A class of multivariate polynomial convolutions (and applications)

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Abstract

We prove two “master” convolution theorems for multivariate determinantal polynomials. The methods used include basic properties of what we call a “minor-orthogonal” ensemble as well as properties of the mixed discriminant of matrices. We also give applications, including a rederivation of a result of Barvinok on computing the permanent of a low rank matrix and a polynomial convolution corresponding to the unitarily invariant addition of generalized singular values.

Keywords: Polynomial convolutions, random matrices.

1 Introduction

The primary goal of this paper is to prove two “master” convolution theorems for determinantal polynomials. One unavoidable fact of general matrices is that they are two dimensional objects, containing both a width and a length. While the dimensions of a matrix tend not to appear explicitly in basic linear algebra formulas, they seem to have a more direct role in the context of random matrices.

One obvious example of this is the Wishart ensemble: let $W = XX^*$ where $X$ is an $n \times m$ random matrix with independent real Gaussian entries. Even though $W$ is, itself, an $n \times n$ matrix (no $m$ involved), the joint eigenvalue has the form

$$\mu_W(\lambda_1, \ldots, \lambda_n) \propto e^{-1/2 \sum_i \lambda_i} \prod_i \lambda_i^{(m-n-1)/2} \prod_{i < j} |\lambda_i - \lambda_j|$$

where the value of $m$ is considered to be a measure of “degrees of freedom” [5]. In this note, we will refer to the value of $m$ in this example as a local dimension (since the $m$ disappears after the product is taken) and to $n$ as a global dimension (since the product matrix still has $n$ as a dimension). Similar to the case of the Wishart ensemble, polynomial convolutions will depend on both local and global parameters, and so the aim will be to find methods for computing both in the most general case possible.

To state the results explicitly, we first introduce some notations: let $\mathcal{M}_{m,n}$ denote the collection of $m \times n$ matrices\(^1\). We will use the standard notation for multivariate polynomials: for $\alpha \in \mathbb{N}^k$,
we write
\[ x^\alpha := \prod_{i=1}^k x_i^{\alpha_i} \quad \text{and} \quad \alpha! := \prod_{i=1}^k \alpha_i! \]

Given two degree \( n \) homogeneous polynomials
\[ p(x_1, \ldots, x_k) = \sum_{\alpha \in \mathbb{N}^k} p_\alpha x^\alpha \quad \text{and} \quad q(x_1, \ldots, x_k) = \sum_{\alpha \in \mathbb{N}^k} q_\alpha x^\alpha, \]

we define the \( \ast \)-convolution of \( p \) and \( q \) to be
\[ [p \ast q](x_1, \ldots, x_k) = \frac{1}{n!} \sum_{\alpha \in \mathbb{N}^k} p_\alpha q_\alpha \alpha! x^\alpha. \quad (1) \]

We also define an operator on multivariate polynomials that operates on pairs of variables (all other variables considered fixed). For integers \( i, j, m \), we define
\[ L_m^{x,y} [x^i y^j] = \begin{cases} \frac{(m-i)(m-j)}{m!(m-i-j)!} x^i y^j & \text{for } i + j \leq m \\ 0 & \text{otherwise} \end{cases} \quad (2) \]

and extend linearly to generic multivariate polynomials. Our first major theorem shows the effect of averaging over a local dimension:

**Theorem 1 (Local).** For integers \( d, m \) and variables \( x, y \) let
- \( R \in \mathcal{M}_{m,m} \) be a uniformly distributed signed permutation matrix
- \( A_1, A_2 \in \mathcal{M}_{d,m} \) and \( B_1, B_2 \in \mathcal{M}_{m,d} \) and \( U \in \mathcal{M}_{d,d} \) be matrices that are independent from \( R \) and do not contain the variables \( x \) and \( y \) (but could contain other variables).

Then
\[ \mathbb{E}_R \{ \det [U + (x A_1 + y A_2 R)(B_1 + R^T B_2)] \} = L_m^{x,y} \{ \det [U + x A_1 B_1 + y A_2 B_2] \} \]

The second main theorem then shows the effect of averaging over a global dimension:

**Theorem 2 (Global).** For integers \( n, d \), let \( A_1, \ldots, A_n, B_1, \ldots, B_n \in \mathcal{M}_{d,d} \) and set
\[ p(x_1, \ldots, x_n) = \det \left[ \sum_i x_i A_i \right] \quad \text{and} \quad q(x_1, \ldots, x_n) = \det \left[ \sum_i x_i B_i \right]. \]

If \( Q \in \mathcal{M}_{d,d} \) is a uniformly distributed signed permutation matrix, then
\[ \mathbb{E}_Q \left\{ \det \left[ \sum_i x_i A_i Q B_i Q^T \right] \right\} = [p \ast q](x_1, \ldots, x_n) \]

\[ ^2 \text{Note that this is not the same as the Schur–Hadamard convolution, which is defined by} \]
\[ x^\alpha \ast x^\beta = \delta_{(\alpha+\beta),\eta} \]

where \( \eta \in \mathbb{N}^k \) contains the maximum degree of each variable in \( p \) and \( q \) (see [2]).
These theorems can then be used iteratively to compute more complicated convolutions (we give examples of this in Sections 6.2.4 and 7).

The paper will proceed by first reviewing some of the basic combinatorial and linear algebraic tools that we will need (Section 2.2). We will then introduce a type of random matrix ensemble which we call minor-orthogonal and prove some basic properties (Section 3). Among these properties will be the fact that a uniformly distributed signed permutation matrix is minor-orthogonal (Lemma 8) In Section 4, we will give a proof of Theorem 1 in the more general context of minor-orthogonal matrices. Unfortunately, we are not able to prove Theorem 2 in similar generality. Instead, we present a proof of Theorem 2 specific to signed permutation matrices in Section 5. We give some examples of applications (reproducing known results) in Section 6 and then give the main application (the introduction of an additive convolution for generalized singular values) in Section 7. Finally, we discuss some open problems in Section 8.

2 The tools

We start by giving the definitions and constructs that we will use.

2.1 General

For a statement $S$, we will use the Dirac delta function

$$\delta\{S\} = \begin{cases} 1 & \text{if } S \text{ is true} \\ 0 & \text{if } S \text{ is false} \end{cases}.$$  

We write $[n]$ to denote the set $\{1, \ldots, n\}$ and for a set $S$, we write $\binom{S}{k}$ to denote the collection of subsets of $S$ that have exactly $k$ elements. For example,

$$\binom{[4]}{2} = \{\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}\}.$$  

When our sets contain integers (which they always will), we will consider the set to be ordered from smallest to largest. Hence, for example, if $S$ contains the elements $\{2, 5, 3\}$, then we will write

$$S = \{s_1, s_2, s_3\} \text{ where } s_1 = 2, s_2 = 3, s_3 = 5.$$  

Now let $S = \{s_1, \ldots, s_k\} \in [n]^k$. For a set $W \in \binom{[k]}{j}$ with $j \leq k$, we will write

$$W(S) = \{s_i : i \in W\}.$$  

Lastly, for a set of integers $S$, we will write

$$\|S\|_1 = \sum_{s \in S} s$$  

and note that (as is easy to check)

$$(-1)^{\|S+T\|_1} = (-1)^{\|S\|_1 + \|T\|_1}.$$  

Example 3. For $W = \{1, 3\}$ and $S = \{2, 4, 5\}$ we have

$$W(S) = \{2, 5\} \text{ and } \|W\|_1 = 1 + 3 = 4 \text{ and } \|S\|_1 = 2 + 4 + 5 = 11.$$  

3
2.2 Matrices

Given a matrix \( A \in M_{n,n} \) and sets \( S \in \binom{[n]}{k} \) and \( T \in \binom{[n]}{k} \), we will write the \((S,T)\)-minor of \( A \) as

\[
[A]_{S,T} = \det \{a_{i,j}\}_{i \in S, j \in T}.
\]

By definition, we will set \([A]_{\emptyset,\emptyset} = 1\). There are well-known formulas for the minor of a product of matrices \([9]\) as well as the minor of a sum of matrices \([14]\):

**Theorem 4.** For integers \( m, n, p, k \) and matrices \( A \in M_{m,n} \) and \( B \in M_{n,p} \), we have

\[
[AB]_{S,T} = \sum_{U \in \binom{[n]}{k}} [A]_{S,U} [B]_{U,T}.
\]

(3)

for any sets \( S \in \binom{[m]}{k} \) and \( T \in \binom{[p]}{k} \).

**Theorem 5.** For integers \( n, k \) and matrices \( A, B \in M_{n,n} \), we have

\[
[A + B]_{S,T} = \sum_{i} \sum_{U, V \in \binom{[k]}{i}} (-1)^{|U(S) + V(T)|} [A]_{U(S), V(T)} [B]_{U(S), V(T)}
\]

(4)

for any sets \( S, T \in \binom{[n]}{k} \).

We will also make use of the **mixed discriminant**: for an integer \( n \), let \( X_1, \ldots, X_n \in M_{n,n} \). The **mixed discriminant** of these matrices is then defined as

\[
D(X_1, \ldots, X_n) = \frac{1}{n!} \frac{\partial^n}{(\partial t_1) \cdots (\partial t_n)} \det \left[ \sum_i t_i X_i \right]
\]

Note that \( \det \left[ \sum_i t_i X_i \right] \) is a degree \( n \) homogeneous polynomial, so the derivative is precisely the coefficient of \( t^\alpha \) where \( \alpha = (1, 1, 1, \ldots, 1) \).

The mixed discriminant has the following well-known properties (all following directly from the definition):

**Lemma 6.** Let \( X_1, \ldots, X_n, Y \in M_{n,n} \) and let \( \{u_i\}_{i=1}^n \), \( \{v_i\}_{i=1}^n \) be vectors of length \( n \). Then

1. \( D(aX_1 + bY, X_2, \ldots, X_n) = aD(X_1, X_2, \ldots, X_n) + bD(Y, X_2, \ldots, X_n) \) for all scalars \( a, b \)
2. \( D(X_1, X_2, \ldots, X_n) = D(X_{\pi(1)}, X_{\pi(2)}, \ldots, X_{\pi(n)}) \) for all permutations \( \pi \)
3. \( D(X_1Y, X_2Y, \ldots, X_nY) = D(YX_1, YX_2, \ldots, YX_n) = \det [Y] D(X_1, X_2, \ldots, X_n) \)
4. \( D(u_1v_1^T, u_2v_2^T, \ldots, u_nv_n^T) = \det [u_1 u_2 \ldots u_n] \det [v_1 v_2 \ldots v_n] \)

Note that properties 1. and 2. combine to show that the mixed discriminant is multilinear (that is, it is a linear function with respect to each of its inputs).

We will leave the discussion of generalized singular values to Section 7.
3 Minor-orthogonality

We will say that a random matrix $R \in \mathcal{M}_{m,n}$ is minor-orthogonal if for all integers $k, \ell \leq \min\{m, n\}$ and all sets $S, T, U, V$ with $|S| = |T| = k$ and $|U| = |V| = \ell$, we have

$$\mathbb{E}_R \{ [R]_{S,T} [R^T]_{U,V} \} = \frac{1}{(\max\{m,n\})^2} \delta_{(S=V)} \delta_{(T=U)}.$$

Given a minor-orthogonal ensemble $R$ it is easy to see from the definition that

1. $R^T$ is minor orthogonal

2. any submatrix that preserves the largest dimension of $R$ is minor orthogonal

**Lemma 7.** If $R$ is minor-orthogonal and $Q$ is a fixed matrix for which $QQ^T = I$, then $QR$ is minor-orthogonal.

**Proof.** For any sets $S, T$ with $|S| = |T| = k$, we have

$$[QR]_{S,T} = \sum_{|W|=k} [Q]_{S,W} [R]_{W,T}$$

so for $|U| = |V| = \ell$, we have

$$\mathbb{E}_R \{ [QR]_{S,T} [(QR)^T]_{U,V} \} = \mathbb{E}_R \left\{ \sum_{|W|=k} \sum_{|Z|=\ell} [Q]_{S,W} [R]_{W,T} [R^T]_{U,Z} [Q^T]_{Z,V} \right\}$$

$$= \sum_{|W|=k} \sum_{|Z|=\ell} [Q]_{S,W} [Q^T]_{Z,V} \frac{1}{(\max\{m,n\})^2} \delta_{(W=Z)} \delta_{(T=U)}$$

$$= \sum_{|W|=k} \frac{1}{(\max\{m,n\})} [Q]_{S,W} [Q^T]_{W,V} \delta_{(T=U)}$$

$$= \frac{1}{(\max\{m,n\})} \delta_{S=V} \delta_{T=U}.$$

where the last line comes from the fact that $[I]_{S,V} = \delta_{S=V}$. \hfill \Box

**Lemma 8.** The collection of $n \times n$ signed permutation matrices (under the uniform distribution) is minor-orthogonal.

**Proof.** We can write a uniformly random signed permutation matrix $Q$ as $Q = E_\chi P_\pi$ where $P_\pi$ is a uniformly random permutation matrix and $E_\chi$ is a uniformly random diagonal matrix with $\{\pm 1\}$ on the diagonal (and the two are independent). Hence for $|S| = |T| = k$ and $|U| = |V| = \ell$, we have

$$\mathbb{E}_Q \{ [Q]_{S,T} [Q^T]_{U,V} \} = \mathbb{E}_{E_\chi,P_\pi} \left\{ [E_\chi P_\pi]_{S,T} [P_\pi^T]_{U,V} E_\chi \right\}$$

$$= \sum_{|W|=k} \sum_{|Z|=\ell} \mathbb{E}_{E_\chi,P_\pi} \left\{ [E_\chi]_{S,W} [P_\pi]_{W,T} [P_\pi^T]_{U,Z} [E_\chi]_{Z,V} \right\}.$$ 

$$= \mathbb{E}_{E_\chi,P_\pi} \left\{ [E_\chi]_{S,S} [P_\pi]_{S,T} [P_\pi^T]_{U,V} [E_\chi]_{V,V} \right\}$$

$$= \mathbb{E}_{E_\chi} \left\{ \prod_{i \in S} \chi_i \prod_{j \in V} \chi_j \right\} \mathbb{E}_{P_\pi} \left\{ [P_\pi]_{S,T} [P_\pi^T]_{U,V} \right\}.$$
where the penultimate line uses the fact that a diagonal matrix $X$ satisfies $[X]_{A,B} = 0$ whenever $A \neq B$. Now the $\chi_i$ are uniformly distributed $\{\pm 1\}$ random variables, so $$\mathbb{E}_X \left\{ \prod_{i \in S} \chi_i \prod_{j \in V} \chi_j \right\} = \delta_{\{S=V\}}$$ and so we have $$\mathbb{E}_Q \left\{ [Q]_{S,T}[Q^T]_{U,V} \right\} = \mathbb{E}_\pi \left\{ [P_{\pi}]_{S,T}[P_{\pi}^T]_{U,V} \right\} \delta_{\{S=V\}}$$ $$= \mathbb{E}_\pi \left\{ [P_{\pi}]_{S,T}[P_{\pi}]_{S,U} \right\} \delta_{\{S=V\}}$$ Furthermore, $[P_{\pi}]_{S,T} = 0$ except when $T = \pi(S)$, so in order for both $[P_{\pi}]_{S,T}$ and $[P_{\pi}]_{S,U}$ to be nonzero simultaneously requires $U = T$. In the case that $U = T = \pi(S)$, $[P_{\pi}]_{S,T} = \pm 1$, and so we have $$\mathbb{E}_Q \left\{ [Q]_{S,T}[Q^T]_{U,V} \right\} = \mathbb{E}_\pi \left\{ [P_{\pi}]_{S,T}^2 \right\} \delta_{\{S=V\}} \delta_{\{T=U\}}$$ $$= \mathbb{E}_\pi \left\{ \delta_{\{\pi(S)=T\}} \right\} \delta_{\{S=V\}} \delta_{\{T=U\}}$$

But it is an easy exercise to check that the probability that a permutation length $n$ maps a set $S$ to a set $T$ with $|S| = |T| = k$ is $$\frac{k!(n-k)!}{n!} = \frac{1}{\binom{n}{k}}$$ and so for $|S| = |T| = k$, we have $$\mathbb{E}_Q \left\{ [Q]_{S,T}[Q^T]_{U,V} \right\} = \frac{1}{\binom{n}{k}} \delta_{\{S=V\}} \delta_{\{T=U\}}$$ as required.

As mentioned at the end of Section 1, Lemma 8 can be extended to show that any suitably symmetric group containing the signed permutation matrices is minor-orthogonal. An example of this is given in Corollary 9:

Corollary 9. The collection of $n \times n$ orthogonal matrices (under the Haar measure) is minor-orthogonal.

Proof. Let $R$ be a Haar distributed random orthogonal matrix. By definition, $RQ$ is also Haar distributed for any fixed orthogonal matrix $Q$, and so (in particular) this holds when $Q$ is a signed permutation matrix. Hence by Lemma 7 $$\mathbb{E}_R \left\{ [R]_{S,T}[R^T]_{U,V} \right\} = \mathbb{E}_R \left\{ [RQ]_{S,T}[(RQ)^T]_{U,V} \right\}$$ and so $$\mathbb{E}_R \left\{ [R]_{S,T}[R^T]_{U,V} \right\} = \mathbb{E}_{R,Q} \left\{ [R]_{S,T}[R^T]_{U,V} \right\} = \mathbb{E}_{R,Q} \left\{ [RQ]_{S,T}[(RQ)^T]_{U,V} \right\}$$ where we are now considering $Q$ to be drawn uniformly and independently from the collection of signed permutation matrices. By Lemma 8, $Q$ is minor-orthogonal and so, for fixed $R$, Lemma 7 implies that $RQ$ is minor-orthogonal. So $$\mathbb{E}_R \left\{ \mathbb{E}_Q \left\{ [RQ]_{S,T}[(RQ)^T]_{U,V} \right\} \right\} = \mathbb{E}_R \left\{ \frac{1}{\binom{n}{k}} \delta_{\{S=V\}} \delta_{\{T=U\}} \right\} = \frac{1}{\binom{n}{k}} \delta_{\{S=V\}} \delta_{\{T=U\}}$$ as required.\[\square\]
4 The local theorem

The goal of this section is to prove Theorem 1. In fact we will prove something more general — that Theorem 1 holds when $R$ is any minor-orthogonal ensemble. For the remainder of the section, we will assume the following setup: we are given fixed integers $d, m$ and variables $x, y$ and the following matrices:

- $R \in \mathcal{M}_{m,m}$, a uniformly distributed signed permutation matrix
- $A_1, A_2 \in \mathcal{M}_{d,m}$ and $B_1, B_2 \in \mathcal{M}_{m,d}$ and $U \in \mathcal{M}_{d,d}$ all of which are independent from $R$ and do not contain the variables $x$ and $y$ (but could contain other variables).

**Lemma 10.** Let $k, \ell \leq d$ be nonnegative integers, and let $S, T \in \binom{[d]}{k}$ and $U, V \in \binom{[d]}{\ell}$. Then

$$\mathbb{E}_R \{ [A_1 R B_1]_{S,T} [A_2 R^T B_2]_{U,V} \} = \frac{1}{\binom{m}{k}} [A_1 B_2]_{S,V} [A_2 B_1]_{U,T} \delta_{k=\ell}$$

**Proof.** By (3) we have

$$\mathbb{E}_R \{ [A_1 R B_1]_{S,T} [A_2 R^T B_2]_{U,V} \} = \frac{1}{\binom{m}{k}} \sum_{W,X} \sum_{s,w \in \binom{[m]}{k}, y,z \in \binom{[m]}{\ell}} [A_1]_{w} [B_1]_{y} [A_2]_{Z,V} \delta_{W=Z} \delta_{X=Y}$$

$$= \frac{1}{\binom{m}{k}} \sum_{W,X} [A_1]_{s,W} [B_1]_{x,T} [A_2]_{u,X} [B_2]_{W,V} \delta_{k=\ell}$$

$$= \frac{1}{\binom{m}{k}} [A_1 B_2]_{S,V} [A_2 B_1]_{U,T} \delta_{k=\ell}$$

- \square

**Lemma 11.** Let $k \leq d$ be nonnegative integers, and let $S, T \in \binom{[d]}{k}$ and consider the polynomials

$$p(x, y) = \mathbb{E}_R \{ [x A_1 R B_1 + y A_2 R^T B_2]_{S,T} \} = \sum_i p_i x^i y^{k-i}$$

and

$$q(x, y) = [x A_1 B_2 + y A_2 B_1]_{S,T} = \sum_i q_i x^i y^{k-i}.$$

Then

$$p_i = \delta_{i=k/2} \frac{(-1)^i}{\binom{m}{i}} q_i$$

**Proof.** We have by (4)

$$[x A_1 R B_1 + y A_2 R^T B_2]_{S,T} = \sum_i \sum_{W,X} (-1)^{||W+X||_1} x^{|W|} y^{|W|} [A_1 R B_1]_{W(S), X(T)} [A_2 R^T B_2]_{W(S), X(T)}$$

so by Lemma 10 we will have

$$\mathbb{E}_R \{ [x A_1 R B_1 + y A_2 R^T B_2]_{S,T} \} = 0$$
whenever $|W| \neq |W|$. To complete the lemma, it therefore remains to show that, for $k = 2t$, we have $p_t = \frac{(-1)^k}{t^3} q_t$.

Using (4) and Lemma 10 again, we have

$$p_t = \frac{1}{m} \sum_{w,x \subseteq \binom{[k]}{t}} (-1)^{|W + X|} \{A_1 B_2 | W(S), X(T) \} \{A_2 B_1 | W(S), X(T) \}$$

$$= \frac{1}{m} \sum_{w,x \subseteq \binom{[k]}{t}} (-1)^{|W + X|} \{A_1 B_2 | W(S), X(T) \} \{A_2 B_1 | W(S), X(T) \}$$

wheras

$$q_t = \sum_{w,x \subseteq \binom{[k]}{t}} (-1)^{|W + X|} \{A_1 B_2 | W(S), X(T) \} \{A_2 B_1 | W(S), X(T) \}$$

Now using the fact that

$$||W||_1 + ||W||_1 = \sum_{i=1}^{k} i = \binom{k+1}{2}$$

we get $(-1)^{|W + X|} = (-1)^{|W + X|} (-1)^{\binom{k+1}{2}}$ and so (5) and (6) combine to give

$$p_t = \frac{(-1)^{\binom{k+1}{2}}}{m} q_t.$$ 

Hence it remains to show $(-1)^{\binom{k+1}{2}} = (-1)^t$. However, (since $k = 2t$) we have

$$(-1)^{\binom{k+1}{2}} = (-1)^{t(2t+1)} = (-1)^{2t^2 + t} = (-1)^t$$

finishing the proof.

**Corollary 12.** For any sets $S, T$ with $|S| = |T|$, 

$$E_R \{ [xB_1 + yA_2 R B_2]_{S,T} \} = \sum_{i} (-1)^i \frac{(m-i)!}{m!} (\partial_{x}^i (\partial_{y})^i [w x A_1 B_1 + y z A_2 B_2]_{S,T})_{w=z=0}$$

**Corollary 13.** For any sets $S, T$ with $|S| = |T|$, 

$$E_R \{ [(x A_1 + y A_2 R) (B_1 + R^T B_2)]_{S,T} \} = \sum_{i} (-1)^i \frac{(m-i)!}{m!} (\partial_{x}^i (\partial_{y})^i [w x A_1 B_1 + y z A_2 B_2]_{S,T})_{w=z=1}$$

We are now in a position to prove Theorem 1:

**Proof of Theorem 1.** By (4), it suffices to show

$$E_R \{ [(x A_1 + y A_2 R) (B_1 + R^T B_2)]_{S,T} \} = L_{m,y}^{x} \{ [x A_1 B_1 + y A_2 B_2]_{S,T} \}$$

for all $|S| = |T| = k$. For $k > m$, both sides of (7) are 0, so we restrict to the case $k \leq m$. Using Corollary 13, (7) is equivalent to showing

$$L_{m,y}^{x} \{ [x A_1 B_1 + y A_2 B_2]_{S,T} \} = \sum_{i} (-1)^i \frac{(m-i)!}{m!} (\partial_{x}^i (\partial_{y})^i [w x A_1 B_2 + y z A_2 B_1]_{S,T})_{w=z=1}.$$
Using (4) again, we have
\[ L^{x,y}_m \{ [x A_1 B_1 + y A_2 B_2]_{S,T} \} = \sum_{i} \sum_{w,x \subseteq \{i\}^k} (-1)^{|W+X|} L^{x,y}_m \{ x^i y^{k-i} \} [A_1 B_1]_{W(S),X(T)} [A_2 B_2]_{W(S),X(T)} \]
so the coefficient of \( x^i y^{k-i} \) is
\[
\sum_{w,x \subseteq \{i\}^k} (-1)^{|W+X|} \frac{(m-i)! (m-k+i)!}{m! (m-k)!} [A_1 B_1]_{W(S),X(T)} [A_2 B_2]_{W(S),X(T)}.
\]
On the other hand, we have
\[
\sum_j (-1)^j \frac{(m-j)!}{m! j!} (\partial_w)^i (\partial_z)^j [wx A_1 B_1 + yz A_2 B_2]_{S,T} \bigg|_{u=z=1}
\]
\[= \sum_{i,j} (-1)^j \frac{(m-j)!}{m! j!} (\partial_w)^i (\partial_z)^j \sum_{w,x \subseteq \{i\}^k} (-1)^{|W+X|} (wx)^i (yz)^j [A_1 B_1]_{W(S),X(T)} [A_2 B_2]_{W(S),X(T)}
\]
\[= \sum_{i,j} (-1)^j \frac{(m-j)!}{m! j!} \sum_{w,x \subseteq \{i\}^k} (-1)^{|W+X|} \frac{i!}{(i-j)!} \frac{(k-i)!}{(k-i-j)!} x^i y^{k-i} [A_1 B_1]_{W(S),X(T)} [A_2 B_2]_{W(S),X(T)}
\]
so the coefficient of \( x^i y^{k-i} \) is
\[
\sum_{w,x \subseteq \{i\}^k} (-1)^{|W+X|} [A_1 B_1]_{W(S),X(T)} [A_2 B_2]_{W(S),X(T)} \sum_j (-1)^j \frac{(m-j)!}{m! j!} \frac{i!}{(i-j)!} \frac{(k-i)!}{(k-i-j)!}
\]
So it suffices to show
\[
\sum_j (-1)^j \frac{(m-j)!}{m! j!} \frac{i!}{(i-j)!} \frac{(k-i)!}{(k-i-j)!} = \frac{(m-i)! (m-k+i)!}{m! (m-k)!}
\]
or, after substituting \( i \leftarrow a \) and \( b \leftarrow k-i \), to show
\[
\sum_j (-1)^j \frac{(m-j)!}{m! j!} \frac{a!}{(a-j)!} \frac{b!}{(b-j)!} = \frac{(m-a)! (m-b)!}{m! (m-a-b)!}
\]
whenever \( a+b \leq m \) (our assumption). However this follows directly from standard theorems on generalized binomials:
\[
\sum_i (-1)^i \frac{(m-i)!}{m! i!} \frac{a!}{(a-i)!} \frac{b!}{(b-i)!} = \frac{b! (m-b)!}{m!} \sum_i (-1)^i \frac{(a)!}{i!} \frac{(m-i)!}{(b-i)!}
\]
\[= \frac{b! (m-b)!}{m!} (1)^b \sum_i (-1)^i \frac{(a)!}{i!} \frac{(-(m-b+1))}{b-i}
\]
\[= \frac{b! (m-b)!}{m!} (1)^b \frac{(-(m-b-a+1))}{b}
\]
\[= \frac{b! (m-b)!}{m!} \frac{(m-a)}{b}
\]
\[= \frac{(m-a)! (m-b)!}{(m-a-b)! m!}
\]
as required. \( \square \)
5 The global theorem

The goal of this section is to prove Theorem 2. Using the multilinearity of the mixed discriminant, one can show that Theorem 2 is equivalent to the following theorem (note that \( n! = D(I, \ldots, I) \) and so this has the familiar form of zonal spherical polynomials — see [10]).

**Theorem 14.** Let \( A_1, \ldots, A_n, B_1, \ldots, B_n, Q \in \mathcal{M}_{n,n} \) where \( Q \) is a uniformly distributed signed permutation matrix. Then

\[
E_Q \left\{ D(A_1 QB_1 Q^T, A_2 QB_2 Q^T, \ldots, A_n QB_n Q^T) \right\} = \frac{1}{n!} D(A_1, A_2, \ldots, A_n) D(B_1, B_2, \ldots, B_n). \tag{8}
\]

**Proof.** We start by noticing that, by the multilinearity of the mixed discriminant, it suffices to prove the theorem when the \( A_i, B_i \) are the basis elements \( \{e_i e_j^T\}_{i,j=1}^n \). So let \( A_i = e_{w_i} e_{x_i}^T \) and \( B_i = e_{y_i} e_{z_i}^T \) where \( w_i, x_i, y_i, z_i \in [n] \). Then for each \( Q \) we have

\[
D(A_1 QB_1 Q^T, \ldots, A_n QB_n Q^T) = det \left[ Q \right] D(e_{w_1} e_{x_1}^T, e_{y_1} e_{z_1}^T, \ldots, e_{w_n} e_{x_n}^T, e_{y_n} e_{z_n}^T)
\]

where each \( e_{x_i}^T Q e_{y_i} \) is a scalar (and so can be factored out). Hence

\[
D(A_1 QB_1 Q^T, \ldots, A_n QB_n Q^T) = det \left[ Q \right] \left( \prod_i e_{x_i}^T Q e_{y_i} \right) D(e_{w_1} e_{x_1}^T, \ldots, e_{w_n} e_{x_n}^T)
\]

\[= det \left[ Q \right] \left( \prod_i e_{x_i}^T Q e_{y_i} \right) det [e_{w_1} \ldots e_{w_n}] det [e_{x_1} \ldots e_{x_n}]
\]

On the other hand,

\[D(A_1, A_2, \ldots, A_n) = det [e_{w_1} \ldots e_{w_n}] det [e_{x_1} \ldots e_{x_n}]
\]

and similarly for \( B \), so we find that (8) is equivalent to showing

\[
E_Q \left\{ det \left[ Q \right] \prod_i e_{x_i}^T Q e_{y_i} \right\} = \frac{1}{n!} det [e_{x_1} \ldots e_{x_n}] det [e_{y_1} \ldots e_{y_n}]. \tag{9}
\]

We now decompose\(^3\) each \( Q \) as \( Q = P_\pi E_\chi \) where \( P_\pi \) is a permutation matrix and \( E_\chi \) is a diagonal matrix with diagonal entries \( \chi_1, \ldots, \chi_n \in \{\pm 1\} \). Hence \( det \left[ Q \right] = (\prod_i \chi_i) det [P_\pi] \) and

\[
e_{x_i}^T Q e_{y_i} = e_{x_i}^T P_\pi E_\chi e_{y_i} = \chi_y e_{x_i} P_\pi e_{y_i} = \chi_y \delta_{\{x_i = \pi(y_i)\}}
\]

and so

\[
E_Q \left\{ det \left[ Q \right] \prod_i e_{x_i}^T Q e_{y_i} \right\} = \frac{1}{n!} \sum_\pi det [P_\pi] \left( \prod_i \delta_{\{x_i = \pi(y_i)\}} \right) E_{\chi_1, \ldots, \chi_n} \left\{ \prod_i \chi_i \chi_y \right\}
\]

Now it is easy to see that

\[
E_{\chi_1, \ldots, \chi_n} \left\{ \prod_i \chi_i \chi_y \right\} = 1
\]

\(^3\)This is where we lose the generality of minor-orthogonal ensembles.
whenever the \( y_i \) are distinct (that is, form a permutation of \([n]\)) and 0 otherwise. For distinct \( y_i \), it should then be clear that
\[
\left( \prod_i \delta_{\{x_i = \pi(y_i)\}} \right) = 0
\]
unless the \( x_i \) are also distinct. Of course, this also holds for \( \det [e_{x_1} \ldots e_{x_n}] \) and \( \det [e_{y_1} \ldots e_{y_n}] \) and so (9) is true whenever the \( x_i \) or \( y_i \) are not distinct (as both sides are 0).

Thus it remains to consider the case when \( x_i = \sigma(i) \) for some \( \sigma \) and \( y_i = \tau(i) \) for some \( \tau \). That is, we must show
\[
\sum_{\pi} \det [P_{\pi}] \left( \prod_i \delta_{\{\sigma(i) = \pi(\tau(i))\}} \right) = \det [P_{\sigma}] \det [P_{\tau}]
\]
for all permutations \( \tau \) and \( \sigma \). But now it is easy to see that \( \left( \prod_i \delta_{\{\sigma(i) = \pi(\tau(i))\}} \right) = 0 \) for all permutations \( \pi \) except for one: \( \pi = \sigma \circ \tau^{-1} \). Hence
\[
\sum_{\pi} \det [P_{\pi}] \left( \prod_i \delta_{\{\sigma(i) = \pi(\tau(i))\}} \right) = \det [P_{\sigma \circ \tau^{-1}}] = \det [P_{\sigma}] \det [P_{\tau^{-1}}]
\]
where
\[
\det [P_{\tau^{-1}}] = \det [P_{\tau}^{-1}] = \det [P_{\tau}] = \det [P_{\tau}^T],
\]
proving (10), which in turn proves the remaining (nonzero) cases of (9), and therefore the theorem. \( \square \)

6 Applications

In this section, we list some direct applications of the main theorems.

6.1 Permanents of Low Rank Matrices

Our first application is an algorithm for computing permanents of low rank matrices that was originally discovered by Barvinok [1] using similar tools. Barvinok’s algorithm takes advantage of a well known connection between permanents and mixed discriminants: the permanent of a matrix \( M \in \mathcal{M}_{n,n} \) is the mixed discriminant \( D(A_1, \ldots, A_n) \) where each \( A_i \) is a diagonal matrix with diagonal matching the \( i \)th column of \( M \).

When working with diagonal matrices, the signed permutation matrices behave in a particularly nice way: the \( \pm 1 \) entries in \( Q \) and \( Q^T \) cancel, so one can reduce such formulas to an average over (unsigned) permutation matrices.

Corollary 15. Let \( \{A_i\}_{i=1}^k \) and \( \{B_i\}_{i=1}^k \) be \( n \times n \) diagonal matrices and consider the polynomials
\[
p(x_1, \ldots, x_k) = \det \left[ \sum_i x_i A_i \right] \quad \text{and} \quad q(x_1, \ldots, x_k) = \det \left[ \sum_i x_i B_i \right]
\]
Then
\[
\frac{1}{n!} \sum_{P \in \mathcal{P}_n} \det \left[ \sum_i x_i A_i P B_i P^T \right] = [p \ast q](x_1, \ldots, x_k)
\]
Given a vector \( v \), let \( \text{diag}(v) \) denote the diagonal matrix whose diagonal entries are \( v \). Note that if \( A_i = \text{diag}(a_i) \) and \( B_i = \text{diag}(b_i) \) for each \( i \), then

\[
\sum_{P \in \mathcal{P}_n} \det \left[ \sum_i x_i A_i P B_i P^T \right] = \text{perm} \left[ \sum_i x_i a_i b_i^T \right].
\]

**Algorithm to find** \( \text{perm} \left[ \sum_{i=1}^k a_i b_i^T \right] \) for \( a_i, b_i \in \mathbb{R}^n \).

1. Form \( A_i = \text{diag}(a_i) \) and \( B_i = \text{diag}(b_i) \).
2. Compute \( p(x_1, \ldots, x_k) = \det \left[ \sum_{i=1}^k x_i A_i \right] \) and \( q(x_1, \ldots, x_k) = \det \left[ \sum_{i=1}^k x_i B_i \right] \)
3. Compute \( [p \ast q](x_1, \ldots, x_k) \)
4. \( \text{perm} \left[ \sum_i a_i b_i^T \right] = n! [p \ast q](1, \ldots, 1) \).

The complexity of this algorithm depends primarily on the number of terms in the polynomials \( p \) and \( q \), which (in general) will be the number of nonnegative integer solutions to the equation \( \sum_{i=1}^k t_i = n \), which is known to be \( \binom{n+k-1}{k-1} \) (see [17]).

### 6.2 Other convolutions

In this section, we show that the standard univariate convolutions defined in [13] can each be derived from the main theorems.

#### 6.2.1 Additive convolution of eigenvalues

Given matrices \( A, B \in M_{d,d} \) and polynomials

\[
p(x) = \det [xI - A] \quad \text{and} \quad q(x) = \det [xI - B]
\]

the *additive convolution* of \( p \) and \( q \) can be written as

\[
[p \boxplus q](x) = \mathbb{E}_Q \left\{ \det \left[ xI - A - QBQ^T \right] \right\}
\]

where \( Q \) can be chosen to be any minor-orthogonal ensemble (see [11]). This can be achieved by setting

\[
\hat{p}(x, y, z) = \det [xI + yA + zI] \quad \text{and} \quad \hat{q}(x, y, z) = \det [xI + yI + zB]
\]

and applying Theorem 2 to get

\[
\hat{p} \ast \hat{q} = \mathbb{E}_R \left\{ \det \left[ xI + yA + zRBR^T \right] \right\}
\]

The formula for \([p \boxplus q]\) follows by setting \( y = z = -1 \).
6.2.2 Multiplicative convolution of eigenvalues

Given matrices $A, B \in \mathcal{M}_{d,d}$ and polynomials $p(x) = \det [xI - A]$ and $q(x) = \det [xI - B]$

the *multiplicative convolution* of $p$ and $q$ can be written as

$$ [p \boxtimes q](x) = \mathbb{E}_Q \{ \det [xI - AQBQ^T] \} $$

where $Q$ can be chosen to be any minor-orthogonal ensemble (see [11]). Given matrices $A$ and $B$, this can be achieved by setting

$$ \hat{p}(x, y) = \det [xI + yA] \quad \text{and} \quad \hat{q}(x, y) = \det [xI + yB] $$

and applying Theorem 2 to get

$$ \hat{p} \star \hat{q} = \mathbb{E}_Q \{ \det [xI + yAQBQ^T] \}. $$

The formula for $[p \boxtimes q]$ follows by setting $y = -1$.

6.2.3 Additive convolution of singular values

Given matrices $A, B \in \mathcal{M}_{d,n+d}$ and polynomials $p(x) = \det [xI - AA^T]$ and $q(x) = \det [xI - BB^T]$

the *rectangular additive convolution* of $p$ and $q$ can be written as

$$ [p \boxplus q](x) = \mathbb{E}_{Q,R} \{ \det [xI - (A + QBR)(A + QBR)^T] \} $$

where $Q$ and $R$ can be chosen to be any (independent) minor-orthogonal ensembles of the appropriate size (see [11]). This can be achieved by setting

$$ \hat{r}(x, y, z) = \det [xI + (yA + zQBR)(A^T + R^TB^TQ^T)]. $$

Assuming $Q$ and $R$ are independent, we can do the expectation in $R$ using Theorem 1 to get

$$ \mathbb{E}_R \{ \hat{r}(x, y, z) \} = L_{m}^{y,z} \{ \det [xI + yAA^T + zQBB^TQ^T] \}. $$

and then we can compute the remaining expectation

$$ \mathbb{E}_Q \{ \det [xI + yAA^T + zQBB^TQ^T] \} $$

in terms of $p$ and $q$ using the method in Section 6.2.1.
6.2.4 Multiplicative convolution of non-Hermitian eigenvalues

For the purpose of studying the eigenvalues of non-Hermitian matrices, one could use the polynomial convolutions in the previous sections, but one quickly realizes that they do not hold as much information as one would like. This is due in part to the fact that, unlike in the Hermitian case, there can be nontrivial relations between the left eigenvectors and right eigenvectors of a non-Hermitian matrix (we refer the interested reader to [15] where a multivariate theory is developed). However it is well known that a non-Hermitian matrix $A$ can be written as $A = H + K$ where

$$H = \frac{A + A^*}{2} \quad \text{and} \quad K = i\frac{A - A^*}{2}$$

are both Hermitian. One can then consider the multivariate polynomial

$$p(x, y, z) = \det [xI + yH + zK]$$

for which an additive convolution follows easily from the Hermitian version in Section 6.2.1. The multiplicative version, however, is more complicated. Given pairs of Hermitian matrices $(H_1, K_1)$ and $(H_2, K_2)$, one would like to “convolve” these matrices in a way that preserves the dichotomy between real and imaginary parts. One such possibility would be the polynomial

$$r(x, y, z) = E_Q \{ \det [xI + yH_1QH_2Q^* - K_1QK_2Q^* + z(H_1QK_2Q^* + K_1QH_2Q^*)] \}$$

but it is not clear (a priori) that the coefficients of this polynomial are functions of the coefficients of the polynomials

$$p_1(x) = \det [xI + yH_1 + zK_1] \quad \text{and} \quad p_2(x) = \det [xI + yH_2 + zK_2].$$

However it is easy to compute $r(x, y, z)$ using Theorem 2. Letting

$$q_1(x, a, b, c, d) = \det [xI + aH_1 + bH_1 + cK_1 + dK_1]$$

$$q_2(x, a, b, c, d) = \det [xI + aH_2 + bK_2 + cH_2 + dK_2]$$

we have that

$$[q_1 \ast q_2](x, a, b, c, d) = E_Q \{ \det [xI + aH_1QH_2Q^* + bH_1QK_2Q^* + cK_1QH_2Q^* + dK_1QK_2Q^*] \}$$

and so $r(x, y, z) = [q_1 \ast q_2](x, y, -y, z, z)$.

7 An additive convolution for generalized singular values

There are three standard ensembles that one studies in random matrix theory: the Wigner ensemble, Wishart ensemble, and Jacobi ensemble [5]. All are alike in that they can be derived from matrices with independent Gaussian entries; the difference between them, as was first noted by Edelman [4], can be paralleled to different matrix decompositions. The Wigner ensemble is Hermitian and the relevant distribution is the eigenvalues distribution. The Wishart ensemble is often thought of as a Hermitian ensemble (with an eigenvalue distribution) but in some sense the more natural way to view it is as a distribution on singular values (which, as the first step in calculate them, you form a
Hermitian matrix). The Jacobi ensemble, in this ansatz, is most naturally viewed as a distribution on “generalized singular values.”

One can attempt to explore this trichotomy further by studying how the eigenvalues/singular values/generalized singular values of matrices behave with respect to more general matrix operations (and more general random matrices). This is one of the motivations behind the polynomial convolutions mentioned in Section 6: the convolution in Section 6.2.1 computes statistics concerning the eigenvalues of a unitarily invariant sum, whereas the convolution in Section 6.2.3 does similarly in the case of singular values. The purpose of this section is to introduce a polynomial that can be used to study the final case: a unitarily invariant addition of generalized singular values. While the previous two could be accomplished using univariate convolutions, it will become clear that this is not possible for the general singular value decomposition. For those interested in other aspects of the GSVD should consult the references [6, 18].

Before jumping into a discussion regarding the generalized singular value decomposition, it will be useful for us to recall the definition of the pseudo-inverse of a matrix. Given any matrix $X \in \mathcal{M}_{m,n}$ with rank $r$, the normal singular value decomposition of matrices allows us to write $X = U \Sigma V^T$ where

- $U \in \mathcal{M}_{m,r}$ satisfies $U^TU = I$
- $V \in \mathcal{M}_{n,r}$ satisfies $V^TV = I$
- $\Sigma \in \mathcal{M}_{r,r}$ is diagonal and invertible.

The pseudo-inverse of $X$ (written $X^\dagger \in \mathcal{M}_{n,m}$) is then defined to be $X^\dagger = V \Sigma^{-1} U^T$. The name “pseudo-inverse” comes from the fact that

- $XX^\dagger \in \mathcal{M}_{m,m}$ is the projection onto the column space of $V$, and
- $X^\dagger X \in \mathcal{M}_{n,n}$ is the projection onto the column space of $U$.

So, in particular, if $n = r$ then $X^\dagger X = I_n$ and if $m = n = r$ then $X$ is invertible and $X^\dagger = X^{-1}$.

Now fix integers $n_1, n_2, m$ and let $M \in \mathcal{M}_{(n_1 + n_2),m}$ have rank $r$ and block structure

$$M = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}$$

The generalized singular value decomposition (GSVD) provides a decomposition of $M_1$ and $M_2$ as

$$M_1 = U_1 CH \quad \text{and} \quad M_2 = U_2 SH$$

where

- $U_1 \in \mathcal{M}_{n_1,r}$ and $U_2 \in \mathcal{M}_{n_2,r}$ satisfy $U_1^TU_1 = U_2^TU_2 = I_r$
- $C, S \in \mathcal{M}_{r,r}$ are positive semidefinite diagonal matrices with $C^TC + S^TS = I$, and
- $H \in \mathcal{M}_{r,m}$ is some matrix with rank $r$. 

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In particular, the diagonal entries of $C$ and $S$ satisfy $c_i^2 + s_i^2 = 1$, and as such, the matrices $C$ and $S$ are often referred to as cosine and sine matrices. Note that when $M_2$ has rank $r$, the matrix $S$ will be invertible and then

$$M_1 M_2^\dagger = (U_1 C H)(U_2 S H)^\dagger = U_1 C S^{-1} U_2$$

will be the (usual) SVD of $M_1 M_2^\dagger$, the reason for the nomenclature “generalized” SVD.

When $M$ has rank $m$, there is an easy way to find the generalized singular values without needing to form the entire decomposition. Letting $W_1 = M_1^T M_1$ and $W_2 = M_2^T M_2$, the GSVD implies that

$$W = (W_1 + W_2)^{-1/2} W_1 (W_1 + W_2)^{-1/2}$$

is a positive semidefinite Hermitian matrix which is unitarily similar to $C^T C$ (all of whose eigenvalues are in the interval $[0, 1]$). Thus generalized singular values can be found directly from the characteristic polynomial

$$\det \left[ x I - (W_1 + W_2)^{-1/2} W_1 (W_1 + W_2)^{-1/2} \right] = \det \left[ (W_1 + W_2)^{-1} \right] \det \left[ (x - 1) W_1 + x W_2 \right]$$

So now assume we are given $M, N \in \mathcal{M}(n_1 + n_2, m)$ with block structure

$$M = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} N_1 \\ N_2 \end{bmatrix}$$

and we form the random matrix

$$P = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} M_1 + R_1 N_1 Q \\ N_2 + R_2 N_2 Q \end{bmatrix}$$

where $R_1, R_2, Q$ are independent signed permutation matrices of the appropriate sizes. Then the natural question is: what (if anything) can we say about the generalized singular values of $P$ given the generalized singular values of $M$ and $N$.

By what we observed in (12), this means finding a correspondence between the polynomials

$$\det \left[ (x - 1) M_1 + x M_2 \right] \quad \text{and} \quad \det \left[ (x - 1) N_1 + x N_2 \right] \quad \text{and} \quad \det \left[ (x - 1) P_1 + x P_2 \right]$$

The obvious first attempt is to consider the polynomials

$$p(x) = \det \left[ x A_1^T A_1 + A_2^T A_2 \right].$$

However when one starts to perturb $A_1$ and $A_2$ independently, one quickly realizes that simply knowing the generalized singular values are not enough — information about $A_1$ and $A_2$ themselves is needed. This motivates using a polynomial that keeps $A_1$ and $A_2$ independent (to some extent), which leads to the following definition:

Given $A \in \mathcal{M}(n_1 + n_2, m)$ with block structure

$$A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$$

we define the generalized singular value characteristic polynomial (GSVCP) to be

$$p_A(x, y, z) = \det \left[ x I + y A_1^T A_1 + z A_2^T A_2 \right]$$

The next theorem shows that (13) defines a valid convolution — that is, one can compute the GSVCP of a unitarily invariant sum of matrices from the GSVCPs of the summands.
Theorem 16. Let $M, N, W \in \mathcal{M}_{n_1 + n_2, m}$ with block structure

\[
M = \begin{bmatrix} M_1 & n_1 \\ M_2 & n_2 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} N_1 & n_1 \\ N_2 & n_2 \end{bmatrix} \quad \text{and} \quad W = \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} = \begin{bmatrix} M_1 + R_1 N_1 Q \\ M_2 + R_2 N_2 Q \end{bmatrix}
\]

where $R_1, R_2, Q$ are independent, uniformly distributed, signed permutation matrices of the appropriate sizes and let

\[
p_M(x, y, z) = \det [xI + yM_1^T M_1 + zM_2^T M_2] = \sum_{j,k} \frac{x^{m-j-k} y^j z^k}{(m-j-k)! (n_1-j)! (n_2-k)!} p_{jk}
\]

\[
p_N(x, y, z) = \det [xI + yN_1^T N_1 + zN_2^T N_2] = \sum_{j,k} \frac{x^{m-j-k} y^j z^k}{(m-j-k)! (n_1-j)! (n_2-k)!} q_{jk}
\]

\[
p_W(x, y, z) = \det [xI + yW_1^T W_1 + zW_2^T W_2] = \sum_{j,k} \frac{x^{m-j-k} y^j z^k}{(m-j-k)! (n_1-j)! (n_2-k)!} r_{jk}
\]

be their GSVCPs, where each $r_{jk}$ is a random variable. Then

\[\mathbb{E}_{Q, R_1, R_2} \{ r_{jk} \} = \begin{cases} \frac{1}{m! n_1 n_2} \sum_{\beta=0}^k \sum_{\delta=0}^k p_{\beta, \delta} q_{j-\beta, k-\delta} & \text{for } j \leq n_1, k \leq n_2, j + k \leq m \\ 0 & \text{otherwise} \end{cases}\]

Proof. We start by changing variables to match Theorem 1: let $f(x, s, t, u, v)$ denote the polynomial

\[\mathbb{E}_{Q, R_1, R_2} \{ \det [xI + (sM_1 + tR_1 N_1 Q)^T (M_1 + R_1 N_1 Q) + (uM_2 + vR_2 N_2 Q)^T (M_2 + R_2 N_2 Q)] \}.
\]

We now do the expectations separately, starting with $R_2$ and then $R_1$. By Theorem 1, we get

\[f(x, s, t, u, v) = L_{n_2}^{u,v} \mathbb{E}_{Q, R_1} \{ \det [xI + (sM_1 + tR_1 N_1 Q)^T (M_1 + R_1 N_1 Q) + (uM_2 + vR_2 N_2 Q)^T (M_2 + R_2 N_2 Q)] \}.
\]

By Theorem 2 we have

\[\mathbb{E}_Q \{ \det [xI + (sM_1^T M_1 + tQ^T N_1^T N_1 Q) + (uM_2^T M_2 + vQ^T N_2^T N_2 Q)] \} = [g * h](x, s, t, u, v)
\]

where

\[g(x, s, t, u, v) = \det [xI + sM_1^T M_1 + tI + uM_2^T M_2 + vI] = p_M(x + t + v, s, u)
\]

and

\[h(x, s, t, u, v) = \det [xI + sI + tN_1^T N_1 + uI + vN_2^T N_2] = p_N(x + s + u, t, v)
\]

are each $m$-homogeneous polynomials. We can now go in the reverse direction to compute $f(x, s, t, u, v)$ from the expansions in the hypothesis. Firstly, we have

\[g(x, s, t, u, v) = \sum_{j,k} \frac{(x + t + v)^{m-j-k} s^j u^k}{(m-j-k)! (n_1-j)! (n_2-k)!} p_{jk}
\]

\[= \sum_{j,k} \sum_{a,b} \frac{x^{m-j-k-a-b} s^j u^k}{(m-j-k-a-b)! a! b! (n_1-j)! (n_2-k)!} p_{jk}
\]

\[= \sum_{\alpha + \beta + \gamma + \delta + \sigma = m} \frac{x^\alpha s^\beta t^\gamma u^\delta v^\sigma}{\alpha! (n_1-\beta)! (n_2-\delta)! \gamma! \sigma!} p_{\beta \delta}
\]

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and similarly

\[
h(x, s, t, u, v) = \sum_{j,k} \frac{(x + s + u)^{m-j-k}}{(m-j-k)! (n_1-j)! (n_2-k)!} t^j v^k
\]

\[
= \sum_{j,k} \sum_{a,b} [m-j-k-a-b] a! b! (n_1-j)! (n_2-k)! q_{jk}
\]

\[
= \sum_{\alpha+\beta+\gamma+\delta+\sigma=m} x^{\alpha} s^{\beta} t^{\gamma} u^{\delta} v^{\sigma} \frac{\alpha!(n_1-\gamma)!(n_2-\sigma)!\beta!\delta!q_{\gamma\sigma}}{\alpha!(n_1-\beta)!(n_2-\delta)!}
\]

Hence by definition of the star product, we have

\[
[g * h](x, s, t, u, v) = \frac{1}{m!} \sum_{\alpha,\beta,\gamma,\delta,\sigma} x^{\alpha}s^{\beta}t^{\gamma}u^{\delta}v^{\sigma} \frac{\alpha!(n_1-\beta)!(n_2-\gamma)!}{\alpha!(n_1-\beta)!(n_2-\delta)!(n_2-\sigma)!} p_{\beta\delta q_{\gamma\sigma}}
\]

and so

\[
f(x, s, t, u, v) = L_{n_1}^s L_{n_2}^u \{ [g * h](x, s, t, u, v) \}
\]

\[
= \frac{1}{m!n_1!n_2!} \sum_{\alpha+\beta+\gamma+\delta+\sigma=m, \beta+\gamma \leq n_1, \delta+\sigma \leq n_2} x^{\alpha}s^{\beta}t^{\gamma}u^{\delta}v^{\sigma} \frac{\alpha!(n_1-\beta-\gamma)!(n_2-\delta-\sigma)!}{\alpha!(n_1-\beta)!\alpha!(n_2-\delta)!} p_{\beta\delta q_{\gamma\sigma}}
\]

Putting all of this together, the polynomial we are interested in is

\[
E_{Q,R_1,R_2} \{ p_{W}(x, y, z) \} = f(x, y, y, z, z)
\]

\[
= \frac{1}{m!n_1!n_2!} \sum_{\alpha+\beta+\gamma+\delta+\sigma=m, \beta+\gamma \leq n_1, \delta+\sigma \leq n_2} x^{\alpha} y^{\beta} z^{\gamma} \frac{\alpha!(n_1-\beta-\gamma)!}{\alpha!(n_1-\beta)!\alpha!(n_2-\delta)!} p_{\beta\delta q_{\gamma\sigma}}
\]

\[
= \frac{1}{m!n_1!n_2!} \sum_{j+k \leq m} x^{m-j-k} y^{j} z^{k} \frac{\alpha!(n_1-\beta-\gamma)!}{\alpha!(n_1-\beta)!\alpha!(n_2-\delta)!} \sum_{\beta=0, \delta=0}^{j+k} p_{\beta\delta q_{j-k,\beta,k-\delta}}
\]

as required.

Note that while Theorem 16 considers fixed matrices $M$ and $N$, one can easily extend it to random matrices using linearity of expectation. We finish the section by observing that the convolution described by Theorem 16 has a remarkably simple form when the polynomials are expressed in the context of differential operators.

**Corollary 17.** Let $p_M, p_N, p_W$ be the polynomials in Theorem 16 and let $P, Q$ be bivariate polynomials for which

\[
y^{n_1} z^{n_2} p_M(x, 1/y, 1/z) = P(\partial_x \partial_y, \partial_x \partial_z) \{ x^{m} y^{n_1} z^{n_2} \}
\]

and

\[
y^{n_1} z^{n_2} p_N(x, 1/y, 1/z) = Q(\partial_x \partial_y, \partial_x \partial_z) \{ x^{m} y^{n_1} z^{n_2} \}.
\]

Then

\[
E_{R_1,R_2,Q} \{ y^{n_1} z^{n_2} p_W(x, 1/y, 1/z) \} = P(\partial_x \partial_y, \partial_x \partial_z) Q(\partial_x \partial_y, \partial_x \partial_z) \{ x^{m} y^{n_1} z^{n_2} \}.
\]
Corollary 17 suggests that if $p_A(x, y, z)$ is the polynomial in (13), then a more reasonable polynomial to consider would be

$$q_A(x, y, z) = y^{n_1}z^{n_2}p_A(x, 1/y, 1/z) = y^{n_1-m}z^{n_2-m}\det [xyzI + zA_1^TA_1 + yA_2^TA_2].$$

Another advantage to this alternative form is that there is a more direct matrix model that one can work with, as one can easily check that

$$\det \begin{bmatrix} xI_m & A_1^T & A_2^T \\ A_1 & yI_{n_1} & 0 \\ A_2 & 0 & zI_{n_2} \end{bmatrix} = y^{n_1-m}z^{n_2-m}\det [xyzI - zA_1^TA_1 - yA_2^TA_2].$$

Obviously the two are simple transformations from each other; we mention it because particular applications can be more well suited to one versus the other.

8 Open Problems

The proof presented in Section 4 applies more generally than Theorem 1 in the respect that it holds for all minor-orthogonal ensembles. We suspect that Theorem 2 has a similar generalization, but have not able to prove it. This is not much of a hindrance when it comes to theoretical applications: the majority of the minor-orthogonal ensembles that one comes across in random matrix theory contain the signed permutation matrices as a subgroup and so Theorem 2 can be extended them by the averaging argument in Corollary 9. The one notable situation where this is not the case is that of the uniform distribution over the standard representation of $S_{n+1}$ (what you get when you turn the collection of $(n+1) \times (n+1)$ permutation matrices into $n \times n$ matrices by projecting each one orthogonally to the constant vector). This is (as far as the author knows) the minor-orthogonal ensemble with the smallest support and so is often useful for computational purposes.

Those familiar with the connection between polynomial convolutions and free probability (see, for example, [12]), might recognize the convolutions in Sections 6.2.1, 6.2.2, and 6.2.3 as the “finite free” versions of the additive, multiplicative, and rectangular convolutions from classical free probability. The operators from free probability have known closure properties (they map distributions on the real line to distributions on the real line) and so one might hope the same is true for the finite analogues. This turns out to be true: the convolution in Section 6.2.1 maps Hermitian determinantal representations to Hermitian determinantal representations and the ones in Sections 6.2.2 and 6.2.3 map positive semidefinite representations to semidefinite representations. In the multivariate case, one can show (using a powerful theorem of Helton and Vinnikov [8]) that the convolution in Section 7 preserves positive semidefinite representations as well. Continuing the analogy, the operator $\ast$ would be the natural finite analogue of the box product $\boxtimes$ from free probability [16] and so one might hope that it, too, has some sort of closure property. However this seems to be completely open. It would therefore be both useful and interesting to understand the conditions under which the operations in this paper can be shown to preserve some (any) class of polynomials.

While the expected characteristic polynomial of a random matrix gives you some information, it will (in general) not be enough to characterize the eigenvalue distribution of the underlying random matrix. However, there is a natural way to “assign” an eigenvalue distribution to a convolution — the one which is uniformly distributed over the roots of the polynomial (the fact that polynomial convolutions preserve real stability imply that this will be a valid distribution on the real line).
It is still not known exactly how the uniform-over-roots distributions derived from polynomial convolutions relate to the actual distributions of the underlying random matrices. The one area that seems to show the most striking resemblances to this is that of free probability, which one can view as the study of the limiting distributions of random matrix theory as the dimension approaches $\infty$. There is some speculation (mostly by this author) that polynomial convolutions represent the limiting distributions of random matrix theory as some other parameter (usually referred to as $\beta$) approaches $\infty$. There is some evidence supporting this idea [7], but in many cases it is not clear how to even define such a limit formally. Understanding this relationship better, however, is certainly an interesting open problem (and one that remains fairly wide open).

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