Dipole modulation of cosmic microwave background temperature and polarization

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Abstract. We propose a dipole modulation model for the Cosmic Microwave Background Radiation (CMBR) polarization field. We show that the model leads to correlations between $l$ and $l + 1$ multipoles, exactly as in the case of temperature. We obtain results for the case of $TE$, $EE$ and $BB$ correlations. An anisotropic or inhomogeneous model of primordial power spectrum which leads to such correlations in temperature field also predicts similar correlations in CMBR polarization. We analyze the CMBR temperature and polarization data in order to extract the signal of these correlation between $l$ and $l + 1$ multipoles. Our results for the case of temperature using the latest PLANCK data agree with those obtained by an earlier analysis. A detailed study of the correlation in the polarization data is not possible at present. Hence we restrict ourselves to a preliminary investigation in this case.

Keywords: CMBR polarisation, cosmology of theories beyond the SM, CMBR theory

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1 Introduction

The Cosmic Microwave Background Radiation (CMB) shows a hemispherical power asymmetry, i.e. the power in the two hemispheres is significantly different [1–8]. A dipole modulation of a statistically isotropic signal provides a useful parametrization of the observed power asymmetry. According to the model, the observed temperature fluctuation $\Delta \tilde{T}$ along a direction $\hat{n}$ is expressed as [9–12],

$$\Delta \tilde{T}(\hat{n}) = \Delta T(\hat{n}) \left(1 + A\hat{\lambda}_1 \cdot \hat{n}\right),$$  \hspace{1cm} (1.1)

where $\Delta T(\hat{n})$ is a statistically isotropic field, $A$ the dipole amplitude and $\hat{\lambda}_1$ the dipole direction. Throughout this paper we shall denote the observed fields, which are assumed to have some contribution due to dipole modulation, with a tilde and the corresponding fields in an isotropic model without a tilde. Choosing our axes such that $\hat{\lambda}_1$ is along $\hat{z}$, eq. (1.1) can be written as

$$\Delta \tilde{T}(\hat{n}) = \Delta T(\hat{n}) (1 + A \cos \theta).$$ \hspace{1cm} (1.2)

As shown in [11, 13], the two point correlation of such a modulated temperature field would show correlations between $l$ and $l + 1$.

The dipole modulation effect is absent at high $l$ [14–16]. The effect is also not seen in the large scale structure surveys [17, 18] which provides further evidence that it should decay at large values of the wavenumber $k$ or equivalently at high $l$. Hence any primordial power spectrum model of the observed dipole modulation effect must show a scale dependent power asymmetry which decays at high $k$ [16, 19].
There also exist several other interesting signals of large scale anisotropy in CMB and other data sets. In particular the CMB quadrupole and octopole show alignment with one another. The preferred axis for both these multipoles points roughly in the direction of the CMB dipole [20–22]. It is interesting that the same direction also shows up in the study of orientations of radio polarizations of distant galaxies [23] as well as in the alignment of optical polarization of quasars [21, 24–26]. Furthermore the radio source count as well as brightness shows a dipole anisotropy in this direction with amplitude significantly larger than expected on the basis on local motion [27–30]. The polarized radio flux also shows a dipole in the same direction with an even larger deviation in comparison to the predicted result [31]. Another interesting signal in the CMB data is the presence of parity asymmetry [32, 33]. The high $z$ supernovae, however, do not show a statistically significant signal of anisotropy [34–37]. This might impose some constraints on the models which attempt to explain the observed large scale anisotropies. We point out that the anisotropy in the cosmological parameters arising due to the dipole modulation effect is found to be relatively small [38, 39]. Hence the absence of anisotropy in the supernova data may not be in conflict with the observed dipole modulation in CMB data. Furthermore the amplitude of dipole anisotropy seen in radio source count, brightness and polarized flux is quite small and leads to a statistically significant signal only in very large data sets. Hence, in this case also, any physical model which may explain these observations is likely to produce only a small anisotropy in cosmological parameters and may not lead to an observable signal in the current supernova data. However as more data accumulates, we may expect to observe a significant signal in the supernova data set also. In any case, more work is required to determine a theoretical model which is consistent with all observations.

If the observed signal of hemispherical anisotropy or equivalently dipole modulation is related to a physical effect, we expect a similar signal to be present in polarization fields too. Several studies have associated this effect with a primordial inhomogeneous [3, 16, 17, 40–42] or anisotropic [16, 43, 44] model. Such models lead to a modification of the primordial power spectrum which culminates in depicting correlations between different multipoles similar to those predicted by eq. (1.1). In a recent paper it has been shown that such a primordial model also leads to correlations between $l$ and $l + 1$ of the polarization fields [16]. In this paper we propose a dipole modulation model for the CMBR polarization field, analogous to eq. (1.1). Such a model is useful to empirically characterize the observed hemispherical anisotropy that might be present in the polarization data, irrespective of the physical cause of its origin. We show that this model leads to correlations between the $l$ and $l + 1$ multipoles for the polarization fields. We also determine the explicit form of these correlations.

We search for such correlations in the recently released Planck experiment data in both temperature and polarization signals and compare them with previous results and predictions. In the case of polarization a detailed study is not possible due to difficulty in handling and interpretation of the noise files. Furthermore the current polarization data at low $l$ is not reliable [45, 46]. Hence in this case we restrict ourselves to a preliminary investigation.

2 Test for dipole modulation

The modulated temperature field is given by eq. (1.2). This being a field on a sphere, can be expanded in spherical harmonics as

$$
\Delta \tilde{T}(\hat{n}) = \sum_{l,m} \hat{a}_{lm}^T Y_{lm}(\hat{n})
$$

(2.1)
The two point correlation of the temperature field in multipole space can be written as

$$
\langle \tilde{a}_{lm}^T \tilde{a}_{l'm'}^{T*} \rangle = \int d\Omega \, d\Omega \, Y_{lm}^*(\hat{n}) \, Y_{l'm'}(\hat{n}') \langle \Delta T(n) \Delta T(n') \rangle
$$

(2.2)

As shown in [13], using eq. (1.2) we obtain

$$
\langle \tilde{a}_{lm}^T \tilde{a}_{l'm'}^{T*} \rangle = \tilde{C}_l^T \delta_{ll'} \delta_{mm'} + A \left( \tilde{C}_l^T + C_l^T \right) \xi_{lm,l'm'}^0
$$

(2.3)

where

$$
\xi_{lm,l'm'}^0 = \delta_{mm'} \left[ \sqrt{\frac{(l+m+1)(l-m+1)}{(2l+3)(2l+1)}} \delta_{l,l+1} + \sqrt{\frac{(l+m)(l-m)}{(2l+1)(2l-1)}} \delta_{l,l-1} \right]
$$

(2.4)

In eq. (2.3), the first term on r.h.s. corresponds to the isotropic part of the correlation $\langle \tilde{a}_{lm}^T \tilde{a}_{l'm'}^{T*} \rangle_{iso}$ and the second term is the contribution of the modulation, $\langle \tilde{a}_{lm}^T \tilde{a}_{l'm'}^{T*} \rangle_{mod}$. The temperature multipoles power $C_l^T$ do not get any contribution from the modulation term [13].

This is found to be true also for the power in the polarization fields, to be discussed later. Hence we do not denote it with a tilde. We follow [13] and seek correlations between $l$ and $l+1$ multipoles, which can be expressed as,

$$
\langle \tilde{a}_{lm}^T \tilde{a}_{l+1m}^{T*} \rangle = A \left( C_{l+1}^T + C_l^T \right) \sqrt{\frac{(l+m+1)(l-m+1)}{(2l+3)(2l+1)}}
$$

(2.5)

Theoretical models used to explain the dipole modulation of the temperature field predict a similar correlation between $l$ and $l+1$ multipoles in the CMB E-mode polarization field [16, 47] in the same direction. These predictions may be tested in future by determining these correlations in the CMB polarization field.

A detailed discussion of CMB polarization is contained in [48, 49] and here we use the notation of [48]. The CMB polarization field is characterized by two Stokes parameters $\hat{Q}$ and $\hat{U}$, while the temperature fluctuation field corresponds to Stokes’ parameter $\hat{I}$. Here $\hat{Q}$ and $\hat{U}$ denote the dipole modulated polarization fields. Under a rotation by an angle $\psi$, the temperature field transforms as a scalar, while combinations of $\hat{Q}$ and $\hat{U}$ behave as spin $±2$ fields on a sphere, viz.

$$
(\hat{Q} \pm i\hat{U})'(\hat{n}) = e^{±2i\psi}(\hat{Q} \pm i\hat{U})(\hat{n}),
$$

(2.6)

and can be expanded in spin $±2$ harmonics as

$$
(\hat{Q} \pm i\hat{U})(\hat{n}) = \sum_{lm} \tilde{a}_{±2,lm} \, Y_{lm}(\hat{n}).
$$

(2.7)

Using the spin raising and lowering operators $\tilde{\sigma}$ and $\tilde{\sigma}$, spin 0 objects can be constructed from $\hat{Q}$ and $\hat{U}$ fields [50]. Using $\sigma$ and $\tilde{\sigma}$ suitably on eq. (2.7) we get

$$
\tilde{\sigma}^2(\hat{Q} + i\hat{U})(\hat{n}) = \sum_{lm} \sqrt{\frac{(l+2)!}{(l-2)!}} \tilde{a}_{2,lm} \, Y_{lm}(\hat{n})
$$

(2.8)

$$
\sigma^2(\hat{Q} - i\hat{U})(\hat{n}) = \sum_{lm} \sqrt{\frac{(l+2)!}{(l-2)!}} \tilde{a}_{-2,lm} \, Y_{lm}(\hat{n}).
$$

(2.9)
Finally the standard $E$ and $B$ mode polarization field can be expressed as,

\[
\tilde{E}(\hat{n}) = -\frac{1}{2} \left[ \tilde{\alpha}^2 (\tilde{Q} + i \tilde{U}) + \tilde{\alpha}^2 (\tilde{Q} - i \tilde{U}) \right] = \sum_{lm} \sqrt{\frac{(l + 2)!}{(l - 2)!}} \tilde{a}_{lm}^E Y_{lm}(\hat{n}) \tag{2.10}
\]

\[
\tilde{B}(\hat{n}) = i \frac{1}{2} \left[ \tilde{\alpha}^2 (\tilde{Q} + i \tilde{U}) - \tilde{\alpha}^2 (\tilde{Q} - i \tilde{U}) \right] = \sum_{lm} \sqrt{\frac{(l + 2)!}{(l - 2)!}} \tilde{a}_{lm}^B Y_{lm}(\hat{n}) \tag{2.11}
\]

The coefficients $\tilde{a}_{lm}^E$ and $\tilde{a}_{lm}^B$ are defined as:

\[
\tilde{a}_{lm}^E = -\frac{1}{2} (\tilde{a}_{2lm} + \tilde{a}_{-2lm}) \tag{2.12}
\]

\[
\tilde{a}_{lm}^B = \frac{i}{2} (\tilde{a}_{2lm} - \tilde{a}_{-2lm}) \tag{2.13}
\]

The $\tilde{a}_{lm}^E$ s define the E-mode polarization in multipole space and are unchanged under parity transformation in contrast to $\tilde{a}_{lm}^B$ s which do change sign under such a transformation. The scalar fields defined in eqs. (2.10) and (2.11) are the real space constructs of the E-mode and B-mode polarizations representing the irrotational and curl components of the CMB polarizations respectively. We define the auto correlation of the E field and cross correlation of the E and T fields as

\[
C_{EE}^l = \frac{1}{2l + 1} \sum_{m} \langle \tilde{a}_{lm}^E \tilde{a}_{lm}^{E*} \rangle \tag{2.14}
\]

\[
C_{TE}^l = \frac{1}{2l + 1} \sum_{m} \langle \tilde{a}_{lm}^E \tilde{a}_{lm}^{T*} \rangle \tag{2.15}
\]

In order to study the $l$ and $l + 1$ correlations we construct

\[
C_{XX}^{l,l+1} = \frac{l(l + 1)}{2l + 1} \sum_{m=-l}^{m=+l} \langle \tilde{a}_{lm}^X \tilde{a}_{lm+1}^{X*} \rangle \tag{2.16}
\]

and define our statistics as

\[
S_{XX}^H = \sum_{l} C_{XX}^{l,l+1} \tag{2.17}
\]

Here $X$ can be either $T$ or $E$ giving us $TT$, $EE$, $TE$ and $ET$ correlations. We search the direction for which the statistic $S_{XX}^H$ maximizes in each of the maps. We also define a quantity $R$ as the ratio of the anisotropic part to isotropic part, i.e.,

\[
R = \frac{\sum_{l=1}^{l_{max}} C_{XX}^{l,l+1}}{\sum_{l=1}^{l_{max}} l(l + 1)C_{II}^{l,l+1}} \tag{2.18}
\]

This may be seen as a measure of the fraction of the anisotropic effect to the isotropic power.

### 3 Dipole modulation in polarization

The dipole modulated polarization fields are denoted by $\tilde{Q}(\hat{n})$ and $\tilde{U}(\hat{n})$ where $\hat{n} \equiv (\theta, \phi)$. We also define

\[
\tilde{x}_\pm(\hat{n}) = \tilde{Q}(\hat{n}) \pm i \tilde{U}(\hat{n}) \tag{3.1}
\]
\[ \alpha_+ (\hat{n}) = Q(\hat{n}) \pm iU(\hat{n}) \] 
where \( Q \) and \( U \) are the standard unmodulated fields in an isotropic model, \( \tilde{\alpha}_- = \tilde{\alpha}_+^* \) and \( \alpha_- = \alpha_+^* \). The preferred direction \( \lambda \) is taken to be the same for both \( Q \) and \( U \) as well as the temperature field [16]. In analogy with temperature, we propose the following model for dipole modulation of polarization:

\[
\tilde{\alpha}_+ (\hat{n}) = \alpha_+ (\hat{n}) \left( 1 + A_P \lambda \cdot \hat{n} \right), \\
\tilde{\alpha}_- (\hat{n}) = \alpha_- (\hat{n}) \left( 1 + A_P^* \lambda \cdot \hat{n} \right). 
\] (3.2)

Here \( A_P = A_1 + iA_2 \) is a complex parameter. We choose our coordinates such that \( \lambda = \hat{\epsilon} \) and hence \( \lambda \cdot \hat{n} = \cos \theta \). In terms of the Stokes’ parameters, we obtain

\[
\tilde{Q} = Q(1 + A_1 \cos \theta) - U A_2 \cos \theta \\
\tilde{U} = QA_2 \cos \theta + U (1 + A_1 \cos \theta) 
\] (3.3)

Using eqs. (2.7) and (2.13) for the modulated polarization fields, we obtain

\[
\tilde{\alpha}_\pm = - \sum_{lm} (\tilde{a}_{lm}^E \pm i\tilde{a}_{lm}^B) \pm_2 Y_{lm}, 
\] (3.4)

where \( \tilde{a}_{E,lm} \) and \( \tilde{a}_{B,lm} \) denote the harmonic coefficients of the modulated fields. Inverting the above equation we obtain

\[
- (\tilde{a}_{lm}^E \pm i\tilde{a}_{lm}^B) = \int \tilde{\alpha}_\pm (\hat{n}) \pm_2 Y_{lm}^* (\hat{n}) d\Omega. 
\] (3.5)

This leads to

\[
\tilde{a}_{lm}^E = \frac{1}{2} \int [\alpha_+ (\hat{n}) (1 + A_P \cos \theta)Y_{lm}^*(\hat{n}) + \alpha_- (\hat{n})(1 + A_P^* \cos \theta) - 2Y_{lm}^*(\hat{n})] d\Omega, 
\] (3.6)

where we have used eq. (3.2). We also obtain a similar equation for \( \tilde{a}_{lm}^B \).

### 3.1 Correlations of the dipole modulated polarization field

The two point correlations of the dipole modulated \( E \) field harmonic coefficients can be expressed as,

\[
\langle \tilde{a}_{lm}^E \tilde{a}_{l'm'}^{E*} \rangle = \frac{1}{4} (I_1 + I_2 + I_3 + I_4) 
\] (3.7)

where

\[
I_1 = \int d\Omega d\Omega' \langle \alpha_+ (\hat{n}) \alpha_- (\hat{n}') \rangle (1 + A_P \cos \theta) (1 + A_P^* \cos \theta') Y_{lm}^*(\hat{n}) Y_{l'm'}^*(\hat{n}'), \\
I_2 = \int d\Omega d\Omega' \langle \alpha_+ (\hat{n}) \alpha_+ (\hat{n}') \rangle (1 + A_P \cos \theta) (1 + A_P^* \cos \theta') Y_{lm}^*(\hat{n}) - 2Y_{l'm'}^*(\hat{n}'), \\
I_3 = \int d\Omega d\Omega' \langle \alpha_- (\hat{n}) \alpha_- (\hat{n}') \rangle (1 + A_P^* \cos \theta) (1 + A_P \cos \theta') - 2Y_{lm}^*(\hat{n}) Y_{l'm'}^*(\hat{n}'), \\
I_4 = \int d\Omega d\Omega' \langle \alpha_- (\hat{n}) \alpha_+ (\hat{n}') \rangle (1 + A_P^* \cos \theta) (1 + A_P \cos \theta') - 2Y_{lm}^*(\hat{n}) - 2Y_{l'm'}^*(\hat{n}').
\]
The two point correlations appearing on the right hand side of these equations can be written as:

\[
\langle \alpha_+ (\hat{n}) \alpha_- (\hat{n}') \rangle = \sum_{l' m'} \left( C^{EE}_{l' m'} + C^{BB}_{l' m'} \right) 2 Y_{l' m'} (\hat{n}) 2 Y_{l' m'}^* (\hat{n}'),
\]

\[
\langle \alpha_+ (\hat{n}) \alpha_+ (\hat{n}') \rangle = \sum_{l' m'} \left( C^{EE}_{l' m'} - C^{BB}_{l' m'} \right) (-1)^{m'} 2 Y_{l' m'} (\hat{n}) 2 Y_{l' m'}^* (\hat{n}'),
\]

\[
\langle \alpha_- (\hat{n}) \alpha_- (\hat{n}') \rangle = \sum_{l' m'} \left( C^{EE}_{l' m'} - C^{BB}_{l' m'} \right) (-1)^{m'} 2 Y_{l' m'}^* (\hat{n}) 2 Y_{l' m'}^* (\hat{n}'),
\]

\[
\langle \alpha_- (\hat{n}) \alpha_+ (\hat{n}') \rangle = \sum_{l' m'} \left( C^{EE}_{l' m'} + C^{BB}_{l' m'} \right) 2 Y_{l' m'}^* (\hat{n}) 2 Y_{l' m'}^* (\hat{n}').
\]

where we have used, \( \langle a^E_{lm} a^E_{l'm'} \rangle = C^E_{l'l'} \delta_{m'm'} \), \( \langle a^B_{lm} a^B_{l'm'} \rangle = C^B_{l'l'} \delta_{m'm'} \), \( \langle a^E_{lm} a^B_{l'm'} \rangle = 0 \) and \( -2 Y_{lm}^* = (-1)^{m} 2 Y_{l(-m)} \). Here \( C^E \) and \( C^B \) represent the isotropic power spectrum corresponding to \( E \) or \( B \) modes respectively. As we shall see the anisotropic model does not contribute to the power spectrum. Hence these also represent the power of the tilde fields.

Substituting the resulting expressions of \( I_l \) in eq. (3.7), we obtain

\[
\langle \hat{a}^E_{lm} \hat{a}^E_{l'm'} \rangle = C^E_{l'l'} \delta_{l'lm'} + \frac{1}{4} (M_1 + M_2 + M_3 + M_4)
\]

where \( M_i \) represent the corrections due to dipole modulation and are given by,

\[
\begin{align*}
M_1 &= \sum_{l' m'} \left( C^{EE}_{l'l'} + C^{BB}_{l'l'} \right) \int d\Omega d\Omega' (A_P \cos \theta + A_P \cos \theta') 2 Y_{l' m'} (\hat{n}) 2 Y_{l' m'}^* (\hat{n}') 2 Y_{l m}^* (\hat{n}) 2 Y_{l m'} (\hat{n}'), \\
M_2 &= \sum_{l' m'} \left( C^{EE}_{l'l'} - C^{BB}_{l'l'} \right) (-1)^{m'+m} \int d\Omega d\Omega' \left( A_P \cos \theta + A_P \cos \theta' \right) 2 Y_{l' m'} (\hat{n}) 2 Y_{l' m'}^* (\hat{n}') \times 2 Y_{l m}^* (\hat{n}) 2 Y_{l m'} (\hat{n}'), \\
M_3 &= \sum_{l' m'} \left( C^{EE}_{l'l'} - C^{BB}_{l'l'} \right) (-1)^{m'+m} \int d\Omega d\Omega' \left( A_P \cos \theta + A_P \cos \theta' \right) 2 Y_{l' m'}^* (\hat{n}') 2 Y_{l' m'}^* (\hat{n}) \times 2 Y_{l(-m)} (\hat{n}) 2 Y_{l m'} (\hat{n}'), \\
M_4 &= \sum_{l' m'} \left( C^{EE}_{l'l'} + C^{BB}_{l'l'} \right) (-1)^{m'+m} \int d\Omega d\Omega' \left( A_P \cos \theta + A_P \cos \theta' \right) 2 Y_{l' m'}^* (\hat{n}) 2 Y_{l' m'}^* (\hat{n}') \times 2 Y_{l(-m)} (\hat{n}) 2 Y_{l m'}^* (\hat{n}'),
\end{align*}
\]

where we have assumed that the modulation parameters \( A_1 \) and \( A_2 \) are small and dropped higher order terms. We can evaluate these integrals by using

\[
\int_0^{2\pi} \int_0^\pi 2 Y_{l m} (\hat{n}) 2 Y_{l' m'} (\hat{n}) d\Omega = \delta_{l'l'} \delta_{m'm'}.
\]

Furthermore we define

\[
\mathbb{I} (l, m, l', m') = \int_0^{2\pi} \int_0^\pi 2 Y_{l m} (\hat{n}) 2 Y_{l' m'} (\hat{n}) \cos \theta d\Omega = \delta_{m'm'} \mathbb{I} (l, l, m).
\]
This integral can be expressed in terms of the Wigner 3-j symbols by using
\[
Y_{10}(\hat{n}) = \sqrt{\frac{3}{4\pi}} \cos \theta .
\]

(3.11)

We obtain
\[
\mathbb{I} (l, m, l', m') = (-1)^{m'} \sqrt{(2l + 1)(2l' + 1)} \begin{pmatrix} l' & l & 1 \\ 2 & -2 & 0 \\ -m' & m & 0 \end{pmatrix}
\]

(3.12)

The Wigner 3-j symbol obeys the condition,
\[
\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = 0 \quad \text{if} \quad |l_1 - l_2| > l_3
\]

(3.13)

Using this we find that
\[
\mathbb{K} (l, l', m) = 0 \quad \text{if} \quad l' > l + 1 \quad \text{and} \quad l' < l - 1
\]

(3.14)

For the remaining cases, \( l = l' \) and \( l' = l \pm 1 \), it can be expressed as,
\[
\mathbb{K} (l, l', m) = (-1)^{l+l'} H (l, l', m) \bigcup (l, l', m),
\]

(3.15)

where
\[
H (l, l', m) = -2 \frac{\sqrt{(2l + 1)(2l' + 1)} (l - m)! (l + m)! (l' - m)! (l' + m)!}{(l + 2)! (l - 2)! (l' + 2)! (l' - 2)!}.
\]

\( \bigcup (l, l', m) = (2l' + 2)! (l' - 2)! \delta_{l+1,l'} + 2m(2l)!(l + 2)! (l - 2)! \delta_{l,l'} + (2l')!(l - 2)! (l' + 2)! \delta_{l-1,l} + (l + m)! (l - m)! (2l + 2)! (l' + m)! (l' - m)! (2l + 1)!
\]

The function \( \mathbb{K} (l, l', m) \) is explicitly evaluated in the next subsection. We can now write the integrals \( M_i \) as
\[
M_1 = (-1)^{l+l'} \delta_{mm'} H (l', l, m) \bigcup (l', l, m) \left[ A_P (C_{l'}^{EE} + C_{l'}^{BB}) + A_P^* (C_l^{EE} + C_l^{BB}) \right],
\]

\[
M_2 = (-1)^{l+l'} \delta_{mm'} A_P [H (l, l', -m) \bigcup (l, l', -m) (C_l^{EE} - C_l^{BB}) + H (l', l, m) \bigcup (l', l, m) (C_{l'}^{EE} - C_{l'}^{BB})],
\]

\[
M_3 = (-1)^{l+l'} \delta_{mm'} A_P^* [H (l, l', -m) \bigcup (l, l', -m) (C_l^{EE} - C_l^{BB}) + H (l', l, m) \bigcup (l', l, m) (C_{l'}^{EE} - C_{l'}^{BB})],
\]

\[
M_4 = (-1)^{l+l'} \delta_{mm'} H (l, l', -m) \bigcup (l, l', -m) \left[ A_P (C_{l'}^{EE} + C_{l'}^{BB}) + A_P (C_l^{EE} + C_l^{BB}) \right].
\]

where we have used
\[
\bigcup (l, l', m) = \bigcup (l', l, m)
\]

and
\[
H (l, l', m) = H (l', l, m) = H (l, l', -m) = H (l', l, -m).
\]

This finally leads to
\[
\langle \hat{a}^E_{im} \hat{a}^{E*}_{im'} \rangle = C_l^{EE} \delta_{il'} \delta_{mm'} + \delta_{mm'} \frac{1}{2} \left[ \mathbb{K} (l, l', m) (A_P C_{l'}^{EE} + A_P C_l^{EE}) + \mathbb{K} (l, l', -m) (A_P C_{l'}^{EE} + A_P C_l^{EE}) \right].
\]
The first term on the right hand side of this equation is the standard contribution due to an isotropic field. The second term arises due to dipole modulation. In this term only contributions linear in the dipole parameters $A_1$ and $A_2$ have been kept. We see that the modulation model, eq. (3.2), leads to correlations between multipoles $l$ and $l + 1$ besides also leading to additional contributions proportional to $\delta_{mm'} \delta_{ll'}$. However the latter contributions cancel out after summing over $m$ and can be ignored. Hence after summing over $m$ the dipole modulation term leads to correlations only between $l$ and $l + 1$. We also notice that the terms proportional to $\delta_{l+1,l'}$ and $\delta_{l-1,l'}$ in eq. (3.15) are symmetric under the interchange $m \leftrightarrow -m$. Hence we deduce that $\mathcal{K}(l, l', m) = \mathcal{K}(l, l', -m)$. Using this we obtain

$$\langle \tilde{a}_{lm}^* \tilde{a}_{l'm'} \rangle = C_l^{EE} \delta_{ll'} \delta_{mm'} + \delta_{mm'} A_1 \mathcal{K}(l, l', m) \left( C_{l'}^{EE} + C_{l'}^{EE} \right),$$

(3.16)

where we have ignored the contributions proportional to $\delta_{ll'}$ in the anisotropic term since they cancel out after summing over $m$. Similarly for the $B$ mode polarization, we obtain

$$\langle \tilde{a}_{lm}^B \tilde{a}_{l'm'}^B \rangle = C_l^{BB} \delta_{ll'} \delta_{mm'} + \delta_{mm'} A_1 \mathcal{K}(l, l', m) \left( C_{l'}^{BB} + C_{l'}^{BB} \right).$$

(3.17)

The dipole modulation model, eq. (3.2), is very useful for characterizing a signal of anisotropic power that might exist in the polarization data. It allows an empirical parametrization of such a signal. Furthermore it can be used to perform simulations which are required for a statistical study of the anisotropy. We explicitly demonstrate this in the present paper. We also point out that an alternate model in which we may directly introduce a dipole modulation in the $E$ mode polarization simply does not work.

The correlations of the $E$ and $B$ mode multipoles, eqs. (3.16) and (3.17) depend only on the parameter $A_1$ and are independent of $A_2$. Hence we can directly extract $A_1$ by studying the $E$ mode correlations.

A similar calculation for the TE mode correlations leads to the following result

$$\langle \tilde{a}_{lm}^{TE} \tilde{a}_{l'm'}^{TE} \rangle = C_l^{TE} \delta_{ll'} \delta_{mm'} + \delta_{mm'} A_1 \mathcal{K}(l, l', m)$$

$$+ \frac{1}{2} A_2 C_l^{TE} \delta_{mm'} \mathcal{K}(l, l', m)$$

$$+ \frac{1}{2} A_2 C_l^{TE} \delta_{mm'} \mathcal{K}(l, l', m)$$

(3.18)

In this case also, after summing over $m$, the $\delta_{ll'}$ term in $\mathcal{K}(l, l', m)$ and $\mathcal{K}(l, l', -m)$ drops out. Hence it can be ignored and the remaining terms are symmetric under $m \leftrightarrow -m$. Therefore we can express this result as

$$\langle \tilde{a}_{lm}^{TE} \tilde{a}_{l'm'}^{TE} \rangle = C_l^{TE} \delta_{ll'} \delta_{mm'} + A_1 C_l^{TE} \delta_{mm'} \mathcal{K}(l, l', m)$$

(3.19)

### 3.2 Calculation of the polarization correlations

We next explicitly evaluate the integral in eq. (3.10). The spin 2 harmonics [51] can be expressed as

$$2Y_{lm} = (-1)^m e^{im\phi} \sqrt{\frac{(2l+1)(l-m)!(l+m)!}{4\pi(l+2)!(l-2)!}}$$

$$\times \sum_{r=0}^{l-2} (-1)^{l-r} \binom{l-2}{r} \binom{l+2}{r+2-m} \sin \frac{\theta}{2}^{2l-2r-2+m} \cos \frac{\theta}{2}^{2r+2-m}.$$
The $\phi$ integration in eq. (3.10) leads to the factor $2\pi\delta_{mm'}$. The $\theta$ integral is evaluated by using the identity
\[
\int_0^\pi d\theta \cos \theta \sin \theta \sin^m \left( \frac{\theta}{2} \right) \cos^n \left( \frac{\theta}{2} \right) = 2 \frac{\Gamma \left( \frac{m+2}{2} \right) \Gamma \left( \frac{n+4}{2} \right)}{\Gamma \left( \frac{m+n+6}{2} \right)} - 2 \frac{\Gamma \left( \frac{m+4}{2} \right) \Gamma \left( \frac{n+2}{2} \right)}{\Gamma \left( \frac{m+n+6}{2} \right)}. \quad (3.21)
\]
which can be derived by using [52]
\[
\int_0^{\pi/2} \sin^m \theta \cos^n \theta d\theta = \frac{\Gamma \left( \frac{m+1}{2} \right) \Gamma \left( \frac{n+1}{2} \right)}{2\Gamma \left( \frac{m+n+2}{2} \right)}.
\]
The function $I(l, m, l', m')$ defined in eq. (3.10) can now be expressed as
\[
I(l, m, l', m') = (-1)^{l+l'} \delta_{mm'}H(l, l', m) \cup (l', m), \quad (3.22)
\]
where
\[
\cup (l', m) = -\frac{(l-2)! (l+2)! (l'+2)! (l'-2)!}{2 (l'+l+2)!} S(l, l', m), \quad (3.23)
\]
\[
S(l, l', m) = \sum_{r=0}^{l-2} \sum_{t=0}^{l'-2} F(l, l', r, t, m)
\]
and
\[
F(l, l', r, t, m) = \frac{(-1)^{r+t} (l+l'-r-t+m-2)! (r+t-m+2)! (2r+2t-2m+4-l-l')}{r! t! (l-2-r)! (l'+2-t)! (r+2-m)! (l+2-m)! (l-r+m)! (l'-t+m)!}.
\]  
(3.25)

We next show that for $l' = l$,
\[
S(l, l', m) = \frac{-4m (2l)!}{l (l+m)! (l-m)! (l+2)! (l-2)!}, \quad (3.26)
\]
Proof: for $l = l'$,
\[
S(l, l', m) = \sum_{r=0}^{l-2} (-1)^{l+r+2} (l-r+m) \frac{(-1)^{l-2}}{(l-2)! (l+2)! (l-2)!} \mathbb{P}. \quad (3.27)
\]
where
\[
\mathbb{P} = \sum_{t=0}^{l-2} (-1)^t \binom{l-2}{t} (2r+2t-2m+4-2t) \prod_{s=1}^{l-2-r} (l-t+m+s) \prod_{v=1}^{t} (t-m+2+v). \quad (3.28)
\]
We next express the two products as,
\[
\prod_{s=1}^{l-2-r} (l-t+m+s) \prod_{v=1}^{t} (t-m+2+v) = (-1)^{l-2-r} \left[ t^{l-2} + a_1 t^{l-3} + a_2 t^{l-4} + \ldots + a_{l-2} \right],
\]
where $a_i \in \mathbb{Z}$. Eq. (3.28) now becomes,
\[
\mathbb{P} = (-1)^{l-2-r} \sum_{t=0}^{l-2} (-1)^t \binom{l-2}{t} (2r+2t-2m+4-2t) \left[ t^{l-2} + a_1 t^{l-3} + a_2 t^{l-4} + \ldots + a_{l-2} \right].
\]
By using eq. (A.3) we find that only two terms, i.e. those proportional to $t^{l-1}$ and $t^{l-2}$, contribute. Thus we obtain

$$P = (-1)^{l-2-r} \sum_{t=0}^{l-2} (-1)^{l} \binom{l-2}{t} \left[ (2r - 2m + 4 - 2l + 2a_1)t^{l-2} + 2t^{l-1} \right].$$

(3.29)

The constant $a_1$ can be determined by using the result that if

$$\prod_{i=1}^{n} (x + \alpha_i) = x^n + x^{n-1}a_1 + \ldots + a_n,$$

then $a_1 = \sum_{i=1}^{n} \alpha_i$. Thus we obtain

$$a_1 = \sum_{s=1}^{l-2-r} (-l - m - s) + \sum_{r=1}^{r} (-m + 2 + v) = \frac{1}{2} \left[ -3l^2 + 7l - 2ml + 4m - 2 + 2r(2l + 1) \right].$$

Using eq. (A.3) we can express eq. (3.29) as,

$$P = (-1)^{-r} (l - 2)! \left[ -2l^2 + 2l - 2ml + 2m + 4r(l + 1) \right].$$

Substituting in eq. (3.27) we obtain

$$S(l, t', m) = \sum_{r=0}^{l-2-r} \left( \frac{l + 2}{l - r + m} \right) \left( \frac{l - 2}{r} \right) \left[ \frac{-2l^2 + 2l - 2ml + 2m + 4r(l + 1)}{(l + 2)! (l - 2)!} \right].$$

This sum can be divided into two parts

$$\frac{-2l^2 + 2l - 2ml + 2m}{(l + 2)! (l - 2)!} \sum_{r=0}^{l-2-r} \left( \frac{l + 2}{l - r + m} \right) \left( \frac{l - 2}{r} \right) + \frac{4(l + 1)}{(l + 2)! (l - 2)!} \sum_{r=0}^{l-2-r} \left( \frac{l + 2}{l - r + m} \right) \left( \frac{l - 2}{r} \right).$$

In the second sum $r = 0$ does not contribute. Hence after some simplifications, it can be re-expressed as,

$$\frac{4(l - 2)(l + 1)}{(l + 2)! (l - 2)!} \sum_{r=0}^{l-3-r} \left( \frac{l + 2}{l - r + m} \right) \left( \frac{l - 3}{r} \right).$$

We can evaluate both of these sums by using the Vandermonde Convolution property of binomial coefficients [53] which can be stated as,

$$\sum_{k=0}^{m} \binom{m}{k} \binom{p}{n-k} = \binom{m+p}{n}, m + p \geq n \& m, n, p \geq 0.$$

We finally obtain

$$S(l, t', m) = \frac{-2l^2 + 2l - 2ml + 2m + 4(l + 1)(l - 2)(2l - 1)!}{(l + m)! (l - m)! (l + 2)! (l - 2)!} + \frac{4(l + 1)(l - 2)(2l - 1)!}{(l + 2)! (l - 2)! (l - m)! (l - 1 + m)!} - \frac{4m(2l)!}{l(l + m)! (l - m)! (l + 2)! (l - 2)!},$$

(3.30)

which is the desired result and leads to $U(l, t', m)$ given in eq. (3.15) for the case $l = l'$. 

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We next show that

\[
S(l, l', m) = \begin{cases} 
\frac{2(2l')!}{(l + m)!(l-m)!(l' + 2)!} & l' = l + 1 \\
\frac{2(2l)'}{l' + m)!(l' - m)!(l' - 2)!} & l' = l - 1 
\end{cases}
\] (3.31)

Proof: we first consider the case \(l' = l + 1\). We can write eq. (3.25) as

\[
\mathcal{F}(l, l', r, t, m) = (-1)^{r+t} \left( \frac{l + 2}{l - r + m} \right) \left( \frac{l' - 2}{t} \right) \frac{(2r + 2t - 2m + 4 - l - l')}{(l' - 2)!(l + 2)!} \times \left[ \frac{(l + l' - r - t + m - 2)!}{(l - r - 2)!(l' - t + m)!} \right] \frac{(r + t - m + 2)!}{r!(t + 2 - m)!}.
\] (3.32)

After simplification of the terms in the two square brackets, this becomes

\[
\frac{(-1)^{r+t}(l + 2)!(l' - 2)!}{(l' - 2)!(l + 2)!} \left[ (2r + 2t - 2m + 4 - l - l') \prod_{s=1}^{l-2-r} (l' - t + m + s) \prod_{v=1}^{r} (t - m + v) \right].
\]

We can write the term in the square brackets above as \(a_0 t^{l-1} + a_1 t^{l-2} + \ldots + a_{l-1}\), where \(a_i \in \mathbb{Z}\). Keeping \(r\) fixed, the sum over \(t\) in eq. (3.24) yields,

\[
\sum_{t=0}^{l' - 2} \mathcal{F}(l, l', r, t, m) = \sum_{t=0}^{l' - 2} (-1)^{t} \left( \frac{l' - 2}{t} \right) \left[ a_0 t^{l-1} + a_1 t^{l-2} + \ldots + a_{l-1} \right].
\] (3.33)

By using eq. (3.32) on the left hand side and by comparing both sides, we find that \(a_0\) is equal to \(2(-1)^{l-2-r}\). Using the second case of eq. (A.3) we obtain

\[
\sum_{t=0}^{l' - 2} \mathcal{F}(l, l', r, t, m) = -2 (1)^r (l - 1)! = -2 (1)^r (l - 1)!
\]

Finally the sum over \(r\), after simplification, yields

\[
S(l, l', m) = -\frac{-2}{(l + 2)!(l - 2)!} \left[ \sum_{r=0}^{l-2} \left( \frac{l + 2}{l - r + m} \right) \left( \frac{l - 2}{r} \right) \right] = \frac{-2(2l)!}{(l + 2)!(l - 2)!(l + m)!(l - m)!},
\]

where we have again used the Vandermonde Convolution property. A similar analysis for the case \(l' = l - 1\) yields

\[
S(l, l', m) = -\frac{-2(2l')!}{(l' + 2)!(l' - 2)!(l' + m)!(l' - m)!}.
\]

These lead to the result for \(\mathcal{U}(l, l', m)\) given in eq. (3.15) for the cases \(l' = l \pm 1\).

4 Data analysis

For the case of temperature data, we perform a detailed analysis of the signal. This allows us to obtain updated results for the statistic \(S_{TT}^T\) with the 2015 Planck data. We studied this statistic in the multipole ranges \(2 \leq l \leq 64\), \(30 \leq l \leq 64\) and \(30 \leq l \leq 100\) in the Planck
2015 CMB intensity maps. The dipole modulation signal in the CMB temperature map was observed in the multipole range 2–64 [42].

For the case of CMB polarization a detailed analysis is not possible at this stage since the data for low \( l \) is unreliable. Furthermore we are unable to properly simulate the noise corresponding to PLANCK detectors. Hence, even for large \( l \), it is not possible for us to obtain a reliable estimate of the errors and the significance of the signal. For this reason we confine ourselves to a preliminary analysis of the polarization signal. However it may still serve a useful purpose in revealing the preferred direction indicated by data. We searched for modulation signal in the multipole ranges 40–100, 40–125, 50–100, 50–125, 50–150 and 50–200. The lower limit of \( l = 40 \) was chosen since the data for lower \( l \) is so far poorly understood.

4.1 Planck 2015 temperature data analysis

We have performed our analysis on both Commander and SMICA IQU maps. For SMICA maps, we have performed the analysis after masking the maps and then inpainting the masked maps using the MRS package of iSAP software or alternatively masking and then filling the masked portion of the map with isotropic data generated using the CAMB simulation package with Planck 2015 parameter set. The data analysis was performed with HEALPix software [54].

The analysis of the Commander and SMICA inpainted maps was performed identically. The maps were downgraded, without any smoothing to either \( N_{\text{side}} = 32 \), for multipole ranges 2–64 and 30–64, or \( N_{\text{side}} = 64 \), for 30–100 multipole range and the analysis was performed on these downgraded maps. We removed the dipole and monopole from CMB intensity maps and calculated the quantities \( \sum_l C_{TT}^{l(l+1)} \) and \( \sum_l l(l+1)C_{TT}^l \) over the aforementioned multipole ranges.

We also used another method to analyze the SMICA maps. In this case the SMICA temperature maps were first masked with their respective masks. The pixels which were masked were filled with data from temperature simulated maps generated using lensed scalar \( C_l \) values generated with the CAMB Boltzmann solver [55] for the Planck 2015 parameters [56], and the Synfast program from the HEALPix package. The masked and filled temperature maps have \( N_{\text{side}} = 2048 \), same as that of the original maps. These maps were smoothed with a FWHM equal to 3 times the pixel size of the low resolution map with \( N_{\text{side}} = 256 \) before downgrading the map to remove the discontinuities at the mask boundary. The smoothed maps are then degraded to \( N_{\text{side}} = 256 \) followed by a further degradation to \( N_{\text{side}} = 32 \) or \( N_{\text{side}} = 64 \), depending on the multipole range, for final analysis. We generated 100 such filled maps and the individual results were found to depend on the random realization used to fill the masked portions of the sky. The results presented here are the mean values for the statistic \( S_{HH}^{TT} \) and direction for which it maximizes. The SMICA temperature maps masked and reconstructed using these two procedures, i.e. inpainting and random filling, are shown in figure 1.

We have fitted the TT mode values of \( S_{HH} \) in order to extract the value of the dipole modulation amplitude \( A \) of eq. (1.2). We simulated 100 isotropic CMB maps using Planck 2015 parameters with \( N_{\text{side}} = 512 \) (\( N_{\text{side}} = 1024 \) for SMICA filled analysis). These maps were rotated to have the z-axis pointing along the direction of maximum statistic and they were modulated using eq. (1.1). The modulated maps were downgraded to \( N_{\text{side}} = 32 \). Each of the downgraded simulated maps were fitted for the value of \( A \) that would give the value of \( S_{HH} \) closest
Figure 1. Left: SMICA inpainted temperature map. Right: SMICA temperature map with the masked portions filled by isotropic randomly generated data. Both maps are for $N_{\text{side}} = 32$ and the temperature is given in units of K.

| Map               | $S_{TT}^H$ in $10^{-2}$ mK$^2$ | $A$             | $(l, b)$                        | P-value  |
|-------------------|-------------------------------|-----------------|--------------------------------|----------|
| Commander         | $2.55 \pm 0.68$              | $0.082 \pm 0.018$ | $(232^\circ \pm 18^\circ, -14^\circ \pm 18^\circ)$ | 0.20%    |
| SMICA(inp.)       | $2.39 \pm 0.70$              | $0.069 \pm 0.013$ | $(236^\circ \pm 27^\circ, -11^\circ \pm 20^\circ)$ | 0.70%    |
| SMICA(filled)     | $2.44 \pm 0.71$              | $0.078 \pm 0.019$ | $(242^\circ \pm 16^\circ, -17^\circ \pm 20^\circ)$ | 0.50%    |

Table 1. The maximum TT Mode $S_H$ values along with the dipole modulation parameter $A$, the preferred direction of maximization and the P-value.

to the one observed in data along the best fit direction. We averaged over 100 best fit values of $A$ obtained by this method giving the modulation amplitude for the results given in table 1.

To estimate the error in $S_{TT}^H$ we generated 1000 maps at $N_{\text{side}} = 512$ ($N_{\text{side}} = 1024$ for SMICA filled analysis) and modulated the maps with the best-fit value of $A$ along the direction of maximum statistic using relation (1.2). The modulated CMB maps were downgraded to $N_{\text{side}} = 32$ or $N_{\text{side}} = 64$ depending on the multipole range under consideration and $S_{TT}^H$ was calculated along the direction of modulation. The standard deviation of the 1000 simulated maps gives the error in $S_{TT}^H$. For the masked and filled SMICA maps, the process of filling the masked region of the SMICA maps with isotopic data is expected to introduce bias in the obtained results for $S_H$. This bias correction is relatively small [42] and we ignore it in our analysis. It is expected to enhance the signal by about 8%.

The error estimation in the preferred direction was performed by simulating 50 isotropic temperature maps. We modulated these maps with the best-fit value of $A$ along the observed direction in the data. For each simulated map we determine the direction along which the statistic $S_{TT}^H$ maximises. The standard deviation of resulting direction parameters provide an estimate of the required error.

Finally to test the significance of our results we simulate 2000 isotropic CMB maps for 2–64 multipole range and 500 maps for the other multipole ranges, with Planck 2015 parameters and search for the direction along which the statistic $S_{TT}^H$ maximizes. The values of $S_H$ obtained by this process is used to obtain probability distribution of the statistic $S_H$ for the isotropic hypothesis. The histogram is given in figure 2. P-values for individual results were obtained as the percentage of simulation results that equal or exceed the observed result for the statistic.
4.2 Planck 2015 polarization data analysis

For the case of polarization, as explained above, we are unable to perform a detailed analysis of dipole modulation. Here we confine ourselves to simply making an estimate of the statistic $S_H^{EE}$ and the corresponding preferred direction using the Commander map. We do not make an attempt to compute the statistical significance of the extracted signal. In the absence of such an analysis we do not have any information about the reliability of the extracted signal. Here we are performing this analysis only to show the utility of our formalism and not to claim the existence of an effect in CMB polarization. Since the low $l$ multipoles are not expected to be reliable we confine our study to the multipole ranges 40–100, 40–125, 50–100 and 50–125. Furthermore we use the dipole modulation model for polarization in order to generate simulated maps which display polarization power anisotropy and to determine the distributions of the corresponding statistic $S_H$ for the $E$ mode polarization.

5 Results

In this section we first present the results for the temperature analysis and later those of polarization.

5.1 Temperature

The TT mode results for multipole range $2 \leq l \leq 64$ are summarized in table 1. As stated in the previous section the dipole modulation in the CMB temperature signal is present in
lower multipoles up to \( l = 64 \). Our primary TT mode results are for this multipole range. We can compare our results with [42] to look for changes between the Planck 2013 and Planck 2015 data. The results for SMICA filled maps for Planck 2013 data are: \( S_{TT}^{S} = (2.1 \pm 0.5) \times 10^{-2}\text{mK}^2 \); a bias corrected value of \((2.3 \pm 0.6) \times 10^{-2}\text{mK}^2 \), along \((229^\circ, -16^\circ)\) with \( A = 0.074 \pm 0.019 \). Comparing the 2013 and 2015 SMICA filled map results we notice agreement in both the values of \( S_{TT}^{S} \) and direction of maximization, while the value of \( A \) is also comparable within the error limits. The results for SMICA inpainted map for Planck 2013 data are: \( S_{TT}^{S} = (2.7 \pm 0.7) \times 10^{-2}\text{mK}^2 \), along \((232^\circ, -12^\circ)\). For SMICA inpainted maps too, we find good agreement with previous results. We also note that the 2015 results have smaller P-values when compared with isotropic ΛCDM simulations generated using 2015 Planck parameters. We can compare our results with Planck Collaboration’s analysis of dipole modulation [46]. Planck 2015 best fit values of modulation amplitude \( A \) is \( 0.063^{+0.025}_{-0.013} \) for Commander maps and \( 0.062^{+0.026}_{-0.013} \) for SMICA map. The direction of modulation is \((213^\circ, -26^\circ) \pm 28^\circ \) for both SMICA and Commander maps. Both the modulation amplitude and the direction agree with our results within the quoted errors. Our results are also consistent with those obtained in [57].

It has been found by an earlier analysis that the hemispherical anisotropy effect rapidly dies out when the lower multipoles are excluded or when summed to higher multipoles [14, 15, 46, 58]. In order to study the multipole dependence we also investigate the effect in the multipole ranges 30–64 and 30–100. The \( S_{TT}^{S} \) obtained in 30–64 for Commander map is contained in table 2. In this case we do not find a significant signal of \( l, l+1 \) correlation. The temperature map \( S_{TT}^{S} \) is very much in agreement with expectations from isotropic theory. We also notice that the \( S_{TT}^{S} \) maximizes along directions which change from one range to another. This can also be seen from figure 3. These maps show \( S_{TT}^{S} \) for different directions in ranges 2–64, 30–64 and 30–100. It can be seen that left to right the dipole pattern becomes less distinct.

The SMICA inpainted TT mode results are as follows:

- In the range 30–64, \( S_{TT}^{S} \) maximizes at \((1.24 \pm 0.43) \times 10^{-2}\text{mK}^2 \) along \((191^\circ \pm 27^\circ, -4^\circ \pm 28^\circ) \) with a P-value of 45% and \( A = 0.059 \pm 0.019 \).
| $l$ Range | $S_{EE}^{S}$ in $10^{-6}$ mK$^2$ | $(l, b)$ | $R$ |
|-----------|-------------------------------|----------|-----|
| 40–100    | 6.6                           | (260$^\circ$, 44$^\circ$) | 0.031 |
| 40–125    | 9.4                           | (291$^\circ$, 14$^\circ$) | 0.023 |
| 50–100    | 6.6                           | (286$^\circ$, 15$^\circ$) | 0.033 |
| 50–125    | 9.5                           | (291$^\circ$, 14$^\circ$) | 0.025 |

Table 3. EE mode $S_{H}$ results for Planck Commander 2015 maps in different multipole ranges.

- For 30–100 range, $S_{EE}^{TT}$ maximizes at $(1.45 \pm 0.71) \times 10^{-2}$ mK$^2$ along $(207^\circ \pm 38^\circ, -18^\circ \pm 27^\circ)$ with a P-value of 41.2% and $A = 0.027 \pm 0.013$.

The SMICA filled map temperature results are:

- In 30–64 range, $S_{EE}^{TT}$ maximizes at $(1.00 \pm 0.43) \times 10^{-2}$ mK$^2$ along $(228^\circ \pm 28^\circ, -9^\circ \pm 27^\circ)$ with a P-value of 66% and $A = 0.055 \pm 0.021$.

- The 30–100 range $S_{EE}^{TT}$ maximizes at $(1.64 \pm 0.77) \times 10^{-2}$ mK$^2$ along $(238^\circ \pm 34^\circ, -11^\circ \pm 28^\circ)$ with a P-value of 27% and $A = 0.031 \pm 0.014$.

Both SMICA inpainted and filled maps results show similar patterns as Commander map results.

### 5.2 Polarization

In this section we present our results for polarization. As explained above, we do not make an attempt to compute either the errors or the significance of this effect due to uncertainties in the noise simulation in polarization. We perform this analysis only to demonstrate how our formalism may be used when reliable data may become available. Furthermore we present the results of a simulation in order to illustrate the utility of the polarization dipole modulation model, eq. (3.2).

In table 3 we give the results for $S_{EE}^{S}$, the preferred direction and the ratio $R$ for the $E$ mode polarization. As emphasized earlier, this signal may simply arise due to a statistical fluctuation since we do not know the statistical significance of the effect. In any case, the result for the case of the multipole range 40–100 is mildly interesting. This is because the preferred direction aligns closely with the CMBR dipole. A preferred direction similar to the one obtained in the range 40–100 has been seen in many other studies [21, 22], including the radio polarization dipole axis [23], the CMB quadrupole — octopole alignment axis [20], the NVSS dipole [27–30], the radio polarization flux dipole axis [31] as well as the optical polarizations from distant quasars [21, 24–26]. It is possible that the present signal in $E$ mode polarization in this range arises from some residual contamination of the low $l$ noise systematic bias. Alternatively it may be a signal of some astrophysical or cosmological effect. This may be settled by future refinements in data.

For the higher multipole ranges the preferred direction starts to deviate. For example, in the multipole range 40–125, 50–100, 50–125 it lies closer to the galactic plane. This might be an indication that it is moving closer to the axis obtained in the case of temperature. Alternatively, since it lies very close to the galactic plane, the signal in this range might be dominated by foregrounds. The plots of $S_{EE}^{S}$ for the multipole range 40–100 and 40–125 are shown in figure 4.
Figure 4. The $S_{\mu}^{EE}$ sky maps for Commander data in multipole ranges 40–100 (left) and 40–125 (right).

Figure 5. The $S_{H}^{EE}$ histogram for the simulated dipole modulated $E$ mode polarization maps with the parameter $A_1 = 0.05$.

We next generate 1000 simulated $Q$ and $U$ mode polarization maps which include dipole modulation with parameter $A_1 = 0.05$ for the multipole range 40–100. We compute the statistics $S_{H}^{EE}$ for these simulated maps. The resulting distribution of this statistics is shown in figure 5. We find that the distribution is close to normal. The parameter chosen is not too far from what might be required to fit the $E$ mode results given in table 3.

6 Conclusions

This work has updated the previous results for dipole modulation in CMB temperature field. We find that the dipole modulation of the temperature field, observed in WMAP data and Planck 2013 data is also present in Planck 2015 data. We have computed the quantity $S_{TT}$ and the results are found to be in agreement with the values obtained from the previous
Our result for modulation parameter $A$ and modulation direction are compatible with Planck 2015 results [46].

We have presented a preliminary analysis of the dipole modulation in the CMBR polarization. In this case we confine our analysis only to the multipole values $l > 40$ since the lower $l$ values are not expected to be reliable. The detector noise is expected to make considerable contribution in this case. Since this information is not available to us, we do not attempt to compute the significance of the signal or the associated error in extracted parameters. In any case our preliminary investigation reveals some interesting results about the direction of the signal. We find that for the range of multipoles 40–100 the preferred direction is close to the CMBR dipole and for a slightly higher range 50–100 or 40–125, the direction shifts closer to the galactic plane. The direction found for the lower range of multipoles is somewhat interesting since a similar direction has been found in several other data sets [20–23, 27–31].

For the case of polarization we also propose a dipole modulation model, eq. (3.2). This is a useful model which generalizes the temperature dipole modulation [9–12], eq. (1.1), to the case of polarization. We find that this model leads to correlations between $l$ and $l + 1$ multipoles of the polarization field. We determine the form of these correlations for the case of $EE$, $BB$ and $TE$ fields. We show the utility of this model by creating simulated polarization maps with non-zero values of the dipole modulation parameter, $A_1$. We find that distribution of the resulting statistic for the case of $E$ mode polarization is well described by a Gaussian.

We expect that this dipole modulation model and our proposed analysis procedure may be very useful in the study of anisotropy which might exist in the CMBR polarized field.

## A Derivation of properties of binomial coefficients

In this appendix we derive the mathematical results required in proving the results in section 3.2. Let $f(n, p)$ be defined as

$$f(n, p) = \sum_{r=0}^{n} r^p (-1)^r \binom{n}{r}$$ \hspace{1cm} (A.1)

where $n, p \geq 0$. This function satisfies the recurrence relation

$$f(n, p + 1) = \begin{cases} (-n) \sum_{q=0}^{p} f(n - 1, q) \binom{p}{q} & n \neq 0 \\ 0 & n = 0 \end{cases}.$$ \hspace{1cm} (A.2)

Proof: the result for $n = 0$ can be verified by direct substitution. For $n > 0$, we can write $f(n, p + 1)$ as

$$f(n, p + 1) = n \sum_{r=0}^{n} (-1)^r r^p \frac{(n - 1)!}{(n - r)!(r - 1)!}.$$ \hspace{1cm} (A.3)

On the right hand side the term corresponding to $r = 0$ is zero due to the factor $(r - 1)!$ in the denominator. Hence we can start the sum from $r = 1$. Setting $r - 1 = t$, we obtain

$$f(n, p + 1) = -n \sum_{t=0}^{n-1} (-1)^t (1 + t)^p \frac{(n - 1)!}{(n - t - 1)!!} = -n \sum_{t=0}^{n-1} \binom{p}{t} \sum_{q=0}^{p} (-1)^q \binom{n - 1}{t} t^q,$$

where we have used the binomial theorem. Furthermore

$$f(n - 1, q) = \sum_{t=0}^{n-1} (-1)^t \binom{n - 1}{t} t^q,$$
and hence
\[ f(n, p + 1) = -n \sum_{q=0}^{p} f(n - 1, q) \binom{p}{q}. \]

This proves the result in eq. (A.2) for \( n \neq 0 \).

The function \( f(n, p) \) defined in eq. (A.1) is given by
\[
f(n, p) = \begin{cases} 
0 & p < n, \, n \neq 0 \\
(-1)^n n! & p = n, \, n \geq 0 \\
\frac{n(n+1)}{2}(-1)^n n! & p = n + 1, \, n \geq 0 
\end{cases}
\] (A.3)

Proof: the first part can be proven by using the Corollary 2 of [59] with \( x = 0 \) and \( p = n - j \). The result follows since \( p < n \) when \( 1 \leq j \leq n \). Furthermore the second part can be obtained by using Theorem 1 of [59] with \( x = 0 \).

Finally we consider the last part. By using eq. (A.2) we find that for \( n = 0, \, p = n + 1, \) \( f(0, p) = 0 \), which agrees with the result given in eq. (A.3). We next consider \( n > 0 \). By using eq. (A.2) we obtain
\[ f(n, n + 1) = -n \sum_{q=0}^{n} f(n - 1, q) \binom{n}{q} = -n \left[ f(n - 1, n - 1) \binom{n}{n-1} + f(n - 1, n) \right]. \]

That is, the only nonvanishing terms in this sum are obtained by setting \( q = n - 1 \) and \( q = n \).

Using the second case of eq. (A.3) this can be further simplified in the form of the following recurrence relation:
\[ P(n) = -n \left[ -(-1)^n n! + P(n - 1) \right], \] (A.4)
where \( P(n) = f(n, n + 1) \). We next proceed by induction. For \( n = 1 \), by using A.1 we obtain \( P(1) = f(1, 2) = -1 \), which agrees with the result given in eq. (A.3). We next assume that this result is true for \( n = k \), i.e.,
\[ P(k) = \frac{k(k+1)}{2}(-1)^k k! \]
and show that it is also true for \( P(k + 1) \). By using recurrence relation A.4 we obtain
\[ P(k + 1) = -(k + 1) \left[ -(-1)^{k+1} (k + 1)! + \frac{k(k+1)}{2}(-1)^k k! \right] = \frac{(k+1)(k+2)}{2}(-1)^{k+1} (k + 1)! \]
which agrees with eq. (A.3). Hence the third part is also proven by induction.

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