Line antiderivations over local fields and their applications.

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Abstract

A non-Archimedean antiderivational line analog of the Cauchy-type line integration is defined and investigated over local fields. Classes of non-Archimedean holomorphic functions are defined and studied. Residues of functions are studied, Lorent series representations are described. Moreover, non-Archimedean antiderivational analogs of integral representations of functions and differential forms such as the Cauchy-Green, Martinelli-Bochner, Leray, Koppelman and Koppelman-Leray formulas are investigated. Applications to manifold and operator theories are studied.

Keywords: local field, non-Archimedean Cauchy-type integration, differential forms, integral representations

1 Introduction

Line (Cauchy) integration is the cornerstone in the complex analysis and integral formulas of functions and differential forms such as the Cauchy-Green, Martinelli-Bochner, Leray, Koppelman and Koppelman-Leray formulas play very important role in it and in analysis on complex manifolds and theory of Stein and Kähler manifolds and theory of holomorphic functions (see, for example, [8, 25]). In the non-Archimedean case there is not so developed
analog of complex analysis. Though there are few works devoted to non-Archimedean holomorphic functions over the complex non-Archimedean field $\mathbb{C}_p$ and the Levi-Civitá fields, which are not locally compact (see [11, 2] and references therein). In that work M.M. Vishik and M. Berz have obtained analogs of residues and the Cauchy formula, but integrals that they have used were of combinatorial-algebraic nature and they have operated with power series mainly for their analogs of holomorphic functions. On the other hand, there is not any measure equivalent with the Haar measure on such non-locally compact fields because of the A. Weil [28] theorem stating that the existence of such nontrivial measure on a topological group implies its local compactness. This article is devoted to others non-Archimedean analogs of integral representation theorems, that were not yet considered by others authors. Moreover, this article operates with locally compact non-Archimedean fields of characteristic zero (local fields) and the corresponding analogs of complex planes. Apart from the classical case in the non-Archimedean case there is not any indefinite integral. Instead of it antiderivation operators by Schikhof [22] are used.

It is necessary to note that in this article are considered not only manifolds treated by the rigid geometry, but much wider classes continuing the previous work [14]. For them the existence of an exponential mapping is proved. A rigid non-Archimedean geometry serves mainly for needs of the cohomology theory on such manifolds, but it is too restrictive and operates with narrow classes of analytic functions [7]. It was introduced at the beginning of sixties of the 20-th century. Few years later wider classes of functions were investigated by Schikhof [22]. In this paper classes of functions and antiderivation operators by Schikhof and their generalizations from works [13, 12] are used.

Section 2 is devoted to the definition and investigations of the non-Archimedean analogs of the line integration over local fields. Classes of non-Archimedean holomorphic functions are defined and studied. For this specific non-Archimedean geometry’s definitions and theorems are given (see also definitions and notations in [12, 13, 14, 15, 17]). It is necessary to note that definitions, formulations of theorems, propositions, etc. and their proofs differ substantially from the classical case (over $\mathbb{C}$). Residues of functions are studied, Lorent series representations are described. In Section 3 non-Archimedean antiderivational analogs of integral representations of functions and differential forms such as the Cauchy-Green, Martinelli-Bochner, Leray,
Koppelman and Koppelman-Leray formulas are investigated. These studies are accomplished on domains in finite dimensional Banach spaces over local fields and also on manifolds over local fields. All results of this paper are obtained for the first time. Finally, applications of the obtained results to the theory of non-Archimedean manifolds and linear operators in non-Archimedean Banach spaces are outlined. In works of Vishik (see [11] and references therein) the theory of non-Archimedean (Krasner) analytic operators with compact spectra in $\mathbb{C}_p$ was developed. In this article operators may have noncompact spectra in a field $L$ such that $\mathbb{Q}_p \subset L$ (may be also $L \supset \mathbb{C}_p$ and $L \neq \mathbb{C}_p$) continuing the investigation of [18].

2 Line antiderivation over local fields

To avoid misunderstandings we first present our specific definitions.

2.1. Notation and Remarks. Let $K$ denotes a local field, that is, a finite algebraic extension of the field $\mathbb{Q}_p$ of $p$-adic numbers with a norm extending that of $\mathbb{Q}_p$ [27]. Denote by $\mathbb{C}_p$ the field of complex numbers with the norm extending that of $\mathbb{Q}_p$ [10]. If $i \in K$ take $\alpha \in \mathbb{C}_p \setminus K$ such that there exists $\tilde{m} \in \mathbb{N}$ with $\alpha^{\tilde{m}} \in K$, where $\tilde{m}$ is such a minimal natural number, $\tilde{m} = \tilde{m}(\alpha)$, $i := (-1)^{1/2}$. If $i \notin K$ take $\alpha = i$. Denote by $K(\alpha)$ a local field which is the extension of $K$ with the help of $\alpha$.

Suppose $U$ is a clopen compact perfect (that is, dense in itself) subset in $K$ and $\sigma$ is its approximation of the identity: there is a sequence of maps $\sigma_l : U \to U$, where $0 \leq l \in \mathbb{Z}$, such that

(i) $\sigma_0$ is constant;
(ii) $\sigma_l \circ \sigma_n = \sigma_n \circ \sigma_l = \sigma_n$ for each $l \geq n$;
(iii) there exists a constant $0 < \rho < 1$ such that for each $x, y \in U$ the inequality $|x - y| < \rho^n$ implies $\sigma_n(x) = \sigma_n(y)$;
(iv) $|\sigma_n(x) - x| < \rho^n$ for each integer $n \geq 0$. Consider spaces $C^n(U, L)$ of all $n$-times continuously differentiable in the sense of difference quotients functions $f : U \to L$, where $L$ is a field containing $K$ with the multiplicative norm $|\cdot|_L$ which is the extension of the multiplicative norm $|\cdot|_K$ in $K$. Then there exists an antiderivation:

(1) $\nu P^n : C^{n-1}(U, L) \to C^n(U, L)$ given by the formula:
(2) $\nu P^n f(x) := \sum_{l=0}^{\infty} \sum_{j=0}^{n-1} f^{(j)}(x_l)(x_{l+1} - x_l)^{j+1}/(j + 1)!$,
where $x_l := \sigma_l(x)$, $x \in U$, $n \geq 1$ (see §80 [22]). Formula (2) shows, that
if $U^m$ is defined on $C^{n-1}(U, K)$, then it is defined on $C^{n-1}(U, Y)$ for each field $L$ which is complete relative to its norm such that $K \subset L$ and a Banach space $Y$ over $L$.

Since $P^m$ is the $L$-linear operator, then there exists the $L$-linear space $pC^m(U, Y) := P^m(C^{n-1}(U, Y))$, put $pC^m(U, Y) := pC^m_0(U, Y) \oplus Y$, where $n \geq 1$, $Y$ is a Banach space over $L$. For a clopen subset $\Omega$ in $(K \oplus \alpha K)^m$ such that $\Omega \subset U^m \times U^m$ consider the antiderivation $\alpha P^m f(z)$ as the restriction of $U^m \times U^m P^m f(z)$ on $\Omega$.

(3) $\alpha P^m f(z) := U^m \times U^m P^m|_\Omega f(z) = U^m \times U^m P^m f(z)\chi_\Omega(z)$, where

(4) $U^m \times U^m P^m f(z) := U P_{x_1}^m \cdots U P_{x_m}^m U P_{y_1}^m \cdots U P_{y_m}^m f(z)$,

$\chi_\Omega(z)$ denotes the characteristic function of $\Omega$, $\chi_\Omega(z) = 1$ for each $z \in \Omega$, $\chi_\Omega(z) = 0$ for each $z \in K^{2m} \setminus \Omega$, $z = (x, y)$, $x, y \in U^m \subset K^m$, $x = (x_1, \ldots, x_m)$, $x_1, \ldots, x_m \in K$, $U P_{x_i}^m$ means the antiderivation by the variable $x_i$. This is correct, since each $f \in C^{0,n-1}(\Omega, L) := C((0, n - 1), \Omega \to L)$ (see §1.2.4 [12]) has a $C^{0,n-1}$-extension on $U^m \times U^m$, for example, $f|_{U^m \times U^m \setminus \Omega} = 0$.

This means, that $U^m \times U^m P^m f(z)$ is the antiderivation defined with the help approximation of the unity on $U^m \times U^m$ such that $U^m \times U^m \sigma = (U \sigma, \ldots, U \sigma)$.

The condition of compactness of $\Omega$ is not very restrictive, since each locally compact subset in $(K \oplus \alpha K)^m$ has a one-point (Alexandroff) compactification which is totally disconnected and hence homeomorphic to a clopen subset in $(K \oplus \alpha K)^m$ (see §3.5 and Theorem 6.2.16 about universality of the Cantor cube in [5]). If $\rho(z_1, z_2) := |z_1 - z_2|$ is the metric in $(K \oplus \alpha K)^m$, then the metric $\rho'(z_1, z_2) := \rho(z_1, z_2)/[1 + \rho(z_1, z_2)]$ has the extension on the one-point compactification $A(K \oplus \alpha K)^m := (K \oplus \alpha K)^m \cup \{A\}$, where $A$ is a singleton. If $Y$ is a metric space with a metric $\rho$, then $B(Y, y, r) := \{z \in Y : \rho(z, y) \leq r\}$ denotes the ball of radius $r > 0$ and containing a point $y \in Y$.

2.2. Notes and Definitions. 1. For a local field $K$ there exists a prime $p$ such that $K$ is a finite algebraic extension of $Q_p$. In view of Theorems 1.1, 4.6 and Proposition 4.4 [27] there exists a prime element $\pi \in K$ such that $P = \pi R = R\pi$, $R/P$ is a finite field $F_p^n$ consisting of $p^n$ elements for some $n \in N$ [27], $mod_K(\pi) := q^{-1}$ and $\Gamma_K := mod(K)$, where $mod_K$ is the modular function of $K$ associated with the nonnegative Haar measure $\mu$ on $K$ such that $\mu(xS) = mod_K(x)\mu(S)$ for each $0 \neq x \in K$, $mod_K(0) := 0$ and each Borel subset $S$ in $K$ with $\mu(S) < \infty$, $P := \{x \in K : |x| < 1\}$, $R := B(K, 0, 1)$. Then each $x \in K$ can be written in the form $x = \sum_{i=1}^n x_i\pi^i$, where $x_i \in \{0, \theta_1, \ldots, \theta_{p^n-1}\}$, $\min_{x_i \neq 0} l := -ord_K(x) > -\infty$, $\theta_0 = P, \theta_1 = P, \ldots, \theta_{p^n-1} = P$ is
the disjoint covering of $R$, $\theta_0 := 0$. Consider in $K$ the linear ordering $a \triangle b$ if $a_k = b_k, \ldots, a_s = b_s, a_{s+1} < b_{s+1}$, where $a, b \in K$, by our definition $\theta_s < \theta_e$ for each $s < v$, $k := \min(\text{ord}_K(a), \text{ord}_K(b))$. In $B(K, 0, 1)$ the largest element relative to such linear ordering is $\beta := \sum_{l=0}^{\infty} \theta_{(p-1)} \pi^l = \theta_{(p-1)}/(1 - \pi)$.

Though this linear ordering is preserved neither by additive nor by multiplicative structures of $K$ it is useful (see, for example, [20] and §62 [22]).

2.2.2. Let $v_0, \ldots, v_k \in K(\alpha)^m$ such that vectors $v_1 - v_0, \ldots, v_k - v_0$ are $K$-linearly independent, then the subset $s := [v_0, \ldots, v_k] := \{z \in K(\alpha)^m : z = a_0v_0 + \ldots + a_kv_k; a_0 + \ldots + a_k = 1; a_0, \ldots, a_k \in B(K, 0, 1)\}$ is called the simplex of dimension $k$ over $K$, $k = \dim_K s$. A polyhedron $P$ is by our definition the union of a locally finite family $\Psi_P$ of simplexes. For compact $P$ a family $\Psi_P$ can be chosen finite. An oriented $k$-dimensional simplex is a simplex together with a class of linear orderings of its vertices $v_0, \ldots, v_k$.

Two linear orderings are equivalent if they differ on an even transposition of vertices. For a simplicial complex $S$ let $C_q(S)$ be an Abelian group generated by simplices $s^q$ of dimension $q$ over $K$ and relations $s^q_1 + s^q_2 = 0$, if $s^q_1$ and $s^q_2$ are differently oriented simplexes (see the real case in Chapter 4 [26]). Then there exists the homomorphism $\partial_q : C_q(S) \to C_{q-1}(S)$ such that $\partial_q[v_0, \ldots, v_q] := \sum_{i=0}^{q}(-1)^i[v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_q]$ and $\partial_q[v_0, \ldots, v_q]$ is called the oriented $K$-boundary of $s^q$.

2.2.3. A clopen compact subset $\Omega$ in $(K + \alpha K)^m$ is totally disconnected and its topological boundary is empty. Nevertheless, using the following affine construction it is possible to introduce convention about certain curves and boundaries which will serve for the antidierivation operators.

Let $\Omega$ be a locally $K$-convex subset in $K(\alpha)^m$ for which there exists a sequence $\Omega_n$ of polyhedra with $\Omega_n \subset \Omega_{n+1}$ for each $n \in \mathbb{N}$, $\Omega = c.l.(\bigcup_n \Omega_n)$, where $c.l.(S)$ denotes the closure of a subset $S$ in $K(\alpha)^m$. Suppose each $\Omega_n$ is the union of simplices $s_{j,n}$ with vertices $v^j_{0,n}, \ldots, v^j_{k,n}$, $j = 1, \ldots, n(b(n) \in \mathbb{N}$, moreover, $dim_K(s_{j,n} \cap s_{j',n}) < k$ for each $j \neq j'$ and each $n$, where $k > 0$ is fixed. Then define the oriented $K$-border $\partial \Omega_n := \sum_{j,l}(-1)^l[v^j_{0,n}, \ldots, v^j_{l-1,n}, v^{j+1}_{l+1,n}, \ldots, v^j_{k,n}]$.

Consider $\Omega_n$ for each $n$ such that if $dim_K(s_{j,n} \cap s_{j',n}) = k - 1$ for some $j \neq j'$, then $s_{j,n} \cap s_{j',n} = [v^j_{0,n}, \ldots, v^j_{l-1,n}, v^j_{l+1,n}, \ldots, v^j_{k,n}] = [v^{j'}_{0,n}, \ldots, v^{j'}_{l-1,n}, v^{j'}_{l+1,n}, \ldots, v^{j'}_{k,n}]$ and $(l - l')$ is odd. For each $n$ choose a set of vertices generating $\Omega_n$ of minimal cardinality and such that the sequence $\{\partial \Omega_n : n\}$ converges relative to the distance function $d(S, B) := \max(\sup_{x \in S} \rho(x, B), \sup_{b \in B} \rho(b, S))$, where $\rho(x, B) := \inf_{b \in B} \rho(x, b)$ and $\rho(x, b) := |x - b|$. Then by our definition
\[ \partial \Omega := \lim_{n \to \infty} \partial \Omega_n. \]

Evidently, each clopen compact subset \( \Omega \) has such decomposition into simplices and the described \( \partial \Omega \), since \( \Omega \) is the finite union of balls, but for two balls \( B_1 \) and \( B_2 \) in \( (K \oplus \alpha K)^m \) either \( B_1 \subset B_2 \) or \( B_2 \subset B_1 \) or \( B_1 \cap B_2 = \emptyset \) due to the ultrametric inequality and each ball \( B \) has such decomposition into simplices as described above.

**2.2.4.** We say that a subset \( \Omega \) in \( A(K \oplus \alpha K)^m \) encompasses a point \( z \) if \( z \in \Omega. \)

For the unit ball relative to the metric \( \rho(z_1, z_2) := |z_1 - z_2| \) let its non-Archimedean canonical oriented \( K \)-border \( \partial_c B_\rho(K \oplus \alpha K, 0, 1) \) be given by the set \( \{(-\beta, -\beta), (\beta, -\beta)] \cup [(\beta, \beta), (-\beta, \beta)] \cup [(-\beta, \beta), (-\beta, -\beta)] \cup [0, 1) \} \) for each \( a, b \in K \oplus \alpha K. \) Then \( \partial_c B((K \oplus \alpha K)^m, 0, 1) := \bigcup_{l=1}^m B((K \oplus \alpha K, 0, 1)^l \times \partial_c B((K \oplus \alpha K, 0, 1)^l \times B((K \oplus \alpha K)^m, z, q^k) := z + \pi^{-k} \partial_c B((K \oplus \alpha K)^m, 0, 1) \). This is the particular case of §2.2.

A continuous mapping \( \gamma : B(K, 0, 1) \to A(K(\alpha))^m \) is called a path. We say that \( \gamma \) encompass a point \( z \in A(K(\alpha))^m \), if

1. \( z \in \Omega, \) where \( \partial \Omega = \gamma, \dim_K \Omega = 2, \)
2. \( z \notin \gamma(B(K, 0, 1)), \)
3. \( |z| < \max_{\theta \in B(K, 0, 1)} |\gamma(\theta)| \) for \( z \neq A, \sup_{\theta \in B(K, 0, 1)} |\gamma(\theta)| < \infty \) for \( z = A. \)

A path \( \gamma \) we call a locally affine, if there exists a finite partition \( Z \) of \( \gamma(B(K, 0, 1)) \) such that \( \gamma = \bigcup_{l=1}^m \tau_l, \) where \( \tau_l := [z_{l-1}, z_l] \) for each \( l = 1, \ldots, n. \) We consider the family \( \mathcal{F}_a \) of all paths \( \gamma \) for which there exists a sequence \( \{\gamma_n : n \} \subset \mathcal{F}_a \) converging relative to the distance function \( d'(S, B) := \max(\sup_{x \in S} \rho'(x, B), \sup_{b \in B} \rho'(b, S)) \) to \( \gamma \) in \( A(K(\alpha))^m, \rho' \) and such that there there exists a homeomorphism \( \nu \) of \( \gamma(B(K, 0, 1)) \) with \( B(K, 0, 1) \) and \( \nu \) is a piecewise \( pC^{q+1} \)-diffeomorphism with it, where \( \mathcal{F}_a \) denotes the family of all locally affine paths, \( q \in N. \) In addition we take \( \Omega \) and \( \gamma \) such that \( \gamma = \partial \Omega \) in accordance with §2.2.

Since \( A(K(\alpha))^m \) and \( A(K \oplus \alpha K)^m \) are compact, then a clopen compact set \( \Omega \) in \( A(K(\alpha))^m \) or in \( A(K \oplus \alpha K)^m \) is homeomorphic with a clopen compact subset \( \kappa(\Omega) \) in \( K(\alpha)^m \) or \( (K \oplus \alpha K)^m \) respectively (see Theorem 6.2.16 and Corollary 6.2.17 about universality of the Cantor cube for zero dimensional spaces [5]), where \( \kappa : \Omega \to \kappa(\Omega) \) is the homeomorphism. Therefore, we can consider \( \Omega P^n, \partial \Omega \) and \( \partial_\Omega P^n \) induced by \( \kappa \) of such sets \( \Omega \) also.

**2.2.5.** Let \( M \) be a \( C^{q+1}\)-manifold of dimension \( k \) over \( K \) such that
\begin{align*}
\xi &= (q, n-1), \text{ where spaces } C^\xi(K^a, K^b) := C(\xi, K^a \to K^b) \text{ and } C^\xi\text{-manifolds and uniform spaces } C^\xi(M, N) \text{ of all } C^\xi\text{-mappings } f : M \to N \text{ were defined in §I.2.4 [12], } 0 \leq q \in \mathbb{Z}, 0 < n \in \mathbb{Z}, \quad P_{l_{0}}^{\xi+(0,1)}(\Omega, L^b) := P_{l_{0}}^{\xi}(C^\xi(\Omega, L^b)) \quad \text{and} \quad P_{l_{0}}^{\xi+(0,1)}(\Omega, L^b) := P_{l_{0}}^{\xi+(0,1)}(\Omega, L^b) \oplus L^b \text{ was described in Lemma 3.4 [14]. Suppose that charts } (V_j, \phi_j) \text{ of the atlas } At(M) \text{ of } M \text{ are such that } V_j \text{ are clopen in } M, \quad \cup_j V_j = M, \phi_j : V_j \to \phi_j(V_j) \subset U^k \text{ are homeomorphisms on clopen subsets in } U^k, \text{ where } \phi_{i,j} := \phi_i \circ \phi_j^{-1} \in P_{\phi_j(V_j)} C^\xi(W_{i,j}, K^k) \text{ for each } i \neq j \text{ with } U_i \cap U_j \neq \emptyset \text{ and each coordinate } x_t \text{ induced from } K, \quad l = 1, \ldots, k, W_{i,j} := \text{dom}(\phi_{i,j}), \quad P_{\phi_j(V_j)} C^\xi(W_{i,j}, K^k) \text{ define paralleliped } \phi_{i,j} \text{ for } k \text{ differential } C^{0,n-1}\text{-form } \omega \text{ on } \phi(\tau) \text{ define}
\phi(\tau) P^n \omega := \tau P^n \phi^* \omega, \quad \text{where } \phi^* \omega \text{ is the pull back of } \omega \text{ such that}
\phi(\tau) P^n \omega = 0 \text{ for } dim K^k \neq k,
\text{since } \omega = 0 \text{ for } k > dim K^k M. \text{ Without loss of generality take } 0 \in U \text{ and } \sigma_0(0) = 0, \text{ then } \sigma_l(0) = 0 \text{ for each } l \in \mathbb{N}, \text{ consequently, } \quad \tau P^n |_{\{0\}} = 0. \text{ Therefore, } \quad \tau P^n |_{\{U^m \cap K^k \times \{0\}^{n-k}}} \omega = 0 \text{ for } k < dim K^k \Omega = m. \text{ Each such paralleliped is the finite union of simplices satisfying conditions of §2.2.3. The orientation of } \partial \tau \text{ is induced by the orientations of constituting its simplices which are consistent. Consider such parallelipedes } \tau_{j,q,l} \text{ with } l = 1, \ldots, b(q) \in \mathbb{N} \text{ and}
\begin{align*}
(ii) & \quad \dim K^k(\tau_{j,q,l} \cap \tau_{j,q,l'}) < k \text{ for each } l \neq l' \text{ and} \\
(iii) & \quad d(\cup_{q} \kappa_{j,q}) = \phi_j(V_j), \text{ where} \\
(iv) & \quad \cup_{l=1}^{b(q)} \tau_{j,q,l} =: \kappa_{j,q}, \\
(v) & \quad \lim_{q \to \infty} \max_l d(\tau_{j,q,l}) = 0.
\end{align*}
\end{align*}
Since $\tau_{j,q,l}P^n v + \tau_{j,q,l}P^n v = \tau_{j,q,l}\tau_{j,q,l}P^n v$ for each differential $C^{(0,n-1)} - k$-form $v$ with support in $U^k$ and each $l \neq l'$ and $U_k P^n$ is the continuous operator from $C^{(0,n)}(U^k, L)$ to $C^{(0,n)}(U^k, L)$ then there exists

\[(vi) \lim_{q \to \infty} \sum_{l=1}^{b(q)} \tau_{j,q,l}P^n v =: \phi_j(V_j)P^n v.\]

Using transition mappings $\phi_{i,j}$ and considering clopen disjoint covering

\[(vii) \ W_j := V_j \setminus \bigcup_{i=1}^{j-1} V_j \text{ of } M \text{ we get} \]

\[(viii) \ M P^n w = \sum_j w_j P^n \text{ independent on the choice of local coordinates in } M.\]

Mention, that since $|\beta| = 1$, then $B(K^l, z, r)$ can be represented as the parallelepiped with the described above $K$-boundary $\partial_k B(K^l, z, r)$ due the ultrametric inequality. Due to $(vi - viii)$ there is defined $\gamma P^n v$ for locally affine path $\gamma$, which is the $pC^n$-manifold, that will be supposed henceforth.

Each compact manifold $M$ has a finite dimension over $K$ and using $W_j$ we get an embedding into $K^b$ for some $b \in K$. Let $\phi : \Omega \to M$ be such that $\phi$ is surjective and bijective, $\phi$ and $\phi^{-1} \in sC^{(1,0)}$, which means that $\phi_j \circ \phi \in sC^{(1,0)}(\phi^{-1}(V_j), k^k)$ and $\phi^{-1} \circ \phi_j^{-1} \in sC^{(1,0)}(\phi_j(V_j), k^k)$ for each $j$, where $\phi^{-1}(M) = \Omega \subset U^k$ satisfies conditions of §2.2.3. Such $\phi$ we call the $sC^{(1,0)}$-diffeomorphism. Then $M$ is oriented together with $\Omega$. Then $\partial M := \phi(\partial \Omega)$ is the oriented boundary. We also can consider the analytic manifold $M$ and the analytic diffeomorphism $\phi$. Each compact $C^\xi$-manifold $M$ can be supplied with the analytic manifold structure using a disjoint covering refined into $At(M)$.

**2.2.6. Theorem.** Let $M$ be a compact $sC^\xi$ or $pC^\xi$-manifold over the local field $K$ with dimension $\dim_K M = k$ and an atlas $At(M) = \{(V_j, \phi_j) : j = 1, \ldots, n\}$, where $\xi = (q, n)$, $1 \leq q \in N$, $0 \leq n \in Z$, then there exists a $sC^\xi$ or $pC^\xi$-embedding of $M$ into $K^{nk}$ respectively.

**Proof.** Let $(V_j, \phi_j)$ be the chart of the atlas $At(M)$, where $V_j$ is clopen in $M$, hence $M \setminus V_j$ is clopen in $M$. Therefore, there exists a $sC^\xi$ or $pC^\xi$-mapping $\psi_j$ of $M$ into $K^k$ such that $\psi_j(M \setminus V_j) = \{x_j\}$ is the singleton and $\psi_j : V_j \to \psi_j(V_j)$ is the $sC^\xi$ or $pC^\xi$-diffeomorphism onto the clopen subset $\psi_j(V_j)$ in $K^k$ correspondingly, $x_j \in K^k \setminus \psi_j(V_j)$, since the operator $M P^n$ is $K$-linear, $M P^n 0 = 0$ and the covering $\{V_j : j\}$ of $M$ has a disjoint finite refinement $\{W_k : k\}$ such that $P^n_{x_l} f = P^n_{x_l} \sum_k f \chi_{W_k} = \sum_k P^n_{x_l} f \chi_{W_k}$ for each $f \in C^{(q,n)}(M, K)$ and each coordinate $x_l$ (see §2.1 and §2.2.5). Then the mapping $\psi(z) := (\psi_1(z), \ldots, \psi_n(z))$ is the embedding into $K^{nk}$, since the rank $\text{rank}[d_z \psi(z)] = k$ at each point $z \in M$, because $\text{rank}[d_z \psi_j(z)] = k$ for
each $z \in V_j$ and $\dim_K \psi(V_j) \leq \dim_K M = k$. Moreover, $\psi(z) \neq \psi(y)$ for each $z \neq y \in V_j$, since $\psi_j(z) \neq \psi_j(y)$. If $z \in V_j$ and $y \in M \setminus V_j$, then there exists $l \neq j$ such that $y \in V_l \setminus V_j$, $\psi_j(z) \neq \psi_j(y) = x_j$.

2.3.1. Theorem. Let $M$ be a compact oriented manifold over $K$ of dimension $\dim_K M = k > 0$ with an oriented boundary $\partial M$ and let $w$ be a differential $(k - 1)$-form as in §2.2.5 such that its pull back $\phi^* w$ is a differential $(k - 1)$ $sC^{(1,n-1)}$-form, then

\[ \int_M P^m dw = \int_{\partial M} P^n w. \]

Proof. Since $M$ is the manifold of $\dim_K M = k > 0$, then $M$ is dense in itself and compact, hence $\Omega$ is dense in itself and compact (see Chapter 1 and Theorems 3.1.2, 3.1.10 [5]) and the approximation of the identity can be applied to $\Omega$. In view of Formulas 2.1.(1 - 4) and 2.5.(1, 2) on the space of $C^\xi$ differential forms operators $\mathcal{U}_q$ and $\mathcal{U}_x$, commute for each $1 \leq q, s \leq k$. Then

(i) $\mathcal{U}_q f^b_a = - \mathcal{U}_q f^b |_a$, where $\mathcal{U}_q f^b |_a := \mathcal{U}_q f(b) - \mathcal{U}_q f(a)$. Using conditions imposed on the manifold $M$, partitions of $\Omega_n$ into unions of parallelepipeds, which are finite unions of simplices as in §2, Formula (i) and 2.5.(1), also using the limit 2.5.(vi) and Formula 2.5.(viii), it is sufficient to verify Formula (1) for a parallelepiped and an arbitrary term\n
$\psi := f(z)dz_1 \wedge \ldots \wedge dz_q - 1 \wedge dz_k$ corresponding to the differential $(k - 1)$ $sC^{(1,n-1)}$-form $\phi^* w$. Consider in $K$ the standard orthonormal base $e_1, \ldots, e_k$, where $e_l := (0, \ldots, 0, 1, 0, \ldots, 0)$ is the vector with 1 in $l$-th place. Without loss of generality using limits we can take the parallelepipeds $\tau = [v_0, v_1] \times \cdots \times [v_{k-1}, v_k]$ with $v_l - v_{l-1} = \lambda_l e_l$ for each $l = 1, \ldots, k$, where $0 \neq \lambda_l \in K$. Therefore, $df(z) = (-1)^{q-1}(\partial f(z)/\partial z_q)dz_1 \wedge \ldots \wedge dz_k$. Since $f \in sC^{(1,n-1)}$, then $\mathcal{U}_q f^b(\partial f(z)/\partial z_q) |_a = f(z_1, \ldots, z_{q-1}, b, z_{q+1}, \ldots, z_k) - f(z_1, \ldots, z_{q-1}, a, z_{q+1}, \ldots, z_k)$ for each $l = 1, \ldots, k$. Consequently,

(ii) $\mathcal{U}_q f^b |_a = (-1)^{q-1}[v_0, v_1] \times \cdots \times [v_{q-2}, v_{q-1}] \times [v_{q+1}, v_{q+2}] \times \cdots \times [v_{k-1}, v_k] P^n dz_1 \wedge \ldots \wedge dz_{q-1} \wedge dz_{q+1} \wedge \ldots \wedge dz_k$.

$\int_{\tau} \mathcal{U}_q f^b |_a = (-1)^{q-1} [v_0, v_1] \times \cdots \times [v_{q-2}, v_{q-1}] \times [v_{q+1}, v_{q+2}] \times \cdots \times [v_{k-1}, v_k] P^n \{ f(z_1, \ldots, z_{q-1}, v_q, dz_{q+1}, \ldots, dz_k) - f(z_1, \ldots, z_{q-1}, v_{q-1}, dz_{q+1}, \ldots, dz_k) \} dz_1 \wedge \ldots \wedge dz_{q-1} \wedge dz_{q+1} \wedge \ldots \wedge dz_k$

for each $q = 1, \ldots, k$. In view of 2.5.(2) antiderivations of $\psi$ by others pieces $(-1)^{q-1} [v_0, v_1] \times \ldots [v_{q-2}, v_{q-1}] \times \{ \{ v_s \} - \{ v_{s-1} \} \} \times [v_s, v_{s+1}] \times \ldots [v_{k-1}, v_k]$ corresponding to $s \neq q$ of the $K$-border are zero.

2.3.2. Corollary. Let $M$ be a compact oriented manifold over $K$ of dimension $\dim_K M = k > 0$ with an oriented boundary $\partial M$ and let $w$ be a differential $(k - 1)$ $C^{(1,n-1)}$-form as in §2.2.5 such that its pull back $\phi^* w = \ldots$
\[ \sum_{j_1, \ldots, j_{k-1}} f_{j_1, \ldots, j_{k-1}} dz_{j_1} \wedge \ldots \wedge dz_{j_{k-1}} \text{ has } f_{j_1, \ldots, j_{k-1}} \text{ in } p_{z_j} C^n(U, L) \text{ by the variable } z_j \text{ for each } j \text{ such that } j \in \{1, \ldots, k\} \setminus \{j_1, \ldots, j_{k-1}\}, \text{ then} \]

\[ \text{(1)} \quad M^P d\omega = \partial M^P \omega. \]

**Proof.** Repeating the proof of Theorem 3.1 for each term \( f_{j_1, \ldots, j_{k-1}} d\omega \wedge \ldots \wedge d\omega \) of \( \omega \) and applying Formulas 3.1.(i, ii) we get the statement of this corollary.

### 2.4.1. Remarks and Notations.

Let \( f \in C^1(K(\alpha), Y) \), where \( Y \) is a Banach space over \( L, L \) is a field containing \( K(\alpha) \) such that \( L \) is complete relative to its uniformity, the multiplicative norm in \( L \) is the extension of the multiplicative norm in \( K(\alpha) \). As the Banach space \( K(\alpha) \) over \( K \) is isomorphic with \( K^r \), where \( 2 \leq r \in \mathbb{N} \). Consider such structure over \( K \).

Then each \( \zeta \in K(\alpha) \) we write in the form \( \zeta = x + \alpha y \), where \( x \in K, y \in K^{r-1} \). Denote by \( \bar{\zeta} := x - \alpha y \) the so called conjugate element to \( \zeta \). Then \( x = (\zeta + \bar{\zeta})/2 \) and \( y = (\zeta - \bar{\zeta})/(2\alpha) \). Therefore,

\[
\begin{align*}
(i) \quad & \partial f(\zeta, \bar{\zeta})/\partial x = \partial f(\zeta, \bar{\zeta})/\partial \zeta + \alpha \partial f(\zeta, \bar{\zeta})/\partial \bar{\zeta} \quad \text{and} \\
(ii) \quad & \partial f(\zeta, \bar{\zeta})/\partial y = \alpha \partial f(\zeta, \bar{\zeta})/\partial \zeta - \alpha \partial f(\zeta, \bar{\zeta})/\partial \bar{\zeta}, \text{ consequently,} \\
(iii) \quad & \partial f(\zeta, \bar{\zeta})/\partial \zeta = [\partial f(\zeta, \bar{\zeta})/\partial x + \alpha^{-1} \partial f(\zeta, \bar{\zeta})/\partial y]/2 \quad \text{and} \\
(iv) \quad & \partial f(\zeta, \bar{\zeta})/\partial \bar{\zeta} = [\partial f(\zeta, \bar{\zeta})/\partial x - \alpha^{-1} \partial f(\zeta, \bar{\zeta})/\partial y]/2.
\end{align*}
\]

In particular, the external differentiation of differential \( C^1 \)-forms \( \omega \) on a clopen subset \( \Omega \) in \( (K \oplus \alpha K)^m \) has the form

\[
\begin{align*}
(v) \quad & \text{d} \omega = \partial \omega + \bar{\partial} \omega, \quad \text{where} \\
(vi) \quad & \omega = \sum_{I,J} w_{I,J}(\zeta, \bar{\zeta}) d\zeta^I \wedge d\bar{\zeta}^J, \\
(vii) \quad & \partial \omega = \sum_{I,J} \partial w_{I,J}(\zeta, \bar{\zeta}) d\zeta^I \wedge d\zeta^J, \\
(viii) \quad & \bar{\partial} \omega = (-1)^{|I|} \sum_{I,J} \partial w_{I,J}(\zeta, \bar{\zeta}) d\bar{\zeta}^I \wedge d\bar{\zeta}^J, \text{ where } d\zeta^I \equiv d\zeta_{I_1} \wedge \ldots \wedge d\zeta_{I_p}, d\bar{\zeta}^J \equiv d\bar{\zeta}_{J_1} \wedge \ldots \wedge d\bar{\zeta}_{J_q}, \quad 1 \leq I_1 < \ldots < I_p \leq m, \quad 1 \leq J_1 < \ldots < J_q \leq m, \\
\text{such that } \omega \text{ is the } (b, c) \text{-form with coefficients } w_{I,J} \in C^1(\Omega, Y), |I| := b.
\end{align*}
\]

If \( r > 2 \), then the differential s-form \( \omega \) can be written as

\[
\begin{align*}
(ix) \quad & \omega = \sum_{I,J,l=s} w_{I,J} d\zeta^I \wedge \ldots \wedge d\zeta^J, \quad z = (z_1, \ldots, z_m), \quad z_l \in K \text{ for each } l = 1, \ldots, rm, \\
dz^I := dz_{I_1} \wedge \ldots \wedge dz_{I_p}, \quad 1 \leq I_1 < \ldots < I_p \leq rm. \quad \text{Let } \Lambda(K(\alpha)^m) \text{ denote the Grassmann algebra (exterior algebra) of } K(\alpha)^m, \text{ where } K(\alpha) \text{ is considered as a } K \text{-linear space, } \Lambda(K(\alpha)^m) = \bigoplus_{l=0}^{rm} \Lambda^l(K(\alpha)^m). \text{ Then } \omega \in C^x(\Omega, L(\Lambda(K(\alpha)^m), Y)) \text{ is the differential form, since the space } (K(\alpha)^m)^* \text{ of } K \text{-linear functionals on } K(\alpha)^m \text{ is the space isomorphic with } K(\alpha)^m \text{ due to discreteness of } \Gamma_K, \text{ where } L(\Lambda(K(\alpha)^m), Y) \text{ is the Banach space of } K \text{-linear operators from } \Lambda(K(\alpha)^m) \text{ into } Y. \\
\text{Consider } \omega \text{ such that } \omega \subset E, \text{ where } E := \{z \in K(\alpha) : |z| < p^{1/(1-p)}\}, \text{ since exp is the bijective analytic function on } E, \text{ therefore, we put}
\end{align*}
\]
(x) \ \exp(\omega) = \Omega, \text{ that is, } \omega = \log(\Omega) \text{ for } \Omega \subset 1 + E \text{ (see §§25 and 44 [22]). Henceforth, if on a manifold } M \text{ there will be considered functions } f \text{ having the property } \bar{\partial}f = 0, \text{ then it will be supposed that } \bar{\partial}\phi_{i,j} = 0 \text{ for each transition mapping } \phi_{i,j}, \text{ if another will not be specified.}

Consider an extension of \log. Denote by \( C_p^+: = \{ z \in C_p : |z - 1| < 1 \} \) and \( K(\alpha)^+: = K(\alpha) \cap C_p^+ \). Then \( K(\alpha)^+ \) is the Abelian subgroup in the additive group \( C_p^+ \) and \( C_p^\times := C_p \setminus \{0\} \) is the Abelian multiplicative group. The group \( C_p^\times \) is divisible, that is, for each \( y \in C_p^\times \) and each \( n \in \mathbb{N} \) there exists \( x \in C_p^\times \) such that \( x^n = y \). Let \( X \) be a proper divisible subgroup in \( C_p^\times \) such that \( C_p^\times \subset X \). Let \( G \) be a subgroup generated by \( X \) and \( y \in C_p^\times \setminus X \). Suppose \( y^n \notin X \) for each \( n \in \mathbb{N} \), then for each \( g \in G \) there exist unique \( n \in \mathbb{Z} \) and \( x \in X \) such that \( g = y^n x \). Choose \( z \in C_p \), then put \( \log(g) := nz + \log(x) \). The second possibility is: \( y^n \in X \) for some \( n \in \mathbb{N}, \ n > 1 \). For each \( g \in G \) there exist unique \( n \in \{0, 1, ..., m - 1\} \) and \( x \in X \) such that \( g = y^n x \), where \( m := \min_{y^n \in X, n \in \mathbb{N}} n \). Since \( C_p \) is divisible, there exists \( z \in C_p \) such that \( z_m = \log(y_m) \), therefore, define \( \log(g) := nz + \log(x) \). Using the Zorn’s Lemma we can extend \log from \( C_p^+ \) to \( C_p^\times \). In particular we can consider values of \( \log(i) \) and \( \log(\alpha) \) using identities \( \log(1) = 0, \ i^4 = 1, \ \alpha^m \in K, \ \alpha^n = 1 \) for some minimal \( n \in \mathbb{N} \). In view of Theorem 45.9 [22] we can choose an infinite family of branches of \( \log \) indexed by \( \mathbb{Z} \). For the convenience put \( \log(0) := A \). From the consideration above it follows, that the extension \( \exp \) of \exp on \( C_p \) and the extension \( \log \) of \log on \( C_p \setminus \{0\} \) can be chosen such that directed going (defined by going from 0 to \( \beta \) in linearly ordered \( B(K, 0, 1) \), see §2.2) by the oriented loop \( \partial_c B(K, 0, p^{-2}) \) changes a branch \( n \log \) of \log in the following manner: \( n+1 \log(x) = n \log(x) \). This is possible, since \( C_p \) and \( C \) are isomorphic fields [10], also points \( p^2(1, -1), \ p^2(1, 1) \) and \( p^2(1, -1) \) belong to \( \partial_c B(K, 0, p^{-2}) \).}

2.4.2. Theorem. Let \( M \) be a compact \( sC^{(q,n)} \)-manifold over \( K \) satisfying conditions of §2.2.5 and §2.4.1 for which \( \phi^{-1}(M) = \Omega \subset K(\alpha) \) with a \( K \)-boundary \( \gamma := \partial M, \dim_K M = 2, 2 \leq r \in \mathbb{N}, 0 \leq q \in \mathbb{Z}, 1 \leq n \in \mathbb{N} \), then there exists a constant \( 0 \neq C := C_n(\alpha) \in K(\alpha) \), such that

\[
\begin{align*}
(1) \quad f(z) &= C^{-1} \partial M P^n \{ f(\zeta)(\zeta - z)^{-1}d\zeta \} - C^{-1} M P^n \{ (\bar{\partial} f \wedge d\zeta )/(\zeta - z) \} \\
(2) \quad f_1(z + \exp(\eta)) &=: \psi(\eta) \in sC^{(1,n-1)}(\omega, Y) \text{ and each marked } z \in M \text{ encompassed by } \gamma,
\end{align*}
\]
for each $\epsilon = \epsilon_j, 0 < \epsilon_j$ for each $j \in \mathbb{N}, \{\epsilon_j : j\}$ is a sequence in $\Gamma_K$ with $\lim_{j \to \infty} \epsilon_j = 0$, $f_1 := f \circ \phi$, where $\omega := \omega(z) := \{\eta \in K(\alpha) : z + \exp(\eta) \in \Omega\}, \omega_i := \omega \setminus \log(B(K(\alpha), z, \epsilon))$, $z \in \Omega$. Moreover, $C_n(\alpha) = C_1(\alpha) = \delta$ for each $n \in \mathbb{N}$.

**Proof.** Using the $C^{(n,m)}$ diffeomorphism $\phi$ reduce the proof to the case of $f$ on $\Omega$. Consider the differential form $\omega := f(\zeta)(\zeta - z)^{-1}d\zeta$ on $\Omega \setminus \{z\}$, then $d\omega = - (\zeta - z)^{-1} (\partial f / \partial \zeta) d\zeta \wedge d\zeta$. Let $s \in \mathbb{Z}$ be such that $\inf_{\zeta \in \partial \Omega} |\zeta - z| = |\pi|^s$. Take the change of variables $\zeta = z + \exp(\eta)$, hence $(\zeta - z)^{-1}d\zeta = d\eta$; also take $l > s$, then from Corollary 2.3.2 and $\S 2.2.5$, 2.4.1 it follows

(i) $\Omega \cap B(K(\alpha), z, |\pi|^l) \cap P^n d\omega = \partial \Omega \cap P^n w - \partial \Omega \cap B(K(\alpha), z, |\pi|^l) \cap P^n w,$

since $f_1(z + \exp(\eta)) = \psi(\eta) \in C^{(1,n-1)}(\omega_\epsilon, Y)$ and from $\psi \in C^{(1,n-1)}(\omega_\epsilon, Y)$ it follows $\psi(x, y) \in P_{x} C^{(n,0)}(\omega_\epsilon,x, Y)$ and $\psi(x, y) \in P_{y} C^{(n,0)}(\omega_\epsilon,y, Y)$ for each $\epsilon = \epsilon_j$ and for each $x, y$, where $z = (x, y)$, $\omega_\epsilon,x = \pi_x(\omega_\epsilon)$, $\omega_\epsilon,y = \pi_y(\omega_\epsilon)$, $\pi(x) = K \oplus \alpha K \rightarrow K$ and $\pi(x) = K \oplus \alpha K \rightarrow K$ are projections, $Y$ is a Banach space over $L$ such that $K(\alpha) \subset L$. The differential form $\omega$ can be written as $\omega = f(\zeta) d\log(\zeta - z)$. From $\log(xz) = \log(x) + \log(z)$ for each $x, z \in C_p^\infty$ it follows, that directed going by the oriented loop $\partial \Omega \cap B(K(\alpha), 0, |\pi|^l)$ changes a branch $\eta \log(\zeta)$ on $\Omega$ in the following manner: $| \log(x) - \eta \log(x) | = \delta \neq 0$ for each $s \in \mathbb{Z}$. In view of $\S 2.4.1$ there exists

$\lim_{l \to \infty} \partial \Omega \cap B(K(\alpha), z, |\pi|^l) \cap P^n w =: C_n(\alpha) f(z)$. Finally $\Omega \cap P^n((\zeta - z)^{-1} \partial f(\zeta) \wedge d\zeta) = \Omega \cap P^n((\zeta - z)^{-1} (\partial f(\zeta)/\partial \zeta) d\zeta \wedge d\zeta)$, where for short we write $f = f(\zeta) = f(\zeta, \bar{\zeta})$.

In view of Formulas 2.1.1(2, 3) and the non-Archimedean Taylor formula for $C^n$-functions (see Theorem 29.4 [22])

$\partial \Omega \cap B(K(\alpha), z, |\pi|^l) \cap P^n((\zeta - z)^{-1} d\zeta) = \partial \Omega \cap B(K(\alpha), z, |\pi|^l) \cap P^n d\log(\zeta - z) + \epsilon(\pi^l)$

such that there exists a constant $0 < b < \infty$ for which $|\epsilon(\pi^l)| \leq b |\pi|^l$ for each $l \in \mathbb{N}$. On the other hand, due to the Taylor formula for $C^1$-functions and Formulas 2.1.1(2, 3):

$\partial \Omega \cap B(K(\alpha), z, |\pi|^l) \cap P^n d\log(\zeta - z) = \delta + \eta(\pi^l),$

where $\lim_{l \to \infty} \eta(\pi^l) = 0$. Therefore, $C_n(\alpha) = C_1(\alpha) = \delta \neq 0$.

**2.4.3. Corollary.** Let suppositions of Theorem 2.4.2 be satisfied for each $z \in M$ encompassed by $\partial M$, then $\partial f(z)/\partial \bar{z} = 0$ for each $z \in M$ encompassed by $\partial M$ if and only if

(1) $f(z) = C^{-1} \partial M \cap P^n \{f(\zeta)(\zeta - z)^{-1} d\zeta\}$

for each $z \in M$ encompassed by $\partial M$. 

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Proof. If $\partial f/\partial \zeta = 0$ on $M$, then the second term in Formula 2.4.2.(1) is equal to zero, that gives Formula 2.4.3.(1). Vice versa, let Formula 2.4.3.(1) be satisfied for each $z \in M$ encompassed by $\partial M$. Since $(\partial z/\partial \zeta) = 0$, $\partial \zeta/\partial z = 0$, then $\partial((\zeta - z)^{-1})/\partial \zeta = 0$, consequently, $\partial f(\zeta)/\partial \zeta = 0$.

2.4.4. Corollary. Let suppositions of Theorem 2.4.2 be satisfied for each $z \in M$ encompassed by $\partial M$ and $\partial f(\zeta)/\partial \zeta = 0$ for each $z \in M$ encompassed by $\partial M$, then $f$ is locally $z$-analytic in a neighbourhood of each point $z$ in $M$ encompassed by $\partial M$.

Proof. Using the mapping $\phi$ we can consider $\Omega$ instead of $M$. Let $z \in \Omega$ and $B(K \oplus \alpha K, z, R) \subset \Omega$ such that $0 < R < \inf\{|z - y|: y \in \partial \Omega\}$. Consider $x \in B(K \oplus \alpha K, z, R/p)$, then

$(\zeta - x)^{-1} = (\zeta - z + z - x)^{-1} = (\zeta - z)^{-1} \sum_{l=0}^{\infty} (x - z)^l/(\zeta - z)^l \in K(\alpha)$,

where $\zeta \in B(K \oplus \alpha K, z, R)$. Applying Formula 2.4.3.(1) we get:

(i) $f(x) = C^{-1} \partial_z \partial^n ((\zeta - x)^{-1} f(\zeta) d\zeta) = C^{-1} \sum_{l=0}^{\infty} (x - z)^l \partial_z \partial^n ((\zeta - z)^{-l-1} f(\zeta) d\zeta)$

for each $x \in B(K \oplus \alpha K, z, R/p)$, since

(ii) $|\partial_z \partial^n ((\zeta - z)^{-l-1} f(\zeta) d\zeta)| \leq \|f\|_{C^{n-1}(\partial_z \partial^n K(\alpha))} \max_{j,s=0,...,n-1}(R^{j-l-s}/(j+1)!)$

and the series is uniformly converging on $B(K \oplus \alpha K, z, R/p)$, where $B = B(K \oplus \alpha K, z, R)$, $K \oplus \alpha K \subset K(\alpha)$, hence $f(x)$ is locally $x$-analytic.

2.4.5. Definition. Let $\Omega$ be as in §2.2.3. Two paths $\gamma_0 : B(K, 0, 1) \to \Omega$ and $\gamma_1 : B(K, 0, 1) \to \Omega$ with common ends $\gamma_0(0) = \gamma_1(0) = a$, $\gamma_0(\beta) = \gamma_1(\beta) = b$ are called affine homotopic in $\Omega$, if there exists a continuous mapping $\gamma(x; y) : B(K, 0, 1)^2 \to \Omega$ such that

(i) $\gamma(0, y) = \gamma_0(y)$, $\gamma(\beta, y) = \gamma_1(y)$ for each $y \in B(K, 0, 1)$,

(ii) $\gamma(x, 0) = a$, $\gamma(x, \beta) = b$ for each $x \in B(K, 0, 1)$,

(iii) there exists a sequence $\{\gamma_n(x, y) : n \in \mathbb{N}\}$ of continuous mappings, $\gamma_n : B(K, 0, 1)^2 \to \Omega$ such that each $\gamma_n$ is locally affine and $\{\gamma_n : n\}$ converges uniformly to $\gamma$ on $B(K, 0, 1)^2$, where $\gamma_n(x, y) = (1 - x/\beta)\gamma_n(0, y) + x\gamma_n(\beta, y)/\beta$ for each $x \in B(K, 0, 1)$, $\gamma_n(0, y)$ and $\gamma_n(\beta, y)$ are locally affine (see §2.2.4). In particular, for $a = b$ this produces the definition of affine homotopic loops. We call $\Omega$ (or $M$) affine homotopic to a point, if $\partial \Omega$ (or $\partial M$ respectively) is affine homotopic to a point $z$ in $\Omega$ (or $z$ in $M$ correspondingly, see §2.2.5).

2.4.6. Theorem. Let conditions of Theorem 2.4.2 be satisfied for each $z \in M$ and let $M$ be affine homotopic to a point, where $\partial f(\zeta, \bar{z})/\partial \bar{z} = 0$ for each $z \in M$ encompassed by $\partial M$. Then
(1) \( \gamma_0 P^n [fd\zeta] = \gamma_1 P^n [fd\zeta] \)
for each two paths \( \gamma_0 \) and \( \gamma_1 \) which are affine homotopic in \( M \).

**Proof.** Using the diffeomorphism \( \phi \) we can consider \( \Omega \) instead of \( M \). For each \( \epsilon > 0 \) there exists a finite partition of a suitable subset \( \Omega_\epsilon \) into finite union of parallelepipeds of diameter less, than \( \epsilon \) in the proof of Theorem 2.3.1, where \( \Omega_\epsilon \subset \{ z \in \Omega : d(z, \partial \Omega) < \epsilon \} \), \( d(\cup_{\epsilon > 0} \Omega_\epsilon) = \Omega \). In view of Corollary 2.4.3 \( 0 = f(z)(z - \zeta)|_{z = \zeta} = C(\alpha)^{-1} \partial \zeta P^n [f(\zeta)d\zeta] \) for each such parallelepiped \( \xi \). Therefore, there exists a sequence \( \{ \gamma_l : l \} \) of affine homotopy such that \( \gamma_l(0, y) \) and \( \gamma_l(\beta, y) \) are contained in the union \( \bigcup_{\xi \subseteq \Omega} \partial \xi \) for each \( \xi = |\pi|^l \), \( l \in \mathbb{N} \). Since \( \gamma_l(0, s) P^n [fd\zeta] = \gamma_l(\beta, s) P^n [fd\zeta] \) for each \( l \) and taking \( l \) tending to the infinity we get (1) due to continuity of the operator \( P^n \).

**2.4.7. Corollary.** Let \( f \) satisfies conditions of Corollary 2.4.4 with \( \Omega = B(K \oplus \alpha K, z, R) \). Then

(i) \( |f(x)| \leq |C|^{-1} \max_{j,s=0,...,n-1}(\|f(j-s)\|_{C^0(\partial B)}, Y)s! \left( \frac{j}{s} \right) R^{j-s-1}/(j+1)! \)

\[ \leq |C|^{-1} \|f\|_{C^n(\partial B)} \max_{j,s=0,...,n-1}(R^{j-s-1}/(j+1)!). \]

**Proof.** From

\[ \partial^j f(\zeta)(\zeta - x)^{-1} = \sum_{s=0}^j s!(\frac{j}{s}) (-1)^s f(j-s)(\zeta - x)^{-1-s} \]

and \( |\zeta|_l + |\zeta| \leq R \) on \( \partial B \) and Section 2.1 it follows Inequality (i).

**2.4.8. Remark.** The field \( K \) is locally compact, then \( T_q \) is not contained in \( K \), where \( T_q \) is a group of all \( q^n \)-roots \( b \) of the unity: \( b^q = 1 \), \( q \in \mathbb{N} \), \( q \) is the prime number, since \( dim_{Q_p} Q_p T_q = \infty \) for \( Q_p \subset K \) and \( K \) would be nonlocally compact whenever \( T_q \subset K \), which is impossible by the supposition on \( K \). Therefore, there exists \( \min\{ s \in \mathbb{N} : b^s \in K, b \notin K \} \), where \( b \neq 1 \) is the \( q^{s+1} \)-root of the unity \}. Hence there exists \( \zeta \in K \) such that \( \zeta^{1/q} \notin K \). In particular, it is true for \( q = 2 \). Therefore, each local field \( K \) has a quadratic extension \( K(\alpha) \) such that \( \alpha \notin K \). In the particular case \( K = Q_p \) there exists the finite field \( F_p := R/P \) (see Section 2.2.1). Then \( F_p \setminus \{ 0 \} \) is the multiplicative group consisting of \( p - 1 \) elements. If \( p = 4n + 1 \), where \( 1 \leq n \in \mathbb{N} \), then \( Q_p \) contains \( i = (-1)^{1/2} \).

**2.5.1. Lemma.** If \( f \) is locally \( z \)-analytic on \( M \), where \( M \) is a locally compact \( C^{(0,n)} \)-manifold satisfying conditions of Section 2.5.2 and 2.4.1, \( \phi^{-1}(M) = \Omega \subset K(\alpha) \), \( dim_K M = 2, 2 \leq r \in \mathbb{N} \), then \( \partial f(z, \bar{z})/\partial \bar{z} = 0 \) on \( M \).

**Proof.** Using the diffeomorphism \( \phi \) we can consider \( \Omega \) instead of \( M \). Since for each \( z \in \Omega \) there exists \( 0 < R < \infty \) such that \( B := B(K \oplus \alpha K, z, R) \subset \Omega \) and \( f(z, \bar{z}) = \sum_{k=0}^\infty (\zeta - z)^k f_k \) on \( B \), where \( f_k \in Y \), then there exist \( \partial f/\partial z \) and \( \partial f/\partial \bar{z} = 0 \) on \( B \). Since \( z \in \Omega \) is arbitrary and such balls
for each submanifold \( V \) arbitrary, then \( \partial f / \partial \bar{z} = 0 \) on \( \Omega \).

**2.5.2. Remark.** Let \( n \geq 1 \), then \( (d/dz) \Omega P^n = I : C^{n-1}(\Omega, L) \to C^{n-1}(\Omega, L) \). But \( P^n d/dz \neq I \) on \( C^m(\Omega, L) \), where \( P^n d/dz : C^m(\Omega, L) \to C^m(\Omega, L) \). If \( \rho C^n(\Omega, L) \) would be dense in \( C^n(\Omega, L) \), then \( P^n d/dz \) would have the continuous extension \( I \) on \( C^n(\Omega, L) \), since \( P^n d/dz \) is the continuous operator from \( C^n \) into \( C^n \) and \( P^n d/dz \mid \rho C^n_1 = I \). Therefore, \( \rho C^n(\Omega, L) \) is not dense in \( C^n(\Omega, L) \). On the other hand, \( C^1(\Omega, L) = \rho C^n_1(\Omega, L) \oplus N^1 \), where \( N^1 := \{ f \in C^1 : f' = 0 \} \) is the closed \( L \)-linear subspace in \( C^1 \) (see Theorem 5.1 and Corollary 5.5 [23]).

**2.5.3. Theorem.** Let \( f \) be a function on \( M \) over \( K \) satisfying Conditions 2.2.5 and 2.4.1 and \( \gamma \) be a loop in \( M \) satisfying Conditions 2.2.4 and let \( \gamma \) be affine homotopic to a point in \( M \), \( \dim K M = 2 \), \( \phi^{-1}(M) = \Omega \subset K(\alpha) \), \( 2 \leq r \in N \), \( f \) satisfies Condition 2.4.2(2) for each \( z \in M \) and \( \partial f(z, \bar{z}) / \partial \bar{z} = 0 \) on \( M \), then \( \gamma P^n f = 0 \).

**Proof.** Let \( V \) be a submanifold in \( M \) such that \( \partial V = \gamma \). In view of Theorem 2.4.6 \( \gamma P^n [d \zeta] = \gamma f P^n [d \zeta] \), where \( \gamma \) and \( \gamma \epsilon \) are affine homotopic and \( 0 < \text{diam}(\gamma \epsilon) < \epsilon \). In view of continuity of the operator \( P^n \) there exists \( \lim_{n \to \infty} \gamma f P^n [d \zeta] = 0 \).

**2.5.4. Theorem.** If \( f \) satisfies Condition 2.4.2(2), a manifold \( M \) over \( K \) satisfies Conditions 2.2.5 and 2.4.1 and \( M \) is affine homotopic to a point, \( \dim K M = 2 \), \( 2 \leq r \in N \) and \( \gamma P^n f = 0 \) for each loop \( \gamma \) in \( M \) satisfying Conditions 2.2.4, then \( \partial f(z, \bar{z}) / \partial \bar{z} = 0 \) for each \( z \in M \) encompassed by \( \partial M \).

**Proof.** Using the diffeomorphism \( \phi \) we can consider \( \Omega \) instead of \( M \). Choose a marked point \( z_0 \) in \( M \). Let \( \eta \) be a path joining points \( z_0 \) and \( z \) and satisfying Conditions 2.2.4. From \( \eta P^n f = 0 \) it follows, that \( \eta P^n f \) does not depend on \( \eta \) besides points \( z_0 = \eta(0) \) and \( z = \eta(\beta) \), since each two points in \( \Omega \) can be joined by an affine path, hence it is possible to put \( F(z) := \eta_{\eta(0)=z_0; \eta(\beta)=z} P^n f \) such that \( F \) is a function on \( \Omega \). In view of Formulas 2.4.1.(i - iv),

\[ (i) \quad \partial F(z) / \partial z = f(z). \]

In view of theorem 2.4.2

\[ (ii) \quad 0 = \gamma P^n f(\zeta) d\zeta = -C^{-1} \int P^n((\partial f(\zeta, \bar{\zeta}) / \partial \bar{\zeta}) d\zeta \wedge d\bar{\zeta}) \]

for each submanifold \( V \) in \( M \) with the loop \( \gamma = \partial V \), \( \dim K V = 2 \). Since \( V \) is arbitrary, then \( \partial f(z, \bar{z}) = 0 \) at each point \( z \in M \) encompassed by \( \partial M \).

**2.5.5. Corollary.** Let conditions of Theorem 2.5.4 be satisfied, then \( f \) has an antiderivative \( F \) such that \( F' = f \) on \( M \).

**2.6.1. Lemma.** Let \( \Omega \) be a clopen compact subset in \( K^m \), then for
each \( y \in \Omega \) there exists a ball \( B \) such that \( y \in B \subset \Omega \) and \( \mathop{PC}^\xi(\Omega, Y)|_B = \mathop{PC}^\xi(\partial M, Y) \) and \( \mathop{SC}^\xi(\Omega, Y)|_B = \mathop{SC}^\xi(\partial M, Y) \) for each \( \xi \), where \( \mathop{PC}^\xi(\Omega, Y)|_B := \{ g|_B : g \in \mathop{PC}^\xi(\Omega, Y) \} \) and \( \mathop{SC}^\xi(\Omega, Y)|_B := \{ g|_B : g \in \mathop{SC}^\xi(\Omega, Y) \} \), \( Y \) is a Banach space over \( L \), \( \xi = (t, n) \), \( 0 \leq n \in \mathbb{Z}, 0 \leq t \in \mathbb{Z}, 1 \leq n \) for \( \mathop{PC}^\xi \), \( 1 \leq t \) for \( \mathop{SC}^\xi \).

**Proof.** Let \( \sigma \) be an approximation of the unity in \( U \). In view of §2.1 it is sufficient to consider the case \( m = 1 \). Choose \( R = \rho^{s+1/2} \) for sufficiently large \( s \in \mathbb{N} \) such that \( \rho_{(0, R)} \). If \( x \in B(K, y, R) \), then \( \sigma_s(x) = \sigma_s(y) \) due to 2.1.(iii). From Formula 2.1.(ii) it follows, that \( \sigma_i(x) = \sigma_i(y) \) for each \( l < s \). Moreover, \( \sigma_l(x) = x_1 \in B \) for each \( l \geq s \), since \( \rho^{s+1} \) is a Banach space over \( \mathbb{R} \). 

From Formula 2.1.(3) and \( \chi_{\Omega} = \chi_{\Omega|B} \) the statement of this lemma follows.

2.6.2. **Definition.** Let a manifold \( M \) be satifying Conditions 2.4.2, \( f \in C^{(q,n)}(M, Y), 0 \leq q \in \mathbb{Z}, 1 \leq n \in \mathbb{N} \), \( Y \) is a Banach space over \( L \), \( K(\alpha) \subset L \). Then put in the sence of distributions:

\[ (i) \quad M^n(fg) := -M^n(gf) \]

for each \( g \in sC^{(1,n-1)}(M, Y) \) with \( \text{supp}(g) \subset M := \{ z \in M : z \text{ is encompassed by } \partial M \} \), where \( M \rightarrow K(\alpha)^N \) (see Theorem 2.2.6), \( Y^* \) is the topologically dual space of all \( L \)-linear continuous functionals \( \theta : Y \rightarrow L \), the valuation group \( \Gamma_L \) of \( L \) is discrete.

2.6.3. **Theorem.** Let a manifold \( M \) satify Conditions 2.4.2 and let \( f \) satisfy 2.4.2.(2) for each \( z \in M \), then the function

\[ (1) \quad u(z) := \varepsilon C_n(\alpha)^{-1} \partial M^n[f(\zeta)(\zeta - z)^{-1} d\zeta] - C_n(\alpha)^{-1} M^n[f(\zeta)(\zeta - z)^{-1} d\zeta] \]

is a solution of the equation

\[ (2) \quad \partial u(z)/\partial \bar{z} = f(z) \]

in the sence of distributions for each \( z \in M \) encompassed by \( \partial M \).

**Proof.** The space \( \mathop{PC}^{(q,n)}(M, Y) \) is dense in \( C^{(q,n-1)}(M, Y) \). Indeed, for each \( \delta > 0 \) and each continuous function \( f \circ \phi \) on \( \Omega \) or a continuous partial difference quotient \( w_\phi := \mathbf{\Phi}^{(q)}f \circ \phi(x; h_1^\otimes s_1, ..., h_m^\otimes s_m; \zeta_1, ..., \zeta_q) \) on a domain contained \( \Omega^{n+1} \times B(K, 0, 1)^t \) with \( 0 \leq t \leq (n-1)m, 0 \leq s_j \leq n \) for each \( j = 1, ..., m, t = s_1 + ... + s_m, x, x + \zeta_jh_j \in \Omega, h_j \in V, \zeta_j \in B(K, 0, 1), \)
$V$ is a neighbourhood of 0 in $K^n$, $\Omega + V \subset \Omega$ (see [12]), $m := \dim_K M$, there exists a finite partition of $\Omega^{n+1}$ into disjoint union of balls $B_t$ such that on each $B_t$ the variation $\var(w_t) := \sup_{x,y \in B_t} |w_q(x) - w_q(y)| < \delta$, since $M$ is compact and for each covering of $M$ by such balls there exists a finite subcovering. Therefore, in $C(q,n-1)(\Omega, Y)$ the subspace $\Sigma \subset C(q,n-1)(\Omega, Y)$ of all $C(q,n-1)(\Omega, Y)$-functions $f$ such that $w_{(n-1)m}$ corresponding to $f$ is locally constant on the diagonal $\Delta \Omega^{(n-1)m+1} := \{(y_1, \ldots, y_{(n-1)m+1}) \in \Omega^{(n-1)m+1} : y_1 = \ldots = y_{(n-1)m+1}\}$ is dense in $C(q,n-1)(\Omega, Y)$ for each $\alpha \in \Omega^{(n-1)m+1}$. Since the operator $\Omega P^n$ is continuous, then $\Omega P^n(\Sigma \subset C(q,n-1)(\Omega, Y))$ is dense in $C(q,n-1)(\Omega, Y)$ and $\Omega P^n$ is continuous, then the family of functionals $\{ \Omega P^n(f) : f \in C(q,n-1)(\Omega, Y) \}$ is dense in $C(q,n-1)(\Omega, Y)$. From $\Omega P^n(f) \in C(q,n-1)(\Omega, Y)$ it follows, that $\Omega P^n(f) \in C(q,n-1)(\Omega, Y)$ is dense in $C(q,n-1)(\Omega, Y)$. In particular, take $L$ such that $K(\omega) \subset L$. Since $pC(0,n)(M,Y) \subset C(1,n)(M,Y)$ it follows, that $pC(0,n)(M,Y) \subset C(1,n)(M,Y)$ separates points of $C(q,n-1)(\Omega, Y)$, since $Y$ separates points of $Y$ for discrete $\Gamma_L$ (see Theorem 4.15 in [21]). In view of Formula 2.6.2(i) it is sufficient to prove this theorem for $f \circ \phi(z + E\exp(\eta)) =: \psi(\eta) \in pC(0,n)(\omega, x, Y) \cap pC(0,n)(\omega, y, Y)$ for each $\epsilon = \epsilon_j$, where $\omega(z) := \omega := \{\eta \in K(\omega) : z + E\exp(\eta) \in \Omega\}$. In $\omega, \omega = \omega \setminus \log(B(K(\omega)), z, \epsilon)$, $\epsilon := \epsilon_j, \omega, x = \pi_{\omega}(\omega), \omega, y := \pi_{\omega}(\omega)$.

Using the diffeomorphism $\phi$ we can consider $\Omega$ instead of $M$. Choose a clopen ball $B := B(K \oplus K(\omega), z_0, R) \subset \Omega$ containing a point $z_0 \in \Omega$ and its characteristic function $\chi := \chi_B$. Then $(f\chi)_1 \in pC(0,n)(b_x, Y) \cap pC(0,n)(b_y, Y)$ for suitable $0 < R < \infty$, where $b := \{\eta \in K(\omega) : z_0 + E\exp(\eta) \in B\}$, $b_x := \pi_y(b), b_y := \pi_y(b)$ (see Lemma 2.6.1). Using the affine mapping $z \mapsto (z - z_0)$ we can consider 0 instead of $z_0$. Then $B$ is the additive group. We can take $R > 0$ sufficiently small such that each point of $B$ is encompassed by $\partial \Omega$. Therefore,

(3) $u = u_1 + u_2$, where

(4) $u_1(z) := C^{-1} \int_{\partial \Omega} P_n[f(\zeta)(\zeta - z)^{-1} d\zeta] - C^{-1} \int_{\partial \Omega} P_n[\chi(\zeta)f(\zeta)(\zeta - z)^{-1} d\zeta \wedge d\zeta]$.

(5) $u_2(z) := -C^{-1} \int_{\partial \Omega} P_n[(1 - \chi(\zeta))f(\zeta)(\zeta - z)^{-1} d\zeta \wedge d\zeta]$.

From (5) it follows, that $\partial u_2(z)/\partial \zeta = 0$ on $B$. From (4) it follows

$u_1(z) = C^{-1} \int_{\partial \Omega} P_n[f(\zeta)(\zeta - z)^{-1} d\zeta]
-C^{-1} \int_{\partial \Omega} P_n[\chi(\zeta + z)f(\zeta + z)(\zeta - z)^{-1} d\zeta \wedge d(\zeta + z)]$

for each $z \in B$, since $B + B = B \subset \Omega$. Since $\partial_B B$ encompasses $z$ and
\[ a_B P_n[f(\zeta)(\zeta - z)^{-1}] d\zeta = \sum_{l=0}^{\infty} (z - y)^l a_B P_n[f(\zeta)(\zeta - y)^{-l-1}] d\zeta \]
for each \( z \in B \) with \( |z - y| < R \) due to Formula 2.4.4.(ii) for this antiderivative, then

(6) \( \partial \{ a_B P_n[f(\zeta)(\zeta - z)^{-1}] d\zeta \} / \partial \bar{z} = 0. \) In view of Formulas 2.1.(1 - 4) and §2.2.5

\[ \partial u_1(z) / \partial \bar{z} = C^{-1} \partial_\Omega P_n[f(\zeta)(\zeta - z)^{-1}] \]

\[ -C^{-1} \partial_\Omega P_n\{[\partial_\zeta + \zeta \partial_\bar{\zeta}] f(\zeta + \bar{\zeta}) \zeta^{-1} \wedge d(\zeta + \bar{\zeta})\}, \]

\( \partial u_1(z) / \partial \bar{z} = C^{-1} \partial_\Omega P_n[f(\zeta)(\zeta - z)^{-1}] \)

\[ -C^{-1} \partial_\Omega P_n\{[\partial_\zeta (\zeta f(\zeta)) \wedge d\zeta](\zeta - z)^{-1}\}. \]

In view of Theorem 2.4.2 we get the statement of this theorem, since \( B \) and \( y \) are arbitrary forming covering of each point \( z \in \Omega \) encompassed by \( \partial \Omega \).

2.7.1. Definition. Let \( M \) be a manifold over \( K \) satisfying Conditions 2.4.2. If \( f \in C^{(q,n-1)}(M,Y) \) and for each loop \( \gamma \) in \( M \) \( \gamma^* P_n f = 0 \), then we call \( f \) \((q,n)\)-antiderivationally holomorphic on \( M \), where \( Y \) is a Banach space over \( L, \) \( K(\alpha) \subset L. \) If \( f \in C^{(q,n)}(M,Y) \) and \( \tilde{f}(z) = 0 \) for each \( z \in M \), then \( f \) we call derivationally \((q,n)\)-holomorphic.

2.7.2. Theorem. Let \( \Omega \) be a clopen compact subset in \((K \oplus \alpha K)^m\). Consider the following conditions.

(i) \( f \) satisfies 2.4.2.(2) and \( \partial f(z) = 0 \) for each \( z \in \Omega \) with \( z_j \) encompassed by \( \partial \Omega_j \) for each \( j = 1, \ldots, m, \) where \( \Omega_j = \pi_j(\Omega), \pi_j(\zeta_j) = \zeta_j \) for each \( \zeta = (\zeta_1, \ldots, \zeta_m), \zeta_j \in K \oplus \alpha K. \)

(ii) \( f \) is \((0,n)\)-antiderivationally holomorphic on

\( \bar{\Omega} := \{ z \in \Omega : z_j \text{ is encompassed by } \partial \Omega_j \text{ for each } j \}. \)

(iii) \( f \in C^{(0,n-1)}(\Omega,Y) \) and for each polydisc \( B = B_1 \times \cdots \times B_m \subset \Omega, \) \( B_j = B(K(\alpha),z_{0,j},R_j) \) for each \( j = 1, \ldots, m, \) \( f(z) \) is given by the antiderivative

(1) \( f(z) = C(\alpha)^{-m} \partial B_1 P_n \cdots \partial B_m P_n[f(\zeta)((\zeta_1 - z_1)^{-1}(\zeta_m - z_m)^{-1} d\zeta_1 \wedge \cdots \wedge \partial B_j \text{ for each } j. \)

(iv) \( f \) is locally \( z \)-analytic, that is,

(2) \( f(z) = \sum_k a_k(z - \zeta)^k \) in some neighbourhood of \( \zeta \in \bar{\Omega}, \) \( a_k \in Y, \)

\( k = (k_1, \ldots, k_m), 0 \leq k_j \in Z, z^k := z_1^{k_1} \cdots z_m^{k_m}, z = (z_1, \ldots, z_m), z_j \in K(\alpha). \)

(v) \( f \in C^\infty(\Omega,Y). \)

(vi) \( f \in C^{(0,n-1)}(\Omega,Y) \) and for every polydisc \( B \) as in (iii) and each multiorder \( k \) as in (iv) derivatives are given by

(3) \( \partial^k f(z) = k! C(\alpha)^{-m} \partial B_1 P_n \cdots \partial B_m P_n[f(\zeta)((\zeta_1 - z_1)^{-k_1-1}(\zeta_m - z_m)^{-k_m-1} d\zeta_1 \wedge \cdots \wedge \partial B_j \text{ for each } j, \partial^k f(z) \text{ given by } \)

(vii) the coefficients in Formula (2) are determined by the equation:
(4) $a_k = \partial^k_z f(z)/k!$.

(viii) The power series (2) converges uniformly in each polydisc $B \subset \tilde{\Omega}$ with sufficiently small $b := \max(R_1, \ldots, R_m, 1)$.

Then from (i) Properties (ii -- viii) follow. Properties (iii) and (vi) are equivalent. From (iii) Properties (iv, v, vii, viii) follow. In the subspace
\[
\{f \in C^{(0,n-1)}(\Omega, Y) : |f(z_1, \ldots, z_k + E \exp(\eta), z_{l+1}, \ldots, z_m) - \psi_i(\eta)| < \delta\}
\]
where $\omega_l := \omega_l(z) := \{\eta \in K(\alpha) : (z_1, \ldots, z_k + E \exp(\eta), z_{l+1}, \ldots, z_m) \in \Omega\}$, $\omega_l := \omega_l \setminus \log(B(K(\alpha), z_l, \epsilon))$, $z \in \Omega$, $Y$ is a Banach space over $L$ such that $K(\alpha) \subset L$, Properties (i -- iv) are equivalent.

**Proof.** From (i) it follows (iv) due to repeated application of Corollary 2.4.4. From (i) it follows (iii) due to repeated application of Corollary 2.4.3. Others statements follow from Theorems 2.5.3, 2.5.4, Lemma 2.5.1 and Formulas (i, ii) in §2.4.4, since from Formula (3) it follows
\[
(5) |\partial^l_z f(z)/|k!| \leq |C(\alpha)|^{-m} \sup_{\zeta \in \Omega} |\partial^l_z f(\zeta)| \max_{|b| - |l| / (l + \epsilon)} |b| < \infty,
\]
where $l = (l_1, \ldots, l_m)$, $0 \leq l_j \in Z$, $l_j < n$ for each $j = 1, \ldots, m$, $|l| = l_1 + \ldots + l_m$, $\epsilon := (1, \ldots, 1) \in Z^m$, $b := \max(R_1, \ldots, R_m)$. The series (2) with $a_k$ given by (4) converges uniformly in $B$, when $\lim_k |a_k|^{1/|k|} b < 1$.

**2.7.3. Corollary.** Spaces $C^{(a)}(\Omega, K(\alpha))$ of locally analytic functions $f : \Omega \rightarrow K(\alpha)$ and the space $C^{(q,n),d}(\Omega, K(\alpha))$ of all derivationally $(q, n)$-holomorphic functions are rings. If $f$ is derivationally $(q, n)$-holomorphic and $f \neq 0$ on $\Omega$, then $1/f$ is derivationally $(q, n)$-holomorphic on $\Omega$.

**2.7.4. Corollary.** If $f$ satisfies Condition 2.4.2.(2) and there exists $\zeta \in \Omega$ encompassed by $\partial \Omega$ such that $\partial_z^k f(\zeta) = 0$ for each $k$, then there exists a polydisc $B \subset \tilde{\Omega}$ (see §2.7.2) such that $f = 0$ on $B$.

**Proof.** In view of Theorems 2.7.2 there exists a polydisc $B$ such that on it Formulas 2.7.2.(2,4) are accomplished.

**2.7.5. Remark.** $u P^a \zeta^{|b|} = (b^{k+1} - a^{k+1})/(k + 1)$ for each $a \neq b \in U$, where $k > 0$. In view of Corollary 54.2 and Theorem 54.4 [22] and [1] the spaces $C^{(a)}(\Omega, K(\alpha)) \cap C^{((q,n),d)}(\Omega, K(\alpha))$, $C^{(a)}(\Omega, K(\alpha)) \cap PC^{(q,n)}(\Omega, K(\alpha))$, $\{f \in C^{(a)}(\Omega, K(\alpha)) : f \text{ is } (q, n)\text{-antiderivationally holomorphic} \}$ are infinite dimensional over $K(\alpha)$, since the condition of the local analyticity means that the expansion coefficients $a(m, f)$ of the function $f$ in the Amice polynomial basis $\bar{Q}_m$ are such that $\lim_{|m| \rightarrow \infty} a(m, f)/P_m(\bar{u}(m)) = 0$, where $P_m$ are definite polynomials (see Formulas 2.6.(i -- iii) [16]).

**2.7.6. Theorem.** Let $\Omega$ and $f$ be as in 2.7.2.(i). If $\zeta$ is zero of $f$ such that $f$ does not coincide with 0 on each neighbourhood of $\zeta$, then there exists
$n \in \mathbb{N}$ such that

$$(1) \quad f(z) = (z - \zeta)^n g(z),$$

where $g$ is analytic and $\phi \neq 0$ on some neighbourhood of $z$.

**Proof.** In view of Theorem 2.7.2 there exists a neighborhood $V$ of $\zeta$ such that $f$ has a decomposition into converging series 2.7.2.(2). If $a_k = 0$ for each $k$, then $f|_V = 0$. Therefore, there exists a minimal $k$ denoted by $l$ such that

$$f(z) = \sum_{k=0}^{\infty} a_k (z - \zeta)^k,$$

put $g(z) = \sum_{k=0}^{\infty} a_{k+l} (z - \zeta)^k$.

Since $a_l \neq 0$, then there exists a neighborhood $\zeta \in W \subset V$ such that $g|_W \neq 0$.

2.7.7. **Theorem.** Let $\Omega$ and two functions $f_1$ and $f_2$ be satisfying 2.7.2.(i) such that $f_1(z) = f_2(z)$ for each $z \in E$, where $E \subset \Omega$ and $E$ contains a limit point $\zeta \in E'$. Then there exists a clopen subset $W$ in $\Omega$ such that $\zeta \in W$ and $f_1|_W = f_2|_W$.

**Proof.** In view of Theorem 2.7.6 $f|_W = 0$ for some clopen $W$ in $\Omega$, where $\zeta \in W$.

2.7.8. **Theorem.** Let $f$ satisfy 2.4.2.(2) and $f$ be derivationally $(0, n)$-holomorphic on $\Omega := \{z \in (\mathbb{K} \oplus \alpha \mathbb{K})^m : R_1 \leq |z - \xi| \leq R_2\}$, where $0 < R_1 < R_2 < \infty$, $R_1$ and $R_2 \in \Gamma_{\mathbb{K}}$. Then

$$(1) \quad f(z) = \sum_k a_k (z - \zeta)^k$$

for each $z \in \Omega$ with $R_2 > |z| > R_1$, where

$$(2) \quad a_k = C(\alpha)^{-m} \partial_{\alpha B_{P,1}}^n \cdots \partial_{\alpha B_{P,m}}^n \left[ (\zeta_1 - \xi_1)^{-k_1-1} \cdots (\zeta_m - \xi_m)^{-k_m-1} f(\zeta) d\zeta_1 \wedge \cdots \wedge d\zeta_m \right]$$

for each $k \in \mathbb{Z}^m$, $R_1 < R < R_2$, $B_{R,l} := \{z_l \in \mathbb{K} \oplus \alpha \mathbb{K} : |z_l - \xi_l| \leq R\}$, $k = (k_1, ..., k_m)$, $k_1 \in \mathbb{Z}$, $l = 1, ..., m$.

**Proof.** Let $\pi_l(z) = z_l$ for each $z = (z_1, ..., z_m) \in (\mathbb{K} \oplus \alpha \mathbb{K})^m$, where $z_l \in \mathbb{K} \oplus \alpha \mathbb{K}$. Then $\pi_l(\Omega) = \{z_l \in \mathbb{K} \oplus \alpha \mathbb{K} : R_1 \leq |z_l - \xi_l| \leq R_2\}$. To prove the theorem consider $f$ by each variable $z_l$. That is, consider $z = z_l$ and $m = 1$. Let $R_3$ and $R_4$ be such that $R_1 < R_3 < R_4 < R_2$ and $z \in W \subset \Omega$, where $W = \{z \in \mathbb{K} \oplus \alpha \mathbb{K} : R_3 \leq |z - \xi| \leq R_4\}$. In view of Theorems 2.4.6 and 2.7.2.(iv, vi)

$$(3) \quad f(z) = C(\alpha)^{-1} \partial_{\alpha B_{P}}^n [f(\zeta)(\zeta - z)^{-1} d\zeta]$$

$= C(\alpha)^{-1} \partial_{\alpha B_{R_4}}^n \partial_{\alpha B_{R_3}}^n [f(\zeta)(\zeta - z)^{-1} d\zeta] - C(\alpha)^{-1} \partial_{\alpha B_{R_3}}^n \partial_{\alpha B_{P}}^n [f(\zeta)(\zeta - z)^{-1} d\zeta]$, since $W$ is the union $W_1 \cup W_2$, $\dim_{\mathbb{K}}(W_1 \cap W_2) = 1$, where $W_1$ and $W_2$ satisfy 2.4.6 and 2.7.2. The part of the path $\gamma_{1,2}$ in $W_1 \cap W_2$ joining $\partial_{\alpha B_{R_4}}$ with $\partial_{\alpha B_{R_3}}$ and forming two paths $\gamma_1$ and $\gamma_2$ affine homotopic to points in $W_1$ and $W_2$, $\gamma_1 \subset W_1$, $\gamma_2 \subset W_2$, such that $\gamma_{1,2}$ is being gone twice in one and opposite directions. This gives (3). For each $\zeta \in \partial_{\alpha B_{R_4}}$ we have $|(z - \xi)(\zeta - \xi)| < 1,$
hence \((\zeta - z)^{-1} = \sum_{k=0}^{\infty} (z - \xi)^k(\zeta - \xi)^{-k-1}\) and inevitably
\[
\partial_{\partial B_{R_3}} P^n[f(\zeta)(\zeta - z)^{-1}d\zeta] = \sum_{k=0}^{\infty} a_k(z - \xi)^k,
\]
where \(a_k = C(\alpha)^{-1} \partial_{\partial B_{R_3}} P^n[f(\zeta)(\zeta - \xi)^{-k-1}d\zeta]\) for each \(0 \leq k \in \mathbb{Z}\).

If \(\zeta \in \partial_{\partial B_{R_3}}\), then \(|(\zeta - \xi)(z - \xi)^{-1}| < 1\) and \((\zeta - z)^{-1} = -\sum_{k=1}^{\infty} (\zeta - \xi)^{-k-1}(z - \xi)^{-k}\), hence due to continuity of \(P^n\)
\[-C(\alpha)^{-1} \partial_{\partial B_{R_3}} P^n[f(\zeta)(\zeta - z)^{-1}d\zeta] = \sum_{k=1}^{\infty} a_{-k}(z - \xi)^{-k},
\]
where \(a_{-k} = C(\alpha)^{-1} \partial_{\partial B_{R_3}} P^n[f(\zeta)(\zeta - \xi)^{-k-1}d\zeta]\) for each \(k \geq 1\). In view of
Theorem 2.4.6 we get Formula (2).

2.7.9. Definitions. A point \(z \in A(K \oplus \alpha K)\) is called an isolated critical point of a function \(f\), if there exists a set \(B(K \oplus \alpha K, z, R) \setminus \{z\}\) for \(z \neq A\), and \(\{\zeta \in K \oplus \alpha K : R < |\zeta| < \infty\}\) for \(z = A\), on which \(f\) is \((q, n)\)-antiderivationally holomorphic. An isolated critical point \(z\) of the function \(f\) is called removable, if there exists a limit \(\lim_{\zeta \to z} f(\zeta) = g \in Y\); it is called a pole if there exists \(\lim_{\zeta \to z} \|f(\zeta)\| = \infty\); it is called essentially critical point, if there exists neither finite nor infinite limit point, when \(\zeta\) tends to \(z\).

2.7.10. Theorem. Let \(f\) satisfy 2.7.2.(i) on \(\Omega \setminus \{z\}\). A point \(z \in K \oplus \alpha K\)
is removable if and only if decomposition 2.7.8.(1) does not contain the main part:
\[(i) f(\zeta) = \sum_{k=0}^{\infty} a_k(\zeta - z)^k.\]

2.7.11. Theorem. Let \(f\) satisfy 2.7.2.(i) on \(\Omega \setminus \{z\}\). An isolated critical point \(z \in K \oplus \alpha K\) is a pole if and only if the main part of series 2.7.8.(1) contains only a finite and positive number of nonzero terms:
\[(i) f(\zeta) = \sum_{k=-N}^{\infty} a_k(\zeta - z)^k, N > 0.\]

2.7.12. Theorem. Let \(f\) satisfy 2.7.2.(i) for \(Y = K(\alpha)\) on \(\Omega \setminus \{z\}\). An isolated critical point \(z\) of \(f\) is essentially critical if and only if the main part of series 2.7.8.(1) in a neighborhood of \(z\) contains an infinite family \(\{a_k \neq 0 : k < 0\}\). If \(z\) is an essentially critical point of \(f\), \(r = 2\), that is, \(K(\alpha) = K \oplus \alpha K\), then for each \(\xi \in AK(\alpha)\) there exists a sequence \(\{z_n : n \in N\}\), \(\lim_{n \to \infty} z_n = z\) such that \(\lim_{n \to \infty} f(z_n) = \xi\).

The proof of these latter three theorems is analogous to the classical case (see, for example, §II.7 [25]) due to the given above Theorems 2.7.2 and 2.7.8.

2.7.13. Definition. Let \(f \in C^{(q,n)}(\Omega, Y)\) and \(B := B(K \oplus \alpha K, z, R) \subset \Omega, 0 < R < \infty, f\) is \((q, n)\)-antiderivationally holomorphic on \(B \setminus \{z\}\), then
\[(i) \text{res}_z f := C(\alpha)^{-1} \partial_{\partial B} P^n[f(\zeta)d\zeta]\]
is called the residue of \(f\), where \(Y\) is a Banach space over \(L\) such that
K(α) ⊂ L.

2.7.14. Theorem. Let f satisfy 2.7.2.(i) on Ω \ ∪'\limits_{l=1}^\nu \{z_l\} such that ∂Ω does not contain critical points z_l of f and all of them are encompassed by ∂Ω, ν ∈ N. Then

(i) \( \partial_\Omega P^n[f(\zeta)d\zeta] = C(\alpha) \sum_{z_i \in \Omega} \text{res}_zf \),
where \( \text{res}_zf \) is independent of n and R in 2.7.13,
(ii) \( \text{res}_zf = a_{-1}, a_k \) is as in 2.7.8.(1).

Proof. In view of Theorems 2.4.2 and 2.4.6 \( C(\alpha) = C_n(\alpha) \) is independent of n and \( \text{res}_zf \) is independent of n and R. From 2.7.8.(2) it follows (ii).

2.7.15. Definition. Let \( f \in C^{(q,n)}(\Omega, Y) \) and let \( A \in \Omega \subset A(K \oplus \alpha K) \) be the isolated critical point of f, put
\( \text{res}_Af := -C(\alpha)^{-1} \partial_B P^n[f(\zeta)d\zeta] \).

2.7.16. Theorem. Let f satisfy 2.7.2.(i) on \((K \oplus \alpha K) \setminus \bigcup_{l=1}^{\nu} \{z_l\}\), then
(i) \( \text{res}_Af + \sum_{l=1}^{\nu} \text{res}_zf = 0 \).

Proof. Take a ball \( B_R := B(K \oplus \alpha K, 0, R) \) of sufficiently large 0 < R_0 < R < \infty such that it contains all \( \{z_l : l = 1, ..., \nu\} \), \( \Omega = A(K \oplus \alpha K) \) and \( \kappa(\Omega) \subset K \oplus \alpha K \) (see §2.2.4). In view of Theorem 2.7.14
(ii) \( \partial_B P^n[f(\zeta)d\zeta] = \sum_{l=1}^{\nu} \text{res}_zf \)
and it is independent of R for each \( R > R_0, R < \infty \). In accordance with Definition 2.7.15 and Theorem 2.4.6
(iii) \( \partial_B P^n[f(\zeta)d\zeta] = -\text{res}_Af \).

Therefore, from (ii, iii) it follows (i).

2.7.17. Definitions. Let \( f \in C^{(q,n)}(\Omega, K(\alpha)) \) and let f be \((q, n)\)-antiderivationally holomorphic on \( B(K \oplus \alpha K, z, R) \setminus \{z\} \), where \( \Omega \subset K \oplus \alpha K, f(z) \neq 0 \). Then
\( \text{res}_zf'(z)/f(z) \) is called the logarithmic residue of f at the point z. Let us count each zero and pole of f a number of times equal to its order.

A function f is called \((q, n)\)-antiderivationally meromorphic, if it is \((q, n)\)-antiderivationally holomorphic on \( \Omega \) besides a set of poles.

2.7.18. Theorem. Let f be \((q, n)\)-antiderivationally meromorphic on \( \Omega \) and let Log(f) satisfy 2.7.2.(i) on \( \Omega \setminus \bigcup_{l=1}^{\nu} \{z_l\} \), where \( z_l \) is the pole of f for each \( l = 1, ..., \nu \), all zeros and poles of f are encompassed by ∂Ω, \( Y = K(\alpha) \).

Then
(1) \( N - P = C(\alpha)^{-1} \partial_\Omega P^n[d\text{Log}(f(\zeta))] \),
where N and P denote total numbers of zeros and poles in \( \Omega \).

Proof. Since \( \Omega \) is compact, then N and P are finite. In view of Theorem 2.7.8
(2) \( C(\alpha)^{-1} \partial_{\Omega} P^m[d\log(f(\zeta))] = \sum_{l=1}^\nu \text{res}_{l} \text{Log}(f) + \sum_{l=1}^\mu \text{res}_{\xi} \text{Log}(f) \),
where \( z_l \) is the pole of \( f \) and \( \xi_l \) is the zero of \( f \) for each \( l \). On the other hand,
\[
\frac{f'(z)}{f(z)} = [k(z - \xi_l)^{k-1}\phi(z) + (z - \xi_l)^k\phi'(z)](z - \xi_l)^{-k}/\phi(z) = (z - \xi_l)^{-1}[k\phi(z) + (z - \xi_l)\phi'(z)]/\phi(z),
\]
where \( f(z) = (z - \xi_l)^k\phi(z) \), \( k = k_l \) is the order of zero \( \xi_l \), \( \phi(z) \neq 0 \) in a neighborhood of \( \xi_l \). Therefore,
\[
(3) \text{res}_{\xi} \text{Log}(f) = k_l \text{ and res}_{z_l} \text{Log}(f) = -s_l,
\]
where \( s_l \) is the order of pole \( z_l \). Hence, from (2, 3) it follows Formula (1).

**2.8. Theorem.** Let \( \Omega \) be a clopen compact subset in, \( B((K\oplus aK)^m, y, R) \), \( 0 < R < p^{1/(1-p)} \). Suppose

(i) \( f_j(z_1, ..., z_{l-1}, z_l + \text{Exp}(\eta), z_{l+1}, ..., z_m) := \psi_{j,\epsilon}(\eta) : \omega_{\xi,\epsilon} \rightarrow Y \) belongs to \( SC(\nu, n-1)(\omega_{\xi,\epsilon}, Y) \) for each \( j, l = 1, ..., m \) and each \( z = (z_1, ..., z_m) \in \Omega \),
where \( \omega = \{ \eta \in K(\alpha) : (z_1, ..., z_{l-1}, z_l + \text{Exp}(\eta), z_{l+1}, ..., z_m) \in \Omega \}, \omega_{\xi,\epsilon} = \omega \setminus \text{Log}(B(K(\alpha), z_l, \epsilon)) \), \( \epsilon = \epsilon_k, 0 < \epsilon_k \) for each \( k \in N \), \( \lim_{k \rightarrow \infty} \epsilon_k = 0 \), \( 0 \leq q \in Z, 1 \leq n \in N \), \( Y \) is a Banach space over \( L \).
Assume:

(1) \( \partial f_j / \partial z_l = \partial f_l / \partial z_l \) for each \( j, l = 1, ..., m \).

Then there exists \( u \in C(\nu, n-1)(\Omega, K(\alpha)) \) such that

(2) \( \partial u(z) / \partial z_l = f_j(z) \) for each \( j = 1, ..., m \) and each \( z \in \bar{\Omega} \) (see \( \S 2.7.2 \)).

**Proof.** Define

(3) \( u(z) := C(\alpha)^{-1} \sum_{j=1}^m \bar{z}_j \partial f_j P^m[f_j(z_1, ..., z_{j-1}, \zeta, z_{j+1}, ..., z_m)(\zeta - z_j)^{-1}d\zeta] \)
\( -C(\alpha)^{-1} \Omega P^m[f_1(\zeta, z_2, ..., z_m)(\zeta - z_1)^{-1}d\zeta \wedge d\zeta]. \)
Hence \( u(z) = C(\alpha)^{-1} \sum_{j=1}^m \bar{z}_j \partial f_j P^m[f_j(z_1, ..., z_{j-1}, \zeta, z_{j+1}, ..., z_m)(\zeta - z_j)^{-1}d\zeta] \)
\( + C(\alpha)^{-1} \sum_{j=1}^m z_j \partial f_j P^m[(\chi_{\Omega_j} f_j)(z_1 - \eta, z_2, ..., z_m)\eta^{-1}d(z_1 - \eta) \wedge d(z_1 - \eta)]. \)
where \( \eta := z_1 - \zeta \) and we can take \( U = B(K, 0, R) \) such that \( U^2 + U^2 = U^2 \) and \( U^2 \) is the additive group, \( \Omega_j := \{ \xi \in U^2 : (z_1, ..., z_{j-1}, \xi, z_{j+1}, ..., z_m) \in \Omega \}. \) Therefore,
\( u \in C(\nu, n-1)(\Omega, Y) \). Then
\[
\partial u / \partial \bar{z}_j = C(\alpha)^{-1} \partial f_j P^m[f_j(z_1, ..., z_{j-1}, \zeta, z_{j+1}, ..., z_m)(\zeta - z_j)^{-1}d\zeta] \]
\( -C(\alpha)^{-1} \Omega P^m[(\chi_{\Omega_j} \partial f_j(z_2, ..., z_m) / \partial \bar{z}_j)(\zeta - z_1)^{-1}d\zeta \wedge d\zeta]. \)
In view of Condition (1) and Formula 2.6.3.(6) and
\[
f_j(z) = C(\alpha)^{-1} \partial f_j P^m[f_j(z_1, ..., z_{j-1}, \zeta, z_{j+1}, ..., z_m)(\zeta - z_j)^{-1}d\zeta] \]
\( -C(\alpha)^{-1} \Omega P^m[(\chi_{\Omega_j} \partial f_j(z_2, ..., z_m) / \partial \bar{z}_j)(\zeta - z_1)^{-1}d\zeta \wedge d\zeta] \) (see Theorem 2.4.2)
it follows (2).
3 Antiderivational representations of functions and differential forms

3.1. Remark and Notation. Let \( \Omega \) be a clopen compact subset in \((K + \alpha K)^m\). Put

\[
(1) \ w(z, \zeta) := \sum_{j=1}^{m} (-1)^{j+1} (\zeta_j - z_j)^{-1} d\zeta_j \wedge t \neq j \ [(\xi(\zeta - \bar{z}))^{-1} d\xi_l (\zeta - \bar{z}) \wedge (\xi(\zeta - z))^{-1} d\xi_l (\zeta - z)]
\]

for each \( z \neq \zeta \in \Omega^2 \), where \((\alpha')^j \neq \alpha^j \) for each \( j = 1, ..., q'; \ t = 1, ..., q \), \( m \leq q' \leq \tilde{m}(\alpha') \), \( r \leq q \leq \tilde{m}(\alpha) \) (see \S 2.1 and 2.4.1); there are constants \( 0 < \epsilon_1 < \epsilon_2 < \infty \) such that

\[
(2) \ \epsilon_1 |\pi|^{-s}|z_\xi| \leq |\log(\xi(\zeta))| \leq \epsilon_2 |\pi|^{-s}|z_\xi| \quad \text{and} \quad \xi(\zeta) \neq 0 \quad \text{for each} \quad \zeta \in \Omega - z, \quad \xi(0) = 1, \quad \text{where} \quad z \in \Omega, \ \xi(\zeta) \in C^{q,n}(\Omega - z, C_p), \ \log(\xi(\zeta)) \in (K + \alpha K)^m \quad \text{for each} \quad \zeta \in \Omega - z, \quad \text{here the embeddings are used:} \quad (K + \alpha K)^m \hookrightarrow (K(\alpha))^m \hookrightarrow K(\alpha, \alpha') \hookrightarrow C_p.
\]

(3) \( \xi \) is such that \( d_\zeta w(z, \zeta) = 0 \) on \( \Omega \setminus \{z\} \);

(4) \( s := s(\zeta) := -ord_{K(\alpha, \alpha')}(\zeta) \) for each \( \zeta \in \Omega - z, \ z_j \neq \zeta_j \) for each \( j, \) \( \alpha' \) is the root of 1 in \( C_p \) such that \( K(\alpha)^m \) is embedded into \( K(\alpha, \alpha') = (K(\alpha))(\alpha'), \ z|_{K(\alpha, \alpha')} = |\pi|^{-ord_{K(\alpha, \alpha')}} \), \( \pi \) is the same as in \S 2.1;

(5) \( \lim_{\xi \to \infty} \partial_{\xi} B^j(K + \alpha K)^{m,n} P^n [w(z, \zeta)] =: q_m \neq 0. \) If \( f \) is a 1-form of class \( C^{0,n-1} \) we define:

\[
(6) \ (B_0^m f)(z) := q_0^{-1} \Omega^n [f(\xi) \wedge w(z, \zeta)] \quad \text{for each} \quad z \in \Omega \text{ encompassed by} \ \partial \Omega.
\]

If \( f \in C^{0,n-1}(\Omega, Y) \), we define

\[
(7) \ (B_0^m f)(z) := q_0^{-1} \xi \in \partial \Omega \ \Omega^n [f(\xi) w(z, \zeta)] \quad \text{for each} \quad z \in \Omega \text{ encompassed by} \ \partial \Omega, \quad \text{where} \quad Y \text{ is a Banach space over} \ L \quad \text{such that} \quad K(\alpha) \subset L.
\]

3.2. Theorem. Let \( \Omega \) be a clopen compact subset in \( B((K + \alpha K)^m, y, R) \), \( 0 < R < p^{1/(1-p)} \), \( B_{\partial \Omega}^n, B_{\partial \Omega}^n \) be given by \S 3.1, \( f \in C^{q+1,n-1}(\Omega, Y) \). Then

\[
(1) \ f(z) = (B_{\partial \Omega}^m f)(z) - (B_{\partial \Omega}^m \partial f)(z)
\]

for each \( z = (z_1, ..., z_m) \in \Omega \) (see \S 2.7.2) such that

(2) \( (f w)(z_1, ..., z_{l-1}, z_l + Exp(\eta), z_{l+1}, ..., z_m) := \psi_l(\eta) \in sC^{q+1,n-1}(\omega_l, L(\Lambda K(\alpha), Y)) \) for each \( l = 1, ..., m \) and each \( \epsilon = \epsilon_j, \) where \( \omega_l := \{\eta \in K(\alpha) : (z_1, ..., z_{l-1}, z_l + Exp(\eta), z_{l+1}, ..., z_m) \in \Omega\} \), \( \omega_l, \epsilon := \omega_l \setminus Log(B(\Lambda K(\alpha), z_l, \epsilon)) \), \( 0 < \epsilon_j \) for each \( j \in \mathbb{N}, \lim_{j \to \infty} \epsilon_j = 0, \ 0 \leq q \in \mathbb{Z}, \ 1 \leq n \in \mathbb{N} \) (see \S 2.4.1).

Proof. Fix \( z \in \Omega \). In the particular case \( \xi(\zeta - \bar{z}) = Exp(\pi^{-s}(\zeta - \bar{z})) \) properties 3.1.(2, 3) are satisfied and \( q_m = C(\alpha)m(2\alpha)^{m-1} \) due to Formulas
2.1. (2, 3), §2.2.4 and §2.4.2, since \( d\bar{z} \wedge dz = 2adx \wedge dy \) and \( U^P_n[dx]\|_a^2 = b - a \) for each \( a, b \in U \), where \( z = x + iy, x, y \in U \subset K \). Therefore, the family of such \( \xi \) and \( w \) satisfying Conditions 3.1.(1–5) is nonvoid. In view of 3.1.(3) \( d\xi w(z, \zeta) = 0 \) on \( \Omega \setminus \{z\} \), hence \( d(f(\zeta)w(z, \zeta)) = \nabla f(\zeta) \wedge w(z, \zeta) \) on \( \Omega \setminus \{z\} \), since \( \partial f(\zeta) \wedge w(z, \zeta) = 0 \) on \( \Omega \setminus \{z\} \). From Corollary 2.3.2 it follows, that there exists \( \delta > 0 \) such that for each \( 0 < \epsilon < \delta \), \( \epsilon \in \Gamma_K \), there is the inclusion \( B(\epsilon) := B((K \oplus \alpha K)^m, z, \epsilon) \subset \Omega \) and the equality is satisfied:

\[
\zeta \in \partial B(\epsilon) P^m[f(\zeta)w(z, \zeta)] = \partial \Omega P^m[f(\zeta)w(z, \zeta)] - \Omega P^n[\nabla f(\zeta) \wedge w(z, \zeta)]
\]

for each \( z \in \Omega \) and satisfying Condition 2.3.(2), where \( \Omega(\epsilon) := \{\zeta \in \Omega : |\zeta - z| \geq \epsilon\} \). In the particular case, \( \xi(\zeta - \bar{z}) = \exp(\pi^{-s}(\zeta - \bar{z})) \) due to 2.3.2:

\[
\zeta \in \partial B(\epsilon) P^m[f(\zeta)w(z, \zeta)] = \pi^{-2(m-1)s} \zeta \in \partial B(\epsilon) P^m[f(\zeta)] \sum_{j=1}^{m} (-1)^{j+1} (\zeta_j - z_j)^{-1} d\zeta \wedge_{l \neq j} (d\bar{\zeta}_l \wedge d\zeta_l) \]

where \( z^l = (z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_m) \), \( x_{2l-1}, x_{2l} \in U \), \( z_l = x_{2l-1} + \alpha x_{2l} \) for each \( l = 1, \ldots, m \). Therefore, there exists

\[
\lim_{\epsilon \to 0} \zeta \in \partial B(\epsilon) P^m[f(\zeta)w(z, \zeta)] = f(z)q_m
\]

due to 3.1.(2, 5), since there exists \( C = const > 0 \), \( C < \infty \), such that

\[
|B(\epsilon) P^m[f(\zeta) - f(z)]w(z, \zeta)] \leq C\|f(\zeta) - f(z)\|_{C^{(\alpha,n-1)}(\Omega(\epsilon), Y)}
\]

for each \( 0 < \epsilon < \delta \).

**3.3. Corollary.** Let \( \Omega \) and \( f \) be as in Theorem 3.2 and \( f \) be derivationally \((q, n)\)-holomorphic on \( \Omega \), then

\[
(1) \ f(z) = (B^P_{\partial \Omega} f)(z) \text{ for each } z \in \hat{\Omega} \text{ (see §2.7.2)}.
\]

**3.4. Remark.** For \( m = 1 \) Formula 3.2.(1) is Formula 2.4.2.(1), which are the non-Archimedean analogs of the Martinelli-Bochner and Cauchy-Green formulas respectively (see for comparison the classical complex case in [8]).

**3.5. Definitions and Notations.** Consider a clopen compact \( \Omega \subset (K \oplus \alpha K)^m \) and a \( C^{(q,n+1)} \)-function \( v : \Omega \times (\partial \Omega)^\delta \to (K \oplus \alpha K)^m \), \( v = (v_1, \ldots, v_m) \), \( 0 < \delta < \infty \), \( v = v(z, \zeta) \), \( z \in \Omega \), \( \zeta \in (\partial \Omega)^\delta \), \( \Psi := \{z \in X : d(z, \Psi) < \epsilon\} \) for a topological space \( X \) with a metric \( d \) and a subset \( \Psi \subset X \), \( d(z, \Psi) := \inf_{x \in \Psi} d(z, x) \), \( 0 < \epsilon, 0 \leq q \in Z \), \( 1 \leq n \in N \). Suppose

\[
\tilde{\phi}(s) := -ord_{K(\alpha, \alpha')} v(z, \zeta), s = -ord_{K(\alpha, \alpha')} (\zeta - \bar{z}) \text{ such that}
\]

\[
(1) \ \lim_{s \to \infty} \tilde{\phi}(s) = \infty
\]

for each \( z \in \Omega \) and \( \zeta \in (\partial \Omega)^\delta \). Put

\[
\eta^v(z, \zeta, \lambda) := (1 - \lambda/\beta) \xi(v(z, \zeta)) + \lambda \xi(\zeta - \bar{z})/\beta,
\]

where \( \lambda \in B(K, 0, 1) \). Impose the condition:
(2) \( \wedge_{k=1}^{m} d\zeta v_{k}(z, \zeta) \wedge_{j=1}^{m} d\zeta_{j} \neq 0 \) and \( \eta^{\nu}(z, \zeta, \lambda) \neq 0 \) for each \( z \in \Omega \) and \( \zeta \in (\partial \Omega)^{\delta} \) and \( \lambda \in B(K, 0, 1) \). Let also

(3) \( \psi(z, \zeta) := \sum_{j=1}^{m} (-1)^{j+1}(\zeta_{j} - z_{j})^{-1}d\zeta_{j} \wedge_{k \neq j} (\zeta^{(v(z, \zeta)))^{-1}}d\zeta_{k}(v(z, \zeta)) \wedge (\xi(\zeta - z))^{-1}d\zeta_{k}(\zeta - z))].

If \( f \in C^{0,n-1}(\partial \Omega, Y) \), we set:

(4) \( (L_{\partial \Omega}^{v,n} f)(z) := q_{m}^{-1} \zeta \in \partial \Omega \omega_{m}^{n}[f(\zeta)\psi(z, \zeta)] \)

for each \( z \in \Omega \), where \( Y \) is a Banach space over \( L \) such that \( K(\alpha) \subset L \). Put also:

(5) \( \gamma(z, \zeta, \lambda) := \sum_{j=1}^{m} (-1)^{j+1}(\zeta_{j} - z_{j})^{-1}d\zeta_{j} \wedge_{k \neq j} (\zeta^{(v(z, \zeta)))^{-1}}d\zeta_{k}(v(z, \zeta)) \wedge (\xi(\zeta - z))^{-1}d\zeta_{k}(\zeta - z))]

for each \( z \in \Omega, \zeta \in (\partial \Omega)^{\delta}, \lambda \in B(K, 0, 1) \). If \( f \) is a \( C^{0,n-1} \)-form on \( \partial \Omega \) put:

(6) \( (R_{\partial \Omega}^{v,n} f)(z) := q_{m}^{-1} \zeta \in \partial \Omega \omega_{m}^{n}[f(\zeta)\gamma(z, \zeta, \lambda)] \)

for each \( z \in \Omega \). Suppose that \( v \) is such that

(7) \( d_{z, \zeta}\psi(z, \zeta) = 0 \) for each \( \zeta \neq z \). In particular, if \( v(z, \zeta) = \bar{\zeta} - \bar{z} \), then

\( L_{\partial \Omega}^{v,n} f = B_{\partial \Omega}^{n} f \) and \( R_{\partial \Omega}^{v,n} f = 0 \),

since \( \gamma \) is the \( (m, m) \)-form by \( (\zeta, \zeta) \) for \( v(z, \zeta) = \bar{\zeta} - \bar{z} \).

3.6. Theorem. Let \( \Omega \) be a clopen compact subset in \( (K + \alpha K)^{m} \) and let \( v(z, \zeta), L_{\partial \Omega}^{v,n} \) and \( R_{\partial \Omega}^{v,n} \) be given by §3.5, \( f \in C^{q+1,n-1}(\Omega, Y) \). Then

(1) \( f(z) = (L_{\partial \Omega}^{v,n} f)(z) - (R_{\partial \Omega}^{v,n} \bar{\partial} f)(z) - (B_{\partial \Omega}^{n} \bar{\partial} f)(z) \)

for each \( z \in \bar{\Omega} \) (see §2.7.2) such that

(2) \( (f \gamma)(z_{1}, \ldots, z_{t-1}, z_{t} + Exp(\eta), z_{t+1}, \ldots, z_{m}) := \psi(\eta) \)

\( \in \mathcal{C}^{q+1,n-1}(\omega_{t, \zeta}, L(\Lambda K(\alpha), Y)) \)

for each \( l = 1, \ldots, m, \epsilon, \eta \in \epsilon_{i} \), where \( \omega_{t} := \omega_{t}(\epsilon) := \{ \eta \in K(\alpha) : (z_{1}, \ldots, z_{t-1}, z_{t} + Exp(\eta), z_{t+1}, \ldots, z_{m}) \in \Omega \}, \omega_{t, \zeta} := \omega_{t} \setminus \log(B(K(\alpha), z_{t}, \epsilon), 0 < \epsilon \in K \) for each \( j \in \mathbb{N}, \lim_{j \to \infty} \epsilon_{j} = 0, 0 \leq \eta \in \mathbb{Z}, 1 \leq n \in \mathbb{N} \) (see §2.4.1).

Proof. The using of Theorem 3.2 reduces the proof to that of the formula:

(3) \( (R_{\partial \Omega}^{v,n} \bar{\partial} f)(z) = (L_{\partial \Omega}^{v,n} f)(z) - (B_{\partial \Omega}^{n} f)(z) \)

for each \( z \in \bar{\Omega} \) and satisfying Condition (2). In view of 3.2(2) and 3.5(7) we have \( d_{\zeta, \lambda}\gamma(z, \zeta, \lambda) = 0 \), since \( d_{\zeta, \lambda}[d_{\zeta, \lambda}(\eta^{\nu})] = 0 \). Therefore, \( d_{\zeta, \lambda} f(\zeta) \gamma(z, \zeta, \lambda) = (\bar{\partial} f(\zeta)) \wedge \gamma(z, \zeta, \lambda), \) since \( (\bar{\partial} f) \wedge \gamma = 0 \). From 3.5(3, 5) it follows that

\( \gamma(z, \zeta, \lambda) \vert_{\lambda = 0} = \psi(z, \zeta) \) and

\( \gamma(z, \zeta, \lambda) \vert_{\lambda = \beta} = \sum_{j=1}^{m} (-1)^{j+1}(\zeta_{j} - z_{j})^{-1}d\zeta_{j} \wedge_{k \neq j} (\zeta^{(v(z, \zeta)))^{-1}}d\zeta_{k}(v(z, \zeta)) \wedge (\xi(\zeta - z))^{-1}d\zeta_{k}(\zeta - z)) \)

Mention that \( \lambda = P^{n}1_{0} \), hence \( \lambda \in PC^{q,n}(B(K, 0, 1), K) \). Then by the degree reasons
\[ \zeta \in \partial \Omega \] 
\[ P^n[f \gamma |_{\lambda = \beta}] = \zeta \in \partial \Omega \] 
\[ P^n[f w], \] 
where \( w \) is given by 3.1.(1), \( \dim K \Omega = 2m, \dim K \partial \Omega = 2m - 1. \) Then
\[ q_m^{-1} \zeta \in \partial \Omega, \lambda \in B P^n \{ d_{\xi, \lambda} [f(\zeta) \gamma(z, \zeta, \lambda)] \} \]
\[ = q_m^{-1} \zeta \in \partial \Omega, \lambda \in B P^n \{ (\partial f(\zeta) \wedge \gamma(z, \zeta, \lambda)) = (R_{\partial \Omega} w, \tilde{f})(z) \] 
for each \( z \in \tilde{\Omega}, \) where \( B := B(K, 0, 1) \). On the other hand,
\[ \partial((\partial \Omega) \times B) = (-1)^{2m-1}((\partial \Omega) \times \{\beta\} - (\partial \Omega) \times \{0\}) = (\partial \Omega) \times \{\beta\} + (\partial \Omega) \times \{0\}. \] 
In view of Corollary 2.3.2
\[ \partial((\partial \Omega) \times B) P^n \{ d_{\xi, \lambda} [f \gamma] \} = - \partial \Omega P^n [f w] + \partial \Omega P^n [f \psi], \]
hence Formula (3) is accomplished.

### 3.7. Corollary
Let \( f \) be as in Theorem 3.6 and \( \tilde{f} = 0 \) on \( \Omega \), then
\[ f(z) = (L^n_{\partial \Omega} f)(z) \] 
for each \( z \in \tilde{\Omega} \) (see §2.7.2) and satisfying Condition 3.6.(2).

### 3.8. Definitions and Remarks
Let \( \Omega \) be a clopen compact subset in \((K \oplus \alpha K)^m, \) consider the differential form:

(1) \[ \tilde{w}(z, \zeta) := \sum_{j=1}^{m} (-1)^{j+1} (\zeta_j - z_j)^{-1} d\zeta_j \wedge \prod_{j \neq j}cip([\zeta_j(\zeta - z)]^{-1} \partial \zeta_j z \zeta_j(\zeta - z) \wedge (\zeta_j(\zeta - z))^{-1} d\zeta_j(\zeta - z)]. \] 
Let \( M \) be a compact manifold over \( K \) and \( \phi : \Omega \to M \to (K(\alpha))^{N} \) be a \( s C^{(q+1, n-1)} \)-diffeomorphism (see §2.2.5). Then the diffeomorphism \( \phi_{*} w \) of the differential form \( w \) is the differential form on \( M. \) Consider these differential forms on \( M \) and denote them by the same notation, since \( \{\phi(\zeta_j) : j\} \) are coordinates in \( M. \) Therefore, Theorems 3.2, 3.6 and Corollaries 3.3, 3.7 are true for \( M \) also due to Theorem 2.3.1 and Corollary 2.3.2, where \( M := \phi^{-1}(\tilde{\Omega}) \) (see §§2.2.5 and 2.7.2). If \( f \) is a \( C^{(0, n-1)} \)-differential form on \( M, \) then we define:

(2) \[ (B^t_{M} f)(z) := q_m^{-1} (\zeta \in M) P^n [f(\zeta) \wedge \tilde{w}(z, \zeta)] \] 
for each \( z \in M \) encompassed by \( \partial M. \) If \( f \) is a \( C^{(0, n-1)} \)-differential form on \( M, \) then we define:

(3) \[ (B^t_{\partial M} f)(z) := q_m^{-1} (\zeta \in \partial M) P^n [f(\zeta) \wedge \tilde{w}(z, \zeta)] \] 
for each \( z \in M \) encompassed by \( \partial M. \) Write \( \tilde{w} \) as:

(4) \[ \tilde{w}(z, \zeta) = \sum_{t=0}^{m-1} \Upsilon_t(z, \zeta), \] 
where \( \Upsilon_t \) is of bedegree \((0, t)\) in \( z \) and of bedegree \((m, m - t - 1)\) in \( \zeta. \) Decompose \( f \) as:

(5) \[ f = \sum_{l+s = \deg(f)} f_{(l, s)} \] 
where \( f_{(l, s)} \) is the \((l, s)\)-form on \( M. \) Then \( f_{(l, s)}(\zeta) \wedge \sum_{j=1}^{m} d\zeta_j = 0 \) for each \( l > 0, \) hence

\[ B^t_{\partial M} f = B^t_{M} f_{(0, \deg(f))}. \] 
On the other hand, \( f(\zeta) \wedge \Upsilon_t(z, \zeta) = 0, \) if \( \deg(f) > q + 1; \)
for each \( z \)
\( R_{n}^{\varphi}f = \zeta \in M \) \( P_{n}^{\varphi}[f(\zeta) \wedge \Upsilon_{i}(z, \zeta)] = 0 \), when \( \deg(f) < q + 1 \) by the definition of the antiderivation. Therefore,

(6) \( B_{n}^{\varphi}f = \zeta \in M \) \( P_{n}^{\varphi}[f(0, \deg(f)) (\zeta) \wedge \Upsilon_{\deg(f) - 1}(z, \zeta)] \) for \( 1 \leq \deg(f) \leq m \),

(7) \( B_{n}^{\varphi}f = 0 \) for \( \deg(f) = 0 \) or \( \deg(f) > m \), similarly

(8) \( B_{n}^{\varphi}f = \zeta \in \partial M \) \( P_{n}^{\varphi}[f(0, \deg(f)) (\zeta) \wedge \Upsilon_{\deg(f)}(z, \zeta)] \) for \( 0 \leq \deg(f) \leq m - 1 \),

(9) \( B_{n}^{\varphi}f = 0 \) for \( \deg(f) \geq m \),

hence \( B_{n}^{\varphi}f \) is of bidegree \( (0, \deg(f) - 1) \); \( P_{\partial M}^{\varphi}f \) is of bidegree \( (0, \deg(f)) \).

Using the notation of §3.5 define:

(10) \( \tilde{\psi}(z, \zeta) := \sum_{i=1}^{n}(-1)^{j+1}(\zeta_{j} - z_{j})^{-1}d\zeta_{j} \wedge_{k \neq j}[(\xi(v(z, \zeta)))^{-1}\bar{d}_{z, \zeta} \xi_{k}(v(z, \zeta)) \wedge \langle \xi(\zeta - z) \rangle^{-1}d\xi_{k}(\zeta - z)] \).

(11) \( \tilde{\gamma}(z, \zeta, \lambda) := \sum_{j=1}^{n}(-1)^{j+1}(\zeta_{j} - z_{j})^{-1}d\zeta_{j} \wedge_{k \neq j}[(\eta^{v}(z, \zeta, \lambda))^{-1}(\bar{d}_{z, \zeta} + d_{\lambda})\eta^{v}(z, \zeta, \lambda) \wedge \langle \xi(\zeta - z) \rangle^{-1}d\xi_{k}(\zeta - z)] \).

If \( f \in C^{(0,n-1)}(\partial M, L(\Lambda K(\alpha, \gamma))) \) put:

(12) \( (L_{\partial M}^{v,n}f)(z) := q_{m}^{-1} \zeta \in M \) \( P_{n}^{\varphi}[f(\zeta) \wedge \tilde{\psi}(z, \zeta)] \) and

(13) \( (R_{\partial M}^{v,n}f)(z) := q_{m}^{-1} \zeta \in M, \lambda \in B \) \( P_{n}^{\varphi}[f(\zeta) \wedge \tilde{\gamma}(z, \zeta, \lambda)] \)

for each \( z \in M := \phi^{-1}(\Omega) \) (see §§2.2.5 and 2.7.2). There exists the decomposition:

(14) \( \tilde{\gamma}(z, \zeta, \lambda) = \sum_{i=0}^{n-1} \Upsilon_{i}^{v}(z, \zeta, \lambda) \),

where \( \Upsilon_{i}^{v}(z, \zeta, \lambda) \) is of bidegree \( (0, t) \) in \( z \) and \( f \) is of bidegree \( (m, m - t - 1) \) in \( (\zeta, \lambda) \). Let \( f \) be a bounded differential form on \( \partial M, f = \sum f_{(l,s)} \), then

(15) \( R_{\partial M}^{v,n}f = \zeta \in \partial M, \lambda \in B \) \( P_{n}^{\varphi}[f(0, \deg(f)) (\zeta) \wedge \Upsilon_{v_{\deg(f)-1}}^{\varphi}(z, \zeta, \lambda)] \) if \( 1 \leq \deg(f) \leq m \),

(16) \( R_{\partial M}^{v,n}f = 0 \) if \( \deg(f) = 0 \) or \( \deg(f) > m \). Similarly,

(17) \( \psi(z, \zeta) = \sum_{i=0}^{n-1} \Upsilon_{i}^{v}(z, \zeta, \lambda) \),

where \( \Upsilon_{i}^{v}(z, \zeta, \lambda) \) is of bidegree \( (0, t) \) in \( z \) and of bidegree \( (m, m - t - 1) \) in \( \zeta \), hence

(18) \( L_{\partial M}^{v,n}f = \zeta \in \partial M \) \( P_{n}^{\varphi}[f(0, \deg(f)) \wedge \Upsilon_{v_{\deg(f)}}^{\varphi}(z, \zeta)] \) if \( \deg(f) \leq m - 1 \),

(19) \( L_{\partial M}^{v,n}f = 0 \) if \( \deg(f) \geq m \). If \( v(z, \zeta) = \bar{\zeta} - \bar{z} \), then \( L_{\partial M}^{v,n}f = B_{\partial M}^{n}f \).

### 3.9. Theorem

Let \( M \) be a compact manifold over \( K \) and let \( B_{M}^{n}f \) and \( B_{\partial M}^{n}f \) be given by §3.8. Suppose that \( f \) is the \( C^{(q+1,n-1)}-0, t) \)-form, \( 0 \leq t \leq m \). Then

(1) \( (-1)^{t}f(z) = (B_{\partial M}^{n}f)(z) - (B_{M}^{n}\bar{d}f)(z) + (\bar{d}B_{M}^{n}f)(z) \)

for each \( z \in M \) such that

(2) \( (f \wedge \bar{w}) \circ \phi(z_{1}, ..., z_{l-1}, z_{l} + \text{Exp}(\eta), z_{l+1}, ..., z_{m}) =: \tilde{\psi}_{l}(\eta) \)
\[ \in sC^{(q+1,n-1)}(\omega_{l,\epsilon}, L(\Lambda(K(\alpha)), Y)) \text{ for each } l = 1, ..., m \text{ and each } \epsilon = \epsilon_j, \]

where \( \omega_l := \{ \eta \in K(\alpha) : (z_1, ..., z_{l-1}, z_l + \text{Exp}(\eta), z_{l+1}, ..., z_m) \in \Omega \} \), \( \omega_{l,\epsilon} := \omega_l \setminus \log(B(K(\alpha), z_l, \epsilon)) \), \( 0 < \epsilon_j \) for each \( j \in \mathbb{N} \), \( \lim_{j \to \infty} \epsilon_j = 0 \), \( 0 \leq q \in \mathbb{Z} \), \( 1 \leq n \in \mathbb{N} \).

**Proof.** Using the diffeomorphism \( \phi \) it is possible reduce the case to \( \Omega \subset (K \oplus \alpha K)^m \). If \( q = 0 \), then by 3.8.7 \( B^n_{\Omega}(\bar{\partial}f) = 0 \) and \( f = B^n_{\partial\Omega}f - B^n_{\partial\Omega}(\bar{\partial}f) \) is Formula 3.2.(1). Since (2) is satisfied, \( v \) and \( \xi \in C^{(q,n)} \), then \( B^n_{\partial\Omega}f \) and \( B^n_{\partial\Omega}(\bar{\partial}f) \) are in \( sC^{(q,n)}(\omega_{l,\epsilon}, L(\Lambda(K(\alpha)), Y)) \) for each \( l = 1, ..., m \), \( \epsilon = \epsilon_j \). From the definition of \( B^n_{\Omega} \) it follows, that \( \sup \| \Phi^n(B^n_{\partial\Omega}(\bar{\partial}f))(z; h^{\xi}_1, ..., h^{\xi}_m; \xi_1, ..., \xi_m) - \Phi^n(B^n_{\partial\Omega}(\bar{\partial}f))(y; h^{\xi}_1, ..., h^{\xi}_m; \xi_1, ..., \xi_m) \|_{C^{(q,n)}} \leq \frac{C_1\|f\|_{C^{(q,n)}}}{1 - \pi^{-2\epsilon_m}} \), where \( s = s(\xi, z) \), \( u = u_1 + ... + u_m \), \( 0 \leq u_t \leq n \), hence \( B^n_{\partial\Omega}(\bar{\partial}f) \in C^{(q,n)}(M, L(\Lambda(K(\alpha)), Y)) \). Analogously, \( B^n_{\partial\Omega}f \) and \( B^n_{\partial\Omega}(\bar{\partial}f) \) are in \( C^{(q,n)}(M, L(\Lambda(K(\alpha)), Y)) \). It remains to prove, that in the sense of distributions:

\[
(\partial B^n_{\Omega}(\bar{\partial}f))(z) = (-1)^{\epsilon}f(z) - (B^n_{\partial\Omega}(\bar{\partial}f))(z) + (B^n_{\partial\Omega}(\bar{\partial}f))(z)
\]

for each \( z \in \Omega \) and satisfying Condition (2). This means, that for each \( sC^{(q+1,n-1)} \)-form \( \nu \), \( \text{supp}(\nu) \subset \bar{\Omega} \), there is satisfied the equality:

\[
(3) \quad (-1)^{\epsilon} \Omega^{P^n}[B^n_{\partial\Omega}(\bar{\partial}f) \wedge \bar{\nu}] = (-1)^{\epsilon} \Omega^{P^n}[f \wedge \nu]
\]

and \( \Omega^{P^n}[B^n_{\partial\Omega}(\bar{\partial}f) \wedge \bar{\nu}] \). In view of Formulas 3.8.(6, 8) \( B^n_{\Omega}f \) and \( B^n_{\partial\Omega}(\bar{\partial}f) \) are of bidegree \((0, t)\) and \( B^n_{\partial\Omega}f \) is of bidegree \((0, t-1)\), we can assume that \( \nu \) is of bidegree \((m, m-t)\). Then (3) takes the form:

\[
(4) \quad (-1)^{\epsilon} \sum_{(z, \zeta) \in \Omega} P^n[f(\zeta) \wedge \bar{w}(z, \zeta) \wedge \bar{\nu}(\zeta)] = (-1)^{\epsilon} \sum_{(z, \zeta) \in \Omega} P^n[f(z) \wedge \nu(z)]
\]

for \( \zeta \neq z \), since \( \bar{\Phi}(\zeta) = 0 \), \( d^2 = 0 \).

Then all monomials in \( \bar{\Phi}(z, \zeta) = \bar{w}(z, \zeta) \) contain at least one of the differentials \( dz_1, ..., dz_m \). For \( \nu(z) \) of bidegree \((m, m-t)\) it contains the factor \( d\zeta_1 \wedge ... \wedge d\zeta_m \), hence from (5, 6) it follows:

\[
(5) \quad \bar{\Phi}(z, \zeta) = 0 \quad \text{for } \zeta \neq z \quad \text{and } \bar{\Phi}(\zeta-z) = 0 \quad \text{then} \quad d\zeta = d\zeta^2 = 0.
\]

Then all monomials in \( \bar{\Phi}(z, \zeta) = \bar{w}(z, \zeta) \) contain at least one of the differentials \( dz_1, ..., dz_m \). For \( \nu(z) \) of bidegree \((m, m-t)\) it contains the factor \( d\zeta_1 \wedge ... \wedge d\zeta_m \), hence from (5, 6) it follows:

\[
(7) \quad \bar{\Phi}(z, \zeta) = d_{\zeta}(\bar{\Phi}(z, \zeta) \wedge \nu(z)) = (-1)^{2m-1} (\bar{\Phi}(z, \zeta) \wedge d\nu(z)) = -\bar{w}(z, \zeta) \wedge \bar{\nu}(z)
\]

for \( \zeta \neq z \), with \( \bar{w}(z, \zeta) \) contains the factor \( d\zeta_1 \wedge ... \wedge d\zeta_m \). Hence (7) implies:

\[
(8) \quad d_{\zeta}(f(\zeta) \wedge \bar{w}(z, \zeta) \wedge \nu(z)) = (\bar{\Phi}(z, \zeta) \wedge \bar{w}(z, \zeta) \wedge \nu(z) - (-1)^{\epsilon} f(\zeta) \wedge \bar{w}(z, \zeta) \wedge \bar{\nu}(z)) \text{ for } \zeta \neq z.
\]
Therefore, taking $R$ \(> d \) then since $2$ and 2, ..., where $\Omega$ for each $l$ such that $\Omega \in (K \oplus \alpha K)^m$ for each $l$. Let $\tilde{f}_I(z, \zeta)$ be a compact manifold and let $M : (K \oplus \alpha K)^m \rightarrow (K \oplus \alpha K)^m$. The differential form $\nu(z)$ contains the factor $dz_1 \wedge ... \wedge dz_m$, hence $\tilde{w}(z, \zeta) \wedge \nu(z) = w(z, \zeta) \wedge \nu(z)$ and $T^*(f(z) \wedge \tilde{w}(z, \zeta) \wedge \nu(z)) = \sum_{|I|=t} f_I(z + \zeta) d(z + \zeta)^m \wedge w(z, \zeta) \wedge \nu(z)$, where $T^*$ is the pull-back operator on differential forms (see §2.2.5). The degree of $w(z, \zeta)$ is $2m - 1$ and $2m - 1 = dim_K(\partial B^{-1})$, consequently, $d(z + \zeta)^m \wedge w(z, \zeta)|_{(\partial B^{-1}) \times (K \oplus \alpha K)^m} = d(z + \zeta)^m \wedge w(z, \zeta)|_{(\partial B^{-1}) \times (K \oplus \alpha K)^m}$.

Therefore, taking $R > 0$ such that $\Omega \in B((K \oplus \alpha K)^m, 0, R) = B_R$:

\[ (\partial U(e))^{-1} P^n[f(z) \wedge \tilde{w}(z, \zeta)] - \partial U(e) P^n[f(z) \wedge \tilde{w}(z, \zeta) \wedge \nu(z)] = \partial U(e) P^n[f(z) \wedge \tilde{w}(z, \zeta) \wedge \nu(z)] \]

for $|I| = t$, $\zeta \in \partial U(e)$, where $\partial U(e) = (\partial B^{-1}) \times (K \oplus \alpha K)^m$, and inevitably $\lim_{\partial U(e)} P^n[f(z) \wedge \tilde{w}(z, \zeta) \wedge \nu(z)] = (\partial U(e))^{-1} P^n[f(z) \wedge \tilde{w}(z, \zeta) \wedge \nu(z)]$.

3.10. **Theorem.** Let $M$ be a compact manifold and let $L_{\alpha M}^n, R_{\alpha M}^n, B_M^n$ be given by §3.8. Suppose $f$ is the $C^2$-differential form, $0 \leq t \leq m$. Then

\[ (\partial U(e))^{-1} P^n[f(z) \wedge \tilde{w}(z, \zeta) \wedge \nu(z)] = (\partial U(e))^{-1} P^n[f(z) \wedge \tilde{w}(z, \zeta) \wedge \nu(z)] \]

for each $z \in M$ such that

\[ (\partial U(e))^{-1} P^n[f(z) \wedge \tilde{w}(z, \zeta) \wedge \nu(z)] \]

for each $l = 1, ..., m$ and each $\epsilon = \epsilon_j$, where $\omega_l := \{ \eta \in K(\alpha) : (z_1, ..., z_l, z_{l+1}, ..., z_m) \in \Omega \}$, $\omega_l := \omega_l \setminus \Omega$, $\epsilon_j > 0$ for each $j \in N$, $\lim_{j \to \infty} \epsilon_j = 0$, $0 \leq q \in Z$, $1 \leq n \in N$.

**Proof.** If $\nu(z, \zeta) = \tilde{z} - \bar{z}$, then $L_{\alpha M}^n = B_{\alpha M}^n, R_{\alpha M}^n = 0$ and Formula
3.10. (1) reduces to Formula 3.9. (1). If \( t = 0 \), then by 3.8. (7, 16) \( B^n_M f = 0 \) and \( R^{v,n}_{\partial M} f = 0 \), hence 3.10. (1) reduces to 3.6. (1). Assume \( 1 \leq t \leq m \). In view of §§3.8 and 3.9 \( L^{v,n}_{\partial M}, R^{v,n}_{\partial M} \partial f, B^n_M \partial f, \partial R^{v,n}_{\partial M} f \) and \( \partial B^n_M f \) are in \( C^{(q,n)}(M, L(\Lambda K(\alpha), Y)) \). Using the diffeomorphism \( \phi \) consider \( \Omega \) instead of \( M \). In view of 3.9. (1) it remains to prove:

(3) \( \partial (R^{v,n}_{\partial M} f)(z) = (B^n_M f)(z) - L^{v,n}_{\partial M} f(z) + (R^{v,n}_{\partial M} \partial f)(z) \)

for each \( z \in \Omega \) and satisfying Condition (2). Consider the differential form:

(4) \( \kappa := \sum_{j=1}^m (-1)^{j+1}(\zeta_j - z_j)^{-1} d\zeta_j \wedge_{k \neq j} [(\eta^v(z, \zeta, \lambda)^{-1} d_z, z, \zeta, \lambda) \wedge (\xi(z - z))^{-1} d_{\xi}(z - z)] \).

In accordance with §§3.1 and 3.5 \( \xi \) and \( v \) are of class of smoothness \( C^{(q,n)} \), hence \( \kappa \) and \( \tilde{\gamma} \) belong to \( C^{(q,n)}(W, L(\Lambda K(\alpha), Y)) \) for suitable clopen \( W \subset \Omega \times (K \oplus \alpha K)^m \times B(\hat{K}, 0, 1) \) such that \( \Omega \times (\partial \Omega) \times B(\hat{K}, 0, 1) \subset W \), \( \zeta \neq z \). Condition 3.5. (7) is satisfied for \( \xi(\zeta - \zeta) = \exp(\pi^{-s}(\zeta - \zeta)) \) and \( v(z, \zeta) \) such that \( \xi(v(z, \zeta)) - \exp(\pi^{-s}(v(z, \zeta))) \), where \( \phi(s) \) is given by §§3.5 and satisfies Condition 3.5. (7). Therefore, the family of such differential forms \( \psi \) and \( w \) is nonvoid. In view of Conditions 3.1. (3) and 3.5. (7) in the sense of distributions:

(5) \( d_z, z, \zeta, \lambda, \kappa = 0 \) on \( W \), that is, \( z \in \Omega P^n[(d_z, z, \zeta, \lambda, \kappa) \wedge v] = 0 \) for each \( v \) as above.

From \( \partial_z \kappa = 0 \) and (5) it follows

(6) \( (\partial_z, z, z, \zeta, \lambda, \kappa) = 0 \), together with \( \partial_z (\tilde{\gamma}) = 0 \) it implies:

since \( (\kappa - \tilde{\gamma}) \) contains a factor \( \tilde{\gamma} \) and \( \partial_z (\kappa - \tilde{\gamma}) = 0 \). The monomials in \( (\kappa - \tilde{\gamma}) \) with respect to \( d_z, d_{\tilde{\gamma}}, d_{\zeta}, d_{\tilde{\gamma}} \) and \( d\lambda \) and, consequently, in \( (\partial_z, z, \zeta, \lambda, \kappa) \) contain at least one of the differentialaials \( d_z, d_{\tilde{\gamma}}, ..., d_z, d_z \) as a factor. The same is true for \( \partial_z (\tilde{\gamma}) \). The monomials in \( (\partial_z, z, \zeta, \lambda, \kappa) \) do not contain any of the differentials \( d_z, d_{\tilde{\gamma}} \). Hence from (6) it follows, that

(7) \( d_{z, \zeta, \lambda, \kappa}(f \wedge \tilde{\gamma}) = (\partial_{\zeta} + d_{\lambda})(f \wedge \tilde{\gamma}) = (\partial_f) \wedge \tilde{\gamma} + (-1)^{\ell} f \wedge (\partial_{\zeta} + d_{\lambda})(\tilde{\gamma}) \).

The applying of Corollary 2.3.2 and Formula (7) to the differential form \( f \wedge \tilde{\gamma} \) on \( (\partial \Omega) \times B \), where \( B := B(\hat{K}, 0, 1) \), gives

(8) \( \tilde{\gamma} \) is the solution of \( \tilde{\gamma} \) with initial data \( \tilde{\gamma} |_{\partial \Omega} = \psi, \tilde{\gamma} |_{\partial \Omega} = \bar{w} \) and Formula (8) is equivalent to Formula (3) due to Formulas 3.8. (3, 12, 13).

3.11. Corollary. Let \( M \) and \( f \) be as in Theorem 3.10 and \( \partial v/\partial \bar{z} = 0 \) on \( M \). For \( t = 1, ..., m \) put
(1) \( T^n_t := (-1)^t(R^n_{\partial M} + B^n_M) \). Then
\[
\partial z = 0 \text{ the monomials in } \Upsilon \text{ of bidegree } (0, t) \text{ in } z \text{ vanish if } t \geq 1.
\]
Therefore, \( L^{v,n}_{\partial M} f = 0 \) and (2) follows from 3.10.(1). Then from (2) it follows \( \partial u(z) = f(z) \) if \( \partial f(z) = 0 \) for each \( z \in \tilde{M} \) and satisfying 3.10.(2), where \( u = T^n_t f \).

**Proof.** In view of Formula 3.8.(18) \( L^{v,n}_{\partial M} f = \partial_M P^a[f \wedge \Upsilon^n] \). Since \( \partial v(z, \xi)/\partial z = 0 \) the monomials in \( \Upsilon^n \) of bidegree \( (0, t) \) in \( z \) vanish if \( t \geq 1 \).

3.12. **Definitions.** Let \( M \) be a manifold over \( K \) satisfying 2.4.2 with \( (q+1, n) \)-antiderivationally holomorphic \( sC^{(q+1,n-1)} \)-transition maps \( \phi_i \circ \phi_j^{-1} \) between charts \( (U_i, \phi_i) \) and \( (U_j, \phi_j) \) for each \( U_i \cap U_j \neq \emptyset \) and let \( GL(N, K(\alpha)) \) be the group of invertible \( N \times N \)-matrices with entries in \( K(\alpha) \).

(1). A \((q+1, n)\)-antiderivationally holomorphic vector bundle over \( K(\alpha) \) of \( K(\alpha) \) dimension \( N \) over \( M \) is a \( sC^{(q+1,n-1)} \)-vector bundle over \( M \) with the characteristic fibre \( (K(\alpha))^N \) and with \((q+1, n)\)-antiderivationally holomorphic atlas of local trivializations of \( B \), that is, with a family \( \{U_j, h_j\} \) such that \( \{U_j\} \) is a (cl)open covering of \( M \), for each \( j \), \( h_j \) is a \( sC^{(q+1,n-1)} \)-bundle isomorphism from \( B|_{U_j} \) onto \( U_j \times (K(\alpha))^N \); the corresponding transition mappings \( g_{ij} : U_i \cap U_j \rightarrow GL(N, K(\alpha)) \) defined by \( (z, g_{ij}(z)v) = h_i \circ h_j^{-1}(z, v) \), \( z \in U_i \cap U_j, v \in (K(\alpha))^N \) are \((q+1, n)\)-antiderivationally holomorphic \( sC^{(q+1,n-1)} \)-mappings. Equipped with the atlas \( \{B|_{U_j}, h_j\} \) the bundle \( B \) gets the structure of the \( sC^{(q+1,n-1)} - (q+1, n) \)-antiderivationally holomorphic manifold.

(2). A \( sC^{(q+1,n-1)} \)-bundle homomorphism between \( sC^{(q+1,n-1)} - (q+1, n) \)-antiderivationally holomorphic vector bundles \( B_1 \) and \( B_2 \) is called \( sC^{(q+1,n-1)} - (q+1, n) \)-antiderivationally holomorphic if it is \( sC^{(q+1,n-1)} - (q+1, n) \)-antiderivationally holomorphic as a map between the \( sC^{(q+1,n-1)} - (q+1, n) \)-antiderivationally holomorphic manifolds \( B_1 \) and \( B_2 \). Similarly is defined a \( sC^{(q+1,n-1)} - (q+1, n) \)-antiderivationally holomorphic section of a \( K(\alpha) \) \( sC^{(q+1,n-1)} - (q+1, n) \)-antiderivationally holomorphic vector bundle.

(3). A \( sC^{(q+1,n-1)} - (q+1, n) \)-antiderivationally holomorphic vector bundle over \( M \) is called \( sC^{(q+1,n-1)} - (q+1, n) \)-antiderivationally holomorphically trivial if there exists a \( sC^{(q+1,n-1)} - (q+1, n) \)-antiderivationally holomorphic bundle isomorphism from \( B \) onto \( M \times (K(\alpha))^N \). \( B \) is called \( sC^{(q+1,n-1)} - (q+1, n) \)-antiderivationally holomorphically trivial over a (cl)open set \( U \subset M \).
if \( B|_U \) is \( sC^{(q+1,n-1)} - (q + 1, n) \)-antiderivationally holomorphically trivial. A \( sC^{(q+1,n-1)} - (q + 1, n) \)-antiderivationally holomorphic trivialization of \( B \) (over \( U \)) is a \( sC^{(q+1,n-1)} - (q + 1, n) \)-antiderivationally holomorphic bundle isomorphism from \( B \) onto \( M \times (K(\alpha))^N \) \((B|_U \) onto \( U \times (K(\alpha))^N \)).

(4). A \( K(\alpha) \)-valued differential form of degree \( r \) over \( M \) can be defined as a section of the vector bundle \( \Lambda^r T^*(M)_{K(\alpha)} \), where \( T^*(X)_{K(\alpha)} \) is the \( K(\alpha) \) cotangent bundle of \( M \) over scalars \( b \in K(\alpha) \) (see [14]). A differential form of degree \( r \) with values in a \( sC^{(q+1,n-1)} - (q + 1, n) \)-antiderivationally holomorphic bundle (or a \( B \)-valued differential form) over \( M \) is a section of the bundle \( \Lambda^r (T^*(M)_{K(\alpha)}) \otimes_{K(\alpha)} B \).

If \( \{U_j : j \in J\} \) is a (cl)open covering of \( M \) such that \( B \) is \( sC^{(q+1,n-1)} - (q + 1, n) \)-antiderivationally holomorphically trivial over each \( U_j \) and \( \{g_{i,j} : i,j \in J\} \) is the corresponding system of transition functions, then a differential form with values in \( M \) can be identified with a system \( \{f_j\} \) of \( N \)-tuplets of differential forms on \( U_j \) such that \( f_i = g_{i,j}f_j \) over \( U_i \cap U_j \) for each \( i,j \in J \). A differential form \( f \) with values in \( B \) is called a \((0,t)\)-form, \( sC^{(q+1,n-1)} - (0,t) \)-form, etc. If for each (cl)open subset \( U \subset M \), where \( B \) is \( sC^{(q+1,n-1)} - (q + 1, n) \)-antiderivationally holomorphically trivial, the corresponding \( N \)-tuple of differential forms on \( U \) consists of \((0,t)\)-forms, \( sC^{(q+1,n-1)} - (0,t) \)-form, etc. Each \((s,t)\)-form with values in a \( sC^{(q+1,n-1)} - (q + 1, n) \)-antiderivationally holomorphic vector bundle can be identified with some \((0,t)\)-forms with values in some other \( n \)-antiderivationally holomorphic vector bundle.

3.13. Definition. Let \( M \) be a \( sC^{(q+1,n-1)} - (q + 1, n) \)-antiderivationally holomorphic manifold, let \( B \) be a \( sC^{(q+1,n-1)} - (q + 1, n) \)-antiderivationally holomorphic vector bundle over \( M \) and \( \{U_j : j \in J\} \) be a (cl)open covering of \( M \), where \( J \) is a set. A derivationally \((q+1,n-1)\)-holomorphic Cousin data in \( M \) means a system \( \{f_{i,j} : i,j \in J\} \) of derivationally \((q+1,n-1)\)-holomorphic sections \( f_{i,j} : U_i \cap U_j \to B \) such that \( f_{i,j} + f_{j,k} = f_{i,k} \) in \( U_i \cap U_j \cap U_k \) for each \( i,j,k \in J \). The corresponding Cousin problem consists in finding a system \( \{f_j : j \in J\} \) of derivationally \((q+1,n-1)\)-holomorphic sections \( f_j : U_j \to B \) such that \( f_{i,j} = f_i - f_j \) in \( U_i \cap U_j \) for each \( i,j \in J \).

3.14. Theorem. Let \( M \) be a \( sC^{(q+1,n-1)} - (q + 1, n) \)-antiderivationally holomorphic manifold and let \( B \) be a \( sC^{(q+1,n-1)} - (q+1,n) \)-antiderivationally holomorphic vector bundle over \( M \). Consider two conditions:

(1) each derivationally \((q + 2,n-1)\)-holomorphic Cousin problem in \( B \) has a solution;
(2) each $B$-valued \( sC^{(q+1,n-1)} - (0,1) \)-form on $M$ such that \( \bar{\partial} f = 0 \) on $M$ has a section $u : M \to B$ such that $\bar{\partial} u = f$ on $M$.

Then from (1) it follows (2). From (2) it follows (1) in the class $u \in C^{(q+2,n-1)}$ and $\bar{\partial} u \in sC^{(q+1,n-1)}$ is $(q+1,n-1)$-antiderivationally holomorphic.

**Proof.** (1) $\implies$ (2). At first, $f$ is an $sC^{(q+1,n-1)}$-form means, that there exists a refinement \( \{U'_{k} : j \} \) of \( \{U_{j} \} \) consisting of clopen $U'_{k}$ such that $g_{k}(U'_{k})$ is bounded in $(K(\alpha))^{N}$ and $f|_{U'_{k}} \in sC^{(q+1,n-1)}$, where $At'(M) = \{(U'_{k},g_{k}) : k \}$. Choose $At'(M)$ such that $\dot{U} \cup U'_{k} = M$. Denote \( \{U'_{k} : k \} \) by \( \{U_{j} : j \in J \} \) also such that $\partial U'_{k}$ satisfies condition of Theorem 2.8 up to the $sC^{(q+1,n-1)}$-diffeomorphism. Then 2.8 on each $U_{j}$ gives a solution $u_{j}$ such that $(u_{i} - u_{j})$ are derivationally $(q+2,n-1)$-holomorphic on $U_{i} \cap U_{j}$ and form derivationally $(q+2,n-1)$-holomorphic Cousin data in $B$. According to (1) there exists a derivationally $(q+2,n-1)$-holomorphic section $h_{j} : U_{j} \to B$ such that $u_{i} - u_{j} = h_{i} - h_{j}$ in $U_{i} \cap U_{j}$. Set $u := u_{i} - h_{i}$ in $U_{i}$ for each $j \in J$.

(2) $\implies$ (1) in the class $u \in C^{(q+2,n-1)}$ and $\bar{\partial} u \in sC^{(q+1,n-1)}$ is $(q+1,n-1)$-antiderivationally holomorphic. Characteristic functions of clopen compact subsets belong to $C^{\infty}$. It is possible to take a refinement $At'(M)$ of $At(M)$ such that its charts be satisfying Lemma 2.6.1, that is, $g_{k}(U'_{k})$ are balls satisfying 2.6.1. Choose $At'(M)$ such that $\cup_{k} \bar{U}'_{k} = M$. Denote it also by $At(M)$. Since $M$ is metrizable it has an atlas consisting of clopen compact charts, hence $M$ has a $C^{\infty}$-partition of unity, $\chi_{k} := \chi_{U'_{k}}$. For each $i$ and $j$ $f_{k,j}$ is \( sC^{(q+1,n-1)} - (q+1,n-1) \)-antiderivationally holomorphic, hence $\chi_{k}f_{k,j}$ is also by Lemma 2.6.1 \( sC^{(q+1,n-1)} - (q+1,n-1) \)-antiderivationally holomorphic for suitable refinement \( \{U_{j} : j \in J \} \), since $\chi_{k}f_{k,j} = f_{k,j}|_{\text{dom}(f_{k,j})}$, $\bar{\partial}(\chi_{k}f_{k,j}) = 0$. Set $\theta_{j} := -\sum_{k} \chi_{k}f_{k,j}$ in $U_{j}$, hence by Theorem 2.7.2 there exists a $C^{(q+2,n-1)}$-solution of the Cousin problem: $f_{i,j} := \sum_{k} \chi_{k}(f_{i,k} + f_{k,j}) = \theta_{i} - \theta_{j}$ in $U_{i} \cap U_{j}$; $\bar{\partial}\theta_{j} = \bar{\partial}\theta_{j} = \bar{\partial}\theta_{j}$ in $U_{i} \cap U_{j}$. Hence by (2) there exists a section $u : M \to B$ such that $\bar{\partial} u = \bar{\partial}\theta_{j}$ in $U_{j}$.

The setting $h_{j} = \theta_{j} - u$ in $U_{j}$ provides (1).

**3.15. Remark.** Formulas 3.6.(1), 3.9.(1), 3.10.(1) are the non-Archimedean analogs of the Leray, Koppelman and Koppelman-Leray formulas correspondingly.

**3.16. Notes and Definitions.** The local field $K$ is the disjoint union of balls $B(K,z_{j},R)$ for a given $0 < R < \infty$, where $z_{j} \in K$ for each $j \in \mathbb{N}$. Therefore, the antiderivation operators $B_{j}P^{n}$ on $B_{j} := B(K,z_{j},R)$ induce
the antiderivation operator \( \kappa^m \) on \( K \) such that

\[
(1) \quad \kappa(P^n[f](y)) = \sum_{j=1}^{\infty} (B_jP^n[f(x)])(y)
\]
on \( C^{(q,n-1)}(K,L) \), where \( K \subset L \subset C_p, L \) is a field complete relative to its uniformity. Then

\[
p_C^{(q,n)}(K^1, Y) := \kappa^m(C^{(q,n-1)}(K^1, Y)) \oplus Y \text{ and } \quad s_C^{(q+1,n-1)}(K^1, Y) := \{ g \in C^{(q+1,n-1)}(K^1, Y) : g(x_1, ..., x_l) \}
\]
in \( p_{x_1}C^{(q+1,n-1)}(K^1, Y) \) for each \( j = 1, ..., l \),

\[
p_{x_1}C^{(q+1,n-1)}(K^1, Y) := \kappa^m(C^{(q,n-1)}(K^1, Y)) \oplus Y,
\]
where \( C^{(q,n-1)}(K^1, Y) \) and \( p_C^{(q,n)}(K^1, Y) \) are supplied with the inductive limit topologies induced by the embeddings \( C^{(q,n-1)}(B(K^1, z, R'), Y) \rightarrow C^{(q,n-1)}(K^1, Y), 0 < R' < \infty, \).

Therefore, in the standard way we get the definition of a locally compact manifold \( M \) over \( K \) of class \( p_C^{(q,n)} \) or \( s_C^{(q+1,n-1)} \), that is, transition mappings of charts \( \phi_{i,j} \in p_C^{(q,n)} \) or \( \phi_{i,j} \in s_C^{(q+1,n-1)} \), where \( V_j \) is clopen in \( M \), \( \phi_j(V_j) \) is clopen in \( K^1 \), \( 1 \leq l \in N \), \( l = \dim M \) (see §2.2.5).

Using charts and \( p_C^{(q,n)}(K^1, K^m) \) or \( s_C^{(q+1,n-1)}(K^1, K^m) \) we get the uniform space \( p_C^{(q,n)}(M, N) \) or \( s_C^{(q+1,n-1)}(M, N) \) of all mappings \( g : M \rightarrow N \) of class \( p_C^{(q,n)} \) or \( s_C^{(q+1,n-1)} \) respectively, where \( M \) is the \( p_C^{(q,n)} \)-manifold or \( s_C^{(q+1,n-1)} \)-manifold on \( K^1 \) and \( N \) is the \( p_C^{(q,n)} \)-manifold or \( s_C^{(q+1,n-1)} \)-manifold on \( K^m \) correspondingly, that is, \( \psi_1 \circ g \circ \phi_1^{-1} \) is of class \( p_C^{(q,n)} \) or \( s_C^{(q+1,n-1)} \) for each \( i \) and \( j \) such that its domain is nonempty, where \( At(M) = \{ (V_j, \phi_j) : j \} \), \( At(N) = \{ (W_j, \psi_j) : j \} \). The uniformity in \( p_C^{(q,n)}(K^1, K^m) \) or \( s_C^{(q+1,n-1)}(K^1, K^m) \) induces the uniformity in \( p_C^{(q,n)}(M, N) \) or \( s_C^{(q+1,n-1)}(M, N) \) respectively (see Remark 2.4 [16]).

For a locally compact manifold \( M \) over \( K \) of class \( p_C^{(q,n)} \) or \( s_C^{(q+1,n-1)} \) let \( Diff^{(q,n)}(M) \) or \( Diff^{(q+1,n-1)}(M) \) denotes a family of all diffeomorphisms \( f : M \rightarrow M, f(M) = M, (f-i) \in p_C^{(q,n)} \) and \( (f^{-1}-i) \in p_C^{(q,n)} \) or \( (f-i) \in s_C^{(q+1,n-1)} \) and \( (f^{-1}-i) \in s_C^{(q+1,n-1)} \) respectively, where \( id(z) = z \) for each \( z \in M, M \rightarrow K^N \), \( p_C^{(q,n)}(M, N) \mid p_C^{(q,n)}(M, K^N) \), \( s_C^{(q+1,n-1)}(M, N) \mid s_C^{(q+1,n-1)}(M, K^N) \) such that \( (f-i) \) is correctly defined, \( N \in N \).

3.17. Theorem. (1) The uniform spaces \( Diff^{(q,n)}(M) \) and \( Diff^{(q+1,n-1)}(M) \) are the topological groups for each \( 0 \leq q \in Z, 1 \leq n \in N \).

(2) They have embeddings as clopen subsets into \( p_C^{(q,n)}(M, N) \) and into
\[ sC^{(q+1,n-1)}(M, M) \text{ respectively.} \]

(3) The uniform spaces \( pC^{(q,n)}(M, N), \ sC^{(q+1,n-1)}(M, N), \text{ Diff} P^{(q,n)}(M) \) and \( \text{Diff} S^{(q+1,n-1)}(M) \) are complete and separable.

(4) The groups \( \text{Diff} P^{(q,n)}(M) \) and \( \text{Diff} S^{(q+1,n-1)}(M) \) are ultrametizable, when \( M \) is compact.

(5) The uniform spaces \( \text{Diff} P^{(q,n)}(M) \) and \( \text{Diff} S^{(q+1,n-1)}(M) \) have the infinite-dimensional manifolds structures over \( K \).

Proof. At first prove, that compositions of diffeomorphisms preserve classes \( pC^{(q,n)}(M, M) \) and \( sC^{(q+1,n-1)}(M, M) \) respectively. For this consider two diffeomorphisms \( \psi, \phi \in \text{Diff} P^{(q,n)}(U^m) \) or \( \text{Diff} S^{(q+1,n-1)}(U^m) \) simultaneously. A diffeomorphism \( \phi \) is called the simplest diffeomorphism, if it has the coordinate form:

\[
x_j = \phi_j(y_1, \ldots, y_m) = y_j \text{ for each } j = 1, \ldots, k - 1, k + 1, \ldots, m,
\]

\[
x_k = \phi_k(y_1, \ldots, y_m) = \phi(y_1, \ldots, y_k, \ldots, y_m), \text{ where } x_j, y_j \in U, x = (x_1, \ldots, x_m), m = \dim K M. \text{ Suppose such marked number } k \text{ is for } \phi \text{ and } l \text{ is for } \psi. \text{ To prove } \phi \circ \psi \in \text{Diff} P^{(q,n)}(U^m) \text{ or } \text{Diff} S^{(q+1,n-1)}(U^m) \text{ it is sufficient to verify that } \{ \phi_k(y_1, \ldots, y_{k-1}, \psi_1(y_1, \ldots, y_m), y_{k+1}, \ldots, y_m) - y_k \} \text{ is in } \]

\( pC^{(q,n)}(U^m, K) \) or \( sC^{(q+1,n-1)}(U^m, K) \) correspondingly.

In \( C^0(U^m, K^m) \) there exists the polynomial Amice base \( \{ \bar{Q}_n(x) : n \in \mathbb{N}_o \} \) and it is also the base in \( C^{(q,n)}(U^m, K^m) \), where \( \mathbb{N}_o := \{ j : 0 \leq j \in \mathbb{Z} \} \) (see [1, 16]). The linear ordering \( \Delta \) in \( K \) induces the linear ordering \( \Delta \) in \( K^m \) and hence in \( U^m \): \( x \Delta y \) if and only if \( x_1 = y_1, \ldots, x_{j-1} = y_{j-1}, x_j \Delta y_j \), where \( 1 \leq j \leq m \), \( y_j \in K \), \( y = (y_1, \ldots, y_m) \) (see §2.2.1). Take in particular, \( U = B(K, 0, 1) \). Then \( (\beta, \beta) \) is the largest element in \( U^m \). Let \( Z_K := \{ z \in K : z = \Sigma_{t=0}^l z_t \pi_t, 0 \leq t \in Z, z_t \in \{ 0, \theta_1, \ldots, \theta_{p-1} \} \} \), then \( Z_K \) is dense in \( B(K, 0, 1) \) and \( Z_K \) is countable. There are decompositions

\[
\begin{align*}
(i) \quad & \psi(y) = \sum_{n \in \mathbb{N}_o} a(n, \psi) \bar{Q}_n(y) \quad \text{and} \\
(ii) \quad & \phi_k(y) = \sum_{n \in \mathbb{N}_o} a(n, \phi_k) \bar{Q}_n(y),
\end{align*}
\]

where \( a(n, \psi) \) and \( a(n, \phi_k) \in K \). In view of the conditions imposed on \( \psi \) and \( \phi_k \) and continuity of the \( K \)-linear operators \( U P^m_{z_j} \):

\[
(iii) \quad \phi_k(y) = \{ \sum_{n \in \mathbb{N}_o} a(n, \partial \phi_k(y)/\partial y_j)( U P^m_{y_j} \bar{Q}_n(y)) \}_{y_j=0}^{y_j=0}
\]

for each \( j = 1, \ldots, m \) and analogously for \( \psi \), where \( y_{j,0} \) and \( y_j \in U \). To show \( (\phi_k(y_1, \ldots, y_{k-1}, \psi_1(y), y_{k+1}, \ldots, y_m) - y_k) \in sC^{(q+1,n-1)}(U^m, K) \) it is sufficient to find \( h_j : U^m \to K \) such that

\[
(iv) \quad U P^m_{y_j} h_j |_{y_j=0} = -h_{j,0} + \phi_k(y_1, \ldots, y_{k-1}, \psi_1(y), y_{k+1}, \ldots, y_m) - y_k
\]
for each \( j = 1, \ldots, m \), where \( h_{j,0} \in \mathbb{K} \).

From (iii) and continuity of the \( \mathbb{K} \)-linear operator \( uP_{y_j}^n \) it follows, that to resolve (iv) it is sufficient to find a solution of the problem:

\[(v)\quad uP_{y_j}^nh_{y_j,0}^{[y_j]} = (uP_{y_j}^ny^t|^{[y_j]}) \ldots (uP_{y_j}^ny^t|^{[y_j]}) \]

for each \( l \in \mathbb{N} \) and each \( t^k = (t_1^k, \ldots, t_m^k) \in \mathbb{N}_0^m \), \( k = 1, \ldots, l \), \( y^t = y_1^t \ldots y_m^t \).

On the other hand,

\[(vi)\quad uP^n_z^t = \sum_{0 \leq j \leq n-1, k \in \mathbb{N}_0} t(t-1)(t-j+1)z_k^t / (j+1)!, \]

where \( z \in U \), \( t \in \mathbb{N} \), \( j \in \mathbb{Z} \). Moreover, \( (\partial/\partial y_j) uP_{y_j}^n |_{(\mathcal{C}(q_n,1)(U_m,\mathbb{K}))} = I \), hence Equation (v) can be simplified in the considered class of \( p_{y_j}C_0^{(q+1,n-1)}(U_m,\mathbb{K}) \)-functions acting on both sides of (v) by \( (\partial/\partial y_j) \). For each \( z \in \mathbb{Z}_K \) there exists a solution \( \underline{z} \) of (v) for each \( y \in U \) such that \( y \Delta z \), since the set \( \{ u \in \mathbb{Z}_K : u \Delta z \} \) is finite. In view of (vi) and §2.1 this family \( \{ \underline{z}h(y) : z \in \mathbb{Z}_K \} \) can be chosen consistent, that is, \( \underline{z}h(y) = \eta h(y) \) for each \( y \) such that \( y \Delta \min(z, \eta) \). Therefore, there exists

\[(vii)\quad h = \lim_{z \to \beta} \underline{z}h \]

such that (v) is satisfied for each \( y \in U \). In particular, \( id \in \mathcal{S}C^{(q+1,n-1)}(M, \mathbb{M}) \).

For the class \( p_{C^{(q,n)}}(U_m,\mathbb{K}) \) it is sufficient to find solution of the problem

\[(viii)\quad uP^n_{y_1 \ldots y_m}h(y) = (uP^n_{y_1}y^1) \ldots (uP^n_{y_m}y^m) \]

for each \( l \in \mathbb{N} \) and each \( t^k \in \mathbb{N}_0^m \), \( k = 1, \ldots, l \), \( |t| := t_1 + \ldots + t_m \geq 1 \). In view of (vi) and §2.1 and \( uP^n_{y_1} = uP^n_{y_2} \ldots uP^n_{y_m} \) there exists a consistent family \( \underline{z}h \) satisfying (viii) for each \( z \in \mathbb{Z}_K^m \) and each \( y \Delta z \) such that \( \underline{z}h(y) = \eta h(y) \) for each \( y \Delta \min(z, \eta) \), where \( \eta \in \mathbb{Z}_K^m \), since the set \( \{ u \in \mathbb{Z}_K^m : u \Delta z \} \) is finite, \( (\partial/\partial y_1) \ldots (\partial/\partial y_m) uP^n_{y_1 \ldots y_m} |_{(\mathcal{C}(q_n,1)(U_m,\mathbb{K}))} = I \) and the acting by \( (\partial/\partial y_1) \ldots (\partial/\partial y_m) \) on both sides of Equation (viii) simplifies it in the class of \( p_{C^{(q,n)}}(U_m,\mathbb{K}) \)-functions. Then

\[(ix)\quad h = \lim_{z \to (\beta_1, \ldots, \beta_\mathbb{Z})} \underline{z}h \]

is the solution of (viii). Therefore, \( (\phi \circ \psi(y) - y) \) and \( (\phi \circ \psi^{-1}(y) - y) \) belong to \( \mathcal{S}C^{(q+1,n-1)} \) or \( p_{C^{(q,n)}} \) correspondingly. The proof above also shows, that if a bijective surjective \( \psi \) is in \( p_{C^{(q,n)}}(M, \mathbb{M}) \) or in \( \mathcal{S}C^{(q+1,n-1)}(M, \mathbb{M}) \), then \( \psi^{-1} \) is in \( p_{C^{(q,n)}}(M, \mathbb{M}) \) or in \( \mathcal{S}C^{(q+1,n-1)}(M, \mathbb{M}) \) respectively, by solving the equation of the type \( v(id(y) + g(y)) = -g(y) \) relative to the function \( v \) for known \( g := \psi - id \). Hence using charts \( (V_j, \phi_j) \) of \( \tilde{A}t(M) \) such that \( \tilde{\phi}_j(V_j) = B \subset U^m + z_j \) with suitable \( z_j \in \mathbb{K}^m \) for each \( j \) and \( \tilde{A}t(M) \) is the refinement of \( At(M) \) and \( B \) satisfies Lemma 2.6.1 (or applying the above proof to \( B \) instead of \( U^m \)), we get that \( (\phi_1 \circ \phi \circ \psi^k \circ \phi^{-1}(y) - y) \) belongs to \( p_{C^{(q,n)}} \) or \( \mathcal{S}C^{(q+1,n-1)} \) respectively on its domain for each \( l \) and \( j \), where \( k = 1 \) or \( k = -1 \). Together with Lemma 2.6.1 it provides
\( \phi \circ \psi^k \in \text{Diff} P(q,n)(M) \) or \( \phi \circ \psi^k \in \text{Diff} S(q+1,n-1)(M) \) correspondingly for each \( k \in \{-1, 1\} \).

If \( M \) is compact, then \( pC(q,n)(M,Y) \) is normable for a Banach space \( Y \) over \( \mathbb{L}, \mathbb{K} \subset \mathbb{L} \) (see analogously Lemma 3.4 [14]). Let \( V = B(C(q,n-1)(M,Y), 0, 1) \), consider \( W := \{ f \in C(q+1,n-1)(M,Y) : f(x_1, ..., x_m) \in p_{x_j}C(q+1,n-1)(M,Y) \cap (P_{x_j}(V) \oplus Y) \) for each \( j = 1, ..., m \). In view of \( \mathbb{K} \)-convexity of \( V \) the set \( W \) is absolutely \( \mathbb{K} \)-convex (disked) and \( W \) is absorbing in \( SC(q+1,n-1)(M,Y) \), since \( P_{x_j} \) are continuous \( \mathbb{K} \)-linear and \( V \) is absorbing in \( C(q,n-1)(M,Y) \). Then \( W \) is bounded in the weak topology in \( SC(q+1,n-1)(M,Y) \). Therefore, the Minkowski functional on \( SC(q+1,n-1)(M,Y) \) generated by \( W \) induces a norm in \( SC(q+1,n-1)(M,Y) \) (see Exer. 6.204 [19]). Each space \( p_{x_j}C(q+1,n-1)(M,Y) \) is complete (see analogously with Lemma 3.4 [14]), since \( Y \) is complete.

Consider the \( \mathbb{K} \)-linear space \( \Psi_j := p_{x_j}C(q+1,n-1)(M,Y) \cap SC(q+1,n-1)(M,Y) \) and topologies \( \tau_{\phi,j} \) on \( p_{x_j}C(q+1,n-1)(M,Y) \) and \( \tau_S \) on \( SC(q+1,n-1)(M,Y) \) induced by norms in these spaces, then \( \tau_S|\Psi_j \subset \tau_{\phi,j} \) for each \( j \) due to continuity of \( P_{x_j} \) (for \( M \) supplied with coordinates \( x_j \)) due to \( pC(q,n) \) or \( SC(q+1,n-1) \), diffeomorphism with \( \Omega \) as in §2.2.5 and definition of \( \tau_S \), since \( \ker(P_{x_j}) = \{0\} \) and due to the open mapping Theorem (14.4.1) [19] there exists the continuous \( \mathbb{K} \)-linear operator \( (P_{x_j})^{-1} : (p_{x_j}C(q+1,n-1)(M,Y), \tau_{\phi,j}) \to (C(q,n-1)(M,Y), \| * \|_{C(q,n-1)(M,Y)}) \), consequently,

\[
(P_{x_j})^{-1} : (\Psi_{j,0}, \tau_S|\Psi_{j,0}) \to (C(q,n-1)(M,Y), \| * \|_{C(q,n-1)(M,Y)}),
\]

is continuous, where \( \Psi_{j,0} := \Psi_j \cap p_{x_j}C_0(q+1,n-1)(M,Y) \), \( \Psi_j = \Psi_{j,0} \oplus Y \). Hence \( SC(q+1,n-1)(M,Y) \) is complete relative to the above norm.

For noncompact \( M \) using a refinement \( At'(M) \) consisting of compact charts \((V', j', \phi'_j)\) and the strict inductive limits of \( pC(q,n)(\bigcup_{j=1}^l V'_j, Y) \) or \( SC(q+1,n-1)(\bigcup_{j=1}^l V'_j, Y, l \in \mathbb{N}) \), we get that \( pC(q,n)(M,Y) \) and \( SC(q+1,n-1)(M,Y) \) are complete relative to their uniformities (see Theorems (12.1.6) and (12.1.8) [19]). In view of Theorem (12.1.4) [19] these spaces are separable.

Let \((f - id) \in pC(q,n)(M,M) \) or \((f - id) \in SC(q+1,n-1)(M,M) \) such that \( M \) is compact and \( \max_{j,l} \| f_{i,j} - id_{i,j} \| < 1 \), where \( f_{i,j} := \phi_l \circ f \circ \phi_j^{-1} \), \( \text{dom}(f_{i,j}) =: U_{i,j} \), \( \| * \| \) is taken of the space \( pC(q,n)(U_{i,j}, \mathbb{K}^m) \) or \( SC(q+1,n-1)(U_{i,j}, \mathbb{K}^m) \). In view of the ultrametric inequality \( f_{i,j} \) is the isometry, since

\[
\| f_{i,j} - id_{i,j} \| = \sup_n \| a(n, f_{i,j} - id_{i,j}) \| \| Q_n \|,
\]

where \( \| * \| \) is the norm in \( pC(q,n)(U_{i,j}, \mathbb{K}^m) \) or in \( SC(q+1,n-1)(U_{i,j}, \mathbb{K}^m) \) respectively induced by the
norm in $C^{(q,n-1)}(U_{i,j}, K^m)$ and the Minkowski functional as above. Then $\|g_{k,t} \circ f_{i,j} - id_{k,j}\| \leq \max(\|g_{k,t} \circ f_{i,j} - f_{i,j}\|, \|f_{i,j} - id_{i,j}\|)$. Using partial difference quotients and $P^n$ and expansion coefficients in the Amice base we get, that

$$\max_{i,j} \|f_{i,j}^{-1} - id_{i,j}\| \leq C \max_{i,j} \|f_{i,j} - id_{i,j}\|,$$

$C = const > 0$ is independent of $f$ (see the proof of Theorem 2.6 [16]), consequently, $Diff^{(q,n)}(M)$ and $Diff^{(q+1,n-1)}(M)$ are topological groups. For noncompact $M$ having $At(M)$ with compact charts and using the strict inductive limit topology we can take an entourage of the diagonal in $pC^{(q,n)}(M, M)^2$ or in $sC^{(q+1,n-1)}(M, M)^2$ of the form $\{f : \|f_{i,j} - id_{i,j}\| \leq |\pi| \text{ for each } l, j \in \lambda\}$, where $\lambda$ is a finite subset in $N$. In view of Theorem A.4 [15] there exists the inverse mapping $f_{i,j}^{-1}$, which is the local diffeomorphism, when $dom(f_{i,j}) \neq \emptyset$. Then $f|_W = id|_W$ for $W := M \setminus \bigcup_{j \in \lambda} V_j$ for each $f \in Diff^{(q,n)}(M)$ and $Diff^{(q+1,n-1)}(M)$ with $W$ dependent on $f$, where $supp(f) := cl(\{x \in M : f(x) \neq x\})$ is compact, a finite subset $\lambda$ of $N$ is such that $supp(f) \subset \bigcup_{j \in \lambda} V_j$. This implies that $f(M) = M$ and $f^{-1}(M) = M$, consequently, $Diff^{(q,n)}(M)$ and $Diff^{(q+1,n-1)}(M)$ are neighborhoods of $id$ in $pC^{(q,n)}(M, M)$ and in $sC^{(q+1,n-1)}(M, M)$ respectively, left shifts in these groups $L_g := g^{-1}f$ imply that these groups are open in the corresponding to them spaces. Since $Diff^{(q,n)}(M)$ and $Diff^{(q+1,n-1)}(M)$ are complete, then they are clopen in $pC^{(q,n)}(M, M)$ and $sC^{(q+1,n-1)}(M, M)$ correspondingly (see Theorem 8.3.6 [5]).

Finally, statements (4, 5) follow from the proofs of Theorems 2.4 and 3.6 [16] modified for the considered here classes of smoothness.

3.18. Remark and Definition. Let $M$ and $N$ be two locally compact $C^{(q,n)}$-manifolds over $K$ and $f \in C^1(M, N)$, $dimK M =: m_M$, $dimK N =: m_N$. Denote by $E := E(f) := \{z \in M : rang(d_z f) < m_N\}$ and this set is called the set of critical values of $f$. The nonnegative Haar measure $\nu$ on $K^{m_N}$ as the additive group induces the measure $\mu$ on $N$ with the help of charts, since $At(N)$ has a disjoint refinement, where $\nu$ is normalized by the condition $\nu(B(K^{m_N}, 0, 1)) = 1$.

3.19. Theorem. Let $f : M \to N$ be a $C^l$-mapping of a $sC^{(q+1,n-1)}$-manifold $M$ into a $sC^{(q+1,n-1)}$-manifold $N$, where $l > \max(m_M, m_N)$. Then $\mu(f(E)) = 0$ (see §3.18).

Proof. Using the charts of atlases it is sufficient to prove the theorem for $f : U \to K^{m_N}$, where $U$ is an open subset in $K^{m_N}$. For $m_M = 0$ and
m_N = 0 the statement is evident, therefore, consider m_M ≥ 1 and m_N ≥ 1. Put E := \{y ∈ U : f^{(j)}(y) = 0 for each j ≤ i\}, hence E ⊃ E_1 ⊃ E_2 ⊃ .... To finish the proof use the following two lemmas.

3.19.1. Lemma. \( \mu(f(E \setminus E_1)) = 0 \).

Proof. Consider n ≥ 2, since for n = 1 there is only one partial derivative and from y ∈ E it follows y ∈ E_1. Let y ∈ E \ E_1, then there exists a nonzero partial derivative, for example, \( \frac{∂f_1(x)}{∂x_1} \) at the point x = y. There exists a mapping \( h : U \rightarrow K^{mN} \) such that \( h(x) := (f_1(x), x_2, ..., x_{m_N}) \) for which \( \text{rang}(dh(y)) = m_N \). In view of Theorem A.4 [15] the mapping h is the diffeomorphism of some open \( V = V(y) \subset U \) onto a neighborhood \( W \ni z := h(y) \). The set \( E' \) of critical points for \( g := f \circ h^{-1} : W \rightarrow K^{mN} \) coincides with \( h(V \cap E) \), that is, \( g(E') = f(V \cap E) \). Consider the family \( g^t : (\{t\} \times K^{mM-1}) \cap W \rightarrow \{t\} \times K^{mN-1} \), where \( t \in B(K, 0, 1) \). The point b is critical for \( g^t \) if and only if it is critical for g. In view of the induction hypothesis \( \mu(g^t(E(g^t))) = 0 \) in \( \{t\} \times K^{mM-1} \), hence \( \mu(g(E')) = \mu(g(E')) \cap (\{t\} \times K^{mN-1}) = 0 \) for each \( t \in B(K, 0, 1) \). From the Fubini theorem in \( L^1(K^m, \mu, R) \) it follows, that \( \mu(g(E')) = 0 \).

3.19.2. Lemma. \( \mu(f(E_k)) = 0 \) for each k such that \( 1 ≤ k < l \).

Proof. Take a covering of \( E_k \) by a countable number of balls of radius \( δ > 0, δ ≤ δ_0 \), where \( δ_0 > 0 \) is sufficiently small. Take one of these balls B. From the definition of \( E_k \) and the Taylor formula (see Theorem 29.4 [22] and Theorem A.5 [15]) it follows, that \( f(x + h) = f(x) + R(x, h) \), where \( \|R(x, h)\| ≤ b\|h\|^{l+1} \), \( x \in E_k, x + h \in B, b ≤ \|f\|_{C^{l+1}(U, K^{mN})} < \infty \) for a compact clopen \( U \subset K^{mM} \). Divide B into a disjoint union of \( q^{kM} \) balls of radius \( δ/q, q = p^{-n} \). Let \( B_1 \) be a ball of this partition such that \( B_1 \ni x \). Then each \( y \in B_1 \) has the form \( y = x + h, \) where \( |h| ≤ δ/q \). Then \( f(B_1) \subset B(K^{mN}, f(x), b/q^{k+1}) \), consequently, \( f(E_k \cap B) \) is contained in the union of \( q^{kN} \) balls \( B_j \) having \( \sum_j \mu(B_j) ≤ q^{kN}(b/q^{k+1})^{m_N} = b^{m_N}q^{-m_Nk} \). Then \( \lim_{q \to \infty} b^{m_N}q^{-m_Nk} = 0 \).

Therefore, Lemmas 3.19.1, 2 finish the proof of Theorem 3.19.

3.19.3. Corollary. The set \( N \setminus f(E) \) is dense in N, where \( f \in C^l(M, N) \) and \( l > \max(m_M, m_N) \).

3.19.4. Corollary. If \( \text{dim}_K M < \text{dim}_K N \), then \( \mu(f(M)) = 0 \).

3.20. Definitions. A \( C^1 \)-mapping \( f : M \rightarrow N \) is called an immersion, if \( \text{rang}(df|_x : T_x M \rightarrow T_{f(x)}N) = m_M \) for each \( x \in M \). An immersion \( f : M \rightarrow N \) is called an embedding, if f is bijective.

3.21. Theorem. Let M be a compact \( sC^{(q+1,n-1)} \) or \( pC^{(q,n)} \)-manifold
over a local field \( K \), \( \dim_K M = m < \infty \). Then there exists a \( sC(q+1,n-1) \) or \( pC(q,n) \)-embedding \( \tau : M \hookrightarrow K^{2m+1} \) and a \( sC(q+1,n-1) \) or \( pC(q,n) \)-immersion \( \theta : M \to K^{2m} \) corresponding. Each continuous mapping \( f : M \to K^{2m+1} \) or \( f : M \to K^{2m} \) can be approximated by \( \tau \) or \( \theta \) relative to the norm \( \| * \|_{C^0} \).

**Proof.** Let \( M \hookrightarrow K^N \) be the \( sC(q+1,n-1) \) or \( pC(q,n) \)-embedding of Theorem 2.2.6. Consider the bundle of all \( K \) straight lines in \( K^N \). They compose the projective space \( KP^{N-1} \). Fix the standard orthonormal (in the non-Archimedean sense) base \( \{ e_1, ..., e_N \} \) in \( K^N \) and projections on \( K \)-linear subspaces relative to this base \( P^l(x) := \sum_{j \in L} x_j e_j \) for the \( K \)-linear span \( L = \text{span}_K \{ e_i : i \in \Lambda_L \} \), \( \Lambda_L \subset \{ 1, ..., N \} \), where \( x = \sum_{j=1}^N x_j e_j \), \( x_j \in K \) for each \( j \). In this base consider the function \( (x,y) := \sum_{j=1}^N x_j y_j \). Let \( l \in KP^{N-1} \), take a \( K \)-hyperplane denoted by \( K_l^{N-1} \) and given by the condition: \( (x,[l]) = 0 \) for each \( x \in K_l^{N-1} \), where \( 0 \neq [l] \in K^N \) characterises \( l \). Take \( \| [l] \| = 1 \). Then the orthonormal base \( \{ q_1, ..., q_{N-1} \} \) in \( K_l^{N-1} \) and together with \( [l] =: q_N \) composes the orthonormal base \( \{ q_1, ..., q_N \} \) in \( K^N \) (see also [21]). This provides the projection \( \pi_l : K^N \to K_l^{N-1} \) relative to the orthonormal base \( \{ q_1, ..., q_N \} \). The operator \( \pi_l \) is \( K \)-linear, hence \( \pi_l \in sC(q+1,n-1) \), since \( P^n \) is the \( K \)-linear operator: \( \lim_{x_j} \lambda e_j b = \lambda (b-a)e_j \) for each \( \lambda \in K \) and \( a,b \in U \), \( j = 1, ..., N \).

To construct an immersion it is sufficient, that each projection \( \pi_l : T_x M \to K_l^{N-1} \) has \( \ker[\text{d}(\pi_l(x))] = \{ 0 \} \) for each \( x \in M \). The set of all \( x \in M \) for which \( \ker[\text{d}(\pi_l(x))] \neq \{ 0 \} \) is called the set of forbidden directions of the first kind. Forbidden are those and only those directions \( l \in KP^{N-1} \) for which there exists \( x \in M \) such that \( l' \subset T_x M \), where \( l' = [l] + z, z \in K^N \). The set of all forbidden directions of the first kind forms the \( C(q,n-1) \)-manifold \( Q \) of dimension \( 2m-1 \) with points \( (x,l) \), \( x \in M \), \( l \in KP^{N-1} \), \( [l] \in T_x M \), where \( C(q,n) \subset C(q+1,n-1) \) for each \( n \geq 1, q \geq 0 \). Take \( g : Q \to KP^{N-1} \) given by \( g(x,l) := l \). Then \( g \) is of class \( C(q,n-1) \). In view of Theorem 3.19 \( \mu(g(Q)) = 0 \), if \( N - 1 > 2m - 1 \), that is, \( 2m < N \). In particular, \( g(Q) \) is not contained in \( KP^{N-1} \) and there exists \( l_0 \notin g(Q) \), consequently, there exists \( \pi_{l_0} : M \to K_{l_0}^{N-1} \). Since \( sC(q+1,n-1) \) or \( pC(q,n) \) respectively is dense in \( C(q,n-1) \), then there exists a mapping \( \kappa \) such that \( \kappa \in sC(q+1,n-1) \) or \( \kappa \in pC(q,n) \) is sufficiently close to \( \pi_{l_0} \) relative to \( \| * \|_{C^1} \) correspondingly such that \( \kappa \circ \theta \) is the immersion, since \( M \) is compact. In view of Theorem 3.17 the composition \( \kappa \circ \theta \) is of class \( sC(q+1,n-1) \) or \( pC(q,n) \) correspondingly. This procedure can be prolonged, when \( 2m < N - k \), where \( k \) is the number of
the step of projection. Hence \( M \) can be immersed in \( K^{2m} \).

Consider now the forbidden directions of the second type: \( l \in KP^{N-1} \), for which there exist \( x \neq y \in M \) simultaneously belonging to \( l \) after suitable parallel translation \([l] \mapsto [l] + z, z \in K^N \). The set of the forbidden directions of the second type forms the manifold \( S := M^2 \setminus \Delta \), where \( \Delta := \{(x, x) : x \in M\} \). Consider \( \psi : S \to KP^{N-1} \), where \( \psi(x, y) \) is the straight \( K \)-line with the direction vector \([x, y] \) of the class \( \mathbb{S} \). Consider \( \pi_0 : M \to K^{N-1} \). Since \( sC(q+1,n-1) \) or \( pC(q,n) \) correspondingly is dense in \( C(q,n-1) \), then there exists a mapping \( \kappa \) such that \( \kappa \in sC(q+1,n-1) \) or \( \kappa \in pC(q,n) \) is sufficiently close to \( \pi_0 \) relative to \( \| * \|_{C^1} \) such that \( \kappa \circ \tau \) is the embedding, since \( M \) is compact. In view of Theorem 3.17 the composition \( \kappa \circ \tau \) is of class \( sC(q+1,n-1) \) or \( pC(q,n) \) correspondingly. This procedure can be prolonged, when \( 2m + 1 < N - k \), where \( k \) is the number of the step of projection. Hence \( M \) can be embedded into \( K^{2m+1} \).

3.21.2. Remark. Theorems 3.19 and 3.21 are non-Archimedean analogs of the Sard’s and Witney’s theorems. In Theorem 3.21 classes of smoothness globally on \( M \) are important. Theorem 3.21 justifies the considered class of manifolds \( M \) in the theorems above about antiderivational representations of functions.

3.22. Note and Definition. The proof of Theorem 3.17 shows, that the family of all diffeomorphisms of \( M \) of the class \( pC((t, s)) \) as it was defined slightly different in [14] also form the topological group. Moreover, spaces \( pC((t, s), \Omega \to Y) := P(l, s)[C((t, s - 1), \Omega \to Y)] \oplus Y \) and \( pC((t, s))(M, Y) \) are topologically \( K \)-linearly isomorphic, where \( l = [t] + 1 \), \( [t] \) is the integer part of \( t \), \( 0 \leq t \leq 1 \), though the antiderivations operators \( P(l, s) \) on a clopen subset \( X' = \Omega \) in \( B(K^m, 0, 1) \) (see §2.11 [12]) and \( \alpha P^s \) above (see §§2.1, 2.2.5) are different.

Define by induction spaces \( 1_s C^{\xi+(l,0)}(\Omega, Y) := \{ f \in C^{\xi+(l,0)}(\Omega, Y) : f(x_1, ..., x_m) \in uP^{m+1}/(l-1)^{-1}C^{\xi+(l-1,0)}(\Omega, Y)) \oplus Y \) for each \( j = 1, ..., m \}, \) where \( l \in \mathbb{N} \). \( 1_s C^{\xi+(1,0)}(\Omega, Y) := sC^{\xi+(1,0)}(\Omega, Y), 2_s C^{\xi}(\Omega, Y) := C^{\xi}(\Omega, Y) \).

3.23. Theorem. Let \( M \) be a \( 1_s C^{(q+1,n-1)} \)-manifold over \( K \) with \( l \geq 2 \), then there exists a clopen neighborhood \( \tilde{M} \) of \( M \) in \( \tilde{T}M \) and an exponential \( 1_s C^{(q+1,n-1)} \)-mapping \( \exp : \tilde{T}M \to M \) of \( \tilde{T}M \) on \( M \).

Proof. As in the proof of Theorem 3.7 [14] it can be shown that the
non-Archimedean geodesic equation $\nabla \dot{c} = 0$ with initial conditions $c(0) = x_0$, $\dot{c}(0) = y_0$, $x_0 \in M$, $y_0 \in T_{x_0} M$ has a unique $^l_S C^{(q+l,n-1)}$-solution, $c : B(K,0,1) \to M$. For a chart $(U_j, \phi_j)$ containing $x$, put $\psi_j(b) = \phi_j \circ c(b)$, then

$$\psi_j(b) = \phi_j(x_0) + u_P^{q+l+n}(y_0 + u_P^{q+l+n-1}f),$$

for each loop $\gamma$ for each loop $\lambda$. It remains to verify that $\psi_j(\beta; x_0, y_0)$ is of class of smoothness $^l_S C^{(q+l,n-1)}$, where $0 < \delta$, $\bar{x}_0 = \phi_j(x_0) \in V_1 \subset V_2 \subset \phi_j(U_j)$, $V_1$ and $V_2$ are clopen, $\delta$ and $V_1$ are sufficiently small, that to satisfy the inclusion $\psi_j(\beta; x_0, y_0) \in V_2$ for each $(\bar{x}_0, y_0) \in V_1 \times B(K^m, 0, \delta)$.

**3.24. Theorem.** Let $\Omega = \Omega_1 \times \cdots \times \Omega_m$ be a polydisk in $(K \oplus \alpha K)^m$ and let

$$s\bar{\mathcal{C}}^{(q+1,n-1)}(\Omega, K(\alpha)) := \{ f \in s\mathcal{C}^{(q+1,n-1)}(\Omega, K(\alpha)) : \bar{\partial} f = 0 \text{ on } \Omega \},$$

then $s\bar{\mathcal{C}}^{(q+1,n-1)}(\Omega, K(\alpha))|_{\bar{\Omega}}$ is the algebra over $K$, where $\bar{\Omega} := \{ z \in \Omega : z_j \text{ is encompassed by } \partial \Omega_j \}$ for each $j = 1, \ldots, m$, $z = (z_1, \ldots, z_m)$.

**Proof.** Evidently $s\bar{\mathcal{C}}^{(q+1,n-1)}(\Omega, K(\alpha))$ is the $K$-linear space, since $\bar{\partial}(\lambda f) = \lambda \bar{\partial} f$ for each $\lambda \in K$ and $\bar{\partial}(f+g) = \bar{\partial} f + \bar{\partial} g$ for each $f, g \in s\mathcal{C}^{(q+1,n-1)}(\Omega, K(\alpha))$.

It remains to verify that $fg|_{\bar{\Omega}} \in s\bar{\mathcal{C}}^{(q+1,n-1)}(\Omega, K(\alpha))|_{\bar{\Omega}}$ for each $f$ and $g \in s\bar{\mathcal{C}}^{(q+1,n-1)}(\Omega, K(\alpha))$, where as in §2.6.1 $s\bar{\mathcal{C}}^{(q+1,n-1)}(\Omega, K(\alpha))|_{\bar{\Omega}} = \{ h|_{\bar{\Omega}} : h \in s\bar{\mathcal{C}}^{(q+1,n-1)}(\Omega, K(\alpha)) \}$. In view of Theorem 2.7.2.(i) if $f$ and $g \in s\bar{\mathcal{C}}^{(q+1,n-1)}(\Omega, K(\alpha))$, then $f$ and $g$ are locally $z$-analytic on $\bar{\Omega}$, consequently, $fg$ is locally $z$-analytic on $\bar{\Omega}$. In view of Formula 2.7.8.(2) or by the direct computation:

(i) $\text{res}_{\xi}(z - \xi)^j = 0$ for each $-1 \neq j \in \mathbb{Z}$ and each $\xi \in \bar{\Omega}$, since $\text{res}_{\xi} h = 0$ for each $h$ having a decomposition 2.7.8.(1) with $a_{-1} = a_{-1}(h) = 0$, indeed it is true for the particular $h(\beta) = h(0)$ for a loop $\gamma$ encompassing 0 and such that $h(x) := \text{Exp}[j\text{Log}(\gamma(x))]$ and $j \in \mathbb{Z}$, $x \in B(K,0,1)$, that leads to the general case.

On the other hand, $(fg)'(z)$ also is locally $z$-analytic on $\bar{\Omega}$. Therefore,

(ii) $\gamma_j P^n[(f g)'(z) dz_j] = 0$ and particularly

$$\gamma_j P^n[\partial (f(z) g(z))/\partial z_j dz_j] = 0$$

for each loop $\gamma_j$ in $\Omega_j$ encompassed by $\partial \Omega_j$ (see Theorem 2.5.3), where $z = (z_1, \ldots, z_m)$, $\Omega = \Omega_1 \times \cdots \times \Omega_m$, $\Omega_j$ is a ball in $K \oplus \alpha K$, $z_j \in K \oplus \alpha K$ for each $j = 1, 2, \ldots, m$. Then

$$\partial (\gamma_j P^n[(\partial (f(z) g(z))/\partial z_j) dz_j]) / \partial z_j = \partial (f g)(z)/\partial z_j$$

and
\[ \gamma_j P^n[(\partial (fg)(z)/\partial z_j)dz_j] = (fg)(z_1, ..., z_{j-1}, \gamma_j(\beta), z_{j+1}, ..., z_m) \]
\[-(fg)(z_1, ..., z_{j-1}, \gamma_j(0), z_{j+1}, ..., z_m). \]
Moreover, \[ \gamma_j P^n[h_j(z)dz_j] = B P^n[h_j(z_1, ..., z_{j-1}, \gamma_j(\zeta), z_{j+1}, ..., z_m)dz_j(\zeta), \]
\[ \gamma_j P^n[(\partial (fg)(z)/\partial z_j)dz_j] = B P^n[v_j(z_1, ..., z_{j-1}, \gamma_j(\zeta), z_{j+1}, ..., z_m)dz_j(\zeta)] \]
\[ \partial( B P^n[h_j(z_1, ..., z_{j-1}, \gamma_j(\zeta), z_{j+1}, ..., z_m)dz_j(\zeta)]/\partial \zeta \]
\[ = h_j(z_1, ..., z_{j-1}, \gamma_j(\zeta), z_{j+1}, ..., z_m)\gamma_j'(\zeta), \]
\[ \partial( B P^n[v_j(z_1, ..., z_{j-1}, \gamma_j(\zeta), z_{j+1}, ..., z_m)dz_j(\zeta)]/\partial \zeta \]
\[ = v_j(z_1, ..., z_{j-1}, \gamma_j(\zeta), z_{j+1}, ..., z_m)\gamma_j'(\zeta), \gamma_j \in pC^n(B, K(\alpha)) \]
where \( v_j(z) := \partial (fg)(z)/\partial z_j, \zeta \in B := B(0,1) \). Proceeding as in the proof of Theorem 3.17 with the help of (i, ii) and Equation 2.2.5.1 (see Equations 3.17.iii-viii) find \( h_j \in C^{(q-n-1)}(\Omega, K(\alpha)) \) such that \( h_j \) is locally \( z \)-analytic and \[ \gamma_j P^n[h_j(z_1, ..., z_{j-1}, \zeta, z_{j+1}, ..., z_m)dz_j(\zeta)] = (fg)(z) - (fg)(z_0) \]
for each \( z \in \Omega \) and each \( j = 1, ..., m \), where \( \gamma_j \) is a path with \( \gamma_j(0) = z_{j,0} \in \Omega_j, \gamma_j(\beta) = z_j \)
for each \( j = 1, ..., m \). This means, that \( (f,g) \in sC^{(q+1,n-1)}(\Omega, K(\alpha))|_{\tilde{\Omega}} \), since \[ \partial \gamma_j P^n[h_j(z_1, ..., z_{j-1}, \zeta, z_{j+1}, ..., z_m)dz_j(\zeta)]/\partial \zeta_j \]
\[ = \partial \gamma_j P^n[h_j(z_1, ..., z_{j-1}, \zeta, z_{j+1}, ..., z_m)dz_j(\zeta)]/\partial z_j = h_j(z) \]
\[ \partial \gamma_j P^n[h_j(z_1, ..., z_{j-1}, \zeta, z_{j+1}, ..., z_m)dz_j(\zeta)]/\partial y_j = \alpha h_j(z) \]
(see Formulas 2.4.1.i, ii) such that \( v x_j h_j x_j, y_j, y_j \) and \( v y_j h_j y_j \) as particular cases of \( \gamma_j \) along axes \( x_j \)
and \( y_j \) give the desired result.

3.25. Corollary. The space \( sC^{(q+1,n-1)}(\Omega, K(\alpha))|_{\tilde{\Omega}} \) contains all locally \( z \)-analytic functions on \( \Omega \).

Proof. Mention that \( 1 \in C^{(q,n-1)}(\Omega, K(\alpha)) \) and \( v P^n x x_0 = x - x_0, \]
\( v y^n y y_0 = y - y_0, \gamma_j P^n = \gamma_j(\beta) - \gamma_j(0) = z_j - z_{j,0} \),
where \( \gamma_j \in \Omega_j \), hence \( z_j - z_{j,0} \in sC^{(q+1,n-1)}(\Omega, K(\alpha)) \) for each \( z_j \) and \( z_{j,0} \in \Omega_j = \pi_j(\tilde{\Omega}_j) \).
It is possible to take \( \gamma_j \) contained in balls \( B \) such that \( B \subset \Omega_j \). Therefore, \[ \gamma_j P^n \sum_{k=1}^k x B_k = \sum_{k=1}^k a_t \gamma_j P^n x B_k \in sC^{(q+1,n-1)}(\Omega_j, K(\alpha)) \]
where \( B_k \) are balls satisfying conditions of Lemma 2.6.1, \( a_t \in K(\alpha), k \in N \). In view of Theorem 3.24 each polynomial in \( z \) belongs to \( sC^{(q+1,n-1)}(\Omega, K(\alpha))|_{\tilde{\Omega}} \). The using of expansions into series by \( z \) of locally \( z \)-analytic functions and limits of sequences of polynomials in \( z \) and Lemma 2.6.1 leads to the conclusion that each locally \( z \)-analytic function on \( \tilde{\Omega} \) belongs to \( sC^{(q+1,n-1)}(\Omega, K(\alpha))|_{\tilde{\Omega}} \).

3.26. Note. From Corollary 3.25 it follows, that a \( sC^{(q+1,n-1)} \)-manifold \( M \) is locally \( z \)-analytic manifold and there exists a refinement \( At(M) = \{(\tilde{U}_j, \tilde{\phi}_j) : j \} \) of \( At(M) \) such that transition mappings \( \tilde{\phi}_k \circ \tilde{\phi}_j^{-1} \) are \( z \)-analytic for each \( \tilde{U}_j \cap \tilde{U}_l \neq \emptyset \). If \( f \) is \( z \)-analytic on \( \tilde{\Omega} \), then \( f' \) is \( z \)-analytic on \( \tilde{\Omega} \).
Therefore, there exists a family \( \hat{\mathcal{Y}} \) of the cardinality \( card(\hat{\mathcal{Y}}) = c := card(R) \)
of all functions \( f \in s\mathcal{C}^{(q+1,n-1)}(\Omega, K(\alpha)) \) and \( f \) is not \( z \)-analytic on \( \tilde{\Omega} \), since a locally \( z \)-analytic function is not necessarily \( z \)-analytic. For example, take 

\[ h \in C^{(q,n-1)}(\Omega, K(\alpha)) \]

locally \( z \)-analytic on \( \tilde{\Omega} \) and nonanalytic on \( \Omega \) and put 

\[ f(z) = P^n h, \]

where \( \gamma(0) = z_0, \gamma(\beta) = z, \Omega \) is a polydisc (see §3.25). Indeed, each locally polynomial in \( z \) nonpolynomial \( h : \Omega \to K(\alpha) \) and its iterated antiderivatives along paths 

\[ h_k(z) := P^k h_{k-1}, k = 1, \ldots, n, \]

\( h_0 := h \), up to order \( n \) fit this construction. For nonlocally compact fields there is the theory of analytic elements [6].

3.27. Corollary. Let \( L \) be a non-Archimedean field such that \( K(\alpha) \subset L \) with a valuation \( | \ast |_L \) extending that of \( K(\alpha) \) and let \( L \) be complete relative to \( | \ast |_L \). Suppose \( \Omega \) is a clopen compact subset in \((K \oplus \alpha K)^m\) and \( Y \) is a Banach space over \( L \). Then \( f \in s\mathcal{C}^{(q+1,n-1)}(\Omega, Y) \) if and only if there exists an open subset \( W \) in \( L^m \) and a locally \( z \)-analytic function \( F \) on \( W, z \in W \), such that \( W \cap (K \oplus \alpha K)^m \supset \tilde{\Omega} \) and \( F|_{\tilde{\Omega}} = f \).

Proof. The valuation group \( \Gamma_{K(\alpha)} \) is discrete, hence \( Y \) as the \( K(\alpha) \)-linear space has an orthonormal base \( \{ e_j : j \in \Lambda \} \), where \( \Lambda \) is a set (see Chapter 5 [21]). Therefore, \( F : W \to Y \) has the decomposition 

\[ F(z) = \sum_{j \in \Lambda} F_j(z) e_j, \]

where \( F_j : W \to K(\alpha) \). Since \( F \) is locally \( z \)-analytic, then \( f \) is locally \( z \)-analytic on \( \tilde{\Omega} \) and in accordance with Corollary 3.25 

\[ f \in s\mathcal{C}^{(q+1,n-1)}(\Omega, Y)|_{\tilde{\Omega}}. \]

Vice versa, if \( f \in s\mathcal{C}^{(q+1,n-1)}(\Omega, Y)|_{\tilde{\Omega}} \), then by Theorem 2.7.2 \( f \) is locally \( z \)-analytic on \( \tilde{\Omega} \), consequently, for each \( \zeta \in \tilde{\Omega} \) there exists a ball 

\[ B(K(\alpha), \zeta, R(\zeta)) \]

with \( 0 < R(\zeta) < \infty \) on which the power series 2.7.2(2) is uniformly convergent, that is, 

\[ \lim_{|k| \to \infty} |a_k|_{L^k} = 0, \]

hence this series is uniformly convergent on \( B(L, \zeta, R(\zeta)) \) also. Put 

\[ W = \bigcup_{\zeta \in \tilde{\Omega}} B(L, \zeta, R(\zeta)). \]

3.28. Definition and Note. Let \( L, \Omega = \Omega(f), W = W(f) \) be satisfying conditions of §3.27 with \( m = 1 \). Let also \( T \in L(Y) \) be a bounded \( L \)-linear operator on a Banach space \( Y \) over \( L \) with a nonvoid spectrum \( \sigma(T) := \{ b \in L : (bI - T) \) is not invertible in \( L(Y) \} \) (see Chapter 6 in [21]), where \( L(X, Y) \) is the Banach space of all bounded \( L \)-linear operators \( T : X \to Y \) for Banach spaces \( X \) and \( Y \) over \( L \), 

\[ \|T\| := \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|}, \quad L(Y) := L(Y, Y). \]

Suppose in addition, that for each \( z \in W \) with \( \text{dist}(z, \Omega) < \infty \) there exist 

\[ R \geq \text{dist}(z, \Omega) \]

and \( \zeta \in \Omega \) such that 

\[ B(L, z, R) \subset W \]

and 

\[ B(K \oplus \alpha K, \zeta, R) \subset \Omega. \]

Denote by \( \mathcal{F}(T) \) a family of all functions \( f \) with 

\[ \psi_f \in s\mathcal{C}^{(q+1,n-1)}(\omega, L) \]

where \( W \) is a clopen neighborhood of \( \sigma(T), W = W(f), \Omega = W \cap (K \oplus \alpha K) \neq \emptyset, 0 < \text{dist}(\partial \Omega, \sigma(T)) := \inf_{z \in \partial \Omega} \text{dist}(z, \sigma(T)), \]

\[ \text{dist}(z, G) := \inf_{y \in G} |z - y| \]

for \( G \subset L \) and \( z \in L \), \( 0 \leq q \in \mathbb{Z}, 1 \leq n \in \mathbb{N}, \psi_f(\eta) := f(z + \text{Exp}(\eta)), \omega := \omega(z) := \{ \eta \in K(\alpha) : z + \text{Exp}(\eta) \in \Omega \}, z \in \Omega, \]

\[ 45 \]
\( \omega_\epsilon = \omega \setminus \text{Log}(B(K(\alpha), z, \epsilon)), \epsilon = \epsilon_j, \epsilon_j > 0 \) for each \( j \in \mathbb{N} \), \( \lim_{j \to \infty} \epsilon_j = 0 \), there exists a locally \( z \)-analytic function \( \Psi_f \) on \( W \) such that \( \Psi_f|_\Omega = \psi_f \) (see §3.27). Put

(1) \( f(T) = C(\alpha)^{-1} \partial_\Omega P^n[f(\zeta)R(\zeta; T)d\zeta] \), where \( R(\zeta; T) = (\zeta I - T)^{-1} \) for \( \zeta \in \rho(T) := L \setminus \sigma(T) \) and the antiderivative is supposed to be convergent in the strong operator topology sense, that is, \( \partial_\Omega P^n[f(\zeta)R(\zeta; Tyd\zeta] \) converges for each \( y \in Y \). There are others definitions of spectral sets (see Chapter 6 [21]), but this one is used here.

3.29. Theorem. Let \( \sigma(T) \neq \emptyset, \sigma(T) \subset L, f, g \in \mathcal{F}(T), a, b \in L \) (see §3.28). Then

(i) \( af + bg \in \mathcal{F}(T) \) and \( aF(T) + b(T) = (af + bg)(T) \);

(ii) \( fg \in \mathcal{F}(T) \) and \( f(T)g(T) = (fg)(T) \);

(iii) if \( f(z) = \sum_{k=0}^{\infty} a_k z_k \) on \( W(f) \) such that \( W(f) \supset \sigma(T) \), then \( f(T) = \sum_{k=0}^{\infty} a_k T^k \).

Proof. Definition 3.28 is correct, since \( xI - T \) is invertible in \( L(Y) \) for each \( x \in \rho(T) := L \setminus \sigma(T) \), hence \( \rho_\sigma(T) := \sup_{x \in \sigma(T)} |x| \leq \|T\| \), where \( \rho(T) \) is open in \( L \) and \( R(x; T) \) is locally \( x \)-analytic on \( \rho(T) \) (see Chapter VII in [4] and Chapter 6 in [21]).

(i). It follows from Definition 3.28 and Corollary 3.27.

(ii). In view of Corollary 3.27 and Theorem 3.24 \( fg \in \mathcal{F}(T) \), since \( W(f) \cap W(g) =: W(fg) \supset \sigma(T) \). Without loss of generality take \( \Omega(f) \) encompassed by \( \partial \Omega(g) \) shrinking \( \Omega(f) \) a little if necessary such that \( W(f) \supset \sigma(T), W(f) \subset W(g) \). Then

\[
\begin{align*}
  f(T)g(T) &= C(\alpha)^{-2} \int_{\zeta \in \partial \Omega(f)} P^n[f(\zeta)R(\zeta; T)d\zeta] \int_{\kappa \in \partial \Omega(g)} P^n[g(\kappa)R(\kappa; T)d\kappa] \\
  &= C(\alpha)^{-2} \int_{\zeta \in \partial \Omega(f)} P^n[f(\zeta)g(\kappa)\{R(\zeta; T)R(\kappa; T)\}d\zeta] \\
  &= C(\alpha)^{-2} \int_{\zeta \in \partial \Omega(f)} P^n[f(\zeta)g(\kappa)(\kappa - \zeta)^{-1}d\zeta]
\end{align*}
\]

On the other hand, \( R(\zeta; T)R(\kappa; T) = (R(\zeta; T) - R(\kappa; T))(\kappa - \zeta)^{-1} \). Therefore,

\[
\begin{align*}
  (1) \ f(T)g(T) &= C(\alpha)^{-2} \int_{\zeta \in \partial \Omega(f)} P^n[f(\zeta)R(\zeta; T)\{\kappa \in \partial \Omega(g)P^n[g(\kappa)(\kappa - \zeta)^{-1}d\kappa]\}]d\zeta \\
  &= C(\alpha)^{-2} \int_{\zeta \in \partial \Omega(f)} P^n[f(\zeta)R(\zeta; T)\{\zeta \in \partial \Omega(g)P^n[f(\zeta)(\kappa - \zeta)^{-1}d\kappa]\}]d\zeta
\end{align*}
\]

The second term on the right hand side of (1) is zero, since \( \partial \Omega(f) \) is encompassed by \( \partial \Omega(g) \), \( \zeta \in \partial \Omega(f) \) (see Formulas 2.72.(2 – 4)). Hence

\[
\begin{align*}
  f(T)g(T) &= C(\alpha)^{-1} \int_{\zeta \in \partial \Omega(f)} P^n[f(\zeta)g(\zeta)R(\zeta; T)d\zeta] = (f)g(T). \\
  (iii) \text{ It follows from Definition 3.28 and Formulas 2.72.(2 – 4) applied to } f(\zeta)R(\zeta; Ty) \text{ for each } y \in Y.
\end{align*}
\]

3.30. Theorem. Let \( \sigma(T) \neq \emptyset, \sigma(T) \subset L, f \in \mathcal{F}(T) \) (see §3.28). Then \( f(\sigma(T)) = \sigma(f(T)) \).
3.31. Theorem. Let $\sigma(T) \neq \emptyset$, $\sigma(T) \subset L$, $f \in \mathcal{F}(T)$, $g \in \mathcal{F}(f(T))$ (see §3.28) and $h(z) := g(f(z))$ for each $z \in f^{-1}[W(g) \cap f(W(f))]$. Then $h \in \mathcal{F}(T)$ and $h(T) = g(f(T))$.

Proof. Theorem 3.30 follows from Theorem 3.29 analogously to Theorems VII.3.10 [4] and 3.3.6 [9]. The function $f$ is locally $z$-analytic on $W(f)$, $g$ is locally $z$-analytic on $W(g)$, hence $h$ is locally $z$-analytic on $f^{-1}[W(g) \cap f(W(f))]$. In view of Theorem 3.30 $\sigma(f(T)) \subset f(W(f)) \cap W(g)$, hence $h$ is defined on open $W(h)$ such that $W(h) \supset \sigma(T)$. Without loss of generality take $W(g) \supset f(W(f))$. Put

$$S(\kappa) = C(\alpha)^{-1} \zeta \partial(\mathcal{O}(f)) P^n[R(\zeta; T)(\kappa - f(\zeta))^{-1}d\zeta],$$

then in accordance with Theorems 3.29 and 2.4.6 (applied on pieces of $\Omega(f)$ affine homotopic to points) $(\kappa I - T)S(\kappa) = S(\kappa)(\kappa I - T) = I$, consequently, $S(\kappa) = R(\kappa; T)$. Therefore,

$$g(f(T)) = C(\alpha)^{-1} \zeta \partial(\mathcal{O}(f)) P^n[g(\kappa)R(\kappa; f(T))d\kappa]$$

$$= -C(\alpha)^{-2} \zeta \partial(\mathcal{O}(f)) P^n[g(\kappa)R(\zeta; T)(\kappa - f(\zeta))^{-1}d\zeta]d\kappa]$$

$$= C(\alpha)^{-1} \zeta \partial(\mathcal{O}(f)) P^n[R(\zeta; T)g(\kappa)R(\zeta; T)d\zeta] = h(T).$$

3.32. Proposition. Let $f_k \in \mathcal{F}(T)$ for each $k \in N$ (see §3.28) and there exists a clopen subset $W$ in $L$ such that $\sigma(T) \subset W \subset \cap_{n=1}^\infty W(f_n)$. If $f_k$ converges to $f$ uniformly on $W$, then $f_n(T)$ converges to $f(T)$ uniformly on each totally bounded subset in $Y$.

Proof. There exists a sequence $C(\alpha)^{-1} \zeta \partial P^n[f_k(\zeta)R(\zeta; T)d\zeta] \in L(X, Y)$ in the topology of pointwise convergence, where $L(X, Y)$ denotes the Banach space of continuous $L$-linear operators $S : X \to Y$ for two Banach spaces $X$ and $Y$ over $L$. In view of Theorem (11.6.3) and Example 11.202.(g) [19] this sequence converges to a $L$-linear operator on $Y$ uniformly on each totally bounded subset in $Y$.

3.33. Definition. A point $z_0 \in \sigma(T)$ is called an isolated point of a spectrum $\sigma(T)$, if there exists a neighborhood $U$ of $z_0$ such that $\sigma(T) \cap U = \{z_0\}$, where $U$ satisfies the same conditions of §3.28 as $W$. An isolated point $z_0 \in \sigma(T)$ is called a pole of an operator $T$ or a pole of a spectrum, if a mapping $R(\zeta; T)$ has a pole at $z_0$. An order $j(z_0)$ of a pole $z_0$ is an order of $z_0$ as a pole of $R(\zeta; T)$.

3.34. Theorem. Let $f, g \in \mathcal{F}(T)$ (see §3.28). Then $f(T) = g(T)$ if and only if $f(\zeta) = g(\zeta)$ on a clopen $W$ such that $\sigma(T) \setminus \bigcup_{\lambda \in \Lambda} \{z_\lambda\} \subset W \subset L$, where $z_\lambda \in \Omega \subset K \oplus \alpha K$ is a pole for each $l \in \Lambda$, $\Lambda$ is a finite set and $(f - g)$ at $z_l$ has zero of order not less than $j(z_l)$ for each $l = 1, \ldots, k$.

Proof. Without loss of generality take $g = 0$ and let $f = 0$ on $W \setminus
\[ f(T) = C(\alpha)^{-1} \sum_{l \in \Lambda} \partial_{\partial T} P^n[f(\zeta)R(\zeta; T)d\zeta], \quad \text{where } B_l := B(\mathbf{K} + \alpha \mathbf{K}, z_l, R_l), \]

\[ 0 < R_l < \infty \text{ and } B_l \cap \sigma(T) = \{z_l\} \text{ for each } l \in \Lambda. \]

Since \( f(\zeta)R(\zeta; T) \) is regular on \( B_l \), then by Theorem 2.7.14 \( f(T) = 0 \). Vice versa, let \( f(T) = 0 \), then by Theorem 3.30 \( f(\sigma(T)) = 0 \). The set \( \sigma(T) \cap (\mathbf{K} + \alpha \mathbf{K}) \) is compact and it can be covered by a finite union of balls \( B(\mathbf{K} + \alpha \mathbf{K}, \zeta_j, R_j), 0 < R_j < \infty \). If \( B(\mathbf{K} + \alpha \mathbf{K}, \zeta_j, R_j) \cap \sigma(T) \) is infinite, then for each limit point \( x \) of the latter set there exists a clopen neighborhood \( V_x \) on which \( f|_{V_x} = 0 \) (see Theorem 2.7.7). Therefore, \( \sigma(T) \cap (\mathbf{K} + \alpha \mathbf{K}) \setminus \bigcup_x V_x \) consists of a finite number of isolated points \( \{\lambda_l : l = 1, \ldots, k\} \), since \( \Omega \supset (\mathbf{K} + \alpha \mathbf{K}) \cap \sigma(T) \), \( \Omega \) is compact. Let \( f \) is not zero on any neighborhood of \( \lambda_1 \). Since \( \lambda_1 \in \sigma(T) \) and \( f(\sigma(T)) = \{0\} \), then \( f \) has a zero of finite order \( j \), hence \( \gamma_j(z) = (\lambda_1 - z)^j / f(z) \) is locally \( z \)-analytic on a neighborhood of \( \lambda_1 \). From the proof of Theorem 3.24 it follows, that

1. \[ R(\zeta; T) = \sum_{m=0}^{\infty} a_m(\lambda_1 - \zeta)^m \]
on \( B(\epsilon) := B(\mathbf{K} + \alpha \mathbf{K}, \lambda_1, \epsilon) \) for a sufficiently small \( 0 < \epsilon < \infty \), where

2. \[ a_{-m} = -C(\alpha)^{-1} \partial_{\partial T} P^n[(\lambda_1 - \zeta)^{m-1}R(\zeta; T)d\zeta] = -(\lambda_1 I - T)^{m-1}h(T), \]
h\( (T) \) denotes a function equal to 1 on \( B(\epsilon) \) and zero on a neighborhood of \( \lambda_1 \) for each \( l \neq 1 \) such that \( \psi(\eta) = h(z + \text{Exp}(\eta)) \) satisfies 3.28, that is, possible due to Lemma 2.6.1 and Corollary 3.27, since \( \text{Exp}(\eta) \) is locally \( \eta \)-analytic. Then \( a_{-m} = -(\lambda_1 I - T)^m h(T) = 0 \) for each \( m \geq j \).

3.35. Definition and Note. A subset \( V \) of \( \sigma(T) \) clopen in \( \sigma(T) \) is called a spectral set if it has a clopen neighborhood \( W_V \) satisfying the same conditions of §3.28 as \( W \) and \( W_V \cap (\sigma(T) \setminus V) = \emptyset \). In accordance with Lemma 2.6.1 and Theorem 3.24 consider \( f \in \mathcal{F}(T) \) such that \( f|_V = 1 \) and \( f|_{\sigma(T) \setminus V} = 0 \), which is possible due to Corollary 3.27, since \( \text{Exp}(\eta) \) is locally \( \eta \)-analytic. Put \( E(V; T) := f(T) \). In view of Theorem 3.34 \( E(V; T) \) depends on \( V \), but not on a concrete choice of \( f \) from its definition. If \( V \cap \sigma(T) = \emptyset \), put \( E(V; T) = 0 \). Write also \( E(z; T) := E(\{z\}; T) \) for a singleton \( \{z\} \). An index \( j = j(z) \) of \( z \in L \) is the smallest integer \( j \) such that \( (z I - T)^j y = 0 \) for each \( y \in Y \) with \( (z I - T)^{j+1} y = 0 \).

3.36. Theorem. Let \( T, W, \Omega, \mathbf{K}(\alpha) \) be as in §§3.27, 3.28. If \( z_0 \) is a pole of \( T \) of order \( j \), then \( z_0 \in \Omega \) has the index \( j \). An isolated point \( z_0 \in \sigma(T) \) is a pole of order \( j \) if and only if

1. \[ (z_0 I - T)^j E(z_0; T) = 0, \]
2. \[ (z_0 I - T)^{j-1} E(z_0; T) \neq 0. \]

Proof. In view of Formulas 3.34.(1,2) \( z_0 \) is a pole of order \( j \) if and
only if (i) is satisfied, since $a_{-m-1} = -(z_0 I - T)^m E(z_0; T)$. The rest of the proof is analogous to that of Theorem VII.3.18 [4] due to Corollary 3.27 and Theorem 3.24.

In view of Theorem 3.29.(ii) 
\[ E(V; T)E(V; T) = E(V; T) \]
that is, $E(V; T)$ is the projection operator on $Y$ (see Chapter 3 [21]).

3.37. **Theorem.** Let $f \in \mathcal{F}(T)$ (see §3.28) and let $V$ be a spectral set of $f(T)$. Then $\sigma(T) \cap f^{-1}(V)$ is the spectral set of $T$ and $E(V; f(T)) = E(f^{-1}(V); T)$.

**Proof.** Let $h_V \in \mathcal{F}(T)$ such that $h_V(z) = 1$ on a neighborhood $V_1$ of $V$, $h_V(z) = 0$ on a neighborhood $V_2$ of $\sigma(f(T)) \setminus V_1$, where $V_1 \cap V_2 = \emptyset$, which is possible due to Theorem 3.24, Corollary 3.27 and Lemma 2.6.1, since $E(x)p(i)$ is locally $\eta$-analytic. Then $h_V(f(T)) = E(V; f(T))$. In view of Theorem 3.30 $\sigma(T) = f^{-1}(V) \cup f^{-1}(\sigma(f(T)) \setminus V)$, where $f^{-1}(V) \cap f^{-1}(\sigma(f(T)) \setminus V) = \emptyset$. Since $f$ is continuous, then $f^{-1}(V)$ and $f^{-1}(\sigma(T) \setminus V)$ are clopen in $\sigma(T)$. Therefore, $\sigma(T) \cap f^{-1}(V) =: \mathcal{Y}$ is the spectral set of $T$. Put $t_T(z) := h_V(f(z))$, then $E(\mathcal{Y}; T) = t_T(T)$, since $t_T \in \mathcal{F}(T)$ due to Corollary 3.27. From Theorem 3.31 it follows, that $E(V; f(T)) = E(\mathcal{Y}; T) = E(f^{-1}(V); T)$.

3.38. **Remark.** In the non-Archimedean case the Gelfand-Naimark Theorem (IX.3.7 [4]) is not true (see Chapter 6 [21]). Therefore, the existence of the projection operator $E(V; T)$ for each spectral set $V$ does not imply a spectral projection-valued measure decomposition of $T$ (see also [18]). Here is considered a particular class of operators satisfying conditions of §3.28 for which the operator $E(V; T)$ is defined for each spectral set $V$, $V \subset \sigma(T)$. Put $Y_V := E(V; T)Y$. In view of Theorem 3.29.(ii) and §3.35 $TY_V \subset Y_V$, where $Y_V$ is the $L$-linear subspace in $Y$, since $E(V; T)$ is $L$-linear, denote $T_V := T|_{Y_V}$.

3.39. **Theorem.** Let $V$ be a spectral set of $\sigma(T) \neq \emptyset$ (see §3.28). Then $\sigma(T_V) = V$. If $f \in \mathcal{F}(T)$, then $f \in \mathcal{F}(T_V)$ and $f(T_V) = f(T_V)$. A point $z_0 \in V \cap \Omega$ is the pole of $T$ of order $j$ if and only if $z_0 \in \Omega$ is the pole of $T_V$ of order $j$.

**Proof.** Take a marked point $z \in V$ and suppose $z \notin \sigma(T_V)$. In view of Corollary 3.27 there exists a function $g \in \mathcal{F}(T)$ such that $g|_{V_1} = 0$ on a neighborhood $V_1$ of $V$ and $g(\zeta) = (z_0 - \zeta)^{-1}$ for each $\zeta \in V_2$, where $V_2$ is open in $L$, $V_1 \cap V_2 = \emptyset$, $V_2 \supset \sigma(T) \setminus V$. In view of Theorem 3.29.(ii) $g(T)(zI - T) = (zI - T)g(T) = I - E(V; T)$. Then $V \subset \sigma(T_V)$ as in Theorem VII.3.20 [4].
Vice versa, let $z \notin V$. Consider $h \in \mathcal{F}(T)$ (see §3.28) such that $h(\zeta)|_{V_1} = (z - \zeta)^{-1}$ and $h|_{V_2} = 0$, where $V_1$ is chosen such that $z \notin V_1$, $V_1$ is a neighborhood of $V$, $V_2$ is as above. Then by Theorem 3.29.(ii) $h(T)(zI - T)h(T) = E(V; T)$. Therefore, $h(T)|_{(zI - T)h(T)} = (zI - T)_V h(T)_V = I_V$, since $z \notin \sigma(T_V)$, consequently, $(zI - T)_V \subset V$ and $R(z; T_V) = R(z; T)_V$. Take $f \in \mathcal{F}(T)$ and a neighborhood $W$ of $\sigma(T)$ as in §3.28. Then $f(T)_V = C(\alpha)^{-1} \partial_0 P^n[f(z)R(z; T)_Vdz] = f(T_V)$ and $E(z; T)E(V; T) = E(z; T)$ for each $z \in V$, hence $(zI - T)_V^k E(z; T) = (zI - T)_V^k E(z; T)$ for each $k \in \mathbb{N}$. In view of Theorem 3.36 $z_0 \in \Omega \cap V$ is a pole of $T$ of order $j$ if and only if it is a pole of $T_V$ of order $j$.

3.40. Corollary. The mapping $E \mapsto E(V; T)$ is the isomorphism of the algebra $\mathcal{Y}$ of all clopen spectral subsets $V$ of $\sigma(T)$ satisfying conditions of §3.28 on the Boolean algebra $\{E(V; T) : V \in \mathcal{Y}\}$.

Proof. In view of Theorem 3.29 the mapping $V \mapsto E(V; T)$ is the homomorphism. If $E(V; T) = 0$, then $Y_V = 0$ and $\sigma(T_V) = \emptyset$, hence $V = \sigma(T_V) = \emptyset$ by Theorem 3.39. If $V_1, V_2 \in \mathcal{Y}$, then evidently $W_{V_1} \cup W_{V_2}$ and $W_{V_1} \cap W_{V_2}$ (for $V_1 \cap V_2 \neq \emptyset$) satisfy conditions of §3.28 as $W$. Consider $\sigma(T \setminus V)$ for $V \in \mathcal{Y}$, then $W_V \cap (\sigma(T \setminus V) = \emptyset$ (see §3.25), hence $W \setminus W_V$ satisfies conditions of §3.28 as $W$, since each two balls in $\mathcal{L}$ are either disjoint or one of them is contained in another. Therefore, $\mathcal{Y}$ is the Boolean algebra and hence $\{E(V; T) : V \in \mathcal{Y}\}$ is the Boolean algebra.

3.41. Note. In sections 3.28-3.40 it can be taken the generalization instead of $\Omega$ for a manifold $M$ which is $\mathcal{C}^{(q+1,n-1)}$-diffeomorphic with $\Omega$.

References

[1] Y. Amice. Interpolation p-adique. Bull. Soc. Math. France 92(1964), 117-180.

[2] M. Berz. Cauchy theory on Levi-Civita fields. in: Contemporary Mathem. 319 (2003), 39-52. Ultrametric functional analysis. Seventh international conference on p-adic functional analysis. June 17-21, 2002. Nijmegen. Ed. W.H. Schikhof, et. al.

[3] N. Bourbaki. Differentiable and analytical manifolds (Moscow: Mir,1975).
[4] N. Dunford, J.T. Schwartz. Linear operators. V.V. 1, 2 (New York: Interscience Publishers, 1958, 1963).

[5] R. Engelking. General topology (Moscow: Mir, 1986).

[6] A. Escassut. Analytic elements in p-adic analysis (Singapore: World Scientific, 1995).

[7] J. Fresnel, M. van der Put. "Géométrie analytique rigide et applications" (Boston: Birkhäuser, 1981).

[8] G. Henkin, J. Leiterer. Theory of functions on complex manifolds (Basel: Birkhäuser-Verlag, 1984).

[9] R.V. Kadison, J.R. Ringrose. Fundamentals of the theory of operator algebras (New York: Academic Press, 1983).

[10] N. Koblitz. p-adic numbers, p-adic analysis and zeta functions (New York: Springer-Verlag, 1977).

[11] N. Koblitz. p-adic analysis: a short course on recent work. Lond. Math. Soc., Lecture Notes Series. 46 (1980).

[12] S.V. Ludkovsky. Quasi-invariant measures on non-Archimedean groups and semigroups of loops and paths, their representations. Ann. Math. B. Pascal. 7: 2 (2000), 19-53, 55-80.

[13] S.V. Ludkovsky. Measures on groups of diffeomorphisms of non-Archimedean manifolds, representations of groups and their applications. Theoret. and Math. Phys. 119: 3 (1999), 698-711.

[14] S.V. Ludkovsky. Non-Archimedean stochastic processes on non-Archimedean manifolds. Los Alamos Nat. Lab. preprint 40 pages, math.GM/0212296, December 2002.

[15] S.V. Ludkovsky. Quasi-invariant and pseudo-differentiable measures with values in non-Archimedean fields on a non-Archimedean Banach space. J. Mathem. Sci. 112 (2002) (previous variants: Intern. Centre for Theor. Phys. Preprint IC/96/210, October 1996; Los Alamos Nat. Lab. Preprint math.GM/0106170).
[16] S.V. Ludkovsky. A structure and representations of diffeomorphism groups of non-Archimedean manifolds. Southeast Asian Bull. Math. 26 (2003), 975-1004.

[17] S.V. Ludkovsky. Embeddings of non-Archimedean Banach manifolds into non-Archimedean Banach spaces. Uspek. Mat. Nauk. 53: 5 (1998), 241-242.

[18] S.V. Ludkovsky, B. Diarra. Spectral integration and spectral theory for non-Archimedean Banach spaces. Intern. J. of Math. and Mathematics Sciences 31: 7 (2002), 421-442.

[19] L. Narici, E. Beckenstein. Topological vector spaces (New York: Marcel Dekker Inc., 1985).

[20] M. van der Put. Algèbres de fonctions continues p-adiques. Indagations Math. A 30: 4 (1968), 401-420.

[21] A.C.M. van Rooij. Non-Archimedean functional analysis (New York: Marcel Dekker Inc., 1978).

[22] W.H. Schikhof. Ultrametric calculus (Cambr.: Cambr. Univ. Press, 1984).

[23] W.H. Schikhof. Non-Archimedean calculus (Nijmegen: Math. Inst., Cath. Univ., Report 7812, 1978).

[24] W.H. Schikhof. The set of derivatives in a non-Archimedean field. Mathem. Annalen. 216(1975), 67-70.

[25] B.V. Shabat. An introduction to the complex analysis (Moscow: Nauka, 1985).

[26] E. Spanier. Algebraic topology (Moscow: Mir, 1971).

[27] A. Weil. Basic number theory (Berlin: Springer-Verlag, 1973).

[28] A. Weil. L’integration dans les groupes topologiques et ses applications. Actual. Scient. et Ind. 869 (Paris: Herman, 1940).
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