POWER CONCAVITY AND DIRICHLET HEAT FLOW

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ABSTRACT. We show that log-concavity is the weakest power concavity preserved by the Dirichlet heat flow in $N$-dimensional convex domains, where $N \geq 2$ (indeed, we prove that starting with a negative power concave initial datum may result in losing immediately any reminiscence of concavity). Jointly with what we already know, i.e. that log-concavity is the strongest power concavity preserved by the Dirichlet heat flow, we see that log-concavity is indeed the only power concavity preserved by the Dirichlet heat flow.

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1. Introduction

Let $\Omega$ be a convex set in $\mathbb{R}^N$ and $\alpha \in [-\infty, \infty]$. A nonnegative function $u$ in $\Omega$ is said $\alpha$-concave in $\Omega$ if

$$u((1 - \mu)x + \mu y) \geq \begin{cases} 
\max\{u(x), u(y)\} & \text{if } \alpha = -\infty, \\
[(1 - \mu)u(x)^\alpha + \mu u(y)^\alpha]^{\frac{1}{\alpha}} & \text{if } \alpha \notin \{0, \pm\infty\}, \\
u(x)^{1-\mu}u(y)^\mu & \text{if } \alpha = 0, \\
\min\{u(x), u(y)\} & \text{if } \alpha = -\infty, 
\end{cases}$$

for $\mu \in (0, 1)$ and $x, y \in S_u$, where $S_u := \{x \in \Omega : u(x) > 0\}$.

Neglecting the trivial case when $S_u = \emptyset$ (i.e. when $u$ identically vanishes), this is equivalent to the following: $S_u$ is a (non empty) convex set and

(i) $u$ is a positive constant in $S_u$, if $\alpha = +\infty$;
(ii) $F_\alpha(u)$ is concave in $S_u$, if $\alpha \in \mathbb{R}$, where

$$F_\alpha(t) := \int_1^t s^{\alpha - 1} ds = \begin{cases} 
t^{\alpha - 1} & \text{if } \alpha \neq 0, \\
\log t & \text{if } \alpha = 0; 
\end{cases}$$

(iii) the level sets $\{x \in \Omega : u(x) > \lambda\}$ are convex for $\lambda > 0$ if $\alpha = -\infty$.

The case $\alpha = 1$ clearly corresponds to usual concavity and $\alpha = 0$ corresponds to log-concavity, while the case $-\infty$ is usually referred to as quasi-concavity and power concavity is a generic term for $\alpha$-concavity with $\alpha \in [-\infty, +\infty]$. Power concavity has the following nice property:

- if $u$ is $\alpha$-concave in a convex set $\Omega$, then $u$ is $\beta$-concave in $\Omega$ for $\beta \leq \alpha$.

This property establishes a hierarchy among power concavities, so that quasi-concavity is the weakest one while $(+\infty)$-concavity is the strongest one.

Power concavity is a useful variation of concavity and it has been largely studied in the framework of elliptic and parabolic equations. Here we are mainly concerned with a classical result by Brascamp and Lieb \[2\]: log-concavity is preserved by the Dirichlet heat flow. More precisely, they proved the following:

- Let $e^{t\Delta \phi}$ be a bounded nonnegative solution to

\[
\begin{align*}
\partial_t u &= \Delta u & \text{in } & \Omega \times (0, \infty), \\
0 &= u & \text{on } & \partial \Omega \times (0, \infty) \text{ if } \partial \Omega \neq \emptyset, \\
\phi(x) &= u(x, 0) & \text{in } & \Omega,
\end{align*}
\]

where $\Omega$ is a convex domain in $\mathbb{R}^N$ and $\phi$ is a bounded nonnegative function in $\Omega$.

Then $e^{t\Delta \phi}$ is log-concave in $\Omega$ for $t > 0$ if $\phi$ is log-concave in $\Omega$.

See also \[9\] and \[16\] for later different proofs. (See e.g. \[7, 8, 11, 12, 14, 15\] and references therein for more informations on related topics.)

The main aim of this paper is to investigate the sharpness of the results by Brascamp and Lieb, asking whether the heat transfer preserves any other power concavity, weaker or stronger than log-concavity. In \[13\] we proved that this does not happen for any $\alpha$-concavity with $\alpha > 0$, then log-concavity is the strongest power concavity which the Dirichlet heat flow transmits from time 0 to any $t > 0$. Indeed in \[13\] Theorem 1.5 we proved, more generally, that log-concavity is the strongest conceivable concavity property (we mean among what we
call $F$-concavities, see below) which is preserved by the Dirichlet heat flow for $t > 0$. Then, what remains still open and we face in this paper is the following question:

(Q) What is the weakest power concavity preserved by the Dirichlet heat flow?

In connection to question (Q), we recall the following results.

**Proposition 1.1.** Let $\Omega$ be a convex domain in $\mathbb{R}^N$.

1. Let $N = 1$. Then $e^{t\Delta}\phi$ is quasi-concave in $\Omega$ for $t > 0$ if $\phi$ is quasi-concave in $\Omega$.
2. Let $N \geq 2$ and $T > 0$. Then there exists $\phi \in C_0(\Omega)$ such that $\phi$ is $\alpha$-concave in $\Omega$ for some $\alpha \in (-\infty, 0)$ and $e^{t\Delta}\phi$ is not quasi-concave in $\Omega$ for some $t \in (0, T)$.

See [11] and [9] for assertion (1). See [10, Theorem 4.1] for assertion (2).

By assertion (1), the answer to question (Q) in the one dimensional case is quasi-concavity. Assertion (2) gives a partial answer to question (Q) for $N \geq 2$: for every convex domain, there exists at least a negative power concavity which is not preserved by the Dirichlet heat flow; more dramatically, there exist some $\alpha < 0$ and some $\alpha$-concave initial data $\phi$ such that $e^{t\Delta}\phi$ loses every reminiscence of concavity (even quasi-concavity) almost immediately. In this paper we show that this in fact happens for every $\alpha < 0$, giving a complete answer to question (Q) for $N \geq 2$ in the framework of power concavity. Our main result is the following.

**Theorem 1.1.** Let $N \geq 2$, $\Omega$ be a convex domain in $\mathbb{R}^N$ and $\alpha < 0$. Then, for every $T > 0$, there exists an $\alpha$-concave function $\phi$ in $\Omega$ such that $e^{t\Delta}\phi(\cdot)$ is not quasi-concave in $\Omega$ for some $t \in (0, T)$. Furthermore, $\phi$ can always be chosen continuous and compactly supported.

We recall that similar investigations about the disrupting of concavity along parabolic flows have been studied also in the case of the one-phase Stefan problem [3, Theorem 1.1] (where not even log-concavity is in general preserved), in the case of porous medium equation [5, Theorem 1.1] and [10, Theorem 1.1] (with sharp results in some cases) and for the Dirichlet heat flow in ring shaped domains [3, Theorem 1.1].

Finally, let us recall that the notion of power concavity can be generalized to $F$-concavity by substituting $F_\alpha$ with any strictly increasing $F$ in (ii) above (see [13]). It is also possible to introduce an order between $F$-concavities and it is natural to ask whether there exists any $F$-concavity, different (possibly weaker) than log-concavity, which is preserved (or at least not completely destroyed) by the heat transfer. This question remains open.

2. **Proof of Theorem 1.1**

The proof of Theorem 1.1 is divided into three steps. Step 1 is crucial in the proof of Theorem 1.1 and the arguments in Steps 2 and 3 are modifications of those in the proof of [10, Theorem 4.1]. In what follows, by $C$ we denotes generic positive constants and they may have different values also within the same line.

Let $\alpha < 0$ and set $\beta := 1/|\alpha| > 0$.

For $x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}$ and $r > 0$, set

\[ B(x, r) := \{ y \in \mathbb{R}^N : |y - x| < r \}, \]

\[ B'(x', r) := \{ y' \in \mathbb{R}^{N-1} : |y' - x'| < r \}. \]
**Step 1:** Consider the case of $\Omega = \mathbb{R}^N$. Set

$$u(x, t) := (4\pi t)^{-\frac{N}{2}} \int_{\mathbb{R}^N} e^{-\frac{|x-y|^2}{4t}} \phi(y) \, dy,$$

where

$$\phi(x', x_N) = (1 + |x'|)^{-\beta} \quad \text{if} \quad x_N \leq 0, \quad \phi(x', x_N) = 0 \quad \text{if} \quad x_N > 0.$$

**Figure 1.** The initial datum $\phi(\alpha = -1/2)$ and its level sets.

Then $\phi$ is $\alpha$-concave in $\mathbb{R}^N$. First, we show that $u(\cdot, 1)$ is not quasi-concave in $\mathbb{R}^N$. For any $t > 0$, by (2.1) and (2.2) we have

- $u(x', x_N, t)$ is monotone decreasing with respect to $x_N$ for fixed $x' \in \mathbb{R}^{N-1}$;
- $0 < u(x, t) \leq u(y, t)$ if $|x'| \geq |y'|$ and $x_N = y_N$.

Since

$$\frac{|x'|}{2} \leq |y'| \leq \frac{3|x'|}{2} \quad \text{if} \quad |y' - x'| < \frac{|x'|}{2},$$

it follows from (2.1) that

$$|\nabla_{x_N} u(x', 0, 1)|$$

$$= (4\pi)^{-\frac{N}{2}} \int_{\mathbb{R}^{N-1}} e^{-\frac{|x'-y'|^2}{4}} (1 + |y'|)^{-\beta} \, dy'$$

$$\geq (4\pi)^{-\frac{N}{2}} \int_{B'(x', |x'|/2)} e^{-\frac{|x'-y'|^2}{4}} (1 + |y'|)^{-\beta} \, dy'$$

$$\geq C^{-1}(1 + |x'|)^{-\beta} \int_{B'(x', |x'|/2)} e^{-\frac{|x'-y'|^2}{4}} \, dy' \geq C^{-1}(1 + |x'|)^{-\beta}$$

for $x' \in \mathbb{R}^{N-1}$. 
Moreover we have
\[
|\nabla_{x'} u(x', 0, 1)|
\leq (4\pi)^{-\frac{N}{2}} \int_{-\infty}^{0} e^{-\frac{y^2}{4}} \, dy \int_{\mathbb{R}^{N-1}} e^{-\frac{(y')^2}{4}}(1 + |y'|)^{-\beta-1} \, dy'
\]
(2.4) \[\leq C \left( \int_{B'(x', |x'|/2)} + \int_{\mathbb{R}^{N-1}\setminus B'(x', |x'|/2)} \right) \times e^{-\frac{(y')^2}{4}}(1 + |y'|)^{-\beta-1} \, dy'
\leq C \left( 1 + \frac{|x'|}{2} \right)^{-\beta-1} \int_{B'(x', |x'|/2)} e^{-\frac{(y')^2}{4}} \, dy' + Ce^{-\frac{|x'|^2}{8}} \int_{\mathbb{R}^{N-1}} e^{-\frac{|y'|^2}{8}} \, dy'
\leq C(1 + |x'|)^{-\beta-1}
\]
for \(x' \in \mathbb{R}^{N-1}\), where \(\nabla_{x'} := (\partial_{x_1}, \ldots, \partial_{x_{N-1}})\).

Assume that \(u(\cdot, 1)\) is quasi-concave in \(\mathbb{R}^{N}\). Let \(\epsilon > 0\) be small enough. Then
\[D_{\epsilon} := \{(x', x_1) \in \mathbb{R}^{N-1} \times (0, \infty) : u(x', x_1) > \epsilon\}\]
is convex for \(\epsilon > 0\). Furthermore, we find \(L_{\epsilon} > 0\) and a function \(\zeta_{\epsilon}\) in \([0, L_{\epsilon}]\) with the following properties:

- \(D_{\epsilon} = \{(x', x_1) \in B'(0, L_{\epsilon}) \times (0, \infty) : 0 < x_1 < \zeta(\epsilon|x'|)\}\);
- \(\zeta_{\epsilon}(r) > 0\) if \(0 \leq r < L_{\epsilon}\) and \(\zeta_{\epsilon}(L_{\epsilon}) = 0\);
- \(u(x', \zeta_{\epsilon}(\epsilon|x'|), 1) = \epsilon\) for \(x \in \mathbb{R}^{N-1}\) with \(|x'| \leq L_{\epsilon}\).

By (2.3) and (2.4), applying the implicit function theorem, we see that \(\zeta_{\epsilon}\) is a \(C^1\)-function in a neighborhood of \(r = L_{\epsilon}\) and
\[|\zeta_{\epsilon}'(r)| = \left| \frac{\nabla_{x'} u(x', 0, 1)}{\nabla_{x_1} u(x', 0, 1)} \right| \leq C(1 + L_{\epsilon})^{-1}\]
for small enough \(\epsilon > 0\). This together with the convexity of \(D_{\epsilon}\) implies that
(2.5) \[\zeta_{\epsilon}(0) \leq L_{\epsilon} |\zeta_{\epsilon}'(L_{\epsilon})| \leq C\]

On the other hand, it follows that
\[
u(0, x_1, 1) = (4\pi)^{-\frac{N}{2}} \int_{\mathbb{R}^{N}} e^{-\frac{|y'|^2+|x_1-y'||x_1|}{4}} \phi(y) \, dy
\]
\[= (4\pi)^{-\frac{N}{2}} \int_{\mathbb{R}^{N}} e^{-\frac{|x_1-y'||x_1|}{4}} \, dy \int_{\mathbb{R}^{N-1}} e^{-\frac{|y'|^2}{4}}(1 + |y'|)^{-\beta} \, dy
\geq Ce^{-\frac{x_1^2}{2}} \int_{\mathbb{R}^{N-1}} e^{-\frac{|y'|^2}{4}}(1 + |y'|)^{-\beta} \, dy
\]
for \(x_1 \in \mathbb{R}\). This implies that
\[Ce^{-\frac{\zeta_{\epsilon}(0)^2}{2}} \leq \epsilon,\]
that is,
\[\zeta_{\epsilon}(0) \geq \sqrt{2|\log(C\epsilon)|}\]
for small enough \(\epsilon > 0\). This contradicts (2.5) for small enough \(\epsilon > 0\). Thus \(u(\cdot, 1)\) is not quasi-concave in \(\mathbb{R}^{N}\) and we find \(X_{\epsilon}, Y_{\epsilon} \in \mathbb{R}^{N-1} \times (0, \infty)\) such that
(2.6) \[u(X_{\epsilon}, 1) > \epsilon, \quad u(Y_{\epsilon}, 1) > \epsilon, \quad u\left(\frac{X_{\epsilon} + Y_{\epsilon}}{2}, 1\right) < \epsilon.\]
To complete the proof of the first part of Theorem 1.1 in the case of $\Omega = \mathbb{R}^N$, for $\ell > \sqrt{1/T}$ let

$$\phi_\ell(x) := \phi(\ell x) \quad \text{and} \quad u_\ell(x, t) := u(\ell x, \ell^2 t).$$

Then $u_\ell$ solves problem (1.1) with initial datum $\phi_\ell$, which is $\alpha$-concave in $\mathbb{R}^N$, while $u(\cdot, \ell^{-2})$ is not quasi-concave.

**Step 2:** We show that we can repeat Step 1 with a continuous compactly supported initial datum.

For any $n = 1, 2, \ldots$, set

$$\xi_n := -(n + n^{-1})e_N, \quad \phi_n(x) := \phi(x) \chi_{B(\xi_n, n)}(x), \quad u_n(x, t) := (e^{\ell A_n} \phi_n)(x).$$

Then

$$\lim_{n \to \infty} \sup_{x \in B(0, R)} |u_n(x, t) - u(x, t)| = 0$$

for $R > 0$ and $t > 0$. This together with (2.6) implies that

$$(2.7) \quad u_n(X_*, 1) > \epsilon, \quad u_n(Y_*, 1) > \epsilon, \quad u_n \left( \frac{X_* + Y_*}{2}, 1 \right) < \epsilon,$$

for large enough $n$.

Let $n$ be a large enough integer such that (2.7) holds. For any $m = 1, 2, \ldots$, set

$$\eta_m(x) := \begin{cases} 0 & \text{if } |x - \xi_n| < n - m^{-1}, \\ (n - |x - \xi_n|)^{-1} - m & \text{if } n - m^{-1} \leq |x - \xi_n| < n. \end{cases}$$

Then $\eta_m(x)$ is a convex function in $B(\xi_n, n)$. On the other hand, since

$$\phi_n(x)^\alpha = 1 + |x'|, \quad x \in B(\xi_n, n),$$

the function $\phi_n(x)^\alpha + \eta_m(x)$ is convex in $B(\xi_n, n)$. Set

$$u_{m,n}(x, t) := (e^{\ell A_n} \phi_{m,n})(x),$$

where

$$\phi_{m,n}(x) := \begin{cases} (\phi_n(x)^\alpha + \eta_m(x))^{\frac{1}{\alpha}} & \text{if } x \in B(\xi_n, n), \\ 0 & \text{otherwise}. \end{cases}$$

Then $\phi_{m,n}(x)$ is $\alpha$-concave in $\mathbb{R}^N$ and it satisfies

$$(2.8) \quad \phi_{m,n} \in C_c(\mathbb{R}^N), \quad \sup \phi_{m,n} = B(\xi_n, n),$$

$$\lim_{m \to \infty} \sup_{x \in B(\xi_n, n)} |\phi_{m,n}(x) - \phi_n(x)| = 0.$$

This implies that

$$\lim_{m \to \infty} \sup_{|x| < R} |u_{m,n}(x, t) - u_n(x, t)| = 0$$

for $R > 0$ and $t > 0$. This together with (2.7) implies that

$$(2.9) \quad u_{m,n}(X_*, 1) > \epsilon, \quad u_{m,n}(Y_*, 1) > \epsilon, \quad u_{m,n} \left( \frac{X_* + Y_*}{2}, 1 \right) < \epsilon,$$

for large enough $m$. Thus $u_{m,n}(\cdot, 1)$ is not quasi-concave in $\mathbb{R}^N$ and this settles the case $T > 1$. Then we can argue as before for the case $T \leq 1$. 

Step 3: We consider the case of $\Omega \neq \mathbb{R}^N$ and complete the proof of Theorem 1.1. Without loss of generality, we can assume

$$0 \in \Omega.$$  

For $\ell = 1, 2, \ldots$, set $\Omega_{\ell} := \ell \Omega$. By (2.10) and the convexity of $\Omega$, we see that

$$\Omega_{\ell} \subset \Omega_{\ell+1}, \quad \bigcup_{\ell=1}^{\infty} \Omega_{\ell} = \mathbb{R}^N.$$  

Let $m$ be a large enough integer such that (2.9) holds. Let $U_{\ell}$ be the solution to the problem

$$\begin{cases}
\partial_t U = \Delta U & \text{in } \Omega_{\ell} \times (0, \infty), \\
U(x, t) = 0 & \text{on } \partial \Omega_{\ell} \times (0, \infty), \\
U(x, 0) = \phi_{m,n}(x) & \text{in } \Omega_{\ell}.
\end{cases}$$

Then we have

$$\lim_{\ell \to \infty} \sup_{B(0,R)} |u_{m,n}(x, t) - U_{\ell}(x, t)| = 0$$

for $R > 0$ and $t > 0$. By (2.8), (2.9) and (2.11) we find a large enough integer $\ell$ such that

$$U_{\ell}(X, 1) > \epsilon, \quad U_{\ell}(Y, 1) > \epsilon, \quad U_{\ell}\left(\frac{X+Y}{2}, 1\right) < \epsilon, \quad \phi_{m,n} \in C_c(\Omega_{\ell}).$$

Finally we set

$$u(x, t) := U_{\ell}(\ell x, \ell^2 t), \quad \phi(x) := \phi_{m,n}(\ell x) \in C_c(\Omega).$$

Then $u$ satisfies (1.1) and $\phi$ is $\alpha$-concave in $\Omega$. Furthermore, it follows from (2.12) that $u(\cdot, \ell^{-2})$ is not quasi-concave in $\Omega$. Thus Theorem 1.1 follows in the case of $\Omega \neq \mathbb{R}^N$, and the proof of Theorem 1.1 is complete.

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