BOHR–ROGOSINSKI RADIUS FOR ANALYTIC FUNCTIONS

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ABSTRACT. There are a number of articles which deal with Bohr’s phenomenon whereas only a few papers appeared in the literature on Rogosinski’s radii for analytic functions defined on the unit disk \(|z| < 1\). In this article, we introduce and investigate Bohr-Rogosinski’s radii for analytic functions defined for \(|z| < 1\). Also, we prove several different improved versions of the classical Bohr’s inequality. Finally, we also discuss the Bohr-Rogosinski’s radius for a class of subordinations. All the results are proved to be sharp.

1. Introduction and Preliminaries

The classical one-variable theorem of Bohr about power series (after subsequent improvements due to M. Riesz, I. Schur and F. Wiener) states that if \(f\) is a bounded analytic function on the unit disk \(\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}\), with the Taylor expansion \(\sum_{k=0}^{\infty} a_k z^k\), then the Bohr sum \(B_f(r)\) satisfies the classical Bohr inequality

\[
B_f(z) := \sum_{k=0}^{\infty} |a_k| r^k \leq \|f\|_{\infty} \quad \text{for} \quad |z| = r \leq 1/3,
\]

and the constant \(1/3\) is sharp. See for example, the recent survey on this topic by Abu-Muhanna et al. \[3\] and the references therein. Besides the Bohr radius, there is also the notion of Rogosinski radius \([9–11]\) which is described as follows: If \(f(z) = \sum_{k=0}^{\infty} a_k z^k\) is an analytic function on \(\mathbb{D}\) such that \(|f(z)| < 1\) in \(\mathbb{D}\), then for every \(N \geq 1\), we have \(|s_N(z)| < 1\) in the disk \(|z| < 1/2\) and this radius is sharp, where \(s_N(z) = \sum_{k=0}^{N-1} a_k z^k\) denotes the partial sums of \(f\). For our investigations, it is natural to introduce a new quantity, which we call Bohr-Rogosinski sum \(R^I_N(z)\) of \(f\) defined by

\[
R^I_N(z) := |f(z)| + \sum_{k=N}^{\infty} |a_k| r^k, \quad |z| = r.
\]

We remark that for \(N = 1\), this quantity is related to the classical Bohr sum in which \(f(0)\) is replaced by \(f(z)\). Clearly,

\[
|s_N(z)| = \left| f(z) - \sum_{k=N}^{\infty} a_k z^k \right| \leq R^I_N(z)
\]

2000 Mathematics Subject Classification. Primary: 30A10, 30H05, 30C35; Secondary: 30C45.

Key words and phrases. Bounded analytic functions, univalent functions, Bohr radius, Rogosinski radius, Schwarz-Pick Lemma, and subordination.

File: KaPoBohrRogo8\final\tex, printed: 21-3-2018, 8.32.
and thus, the validity of Bohr-type radius for $R_N'(z)$ gives Rogosinski radius in the case of bounded analytic functions. Hence, Bohr-Rogosinski’s sum is related to Rogosinski’s characteristic. As with the classical situation of Bohr radius, it is natural to obtain Bohr-Rogosinski radius.

In Section 2, we state and prove our first main result of this article which connects these radii. In Section 3 several improved versions of Bohr’s inequality are stated and their proofs are presented in Section 4. The notion of Bohr’s radius, initially defined for analytic functions from the unit disk $D$, was generalized by authors to include mappings from $D$ to some other domains $Ω$ in $\mathbb{D}$ ( [12][1]). In Section 5 we also consider Bohr–Rogosinski radius as a generalization to a class of subordinations.

2. Bohr-Rogosinski radius for analytic mappings

**Theorem 1.** Suppose that $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is analytic in $D$ and $|f(z)| < 1$ in $D$. Then

$$|f(z)| + \sum_{k=N}^{\infty} |a_k| r^k \leq 1 \text{ for } r \leq R_N,$$

where $R_N$ is the positive root of the equation $ψ_N(r) = 0$, $ψ_N(r) = 2(1+r)r^N - (1-r)^2$. The radius $R_N$ is best possible. Moreover,

$$|f(z)|^2 + \sum_{k=N}^{\infty} |a_k| r^k \leq 1 \text{ for } r \leq R_N',$$

where $R_N'$ is the positive root of the equation $(1+r)r^N - (1-r)^2 = 0$. The radius $R_N'$ is best possible.

**Proof.** By assumption $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is analytic in $D$ and $|f(z)| < 1$ in $D$. Since $f(0) = a_0$, it follows that for $z = re^{iθ} \in D$,

$$|f(z)| \leq \frac{r + |a_0|}{1 + |a_0|r} \text{ and } |a_k| \leq 1 - |a_0|^2 \text{ for } k = 1, 2, \ldots,$$

where the first inequality is a well-known consequence of Schwarz-Pick Lemma (often referred as Lindelöf’s inequality) while the second one is a well-known result due to F.W. Wiener (see also [5]). Using the last two inequalities, we have

$$|f(z)| + \sum_{k=N}^{\infty} |a_k| r^k \leq \frac{r + |a_0|}{1 + |a_0|r} + (1 - |a_0|^2) \frac{r^N}{1 - r},$$

which is less than or equal to 1 provided $ϕ_N(r) \leq 0$, where

$$ϕ_N(r) = (r + |a_0|)(1 - r) + (1 - |a_0|^2)r^N(1 + |a_0|r) - (1-r)(1 + |a_0|r)$$

$$= (1 - |a_0|)((1 + |a_0|)(1 + |a_0|r)r^N - (1-r)^2)$$

$$\leq (1 - |a_0|)[2(1+r)r^N - (1-r)^2], \text{ since } |a_0| < 1.$$

Now, $ϕ_N(r) \leq 0$ if $ψ_N(r) := 2(1+r)r^N - (1-r)^2 \leq 0$ which holds for $r \leq R_N$. The first part of the theorem follows.
To show the sharpness of the number $R_N$, we let $a \in [0, 1)$ and consider the function

\begin{equation}
\tag{3}
f(z) = \frac{a - z}{1 - az} = a - (1 - a^2) \sum_{k=1}^{\infty} a^{k-1} z^k, \quad z \in \mathbb{D}.
\end{equation}

For this function, we find that

\begin{equation}
\tag{4}
|f(-r)| + \sum_{k=N}^{\infty} |a_k| r^k = \frac{r + a}{1 + ar} + (1 - a^2) \frac{a^{N-1} r^N}{1 - ar}.
\end{equation}

The last expression is bigger than 1 if and only if

\((1 - a)[(1 + a)(1 + ar) a^{N-1} r^N - (1 - r)(1 - ar)] > 0.\)

Note that the expression (4) is less than or equal to 1 for all $a \in [0, 1)$, only in the case when $r \leq R_N$. Finally, allowing $a \to 1$ in the last inequality shows that the expression (4) is bigger than 1 if $r > R_N$. This proves the sharpness.

Next, we verify the inequality (2). In this case, simple computation shows that

\begin{equation}
\left|f(z)^2 + \sum_{k=N}^{\infty} |a_k| r^k\right| \leq \left(\frac{r + |a_0|}{1 + |a_0| r}\right)^2 + (1 - |a_0|^2) \frac{r^N}{1 - r}
= 1 + \frac{(1 - |a_0|^2)[r^N(1 + |a_0| r)^2 - (1 - r)^2(1 + r)]}{(1 - r)(1 + |a_0| r)^2}
\end{equation}

and the last expression is non-positive if and only if

\[r^N(1 + |a_0| r)^2 - (1 - r)^2(1 + r) \leq 0.\]

Since $|a_0| < 1$, the last inequality is guaranteed by the condition

\[-(1 - r)^2 + r^N(1 + r) \leq 0\]

which gives $r \leq R'_{N}$, where $R'_{N}$ is as in the statement of the theorem. Note that for $N = 1$, this condition is equivalent to $-1 + 2r + 3r^2 \leq 0$ and we obtain $r \leq R'_1 = 1/3$.

To prove the sharpness of the number $R'_{N}$, we consider the function $f(z)$ defined by (3) and for this function we observe that

\begin{equation}
\tag{5}
|f(-r)|^2 + \sum_{k=N}^{\infty} |a_k| r^k = \left(\frac{r + a}{1 + ar}\right)^2 + (1 - a^2) \frac{a^{N-1} r^N}{1 - ar}
\end{equation}

which is bigger than 1 for all $a \in [0, 1)$ provided

\[(1 + ar)^2 a^{N-1} r^N - (1 - r^2)(1 - ar) > 0.\]

Again, allowing $a \to 1$, it follows that the expression (5) is bigger than 1 if $r > R'_{N}$. This proves the sharpness and we complete the proof of Theorem 1.

It follows from the Maximum principle that the Bohr–Rogosinski radius is always less than or equal to the Bohr radius. Clearly, Rogosinski radius is always bigger than or equal to the Bohr–Rogosinski radius.

It is easy to see that $R_1 = \sqrt{5} - 2$ and $R'_1 = 1/3$. Also, we remark that the numbers $R_N$ and $R'_{N}$ in Theorem 1 both approach 1 as $N \to \infty$ so that Bohr-Rogosinski’s radius in both cases tend to 1 as $N \to \infty$. We can easily get the following result and, since the proof of it follows on the similar lines of the proof of Theorem 1 we omit its details.
Theorem 2. Suppose that \( f(z) = \sum_{k=0}^{\infty} a_k z^k \) is analytic in \( \mathbb{D} \) such that \( |f(z)| \leq 1 \) in \( \mathbb{D} \). Then for each \( m, N \in \mathbb{N} \), we have
\[
|f(z^m)| + \sum_{k=N}^{\infty} |a_k| r^k \leq 1 \quad \text{for} \quad r \leq R_{m,N},
\]
where \( R_{m,N} \) is the positive root of the equation
\[
2r^N(1 + r^m) - (1 - r)(1 - r^m) = 0,
\]
and the number \( R_{m,N} \) cannot be improved. Moreover,
\[
\lim_{N \to \infty} R_{m,N} = 1 \quad \text{and} \quad \lim_{m \to \infty} R_{m,N} = A_N,
\]
where \( A_N \) is the positive root of the equation \( 2r^N = 1 - r \). Also, \( A_1 = 1/3 \) and \( A_2 = 1/2 \).

3. Improved Bohr’s inequality for analytic mappings

Next, we state several different improved versions of Bohr’s inequality.

Theorem 3. Suppose that \( f(z) = \sum_{k=0}^{\infty} a_k z^k \) is analytic in \( \mathbb{D} \), \( |f(z)| \leq 1 \) in \( \mathbb{D} \) and \( S_r \) denotes the area of the image of the subdisk \( |z| < r \) under the mapping \( f \). Then
\[
B_1(r) := \sum_{k=0}^{\infty} |a_k| r^k + \frac{16}{9} \left( \frac{S_r}{\pi} \right) \leq 1 \quad \text{for} \quad r \leq \frac{1}{3}
\]
and the numbers 1/3 and 16/9 cannot be improved. Moreover,
\[
B_2(r) := |a_0|^2 + \sum_{k=1}^{\infty} |a_k| r^k + \frac{9}{8} \left( \frac{S_r}{\pi} \right) \leq 1 \quad \text{for} \quad r \leq \frac{1}{2}
\]
and the constants 1/2 and 9/8 cannot be improved.

Theorem 4. Suppose that \( f(z) = \sum_{k=0}^{\infty} a_k z^k \) is analytic in \( \mathbb{D} \) and \( |f(z)| \leq 1 \) in \( \mathbb{D} \). Then
\[
|a_0| + \sum_{k=1}^{\infty} \left( |a_k| + \frac{1}{2} |a_k|^2 \right) r^k \leq 1 \quad \text{for} \quad r \leq \frac{1}{3}
\]
and the numbers 1/3 and 1/2 cannot be improved.

Theorem 5. Suppose that \( f(z) = \sum_{k=0}^{\infty} a_k z^k \) is analytic in \( \mathbb{D} \) and \( |f(z)| \leq 1 \) in \( \mathbb{D} \). Then
\[
\sum_{k=0}^{\infty} |a_k| r^k + |f(z) - a_0|^2 \leq 1 \quad \text{for} \quad r \leq \frac{1}{3}
\]
and the number 1/3 cannot be improved.

Finally, we also prove the following sharp inequality.

Theorem 6. Suppose that \( f(z) = \sum_{k=0}^{\infty} a_k z^k \) is analytic in \( \mathbb{D} \) and \( |f(z)| \leq 1 \) in \( \mathbb{D} \). Then
\[
|f(z)|^2 + \sum_{k=1}^{\infty} |a_k|^2 r^{2k} \leq 1 \quad \text{for} \quad r \leq \sqrt{\frac{11}{27}}
\]
and this number cannot be improved.
4. PROOFS OF THEOREMS 3, 4, 5 AND 6

For the proof of Theorem 3, we need the following lemma, especially when \(0 < r \leq 1/2\).

**Lemma 1.** Let \(|b_0| < 1\) and \(0 < r \leq 1/\sqrt{2}\). If \(g(z) = \sum_{k=0}^{\infty} b_k z^k\) is analytic and satisfies the inequality \(|g(z)| < 1\) in \(\mathbb{D}\), then the following sharp inequality holds:

\[
\sum_{k=1}^{\infty} k|b_k|^2 r^{2k} \leq r^2 \frac{(1 - |b_0|^2)^2}{(1 - |b_0|^2 r^2)^2}.
\]

**Proof.** Let \(b_0 = a\). Then, it is easy to see that the condition on \(g\) can be rewritten in terms of subordination as

\[
g(z) = \sum_{k=0}^{\infty} b_k z^k \prec \phi(z) = a - (1 - |a|^2) \sum_{k=1}^{\infty} (\bar{a})^{k-1} z^k, \quad z \in \mathbb{D},
\]

where \(\prec\) denotes the usual subordination (see [6, 7]). Note that \(\phi\) is analytic in \(\mathbb{D}\) and \(|\phi(z)| < 1\) for \(z \in \mathbb{D}\). The subordination relation (10) gives

\[
\sum_{k=1}^{\infty} k|b_k|^2 r^{2k} \leq (1 - |a|^2)^2 \sum_{k=1}^{\infty} k|a|^{2(k-1)} r^{2k} = r^2 \frac{(1 - |a|^2)^2}{(1 - |a|^2 r^2)^2}
\]

from which we arrive at the inequality (11) which proves Lemma 1. For \(0 < r \leq 1/\sqrt{2}\), it is important to note here that the sequence \(\{kr^{2k}\}\) is non-increasing for all \(k \geq 1\) so that we were able to apply the classical Goluzin’s inequality [7] (see also [6, Theorem 6.3]) which extends the classical Rogosinski inequality.

**Proof of Theorem 3.** Since the left hand side of (6) is an increasing function of \(r\), it is enough to prove it for \(r = 1/3\). Therefore, we set \(r = 1/3\). Moreover, the present authors in the proof of Theorem 1 in [8] proved the following inequalities:

\[
\sum_{k=1}^{\infty} |a_k|r^{k} \leq \begin{cases} A(r) := \frac{1 - |a_0|^2}{1 - r|a_0|} & \text{for } |a_0| \geq r \\ B(r) := \frac{\sqrt{1 - |a_0|^2}}{\sqrt{1 - r^2}} & \text{for } |a_0| < r. \end{cases}
\]

Note that \(|a_k| \leq 1 - |a_0|^2\) for \(k \geq 1\) and, from the definition of \(S_r\), we see that

\[
\frac{S_r}{\pi} = \frac{1}{\pi} \int_{|z|<r} |f'(z)|^2 dx dy = \sum_{k=1}^{\infty} k|a_k|^2 r^{2k}
\]

\[
\leq (1 - |a_0|^2)^2 \sum_{k=1}^{\infty} kr^{2k} = (1 - |a_0|^2)^2 \frac{r^2}{(1 - r^2)^2}.
\]
At first we consider the case $|a_0| \geq r = 1/3$. In this case, using (11) and (12), we have
\[
B_1(r) = |a_0| + \sum_{k=1}^{\infty} |a_k|r^k + \frac{16}{9\pi}Sr \leq |a_0| + A(1/3) + \frac{16}{9\pi}S_{1/3} \leq |a_0| + \frac{1 - |a_0|^2}{3 - |a_0|} + \frac{(1 - |a_0|^2)^2}{4} = 1 - \frac{(1 - |a_0|^2)(5 - |a_0|^2)}{4(3 - |a_0|)} \leq 1.
\]

Next we consider the case $|a_0| < r = 1/3$. Again, using (11) and (12), we deduce that
\[
B_1(r) = \sum_{k=0}^{\infty} |a_k|r^k + \frac{16}{9\pi}Sr \leq |a_0| + B(1/3) + \frac{16}{9\pi}S_{1/3} \leq |a_0| + \sqrt{\frac{1 - |a_0|^2}{\sqrt{8}}} + \frac{(1 - |a_0|^2)^2}{4} \leq \frac{1}{3} + \frac{1}{\sqrt{8}} + \frac{1}{4} < 1 \quad \text{(since } |a_0| < 1/3)\]

and the desired inequality (6) follows.

To prove that the constant $\frac{16}{9\pi}$ is sharp, we consider the function $f$ given by (3). For this function, straightforward calculations show that
\[
\sum_{k=0}^{\infty} |a_k|r^k + \frac{\lambda}{\pi}Sr = a + r \frac{1 - a^2}{1 - ra} + \lambda(1 - a^2)^2 \frac{r^2}{(1 - a^2 r^2)^2}.
\]

In the case $r = 1/3$ the last expression becomes
\[
a + \frac{1 - a^2}{3 - a} + 9\lambda \frac{(1 - a^2)^2}{(9 - a^2)^2} = 1 - \frac{2(1 - a)^3(19 + 12a + a^2)}{(a^2 - 9)^2} + (9\lambda - 16) \frac{(1 - a^2)^2}{(9 - a^2)^2}
\]

which is obviously bigger than 1 in case $\lambda > 16/9$ and $a \to 1$. The proof of the first part of Theorem 3 is complete.

Let us now verify the inequality (7). To do it we will use the method presented above and Lemma 1 for $r \leq 1/2$. From Lemma 1 it follows that
\[
(13) \quad \frac{S_r}{\pi} \leq (1 - |a_0|^2)^2 \frac{r^2}{(1 - |a_0|^2 r^2)^2}, \quad r \leq 1/2.
\]

Let $r \leq 1/2$ and we first consider the case $|a_0| \geq 1/2$. Then, using (11) and (13), we obtain that
\[
B_2(r) = |a_0|^2 + \sum_{k=1}^{\infty} |a_k|r^k + \frac{9}{8\pi}Sr \leq |a_0|^2 + A(1/2) + \frac{9}{8\pi}S_{1/2} \leq |a_0|^2 + \frac{1 - |a_0|^2}{2 - |a_0|} + \frac{4(1 - |a_0|^2)^2}{(4 - |a_0|^2)^2}
\]
\[
= 1 - \frac{(1 - |a_0|^2)(1 + |a_0|(7 + 6|a_0| + 2|a_0|^2))}{2(4 - |a_0|^2)^2} \leq 1
\]

Now we consider the case $|a_0| < 1/2$. In this case we have

$$B_2(r) \leq |a_0|^2 + B(1/2) + \frac{9}{8\pi} S_{1/2}$$

$$\leq |a_0|^2 + \frac{\sqrt{1 - |a_0|^2}}{\sqrt{3}} + \frac{4(1 - |a_0|^2)^2}{(4 - |a_0|^2)^2}$$

$$\leq \frac{1}{\sqrt{3}} + |a_0|^2 + \frac{4(1 - |a_0|^2)^2}{(4 - |a_0|^2)^2}$$

$$\leq \frac{1}{\sqrt{3}} + \frac{41}{100} - \frac{(1 - 4|a_0|^2)(256 - 104|a_0|^2 + 25|a_0|^4)}{100(|a_0|^2 - 4)^2}$$

which is less than 1. The sharpness of the constant $9/8$ can be established as in the previous case and thus, we omit the details. The proof of the theorem is complete. □

Proof of Theorem 4. Let $A(r)$ and $B(r)$ be defined as in (11). Furthermore, the present authors in [8] demonstrated the following inequality for the coefficients of $f$:

$$\sum_{k=1}^{\infty} |a_k|^2 r^k \leq \frac{r(1 - |a_0|^2)^2}{1 - |a_0|^2 r}.$$  

(14)

As remarked in the proof of earlier theorems, it suffices to prove the inequality [8] for $r = 1/3$ and thus, we may set $r = 1/3$ in the proof below. At first we consider the case $|a_0| \geq 1/3$ so that

$$\sum_{k=0}^{\infty} |a_k|^2 r^k + \frac{1}{2} \sum_{k=1}^{\infty} |a_k|^2 r^k \leq |a_0| + A(1/3) + \frac{(1 - |a_0|^2)^2}{6 - 2|a_0|^2}$$

$$= |a_0| + \frac{1 - |a_0|^2}{3 - |a_0|} + \frac{(1 - |a_0|^2)^2}{6 - 2|a_0|^2}$$

$$= 1 - \frac{(1 - |a_0|^2)^2}{2} \leq 1 \quad \text{since } |a_0| \leq 1.$$

Similarly, for the case $|a_0| < 1/3$, we have

$$\sum_{k=0}^{\infty} |a_k|^2 r^k + \frac{1}{2} \sum_{k=1}^{\infty} |a_k|^2 r^k \leq |a_0| + B(1/3) + \frac{(1 - |a_0|^2)^2}{6 - 2|a_0|^2}$$

$$\leq |a_0| + \frac{\sqrt{1 - |a_0|^2}}{\sqrt{8}} + \frac{(1 - |a_0|^2)^2}{6 - 2|a_0|^2}$$

$$\leq \frac{1}{3} + \frac{1}{\sqrt{8}} + \frac{1}{6}$$

$$< 1 \quad \text{since } |a_0| < 1/3$$

which concludes the proof of Theorem 4 since the proof of sharpness follows similarly. □
Proof of Theorem 5. Let $A(r)$ and $B(r)$ be defined as in (11). Also, we may let $r = 1/3$. Accordingly, we first consider the case $|a_0| \geq 1/3$ so that

$$\sum_{k=0}^{\infty} |a_k|^r + |f(z) - a_0|^2 \leq |a_0| + A(1/3) + A(1/3)^2$$

$$= |a_0| + \frac{1 - |a_0|^2}{3 - |a_0|} + \frac{(1 - |a_0|^2)^2}{(3 - |a_0|)^2}$$

$$= 1 - \frac{(1 - |a_0|^2)^3 (5 + |a_0|)}{(3 - |a_0|)^2} \leq 1 \quad \text{(since } |a_0| \leq 1).$$

Next, we consider the case $|a_0| < 1/3$ so that

$$\sum_{k=0}^{\infty} |a_k|^r + |f(z) - a_0|^2 \leq |a_0| + B(1/3) + B(1/3)^2$$

$$= |a_0| + \sqrt{1 - |a_0|^2} + \frac{1 - |a_0|^2}{8}$$

$$\leq \frac{1}{3} + \frac{1}{\sqrt{8}} + \frac{1}{8} < 1.$$

This concludes the proof of Theorem 5 and the sharpness follows similarly. □

Proof of Theorem 6. Using (14) (see [8, Lemma 1]) and the classical inequality for $|f(z)|$, we have

$$|f(z)|^2 + \sum_{k=1}^{\infty} |a_k|^2 r^{2k} \leq \left( \frac{r + |a_0|}{1 + r |a_0|} \right)^2 + \frac{r^2 (1 - |a_0|^2)^2}{1 - |a_0|^2 r^2}.$$

For $r = \sqrt{11/27}$, the last expression on the right gives

$$1 - \frac{3(1 - |a_0|^2)}{(9 + \sqrt{33} |a_0|^2)^2} (135 - 66\sqrt{33} |a_0| + 66\sqrt{33} |a_0|^3 + 121|a_0|^4).$$

and straightforward calculations show that this expression is less than or equal to 1 for all $|a_0| \leq 1$. The example

$$f(z) = \frac{z + a}{1 + az}$$

with $a = \sqrt{3/11}$ shows that $r = \sqrt{11/27}$ is sharp. This completes the proof. □

5. Bohr-Rogosinski’s radius for a class of subordinations

We may generalize Bohr-Rogosinski’s radius, defined in Section 4 for mappings from $D$ to itself, by writing Bohr-Rogosinski inequality in the equivalent form

$$\sum_{k=1}^{\infty} |b_k|^r \leq 1 - |g(z)| = \text{dist}(g(z), \partial D).$$

Observe that the number $1 - |g(z)|$ is the distance from the point $g(z)$ to the boundary $\partial D$ of the unit disk $D$. Using this “distance form” formulation of the Bohr-Rogosinski inequality, the notion of the Bohr-Rogosinski radius can be generalized to the class of
functions \( f \) analytic in \( \mathbb{D} \) which take values in a given domain \( \Omega \). For our formulation, we shall use the notion of subordination.

As in the case of Bohr phenomenon \([1]\), for a given \( f \), it is natural to introduce \( S(f) = \{ g : g < f \} \) and \( \Omega = f(\mathbb{D}) \). We say that the family \( S(f) \) has a **Bohr-Rogosinski phenomenon** if there exists an \( r_f, 0 < r_f \leq 1 \), such that whenever \( g(z) = \sum_{k=0}^{\infty} b_k z^k \in S(f) \), we have

\[
|g(z)| + \sum_{k=1}^{\infty} |b_k| r^k \leq |f(0)| + \text{dist}(f(0), \partial \Omega)
\]

for \(|z| = r < r_f\). We observe that if \( f(z) = (a_0 - z)/(1 - \bar{a}_0 z) \) with \(|a_0| < 1\), and \( \Omega = \mathbb{D} \), then we have

\[
\text{dist}(f(0), \partial \Omega) = 1 - |f(0)|,
\]

which means that (15) holds with \( r_f = \sqrt{5} - 2 \), according to Theorem 1. In view of this observation, we say that the family \( S(f) \) satisfies the **classical** Bohr-Rogosinski phenomenon if (15) holds for \(|z| = r < \sqrt{5} - 2 \) with \( 1 - |g(z)| \) in place of \( \text{dist}(g(z), \partial f(\mathbb{D})) \). Hence the distance form allows us to extend Bohr-Rogosinski’s theorem to a variety of distances provided the Bohr-Rogosinski phenomenon exists.

**Theorem 7.** If \( f, g \) are analytic in \( \mathbb{D} \) such that \( f \) is univalent in \( \mathbb{D} \) and \( g \in S(f) \), then inequality (15) holds with \( r_f = 5 - 2\sqrt{6} \approx 0.101021 \). The sharpness of \( r_f \) is shown by the Koebe function \( f(z) = z/(1 - z)^2 \).

**Proof.** Let \( g(z) = \sum_{k=0}^{\infty} b_k z^k < f(z) \), where \( f \) is a univalent mapping of \( \mathbb{D} \) onto a simply connected domain \( \Omega = f(\mathbb{D}) \). Then it is well known that (see, for instance, [3, 7]) for all \( z \in \mathbb{D} \) and \( k \geq 1 \),

\[
\frac{1}{4} |f'(z)|(1 - |z|^2) \leq \text{dist}(f(z), \partial \Omega) \leq |f'(z)|(1 - |z|^2) \quad \text{and} \quad |b_k| \leq k |f'(0)|.
\]

It follows that \(|b_k| \leq 4k \text{dist}(f(0), \partial \Omega) = 4k \text{dist}(g(0), \partial \Omega)\), for \( k \geq 1 \), and thus

\[
\sum_{k=1}^{\infty} |b_k| r^k \leq 4 \text{dist}(f(0), \partial \Omega) \sum_{k=1}^{\infty} k r^k = \text{dist}(f(0), \partial \Omega) \frac{4r}{(1 - r)^2}.
\]

Moreover, because \( g < f \), it follows that

\[
|g(z) - g(0)| \leq |a_1| \frac{r}{(1 - r)^2} \leq \text{dist}(f(0), \partial \Omega) \frac{4r}{(1 - r)^2}
\]

so that (since \( g(0) = f(0) \))

\[
|g(z)| \leq |f(0)| + \text{dist}(f(0), \partial \Omega) \frac{4r}{(1 - r)^2}.
\]

By (17) and (18), we deduce that

\[
|g(z)| + \sum_{k=1}^{\infty} |b_k| r^k \leq |f(0)| + \text{dist}(f(0), \partial \Omega) \frac{8r}{(1 - r)^2} \leq |f(0)| + \text{dist}(f(0), \partial \Omega)
\]

provided \( 8r \leq (1 - r)^2 \). This gives the condition \( r \leq 5 - 2\sqrt{6} \). When \( f(z) = z/(1 - z)^2 \), we obtain \( \text{dist}(f(0), \partial \Omega) = 1/4 \) and a simple calculation gives the sharpness.

\( \square \)
In the case of univalent convex function $f$ with $g(z) = \sum_{k=0}^{\infty} b_k z^k \prec f(z)$, the equation (16) takes that form (see, for instance, [6, 7]),

$$
\frac{1}{2} |f'(z)|(1 - |z|^2) \leq \text{dist}(f(z), \partial \Omega) \leq |f'(z)|(1 - |z|^2) \quad \text{and} \quad |b_k| \leq |f'(0)| \quad \text{for} \quad k \geq 1
$$

and thus, it is easy to see that Theorem 8 takes the following form. Note that when $f(z) = z/(1 - z)$, we have $\text{dist}(f(0), \partial \Omega) = 1/2$.

**Theorem 8.** If $f, g$ are analytic in $D$ such that $f$ is convex (univalent) in $D$ and $g \in S(f)$, then inequality (15) holds with $r_f = 1/5$. The sharpness of $r_f$ is shown by the convex function $f(z) = z/(1 - z)$.

**Acknowledgements.** The research of the first author was supported by Russian foundation for basic research, Proj. 17-01-00282, and the research of the second author was supported by the project RUS/RFBR/P-163 under Department of Science & Technology (India). The second author is currently on leave from the IIT Madras.

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