Fractal patterns related to dividing coins

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Abstract

The present paper formulates and solves a problem of dividing coins. The basic form of the problem seeks the set of the possible ways of dividing coins of face values 1, 2, 4, 8, . . . between three people. We show that this set possesses a nested structure like the Sierpinski-gasket fractal. For a set of coins with face values power of \(r\), the number of layers of the gasket becomes \(r\). A higher-dimensional Sierpinski gasket is obtained if the number of people is more than three. In addition to Sierpinski-type fractals, the Cantor set is also obtained in dividing an incomplete coin set between two people.

Keywords: iterated function system, coin dividing, Sierpinski gasket, Cantor set

1. Introduction

Money is a mathematical system so familiar to us. It is a very good instance of combinatorics and discrete mathematics.

Telser [1] considered the problem of optimal currency, and deduced that the optimal currency system consists of denominations with face values of 1, 3, 9, . . . , \(3^m - 1\). This result comes from the problem of Bachet which seeks the smallest number of weights so that they can weigh any integer quantity on a two-pan balance. Many actual currency systems take their average multiples close to three: 2.8 by Wynne [2] and 2.60 by Tschoegl [3]. This fact is surprising because most currencies are established based on the decimal system, which is not compatible with Telser’s power-of-three theory, and because individual customs and culture of each country are reflected to its currency.

One of the classical and famous problems is the Frobenius coin problem, which seeks the largest amount of money that cannot be made using only coins of specified denominations [4]. The solution to this problem is called the Frobenius number. For only two types of coins of denominations \(a_1\) and \(a_2\) which are relatively prime, the Frobenius number is given by \(a_1a_2 - a_1 - a_2\) [5]. However, closed expressions are not known for more than two types of coins, and the problem was found to be NP-hard [6].

Another classical problem about a coin is the change-making problem. It asks how a given amount of money can be made with the least number of coins of given denominations. This problem is a variant of the knapsack problem [7], and is also an NP-hard problem [8]. The greedy algorithm [9] and other heuristic methods such as dynamic programming [10] give the optimal solution in some cases. A coin system is said to be canonical if the greedy algorithm works correctly [11], and almost all currencies in the world are arranged to be canonical. Even with a canonical coin system, which is easy to study, the change-making process possesses a rich mathematical

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structure. If we repeatedly pay money so that the number of coins in the purse after each payment is minimized, a fractal pattern is obtained from a sequence of change amounts \[12, 13\].

The present article develops a generation mechanism of a fractal set in the form of dividing coins between people. The problem is very simple in appearance, but the possible ways of division form a nontrivial fractal pattern like the Sierpinski gasket and Cantor set. The basic situation, discussed in \[12\], is that three people divide coins of face values \(1, 2, 4, \ldots, 2^{m-1}\), and the Sierpinski gasket appears as the attractor. The emergence of the Sierpinski gasket has been previously reported \[14\], but in the present article we rigorously formulate the fractal structure and diverse generalizations in the framework of iterated function systems. In addition to the Sierpinski gasket, the Cantor set also appears in a specific case.

The notion of fractal was invented by Mandelbrot \[15\] to measure the morphology of natural objects. Fractal theory has been applied to various phenomena such as in physics, economics, and biology. In naive description, a fractal object is created by defining a starting shape (an \textit{initiator}) and replacing each part with another shape called a \textit{generator}, ad infinitum \[16\]. Mathematically, a fractal is defined using an iterated function system (IFS for short). Here we briefly review construction of the Sierpinski gasket and Cantor set (see Barnsley \[17\] for detail). Let \(p_1, p_2, p_3\) be three points in \(\mathbb{R}^d\) which are not collinear. Let three contraction maps \(f_1, f_2, f_3 : \mathbb{R}^d \rightarrow \mathbb{R}^d\) be given by \(f_i(x) = (x + p_i)/2\) for \(i = 1, 2, 3\). The set of contraction maps \(\{f_1, f_2, f_3\}\) is called an IFS. We let \(H(\mathbb{R}^d)\) denote the collection of all nonempty compact subsets of \(\mathbb{R}^d\).

It is well known that \(H(\mathbb{R}^d)\) is a complete metric space with Hausdorff metric \[17\]. The IFS \(\{f_1, f_2, f_3\}\) defines a map \(F : H(\mathbb{R}^d) \ni K \mapsto f_1(K) \cup f_2(K) \cup f_3(K) \in H(\mathbb{R}^d)\) which is a contraction on \(H(\mathbb{R}^d)\). The Sierpinski gasket \(\triangle \in H(\mathbb{R}^d)\) is defined as the unique fixed point of \(F\) which obeys \(\triangle = F(\triangle) = f_1(\triangle) \cup f_2(\triangle) \cup f_3(\triangle)\) existentially and uniqueness are guaranteed by the contraction mapping theorem (or the Banach fixed-point theorem). The fixed point \(\triangle\) also satisfies \(\triangle = \lim_{m \to \infty} F^m(K)\) for any \(K \in H(\mathbb{R}^d)\); in this sense, \(\triangle\) is called the \textit{attractor} of the IFS. The three points \(p_1, p_2,\) and \(p_3\) are at the three corners of \(\triangle\). In a similar way, the Cantor set between \(p_1\) and \(p_2\) is the attractor of an IFS \(\{g_1, g_2 : \mathbb{R}^d \rightarrow \mathbb{R}^d\}\) where \(g_i(x) = (x + p_i)/3\) for \(i = 1, 2\).

2. Formulation of the problem and a basic solution

This paper treats of a problem related to dividing a set of coins by some “players”. In order to specify the set of coins, we introduce a notation

\[
\begin{pmatrix}
  v_1 & \cdots & v_m \\
  c_1 & \cdots & c_m
\end{pmatrix},
\]

where \(v_1, \ldots, v_m\) is the list of face values of the coins, and \(c_i\) is the number of coins of face value \(v_i\). For a coin set \(S = \begin{pmatrix}
  v_1 & \cdots & v_m \\
  c_1 & \cdots & c_m
\end{pmatrix}\), we define \(|S|\) as the total amount of coins in \(S\), given by

\[
|S| := \sum_{i=1}^{m} v_i c_i.
\]

For example, the set of coins

\[
S = \begin{pmatrix}
  1 & 10 \\
  2 & 1
\end{pmatrix}
\]

consists of two pennies (1 cent coins) and one dime (10 cent coin). When two players \(A\) and \(B\) divide these coins, the division of these coins is expressed by a pair \((n_A, n_B)\) of money amounts that \(A\) and \(B\) respectively receive. There are six ways to divide the coins in this case:

\[
(n_A, n_B) = (0, 12), (1, 11), (2, 10), (10, 2), (11, 1), (12, 0).
\]
Note that we admit cases in which some players receive no coins. In this paper, we mainly focus on the coin-dividing problem between three players.

We start from the most basic form of the problem. Let us suppose a “binary” currency system

$$S_{2,m} = \begin{pmatrix} 1 & 2 & 4 & \cdots & 2^{m-1} \\ 1 & 1 & 1 & & 1 \end{pmatrix},$$

and three players $A$, $B$, and $C$ divide these coins. Each way of division is represented by lining each player’s share as the triplet $(n_A, n_B, n_C)$, which defines a point in a three-dimensional space. We study the structure of the possible points $(n_A, n_B, n_C)$.

This problem can be described by weights. If there are $m$ weights of $1, 2, \ldots, 2^{m-1}$ grams, and one separates them into three groups (admitting one or two null groups), then we study the shape of the possible groupings. The equivalence of coins and weights resembles Telser’s theory of optimal currency (see the beginning of §).

Formally, an orthonormal basis $\{ e_A, e_B, e_C \}$ is taken so that $(n_A, n_B, n_C) = n_A e_A + n_B e_B + n_C e_C$. By definition each division satisfies $n_A + n_B + n_C = 2^m - 1$ and $n_A, n_B, n_C \geq 0$. Hence, all the possible points $(n_A, n_B, n_C)$ are confined in a triangular area spanned by $(2^m - 1)e_A$, $(2^m - 1)e_B$, and $(2^m - 1)e_C$. Let $\Delta_m(\subset \mathbb{R}^3)$ denote the set of the points of possible division $(n_A, n_B, n_C)$. For the simplest case $m = 1$, three players divide only one coin of face value 1. The possible ways of division are $(n_A, n_B, n_C) = (1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$, and they form an equilateral triangle in the three dimensional space (Fig. 1(a)):

$$\Delta_1 = \{ (1, 0, 0), (0, 1, 0), (0, 0, 1) \} = \{ e_A, e_B, e_C \}.$$

Next we take into account the coin of face value 2 to advance to $m = 2$. If the player $A$ receives this coin, the set of possible division is obtained by considering who receives the coin of face value 1. This set is written as $\{(3, 0, 0), (2, 1, 0), (2, 0, 1)\} = \Delta_1 + 2e_A$, which is translation of $\Delta_1$. Similar expressions hold for $B$ and $C$, so $\Delta_2$ is given by

$$\Delta_2 = (\Delta_1 + 2e_A) \cup (\Delta_1 + 2e_B) \cup (\Delta_1 + 2e_C).$$

That is, $\Delta_2$ consists of three copies of $\Delta_1$, each of which is placed at one of the three corners (see Fig. 1(b) for reference). Similarly, by considering who receives the coin of face value $2^{m-1}$, $\Delta_m$ is inductively given by

$$\Delta_m = (\Delta_{m-1} + 2^{m-1}e_A) \cup (\Delta_{m-1} + 2^{m-1}e_B) \cup (\Delta_{m-1} + 2^{m-1}e_C).$$

(1)
For each \( m \geq 2 \), three copies of \( \triangle_{m-1} \) are placed triangularly to form \( \triangle_m \). By setting \( \triangle_0 = \{(0,0,0)\} \) for convenience, Eq. (1) holds also for \( m = 1 \).

Intuitively, \( \triangle_m \) achieves self-similarity as \( m \) increases. Actually, \( \triangle_7 \), shown in Fig. 1 (c), is almost indistinguishable from the genuine Sierpinski gasket. The connection between \( \triangle_m \) and the Sierpinski gasket is formulated as follows. We set three contraction maps \( f_A, f_B, f_C : \mathbb{R}^3 \to \mathbb{R}^3 \) given by

\[
f_P(x) = \frac{x + e_P}{2} \quad (P = A, B, C).
\]

As stated in [1] the attractor \( \triangle \in \mathcal{H}(\mathbb{R}^3) \) of an IFS \( \{f_A, f_B, f_C\} \) is the Sierpinski gasket spanned between \( e_A, e_B, \) and \( e_C \). On the other hand, multiply \( 2^{-m} \) to Eq. (1) to get

\[
2^{-m} \triangle_m = \left( \frac{2^{-(m-1)} r \triangle_{m-1} + e_A}{2} \right) \cup \left( \frac{2^{-(m-1)} r^2 \triangle_{m-1} + e_B}{2} \right) \cup \left( \frac{2^{-(m-1)} r^{m-1} \triangle_{m-1} + e_C}{2} \right) = f_A(2^{-(m-1)} \triangle_{m-1}) \cup f_B(2^{-(m-1)} \triangle_{m-1}) \cup f_C(2^{-(m-1)} \triangle_{m-1}) = F(2^{-(m-1)} \triangle_{m-1}),
\]

where \( F \) is the contraction map on \( \mathcal{H}(\mathbb{R}^3) \) induced from the IFS. That is, adding the \( (m+1) \)st coin to the coin set \( S_{2,m} \) is directly represented as the action of \( F \). Hence \( 2^{-m} \triangle_m = F^m(\triangle_0) \). Because \( \triangle_0 = \{(0,0,0)\} \in \mathcal{H}(\mathbb{R}^3) \), it follows from the contraction mapping theorem that

\[
\triangle = \lim_{m \to \infty} F^m(\triangle_0) = \lim_{m \to \infty} 2^{-m} \triangle_m.
\]

This is the result which connects the coin-dividing problem and the Sierpinski gasket.

3. Generalization of the coin system

In this section we consider a generalized coin-dividing problem where the coin system is not powers of two. We give the set of coins

\[
S_{r,m} = \left( \begin{array}{cccc}
1 & r & r^2 & \cdots & r^{m-1} \\
1 & r-1 & r-1 & \cdots & r-1
\end{array} \right),
\]

where \( r \) is an integer equal to or greater than two, and \( |S_{r,m}| = r^m - 1 \).

Before working on the problem of \( S_{r,m} \), we revise Eq. (1) to derive a useful expression. By means of the result \( \triangle_1 = \{e_A, e_B, e_C\} \), Eq. (1) is rewritten as

\[
\triangle_m = \bigcup_{q \in \triangle_1} (\triangle_{m-1} + 2^{m-1}q)
\]

for \( m \geq 2 \). A similar formula is valid for \( S_{r,m} \), and hence the set \( \triangle_{r,m} \) of the possible points corresponding to \( S_{r,m} \) is inductively given by

\[
\triangle_{r,1} = \{q_A e_A + q_B e_B + q_C e_C | q_A, q_B, q_C \in \mathbb{N} \cup \{0\}, q_A + q_B + q_C = r - 1\},
\]

\[
\triangle_{r,m} = \bigcup_{q \in \triangle_{r,m-1}} (\triangle_{r,m-1} + r^{m-1}q).
\]

The set \( \triangle_{r,1} \) consists of \( r(r+1)/2 \) points which form a triangle, and similarly \( \triangle_{r,m} \) consists of \( r(r+1)/2 \) copies of \( \triangle_{r,m-1} \) arranged triangularly.
Figure 2: The sets of possible points \((n_A, n_B, n_C)\) of the coin set \(S_{3,4}\) (a), \(S_{4,3}\) (b), and \(S_{5,3}\) (c). They resemble the Sierpinski gasket, but the number of layers is equal to \(r\).

By multiplying \(r^{-m}\) to Eq. (4), we have

\[
r^{-m} \Delta_{r,m} = \bigcup_{q \in \Delta_{r,1}} f_q(r^{-(m-1)} \Delta_{r,m-1}),
\]

where \(f_q : \mathbb{R}^3 \ni x \mapsto (x + q)/r \in \mathbb{R}^3\) is a contraction map. By the contraction mapping theorem, \(r^{-m} \Delta_{r,m}\) converges to an attractor of IFS \(\{f_q | q \in \Delta_{r,1}\}\), as \(m \to \infty\). This attractor is the Sierpinski gasket with \(r\) layers. We depict the coin-dividing plot corresponding to \(r = 3, 4,\) and \(5\) in Fig. 2. The plot is essentially the same as a class of Pascal-Sierpinski gaskets if \(r\) is prime— the Pascal-Sierpinski gasket of order \(p\) is obtained by coloring the number not divisible by \(p\) in Pascal’s triangle \([19]\).

More generally, we further generalize the coin set

\[
S_{r,c,m} = \begin{pmatrix} 1 & r & \cdots & r^{m-1} \\ c & c & \cdots & c \end{pmatrix},
\]

and the corresponding set \(\Delta_{r,c,m}\) of possible division of coins. Obviously, the point set \(\Delta_{r,c,m}\) lies on the plane given by \(n_A + n_B + n_C = |S_{r,c,m}| = c(r^m - 1)/(r - 1)\). Furthermore, the coin-dividing problem is symmetric with respect to the three players \(A, B,\) and \(C\). Hence \(\Delta_{r,c,m}\) is invariant under the action of any permutation of \((A, B, C)\). That is, \(\Delta_{r,c,m}\) has left-right symmetry and three-fold rotational symmetry.

The inductive formula of \(S_{r,c,m}\) is

\[
\Delta_{r,c,m} = \bigcup_{q \in \Delta_{c+1,1}} (\Delta_{r,c,m-1} + r^{m-1}q),
\]

where \(\Delta_{c+1,1}\) is given by Eq. (3). The structure of the attractor is classified into three cases according to the values of \(r\) and \(c\).

1. If \(c < r - 1\), some money amounts less than \(|S_{r,c,m}|\) cannot be made by picking out coins from \(S_{r,c,m}\). In fact, the money amount \(r - 1\) needs \(r - 1\) coins of face value 1, but there are only \(c\) \((< r - 1)\) coins of face value 1 in \(S_{r,c,m}\). Reflecting this, the attractor becomes totally disconnected.
2. If \(c = r - 1\), the attractor is the Sierpinski gasket with \(r\) layers. It belongs to the class of finitely-ramified fractals, which means that any subset of the fractal can be disconnected by removing a finite number of points.
3. If \( c > r - 1 \), some money amounts can be made by more than one way; for example, the money amount \( r \) can be made by one coin of face value \( r \) or by \( r \) coins of face value 1. The attractor becomes infinitely ramified.

Figure 3 shows the structural difference by the magnitude of \( c \) and \( r - 1 \).

The Hausdorff dimension of the corresponding attractor is easily derived for \( c \leq r - 1 \). The whole pattern can be decomposed into the \((c+1)(c+2)/2\) smaller copies of itself scaled by a factor \( r \), so the similarity dimension \( D_{r,c} \), depending on \( r \) and \( c \), is given by

\[
D_{r,c} = \frac{\ln((c+1)(c+2)/2)}{\ln r}.
\]

Since \((c+1)(c+2)/2 < r^2\) for any \( r \geq 2 \) and \( c \leq r - 1 \), we have \( D_{r,c} < 2 \). In particular, \( D_{2,1} = \ln 3 / \ln 2 \) (\( \approx 1.585 \)) is the Hausdorff dimension of the ordinary Sierpinski gasket. We recall here that \((c+1)(c+2)/2\) is the number of contraction maps of the IFS, and \(1/r\) is the contraction ratio of each map in the IFS. For \( c > r - 1 \), the dimension is not calculated easily because of overlap. The Hausdorff dimension of an overlapping Sierpinski gasket has been derived only in restricted situations [20].

In the case of \( c > r - 1 \), a point of \( \Delta_{r,c,m} \) can correspond to more than one ways of coin division. We can decompose \( \Delta_{r,c,m} \) into subsets according to this multiplicity. Figure 4 shows an example of this decomposition where \((c, r, m) = (3, 3, 4)\). The highest multiplicity in this case is nine, so \( \Delta_{r,c,m} \) splits into nine subsets, each of which has a fractal shape. A subset becomes sparse in high multiplicity, and we only present the lowest four subsets in the figure. We expect that this multi-level structure gives insight to analyses of an overlapping Sierpinski gasket.

For reference, we consider the coin system of the US dollar: 1, 5, 10, 25, 50 cent coins. We choose a set of coins as

\[
S_{\text{cent}} = \begin{pmatrix} 1 & 5 & 10 & 25 & 50 \\ 4 & 1 & 2 & 1 & 1 \end{pmatrix},
\]

so that each money amount from 1 to \( |S_{\text{cent}}| = 104 \) cents can be made by some coins of \( S_{\text{cent}} \). The set of possible division is shown in Fig. 5, which is obtained by enumerating all division directly. A hierarchical structure can be seen but not perfectly. Breaking of the hierarchical structure is due to the property that some money amounts can be made in two ways, e.g., \( 25 = 25 \times 1 = 5 \times 1 + 10 \times 2 \) and \( 50 = 50 \times 1 = 5 \times 1 + 10 \times 2 + 25 \times 1 \).
4. Generalization of the number of players

In this section, the number of players is changed. For simplicity, we mainly deal with the binary coin set $S_{2,m}$ as in the most basic type of the problem stated in §2.

We start from the coin dividing between four players $A$, $B$, $C$, and $D$. Each way of division, written as a quadruplet $(n_A, n_B, n_C, n_D)$, gives a point in a four-dimensional space, and we represent the point as $(n_A, n_B, n_C, n_D) = n_A e_A + n_B e_B + n_C e_C + n_D e_D$. Let us denote by $\triangle^{(4)}_m$ the point set corresponding to the possible division of the coins. The superscript “$^{(4)}$” represents explicitly the number of players.
By using an argument similar to that in [2] $\Delta_m^{(4)}$ is determined inductively as

$$\Delta_1^{(4)} = \{ e_A, e_B, e_C, e_D \},$$

$$\Delta_m^{(4)} = (\Delta_{m-1}^{(4)} + 2^{m-1} e_A) \cup (\Delta_{m-1}^{(4)} + 2^{m-1} e_B) \cup (\Delta_{m-1}^{(4)} + 2^{m-1} e_C) \cup (\Delta_{m-1}^{(4)} + 2^{m-1} e_D).$$

The first equation signifies that $\Delta_1^{(4)}$ consists of four points corresponding to who receives the coin of face value 1. $\Delta_1^{(4)}$ forms the vertices of a regular tetrahedron whose sides are $\| e_A - e_B \| = \| e_A - e_C \| = \cdots = \| e_C - e_D \| = \sqrt{2}$. The second equation signifies that $\Delta_m^{(4)}$ is made up of four subsets corresponding to who receives the coin of face value $2^{m-1}$, which generates a nested structure of tetrahedra.

$\Delta_m^{(4)}$ is a point set in the four-dimensional space, but it lies on a three-dimensional affine hyperplane given by $n_A + n_B + n_C + n_D = 2^m - 1$. Hence we can visualize $\Delta_m^{(4)}$ by taking a three-dimensional coordinate suitably. Setting three vectors $u_1 := e_A - e_D, u_2 := e_B - e_D, \text{ and } u_3 := e_C - e_D$, we get

$$n_A e_A + n_B e_B + n_C e_C + n_D e_D = n_A (e_A - e_D) + n_B (e_B - e_D) + n_C (e_C - e_D) + (2^m - 1) e_D = n_A u_1 + n_B u_2 + n_C u_3 + (2^m - 1) e_D.$$

That is, $(n_A, n_B, n_C, n_D) \in \text{span}\{ u_1, u_2, u_3 \} + (2^m - 1) e_D$. One can find an orthonormal basis $\{ v_1, v_2, v_3 \}$ of subspace $\text{span}\{ u_1, u_2, u_3 \}$ by the Gram-Schmidt process, as

$$v_1 = \frac{u_1}{\sqrt{2}}, \quad v_2 = \frac{2 u_2 - u_1}{\sqrt{6}}, \quad v_3 = \frac{3 u_3 - u_2 - u_1}{2 \sqrt{3}}.$$

Therefore, we have an isometric embedding

$$\Delta_m^{(4)} \ni (n_A, n_B, n_C, n_D) \mapsto \left( \frac{2 n_A + n_B + n_C}{\sqrt{2}}, \frac{3 n_B + n_C}{\sqrt{6}}, \frac{2 n_C}{\sqrt{3}} \right) \in \mathbb{R}^3. \quad (5)$$

The $i$-th component on the right-hand side is the scalar product of $n_A u_1 + n_B u_2 + n_C u_3$ and $v_i$ ($i = 1, 2, 3$). In Fig. (a), we illustrate $\Delta_m^{(4)}$ with $m = 6$ embedded in a three-dimensional space by using mapping (5). $\Delta_m^{(4)}$ is a finite approximation of the three-dimensional Sierpinski gasket or so-called Sierpinski tetrahedron, which is intuitively obtained by iterating the removal process shown in Fig. (b). By analogy with the results in the previous section, the change of a coin system affects the number of layers of the Sierpinski tetrahedron.

The coin-dividing problem between $s(>4)$ players is considered as well. Taking an orthonormal basis $\{ e_1, \ldots, e_s \}$ of an $s$-dimensional space, we have a recursive equation of $\Delta_m^{(s)}$

$$\Delta_1^{(s)} = \{ e_1, \ldots, e_s \}, \quad \Delta_m^{(s)} = \bigcup_{s=1}^s (\Delta_{m-1}^{(s)} + 2^{m-1} e_s).$$

$\Delta_m^{(s)}$ becomes $(s - 1)$-dimensional Sierpinski gasket as $m$ increases, but we cannot visualize such a high-dimensional object.

Of course, we can treat of the coin-dividing problem between one or two players. For the dividing by one player $A$, there is only one trivial dividing of which all coins go to $A$; therefore, $\Delta_1^{(1)}$ is a one-point set. As for the dividing between two players $A$ and $B$, the solution is $\Delta_2^{(2)} = \{(n_A, n_B) | n_A = 0, 1, \ldots, 2^m - 1, n_A + n_B = 2^m - 1\}$, consisting of $2^m$ points which are equally spaced. The inductive relation in this case is

$$\Delta_m^{(2)} = (\Delta_{m-1}^{(2)} + 2^{m-1} e_A) \cup (\Delta_{m-1}^{(2)} + 2^{m-1} e_B).$$

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which is converted to

\[ 2^{-m} \Delta_m^{(2)} = f_A(2^{-(m-1)} \Delta_{m-1}) \cup f_B(2^{-(m-1)} \Delta_{m-1}), \]

where \( f_A \) and \( f_B \) are given in Eq. (2). It is easy to find that the attractor of IFS \( \{ f_A, f_B \} \) is a line segment \( \{(n_A, n_B) | n_A \geq 0, n_B \geq 0, n_A + n_B = 1 \} \). In conclusion, nontrivial fractal pattern is not observed in these two cases. However, we explain in the next section that coin-dividing between two players with a suitable coin set can generate the Cantor set.

5. Cantor set between two players

In the previous section, no interesting fractal structure emerges in coin dividing between two players. We see in this section that the Cantor set is obtained if the coin set is not complete.

Let us study on trial the coin-dividing problem by two players \( A \) and \( B \) with the set of coins

\[ S_{3,1,m} = \begin{pmatrix} 1 & 3 & 9 & \cdots & 3^{m-1} \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix}. \]

Some money amounts, e.g., 2, 5, 6, 7, and 8, cannot be made using these coins—this incompleteness is a different property from \( S_{2,m} (= S_{2,1,m}) \). We let \( C_m \) denote the set of pairs \( (n_A, n_B) \) of dividing coins \( S_{3,1,m} \).

The inductive relation of \( C_m \) is

\[ C_m = (C_{m-1} + 3^{m-1} e_A) \cup (C_{m-1} + 3^{m-1} e_B), \]

and is equivalently

\[ 3^{-m} C_m = \left( \frac{3^{-(m-1)} C_{m-1} + e_A}{3} \right) \cup \left( \frac{3^{-(m-1)} C_{m-1} + e_B}{3} \right) = g_A(3^{-(m-1)} C_{m-1}) \cup g_B(3^{-(m-1)} C_{m-1}), \]

where \( g_A(x) = (x + e_A)/3 \) and \( g_B(x) = (x + e_B)/3 \) are contraction maps. The set \( 3^{-m} C_m \) converges to the attractor of IFS \( \{ g_A, g_B \} \), and as stated in [11] it is the Cantor set spanned between \( e_A \) and \( e_B \). In addition, \( C_m \) can be regarded as a subset of \( \Delta_{3,1,m} \) formed by the points of \( n_C = 0 \), or the intersecting points of \( \Delta_{3,1,m} \) and the \((n_A, n_B)\) plane (see Fig. 7 for reference).
Figure 7: (a) The possible division of \(S_{3,1,4}\). For ease of display, two axes are chosen to \((2n_A, 2n_B)\). (b) The intersecting points of \(\triangle_{3,1,4}\) and the \((n_A, n_B)\)-plane is \(C_4\).

Let us study more about the coin-dividing problem and the Cantor set. We introduce a mapping \(\varphi_m : C_m \ni (n_A, n_B) \mapsto 2 \cdot 3^{-m}n_A \in [0, 1]\). \(\varphi_m\) is the composite mapping of a projection \(\pi : (x, y) \mapsto x\) and a scaling function \(\rho_m : x \mapsto 2x/3^m\). By definition, money amount \(n_A\) is written as \(n_A = \sum_{k=0}^{m-1} \chi_k 3^k\), where \(\chi_k \in \{0, 1\}\) is the indicator of whether the coin of face value \(3^k\) goes to \(A\) or not. Thus,

\[
\varphi_m(C_m) = \left\{ \sum_{k=0}^{m-1} \frac{2\chi_k}{3^{m-k}} \mid \chi_k \in \{0, 1\} \right\}.
\]

\(\varphi_m(C_m)\) consists of numbers within \([0, 1]\) whose base-3 representations are up to \(m\) digits with entirely 0s and 2s, and formally \(\lim_{m \to \infty} \varphi(C_m) \subset [0, 1]\) consists of numbers whose base-3 representation are entirely 0s and 2s. As is known well [21], this is identical with another definition of the Cantor set.

Generalization to the coin set

\[
S_{r,1,m} = \left( \begin{array}{cccc}
1 & r & r^2 & \cdots & r^{m-1} \\
1 & 1 & 1 & \cdots & 1
\end{array} \right)
\]

is straightforwardly. By way of the inductive relation for the set \(C_{r,m}\) of division, \(r^{-m}C_{r,m}\) converges to a Cantor-like fractal whose Hausdorff dimension is

\[
D_r = \frac{\ln 2}{\ln r}.
\]

In particular, \(D_3 = \ln 2/\ln 3 \approx 0.631\) is the dimension of the Cantor set. Moreover, \(D_2 = \ln 2/\ln 2 = 1\) is consistent with the result that the attractor of coin-dividing of \(S_{2,m}(= S_{2,1,m})\) becomes a line segment, as stated at the end of the previous section.

6. Conclusion

In this article, we have treated of the coin-dividing problem which seeks the set of the possible division of a set of coins. By considering an appropriate scaling limit, the set of points in dividing of coins \(S_{r,c,m}\) converges to a fractal set as \(m\) tends to infinity. This result follows from a remarkable property that increment of the
coin types $m$ represents the action of an iterated function system. The parameters $r$ and $c$ of $S_{r,c,m}$ are related to the contraction ratio and the number of maps of the iterated function system, respectively. Depending on the magnitude of $c$ and $r - 1$, the coin-dividing fractal belongs to one of three classes: totally disconnected if $c < r - 1$, finitely ramified if $c = r - 1$, and infinitely ramified if $c > r - 1$. In particular, we have obtained the Sierpinski gasket when three players divide $S_{2,1,m}$, and the Cantor set when two players divide $S_{3,1,m}$.

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References

[1] L. G. Telser, Optimal denominations for coins and currency, Econ. Lett. 49 (1995) 425–427.
[2] M. A. Wynne, More on optimal denominations for coins and currency, Econ. Lett. 55 (1997) 221–225.
[3] A. E. Tschoegl, The optimal denomination of currency, J. Money, Credit and Banking 29 (1997) 546–554.
[4] J. L. Ramírez-Alfonsín, The Diophantine Frobenius Problem, Oxford University Press, Oxford, 2005.
[5] J. J. Sylvester, Question 7382, Mathematical Questions from the Educational Times 41 (1884) 21.
[6] J. L. Ramírez-Alfonsín, Complexity of the Frobenius problem, Combinatorica 16 (1996) 143–147.
[7] H. Kellerer, U. Pferschy, and D. Pisinger, Knapsack Problems, Springer, Berlin, 2004.
[8] G. S. Lueker, Two NP-complete problem in nonnegative integer programming, Report 178, Computer Science Laboratory, Princeton University, 1975.
[9] M. J. Magazine, G. L. Nemhauser, and L. E. Trotter, Jr., When the greedy solution solves a class of knapsack problems, Oper. Res. 23 (1975) 207–217.
[10] S. Martello and P. Toth, An exact algorithm for large unbounded knapsack problems, Oper. Res. Lett. 9 (1990) 15–20.
[11] D. Pearson, A polynomial-time algorithm for the change-making problem, Oper. Res. Lett. 33 (2005) 231–234.
[12] K. Yamamoto and Y. Yamazaki, Fractal behind coin-reducing payment, Chaos, Solitons & Fractals 45 (2012) 1058–1066.
[13] K. Yamamoto and Y. Yamazaki, Multifractal aspects of an efficient change-making process, Fractals 21 (2013) 1350014.
[14] K. Yamamoto, Emergence of the Sierpinski gasket in coin-dividing problems, J. Stat. Phys. 152 (2013) 534–540.
[15] B. B. Mandelbrot, The Fractal Geometry of Nature, WH Freeman, San Francisco, 1982.
[16] J. Feder, Fractals, Plenum, New York, 1988.
[17] M. F. Barnsley, Fractals Everywhere, Academic Press, Boston, 1993.

[18] T. W. Gamelin and M. A. Mnatsakanian, Arithmetic based fractals associated with Pascal’s triangle, Publ. Mat. 49 (2005) 329–349.

[19] N. S. Holter, A. Lakhtakia, V. K. Varadan, V. V. Varadan, and R. Messier, On a new class of planar fractals: the Pascal-Sierpinski gaskets, J. Phys. A: Math. Gen. 19 (1986) 1753–1759.

[20] D. Broomhead, J. Montaldi, and N. Sidorov, Golden gaskets: variations on the Sierpinski sieve, Nonlinearity 17 (2004) 1455–1480.

[21] H.-O. Peitgen, H. Jürgens, and D. Saupe, Chaos and Fractals: New Frontiers of Science, Springer, New York, 2004.