Type-II Weyl Points in Three-Dimensional Cold Atom Optical Lattices

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Topological Lifshitz phase transition characterizes an abrupt change of the topology of the Fermi surface through a continuous deformation of parameters. Recently, Lifshitz transition has been predicted to separate two types of Weyl points: type-I and type-II (or called structured Weyl points), which has attracted considerable attention in various fields. Although recent experimental investigation has seen a rapid progress on type-II Weyl points, it still remains a significant challenge to observe their characteristic Lifshitz transition. Here, we propose a scheme to realize both type-I and type-II Weyl points in three-dimensional ultracold atomic gases by introducing an experimentally feasible configuration based on current spin-orbit coupling technology. In the resultant Hamiltonian, we find three degenerate points: two Weyl points carrying a Chern number $-1$ and a four-fold degenerate point carrying a Chern number $2$. Remarkably, by continuous tuning of a convenient experimental knob, all these degenerate points can transition from type-I to type-II, thereby providing an ideal platform to study different types of Weyl points and directly probe their Lifshitz phase transition.

PACS numbers: 03.75.Ss, 37.10.Jk, 03.65.Vf, 03.75.Lm

I. INTRODUCTION

Distinct from conventional phase transitions driven by spontaneous symmetry breaking, Lifshitz phase transition is driven by an abrupt change of the topology of the Fermi surface\textsuperscript{1}. Recently, it has been predicted in three-dimensional (3D) condensed matter materials that type-I Weyl fermions (i.e., Weyl points)\textsuperscript{2–5} can transition to type-II\textsuperscript{6} (or called structured Weyl points\textsuperscript{7}) through Lifshitz transition\textsuperscript{6,7}. Although type-I Weyl fermions\textsuperscript{8} were initially predicted in particle physics, type-II Weyl fermions may not be allowed to exist there due to Lorentz symmetry, which is absent in condensed matter materials. Such Lifshitz transition is perceived as the change of a type-I Weyl point’s single point Fermi surface to a type-II’s open Fermi surface consisting of particle and hole pockets along with a touching point\textsuperscript{6,7}. While remarkable experimental and theoretical progress has been reported recently on type-II Weyl semimetals and their properties\textsuperscript{6,7,9–25}, it remains a significant challenge in solid-state materials to observe the characteristic Lifshitz phase transition between type-I and type-II Weyl points.

Ultracold atomic gases provide an ideal platform to observe the Lifshitz transition because of their high controllability. And recent experiments on one-dimensional and two-dimensional (2D) spin-orbit coupling\textsuperscript{26–35} in ultracold atomic gases have further paved the way for discovering novel topological quantum states\textsuperscript{36–38}. Although type-I and type-II Weyl points were proposed in quasiparticle spectra of spin-orbit coupled Fermi superfluids\textsuperscript{7,39–43}, realization of such superfluids is still a big challenge with current experimental technology\textsuperscript{44–48}. A more feasible experimental scheme is to realize Weyl points in the single-particle spectra without the need of low-temperature Fermi superfluids. While some proposals have been made concerning type-I Weyl points\textsuperscript{44–48}, the scheme to realize both type-I and type-II Weyl points and their Lifshitz transition in the single-particle spectra of cold atoms is still lacking and highly desired.

In this paper, we propose a novel 3D model to realize both type-I and type-II Weyl points in the single-particle spectra of cold atoms in 2D optical lattices. In this model, we find two Weyl points carrying a Chern number $-1$ and a four-fold degenerate point carrying a Chern number $2$, protected by 2D pseudo-time-reversal (2D-PTRS), 2D inversion (2D-IS), and combined rotational symmetries. By continuous tuning of a Zeeman field, all these degenerate points can experience the Lifshitz transition from type-I to type-II. Furthermore, we find a laser-atom coupling configuration to implement the model based on the current experimental technology that realizes the required spin-orbit coupling. Due to the controllability of the Zeeman field through continuous tuning of a convenient experimental knob, this system offers a unique opportunity to observe the characteristic topological Lifshitz quantum phase transition between type-I and type-II Weyl points.

II. MODEL HAMILTONIAN

We start by briefly reviewing the concept of type-I and type-II Weyl points that are described by an effective Hamiltonian $H_W = v_0 k_z + \sum_{\mu=x,y,z} v_\mu k_\mu \sigma_\mu$, where $k_\mu$ ($\sigma_\mu$) denote momenta (Pauli matrices) and $v_0, v_\mu$ are real parameters. Its energy spectrum is given by $E_\pm(k) = v_0 k_z \pm \sqrt{\sum_{\mu=x,y,z} v_\mu^2 k_\mu^2}$ with $\pm$ labeling particle and hole bands. When $|v_0| < |v_z|$, the energy of the
discretize

particle (hole) band is positive (negative), except at the touching point where the energy vanishes. This touching point is dubbed type-I Weyl point. When \(|v_0| > |v_2|\), in certain regions, the energy goes negative for the particle band and positive for the hole band, leading to an open Fermi surface besides a touching point at \(E_z(\mathbf{k} = 0) = 0\). This structure is dubbed type-II Weyl point.

To realize both type-I and type-II Weyl points with cold atoms, we consider the following Hamiltonian that describes atoms in 2D optical lattices

\[
H' = \frac{\mathbf{p}^2}{2m} + \sum_{\nu=x,y} V_\nu \sin^2(k_{L\nu} r_\nu) + h_z \sigma_z + V_{SO},
\]

where \(\mathbf{p} = -i\hbar \nabla\) is the momentum operator, \(m\) is the mass of atoms, \(V_\nu (\nu = x, y)\) denote the strength of optical lattices with the period being \(a_\nu = \pi/k_{L\nu}\) along the \(\nu\) direction, \(h_z\) is the Zeeman field, \(\sigma_\nu\) are Pauli matrices for spins, and \(V_{SO}\) is a laser-induced spin-orbit coupling term taking the form

\[
V_{SO} = \Omega_{SO}(M_x + i M_y)e^{ik_{L\nu} r_\nu}\frac{1}{\uparrow} \uparrow + \text{H.c.}
\]

with \(M_x = \sin(k_{Lx} r_x) \cos(k_{Ly} r_y)\), \(M_y = \sin(k_{Ly} r_y) \cos(k_{Lx} r_x)\), \(\Omega_{SO}\) proportional to the laser strength, and \(k_{L\nu}\) with \(\nu = x, y, z\) determined by lasers’ wave vector along the \(\nu\) direction. Later, we will describe the laser configuration that directly realizes the Hamiltonian (1), in particular the spin-orbit coupling term \(V_{SO}\). Employing a unitary transformation with \(U = e^{(-i k_{L\nu} r_\nu/2)} \uparrow(\downarrow) \uparrow(\downarrow) + e^{i k_{L\nu} r_\nu/2} \downarrow(\uparrow) \downarrow(\uparrow)\) yields \(H = U H' U^{-1}\) with

\[
H = \frac{\hbar^2 k_z^2}{2m} + h_z \sigma_z + H_{2D},
\]

where \(k_z = p_z/\hbar\), \(\tilde{h}_z = h^2 k_z^2/(2m) + h_z\), and the 2D Hamiltonian \(H_{2D}\) in the \((x, y)\) plane is expressed as

\[
H_{2D} = \sum_{\nu=x,y} \left[ \frac{p_\nu^2}{2m} + V_\nu \sin^2(k_{L\nu} r_\nu) \right] + [\Omega_{SO}(M_x + i M_y)\frac{1}{\uparrow} \uparrow + \text{H.c.}].
\]

To see how the Weyl points emerge in this model, we discretize \(H_{2D}\) and study its physics in the tight-binding model (see Appendix A for details of discretization). The tight-binding form of \(H\) can be written as [let us first neglect \(\hbar^2 k_z^2/(2m)\) and focus on type-I Weyl points]

\[
H_{TB} = \sum_{k_z \mathbf{x}} \left[ \hbar \hat{z} \hat{c}_{k_z,\mathbf{x}} \sigma_z \hat{c}^\dagger_{k_z,\mathbf{x}} + \sum_{\nu=x,y} (-t_\nu \hat{c}_{k_z,\mathbf{x}} \sigma_\nu \hat{c}^\dagger_{k_z,\mathbf{x} + \mathbf{g}_\nu} + (1 + \tau) t_{SO\nu} \hat{c}_{k_z,\mathbf{x}} \sigma_\nu \hat{c}^\dagger_{k_z,\mathbf{x} + \mathbf{g}_\nu} + \text{H.c.}) \right],
\]

where \(\mathbf{g}_\nu = a_\nu \mathbf{e}_\nu\), \(\hat{c}_{k_z,\mathbf{x}} = (\hat{c}_{k_z,\mathbf{x}}^\uparrow, \hat{c}_{k_z,\mathbf{x}}^\downarrow)\) with \(\hat{c}_{k_z,\mathbf{x}}^{\uparrow,\downarrow}\) being the creating (annihilating) operator and \(\mathbf{x} = (x, y)\). \(t_\nu\) and \(t_{SO\nu}\) denote the tunneling and spin-orbit coupling strength along the \(\nu\) direction.

Different from the well-known 2D Chern insulator \([49, 50]\), this Hamiltonian involves the position dependent spin-orbit coupling, and we thus need to choose a unit cell consisting of two sites: \(A\) and \(B\) [as shown in Fig. 1(a)] and the Hamiltonian in the momentum space in the new basis \(\Psi(\mathbf{k})^T = (e^{ik_z a_x} \hat{A}_{k x}^\dagger e^{ik_z a_y} \hat{A}_{k y})\). \(\hat{B}_{k z}\) reads

\[
H_{TB}(\mathbf{k}) = \hbar \sigma_z - h_t \sigma_x + \tau (-d_x \sigma_x + d_y \sigma_y),
\]

where \(h_t = \sum_{\nu=x,y} t_\nu \cos(k_{L\nu} a_\nu)\), \(d_x = 2t_{SOx} \sin(k_{Lx} a_x)\) and \(d_y = -2t_{SOy} \sin(k_{Ly} a_y)\); \(\tau\) are Pauli matrices acting on \(A, B\) sublattices. In the \((k_x, k_y)\) plane, \(H_{TB}(k_x, k_y)\) \((k_x = k_x \mathbf{e}_x + k_y \mathbf{e}_y)\) respects 2D-IS: \(\pi T \neq H_{TB}(k_x, k_y)\). \(H_{TB}(k_x, k_y)\) is the one for the monolayer model. \(\hbar \tau \sigma_z H_{TB}(k_x, k_y) T\sigma_z = H_{TB}(-k_x, k_y)\) and when \(\tilde{h}_z = 0\), \(H_{TB}(k_x, k_y)\) for \(T\sigma_z H_{TB}(k_x, k_y) T^{-1} = H_{TB}(-k_x, k_y)\) with \(T = \sigma_z \tau_x \sigma_\nu \mathbf{K}\) being the complex conjugate operator. These two symmetries guarantee that the spectrum in this specific plane (\(\hbar = 0\)) is at least doubly degenerate, implying that the touching point, if exists, is four-fold degenerate. We note that in the continuous model 2D-IS corresponds to \(\mathbb{I}_c \mathbb{H}_c^{-1} = H(-r_x + \pi/k_{Lx}, r_y - r_z)\), with the inversion center located at \((\pi/(2k_{Lz}), 0)\) and 2D-PTRS corresponds to \(T \mathbb{H}_{2D} T^{-1} = H_{2D}(-r_x, -r_y)\).
Specifically, the eigenvalues of $H_{FB}(k)$ read $E_k = \pm \sqrt{d_1^2 + (h_t \pm h_z)^2}$ with $d_1^2 = d_x^2 + d_y^2$, which supports the above symmetry analysis that the energy band at each $k$ is doubly degenerate without $h_z$. Clearly, when $d_x = d_y = 0$ and $h_t \pm h_z = 0$, there emerge degenerate points. This requires $(k_xa_x, k_ya_y) = (0, \pi)$ or $(0, 0)$. In the former case, a single degenerate point appears at $k_w^0 = -2m\pi h_t/(\hbar^2 k_{Lz}^2)$ if $t_x = t_y$ (thus $h_t = 0$ at this point) as a result of a combined rotational symmetry (i.e., $U_4H_{FB}U_4^{-1} = H$ where $U_4 = S_1C_4$ with $S_1 = e^{i\sigma_y\pi/4}$ and $C_4$ being the four-fold rotational operator along $z$ when $V_x = V_y$ and $k_{Lz} = k_{Ly}$) readily achievable in experiments; in this plane, both 2D-PTRS and 2D-IS are preserved and this point’s degeneracy is therefore four-fold (also seen from the eigenvalues). The dispersion is linear along all three momenta directions as visually shown in Fig. 1(c). Compared with a Dirac point in a Hamiltonian with both TRS and IS in 3D [51, 52], in our case, two similar symmetries (2D-PTRS and 2D-IS) are both respected only in the plane $h_t = 0$. In fact, the four-fold degenerate point can be viewed as consisting of two Weyl points with the same Chern number. To demonstrate this, let us write down the effective Hamiltonian near the point, which, after a unitary transformation, reads

$$H(q) \sim (v_z q_z \sigma_z + v_x q_x \sigma_x + v_y q_y \sigma_y)\tau_0,$$  \hspace{1cm} (7)

where $v_z = \hbar^2 k_{Lz}/2m$, $v_x = 2t_S a_x$, and $v_y = -2t_S a_y/\tau_0$; $\tau_0$ is a $2 \times 2$ identity matrix; the momenta $q$ are measured with respect to the degenerate point. It is clear that each spinor corresponds to a Weyl point with the same chirality [51].

To characterize the topological charge of a degenerate point, we define the first Chern number

$$C = \frac{1}{2\pi} \sum_{n=1,2} \iint_S \Omega_n(k) \cdot dS,$$  \hspace{1cm} (8)

where the surface $S$ encloses a considered degenerate point, and $\Omega_n(k) = (\nabla_k v_n(k)) \times (\nabla_k u_n(k))$ is the Berry curvature [52] for the $n$-th band with $|u_n(k)|$ being its wave function. For our parameters, a direct calculation yields $C = 2$ for the above discussed four-fold degenerate point.

In the latter case where $(k_{w \pm} a_x, k_{w \pm} a_y) = (0, 0)$, two degenerate points occur at $k_{w \pm} a_z = 2m\pi/4t \pm h_z/(\hbar^2 k_{Lz}^2)$ with $t = (t_x + t_y)/2$. Their degeneracy is double instead of four-fold (owing to the breaking of 2D-PTRS), and their dispersion is also linear in all three momentum directions. Our calculation demonstrates $C = -1$ for either of them. Due to the existence of the energy $\hbar^2 k_{Lz}^2/(2m)$, the energy at these two Weyl points is different except when $h_z = 0$. Moreover, while we consider single-particle physics, a crude estimate using mean-field analysis suggests that weak repulsive short-range interactions may shift the locations of Weyl and four-fold degenerate points along $z$ and cause the transition between the type-I and type-II but not destroy them (see Appendix B).

![FIG. 2: (Color online) (a) Phase diagram with respect to $h_z$ and $t$ ($t = t_x = t_y$), in which $W_I(W_{II})$ with $\beta = 0, +, -$ represent the phase with a type-I (type-II) Weyl or four-fold degenerate point located at $k_w^0$. The dashed line shows the case for $t = 0.05E_R$. (a)(b) Spectra as a function of $k_w a_z$ for $k_y = 0$ under open boundary condition along the $x$ direction. In (a), $h_z = 0$ and the Weyl points are all type-I; in (b), $h_z = 0.4E_R$ and a Weyl point on the left side is type-II and all others are type-I. The black lines denote surface states.](image)

To illustrate that these type-I Weyl and four-fold degenerate points can transition to type-II, we include $\hbar^2 k_{Lz}^2/(2m)$ and expand the Hamiltonian near such points, e.g., $(0, \pi/a_y, k_w^0)$,

$$H(q) \sim (v_0 \tau_0 q_z + v_x q_x \sigma_x + v_y q_y \sigma_y)\tau_0$$  \hspace{1cm} (9)

with $v_0 = -2h_z/k_{Lz}$. Remarkably, when $|h_z| > \hbar^2 k_{Lz}^2/(4m)$ (i.e., $|v_0| > v_z$), at certain regions, the energy of both particle bands goes negative while that of both hole bands goes positive [as visually displayed in Fig. 1(d)], indicating that the four-fold degenerate point becomes type-II. Similarly, the type-I Weyl point at $k_w^+$ becomes type-II when $h_z < 4t - \hbar^2 k_{Lz}^2/(4m)$ or $h_z > 4t + \hbar^2 k_{Lz}^2/(4m)$, and the point at $k_w^-$ becomes type-II when $h_z < -4t + \hbar^2 k_{Lz}^2/(4m)$ or $h_z > -4t - \hbar^2 k_{Lz}^2/(4m)$. In Fig. 2(a), we map out the phase diagram displaying the following phases: all degenerate points are type-I, all are type-II, and partial type-I and partial type-II. In cold atoms, $h_z$ can be easily tuned by changing two-photon detuning to continuously drive the transition among different phases, thereby providing an ideal platform to directly observe the Lifshitz transition.

We now turn to the study of surface states in this system. In Fig. 2, the energy spectra with respect to $k_w a_z$ for $k_y = 0$ are plotted: (a) $h_z = 0$ with type-I degenerate points and (b) $h_z = 0.4E_R$ with a type-II point. It shows that in both cases there emerge surface states (called Fermi arc) connecting the four-fold degenerate point at the center with two other Weyl points on two sides. The spectra in Fig. 2(b) also illustrate the feature of the type-II Weyl point at $k_w a_z/\pi = -0.62$ that all the particle and hole bands near the point at each $k_z$ are pos-
itive or negative with respect to $E_{W-}$, the energy at the point.

Apart from the model that we have discussed, if we choose a simplified scheme with $M_{x} = \sin(k_{Lx}r_{x})e^{ik_{Lz}r_{z}}$ and $M_{y} = \sin(k_{Ly}r_{y})e^{-ik_{Lz}r_{z}}$, we can still obtain both type-I and type-II Weyl points. However, while it still respects 2D-PTRS, it breaks 2D-IS, splitting the four-fold degenerate point into two doubly degenerate ones (see Appendix A). To satisfy these symmetry requirements, we need to add an additional term into the model (ii) $\tau_{z}(\alpha_{1}a_{y} - \alpha_{2}a_{x})$ which respects 2D-PTRS but breaks 2D-IS. If $V_{x} = V_{y}$ and $k_{Lx} = k_{Ly}$, we have $\alpha_{1} = \alpha_{2} = \alpha$ due to a symmetry requirement (see Appendix A), and the spectrum is $E_{k} = \pm\sqrt{(d_{x} + \lambda\alpha)^{2} + (d_{y} - \lambda\alpha)^{2} + (h_{z} - \lambda h_{z})^{2}}$ with $\lambda = \pm 1$, and Weyl points occur at $[k_{x}^{W}a, k_{y}^{W}a, k_{z}^{W}a] = [\lambda\delta\theta, \pi + \lambda\delta\theta, -2\pi\eta h_{z}/(h_{x}^{2}k_{z}^{2})]$ (or $[\lambda\delta\theta, -\lambda\delta\theta, 2\pi\eta(4\lambda\cos\delta\theta - h_{z})/(h_{x}^{2}k_{z}^{2})]$) with $\delta\theta = -\sin^{-1}(\tan(2\lambda\delta\theta))$. Moreover, when $|h_{z}| > R_{LZ}/(2m)$ ($|h_{z}| - 4\lambda\cos\delta\theta > h_{x}^{2}k_{z}^{2}/2m$), the former (latter) Weyl points become type-II.

III. REALIZATION OF TYPE-II WEEY POINTS

To realize the Hamiltonian (ii), we propose an experimental scheme (as shown in Fig. 3) that is based on a modification of the experimental configuration in Ref. 33. We consider two hyperfine states of alkali atoms such as $^{85}$K and $^{87}$Rb and employ two independent Raman processes to create the spin-dependent optical lattices. Each Raman process involves two linearly polarized blue-detuned Raman laser beams with the polarization being along $y$ (parallel to the magnetic field) and $x$, respectively. Each pair of Raman laser beams is characterized by a pair of Rabi frequencies $|\Omega_{1} = \Omega_{10}\sin(k_{Lx}r_{x})e^{-ik_{Lz}r_{z}}/2, \Omega_{2} = \Omega_{20}\cos(k_{Ly}r_{y})e^{ik_{Lz}r_{z}}/2]$ and $|\Omega_{1}' = \Omega_{10}\sin(k_{Lx}r_{x})e^{-i(k_{Lz}r_{z})/2}, \Omega_{2}' = \Omega_{20}\cos(k_{Ly}r_{y})e^{i(k_{Lz}r_{z})/2}]$, respectively. They form a standing wave along $x$ or $y$ but remain a plane wave along $z$. Since the laser beam $\Omega_{1}'$ ($\Omega_{2}'$) is obtained by reflecting the beam $\Omega_{1}$ ($\Omega_{2}$) by mirrors, they possess the same frequency $\omega_{1}$ ($\omega_{2}$), and no phase locking is required 53, 54. The laser beam $\Omega_{2}$ with a different frequency $\omega_{2}$ is generated by applying an acoustic-optical modulator on a laser beam split from the first laser beam. Each pair of Raman laser beams couple two hyperfine states independently, leading to the spin-orbit coupling term $V_{SO}$ with $V_{SO} = \Omega_{10}/\Omega_{20}/\Delta_{0}$. Moreover, owing to the stark effects, these lasers also create optical lattices along the $x$ and $y$ directions: $V_{x}\sin^{2}(k_{Lx}r_{x})$ and $V_{y}\sin^{2}(k_{Ly}r_{y})$ with $V_{x} = V_{y} = 2(\Omega_{10}^{2} - |\Omega_{20}|^{2})/\Delta_{0}$. The Zeeman field $h_{z}$ is generated by the two-photon detuning $\delta$ through $h_{z} = \delta/2$ as shown in Fig. 3.

In comparison, a similar laser configuration can also realize the simplified spin-orbit coupling scheme that we mentioned before. In this case, we further simplify the laser configuration [as shown in Fig. 3(c)] so that the Rabi frequencies take the form $\Omega_{2} = 2\Omega_{20}e^{ik_{Lz}r_{z}}/2 + ik_{Lz}r_{z}$ and $\Omega_{2}' = i\Omega_{20}e^{ik_{Lz}r_{z}}/2 - ik_{Lz}r_{z}$, corresponding to plane waves for the second set of laser beams.

We may choose either $^{85}$K (fermions) or $^{87}$Rb (bosons) atoms for observation of Weyl points in experiments. Here we take $^{85}$K as an example. With a blue-detuned laser beam at wavelength 764 nm (corresponding to $\Delta_{c} = 2\pi \times 1.38$ THz) 23, the recoil energy $E_{R}/h = 2\pi \times 8.5$ kHz. If we choose a geometry with $k_{Lx} = k_{Ly} = k_{R}\sin\theta$ and $k_{Lz} = 2k_{R}\cos\theta$ with $\theta = 60^\circ$ (the angle between laser beams and $z$ axis), $\Omega_{10} = 2\pi \times 0.15$ GHz, $\Omega_{20} = \Omega_{10}/3$, we have $V_{x} = V_{y} = 3.7E_{R}$ and $\Omega_{SO} = 0.7E_{R}$. With these experimental parameters, the tight-binding parameters are $t_{x} = t_{y} = 0.05E_{R}$, and $t_{SOx} = -t_{SOy} = 0.028E_{R}$. $\delta$ can be readily tuned from zero and when $\delta$ crosses 0.53$E_{R}$, we will observe a Lifshitz-type quantum phase transition from type-I to type-II Weyl points.

To detect the Weyl points of fermionic atoms and their Lifshitz phase transition, one can measure their linear spectra along all three momenta directions by momentum resolved radio-frequency spectroscopy, which has been utilized for observation of a 2D Dirac cone in spin-orbit-coupled atomic gases 31, 32. The Lifshitz transition is directly reflected by a sharp change of the dispersion of particle or hole bands along $k_{z}$ near a Weyl point so that the slopes of their spectra have the same sign. For bosonic atoms such as $^{87}$Rb, although the Fermi surface does not exist, there is still the band structure with a touching point. One may consider driving a BEC across a Weyl point by a constant force $F$ and measuring the Landau-Zener tunneling probability 48, 55–57, which is $P_{LZ} = e^{-\pi E_{R}^{2}/(4av^{2}F)}$ with $E_{R}$ being the gap between the considered particle and hole bands and $v$ being the velocity of the BEC 58. Therefore, the gap closing at a Weyl point is signalled by a peak of the Landau-Zener tunneling probability. When the BEC bypasses a type-I (type-II) Weyl point, a finite fraction of atoms remains in the hole band and these atoms move in the opposite (same) direction along $z$ compared with those tunneling into a higher band. This different behavior can be uti-
lized to measure the Lifshitz transition.

In summary, we have proposed a scheme well based on the current experimental technology to realize both type-I and type-II Weyl points in the single-particle spectra with cold atoms in an optical lattice. The proposed system offers a unique opportunity to observe and study the topological Lifshitz-type quantum phase transition from type-I to type-II Weyl points by continuously tuning one of the experimental knobs.

Acknowledgments

We thank S.-T. Wang and S. A. Yang for helpful discussions. This work was supported by the ARL, the IARPA LogiQ program, and the AFOSR MURI program.

Appendix A: DERIVATION OF TIGHT-BINDING MODEL

In this appendix, we derive the tight-binding model from the continuous model \( H \) in Eq. (3) in the main text and compare their spectra to verify the tight-binding model’s reliability.

Let us first focus on the discretization of \( H_{2D} \), which can be written as in the second quantization language

\[
H_{11} = \int d\mathbf{r} \hat{\psi}^\dagger(\mathbf{r}) H_{2D} \hat{\psi}(\mathbf{r}),
\]

where \( \mathbf{r} \) is restricted to the \((x, y)\) plane and \( \hat{\psi}(\mathbf{r}) = [\hat{\psi}_\uparrow(\mathbf{r}) \hat{\psi}_\downarrow(\mathbf{r})]^T \) with \( \hat{\psi}_\sigma(\mathbf{r}) \) \((\hat{\psi}^\dagger(\mathbf{r}))\) annihilating (creating) an atom with spin \( \sigma \) \((\uparrow, \downarrow)\) located at \( \mathbf{r} \). They satisfy the anti-commutation or commutation relation \([\hat{\psi}_\sigma(\mathbf{r}), \hat{\psi}^\dagger(\mathbf{r}')]_\pm = \delta_{\sigma,\sigma'}\delta(\mathbf{r} - \mathbf{r}')\) for fermionic atoms \((+)\) or bosonic atoms \((-)\), respectively. We expand the field operator using local Wannier functions

\[
\hat{\psi}_\sigma(\mathbf{r}) = \sum_{n,j_x,j_y} W_{n,j_x,j_y} \hat{c}_{n,j_x,j_y,\sigma},
\]

where \( W_{n,j_x,j_y} \) is the Wannier function for \( \Omega_{SO} = 0 \) located at the site \((j_x, j_y)\) for the \( n\)-th band, and \( \hat{c}_{n,j_x,j_y,\sigma} \) \((\hat{c}^\dagger_{n,j_x,j_y,\sigma})\) annihilates (creates) an atom located at the state \( W_{n,j_x,j_y} \) with spin \( \sigma \). Let us focus on the physics in the lowest band and thus assume \( n = 1 \), thereby simplifying the above expression

\[
\hat{\psi}_\sigma(\mathbf{r}) \approx \sum_{j_x,j_y} W_{j_x,j_y} \hat{c}_{j_x,j_y,\sigma},
\]

where \( W_j = W^x_j(r_x)W^y_j(r_y) \) with \( W^\nu_j(r_\nu) = W^\nu(r_\nu - j_\nu a_\nu) \) being the Wannier function along \( \nu \). We note that although this is an approximation, it proves to be qualitatively correct and we will verify it by comparing the spectra obtained by solving the continuous model and the tight-binding one. Using this expansion, we obtain the following tight-binding model without \( V_{SO} \)

\[
H_t = \sum_{j_x,j_y,\sigma} \left( t_{x} \hat{c}_{j_x,j_y,\sigma}^\dagger \hat{c}_{j_x+1,j_y,\sigma} + t_{y} \hat{c}_{j_x,j_y+1,\sigma}^\dagger \hat{c}_{j_x,j_y,\sigma+1,\sigma} \right) + \text{H.c.},
\]

where \( t_\nu \) with \( \nu = x, y \) denote the hopping amplitudes defined as

\[
t_\nu = - \int dr_\nu W^\nu_j \left[ \frac{p_\nu^2}{2m} + V_\nu \sin^2(k_L r_\nu) \right] W^\nu_j + 1. \tag{5}\]

We approximately derive the tight-binding term contributed by the spin-dependent lattice as follows

\[
H_{SO} = \Omega_{SO} \int d\mathbf{r} \hat{\psi}_\uparrow(\mathbf{r}) (M_x + iM_y) \hat{\psi}_\downarrow(\mathbf{r}) + \text{H.c.},
\]

\[
\approx \Omega_{SO} \sum_{j_x,j_y} \sum_{j'_x,j'_y} \left[ t_{SOx}^{(j_x,j_y),(j'_x,j'_y)} \hat{c}_{j_x,j_y,\uparrow}^\dagger \hat{c}_{j'_x,j'_y,\downarrow} + t_{SOy}^{(j_x,j_y),(j'_x,j'_y)} \hat{c}_{j_x,j_y,\downarrow}^\dagger \hat{c}_{j'_x,j'_y,\uparrow} \right] + \text{H.c.},
\]

where

\[
t_{SOx}^{(j_x,j_y),(j'_x,j'_y)} = \int d\mathbf{r} W_{j_x,j_y} \sin(k_{Lx} r_x) \cos(k_{Ly} r_y) W_{j'_x,j'_y},
\]

\[
t_{SOy}^{(j_x,j_y),(j'_x,j'_y)} = - \int d\mathbf{r} W_{j_x,j_y} \cos(k_{Lx} r_x) \sin(k_{Ly} r_y) W_{j'_x,j'_y}.
\]

Note that we have added a minus sign in the definition of \( t_{SOy} \) in order to write the Hamiltonian in a compact form.

Employing the condition \( W^\nu_0(r_\nu) = W^\nu_0(-r_\nu) \) given that one of the optical wells is located at \( \mathbf{r} = (0,0) \) when

\[\text{}}\]
To compare these spectra in detail, we further plot them (with solid black and dashed cyan lines denoting the spectra of continuous and tight-binding model, respectively) around a degenerate point \( (k_x, k_y) \) in Eq. (5) in the main text. Here, we choose the energy at the degenerate point to the energy at the same point of the spectra of the tight-binding model.

\[ \Omega_{SO} = 0, \]

we get

\[ t^{(0,0),(0,1)}_{SOx} (j_x,j_y)_{j_x,j_y} = 0, \]

\[ t^{(0,0),(1,0)}_{SOy} (j_x,j_y)_{j_x,j_y} = 0, \]

\[ t^{(0,0),(0,1)}_{SOy} (j_x,j_y)_{j_x,j_y+1} = 0, \]

where the last two relations are obtained because \( \sin(k_{Lx} (r_y + a_y)) = -\sin(k_{Lx} r_y) \) and \( \cos(k_{Lx} (r_y + a_y)) = -\cos(k_{Lx} r_y) \). Therefore, if we only consider the nearest-neighbor hopping, we obtain the following spin-orbit coupling term of the tight-binding model:

\[
H_{SO} \approx \Omega_{SO} \sum_{j_x,j_y} \left[ \hat{c}^\dagger_{j_x,j_y,\uparrow} \hat{c}_{j_x+1,j_y,\uparrow} t^{(j_x,j_y)}_{SOx} (j_x+1,j_y) + \hat{c}^\dagger_{j_x,j_y,\downarrow} \hat{c}_{j_x+1,j_y,\downarrow} t^{(j_x,j_y)}_{SOx} (j_x+1,j_y) \\
- i \hat{c}^\dagger_{j_x,j_y,\uparrow} \hat{c}_{j_x+1,j_y,\downarrow} t^{(j_x,j_y)}_{SOy} (j_x+1,j_y) - i \hat{c}^\dagger_{j_x,j_y,\downarrow} \hat{c}_{j_x+1,j_y,\uparrow} t^{(j_x,j_y)}_{SOy} (j_x+1,j_y) \right] + \text{H.c.},
\]

\[
= \Omega_{SO} \sum_{j_x,j_y} (-1)^{j_x+j_y} \left[ t^{(0,0),(1,0)}_{SOx} (j_x,j_y) \hat{c}^\dagger_{j_x,j_y+1,\uparrow} \hat{c}_{j_x+1,j_y,\uparrow} - \hat{c}^\dagger_{j_x,j_y+1,\downarrow} \hat{c}_{j_x+1,j_y,\downarrow} \right] + \text{H.c.},
\]

\[
= \sum_x \sum_{\nu=x,y} (-1)^{j_x+j_y} t^{(0,0),(0,1)}_{SOx} (j_x,j_y) \hat{c}^\dagger_{j_x,j_y+1,\nu} \hat{c}_{j_x+1,j_y,\nu} + \text{H.c.},
\]

where in the last step we have recast the Hamiltonian into a compact form by defining \( t_{SO\nu} = \Omega_{SO} t^{(0,0),(0,1)}_{SO\nu} \), \( \mathbf{g}_\nu = a_\nu \mathbf{e}_\nu \), \( \hat{c}^\dagger_{x,\sigma} = (\hat{c}^\dagger_{x,\uparrow}, \hat{c}^\dagger_{x,\downarrow}) \) with \( \hat{c}^\dagger_{x,\sigma} \equiv \hat{c}^\dagger_{j_x,j_y,\sigma} \) and \( \mathbf{x} = j_x a_x \mathbf{e}_x + j_y a_y \mathbf{e}_y \). When \( V_x = V_y \) and \( k_{Lx} = k_{Ly} \), \( t_{SOx} = -t_{SOy} \). In 3D, after replacing \( \hat{c}_{x,\sigma} \) with \( \hat{c}_{x,x,\sigma} \) and including \( H_t \) and the Zeeman field term, we obtain \( H_{TB} \) in Eq. (5) in the main text.
To verify the reliability of the tight-binding model, in Fig. 4 and Fig. 5 we compare the energy spectra of the tight-binding model with those of the continuous model, which are numerically calculated using Fourier series expansion of a Bloch function. In Fig. 4(a1) and (b1), we present the spectra of the continuous model, which qualitatively agree with their tight-binding counterparts in Fig. 1(c) and (d) of the main text. To see their difference more quantitatively, we plot their spectra around a degenerate point \([k_x = 0, k_y a/\pi = 1]\) for the first row panel and \((k_x a/\pi = -0.024, k_y a/\pi = 0.976)\) denoted by the red square in (b1) and (b2) for the second row panel as a function of \(k_x\) and \(k_y\) in (a3,b3) and (a4,b4), respectively. Here, we choose \(V_c = V_y = 3.7E_R\), \(\Omega_{SO} = 0.7E_R\), and \(h_2 = 0\), corresponding to the tight-binding model with \(t_x = t_y = 0.058E_R\), and \(t_{SO} = 0.028E_R\). For comparison, we have shifted the spectra of the continuous model so that the energy at the degenerate point is zero.

FIG. 5: (Color online) Single-particle spectra in the \((k_x, k_y)\) plane for \(k_z = 0\), obtained by ab initio theory in (a1) and (b1) and by diagonalizing the tight-binding model in (a2) and (b2). (a1-a4) correspond to the model with \(M_x = \sin(k_{Ly} r_y)\) and \(M_y = \sin(k_{Lx} r_x)\), and (b1-b4) to the simplified model with \(M_x = \sin(k_{Lx} r_x e^{i k_{Ly} r_y})\) and \(M_y = \sin(k_{Ly} r_y e^{-i k_{Lx} r_x})\). To compare these spectra in detail, we further plot them (with solid black and dashed cyan lines denoting the spectra of continuous and tight-binding model, respectively) around a degenerate point \([k_x = 0, k_y a/\pi = 1]\) for the first row panel and \((k_x a/\pi = -0.024, k_y a/\pi = 0.976)\) denoted by the red square in (b1) and (b2) for the second row panel as a function of \(k_x\) and \(k_y\) in (a3,b3) and (a4,b4), respectively. Here, we choose \(V_c = V_y = 3.7E_R\), \(\Omega_{SO} = 0.7E_R\), and \(h_2 = 0\), corresponding to the tight-binding model with \(t_x = t_y = 0.058E_R\), and \(t_{SO} = 0.028E_R\). For comparison, we have shifted the spectra of the continuous model so that the energy at the degenerate point is zero.

Appendix B: ANALYSIS OF MANY-BODY EFFECTS

In this section, we make a crude estimate of many-body effects on the degenerate points in the presence of weak repulsive atom-atom interactions for fermionic atoms. For alkali atoms, the interaction readily tuned by Feshbach resonances is short-range and can be written as

\[ V_{\text{int}}(r) = \frac{\hbar^2}{2m} \alpha (\sigma_x - \sigma_y) \]

that preserves 2D-PTRS but lacks 2D-IS. When \(V_c = V_y\) and \(k_{Lx} = k_{Ly}\), we have \(\alpha_1 = \alpha_2 = \alpha\) because the 2D continuous Hamiltonian respects a \(\Pi_C\) symmetry, i.e., \(\Pi_C H_{2D} \Pi_C^{-1} = H_{2D}\) with \(M H_{2D}(x, y) M^{-1} = H_{2D}(y, x)\), the representation of this symmetry in the momentum space corresponds to \(\Pi^\dagger H_{TB}^T(k) \Pi^{-1} = H_{TB}(k_x, k_y) [H_{TB}^T(k_x, k_y)]^{-1}\) with \(\Pi = (\sigma_x - \sigma_y) / \sqrt{2}\), indicating that the additional term must take the form of \(\sigma_x - \sigma_y\). In Fig. 4(b1-b2), we plot the spectra of the continuous and tight-binding models and both figures illustrate that a four-fold degenerate touching point splits into two doubly degenerate ones. Our further comparison around a touching point in Fig. 5(b3-b4) shows the quantitative agreement of the latter with the former.
with $g$ denotes the strength of interactions proportional to the $s$-wave scattering length. For weak interactions, using mean-field approximations yields

$$H_{\text{Int}} = g \sum_{k, k', \mathbf{Q}} \sum_{\mathbf{x}} \hat{c}_{k, \mathbf{x} \uparrow}^\dagger \hat{c}_{k, \mathbf{x} \uparrow} + g \sum_{k, \mathbf{k}, \mathbf{k}', \mathbf{Q}} \hat{c}_{k, \mathbf{x} \uparrow}^\dagger \hat{c}_{k, \mathbf{x} \uparrow} + Q \hat{c}_{k, \mathbf{x} \uparrow}^\dagger \hat{c}_{k, \mathbf{x} \uparrow} + Q \hat{c}_{k, \mathbf{x} \uparrow}^\dagger \hat{c}_{k, \mathbf{x} \uparrow}, \quad (A15)$$

where $N$ is the number of sites in the $(x, y)$ plane,

$$m_{z, \sigma, D} = g \sum_{k_z} \langle \mathbf{D}_{k_z} \mathbf{x} \sigma \mathbf{D}^\dagger_{k_z} \mathbf{x} \sigma \rangle, \quad (A17)$$

$$m_{||, D} = m_{x, D} + i m_{y, D}$$

and

$$H_{\text{Int}}^M (k) = m_{z, 1} + m_{z, 2} \sigma_z + m_{x, 1} \sigma_x + m_{y, 1} \sigma_y + m_{z, 3} \tau_z + m_{z, 4} \sigma_z \tau_z + m_{x, 2} \sigma_x \tau_z + m_{y, 2} \sigma_y \tau_z, \quad (A19)$$

with

$$m_{z, 1} = \frac{1}{4} (m_{z, \uparrow, A} + m_{z, \uparrow, B} + m_{z, \downarrow, B}) \quad (A20)$$

$$m_{z, 2} = \frac{1}{4} (m_{z, \uparrow, A} - m_{z, \uparrow, B} + m_{z, \downarrow, B}) \quad (A21)$$

$$m_{z, 3} = \frac{1}{4} (m_{z, \uparrow, A} + m_{z, \uparrow, B} - m_{z, \downarrow, B}) \quad (A22)$$

$$m_{z, 4} = \frac{1}{4} (m_{z, \downarrow, A} - m_{z, \downarrow, B} + m_{z, \uparrow, B}) \quad (A23)$$

If we consider using the ground state of non-interacting fermionic atoms as the initial state for iteration while searching for the many-body ground state, we have $m_{z, \sigma, D} = m_{z, \sigma, B}$ and $m_{x, A} = m_{x, B} = m_{y, A} = m_{y, B} = 0$, so that $m_{z, 3} = m_{z, 4} = m_{z, 1} = m_{y, 1} = m_{x, 2} = m_{y, 2} = 0$. Based on this argument, we have

$$H_{\text{Int}} (k) = m_{z, 1} + m_{z, 2} \sigma_z. \quad (A28)$$

Clearly, the presence of interactions may induce an effective Zeeman field, which will shift the locations of degenerate points along the $z$ direction and may cause the transition between the type-I and type-II, but will neither destroy Weyl nor four-fold degenerate points.

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[1] I. M. Lifshitz, Anomalies of electron characteristics of a metal in the high pressure region, Sov. Phys. JETP 11, 1130 (1960).
[2] G. E. Volovik, The Universe in a Helium Droplet (Clarendon Press, Oxford, 2003).
[3] S. Murakami, Phase transition between the quantum spin Hall and insulator phases in 3D: emergence of a topological gapless phase, New Journal of Physics 9, 356 (2007).
[4] X. Wan, A. M. Turner, A. Vishwanath, and S. Y. Savrasov, Topological semimetal and Fermi-arc surface states in the electronic structure of pyrochlore iridates, Phys. Rev. B 83, 205101 (2011).
[5] O. Vafek and A. Vishwanath, Dirac Fermions in Solids: From High-Tc Cuprates and Graphene to Topological Insulators and Weyl Semimetals, Annual Review of Condensed Matter Physics 5, 83 (2014).
[6] A. A. Soluyanov, D. Gresch, Z. Wang, Q. Wu, M. Troyer, X. Dai, and B. A. Bernevig, Type-II Weyl semimetals, Nature 527, 495 (2015).
[7] Y. Xu, F. Zhang, and C. Zhang, Structured Weyl Points in Spin-Orbit Coupled Fermionic Superfluids, Phys. Rev. Lett. 115, 265304 (2015).
[8] H. Weyl, Zeitschrift für Physik 56, 330 (1929).
[9] E. J. Bergholtz, Z. Liu, M. Trescher, R. Moessner, and M. Udagawa, Topology and Interactions in a Frustrated Slab: Tuning from Weyl Semimetals to $C > 1$ Fractional Chern Insulators, Phys. Rev. Lett. 114, 016806 (2015).
[10] Z. Yu, Y. Yao, and S. A. Yang, Unusual Magnetoresponse in Type-II Weyl Semimetals, arXiv:1604.04030.
[11] M. Udagawa and E. J. Bergholtz, Field-Selective Anomaly and Chiral Charge Reversal in Type-II Weyl Materials, arXiv:1604.08457.
[12] G. E. Volovik, Lifshitz transitions via the type-II Dirac and type-II Weyl points, arXiv:1604.00849.
[13] A. A. Zyuzin and R. P. Tiwari, Intrinsic Anomalous Hall Effect in Type-II Weyl Semimetals, arXiv:1601.00800.
T.-R. Chang, S.-Y. Xu, G. Chang, C.-C. Lee, S.-M. Huang, B. Wang, G. Bian, H. Zheng, D. S. Sanchez, I. Belopolski, N. Alidoust, M. Neupane, A. Bansil, H.-T. Jeng, H. Lin, and M. Zahid Hasan, Prediction of an arc-tunable Weyl Fermion metallic state in MoV1-xTe2, Nat. Commun. 7, 10639 (2016).

K. Koepernik, D. Kasinathan, D. V. Efremov, S. Khim, M. Suzuki, R. Arita, Y. Wu, D. Mou, H. Cao, J. Yan, N. Trivedi, and A. Kaminski, Spectroscopic evidence for type II Weyl semimetal state in MoTe2, arXiv:1603.06182.

S.-Y. Xu, N. Alidoust, G. Chang, H. Lu, B. Singh, I. Belopolski, D. Sanchez, X. Zhang, G. Bian, H. Zheng, M.-A. Husnan, Y. Bian, S.-M. Huang, C.-H. Hsu, T.-R. Chang, H.-T. Jeng, A. Bansil, V. N. Strocov, H. Lin, S. Jia, and M. Z. Hasan, Discovery of Lorentz-violating Weyl fermion semimetal state in LaAlGe materials, arXiv:1603.07518.

K. Deng, G. Wan, P. Deng, K. Zhang, S. Ding, E. Wang, M. Yan, H. Huang, H. Zhang, Z. Xu, J. Denlinger, A. Fedorov, H. Yang, W. Duan, H. Yao, Y. Wu, S. Fan, H. Zhang, X. Chen, and S. Zhou, Experimental observation of topological Fermi arcs in type-II Weyl semimetal MoTe2, arXiv:1603.08508.

J. Liu, H. Wang, C. Fang, L. Fu, and X. Qian, Van der Waals Stacking Induced Topological Phase Transition in Layered Ternary Transition Metal Chalcogenides, arXiv:1605.03903.

Z. Wu, L. Zhang, W. Sun, X.-T. Xu, B.-Z. Wang, S.-C. Chen, D. Li, Q. Zhou, and J. Zhang, Experimental realization of a two-dimensional synthetic spin-orbit coupling in ultracold Fermi gases, Nat. Phys. 12, 540 (2016).

Z. Meng, L. Huang, P. Peng, D. Li, L. Chen, Y. Xu, C. Zhang, P. Wang, and J. Zhang, Experimental observation of topological band gap opening in ultracold Fermi gases with two-dimensional spin-orbit coupling, arXiv:1511.08092.

Z. Wu, L. Zhang, W. Sun, X.-T. Xu, B.-Z. Wang, S.-C. Ji, Y. Deng, S. Chen, X.-J. Liu, and J.-W. Pan, Realization of Two-Dimensional Spin-orbit Coupling for Bose-Einstein Condensates, Science 354, 83 (2016).

N. Q. Burdick, Y. Tang, and B. L. Lev, A long-lived spin-orbit-coupled dipolar Fermi gas, arXiv:1605.03211.

J. Li, W. Huang, B. Shteynas, S. Burchesky, F. C. Top, E. Su, J. Lee, A. O. Jamison, and W. Ketterle, Spin-Orbit Coupling and Spin Textures in Optical Superlattices, arXiv:1606.03514.

I. Bloch, J. Dalibard, and S. Nascimbene, Quantum simulations with ultracold quantum gases, Nat. Phys. 8, 267 (2012).

V. Galitski and I. B. Spielman, Spin-orbit coupling in atomic gases, Nature 495, 49 (2013).

N. Goldman, J. C. Budich, and P. Zoller, Topological quantum matter with ultracold gases in optical lattices, Nat. Phys. 12, 639 (2016).

M. Gong, S. Tewari, and C. Zhang, BCS-BEC Crossover and Topological Phase Transition in 3D Spin-Orbit Coupled Degenerate Fermi Gases, Phys. Rev. Lett. 107, 195303 (2011).

K. Seo, C. Zhang, and S. Tewari, Thermodynamic signatures for topological phase transitions to Majorana and Weyl superfluids in ultracold Fermi gases, Phys. Rev. A 87, 063618 (2013).

Y. Xu, R.-L. Chu, and C. Zhang, Anisotropic Weyl Fermions from the Quasiparticle Excitation Spectrum of a 3D Fulde-Ferrell Superfluid, Phys. Rev. Lett. 112, 136402, (2014).

H. Hu, L. Dong, Y. Cao, H. Pu, and X.-J. Liu, Gapless topological Fulde-Ferrell superfluidity induced by an in-plane Zeeman field, Phys. Rev. A 90, 033624 (2014).

Y. Xu and C. Zhang, Topological Fulde-Ferrell superfluids of a spin-orbit coupled Fermi gas, Int. J. Mod. Phys. B 29, 1530001 (2015).
[44] B. M. Anderson, G. Juzeliunas, V. M. Galitski, and I. B. Spielman, Synthetic 3D Spin-Orbit Coupling, Phys. Rev. Lett. 108, 235301 (2012).
[45] J. H. Jiang, Tunable topological Weyl semimetal from simple-cubic lattices with staggered fluxes, Phys. Rev. A 85, 033640 (2012).
[46] S. Ganeshan, and S. Das Sarma, Constructing a Weyl semimetal by stacking one dimensional topological phases, Phys. Rev. B 91, 125438 (2015).
[47] T. Dubček, C. J. Kennedy, L. Lu, W. Ketterle, M. Soljačić, and Hrvoje Buljan, Weyl points in three-dimensional optical lattices: synthetic magnetic monopoles in momentum space, Phys. Rev. Lett. 114, 225301 (2015).
[48] W.-Y. He, S. Zhang, and K. T. Law, The realization and detection of Weyl semimetals in cold atomic systems, arXiv:1501.02348.
[49] X.-L. Qi, T. L. Hughes, and S.-C. Zhang, Topological field theory of time-reversal invariant insulators, Phys. Rev. B 78, 195424 (2008).
[50] H. Shapourian and T. L. Hughes, Phase diagrams of disordered Weyl semimetals, Phys. Rev. B 93, 075108 (2016).
[51] Y. Chen, Y. Xie, S. A. Yang, H. Pan, F. Zhang, M. L. Cohen, and S. Zhang, Spin-orbit-free Weyl-loop and Weyl-point semimetals in a stable three-dimensional carbon allotrope, Nano Lett. 15, 6974 (2015).
[52] D. Xiao, M.-C. Chang, and Q. Niu, Berry phase effects on electronic properties, Rev. Mod. Phys. 82, 1950 (2010).
[53] X.-J. Liu, K. T. Law, and T. K. Ng, Realization of 2D Spin-Orbit Interaction and Exotic Topological Orders in Cold Atoms, Phys. Rev. Lett. 112, 086401 (2014).
[54] Y. Xu and C. Zhang, Dirac and Weyl rings in three-dimensional cold-atom optical lattices, Phys. Rev. A 93, 063606 (2016).
[55] L. Tarruell, D. Greif, T. Uehlinger, G. Jotzu, and T. Esslinger, Creating, moving and merging Dirac points with a Fermi gas in a tunable honeycomb lattice, Nature (London) 483, 302 (2012).
[56] L.-K. Lim, J.-N. Fuchs, and G. Montambaux, Bloch-Zener Oscillations across a Merging Transition of Dirac Points, Phys. Rev. Lett. 108, 175303 (2012).
[57] Y.-Q. Wang and X.-J. Liu, A Scaling Behavior of Bloch Oscillation in Weyl Semimetals, arXiv:1605.02671.
[58] L. D. Landau, Phys. Z. Sowjetunion 2, 46 (1932); C. Zener, Proc. R. Soc. A 137, 696 (1932).