

| Algorithm          | Frequency |
|--------------------|-----------|
| Alg-EF1+PO         | 0.3       |
| RoundRobin         | 0.8       |
| MNW                | 0.5       |
| EnvyGraph          | 0.2       |

### Number of goods that must be hidden in the worst-case (max over all 100 instances)

| Algorithm          | Number of Goods |
|--------------------|-----------------|
| Alg-EF1+PO         | 10              |
| RoundRobin         | 15              |
| MNW                | 20              |
| EnvyGraph          | 5               |

### Table 4: Comparing various EF1 algorithms over synthetically generated binary instances with \( v_{i,j} \sim \text{Ber}(0.7) \) i.i.d.

| Algorithm          | Agents  |
|--------------------|---------|
| Alg-EF1+PO         | 5       |
| RoundRobin         | 10      |
| MNW                | 15      |
| EnvyGraph          | 20      |

### Figure 3: Distribution of the Spliddit data. The color of each cell denotes the number of instances in the dataset with the corresponding number of goods, \( m \), on the X axis, and number of agents, \( n \), on the Y axis.

### 7.4 Additional Experimental Results

Table 4 presents additional results for the synthetic data used in Section 5.1 (i.e., binary valuations with \( v_{i,j} \sim \text{Ber}(0.7) \) i.i.d.). This time, we compare the algorithms in terms of their (a) normalized worst-case regret (over the 100 instances), (b) the frequency with which the algorithms compute envy-free outcomes, and (c) the worst-case number of goods that must be hidden by each algorithm. The trend is similar to that in Section 5.1, with Alg-EF1+PO and RoundRobin outperforming MNW and EnvyGraph. Table 5 presents similar results for Bernoulli parameter 0.5. Finally, Figure 3 illustrates the distribution of the Spliddit data. As can be seen, a large fraction of instances have between 3 and 6 agents and between 3 and 15 goods, with a sharp spike at \( n = 3 \) and \( m = 6 \).
Table 5: Comparing various EF1 algorithms over synthetically generated binary instances with $v_{i,j} \sim \text{Ber}(0.5)$ i.i.d.