Quark Masses and Mixing with $A_4$ Family Symmetry

Ernest Ma$^a$, Hideyuki Sawanaka$^b$, and Morimitsu Tanimoto$^c$

$^a$ Physics Department, University of California, Riverside, California 92521, USA
$^b$ Graduate School of Science and Technology, Niigata University, 950-2181 Niigata, Japan
$^c$ Department of Physics, Niigata University, 950-2181 Niigata, Japan

Abstract

The successful $A_4$ family symmetry for leptons is applied to quarks, motivated by the quark-lepton assignments of SU(5). Realistic quark masses and mixing angles are obtained, in good agreement with data. In particular, we find a strong correlation between $|V_{ub}|$ and the CP phase $\beta$, thus allowing for a decisive future test of this model.
Since the original papers [1, 2] on the application of the non-Abelian discrete symmetry $A_4$ to quark and lepton families, much progress has been made in understanding the case of tribimaximal mixing [3] for neutrinos in a number of specific models [4]. As for quarks, the generic prediction [2] is that its mixing matrix is just the unit matrix, which can become realistic only if small mixing angles can be generated by interactions beyond those of the Standard Model, such as in supersymmetry [5]. Other ideas of quark mixing include the judicious addition of terms which break the $A_4$ (as well as the residual $Z_3$) symmetry explicitly [6]. In this paper, motivated by the quark-lepton assignments of SU(5), we study a new alternative scenario, where realistic quark masses and mixing angles are obtained, entirely within the $A_4$ context.

In SU(5) grand unification, the $5^*$ representation contains the lepton doublet $(\nu, l)$ and the quark singlet $d^c$, whereas the $10$ representation contains the lepton singlet $l^c$ and the quark doublet $(u, d)$ and singlet $u^c$. In the successful $A_4$ model for leptons, $(\nu_i, l_i)$ transform as a $3$ whereas $l_i^c$ transform as $1, 1', 1''$. This means that we should choose

$$d_i^c \sim 3, \quad u_i^c, \ (u_i, d_i) \sim 1, 1', 1''.$$  

(1)

Assuming as in the leptonic case three Higgs doublets $\Phi_i = (\phi_i^+ , \phi_i^0)$ transforming as $3$ under $A_4$, the relevant Yukawa couplings linking $d_i$ with $d_j^c$ are given by

$$h_1 d_1 (d_1^c \phi_1^0 + d_2^c \phi_2^0 + d_3^c \phi_3^0) + h_2 d_2 (d_1^c \phi_1^0 + \omega d_2^c \phi_2^0 + \omega^2 d_3^c \phi_3^0) + h_3 d_3 (d_1^c \phi_1^0 + \omega^2 d_2^c \phi_2^0 + \omega d_3^c \phi_3^0),$$  

(2)

resulting in the $3 \times 3$ mass matrix:

$$M_{dd^c} = \begin{pmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \begin{pmatrix} v_1 & 0 & 0 \\ 0 & v_2 & 0 \\ 0 & 0 & v_3 \end{pmatrix},$$  

(3)

where $\omega = \exp(2\pi i/3)$, $h_i$ are three independent Yukawa couplings, and $v_i$ are the vacuum expectation values of $\phi_i^0$. (For details of the $A_4$ multiplication rules, see for example the
original papers [1, 2] or the more recent review [7].) On the other hand, the Higgs doublets linking $u$ with $u^c$ must be different because of the latter’s $A_4$ assignments. We choose here two Higgs doublets transforming as $1'$ and $1''$, then

$$
\mathcal{M}_{uu^c} = \begin{pmatrix}
0 & \mu_2 & \mu_3 \\
\mu_2 & m_2 & 0 \\
\mu_3 & 0 & m_3
\end{pmatrix},
$$

(4)

where $m_2, \mu_3$ come from $1'$ and $m_3, \mu_2$ from $1''$. This matrix is also symmetric because of the usually assumed SU(5) decomposition of $10 \times 10 \times 5 \to 1$.

In minimal SU(5), there is just one $5$ representation of Higgs bosons, yielding thus only two invariants, i.e. $10 \times 10 \times 5 \to 1$ (for the $uu^c$ mass matrix) and $5^* \times 10 \times 5^* \to 1$ (for the $ll^c$ and $d^c d$ mass matrices). The second invariant implies $m_\tau = m_b$ at the unification scale which is phenomenologically desirable, but also $m_\mu = m_s$ and $m_\epsilon = m_d$ which are not. To decouple the $ll^c$ and $d^c d$ mass matrices, we follow the usual strategy of using both $5^*$ and $45$ representations of Higgs bosons, so that one linear combination couples to only leptons, and the other only to quarks. Both transform as $3$ under $A_4$. There are also two $5$ representations transforming as $1'$ and $1''$ under $A_4$ which couple only to $uu^c$, which must still be symmetric.

In the limit $|\mu_2| << |m_2|$ and $|\mu_3| << |m_3|$, we obtain the three eigenvalues of $\mathcal{M}_{uu^c}$ as

$$
m_t \simeq |m_3|, \quad m_c \simeq |m_2|, \quad m_u \simeq \left| \frac{\mu_2^2}{m_2} + \frac{\mu_3^2}{m_3} \right|
$$

(5)

with mixing angles

$$
V_{uc} \simeq \frac{\mu_2}{m_2}, \quad V_{ut} \simeq \frac{\mu_3}{m_3}, \quad V_{ct} \simeq 0.
$$

(6)

In the $d$ sector, we note first that

$$
\mathcal{M}_{dd^c} \mathcal{M}_{dd^c}^\dagger = \begin{pmatrix}
A|h_1|^2 & B^* h_1 h_2^* & B h_1 h_3^* \\
B h_1^* h_2 & A|h_2|^2 & B^* h_2 h_3^* \\
B^* h_1^* h_3 & B h_2^* h_3 & A|h_3|^2
\end{pmatrix},
$$

(7)
where
\[ A = |v_1|^2 + |v_2|^2 + |v_3|^2, \quad B = |v_1|^2 + \omega |v_2|^2 + \omega^2 |v_3|^2. \] (8)

Its eigenvalue equation is
\[
\lambda^3 - \lambda^2 (|v_1|^2 + |v_2|^2 + |v_3|^2) (|h_1|^2 + |h_2|^2 + |h_3|^2) - 27|v_1|^2 |v_2|^2 |v_3|^2 |h_1|^2 |h_2|^2 |h_3|^2 \\
+ 3\lambda (|v_1|^2 |v_2|^2 + |v_1|^2 |v_3|^2 + |v_2|^2 |v_3|^2) (|h_1|^2 |h_2|^2 + |h_1|^2 |h_3|^2 + |h_2|^2 |h_3|^2) = 0. \] (9)

If \(|v_1| = |v_2| = |v_3| = |v|\) as assumed in the original papers and all those of Ref. [4], then \(A = 3|v|^2, B = 0\), and the three eigenvalues are simply \(3|h_{1,2,3}|^2 |v|^2\). We choose them instead to be different, but we still assume \(|h_3|^2 >> |h_2|^2 >> |h_1|^2\). In that case, we find
\[
m_b^2 \simeq \frac{(|v_1|^2 + |v_2|^2 + |v_3|^2)|h_3|^2}{|v_1|^2 + |v_2|^2 + |v_3|^2}, \] (10)
\[
m_s^2 \simeq \frac{3(|v_1|^2 |v_2|^2 + |v_1|^2 |v_3|^2 + |v_2|^2 |v_3|^2)|h_2|^2}{|v_1|^2 + |v_2|^2 + |v_3|^2}, \] (11)
\[
m_d^2 \simeq \frac{9|v_1|^2 |v_2|^2 |v_3|^2 |h_1|^2}{|v_1|^2 |v_2|^2 + |v_1|^2 |v_3|^2 + |v_2|^2 |v_3|^2}, \] (12)

and the mixing angles are given by
\[
V_{sb} \simeq \left( \frac{B^*}{A} \right) \frac{h_2}{h_3}, \] (13)
\[
V_{db} \simeq \left( \frac{B}{A} \right) \frac{h_1}{h_3}, \] (14)
\[
V_{ds} \simeq \left( \frac{AB^* - B^2}{A^2 - |B|^2} \right) \frac{h_1}{h_2}, \] (15)

thereby requiring the condition
\[
\left| \frac{V_{ds} V_{sb}}{V_{db}} \right| \simeq \left| \frac{AB^* - B^2}{A^2 - |B|^2} \right|, \] (16)

which has unity as an upper bound. Using current experimental values for the left-hand side, we see that quark mixing in the \(d\) sector alone cannot explain the observed \(V_{CKM}\). Taking into account \(V_u\), we then have \(V_{CKM} = V_u^\dagger V_d\). Hence
\[
V_{us} \simeq V_{ds} - V_{uc}, \] (17)
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\[ V_{cb} \simeq V_{sb}, \quad (18) \]
\[ V_{ub} \simeq V_{db} - V_{uc}V_{sb} - V_{ut}. \quad (19) \]

We show in the following how all quark masses and mixing angles are realistically obtained in this model.

We note first that our quark mass matrices are restricted by our choice of \( A_4 \) representations to have only 5 independent parameters each. In the down sector, the Yukawa couplings \( h_{1,2,3} \) can all be chosen real, \( A \) is just an overall scale, and \( B \) is complex. The 5 independent parameters can be chosen as the 3 quark masses, and 2 angles. In the up sector, we can choose \( \mu_2 \) and \( m_{2,3} \) to be real, with \( \mu_3 \) complex. The 5 independent parameters can be chosen as the 3 quark masses, 1 angle, and 1 phase. Since we also have 10 observables (6 quark masses, 3 angles, and 1 phase), it may appear that a fit is not so remarkable. However, the forms of the 2 mass matrices are very restrictive, and it is by no means trivial to obtain a good fit. Indeed, we find that \( V_{ub} \) is strongly correlated with the CP phase \( \beta \). If we were to fit just the 6 masses and the 3 angles, the structure of our mass matrices would allow only a very narrow range of values for \( \beta \) at each value of \( |V_{ub}| \). This means that future more precise determinations of these two parameters will be a decisive test of this model.

Our quark mass matrices are given at the SU(5) unification scale in principle. However, the \( A_4 \) flavor symmetry is spontaneously broken at the electroweak scale. Therefore, the forms of our mass matrices are not changed except for the magnitudes of the Yukawa couplings between the unification and electroweak scales. Our numerical analyses are presented at the electroweak scale.

In order to fit the ten observables (six quark masses, three CKM mixing angles and one phase), 1,000,000 random numbers have been generated for the ten parameters of our model. We then choose the parameter sets which are allowed by the experimental data. First we show the prediction of \( |V_{ub}| \) versus \( \beta \) in Figure 1, with the following nine experimental inputs
Figure 1: Plot of allowed values in the $\beta - |V_{ub}|$ plane, where the value of $\beta$ is expressed in radians. The horizontal and vertical lines denote experimental bounds at 90% C.L.

\[8, 9, 10\]:

\[m_u = 0.9 \sim 2.9 \text{ (MeV)}, \quad m_c = 530 \sim 680 \text{ (MeV)}, \quad m_t = 168 \sim 180 \text{ (GeV)},\]
\[m_d = 1.8 \sim 5.3 \text{ (MeV)}, \quad m_s = 35 \sim 100 \text{ (MeV)}, \quad m_b = 2.8 \sim 3 \text{ (GeV)},\]
\[|V_{us}| = 0.221 \sim 0.227, \quad |V_{cb}| = 0.039 \sim 0.044, \quad J_{CP} = (2.75 \sim 3.35) \times 10^{-5},\]

which are given at the electroweak scale. Here $J_{CP}$ is the Jarlskog invariant \[11\]. We see that the experimental allowed region of $\beta$ (0.370 $\sim$ 0.427 radian at 90% C.L.) \[10\] corresponds to $|V_{ub}|$ in the range 0.0032 $\sim$ 0.0044, which is consistent with the experimental value of $|V_{ub}| = 0.0029 \sim 0.0045$. Thus our model is able to reproduce realistically the experimental data of quark masses and the CKM matrix.

Precisely measured heavy quark masses and CKM matrix elements are expected in future experiments and precise light quark masses are expected in future lattice evaluations. If the allowed regions of the current data shown in Eq. \[20\] are reduced, the correlation between
$|V_{ub}|$ and $\beta$ will become stronger. We show in Figure 2 the case where the experimental data are restricted to some very narrow ranges about their central values:

$$m_u/m_d = 0.5 \sim 0.6, \quad m_c = 600 \sim 610 \text{ (MeV)}, \quad m_t = 172 \sim 176 \text{ (GeV)},$$  
$$m_d = 3.4 \sim 3.6 \text{ (MeV)}, \quad m_s/m_d = 18 \sim 20, \quad m_b = 2.85 \sim 2.95 \text{ (GeV)}, \quad (21)$$  
$$|V_{us}| = 0.221 \sim 0.227, \quad |V_{cb}| = 0.041 \sim 0.042, \quad J_{CP} = (3.0 \sim 3.1) \times 10^{-5}.$$  

Here we use the tighter constraints on the mass ratios of light quarks, i.e. $m_u/m_d$ and $m_s/m_d$, consistent with the well-known successful low-energy sum rules $[12]$. Clearly, future more precise determinations of $|V_{ub}|$ and $\beta$ will be a sensitive test of our model.

We should also comment on the hierarchy of $h_i$ and $v_i$. The order of $h_i$ are fixed by quark mixings. The ratio of $h_2/h_3 \simeq \lambda^2$ is required by the $V_{cb}$ mixing ($\lambda \simeq 0.22$), on the other hand, $h_1/h_2 \simeq \lambda$ comes from $V_{us}$. Once $h_i$ are fixed, quark masses determine the hierarchy of $v_i$ as follows: $v_1/v_3 \simeq \lambda^2$ and $v_2/v_3 \simeq \lambda \sim \lambda^{1/2}$. These hierarchies of $h_i$ and $v_i$ are also
consistent with the magnitude of $J_{CP}$, which is given by

$$J_{CP} \simeq \frac{\sqrt{3}}{2} \left( \frac{v_2^2 - v_1^2}{v_2^2 + v_1^2} \right) \left( 1 + \frac{\text{Re}(\mu_3) + \sqrt{3} \text{Im}(\mu_3)}{m_t h_3} \right).$$ (22)

In summary, the $A_4$ family symmetry (which has been successful in understanding the mixing pattern of neutrinos) is applied successfully as well to quarks, motivated by the quark-lepton assignments of SU(5). The Yukawa interactions of this model are invariant under $A_4$, but the vacuum expectation values of the Higgs scalars are allowed to be arbitrary. The resulting $u$ and $d$ quark mass matrices have 5 parameters each, with different specific structures. Realistic quark masses and mixing angles are obtained, in good agreement with data. In particular, we find a strong correlation between $|V_{ub}|$ and the CP phase $\beta$, thus allowing for a decisive future test of this model.

This work was supported in part by the U. S. Department of Energy under Grant No. DE-FG03-94ER40837 and the Grant-in-Aid for Science Research, Ministry of Education, Science and Culture, Japan (No. 17540243). EM thanks the Department of Physics, Niigata University for hospitality during a recent visit.
References

[1] E. Ma and G. Rajasekaran, Phys. Rev. D64, 113012 (2001).

[2] K. S. Babu, E. Ma, and J. W. F. Valle, Phys. Lett. B552, 207 (2003).

[3] P. F. Harrison, D. H. Perkins, and W. G. Scott, Phys. Lett. B530, 167 (2002). See also X.-G. He and A. Zee, Phys. Lett. B560, 87 (2003).

[4] See for example E. Ma, Phys. Rev. D70, 031901R (2004); Phys. Rev. D72, 037301 (2005); D73, 057304 (2006); G. Altarelli and F. Feruglio, Nucl. Phys. B720, 64 (2005); B741, 215 (2006); K. S. Babu and X.-G. He, hep-ph/0507217.

[5] K. S. Babu, B. Dutta, and R. N. Mohapatra, Phys. Rev. D60, 095004 (1999).

[6] E. Ma, Mod. Phys. Lett. A17, 627 (2002); X.-G. He, Y.-Y. Keum, and R. R. Volkas, JHEP 0604, 039 (2006).

[7] E. Ma, hep-ph/0409075

[8] H. Fritzsch and Z. Xing, Prog. Part. Nucl. Phys. 45, 1 (2000).

[9] Particle Data Group, http://pdg.lbl.gov/ (2006).

[10] J. Charles (CKM fitter group), hep-ph/0606046.

[11] C. Jarlskog, Phys. Rev. Lett. 55, 1039 (1985).

[12] H. Leutwyler, Phys. Lett. B378, 313 (1996), hep-ph/9609467

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