Optimal Guidance Law for Intercepting the Active Defense Aircraft with Terminal Angle Constraint

Hao Zhang, Yiqun Zhang*, Pengfei Zhang

Beijing Institute of Electronic System Engineering, Yongding Road, Haidian District, Beijing, P R China

*Email: zhanghaoibit@163.com

Abstract. For the scenario where the target can actively defend against the attacking missile, an optimal guidance law for the attacking missile that can simultaneously evade the defending missile and intercept the target is proposed. To increase lethality of the attacking missile, the terminal angle constraint is considered. Assuming that the defending missile and the target is using a known control strategy, the guidance law design problem for the attacking missile is formulated as an optimal control problem with inequality constraint. General solution of the problem is derived for arbitrary order dynamics of the adversaries. Then first-order dynamics of the three players are considered and an analytic solution is presented. Numerical simulations are carried out to demonstrate the high performance of the proposed guidance law.

1. Introduction

To improve the probability of survival, the target (e.g., an aircraft) which is aware it is being tracked by an attacking missile, fires a defending missile to intercept the attacking missile while performing maneuvers to evade. This type of engagement scenario is called as the attacker-target-defender (ATD) scenario which has received a lot of attention in recent years. Different from the traditional one-on-one engagement, this scenario consists three agents with mutually conflicting objectives.

This scenario was first proposed in [1], where a minimum range for the defending missile was calculated to intercept the attacking missile within a specified distance. Then in [2], the expression of the range for the defending missile was simplified. Unfortunately, these results were based on constant bearing trajectories which were unrealistic.

In [3], a triangle intercept guidance law was proposed. This guidance law can degenerate the triangle formed by the ATD into a straight line, so that the attacker can be intercepted by the defender before it approaches the target. In [4-5], modified command to line-of-sight (LOS) intercept guidance strategies, based on velocity error feedback and optimal control theory, respectively, were presented. It should be noted that the control strategies in [3-5] do not take into account cooperation between the target and defender. In [6], kinematics of LOS guidance with a moving/maneuvering platform were analysed. Then, using the kinematic results, a cooperative guidance law was proposed to maximize the attacker’s lateral acceleration.

In [7], assuming that the attacker is using a known linear missile guidance strategy, a cooperative guidance strategy was proposed for the defender and the target with arbitrary order dynamics. The optimal evasion strategy for the target was also discussed. In [8], three types of cooperative guidance strategies were derived for different cooperation scheme. In the first case, the target–defender team uses the optimal cooperative strategy which is called two-way cooperation. In the other two cases, one
of them maneuvers independently, and notifies the other its control strategy. In [9], guidance strategies of the defender and the target were designed separately by minimizing the control effort and specifying the miss distance. Further in [10], combined and cooperative minimum-effort guidance strategies were derived for the defender and the target.

In [11], a two-team dynamic game was used to formulate the problem, the lady and the body-guard as one team, the bandit as another team. The bandit’s goal is to capture the lady while the lady is trying to evade. The body-guard is trying to intercept the bandit before he reaches close to the lady. In [12], the game was reformulated as the ATD problem. Based on the linear quadratic differential game (LQDG) theory, closed loop Nash equilibrium strategies were derived for the three players. In [13], arbitrary order dynamics of the three players were considered and analytical solution to the LQDG problem was derived. In addition, nonlinear numerical simulations were carried out to demonstrate the theoretical analysis.

Most of the guidance strategies were derived under the linearization assumption. While in [14-17], the guidance strategies were presented in a nonlinear framework, which can be applied to any engagement geometry. In [14] and [15], assuming that the attacker is using PN or pure pursuit guidance law, a cooperative guidance strategy, was proposed which can maximize separation between the attacker and target. However, solving a two-point boundary value problem will bring a lot of calculation. In [16], based on the geometric properties of the ATD problem, optimal strategies were proposed in the attacker’s winning region. However, optimality was not strictly proven in this paper. This issue was addressed in [17], by finding the solution of the Hamilton-Jacobi-Isaacs (HJI) Partial Differential Equation (PDE).

This paper presents the guidance strategy for the attacker, derived in optimal control theory framework, which simultaneously achieves evasion from the defender with specified miss distance and perfect interception of the target. In order to increase lethality of the attacker, the intercept angle constraint is also considered. Such issue has been studied earlier in [18], which was considered for the standard one-on-one engagement. To the best of the authors’ knowledge, imposing a terminal intercept angle for the attacker-target interception was not considered earlier in the ATD scenario. In [19], similar issue was studied but the intercept angle constraint was taken into account for the defender-attacker interception which is different from the work in this paper.

2. Models Derivation and Problem Formulation

2.1. Nonlinear Kinematics

A schematic view of the planar endgame geometry of adversaries is shown in figure 1, where $X_1 - O_1 - Y_1$ is a Cartesian inertial reference frame.

![Figure 1. Planar ATD engagement geometry.](image)

The attacking missile, evading target, and defending missile are denoted as A, T, and D, respectively. The flight path angles, speed, and lateral acceleration of adversaries are denoted as $\gamma_i$, $V_i$, and $a_i$.
and \(a_i, \ i \in \{A, T, D\}\). The variables \(r_{AT}\) and \(r_{AD}\) represent the relative ranges between the pairs A-T and A-D. The variables \(\lambda_{AT}\) and \(\lambda_{AD}\) denote the LOS angles between the pairs A-T and A-D. The initial LOS between A and T is denoted as LOS\(_{AT}\); and similarly, the initial LOS between A and D is denoted as LOS\(_{AD}\).

In the polar coordinate system \((r, \lambda)\), the nonlinear kinematics for the engagement between the target and attacker are

\[
\begin{align*}
\dot{r}_{AT} &= -V_A \cos(\gamma_A + \lambda_{AT}) - V_T \cos(\gamma_T - \lambda_{AT}) \\
\dot{\lambda}_{AT} &= -V_A \sin(\gamma_A + \lambda_{AT}) - V_T \sin(\gamma_T - \lambda_{AT}) \tag{1}
\end{align*}
\]

In the same way, the kinematics between the defender and attacker are

\[
\begin{align*}
\dot{r}_{AD} &= -V_A \cos(\gamma_A + \lambda_{AD}) - V_D \cos(\gamma_D - \lambda_{AD}) \\
\dot{\lambda}_{AD} &= -V_A \sin(\gamma_A + \lambda_{AD}) - V_D \sin(\gamma_D - \lambda_{AD}) \tag{4}
\end{align*}
\]

The derivative of the flight path angles for the three entities can be written as

\[
\dot{\gamma}_i = a_i / V_i; \quad i \in \{A, T, D\} \tag{5}
\]

Additionally, arbitrary order state equations can be used to express the dynamics of the adversaries

\[
\begin{align*}
\dot{x}_i &= A_i x_i + b_i u_i^c; \quad i \in \{A, T, D\} \\
a_i &= c_i x_i + d_i u_i^c
\end{align*} \tag{6}
\]

where \(x_i\) represents the state variable of the adversary’s internal dynamics, and \(u_i^c\) is its guidance command norm to the velocity vector.

### 2.2. Linearized Kinematics for Guidance Law Derivation

There are two collision triangles in this engagement with three entities. First, between the attacker and the target and second between the defender and the attacker. Assuming that the engagement takes place in the terminal guidance phase, so the three entities flight near these initial collision triangles. In this case, the linearization along the initial LOS is justified.

In figure 1, the variable \(y_{AT}\) represents the relative distance between A and T, perpendicular to LOS\(_{AT}\); and in the same way, the variable \(y_{AD}\) represents the relative distance between the A and D, perpendicular to LOS\(_{AD}\).

As a matter of convenience, the attacker’s and the target’s accelerations normal to LOS\(_{AT}\) are denoted as \(a_{AN1}\) and \(a_{TN}\), respectively; and similarly, the attacker’s and the defender’s accelerations normal to LOS\(_{AD}\) are denoted as \(a_{AN2}\) and \(a_{DN}\). Their relationship with the original lateral accelerations is as follows

\[
\begin{align*}
a_{AN1} &= a_A \cos(\gamma_A + \lambda_{AT}) = c_A x_A \cos(\gamma_A + \lambda_{AT}) + d_A u_A \tag{7}
\end{align*}
\]

\[
\begin{align*}
a_{AN2} &= a_A \cos(\gamma_A + \lambda_{AD}) = c_A x_A \cos(\gamma_A + \lambda_{AD}) + d_A u_A \cos(\gamma_A + \lambda_{AD}) / \cos(\gamma_A + \lambda_{AT}) \tag{8}
\end{align*}
\]

\[
\begin{align*}
a_{TN} &= a_T \cos(\gamma_T - \lambda_{AT}) = c_T x_T \cos(\gamma_T - \lambda_{AT}) + d_T u_T \tag{9}
\end{align*}
\]

\[
\begin{align*}
a_{DN} &= a_D \cos(\gamma_D - \lambda_{AD}) = c_D x_D \cos(\gamma_D - \lambda_{AD}) + d_D u_D \tag{10}
\end{align*}
\]

where \(u_A, u_T, \) and \(u_D\) are the respective guidance commands normal to the corresponding initial LOS, and they satisfy

\[
\begin{align*}
u_A &= u_A^c \cos(\gamma_A + \lambda_{AT}) \tag{11}
\end{align*}
\]

\[
\begin{align*}
u_T &= u_T^c \cos(\gamma_T - \lambda_{AT}) \tag{12}
\end{align*}
\]
\[ u_d = u'_d \cos(\gamma_{d_0} - \lambda_{AD}) \] (13)

The state vector of the linearized kinematics is
\[
x = \begin{bmatrix} x_{AT}^T \ x_{AD}^T \ x_A^T \ x_N \end{bmatrix}^T
\] (14)

where
\[
x_{AT} = \begin{bmatrix} y_{AT}^T \ y_{AT}^T \ x_T^T \end{bmatrix}^T
\] (15)
\[
x_{AD} = \begin{bmatrix} y_{AD}^T \ y_{AD}^T \ x_D^T \end{bmatrix}^T
\] (16)
\[
x_N = \gamma_T + \gamma_A
\] (17)

The combined equations of motion for the three entities can be written as
\[
\dot{x} = \begin{bmatrix} \dot{y}_{AT} = \dot{y}_{AT} \\
\dot{y}_{AT} = d_{TN} - a_{AN} \\
x_T = A_x x_T + b_x u_T \cos(\gamma_{T_0} - \lambda_{AT}) \\
\dot{y}_{AD} = \dot{y}_{AD} \\
\dot{y}_{AD} = a_{AN} - a_{DN} \\
x_D = A_x x_D + b_x u_D \cos(\gamma_{D_0} - \lambda_{AD}) \\
\dot{x_A} = \dot{x_A} + b_x u_A \cos(\gamma_{A_0} + \lambda_{AT}) \\
x_N = a_T / V_T + d_A / V_A
\end{bmatrix}
\] (18)

Substituting (7)-(10) into (18), we obtain the final state equations, written as
\[
\dot{x} = Ax + B_T u_T + B_D u_D + B_A u_A
\] (19)

where
\[
A = \begin{bmatrix} A_{AT} & [0] & E_{AT} & [0] \\
[0] & A_{AD} & E_{AD} & [0] \\
[0] & [0] & A_A & [0] \\
E_{TN} & [0] & E_{AN} & 0 \end{bmatrix} ;
B_T = \begin{bmatrix} B_{AT} \\
[0] \\
[0] \\
0 \end{bmatrix} ;
B_D = \begin{bmatrix} B_{AD} \\
[0] \\
[0] \\
0 \end{bmatrix} ;
B_A = \begin{bmatrix} 0 \\
-d_A \cos(\gamma_{A_0} + \lambda_{AT}) \\
0 \\
0 \end{bmatrix}
\]

Note that \([0]\) represents a matrix of zeros with appropriate dimensions.

The output vector of the system is
\[
y = \begin{bmatrix} y_{AT} \ y_{AD} \ x_N \end{bmatrix}^T
\] (20)

The output equation is
\[
y = Cx
\] (21)
where \( C = \begin{bmatrix} e_{AT}^T & e_{AD}^T & e_{N}^T \end{bmatrix}^T; \quad e_{AT} = [1 \ 0] ; \quad e_{AD} = [0 \ 1] ; \quad e_{N} = [0 \ 1] \). 

2.3. Timeline
The initial range between the attacker and the target is \( r_{AT} \). Similarly, between the attacker and the defender it is \( r_{AD} \). Under the linearization assumption of small deviations from the initial collision triangle, closing speeds of attacker-target \( V_{AT} \) and attacker-defender \( V_{AD} \) are assumed to be constant. Then, the interception time can be written as
\[
t_{jAT} = \frac{r_{AT}}{V_{AT}} \tag{22}
\]
where
\[
V_{AT} = V_A \cos (\gamma_A + \dot{\lambda}_{AT}) + V_t \cos (\gamma_t - \dot{\lambda}_{AT}) \tag{23}
\]
and similarly
\[
t_{jAD} = \frac{r_{AD}}{V_{AD}} \tag{24}
\]
where
\[
V_{AD} = V_A \cos (\gamma_A + \dot{\lambda}_{AD}) + V_D \cos (\gamma_A - \dot{\lambda}_{AD}) \tag{25}
\]
Note that \( t_{jAD} < t_{jAT} \).
The times-to-go of the attacker-target and attacker-defender engagements are
\[
t_{tAT} = t_{jAT} - t \tag{26}
\]
\[
t_{tAD} = t_{jAD} - t \tag{27}
\]
2.4. Formulation of the Optimization Problem
It is assumed that the defender uses a known linear guidance law, which can be written as
\[
u_D = F_D(t)x \tag{28}
\]
and that the target’s future maneuver strategy \( u_t \) is constant. Thus, the state equation (19) becomes
\[
\dot{x} = A_{PD}(t)x + B_A u_A(t) + B_{AD} u_D(t) \tag{29}
\]
where \( A_{PD}(t) = A + B_D F_D(t) \).
The initial condition of differential equation (29) is
\[
x(t_0) = x_0 \tag{30}
\]
The quadratic cost function needs to be minimized is
\[
J = \frac{a}{2} y_{AT}^2(t_{jAT}) + \frac{b}{2} \left[ x_N(t_{jAT}) - x_N^* \right]^2 + \frac{1}{2} \int_{t_0}^{t_{jAT}} u_A^2(t)dt \tag{31}
\]
where \( a \) and \( b \) are nonnegative weight coefficients. If \( a \to \infty \) and \( b \to \infty \), a perfect interception guidance law with terminal angle constraint can be obtained. In our study, we concentrate on this limiting case. Thus, \( y_{AT}(t_{jAT}) = 0, \ x_N(t_{jAT}) = x_N^* \), and the quadratic cost function reduces to
\[
J = \frac{1}{2} \int_{t_0}^{t_{jAT}} u_A^2(t)dt \tag{32}
\]
As the attacker is being intercepted by the defender, the premise for the attacker to hit the target is that it can survive in the attacker-defender engagement. This can be defined by the condition
\[
\left| y_{AD}(t_{jAD}) \right| \geq L_{AD} \tag{33}
\]
Here, \( L_{AD} \) defines the lethality region of the defender.
Equations (29)-(33) constitute a linear quadratic optimal control problem with inequality constraint.

3. Optimal Guidance Law Formulation of the Optimization Problem
3.1. Order Reduction

In order to simplify the derivation of the problem and reduce the order of original state equation, we use the terminal projection transformation, introduced by Bryson and Ho [20]. Let us define three zero-effort quantities as new state variables.

\[ Z_{AT}(t) = e_{AT} \Phi_{FD}(t_{jAT}, t)x(t) + e_{AT} \int_{t_0}^{t_{jAT}} \Phi_{FD}(t_{jAT}, \tau)B_{A}u_A(\tau)d\tau \]
\[ Z_{AD}(t) = e_{AD} \Phi_{FD}(t_{jAD}, t)x(t) \]
\[ Z_{N}(t) = e_{N} \Phi_{FD}(t_{jAT}, t)x(t) + e_{N} \int_{t_0}^{t_{jAT}} \Phi_{FD}(t_{jAT}, \tau)B_{N}u_N(\tau)d\tau \]

where \( \Phi_{FD}(t, \tau) \) is the state transition matrix related to \( A_{FD}(t) \).

The three new state variables have important physical meanings. \( Z_{AT}(t) \) and \( Z_{AD}(t) \) are zero-effort miss distances for engagement between the pairs A-T and A-D, respectively. \( Z_{N}(t) \) is zero-effort angle for the A-T engagement.

The derivative of these new state variables are

\[ \dot{Z}_{AT} = \alpha_A(t)u_A \]
\[ \dot{Z}_{AD} = \beta_A(t)u_A \]
\[ \dot{Z}_{N} = \gamma_A(t)u_A \]

where

\[ \alpha_A(t) = e_{AT} \Phi_{FD}(t_{jAT}, t)B_A \]
\[ \beta_A(t) = e_{AD} \Phi_{FD}(t_{jAD}, t)B_A \]
\[ \gamma_A(t) = e_{N} \Phi_{FD}(t_{jAT}, t)B_N \]

In order to reflect the defender’s disappearance after termination of the engagement with the attacker, we force \( Z_{AD}(t) = Z_{AD}(t_{jAD}) \) for \( t_{jAD} < t \leq t_{jAT} \), which means that \( \beta_A(t) \equiv 0 \) for \( t_{jAD} < t \leq t_{jAT} \).

The initial conditions of the new differential equations (35) are

\[ Z_{AT}(t_0) = e_{AT} \Phi_{FD}(t_{jAT}, t_0)x_0 + e_{AT} \int_{t_0}^{t_{jAT}} \Phi_{FD}(t_{jAT}, \tau)B_{A}u_A(\tau)d\tau \]
\[ Z_{AD}(t_0) = e_{AD} \Phi_{FD}(t_{jAD}, t_0)x_0 \]
\[ Z_{N}(t_0) = e_{N} \Phi_{FD}(t_{jAT}, t_0)x_0 + e_{N} \int_{t_0}^{t_{jAT}} \Phi_{FD}(t_{jAT}, \tau)B_{N}u_N(\tau)d\tau \]

Using these new variables, the terminal conditions become

\[ Z_{AT}(t_{jAT}) = 0 \]
\[ Z_{N}(t_{jAT}) = x_N^\epsilon \]

Also, the inequality constraint from equation (33) can be expressed as

\[ \left| Z_{AD}(t_{jAT}) \right| \geq L_{AD} \]

Note that the quadratic cost function from equation (32) is not changed.

3.2. Analytical Solution for Arbitrary Order Dynamics

Since for \( t_{jAD} < t \leq t_{jAT} \), we actually have a standard one-on-one engagement, which was solved in [18]. Therefore, we only need to determine the optimal guidance law for \( t_0 \leq t \leq t_{jAD} \).

To solve the optimization problem with inequality constraint, we first consider the equality constraint problem. Let \( Z_{AD}(t_{jAT}) = z_\phi \), where \( z_\phi \) is fixed but an arbitrary real number.

The Hamiltonian of the problem is
\[ H = \frac{1}{2} u_0^2 + \lambda_1 \dot{Z}_{AT} + \lambda_2 \dot{Z}_{AD} + \lambda_3 \dot{Z}_N \]  

\[ \lambda_1 = -\frac{\partial H}{\partial Z_{AT}} = 0 \]
\[ \lambda_2 = -\frac{\partial H}{\partial Z_{AD}} = 0 \]
\[ \lambda_3 = -\frac{\partial H}{\partial Z_N} = 0 \]  

The co-state equations are given by

\[ \dot{\lambda}_1 = -\lambda_1 \dot{Z}_{AT} \]
\[ \dot{\lambda}_2 = -\lambda_1 \dot{Z}_{AD} \]
\[ \dot{\lambda}_3 = -\lambda_3 \dot{Z}_N \]  

Thus, the co-states \( \lambda_1 \), \( \lambda_2 \) and \( \lambda_3 \) are constants.

The optimal controller for the attacker satisfies

\[ \frac{\partial H}{\partial u_A} = 0 \Rightarrow u_A(t) = -\left[ \lambda_1 \alpha_A(t) + \lambda_2 \beta_A(t) + \lambda_3 \gamma_A(t) \right] \]  

Substituting equation (42) into equation (35), and integrating from \( t_0 \) to \( t_{AT} \), yields the following equations

\[ Z_{AT}(t_{AT}) - Z_{T0} = -\lambda_1 \int_{t_0}^{t_{AT}} \dot{Z}_{AT}(t) \, dt - \lambda_2 \int_{t_0}^{t_{AT}} \dot{Z}_{AD}(t) \, dt - \lambda_3 \int_{t_0}^{t_{AT}} \dot{Z}_N(t) \, dt \]
\[ Z_{AD}(t_{AT}) - Z_{D0} = -\lambda_1 \int_{t_0}^{t_{AT}} \dot{Z}_{AD}(t) \, dt - \lambda_2 \int_{t_0}^{t_{AT}} \dot{Z}_{AD}(t) \, dt - \lambda_3 \int_{t_0}^{t_{AT}} \dot{Z}_N(t) \, dt \]
\[ Z_N(t_{AT}) - Z_{N0} = -\lambda_1 \int_{t_0}^{t_{AT}} \dot{Z}_N(t) \, dt - \lambda_2 \int_{t_0}^{t_{AT}} \dot{Z}_N(t) \, dt - \lambda_3 \int_{t_0}^{t_{AT}} \dot{Z}_N(t) \, dt \]  

Using the terminal conditions (38), equation (43) can be written as

\[ \begin{bmatrix} -Z_{T0} \\ z_{D} - Z_{D0} \\ x_{N} - Z_{N0} \end{bmatrix} = -H(t_0) \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} \]  

where

\[ H(t_0) = \begin{bmatrix} \int_{t_0}^{t_{AT}} \dot{Z}_{AT}(t) \, dt & \int_{t_0}^{t_{AT}} \dot{Z}_{AD}(t) \, dt & \int_{t_0}^{t_{AT}} \dot{Z}_N(t) \, dt \\ \int_{t_0}^{t_{AT}} \dot{Z}_{AD}(t) \, dt & \int_{t_0}^{t_{AT}} \dot{Z}_{AD}(t) \, dt & \int_{t_0}^{t_{AT}} \dot{Z}_N(t) \, dt \\ \int_{t_0}^{t_{AT}} \dot{Z}_N(t) \, dt & \int_{t_0}^{t_{AT}} \dot{Z}_N(t) \, dt & \int_{t_0}^{t_{AT}} \dot{Z}_N(t) \, dt \end{bmatrix} \]  

If \( \alpha_A(t) \), \( \beta_A(t) \) and \( \gamma_A(t) \) are linearly independent, the matrix \( H(t_0) \) is invertible, then

\[ \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = -H^{-1}(t_0) \begin{bmatrix} -Z_{T0} \\ z_{D} - Z_{D0} \\ x_{N} - Z_{N0} \end{bmatrix} \]  

Thus, the optimal controller can be written as

\[ u_A(t) = \begin{bmatrix} \alpha_A(t) \\ \beta_A(t) \\ \gamma_A(t) \end{bmatrix} H^{-1}(t_0) \begin{bmatrix} -Z_{T0} \\ z_{D} - Z_{D0} \\ x_{N} - Z_{N0} \end{bmatrix} \]  

Substituting equation (47) into equation (32), the value of the cost function becomes
The attacker’s optimal guidance strategy is to minimize the cost function under the constraint (39). Obviously, $J$ is a quadratic function of $z_{D}$, so we can choose $z_{D}$ that minimize $J$ on the interval $[L_{AD}, + \infty)$. After some straightforward manipulations, $z_{D}^{*}$ can be obtained

$$z_{D}^{*} = Z_{D0} + \frac{\phi_{21}(t_0)}{\phi_{22}(t_0)} Z_{T0} - \frac{\phi_{31}(t_0)}{\phi_{22}(t_0)} (x_{N}^{c} - Z_{N0}) - \psi \left[ Z_{D0} + \frac{\phi_{21}(t_0)}{\phi_{22}(t_0)} Z_{T0} - \frac{\phi_{31}(t_0)}{\phi_{22}(t_0)} (x_{N}^{c} - Z_{N0}) \right]$$

where

$$\psi(x) = \begin{cases} 
+ \infty, & x < 0 \\
0, & x \geq L_{AD} \\
-x - L_{AD}, & 0 \leq x < L_{AD}
\end{cases}$$

and the expression of $\phi_{i}^{*}(\cdot)$ will be given later.

The optimal guidance law is obtained by substituting $z_{D}^{*}$ into equation (47)

$$u_{A}(t) = \frac{1}{\Delta(t)} \left[ \alpha_{A}(t) \frac{\phi_{21}^{2}(t_0) - \phi_{11}(t_0) \phi_{22}(t_0)}{\phi_{22}(t_0)} + \gamma_{A}(t) \frac{\phi_{21}(t_0) \phi_{23}(t_0) - \phi_{22}(t_0) \phi_{32}(t_0)}{\phi_{22}(t_0)} \right] Z_{T0}$$

$$- \frac{1}{\Delta(t)} \left[ \alpha_{A}(t) \frac{\phi_{21}^{2}(t_0) - \phi_{11}(t_0) \phi_{22}(t_0)}{\phi_{22}(t_0)} + \gamma_{A}(t) \frac{\phi_{21}(t_0) \phi_{23}(t_0) - \phi_{22}(t_0) \phi_{32}(t_0)}{\phi_{22}(t_0)} \right] (Z_{N0} - x_{N}^{c})$$

$$- \frac{1}{\Delta(t)} \left[ \alpha_{A}(t) \phi_{21}(t_0) + \beta_{A}(t) \phi_{22}(t_0) + \gamma_{A}(t) \phi_{32}(t_0) \right] \psi \left[ Z_{D0} + \frac{\phi_{21}(t_0)}{\phi_{22}(t_0)} Z_{T0} + \frac{\phi_{32}(t_0)}{\phi_{22}(t_0)} (Z_{N0} - x_{N}^{c}) \right]$$

where $\Delta(\cdot) = \det H (\cdot)$ and the terms $\phi_{i}(\cdot)$ are given by

\[
\begin{align*}
\phi_{11}(\cdot) &= h_{22}(\cdot) h_{33}(\cdot) - h_{23}(\cdot) \\
\phi_{22}(\cdot) &= h_{13}(\cdot) h_{33}(\cdot) - h_{23}(\cdot) h_{32}(\cdot) \\
\phi_{33}(\cdot) &= h_{13}(\cdot) h_{22}(\cdot) - h_{12}(\cdot) h_{33}(\cdot) \\
\phi_{23}(\cdot) &= h_{13}(\cdot) h_{32}(\cdot) - h_{12}(\cdot) h_{33}(\cdot) \\
\phi_{32}(\cdot) &= h_{13}(\cdot) h_{22}(\cdot) - h_{12}(\cdot) h_{33}(\cdot)
\end{align*}
\]

$h_{ij}(\cdot)$ is the element in the i-th row and j-th column of the matrix $H (\cdot)$.

Because the initial time $t_0$ can be chosen arbitrarily for $t_0 \leq t_{AD}$, the optimal guidance law can be written in closed-loop form with an arbitrary time $t$. Then, for $t \leq t_{AD}$, we can write the closed-loop optimal guidance law as follows

$$u_{A}(t) = \frac{1}{\Delta(t)} \left[ \alpha_{A}(t) \frac{\phi_{21}^{2}(t) - \phi_{11}(t) \phi_{22}(t)}{\phi_{22}(t)} + \gamma_{A}(t) \frac{\phi_{21}(t) \phi_{23}(t) - \phi_{22}(t) \phi_{32}(t)}{\phi_{22}(t)} \right] Z_{T0}(t)$$

$$- \frac{1}{\Delta(t)} \left[ \alpha_{A}(t) \frac{\phi_{21}^{2}(t) - \phi_{11}(t) \phi_{22}(t)}{\phi_{22}(t)} + \gamma_{A}(t) \frac{\phi_{21}(t) \phi_{23}(t) - \phi_{22}(t) \phi_{32}(t)}{\phi_{22}(t)} \right] (Z_{N}(t) - x_{N}^{c})$$

$$- \frac{1}{\Delta(t)} \left[ \alpha_{A}(t) \phi_{21}(t) + \beta_{A}(t) \phi_{22}(t) + \gamma_{A}(t) \phi_{32}(t) \right] \psi \left[ Z_{AD}(t) + \frac{\phi_{21}(t)}{\phi_{22}(t)} Z_{T0}(t) + \frac{\phi_{32}(t)}{\phi_{22}(t)} (Z_{N}(t) - x_{N}^{c}) \right]$$

and for $t_{AD} < t \leq t_{AT}$, according to [18], the optimal guidance law is
\[ u_A(t) = \frac{1}{\Delta(t)} \left[ -\alpha_A(t)h_3(t) + \gamma_A(t)h_3(t) \right] Z_{\mathcal{X}_T}(t) + \frac{1}{\Delta(t)} \left[ \alpha_A(t)h_3(t) - \gamma_A(t)h_1(t) \right] \left( Z_N(t) - x_N^t \right) \] (54)

3.3. Guidance Law for First-Order Dynamics

In the previous subsection, the attacker’s optimal guidance strategy is derived for arbitrary order dynamics. Here, we concentrate on the first-order dynamics case. The critical step is to solve the expression of \( \alpha_A(t), \beta_A(t), \gamma_A(t), Z_{\mathcal{X}_T}(t), Z_{\mathcal{X}_D}(t), Z_N(t) \).

Define

\[
X_T(t) = \Phi_{FD}^T(t_{jAT}, t) c_{AT}^T
\]
\[
X_D(t) = \Phi_{FD}^T(t_{jAD}, t) c_{AD}^T
\]
\[
X_N(t) = \Phi_{FD}^T(t_{jAT}, t) c_N^T
\]

Then

\[
\alpha_A(t) = X_T^T(t) B_A
\]
\[
\beta_A(t) = X_D^T(t) B_A
\]
\[
\gamma_A(t) = X_N^T(t) B_A
\]

and

\[
Z_{\mathcal{X}_T}(t) = X_T^T(t) x(t) + \int_{t_{jAT}}^{\infty} X_T^T(\tau) B_x u_x(\tau) d\tau
\]
\[
Z_{\mathcal{X}_D}(t) = X_D^T(t) x(t)
\]
\[
Z_N(t) = X_N^T(t) x(t) + \int_{t_{jAT}}^{\infty} X_N^T(\tau) B_x u_x(\tau) d\tau
\]

According to the properties of the state transition matrix, the differential equations and initial conditions satisfied by \( X_T(t), X_D(t), X_N(t) \) are as follows

\[ \begin{aligned}
\frac{dX_T}{dt_{jAT}} &= -X_T = A_{FD}^T(t_{jAT} - t_{jAT}) X_T \\
X_T(t_{jAT} = 0) &= X_T(t = t_{jAT}) = c_{AT}^T
\end{aligned} \] (58)

\[ \begin{aligned}
\frac{dX_D}{dt_{jAD}} &= -X_D = A_{FD}^T(t_{jAD} - t_{jAD}) X_D \\
X_D(t_{jAD} = 0) &= X_D(t = t_{jAD}) = c_{AD}^T
\end{aligned} \] (59)

\[ \begin{aligned}
\frac{dX_N}{dt_{jAT}} &= -X_N = A_{FD}^T(t_{jAT} - t_{jAT}) X_N \\
X_N(t_{jAT} = 0) &= X_N(t = t_{jAT}) = c_N^T
\end{aligned} \] (60)

Note that here we take the derivative with respect to time-to-go, not time.

For the case that all of the adversaries have first-order dynamics, the matrices in equation (6) become

\[
A_i = \frac{1}{\tau_i}, \ b_i = \frac{1}{\tau_i}, \ c_i = 1, \ d_i = 0; \ i \in \{ A, T, D \}
\] (61)

It is assumed that the defender uses proportional navigation guidance law, that is

\[
u_D = \frac{X_D^T}{t_{jAD}^2} (y_{\mathcal{X}_D} + \gamma_{\mathcal{X}_D} t_{jAD})
\] (62)

The state variables for first-order dynamics are

\[
x = [y_{\mathcal{X}_T} \ y_{\mathcal{X}_D} \ a_T \ y_{\mathcal{X}_D} \ a_T \ a_D \ a_A \ x_N]^{T}
\] (63)

The state-space equation is
\[ \dot{x} = A_{FD}(t)x + B_{f}u_{f}(t) + B_{A}u_{A}(t) \]
\[ y = Cx \] (64)

where
\[
A_{FD} = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \cos(y_{D0} - \lambda_{AT}) & 0 & 0 & 0 & -\cos(y_{a0} + \lambda_{AT}) & 0 \\
0 & 0 & -\frac{1}{\tau_{T}} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\cos(y_{D0} - \lambda_{AD}) & \cos(y_{a0} + \lambda_{AD}) & 0 \\
0 & 0 & 0 & \frac{1}{\tau_{D}} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{V_{T}} & 0 \\
\end{bmatrix} \\
A_{FD64} = N_{p}'f\left[\tau_{p}'\tau_{D}'\cos(y_{D0} - \lambda_{AD})\right] ; \quad A_{FD65} = N_{p}'f\left[\tau_{p}'\tau_{D}'\cos(y_{D0} - \lambda_{AD})\right] ; \\
B_{f} = \begin{bmatrix}
0 & 0 & \frac{1}{\tau_{T}} & \cos(y_{T0} - \lambda_{AT}) & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{\tau_{D}} & \cos(y_{D0} - \lambda_{AD}) & 0 & 0 & 0 & 0 \\
\end{bmatrix} ; \quad B_{A} = \begin{bmatrix}
0 & 0 & 0 & \frac{1}{\tau_{A}} & \cos(y_{A0} + \lambda_{AT}) & 0 \\
\end{bmatrix} ; \\
C = \begin{bmatrix}
c_{AT}' & c_{AD}' & c_{N}' \\
\end{bmatrix} ; \\
c_{AT} = [1 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0] ; \quad c_{AD} = [0 \; 0 \; 0 \; 1 \; 0 \; 0 \; 0 \; 0] ; \quad c_{N} = [0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 1] .
\]

For convenience, the navigation constant of the defender is chosen to be \( N_{p}' = 3 \).

With these expressions, the differential equations (58)-(60) can be solved, and the solutions are
\[ X_{T}(t) = \begin{bmatrix} 1 & t_{SAT} \cos(y_{T0} - \lambda_{AT}) \tau_{D}^{2} f(t_{SAT} / \tau_{T}) & 0 & 0 & 0 & -\cos(y_{a0} + \lambda_{AT}) \tau_{D}^{2} f(t_{SAT} / \tau_{A}) & 0 \end{bmatrix}^{T} \] (65)
\[ X_{D}(t) = \begin{bmatrix} 0 & 0 & 0 & x_{D4}(t) & t_{SAT} x_{D4}(t) & x_{D6}(t) & x_{D7}(t) & 0 \end{bmatrix}^{T} \] (66)
\[ X_{N}(t) = \begin{bmatrix} 0 & 0 & \tau_{T} (1 - e^{-\tau_{T} / \tau_{T}}) / V_{T} & 0 & 0 & \tau_{A} (1 - e^{-\tau_{A} / \tau_{A}}) / V_{A} & 1 \end{bmatrix}^{T} \] (67)

where
\[ f(\zeta) = e^{-\zeta} + \zeta - 1 ; \quad x_{D4}(t) = \begin{cases} 1 - \frac{1}{2} t_{SAT} / \tau_{D} \end{cases} e^{-\tau_{D} / \tau_{D}} ; \quad x_{D6}(t) = \frac{1}{6} \cos(y_{D0} - \lambda_{AD}) (t_{SAT} / \tau_{D} - 3) \tau_{D}^{2} e^{-\tau_{D} / \tau_{D}} ; \]
\[ x_{D7}(t) = \begin{cases} \cos(y_{a0} + \lambda_{AD}) \left( \frac{1}{2} - \frac{t_{SAT}}{6 \tau_{A}} \right) \tau_{D}^{2} e^{-\tau_{D} / \tau_{D}} , \quad \tau_{A} = \tau_{D} \end{cases} \]
\[ \alpha_{A}(t) = -\tau_{A} f(t_{SAT} / \tau_{A}) \] (68)
\[ \beta_{A}(t) = x_{D7}(t) / \left[ \tau_{A} \cos(y_{a0} + \lambda_{AT}) \right] \] (69)
\[ \gamma_{A}(t) = \left( 1 - e^{-\tau_{SAT} / \tau_{A}} \right) / \left[ \left( V_{A} \cos(y_{a0} + \lambda_{AT}) \right) \right] \] (70)

According to equations (56)-(57)
\[ Z_{AT}(t) = y_{AT} + \dot{y}_{AT} t_{SAT} + a_{z} \cos(y_{A_{z}} - \lambda_{A_{z}}) \tau_{A} f(t_{SAT} / \tau_{A}) - a_{z} \cos(y_{A_{z}} + \lambda_{A_{z}}) \tau_{A} f(t_{SAT} / \tau_{A}) + u_{t} \left[ \frac{1}{2} t_{SAT} - \tau_{A} f(t_{SAT} / \tau_{A}) \right] \] 
\[ Z_{AD}(t) = y_{AD} x_{DA}(t) + \dot{y}_{AD} x_{DA}(t) + a_{x} x_{DA}(t) + a_{x} x_{DD}(t) \] 
\[ Z_{A}(t) = x_{A} + a_{r} r_{t} (1 - e^{-r_{t}/\tau_{t}}) / V_{A} + a_{r} r_{t} (1 - e^{-r_{t}/\tau_{t}}) / V_{A} + u_{t} r_{t} f(t_{SAT} / \tau_{A}) \left[ V_{A} \cos(y_{A_{z}} - \lambda_{A_{z}}) \right] \]

4. Simulation Results

In this section, the performance of the guidance law proposed in the previous section is investigated via numerical simulations. The simulations are based on perfect information and linear mode of the engagement, so the simulation results would be close to the results of theoretical analysis.

The engagement is initiated in an head-on setting, that is to say, initially the adversaries move along the initial LOS approaching each other. The initial position of the attacker is assumed to be at \((x_{A_{i}}, y_{A_{i}}) = (0m, 0m)\). Because the defender is launched from the target platform, the initial positions of the target and the defender are the same, which is assumed to be at \((x_{T_{i}}, y_{T_{i}}) = (x_{D_{i}}, y_{D_{i}}) = (10800m, 0m)\).

The target speed is 300m/s and the speed of both the attacker and the defender is 600m/s. The first-order time constant of the attacker, target and defender is 0.1s, 1s and 0.8s, respectively. In addition, the navigation constant of the defender is chosen to be \(N_{D} = 3\), and the target maintain a constant maneuver \(u_{t} = 3g\).

Figure 2 presents the trajectories for a fixed evasion miss distance and different requirements on the intercept angle. We can observe that the attacker’s trajectories are shaped to evade the defender first and then intercept the target with terminal angle constraints.

![Figure 2. Adversaries trajectories for various intercept angles; \(L_{AD} = 100m\).](image-url)

In order to understand this behaviour of the attacker, we can take a look at Figure 3 which plot the guidance command profiles of the attacker throughout the engagement. It can be seen clearly that the guidance command changed significantly after the attacker-defender engagement. In the first phase, the primary objective of the attacker is to evade the defender with specified miss distance, and this requires the attacker to perform large maneuvers and jerks. In the second phase, the engagement becomes the standard one-on-one engagement, the maneuver demand on the attacker to intercept the target with angle constraint vary linearly but is smaller and smoother than that in the first phase.
It can be observed that the guidance command in the terminal phase of the attacker-defender engagement depends strongly on the variation of $L_{AD}$ and not very much on the required intercept angle. While, in the attacker-target engagement, the guidance command is mainly dependent on the required intercept angle.

The zero-effort quantities for different requirements on the evasion miss distance or the intercept angle are shown in Figure 4, we can see that evasion from the defender with specified miss distance and perfect interception of the target with terminal angle constraint are indeed achieved by this guidance law.
(a) $Z_{AT}$ for various specified miss distances; $x_N = 60^\circ$.

(b) $Z_{AD}$ for various specified miss distances; $x_N = 60^\circ$. 
5. Conclusions

In this paper, an optimal guidance law for the attacking missile is proposed in the ATD scenario. The terminal intercept angle constraint of the attacker with the target is also considered. The first two terms of the optimal control strategy are linear with respect to the zero-effort quantities, while the third term is the nonlinear function of the zero-effort quantities. Using the guidance law two objectives can be achieved simultaneously, which are evasion from the defender with specified miss distance; perfect interception of the target with perfect terminal intercept angle. To achieve the first objective, higher acceleration will be required, the peak accelerations increased as the required evasion miss distance increased. For the second objective, the required acceleration is smoother and its value is mainly dependent on the required intercept angle. Numerical simulations show that the proposed guidance law is consistent with the theory.

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