Abstract

We study the ring generated over a field of characteristic 0 by noncommuting indeterminates \( \{x_1, x_2, \ldots, x_n\} \) subject only to the relations \( x_i \sigma_k = \sigma_k x_i \), \( i, k = 1, 2, \ldots, n \), and their consequences, where \( \sigma_k = \sigma_k(x_1, x_2, \ldots, x_n) \) is the k-th elementary polynomial in the noncommuting variables \( x_i \). We assume \( n \geq 3 \) throughout.

1. BASICS

Let \( K \) represent a commutative field of characteristic 0, let \( x_1, x_2, \ldots, x_n \) be algebraically independent noncommuting indeterminates over \( K \), and denote by \( P = K < x_1, x_2, \ldots, x_n > \) the free \( K \)-ring generated over \( K \) by the \( x_i \). Each \( x_i \) commutes with every element of \( K \).

**Monomials** \( u \in P \) have the form

\[
u = x_{i(1)}^{j(1)} x_{i(2)}^{j(2)} \cdots x_{i(\ell)}^{j(\ell)},\]

and will be understood to satisfy the following conditions: the \( j(\cdot) \) are integers \( \geq 1 \), \( i(\cdot) \in \{1, 2, \ldots, n\} \), and \( i(p) \neq i(q) \) when \( |p - q| = 1 \). We call \( \{j(1), j(2), \ldots, j(\ell)\} \) the exponent sequence of \( u \), the sequence of subscripts \( \{i(1), i(2), \ldots, i(\ell)\} \) its complexion, \( j(1) + j(2) + \cdots + j(\ell) \) its degree, and \( \ell \) its length. We denote the set of all monomials by \( Q; Q \cup \{1\} \) is a vector space basis of \( P \) over \( K \).

We define the **elementary polynomials**, \( \sigma_k(x_1, x_2, \ldots, x_n) \), as follows. Set \( \sigma_0 = 1, \sigma_k = 0 \) when \( k < 0 \) or \( k > n \), and, for, \( k = 1, 2, \ldots, n \),

\[
\sigma_k(x_1, x_2, \ldots, x_n) = \sum (x_{i(1)} x_{i(2)} \cdots x_{i(k)} : i(\cdot) \in \{1, 2, \ldots, n\}, i(1) < i(2) < \cdots < i(k))
\]

Each \( \sigma_k, k = 1, 2, \ldots, n \), has \( \binom{n}{k} \) monomials where \( \binom{n}{k} = n!/(n - k)!k! \) is the usual binomial coefficient. We may break up the sum (2) into \( n - k + 1 \) subsums according as the monomials involved begin with \( x_1, x_2, \ldots \).
In more traditional form,

\[
\begin{cases}
\sigma_k = \sum (x_1 x_{i(2)} \cdots x_{i(k)} : 1 < i(2) < \ldots < i(k)) & \binom{n-1}{k-1} 
\sigma_k = \sum (x_2 x_{i(2)} \cdots x_{i(k)} : 2 < i(2) < \ldots < i(k)) & \binom{n-2}{k-1} 
\vdots 
+x_{n-k+1} x_{n-k+2} \cdots x_n & \binom{k-1}{k-1} = 1 \text{ monomial}
\end{cases}
\]

In more traditional form,

\[
\sigma_1(x_1, x_2, \ldots, x_n) = x_1 + x_2 + \cdots + x_n \\
\sigma_2(x_1, x_2, \ldots, x_n) = (x_1 x_2 + x_1 x_3 + \cdots + x_1 x_n) + (x_2 x_3 + \cdots + x_2 x_n) + \cdots + x_{n-1} x_n \\
\vdots \\
\sigma_{n-1}(x_1, x_2, \ldots, x_n) = (x_1 x_2 \cdots x_{n-1} + x_1 x_2 \cdots x_{n-2} x_n + \cdots + x_1 x_3 \cdots x_n) + x_2 x_3 \cdots x_n \\
\sigma_n(x_1, x_2, \ldots, x_n) = x_1 x_2 \cdots x_n
\]

In the commutative case, these polynomials are the same as the "elementary symmetric functions" (as commonly written), but here the variables do not commute so specification of the order of the variables is necessary.

We may rewrite equation (3) as follows,

\[
\sigma_k(x_1, x_2, \ldots, x_n) = \sum \left( x_1 x_{i(2)} \cdots x_{i(k)} : i(\cdot) \in \{2, 3, \ldots, n\}, i(2) < i(3) < \ldots < i(k) \right) \\
+ \sum \left( x_{i(1)} x_{i(2)} \cdots x_{i(k)} : i(\cdot) \in \{2, 3, \ldots, n\}, i(1) < i(2) < \ldots < i(k) \right)
\]

and deduce therefrom the first recursion formula for \( \sigma_k \)

(I) \[ \sigma_k(x_1, x_2, \ldots, x_n) = x_1 \sigma_{k-1}(x_2, x_3, \ldots, x_n) + \sigma_k(x_2, x_3, \ldots, x_n), \]

\( k = 1, 2, \ldots n \)

which is valid over the \( k \)-range indicated because we have set \( \sigma_0 = 1 \) and \( \sigma_k = 0 \) when \( k > \# \) of variables.

We may derive a companion formula to (2) where we factor out the last variable \( x_n \) instead of the first, and, doing so, we get the second recursion formula for \( \sigma_k \).

(II) \[ \sigma_k(x_1, x_2, \ldots, x_n) = \sigma_{k-1}(x_1, x_2, \ldots, x_{n-1}) x_n + \sigma_k(x_1, x_2, \ldots, x_{n-1}), \]

\( k = 1, 2, \ldots, n \)

We are concerned here with the action on the free ring \( P \) of the group \( G \) of circular permutations by which I mean the cyclic subgroup of the full permutation group \( S_n \) on \( n \) letters generated by \( (12\ldots n) \). The action of \( g \in G \) on the monomial (1) is given by

\[
u^g = x_{i(1)g}^{j(1)} x_{i(2)g}^{j(2)} \cdots x_{i(\ell)g}^{j(\ell)}.\]
Note that $G$ acts only on the complexion of $u$; its exponent sequence does not change. The action of $G$ extends by linearity to polynomials. The coefficient field $K$ is pointwise fixed by $G$. A polynomial $p \in P$ is invariant under $G$ when $p^g = p$ for all $g \in G$. As $G$ is cyclic, it is enough that $p^{g(1)} = p$ for the generator $g(1) = (12...n)$ of $G$.

**Theorem 1.1.** In the free ring $P$, the (two-sided) ideal generated by the $\sigma_k(x_1, x_2, ..., x_n) - \sigma_k(x_1, x_2, ..., x_n)^g$, $k = 1, 2, ..., n, \forall g \in G$, is the same as that generated by the commutators $[x_i, \sigma_k] = x_i\sigma_k(x_1, x_2, ..., x_n) - \sigma_k(x_1, x_2, ..., x_n)x_i, 1 \leq i, k \leq n$. Or, what amounts to the same thing, the set of relations $\{\sigma_k^n = \sigma_k : 1 \leq k \leq n, \forall g \in G\}$ is equivalent to the set $\{x_i\sigma_k = \sigma_kx_i : 1 \leq i, k \leq n\}$.

**Proof.** The following notations will be useful. Denote by $g(1) = (12...n)$ the generator of the group $G$ of circular permutations, and by $g(i)$ the $i$-th power of $g(1)$. So $G = \{1 = g(0), g(1), g(2), ..., g(n - 1)\}$. The recursion formulas (I) and (II) hold for any set of $n$ subscripted letters as long as their order there is maintained. In particular, we may apply $g(i)$ to them to get

$I(i)$ \quad $\sigma_k^{g(i)} = x_{i+1}\sigma_{k-1}(x_{i+2}, ..., x_i) + \sigma_k(x_{i+2}, ..., x_i)$

$I(i+1)$ \quad $\sigma_k^{g(i+1)} = \sigma_{k-1}(x_{i+2}, ..., x_i)x_{i+1} + \sigma_k(x_{i+2}, ..., x_i)$

for $k = 1, 2, ..., n$ and $i = 0, 1, ..., n - 1$. (In the complexions we work mod $n$.) Denote by $I$ the ideal generated by the $\sigma_k - \sigma_k^g$, and by $I'$ that generated by the $[x_i, \sigma_k]$. We aim to show that $I = I'$. (In this paper “ideal” means two-sided ideal.)

$I \subset I'$: The recursion $I(0)$ gives

$$\sigma_k = x_1\sigma_{k-1}(x_2, ..., x_n) + \sigma_k(x_2, ..., x_n), k = 1, 2, ..., n$$

so

$$[x_1, \sigma_k] = [x_1, x_1\sigma_{k-1}(x_2, ..., x_n)] + [x_1, \sigma_k(x_2, ..., x_n)]$$

$$= x_1[x_1, \sigma_{k-1}(x_2, ..., x_n)] + [x_1, \sigma_k(x_2, ..., x_n)]$$

(using $[a, bc] = [a, b]c + b[a, c]$). Put $k = 1$. As $\sigma_0 = 1$ we get

$$[x_1, \sigma_1] = [x_1, \sigma_1(x_2, ..., x_n)]$$

so the latter commutator belongs to $I'$. Put $k = 2$.

$$[x_1, \sigma_2] = x_1[x_1, \sigma_1(x_2, ..., x_n)] + [x_1, \sigma_2(x_2, ..., x_n)].$$

We have just shown that the first term on the right belongs to $I'$, hence also $[x_1, \sigma_2(x_2, ..., x_n)]$. Continuing, we establish that

$$[x_1, \sigma_k(x_2, ..., x_n)] \in I', \quad k = 1, 2, ..., n$$
The case $k = n$ can be included because $\sigma_k = 0$ when $k > \# \text{ of variables.}$) Now go to II(1),

$$\sigma_k^{(1)} = \sigma_{k-1}(x_2, \ldots, x_n)x_1 + \sigma_k(x_2, \ldots, x_n),$$

and subtract I(0)-II(1),

$$\sigma_k - \sigma_k^{(1)} = [x_1, \sigma_{k-1}(x_2, \ldots, x_n)] \in I', \; k = 1, 2, \ldots, n.$$

That is the first step. If we had taken the trouble to establish in advance that $(I')^g \subset I'$, we would be done. But this direct method seems preferable, so, we proceed to the second step. From $\sigma_k - \sigma_k^{(1)} \in I'$ we get $[x_i, \sigma_k - \sigma_k^{(1)}] = [x_i, \sigma_k] - [x_i, \sigma_k^{(1)}] \in I'$, hence $[x_i, \sigma_k^{(1)}] \in I', \; i, k = 1, 2, \ldots, n$. The recursion I(1) gives

$$\sigma_k^{(1)} = x_2\sigma_{k-1}(x_3, \ldots, x_1) + \sigma_k(x_3, \ldots, x_1)$$

so

$$[x_2, \sigma_k^{(1)}] = x_2[x_2, \sigma_{k-1}(x_3, \ldots, x_1)] + [x_2, \sigma_k(x_3, \ldots, x_1)].$$

Put $k = 1, k = 2$, etc, as before, to show that $[x_2, \sigma_k(x_3, \ldots, x_1)] \in I', \; k = 1, 2, \ldots, n$. Then subtract I(1)-II(2),

$$\sigma_k^{(1)} - \sigma_k^{(2)} = [x_2, \sigma_{k-1}(x_3, \ldots, x_1)] \in I', \; k = 1, 2, \ldots, n,$$

which completes the second step. Continue, to get

$$\sigma_k^{(i)} - \sigma_k^{(i+1)} \in I', \; k = 1, 2, \ldots, n, \; i = 0, 1, \ldots, n - 1$$

which shows $I \subseteq I'$

$I' \subseteq I$. We need to show that all commutators $[x_i, \sigma_k]$ belong to the ideal I generated by the $\sigma_k - \sigma_k^{(i)}$. Put $i = 0$ in I(i) and II(i+1),

I(0) \hspace{1cm} \sigma_k = x_1\sigma_{k-1}(x_2, \ldots, x_n) + \sigma_k(x_2, \ldots, x_n)

II(1) \hspace{1cm} \sigma_k^{(1)} = \sigma_{k-1}(x_2, \ldots, x_n)x_1 + \sigma_k(x_2, \ldots, x_n)

and subtract to get

$$\sigma_k - \sigma_k^{(1)} = [x_1, \sigma_{k-1}(x_2, \ldots, x_n)], \; k = 1, 2, \ldots, n.$$
\[ [x_1, \sigma_k] = x_1(\sigma_k - \sigma_k^{g(1)}) + (\sigma_{k+1} - \sigma_k^{g(1)}), k = 1, 2, ..., n, \]

which shows that \([x_1, \sigma_k] \in I, k = 1, 2, ..., n.\) Next, put \(i = 1\) in \(I(i)\) and \(I(i+1)\)

\[
I(1) \quad \sigma_k^{g(1)} = x_2\sigma_{k-1}(x_3, ..., x_1) + \sigma_k(x_3, ..., x_1) \\
I(2) \quad \sigma_k^{g(2)} = \sigma_{k-1}(x_3, ..., x_1)x_2 + \sigma_k(x_3, ..., x_1),
\]

and subtract to get

\[
\sigma_k^{g(1)} - \sigma_k^{g(2)} = [x_2, \sigma_{k-1}(x_3, ..., x_1)] \quad k = 1, 2, ..., n.
\]

Then, using \(I(1)\),

\[
[x_2, \sigma_k^{g(1)}] = x_2(\sigma_k^{g(1)} - \sigma_k^{g(2)}) + (\sigma_{k+1} - \sigma_k^{g(1)})^{(2)}.
\]

Then write \(\sigma_k = \sigma_k^{g(1)} + (\sigma_k - \sigma_k^{g(1)})\) to get \([x_2, \sigma_k] = x_2(\sigma_k^{g(1)} - \sigma_k^{g(2)}) + (\sigma_{k+1} - \sigma_k^{g(1)}) + [x_2, (\sigma_k - \sigma_k^{g(1)})] \quad k = 1, 2, ..., n.\) which shows that \([x_2, \sigma_k] \in I, k = 1, 2, ..., n.\)

The remaining steps follow the same pattern. That completes the proof of Theorem 1.1.

The proof of Theorem 1.1 applies without change to the free \(\mathbb{Z}\)-ring \(\mathbb{Z} < x_1, x_2, ..., x_n >\) where \(\mathbb{Z} \subseteq K\) is the ring of rational integers. As no constant term is involved, we have

**Corollary 1.2** Let \(B\) be any ring, \(b_1, b_2, ..., b_n\) any finite number of elements from \(B.\) The following two conditions are equivalent.

(i) Each \(\sigma_k(b_1, b_2, ..., b_n), k = 1, 2, ..., n,\) is invariant under circular permutations of the \(b_i.\)

(ii) Each \(\sigma_k(b_1, b_2, ..., b_n)\) commutes with each \(b_i, i, k = 1, 2, ..., n.\)

We shall denote by \(I\) the common ideal described in Theorem 1.1, and shall denote by \(R'\) the quotient ring \(P/I.\) We shall work with \(R'\) as a polynomial ring with relations (those specified in Theorem 1.1). We shall also make frequent use of the fact that a polynomial in \(R'\) is zero there if and only if it, exactly as written, when considered in the free ring \(P,\) belongs to the ideal \(I.\) Because a polynomial in \(P\) belongs to \(I\) if and only if its homogeneous components do, a polynomial in \(R'\) will be zero there if and only if its homogeneous components are separately zero. Thus two polynomials in \(R'\) will be equal exactly when their homogenous components are separately equal, and monomials of different degree cannot be equal. But monomials of the same degree can be; for example when \(n = 3\) \(x_1x_2, x_3 = x_2x_3x_1 = x_3x_1x_2(= \sigma_3)\). Further the same element of \(R'\) may have different polynomial representations. Again, when
\[ n = 3, x_1x_2 + x_1x_3 + x_2x_3 = x_2x_3 + x_2x_1 + x_3x_1 (= \sigma_2) \text{ so, in } R', x_1x_2 + x_1x_3 - x_2x_1 - x_3x_1 = 0. \]

The general homogeneous polynomial of degree \( w \) in \( I \) is a \( K \)-linear combination of polynomials \( u[x_i, \sigma_k]v \) where \( u \) and \( v \) are monomials of degree \( r \) and \( s \) respectively (say) and \( w = r + s + k + 1 \). As \( k \geq 1, w \geq 2 \), so \( I \) contains no linear polynomials. Hence, in \( R' \), the variables \( x_i \) are \( K \)-linearly independent. If \( w = 2 \), then \( r = s = 0 \) and \( k = 1 \) in which case the generators \( [x_i, \sigma_k] \) are just sums of the \( [x_i, x_j] \). We shall use the notations \( [i, j] = x_ix_j - x_jx_i \) for these additive commutators, and we need only consider \( [i, j] \) when \( i < j \). We shall refer to commutators \( [i, i + 1] \) as “diagonal”, and \( [i, j] \) when \( j > i + 1 \) as “off-diagonal”.
**Theorem 1.3** The off-diagonal commutators \([i, j + 1], 1 \leq i < j < n\) are \(K\)-linearly independent in \(R'\). The diagonal commutators are expressed in terms of the off-diagonal by the formulas

\[
[k, k - 1] = \sum_{p=1}^{k-2} \sum_{j=k}^{n} [p, j] - \sum_{j=k+1}^{n} [k - 1, j], \tag{4}
\]

\(k = 2, 3, ..., n\), where we use the convention that sums whose lower limit exceeds the upper are given the value 0.

Under the stated convention, the first sum vanishes when \(k = 2\), and the last does when \(k = n\). Hence, when \(k = 2\), formula (4) becomes

\[
[2, 1] = \sum_{j=3}^{n} [1, j],
\]

and, when \(k = n\),

\[
[n, n - 1] = \sum_{p=1}^{n-2} [p, n].
\]

Note also this consequence when \(n = 3\) : \([1, 2] = [2, 3] = [3, 1]\). (Remember \(n \geq 3\) is assumed throughout.)

**Proof:** As commutators are homogeneous of degree 2, the assertion that a \(K\)-linear combination of off-diagonal commutators equals zero in \(R'\) takes the form

\[
\sum_{1 \leq i < j < n} a_{ij}[i, j + 1] = \sum_{i=1}^{n} b_i (x_i \sigma_1 - \sigma_1 x_i)
\]

in the free ring \(P\), where the \(a\)'s and \(b\)'s belong to \(K\). An easy computation evaluates the right side and we get

\[
\sum_{1 \leq i < j < n} a_{ij}[i, j + 1] = \sum_{i=1}^{n} b_i \sum_{\ell=i}^{n} [i, \ell]. \tag{5}
\]

Because the left side of (5) consists solely of off-diagonal commutators, it does not contain either \([1, 2]\) or \([2, 1]\). The right side however contains \(b_2[2, 1] + b_1[1, 2]\). As we are working in a free ring we must have \(b_1 = b_2\). Arguing similarly with \([2, 3]\) we get \(b_2 = b_3\). And so on. The right side of (5) thus becomes
which is zero because we are summing over all entries of a skew symmetric matrix. Thus the right side of (5) is zero in the free ring \( P \). And, as we are working in the free ring, it follows that all \( a_{ij} \) must be zero. This proves the linear independence of the off-diagonal commutators in \( R' \). Next we turn to equation (4) expressing the diagonal commutators in terms of the off-diagonal. In the ring \( R' \) we have the identity

\[ \sum_{p=1}^{n} a_p (x_p \sigma_1 - \sigma_1 x_p) = 0 \]

for any selection of \( a_p \in K \), because \( x \sigma = \sigma x \) holds identically in \( R' \). The above equation may be recast in the form

\[ \sum_{1 \leq p < q \leq n} (a_p - a_q) [p, q] = 0. \]  

In (6), put \( a_1 = 1 \), and \( a_p = 0, p \geq 2 \) to get \([1, 2] + [1, 3] + \cdots + [1, n] = 0\) which we may solve for \([1, 2]\) in terms of off-diagonal commutators. Next put \( a_1 = a_2 = 1, \ a_p = 0, p \geq 3 \) and solve for \([2, 3]\). Continue, to get \( n - 1 \) equations for the \([k - 1, k]\) which constitute the system of equations (4). We have \( n - 1 \) independent equations for \( n(n - 1)/2 \) unknowns, and we have solved for the \( n - 1 \) diagonal commutators in terms of the \((n - 1)(n - 2)/2\) independent off-diagonal ones. That completes the proof of Theorem 1.3.

**Corollary 1.4.** In \( R' \), \( x_i x_j \neq x_j x_i \) when \( i \neq j \).

**Proof.** As the off-diagonal commutators are \( K \)-linearly independent, none of them can be zero. And formula (4) shows that no diagonal commutators can be zero either lest we get a nontrivial linear combination of the off-diagonals equal to zero. \( \diamond \)

We make no use in the sequel of the following corollary, so I register it here without proof (the “\( j \)” in front of the bracket is the integer \( j \)).

**Corollary 1.5.** For the sum of the diagonal commutators in \( R' \) we have

\[ \sum_{k=2}^{n} [k, k - 1] = \sum_{p=1}^{n-2} \sum_{j=2}^{n-p} j [p, j + p]. \]

The next corollary that describes a canonical form for the homogeneous quadratic in \( R' \) can be deduced from Theorem 1.3 in a straightforward way, so that proof is left to the
Corollary 1.6. Every homogeneous quadratic polynomial \( p \in R' \) may be written

\[
p = \sum_{i=1}^{n} a_i x_i^2 + \sum_{1 \leq i < j \leq n} b_{ij} x_i x_j + \sum_{1 \leq i < j < n} c_{ij}[i, j + 1]
\]

where the coefficients (in \( K \)) are uniquely determined by \( p \).

Our construction of the ring \( R' \) can be viewed as parallel to a construction of the free commutative ring. One may show that, in the free ring \( P = K < x_1, x_2, ..., x_n > \), the set of relations \( \{ x_i x_j = x_j x_i : i, j = 1, 2, ..., n \} \) is equivalent to the set \( \{ \sigma_k = \sigma_k^\pi : \forall \pi \in S_n \} \).

Our Theorem 1.1 shows that if we restrict to invariance under only the subgroup \( G = < (12...n) > \) of \( S_n \), then the corresponding commutativity is \( x_i \sigma_k = \sigma_k x_i \). Each commutator \([x_i, \sigma_k]\) can be expressed in terms of the \([x_i, x_j]\), so the ideal \( J \) of the free ring \( P \) generated by the \([x_i, x_j]\) contains \( I \), and contains it properly when \( n \geq 3 \) by Corollary 1.4. (When \( n = 2 \), \( J = I \) which is why we assume \( n \geq 3 \) throughout.) Hence, if \( A = K[x_1, x_2, ..., x_n] \) is the free abelian ring in \( n \) commuting indeterminates, we have

\[
A \cong P/I \cong (P/I)/(J/I)
\]

or

\[
A \cong R'/\langle J/I \rangle
\]

and thus the following diagram where \( \omega \) denotes the homomorphism \( P \to P/J \cong A \), \( \varphi : P \to P/I \cong R' \), and \( \Psi : R' \to R'/\langle J/I \rangle \approx A \).

\[
P = K < x_1 x_2, ..., x_n > \text{ the free ring}
\]

\[
R'
\]

\[
\omega = \Psi \circ \varphi
\]

\[
A = K[x_1, x_2, ..., x_n] \text{ the free abelian ring}
\]

Figure 1

Theorem 1.7

(i) A necessary condition that \( p(x_1, x_2, ..., x_n) = 0 \) in \( R' \) is that it can be brought to zero by assuming that the \( x_i \) mutually commute.
(ii) In \( R' \) the elementary polynomials \( \sigma_k(x_1, x_2, ..., x_n), k = 1, 2, ..., n \) are algebraically independent over \( K \), thus generate a commutative domain \( W' = K[\sigma_1, \sigma_2, ..., \sigma_n] \) within the center of \( R' \).

**Proof.** As for (i), if \( p = 0 \) in \( R' \), then \( \Psi(p) = 0 \) in the free abelian ring \( A \). So if \( p \) cannot be brought to zero by assuming that its variables commute, then it cannot be zero in \( R' \). As for (ii), note first that the \( \sigma_k \) are central in \( R' \) by construction. If \( p(\sigma_1, \sigma_2, ..., \sigma_n) \in R' \) is a \( K \)-coefficient polynomial in the \( \sigma_k \) (nontrivial), then \( \Psi(p) \) is the same polynomial which now lies in the free abelian ring \( A = K[x_1, x_2, ..., x_n] \) where we know the “elementary symmetric functions” \( \sigma_k \) are algebraically independent. So \( \Psi(p) \neq 0 \) in \( A \), hence \( p \neq 0 \) in \( R' \).

Look for a moment at the commutative case. The commutative domain \( A = K[x_1, x_2, ..., x_n] \) is contained in the field \( F = K(x_1, x_2, ..., x_n) \) of rational functions in the \( x_i \). The field \( F \) contains the field \( W = K(\sigma_1, \sigma_2, ..., \sigma_n) \) of rational functions in the algebraically independent elementary symmetric functions \( \sigma_k \), and \( F \) is a Galois extension of \( W \) with Galois group \( S_n \). \( F/W \) is the splitting field of \( y^n - \sigma_1 y^{n-1} + ... + (-1)^n \sigma_n \). What follows is a noncommutative analogue of this set-up.

**Definition 1.8.** By the symbol \( R \) we shall denote the extension of \( R' \) constructed as described below. \( R \) contains \( R' \) as a subring, and contains the field \( W = K(\sigma_1, \sigma_2, ..., \sigma_n) \) within its center.

The method of construction of the field of fractions of a commutative domain works without change to produce the ring \( R \) described above. \( R \) consists of equivalence classes of pairs \( (r, \lambda), r \in R', 0 \neq \lambda \in W = K[\sigma_1, \sigma_2, ..., \sigma_n] \) under the equivalence relation \( (r_1, \lambda_1) = (r_2, \lambda_2) \iff r_1 \lambda_2 = r_2 \lambda_1 \). The ring operations are defined as usual, and the map \( r \mapsto (r, 1) \) embeds \( R' \) in \( R \). In working with \( R \) we shall write \( \sigma^{-1} \) for \( (1, \sigma) \), etc.

The ring \( R \) is the focus of this paper. We note the following easily verified facts about \( R \):

1. Every \( x_i \in R \) is a root of the polynomial \( f(y) = y^n - \sigma_1 y^{n-1} + \sigma_2 y^{n-2} - \cdots + (-1)^n \sigma_n \) (\( y \) a central indeterminate) which factors in \( R \) as \( f(y) = (y - x_1)(y - x_2) \cdots (y - x_n) \) where the \( x_i \) may be circularly permuted. (2) Every \( x_i \) has an inverse in \( R \) given by \( x_i^{-1} = (-1)^{n+1} \sigma_n^{-1}(x^{n-1} - \sigma_1 x^{n-2} + \cdots + (-1)^{n-1} \sigma_{n-1}) \) (\( x = x_i \)), and \( x_i^{-1} \) is also a root of an \( n \)th degree polynomial, coefficients in \( W \). (3) Every monomial has an inverse in \( R \).

I cannot however settle this basic question: Is \( R \) a domain? I have very little to offer on either side of this question, but, hoping for an affirmative answer, I state it as

**Conjecture 1.9.** \( R \) is a domain.

A key would seem to be to establish whether or not a product of commutators \( [x_i, x_j], i \neq j \), can be zero in \( R \).

2. **INVASUNITIE UNDER** \( G = < (12...n) > \)
In constructing the ring $R$ we have forced the elementary polynomials $\sigma_k$ to be fixed under the group $G$ of circular permutations. $R$ will contain other polynomials fixed by $G$, namely the images in $R' = P/I$ of those polynomials in the free ring $P$ already fixed by $G$ (because $I^g = I$ for all $g$ in $G$). So any information about the fixed ring of $P$ should, via the map $\varphi : P \to R' \subseteq R$ provide information about $R$. In [1] Bergman and Cohn have solved a much more general problem. They consider the free product of a family of copies of a ring over a skew field, and determine the structure of the subring fixed under the action of any group that acts with finite orbits. I am very grateful to Prof. Bergman for bringing this paper to my attention, and for providing valuable advice for its application to this particular case. In this section I will effect the specialization of [1] to our situation, the determination of the structure of the subring of the free product of $n$ copies of $K[x]$ fixed under the action of the group of circular permutations. For proofs, refer to [1] and the paper [2] by Wolf.

We denote by $S$ the subring of the free ring $P$ consisting of polynomials fixed by $G$. Given a monomial $u \in P$, the orbit polynomial $\overline{u}$ of $u$ is

$$\overline{u} = \sum (u^g : g \in G),$$

and consists of exactly $n$ different monomials. The orbit polynomials, together with 1, constitute a vector-space basis of $S$ over $K$. If $\overline{u}$ and $\overline{v}$ are two orbit polynomials, their product $\overline{u} \overline{v}$ will also be invariant under $G$ so will be a unique $K$-linear combination of orbit polynomials. That $K$-linear combination can be explicitly described: In the term-by-term product $\overline{u} \overline{v}$, all $n^2$ monomials $u^gv^h$ are different, and when rearranged so that each grouping consists of monomials carried into each other by the action of $G$, each grouping is an orbit polynomial, and their sum (all coefficients 1) is the unique $K$-linear representation of $\overline{u} \overline{v}$.

Next we introduce an ordering that is the key to proving that the fixed ring $S$ is freely generated by certain explicitly given orbit polynomials. Bergman and Cohn, dealing with their more general situation described earlier, order the complexions (finite nonrepetitive sequences from the index set, here $\{1, 2, ..., n\}$). For the polynomial ring under consideration here, I found it more convenient to order the monomials directly, rather than their complexions, using the first ordering given by Wolf [2; Sec. 3]. Her ordering has all the essential properties of Bergman and Cohn’s “orbital ordering” [1; pp 526-527], and works perfectly well for the group of circular permutations even though applied originally to the full permutation group.

**Definition 2.1.** [2; sec.3]. For monomials

$$u = x_{i(1)}^{j(1)} x_{i(2)}^{j(2)} \cdots x_{i(\ell)}^{j(\ell)} \quad v = x_{i'(1)}^{j'(1)} x_{i'(2)}^{j'(2)} \cdots x_{i'(m)}^{j'(m)}$$

in the free ring $P$, set $u > v$ when

(i) $\deg u > \deg v$, or
(ii) \( \deg u = \deg v \), the exponent sequences of \( u \) and \( v \) are the same (so \( \ell = m \)), and “the first nonzero difference in subscripts \( i - i' \) is less than zero”, or

(iii) \( \deg u = \deg v \), the exponent sequences of \( u \) and \( v \) are different, and “the first nonzero difference in exponents \( j - j' \) is greater than zero”. (Quoted phrases from Wolf [2])

So monomials of the same degree are ordered lexicographically by their complexions when they have the same exponent sequence, and anti-lexicographically by their exponent sequences when these are different. Hence, for example, when \( n = 4 \), \( x_1x_2^2x_3 > x_1x_2^2x_4 \) and \( x_1x_2^3 > x_1x_2x_3 \). One checks directly that Definition 2.1 provides a total ordering of the set of monomials.

Following Bergman and Cohn [1] we denote by \( Q_0 \) the set of monomials that are maximal in their \( G \)-orbits. Given a monomial \( u \), the \( n \) (different) monomials in its \( G \)-orbit (equivalently, the \( n \) monomials in its orbit polynomial \( \bar{u} \)) all have the same exponent sequence, hence are ordered lexicographically by their complexions according to 2.1(ii). Thus \( Q_0 \) consists exactly of those monomials that begin with a power of \( x_1 \). Given monomials \( u, v \) in \( Q_0 \), define \( u \cdot v \) to be the largest monomial among the \( n^2 \) monomials in the (ordinary) product \( \bar{u} \bar{v} \) of their orbit polynomials. Define also \( 1 \cdot u = u = 1 \cdot u \). The product \( \bar{u} \bar{v} \) is invariant, thus a sum (all coefficients 1) of orbit polynomials. Then \( u \cdot v \) is the largest among the maximal monomials of these orbit polynomials. This shows that.

\[ u \cdot v \in Q_0, \text{ and, upon further analysis;} \text{ provides a simple procedure for computing } u \cdot v \text{ the monomial } u \cdot v \text{ is the ordinary product of the largest monomial } y \text{ in } \bar{u} \text{ multiplied on the right by the (only) monomial in } \bar{v} \text{ whose first indeterminate equals the last one in } y. \]

For example, \( \bar{u}_1 = x_1 + x_2 + \cdots + x_n \) and \( \bar{v}_2 = x_1x_2 + x_2x_3 + \cdots + x_nx_1 \), so \( (x_1) \cdot (x_1x_2) = x_1^2x_2 \).

**Theorem 2.2.** Under the product “”, \( Q_0 \cup \{1\} \) is an associative semigroup whose atoms are precisely those monomials in \( Q_0 \) whose indeterminates occur only to the first power. \( Q_0 \cup \{1\} \) is freely generated by its atoms.

An “atom” in the semigroup \( Q_0 \cup \{1\} \) is a monomial that cannot be written as the product of two monomials. The product \( u \cdot v \) of two monomials \( u, v \) in \( Q_0 \) computed as described directly before the theorem, will always contain the square (at least) of an indeterminate. Hence a monomial whose indeterminates occur only to the first power cannot be written as a product, so is an atom. Conversely, suppose that \( u \in Q_0 \) is an atom. If \( u \) contained an indeterminate to a power greater than one, then \( u \) can be “broken” at that point, and written as product. For example, when \( n = 3 \), \( x_1x_2x_3^2x_2 = (x_1x_2^3) \cdot (x_1x_2^2x_3) \). So, if, \( u \) is an atom, then its indeterminates occur only to the first power.

“Freely generated by its atoms” means that every monomial in \( Q_0 \) is a product of atoms, and that representation is unique: if \( a_i, b_j \) are atoms, and \( a_1 \cdot a_2 \cdot \cdots \cdot a_k = b_1 \cdot b_2 \cdot \cdots \cdot b_\ell \), then \( k = \ell \), and \( a_i = b_i, i = 1, 2, \ldots, k \). Theorem 2.2 corresponds to Proposition 2.1 in [1].

A listing of atoms would begin
deg 1 \quad x_1 \\
deg 2 \quad x_1x_2, x_1x_3, ..., x_1x_n \\
deg 3 \quad x_1x_2x_1, x_1x_2x_3, ..., x_1x_2x_n, ..., x_1x_nx_1, ..., x_1x_nx_{n-1}

There are \((n - 1)^{d-1}\) atoms of degree \(d\). Here are some examples of factorizations into atoms. Write \((x_1)^a\) for \((x_1) \cdot (x_1) \cdot ... \cdot (x_1)\) \(a\) times.

\[
x_1^aq^b = (x_1)^{a-1} \cdot (x_1x_i) \cdot (x_1)^{b-1} \quad i \neq 1
\]
\[
x_1^aq^bh^c = (x_1)^{a-1} \cdot (x_1x_i) \cdot (x_1)^{b-2} \cdot (x_1x_k) \cdot (x_1)^{c-1} \quad ; i \neq j \quad i \neq 1
\]

Here \(b \geq 2\) and \(k = j^h\) where \(h = g^{-1}\) and \(g \in G\) determined by \(1^g = i\).

\[
x_1^aq^bh^c = (x_1)^{a-1} \cdot (x_1x_i) \cdot (x_1)^{c-1} \quad i \neq j \quad i \neq 1
\]
Theorem 2.3. Every polynomial in the free ring \( P = K < x_1, x_2, ..., x_n > \) that is invariant under the action of the group \( G = < (12...n) > \) of circular permutations is uniquely a polynomial in orbit polynomials \( \overline{a}_i \) where the \( a_i \) are atoms in \( Q_0 \).

This result corresponds to Theorem 1 and Corollary 1 in [1]. Theorem 2.3 is implemented exactly as in the commutative case. Given a \( G \)-invariant polynomial, it is uniquely a linear combination of orbit polynomials, so it is enough to deal with orbit polynomials. Let \( u \) be an orbit polynomial where the monomial \( u \) is chosen to be maximal in its orbit, so \( u \in Q_0 \). Factor \( u \) into a product of atoms:

\[
u = a_1 \cdot a_2 \cdot \cdots \cdot a_k \text{, } a_i \text{ atoms in } Q_0, \text{ repetitions possible}
\]

As an ordinary product of monomials, we have

\[
u = a_1 a_2^g \cdots a_k^h \text{ where } g, h \in G.
\]

Now the product \( \overline{a}_1 \overline{a}_2 \cdots \overline{a}_k \) of the orbit polynomials \( \overline{a}_i \) is itself invariant, so is a sum of orbit polynomials. Among the maximal monomials of these, \( u \) is biggest. Hence \( \overline{u} - \overline{a}_1 \overline{a}_2 \cdots \overline{a}_k \) is a sum of orbit polynomials whose maximal monomials are all smaller than \( u \). Repeat the procedure with the largest among these. The process will end in a finite number of steps, leaving \( u \) expressed as a polynomial in the \( a_i \). Here are a couple of examples of representations of orbit polynomials as polynomials in the \( \overline{a}_i \). I have taken \( n = 3 \) for simplicity.

\[
\overline{x}_1^2 = x_1^2 + x_2^2 + x_3^2 = (\overline{x}_1)^2 - (\overline{x}_1 x_2 + \overline{x}_1 x_3)
\]

\[
\overline{x}_1^3 = x_1^3 + x_2^3 + x_3^3 = (\overline{x}_1)^3 + (\overline{x}_1 x_2 \overline{x}_1 + \overline{x}_1 x_2 x_3 + \overline{x}_1 x_3 x_1 + \overline{x}_1 x_3 x_2)
\]

\[
- ((\overline{x}_1)(\overline{x}_1 x_2) + (\overline{x}_1 x_2)(\overline{x}_1) + (\overline{x}_1)(\overline{x}_1 x_3) + (\overline{x}_1 x_3)(\overline{x}_1))
\]

So we have an explicit description of the polynomials in the free ring \( P \) that are fixed under \( G \): each is uniquely a polynomial in the orbit polynomials \( \overline{a} \) where the \( a \)'s are monomials that begin with \( x_1 \) and whose indeterminates occur only to the first power. Each such polynomial, considered now in \( R' = P/I \), remains invariant there (because \( I^g = I \) for all \( g \) in \( G \)), and has the same representation as a polynomial in the \( \overline{a} \), but no longer unique. Moreover, all invariant polynomials in \( R \) arise this way.

Theorem 2.4 Every \( K \)-coefficient polynomial in \( R \) fixed by \( G \) is a \( K \)-coefficient polynomial in the orbit polynomials \( \overline{a} = \sum (a^g : g \in G) \), where the monomials \( a \) begin with \( x_1 \) and have all their indeterminates to the first power.

Proof. Given \( p \in R \) fixed by \( G \) consider the same polynomial in the free ring \( P \). It may or may not be invariant there, but \( \overline{p} = \sum (p^g : g \in G) \) is. Apply \( \varphi : P \to P/I : \varphi(\overline{p}) = \overline{p} = \overline{p} \).
\[ \sum \varphi(p^\theta) = \sum \varphi(p)^q = np, \text{ so } p = (1/n)\varphi(p^\theta) \text{ expresses } p \text{ in the form desired}. \]

For example, when \( n = 3 \), \( p = x_1x_2 - x_2x_1 \) is invariant in \( R \) but not in \( P \). Write \( \overline{p} = x_1x_2 - x_1x_3 \), whence \( x_1x_2 - x_2x_1 = (1/3)(x_1x_2 - x_1x_3) \) in \( R \).

The \( \sigma_k \) are invariant in \( R \), so they fall under Theorem 2.4. In this case the \( \alpha^\prime s \) that appear have increasing complexities, and the representation is reasonably explicit:

**Corollary 2.5.** The \( \sigma_k(x_1, x_2, \ldots, x_n), k = 1, 2, \ldots, n \), can be written in \( R \) as a linear combination of orbit polynomials with positive integer coefficients as follows

\[ n\sigma_k = \sum (\alpha_j x_1^{i(2)} \cdots x_i^{(k)}): i(\cdot) \in \{2, 3, \ldots, n\}, 1 < i(2) < \cdots < i(k) \]

where the positive integers \( \alpha_j \) are easily computed.

The result follows from formula (2) of Sec 1; I omit the details. Here is a sample calculation for \( n = 4 \).

\[
\begin{align*}
\sigma_1 &= \overline{x_1} \\
4\sigma_2 &= 3x_1x_2 + 2x_1x_3 + x_1x_4 \\
4\sigma_3 &= 2x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 \\
4\sigma_4 &= x_1x_2x_3x_4
\end{align*}
\]

So, just as in the free ring \( P \), every invariant polynomial in \( R \) is a polynomial in the orbit polynomials \( \overline{a} \), where the monomials \( a \) begin with \( x_1 \) and have all their indeterminates to the first power. In \( P \), these orbit polynomials are algebraically independent, and there are infinitely many of them. In \( R \), the central issue regarding its fixed ring is to select from these “residual atoms” a smaller generating set, hopefully finite. When \( n = 3 \) it turns out that one will do (next section).

**THE CASE \( n = 3 \).**

We assume \( n = 3 \) throughout this section. So here we are dealing with \( R^\prime = P/I \) where \( I \) is the ideal of the free ring \( P = K < x_1, x_2, x_3 > \) generated by the four polynomials

\[
\begin{align*}
A &= \sigma_2 - \sigma_2^{(1)} = [1, 2] + [1, 3] \\
B &= \sigma_2^{(1)} - \sigma_2^{(2)} = [2, 3] + [2, 1] \\
C &= \sigma_3 - \sigma_3^{(1)} = x_1x_2x_3 - x_2x_3x_1 = [1, 23] = x_2[1, 3] + [1, 2]x_3 \\
D &= \sigma_3^{(1)} - \sigma_3^{(2)} = x_2x_3x_1 - x_3x_1x_2 = [2, 31] = x_3[2, 1] + [2, 3]x_1
\end{align*}
\]

Our augmented ring \( R \) contains the field \( W = K(\sigma_1, \sigma_2, \sigma_3) \) within its center, the elementary polynomials \( \sigma_1 = x_1 + x_2 + x_3, \ \sigma_2 = x_1x_2 + x_1x_3 + x_2x_3, \ \sigma_3 = x_1x_2x_3 \) being
algebraically independent over $K$ and invariant in $R$ under the group $G = \langle (123) \rangle$.

**Definition 3.1.** Denote by $c$ the common value $[1,2] = [2,3] = [3,1]$ which is nonzero and invariant (refer to Theorem 1.3).
Lemma 3.2 In the ring $R$,

(i) $c^3$, and the two orbit polynomials $x_1 x_2 x_1$ and $x_1 x_3 x_1$, are each nonzero, central, and invariant. None belong to $W$.

(ii) Neither $c$ nor the orbit polynomial $x_1 x_2$ commutes with $x_1, x_2, or x_3$. Each has degree 3 over $W[c^3]$.

Proof. (i) The relations $A = B = C = D = 0$ hold in $R$. $C = 0$ gives $x_2 c = c x_3$, and $D = 0 \ x_3 c = c x_1$. Applying $G$ we get $x_1 c = c x_2$. These three in turn yield $x_1 c^2 = c^2 x_3, \ x_2 c^2 = c^2 x_1$, and $x_3 c^2 = c^2 x_2$. From these we get $x_i c^3 = c^3 x_i, \ i = 1, 2, 3$, thus $c^3$ is central. As for $c^3 \neq 0$, the four relations above yield all identities that hold in $R$. We have just shown that $C = 0$ and $D = 0$ are equivalent to $x_i c = c x_{i+1} (4 \equiv 1)$. The relations $A = 0, B = 0$ are equivalent to $[1, 2] = [2, 3] = [3, 1]$. Clearly $c^3 = 0$ is not a consequence of these. Obviously $c^3$ is invariant because $c$ is. As for $c^3 \notin W$, suppose that it were. Then $c^3$ would equal a rational function in $\sigma_1, \sigma_2, \sigma_3$ whose numerator would not be zero because $c^3 \neq 0$. That same equality will hold when the variables $x_i$ are allowed to commute (Theorem 1.7(i)). Then $c^3 = 0$ but the right side remains unchanged which contradicts the algebraic independence of the $\sigma_i$. Hence $c^3 \notin W$. As for the orbit polynomials $x_1 x_2 x_1$ and $x_1 x_3 x_1$ they are obviously invariant and neither is zero because they cannot be brought to zero by allowing the $x_i$ to commute. And neither can lie in $W$ because, for example, were $x_1 x_2 x_1, \in W$ then it would equal a rational function in the $\sigma_k$. But, allowing the variables to commute, we would have $x_1^2 x_2 + x_2^2 x_3 + x_3^2 x_1$ equal to a quantity invariant under $S_3$, a contradiction. Finally, as for their centrality, we need only consider one by virtue of the relation $x_1 x_2 x_1 + x_1 x_3 x_1 = \sigma_1 \sigma_2 - 3 \sigma_3$. To check this, argue as follows: $x_1 x_2 x_1 + x_1 x_3 x_1 = x_1 (x_2 + x_3) x_1 = x_1 (\sigma_1 - x_1) x_1 = \sigma_1 x_1^2 - x_1^3 = \sigma_2 x_1 - \sigma_3$, the last step because $x_1$ is a root of $y^3 - \sigma_1 y^2 + \sigma_2 y - \sigma_3$. Then pass to the orbit polynomials. We shall work with $x_1 x_2 x_1$. Because it is invariant, to prove that it is central, it is enough to prove that it commutes with $x_1$; i.e. that $[x_1, x_1 x_2 x_1] = 0$. In the following computation remember that, in $R, \sigma_3$ is central and invariant so $\sigma_3 = x_1 x_2 x_3 = x_2 x_3 x_1 = x_3 x_1 x_2$. We shall use the simplifying notation $[x_1, x_1 x_2 x_1] = [1, 121]$.

$$[1, 121] = [1, (12)1] + [1, (23)2] + [1, (31)3]$$

$$= [1, 12] 1 + 23 [1, 2] + [1, 23] 2 + 31 [1, 3] + [1, 31] 3$$

$$= 1 \ 12 \ 23 + 23 [1, 2] + 31 [1, 3] + [1, 31] 13 \ (1, 23) = 0$$

$$= x_1 c x_1 + x_2 x_3 c - x_3 x_1 c - c x_1 x_3$$

$$= x_1 x_3 c + x_2 x_3 c - x_3 x_1 c - x_3 x_2 c$$

$$= ([1, 3] + [2, 3]) c = (-c + c) c = 0 c = 0.$$ 

Hence $x_1 x_2 x_1$ and $x_1 x_3 x_1$ are central. The fact that $c^3$, which has degree 6, and the cubics $x_1 x_2 x_1$ and $x_1 x_3 x_1$ are central contrasts with the fact that every central quadratic polynomial in $R$ is a $K$-coefficient polynomial in $\sigma_1$ and $\sigma_2$ (proof omitted).

(ii) Because $x_1 x_2$ and $c$ are related by the easily verified formula $x_1 x_2 = c + \sigma_2$, it is enough to deal with $c$. And, as $c$ is invariant, we need only show that it does not commute with $x_1$. Note that if we knew that $R$ was a domain (Conjecture 1.9), the proof would be trivial: from $x_1 c = c x_1$ and $x_1 c = c x_2$ we would have $c(x_1 - x_2) = 0$ a contradiction as neither is
zero. To avoid relying on the truth of Conjecture 1.9 at this point, I supply a considerably longer proof which goes as follows. Use $c = [2, 3]$ and compute $[1, c] = [1, [2, 3]] = -x_1 x_3 x_2 + x_3 x_2 x_1$. So it is a matter of showing that $x_1 x_3 x_2 - x_3 x_2 x_1 \neq 0$ in $R$. I shall prove this by showing that, in the free ring $P$, $x_1 x_3 x_2 - x_3 x_2 x_1 \notin I$. A homogeneous cubic will belong to $I$ if and only if it can be written as a $K$-linear combination of the following four polynomials: $qAr$, $sBt$, $uCv$, and $yDz$, where the eight monomials involved must satisfy $\deg q + \deg r = \deg s + \deg t = 1$ and $\deg u = \deg v = \deg y = \deg z = 0$. So we have the following six expressions:

\[
(\alpha_1 + \alpha_2 + \alpha_3) (12 - 21 + 13 - 31)
(\beta_1 + \beta_2 + \beta_3) (12 - 21 + 13 - 31)
(\gamma_1 + \gamma_2 + \gamma_3) (23 - 32 + 21 - 12)
(\delta_1 + \delta_2 + \delta_3) (23 - 32 + 21 - 12)
\mu (123 - 231)
\sigma (231 - 312)
\]

where I have written 12 for $x_1 x_2$, etc, and the greek letters stand for elements of $K$. There are 27 monomials of degree 3. We may disregard the three that have exponent sequence 3. I have listed below the remaining 24 in decreasing order.

| Exponent sequence | Monomials |
|-------------------|-----------|
| 21                | $1^2 2 > 1^2 3 > 2^2 1 > 2^2 3 > 3^2 1 > 3^2 2$ |
| $\lor$            | $1^2 2 > 1^2 3 > 2^2 1 > 2^2 3 > 3^2 1 > 3^2 2$ |
| 12                | $1^2 2 > 1^2 3 > 2^2 1 > 2^2 3 > 3^2 1 > 3^2 2$ |
| $\lor$            | $1^2 2 > 1^2 3 > 2^2 1 > 2^2 3 > 3^2 1 > 3^2 2$ |
| 111               | $1^2 1 > 1^2 3 > 1^3 1 > 1^3 2 > 1^2 3 > 2^3 1 > 2^3 2 > 31 2 > 32 1 > 31 3 > 32 1 > 32 3$ |

Only 132-321 appears on the right side of our equation, so all other monomials must have coefficient zero. Beginning with the monomials with exponent sequence 21 we get

\[
(\alpha_1 - \gamma_1) 1^2 2 + \alpha_1 1^2 3 + (-\alpha_2 + \gamma_2) 2^2 1 - \gamma_2 2^2 3 + (-\alpha_3) 3^2 1 + (-\gamma_3) 3^2 2
\]

which implies $\alpha_1 = \alpha_2 = \alpha_3 = \gamma_1 = \gamma_2 = \gamma_3 = 0$. Working similarly with exponent sequence 12 we get $\beta_1 = \beta_2 = \beta_3 = \delta_1 = \delta_2 = \delta_3 = 0$. So we are left with

\[
\mu (123 - 231) + \sigma (231 - 312) = 132 - 321
\]

which is clearly impossible. Thus our result: $c$, so also $\overline{x_1 x_2} = c + \sigma_2$, commutes with neither $x_1$, $x_2$, nor $x_3$. As for the $Z$-coefficient cubic satisfied by $\overline{x_1 x_2}$, just cube the equation $\overline{x_1 x_2} - \sigma_2 = c$ to get $(\overline{x_1 x_2})^3 - 3\sigma_2 (\overline{x_1 x_2})^2 + 3\sigma_2 (\overline{x_1 x_2}) - (\sigma_2^2 + c^3) = 0$. ◊
Definition 3.3. Let $Z = W[c^3, x_1, x_2, x_1]$ which is a central subring of $R$ pointwise fixed by $G$. (The bracket $[ ]$ here denotes adjunction, not the commutator.) Denote by $S$ the subring of $R$ consisting of all polynomials fixed by $G$. 
**Theorem 3.4.** Every \( p \in S \) has the form \( z_0 + z_1c + z_2c^2 \) for some \( z_i \in Z \). Accordingly

(i) \( S \) is a commutative degree 3 extension of \( Z \), and 

(ii) For every \( p \in R \), \( \overline{p} = \sum(p^g : g \in G) \) is either central or is a root of a cubic, coefficients in \( Z \).

**Proof.** In this proof we shall refer to monomials in \( R \) that begin with \( x_1 \) and have all their indeterminates to the first power as “residual atoms”. These are not atoms in any technical sense - this is just a handy notation. In Theorem 2.4 we have shown that every \( K \)-coefficient polynomial in \( R \) that is invariant under \( G \) is a \( K \)-coefficient polynomial in the orbit polynomials of residual atoms. So it is enough to prove that every orbit polynomial of a residual atom is a polynomial in \( x_1x_2 \), coefficients in \( Z \). We begin by establishing this for the residual atoms of degree \( \leq 3 \):

\[
\begin{align*}
\overline{x_1} &= \sigma_1 \in Z \\
\overline{x_1x_3} &= 3\sigma_2 - 2\overline{x_1x_2} = 3\sigma_2 - 2(c + \sigma_2) = \sigma_2 - 2c \notin Z \\
\overline{x_1x_2x_1} &= \overline{3\sigma_3} \in Z \\
\overline{x_1x_3^4} &= \sigma_1\sigma_2 - 3\sigma_3 - \overline{x_1x_2x_1} \in Z \\
\overline{x_1x_2x_2} &= \sigma_1\sigma_2 + 3\sigma_3 - (x_1x_2) \sigma_1 = 3\sigma_3 - c\sigma_1 \notin Z
\end{align*}
\]

Relations 1,3, and 4 are evident; number 5 was done in the proof of Lemma 3.2. We begin with 2:

\[
3\sigma_2 - 2\overline{x_1x_2} = \sigma_2 + 2(\sigma_2 - \overline{x_1x_2}) = \sigma_2 - 2c = \sigma_2 - c - c \\
= \sigma_2 - (x_1x_2 - x_2x_1) - (x_2x_3 - x_3x_2) \\
= x_1x_2 + x_1x_3 + x_2x_3 - x_1x_2 + x_2x_1 - x_2x_3 + x_3x_2 \\
= x_1x_3 + x_2x_1 + x_3x_2 = x_1x_3.
\]

Finally number 6, which uses \( c = x_2x_3 - x_3x_2 = \overline{x_1x_2} - \sigma_2 \):

\[
\begin{align*}
x_1x_3x_2 &= x_1(x_2x_3 - c) = x_1(x_2x_3 + \sigma_2 - \overline{x_1x_2}) \\
&= x_1x_2x_3 + x_1\sigma_2 - x_1(\overline{x_1x_2}) \, \text{, whence} \\
\overline{x_1x_3x_2} &= \overline{x_1x_2x_3} + \overline{x_1\sigma_2} - \overline{x_1(\overline{x_1x_2})} \\
&= 3\sigma_3 + \sigma_1\sigma_2 - \overline{\sigma_1}\overline{x_1x_2}.
\end{align*}
\]

Let us call a residual atom a reducible if \( \overline{\sigma} \) is a \( Z \)-coefficient polynomial in orbit polynomials of residual atoms of degree strictly less than that of \( a \). We shall prove that every residual atom of degree \( \geq 4 \) is reducible which will establish Theorem 3.4. A residual atom \( a \) of degree \( \geq 4 \) will begin 121..., 123..., 131..., or 132... . If \( a = 121u \) then the monomial \( u \) must begin with either 2 or 3. In the first case use 12 = \( \sigma_2 - 13 - 23 \) to get \( a = 1212... = (\sigma_2 - 13 - 23)12.. = (12)\sigma_2 - 1312... - 2312... \). In \( R, \sigma_3 = 312, \) so we get \( \overline{\sigma} = (12)\sigma_2 - \sigma_3(1...) - \sigma_3(2...) \). In the second case \( a = 1213... = 1(-c + 12)3... = -1c3... - 1(123)... = -c23... - \sigma_3(1...) = -(x_1x_2 - \sigma_2)23... - \sigma_3(1...) \) So
\[ \sigma = -(x_1x_2)(23...) - \sigma_2(23...) - \sigma_3(1...). \] If \( a = 123u \) then \( \sigma = \sigma_3\sigma \). If \( a = 131u \) and \( u \) begins with 2, then \( a = 1312... = \sigma_3(1...) \). If \( u \) begins with 3, then use \( \sigma_2 = 23+21+31 \) to write \( a = 1(\sigma_2 - 23 - 21)3... = \sigma_2(3...) - (123)(3...) - 1213... \). Then \( \sigma = \sigma_3(3...) - \sigma_3(3...) - 1213... \) and the last term begins with 121 so is reducible by our first case. Finally, if \( a = 132u \) then we write \( 13 = 31 - c = 31 - T2 + \sigma_2 \) to get \( a = (31 - T2 + \sigma_2)2u = 312u - (T2)(2u) + \sigma_2(2u) \) whence \( \sigma = \sigma_3\sigma - (T2)(2u) + \sigma_2(2u) \).}

**Theorem 3.5.** If \( R \) is a domain, then

(i) its center is \( Z \),

(ii) both \( c \) and \( x_1x_2x_1 \) are algebraically independent transcendentals over \( W \), and

(iii) when \( R \) is enlarged as described below, every nonzero invariant polynomial is invertible in \( R \).

**Proof.** (i) If \( p \) is central it commutes with \( c \) so \( cp = pc \). Multiplication on the right by \( c \) has the effect of applying \( g = (123) \), \( pc = cp^g \), whence \( c(p - p^g) = 0 \) so that \( p \) is invariant. Then \( p = z_0 + z_1c + z_2c^2 \) must commute with \( x_1 \) which forces \( z_1 = z_2 = 0 \)

(ii) We shall first prove that \( c \) and \( x_1x_2x_1 \) are individually transcendental over \( W \). Suppose that \( \sigma \) were a root of a monic polynomial, \( \sum a_jc^j = 0 \), where the \( a_j \in W \). If this polynomial has no constant term, we may factor out a power of \( c \) to get \( c^k\sum b_jc^j = 0 \) where the latter polynomial now has a nonzero constant term. It must itself be zero as we are working in a domain and \( c \neq 0 \). Now allow the \( x_i \) to commute. All \( c \)'s will vanish leaving our constant term equal to zero contradicting the algebraic independence of the “elementary symmetric functions” in commuting variables. The proof for \( x_1x_2x_1 \) is similar except that, when the \( x_i \) are allowed to commute, we reach the contradiction that a polynomial in \( x_1^2x_2 + x_2^2x_3 + x_3^2x_1 \), which is not invariant under \( S_3 \), equals a rational function in the \( \sigma_k \), which is. Finally, the algebraic independence of \( c \) and \( x_1x_2x_1 \). Set \( d = x_1x_2x_1 \) for convenience. Let \( \sum a_{ij}\sigma^i \) be a monic polynomial in which \( c \) and \( d \) both actually appear, \( a_{ij} \in W \). If there is no constant term we may factor out \( c^r d^s \) (one of the integers \( r \) or \( s \) may be zero) to get \( c^r d^s \sum b_{ij}\sigma^i \) where the latter polynomial does have a nonzero constant term. If the original polynomial is zero, so is the latter. Then argue as before.

(iii) Owing to the fact that \( c^3 \) and \( x_1x_2x_1 \) are central algebraically independent transcendentals over \( W \), we may enlarge \( R \) using the same method described in the paragraph following Definition 1.8 to a ring that has the field of fractions \( W(c^3, x_1x_2x_1) \) as its center. Let us use the same symbol \( R \) for this enlarged ring, and the same symbol \( Z \) for its center \( W(c^3, x_1x_2x_1) \) which we note has transcendence degree 5 over \( K \). Then, in this \( R \), every nonzero invariant polynomial either lies in the center \( Z \), in which case it is invertible, or satisfies a cubic with nonzero constant term that lies in \( Z \), thus invertible in this case as well. 

I have not been able to answer the natural follow-up question to Theorem 3.5: is \( R \) itself a field? More to the point, does this method of construction (forming the quotient of the free ring \( K < x_1, x_2, ..., x_n > \) by the ideal \( I \) generated by the \([x_i, \sigma_k]\)) provide a way to construct finite-dimensional fields? If so, do said fields have any novel properties? All this is related to the long standing open question: is every field of degree 5 cyclic?
References

[1] G. M. Bergman and P. M. Cohn, “Symmetric elements in free powers of rings”, J. London Math. Soc., (2), 1 (1969), 525-534.

[2] M. C. Wolf, “Symmetric functions of noncommutative elements”, Duke Math. J., 2(1936), 626-637.