MULTIPICLITY AND CONCENTRATION OF SOLUTIONS FOR NONLINEAR FRACTIONAL ELLIPTIC EQUATIONS WITH STEEP POTENTIAL

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Abstract. In this article, we prove the existence, multiplicity and concentration of non-trivial solutions for the following indefinite fractional elliptic equation with concave-convex nonlinearities:

\[
\begin{cases}
(-\Delta)^{\alpha} u + V_\lambda(x)u = a(x)|u|^{q-2}u + b(x)|u|^{p-2}u & \text{in } \mathbb{R}^N, \\
 u \geq 0 & \text{in } \mathbb{R}^N,
\end{cases}
\]

where \(0 < \alpha < 1\), \(N > 2\alpha\), \(1 < q < 2 < p < 2^*_\alpha\) with \(2^*_\alpha = 2N/(N - 2\alpha)\), the potential \(V_\lambda(x) = \lambda V^+(x) - V^-(x)\) with \(V^\pm = \max\{\pm V, 0\}\) and the parameter \(\lambda > 0\). Our multiplicity results are based on studying the decomposition of the Nehari manifold.

1. Introduction. The fractional Schrödinger equation is a fundamental equation of fractional quantum mechanics. It was discovered by Nick Laskin [18, 19] as a result of extending the Feynman path integral, from the Brownian-like to Lévy-like quantum mechanical paths.

In this paper, we consider the existence, multiplicity and concentration of non-trivial solutions of the following concave-convex elliptic equations involving fractional Laplacian:

\[
\begin{cases}
(-\Delta)^{\alpha} u + V_\lambda(x)u = a(x)|u|^{q-2}u + b(x)|u|^{p-2}u & \text{in } \mathbb{R}^N, \\
 u \geq 0 & \text{in } \mathbb{R}^N,
\end{cases}
\]

where \(0 < \alpha < 1\), \(N > 2\alpha\), \(1 < q < 2 < p < 2^*_\alpha\) with \(2^*_\alpha = 2N/(N - 2\alpha)\), the potential \(V_\lambda(x) = \lambda V^+(x) - V^-(x)\) with \(V^\pm = \max\{\pm V, 0\}\) and the parameter \(\lambda > 0\). Here \((-\Delta)^{\alpha}\) is the fractional Laplacian defined, up to a normalization constant, as

\[
(-\Delta)^{\alpha} u(x) = P.V. \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2\alpha}} dy,
\]

for \(x \in \mathbb{R}^N\), where P.V. denotes the principal value of the integral.

Concerning the functions \(a(x), b(x)\) and \(V(x)\), we may assume that

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Sobolev inequalities, we have considered a nonlinear Schrödinger equation. The potential \( V \) for any \( u \),

\[
\|a^+\|_{L^q(R^N)}>0 \text{ where } a^+(x) = \max\{a(x),0\}.
\]

(b1) \( b \in L^\infty(R^N) \) with \( \|b^+\|_{L^\infty(R^N)}>0 \), where \( b^+(x) = \max\{b(x),0\} \).

(V1) \( V^+ \) is a continuous function on \( R^N \) and \( V^- \in L^{N/2}(R^N) \).

(V2) there exists \( \kappa > 0 \) such that the set \( \{V^+ < \kappa\} = \{x \in R^N : V^+(x) < \kappa\} \) is nonempty and has finite measure.

(V3) \( \Omega = \text{int}\{x \in R^N : V^+(x) = 0\} \) is nonempty and has a smooth boundary with \( \Omega = \text{int}\{x \in R^N : V^+(x) = 0\} \).

(V4) there exists a constant \( \mu_0 > 1 \) such that

\[
\mu_1(\lambda) := \inf_{u \in H^\alpha(R^N) \setminus \{0\}} \frac{\int_{R^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2\alpha}} \, dx \, dy + \int_{R^N} \lambda V^+ u^2 \, dx}{\int_{R^N} V^- u^2 \, dx} \geq \mu_0,
\]

for all \( \lambda > 0 \), where \( H^\alpha(R^N) \) is the fractional Sobolev space.

Remark 1.1. By [7], there exists a sharp constant \( S_\alpha > 0 \) such that

\[
\|u\|_{L^{2_\alpha}(R^N)} \leq S_\alpha \int_{R^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2\alpha}} \, dx \, dy,
\]

for any \( u \in H^\alpha(R^N) \). Then by the condition (V1) – (V3) and the Hölder and Sobolev inequalities, we have

\[
\frac{\int_{R^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2\alpha}} \, dx \, dy + \int_{R^N} \lambda V^+ u^2 \, dx}{\int_{R^N} V^- u^2 \, dx} \geq \frac{\int_{R^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2\alpha}} \, dx \, dy}{\|V^\alpha\|_{L^{N/2}} (\int_{R^N} |u|^2 \alpha^dx)^{2/N}} \geq \frac{\int_{R^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2\alpha}} \, dx \, dy}{\|V^-\|_{L^{N/2}} S_\alpha^{-2} \int_{R^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2\alpha}} \, dx \, dy} \geq \frac{S_\alpha^2}{\|V^-\|_{L^{N/2}}},
\]

for all \( \lambda \geq 0 \), which implies that if \( \|V^-\|_{L^{N/2}} \leq S_\alpha^2 \), then

\[
\mu_1(\lambda) \geq \frac{S_\alpha^2}{\|V^-\|_{L^{N/2}}} > 1 \text{ for all } \lambda > 0.
\]

Therefore, the condition (V4) holds.

This type of assumptions was first introduced by Bartsch and Wang [4] and they considered a nonlinear Schrödinger equation. The potential \( V_\lambda \) with \( V \) satisfies (V1) – (V3) is called as the steep well potential.

In recent years, problem (1.1) with \( \alpha = 1 \) (that is, the Laplace case) has been widely studied under variant assumptions on \( V(X) \) and \( b(x) \). Most of the literature has focused on the problem for \( V \) being a positive constant and \( b \) being a positive weight function. Existence and multiplicity results have been obtained in many papers, see for example [4, 5] and references therein. Recently, Cheng and Wu [8] consider the potential \( V_\lambda \) which can be sign-changed, that is, \( V_\lambda = \lambda V^+ - V^- \), and they obtained the multiplicity and concentration results with concave-convex nonlinearity terms. In our paper, we would like to extend Cheng and Wu’s results to fractional Laplacian case, that is, equation (1.1).

Our first result is
Theorem 1.1. Assume that $2 < p < 2^*_a$ and the functions $a$, $b$ and $V$ satisfy the assumptions (A1), (B1) and (V1)-(V4) as well as the following condition:

(V5) \[ 0 < |\{V^+ < \kappa\}|^{\frac{2(p-q)(2^*_a-p)}{q}} \cdot \left( \frac{q(p-2)}{2\|a^+\|_{L^p}} \right)^{p-2} \left( \frac{2-q}{\|b^+\|_{L^q}} \right)^{2-q} \left( \frac{(\mu_0 - 1)S^2_0}{\mu_0(p-q)} \right)^{p-q}. \]

Then, there exists $\Lambda_0 \geq 0$ such that problem (1.1) has at least one non-trivial non-negative solutions if $\lambda \in (\Lambda_0, \infty)$.

For the multiplicity result, we need more conditions on $b(x)$. Therefore, our second main result is

Theorem 1.2. Assume that $2 < p < 2^*_a$ and the functions $a$, $b$ and $V$ satisfy the assumptions (A1), (B1) and (V1)-(V5) as well as the following condition:

(B2) there exists a nonempty open set $\Omega_0 \subset \Omega$ such that $b(x) > 0$ in $\Omega_0$. Then there exists $\tilde{\Lambda}_0 > 0$ such that for every $\lambda > \tilde{\Lambda}_0$, equation (1.1) has at least two non-trivial non-negative solutions $u_\lambda^+$ and $u_\lambda^-$. 

Recently, concentration phenomena for the fractional Schrödinger equation has been attracted many attentions, see for example [9, 10, 16, 27] and references therein. When $V_\lambda = \lambda V^+$, Torres [27] consider the following problem 

\[
\begin{align*}
\left\{ \begin{array}{ll}
(-\Delta)^\alpha u + \lambda V^+(x) = f(x,u) & \text{in } \mathbb{R}^N, \\
0 & \text{in } \mathbb{R}^N.
\end{array} \right.
\end{align*}
\]

When $f(x,u)$ satisfies some sub-quadratic assumptions as $|u| \to \infty$, they proved existence and concentration results. In our paper, we consider equation (1.1) with a more general potential $V$ and convex-concave nonlinearity. Furthermore, we also obtain the following concentration result.

Theorem 1.3. Let $u_\lambda^+$ and $u_\lambda^-$ be the solutions in Theorem 1.2. Then $u_\lambda^+ \to u_0^+$ and $u_\lambda^- \to u_0^-$ in $H^\alpha(\mathbb{R}^N)$ as $\lambda \to \infty$, where $H^\alpha(\mathbb{R}^N)$ is the fractional Sobolev space and $u_0^\pm \in X_0^\alpha(\Omega)$ are solutions of 

\[
\begin{align*}
\left\{ \begin{array}{ll}
(-\Delta)^\alpha u - \lambda V^-(x) = a(x)|u|^{q-2}u + b(x)|u|^{p-2}u & \text{in } \Omega, \\
0 & \text{in } \mathbb{R}^N \setminus \Omega.
\end{array} \right.
\end{align*}
\]

with $X_0^\alpha(\Omega) = \{ u \in H^\alpha(\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega \}$. Furthermore, we have the following:

(1) $u_0^-$ is a non-trivial non-negative solution of equation (1.3)
(2) if $a \leq 0$ in $\Omega$, then $u_0^+ \equiv 0$;
(3) if there is a nonempty open set $\Omega_a \subset \Omega$ such that $a(x) > 0$ in $\Omega_a$, then $u_0^+$ is a non-trivial non-negative solution of equation (1.3);
(4) $u_0^- \not\equiv u_0^+$.

The convex-concave problem involving the fractional Laplacian related to problem (1.1) has been studied by Barrios et al. [2], Quaas and Xia [22] and references therein. One can also define a fractional power of the Laplacian using spectral decomposition. Problem (1.1) for the spectral fractional Laplacian has been treated in [3]. In [17], Felmer et al. considered the existence of positive solutions for (1.1) when $V^+(x) \equiv 1$, $V^-(x) \equiv 0$ and $\lambda = 1$.

We use variational methods to find non-trivial non-negative solutions of equation (1.1). As it is well known, when one uses the variational methods to find the
critical points of the functional, some geometry structures are needed such as the mountain pass structure, the linking structures and so on. For problem (1.1), the main difficulty lies in the functional may not possess such structures since the sign-changing weight. In order to overcome this difficulty, we turn to another approach, that is, the Nehari manifold, which was introduced by Nehari in [20] and has been widely used in the literature, for example [26, 1, 28, 29, 6, 30, 11] and references therein. The main idea of these articles lies in dividing the Nehari manifold into three parts and considering the infima of the functional on each part. Precisely, the Nehari manifold for $\Phi_\lambda(u)$ which is the functional of equation (1.1) (see Section 2), is defined as

$$N_\lambda = \{ u \in X : \langle \Phi'_\lambda(u), u \rangle = 0 \}.$$ 

It is clear that all critical points of $\Phi_\lambda$ must lie on $N_\lambda$, as we will see below, local minimizers on $N_\lambda$ are usually critical points of $\Phi_\lambda$. By consider the fibering map $h_u(t) = \Phi_{\lambda}(tu)$, we can divide that $N_\lambda$ into three subsets $N_\lambda^+, N_\lambda^-$ and $N_\lambda^0$ which correspond to local minima, local maxima and points of inflexion of fibering maps. Then we can find that $N_\lambda^0 = \emptyset$ if $\lambda \in (\Lambda_0, \infty)$ and meanwhile there exists at least one non-trivial non-negative solution in $N_\lambda^+$ and $N_\lambda^-$ respectively.

This article is organized as follows. In Section 2 we give some notations and preliminaries for the Nehari manifold. Sections 3 and 4 are devoted to prove the existence and multiplicity of non-trivial non-negative solutions of equation (1.1), Theorem 1.1 and Theorem 1.2, respectively. We prove Theorem 1.3 in Section 5.

2. Variational setting and preliminaries. In this section, we give a few results that we are later going to use for the proofs of the main results. For $\alpha \in (0, 1)$, we define $D^{\alpha,2}(\mathbb{R}^N)$ as the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to

$$[u]_{H^{\alpha}(\mathbb{R}^N)}^2 = \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2\alpha}} dxdy,$$

that is,

$$D^{\alpha,2}(\mathbb{R}^N) = \left\{ u \in L^{2^*_\alpha}(\mathbb{R}^N) : [u]_{H^{\alpha}(\mathbb{R}^N)} < \infty \right\}.$$

Now, let us introduce the fractional Sobolev space

$$H^{\alpha}(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) : \frac{u(x) - u(y)}{|x - y|^{N+2\alpha}} \in L^2(\mathbb{R}^{2N}) \right\}$$

equipped with the natural norm

$$\|u\|_{H^{\alpha}(\mathbb{R}^N)} = \left( [u]_{H^{\alpha}(\mathbb{R}^N)}^2 + \|u\|_{L^2(\mathbb{R}^N)}^2 \right)^{1/2}.$$

Next, we give the variational setting for equation (1.1) following [14], and we establish the compactness conditions. Let

$$X = \left\{ u \in H^{\alpha}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V^+ u^2 dx < \infty \right\}$$

be equipped with the inner product and norm

$$\langle u, v \rangle = \int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2\alpha}} dxdy + \int_{\mathbb{R}^N} V^+ uv dx,$$

$$\|u\|_X = \langle u, u \rangle^{1/2}.$$
For $\lambda > 0$, we also need the following scalar product and norm
\[
\langle u, v \rangle_{\lambda} = \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2\alpha}} \, dx \, dy + \lambda \int_{\mathbb{R}^N} V^+ u v \, dx,
\]
\[
\|u\|_{\lambda} = \langle u, v \rangle_{\lambda}^{1/2}.
\]
It is clear that $\|u\|_X \leq \|u\|_{\lambda}$ for $\lambda \geq 1$. Moreover, by condition (V4),
\[
\|u\|^2_{\lambda} \geq \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2\alpha}} \, dx \, dy + \int_{\mathbb{R}^N} V^+ \lambda u^2 \, dx \geq \frac{\mu_0 - 1}{\mu_0} \|u\|^2_{\lambda},
\]
for all $\lambda \geq 0$.

We use the variational methods to find solutions of equation (1.1). Associated with the equation (1.1), we consider the energy functional $\Phi_\lambda : X \to \mathbb{R}$,
\[
\Phi_\lambda(u) = \frac{1}{2} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2\alpha}} \, dx \, dy + \frac{1}{2} \int_{\mathbb{R}^N} V^+ \lambda u^2 \, dx - \frac{1}{q} \int_{\mathbb{R}^N} a|u|^q \, dx - \frac{1}{p} \int_{\mathbb{R}^N} b|u|^p \, dx.
\]
We refer reader to [13, 24] for more details of fractional Sobolev space and variational methods.

Since the energy functional $\Phi_\lambda$ is not bounded below on $X$, it is useful to consider the functional on the Nehari manifold
\[
\mathcal{N}_\lambda = \{ u \in X \setminus \{0\} \mid \langle \Phi'_\lambda(u), u \rangle = 0 \}.
\]
Thus, $u \in \mathcal{N}_\lambda$ if and only if
\[
\|u\|^2_{\lambda} - \int_{\mathbb{R}^N} V^- u^2 \, dx - \frac{1}{q} \int_{\mathbb{R}^N} a|u|^q \, dx - \frac{1}{p} \int_{\mathbb{R}^N} b|u|^p \, dx = 0.
\]
We notice that $\mathcal{N}_\lambda$ contain every non-zero solution of equation (1.1).

Next, we define the Palais-Smale (PS)-sequences and (PS)-condition in $X$ for $\Phi_\lambda$ as follows.

**Definition 2.1.**  (1) For $\beta \in \mathbb{R}$, a sequence $\{u_n\}$ is a $(PS)_\beta$-sequence in $X$ for $\Phi_\lambda$ if $\Phi_\lambda(u_n) = \beta + o(1)$ and $\Phi'_\lambda(u_n) = o(1)$ strongly in $X^{-1}$ as $n \to \infty$, where $X^{-1}$ is the dual space of $X$.

(2) $\Phi_\lambda$ satisfies the $(PS)_\beta$-condition in $X$ if every $(PS)_\beta$-sequence in $X$ for $\Phi_\lambda$ contains a strongly convergent subsequence.

Since the energy functional $\Phi_\lambda$ is not bounded below on $X$, it is useful to consider the functional on the Nehari manifold $\mathcal{N}_\lambda$. Set
\[
\Lambda_0 := \frac{S_2^2}{\kappa} \left| \{ V^+ < \kappa \} \right|^{\frac{\alpha - 2}{\alpha}} S_\alpha^{-p} \|u\|_{\lambda}^p,
\]
then we have the following estimate.

**Lemma 2.1.** Assume that (V1) – (V2) and $\lambda \geq \Lambda_0$, then
\[
\int_{\mathbb{R}^N} |u|^p \, dx \leq \left| \{ V^+ < \kappa \} \right|^{\frac{\alpha - p}{\alpha} S_\alpha^{-p} \|u\|_{\lambda}^p, \text{ for } p \in [2, 2^*_\alpha).}
\]
Proof. By the Hölder inequality, for any \( p \in [2, 2^*_s) \), we have,
\[
\int_{\mathbb{R}^N} |u|^p \, dx = \int_{\mathbb{R}^N} |u|^{\frac{2^*_s-p}{2^*_s-p}} \cdot |u|^{\frac{p-2}{2^*_s-p}} \, dx \leq \left( \int_{\mathbb{R}^N} |u|^2 \, dx \right)^{\frac{2^*_s-p}{2}} \left( \int_{\mathbb{R}^N} |u|^{2^*_s} \, dx \right)^{\frac{p-2}{2}}.
\]
Moreover, using the conditions (V1) and (V2), and by Hölder and Sobolev inequalities, we also have
\[
\int_{\mathbb{R}^N} |u|^2 \, dx = \int_{\{V^+ \geq \kappa\}} |u|^2 \, dx + \int_{\{V^+ < \kappa\}} |u|^2 \, dx \leq \frac{1}{\kappa} \int_{\mathbb{R}^N} V^+ |u|^2 \, dx + |\{V^+ < \kappa\}| \left( \int_{\mathbb{R}^N} |u|^{2^*_s} \, dx \right)^{\frac{1}{2^*_s}} \leq \frac{1}{\kappa \lambda} \|u\|_\alpha^2 + |\{V^+ < \kappa\}| \left( \int_{\mathbb{R}^N} |u|^{2^*_s} \, dx \right)^{\frac{1}{2^*_s}} \leq \max \left\{ \frac{1}{\lambda \kappa}, |\{V^+ < \kappa\}| \left( \int_{\mathbb{R}^N} |u|^{2^*_s} \, dx \right)^{\frac{1}{2^*_s}} \right\} \|u\|_\alpha^2 \leq |\{V^+ < \kappa\}| \left( \frac{2^*_s-2}{2^*_s} \right) S_\alpha^{-2} \|u\|_\alpha^2,
\]
since \( \lambda \geq \Lambda_0 \), and
\[
\int_{\mathbb{R}^N} |u|^{2^*_s} \, dx \leq S_\alpha^{-2} \|u\|_{H^s(\mathbb{R}^N)}^{2^*_s} \leq S_\alpha^{-2} \|u\|_\alpha^{2^*_s}.
\]
Therefore,
\[
\int_{\mathbb{R}^N} |u|^p \, dx \leq |\{V^+ < \kappa\}| \left( \frac{2^*_s-p}{2^*_s} \right) S_\alpha^{-p} \|u\|_\alpha^p.
\]
We have the following results.

**Lemma 2.2.** The energy functional \( \Phi_\lambda \) is coercive and bounded below on \( \mathcal{N}_\lambda \). Moreover, we have
\[
\Phi_\lambda(u) \geq -\frac{2 - q}{2pq} \left( \frac{\mu_0}{(\mu_0 - 1)(p-2)} \right)^{\frac{q}{2-q}} \left( \frac{(p-q) \{V^+ < \kappa\} \|a^+\|_{L^q}^q}{S_\alpha^q} \right)^{\frac{2}{2-q}}.
\]

**Proof.** If \( u \in \mathcal{N}_\lambda \), then, by Hölder inequality, (2.1) and Lemma 2.1, we have
\[
\Phi_\lambda(u) = \left( \frac{1}{2} - \frac{1}{p} \right) \left( \|u\|_{H^s(\mathbb{R}^N)}^2 + \int_{\mathbb{R}^N} V_\lambda u^2 \, dx \right) - \left( \frac{1}{q} - \frac{1}{p} \right) \int_{\Omega} a(x)|u|^q \, dx \geq \left( \frac{1}{2} - \frac{1}{p} \right) \frac{\mu_0 - 1}{\mu_0} \|u\|_\alpha^2 \left( \frac{1}{q} \right) \|a^+\|_{L^q(\mathbb{R}^N)} \left( {\frac{2^*_s-p}{2^*_s}} \right) S_\alpha^{-q} \|u\|_\alpha^q \quad (2.3)
\]
\[
\geq -\frac{2 - q}{2pq} \left( \frac{\mu_0}{(\mu_0 - 1)(p-2)} \right)^{\frac{q}{2-q}} \left( \frac{(p-q) \{V^+ < \kappa\} \|a^+\|_{L^q}^q}{S_\alpha^q} \right)^{\frac{2}{2-q}}.
\]
Thus \( \Phi_\lambda \) is coercive and bounded below on \( \mathcal{N}_\lambda \). \( \square \)
The Nehari manifold $\mathcal{N}_\lambda$ is closely related to the behaviour of the function of the form $h_u : t \to \Phi_\lambda(tu)$ for $t > 0$. Such maps are known as fibering maps that date back to the fundamental works \cite{21, 23, 12}. If $u \in X$, we have

$$h_u(t) = \frac{t^2}{2} \left( \|u\|_A^2 - \int_{\mathbb{R}^N} V^- u^2 \, dx \right) - \frac{t^q}{q} \int_{\mathbb{R}^N} a|u|^q \, dx - \frac{t^p}{p} \int_{\mathbb{R}^N} b|u|^p \, dx;$$

$$h_u'(t) = t \left( \|u\|_A^2 - \int_{\mathbb{R}^N} V^- u^2 \, dx \right) - t^{q-1} \int_{\mathbb{R}^N} a|u|^q \, dx - t^{p-1} \int_{\mathbb{R}^N} b|u|^p \, dx;$$

$$h_u''(t) = \|u\|_A^2 - \int_{\mathbb{R}^N} V^- u^2 \, dx - (q-1)t^{q-2} \int_{\mathbb{R}^N} a|u|^q \, dx - (p-1)t^{p-2} \int_{\mathbb{R}^N} b|u|^p \, dx.$$

We observe that

$$th_u'(t) = \|tu\|_A^2 - \int_{\mathbb{R}^N} V^-(tu)^2 \, dx - \int_{\mathbb{R}^N} a|tu|^q \, dx - \int_{\mathbb{R}^N} b|tu|^p \, dx$$

and thus, for $u \in X \setminus \{0\}$ and $t > 0$, $h_u'(t) = 0$ if and only if $tu \in \mathcal{N}_\lambda$, that is, positive critical points of $h_u$ correspond points on the Nehari manifold. In particular, $h_u'(1) = 0$ if and only if $u \in \mathcal{N}_\lambda$. So it is natural to split $\mathcal{N}_\lambda$ into three parts corresponding local minimal, local maximum and points of inflection. Accordingly, we define

$$\mathcal{N}_\lambda^+ = \{ u \in \mathcal{N}_\lambda \mid h_u''(1) > 0 \};$$

$$\mathcal{N}_\lambda^0 = \{ u \in \mathcal{N}_\lambda \mid h_u''(1) = 0 \};$$

$$\mathcal{N}_\lambda^- = \{ u \in \mathcal{N}_\lambda \mid h_u''(1) < 0 \}.$$

Next, we establish some basic properties of $\mathcal{N}_\lambda^+, \mathcal{N}_\lambda^0$, and $\mathcal{N}_\lambda^-$. 

**Lemma 2.3.** Suppose that $u_0$ is a local minimizer of $\Phi_\lambda$ on $\mathcal{N}_\lambda$ and $u_0 \not\in \mathcal{N}_\lambda^0$. Then $\Phi_\lambda'(u_0) = 0$ in $X^{-1}$.

**Proof.** If $u_0$ is a local minimizer for $\Phi_\lambda$ on $\mathcal{N}_\lambda$, then $u_0$ is a solution of the optimization problem

$$\text{minimizer } \Phi_\lambda(u) \text{ subject to } J(u) = 0,$$

where $J(u) = \|u\|_A^2 - \int_{\mathbb{R}^N} V^- u^2 \, dx - \int_{\mathbb{R}^N} a|u|^q \, dx - \int_{\mathbb{R}^N} b|u|^p \, dx$. Hence, by the theory of Lagrange multipliers, there exists $\mu \in \mathbb{R}$ such that $\Phi_\lambda'(u_0) = \mu J'(u_0)$. Thus we have

$$\langle \Phi_\lambda'(u_0), u_0 \rangle = \mu \langle J'(u_0), u_0 \rangle. \quad (2.4)$$

Since $u_0 \in \mathcal{N}_\lambda$, we have that $\|u_0\|_A^2 - \int_{\mathbb{R}^N} V^- u_0^2 \, dx - \int_{\mathbb{R}^N} a|u_0|^q \, dx - \int_{\mathbb{R}^N} b|u_0|^p \, dx = 0$. Hence,

$$\langle J'(u_0), u_0 \rangle = 2\|u_0\|_A^2 - 2 \int_{\mathbb{R}^N} V^- u_0^2 \, dx - q \int_{\mathbb{R}^N} a|u_0|^q \, dx - p \int_{\mathbb{R}^N} b|u_0|^p \, dx$$

$$= \|u_0\|_A^2 - \int_{\mathbb{R}^N} V^- u_0^2 \, dx - (q-1) \int_{\mathbb{R}^N} a|u_0|^q \, dx - (p-1) \int_{\mathbb{R}^N} b|u_0|^p \, dx.$$

So, if $u_0 \not\in \mathcal{N}_\lambda^0$, $\langle J'(u_0), u_0 \rangle \neq 0$ and thus $\mu = 0$ by (2.4). Hence, we complete the proof. \qed
Thus, we have that
\[
h''_u(1) = \|u\|_\lambda^2 - \int_{\mathbb{R}^N} V^- u^2 \, dx - (q - 1) \int_{\mathbb{R}^N} a|u|^q \, dx - (p - 1) \int_{\mathbb{R}^N} b|u|^p \, dx
\]
\[
= (2 - p) \left( \|u\|_\lambda^2 - \int_{\mathbb{R}^N} V^- u^2 \, dx \right) - (q - p) \int_{\mathbb{R}^N} a|u|^q \, dx \tag{2.5}
\]
\[
= (2 - q) \left( \|u\|_\lambda^2 - \int_{\mathbb{R}^N} V^- u^2 \, dx \right) - (p - q) \int_{\mathbb{R}^N} b|u|^p \, dx. \tag{2.6}
\]

Then we have following result.

**Lemma 2.4.** (1) For any \( u \in N_\lambda^+ \cup N_\lambda^0 \), we have \( \int_{\mathbb{R}^N} a|u|^q \, dx > 0 \);
(2) For any \( u \in N_\lambda^- \), we have \( \int_{\mathbb{R}^N} b|u|^p \, dx > 0 \).

**Proof.** By the definitions of \( N_\lambda^+ \) and \( N_\lambda^0 \), it is easy to get that \( \int_{\mathbb{R}^N} a|u|^q \, dx > 0 \) from (2.5). Similarly, the definition of \( N_\lambda^- \) and (2.6) imply that \( \int_{\mathbb{R}^N} b|u|^p \, dx > 0 \).

Let \( \Lambda_0 \) be as in (2.2). Then we have the following result.

**Lemma 2.5.** Suppose that the functions \( a, b \) and \( V \) satisfy the conditions (A1), (B1) and (V1) – (V5). Then, for each \( \lambda \geq \Lambda_0 \), we have \( N_\lambda^0 = \emptyset \).

**Proof.** We prove it by contradiction arguments. Suppose that there exists \( \lambda \geq \Lambda_0 \) such that \( N_\lambda^0 \neq \emptyset \). Then, for \( u_0 \in N_\lambda^0 \), by (2.5), (2.1), Lemma 2.1 and the Hölder inequality, we have
\[
\frac{\mu_0 - 1}{\mu_0} \|u\|_\lambda^2 \leq \frac{p - q}{p - 2} \int_{\mathbb{R}^N} a|u|^q \, dx \leq \frac{p - q}{p - 2} \int_{\mathbb{R}^N} a^+|u|^q \, dx
\]
\[
\leq \frac{p - q}{p - 2} \|a^+\|_{L^r(\mathbb{R}^N)} \{V^+ < \kappa\}^{\frac{q}{p-2}} \, S_\alpha^{-q} \|u\|_\lambda^q
\]
and so
\[
\|u\|_\lambda \leq \left( \frac{\mu_0(p - q)}{(\mu_0 - 1)(p - 2)} \|a^+\|_{L^r(\mathbb{R}^N)} \{V^+ < \kappa\}^{\frac{q}{p-2}} \, S_\alpha^{-q} \right)^{1/(2 - q)}. \tag{2.7}
\]
Similarly, by (2.6), (2.1), Lemma 2.1 and the Hölder inequality, we have
\[
\frac{(\mu_0 - 1)(2 - q)}{\mu_0(p - q)} \|u\|_\lambda^2 \leq \int_{\mathbb{R}^N} b|u|^p \, dx \leq \|b^+\|_{L^\infty(\mathbb{R}^N)} \{V^+ < \kappa\}^{\frac{q}{p-2}} \, S_\alpha^{-p} \|u\|_\lambda^p.
\]
Thus,
\[
\|u\|_\lambda \geq \left( \frac{(\mu_0 - 1)(2 - q)S_\alpha^p}{\mu_0(p - q)\|b^+\|_{L^\infty(\mathbb{R}^N)} \{V^+ < \kappa\}^{\frac{q}{p-2}} \, S_\alpha^{-p}} \right)^{1/(p-2)}. \tag{2.8}
\]
Hence, combining (2.7) and (2.8), we must have
\[
|\{V^+ < \kappa\}| \geq \left[ \frac{q}{p-2} \right]^{p-2} \left( \frac{2 - q}{p-2} \right)^{2 - q} \left( \frac{(\mu_0 - 1)S_\alpha^2}{\mu_0(p - q)} \right)^{p-q} \left( \frac{q}{2(p-q)(2-q)} \right),
\]
which is a contradiction. This completes the proof. \(\square\)
In order to get a better understanding of the Nehari manifold and the fibering maps, we consider the function \( m_u : \mathbb{R}^+ \to \mathbb{R} \) defined by

\[
m_u(t) = t^{2-q} \left( \|u\|^2_{\lambda} - \int_{\mathbb{R}^N} V^{-u^2}dx \right) - t^{p-q} \int_{\mathbb{R}^N} b|u|^pdx \quad \text{for } t > 0.
\]  

(2.9)

It is clear that \( tu \in \mathcal{N}_\lambda \) if and only if \( m_u(t) = \int_{\mathbb{R}^N} a|u|^q \). Moreover,

\[
m'_u(t) = (2 - q)t^{1-q} \left( \|u\|^2_{\lambda} - \int_{\mathbb{R}^N} V^{-u^2}dx \right) - (p-q)t^{p-q-1} \int_{\mathbb{R}^N} b|u|^pdx
\]  

(2.10)

and it is easy to see that, if \( tu \in \mathcal{N}_\lambda \), then \( t^{q-1}m'_u(t) = h''_u(t) \). Hence \( tu \in \mathcal{N}_\lambda^+ \) (or \( \mathcal{N}_\lambda^- \)) if and only if \( m'_u(t) > 0 \) (or \(< 0 \).

For every \( u \in X \setminus \{0\} \) with \( \int_{\mathbb{R}^N} b|u|^pdx > 0 \), we let

\[
t_{\max,\lambda}(u) = \left( \frac{(2 - q) \left( \|u\|^2_{\lambda} - \int_{\mathbb{R}^N} V^{-u^2}dx \right)}{(p-q) \int_{\mathbb{R}^N} b|u|^pdx} \right)^{\frac{1}{q-1}} > 0,
\]  

(2.11)

which leads the following lemma.

**Lemma 2.6.** Assume that the functions \( a, b \) and \( V \) satisfy the conditions (A1), (B1) and (V1)–(V5). Then, for each \( \lambda \geq \Lambda_0 \) and \( u \in X \setminus \{0\} \) with \( \int_{\mathbb{R}^N} b|u|^pdx > 0 \), we have that

1. if \( \int_{\mathbb{R}^N} a|u|^qdx \leq 0 \), then there exists a unique \( t^- = t^-(u) > t_{\max,\lambda}(u) \) such that \( t^-u \in \mathcal{N}_{\lambda}^- \) and
   \[
   \Phi_{\lambda}(t^-u) = \sup_{t \geq 0} \Phi_{\lambda}(tu).
   \]

(2.12)

2. if \( \int_{\mathbb{R}^N} a|u|^qdx > 0 \), then there exists a unique \( 0 < t^+ = t^+(u) < t_{\max,\lambda}(u) < t^- \) such that \( t^+u \in \mathcal{N}_{\lambda}^+ \), \( t^-u \in \mathcal{N}_{\lambda}^- \) and
   \[
   \Phi_{\lambda}(t^+u) = \inf_{0 \leq t \leq t_{\max,\lambda}(u)} \Phi_{\lambda}(tu), \quad \Phi_{\lambda}(t^-u) = \sup_{t \geq t^+} \Phi_{\lambda}(tu).
   \]

(2.13)

**Proof.** By (2.10), we know \( t_{\max,\lambda} \) is the unique critical point of \( m_u \) and \( m_u \) is strictly increasing on \((0, t_{\max,\lambda})\) and strictly decreasing on \((t_{\max,\lambda}, \infty)\) with

\[
\lim_{t \to \infty} m_u(t) = -\infty.
\]

Moreover, by (2.11), Hölder inequality and the same arguments as in the proof of Lemma 2.5, we have that

\[
m_u(t_{\max,\lambda}) \geq \left[ \frac{2 - q}{p-q} - \frac{2 - q}{p-q} \right] \left[ \frac{p - 2}{p-q} \right] \left( \mu_0 - 1 \right) \left( \int_{\mathbb{R}^N} V^{-u^2}dx \right)^{\frac{p-q}{2-q}} \left( \int_{\mathbb{R}^N} b|u|^pdx \right)^{\frac{p-q}{2-q}} \geq \int_{\mathbb{R}^N} a|u|^qdx.
\]

Next, we fix \( u \in X \setminus \{0\} \). Suppose that \( \int_{\mathbb{R}^N} a|u|^qdx \leq 0 \). Then \( m_u(t) = \int_{\mathbb{R}^N} a|u|^q \) has unique solution \( t^- > t_{\max,\lambda} \) and \( m'_u(t^-) < 0 \). Hence \( h_u \) has a unique turning point at \( t = t^- \) and \( h''(t^-) < 0 \). Thus \( t^-u \in \mathcal{N}_{\lambda}^- \) and (2.12) holds.

Suppose \( \int_{\mathbb{R}^N} a|u|^qdx > 0 \). Since \( m_u(t_{\max,\lambda}) > \int_{\mathbb{R}^N} a|u|^qdx \), the equation \( m_u(t) = \int_{\mathbb{R}^N} a|u|^q \) has exactly two solutions \( 0 < t^+ < t_{\max,\lambda}(u) < t^- \) such that \( m'_u(t^+) > 0 \) and \( m'_u(t^-) < 0 \). Hence, there are two multiplies of \( u \) lying in \( \mathcal{N}_{\lambda}^+ \), that is, \( t^+u \in \mathcal{N}_{\lambda}^+ \) and \( t^-u \in \mathcal{N}_{\lambda}^- \).
and \( t^{-1} u \in \mathcal{N}_{\lambda}^- \). Thus \( h_u \) has turning points at \( t = t^+ \) and \( t = t^- \) with \( h''(t^+) < 0 \) and \( h''(t^-) < 0 \). Thus, \( h_u \) is decreasing on \((0, t^+)\), increasing on \((t^-, t^+)\) and decreasing on \((t^+, \infty)\). Hence (2.13) holds.

By Lemma 2.6, we can observe that \( \mathcal{N}_{\lambda}^+ \) and \( \mathcal{N}_{\lambda}^- \) are non-empty. For \( \lambda > \Lambda_0 \), by Lemma 2.5, we can write \( \mathcal{N}_{\lambda} = \mathcal{N}_{\lambda}^+ \cup \mathcal{N}_{\lambda}^- \) and Lemma 2.2, we can define

\[
c_\lambda^+ = \inf_{u \in \mathcal{N}_{\lambda}^+} \Phi_{\lambda}(u) \quad \text{and} \quad c_\lambda^- = \inf_{u \in \mathcal{N}_{\lambda}^-} \Phi_{\lambda}(u).
\]

**Lemma 2.7.** Suppose that the functions \( a, b \) and \( V \) satisfy the conditions (A1), (B2), and (V1) – (V5). Then, for each \( \lambda \geq \Lambda_0 \), there exists \( C_0 \) such that \( c_\lambda^+ < 0 < C_0 < c_\lambda^- \). In particular, \( c_\lambda^+ = \inf_{u \in \mathcal{N}_{\lambda}} \Phi_{\lambda}(u) \).

**Proof.** (1) Let \( u \in \mathcal{N}_{\lambda}^+ \subset \mathcal{N}_{\lambda} \). Then, by (2.5), we have

\[
\int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2\alpha}} \, dx \, dy + \int_{\mathbb{R}^N} V_{\lambda} u^2 \, dx < \frac{p - q}{p - 2} \int_{\Omega} a|u|^q \, dx.
\]

Hence, by (2.1) and Lemma 2.4, we have

\[
\Phi_{\lambda}(u) = \left( \frac{1}{2} - \frac{1}{p} \right) \left( \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2\alpha}} \, dx \, dy + \int_{\mathbb{R}^N} V_{\lambda} u^2 \, dx \right) - \frac{1}{q - 1} \int_{\Omega} a|u|^q \, dx < - \frac{(p - 2)(2 - q)(\mu_0 - 1)}{2pq\mu_0} \|u\|_{\lambda}^2 < 0.
\]

Thus, \( c_\lambda^+ < 0 \).

(2) Let \( u \in \mathcal{N}_{\lambda}^- \). Then, by (2.1), (2.6) and the Sobolev inequality, we have

\[
\frac{(\mu_0 - 1)(2 - q)}{\mu_0(p - q)} \|u\|_{\lambda}^2 \leq 2 - q \left( \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2\alpha}} \, dx \, dy + \int_{\mathbb{R}^N} V_{\lambda} u^2 \, dx \right) \leq \int_{\Omega} b|u|^p \, dx \leq \|b^+\|_{L^\infty} \|\{V^+ < \kappa\}\|^{\frac{2q - p}{2q}} S_{\alpha}^{q-\frac{2q - p}{2q}} \|u\|_\lambda^p
\]

and so

\[
\|u\|_\lambda > \left[ \frac{(\mu_0 - 1)(2 - q)S_{\alpha}^p}{\mu_0(p - q)} \|b^+\|_{L^\infty} \|\{V^+ < \kappa\}\|^{\frac{2q - p}{2q}} S_{\alpha}^{q-\frac{2q - p}{2q}} \right]^{\frac{1}{2q - p}} := \hat{C}_0.
\]

Therefore, by (2.3), we know

\[
\Phi_{\lambda}(u) \geq \left( \frac{1}{2} - \frac{1}{p} \right) \frac{\mu_0 - 1}{\mu_0} \|u\|_{\lambda}^2 - \left( \frac{1}{q} - \frac{1}{p} \right) \|a^+ \|_{L^{q^*}(\mathbb{R}^N) \|\{V^+ < \kappa\}\|^{\frac{2q - p}{2q}} S_{\alpha}^{q}} \|u\|_{\lambda}^q.
\]

Thus, if \( \lambda \geq \Lambda_0 \), then \( c_\lambda^- > \hat{C}_0 \) for some \( \hat{C}_0 \). This completes the proof. \( \square \)
3. Proof of Theorem 1.1. In this section, we prove Theorem 1.1 by variational methods. We establish the existence of a local minimum for $\Phi_\lambda$ on $N_\lambda^+$.

**Theorem 3.1.** Suppose that $2 < p < 2_\alpha^*$ and the functions $a$, $b$ and $V$ satisfy the conditions (A1), (B1) and (V1) – (V5). Then, for each $\lambda \geq \Lambda_0$, the functional $\Phi_\lambda$ has a minimizer $u_\lambda^+$ in $N_\lambda^+$ satisfying that

1. $\Phi_\lambda(u_\lambda^+) = c_\lambda^+ = \inf_{u \in N_\lambda^+} \Phi_\lambda(u)$;
2. $u_\lambda^+$ is a non-trivial non-negative solution of (1.1).

**Proof.** By Lemma 2.7 and the Ekeland variational principle [15], there exists $\{u_n\} \subset N_\lambda^+$ such that it is a $(PS)_{c_\lambda^+}$-sequence for $\Phi_\lambda$. Moreover, $\{u_n\}$ is bounded in $X$ by Lemma 2.2. Therefore, there exists a subsequence of $\{u_n\}$ (we still denote as $\{u_n\}$) and $u_\lambda^+ \in X$ such that $u_n \rightharpoonup u$ in $X$ and $u_n \rightarrow u$ in $L^r_{loc}(\mathbb{R}^N)$ with $2 \leq r < 2_\alpha^*$. Moreover, $\Phi_\lambda'(u_\lambda^+) = 0$.

Next, we show that $u_\lambda^+ \neq 0$. Suppose the contrary, then by (2.1), the condition (A1), the Egoroff theorem and the Hölder inequality, we have

$$\int_{\mathbb{R}^N} a|u_n|^q \, dx \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty,$$

which implies that

$$\int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^2}{|x - y|^{N + 2\alpha}} \, dx \, dy + \int_{\mathbb{R}^N} V_\lambda u_n^2 \, dx = \int_{\mathbb{R}^N} b|u_n|^p \, dx + o(1)$$

and

$$\Phi_\lambda(u_n) = \frac{1}{2} \left( \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^2}{|x - y|^{N + 2\alpha}} \, dx \, dy + \int_{\mathbb{R}^N} V_\lambda u_n^2 \, dx \right)$$

$$- \frac{1}{q} \int_{\mathbb{R}^N} a|u_n|^q \, dx - \frac{1}{p} \int_{\mathbb{R}^N} b|u_n|^p \, dx$$

$$\geq \frac{(p - 2)(\mu_0 - 1)}{2p\mu_0} ||u_n||_2^2 + o(1),$$

this contradicts $\lim_{n \rightarrow \infty} \Phi_\lambda(u_n) = c_\lambda^+ < 0$. Hence $\int_{\mathbb{R}^N} a|u_n| \, dx \neq 0$. In particular, $u_\lambda^+$ is a nontrivial solution of equation (1.1).

Now, we prove that $u_n \rightarrow u_\lambda^+$ in $X$. Suppose the contrary, then by (2.1),

$$\int_{\mathbb{R}^N} \frac{|u_\lambda^+(x) - u_\lambda^+(y)|^2}{|x - y|^{N + 2\alpha}} \, dx \, dy + \int_{\mathbb{R}^N} V_\lambda(u_\lambda^+)^2 \, dx$$

$$< \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^2}{|x - y|^{N + 2\alpha}} \, dx \, dy + \int_{\mathbb{R}^N} V_\lambda u_n^2 \, dx,$$

and so

$$\inf_{u \in N_\lambda^+} \Phi_\lambda(u) \leq \Phi_\lambda(u_\lambda^+)$$

$$= \left( \frac{1}{2} - \frac{1}{p} \right) \left( \int_{\mathbb{R}^N} \frac{|u_\lambda^+(x) - u_\lambda^+(y)|^2}{|x - y|^{N + 2\alpha}} \, dx \, dy + \int_{\mathbb{R}^N} V_\lambda(u_\lambda^+)^2 \, dx \right)$$

$$- \left( \frac{1}{q} - \frac{1}{p} \right) \int_{\mathbb{R}^N} a|u_\lambda^+|^q \, dx$$

$$< \lim_{n \rightarrow \infty} \Phi_\lambda(u_n) = c_\lambda^+,$$
which contradicts \( \inf_{u \in \mathcal{N}} \Phi_\lambda(u) = c_1^+ \). Therefore, \( u_n \to u_\lambda^+ \) in \( X \) and \( \Phi_\lambda(u_\lambda^+) = c_1^+ \). Moreover, \( \Phi_\lambda(u_\lambda^+) \leq \Phi_\lambda(u_\lambda) \) (see (A.11) in [25]) and \( |u_\lambda^+| \in \mathcal{N}_\lambda^+ \), by Lemma 2.3, we may assume \( u_\lambda^+ \) is a non-trivial non-negative solution of (1.1).

**Proof of Theorem 1.1.** Theorem 1.1 is a direct conclusion of Theorem 3.1.

4. **Proof of Theorem 1.2.** This section is devoted to proof of Theorem 1.2. We begin this section by proving the following proposition provides a precise description for the (PS)-sequence of \( \Phi_\lambda \).

**Proposition 4.1.** Suppose that \( 2 < p < 2^*_\alpha \) and the functions \( a, b \) and \( V \) satisfy the conditions (A1), (B1) and (V1) \(-\) (V2). Then for each \( \beta > 0 \) there exits \( \Lambda_0 = \Lambda(\beta) > 0 \) such that \( \Phi_\lambda \) satisfies the \((PS)_c\)-condition in \( X \) for all \( c < \beta \) and \( \lambda > \Lambda_0 \).

**Proof.** Let \( \{u_n\} \) be a \((PS)_c\)-sequence with \( c < \beta \). By Lemma 2.2, there exists constant \( C(\lambda) \) such that \( \|u_n\|_\lambda \leq C(\lambda) \). Hence, there has a subsequence which still denoted as \( \{u_n\} \) and \( u_0 \in X \) such that \( u_n \to u_0 \) in \( X \) and \( u_n \to u_0 \) in \( L^r_{loc}(\mathbb{R}^N) \) with \( 2 \leq r < 2^*_\alpha \). Moreover, \( \Phi_\lambda(u_0) = 0 \). Then, by (A1), we have

\[
\int_{\mathbb{R}^N} a|u_n|^q dx \to \int_{\mathbb{R}^N} a|u_0|^q dx. \tag{4.1}
\]

Next, we claim that \( u_n \to u_0 \) in \( X \). In fact, let \( v_n = u_n - u_0 \). Using (V2), we have

\[
\int_{\mathbb{R}^N} v_n^2 dx = \int_{\{v \geq \kappa\}} v_n^2 dx + \int_{\{v < \kappa\}} v_n^2 dx \leq \frac{1}{\lambda \kappa} \|v_n\|_\lambda^2 + o(1).
\]

Then, by the Hölder and Sobolev inequalities, we have

\[
\int_{\mathbb{R}^N} |v_n|^p dx \leq \left( \int_{\mathbb{R}^N} |v_n|^2 dx \right)^{\frac{2p}{2p-2}} \left( \int_{\mathbb{R}^N} |v_n|^{2^*_\alpha} dx \right)^{\frac{2^*_\alpha - 2}{2^*_\alpha}} \\
\leq \left( \frac{1}{\lambda \kappa} \|v_n\|_\lambda^2 \right)^{\frac{2^*_\alpha - 2}{2^*_\alpha}} \left( S_{\alpha} \int_{\mathbb{R}^{2N}} |u(x) - u(y)|^{2^*_\alpha} \frac{dx dy}{|x-y|^{N+2\alpha}} \right)^{\frac{2^*_\alpha - 2}{2^*_\alpha}} + o(1) \\
\leq \left( \frac{1}{\lambda \kappa} \right)^{\frac{2^*_\alpha - 2}{2^*_\alpha}} \left( S_{\alpha} \int_{\mathbb{R}^N} |v_n|^p dx \right)^{\frac{2^*_\alpha - 2}{2^*_\alpha}} + o(1). \tag{4.2}
\]

By (A1), (B1) and the Brezis-Lieb Lemma, we have

\[
\Phi_\lambda(v_n) = \Phi_\lambda(u_n) - \Phi_\lambda(u_0) + o(1) \quad \text{and} \quad \Phi'_\lambda(v_n) = o(1).
\]

Hence, this together with (4.1) and Lemma 2.1, we have

\[
\left( \frac{1}{2} - \frac{1}{p} \right) \int_{\mathbb{R}^N} b|v_n|^p dx = \left( \frac{1}{2} - \frac{1}{p} \right) \int_{\mathbb{R}^N} b|v_n|^p dx - \left( \frac{1}{q} - \frac{1}{2} \right) \int_{\mathbb{R}^N} a|v_n|^q dx + o(1) \\
= \left( \frac{1}{2} - \frac{1}{p} \right) \int_{\mathbb{R}^N} b|v_n|^p dx - \frac{1}{2} \langle \Phi'_\lambda(v_n), v_n \rangle + o(1) \\
= c - \Phi_\lambda(u_0) + o(1) \\
\leq \beta + C_0 + o(1),
\]
Thus, there exists $\bar{\Lambda}$ where

\[
(1) \quad \Phi(\bar{\Lambda}) \leq \Phi(\lambda).
\]

and it satisfies $\Phi(\lambda) < c^-$. Moreover, $\Phi(\lambda) \leq \Phi(\bar{\Lambda})$ (see (A.11) in [25]) and $|u^\lambda| \in \mathcal{N}_\lambda^+$, by Lemma 2.3, we may assume $u^\lambda$ is a non-trivial non-negative solution of (1.1). \qed
Proof of Theorem 1.2. By Theorems 3.1 and 4.1 and Lemma 2.7, equation (1.1) has two non-trivial non-negative solution \( u_\lambda^+ \) and \( u_\lambda^- \) such that \( u_\lambda^+ \in \mathcal{N}_\lambda^+ \) and \( u_\lambda^- \in \mathcal{N}_\lambda^- \) with
\[
\Phi_\lambda(u_\lambda^+) = c_\lambda^+ < C_0 < \Phi_\lambda(u_\lambda^-) = c_\lambda^-.
\]
This completes the proof of Theorem 1.2. \( \square \)

5. Proof of Theorem 1.3. In this section we consider the concentration of solutions and give the proof of Theorem 1.3.

Proof of Theorem 1.3. We follow the argument in [5] (see also [27]). For any sequence \( \lambda_n \to \infty \), let \( u_n^\pm := u_{\lambda_n}^\pm \) be the critical points of \( \Phi_{\lambda_n} \) obtained in Theorem 1.2.

Since
\[
\beta_0 \geq c_\lambda^+ = \Phi_\lambda(u_\lambda^+)
\]
\[
\geq \frac{p - 2}{2p} \| u_n^\pm \|_{L_\infty}^2 - \frac{p - q}{pq} \| V^+ < \kappa \|^{\frac{2(p - q - p)}{pq}} \| \alpha_p \|_{L^\infty}^2 \| u_n^\pm \|_{L^\infty}^q,
\]
then
\[
\| u_n^\pm \|_{L^\infty} \leq C_0,
\]
where the constant \( C_0 \) is independent of \( \lambda_n \). Therefore, we may assume that \( u_n^\pm \to u_0^\pm \) in \( X \) and \( u_{\lambda_n}^\pm \to u_0^\pm \) in \( L^r_{\text{loc}}(\mathbb{R}^N) \) for \( 2 \leq r < 2\alpha^* \). By Fatou’s lemma, we have
\[
\int_{\mathbb{R}^N} V^+(u_0^\pm)^2 dx \leq \liminf_{n \to \infty} \int_{\mathbb{R}^N} V^+(u_n^\pm)^2 dx \leq \liminf_{n \to \infty} \frac{\| u_n^\pm \|_{L_\infty}^2}{n} = 0,
\]
then \( u_0^\pm = 0 \) a.e. in \( \mathbb{R}^N \setminus \bar{\Omega} \) and \( u_0^\pm \in X_0^\alpha(\Omega) \) by (V3), where \( X_0^\alpha(\Omega) = \{ u \in H^\alpha(\mathbb{R}^N) : u = 0 \) a.e. in \( \mathbb{R}^N \setminus \Omega \}. Since \( \langle \Phi'(u_n^\pm), \phi \rangle = 0 \) for any \( \phi \in C_0^\infty(\Omega) \), it is easy check that
\[
\int_{\mathbb{R}^N} (u_0^\pm(x) - u_0^\pm(y))(\phi(x) - \phi(y)) dx dy - \int_{\mathbb{R}^N} V u_0^\pm \phi dx
\]
\[
= \int_{\mathbb{R}^N} a|u_0^\pm|^{p-2} u_0^\pm \phi dx + \int_{\mathbb{R}^N} |u_0^\pm|^{p-2} u_0^\pm \phi dx,
\]
that is, \( u_0^\pm \) is a weak solution of equation (1.3) by the density of \( C_0^\infty(\Omega) \) in \( X_0^\alpha(\Omega) \).

Next, we show that \( u_{\lambda_n}^\pm \to u_0^\pm \) in \( L^r(\mathbb{R}^N) \) for \( 2 < r < 2\alpha^* \). Otherwise, by Lions vanishing lemma [17], there exists \( \delta > 0, \rho > 0 \) and \( x_n \in \mathbb{R}^N \) such that
\[
\int_{B_\rho(x_n)} (u_n^\pm - u_0^\pm)^2 dx \geq \delta.
\]
Moreover, let \( |x_n| \to \infty \), then \( |B_\rho(x_n) \cap \{ V < \kappa \}| \to 0 \). By the Hölder inequality, we have
\[
\int_{B_\rho(x_n) \cap \{ V < \kappa \}} (u_n^\pm - u_0^\pm)^2 dx \to 0.
\]
Therefore,
\[
\|u_n^+\|^2_{\lambda_n} \geq \lambda_n \int_{B_{\rho}(x_n) \cap \{V \geq \kappa\}} (u_n^+)^2 \, dx
\]
\[
= \lambda_n \left( \int_{B_{\rho}(x_n) \cap \{V \geq \kappa\}} (u_n^+ - u_0^+)^2 \, dx + \int_{B_{\rho}(x_n) \cap \{V \geq \kappa\}} (u_0^+)^2 \, dx \right) + o(1)
\]
\[
\geq \lambda_n \left( \int_{B_{\rho}(x_n)} (u_n^+ - u_0^+)^2 \, dx - \int_{B_{\rho}(x_n) \cap \{V < \kappa\}} (u_0^+)^2 \, dx \right) + o(1) 
\rightarrow 0,
\]
which contradicts (5.1). Moreover, by assumption (A1), the Hölder inequality and \(u_n^+ \to u_0^+\) in \(L^p(\mathbb{R}^N)\),
\[
\int_{\mathbb{R}^N} a|u_n^+|^q \, dx \to \int_{\mathbb{R}^N} a|u_0^+|^q \, dx. \tag{5.2}
\]
Now, show that \(u_n^+ \to u_0^+\) in \(X\). Since
\[
\langle \Phi'(u_n^+), u_n^+ \rangle = \langle \Phi'(u_0^+), u_0^+ \rangle = 0,
\]
then we have
\[
\|u_n\|^2_{\lambda_n} = \int_{\mathbb{R}^N} a|u_n^+|^q \, dx + \int_{\mathbb{R}^N} b|u_n^+|^p \, dx \tag{5.3}
\]
and
\[
\langle u_n^+, u_0^+ \rangle_{\lambda_n} = \int_{\mathbb{R}^N} a|u_n^+|^{q-2} u_n^+ u_0^+ \, dx + \int_{\mathbb{R}^N} b|u_n^+|^{p-2} u_n^+ u_0^+ \, dx \tag{5.4}
\]
By (5.2)–(5.4), we have
\[
\lim_{n \to \infty} \|u_n^+\|^2_{\lambda_n} = \lim_{n \to \infty} \langle u_n^+, u_0^+ \rangle_{\lambda_n} = \lim_{n \to \infty} \|u_n^+\|_{\lambda_n}^2 = \|u_0^+\|^2.
\]
On the other hand, the weakly lower semi-continuity of norm yields
\[
\|u_0^+\|^2 \leq \liminf_{n \to \infty} \|u_n^+\|^2 \leq \liminf_{n \to \infty} \|u_n^+\|^2_{\lambda_n},
\]
then \(u_n^+ \to u_0^+\) in \(X\). By (2.6) and the fact \(u_n^- \neq 0\), we have
\[
\|u_n^-\|^2 \leq \|u_n^+\|^2_{\lambda_n} < \frac{p-q}{2-q} \|b^+\|_{L^\infty} \int_{\mathbb{R}^N} |u_n^-|^q \, dx \leq \frac{p-q}{2-q} \|b^+\|_{L^\infty} C \|u_n^-\|^p,
\]
for \(n\) large enough, which implies \(u_0^- \neq 0\).

To proceed, we consider the following two case:

Case I \((a \leq 0 \text{ on } \Omega)\): by (2.5),
\[
\|u_n^+\|^2 \leq \|u_n^+\|^2_{\lambda_n} < \frac{p-q}{p-2} \int_{\mathbb{R}^N} a|u_n^+|^q \, dx.
\]
This implies that
\[
\|u_0^+\|^2 \leq \frac{p-q}{p-2} \int_{\mathbb{R}^N} a|u_0^+|^q \, dx \leq 0,
\]
and thus \(u_0^+ \equiv 0\).

Case II \((a > 0 \text{ on } \Omega)\): we may choose \(\psi \in C_0^\infty(\Omega_a)\) with
\[
\int_{\mathbb{R}^{2N}} \frac{|\psi(x) - \psi(y)|^2}{|x-y|^{N+2\alpha}} \, dx \, dy - \int_{\Omega_a} V^{-\psi^2} \, dx > 0.
\]
such that
\[ h_\psi(t) = \Phi_{\lambda_n}(t\psi) \]
\[ = \frac{t^2}{2} \left( \int_{\mathbb{R}^N} \frac{|\psi(x) - \psi(y)|^2}{|x - y|^{N+2\alpha}} \, dx \, dy - \int_{\Omega_n} V^{-\psi^2} \, dx \right) 
- \frac{t^q}{q} \int_{\Omega_n} a|\psi|^q \, dx 
- \frac{t^p}{p} \int_{\Omega_n} b|\psi|^p \, dx \]

have \( t_0^+ > 0 \) and \( \gamma_0 < 0 \) are independent of \( n \) such that \( t_0^+ \psi \in \mathcal{N}_{\lambda_n}^+ \) for all \( n > 0 \) and
\[ \inf_{0 < t < t_0^+} h_\psi(t) = h_\psi(t_0^+) = \gamma_0 < 0. \]

This implies that \( \Phi_{\lambda_n}(u_0^+) = c_{\lambda_n}^+ \leq \gamma_0. \) Therefore,
\[ \frac{1}{2} \left( \int_{\mathbb{R}^N} \frac{|u_0^+(x) - u_0^+(y)|^2}{|x - y|^{N+2\alpha}} \, dx \, dy - \int_{\mathbb{R}^N} V^{-u_0^+(y)^2} \, dx \right) 
- \frac{1}{q} \int_{\mathbb{R}^N} a|u_0^+|^q \, dx 
- \frac{1}{p} \int_{\mathbb{R}^N} b|u_0^+|^p \, dx \]
\[ \leq \gamma_0 < 0, \]

which implies that \( u_0^+ \neq 0. \)

We complete the proof by showing that \( u_0^+ \) and \( u_0^- \) are distinct. Since \( \Phi_{\lambda_n}(u_0^+) = c_{\lambda_n}^+ < 0 \) and \( \Phi_{\lambda_n}(u_0^-) = c_{\lambda_n}^+ > C_0 > 0, \) then we have \( u_0^+ \neq u_0^- \). This completes the proof.

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