A two-level stochastic collocation method for semilinear elliptic equations with random coefficients

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Abstract In this work, we propose a novel two-level discretization for solving semilinear elliptic equations with random coefficients. Motivated by the two-grid method for deterministic partial differential equations (PDEs) introduced by Xu [33], our two-level stochastic collocation method utilizes a two-grid finite element discretization in the physical space and a two-level collocation method in the random domain. In particular, we solve semilinear equations on a coarse mesh $T_H$ with a low level stochastic collocation (corresponding to the polynomial space $P_P$) and solve linearized equations on a fine mesh $T_h$ using high level stochastic collocation (corresponding to the polynomial space $P_p$). We prove that the approximated solution obtained from this method achieves the same order of accuracy as that from solving the original semilinear problem directly by stochastic collocation method with $T_h$ and $P_p$. The two-level method is computationally more efficient, especially for nonlinear problems with high random dimensions. Numerical experiments are also provided to verify the theoretical results.

Keywords Semilinear problems · random coefficients · two-grid · stochastic collocation

1 Introduction

Stochastic partial differential equations (SPDEs), especially nonlinear SPDEs, provide mathematical models for the quantification of uncertainties in many complex physical and engineering applica-
tions. Some examples include the propagation of uncertainties associated with input parameters (such as the coefficients, forcing terms, boundary conditions, geometry of the domain etc.) to certain output quantities of interests, see e.g., flow in heterogeneous porous media [22], thermo-fluid processes [16, 20], flow-structure interactions [32]. More applications of the nonlinear SPDEs in physics and mechanics can be found in [1, 5].

Numerical methods dealing with SPDEs can be roughly categorized as either intrusive or non-intrusive types. The stochastic Galerkin (SG) method based on the polynomial chaos expansion [4, 23, 24, 30, 31, 32, 37] is considered as an intrusive method since it results in coupled systems which cannot be solved directly by the corresponding deterministic solvers. The SG method applies Galerkin projection to discretize the stochastic space and uses standard finite element discretization in the physical space. It is often advantageous over non-intrusive approaches in terms of the computational efficiency when efficient solvers are available. It provides an exponentially convergent approximation when the solution of stochastic problem is smooth with respect to the random variables. However, it is reported that for some nonlinear problems, the stochastic Galerkin method may not be as efficient as non-intrusive stochastic methods [36]. The Monte Carlo (MC) method [6, 23, 27, 36] is the most widely used non-intrusive method based on the sampling techniques. MC method is attractive because its convergence is independent of the stochastic dimension. On the other hand, its rate of convergence is rather slow, proportional to $1/\sqrt{N}$ with $N$ being the number of samples. Another non-intrusive type method, stochastic collocation (SC) method [3, 15, 21, 25, 29, 36], has recently gained popularity. The stochastic collocation method shares the same exponential convergence property as the stochastic Galerkin method. Moreover, stochastic collocation method only requires solving the deterministic problem on a set of collocation points, hence existing efficient and robust solvers for these problems are applicable. Both the stochastic Galerkin and stochastic collocation method suffer from the curse of dimensionality.

The purpose of this study is to improve the efficiency of the stochastic collocation method for solving semilinear SPDEs. Our motivation comes from the two-grid finite element discretization proposed by Xu [33, 34] for the nonsymmetric, indefinite and nonlinear elliptic problems. The main idea of two-grid method is based on the observation that a very coarse grid space is sufficient for some nonsymmetric, indefinite and/or nonlinear problems that are dominated by their symmetric, positive and/or linear parts. Later, the method has been applied to solve semilinear elliptic eigenvalue problems [10, 35], nonlinear parabolic differential equations [8, 9, 11, 12], Navier-Stokes equations [2, 14, 17, 19, 28], magnetohydrodynamics system [18], etc.

In order to generalize the two-grid technique for solving semilinear SPDEs, we shall utilize two meshes in the physical domain and two levels of collocation points in the random domain. Furthermore, to minimize the computational cost, we use fine mesh for spatial discretization when approximating the stochastic variables with high order polynomial space, and use coarse mesh in spatial space with low order polynomial space in stochastic space. More precisely, our method consists of two steps, i.e., we first solve a nonlinear problems using the coarse mesh and low level stochastic collocation, then solve a corresponding linearized problems on the fine mesh with high level stochastic collocation. The resulting two-level discretization method is computationally more efficient. Moreover, we prove that the solution obtained from two-level approach has the same order of accuracy as that from solving nonlinear problems directly using fine mesh and high level collocation points. We verify the theoretical results by several numerical examples.

The rest of the paper is organized as follows. In Section 2, we introduce the model problem and some notations. The two-level method is described in detail in Section 3. In Section 4, we estimate the error of the approximated solution. Finally, in Section 5, numerical experiments are given to verify the theoretical results.
2 Model problem and weak formulation

In this work, we investigate the following semilinear elliptic problem with random coefficient

\[
\begin{align*}
-\nabla \cdot (a(\omega, x)\nabla u(\omega, x)) + f(\omega, x, u(\omega, x)) &= 0, \quad x \in D, \\
u(\omega, x) &= 0, \quad x \in \partial D,
\end{align*}
\]

(2.1)

where \( D \subset \mathbb{R}^d \) is a bounded domain, \( \partial D \), the boundary of \( D \), is either smooth or convex and piecewise smooth, the diffusion coefficient \( a \) is a real-valued random field defined on \( D \), i.e., for each \( x \in D \), \( a(\cdot, x) : \Omega \rightarrow \mathbb{R} \) is a random variable with respect to a suitable probability space \((\Omega, \mathcal{F}, \mathbb{P})\). Here \( \Omega \) is the set of elementary events, \( \mathcal{F} \) is the \( \sigma \)-algebra and \( \mathbb{P} : \Omega \rightarrow [0, 1] \) is a probability measure. We assume that \( a \) is bounded and uniformly coercive, i.e., there exist \( a_{\min}, a_{\max} \in (0, \infty) \), such that

\[
\mathbb{P}(\omega \in \Omega : a(\omega, x) \in [a_{\min}, a_{\max}], \forall x \in \bar{D}) = 1.
\]

(2.2)

Here, we also assume \( f(\omega, x, u(\omega, x)) \) is sufficiently smooth. For brevity, we shall drop the dependence of variables \( \omega, x \) in \( f(\omega, x, u) \) in the following exposition. We also note that here and later in this paper the gradient operator, \( \nabla \), always represents differentiation with respect to \( x \) only.

We introduce some notations which will be used later. Let \( \langle \cdot, \cdot \rangle \) be the inner product of \( L^2(D) \), \( \mathcal{W}_p(D) \) denotes the standard Sobolev space with norm \( \| \cdot \|_p \) given by

\[
\|u\|_p = \left( \int_D \| \nabla^m u \|^p \, dx \right)^{\frac{1}{p}},
\]

when \( p = 2 \), we denote \( \mathcal{H}^m(D) = \mathcal{W}_m^2(D) \). Let \( \mathcal{H}_D(D) \) be the subspace of \( \mathcal{H}^m(D) \) consisting of all the functions with vanishing trace on \( \partial D \). \( \| \cdot \|_m = \| \cdot \|_{m,2} \) and \( \| \cdot \| = \| \cdot \|_{0,2} \). The following Sobolev inequality is well known and is important for the theoretical analysis in Section 4

\[
\|u\|_{0,p} \lesssim \|u\|_1 \quad (d = 2 \text{ and } 1 \leq p < \infty).
\]

(2.3)

where the notation “\( \lesssim \)” is equivalent to “\( \leq C \)” for some positive constant \( C \). We assume that problem (2.1) has at least one solution \( u(\omega, x) \in \mathcal{H}_D(D) \cap H^2(D) \) for each parameter \( \omega \in \Omega \).

To introduce the stochastic discretization, we first approximate the input random field \( a(\omega, x) \) by a truncated Karhunen-Lo`eve (KL) expansion

\[
a(\omega, x) \approx a_N(\omega, x) = a_N(Y_1(\omega), Y_2(\omega), \ldots, Y_N(\omega), x) \\
= \bar{a}(x) + \sum_{n=1}^N \sqrt{\lambda_n}b_n(x)Y_n(\omega),
\]

where \( \bar{a}(x) \) is the mean value of \( a(\omega, x), \{Y_1, Y_2, \ldots, Y_N\} \) are uncorrelated and identically distributed random variables with zero mean and unit variance. For simplicity, we assume that \( \{Y_1, Y_2, \ldots, Y_N\} \) are independent and \( \{\rho_n\}_{n=1}^N \) are the probability density functions of the random variables \( \{Y_n\}_{n=1}^N \). \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_i \geq \ldots \geq 0 \) and \( \{b_n(x)\}_{n=1}^N \subset L^2(D) \) are the eigenvalues and eigenfunctions of the symmetric positive semidefinite Fredholm operator \( C_0 : L^2(D) \rightarrow L^2(D) \) defined by

\[
(C_0g)(x) = \int_D Cov_a(x,x')g(x')dx',
\]

with \( Cov_a \) being a given continuous covariance function. The truncated KL expansion is optimal in the sense that it obtains the smallest mean square error among all approximations of \( a \) in \( N \) uncorrelated random variables [13].

Let \( \Gamma_n = Y_n(\Omega) \) be the image of \( Y_n \), \( \Gamma = \bigcap_{n=1}^N \Gamma_n \). The random variables \( \{Y_1, Y_2, \ldots, Y_N\} \) have a joint probability density function \( \rho = \bigcap_{n=1}^N \rho_n \). By Doob-Dynkin’s Lemma [26], the solution \( u(\omega, x) \) can be represented by \( u(Y_1(\omega), Y_2(\omega), \ldots, Y_N(\omega), x) \). Let \((\Gamma, B^N, \rho_{dy})\) be a probability space with \( B^N \)
being the \( \sigma \)-algebra associated with the set of outcomes \( \Gamma \). The expectation of a random variable \( \mu(y) \in (\Gamma, \mathcal{B}^\Gamma, \mu(dy)) \) is \( E(\mu(y)) = \int_\Gamma \mu(y)\rho(y)dy \) and variance is \( \text{Var}(\mu(y)) = \int_\Gamma \mu^2(y)\rho(y)dy - [\int_\Gamma \mu(y)\rho(y)dy]^2 \). Thus, after replacing the diffusion coefficient \( a \) by the truncated KL expansion \( a_N \), (2.1) can be written as the following parametrized problem with N-dimensional parameter

\[
\begin{align*}
-\nabla \cdot (a_N(y,x)\nabla u(y,x)) + f(u(y,x)) &= 0, \quad x \in D, \\
u(y,x) &= 0, \quad x \in \partial D.
\end{align*}
\]

For a.e. \( y \in \Gamma \), we assume that problem (2.4) has at least one solution in \( H^1_0(D) \cap H^2(D) \) and the linearized operator \( L_u := -\nabla \cdot (a_N \nabla) + f'(u) \) is nonsingular. As a result of this assumption, for a.e. \( y \in \Gamma \), \( L_u : H^2(D) \cap H^1_0(D) \rightarrow L^2(D) \) is a bijection and satisfies

\[
\|w\|_2 \leq C\|L_uw\|, \quad \forall w \in H^2(D) \cap H^1_0(D).
\]

We also denote by \( L^2_p(\Gamma) \) the Hilbert space with inner product defined as

\[
(p,q)_{L^2_p(\Gamma)} = \int_D p(y)q(y)\rho(y)dy, \quad \forall p,q \in L^2_p(\Gamma),
\]

and introduce the following tensor product spaces

\[
L^2_p(\Gamma) \otimes H^1_0(D) = \{u(y,x) : \Gamma \times D \rightarrow \mathbb{R} | \text{u(y,\cdot) } \in H^1_0(D), \text{a.e. on } \Gamma, \text{ and u(\cdot,x) } \in L^2_p(\Gamma), \text{a.e. on } D \},
\]

\[
L^2_p(\Gamma) \otimes L^2(D) = \{u(y,x) : \Gamma \times D \rightarrow \mathbb{R} | \text{u(y,\cdot) } \in L^2(D), \text{a.e. on } \Gamma, \text{ and u(\cdot,x) } \in L^2_p(\Gamma), \text{a.e. on } D \},
\]

endowed with the inner products defined by: \( \forall u,v \in L^2_p(\Gamma) \otimes H^1_0(D), \ p,q \in L^2_p(\Gamma) \otimes L^2(D) \)

\[
(u,v)_{L^2_p(\Gamma) \otimes H^1_0(D)} = E[\nabla u, \nabla v] = \int_\Gamma \left( \int_D \nabla u(y,x) \cdot \nabla v(y,x) dy \right) \rho(y)dy,
\]

\[
(p,q)_{L^2_p(\Gamma) \otimes L^2(D)} = E[p,q] = \int_\Gamma \left( \int_D p(y,x) \cdot q(y,x) dx \right) \rho(y)dy.
\]

We assume that problem (2.4) has at least one solution \( u \in L^2_p(\Gamma) \otimes (H^1_0(D) \cap H^2(D)) \).

We also need the following tensor product space in our analysis

\[
L^2_p(\Gamma) \otimes L^p(D) = \{u(y,x) : \Gamma \times D \rightarrow \mathbb{R} | \text{u(y,\cdot) } \in L^p(D), \text{a.e. on } \Gamma, \text{ and u(\cdot,x) } \in L^2_p(\Gamma), \text{a.e. on } D \},
\]

and the corresponding norm is defined by

\[
\|u\|_{L^2_p(\Gamma) \otimes L^p(D)}^2 = E(\|u\|^2_{L^p(D)}).
\]

Similarly, we can define \( L^2_p(\Gamma) \otimes W^{1,p}(D) \) and \( L^2_p(\Gamma) \otimes L^{\infty}(D) \).

For convenience, we denote \( V_p := L^2_p(\Gamma) \otimes H^1_0(D) \) and use notation \( u(y) \) in the following whenever we want to highlight the dependence on the parameter \( y \).

The weak formulation of problem (2.4) is given by

\[
\int_D a_N(y)\nabla u(y)\nabla wx + \int_D f(u(y))wx = 0, \quad \forall w \in H^1_0(D), \rho - \text{a.e. in } \Gamma.
\]
and the weak formulation for the linearized problem of (2.4) can be represented as: for some \( v \in L^2_p(\Gamma) \otimes W^{1,p}(D) \)

\[
\int_D a_N \nabla u \nabla v \, dx + \int_D f'(v)u \, v \, dx = \int_D (-f(v) + f(v)v) \, v \, dx, \quad \forall v \in H^1_0(D), \rho - \text{a.e. in } \Gamma. \tag{2.6}
\]

Following [3], we make an assumption that the coefficient \( a_N \) and \( f(u) \) admit a smooth extension on the \( \rho \)-zero measure sets. Then, equation (2.5), (2.6) can be extended a.e. in \( \Gamma \) with respect to the Lebesgue measure.

3 Two-level discretization for semilinear SPDEs

In this section, we first describe the stochastic collocation method following [3]. Then, we present the two-level stochastic collocation method for solving semilinear PDEs with random coefficients.

3.1 Stochastic collocation method

We first introduce the finite dimensional subspace \( V_{p,h} \subset V_p \) given by \( \mathcal{P}_p(\Gamma) \otimes X_h(D) \), where

- \( \mathcal{P}_p(\Gamma) = \bigoplus_{n=1}^{N_p} \mathcal{P}_{p_n}(\Gamma_n) \) is the span of the tensor product polynomials with degree at most \( p = (p_1, p_2, \cdots, p_N) \), and

\[
\mathcal{P}_{p_n}(\Gamma_n) = \text{span}\{y_n^m, m = 0, 1, \cdots, p_n\}, \quad n = 1, 2, \cdots, N.
\]

Therefore, the dimension of \( \mathcal{P}_p \) is \( N_p = N^{p_1} \). 

- \( X_h(D) = \text{span}\{\phi_1, \phi_2, \cdots, \phi_N\} \) is a finite element space of dimension \( N_h \), where \( \phi_1, \phi_2, \cdots, \phi_N \) are piecewise polynomials defined on a quasi-uniform triangulation \( \mathcal{T}_h \) with mesh size \( h \).

We first introduce a semi-discrete approximation \( u_h : \Gamma \to X_h(D) \), i.e., for a.e. \( y \in \Gamma \), \( \forall w \in X_h(D) \),

\[
\int_D a_N(y) \nabla u_h(y) \cdot \nabla w \, dx + \int_D f(u_h(y)) \, w \, dx = 0. \tag{3.1}
\]

Similarly, for a.e. \( y \in \Gamma \), we also introduce the semi-discrete approximate \( u^h \) of the linearized equation (2.6) satisfying

\[
\int_D a_N \nabla u^h \nabla w \, dx + \int_D f'(v)u^h \, w \, dx = \int_D (-f(v) + f(v)v) \, w \, dx, \quad \forall w \in X_h(D). \tag{3.2}
\]

Next, we collocate equation (3.1) on the roots of orthogonal polynomials with respect to the weight \( \rho \) and build the fully discrete solution \( u_{h,p} \in \mathcal{P}_p(\Gamma) \otimes X_h(D) \) by interpolating in \( y \) with the collocated solutions, i.e.

\[
u_{h,p}(y, x) = \sum_{k=1}^{N_p} u_h(\hat{y}_k, x)\psi_k(y),
\]

where \( u_h(\hat{y}_k, \cdot) \) is the solution of (3.1) at the collocation point \( \hat{y}_k = (y_{1,k_1}, y_{2,k_2}, \cdots, y_{N,N_k}) \) and \( \{\psi_k(y)\}_{k=1}^{N_p} \) are the Lagrange basis with respect to the collocation points \( \{\hat{y}_k\}_{k=1}^{N_p} \). Using the Lagrange interpolation operator \( \mathcal{I}_p : C^0(\Gamma; H^1_0(D)) \to \mathcal{P}_p(\Gamma) \otimes H^1_0(D) \), defined by

\[
(\mathcal{I}_p v)(y) = \sum_{k=1}^{N_p} v(\hat{y}_k)\psi_k(y), \quad \forall v \in C^0(\Gamma; H^1_0(D)),
\]

we have \( u_{h,p} = \mathcal{I}_p u_h \).
3.2 Two-level stochastic collocation method

In this subsection, we shall present a two-level discretization scheme for semilinear elliptic equations with random coefficients based on two tensor product spaces $\mathcal{P}_P(\Gamma) \otimes \mathcal{X}_H(D)$ and $\mathcal{P}_P(\Gamma) \otimes \mathcal{X}_h(D)$. The idea of the two-level method is to reduce a nonlinear SPDE problem into a linear SPDE problem by solving a nonlinear SPDE problem on a much small space. The method is described in detail as follows.

Two-level discretization

- **Step 1:** on the coarse mesh $\mathcal{T}_H$, we solve the semilinear equation on a small number of collocation points. More precisely, for $k = 1, 2, \ldots, N_P$, find $u_H(\hat{y}_k, \cdot)$ on the coarse mesh such that

\[
(a_N(\hat{y}_k, \cdot)\nabla u_H(\hat{y}_k, \cdot), \nabla w) + (f(u_H(\hat{y}_k, \cdot)), w) = 0, \quad \forall w \in \mathcal{X}_H(D), \quad (3.3)
\]

where $\{\hat{y}_k\}_{k=1}^{N_P}$ is the set of collocation points corresponding to polynomial space $\mathcal{P}_P(\Gamma)$. The approximated solution of $(2.5)$ in $\mathcal{P}_P(\Gamma) \otimes \mathcal{X}_H(D)$ is given by

\[
u_{H,P}(y, x) = (\mathcal{I}_P u_H)(y) = \sum_{k=1}^{N_P} u_H(\hat{y}_k, x)\psi_k^P(y),
\]

where $\{\psi_k^P\}_{k=1}^{N_P}$ are Lagrange basis functions of $\mathcal{P}_P(\Gamma)$.

- **Step 2:** We solve the following linearized problem on a larger set of collocation points. Namely, find $u^h(\hat{y}_k, \cdot)$ on the fine mesh $\mathcal{T}_h$ such that

\[
(a_N(\hat{y}_k, \cdot)\nabla u^h(\hat{y}_k, \cdot), \nabla w) + (f'(u_H(\hat{y}_k, \cdot))u^h(\hat{y}_k, \cdot), w)
= (-f(u_H(\hat{y}_k, \cdot)) + f'(u_H(\hat{y}_k, \cdot))u_H(\hat{y}_k, \cdot), w), \quad \forall w \in \mathcal{X}_h(D), \quad (3.4)
\]

where $\{\hat{y}_k\}_{k=1}^{N_p}$ is the set of collocation points corresponding to $\mathcal{P}_p(\Gamma)$. Finally, the two-level solution $u^{h,p}$ is given by

\[
u_{h,p} = (\mathcal{I}_P u^h)(y) = \sum_{k=1}^{N_P} u^h(\hat{y}_k, x)\psi_k^p(y),
\]

where $\{\psi_k^p\}_{k=1}^{N_P}$ are the Lagrange basis functions of $\mathcal{P}_p(\Gamma)$.

We use Newton’s method for the semilinear system $(3.3)$, i.e., for each collocation point, starting from an initial guess $u^0_H$, and for $l = 0, 1, \ldots$, we solve

\[
b_N \nabla u^{l+1}_H, \nabla w) + (f(u^l_H) + f'(u^l_H)(u^{l+1}_H - u^l_H), w) = 0, \quad \forall w \in \mathcal{X}_H(D). \quad (3.5)
\]

Let $\{\phi_j^H\}_{j=1}^{N_H}$ be the finite element basis functions on triangulation $\mathcal{T}_H$, $A, J_i$ be the matrices whose entries are given by

\[
A_{ij} = (a_N \nabla \phi_j^H, \nabla \phi_i^H), \quad (J_i)_{ij} = (f'(u^l_H)\phi_j^H, \phi_i^H),
\]
and $F_i$ be the right hand side vector with $(F_i)_i = (-f(u_H^t) + f'(u_H^t)u_H^t, \phi^H_i)$. Then, for each collocation point, Newton iteration (3.5) can be written as

$$U_H^{t+1} = U_H^t + (A + J_t)^{-1}((F_i - (A + J_t)U_H^t)),$$

(3.6)

where $U_H^{t+1} = \sum_{j=1}^{N_H} (U_H^{t+1})_j \phi^H_j$.

The advantage of the two-level discretization is that we only need to solve a small number of semilinear equations on the coarse mesh in addition to solving linearized problems on the fine mesh. In fact, we solve $N_P(= \Pi^{N}_{n=1}(p_n + 1))$ semilinear equations on the coarse mesh and $N_P(= \Pi^{N}_{n=1}(p_n + 1))$ linearized equations on the fine mesh. From the analysis given in Section 4, when choosing $P_n = p_n/2$ ($n = 1, 2, \ldots, N$) and $H = \frac{h^1}{4}$ in the two-level discretization, the resulting approximated solution has the same order of accuracy as that obtained from the standard stochastic collocation method on mesh $T_h$ and tensor-product polynomial space $P_p(G)$. For the standard stochastic collocation method, we need to solve $N_P$ semilinear equations. Roughly speaking, this corresponds to solving $kN_P$ linear equations on the fine mesh if we assume the number of Newton iteration is $k$ for each collocation point. Hence, the two-level method saves $(k-1)N_P$ linear solves on the fine mesh at the expense of $kN_P$ linear solves on the coarse mesh. When $h >> H$ and $N$ is big, the computational savings is enormous. For example, if the fine mesh size $h = 2^{-12}$ which gives dim $X_h \approx 1.7 \times 10^7$, choosing $H = \frac{h^1}{4}$ gives dim $X_H \approx 49$; if the number of random variables $N = 8$, the polynomial degree of each random dimension in $P_p$ is $p_n = 4$ (for $n = 1, 2, \ldots, N$) which gives $N_P \approx 3.9 \times 10^5$, choosing $P_n = p_n/2$ gives $N_P \approx 6.5 \times 10^3$, so the two-level method saves the solving of approximately $10^5$ linear systems of equations with dimension $10^7$.

4 Convergence analysis

We shall now derive some error estimates for the two-level discretization introduced in Section 3. We first give the following error estimate for the semi-discrete solution by finite element methods.

**Lemma 1** Let $u_h : \Gamma \to X_h(D)$ be the semi-discrete finite element solution satisfying (3.1).

Then,

$$\|u - u_h\|_{L^2(\Gamma) \otimes L^p(D)} + h\|u - u_h\|_{L^2(\Gamma) \otimes W^{1,p}(D)} \lesssim h^2 \|u\|_{L^2(\Gamma) \otimes W^{2,p}(D)},$$

and

$$\|u - u_h\|_{L^2(\Gamma) \otimes L^\infty(D)} \lesssim h^2 |\log h| \|u\|_{L^2(\Gamma) \otimes W^{2,\infty}(D)},$$

$$\|u - u_h\|_{L^2(\Gamma) \otimes L^\infty(D)} \lesssim h \|u\|_{L^2(\Gamma) \otimes W^{2,\infty}(D)}.$$

**Proof** It follows directly from the result of the corresponding deterministic problem [33].

For the linearized operator $L_v(y) = -\nabla \cdot (a_N(y) \nabla) + f'(v(y))$, we have the following property.

**Lemma 2** There exists a constant $\delta > 0$ such that for any given $v \in L^2(\Gamma) \otimes (H^2_0(D) \cap L^\infty(D))$ with $\|u - v\|_{L^2(\Gamma) \otimes L^\infty(D)} \leq \delta$, and for a.e. $y \in \Gamma$ given,

- $L_v(y) : H^2(D) \cap H^1_0(D) \to H^2(D) \cap H^1_0(D)$ is bijective and there exists a constant $C = C(\delta)$, such that

$$\|w\|_{H^2(D)} \leq C(\delta) \|L_v(y) w\|_{L^2(D)}, \quad \forall w \in H^1_0(D) \cap H^2(D).$$
If \( h \) is sufficiently small, there exists a constant \( c(\delta) \) such that

\[
\sup_{\chi \in \mathcal{X}_h(D)} \frac{A_v(y)(w_h, \chi)}{||\chi||_1} \geq c(\delta)||w_h||_1,
\]

where \( A_v(y)(w_h, \chi) = (a_N(y)\nabla w_h, \nabla \chi) + (f'(v(y))w_h, \chi) \), and \( w_h \in \mathcal{X}_h(D) \).

Let \( u^{h,p} \) be the two-level solution, we have

\[
|u - u^{h,p}| = |u - u^h| + |u^h - u^{h,p}| = |u - u^h| + (u^h - I_p(u^h)),
\]

where \( u^h \) is the semi-discrete solution satisfying the equation (3.2) and \( u^h - I_p(u^h) \) is the Lagrange interpolation error. We first present the following results of the interpolation operator \( I_p \) which are shown in [3].

**Lemma 3** There exists positive constants \( r_n, n = 1, 2, \cdots, N, \) independent of \( h \) and \( p, \) such that

\[
||u_h - I_p u_h||_{L_2(\Gamma) \otimes \mathcal{H}_0^1(D)} \lesssim \sum_{n=1}^{N} \beta_n(p_n)e^{-r_n p_n \theta_n},
\]

where

- if \( \Gamma_n \) is bounded
  \[
  \begin{cases}
  \theta_n = \beta_n = 1; \\
  r_n = \log \left[ \frac{2\tau_n}{|\Gamma_n|} \left( 1 + \sqrt{1 + \frac{|\Gamma_n|^2}{4\tau_n^2}} \right) \right],
  \end{cases}
  \]

- if \( \Gamma_n \) is unbounded
  \[
  \begin{cases}
  \theta_n = 1/2, \quad \beta_n = \mathcal{O}(\sqrt{\tau_n}); \\
  r_n = \tau_n \delta_n,
  \end{cases}
  \]

\( \tau_n \) is the distance between \( \Gamma_n \) and the nearest singularity in the complex plane and \( \delta_n \) is strictly positive value when \( \Gamma_n \) is unbounded such that the joint probability density \( \rho \) satisfies

\[
\rho(y) \leq C_\rho e^{-\sum_{n=1}^{N}(\delta_n y_n)^2}, \quad \forall y \in \Gamma,
\]

for some \( C_\rho > 0. \)

After using Lemma 3 and (2.3), we have the following estimation

\[
||u_H - I_p u_H||_{L_2^2(\Gamma) \otimes \mathcal{L}_p(D)} \lesssim \sum_{n=1}^{N} \beta_n^2(p_n)e^{-2r_n p_n \theta_n^2}, \tag{4.1}
\]

**Theorem 1** Let \( I_p(u^h) \) be the Lagrange interpolation of \( u^h \) with \( N_p \) collocation points, then, the following result holds

\[
||u^h - I_p u^h||_{L_2^2(\Gamma) \otimes \mathcal{H}_0^1(D)} \lesssim \sum_{n=1}^{N} \beta_n(p_n)e^{-r_n p_n \theta_n^2},
\]

where parameters \( \beta_n, r_n, \theta_n \) are defined in Lemma 3.

Applying Lemma 1, we only need to estimate \( u_h - u^h \) in order to bound \( u - u^h \).
Theorem 2 Let \( u^h \) be the semi-discrete solution satisfying (3.2) and \( u_h \) be the semi-discrete solution of equation (3.1). Then,

\[
\|u_h - u^h\|_{L^2(\Gamma) \otimes H^1_0(D)} \lesssim H^4 + \sum_{n=1}^{N} \beta_n^2(P_n) e^{-2r_n P_n^a},
\]

Proof Choose \( v = u_{H,P} \) in (3.2) and subtract (3.2) from (3.1), we have, for a.e. \( y \in \Gamma \) and for any \( w \in X_h(D) \), the following equation holds.

\[
(a_N(y) \nabla(u_h - u^h)(y), \nabla w) + (f(u_h(y)) - f(u_{H,P}(y)) + f'(u_{H,P}(y))(u^h - u_{H,P})(y), w) = 0,
\]

or

\[
A_{u,H,P}(y)((u_h - u^h)(y), w) = (\beta(u_h(y) - u_{H,P}(y))^2, w),
\]

where

\[
\beta = -\int_0^1 (1 - t) f''(u_{H,P}(y) + t(u_h - u_{H,P})(y)) dt.
\]

By assumption, it is easy to see that \( \beta \) is a uniformly bounded function on \( \bar{D} \). From the Hölder inequality and (2.3), we get

\[
(\beta(u_h(y) - u_{H,P}(y))^2, w) \lesssim \|(u_h(y) - u_{H,P}(y))^2\|_{0,2} \|w\|_{0,2} \lesssim \|u_h(y) - u_{H,P}(y)\|^{2}_{0,2} \|w\|_1.
\]

Applying Lemma 2, for a.e. \( y \in \Gamma \), we have

\[
\|(u_h - u^h)(y)\|_1 \lesssim \sup_{w \in X_h(D)} \frac{A_{u,H,P}(y)((u_h - u^h)(y), w)}{\|w\|_1} \geq \sup_{w \in X_h(D)} \frac{(\beta(u_h(y) - u_{H,P}(y))^2, w)}{\|w\|_1}
\]

\[
\lesssim \|(u_h - u_{H,P})(y)\|^{2}_{0,2}
\]

\[
\lesssim \left( \|(u_h - u_{H})(y)\|_{0,P} + \|(u_h - u_{H,P})(y)\|_{0,P} \right)^2.
\]

By Cauchy-Schwartz inequality, we get

\[
\|u_h - u^h\|^2_{L^2(\Gamma) \otimes H^1_0(D)} = \int_{\Gamma} \int_D |\nabla(u_h - u^h)|^2 dy dx = \int_{\Gamma} \|u_h - u^h\|^2_{1,1} dy
\]

\[
\lesssim \int_{\Gamma} \left( \|u_h - u_{H}\|_{0,2} + \|u_{H} - \mathcal{I}_P u_{H}\|_{0,2} \right)^4 dy
\]

\[
\lesssim \int_{\Gamma} \left( \|u_h - u_{H}\|_{4,2} + \|u_{H} - \mathcal{I}_P u_{H}\|_{4,2} \right) dy.
\]

Since

\[
\|w\|_{0,2} \lesssim \|w\|_{0,2} = \|w\|_{0,2}^2,
\]

we get

\[
\|u_h - u^h\|^2_{L^2(\Gamma) \otimes H^1_0(D)} \lesssim \left( \int_{\Gamma} \|u_h - u_{H}\|_{0,2}^2 dy + \int_{\Gamma} \|u_{H} - \mathcal{I}_P u_{H}\|_{0,2}^2 dy \right)
\]

\[
= \left( \int_{\Gamma} \|(u_h - u_{H})^2\|_{0,2}^2 dy + \int_{\Gamma} \|(u_{H} - \mathcal{I}_P u_{H})^2\|_{0,2} dy \right).}
\]
Hence
\[ \|u_h - u^h\|^2_{L^2(\Gamma) \otimes H^1_0(D)} \leq \|(u_h - u^H)^2\|^2_{L^2(\Gamma) \otimes L^p(D)} + \|(u_H - I_P u^H)^2\|^2_{L^2(\Gamma) \otimes L^p(D)} \]
\[ \lesssim \|u_h - u_H\|^4_{L^2(\Gamma) \otimes L^p(D)} + \|u_H - I_P u^H\|^4_{L^2(\Gamma) \otimes L^p(D)} \]
\[ \lesssim \left( \|u_h - u_H\|^2_{L^2(\Gamma) \otimes L^p(D)} + \|u_H - I_P u^H\|^2_{L^2(\Gamma) \otimes L^p(D)} \right)^2. \]

By Lemma 1, it is easy to get
\[ \|u_h - u_H\|_{L^2(\Gamma) \otimes L^p(D)} \lesssim H^2 \|u\|_{L^2(\Gamma) \otimes W^{2,p}(D)}. \quad (4.3) \]

Notice \( \|u_H - u_{H,P}\|_{L^2(\Gamma) \otimes L^p(D)} = \|u_H - I_P (u_H)\|_{L^2(\Gamma) \otimes L^p(D)}, \) from (4.3) and (4.1), we have
\[ \|u_h - u^h\|_{L^2(\Gamma) \otimes H^1_0(D)} \lesssim H^4 + \sum_{n=1}^{N} \beta_n^2(P_n)e^{-2r_n p_n^\alpha}. \]

Finally, we get the following error estimates for the two-level solution.

**Theorem 3** Let \( u^{h,P} \) be the two-level solution and \( u \) be the exact solution of (2.4). Then, we have
\[ \|u - u^{h,P}\|_{L^2(\Gamma) \otimes H^1_0(D)} \lesssim h + H^4 + \sum_{n=1}^{N} \beta_n^2(P_n)e^{-2r_n p_n^\alpha} + \sum_{n=1}^{N} \beta_n(p_n)e^{-r_n p_n^\alpha}, \quad (4.4) \]
\[ \|u - u^{h,P}\|_{L^2(\Gamma) \otimes L^2(D)} \lesssim h^2 + H^4 + \sum_{n=1}^{N} \beta_n^2(P_n)e^{-2r_n p_n^\alpha} + \sum_{n=1}^{N} \beta_n(p_n)e^{-r_n p_n^\alpha}. \quad (4.5) \]

where \( \beta_n, r_n, \theta_n \) are constants from Lemma 3.

**Proof** Estimation (4.4) follows from Lemma 1, Theorem 1 and 2. For (4.5), we have
\[ \|u - u^{h,P}\|_{L^2(\Gamma) \otimes L^2(D)} \lesssim \|u - u_h\|_{L^2(\Gamma) \otimes L^2(D)} + \|u_h - u^h\|_{L^2(\Gamma) \otimes L^2(D)} + \|u_h - I_P u^h\|_{L^2(\Gamma) \otimes L^2(D)} \]
\[ \lesssim \|u - u_h\|_{L^2(\Gamma) \otimes L^2(D)} + \|u_h - u^h\|_{L^2(\Gamma) \otimes H^1_0(D)} + \|u_h - I_P u^h\|_{L^2(\Gamma) \otimes L^2(D)} \]
\[ \lesssim h^2 + H^4 + \sum_{n=1}^{N} \beta_n^2(P_n)e^{-2r_n p_n^\alpha} + \sum_{n=1}^{N} \beta_n(p_n)e^{-r_n p_n^\alpha}. \]

5 Numerical experiments

In this section, we present some numerical experiments to verify the theoretical results given in Section 4. Our model problem is
\[ -\nabla \cdot (a \nabla u) + u^3 + g = 0, \quad \text{in} \ \Gamma \times D, \]
\[ u = 0, \quad \text{on} \ \Gamma \times \partial D, \]
with \( \Gamma = D = [-1, 1]^2. \) We first choose a particular diffusion coefficient such that the exact solution is available. Then, we consider a case with random coefficient approximated by truncated KL expansion.

We use piecewise linear finite element method for the spatial discretization. To see the order of accuracy in physical space, we choose mesh size pairs \((H,h) = (\frac{1}{2}, \frac{1}{4}), (\frac{1}{4}, \frac{1}{16}), (\frac{1}{8}, \frac{1}{32}), \) and \((\frac{1}{16}, \frac{1}{64}), \)

and fix the polynomial space pair with \((P,p) = (4, 8)\) (note that \((P,p) = (4, 8)\) means \(P_n = 4, p_n = 8\) for \(n = 1, \cdots, N\)) such that the overall approximation error is dominated by the spatial discretization error. Similarly, for the order of accuracy in the stochastic domain, we choose a fixed mesh size pair \((H, h) = (\frac{1}{16}, \frac{1}{1024})\) and consider the polynomial space pairs with \((P,p) = (1, 2), (2, 4), (3, 6), (4, 8)\). The stopping criterion for Newton iteration is chosen to be the relative error between two adjacent iterates less than a prescribed tolerance, i.e.,

\[
\frac{\|U_{H}^{n+1} - U_{H}^{n}\|}{\|U_{H}^{n+1}\|} \leq \epsilon,
\]

where \(\epsilon = 10^{-2}\) is used in our numerical tests. For the linear system of equations, we use the algebraic multigrid method with tolerance \(10^{-9}\).

The numerical experiments are conducted on a desktop computer with 3.5 GHz 6-core Intel Xeon E5 CPU and 16 GB 1867 MHz DDR3 memory. The MATLAB finite element package iFEM is used for the implementation [7].

**Example 1:** We choose the following random coefficient

\[
a(Y_1(\omega), Y_2(\omega), x_1, x_2) = 3 + Y_1(\omega) + Y_2(\omega),
\]

where the random variables \(Y_n(\omega), \ (n = 1, 2)\) are independent and identically distributed, satisfying uniform distribution. The collocation points are zeros of the Legendre polynomials. The function \(g(\omega, x)\) is chosen such that the exact solution is

\[
u(Y_1(\omega), Y_2(\omega), x_1, x_2) = \frac{1}{a(Y_1, Y_2, x_1, x_2)} \sin \pi x_1 \sin \pi x_2.
\]

From the left figure of Fig. 5.1, we observe that the convergence order is \(O(h^2)\) (equivalently, \(O(N^{-1})\) where \(N\) is the total number of degrees of freedom in space) in the \(L^2_p(\Gamma) \otimes L^2(\Omega)\) norm and is \(O(h)\) (equivalently, \(O(N^{-0.5})\)) in the \(L^2_p(\Gamma) \otimes H^1_0(D)\) norm which are consistent with the theory. The right figure of Fig. 5.1 shows that the error decays exponentially with respect to the polynomial degree \(p\) which is also consistent with the theoretical results. Numerical results presented in Table 5.2 show that two-level solution has the same accuracy as the standard stochastic collocation solution using \(T_h(D)\) and \(P_p(\Gamma)\) in \(L^2_p(\Gamma) \otimes L^2(\Omega)\) and \(L^2_p(\Gamma) \otimes H^1_0(D)\) norms when \(h = H^2\). Table 5.1 demonstrates that accuracy of the two-level solution is the same as the fine level stochastic collocation solution in \(L^2_p(\Gamma) \otimes H^1_0(D)\) when \(h = H^4\).
Table 5.1: Example 1: approximate errors with \((H,h) = (1/4,1/256), (P,p) = (4,8)\)

| \(L^2_p(\Gamma) \otimes \mathcal{L}^2(D)\) norm | \(L^2_p(\Gamma) \otimes H^1_0(D)\) norm |
|----------------|----------------|
| \(u-u_{H,p}\) | 0.0167          | 0.0417          |
| \(u-u_{h,p}\)  | 1.2760E - 4     | 0.0105          |
| \(u-u_{h,p}\)  | 1.5960E - 5     | 0.0105          |

Table 5.2: Example 1: approximate errors with \((H,h) = (1/16,1/256), (P,p) = (4,8)\)

| \(L^2_p(\Gamma) \otimes \mathcal{L}^2(D)\) norm | \(L^2_p(\Gamma) \otimes H^1_0(D)\) norm |
|----------------|----------------|
| \(u-u_{H,p}\) | 0.0042          | 0.1608          |
| \(u-u_{h,p}\)  | 1.6187E - 5     | 0.0105          |
| \(u-u_{h,p}\)  | 1.5940E - 5     | 0.0105          |

**Example 2:** We choose

\[
a(\omega, x) = \bar{a}(x) + \sum_{n=1}^{2} \sqrt{\lambda_n} b_n(x) Y_n(\omega),
\]

where \(\rho = 0.25\), \(\bar{a}(x) = 1\), \(\lambda_n, b_n(x)\) are the eigenvalues and eigenfunctions corresponding to the covariance function \(\text{Cov}_a(x,x') = \sigma^2 \exp(-|x-x'|)\) with \(\sigma = 0.4\). We choose \(g = 2(0.5 - |x|^2)\).

Since the exact solution of this example is not available, we construct a “reference” solution numerically by using a very fine mesh \((h = 1/1024)\) and a polynomial space of very high degree \((p = 10)\) and denote it by \(u^* = u_{h,p}\). It can be seen from the left figure of Fig. 5.2 that the accuracy in \(L^2_p(\Gamma) \otimes \mathcal{L}^2(D)\) norm is of the optimal order (i.e., \(O(h^2)\)) while the convergence in \(L^2_p(\Gamma) \otimes H^1_0(D)\) norm is better than what the theory predicted. This may be due to the supercloseness phenomenon commonly observed in the finite element computations. The right graph of Fig. 5.2 shows the exponential decay with respect to the polynomial degree \(p_n\). Same conclusions can be drawn from Table 5.3 and Table 5.4 as Example 1.
A two-level stochastic collocation method

L₂(Γ) ⊗ L₂(D) - norm
L₂(Γ) ⊗ H₁₀(D) - norm

| | | |
|---|---|---|
| u^* - u_{H,P} | 1.4156E - 4 | 0.0013 |
| u^* - u_{h,p} | 5.4499E - 7 | 8.2963E - 6 |
| u^* - u_{h,p} | 5.4836E - 7 | 8.2988E - 6 |

Table 5.3: Example 2: approximate errors with \((H, h) = (1/16, 1/256)\), \((P, p) = (4, 8)\)

L₂(Γ) ⊗ L₂(D) - norm
L₂(Γ) ⊗ H₁₀(D) - norm

| | | |
|---|---|---|
| u^* - u_{H,P} | 0.0016 | 0.0095 |
| u^* - u_{h,p} | 1.2712E - 6 | 8.7388E - 6 |
| u^* - u_{h,p} | 5.4836E - 7 | 8.2988E - 6 |

Table 5.4: Example 2: approximate errors with \((H, h) = (1/4, 1/256)\), \((P, p) = (4, 8)\)

6 Conclusion

In this work, we study the stochastic collocation method for solving semilinear elliptic equation with random coefficient. A novel two-level discretization technique is proposed to improve the efficiency of the standard stochastic collocation method. We analyze the convergence of this two-level discretization scheme and prove that when choosing the discretization parameters \(h, H, p, P\) appropriately, the two-level solution has the same order of accuracy as the fine level stochastic collocation solution. We also verify the theoretical results by several numerical examples. The main advantage of the two-level approach is that it reduces the computational complexity significantly.

Although we have restricted ourselves to the semilinear elliptic problems, the two-level method presented here is certainly applicable to a larger class of nonlinear stochastic partial differential equations. Also, besides the full tensor product polynomial spaces used here, we can also consider sparse grid polynomial spaces which are suitable for problems with high stochastic dimension [25]. Moreover, it may be possible to generalize this idea to utilize multiple levels of tensor product spaces when solving stochastic partial differential equations. This will be left as a future work.

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