ILL-POSEDNESS OF THE NAVIER-STOKES EQUATIONS IN A CRITICAL SPACE IN 3D

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Abstract. We prove that the Cauchy problem for the three dimensional Navier-Stokes equations is ill posed in $\dot{B}^{-1,\infty}_{\infty}$ in the sense that a “norm inflation” happens in finite time. More precisely, we show that initial data in the Schwartz class $\mathcal{S}$ that are arbitrarily small in $\dot{B}^{-1,\infty}_{\infty}$ can produce solutions arbitrarily large in $\dot{B}^{-1,\infty}_{\infty}$ after an arbitrarily short time. Such a result implies that the solution map itself is discontinuous in $\dot{B}^{-1,\infty}_{\infty}$ at the origin.

1. Introduction

In this paper we address a long standing open problem concerning well-posedness of the three dimensional Navier-Stokes equations in the largest critical space $\dot{B}^{-1,\infty}_{\infty}$ and prove that the Cauchy problem for the three dimensional Navier-Stokes equations is ill posed in $\dot{B}^{-1,\infty}_{\infty}$.

The Navier-Stokes equations for the incompressible fluid in $\mathbb{R}^3$ are given by

\begin{align}
\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p &= \nu \Delta u, \\
\nabla \cdot u &= 0,
\end{align}

and the initial condition

\begin{equation}
\tag{1.3}
u(x,0) = u_0(x),
\end{equation}

for the unknown velocity vector $u = u(x,t) \in \mathbb{R}^3$ and the pressure $p = p(x,t) \in \mathbb{R}$, where $x \in \mathbb{R}^3$ and $t \in [0,\infty)$.

We adapt the standard notion of well-posedness. More precisely, a Cauchy problem is said to be \textit{locally well-posed} in $Z$ if for every initial data $u_0(x) \in Z$ there exists a time $T = T(\|u_0\|_Z) > 0$ such that a solution to the initial value problem exists in the time interval $\{0,T\}$, is unique in a certain Banach space of functions $Y \subset C([0,T];Z)$ and the solution map from the initial data $u_0$ to the solution $u$ is continuous from $Z$ to $C([0,T];Z)$. If $T$ can be taken

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arbitrarily large we say that the Cauchy problem is *globally well-posed*. Also we say that the Cauchy problem is *ill-posed* if it is not well-posed. Having such a definition of ill-posedness it is clear that the problem may be ill-posed due to different reasons ranging from a failure of a solution map to be continuous to a more serious type of ill-posedness such as a blow-up in finite time. Here we shall establish an ill-posedness of the Navier-Stokes initial value problem (1.1) - (1.3) via proving a finite time blow-up for solutions to the Navier-Stokes equations in the largest critical space, the Besov space $\dot{B}_{\infty,\infty}^{-1,\infty}$.

In order to understand the role of the space $\dot{B}_{\infty,\infty}^{-1,\infty}$ in the analysis of the Navier-Stokes equations we recall the scaling property of the equations first. It is easy to see that if the pair $(u(x,t), p(x,t))$ solves the Navier-Stokes equations (1.1) then $(u_\lambda(x,t), p_\lambda(x,t))$ with

$$
u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t),$$
$$p_\lambda(x,t) = \lambda^2 p(\lambda x, \lambda^2 t)$$

is a solution to the system (1.1) with the initial data

$$u_0 \lambda = \lambda u_0(\lambda x).$$

The spaces which are invariant under such a scaling are called critical spaces for the Navier-Stokes. Examples of critical spaces for the Navier-Stokes in 3D are:

$$\dot{H}^{\frac{1}{2}} \hookrightarrow L^3 \hookrightarrow B^{-1+\frac{3}{p},\infty}_{p,p<\infty} \hookrightarrow BMO^{-1} \hookrightarrow \dot{B}_{\infty,\infty}^{-1,\infty}. \quad (1.4)$$

Kato [9] initiated the study of the Navier-Stokes equations in critical spaces by proving that the problem (1.1)-(1.3) is locally well-posed in $L^3$ and globally well-posed if the initial data are small in $L^3(\mathbb{R}^3)$. The study of the Navier-Stokes equations in critical spaces was continued by many authors, see, for example, [8, 17, 2, 16]. In particular, in 2001 Koch and Tataru [12] proved the global well-posedness of the Navier-Stokes equations evolving from small initial data in the space $BMO^{-1}$. The space $BMO^{-1}$ has a special role since it is the largest critical space among the spaces listed in (1.4) where such existence results are available.

The importance of considering the three dimensional Navier-Stokes equations in the Besov space $\dot{B}_{\infty,\infty}^{-1,\infty}$ is related to the fact that all critical spaces for the 3D Navier-Stokes equations are embedded in the same function space, $\dot{B}_{\infty,\infty}^{-1,\infty}$. A proof of this embedding could be found in, for example, [3]. It has been a long standing problem to determine if the Navier-Stokes initial value problem is well-posed in the space $\dot{B}_{\infty,\infty}^{-1,\infty}$. The problem is stated as a conjecture in [3] and [14].

An indication that the Navier-Stokes initial value problem might be ill-posed in the largest critical space is given in [15], where Montgomery-Smith proved
a finite time blow-up for solutions of a simplified model for the Navier-Stokes equations in the space $\dot{B}_\infty^{-1,\infty}$. The work [15] suggests that the applications of a fixed point argument that are available up to now are not likely to produce an existence result for the Navier-Stokes equations themselves in the largest critical space, but it does not prove this for the actual Navier-Stokes equations.

In this paper we prove that the actual Navier-Stokes system is ill-posed in $\dot{B}_\infty^{-1,\infty}$ in the sense that there is a so called “norm inflation” (for similar results in the context of NLS see, e.g. [5]). Here by a “norm inflation” we mean that initial data in the Schwartz class $S$ that are arbitrarily small in $\dot{B}_\infty^{-1,\infty}$ can produce solutions arbitrarily large in $\dot{B}_\infty^{-1,\infty}$ after an arbitrarily short time. Such a result implies that the solution map itself is discontinuous in $\dot{B}_\infty^{-1,\infty}$ at the origin. More precisely, our “norm inflation” result can be formulated in the following way:

**Theorem 1.1.** For any $\delta > 0$ there exists a solution $(u, p)$ to the Navier-Stokes equations (1.1) - (1.3) and $0 < t < \delta$ such that $u(0) \in S$

$$\|u(0)\|_{\dot{B}_\infty^{-1,\infty}} \leq \delta,$$

with

$$\|u(t)\|_{\dot{B}_\infty^{-1,\infty}} > \frac{1}{\delta}.$$  

We remark that similar programs of establishing ill-posedness have been successfully carried out in the context of the nonlinear dispersive equations, see for example work of Bourgain [1], Kenig, Ponce, Vega [11], Christ-Colliander-Tao [5], [6].

The main idea of our approach is to choose initial data $u_0$ in $\dot{B}_\infty^{-1,\infty} \cap S$ so that when they evolve in time a certain part of the solution will become arbitrarily large in finite time. More precisely, we write a solution to the Navier-Stokes equations (1.1) - (1.3) as

$$u = e^{t \Delta} u_0 - u_1 + y,$$

where $u_1$ is the first approximation of the solution to the corresponding linear equation and is given by

$$u_1(x, t) = \int_0^t e^{(t-\tau) \Delta} \mathbb{P}(e^{\tau \Delta} u_0 \cdot \nabla) e^{\tau \Delta} u_0 \, d\tau,$$

where $\mathbb{P}$ denotes the projection on divergence free vector fields. We decompose $u_1$ as $u_1 = u_{1,0} + u_{1,1}$, so that the piece $u_{1,0}$ gets arbitrarily large in finite time. On the other hand, we obtain a PDE that $y$ solves, thanks to which we control $e^{i \Delta} u_0 - u_{1,1} + y$ in the space $X_T$ that was introduced in [12] by Koch and Tataru (see Section 2 for a precise definition of $X_T$).
We note that recently Chemin and Gallagher [4] established global existence of solutions for the Navier-Stokes equations evolving from arbitrary large initial data in $\dot{B}^{-1,\infty}_\infty$ under the assumption of a certain nonlinear smallness on the initial data. Since the initial data that we exhibit do not appear to satisfy this nonlinear smallness condition, our work could be understood as a complement of [4].

After we completed the present paper we learned about the recent work of Germain [7] where he proves an instability result for the Navier-Stokes equations in $\dot{B}^{-1,q}_\infty$, for $q > 2$ by showing that the map from the initial data to the solution is not in the class $C^2$. We remark that [7] does not treat a norm inflation phenomenon.

Organization of the paper. In section 2 we introduce the notation that shall be used throughout the paper. Also in Section 2 we recall the result of Koch and Tataru [12]. In section 3 we present a proof of Theorem 1.1.

2. Preliminaries

2.1. Notation. We shall denote by $A \lesssim B$ an estimate of the form $A \leq CB$ with some constant $C$. Throughout the paper, $i$th coordinate ($i = 1, 2, 3$) of a vector $x \in \mathbb{R}^3$ will be denoted by $x^i$.

We recall that the Besov space $\dot{B}^{-1,\infty}_\infty$ is equipped with the norm

$$
\|f(\cdot)\|_{\dot{B}^{-1,\infty}_\infty} = \sup_{t > 0} t^{\frac{2}{3}} \|e^{t\Delta} f(\cdot)\|_{L^\infty}.
$$

2.2. The result of Koch and Tataru. Here we recall the result of Koch and Tataru [12] that establishes the global well-posedness of the Navier-Stokes equations evolving from small initial data in the space $\text{BMO}^{-1}$.

First, let us recall the definition of the space $\text{BMO}^{-1}$ as given in [12]:

$$
\|f(\cdot)\|_{\text{BMO}^{-1}} = \sup_{x_0, R} \left( \frac{1}{|B(x_0, \sqrt{R})|} \int_0^R \int_{B(x_0, \sqrt{R})} |e^{t\Delta} f(y)|^2 \, dy \, dt \right)^{\frac{1}{2}}.
$$

(2.1)

In [12] Koch and Tataru proved the following existence theorem:
Theorem 2.1. The Navier-Stokes equations (1.1) - (1.3) have a unique global solution in $X$

$$\|u(\cdot, \cdot)\|_X = \sup_t t^{\frac{1}{2}} \|u(\cdot, t)\|_{L^\infty}$$

$$+ \sup_{x_0, R} \left( \frac{1}{|B(x_0, \sqrt{R})|} \int_0^R \int_{B(x_0, \sqrt{R})} |u(y, t)|^2 \, dy \, dt \right)^{\frac{1}{2}},$$

for all initial data $u_0$ with $\nabla \cdot u_0 = 0$ which are small in $BMO^1$.

Let $T \in (0, \infty]$. We denote by $X_T$ the space equipped with the norm

$$\|u(\cdot, \cdot)\|_{X_T} = \sup_{0 < t < T} t^{\frac{1}{2}} \|u(\cdot, t)\|_{L^\infty}$$

$$+ \sup_{x_0} \sup_{0 < R < T} \left( \frac{1}{|B(x_0, \sqrt{R})|} \int_0^R \int_{B(x_0, \sqrt{R})} |u(y, t)|^2 \, dy \, dt \right)^{\frac{1}{2}}.$$

Now let $P$ denote the projection on divergence free vector fields. As shown in [12], see also [13], the bilinear operator

$$B(u, v) = \int_0^t e^{(t-\tau)\Delta} P((u \cdot \nabla)v) \, d\tau,$$  \hspace{1cm} (2.2)

maps $X_T \times X_T$ into $X_T$. More precisely,

$$\|B(u, v)\|_{X_T} \lesssim \|u\|_{X_T} \|v\|_{X_T}. \hspace{1cm} (2.3)$$

3. Proof of Theorem 1.1

We rewrite the Navier-Stokes equations (1.1) in the following way:

$$u = e^{t\Delta} u_0 - u_1 + y,$$  \hspace{1cm} (3.1)

where

$$u_1(x, t) = B(e^{t\Delta} u_0(x), e^{t\Delta} u_0(x)), \hspace{1cm} (3.2)$$

and $y$ satisfies the following equation:

$$\partial_t y - \Delta y + G_1 + G_2 + G_3 = 0,$$  \hspace{1cm} (3.3)

where

$$G_1 = P[(e^{t\Delta} u_0 \cdot \nabla)y + (u_1 \cdot \nabla)y + (y \cdot \nabla)e^{t\Delta} u_0 + (y \cdot \nabla)u_1],$$

$$G_2 = P[(y \cdot \nabla)y],$$

$$G_3 = P[(e^{t\Delta} u_0 \cdot \nabla)u_1 + (u_1 \cdot \nabla)e^{t\Delta} u_0 + (u_1 \cdot \nabla)u_1].$$
We shall choose initial data \( u_0 \) in such a way that when they evolve in time, the part of the solution \( u_1 \) will become arbitrarily large in \( \dot{B}_{-1,\infty}^\infty \) at certain time \( T \), while we will be able to control the behavior of \( y \) in the space \( X_T \).

### 3.1. Choice of initial data.

Fix small numbers \( T > 0, \delta > 0 \) and a large number \( Q > 0 \) (eventually \( T \to 0, \delta \to 0 \) and \( Q \to \infty \)). Let \( \eta \in \mathbb{S}^2 \). Let \( r = r(Q) \) be a large integer (to be specified). We choose the initial data as follows:

\[
 u_0 = \frac{Q}{\sqrt{r}} \sum_{s=1}^{r} |k_s|[v_s \cos(k_s \cdot x) + v'_s \cos(k'_s \cdot x)],
\]

(3.4)

where

1. The vectors \( k_s \in \mathbb{R}^3 \) are parallel to a given vector \( k_0 \in \mathbb{R}^3 \) and \( k'_s \in \mathbb{R}^3 \) is defined by

\[
 k_s - k'_s = \eta.
\]

Furthermore, we take \( |k_0| \) large (depending on \( Q \)) and \( |k_s| \) (1 ≤ \( s \) ≤ \( r \)) very lacunary. For example,

\[
 |k_s| = 2^s|k_0| |k_{s-1}|, \quad s = 2, 3, ..., r.
\]

2. \( v_s, v'_s \in \mathbb{S}^2 \) such that

(a)

\[
 k_s \cdot v_s = 0 = k'_s \cdot v'_s.
\]

Note that (3.6) implies that \( \text{div} \ u_0 = 0 \).

(b) By (3.5) we may ensure that

\[
 v_s \approx v'_s \approx v \in \mathbb{S}^2.
\]

We require that

\[
 \eta \cdot v_s = \eta \cdot v'_s = \eta \cdot v = \frac{1}{2}.
\]

(3.7)

It is obvious from (3.4) that

\[
 \|u_0\|_{\dot{B}_{-1,\infty}^\infty} \sim \frac{Q}{\sqrt{r}} < \delta
\]

for appropriate \( r \).
3.2. Analysis of $u_1$. Now we analyze $u_1$ with a goal to split it into two pieces $u_{1,0}$ and $u_{1,1}$ such that the upper bound on $u_{1,0}$ in the Besov space $\dot{B}^{-1,\infty}_{\infty}$ is roughly $Q^2$ on a certain time interval.

For the initial data $u_0$ given by (3.4), $e^{\tau\Delta}u_0$ can be written as follows

$$e^{\tau\Delta}u_0 = \frac{Q}{\sqrt{r}} \sum_{s=1}^{r} |k_s| \left( v_s \cos(k_s \cdot x)e^{-|k_s|^2 r} + v'_s \cos(k'_s \cdot x)e^{-|k'_s|^2 r} \right).$$

(3.8)

Hence we can calculate $(e^{\tau\Delta}u_0 \cdot \nabla)e^{\tau\Delta}u_0$ via its coordinates as follows:

$$( (e^{\tau\Delta}u_0 \cdot \nabla) e^{\tau\Delta}u_0 )^i = \sum_j \partial_j [(e^{\tau\Delta}u_0)^i (e^{\tau\Delta}u_0)^j]$$

$$\sim N_1^i + N_2^i + N_3^i,$$

where

$$N_1^i = \frac{Q^2}{r} \sum_{s=1}^{r} |k_s|^2 e^{-2|k_s|^2 r} \sin(\eta \cdot x) [(\eta \cdot v_s) v_s^i + (\eta \cdot v_s)(v'_s)^i]$$

$$N_2^i = \frac{Q^2}{r} \sum_{s=1}^{r} |k_s|^2 e^{-(|k_s|^2 + |k'_s|^2) r} \sin((k_s + k'_s) \cdot x) \times$$

$$\times \left[ ((k_s + k'_s) \cdot v_s) v_s^i + ((k_s + k'_s) \cdot v_s)(v'_s)^i \right]$$

$$N_3^i = \frac{Q^2}{r} \sum_{s \neq s'} |k_s| |k_{s'}| e^{-(|k_s|^2 + |k_{s'}|^2) r} \sin((k_s \pm k_{s'}) \cdot x) \times$$

$$\times \left[ ((k_s \pm k_{s'}) \cdot v_{s'}) v_{s'}^i + ((k_s \pm k_{s'}) \cdot v_s v_{s'}^i \right] + \text{similar terms}. $$

We consider contributions to $u_1$ coming from each of three terms $N_1$, $N_2$, $N_3$. Contributions coming from $N_1$ can be estimated by integrating in time and using (3.7) as follows

$$\int_0^t e^{(t-\tau)\Delta} N_1 \ d\tau$$

$$\sim \frac{Q^2}{r} \sum_{s=1}^{r} |k_s|^2 \left[ \int_0^t e^{-(t-\tau)|\eta|^2 - 2|k_s|^2 r} d\tau \right] \sin(\eta \cdot x) \left[ (\eta \cdot v_s') v_s + (\eta \cdot v_s) v'_s \right]$$

$$\sim Q^2 \sin(\eta \cdot x) v,$$

for

$$\frac{1}{|k_1|^2} \ll T \ll 1.$$  

(3.10)

Therefore, recalling (3.7)

$$\| \int_0^t e^{(t-\tau)\Delta} \mathfrak{P}(N_1) \ dt \|_{B^{-1,\infty}_{\infty}} \sim Q^2.$$

(3.11)
Also
\[
\| \int_0^t e^{(t-\tau)\Delta \mathcal{P}(N_1)} \ dt \|_{X_T} \lesssim \sqrt{T} Q^2. \tag{3.12}
\]

Now consider contributions to \( u_1 \) coming from \( N_3 \).

\[
\left| \int_0^t e^{(t-\tau)\Delta N_3} \ dt \right| \leq \frac{Q^2}{r} \sum_{s=1}^r \sum_{s' < s} |k_s| |k_s'| \left| \int_0^t e^{-(t-\tau)|k_s\pm k_s'|^2-(|k_s|^2+|k_s'|^2)t} \ d\tau \right| \times
\]
\[
\times |\sin((k_s \pm k_s') \cdot x)| \ O(|k_s|)
\]
\[
\lesssim \frac{Q^2}{r} \sum_{s=1}^r \sum_{s' < s} |k_s| |k_s'| \left| e^{-(|k_s|^2+|k_s'|^2)t} - e^{-|k_s\pm k_s'|^2t} \right| |k_s \pm k_s'|^2 \ O(|k_s|)
\]
\[
\lesssim \frac{Q^2}{r} \sum_{s=1}^r \sum_{s' < s} |k_s| |k_s'| e^{-\frac{1}{2}|k_s'|^2t} \ t \ O(|k_s|) \tag{3.13}
\]
\[
\lesssim \frac{Q^2}{r} \sum_{s=1}^r |k_{s-1}| e^{-\frac{1}{2}|k_{s-1}|^2t}, \tag{3.14}
\]

where to obtain (3.13) we use the boundedness of the function \( g(t) = \frac{1-e^{-\lambda t}}{\lambda t} \), with \( \lambda > 0 \), while to obtain (3.14) we use the boundedness of the function \( h(t) = \mu t e^{-ut} \), with \( \mu > 0 \) and we replace \( e^{-k_s^2} \) by \( e^{-\frac{1}{2}c^2} \) for some \( l \). We also use the lacunarity of the sequence \( |k_s| \).

Thus (3.14) implies that
\[
\| \int_0^t e^{(t-\tau)\Delta \mathcal{P}(N_3)} \ dt \|_{X_T} \leq \frac{Q^2}{r} \sum_{s=1}^r \frac{|k_{s-1}|}{|k_s|} + \frac{Q^2}{r} \sup_{t<T} \left\{ \int_0^t \left[ \sum_{s=1}^r |k_{s-1}| e^{-\frac{1}{2}|k_s|^2t} \right]^2 \ d\tau \right\}^{\frac{1}{2}}
\]
\[
\lesssim \frac{Q^2}{r} \sum_{s=1}^r \frac{|k_{s-1}|}{|k_s|}
\]
\[
< \frac{Q^2}{r}, \tag{3.15}
\]
agian by lacunarity of \( |k_s| \).

Next we estimate the contribution coming from \( N_2 \). Clearly, recalling (3.5)
\[
\int_0^t e^{(t-\tau)\Delta N_2} \ dt \sim \frac{Q^2}{r} \left\{ \sum_{s=1}^r O(|k_s| e^{-|k_s|^2t}) \sin(k_s + k_s') \cdot x \right\}.
\]
Therefore
\[
\| \int_0^t e^{(t-\tau)\Delta} \mathcal{P}(N_2) \, dt \|_{X_T} \lesssim \frac{Q^2}{r} \sup_{t>0} \left| \sum_{s=1}^r \frac{1}{t^\frac{1}{2}} |k_s| e^{-|k_s|^2 t} \right|
+ \frac{Q^2}{r} \sup_{R>0} \left\{ \int_0^R \left[ \sum_{|k_s| > \frac{1}{\sqrt{r}}} |k_s|^2 e^{-|k_s|^2 t} \right] \, dt + \int_0^R \left( \sum_{|k_s| \leq \frac{1}{\sqrt{r}}} |k_s| \right)^2 \, dt \right\}^\frac{1}{2}
\lesssim \frac{Q^2}{r} + \frac{Q^2}{r} (r + 1)^\frac{1}{2}
\lesssim \frac{Q^2}{\sqrt{r}},
\] (3.16)

using the fact that \( \sqrt{t} \sum_s |k_s|^2 e^{-|k_s|^2 t} \lesssim 1 \) and making the appropriate splitting to bound the second term in \( \| \cdot \|_{X_T} \).

Hence we can decompose \( u_1 \) as follows
\[
u_1 = u_{1,0} + u_{1,1},
\]
where
\[
\| u_{1,0} \|_{B^{-1,\infty}} \sim Q^2 \text{ for } \frac{1}{|k_1|^2} \ll t \ll 1,
\]
\[
\| u_{1,0} \|_{X_T} \lesssim \sqrt{T} Q^2,
\]
\[
\| u_{1,1} \|_{X_T} \lesssim \frac{Q^2}{\sqrt{r}},
\]
(3.17)

3.3. **Analysis of \( y \).** Now we analyze the remaining part of the solution, which we denoted by \( y \). The main idea is to control \( y \) using the space of Koch and Tataru \( X_T \).

Consider time-intervals
\[
0 < T_1 < T_2 < \ldots < T_\beta, \quad \beta = Q^3
\]
with
\[
T_\alpha^{-1} = |k_{r_\alpha}|^2
\]
(3.18)
\[
r_\alpha = r - \alpha Q^{-3} r, \quad \alpha = 1, 2, \ldots
\]
(3.19)

In particular, \( r_\beta = 0 \) and \( T_\beta^{-1} = |k_0|^2 \).
For $t \geq T_{\alpha}$ the equation for $y$ can be written in the integral form as

$$y(t) = e^{(t-T_{\alpha})\Delta} y(T_{\alpha}) - \int_{T_{\alpha}}^{t} e^{(t-\tau)\Delta} \left[ G_1 + G_2 + G_3 \right](\tau) \, d\tau,$$

(3.20)

where $G_i, \ i = 1, 2, 3$ are given by (3.3).

Also

$$y(T_{\alpha}) = \int_{0}^{T_{\alpha}} e^{(T_{\alpha}-\tau)\Delta} \left[ G_1 + G_2 + G_3 \right](\tau) \, d\tau.$$

Therefore

$$e^{(t-T_{\alpha})\Delta} y(T_{\alpha}) = \int_{0}^{T_{\alpha}} e^{(t-\tau)\Delta} \left[ G_1 + G_2 + G_3 \right](\tau) \, d\tau$$

$$= \int_{0}^{t} e^{(t-\tau)\Delta} \left[ G_1 + G_2 + G_3 \right](\tau) \chi_{[0,T_{\alpha}]}(\tau) \, d\tau,$$

(3.21)

where $\chi_{[0,T_{\alpha}]}$ is a characteristic function of the interval $[0, T_{\alpha}]$.

Now we substitute (3.21) in (3.20) to obtain

$$\|y\|_{X_{T_{\alpha}+1}} \leq I + II,$$

(3.22)

where

$$I = \| \int_{0}^{t} e^{(t-\tau)\Delta} \left[ G_1 + G_2 + G_3 \right](\tau) \chi_{[0,T_{\alpha}]}(\tau) \, d\tau \|_{X_{T_{\alpha}+1}}$$

(3.23)

and

$$II = \| \int_{0}^{t} e^{(t-\tau)\Delta} \left[ G_1 + G_2 + G_3 \right](\tau) \chi_{[T_{\alpha},T_{\alpha}+1]}(\tau) \, d\tau \|_{X_{T_{\alpha}+1}}.$$

(3.24)

Next we use the bilinear estimate (2.3) on the terms in $G_1$, $G_2$ and $G_3$ to obtain an upper bound on $I$ and $II$ respectively. Before we obtain an upper bound on $I$, we estimate $\|e^{t\Delta}u_0\|_{X_{T_{\alpha}}}$. From (3.8) we have

$$e^{t\Delta}u_0 \approx \frac{Q}{\sqrt{r}} \sum_{s \leq r} |k_s| v_s \cos(k_s \cdot x) e^{-|k_s|^2t}.$$
We estimate \( \|e^{t\Delta}u_0\|_{X^T_{\alpha}} \) as follows

\[
\|e^{t\Delta}u_0\|_{X^T_{\alpha}} \leq \frac{Q}{\sqrt{T}} \sup_{t < T_{\alpha}} \sqrt{t} \sum_{s \leq r} |k_s| e^{-k_s^2t} \\
+ \frac{Q}{\sqrt{T}} \sup_{x_0, \ 0 < t < T_{\alpha}} \left( t^{-3/2} \int_0^t \int_{|x-x_0| < \sqrt{r}} \left| \sum_{s \leq r} |k_s| v_s \cos(k_s \cdot x) e^{-k_s^2\tau} \right|^2 d\tau \right)^{1/2} \\
\leq \frac{Q}{\sqrt{T}} + \frac{Q}{\sqrt{T}} (r + 1)^{1/2} \\
\leq Q,
\]

(3.26) similarly to (3.16). Hence

\[
\|e^{t\Delta}u_0\|_{X^T_{\alpha}} \leq Q.
\]

(3.27)

Now we are ready to estimate \( I \) using (3.3) and the bilinear estimate (2.3):

\[
I \leq \left( \|e^{t\Delta}u_0\|_{X^T_{\alpha}} + \|u_1\|_{X^T_{\alpha}} + \|y\|_{X^T_{\alpha}} \right) \|y\|_{X^T_{\alpha}} \\
+ \left( \|e^{t\Delta}u_0\|_{X^T_{\alpha}} + \|u_1\|_{X^T_{\alpha}} \right) \|u_1\|_{X^T_{\alpha}} \\
\leq \left( Q + Q^2T_{\alpha}^{1/2} + \frac{Q^2}{\sqrt{T}} \right) \|y\|_{X^T_{\alpha}} \\
+ \left( Q + Q^2T_{\alpha}^{1/2} + \frac{Q^2}{\sqrt{T}} \right) \left( Q^2T_{\alpha}^{1/2} + \frac{Q^2}{\sqrt{T}} \right),
\]

(3.28)

where to obtain (3.28) we used (3.27) and (3.17).

In order to obtain an upper bound on \( II \), first, we estimate \( \|e^{t\Delta}u_0\chi_{[T_{\alpha}, T_{\alpha}+1]}(t)\|_{X^T_{\alpha+1}} \).

More precisely, from (3.8) we have

\[
(e^{t\Delta}u_0) \chi_{[T_{\alpha}, T_{\alpha}+1]}(t) \approx L_1 + L_2,
\]

(3.29)

where

\[
L_1 = \frac{Q}{\sqrt{T}} \sum_{s < r_{\alpha+1}} |k_s| v_s \cos(k_s \cdot x) \chi_{[T_{\alpha}, T_{\alpha}+1]}(t)
\]

and

\[
L_2 = \frac{Q}{\sqrt{T}} \sum_{s = r_{\alpha+1}}^{r_{\alpha}} |k_s| v_s \cos(k_s \cdot x) e^{-|k_s|^2t} \chi_{[T_{\alpha}, T_{\alpha}+1]}(t).
\]
We estimate \( L_1 \) keeping in mind that, thanks to (3.18), \( T_{\alpha+1} = |k_{r_{\alpha+1}}|^{-2} \):

\[
\|L_1\|_{X^{T_{\alpha+1}+1}} \leq \frac{Q}{\sqrt{r}} T_{\alpha+1}^{1/2} |k_{r_{\alpha+1}}|^{-1} \\
+ \frac{Q}{\sqrt{r}} \sup_{x,t} \left( t^{-3/2} \int_0^t \int_{|x-x_0|<\sqrt{r}} \left| k_s \chi_{[T_{\alpha},T_{\alpha+1}]}(\tau) \right|^2 \, dx \, d\tau \right)^{1/2} \\
\leq \frac{Q}{\sqrt{r}} |k_{r_{\alpha+1}}|^{-1} + \frac{Q}{\sqrt{r}} (T_{\alpha+1} |k_{r_{\alpha+1}}|^2)^{1/2} \\
< \frac{Q}{\sqrt{r}}. \tag{3.30}
\]

We estimate \( L_2 \) as follows

\[
\|L_2\|_{X^{T_{\alpha+1}+1}} \leq \frac{Q}{\sqrt{r}} \sup_t \sqrt{t} \sum_{s=r_{\alpha+1}}^{r_{\alpha}} |k_s| e^{-k_s^2 t} \\
+ \frac{Q}{\sqrt{r}} \sup_{x,t} \left( t^{-3/2} \int_0^t \int_{|x-x_0|<\sqrt{r}} \left| k_s v_s \cos(k_s \cdot x) e^{-k_s^2 \tau} \chi_{[T_{\alpha},T_{\alpha+1}]}(\tau) \right|^2 \, dx \, d\tau \right)^{1/2} \\
\leq \frac{Q}{\sqrt{r}} + \frac{Q}{\sqrt{r}} (r_{\alpha} - r_{\alpha+1})^{1/2} \\
= \frac{Q}{\sqrt{r}} + \frac{Q}{\sqrt{r}} (Q^{-3} r)^{1/2} \tag{3.31} \\
\leq Q^{-1/2}, \tag{3.32}
\]

where to obtain (3.31) we used (3.19). Hence we combine (3.29), (3.30) and (3.32) to conclude

\[
\|(e^{t\Delta} u_0) \chi_{[T_{\alpha},T_{\alpha+1}]}(t)\|_{X^{T_{\alpha+1}}} \lesssim Q^{-1/2}. \tag{3.33}
\]

Also we recall that (3.17) implies

\[
\|u_1 \chi_{[T_{\alpha},T_{\alpha+1}]}(t)\|_{X^{T_{\alpha+1}}} \lesssim Q^2 T_{\alpha+1}^{1/2} + \frac{Q^2}{\sqrt{r}}. \tag{3.34}
\]
Now we are ready to find an upper bound on $II$ by employing the bilinear estimate (2.3):

$$
II \lesssim \left( \left\| (e^{t \Delta} u_0) \chi_{[T_\alpha, T_\alpha+1]}(t) \right\| x_{T_\alpha+1} + \left\| u_1 \chi_{[T_\alpha, T_\alpha+1]}(t) \right\| x_{T_\alpha+1} + \left\| y \right\| x_{T_\alpha+1} \right) \left\| y \right\| x_{T_\alpha+1} \\
+ \left( \left\| (e^{t \Delta} u_0) \chi_{[T_\alpha, T_\alpha+1]}(t) \right\| x_{T_\alpha+1} + \left\| u_1 \chi_{[T_\alpha, T_\alpha+1]}(t) \right\| x_{T_\alpha+1} \right) \left\| u_1 \chi_{[T_\alpha, T_\alpha+1]}(t) \right\| x_{T_\alpha+1} \\
\leq \left( Q^{-1/2} + Q^2 T_{\alpha+1}^{1/2} + \frac{Q^2}{\sqrt{T}} + \left\| y \right\| x_{T_\alpha+1} \right) \left\| y \right\| x_{T_\alpha+1} \\
+ \left( Q^{-1/2} + Q^2 T_{\alpha+1}^{1/2} + \frac{Q^2}{\sqrt{T}} \right) \left( Q^2 T_{\alpha+1}^{1/2} + \frac{Q^2}{\sqrt{T}} \right),
$$

where to obtain (3.35) we used (3.33) and (3.34).

Having in mind that $T_\alpha < T_{\alpha+1} < T$ and that $T$ will be chosen to satisfy (3.45), we combine (3.22), (3.28) and (3.35) to obtain

$$
\left\| y \right\| x_{T_{\alpha+1}} \lesssim Q^{-1/2} \left\| y \right\| x_{T_{\alpha+1}} + \left\| y \right\| x_{T_{\alpha+1}}^2 + Q^3 \left( \frac{1}{\sqrt{T}} + T_{\alpha+1}^{1/2} \right) + Q \left\| y \right\| x_{T_\alpha}.
$$

Thus

$$
\left\| y \right\| x_{T_{\alpha+1}} \lesssim Q^3 \left( \frac{1}{\sqrt{T}} + T_{\alpha+1}^{1/2} \right) + Q \left\| y \right\| x_{T_\alpha}.
$$

(3.36)

Iterating (3.36) gives

$$
\left\| y \right\| x_{T_\beta} \lesssim Q^{\beta+3} \left( \frac{1}{\sqrt{T}} + T_{\beta}^{1/2} \right).
$$

(3.37)

Now we take $T > T_\beta$ and write (3.20) and (3.21) with $\alpha = \beta$. Thus

$$
\left\| y \right\| x_T \leq I_\beta + II_\beta
$$

(3.38)

where

$$
I_\beta = \left\| \int_0^t e^{(t-\tau)\Delta} \chi_{[0,T_\beta]}(\tau) \chi_{[T_\beta,T]}(\tau) d\tau \right\| x_T
$$

(3.39)

and

$$
II_\beta = \left\| \int_0^t e^{(t-\tau)\Delta} \chi_{[0,T_\beta]}(\tau) \chi_{[T_\beta,T]}(\tau) d\tau \right\| x_T.
$$

(3.40)
We obtain an upper bound on $I_{\beta}$ by using (3.3) and the bilinear estimate (2.3):
\[
I_{\beta} \lesssim \left( \| (e^{t\Delta} u_0) \|_{X_{T\beta}} + \| u_1 \|_{X_{T\beta}} + \| y \|_{X_{T\beta}} \right) \| y \|_{X_{T\beta}}
+ \left( \| (e^{t\Delta} u_0) \|_{X_{T\beta}} + \| u_1 \|_{X_{T\beta}} \right) \| u_1 \|_{X_{T\beta}}
\leq \left( Q + Q^2 T_{\beta}^{1/2} + \frac{Q^2}{\sqrt{T}} + Q^3 \left( \frac{1}{r} + T_{\beta} \right)^{1/2} \right) Q^{2} \left( \frac{1}{r} + T_{\beta} \right)^{1/2}
\]
\[
\quad + \left( Q + Q^2 T_{\beta}^{1/2} + \frac{Q^2}{\sqrt{T}} \right) \left( Q^2 T_{\beta}^{1/2} + \frac{Q^2}{\sqrt{T}} \right).
\]
(3.41)

We rely here on (3.27), (3.17) and (3.37).

Recalling that $T_{\beta} = |k_0|^{-2}$ and choosing $r$ and $|k_0|$ large enough, it follows from (3.41) and (3.42) that
\[
I_{\beta} \lesssim r^{-1/3} + |k_0|^{-1/2}.
\]
(3.43)

Also
\[
II_{\beta} \lesssim \left( \| (e^{t\Delta} u_0) \chi_{[T, T_{\beta}]}(t) \|_{X_T} + \| u_1 \|_{X_T} + \| y \|_{X_T} \right) \| y \|_{X_T}
+ \left( \| (e^{t\Delta} u_0) \chi_{[T, T_{\beta}]}(t) \|_{X_T} + \| u_1 \|_{X_T} \right) \| u_1 \|_{X_T}
\leq \left( |k_1| e^{-\frac{|k_1|^2}{|u_1|^2}} + Q^2 T_{\beta}^{1/2} + \frac{Q^2}{\sqrt{T}} + \| y \|_{X_T} \right) \| y \|_{X_T}
+ \left( |k_1| e^{-\frac{|k_1|^2}{|u_1|^2}} + Q^2 T_{\beta}^{1/2} + \frac{Q^2}{\sqrt{T}} \right)^2
\]
where to obtain (3.44) we used (3.8) and (3.17).

Let us also assume that
\[
T < Q^{-8}.
\]
(3.45)

Since $|k_1| > |k_0|^2$, (3.44) implies
\[
II_{\beta} < (o(1) + \| y \|_{X_T}) \| y \|_{X_T} + 2Q^4 T.
\]
(3.46)

Therefore, from (3.43) and (3.46)
\[
\| y \|_{X_T} < 3Q^4 T
\]
(3.47)

implying
\[
\| y \|_{L^\infty} \leq T^{-\frac{1}{4}} \| y \|_{X_T} < 3Q^4 T^{\frac{1}{4}}.
\]
(3.48)

Now we combine (3.1), (3.17) and (3.48) to conclude that
\[
\| u(T) - e^{T\Delta} u_0 \|_{B_{\infty, 1}^{-1, \infty}} \geq Q^2 - \| u_{1,1} \|_{L^\infty} - \| y \|_{L^\infty} > Q^2 \left( 1 - \frac{1}{\sqrt{T}} \right) - 3Q^2 T^{\frac{1}{4}}
\]

and
\[ \| u(T) \|_{\dot{B}^{-1,\infty} \cap L^2} > \frac{1}{2} Q^2. \]  

(3.49)

Consequently we proved that for all \( \delta > 0 \)
\[ \sup_{\| u(0) \|_{\dot{B}^{-1,\infty} \cap L^2} \leq \delta} \sup_{0 < t < \delta} \| u(t) \|_{\dot{B}^{-1,\infty} \cap L^2} = \infty. \]  

(3.50)

REFERENCES

[1] J. Bourgain, Periodic Korteweg De Vries Equations with Measures as Initial Data, Sel. Math. New. Ser. 3 (1993), 115–159.
[2] M. Cannone, A generalization of a theorem by Kato on Navier-Stokes equations, Rev. Mat. Iberoam. 13, No. 3 (1997), 515–541.
[3] M. Cannone, Harmonic analysis tools for solving the incompressible Navier-Stokes equations, Handbook of mathematical fluid dynamics. Vol. III, North-Holland, Amsterdam (2004), 161–244.
[4] J.-Y. Chemin and I. Gallagher, Wellposedness and stability results for the Navier-Stokes equations in \( \mathbb{R}^3 \), To appear in Annales de l’Institut Henri Poincaré, Analyse Non Linéaire.
[5] M. Christ, J. Colliander and T. Tao, Ill-posedness for nonlinear Schrödinger and wave equations, To appear in Annales Henri Poincaré.
[6] M. Christ, J. Colliander and T. Tao, Instability of the periodic nonlinear Schrodinger equation, Preprint (2003).
[7] P. Germain, The second iterate for the Navier-Stokes equation, Preprint (2008), arXiv:0806.4525.
[8] Y. Giga and T. Miyakawa, Navier-Stokes flow in \( \mathbb{R}^3 \) with measures as initial vorticity and Morrey spaces, Comm. Partial Differential Equations 14, No. 5 (1989), 577–618.
[9] T. Kato, Strong \( L^p \)-solutions of the Navier-Stokes equations in \( \mathbb{R}^m \) with applications to weak solutions, Math. Zeit. 187 (1984), 471–480.
[10] T. Kato and H. Fujita, On the non-stationary Navier-Stokes system, Rend. Sem. Mat. Univ. Padova 32 (1962), 243–260.
[11] C.E. Kenig, G. Ponce, and L. Vega, On the ill-posedness of some canonical dispersive equations, Duke Math J. 106 (2001), 617–633.
[12] H. Koch and D. Tataru, Well Posedness for the Navier-Stokes equations, Adv. Math. 157 (2001), 22–35.
[13] P.G. Lemarié-Rieusset, Une remarque sur l’analyticité des solutions milds des équations de Navier-Stokes dans \( \mathbb{R}^3 \), C. R. Acad. Sci. Paris 330 Série I (2000), 183–186.
[14] Y. Meyer, Wavelets, paraproducts and Navier-Stokes equations, Current Developments in Mathematics 1996, International Press Cambridge MA (1999), 105–212.
[15] S. Montgomery-Smith, Finite time blow up for a Navier-Stokes like equation, Proc. Amer. Math. Soc. 129, No. 10, (2001), 3025–3029.
[16] F. Planchon, Global strong solutions in Sobolev or Lebesgue spaces to the incompressible Navier-Stokes equations in \( \mathbb{R}^3 \), Ann. Inst. Henri Poincare, Anal. Non Lineaire 163, no. 3 (1996), 319–336.
[17] M. Taylor, Analysis on Morrey spaces and applications to Navier-Stokes equation, Comm. Partial Differential Equations 17 (1992), 1407–1456.
