A generalized solution procedure for in-plane free vibration of rectangular plates and annular sectorial plates

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A generalized solution procedure is developed for in-plane free vibration of rectangular and annular sectorial plates with general boundary conditions. For the annular sectorial plate, the introduction of a logarithmic radial variable simplifies the basic theory and the expression of the total energy. The coordinates, geometric parameters and potential energy for the two different shapes are organized in a unified framework such that a generalized solving procedure becomes feasible. By using the improved Fourier–Ritz approach, the admissible functions are formulated in trigonometric form, which allows the explicit assembly of global mass and stiffness matrices for both rectangular and annular sectorial plates, thereby making the method computationally effective, especially when analysing annular sectorial plates. Moreover, the improved Fourier expansion eliminates the potential discontinuity of the original normal and tangential displacement functions and their derivatives in the entire domain, and accelerates the convergence. The generalized Fourier–Ritz approach for both shapes has the characteristics of generality, accuracy and efficiency. These features are demonstrated via a few numerical examples.

1. Introduction

The in-plane vibration of built-up structures is found to have a significant effect on the sound radiation and transmission of vibration energies [1,2]. In-plane vibration analysis is also important when inspecting the hulls of ships under the impacts...
of boundary flow, and similarly for studying the dynamic behaviours of composite shells of an aeroplane flying through turbulence [3]. Therefore, a better understanding in the in-plane vibration behaviours of plates is important in the design of similar structures. For the in-plane vibration of plate structures, several analytical solutions are developed, e.g. the variational method by Kantorovich–Krylov in [4], the superposition method by Gorman [5], the direct separation of variables and eigenvalue-problem approach by Xing and Liu [3,6], the strong form of the governing equation solved via a two-dimensional improved Fourier series by Du et al. [7] and the Ritz method based on a set of trigonometric functions by Dozio [8], just to name a few. Bardell et al. [9] presented, for the first time, the in-plane frequencies of rectangular plates under many combined boundary conditions. Some new methods have also been presented recently: for example, the dynamic stiffness method by Nefovska-Danilovic & Petronijevic [10], and the solution for free vibration of thin rectangular plates with elastic boundary and internal line supports based on improved Fourier–Ritz method by Shi et al. [11]. For a circular disk, Onoe [12,13] presented an exact solution on the basis of Love’s theory. Chen & Liu [14] proposed a general solution of the governing differential equations for thin plates in different shapes with boundary conditions satisfied in a least-square sense. Holland [15] as well as Farag & Pan [16] adopted the trigonometric and Bessel functions to study in-plane vibration of circular plates. A few different methods are also proposed: the transfer matrix method by Irie et al. [17], the generalized Rayleigh–Ritz method by Bashmal et al. [18], Hamilton’s principle by Park [19], the variational approximation procedure by Seok & Tiersten [20] and the stress–strain–displacement expressions by Ravari & Forouzan [21]. Vladimir et al. [22] applied the potential theory to study the free in-plane vibration of rectangular, annular and circular plates. Kim et al. [23] discussed the in-plane vibration of a circular plate based on the assumption that the mode shapes are dependent on the number of nodal diameters. Singh & Muhammad [24] presented a numerical method to study the free in-plane vibration of the isotropic non-rectangular plate, in which the plate is meshed and the displacement field and the coordinate field are interpolated separately. Wang et al. [25] used a modified Fourier–Ritz approach [26,27] to solve the free in-plane vibration of orthotropic circular, annular and sectorial plates subjected to general boundary conditions. Recently, the modified Fourier series technique has been extended to study the in-plane vibration of plate and shell structures with general boundary conditions by the modified Ritz method [25,28–31].

However, in this large volume of literature, plates of different shapes are always treated separately and solved by different approaches. No generalized solution has been presented for both rectangular and circular shapes. In this work, the basic model for rectangular plates is briefly reviewed, then the basic theory for annular sectorial plates is modified by introducing a logarithmic radial variable, the theories are then formulated in a unified framework to account for both rectangular and annular sectorial plates. A few numerical examples are presented to demonstrate the versatility of the generalized approach.

2. Theoretical formulations

2.1. Orthotropic rectangular plates

The Rayleigh–Ritz method combined with the artificial spring technique [11] is briefly introduced here for analysing in-plane vibration problems of orthotropic rectangular plates.

Consider an orthotropic plate with length $a$ and width $b$ as shown in Figure 1. On all the sides of the plate two groups of boundary elastic springs are arranged along the normal and tangential directions, to simulate the boundary conditions. By assigning the stiffness of the boundary springs with various values, we can impose different boundary conditions on the mid-surface of the plate edges. For the orthotropic rectangular plate, based on the strain–stress relationship, the boundary conditions can be expressed as

$$
k_{xx}^{U}u = (-1)^{i} \left( A_{xx} \frac{\partial u}{\partial x} + A_{xy} \frac{\partial v}{\partial y} \right),
k_{yy}^{V}v = (-1)^{i} G_{xy} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \quad (i = 0, 1) \quad (2.1)
$$

and

$$
k_{yy}^{U}u = (-1)^{i} \left( A_{yy} \frac{\partial u}{\partial y} + A_{xy} \frac{\partial v}{\partial x} \right),
k_{xx}^{V}v = (-1)^{i} G_{xy} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \quad (i = 0, 1), \quad (2.2)
$$

where $A_{xx} = E_{x}/1 - \mu_{x}\mu_{y}$, $A_{yy} = E_{y}/1 - \mu_{y}\mu_{x}$, $A_{xy} = A_{yx} = \mu_{x}E_{y}/1 - \mu_{x}\mu_{y}$. $E_{x}$ and $E_{y}$ are the in-plane stretch stiffness, $G_{xy}$ is the shear Young’s modulus, $E_{z}$ and $\mu_{z}$ ($z = x$ or $y$) are Young’s moduli and Poisson’s ratio in the $x$ and $y$ directions, respectively. The parameter $k_{ij}^{\delta}$ stands for the attached spring stiffness, with its superscript $\delta = U, V$ indicating the $x$ and $y$ directions and the subscript $\gamma = x0, y0, x1, y1$ referring to the left, bottom, right, and top edges of the corresponding spring, respectively. For example, $k_{x0}^{U}$ denotes the spring stiffness in the $x$ direction along the edge at $x = 0$. A clamped boundary can be readily obtained...
Figure 1. An orthotropic rectangular plate with arbitrary in-plane elastic supports.

by setting the spring coefficients to infinity for both the normal and tangential restraining springs. The total potential energy of the plate, consisting of the strain energy of the plate and the potential energy stored in the boundary springs, can be expressed as:

\[ V = \frac{h}{2} \int_{0}^{a} \int_{0}^{b} \left[ A_{xx} \left( \frac{\partial u}{\partial x} \right)^2 + A_{yy} \left( \frac{\partial v}{\partial y} \right)^2 + 2A_{xy} \left( \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} + G_{xy} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 \right) \right] \, dx \, dy \]

\[ + \frac{1}{2} \int_{0}^{a} \left[ (k_{xx}u^2 + k_{xy}v^2) \right]_{x=0} + \left[ (k_{xx}u^2 + k_{xy}v^2) \right]_{x=a} \, dy \]

\[ + \frac{1}{2} \int_{0}^{b} \left[ (k_{yy}u^2 + k_{yy}v^2) \right]_{y=0} + \left[ (k_{yy}u^2 + k_{yy}v^2) \right]_{y=b} \, dx. \]  

(2.3)

The kinetic energy \( T \) of the plate is expressed as:

\[ T = \rho \frac{h}{2} \int_{0}^{a} \int_{0}^{b} \left[ \left( \frac{\partial u}{\partial t} \right)^2 + \left( \frac{\partial v}{\partial t} \right)^2 \right] \, dx \, dy. \]  

(2.4)

where \( \rho \) is the plate mass density. Considering an harmonic motion with frequency \( \omega \), i.e.

\[ u(x, y, t) = u(x, y)e^{\omega t} = u e^{j\omega t} \]

and

\[ v(x, y, t) = v(x, y)e^{\omega t} = v e^{j\omega t}, \]

(2.5)

the maximum strain energy \( V_{\text{max}} \) and the maximum kinetic energy \( T_{\text{max}} \) for the plate are

\[ V_{\text{max}} = \frac{h}{2} \int_{0}^{a} \int_{0}^{b} \left[ A_{xx} \left( \frac{\partial u}{\partial x} \right)^2 + A_{yy} \left( \frac{\partial v}{\partial y} \right)^2 + 2A_{xy} \left( \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} + G_{xy} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 \right) \right] \, dx \, dy \]

\[ + \frac{1}{2} \int_{0}^{a} \left[ (k_{xx}u^2 + k_{xy}v^2) \right]_{x=0} + \left[ (k_{xx}u^2 + k_{xy}v^2) \right]_{x=a} \, dy \]

\[ + \frac{1}{2} \int_{0}^{b} \left[ (k_{yy}u^2 + k_{yy}v^2) \right]_{y=0} + \left[ (k_{yy}u^2 + k_{yy}v^2) \right]_{y=b} \, dx. \]  

(2.6)

and

\[ T_{\text{max}} = \frac{\rho h \omega^2}{2} \int_{0}^{a} \int_{0}^{b} (u^2 + v^2) \, dx \, dy, \]  

(2.7)

respectively. By using the Ritz method, the energy function is defined by Lagrangian function as

\[ L = V - T. \]  

(2.8)

2.2. Orthotropic annular sectorial plates

Consider an orthotropic annular sectorial plate with uniform thickness \( h \), inner radius \( R_0 \), outer radius \( R_1 \), and sector angle \( \phi \) as shown figure 2a. For general supported orthotropic annular sectorial plates,
based on the force equilibrium relationship at the four edges, the boundary conditions corresponding to the elastic spring can be expressed as:

\[
k^{ii}_{ri}u = A_{rr} \frac{\partial u}{\partial r} + A_{r\theta} \frac{1}{r} \left( u + \frac{\partial v}{\partial \theta} \right), \quad k^{ii}_{ri}v = G_{r\theta} \left( \frac{1}{r} \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} \right) \quad (i = 0, 1)
\]

and

\[
k^{ii}_{ri}u = A_{\theta \theta} \frac{\partial u}{\partial \theta} + A_{\theta \theta} \frac{1}{r} \left( u + \frac{\partial v}{\partial \theta} \right), \quad k^{ii}_{ri}v = -G_{\theta \theta} \left( \frac{1}{r} \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} \right) \quad (i = 0, 1),
\]

where \( A_{rr} = E_{r}/1 - \mu_{r} \mu_{\theta} \), \( A_{r\theta} = E_{r}/1 - \mu_{r} \mu_{\theta} \), \( A_{\theta \theta} = \mu_{r} E_{\theta}/1 - \mu_{r} \mu_{\theta} \) are the in-plane stretch stiffness, \( G_{r\theta} \) is the shear Young’s modulus, \( E_{z} \) and \( \mu_{z} \) (\( z = r \) or \( \theta \)) are Young’s moduli and Poisson’s ration in the \( r \) and \( \theta \) directions of the orthotropic plate, respectively.

Arbitrary elastic support conditions are imposed by assigning massless normal and tangential springs along each edge. Similar to the case of a rectangular plate, the symbol \( k^{ii}_{ri} \) denotes the spring restraining stiffness value, with \( \delta = U, V \) denoting the \( r \) and \( \theta \) directions and \( \gamma = R_{0}, \theta_{0}, R_{1}, \theta_{1} \) referring to the inner, bottom, outer, and upper edges, respectively.

The strain energy \( V_{pl} \) of the sectorial plate is written as

\[
V_{pl} = \frac{h}{2} \int_{R_{0}}^{R_{1}} \int_{0}^{\theta_{1}} \left[ A_{rr} \left( \frac{\partial u}{\partial r} \right)^{2} + A_{\theta \theta} \left( \frac{\partial u}{\partial \theta} \right)^{2} + 2A_{r\theta} \frac{\partial u}{\partial r} \left( \frac{\partial v}{\partial \theta} - u \right) + G_{r\theta} \left( \frac{1}{r} \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} \right)^{2} \right] rdrd\theta.
\]

The potential energy \( V_{s} \) stored in the boundary springs is obtained as

\[
V_{s} = \frac{1}{2} \int_{0}^{\theta_{0}} \left[ R_{0}(h_{1}^{U}u_{2}^{2} + h_{1}^{V}v_{2}^{2})]_{\rho = \text{inlet}} + R_{1}(h_{1}^{U}u_{2}^{2} + h_{1}^{V}v_{2}^{2})]_{\rho = \text{outlet}} \right] d\theta
+ \frac{1}{2} \int_{R_{0}}^{R_{1}} \left[ (h_{0}^{U}u_{2}^{2} + h_{0}^{V}v_{2}^{2})]_{\theta = 0} + (h_{1}^{U}u_{2}^{2} + h_{1}^{V}v_{2}^{2})]_{\theta = \theta_{1}} \right] rdr.
\]

The total potential energy is

\[
V = \frac{h}{2} \int_{R_{0}}^{R_{1}} \int_{0}^{\theta_{1}} \left[ A_{rr} \left( \frac{\partial u}{\partial r} \right)^{2} + A_{\theta \theta} \left( \frac{\partial v}{\partial \theta} \right)^{2} + 2A_{r\theta} \frac{\partial u}{\partial r} \left( \frac{\partial v}{\partial \theta} - u \right) + G_{r\theta} \left( \frac{1}{r} \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} \right)^{2} \right] rdrd\theta
+ \frac{1}{2} \int_{0}^{\theta_{0}} \left[ R_{0}(h_{1}^{U}u_{2}^{2} + h_{1}^{V}v_{2}^{2})]_{\rho = \text{inlet}} + R_{1}(h_{1}^{U}u_{2}^{2} + h_{1}^{V}v_{2}^{2})]_{\rho = \text{outlet}} \right] d\theta
+ \frac{1}{2} \int_{R_{0}}^{R_{1}} \left[ (h_{0}^{U}u_{2}^{2} + h_{0}^{V}v_{2}^{2})]_{\theta = 0} + (h_{1}^{U}u_{2}^{2} + h_{1}^{V}v_{2}^{2})]_{\theta = \theta_{1}} \right] rdr.
\]

The kinetic energy \( T \) is expressed as:

\[
T = \frac{ph}{2} \int_{R_{0}}^{R_{1}} \int_{0}^{\theta_{1}} \left[ \left( \frac{\partial u}{\partial t} \right)^{2} + \left( \frac{\partial v}{\partial t} \right)^{2} \right] rdrd\theta,
\]
where $\rho$ is the plate mass density. Considering an harmonic motion with frequency $\omega$, i.e.

$$u(r, \theta, t) = \bar{u}(r, \theta)e^{i\omega t} = \tilde{u}e^{i\omega t}$$

and

$$v(r, \theta, t) = v(r, \theta)e^{i\omega t} = \tilde{v}e^{i\omega t},$$

the maximum strain energy $V_{\text{max}}$ and the maximum kinetic energy $T_{\text{max}}$ for the plate are

$$V_{\text{max}} = \frac{h}{2} \int_{R_0}^{R_1} \left[ A_{rr} \left( \frac{\partial \tilde{u}}{\partial r} \right)^2 + A_{\theta \theta} \left( \frac{\partial \tilde{u}}{\partial \theta} \right)^2 + 2A_{r\theta} \frac{\partial \tilde{u}}{\partial r} \frac{\partial \tilde{v}}{\partial \theta} + \tilde{u} \right] 2\pi r dr d\theta + \frac{1}{2} \int_0^\phi \left[ R_0(k_{r0}^l \tilde{v}^2 + k_{\theta0}^l \tilde{u}^2)_{\xi=R_0} + R_1(k_{r1}^l \tilde{v}^2 + k_{\theta1}^l \tilde{u}^2)_{\xi=R_1} \right] d\theta$$

$$+ \frac{1}{2} \int_{R_0}^{R_1} \left[ (k_{r0}^l \tilde{u}^2 + k_{\theta0}^l \tilde{v}^2)_{\theta=0} + (k_{r1}^l \tilde{u}^2 + k_{\theta1}^l \tilde{v}^2)_{\theta=\phi} \right] dr$$

and

$$T_{\text{max}} = \frac{\rho h \omega^2}{2} \int_{R_0}^{R_1} \left( \tilde{u}^2 + \tilde{v}^2 \right) 2\pi r dr d\theta,$$

respectively.

To simplify the expressions in equation (2.16), a logarithmic radial variable is introduced based on the work of Yao et al. [32]

$$\zeta = \ln \left( \frac{r}{R_0} \right).$$

Figure 2b shows the illustration of the plate using the logarithmic radial variable. The following relationships can be obtained

$$\frac{dr}{dz} = \frac{1}{r} \frac{dS(r, \theta)}{dz},$$

where $S(r, \theta)$ stands for an arbitrary function varying with $r$ and $\theta$, and $S_1(\zeta, \theta)$ is the function $S(r, \theta)$ written in terms of $\zeta$ and $\theta$. Substituting equation (2.19) into equations (2.13) and (2.14), we have

$$V_{\text{max}} = \frac{h}{2} \int_{L_R}^{L_R} \left[ A_{rr} \left( \frac{\partial \tilde{u}}{\partial \zeta} \right)^2 + A_{\theta \theta} \left( \frac{\partial \tilde{u}}{\partial \theta} \right)^2 + 2A_{r\theta} \frac{\partial \tilde{u}}{\partial \zeta} \frac{\partial \tilde{v}}{\partial \theta} + \tilde{u} \right] 2\pi \zeta d\zeta d\theta + \frac{1}{2} \int_0^\phi \left[ R_0(k_{r0}^l \tilde{v}^2 + k_{\theta0}^l \tilde{u}^2)_{\zeta=0} + R_1(k_{r1}^l \tilde{v}^2 + k_{\theta1}^l \tilde{u}^2)_{\zeta=\phi} \right] d\theta$$

$$+ \frac{1}{2} \int_0^\phi \left[ R_0(k_{r0}^l \tilde{u}^2 + k_{\theta0}^l \tilde{v}^2)_{\theta=0} + R_1(k_{r1}^l \tilde{u}^2 + k_{\theta1}^l \tilde{v}^2)_{\theta=\phi} \right] d\zeta$$

and

$$T_{\text{max}} = \frac{\rho h \omega^2}{2} \int_0^{L_R} \left( \tilde{u}^2 + \tilde{v}^2 \right) e^{2\phi} d\zeta d\theta,$$

respectively, where $L_R = \ln(R_1/R_0)$.

### 2.3. Generalized theories for rectangular and annular sectorial plates

Similarities between the theories for rectangular and annular sectorial plates are summarized in this section. First, we formulate the unified expressions for the potential energy and the kinematic energy as

$$V_{\text{max}} = \frac{h}{2} \int_0^{\xi_{\text{max}}} \int_0^{\eta_{\text{max}}} \left[ A_{11} \left( \frac{\partial \tilde{u}}{\partial \xi} \right)^2 + A_{22} \left( \frac{\partial \tilde{v}}{\partial \eta} \right)^2 + 2A_{12} \frac{\partial \tilde{u}}{\partial \xi} \frac{\partial \tilde{v}}{\partial \eta} + A_{0} \left( \frac{\partial \tilde{u}}{\partial \xi} + \frac{\partial \tilde{v}}{\partial \eta} - \tilde{v} \right)^2 \right] d\xi d\eta$$

$$+ \frac{1}{2} \int_0^{\xi_{\text{max}}} \left[ e^{\text{ln} R_0(k_{r0}^l \tilde{v}^2 + k_{\theta0}^l \tilde{u}^2)}_{\xi=0} + e^{\text{ln} R_1(k_{r1}^l \tilde{v}^2 + k_{\theta1}^l \tilde{u}^2)}_{\xi=\phi} \right] d\eta$$

$$+ \frac{1}{2} R_0 \int_0^{\eta_{\text{max}}} \left[ (k_{r0}^l \tilde{u}^2 + k_{\theta0}^l \tilde{v}^2)_{\theta=0} + (k_{r1}^l \tilde{u}^2 + k_{\theta1}^l \tilde{v}^2)_{\theta=\phi} \right] e^{2\phi} d\xi$$

and

$$T_{\text{max}} = \frac{\rho h \omega^2}{2} \int_0^{\xi_{\text{max}}} \int_0^{\eta_{\text{max}}} \left( \tilde{u}^2 + \tilde{v}^2 \right) e^{2\phi} d\xi d\eta,$$
respectively, where the shape parameter $p$ is defined as

$$p = 0 \quad \text{(for rectangular plate)}$$

$$p = 1 \quad \text{(for annular sectorial plate).}$$

The material parameters are written in generalized forms as $A_{11}, A_{12}, A_{22}, G_{12}, E_1, E_2, \mu_1, \mu_2$ and the maximum lengths in $\xi$ and $\eta$ directions are $\xi_{\text{max}}$ and $\eta_{\text{max}}$, respectively. Table 1 shows the corresponding variables for rectangular and annular sectorial shapes, respectively.

The displacements can be written as

$$u(\xi, \eta) = \sum_{m=1}^{M} \sum_{n=1}^{N} a_{mn} f_m(\xi) g_n(\eta)$$

and

$$v(\xi, \eta) = \sum_{m=1}^{M} \sum_{n=1}^{N} b_{mn} f_m(\xi) g_n(\eta),$$

where $a_{mn}$ and $b_{mn}$ are unknown coefficients, $f_m(\xi)$ and $g_n(\eta)$ are appropriate admissible functions, and $M$ and $N$ are the numbers of truncated terms in the series expansion. The proposed solution can be of arbitrary precision depending on the number of terms used in the series. In this work, the simple trigonometric series for constructing the two-direction displacements are selected as

$$f_m(\xi) = \begin{cases} \sin\left(\frac{m\pi}{\xi_{\text{max}}}\right), & m = 1, 2 \\ \cos\left(\frac{(m-3)\pi}{\xi_{\text{max}}}\right), & m \geq 3 \end{cases}$$

and

$$g_n(\eta) = \begin{cases} \sin\left(\frac{n\pi}{\eta_{\text{max}}}\right), & n = 1, 2 \\ \cos\left(\frac{(n-3)\pi}{\eta_{\text{max}}}\right), & n \geq 3 \end{cases}$$

Substitution of equation (2.25) into equations (2.22) and (2.23), and minimizing the energy function $L = U_{\text{max}} - T_{\text{max}}$ with respect to the unknown coefficients $a_{mn}$ and $b_{mn}$ yield the following eigenvalue equations

$$\sum_{r=1}^{M} \sum_{s=1}^{N} \left[ K_{mnrs}^{uu} a_r + K_{mnrs}^{uv} b_r - \omega^2 M_{mnrs} a_r \right] = 0$$

and

$$\sum_{r=1}^{M} \sum_{s=1}^{N} \left[ K_{mnrs}^{uu} a_r + K_{mnrs}^{uv} b_r - \omega^2 M_{mnrs} b_r \right] = 0.$$
where the elements of the global stiffness matrix \( K_{mrr}^{uu}, K_{mrr}^{uv}, K_{mrr}^{vu}, K_{mrr}^{vv} \) are given by

\[
K_{mrr}^{uu} = A_{11} I_{mr}^{11} + G_{12} I_{mr}^{11} + p A_{12} I_{mr}^{10} + p A_{22} I_{mr}^{10} \left( I_{mr}^{10} + I_{mr}^{01} \right) + e^{\gamma \ln R_0} \left( E_{mr}^{01} + E_{mr}^{10} \right) + e^{\gamma \ln R_1} f_{mr}^{01} f_{mr}^{10} I_{mr}^{11} \right) + e^{\gamma \ln R_1} f_{mr}^{11} f_{mr}^{10} I_{mr}^{10} \right)
\]

\[
K_{mrr}^{uv} = p A_{22} I_{mr}^{10} + A_{12} I_{mr}^{01} + G_{12} (I_{mr}^{10} - p I_{mr}^{01}) \right)
\]

\[
K_{mrr}^{vu} = A_{22} I_{mr}^{10} + G_{12} (p I_{mr}^{01} - p I_{mr}^{10} + I_{mr}^{10}) \right) + e^{\gamma \ln R_0} (E_{mr}^{01} + E_{mr}^{10}) + e^{\gamma \ln R_1} f_{mr}^{11} f_{mr}^{10} I_{mr}^{11} \left)
\]

(2.28)

and

where \( \rho^2 \) is replaced by \( \rho \) because \( \rho = p^2 \), and the following quantities are defined:

\[
E_{mr}^{01} = f_m(0)f_r(0), E_{mr}^{10} = g_n(0)g_r(0) \right)
\]

(2.29)

and

\[
p_{mr}^\delta = \int_0^{\xi_{\max}} \frac{d^2 f_m}{d\xi^2} f_r d\xi, I_{mr}^\delta = \int_0^{\eta_{\max}} \frac{d^2 g_n}{d\eta^2} g_r(\eta) d\eta \right)
\]

(2.30)

and

\[
P_{mr} = \int_0^{\xi_{\max}} f_m e^{2d\xi} d\xi.
\]

(2.31)

The elements of the global mass matrix \( M_{mrr}^{uu}, M_{mrr}^{uv} \) are

\[
M_{mrr}^{uu} = \rho \rho e^{2g\ln R_0} P_{mr}^{01} I_{mr}^{10} I_{mr}^{01} \right)
\]

and

\[
M_{mrr}^{vv} = \rho \rho e^{2g\ln R_1} P_{mr}^{11} I_{mr}^{10} I_{mr}^{01} \right)
\]

(2.32)

When the spring stiffness of the elastic boundary is uniform along all the boundary edges, it can be found that

\[
f_r^{\delta} = k_r^\delta I_{mr}^{01} f_r^{\delta} = k_r^\delta I_{mr}^{01} \right)
\]

(2.33)

Equation (2.27) can also be written in the matrix form:

\[
\left\{ \begin{array}{cc}
K_{uu} & K_{uv} \\
K_{vu} & K_{vv}
\end{array} \right\} \left\{ \begin{array}{c}
A^u \\
B^v
\end{array} \right\} = 0,
\]

(2.34)

where

\[
A^u = \left\{ a_{11}, a_{12}, a_{13}, \ldots, a_{1N_1}, a_{21}, \ldots, a_{1N_2}, \ldots, a_{MN} \right\}
\]

and

\[
B^v = \left\{ b_{11}, b_{12}, b_{13}, \ldots, b_{1N_1}, b_{21}, \ldots, b_{1N_2}, \ldots, b_{MN} \right\}
\]

(2.35)

Equation (2.34) corresponds to an eigenvalue problem, whose eigenvalues correspond to the frequencies of the in-plane free vibration of the plates. Substitution of the obtained eigenvectors into equation (2.25) yields the corresponding mode shapes.

In the present method, the analytic form of the integrals involved in the mass and stiffness matrix can be obtained. For the case of uniform spring stiffness, when equation (2.26) is selected as admissible functions, the formulations of all the sub-matrices in equation (2.34), such as \( K_{uu}, K_{vv}, K_{uv}, M_{uu}, M_{vv} \), can all be obtained explicitly with the aid of the basic integration formulae in the electronic supplementary material. Further, according to the form of the admissible functions selected, these matrixes can also be assembled by four block sub-matrices as

\[
Z = \left[ \begin{array}{cc}
Z_{11} & Z_{12} \\
Z_{21} & Z_{22}
\end{array} \right],
\]

(2.36)

where \( Z \) stands for \( K_{uu}, K_{vv}, K_{uv}, M_{uu}, M_{vv} \), and \( Z_{11}, Z_{12}, Z_{21}, Z_{22} \), are \( 2 \times 2, 2 \times (MN - 2), (MN - 2) \times 2, (MN - 2) \times (MN - 2) \) matrices, respectively. If the chosen admissible functions in equation (2.26) do not include the first two items, this corresponds to the case of normal Fourier expansion for the in-plane displacements, and the matrix \( Z \) degenerates to \( Z_{22} \) in equation (2.36).

When calculating the sub matrix \( Z_{ij}, (i, j = 1, 2) \) for the improved Fourier series, the base functions \( f_m(\xi), f_r(\xi), g_n(\eta) \) and \( g_r(\eta) \) are all the trigonometric functions, and therefore the orthogonal characteristics
of the Fourier series can be exploited

\[
\begin{align*}
\int_0^L \cos(\lambda_nx) \cos(\lambda_mx) dx &= 0, m \neq n, \\
\int_0^L \sin(\lambda_nx) \sin(\lambda_mx) dx &= 0, m \neq n, \\
\int_0^L \sin(\lambda_nx) \cos(\lambda_mx) dx &= 0, m = n,
\end{align*}
\]

(2.37)

and

where \(\lambda_n = m\pi/L, \lambda_m = n\pi/L\). This orthogonal property is useful in obtaining the elements of matrix \(Z\) and many of which are found to be zero.

### 3. Numerical examples and discussions

Four different materials are used in the examples, and their properties are listed in Table 2. The shear elasticity \(G_{xy}\) is defined as

\[
G_{xy} = \frac{E_y/2}{(1 - \mu_x \mu_y)} \quad \text{(for rectangular plate)}
\]

(3.1)

\[
G_{r\theta} = \frac{A_{rr}/2}{(1 - \mu_r \mu_\theta)} \quad \text{(for annular sectorial plate)}.
\]

The boundary conditions of rectangular and annular sectorial plates are denoted by a four-letter symbol, with each letter standing for the boundary condition of one edge starting from the left edge at \(\xi = 0\). For example, for a rectangular plate, the symbol \(E^1CFS^1\) denotes \(E^1\) type elastic boundary condition at \(x = 0\), clamped boundary condition at \(y = 0\), free boundary condition at \(x = a\), and \(S^1\) type simply supported at \(y = b\). For an annular sectorial plate, the same symbol denotes the corresponding boundary conditions at \(r = R_0, \theta = 0, r = R_1, \theta = \phi\), respectively. The stiffness values of the boundary spring are listed in Table 3 for the different types of boundary conditions. \(10^4\) and \(10^5\) are shown to be appropriate values for the non-dimensional spring stiffness for clamped boundary condition in rectangular and annular sectorial plates, respectively [11].

#### 3.1. In-plane vibration of rectangular plates

In the following calculations, all the terms of the Fourier series for the displacement fields are truncated into \(M = N = 10\). As far as the accuracy of the present method is concerned, the converged solutions of the present method are in excellent agreement with both the reference data and the finite-element results. Table 4 gives the normalized frequency parameter \(\Omega = \omega b (\rho / G_{xy})^{1/2} / \pi\) for rectangular plates under different boundary conditions \((E_x/E_y = 2)\), which agree very well with data from the literature. Table 5 considers the influence of different values of \(E_x/E_y\) for a \(S^2CS^2F\) orthotropic rectangular plate. The data obtained in [6,11] are also provided for comparison. All the results are in good agreement with data in previous studies.

#### 3.2. Annular sectorial plates

In this section, the convergence of the method is studied first, followed by some benchmark examples that demonstrate the excellent accuracy and reliability of the current approach, and then some computational examples using the generalized model are presented. Unless otherwise stated, the values of the following variables are used: \(R_1 = 1\) m, \(\phi = 90^\circ\), and \(h/R_1 = 0.001\).
Table 3. Non-dimensional spring stiffness values for general boundary conditions.

| shape         | boundary condition | at $\xi = \text{constant}$ | at $\eta = \text{constant}$ |
|---------------|---------------------|----------------------------|----------------------------|
|               |                     | $\Gamma_{x}$ | $\Gamma_{y}$ | $\Gamma_{x}$ | $\Gamma_{y}$ | $\Gamma_{x}$ | $\Gamma_{y}$ | $\Gamma_{x}$ | $\Gamma_{y}$ | $\Gamma_{x}$ | $\Gamma_{y}$ |
|               |                     | $\sigma_{x} = 0, \tau_{xy} = 0$ | 0 | 0 | $\sigma_{y} = 0, \tau_{xy} = 0$ | 0 | 0 |
| rectangular   | free (F)            | $u = 0, v = 0$ | 10^4 | 10^4 | $u = 0, v = 0$ | 10^4 | 10^4 |
|               | clamped (C)         | $v = 0, \sigma_{x} = 0$ | 0 | 10^4 | $u = 0, \sigma_{y} = 0$ | 0 | 10^4 |
|               | simple-support (S^1) | $u = 0, \tau_{xy} = 0$ | 10^4 | 0 | $u = 0, \tau_{xy} = 0$ | 10^4 | 0 |
|               | simple-support (S^2) | $u = 0, \tau_{xy} = 0$ | 10^4 | 0 | $u = 0, \tau_{xy} = 0$ | 10^4 | 0 |
|               | elastic 1 (E^1)     | $u = 0, \tau_{xy} = 0$ | 10^4 | 0 | $u = 0, \tau_{xy} = 0$ | 10^4 | 0 |
|               | elastic 2 (E^2)     | $u = 0, \tau_{xy} = 0$ | 10^4 | 0 | $u = 0, \tau_{xy} = 0$ | 10^4 | 0 |
|               | elastic 3 (E^3)     | $u = 0, \tau_{xy} = 0$ | 10^4 | 0 | $u = 0, \tau_{xy} = 0$ | 10^4 | 0 |
| annular sectorial | free (F)       | $\sigma_{r} = 0, \tau_{r\theta} = 0$ | 0 | 0 | $\sigma_{r} = 0, \tau_{r\theta} = 0$ | 0 | 0 |
|               | clamped (C)         | $u = 0, v = 0$ | 10^4 | 10^4 | $\sigma_{r} = 0, \tau_{r\theta} = 0$ | 10^4 | 10^4 |
|               | simple-support (S^1) | $u = 0, \sigma_{r} = 0$ | 0 | 10^4 | $\sigma_{r} = 0, \tau_{r\theta} = 0$ | 10^4 | 0 |
|               | simple-support (S^2) | $u = 0, \tau_{r\theta} = 0$ | 10^4 | 0 | $\sigma_{r} = 0, \tau_{r\theta} = 0$ | 10^4 | 0 |
|               | elastic 1 (E^1)     | $u = 0, \tau_{r\theta} = 0$ | 10^4 | 0 | $\sigma_{r} = 0, \tau_{r\theta} = 0$ | 10^4 | 0 |
|               | elastic 2 (E^2)     | $u = 0, \tau_{r\theta} = 0$ | 10^4 | 0 | $\sigma_{r} = 0, \tau_{r\theta} = 0$ | 10^4 | 0 |
|               | elastic 3 (E^3)     | $u = 0, \tau_{r\theta} = 0$ | 10^4 | 0 | $\sigma_{r} = 0, \tau_{r\theta} = 0$ | 10^4 | 0 |

Table 4. Normalized frequency parameter for square plates with various boundary conditions ($E_i/E_f = 2$).

| B.C.          | mode no. | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---------------|----------|---|---|---|---|---|---|---|---|---|
| $S^1S^2S^3C$  | 0.7070   | 1.1617 | 1.9170 | 2.0777 | 2.1209 | 2.3778 | 2.6047 | 2.8188 | 3.0500 |
|               | 0.7070   | 1.1619 | 1.9177 | 2.0779 | 2.1213 | 2.3789 | 2.6052 | 2.8200 | 3.0508 |
| $S^1S^2S^3C$  | 1.4141   | 1.5223 | 2.0081 | 2.2931 | 2.3491 | 2.8060 | 2.8281 | 2.9749 | 3.1979 |
|               | 1.4142   | 1.5228 | 2.0815 | 2.2932 | 2.3493 | 2.8060 | 2.8284 | 2.9750 | 3.1975 |
| $S^1S^2S^3C$  | 1.4142   | 1.6169 | 2.1208 | 2.3235 | 2.7838 | 2.8272 | 2.8513 | 3.0447 | 3.2114 |
|               | 1.4141   | 1.6175 | 2.1222 | 2.3238 | 2.7853 | 2.8284 | 2.8522 | 3.0457 | 3.2116 |
| $S^2S^3S^4C$  | 0.8956   | 1.4142 | 1.4322 | 1.8517 | 2.0708 | 2.1814 | 2.3409 | 2.7901 | 2.8151 |
|               | 0.8957a  | 1.4142 | 1.4324 | 1.8517 | 2.0720 | 2.1818 | 2.3417 | 2.7893 | 2.8158 |
| $S^2S^3S^4C$  | 0.7070   | 1.0299 | 1.8749 | 1.8759 | 2.1209 | 2.2721 | 2.4267 | 2.5573 | 2.7935 |
|               | 0.7070b  | 1.0301 | 1.8746 | 1.8768 | 2.1213 | 2.2731 | 2.4279 | 2.5577 | 2.7932 |
| $S^2S^3S^4C$  | 0.8085   | 1.3921 | 1.6055 | 2.0015 | 2.2484 | 2.3801 | 2.5611 | 3.0011 | 3.0212 |
|               | 0.8087b  | 1.3928 | 1.6058 | 2.0019 | 2.2488 | 2.3809 | 2.5615 | 3.0016 | 3.0219 |
| $S^2S^3S^4C$  | 1.0607   | 1.4256 | 1.8428 | 1.8806 | 2.939 | 2.6480 | 2.7361 | 2.9937 | 3.0715 |
|               | 1.061b   | 1.4261 | 1.8433 | 1.8811 | 2.5402 | 2.6482 | 2.7364 | 2.9942 | 3.0720 |
| $S^2S^3S^4C$  | 1.3859   | 1.4422 | 1.9311 | 2.2449 | 2.7033 | 2.7563 | 2.7639 | 3.2421 | 3.3282 |
|               | 1.3864b  | 1.4426 | 1.9315 | 2.2452 | 2.7036 | 2.7566 | 2.7648 | 3.2425 | 3.3286 |
| $S^2S^3S^4F$  | 0.4029   | 1.0563 | 1.3770 | 1.3972 | 1.7724 | 1.8627 | 2.3221 | 2.5317 | 2.5750 |
|               | 0.4029b  | 1.0566 | 1.3771 | 1.3974 | 1.7727 | 1.8629 | 2.3219 | 2.5320 | 2.5753 |
| $S^2S^3S^4F$  | 0.6981   | 1.0070 | 1.4124 | 1.4232 | 1.9817 | 2.1600 | 2.3307 | 2.3924 | 2.8964 |
|               | 0.6983b  | 1.0073 | 1.4126 | 1.4234 | 1.9821 | 2.1603 | 2.3303 | 2.3927 | 2.8968 |

a Results in italic font from [11].
b Results in italic font from [6].
In the convergence study, choosing an appropriate number of terms in the truncated series is important. Table 6 shows the first four non-dimensional frequency parameters of annular sectorial plates with free boundary condition at all the edges for different number of terms. The inner–outer radius ratio is $R_0/R_1 = 1/2$. Table 6 shows the trend of the frequency parameters with increasing number of terms in the truncated series. $10 \times 10$ terms in the truncated series give satisfactory accuracy as will be shown in the numerical examples in tables 7 and 8.

In table 7, the results of an annular plate with different classical boundary conditions are compared to data from the literature. The results by ABAQUS-V6.12 and ANASYS-V14.5 are from [25]. Table 7 shows excellent agreement between the current model and existing data. These agreements prove that the present method with the adoption of the logarithmic radial variable is accurate and efficient in solving the in-plane vibration problems for annular sectorial plates. The presented method improves the efficiency for the sectorial plate in two aspects. First, the stiffness matrix and the mass matrix in equation (2.34) has

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Table 5. Normalized frequency parameter for orthotropic square plates of different stiffness ratios $E_x/E_y$ under $S^1CS^2F$ boundary condition.

| $E_x/E_y$ | mode no. | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|----------|----------|---|---|---|---|---|---|---|---|---|----|
| 3        |          | 0.7070 | 1.0717 | 1.9367 | 2.0823 | 2.1210 | 2.5607 | 2.6650 | 2.7598 | 2.8758 | 3.4396 |
|          |         | 0.7070 | 1.0719 | 1.9364 | 2.0827 | 2.1213 | 2.5624 | 2.6651 | 2.7612 | 2.8726 | 3.4389 |
| 6        |          | 0.7071 | 1.1243 | 2.0061 | 2.1212 | 2.2154 | 2.7743 | 2.9626 | 3.3606 | 3.5072 | 3.5353 |
|          |         | 0.7070 | 1.1243 | 2.0044 | 2.1213 | 2.2156 | 2.7737 | 2.9541 | 3.3614 | 3.5099 | 3.5355 |
| 9        |          | 0.7071 | 1.1461 | 2.0336 | 2.1212 | 2.2532 | 2.8129 | 2.9956 | 3.5101 | 3.5354 | 3.5760 |
|          |         | 0.7074 | 1.1460 | 2.0310 | 2.1213 | 2.2533 | 2.8119 | 2.9866 | 3.5103 | 3.5355 | 3.5717 |

*Results in italic font from [11].

Table 6. Normalized frequency parameters for annular plates with complete free boundary conditions.

| $M \times N$ | mode no. | 1 | 2 | 3 | 4 | 5 | 6 |
|-------------|----------|---|---|---|---|---|---|
| 7 $\times$ 7 |          | 1.0316 | 1.7348 | 2.0502 | 3.0598 | 3.1928 | 3.4189 |
| 8 $\times$ 8 |          | 1.0303 | 1.7348 | 2.0502 | 3.0597 | 3.1857 | 3.4101 |
| 9 $\times$ 9 |          | 1.0303 | 1.7348 | 2.0497 | 3.0597 | 3.1827 | 3.4096 |
| 10 $\times$ 10 |         | 1.0302 | 1.7347 | 2.0485 | 3.0597 | 3.1820 | 3.4086 |
| 11 $\times$ 11 |        | 1.0302 | 1.7347 | 2.0485 | 3.0597 | 3.1820 | 3.4086 |

Table 7. Frequency parameters for annular sectorial plate with various classical boundary condition. Note: $FE_{AB}$ represents results obtained from ABAQUS; $FE_{AN}$ represents results from ANASYS.

| B.C.   | method | mode no. | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|--------|--------|----------|---|---|---|---|---|---|---|---|
| CCCC   | present |          | 3.3677 | 4.4791 | 5.8203 | 5.9742 | 6.7084 | 7.2168 | 7.7834 | 8.6131 |
|        | $FE_{AB}$ [25] | 3.3692 | 4.4807 | 5.8240 | 5.9810 | 6.7152 | 7.2240 | 7.7894 | 8.626 |
| $S^1S^1S^2S^1$ | present | | 1.5882 | 3.063 | 3.6308 | 4.4535 | 5.8097 | 6.0961 | 4.9339 | 6.4738 |
|        | $FE_{AN}$ [25] | 1.5884 | 3.0641 | 3.6320 | 4.4572 | 4.9362 | 5.8163 | 6.1036 | 6.4793 |
| $S^2S^2S^2S^2$ | present | | 1.3469 | 2.3639 | 2.8909 | 3.4244 | 3.5297 | 4.5766 | 4.7263 | 5.7089 |
|        | $FE_{AN}$ [25] | 1.347 | 2.3646 | 2.8920 | 3.4255 | 3.5307 | 4.5787 | 4.7307 | 5.7165 |
| FFFF   | present | | 1.0311 | 1.7367 | 2.0504 | 3.0618 | 3.1817 | 3.4099 | 4.303 | 4.5766 |
|        | $FE_{AN}$ [25] | 1.0312 | 1.7370 | 2.0513 | 3.0630 | 3.1835 | 3.4108 | 4.3054 | 4.5788 |
the explicit form, which does not need the tedious numerical integration process [25], so it is helpful in generating the global matrices. Second, in equation (2.25), the terms in the truncated series of this paper are \( M = N = 10 \), while the chosen number of \( M \) and \( N \) are 15 in [25]. The present method significantly reduces the computation cost of the eigen-problems for the matrix dimension in equation (2.34) is \( 4/9 \) of that in [25], which does not use the logarithmic radial variable.

Most existing techniques can only handle one kind of the boundary condition and geometrical shape, but the current method can easily accommodate changes in geometry and boundary conditions.

3.3. In-plane vibration of rectangular and annular sectorial plates problems solved by the generalized model

In this section, the generalized model is used in several numerical examples. The parameters \( \xi_{\text{max}} \) and \( \eta_{\text{max}} \) are assigned the same values for plates of different shapes. For example, when the aspect ratio of the outer–inner radius for the annular sectorial plate is \( R_1/R_0 = 2 \), the length ratio of the rectangular plate is \( b/a = \ln(2) \). In the \( \eta \) direction, the width of the rectangular plate is set to be equal to the angle of the sectorial plate, e.g. \( b = \phi = \pi/2 \). The calculated frequency parameters for the two shapes are listed in table 8. It is shown from table 8 that the generalized model accommodates both the rectangular and annular sectorial shapes, and the results agree very well with data from the literature [24,25,28].

4. Concluding remarks

(1) A variable transformation by adopting the logarithmic radial variable significantly simplifies the basic theory for in-plane vibration of annular sectorial plate. This simplification makes it possible to formulate the basic theories for annular sectorial and rectangular plates in a uniform framework.

(2) In the generalized model, the improved Fourier–Ritz expansion of the displacements are expressed by admissible trigonometric functions. Due to the orthogonality of the modified
Fourier series, the global stiffness and mass matrices can be obtained explicitly by using the integration formulae in the electronic supplementary material.

(3) The appropriate spring stiffness values for various boundary conditions for rectangular and annular sectorial plates are discussed and provided in table 3.

(4) The number of terms in the truncated Fourier series for displacement fields are $10 \times 10$ for both shapes, and the numerical results show that the error of the present generalized model is universally less than 0.5%.

Data accessibility. Supporting data in this work can be accessed via the following links: http://dx.doi.org/10.5061/dryad.hd7tm [33], http://qishigreat.blog.hexun.com/111979175_d.html.

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References

1. Lyon RH. 1986 In-plane contribution to structural noise transmission. Noise Control Eng. J. 26, 22–27. (doi:10.3977/12872699)

2. Beric A. 1996 An assessment of the effects of in-plane vibrations on the energy flow between coupled plates. J. Sound Vib. 191, 661–680. (doi:10.1016/0022-460X(96)00149)

3. Liu B, Xing Y. 2011 Comprehensive exact solutions for free in-plane vibrations of orthotropic rectangular plates. Eur. J. Mech. A Solids 30, 363–395. (doi:10.1016/j.euromechsol.2011.01.003)

4. Wang G, Wereley NM. 2002 Free in-plane vibration of rectangular plates. AIAA J. 40, 955–959. (doi:10.2514/2.1732)

5. Gorman D. 2004 Free in-plane vibration analysis of rectangular plates by the method of superposition. J. Sound Vib. 272, 831–851. (doi:10.1016/S0022-460X(03)00421-8)

6. Xing Y, Liu B. 2009 Exact solutions for the free in-plane vibrations of rectangular plates. Int. J. Mech. Sci. 51, 246–255. (doi:10.1016/j.ijmecsci.2008.12.009)

7. Du J, Li WL, Jin G, Yang T, Liu Z. 2007 An analytical method for the in-plane vibration analysis of rectangular plates with elastically restrained edges. J. Sound Vib. 306, 908–927. (doi:10.1016/j.jsv.2007.06.011)

8. Dozio L. 2010 Free in-plane vibration analysis of rectangular plates with arbitrary elastic boundaries. Mech. Res. Commun. 37, 627–635. (doi:10.1016/j.mechrescom.2010.09.003)

9. Bardell N, Langley R, Dunjod J. 1996 On the free in-plane vibration of isotropic rectangular plates. J. Sound Vib. 191, 459–467. (doi:10.1016/0022-460X(96)01394)

10. Nefovska-Danilovic M, Petronijevic M. 2015 In-plane free vibration and response analysis of isotropic rectangular plates using the dynamic stiffness method. Comput. Struct. 152, 82–95. (doi:10.1016/j.compstruc.2015.02.001)

11. Shi D, Wang Q, Shi X, Pang F. 2015 A series solution for the in-plane vibration analysis of orthotropic rectangular plates with non-uniform elastic boundary constraints and internal line supports. Arch. Appl. Mech. 85, 51–73. (doi:10.1007/s00419-014-0899-x)

12. O Too M. 1956 Contour vibrations of isotropic circular plates. J. Acoust. Soc. Am. 28, 1158–1162. (doi:10.1121/1.1908579)

13. O Too M. 1958 Gravest contour vibration of thin anisotropic circular plates. J. Acoust. Soc. Am. 30, 634–638. (doi:10.1121/1.1909720)

14. Chen SS, Liu TL. 1975 Extensive vibration of thin plates of various shapes. J. Acoust. Soc. Am. 58, 828–831. (doi:10.1121/1.380729)

15. Holland R. 1966 Numerical studies of elastic-disk contour modes lacking axial symmetry. J. Acoust. Soc. Am. 40, 1051–1057. (doi:10.1121/1.1910187)

16. Farag N, Pan J. 2003 Modal characteristics of in-plane vibration of circular plates clamped at the outer edge. J. Acoust. Soc. Am. 113, 1935–1946. (doi:10.1121/1.1553456)

17. Irene T, Yamada G, Muramoto Y. 1984 Natural frequencies of in-plane vibration of annular plates. J. Sound Vib. 97, 171–175. (doi:10.1016/0022-0460(84)90479-6)

18. Bashmal S, Bhart R, Rakheja S. 2009 In-plane free vibration of circular annular disks. J. Sound Vib. 322, 216–226. (doi:10.1016/j.jsv.2008.11.024)

19. Park CI. 2008 Frequency equation for the in-plane vibration of a clamped circular plate. J. Sound Vib. 313, 325–333. (doi:10.1016/j.jsv.2007.07.014)

20. Seok J, Tiersten H. 2004 Free vibrations of annular sector cantilever plates. Part 2: In-plane motion. J. Sound Vib. 271, 773–787. (doi:10.1016/S0022-0460(03)00415-2)

21. Ravati MK, Forouzan M. 2011 Frequency equations for the in-plane vibration of orthotropic circular annular plate. Arch. Appl. Mech. 81, 1307–1322. (doi:10.1007/s00419-010-0484-6)

22. Vladimir N, Hadžić N, Seničarović I. 2014 Potential theory of in-plane vibrations of rectangular and circular plates. Int. J. Eng. Model. 27, 69–84.

23. Kim C-B, Cho HS, Bom HG. 2012 Exact solutions of in-plane natural vibration of a circular plate with outer edge restrained elastically. J. Sound Vib. 331, 2173–2189. (doi:10.1016/j.jsv.2011.12.027)

24. Singh A, Muhammad T. 2004 Free in-plane vibration of isotropic non-rectangular plates. J. Sound Vib. 273, 219–221. (doi:10.1016/S0022-460X(03)00496-6)

25. Wang G, Shi D, Liang Q, Ahmad F. 2016 A unified solution for free in-plane vibration of orthotropic circular, annular and sector plates with general boundary conditions. Appl. Math. Model. 40, 9228–9235. (doi:10.1016/j.apm.2016.06.005)

26. Li WL. 2000 Free vibrations of beams with general boundary conditions. J. Sound Vib. 237, 799–725. (doi:10.1006/jsvi.2000.3950)

27. Li WL. 2002 Comparison of Fourier sine and cosine series expansions for beams with arbitrary boundary conditions. J. Sound Vib. 255, 185–194. (doi:10.1016/j.jsv.2001.4080)

28. Shi X, Li W, Shi D. 2014 Free in-plane vibrations of annular sector plates with elastic boundary supports. In Proceedings of the 2014 Meetings on Acoustics, pp. 1–5. Acoustical Society of America.

29. Zhang K, Shi D, Teng X, Zhao Y, Liang Q. 2015 A series solution for the in-plane vibration of sector plates with arbitrary inclusion angles and boundary conditions. J. Vibration Eng. 27, 870–882.

30. Chen Y, Jin G, Liu Z. 2014 Flexural and in-plane vibration analysis of elastically restrained thin rectangular plate with cutout using Chebyshev—Lagrange method. Int. J. Mech. Sci. 89, 264–278. (doi:10.1016/j.ijmecsci.2014.09.006)

31. Wang Q, Shi D, Liang Q, Shi X. 2016 A unified solution for vibration analysis of functionally graded circular, annular and sector plates with general boundary conditions. Compos. B Eng. 88, 264–294. (doi:10.1016/j.compositesb.2015.10.043)

32. Yao W, Zhong W, Lim CW. 2009 Symplectic elasticity. Singapore: World Scientific.

33. Bao S, Wang S. 2017 Data from: A generalized solution procedure for in-plane free vibration of rectangular plates and annular sectoral plates. Dryad Digital Repository. (http://dx.doi.org/10.5061/dryad.hd7tm)