Bounds of the accuracy of the normal approximation to the distributions of random sums under relaxed moment conditions

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Abstract: Bounds of the accuracy of the normal approximation to the distribution of a sum of independent random variables are improved under relaxed moment conditions, in particular, under the absence of moments of orders higher than the second. These results are extended to Poisson-binomial, binomial and Poisson random sums. Under the same conditions, bounds are obtained for the accuracy of the approximation of the distributions of mixed Poisson random sums by the corresponding limit law. In particular, these bounds are constructed for the accuracy of approximation of the distributions of geometric, negative binomial and Poisson-inverse gamma (Sichel) random sums by the Laplace, variance gamma and Student distributions, respectively. All absolute constants are written out explicitly.

Key words and phrases: central limit theorem, normal distribution, convergence rate estimate, Lindeberg condition, uniform distance, Poisson-binomial distribution, Poisson-binomial random sum, binomial random sum, Poisson random sum, mixed Poisson random sum, geometric random sum, negative binomial random sum, Poisson-inverse gamma random sum, Laplace distribution, variance gamma distribution, Student distribution, absolute constant

1 Introduction

1.1 The history of the problem and aims of the paper

Let \( X_1, X_2, \ldots \) be independent random variables with \( \mathbb{E} X_i = 0 \) and \( 0 < \mathbb{E} X_i^2 \equiv \sigma_i^2 < \infty \), \( i = 1, 2, \ldots \). For \( n \in \mathbb{N} \) denote \( S_n = X_1 + \ldots + X_n \), \( B_n^2 = \sigma_1^2 + \ldots + \sigma_n^2 \). Let \( \Phi(x) \) be the standard normal distribution function,

\[
\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-z^2/2} dz, \quad x \in \mathbb{R}.
\]

Denote

\[
\Delta_n = \sup_x |\mathbb{P}(S_n < xB_n) - \Phi(x)|.
\]

Let \( \mathcal{G} \) be the class of real functions \( g(x) \) of \( x \in \mathbb{R} \) such that

- \( g(x) \) is even;
- \( g(x) \) is nonnegative for all \( x \) and \( g(x) > 0 \) for \( x > 0 \);
- \( g(x) \) and \( xg(x) \) do not decrease for \( x > 0 \).

In 1963 M. Katz proved that, whatever \( g \in \mathcal{G} \) is, if the random variables \( X_1, X_2, \ldots \) are identically distributed and \( \mathbb{E} X_1^2 g(X_1) < \infty \), then there exists a finite positive constant \( C_1 \) such that

\[
\Delta_n \le C_1 \cdot \frac{\mathbb{E} X_1^2 g(X_1)}{\sigma_1^2 g(\sigma_1 \sqrt{n})},
\]

(1)

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In 1965 this result was generalized by V. V. Petrov [10] to the case of non-identically distributed summands (also see [11]): whatever \( g \in G \) is, if \( \mathbb{E}X_i^2 g(X_i) < \infty, i = 1, \ldots, n \), then there exists a finite positive constant \( C_2 \) such that

\[
\Delta_n \leq \frac{C_2}{B_n^2} \sum_{i=1}^{n} \mathbb{E}X_i^2 g(X_i).
\] (2)

Everywhere in what follows the symbol \( \mathbb{I}(A) \) will denote the indicator function of an event \( A \). For \( \varepsilon \in (0, \infty) \) denote

\[
L_n(\varepsilon) = \frac{1}{B_n^2} \sum_{i=1}^{n} \mathbb{E}X_i^2 \mathbb{I}(|X_i| \geq \varepsilon B_n), \quad M_n(\varepsilon) = \frac{1}{B_n^2} \sum_{i=1}^{n} \mathbb{E}|X_i|^3 \mathbb{I}(|X_i| < \varepsilon B_n).
\]

In 1966 L. V. Osipov [9] proved that there exists a finite positive absolute constant \( C_3 \) such that for any \( \varepsilon \in (0, \infty) \)

\[
\Delta_n \leq C_3 \left[ L_n(\varepsilon) + M_n(\varepsilon) \right]
\] (3)

(also see [12], Chapt V, Sect. 3, theorem 7). This inequality is of special importance. Indeed, it is easy to see that

\[
M_n(\varepsilon) \leq \frac{\varepsilon}{B_n^2} \sum_{i=1}^{n} \mathbb{E}X_i^2 \mathbb{I}(|X_i| < \varepsilon B_n) \leq \varepsilon.
\]

Hence, from (3) it follows that for any \( \varepsilon \in (0, \infty) \)

\[
\Delta_n \leq C_3(\varepsilon + L_n(\varepsilon)).
\] (4)

But, as is well known, the Lindeberg condition

\[
\lim_{n \to \infty} L_n(\varepsilon) = 0 \quad \text{for any} \quad \varepsilon \in (0, \infty)
\]

is a criterion of convergence in the central limit theorem. Therefore, in terminology proposed by V. M. Zolotarev [35], bound (4) is natural, since it relates the convergence criterion with the convergence rate and its left-hand and right-hand sides converge to zero or diverge simultaneously.

In 1968 inequality (3) in a somewhat more general form was re-proved by W. Feller [22], who used the method of characteristic functions to show that \( C_3 \leq 6 \).

A special case of (3) is the inequality

\[
\Delta_n \leq C_3'[L_n(1) + M_n(1)].
\] (5)

In the book [11] it was demonstrated that \( C_3 \leq 2C_3' \).

For identically distributed summands inequality (5) takes the form

\[
\Delta_n \leq \frac{C_4}{\sigma_1^2} \min \left\{ 1, \frac{|X_1|}{\sigma_1 \sqrt{n}} \right\}.
\] (6)

In the papers [29, 30] L. Paditz showed that the constant \( C_4 \) can be bounded as \( C_4 < 4.77 \). In 1986 in the paper [31] he noted that with the account of lemma 12.2 from [2], using the technique developed in [29, 30], the upper bound for \( C_4 \) can be lowered to \( C_4 < 3.51 \).

In 1984 A. Barbour and P. Hall [18] proved inequality (5) by Stein’s method and, citing Feller’s result mentioned above, stated that the method they used gave only the bound \( C_3' \leq 18 \) (although the paper itself contains only the proof of the bound \( C_3' \leq 22 \)). In 2001 L. Chen and K. Shao published the paper [19] containing no references to Paditz’ papers [29, 30, 31] in which the proved inequality (5) by Stein’s method with the absolute constant \( C_3' = 4.1 \).

In 2011 V. Yu. Korolev and S. V. Popov [26] showed that there exist universal constants \( C_1 \) and \( C_2 \) which do not depend on a particular form of \( g \in \mathcal{G} \), such that inequalities (1), (2), (5) and (6) are valid.
with $C_1 = C_4 \leq 3.0466$ and $C_2 = C'_2 \leq 3.1905$. This result was later improved by the same authors in the papers [6, 7], where it was shown that $C_1 = C_2 = C_4 = C'_3 \leq 2.011$.

Moreover, in the paper [7] lower bounds were established for the universal constants $C_1$ and $C_2$. Namely, let $g$ be an arbitrary function from the class $\mathcal{G}$. Denote by $\mathcal{H}_g$ the set of all random variables $X$ satisfying the condition $E X^2 g(X) < \infty$. Denote

$$C^* = \sup_{g \in \mathcal{G}} \sup_{X_i \in \mathcal{H}_g} \frac{\Delta_n B_n^2 g(B_n)}{\sum_{i=1}^n E X_i^2 g(X_i)}.$$ 

It is easily seen that $C^*$ is the least possible value of the absolute constant $C_2$ that provides the validity of inequality (2) for all functions $g \in \mathcal{G}$ at once. In the paper [7] it was proved that

$$C^* \geq \sup_{z > 0} \left| \frac{1}{1 + z^2} - \Phi(-z) \right| = 0.54093\ldots$$

The aim of the present paper is to improve and extend the results mentioned above. First, we will show that one can take $C_3 = C'_3$. Second, we will sharpen the upper bounds of the absolute constants mentioned above. Third, we will extend these results to Poisson-binomial, binomial and Poisson random sums. Under the same conditions, bounds will be obtained for the accuracy of the approximation of the distributions of mixed Poisson random sums by the corresponding limit law. In particular, we will construct these bounds for the accuracy of approximation of the distributions of geometric, negative binomial and Poisson-inverse gamma (Sichel) random sums by the Laplace, variance gamma and Student distributions, respectively. All absolute constants will be written out explicitly.

Along with purely theoretical motivation to sharpen and generalize known results, there is a somewhat practical interest in the problems considered below. Poisson-binomial, binomial and mixed Poisson (first of all, geometric) random sums are widely used as stopped-random-walk models in many fields such as financial mathematics (Cox–Ross–Rubinstein binomial random walk model for option pricing [20]), insurance (Poisson random sums as total claim size in dynamic collective risk models [21]), binomial random sums as total claim size in static portfolio risk models, geometric sums in the Pollaczek–Khinchin–Beekman representation of the ruin probability within the framework of the classical risk process [24], reliability theory for modeling rare events [24]. It is now a tradition to admit that the distributions of elementary jumps of these random walks may have very heavy tails. The problems considered in the present paper correspond to the situation where the tails may be as heavy as possible for the normal approximation to be still adequate. Moreover, the bounds obtained in this paper partly give an answer to the questions how heavy these tails can be for the normal approximation (or scale-mixed normal approximation) to be reasonable.

The paper is organized as follows. In Section 1.1 we prove that in inequalities (1)–(5) the absolute constants coincide and that the values of these constants are determined by that of $C'_3$. In Section 2 the upper bound of $C'_3$ is sharpened. In Section 3 the analogs of inequalities (1), (2), (3) and (6) are proved for Poisson-binomial and binomial random sums. In Section 4 the results obtained in Section 3 are used to construct the analogs of (1) and (6) for Poisson random sums. The results of Section 4 are used in Section 5 to obtain bounds for the accuracy of the approximation of the distributions of mixed Poisson random sums by the corresponding limit law. In particular, here these bounds are constructed for the accuracy of approximation of the distributions of geometric, negative binomial and Poisson-inverse gamma random sums by the Laplace, variance gamma and Student distributions, respectively.

### 1.2 On the coincidence of the absolute constants in inequalities (1)–(5)

The main result of this section is the following statement.

**Lemma 1.** For any $\varepsilon \in (0, \infty)$

$$L_n(1) + M_n(1) \leq L_n(\varepsilon) + M_n(\varepsilon).$$

(7)
Proof. For $\varepsilon = 1$ the statement is trivial. Let $\varepsilon < 1$. Then

$$L_n(1) + M_n(1) = L_n(\varepsilon) + M_n(\varepsilon) +$$

$$+ \frac{1}{B_n^3} \sum_{j=1}^{n} |X_j|^3 I(\varepsilon B_n \leq |X_j| < B_n) - \frac{1}{B_n^2} \sum_{j=1}^{n} E X_j^2 I |X_j| < B_n.$$ But

$$\frac{1}{B_n^3} \sum_{j=1}^{n} |X_j|^3 I(\varepsilon B_n \leq |X_j| < B_n) - \frac{1}{B_n^2} \sum_{j=1}^{n} E X_j^2 I |X_j| < B_n \leq$$

$$\leq \frac{1}{B_n^2} \sum_{j=1}^{n} E X_j^2 I(\varepsilon B_n \leq |X_j| < B_n) - \frac{1}{B_n^2} \sum_{j=1}^{n} E X_j^2 I |X_j| < \varepsilon B_n = 0,$$

therefore, in the case $\varepsilon < 1$ inequality (7) is proved.

Let now $\varepsilon > 1$. Then

$$L_n(1) + M_n(1) = L_n(\varepsilon) + M_n(\varepsilon) +$$

$$+ \frac{1}{B_n^2} \sum_{j=1}^{n} E X_j^2 I(\varepsilon B_n \leq |X_j| < \varepsilon B_n) - \frac{1}{B_n^2} \sum_{j=1}^{n} |X_j|^3 I(\varepsilon B_n \leq |X_j| < \varepsilon B_n).$$ But

$$\frac{1}{B_n^2} \sum_{j=1}^{n} E X_j^2 I(\varepsilon B_n \leq |X_j| < \varepsilon B_n) - \frac{1}{B_n^2} \sum_{j=1}^{n} |X_j|^3 I(\varepsilon B_n \leq |X_j| < \varepsilon B_n) \leq$$

$$\leq \frac{1}{B_n^2} \sum_{j=1}^{n} E X_j^2 I(\varepsilon B_n \leq |X_j| < \varepsilon B_n) - \frac{1}{B_n^2} \sum_{j=1}^{n} E X_j^2 I |X_j| < \varepsilon B_n = 0,$$

that is, the statement of the lemma holds for $\varepsilon > 1$ as well.

Corollary 1. The absolute constants in inequalities (3), (4), (5) and (6) can be taken identical, that is, if inequality (5) holds with $C_3' \leq C_0$, then inequalities (3), (4) and (6) hold with $C_3 \leq C_0$ and $C_4 \leq C_0$.

Remark 1. In the paper [7] it was shown that if inequality (5) holds with $C_3' \leq C_0$, then inequalities (1) and (2) hold with $C_i \leq C_0$, $i = 1, 2$.

So, in the evaluation of the constants in the above inequalities, the constant $C_3'$ in inequality (5) plays the determining role: if a particular upper bound $C_3' \leq C_0$ is known, then in all the other inequalities (1)–(4) and (6) one can let $C_i \leq C_0$, $i = 1, 2, 3, 4$. That is the reason for us to focus on sharpening the upper bound for $C_3'$.

2 Sharpening of the upper bound for the constant $C_3'$

2.1 Auxiliary results

For $x \geq 0$, $n \in \mathbb{N}$ and $i = 1, \ldots, n$ denote

$$Y_i(x) = B_n^{-1} X_i I(|X_i| < (1 + x) B_n), \quad Y_i = Y_i(0), \quad W_n(x) = \sum_{i=1}^{n} Y_i(x), \quad W_n = W_n(0).$$

Since $EX_i = 0$, we have

$$|EX_i I(|X_i| < (1 + x) B_n)| = |EX_i I(|X_i| \geq (1 + x) B_n)|.$$ (8)
By the definition of the random variables $Y_i(x)$ the relation
\begin{equation}
\sum_{i=1}^{n} \mathbb{E}Y_i^2(x) \leq \frac{1}{B_n^2} \sum_{i=1}^{n} \mathbb{E}X_i^2 = 1
\end{equation}
holds. Denote
$$K = \frac{17 + 7\sqrt{7}}{27} < 1.3156.$$  

**Lemma 2.** 1°. For any $n \in \mathbb{N}$, $x \geq 0$ and $p \in [1, K]$ there holds the inequality
\begin{equation}
\sum_{i=1}^{n} \mathbb{E}|Y_i(x) - \mathbb{E}Y_i(x)|^3 \leq \min \left\{ KM_n(1 + x), pM_n(1 + x) + \frac{(5 - p)L_n(1 + x)}{1 + x} \right\}.
\end{equation}

2°. For any $n \in \mathbb{N}$ and $x \geq 0$ there hold the inequalities
$$1 - 2L_n(1 + x) \leq DW_n(x) \leq 1.$$  

3°. Let $M_n(1) = \gamma L_n(1)$, $\gamma \geq 0$. Then for any $n \in \mathbb{N}$ there holds the inequality
\begin{equation}
\sum_{i=1}^{n} \mathbb{E}|Y_i - \mathbb{E}Y_i|^3 \leq L_n(1) \min \{K\gamma, \gamma + 4\}.
\end{equation}

The proof based on the results of [23, 3] and [8] was given in [6].

**Lemma 3.** 1°. Let $q > 0$. Then
\begin{equation}
\sup_{x} |\Phi(qx) - \Phi(x)| = \left| \Phi\left( q \sqrt{\frac{\ln q^2}{q^2 - 1}} \right) - \Phi\left( \sqrt{\frac{\ln q^2}{q^2 - 1}} \right) \right| \leq \sqrt{\frac{(q - 1) \ln q}{\pi(q + 1)}} \exp \left\{ - \min(1, q) \frac{\ln q}{q^2 - 1} \right\} \leq \frac{1}{\sqrt{2\pi e}} \left( \max \left\{ q, \frac{1}{q} \right\} - 1 \right).
\end{equation}

2°. Let $a \in \mathbb{R}$. Then
\begin{equation}
\sup_{x} |\Phi(x + a) - \Phi(x)| = 2\Phi\left( \frac{|a|}{2} \right) - 1 \leq \frac{|a|}{\sqrt{2\pi}}.
\end{equation}

The elementary proof of this lemma is based on the Lagrange formula and the easily verifiable fact: if $F(x)$ and $G(x)$ are two differentiable distribution functions, then $\sup_{x} |F(x) - G(x)|$ is attained at those points $x$, where $F'(x) = G'(x)$ (also see [11], p. 143).

**Lemma 4.** Assume that $L_n(1) \leq A$ for some $A \in (0, \frac{1}{2})$. Let
$$B(A) = \frac{2}{(1 + \sqrt{1 - 2A})\sqrt{1 - 2A}}.$$  

Then
$$1 \leq \frac{1}{\sqrt{D W_n}} \leq 1 + B(A)L_n(1).$$  

For the proof see [6].

**Lemma 5.** Let $X$ be a random variable with $\mathbb{E}X^2 < \infty$. Then
\begin{equation}
\sup_{x} \left| P\left( \frac{X - \mathbb{E}X}{\sqrt{\mathbb{D}X}} < x \right) - \Phi(x) \right| \leq \sup_{z > 0} \left| \frac{1}{1 + z^2} - \Phi(-z) \right| = 0.54093\ldots
\end{equation}

For the proof see, e. g., the book [2] and the papers [11, 7].
2.2 General case

**Theorem 1.** Let \( n \in \mathbb{N} \), the random variables \( X_1, \ldots, X_n \) be independent, \( \mathbb{E}X_i = 0 \) and \( 0 < \mathbb{E}X_i^2 < \infty \), \( i = 1, \ldots, n \). Let \( \gamma = M_n(1)/L_n(1) \). Then there exists a finite positive number \( C_1(\gamma) \) depending only on \( \gamma \) such that

\[
\Delta_n \leq (1 + \gamma)C_1(\gamma)L_n(1).
\]

Moreover, the upper bounds for \( C_1(\gamma) \) are presented in table 1.

| \( \gamma \)  | \( C_1(\gamma) \leq \) | \( \gamma \)  | \( C_1(\gamma) \leq \) | \( \gamma \)  | \( C_1(\gamma) \leq \) |
|-------------|------------------|-------------|------------------|-------------|------------------|
| \( \gamma \geq 0 \) | 1.8627            | \( \gamma \geq 1 \) | 1.5605            | \( \gamma \geq 10 \) | 0.9393          |
| \( \gamma \geq 0.1 \) | 1.8587            | \( \gamma \geq 2 \) | 1.3488            | \( \gamma \geq 100 \) | 0.6067          |
| \( \gamma \geq 0.5 \) | 1.7244            | \( \gamma \geq 5 \) | 1.0836            | \( \gamma \to \infty \) | 0.5583          |

Table 1: Upper bounds for \( C_1(\gamma) \).

**Corollary 2.** Under the conditions of theorem 1, inequalities (2) – (5) hold with \( C_2 = C_3 = C_3' \leq 1.8627 \).

**Proof of theorem 1.** For any \( y \in \mathbb{R} \) the event \( \{ S_n < yB_n \} \) implies the event

\[
\{ W_n < y \} \cup \{|X_1| \geq B_n \} \cup \ldots \cup \{|X_n| \geq B_n \},
\]

whereas the event \( \{ W_n < y \} \) implies the event

\[
\{ S_n < yB_n \} \cup \{|X_1| \geq B_n \} \cup \ldots \cup \{|X_n| \geq B_n \}.
\]

Therefore,

\[
\sup_y |\mathbb{P}(S_n < yB_n) - \mathbb{P}(W_n < y)| \leq \sum_{i=1}^n \mathbb{P}(|X_i| \geq B_n).
\]

Hence, for any \( y \in \mathbb{R} \)

\[
\Delta_n \leq Q_1 + Q_2 + Q_3, \quad (10)
\]

where

\[
Q_1 = \sup_y \left| \mathbb{P} \left( \frac{W_n - \mathbb{E}W_n}{\sqrt{D W_n}} < \frac{y - \mathbb{E}W_n}{\sqrt{D W_n}} \right) - \Phi \left( \frac{y - \mathbb{E}W_n}{\sqrt{D W_n}} \right) \right|,
\]

\[
Q_2 = \sup_y \left| \Phi \left( \frac{y - \mathbb{E}W_n}{\sqrt{D W_n}} \right) - \Phi(y) \right|, \quad Q_3 = \sum_{i=1}^n \mathbb{P}(|X_i| \geq B_n).
\]

Consider \( Q_1 \). By virtue of the Berry–Esseen inequality with the best known upper bound of the absolute constant (see [13]) we have

\[
Q_1 \leq \frac{0.5583}{(D W_n)^{3/2}} \sum_{i=1}^n \mathbb{E}|Y_i - \mathbb{E}Y_i|^3.
\]

Assume that \( L_n(1) \leq A < \frac{1}{2} \). Then in accordance with statements 2° and 3° of lemma 2

\[
Q_1 \leq \frac{0.5583 \cdot \min\{K_\gamma, \gamma + 4\} L_n(1)}{(1 - 2A)^{3/2}}. \quad (11)
\]

Consider \( Q_2 \). We obviously have

\[
Q_2 = \sup_y \left| \Phi \left( \frac{y - \mathbb{E}W_n}{\sqrt{D W_n}} \right) - \Phi(y) \right| \leq
\]

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Finally, by the Markov inequality

\[ \Phi \left( \frac{y - EW_n}{\sqrt{DW_n}} \right) - \Phi \left( y - EW_n \right) + \sup_y \left| \Phi(y - EW_n) - \Phi(y) \right| = \sup_y \left| \Phi \left( \frac{y}{\sqrt{DW_n}} \right) - \Phi \left( y \right) \right| + \sup_y \left| \Phi(y - EW_n) - \Phi(y) \right| \equiv Q_{21} + Q_{22}. \]

According to statement 2° of lemma 2, $DW_n \leq 1$. Therefore, by virtue of statement 1° of lemma 3 and lemma 4, there holds the inequality

\[ Q_{21} \leq \frac{1}{\sqrt{2\pi e}} \left( \frac{1}{\sqrt{DW_n}} - 1 \right) \leq \frac{2L_n(1)}{\sqrt{2\pi e(1 - 2A)(1 + \sqrt{1 - 2A})}. \] (12)

Consider $Q_{22}$. By virtue of (8) we have

\[ |EW_n| = \left| \sum_{i=1}^{n} EY_i \right| \leq \frac{1}{B_n} \sum_{i=1}^{n} |EX_i|I(|X_i| < B_n) = \frac{1}{B_n} \sum_{i=1}^{n} |EX_i|I(|X_i| \geq B_n) \leq \frac{1}{B_n} \sum_{i=1}^{n} E|X_i|I(|X_i| \geq B_n) \leq \frac{1}{B_n^2} \sum_{i=1}^{n} E|X_i|^2I(|X_i| \geq B_n) = L_n(1). \]

Therefore, by statements 2° of lemma 2 and 2° of lemma 3,

\[ Q_{22} \leq \frac{L_n(1)}{\sqrt{2\pi}}. \] (13)

From (12) and (13) it follows that

\[ Q_2 \leq \frac{L_n(1)}{\sqrt{2\pi}} \left( 1 + \frac{2}{\sqrt{e(1 - 2A)(1 + \sqrt{1 - 2A})}} \right). \] (14)

Finally, by the Markov inequality

\[ Q_3 = \sum_{i=1}^{n} P(|X_i| \geq B_n) \leq \frac{1}{B_n^2} \sum_{i=1}^{n} EX_i^2I(|X_i| \geq B_n) = L_n(1). \] (15)

So, from (10), (11), (14) and (15) we obtain

\[ \Delta_n \leq L_n(1) \left[ 1 + \frac{1}{\sqrt{2\pi}} \left( 1 + \frac{2}{\sqrt{e(1 - 2A)(1 + \sqrt{1 - 2A})}} \right) + 0.5583 \cdot \min \{ K\gamma, \gamma + 4 \} \right]. \] (16)

Introduce the function

\[ H_1(\gamma, A) = 1 + \frac{1}{\sqrt{2\pi}} \left( 1 + \frac{2}{\sqrt{e(1 - 2A)(1 + \sqrt{1 - 2A})}} \right) + 0.5583 \cdot \min \{ K\gamma, \gamma + 4 \} \] (17)

For any $0 \leq A < \frac{1}{2}$ we have the inequality

\[ \Delta_n \leq L_n(1) \cdot \max \left\{ H_1(\gamma, A), \frac{0.541}{A} \right\}. \]

This follows from (16) if $L_n(1) \leq A$ and from lemma 5 otherwise.

Now, with the account of the equality

\[ L_n(1) = \frac{(L_n(1) + \gamma L_n(1))}{1 + \gamma} = \frac{L_n(1) + M_n(1)}{1 + \gamma}, \]

we obtain
we have

\[ C_1(\gamma) \leq \min_{0 \leq A < \frac{\gamma}{4}} \max \left\{ \frac{H_1(\gamma, A)}{1 + \gamma}, \frac{0.541}{A(1 + \gamma)} \right\}. \]

The computation by this formula yield the values presented in table 1. Note that the first function of \( A \) inside the minimax is increasing whereas the second one is decreasing. Hence, the value of the minimax is delivered by the unique solution of the equation

\[ \frac{H_1(\gamma, A)}{1 + \gamma} = \frac{0.541}{A(1 + \gamma)}. \]

For \( \gamma > 13 \) (we have \( \gamma + 4 < K\gamma \)) both functions decrease in \( \gamma \), that is, the minimax value decreases. Therefore, the corresponding part of table 1 is obtained by the evaluation of the bound for \( C_1(\gamma) \) at one point. The part of table 1 corresponding to \( 0 \leq \gamma \leq 13 \) is obtained by numerical optimization of a finite interval. The theorem is proved.

2.3 Special cases

Using the best current upper bound \( C_0 \leq 0.4690 \) for the absolute constant in the Berry–Esseen inequality for identically distributed summands (see [15]), the following statement can be obtained in the way similar to the proof of theorem 1.

**Theorem 2.** In addition to the assumptions of theorem 1, let the random variables \( X_1, X_2, \ldots \) be identically distributed. Then there exists a finite positive number \( C_2(\gamma) \) depending only on \( \gamma \) such that

\[ \Delta_n \leq (1 + \gamma)C_2(\gamma)L_n(1). \]

Moreover, the upper bounds for \( C_2(\gamma) \) are presented in table 2.

| \( \gamma \) | \( C_2(\gamma) \) | \( \gamma \) | \( C_2(\gamma) \) | \( \gamma \) | \( C_2(\gamma) \) |
|-------|-------|-------|-------|-------|-------|
| \( \gamma \geq 0 \) | 1.8546 | \( \gamma \geq 1 \) | 1.4793 | \( \gamma \geq 10 \) | 0.8292 |
| \( \gamma \geq 0.1 \) | 1.8338 | \( \gamma \geq 2 \) | 1.2540 | \( \gamma \geq 100 \) | 0.5147 |
| \( \gamma \geq 0.5 \) | 1.6608 | \( \gamma \geq 5 \) | 0.9781 | \( \gamma \to \infty \) | 0.4690 |

Table 2: Upper bounds for \( C_2(\gamma) \).

**Proof.** Using the reasoning similar to that used to prove theorem 1, it is easy to see that

\[ C_2(\gamma) \leq \min_{0 \leq A < \frac{\gamma}{4}} \max \left\{ \frac{H_2(\gamma, A)}{1 + \gamma}, \frac{0.541}{A(1 + \gamma)} \right\}, \]

where

\[ H_2(\gamma, A) = 1 + \frac{1}{\sqrt{2\pi \gamma}} \left( 1 + \frac{2}{\sqrt{\pi(1 - 2A)(1 + \sqrt{1 - 2A})}} \right) + \frac{0.4690 \cdot \min\{K\gamma, \gamma + 4\}}{(1 - 2A)^{3/2}}. \]

The computations by these formula yield the values of the upper bounds for \( C_2(\gamma) \) presented in table 2. The theorem is proved.

**Corollary 3.** Under conditions of theorem 2, inequalities (1) and (6) hold with \( C_1 = C_4 \leq 1.8546 \).

**Theorem 3.** In addition to the conditions of theorem 1, let the random variables \( X_1, X_2, \ldots \) have symmetric distributions. Then there exists a finite positive number \( C_3(\gamma) \) depending only on \( \gamma \) such that

\[ \Delta_n \leq (1 + \gamma)C_3(\gamma)L_n(1). \]

Moreover, the upper bounds for \( C_3(\gamma) \) are presented in table 3.
Table 3: Upper bounds of $C_3(\gamma)$.

| $\gamma$ | $C_3(\gamma) \leq$ |
|---|---|
| $\gamma \geq 0$ | 1.5769 |
| $\gamma \geq 0.1$ | 1.5749 |
| $\gamma \geq 0.5$ | 1.4532 |
| $\gamma \geq 1$ | 1.3033 |
| $\gamma \geq 2$ | 1.1115 |
| $\gamma \geq 5$ | 0.8729 |
| $\gamma \geq 10$ | 0.7433 |
| $\gamma \geq 100$ | 0.5808 |
| $\gamma \rightarrow \infty$ | 0.5583 |

Corollary 4. Under the conditions of theorem 3, inequalities (2) – (5) hold with $C_2 = C_3 = C_3' \leq 1.5769$.

The proof of theorem 3. In the case under consideration instead (8) we have

$$Q_1 \leq \frac{0.5583 M_n(1)}{(1 - 2A)^{3/2}}.$$ 

and $Q_{22} = 0$, since $E W_n = 0$. Therefore, the bound

$$\Delta_n \leq L_n(1) \left( 1 + \frac{2}{\sqrt{2\pi e (1 - 2A)(1 + \sqrt{1 - 2A})}} \right) + \frac{0.5583 M_n(1)}{(1 - 2A)^{3/2}}$$

holds. Thus,

$$C_3(\gamma) \leq \min_{0 \leq A < \frac{1}{2}} \max_{\gamma \geq 1} \left\{ \frac{H_3(\gamma, A)}{1 + \gamma}, \frac{0.541}{A(1 + \gamma)} \right\},$$

where

$$H_3(\gamma, A) = 1 + \frac{2}{\sqrt{2\pi e (1 - 2A)(1 + \sqrt{1 - 2A})}} + \frac{0.5583}{(1 - 2A)^{3/2}}.$$ 

The computations by the above formulas yield the values of the upper bounds for $C_3(\gamma)$ presented in table 3. The theorem is proved.

Theorem 4. In addition to the conditions of 3, let the random variables $X_1, X_2, \ldots$ be identically distributed. Then there exists a finite positive number $C_4(\gamma)$ depending only on $\gamma$ such that

$$\Delta_n \leq (1 + \gamma) C_4(\gamma) L_n(1).$$

Moreover, the upper bounds for $C_4(\gamma)$ are presented in table 4.

| $\gamma$ | $C_4(\gamma) \leq$ |
|---|---|
| $\gamma \geq 0$ | 1.5645 |
| $\gamma \geq 0.1$ | 1.5534 |
| $\gamma \geq 0.5$ | 1.4018 |
| $\gamma \geq 1$ | 1.2388 |
| $\gamma \geq 2$ | 1.0373 |
| $\gamma \geq 5$ | 0.7915 |
| $\gamma \geq 10$ | 0.6591 |
| $\gamma \geq 100$ | 0.4923 |
| $\gamma \rightarrow \infty$ | 0.4690 |

Table 4: Upper bound for $C_4(\gamma)$.

Corollary 5. Under the conditions of theorem 4, inequalities (1) and (6) hold with $C_1 = C_4 \leq 1.5645$.

Proof of theorem 4. In the case under consideration

$$C_4(\gamma) \leq \min_{0 \leq A < \frac{1}{2}} \max_{\gamma \geq 1} \left\{ \frac{H_4(\gamma, A)}{1 + \gamma}, \frac{0.541}{A(1 + \gamma)} \right\},$$

where

$$H_4(\gamma, A) = 1 + \frac{2}{\sqrt{2\pi e (1 - 2A)(1 + \sqrt{1 - 2A})}} + \frac{0.4690}{(1 - 2A)^{3/2}}.$$ 

The computations by the above formulas yield the values of the upper bounds for $C_4(\gamma)$ presented in table 4. The theorem is proved.
3 The accuracy of the normal approximation to the distributions of Poisson-binomial random sums

From this point on let $X_1, X_2, \ldots$ be independent identically distributed random variables with $\mathbb{E}X_i = 0$ and $0 < \mathbb{E}X_i^2 \equiv \sigma^2 < \infty$. Let $p_j \in (0, 1]$ be arbitrary numbers, $j = 1, 2, \ldots$. For $n \in \mathbb{N}$ denote $\theta_n = p_1 + \ldots + p_n$, $p_n = (p_1, \ldots, p_n)$. The distribution of the random variable

$$N_{n, p_n} = \xi_1 + \ldots + \xi_n,$$

where $\xi_1, \ldots, \xi_n$ are independent random variables such that

$$\xi_j = \begin{cases} 1 & \text{with probability } p_j, \\ 0 & \text{with probability } 1 - p_j, \end{cases}, \quad j = 1, \ldots, n,$$

is usually called Poisson-binomial distribution with parameters $n; p_n$. Assume that for each $n \in \mathbb{N}$ the random variables $N_{n, p_n}, X_1, X_2, \ldots$ are jointly independent. The main objects considered in this section are Poisson-binomial random sums of the form

$$S_{N_{n, p_n}} = X_1 + \ldots + X_{N_{n, p_n}},$$

As this is so, if $N_{n, p_n} = 0$, then we assume $S_{N_{n, p_n}} = 0$.

For $j \in \mathbb{N}$ introduce the random variables $\tilde{X}_j$ by setting

$$\tilde{X}_j = \begin{cases} X_j & \text{with probability } p_j, \\ 0 & \text{with probability } 1 - p_j. \end{cases}$$

If the common distribution function of the random variables $X_j$ is denoted $F(x)$ and the distribution function with a single unit jump at zero is denoted $E_0(x)$, then, as is easily seen,

$$\mathbb{P}(\tilde{X}_j < x) = p_j F(x) + (1 - p_j) E_0(x), \quad x \in \mathbb{R}, \ j \in \mathbb{N}.$$

It is obvious that $\mathbb{E}\tilde{X}_j = 0,$

$$\mathbb{D}\tilde{X}_j = \mathbb{E}\tilde{X}_j^2 = p_j \sigma^2. \quad (18)$$

In what follows the symbol $\overset{d}{=} \text{ will denote coincidence of distributions.}$

**Lemma 6.** For any $n \in \mathbb{N}$ and $p_j \in (0, 1]$

$$S_{N_{n, p_n}} \overset{d}{=} \tilde{X}_1 + \ldots + \tilde{X}_n, \quad (19)$$

where the random variables on the right-hand side of (19) are independent.

**Proof.** The characteristic functions of the left-hand and right-hand sides of (19) have the following forms

$$\varphi_{S_{N_{n, p_n}}}(t) = \sum_{k=0}^{n} \varphi_{X_1 + \ldots + X_k}(t) \mathbb{P}(N_{n, p_n} = k) \quad \text{and} \quad \varphi_{\tilde{X}_1 + \ldots + \tilde{X}_n}(t) = \prod_{j=1}^{n} [p_j \varphi_{X_j}(t) + (1 - p_j)].$$

It suffices to make sure that the characteristic functions of the left-hand and right-hand sides of (19) coincide.

We will use the method of mathematical induction. Basis: $n = 1$.

$$p_1 \varphi_{X_1}(t) + (1 - p_1) = p_1 \varphi_{X_1}(t) + (1 - p_1).$$
Note that the right-hand sides of the above chain of equalities coincide. The lemma is proved.

Denote
\[ n(19) \]
coincide with \( \Delta \), then they also coincide with \( n = m + 1 \).

\[
\prod_{j=1}^{m+1} [p_j \varphi X_j(t) + (1 - p_j)] = \prod_{j=1}^{m} [p_j \varphi X_j(t) + (1 - p_j)](p_{m+1} \varphi X_{m+1}(t) + (1 - p_{m+1})) =
\]

\[
= (1 - p_{m+1}) \sum_{k=0}^{m} \varphi X_1 + \ldots + X_k(t)P(N_{m,p_m} = k) + p_{m+1} \varphi X_{m+1}(t) \sum_{k=0}^{m} \varphi X_1 + \ldots + X_k(t)P(N_{m,p_m} = k) =
\]

\[
= (1 - p_{m+1}) \sum_{k=0}^{m} \varphi X_1 + \ldots + X_k(t)P(N_{m,p_m} = k) + p_1 \ldots p_{m+1} \varphi X_1 + \ldots + X_{m+1} +
\]

\[
+ p_{m+1} \varphi X_{m+1}(t) \sum_{k=0}^{m-1} \varphi X_1 + \ldots + X_k(t)P(N_{m,p_m} = k) =
\]

= (note that \( \varphi X_1(t) = \ldots = \varphi X_{m+1}(t) \) and transform the last term) =

\[
= (1 - p_{m+1}) \sum_{k=0}^{m} \varphi X_1 + \ldots + X_k(t)P(N_{m,p_m} = k) + p_1 \ldots p_{m+1} \varphi X_1 + \ldots + X_{m+1} +
\]

\[
+ p_{m+1} \sum_{k=0}^{m} \varphi X_1 + \ldots + X_k(t)P(N_{m,p_m} = k - 1).
\]

On the other hand,

\[
\sum_{k=0}^{m+1} \varphi X_1 + \ldots + X_k(t)P(N_{m+1,p_{m+1}} = k) =
\]

\[
= \sum_{k=0}^{m} \varphi X_1 + \ldots + X_k(t)P(N_{m+1,p_{m+1}} = k) + p_1 \ldots p_{m+1} \varphi X_1 + \ldots + X_{m+1} =
\]

\[
= \sum_{k=0}^{m} \varphi X_1 + \ldots + X_k(t)P(\{N_{m,p_m} = k \cap \xi_{m+1} = 0\} \cup \{N_{m,p_m} = k - 1 \cap \xi_{m+1} = 1\}) +
\]

\[
+ p_1 \ldots p_{m+1} \varphi X_1 + \ldots + X_{m+1} =
\]

\[
= (1 - p_{m+1}) \sum_{k=0}^{m} \varphi X_1 + \ldots + X_k(t)P(N_{m,p_m} = k) + p_{m+1} \sum_{k=0}^{m} \varphi X_1 + \ldots + X_k(t)P(N_{m,p_m} = k - 1) +
\]

\[
+ p_1 \ldots p_{m+1} \varphi X_1 + \ldots + X_{m+1}.
\]

Note that the right-hand sides of the above chain of equalities coincide. The lemma is proved.

With the account of (18) and (19) it is easy to notice that

\[
DS_{N_n,p_n} = \theta_n \sigma^2.
\]

(20)

Denote

\[
\Delta_{n,p_n} = \sup_x |P(S_{N_n,p_n} < x\sqrt{\theta_n}) - \Phi(x)|.
\]

Theorem 5. For any \( n \in \mathbb{N} \) and \( p_j \in (0,1] \), \( j \in \mathbb{N} \),\n
\[
\Delta_{n,p_n} \leq \frac{1.8627}{\sigma^2} EX^2_1 \min \left\{ 1, \frac{|X_1|}{\sigma \sqrt{\theta_n}} \right\},
\]

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Proof. By virtue of lemma 6 and relation (20) we have
\[ \Delta_{n,p_n} = \sup_x |P(\tilde{X}_1 + \ldots + \tilde{X}_n < x\sqrt{\theta_n}) - \Phi(x)|, \]
and for the latter expression we can use the bound given in theorem 1:
\[ \sup_x |P(\tilde{X}_1 + \ldots + \tilde{X}_n < x\sqrt{\theta_n}) - \Phi(x)| \leq \]
\[ 1.8627 \left[ \frac{1}{\sigma^2 \theta_n} \sum_{j=1}^{n} E[|X_j| \geq \sigma \sqrt{\theta_n}] + \frac{1}{\sigma^3 \theta_n^{3/2}} \sum_{j=1}^{n} E[|X_j|^{3/2} \mathbb{I}(|X_j| < \sigma \sqrt{\theta_n})] \right] = \]
\[ 1.8627 \left[ \frac{1}{\sigma^2 \theta_n} \sum_{j=1}^{n} p_j E[|X_j| \geq \sigma \sqrt{\theta_n}] + \frac{1}{\sigma^3 \theta_n^{3/2}} \sum_{j=1}^{n} p_j E[|X_j|^{3/2} \mathbb{I}(|X_j| < \sigma \sqrt{\theta_n})] \right] = \]
\[ 1.8627 \left[ \frac{1}{\sigma^2 \theta_n} \sum_{j=1}^{n} p_j E[|X_1| \geq \sigma \sqrt{\theta_n}] + \frac{1}{\sigma^3 \theta_n^{3/2}} \sum_{j=1}^{n} p_j E[|X_1|^{3/2} \mathbb{I}(|X_1| < \sigma \sqrt{\theta_n})] \right] = \]
\[ 1.8627 \left[ \frac{1}{\sigma^2 \theta_n} E[|X_1| \geq \sigma \sqrt{\theta_n}] + \frac{1}{\sigma^3 \theta_n^{3/2}} E[|X_1|^{3/2} \mathbb{I}(|X_1| < \sigma \sqrt{\theta_n})] \right] = \]
\[ 1.8627 \left[ \frac{1}{\sigma^2 \theta_n} E[X_1^2 \min \{ \sigma \sqrt{\theta_n}, |X_1| \}] + \frac{1.8627}{\sigma^2} E[X_1^2 \min \{ 1, \frac{|X_1|}{\sigma \sqrt{\theta_n}} \}] \right], \]
Q. E. D.

Theorem 6. Under the conditions of theorem 5, whatever function \( g \in \mathcal{G} \) is such that \( E X_1^2 g(X_1) < \infty \), there holds the inequality
\[ \Delta_{n,p_n} \leq 1.8627 \frac{E X_1^2 g(X_1)}{\sigma^2 g(\sigma \sqrt{\theta_n})}. \]

Proof. Let \( g \) be an arbitrary function from the class \( \mathcal{G} \). With the account of the properties of a function \( g \in \mathcal{G} \) it is easy to see that
\[ E X_1^2 \mathbb{I}(|X_1| \geq \sigma \sqrt{\theta_n}) = E X_1^2 g(X_1) \frac{g(X_1)}{g(\sigma \sqrt{\theta_n})} \mathbb{I}(|X_1| \geq \sigma \sqrt{\theta_n}) \leq \frac{1}{g(\sigma \sqrt{\theta_n})} E X_1^2 g(X_1) \mathbb{I}(|X_1| \geq \sigma \sqrt{\theta_n}) \] (21)
and
\[ E X_1^2 \mathbb{I}(|X_1| < \sigma \sqrt{\theta_n}) = E X_1^2 g(X_1) \frac{|X_1|}{g(X_1)} \mathbb{I}(|X_1| < \sigma \sqrt{\theta_n}) \leq \frac{\sigma \sqrt{\theta_n}}{g(\sigma \sqrt{\theta_n})} E X_1^2 g(X_1) \mathbb{I}(|X_1| < \sigma \sqrt{\theta_n}). \] (22)
Substituting these estimates into the inequality
\[ \Delta_{n,p_n} \leq 1.8627 \left[ \frac{1}{\sigma^2} E X_1^2 \mathbb{I}(|X_1| \geq \sigma \sqrt{\theta_n}) + \frac{1}{\sigma^3 \sqrt{\theta_n}} E|X_1|^{3/2} \mathbb{I}(|X_1| < \sigma \sqrt{\theta_n}) \right] \] (23)
obtained in the proof of theorem 5, we have
\[ \Delta_{n,p_n} \leq 1.8627 \left[ \frac{1}{\sigma^2} E X_1^2 g(X_1) \mathbb{I}(|X_1| \geq \sigma \sqrt{\theta_n}) + E X_1^2 g(X_1) \mathbb{I}(|X_1| < \sigma \sqrt{\theta_n}) \right] = 1.8627 \frac{E X_1^2 g(X_1)}{\sigma^2 g(\sigma \sqrt{\theta_n})}. \]
The theorem is proved.

In particular, if \( p_1 = p_2 = \ldots = p \), then the Poisson-binomial distribution with parameters \( n \in \mathbb{N} \) and \( p_n \) becomes the classical binomial distribution with parameters \( n \) and \( p \):
\[ N_{n,p} \overset{d}{=} N_{n,p}, \quad P(N_{n,p} = k) = C_n^k p^k (1 - p)^{n-k}, \quad k = 0, \ldots, n. \]

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In this case $\theta_n = np$, so that $DS_{N_n,p} = np\sigma^2$. Denote

$$\Delta_{n,p} = \sup_x |P(S_{N_n,p} < x\sigma\sqrt{np}) - \Phi(x)|.$$ 

Estimates of the accuracy of the normal approximation to the distributions of binomial random sums (under traditional conditions of the existence of the third moments of summands) were considered in the paper [33], where a conventional approach was used which is based on the direct application of the total probability formula and does not involve representation (19). Hence, in [33] estimates were obtained with the structure far from being optimal, containing unnecessary terms and unreasonably large values of absolute constants.

Theorems 2 and 5 imply

**Corollary 6.** For any $n \in \mathbb{N}$ and $p \in (0, 1]$ 

$$\Delta_{n,p} \leq \frac{1.8546}{\sigma^2} \mathbb{E}X_1^2 \min \left\{ 1, \frac{|X_1|}{\sigma\sqrt{np}} \right\}.$$

Theorems 2 and 6 imply

**Corollary 7.** Under the conditions of theorem 5, whatever function $g \in \mathcal{G}$ is such that $\mathbb{E}X_1^2g(X_1) < \infty$, for any $n \in \mathbb{N}$ and $p \in (0, 1]$ there holds the inequality

$$\Delta_{n,p_n} \leq \frac{1.8546}{\sigma^2} \mathbb{E}X_1^2g(X_1) \frac{g(\sigma\sqrt{np})}{\sigma g(\sigma\sqrt{np})}.$$

**4 The accuracy of the normal approximation to the distributions of Poisson random sums**

In addition to the notation introduced above, let $\lambda > 0$ and $N_\lambda$ be the random variable with the Poisson distribution with parameter $\lambda$:

$$P(N_\lambda = k) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k \in \mathbb{N} \cup \{0\}.$$ 

Assume that for each $\lambda > 0$ the random variables $N_\lambda, X_1, X_2, \ldots$ are jointly independent. Consider the Poisson random sum

$$S_{N_\lambda} = X_1 + \ldots + X_{N_\lambda}.$$ 

If $N_\lambda = 0$, then we set $S_{N_\lambda} = 0$. It is easy to see that $\mathbb{E}S_\lambda = 0$ and $DS_\lambda = \lambda\sigma^2$. The accuracy of the normal approximation to the distributions of Poisson random sum was considered by many authors, see the historical surveys in [27, 14]. However, the authors are unaware of any analogs of the Katz–Osipov-type inequalities (1) and (6) under relaxed moment conditions.

We will obtain a bound for

$$\Delta_\lambda = \sup_x |P(S_\lambda < x\sqrt{\lambda\sigma}) - \Phi(x)|.$$ 

For this purpose fix $\lambda$ and along with $N_\lambda$ consider the random variable $N_{n,p}$ having the binomial distribution with arbitrary parameters $n$ and $p \in (0, 1]$ such that $np = \lambda$. As this is so, the reasoning used above implies that

$$DS_{N_\lambda} = DS_{N_{n,p}} = \sigma^2\lambda = \sigma^2np.$$ 

Therefore, by the triangle inequality, in accordance with corollary 6 we have

$$\Delta_\lambda \leq \Delta_{n,p} + \sup_x |P(S_{N_\lambda} < x) - P(S_{N_{n,p}} < x)| \leq$$
Estimate the second term on the right-hand side of (24) by the Prokhorov inequality [13] (also see [17], p. 76), according to which

$$\sum_{k=0}^{\infty} |P(N_{n,p} = k) - P(N_\lambda = k)| \leq 2 \min\{2, \lambda\},$$

and obtain that for any $n$ and $p$ such that $np = \lambda$, there holds the inequality

$$\Delta_\lambda \leq \frac{1.8546 \cdot \mathbb{E}X_1^2 \min\{1, \frac{|X_1|}{\sigma \sqrt{n_p}}\}}{\sigma^2} + 2 \min\{2, \lambda\}.$$  \hspace{1cm} (25)

Now, putting in (25) $p = \lambda/n$ and letting $n \to \infty$, we obtain the final result:

**Theorem 7.** For any $\lambda > 0$

$$\Delta_\lambda \leq \frac{1.8546 \cdot \mathbb{E}X_1^2 \min\{1, \frac{|X_1|}{\sigma \sqrt{\lambda}}\}}{\sigma^2}.$$  \hspace{1cm} (26)

Using inequalities (21) – (23) to estimate $\Delta_{n,p}$ in (24), we obtain the following result.

**Theorem 8.** Whatever function $g \in \mathcal{G}$ is such that $\mathbb{E}X_1^2 g(X_1) < \infty$, there holds the inequality

$$\Delta_\lambda \leq \frac{1.8546 \cdot \mathbb{E}X_1^2 g(x_1)}{\sigma^2 g(\sigma \sqrt{\lambda})}.\hspace{1cm} (27)$$

**Remark 2.** The upper bound of the absolute constant used in theorem 8 is uniform over the class $\mathcal{G}$. In specific cases this bound can be considerably sharpened. For example, it is obvious that $g(x) \equiv |x| \in \mathcal{G}$. For such a function $g$ inequality (26) takes the form of the classical Berry–Esseen inequality for Poisson random sums, the best current upper bound for the absolute constant in which is given in [16]:

$$\Delta_\lambda \leq 0.3031 \frac{\mathbb{E}|X_1|^3}{\sigma^3 \sqrt{\lambda}}.\hspace{1cm} (28)$$

## 5 Convergence rate estimates for mixed Poisson random sums

### 5.1 General results

In this section we extend the results of the preceding section to the case where the random number of summands has the mixed Poisson distribution. For convenience, in this case we introduce an “infinitely large” parameter $n \in \mathbb{N}$ and consider random variables $N_n^*$ such that for each $n \in \mathbb{N}$

$$P(N_n^* = k) = \int_0^\infty e^{-\lambda} \frac{\lambda^k}{k!} dP(\Lambda_n < \lambda), \quad k \in \mathbb{N} \cup \{0\},$$  \hspace{1cm} (28)

for some positive random variable $\Lambda_n$. For simplicity $n$ may be assumed to be the scale parameter of the distribution of $\Lambda_n$ so that $\Lambda_n = n\Lambda$ where $\Lambda$ is some positive “standard” random variable in the sense, say, that $\mathbb{E}\Lambda = 1$ (if the latter exists).
Assume that for each \( n \in \mathbb{N} \) the random variable \( N^*_n \) is independent of the sequence \( X_1, X_2, \ldots \). As above, let \( S_{N^*_n} = X_1 + \ldots + X_{N^*_n} \) and if \( N^*_n = 0 \), then \( S_{N^*_n} = 0 \).

From (28) it is easily seen that, if \( \mathbb{E} \lambda_n < \infty \), then \( \mathbb{E} N^*_n = \mathbb{E} \lambda_n \) so that \( DS_n = \sigma^2 \mathbb{E} \lambda_n \).

Let \( N^*_n \) be the random variable with the Poisson distribution with parameter \( \lambda \) independent of \( X_1, X_2, \ldots \). For any \( x \in \mathbb{R} \) we have

\[
P(S_{N^*_n} \leq x \sigma \sqrt{\mathbb{E} \lambda_n}) = \sum_{k=0}^{\infty} P(N^*_n = k) P(S_k \leq x \sigma \sqrt{\mathbb{E} \lambda_n}) = \\ = \sum_{k=0}^{\infty} P(S_k \leq x \sigma \sqrt{\mathbb{E} \lambda_n}) \int_0^{\infty} P(N^*_k = k) dP(\lambda_k < \lambda) = \\ = \int_0^{\infty} P(S_{N^*_k} < x \sigma \sqrt{\mathbb{E} \lambda_n}) dP(\lambda_k < \lambda) = \int_0^{\infty} P\left(S_{N^*_k} < x \sqrt{\frac{\mathbb{E} \lambda_n}{\lambda}}\right) dP(\lambda_k < \lambda) = \\ = \int_0^{\infty} \Phi\left(x \sqrt{\frac{\mathbb{E} \lambda_n}{\lambda}}\right) dP(\lambda_k < \lambda) + \int_0^{\infty} P\left(S_{N^*_k} < x \sqrt{\frac{\mathbb{E} \lambda_n}{\lambda}}\right) dP(\lambda_k < \lambda) - \int_0^{\infty} \Phi\left(x \sqrt{\frac{\mathbb{E} \lambda_n}{\lambda}}\right) dP(\lambda_k < \lambda). \tag{29}
\]

From (29) it follows that

\[
\Delta^* = \sup_x P(S_{N^*_k} < x \sigma \sqrt{\mathbb{E} \lambda_n}) - \int_0^{\infty} \Phi\left(x \sqrt{\frac{\mathbb{E} \lambda_n}{\lambda}}\right) dP(\lambda_k < \lambda) \leq \\
\leq \sup_0^{\infty} P\left(S_{N^*_k} < x \sqrt{\frac{\mathbb{E} \lambda_n}{\lambda}}\right) - \Phi(x) dP(\lambda_k < \lambda) \leq \int_0^{\infty} \Delta \lambda dP(\lambda_k < \lambda). \tag{30}
\]

Now, if to estimate the integrand \( \Delta \lambda \) in (30) we use theorem 7 and recall the notation \( F(x) = P(X < x) \), then by the Fubini theorem we arrive at the representation

\[
\Delta^*_n \leq \frac{1.8546}{\sigma^2} \int_0^{\infty} \mathbb{E} X^2 \min\left\{1, \frac{|X|}{\sigma \sqrt{\lambda}}\right\} dP(\lambda_k < \lambda) = \frac{1.8546}{\sigma^2} \int_0^{\infty} \int_0^{\infty} \min\left\{1, \frac{|x|}{\sigma \sqrt{\lambda}}\right\} dF_1(x) dP(\lambda_k < \lambda) = \\ = \frac{1.8546}{\sigma^2} \int_0^{\infty} \int_0^{\infty} \min\left\{1, \frac{|x|}{\sigma \sqrt{\lambda}}\right\} dP(\lambda_k < \lambda) dF_1(x). \tag{31}
\]

For \( x \in \mathbb{R} \) introduce the function

\[
G_n(x) = \mathbb{E} \min\left\{1, \frac{|x|}{\sigma \sqrt{\lambda_n}}\right\} = P\left(\lambda_n < \sigma^2 \frac{x^2}{\sigma^2}\right) + \frac{|x|}{\sigma} \mathbb{E} \frac{1}{\sqrt{\lambda_n}} P\left(\lambda_n > \frac{x^2}{\sigma^2}\right). \tag{32}
\]

The expectation in (32) exists since the random variable under the expectation sign is bounded by 1. Of course, the particular form of \( G_n(x) \) depends on the particular form of the distribution of \( \lambda_n \). From (30), (31) and (32) we obtain the following statement.

**Theorem 9.** If \( \mathbb{E} \lambda_n < \infty \), then

\[
\Delta^*_n \leq \frac{1.8546}{\sigma^2} \mathbb{E} X^2 G_n(X_1) = \frac{1.8546}{\sigma^2} \mathbb{E} X^2 \min\left\{1, \frac{|X|}{\sigma \sqrt{\lambda_n}}\right\} = \\ = \frac{1.8546}{\sigma^2} \mathbb{E} X^2 \mathbb{P}\left(|X| \geq \sigma \sqrt{\lambda_n}\right) + \mathbb{E} \frac{|X|}{\sigma \sqrt{\lambda_n}} \mathbb{I}\left(|X| < \sigma \sqrt{\lambda_n}\right),
\]

where the random variables \( X_1 \) and \( \lambda_n \) are assumed independent.

In the subsequent sections we will consider special cases where \( \lambda_n \) has the exponential, gamma and inverse gamma distributions.
5.2 Estimates of the rate of convergence of the distributions of geometric random sums to the Laplace law

In this section we consider sums of a random number of independent random variables in which the number of summands \( N_n^* \) has the geometric distribution with parameter \( p = \frac{1}{1+n}, \ n \in \mathbb{N} \):

\[
P(N_n^* = k) = \frac{1}{n + 1} \left( \frac{n}{n + 1} \right)^k, \quad k \in \mathbb{N} \cup \{0\}.
\] (33)

As usual, we assume that for each \( n \in \mathbb{N} \) the random variables \( N_n^*, X_1, X_2, \ldots \) are independent. We again use the notation \( S_{N_n^*} = X_1 + \ldots + X_{N_n^*} \). If \( N_n^* = 0 \), then we set \( S_{N_n^*} = 0 \). It is easy to see that \( \mathbb{E} N_n^* = n, \ \mathbb{D} S_{N_n^*} = n\sigma^2 \). Note that for any \( k \in \mathbb{N} \cup \{0\} \)

\[
P(N_n^* = k) = \frac{1}{n + 1} \int_0^\infty P(N_\lambda = k) \exp \left\{ -\frac{\lambda}{n} \right\} d\lambda,
\]

where \( N_\lambda \) is the random variable with the Poisson distribution with parameter \( \lambda \). This means that for \( N_n^* \) representation (28) holds with \( \Lambda_n \) being an exponentially distributed random variable with parameter \( \frac{1}{n} \).

In what follows we will use traditional notation

\[
\Gamma(\alpha, z) \equiv \int_z^{\infty} y^{\alpha-1} e^{-y} dy, \quad \gamma(\alpha, z) \equiv \int_0^z y^{\alpha-1} e^{-y} dy, \quad \Gamma(\alpha) \equiv \Gamma(\alpha, 0) = \gamma(\alpha, \infty),
\]

for upper incomplete gamma-function, lower incomplete gamma-function and gamma-function itself, respectively, where \( \alpha > 0, \ z > 0 \).

In the case under consideration

\[
\frac{1}{n} \int_0^\infty \Phi \left( x, \sqrt{\frac{n}{\lambda}} \right) \exp \left\{ -\frac{\lambda}{n} \right\} d\lambda = \int_0^\infty \Phi \left( \frac{y}{\sqrt{\lambda}} \right) e^{-y} dy = \mathcal{L}(x),
\]

where \( \mathcal{L}(x) \) is the Laplace distribution function corresponding to the density

\[
\ell(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad x \in \mathbb{R}
\]

(see, e. g., lemma 12.7.1 in [5]).

At the same time, the function \( G_n(x) \) (see (32)) has the form

\[
G_n(x) = 1 - \exp \left\{ -\frac{x^2}{n\sigma^2} \right\} + \frac{|x|}{n\sigma^2} \int_{x^2/n\sigma^2}^{\infty} e^{-\lambda/n} \sqrt{\lambda} d\lambda = \gamma(1, \frac{x^2}{n\sigma^2}) + \frac{|x|}{\sigma \sqrt{n}} \Gamma \left( \frac{1}{2}, \frac{x^2}{n\sigma^2} \right),
\]

So, from theorem 9 we obtain the following result.

**Corollary 8.** Let \( N_n^* \) have the geometric distribution (33). Then

\[
\sup_x \left| P(S_{N_n^*} < x\sigma \sqrt{n}) - \mathcal{L}(x) \right| \leq \frac{1.8546}{\sigma^2} \left\{ \mathbb{E} \left[ X_1^2 \gamma \left( 1, \frac{X_1^2}{n\sigma^2} \right) \right] + \frac{1}{\sigma \sqrt{n}} \mathbb{E} \left[ |X_1|^3 \Gamma \left( \frac{1}{2}, \frac{X_1^2}{n\sigma^2} \right) \right] \right\}.
\]
5.3 Estimates of the rate of convergence of the distributions of negative binomial random sums to the variance-gamma law

The case more general than that considered in the preceding section is the case of negative binomial random sums.

Let \( r > 0 \) be an arbitrary number. Assume that representation (28) holds with \( \Lambda_n \) being a gamma-distributed random variable with the density

\[
p(\lambda) = \frac{\lambda^{r-1} e^{-\lambda/n}}{n^r \Gamma(r)} \quad \lambda > 0.
\]

Then the random variable \( N_n^* \) has the negative binomial distribution with parameters \( r \) and \( \frac{1}{1+n} \):

\[
P(N_n^* = k) = \frac{1}{n^r \Gamma(r)} \int_0^\infty e^{-\lambda} \frac{\lambda^k}{k!} \lambda^{r-1} e^{-\lambda/n} d\lambda = \frac{\Gamma(r+k)}{\Gamma(r) k!} \left( \frac{n}{1+n} \right)^r \left( 1 + \frac{n}{1+n} \right)^k, \quad k \in \mathbb{N} \cup \{0\}. \tag{34}
\]

Let \( V_r(x) \equiv \frac{1}{\Gamma(r)} \int_0^\infty \Phi \left( \frac{x}{\sqrt{\lambda}} \right) \lambda^{r-1} e^{-\lambda/n} d\lambda, \quad x \in \mathbb{R} \),

be the symmetric variance-gamma distribution with shape parameter \( r \) (see, e.g., [28]).

In the case under consideration \( E N_n^* = E \Lambda_n = nr \) so that \( DS_{N_n^*} = nr \sigma^2 \) and for any \( x \in \mathbb{R} \)

\[
\int_0^\infty \Phi \left( \frac{x \sqrt{\lambda}}{\sqrt{\lambda}} \right) dP(\Lambda < \lambda) = \frac{1}{n^r \Gamma(r)} \int_0^\infty \Phi \left( \frac{x \sqrt{\lambda}}{\sqrt{\lambda}} \right) \lambda^{r-1} e^{-\lambda/n} d\lambda = \frac{1}{\Gamma(r)} \int_0^\infty \Phi \left( \frac{x \sqrt{\lambda}}{\sqrt{\lambda}} \right) \lambda^{r-1} e^{-\lambda/n} d\lambda = V_r(x \sqrt{\lambda}).
\]

Here the function \( G_n(x) \) (see (32)) has the form

\[
G_n(x) = \frac{1}{n^r \Gamma(r)} \int_0^{x^2/\sigma^2} \lambda^{r-1} e^{-\lambda/n} d\lambda + \frac{|x|}{\sigma n \Gamma(r)} \int_{x^2/\sigma^2}^\infty \lambda^{r-3/2} e^{-\lambda/n} d\lambda = \frac{1}{\Gamma(r)} \left[ \gamma \left( r, \frac{x^2}{\sigma^2} \right) + \frac{|x|}{\sigma \sqrt{n}} \Gamma \left( r - \frac{1}{2}, \frac{x^2}{\sigma^2} \right) \right].
\]

So, from theorem 9 we obtain the following result.

**Corollary 9.** Let \( N_n^* \) have the negative binomial distribution (34). Then

\[
\sup_x |P(S_{N_n^*} < x \sigma \sqrt{n}) - V_r(x)| \leq \frac{1.8546}{\sigma^2 \Gamma(r)} \left( E \left[ X_1^2 \gamma \left( r, \frac{X_1^2}{\sigma^2} \right) \right] + \frac{1}{\sigma \sqrt{n}} E \left[ |X_1|^3 \Gamma \left( r - \frac{1}{2}, \frac{X_1^2}{\sigma^2} \right) \right] \right).
\]

5.4 Estimates of the rate of convergence of the distributions of Poisson-inverse gamma random sums to the Student distribution

Let \( r > 1 \) be an arbitrary number. Assume that representation (28) holds with \( \Lambda_n \) being an inverse-gamma-distributed random variable with parameters \( \xi \) and \( \eta \) having the density

\[
p(\lambda) = \frac{n^{r/2} \lambda^{-r/2 - 1}}{2^{r/2} \Gamma(\xi)} \exp \left\{ - \frac{n}{2\lambda} \right\}, \quad \lambda > 0.
\]
Then the random variable $N_n^\ast$ has the so-called Poisson-inverse gamma distribution:

$$
P(N_n^\ast = k) = \frac{n^{r/2}}{2^{r/2}\Gamma(\frac{r}{2})} \int_0^\infty e^{-\lambda} \lambda^k k! \lambda^{-r/2-1} \exp \left\{ -\frac{n}{2\lambda} \right\} d\lambda, \quad k \in \mathbb{N} \cup \{0\},
$$

(35)

which is a special case of the so-called Sichel distribution see, e. g., [32, 34]. In this case

$$\text{E} \Lambda_n = \frac{n}{r-2}$$

so that

$$\text{D} S_n^\ast = \frac{n\sigma^2}{r-2}.$$  

Nevertheless, we will normalize random sums not by their mean square deviations, but by slightly different and asymptotically equivalent quantities $\sigma \sqrt{n/r}$.

As is known, if $\Lambda_n$ has the inverse gamma distribution with parameters $\frac{r}{2}$ and $\frac{n}{2}$, then $\Lambda_n^{-1}$ has the gamma distribution with the same parameters. Therefore, we have

$$\frac{n^{r/2}}{2^{r/2}\Gamma(\frac{r}{2})} \int_0^\infty \Phi(x \sqrt{\frac{n}{r\lambda}}) \lambda^{-r/2-1} \exp \left\{ -\frac{n}{2\lambda} \right\} d\lambda =$$

$$= \frac{1}{2^{r/2}\Gamma(\frac{r}{2})} \int_0^\infty \Phi(x \sqrt{\frac{\lambda}{r}}) \lambda^{r/2-1} e^{-\lambda/2} d\lambda = T_r(x), \quad x \in \mathbb{R},$$

where $T_r(x)$ is the Student distribution function with parameter $r$ (r «degrees of freedom») corresponding to the density

$$t_r(x) = \frac{\Gamma\left(\frac{r+1}{2}\right)}{\sqrt{\pi r} \Gamma\left(\frac{r}{2}\right)} \left(1 + \frac{x^2}{r}\right)^{-(r+1)/2}, \quad x \in \mathbb{R},$$

see, e. g., [1].

In this case the function $G_n(x)$ (see (32)) has the form

$$G_n(x) = \text{P}\left(\Lambda_n^{-1} > \frac{\sigma^2}{x^2}\right) + \frac{|x|}{\sigma} \text{E} \sqrt{\Lambda_n^{-1}} \mathbb{I}\left(\Lambda_n^{-1} \leq \frac{\sigma^2}{x^2}\right) =$$

$$= \frac{n^{r/2}}{2^{r/2}\Gamma(\frac{r}{2})} \int_0^\infty \lambda^{r/2-1} e^{-\lambda/2} d\lambda + \frac{|x| n^{r/2}}{2^{r/2}\sigma \Gamma(\frac{r}{2})} \int_0^{\sigma^2/x^2} \lambda^{(r-1)/2} e^{-\lambda/2} d\lambda =$$

$$= \frac{1}{\Gamma\left(\frac{r}{2}\right)} \left[ \Gamma\left(\frac{r}{2}, \frac{n\sigma^2}{2x^2}\right) + |x| \sqrt{\frac{n}{\sigma}} \sqrt{\gamma\left(\frac{r+1}{2}, \frac{n\sigma^2}{2x^2}\right)} \right],$$

where $\gamma(\cdot, \cdot)$ and $\Gamma(\cdot, \cdot)$ are the lower and upper incomplete gamma-functions, respectively. So, from theorem 9 we obtain the following result.

**COROLLARY 10.** Let $N_n^\ast$ have the Poisson-inverse gamma distribution (35). Then

$$\Delta_n^\ast = \sup_{x} | \text{P}\left(S_{N_n^\ast} < x \sigma \sqrt{\frac{n}{r}} - T_r(x)\right) | \leq 1.8546 \frac{\sqrt{n}}{\sigma^2 \Gamma(\frac{r}{2})} \left\{ \text{E} X_1^2 \Gamma\left(\frac{r}{2}, \frac{n\sigma^2}{2X_1^2}\right) \right\} + \frac{1}{\sigma} \sqrt{\frac{n}{2}} \text{E} \left[ X_1^3 \gamma\left(\frac{r+1}{2}, \frac{n\sigma^2}{2X_1^2}\right) \right].$$
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