APPROXIMATION OF THE SEMIGROUP GENERATED BY THE ROBIN LAPLACIAN IN TERMS OF THE GAUSSIAN SEMIGROUP

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Abstract. For smooth bounded open sets in euclidean space, we construct corresponding contractive linear extension operators for the space of continuous functions which preserve regularity of functions in the domain of the Robin Laplacian. We also prove a Trotter-like approximation for the semigroup generated by the Laplacian subject to Robin boundary conditions in terms of these extension operators. The limiting case of Dirichlet boundary conditions is treated separately.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded open set. Here and in the following, “smooth” means “of class $C^\infty$”, although the main results remain true under slightly milder regularity assumptions. On such a set we consider the (autonomous, homogeneous) diffusion equation

$$
\begin{cases}
  u_t = \Delta u, & \text{on } (0, \infty) \times \Omega, \\
  \frac{\partial u}{\partial \nu}(t, z) = -\beta(z)u(t, z), & \text{for } t > 0 \text{ and } z \in \partial \Omega, \\
  u(0, x) = u_0(x), & \text{for } x \in \Omega,
\end{cases}
$$

(1)

subject to Robin boundary conditions. Here $u_0 \in C(\overline{\Omega})$ is an arbitrary initial function, $\beta$ is a non-negative smooth function on $\partial \Omega$ which does not depend on $t$ and $\frac{\partial u}{\partial \nu}$ denotes the directional derivative of $u$ along the outwards pointing unit normal of $\Omega$. We remark that this setting includes Neumann boundary conditions for $\beta \equiv 0$. A mild solution of (1) is a function $u \in C([0, \infty); C(\overline{\Omega}))$ such that $\int_0^t u(s)ds \in D(\Delta_R)$ and

$$u(t) = u_0 + \Delta_R \int_0^t u(s)ds$$

for every $t \geq 0$. Note that we have incorporated the Robin boundary conditions

$$\frac{\partial u}{\partial \nu} + \beta u = 0 \text{ on } \partial \Omega$$

(2)
into the domain
\[
D(\Delta_R) := \left\{ u \in H^1(\Omega) \cap C(\overline{\Omega}) \mid \Delta u \in C(\overline{\Omega}), \right. \\
\Delta_R u := \Delta u
\]
of the Laplacian on $\Omega$ subject to Robin boundary conditions.

It is known that for every non-negative, bounded, measurable function $\beta$ and every initial value $u_0 \in C(\overline{\Omega})$ there exists a unique mild solution to problem (1). In fact, Warma proved that under the above assumptions $\Delta_R$ generates a $C_0$-semigroup $T_R(t)$ on $C(\overline{\Omega})$ [14], and it follows from the general theory of semigroups that then $u(t) = T_R(t)u_0$ is the unique mild solution of the corresponding homogeneous abstract Cauchy problem [4, Proposition II.6.4]; note that Warma’s proof remains valid for arbitrary non-negative functions $\beta \in L^\infty(\partial\Omega)$.

If we want to calculate this solution numerically, a typical problem is how to handle the boundary conditions. To fix the general ideas, let $\Omega = (0,1)$ and assume that we want to apply an explicit finite difference method. Then one replaces the derivatives $u_t$ and $u_{xx}$ by appropriate difference quotients and successively calculates approximations $u(t_n, x_j)$ of the exact solution $u(t, x)$ by the relation
\[
u(t_{n+1}, x_j) - u(t_n, x_j) \quad \frac{k}{h^2} = u(t_n, x_{j+1}) - 2u(t_n, x_j) + u(t_n, x_{j-1}),
\]
where $t_n = n \cdot k$ and $x_j = j \cdot h$ for given small numbers $k, h > 0$. Note, however, that this cannot be directly applied to calculate $u(t_{n+1}, 0)$ and $u(t_{n+1}, 1)$ because $u(t_n, -h)$ and $u(t_n, 1+h)$ are not defined. For Dirichlet boundary conditions, we can assign $u(t_n, -h) := u(t_n, 1+h) := 0$. On the other hand, for Neumann boundary conditions the situation is not as simple. One common technique is to use
\[
u(t_n, -h) := u(t_n, h) \quad \text{and} \quad u(t_n, 1+h) := u(t_n, 1-h)
\]
in the calculations, which comes from a second order accurate approximation of the derivative at the boundary, see [14, Section 8.3].

The aim of the article at hand is to justify the use of (3) for Neumann boundary conditions from a semigroup perspective, showing that the corresponding continuous method converges to the exact solution as $k \to 0$, and to extend it to the more general case of Robin boundary conditions. More precisely, we construct an extension operator $E_\beta$ from $C(\overline{\Omega})$ to $C_0(\mathbb{R}^N)$ which depends only on $\Omega$ and $\beta$ (but not on $t$) and resembles a continuous version of (3) if $\beta = 0$, such that $E_\beta$ is a contraction and that $E_{\beta\mu}$ is sufficiently regular whenever $u \in D(\Delta_R)$; we refer to Corollary [12] for the precise statement. For operators $E_\beta$ satisfying these two assumptions, we prove the Trotter-like (compare to [12]) approximation result
\[
T_R(t)u = \lim_{n \to \infty} \left( R_{G_0(t/n)}E_\beta \right)^n u \quad \text{in } C(\overline{\Omega}) \quad \text{for every } u \in C(\overline{\Omega})
\]
uniformly on $[0,T]$ for every $T > 0$, where $G_0(t)$ denotes the Gaussian semigroup on $C_0(\mathbb{R}^N)$ and $R \colon C_0(\mathbb{R}^N) \to C(\overline{\Omega})$ is the restriction operator $Ru = u|_\Omega$. This shows how Robin boundary conditions can be incorporated into a numerical solver such that the numerical solutions converge uniformly on $[0,T] \times \overline{\Omega}$, at least if error introduced by space discretization is neglected.
2. Notation and Preliminary Results

It is well-known that for smooth boundary a neighborhood of \( \partial \Omega \) can be parametrized by the outwards pointing unit normal \( \nu \). Because certain features of the parametrization are needed later on, we state this result in the formulation we want to use and prove it for the sake of completeness.

**Proposition 1.** Let \( \delta > 0 \) and

\[
T : \partial \Omega \times (-\delta, \delta) \to \mathbb{R}^N, \quad (p, t) \mapsto p + t\nu(p).
\]

For small \( \delta > 0 \), \( T \) is a smooth diffeomorphism onto a neighborhood of \( \partial \Omega \) in \( \mathbb{R}^N \).

**Remark 2.**

(a) Let \( x_0 \in \partial \Omega \) be arbitrary. By definition of “smooth boundary”, \( \Omega \) can locally at \( x_0 \) be represented as the subgraph of a smooth function \( \varphi : U \to \mathbb{R} \) \((U \subset \mathbb{R}^{N-1})\) up to rotation. Assume for the moment that no rotation is needed. Then

\[
\partial \Omega \cap V = \left\{ \left( z, \varphi(z) \right) \mid z \in U \right\}
\]

for an open set \( V \subset \mathbb{R}^N \). Thus \( z \mapsto \left( \frac{z}{\varphi(z)} \right) \) is a bijection of an open subset of \( \mathbb{R}^{N-1} \) onto a neighborhood of \( x_0 \) in \( \partial \Omega \). Using these mappings for all \( x \in \partial \Omega \) as charts we make \( \partial \Omega \) into a smooth manifold built upon the subspace topology induced by \( \mathbb{R}^N \). Thus we can look at \( T \) as a mapping from a manifold to an euclidean space. As usual we say that \( T \) is smooth if the composition \( T^* \) of \( T \) with a chart is smooth, i.e., if

\[
T^* : U \times (-\delta, \delta) \to \mathbb{R}^N, \quad (z, t) \mapsto \left( \frac{z}{\varphi(z)} \right) + t\nu\left( \frac{z}{\varphi(z)} \right)
\]

is smooth as a mapping between euclidean spaces.

(b) Using charts, the outwards pointing unit normal \( \nu \) can be written as

\[
\nu\left( \frac{z}{\varphi(z)} \right) = \pm \left( \begin{pmatrix} -\nabla \varphi(z) \\ 1 \end{pmatrix} \right)^T \begin{pmatrix} -\nabla \varphi(z) \\ 1 \end{pmatrix}^{-1} \left( \begin{pmatrix} -\nabla \varphi(z) \\ 1 \end{pmatrix} \right)^T.
\]

To see this, note that for \( x \in \partial \Omega \) the direction of \( \nu(x) \) is uniquely described by the property that for every smooth curve \( \xi \) in \( \partial \Omega \) satisfying \( \xi(0) = x = \left( \frac{\varphi(z)}{\varphi(z)} \right) \) the vectors \( \nu(x) \) and \( \xi'(0) \) are orthogonal. Since by (5), locally \( \xi(t) = \left( \frac{\zeta(t)}{\varphi(\zeta(t))} \right) \), where \( \zeta(0) = z \), (7) follows from the identity

\[
\left( -\nabla \varphi(z) \\ 1 \right) \left( \begin{pmatrix} \zeta'(0) \\ \varphi(\zeta'(0)) \end{pmatrix} \cdot \nabla \varphi(\zeta'(0)) \right) = -\nabla \varphi(z) \cdot \zeta'(0) + \nabla \varphi(z) \cdot \zeta'(0) = 0.
\]
Proof of Proposition \[\text{[2]}\] Let \(x \in \partial \Omega\) be arbitrary. Working locally near \(x\), for simplicity we may assume without loss of generality that there exists \(\varphi\) be as in the previous remark, i.e., that no rotation is needed for \(\Omega\) to be the subgraph of a smooth function. Then \(x = (\varphi(z), z)\) for some \(z\). Using \([\text{1}]\), the derivative of \(T^*\) is

\[
T^*(z, 0) = \begin{pmatrix} I & -c\nabla \varphi(z) \end{pmatrix}.
\]

Here, \(c = \pm |(-\nabla \varphi(z))|^{-1} \neq 0\). In particular, we obtain

\[
\det T^*(z, 0) = c \cdot \det \begin{pmatrix} I & -\nabla \varphi(z) \end{pmatrix} = c \cdot \det \begin{pmatrix} I & -\nabla \varphi(z) \end{pmatrix} \neq 0
\]

by applying the Gauss-Jordan elimination algorithm. The inverse function theorem asserts that \(T^*\) and hence \(T\) is locally a smooth diffeomorphism. Because \(x \in \partial \Omega\) was arbitrary, all that remains to show is that \(T\) is injective if \(\delta > 0\) is small enough.

By the above argument for every \(x \in \partial \Omega\) there exists an open neighborhood \(O_x\) of \(x\) in \(\partial \Omega\) and \(\delta_x > 0\) such that \(T\) is a smooth diffeomorphism from \(O_x \times (-\delta_x, \delta_x)\) to a neighborhood of \(x\) in \(\mathbb{R}^N\). By compactness of \(\partial \Omega\) we can choose finitely many such \(O_{x_i}, (i = 1, \ldots, m)\) which already cover \(\partial \Omega\). It is easily proved by contradiction that we can find \(\delta > 0\) such that for every \(x \in \partial \Omega\) there exists an index \(k(x) \in \{1, \ldots, m\}\) with the property that \(B_{4\delta}(x) \cap \partial \Omega \subset O_{x_{k(x)}}\), where \(B_r(a)\) denotes the open ball with center \(a\) and radius \(r\). We pick \(\delta\) such that \(\delta < \delta_{x_i}\) for all \(i = 1, \ldots, m\).

For this choice of \(\delta\), \(T\) is injective. To see this, let \(T(y_1, t_1) = T(y_2, t_2)\) where \(y_1, y_2 \in \partial \Omega\) and \(t_1, t_2 \in (-\delta, \delta)\). We estimate

\[
0 = |T(y_1, t_1) - T(y_2, t_2)| = |y_1 - y_2| - (|t_1\nu(y_1)| + |t_2\nu(y_2)|) \geq |y_1 - y_2| - 2\delta.
\]

This shows \(|y_1 - y_2| \leq 2\delta\) and thus \(y_2 \in B_{4\delta}(y_1)\), hence \(y_1, y_2 \in O_{x_{k(x)}}\), \(k = k(y_2)\).

By construction, \(T\) is injective on \(O_k \times (-\delta, \delta)\), hence \(y_1 = y_2\) and \(t_1 = t_2\), proving the claim. \(\square\)

**Lemma 3.** The set \(D := D(\Delta_R) \cap C^\infty(\overline{\Omega})\) is an operator core for \(\Delta_R\), i.e., \(D\) is dense in \(D(\Delta_R)\) with respect to the graph norm.

**Proof.** As \(\Delta_R\) is a generator, the space \(\cap_{n \in \mathbb{N}} D(\Delta_R^n)\) is a core for \(\Delta_R\) \([\text{[4]}\] Proposition II.1.8\]. Moreover,

\[
R(1, \Delta_R) \left(H^m(\Omega) \cap C(\overline{\Omega})\right) \subset H^{m+2}(\Omega)
\]

for every \(m \in \mathbb{N}_0\) by the regularization properties of elliptic operators \([\text{[6]}\] Remark 2.5.1.2\]. By a standard Sobolev embedding theorem \([\text{[5]}\] Section 5.6\],

\[
D(\Delta_R^n) = R(1, \Delta_R)^n C(\overline{\Omega}) \subset H^{2n}(\Omega) \subset C^{2n-\left[\frac{n}{2}\right]-1}(\overline{\Omega})
\]

for all \(n > \frac{N}{2}\). Letting \(n \to \infty\) we obtain the assertion. \(\square\)

We remark that for \(u \in C^\infty(\overline{\Omega})\) the normal derivative exists in the classical sense. For these functions, \(u \in D(\Delta_R)\) is equivalent to \([\text{[2]}\], and we will frequently use the boundary condition in this way.

Let \(G_2(t)\) denote the \(C_0\)-semigroup on \(L^2(\mathbb{R}^N)\) with generator

\[
D(\Delta_2) := \{u \in L^2(\mathbb{R}^N) | \Delta u \in L^2(\mathbb{R}^N)\},
\]

\[
\Delta_2 u := \Delta u.
\]
The semigroup \( G_2(t) \) leaves the space \( C_0(\mathbb{R}^N) \cap L^2(\mathbb{R}^N) \) invariant, and its restriction extends continuously to a positive, contractive \( C_0 \)-semigroup on \( C_0(\mathbb{R}^N) \), denoted by \( G_0(t) \). The generator of this semigroup is

\[
D(\Delta_0) := \{ u \in C_0(\mathbb{R}^N) | \Delta u \in C_0(\mathbb{R}^N) \},
\]

\[
\Delta_0 u := \Delta u.
\]

We will refer to both semigroups as the Gaussian semigroup. For more details about the Gaussian semigroup, we refer to [1, Chapter 3.7].

3. Extension Operator

Given a smooth bounded open set \( \Omega \) and a smooth function \( \beta : \partial \Omega \to \mathbb{R}_+ \), we construct an extension operator \( E_\beta \) which satisfies the assumptions under which we will prove \( \text{(4)} \) in Section 4. For \( \beta = 0 \), the operator is similar to, but slightly simpler than the extension operator in [5, Section II.5.4]. However, the properties which we prove here may also be of independent interest.

For the whole section, let \( \delta \) and \( T \) be as in Proposition 4. We start by fixing a “kinking function” \( \varrho \). First choose a function \( \varrho_1 \) having the following properties.

(a) \( \varrho_1 \in C^\infty([0, \infty) \times [0, \infty)) \)
(b) \( 0 \leq \varrho_1(\gamma, t) \leq 1 \) for all \( \gamma, t \geq 0 \)
(c) \( \varrho_1(\gamma, t) = 0 \) for all \( t \geq \frac{1}{2} \) and \( \gamma \geq 0 \)
(d) \( \varrho_1(\gamma, 0) = 1 \) for all \( \gamma \geq 0 \)
(e) \( \varrho_1(\gamma, t) = -2 \gamma \) for all \( \gamma \geq 0 \)
(f) \( \frac{\partial^2}{\partial t^2} \varrho_1(\gamma, 0) = 4 \gamma^2 \) for all \( \gamma \geq 0 \)

Here \( \frac{\partial}{\partial t} \varrho_1 \) denotes the partial derivative of \( \varrho_1 \) with respect to the second argument.

For example, we may choose \( \varrho_1(\gamma, t) := \exp(-2\gamma t) \chi(t) \) where \( \chi \) is a smooth cut-off function such that \( \chi \equiv 1 \) near 0.

Now define \( \varrho : \Omega^C \to \mathbb{R} \) to be

\[
\varrho(x) := \begin{cases} 
\varrho_1(\beta(z), t), & \text{if } x = T(z, t), 0 \leq t < \delta, \\
0, & \text{otherwise}.
\end{cases}
\]

Note that \( \varrho \) is well-defined since \( T \) is injective, and it is smooth by construction.

**Definition 4** (Reflection at the Boundary). Let \( x \in T(\partial \Omega \times (-\delta, \delta)), x = T(z, t) \). We call \( \tilde{x} := T(z, -t) \) the (orthogonal) reflection of \( x \) at the boundary \( \partial \Omega \). For a function \( u : \Omega \to \mathbb{R} \) we define the reflected function

\[
\tilde{u} : T(\partial \Omega \times (0, \delta)) \to \mathbb{R}, \quad \tilde{u}(x) := u(\tilde{x}).
\]

We define the extension operator \( E_\beta \) belonging to \( \beta \) as

\[
E_\beta : C(\Omega) \to C_0(\mathbb{R}^N), \quad E_\beta u := \begin{cases} u, & \text{on } \Omega, \\
\varrho \tilde{u}, & \text{on } \Omega^C.
\end{cases}
\]  

(9)

Here \( \varrho \tilde{u} \) is understood to be 0 outside \( T(\partial \Omega \times (0, \delta)) \) because \( \varrho \) equals 0 in that region.

**Lemma 5.** The operator \( E_\beta \) is well-defined, linear, positive, contractive and an extension operator, i.e., \( RE_\beta = I \), where \( R : C_0(\mathbb{R}^N) \to C(\Omega), u \mapsto u|_\Omega \).
Proof. Let \( u \in C(\overline{\Omega}) \). By property (1), the function \( E_\beta u \) is continuous on \( \mathbb{R}^N \). Since it has compact support, \( E_\beta u \in C_0(\mathbb{R}^N) \). Positivity and contractivity follow from property (4). The other two properties are obvious from (9).

We now turn towards a more interesting property of \( E_\beta \): we prove that it maps \( D \) as defined Lemma 3 into \( D(\Delta_0) \). This extensive calculation is split into several lemmata. Most calculations will be carried out in local coordinates, i.e., locally at \( x_0 = T^*(z_0, 0) \in \partial \Omega \), where we represent all functions with respect to the charts as follows. Here \( T^* \) is defined as in (6).

\[
\begin{align*}
 u^*(z, t) &:= u(T^*(z, t)) & \tilde{u}^*(z, t) &:= \tilde{u}(T^*(z, t)) \\
 \beta^*(z) &:= \beta(T^*(z, 0)) & \varrho^*(z, t) &:= \varrho(T^*(z, t)) = \varrho_1(\beta^*(z), t)
\end{align*}
\]

In the following we will adhere to the usual notation for normal derivatives, i.e., \( \frac{\partial g}{\partial \nu} \) denotes the directional derivative of \( g \) along the outwards pointing unit normal with respect to the domain of \( g \). Note that for functions defined on \( \Omega \) this means that \( \frac{\partial g}{\partial \nu} = -\nabla g \cdot \nu \), where \( \nu \) always denotes the outwards pointing unit normal of \( \Omega \).

Lemma 6. Let \( u \in D \). Then \( E_\beta u \in D(\Delta_2) \) and \( (\Delta_2(Eu)) \big|_{\Omega} = \Delta_0 u \).

Proof. The continuous function \( E_\beta u \) has compact support, hence \( E_\beta u \in L^2(\mathbb{R}^N) \). Moreover, \( E_\beta u \) is smooth away from \( \partial \Omega \) being the composition of smooth functions. Thus the measurable function

\[
f := \begin{cases} 
\Delta u, & \text{on } \Omega, \\
\Delta(\tilde{u} \tilde{u}), & \text{on } \Omega^c.
\end{cases}
\]

is defined outside \( \partial \Omega \) which is a set of measure zero. As \( u \) and \( \tilde{u} \tilde{u} \) are smooth up to \( \partial \Omega \), \( f \) is bounded. Note that \( f \) has compact support, hence \( f \in L^2(\mathbb{R}^N) \).

For the assertion of the lemma, it remains to show that \( f = \Delta(Eu) \) in the sense of distributions. For this we calculate the (classical) normal derivative of \( \tilde{u} \tilde{u} \) using that \( u \) satisfies (2). For \( z \in \partial \Omega \) we have

\[
\frac{\partial \tilde{u}}{\partial \nu}(z) = -\lim_{h \to 0} \frac{\tilde{u}(z + h\nu(z)) - \tilde{u}(z)}{h} = -\lim_{h \to 0} \frac{u(z + h\nu(z)) - u(z)}{h} = -\beta(z)u(z)
\]

and

\[
\frac{\partial \varrho}{\partial \nu}(z) = -\lim_{h \to 0} \frac{\varrho(z + h\nu(z)) - \varrho(z)}{h} = -\lim_{h \to 0} \frac{\varrho_1(\beta(z), h) - \varrho_1(\beta(z), 0)}{h} = 2\beta(z)
\]

This implies

\[
\frac{\partial(\tilde{u} \tilde{u})}{\partial \nu}(z) = \varrho(z) \frac{\partial \tilde{u}}{\partial \nu}(z) + \frac{\partial \varrho}{\partial \nu}(z) \tilde{u}(z) = \frac{\partial u}{\partial \nu}(z) + \frac{\partial \varrho}{\partial \nu}(z) u(z) = \beta(z)u(z).
\]
Now let \( \varphi \in D(\mathbb{R}^N) \) be an arbitrary test function. From the above calculations, the classical Green formula [2, Section II.1.3] and \( \varrho \tilde{u} \mid_{\partial \Omega} = u \mid_{\partial \Omega} \), we obtain

\[
\int_{\mathbb{R}^N} (E\beta u) \Delta \varphi = \int_{\Omega} u \Delta \varphi + \int_{\Omega^c} \varrho \tilde{u} \Delta \varphi
\]

\[
= \int_{\Omega} \Delta u \varphi + \int_{\partial \Omega^+} \left( \frac{\partial \varphi}{\partial \nu} - \frac{\partial u}{\partial \nu} \varphi \right) d\sigma
\]

\[
+ \int_{\Omega^c} \Delta(\varrho \tilde{u}) \varphi + \int_{\partial \Omega^-} \left( \varrho \tilde{u} \frac{\partial \varphi}{\partial \nu} - \varrho \frac{\partial(\varrho \tilde{u})}{\partial \nu} \varphi \right) d\sigma
\]

\[
= \int_{\Omega} \Delta \varphi + \int_{\Omega^c} \Delta(\varrho \tilde{u}) \varphi = \int_{\mathbb{R}^N} f \varphi,
\]

where \( (\partial \Omega)^+ \) is understood as the (oriented) boundary of \( \Omega \), whereas \( (\partial \Omega)^- \) denotes the (oriented) boundary of \( \Omega^c \). This shows \( \Delta(E\beta u) = f \) in the sense of distributions. \( \square \)

**Remark 7.** The above lemma tells us that \( \Delta(E\beta u) \) is a function. To see that \( E\beta u \in D(\Delta_0) \), it remains to show that \( \Delta(E\beta u) \in C_0(\mathbb{R}^N) \). We already know that \( \Delta(E\beta u) \) has compact support and is smooth on \( \mathbb{R}^N \setminus \partial \Omega \). Thus it suffices to show that the function can continuously be extended to \( \partial \Omega \). This is a local property. In fact, since we already know that the limits from the inside and the outside both exist, it suffices to show that \( \Delta u(x_0) = \Delta(\varrho \tilde{u})(x_0) \) for every \( x_0 \in \partial \Omega \).

Let \( x_0 \in \partial \Omega \) be fixed. To simplify notation, we may assume that \( \nu(x_0) = e_N \) without loss of generality, exploiting the rotational invariance of the Laplacian. Here and in the following, \( e_n \) denotes the \( n \)th unit vector in \( \mathbb{R}^N \). Moreover, since we treat the problem locally, we may work in local coordinates, \( x_0 = T^*(z_0,0) \).

We start by calculating the partial derivatives of \( T^* \), where \( T^* \) is defined as in [6].

**Lemma 8.** For \( n \in \{1, \ldots, N\} \),

\[
\left( \frac{\partial}{\partial x_n} \left( T^{*-1} \right) \right)(x_0) = e_n,
\]

\[
\left( \frac{\partial^2}{\partial x_n^2} \left( T^{*-1} \right) \right)(x_0) = \begin{cases} 
0 & \text{if } n \neq N, \\
-\frac{\partial^2}{\partial z_n^2} \varphi(z_0) & \text{if } n = N.
\end{cases}
\]

**Proof.** The assumption \( \nu(z_0) = e_N \) implies \( \nabla \varphi(z_0) = 0 \) due to [7]. As in [8], this shows \( T^{*-1}(z_0,0) = I \). By the inverse function theorem,

\[
\left( T^{*-1} \right)'(x) = \left( T^{*'} \left( T^{*-1}(x) \right) \right)^{-1}.
\]

For the partial derivatives at \( x_0 \) this means

\[
\left( \frac{\partial}{\partial x_n} \left( T^{*-1} \right) \right)(x_0) = T^{*'}(z_0,0)^{-1} e_n = I e_n = e_n.
\]
To calculate the second derivatives, we employ a differentiation rule for matrices, \( \frac{d}{dA(t)^{-1}} = -A(t)^{-1}(A'(t)A^{-1}(t)) \).

\[
\left( \frac{\partial^2}{\partial x_n^2} \left( T^*-1 \right) \right)(x) = \left( \frac{\partial}{\partial x_n} \left( T^* (T^*-1(x))^{-1} \right) \right) \cdot e_n
\]

\[
= - \left( T^* (T^*-1(x))^{-1} \right) \left( \frac{\partial}{\partial x_n} \left( T^* (T^*-1(x))^{-1} \right) \right) \cdot (T^* (T^*-1(x))^{-1}) e_n
\]

If we denote the entries of \( T^* \) by \( t_{ij} \) \( (i, j = 1, \ldots, N) \), we can proceed as follows.

\[
\frac{\partial}{\partial x_n} \left( t_{ij} \left( T^*-1(x) \right) \right) = \nabla t_{ij} \left( T^*-1(x) \right) \cdot \left( \frac{\partial}{\partial x_n} \left( T^*-1 \right) \right) (x)
\]

For \( x = x_0 \) this yields

\[
\frac{\partial}{\partial x_n} t_{ij} \left( T^*-1(x_0) \right) = \nabla t_{ij}(z_0, 0)e_n = \frac{\partial}{\partial z_n} t_{ij}(z_0, 0),
\]

where for notational simplicity we use \( z_N \) as an alias for the variable \( t \). Inserting this expression into the above identity, we arrive at

\[
\left( \frac{\partial^2}{\partial x_n^2} \left( T^*-1 \right) \right)(x_0) = - \left( \frac{\partial}{\partial z_n} t_{ij}(z_0, 0) \right)_{i,j=1,\ldots,N} e_n
\]

\[
\quad = - \left( \frac{\partial}{\partial z_n} t_{in}(z_0, 0) \right)_{i=1,\ldots,N} = - \frac{\partial^2}{\partial z_n^2} T^*(z_0, 0).
\]

In combination with formula (4) this finishes the proof.

Having the derivatives of the charts at hand, we are able to calculate all derivatives in local coordinates.

**Lemma 9.** Let \( f \) and \( f^* \) be functions such that locally \( f^*(z, t) = f(T(z, t)) \). Then

\[
\nabla f(x_0) = \nabla f^*(z_0, 0),
\]

\[
\Delta f(x_0) = \sum_{n=1}^{N-1} \frac{\partial^2 f^*(z_0, 0)}{\partial z_n^2} + \frac{\partial^2}{\partial t^2} f^*(z_0, 0) - \frac{\partial f^*(z_0, 0)}{\partial t} \sum_{n=1}^{N-1} \frac{\partial^2}{\partial z_n^2} \varphi(z_0).
\]

In particular,

\[
\nabla \varphi(x_0) = (0, -2\beta(x_0)), \quad \Delta \varphi(x_0) = 4\beta(x_0)^2 + 2\beta(x_0) \sum_{n=1}^{N-1} \frac{\partial^2}{\partial z_n^2} \varphi(z_0).
\]

**Proof.** Differentiating \( f(x) = f^*(T^*-1(x)) \) we obtain

\[
\frac{\partial}{\partial x_n} f(x) = \left( \nabla f^* \right) \left( T^*-1(x) \right) \left( \frac{\partial}{\partial x_n} \left( T^*-1 \right) \right) (x),
\]

\[
\frac{\partial^2}{\partial x_n^2} f(x) = \left( \frac{\partial}{\partial x_n} \left( T^*-1 \right) \right)^T (x) H_{f^*} \left( T^*-1(x) \right) \left( \frac{\partial}{\partial x_n} \left( T^*-1 \right) \right) (x)
\]

\[+ \left( \nabla f^* \right) \left( T^*-1(x) \right) \left( \frac{\partial^2}{\partial x_n^2} \left( T^*-1 \right) \right) (x),
\]

where \( H_{f^*} = \left( \frac{\partial^2 f^*}{\partial z_i \partial z_j} \right)_{i,j=1,\ldots,N} \) denotes the Hessian matrix of \( f^* \). By using Lemma 8 and summing up, we arrive at the desired formulae for \( x = x_0 \).
Concerning $\varrho$ we remark that $\varrho^*(z,0) = \varrho(\beta(z),0) = 1$ implies $\frac{\partial}{\partial t^*} \varrho^*(z_0,0) = 0$ ($n = 1, \ldots, N - 1$). On the other hand, the derivatives with respect to $t$ equal
\[
\frac{\partial}{\partial t^*} \varrho^*(z_0,0) = \frac{\partial}{\partial t} \varrho_1(\beta^*(z_0),0) = -2\beta^*(z_0) = -2\beta(x_0),
\]
\[
\frac{\partial}{\partial t^2} \varrho^*(z_0,0) = \frac{\partial}{\partial t^2} \varrho_1(\beta^*(z_0),0) = 4\beta^*(z_0)^2 = 4\beta(x_0)^2.
\]
With this information, the formulae for $\varrho$ follow from the general formulae. □

Finally, it is easy to calculate the relation between the derivatives of the function and its reflection at the boundary in local coordinates. It suffices to observe that
\[
\tilde{u}^*(z,t) = \tilde{u}(T(z,t)) = u(T(z,-t)) = u^*(z,-t).
\]
From this we deduce the following formulae.
\[
\tilde{u}^*(z,t) = u^*(z,-t)
\]
\[
\frac{\partial}{\partial z_n} \tilde{u}^*(z,t) = -\frac{\partial}{\partial z_n} u^*(z,-t)
\]
\[
\frac{\partial^2}{\partial z_n^2} \tilde{u}^*(z,t) = \frac{\partial^2}{\partial z_n^2} u^*(z,-t)
\]
\[
\frac{\partial^2}{\partial t^2} \tilde{u}^*(z,t) = \frac{\partial}{\partial t^2} u^*(z,-t)
\]
Now we are ready to prove continuity of $\Delta(E_\beta u)$ at $x_0$.

**Proposition 10.** For every $u \in D$, $\Delta u(x_0) = \Delta(\varrho \tilde{u})(x_0)$.

*Proof.* Note that
\[
\frac{\partial}{\partial t} \tilde{u}^*(z_0,0) = -\frac{\partial}{\partial t} u^*(z_0,0) = -\frac{\partial u}{\partial \nu}(x_0) = \beta(x_0)u(x_0) = \beta(x_0)\tilde{u}(x_0).
\]
We use the formulae of this section to obtain the desired identity.
\[
\Delta(\varrho \tilde{u})(x_0) = \Delta \varrho(x_0) \tilde{u}(x_0) + 2\nabla \varrho(x_0) \cdot \nabla \tilde{u}(x_0) + \varrho(x_0) \Delta \tilde{u}(x_0)
\]
\[
= 4\beta(x_0)^2 \tilde{u}(x_0) + 2\beta(x_0)\tilde{u}(x_0) \sum_{n=1}^{N-1} \frac{\partial^2}{\partial z_n^2} \varphi(z_0) - 4\beta(x_0) \frac{\partial}{\partial t} \tilde{u}^*(z_0,0)
\]
\[
+ \sum_{n=1}^{N-1} \frac{\partial^2}{\partial z_n^2} \tilde{u}^*(z_0,0) + \frac{\partial^2}{\partial t^2} \tilde{u}^*(z_0,0) - \frac{\partial}{\partial t} \tilde{u}^*(z_0,0) \sum_{n=1}^{N-1} \frac{\partial^2}{\partial z_n^2} \varphi(z_0)
\]
\[
= \sum_{n=1}^{N-1} \frac{\partial^2}{\partial z_n^2} u^*(z_0,0) + \frac{\partial^2}{\partial t^2} u^*(z_0,t) - \frac{\partial}{\partial t} u^*(z_0,0) \sum_{n=1}^{N-1} \frac{\partial^2}{\partial z_n^2} \varphi(z_0)
\]
\[
= \Delta u(x_0).
\]
□

The following theorem is the main result of this section. As explained in Remark 7 it follows by combining Lemma 6 and the last proposition. Even though Theorem 11 is also true for the usual extension operator for Lipschitz domains [10 VI, §3, Theorem 5], that operator fails to be contractive and thus is more difficult to handle for the application in Section 4.

**Theorem 11.** The operator $E_\beta$ maps $D$ into $D(\Delta_0)$.

**Corollary 12.** The operator $E_\beta$ maps $D(\Delta_R)$ into $D(\Delta_0)$. 
Proof. There exists a constant $C > 0$ satisfying $\|E_\beta u\|_{D(\Delta_0)} \leq C \|u\|_{D(\Delta_0)}$ for all $u \in D$. To see this, note that on $\mathbb{R}^d$

$$\|\Delta(\tilde{u} \varphi)\|_\infty = \|\Delta \tilde{u} \varphi + 2 \nabla \tilde{u} \nabla \varphi + \tilde{u} \Delta \varphi\|_\infty 
\leq \|\Delta \tilde{u}\|_\infty \|\varphi\|_\infty + 2 \|\nabla \varphi\|_\infty (\varepsilon \|\Delta \tilde{u}\|_\infty + \|\tilde{u}\|_\infty) + \|\tilde{u}\|_\infty \|\Delta \varphi\|_\infty$$

for every $\varepsilon > 0$. Similarly, $\|\tilde{u}\|_\infty \leq \|u\|_\infty$ and $\|\Delta \tilde{u}\|_\infty \leq \tilde{C} (\|\Delta u\|_\infty + \|u\|_\infty)$, using the definition of $\tilde{u}$ as a composition of $u$ and a function involving $T$, where $\tilde{C} > 0$ depends only depends on a estimate on the derivatives of $T$. Noting that $\varphi$ and its derivatives are bounded by assumption, we see that there exists a $C > 0$ as claimed.

As $D$ is a core of $\Delta_R$, the above estimate shows that there exists a unique continuous extension of $E_\beta|_D$ to $D(\Delta_R)$, and that this operator still takes values in $D(\Delta_0)$. Because $D(\Delta_R)$ is continuously embedded into $C(\overline{\Omega})$ and $E_\beta$ is continuous on $C(\overline{\Omega})$, this extension agrees with $E_\beta|_{D(\Delta_0)}$. Thus the claim is proved. \qed

4. APPROXIMATION RESULT

In this section, we prove that if $E_\beta : C(\overline{\Omega}) \to C_0(\mathbb{R}^N)$ is a contractive extension operator mapping an operator core $D$ for $\Delta_R$ into $D(\Delta_0)$, then formula (4) holds. Note that the operator defined in (4) has the properties as shown in the preceding section. The tool we use for the proof is the following approximation result for semigroups due to Chernoff.

**Theorem 13 (H Theorem III.5.2).** Let $X$ be a Banach space. Consider a function $V : [0, \infty) \to \mathcal{L}(X)$ satisfying $V(0) = I$ and $\|V(t)^m\| \leq M$ for all $t \geq 0$, $m \in \mathbb{N}$ and some $M \geq 1$. Assume that

$$Ax := \lim_{h \to 0} \frac{V(h)x - x}{h}$$
exists for all on $x \in D \subset X$, where $D$ and $(\lambda_0 - A)D$ are dense subspaces in $X$ for some $\lambda_0 > 0$.

Then $(A, D)$ is closable and $\overline{A}$ generates a bounded $C_0$-semigroup $T(t)$, which is given by

$$T(t)x = \lim_{n \to \infty} (V(t/n))^n x$$

for every $x \in X$ locally uniformly with respect to $t \geq 0$.

We apply the theorem by setting

$$X := C(\overline{\Omega}), \quad V(t) := RG_0(t)E_\beta, \quad A := \Delta_R. \quad (10)$$

As $D$ is an operator core for $\Delta_0$, the density conditions are fulfilled because $\lambda - \Delta_R$ is an isomorphism between $D(\Delta_R)$ with the graph norm and $C(\overline{\Omega})$ for every $\lambda > 0$.

**Theorem 14.** Let $E_\beta : C(\overline{\Omega}) \to C_0(\mathbb{R}^N)$ be a contractive extension operator which maps an operator core $D$ for $\Delta_R$ into $D(\Delta_0)$. Then formula (4) holds true.

Proof. We check the conditions of Chernoff’s product formula with the choices made in (4). The fact $V(0) = I$ is equivalent to $E_\beta$ being an extension operator. Since all three of their factors are contractions, the operators $V(t)$ are contractions for every $t \geq 0$, thus $\|V(t)^m\| \leq 1$; in particular $V(t)$ is a bounded operator for every $t \geq 0$. The density assumptions on $D$ are fulfilled because $D$ is an operator core for $\Delta_R$. 

Now let $u \in D$ be arbitrary. By assumption, $E_{\beta}u \in D(\Delta_0)$. By definition of the infinitesimal generator,
\[
\frac{V(h)u - u}{h} = R \frac{G(h)(E_{\beta}u) - E_{\beta}u}{h} \rightarrow R \Delta_0 E_{\beta}u. \quad (h \to 0)
\]
Since the function $E_{\beta}u$ agrees with $u$ on $\Omega$, they represent the same distribution acting on the test functions $D(\Omega)$. This means that they have the same distributional derivatives, hence $R \Delta_0 E_{\beta}u = \Delta_R u$. Having checked all the conditions of Theorem 13 we deduce that indeed (4) holds true. □

**Corollary 15.** Formula (4) holds true for the operator $E_{\beta}$ defined in (9).

**Remark 16.** As a special case, we may choose $\beta = 0$. Then $\Delta_R = \Delta_N$ is the Laplacian with Neumann boundary conditions. In this case, $E_0$ is the reflection without “kinking”, corresponding to certain numeric schemes where Neumann boundary conditions are realized as in (8). A different extension operator for Neumann boundary conditions would be given by extending constantly along the outwards pointing unit normal and again multiplying by a cut-off function. This corresponds to a first-order accurate boundary condition approximation, see again [11, Section 8.3]. Although this might seem more natural at first, it is not obvious whether formula (4) is true for this choice of $E_{\beta}$. Unfortunately, Chernoff’s theorem cannot be applied again because $\Delta E_{\beta}u$ fails to be continuous at $\partial \Omega$ as can easily be seen.

### 5. Dirichlet Boundary Conditions

Next we treat the model problem of an elliptic operator on a bounded set, the Laplacian with Dirichlet boundary conditions. Typically, all results about elliptic operators are much simpler for this special case. Surprisingly, for the aim of this article there arise completely different problems than for Robin and Neumann boundary conditions. This is the reason why we consider it worthwhile to treat this operator in detail.

The Laplacian with Dirichlet boundary conditions defined by
\[
D(\Delta_D) := \{ u \in C_0(\Omega) | \Delta u \in C_0(\Omega) \}, \quad \Delta_D u := \Delta u
\]
generates an positive, contractive $C_0$-semigroup $T_D(t)$ on $C_0(\Omega)$ [1, Theorem 6.1.8]. Formally, the boundary conditions (2) become the Dirichlet boundary conditions $u = 0$ on $\partial \Omega$ in the limit $\beta \to \infty$. This observation can be made precise, cf. [13, Proposition 3.5.3]. As we want to prove an analogue of (4) for $T_D(t)$, we have to define an appropriate extension operator $E_\infty$ for $\beta = \infty$. Taking the limit in (9), we arrive at
\[
E_\infty : C_0(\Omega) \to C_0(\mathbb{R}^N), \quad E_\infty u := \begin{cases} u & \text{on } \Omega, \\ 0 & \text{on } \Omega^C. \end{cases}
\]
Note that we had to replace $C(\overline{\Omega})$ by $C_0(\Omega)$ as we require $E_\infty u$ to be continuous. Unfortunately, we cannot simply replace $E_\beta$ by $E_\infty$ in formula (4) because the iteration scheme does not remain in $C_0(\Omega)$, hence leaving the domain of $E_\infty$.

However, the analogue formula is well-defined (and true) in the $L^2$-context. To see this, note that $L^2(\Omega)$ is a closed subspace of $L^2(\mathbb{R}^N)$ if we consider its functions to be extended by zero. Then the identity mapping takes the role of $E_\infty$, and the
restriction becomes multiplication with 1\( \Omega \). Thus, the analogue of formula (4) for Dirichlet boundary conditions reads

\[ T_{D,2}(t)u = \lim_{n \to \infty} \left( \mathbb{1}_\Omega G_2 \left( \frac{t}{n} \right) \right)^n u \text{ in } L^2(\Omega) \text{ for every } u \in L^2(\Omega), \tag{11} \]

where \( T_{D,2} \) denotes the Dirichlet semigroup on \( L^2(\Omega) \) generated by the Laplacian on \( L^2(\Omega) \) with domain \( H_0^1(\Omega) \cap H^2(\Omega) \). Indeed, formula (11) remains true even if \( \Omega \) has merely Lipschitz regular boundary, cf. [8].

It is interesting to note that (11) cannot be proved using Chernoff’s product formula in the way we did in Section 4. For this, a dense subspace of \( H^1_0(\Omega) \cap H^2(\Omega) \) would have to be contained in \( H^2(\mathbb{R}^N) \), where both spaces carry the graph norm of the Laplacian. But then, continuity asserts \( H^1_0(\Omega) \cap H^2(\Omega) \subset H^2(\mathbb{R}^N) \). However, this cannot be true. In fact, a function in \( C_0(\Omega) \cap C^\infty(\Omega) \subset H^1_0(\Omega) \cap H^2(\Omega) \) whose normal derivative does not vanish is not an element of \( H^2(\mathbb{R}^N) \).

Despite those problems, it is possible to prove a similar result in the same spirit as in Section 4 even in \( C_0(\Omega) \). For this, we need to replace \( \mathbb{1}_\Omega \) by a sequence of smooth interior cut-off functions. But we have to assure that they exhaust \( \Omega \) sufficiently fast compared to the decay of functions in a core for \( \Delta_D \). So we start by a investigation of that decay.

**Lemma 17.** Given a Dirichlet regular bounded set \( \Omega \), there exists \( m \in \mathbb{N} \) having the following property. Given \( t > 0 \), there exists a neighborhood \( U_t \) of \( \partial \Omega \) such that the estimate

\[ |u(x)| \leq t^2 \|(I - \Delta_D)^m u\|_{\infty} \]

holds for every \( x \in \Omega \cap U_t \) and every \( u \in D(\Delta_D^m) \).

**Proof.** It is well-known that \( T_D(t) \) has a kernel representation with a continuous non-negative symmetric kernel \( k_v(x, y) \) which vanishes on \( \partial \Omega \) and is dominated by the Gaussian kernel [3, Section 3.4]. Let \( m > \frac{2}{t^2} \) be fixed. The integral formula for powers of the resolvent [4, Corollary 2.1.11] shows that \( (I - \Delta_D)^{-m} \) is a positive kernel operator with the continuous non-negative symmetric kernel

\[ k(x, y) = \int_0^\infty \frac{s^{m-1}}{(m-1)!} e^{-s} k_s(x, y) \, ds \]

which vanishes for \( x \in \partial \Omega \). Using compactness of \( \Omega \) and \( \partial \Omega \) we deduce that for any \( \varepsilon > 0 \) there exists a neighborhood \( S_\varepsilon \) of \( \partial \Omega \) such that \( x \in S_\varepsilon \) implies \( k(x, y) \leq \varepsilon \) for all \( y \in \Omega \). Define \( U_t := S_\varepsilon \), where \( \varepsilon := \frac{t^2}{|\Omega|} \).

Now fix \( u \in D(\Delta_D^m) \) and define \( v := (I - \Delta_D)^m u \in C_0(\Omega) \). For \( x \in \Omega \cap U_t \), i.e., \( x \in \Omega \cap S_\varepsilon \), we obtain

\[ |u(x)| \leq (I - \Delta_D)^{-m} |v| = \int_\Omega k(x, y) |v(y)| \, dy \leq \varepsilon \|v\|_{\infty} = t^2 \|(I - \Delta_D)^m u\|_{\infty}. \]

This concludes the proof. \( \square \)

We have already explained why we cannot use \( E_\infty \) as extension operator. Instead, we choose

\[ E_D: C_0(\Omega) \to C_0(\mathbb{R}^N), \quad E_D u := \begin{cases} u, & \text{on } \overline{\Omega}, \\ -\theta \tilde{u}, & \text{on } \overline{\Omega}^c, \end{cases} \]
similarly to (1). Here \( \varrho \) denotes a cut-off function that equals 1 near \( \partial \Omega \). Using the same ideas as in Section 3 it can be shown that \( E_D \) is a contractive extension operator that maps \( D(\Delta_D) \cap C^\infty(\overline{\Omega}) \) into \( D(\Delta_0) \). In fact, the main difference to Section 3 is that we know \( \Delta u \in C_0(\Omega) \) for \( u \in D(\Delta_D) \) which makes it easy to check the continuity of \( \Delta(E_D u) \), significantly shortening the chain of arguments.

Now let \( m \) be as in the above lemma, and choose a family \((U_t)_{t \geq 0}\) as in the lemma. For every \( t > 0 \) we fix a suitable cut-off function \( \chi_t \in C_0(\Omega) \) satisfying \( 0 \leq \chi_t \leq 1 \), and \( \chi_t(x) = 1 \) if \( x \in \overline{\Omega} \setminus U_t \). Moreover, define \( \chi_0 := 1_{\overline{\Omega}} \). To simplify notation, we use the multiplication operator \( \chi_t \) as an operator from \( C_0(\mathbb{R}^N) \) to \( C_0(\Omega) \) and \( \chi_0 \) as the restriction from \( C_0(\mathbb{R}^N) \) to \( C(\Omega) \), whenever they are applied to functions in \( C_0(\mathbb{R}^N) \).

We remark that in view of the kernel of \( T_D(t) \) \((t > 0)\) being strictly positive in the interior of \( \Omega \) due to the strong maximum principle, it can be seen that for every compact set \( K \subset \Omega \) there exists \( t_0 > 0 \) such that \( U_t \) and \( K \) are disjoint whenever \( t < t_0 \), implying that \( \chi_t \to 1_{\Omega} \) pointwise as \( t \to 0 \). In this sense, the next result is another flavor of formula (11).

**Theorem 18.** Let \( m \in \mathbb{N} \) and \((\chi_t)_{t \geq 0}\) be as above. Then

\[
T_D(t)u = \lim_{n \to \infty} \left( \chi_t G_0(t) E_D \right)^n u
\]

for every \( u \in C_0(\Omega) \) uniformly on \([0, T]\) for every \( T > 0 \).

**Proof.** We apply Theorem 13 to the operators

\[
V(t): C_0(\Omega) \to C_0(\Omega), \quad u \mapsto \chi_t G_0(t) E_D u.
\]

The properties \( V(0) = I \) and \( \|V(t)^n\| \leq 1 \) for every \( t \geq 0 \) and \( n \in \mathbb{N} \) are obvious from the properties of \( E_D \) and the Gaussian semigroup. Let \( D := D(\Delta_D) \cap C^\infty(\overline{\Omega}) \), which is a core for \( \Delta_D \). This choice makes the density conditions automatic once we show that the limit operator is \( \Delta_D \).

It only remains to prove the convergence to \( \Delta u \) on \( D \). For this, let \( u \in D \). In particular \( u \in D(\Delta_D) \), thus \( \Delta u \in C_0(\Omega) \). Note that

\[
\left\| \frac{V(t)u - u}{t} - \Delta u \right\| = \left\| \chi_t G_0(t) \frac{E_D u - u}{t} - \Delta u \right\| \\
\leq \left\| \chi_t \left( \frac{G_0(t) E_D u - E_D u}{t} - \Delta u \right) \right\| + \left\| \frac{\chi_t E_D u - u}{t} \right\| + \left\| \chi_t \Delta u - \Delta u \right\|.
\]

We estimate the three summands separately. After estimating \( \chi_t \) by \( 1 \) in the first expression, convergence to zero follows from \( E_D u \in D(\Delta_0) \) and the fact that \( \Delta_0(E_D u) = \Delta u \) on \( \Omega \). The third summand can be estimated by \( \sup_{x \in U_t} 2|\Delta u(x)| \) using that \( \chi_t = 1 \) on \( \overline{\Omega} \setminus U_t \). But since we assumed that \( U_t \) leaves any compact set \( K \subset \Omega \) for small \( t \), this expression becomes small as \( t \to 0 \) because \( \Delta u \in C_0(\Omega) \).

The second summand can be estimated with the help of Lemma 17. We obtain

\[
\left\| \frac{\chi_t E_D u - u}{t} \right\| = \frac{1}{t} \left\| \chi_t u - u \right\| \leq \frac{2}{t} \sup_{x \in U_t} |u(x)| \leq 2t \left\| (I - \Delta) mu \right\| \to 0
\]

as \( t \to 0 \). Together, these three estimates show the convergence of the difference quotient to \( \Delta u \) as \( t \) tends to zero. We have checked the assumptions of Chernoff’s product formula, thus proving the claim of the theorem. \( \square \)
6. Conclusion

It is a direct consequence of (1) that $T_R(t)$ is a positive semigroup. Because the operators on the right are $L^\infty$-contractive, it is also clear that $T_R(t)$ is $L^\infty$-contractive, thus submarkovian. In the same way other properties of the limiting semigroup can be deduced by such an approximation formula, as long as they are preserved when taking limits in the strong operator topology. To obtain further properties, it might help to modify the formula a little bit.

So far, we have only considered the Gaussian semigroup as underlying tool. However, it can be seen from the proofs that actually we used only few properties of the Gaussian semigroup. More precisely, we only used that $G_0(t)$ is a contraction on $C_0(\mathbb{R}^N)$ and that any continuous function $u$ such that the support of $u$ is contained in a given neighborhood of $\Omega$ and $\Delta u$ is continuous on $\mathbb{R}^N$ is in the domain of the generator of $G_0(t)$. Thus we could replace $G_0(t)$ with other semigroups, for example with the semigroup generated by the Laplacian with Dirichlet or Neumann boundary conditions, on a larger bounded set $\Omega' \subset \mathbb{R}^N$. Then (1) becomes an approximation formula where the approximating operators are compact. Note, however, that this does not imply that $T_R(t)$ is compact, as the limit is only in the strong operator topology.

Similarly, we can try to approximate $T_R(t)$ only in terms of operators on $C(\overline{\Omega})$, i.e., without any extension to $\mathbb{R}^N$, to obtain an intrinsic approximation. The most natural Trotter-like candidate of this kind would be $S_n(t) := (T_D(\frac{\alpha}{n}) T_N(\frac{\beta}{n}))^n$, where $\alpha$ and $\beta$ are positive numbers such that $\alpha + \beta = 1$ and $T_D(t)$ and $T_N(t)$ denote the Dirichlet and Neumann semigroups on $C(\overline{\Omega})$, respectively. It is known, however, that $\lim S_n(t) = T_D(t)$ in the strong operator topology on $L^2(\Omega)$, whenever $\alpha > 0$, see [7].

Recall that regarding the extension operator we used only two of its properties in Section 4, namely contractivity and some regularity of the extended function. By definition of the extension operator, contractivity came for free. This is due to the rather special definition of $E_\beta$ and the choice of spaces and is a very convenient prerequisite for the application of Theorem 13 although not a necessary one.

Assume that we replace $E_\beta$ by some other, non-contractive extension operator $E$. This is a natural consideration because most extension operators are non-contractive. In fact, it is easy to see that no operator extending $C^1(\overline{\Omega})$ to $C^1(\mathbb{R}^N)$ can be contractive. This also shows that it is a very special property for an extension operator to be contractive and to preserve the regularity of functions in $D(\Delta_R)$.

For such an extension operator $E$, it is considerably more difficult to check whether $(RG_1(t)E)^n$ is uniformly bounded in operator norm with respect to $n$. Because it is hard to control such iterated applications of the Gaussian semigroup, one could try to estimate each factor separately. Then one has to show that $\|RG_0(t)E\| \leq 1 + cf$ for some $c > 0$, leading to the upper bound $e^{ct}$. The short time diffusion through the boundary, however, is of order $O(\sqrt{t})$, see [9]. This is why in general only estimates of the kind $\|RG_0(t)E\| = 1 + O(\sqrt{t})$ can be obtained.

Almost the same reasoning applies if $C(\overline{\Omega})$ is replaced by an $L^p$-space, for example by $L^1(\Omega)$. As $\Omega$ is bounded, uniform convergence already implies convergence in $L^1(\Omega)$, hence

$$T_R(t)u = \lim_{n \to \infty} (RG_0(t)E_\beta)^n u \text{ in } L^1(\Omega) \text{ for every } u \in C(\overline{\Omega})$$

...
by what we have already shown. If we want to extend this result to \( u \in L^1(\Omega) \), it suffices to show that the approximating operators \( (RG_0(t)E_\beta)^n \) remain bounded in the norm of operators on \( L^1(\Omega) \). Here again, there arise difficulties which are similar to those mentioned in the preceding paragraph because no non-trivial extension operator from \( L^1(\Omega) \) to \( L^1(\mathbb{R}^N) \) is contractive. This shows that for our applications the space \( C(\Omega) \) has significant advantages.

A related question is whether (4) remains true if the assumption \( \beta \geq 0 \) is dropped. We mention that it can be seen that for any \( \beta \in L^\infty(\partial\Omega) \) the Laplacian with Robin boundary conditions is the generator of a semigroup on \( L^2(\Omega) \) and thus this question makes sense. But if we define \( E_\beta \) as in Section 3, we do not even in \( C(\Omega) \) obtain a contraction if \( \beta(z) < 0 \) for a point \( z \in \partial\Omega \), causing the same problems again. Moreover, it is clear that the assumption \( \|(RG_0(t)E_\beta)^n\| \) which is needed for Theorem 13 cannot be fulfilled since the candidate limit semigroup \( T_R(t) \) will not be bounded. But the latter is merely a problem of rescaling, compare [4, Corollary III.5.3].

It should be possible to extend the results to smooth unbounded open sets \( \Omega \) without difficulties because most arguments are local. However, the other calculations become even more technical. This is why we have restricted ourselves to bounded domains.

On the other hand, choosing a different (contractive) extension operator will usually change the situation completely. For example, Theorem 13 cannot be applied for the constant extension as in Remark 16 reflecting the fact that a worse numerical approximation of the normal derivative leads to worse convergence behavior. But that extension operator can be defined even for convex domains without any smoothness assumptions, which might provide an alternative approximation scheme for less smooth domains. It is easy to come up with various other extension operators when trying to find an approximation formula such as (4) for (not necessarily convex) sets with non-smooth boundary. This is ongoing work and might be the topic of a future publication.

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