Axion Electrodynamics in Topological Insulators for beginners.

Part I: Introduction and basic equations
Part II: Images of a charge close to an interface ordinary insulator-topological insulator
Part III: Appendices on boundary conditions, the electrostatic image method and on electrodynamic atomic units

Josep Planelles, Dept. Química-Física i Analítica, Universitat Jaume I
December 5, 2022

1 Introduction

According to band theory, insulators are defined as those materials that have a gap between the occupied valence and the vacant conduction bands, while conductors do not have a gap (either, because the last band is semi-occupied or because the conduction and valence bands overlap).

In 2006 unusual electromagnetic properties were observed in a $CdTe\mid HgTe\mid CdTe$ quantum well: this quantum well behaves in bulk as an insulator but it was observed electric current across the interface, i.e., it behaves like a conductor in surface. The most striking feature is that this behavior cannot be accounted by the Maxwell’s equations.

Earlier, in 1987, Frank Wilczek[2] suggested the possible behavior of this kind of materials and pointed out that it could be explained by means the axion electrodynamics that himself[3] and Steven Weinberg[4] developed to understand the violation of combined symmetries of charge conjugation and parity in the strong interactions. The name axion is related to the name of the particle associated to this peculiar field.

In a relatively recent paper, Qi and Zhang[5] present and discuss experimental results on the $CdTe\mid HgTe\mid CdTe$ quantum well that shows an almost infinite resistance (i.e. behaves like an insulator) if the $HgTe$ thickness is smaller than a critical distance $d \sim 6.5\text{nm}$, whereas its resistance is small and displays the typical plateaus of the quantum Hall effect for $d > 6.5\text{nm}$.

Axion electrodynamics can account for this behavior. To this end, a $\theta$ parameter, related to the Berry phase and the Chern number, is introduced. This parameter, called magneto-electric polarizability, is a piece-wise constant function. Its value is $\theta = 0$ for ordinary and $\theta = \pi$ for time-reversal symmetry topological insulators, as e.g. $HgTe$. Axion electrodynamics with $\theta(x,t) = 2b \cdot x - 2b_0t$ also describes Weyl semimetals[12] or, in general, the electrodynamics of magneto-electric media. A constant axion angle, $\theta(x,t) = k$, implies spatial and temporal translation symmetry conservation.\cite{8}
2 $\theta$-electrodynamics

Electromagnetism in material media is described by Maxwell equations. In differential form and a.u. they can be written as (see a comment on electrodynamics atomic units in appendix 3):

\[ \nabla \cdot \mathbf{D} = 4\pi \rho \quad \text{(Gauss law)} \]
\[ \nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} \quad \text{(Faraday law)} \]
\[ \nabla \cdot \mathbf{B} = 0 \quad \text{(Gauss law for magnetism)} \]
\[ \nabla \times \mathbf{H} = \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t} + \frac{4\pi}{c} \mathbf{J} \quad \text{(Ampere law)} \]

We should add to these equations the $\mathbf{D}, \mathbf{H}$ constitutive relations in terms of $\mathbf{E}, \mathbf{B}$. For linear material media these relations are simple: $\mathbf{D} = \varepsilon \mathbf{E}, \mathbf{H} = \frac{\mathbf{B}}{\mu}$, with $\varepsilon, \mu$ the dielectric constant and magnetic permeability. For isotropic materials $\varepsilon$ and $\mu$ are constants, while in anisotropic materials are tensors, eventually coordinate-dependent.

In topological media, where magneto-electric effects take place, the electric displacement vector $\mathbf{D}$ is modified by the magnetic induction $\mathbf{B}$ and the magnetic field intensity $\mathbf{H}$ of is in turn influenced by the electric field. Then, the relations $\mathbf{D} = \varepsilon \mathbf{E}, \mathbf{B} = \mu \mathbf{H}$ must be modified:

\[ \mathbf{D} = \varepsilon \mathbf{E} - \frac{\theta \alpha}{\pi} \mathbf{B} \quad \mathbf{H} = \frac{\mathbf{B}}{\mu} + \frac{\theta \alpha}{\pi} \mathbf{E} \]

where $\alpha = 1/137$ is the fine-structure constant and $\theta$ an additional parameter that can be considered at the same level as the permittivity $\varepsilon$ or the permeability $\mu$. In topological media $\theta = \pi$ while in ordinary media $\theta = 0$ (and we recover ordinary Maxwell equations).

These are the $\mathbf{D}, \mathbf{H}$ constitutive equations as reported by Nogueira and van der Brink. It should be mentioned that opposite sign for the axion term can be found in the literature (see e.g. [11]). All the same, as pointed out by Vazifeh and Franz, what allows the T- and P-invariant insulators to possess an axion term with $\theta = \pi$ is the $2\pi$ periodicity of the axion action in parameter $\theta$. Consequently, $\theta = \pi$ and $\theta = -\pi$ are two equivalent points and describe a T- and P-invariant system. Then, what about the axion term sign? As pointed out by Zirnstein and Rosenow the response of a time-reversal-symmetric system always has to be time-reversal-symmetric. For this reason, the idea that the axion action describes the response of a finite time-reversal-symmetric topological insulator is a misconception and is incorrect. In other word, a topological system with periodic boundary conditions, the axion action generates no classical response, while in an open system i.e., a finite system separated by a border from ordinary material, its effect is canceled by the response of the topologically protected surface boundary state. Then, in order to get axion response we should break time-reversal. Actually, we need a setup such that time-reversal symmetry is broken only on the surface of the topological insulator, but is preserved in the bulk. This can be achieved e.g. by doping the topological insulator surface with magnetic impurities or by attaching on the surface a shell of another material with ordered magnetization (proximity effect). Then, since time-reversal symmetry is preserved in the bulk, we still get $\theta = \pi$. Meanwhile, since it is not on the surface, we can get axion response. In this case, the aforementioned axion sign is determined by the direction of the surface magnetization, $\text{sign}[\mathbf{M} \cdot \mathbf{n}]$, with $\mathbf{n}$ a surface unit vector pointing out of the topological insulator. [14, 15, 16]

These modifications in the $\mathbf{D}, \mathbf{H}$ constitutive relations entail changes of two Maxwell equations, as we will show later. But first we will try to outline on the natural appearance of this extra term in electromagnetism. To this end, lets recall the electromagnetic energy expression:

\[ W = \frac{1}{2} (\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H}) = \frac{1}{2} \varepsilon \mathbf{E}^2 + \frac{1}{2} \frac{1}{\mu} \mathbf{B}^2 \]
On the other hand, the Lagrangian is:

\[ \mathcal{L} = \frac{1}{2} \epsilon E^2 - \frac{1}{2} \mu B^2 \]

From this Lagrangian, Maxwell equations can be obtained by means the Euler-Lagrange variational calculus (see e.g. Civelek et al.\[18\]).

We observe that the Lagrangian is quadratic with respect to the electromagnetic field. F. Wilczek\[2\] pointed out that there exists an additional quadratic term missing in the previous Lagrangian:

\[ \Delta \mathcal{L} = \kappa \theta \mathbf{E} \cdot \mathbf{B} \]

By including this term, Wilczek obtains the axion electrodynamics equations. Here, however, we pursue a less elegant but simpler derivation: we incorporate, as already said, the \(D, H\) constitutive relations in a \(\theta \neq 0\) medium. By looking to Maxwell equations we can see:

\[
\begin{align*}
\nabla \cdot D &= 4\pi \rho \quad \text{(changes as D changes)} \\
\nabla \times E &= -\frac{1}{c} \frac{\partial B}{\partial t} \quad \text{(does not change)} \\
\nabla \cdot B &= 0 \quad \text{(does not change)} \\
\n\nabla \times H &= \frac{1}{c} \frac{\partial D}{\partial t} + \frac{4\pi}{c} J \quad \text{(changes as H and D changes)}
\end{align*}
\]

We add now the constitutive relations\[3\]

\[
\begin{align*}
D &= \epsilon E - \frac{\theta \alpha}{\pi} B \\
H &= B + \frac{\theta \alpha}{\pi} \mu E
\end{align*}
\]

From equations (1), (5) we can write \(\nabla \cdot (\epsilon E) = 4\pi \rho + \frac{\alpha}{\pi} \nabla \cdot (\theta \mathbf{B})\). Since \(\nabla \cdot B = 0\), eq. (2), we finally rewrite eq. (1) as:

\[ \nabla \cdot (\epsilon E) = 4\pi \rho + \frac{\alpha}{\pi} \nabla \theta \mathbf{B} \]

In a similar way, from equations (4), (5) and (6) we have:

\[
\begin{align*}
\nabla \times \left( \frac{1}{\mu} B \right) - \frac{1}{c} \frac{\partial (\epsilon E)}{\partial t} &= \frac{4\pi}{c} J - \frac{1}{c} \frac{\alpha}{\pi} \frac{\partial (\theta \mathbf{B})}{\partial t} - \frac{\alpha}{\pi} \nabla \times (\theta \mathbf{E})
\end{align*}
\]

Incorporating the axion into Maxwell’s equations has the effect of "rotating" the electric and magnetic fields into each other:

\[
\begin{align*}
\begin{pmatrix} E' \\ cB' \end{pmatrix} = \begin{pmatrix} E - \frac{\theta \alpha}{\pi} cB \\ cB + \frac{\theta \alpha}{\pi} \mu E \end{pmatrix} = \begin{pmatrix} E - \frac{\theta \alpha}{\pi} cB + \frac{\theta \alpha}{\pi} cB \\ cB + \frac{\theta \alpha}{\pi} \mu E \end{pmatrix} = \begin{pmatrix} \cos \xi & -\sin \xi \\ \sin \xi & \cos \xi \end{pmatrix} \begin{pmatrix} E \\ cB + \tan \xi cB \end{pmatrix} = \begin{pmatrix} E - \tan \xi cB \\ cB + \tan \xi E \end{pmatrix}
\end{align*}
\]

with \(\tan \xi = \frac{\theta \alpha}{\pi\epsilon \mu} \neq \frac{\theta \alpha}{\pi\epsilon} \sqrt{\mu} = \frac{\theta \alpha}{\pi} \mu \frac{1}{\sqrt{\epsilon \mu}} = \frac{\theta \alpha}{\pi} \mu c\), the mixing angle \(\xi\) depending then on the axion field strength \(\theta\) and the coupling constants.

---

\[3\] Incorporating the axion into Maxwell’s equations has the effect of “rotating” the electric and magnetic fields into each other:
Now we carry out the two last terms derivatives:

\[-\frac{1}{c} \frac{\partial (\theta B)}{\partial t} - \frac{\alpha}{\pi} \nabla \times (\theta E) = -\frac{1}{c} \frac{\partial \theta}{\partial t} \left( B + \theta \frac{\partial B}{\partial t} \right) - \frac{\alpha}{\pi} (\nabla \theta \times E + \theta \nabla \times E) \]

\[= \frac{\alpha}{\pi} \theta \left( \frac{1}{c} \frac{\partial B}{\partial t} - \nabla \theta \right) - \frac{\alpha}{\pi} \left( \frac{1}{c} \frac{\partial}{\partial t} \left( B + \nabla \theta \times E \right) \right) \]

Taking into account eq (2), the first bracket in the last equation must be zero. Then, eq (4) turns into:

\[\nabla \times \left( \frac{1}{\mu} B \right) - \frac{1}{c} \frac{\partial (\epsilon E)}{\partial t} = \frac{4\pi}{c} J - \frac{1}{c} \frac{\alpha}{\pi} B - \frac{\alpha}{\pi} \nabla \theta \times E \]

The new equations (7) and (8), replacing eqs. (1) and (4), suggest the definition of effective charges and current densities. Thus, from (7) we define the effective charge \(\rho_\theta\):

\[4\pi \rho_\theta = \frac{\alpha}{\pi} \nabla \theta \cdot B \rightarrow \rho_\theta = \frac{\alpha}{4\pi^2} \nabla \theta \cdot B \] (9)

and from (8) the effective current density \(J_\theta\):

\[\frac{4\pi}{c} J_\theta = \frac{\alpha}{\pi} \left( -\frac{1}{c} \frac{\partial \theta}{\partial t} B - \nabla \theta \times E \right) \rightarrow \left. J_\theta = \frac{\alpha}{4\pi^2} \left( -\frac{\partial \theta}{\partial t} B - c \nabla \theta \times E \right) \right| \]

(10)

Should \(\theta = 0\) then, equations (7) and (8) goes back to (1) and (4). Should be \(\theta\) coordinates and time independent, i.e. should be \(\theta\) a constant, then \(\rho_\theta = J_\theta = 0\) and we return again to the Maxwell equations.

As said above, \(\theta = 0\) in a vacuum or in an ordinary insulator, while \(\theta = \pi\) for time reversal topological insulators. Therefore, \(\theta\) is a piecewise constant function (see figure).

If \(\theta(z)\) is a step function, then \(\nabla \theta = \pi \delta(z_0) n\), where \(n\) is a unit vector in the \(z\) direction and \(\delta(z_0)\) is the Dirac delta (see note[19]). Therefore, since \(\theta\) is time-independent, eq. (10) leads to:

\[J_\theta = -\frac{c\alpha}{4\pi} \delta(z_0) n \times E \] (11)

In a similar way, from eq. (9) we have:

\[\rho_\theta = \frac{\alpha}{4\pi} \delta(z_0) n \cdot B \] (12)

The set of equations (7), (2), (3), (8) are those introduced by F. Wilczek[2] to define axion electrodynamics and are broadly used (see e.g. equation 39 in [15], or equation 68 in [8] etc.). However, Luca Visinelli[20]

\[\text{Please note that both, } \nabla \theta \text{ and the unitary } n \text{ vector, point from ordinary (} \theta = 0\text{) to topological (} \theta = \pi\text{) insulator.} \]
points out that Maxwell equations for an electromagnetic field show an internal symmetry, known as the duality transformation,

\[
\begin{pmatrix}
E' \\
B'
\end{pmatrix}
= \begin{pmatrix}
\cos \xi & \sin \xi \\
-\sin \xi & \cos \xi
\end{pmatrix}
\begin{pmatrix}
E \\
B
\end{pmatrix}
\]

and that whenever a pseudoscalar axion-like field \( \theta = \theta(x) \) is introduced in the theory, the dual symmetry is spontaneously and explicitly broken. He relates this broken symmetry to the fact that the introduction of an axion-like interaction with the electromagnetic field only modifies two of the four Maxwell equations (Gauss and Ampere laws), but not the remaining two equations (Faraday and Gauss law for B). The requirement that the electric and magnetic fields must satisfy also the above duality relation along with Gauss and Ampere laws for axion electrodynamics leads him we obtain new terms that also modify Faraday law and Gauss law for B (see eq. 26 in [20]. See also [21]).
Part II:

Images of a charge close to an interface ordinary insulator-topological insulator

3 Image charge in topological insulators

3.1 Boundary conditions at the interface with a topological insulator

In Appendix 1 we obtain the boundary conditions (BCs) in a.u. for electrostatics and magnetostatics (i.e., for time-independent fields). In particular, in absence of free charge and current, $D_\perp = D_\perp$, $B_\perp = B_\perp$, $H_\parallel = H_\parallel$ and $E_\parallel = E_\parallel$. By injecting the constitutive relation (5) in the first boundary condition we find:

$$\epsilon_1 E_\perp - \frac{\theta_1 \alpha}{\pi} B_\perp = \epsilon_2 E_\perp - \frac{\theta_2 \alpha}{\pi} B_\perp$$  (13)

Assuming the border surface at $z = 0$, the second boundary condition, $B_\perp(z = 0) = B_\perp(z = 0) = B_z$ yields,

$$\epsilon_1 E_\perp - \epsilon_2 E_\perp = (\theta_1 - \theta_2) \frac{\alpha}{\pi} B_z$$  (14)

In particular, if medium 1 ($z < 0$) is ordinary ($\theta = 0$) and medium 2 ($z > 0$) topological ($\theta = \pi$), then

$$\epsilon_1 E_\perp^{\text{ord}} = \epsilon_2 E_\perp^{\text{top}} - \alpha B_z$$  (15)

In a similar way, the third boundary conditions with the constitutive relation (6) yield:

$$\frac{1}{\mu_1} B_\parallel + \frac{\theta_1 \alpha}{\pi} E_\parallel = \frac{1}{\mu_2} B_\parallel + \frac{\theta_2 \alpha}{\pi} E_\parallel$$  (16)

Assuming again the border surface at $z = 0$, the fourth boundary condition, $E_\parallel(z = 0) = E_\parallel(z = 0) = E_\parallel$, injected in eq. (16), yields,

$$\frac{1}{\mu_1} B_\parallel - \frac{1}{\mu_2} B_\parallel = (\theta_2 - \theta_1) \frac{\alpha}{\pi} E_\parallel$$  (17)

In particular, if medium 1 ($z < 0$) is ordinary ($\theta = 0$) and medium 2 ($z > 0$) topological ($\theta = \pi$), then

$$\frac{1}{\mu_1} B_\parallel^{\text{ord}} = \frac{1}{\mu_2} B_\parallel^{\text{top}} + \alpha E_\parallel.$$  (18)

3.2 Images of an electric charge close to the interface between ordinary and topological insulator

We assume a static problem i.e., without magnetic induction $B$ or electric field $E$ temporary dependence, and that there are no free charges and currents. Then, except at the interface, Maxwell equations (1), (2), (3), (4) become $\nabla \cdot D = 0$, $\nabla \cdot B = 0$, $\nabla \times E = 0$ and $\nabla \times H = 0$. Should the curl of a vector field be zero, then it can be written as the gradient of a scalar field. Therefore, from $\nabla \times E = 0$ we conclude that $E = -\nabla V$. On the other hand, from $\nabla \times H = 0$ and the constituent equation (6), it follows that $\nabla \times \left(\frac{B}{\mu} + \frac{\theta_0}{\pi} E\right) = 0$ and then $B + \frac{\theta_0}{\pi} E = -\nabla W$ or, in an equivalent way, $B = -\nabla \left[\mu \left(W - \frac{\theta_0}{\pi} V\right)\right] = -\nabla U$. Therefore, we can define electric and magnetic potentials whose opposite sign gradients yield the field.
Let $Q$ be an electric charge located at $(0,0,a)$ in an $(\epsilon_1, \mu_1)$ ordinary medium in contact with an $(\epsilon_2, \mu_2, \alpha)$ topological insulator ($\theta = 0, \pi$ for ordinary and topological insulators, respectively). Assume the boundary at $z = 0$. According to the image method (see Appendix 2), in order to calculate the potential in the ordinary insulator (zone 1, $z > 0$) we must add a fictitious charge at $(0,0,-a)$ while to do it in the topological insulator (zone 2, $z < 0$) we must add the fictitious charge at $(0,0,a)$. We have learned that a magnetic field $\mathbf{B}$ induces a surface charge $\sigma = \frac{\alpha}{4\pi} \mathbf{n} \cdot \mathbf{B}$ while an electric field (e.g. that from the electric source $Q$) generates surface currents $\mathbf{J}_s = -\frac{c\alpha}{4\pi} \mathbf{n} \times \mathbf{E}$. The axial symmetry of the $Q$-generated electric field leads us to conclude from $\mathbf{J}_s$ formulae that circular currents are generated around the axis joining $Q$ with the interface (see Figure).

This surface current, generated by $Q$ located at $z > 0$, originates a magnetic field on the other side of the interface, $z < 0$, proportional to the $Q$-electric field. Therefore proportional to $\frac{Q}{\epsilon_1}$. This magnetic field is equivalent to the magnetic field generated by a magnetic monopole $p_1$ located at $(0,0,b)$. Additionally, the surface current generates another magnetic field in the zone $z > 0$, also proportional to the electric field, equivalent to that generated by a magnetic monopole $p_2$ located, for sake of symmetry, at $(0,0,-b)$.

The electric and magnetic potentials in both regions, according to the image method (Appendix 2) are:

\[
\begin{align*}
V(x, z > 0) &= \frac{Q/\epsilon_1}{\sqrt{x^2 + (z-a)^2}} + \frac{q}{\sqrt{x^2 + (z+a)^2}} \quad (19) \\
V(x, z < 0) &= \frac{Q/\epsilon_1}{\sqrt{x^2 + (z-a)^2}} + \frac{q}{\sqrt{x^2 + (z+a)^2}} \quad (20) \\
U(x, z > 0) &= \frac{p_2}{\sqrt{x^2 + (z+b)^2}} \quad (21) \\
U(x, z < 0) &= \frac{p_1}{\sqrt{x^2 + (z-b)^2}} \quad (22)
\end{align*}
\]

The lack of symmetry of the magnetic field (axial vector) with respect to a horizontal plane lead us to write $p_1$ and $p_2$ as unknowns, with the expectation to find out $p_1 = -p_2$.\footnote{The lack of symmetry of the magnetic field (axial vector) with respect to a horizontal plane lead us to write $p_1$ and $p_2$ as unknowns, with the expectation to find out $p_1 = -p_2$.}
The $B_\perp = -\frac{\partial U}{\partial x}$ continuity across the boundary means that $(\frac{\partial U(z>0)}{\partial x})_0 = (\frac{\partial U(z<0)}{\partial x})_0$. Then:

$$p_2 \frac{(z^0 + b)}{[x^2 + (z^0 + b)^2]^{3/2}} = p_1 \frac{(z^0 - b)}{[x^2 + (z^0 - b)^2]^{3/2}} \rightarrow p_1 = -p_2$$

Continuity at $z = 0$ of $D_\perp$, eq. (15), i.e., $\epsilon_1 E_{1\perp} = \epsilon_2 E_{2\perp} - \alpha B_z$, with $B_z = -(\frac{\partial U}{\partial x})_0$ and $E_\perp = -(\frac{\partial V}{\partial x})_0$, leads to:

$$\epsilon_1 \left( \frac{Q}{\epsilon_1} \frac{(z^0 - a)}{[x^2 + (z^0 - a)^2]^{3/2}} + q \frac{(z^0 + a)}{[x^2 + (z^0 + a)^2]^{3/2}} \right) = \epsilon_2 \left( \frac{Q}{\epsilon_2} \frac{(z^0 - a)}{[x^2 + (z^0 - a)^2]^{3/2}} + q \frac{(z^0 + a)}{[x^2 + (z^0 + a)^2]^{3/2}} \right)$$

$$\rightarrow Q (1 - \frac{\epsilon_2}{\epsilon_1}) \frac{(-a)}{[x^2 + a^2]^{3/2}} + \frac{aq}{[x^2 + a^2]^{3/2}} (\epsilon_1 + \epsilon_2) = -\alpha p_1 \forall x$$

$$a = b$$

$$\rightarrow Q \left( \frac{\epsilon_2 - \epsilon_1}{\epsilon_1} \right) + (\epsilon_1 + \epsilon_2) q = -\alpha p_1 \rightarrow p_1 = -\frac{Q}{\alpha} \frac{\epsilon_2 - \epsilon_1}{\epsilon_1} - \frac{q}{\alpha} (\epsilon_1 + \epsilon_2)$$

(24)

The $H_\parallel$ continuity at $z = 0$, eq. (18), i.e., $\frac{1}{\mu_1} B_{1\parallel} = \frac{1}{\mu_2} B_{2\parallel} + \alpha E_\parallel$, with $p_1 = -p_2$, $B_{\parallel} = -(\frac{\partial U}{\partial x})_{z=0}$, $E_\parallel = -(\frac{\partial V}{\partial x})_{z=0}$ yields (at $z = 0$):

$$\frac{1}{\mu_1} \frac{x p_1}{[x^2 + a^2]^{3/2}} = \frac{1}{\mu_2} \frac{x (-p_1)}{[x^2 + a^2]^{3/2}} + \alpha \left( \frac{Q}{\epsilon_1} \frac{x}{[x^2 + a^2]^{3/2}} + \frac{x}{[x^2 + a^2]^{3/2}} \right)$$

$$\rightarrow (\frac{1}{\mu_1} + \frac{1}{\mu_2}) p_1 = \alpha \left( \frac{Q}{\epsilon_1} + q \right)$$

$$p_1 = \frac{\alpha}{\frac{1}{\mu_1} + \frac{1}{\mu_2}} \left( \frac{Q}{\epsilon_1} + q \right)$$

(25)

From equations (24) and (25) it follows:

$$\frac{Q}{\alpha} \frac{\epsilon_2 - \epsilon_1}{\epsilon_1} + \frac{q}{\alpha} (\epsilon_1 + \epsilon_2) = -\frac{\alpha}{\frac{1}{\mu_1} + \frac{1}{\mu_2}} \left( \frac{Q}{\epsilon_1} + q \right)$$

$$\rightarrow \frac{Q}{\epsilon_1} \left( \frac{\epsilon_2 - \epsilon_1}{\alpha} + \frac{\alpha}{\frac{1}{\mu_1} + \frac{1}{\mu_2}} \right) + q \left( \frac{\epsilon_1 + \epsilon_2}{\alpha} + \frac{\alpha}{\frac{1}{\mu_1} + \frac{1}{\mu_2}} \right) = 0$$

$$\rightarrow q = -\frac{\frac{\epsilon_2 - \epsilon_1}{\alpha} + \frac{\alpha}{\frac{1}{\mu_1} + \frac{1}{\mu_2}}}{\frac{\epsilon_1 + \epsilon_2}{\alpha} + \frac{\alpha}{\frac{1}{\mu_1} + \frac{1}{\mu_2}}}$$

(26)

In the derivation we have calculated $B_z$ as the $U(z > 0)$ derivative at $z = 0$. The same result is achieved with $U(z < 0)$ since a double change of sign (monopole charge and position) compensates.
By identifying $\alpha^2$ with $4P_3^2\alpha^2$ (for $P_3 = \pm \frac{1}{2}$ i.e. $4P_3^2 = 1$, see [14] page 1185) we can check that Eq. (26) matches eq. 3.2 and eq. (27) matches eq. 3b in [14].
Like images of an electric charge, a magnetic monopole \( P \) generates an electric \( (V) \) and magnetic \( (U) \) potentials given by:

\[
U_1(x, z > 0) = \frac{\mu_1 P}{\sqrt{x^2 + (z-a)^2}} + \frac{p_2}{\sqrt{x^2 + (z+a)^2}} \tag{28}
\]
\[
U_2(x, z < 0) = \frac{\mu_1 P}{\sqrt{x^2 + (z-a)^2}} + \frac{p_1}{\sqrt{x^2 + (z-a)^2}} \tag{29}
\]
\[
V_1(x, z > 0) = \frac{q}{\sqrt{x^2 + (z+b)^2}} \tag{30}
\]
\[
V_2(x, z < 0) = \frac{q}{\sqrt{x^2 + (z-b)^2}} \tag{31}
\]

with boundary conditions (BCs):

\[
\epsilon_1 \left( \frac{\partial V_1}{\partial z} \right)_0 = \epsilon_2 \left( \frac{\partial V_2}{\partial z} \right)_0 - \alpha \left( \frac{\partial U}{\partial z} \right)_0 
\tag{33}
\]
\[
\frac{1}{\mu_1} \left( \frac{\partial U_1}{\partial x} \right)_0 = \frac{1}{\mu_2} \left( \frac{\partial U_2}{\partial x} \right)_0 + \alpha \left( \frac{\partial V}{\partial x} \right)_0 \tag{34}
\]

where since \( \left( \frac{\partial U_1}{\partial z} \right)_0 = \left( \frac{\partial U_2}{\partial z} \right)_0 \) and \( \left( \frac{\partial U_1}{\partial x} \right)_0 = \left( \frac{\partial V}{\partial x} \right)_0 \), we can employ \( U \) and \( V \) of either region.

From the first BC, eq. (33), we have:

\[
\epsilon_1 q b = \epsilon_2 q b + \alpha \left( \frac{\mu_1 P a}{x^2 + b^2} + \frac{p_1 a}{x^2 + a^2} \right) \forall x \rightarrow a = b \tag{35}
\]

and coming back to this equation with \( a = b \) it follows:

\[
\epsilon_1 q = -\epsilon_2 q + \alpha (\mu_1 P + p_1) \rightarrow q = \frac{\alpha}{\epsilon_1 + \epsilon_2} (p_1 + \mu_1 P) \tag{36}
\]

From the second BC, eq. (34), we find out:

\[
\frac{1}{\mu_1} \left( \frac{\mu_1 P x}{x^2 + a^2} + \frac{p_2 x}{x^2 + a^2} \right) = \frac{1}{\mu_2} \left( \frac{\mu_1 P x}{x^2 + a^2} + \frac{p_1 x}{x^2 + a^2} \right) + \alpha \frac{q x}{x^2 + a^2} \forall x
\]

\[
\rightarrow P + \frac{p_2}{\mu_1} = \frac{\mu_1}{\mu_2} P \left( 1 - \frac{p_1}{\mu_2} \right) + \frac{p_1}{\mu_2} \left( 1 - \frac{p_1}{\mu_2} \right) \rightarrow p_1 = -p_2 \tag{37}
\]

The boundary condition \( E_2 = E_1 \) turns to be an identity which brings nothing. Finally the boundary condition \( B_{1\perp} = B_{2\perp} \) with \( B_\perp = -\frac{\partial U}{\partial z} \) leads to:

\[
\frac{\mu_1 P (x^0 - a)}{x^2 + a^2} + \frac{p_2 (x^0 + a)}{x^2 + a^2} = \frac{\mu_1 P (x^0 - a)}{x^2 + a^2} + \frac{p_1 (x^0 - a)}{x^2 + a^2} \rightarrow p_1 = -p_2 \tag{38}
\]
Now, by combining eqs. (37) and (38), taking into account (36), we find:

\[ P \left( 1 - \frac{\mu_1}{\mu_2} \right) - p_1 \left( \frac{1}{\mu_1} + \frac{1}{\mu_2} \right) - \frac{\alpha^2}{\epsilon_1 + \epsilon_2} (p_1 + \mu_1 P) = 0 \]

\rightarrow P \left( 1 - \frac{\mu_1}{\mu_2} - \frac{\alpha^2 \mu_1}{\epsilon_1 + \epsilon_2} \right) - p_1 \left( \frac{1}{\mu_1} + \frac{1}{\mu_2} + \frac{\alpha^2}{\epsilon_1 + \epsilon_2} \right) = 0

\rightarrow p_1 = P \frac{1 - \frac{\mu_1}{\mu_2} - \frac{\alpha^2 \mu_1}{\epsilon_1 + \epsilon_2}}{\frac{1}{\mu_1} + \frac{1}{\mu_2} + \frac{\alpha^2}{\epsilon_1 + \epsilon_2}} \quad (39)

Note that if the insulator is not topological but ordinary, then we should take \( \alpha = 0 \) and obtain \( p_1 = \mu_1 P \frac{\mu_2 - \mu_1}{\mu_1 + \mu_2} \) which is the image of a monopole in front of an interface between two different ordinary media.

By injecting \( p_1 \), eq. (39), into \( q \) given by equation (36) we get:

\[ q = \frac{\alpha}{\epsilon_1 + \epsilon_2} \mu_1 P \frac{\frac{1}{\mu_1} - \frac{1}{\mu_2} - \frac{\alpha^2}{\epsilon_1 + \epsilon_2} + 1}{\frac{1}{\mu_2} + \frac{1}{\mu_1} + \frac{\alpha^2}{\epsilon_1 + \epsilon_2}} \quad (40) \]

We can simplify eqs. (39) i (40) as follow:

\[ p_1 = \mu_1 P \frac{\frac{1}{\mu_1} - \frac{1}{\mu_2} + \alpha^2}{\frac{1}{\mu_2} + \frac{1}{\mu_1} + \alpha^2} \quad (41) \]

\[ q = \frac{2 P}{\frac{1}{\mu_1} + \frac{1}{\mu_2} + \alpha^2} \quad (42) \]

The result is a perfect analogy of the image of an electric charge, with the detail that the image \( q \) of the charge \( Q \), eq. [26], is proportional to \( \frac{1}{\mu_1} + \frac{1}{\mu_2} \) while the image \( p \) of the monopole \( P \), eq. [42], is proportional to \( \frac{1}{\mu_1} - \frac{1}{\mu_2} \). Dielectric constants and magnetic permeabilities exchange their role.

**Exercise**

An electric charge is located in the central region of a quantum well build up by a topological insulator surrounded by an ordinary insulator. (a) Calculate the electrical potential generated in the central region, assuming that both the insulator of the central region and that of the surrounding barriers, are ordinary. Check that the result agrees with equations (2.8) and (2.14) by Kumagai and Takagahara. [22] (b) Calculate the electrical and magnetic potentials under the assumption that the insulator of the central region is topological. Check that by forcing \( \alpha = 0 \) the magnetic potential goes to zero and the electric potential matches that obtained in the previous section.

**References**

[1] B. Andrei Bernevig, Taylor L. Hughes, and Shou-Cheng Zhang, Science 314 (2006) 1757.
[2] F. Wilczek, Phys. Rev. Lett. 58 (1987) 1799.
[3] F. Wilczek, Phys. Rev. Lett. 40 (1978) 279.
[4] S. Weinberg, Phys. Rev. Lett. 40 (1978) 223.
[5] X-L Qi and S-C Zhang, Physics Today 63 (2010) 33.
[6] M. M. Vazifeh and M. Franz, Phys. Rev. Lett. 111 (2013) 027201.
[7] T. H. O’Dell, The electrodynamics of magneto-electric media, American Elsevier Pub. Co., New York 1970.
[8] See e.g. section 3, after eq. 62, in X-L Qi and S-C Zhang, Rev. Mod. Phys. 83 (2011) 1057.
[9] A. Martín-Ruiz, M. Cambiaso and L. F. Urrutia, Int. J. Mod. Phys. A 34 (2019) 1941002.
[10] F.V. Nogueira and J. van der Brink, Phys. Rev. Research 4 (2022) 013074.
[11] A. Martín-Ruiz, O. Rodríguez-Tzompantzi, J. R. Maze, and L. F. Urrutia, Phys. Rev. B 100 (2019) 042124.
[12] M. M. Vazifeh and M. Franz, Phys. Rev. B 82 (2010) 233103.
[13] H. G. Zirnstein and B. Rosenow, Phys. Status Solidi B 257 (2020) 1900698.
[14] X-L Qi, R. Li, J. Zang and S-C Zhang, Science 329 (2009) 1184.
[15] A. Sekine and K. Nomura, J. Appl. Phys. 129 (2021) 141101.
[16] W. H. Campos, W. A. Moura-Melo and J. M. Fonseca, Phys. Lett. A 381 (2017) 417.
[17] The electromagnetic energy density $W = \frac{1}{2} \mu \mathbf{B}^2 + \frac{1}{2} \epsilon \mathbf{E}^2$ can formally be identified with a momentum $\mathbf{B}$, mass $\mu$, coordinate $\mathbf{E}$ and force constant $\epsilon$ harmonic oscillator, so that energy $W$ is a sum of kinetic $T$ and potential $V$ energies. By convention, Lagrangian is defined as $\mathcal{L} = T - V$, but can also be defined with opposite sign $\mathcal{L} = V - T = \frac{1}{2} \epsilon \mathbf{E}^2 - \frac{1}{2} \mu \mathbf{B}^2$ for both definitions lead to the same Euler-Lagrange equation $\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} \right) - \left( \frac{\partial \mathcal{L}}{\partial \mathbf{q}} \right) = 0$.
[18] C. Civelek and T. F. Bechteler, Int. J. of Eng. Sci. 46 (2008) 1218.
[19] Dirac delta is defined as the derivative of the Heaviside function $\{H(z > 0) = 1, \ H(z < 0) = 0\}$, $\delta(z) = \frac{d}{dz} H(z)$. Since $\theta = \pi H(z_0)$, then $\frac{d\theta}{dz} = \pi \frac{dH}{dz} = \pi \delta(z_0)$ and therefore $\nabla \theta = \pi \delta(z_0) \mathbf{n}$.
[20] L. Visinelli, Mod. Phys. Lett. A 28 (2013) 135062.
[21] Since for image charges calculation only equations (7) and (8) are employed, Visinelli proposal[20] does not affect this kind of calculation.
[22] M. Kumagai and T. Takagahara, Phys. Rev. B 40 (1989) 12359.
4 Appendix 1: Boundary conditions at the interface of ordinary insulators

Maxwell’s equations in integral form (and MKS rational system) are:

$$\iint D \cdot dS = \iiint \rho \, dv$$
$$\iint B \cdot dS = \iiint \nabla \cdot B \, dv = 0$$
$$\iint E \cdot dS = \iiint \nabla \times E \cdot dS = \oint E \cdot d\ell = - \iiint \frac{\partial B}{\partial t} \cdot dS$$
$$\iint H \cdot dS = \iiint \nabla \times H \cdot dS = \oint H \cdot d\ell = \iint J \cdot dS + \iiint \frac{\partial D}{\partial t} \cdot dS$$

(43)

Should we consider time independence, the last two equations including time derivatives are simplified. Then, Maxwell equations become:

$$\iint D \cdot dS = \iiint \rho \, dv$$
$$\iint B \cdot dS = 0$$
$$\oint E \cdot d\ell = 0$$
$$\oint H \cdot d\ell = \iint J \cdot dS$$

(44)

On left in the Figure we show a slab of area $\Delta S$ and infinitesimal height $dx$. We rewrite the integral $\iint B \cdot dS = 0$ for this slab as:

$$0 = \iint B \cdot dS = \iint B \cdot n \, dS = (B_2 \cdot n - B_1 \cdot n) \Delta S = (B_2 - B_1) \cdot n \Delta S \rightarrow [B_{2\perp} = B_{1\perp}]$$

In a similar way, $\iint D \cdot dS = (D_2 - D_1) \cdot n \Delta S$. On the other hand $\iiint \rho \, dv = \iiint \rho \, dx \, \Delta S = \sigma \Delta S$. Then,

$$(D_2 - D_1) \cdot n = \sigma \rightarrow [D_{2\perp} = D_{1\perp} + \sigma]$$

On the right hand side of the figure it is drawn a circuit in the shape of a rectangle with a long side $\Delta \ell$ and a narrow infinitesimal side $dx$. The circulation of the electric field $\oint E \cdot d\ell = 0$ in this circuit is:

$$0 = \oint E \cdot d\ell = (E_2 - E_1) \cdot \Delta \ell \rightarrow [E_{2\parallel} = E_{1\parallel}]$$

In a similar way, $\oint H \cdot d\ell = (H_2 - H_1) \cdot \Delta \ell$. Also, $\iint J \cdot dS = \iint J \cdot (dx \Delta \ell) \tau = (\iint J \, dx) \cdot \tau \Delta \ell = J_s \Delta \ell$, where $J_s$ represents surface current at the interface. Therefore:

$$[H_{2\parallel} = H_{1\parallel} + J_s]$$
4.1 Boundary conditions at the interface of ordinary insulators in a.u.

Maxwell equations in differential form and a.u. reads:
\[
\nabla \cdot \mathbf{D} = 4\pi \rho \quad \text{(Gauss law)}
\]
\[
\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} \quad \text{(Faraday law)}
\]
\[
\nabla \cdot \mathbf{B} = 0 \quad \text{(Gauss law for magnetism)}
\]
\[
\nabla \times \mathbf{H} = \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t} + \frac{4\pi}{c} \mathbf{J} \quad \text{(Ampere law)}
\]

Therefore, \(4\pi\) must be added to the \(\mathbf{D}\) boundary condition and \(\frac{4\pi}{c}\) to the \(\mathbf{H}\) boundary condition:
\[
\mathbf{D}_{2\perp} = \mathbf{D}_{1\perp} + 4\pi \sigma \\
\mathbf{H}_{2\parallel} = \mathbf{H}_{1\parallel} + \frac{4\pi}{c} \mathbf{J}_s
\]

5 Appendix 2: Electrostatic Image Method

Let \(Q\) be a charge in a dielectric constant \(\varepsilon_1\) medium. \(Q\) polarizes this medium. Should the medium be infinite then the result is that \(Q\) plus the medium generate the same electric field as that of an effective charge \(Q/\varepsilon_1\) in a vacuum. If there is an interface separating two different polarizability media, i.e., with different dielectric constant (see figure), on the surface of the upper medium appears the lower ends of the last dipole layer induced in this medium which are not compensated with the first dipole layer induced in the lower medium. The result is that the charge \(Q\) located at a distance \(a\) from the interface plus the two media generate the same field as that generated in a vacuum by an effective charge \(Q/\varepsilon_1\) located where the charge \(Q\) is located plus a surface density \(\sigma\) located at the interface between the media. The charge distribution \(\sigma\) generates in the upper medium the same field as that generated by a charge \(q_1\) located symmetrically in the lower medium at a distance \(b\), while in the lower medium it generates the same field than that generated by a charge \(q_2\) located in the same vertical as \(Q\) (for symmetry reasons) at a distance \(c\) from the interface.

\[
\begin{align*}
V(z > 0) &= \frac{Q/\varepsilon_1}{\sqrt{x^2 + (z-a)^2}} + \frac{q_1}{\sqrt{x^2 + (z+b)^2}} \
V(z < 0) &= \frac{Q/\varepsilon_1}{\sqrt{x^2 + (z-a)^2}} + \frac{q_2}{\sqrt{x^2 + (z-c)^2}}
\end{align*}
\]

To determine the values \(b, c, q_1, q_2\) we apply the BCs obtained in Appendix 1. From BC \(E_{2\parallel} = E_{1\parallel}\), with \(E_{\parallel} = -\frac{\partial V}{\partial z}\) we find:
\[
E_{1\parallel}(z > 0) = \frac{Q}{\varepsilon_1} \frac{x}{\sqrt{x^2 + (z-a)^2}} + q_1 \frac{x}{\sqrt{x^2 + (z+b)^2}}
\]
E_2|| (z < 0) = \frac{Q}{\epsilon_1} \frac{x}{[x^2 + (z-a)^2]^{3/2}} + q_2 \frac{x}{[x^2 + (z-c)^2]^{3/2}} \quad (48)

Equating the two equations at z = 0 we find that:

\frac{q_1}{[x^2 + b^2]^{3/2}} = \frac{q_2}{[x^2 + c^2]^{3/2}} \rightarrow q_1 = \frac{[x^2 + b^2]^{3/2}}{[x^2 + c^2]^{3/2}} \quad (49)

As the ratio \frac{q_1}{q_2} is constant, so must be the fraction \frac{[x^2 + b^2]^{3/2}}{[x^2 + c^2]^{3/2}}. Then, b = c and therefore q_1 = q_2.

From b = c, q_1 = q_2 = q, BC D_{2\perp} = D_{1\perp} + \sigma, taking into account that there is no free surface charge (\sigma = 0), and that D_{\perp} = \epsilon E_{\perp}, with E_{\perp} = -\frac{\partial V}{\partial z}, we have:

E_{1\perp} (z > 0) = \frac{Q}{\epsilon_1} \frac{z-a}{[x^2 + (z-a)^2]^{3/2}} + q \frac{z+b}{[x^2 + (z+b)^2]^{3/2}} \quad (50)

E_{2\perp} (z < 0) = \frac{Q}{\epsilon_1} \frac{z-a}{[x^2 + (z-a)^2]^{3/2}} + q \frac{z-b}{[x^2 + (z-b)^2]^{3/2}} \quad (51)

Equating \epsilon_1 E_{1\perp} = \epsilon_2 E_{2\perp} at z = 0 we find:

\begin{align*}
-Q \frac{a}{[x^2 + a^2]^{3/2}} + \epsilon_1 q \frac{b}{[x^2 + b^2]^{3/2}} &= -\frac{\epsilon_2}{\epsilon_1} Q \frac{a}{[x^2 + a^2]^{3/2}} - \epsilon_2 q \frac{b}{[x^2 + b^2]^{3/2}} \quad (52)
\end{align*}

Grouping factors:

\begin{align*}
Q \left(1 - \frac{\epsilon_2}{\epsilon_1}\right) \frac{a}{[x^2 + a^2]^{3/2}} &= q(\epsilon_1 + \epsilon_2) \frac{b}{[x^2 + b^2]^{3/2}} \rightarrow \frac{1}{\epsilon_1} Q \left(\frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + \epsilon_2}\right) \frac{a}{[x^2 + a^2]^{3/2}} = \frac{[x^2 + a^2]^{3/2}}{[x^2 + b^2]^{3/2}}. \quad (53)
\end{align*}

As the first term of the last equation is a constant, so must be the second, which implies a = b, a result that entails:

\begin{align*}
q &= \frac{1}{\epsilon_1} \left(\frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + \epsilon_2}\right) Q \quad (54)
\end{align*}

In summary, the electric potential generated by a charge Q in a medium of dielectric constant \epsilon_1 located at a distance a from the interface with another medium of dielectric constant \epsilon_2 is the same to that generated in a vacuum by an effective charge \(Q^* = Q/\epsilon_1\) located at Q site plus that generated by a image charge \(q = \frac{1}{\epsilon_1} \left(\frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + \epsilon_2}\right) Q\) located symmetrically in the other medium at the same distance from the interface. In addition, Q also generates in the other medium a potential like the one generated in a vacuum by an effective charge \(Q^* = Q/\epsilon_1\) plus an effective charge \(q = \frac{1}{\epsilon_1} \left(\frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + \epsilon_2}\right) Q\) both located at the Q site.
6 Appendix 3: On the electrodynamics atomic units

There is some ambiguity in defining the atomic units of magnetic field and the own Maxwell equations. Within the Lorentz force convention, the Maxwell equations reads:

\[
\begin{align*}
\nabla \cdot D &= 4\pi \rho \\
\nabla \cdot B &= 0 \\
\n\nabla \times E &= -\frac{\partial B}{\partial t} \\
\n\nabla \times H &= \frac{\partial D}{\partial t} + 4\pi J
\end{align*}
\]

(55) (56)

with the constitutive relations \( D = E + 4\pi P \), \( H = c^2 B - 4\pi M \). In the last equation the value \( c \) of the speed of light in a.u. is numerically equal to \( \frac{1}{\alpha} \), with \( \alpha = 1/137.036 \) being the dimensionless fine-structure constant. These equations are complemented by the Lorentz force law: \( F = e (E + v \times B) \).

Within the Gaussian convention, the Maxwell equations are:

\[
\begin{align*}
\nabla \cdot D &= 4\pi \rho \\
\nabla \cdot B &= 0 \\
\n\nabla \times E &= -\alpha \frac{\partial B}{\partial t} \\
\n\nabla \times H &= \alpha \frac{\partial D}{\partial t} + 4\pi \alpha J
\end{align*}
\]

(57) (58)

with the constitutive relations \( D = E + 4\pi P \), \( H = B - 4\pi M \), which looks like those of the cgs-Gaussian system. In addition, we have the Lorentz force law: \( F = e (E + \frac{v}{c} \times B) \).

Please note that the formal replacement \( \alpha B \rightarrow B \), \( \frac{H}{\alpha} \rightarrow H \), \( \frac{M}{\alpha} \rightarrow M \) in the Gaussian convention equations retrieves those of the Lorentz convention.

Care should we have in using either convention. For example, the atomic unit for magnetic field in the Lorentz force convention is \( 1 \text{ a.u.}_{LF} \approx 2.35 \cdot 10^5 \text{T} \) while \( 1 \text{ a.u.}_G \approx 1.72 \cdot 10^7 \text{G} \). Since \( 1 \text{T} = 10^4 \text{G} \) we can check that \( \frac{1 \text{ a.u.}_{LF}}{1 \text{ a.u.}_G} = \frac{1}{\alpha} \).

It would be instructive to have a look to the units guide provided by Andrea Dal Corso at the address: https:\/\people.sissa.it\～dalcorso\notes\units.pdf  The text is motivated by the conversion factors implemented in the QUANTUM ESPRESSO (https:\/\www.quantum-espresso.org/) but is of general interest.

6.1 A bit more on electrodynamics units

In order to dig on the different atomic units we start from two fundamental laws, the Coulomb force between electrical charges, \( F = k_c \frac{q_1 q_2}{r^2} \) and the Biot-Savart force between infinitesimal \( dl_1 \), \( dl_2 \) fragments of wires supporting current of intensity \( i_1 \) and \( i_2 \), respectively, \( F = k_b \frac{i_1 i_2}{r^2} dl_1 \cdot dl_2 \). The constants \( k_c \) and \( k_b \) are related by the equation,

\[
k_c/k_b = c^2,
\]

(59)

with \( c \) the speed of light.

We can assume this relationship as an experimental fact. All the same, it can be derived since, according to theory of relativity, magnetism is just electric interaction between moving charges (see e.g. chap. 5 in E.M. Purcell and J. Morin, Electricity and Magnetism, Cambridge University Press 2013)\footnote{We can provide a simple, rather oversimplified, approach that can help to have a taste of this. To this end we consider \( S_0 \), a lab frame of reference attached to the positive cores of a wire supporting a current density \( i_1 \), carried by negative electrons. In this inertial frame we observe a neutral wire, i.e., the linear charge densities of cores \( \rho_+ \) and electrons \( \rho_- \) are equal: \( \rho_+ = \rho_- \). We employ the notation \( L_0^+ \) for the observed proper length between consecutive cores and \( L^- \) for the observed length between}. Once we have
settled eq. (59), there are several options to define these constants, yielding the different unit systems. The cgs Gaussian unit system assumes \( k_c = 1 \) and, according to \( (59) \), \( k_b = 1/c^2 \). The same does the atomic unit system in the so-called Gaussian convention. However, while \( k_c \) has the same numerical value in both unit systems, taking into account that \( c = 2.89 \cdot 10^{10} \text{ cm/s} \) and \( c = 137.036 \text{ a.u.} \), \( k_b \) has a different numerical value in either system.

On the other hand, the international SI or MKS unit system assumes \( k_c = \frac{1}{4\pi\epsilon_0} \) and \( k_b = \frac{\mu_0}{4\pi} \) with \( \epsilon_0 = 8.854 \cdot 10^{-12} \text{ C}^2/\text{N} \cdot \text{m}^2 \), \( \mu_0 = 4\pi \cdot 10^{-7} \text{ N/A}^2 \). Again, \( k_c \) and \( k_b \) have the same numerical value in both unit systems.

Another challenge is the transformation of electromagnetic equations between both unit systems. To this end we first invoke the Coulomb equation:

\[
\frac{q_1^G q_2^G}{r^2} \quad \text{v.s.} \quad \frac{1}{4\pi\epsilon_0} \frac{q_1^{SI} q_2^{SI}}{r^2} = \frac{q_1^{G} q_2^{G}}{\sqrt{4\pi\epsilon_0} \sqrt{4\pi\epsilon_0}} \frac{q_1^{SI} q_2^{SI}}{r^2}
\]

where \( r \) must be expressed either, in cm (Gaussian formula) or in m (SI formula). Then, we see that

\[
q^{SI} = \sqrt{4\pi\epsilon_0} q^{G}.
\]  

(66)

consecutive electrons (remember: the proper length \( L_0 \) is the largest, so that the length \( L \) observed from another inertial frame \( S \), moving at the speed \( v \), where the system is not seen at rest, is related to \( L_0 \) by \( L = L_0 \sqrt{1 - v^2/c^2} \)). We can write \( \rho_+ = \frac{e}{L_0} \) and \( \rho_- = -\frac{e}{L_0} \), with \( e \) representing the absolute value of the electron charge. From \( \rho_+ = -\rho_- \) we conclude

\[
L_0^+ = L^-
\]  

(60)

Next, we consider a second inertial frame \( S' \) attached to electrons. Then, relative to \( S_0 \), it has a speed \( v \). In this new frame the distance between consecutive moving cores is \( L^+ = L_0 \sqrt{1 - v^2/c^2} \) and that of electrons \( L'^+ \). Interestingly, from this inertial frame we see a net density charge in the wire:

\[
\rho' = \frac{e}{L^+} = \frac{e}{L_0^+} \frac{1}{\sqrt{1 - v^2/c^2}} - \frac{e}{L^+} \sqrt{1 - v^2/c^2}
\]  

(61)

With eq. (60) we have:

\[
\rho' = \frac{e}{L_0^+} \frac{1}{\sqrt{1 - v^2/c^2}} (1 - 1 + \frac{v^2}{c^2}) = \frac{e}{L^+} \frac{v^2}{c^2}
\]  

(62)

Since the density of cores observed in \( S' \) is \( \rho = \frac{e}{L^+} \), we can write:

\[
\rho' = \frac{\rho^2}{c^2}
\]  

(63)

The cores are observed to move in \( S' \) at speed \( v \) then, the current intensity observed is \( i = \frac{dq}{dt} = \rho \frac{dl}{dt} = \rho v \), so that \( \rho' = i \frac{v}{c} \).

A wire-length \( dl \) with a non-zero density charge \( \rho' \) yields an electrical field,

\[
E = \frac{k_c}{r^2} \rho' dl = \frac{k_c}{r^2} i dl \frac{v}{c}.
\]  

(64)

Let’s write \( i_1 \) and \( dl_1 \) instead of \( i \) and \( dl \) to indicate that they correspond to a given wire that we refer to as wire 1.

Consider next that \( S' \) (attached to electrons of wire 1) sees a net charge \( dq_2 = \rho_2 dl_2 \) moving along wire 2 at speed \( v \). Then \( S' \) sees a current intensity \( i_2 = \rho_2 v \) in wire 2 and exert on it a force:

\[
F = E dq_2 = \frac{k_c}{r^2} i_1 dl_1 \frac{v}{c} \rho_2 dl_2 = \frac{k_c}{r^2} \frac{i_1 i_2}{c^2} dl_1 dl_2,
\]  

(65)

that, if compared to the Biot-Savart force, \( F = k_b \frac{i_1 i_2}{r^2} dl_1 dl_2 \), lead us to say: \( \frac{k_c}{r^2} = k_b \).
We consider next the electric field:

\[
E^{SI} = \frac{1}{4\pi\varepsilon_0} \frac{q^{SI}}{r^2} = \frac{1}{\sqrt{4\pi\varepsilon_0}} \frac{q^G}{r^2} = \frac{1}{\sqrt{4\pi\varepsilon_0}} E^G
\]  

(67)

In a similar way, we derive the potential:

\[
V^{SI} = \frac{V^G}{\sqrt{4\pi\varepsilon_0}}
\]  

(68)

In dielectric media, the Coulomb law turns into

\[
F = \frac{1}{4\pi\varepsilon_0} \frac{q^{SI} q^{SI}}{r^2} = \frac{1}{\varepsilon} \frac{q^G q^G}{r^2}. \quad \text{Then,}
\]

\[
\varepsilon^{SI} = \varepsilon_0 \varepsilon^G = \varepsilon_0 \varepsilon' \quad \text{i.e.} \quad \varepsilon^G = \varepsilon'
\]  

(69)

Finally,

\[
D^{SI} = \varepsilon^{SI} E^{SI} = \varepsilon_0 \varepsilon^G \frac{1}{\sqrt{4\pi\varepsilon_0}} E^G = \sqrt{\frac{\varepsilon_0}{4\pi}} D^G.
\]  

(70)

In the magnetic equations we have \(\frac{\mu_0}{4\pi}\) instead of \(\frac{1}{4\pi\varepsilon_0}\) and poles instead of charges. Then, proceeding in a similar way, we find out:

\[
p^{SI} = \sqrt{\frac{4\pi}{\mu_0}} p^G
\]  

(71)

\[
B^{SI} = \sqrt{\frac{\mu_0}{4\pi}} B^G
\]  

(72)

\[
U^{SI} = \sqrt{\frac{\mu_0}{4\pi}} U^G
\]  

(73)

\[
\mu^{SI} = \mu_0 \mu^G
\]  

(74)

\[
H^{SI} = \frac{1}{\sqrt{4\pi\mu_0}} H^G
\]  

(75)

A complete table with Gaussian-SI equivalences and constant numerical values is reported by wikipedia (https://en.wikipedia.org/wiki/Gaussian_units).

With these equation we can translate eqs. in section 3.2 from Gaussian (or atomic unit in Gaussian convention) to SI units. I enclose, next, some examples.

1. Equation (21):

   \[
   U^G(x, z > 0) = \frac{p^G}{\sqrt{x^2 + (z+b)^2}} \rightarrow U^{SI} = \sqrt{\frac{\mu_0}{4\pi}} U^G = \sqrt{\frac{\mu_0}{4\pi}} \sqrt{\frac{\mu_0}{4\pi}} p^{SI}_{11} = \frac{\mu_0}{4\pi} p^{SI}_{11} \]

2. Boundary condition \(D^{SI}_{1\perp} = D^{G}_{1\perp} - \alpha B^{G}_{2\perp}\) yields \(D^{SI}_{1\perp} = D^{SI}_{2\perp} - \frac{\alpha}{\mu_0} \alpha B^{SI}_{2\perp}\)

   i.e. \(\epsilon^{SI}_{1\perp} E^{SI}_{1\perp} = \epsilon^{SI}_{2\perp} E^{SI}_{2\perp} - \frac{\alpha}{\mu_0} \alpha B^{SI}_{2\perp}\).

3. Boundary condition \(H^{SI}_{1||} = H^{G}_{2||} + \alpha E^{G}_{2||}\) yields \(H^{SI}_{1||} = H^{SI}_{2||} + \frac{\alpha}{\mu_0} \alpha E^{SI}_{2||}\)

   i.e. \(\frac{1}{\mu_0^{SI}} B^{SI}_{1||} = \frac{1}{\mu_0^{G}} B^{G}_{2||} + \frac{\alpha}{\mu_0} \alpha E^{SI}_{2||}\).

4. Equation (25): \(p^{G}_1 = \frac{\mu_0}{\mu_1^{G}} \frac{\alpha}{\mu_0^{G}} \left( \frac{Q^{G}_1}{\epsilon_1} + q^G \right)\) with \(p^{SI}_1 = \sqrt{\frac{4\pi}{\mu_0}} \frac{Q^{SI}_1}{\epsilon_1} + \frac{1}{\sqrt{4\pi\varepsilon_0}} q^{SI} = \sqrt{\frac{\varepsilon_0}{\mu_0}} \frac{\alpha}{\mu_0} \frac{\alpha}{\mu_0} \left( \frac{Q^{SI}_1}{\epsilon_1} + q^{SI} \right)\) yields,

\[
p^{SI}_1 = \sqrt{\frac{4\pi}{\mu_0}} \frac{\alpha}{\mu_1^{G}} \frac{\alpha}{\mu_0^{G}} \left( \frac{Q^{SI}_1}{\epsilon_1} + \frac{1}{\sqrt{4\pi\varepsilon_0}} q^{SI} \right) = \sqrt{\frac{\varepsilon_0}{\mu_0}} \frac{\alpha}{\mu_0} \frac{\alpha}{\mu_0} \left( \frac{Q^{SI}_1}{\epsilon_1} + q^{SI} \right)
\]
5. Equation (26): \( q^G = \frac{1}{\sqrt{4\pi\epsilon_0}} q^{SI} = \frac{Q^G}{\epsilon_1'} \left( \frac{1}{\mu_1'} + \frac{1}{\mu_2'} \right) (\epsilon_1'^{-\alpha} - \epsilon_2'^{-\alpha}) - \alpha^2 \) yields, 
\[
q^{SI} = \epsilon_0 Q^{SI} \left( \frac{1}{\mu_1'} + \frac{1}{\mu_2'} \right) (\epsilon_1'^{-\alpha} - \epsilon_2'^{-\alpha}) - \frac{\alpha^2}{\epsilon_0} \]

6. Finally, Equation (27): \( p^G_1 = \alpha \frac{2Q^G}{\epsilon_1' + \epsilon_2'} (\epsilon_1'^{-\alpha} + \epsilon_2'^{-\alpha}) + \alpha \) yields, 
\[
p^{SI}_1 = \frac{1}{\mu_0} \left( \sqrt{\frac{\alpha}{\mu_0}} \right) \left( \frac{2Q^{SI}}{\epsilon_1' + \epsilon_2'} (\epsilon_1'^{-\alpha} + \epsilon_2'^{-\alpha}) + \alpha \right)
\]

As far as the dimensionless fine-structure constant \( \alpha \), it has the same numerically value in all unit systems \( (\alpha = 1/137.036) \), but presents different formulas in different systems. Thus, \( \alpha^{SI} = \frac{\epsilon^2}{4\pi\epsilon_0\hbar c} = 7.297 \cdot 10^{-3} \). Since \( \epsilon_0^G = \frac{1}{4\pi} \), then \( \alpha^G = \frac{\epsilon^2}{\hbar c} = 7.297 \cdot 10^{-3} \). Finally, from \( e = h = 1 \) a.u., \( \alpha^{a.u.} = \frac{1}{c} = 7.297 \cdot 10^{-3} \).