A Note on the DP-Chromatic Number of Complete Bipartite Graphs

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Abstract

DP-coloring (also called correspondence coloring) is a generalization of list coloring recently introduced by Dvořák and Postle. Several known bounds for the list chromatic number of a graph $G$, $\chi_l(G)$, also hold for the DP-chromatic number of $G$, $\chi_{DP}(G)$. On the other hand, there are several properties of the DP-chromatic number that shows that it differs with the list chromatic number. In this note we show one such property. It is well known that $\chi_l(K_{k,t}) = k + 1$ if and only if $t \geq k^k$. We show that $\chi_{DP}(K_{k,t}) = k + 1$ if $t \geq 1 + (k^k/k!)(\log(k!) + 1)$, and we show that $\chi_{DP}(K_{k,t}) < k + 1$ if $t < k^k/k!$.

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1 Introduction

In this note all graphs are nonempty, finite, simple graphs unless otherwise noted. Generally speaking we follow West [14] for terminology and notation. For this note the set of natural numbers is $\mathbb{N} = \{1, 2, 3, \ldots \}$. The natural log function is denoted log. Given a set $A$, $\mathcal{P}(A)$ is the power set of $A$. Also, for any $k \in \mathbb{N}$, $[k] = \{1, 2, 3, \ldots , k\}$. If $G$ is a graph and $S, U \subseteq V(G)$, we use $G[S]$ for the subgraph of $G$ induced by $S$, and we use $E_G(S, U)$ for the subset of $E(G)$ with one endpoint in $S$ and one endpoint in $U$. Also, if $v \in V(G)$ we use $N_G(v)$ for the set of neighbors of $v$ in $G$.

1.1 List Coloring

List coloring is a well known variation on the classic vertex coloring problem, and it was introduced independently by Vizing [12] and Erdős, Rubin, and Taylor [8] in the 1970’s. In the classic vertex coloring problem we wish to color the vertices of a graph $G$ with as few colors as possible so that adjacent vertices receive different colors, a so-called proper coloring. The chromatic number of a graph, denoted $\chi(G)$, is the smallest $k$ such that $G$ has a proper coloring that uses $k$ colors. For list coloring, we associate a list assignment, $L$, with a graph $G$ such that each vertex $v \in V(G)$ is assigned a list of colors $L(v)$ (we say $L$
is a list assignment for $G$). The graph $G$ is $L$-colorable if there exists a proper coloring $f$ of $G$ such that $f(v) \in L(v)$ for each $v \in V(G)$ (we refer to $f$ as a proper $L$-coloring of $G$). A list assignment $L$ is called a $k$-assignment for $G$ if $|L(v)| = k$ for each $v \in V(G)$.

The list chromatic number of a graph $G$, denoted $\chi_L(G)$, is the smallest $k$ such that $G$ is $L$-colorable whenever $L$ is a $k$-assignment for $G$. We say $G$ is $k$-choosable if $k \geq \chi_L(G)$.

It is immediately obvious that for any graph $G$, $\chi(G) \leq \chi_L(G)$. Erdős, Rubin, and Taylor [8] studied the equitable choosability of $K_{m,m}$ and observed that if $m = \binom{2k-1}{k}$, then $\chi_L(K_{m,m}) > k$. The following related result is often attributed to Vizing [12] or Erdős, Rubin, and Taylor [8], but it is best described as a folklore result.

**Theorem 1.** For $k \in \mathbb{N}$, $\chi_L(K_{k,t}) = k + 1$ if and only if $t \geq k^k$.

We study the analogue of Theorem 1 for DP-coloring.

### 1.2 DP-coloring

Dvořák and Postle [7] introduced DP-coloring (they called it correspondence coloring) in 2015 in order to prove that every planar graph without cycles of lengths 4 to 8 is 3-choosable. Intuitively, DP-coloring is a generalization of list coloring where each vertex in the graph still gets a list of colors but identification of which colors are different can vary from edge to edge.

Following [5], we now give the formal definition. Suppose $G$ is a graph. A cover of $G$ is a pair $\mathcal{H} = (L, H)$ consisting of a graph $H$ and a function $L : V(G) \to \mathcal{P}(V(H))$ satisfying the following four requirements:

1. the sets $\{L(u) : u \in V(G)\}$ form a partition of $V(H)$;
2. for every $u \in V(G)$, the graph $H[L(u)]$ is complete;
3. if $E_H(L(u), L(v))$ is nonempty, then $u = v$ or $uv \in E(G)$;
4. if $uv \in E(G)$, then $E_H(L(u), L(v))$ is a matching (the matching may be empty).

Suppose $\mathcal{H} = (L, H)$ is a cover of $G$. We say $\mathcal{H}$ is $k$-fold if $|L(u)| = k$ for each $u \in V(G)$. An $\mathcal{H}$-coloring of $G$ is an independent set in $H$ of size $|V(G)|$. It is immediately clear that $I \subseteq V(G)$ is an $\mathcal{H}$-coloring if and only if $|I \cap L(u)| = 1$ for each $u \in V(G)$.

The DP-chromatic number of a graph $G$, $\chi_{DP}(G)$, is the smallest $k \in \mathbb{N}$ such that $G$ admits an $\mathcal{H}$-coloring for every $k$-fold cover $\mathcal{H}$ of $G$. Suppose we wish to prove $\chi_{DP}(G) \leq k$. Since every $k$-fold cover of $G$ is isomorphic to a subgraph of some $k$-fold cover, $\mathcal{H}' = (L', H')$, of $G$ with the property that $E_{H'}(L'(u), L'(v))$ is a perfect matching whenever $uv \in E(G)$, we need only show that $G$ has an $\mathcal{H}$-coloring whenever $\mathcal{H} = (L, H)$ is a $k$-fold cover of $G$ such that $E_H(L(u), L(v))$ is a perfect matching for each $uv \in E(G)$.

Given a list assignment, $L$, for a graph $G$, it is easy to construct a cover $\mathcal{H}$ of $G$ such that $G$ has an $\mathcal{H}$-coloring if and only if $G$ has a proper $L$-coloring (see [5]). It follows that $\chi_L(G) \leq \chi_{DP}(G)$. This inequality may be strict since it is easy to prove that $\chi_{DP}(C_n) = 3$ whenever $n \geq 3$, but the list chromatic number of any even cycle is 2 (see [5] and [8]).

We now briefly discuss some similarities between the DP-coloring and list coloring. First, notice that like $k$-choosability, the graph property of having DP-chromatic number at most $k$ is monotone. It is also clear that, as in the context of list coloring, if $\chi_{DP}(G) = k$, then an $\mathcal{H}$-coloring of $G$ exists whenever $\mathcal{H}$ is an $m$-fold cover of $G$ with $m \geq k$. The
Proof. Proposition 2. This means that \( \mu \) the smallest natural number \( k \) such that each vertex \( v_i \) has at most \( d - 1 \) neighbors among \( v_1, v_2, \ldots, v_{i-1} \). It is easy to prove that \( \chi_t(G) \leq \chi_{DP}(G) \leq \text{col}(G) \).

Thomassen [13] famously proved that every planar graph is 5-choosable, and Dvořák and L. Postle [7] observed that the DP-chromatic number of every planar graph is at most 5. Also, Molloy [11] recently improved a theorem of Johansson, and showed that every triangle-free graph \( G \) with maximum degree \( \Delta(G) \) satisfies \( \chi_t(G) \leq (1 + o(1))\Delta(G)/\log(\Delta(G)) \). Bernshteyn [4] subsequently showed that this bound also holds for the DP-chromatic number.

On the other hand, Bernshteyn [3] showed that if the average degree of a graph \( G \) is \( d \), then \( \chi_{DP}(G) = \Omega(d/\log(d)) \). This is in stark contrast to the celebrated result of Alon [1] which says \( \chi_t(G) = \Omega(\log(d)) \). It was also recently shown in [5] that there exist planar bipartite graphs with DP-chromatic number 4 even though the list chromatic number of any planar bipartite graph is at most 3 [2]. A famous result of Galvin [9] says that if \( G \) is a bipartite multigraph and \( L(G) \) is the line graph of \( G \), then \( \chi_t(L(G)) = \chi(L(G)) = \Delta(G) \).

However, it is also shown in [5] that every \( d \)-regular graph \( G \) satisfies \( \chi_{DP}(L(G)) \geq d + 1 \).

1.3 Outline of Results and an Open Question

In this note we present some results on the DP-chromatic number of complete bipartite graphs. By what was mentioned in the previous subsection, we know that if \( k, t \in \mathbb{N} \), \( \chi_{DP}(K_{k,l}) \leq \text{col}(K_{k,l}) \leq k + 1 \). For the remainder of this note, for each \( k \in \mathbb{N} \), let \( \mu(k) \) be the smallest natural number \( l \) such that \( \chi_{DP}(K_{k,l}) = k + 1 \). We have that \( \mu(k) \) exists for each \( k \in \mathbb{N} \) since we know by Theorem 1

\[
k + 1 = \chi_t(K_{k,k}) \leq \chi_{DP}(K_{k,k}) \leq k + 1.
\]

This means that \( \mu(k) \leq k^k \) for each \( k \in \mathbb{N} \). The following proposition is also clear.

Proposition 2. For \( k \in \mathbb{N} \), \( \chi_{DP}(K_{k,t}) = k + 1 \) if and only if \( t \geq \mu(k) \)

Proof. If \( t \geq \mu(k) \), \( k + 1 = \chi_{DP}(K_{k,\mu(k)}) \leq \chi_{DP}(K_{k,t}) \leq k + 1 \) since \( K_{k,\mu(k)} \) is a subgraph of \( K_{k,t} \). Conversely, if \( \chi_{DP}(K_{k,t}) = k + 1 \), then \( \mu(k) \leq t \) by the definition of \( \mu(k) \). \( \square \)

Computing \( \mu(k) \) is easy when \( k = 1, 2 \). Clearly, \( \mu(1) = 1 \). Also, \( \mu(2) = 2 \) follows from the fact that \( \chi_{DP}(K_{2,1}) \leq \text{col}(K_{2,1}) = 2 \), and the fact that \( K_{2,2} \) is a 4-cycle which implies \( \chi_{DP}(K_{2,2}) = 3 \). We have a tedious argument that shows \( \mu(3) = 6 \), which for the sake of brevity, we do not present in this note. The following question lead to the discovery of both results in this note.

Question 3. For each \( k \geq 4 \), what is the exact value of \( \mu(k) \)?

We obtain an upper bound and lower bound on \( \mu(k) \). Our first result gives us a lower bound.

Theorem 4. For \( k \in \mathbb{N} \), if \( t < \frac{k^k}{k!} \), then \( \chi_{DP}(K_{k,t}) < k + 1 \).

Theorem 3 tells us that \( [k^k/k!] \leq \mu(k) \) notice this lower bound is tight for \( k = 1, 2 \), and it is 1 away from being tight for \( k = 3 \). We then use a simple probabilistic argument to prove our second result which gives us an upper bound on \( \mu(k) \).
Theorem 5. For $t \in \mathbb{N}$ let

$$m = t + \left\lfloor \frac{k^t}{k!} \right\rfloor$$

Then, $\chi_{DP}(K_{k,m}) = k + 1$.

Theorems 4 and 5 imply

$$\left\lfloor \frac{k^t}{k!} \right\rfloor \leq \mu(k) \leq 1 + \frac{k^t(\log(k!) + 1)}{k!}.$$ 

We suspect that $\lfloor k^t/k! \rfloor$ is closer to the exact value of $\mu(k)$ than the upper bound.

2 Proofs of Results

In this section we prove Theorems 4 and 5. We begin with a definition. Suppose that $\mathcal{H} = (L, H)$ is a $k$-fold cover of $G$. For any $v \in V(G)$, we say an independent set, $I$, in $H[\bigcup_{u \in N_G(v)} L(u)]$ is bad for $v$ if $|I| = |N_G(v)|$ and for each $w \in L(v)$, $w$ is adjacent to some vertex in $I$. Notice that if $I$ is bad for $v$, then an $\mathcal{H}$-coloring of $G$ cannot contain $I$.

In this section we often have $G = K_{k,t}$, and we always suppose $G$ has bipartition $X = \{v_1, v_2, \ldots, v_k\}$, $Y = \{u_1, u_2, \ldots, u_t\}$. We now mention an idea used frequently in this section. Notice that if $\mathcal{H} = (L, H)$ is a $k$-fold cover $G = K_{k,t}$, then there are precisely $k^t$ independent sets of size $k$ in $H[\bigcup_{v \in X} L(v)]$. If all of these independent sets are bad for at least one vertex in $Y$, then there is no $\mathcal{H}$-coloring of $G$. We now prove a lemma which gives us a bound on how many independent sets of size $k$ in $H[\bigcup_{v \in X} L(v)]$ can be bad for a vertex in $Y$.

Lemma 6. Suppose $G$ is a graph, $v \in V(G)$, and $|N_G(v)| = k$. Suppose that $\mathcal{H} = (L, H)$ is a $k$-fold cover of $G$. Then, there are at most $k!$ distinct independent sets in $H[\bigcup_{u \in N_G(v)} L(u)]$ that are bad for $v$.

Proof. The result is obvious when $k = 1$. So, suppose $k \geq 2$. We let $H' = H[\bigcup_{u \in N_G(v)} L(u)]$. Suppose that $N_G(v) = \{v_1, v_2, \ldots, v_k\}$.

Let $\mathcal{C}$ denote the set of bijective functions from $[k]$ to $L(v)$. Let $\mathcal{I}$ denote the set of all independent sets in $H'$ that are bad for $v$. We are done if $\mathcal{I} = \emptyset$, so we assume $\mathcal{I} \neq \emptyset$. We now define an injective mapping, $f : \mathcal{I} \to \mathcal{C}$. For $I \in \mathcal{I}$ suppose that $I = \{u_1, u_2, \ldots, u_k\}$ where $u_i \in L(v_i)$ (we know that $|I \cap L(v_i)| = 1$ for each $i \in [k]$). Suppose that for each $i \in [k]$, $w_i$ is the one vertex in $L(v_i)$ to which $u_i$ is adjacent. Then, let $\sigma_I : [k] \to L(v)$ be the function defined by $\sigma_I(i) = w_i$. Since $I$ is bad for $v$, we know that $\sigma_I \in \mathcal{C}$. So, we can let $f(I) = \sigma_I$.

To see that $f$ is injective, suppose that $I = \{u_1, u_2, \ldots, u_k\}$ and $I' = \{u'_1, u'_2, \ldots, u'_k\}$ are distinct elements of $\mathcal{I}$ where $u_i, u'_i \in L(v_i)$ for each $i \in [k]$. This means that there must be a $j \in [k]$ such that $u_j \neq u'_j$. Since $E_H(L(v_j), L(v_j))$ is a matching, we know that $u_j$ and $u'_j$ are adjacent to distinct vertices in $L(v_j)$. Thus, $f(I) \neq f(I')$. The fact that $f$ is injective immediately implies that $|\mathcal{I}| \leq |\mathcal{C}| = k!$. 

We are now ready to prove Theorem 4.
Proof. We suppose $k \geq 3$ since the result is clear for $k = 1, 2$. We also assume $t \in \mathbb{N}$ since the result is clear when $t = 0$. Suppose $G = K_{k,t}$.

Let $\mathcal{H} = (L, H)$ be an arbitrary $k$-fold cover of $G$. Let $H' = H[ \bigcup_{i=1}^{k} L(v_i) ]$. It is clear that there are $k^k$ independent sets of size $k$ in $H'$. Moreover, we know from Lemma 6 that there are at most $k!$ independent sets in $H'$ that are bad for $u_j$ for each $j \in [t]$. Since

$$k^k - t(k!) > 0,$$

there is an independent set, $I$ in $H'$ such that $|I| = k$ and $I$ is not bad for any vertex in $Y$. Thus, for each $j \in [t]$, we can find a $w_j \in L(u_j)$ that is not adjacent to any vertex in $I$. Finally, $I \cup \{w_1, w_2, \ldots, w_t\}$ is an $\mathcal{H}$-coloring of $G$.

We now prove Theorem 5.

Proof. We suppose $k \geq 2$ since the result is clear for $k = 1$. Suppose $G = K_{k,t}$. We form a $k$-fold cover of $G$ by the following (partially random) process. We begin by letting $L(v_i) = \{(v_i, l) : l \in [k]\}$ and $L(u_j) = \{(u_j, l) : l \in [k]\}$ for each $i \in [k]$ and $j \in [t]$. Let graph $H$ have vertex set

$$\left( \bigcup_{i=1}^{k} L(v_i) \right) \bigcup \left( \bigcup_{j=1}^{t} L(u_j) \right).$$

Also, draw edges in $H$ so that $H[L(v)]$ is a clique for each $v \in V(G)$. Finally, for each $i \in [k]$ and $j \in [t]$, uniformly and randomly choose a perfect matching between $L(v_i)$ and $L(u_j)$ from the $k!$ possible perfect matchings. It is easy to see that $\mathcal{H} = (L, H)$ is a $k$-fold cover of $G$.

Note that there are exactly $k^k$ independent sets of size $k$ in $H[ \bigcup_{i=1}^{k} L(v_i) ]$. Suppose we name the $k^k$ functions from $[k]$ to $[k]$ by $f_1, f_2, f_3, \ldots, f_{k^k}$. Then the $k^k$ independent sets of size $k$ in $H[ \bigcup_{i=1}^{k} L(v_i) ]$ are precisely: $I_1, I_2, \ldots, I_{k^k}$ where $I_i = \{(u_i, f_i(l)) : l \in [k]\}$.

Suppose that for each $i \in [k^k]$, $E_i$ is the event that $I_i$ is not bad for any vertex in $Y$. For any vertex $u \in Y$, it is easy to see that the probability that $I_i$ is bad for $u$ is

$$\frac{k!((k-1)!)^k}{(k!)^k} = \frac{k!}{k^k}.$$

Thus, $P[E_i] = (1 - \frac{k!}{k^k})^t$. Let $X_i$ be the random variable that is 1 when $E_i$ occurs, and it is 0 otherwise. Let $X = \sum_{i=1}^{k^k} X_i$. By linearity of expectation,

$$E[X] = k^k \left( 1 - \frac{k!}{k^k} \right)^t.$$

Let $z = E[X]$. We can find a $k$-fold cover, $\mathcal{H}' = (L', H')$, of $G$ such that at most $z$ of the independent sets of size $k$ in $H'[ \bigcup_{i=1}^{k} L'(v_i) ]$ are not bad for any vertex in $Y$. Suppose we call such independent sets: $I_{a_1}, I_{a_2}, \ldots, I_{a_r}$ (we know $r \leq z$).

Starting with $G$, we create a copy of $K_{k,t+r}$, called $M$, by adding $r$ new vertices, $w_1, \ldots, w_r$, to $Y$. We construct a $k$-fold cover of $M$ starting from $\mathcal{H}'$ as follows. With each $w_i$ we associate $k$ vertices, $L''(w_i)$, and we add these vertices to $H'$ along with edges so that the vertices in $L''(w_i)$ are pairwise adjacent. Then, for $i \in [r]$, we create a matching between $L'(v_j)$ and $L''(w_i)$ for each $j \in [k]$ so that $I_{a_i}$ is bad for $w_i$. The result is a $k$-fold cover, $\mathcal{H}''$, of $M$ with the property that there is no $\mathcal{H}''$-coloring of $M$. Thus, $k+1 = \chi_{DP}(K_{k,t+r}) \leq \chi_{DP}(K_{k,m})$.
Letting $t = \lceil k^k \log(k!) / k! \rceil$, we note

$$t + \left( k^k \left( 1 - \frac{k!}{k^k} \right)^t \right) \leq \left[ \frac{k^k \log(k!)}{k!} \right] + \frac{k^k}{k!} \leq 1 + \frac{k^k (\log(k!) + 1)}{k!}$$

and the upper bound on $\mu(k)$ mentioned in the previous section follows.

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