Finite volume method for a coupled Stokes-Darcy problem

Staïli Naoufal

Abstract

In this paper we propose a finite volume method to solve the coupled Stokes-Darcy problem using steady Stokes equations for the fluid region and Darcy equations for the porous region.

At the contact interface between the fluid region and the porous media we imposed two conditions. The first one is the normal continuity of the velocity, while the second one is the continuity of the pressure. Furthermore, due to the lack of information about both the velocity and the pressure on the interface, we will use schwarz domain decomposition method.

In Darcy equations, the tensor of permeability will be considered as variable, since it depends on both the properties of the porous medium and the viscosity of the fluid. Numerical examples are presented to demonstrate the efficiency of the proposed method.

Keywords: Stokes, Darcy, coupled problem, finite volume method, Schwarz, domain decomposition method

1. Introduction

We focus our study on the flow of a fluid in a steady state, this flow is governed by Darcy’s law, coupled with Stokes equation, this model describes macroscopic properties of filtration processes that find many important applications in porous media problems, such as water flowing across semi-permeable soil, transport of contaminants through rivers into the aquifers or oil filtering through sand and rocks.

Let Ω₁ and Ω₂ be two non-intersecting open bounded subsets of $\mathbb{R}^2$, with lipschitzian boundaries $\partial \Omega_1$ and $\partial \Omega_2$. We assume that their boundaries have a non-empty intersection denoted by $\Gamma := \partial \Omega_1 \cap \partial \Omega_2$ and to which we refer as the interface between $\Omega_1$ and $\Omega_2$. The remaining parts of the boundaries are $\Gamma_1 := \partial \Omega_1 - \Gamma$ and $\Gamma_2 := \partial \Omega_2 - \Gamma$. (see fig1)
The Stokes equation govern the fluid flow in $\Omega_1$:

$$
\begin{align*}
-\nabla \cdot T(u_1, p_1) &= f_1 \quad \text{in} \quad \Omega_1, \\
\nabla \cdot u_1 &= 0 \quad \text{in} \quad \Omega_1,
\end{align*}
$$

(1.1)

where $u_1$ and $p_1$ are the fluid velocity and pressure, $T(u_1, p_1) := -p_1 I + 2\mu D(u_1)$ is the stress tensor, $D(u_1) := \frac{1}{2}(\nabla u_1 + \nabla u_1^T)$ is the deformation rate tensor and $\mu \geq 0$ is the kinematic viscosity of the fluid.

The Darcy equations govern the porous media flow in $\Omega_2$:

$$
\begin{align*}
\mu u_2 + K \nabla p_2 &= 0 \quad \text{in} \quad \Omega_2, \\
\nabla \cdot u_2 &= f_2 \quad \text{in} \quad \Omega_2,
\end{align*}
$$

(1.2)

where $u_2$ and $p_2$ are the velocity and pressure, $K$ is the Darcy permeability.

On the interface $\Gamma$, we impose the following interface conditions:

$$
\begin{align*}
u_1, n_1 + u_2, n_2 = 0 \quad \text{on} \quad \Gamma, \\
p_1 = p_2 \quad \text{on} \quad \Gamma,
\end{align*}
$$

where $n_1$ and $n_2$ denote the unit outward normal vector on $\partial \Omega_1$ and $\partial \Omega_2$, in particularity, $n_1 = -n_2$ on $\Gamma$.

We assume Dirichlet boundary conditions are satisfied on $\Gamma_1$ and $\Gamma_2$:

$$
\begin{align*}
u_1 &= g_1 \quad \text{on} \quad \Gamma_1, \\
u_2 &= g_2 \quad \text{on} \quad \Gamma_2, \\
p_1 &= h_1 \quad \text{on} \quad \Gamma_1, \\
p_2 &= h_2 \quad \text{on} \quad \Gamma_2.
\end{align*}
$$

Other type of boundary conditions could have been chosen as well.

Considerable efforts has been done to develop modelling and simulating this interaction, coupling between ground fluid and surface fluid is a very interesting subject that we found in many serious problems currently facing the world, and great deal of works has been directed to develop methods of solving Stokes-Darcy coupled problem, been motivated by a variety of applications such as bioengineering[10], industry[8, 11], environment research[7], and specially to increasing more efficient numerical methods.
This problem has been studied from mathematical and numerical analysis\cite{15, 1, 11, 12, 7, 6, 5}. In\cite{7}, M. Discacciati and A. Quarteroni introduced a differentiel system based on the coupling of the Navier-Stokes equations and the Darcy equation, they formulate the problem as an interface problem, they propose a way of solving the coupled problem iteratively, by solving one problem at each step.

In\cite{6} M. Discacciati consider a new approach to characterize preconditioners for the nonlinear Stokes-Darcy problem, then propose a general nonlinear domain decomposition strategy for the resolution of the interface problem of nonlinear Navier-Stokes/Darcy coupling.

In\cite{4} Prince chidyagwai and BÃťatrice RiviÃĺre analyzed two models for the coupled problem Stokes-Darcy, using continious finite elements in the incompressible flow region and discontinous finite elements in the porous medium.

Other works have presented the same problem with more or less generality\cite{14, 5, 2, 9}, interested in the choice of the mesh, the different conditions of the edge or the different coupling conditions on the interface.

The before-mentioned works used finite element method to treat the problem (2). Our first objective is to elaborate a finite volumes scheme for the coupled Stokes-anisotropic Darcy problem, since the permeability is presented as variable in our case, the classical finite volumes method won’t be able to treat this type of equation, to do so, we used the DDFV scheme. Also note that the transmission conditions between the two domains naturally intervenes in the variational formulation in all works which used the finite elements method, what is not the case in the finite volumes method. then our second objective is to use the schwartz’s method to overcome this difficulty.

The paper is organized as follows. In section 2, we first give the finite-volume notation associated with the particular geometry of the domain, we introduce the Finite Volume scheme and define the associated discrete operators. In section 3 we study a finite volume approximation of the incompressible Darcy-Stokes coupled problem in the whole domaine $\Omega = \Omega_1 \cup \Omega_2$, and we propose the method of schwarz to treat the interface. and we give some numerical tests.

Finally, we give a conclusion and a perspective of the next problem we can hold.

2. Framework

2.1. Meshes and notations.

We recall here the main notations and definitions for DDFV. A Discrete Duality Finite Volume mesh $\mathcal{T}$ is constituted by a primal mesh $\mathcal{M} \cup \partial \mathcal{M}$ and a dual mesh $\mathcal{M}^* \cup \partial \mathcal{M}^*$ (fig 2).

The interior primal mesh $\mathcal{M}$ is a set of disjoint open polygonal control volumes $K \subset \Omega$ such that $\cup \mathcal{K} = \overline{\Omega}$. We denote by $\partial \mathcal{M}$ the set of edges of the control volumes in $\mathcal{M}$ included in $\partial \Omega$, which we consider as degenerate control volumes.

- To each control volume $K \in \mathcal{M}$, we associate a point $x_k$. Even though many choices are possible, in this paper, we always assume $x_k$ to be the mass center of $K$.
- To each degenerate control volume $K \in \partial \mathcal{M}$, we associate the point $x_k$ equal the midpoint of the control volume $K$.

This family of points is denoted by $X = \{x_K, K \in \mathcal{M} \cup \partial \mathcal{M}\}$.

Let $X^*$ denote the set of the vertices of the primal control volume in $\mathcal{M}$ that we split into $X^* = X^*_{\text{int}} \cup X^*_{\text{ext}}$ where $X^*_{\text{int}} \cap \partial \Omega = \emptyset$ and $X^*_{\text{ext}} \subset \partial \Omega$. With any points $x^*_{K*} \in X^*_{\text{int}}$ (resp. $x^*_{K*} \in X^*_{\text{ext}}$), we associate the polygon $K^* \in \mathcal{M}^*$ (resp. $K^* \in \partial \mathcal{M}^*$) whose vertices are \{ $x_K \in X$, such that $x_{K*} \in \overline{K}, K \in \mathcal{M}$ \} (resp. \{ $x_K \in X$, such that $x_{K*} \in \overline{K}, K \in (\mathcal{M} \cup \partial \mathcal{M})$ \}) sorted with respect to the clockwise order of the corresponding control volumes.
This defines the set $\mathcal{M}^* \cup \partial \mathcal{M}^*$ of the dual control volumes.

For all control volumes $K$ and $L$, we assume that $\partial K \cap \partial L$ is either empty or a common vertex or an edge of the primal mesh denoted by $\sigma = K|L$. We note by $E$ the set of such edges.

We also note $\sigma^* = K^*|L^*$ for the corresponding dual definitions.

Given the primal and dual control volumes, we define the diamonds cells $D_{\sigma,\sigma^*}$ being the quadrangles whose diagonals are a primal edge $\sigma = K|L = (x_K, x_L)$ and a corresponding dual edge $\sigma^* = K^*|L^* = (x_K^*, x_L^*)$, (see ??). Note that the diamond cells are not necessarily convex. If $\sigma \in E \cap \partial \Omega$, the quadrangle $D_{\sigma,\sigma^*}$ degenerate into triangle. The set of diamond cells is denoted by $\mathcal{D}$ and we have $\Omega = \cup_{D \in \mathcal{D}} D$.

Notations For any primal control volume $K \in \mathcal{M} \cup \partial \mathcal{M}$, we note:

- $m_K$ its Lebesgue measure,
- $E_K$ the set of its edges (if $K \in \mathcal{M}$), or the one-element set $\{K\}$ if $K \in \partial \mathcal{M}$.
- $\mathcal{D}_K = \{D_{\sigma,\sigma^*} \in \mathcal{D}, \sigma \in E_K\}$,
- $h_K$ its diameter.

We will also use the corresponding dual notations: $m_K^*$, $E_K^*$, $D_K^*$ and $h_K^*$.

For a diamond cell $D = D_{\sigma,\sigma^*}$ whose vertices are $(x_K, x_K^*, x_L, x_L^*)$ (see ??), we note:

- $m_\sigma$ the length of the primal edge $\sigma$,
- $m_{\sigma^*}$ the length of the dual edge $\sigma^*$,
- $\vec{n}_\sigma$ the unit vector normal to $\sigma$ oriented from $x_K$ to $x_L$,
- $\vec{n}_{\sigma^*}$ the unit vector normal to $\sigma^*$ oriented from $x_K^*$ to $x_L^*$,
- $h_D$ its diameter,
- $m_D$ its measure.
We define the set of boundary diamond cell $\mathcal{D}_{\text{ext}}$ as the set of diamond cells which possess one side included in $\partial \Omega$; the set of interior diamond cells is thus $\mathcal{D}_{\text{int}} = \mathcal{D} \setminus \mathcal{D}_{\text{ext}}$.

Mesh regularity measurement. Let $\text{size}(T)$ be the maximum of the diameter of the diamond cells in $\mathcal{D}$. We introduce a positive number $\text{reg}(T)$ that measures the regularity of a given mesh and is useful to perform the convergence analysis of finite volume schemes:

$$\text{reg}(T) := \max \left( N, N^*; \max_{D \in \mathcal{D}} \frac{m_D m_{\sigma^*}}{m_D}, \max_{K \in \mathcal{M}, D \in \mathcal{D}_K} \frac{h^K}{h_D}, \max_{K^* \in \mathcal{M}^* \cap \partial \mathcal{M}^*} \frac{h^{K^*}}{h_D}, \max_{D \in \mathcal{D}^*} \frac{h_D}{\sqrt{m_D}}, \max_{K^* \in \mathcal{M}^* \cap \partial \mathcal{M}^*} \frac{h^{K^*}}{\sqrt{m_{K^*}}}, \max_{K \in \mathcal{M}} \frac{h_K}{\sqrt{m_K}} \right),$$

Where $N$ and $N^*$ are the maximum of edges of each primal cell and the maximum of edges incident to any vertex. The number $\text{reg}(T)$ should be uniformly bounded when $\text{size}(T) \to 0$ for the convergence results to hold.

2.2. Discrete unknowns and discrete mean-value projection.

The DDFV method for the coupled Stokes-Darcy problem requires staggered unknowns. It associates to any primal cell $K \in \mathcal{M} \cup \partial \mathcal{M}$ an unknown value $u_K \in \mathbb{R}^2$ and to any dual cell $K^* \in \mathcal{M}^* \cup \partial \mathcal{M}^*$ an unknown value $u_{K^*} \in \mathbb{R}^2$ for the velocity, to any dual $D \in \mathcal{D}$ an unknown value $p^D \in \mathbb{R}$ for the pressure. These unknowns are collected in the families.

$$u^\mathcal{D} = \left( (u_k)_{k \in (\mathcal{M} \cup \partial \mathcal{M})}, (u_{k^*})_{k^* \in (\mathcal{M}^* \cup \partial \mathcal{M}^*)} \right) \in (\mathbb{R}^2)^{\mathcal{D}}$$

$$p^\mathcal{D} = ((p^D)_{D \in \mathcal{D}})$$

We define now the mean-value projection for any vector field $v \in (H^1(\Omega))^2$, 

Figure 2: Diamond cell $\mathcal{D}_{\sigma, \sigma^*}$
\[ P^\partial_{\Omega} m v = \left( \frac{1}{\text{m}_{Bk}} \int_{Bk} v(x) \, dx \right)_{k \in \partial \mathcal{M}}, \quad \frac{1}{\text{m}_{Bk^*}} \int_{Bk^*} v(x) \, dx \right)_{k^* \in \partial \mathcal{M}^*}. \]

\[ P^M_{m} v = \left( \frac{1}{\text{m}_{k}} \int_{k} v(x) \, dx \right)_{k \in \mathcal{M}}, \quad P^{M^*}_{m} v = \left( \frac{1}{\text{m}_{k^*}} \int_{k^*} v(x) \, dx \right)_{k^* \in \mathcal{M}^*}. \]

We finally gather these projections in the following notation
\[ P^T_{m} v = \left( P^M_{m} v, P^{M^*}_{m} v, P^\partial_{\Omega} m v \right), \quad \forall v \in (H^1(\Omega))^2. \]

We specify a subset of \((\mathbb{R}^2)^\sharp\) needed to take into account the Dirichlet boundary conditions:
\[ \mathcal{E}_g = \left\{ v^\sharp \in (\mathbb{R}^2)^\sharp : v_k = \left( P^\partial_{\Omega} m g \right)_{k}, \forall K \in \partial \mathcal{M} \text{ and } v_{k^*} = \left( P^\partial_{\Omega} m g \right)_{k^*}, \forall K^* \in \partial \mathcal{M}^* \right\} \]

### 2.3. Discrete operators

In this subsection, we define the discrete operators, gradient and divergence, which are needed in order to write and analyse the DDFV scheme, see [13]

**Definition 2.1.** We define the discrete gradient operator \( \nabla^D \) mapping vector field of \((\mathbb{R}^2)^\sharp\) into matrix fields of \((\mathcal{M}_2(\mathbb{R}))^\sharp\) as follows
\[ \nabla^D u^\sharp = \frac{1}{\sin(\alpha_{D})} \left( \frac{u_l - u_k}{m_{\sigma^*}} \otimes \hat{n}_{\sigma k} + \frac{u_{l^*} - u_{k^*}}{m_{\sigma^*}} \otimes \hat{n}_{\sigma^* k^*} \right), \]
\[ \nabla^D u^\sharp = \frac{1}{2m_{D}} \left( m_{\sigma}(u_l - u_k) \otimes \hat{n}_{\sigma k} + m_{\sigma^*}(u_{l^*} - u_{k^*}) \otimes \hat{n}_{\sigma^* k^*} \right). \]

**Definition 2.2.** We define the discrete gradient operator \( \nabla^T \) mapping a scalar fields \( \mathbb{R}^D \) into vector fields in \( \mathcal{E}_0 \) as follows
\[ \nabla^T p^\sharp = div^T (p^\sharp Id), \quad \forall p^\sharp \in \mathbb{R}^\sharp \]

**Definition 2.3.** We define the discrete divergence operator \( div^\sharp \) mapping vector field of \((\mathbb{R}^2)^\sharp\) into scalar fields in \( \mathbb{R}^\sharp \), as follows
\[ div^k \xi^\sharp = \frac{1}{m_{k}} \sum_{D_{\sigma,\sigma^*} \in \mathcal{D}_k} m_{\sigma} \xi^D \cdot \hat{n}_{\sigma k}, \forall k \in \mathcal{M}, \]
\[ div^{k^*} \xi^\sharp = \frac{1}{m_{k^*}} \sum_{D_{\sigma,\sigma^*} \in \mathcal{D}_{k^*}} m_{\sigma^*} \xi^D \cdot \hat{n}_{\sigma^* k^*}, \forall k^* \in \mathcal{M}^* \]
\[ div^{k^*} \xi^\sharp = \frac{1}{m_{k^*}} \left( \sum_{D_{\sigma,\sigma^*} \in \mathcal{D}_{k^*}} m_{\sigma^*} \xi^D \cdot \hat{n}_{\sigma^* k^*} + \sum_{D_{\sigma,\sigma^*} \in \mathcal{D}_{k^*} \cap \mathcal{D}_{\text{ext}}} d_{k^*} d_{\xi^D} \cdot \hat{n}_{\sigma k} \right), \forall k^* \in \partial \mathcal{M}^* \]
The discrete divergence of a vector field of \((\mathbb{R}^2)^T\): \(\text{div}^D: u^T \in (\mathbb{R}^2)^T \rightarrow (\text{div}^D u^T)_D \in \mathbb{R}^D\) is defined as follow:

\[
\text{div}^D u^T = \text{Tr}(\nabla^D u^T) = \frac{1}{2m} \left( m_\sigma (u_l - u_k) \cdot \vec{n}_\sigma_k + m_\sigma^* (u_l^* - u_k^*) \cdot \vec{n}_\sigma^*_k \right), \quad \forall D \in \mathcal{D}
\]

3. Main results

3.1. Discretization of coupled Stokes-Darcy problem

We propose to resolve the coupled Stokes-Darcy problem using the finite volumes method. We give the standart problem:

As claimed in framework, we approximate the velocity on both vertices and centers of primal control volumes and the pressure on the diamond cells. We integrate the momentum conservation law of problem (1) on the primal mesh \(\mathcal{M}\) and on the interior dual mesh \(\mathcal{M}^*\). The mass conservation equation is directly approached on the diamond mesh using the discrete operator \(\text{div}^D\). We impose the Dirichlet boundary conditions on \(\partial \mathcal{M}\) and on \(\partial \mathcal{M}^*\). Assuming that \(\mu = 1\), the scheme for the problem (1) reads as follows:

\[
\begin{align*}
\text{Find } u^T, p^T \in E_g \text{ such that,} \\
\text{div}^M (\nabla^S u^T + p^T I) &= f^M_1, \\
\text{div}^{M*} (\nabla^S u^T + p^T I) &= f_1^{M*}, \\
\text{div}^D (u^T) &= 0, \\
\end{align*}
\]

with \(f^M = P^M_m f\) and \(f^{M*} = P^{M*}_m f\).

Combining Darcy’s law with the conservation of mass equation in (2) and assuming that \(\mu = 1\), we get the following equation:

\[
\begin{align*}
\begin{cases}
\quad u_2 = -K \nabla p_2 & \text{in } \Omega_2, \\
-p_2 = 0, \quad u_2 = g_2 & \text{on } \Gamma_2,
\end{cases}
\end{align*}
\]

The scheme for the problem 2 reads as follows:

\[
\begin{align*}
\begin{cases}
\quad u_2 = -K \nabla p_2^{T} & \text{in } \Omega_2, \\
-p_2^{T} = 0, \quad u_2 = g_2^{T} & \text{on } \Gamma_2,
\end{cases}
\end{align*}
\]

Note the absence of information of viscosity and pressure on the interface \(\Gamma\), and to overcome this problem, we have used the schwarz method. The algorithm consists in constructing sequences \((u_{1n})_n, (u_{2n})_n \subset E_g\) and \((p_{1n})_n, (p_{2n})_n \subset E_0\).

Choose Initializations \(u_0^1, u_0^2, p_0^1\) and \(p_0^2\).
For all $n \geq 0$, Perform successively:

\[
\begin{align*}
\begin{cases}
\text{Find } (u_{n+1}^1)^T \in E_g \text{ and } (p_{n+1}^1)^D \in E_0 \text{ such that,} \\
\quad \text{div}^M(\nabla^D (u_{n+1}^1)^T + (p_{n+1}^1)^D \text{Id}) = f^M_1 \text{ in } \Omega_1, \\
\quad \text{div}^{M^*}(\nabla^D (u_{n+1}^1)^T + (p_{n+1}^1)^D \text{Id}) = f^{M^*}_1 \text{ in } \Omega_1, \\
\quad \text{div}^{D^*}(u_{n+1}^1)^T = 0 \text{ in } \Omega_1, \\
\quad u_{n+1}^1 = g_1 \text{ on } \Gamma_1, \\
\quad u_{n+1}^1 = u_n, \quad (p_{n+1}^1) = p_n^1 \text{ on } \Gamma,
\end{cases}
\end{align*}
\]  

\[
\begin{align*}
\begin{cases}
\text{Find } (u_{n+1}^2)^T \in E_g \text{ and } (p_{n+1}^2)^T \in E_0 \text{ such that,} \\
\quad (u_{n+1}^2)^T = -K.\nabla^D (p_{n+1}^2)^T \text{ in } \Omega_2, \\
\quad \text{div}^M(K.\nabla^D (p_{n+1}^2)^T) = f^M_2 \text{ in } \Omega_2, \\
\quad \text{div}^{M^*}(K.\nabla^D (p_{n+1}^2)^T) = f^{M^*}_2 \text{ in } \Omega_2, \\
\quad p_{n+1}^2 = 0, \quad u_{n+1}^2 = g_2 \text{ on } \Gamma_2, \\
\quad u_{n+1}^2 = u_{n+1}^1, \quad (p_{n+1}^2) = p_{n+1}^1 \text{ on } \Gamma,
\end{cases}
\end{align*}
\]

Stop at convergence when the velocity $u_n^1$ and $u_n^2$ are sufficiently close (resp. the pressure) on $\Gamma$.

3.2. A numerical examples

We give a DDFV mesh for the whole domain $\Omega = \Omega_1 \cup \Omega_2$, with mention of the interface $\Gamma$ of contact between the two regions:

![DDFV mesh](image)

Figure 3: A DDFV mesh $\mathcal{T}$ of the whole domain $\Omega$

We present the subdomains $\Omega_1$ and $\Omega_2$ with two independent DDFV meshes $\mathcal{T}_1$ and $\mathcal{T}_2$. In Darcy region we consider a well, proportionnally with big dimensions just for visuality cause:

We say that de two meshes are compatible in the following sense.

**Definition 3.1.** ([3]) The two meshes $\mathcal{T}_1$ and $\mathcal{T}_2$ are compatible if they verify the following two conditions:
1. The two meshes have the same vertices on $\Gamma$. This implies in particular that the two meshes have the same degenerate control volumes on $\Gamma$, i.e., $\partial M_{1,\Gamma} = \partial M_{2,\Gamma}$.

2. The center $x_\sigma$ of a degenerate interface control volume $\sigma = [x_{K^*}, x_{L^*}] \in \partial M_{1,\Gamma} = \partial M_{1,\Gamma}$ is the intersection of $(x_{K^*}, x_{L^*})$ and $(x_{K_1}, x_{K_2})$, where $K_1 \in M_1$ and $K_2 \in M_2$ are the two primal control volumes such that $\sigma \subset \partial K_1$ and $\sigma \subset \partial K_2$.

**Examples**

**Example 1** Let $\Omega = \Omega_1 \cup \Omega_2$ with $\Omega_1 = \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq 1$ and $0 \leq y \leq 1-x\}$ and $\Omega_2 = \Omega^*|C$, such that $\Omega^* = \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq 1$ and $1-x \leq y \leq 1\}$ and $C = \{(x,y) \in \mathbb{R}^2 : (x-0.2)^2 + (y-0.2)^2 = (0.1)^2\}$.

For simplicity we take the kinematic viscosity of the fluid $\mu = 1$ and so that we can verify our method we propose in a first example that the tensor of permeability $K = I$.

We propose the following problem in $\Omega_1$:

$$
\begin{align*}
-\Delta u_1 + \nabla p_1 &= f_1 \quad \text{in} \quad \Omega_1, \\
\nabla \cdot u_1 &= 0 \quad \text{in} \quad \Omega_1, \\
\tilde{u}_1 &= u_1^\text{eff} \quad \text{and} \quad \tilde{p}_1 = p_1^\text{eff} \quad \text{on} \quad \Gamma_1,
\end{align*}
$$

(3.6)

Such that

$$f_1(x,y) = \left( (2x - 1)(1-y) \right) ; \quad x(1-x) + (y-1)^2 - 4$$

In $\Omega_2$ we consider Darcy problem:
Figure 6: Exact solution and numerical solution of pressure in $\Omega_1$

\[
\begin{cases}
    u_2 + K \nabla p_2 &= 0 \text{ in } \Omega_2, \\
    \nabla \cdot u_2 &= f_2 \text{ in } \Omega_2, \\
    u_2 = u_2^{ex} \text{ and } p_2 = p_2^{ex} \text{ on } \Gamma_2 \cup \partial C,
\end{cases}
\]  

(3.7)

Such that $f_2 = 0$

We present the exact solution of pressure and viscosity:

\[
u_1^{ex}(x, y) = u_2^{ex}(x, y) = \left((2x - 1)(y - 1) \ ; \ x(x - 1) - (y - 1)^2\right), \text{ for } (x, y) \in \Omega_1
\]

\[
p_1^{ex}(x, y) = p_2^{ex}(x, y) = \left(x(1 - x)(y - 1) + \frac{(y - 1)^3}{3}\right), \text{ for } (x, y) \in \Omega,
\]

We give the exact solution and the numerical solution in $\Omega_1$:

We give the exact solution and the numerical solution in $\Omega_2$:

We present the numerical approximation of pressure in $\Omega_1$ and $\Omega_2$:

Then we present the numerical approximation of pressure in the whole domain $\Omega$

The errors between the exact solution and the numerical solution for pressure and velocity:

| $h$ mesh | primal cells | dual cells | diamond cells | $\|U_{ex} - U_{nu}\|_{\infty}$ | $\|U_{ex} - U_{nu}\|_{L^2}$ |
|----------|--------------|------------|---------------|-----------------------------|-----------------------------|
| 0.1155   | 334          | 188        | 521           | 0.0867                      | 0.1467                      |
| 0.0578   | 1336         | 709        | 2044          | 0.0298                      | 0.0744                      |
| 0.0289   | 5344         | 2753       | 8096          | 0.0141                      | 0.0389                      |
| 0.0142   | 20992        | 10657      | 31648         | 0.0071                      | 0.0215                      |

Example 2

We keep the same domains $\Omega_1$ and $\Omega_2$ but this time we propose the tensor of permeability symmetric positive definite in $\Omega$ as:
Figure 7: Exact solution and numerical solution of velocity in $\Omega_1$

Figure 8: Exact solution and numerical solution of pressure in $\Omega_2$

Figure 9: Exact solution and numerical solution of velocity in $\Omega_2$
Figure 10: Stokes approximation of pressure and Darcy approximation of pressure

Figure 11: Stokes-Darcy approximation of pressure

Figure 12: Exact solution and numerical solution of pressure in Ω
In $\Omega_1$, we propose the following problem:

$$
\begin{cases}
-\Delta u_1 + \nabla p_1 &= f_1 \quad \text{in} \quad \Omega_1, \\
\nabla \cdot u_1 &= 0 \quad \text{in} \quad \Omega_1, \\
u_1 = U \quad \text{and} \quad p_1 = P \quad \text{on} \quad \Gamma_1,
\end{cases}
(3.8)
$$

And we propose the Darcy problem in $\Omega_2$:

$$
\begin{cases}
u_2 + K \nabla p_2 &= 0 \quad \text{in} \quad \Omega_2, \\
\nabla \cdot u_2 &= 0 \quad \text{in} \quad \Omega_2, \\
u_2 = U \quad \text{and} \quad p_2 = P \quad \text{on} \quad \Gamma_2 \cup \partial C,
\end{cases}
(3.9)
$$

Such that:

$$f_1 = \frac{x^2 y(x - 1)}{(x + 1)(y + 1) - x^2 y^2} \cdot \frac{x^3 - x + 2}{(x + 1)(y + 1) - x^2 y^2},$$

$$P = x(x - 1)(y - 1) - \frac{(y - 1)^3 - 2}{3} \quad \text{and} \quad U = (0, x(1 - x))$$

4. Conclusions

We presented the Discrete Duality Finite Volume method for the coupling of Stokes and Darcy flow, by the use of Schwarz method to threat the interface problem. We presented an error norm estimate which is optimal with respect to the approximation spaces. We hope finding a more strong model by including more conditions on the interface $\Gamma$, such as Beavers-Joseph-Saffeman which will be postponed to a future article.
Figure 14: Numerical solution of pressure in $\Omega$

Figure 15: Numerical solution of velocity in $\Omega$
References

[1] Yassine Boubendir and Svetlana Tlupova, *StokesâŠ†Darcy boundary integral solutions using preconditioners*, Journal of Computational Physics 228 (2009), no. 23, 8627–8641 (en).

[2] Mingchao Cai, Mo Mu, and Jinchao Xu, *Preconditioning techniques for a mixed Stokes/Darcy model in porous media applications*, Journal of Computational and Applied Mathematics 233 (2009), no. 2, 346–355 (en).

[3] Claire Chainais-Hillairet, Stella Krell, and Alexandre Mouton, *Study of discrete duality finite volume schemes for the Peaceman model*, SIAM Journal on Scientific Computing 35 (2013), no. 6, A2928–A2952.

[4] Prince Chidyagwai and Bâ†brete Riviâ†re, *On the solution of the coupled NavierâŠ†Stokes and Darcy equations*, Computer Methods in Applied Mechanics and Engineering 198 (2009), no. 27-28, 2596–2605 (en).

[5] M.R. Correa and A.F.D. Loula, *A unified mixed formulation naturally coupling Stokes and Darcy flows*, Computer Methods in Applied Mechanics and Engineering 198 (2009), no. 33-36, 2710–2722 (en).

[6] Marco Discacciati, *Domain decomposition methods for the coupling of surface and groundwater flows*, Ph.D. thesis, École Polytechnique Fédérale de Lausanne, 2004.

[7] Marco Discacciati, Edie Miglio, and Alfio Quarteroni, *Mathematical and numerical models for coupling surface and groundwater flows*, Applied Numerical Mathematics 43 (2002), no. 1, 57–74.

[8] Mustapha Hellou, Juan Martinez, and Mohamed. El Yazidi, *Stokes flow through microstructural model of fibrous media*, Mechanics Research Communications 31 (2004), no. 1, 97–103.

[9] G. Kanschat and B. Riviâ†re, *A strongly conservative finite element method for the coupling of Stokes and Darcy flow*, Journal of Computational Physics 229 (2010), no. 17, 5933–5943 (en).

[10] A. R. A. Khaled and K. Vafai, *The role of porous media in modeling flow and heat transfer in biological tissues*, International Journal of Heat and Mass Transfer 46 (2003), no. 26, 4989–5003.

[11] J. M. V. A. Koelman and M. Neveu, *Darcy flow in porous media: Cellular automata simulations*, Numerical Methods for the Simulation of Multi-Phase and Complex Flow, Lecture Notes in Physics, no. 398, Springer Berlin Heidelberg, 1992, DOI: 10.1007/BFb0022313, pp. 136–145 (en).

[12] William J. Layton, Friedhelm Schieweck, and Ivan Yotov, *Coupling fluid flow with porous media flow*, SIAM Journal on Numerical Analysis 40 (2002), no. 6, 2195–2218.

[13] Pascal Omnes, *Développement et analyse de mélthodes de volumes finis*, Ph.D. thesis, Université Paris-Nord-Paris XIII, 2010.

[14] Hongxing Rui and Ran Zhang, *A unified stabilized mixed finite element method for coupling Stokes and Darcy flows*, Computer Methods in Applied Mechanics and Engineering 198 (2009), no. 33-36, 2692–2699 (en).

[15] J.M. Urquiza, D. N’Dri, A. Garon, and M.C. Delfour, *Coupling Stokes and Darcy equations*, Applied Numerical Mathematics 58 (2008), no. 5, 525–538 (en).