WHEN IS THE SUM OF TWO CLOSED SUBGROUPS CLOSED IN A LOCALLY COMPACT ABELIAN GROUP?

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Abstract. Locally compact abelian groups are classified in which the sum of any two closed subgroups is itself closed. This amounts to reproving and extending results by Yu. N. Mukhin from 1970. Namely we contribute a complete classification of all totally disconnected locally compact abelian groups with $X + Y$ closed for any closed subgroups $X$ and $Y$.

1. Introduction and Main Results

It was R. Dedekind, who in 1877 proved the modular law for the subgroup lattice of a certain abelian group (see [3]). However, his proof works for any abelian group. In 1970 Yu. N. Mukhin investigated the analogous property for locally compact abelian groups in [10]. The closed subgroup lattice $L(G)$ of a topological group is its set of closed subgroups endowed with join given as $A \vee B := \langle A \cup B \rangle$ and meet as $A \wedge B := A \cap B$ for $A$ and $B$ any closed subgroups. Then $G$ is a topologically modular group provided the modular law holds for any closed subgroups $A$, $B$ and $C$ of $G$ with $A$ a subgroup of $C$:

$$A \vee (B \wedge C) = (A \vee B) \wedge C$$

Note that a group is modular if, and only if, its lattice of closed subgroups does not contain a sublattice isomorphic to $E_5$ (cf. [13] 2.1.2 Theorem), i.e., geometrically, a pentagon (see also Remark 2.8 below).
Mukhin, in the same paper, also classifies all locally compact abelian groups $G$ for which $X + Y$ is closed whenever $X$ and $Y$ are closed subgroups of $G$ and we will call any locally compact abelian group $G$ with this property strongly topologically quasihamiltonian. It follows from the definitions that every strongly topologically quasihamiltonian group is topologically modular.

We shall derive his results with slightly different approach, but essentially the same methods of proof. Our motivation comes from studying nonabelian locally compact groups satisfying an analogous property, see [5].

Let us fix some notation. We mostly use the notation from [9]. The $\mathbb{Z}$-rank of a discrete abelian group $A$ is the torsion free rank, i.e., the $\mathbb{Q}$-dimension of $A \otimes \mathbb{Z} \mathbb{Q}$. When $A$ is torsion-free then the $\mathbb{Z}$-rank of $A$ is the dimension of its dual $\hat{A}$ (see e.g. Theorem 8.22 in [9]). We shall call a locally compact abelian group $A$ periodic if it is both totally disconnected and the union of its compact subgroups. We shall use additive notation unless stated differently. For $p$ a prime, an element $x$ in a locally compact abelian group $G$ with $p^k x \to 0$ as $k$ tends to infinity, is called a $p$-element. As discussed on page 48 in [9] this definition is equivalent to saying that $\langle x \rangle$ is a pro-$p$ group. In a periodic locally compact abelian group $A$ the set of $p$-elements is a closed subgroup $A_p$ – its $p$-primary component (or $p$-Sylow subgroup, see [9] Definitions 8.7.). In a periodic abelian group, for a set of primes $\pi$ the $\pi$-primary component $A_\pi$ of a periodic group $A$ is defined to be the subgroup of $A$ topologically generated by all $p$-primary components with $p \in \pi$. For a periodic group $A$ we denote by $\pi(A)$ the set of all primes $p$ with $A_p$ not trivial. If $\pi(A) = \{p\}$, for a single prime $p$, then $A$ is a $p$-group. For a compact group our definition agrees with [9, Definition 8.7]. For any fixed prime $p$ the kernel of the map $x \mapsto px$ is the $p$-socle of $A$ and will be denoted by $\text{socle}_p(A)$ or just by $\text{socle}(A)$ if there is no danger of confusion (see [9, Definition A1.20].).

A locally compact abelian group $A$ will be termed finitely generated if there is a finite subset $X$ of $A$ which generates $A$ topologically, i.e., $A = \langle X \rangle$. We say that a locally compact abelian $p$-group $A$ has finite $p$-rank
n if, and only if, every finitely generated closed subgroup \( H \) has a set of topological generators of cardinality \( n \) and, in addition, \( A \) contains a finitely generated subgroup which cannot be generated topologically with fewer elements. If \( G \) contains finitely generated subgroups of arbitrary large rank then the \( p \)-rank of \( G \) is said to be infinite. For a finitely generated compact \( p \)-group \( A \) this definition agrees with \( d(A) \), the minimum cardinality of a topological set of generators of \( A \), as given in [12, p. 43], i.e., \( d(A) = \text{rank}_p(A) \). Moreover, by [12, Proposition 4.3.6], for any closed subgroup \( H \) of \( A \) one has \( d(H) \leq d(A) \). Therefore for any finite \( p \)-rank abelian \( p \)-group \( G \) and closed subgroup \( H \) one has \( \text{rank}_p(H) \leq \text{rank}_p(G) \).

It is a consequence of [6, Lemma 3.9] that our definition is equivalent to the one given by Čarin in [2]. If \( A \) is not topologically finitely generated, we say that \( A \) has infinite \( p \)-rank. A proposal for defining the \( p \)-rank of an arbitrary locally compact abelian \( p \)-group has been recently made in [7, Section 10], which got reproduced in [6, 3.10, p. 93]. For a prime \( p \) the symbols \( \mathbb{Q}_p, \mathbb{Z}_p, \mathbb{Z}(p\infty) \) denote respectively the additive group of the field of \( p \)-adic rationals, its closed subgroup of \( p \)-adic integers, and the factor group \( \mathbb{Q}_p/\mathbb{Z}_p \) (see [9, p. 28 ff]).

The set of elements of finite order of an abelian group \( G \) will be denoted by \( \text{tor}(G) \) and we let \( \text{div}(G) \) stand for the largest divisible subgroup of \( G \). The main properties of \( \text{tor}(G) \) and \( \text{div}(G) \) are discussed in [9, Appendix 1].

Our main results are as follows:

**Theorem 1.1.** The following statements for a locally compact abelian \( p \)-group \( G \) are equivalent:

(a) \( G \) is topologically modular.

(b) For \( U \) an open compact subgroup exclusively one of the following holds

(b.1) \( U \) has finite \( p \)-rank. Then \( \text{tor}(G) \) is discrete and \( G/\text{tor}(G) \) has finite \( p \)-rank.

(b.2) \( U \) has infinite \( p \)-rank. Then \( \text{div}(G) \) is closed, \( G/U \) and \( \text{div}(G) \) both have finite \( p \)-rank, and, \( G/\text{div}(G) \) is compact.

(c) \( G \) is strongly topologically quasihamiltonian.

**Theorem 1.2.** A periodic locally compact abelian group \( G \) is topologically modular if, and only if, for every prime \( p \) the respective \( p \)-component is topologically modular.

The next two results exhibit the structure of any torsion strongly topologically quasihamiltonian group and the splitting of the torsion subgroup in topologically modular groups.
Theorem 1.3. Let $A$ be a locally compact abelian torsion group. The following statements are equivalent:

(a) The group $A$ is topologically modular.
(b) There is a partition 
$$\pi(A) = \delta \cup \phi$$
and all of the following holds:
(b.1) The set of primes $\phi$ is finite and 
$$A_\phi = D_\phi \oplus V_\phi$$
for $D_\phi$ discrete and divisible and $V_\phi$ compact and open in $G$.
(b.2) $A_\delta$ is a discrete subgroup of $A$.
(c) The group $A$ is strongly topologically quasihamiltonian.

Theorem 1.4. Let $G$ be a totally disconnected locally compact abelian group – neither discrete nor periodic.
Then the following statements are equivalent.

(a) $G$ is topologically modular.
(b) All of the following holds:
(b.1) $T = \text{tor}(G) = \text{comp}(G)$ is open in $G$ and $G/T$ is discrete and torsion-free of finite $\mathbb{Z}$-rank.
(b.2) The torsion subgroup $T = \text{tor}(G)$ is strongly topologically quasihamiltonian.
Moreover, if $N$ is any closed subgroup of $G$, contained in $T$ then \( \text{tor}(G/N) = T/N \) and (b) holds for $G/N$ with $T$ replaced by $T/N$.
(c) $G$ is strongly topologically quasihamiltonian.

The preceding result corrects [6, Theorem 14.32(b.3)].
Using Pontryagin duality (see [9, Chapter 7]) we shall deduce a structure theorem for locally compact abelian topologically modular groups with nontrivial connected components, see Theorem 3.8.

The fact that not every periodic nondiscrete topologically modular group is strongly topologically quasihamiltonian, will be shown in Lemma 2.17. A locally compact abelian group $A$ is inductively monothetic provided every finite subset of $A$ is contained in a monothetic subgroup (see Definition 4.10 below):

Theorem 1.5. For a locally compact abelian periodic group $A$ and open compact subgroup $U$ the following statements are equivalent:

(A) $A$ is strongly topologically quasihamiltonian.
(B) There is a partition of $\pi(A)$ into 4 disjoint subsets $\delta$, $\gamma$, $\phi$, and $\mu$ and all of the following holds:
(i) $\delta := \{ p \in \pi(A) : A_p \cap U = \{0\} \}$ and $A_\delta$ is a discrete subgroup of $A$. 
(ii) $\gamma := \{p \in \pi(A) : A_p \leq U\}$ and $A_\gamma$ is a profinite subgroup of $A$.

(iii) $\phi := \{p \in \pi(A) \setminus \{\delta \cup \gamma\} : \text{rank}_p(A_p) \geq 2\}$. The set $\phi$ is finite and for all $p \in \phi$ the $p$-Sylow subgroup $A_p$ is strongly topologically quasihamiltonian.

(iv) $A_\mu$ is inductively monothetic.

(v) $A = A_\delta \oplus A_\gamma \oplus A_\phi \oplus A_\mu$ topologically and algebraically.

This result, we feel, is our genuine contribution. Namely, for periodic topologically modular groups Theorem 2 in [10] and its proof seem not to lead to a proof of our description of periodic nondiscrete strongly topologically quasihamiltonian groups. We also correct [6, Theorem 14.22(B)], where $\gamma$ and $A_\gamma$ are missing in the decomposition.

The concluding Section 4 contains several consequences.

2. Preliminaries

In a number of places we shall need a fact about compact abelian torsion groups, [9, Corollory 8.9], which we rephrase here:

**Proposition 2.1.** The following statements about a compact abelian group $G$ are equivalent:

(a) $G$ is a torsion group.
(b) $G$ is profinite and has finite exponent.
(c) $G = \prod_{p \in S} G_p$ is the cartesian product of compact $p$-groups of finite exponent for $p$ in a finite set $S$.

We have the following observation:

**Lemma 2.2.** Every strongly topologically quasihamiltonian group is topologically modular.

**Proof.** The equality $(A \lor B) \land C = A \lor (B \land C)$ for closed subgroups $A \subseteq C$ and $B$ follows from the containments

$(A \lor B) \land C = (A + B) \cap C \subseteq A + (B \cap C) = A \lor (B \land C)$,

$A \lor (B \land C) = A + (B \cap C) \subseteq (A + B) \cap C = (A \lor B) \land C$.

□

A result by Čarin (see [2, Theorem 5]) and a mild generalisation in [7] will be used frequently.

**Proposition 2.3.** For a locally compact abelian $p$-group $G$ the following conditions are equivalent:

(1) $G$ has finite $p$-rank.
(2) There is a compact open subgroup $U$ such that both $\text{rank}_p(U)$ and $\text{rank}_p(G/U)$ are finite.

(3) There is a compact open subgroup $U$ such that $U \cong \mathbb{Z}_p^m \oplus F$ for a nonnegative integer $m \in \mathbb{N}_0$ and a finite abelian group $F$ and that the $p$-socle $\text{socle}_p(G/U)$ is isomorphic to $\mathbb{Z}(p)^n$ for some $n \in \mathbb{N}_0$.

(4) There is a natural number $r$ such that every finitely generated subgroup of $G$ can be generated by at most $r$ elements.

(5) There are nonnegative integers $m$, $n$, $k$, and a finite $p$-group $F$ such that algebraically and topologically $G \cong \mathbb{Q}_p^m \oplus \mathbb{Z}(p^\infty)^n \oplus \mathbb{Z}_p^k \oplus F$.

Lemma 2.4 ([7, Lemma 3.6]). For a locally compact abelian $p$-group $G$ the following statements are equivalent:

(a) $G$ is finitely generated.
(b) $G$ is compact and has finite $p$-rank.
(c) There are $m \geq 0$ and a finite abelian $p$-group $F$ such that $G$ is algebraically and topologically isomorphic to $\mathbb{Z}_p^m \oplus F$.

Lemma 2.5. Let $G$ be a locally compact abelian torsion-free $p$-group containing a compact open subgroup $U$ of finite $p$-rank. Then

$$\text{rank}_p(G) = \text{rank}_p(U).$$

Proof. By the definition of the $p$-rank we need to prove that every finitely generated subgroup $T$ of $G$ satisfies $d(T) = \text{rank}_p(T) \leq r := \text{rank}_p(U)$. Lemma 2.4 implies that $T$ is compact and so is $T + U$. Because of $\text{rank}_p(T) \leq \text{rank}_p(T + U)$ it will suffice to prove $\text{rank}_p(T + U) \leq r$, i.e., we may assume $U \leq T$. Since $T$ is compact and $U$ is an open subgroup there is $k \geq 0$ such that $|T/U| = p^k$. The homomorphism $\phi : T \to U$ sending $t \in T$ to $p^k t$ is continuous and injective and therefore the compact subgroup $\phi(T) = p^k T \cong T$ algebraically and topologically. Deduce from this that

$$\text{rank}_p(T) = \text{rank}_p(p^k T) \leq r = \text{rank}_p(U) \leq \text{rank}_p(T),$$

showing $\text{rank}_p(T) = r$, as desired. \qed

We record a well known fact, see e.g. [1] 2.13 Corollary]:

Lemma 2.6. A locally compact abelian group $G$ is a $p$-group if, and only if, its dual $\hat{G}$ is.

Remark 2.7. As has been said above, a lattice is modular if, and only if, it is $E_5$-free (see [13] 2.1.2 Theorem]. The absence of $E_5$ in the closed subgroup lattice is inherited by closed subgroups and quotient groups. Hence closed subgroups and quotient groups of a topologically
modular group are topologically modular groups. Moreover, a locally compact abelian group $G$ is topologically modular if, and only if, its Pontryagin dual $\hat{G}$ is topologically modular (the latter fact follows from applying the Annihilator Mechanism, see [9, p. 314]).

However, the class of topologically modular groups fails to be closed under the formation of strict projective limits and (local) products as the following example, due to Mukhin shows (see [10] which will be reproduced in Example 2.11 below).

**Remark 2.8.** Every group $G$ that is not topologically modular must contain closed subgroups $A$, $B$, and $C$, where $A \subseteq C$, the meet $B \wedge C$ is a proper subgroup of $A$, and, $C$ is a proper subgroup of the join $A \vee B$. Then the five closed subgroups

\[(\dagger) \quad A \vee B, \quad C, \quad A, \quad B, \quad B \wedge C\]

form a subgroup sublattice of the lattice of closed subgroups of $G$ or equivalently, the five groups in Eq. (\dagger) are all pairwise different and $B \cap C \subseteq A$.

Indeed, if the closed subgroups $X \subset Z$ and $Y$ do not satisfy the modular identity then $A := X \vee (Y \wedge Z)$, $B := Y$, and, $C := (X \vee Y) \wedge Z$ serve the purpose.

This observation provides a simple method for exhibiting important examples of locally compact abelian groups not topologically modular.

**Example 2.9.** Let $G := \mathbb{R}$ be the reals and fix subgroups $C := \mathbb{Z}$, $A := 2\mathbb{Z}$, and, $B := \sqrt{2}\mathbb{Z} = \{z\sqrt{2} : z \in \mathbb{Z}\}$. Then $A \vee B = \mathbb{R}$ by the density of $2\mathbb{Z} + \sqrt{2}\mathbb{Z} = 2(\mathbb{Z} + \sqrt{2}/2\mathbb{Z})$. Moreover, $B \wedge C = \{0\}$ is contained in $A$.

**Example 2.10.** Let $p$ be a prime and $G = \mathbb{Z} \oplus K$ be the topological direct sum of a discrete group $Z \cong \mathbb{Z}$ and an infinite compact monothetic group $K$. Suppose that $K = pK$, i.e., $K$ is $p$-divisible.

Fix a topological generator $k$ of $K$ and a generator $t$ of $Z$. Let $C := Z$, $A := p\mathbb{Z}$, and, $B := \{zt + zk : z \in \mathbb{Z}\}$ and observe that it is the graph of the homomorphism $f : Z \to K$ sending the generator $t$ of $Z$ to the generator $k$ of $K$. Hence $B$ is discrete. For proving $G = A \vee B$ observe first that $pK = K$ implies that $\langle pk \rangle = K$. Then $A + B$ contains all elements of the form $put + v(t + k) = put + vt + vk$ for $u$ and $v$ in $\mathbb{Z}$. Select $u := 1$ and $v := -p$ in order to see that $pk \in A + B$. Therefore $A \vee B = A + B$ contains $\langle pk \rangle = K$ and hence

\[(1) \quad (A \vee B) \wedge C = G \cap C = C = Z.\]
Suppose there exists an integer $z \in \mathbb{Z}$ such that $z(t + k) = zt + zk \in B \cap C$. This implies $zk = 0$. However, $K = \langle k \rangle$ is an infinite monothetic group and thus $k$ cannot be a torsion element. Thus $z = 0$ and therefore $B \cap C = \{0\}$, so that taking Eq. (11) into account,

$$A \vee (B \cap C) = A \neq C = (A \vee B) \wedge C$$

follows. Thus $G$ is not topologically quasihamiltonian.

**Example 2.11.** Let $S := \mathbb{Z}(p)^{(\mathbb{N})}$ and $P := \mathbb{Z}(p)^{\mathbb{N}}$ and form $G := S \oplus P$, the topological direct sum. Let $\iota : S \rightarrow P$ be the canonical dense embedding of $S$ in $P$ and $K \cong \mathbb{Z}(p)$ a finite subgroup of $P$ intersecting $\iota(S)$ trivially. Such $K$ can be provided by the subgroup of all constant maps $\mathbb{N} \rightarrow \mathbb{Z}(p)$. Define closed subgroups $C := S \oplus K$, $A := S$, and, $B := \{(s + \iota(s)) : s \in S\}$. Then $B$ is algebraically and topologically isomorphic to the graph of the function $\iota$ and hence a discrete subgroup of $G$. Then, for $x$ to belong to $B \cap C$ it is necessary and sufficient that there are $s, s' \in S$ an $k \in K$ with

$$x = s + \iota(s) = s' + k.$$

Since $K \cap \iota(S) = \{0\}$ we must have $\iota(s) = s = k = 0$. Hence $B \wedge C = \{0\}$. Since $A + B = S + \iota(S)$ and $\iota(S) = P$ one finds $A \vee B = A + B = S + P = G$.

For describing the next examples, and also later, for the proof of Theorem 3.1, we need to recall the notion of *local product* of locally compact groups.

**Definition 2.12.** Let $(G_j)_{j \in J}$ be a family of locally compact groups and assume that for each $j \in J$ the group $G_j$ contains a compact open subgroup $C_j$. Let $P$ be the subgroup of the cartesian product of the $G_j$ containing exactly those $J$-tuples $(g_j)_{j \in J}$ of elements $g_j \in G_j$ for which the set $\{j \in J : g_j \notin C_j\}$ is finite. Then $P$ contains the cartesian product $C := \prod_{j \in J} C_j$ which is a compact topological group with respect to the Tychonoff topology. The group $P$ has a unique group topology with respect to which $C$ is an open subgroup. Now the *local product* of the family $((G_j, C_j))_{j \in J}$ is the group $P$ with this topology, and it is denoted by

$$P = \prod_{j \in J}^{\text{loc}} (G_j, C_j).$$

Finally, when $G = \prod_{i \geq 1}^{\text{loc}} (G_i, C_i)$ is a local product and $G_i \cong A$ and $C_i \cong B$ algebraically and topologically then we shall denote $G$ by $(A, B)^{\text{loc}, N}$. 
Example 2.13. Let us show that the local product
\[ L := (\mathbb{Z}(p^2), p\mathbb{Z}(p^2))^{\text{loc},\mathbb{N}} \]
cannot be topologically modular.

We are going to show that a closed subgroup of a Hausdorff quotient group of \( L \) is not topologically modular and hence \( L \) is not topologically modular by Remark 2.7. Select an infinite subset \( I \) of \( \mathbb{N} \) with infinite complement \( J := \mathbb{N} \setminus I \). Then there is a topological and algebraic isomorphism
\[ L \cong L_I \oplus L_J, \]
where \( L_I := (\mathbb{Z}(p^2), p\mathbb{Z}(p^2))^{\text{loc},I} \) and \( L_J := (\mathbb{Z}(p^2), p\mathbb{Z}(p^2))^{\text{loc},J} \) are both algebraically and topologically isomorphic to \( L \). The socles \( S_I \) and \( S_J \) of respectively \( L_I \) and \( L_J \) are compact and open therein and isomorphic to \( \mathbb{Z}(p)^\mathbb{N} \). Since \( L_I/S_I \cong \mathbb{Z}(p)^{(\mathbb{N})} \) the subquotient \( L_I/S_I \oplus S_J \) of \( L \) is algebraically and topologically isomorphic to \( \mathbb{Z}(p)^{(\mathbb{N})} \oplus \mathbb{Z}(p)^\mathbb{N} \), which is not topologically modular, as has been shown in Example 2.11.

Lemma 2.14. Let \( C \) be a compact monothetic not torsion group. Then there is a prime \( p \) and a monothetic subgroup \( K \) of \( C \) with \( pK = K \) and \( K \) is not torsion.

Proof. If the connected component \( C_0 \) of \( C \) is not trivial we may choose \( K := C_0 \). Since \( C_0 \) is divisible, for any prime \( p \), \( pK = K \). Since the weight of \( C_0 \) does not exceed the weight of \( C \) infer from \([8, (25.17)\) Theorem] that \( C_0 \) is monothetic.

Next assume \( C_0 = \{0\} \). Then \( C \) is profinite and abelian and hence pronilpotent. Making use of \([12, Proposition 2.3.8]\), we deduce that \( C = \prod_p C_p \) is the cartesian product of its \( p \)-Sylow subgroups. If there is a prime \( p \) with \( C_p = \{0\} \) then \( K := C \) serves the purpose. Now assume that \( C_p \neq \{0\} \) holds for all primes \( p \). Select any prime \( p \) and note that by Proposition 2.11 the closed subgroup \( K := \prod_{q \neq p} C_q \) cannot be torsion. Certainly \( pK = K \).

Lemma 2.15. Let \( G \) be a locally compact abelian topologically modular group. Then
(a) The connected component \( G_0 \) of \( G \) is compact and \( \text{comp}(G) \) is an open subgroup of \( G \).
(b) If \( \text{comp}(G) \) is a proper subgroup of \( G \) then \( \text{comp}(G) = \text{tor}(G) \).
(c) If \( U \) is any open compact subgroup then \( G/U \) has finite \( \mathbb{Z} \)-rank.

Proof. (a) Since \( G \) is topologically modular so is, by Remark 2.7, the connected component \( G_0 \). By the Vector Splitting Theorem (see \([9]\)
Theorem 7.57)) there is \( n \geq 0 \) and a compact connected subgroup \( K \) such that
\[
G_0 = \mathbb{R}^n \oplus K.
\]
If \( G_0 \) were not compact then \( n > 0 \) and hence there is a closed subgroup \( R \cong \mathbb{R} \) of \( G_0 \) which must be topologically modular, contradicting the findings in Example 2.9. Hence \( G_0 = K \) is compact. The factor group \( G/G_0 \) is totally disconnected and thus contains a compact open subgroup, say \( C \). The latter gives rise to an open compact subgroup of \( G \). Therefore \( \text{comp}(G) \) is open.

(b) Since \( \text{comp}(G) < G \), (a) implies that the factor group \( G/\text{comp}(G) \) is discrete and torsion-free. Therefore one can find a discrete subgroup \( Z \cong \mathbb{Z} \) of \( G \). Suppose \( G \) to contain an element \( c \) with \( C := \langle c \rangle \) compact and not torsion. Then \( C \) is an infinite monothetic subgroup. Lemma 2.14 provides a prime \( p \) and a monothetic infinite subgroup \( K \) of \( C \) with \( \bar{K} = pK \). Remark 2.7 shows that the closed subgroup \( Z \oplus K \) must be topologically modular. This leads to a contradiction in light of Example 2.10.

(c) If, for some open compact subgroup \( U \) of \( G \), the factor group \( G/U \) has infinite \( \mathbb{Z} \)-rank then \( G \) contains a closed subgroup \( S \cong \mathbb{Z}(N) \oplus U \). By (b) \( U \) is a compact torsion group and by Proposition 2.1 it is the cartesian product \( U = \prod_{p \in S} U_p \) of compact finite exponent \( p \)-groups for a finite set \( S \) of primes. Since \( U \) is assumed to be infinite (else \( G \) would be discrete) there is \( p \in S \) with \( U_p/pU_p \) infinite. Since \( G \) is by assumption topologically modular, so is \( R := \mathbb{Z}(N) \oplus U_p/pU_p \).

Let \( (V_i)_{i \in \mathbb{N}} \) be a properly descending sequence of open subgroups of \( R \) all contained in \( U_p/pU_p \) and let \( V := \bigcap_{i \geq 1} V_i \) denote the intersection. Then \( (U_p/pU_p)/V \) is first countable and has exponent \( p \). Therefore \( (U_p/pU_p)/V \cong \mathbb{Z}(p)^N \) and hence topologically and algebraically
\[
R/V \cong \mathbb{Z}(N) \oplus \mathbb{Z}(p)^N.
\]
Letting \( A = \mathbb{Z}(N) \) denote the first direct summand we may factor \( pA \) and obtain the topologically modular \( p \)-group
\[
(R/V)/pA \cong \mathbb{Z}(p)^{(N)} \oplus \mathbb{Z}(p)^{N}
\]
contradicting our findings in Example 2.11.

If a periodic group is the topological direct sum of groups \( G \) and \( H \) and \( \pi(G) \cap \pi(H) = \emptyset \) it will be enough to ensure that each factor is strongly topologically quasihamiltonian, in order to prove that \( G \oplus H \) is strongly topologically quasihamiltonian.
Lemma 2.16. If $G$ and $H$ are both periodic strongly topologically quasihamiltonian groups and $\pi(G) \cap \pi(H) = \emptyset$ then their topological direct sum $G \oplus H$ is a strongly topologically quasihamiltonian group.

Proof. Put $\pi := \pi(G)$ and $\sigma := \pi(H)$. For closed subgroups $X$ and $Y$ there is a corresponding decomposition

$$X = X_\pi \oplus X_\sigma, \ Y = Y_\pi \oplus Y_\sigma.$$ 

Then $X_\pi + Y_\pi$ and $Y_\sigma + Y_\sigma$ are both closed subgroups in respectively $G$ and $H$ by our assumptions. Hence

$$X + Y = (X_\pi + Y_\pi) \oplus (X_\sigma + Y_\sigma)$$

is a closed subgroup of $G \oplus H$. $\square$

The following fact has already been observed in [11, Remark 2].

Lemma 2.17. Let $I$ be a nonempty index set and select for every $i \in I$ a prime $p_i$. Set

$$A := \bigoplus_{i \in I} \mathbb{Z}(p_i) \times \prod_{i \in I} \mathbb{Z}(p_i).$$

Then $A$ is strongly topologically quasihamiltonian if, and only if, $I$ is finite.

Proof. Suppose that $A$ is strongly topologically quasihamiltonian. Let $c_i$ be a topological generator of $\mathbb{Z}(p_i)$ in the profinite factor

$$C := \prod_{i \in I} \mathbb{Z}(p_i)$$

of $A$ and $b_i$ for $\mathbb{Z}(p_i)$ in the discrete factor

$$B := \bigoplus_{i \in I} \mathbb{Z}(p_i)$$

of $A$. Then

$$A = B \oplus C$$

where $C$ is a compact open subgroup of $A$. For $i \in I$ set $a_i := b_i + c_i$ and define $X := \langle a_i : i \in I \rangle$. View $X$ as the graph of the obvious injection $\iota : B \to C$ in $B \times C$. A graph of any continuous function is always homeomorphic to the domain and therefore $X$ is a discrete subgroup of $G$. Set $Y := B$. Then

$$X + Y = \langle c_i : i \in I \rangle + \langle b_i : i \in I \rangle = B + \iota(B)$$

is dense in $B \times C$ and is therefore closed if, and only if, $I$ is finite. $\square$
Remark 2.18. If $I$ is infinite countable and the primes $p_i$ are pairwise different then we will show later, in Theorem 1.2, that $A$ is topologically modular and not strongly topologically quasihamiltonian. Note that $A$ is the projective limit with compact kernels of discrete strongly topologically quasihamiltonian groups.

The good properties of the class of strongly topologically quasihamiltonian groups are the following ones.

Proposition 2.19. Let $\mathcal{X}$ be either the class of topologically modular groups or of strongly topologically quasihamiltonian groups. Then $\mathcal{X}$ is closed under
(a) passing to closed subgroups; and
(b) passing to factor groups modulo closed normal subgroups.

For its proof we first establish an elementary fact.

Lemma 2.20. Let $G$ be a topological group and $N$ a closed normal subgroup. Then any subgroup $S$ containing $N$ is closed in $G$ if and only if $S/N$ is a closed subgroup of $G/N$.

Proof. Let $\phi : G \to G/N$ denote the quotient map. Then $\phi(S) \subseteq G/N$ is closed if, and only if, $S + N = \phi^{-1}(\phi(S))$ is closed in $G$. □

Proof of Proposition 2.19. When $\mathcal{X}$ is the class of all topologically modular groups then (a) and (b) follow from the fact that the lattice of closed subgroups must not contain the graph $E_5$.

We turn to $\mathcal{X}$ being the class of strongly topologically quasihamiltonian groups. Let $G \in \mathcal{X}$ and $L$ a closed subgroup. Then the product of any two closed subgroups of $L$ is a closed subgroup of $G/N$ and hence of $L$. Thus $L$ is strongly topologically quasihamiltonian.

That $G/N$ is strongly topologically quasihamiltonian follows from Lemma 2.20 □

A fact about certain $p$-groups of exponent $p^2$ and the local product

\[(*) \quad L := (\mathbb{Z}(p^2), p\mathbb{Z}(p^2))^{\text{loc}, \mathbb{N}}\]

will be needed.

From Example 2.13 it should be clear that we are looking for information which secures that a locally compact abelian $p$-group may have a quotient which contains a subgroup isomorphic to $L$ in Eq. (*)

The following discussion serves this purpose

The group $L$ has two significant components, namely,

\[P := p\mathbb{Z}(p^2)^{\mathbb{N}},\]
the socle of \( L \), a compact open characteristic subgroup, and
\[
F := \mathbb{Z}(p^2)^{(\mathbb{N})},
\]
a noncharacteristic dense countable subgroup such that
\[
L := F + P, \text{ and } p \cdot F = F \cap P, \text{ dense in } P.
\]
We observe that we have a basis of compact open zero neighborhoods
\[
P_m := p\mathbb{Z}(p^2)^{\{n \in \mathbb{N} : m \leq n\}}, \ m \in \mathbb{N}
\]
in \( L \), and an ascending union of discrete finite subgroups \( F_0 = \{0\} \) and
\[
F_m := \mathbb{Z}(p^2)^{\{n \in \mathbb{N} : n < m\}}, \ m \in \mathbb{N}
\]
such that
\[
F = \bigcup_{m \in \mathbb{N}} F_m \quad \text{and} \quad L = \bigcup_{m \in \mathbb{N}} (F_m \oplus P_m) = \colim_{m \in \mathbb{N}} F_m \oplus P_m,
\]
and
\[
(\forall m \in \mathbb{N}) \ (p \cdot F_m \oplus P_m) = P.
\]

Finding a copy of \( L \) in a \( p \)-group \( G \) now amounts to finding cyclic subgroups \( Z_k \) isomorphic \( \mathbb{Z}(p^2) \) matching these configurations.

**Lemma 2.21.** Assume that a locally compact abelian \( p \)-group \( G \) has a descending basis of \( 0 \)-neighborhoods of compact open subgroups \( V_1 \supseteq V_2 \supseteq V_3 \supseteq \cdots \) and a family \( \{Z_k : k \in \mathbb{N}\} \) of subgroups with isomorphisms \( \zeta_k : \mathbb{Z}(p^2) \to Z_k \) satisfying the following conditions
(a) \( V_1 \) has exponent \( p \).
(b) The sum \( Z_1 + \cdots + Z_m + V_{m+1} \) is direct (algebraically and topologically) for \( m = 1, 2, \ldots \).
(c) \( p \cdot Z_k \subseteq V_k \) for \( k = 1, 2, \ldots \).

Set \( F_G := \sum_{k \in \mathbb{N}} Z_k \), and \( L_G := \overline{F_G} \), further \( P_G := \overline{p \cdot F_G} \subseteq V_1 \). Then \( L_G \) is equal to the sum \( F_G + P_G \) and all of the following holds:
(i) there is an algebraic isomorphism \( \eta_F : \mathbb{Z}(p^2)^{(\mathbb{N})} \to F_G \) such that the restriction to the \( k \)-th summand is \( \zeta_k \).
(ii) There is an isomorphism of compact groups \( \eta_P : p\mathbb{Z}(p^2)^{\mathbb{N}} \to P_G \) such that the restriction to the \( k \)-th factor is \( \zeta_k(p\mathbb{Z}(p^2)) \).
(iii) There is an isomorphism of topological groups \( \eta_L : L \to L_G \).

**Proof.** There is no loss of generality to assume \( G = L_G \). For each \( k \) we have an isomorphism \( \zeta_k : \mathbb{Z}(p^2) \to Z_k \) by the definition of \( Z_k \).

Conclusion (i) follows from Assumption (b). Since \( F_G \) is dense in \( L_G = G \) we have that \( F_G \cap V_1 \) is dense in the open set \( V_1 \), i.e., \( F_G \cap V_1 = V_1 \).
Since $V_1$ has exponent $p$ the equation $F_G \cap V_1 = pF_G$ follows from (c). Then
\[(2) \quad V_1 = \overline{F_G \cap V_1} = \overline{pF_G} = P_G\]
is open in $G$. As a consequence by the density of $F_G$ in $L_G$ we have
\[(3) \quad G = F_G + P_G.\]
Furthermore,
\[(\forall k \in \mathbb{N}) \quad Z_k \cap P_G = Z_k \cap V_1 \cap F_G = Z_k \cap pF_G = pZ_k.\]

We now prove (ii).
Let us set $F_G,k := \bigoplus_{1 \leq j \leq k} Z_j$. We claim that for any $k \geq 1$
\[P_G = pF_G,k \oplus V_{k+1}.\]
Passing in (c) on both sides to the union and noting that $(V_l)_{l \geq 1}$ is decreasing one obtains
\[\bigcup_{l \geq 1} pZ_l = \left( \bigcup_{1 \leq l \leq k} pZ_l \right) \cup \left( \bigcup_{l \geq k+1} pZ_k \right) \subseteq \left( \bigcup_{1 \leq l \leq k} pZ_k \right) \cup \left( \bigcup_{l \geq k+1} V_l \right) = \left( \bigcup_{1 \leq l \leq k} pZ_k \right) \cup V_{k+1}.\]
Observing that the set on the left hand side generates $pF_G$ and, taking (b) into account, one arrives at $pF_G \leq pF_G,k \oplus V_{k+1}$.

Hence $P_G = pF_G \leq pF_G,k \oplus V_{k+1}$. For proving the converse containment, we take Eq. (2) into account and observe
\[pF_G,k \oplus V_{k+1} \leq pF_G + V_1 = P_G.\]
Thus,
\[P_G = pF_G,k \oplus V_{k+1},\]
that is, there is a projection $p_k : P_G \to pF_G,k$, and for each $k \geq 2$ there is a canonical projection $\phi_k : pF_G,k \to pF_G,k-1$ with kernel $pZ_k$. So $(pF_G,k, \phi_k)_{k \in \mathbb{N}}$ forms an inverse system with projective limit
\[\lim_{\leftarrow k} pF_G,k = \prod_{k \geq 1} pZ_k.\]
Let us prove the equality
\[\phi_k \circ p_k = p_{k-1}, \quad k \geq 2.\]
As $P_G = pF_G,k \oplus V_{k+1}$ we may decompose $x \in P_G$ as $x = pf + v$ for $f \in F_G,k = \bigoplus_{1 \leq j \leq k} Z_j$ and $v \in V_{k+1}$. Therefore
\[(\phi_k \circ p_k)(pf + v) = \phi_k(pf).\]
Decomposing $f = f_1 + z_k$ for some $f_1 \in F_G,k-1$ and $z_k \in Z_k \leq V_k$, the expression on the right yields
\[\phi_k(pf) = \phi_k(pf_1 + pz_k) = pf_1 = p_{k-1}(pf_{k-1} + pz_k + v) = p_{k-1}(x),\]
as needed. Therefore, by the universal property of the limit, there is a unique morphism

$$
\phi: P_G \to \lim_{\longleftarrow k} pF_{G,k} = \prod_k pZ_k.
$$

Since all morphisms \((p_k)_{k \geq 1}\) are surjective, so is \(\phi\) and and since these morphisms separate the points, \(\phi\) is an isomorphism of compact groups. By the definition of \(Z_k\) we have an isomorphism \(\alpha: p\mathbb{Z}(p^2)^N \to \prod_{k \geq 1} pZ_k\) so that the restriction and corestriction to the \(k\)-th factor agrees with \(\zeta_k|p\mathbb{Z}(p^2): p\mathbb{Z}(p^2) \to pZ_k\). Thus, as has been claimed, \(\eta_F = \phi^{-1} \circ \alpha: p\mathbb{Z}(p^2)^N \to P_G\) is an isomorphism mapping the \(k\)-th factor of \(p\mathbb{Z}(p^2)^N\) to \(pZ_k \subseteq P_G\).

For a proof of (iii) denote \(\mathbb{Z}(p^2)^{(N)}\) by \(F\) and note that it is a free \(\mathbb{Z}(p^2)\)-module. Therefore there is a homomorphism \(\eta_F: F \to \sum_{k \geq 1} Z_k\) which restricts to \(\zeta_k\) on the \(k\)-th direct summand of \(F\). Take an element \(z = (z_n)_{n \in \mathbb{N}} \in \mathbb{Z}(p^2)^{(N)} \cap p\mathbb{Z}(p^2)\). Then \(\eta_F(z) = \sum_{n \in \mathbb{N}} \zeta_n(z_n)\) by (i) in view of the definition of a direct sum. Therefore the restrictions of respectively \(\eta_F\) and \(\eta_P\) from (ii) to \(pF = (p\mathbb{Z}(p^2))^{(N)}\) agree. Setting \(P := (p\mathbb{Z}(p^2))^{(N)}\) one observes that \(\eta_F\) and \(\eta_P\) agree on \(F \cap P\) and thus define a unique algebraic morphism \(\eta: L = F + P \to F_G + P_G = G\), where \(F_G\) and \(P_G\) are as in Eq. (3).

Since \(\eta\) agrees on the open subgroup \(P\) with the continuous and open map \(\eta_P\) it is continuous and open. Since \(\eta_F\) is an isomorphism, \(F_G\) is in the image of \(\eta\). Similarly, \(P_G\) is in the image of \(\eta\) as well. Hence \(\eta\) is surjective. If \(\eta(z) = 0\), and \(z \in F\), then \(0 = \eta(z) = \eta_P(z)\) implies \(z = 0\) since \(\eta_F\) is injective. If \(z \in P\) then \(0 = \eta_F(z) = \eta_P(z)\) and by (ii) we must have \(z = 0\). If \(z = f + v\) for \(f \in F\) and \(v \in P\) then \(0 = \eta(z) = \eta(f) + \eta(v)\) implies \(0 = p\eta(f) = \eta(pf)\) so that from \(pf \in P\) and \(\eta|_P = \eta_F\) we may deduce \(pf = 0\). Therefore \(f\) itself belongs to \(P\) and thus \(\eta(f + v) = 0\) implies \(z = f + v = 0\). Hence \(\eta\) is injective and thus is an isomorphism of topological groups. This completes the proof.

\[\square\]

**Proposition 2.22.** Let \(U\) be a closed totally disconnected subgroup of a compact connected \(n\)-dimensional abelian group \(G\). Then, for every \(p \in \pi(G)\), the \(p\)-rank of the \(p\)-Sylow subgroup \(U_p\) of \(U\) is bounded by \(n\).

In particular, every subgroup of \(G\) of finite exponent is finite.

**Proof.** By [2] Corollary 8.24(iv) we have \(\dim(G/U) = \dim(G) = n\). Observing that \(G\) is connected, duality applied to the exact sequence

\[\{0\} \to U \to G \to G/U \to \{0\}\]
renders an exact sequence
\[ \{0\} \to \hat{G}/\hat{U} \to \hat{G} \to \hat{U} \to \{0\} \]
where the second and the third term are torsion-free groups of \( \mathbb{Z} \)-rank \( n \) and \( \hat{U} \) is a discrete torsion group. Since \( \hat{G} \) is a subgroup of \( \mathbb{Q}^n \) and \( \hat{G}/\hat{U} \) must contain a subgroup \( L \cong \mathbb{Z}^n \) it follows that \( \hat{U} \) must be a subgroup of a quotient of
\[ \mathbb{Q}^n / L \cong \bigoplus_p \mathbb{Z}(p^\infty)^n. \]

Therefore the \( p \)-rank of \( \mathbb{Q}^n / L \) does not exceed \( n \). Hence \( \hat{U} \) has \( p \)-rank not exceeding \( n \). Recalling from [7, Definition 3.1] that the \( p \)-rank of \( U_p \) is precisely the \( p \)-rank of the socle of \( \hat{U}_p \) we arrive at \( \text{rank}_p(U_p) \leq n \), as needed.

For proving the second statement, let \( E \) be a subgroup of \( G \) of finite exponent, say \( e \). Then its closure \( \overline{E} \) also has exponent \( e \) and thus there is no loss of generality to assume that \( E \) is closed and hence compact. Therefore Proposition 2.21 implies that \( E = \prod_{p \in S} E_p \) for a finite set of \( S \) of primes and, moreover, \( E_p \) has finite exponent. By the first part of the proof we know that \( \text{rank}_p(E_p) \) must be finite, and therefore, \( E_p \) is finite for every \( p \in S \), and so is \( E \).

**Corollary 2.23.** Suppose \( G \) is a locally compact abelian group with compact finite dimensional connected component \( G_0 \) and suppose that \( \text{tor}(G/G_0) \) is a discrete subgroup of \( G/G_0 \) and has finite exponent, say \( e \). Then \( E := \{ x \in G : ex = 0 \} \) is a discrete subgroup of \( G \).

**Proof.** By [8] (5.23) Lemma] the isomorphism \( (E + G_0)/G_0 \to E/(E \cap G_0) \) maps open sets to open sets and since \( (E + G_0)/G_0 \) is discrete we may conclude that \( E/(E \cap G_0) \) is discrete. Proposition 2.22 shows that \( F := E \cap G_0 \) is finite. Since \( E \) is a closed totally disconnected subgroup of \( G \) it contains a compact open subgroup \( V \) with \( V \cap F = \{0\} \). Therefore the discrete compact and hence finite group \( (V + G_0)/G_0 \) maps onto \( V/(V \cap G_0) \) by the above map showing that \( V \) itself is finite. Thus \( E \) is a discrete subgroup of \( G \).

**Lemma 2.24.** Let \( G \) be a locally compact abelian group and \( X \) and \( Y \) be closed subgroups. Let \( U_X \) and \( U_Y \) be compact subgroups of respectively \( X \) and \( Y \). Then \( \tilde{X} := X + U_Y \) and \( \tilde{Y} := Y + U_X \) are closed and \( \tilde{X} + \tilde{Y} = X + Y \). Letting \( K := U_X + U_Y \), the subgroup \( X + Y \) is closed in \( G \) if, and only if, \( \tilde{X}/K + \tilde{Y}/K \) is closed in \( G/K \).
If there is a compact subgroup $U$ with $U_X = X \cap U$ and $U_Y = Y \cap U$ then we have $\tilde{X} \cap U = \tilde{Y} \cap U = K$. If, in addition, $U$ is open, then $X/(X \cap U)$ is a discrete subgroup of $G/(X \cap U)$.

**Proof.** Since $K = U_X + U_Y$ is the sum of compact subgroups of $G$ it is compact. Certainly

$$\tilde{X} + \tilde{Y} = X + U_Y + Y + U_Y = X + U_X + Y + U_Y = X + Y.$$  

Now the result follows from Lemma [2.20](#).

For proving the second statement, we only show that $\tilde{X} \cap U = K$, as the equality $\tilde{Y} \cap U = K$ can be proved along the same lines. By construction, $K \leq \tilde{X} \cap U$. Now fix $z \in \tilde{X} \cap U$. Then there are $x \in X$ and $y \in Y \cap U = Y_U$ with $z = x + y$. Since $y \in K \leq U$ and $z \in U$ we can conclude $x \in X \cap U = X_U$. Hence $z = x + y \in X_U + K = K$. Thus $\tilde{X} \cap U = K$.

Suppose that $U$ is compact open. Then $(X+U)/U$ is discrete and [8, (5.32) Theorem] implies that the isomorphism $(X+U)/U \to X/(X \cap U)$ maps open subsets of $(X+U)/U$ to open subsets of $X/(X \cap U)$. Hence $X/(X \cap U)$ is a discrete subgroup of $G/(X \cap U)$.

**Lemma 2.25.** Let $G$ be a locally compact abelian group. Suppose $K$ is a compact subgroup of $G$ and $X$ a closed subgroup of $G$. Then $X + K$ is closed in $G$. Furthermore, $X$ is compact if, and only if, $(X + K)/K$ is compact.

**Proof.** Certainly $X + K$ is closed and if $X$ is compact, so is $(X + K)/K$. On the other hand $X + K$ is locally compact abelian, so if $(X + K)/K$ is compact then so is $X + K$ by [8, (5.25) Theorem] and therefore $X$ is compact.

**Lemma 2.26.** Let $H$ be a closed subgroup of a locally compact abelian group $A$ satisfying the following premises:

(a) $A$ is periodic and $\phi := \pi(A)$ is finite.
(b) $E := \text{tor}(A)$ is discrete and has finite exponent, say $e$.
(c) $A/E$ is finitely generated.

Then, algebraically and topologically, $H = Z \oplus E_H$ for $Z$ torsion-free and finitely generated and $E_H = \text{tor}(H)$.

**Proof.** We first claim that $A = X \oplus E$ for $X$ a finitely generated torsion-free subgroup of $A$. Since $\phi$ is finite and, algebraically and topologically, $A = \bigoplus_{p \in \phi} A_p$ for $A_p$ the $p$-primary subgroup of $A$, it will suffice to prove the claim under the additional assumption that $\phi = \{p\}$ and thus $A$ is a locally compact abelian $p$-group.
According to (c), $A/E$ is finitely generated. Hence Lemma 2.4 implies for some $r \geq 0$ the topological isomorphism $A/E \cong \mathbb{Z}_p^r$. Lifting topological generators of $A/E$ to $A$ gives rise to a closed subgroup $X \cong \mathbb{Z}_p^r$ of $A$ with $A = X \oplus E$. The claim holds.

As during the proof of the claim, for establishing the statements about $H$, we may assume that $H$ is a $p$-group for some prime $p$. Certainly $E_H = \text{tor}(H) = H \cap \text{tor}(A)$ is discrete. The continuous epimorphism $\phi : A \to A/E \cong \mathbb{Z}_p^r$ restricts to a map $\chi : H \to A/E$ with kernel $E_H = \text{tor}(H)$. The induced homomorphism $\bar{\chi}$ from the compact group $H/E_H$ to $A/E$ is continuous and renders a compact image, say $L$, in $A/E$. It follows from the compactness of $H/E_H$ and $A/E$ that $\bar{\chi} : H/E_H \to L$ is an isomorphism of topological groups and hence $H/E_H$ must be finitely generated. Therefore, replacing in the above claim $A$ by $H$, we can deduce the existence of a finitely generated subgroup $Z$ of $H$ with $H = Z \oplus E_H$ algebraically and topologically. \hfill $\Box$

3. Proving the Main Results

We shall proceed in three subsections, dealing first with $p$-groups, then with totally disconnected ones, and finally with groups having nontrivial connected components.

3.1. $p$-Groups. Let us first describe the structure of an abelian topologically modular $p$-group.

**Theorem 3.1** (Mukhin, see [10]). A locally compact abelian topologically modular $p$-group $G$ satisfies one of the following conditions:

(a) $G$ contains an open compact subgroup of finite $p$-rank. Then the torsion subgroup $T = \text{tor}(G)$ of $G$ is discrete and $G/T$ has finite $p$-rank.

(b) There is an open compact subgroup $U$ of $G$ with infinite $p$-rank. Then $G/U$ has finite $p$-rank and $G$ contains a closed subgroup $D$ of finite $p$-rank with compact factor group $G/D$. In particular, $D$ can be taken to be $\text{div}(G)$.

**Proof.** Suppose first that the premise of (a) is valid, i.e., $\text{rank}_p(U)$ is finite for some open compact subgroup $U$ of $G$. Then, taking Proposition 2.3 into account, we may replace $U$ by a suitable of its open subgroups and achieve that $U$ is torsion-free. Therefore $\text{tor}(G)$ must be a discrete subgroup. As the torsion-free group $G/\text{tor}(G)$ contains the open subgroup $(U + \text{tor}(G))/\text{tor}(G)$ and the latter has finite $p$-rank, Lemma 2.5 implies that $G/T = G/\text{tor}(G)$ has finite $p$-rank.
Let us assume the premise of (b) now. Suppose, by way of contradiction, the p-rank of $G/U$ to be infinite. We shall derive a contradiction from this by showing that a local product isomorphic to the one in Eq. (11) can be manufactured to be a factor group of a closed subgroup of $G$, being topologically modular by Remark 2.7 and then refer to Example 2.13.

Claim 1: One can assume $U$ to have exponent $p$.

With $G$ also $G/pU$ is topologically modular. Consider instead of $G$ and $U$ the factor group $G/pU$ and its open compact subgroup $U/pU$ which still has infinite $p$-rank.

Claim 2: One can assume $U$ to be first countable and hence metric. Furthermore, one can arrange $G/U \cong \mathbb{Z}(p)^{N}$ and $U \cong \mathbb{Z}(p)^{N}$.

Moreover, every open subgroup $V$ of $U$ is isomorphic to $U$. In particular, $V$ is metrizable, has infinite $p$-rank, and, has exponent $p$.

There is a strictly decreasing sequence $(V_k)_{k \geq 1}$ of open subgroups of $U$. Letting $V := \bigcap_{k \geq 1} V_k$ we pass from $(G, U)$ to $(G/V, U/V)$. Then $G/V$ is topologically modular and $U/V$ is first countable and infinite of exponent $p$. Therefore $U/V$ has infinite $p$-rank and is thus isomorphic to $\mathbb{Z}(p)^{N}$. By replacing $G$ with the inverse image of socle$(G/U)$ under projection $G \to G/U$ we achieve $G/U \cong \mathbb{Z}(p)^{N}$.

The “moreover” statement follows from $U \cong \mathbb{Z}(p)^{N}$ and [12, Theorem 4.3.8].

Claim 3: One can assume that $pG \cap U$ is open in $G$. Moreover, for any open subgroup $V$ of $U$ the intersection $pG \cap V \neq \{0\}$.

If $T := pG \cap U$ is not open in $G$ it must have infinite index in $U$. By Claim 2 the group $G$ has exponent $p^2$ and $pG \leq U$. Therefore the factor group $G/T$ has exponent $p$ and, as $G/U$ is infinite, so is $(G/T)/(U/T)$. Since $G/T$ may be considered a GF$(p)$-vector space the open subgroup $U/T$ admits a complement, say $S$. Thus algebraically and topologically

$$G/T \cong S \oplus U/T.$$ 

Select in $G/T$ a countable subgroup $\Sigma_p$ of $S$ isomorphic to $\cong \mathbb{Z}(p)^{N}$. Since $U/T$ is a compact group of exponent $p$ (by Claim 1) and metrizable (by Claim 2), it follows that it is topologically isomorphic to $\mathbb{Z}(p)^{N}$. Hence it turns out that $\Sigma_p \oplus U/T$ is a factor group of a closed subgroup of $G$ and therefore is topologically modular by Remark 2.7. This contradicts the finding in Example 2.11.

For proving the second statement, suppose, by way of contradiction that $pG \cap V = \{0\}$ for some open subgroup $V$ of $U$. The (purely
algebraic) isomorphism

$$pG \cong pG/(pG \cap V) \cong (pG + V)/V$$

and the fact that $V$ and $pG + V$ are open subgroups of $U$ implies that
$pG$ must be finite. But then the open subgroup $pG \cap U$ would be finite
and $G$ would be discrete, a contradiction.

Claim 4: There is a sequence $(F_k, V_k)_{k \geq 1}$ of pairs of compact sub-
groups of $G$ where for all $k \geq 1$
(1) $F_k$ is finite, $V_k$ is open, and, $F_k \cap V_{k+1} = \{0\}$; and
(2) $F_k \cap V_k = \langle px_k \rangle$ for some nontrivial element $x_k \in G$; and
(3) $F_k \subseteq F_{k+1}$ and $V_k \subseteq V_{k+1}$ and $\bigcap_{k \geq 1} V_k = \{0\}$.

We proceed by induction on $k$ and recall that for all $k \geq 1$ the
subgroups $V_k$ of $U$ will be separable of infinite $p$-rank, and, have exponent
$p$ (see Claim 2). For $k = 1$ let $V_1$ be the open subgroup $pG \cap U$ (see
Claim 3). Then there exists $x_1 \in G$ of order $p^2$ with $px_1 \in V_1$ and we
set $F_1 := \langle x_1 \rangle$.

Suppose $(F_i, V_i)$ have been found for $1 \leq i \leq k$. Then $F_k$ is finite and
hence there exists an open subgroup $V_{k+1}$ contained in $V_k$ with $F_k \cap
V_{k+1} = \{0\}$. By Claim 3 the intersection $pG \cap V_{k+1} \neq \{0\}$. Therefore
one can find $x_{k+1} \in G$ of order $p^2$ with $px_{k+1} \in V_{k+1}$. Set $F_{k+1} :=
F_k \oplus \langle x_{k+1} \rangle$.

(1) and the first statement of (3) are now clear from the construction.
For proving (2) suppose $x \in F_{k+1} \cap V_{k+1}$. Then $x = f_k + \lambda x_{k+1}$ for
some $f_k \in F_k$ and $0 \leq \lambda \leq p^2 - 1$. Since $x \in V_{k+1}$ we must have
$0 = px = p f_k + p\lambda x_{k+1}$ and $p f_k = -\lambda px_{k+1} \in F_k \cap V_{k+1} = \{0\}$ (by
the construction of $V_{k+1}$). Hence $\lambda = p\mu$ for some $0 \leq \mu \leq p - 1$. This
implies $x = \lambda x_{k+1} = \mu px_{k+1} \in \langle px_{k+1} \rangle$.

Selecting at each step $V_i$ small enough one can achieve the second
statement in (3), namely $\bigcap_{i \geq 1} V_i = \{0\}$.

Setting in Claim 4 for all $k \in \mathbb{N}$ respectively $Z_k := \langle x_k \rangle$ and $F :=
\langle Z_k : k \geq 1 \rangle$ shows that the assumptions of Lemma 2.21 hold. There-
fore there is a closed subgroup $L$ of $G$ topologically and algebraically
isomorphic to the group in Eq. (2). We have reached a contradiction
and therefore the $p$-rank of $G/U$ must be finite.

For proving the remaining assertions of (b) let us return to the original
meaning of $G$ and its open compact subgroup $U$ of infinite $p$-rank.
By what we just proved, the factor group $G/U$ has finite $p$-rank. Thus,
according to Proposition 2.3, $G/U \cong \mathbb{Z}(p^\infty)^m \oplus F$, for some $m \geq 0$
and finite group $F$. Replacing $U$ by the preimage of $F$ in $G$ allows to
have $F = \{0\}$. Lemma 2.6 implies that $\overline{G}$ is topologically modular and
duality theory applied to the short exact sequence $U \to G \to \mathbb{Z}(p^{\infty})^m$ implies that $\hat{G}$ must contain an open compact subgroup $\cong \mathbb{Z}_p^m$.

Hence $\hat{G}$ satisfies the premise of (a) and therefore $\text{tor}(\hat{G})$ is a discrete subgroup of $\hat{G}$ with $\hat{G}/\text{tor}(\hat{G})$ being torsion-free of finite $p$-rank. Therefore Proposition 2.3(5) implies the existence of $k \geq 0$ and $l \geq 0$ such that

$$\hat{G}/\text{tor}(\hat{G}) \cong \mathbb{Q}_p^k \oplus \mathbb{Z}_p^l.$$  

Duality applies and yields topological isomorphisms

$$D := \text{tor}(\hat{G}) \cong (\hat{G}/\text{tor}(\hat{G})) \cong \mathbb{Q}_p^k \oplus \mathbb{Z}(p^{\infty})^l.$$  

Hence $D$ is a closed divisible subgroup of $G$ having finite $p$-rank, with compact factor group $G/D$. \hfill \Box

**Lemma 3.2.** Let a locally compact abelian $p$-group $G$ contain a finitely generated open subgroup $U$. Then $G$ is strongly topologically quasihamiltonian.

**Proof.** Since $G$ is periodic the finitely generated subgroup $U$ is compact. Making use of Proposition 2.3 and Theorem 3.1(a) we can pass to a smaller open subgroup of $U$ and achieve $U$ to be torsion-free. Moreover, since the smaller subgroup is open in the finitely generated subgroup $U$ it is itself finitely generated (see e.g. [12, Proposition 2.5.5]). Then $\text{tor}(G)$ turns out to be a discrete, hence closed subgroup of $G$. Let $X$ and $Y$ be closed subgroups of $G$. Since $U$ is compact and open, we can pass to a factor group of $G$ which still satisfies the premises of the lemma, and thereby modifying $X$ and $Y$ by making use of Lemma 2.24 and achieve that $X$ and $Y$ are both discrete subgroups of the modified group $G$. Therefore $X + Y$ is contained in the discrete subgroup $\text{tor}(G)$ and is thus closed in $G$. \hfill \Box

**Theorem 3.3.** The following statements about a locally compact abelian $p$-group $G$ are equivalent:

(a) $G$ is topologically modular.

(b) $G$ is strongly topologically quasihamiltonian.

**Proof.** In light of Lemma 2.2, we only need to show that (b) is a consequence of (a).

Thus assume (a) and let $U$ be an open compact subgroup of $G$. If $\text{rank}_p(U)$ is finite then (b) is a consequence of Lemma 3.2. We assume from now on that the $p$-rank of $U$ is infinite.

Let $X$ and $Y$ be closed subgroups of $G$. Making use of Lemma 2.24 with $K := X \cap U + Y \cap U$ by modifying $X$ and $Y$ and replacing $G$ by $G/K$ (which is still topologically modular by Proposition 2.19) we
can arrange $X \cap U = Y \cap U = \{0\}$. If $\text{rank}_p(U)$ is finite then $G$ is strongly topologically quasihamiltonian by Lemma 3.2. Otherwise the $p$-rank of $U$ is infinite and therefore Theorem 3.1(b) implies that the discrete group $G/U$ is finite. Moreover, the embeddings $X \to G/U$ and $Y \to G/U$ are embeddings of discrete groups. Hence the $p$-ranks of $X$ and $Y$ are finite and therefore the $p$-rank of $X + Y$ is finite (considered without topology). Thus $X + Y$ has finite socle. Making our open compact subgroup $U$ small enough and observing that the $p$-rank of $U$ is then still infinite allows us to assume that $\text{socle}(X + Y)$ and hence $X + Y$ intersect $U$ trivially. Therefore $X + Y$ is a discrete subgroup of $G$ and hence is closed. Thus (b) holds. □

**Lemma 3.4.** Let $G$ be a locally compact abelian $p$-group containing a subgroup $D$ which algebraically is isomorphic to $\mathbb{Z}(p^\infty)^k$ for some $k \geq 1$. Then $D$ is a discrete and hence closed subgroup of $G$.

**Proof.** There is no loss of generality to assume $G = \overline{D}$. Let $U$ be a compact open subgroup of $G$. We claim that $D \cap U$ must have finite exponent. In order to see this we remark that $D \cap U$, as an abstract group, has finite $p$-rank at most $r$ and thus, algebraically

$$D \cap U \cong \mathbb{Z}(p^\infty)^l \oplus F$$

for some $0 \leq l \leq r$ and finite $F$. Since $U$ is a compact $p$-group it is reduced and cannot contain a divisible subgroup. Hence $l = 0$ and hence $D \cap U$ is finite. Since $D \cap U = F$ is dense in $U$ we may conclude that $U$ itself is finite and thus $D$ must be a discrete subgroup of $G$. □

We are ready for proving the first main result.

**Proof of Theorem 1.1.** Let (a) be true. Then (b) follows from Theorem 3.1. Assume (b.1). This condition holds for closed subgroups and factor groups. Let $X$ and $Y$ be any closed subgroups. Then, letting $U_X$ and $U_Y$ be open compact subgroups of respectively $X$ and $Y$, and taking Lemma 2.24 into account, one may pass to the factor group $H := G/K$ and hence assume that $X$ and $Y$ intersect $U$ trivially. Hence $X$ and $Y$ can be assumed to be discrete and are therefore torsion subgroups of $H$. Since $\text{tor}(H)$ is discrete conclude that $X + Y$ is discrete and hence closed.

Assume (b.2). Let $X$ and $Y$ be any closed subgroups and, similarly as before, setting $U_X := U \cap X$ and $U_Y := U \cap Y$, and, using Lemma 2.24 replace $X$ and $Y$ respectively by $X + U_Y$ and $Y + U_X$, and, factor $U_X + U_Y$ in $G$, we can achieve that $X$ and $Y$ can be assumed to be discrete subgroups of $G$. If $U/(U_X + U_Y)$ has finite $p$-rank, we may apply
the reasoning of case (b.1). Let us assume now that after factoring
$U_X + U_Y$ that $U/(U_X + U_Y)$ still has infinite $p$-rank. Simplifying notation
we let $G$ and $U$ and $X$ and $Y$ denote the respective factor groups.
Since $(X + U)/U \cong X/(X \cap U) \cong X$ is discrete and has finite $p$-rank
by condition (b.2) and a similar statement holds for $Y$, we can assume

$$X = D_X + F_X, \quad Y = D_Y + F_Y$$

for finite groups $F_X$ and $F_Y$ and finite $p$-rank torsion divisible subgroups
$D_X$ and $D_Y$ of $G$. Since

$$X + Y = (D_X + D_Y) + (F_X + F_Y)$$

it will suffice to prove that $\Delta := D_X + D_Y$ is closed and to note that
adding the finite summand $F_X + F_Y$ renders again a closed subgroup
of $G$. Since $\Delta$, as an abstract group, is isomorphic to $\mathbb{Z}(p^\infty)^k$ for some
$k \geq 1$ it follows from Lemma \[3.4\] that $\Delta$ and hence $X + Y$ is closed.

Thus (b) implies (c).

That (c) implies (a) has been shown in Theorem \[3.3\] \hfill $\Box$

More can be said if $G$ is torsion.

**Corollary 3.5.** Let $G$ be a locally compact abelian nondiscrete torsion
$p$-group. Then $G$ is topologically modular if, and only if, the maximal
divisible subgroup $D := \text{div}(G)$ is discrete and has finite $p$-rank and
there is a compact open, and hence reduced, subgroup $R$ such that

$$G = R \oplus D$$

algebraically and topologically.

**Proof.** We discuss the cases (a) and (b) in Theorem \[3.1\] If the premise
in (a) holds then, since $U$ is a compact torsion group having finite
exponent and finite $p$-rank, it is finite. Then $G$ would have to be
discrete, contrary to our assumptions. Thus $G$ satisfies premise (b)
in Theorem \[3.1\] Therefore the $p$-rank of $G/U$ is finite and $D$ is a
finite $p$-rank divisible subgroup – hence $D$ is a discrete subgroup of
$G$. Passing to a smaller open compact subgroup of $U$ if necessary, \[12\]
Proposition 2.5.5] ensures finite generation, and, taking Proposition 2.3
into account, we can in addition assume $D \cap U = \{0\}$. Using a result
of R. Baer (see \[14\] Theorem 22.2]), one can find a reduced subgroup $R$
of $G$ containing the reduced subgroup $U$ with $G = D \oplus R$. Necessarily
$R$ is open in $G$ and $G = D \oplus R$ algebraically and topologically. Hence
$R$ is itself compact.

Since $R$ is open and $D \cap R = \{0\}$ we infer that $D$ is discrete. \hfill $\Box$
3.2. **Totally Disconnected LCA-Groups.** We treat the case when \( G \) is periodic first and later turn to totally disconnected but not periodic \( G \).

**Proof of Theorem 1.3.**\((a)\Rightarrow(b)\). Assume first that \( A \) is topologically modular. Select an open compact subgroup, say \( U \), of \( A \). Since \( A \) is torsion, \( U \) is a compact abelian torsion group, and therefore the set \( \phi := \pi(U) \) must be finite. Put \( \delta := \pi(A) \setminus \phi \). From \( A_{\delta} \cap U = \{0\} \) it follows that \( A_{\delta} \) is a discrete subgroup of \( A \), proving (b.2).

Next observe that \( A_\phi = \bigoplus_{p \in \phi} A_p \) is a direct sum since \( \phi \) is finite. Corollary 3.5 (in conjunction with Theorem 3.1) implies that for each \( p \) in \( \phi \) there is a decomposition \( A_p = D_p \oplus V_p \), with \( D_p \) a divisible finite \( p \)-rank subgroup of \( A \) and \( V_p \) compact. Thus there is a decomposition \( A_\phi = D_\phi \oplus V_\phi \)

where \( D_\phi := \bigoplus_{p \in \phi} D_p \) is a discrete divisible subgroup, and, \( V_\phi := \bigoplus_{p \in \phi} V_p \) is compact. Thus also (b.1) is established.

\((b)\Rightarrow(c)\). Assume (b) to hold. Apply Lemma 2.16 to \( A_\delta \) and the finitely many factors \( A_p \) for \( p \in \phi \). Then \( A_\delta \) being discrete, is strongly topologically quasihamiltonian and by Corollary 3.5 (in conjunction with Theorem 3.1) so is \( A_p \) for every \( p \) in \( \phi \).

\((c)\Rightarrow(a)\). Assume (c) to hold. Then, using Lemma 2.2 (a) follows. \(\square\)

**Proof of Theorem 1.2.** If \( G \) is topologically modular, then, for every \( p \in \pi(G) \), the \( p \)-component is a factor group of \( G \) and hence, taking Remark 2.7 into account, is topologically modular.

Let us sketch how to prove the converse, for more details see [10]. For any closed subgroup \( H \) of \( G = \prod_{p \in \pi} (G_p, C_p) \) one has

\[ H = \langle H_p : p \in \pi \rangle = \prod_{p \in \pi} (H_p, H_p \cap C_p). \]

It then follows that

\[ X \vee Y = \langle X_p \vee Y_p : p \in \pi \rangle \quad \text{and} \quad X \wedge Y = \langle X_p \wedge Y_p : p \in \pi \rangle. \]
Using these equalities and the fact that $X \leq Z$ if, and only if, $X_p \leq Z_p$ holds for all $p \in \pi$, one derives from
\[ \forall p \in \pi : (X_p \vee Y_p) \wedge Z_p = X_p \vee (Y_p \wedge Z_p) \]
that $G$ is topologically modular. \hfill \Box

**Proof of Theorem 1.4.** Remark first that $G$ is totally disconnected and locally compact abelian. Therefore a compact open subgroup $U$ exists. Since $G$ is neither discrete nor periodic $U$ is infinite.

(a) $\Rightarrow$ (b):

(b.1) Since $G$ is not periodic $\text{comp}(G)$ is a proper subgroup of $G$. Therefore Lemma 2.15(b) shows that $\text{tor}(G) = \text{comp}(G)$. By Lemma 2.15(c), the $\mathbb{Z}$-rank of $G/U$ is finite. Since $U \leq \text{comp}(G)$ both subgroups are open and hence $G/U \to G/\text{comp}(G)$ is a quotient map of discrete groups. This implies that the $\mathbb{Z}$-rank of $G/\text{tor}(G)$ does not exceed the $\mathbb{Z}$-rank of $G/U$ and is therefore finite.

(b.2)

Since, by (b.1), $T = \text{tor}(G)$ is a topologically modular torsion group it follows from Theorem 1.3 that $T$ is strongly topologically quasihamiltonian.

Let us prove the extra statement about $G/N$ with $N$ a closed subgroup of $T$. Certainly $\text{tor}(G/N) = T/N$ because $N \leq T = \text{tor}(G)$. Since $N$ is a closed subgroup of $G$ it follows that $G/N$ is topologically modular. As $N \leq T$ it follows that $G/N$ cannot be periodic. Indeed, by the second isomorphism theorem $G/T \cong (G/N)/(T/N)$ must be torsion free of finite $\mathbb{Z}$-rank and $\text{tor}(G/N) = T/N$ is strongly topologically quasihamiltonian by Proposition 2.19. Thus the factor group $G/N$ will enjoy all the properties listed in (b).

(b) $\Rightarrow$ (c):

Fix closed subgroups $X$ and $Y$ of $G$. During the proof we shall modify $X$ and $Y$ and factor some closed subgroup $N$ of $\text{tor}(G)$ with $N \leq X \cap Y$. By the additional statement in (b) $G/N$ still enjoys the properties in (b) and thus $X + Y$ will be closed if, and only if, $(X + Y)/N$ is closed in $G/N$, by Lemma 2.20.

Thus we need to show that $X + Y$ is closed and first consider the cases:

α.) $X$ and $Y$ are both torsion.

β.) $Y$ is torsion.

α.) Since $X + Y \leq T = \text{tor}(G)$ and the latter is strongly topologically quasihamiltonian by (b), conclude that $X + Y$ must be closed. Since $\phi$ is finite conclude from $X + Y = \bigoplus_{p \in \phi}(X_p + Y_p)$ being topologically
isomorphic to the direct product of the $p$-primary groups $(X_p + Y_p)$ that $X + Y$ is closed.

$\beta$. We already know that $\text{tor}(X) + Y = \text{tor}(X) + \text{tor}(Y)$ is closed, and we first remark that $X + Y$ is closed, if and only if, $(X + (\text{tor}(X) + Y))/\text{tor}(X)$ and $\text{tor}(X)$ are closed, in light of Lemma 2.20. Therefore observing that $X/\text{tor}(X) = X/\text{tor}(X) \cap X \cong (X + \text{tor}(G))/\text{tor}(G)$ is discrete and of finite $\mathbb{Z}$-rank, we may factor the closed subgroup $\text{tor}(X)$ and hence assume that $X$ is torsion-free, has finite $\mathbb{Z}$-rank, and, that $Y$ is torsion. Moreover, due to the additional statement in (b), our modified group $G$ still satisfies (b). Since $\text{tor}(G)$ is open in $G$ and $X \cap \text{tor}(G) = \{0\}$ we have that $M := X + \text{tor}(G)$ is open and a direct sum $M = X \oplus \text{tor}(G)$. Since $Y$ is a closed subgroup of $\text{tor}(G)$ deduce that $X \oplus Y$ is closed in $M$ and hence in $G$.

For finishing the proof of “(b)⇒(c)” let $X$ and $Y$ now be arbitrary closed subgroups of $G$. Then $X + \text{tor}(Y)$ and $Y + \text{tor}(X)$ are closed subgroups of $G$ and $X + Y$ is closed if, and only if, $(X + \text{tor}(Y)) + (Y + \text{tor}(X))$ is closed. Since

$$\text{tor}(X + \text{tor}(Y)) = \text{tor}(Y + \text{tor}(X)) = \text{tor}(X) + \text{tor}(Y)$$

we may factor $\text{tor}(X) + \text{tor}(Y)$ and, taking the additional statement in (b) into account, in the sequel assume that both, $X$ and $Y$, are torsion-free and hence discrete subgroups with finite $\mathbb{Z}$-rank. Therefore, if $r$ is the sum of the $\mathbb{Z}$-ranks of $X$ and $Y$, it turns out that every \textit{algebraically} finitely generated subgroup of $X + Y$ can be generated by at most $r$ elements. This property holds in particular for $C := (X + Y) \cap U$ for $U$ any open compact subgroup of $G$. Since $U \leq \text{tor}(G)$ it is a compact torsion group and has therefore finite exponent; and so has its subgroup $C$. Therefore $C$ is finite. Hence, passing to a smaller open compact subgroup $V$ of $U$, one can arrange $(X + Y) \cap V = \{0\}$. Therefore $X + Y$ is a discrete and hence closed subgroup of $G$.

Certainly (c) implies (a), by Lemma 2.2. □

**Remark 3.6.** The group $G$ in Theorem 1.4 need not be a split extension of $\text{tor}(G)$ by $L$ – even if $G$ is discrete, as has been demonstrated by [9, Appendix 1, Theorem A1.32].

**Corollary 3.7.** Let the totally disconnected nonperiodic locally compact abelian strongly topologically quasihamiltonian group $G$. Then every torsion-free subgroup is discrete and hence closed. Moreover, every algebraically finitely generated subgroup is discrete and hence closed.
Proof. Since $G$ is not periodic the subgroup $\text{comp}(G)$ is open and by the premises it agrees with $T = \text{tor}(G)$. Therefore, for $H$ any torsion-free subgroup of $G$, one has $T \cap H = \{0\}$ showing that $H$ is a discrete subgroup of $G$. The second statement follows from the first one and the structure of algebraically finitely generated abelian groups. □

3.3. Groups with a Nontrivial Connected Component. Via Pontryagin duality Theorem 1.4 implies at once the following structure theorem.

**Theorem 3.8.** The following statements about a locally compact abelian group $G$, neither compact nor discrete, and with nontrivial component $G_0$, are equivalent:

(a) $G$ is topologically modular.

(b) There are a finite set of primes $\phi$ and a disjoint set $\delta$ of primes and all of the following statements hold:

(b.1) The component $G_0$ is a finite dimensional compact connected subgroup of $G$.

(b.2) $G/G_0$ algebraically and topologically decomposes as

$$G/G_0 = F_\phi \oplus Z_\phi \oplus Z_\delta \oplus S_\delta,$$

where $F_\phi$ is discrete of finite exponent, $Z_\phi = \prod_{p \in \phi} Z_p$ is torsion-free and for every $p \in \phi$ one has $\text{rank}_p(Z_p)$ finite. Moreover, $Z_\delta = \prod_{p \in \delta} Z_p$ is compact and torsion-free, and, for every $p \in \delta$, the $p$-primary subgroup $Z_p \cong Z_p^{m_p}$, where $m_p$ is some cardinal. The subgroup $S_\delta$ is compact and coreduced (i.e., its dual is reduced).

(b.3) The preimage, say $K$, of $Z_\phi \oplus Z_\delta$ under the canonical epimorphism from $G$ onto $G/G_0$ is a split extension of $G_0$ by $Z_\phi \oplus Z_\delta$, i.e., algebraically and topologically

$$K \cong G_0 \oplus Z_\phi \oplus Z_\delta$$

and the factor group $G/K$ is algebraically and topologically isomorphic to $F_\phi \oplus S_\delta$.

(c) $G$ is strongly topologically quasihamiltonian.

Proof. Suppose (a).

Claim 1: The dual $\hat{G}$ satisfies the premise of Theorem 1.4.

Remark 2.7 implies that $\hat{G}$ is topologically modular. Since $G$ is neither discrete nor compact so is $\hat{G}$ by duality. Therefore, in particular, $\text{comp}(\hat{G}) < \hat{G}$. Now Lemma 2.15(a) and (b) together imply that $(\hat{G})_0$ is compact and $\text{comp}(\hat{G}) = \text{tor}(\hat{G})$. Then $(\hat{G})_0$ is a compact torsion
group and therefore is trivial (see [9, Corollary 8.5(a)(e)]). Hence \( \hat{G} \) is totally disconnected and Claim 1 is established.

**Claim 2:** (b.1) holds.

By [9, Corollary 7.69] we have that \( \hat{G}_0 \cong \hat{G} / \text{tor}(\hat{G}) \). The last statement in Theorem [1, b.1] shows that \( \hat{G}_0 \) has finite \( \mathbb{Z} \)-rank. Therefore [9, Theorem 8.22] implies that \( G_0 \) has finite dimension. Whence Claim 2 follows.

**Claim 3:** (b.2) holds.

Claim 1 shows that \( \text{tor}(\hat{G}) \) satisfies Theorem [1.3(a)]. Therefore Theorem [1.3(b.2)] applied to \( \text{tor}(\hat{G}) \) yields a finite set \( \phi \) of primes and

\[
\text{tor}(\hat{G}) = D_{\phi} \oplus E \oplus D_\delta \oplus R_\delta
\]

where \( D_{\phi} \) is discrete, divisible, and of finite \( p \)-rank for all \( p \in \phi \), \( E \) is a compact subgroup and \( \pi(E) \subseteq \phi \), \( D_\delta \) is a discrete divisible torsion group, and, \( R_\delta \) is reduced discrete subgroup. Moreover \( \pi(D_\delta) \cup \pi(R_\delta) \) intersects \( \phi \) trivially.

Dualization yields a decomposition

\[
G/G_0 \cong \text{tor}(\hat{G}) \cong Z_\phi \oplus F_\phi \oplus Z_\delta \oplus S_\delta
\]

where, for respective \( p \)-Sylow subgroups \( Z_p \cong \mathbb{Z}_{p^m} \) for cardinalities \( m_p \) and \( p \in \phi \cup \delta \), we have \( Z_\phi = \prod_{p \in \phi} Z_p \) and \( Z_\delta = \prod_{p \in \delta} Z_p \). Moreover, for \( p \in \phi \) the cardinality \( m_p \) is finite. Furthermore, \( F_\phi \cong \hat{E} \) is discrete and has finite exponent and \( S_\delta \cong \hat{R_\delta} \) is compact and since \( R_\delta \) is reduced \( S_\delta \) is coreduced.

Thus Claim 2 is established.

Let us show that (b.3) holds. Note that \( K \) is compact and \( K/G_0 \cong Z_\phi \oplus Z_\delta \) is torsion-free and compact. Therefore [3, (25.30)(b)] implies a splitting, i.e., algebraically and topologically

\[
K \cong G_0 \oplus Z_\phi \oplus Z_\delta.
\]

Hence (b.3) holds.

(b) \(\Rightarrow\) (c). We start with a simple observation:

**Claim 1:** Let \( K \) be any closed subgroup of \( G_0 \). Then the factor group \( G_0/K \) is finite dimensional and \( G/G_0 \cong (G/K)/(G_0/K) \) is a topological isomorphism.

As \( G_0 \) has finite dimension so has \( G_0/K \). The second isomorphism theorem yields the second statement of the Claim.

**Claim 2:** We may assume \( G_0 \cap X = G_0 \cap Y = \{0\} \).
We may use Lemma 2.24 with $G_0$ playing the role of $U$, modify $X$ and $Y$ according to the lemma, make use of Claim 1 where we let $K := X \cap G_0 + Y \cap G_0$, pass to the factor group $G/K$. Then certainly $X \cap G_0 = Y \cap G_0 = \{0\}$. Observe that we still have a decomposition of $G/G_0$ as in (b.2). Claim 2 holds.

Claim 3: Let $C$ be any closed subgroup $G$ with $C \cap G_0 = \{0\}$ and let $	ilde{C} := (C + G_0)/G_0$. Let $p : G \to G/G_0$ be the canonical projection and $\phi_C : \tilde{C} \to C$ the canonical algebraic isomorphism. All of the following holds:

(i) For any closed subgroup $\tilde{L}$ of $\tilde{C}$ one has $\phi_C(\tilde{L}) = p^{-1}(\tilde{L}) \cap C$. Moreover, if $\tilde{L}$ is compact, then so is $\phi_C(\tilde{L})$.

(ii) For $\sigma \subseteq \pi(\tilde{C})$ and $\tilde{C}_\sigma$ the $\sigma$-primary subgroup of $\tilde{C}$ one has $C_\sigma = \phi_C(\tilde{C}_\sigma)$.

(iii) There is an algebraic and topological direct decomposition $\tilde{C} = \tilde{Z}_C \oplus \tilde{F}_C \oplus \tilde{C}_\delta$ with $\tilde{F}_C = \text{tor}(\tilde{C}_\delta)$ and $\tilde{Z}_C$ a finitely generated torsion-free subgroup of $\tilde{C}_\delta$.

(iv) If $C$ is torsion and only contains $\phi$-elements, it is discrete and has finite exponent.

(v) The map $\phi_C$ induces algebraic and topological isomorphisms

$Z_C := \phi_C(\tilde{Z}_C) \cong \tilde{Z}_C$, $F_C := \text{tor}(C_\phi) \cong \tilde{F}_C$ and $C_\delta = \phi_C(\tilde{C}_\delta) \cong \tilde{C}_\delta$

and the subgroups $Z_C$ and $C_\delta$ are compact.

(vi) $C \cong \tilde{C}$ algebraically and topologically.

(i) Let $i_C : C \to G$ be the canonical embedding and $j_C : C \to \tilde{C}$ be the inverse of $\phi_C$. Then, as $j_C = p \circ i_C$, the map $j_C$ is continuous and one finds

$$\phi_C(\tilde{L}) = j_C^{-1}(\tilde{L}) = i_C^{-1}p^{-1}(\tilde{L}) = p_C^{-1}(\tilde{L}) \cap C.$$ 

Suppose that in addition $\tilde{L}$ is compact. Then, applying Lemma 2.24 to $p_C^{-1}(\tilde{L})$ where $G_0$ plays the role of $K$ we find that $p_C^{-1}(\tilde{L})$ is compact. Therefore so is the intersection $\phi_C(\tilde{L}) = p_C^{-1}(\tilde{L}) \cap C$.

(ii) Pick $x \in C_\sigma$. The algebraic isomorphism $\phi_C : \tilde{C} \to C$ induces a topological isomorphism

$$\langle x \rangle \cong (\langle x \rangle + G_0)/G_0 = \langle \phi_C(x) \rangle \leq C.$$ 

Thus $\phi_C(\tilde{C}_\sigma) \leq C_\sigma$. On the other hand, since $\phi_C : \tilde{C} \to C$ is an algebraic isomorphism with continuous inverse $j_C$, it follows that $j_C(C_\sigma) \leq \tilde{C}_\sigma$, and applying $\phi_C$ on both sides renders $C_\sigma \leq \tilde{C}_\sigma$. Therefore $\phi_C(\tilde{X}_\sigma) = C_\sigma$. 
(iii) The decomposition of $G/G_0$ in (b.2) in conjunction with Lemma 2.26 applied to the $\phi$-component of $\tilde{C}$ implies that

$$\tilde{C} = \tilde{Z}_C \oplus \tilde{F}_C \oplus \tilde{C}_\delta$$

for $\tilde{Z}_C$ finitely generated torsion-free with $\pi(\tilde{Z}_C) \subseteq \phi$, $\tilde{F}_C$ a discrete subgroup of finite exponent, and, $\tilde{Z}_C$ and $\tilde{C}_\delta$ profinite groups with $\pi(X_\delta) \subseteq \delta$.

(iv) Observe first that $\tilde{C} := (C + G_0)/G_0$ is discrete being contained in the discrete subgroup $\text{tor}(G/G_0)_\phi$. Thus there is an open compact subgroup $U$ of $G/G_0$ with $\tilde{C} \cap U = \{0\}$. Let $p : G \to G/G_0$ be the canonical projection. Then $W := p^{-1}(U)$ is open and

$$W \cap C \leq (W \cap (C + G_0)) \cap C \leq G_0 \cap C = \{0\}$$

showing that $C$ is indeed a discrete subgroup of $G$. Since $\text{tor}(G)_\phi$ has finite exponent so has $\tilde{C}$ and thus also $C$.

(v) Setting $\sigma := \phi$ in (ii) implies $C_\phi = \phi_C(\tilde{C}_\phi)$ and using (i) with $\tilde{L} := \tilde{Z}_C$ one obtains a topological isomorphism $Z_C := \phi_C(\tilde{Z}_C) \cong \tilde{Z}_C$ since $\tilde{Z}_C$ and hence $Z_C$ are finitely generated and hence compact.

For proving the second equation we note that by (iv) $\text{tor}(C_\phi)$ is discrete and hence

$$F_C := \phi_C(\tilde{F}_\phi) = \phi_C(\text{tor}(\tilde{C}_\phi)) = \text{tor}(C_\phi).$$

The third topological isomorphism follows by letting $\sigma := \delta$ and making use of (i) in order to see that $C_\delta$ is compact.

(vi) Since $G_0 \cap C = \{0\}$ so that $C/G_0 \cap C = C$, [8, (5.32) Theorem] implies that the map $\phi_C : \tilde{C} = (C + G_0)/G_0 \to C$ is an algebraic isomorphism carrying open sets to open sets. The compact subgroup $\tilde{Z}_C \oplus \tilde{C}_\delta$ is open in $C$ and therefore, taking (i) into account, its image

$$\phi_C(\tilde{Z}_C \oplus \tilde{C}_\delta) = Z_C \oplus C_\delta$$

is an open compact subgroup of $C$. Thus the restriction of $\phi_C$ to $\tilde{Z}_C \oplus \tilde{C}$ is a topological isomorphism onto an open subgroup of $C$ and therefore $\phi_C$ is a topological isomorphism. Thus (vi) and hence Claim 3 are established.

Let us finish proving “(b)$\Rightarrow$(c)”. Applying Claim 3(vi) to $X$ and $Y$ separately one finds compact groups $Z_X$, $Z_Y$, $X_\delta$, $Y_\delta$ and discrete torsion subgroups $F_X$ and $F_Y$ with $\pi(F_X) \cup \pi(F_Y) \subseteq \phi$ such that algebraically and topologically

$$X = Z_X \oplus F_X \oplus X_\delta, \quad Y = Z_Y \oplus F_Y \oplus Y_\delta.$$
The sum of compact subgroups
\[ V := Z_X + Z_Y + X_\delta + Y_\delta \]
is compact. Therefore
\[ X + Y = (F_X + F_Y) + V \]
will turn out to be closed if we can show that \( F_X + F_Y \) is closed.

From (iv) it follows that \( F_X \) and \( F_Y \) are discrete torsion groups of
finite exponent and hence closed subgroups of \( G \) and Corollary \[2.23\] implies the finiteness of \((F_X + F_Y) \cap G_0\). Now we may use Claim 1
with \( K := (F_X + F_Y) \cap G_0 \) and assume \((F_X + F_Y) \cap G_0 = \{0\}\). Since
\( \pi(F_X + F_Y) \subseteq \phi \) and \( F_X + F_Y \) is torsion deduce from (iv) that indeed
\( F_X + F_Y \) is closed.

Therefore (c) holds.

\[(c) \Rightarrow (a). \text{ This follows from Lemma} \[2.2] \quad \Box \]

The preceding result corrects Theorem \[6, \text{Theorem 14.34(ii)}\].

4. SOME CONSEQUENCES

From Corollary \[3.5\] one obtains refined structure results for reduced,
for torsion, and, for divisible \( p \)-groups.

Corollary 4.1. Let \( G \) be a reduced locally compact abelian torsion \( p \)-
group. To be strongly topologically quasihamiltonian it is necessary and
sufficient that \( G \) is either discrete or compact.

Corollary 4.2. Let \( G \) be a locally compact abelian torsion strongly
topologically quasihamiltonian \( p \)-group. Then either \( G \) is discrete or
\( \text{div}(G) \) has finite \( p \)-rank.

In particular \( \text{div}(G) \) is a closed subgroup of \( G \).

A referee provided the proof of the preceding proposition and actu-
ally proved the following very general result:

Proposition 4.3. Let \( G \) be a topological group with a subgroup topology
and \( D \) a discrete divisible subgroup. Then \( G = D \oplus R \) algebraically and
topologically for some open subgroup \( R \).

Proof. Since \( D \) is discrete there is an open subgroup \( U \) of \( G \) intersecting
\( D \) trivially and hence \( D + U = D \oplus U \) algebraically and topologically.
The canonical epimorphism \( \phi : D + U \to D \), by the universal property of
divisibility extends to a homomorphism \( \phi' : G \to D \). The latter
agrees with \( \phi \) on the open subgroup \( D \oplus U \) of \( G \) so that \( \phi' \) is continuous.
Therefore
\[ G = D \oplus R \]
for \( R := \ker(\phi') \).

For the \( p \)-group case the following immediate consequence will be helpful.

**Corollary 4.4.** Let \( G \) be a strongly topologically quasihamiltonian group having discrete maximal divisible subgroup \( D := \text{div}(G) \). Then \( G = D \oplus R \) for a reduced subgroup \( R \) algebraically and topologically.

An additional consequence may be concluded.

**Corollary 4.5.** Let \( G \) be a locally compact abelian strongly topologically quasihamiltonian nondiscrete torsion \( p \)-group. Then, for \( C \) compact subgroup of \( G \),

\[
(\text{div}(G) + C)/C = \text{div}(G/C).
\]

**Proof.** Theorem 3.3 implies that \( G \) is topologically modular and Corollary 3.5 shows that the maximal divisible subgroup \( D := \text{div}(G) \) of \( G \) is discrete and has finite \( p \)-rank. Moreover, \( G = D \oplus R \), for some compact open torsion subgroup \( R \) of \( G \). By Proposition 2.1 \( R \) and hence \( R/R \cap (D + C) \) have finite exponent. Because \( (D + C)/C \) is divisible we have \( (D + C)/C \leq \text{div}(G/C) \). In order to show that \( (D + C)/C \) is the **maximal** divisible subgroup of \( G/C \) consider the algebraic isomorphisms

\[
(G/C)/((D+C)/C) \cong G/(D+C) = (R + D + C)/(D + C) \cong R/(R \cap (D + C)).
\]

It follows that \( (G/C)/((D+C)/C) \) has finite exponent and is hence reduced showing the desired containment \( \text{div}(G/C) \leq (D + C)/C \). \( \Box \)

Every compact or discrete abelian \( p \)-group clearly is strongly topologically quasihamiltonian. Therefore we first concentrate on groups neither compact nor discrete.

**Proposition 4.6.** The following statements about a locally compact abelian reduced \( p \)-group \( G \), neither discrete nor compact, are equivalent:

(a) \( G \) is strongly topologically quasihamiltonian.

(b) \( G \) contains an open finitely generated subgroup.

**Proof.** Suppose (a). Then by Theorem 3.3 \( G \) is topologically modular and therefore either (a) or (b) of Theorem 3.1 must hold. In the latter case \( G \) would have to be compact since \( \text{div}(D) = \{0\} \). Hence case (a) of Theorem 3.1 holds and therefore, as desired, \( G \) has an open finitely generated subgroup.

That (b) implies (a) is an immediate consequence of Lemma 3.2. \( \Box \)
Remark 4.7. An example of a reduced locally compact abelian $p$-group which is strongly topologically quasihamiltonian and neither compact nor discrete can be found in [6, Remark 14.6].

Lemma 4.8. Let $G$ be a non-discrete locally compact abelian $p$-group with a finitely generated open subgroup $U$. Then the maximal divisible subgroup $D$ is closed and $G$ algebraically and topologically decomposes

$$G = D \oplus R,$$

for $R$ a closed reduced subgroup of $G$.

Proof. Lemma [3,2] implies that $G$ is strongly topologically quasihamiltonian. Since $U$ is finitely generated Lemma 2.4 implies that $U$ is algebraically and topologically isomorphic to $V \oplus F$ for some finite subgroup $F$ and $V$ open and torsion-free. If $V = \{0\}$ the group $G$ is discrete, a contradiction. Thus $V \neq \{0\}$ and, replacing $U$ by $V$ we can arrange that $U$ is torsion-free. Then certainly $D \cap U$ is finitely generated and hence by Proposition 3.76 in [6] $D$ is closed and thus $X := D \cap U = \overline{D \cap U}$ is finitely generated.

Since $U/X$ is finitely generated, by Lemma 2.4 there are a finite $p$-group $F$, some $r \geq 0$, and, a closed subgroup $W$ of $U$ containing $X$ such that $U/X = F \oplus W/X$ and $W/X \cong \mathbb{Z}_p^r$. Lifting $r$ generators of $\mathbb{Z}_p^r$ to $W$ yields a closed subgroup $Z \cong \mathbb{Z}_p^r$ of $W$ such that

$$W = X \oplus Z = (D \cap U) \oplus Z.$$

The endomorphism $\eta : W \to D \cap U$ extends to a continuous homomorphism $\tilde{\eta} : W + D \to D$ which restricts to the identity on $D$. Therefore, setting $R := \ker(\tilde{\eta})$

$$G = D \oplus R$$

is a splitting. \hfill \Box

Now we offer a new and short argument for the following result of Mukhin [10].

Proposition 4.9. A divisible locally compact abelian $p$-group $G$ is strongly topologically quasihamiltonian if and only if, for some set $I$ and nonnegative integer $m$,

$$G \cong \mathbb{Z}(p^\infty)^{(I)} \oplus \mathbb{Q}_p^m$$

algebraically and topologically.

Proof. Assume first that $G$ is strongly topologically quasihamiltonian. If $G$ is discrete it has the described structure for $m = 0$. Thus we may assume $G$ not to be discrete. Fix an open compact subgroup $U$ of $G$. Then $G/pU$ is a torsion strongly topologically quasihamiltonian group
and Corollary 3.5 (in conjunction with Theorem 3.1) implies that $G/pU$ is either discrete or has finite $p$-rank. In either case $U/pU$ is finite and hence $U$ has finite $p$-rank. Therefore, by Proposition 2.3,

$$U \cong F \oplus \mathbb{Z}_p^m$$

for some nonnegative integer $m$ and a finite subgroup $F$ of $G$. Choosing the open compact subgroup $U$ small enough, we can arrange

$$U \cap \text{tor}(G) = \{0\}.$$  

Hence the subgroup $\text{tor}(G)$ is discrete and divisible and is therefore, as a consequence of Lemma 4.4 in [7], topologically and algebraically a direct summand of $G$ intersecting $U$ trivially. Then

$$G = \text{tor}(G) \oplus D_U,$$

with $D_U \cong \mathbb{Q}_p^m$ a divisible hull of $U$.

Since $\text{tor}(G)$ is discrete and divisible there is a set $I$ with

$$\text{tor}(G) \cong \mathbb{Z}(p^\infty)^{(I)}.$$  

Conversely, suppose

$$G \cong \mathbb{Z}(p^\infty)^{(I)} \oplus \mathbb{Q}_p^m$$

for $I$ some set and $m \geq 0$. Since the $p$-rank of the open summand $\mathbb{Q}_p^m$ is finite (equal to $m$) there is an open finitely generated subgroup $U$ of $\mathbb{Q}_p^m$ and hence of $G$. Lemma 3.2 implies that $G$ is strongly topologically quasihamiltonian. \hfill $\square$

Next we turn to providing a full classification of the periodic strongly topologically quasihamiltonian groups, Theorem 1.5.

**Definition 4.10.** A periodic locally compact abelian group $G$ is *inductively monothetic*, provided every finitely generated subgroup $H$ of $G$ can be topologically generated by a single element.

Note that a periodic locally compact abelian group $G$ is inductively monothetic if, and only if, for every $p \in \pi(G)$ the $p$-primary subgroup $G_p$ has $p$-rank 1. (see Subsection 3.1).

That inductively monothetic groups are always strongly topologically quasihamiltonian could be read off from Mukhin’s characterization of abelian strongly topologically quasihamiltonian groups, see [10], a very elementary proof of this fact follows.

**Proposition 4.11.** Every periodic inductively monothetic group $H$ is strongly topologically quasihamiltonian.
Proof. Assume first that $H$ is a $p$-group. Then, as a consequence of Proposition 2.3, $H$ is isomorphic to either the additive group of the $p$-adic field $\mathbb{Q}_p$, or to the additive group of the $p$-adic integers $\mathbb{Z}_p$, or to a finite cyclic $p$-group, or to Prüfer’s $p$-group $\mathbb{Z}(p^\infty)$. Then, for $X$ and $Y$ any closed subgroups of $H$, either $X \subseteq Y$ or $Y \subseteq X$ must hold. But then $X + Y$ agrees with one of the closed subgroups $Y$ or $X$ and is hence a closed subgroup of $H$.

Let $H$ now be arbitrary and consider closed subgroups $X$ and $Y$. Put $\pi := \{ p \in \pi(H) : X_p \subseteq Y_p \}$. Then, letting $\pi' := \pi(X) \setminus \pi$, there are direct sum decompositions

$$X = X_\pi \oplus X_{\pi'} \quad \text{and} \quad Y = Y_\pi \oplus Y_{\pi'}.$$ 

For proving $X + Y$ to be a closed subgroup of $H$, in light of Lemma 2.16, it suffices to prove closedness of the two subgroups $X_{\pi'} + Y_{\pi'}$. In the first case the group in question agrees with the closed subgroup $Y_{\pi'}$ and in the second one with the closed subgroup $X_{\pi'}$. Hence $X + Y$ is a closed subgroup. Thus $H$ is strongly topologically quasihamiltonian. □

Here is a complete description of all torsion locally compact abelian strongly topologically quasihamiltonian groups.

We can now complete Mukhin’s classification of abelian strongly topologically quasihamiltonian groups. For a profinite abelian group $U$ the Frattini subgroup $\Phi(U)$ is the intersection of all maximal open subgroups of $U$. As pointed out in [12] the Frattini subgroup of $G = \prod_p A_p$ takes the form

$$\Phi(A) = \prod_p \Phi(A_p) = \prod_p pA_p.$$ 

An elementary fact about locally compact abelian $p$-groups will be needed.

**Lemma 4.12.** Let $A$ be a locally compact abelian $p$-group containing properly an open compact subgroup $U$ of exponent $p$. If $A$ is not inductively monothetic then there are elements $a \in A \setminus U$ and $0 \neq b \in U$ with $\langle a \rangle \cap \langle b \rangle = \{0\}$.

**Proof.** Since the compact open subgroup $U$ has exponent $p$ the group $A$ is torsion and $U \leq \text{socle}(A)$. If there is $a \in \text{socle}(A) \setminus U$ then $\langle a \rangle + U = \langle a \rangle \oplus U$ and we may pick $0 \neq b \in U$ in order to have $\langle a \rangle \cap \langle b \rangle = \{0\}$. Suppose next that $\text{socle}(A) = U$. Pick any $a \in A \setminus U$. Since $A$, by assumption, is not inductively monothetic and is torsion $U = \text{socle}(A)$ cannot be cyclic and hence $U$ must contain a subgroup
\[
L = \langle x, y \rangle \cong \mathbb{Z}(p) \oplus \mathbb{Z}(p). \quad \text{There must be } b \in L \text{ not belonging to } \text{socle}(\langle a \rangle). \quad \text{Then } \langle a \rangle \cap \langle b \rangle = \text{socle}(\langle a \rangle) \cap \langle b \rangle = \{0\}. \quad \square
\]

We come to prove our addition to the classification results in [10].

Proof of Theorem 1.5. Assume (A). Then certainly \( A_\delta \cap U = \{0\} \) and hence \( A_\delta \) is a discrete \( \delta \)-Sylow subgroup of \( A \). Since \( A_\gamma \leq U \) conclude that \( A_\gamma \) is profinite and so (ii) holds. It follows that

\[
A = A_\delta \times A_\gamma \times A_{\delta'},
\]

for \( \delta' := \pi(A) \setminus \{\delta \cup \gamma\} \). algebraically and topologically. Thus (i) is established.

For establishing (iii) and (iv) we may restrict ourselves to the case \( \delta = \gamma = \emptyset \), i.e., \( A_\delta = A_\gamma = \{0\} \) from now on.

Since, by Proposition 2.19, every subgroup and quotient of a strongly topologically quasihamiltonian group again is a strongly topologically quasihamiltonian group, passing to subgroups of quotients of \( A_\phi \) renders strongly topologically quasihamiltonian groups. For proving that \( \phi \) must be finite we may factor \( \Phi(U_\phi) = \prod_{p \in \phi} pU_p \) and achieve that \( U_p \) has exponent \( p \) only. By using Lemma 4.12 one can find for every \( p \in \phi \) elements \( a_p \in A_p \setminus U_p \) of order at most \( p^2 \) and \( 0 \neq b_p \in U_p \) with \( \langle a_p \rangle \cap \langle b_p \rangle = \{0\} \). The closed subgroup

\[
L := \langle a_p, b_p : p \in \phi \rangle
\]

of \( A_\phi \) is still strongly topologically quasihamiltonian. Observe that \( U \cap L = \{pa_p, b_p : p \in \phi\} \) and, after factoring in it the closed subgroup \( N \) generated by all elements of the form \( pa_p \) we find an algebraic and topological isomorphism

\[
L/N \cong \bigoplus_{p \in \phi} \mathbb{Z}(p) \oplus \prod_{p \in \phi} \mathbb{Z}(p).
\]

Since \( L/N \) is strongly topologically quasihamiltonian deduce from Lemma 2.17 that \( \phi \) must be finite. Hence (iii) holds.

(iv) is an immediate consequence of the fact that \( p \in \mu \) if and only if \( \text{rank}_p(A_p) = 1 \) if and only if \( A_p \) is inductively monothetic.

Finally, \( A \) certainly is the cartesian product of the Sylow subgroups \( A_\delta, A_\phi \) and \( A_\mu \). Hence (B) holds.

Assume now (B). In light of Lemma 2.16 it will suffice to prove that \( A_\delta, A_\gamma, A_\phi, \) and \( A_\mu \) are strongly topologically quasihamiltonian. For \( A_\delta \) this is obvious since \( A_\delta \) is discrete. Since \( A_\gamma \) is compact it is strongly topologically quasihamiltonian. For every \( p \) in the finite set \( \phi \)
we know from (iii) that $A_p$ is strongly topologically quasihamiltonian. Thus applying Lemma 2.16 to the finite product

$$A_\phi = \prod_{p \in \phi} A_p$$

shows that $A_\phi$ is strongly topologically quasihamiltonian. Finally observe that $A_\mu$ has $p$-rank 1 $p$-Sylow subgroups for all $p \in \mu$. □

One may ask under which conditions on a locally compact abelian group $G$ the group is topologically modular if, and only if, $G$ is strongly topologically quasihamiltonian. When $G$ is discrete then, as has been mentioned in the introduction, Dedekind [3] proved that $G$, equipped with the discrete topology, is topologically modular and it certainly is strongly topologically quasihamiltonian. We conclude our work by contributing to the above question.

**Theorem 4.13.** Let $G$ be a nonperiodic totally disconnected locally compact abelian group. The following statements are equivalent:

(a) $G$ is topologically modular.

(b) $G$ is strongly topologically quasihamiltonian.

**Proof.** Since, by Lemma 2.2, (b) implies (a) we only need to deduce (b) from (a).

Thus assume that $G$ is topologically modular. If $G$ is compact then $G$ is certainly strongly topologically quasihamiltonian else $G$ is neither discrete nor compact and the result follows from Theorem 1.4. □

Let us summarize the findings of Theorems 3.3, 3.8, and 4.13 and keep in mind Lemma 2.17 and Remark 2.18:

**Corollary 4.14.** Under the following conditions a nondiscrete locally compact abelian group $G$ is topologically modular if, and only if, $G$ is strongly topologically quasihamiltonian.

(i) $G$ is a $p$-group.

(ii) $G$ is totally disconnected but not periodic.

(iii) $G_0$ is not trivial.

Moreover, whenever $G$ satisfies one of the conditions (i)–(iii) then the Pontryagin dual $\hat{G}$ is strongly topologically quasihamiltonian if, and only if, $G$ is strongly topologically quasihamiltonian.

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