Two-flavour Schwinger model with dynamical fermions in the Lüscher formalism

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We report preliminary results for 2D massive QED with two flavours of Wilson fermions, using the Hermitian variant of Lüscher’s bosonization technique. The chiral condensate and meson masses are obtained. The simplicity of the model allows for high statistics simulations close to the chiral and continuum limit, both in the quenched approximation and with dynamical fermions.

1. Schwinger model

Studies of algorithms for dynamical fermions are very time-consuming, therefore we propose the massive Schwinger model (QED in two dimensions) as a low-cost laboratory. We start from the Euclidean continuum Lagrangean

\[ \mathcal{L} = \frac{1}{4} F^2 + \sum_{a=1}^{2} \left[ \bar{\psi}^a (\partial_\mu + igA_\mu + m) \psi^a \right] \]

with two degenerate flavours of fermions. For \( m = 0 \) it is exactly soluble [1] and has a known expansion in the mass [2]. Close to the chiral limit, we expect three light states (‘pions’), massive mesons and a non-zero fermion condensate.

The lattice version is defined using compact link variables \( U_\mu(x) \in U(1) \) with the standard plaquette action \( S_Q[U] \) and Wilson fermions with operator \( M \). Integration over the fermion variables results in a positive effective action for the gauge fields:

\[ P_{\text{eff}}[U] \propto \det M^2 e^{-S_Q[U]} \]

2. Lüscher’s local bosonic theory

As an alternative to the Hybrid Monte Carlo algorithm, M. Lüscher proposed a local bosonic formulation [3]. Let \( Q = c \gamma_5 M = Q^1 \) be scaled so that its eigenvalues are in \([-1, 1]\) and \( P_n(s) \) a polynomial of even degree \( n \) which approximates 1/s in \((0, 1]\). Its roots \( z_k \) \( (k = 1 \ldots n) \) come in complex conjugate pairs and determine \( \sqrt{z_k} = \mu_k + i \nu_k \) \((\nu_k > 0)\). Then

\[ P_{\text{eff}}[U] \propto \det Q^2 e^{-S_Q[U]} \]

\[ = \det[Q^2 P_n(Q^2)] [\det P_n(Q^2)]^{-1} e^{-S_Q[U]} \]

\[ \propto C(Q^2) e^{-S_Q[U]} \]

\[ \int D\phi e^{-\sum_k [z_k(Q\phi_k)^2 + \nu_k^2 \phi_k^2]} \]

with \( C(Q^2) = \det[Q^2 P_n(Q^2)] \approx 1 \) and \( n \) complex bosonic Dirac fields \( \phi_k \).

3. Implementation

We chose as approximation polynomials \( P_n(s) \) the Chebychev polynomials proposed by Bunk et al. [3]. The convergence of \( P_n(s) \rightarrow 1/s \) as \( n \rightarrow \infty \) is exponential and uniform for \( s \in [\epsilon, 1] \), introducing a parameter \( \epsilon \) into the algorithm.

The updating process consists of exact heat bath sweeps for the \( \phi \)'s and \( U \)'s followed by a number of over-relaxation iterations. We confirmed, in our preliminary runs, that the reflection sweeps for the \( \phi \)'s and \( U \)'s have to be combined in pairs, as was observed before [4]. Finally, the approximation \( C \approx 1 \) has to be controlled.

To this end we compute the lowest 8 eigenvalues of \( Q^2 \), use them to estimate the change in \( C \) and apply a global Metropolis correction step with acceptance probability \( \min[1, \frac{C}{C'}] \). This will be improved in the future.

In order to avoid trouble with bad pseudo-random numbers we use Lüscher’s high-quality random number generator [5].
4. Observables

We measured averages of local fermion bilinears with a noisy estimator scheme, using random spinors $\eta(x) = \pm 1$ as sources for the CG solver. $\langle \bar{\psi}\gamma_5\psi \rangle$ and $\langle \bar{\psi}\gamma_\mu\psi \rangle$ were checked to vanish within errors. The fermion condensate $\langle \bar{\psi}\psi \rangle$ remains nonzero even in the chiral limit because Wilson fermions break chiral invariance explicitly. Our MC results are presented below.

For the determination of meson masses operators with various quantum numbers are defined:

- **flavour triplet** - ($\bar{\psi}\gamma_5\tau\psi$) ‘π’, ($\bar{\psi}\tau\psi$) ‘$a_0$’,

- **flavour singlet** - ($\bar{\psi}\gamma_5\psi$) ‘η’, ($\bar{\psi}\psi$) ‘$f_0$’.

Moreover, insertion of a $\gamma_0$, e.g. ($\bar{\psi}\gamma_0\gamma_5\tau\psi$) for the π, leads to alternative operators with the same quantum numbers in the rest frame. This exhausts the Dirac algebra in two dimensions.

The calculation of temporal correlators $\Gamma(\Delta t)$ involved point sources at randomly chosen positions $(x,t)$ and summation over spatial displacements $y$ in the other time slice $(y,t + \Delta t)$ to project out the zero momentum states. As to the flavour–singlet channels, the disconnected piece was subtracted with the aid of the noisy inversions performed earlier for the measurement of the condensate.

5. Calculations

5.1. Dynamical condensates

The condensate with dynamical fermions was calculated on 16x16 lattices for $\beta = 2.0$, $\kappa = 0.20 \ldots 0.275$ and on 16x32 lattices for $\beta = 10.0$, $\kappa = 0.20 \ldots 0.25$.

The Lüscher algorithm required a small number ($n = 20 \ldots 40$) of boson fields only and allowed for very high values of $\epsilon = 0.01 \ldots 0.1$. With moderate statistics ($200 \ldots 400$ measurements), we reproduced the results obtained with a Hybrid Monte Carlo by the Graz group, see Fig. 1.

5.2. Quenched masses

Meson masses in the quenched approximation were obtained on 16x32 lattices for $\beta = 2.0, 6.0, 10.0, 20.0$ and $\kappa = 0.20 \ldots 0.275$ performing high-statistics runs with about 2000 independent measurements each. In case of $\beta = 6.0$, Fig. 2 shows the clear signal for the pion mass decreasing as $\kappa$ is increased, its alternative operator gives masses which agree within errors. For the η and its variant, consistency is also verified with larger errors. The higher states give noisy results.

5.3. Dynamical masses

With dynamical fermions, Irving et al. gave a mass of 0.369(3) for a 32x32 lattice with $\beta = 2.29$, $\kappa = 0.26$. We checked our implementation of the dynamical fermion update with $n = 40$ and obtained $m = 0.377(4)$ from about 2000 measurements.

More masses were determined on 16x32 lattices for $\beta = 10.0$, $\kappa = 0.20 \ldots 0.25$ and are shown in Fig. 3. We observe the same qualitative behaviour as in the quenched case.

6. Conclusions and outlook

The two–flavour Schwinger model has physical properties similar to QCD in four dimensions. It is much easier to simulate even with dynami-
cal fermions and allows to determine observables with high precision. The preliminary results presented above are encouraging.

The next steps will be to include standard versions of preconditioning, a test of the non-hermitean variant of the polynomial approximation and a noisy estimator which makes the Metropolis acceptance step exact.

As to the correlation functions, different noisy estimator schemes will be tested and their efficiency compared.

Finally, the Schwinger model will be used to tune the boson algorithm and to determine how the computational cost scales in the continuum limit $am_{\eta} \rightarrow 0$ with $m_{\pi}/m_{\eta}$ and $Lm_{\eta}$ kept fixed. Although larger lattices than those used so far will be necessary, the cost will be lower by orders of magnitude as compared to the case of QCD in four dimensions. This makes the Schwinger model a reasonable testing ground for dynamical fermion algorithms.

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