An analogue of the Szemeredi Regularity Lemma
for bounded degree graphs

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Abstract

We show that a sufficiently large graph of bounded degree can be decomposed into quasihomogeneous pieces. The result can be viewed as a “finitarization” of the classical Farrell-Varadarajan Ergodic Decomposition Theorem.

1 Introduction

In order to state our result we need to recall some basic definitions. Let $\text{Graph}_d$ denote the set of all connected finite simple graphs $G$ (up to isomorphism) for which $\text{deg}(x) \leq d$ for every $x \in V(G)$. For a graph $G$ and $x, y \in V(G)$ let $d_G(x, y)$ denote the distance of $x$ and $y$, that is the length of the shortest path from $x$ to $y$. A rooted $(r, d)$-ball is a graph $G \in \text{Graph}_d$ with a marked vertex $x \in V(G)$ called the root such that $d_G(x, y) \leq r$ for every $y \in V(G)$. By $U_{r,d}$ we shall denote the set of rooted $(r, d)$-balls (up to rooted isomorphism).

If $G \in \text{Graph}_d$ is a graph and $x \in V(G)$ then $B_r(x) \in U_{r,d}$ shall denote the rooted $(r, d)$-ball around $x$ in $G$. For any $\alpha \in U_{r,d}$ and $G \in \text{Graph}_d$ we define the set $T(G, \alpha) \overset{\text{def}}{=} \{x \in V(G) : B_r(x) \cong \alpha\}$ and let $p_G(\alpha) = \frac{|T(G, \alpha)|}{|V(G)|}$.

Fix an enumeration of all possible $(r, d)$-balls, for every $r \geq 1$: $(\alpha_1, \alpha_2, \ldots)$. Let us define the statistical distance of two graphs $G$ and $H$ as

$$d_s(G, H) = \sum_{i=1}^{\infty} \frac{1}{2^i} |p_G(\alpha_i) - p_H(\alpha_i)|.$$ 

It is easy to see that $d_s(G, H)$ defines a metric on $\text{Graph}_d$. We define $d_s$ for not necessary connected graphs as well. In this case $d_s$ defines a pseudo-distance.
Remark 1.1 Note that if $F_1, F_2, \ldots, F_k$ are finite connected graphs then for any $\epsilon > 0$ there exists a $\delta > 0$ such that if $d_\delta(G,H) \leq \delta$ then for any $1 \leq i \leq k$ 
\[ |\text{dens}(F_i,G) - \text{dens}(F_i,H)| \leq \epsilon, \]
where
\[ \text{dens}(F_i,G) = \frac{\text{the number of subgraphs of } G \text{ isomorphic to } F_i}{|V(G)|}. \]

The number $\text{dens}(F_i,G)$ is called the “sparse” $F_i$-density of $G$.

If $J \subset G$ is a spanned subgraph then $E(J,G,J)$ denotes the number of edges between the vertices of $J$ and $G \setminus J$. The following is our key definition.

Definition 1.1 $G \in \text{Graph}_d$ is called $(\epsilon, \lambda, \delta)$-quasihomogeneous, if for any spanned subgraph $H \subset G$ such that 
\[ \lambda \leq \frac{|V(H)|}{|V(G)|} \]
\[ E(H,G \setminus H) \leq \epsilon|V(G)| \]

we have $d_\epsilon(G,H) \leq \delta$.

Informally speaking $G$ is “quasihomogeneous” if for any large enough spanned graph $H$ which is connected to $G \setminus H$ by only a small amount of edges, the subgraph densities of $G$ and $H$ are very similar. Now let us recall the regularity lemma of Szemeredi for dense graphs. Let $G$ be a graph and $X \subset V(G)$, $Y \subset V(G)$ be disjoint subsets. The density of the pair $X,Y$ is 
\[ p(X,Y) := \frac{|E(X,Y)|}{|X||Y|}. \]

The pair $X,Y$ is called $\epsilon$-quasirandom if for any subsets $A \subset X$, $B \subset Y$, 
\[ |A| \geq \epsilon|X|, |B| \geq \epsilon|Y| \]
\[ |p(A,B) - p(X,Y)| < \epsilon. \]

Now we have a similar subgraph counting principle as in Remark 1.1. If $F_1$, $F_2, \ldots, F_k$ are fixed simple connected graphs then for any $\gamma > 0$ there exists a $\epsilon > 0$ such that if $X,Y$ is a $\epsilon$-quasirandom pair, and $A,B$ are subsets as above then for any $1 \leq i \leq k$:
\[ |\text{density}(F_i,E(X,Y)) - \text{density}(F_i,E(A,B))| \leq \gamma, \]
where
\[ \text{density}(F_i,G) = \frac{\text{the number of subgraphs of } G \text{ isomorphic to } F_i}{|V(G)|^{||F_i||}}. \]

The number density($F_i,G$) is called the $F_i$-density of $G$. That is quasirandomness also implies a certain kind of quasihomogeneity. According to the Szemeredi Regularity Lemma for any $\epsilon > 0$ there exists $K(\epsilon) > \frac{1}{\epsilon}$ and $N(\epsilon) > 1$ such that for any graph $G$ with $|V(G)| \geq N(\epsilon)$ one can remove $\epsilon|V(G)|^2$ edges from $G$ such that the vertices of the remaining graph $G'$ can be partitioned into $\frac{1}{2} \leq K \leq K(\epsilon)$ parts (of almost equal size) $V(G') = V(G'_1) \cup V(G'_2) \cup \cdots \cup V(G'_{K(\epsilon)})$ and all pairs $(G'_i, G'_j)$ are $\epsilon$-quasirandom. Our main theorem might be considered as a bounded degree analogue of the Szemeredi Regularity Lemma:
**Theorem 1** For every $\delta > 0, \lambda > 0$ there exist positive integers $K = K(\delta, \lambda)$, $N = N(\delta, \lambda)$ and a positive constant $\varepsilon = \varepsilon(\delta, \lambda) < \delta$ for which the following hold:

a) If $G \in \text{Graph}_d$ is a finite connected graph that has at least $N$ vertices, then it can be partitioned into $K + 1$ disjoint graphs $G = G_1 \cup G_2 \cup \ldots G_K \cup G_\emptyset$ by deleting at most $\delta|E(G)|$ edges such that

- $G_\emptyset$ is an edge-less graph, $\frac{|V(G_\emptyset)|}{|V(G)|} < \delta$
- Either $\frac{|V(G_i)|}{|V(G)|} > \frac{\delta^2}{10\pi K}$ or $V(G_i)$ is empty.
- All non-empty parts are $(\varepsilon, \lambda, \delta)$-quasihomogeneous.

b) For any $\sigma > 0$ there is a $\tau > 0$ such that whenever $d_s(G, H) < \tau$ then there exist partitions for which $d_s(G_i, H_i) < \sigma$ and $\left| \frac{|V(G_i)|}{|V(G)|} - \frac{|V(H_i)|}{|V(H)|} \right| < \sigma$ for all $i$'s where either $G_i$ or $H_i$ is non-empty.

The second statement of the theorem can be interpreted that if two graphs are close in terms of “sparse” subgraph densities then they have similar quasihomogeneous partitions. One should note that according to the result of Borgs, Chayes, Lovasz, T.Sós and Vesztergombi [3] if two dense graphs are close in terms of their subgraphs densities then they have similar Szemeredi partitions. The proof of the theorem is based on a “finitarization” of the Farrell-Varadarajan Ergodic Decomposition Theorem. The necessary background on ergodic theory and its connections to graph theory shall be surveyed in Section 2. The proof of Theorem 1 shall be given in Section 4.

## 2 Ergodic theory

### 2.1 Borel equivalence relations and invariant measures

In this subsection we recall some basic notions from Chapter I. of [5] on countable Borel equivalence relations. Let $X$ be a compact metric space, $E \subset X \times X$ is a countable Borel equivalence relation if all the equivalence classes are countable and $E$ is a Borel subset of $X \times X$. Typical example of a Borel equivalence relation is the orbit equivalence relation of a Borel action of a countable group. As a matter of fact according to the theorem of Feldman and Moore any Borel equivalence relation can be described this way. A Borel probability measure $\mu$ on $X$ is $E$-invariant if its invariant under a countable group action that defines $E$. Note that in this case $\mu$ is invariant under all the group action that defines $E$. Equivalently, $\mu$ is invariant if for any Borel isomorphism $f : X \rightarrow X$ $f_*(\mu) = \mu$, that is if $A \subseteq X$ is a Borel-set then $\mu(f^{-1}(A)) = \mu(A)$. The space of invariant probability measures is denoted by $\mathcal{I}_E$. A measurable set
A ⊆ X is called E-invariant if for any \( x \in A \) and \( y \in X \), \( x \sim_E y \) \( y \in A \). The invariant measure \( \mu \) is called ergodic if the \( \mu \)-measure of any invariant set is either 0 or 1. The space of ergodic probability measures is denoted by \( EI_E \). Note that the set of probability measures on \( X \), \( P(X) \) is compact convex set of the topological vectorspace of all measures in the weak-topology (Banach-Alaoglu Theorem). The space \( I_E \) is a convex subset of \( P(X) \) and \( EI_E \) can be identified as the set of extremal points in \( I_E \).

Our main tool will be the following well known result (see e.g. [5]):

**Proposition 2.1 (Ergodic Decomposition – Farrell, Varadarajan)** Let \( E \) be a countable Borel equivalence relation on \( X \). Then \( I_E, EI_E \) are Borel sets in the standard Borel space \( P(X) \) of probability measures on \( X \). Now suppose \( I_E \neq \emptyset \). Then \( EI_E \neq \emptyset \), and there is a Borel surjection \( \pi : X \rightarrow EI_E \) such that

a) \( \pi \) is \( E \)-invariant,

b) if \( X_e = \{ x : \pi(x) = e \} \), then \( e(X_e) = 1 \) and \( E|X_e \) has a unique invariant measure, namely \( e \), and

c) if \( \mu \in I_E \), then \( \mu = \int \pi(x) d\mu(x) \).

Moreover, \( \pi \) is uniquely determined in the sense that, if \( \pi' \) is another such map, then \( \{ x : \pi(x) \neq \pi'(x) \} \) is null with respect to all measures in \( I_E \).

We need a simple observation on the space \( P(X) \). The space \( P(X) \) is metrizable in the weak-topology by

\[
d_X(\mu, \nu) := \sum_{n=1}^{\infty} \frac{1}{2^n}|(\mu - \nu)(f_n)|,
\]

where \( \{f_n\}_{n=1}^{\infty} \) is a countable dense set in the unit ball of \( C(X) \) (the space of continuous functions). Here \( \mu(f) \) denotes \( \int_X f d\mu \).

**Lemma 2.1**

- If \( \mu_n \rightarrow \mu, \nu_n \rightarrow \nu \) in \( P(X) \) and \( \lambda_n \rightarrow \lambda \) in \( \mathbb{R} \) then \( \sum_{n=1}^{\infty} \lambda_n \mu_n + (1 - \lambda_n) \nu_n \rightarrow \lambda \mu + (1 - \lambda) \nu \).

- If \( T \subset P(X) \) is an arbitrary subset then if \( p \) is in the convex hull of \( T \) (the closure of the finite convex combinations in \( T \)) then \( d_X(p, T) \leq \text{diam}(T) \).

That is:

\[
\text{diam}(\text{hull}(T)) \leq 3\text{diam}(T).
\]

**Proof.** The first statement is a straightforward consequence of the definition of weak convergence. Now let \( w, v_1, v_2, \ldots, v_k \in T, \lambda_i \geq 0, \sum_{i=1}^{k} \lambda_i = 1 \). It is enough to prove that

\[
d_X(\sum_{i=1}^{k} \lambda_i v_i, w) \leq \sum_{i=1}^{k} \lambda_i d_X(v_i, w).
\]
\[ d_X \left( \sum_{i=1}^{k} \lambda_i v_i, w \right) = \sum_{n=1}^{\infty} \frac{1}{2^n} \left| \left( \sum_{i=1}^{k} \lambda_i v_i \right) - w \right| (f_n) \leq \sum_{n=1}^{\infty} \frac{1}{2^n} \sum_{i=1}^{k} \lambda_i \left| (v_i - w) (f_n) \right| = \sum_{i=1}^{k} \lambda_i d_X (v_i, w). \]

2.2 Borel-graphings

Let \( X \) be a standard Borel-space and \( R \subset X^2 \) be a Borel-set which is a symmetric and irreflexive relation. This structure is called a Borel-graphing and denoted by \( G = (X, R) \). If \( xRy \) then we say that \( x \sim y \) is an edge of \( G \). Let \( \Gamma \) be a discrete group acting on a space \( X \) in a Borel way and let \( S \) be a generating system of \( \Gamma \). Let \( x \) and \( y \) be in relation \( R \) if \( x \neq y \) and \( sx = y \) for some \( s \in S \). Then \( R \) is a Borel-graphing. The connected components of \( G \) are countable graphs on the orbit of the \( \Gamma \)-action. We shall use the following result of Kechris, Solecki and Todorcevic [6]: Any Borel-graphing with vertex degree bound \( d \) has a Borel-coloring by \( d + 1 \)-colors. That is there is a partition of \( X \) into \( d + 1 \) Borel pieces such that if two points are in the same piece, then they are not adjacent in \( G \).

2.3 Limits of graph sequences

In this subsection we briefly recall the notion of weak graph convergence from [2]. A graph sequence \( G = \{G_n\}_{n=1}^{\infty} \subset \text{Graph}_d \) is weakly convergent if \( \lim_{n \to \infty} |V(G_n)| = \infty \) and for every \( r \) and every \( \alpha \in U_{r,d} \) the limit \( \lim_{n \to \infty} p_{G_n}(\alpha) \) exists.

Let \( \text{Gr}_d \) denote the set of all countable, connected rooted graphs \( G \) for which \( \deg(x) \leq d \) for every \( x \in V(G) \). If \( G, H \in \text{Gr}_d \) let \( d_G(G, H) = 2^{-r} \), where \( r \) is the maximal number such that the \( r \)-balls around the roots of \( G \) resp. \( H \) are rooted isomorphic. The distance \( d_G \) makes \( \text{Gr}_d \) a compact metric space. Given an \( \alpha \in U_{r,d} \) let \( T(\text{Gr}_d, \alpha) = \{(G, x) \in \text{Gr}_d : B_r(x) \cong \alpha\} \). The sets \( T(\text{Gr}_d, \alpha) \) are closed open sets that generate the Borel structure of \( \text{Gr}_d \).

We can equip \( \text{Gr}_d \) with an equivalence relation \( E \): two rooted graphs \( G, H \) are equivalent \( (G \sim_E H) \) if they are isomorphic as graphs (but this isomorphism need not respect the root). It is easy to see that \( E \) is a countable Borel equivalence relation. Also, convergent graph sequences define a limit measure \( \mu_G \) on \( \text{Gr}_d \), where \( \mu_G(T(\text{Gr}_d, \alpha)) = \lim_{n \to \infty} p_{G_n}(\alpha) \). Note however that \( \mu \) is not necessary an invariant measure on \( \text{Gr}_d \). That is why we need the notion of \( C \)-graphs.

2.4 \( C \)-graphs and the space \( \text{CGr}_d \)
In this subsection we extend our definitions for edge-colored graphs. A C-graph is a graph with edges properly colored by the set \(\{c_1, c_2, \ldots, c_{d^2+1}\}\). That is each edge is labeled by an element of \(\{c_1, c_2, \ldots, c_{d^2+1}\}\) and incident edges are labeled differently. The reason why we use \(\binom{d^2+1}{2}\) colors will be made clear in the last section. We shall denote by \(\text{C-Graph}_d\) the set of finite connected C-graphs (up to colored isomorphisms).

Let \(V^{r,d}\) be the set of rooted \((r,d)\)-balls edge-colored by \(\{c_1, c_2, \ldots, c_{d^2+1}\}\).

Remark 2.1 We introduce a metric for \(\text{C-Graph}_d\) as in the Introduction. Let \(\Psi : \text{C-Graph}_d \to \text{Gr}_d\) be the isomorphism classes of all connected countable rooted \(\text{C-Graph}_d\). We call \(\mu\) the limit measure of \(\{CG_n\}_{n=1}^\infty\). That is each edge is labeled by an element of \(\{c_1, c_2, \ldots, c_{d^2+1}\}\), hence Lemma 2.1 applies. Note that there is a natural metric on \(\text{C-Graph}_d\). If \(\beta \in \text{C-Graph}_d\) then

\[
d_C(X, Y) = 2^{-\tau},
\]

where \(\tau\) is the maximal number such that \(B^r(x) \cong B^r(y)\), where \(x\) is the root of \(X\), \(y\) is the root of \(Y\). The subsets \(T(\text{C-Graph}_d, \beta) : \beta \in V^{r,d}, r \in \mathbb{N}\) are closed-open sets and generate the Borel-structure of \(\text{C-Graph}_d\).

We define the statistical distance of two C-graphs \(CG\) and \(CH\) as

\[
d_s(CG, CH) = \sum_{i=1}^\infty \frac{1}{2i} |\mu(CG, \beta_i) - \nu(CH, \beta_i)|,
\]

where \((\beta_1, \beta_2, \ldots)\) is an enumeration of all the edge-colored \((r,d)\)-balls, for all \(r \geq 1\).

Let \(\text{C-Graph}_d\) be the isomorphism classes of all connected countable rooted C-graphs with vertex degree bound \(d\). For \(\beta \in V^{r,d}\) we define \(T(\text{C-Graph}_d, \beta) = \{x \in \text{C-Graph}_d : B_r(x) \cong \alpha\}\). Again, we have a natural metric on \(\text{C-Graph}_d\). If \(X, Y \in \text{C-Graph}_d\) then

\[
d_C(X, Y) = 2^{-\tau},
\]

where \(\tau\) is the maximal number such that \(B^r(x) \cong B^r(y)\), where \(x\) is the root of \(X\), \(y\) is the root of \(Y\). The subsets \(T(\text{C-Graph}_d, \beta) : \beta \in V^{r,d}, r \in \mathbb{N}\) are closed-open sets and generate the Borel-structure of \(\text{C-Graph}_d\).

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where \((\beta_1, \beta_2, \ldots)\) is an enumeration of all the edge-colored \((r,d)\)-balls, for all \(r \geq 1\).

Remark 2.1 We introduce a metric for \(P(\text{C-Graph}_d)\) the following way. Let \((\beta_1, \beta_2, \ldots)\) be an enumeration of the classes \(V^{r,d}, r \geq 1\). Let

\[
d_{\text{C-Graph}_d}(\mu, \nu) := \sum_{i=1}^\infty \frac{1}{2i} |\mu(T(\text{C-Graph}_d, \beta_i)) - \nu(T(\text{C-Graph}_d, \beta_i))|.
\]

Clearly \(d_{\text{C-Graph}_d}\) metrize the weak-topology of \(P(\text{C-Graph}_d)\). Also, \(\mu(T(\text{C-Graph}_d, \beta_i))) = \int_{\text{C-Graph}_d} 1_{T(\text{C-Graph}_d, \beta_i)} \, d\mu\), hence Lemma 2.1 applies. Note that there is a natural forgetting map \(\Psi : \text{C-Graph}_d \to \text{Gr}_d\). Again we can consider an enumeration \((\alpha_1, \alpha_2, \ldots)\) of the classes \(U^{r,d}, r \geq 1\) as in the Introduction. Let

\[
d_{\text{Gr}_d}(\mu, \nu) := \sum_{i=1}^\infty \frac{1}{2i} |\mu(T(\text{Gr}_d, \alpha_i)) - \nu(T(\text{Gr}_d, \alpha_i))|.
\]

The distance \(d_{\text{Gr}_d}\) metrize the weak-topology of \(P(\text{Gr}_d)\). By compactness, for any \(\epsilon > 0\), there exists \(\delta > 0\) such that if \(d_{\text{C-Graph}_d}(\mu, \nu) < \delta\) then \(d_{\text{Gr}_d}(\Psi_* \mu, \Psi_* \nu) < \epsilon\).
Also note that for any finite $C$-graph $C_G$ one can associate a measure on $CGr_d$ concentrated on finitely many points. Simply define $\mu_{CG}(T(CGr_d, \beta)) := pCG(\beta)$ for any $\beta \in V^{r,d}$. The same way for any finite graph $G$ one can associate a measure $\mu_G$ on $Gr_d$. Notice that

\[ d_s(G, H) = d_{Gr_d}(\mu_G, \mu_H). \]

Thus the statistical distance is just the distance of the two associated measures

\[ d_s(CG, CH) = d_{CGr_d}(\mu_{CG}, \mu_{CH}). \]

In other words, weak convergence of graphs actually means the weak convergence of the associated measures. Note that if $G$ is a finite graph and $G'$ is the graph consisting of 2 disjoint copies of $G$ then $\mu_G = \mu_{G'}$.

The equivalence relation $E$ can be extended to $CGr_d$ in an obvious way. We shall denote this Borel-equivalence relation by $E_C$. However in this case, we have a natural continuous group action that defines the equivalence relation. Namely, let $F^2_{(d^2+1)}$ be the free product of $\binom{d^2+1}{2}$ elements of order 2, with generators $\{c_1, c_2, \ldots, c_{(d^2+1)}\}$. The generators act on $CGr_d$ the obvious way: If $CG$ is an element of $CGr_d$ with root $x$ and $c_i$ is a generator, then $c_i(CG)$ has the same graph as $CG$ but with root $y$, where $y$ is the endpoint of the edge colored by $c_i$ (if there is no such edge, then $CG$ is fixed by $c_i$). Obviously, this action of $F^2_{(d^2+1)}$ is continuous and the orbits of the action are exactly the equivalence classes. Since the action is continuous, the space of invariant measures $I_C$ is compact. Note that any finite $C$-graph $CG$ defines an invariant point-measure, hence all the limit measures are invariant measures. This is the advantage of using $C$-graphs instead of colorless graphs.

**Remark 2.2** It is worth to mention that connected finite $C$-graphs always define ergodic measures and it is easy to construct sequence of connected finite graphs converging a non-ergodic limit measure.

In Section 5 we shall prove that if $\{G_n\}$ is a convergent graph sequence then we have a convergent sequence $\{CG_n\}$ of $\binom{d^2+1}{2}$-colorings of $\{G_n\}$.

### 3 The Three Lemmas

The three ingredients of our proof of Theorem 1 are the Stability Lemma, the Decomposition Lemma and the Homogeneity Lemma.

#### 3.1 The Stability Lemma

The edit distance $ed(G, H)$ of two graphs $G, H$ on the same set of vertices is the minimum number of edges that has to be deleted from the graphs to make them identical, divided by the number of vertices. The edit distance can also
be interpreted for $C$-graphs the same way, except here we require that the two graphs become labeled-identical after the removal of the edges.

**Lemma 3.1 (The Stability Lemma)**

a) For any $\omega > 0$ there is a $\varepsilon > 0$ such that $ed(G, H) < \varepsilon$ implies $d_s(G, H) < \omega$ for all graphs $G, H$.

b) For any $\omega > 0$ there is a $\varepsilon > 0$ such that $ed(CG, CH) < \varepsilon$ implies $d_s(CG, CH) < \omega$ for all $C$-graphs $CG, CH$.

**Proof.** We prove the first part here, the proof of the second part is basically identical. Let us suppose that $ed(G, H) < \varepsilon$. Let us fix a natural number $r$.

Those vertices that have different $r$-neighborhoods in $G$ and $H$ must be “close” to a deleted edge in one of the graphs. Thus an upper estimate for the number of such points is $2 \cdot 2^{dn \varepsilon} \cdot d^{r-1}$. Thus if $\alpha \in U^r, d$ then $|p_G(\alpha) - p_H(\alpha)| \leq 4d^r \varepsilon$.

Let us choose $i_0$ so that $1/2i_0 < \omega/4$. In our enumeration of neighborhoods $\alpha_i$ let $r_0$ be the largest occurring radius among the first $i_0$ elements. Finally let us choose $\varepsilon$ to be smaller than $\omega/8d^{r_0}$. With these choices we have

$$d_s(G, H) = \sum_{i=1}^{\infty} \frac{1}{2^i} |p_G(\alpha_i) - p_H(\alpha_i)| =$$

$$= \sum_{i=1}^{i_0} \frac{1}{2^i} |p_G(\alpha_i) - p_H(\alpha_i)| + \sum_{i=i_0+1}^{\infty} \frac{1}{2^i} |p_G(\alpha_i) - p_H(\alpha_i)| \leq$$

$$\leq \sum_{i=1}^{i_0} \frac{1}{2^i} 4d^{r_0} \varepsilon + \sum_{i=i_0+1}^{\infty} \frac{2}{2^i} \leq \omega/2 + \omega/2 = \omega \quad (1)$$

that completes the proof.  

**3.2 The Decomposition Lemma**

**Definition 3.1** A $K$-splitting of a convergent graph sequence $G = \{G_n\}_{n=1}^{\infty}$ with limit measure $\mu$ is a collection of $K$ graph sequences $G^i = \{G_n^i\}_{n=1}^{\infty}$, $1 \leq i \leq K$ obtained by removing some edges from $G$ that satisfies the following properties:

a) $G_n^i$ $(1 \leq i \leq K)$ are vertex-disjoint spanned subgraphs of $G_n$.

b) $\lim_{n \to -\infty} \frac{|E(G_n^i, G_n^j)|}{V(G_n)} = 0$ for every $i \neq j$, that is the ratio of removed edges tends to 0.
c) For any \( 1 \leq i \leq K \), \( \lim_{n \to \infty} \frac{|V(G_i^n)|}{|V(G_n)|} = a_i \) exists.

\[ \text{d) Either } \lim_{n \to \infty} \frac{|V(G_i^n)|}{|V(G_n)|} = 0 \text{ or } \{G_i^n\} \text{ is a convergent graph sequence with limit measure } \mu_i. \]

The exact same notion can be defined for sequences of C-graphs.

**Proposition 3.1** In a K-splitting \( \sum_{i, a_i \neq 0} a_i \cdot \mu_i = \mu. \)

**Proof.** Let \( H_n = \bigcup_{i=1}^{K} G_i \) be the graph obtained from \( G_n \) by removing the necessary edges. From part b) of the definition of a splitting we have that \( \text{ed}(G_n, H_n) \to 0 \), so by the Stability Lemma \( d_i(G_n, H_n) \to 0 \) hence \( H_n \) also converges to \( \mu \). Now for a fixed neighborhood type \( \alpha \) we have \( p_{H_n}(\alpha) = \sum_{G_i} \frac{|V(G_i^n)|}{|V(G_n)|} \cdot p_{G_i}(\alpha) \). The left hand side converges to \( \mu(T(\text{Gr}_d, \alpha)) \), while each term on the right hand sides converges to \( a_i \cdot \mu_i(\text{Gr}_d, \alpha) \), even if \( a_i = 0 \). So \( \mu = \sum a_i \cdot \mu_i. \)

Now let us consider the convex compact space \( \mathcal{I}_C \) of invariant measures on \( \text{CGr}_d \) and the set of its extremal points, that is the set of ergodic invariant measures \( \mathcal{E} \mathcal{I}_C \) (note that \( \mathcal{E} \mathcal{I}_C \) is non-compact).

**Lemma 3.2 (The Decomposition Lemma)** Let \( \text{CG} \) be a C-graph sequence that converges weakly to an invariant measure \( \mu \) on \( \text{CGr}_d \). Let \( Z_1, \ldots, Z_L \) be a Borel-partition of \( \mathcal{E} \mathcal{I}_C \). Then we have a K-splitting \( \{\text{CG}^1, \ldots, \text{CG}^K\} \) of \( \text{CG} \) such that \( \mu_i \in \text{hull}(Z_i) \) whenever \( a_i \neq 0 \).

**Proof.** Let \( Y_i = \pi^{-1}(Z_i) \), where \( \pi \) is the Borel-surjection in the Farrell-Varadarajan Theorem. The sets \( Y_i \) are \( E_C \)-invariant Borel-subsets of \( \text{CGr}_d \). Let \( \mu \) denote the limit measure of \( \text{CG} \). For any natural number \( M > 0 \) one can approximate the partition \( \{Y_i\}_{i=1}^{K} \) with a slightly perturbed partition \( \{Y_i^M\}_{i=1}^{K} \), where

- \( \lim_{i \to \infty} \mu(Y_i \triangle Y_i^M) = 0 \) for any \( 1 \leq i \leq K \).
- Each \( Y_i^M \) is a closed-open set in the form

\[ Y_i^M = \bigcup_{j=1}^{L_{i,M}} T(\text{CGr}_d, \alpha_j^i), \]

where \( \alpha_j^i \in V^{M,d}. \)

That is if \( x \in V(\text{CG}_n) \) then by looking at the \( M \)-neighbourhood of \( x \) and the colors of its edges one can decide for which \( 1 \leq i \leq K \); \( x \in T(\text{CG}_n, Y_i^M) \). Note that we use the notation \( T(\text{CG}_n, Y_i^M) \) for \( \bigcup_{j=1}^{L_{i,M}} T(\text{CG}_n, \alpha_j^i) \). Let \( \partial Y_i^M \) denote the set of graphs \( X \) in \( Y_i^M \) such that if we move the root of \( X \) to one of its neighbours the resulting rooted C-graph is not in \( Y_i^M \). Since \( Y_i \) is \( E_C \)-invariant set and \( \mu(Y_i \triangle Y_i^M) \to 0 \), we have \( \lim_{M \to \infty} \mu(\partial Y_i^M) = 0 \). Note that \( \partial Y_i^M \) is still a closed open set in the form \( \bigcup_{j=1}^{R_{i,M}} T(\text{CG}_d, \beta_j^i) \), where \( \beta_j^i \in V^{M+1,d} \).
Hence \( \lim_{n \to \infty} p_{CG_n}(\partial Y_i^M) = \mu(\partial Y_i^M) \). Observe that \( T(CG_n, \partial Y_i^M) \) is the set of vertices \( x \in V(CG_n) \) such that \( x \in T(CG_n, Y_i^M) \) but \( y \notin T(CG_n, Y_i^M) \) for a neighbour of \( x \). Let

\[
Q_M = \{ \alpha \in V^{M,d} \mid \mu(T(CG_{d, \alpha}) \neq 0) \}
\]

\[
R_M = \{ \alpha \in V^{M,d} \mid \mu(T(CG_{d, \alpha}) = 0) \}
\]

Since \( CG \) is a convergent \( C \)-graph sequence there exists a natural number \( n_M \)

such that if \( n \geq n_M \) then

- \( p_{CG_n}(\partial Y_i^M) \leq \mu(\partial Y_i^M) + \frac{1}{M} \) for any \( 1 \leq i \leq K \).
- \( (1 - \frac{1}{M})\mu(T(CG_{d, \alpha})) < p_{CG_n}(\alpha) < (1 + \frac{1}{M})\mu(T(CG_{d, \alpha})) \) for any \( \alpha \in Q_M \).
- \( p_{CG_n}(\alpha) \leq \frac{1}{M} |V^{M,d}|, \) for any \( \alpha \in R_M \).

If \( n_M \leq n < n_{M+1} \) let us partition \( V(CG_n) \) as

\[
V(CG_n) = \bigcup_{i=1}^{K} V(CG_i^M) = \bigcup_{i=1}^{K} T(CG_n, Y_i^M).
\]

According to our conditions on \( n_M \), if \( n > n_M \) then

\[
\left| \frac{|T(CG_n, \alpha)|}{|V(CG_n)|} - \mu(T(CG_{d, \alpha})) \right| \leq \frac{1}{M} \mu(T(CG_{d, \alpha})) \quad \text{if} \quad \alpha \in Q_M. \tag{2}
\]

\[
\left| \frac{|T(CG_n, \alpha)|}{|V(CG_n)|} \right| \leq \frac{1}{M} \frac{1}{|V^{M,d}|} \quad \text{if} \quad \alpha \in R_M. \tag{3}
\]

\[
\left| \sum_{1 \leq i, j \leq K} \frac{E(CG_i^M, CG_j^M)|}{|V(CG_n)|} \right| \leq Kd \left( \mu(\partial Y_i^M) + \frac{1}{M} \right) \tag{4}
\]

**Lemma 3.3** For any \( r > 1 \) and \( \delta > 0 \) there exists \( M > 0 \) such that for any \( \beta \in V^{r,d} \) and \( 1 \leq i \leq K \) if \( \mu(Y_i) \neq 0 \) then

\[
\left| \frac{|T(CG_i^M, \beta)|}{|V(CG_i^M)|} - \frac{\mu(T(CG_{d, \beta}) \cap Y_i)}{\mu(Y_i)} \right| < \delta.
\]

**Proof.** Let \( M > r \). Then for any \( 1 \leq i \leq K \) and for any graph \( G \), we have decompositions

\[
T(CG, \beta) = \bigcup_{i=1}^{K} \bigcup_{j=1}^{N_i} T(CG, \alpha_j^i),
\]

where \( \alpha_j^i \in V^{M,d} \) and \( \bigcup_{j=1}^{N_i} T(CG, \alpha_j^i) = T(CG, Y_i^M) \cap T(CG, \beta) \). Also

\[
T(CG_{d, \beta}) = \bigcup_{i=1}^{K} \bigcup_{j=1}^{N_i} T(CG_{d, \alpha_j^i}),
\]

and \( \bigcup_{j=1}^{N_i} T(CG_{d, \alpha_j^i}) = Y_i^M \cap T(CG_{d, \beta}) \).
Since \( \mu(T(\mathbf{CG}_d, \beta) \cap Y_i) \rightarrow \mu(T(\mathbf{CG}_d, \beta) \cap Y_i) \) by the estimates (2) and (3) one can immediately see that for any \( \kappa > 0 \) there exists \( M > 0 \) such that if \( n > n_M \) and \( \beta \in V^{r,d} \) then for any \( 1 \leq i \leq K \), where \( \mu(Y_i) \neq 0 \):

\[
\left| \frac{|V(CG_{i,M}^n) \cap T(CG_n, \beta)|}{|V(CG_n)|} \frac{\mu(T(\mathbf{CG}_d, \beta) \cap Y_i)}{\mu(Y_i)} \right| < \kappa \quad (5)
\]

That is, if \( M \) is large enough then if \( n > n_M \) and \( \beta \in V^{r,d} \) then for any \( 1 \leq i \leq K \), where \( \mu(Y_i) \neq 0 \):

\[
\left| \frac{|V(CG_{i,M}^n) \cap T(CG_n, \beta)|}{|V(CG_n)|} \frac{\mu(T(\mathbf{CG}_d, \beta) \cap Y_i)}{\mu(Y_i)} \right| < \frac{\delta}{3} \quad (6)
\]

Hence it is enough to show that if \( M \) is large enough then for any \( 1 \leq i \leq K \) and \( \beta \in V^{r,d} \)

\[
\left| \frac{(V(CG_{i,M}^n) \cap T(CG_n, \beta)) \Delta T(CG_{i,M}^n, \alpha)}{|V(G_n)|} \right| < \frac{\delta}{3} \quad (7)
\]

Observe that if \( x \in (V(CG_{i,M}^n) \cap T(CG_n, \beta)) \Delta T(CG_{i,M}^n, \beta) \), then

\[
B_r(x) \cap T(CG_n, \partial Y_i^M) \neq \emptyset \quad (8)
\]

Since \( \frac{|T(CG_n, \partial Y_i^M)|}{|V(G_n)|} \leq \mu(\partial Y_i^M) + \frac{1}{M} \), for any \( \eta > 0 \) one can choose large enough \( M \) so that

\[
\frac{|B_r(T(CG_n, \partial Y_i^M))|}{|V(G_n)|} \leq \eta.
\]

Therefore our lemma follows. \( \blacksquare \)

By our lemma if \( CG_{i,n} = CG_{i,M}^n \), where \( n_M < n \leq n_{M+1} \), \( \lim_{n \rightarrow \infty} \frac{|V(CG_{i,n})|}{|V(CG_n)|} \neq 0 \) then \( \{CG_{i,n}\}_{n=1}^{\infty} \) is convergent and the limit measure is \( \mu_i \), where \( \mu_i(U) = \frac{\mu(U(Y_i))}{\mu(Y_i)} \). By the Farrell-Varadarajan Theorem

\[
\mu_i = \frac{\int_Z \pi(x) d\mu(x)}{\mu(Y_i)}
\]

that is

\[
\mu_i = \int_{Z_i} p d\nu_i(p),
\]

where \( \nu_i \) is a Borel-probability measure on \( Z_i \). Thus \( \mu_i \) is the barycenter of \( \nu_i \), hence \( \mu_i \in \text{hull}(Z_i) \). \( \blacksquare \)

### 3.3 The Homogeneity Lemma

**Lemma 3.4 (The Homogeneity Lemma)** Let \( 0 < \delta < 1 \), \( 0 < \lambda < 1 \) be real numbers and let \( \delta' < \delta \) be the constant in Remark 2.1 that is if for two \( C \)-graphs \( d_s(CG, CH) < \delta' \) then \( d_s(G, H) < \delta \) holds for the underlying graphs. Let
$Z \subset E\mathcal{I}_C$, such that $\text{diam}(Z) < \frac{1}{4}\delta'$. Let \( \{CG_n\}_{n=1}^{\infty} \) be a sequence of $C$-graphs converging to a measure $\mu$ supported on $\pi^{-1}(Z)$. Then there exists $0 < \epsilon < \delta$ and $N > 0$ such that if $n \geq N$ then the graph $G_n$ is $(\epsilon, \lambda, \delta)$-homogeneous.

**Proof.** Suppose that the Lemma does not hold. Then we have a sequence of graphs $\{CG_n\}_{n=1}^{\infty}$ such that $G_n$ are not $(\frac{1}{n}, \lambda, \delta)$-homogeneous. Thus $CG_n$ are not $(1/n, \lambda, \delta')$-homogeneous by the choice of $\delta'$. Therefore for each $CG_n$ there is a spanned subgraph $CH_n \subset CG_n$ such that it is not small: $|V(CH_n)| \geq \lambda|V(CG_n)|$, it has only few edges going out of it: $|E(CH_n, CG_n \setminus CH_n)| \leq |V(CG_n)|/n$, but still it is not similar to the big graph: $d_s(CH_n, CG_n) > \delta'$.

We may again choose a subsequence so that $CH_n, CG_n \setminus CH_n$ convergent $C$-graph sequences and $\frac{|V(CH_n)|}{|V(CG_n)|}$ is a convergent sequence of real numbers. The $C$-graphs converge to invariant measures $\mu_1, \mu_1' \in \mathcal{I}_C$ while $\frac{|V(CH_n)|}{|V(CG_n)|}$ converges to some real number $\lambda \leq a \leq 1$. Therefore we obtained a 2-splitting of the $CG_n$ sequence. By Proposition 3.1 we know that $\mu_1 = a \cdot \mu_1' + (1-a) \cdot \mu_1''$. The splitting was chosen such that $d_s(CH_n, CG_n) > \delta'$, so in the limit we have $d_{CG_n}(\mu_1, \mu_1') \geq \delta'$. On the other hand since $\mu_1$ was entirely supported on $\pi^{-1}(Z)$, the same must hold for $\mu_1'$ and $\mu_1''$. Since $Z$ has diameter strictly less than $\frac{1}{2}\delta'$ by Lemma 2.1 its convex hull has diameter less than $\delta'$. This leads to a contradiction.

## 4 The proof of Theorem 1

Again, let $\delta' > 0$ be small enough such that for any two $C$-graphs and their underlying regular graphs $d_s(CG, CH) < \delta'$ implies $d_s(G, H) < \delta$. The space $\mathcal{I}_C$ is compact, hence it can be split into $K = K(\delta)$ Borel pieces each of diameter strictly less than $\delta'/3$. Let us denote by $Z_i$ the intersection of the $i$-th piece with $E\mathcal{I}_C$. Let us suppose that for this choice of $K$ there is no good choice for $N$ and $\epsilon$. That means we have graphs $G_n$ with $|V(G_n)| \geq n$ such that $G_n$ does not have the desired decomposition into pieces that are $(1/n, \lambda, \delta)$-homogeneous.

Let $\{CG_n\}_{n=1}^{\infty}$ be $(\frac{\delta}{2}+1)$-colorings of $G_n$. By Theorem 2 we can assume that $\{CG_n\}_{n=1}^{\infty}$ converge to an invariant measure $\mu \in \mathcal{I}_C$. By the Decomposition Lemma we have a $K$-splitting $\{CG_n^i\}_{i=1}^{K}$ such that the limit measure $\mu_i$ of $CG_n^i$ (whenever $a_i$ in the Decomposition Lemma is non-zero) is entirely supported on $\pi^{-1}(\text{hull } Z_i) \subset \text{CG}_{K}$.

Since in a $K$-splitting the ratio of deleted edges tends to 0, all but a finite number of the $CG_n$’s are split by removing less than $\frac{\delta}{10}d_{CG_n}$ edges. Let us further remove all edges from those parts $CG_n^i$ for which

$$\frac{|V(CG_n^i)|}{|V(CG_n)|} \leq \frac{\delta}{10d_{CG_n}}$$

and put these parts into $CG_n^0$. The number of edges removed in this step is at most $\frac{\delta}{10d_{CG_n}}|V(CG_n)|$, so in total we still have not removed more than $\delta|V(CG_n)|$. 

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edges. Lastly, the empty parts together clearly contain less than $\delta|V(CG_n)|$ vertices.

We can use this partition of the sequence $\{CG_n\}_{n=1}^{\infty}$ on the original graph sequence $\{G_n\}_{n=1}^{\infty}$ to obtain a candidate for the partition required in the theorem: we didn’t remove too many edges, all parts are big or empty and the empty parts are small altogether. Still our graphs are counterexamples to the theorem, so for each $n$, one of the non-empty parts must not be $(1/n, \lambda, \delta)$-homogeneous. This is in contradiction with the Homogeneity Lemma.

Now we prove part (b). If we have two convergent graph sequences $G_n, H_n$ for which $d_r(G_n, H_n) \to 0$ then by our Theorem 2 there exist convergent $d + 1$-colorings of them $\{CG_n\}_{n=1}^{\infty}$ and $\{CH_n\}_{n=1}^{\infty}$ converging to the same limit measure $\mu$.

Now we can write $\mu = \sum_i \alpha_i \mu_i$ and partition $G_n$ and $H_n$ just as in the proof of part a). For a fixed index $i$ the $CG^i_n$ will converge to $\mu$ if $\alpha_i > \delta/10dK$ and will be empty otherwise. The same holds for the $CH^i_n$’s. Hence for large $n$ the same parts will remain non-empty, and they will both converge to $\mu_i$, while the ratios $\frac{|V(CG^i_n)|}{|V(CH^i_n)|}$ will both converge to $\alpha_i$.

So if for a fixed $\sigma > 0$ the statment were false, we could choose a pair of graphs $G_n, H_n$ which would provide a counterexample for $\tau = 1/n$. However a convergent subsequence of these graphs would contradict the observation of the previous paragraph. Thus part b) follows.

5 The Coloring Theorem

In this section we prove a folklore conjecture: a convergent graph sequence can be vertex- resp. edge-colored properly to obtain a convergent sequence of vertex- resp. edge colored graphs. This result is used in the proof of Theorem 1.

5.1 $B$-graphs and the space $BGr_d$

In this subsection we recall the notion of $B$-graphs from [4]. Let $B = \{0, 1\}^N$ be the Bernoulli space of 0-1-sequences with the standard product measure $\nu$. A rooted $B$-graph is a rooted graph $G$ and a function $\tau_G : V(G) \to B$. Two rooted $B$-graphs $G$ and $H$ are said to be isomorphic if there exists a rooted isomorphism $\psi : V(G) \to V(H)$ such that $\tau_H(\psi(x)) = \tau_G(x)$ for every $x \in V(G)$. The set of isomorphism classes of all countable rooted $B$-graphs with degree bound $d$ is denoted by $BGr_d$.

Let $U^{k,r,d}$ denote the set of isomorphism classes of rooted $r$-balls with degree bound $d$ and vertices labeled with $k$ $(0, 1)$-digits. For a $B$-graph $BG$ and a vertex $x \in V(BG)$ by $B^k_r(x) \in U^{k,r,d}$ we shall denote the rooted $r$-ball around $x$ with the labels truncated to the first $k$ digits. For any $\alpha \in U^{k,r,d}$ and a $B$-graph $BG$ we define the set $T(BG, \alpha) \overset{\text{def}}{=} \{x \in V(G) : B^k_r(x) \cong_B \alpha\}$ and define $p_{BG}(\alpha) \overset{\text{def}}{=} \frac{|T(BG, \alpha)|}{|V(G)|}$. For $\alpha \in U^{k,r,d}$ let us define $T(BGr_d, \alpha) = \{x \in BGr_d : B^k_r(x) \cong \alpha\}$.
Again, we have a natural metric on $\text{BGr}_d$. If $X, Y \in \text{BGr}_d$ then
\[ d_b(X, Y) = 2^{-r}, \]
where $r$ is the maximal number such that $B^+_{r}(x) \cong B^+_{r}(y)$, where $x$ is the root of $X$, $y$ is the root of $Y$. The subsets $T(\text{Gr}_d, \alpha) : \alpha \in U^{k,r,d}, k, r \in \mathbb{N}$ are closed-open sets and generate the Borel-structure of $\text{BGr}_d$.

Let $\{\text{BG}_n\}_{n=1}^{\infty}$ be a sequence of $B$-graphs. We say that $\{\text{BG}_n\}_{n=1}^{\infty}$ converges if for any $\alpha \in U^{k,r,d}$, $\lim_{n \to \infty} p_{\text{BG}_n}(\alpha) = \mu(T(\text{BG}_d, \alpha))$ exists. In this case $\mu$ is a Borel-measure on $\text{BGr}_d$. We call $\mu$ the limit measure of $\{\text{BG}_n\}_{n=1}^{\infty}$.

We can consider the Borel-set $\text{BGr}_d$ of such rooted $B$-graphs where all the vertex labels are different. Again, we have the natural equivalence relation on this $B$-graphs defined by a natural Borel group action (see [4]) making $\text{BGr}_d$ a Borel-graphing (see Section 2). The following proposition is the straightforward consequence of Proposition 2.2 and Corollary 5.1 of [4]. See also [1] Example 9.9.

**Proposition 5.1** Let $\{G_n\}_{n=1}^{\infty}$ be a convergent graph sequence. Let $\{\text{BG}_n\}_{n=1}^{\infty}$ be a uniformly random $B$-labelling of the vertices of $G_n$. Then with probability 1 $\{\text{BG}_n\}_{n=1}^{\infty}$ converges to an $I_B$-invariant measure $\mu$ supported entirely on $\text{BGr}_d$.

Let $b_1, b_2, b_3, \ldots, b_k$ be elements of $B$ such that $b_i$ is everywhere zero accept in the $i$-th digit. We call a $B$-graph a $B_k$-graph if all its vertex labels are from this set and any two adjacent vertex is labelled differently. Of course, the limit measure of such graphs are concentrated on those elements of $\text{BGr}_d$ that are vertex labelled by elements of $b_1, b_2, b_3, \ldots, b_k$. We denote this compact subspace by $\text{BGr}_d^k$.

### 5.2 Borel-colorings

According to the theorem of Kechris, Solecki and Todorcevic [5] we have a Borel-coloring $c$ of $\text{BGr}_d$ by $d+1$-colors. Note that this gives us a proper coloring of the vertices of each rooted $B$-graphs with $b_1, b_2, b_3, \ldots, b_{d+1}$. Thus each vertex has two labels one from just $B$, the second is from the set $b_1, b_2, b_3, \ldots, b_{d+1}$. Now let $\mu$ be a measure on $\text{BGr}_d$ then we associate a measure $\mu^{c}_{d+1}$ on $\text{BGr}_d$ concentrated on $\text{BGr}_d^{d+1}$. Indeed, let $\alpha \in U^{d+1,r,d+1}$ a rooted graph labeled only by the elements of $b_1, b_2, b_3, \ldots, b_{d+1}$. Define $\mu^{c}_{d+1}(T(\text{BG}_d, \alpha)) := r_\alpha$, where $r_\alpha$ is the $\mu$-measure of such rooted graphs $G$ in $\text{BGr}_d$ such that the $d+1$-coloring of the $r$-ball around the root of $G$ is just $\alpha$.

**Proposition 5.2** Let $\{G_n\}_{n=1}^{\infty} \subset \text{Gr}_d$ be a convergent graph sequence. Let $\{\text{BG}_n\}_{n=1}^{\infty}$ be a random $B$-labelling of the graphs converging to a measure $\mu$ supported entirely on $\text{BGr}_d$ as in Proposition 5.1. Let $\mu^{c}_{d+1}$ be the associated measure. Then there exist proper vertex colorings of the graphs by $b_1, b_2, b_3, \ldots, b_{d+1}$, such that the resulting $B_{d+1}$-graphs $\{\text{BG}_{d+1,n}^{c}\}_{n=1}^{\infty}$ converge to $\mu^{c}_{d+1}$.
Proof. We denote by $H^{d+1,r,d}$ the set of rooted balls properly vertex-colored by the set $\{b_1, b_2, \ldots, b_{d+1}\}$. Clearly, it is enough to prove that for any $r > 1$ and $\varepsilon > 0$ there exist proper colorings of $\{G_n\}_{n=1}^\infty$ by $\{b_1, b_2, \ldots, b_{d+1}\}$ such a way that if $n$ is large enough then

$$|p_{G_n}(\gamma) - \mu_{d+1}(T(\mathcal{B}Gr_d, \gamma))| < \varepsilon$$

(9)

for any $\gamma \in H^{d+1,r,d}$. Note that $p_{G_n}(\gamma) = \frac{|T(G_n, \gamma)|}{V(G_n)}$, where $T(G_n, \gamma)$ is the set of vertices $x$ for which the colored $r$-neighborhood of $x$ is rooted isomorphic to $\gamma$. From now on we fix an $r > 1$ and an $\varepsilon > 0$.

For $\gamma \in H^{d+1,r,d}$ let $W_\gamma$ be the Borel-set of points in $\mathcal{B}Gr_d$ such that $B_r(x)$ is isomorphic to $\gamma$ under the $c$-coloring. Clearly,

$$\mathcal{B}Gr_d = \bigcup_{\gamma \in H^{d+1,r,d}} W_\gamma$$

form a Borel-partition. We approximate the Borel-sets $W_\gamma$ by closed-open sets the following way. For each $M \geq 1$ let $Y_\gamma^M = \bigcup_{i=1}^{L_{M,\gamma}} T(\mathcal{B}Gr_d, \alpha_i^{M,\gamma})$ where $\alpha_i^{M,\gamma} \in U^{M,\gamma,d}$ and

- for any $\gamma \in H^{d+1,r,d}$, $\mu(W_\gamma \triangle Y_\gamma^M) \leq e(M)$
- $Y_\gamma^M \cap Y_\delta^M = \emptyset$ if $\gamma \neq \delta$
- $\lim_{M \to \infty} e(M) = 0$.

Then $Y_\gamma^M$ are disjoint Borel-sets and $\mu(\mathcal{B}Gr_d \setminus \bigcup_{\gamma \in H^{d+1,r,d}} (Y_\gamma^M \cap W_\gamma)) \leq e(M) |H^{d+1,r,d}|$.

Note that we immediately have a (not necessarily proper) coloring $c_M : \mathcal{B}Gr_d \to \{b_1, b_2, \ldots, b_{d+2}\}$. Simply, let $c_M(x) = b_i$ if $x \in T(\mathcal{B}Gr_d, \alpha_j^{M,\gamma})$ and the color of the root of $\gamma$ is $b_i$. Let $c_M(x) = b_{d+2}$ if $x \in \mathcal{B}Gr_d \setminus \bigcup_{\gamma \in H^{d+1,r,d}} Y_\gamma^M$. Thus $c_M$ is a step-function approximation of the Borel-function $c$ depending only on the first $M$ digits of the labels of the $M$-neighborhoods of the points. Observe that if $x \in \bigcup_{\gamma \in H^{d+1,r,d}} (Y_\gamma^M \cap W_\gamma)$ then $c_M(x) = c(x)$. Consequently, for any $\gamma \in H^{d+1,r,d}$,

$$\lim_{M \to \infty} \mu_{d+1}^{c_M}(T(\mathcal{B}Gr_d, \gamma)) = \mu_{d+1}^{c}(T(\mathcal{B}Gr_d, \gamma)).$$

That is if $M$ is large enough, then

$$|\mu_{d+1}^{c_M}(T(\mathcal{B}Gr_d, \gamma)) - \mu_{d+1}^{c}(T(\mathcal{B}Gr_d, \gamma))| < \frac{\varepsilon}{10}$$

for any $\gamma \in H^{d+1,r,d}$. It is enough to construct of proper vertex colorings of the graphs $\{G_n\}_{n=1}^\infty$ such that if $n$ is large enough then

$$|p_{G_n}(\gamma) - \mu_{d+1}^{c_M}(T(\mathcal{B}Gr_d, \gamma))| < \frac{\varepsilon}{10},$$

for any $\gamma \in H^{d+1,r,d}$. We call $\alpha \in U^{M,\gamma,d}$ nice if $\alpha = \alpha_i^{M,\gamma}$ for some $\gamma \in H^{d+1,r,d}$. Also, we call $\beta \in U^{M,\gamma,d}$ nice if the following holds:
\[ \text{if } s \in B_r(x), \text{ where } x \text{ is the root of } \beta, \text{ then } \beta_M(s) \text{ is a nice element of } U_{M,M,d}. \]

Here, \( \beta_M \) denotes the restriction of \( \beta \) onto the \( M \)-ball around \( s \). Notice that if \( x \in B_r(x) \) and \( x \in T(BGr_d, \beta) \), where \( \beta \in U_{M,M+r,d} \) is nice, then the \( c_M \)-coloring of the \( r \)-neighborhood of \( x \) depends only on \( \beta \). Let \( N_{M,M+r,d} \) be the set of nice balls and \( N_{M,M+r,d}^{\gamma} \) is the set of those nice balls that the corresponding \( c_M \)-coloring of the \( r \)-neighborhood of the root \( x \) is isomorphic to \( \gamma \). Now pick a large enough \( M \) such that

\[ |\mu_{d+1}^{c_M}(T(BGr_d, \gamma)) - \mu_{d+1}^{\gamma}(T(BGr_d, \gamma))| < \frac{\varepsilon}{10} \text{ for any } \gamma \in H^{d+1,r,d}. \]

Then for any \( \gamma \in H^{d+1,r,d} \)

\[ 0 < \mu_{d+1}^{c_M}(T(BGr_d, \gamma)) - \sum_{\beta \in N_{M,M+r,d}^{\gamma}} \mu(T(BGr_d, \beta)) < \frac{\varepsilon}{10}. \]

Now we construct our proper vertex-colorings of the graph sequence \( \{G_n\}_{n=1}^{\infty} \). Let \( \{BG_n\}_{n=1}^{\infty} \) is a sequence of \( B \)-colorings such that \( \mu_{BG_n} \) weakly converges to \( \mu \). Let \( M \) be the number determined in the previous paragraph. If \( p \in T(BG_n, \alpha), \alpha \in N_{M,M,d}^{\gamma} \) for a certain \( \gamma \) then let us color \( p \) by the color of the root of \( \gamma \). We color the remaining vertices arbitrarily to obtain a proper coloring of the underlying graph \( G_n \). Then if \( n \) is large enough

\[ |p_{BG_n}(\beta) - \mu(T(BGr_d, \beta))| < \frac{\varepsilon}{10} \text{ for any } \beta \in N_{M,M+r,d}^{\gamma}. \]

The ratio of vertices \( x \) for which the \( M \)-neighbourhood of \( x \) is not nice is less than \( \frac{\varepsilon}{5} \).

Observe that if \( x \in T(BG_n, \beta), \beta \in N_{M,M+r,d}^{\gamma} \) then in the vertex colored graph \( B_r(x) \) will be isomorphic to \( \gamma \). Therefore (9) holds if \( n \) is large enough.

5.3 Edge-colorings

The goal of this subsection is to prove the following theorem.

**Theorem 2** Let \( \{G_n\}_{n=1}^{\infty} \subset Graph_d \) be a convergent graph sequence. Then we have colorings of the graphs by \( (d^2+1) \) colors such that the resulting graph sequence \( \{CG_n\}_{n=1}^{\infty} \) is convergent as well. In particular, if \( \{G_n\}_{n=1}^{\infty} \) and \( \{H_n\}_{n=1}^{\infty} \) are two convergent graph sequences that have the same limit measure, then there exist colorings of the graph sequences \( \{CG_n\}_{n=1}^{\infty} \) and \( \{CH_n\}_{n=1}^{\infty} \) converging to the same measure on \( CGr_d \).
Proof. Let the graph $H_n$ be defined the following way.

- $V(H_n) = V(G_n)$.
- $(x, y) \in E(H_n)$ if $x \neq y$ and $d_{G_n}(x, y) \leq 2$.

Clearly, $\{H_n\}_{n=1}^{\infty} \subset \text{Graph}_{d_2}$ is a convergent graph sequence. Thus, by our previous proposition we have convergent vertex-colorings of $H_n$ by the colors $\{b_1, b_2, \ldots, b_{d_2+1}\}$. Let $a_1, a_2, \ldots, a_{d_2+1}$ be the set of pairs of different elements of $\{b_1, b_2, \ldots, b_{d_2+1}\}$. Color an edge $(x, y)$ by $a_i$ if $a_i$ is the pair of colors of $x$ and $y$ in the vertex colorings of $H_n$. Obviously, we obtain a properly edge-colored graph sequence $\{CG_n\}_{n=1}^{\infty}$ and this sequence is convergent as well.

Note: Balazs Szegedy and Omer Angel informed us that they also proved part (a) of Theorem 1., using a different argument [7].

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