Research Article

Monotone Iterative Technique for Conformable Fractional Differential Equations with Deviating Arguments

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This paper is concerned with the existence of extremal solutions for periodic boundary value problems for conformable fractional differential equations with deviating arguments. We first build two comparison principles for the corresponding linear equation with deviating arguments. With the help of new comparison principles, some sufficient conditions for the existence of extremal solutions are established by combining the method of lower and upper solutions and the monotone iterative technique. As an application, an example is presented to enrich the main results of this article.

1. Introduction

In recent years, people have been paying attention to the progress of the fractional differential equations. In fact, it is the generalization of the ordinary differential equations to a noninteger order. Significantly, fractional differential equations appear more frequently in different fields of science and engineering, such as viscoelasticity, circuit, and neuron modeling [1–3]. Gradually, fractional differential equations are increasingly regarded as effective assistants. We have observed that many papers are exploring the existence of solutions of boundary value problems for fractional differential equations by using nonlinear functional analysis methods such as fixed point theorems, fixed point index on cone, variational methods and critical point theory, the theory of Mawhin coincidence degree, and the upper and lower solution method; see the monographs of Kilbas et al. [1], Podlubny [2], Diethem [3], the papers [4–26], and the references therein. Among them, the monotone iterative technique is an ingenious and effective method that offers theoretical, as well constructive existence results for nonlinear problems via linear iterates [9–15, 17, 23, 26]. It yields monotone sequences that converge to the extremal solutions in a sector generated by the upper and lower solutions. For example, the authors of [22] adopted the method of monotone iteration combined with the method of upper and lower solutions to consider the following system of nonlinear fractional differential equations:

\[
\begin{cases}
D_0^\alpha v(t) = f(t, v(t), w(t)), & t \in (0, T], \\
D_0^\alpha w(t) = g(t, w(t), v(t)), & t \in (0, T], \\
|t_1-\alpha v(t)|_{t=0} = x_0, \\
|t_1-\alpha w(t)|_{t=0} = y_0,
\end{cases}
\]

where \(0 < T < \infty\), \(f, g \in C([0, T] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})\), \(x_0, y_0 \in \mathbb{R}\), and \(x_0 \leq y_0\). In addition, [15, 24] used these methods to study the initial value problems for nonlinear fractional differential equations with no deviating arguments. On the basis of [22], Jian et al. [13] successfully investigated the following nonlinear fractional order differential systems with deviating arguments:
\[
\begin{align*}
D^\delta v(t) &= f(t, v(t), \nu(t)), w(t), w(\nu(t)), \quad t \in (0, 1], \\
D^\delta w(t) &= g(t, w(t), w(\nu(t)), \nu(t), \nu(\nu(t))), \quad t \in (0, 1], \\
t^{1-\alpha} v(t)|_{t=0} &= x_0, \\
t^{1-\alpha} w(t)|_{t=0} &= y_0,
\end{align*}
\]

where \( \theta \in C([0, 1], [0, 1]) \). They introduce two well-defined monotone sequences that converge to the solution of the system and, then, establish the existence and uniqueness of the solution of the system. Finally, a numerical iterative scheme is introduced to obtain an accurate approximate solution for the systems.

Motivated by the abovementioned papers, in this paper, we devote ourselves to the existence of solutions to the following boundary value problems with deviation arguments:

\[
\begin{align*}
\mathcal{D}^\delta \phi(t) &= f(t, \phi(t), \phi(\theta(t))), \quad t \in [0, T], \\
\phi(0) &= \phi(T),
\end{align*}
\]

where \( \delta \in (0, 1] \), \( \theta \in C([0, 1], [0, 1]) \), and \( f \in C([0, 1] \times \mathbb{R} \times \mathbb{R}) \), and \( \mathcal{D}^\delta \phi \) is the conformable fractional derivative of order \( \delta \). The conformable fractional calculus which was introduced in the work of Khalil et al. [27], then developed by Abdeljawad [28], have been receiving a lot of attention due to the wide application in physics and engineering [29, 30]. The reader is referred to [14, 16, 17, 27–33] and references therein for some recent advances in conformable fractional calculus and its applications.

In this paper, by establishing two comparison results and using the monotone iterative technique combined with the method of upper and lower solutions, some sufficient conditions are presented for the existence of extremal solutions for periodic boundary value problem (3).

### 2. Preliminaries

**Definition 1** (See [27]). Let \( f : [0, +\infty) \rightarrow \mathbb{R} \) and \( t \in (0, 1] \), and the conformable fractional derivative of order \( 0 < \alpha \leq 1 \) is defined by

\[
D_\alpha f(t) = \lim_{\rho \to 0} \frac{f(t + \rho t^{1-\alpha}) - f(t)}{\rho},
\]

for \( t > 0 \), and the conformable fractional derivative at 0 is defined as \( D_\alpha f(0) = \lim_{t \to 0}(D_\alpha f)(t) \). If \( f \) is differentiable, then \( D_\alpha f(t) = t^{1-\alpha} f'(t) \).

**Definition 2** (See [27]). Let \( \alpha \in (0, 1] \). The conformable fractional integral of a function \( f : [0, +\infty) \rightarrow \mathbb{R} \) of order \( \alpha \) is denoted as

\[
I_\alpha f(t) = \int_0^t s^{\alpha-1} f(s) \, ds.
\]

**Lemma 1** (See [32]). Let \( T > 0 \). Assume that \( f \in C(0,T] \) and \( D_\alpha f \in C(0,T] \cap L(0,T) \) with \( 0 < \alpha \leq 1 \). Then, we have

\[
I_\alpha D_\alpha f(t) = f(t) - f(0).
\]

**Lemma 2** (See [27]). Let \( \alpha \in (0, 1] \), \( l_1, l_2, q, R, K \in \mathbb{R} \) and the functions \( f, h \) be \( \alpha \)-differentiable on \( [0, +\infty) \). Then,

\begin{enumerate}
\item \( D_\alpha K = 0 \) for all constant functions \( f(t) = K \)
\item \( D_\alpha (l_1 f + l_2 h) = l_1 D_\alpha f(t) + l_2 D_\alpha h(t) \)
\item \( D_\alpha \rightarrow^\alpha = q^{\alpha} q^\alpha \)
\item \( D_\alpha (f h) = f(t) D_\alpha h(t) + h(t) D_\alpha f(t) \)
\item \( D_\alpha (f h) = ((h D_\alpha f - f D_\alpha h)/h^2) \) when \( h(t) \neq 0 \)
\end{enumerate}

**Lemma 3** (See [34]). Let \( \mathcal{A} : X \rightarrow X \) linear operator, \( r(\mathcal{A}) \) be the spectral radius of \( \mathcal{A} \), and \( \|\mathcal{A}\| = \max_{\|\phi\|=1} \|\mathcal{A} \phi\| \). Then,

\begin{enumerate}
\item \( r(\mathcal{A}) \leq \|\mathcal{A}\| \)
\item if \( r(\mathcal{A}) < 1 \), then \( (\mathcal{I} - \mathcal{A})^{-1} \) exists and \( (\mathcal{I} - \mathcal{A})^{-1} = \sum_{n=0}^{\infty} \mathcal{A}^n \), where \( \mathcal{I} \) stands for the identity operator
\end{enumerate}

It is given that \( T > 0 \). Let \( E = C(0,T] \); then, \( E \) is a Banach space with the norm \( \|x\| = \max_{t \in [0,T]} |x(t)| \).

Let us introduce the following values and functions which will be used in the rest paper.

\[
K_1 = \frac{K}{\delta},
\]

\[
l = e^{-K_1 T^\alpha},
\]

\[
M = \frac{e^{K_1 T^\alpha}}{e^{K_1 T^\alpha} - 1},
\]

\[
\Psi_1(t) = Ml,
\]

\[
\Psi_2(t) = M, \quad t \in [0, T],
\]

\[
\overline{M} = Ml + \frac{\delta N^2 M^2 T^\delta}{K(\delta^2 - N^2 M^2 T^{2\delta})} - \frac{\delta^2 NM}{K(\delta^2 - N^2 M^2 T^{2\delta})},
\]

\[
\overline{M} = Ml + \frac{M^2 T^\delta}{\delta} + \frac{\delta N^2 M^2 T^\delta}{K(\delta^2 - N^2 M^2 T^{2\delta})} - \frac{N^3 M^3 T^{\delta}}{K(\delta^2 - N^2 M^2 T^{2\delta})}.
\]

For the forthcoming analysis, we first consider the following two boundary value problems for a linear differential fractional equations:
Lemma 4. Let $K > 0$, $a \in \mathbb{R}$, and $h \in E$. Then, problem (8) has the unique solution:

$$
\phi(t) = \int_{0}^{T} G(t, s) h(s) ds + a \Psi(t),
$$

(10)

where $\Psi(t) = (1/(1 - e^{-K_i T^3}))e^{-K_i t}$ and

$$
G(t, s) =
\begin{cases}
\frac{e^{K_i t \delta}}{e^{K_i T^3} - 1} - \frac{e^{-K_i (t^\delta - s^\delta)}}{e^{K_i T^3} - 1}, & 0 < s \leq t, \\
\frac{1}{e^{K_i T^3} - 1} - \frac{e^{-K_i (t^\delta - s^\delta)}}{e^{K_i T^3} - 1}, & 0 \leq t < s \leq T.
\end{cases}
$$

(11)

Proof. Multiply both sides of the first equation of (8) by $e^{K_i t}$, namely,

$$
e^{K_i t} \delta \phi(t) + Ke^{K_i t} \phi(t) = e^{K_i t} h(t).
$$

(12)

By using Lemma 2 (d), equation (12) is equivalent to

$$
\delta \left[ e^{K_i t} \phi(t) \right] = e^{K_i t} h(t).
$$

(13)

In view of Lemma 1 and Definition 2, we get

$$
e^{K_i t} \phi(t) - \phi(0) = \int_{0}^{t} s^{\delta - 1} e^{K_i{s^{\delta}}\right] h(s) ds,
$$

(14)

so

$$
\phi(t) = e^{-K_i t} \left[ \phi(0) + \int_{0}^{t} s^{\delta - 1} e^{K_i s} h(s) ds \right].
$$

(15)

The boundary condition $\phi(0) = \phi(T) + a$ leads to

$$
\phi(0) = \phi(T) + a = e^{-K_i T^3} \left[ \phi(0) + \int_{0}^{T} s^{\delta - 1} e^{K_i s} h(s) ds \right] + a.
$$

(16)

Clearly,

$$
\phi(0) = \frac{1}{e^{K_i T^3} - 1} \int_{0}^{T} s^{\delta - 1} e^{K_i s} h(s) ds + \frac{a}{1 - e^{-K_i T^3}}.
$$

(17)

Substituting (17) into (15), it follows that linear problem (8) has the following integral representation of the solution:

$$
\phi(t) = e^{K_i t} \left[ \frac{1}{e^{K_i T^3} - 1} \int_{0}^{T} s^{\delta - 1} e^{K_i s} h(s) ds + \int_{0}^{t} s^{\delta - 1} e^{K_i s} h(s) ds \right] + \frac{a}{1 - e^{-K_i T^3}}.
$$

(18)

This completes the proof.

For all $0 < \delta \leq 1$, Green’s function $G$ admits the following properties:

$$
\frac{1}{e^{K_i T^3} - 1} s^{\delta - 1} \leq G(t, s) \leq \frac{e^{K_i T^3}}{1 - e^{-K_i T^3}} s^{\delta - 1}, \quad t \in [0, T], \ s \in (0, T).
$$

(19)

Namely,

$$
M s^{\delta - 1} \leq G(t, s) \leq M s^{\delta - 1}, \quad t \in [0, T], \ s \in (0, T).
$$

(20)

In addition, for $\Psi$ given in Lemma 4, we can get

$$
\Psi_1(t) = \frac{e^{-K_i T^3}}{1 - e^{-K_i T^3}} \leq \Psi(t) \leq \frac{1}{1 - e^{-K_i T^3}} = \Psi_2(t).
$$

(21)

We define the operator $\mathcal{A}$ on $E$ by

$$
(\mathcal{A}h)(t) = \int_{0}^{T} G(t, s) h(s) ds, \ h \in E.
$$

(22)

It is easy to see that $\mathcal{A}: E \rightarrow E$ is a positive linear continuous operator. \hfill \Box

Lemma 5. $\|\mathcal{A}\| = (1/K)$.

Proof. By direct computation, one has

$$
\int_{0}^{T} G(t, s) ds = \frac{e^{K_i T^3}}{e^{K_i T^3} - 1} \int_{0}^{T} s^{\delta - 1} e^{K_i s} ds + \frac{1}{e^{K_i T^3} - 1} \int_{t}^{T} s^{\delta - 1} e^{K_i s} ds,
$$

$$
\int_{t}^{T} s^{\delta - 1} e^{K_i (s^\delta - t^\delta)} ds = \frac{e^{K_i T^3}}{\delta K_1 \left( e^{K_i T^3} - 1 \right)} \left( 1 - e^{-K_i t^\delta} \right) + \frac{1}{\delta K_1 \left( e^{K_i T^3} - 1 \right)} \int_{e^{K_i T^3} - K_i T^3 - 1}^{1} \frac{1}{K}.
$$

(23)

Then, for any $h \in E$, we have
\[ \|\mathcal{A}h\| = \max_{t \in [0,T]} |(\mathcal{A}h)(t)| \leq \max_{t \in [0,T]} \int_0^T G(t,s)ds \cdot \|h\| = \frac{1}{K} \|h\|, \]

(24)

which implies that \( \|\mathcal{A}\| \leq (1/K) \). On the other hand, take \( h_0(t) \equiv 1 \), then \( h_0 \in E, \|h_0\| = 1 \), and

\[ \|\mathcal{A}h_0\| = \max_{t \in [0,T]} |(\mathcal{A}h_0)(t)| = \int_0^T G(t,s)ds = \frac{1}{K} \|h_0\|. \]

(25)

This yields \( \|\mathcal{A}\| \geq (1/K) \). Therefore, \( \|\mathcal{A}\| = (1/K) \). This completes the proof.

We recall that \( I = e^{-K_1T^3} \). Then, \( l \in (0,1) \). For \( \forall h \in C([0,T], [0, +\infty)) \), it follows from (20) that

\[
(\mathcal{A}h)(t) = \int_0^T G(t,s)h(s)ds \leq M \int_0^T s^{-1}h(s)ds, \quad t \in [0,T].
\]

(26)

The abovementioned two inequalities show that

\[ (\mathcal{A}h)(t) \geq I((\mathcal{A}h)(s), \forall t, s \in [0,T], \forall h \in C([0,T], [0, +\infty))). \]

(27)

Based on the above analysis, we have the following result on (9).

\[ \Box \]

**Lemma 6.** Let \( K > 0 \), \( 0 \leq N < K \), \( a \in \mathbb{R} \), \( \theta \in C([0,T], [0, T]) \), and \( h \in E \). Then, problem (9) has a unique solution.

\[ \Box \]

**Proof.** From Lemma 4, it follows that \( \phi \in E \) is a solution of (9) if and only if

\[ \phi(t) = \int_0^1 G(t,s)[-N\phi(\theta(s)) + h(s)]ds + a\Psi(t). \]

(28)

Now, we introduce an operator \( \mathcal{B} : E \rightarrow E \) as follows:

\[ (\mathcal{B}h)(t) = N\phi(\theta(t)), \quad t \in [0,T]. \]

(29)

It is easy to see that \( \mathcal{B} \) is a positive linear operator with \( \|\mathcal{B}\| = N \). Thus, (28) reduces to

\[ (1 + \mathcal{A}\mathcal{B})\phi(t) = \mathcal{A}h(t) + a\Psi(t). \]

(30)

Note from Lemma 5 that \( \|\mathcal{A}\mathcal{B}\| \leq \|\mathcal{A}\| \cdot \|\mathcal{B}\| = (N/K) < 1 \). Thus, it follows from Lemma 3 that \( (1 + \mathcal{A}\mathcal{B})^{-1} \) exists

\[ (1 + \mathcal{A}\mathcal{B})^{-1} = \sum_{i=0}^\infty (-1)^i (\mathcal{A}\mathcal{B})^i = I - \mathcal{A}\mathcal{B} + (\mathcal{A}\mathcal{B})^2 + \cdots \]

\[ + (-1)^n (\mathcal{A}\mathcal{B})^n + \cdots. \]

(31)

Therefore, the unique solution of (9) is given by

\[ \phi(t) = \sum_{i=0}^\infty (-1)^i (\mathcal{A}\mathcal{B})^i \mathcal{A}h(t) + a \sum_{i=0}^\infty (-1)^i (\mathcal{A}\mathcal{B})^i \Psi(t). \]

(32)

The proof is complete.

Now, we present two comparison results.

\[ \Box \]

**Lemma 7.** Let \( K > 0 \), \( 0 \leq N < K \), \( a \in \mathbb{R} \), and \( \theta \in C([0,T], [0, T]) \). Assume that \( \phi \in E \) satisfies \( D^\delta \phi \in E \) and

\[
\begin{aligned}
\mathcal{D}^\delta \phi(t) &\leq -K\phi(t) - N\phi(\theta(t)), & t \in [0,T], \\
\phi(0) &\leq \phi(T).
\end{aligned}
\]

(33)

Then, \( \phi(t) \leq 0 \) for all \( t \in [0,T] \).

**Proof.** Take \( h(t) = D^\delta \phi(t) + K\phi(t) + N\phi(\theta(t)), a = \phi(0) - \phi(T) \). Then,

\[ h(t) \leq 0, \quad a \leq 0. \]

(34)

Applying Lemma 6, (32) holds, and (32) can be expressed by

\[ \phi(t) = \sum_{i=0}^\infty (\mathcal{A}\mathcal{B})^i (1 - \mathcal{A}\mathcal{B})\mathcal{A}h(t) + a \sum_{i=0}^\infty (\mathcal{A}\mathcal{B})^i (1 - \mathcal{A}\mathcal{B})\Psi(t). \]

(35)

Since \( h \leq 0 \), it implies that \( h_0(t) \equiv (\mathcal{A}h)(0) \geq 0 \). Thus, from (27), we obtain

\[ -\mathcal{A}h \geq lh_0, \quad -\mathcal{A}h \leq \frac{1}{l}h_0. \]

(36)

With the help of positivity of operator \( \mathcal{A}\mathcal{B} \), the definition of operator \( \mathcal{B} \), and (23), we have

\[ -(\mathcal{A}\mathcal{B})\mathcal{A}h \leq \frac{1}{l} (\mathcal{A}\mathcal{B})h_0 = \frac{N}{lK}h_0. \]

(37)

Consequently, we conclude that

\[ (1 - \mathcal{A}\mathcal{B})\mathcal{B}h \leq -lh_0 + \frac{N}{lK}h_0 = \left(1 - \frac{N}{lK}\right)h_0 \leq 0. \]

(38)

On the other hand, by (21), we infer that

\[ (1 - \mathcal{A}\mathcal{B})\Psi(t) = \frac{e^{-K_1T^3}}{1 - e^{-K_1T^3}} - N \int_0^T G(t,s)e^{-K_1(\theta(s))^3}ds \]

\[ \geq \frac{e^{-K_1T^3}}{1 - e^{-K_1T^3}} - N \int_0^T G(t,s)ds, \]

\[ = \frac{1}{1 - e^{-K_1T^3}} \left(1 - \frac{N}{K}\right) \geq 0. \]

(39)

Hence, \( \phi(t) \leq 0 \) holds for all \( t \in [0,T] \) that follow from \( a \leq 0 \) and (35). This completes the proof.

\[ \Box \]

**Lemma 8.** Let \( K > 0 \), \( 0 \leq NMT^\delta < \delta \), \( 0 \leq N < K \), \( M > 0 \), \( \bar{M} > 0 \), and \( \theta \in C([0,T], [0, T]) \). Assume that \( \phi \in E \) satisfies \( D^\delta \phi \in E \) and (33). Then, \( \phi(t) \leq 0 \) for all \( t \in [0,T] \).
Proof. Take again \( h(t) = \varphi(t) + K\phi(t) + N\phi(\theta(t)) \), \( a = \phi(0) - \phi(T) \). Then,
\[
\begin{align*}
h(t) &\leq 0, \\
a &\leq 0. 
\end{align*}
\]
(40)

Applying Lemma 6, (32) holds, and (32) can be expressed by
\[
\phi(t) = \sum_{i=0}^{\infty} (\mathcal{A}^i \phi)(t) - \sum_{i=0}^{\infty} (\mathcal{A}^{2i+1} \phi)(t) + a \sum_{i=0}^{\infty} (\mathcal{A}^{i+1} \phi)(t) .
\]
(41)

Taking notice of the fact that \( h(t) \leq 0 \), by (20), we have
\[
(\mathcal{A} \phi)(t) = \int_{0}^{T} G(t,s) h(s) ds \leq ML \int_{0}^{T} s^{-1} h(s) ds, 
\]
(42)
and for \( n \geq 1 \),
\[
\begin{align*}
(\mathcal{A}^n \mathcal{A} \phi)(t) &= N^{2n} \int_{0}^{T} G(t,s) \left( \int_{0}^{T} G(\theta(s), \tau_{2n-1}) \right. \\
&\quad \left. \int_{0}^{T} G(\theta(\tau_{2n-1}), \tau_{2n-2}) \right) \ldots \\
&\quad \left. \int_{0}^{T} G(\theta(\tau_{2}) \tau_{1}) \right) \\
&\quad \int_{0}^{T} G(\theta(\tau_{2}), \tau_{1}) h(\tau_{1}) d\tau_{1} \ldots \\
&\quad \int_{0}^{T} G(\theta(\tau_{1}), \tau_{0}) h(\tau_{0}) d\tau_{0} \\
&\quad \int_{0}^{T} G(\theta(\tau_{0}), \tau_{0}) h(\tau_{0}) d\tau_{0} \\
&\quad \int_{0}^{T} G(\theta(\tau_{0}), \tau_{0}) h(\tau_{0}) d\tau_{0},
\end{align*}
\]
(43)
and for \( n \geq 1 \),
\[
\begin{align*}
(\mathcal{A}^{2n+1} \mathcal{A} \phi)(t) &= N^{2n+1} \int_{0}^{T} G(t,s) \left( \int_{0}^{T} G(\theta(s), \tau_{2n}) \right. \\
&\quad \left. \int_{0}^{T} G(\theta(\tau_{2}), \tau_{1}) \right) \\
&\quad \int_{0}^{T} G(\theta(\tau_{1}), \tau_{0}) h(\tau_{0}) d\tau_{0} \\
&\quad \int_{0}^{T} G(\theta(\tau_{0}), \tau_{0}) h(\tau_{0}) d\tau_{0} \\
&\quad \int_{0}^{T} G(\theta(\tau_{0}), \tau_{0}) h(\tau_{0}) d\tau_{0},
\end{align*}
\]
(44)
and for \( n \geq 1 \),
\[
\begin{align*}
\sum_{i=0}^{\infty} (\mathcal{A}^{i} \mathcal{A} \phi)(t) - \sum_{i=0}^{\infty} (\mathcal{A}^{2i+1} \mathcal{A} \phi)(t) \leq \left[ ML + \sum_{i=0}^{\infty} N^{2i} M^{2i} (T^{\delta})^{2i-1} \right] \\
\int_{0}^{T} s^{-1} h(s) ds = M \int_{0}^{T} s^{-1} h(s) ds \leq 0.
\end{align*}
\]
(45)

By (20)–(23) and the positivity of operator \( \mathcal{A} \mathcal{B} \), we have
\[
\Psi(t) \geq \frac{e^{-K_{T}T}}{1 - e^{-K_{T}T}} = ML,
\]
(46)
and for \( n \geq 1 \),
\[
\begin{align*}
(\mathcal{A} \mathcal{B} \Psi)(t) &\geq (\mathcal{A} \mathcal{B} \Psi_{1})(t) \\
&= N \int_{0}^{T} G(t,s) \Psi_{1}(s) ds \\
&\quad \int_{0}^{T} G(\theta(s), \tau_{2n-1}) \Psi_{1}(\theta(s)) d\tau_{2} \ldots \\
&\quad \int_{0}^{T} G(\theta(\tau_{1}), \tau_{0}) \Psi_{1}(\theta(\tau_{1})) d\tau_{0} \int_{0}^{T} G(\theta(\tau_{0}), \tau_{0}) h(\tau_{0}) d\tau_{0}.
\end{align*}
\]
(47)
These lead us to
\[
\begin{align*}
\sum_{i=0}^{\infty} (\mathcal{A} \mathcal{B} \phi)(t) - \sum_{i=0}^{\infty} (\mathcal{A} \mathcal{B}^{2i+1} \phi)(t) \leq M L + \sum_{i=0}^{\infty} N^{2i} M^{2i} (T^{\delta})^{2i-1} \\
\int_{0}^{T} s^{-1} h(s) ds = M \int_{0}^{T} s^{-1} h(s) ds \leq 0.
\end{align*}
\]
(48)
These, together with the fact that \( a \leq 0 \), ensure that
are well defined. Furthermore, problems (53) and (54) have a unique solution $u_k$.

Now, we are in the position to prove the existence of the lower and upper solutions of (3).

### 3. Main Results

Now, we are in the position to prove the existence of extremal solutions of (3) by using the monotone iterative method of lower and upper solutions. To this end, we define the lower and upper solutions of (3).

**Definition 3.** A function $u_0 \in E$ satisfying $\mathcal{D}^\delta u_0 \in E$ is called a lower solution of problem (3) if it satisfies
\[
\begin{align*}
\mathcal{D}^\delta u_0 (t) &\leq f (t, u_0 (t), u_0 (\theta (t))), \quad t \in [0, T], \\
u_0 (0) &\leq u_0 (T).
\end{align*}
\]

 Analogously, a function $w_0 \in E$ satisfying $\mathcal{D}^\delta w_0 \in E$ is called an upper solution of (3) if the inequalities
\[
\begin{align*}
\mathcal{D}^\delta w_0 (t) &\geq f (t, w_0 (t), w_0 (\theta (t))), \quad t \in [0, T], \\
w_0 (0) &\geq w_0 (T),
\end{align*}
\]

hold.

**Theorem 1.** Assume that the following conditions hold:

$(H_1) \theta \in C ([0, T], [0, T])$

$(H_2)$: the functions $u_0$ and $w_0$ are lower and upper solutions of problem (3), respectively, such that $u_0 (t) \leq w_0 (t) \text{ on } [0, T]$.

$(H_3)$: $f \in C ([0, T] \times \mathbb{R}^2, \mathbb{R})$ and there exist $K > 0, N \geq 0$ such that
\[
f (t, x, z) - f (t, \bar{x}, \bar{z}) \geq - K (x - \bar{x}) - N (z - \bar{z}),
\]
for all $t \in [0, T], \quad u_0 (t) \leq \bar{x} \leq x \leq w_0 (t), \quad u_0 (t) \leq \bar{z} \leq z \leq w_0 (t)$.

$(H_4)$: the inequality $N \leq K \delta$ holds or the inequalities $N M T^\delta < \delta, N < K, M > 0, M > 0$ hold.

Then, (3) has minimal and maximal solution $u, w$ in the sector $[u_0, w_0]$, which can be obtained by monotone iterative sequences starting from $u_0$ and $w_0$, where $[u_0, w_0] = \{ z \in E : u_0 (t) \leq z (t) \leq w_0 (t), t \in [0, T] \}$.

**Proof:** For $k = 1, 2, \ldots$, let us define
\[
\begin{align*}
\mathcal{D}^\delta u_k (t) + K u_k (t) + N u_k (\theta (t)) &= f (t, u_{k-1} (t), u_{k-1} (\theta (t))) + K u_{k-1} (t) + N u_{k-1} (\theta (t)), \quad t \in [0, T], \\
u_k (0) &= u_0 (T),
\end{align*}
\]

\[
\begin{align*}
\mathcal{D}^\delta w_k (t) + K w_k (t) + N w_k (\theta (t)) &= f (t, w_{k-1} (t), w_{k-1} (\theta (t))) + K w_{k-1} (t) + N w_{k-1} (\theta (t)), \quad t \in [0, T], \\
w_k (0) &= w_0 (T).
\end{align*}
\]

By Lemma 6, for any $k = 1, 2, \ldots$, we know that linear problems (53) and (54) have a unique solution $u_k (t), w_k (t)$, respectively, which implies that the sequences $\{ u_k (t) \}, \{ w_k (t) \}$ are well defined. Furthermore, $u_k (t), w_k (t)$ can be expressed as
\[
\begin{align*}
u_k (t) &= (I + \mathcal{A})^{-1} \mathcal{A} u_{k-1} (t), \\
w_k (t) &= (I + \mathcal{A})^{-1} \mathcal{A} w_{k-1} (t),
\end{align*}
\]

where $\mathcal{F}: E \longrightarrow E$ is a bounded operator defined by
\[
(\mathcal{F} u)(t) = f (t, u(t), u(\theta(t))) + K u(t) + N u(t), \quad u \in E.
\]

By the integral expression of operator $\mathcal{A}$, it is easy to see that $\mathcal{A}$ is completely continuous. Hence, $(I + \mathcal{A})^{-1} \mathcal{A} \mathcal{F}$ is completely continuous.

Firstly, let us prove that
\[
\mathcal{D}^\delta u_0 (t) \geq f (t, w_0 (t), w_0 (\theta (t))), \quad t \in [0, T], \\
w_0 (0) \geq w_0 (T),
\]

which proves the lemma.
Now, let \( v(t) = u_1(t) - w_1(t) \); by \((H_2)\) and \((H_3)\), we obtain
\[
\mathcal{D}^\alpha v(t) = \mathcal{D}^\alpha u_1(t) - \mathcal{D}^\alpha w_1(t),
\]
\[
= f(t, u_0(t), u_0(\theta(t))) - K(u_1(t) - u_0(t)) - N(u_1(\theta(t)) - u_0(\theta(t)))
\]
\[
- f(t, u_1(t), u_1(\theta(t))) + f(t, u_1(t), u_1(\theta(t)))
\]
\[
\leq -K[u_0(t) - u_1(t)] - N[u_0(\theta(t)) - u_1(\theta(t))] - K(u_1(t) - u_0(t))
\]
\[
- N(u_1(\theta(t)) - u_0(\theta(t))) + f(t, u_1(t), u_1(\theta(t))),
\]
\[
= f(t, u_1(t), u_1(\theta(t))),
\]
\[
u(0) = u_1(0) - w_1(0) = u_1(T) - w_1(T) = v(T).
\]

Then, from Lemma 7 or Lemma 8, we get \( v(t) \leq 0 \), which yields \( u_1 \leq w_1 \).

Secondly, we need to show that \( u_1 \) and \( w_1 \) are the lower and upper solutions of problem \((3)\), respectively. In fact, it follows from \((H_2)\) and \((H_3)\) that
\[
\mathcal{D}^\alpha v(t) = \mathcal{D}^\alpha u_k(t) - \mathcal{D}^\alpha w_k(t),
\]
\[
= f(t, u_{k-1}(t), u_{k-1}(\theta(t))) - K(u_k(t) - u_{k-1}(t)) - N(u_k(\theta(t)) - u_{k-1}(\theta(t)))
\]
\[
- f(t, u_k(t), u_k(\theta(t))) + f(t, u_k(t), u_k(\theta(t)))
\]
\[
\leq -K[u_{k-1}(t) - u_k(t)] - N[u_{k-1}(\theta(t)) - u_k(\theta(t))] - K(u_k(t) - u_{k-1}(t))
\]
\[
- N(u_k(\theta(t)) - u_{k-1}(\theta(t))) + f(t, u_k(t), u_k(\theta(t))),
\]
\[
v(0) = u_k(0) - z(0) = u_k(T) - z(T) = v(T),
\]

which show that \( v_1 \) is a lower solution of problem \((3)\). Similarly, we can conclude that \( w_1 \) is an upper solution of problem \((3)\).

Repeating the foregoing arguments, we can prove that the sequences \([u_k(t)]\), \([w_k(t)]\) are lower and upper solutions of problem \((3)\), respectively, and satisfy the following inequality:
\[
u_0 \leq u_1 \leq \cdots \leq u_k \leq \cdots \leq w_k \leq \cdots \leq w_1 \leq w_0.
\]

Obviously, the sequences \([u_k(t)]\), \([w_k(t)]\) are uniformly bounded in \( E \) and by \((55)\) and the complete continuity of operator \((1 + \mathcal{A}^\alpha)\)^{−1} and \( \mathcal{F} \), it follows that \([u_k(t)]\), \([w_k(t)]\) are relatively compact. This, together with the monotonicity of the sequences \([u_k(t)]\), \([w_k(t)]\), guarantees that the sequences \([u_k(t)]\), \([w_k(t)]\) converge uniformly to \( u, \varphi \), respectively, and that \( u, \varphi \in [v_0, w_0] \) are solutions of \((3)\).

Finally, we prove the minimal and maximal property of \( u \) and \( \varphi \) on \([v_0, w_0]\). We assume that \( z \in [v_0, w_0] \) is any solution of \((3)\) and there exists a positive integer \( k \) such that \( u_k(t) \leq z(t) \leq w_k(t) \) for \( t \in [0, T] \).

Let \( v(t) = u_k(t) - z(t) \), then
\[
\text{undoubtedly, } a(t) \leq 0, \text{ namely, } u_k(t) \leq z(t). \text{ By a similar method, we can show that } z(t) \leq w_k(t). \text{ Thus, } u_k \leq z \leq w_k, \text{ } k = 1, 2, \ldots. \text{ It is easy to find that } u(t) \leq z \leq w(t) \text{ when } k \rightarrow \infty. \text{ That is } u, \varphi \text{ are minimal and maximal solutions of } (1) \text{ in the sector } [v_0, w_0]. \text{ The proof is completed.}
\]

Then, by applying Lemma 7 or Lemma 8, we get \( v(t) \leq 0 \), that is \( u_k(t) \leq z(t) \text{ on } [0, T] \). Similarly, we can show that \( z(t) \leq w_k(t) \) on \([0, T]\). Notice that \( u_0(t) \leq z(t) \leq w_0(t) \), then \( u_k(t) \leq z(t) \leq w_k(t) \) hold for every \( k \) from mathematical induction. Hence, by taking \( k \rightarrow + \infty \), we have \( u(t) \leq z(t) \leq w(t) \) on \([0, T]\). The proof is complete. 

Example 1. We consider the following BVP:
equation (63) has the extremal solution in $[\nu_0, w_0]$. Hence, all conditions of Theorem 1 hold. Therefore, equation (63) has the extremal solution in $[\nu_0, w_0]$. Let $u_0(t) = -1, w_0(t) = \sqrt{2}t;\text{ then,}$

\[
\begin{align*}
\mathcal{D}^{(1/2)} u_0(t) &= 0 < -\frac{\sqrt{2}}{60\pi}\cos\frac{1}{4} = f(t, u_0(t), u_0(\theta(t))), \quad t \in [0, 1], \\
u_0(0) &= u_0(1) = -1, \\
\mathcal{D}^{(1/2)} w_0(t) &= 0 > -\frac{1 + \frac{\sqrt{2}t}{3}}{f(t, w_0(t), w_0(\theta(t))), \quad t \in [0, 1], \\
w_0(0) &= w_0(1) = \sqrt{2\pi}.
\end{align*}
\]

This shows that $u_0, w_0$ are lower and upper solutions of (63). On the other hand, it is easy to verify that $(H_4)$ holds for $K = 1$ and $N = (1/60)$. Furthermore, we have

\begin{align*}
K_1 &= 2, \\
l &= e^{-2} \in (0, 1), \\
M &= \frac{1}{1 - e^{-2}} \approx 1.1565, \\
N &= \frac{1}{60} < 1 = K, \\
NMT^d &= 0.0193 < \frac{1}{2} = \delta, \\
M &\approx 0.1388 > 0, \\
\bar{M} &\approx 0.1119 > 0.
\end{align*}

Hence, all conditions of Theorem 1 hold. Therefore, equation (63) has the extremal solution in $[\nu_0, w_0]$.

**Data Availability**

The data used to support the findings of this study are available from the corresponding author upon request.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

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