TOPOLOGICAL COMPUTATION OF THE FIRST MILNOR FIBER COHOMOLOGY OF HYPERPLANE ARRANGEMENTS

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Abstract. We study a topological method to calculate the first Milnor fiber cohomology of a defining polynomial of a reduced projective hyperplane arrangement $X$ of degree $d$. We show the vanishing of a monodromy eigenspace of the first Milnor fiber cohomology with eigenvalue of order $m \geq 2$ if $X \setminus (X^{(m)} \cup X^{(3)})$ or more generally $X \setminus (X^{(m)} \cup X^{(3)} \cup X_d)$ is connected. Here $X^{(3)} := \bigcup_{i<j<k} X_i \cap X_j \cap X_k$ with $X_i$ the irreducible components of $X$, and $X^{(m)}$ is the set of points of $X$ with multiplicity divisible by $m$. This is reduced to the case of a line arrangement in $\mathbb{P}^2$ by Artin’s vanishing theorem, and we use a projection from $\mathbb{P}^2$ to $\mathbb{P}^1$ with center a sufficiently general point of $X_d$.

Introduction

Let $f$ be a reduced polynomial defining a projective hyperplane arrangement $X \subset \mathbb{P}^{n-1}$ of degree $d$. For a calculation of monodromy eigenspaces of its first Milnor fiber cohomology $H^i(F_f, \mathbb{C})_\lambda$ with eigenvalue $\lambda$ of order $m > 1$, we may assume $d/m \in \mathbb{Z}$, since $H^i(F_f, \mathbb{C})_\lambda = 0$ otherwise, where $F_f$ denotes the Milnor fiber of $f$. These eigenspaces are calculated by the corresponding Aomoto complex if some hypothesis from [ESV] is satisfied (although the condition is rather restrictive, and is often unsatisfied), see [BDS, BSY, Fa, Sa2, Sa3], etc. So we consider a topological way to calculate it as in Section 1 below, and get the following.

Theorem 1. We have $H^1(F_f, \mathbb{C})_\lambda = 0$ with $m = \text{ord} \lambda \geq 2$ if $X \setminus (X^{(m)} \cup X^{(3)})$ or more generally $X \setminus (X^{(m)} \cup X^{(3)} \cup X_d)$ is connected. Here $X^{(m)}$ is the set of $P \in X$ with $\text{mult}_P X$ divisible by $m$, and $X^{(3)} := \bigcup_{i<j<k} X_i \cap X_j \cap X_k$ with $X_i$ ($i \in [1, d]$) the irreducible components of $X$.

This can be reduced to the case $n = 3$ by Artin’s vanishing theorem [BBD] together with an iterated general hyperplane cut, and we use a projection $\mathbb{P}^2 \to \mathbb{P}^1$ with center a sufficiently general point of $X_d$, see (2.1) and Section 1 below. Theorem 1 is more or less well-known (see for instance [BDS]) in the case $H^1(F_f, \mathbb{C})_\lambda$ is calculable by the method in [ESV] mentioned above, see also [Ba, PS], etc. Note that $X \setminus (X^{(m)} \cup X^{(3)} \cup X_d)$ is connected if so is $X \setminus (X^{(m)} \cup X^{(3)})$, changing the order of the $X_i$ if necessary. This can be verified by using the dual $(m)$-graph of $X$. (Its vertices correspond to the irreducible components of $X$, and two vertices are connected by an edge if and only if the multiplicity of $X$ at a general point of the intersection of the corresponding two hyperplanes is not divisible by $m$.)

It does not seem easy to find an example with hypothesis of Theorem 1 unsatisfied for $m \geq 4$ (with $d/m \in \mathbb{Z}$) except the generalized Hessian arrangement (where the dual $(4)$-graph has 2 or 4 connected components, see [BDS, Di2, Yu], and (2.2) below). Theorem 1 implies that $H^1(F_f, \mathbb{C})_\lambda = 0$ with $m = \text{ord} \lambda \geq 4$ for any cases as far as calculated except the generalized Hessian. Note that the vanishing of $H^1(F_f, \mathbb{C})_\lambda$ for ord $\lambda \geq 5$ is conjectured in [PS, Conjecture 1.9], see also [Sa3, Problem 2]. (It seems possible to extend Theorem 1 to the case of generalized Hessian except the classical one.)

In Section 1 we study a topological way to describe the Milnor fiber cohomology of line arrangements. In Section 2 we prove the main theorem.
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1. Milnor fiber cohomology of line arrangements

In this section we study a topological way to describe the Milnor fiber cohomology of line arrangements.

1.1. Calculation of the Milnor fiber cohomology. Let $L \subset Y := \mathbb{P}^2$ be a reduced line arrangement with $f$ a defining polynomial of degree $d$. It is well-known that there are local systems $\mathcal{L}^{(k)}$ of rank 1 on $U := Y \setminus L$ ($k \in [0, d-1]$) such that

\[(1.1.1) \quad H^j(F_j, \mathbb{C}) \cong H^j(U, \mathcal{L}^{(k)}) \quad (\lambda = \exp(-2\pi \sqrt{-1}k/d)),\]

and the monodromy around any irreducible component of $L$ is given by multiplication by $\lambda^{-1} = \exp(2\pi \sqrt{-1}k/d)$, see [Di1], [CS], [BS], etc. In particular, we have $\mathcal{L}^{(0)} = \mathbb{C}_U$, and hence $H^j(F_j, \mathbb{C}) = H^j(U, \mathbb{C})$.

Let $\pi : \mathcal{Y} \to Y := \mathbb{P}^2$ be the blow-up along a sufficiently general point $P \in L_d$ with $j_\mathcal{Y} : U \hookrightarrow \mathcal{Y}$ a natural inclusion. Here the $L_i$ ($i \in [1, d]$) are the irreducible components of $L$. Set

$$\mathcal{F} := R(j_\mathcal{Y})_*\mathcal{L}^{(k')}[2] \quad \text{with} \quad k' = d/m \quad \text{or} \quad d - d/m. $$

(Here we use the irreducibility of cyclotomic polynomials implicitly, see also [Sa3].) This is a direct factor of the underlying $\mathbb{C}$-complex of a mixed Hodge module, and has the weight filtration $W$, see [Sa1], [BS].

**Lemma 1.1.** We have the isomorphisms

\[(1.1.2) \quad \text{Gr}_2^W \mathcal{F} = IC_{\mathcal{Y}}\mathcal{L}^{(k')}, \quad \text{Gr}_i^W \mathcal{F} = 0 \quad (i \neq 2, 3).\]

Moreover $\dim_{\mathbb{C}} H^i(\text{Gr}_i^W \mathcal{F}) = km - 2$ if $\pi(y) \in L^{[km]}$ ($k \in \mathbb{Z}_{>0}$) with $j = i - 3 = -1$ or 0, and it vanishes otherwise. Here $L^{[r]} := \{ P \in L \mid \text{mult}_P L = r \}$. \hfill

**Proof.** Let $\sigma : \tilde{\mathcal{Y}} \to \mathcal{Y}$ be the blow-up along all the points of $\pi^{-1}(\text{Sing } L)$ with $j_{\tilde{\mathcal{Y}}} : U \hookrightarrow \tilde{\mathcal{Y}}$ the inclusion. Then (1.1.2) holds with $\mathcal{Y}, \mathcal{F}$ replaced by $\tilde{\mathcal{Y}}, R(j_{\tilde{\mathcal{Y}}})_*\mathcal{L}^{(k')}[2]$, and we have the isomorphism

\[(1.1.3) \quad IC_{\tilde{\mathcal{Y}}}\mathcal{L}^{(k')}[-2] = (j_{\tilde{\mathcal{Y}}})_*\mathcal{L}^{(k')} .\]

Moreover the restriction of $R^j(j_{\tilde{\mathcal{Y}}})_*\mathcal{L}^{(k')}$ to $\sigma^{-1}(y) \cong \mathbb{P}^1$ with $\pi(y) \in \text{Sing } L$ is the direct image of a local system of rank 1 on the complement of $km$ points of $\mathbb{P}^1$ (whose local monodromies are non-trivial) if $\pi(y) \in L^{[km]}$ with $j = 0$ or 1, and it vanishes otherwise. So we get the last assertions of Lemma 1.1 together with the isomorphism

\[(1.1.4) \quad R\sigma_! IC_{\tilde{\mathcal{Y}}}\mathcal{L}^{(k')} = IC_{\mathcal{Y}}\mathcal{L}^{(k')} .\]

This finishes the proof of Lemma 1.1.

1.2. Generic projection. There is a natural projection $\rho : \mathcal{Y} \to C := \mathbb{P}^1$ so that $\mathcal{Y}$ is a $\mathbb{P}^1$-bundle over $C$, and the exceptional divisor of the blow-up is the zero-section. (Here $C$ is identified with the set of lines on $\mathbb{P}^2$ passing through $P$.) There is $c_\infty \in C$ such that

$$\pi(\rho^{-1}(c_\infty)) = L_d. $$

We may assume that the restriction of $\rho$ to $\pi^{-1}(\text{Sing } L \setminus L_d)$ is injective, since $P \in L_d$ is sufficiently general. There are isomorphisms

$$Y \setminus L_d \cong \mathbb{C}^2, \quad C \setminus \{c_\infty\} \cong \mathbb{C},$$

This finishes the proof of Lemma 1.1.
and \( \rho \) is identified with a linear function with constant term. Set
\[
C' := C \setminus \rho'(\operatorname{Sing} L),
\]
where \( \rho' \) is the restriction of \( \rho \) to \( \mathcal{Y} \setminus \pi^{-1}(P) = Y \setminus \{P\} \).

**Lemma 1.2.** We have the vanishing
\[
R^j \rho_* \operatorname{Gr}^W_i \mathcal{F} = 0 \quad \text{unless} \quad (i, j) = (2, -1) \quad \text{or} \quad (3, 0).
\]
For \( c \in C \setminus \{c_\infty\} \cong \mathbb{C} \), there are equalities
\[
\dim_{\mathbb{C}} (R^{-1} \rho_* \operatorname{Gr}^W_2 \mathcal{F})_c = \begin{cases} 
d -2 & \text{if} \quad c \in C' \\
d -3 & \text{if} \quad c \in \rho'(L[\kappa]), \ k/m \in \mathbb{Z}, \\
d -k + 1 & \text{if} \quad c \in \rho'(L[\kappa]), \ k/m \notin \mathbb{Z},
\end{cases}
\]

**Proof.** The assertion for \( i = 3 \) is clear by Lemma 1.1. For \( i = 2 \), it is enough to calculate
\[
H^j(\mathcal{Y}_c, \operatorname{IC}_Y \mathcal{L}(\kappa)^{[k]})|_{\mathcal{Y}_c} = H^j(\mathcal{Y}_c, \operatorname{IC}_Y \mathcal{L}(\kappa')|_{\mathcal{Y}_c}),
\]
using (1.1.4), where \( \mathcal{Y}_c := \rho^{-1}(c), \mathcal{Y}_c := \sigma^{-1}(\mathcal{Y}_c) \) (since \( \rho, \sigma \) are proper). We see that this vanishes for \( j \neq 1 \), and the dimension is as in (1.2.3) by an argument similar to the proof of Lemma 1.1. This finishes the proof of Lemma 1.2.

### 1.3. Description of the first Milnor fiber cohomology

Set
\[
\mathcal{G} := R\rho_* \mathcal{F}.
\]
(This is equal to \( \mathcal{H}^0 R\rho_* \mathcal{F} \) in the notation of [BBD].) It has the weight filtration \( W \) with
\[
\operatorname{Gr}^W_i \mathcal{G} = R\rho_* \operatorname{Gr}^W_i \mathcal{F} \quad (i = 2, 3).
\]
We have
\[
\operatorname{Supp} \operatorname{Gr}^W_3 \mathcal{G} \subset C \setminus C', \quad \mathcal{H}^j \operatorname{Gr}^W_3 \mathcal{G} = 0 \quad (j \neq 0),
\]
and \( \operatorname{Gr}^W_3 \mathcal{G} \) is an intersection complex with coefficients in a local system. More precisely, \( \operatorname{Gr}^W_2 \mathcal{G}[-1] \) is a sheaf in the usual sense on \( C \), and is the open direct image of a local system \( \mathcal{G}' \) of rank \( d - 2 \) on \( C' \), where \( \mathcal{G}' = \mathcal{G}[-1]|_{C'} \). We thus get the following.

**Proposition 1.3.** There are isomorphisms
\[
R^{-1} \rho_* \operatorname{Gr}^W_2 \mathcal{F} = \mathcal{H}^{-1} \mathcal{G},
\]
\[
H^1(F_1, C) \lambda \cong H^{-1}(C, \mathcal{G}) = \Gamma(C', \mathcal{G}'),
\]
where \( \lambda = \exp(\pm 2\pi \sqrt{-1}/m) \). (Note that the monodromy is defined over \( \mathbb{Q} \).)

### 1.4. Description of the vanishing cycles

We have the following.

**Lemma 1.4.** Let \( c \in \rho'(L[\kappa]) \setminus \{c_\infty\} \subset C \setminus \{c_\infty\} \cong \mathbb{C} \). Then
\[
\dim_{\mathbb{C}} \varphi_{t - c, 1} \mathcal{G}' = 1, \quad \dim_{\mathbb{C}} \varphi_{t - c, \neq 1} \mathcal{G}' = 0 \quad \text{if} \quad k/m \in \mathbb{Z},
\]
\[
\dim_{\mathbb{C}} \varphi_{t - c, 1} \mathcal{G}' = 0, \quad \dim_{\mathbb{C}} \varphi_{t - c, \neq 1} \mathcal{G}' = k - 1 \quad \text{if} \quad k/m \notin \mathbb{Z}.
\]
Here \( \varphi_{t - c, 1} \) and \( \varphi_{t - c, \neq 1} \) denote respectively the unipotent and non-unipotent monodromy part of the vanishing cycle functor \( \varphi_{t - c} \) with \( t \) the coordinate of \( C \setminus \{c_\infty\} \cong \mathbb{C} \).

**Proof.** This is shown by calculating the vanishing cycles \( \varphi_{\tilde{\rho}^{-1} t - c}(j_{\mathcal{Y}})_* \mathcal{L}(\kappa) \) on \( \mathcal{Y} \) using (1.1.3–4) together with the commutativity of the nearby and vanishing cycle functors with the direct image under a proper morphism, where \( \tilde{\rho} := \rho \circ \sigma \).

In the case \( k/m \in \mathbb{Z} \), the direct image sheaf \( (j_{\mathcal{Y}})_* \mathcal{L}(\kappa) \) is a local system on the complement of the proper transform of \( L \) in an open neighborhood of \( \sigma^{-1}(y) \) with \( \pi(y) \in L[\kappa] \). So (1.4.1) follows.
In the case $k/m \notin \mathbb{Z}$, the restriction of $\varphi_{\tilde{p}^{-1}(c)}(j_{Y})_{*}L^{(k')} = 0$ and we have the vanishing

$$\varphi_{\tilde{p}^{-1}(c)}(j_{Y})_{*}L^{(k')} = 0,$$

since we can verify the isomorphism $(j_{Y})_{*}L^{(k')}|_{\tilde{p}^{-1}(c)} \sim \psi_{\tilde{p}^{-1}(c)}(j_{Y})_{*}L^{(k')}$. So we get (1.4.2). This finishes the proof of Lemma 1.4.

1.5. Description of a general stalk of $G'$. Let $c \in C'$. Set

$$Z_c := \pi(\rho^{-1}(c)) \subset Y, \quad Z'_c := Z_c \setminus L, \quad E_c := L^{(k')}|_{Z'_c},$$

so that

$$G'_c = H^1(Z'_c, E_c).$$

Since $Z_c \cong \mathbb{P}^1$ and $d/m \in \mathbb{Z}$, we have a ramified covering

$$p_c : \tilde{Z}_c \to Z_c$$

of degree $m$, which is unramified over $Z'_c$, and is totally ramified over

$$\Xi_c := Z_c \cap L,$$

that is, $|p_c^{-1}(z)| = 1$ if $z \in \Xi_c$. Here $\tilde{Z}_c$ is normal, and hence smooth. Let

$$p'_c : \tilde{Z}'_c \to Z'_c$$

be the restriction of $p_c$ over $Z'_c$, which is an unramified covering of degree $m$. Then we get an isomorphism

$$E_c \cong (p'_{c*})_{*}C_{\tilde{Z}'_c}^{*} = \lambda'^{*},$$

with right-hand side the direct factor of $(p'_{c*})_{*}C_{\tilde{Z}'_c}^{*}$ on which the action of a generator $\gamma$ of the covering transformation group of $p_c$ coincides with multiplication by $\lambda' := \exp(2\pi \sqrt{-1}/m)$ (replacing $\gamma$ if necessary). Here $\gamma^{*}$ is semi-simple, since it has finite order. (Note that a local system of rank 1 on a Zariski-open subset of $\mathbb{P}^1$ is determined by the local monodromies.)

Since $p_c$ is totally ramified at $\Xi_c$, we get by (1.5.1–2) the following.

**Proposition 1.5.** We have the isomorphism

$$G'_c \cong H^1(\tilde{Z}_c, C)_{\gamma'^{*}} = \lambda'^{*},$$

with right-hand side the direct factor of $H^1(\tilde{Z}_c, C)$ on which the action of $\gamma^*$ coincides with multiplication by $\lambda'$.

(The isomorphisms in (1.5.2–3) are canonical up to constant multiplication by $\mathbb{C}^*$.)

1.6. Description of the right-hand side of (1.5.3). We can describe the right-hand side of (1.5.3) using the isomorphisms by Poincaré duality:

$$H^1(\tilde{Z}_c, C) = H^1(\tilde{Z}_c, C)^{\vee} = H_1(\tilde{Z}_c, C).$$

Note that for $\tau, \tau' \in H^1(\tilde{Z}_c, C)$, $\sigma \in H_1(\tilde{Z}_c, C)$, we have

$$\langle \gamma^* \tau, \gamma^* \tau' \rangle = \langle \tau, \tau' \rangle, \quad \langle \gamma^* \tau, \sigma \rangle = \langle \tau, \gamma_s \sigma \rangle.$$

Let $z_i \in Z_c$ with $\{z_i\} = Z_c \cap L_i$ for each line $L_i \subset L$ (where $z_d = P$). Note that $\Xi_c = \{z_1, \ldots, z_d\}$. Choose a path $\xi_i$ between $z_d$ and $z_i$, which is contained in $Z'_c$ except the both ends. Here we may assume $\xi_i \cap \xi_j = \{z_d\}$ ($i \neq j$). Using a sufficiently small circle around $z_i$, we get an element

$$\eta_i \in \pi_1(Z_c \setminus \Xi_c, z_d) \quad \text{with} \quad \Xi_c := \Xi_c \setminus \{z_d\}.$$
In this section we prove the main theorem.

Here $c$ denotes by $L$ with the images of the $e$ where the $(1$ as the covering transformation group of the unramified covering $(1$ this, since $\tilde{Z}$ is a deformation retract of the complement. The assertion for $\sigma_i^{x''}$ is then given by the natural action of $\mu_m \subset \mathbb{C}^\ast$ (by scalar multiplication), since the isomorphism $\Gamma_{p_c} \cong \mu_m$ is given by the generator $\gamma \in \Gamma_{p_c}$.

Proposition 1.6. For any $j \in [1, d-1]$, set $I_j := \{1, \ldots, d-1\} \setminus \{j\}$. Then the $\sigma_i^{x''}$ ($i \in I_j$) form a $\mathbb{C}$-basis of $H_1(\tilde{Z}_c, \mathbb{C})^{\gamma=\lambda''}$.

Proof. Since $\dim H_1(\tilde{Z}_c, \mathbb{C})^{\gamma=\lambda''} = d - 2$ by (1.5.3), the assertion is reduced to showing that the $\sigma_i^{x''}$ ($i \neq j$) span $H_1(\tilde{Z}_c, \mathbb{C})^{\gamma=\lambda''}$ for any $j \in [1, d-1]$. For this it is enough to prove that the $\sigma_i$ ($i \neq j$) generate $H_1(\tilde{Z}_c, \mathbb{C})$ over the group ring $\mathbb{C}[\Gamma_{p_c}]$. (Note that $\Gamma_{p_c} \cong \mu_m$ is generated by $\gamma$.) Here we have the surjection

\[ H_1(\tilde{Z}_c \setminus \{\tilde{z}_j\}, \mathbb{C}) \twoheadrightarrow H_1(\tilde{Z}_c, \mathbb{C}). \]

We see that $\bigcup_{i \neq j} \tilde{\xi}_i$, $\bigcup_{i \neq j} \tilde{\xi}_i$ are deformation retracts of $Z_c \setminus \{z_j\}$ and $\tilde{Z}_c \setminus \{\tilde{z}_j\}$ respectively, where $\tilde{\xi}_i := p_c^{-1}(\xi_i)$. (Indeed, choosing $z_\infty \in Z_c$, its complement $Z_c \setminus \{z_\infty\}$ is identified with $\mathbb{C}$, and the $\xi_i$ can be obtained by modifying appropriately the convex hull of $z_i$ and $z_d$ in the real vector space $\mathbb{C}$. Consider the real half line $\ell_j$ in $\mathbb{C}$ which contains $z_j$ as the end point and such that the real full line containing $\ell_j$ contains $z_d$ although $z_d \notin \ell_j$. In the case $\ell_j$ contains some other $z_i$, we deform $\ell_j$ appropriately. Then the complement in $Z_c$ of an appropriate open neighborhood of the closure of $\ell_j$ in $Z_c$ is a deformation retract of $Z_c \setminus \{z_j\}$, and $\bigcup_{i \neq j} \tilde{\xi}_i$ is a deformation retract of the complement. The assertion for $\bigcup_{i \neq j} \tilde{\xi}_i$, $\tilde{Z}_c \setminus \{\tilde{z}_j\}$ follows from this, since $\tilde{Z}_c$ is an unramified covering over $Z_c'$. The assertion then follows, since $\mu_m$ acts as the covering transformation group of the unramified covering $\tilde{\xi}_i \setminus \{\tilde{z}_i, \tilde{z}_d\} \to \xi_i \setminus \{z_j, z_d\}$. This finishes the proof of Proposition 1.6.

Corollary 1.6. The $\sigma_i^{x''}$ ($i \in [1, d-1]$) span $H_1(\tilde{Z}_c, \mathbb{C})^{\gamma=\lambda''}$, and there are $\alpha_i \in \mathbb{C}^\ast$ with

\[ \sum_{i=1}^{d-1} \alpha_i \sigma_i^{x''} = 0. \]

Hence we get the isomorphism

\[ H_1(\tilde{Z}_c, \mathbb{C})^{\gamma=\lambda''} = (\mathbb{C}^{d-1} \subset \mathbb{C} e_i) / \mathbb{C} (\sum_{i=1}^{d-1} \alpha_i e_i), \]

where the $e_i$ are the canonical generators of the vector space $\mathbb{C}^{d-1}$, and the $\sigma_i^{x''}$ are identified with the images of the $e_i$ in the right-hand side of (1.6.5).

2. Proof of the main theorem

In this section we prove the main theorem.

2.1. Proof of Theorem 1. The assertion is reduced to the line arrangement case by Artin’s vanishing theorem [BBBD] using an iterated general hyperplane cut. So $X$ will be denoted by $L$ and the notation in Section 1 will be used. By Proposition 1.3, it is enough to show that

\[ \Gamma(C', \mathcal{G}') = 0, \text{ or equivalently, } (\mathcal{G}')^G = 0. \]

Here $c \in C'$ as in (1.5), and $G$ is the monodromy group of the local system $\mathcal{G}'$ on $C'$.
By Proposition 1.5 and Corollary 1.6 together with Poincaré duality as in (1.6.1–2), any element of \((G'_c)^G\) can be expressed by

\[ \sum_{i=1}^{d-1} \beta_i \sigma_i^{X''}, \]

where the \(\beta_i \in \mathbb{C}\) are unique up to addition by \(\varepsilon \alpha_i\) with \(\varepsilon \in \mathbb{C}\). We have to show that

\[ (2.1.2) \quad \beta_i/\alpha_i \text{ is independent of } i \in \{1, \ldots, d-1\}. \]

Here \(\{1, \ldots, d-1\}\) is identified with the vertices of the dual \((m)\)-graph. Since the latter is connected by hypothesis, it is enough to show (2.1.2) in the case \(i\) belong to a subset \(I_z\) such that the \(L_i\) \((i \in I_z)\) are the lines in \(L\) passing through some \(z \in L^k\) with \(k/m \notin \mathbb{Z}\), where \(|I_z| = k\).

Choosing a path between \(c\) and \(\rho'(z)\), and taking a sufficiently small circle around \(\rho'(z)\), we get \(\zeta \in \pi_1(C', c)\) going around \(\rho'(z)\). Consider the short exact sequence

\[ (2.1.3) \quad 0 \rightarrow (G'_c)^\zeta \rightarrow G'_c \rightarrow \text{Coker } \iota \rightarrow 0, \]

where the first term is the invariant part of \(G'_c\) under the monodromy action by \(\zeta\). Using (1.2.2), (1.3.3), (1.4.4), the dimensions of the three terms are given respectively by

\[ d-k-1, \quad d-2, \quad k-1. \]

We can verify that the \(\sigma_i^{X''}\) for \(i \in I_z^c := \{1, \ldots, d-1\} \setminus I_z\) belong to \((G'_c)^\zeta\). Then they span it using Proposition 1.6, since \(|I_z^c| = d-k-1\). We thus get by Corollary (1.6) the isomorphism

\[ (2.1.4) \quad \text{Coker } \iota = \left( \bigoplus_{i \in I_z} \mathbb{C} e_i \right) / \mathbb{C} \left( \sum_{i \in I_z} \alpha_i e_i \right), \]

such that the images of the \(\sigma_i^{X''}\) \((i \in I_z)\) in \(\text{Coker } \iota\) are identified with the images of the \(e_i\) in the right-hand side of (2.1.4).

Since \(\sum_{i=1}^{d-1} \beta_i \sigma_i^{X''} \in (G'_c)^G\) belongs to \((G'_c)^\zeta\), its image in \(\text{Coker } \iota\) vanishes. This means that

\[ \sum_{i \in I_z} \beta_i e_i \in \mathbb{C} \left( \sum_{i \in I_z} \alpha_i e_i \right). \]

So we get (2.1.2) for \(i \in I_z\). This finishes the proof of Theorem 1.

2.2. Generalized Hessian arrangements. For a positive integer \(a\), set \(L := \{f = 0\} \subset \mathbb{P}^2\) with

\[ f = xyz \prod_{i,j=0}^{b-1} (\zeta^i x + \zeta^j y + z), \]

where \(b := 4a-1\), \(\zeta := e^{2\pi \sqrt{-1}/b}\). This is the Hessian arrangement if \(a = 1\), see [BDS], [Di2], [Sa3], [Yu], etc. It is called a generalized Hessian arrangement for \(a \geq 2\). We have

\[ d = (4a-1)^2 + 3 = 4(4a^2 - 2a + 1), \]

and \(\text{Sing } L = L^{[4a]} \cup L^{[2]}\) with \(L^{[4a]} = \text{Sing } L \cap \{xyz = 0\}\), see [Sa3], 3.4–5]. The dual \((4)-\)graph has two or four connected components depending on whether \(a \geq 2\) or \(a = 1\). (Note that the conditions \(d/m, 4a/m \in \mathbb{Z}\) imply that \(4/m \in \mathbb{Z}\).)

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