BSE-PROPERTIES OF SECOND DUAL OF BANACH ALGEBRAS

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Abstract. Let A be a commutative semisimple Arens regular unital Banach algebra. The correlation between the BSE-property of the Banach algebra A and its second duals are assessed. It is found and approved that if A is a BSE-algebra, then so is $A^{**}$. The opposite correlation will hold in certain conditions. The correlation of the BSE-norm property of the Banach algebra and its second dual are assessed and examined. It is revealed that, if A is a commutative Arens regular unital Banach algebra where $A^{**}$ is semisimple, then, $A$ and $A^{**}$ are BSE-norm algebra.

1. Introduction

Let A be a Banach algebra. The researcher in [3] and [4], introduced the two Arens multiplications on $A^{**}$ which convert it into a Banach algebra. A Banach algebra is an Arens regular if the Arens multiplications coincide with its second dual.

The Bochner-Schoenberg-Eberlein (BSE) is derived from the famous theorem proved in 1980 by Bochner and Schoenberg for the group of real numbers; [5] and [13]. The researcher in [8], revealed that if G is any locally compact abelian group, then the group algebra $L_1(G)$ is a BSE algebra. The researcher in [13],[15],[16] assessed the commutative Banach algebras that meet the Bochner-Schoenberg-Eberlein- type theorem and explained their properties. They are introduced and assessed in [14] the first and second types of BSE algebras. This concept is expanded in [9] and [11]. The concept of an n-BSE function and Banach algebras $C_{BSE(n)}(\Delta(A))$, for each \( n \in \mathbb{N} \) is introduced in [15], where the correlation between these concepts and the BSE property is assessed. In 1992, a class of commutative Banach algebras was introduced by the same researchers in the form of the following equation:

$$\widehat{A} = C_{BSE}(\Delta(A))$$

They proved that the commutative uniform $C^*$-algebras belong to this category; [15].

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The concept of BSE-norm algebra, where $L_1(G)$ is a BSE-norm algebra, for any locally compact group, Hausdorff, and abelian group $G$ is devised in [16]. Another essential reference for BSE-norm algebras is [7]. The BSE property for some other Banach algebras, including the direct sum of Banach algebras is assessed in [10]. That the Lipschitz algebra $\text{Lip}_\alpha(K, A)$ is a BSE-algebra if and only if $A$ is a BSE-algebra, where $K$ is a compact metric space, $A$ is a commutative unital semisimple Banach algebra, and $0 < \alpha \leq 1$ is proved in [1].

In this article, $A$ is a commutative unital Banach algebra, where $A$ is Arens regular and $A^{**}$ is semisimple. That the Lipschitz algebra $\text{Lip}_\alpha(K, A)$ is a BSE-algebra if and only if $A$ is a BSE-algebra, where $K$ is a compact metric space, $A$ is a commutative unital semisimple Banach algebra, and $0 < \alpha \leq 1$ is proved in [1].

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The BSE-norm property for $A^{**}$ and proving that $A$ and $A^{**}$ are BSE-norm algebras are assessed in this article.

2. Preliminaries

The basic terminologies and the related information on BSE algebras are extracted from [14], [15], and [16].

Let $A$ be a commutative semisimple Banach algebra, and $\Delta(A)$ be the character space of $A$ with the Gelfand topology. In this study, $\Delta(A)$ represents the set of all non-zero multiplicative linear functionals over $A$.

Assume that $C_b(\Delta(A))$ is the space consisting of all complex-valued continuous bounded functions on $\Delta(A)$. A linear operator $T$ on $A$ is named a multiplier if for all $x, y \in A$, $T(xy) = xT(y)$. The set of all multipliers on $A$ will be expressed as $M(A)$. It is obvious that $M(A)$ is a Banach algebra, and if $A$ is an unital Banach algebra, then $M(A) \cong A$. As observed in [12], for each $T \in M(A)$ there exists a unique bounded continuous function $\hat{T}$ on $\Delta(A)$ expressed as:

$$\varphi(Tx) = \hat{T}(\varphi)\varphi(x),$$

for all $x \in A$ and $\varphi \in \Delta(A)$. By setting $\{\hat{T} : T \in M(A)\}$, the $\hat{M(A)}$ is yield.

If the Banach algebra $A$ is semisimple, then the Gelfand map $\Gamma_A : A \to \hat{A}$, $f \mapsto \hat{f}$, is injective or equivalently, and the following equation holds:

$$\bigcap_{\varphi \in \Delta(A)} \ker(\varphi) = \{0\}.$$

A bounded complex-valued continuous function $\sigma$ on $\Delta(A)$ is named a BSE function, if there exists a positive real number $\beta$ in a sense that for every finite complex-number $c_1, \ldots, c_n$, and the same many $\varphi_1, \ldots, \varphi_n$ in $\Delta(A)$ the following inequality

$$| \sum_{j=1}^n c_j \sigma(\varphi_j) | \leq \beta \sum_{j=1}^n c_j \varphi_j \|A^*.$$
The set of all BSE-functions is expressed by $C_{BSE}(\Delta(A))$, where for each $\sigma$, the BSE-norm of $\sigma$, $\|\sigma\|_{BSE}$ is the infimum of all $\beta$s applied in the above inequality. That $(C_{BSE}(\Delta(A)), \|\cdot\|_{BSE})$ is a semisimple Banach subalgebra of $C_b(\Delta(A))$ is in Lemma 1 proved in [14]. Algebra $A$ is named a BSE algebra if it meets the following condition:

$$\hat{M}(A) \cong C_{BSE}(\Delta(A)).$$

If $A$ is unital, then $\hat{M}(A) \cong \hat{A}|_{\Delta(A)}$, indicating that $A$ is a BSE algebra if and only if $C_{BSE}(\Delta(A)) \cong \hat{A}|_{\Delta(A)}$. The semisimple Banach algebra $A$ is named a norm-BSE algebra if there exists some $K > 0$ in a sense that for each $a \in A$, the following holds:

$$\|a\|_A \leq K\|\hat{a}\|_{BSE}$$

Let $A$ be a Banach algebra, $F, G \in A^{**}$, $f \in A^*$ and $x, y \in A$.

1) The first Arens multiplication is defined by:

$$F \square G(f) = F(G.f)$$
$$G.f(x) = G(f.x)$$
$$f.x(y) = f(xy)$$

2) The second Arens multiplication on $A^{**}$ is defined by:

$$F \lozenge G(f) = G(f.F)$$
$$f.F(x) = F(x.f)$$
$$x.f(y) = f(yx)$$

It is obvious that $(A^{**}, \square)$ and $(A^{**}, \lozenge)$ are Banach algebras. If the binary operation $\square$ is the same as the action of $\lozenge$, then $A$ is named the Arens regular Banach algebra. If $A$ is a commutative Banach algebra, then $A^{**}$ is a commutative Banach algebra if and only if $A$ is an Arens regular Banach algebra; [6].

3. BSE algebras and their second Dual

The structure of the BSE functions on $\Delta(A^{**})$ is characterization and the correlations between the BSE property of $A$ and their second duals are assessed.

Let $A$ be a commutative semisimple Banach algebra. Then

(i) $C_{BSE}(\Delta(A))$ equals the set of all $\sigma \in C_b(\Delta(A))$ for which there exists at least one bounded net $(a_\alpha)$ in $A$ with $\lim a_\alpha(\varphi) = \sigma(\varphi)$ for all $\varphi \in \Delta(A)$.

(ii) $C_{BSE}(\Delta(A)) = C_b(\Delta(A)) \cap A^{**}$, where $A^{**}$ is the second dual of $A$.

Proved in [Theorem4, [14]].

**Lemma 1.** Let $A$ be a commutative semisimple Banach algebra. Then

$$<\Delta(A)>^{\omega^*} = A^*$$
**Proof.** Assume that $B = \langle \Delta(A) \rangle^{w^*}$ and $B \subsetneqq A^*$. Define $\Theta : B \to \mathbb{C}$ by $\Theta(b) = 0$ for all $b \in B$. Let $b_0 \in A^* \setminus B$. Then by referring to the Hahn–Banach theorem there exists a $w^*$- continuous linear function $\overline{\Theta} : A^* \to \mathbb{C}$ where $\overline{\Theta}(b_0) = 1$ and $\overline{\Theta}(b) = 0$ for all $b \in B$. Because $(A^*, w^*)$ is a locally convex and $\overline{\Theta}$ is $w^*$-continuous, thus there exists $a \in A$ where $\overline{\Theta} = \hat{a}$. If $\varphi \in \Delta(A)$, the following is yield:

$$\varphi(a) = \hat{a}(\varphi) = \overline{\Theta}(\varphi) = 0$$

Because $A$ is semisimple, $a = 0$. Thus,

$$\overline{\Theta}(b_0) = \hat{a}(b_0) = b_0(0) = 0$$

which is a contradiction; therefore $\langle \Delta(A) \rangle^{w^*} = A^*$.

**Lemma 2.** Let $A$ be a commutative Banach algebra and $\langle \Delta(A) \rangle^{w^*} = A^*$. Then $\Delta(A^{**}) = \Delta(A)^{w^*}$.

**Proof.** Provided in [6]. □

**Corollary 3.** Let $A$ be a commutative semisimple unital Banach algebra. Then

$$\Delta(A^{**}) = \Delta(A)$$

**Proof.** Because $A$ is unital, $\Delta(A)$ is $w^*$- compact, thus, $\Delta(A) = \Delta(A)^{w^*}$. Then by applying Lemma 1 and Lemma 2, the equation $\Delta(A^{**}) = \Delta(A)$ is yield. □

**Remark 1.** Let $A$ be a commutative semisimple unital Banach algebra. Assume that $\varphi_1, \ldots, \varphi_n \in \Delta(A)$ and $c_1, \ldots, c_n \in \mathbb{C}$, then the following is yield:

$$\| \sum_{i=1}^{n} c_i \varphi_i \|_{A^*} = \| \sum_{i=1}^{n} c_i \hat{\varphi}_i \|_{A^{**}}$$

**Remark 2.** Let $A$ be a commutative Arens regular Banach algebra. If $A^{**}$ is semisimple, then $A$ is semisimple. Because $\ker(\varphi) \subseteq \ker(\hat{\varphi})$ and $\text{Rad}(A) \subseteq \text{Rad}(A^{**})$, then $A$ is semisimple, because $A^{**}$ is semisimple.

**Theorem 4.** Let $A$ be a commutative Arens regular semisimple unital Banach algebra. Then the following statement holds:

$$C_{\text{BSE}}(\Delta(A^{**})) \cong C_{\text{BSE}}(\Delta(A))$$

These two as Banach algebras are isometric.

**Proof.** Because $A$ is semisimple, by applying Corollary 3, the $\Delta(A^{**}) = \Delta(A)$ is yield, thus, $C_b(\Delta(A^{**})) \cong C_b(\Delta(A))$, and $A^{****} \mid_{\Delta(A)} \cong A^{**} \mid_{\Delta(A)}$. Consequently by applying [14, Theorem 4], $C_{\text{BSE}}(\Delta(A^{**})) \cong C_{\text{BSE}}(\Delta(A))$ is yield. □
The correlation between the BSE-property of algebra $A$ and its second dual $A^{**}$ is assessed as follows:

**Lemma 5.** Let $A$ be a commutative unital Banach algebra such that $A$ is Arens regular and $A^{**}$ is semisimple. If $A$ is a BSE algebra, so is $A^{**}$.

**Proof.** Let $\Sigma \in C_{BSE}(\Delta(A^{**}))$. Define $\sigma(\varphi) = \Sigma(\hat{\varphi})$ for all $\varphi \in \Delta(A)$. Obviously, $\sigma$ is well defined and $\sigma \in C_{BSE}(\Delta(A))$. By allowing $\varphi_1, \cdots, \varphi_n \in \Delta(A)$ and $c_1, \cdots, c_n \in \mathbb{C}$ the following is yield:

$$\left| \sum_{j=1}^{n} c_j \sigma(\varphi_j) \right| = \left| \sum_{j=1}^{n} c_j \Sigma(\hat{\varphi}_j) \right|$$

$$\leq M \left\| \sum_{j=1}^{n} c_j \hat{\varphi}_j \right\|_{A^{***}}$$

$$= M \left\| \sum_{j=1}^{n} c_j \varphi_j \right\|_{A^{*}}$$

where $M > 0$. Because $C_{BSE}(\Delta(A)) = \hat{A} \mid_{\Delta(A)}$, there exists $a \in A$ where $\sigma = \hat{a}$.

Assume that $\Phi \in \Delta(A^{**})$, thus $\Phi = \hat{\varphi}$ for some $\varphi \in \Delta(A)$. Which yield:

$$\Sigma(\Phi) = \Sigma(\hat{\varphi}) = \sigma(\varphi)$$

$$= \hat{a}(\varphi)$$

$$= \hat{a}(\hat{\varphi})$$

$$= \hat{a}(\Phi)$$

where, $\Sigma = \hat{a}$, therefore $C_{BSE}(\Delta(A^{**})) = \hat{A}^{**} \mid_{\Delta(A)}$. \qed

**Lemma 6.** Let $A$ be a commutative unital Banach algebra where $A$ is Arens regular and $A^{**}$ is semisimple. If $A^{**}$ is a BSE algebra and $\hat{A} \mid_{\Delta(A)} \cong \hat{A}^{**} \mid_{\Delta(A)}$, then $A$ is a BSE-algebra.

**Proof.** Assume that $A^{**}$ is a BSE algebra, the following is yield:

$$C_{BSE}(\Delta(A)) \cong C_{BSE}(\Delta(A^{**}))$$

$$\cong \hat{A}^{**} \mid_{\Delta(A)}$$

$$\cong \hat{A} \mid_{\Delta(A)}$$

thus, $A$ is a BSE-algebra. \qed

At this stage, based on the established prerequisite the primary Theorem, is expressed as follows:

**Theorem 7.** Let $A$ be a commutative unital Banach algebra, such that $A$ is Arens regular and $A^{**}$ is semisimple. Then the following statements are equivalent:
(i) $A$ is a BSE algebra.
(ii) $A^{**}$ is a BSE algebra and $\hat{A} |_{\Delta(A)} \cong \hat{A}^{**} |_{\Delta(A)}$.

Example 1. Let $(K, d)$ be a compact metric space and $A$ be a unital commutative semisimple Banach algebra. Then $A$ is a BSE-algebra if and only if $\text{Lip}(K, A)$ is a BSE-algebra; [1]. In this context, the following statements are equivalent:
(i) $A$ is a BSE algebra.
(ii) $A^{**}$ is a BSE algebra and $\hat{A} |_{\Delta(A)} \cong \hat{A}^{**} |_{\Delta(A)}$.
(iii) $\text{Lip}(K, A)$ is a BSE-algebra.
(iv) $\text{Lip}(K, A^{**})$ is a BSE-algebra and $\hat{A} |_{\Delta(A)} \cong \hat{A}^{**} |_{\Delta(A)}$.

Example 2. Let $X$ be a metric space and $A = \text{Lip}(X)$. Then $A$ is unital and BSE algebra; [2]. By applying Lemma 5, $\text{Lip}(X)^{**}$ becomes a BSE algebra.

Example 3. Let $A$ be a commutative reflexive Banach algebra. Then
$$C_{\text{BSE}}(\Delta(A)) = \hat{A} |_{\Delta(A)}$$
Moreover, $A$ is unital, then $A$ is a BSE algebra.

Proof. According to [Theorem 4, [14]] because $A$ is reflexive, and the following is yield:
$$C_{\text{BSE}}(\Delta(A)) = A^{**} |_{\Delta(A)} \cap C_b(\Delta(A)) = \hat{A} |_{\Delta(A)} \cap C_b(\Delta(A)) = \hat{A} |_{\Delta(A)}$$

4. BSE-norm algebra
The BSE-norm property of the second dual of Banach algebra is assessed and the correlation between the BSE-norm of the Banach algebra $A$ and its second dual is assessed.

Theorem 8. Let $A$ be a commutative unial Banach algebra such that $A$ is Arens regular and $A^{**}$ is semisimple. Then $A^{**}$ is a BSE-norm algebra.

Proof. By applying the open mapping theorem there exists some $M > 0$ where $||F||_{A^{**}} \leq M ||F|_{\Delta(A)}||_{A^{**}}$ for each $F \in A^{**}$, because the map $\Theta: A^{**} \to C_b(\Delta(A))$ given by $\Theta(F) = F|_{\Delta(A)}$ is continuous, linear and
injective, thus:

\[ \| \hat{F} \|_{BSE} = \sup \{ \| \sum_{i=1}^{n} c_i \hat{F}(\Phi_i) \| : \Phi_i \in \Delta(A^{**}), \| \sum_{i=1}^{n} c_i \hat{\Phi}_i \|_{A^{***}} \leq 1 \} \]

\[ = \sup \{ \| \sum_{i=1}^{n} c_i \hat{F}(\varphi_i) \| : \varphi_i \in \Delta(A), \| \sum_{i=1}^{n} c_i \varphi_i \|_{A^{*}} \leq 1 \} \]

\[ = \sup \{ \| \sum_{i=1}^{n} c_i F(\varphi_i) \| : \varphi_i \in \Delta(A), \| \sum_{i=1}^{n} c_i \varphi_i \|_{A^{*}} \leq 1 \} \]

\[ = \| F \|_{\Delta(A)} \|_{BSE} \]

in addition:

\[ \| F \|_{\Delta(A)} \|_{A^{**}} = \sup \{ | F(\varphi) | : \varphi \in \Delta(A) \} \]

\[ \leq \| F \|_{\Delta(A)} \|_{BSE} \]

Then by applying the open mapping theorem, the following is yield:

\[ \| F \|_{A^{**}} \leq M \| F \|_{\Delta(A)} \|_{A^{**}} \leq M \| F \|_{\Delta(A)} \|_{BSE} = M \| \hat{F} \|_{BSE} \]

therefore, \( A^{**} \) is a BSE-norm algebra. \( \square \)

**Theorem 9.** Let \( A \) be a commutative unital Banach algebra, such that \( A \) is Arens regular and \( A^{**} \) is semisimple. Then \( A \) is a BSE-norm algebra.

**Proof.** Because \( A^{**} \) is a BSE-norm algebra, consequently, by definition there exists \( M > 0 \) where \( \| F \|_{A^{**}} \leq M \| \hat{F} \|_{BSE} \) for each \( F \in A^{**} \). If \( a \in A \), the following is yield:

\[ \| a \|_{A} = \| \hat{a} \|_{A^{**}} \leq M \| \hat{a} \|_{BSE} \]

in addition

\[ \| \hat{a} \|_{BSE} = \sup \{ \| \sum_{i=1}^{n} c_i \Phi_i(\hat{a}) \| : \Phi_i \in \Delta(A^{**}), \| \sum_{i=1}^{n} c_i \Phi_i \|_{A^{***}} \leq 1 \} \]

\[ = \| \sum_{i=1}^{n} c_i \hat{\Phi}_i(\hat{a}) \| \| \sum_{i=1}^{n} c_i \hat{\Phi}_i \|_{A^{***}} \leq 1 \} \]

\[ = \| \sum_{i=1}^{n} c_i \varphi_i(a) \| \| \sum_{i=1}^{n} c_i \varphi_i \|_{A^{*}} \leq 1 \} \]

\[ = \| \hat{a} \|_{BSE} \]

therefore, \( \| a \|_{A} \leq M \| \hat{a} \|_{BSE} \). Then \( A \) is a BSE-norm algebra. \( \square \)

**Corollary 10.** Let \( A \) be a commutative unital Banach algebra, such that \( A \) is Arens regular and \( A^{**} \) is semisimple. Then \( A \) and \( A^{**} \) are BSE-norm algebras.

**Example 4.** Let \( A \) be a commutative semisimple reflexive Banach algebra where \( \Delta(A) \) is a discrete space. Then \( A \) is a BSE-norm algebra.
Proof. By applying Example 3, $C_{BSE}(\Delta(A)) = \hat{A}|_{\Delta(A)}$, making $\hat{A}|_{\Delta(A)}$ is a closed BSE-norm space, hence $\Phi: A \rightarrow \hat{A}|_{\Delta(A)}$ is an one-to-one continuous map where $\Phi(A) = \hat{A}|_{\Delta(A)}$. Then by applying the open mapping theorem it is revealed that there exists some $M > 0$ where $\|a\|_A \leq M\|\hat{a}\|_{BSE}$ for each $a \in A$, therefore $A$ is a BSE-norm algebra.

Example 5. 1) If $A$ is a commutative unital finite dimensional Banach algebra, then $A$ is a BSE-norm algebra.
2) Let $G$ be a compact group and $A = l_p(G)$. Then $A$ is a BSE-norm algebra.

Proof. 1) Since $A$ is finite dimensional, $A \cong C^n$, for some positive integer $n$ under equivalent norms. Example 4 implies that $A$ is a BSE-norm algebra.
2) Because $\Delta(A) = \hat{G}$ is a discrete space, $A$ is a semisimple and reflexive algebra, thus by according to Example 4, $A$ is a BSE-norm algebra.

Example 6. Assume that $G$ is a compact group and $A$ is a reflexive semisimple Banach algebra where $\Delta(A)$ is discrete and $1 < p < \infty$. Then $L_p(G, A)$ is a BSE-norm algebra.

Proof. The following can be written:

$$L_p(G, A)^{**} \cong L_q(G, A^*)^*$$

$$\cong L_p(G, A^{**}) \cong L_p(G, A)$$

Consequently $L_p(G, A)$ is reflexive, and
$$\Delta(L_p(G, A)) = \hat{G} \times \Delta(A)$$

Because $G$ is a compact group, and $\Delta(A)$ is discrete, it is concluded that $\Delta(L_p(G, A))$ is discrete. Therefore by applying Example 4, $L_p(G, A)$ is a BSE-norm algebra.

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