Modeling of spatial natural oscillations of axisymmetric systems

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Abstract. A mathematical model, methods and algorithm for estimating the natural vibrations of inhomogeneous axisymmetric systems in a three-dimensional statement are presented in the paper. The problem is solved by the semi-analytical finite element method (FEM). The reliability of the developed methods and algorithms was validated by comparing the results obtained with the exact solution of a number of test problems for three-dimensional bodies, as well as by comparing the results with the results of field experiments. Spatial natural vibrations of inhomogeneous spatial axisymmetric systems, i.e. the models of nuclear power plants (NPPs) shielding shell in a three-dimensional statement were investigated. The results were analyzed to detect mechanical effects and revealed that the presence of multiple (close) eigenfrequencies in real spatial axisymmetric systems is not an exception, but a rule.

Keywords: spatial, axisymmetric, inhomogeneous systems, three-dimensional bodies, shielding shell, multiple eigenfrequencies

1. Introduction
To ensure reliable operation of containment shells of nuclear power plants under various loads, it is necessary first to evaluate their eigenfrequencies, modes and decrement of vibrations. The solution to the problem of natural vibrations is an independent and rather difficult task.

A shell model is often used as a calculation model for containment shells; it does not take into account the complexity of the structure and mainly considers the elastic properties of the structure material.

For instance:
- in reference [1], the frequency equation was investigated. The frequency and modes of vibrations of a shell were analyzed. The dependence of axisymmetric natural vibrations frequency of the system on wave formation in the longitudinal direction was plotted.
- computational models of a containment shell and improved calculation algorithms were proposed in reference [2], the reasons for the appearance and growth of tensile stresses in the walls of the containment shell were established.
- the study in reference [3] provided an assessment of the stress-strain state of a containment shell of the Rostov NPP.
- in reference [4], a general solution of the equation of natural vibrations of the elastic sphere in a three-dimensional formulation in a spherical coordinate system was given.
- the methods for determining dynamic characteristics of the Kalinin NPP reactor section were tested in reference [5]. It was determined that the prevailing frequencies in the vibration spectra correspond to the first mode of vibrations.
- the analysis of reference [6] gave comparative results of dynamic characteristics of containment shells obtained by model investigations and in field tests.
- in [7], [8], the eigenfrequencies of a cylindrical shell were studied under various boundary conditions considering the thickness variation of shells.

In references [9]–[26], information is given related to the assessment of eigenfrequencies, modes of vibration, and stress-strain state (SSS) of various structures with an account for design features and operating conditions.

The above review of well-known publications shows, that the development of effective methods and the study of eigenfrequencies and modes of vibrations of spatial axisymmetric systems in three-dimensional formulations is an urgent task.

2. Methods
2.1. Models, methods and algorithms for estimating eigenfrequencies and vibration modes of inhomogeneous spatial axisymmetric systems

A containment shell is considered (Figure 1); it consists of dome-shaped structure - 1, foundation - 2 and soil base - 3, which occupy volumes $V_1, V_2$ and $V_3$, respectively. The bottom part ($z = 0$) of the soil base is a rigid bed. It is necessary to determine the spatial eigenfrequencies and vibration modes of the system in question of the volume $V = V_1 + V_2 + V_3$. As is known, the eigenfrequencies and vibration modes are determined when solving the problem of natural vibrations of the system under consideration, which occur in the absence of external influences.

![Figure 1. Inhomogeneous spatial axisymmetric system](image-url)

To formulate the problem of natural vibrations of a spatial system (Figure 1), the principle of virtual displacements is used, according to which the sum of the work of all active forces acting on the system, including the inertial forces on virtual displacements, is zero i.e.:

$$
\delta A = - \int_{V_1} \sigma_{ij} \delta e_{ij} dV - \int_{V_2} \sigma_{ij} \delta e_{ij} dV - \int_{V_3} \sigma_{ij} \delta e_{ij} dV - \int_{V_1} \rho_1 \ddot{u} \delta \ddot{u} dV - \int_{V_2} \rho_2 \ddot{u} \delta \ddot{u} dV - \int_{V_3} \rho_3 \ddot{u} \delta \ddot{u} dV = 0
$$

$$i,j = r, z, \varphi$$
Along with (1), to model the strain process, it is necessary to consider:
- kinematic boundary conditions
  \[ \ddot{\mathbf{x}} \in \sum \mathbf{u} : \ddot{\mathbf{u}} = 0, \]  
  (2)
- the Cauchy relation \[ 27 \] in spherical coordinates
  \[ \varepsilon_{rr} = \frac{\partial u_r}{\partial r}, \quad \varepsilon_{\varrho\varphi} = \frac{1}{r} \frac{\partial u_\varphi}{\partial \varphi} + \frac{u_r}{r}, \quad \varepsilon_{zz} = \frac{\partial u_z}{\partial z}, \]
  \[ \varepsilon_{r\varphi} = \frac{1}{2} \left( \frac{1}{r} \frac{\partial u_r}{\partial \varphi} + \frac{\partial u_\varphi}{\partial r} - \frac{u_\varphi}{r} \right), \]
  \[ \varepsilon_{rz} = \frac{1}{2} \left( \frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z} \right), \quad \varepsilon_{\varphi r} = \frac{1}{2} \left( \frac{\partial u_\varphi}{\partial z} + \frac{\partial u_z}{\partial \varphi} \right), \]  
  (3)
- the relation connecting the stress tensors with strain tensors \[ 27 \]:
  \[ \sigma_{ij} = \lambda_n \varepsilon_{kk} \delta_{ij} + 2 \mu_n \varepsilon_{ij} \]
  \[ i, j, k = r, z, \varphi \]  
  (4)
where \( \ddot{\mathbf{u}}, \sigma_i, \varepsilon_{ij} \) are the components of the displacement vector, stress and strain tensors, respectively;
\( \rho_n \) is the density of the material of the system; \( \lambda_n \) and \( \mu_n \) are the Lame constants; \( \delta \mathbf{u}, \delta \varepsilon_{ij} \) are the isochronous variations of displacements and strains; index \( n = 1, 2, 3 \) means the individual parts of the system (i.e. structure, foundation, base) to which this characteristic relates; \( \delta \) is the Kronecker symbol; \( \ddot{x} = \{r, z, \varphi\} \) are the cylindrical coordinates; \( \mathbf{\ddot{u}} = \{u_r, u_z, u_\varphi\} \) are the components of displacement vector.

A nontrivial solution to the variational equation (1), taking into account (2) - (4), is sought in the following form
\[ \ddot{\mathbf{u}}(\ddot{x}, t) = \dddot{\mathbf{u}}^* (\dddot{x}) e^{-i \omega t}, \quad \dddot{\mathbf{u}}^* = \{\dddot{u}_r, \dddot{u}_z, \dddot{u}_\varphi\} \]
(5)
Substitution of (5) into (1) - (4) reduces the problem under consideration to a real variational eigenvalue problem:
\[ - \int_V \sigma_{ij}^* \delta \varepsilon_{ij} dV + \frac{\omega^2}{V} \int_V \rho_n \dddot{u}^* \delta \dddot{u}^* dV = 0, \]
\[ \dddot{x} \in \sum \mathbf{u} : \delta \mathbf{\dddot{u}}^* = 0, \]
(6)
(7)
where \( \sigma_{ij}^* \) is the amplitude of the components of stress tensors.

The natural mode of vibration \( \dddot{u}^* \) is determined from (6) up to a constant multiplier. To eliminate this arbitrariness, an additional condition is introduced to normalize natural modes:
\[ \int_V \rho_n \dddot{u}^* dV = 1. \]
(8)

Now the problem of spatial (three-dimensional) natural vibrations of elastic system is reduced to determining constant \( \omega^2 \) and function \( \dddot{u}^* (\dddot{x}) \) satisfying equations (6), (3), (4), normalizing conditions (8), and kinematic conditions (7) for any value of \( \delta \dddot{u}^* (\dddot{x}) \).
The spatial problem for the system under consideration (Figure 1) is sought [28] by the semi-analytical finite element method, therefore, the solutions are presented in the form of separate harmonics.

\[
\begin{align*}
    u^*_r &= u^0_r(r,z) \cos m\varphi \\
    u^*_{\varphi} &= u^0_{\varphi}(r,z) \sin m\varphi \\
    u^*_z &= u^0_z(r,z) \cos m\varphi 
\end{align*}
\]  

(9)

where \( u^0_r, u^0_{\varphi}, u^0_z \) are the sought for functions, \( m \) is the number of harmonics (\( m = 0,1,2, ... \)).

At \( m = 0 \), the problem splits into two independent ones, the axisymmetric vibration problem:
\[
\begin{align*}
    u^*_r &= u^0_r(r,z), \\
    u^*_{\varphi} &= u^0_{\varphi}(r,z), \\
    u^*_z &= 0
\end{align*}
\]

and the torsional vibration problem:
\[
\begin{align*}
    u^*_r &= 0, \\
    u^*_{\varphi} &= u^0_{\varphi}(r,z).
\end{align*}
\]

At \( m = 1 \), there arises the problem of bending vibrations. In the framework of the one-dimensional theory of the strength of materials, the analogs of the three problems presented are the problems of longitudinal, torsional and transverse vibrations of a beam of circular section.

At \( m \geq 2 \), the three-dimensional problem has no one-dimensional analogs.

Next, for the solution of the problem under consideration, the FEM procedure is used and for the finite element discretization of the region \( V = V_1 + V_2 + V_3 \) occupied by the system (Figure 1), an annular element of a triangular section with a linear approximation of the mixing field \( \tilde{u}^0(r,z) \) inside the element is used.

As a result of the FEM procedure, a variational problem (6) - (7) is reduced to a finite-dimensional problem - to a real algebraic eigenvalue problem:

\[
\begin{pmatrix}
    [K] - \omega^2 [M]\end{pmatrix} \{\vec{u}\} = 0,
\]  

(10)

where \([K], [M]\) are the matrices of stiffness and mass of the system under consideration (Figure 1); \(\omega_\{\vec{u}\}\) are the eigenfrequency and eigenvector of the system.

A nontrivial solution of the algebraic problem (10) for eigenvalues, i.e. the roots \( \lambda = \omega^2 \) of the characteristic determinant (10) are defined by the Muller method [29], and the Gauss method or square root method [30] are used to determine natural modes of vibration \(\{\vec{u}\}\).

The order of system (10) depends on the number of finite elements into which the domain is divided. In the present study, the maximum order of system (10) reached 3000.

Specific calculations were performed on the IBM PC. An author’s certificate of the Intellectual Property Agency under the Ministry of Justice of the Republic of Uzbekistan was obtained for the developed computer program.

3. Results and Discussion

3.1. Accuracy study of the algorithm and program in solving test problems

In this section the solution accuracy of various three-dimensional problems of natural vibrations were examined using the developed algorithms and the calculation program on the IBM PC. The results obtained were compared with the known exact solutions and the results of field experiments.

Problem 1

Consider axisymmetric, torsional, and bending vibrations of a hollow elastic cylinder of finite length \( l \) with external \( r_2 \) and internal \( r_1 \) radii in a three-dimensional statement with a fixed lower part and a free upper part, i.e.:

\[
z=0; \quad \vec{u} = 0
\]

The solutions obtained in a three-dimensional statement are compared with the known exact solutions of one-dimensional problems:

- at \( m = 0 \) in (9), the three-dimensional problem has two one-dimensional equivalents, i.e. the problems of longitudinal and torsional vibrations of a beam of circular section;
- at \( m = 1 \) in (9), the three-dimensional problem has a one-dimensional equivalent, i.e. the problem of transverse natural vibrations of a beam of circular section.

The results obtained using the developed algorithm for solving three-dimensional problems were compared with the exact solution of one-dimensional problems for various ratios of length \( l \) to the outer radius \( r_2 \) of the cylinder. Calculation results are shown in Table 1. In calculations the cylinder material was considered hypothetically, i.e.: \( E = 1.0 \), \( \nu = 0.27 \), \( \rho = 1.0 \); with the radii of the cylinder \( r_2 = 15.0 \), \( r_1 = 10.0 \).

### Table 1. Comparison of the obtained solutions with the exact solution

| № of eigen frequency | Natural vibrations |
|----------------------|-------------------|
|                       | Axisymmetric at \( l/r_2 = 20 \) | Bending at \( l/r_2 = 100 \) | Torsional at \( l/r_2 = 20 \) |
|                       | One-dimensional solution | Three-dimensional solution | One-dimensional solution | Three-dimensional solution | One-dimensional solution | Three-dimensional solution |
| \( \omega_1 \)       | 0.005239            | 0.005246              | 0.000014            | 0.000014               | 0.003285            | 0.003290              |
| \( \omega_2 \)       | 0.0155708           | 0.015721              | 0.000088            | 0.000088               | 0.009856            | 0.009918              |
| \( \omega_3 \)       | 0.026182            | 0.026133              | 0.000247            | 0.000247               | 0.016426            | 0.016532              |
| \( \omega_4 \)       | 0.036660            | 0.036416              | 0.004843            | 0.004832               | 0.022997            | 0.023149              |
| \( \omega_5 \)       | 0.047141            | 0.046892              | 0.000800            | 0.000800               | 0.029568            | 0.029769              |

An analysis of the comparison shows good agreement between the numerical results of three-dimensional problems for a hollow cylinder and the exact solution of one-dimensional problems at certain ratios of the length of the cylinder \( l \) to its radius \( r_2 \).

### Problem 2

The problem of bending natural vibrations of a reinforced concrete ventilation pipe of the Armenian NPP, 150 m high is considered. The problem is solved as a three-dimensional problem using the developed algorithms and a calculation program for the IBM and the obtained solutions are compared with the results of field experiments [31]. The solutions are presented and compared in Table 2.

### Table 2. Comparison of the solutions obtained with the results of field experiments

| № of eigen frequency | Periods (sec) | Solution Methods |
|----------------------|---------------|------------------|
|                       | Field experiments | Three-dimensional solution |
| \( T_1 \)          | 1.63           | 1.598            |
| \( T_2 \)          | 0.51           | 0.487            |
| \( T_3 \)          | 0.20           | 0.211            |
| \( T_4 \)          | -              | 0.131            |
| \( T_5 \)          | -              | 0.089            |

With the results of test problems, it can be stated that the assessment accuracy, the numerical solutions of the problems obtained, their comparison with the exact solutions and results of field experiments show the reliability and validity of the developed algorithm and software for the IBM PC when solving three-dimensional problems of natural vibrations in axisymmetric spatial bodies.

### 3.2. Study of spatial natural vibrations of inhomogeneous three-dimensional systems

In this section, we study the spatial natural vibrations (frequencies and modes) of the model of a containment shell of the nuclear power plant considering its foundation and base (Figure 1) in an elastic three-dimensional statement using the developed methods and the IBM PC software.
In calculations, the following dimensionless physicomechanical characteristics of the containment shell material were taken. The shell material is supposed to be hypothetical, i.e.:

\[ \rho \sqrt{E} = 1 \; ; \; \nu = 0.27 \]

In Figure 2 geometric dimensions of the shell in a dimensionless form are:

\[ H/R_2 = 2.346; \; R_1/R_2 = 0.949; \; r_1/R_2 = 0.54; \; r_2/R_2 = 1.468; \; a/R_2 = 1.662; \; b/R_2 = 0.684; \; c/R_2 = 0.827; \; \text{tet}/R_2 = 0.0506. \]

The spatial natural vibrations of the system were considered (Figure 1), the elastic modulus \( E \) of the foundation and base was taken two orders of magnitude greater than the elastic modulus of the containment shells of the system.

The aim of this calculation is to identify mechanical effects in the process of shell strain when studying close and multiple eigenfrequencies in a structure at various harmonics \( m \) in (9).

Table 3 shows the five first eigenfrequencies of the considered system (Figure 1) for each harmonics \( m = 0,1,2,3, \ldots \).

| Number of harmonics | \( \omega_1 \) | \( \omega_2 \) | \( \omega_3 \) | \( \omega_4 \) | \( \omega_5 \) |
|---------------------|---------------|---------------|---------------|---------------|---------------|
| \( m = 0 \) (axisymmetric) | 0.07961 | 0.12151 | 0.16167 | 0.17672 | 0.23268 |
| \( m = 0 \) (torsional) | 0.06643 | 0.19508 | 0.30926 | 0.41155 | 0.52257 |
| \( m = 1 \) | 0.03437 | 0.10082 | 0.12174 | 0.15619 | 0.19758 |
| \( m = 2 \) | 0.06757 | 0.14434 | 0.15944 | 0.20798 | 0.21199 |
| \( m = 3 \) | 0.06701 | 0.15446 | 0.19411 | 0.23934 | 0.29696 |
| \( m = 4 \) | 0.07708 | 0.16613 | 0.22801 | 0.26138 | 0.34455 |
| \( m = 5 \) | 0.09532 | 0.18388 | 0.25691 | 0.29579 | 0.38139 |
An analysis of structure eigenfrequencies at various values of ratio $H / R_2$ ($H$ is the height of the structure, $R_2$ is its outer radius) within $H / R_2 = 1.5-2.5$ showed that the frequencies and modes are transposed. The greater the ratio $H / R_2$ (within the specified limits), the greater the frequency of torsional vibration in relation to the first - the lowest - frequency of bending vibration. In the case when the ratio $H / R_2$ is closer to the minimum value (within specified limits), the torsional vibration frequency corresponds to the third or fourth eigenfrequency of the structure.

Figure 3 shows the characteristic dependences of the first three eigenfrequencies (of bending, axisymmetric and torsional vibrations) on the relative height of the structure - $H / R_2$. All three frequencies increase with an increase in relative height. The most significant seems to be the difference in the increased rates for different frequencies, as a result of which there appear the points of curves intersection, as at point “A”. At these points, the vibration frequencies become multiple. A similar, more complicated situation arises at higher eigenfrequencies; it may turn out that three curves intersect in the vicinity of one specific parameter value, which relate to frequencies $\omega_i$ corresponding to different harmonics numbers - $m$.

Next, the eigenfrequencies were analyzed in order to find multiple or close frequencies of the system (Figure 1). This analysis was performed for various ratios of the geometric parameters of the system.

The interest in finding close and multiple eigenfrequencies is due to the fact that further studies of the dynamic behavior of spatial axisymmetric systems led to a number of new mechanical effects.

As seen from Table 3, the second axisymmetric frequency is close (multiple) to the third frequency at $m = 1$: $(\omega_2^{m=0} = 0.121517 \approx \omega_3^{m=1} = 0.121740)$.

The value of the first eigenfrequency of torsional vibrations is close to the first eigenfrequencies at $m = 2$ and $m = 3$, which are multiple to each other $(\omega_1^{m=0} = 0.066438 \approx \omega_2^{m=2} = 0.067578 \approx \omega_3^{m=3} = 0.067014)$. The lower eigenfrequencies are also close $(\omega_1^{m=1} = 0.156196 \approx \omega_2^{m=2} = 0.159440 \approx \omega_3^{m=3} = 0.159460)$ as well as others.

For the obtained eigenfrequencies, the corresponding spatial eigenmodes of vibration are constructed in axonometric projection. As an example, Figure 4 shows the obtained fifth eigenmodes of vibration of the containment shells considered at various harmonics $m = 1,2,3,4,5$. 

\begin{center}
\begin{tabular}{ccc}
$\omega_5^{m=1} = 0.19758$ & $\omega_5^{m=2} = 0.21199$ & $\omega_5^{m=3} = 0.29696$
\end{tabular}
\end{center}
Figure 4. Non-axisymmetric natural modes of vibration of containment shells

4. Conclusion
1. The mathematical model, methods and algorithm to assess spatial natural vibrations and the modes of vibrations of containment shells are proposed in a three-dimensional formulation.
2. The adequacy of the mathematical model and the reliability of the methods and algorithms was validated by comparing the results obtained with known exact solutions of test problems.
3. The spatial natural vibrations of the NPP containment shells were investigated in a three-dimensional statement.
4. The presence of close and multiple eigenfrequencies in spatial axisymmetric systems under consideration was identified at certain geometric parameters of the structure.

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