On Almost Complete Subsets of a Conic in $\text{PG}(2, q)$, Completeness of Normal Rational Curves and Extendability of Reed-Solomon Codes

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Abstract

A subset $\mathcal{S}$ of a conic $\mathcal{C}$ in the projective plane $\text{PG}(2, q)$ is called almost complete (AC-subset for short) if it can be extended to a larger arc in $\text{PG}(2, q)$ only by the points of $\mathcal{C} \setminus \mathcal{S}$ and by the nucleus of $\mathcal{C}$ when $q$ is even. New upper bounds on the smallest size $t(q)$ of an AC-subset are obtained, in particular,

$$t(q) < \sqrt{q(3 \ln q + \ln \ln q + \ln 3)} + \sqrt{\frac{q}{3 \ln q}} + 4 \sim \sqrt{3q \ln q};$$

$$t(q) < 1.835 \sqrt{q \ln q}.$$

The new bounds are used to increase regions of pairs $(N, q)$ for which it is proved that every normal rational curve in $\text{PG}(N, q)$ is a complete $(q + 1)$-arc or, equivalently, that no $[q + 1, N + 1, q - N + 1]_q$ generalized doubly-extended Reed-Solomon code can be extended to a $[q + 2, N + 1, q - N + 2]_q$ MDS code.

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1 Introduction

Let $\text{PG}(N, q)$ be the $N$-dimensional projective space over the Galois field $\mathbb{F}_q$ of order $q$. An $n$-arc in $\text{PG}(N, q)$ with $n > N + 1$ is a set of $n$ points such that no $N + 1$ points belong to the same hyperplane of $\text{PG}(N, q)$. An $n$-arc of $\text{PG}(N, q)$ is complete if it is not contained in an $(n + 1)$-arc of $\text{PG}(N, q)$. In $\text{PG}(N, q)$ with $2 \leq N \leq q - 2$, a normal rational curve is any $(q + 1)$-arc projectively equivalent to the arc $\{(1, t, t^2, \ldots, t^N) : t \in \mathbb{F}_q\} \cup \{(0, \ldots, 0, 1)\}$.

For an introduction to projective geometries over finite fields see [1–3].

Let an $[n, k, d]_q$ code be a $q$-ary linear code of length $n$, dimension $k$, and minimum distance $d$. If $d = n - k + 1$, it is a maximum distance separable (MDS) code. The code dual to an $[n, k, n - k + 1]_q$ MDS code is an $[n, n - k, k + 1]_q$ MDS code.

Points (in the homogeneous coordinates) of an $n$-arc in $\text{PG}(N, q)$ treated as columns define a generator matrix of an $[n, N + 1, n - N]_q$ MDS code. If an $n$-arc in $\text{PG}(N, q)$ is complete then the corresponding $[n, N + 1, n - N]_q$ MDS code cannot be extended to an $[n + 1, N + 1, n - N + 1]_q$ MDS code. For properties of linear MDS codes and their equivalence to arcs see e.g. [1–14].

The $j$-th column of a generator matrix of a $[q + 1, N + 1, q - N + 1]_q$ generalized doubly-extended Reed-Solomon (GDRS) code has the form $(v_j, v_j \alpha_j, v_j \alpha_j^2, \ldots, v_j \alpha_j^N)^T$, where $j = 1, 2, \ldots, q$; $\alpha_1, \ldots, \alpha_q$ are distinct elements of $\mathbb{F}_q$; $v_1, \ldots, v_q$ are nonzero (not necessarily distinct) elements of $\mathbb{F}_q$. Also, this matrix contains one more column $(0, \ldots, 0, v)^T$ with $v \neq 0$. The code, dual to a GDRS code, is a GDRS code too.

Points (in the homogeneous coordinates) of a normal rational curve in $\text{PG}(N, q)$ treated as columns define a generator matrix of a $[q + 1, N + 1, q - N + 1]_q$ GDRS code. Proposition 1.1 is well known.

Proposition 1.1. Let $N$ and $q$ be fixed integers with $2 \leq N \leq q - 2$. Moreover, let $q$ be a prime power. The following statements are equivalent:

• Every normal rational curve in $\text{PG}(N, q)$ is a complete $(q + 1)$-arc;
• No $[q + 1, N + 1, q - N + 1]_q$ GDRS code can be extended to a $[q + 2, N + 1, q - N + 2]_q$ MDS code.

Due to Proposition 1.1, all results given below on completeness of normal rational curves can be reformulated in coding theory language for extendability of GDRS codes.

The completeness of normal rational curves and related problems are considered in numerous works starting from Segre’s paper [15] of 1955; see for example [1–20], where
surveys and references can be found. In particular, the following conjecture, connected with the famous Segre’s three problems, is well known.

**Conjecture 1.2.** Let \( 2 \leq N \leq q - 2 \). Every normal rational curve in \( \text{PG}(N, q) \) is a complete \((q + 1)\text{-arc}\) except for the cases \( q \) even and \( N \in \{2, q - 2\} \) when one point can be added to the curve.

**Remark 1.3.** As a comment to Conjecture 1.2 for \( q \) even, note the following. If \( N = 2 \), the point which can be added to a normal rational curve is unique. But if \( N = q - 2 \), there are many points in \( \text{PG}(q - 2, q) \) which extend a normal rational curve to a \((q + 2)\text{-arc}\), see [13, Theorem 3.10] for the geometrical characterization of these points.

**Remark 1.4.** If \( k \geq q \) then an \([n, k, n - k + 1]\)q MDS code has length \( n \leq k + 1 \), see e.g. [10, 11]. For \( 2 \leq N \leq q - 2 \), the well known MDS conjecture assumes that an \([n, N + 1, n - N]_q\) MDS code (or equivalently an \( n\)-arc in \( \text{PG}(N, q) \)) has length \( n \leq q + 1 \) except for the cases \( q \) even and \( N \in \{2, q - 2\} \) when \( n \leq q + 2 \). The MDS conjecture considers all MDS codes (or all arcs) whereas Conjecture 1.2 says only something about normal rational curves (or GDRS codes). If the MDS conjecture holds for some pair \((N, q)\) then Conjecture 1.2 holds too, but in general the reverse is not true.

For many pairs \((N, q)\) Conjecture 1.2 is proved, see [11, 14–20] and the references therein; but in general, completeness of normal rational curves is an open problem. The main known results are given in Table 1 where \( p \) and \( p_0(h) \) are prime. For rows 1–6 of Table 1 in fact, the MDS conjecture is proved. In [5], see row 7 of Table 1 it is proved that a subset of size \( 3(N - 1) - 6 \) of a normal rational curve in \( \text{PG}(N, q) \), \( q \) odd, cannot be extended to an arc of size \( q + 2 \). This means that \( 3N - 3 \leq q + 1 \) (otherwise the curve could not contain a such subset). So, \( N \leq \frac{q + 4}{3} \). The regions of \( N \) in rows 10–11 cover the ones in rows 6–8; we included rows 6–8 in Table 1 as the methods used for them are useful for further research.

For the problem of completeness of normal rational curves we use tools connected with almost complete subsets of a conic in the projective plane \( \text{PG}(2, q) \).

An \( n\)-arc in \( \text{PG}(2, q) \) is a set of \( n \) points no three of which are collinear. A point \( P \) of \( \text{PG}(2, q) \) is covered by an arc \( \mathcal{K} \subset \text{PG}(2, q) \) if \( P \) lies on a bisecant of \( \mathcal{K} \). Throughout the paper, \( \mathcal{C} = \{(1, t, t^2) : t \in \mathbb{F}_q\} \cup \{(0, 0, 1)\} \) is a fixed conic in \( \text{PG}(2, q) \). Any point subset of \( \mathcal{C} \) is an arc. For even \( q \), denote by \( \mathcal{O} \) the nucleus of \( \mathcal{C} \). Let

\[
\mathcal{M}_q := \begin{cases} 
\text{PG}(2, q) \setminus \mathcal{C} & \text{if } q \text{ odd} \\
\text{PG}(2, q) \setminus (\mathcal{C} \cup \{\mathcal{O}\}) & \text{if } q \text{ even}
\end{cases}
\]

**Definition 1.5.** (i) In \( \text{PG}(2, q) \), an almost complete subset of the conic \( \mathcal{C} \) (AC-subset, for short) is a proper subset of \( \mathcal{C} \) covering all the points of \( \mathcal{M}_q \). An \( n\)-AC-subset is an AC-subset of size \( n \).

(ii) An AC-subset is minimal if it does not contain a smaller AC-subset.
Table 1: Pairs \((N, q)\) for which it is proved that every normal rational curve in \(\text{PG}(N, q)\) is a complete \((q + 1)\)-arc

| no. | \(q\) | \(N\) | Reference |
|-----|-------|-------|-----------|
| 1   | \(q = p^{2h+1}, p \geq 3, h \geq 1\) | \(q - \frac{1}{3}\sqrt{pq} + \frac{20}{7p}p - 3 < N \leq q - 3\) | [2] Table 3.4 |
| 2   | \(q = p^h, p \geq 5\) | \(q - \frac{1}{7}\sqrt{q} + 1 < N \leq q - 3\) | [2] Table 3.4 |
| 3   | \(q = p^h \geq 23^2; p \geq 3; q \neq 5^5, 3^6; h\) even for \(q = 3\) | \(q - \frac{1}{2}\sqrt{q} - 1 < N \leq q - 3\) | [2] Table 3.4 |
| 4   | \(q = 2^h, h > 2\) | \(q - \frac{1}{3}\sqrt{q} - \frac{11}{4} < N \leq q - 5\) | [2] Table 3.4 |
| 5   | \(q = p\) | \(2 \leq N \leq p - 1\) | [4, 6, 16, 17] |
| 6   | \(q = p^2\) | \(2 \leq N \leq 2\sqrt{q} - 3\) | [4, 6, 16, 17] |
| 7   | \(q\) odd | \(N \leq \frac{q + 4}{3}\) | [5, Theorem 1.4] |
| 8   | all \(q\) | \(3 \leq N \leq q + 2 - 6\sqrt{q\ln q}\) | [19, Theorem 3.3] |
| 9   | \(q = p^{2h+1}; p \geq p_0(h); p_0(h)\) is the smallest \(\hat{p}\) satisfying \(\sqrt{\hat{p}} > 24\sqrt{(2h + 1)\ln \hat{p}} + \frac{20}{\hat{p}^{h+1}} - \frac{20}{\hat{p}^{2h+1}}\) | \(2 \leq N \leq q - 2\) | [19, Theorem 3.5] |
| 10  | \(q\) odd | \(2 \leq N \leq q - 2 - \sqrt{7(q + 1)\ln q}\) | [18, Theorem 9.2] |
| 11  | \(q\) even | \(3 \leq N \leq q - 1 - \sqrt{7(q + 1)\ln q}\) | [18, Theorem 9.2] |

Note that an AC-subset \(S\) is an arc that can be extended to a larger arc in \(\text{PG}(2, q)\) only by the points of \(C \setminus S\) and by the nucleus \(O\) when \(q\) is even. The term “almost completeness” was introduced in [18, p. 94] for objects in the affine plane \(\text{AG}(2, q)\).

Denote by \(t(q)\) the smallest size of an AC-subset in \(\text{PG}(2, q)\).

In this work we provide new upper bounds on \(t(q)\). This is an open problem. It is addressed, for example, in [19–21]. In [21], by probabilistic methods, it is proved that

\[
t(q) < 6\sqrt{q\ln q}. \tag{1.1}
\]

In [19, Theorem 3.1], using the results and approaches of [20], the following connection between \(t(q)\) and the completeness of normal rational curves is proved:

under the condition

\[
3 \leq N \leq q + 2 - t(q), \tag{1.2}
\]

every normal rational curve in \(\text{PG}(N, q)\) is a complete \((q + 1)\)-arc.

The aims of this paper are as follows: obtain new upper bounds on the smallest size of an AC-subset of a conic in \(\text{PG}(2, q)\); using the bounds, extend regions of pairs \((N, q)\) for which it is proved that every normal rational curve in \(\text{PG}(N, q)\) is a complete \((q + 1)\)-arc.
The paper is organized as follows. In Section 2 the main results of this paper are formulated. In Section 3 we consider an estimate of the number of new covered points in one step of a step-by-step algorithm constructing AC-subsets. In Section 4 implicit and explicit upper bounds on \( t(q) \), based on the results of Section 3, are obtained. In Section 5 computer assisted bounds on \( t(q) \) are studied. In Section 6, new bounds on \( t(q) \) are applied to the problem of completeness of normal rational curves. Finally, in Appendix tables of the smallest known sizes \( t(q) \) of AC-subsets in PG\( (2, q) \) are given.

## 2 The main results

We introduce the following set of prime powers.

\[
Q_1 := \{8 \leq q \leq 139129, \ q = p^m, \ p \text{ prime, } m \geq 2\}. \tag{2.1}
\]

Throughout the paper we denote

\[
\Phi(q) = \sqrt{q(3 \ln q + \ln \ln q + \ln 3)} + \sqrt{\frac{q}{3\ln q}} + 4 \sim 3\sqrt{q \ln q}; \tag{2.2}
\]

\[
\Theta(q) = \begin{cases} 
1.62\sqrt{q \ln q} & \text{for } 8 \leq q \leq 17041 \\
1.635\sqrt{q \ln q} & \text{for } 17041 < q \leq 33013 \\
1.674\sqrt{q \ln q} & \text{for } q \in Q_1 \\
\min\{1.835\sqrt{q \ln q}, \Phi(q)\} & \text{for all } q \geq 5
\end{cases}, \tag{2.3}
\]

where

\[
\min\{1.835\sqrt{q \ln q}, \Phi(q)\} = \begin{cases} 
1.835\sqrt{q \ln q} & \text{for } q < 12755807 \\
\Phi(q) & \text{for } 12755807 \leq q
\end{cases}.
\]

The main result of this paper is Theorem 2.1 based on Theorems 4.10, 4.12, and 5.1.

**Theorem 2.1.** The following upper bound on the smallest size \( t(q) \) of an AC-subset of the conic \( C \) in PG\( (2, q) \) holds:

\[
t(q) < \Theta(q). \tag{2.4}
\]

Similarly to [19], we use upper bounds on \( t(q) \) to prove the completeness of the normal rational curves as arcs in projective spaces. From Theorem 2.1 and [19] Theorems 3.1,3.5 we obtained Corollaries 2.2 and 2.3; see Section 6.

**Corollary 2.2.** Let

\[
3 \leq N \leq q + 2 - \Theta(q). \tag{2.5}
\]

Then every normal rational curve in PG\( (N, q) \) is a complete \( (q + 1) \)-arc.
Corollary 2.3. Let $h \geq 1$ be a fixed integer. Let $p_0(1) = 757$, $p_0(2) = 1399$, $p_0(3) = 2129$, $p_0(4) = 2887$, $p_0(5) = 3623$. Also, for $h \geq 6$ let $p_0(h)$ be the smallest odd prime $p$ satisfying

$$\sqrt{p} > 4c\sqrt{(2h + 1)\ln p} + \frac{29}{4p^{h-0.5}} - \frac{20}{p^{h+0.5}},$$

(2.6)

where $c = 1.62$ for $6 \leq h \leq 19$, $c = 1.635$ for $20 \leq h \leq 28$, $c = 1.835$ for $h \geq 29$.

Then for every odd prime $p \geq p_0(h)$ in $\text{PG}(N, q)$ with $q = p^{2h+1}$, $2 \leq N \leq q-2$, every normal rational curve is a complete $(q+1)$-arc.

Remark 2.4. In (2.6), the term $\frac{29}{4p^{h-0.5}} - \frac{20}{p^{h+0.5}}$ quickly decreases when $h$ grows. Therefore, practically, use of inequality $\sqrt{p} > 16c^2(2h + 1)\ln p$ gives the same result as for (2.6). In particular, we have checked this for $h \leq 16$.

In Section 4 we consider also implicit upper bounds on $t(q)$.

All bounds on $t(q)$ obtained in this paper are better than the bound of (1.1).

Corollaries 2.2 and 2.3 extend regions of pairs $(N, q)$ for which it is proved that every normal rational curve in $\text{PG}(N, q)$ is a complete $(q+1)$-arc.

Corollary 2.2 improves the results of [13, Theorem 9.2], cf. (2.5) and rows 10–11 of Table 1 in (2.5) the region on $N$ values is greater by $\sim 0.8\sqrt{q\ln q}$.

Corollary 2.3 gives essentially smaller values $p_0(h)$ than [19, Theorem 3.5]. By Corollary 2.3 we have $\{p_0(1), p_0(2), \ldots, p_0(16)\} = \{757, 1399, 2129, 2887, 3623, 4621, 5417, 6247, 7079, 7919, 8779, 9629, 10499, 11383, 12253, 13147\}$. For comparison, [19, Theorem 3.5], see row 9 of Table II provides $\{p_0(1), p_0(2), \ldots, p_0(16)\} = \{16831, 29663, 43037, 56747, 70769, 85009, 99431, 114031, 128767, 143651, 158647, 173741, 188953, 204251, 219629, 235091\}$.

3 The number of new covered points in one step of a step-by-step algorithm constructing AC-subsets

Assume that an AC-subset is constructed by a step-by-step algorithm (Algorithm, for short) adding a new point to the subset on every step. As an example, we mention the greedy algorithm that on every step adds to the subset a point providing the maximal possible (for the given step) number of new covered points.

Let $w > 0$ be a fixed integer. Consider the $(w+1)$st step of Algorithm. This step starts from a $w$-subset $K_w \subset C$ constructed in the previous $w$ steps. Let $U(K_w)$ be the subset of points of $M_q$ not covered by the subset $K_w$.

Let the subset $K_w$ consist of $w$ points $A_1, A_2, \ldots, A_w$. Let $A_{w+1} \in C \setminus K_w$ be the point that will be included into the subset in the $(w+1)$st step. Denote by $U(K_w \cup \{A_{w+1}\})$ the subset of points of $M_q$ not covered by the new subset $K_w \cup \{A_{w+1}\}$.

Let $AB$ be the line through points $A$ and $B$. The point $A_{w+1}$ defines a bundle $B(A_{w+1}) = \{A_{1}A_{w+1}, A_{2}A_{w+1}, \ldots, A_{w}A_{w+1}\}$ of $w$ tangents (unisecants) to $K_w$ which are
bisecants of \( C \). In order to obtain the next subset \( K_{w+1} \), we may include to \( K_w \) any of \( q + 1 - w \) points of \( C \setminus K_w \). So, there exist \( q + 1 - w \) distinct points \( A_{w+1} \) and \( q + 1 - w \) distinct bundles. Introduce the set of \( w(q + 1 - w) \) lines

\[
B^\cup_{w+1} = \bigcup_{A_{w+1} \in C \setminus K_w} B(A_{w+1}).
\]

Let \( P^\cup_{w+1} \) be the point multiset consisting of all points of \( B^\cup_{w+1} \). A point that is the intersection of \( m \) lines of \( B^\cup_{w+1} \) has multiplicity \( m \) in \( P^\cup_{w+1} \).

Let \( \Delta(A_{w+1}) \) be the number of the new covered points in the \((w + 1)\)st step. Denote by \( N(A_{w+1}) \) the set of new points covered by \( K_w \cup \{ A_{w+1} \} \). By definition,

\[
N(A_{w+1}) = U(K_w) \setminus U(K_w \cup \{ A_{w+1} \}),
\]

\[
\Delta(A_{w+1}) = \#N(A_{w+1}) = \#U(K_w) - \#U(K_w \cup \{ A_{w+1} \}).
\]

Introduce the point multiset

\[
N^\cup_{w+1} = \bigcup_{A_{w+1} \in C \setminus K_w} N(A_{w+1}) \subset P^\cup_{w+1}.
\]

By the definitions above,

\[
\#N^\cup_{w+1} = \sum_{A_{w+1} \in C \setminus K_w} \Delta(A_{w+1}).
\]

Let \( P \in U(K_w) \subset M_q \) be a point not covered by \( K_w \). Every point of \( M_q \) lies at most on two tangents of \( C \). The rest of lines through this point and the points of \( C \) are bisecants. Therefore, among the \( w \) lines connecting \( P \) with \( K_w \) there are at least \( w - 2 \) bisecants of \( C \). None of those bisecants is a bisecant of \( K_w \) otherwise the point \( P \) would be covered. Hence, all bisecants of \( C \) through \( P \) and \( K_w \) belong to \( B^\cup_{w+1} \). It means that every point of \( U(K_w) \) is included in \( N^\cup_{w+1} \) at least \( w - 2 \) times. So,

\[
\#N^\cup_{w+1} \geq (w - 2) \cdot \#U(K_w). \tag{3.1}
\]

**Remark 3.1.** For even \( q \), every point of \( M_q \) lies on one tangent of \( C \). Therefore for even \( q \), in relation (3.1) we may change \( w - 2 \) by \( w - 1 \). Also, for odd \( q \), an internal point does not belong to any tangent of a conic whereas each of the \( \frac{1}{2} q(q + 1) \) external points lies on two distinct tangents. Hence for odd \( q \), in (3.1) we may change \( (w - 2) \cdot \#U(K_w) \) by \( (w - 2) \cdot \#U(K_w) + 2 \max \{ 0, \#U(K_w) - \frac{1}{2} q(q + 1) \} \). These changes could slightly improve estimates below. However, for simplicity of presentation, we save relation (3.1) as it is.
By the above, the average number, say $\Delta_{w+1}^{\text{aver}}$, of new covered points in a bundle in the $(w+1)$st step is as follows

$$\Delta_{w+1}^{\text{aver}} = \frac{\sum_{A_{w+1} \in C \setminus K_w} \Delta(A_{w+1})}{q + 1 - w} \geq \frac{(w - 2) \cdot U(K_w)}{q + 1 - w}.$$  

Clearly,

$$\max_{A_{w+1} \in C \setminus K_w} \Delta(A_{w+1}) \geq \lceil \Delta_{w+1}^{\text{aver}} \rceil.$$  

So, we have proved the following lemma.

**Lemma 3.2.** For an arbitrary step-by-step algorithm, there exists a point $A_{w+1}$ providing

$$\Delta(A_{w+1}) \geq \left\lceil \frac{(w - 2) \cdot U(K_w)}{q + 1 - w} \right\rceil.$$  

Note that the greedy algorithm always finds the point $A_{w+1}$ with property (3.2).

### 4 Upper bounds on the smallest size of an AC-subset based on properties of step-by-step algorithms

We denote

$$t^*(q) = \frac{t(q)}{\sqrt{q \ln q}}.$$  

Let $t(q) < f(q)$. Then $t^*(q) < f(q)/\sqrt{q \ln q}$. The upper bounds on $t^*(q)$ are more convenient for graphical representation than bounds on $t(q)$. If $f(q)$ is called “Bound L”, say, then we call $f(q)/\sqrt{q \ln q}$ “Bound L*”.

#### 4.1 Implicit bound A

By Section 3,

$$\#U(K_w \cup \{A_{w+1}\}) = \#U(K_w) - \Delta(A_{w+1}) \leq \#U(K_w) - \left\lceil \frac{(w - 2) \cdot U(K_w)}{q + 1 - w} \right\rceil.$$  

(4.1)

Define $U_w$ as an upper bound on $\#U(K_w)$:

$$\#U(K_w) = U_w - \delta \leq U_w; \quad \delta \geq 0.$$  

(4.2)
By (4.1), (4.2),
\[
\#U(K_w \cup \{A_{w+1}\}) \leq U_w - \delta - \left\lceil \frac{(w - 2)(U_w - \delta)}{q + 1 - w} \right\rceil = U_w - \left\lceil \frac{(w - 2)U_w + (q + 3 - 2w)\delta}{q + 1 - w} \right\rceil.
\]

From now on, we suppose
\[q + 3 > 2w.\] (4.3)

Under condition (4.3), it holds that
\[
\#U(K_w \cup \{A_{w+1}\}) = \#U(K_w) - \Delta(A_{w+1}) \leq U_w - \left\lceil \frac{(w - 2)U_w}{q + 1 - w} \right\rceil. \tag{4.4}
\]

Assume that there exists a \(w_0\)-subset \(K_{w_0} \subset C \subset PG(2, q)\) that does not cover at most \(U_{w_0}\) points of \(M_q\). Then, starting from values \(w_0\) and \(U_{w_0}\), one can iteratively apply the relation (4.4) and obtain eventually \(\#U(K_w \cup \{A_{w+1}\}) = 0\) for some \(w\), say \(w_{\text{fin}}\). Clearly, \(w_{\text{fin}}\) depends on \(w_0\) and \(U_{w_0}\), i.e. we have a function \(w_{\text{fin}}(w_0, U_{w_0})\). The size \(k\) of the obtained AC-subset is as follows:
\[k = w_{\text{fin}}(w_0, U_{w_0}) + 1 \text{ under condition } \#U(K_{w_{\text{fin}}(w_0, U_{w_0})} \cup \{A_{w_{\text{fin}}(w_0, U_{w_0})+1}\}) = 0.\]

From the above we have the following theorem.

**Theorem 4.1. (implicit bound \(A(w_0, U_{w_0})\))** Let the values \(w_0\), \(U_{w_0}\), and \(w_{\text{fin}}(w_0, U_{w_0})\) be defined and calculated as above. Let also \(w_{\text{fin}}(w_0, U_{w_0}) < \frac{q+3}{2}\). Then it holds that
\[t(q) \leq w_{\text{fin}}(w_0, U_{w_0}) + 1.\]

It is easily seen that, for any \(q\), there exists a 5-subset \(K_5 \subset C \subset PG(2, q)\) that does not cover \(\#U(K_5) = \#M_q - (10q - 25) \leq U_5 = (q - 5)^2\) points of \(M_q\). The corresponding implicit bound \(A^*(5, (q - 5)^2)\) (i.e. the value \((w_{\text{fin}}(5, (q - 5)^2) + 1)/\sqrt{q \ln q}\) is shown by the third blue curve on Figs. 1 and 2.

**Observation 4.2.** In the region \(7 \leq q \leq 55711\) the implicit bound \(A^*(5, (q - 5)^2)\) tends to increase with the maximal value \(A^*(5, (q - 5)^2) \sim 1.8341\) for \(q = 55711\). In the region \(55711 < q \leq 14000029\) the bound \(A^*(5, (q - 5)^2)\) tends to decrease with the minimal value \(A^*(5, (q - 5)^2) \sim 1.8180\) for \(q = 13995829\), see Fig. 2.
Figure 1: Upper bounds on sizes of AC-subsets divided by $\sqrt{q \ln q}, q \leq 253009$; bound $C^*$ equal to $\Phi(q)/\sqrt{q \ln q}$ (top dashed-dotted red curve); implicit bound $B^*$ (the 2-nd magenta curve); bound (4.21) (dashed red line $y = 1.835$); implicit bound $A^*(5, (q - 5)^2)$ (the 3-rd blue curve); bound (5.6) (dashed red line $y = 1.635$); bound (5.7) (dashed red line $y = 1.674$); the smallest known sizes of AC-subsets divided by $\sqrt{q \ln q}$, i.e. values $t^*(q)$ (bottom black curve). Vertical dashed lines $x = 33013$ and $x = 139129$ mark regions of complete computer search, respectively, for all prime powers $q$ and all non-prime $q$'s.
Figure 2: Upper bounds on sizes of AC-subsets divided by $\sqrt{q \ln q}$, $q \leq 14000029$: bound $C^*$ equal to $\Phi(q)/\sqrt{q \ln q}$ (top dashed-dotted red curve); implicit bound $B^*$ (the 2-nd magenta curve); bound (4.21) (dashed red line $y = 1.835$); implicit bound $A^*(5, (q - 5)^2)$ (the 3-rd blue curve)
4.2 A truncated iterative process

From (4.1) we have that

\[ \#U(K_w \cup \{A_{w+1}\}) = \#U(K_w) - \Delta(A_{w+1}) \leq U_w \left(1 - \frac{w - 2}{q + 1 - w}\right). \]  \hfill (4.5)

Clearly, \( \#U(K_1) \leq U_1 = q^2 \). Using (4.5) iteratively, we obtain

\[ \#U(K_w \cup \{A_{w+1}\}) = U_{w+1} \leq q^2 f_q(w), \]  \hfill (4.6)

where

\[ f_q(w) = \prod_{i=1}^{w} \left(1 - \frac{i - 2}{q + 1 - i}\right). \]  \hfill (4.7)

From now on, we will stop the iterative process when \( \#U(K_w \cup \{A_{w+1}\}) \leq \xi \) where \( \xi \geq 1 \) is some value that we may assign to improve estimates. Note that if some point \( P \in \mathcal{M}_q \) is not covered by \( K_w \cup \{A_{w+1}\} \), one always can find a point \( A_{w+2} \in \mathcal{C} \setminus (K_w \cup \{A_{w+1}\}) \) such that \( P \) is covered by \( K_w \cup \{A_{w+1}, A_{w+2}\} \). It means that after the end of the iterative process we can add at most \( \xi \) points of \( \mathcal{C} \) to the running subset in order to get a \( k \)-AC-subset with size \( k \) satisfying

\[ w + 1 \leq k \leq w + 1 + \xi \]  \hfill (4.8)

**Theorem 4.3.** Let \( \xi \geq 1 \) be a fixed value independent of \( w \). Let \( w < \frac{q^2 + 3}{2} \) satisfy

\[ f_q(w) = \prod_{i=1}^{w} \left(1 - \frac{i - 2}{q + 1 - i}\right) \leq \frac{\xi}{q^2}. \]  \hfill (4.9)

Then it holds that

\[ t(q) \leq w + 1 + \xi. \]  \hfill (4.10)

**Proof.** By (4.6), to provide the inequality \( \#U(K_w \cup \{A_{w+1}\}) \leq \xi \) it is sufficient to find \( w \) such that \( q^2 f_q(w) \leq \xi \). Now (4.10) follows from (4.8). \( \square \)

Clearly, we should choose \( \xi \) such that \( w + 1 + \xi \) is small under condition \( \#U(K_w \cup \{A_{w+1}\}) \leq \xi \).

In order to get more simple forms of upper bounds on \( t(q) \) we will find an upper bound on \( f_q(w) \) of (4.7). To this end we use the Taylor series \( e^{-\alpha} = 1 - \alpha + \frac{\alpha^2}{2} - \frac{\alpha^3}{6} + \ldots \), whence

\[ 1 - \alpha < e^{-\alpha} \text{ for } \alpha \neq 0. \]  \hfill (4.11)
4.3 Implicit bound B

Lemma 4.4. It holds that

\[ f_q(w) = \prod_{i=1}^{w} \left( 1 - \frac{i - 2}{q + 1 - i} \right) < e^{-S}, \]  

(4.12)

where

\[ -w + (q - 1) \ln \frac{q + 1}{q + 1 - w} < S < -w + (q - 1) \ln \frac{q}{q - w}. \]  

(4.13)

Proof. By (4.11),

\[ \prod_{i=1}^{w} \left( 1 - \frac{i - 2}{q + 1 - i} \right) < e^{-S}, \quad S = \sum_{i=1}^{w} \frac{i - 2}{q + 1 - i}. \]

Also,

\[ S = \sum_{i=1}^{w} \frac{i - 2}{q + 1 - i} = \sum_{u=-1}^{w-2} \frac{u}{q - 1 - u} = -w + \sum_{u=-1}^{w-2} \left( \frac{u}{q - 1 - u} + 1 \right) = -w + (q - 1) \sum_{u=-1}^{w-2} \frac{1}{q - 1 - u} = -w + (q - 1) \sum_{t=q+1-w}^{q} \frac{1}{t}. \]

It is well known that

\[ \ln(q + 1) < \sum_{t=1}^{q} \frac{1}{t} < 1 + \ln q. \]

Therefore,

\[ \ln(q + 1) - \ln(q + 1 - w) < \sum_{t=q+1-w}^{q} \frac{1}{t} = \sum_{t=1}^{q} \frac{1}{t} - \sum_{t=1}^{q-w} \frac{1}{t} < \ln q - \ln(q - w). \]

Corollary 4.5. Let \( \xi \geq 1 \) be a fixed value independent of \( w \). Let \( w < \frac{q + 3}{2} \) satisfy

\[ w - (q - 1) \ln \frac{q + 1}{q + 1 - w} \leq \ln \frac{\xi}{q^{2}}. \]

Then it holds that

\[ t(q) \leq w + 1 + \xi. \]

Proof. We substitute (4.12) and (4.13) in (4.9).
Corollary 4.6. (implicit bound B) Let \( w < \frac{q+3}{2} \) satisfy
\[
w - (q-1) \ln \frac{q+1}{q+1-w} \leq \ln \frac{1}{q\sqrt{3q\ln q}}.
\]
Then it holds that
\[
t(q) \leq w + 1 + \sqrt{\frac{q}{3\ln q}}.
\]
Proof. In the assertions of Corollary 4.5 we use \( \xi = \sqrt{\frac{q}{3\ln q}} \). □

The implicit bound \( B^* \) is shown by the second magenta curve on Figs. 1 and 2.

4.4 Explicit bounds

By (4.7) and (4.11), we have
\[
f_q(w) < \prod_{i=1}^{w} \left(1 - \frac{i-2}{q}\right) < \prod_{i=1}^{w} e^{-(i-2)/q} = e^{-w(w^2-3w)/2q} < e^{-(w-2)^2/2q}.
\]

(4.14)

Lemma 4.7. Let \( \xi \geq 1 \) be a fixed value independent of \( w \). The value
\[
\frac{q + 3}{2} > w \geq \sqrt{2q} \sqrt{\ln \frac{q^2}{\xi} + 3}
\]

satisfies inequality (4.9).

Proof. By (4.14), to provide (4.9) it is sufficient to find \( w \) such that
\[
e^{-(w-2)^2/2q} < \frac{\xi}{q^2}.
\]
As \( w \) should be an integer, in (4.15) one is added. Inequality \( w < \frac{q+3}{2} \) is obvious. □

Theorem 4.8. In \( PG(2,q) \) it holds that
\[
t(q) \leq \sqrt{2q} \sqrt{\ln \frac{q^2}{\xi} + \xi + 4}, \quad \xi \geq 1,
\]

(4.16)

where \( \xi \) is an arbitrarily chosen value.

Proof. The assertion follows from (4.10) and (4.15). □
Remark 4.9. We consider the function of $\xi$ of the form

$$
\phi(\xi) = \sqrt{2q} \sqrt{\ln \frac{q^2}{\xi}} + \xi + 4.
$$

Its derivative by $\xi$ is

$$
\phi'(\xi) = 1 - \frac{1}{\xi} \sqrt{\frac{q}{2 \ln \frac{q^2}{\xi}}}.
$$

Put $\phi'(\xi) = 0$. Then

$$
\xi^2 = \frac{q}{4 \ln q - 2 \ln \xi}.
$$

(4.17)

We find $\xi$ in the form $\xi = \sqrt{\frac{q}{c \ln q}}$. By (4.17), $c = 3 + \frac{\ln c + \ln \ln q}{\ln q}$. For simplicity, we choose $c = 3$. Then $\xi = \sqrt{\frac{q}{3 \ln q}}$ and the value

$$
\phi' \left( \sqrt{\frac{q}{3 \ln q}} \right) = 1 - \sqrt{\frac{3 \ln q}{3 \ln q + \ln \ln q + \ln 3}}
$$

is close to zero for growing $q$. Also, it is easy to check the following: $\phi'(1) < 0$ if $q \geq 9$, $\phi'(\xi)$ is an increasing function, $0 < \phi' \left( \sqrt{\frac{q}{3 \ln q}} \right) < \phi'(\sqrt{q}) = 1 - \sqrt{\frac{1}{3 \ln q}}$.

So, the choice $\xi = \sqrt{\frac{q}{3 \ln q}}$ in (4.16) seems to be convenient.

Theorem 4.10. (Bound C) The following upper bound on the smallest size $t(q)$ of an AC-subset in $\text{PG}(2, q)$ holds.

$$
t(q) < \Phi(q) = \sqrt{q(3 \ln q + \ln \ln q + \ln 3)} + \sqrt{\frac{q}{3 \ln q}} + 4 \sim \sqrt{3q \ln q}.
$$

(4.18)

Proof. We substitute $\xi = \sqrt{\frac{q}{3 \ln q}}$ in (4.16). \qed

The bound C* (i.e. the value $\Phi(q)/\sqrt{q \ln q}$) is shown by the top dashed-dotted red curve on Figs. 1 and 2.

Remark 4.11. If in (4.16) we take $\xi = 1$ and $\xi = \sqrt{q}$, we obtain bounds (4.19) and (4.20):

$$
t(q) < 2 \sqrt{q \ln q} + 5.
$$

(4.19)

$$
t(q) < \sqrt{3q \ln q} + \sqrt{q} + 4.
$$

(4.20)

It can be shown that bounds (4.19) and (4.20) are worse than (4.18).

If we put, see Remark 4.9, $c = 3 + \frac{\ln c + \ln \ln q}{\ln q}$, $\xi = \sqrt{\frac{q}{3 \ln q + \ln \ln q + 1}}$, we improve bound (4.18). But, the improvement is unessential whereas the bound takes a lengthy form.
Theorem 4.12. The following upper bound on the smallest size $t(q)$ of an AC-subset in $\text{PG}(2, q)$ holds.

$$t(q) < 1.835\sqrt{q \ln q}.$$ (4.21)

Proof. For $q \leq 12755807$ we checked by computer that the implicit bound $A(5, (q-5)^2) < 1.8341\sqrt{q \ln q}$; so in this region the assertion is provided by the bound $A(5, (q-5)^2)$, see Observation 4.2 and Fig. 2. It is easy to see that $\Phi(q)/\sqrt{q \ln q}$ is a decreasing function of $q$. Moreover, $\Phi(q)/\sqrt{q \ln q} < 1.835$ for $q = 12755807$. So, for $q > 12755807$ the assertion is provided by the bound $C$.

The bound (4.21) is presented by the dashed red line $y = 1.835$ in Figs. 1 and 2.

5 Computer assisted results on $t(q)$ and $t^*(q)$

Let $\overline{t}(q)$ be the smallest known size of an AC-subset in $\text{PG}(2, q)$. Let $\overline{t}(q) = \overline{t}(q)/\sqrt{q \ln q}$. We denote the following sets of values of $q$: $Q_2 := \{5 \leq q \leq 33013, \ q \text{ prime power}\}; Q_3 := \{5 \leq q \leq 32, \ q \text{ prime power}\}; Q_4 := Q_1 \cup \{160801, 208849, 253009\}$. Let $Q_1$ be as in (2.1).

For the set $Q_3$ we obtained by computer search the smallest sizes $t(q)$ of AC-subsets of $\mathcal{C}$ in $\text{PG}(2, q)$, see Table 2. The algorithm, used in the search, fixes a conic, computes all the non-equivalent point subsets of the conic of a certain size (6 in our complete cases) and extends each of them trying to obtain a minimal AC-subset. Each time an example is found only smaller examples are looked for. Minimality is checked explicitly: once we have found an AC-subset we test that deleting from it a point in all possible ways no almost complete subset is obtained. All computations are performed using the system for symbol calculations MAGMA [22].

| $q$  | $t(q)$ | $q$  | $t(q)$ | $q$  | $t(q)$ | $q$  | $t(q)$ | $q$  | $t(q)$ | $q$  | $t(q)$ | $q$  | $t(q)$ |
|------|--------|------|--------|------|--------|------|--------|------|--------|------|--------|------|--------|
| 5    | 5      | 19   | 11     | 11   | 10     | 11   | 9      | 16   | 17     | 19   | 13     | 27   | 17     |
| 7    | 6      | 13   | 14     | 29   | 16     | 25   | 15     | 31   | 19     | 23   | 13     | 32   | 15     |
| 6    | 8      | 12   | 12     | 13   | 19     | 25   | 12     | 14   | 20     | 27   | 17     | 30   | 18     |
| 9    | 6      | 13   | 14     | 27   | 18     | 25   | 13     | 31   | 19     | 24   | 15     | 33   | 20     |
| 11   | 8      | 15   | 15     | 29   | 19     | 27   | 15     | 32   | 20     | 28   | 16     | 35   | 21     |
| 13   | 8      | 16   | 16     | 31   | 20     | 29   | 16     | 33   | 21     | 30   | 17     | 37   | 22     |
| 16   | 9      | 17   | 17     | 32   | 21     | 31   | 17     | 34   | 22     | 31   | 18     | 39   | 23     |
| 17   | 10     |      |        |      |        |      |        |      |        |      |        |      |        |

For the sets $Q_2$ and $Q_4$ we obtained small AC-subsets of $\mathcal{C}$ in $\text{PG}(2, q)$ by computer search³. For it we used step-by-step randomized greedy algorithms similar to those

³The computer search for $q \in Q_2 \cup Q_4$ has been carried out using computing resources of the federal collective usage center Complex for Simulation and Data Processing for Mega-science Facilities at NRC “Kurchatov Institute”, http://ckp.nrcki.ru/
from [23], see also the references therein. Recall that at each step a randomized greedy algorithm maximizes some objective function \( f \), but some steps are executed in a random manner. Also, if one and the same maximum of \( f \) can be obtained in different ways, the choice is made at random. As the value of the objective function, the number of points lying on bisecants of the running subset is considered.

As far as the authors know, sizes of AC-subsets, obtained by the mentioned computer search, are the smallest known. The corresponding values of \( \overline{t}(q) \) are shown by the bottom black curve in Fig. 1. Recall that, \( t^*(q) = \frac{t(q)}{\sqrt{q \ln q}} \).

The values \( \overline{t}(q) \) and \( \overline{t'}(q) \) for \( q \in Q_4 \) and prime \( q \in Q_2 \) are given in Tables 3 and 4, respectively, see Appendix. As values of \( \overline{t}(q) \) are not integers, in Tables 3 and 4 we give rounded values of \( \overline{t}(q) \), moreover we round up. This explains the entry “\( \overline{t}(q) < \)” in the top of columns.

In Table 4, the values \( \overline{t}(q) \) are written for not all \( q \)'s. The rules for entries \( \overline{t}(q) \) are as follows. Assume that the following holds: \( q' < q'' \); the values \( \overline{t}(q') \) and \( \overline{t}(q'') \) are written in Table 4; no value \( \overline{t}(q) \) is written in the table if \( q' < q < q'' \). Then \( \overline{t}(q') \leq \overline{t}(q'') \) and \( \overline{t}(q) \leq \overline{t}(q') \) with \( q' < q < q'' \).

For example, one may take \( q' = 19 \) and \( q'' = 307 \). We see that no value \( \overline{t}(q) \) is written in Table 4 if \( 19 < q < 307 \). We have \( \overline{t}(19) \approx 1.471 < \overline{t}(307) \approx 1.479 \) and \( \overline{t}(q) \leq 1.471 \) with \( 19 < q < 307 \).

So, in Table 4, the blank on place \( \overline{t}(q) \) means that \( \overline{t}(q) \leq \overline{t}(q') \) under the conditions that \( q' < q \), value \( \overline{t}(q') \) is written in the table, and no value \( \overline{t}(q^*) \) is written if \( q' < q^* < q \).

By computer search for the sets \( Q_2 \) and \( Q_4 \), see Tables 3 and 4, we have Theorem 5.1.

**Theorem 5.1.** The following upper bounds on the smallest size \( t(q) \) of an AC-subset of the conic \( C \) in PG(2,\( q \)) hold:

\[
\begin{align*}
\text{\( t(q) < 1.525\sqrt{q \ln q} \),} & \quad 8 \leq q \leq 887, \ q \text{ prime power, } q \neq 11; \quad (5.1) \\
\text{\( t(q) < 1.548\sqrt{q \ln q} \),} & \quad 887 < q \leq 1553, \ q \text{ prime power;} \quad (5.2) \\
\text{\( t(q) < 1.572\sqrt{q \ln q} \),} & \quad 1553 < q \leq 2351, \ q \text{ prime power, } q = 11; \quad (5.3) \\
\text{\( t(q) < 1.585\sqrt{q \ln q} \),} & \quad 2351 < q \leq 4027, \ q \text{ prime power;} \quad (5.4) \\
\text{\( t(q) < 1.620\sqrt{q \ln q} \),} & \quad 4027 < q \leq 17041, \ q \text{ prime power;} \quad (5.5) \\
\text{\( t(q) < 1.635\sqrt{q \ln q} \),} & \quad 17041 < q \leq 33013, \ q \text{ prime power, } q = 7; \quad (5.6) \\
\text{\( t(q) < 1.674\sqrt{q \ln q} \),} & \quad q = p^m, \ p \text{ prime, } m \geq 2, \ q \in Q_1; \quad (5.7) \\
\text{\( t(q) < 1.686\sqrt{q \ln q} \),} & \quad q = 160801, 208849, 253009. \quad (5.8)
\end{align*}
\]
The bounds (5.6), (5.7) are presented by dashed red lines $y = 1.635, y = 1.674$ in Fig. 1.

6 New bounds on $t(q)$ and completeness of normal rational curves

Proof of Corollary 2.2. We substitute the new bounds of Theorem 2.1 in relation (1.2) taken from [19, Theorem 3.1].

Proof of Corollary 2.3. We act analogously to the proof of [19, Theorem 3.5], changing in it $6\sqrt{q \ln q}$ by $c\sqrt{q \ln q}$. As the result we obtain inequality (2.6).

By (2.6), for $c = 1.835, h \geq 29$, we have $p_0(h) \geq 33079 > 33013$; but for $c = 1.835, h \leq 28$, it holds that $p_0(h) \leq 31840 < 33013$. So, by (4.21) and (5.1)–(5.6), we may take $c = 1.835$ for $h \geq 29$ and $c = 1.635$ for $h \leq 28$.

Again we use (2.6). For $c = 1.635, h \geq 20$, we have $p_0(h) \geq 17091 > 17041$; but for $c = 1.635, h \leq 19$, it holds that $p_0(h) \leq 16164 < 17041$. So, by (5.1)–(5.6), we may take $c = 1.635$ for $20 \leq h \leq 28$ and $c = 1.62$ for $h \leq 19$.

Now for $h = 1, \ldots, 5$ we found $p_0(h)$ as a solution of (2.6) taking $c$ on the base Theorem 5.1. For the given $h$, at the beginning we obtain $p_0(h)$ with $c = 1.62$. Then we decrease $c$ using (5.1)–(5.4) and get a smaller $p_0(h)$. For $c = 1.62$ we obtain $p_0(1) = 877, p_0(2) = 1543, p_0(3) = 2273, p_0(4) = 3037, p_0(5) = 3821$. So, we may put $c = 1.525$ for $h = 1, c = 1.548$ for $h = 2, c = 1.572$ for $h = 3, c = 1.585$ for $h = 4, 5$, see (5.1), (5.2), (5.3), and (5.4), respectively. Solutions of inequality (2.6) for these $(c, h)$ are the values $p_0(1), \ldots, p_0(5)$ written in the assertion of the corollary.

Remark 6.1. We can also improve the result of [19, Theorem 3.4]. If in the proof of [19, Theorem 3.4] one uses the new bound $t(q) < 1.835\sqrt{q \ln q}$ instead of (1.1) then the following assertion can be proved: for prime $p \geq 76207$ every normal rational curve in $\text{PG}(N, p)$ with $2 \leq N \leq p - 2$ is a complete $(q + 1)$-arc.

For comparison note that in [19, Theorem 3.4] the value $p > 1007215$ is pointed out.

Of course, due to the results of [1, 16, 17], see row 5 of Table 1 we know that for all primes $p$ normal rational curves in $\text{PG}(N, p)$ are complete.

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Appendix. Tables of the smallest known sizes $\tilde{t}(q)$ of AC-subsets in $\text{PG}(2, q)$

Table 3. The smallest known sizes $\tilde{t}(q)$ of AC-subsets in $\text{PG}(2, q)$ and values $\overline{t^*}(q)$,
$q$ non-prime, $q \in \{8 \leq q \leq 139129, q = p^m, p$ prime, $m \geq 2\} \cup \{160801, 208849, 253009\}$

| $q$ | $p^m$ | $\tilde{t}(q)$ | $q$ | $p^m$ | $\tilde{t}(q)$ | $q$ | $p^m$ | $\tilde{t}(q)$ |
|-----|-------|---------------|-----|-------|---------------|-----|-------|---------------|
| 8   | 2^4   | 6             | 16  | 2^7   | 15            | 25  | 5^2   | 12            |
| 49  | 7^2   | 18            | 121 | 11^2  | 33            | 169 | 13^2  | 41            |
| 289 | 17^2  | 58            | 512 | 2^9   | 84            | 729 | 3^6   | 102           |
| 1024| 2^10  | 127           | 10609| 103^2 | 503           | 1024| 2^10  | 127           |
| 1681| 41^2  | 173           | 1681| 11^3  | 150           | 2187| 3^7   | 203           |
| 2401| 7^4   | 214           | 2401| 13^3  | 203           | 3481| 59^2  | 309           |
| 4489| 67^2  | 309           | 4489| 17^3  | 325           | 5329| 73^2  | 341           |
| 6859| 19^3  | 393           | 6859| 83^2  | 394           | 8192| 2^13  | 435           |
| 8192| 2^13  | 435           | 10609| 103^2 | 503           | 12167| 23^3  | 545           |
| 12167| 23^3 | 545           | 15625| 5^6  | 629           | 15625| 5^6  | 629           |
| 16807| 7^5  | 655           | 16807| 7^5  | 663           | 19321| 13^2  | 712           |
| 19321| 13^2 | 712           | 21287| 3^7  | 203           | 21287| 3^7  | 203           |
| 22801| 151^2| 778           | 22801| 151^2| 778           | 26569| 163^2 | 849           |
| 29791| 31^3 | 904           | 29791| 31^3 | 904           | 32761| 181^2 | 952           |
| 32761| 181^2| 952           | 37249| 193^2 | 1025         | 37249| 193^2 | 1025         |
| 44521| 211^2| 1133          | 44521| 211^2| 1133         | 51529| 227^2 | 1230         |
| 51529| 227^2| 1230          | 57121| 239^2 | 1302         | 57121| 239^2 | 1302         |
| 63001| 251^2| 1378          | 63001| 251^2| 1378         | 63001| 251^2| 1378         |

$\overline{t^*}(q)$ values for non-prime $q$ are given for $q \leq 139129$. For prime $q$, $\overline{t^*}(q)$ is not computed.
Table 3. Continue

| $q$   | $p^m$ | $\bar{t}(q)$ | $\bar{t}(q)$ | $\bar{t}(q)$ |
|-------|-------|---------------|---------------|---------------|
| 68921 | $41^4$| 1451          | 1.656         |               |
| 73441 | $271^2$| 1501         | 1.655         |               |
| 78961 | $281^2$| 1561         | 1.655         |               |
| 83521 | $17^4$| 1614         | 1.659         |               |
| 96721 | $311^2$| 1748         | 1.659         |               |
| 103823 | $47^3$| 1824         | 1.666         |               |
| 117649 | $7^6$| 1954         | 1.668         |               |
| 124609 | $353^2$| 2021        | 1.672         |               |
| 131072 | $2^{17}$| 2077       | 1.672         |               |
| 160801 | $401^2$| 2332       | 1.680         |               |
| 69169 | $263^2$| 1452        | 1.654         |               |
| 76729 | $277^2$| 1541        | 1.659         |               |
| 79507 | $43^3$| 1571        | 1.659         |               |
| 85849 | $293^2$| 1637        | 1.658         |               |
| 97969 | $313^2$| 1761        | 1.660         |               |
| 109561 | $331^2$| 1878        | 1.666         |               |
| 120409 | $347^2$| 1985        | 1.673         |               |
| 128881 | $359^2$| 2058        | 1.672         |               |
| 134689 | $367^2$| 2110        | 1.673         |               |
| 208849 | $457^2$| 2686        | 1.680         |               |
| 72361 | $269^2$| 1489        | 1.655         |               |
| 78125 | $5^7$| 1553        | 1.656         |               |
| 80089 | $283^2$| 1576        | 1.658         |               |
| 94249 | $307^2$| 1723        | 1.659         |               |
| 100489 | $317^2$| 1786        | 1.661         |               |
| 113569 | $337^2$| 1917        | 1.668         |               |
| 121801 | $349^2$| 1999        | 1.674         |               |
| 130321 | $19^4$| 2070        | 1.671         |               |
| 139129 | $373^2$| 2142        | 1.669         |               |
| 253009 | $503^2$| 2991        | 1.686         |               |
Table 4. The smallest known sizes $\overline{t}(q)$ of AC-subsets in $\text{PG}(2, q)$ and values $\overline{\tau}(q)$, $13 \leq q \leq 33013$, $q$ prime

| $q$ | $\overline{t}(q)$ | $\overline{\tau}(q)$ | $\overline{t}(q)$ | $\overline{\tau}(q)$ | $\overline{t}(q)$ | $\overline{\tau}(q)$ | $\overline{t}(q)$ | $\overline{\tau}(q)$ |
|-----|-------------------|-------------------|-------------------|-------------------|-------------------|-------------------|-------------------|-------------------|
| 13  | 8                 | 1.386             | 17                | 1.441             | 19                | 1.471             | 23                | 1.529             |
| 31  | 14                | 37                | 16                | 41                | 16                | 43                | 16                | 47                |
| 53  | 20                | 59                | 21                | 61                | 22                | 67                | 23                | 71                |
| 73  | 24                | 79                | 25                | 83                | 26                | 89                | 24                | 97                |
| 101 | 30                | 103               | 30                | 107               | 31                | 109               | 32                | 113               |
| 127 | 35                | 131               | 36                | 137               | 37                | 139               | 37                | 149               |
| 151 | 39                | 157               | 40                | 163               | 41                | 167               | 42                | 173               |
| 179 | 43                | 181               | 44                | 191               | 45                | 193               | 46                | 197               |
| 199 | 47                | 211               | 48                | 223               | 50                | 227               | 51                | 229               |
| 233 | 51                | 239               | 53                | 241               | 53                | 251               | 54                | 257               |
| 263 | 55                | 269               | 56                | 271               | 57                | 277               | 58                | 281               |
| 283 | 58                | 293               | 59                | 307               | 62                | 1.479             | 311               | 62                |
| 317 | 62                | 331               | 64                | 337               | 65                | 347               | 66                | 349               |
| 353 | 67                | 359               | 68                | 367               | 68                | 373               | 69                | 379               |
| 383 | 70                | 389               | 71                | 397               | 72                | 401               | 73                | 1.489             |
| 419 | 75                | 1.492             | 421               | 74                | 431               | 76                | 433               | 76                | 439               |
| 443 | 77                | 449               | 78                | 457               | 79                | 1.494             | 461               | 79                | 463               |
| 467 | 80                | 479               | 81                | 487               | 82                | 1.494             | 491               | 82                | 499               |
| 503 | 83                | 509               | 84                | 521               | 85                | 523               | 85                | 541               |
| 547 | 88                | 1.499             | 557               | 89                | 1.500             | 563               | 89                | 569               |
| 577 | 91                | 1.503             | 587               | 92                | 1.504             | 593               | 92                | 599               |
| 607 | 93                | 613               | 94                | 617               | 95                | 1.509             | 619               | 95                |
| 641 | 97                | 643               | 97                | 647               | 97                | 653               | 98                | 659               |
| 661 | 99                | 1.512             | 673               | 100               | 677               | 100               | 683               | 101               |
| 701 | 102               | 709               | 103               | 719               | 104               | 727               | 104               | 733               |
| 739 | 105               | 743               | 106               | 751               | 107               | 1.518             | 757               | 107               |
| 769 | 108               | 773               | 108               | 787               | 109               | 797               | 111               | 1.522             |
| 811 | 112               | 821               | 112               | 823               | 113               | 827               | 113               | 829               |
| 839 | 114               | 853               | 115               | 857               | 116               | 1.525             | 859               | 116               |
| 877 | 117               | 881               | 117               | 883               | 118               | 887               | 118               | 907               |
| 911 | 120               | 919               | 120               | 929               | 121               | 937               | 122               | 941               |
| 947 | 123               | 953               | 123               | 967               | 124               | 971               | 125               | 1.530             |
| 983 | 125               | 991               | 126               | 997               | 127               | 1.531             | 1009              | 128               |
| 1019| 128               | 1021              | 128               | 1031              | 129               | 1033              | 129               | 1039              |
| 1049| 130               | 1051              | 131               | 1061              | 131               | 1063              | 132               |
| 1087| 133               | 1091              | 133               | 1093              | 134               | 1097              | 134               |
| 1109| 135               | 1117              | 136               | 1123              | 136               | 1129              | 137               |
| 1153| 138               | 1163              | 139               | 1171              | 139               | 1181              | 140               |
| 1193| 141               | 1201              | 141               | 1213              | 143               |
| 1229| 144               | 1231              | 144               | 1237              | 144               | 1249              | 145               |
| 1277| 147               | 1279              | 147               | 1283              | 147               | 1289              | 148               |
| 1297| 148               | 1301              | 149               | 1.543             | 1303              | 149               | 1307              | 149               |
| 1321| 150               | 1327              | 150               | 1361              | 153               | 1.544             | 1367              | 153               |

$q$ prime
| $q$ | $\bar{t}(q)$ | $\bar{t}(q)$ | $\bar{t}(q)$ | $\bar{t}(q)$ | $\bar{t}(q)$ | $\bar{t}(q)$ |
|-----|-----|-----|-----|-----|-----|-----|
| 1381 | 153 | 1399 | 155 | 1409 | 156 | 1423 | 157 |
| 1427 | 157 | 1429 | 157 | 1439 | 158 | 1447 | 158 |
| 1451 | 159 | 1453 | 159 | 1471 | 160 | 1481 | 160 |
| 1483 | 161 | 1487 | 161 | 1493 | 161 | 1499 | 162 |
| 1511 | 162 | 1523 | 163 | 1531 | 163 | 1543 | 164 |
| 1549 | 165 | 1559 | 166 | 1567 | 166 | 1571 | 166 |
| 1579 | 167 | 1597 | 167 | 1601 | 168 | 1607 | 168 |
| 1609 | 169 | 1619 | 169 | 1621 | 169 | 1627 | 170 |
| 1637 | 171 | 1663 | 172 | 1667 | 172 | 1669 | 172 |
| 1693 | 174 | 1699 | 174 | 1709 | 175 | 1721 | 175 |
| 1723 | 176 | 1741 | 177 | 1747 | 177 | 1753 | 178 |
| 1759 | 178 | 1783 | 179 | 1789 | 180 | 1801 | 181 |
| 1811 | 182 | 1823 | 182 | 1831 | 183 | 1847 | 183 |
| 1861 | 185 | 1871 | 185 | 1873 | 185 | 1877 | 185 |
| 1893 | 186 | 1901 | 187 | 1907 | 187 | 1913 | 188 |
| 1931 | 189 | 1949 | 190 | 1951 | 190 | 1973 | 191 |
| 1999 | 192 | 1993 | 192 | 1997 | 192 | 2003 | 192 |
| 2011 | 193 | 2017 | 193 | 2027 | 194 | 2029 | 194 |
| 2039 | 194 | 2063 | 196 | 2081 | 197 | 2083 | 197 |
| 2087 | 197 | 2089 | 197 | 2099 | 198 | 2111 | 199 |
| 2129 | 200 | 2131 | 200 | 2137 | 200 | 2141 | 201 |
| 2153 | 202 | 2161 | 202 | 2179 | 203 | 2203 | 204 |
| 2213 | 205 | 2221 | 205 | 2237 | 206 | 2239 | 206 |
| 2251 | 206 | 2267 | 208 | 2269 | 207 | 2273 | 208 |
| 2287 | 208 | 2293 | 209 | 2297 | 208 | 2309 | 209 |
| 2333 | 211 | 2339 | 211 | 2341 | 211 | 2347 | 212 |
| 2357 | 213 | 2371 | 212 | 2377 | 212 | 2381 | 213 |
| 2389 | 213 | 2393 | 213 | 2399 | 214 | 2411 | 214 |
| 2423 | 215 | 2437 | 216 | 2441 | 217 | 2447 | 217 |
| 2467 | 218 | 2473 | 219 | 2477 | 219 | 2503 | 219 |
| 2531 | 220 | 2539 | 221 | 2543 | 222 | 2549 | 223 |
| 2551 | 222 | 2579 | 223 | 2591 | 224 | 2593 | 223 |
| 2609 | 224 | 2621 | 225 | 2633 | 225 | 2647 | 227 |
| 2657 | 228 | 2663 | 228 | 2671 | 229 | 2677 | 229 |
| 2687 | 230 | 2689 | 230 | 2693 | 230 | 2699 | 231 |
| 2707 | 231 | 2713 | 231 | 2719 | 231 | 2729 | 232 |
| 2741 | 231 | 2749 | 231 | 2753 | 231 | 2767 | 232 |
| 2789 | 233 | 2791 | 233 | 2797 | 234 | 2801 | 234 |
| 2819 | 235 | 2833 | 235 | 2837 | 236 | 2843 | 236 |
| 2851 | 236 | 2861 | 237 | 2879 | 238 | 2887 | 238 |
| 2897 | 238 | 2909 | 239 | 2917 | 239 | 2927 | 240 |
| 2939 | 241 | 2957 | 241 | 2963 | 241 | 2969 | 242 |
| 2971 | 242 | 3001 | 243 | 3011 | 244 | 3019 | 244 |

Table 4. Continue 1
Table 4. Continue 2

| q     | \( \bar{t}(q) \) | \( \bar{t}(q) \) | \( \bar{t}(q) \) | \( \bar{t}(q) \) | \( \bar{t}(q) \) | \( \bar{t}(q) \) |
|-------|------------------|------------------|------------------|------------------|------------------|------------------|
| 3041  | 245              | 3049 246         | 3061 247         | 3067 248         | 3079 248         |
| 3083  | 249              | 3089 248         | 3109 249         | 3119 249         | 3121 250         |
| 3137  | 250              | 3163 251         | 3167 251         | 3169 252         | 3181 252         |
| 3187  | 253              | 3191 253         | 3203 253         | 3209 254         | 3217 255         |
| 3221  | 254              | 3229 255         | 3251 255         | 3253 255         | 3257 256         |
| 3259  | 256              | 3271 256         | 3299 258         | 3301 258         | 3307 257         |
| 3313  | 258              | 3319 259         | 3323 259         | 3329 258         | 3331 258         |
| 3343  | 260              | 3347 260         | 3359 261         | 3361 260         | 3371 261         |
| 3373  | 261              | 3389 262         | 3391 262         | 3407 262         | 3413 262         |
| 3433  | 264              | 3449 264         | 3457 264         | 3461 265         | 3463 264         |
| 3467  | 265              | 3469 265         | 3491 266         | 3499 266         | 3511 267         |
| 3517  | 267              | 3527 268         | 3529 268         | 3533 268         | 3539 268         |
| 3541  | 268              | 3547 269         | 3557 269         | 3559 269         | 3571 270         |
| 3581  | 270              | 3583 270         | 3593 271         | 3607 271         | 3613 271         |
| 3617  | 272              | 3623 271         | 3631 273         | 3637 273         | 3643 273         |
| 3659  | 274              | 3671 274         | 3673 274         | 3677 274         | 3691 275         |
| 3697  | 275              | 3701 275         | 3709 276         | 3719 276         | 3727 277         |
| 3733  | 277              | 3739 277         | 3761 278         | 3767 279         | 3769 279         |
| 3779  | 279              | 3793 280         | 3797 280         | 3803 280         | 3821 281         |
| 3823  | 281              | 3833 281         | 3847 282         | 3851 282         | 3853 282         |
| 3863  | 282              | 3877 283         | 3881 283         | 3889 283         | 3907 284         |
| 3911  | 284              | 3917 284         | 3919 284         | 3923 284         | 3929 285         |
| 3931  | 285              | 3943 286         | 3947 286         | 3967 286         | 3989 288         |
| 4001  | 288              | 4003 288         | 4007 288         | 4013 288         | 4019 289         |
| 4021  | 288              | 4027 288         | 4049 291         | 4051 290         | 4057 290         |
| 4073  | 291              | 4079 291         | 4091 292         | 4093 292         | 4099 291         |
| 4111  | 292              | 4127 293         | 4129 293         | 4133 294         | 4139 293         |
| 4153  | 294              | 4157 294         | 4159 295         | 4177 295         | 4201 297         |
| 4211  | 297              | 4217 297         | 4219 297         | 4229 298         | 4231 298         |
| 4241  | 298              | 4243 298         | 4253 298         | 4259 298         | 4261 299         |
| 4271  | 299              | 4273 300         | 4283 300         | 4289 299         | 4297 300         |
| 4327  | 301              | 4337 302         | 4339 302         | 4349 302         | 4357 303         |
| 4363  | 303              | 4373 304         | 4391 304         | 4397 305         | 4409 305         |
| 4421  | 306              | 4423 305         | 4441 306         | 4447 307         | 4451 306         |
| 4457  | 307              | 4463 307         | 4481 307         | 4483 308         | 4493 308         |
| 4507  | 309              | 4513 308         | 4517 309         | 4519 309         | 4523 309         |
| 4547  | 310              | 4549 310         | 4561 311         | 4567 311         | 4583 312         |
| 4591  | 311              | 4597 312         | 4603 312         | 4621 313         | 4637 314         |
| 4639  | 314              | 4643 314         | 4649 314         | 4651 314         | 4657 314         |
| 4663  | 315              | 4673 315         | 4679 315         | 4691 316         | 4703 317         |
| 4721  | 317              | 4723 318         | 4729 317         | 4733 317         | 4751 318         |
| 4759  | 319              | 4783 320         | 4787 320         | 4789 320         | 4793 320         |
| 4799  | 320              | 4801 320         | 4813 321         | 4817 321         | 4831 322         |
Table 4. Continue 3

| q   | ℓ(q) | ℓ(q) | q   | ℓ(q) | ℓ(q) | q   | ℓ(q) | ℓ(q) | q   | ℓ(q) | ℓ(q) |
|-----|------|------|-----|------|------|-----|------|------|-----|------|------|
| 4861| 323  |      | 4909| 325  | 1.592| 4943| 326  |      | 4973| 327  |      |
| 5009| 328  |      | 5051| 330  |      | 5099| 331  |      | 5147| 334  | 1.593|
| 5189| 336  | 1.595| 5233| 337  |      | 5281| 338  |      | 5333| 341  |      |
| 5393| 343  |      | 5449| 344  |      | 5501| 346  |      | 5564| 352  |      |
| 5659| 353  |      | 5701| 354  |      | 5743| 356  |      | 5801| 358  |      |
| 5839| 360  | 1.600| 5861| 360  |      | 5937| 373  |      | 6029| 366  |      |
| 6067| 368  |      | 6101| 368  |      | 6143| 370  |      | 6199| 371  |      |
| 6229| 373  |      | 6271| 374  |      | 6311| 375  |      | 6343| 377  |      |
| 6373| 377  |      | 6427| 379  |      | 6481| 381  |      | 6551| 383  |      |
| 6577| 385  |      | 6637| 386  |      | 6679| 388  |      |      |      |      |
Table 4. Continue 4

| q   | $\bar{q}(q)$ | $\bar{q}(q)$ | $\bar{q}(q)$ | $\bar{q}(q)$ | $\bar{q}(q)$ | $\bar{q}(q)$ |
|-----|--------------|--------------|--------------|--------------|--------------|--------------|
| 6709| 388          | 6719         | 389          | 6733         | 389          | 6737         | 389          | 6761         | 390          |
| 6763| 390          | 6779         | 390          | 6781         | 391          | 6791         | 391          | 6793         | 391          |
| 6803| 391          | 6827         | 391          | 6829         | 392          | 6833         | 392          | 6871         | 393          |
| 6841| 392          | 6863         | 393          | 6869         | 393          | 6871         | 393          | 6871         | 393          |
| 6883| 394          | 6907         | 395          | 6911         | 395          | 6917         | 395          | 6917         | 395          |
| 6947| 396          | 6959         | 397          | 6961         | 397          | 6967         | 397          | 6967         | 397          |
| 6971| 398          | 7019         | 399          | 7027         | 398          | 7039         | 399          | 7039         | 399          |
| 7001| 398          | 7057         | 400          | 7079         | 400          | 7103         | 401          | 7103         | 401          |
| 7109| 401          | 7127         | 402          | 7129         | 402          | 7151         | 403          | 7151         | 403          |
| 7159| 403          | 7187         | 404          | 7193         | 404          | 7207         | 405          | 7207         | 405          |
| 7211| 405          | 7219         | 405          | 7229         | 405          | 7237         | 406          | 7237         | 406          |
| 7243| 406          | 7253         | 406          | 7283         | 407          | 7297         | 408          | 7297         | 408          |
| 7307| 407          | 7321         | 408          | 7331         | 409          | 7333         | 409          | 7333         | 409          |
| 7349| 409          | 7385         | 409          | 7403         | 411          | 7411         | 411          | 7411         | 411          |
| 7417| 411          | 7451         | 412          | 7457         | 412          | 7459         | 412          | 7459         | 412          |
| 7477| 413          | 7487         | 413          | 7489         | 413          | 7499         | 414          | 7499         | 414          |
| 7507| 414          | 7523         | 415          | 7529         | 415          | 7537         | 415          | 7537         | 415          |
| 7541| 416          | 7549         | 416          | 7559         | 416          | 7561         | 416          | 7561         | 416          |
| 7573| 416          | 7583         | 417          | 7589         | 416          | 7591         | 417          | 7591         | 417          |
| 7603| 417          | 7607         | 417          | 7621         | 418          | 7639         | 419          | 7643         | 418          |
| 7649| 419          | 7669         | 419          | 7673         | 419          | 7681         | 420          | 7687         | 420          |
| 7691| 420          | 7699         | 420          | 7703         | 420          | 7717         | 421          | 7723         | 420          |
| 7727| 421          | 7741         | 422          | 7753         | 422          | 7757         | 422          | 7759         | 422          |
| 7789| 422          | 7793         | 423          | 7817         | 424          | 7823         | 424          | 7829         | 424          |
| 7841| 424          | 7853         | 425          | 7867         | 426          | 7873         | 426          | 7877         | 425          |
| 7879| 425          | 7863         | 425          | 7901         | 426          | 7907         | 427          | 7919         | 427          |
| 7927| 427          | 7933         | 427          | 7937         | 428          | 7949         | 428          | 7951         | 428          |
| 7963| 428          | 7993         | 429          | 8009         | 429          | 8011         | 428          | 8017         | 429          |
| 8039| 431          | 8053         | 431          | 8059         | 431          | 8069         | 431          | 8081         | 432          |
| 8087| 433          | 8101         | 432          | 8109         | 432          | 8110         | 432          | 8111         | 433          |
| 8117| 433          | 8123         | 433          | 8147         | 434          | 8161         | 434          | 8167         | 434          |
| 8171| 434          | 8179         | 435          | 8191         | 435          | 8209         | 435          | 8219         | 436          |
| 8221| 436          | 8231         | 437          | 8233         | 436          | 8237         | 437          | 8243         | 437          |
| 8263| 437          | 8269         | 437          | 8273         | 438          | 8287         | 438          | 8291         | 437          |
| 8293| 438          | 8297         | 438          | 8311         | 438          | 8317         | 440          | 8329         | 439          |
| 8353| 440          | 8363         | 440          | 8369         | 441          | 8377         | 441          | 8387         | 440          |
| 8389| 442          | 8419         | 442          | 8423         | 442          | 8429         | 443          | 8431         | 442          |
| 8443| 444          | 8447         | 443          | 8461         | 444          | 8467         | 443          | 8501         | 445          |
| 8513| 445          | 8521         | 445          | 8527         | 445          | 8537         | 445          | 8539         | 444          |
| 8543| 446          | 8563         | 446          | 8573         | 447          | 8581         | 447          | 8597         | 448          |
| 8599| 447          | 8609         | 448          | 8623         | 448          | 8627         | 448          | 8629         | 448          |
| 8641| 449          | 8647         | 450          | 8663         | 449          | 8669         | 450          | 8677         | 450          |
| $q$ | $\bar{t}(q)$ | $\bar{t}(q)$ | $\bar{t}(q)$ | $\bar{t}(q)$ | $\bar{t}(q)$ | $\bar{t}(q)$ |
|-----|--------------|--------------|--------------|--------------|--------------|--------------|
| 8681 | 450 | 8689 | 451 | 8693 | 450 | 8699 | 450 | 8707 | 451 |
| 8713 | 450 | 8719 | 451 | 8731 | 451 | 8737 | 451 | 8741 | 452 |
| 8747 | 452 | 8753 | 452 | 8761 | 452 | 8779 | 452 | 8783 | 452 |
| 8803 | 453 | 8807 | 453 | 8819 | 454 | 8821 | 454 | 8831 | 455 |
| 8837 | 454 | 8839 | 455 | 8849 | 455 | 8861 | 455 | 8863 | 456 |
| 8867 | 456 | 8887 | 455 | 8893 | 456 | 8923 | 456 | 8929 | 456 |
| 8933 | 457 | 8941 | 458 | 8951 | 458 | 8963 | 458 | 8969 | 458 |
| 8971 | 458 | 8999 | 459 | 9001 | 459 | 9007 | 458 | 9011 | 459 |
| 9013 | 459 | 9029 | 460 | 9041 | 461 | 9043 | 460 | 9049 | 460 |
| 9059 | 462 | 1.609 | 9067 | 461 | 9091 | 462 | 9103 | 462 | 9109 | 462 |
| 9127 | 463 | 9133 | 464 | 9137 | 464 | 9151 | 463 | 9157 | 464 |
| 9161 | 464 | 9173 | 464 | 9181 | 464 | 9187 | 465 | 9199 | 465 |
| 9203 | 465 | 9209 | 465 | 9221 | 466 | 9227 | 465 | 9239 | 466 |
| 9241 | 466 | 9257 | 467 | 9277 | 467 | 9281 | 468 | 9283 | 467 |
| 9293 | 468 | 9311 | 469 | 9319 | 469 | 9323 | 469 | 9337 | 470 | 1.609 |
| 9341 | 469 | 9343 | 470 | 9349 | 469 | 9371 | 470 | 9377 | 471 |
| 9391 | 471 | 9397 | 470 | 9403 | 470 | 9413 | 471 | 9419 | 471 |
| 9421 | 471 | 9431 | 473 | 1.611 | 9433 | 471 | 9437 | 471 | 9439 | 472 |
| 9461 | 473 | 9463 | 472 | 9467 | 473 | 9473 | 473 | 9479 | 473 |
| 9491 | 473 | 9497 | 474 | 9511 | 474 | 9521 | 474 | 9533 | 475 |
| 9539 | 475 | 9547 | 475 | 9551 | 475 | 9587 | 476 | 9601 | 475 |
| 9613 | 477 | 9619 | 477 | 9623 | 477 | 9629 | 478 | 9631 | 478 |
| 9643 | 477 | 9649 | 479 | 9661 | 478 | 9677 | 478 | 9679 | 479 |
| 9689 | 478 | 9697 | 479 | 9719 | 479 | 9721 | 480 | 9733 | 481 |
| 9739 | 480 | 9743 | 480 | 9749 | 480 | 9767 | 481 | 9769 | 480 |
| 9781 | 482 | 9787 | 481 | 9791 | 482 | 9803 | 482 | 9811 | 482 |
| 9817 | 482 | 9829 | 483 | 9833 | 483 | 9839 | 484 | 9851 | 484 |
| 9857 | 483 | 9859 | 483 | 9871 | 484 | 9883 | 484 | 9887 | 485 |
| 9901 | 485 | 9907 | 486 | 9923 | 485 | 9929 | 485 | 9931 | 485 |
| 9941 | 485 | 9949 | 487 | 9967 | 488 | 1.611 | 9973 | 487 | 10007 | 488 |
| 10009 | 487 | 10037 | 489 | 10039 | 489 | 10061 | 490 | 10067 | 490 |
| 10069 | 489 | 10079 | 490 | 10091 | 490 | 10093 | 490 | 10099 | 491 |
| 10103 | 490 | 10111 | 490 | 10133 | 492 | 10139 | 492 | 10141 | 492 |
| 10151 | 492 | 10159 | 492 | 10163 | 492 | 10169 | 492 | 10177 | 493 |
| 10181 | 493 | 10193 | 492 | 10211 | 494 | 10223 | 494 | 10243 | 495 |
| 10247 | 494 | 10253 | 495 | 10259 | 495 | 10267 | 494 | 10271 | 496 |
| 10273 | 494 | 10289 | 496 | 10301 | 496 | 10303 | 495 | 10313 | 496 |
| 10321 | 496 | 10331 | 498 | 1.612 | 10333 | 497 | 10337 | 496 | 10343 | 497 |
| 10357 | 497 | 10369 | 498 | 10391 | 499 | 10399 | 498 | 10427 | 499 |
| 10429 | 499 | 10433 | 500 | 10453 | 501 | 10457 | 500 | 10459 | 500 |
| 10463 | 501 | 10477 | 500 | 10487 | 501 | 10499 | 501 | 10501 | 503 | 1.614 |
| 10513 | 502 | 10529 | 503 | 10531 | 502 | 10559 | 503 | 10567 | 503 |
| 10589 | 503 | 10597 | 504 | 10601 | 504 | 10607 | 505 | 10613 | 504 |
| \( q \) | \( \bar{t}(q) < \) | \( \bar{t}(q) < \) | \( \bar{t}(q) < \) | \( \bar{t}(q) < \) | \( \bar{t}(q) < \) | \( \bar{t}(q) < \) |
|-----|--------|--------|--------|--------|--------|--------|
| 10627 | 505 | 10631 | 505 | 10639 | 504 | 10651 | 505 | 10657 | 505 |
| 10663 | 507 | 10667 | 505 | 10687 | 506 | 10691 | 506 | 10709 | 507 |
| 10711 | 507 | 10723 | 507 | 10729 | 507 | 10733 | 508 | 10739 | 507 |
| 10753 | 508 | 10771 | 509 | 10781 | 509 | 10789 | 509 | 10799 | 510 |
| 10831 | 510 | 10837 | 511 | 10847 | 511 | 10853 | 511 | 10859 | 511 |
| 10861 | 511 | 10867 | 511 | 10883 | 511 | 10889 | 512 | 10891 | 511 |
| 10903 | 512 | 10909 | 512 | 10937 | 513 | 10939 | 513 | 10949 | 513 |
| 10957 | 514 | 10973 | 514 | 10979 | 514 | 10987 | 515 | 10993 | 515 |
| 11003 | 515 | 11027 | 516 | 11047 | 516 | 11057 | 515 | 11059 | 516 |
| 11069 | 517 | 11071 | 516 | 11083 | 517 | 11087 | 517 | 11093 | 517 |
| 11113 | 518 | 11117 | 518 | 11119 | 518 | 11131 | 518 | 11149 | 519 |
| 11159 | 519 | 11161 | 519 | 11171 | 519 | 11173 | 518 | 11177 | 520 |
| 11197 | 521 | 11213 | 521 | 11239 | 522 | 11243 | 521 | 11251 | 521 |
| 11257 | 522 | 11261 | 521 | 11273 | 522 | 11279 | 522 | 11287 | 522 |
| 11299 | 523 | 11311 | 523 | 11317 | 523 | 11321 | 522 | 11329 | 523 |
| 11351 | 524 | 11353 | 525 | 11369 | 525 | 11383 | 525 | 11393 | 525 |
| 11399 | 525 | 11411 | 525 | 11423 | 526 | 11437 | 527 | 11443 | 527 |
| 11447 | 526 | 11467 | 528 | 11471 | 527 | 11483 | 528 | 11489 | 526 |
| 11491 | 528 | 11497 | 528 | 11503 | 527 | 11519 | 529 | 11527 | 529 |
| 11549 | 530 | 11551 | 528 | 11579 | 530 | 11587 | 530 | 11593 | 531 |
| 11597 | 530 | 11617 | 531 | 11621 | 531 | 11633 | 532 | 11657 | 532 |
| 11677 | 533 | 11681 | 533 | 11689 | 533 | 11699 | 533 | 11701 | 533 |
| 11717 | 534 | 11719 | 534 | 11731 | 534 | 11743 | 534 | 11777 | 536 1.614 |
| 11779 | 536 | 11783 | 536 | 11789 | 535 | 11801 | 536 | 11807 | 536 |
| 11813 | 536 | 11821 | 537 | 11827 | 537 | 11831 | 537 | 11833 | 536 |
| 11839 | 537 | 11863 | 537 | 11867 | 537 | 11887 | 538 | 11897 | 538 |
| 11903 | 538 | 11909 | 539 | 11923 | 539 | 11927 | 539 | 11933 | 539 |
| 11939 | 539 | 11941 | 539 | 11953 | 540 | 11959 | 539 | 11969 | 540 |
| 11971 | 540 | 11981 | 540 | 11987 | 541 | 12007 | 541 | 12011 | 541 |
| 12037 | 542 | 12041 | 542 | 12043 | 542 | 12049 | 542 | 12071 | 543 |
| 12073 | 542 | 12097 | 543 | 12101 | 544 | 12107 | 544 | 12109 | 544 |
| 12113 | 543 | 12119 | 544 | 12143 | 545 | 12149 | 544 | 12157 | 545 |
| 12161 | 545 | 12163 | 544 | 12197 | 546 | 12203 | 546 | 12211 | 547 1.614 |
| 12227 | 545 | 12239 | 546 | 12241 | 546 | 12251 | 547 | 12253 | 546 |
| 12263 | 547 | 12269 | 547 | 12277 | 548 | 12281 | 548 | 12289 | 548 |
| 12301 | 549 | 12323 | 549 | 12329 | 549 | 12343 | 550 | 12347 | 549 |
| 12373 | 549 | 12377 | 550 | 12379 | 550 | 12391 | 551 | 12401 | 550 |
| 12409 | 551 | 12413 | 551 | 12421 | 552 | 12433 | 551 | 12437 | 552 |
| 12451 | 552 | 12457 | 552 | 12473 | 553 | 12479 | 553 | 12487 | 552 |
| 12491 | 554 1.614 | 12497 | 553 | 12503 | 553 | 12511 | 554 | 12517 | 554 |
| 12527 | 554 | 12539 | 555 | 12541 | 555 | 12547 | 554 | 12553 | 555 |
| 12569 | 555 | 12577 | 555 | 12583 | 555 | 12589 | 555 | 12601 | 556 |
| 12611 | 556 | 12613 | 556 | 12619 | 556 | 12637 | 557 | 12641 | 556 |
Table 4. Continue 7

| \( q \) | \( t(q) < \) | \( q \) | \( t(q) < \) | \( q \) | \( t(q) < \) | \( q \) | \( t(q) < \) | \( q \) | \( t(q) < \) |
|---|---|---|---|---|---|---|---|---|---|
| 12647 | 556 | 12653 | 558 | 12659 | 557 | 12671 | 557 | 12689 | 558 |
| 12697 | 558 | 12703 | 558 | 12713 | 559 | 12721 | 559 | 12739 | 559 |
| 12743 | 559 | 12757 | 559 | 12763 | 559 | 12781 | 561 | 12791 | 561 |
| 12799 | 560 | 12809 | 561 | 12821 | 561 | 12823 | 562 | 12829 | 561 |
| 12841 | 560 | 12853 | 562 | 12889 | 563 | 12893 | 563 | 12899 | 563 |
| 12907 | 564 | 12911 | 564 | 12917 | 563 | 12919 | 563 | 12923 | 564 |
| 12941 | 565 | 12953 | 564 | 12959 | 565 | 12967 | 565 | 12973 | 565 |
| 12979 | 565 | 12983 | 565 | 13001 | 566 | 13003 | 565 | 13007 | 566 |
| 13009 | 565 | 13033 | 566 | 13037 | 566 | 13043 | 566 | 13049 | 567 |
| 13063 | 568 | 13093 | 568 | 13099 | 568 | 13103 | 568 | 13109 | 568 |
| 13121 | 569 | 13127 | 567 | 13147 | 569 | 13151 | 570 | 13159 | 569 |
| 13163 | 570 | 13171 | 570 | 13177 | 570 | 13183 | 570 | 13187 | 570 |
| 13217 | 573 | 13219 | 571 | 13229 | 571 | 13241 | 572 | 13249 | 572 |
| 13259 | 573 | 13267 | 572 | 13291 | 574 | 13297 | 573 | 13309 | 573 |
| 13313 | 573 | 13327 | 574 | 13331 | 573 | 13337 | 573 | 13339 | 573 |
| 13367 | 574 | 13381 | 575 | 13397 | 576 | 13399 | 576 | 13411 | 574 |
| 13417 | 576 | 13421 | 576 | 13441 | 577 | 13451 | 576 | 13457 | 578 |
| 13463 | 578 | 13469 | 576 | 13477 | 577 | 13487 | 578 | 13499 | 578 |
| 13513 | 579 | 13523 | 579 | 13537 | 578 | 13553 | 579 | 13567 | 580 |
| 13577 | 580 | 13591 | 581 | 13597 | 580 | 13613 | 580 | 13619 | 582 |
| 13627 | 581 | 13633 | 582 | 13649 | 581 | 13669 | 582 | 13679 | 582 |
| 13681 | 582 | 13687 | 582 | 13691 | 583 | 13693 | 583 | 13697 | 583 |
| 13709 | 583 | 13711 | 583 | 13721 | 582 | 13723 | 582 | 13729 | 583 |
| 13751 | 583 | 13757 | 584 | 13759 | 583 | 13763 | 584 | 13781 | 586 |
| 13789 | 584 | 13799 | 584 | 13807 | 586 | 13829 | 586 | 13831 | 586 |
| 13841 | 585 | 13859 | 586 | 13873 | 587 | 13877 | 586 | 13879 | 586 |
| 13883 | 587 | 13901 | 588 | 13903 | 587 | 13907 | 588 | 13913 | 588 |
| 13921 | 587 | 13931 | 589 | 13933 | 588 | 13963 | 590 | 13967 | 589 |
| 13997 | 590 | 13999 | 590 | 14009 | 590 | 14011 | 590 | 14029 | 591 |
| 14033 | 590 | 14051 | 591 | 14057 | 591 | 14071 | 591 | 14081 | 592 |
| 14083 | 592 | 14087 | 592 | 14107 | 592 | 14143 | 594 | 14149 | 593 |
| 14153 | 593 | 14159 | 593 | 14173 | 595 | 14177 | 595 | 14197 | 593 |
| 14207 | 594 | 14221 | 595 | 14243 | 596 | 14249 | 596 | 14251 | 595 |
| 14281 | 596 | 14293 | 597 | 14303 | 597 | 14321 | 597 | 14323 | 597 |
| 14327 | 599 | 14341 | 598 | 14347 | 598 | 14369 | 599 | 14387 | 600 |
| 14389 | 599 | 14401 | 600 | 14407 | 599 | 14411 | 600 | 14419 | 599 |
| 14423 | 600 | 14431 | 601 | 14437 | 601 | 14447 | 600 | 14449 | 600 |
| 14461 | 602 | 14479 | 601 | 14489 | 603 | 14503 | 601 | 14519 | 602 |
| 14533 | 603 | 14537 | 602 | 14543 | 603 | 14549 | 603 | 14551 | 603 |
| 14557 | 603 | 14561 | 604 | 14563 | 603 | 14591 | 604 | 14593 | 603 |
| 14621 | 605 | 14627 | 605 | 14629 | 604 | 14633 | 605 | 14639 | 605 |
| 14653 | 605 | 14657 | 605 | 14669 | 606 | 14683 | 605 | 14699 | 607 |
| 14713 | 607 | 14717 | 607 | 14723 | 607 | 14731 | 607 | 14737 | 607 |
| \( q \) | \( \bar{t}(q) \) | \( q \) | \( \bar{t}(q) \) | \( q \) | \( \bar{t}(q) \) | \( q \) | \( \bar{t}(q) \) | \( q \) | \( \bar{t}(q) \) |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 14741 | 607 | 14747 | 608 | 14753 | 608 | 14759 | 608 | 14767 | 608 |
| 14771 | 607 | 14779 | 609 | 14783 | 609 | 14797 | 608 | 14813 | 609 |
| 14821 | 609 | 14827 | 609 | 14831 | 610 | 14843 | 610 | 14851 | 610 |
| 14867 | 610 | 14869 | 610 | 14879 | 610 | 14887 | 611 | 14891 | 611 |
| 14897 | 611 | 14923 | 611 | 14929 | 611 | 14939 | 613 | 14947 | 613 |
| 14951 | 612 | 14957 | 612 | 14969 | 612 | 14983 | 613 | 15013 | 614 |
| 15017 | 614 | 15031 | 614 | 15053 | 614 | 15061 | 614 | 15073 | 615 |
| 15077 | 614 | 15083 | 615 | 15091 | 616 | 15101 | 617 | 15107 | 616 |
| 15121 | 616 | 15131 | 617 | 15137 | 617 | 15139 | 616 | 15149 | 618 |
| 15161 | 617 | 15173 | 618 | 15187 | 618 | 15193 | 618 | 15199 | 617 |
| 15217 | 618 | 15227 | 619 | 15233 | 618 | 15241 | 620 | 15259 | 620 |
| 15263 | 620 | 15269 | 619 | 15271 | 620 | 15277 | 619 | 15287 | 621 |
| 15289 | 621 | 15299 | 621 | 15307 | 621 | 15313 | 621 | 15319 | 621 |
| 15329 | 621 | 15331 | 622 | 15349 | 621 | 15359 | 622 | 15361 | 621 |
| 15373 | 622 | 15377 | 623 | 15383 | 622 | 15391 | 623 | 15401 | 622 |
| 15413 | 623 | 15427 | 623 | 15439 | 623 | 15443 | 623 | 15451 | 624 |
| 15461 | 625 | 15467 | 624 | 15473 | 624 | 15493 | 626 | 15497 | 625 |
| 15511 | 625 | 15527 | 626 | 15541 | 625 | 15551 | 626 | 15559 | 626 |
| 15569 | 626 | 15581 | 627 | 15583 | 626 | 15601 | 628 | 15607 | 628 |
| 15619 | 627 | 15629 | 628 | 15641 | 629 | 15643 | 628 | 15647 | 628 |
| 15649 | 628 | 15661 | 629 | 15667 | 629 | 15671 | 628 | 15679 | 628 |
| 15683 | 630 | 15727 | 630 | 15731 | 630 | 15733 | 630 | 15737 | 631 |
| 15739 | 630 | 15749 | 631 | 15761 | 631 | 15767 | 630 | 15773 | 631 |
| 15787 | 631 | 15791 | 631 | 15797 | 631 | 15803 | 632 | 15809 | 633 |
| 15817 | 632 | 15823 | 631 | 15859 | 634 | 15871 | 634 | 15881 | 633 |
| 15887 | 634 | 15889 | 634 | 15901 | 635 | 15907 | 634 | 15913 | 635 |
| 15919 | 635 | 15923 | 635 | 15937 | 635 | 15959 | 634 | 15971 | 636 |
| 15973 | 635 | 15991 | 636 | 16001 | 636 | 16007 | 637 | 16033 | 637 |
| 16057 | 638 | 16061 | 637 | 16063 | 637 | 16067 | 638 | 16069 | 638 |
| 16073 | 638 | 16087 | 639 | 16091 | 637 | 16097 | 639 | 16103 | 639 |
| 16111 | 639 | 16127 | 639 | 16139 | 640 | 16141 | 640 | 16183 | 641 |
| 16187 | 641 | 16189 | 641 | 16193 | 640 | 16217 | 641 | 16223 | 641 |
| 16229 | 641 | 16231 | 642 | 16249 | 642 | 16253 | 642 | 16267 | 642 |
| 16273 | 642 | 16301 | 643 | 16319 | 643 | 16333 | 644 | 16339 | 644 |
| 16349 | 645 | 16361 | 644 | 16363 | 645 | 16369 | 645 | 16381 | 645 |
| 16411 | 645 | 16417 | 646 | 16421 | 645 | 16427 | 646 | 16433 | 645 |
| 16447 | 647 | 16451 | 646 | 16453 | 647 | 16477 | 646 | 16481 | 647 |
| 16487 | 648 | 16493 | 646 | 16519 | 648 | 16529 | 647 | 16547 | 648 |
| 16553 | 649 | 16561 | 648 | 16567 | 648 | 16573 | 649 | 16603 | 649 |
| 16607 | 650 | 16619 | 650 | 16631 | 650 | 16633 | 651 | 16649 | 651 |
| 16651 | 651 | 16657 | 651 | 16661 | 651 | 16673 | 652 | 16691 | 650 |
| 16693 | 651 | 16699 | 652 | 16703 | 652 | 16729 | 653 | 16741 | 653 |
| 16747 | 653 | 16759 | 654 | 16763 | 653 | 16787 | 654 | 16811 | 654 |
**Table 4. Continue 9**

| $q$  | $\text{Tr}(q)$ | $q$  | $\text{Tr}(q)$ | $q$  | $\text{Tr}(q)$ | $q$  | $\text{Tr}(q)$ | $q$  | $\text{Tr}(q)$ |
|------|----------------|------|----------------|------|----------------|------|----------------|------|----------------|
| 16823 654 | < | 16829 654 | < | 16831 654 | < | 16843 655 | < | 16871 656 |
| 16879 656 | < | 16883 655 | < | 16889 655 | < | 16901 656 | < | 16903 657 |
| 16921 657 | < | 16927 657 | < | 16931 657 | < | 16937 657 | < | 16943 656 |
| 16963 657 | < | 16979 657 | < | 16981 658 | < | 16987 658 | < | 16993 657 |
| 17011 659 | < | 17021 659 | < | 17027 659 | < | 17029 658 | < | 17033 659 |
| 17041 659 | < | 17047 661 1.622 | < | 17053 660 | < | 17077 660 | < | 17093 661 |
| 17099 661 | < | 17107 660 | < | 17117 660 | < | 17123 660 | < | 17137 661 |
| 17159 662 | < | 17167 663 | < | 17183 662 | < | 17189 664 | < | 17191 662 |
| 17203 664 | < | 17207 662 | < | 17209 662 | < | 17231 664 | < | 17239 663 |
| 17257 664 | < | 17291 664 | < | 17293 665 | < | 17299 665 | < | 17317 665 |
| 17321 666 | < | 17327 666 | < | 17333 666 | < | 17341 666 | < | 17351 666 |
| 17359 666 | < | 17377 667 | < | 17383 667 | < | 17387 667 | < | 17389 667 |
| 17393 667 | < | 17401 668 | < | 17417 668 | < | 17419 668 | < | 17431 667 |
| 17443 668 | < | 17449 670 1.623 | < | 17467 669 | < | 17471 669 | < | 17477 668 |
| 17483 669 | < | 17489 668 | < | 17491 669 | < | 17497 666 | < | 17509 670 |
| 17519 671 | < | 17539 669 | < | 17551 671 | < | 17569 671 | < | 17573 670 |
| 17579 671 | < | 17581 670 | < | 17597 671 | < | 17599 672 | < | 17609 672 |
| 17623 672 | < | 17627 673 | < | 17657 672 | < | 17659 673 | < | 17669 673 |
| 17681 673 | < | 17683 673 | < | 17707 674 | < | 17713 674 | < | 17729 674 |
| 17737 674 | < | 17747 674 | < | 17749 674 | < | 17761 675 | < | 17783 676 |
| 17789 675 | < | 17791 676 | < | 17807 676 | < | 17827 676 | < | 17837 676 |
| 17839 676 | < | 17851 676 | < | 17863 677 | < | 17881 677 | < | 17891 677 |
| 17903 678 | < | 17909 678 | < | 17911 678 | < | 17921 679 | < | 17923 678 |
| 17929 677 | < | 17939 679 | < | 17957 679 | < | 17959 680 | < | 17971 679 |
| 17977 679 | < | 17981 679 | < | 17987 680 | < | 17989 680 | < | 18013 679 |
| 18041 681 | < | 18043 682 | < | 18047 681 | < | 18049 681 | < | 18059 682 |
| 18061 681 | < | 18077 681 | < | 18089 682 | < | 18097 683 | < | 18119 682 |
| 18121 682 | < | 18127 683 | < | 18131 683 | < | 18133 683 | < | 18143 683 |
| 18149 682 | < | 18169 684 | < | 18181 683 | < | 18191 683 | < | 18199 683 |
| 18211 685 | < | 18217 684 | < | 18223 685 | < | 18229 685 | < | 18233 685 |
| 18251 685 | < | 18253 685 | < | 18257 685 | < | 18269 685 | < | 18287 686 |
| 18289 687 | < | 18301 686 | < | 18307 686 | < | 18311 687 | < | 18313 686 |
| 18329 687 | < | 18341 688 | < | 18353 687 | < | 18367 688 | < | 18371 687 |
| 18379 687 | < | 18397 688 | < | 18401 689 | < | 18413 689 | < | 18427 688 |
| 18433 689 | < | 18439 689 | < | 18443 689 | < | 18451 688 | < | 18457 690 |
| 18461 690 | < | 18481 689 | < | 18493 691 | < | 18503 691 | < | 18517 691 |
| 18521 692 | < | 18523 690 | < | 18539 691 | < | 18541 691 | < | 18553 691 |
| 18583 692 | < | 18587 693 | < | 18593 692 | < | 18617 693 | < | 18637 692 |
| 18661 693 | < | 18671 694 | < | 18679 695 | < | 18691 696 1.624 | < | 18701 694 |
| 18713 696 | < | 18719 694 | < | 18731 696 | < | 18743 695 | < | 18749 695 |
| 18757 696 | < | 18773 697 | < | 18787 697 | < | 18793 697 | < | 18797 697 |
| 18803 697 | < | 18839 699 | < | 18859 698 | < | 18869 698 | < | 18899 698 |
| 18911 699 | < | 18913 698 | < | 18917 699 | < | 18919 699 | < | 18947 700 |
| $q$ | $\bar{t}(q)$ | $q$ | $\bar{t}(q)$ | $q$ | $\bar{t}(q)$ | $q$ | $\bar{t}(q)$ | $q$ | $\bar{t}(q)$ |
|-----|-------------|-----|-------------|-----|-------------|-----|-------------|-----|-------------|
| 18959 | 700 | 18973 | 700 | 18979 | 701 | 19001 | 702 | 19009 | 701 |
| 19013 | 701 | 19031 | 702 | 19037 | 702 | 19051 | 702 | 19069 | 702 |
| 19073 | 702 | 19079 | 703 | 19081 | 703 | 19087 | 703 | 19121 | 704 |
| 19139 | 704 | 19141 | 705 | 19157 | 704 | 19163 | 703 | 19181 | 706 |
| 19183 | 704 | 19207 | 706 | 19211 | 705 | 19213 | 705 | 19219 | 706 |
| 19231 | 705 | 19237 | 705 | 19249 | 706 | 19259 | 707 | 19267 | 706 |
| 19273 | 707 | 19289 | 708 | 19301 | 708 | 19309 | 707 | 19319 | 708 |
| 19333 | 707 | 19373 | 709 | 19379 | 709 | 19381 | 709 | 19387 | 709 |
| 19391 | 709 | 19403 | 709 | 19417 | 710 | 19421 | 709 | 19423 | 710 |
| 19427 | 710 | 19429 | 711 | 19433 | 710 | 19441 | 709 | 19447 | 709 |
| 19457 | 712 | 19463 | 710 | 19469 | 711 | 19471 | 710 | 19477 | 711 |
| 19483 | 711 | 19489 | 711 | 19501 | 712 | 19507 | 710 | 19531 | 712 |
| 19541 | 712 | 19543 | 713 | 19553 | 713 | 19559 | 712 | 19571 | 713 |
| 19577 | 713 | 19583 | 713 | 19597 | 712 | 19603 | 714 | 19609 | 713 |
| 19661 | 714 | 19681 | 714 | 19687 | 715 | 19697 | 715 | 19699 | 716 |
| 19709 | 716 | 19717 | 716 | 19727 | 716 | 19739 | 716 | 19751 | 717 |
| 19753 | 717 | 19759 | 717 | 19763 | 717 | 19777 | 716 | 19793 | 717 |
| 19801 | 719 | 19813 | 718 | 19819 | 719 | 19841 | 718 | 19843 | 718 |
| 19853 | 719 | 19861 | 719 | 19867 | 719 | 19889 | 720 | 19891 | 720 |
| 19913 | 719 | 19919 | 721 | 19927 | 720 | 19937 | 720 | 19949 | 721 |
| 19961 | 721 | 19963 | 721 | 19973 | 720 | 19979 | 721 | 19991 | 721 |
| 19993 | 722 | 19997 | 722 | 20011 | 722 | 20021 | 721 | 20023 | 722 |
| 20029 | 722 | 20047 | 723 | 20051 | 722 | 20063 | 722 | 20071 | 723 |
| 20089 | 723 | 20101 | 724 | 20107 | 724 | 20113 | 724 | 20117 | 724 |
| 20123 | 723 | 20129 | 725 | 20143 | 723 | 20147 | 724 | 20149 | 725 |
| 20161 | 725 | 20173 | 726 | 20177 | 725 | 20183 | 726 | 20201 | 725 |
| 20219 | 726 | 20231 | 727 | 20233 | 727 | 20249 | 727 | 20261 | 727 |
| 20269 | 727 | 20287 | 726 | 20297 | 728 | 20323 | 728 | 20327 | 729 |
| 20333 | 728 | 20341 | 729 | 20347 | 728 | 20353 | 729 | 20357 | 728 |
| 20359 | 728 | 20369 | 729 | 20389 | 730 | 20393 | 729 | 20399 | 730 |
| 20407 | 729 | 20411 | 729 | 20431 | 731 | 20441 | 729 | 20443 | 730 |
| 20477 | 731 | 20479 | 731 | 20483 | 731 | 20507 | 732 | 20509 | 732 |
| 20521 | 733 | 20533 | 732 | 20543 | 732 | 20549 | 732 | 20551 | 733 |
| 20563 | 733 | 20593 | 733 | 20599 | 734 | 20611 | 733 | 20627 | 734 |
| 20639 | 735 | 20641 | 734 | 20663 | 734 | 20681 | 735 | 20693 | 736 |
| 20707 | 736 | 20717 | 735 | 20719 | 736 | 20731 | 736 | 20743 | 737 |
| 20747 | 737 | 20749 | 738 | 20753 | 736 | 20759 | 737 | 20771 | 737 |
| 20773 | 737 | 20789 | 738 | 20807 | 737 | 20809 | 738 | 20849 | 739 |
| 20857 | 740 | 20873 | 740 | 20879 | 740 | 20887 | 739 | 20897 | 739 |
| 20899 | 740 | 20903 | 740 | 20921 | 741 | 20929 | 740 | 20939 | 740 |
| 20947 | 741 | 20959 | 740 | 20963 | 743 | 20981 | 741 | 20983 | 741 |
| 21001 | 742 | 21011 | 741 | 21013 | 742 | 21017 | 741 | 21019 | 741 |
| 21023 | 742 | 21031 | 742 | 21059 | 743 | 21061 | 742 | 21067 | 743 |
Table 4. Continue 11

| q    | $\bar{t}(q)$ | $\overline{T}(q)$ | q    | $\bar{t}(q)$ | $\overline{T}(q)$ | q    | $\bar{t}(q)$ | $\overline{T}(q)$ | q    | $\bar{t}(q)$ | $\overline{T}(q)$ |
|------|--------------|-------------------|------|--------------|-------------------|------|--------------|-------------------|------|--------------|-------------------|
| 21089| 745          |                   | 21101| 745          |                   | 21107| 743          |                   | 21121| 745          |                   |
| 21143| 746          |                   | 21149| 743          |                   | 21157| 745          |                   | 21163| 745          |                   |
| 21179| 745          |                   | 21187| 746          |                   | 21191| 745          |                   | 21193| 745          |                   |
| 21221| 746          |                   | 21227| 746          |                   | 21247| 746          |                   | 21269| 747          |                   |
| 21283| 748          |                   | 21313| 749          |                   | 21317| 748          |                   | 21319| 747          |                   |
| 21341| 749          |                   | 21347| 748          |                   | 21377| 749          |                   | 21379| 749          |                   |
| 21391| 748          |                   | 21397| 749          |                   | 21401| 750          |                   | 21407| 750          |                   |
| 21433| 751          |                   | 21467| 750          |                   | 21481| 750          |                   | 21487| 751          |                   |
| 21493| 751          |                   | 21499| 752          |                   | 21503| 752          |                   | 21517| 752          |                   |
| 21523| 752          |                   | 21529| 752          |                   | 21557| 752          |                   | 21559| 753          |                   |
| 21569| 753          |                   | 21577| 752          |                   | 21587| 753          |                   | 21589| 754          |                   |
| 21601| 753          |                   | 21611| 754          |                   | 21613| 755          |                   | 21617| 754          |                   |
| 21649| 755          |                   | 21661| 754          |                   | 21673| 755          |                   | 21683| 756          |                   |
| 21713| 756          |                   | 21727| 755          |                   | 21737| 756          |                   | 21739| 756          |                   |
| 21757| 756          |                   | 21767| 758          |                   | 21773| 757          |                   | 21787| 758          |                   |
| 21803| 758          |                   | 21817| 758          |                   | 21821| 757          |                   | 21839| 759          |                   |
| 21851| 758          |                   | 21859| 757          |                   | 21863| 760          |                   | 21871| 759          |                   |
| 21893| 759          |                   | 21911| 759          |                   | 21929| 760          |                   | 21937| 758          |                   |
| 21961| 760          |                   | 21977| 761          |                   | 21991| 759          |                   | 21997| 760          |                   |
| 22013| 761          |                   | 22027| 761          |                   | 22031| 762          |                   | 22037| 762          |                   |
| 22051| 763          |                   | 22063| 763          |                   | 22067| 761          |                   | 22073| 762          |                   |
| 22091| 763          |                   | 22093| 763          |                   | 22109| 762          |                   | 22111| 763          |                   |
| 22129| 764          |                   | 22133| 764          |                   | 22147| 763          |                   | 22153| 764          |                   |
| 22159| 764          |                   | 22171| 764          |                   | 22189| 766          |                   | 22193| 765          |                   |
| 22247| 766          |                   | 22259| 765          |                   | 22271| 767          |                   | 22273| 766          |                   |
| 22279| 766          |                   | 22283| 766          |                   | 22291| 767          |                   | 22303| 768          |                   |
| 22343| 768          |                   | 22349| 768          |                   | 22367| 768          |                   | 22369| 768          |                   |
| 22391| 769          |                   | 22397| 769          |                   | 22409| 770          |                   | 22433| 770          |                   |
| 22447| 770          |                   | 22453| 770          |                   | 22469| 769          |                   | 22481| 771          |                   |
| 22501| 771          |                   | 22511| 771          |                   | 22531| 772          |                   | 22541| 771          |                   |
| 22549| 773          |                   | 22567| 773          |                   | 22571| 773          |                   | 22573| 773          |                   |
| 22619| 773          |                   | 22621| 773          |                   | 22637| 773          |                   | 22639| 774          |                   |
| 22651| 773          |                   | 22669| 775          |                   | 22679| 774          |                   | 22691| 774          |                   |
| 22699| 774          |                   | 22709| 776          |                   | 22717| 775          |                   | 22721| 775          |                   |
| 22739| 775          |                   | 22741| 776          |                   | 22751| 775          |                   | 22769| 777          |                   |
| 22783| 775          |                   | 22787| 777          |                   | 22807| 778          |                   | 22811| 777          |                   |
| 22853| 778          |                   | 22859| 777          |                   | 22861| 778          |                   | 22871| 778          |                   |
| 22901| 777          |                   | 22907| 779          |                   | 22921| 779          |                   | 22937| 779          |                   |
| 22961| 780          |                   | 22963| 780          |                   | 22973| 780          |                   | 22993| 780          |                   |
| 23011| 782          |                   | 23017| 781          |                   | 23021| 782          |                   | 23027| 781          |                   |
| 23039| 781          |                   | 23041| 781          |                   | 23053| 782          |                   | 23057| 781          |                   |
| 23063| 782          |                   | 23071| 782          |                   | 23081| 782          |                   | 23087| 783          |                   |
| 23117| 783          |                   | 23131| 783          |                   | 23143| 783          |                   | 23159| 783          |                   |
| 23167| 784          |                   | 23169| 784          |                   | 23179| 784          |                   | 23187| 784          |                   |
### Table 4. Continue 12

|   | $\bar{t}(q)$ | $\bar{t}^{(q)}$ | $\bar{t}^{(q)}$ | $\bar{t}^{(q)}$ | $\bar{t}^{(q)}$ | $\bar{t}^{(q)}$ |
|---|--------------|------------------|------------------|------------------|------------------|------------------|
| 23173 | 784 | 23189 | 784 | 23197 | 785 | 23201 | 784 | 23203 | 784 |
| 23209 | 785 | 23227 | 785 | 23251 | 786 | 23269 | 786 | 23279 | 785 |
| 23291 | 786 | 23323 | 787 | 23339 | 787 | 23357 | 788 | 23369 | 788 |
| 23371 | 788 | 23399 | 789 | 23417 | 788 | 23431 | 788 | 23447 | 789 |
| 23459 | 789 | 23473 | 789 | 23497 | 791 | 23509 | 790 | 23531 | 791 |
| 23537 | 791 | 23539 | 791 | 23549 | 792 | 23557 | 793 | 1.629 | 23561 | 792 |
| 23563 | 792 | 23567 | 792 | 23581 | 791 | 23593 | 792 | 23599 | 792 |
| 23603 | 793 | 23609 | 793 | 23623 | 792 | 23627 | 793 | 23629 | 793 |
| 23633 | 792 | 23663 | 792 | 23669 | 793 | 23671 | 793 | 23677 | 793 |
| 23687 | 794 | 23689 | 794 | 23719 | 794 | 23741 | 794 | 23743 | 794 |
| 23747 | 795 | 23753 | 795 | 23761 | 793 | 23767 | 795 | 23773 | 795 |
| 23789 | 796 | 23801 | 796 | 23813 | 795 | 23819 | 796 | 23827 | 798 | 1.629 |
| 23831 | 796 | 23833 | 797 | 23857 | 796 | 23869 | 796 | 23873 | 797 |
| 23879 | 797 | 23887 | 797 | 23893 | 798 | 23899 | 798 | 23909 | 798 |
| 23911 | 797 | 23917 | 798 | 23929 | 799 | 23957 | 798 | 23971 | 799 |
| 23977 | 799 | 23981 | 800 | 23993 | 800 | 24001 | 800 | 24007 | 799 |
| 24019 | 799 | 24023 | 800 | 24029 | 799 | 24043 | 800 | 24049 | 800 |
| 24061 | 800 | 24071 | 801 | 24077 | 802 | 24083 | 801 | 24091 | 801 |
| 24097 | 801 | 24103 | 802 | 24107 | 802 | 24109 | 802 | 24113 | 803 |
| 24121 | 801 | 24133 | 802 | 24137 | 803 | 24151 | 802 | 24169 | 803 |
| 24179 | 803 | 24181 | 803 | 24197 | 803 | 24203 | 803 | 24223 | 804 |
| 24229 | 802 | 24239 | 804 | 24247 | 804 | 24251 | 805 | 24281 | 805 |
| 24317 | 805 | 24329 | 806 | 24337 | 805 | 24359 | 806 | 24371 | 807 |
| 24373 | 807 | 24379 | 806 | 24391 | 808 | 24407 | 808 | 24413 | 807 |
| 24419 | 807 | 24421 | 808 | 24439 | 808 | 24443 | 807 | 24469 | 808 |
| 24473 | 809 | 24481 | 808 | 24499 | 809 | 24509 | 808 | 24517 | 810 |
| 24527 | 809 | 24533 | 809 | 24547 | 808 | 24551 | 811 | 24571 | 810 |
| 24593 | 811 | 24611 | 811 | 24623 | 811 | 24631 | 811 | 24659 | 812 |
| 24671 | 812 | 24677 | 812 | 24683 | 812 | 24691 | 813 | 24697 | 813 |
| 24709 | 813 | 24733 | 812 | 24749 | 814 | 24763 | 814 | 24767 | 814 |
| 24781 | 814 | 24793 | 815 | 24799 | 814 | 24809 | 814 | 24821 | 816 |
| 24841 | 816 | 24847 | 815 | 24851 | 815 | 24859 | 815 | 24877 | 815 |
| 24889 | 816 | 24907 | 816 | 24917 | 817 | 24919 | 817 | 24923 | 817 |
| 24943 | 817 | 24953 | 818 | 24967 | 818 | 24971 | 818 | 24977 | 818 |
| 24979 | 817 | 24989 | 817 | 25013 | 818 | 25031 | 818 | 25033 | 820 | 1.629 |
| 25037 | 818 | 25057 | 820 | 25073 | 819 | 25087 | 821 | 1.629 | 25097 | 818 |
| 25111 | 821 | 25117 | 820 | 25121 | 820 | 25127 | 821 | 25147 | 820 |
| 25153 | 820 | 25163 | 822 | 25169 | 820 | 25171 | 821 | 25183 | 820 |
| 25189 | 822 | 25219 | 821 | 25229 | 823 | 25237 | 823 | 25243 | 824 | 1.629 |
| 25247 | 822 | 25253 | 822 | 25261 | 822 | 25301 | 823 | 25303 | 824 |
| 25307 | 824 | 25309 | 823 | 25321 | 825 | 25339 | 824 | 25343 | 825 |
| 25349 | 824 | 25357 | 825 | 25367 | 824 | 25373 | 825 | 25391 | 825 |
| \( q \) | \( \bar{r}(q) < \) | \( q \) | \( \bar{r}(q) < \) | \( q \) | \( \bar{r}(q) < \) | \( q \) | \( \bar{r}(q) < \) |
|---|---|---|---|---|---|---|---|
| 25409 | 825 | 25411 | 826 | 25423 | 827 | 25439 | 827 |
| 25453 | 827 | 25457 | 826 | 25463 | 826 | 25469 | 827 |
| 25523 | 829 | 25537 | 827 | 25541 | 829 | 25561 | 828 |
| 25579 | 829 | 25583 | 829 | 25589 | 828 | 25601 | 830 |
| 25609 | 830 | 25621 | 830 | 25633 | 831 | 1.630 | 25639 |
| 25657 | 831 | 25667 | 830 | 25673 | 830 | 25679 | 830 |
| 25699 | 835 | 25699 | 835 | 25699 | 835 | 25703 | 838 |
| 25703 | 831 | 25717 | 831 | 25733 | 832 | 25741 | 834 |
| 25759 | 832 | 25763 | 831 | 25771 | 831 | 25793 | 833 |
| 25801 | 833 | 25819 | 834 | 25841 | 833 | 25847 | 833 |
| 25867 | 834 | 25873 | 833 | 25889 | 835 | 25903 | 835 |
| 25919 | 836 | 25931 | 836 | 25933 | 836 | 25939 | 835 |
| 25951 | 835 | 25969 | 835 | 25981 | 835 | 25997 | 835 |
| 26003 | 836 | 26017 | 836 | 26021 | 837 | 26029 | 838 |
| 26053 | 837 | 26083 | 837 | 26099 | 837 | 26107 | 838 |
| 26113 | 838 | 26119 | 839 | 26141 | 839 | 26153 | 839 |
| 26171 | 840 | 26177 | 841 | 26183 | 840 | 26189 | 841 |
| 26209 | 841 | 26227 | 840 | 26237 | 841 | 26249 | 841 |
| 26261 | 840 | 26263 | 841 | 26267 | 841 | 26293 | 841 |
| 26309 | 841 | 26317 | 843 | 26321 | 841 | 26339 | 843 |
| 26357 | 841 | 26371 | 843 | 26387 | 844 | 26393 | 843 |
| 26407 | 844 | 26417 | 844 | 26423 | 844 | 26431 | 844 |
| 26449 | 845 | 26459 | 845 | 26479 | 845 | 26489 | 845 |
| 26501 | 846 | 26513 | 846 | 26539 | 846 | 26557 | 846 |
| 26573 | 847 | 26591 | 848 | 26597 | 847 | 26627 | 848 |
| 26641 | 848 | 26647 | 847 | 26669 | 849 | 26681 | 849 |
| 26687 | 849 | 26693 | 850 | 26699 | 849 | 26701 | 848 |
| 26713 | 849 | 26717 | 850 | 26723 | 850 | 26729 | 848 |
| 26737 | 849 | 26759 | 851 | 26777 | 851 | 26783 | 850 |
| 26813 | 851 | 26821 | 850 | 26833 | 851 | 26839 | 852 |
| 26861 | 851 | 26863 | 853 | 26879 | 852 | 26881 | 850 |
| 26893 | 852 | 26903 | 853 | 26921 | 852 | 26927 | 853 |
| 26951 | 853 | 26953 | 854 | 26959 | 855 | 26981 | 854 |
| 26993 | 854 | 27011 | 854 | 27017 | 856 | 27031 | 855 |
| 27059 | 856 | 27061 | 854 | 27067 | 856 | 27073 | 856 |
| 27091 | 856 | 27103 | 856 | 27107 | 856 | 27109 | 856 |
| 27143 | 857 | 27179 | 856 | 27191 | 856 | 27197 | 857 |
| 27239 | 859 | 27241 | 859 | 27253 | 858 | 27259 | 859 |
| 27277 | 858 | 27281 | 859 | 27283 | 860 | 27299 | 859 |
| 27337 | 860 | 27361 | 862 | 27367 | 861 | 27397 | 861 |
| 27409 | 861 | 27427 | 861 | 27431 | 862 | 27437 | 862 |
| 27457 | 862 | 27479 | 862 | 27481 | 863 | 27487 | 863 |
| 27527 | 864 | 27529 | 863 | 27539 | 864 | 27541 | 864 |
| 27581 | 865 | 27583 | 864 | 27611 | 866 | 27617 | 864 |
Table 4. Continue 14

| $q$ | $\tilde{t}(q)$ | $\tilde{t}(q)$ | $\tilde{t}(q)$ | $\tilde{t}(q)$ | $\tilde{t}(q)$ |
|-----|----------------|----------------|----------------|----------------|----------------|
| 27647 | 864 | 27653 | 865 | 27673 | 867 | 27693 | 866 | 27691 | 866 |
| 27697 | 867 | 27701 | 866 | 27733 | 868 | 27737 | 867 | 27739 | 867 |
| 27743 | 867 | 27749 | 867 | 27751 | 867 | 27763 | 868 | 27767 | 868 |
| 27773 | 866 | 27779 | 869 | 27791 | 868 | 27793 | 869 | 27799 | 868 |
| 27803 | 869 | 27809 | 868 | 27817 | 868 | 27823 | 868 | 27827 | 868 |
| 27847 | 868 | 27851 | 869 | 27883 | 870 | 27893 | 869 | 27901 | 870 |
| 27917 | 871 | 27919 | 871 | 27941 | 871 | 27943 | 872 | 27947 | 870 |
| 27953 | 872 | 27961 | 872 | 27967 | 871 | 27983 | 872 | 27997 | 871 |
| 28001 | 872 | 28019 | 872 | 28027 | 873 | 28031 | 872 | 28051 | 873 |
| 28057 | 872 | 28069 | 874 | 28081 | 872 | 28087 | 875 | 1.632 | 28097 | 874 |
| 28099 | 873 | 28109 | 874 | 28111 | 874 | 28123 | 873 | 28151 | 876 |
| 28163 | 875 | 28181 | 875 | 28183 | 875 | 28201 | 876 | 28211 | 876 |
| 28219 | 875 | 28229 | 875 | 28277 | 876 | 28279 | 876 | 28283 | 876 |
| 28289 | 875 | 28297 | 876 | 28307 | 877 | 28309 | 878 | 28319 | 877 |
| 28349 | 876 | 28351 | 878 | 28387 | 879 | 28393 | 877 | 28403 | 878 |
| 28409 | 878 | 28411 | 879 | 28429 | 878 | 28433 | 880 | 28439 | 881 |
| 28447 | 880 | 28463 | 879 | 28477 | 880 | 28493 | 881 | 28499 | 881 |
| 28513 | 880 | 28517 | 881 | 28537 | 881 | 28541 | 881 | 28547 | 881 |
| 28549 | 882 | 28559 | 883 | 28571 | 882 | 28573 | 883 | 28579 | 883 |
| 28591 | 882 | 28597 | 882 | 28603 | 884 | 1.632 | 28607 | 882 | 28619 | 882 |
| 28621 | 882 | 28627 | 883 | 28631 | 884 | 28643 | 882 | 28649 | 884 |
| 28657 | 883 | 28661 | 883 | 28663 | 883 | 28669 | 883 | 28687 | 884 |
| 28697 | 884 | 28703 | 883 | 28711 | 885 | 28723 | 884 | 28729 | 884 |
| 28751 | 883 | 28753 | 885 | 28759 | 883 | 28771 | 885 | 28789 | 886 |
| 28793 | 885 | 28807 | 884 | 28813 | 887 | 28817 | 885 | 28837 | 887 |
| 28843 | 884 | 28859 | 886 | 28867 | 886 | 28871 | 886 | 28879 | 888 |
| 28901 | 888 | 28909 | 887 | 28921 | 889 | 28927 | 888 | 28933 | 887 |
| 28949 | 888 | 28961 | 888 | 28979 | 889 | 29009 | 888 | 29017 | 889 |
| 29021 | 890 | 29023 | 888 | 29027 | 890 | 29033 | 891 | 29059 | 891 |
| 29063 | 891 | 29077 | 889 | 29101 | 891 | 29123 | 892 | 29129 | 892 |
| 29131 | 890 | 29137 | 892 | 29147 | 891 | 29153 | 891 | 29167 | 893 |
| 29173 | 893 | 29179 | 893 | 29191 | 892 | 29201 | 894 | 29207 | 893 |
| 29209 | 893 | 29221 | 893 | 29231 | 892 | 29243 | 892 | 29251 | 894 |
| 29269 | 893 | 29287 | 894 | 29297 | 894 | 29303 | 894 | 29311 | 895 |
| 29327 | 894 | 29333 | 896 | 29339 | 895 | 29347 | 895 | 29363 | 895 |
| 29383 | 895 | 29387 | 896 | 29389 | 896 | 29399 | 895 | 29401 | 896 |
| 29411 | 895 | 29423 | 895 | 29429 | 897 | 29437 | 896 | 29443 | 896 |
| 29453 | 896 | 29473 | 898 | 29483 | 898 | 29501 | 898 | 29527 | 898 |
| 29531 | 897 | 29537 | 899 | 29567 | 898 | 29569 | 900 | 29573 | 900 |
| 29581 | 899 | 29587 | 900 | 29599 | 899 | 29611 | 899 | 29629 | 900 |
| 29633 | 900 | 29641 | 900 | 29663 | 900 | 29669 | 900 | 29671 | 901 |
| 29683 | 900 | 29717 | 901 | 29723 | 902 | 29741 | 902 | 29753 | 902 |
| 29759 | 902 | 29761 | 902 | 29789 | 902 | 29803 | 903 | 29819 | 903 |
| $q$ | $\bar{r}(q)$ | $\bar{r}(q)$ | $\bar{r}(q)$ | $\bar{r}(q)$ | $\bar{r}(q)$ | $\bar{r}(q)$ |
|-----|--------------|--------------|--------------|--------------|--------------|--------------|
| 29833 | 903 | 29837 | 903 | 29851 | 904 | 29863 | 905 | 29867 | 903 |
| 29873 | 905 | 29879 | 904 | 29881 | 903 | 29917 | 904 | 29921 | 905 |
| 29927 | 904 | 29947 | 905 | 29959 | 906 | 29983 | 906 | 29989 | 907 |
| 30011 | 907 | 30013 | 906 | 30029 | 907 | 30047 | 906 | 30059 | 908 |
| 30071 | 907 | 30089 | 908 | 30091 | 903 | 30117 | 904 | 30121 | 910 |
| 30109 | 909 | 30113 | 908 | 30119 | 909 | 30133 | 908 | 30137 | 909 |
| 30139 | 909 | 30161 | 910 | 30169 | 910 | 30181 | 909 | 30187 | 910 |
| 30197 | 910 | 30203 | 910 | 30211 | 910 | 30223 | 909 | 30241 | 910 |
| 30253 | 911 | 30259 | 911 | 30269 | 910 | 30271 | 911 | 30293 | 911 |
| 30307 | 912 | 30313 | 913 | 1.633 | 30319 | 912 | 30323 | 911 | 30341 | 911 |
| 30347 | 911 | 30367 | 913 | 30389 | 912 | 30391 | 914 | 30403 | 913 |
| 30427 | 914 | 30431 | 914 | 30449 | 913 | 30467 | 913 | 30469 | 915 |
| 30491 | 914 | 30493 | 914 | 30497 | 914 | 30509 | 915 | 30517 | 915 |
| 30529 | 915 | 30539 | 915 | 30553 | 915 | 30557 | 916 | 30559 | 916 |
| 30577 | 916 | 30593 | 915 | 30631 | 917 | 30637 | 917 | 30643 | 917 |
| 30649 | 916 | 30661 | 915 | 30671 | 917 | 30677 | 918 | 30689 | 917 |
| 30697 | 918 | 30703 | 917 | 30707 | 918 | 30713 | 917 | 30727 | 919 |
| 30757 | 919 | 30763 | 919 | 30773 | 920 | 30781 | 918 | 30803 | 921 |
| 30809 | 919 | 30817 | 921 | 30829 | 920 | 30839 | 921 | 30841 | 920 |
| 30851 | 919 | 30853 | 921 | 30859 | 922 | 1.633 | 30869 | 920 | 30871 | 921 |
| 30881 | 919 | 30893 | 922 | 30911 | 922 | 30931 | 922 | 30937 | 920 |
| 30941 | 922 | 30949 | 921 | 30971 | 922 | 30977 | 921 | 30983 | 923 |
| 31013 | 925 | 31019 | 923 | 31033 | 924 | 31039 | 923 | 31051 | 925 |
| 31063 | 925 | 31069 | 924 | 31079 | 925 | 31081 | 925 | 31091 | 924 |
| 31121 | 926 | 31123 | 926 | 31139 | 925 | 31147 | 924 | 31151 | 927 |
| 31153 | 925 | 31159 | 926 | 31177 | 926 | 31181 | 926 | 31183 | 927 |
| 31189 | 928 | 31193 | 927 | 31219 | 927 | 31223 | 927 | 31231 | 927 |
| 31237 | 928 | 31247 | 927 | 31249 | 928 | 31253 | 927 | 31259 | 927 |
| 31267 | 928 | 31271 | 928 | 31277 | 928 | 31307 | 928 | 31319 | 929 |
| 31321 | 929 | 31327 | 929 | 31333 | 930 | 31337 | 928 | 31357 | 928 |
| 31379 | 929 | 31387 | 930 | 31391 | 930 | 31393 | 930 | 31397 | 930 |
| 31469 | 931 | 31477 | 930 | 31481 | 931 | 31489 | 932 | 31511 | 930 |
| 31513 | 930 | 31517 | 931 | 31531 | 932 | 31541 | 932 | 31543 | 932 |
| 31547 | 933 | 31567 | 934 | 31573 | 933 | 31583 | 933 | 31601 | 933 |
| 31607 | 935 | 1.634 | 31627 | 933 | 31643 | 934 | 31649 | 934 | 31657 | 934 |
| 31663 | 935 | 31667 | 934 | 31687 | 936 | 31699 | 936 | 31721 | 934 |
| 31723 | 937 | 1.635 | 31727 | 935 | 31729 | 935 | 31741 | 936 | 31751 | 937 |
| 31769 | 937 | 31771 | 936 | 31793 | 937 | 31799 | 936 | 31817 | 936 |
| 31847 | 937 | 31849 | 938 | 31859 | 938 | 31873 | 938 | 31883 | 937 |
| 31891 | 938 | 31907 | 939 | 31957 | 939 | 31963 | 938 | 31973 | 939 |
| 31981 | 940 | 31991 | 940 | 32003 | 940 | 32009 | 941 | 32027 | 941 |
| 32029 | 939 | 32051 | 941 | 32057 | 941 | 32059 | 940 | 32063 | 941 |
| 32069 | 940 | 32077 | 941 | 32083 | 941 | 32089 | 941 | 32099 | 941 |
| $q$ | $\overline{t}(q)$ | $\overline{t}(q)$ | $\overline{t}(q)$ | $\overline{t}(q)$ | $\overline{t}(q)$ | $\overline{t}(q)$ |
|-----|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 32117 | 942 | 32141 | 941 | 32143 | 942 | 32159 | 943 |
| 32173 | 943 | 32183 | 943 | 32191 | 941 | 32203 | 944 |
| 32213 | 944 | 32233 | 944 | 32251 | 943 | 32257 | 945 |
| 32261 | 945 | 32297 | 945 | 32303 | 946 | 32309 | 946 |
| 32321 | 945 | 32323 | 945 | 32341 | 947 | 32377 | 946 |
| 32359 | 947 | 32363 | 946 | 32369 | 947 | 32371 | 947 |
| 32381 | 946 | 32401 | 947 | 32413 | 946 | 32423 | 946 |
| 32429 | 947 | 32441 | 946 | 32467 | 947 | 32479 | 948 |
| 32491 | 948 | 32497 | 948 | 32507 | 947 | 32531 | 949 |
| 32533 | 947 | 32537 | 947 | 32563 | 950 | 32569 | 950 |
| 32573 | 949 | 32579 | 949 | 32587 | 950 | 32603 | 950 |
| 32611 | 951 | 32621 | 950 | 32633 | 951 | 32653 | 951 |
| 32687 | 951 | 32693 | 951 | 32707 | 952 | 32713 | 952 |
| 32719 | 951 | 32749 | 953 | 32771 | 953 | 32779 | 953 |
| 32789 | 952 | 32797 | 953 | 32801 | 952 | 32803 | 953 |
| 32833 | 953 | 32839 | 954 | 32843 | 954 | 32869 | 954 |
| 32909 | 956 | 32911 | 955 | 32917 | 956 | 32933 | 955 |
| 32941 | 957 | 1.635 | 32957 | 955 | 32969 | 956 | 32971 | 956 |
| 32987 | 955 | 32993 | 957 | 32999 | 957 | 33013 | 957 |