Perturbation theory of $N$ point-mass gravitational lens systems without symmetry: small mass-ratio approximation

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ABSTRACT

This paper makes the first systematic attempt to determine using perturbation theory the positions of images by gravitational lensing due to arbitrary number of coplanar masses without any symmetry on a plane, as a function of lens and source parameters. We present a method of Taylor-series expansion to solve the lens equation under a small mass-ratio approximation. First, we investigate perturbative structures of a single-complex-variable polynomial, which has been commonly used. Perturbative roots are found. Some roots represent positions of lensed images, while the others are unphysical because they do not satisfy the lens equation. This is consistent with a fact that the degree of the polynomial, namely the number of zeros, exceeds the maximum number of lensed images if $N = 3$ (or more). The theorem never tells which roots are physical (or unphysical). In this paper, unphysical ones are identified. Secondly, to avoid unphysical roots, we re-examine the lens equation. The advantage of our method is that it allows a systematic iterative analysis. We determine image positions for binary lens systems up to the third order in mass ratios and for arbitrary $N$ point masses up to the second order. This clarifies the dependence on parameters. Thirdly, the number of the images that admit a small mass-ratio limit is less than the maximum number. It is suggested that positions of extra images could not be expressed as Maclaurin series in mass ratios. Magnifications are finally discussed.

Key words: gravitational lensing – methods: analytical – stars: general – cosmology: theory.

1 INTRODUCTION

Gravitational lensing has become one of important subjects in modern astronomy and cosmology (e.g. Schneider 2006; Weinberg 2008). It has many applications as gravitational telescopes in various fields ranging from extra-solar planets to dark matter and dark energy at cosmological scales. This paper focuses on gravitational lensing due to a $N$ point-mass system. Indeed, it is a challenging problem to express the image positions as functions of lens and source parameters. There are several motivations. One is that gravitational lensing offers a tool of discoveries and measurements of planetary systems (Schneider & Weiss 1986; Mao & Paczynski 1991; Gould & Loeb 1992; Bond et al. 2004; Beaulieu et al. 2006), compact stars or a cluster of dark objects, which are difficult to probe with other methods. Gaudi et al. (2008) have recently found an analogy of the Sun–Jupiter–Saturn system by lensing. Another motivation is theoretically oriented. One may be tempted to pursue a transit between a particle method and a fluid (mean field) one. For microlensing studies, particle methods are employed because the systems consist of stars, planets or massive compact halo objects (MACHOs). In cosmological lensing, on the other hand, light propagation is considered for the gravitational field produced by inhomogeneities of cosmic fluids, say galaxies or large-scale structures of our Universe (e.g. Refregier 2003, for a review). It seems natural, though no explicit proof has been given, that observed quantities computed by continuum fluid methods will agree with those by discrete particle ones in the limit $N \to \infty$, at least on average, where $N$ is the number of particles.

Related to the problems mentioned above, we should note an astronomically important effect caused by the finiteness of $N$. For most of cosmological gravitational lenses (both of strong and weak ones), a continuum approximation can be safe and has worked well. There exists an exceptional case, however, for which discreteness becomes important. One example is a quasar microlensing due to a point-like lens object, which is possibly a star in a host galaxy (for an extensive review, Wambsganss 2006). A galaxy consists of very large number $N$ particles, and light rays from an object at cosmological distance may have a chance to pass very near one of the point masses. As a consequence of finite-$N$ effect in large $N$ point lenses, anomalous changes in the light curve are observed. For such a quasar microlensing, hybrid approaches are usually employed,
where particles are located in a smooth gravitational field representing a host galaxy. It is thus likely that point-mass approach will be useful also when we study such a finite-N effect at a certain transit stage between a particles system and a smooth one. Along this course, An (2007) investigated a N point lens model, which represents a very special configuration that every point masses are located on regular grid points.

For a N point-mass lens at a general configuration, very few things are known in spite of many efforts. Among known ones is the maximum number of images lensed by N point masses. After direct calculations by Witt (1990) and Mao, Petters & Witt (1997), a careful study by Rhie (2001, for N = 4, 2003 for general N) revealed that it is possible to obtain the maximum number of images as 5(N − 1). This theorem for polynomials has been extended to a more general case including rational functions by Khavinson & Neumann (2006). (See Khavinson & Neumann 2008 for an elegant review on a connection between the gravitational lens theory and the algebra, especially the fundamental theorem of algebra, and its extension to rational functions.)

**Theorem** (Khavinson & Neumann 2006).

Let \( r(z) = p(z)/q(z) \), where \( p \) and \( q \) are relatively prime polynomials in \( z \), and let \( n \) be the degree of \( r \). If \( n > 1 \), then the number of zeros for \( r(z) - z^* \leq 5(n - 1) \). Here, \( z \) and \( z^* \) denote a complex number and its complex conjugate, respectively.

Furthermore, Bayer, Dyer & Giang (2006) showed that in a configuration of point masses, replacing one of the point deflectors by a spherically symmetric distributed mass only introduces one extra image. Hence, they found that the maximum number of images due to \( N \) distributed lensing objects located on a plane is \( 6(N - 1) + 1 \).

Global properties such as lower bounds on the number of images are also discussed in Petters, Levine & Wambsganss (2001) and references therein.

In spite of many efforts on \( N \) lensing objects, functions for image positions are still unknown even for \( N \) point-mass lenses in a general configuration under the thin lens approximation. Hence, it is a challenging problem to express the image positions as functions of lens and source locations. Once such an expression is known, one can immediately obtain magnifications via computing the Jacobian of the lens mapping (Schneider, Ehlers & Falco 1992).

Only for a very few cases such as a single point mass and a singular isothermal ellipsoid, the lens equation can be solved by hand and image positions are known because the lens equation becomes a quadratic or fourth-order one (for a singular isothermal ellipsoid, Asada, Hamana & Kasai 2003). For the binary lens system, the lens equation has the degree of five in a complex variable (Witt 1990). It has the same degree also in a real variable (Asada 2002a; Asada, Kasi & Kasai 2004). This improvement is not trivial because a complex variable brings two degrees of freedom. This single-real-variable polynomial has advantages. For instance, the number of real roots (with vanishing imaginary parts) corresponds to that of lensed images. The analytic expression of the caustic, where the number of images changes, is obtained by the fifth-order polynomial (Asada, Kasai & Kasai 2002). Galois showed, however, that the fifth-order and higher polynomials cannot be solved algebraically (van der Waerden 1966). Hence, no formula for the quintic equation is known. For this reason, some numerical implementation is required to find out image positions (and magnifications) for the binary gravitational lens for a general position of the source. Only for special cases of the source at a symmetric location such as on-axis sources, the lens equation can be solved by hand and image positions are thus known (Schneider & Weiss 1986). For a weak field region, some perturbative solutions for the binary lens have been found (Bozza 1999; Asada 2002b), for instance in order to discuss astrometric lensing, which is caused by the image centroid shifts (for a single mass, Miyamoto & Yoshii 1995; Walker 1995; for a binary lens, Saizadeh, Dalal & Grier 1999; Jeong, Han & Park 1999; Asada 2002b).

If the number of point masses \( N \) is larger than two, the basic equation is much more highly non-linear so that the lens equation can be solved only by numerical methods. As a result, observational properties such as magnifications and image separations have been investigated so far numerically for \( N \) point-mass lenses. This makes it difficult to investigate the dependence of observational quantities on lens parameters.

This paper is the first attempt to seek an analytic expression of image positions without assuming any special symmetry. For this purpose, we shall present a method of Taylor-series expansion to solve the lens equation for \( N \) point-mass lens systems. Our method allows a systematic iterative analysis as shown later.

Under three assumptions of weak gravitational fields, thin lenses and small deflection angles, gravitational lensing is usually described as a mapping from the lens plane on to the source plane (Schneider et al. 1992). Bourassa, Kantowski & Norton (1973) and Bourassa & Kantowski (1975) introduced a complex notation to describe gravitational lensing. Their notation was exclusively used to describe lenses with elliptical or spheroidal symmetry (Borgeest 1983; Bray 1984; Schramm 1990). For \( N \) point lenses, Witt (1990) succeeded in recasting the lens equation into a single-complex-variable polynomial. This is in an elegant form and thus has been often used in investigations of point-mass lenses. An advantage in the single-complex-variable formulation is that we can use some mathematical tools applicable to complex-analytic functions, especially polynomials (Witt 1993; Witt & Petters 1993; Witt & Mao 1995). One tool is the fundamental theorem of algebra: every non-constant single-variable polynomial with complex coefficients has at least one complex root. This is also stated as: every non-zero single-variable polynomial, with complex coefficients, has exactly as many complex roots as its degree, if each root is counted as many times as its multiplicity. On the other hand, in the original form of the lens equation, one can hardly count up the number of images because of non-linearly coupled properties. This theorem, therefore, raises a problem in gravitational lensing. The single-variable polynomial due to \( N \) point lenses has the degree of \( N^2 + 1 \), though the maximum number of images is \( 5(N - 1) \). This means that unphysical roots are included in the polynomial (for detailed discussions on the disappearance and appearance of images near fold and cusp caustics for general lens systems, see also Petters et al. (2001) and references therein). First, we thus investigate explicitly behaviours of roots for the polynomial lens equation from the viewpoint of perturbations. We shall identify unphysical roots. Secondly, we shall re-examine the lens equation, so that the appearance of unphysical roots can be avoided.

This paper is organized as follows. In Section 2, the complex description of gravitational lensing is briefly summarized. The lens equation is embedded into a single-complex-variable polynomial in Section 3. Perturbative roots for the complex polynomial are presented for binary and triple systems in Sections 4 and 5, respectively. They are extended to a case of \( N \) point lenses in Section 6. In Section 7, we re-examine the lens equation in a dual-complex-variables formalism and its perturbation scheme for a binary lens for its simplicity. The perturbation scheme is extended to a \( N \) point lens system in Section 8. Section 9 is devoted to the conclusion.
2 POLYNOMIAL FORMALISM USING COMPLEX VARIABLES

We consider a lens system with \( N \) point masses. The mass and two-dimensional location of each body is denoted as \( M_i \) and the vector \( E_i \), respectively. For the later convenience, let us define the angular size of the Einstein ring as

\[
\theta_E = \sqrt{\frac{4GM_{\text{tot}}D_{LS}}{c^2D_LD_S}},
\]

where \( G \) is the gravitational constant, \( c \) is the light speed, \( M_{\text{tot}} \) is the total mass \( \sum_{i=1}^{N} M_i \) and \( D_{LS} \) and \( D_LD_S \) denote distances between the observer and the lens, between the observer and the source, and between the lens and the source, respectively. In the unit normalized by the angular size of the Einstein ring, the lens equation becomes

\[
\beta = \theta - \sum_{i=1}^{N} v_i \frac{\theta - e_i}{|\theta - e_i|^2},
\]

where \( \beta = (\beta_x, \beta_y) \) and \( \theta = (\theta_x, \theta_y) \) denote the vectors for the position of the source and image, respectively, and we defined the mass ratio and the angular separation vector as \( v_i = M_i / M_{\text{tot}} \) and \( e_i = E_i / \theta_E = (e_{ix}, e_{iy}) \).

In a formalism based on complex variables, two-dimensional vectors for the source, lens and image positions are denoted as \( w = \beta_x + i\beta_y, z = \theta_x + i\theta_y \) and \( e_i = e_{ix} + ie_{iy} \), respectively (see also Fig. 1). By employing this formalism, the lens equation is rewritten as

\[
w = z - \sum_{i=1}^{N} \frac{v_i}{z - e_i} \tag{3},
\]

where the asterisk * means the complex conjugate. The lens equation is non-analytic because it contains both \( z \) and \( z^* \).

3 EMBEDDING THE LENS EQUATION INTO AN ANALYTIC POLYNOMIAL

The complex conjugate of equation (3) is expressed as

\[
w^* = z^* - \sum_{i=1}^{N} \frac{v_i}{z^* - e_i^*} \tag{4}.
\]

This expression can be substituted into \( z^* \) in equation (3) to eliminate the complex variable \( z^* \). As a result, we obtain a \( (N^2 + 1) \)th order analytic polynomial equation as (Witt 1990)

\[
(z - w) \prod_{\ell=1}^{N} (w^* - \epsilon_\ell^*) \prod_{k=1}^{N} (z - \epsilon_k) + \sum_{k=1}^{N} v_k \prod_{j \neq k}^{N} (z - \epsilon_j) = \sum_{i=1}^{N} v_i (z - \epsilon_i) \prod_{\ell=1}^{N} (w - \epsilon_\ell) \prod_{j \neq \ell}^{N} (z - \epsilon_j) \prod_{k=1}^{N} (z - \epsilon_k) \tag{5}.
\]

Equation (A3) in Witt (1990) takes a rather complicated form because of inclusion of non-zero shear \( \gamma \) due to surrounding matter. Bayer et al. (2006) uses a complex formalism in order to discuss the maximum number of images in a configuration of point masses, by replacing one of point deflectors by a spherically symmetric distributed mass. Their lens equation (3) agrees with equation (5). In order to show this agreement, one may use \((-1)^{N+1} = (-1)^{N-1}\). It is worthwhile to mention that equation (5) contains not only all the solutions for the lens equation (2) but also unphysical false roots which do not satisfy equation (2), in price of the manipulation for obtaining an analytic polynomial equation, as already pointed out by Rhie (2001, 2003) and Bayer et al. (2006). Such an inclusion of unphysical solutions can be easily understood by remembering that we get unphysical roots as well as true ones if one takes a square of an equation including the square root. In fact, an analogous thing happens in another example of gravitational lenses such as an isothermal ellipsoidal lens as a simple model of galaxies (Asada et al. 2003).

In general, the mass ratio \( v_i \) satisfies \( 0 < v_i < 1 \), so that it can be taken as an expansion parameter. Without loss of generality, we can assume that the first lens object is the most massive, namely \( m_1 \geq m_i \) for \( i = 2, 3, \ldots, N \). Thus, formal solutions are expressed in Taylor series as

\[
z = \sum_{p_2=0}^{\infty} \sum_{p_3=0}^{\infty} \cdots \sum_{p_N=0}^{\infty} v_2^{p_2} v_3^{p_3} \cdots v_N^{p_N} z_{(p_2)(p_3)\cdots(p_N)}, \tag{6}
\]

where the coefficients \( z_{(p_2)p_3\cdots(p_N)} \) are independent of \( v_i \).

Up to this point, the origin of the lens plane is arbitrary. In the following, the origin of the lens plane is chosen as the location of the mass \( m_1 \), such that one can put \( e_1 = 0 \). This enables us to simplify some expressions and to easily understand their physical meanings, mostly because gravity is dominated by \( m_1 \) in most regions except for the vicinity of \( m_i (i \neq 1) \). Namely, it is natural to treat our problem as perturbations around a single lens by \( m_1 \) (located at the origin of the coordinates).

In numerical simulations or practical data analysis, however, one may use the coordinates in which the origin is not the location of \( m_1 \). If one wishes to consider such a case of \( e_1 \neq 0 \), one could make a translation by \( e_1 \) as \( z \rightarrow z + e_1 \), \( w \rightarrow w + e_1 \) and \( e_i \rightarrow e_i + e_1 \) in our perturbative solutions that are given below.

4 PERTURBATIVE SOLUTIONS FOR A POLYNOMIAL FORMALISM 1: BINARY LENS

In this section, we investigate binary lenses explicitly up to the third order. This simple example may help us to understand the structure of the perturbative solutions. For an arbitrary \( N \) case, expressions of iterative solutions are quite formal (see below).

For simplicity, we denote our expansion parameter as \( m = v_2 \). This means \( v_1 = 1 - m \). We also denote \( e_2 \) simply by \( \epsilon \).
In powers of \( m \), the polynomial equation is rewritten as
\[
\sum_{k=0}^{2} m^k f_k(z) = 0, 
\]
where we defined
\[
\begin{align*}
  f_0(z) &= (z - \epsilon)^3[(w^* - \epsilon^*)z + 1][w^*z^2 - w^*z - w] , \\
  f_1(z) &= (z - w) \\
  &\times \{\epsilon(z - w)(2w^* - \epsilon^*)z + 2\} - \epsilon^*z^2(z - \epsilon) - \epsilon z , \\
  f_2(z) &= \epsilon^2(z - w) .
\end{align*}
\]

We seek a solution in expansion series as
\[
z = \sum_{p=0}^{\infty} m^p z(p) ,
\]

### 4.1 Zeroth order
At \( O(m^0) \), the lens equation becomes the fifth-order polynomial equation as \( f_0 = 0 \). Zeroth order solutions are obtained by solving this. All the solutions are \( \epsilon \) (doublet), \( \alpha_3 \) and \( \alpha_{\pm} \), where we defined
\[
\begin{align*}
  \alpha_3 &= \frac{1}{\epsilon - w^*} , \\
  \alpha_{\pm} &= \frac{w}{2} \left( 1 \pm \sqrt{1 + \frac{4}{w w^*}} \right) .
\end{align*}
\]

One of the roots, \( \alpha_3 \), is unphysical because it does not satisfy equation (2) at \( O(m^0) \). By using all the zeroth order roots including unphysical ones, \( f_0 \) is factorized as
\[
f_0(z) = w^* (w^* - \epsilon^*) (z - \epsilon^3)(z - \alpha_3)(z - \alpha_+)(z - \alpha_-) .
\]

### 4.2 First order
Next, we seek first-order roots. We put \( z = \alpha_{\pm} + m z(1) + O(m^2) \). At the linear order in \( m \), equation (5) becomes
\[
z(1) f_2(\alpha_{\pm}) + f_1(\alpha_{\pm}) = 0 ,
\]
where the prime denotes the derivative with respect to \( z \). Thereby, we obtain a first-order root as
\[
z(1) = \frac{f_1(\alpha_{\pm})}{f_2(\alpha_{\pm})} .
\]
The similar manner cannot be applied to a case of \( \epsilon \) because it is a doublet root with \( f_0(\epsilon) = f_1(\epsilon) = 0 \), while \( f_0(\epsilon) \neq 0 \). At \( O(m^2) \), equation (5) can be factorized as
\[
\begin{align*}
  \{z(1)[(w^* - \epsilon^*)z + 1] + \epsilon\} \\
  &\times \{z(1)[(z - w)(w^* + 1) - \epsilon] + \epsilon(z - w)\} = 0 .
\end{align*}
\]

Hence, we obtain two roots as
\[
\begin{align*}
  z(1) &= \frac{\epsilon}{(\epsilon - w^*) - 1} , \\
  z(1) &= \frac{\epsilon(z - w)}{(z - w)(w^* + 1) - \epsilon} .
\end{align*}
\]

Here, the latter root expressed by equation (16) is unphysical and thus abandoned because it does not satisfy the original lens equation (2). On the other hand, the former root by equation (15) satisfies the equation and thus expresses a physically correct image.

### 4.3 Second order
First, we consider perturbations around zeroth-order solutions of \( \alpha_{\pm} \). At \( O(m^2) \), equation (5) is linear in \( z(2) \) and thus easily solved for \( z(2) \) as
\[
z(2) = -\frac{z^2(1) f_0'(\alpha_{\pm})}{2 f_0(\alpha_{\pm})} + 2 z(1) f_1(\alpha_{\pm}) + 2 f_2(\alpha_{\pm}) .
\]

Next, we investigate a multiple root \( \epsilon \). At \( O(m^3) \), equation (5) becomes linear in \( z(3) \). It is solved as
\[
\begin{align*}
z(3) &= -\frac{z^2(1) f_0''(\epsilon)}{6 f_0'(\epsilon)} + 3 z^2(1) f_1''(\epsilon) + 6 z(1) f_2(\epsilon) .
\end{align*}
\]

### 4.4 Third order
Around zeroth-order solutions of \( \alpha_{\pm} \), equation (5) at \( O(m^3) \) is linear in \( z(2) \) and thus solved as
\[
\begin{align*}
z(3) &= -\frac{z^2(1) f_0''(\epsilon)}{f_0'(\epsilon)} + f_1'(\epsilon) \\
&\times \left[ \frac{1}{2} z^2(1) f_0''(\epsilon) + 2 z(1) f_2(\epsilon) + \frac{1}{24} z^3(1) f_0'''(\epsilon) \\
+ 2 z^2(2) f_1'(\epsilon) + \frac{1}{6} z(1) f_2(\epsilon) \\
+ 2 z(2) f_1'(\epsilon) \right] .
\end{align*}
\]

where we used \( f_2''(\epsilon) = 0 \). Table 1 shows a numerical example of perturbative roots and their convergence.

### 5 Perturbative Solutions for a Polynomial Formalism 2: Triplet Lens
In a binary case, we have only the single parameter \( m \) for the perturbations. For \( N \) point masses, we have to take account of couplings among several expansion parameters. In addition, the degree of the polynomial becomes \( N^2 + 1 \), so that we cannot write down the whole equation. In order to get hints for \( N \) point-mass lenses, in this section, we investigate triple-mass lenses explicitly up to the second order in \( v_1 \) and \( v_2 \).

The polynomial equation is rewritten as
\[
\sum_{p=0}^{3} \sum_{p=0}^{3} [v_2]^3(v_3)^3 g(p_2, p_3)(z) = 0 ,
\]
where we defined
\[
\begin{align*}
g(0, 0)(z) &= (z - \epsilon_2)^3(z - \epsilon_3)^3[w^* - \epsilon^*_2 z + 1] \\
&\times [w^* - \epsilon^*_3 z + 1][w^* z^2 - w^* z - w].
\end{align*}
\]

We seek a solution in expansion series as
\[
z = \sum_{p=0}^{\infty} \sum_{p=0}^{\infty} [v_2]^3(v_3)^3 z(p_2, p_3) .
\]
Table 1. Example of perturbative roots in the single-complex-polynomial: we assume \( v = 0.1, \epsilon = 1 \) and two cases of \( w = 2 \) (on-axis) and \( w = 1 + i \) (off-axis).

| Case 1 | On-axis | \( v = 0.1 \) | \( \epsilon = 1 \) | \( w = 2 \) |
|--------|---------|--------------|----------------|-----------|
| Root   |         | 1            | 2              | 3          | 4          | 5          |
| First  | 2.43921 | -0.389214    | 0.95           | -0.925    | 0.975      |
| Second | 2.43855 | -0.388551    | 0.95           | -0.92063  | 0.974063   |
| Third  | 2.43858 | -0.388519    | 0.949938       | -0.920416 | 0.974016   |
| Polynomial | 2.43858 | -0.388517    | 0.949937       | -0.920413 | 0.974013   |
| Lens eq. | 2.43858 | -0.388517    | 0.949937       | None      | None       |

| Case 2 | Off-axis | \( v = 0.1 \) | \( \epsilon = 1 \) | \( w = 1 + i \) |
|--------|----------|--------------|----------------|----------------|
| Root   |          | 1            | 2              | 3            | 4          | 5          |
| First  | 1.33716+1.40546 i | -0.337158-0.355459 i | 0.95-0.05 i | 0.025-0.925 i | 0.975-0.025 i |
| Second | 1.33632+1.40363 i | -0.336316-0.354881 i | 0.95-0.05 i | 0.02625-0.9225 i | 0.97375-0.02625 i |
| Third  | 1.33634+1.40371 i | -0.336275-0.354839 i | 0.95-0.05025 i | 0.0262813-0.92228 i | 0.973656-0.0263438 i |
| Polynomial | 1.33633+1.40371 i | -0.336272-0.354835 i | 0.950015-0.0502659 i | 0.0262762-0.922254 i | 0.973646-0.0263517 i |
| Lens eq. | 1.33633+1.40371 i | -0.336272-0.354835 i | 0.950015-0.0502659 i | None | None |

5.1 Zeroth order

Zeroth order solutions are obtained by solving the tenth-order polynomial equation as \( g_{0000} = 0 \). The roots are \( \epsilon_2 \) (doublet), \( \epsilon_3 \) (doublet), \( \alpha_3 \), \( \alpha_4 \) and \( \alpha_5 \), where we defined

\[
\begin{align*}
\alpha_3 &= \frac{1}{\epsilon_2 - w}, \\
\alpha_4 &= \frac{1}{\epsilon_3 - w}.
\end{align*}
\]

For the same reason in the binary lens, \( \alpha_3 \) and \( \alpha_4 \) are unphysical, in the sense that it does not satisfy the lens equation (2). By using all the zeroth order roots, \( g_{0000} \) is factorized as

\[
g_{0000}(z) = w^2 (w^* - \epsilon_2^*) (w^* - \epsilon_3^*) (z - \alpha_3^*)(z - \alpha_4^*)(z - \alpha_5^*).
\]

5.2 First order

Here, we seek first-order roots. The image position is expanded as \( z = \alpha^* + v_2 z_{10}(\epsilon_2) + v_3 z_{01}(\epsilon_3) + O(v_2^2, v_3^2, v_2 v_3) \). By making a replacement in notations as \( 2 \rightarrow 3 \), one can construct \( z_{00}(\epsilon_1) \) from \( z_{10}(\epsilon_1) \). Hence, we focus on \( z_{10}(\epsilon_1) \) below. At the linear order in \( v_2 \), equation (5) becomes

\[
z_{10}(\epsilon_1) g'_{0000}(z) + g_{10}(\epsilon_2) = 0.
\]

Thereby we obtain a first-order root as

\[
z_{10}(\epsilon_1) = \frac{g_{10}(\epsilon_2)}{g'_{0000}(\epsilon_2)}.
\]

Hence, we obtain three roots as

\[
\begin{align*}
z_{1}(\epsilon_1) &= \frac{\epsilon_2}{(\epsilon_2^* - w^*) \epsilon_2 - 1}, \\
z_{2}(\epsilon_1) &= \frac{\epsilon_2}{(\epsilon_2^* - w^*) \epsilon_2 - 1}, \\
z_{3}(\epsilon_1) &= \frac{-\epsilon_2 (\epsilon_2^* - w^*)}{(\epsilon_2 - w^*) (w^* \epsilon_2 + 1) - \epsilon_2^*}.
\end{align*}
\]

At the linear order in \( v_2 \), true solutions for the triple lens system has to agree with that for the binary system, when one takes a limit as \( v_3 \to 0 \). Therefore, out of the above three roots, ones expressed by equations (30) and (31) must be abandoned.

6 Perturbative solutions for a polynomial formalism 3: N point-mass lens

In the previous section, we have learned couplings between the second and third masses. Now we are in a position to investigate a lens system consisting of \( N \) point masses.

The polynomial lens equation (5) is expanded as

\[
\begin{align*}
\sum_{p_2=0}^{N} \sum_{p_3=0}^{N} \cdots \sum_{p_N=0}^{N} (v_2)^{p_2} (v_3)^{p_3} \cdots (v_N)^{p_N} \\
\times g_{(p_2,p_3) \cdots (p_N)}(z) = 0.
\end{align*}
\]

For this equation, we seek a solution in expansion series as

\[
z = \sum_{p_2=0}^{\infty} \sum_{p_3=0}^{\infty} \cdots \sum_{p_N=0}^{\infty} (v_2)^{p_2} (v_3)^{p_3} \cdots (v_N)^{p_N} \\
\times z_{(p_2,p_3) \cdots (p_N)}.
\]

6.1 Zeroth order

Zeroth order solutions are obtained by solving the \((N^2 + 1)\)th-order polynomial equation as \( g_{0000} = 0 \). The roots are \( \alpha_i = -1/w_i^*, \alpha_0^*, \)
and $\epsilon_i$ (with multiplicity $= N$) for $i = 2, \ldots, N$, where for later convenience we denoted

$$w_i = w - \epsilon_i.$$  

(34)

Like in the binary lens, $\sigma_i$ is unphysical, in the sense that it does not satisfy the lens equation (2). By using all the zeroth order roots, $g_{(0)}$ is factorized as

$$g_{(0)}(z) = (z - \alpha_+)(z - \alpha_-)w^* \prod_{j=2}^{N} (w_j^*)$$

$$\times \prod_{k=2}^{N} (z - \epsilon_k)^N \prod_{\ell=2}^{N} \left( z + \frac{1}{w_\ell^*} \right),$$  

(35)

where this degree is $N^2 + 1$ in agreement with that of the polynomial equation.

6.2 First order

Next, we seek first-order roots. In the similar manner in the double or triple mass case, we can obtain a first-order root as

$$z_{k(0)} = \frac{-g_{(0)}(\alpha_+)}{g_{(0)}(\alpha_-)} w_\ell^*,$$  

(36)

where $l_k$ denotes that the kth index is the unity, namely $p_k = 1$.

For $N$ point mass lens systems, a root $\epsilon_k$ is multiple with multiplicity $= N$. Without loss of generality, we choose $\epsilon_2$ as a root in the following discussion. Calculations done above for a double or triple mass system suggest that equation (5) at $O(v_2^2)$ can be factorized as

$$\prod_{k=2}^{N} \{ z_{k(0)}(w_\ell^*) e_2 + 1 \} + e_2$$

$$\times \{ z_{k(0)}(w_\ell^*) e_2 + 1 + e_2 \} = 0.$$  

(37)

By using this factorization, we obtain $N$ roots. At the linear order in $v_2$, however, true solutions for the present lens system has to agree with that for the binary system because one can take the limit as $v_p \to 0$ for $p \geq 3$. Therefore, only the $-\epsilon_2/(w_\ell^* \epsilon_2 + 1)$ out of the above $N$ roots is correct for the original lens equation. The same argument is true of any $\epsilon_i$.

7 PERTURBATIVE SOLUTIONS FOR ZZ*-DUAL FORMALISM: BINARY LENS

As shown above, an analytic polynomial formalism is apparently simple. When we solve perturbatively the polynomial equation, however, we find unphysical roots which satisfy the polynomial but does not the original lens equation. In the polynomial formalism, therefore, we are required to check every root and then to pick up only the physical roots satisfying the original lens equation with discarding unphysical ones. It is even worse that the order of the polynomial grows rapidly as $N^2 + 1$, as the number of the lens objects increases. This means that the perturbative structure of the formalism becomes much more complicated as $N$ increases. In this section, we thus investigate another formalism, which allows a more straightforward calculation especially without needing extra procedures such as deleting physically incorrect roots.

First, we focus on a binary case for its simplicity. The lens equation is rewritten as

$$C(z, z^*) = m D(z^*),$$  

(38)

where we defined

$$C(z, z^*) = w - z + \frac{1}{z^*}.$$  

(39)

$$D(z^*) = \frac{1}{z^*} - \frac{1}{z^* - \epsilon^*}.$$  

(40)

One of advantages in this $zz^*$ formulation is that the master equation (38) is linear in $m$. Therefore, counting orders in $m$ can be drastically simplified when we perform iterative calculations. On the other hand, an analytic polynomial is second order in $m$. In fact, practical perturbative computations in the polynomial formalism are quite complicated, in the sense that several different terms ($f_0$, $f_1$ and $f_2$ for a binary case) may make the same order-of-magnitude contributions at each iteration step.

We seek a solution in expansion series as

$$z = \sum_{k=0}^{\infty} m^k z(k).$$  

(41)

The complex conjugate of this becomes

$$z^* = \sum_{k=0}^{\infty} m^k z(k)^*.$$  

(42)

According to these power-series expansions of $z$ and $z^*$, both sides of the lens equation are expanded as

$$C(z, z^*) = \sum_{k=0}^{\infty} m^k C(k),$$  

(43)

$$D(z^*) = \sum_{k=0}^{\infty} m^k D(k),$$  

(44)

where $C(k)$ and $D(k)$ are independent of $m$. At $O(m^k)$, equation (38) becomes

$$C_{(k)} = D_{(k-1)},$$  

(45)

which shows clearly a much simpler structure than a polynomial case such as equations (17) and (18). Equation (40) indicates that $D(z^*)$ has a pole at $z^* = \epsilon^*$. Therefore, we shall discuss two cases of $z(0) \neq \epsilon$ and $z(0) = \epsilon$, separately.

7.1 Zeroth order [$z(0) \neq \epsilon$]

Zeroth order solutions are obtained by solving the equation as

$$C(z(0), z^*(0)) = 0.$$  

(46)

The solution for this is the well-known roots for a single mass lens. In order to help readers to understand the $zz^*$-dual formulation, we shall derive the roots by keeping both $z$ and $z^*$ real. Unless $z(0) \neq \epsilon$, one obtains a quadratic equation for $A$ as

$$w w^* A^2 - w w^* A - 1 = 0.$$  

(47)

The left-hand side is purely real so that the right-hand side must be real. Unless $w = 0$, therefore, one can put $z(0) = A w$ by introducing a certain real number $A$. By substituting $z(0) = A w$ into equation (47), one obtains a quadratic equation for $A$ as

$$w w^* A^2 - w w^* A - 1 = 0.$$  

(48)

This is solved as

$$A = \frac{1}{2} \left( 1 \pm \sqrt{1 + \frac{4}{w w^*}} \right),$$  

(49)

which gives the zeroth-order solution.

In the special case of $w = 0$, equation (47) becomes $|z(0)| = 1$, which is the Einstein ring. In the following, we assume $w \neq 0$. 

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7.2 First order \([z(0) \neq \epsilon]\)

In units of \(z(0)\), the expansion series of \(z\) is normalized as

\[
z = z(0) \sum_{k=0}^{\infty} m^k \sigma_k(t),
\]

where we defined \(\sigma_k = z_k/z(0)\).

First, we investigate a case of \(z(0) \neq \epsilon\). At the linear order in \(m\), equation (38) becomes

\[
z(1) + \frac{z(1)}{z(0)^2} = -\frac{1}{z(0)} + \frac{1}{z(0) - \epsilon},
\]

(51)

In order to solve equation (51), we consider an equation linear in both \(z\) and \(z^*\) as

\[
z + az^* = b,
\]

(52)

for two complex constants \(a, b \in C\).

Unless \(|a| = 1\), the general root for this equation is

\[
z = \frac{b - ab^*}{1 - aa^*},
\]

(53)

This can be verified by a direct substitution of equation (53) into equation (38). If \(|a| = 1\), equation (52) is underdetermined, in the sense that it could not provide the unique root without any additional constraint condition on \(z\) and \(z^*\).

By using directly equation (53), equation (51) is solved as

\[
z(1) = \frac{1}{z(0)^2}\left[ \frac{\epsilon z(0)}{z(0)} + \frac{\epsilon z(0)}{z(0) - \epsilon} \right].
\]

(54)

7.3 Second order \([z(0) \neq \epsilon]\)

At \(O(m^2)\), equation (38) is

\[
z(2) + a_1z^{*}(2) = b_2,
\]

(55)

where we defined

\[
a_2 = \frac{1}{z(0)^2},
\]

(56)

\[
b_2 = -D(1) + \frac{\sigma_1^*}{z(0)}.
\]

(57)

Here, \(D(1)\) is written as

\[
D(1) = \frac{\sigma_1}{z(0)} + \frac{\sigma_1^*}{z(0) - \epsilon}.
\]

(58)

By using the relation (53), equation (55) is solved as

\[
z(2) = \frac{b_2 - a_2b_2^*}{1 - a_2a_2^*}.
\]

(59)

7.4 Third order and \(n\)th order \([z(0) \neq \epsilon]\)

Computations at \(O(m^3)\) are similar to those at \(O(m^2)\) as shown below. At \(O(m^n)\), equation (38) takes a form as

\[
z(n) + a_1z^{*}(n) = b_3,
\]

(60)

where we defined

\[
a_3 = \frac{1}{z(0)^2},
\]

(61)

\[
b_3 = -D(2) + \frac{2\sigma_1^*\sigma_2^* - [\sigma_1^*]^2}{z(0)}.\]

(62)

Here, \(D(2)\) is written as

\[
D(2) = \frac{\sigma_1}{z(0)} + \frac{\sigma_2}{z(0) - \epsilon} + \frac{[\sigma_2^*]^2}{z(0)}.
\]

(63)

Using the relation (53) for equation (60), we obtain

\[
z(3) = \frac{b_3 - a_3b_3^*}{1 - a_3a_3^*}.
\]

(64)

In the similar manner, one can obtain iteratively \(n\)th-order roots \(z(n)\), which obeys an equation in the form of equation (52), and thus can use equation (53) to obtain \(z(n)\).

7.5 Zeroth and first order \([z(0) = \epsilon]\)

Next, we investigate the vicinity of \(z = \epsilon\), which is a pole of \(D\). The other pole of \(D\) is \(z = 0\), which makes also \(C(z, z^*)\) divergent. Therefore, \(z = 0\) and its neighbourhood are abandoned. Let us focus on a root around \(z = \epsilon\).

We assume \(z = \epsilon + m\epsilon + O(m^2)\). Then, the relevant terms in expansion series of \(C\) and \(D\) become

\[
C(0) = w - \epsilon + \frac{1}{\epsilon},
\]

(65)

\[
D(-1) = \frac{1}{\epsilon},
\]

(66)

where the index \(-1\) means that the inverse of \(m\) appears because of the pole at \(\epsilon\). Therefore, the lens equation at \(O(m^0)\) becomes linear in \(\epsilon^*\) without including \(z(1)\). Immediately, it determines \(z(1)\). Its complex conjugate becomes

\[
z(1) = -\frac{\epsilon}{(\epsilon^*)^2} + 1.
\]

(67)

This shows a clear difference between \(z(0) = \epsilon\) and \(z(0) \neq \epsilon\) cases. Equation (51) for the latter case contains both \(z(1)\) and \(z(1)^*\), so that we must use a relation such as equation (53).

7.6 Second, third and \(n\)th order \([z(0) = \epsilon]\)

Next, we consider the lens equation at \(O(m^1)\), namely \(C(1) = D(0)\). This determines \(z(2)\) as

\[
z(2) = [z(1)]^2\left[ C(1) - \frac{1}{\epsilon^*} \right].
\]

(68)

where we may use

\[
C(1) = -z(1) - \frac{z(1)}{(\epsilon^*)^2}.
\]

(69)

Let us consider \(O(m^2)\) to look for \(z(3)\). Equation of \(C(2) = D(1)\) provides \(z(3)^*\) as

\[
z(3)^* = [z(1)]^2C(2) + \frac{[z(1)]^3}{(\epsilon^*)^2} + \frac{[z(2)]^2}{z(1)}.
\]

(70)

where we can use

\[
C(2) = -z(2) - \frac{z(2)}{(\epsilon^*)^2} + \frac{[z(1)]^2}{(\epsilon^*)^3}.
\]

(71)

By the same way, one can obtain perturbatively \(n\)th-order solutions \(z(n)\) around \(z(0) = \epsilon\).
The purpose of this section is to extend the proposed method to a general case of gravitational lensing by an arbitrary number of point masses.

8 PERTURBATIVE SOLUTIONS FOR ZZ'-DUAL FORMALISM 2: LENSING BY N POINT MASS

The lens equation is written as

\[ C(z, z^*) = \sum_{i=2}^{N} v_k D_k(z^*), \] (72)

where \( C(z, z^*) \) was defined by equation (39) and we defined

\[ D_k(z^*) = \frac{1}{z^* - \epsilon_i^*}. \] (73)

\( C(z, z^*) \) and \( D_k(z^*) \) in the lens equation (72) are expanded as

\[ C(z, z^*) = \sum_{p_2=0}^{\infty} \sum_{p_3=0}^{\infty} \cdots \sum_{p_N=0}^{\infty} (v_2)^{p_2}(v_3)^{p_3} \cdots (v_N)^{p_N} \]

\[ \times C_{(p_2,p_3)\cdots(p_N)}(z, z^*), \] (74)

\[ D_k(z^*) = \sum_{p_2=0}^{\infty} \sum_{p_3=0}^{\infty} \cdots \sum_{p_N=0}^{\infty} (v_2)^{p_2}(v_3)^{p_3} \cdots (v_N)^{p_N} \]

\[ \times D_{k(p_2,p_3)\cdots(p_N)}(z^*), \] (75)

where \( C_{(p_2,p_3)\cdots(p_N)} \) and \( D_{k(p_2,p_3)\cdots(p_N)} \) are independent of any \( v_i \). We seek a solution in expansion series as

\[ z = \sum_{p_2=0}^{\infty} \sum_{p_3=0}^{\infty} \cdots \sum_{p_N=0}^{\infty} (v_2)^{p_2}(v_3)^{p_3} \cdots (v_N)^{p_N} z_{(p_2,p_3)\cdots(p_N)}, \] (76)

where \( z_{(p_2,p_3)\cdots(p_N)} \) is a constant to be determined iteratively. The perturbed roots are normalized by the zeroth-order one as

\[ \sigma_{(p_2,p_3)\cdots(p_N)} = \frac{z_{(p_2,p_3)\cdots(p_N)}}{z_{(0)\cdots(0)}}. \] (77)

Equation (73) shows that \( D_k(z^*) \) has a pole at \( z^* = \epsilon_i^* \). Therefore, we shall discuss two cases of \( z_{(0)\cdots(0)} \neq \epsilon_i \) or \( z_{(0)\cdots(0)} = \epsilon_i \), separately.

8.1 Zeroth order \([z_{(0)\cdots(0)} \neq \epsilon_i \text{ for } i = 1, \ldots, N]\)

Zeroth order solutions are obtained by solving the equation as

\[ C(z, z^*) = 0. \] (78)

This has been solved for the binary lens case. The solution is given as

\[ z_{(0)\cdots(0)} = Aw, \] (79)

with the coefficient \( A \) defined by equation (49).
8.2 First order \([z_{(0...0)} \neq \epsilon_i \text{ for } i = 1, \ldots, N]\)

At the linear order in \(v_k\), equation (72) is

\[
C_{(0)}{(-1)}\cdot \cdot \cdot (0) = v_k D_{(0)}{(-1)}\cdot \cdot \cdot (0),
\]

where \(l_k\) denotes that the \(k\)th index is the unity. This equation is rewritten as

\[
z_{(0...-1)}\cdot \cdot \cdot (0) + a_{(0...-1)}\cdot \cdot \cdot (0) \times z_{(0...-1)}\cdot \cdot \cdot (0) = b_{(0...-1)}\cdot \cdot \cdot (0),
\]

where we defined

\[
a_{(0...-1)}\cdot \cdot \cdot (0) = \frac{1}{[z_{(0...0)}]^2},
\]

\[
b_{(0...-1)}\cdot \cdot \cdot (0) = \frac{\epsilon_i^*}{z_{(0...0)}[z_{(0...0)}^2 - \epsilon_i^2]}.
\]

By using equation (53), we obtain

\[
z_{(0...-1)}\cdot \cdot \cdot (0) = \frac{b_{(0...-1)}\cdot \cdot \cdot (0) - a_{(0...-1)}\cdot \cdot \cdot (0) b^*_0{(-1)}\cdot \cdot \cdot (0)}{1 - a_{(0...-1)}\cdot \cdot \cdot (0) a^*_0{(-1)}\cdot \cdot \cdot (0)}.
\]

8.3 Second order \([z_{(0...0)} \neq \epsilon_i \text{ for } i = 1, \ldots, N]\)

Let us consider two types of second-order solutions as \(z_{(0...-2)}\cdot \cdot \cdot (0)\) and \(z_{(0...-1)}\cdot \cdot \cdot (1)\cdot \cdot \cdot (0)\) for \(k \neq \ell\), separately.

First, we shall seek \(z_{(0...-2)}\cdot \cdot \cdot (0)\). At \(O(v_k^2\)\), equation (72) becomes

\[
z_{(0...-2)}\cdot \cdot \cdot (0) + a_{(0...-2)}\cdot \cdot \cdot (0) z^*_0{(-2)}\cdot \cdot \cdot (0) = b_{(0...-2)}\cdot \cdot \cdot (0),
\]

where we defined

\[
a_{(0...-2)}\cdot \cdot \cdot (0) = \frac{1}{[z_{(0...0)}]^2},
\]

\[
b_{(0...-2)}\cdot \cdot \cdot (0) = \frac{D_{(0)}{(-1)}\cdot \cdot \cdot (0)}{[z_{(0...0)}]^2} + \frac{|z^*_0{(-2)}\cdot \cdot \cdot (0)|^2}{z_{(0...0)}},
\]

where \(D_{(0)}{(-1)}\cdot \cdot \cdot (0)\) is written as

\[
D_{(0)}{(-1)}\cdot \cdot \cdot (0) = \frac{1}{2} \left[ \frac{1}{[z_{(0...0)}]^2} - \frac{1}{z_{(0...0)}^2 - \epsilon_i^2} \right].
\]

By using the relation (53) for equation (85), we obtain

\[
z_{(0...-2)}\cdot \cdot \cdot (0) = \frac{b_{(0...-2)}\cdot \cdot \cdot (0) - a_{(0...-2)}\cdot \cdot \cdot (0) b^*_0{(-2)}\cdot \cdot \cdot (0)}{1 - a_{(0...-2)}\cdot \cdot \cdot (0) a^*_0{(-2)}\cdot \cdot \cdot (0)}.
\]

Next, let us determine \(z_{(0...-1)}\cdot \cdot \cdot (1)\cdot \cdot \cdot (0)\). At \(O(v_k v_\ell)\) for \(k < \ell\), equation (72) becomes

\[
z_{(0...-1)}\cdot \cdot \cdot (1)\cdot \cdot \cdot (0) + a_{(0...-1)}\cdot \cdot \cdot (1)\cdot \cdot \cdot (0) z^*_0{(-1)}\cdot \cdot \cdot (1)\cdot \cdot \cdot (0) = b_{(0...-1)}\cdot \cdot \cdot (1)\cdot \cdot \cdot (0),
\]

where we defined

\[
a_{(0...-1)}\cdot \cdot \cdot (1)\cdot \cdot \cdot (0) = \frac{1}{[z_{(0...0)}]^2},
\]

\[
b_{(0...-1)}\cdot \cdot \cdot (1)\cdot \cdot \cdot (0) = -D_{(0)}{(-1)}\cdot \cdot \cdot (0) - D_{(0)}{(-1)}\cdot \cdot \cdot (0) + 2a^*_0{(-1)}\cdot \cdot \cdot (1)\cdot \cdot \cdot (0) a^*_0{(-1)}\cdot \cdot \cdot (1)\cdot \cdot \cdot (0).
\]

Here, \(D_{(0)}{(-1)}\cdot \cdot \cdot (0)\) and \(D_{(0)}{(-1)}\cdot \cdot \cdot (0)\) are written as

\[
D_{(0)}{(-1)}\cdot \cdot \cdot (0) = \frac{1}{[z_{(0...0)}]^2} - \frac{1}{[z^*_0{(-1)}\cdot \cdot \cdot (0)]^2},
\]

\[
D_{(0)}{(-1)}\cdot \cdot \cdot (0) = \frac{1}{[z_{(0...0)}]^2} - \frac{1}{[z^*_0{(-1)}\cdot \cdot \cdot (0)]^2 - \epsilon_i^2}.
\]

By using the relation (53) for equation (90), we obtain

\[
z_{(0...-1)}\cdot \cdot \cdot (1)\cdot \cdot \cdot (0) = \frac{b_{(0...-1)}\cdot \cdot \cdot (1)\cdot \cdot \cdot (0) - a_{(0...-1)}\cdot \cdot \cdot (1)\cdot \cdot \cdot (0) b^*_0{(-1)}\cdot \cdot \cdot (1)\cdot \cdot \cdot (0)}{1 - a_{(0...-1)}\cdot \cdot \cdot (1)\cdot \cdot \cdot (0) a^*_0{(-1)}\cdot \cdot \cdot (1)\cdot \cdot \cdot (0)}.
\]

8.4 Zeroth and first order \([z_{(0...0)} = \epsilon_i]\)

Next, we investigate the vicinity of \(z = \epsilon_i\), which is a pole of \(D_{(0)}\). The other pole of \(D_{(0)}\) is \(z = 0\), which makes \(C(z, z^*)\) divergent. Therefore, \(z = 0\) and its neighbourhood are abandoned. Let us focus on a root around

\[
z_{(0...0)} = \epsilon_i.
\]

If we admitted \(z_{(0...-1)}\cdot \cdot \cdot (0)\) around \(\epsilon_i\) for \(l \neq k\), only the \(D_k\) function would contain the inverse of \(v_k\), which introduces a term at \(O(v_k/v_\ell)\) in the lens equation and leads to inconsistency. Namely, the lens equation prohibits \(z_{(0...-1)}\cdot \cdot \cdot (0)\) around \(\epsilon_i\) for \(l \neq k\). This agrees with the polynomial case. We thus assume \(z = \epsilon_i + v_k z_{(0...-1)}\cdot \cdot \cdot (0) + O(v_k^2)\). Then, we obtain

\[
C_{(0)}{(-0)} = u - \epsilon_i + \frac{1}{\epsilon_i^2},
\]

\[
D_{(0...(-1)}\cdot \cdot \cdot (0)} = -\frac{1}{[z^*_0{(-1)}\cdot \cdot \cdot (0)]^2},
\]

where \(-l_k\) means that the inverse of \(v_k\) appears because of the pole at \(\epsilon_i\). Therefore, the lens equation at \(O(v_k^2)\) becomes linear in \(z_{(0...-1)}\cdot \cdot \cdot (0)\) without including \(z_{(0...-1)}\cdot \cdot \cdot (0)\). Immediately, it determines \(z^*_{(0...-1)}\cdot \cdot \cdot (0)\). Hence, its complex conjugate provides

\[
z_{(0...-1)}\cdot \cdot \cdot (0) = -\frac{\epsilon_i}{(u + \epsilon_i^2)\epsilon_i + 1}.
\]

8.5 Second order \([z_{(0...0)} = \epsilon_i]\)

Here, we consider the lens equation at \(O(v_\ell^2)\), namely \(C_{(0...-1)}\cdot \cdot \cdot (0) = D_{(0...0)}\), where

\[
D_{(0...0)} = \frac{1}{\epsilon_i^2} + \frac{z^*_0{(-2)}\cdot \cdot \cdot (0)}{[z^*_0{(-1)}\cdot \cdot \cdot (0)]^2}.
\]

Hence, we obtain \(z^*_{(0...-2)}\cdot \cdot \cdot (0)\) and thereby its complex conjugate as

\[
z_{(0...-2)}\cdot \cdot \cdot (0) = \left[ z_{(0...-1)}\cdot \cdot \cdot (0) \right]^2 \left[ C^*_{(0...-1)}\cdot \cdot \cdot (0) - \frac{1}{\epsilon_i^2} \right],
\]

where \(C_{(0...-1)}\cdot \cdot \cdot (0)\) becomes

\[
C_{(0...-1)}\cdot \cdot \cdot (0) = \left[ z_{(0...-1)}\cdot \cdot \cdot (0) + \frac{z^*_0{(-1)}\cdot \cdot \cdot (0)}{\epsilon_i^2} \right].
\]
Figure 3. Light curves by two methods. One is based on a numerical case that the lens equation is solved numerically. The other is due to the first order approximation. The top figure shows that the two curves are overlapped, where A denotes the total amplification. The bottom panel shows the residual by the two methods. The residual is defined as the difference between A computed numerically and A in the linear approximation. We assume the source trajectory as $w = 1.4 + it$. Here, the time $t$ is in units of the Einstein cross time, which is defined as $\theta_E/v_\perp$ for the transverse angular relative velocity. The lens parameters are $\nu_2 = 0.1$ and $e = 1$.

Next, we consider a root at $O(\nu_k^{\ell})$, where we can assume $k < \ell$ without loss of generality. At this order, the inverse of $\nu_k$ appears.

The lens equation at $O(\nu_k^{\ell})$ becomes

$$C_{(0)-(1_k)-(\ell)-0} = D_{(0)-(1_k)-(\ell)-0} + D_{(0)-(0)},$$

(103)

where

$$D_{(0)-(1_k)-(\ell)-0} = \frac{z_{(0)-(1_k)-(\ell)-0}}{[z_{(0)-(1_k)-(\ell)-0}]^2}.$$ (104)

Hence, we obtain $z_{(0)-(1_k)-(\ell)-0}$ and thereby its complex conjugate as

$$z_{(0)-(1_k)-(\ell)-0} = \left[z_{(0)-(1_k)-(\ell)-0}\right]^2$$

$$\times \left[C_{(0)-(1_k)-(\ell)-0} - D_{(0)-(0)}\right],$$ (105)

Figure 4. Light curves by two methods. In this figure, we assume a different source trajectory as $w = 0.8 + it$. The lens parameters are the same as $\nu_2 = 0.1$ and $e = 1$ in Fig. 3. The solid curve in the top panel denotes a case when the lens equation is numerically solved. The dotted curve is drawn by using the linear order approximation. The bottom panel shows the residual between the two curves.

where $C_{(0)-(1_k)-(\ell)-0}$ and $D_{(0)-(0)}$ are written as

$$C_{(0)-(1_k)-(\ell)-0} = -z_{(0)-(1_k)-(\ell)-0} - \frac{z_{(0)-(1_k)-(\ell)-0}}{\epsilon_k^2},$$ (106)

$$D_{(0)-(0)} = \frac{1}{\epsilon_k^2} - \frac{1}{\epsilon_k^2 - \epsilon_k'},$$ (107)

This direct computation shows that $z_{(0)-(1_k)-(\ell)-0}$ does not exist because $z_{(0)-(1_k)-(\ell)-0}$ is prohibited in the vicinity of $\epsilon_k$. This is also consistent with the polynomial case at the second order.

8.6 Magnifications

Before closing this paper, it is worthwhile to mention magnifications by N point-mass lensing in the framework of the present perturbation theory that is intended to solve the lens equation to obtain image positions. The amplification factor is the inverse of the Jacobian for
Figure 5. Graph representations of interactions among point masses for images at the second order level. The top and bottom graphs represent a mutually interacting image and a self-interacting one, respectively.

the lens mapping. It is expressed as

\[ A \equiv \left( \frac{\partial \beta}{\partial \theta} \right)^{-1} = \left[ \frac{\partial (w, w^*)}{\partial (\tau, \tau^*)} \right]^{-1} = \left( \frac{\partial w}{\partial \tau} \right)^2 - \left( \frac{\partial w}{\partial \tau^*} \right)^2 \right)^{-1}, \]

(108)

where the terms in the last line can be computed directly by a derivative of equation (3), the lens equation in a complex notation. Amplifications of each image are obtained by substituting its image position into equation (108). Practical numerical estimations may follow this procedure. For illustrating this, Figs 3 and 4 show examples of light curves by a binary lens via the perturbative approach. These curves are well reproduced. However, double peaks due to caustic crossings cannot be reproduced by the present method.

As an approach enabling a simpler argument before going to numerical estimations, we use the functional form of perturbed image positions. In the perturbation theory, lensed images can be split into two groups. One is that their zeroth-order root is not located at a lens object \( (z_0, \theta_0) \neq 0 \). In the other group, zeroth-order roots originate from a lens position at \( \epsilon_k \). We call the former and latter ones mutually interacting and self-interacting images, respectively, because all the lens objects make contributions to mutually interacting images at the linear order as shown by equation (84). On the other hand, self-interacting images are influenced only by the nearest lens object at \( \epsilon_k \) at the linear and even at the second orders as shown by equations (99) and (101). Fig. 5 shows graph representations for the two groups of images.

For the simplicity, we consider stretching of images roughly as \( |\partial z/\partial w| \), though rigorously speaking it must be the amplification. Table 1 and equation (76) mean that the complex derivative becomes

\[ \frac{\partial z}{\partial w} = \frac{\partial z(0)}{\partial w} + \sum_k v_k \frac{\partial z(0)}{\partial w} (1_k \cdot 0_k), \]

(109)

for mutually interacting images and

\[ \frac{\partial z}{\partial w} = v_k \frac{\partial z(0)}{\partial w} (1_k \cdot 0_k), \]

(110)

for self-interacting images, where we used that \( \epsilon_k \) is a constant.

For the simplicity, we assume \( v_k \approx O(1/N) \) for a large \( N \) case. Then, the linear order term in self-interacting images is \( O(1/N) \), and thus they become negligible as \( N \to \infty \). On the other hand, mutually interacting ones have non-vanishing terms even at the zeroth order. Hence, they can play a crucial role.

However, we should take account of a spatial distribution of lens objects. If they are clustering and thus dense at a certain region, then the total flux of light through such a dense region is not negligible any more. Let us denote the fraction of the clustering particles by...
mass gravitational lens systems without symmetries on a plane. The system can be separated into a single mass lens as a background and its perturbation due to the remaining point masses.

First, we investigated perturbative structures of the single-complex-variable polynomial, into which the lens equation is embedded. Some of zeroth-order roots of the polynomial do not satisfy the lens equation and thus are unphysical. This appearance of correct but unphysical roots is consistent with the earlier work on a theorem on the maximum number of lensed images (Rhie 2001, 2003). However, the theorem never tells which roots are physical (or unphysical). What we did is that unphysical roots are identified.

Next, we re-examined the lens equation in the dual-complex-variables formalism to avoid inclusions of unphysical roots. We presented an explicit form of perturbed image positions as a function of source and lens positions. As a key tool for perturbative computations, equation (53) was also found. For readers’ convenience, the perturbative roots are listed in Table 3. If one wishes to go to higher orders, our method will enable one to easily use computer algebra softwares such as MAPLE and MATHEMATICA. This is because it requires simpler algebra (only the four basic operations of arithmetic), compared with vector forms which need extra operations such as inner and outer products.

There are numerous possible applications along the course of the perturbation theory of N point-mass gravitational lens systems. For instance, it will be interesting to study lensing properties such as magnifications by using the functional form of image positions. Furthermore, the validity of the present result may be limited in the weak field regions. It is important also to extend the perturbation theory to a domain near the strong field.

Our method considers only the images which exist in the small mass-ratio limit as $v_1 \to 0$. The number of the images that admit the small mass-ratio limit is less than the maximum number. This suggests that the other images do not have the small mass limit. Therefore, it is conjectured that positions of the extra images could not be expressed as Maclaurin series in mass ratios. This may be implied also by previous works. For instance, the appearance of the maximum number of images for a binary lens requires a finite mass ratio and the caustic crossing (Schneider & Weiss 1986). Regarding this point, further studies will be needed to determine positions of all the images with the maximum number as a function of lens and source parameters.

9 CONCLUSION

Under a small mass-ratio approximation, this paper developed a perturbation theory of N coplanar (in the thin lens approximation) point-
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