MONOTONE SOBOLEV MAPPINGS
of planar domains and surfaces

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Abstract. An approximation theorem of Youngs (1948) asserts that a continuous map between compact oriented topological 2-manifolds (surfaces) is monotone if and only if it is a uniform limit of homeomorphisms. Analogous approximation of Sobolev mappings is at the very heart of Geometric Function Theory (GFT) and Nonlinear Elasticity (NE). In both theories the mappings in question arise naturally as weak limits of energy-minimizing sequences of homeomorphisms. As a result of this, the energy-minimal mappings turn out to be monotone. In the present paper we show that, conversely, monotone mappings in the Sobolev space $W^{1,p}$, $1 < p < \infty$, are none other than $W^{1,p}$-weak (also strong) limits of homeomorphisms. In fact, these are limits of diffeomorphisms. By way of illustration, we establish the existence of energy-minimal deformations within the class of Sobolev monotone mappings for $p$-harmonic type energy integrals.

1. Introduction

There has been recently increasing interest in the Sobolev $W^{1,p}$-homeomorphisms and their weak and strong limits. In planar domains (or surfaces) and $p \geq 2$, these limits turn out to be monotone mappings in the topological sense of C.B. Morrey [42], see Definition 1.1 below. We shall see that, conversely, every $W^{1,p}$-map that is continuous and monotone between Lipschitz domain (this time for any exponent $1 < p < \infty$) can be approximated by homeomorphisms uniformly and strongly in the Sobolev norm; hence, by $C^\infty$-diffeomorphisms, also by piecewise affine homeomorphisms. A motivation for Sobolev mappings comes from the study of extremal problems in Geometric Function Theory (GFT) [4, 20, 22, 25, 27, 29, 30]. Further motivation comes from the fields of materials science such as Nonlinear Elasticity (NE) [2, 5, 11, 39, 47, 48].

Let us begin with a brief analysis of the $p$-harmonic energy of homeomorphisms $f : \mathcal{H}^{1,p}_p(X,Y)$ between bounded domains $X, Y \subset \mathbb{R}^2$.

$$E_p[f] = \int_X |Df(x)|^p \, dx,$$

where $|Df(x)|^2 \triangleq \text{Tr}[D^*f(x) Df(x)]$

Hereafter $Df(x)$ stands for the Jacobian matrix of $f$ (deformation gradient). Denote by $\mathcal{H}_p(X,Y)$ the class of orientation preserving homeomorphisms $f : X^{\text{out}} \to Y$.

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of finite energy and

\[ \mathcal{E}_p(X, Y) \overset{\text{def}}{=} \inf_{f \in \mathcal{H}_p(X, Y)} \int_X |Df(x)|^p \, dx \]

The infimum may or may not be attained. If not, we wish to find a map \( h_o \in W^{1,p}(X, Y) \) in close proximity to \( \mathcal{H}_p(X, Y) \) such that

\[ \int_X |Dh_o(x)|^p \, dx = \mathcal{E}_p(X, Y) \]

It is natural to look for \( h_o \) as \( W^{1,p} \)-weak limit of an energy-minimizing sequence of homeomorphisms \( h_j : X \rightarrow Y \). That this indeed would solve the problem (at least for Lipschitz domains and \( p \geq 2 \)) is far from being obvious. We always have the upper bound

\[ \int_X |Dh_o(x)|^p \, dx \leq \mathcal{E}_p(X, Y) \]

Equality occurs if and only if the minimizing sequence \( h_j \rightharpoonup h_o \) actually converges strongly in \( W^{1,p}(X, \mathbb{R}^2) \), which places \( h_o \) in the closest possible proximity to \( \mathcal{H}_p(X, Y) \). Equality (1.3) is of practical significance. Indeed, once \( h_o \) fails to be injective, it will tell us when to stop the minimizing sequence of homeomorphisms; that is, prior to the conditions favorable for the collapse of injectivity. This is a phenomenon known as interpenetration of matter.

**Monotone Mappings.** The concept is due to C.B. Morrey [42] (1935). The interested reader is referred to the Proceedings of the Conference on Monotone and Open Mappings [40] (1970) and the series of early papers by G.T. Whyburn for further reading.

**Definition 1.1.** Let \( A \) and \( B \) be compact metric spaces. A continuous map \( h : A \rightarrow B \) is said to be monotone if every fiber \( h^{-1}(b) \subset A \) of a point \( b \in B \) is connected. As shown by G.T. Whyburn the preimage \( h^{-1}(C) = \{ a \in A : h(a) \in C \} \) of any connected set \( C \subset B \) is connected in \( A \).

Let us emphasize that monotone mappings are continuous, by the definition.

While manifolds and Riemannian metric tensors are not the primary issues, it is desirable to keep them in mind since the topological aspects really crystallize in the manifold setting. Thus we choose and fix, as reference manifolds, two \( C^1 \)-smooth closed (compact without boundary) oriented Riemannian 2-manifolds \( \mathcal{X} \) and \( \mathcal{Y} \) of the same topological type. We shall consider multiply connected Jordan domains \( X \subset \mathcal{X} \) and \( Y \subset \mathcal{Y} \). Precisely, \( X \) and \( Y \) will be obtained by removing from \( \mathcal{X} \) and \( \mathcal{Y} \) the same number, say \( 0 \leq \ell < \infty \), of closed disjoint topological disks. In fact, any pair of topologically equivalent open \( C^1 \)-smooth surfaces with \( \ell \) boundaries can be obtained in this way. Such are planar multiply connected Jordan domains in \( \mathbb{R}^2 \cong S^2 \subset \mathbb{R}^3 \).

**Theorem 1.2** (Youngs, [51]). Let \( X \subset \mathcal{X} \) and \( Y \subset \mathcal{Y} \) be Jordan domains of the same topological type. A map \( h : X \rightarrow Y \) is monotone if and only if it is a uniform limit of homeomorphisms \( h_j : X \rightarrow Y \), \( j = 1, 2, \ldots \) .

It follows, in particular, that the boundary mapping \( h : \partial X \rightarrow \partial Y \) is monotone as well.
**Sobolev Variant of Youngs’ Approximation Theorem.** The main result:

**Theorem 1.3.** Let \( \mathcal{X} \subset \mathcal{X} \) and \( \mathcal{Y} \subset \mathcal{Y} \) be Jordan domains of the same topological type, \( \mathcal{Y} \) being Lipschitz. For every monotone map \( h : \mathcal{X} \overset{\text{onto}}{\rightarrow} \mathcal{Y} \) in the Sobolev space \( \mathcal{W}^{1,p}(\mathcal{X}, \mathcal{Y}), 1 < p < \infty \), there exists a sequence of monotone mappings \( h_j : \mathcal{X} \overset{\text{onto}}{\rightarrow} \mathcal{Y} \) such that:

(i) \( h_j : \mathcal{X} \overset{\text{onto}}{\rightarrow} \mathcal{Y} \) are homeomorphisms

(ii) \( h_j \rightharpoonup h \) uniformly on \( \mathcal{X} \)

(iii) \( h_j \rightarrow h \) strongly in \( \mathcal{W}^{1,p}(\mathcal{X}, \mathcal{Y}) \)

(iv) \( h_j = h : \partial \mathcal{X} \overset{\text{onto}}{\rightarrow} \partial \mathcal{Y} \), for \( j = 1, 2, ... \)

Let \( \mathcal{M}_p(\mathcal{X}, \mathcal{Y}) \) denote the class of orientation preserving monotone mappings \( f : \mathcal{X} \overset{\text{onto}}{\rightarrow} \mathcal{Y} \) of finite \( p \)-harmonic energy, \( 1 < p < \infty \). Theorem 1.3 implies (by direct method) that there always exists \( h \in \mathcal{M}_p(\mathcal{X}, \mathcal{Y}) \) with smallest \( p \)-harmonic energy. More importantly, the energy of \( h \) equals precisely the infimum of the energy among homeomorphisms:

\[
\int_{\mathcal{X}} |Dh(x)|^p \, dx = \min_{f \in \mathcal{M}_p(\mathcal{X}, \mathcal{Y})} \int_{\mathcal{X}} |Df(x)|^p \, dx = \inf_{f \in \mathcal{H}_p(\mathcal{X}, \mathcal{Y})} \int_{\mathcal{X}} |Df(x)|^p \, dx = E_p(\mathcal{X}, \mathcal{Y})
\]

see Section 6 for a definition of the \( p \)-harmonic integrals on surfaces. In other words, no Lavrentiev Phenomenon occurs in the class of monotone Sobolev mappings.

**Remark 1.4.** It is worth noting that, for an arbitrary pair \( (\mathcal{X}, \mathcal{Y}) \) of topologically equivalent planar domains, diffeomorphisms are dense in \( \mathcal{H}_p(\mathcal{X}, \mathcal{Y}) \), see [24]. Thus one can take for \( h_j \) in Theorem 1.3 a sequence of \( \mathcal{C}^\infty \)-smooth diffeomorphisms. Furthermore in case \( p = 2 \), if one is willing to sacrifice the boundary condition (iv) then the diffeomorphisms \( h_j : \mathcal{X} \overset{\text{onto}}{\rightarrow} \mathcal{Y} \) can be chosen to be homeomorphisms up to the boundary, again denoted by \( h_j : \mathcal{X} \overset{\text{onto}}{\rightarrow} \mathcal{Y} \), see [23].

**Remark 1.5.** It is not difficult to see that, in case of planar Lipshitz domains, a sequence of homeomorphisms \( h_j : \mathcal{X} \overset{\text{onto}}{\rightarrow} \mathcal{Y} \) which converges weakly in \( \mathcal{W}^{1,p}(\mathcal{X}, \mathcal{Y}) \), \( p \geq 2 \), actually converges uniformly to a monotone mapping \( h : \mathcal{X} \overset{\text{onto}}{\rightarrow} \mathcal{Y} \). Thus, in particular, Theorem 1.3 implies that \( h \) can be realized as \( \mathcal{H}^{1,p} \)-strong limit of homeomorphisms. We therefore recover a result in [31]; accordingly, weak sequential closure and strong closure of \( \mathcal{H}_p(\mathcal{X}, \mathcal{Y}) \), \( p \geq 2 \), are the same.

2. **Topological Preliminaries**

This section is intended as a gentle introduction to underlying geometric analysis on planar domains and surfaces. When discussing the Sobolev class \( \mathcal{W}^{1,p}(\mathcal{X}, \mathcal{Y}) \), it is particularly convenient to embed the reference manifolds \( \mathcal{X} \) and \( \mathcal{Y} \) into an Euclidean space. We may, for example, use the Nash-Kuiper embedding theorem [36, 43] which assures that the oriented 2-manifolds are isometrically \( \mathcal{C}^\infty \)-embeddable in \( \mathbb{R}^3 \). Thus we may (and do) assume that

\[ \mathcal{X}, \mathcal{Y} \subset \mathbb{R}^3 ; \text{ inclusion being a } \mathcal{C}^1 \text{- embedding.} \]

Note that for the purpose of defining \( \mathcal{W}^{1,p}(\mathcal{X}, \mathcal{Y}) \) one needs only assume uniform upper and lower bounds on the metric tensors.
When interpreting the conclusions, one might view the surfaces $X$ and $Y$ as thin films in $\mathbb{R}^3$, or flat plates in case $X = Y = \mathbb{R}^2 \cong S^2 \subset \mathbb{R}^3$.

The components of $X \setminus X$ and $Y \setminus Y$ are topological disks, say

$$X \setminus X = X_1 \cup X_2 \cup \ldots \cup X_\ell$$

and

$$Y \setminus Y = Y_1 \cup Y_2 \cup \ldots \cup Y_\ell$$

Their boundaries, denoted by $\partial X_\nu \overset{\text{def}}{=} X_\nu$ and $\partial Y_\nu \overset{\text{def}}{=} Y_\nu$, are exactly the components of $\partial X$ and $\partial Y$,

$$\partial X = X_1 \cup \ldots \cup X_\ell$$

and

$$\partial Y = Y_1 \cup \ldots \cup Y_\ell.$$ 

By convention, $X$ and $Y$ have no boundary if $\ell = 0$, in which case $X = X^-$ and $Y = Y^-$.

**Continuous Functions and Homeomorphisms.** We shall work with various function spaces defined on subsets of the reference manifold $\mathcal{X}$:

- For a compact subset $A \subset X$ we denote by $C(A)$ the space of continuous functions $h : A \rightarrow \mathbb{R}^3$ furnished with the norm:

  $$\|h\|_{C(A)} = \max_{x \in A} |h(x)|$$

  The notation $h \in C(A, B)$ will be used if we want to make explicit the range of $h : A \rightarrow B \subset \mathbb{R}^3$.

- $\mathcal{H}(X, Y)$ consists of orientation preserving homeomorphism $h : X \rightarrow Y$. Every homeomorphism $h : X \rightarrow Y$ gives rise to a one-to-one correspondence between boundary components $X_1, X_2, \ldots, X_\ell$ and $Y_1, Y_2, \ldots, Y_\ell$ by means of cluster limits. We conveniently rearrange the indices so that the boundary correspondence reads as follows:

  $$h : X_\nu \rightarrow Y_\nu, \quad \text{for } \nu = 1, ..., \ell;$$

  This arrangement will tacitly be assumed throughout this paper for homeomorphisms in the class $\mathcal{H}(X, Y)$ and their uniform limits.

- $\mathcal{H}(\bar{X}, \bar{Y})$ consists of homeomorphisms $h : X \rightarrow Y$ which extend continuously to the closure of $X$. Note that the continuous extensions become monotone maps from $\bar{X}$ onto $\bar{Y}$, still denoted by $h : \bar{X} \rightarrow \bar{Y}$. They take $\partial X$ onto $\partial Y$; specifically, $h(X_\nu) = Y_\nu$ for $\nu = 1, ..., \ell$. We refer to $h : X_\nu \rightarrow Y_\nu, \nu = 1, ..., \ell$, as the boundary maps. Each of these boundary maps is monotone.

- $\mathcal{H}(\bar{X}, \bar{Y})$ is the space of homeomorphisms $h : \bar{X} \rightarrow \bar{Y}$.

Similar notation, with the obvious analogous meaning, will be used for spaces of mappings defined on other subsets of $\mathcal{X}$.

**Monotone Mappings Versus Uniform Limits of Homeomorphisms.** Let $A \subset X^-$ and $B \subset Y^-$ be compact. The class

$$\mathcal{M}(A, B) \subset C(A, B)$$

stands for the space of monotone mappings $h : A \rightarrow B$. 
Obviously $\mathcal{M}(A, B) \subsetneq \mathcal{M}(\hat{A}, \hat{B})$. We now invoke the Approximation Theorem of J. W. T. Youngs [51].

**Theorem 2.1.** Let $\mathcal{X}$ and $\mathcal{Y}$ be topologically equivalent compact 2-manifolds.

Then for every monotone map $h: \mathcal{X} \to \mathcal{Y}$ there exists a sequence of homeomorphisms $h_j: \mathcal{X} \to \mathcal{Y}$ converging uniformly to $h$. In symbols,

$$\mathcal{M}(\mathcal{X}, \mathcal{Y}) = \mathcal{H}(\mathcal{X}, \mathcal{Y})$$

This theorem will provide us with a powerful tool when dealing with monotone mappings. We shall appeal to it repeatedly to either recover or refine the well known properties of monotone mappings between Jordan domains $\mathcal{X} \subset \mathcal{X}$ and $\mathcal{Y} \subset \mathcal{Y}$.

For example, Theorem 2.1 readily implies that

**Lemma 2.2.** We have the inclusions

$$\mathcal{H}(\mathcal{X}, \mathcal{Y}) \subset \mathcal{H}(\mathcal{\hat{X}}, \mathcal{\hat{Y}}) \subset \mathcal{M}(\mathcal{X}, \mathcal{Y}) = \mathcal{H}(\mathcal{X}, \mathcal{Y}) \subset C(\mathcal{X}, \mathcal{Y})$$

Moreover, for $h \in \mathcal{M}(\mathcal{X}, \mathcal{Y})$, each boundary map $h: \partial \mathcal{X} \to \partial \mathcal{Y}$ is monotone.

We shall also take advantage of the following refinement of the Modification Theorem in [51].

**Lemma 2.3** (Homeomorphic Extension of a Monotone Boundary Map). Let $\mathcal{X}_o \subset \mathcal{X}$ and $\mathcal{Y}_o \subset \mathcal{Y}$ be simply connected Jordan domains. Then every monotone map $h: \partial \mathcal{X}_o \to \partial \mathcal{Y}_o$ admits a continuous monotone extension $h: \mathcal{\hat{X}}_o \to \mathcal{\hat{Y}}_o$. Such an extension can be further modified to become a homeomorphism between $\mathcal{X}_o$ and $\mathcal{Y}_o$.

**Proof.** First, with the aid of the uniformization theorem, we transform $\mathcal{X}_o$ and $\mathcal{Y}_o$ homeomorphically onto the closed Euclidean disks. We obtain a monotone map between circles, which we extend (in a radial fashion) to a monotone map of the disks. We then lift such an extension back to the reference manifolds $\mathcal{X}$ and $\mathcal{Y}$, completing the proof of the first statement. Then, by ”Modification Theorem” in [51], this extension can be modified to become a homeomorphism between $\mathcal{X}_o$ and $\mathcal{Y}_o$. \qed

**Remark 2.4.** In Theorem 2.1, if one is willing to forgo the univalence of the boundary mappings $h_j: \partial \mathcal{X} \rightleftharpoons \partial \mathcal{Y}$, but only wants them to be injective from $\mathcal{X} \rightleftharpoons \mathcal{Y}$, then we can ensure that $h_j = h$ on $\partial \mathcal{X}$. However, this variant of Youngs’ theorem will not be exercised here.

**Monotone Extension Outside $\mathcal{X}$.**

**Lemma 2.5.** Every $h \in \mathcal{M}(\mathcal{X}, \mathcal{Y})$ extends as a continuous monotone map between reference manifolds.

**Proof.** If $\mathcal{X}$ has no boundary then so does $\mathcal{Y}$. Hence $\mathcal{X} = \mathcal{X}$ and $\mathcal{Y} = \mathcal{Y}$. Now, consider a boundary component $\mathcal{X}_\nu \subset \partial \mathcal{X}$ and the corresponding boundary component $\mathcal{Y}_\nu \subset \partial \mathcal{Y}$. Recall that $h: \mathcal{X}_\nu \to \mathcal{Y}_\nu$ is monotone. Both $\mathcal{X}_\nu$ and $\mathcal{Y}_\nu$ are topological circles on the reference surfaces $\mathcal{X}$ and $\mathcal{Y}$, respectively. Filling these circles with topological open disks, say $\mathcal{X}_\nu \subset \mathcal{X}$ and $\mathcal{Y}_\nu \subset \mathcal{Y}$, results in closed 2-cells in the reference manifolds. The extension is immediate from Lemma 2.3. \qed
Remark 2.6. With little extra efforts one can ensure that such an extension is a homeomorphism between the components $X_\nu$ and $Y_\nu$, for all $\nu = 1, 2, \ldots, \ell$. But there will be no need for this.

We continue to use the same notation $h : X \to Y$ for the monotone extension. Thus $h$ becomes an element of $M(X, Y)$ as well. There will be no need of any regularity of $h : X \to Y$ outside $X$.

**Cells and Pre-cells.** A cell in the reference manifold $Y$ is any simply connected Jordan domain $Q \subset Y$. Its preimage $h^{-1}(Q) \subset X$ under a monotone map $h : X \to Y$ is still simply connected but not necessarily a Jordan domain. We refer to $h^{-1}(Q) \subset X$ as *pre-cell* in $X$. Beware that not every simply connected domain in $X$ comes as a pre-cell for $h$; and that, in general, the closure of a pre-cell may not contain $h^{-1}(\overline{Q})$. In fact, the strict inclusions $h^{-1}(Q) \subsetneq h^{-1}(\overline{Q})$ and $\partial h^{-1}(Q) \subsetneq h^{-1}(\partial Q)$ are typical of monotone mappings. Associated with $X \subset X$, $Y \subset Y$ and the monotone map $h : X \to Y$ are the concepts of internal and boundary cells.

**Definition 2.7.** The term cell in a domain $Y \subset Y$ refers to any simply connected Jordan domain $Q \subset Y$ which satisfies one of the following conditions:

- $Q \Subset Y$, we call it an *internal cell*.
- $\overline{Q} \cap \partial Y = \overline{C}$ is a closed Jordan arc (not a single point). We refer to such $Q$ as *boundary cell* in $Y$ and to $\overline{C}$ as its *external face*. In this case we shall also make use of the set $Q_+ \overset{\text{def}}{=} Q \cup C$.

Every boundary cell $Q \subset Y$ can be extended beyond its external face to become a cell in the reference manifold $Y$. Let $Q^* \subset Y$ be such an extension so $Q = Q^* \cap Y$. Here $\partial Y$ splits $Q^*$ into two simply connected Jordan domains.

To every cell $Q$ in $Y$ (internal or boundary) there corresponds a simply connected domain in $X$, called *pre-cell* in $X$. It is defined by the following rule:

- If $Q \Subset Y$, then we call $U \overset{\text{def}}{=} h^{-1}(Q) \Subset X$ the internal pre-cell in $X$.
- If $\overline{Q} \cap \partial Y \neq \emptyset$, then we call $U \overset{\text{def}}{=} h^{-1}(Q^*) \cap X$ the boundary pre-cell in $X$. It is of no importance which extension $Q^*$ is taken in this formula for $U$. In fact, we always have $U = h^{-1}(Q_+) \cap X$.

The reader is cautioned that in general $U$ is not the preimage of the boundary cell $Q \subset Y$; we have only the inclusion $h^{-1}(Q) \subset U$. Furthermore, the pre-cells in $X$ need not be Jordan domains, and that is why some complications (topological and analytical) are to be expected.

**Lemma 2.8.** The *pre-cells* (both internal and boundary) are simply connected domains in $X$.

**Proof.** We again take advantage of Young’s Approximation Theorem. The lemma holds if $h : X \to Y$ is a homeomorphism. But it also holds if $h$ is monotone; just use the sequence of homeomorphisms converging uniformly to $h$, to represent the pre-cell as union of an increasing sequence of simply connected domains. □

Our next topological fact, which actually strengthens Lemma 2.3, will require some work. Recall that we have extended $h \in M(X, Y)$ to a map in $M(X, Y)$. 
Lemma 2.9 (Homeomorphic Replacement in Pre-cells). Let $Q \subset Y$ be a cell in $Y$ (internal or boundary) and $U \subset X$ the corresponding pre-cell in $X$. Then there exists a monotone map $h_{\alpha} : U \rightarrow Q$ such that

- $h_{\alpha} : U \rightarrow Q$ is a homeomorphism
- $h_{\alpha} \equiv h : X \setminus U \rightarrow Y \setminus Q$

Proof. The proof comes down to monotone mappings between 2-spheres and a result by T. Radó [46] page 66, II.1.47. Let us consider two cases.

Case 1 [internal pre-cell]. Suppose $U = h^{-1}(Q)$ where $Q \subset Y$. Choose and fix slightly larger cell $Q' \subset Y$, which gives us a larger pre-cell $U \subset U' \equiv h^{-1}(Q') \subset X$. We view $U'$ and $Q'$ as open simply connected Riemann surfaces. Note that the map $h : U' \rightarrow \overline{Q'}$ is continuous and takes $\partial U'$ into $\partial Q'$. Denote by $\overline{U'}$ and $\overline{Q'}$ the Alexandroff one-point compactifications of $U'$ and $Q'$, respectively. These are topological 2-spheres. The unique continuous extension $\hat{h} : \overline{U'} \rightarrow \overline{Q'}$ of $h : U' \rightarrow Q'$ remains monotone. At this point we may appeal to [46] page 66, II.1.47, which asserts that there is a monotone map $h_{\alpha} : U' \rightarrow Q'$ which takes $U$ homeomorphically onto $Q$ and agrees with $h$ outside $U$.

Case 2 [boundary pre-cell]. Suppose $U \equiv h^{-1}(Q^*) \subset X$, where $Q^*$ is a cell in $Y$ such that $Q = Q^* \cap Y$ and $Q^* \cap \partial Y = C$ is an open Jordan arc. Denote by $\Omega = h^{-1}(Q^*)$ the pre-cell in $X$ under the map $h : X \rightarrow Y$. As in the previous case we find a monotone mapping $H \equiv h_{\Omega} : X \rightarrow Y$ that agrees with $h : X \setminus \Omega \rightarrow Y \setminus Q^*$ and takes $\Omega^*$ homeomorphically onto $Q^*$. The issue is that $H : \Omega \rightarrow Q^*$ need not take $U = \Omega \cap X$ onto $Q^* \cap Y$ and, even if it does, need not coincide with $h$ outside $U$. The idea is to correct $H$ within $\Omega$. For, we look closely at the crosscut of $\Omega$ by the boundary of $X$, say by the component $X = X_{\nu} \subset \partial X$, for some $\nu = 1, \ldots, k$. Let $\Upsilon = \Upsilon_{\nu}$ denote the corresponding boundary component of $\partial Y$. Recall that the boundary map $h : X \rightarrow \Upsilon$ is also monotone; that is, the preimages of connected sets in $\Upsilon$ are connected in $X$. In this way we have defined an open subarc of $X$,

$$\Gamma = \{ x \in X ; h(x) \in C \}. \quad \text{Note that } \Gamma \subset X, \quad \text{since } C \subset \Upsilon = h(X).$$

This subarc has two endpoints (the limit points), say $a, b \in \partial \Omega$. It is a topological folklore that an open Jordan arc in a simply connected domain $\Omega$, whose endpoints lie $\partial \Omega$, splits $\Omega$ into two simply connected subdomains. These subdomains are:

$$U = \Omega \cap X \quad \text{and} \quad V \equiv \Omega \setminus X,$$

hence a decomposition $\Omega = U \cup \Gamma \cup V$.

Homeomorphism $H : \Omega \rightarrow Q^*$ yields a decomposition of the cell $Q^* \subset Y$; namely,

$$Q^* = U_{\alpha} \cup \Gamma_{\alpha} \cup V_{\alpha} \quad \text{where} \quad U_{\alpha} = H(U), \quad \Gamma_{\alpha} = H(\Gamma) \quad \text{and} \quad V_{\alpha} = H(V).$$

The open Jordan arcs $C \subset Q^*$ and $\Gamma_{\alpha} \subset Q^*$ share common endpoints which we denote by $A = h(a) = H(a) \in \partial Q^*$ and $B = h(b) = H(b) \in \partial Q^*$. We note that $U_{\alpha}$ and $V_{\alpha}$ are Jordan domains for which $\Gamma_{\alpha} \cup \{ A, B \}$ constitutes their common boundary. Precisely, we have

$$\partial U_{\alpha} = \alpha \cup \Gamma_{\alpha}, \quad \partial V_{\alpha} = \beta \cup \Gamma_{\alpha}, \quad \partial U_{\alpha} \cap \partial V_{\alpha} = \Gamma_{\alpha}.$$
where $\alpha$ is the closed sub-arc of $\partial Q^*$ between $A$ and $B$ that lies in $\bar{\gamma}$, and $\beta$ is the closed sub-arc of $\partial Q^*$ between $A$ and $B$ that lies in $\gamma \setminus \gamma$.

Now, just the fact that $h: \Omega \rightrightarrows Q^*$ is continuous and agrees with $H$ on $\partial \Omega$ lets us observe that the mapping $h \circ H^{-1} : Q^* \rightrightarrows Q^*$ (not necessarily injective) extends continuously as a monotone map of $Q^*$ onto itself. Upon such an extension, the boundary map $h \circ H^{-1} : \partial Q^* \rightrightarrows \partial Q^*$ becomes the identity. Let us see how this extension acts on $\partial U_h$. It is still the identity map on $\alpha \subset Q^*$ and it takes the closed subarc $\Gamma_h \subset \partial U_h$ monotonically onto $Q \subset \partial Q^*$. Thus we have a monotone map $h \circ H^{-1} : \partial U_h \rightrightarrows \partial(Q^* \cap Y)$. It is important to observe that both $U_h$ and $Q^* \cap Y$ are Jordan domains. At this stage we appeal to Lemma 2.9. Accordingly, we extend $h \circ H^{-1} : \partial U_h \rightrightarrows \partial(Q^* \cap Y)$ continuously, and as a homeomorphism inside the curves. Denote the extension by $(h \circ H^{-1})^2 : U_h \rightrightarrows Q^* \cap Y$. In summary, we have constructed a continuous monotone map $F : \mathcal{Y} \rightrightarrows \mathcal{Y}$,

$$F = \begin{cases} \text{identity} & \text{in } \mathcal{Y} \setminus Q^* \\ h \circ H^{-1} & \text{in } \mathcal{Y}_h \\ (h \circ H^{-1})^2 & \text{in } \mathcal{U}_h \end{cases}$$

The composition $h_u \overset{\text{def}}{=} F \circ H : \mathcal{X} \rightrightarrows \mathcal{Y}$ is the desired replacement, as claimed in Lemma 2.9.

3. Analytical Requisites

Since the reference manifolds $\mathcal{X}$ and $\mathcal{Y}$ are closed oriented Riemannian 2-manifolds of class $C^1$, we may speak of the Sobolev class $\mathcal{W}^{1,p}(\Omega, \mathcal{Y})$ of mappings $h : \Omega \rightarrow \mathcal{Y}$ defined on any open subset $\Omega \subset \mathcal{X}$. We do not reserve any particular notation of the metric tensors on $\mathcal{X}$ and $\mathcal{Y}$, though we fix them for the rest of this paper. The volume element on $\mathcal{X}$, denoted by $dx$, is the one induced by the metric tensor. We recall the $C^1$-isometric embeddings $\mathcal{X} \subset \mathbb{R}^3$ and $\mathcal{Y} \subset \mathbb{R}^3$.

3.1. Sobolev Mappings Between Surfaces. The Sobolev space $\mathcal{W}^{1,p}(\Omega)$ of real-valued functions on $\Omega \subset \mathcal{X}$ will be endowed with the seminorm

$$\|\phi\|_{\mathcal{W}^{1,p}(\Omega)} \overset{\text{def}}{=} \left( \int_{\Omega} |D\phi(x)|^p \, dx \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty$$

where $|D\phi(x)|$, defined almost everywhere, stands for the norm of the linear tangent map $D\phi(x) : T_x(\Omega) \rightarrow \mathbb{R}$ with respect to the inner product in $T_x(\Omega)$. The Sobolev class $\mathcal{W}^{1,p}(\Omega, \mathcal{Y}) \subset \mathcal{W}^{1,p}(\Omega, \mathbb{R}^3)$ consists of mappings $h = (h^1, h^2, h^3) : \Omega \rightarrow \mathbb{R}^3$ whose coordinate functions $h^1, h^2, h^3$ belong to $\mathcal{W}^{1,p}(\Omega)$ and $h(x) \in \mathcal{Y}$ for almost every $x \in \Omega$. It may be worth reminding the reader that for $1 \leq p < 2 = \dim \Omega$ topology can inhibit the space $\mathcal{C}^1(\Omega, \mathcal{Y})$ from being dense in $\mathcal{W}^{1,p}(\Omega, \mathcal{Y})$. The interested reader is referred to [10, 17, 18, 16] for this issue. This problem, however, disappears completely since our mappings in question are continuous. De facto, when $1 < p < 2$, the continuity assumption of monotone Sobolev mappings is critical for the subsequent arguments; it is superfluous, however, when $p \geq 2$.

Royden $p$-Algebra. The class $\mathcal{A}^p(\Omega) \overset{\text{def}}{=} \mathcal{C}(\overline{\Omega}) \cap \mathcal{W}^{1,p}(\Omega)$ consist of real-valued functions in the Sobolev space $\mathcal{W}^{1,p}(\Omega)$ that are continuous on $\overline{\Omega}$. This is a Banach algebra with respect to the sub-multiplicative norm

$$\|\phi\|_{\mathcal{A}^p(\Omega)} \overset{\text{def}}{=} |\phi|_{\mathcal{C}(\overline{\Omega})} + \|\phi\|_{\mathcal{W}^{1,p}(\Omega)}, \quad \|\phi \cdot \psi\|_{\mathcal{A}^p(\Omega)} \leq \|\phi\|_{\mathcal{A}^p(\Omega)} \cdot \|\psi\|_{\mathcal{A}^p(\Omega)}$$
3.2. p-Harmonic Boundary-Value Problem. We record less familiar aspects of the Dirichlet problem for the p-harmonic equation.

\[
\text{div} |\nabla \phi|^p - 2 \nabla \phi = 0, \quad \text{for } \phi \in W^{1,p}_{\text{loc}}(\Omega), \ 1 < p < \infty
\]

in planar simply connected domains. In general, simply connected domains may have rather odd boundary.

**Lemma 3.1.** Let \( \Omega \subset \mathbb{R}^2 \) be bounded simply connected domain and \( \Phi \in C(\overline{\Omega}) \). Then there exists unique \( \phi \in C(\Omega) \) that is \( p \)-harmonic in \( \Omega \), \( 1 < p < \infty \), and agrees with \( \Phi \) on \( \partial \Omega \). If \( \Phi \in R^p(\Omega) \), then also \( \phi \in R^p(\Omega) \). Moreover,

\[
\phi \in \Phi + W^{1,p}(\Omega) \quad \text{and} \quad \int_{\Omega} |\nabla \phi|^p \leq \int_{\Omega} |\nabla \Phi|^p
\]

Equality occurs if and only if \( \Phi = \phi \).

**Proof.** We will only briefly outline the key points of the proof. For a thorough treatment of the Dirichlet problem we refer the reader to [19] and [38]. By virtue of the Wiener’s criterion the first statement of the lemma holds whenever the complement \( \mathbb{R}^2 \setminus \Omega \) is \( p \)-thick at every boundary point, see [38, Corollary 6.22] and [38, (2.22)] for a formulation of Wiener’s criterion. Simply connected domains indeed satisfy this criterion, a fact not difficult to verify, but it is not explicitly exemplified in the vast literature. For the second statement we minimize the \( p \)-harmonic energy in the class \( \Phi + W^{1,p}(\Omega) \) (so-called variational formulation) which does not require any regularity assumption on the domain \( \Omega \). The only point is to show that the variational solution extends continuously to the boundary and it coincides with \( \Phi \). This again follows by Wiener’s criterion, see [19, Theorem 6.27]. \( \square \)

3.3. p-Harmonic Replacements. Let a map \( F = u + iv : \Omega \to C \simeq \mathbb{R}^2 \), defined in a domain \( \Omega \subset C \), belong to the Sobolev space \( W^{1,p}_{\text{loc}}(\Omega, C) \), \( 1 < p < \infty \).

**Definition 3.2.** \( F \) is said to be \( p \)-harmonic (coordinate-wise) if both coordinate functions \( u \) and \( v \) satisfy the equation (3.1). We shall introduce the unisotropic \( p \)-harmonic energy of \( F \):

\[
\mathcal{E}[F] = \mathcal{E}_p[F] \overset{\text{def}}{=} \int_{\Omega} \left( |\nabla u|^p + |\nabla v|^p \right)
\]

and repeatedly abbreviate this as energy of \( F \); because the exponent \( p \) will remain fixed.

**Remark 3.3.** This terminology is different from what can be found in the literature; the term \( p \)-harmonic mapping is usually reserved for the coupled \( p \)-harmonic system \( \text{div}|Df|^{p-2}Df = 0 \). There is a subtle distinction between these two concepts.

Recall from Lemma 3.1 that for any \( F \in R^p(\Omega, \mathbb{R}^2) \) in a bounded simply connected domain its boundary map \( F : \partial \Omega \to \mathbb{R}^2 \) admits unique \( p \)-harmonic extension to \( \Omega \). The question arises whether such an extension is injective. The answer depends on how \( F \) runs along \( \partial \Omega \). Of course, the boundary map \( F : \partial \Omega \to \mathbb{R}^2 \) must admit at least one homeomorphic extension inside \( \Omega \). This is also sufficient if \( F(\partial \Omega) \) is a convex curve.
Lemma 3.4. Let \( \Omega \subset \mathbb{R}^2 \) be bounded simply connected domain and let \( F \in \mathcal{A}_p(\Omega, \mathbb{R}^2) \) be \( p \)-harmonic (coordinate wise) in \( \Omega \). Suppose there is \( \Psi \in \mathcal{E}(\overline{\Omega}, \mathbb{R}^2) \) that agrees with \( F \) on \( \partial\Omega \) and takes \( \Omega \) homeomorphically onto a convex domain \( \Delta \). Then \( F \) is a \( \mathcal{C}^\infty \)-diffeomorphism of \( \Omega \) onto \( \Delta \). Moreover,
\[
\|\Phi\|_{\mathcal{E}(\Omega, \mathbb{R}^2)} \leq C_{\Omega, \Delta}(\Psi, \|F\|_{\mathcal{E}(\Omega, \mathbb{R}^2)}),
\]
whenever \( \Phi \in F + \mathcal{A}_p(\Omega, \mathbb{R}^2) \).

In this setting \( \Psi \) plays the role of the classical Dirichlet boundary data. It helps to capitalize on the topological properties of \( F \) near the (rather weird) boundary \( \partial\Omega \). We emphasize that \( \Psi \) is not required to have any Sobolev regularity in \( \Omega \).

Remark 3.5. Lemma 3.4 is a \( p \)-harmonic analogue of the celebrated Radó-Kneser-Chouquet Theorem [14]. Under additional assumptions on the domain \( \Omega \), a \( p \)-harmonic (coordinate wise) analogue of Radó-Kneser-Chouquet Theorem was first shown by Alessandrini and Sigalotti [1], see [21] for the isotropic case. The paper [1] is concerned with the domains which satisfy the external condition. This would be redundant, as mentioned in "Remark 3.2" of this paper. However, the essential shortcoming is that neither formal statement nor the proof in case of non-Jordan domains and non homeomorphic boundary data are provided in [1]. Thus we must work out additional arguments.

Proof. We may assume, in addition to the hypotheses above, that \( \Psi \) takes \( \Omega \) diffeomorphically onto \( \Delta \). This is permissible by a theorem of Radó [45], see also [41], which asserts that to every homeomorphism \( \Psi : \Omega \to \mathbb{R}^2 \) and continuous function \( \tau : \Omega \to (0, \infty) \) there corresponds a diffeomorphism \( \Psi_\tau : \Omega \to \mathbb{R}^2 \) such that \( |\Psi_\tau(x) - \Psi_\tau(x')| < \tau(x) \) in \( \Omega \). Therefore, one can replace \( \Psi \) in Lemma 3.4 by a diffeomorphism \( \Psi_\tau \) with \( \tau(x) = \text{dist}(x, \partial\Omega) \).

Now, consider an increasing sequence of smooth convex domains \( \Delta_1 \subset \Delta_2 \subset \ldots \subset \Delta_n \subset \Delta_{n+1} \ldots \subset \Delta \) whose union is \( \Delta \). The corresponding preimages under \( \Psi \) are smooth Jordan domains in \( \Omega \),
\[
\Omega_n := \Psi^{-1}(\Delta_n), \quad \Omega_1 \subset \Omega_2 \subset \ldots \subset \Omega_n \subset \Omega_{n+1} \subset \ldots, \quad \bigcup_{n} \Omega_n = \Omega.
\]

Next, we replace each diffeomorphism \( \Psi : \overline{\Omega} \to \overline{\Delta} \) by a homeomorphism \( \Psi_n : \overline{\Omega_n} \to \overline{\Delta} \) which is \( p \)-harmonic (coordinate-wise) in \( \Omega_n \) (thus a diffeomorphism in \( \Omega_n \)) and agrees with \( \Psi \) on \( \partial\Omega_n \), see Theorem 5.1 in [1]. We just constructed a sequence of continuous mappings \( F_n : \overline{\Omega} \to \overline{\Delta} \) which are homeomorphisms on \( \Omega \) and coincide with \( F \) on \( \partial\Omega \).

\[
F_n = \begin{cases} 
\Psi & \text{in } \Omega \setminus \Omega_n \\
\Psi_n & \text{in } \Omega_n
\end{cases}
\]

This sequence converges uniformly to \( F \). Indeed, for every \( x \in \overline{\Omega} \) we have
\[
|F_n(x) - F(x)| \leq \sqrt{2} \|\Psi - F\|_{\mathcal{E}(\overline{\Omega}, \mathbb{R}^2)} \to 0.
\]

The latter estimate is trivial when \( x \in \overline{\Omega} \setminus \Omega_n \). To see that this also holds for \( x \in \Omega_n \), we argue by a comparison principle [19, Comparison principle 7.6, page 133] in a straightforward way. Namely, the coordinate functions of \( F = (u, v) \) and \( F_n = (u_n, v_n) \), being \( p \)-harmonic in \( \Omega_n \), satisfy
\[
|u_n(x) - u(x)| \leq \|u_n - u\|_{\mathcal{E}(\partial\Omega_n)} \quad \text{and} \quad |v_n(x) - v(x)| \leq \|v_n - v\|_{\mathcal{E}(\partial\Omega_n)}
\]
which yields the desired estimate in (3.4). Concerning the positive sign of the Jacobian determinant of \( F \), we shall appeal to a \( p \)-harmonic variant of Hurwitz Theorem [31, Theorem 4.9]; that is,

**Theorem 3.6.** If a sequence \( F_n : \Omega \to \mathbb{R}^2 \) of \( p \)-harmonic (coordinate-wise) orientation preserving diffeomorphisms converges uniformly to \( F : \Omega \to \mathbb{R}^2 \) in a domain \( \Omega \subset \mathbb{R}^2 \), then either \( J(x,F) > 0 \) everywhere in \( \Omega \) or \( J(x,F) \equiv 0 \) in \( \Omega \).

Let us first exclude a possibility that \( J(x,F) \equiv 0 \) in \( \Omega \). For this, we choose and fix a nonnegative test function \( \eta \in C_0^\infty(\Delta) \) whose integral mean equals 1. Let \( \mathbb{G} \in \Delta \) denote the support of \( \eta \). Since \( F_n \rightharpoonup F \) (uniformly) and \( F(\partial \Omega) = \partial \Delta \) it follows that

\[
\bigcup_{n \geq 1} F_n^{-1}(\mathbb{G}) \subset \Omega.
\]

In particular, \( \bigcup_{n \geq 1} F_n^{-1}(\mathbb{G}) \in \Omega_k \subset \Omega \), for sufficiently large \( k \).

Hence, for \( n > k \), we have

\[
\int_{\Omega_k} \eta(F_n(x)) J(x,F_n) \, dx = \int_{F_n(\Omega_k)} \eta(y) \, dy \geq \int_{\mathbb{G}} \eta(y) \, dy = 1.
\]

Recall that \( F_n \rightharpoonup F \) on \( \Omega_{k+1} \supset \Omega_k \). Since \( F_n \) are \( p \)-harmonic in \( \Omega_{k+1} \), we also have \( DF_n \rightharpoonup DF \) on \( \Omega_k \). This follows from local \( C^{1,\alpha} \)-estimates of \( p \)-harmonic functions [26, 49, 50]. It is now legitimate to pass to the limit in the above estimate to obtain

\[
\int_{\Omega_k} \eta(F(x)) J(x,F) \, dx \geq 1.
\]

In particular, \( J(x,F) \neq 0 \) in \( \Omega_k \). Thus \( J(x,F) > 0 \) everywhere in \( \Omega_k \). But \( k \) can be as large as we wish, so \( J(x,F) > 0 \) everywhere in \( \Omega \).

In particular, \( F \) is a local diffeomorphism in \( \Omega \). On the other hand the map \( F : \Omega \rightharpoonup \Delta \) is a uniform limit of homeomorphisms. Therefore, \( F \) is a global diffeomorphism, completing the proof of Lemma 3.4.

**4. APPROXIMATION INSIDE A GIVEN PRE-CELL**

From now on \( \mathbb{Y} \) is a Lipschitz domain. This means that every point \( y_0 \in \partial \mathbb{Y} \) has a neighborhood \( \mathcal{O} \subset \mathbb{Y} \) and a local \( (C^1\text{-smooth}) \) chart \( \kappa : \mathcal{O} \to \mathbb{R}^2 \) such that \( \kappa(\mathcal{O} \cap \partial \mathbb{Y}) \) is a graph of a Lipschitz function. Regarding the given map \( h \in C(\overline{\mathbb{X}}, \overline{\mathbb{Y}}) \cap W^{1,p}(\mathbb{X}, \mathbb{R}^3) \), \( 1 < p < \infty \), we recall that it extends as a monotone map \( h : \mathbb{X} \rightharpoonup \mathbb{Y} \). No regularity outside \( \overline{\mathbb{X}} \) is required, as this extension will assist us only in the topological aspects of the proof.

As a preliminary step in the proof of Theorem 1.3 we shall approximate \( h : \overline{\mathbb{X}} \rightharpoonup \overline{\mathbb{Y}} \) with Sobolev mappings which are univalent within a given pre-cell, and remain unchanged outside the pre-cell. The approximation is understood by means of the metric in the Royden space \( \mathscr{R}^p(\mathbb{X}, \mathbb{Y}) = C(\overline{\mathbb{X}}, \overline{\mathbb{Y}}) \cap W^{1,p}(\mathbb{X}, \mathbb{R}^3) \); that is, uniformly and strongly in \( W^{1,p}(\mathbb{X}, \mathbb{R}^3) \). Precisely, we have:

**Proposition 4.1.** Let \( \mathcal{Q} \subset \mathbb{V} \) be a cell in \( \mathbb{Y} \), which we assume to be Lipschitz regular, and \( \mathcal{U} \) be its pre-cell in \( \mathbb{X} \). Then there exists a sequence of monotone mappings \( h_j : \overline{\mathbb{X}} \rightharpoonup \overline{\mathbb{Y}} \) such that:

(a) \( h_j \in C(\overline{\mathbb{X}}, \overline{\mathbb{Y}}) \cap W^{1,p}(\mathbb{X}, \mathbb{R}^3) \), \( j = 1, 2, ... \),
(b) \( h_j = h \) on \( \mathbb{X} \setminus \mathcal{U} \),
(c) \( h_j : U \rightharpoonup \mathcal{Q} \) are homeomorphisms,
(d) \( h_j \rightharpoonup h \) uniformly in \( \overline{\mathbb{X}} \),
(e) $h_j \to h$ strongly in $\mathcal{W}^{1,p}(\Omega, \mathbb{R}^3)$,

**Remark 4.2.** Upon the extension $h_j \overset{\text{def}}{=} h : \mathcal{X} \setminus \mathbb{R}^3 \to \mathcal{Y} \setminus \mathbb{Y}$, we obtain monotone mappings between reference manifolds, again denoted by $h_j : \mathcal{X} \overset{\text{auto}}{\to} \mathcal{Y}$.

**Proof.** It will takes 5 steps to complete the proof of Proposition 4.1.

**Step 1. Reduction to simply connected Jordan domains in $\mathbb{S}^2$.** We view $\mathbb{S}^2$ as the extended complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{ \infty \} \simeq \mathbb{R}^2$ equipped with the chordal metric. Choose and fix a simply connected neighborhood $Q' \ni \overline{Q}$ such that $Q' \cap \partial \mathbb{Y}$ becomes a Lipschitz regular Jordan arc. Obviously such $Q'$ does exist. Thus $Q' \cap \mathbb{Y}$ is a boundary cell in $\mathbb{Y}$, even if $Q$ was an internal cell. Denote by $U' = h^{-1}(Q') \subset \mathcal{X}$. Since $h$ is monotone, $U'$ is a simply connected neighborhood of the continuum $h^{-1}(\overline{Q}) \in U'$ and $h : \partial U' \to \partial Q'$. Take a look at the commutative diagram

$$
\begin{array}{ccc}
U' & \longrightarrow & Q' \\
\phi & \longrightarrow & \psi \\
\downarrow & & \downarrow \\
\mathbb{R}^2 & \longrightarrow & \mathbb{R}^2
\end{array}
$$

where $\phi : U' \overset{\text{auto}}{\to} \mathbb{R}^2$ and $\psi : Q' \overset{\text{auto}}{\to} \mathbb{R}^2$ are $C^1$-diffeomorphisms. For the existence of such diffeomorphisms one may appeal to the uniformization theorem [44] and [33, 34, 35]. It asserts that every simply connected Riemann surface is conformally equivalent to either open unit disk, the complex plane or the standard sphere; thus in our case, $C^1$-diffeomorphic to $\mathbb{R}^2$. Note that $\overline{h} = \psi \circ h \circ \phi^{-1} : \mathbb{R}^2 \overset{\text{auto}}{\to} \mathbb{R}^2$ extends continuously to the one-point compactification of the plane; that is, to a monotone mapping of the Riemann sphere onto itself, still denoted by $\overline{h} : \hat{\mathbb{C}} \overset{\text{auto}}{\to} \hat{\mathbb{C}}$, $\overline{h}(\infty) = \infty$. With the aid of stereographic projections we move the sets $X \cap \varphi(U')$ and $Y \cap \psi(Q')$ away from $\infty$. Now the proof of Proposition 4.1 reduces to the case in which

$$
\mathcal{X} = \hat{\mathbb{C}}, \mathcal{Y} = \hat{\mathbb{C}}
$$

$X, Y \subset \mathbb{C}$ are bounded simply connected Jordan domains,

$Y$ and the cell $Q \subset Y$ are Lipschitz domains.

**Step 2. Further reduction.** We shall need $Q \subset Y$ to be convex; for instance, the unit square. For this, since $Q \subset \mathbb{R}^2$ is Lipschitz domain, one might try to use a bi-Lipschitz transformation $F : \mathbb{R}^2 \overset{\text{auto}}{\to} \mathbb{R}^2$, such that $F : Q \overset{\text{auto}}{\to} (0,1) \times (0,1)$.

**Remark 4.3.** Although the existence of such $F$ poses no problem, a caution must be exercised. Every bi-Lipschitz map $F : A \overset{\text{auto}}{\to} \mathbb{B}$ between bounded planar domains and its inverse $F^{-1} : \mathbb{B} \overset{\text{auto}}{\to} A$ induce bounded (nonlinear) composition operators:

$$
F_g : \mathcal{W}^{1,p}(\Omega, A) \to \mathcal{W}^{1,p}(\Omega, \mathbb{B}), \text{ by the rule } F_g(g) \overset{\text{def}}{=} F \circ g,
$$

$$
F^{-1}_h : \mathcal{W}^{1,p}(\Omega, \mathbb{B}) \to \mathcal{W}^{1,p}(\Omega, A), \text{ by the rule } F^{-1}_h(f) \overset{\text{def}}{=} F^{-1} \circ f,
$$

whatever the domain $\Omega \subset \mathbb{R}^2$ is. But in general the continuity of these operators is questionable [15]. Fortunately, there is a satisfactory solution to this puzzle. For Lipschitz domains such as $A = Q$ and $B = (0,1) \times (0,1)$ one can construct special bi-Lipschitz transformation $F : \mathbb{R}^2 \overset{\text{auto}}{\to} \mathbb{R}^2$, $F : A \overset{\text{auto}}{\to} B$, for which the induced
composition operators \( F_2 \) and \( F_2^{-1} \) are indeed continuous. Actual construction of such \( F \) is presented in [31].

Thus we may assume that \( Q = (0, 1) \times (0, 1) \subset Y \). Furthermore, in case of a boundary cell, we assume that its external face \( \partial Q \cap F \) equals \( \{(x, 0), \ 0 \leq x \leq 1\} \).

**Step 3.** Covering by small cells. Given \( \varepsilon > 0 \), we cover \( Q \) by the family of overlapping open squares \( Q_1, Q_2, \ldots, Q_N \) of diameter less then \( \varepsilon \) in which every point in \( Q \) belongs to at most three of the closed squares in this family. In mathematical terms,

- \( Q = Q_1 \cup Q_2 \cdots \cup Q_N \), \( N = N(\varepsilon) \)
- \( \operatorname{diam} Q_i < \varepsilon \), \( i = 1, 2, \ldots, N \)
- \( 1 \leq \sum_{i=1}^{N} \lambda_{C}(z) \leq 3 \), for \( z \in Q \)
- Each \( Q_i \) is either internal or a boundary cell for \( Y \).

Construction of such a cover poses no difficulty. We now consider the corresponding pre-cells in \( U_i = h^{-1}(Q_i) \subset \mathcal{U} \subset X \).

- \( \mathcal{U} = U_1 \cup U_2 \cup \cdots \cup U_N \), \( N = N(\varepsilon) \)
- \( 1 \leq \sum_{i=1}^{N} \chi_{U_i}(x) \leq 3 \), for \( x \in \mathcal{U} \)
- Each \( U_i \) is either internal or a boundary pre-cell in \( X \).

It may be worth pointing out that the pre-cells \( U_i \) can remain very large in diameter as \( \varepsilon \) approaches zero. Typically this occurs when a continuum collapses into a point.

**Step 4.** A chain of \( p \)-harmonic replacements. We are going to construct by induction a chain \( f_1 \sim f_2 \sim \cdots \sim f_N \sim f_{N+1} \) of mappings \( f_i : \hat{C} \overset{\text{onto}}{\rightarrow} \hat{C} \). We set \( f_1 = h \). For the induction step suppose we are given a monotone map \( f_i : X \overset{\text{onto}}{\rightarrow} Y \) of Sobolev class \( W^{1,p}(X, Y) \) and its monotone extension \( f_i : \hat{C} \overset{\text{onto}}{\rightarrow} \hat{C} \). Consider the cell \( Q_i \subset Q \subset Y \) (internal or boundary) and the corresponding pre-cell \( U_i \subset \mathcal{U} \subset X \) for the mapping \( f_i \). By Lemma 2.9, there exists a monotone map \( \Psi_i : \hat{C} \overset{\text{onto}}{\rightarrow} Q_i \) is a homeomorphism and \( \Psi_i \equiv f_i : \hat{C} \setminus U_i \overset{\text{onto}}{\rightarrow} \hat{C} \setminus Q_i \). Then, with the aid of Lemmas 3.1 and 3.4, we can modify \( f_i \) within the pre-cell \( U_i \) to obtain,

\[
   f_{i+1} = \begin{cases} 
   f_i & \text{\( p \)-harmonic of class } f_i + W^{1,p}_0(U_i) \text{ in } \hat{C} \setminus U_i \\
   f_i & \text{in } U_i
   \end{cases}
\]

We emphasize (in view of Lemmas 2.9 and 3.4) that \( f_{i+1} \) takes \( U_i \) homeomorphically onto \( Q_i \), whereas under the map \( f_i \) some points in \( U_i \) may collapse into \( \partial Q_i \).

Here are the essential properties of these mappings. It may be worth reminding the reader that we are dealing with rather weird domains.

- Each \( f_i \) belongs to the Sobolev class \( W^{1,p}(X, Y) \). This is due to Lemma 3.1 which yields \( f_{i+1} - f_i \in W^{1,p}_0(U_i, \mathbb{C}) \).

- The energies are nonincreasing; \( \mathcal{E}_X[f_i] \leq \mathcal{E}_X[f_{i+1}] \leq \ldots \leq \mathcal{E}_X[h] \).
Each \( f_{i+1} \) is locally injective in \( U_1 \cup U_2 \cup \ldots \cup U_i \) (no branch points). This is because when making the \( p \)-harmonic replacement of \( f_i \) we gained new points of local injectivity for \( f_{i+1} \). These are all points in \( U_i \). At the same time we did not lose local injectivity at the points where \( f_i \) was already injective.

\( f_{N+1} : U \rightarrow \varepsilon \) is a homeomorphism, because it is a local homeomorphism and the preimage of any point is a continuum in \( \overline{X} \).

For each \( z \in U \), we have \( |f_{N+1}(z) - h(z)| \leq 3 \varepsilon \). Indeed, by triangle inequality,

\[
|f_{N+1}(z) - h(z)| \leq \sum_{i=1}^{N} |f_{i+1}(z) - f_i(z)|
\]

Let us take a quick look at each term \( |f_{i+1}(z) - f_i(z)| \). If \( z \in U_i \) then both \( f_{i+1} \) and \( f_i \) lie in \( Q_i \); therefore, \( |f_{i+1}(z) - f_i(z)| \leq \text{diam} Q_i \leq \varepsilon \). This term vanishes if \( z \not\in U_i \). But, for a given point \( z \) there can be at most three pre-cells containing \( z \). In other words the above sum consist of at most three nonzero terms, each of which does not exceed \( \varepsilon \).

**Step 5.** Letting \( \varepsilon \) small. We are now ready to proceed to the final construction of the mappings \( h_j : \overline{X} \rightarrow \varepsilon \).

\[ h_j = \begin{cases} 
    f_{N+1}(z), & \text{where } N = N(\varepsilon), \varepsilon = 1/j \text{ if } z \in U \\
    h(z) & \text{if } z \in \overline{X} \setminus U 
\end{cases} \]

Obviously, we have \( |h_j(z) - h(z)| \leq 3/j \) everywhere in \( \overline{X} \). Hence \( h_j \rightrightarrows h \) uniformly in \( \overline{X} \). To complete the proof of Proposition 4.1, we need only verify that \( h_j \rightarrow h \) strongly in \( W^{1,p}(X, \mathbb{R}^2) \). The crucial ingredient is that \( \varepsilon_X[h_j] \leq \varepsilon_X[h] \), for all \( j = 1, 2, \ldots \). In particular, \( h_j \) converge to \( h \) weakly in \( W^{1,p}(X, \mathbb{R}^2) \). Now, the lower semicontinuity of the energy functional yields \( \varepsilon_X[h] \leq \liminf \varepsilon_X[h_j] \leq \liminf \varepsilon_X[h_j] = \varepsilon_X[h] \). This in turn implies, by uniform convexity arguments, that \( h_j \) converge to \( h \) strongly in \( W^{1,p}(X, \mathbb{R}^2) \).

**Remark 4.4.** Here is a careful look at the uniform convexity arguments. Consider the coordinate functions for \( h_j = u_j + i v_j \) and \( h = u + i v \). In view of lower semicontinuity, we have \( \int |\nabla u|^p \leq \liminf \int |\nabla u_j|^p \) and \( \int |\nabla v|^p \leq \liminf \int |\nabla v_j|^p \). Adding these inequalities, we obtain \( \int (|\nabla u|^p + |\nabla v|^p) \leq \liminf \int (|\nabla u_j|^p + |\nabla v_j|^p) = \liminf \varepsilon'[h_j] \leq \varepsilon'[h] = \int (|\nabla u|^p + |\nabla v|^p) \), which is possible only when \( \int |\nabla u_j|^p \leq \liminf \int |\nabla u|^p \) and \( \int |\nabla v_j|^p \leq \liminf \int |\nabla v|^p \). Now we see that \( \nabla u_j \) and \( \nabla v_j \) converge strongly in \( L^p(X, \mathbb{R}^2) \), because the usual normed space \( L^p(X, \mathbb{R}^2) \) is uniformly convex, by Clarkson’s Inequality for vectors in the Euclidean space \( \mathbb{R}^2 \), see [12]. Actually, one could apply Theorem 2 in [13] to infer that the Banach space \( L^p(X, \mathbb{R}^2) \times L^p(X, \mathbb{R}^2) \), equipped with the norm \( \left[ \int_X (|f|^p + |g|^p) \right]^{1/p} \), is uniformly convex as well.

5. **Completing the proof of Theorem 1.3**

We now return to the surfaces \( X \subset \mathcal{X} \) and \( Y \subset \mathcal{Y} \) and construct the mappings \( h_j : \overline{X} \rightarrow Y \) stated in Theorem 1.3. The arguments are similar to those used in
Suppose we are given a monotone map
\begin{equation}
(6.1)
\end{equation}
and
\begin{equation}
\begin{split}
\text{Steps 3,4 and 5. The main difference, however, is that we now choose and fix one particular finite cover of } \mathcal{Y} \text{ by cells. Having Proposition 4.1 in hands there will be no need to partition those cell into smaller cells. We adopt analogous notation. Thus, we let } \mathcal{Y} \subset \mathcal{Y} \text{ be covered by Lipschitz cells } Q_1, Q_2, \ldots, Q_N, \text{ including both internal and boundary cells. This time } N \text{ is fixed for the rest of our proof. In symbols,}
\end{split}
\end{equation}

\begin{itemize}
  \item \( \mathcal{Y} = Q_1 \cup Q_2 \cup \ldots \cup Q_N \)
  \item \( 1 \leq \sum_{i=1}^{N} \chi_{Q_i}(y) \leq N, \quad \text{for } y \in \mathcal{Y} \)
\end{itemize}

Let \( \varepsilon \) be any positive number. As before, we proceed by induction to define a chain \( F_1 \hookrightarrow F_2 \hookrightarrow \ldots \hookrightarrow F_N \hookrightarrow F_{N+1} \) of monotone mappings \( F_i : \mathcal{X} \overset{\text{onto}}{\longrightarrow} \mathcal{Y} \).

The first map is \( F_1 \overset{\text{def}}{=} h \). In the induction step we appeal to Proposition 4.1. Suppose we are given a monotone map \( F_i : \mathcal{X} \overset{\text{onto}}{\longrightarrow} \mathcal{Y} \) of Sobolev class \( W^{1,p}(X,Y) \) and its monotone extension \( F_i : \hat{\mathcal{X}} \overset{\text{onto}}{\longrightarrow} \hat{\mathcal{Y}} \). Consider the cell \( Q_i \subset \mathcal{Y} \) and the corresponding pre-cell \( U_i \subset \mathcal{X} \) for \( F_i \). Then, by Proposition 4.1, there exists a monotone map \( F_{i+1} : \mathcal{X} \overset{\text{onto}}{\longrightarrow} \mathcal{Y} \) such that:

\begin{itemize}
  \item (a) \( F_{i+1} \in C(\mathcal{X},\mathcal{Y}) \cap W^{1,p}(X,R^3) \)
  \item (b) \( F_{i+1} = F_i : \mathcal{X} \setminus U_i \overset{\text{onto}}{\longrightarrow} \mathcal{Y} \setminus Q_i \)
  \item (c) \( F_{i+1} : U_i \overset{\text{onto}}{\longrightarrow} Q_i \) is a homeomorphism,
  \item (d) \( |F_{i+1} - F_i| \leq \frac{\varepsilon}{N} \) everywhere in \( \mathcal{X} \),
  \item (e) \( \|DF_{i+1} - DF_i\|_{L^p(X)} \leq \frac{\varepsilon}{2} \)
\end{itemize}

In each induction step, passing from \( F_i \) to \( F_{i+1} \) we gain injectivity of \( F_{i+1} \) within \( U_i \), and at the same time produce no branch points in the previous precells. Thus \( F_{N+1} : \mathcal{X} \overset{\text{onto}}{\longrightarrow} \mathcal{Y} \) is a local homeomorphism. Arguing as in Step 4, since \( F_{N+1} : \mathcal{X} \overset{\text{onto}}{\longrightarrow} \mathcal{Y} \) is monotone, we see that \( F_{N+1} : \mathcal{X} \overset{\text{onto}}{\longrightarrow} \mathcal{Y} \) is a homeomorphism.

Lastly, by triangle inequality, we obtain
\begin{equation}
\begin{split}
\|F_{N+1} - h\|_{W^{1,p}(X)} \overset{\text{def}}{=} & |F_{N+1} - F_1|_{W^{1,p}(X)} + \|DF_{N+1} - DF_1\|_{L^p(X)} \\
\leq & \sum_{i=1}^{N} |F_{i+1} - F_i|_{W^{1,p}(X)} + \sum_{i=1}^{N} |DF_{i+1} - DF_i|_{L^p(X)} \\
\leq & \varepsilon + \varepsilon = \varepsilon
\end{split}
\end{equation}

The proof of Theorem 1.3 is concluded by setting \( \varepsilon = 1/j \) and \( h_j = F_{N+1} \).

6. Applications to Thin Plates and Films

Let us demonstrate the utility of Theorem 1.3 by establishing the existence of the energy-minimal deformations of thin plates (planar domains) and films (surfaces) for \( p \)-harmonic type energy. Recall the reference manifolds and the Jordan domains \( \mathcal{X} \subset \mathcal{X} \) and \( \mathcal{Y} \subset \mathcal{Y} \). Here we assume that both \( \mathcal{X} \) and \( \mathcal{Y} \) are Lipschitz domains. We examine homeomorphisms \( h : \mathcal{X} \to \mathcal{Y} \) in the Sobolev space \( W^{1,p}(X,Y) \subset W^{1,2}(X,Y) \), \( 2 \leq p < \infty \), and their weak limits. Every homeomorphism \( h \in W^{1,2}(X,Y) \) extends up to the boundary as a continuous monotone map, still denoted by \( h : \mathcal{X} \overset{\text{onto}}{\longrightarrow} \mathcal{Y} \). Moreover, we have a uniform bound of the modulus of continuity in terms of the Dirichlet and the \( \delta_p \)-energy on a surface, see Subsection 6.1 below for the definition of \( \delta_p \).

\begin{equation}
(6.1) \quad |h(x_1) - h(x_2)|^2 \leq \frac{C(X,Y) \delta_2[h]}{\log \left(e + \frac{1}{|x_1-x_2|} \right)} \leq \frac{C_p(X,Y) \left[\delta_p[h]\right]^{2/p}}{\log \left(e + \frac{1}{|x_1-x_2|} \right)}
\end{equation}
For $h$ fixed, its logarithmic modulus of continuity was already known by Lebesgue [37]. However, it is the dependence on the energy of $h$ that we are specifically concerned (to apply limiting arguments). The proof of estimate (6.1) runs along similar lines as for Lipschitz planar domains in [28]. There are, however, routine adjustments necessary to fit the arguments to $2$-dimensional surfaces.

Now a $W^{1,p}$-weakly converging sequence of homeomorphisms between $X$ and $Y$ actually converges uniformly. Based on Theorem 1.3, we have

\[(6.2) \quad \mathcal{H}_p(X, Y) = \mathcal{H}_p(X, Y) = \mathcal{M}_p(X, Y)\]

meaning that (respectively) the strong closure, the sequential weak closure, and the monotone maps in the Sobolev space $W^{1,p}(X, Y)$ are the same thing.

### 6.1. The isotropic $p$-harmonic integral on surfaces

An intrinsic example of the energy-minimal deformations of thin plates and films is furnished by the $p$-harmonic integral, $2 \leq p < \infty$. Suppose we are given a monotone Sobolev map $h \in \mathcal{M}_p(X, Y)$. To almost every point $x \in X$ there corresponds the linear tangent map $Dh(x) : T_x(X) \rightarrow T_y(Y)$, $y = h(x) \in Y$, and its adjoint $D^*h(x) : T_y(Y) \rightarrow T_x(X)$ with respect to the scalar products in $T_x(X)$ and $T_y(Y)$. The Cauchy-Green stress tensor $G_h \overset{\text{def}}{=} [D^*h] \circ [Dh] : T_x(X) \rightarrow T_x(X)$ gives rise to the Hilbert-Schmidt norm of the tangent map, $|Dh| = |\text{Trace} G_h|^{1/2}$. Now the isotropic $p$-harmonic energy of $h$ is defined by:

\[(6.3) \quad \mathcal{E}_p[h] \overset{\text{def}}{=} \int_X |Dh(x)|^p \, dx ,\]

where the area element $dx$ is the one induced by the Riemannian metric in $X$. The term isotropic refers to the fact that the integrand is invariant under the rotations in $T_x(X)$ and $T_y(Y)$. We call $\mathcal{E}_2[h]$ the Dirichlet energy. The energy in (6.3) fits to the following more general scheme:

\[(6.4) \quad \mathcal{E}[h] \overset{\text{def}}{=} \int_X E(x, h, Dh) \, dx , \quad \text{for mappings } h \in \mathcal{M}_p(X, Y) ,\]

where $E(x, y, L)$ is a given real-valued function defined for $x \in X$ and $y \in Y$, and the linear maps $L : T_x(X) \rightarrow T_y(Y)$. We shall impose the following conditions on the energy integral in (6.4): they suffice for the application of the Direct Method in the Calculus of Variations:

- **coercivity:**
  \[ c \int_X |Dh(x)|^p \, dx \leq \int_X E(x, h, Dh) \, dx \leq C \int_X |Dh(x)|^p \, dx . \]

- **continuity in the strong topology of $W^{1,p}(X, Y)$:**
  \[ \mathcal{E}[h] = \liminf \mathcal{E}[h_j] , \quad \text{whenever } h_j \in \mathcal{M}_p(X, Y) \text{ converge strongly to } h . \]

- **lower semicontinuity:**
  \[ \mathcal{E}[h] \leq \liminf \mathcal{E}[h_j] , \quad \text{whenever } h_j \in \mathcal{M}_p(X, Y) \text{ converge weakly to } h . \]

These conditions hold, in particular, for the $p$-harmonic integral in (6.3). Next, choose and fix a homeomorphism $\varphi \in \mathcal{H}_p(X, Y)$ and a compact set $\Gamma \subset \partial X$ (empty in case of traction free problems). We consider the class of Sobolev homeomorphisms,

\[(6.5) \quad \mathcal{H}_p(X, Y; \varphi) \overset{\text{def}}{=} \{ h \in \mathcal{H}_p(X, Y); h|\Gamma = \varphi|\Gamma \text{ (upon extension to } \partial X) \} \]
By Theorem 1.3, its strong and sequential weak closures in $W^{1,p}(X,\mathbb{R}^2)$ are the same and coincide with the monotone mappings of the class:

\[
\mathcal{M}_p(X,Y,\Gamma;\varphi) \doteq \{ h \in \mathcal{M}_p(X,Y) ; h|_{\Gamma} = \varphi|_{\Gamma} \}
\]

Straightforward application of the direct method in the Calculus of Variations yields

**Theorem 6.1.** There always exists the energy-minimal map $h_o \in \mathcal{M}_p(X,Y,\Gamma;\varphi)$ such that

\[
\mathcal{E} \left[ h_o \right] = \min_{h \in \mathcal{M}_p(X,Y,\Gamma;\varphi)} \mathcal{E} \left[ h \right] = \inf_{h \in \mathcal{M}_p(X,Y,\Gamma;\varphi)} \mathcal{E} \left[ h \right] \quad \text{(no Lavrentiev phenomenon)}
\]

**Remark 6.2.** The existence of energy-minimal mappings within $\mathcal{M}_p(X,Y)$, $p \geq 2$, can also be obtained for many realistic variational integrals in NE, including neo-hookean energy;

\[
\mathcal{E}[h] = \int_X \left( |Dh|^2 + \frac{1}{\det Dh} \right) \quad \text{for} \quad h \in \mathcal{M}_2^+(X,Y)
\]

Here $\mathcal{M}_2^+$ stands for mappings in $\mathcal{M}_2(X,Y)$ with positive Jacobian determinant. For $\ell$-connected, $2 \leq \ell < \infty$, Lipschitz domains $X$ and $Y$ the class $\mathcal{M}_2(X,Y)$, as oppose to $\mathcal{H}_2(X,Y)$, is closed under weak convergence in $W^{1,2}(X,Y)$. Applying the direct method in the calculus of variations we see that there always exists an energy-minimal map $h_o \in \mathcal{M}_2^+(X,Y)$. Now, by Theorem 1.3 the minimizer $h_o$ can be approximated uniformly and strongly in $W^{1,2}(X,\mathbb{R}^2)$ by homeomorphisms. The key issue is whether

\[
\mathcal{E}[h_o] = \inf_{h \in \mathcal{M}_2(X,Y)} \mathcal{E}[h]
\]

It is interesting to notice that (6.7) holds if

\[
\frac{1}{\det Dh_o} \in \mathcal{L}^\infty_{\text{loc}}(X).
\]

Indeed, this condition implies that $h_o$ has locally integrable distortion; that is,

\[
|Dh_o(x)|^2 \leq K(x) \det Dh_o(x), \quad \text{where} \quad K \in \mathcal{L}_1^{\text{loc}}(X).
\]

By [32] $h_o$ is open and discrete and, being monotone, is a homeomorphism of class $\mathcal{H}_2(X,Y)$.

7. Afterward

Remarkably, the existence of traction free minimal deformations requires advanced topological arguments, as compared with the classical Dirichlet boundary value problems. Before further reflections let us take a brief look at one more example.

**An Example.** Given a pair of circular annuli $X = \{ z ; r < |z| < R \}$ and $Y = \{ w ; 1 < |w| < \frac{1}{2}(R + R^{-1}) \}$, $r < 1 < R$, the energy-minimal deformation in $\mathcal{M}_2(X,Y)$, unique up to the rotations, is given by:

\[
h(z) = \begin{cases} 
\frac{z}{|z|} , & r < |z| \leq 1 \\
\frac{1}{2} \left( z + \frac{1}{z} \right) , & 1 < |z| < R
\end{cases}
\]

( critical harmonic Nitsche map )
Squeezing of \( \{ z; r < |z| \leq 1 \} \subset X \) into the inner circle of \( Y \) manifests itself. Let us confront it with the solution of the classical Dirichlet problem under the same boundary values as \( h \); that is,

\[
(7.1) \quad h(z) = \begin{cases} 
  r^{-1}z, & \text{for } |z| = r \\
  \frac{1}{2}(1 + R^{-2})z, & \text{for } |z| = R
\end{cases}
\]

The solution is a harmonic mapping

\[
h(z) = Az + \frac{B}{z}, \quad \text{where } A = \frac{R^2 - 2r + 1}{2(R^2 - r^2)} \quad \text{and} \quad B = \frac{r(2R^2 - rR^2 - r)}{2(R^2 - r^2)}
\]

which, as expected, fails to be injective. The point is that, under this harmonic solution the annulus \( X \) folds along the circle \( |z| = \sqrt{\frac{B}{A}} \).

It is axiomatic in NE that folding should not occur under hyperelastic deformations. Uniform limits of hyperelastic deformations do not exhibit foldings as well. It is a common struggle in mathematical models of Nonlinear Elasticity [2, 5, 6, 7, 8, 9, 11] to establish existence of the energy-minimal deformations which comply with the principle of no interpenetration of matter. The phenomenon of squeezing a part of a domain, as described by monotone mappings, is inevitable. It is therefore reasonable to adopt Monotone Sobolev Mappings as legitimate deformations in the mathematical description of 2D-elasticity (cellular mappings in higher dimensions).

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