Exotic nonlinear supersymmetry and integrable systems

Mikhail S. Plyushchay

Departamento de Física, Universidad de Santiago de Chile, Casilla 307, Santiago, Chile
E-mail: mikhail.plyushchay@usach.cl

Abstract

Peculiar properties of many classical and quantum systems can be related to, or derived from those of a free particle. In this way we explain the appearance and peculiarities of the exotic nonlinear Poincaré supersymmetry in reflectionless and finite-gap quantum systems related to the Korteweg-de Vries equation. The same approach is used to explain the origin and the nature of nonlinear symmetries in the perfectly invisible \( PT \)-regularized conformal and superconformal mechanics systems.

Peculiar properties of many classical and quantum systems can be related to, or derived from those of a free particle. The peculiarity of the simplest 1D free particle system is that it has a local integral of motion \( p = -i \frac{d}{dx} \), which separates the left- and right-moving plane waves \( e^{\pm ikx} \) of the same energy, and distinguishes the unique non-degenerate state \( \psi_0 = 1 \) of zero energy at the very edge of the continuous spectrum by annihilating it. This simplest system also is characterized by the \( sl(2,R) \) conformal symmetry that expands up to the Schrödinger symmetry due to the presence of the integral \( p \) [1].

The interesting class of the quantum systems intimately related to the simplest case of the one-dimensional free particle corresponds to reflectionless systems [2]. Applying a generalized Darboux transformation [3] of order \( n \geq 1 \) to the quantum 1D free particle \( H_0 \), a quantum reflectionless system \( H_n \) is generated. Each energy level in the continuous part of the spectrum of \( H_n \) corresponds to a deformed plane wave \( \Psi_{\pm k}(x) = A^- e^{\pm ikx} \) propagating to the left or to the right. The generator of the Darboux transformation \( A^- \) is a differential operator of order \( n \) based here on an appropriately chosen set of the seed states \( \psi_j^s(x), j = 1, \ldots, n \), which are formal (non-physical) eigenstates of \( H_0 \), \( \ker A^- = \text{span} \{ \psi_1^s, \ldots, \psi_n^s \} \).

Reflectionless system \( H_n \) has then \( n \) bound states \( \Psi_j(x) = A^- \tilde{\psi}_j^s, j = 1, \ldots, n \), generated from linearly independent formal eigenstates \( \tilde{\psi}_j^s(x) = \psi_j^s(x) \int d\xi / (\psi_j^s(\xi))^2 \) of \( H_0 \) of the same eigenvalues as the states \( \psi_j^s \) [4]. A non-degenerate state \( \Psi_0(x) = A^- 1 \) at the very edge of the continuos part of the spectrum is a Darboux-transformed non-degenerate state \( \psi_0 = 1 \) of the free particle. All these spectral peculiarities of reflectionless system \( H_n \) are detected by a nontrivial integral of motion \( \mathcal{P} = A^- p A^+, A^+ = (A^-)^\dagger \), which is a differential operator of the odd order \( 2n + 1 \) being a Darboux-dressed momentum operator of the free particle. The operator \( \mathcal{P} \) annihilates all the \( n \) bound states as well as the lowest state \( \Psi_0(x) \) in the continuos part of the spectrum of \( H_n \), and separates the left- and right-moving deformed plane waves \( \Psi_{\pm k}(x) \) of equal energy being eigenstates of the \( \mathcal{P} \) of opposite eigenvalues [5].
Potentials of the quantum reflectionless systems can be promoted to multi-soliton solutions of the Korteweg-de Vries (KdV) equation by exploiting the covariance of its Lax representation with respect to the Darboux transformations [3, 6]. In this way, potential of any reflectionless system with \( n \) bound states represents a snapshot of an \( n \)-soliton solution to the KdV equation, whose temporal evolution corresponds to an isospectral deformation of the reflectionless system. Operator \( \mathcal{P} \) in such a picture is a Lax-Novikov integral of the \( n \)-th stationary equation of the KdV hierarchy [5].

By periodization of reflectionless systems, some finite-gap quantum systems can be obtained, whose potentials are solutions of the stationary equations of the KdV hierarchy. Lax-Novikov integral of an \( n \)-gap quantum system, being differential operator of order \( 2n + 1 \), separates the left- and right-moving Bloch states of the same energy inside the valence and conduction bands, and annihilates all the \( 2n + 1 \) periodic and anti-periodic edge states at the edges of the bands, on which two irreducible non-unitary finite-dimensional representations of the conformal \( \mathfrak{sl}(2, \mathbb{R}) \) algebra are realized [7]. The Darboux covariance of the Lax representation allows to promote the potentials of finite-gap quantum systems to the cnoidal-type solutions of the KdV equation [6].

Darboux transformations also can be applied to the finite-gap systems of the most general form to produce some finite-gap systems completely isospectral to the initial ones, or to generate finite-gap systems with the added arbitrary number of the bound states inside the prohibited zones or at their edges. In the latter case the generated potentials and related to them super-potentials are promoted to solutions of the KdV and modified KdV equations in the form of the soliton defects propagating in a finite-gap background [6].

With the pairs of the quantum systems produced starting from the quantum 1D free particle or a finite-gap system, an exotic nonlinear \( \mathcal{N} = 4 \) supersymmetry can be associated. The emergence of the exotic supersymmetry in such systems is rooted in existence of the momentum integral in the free particle, or the Lax-Novikov integral in a finite-gap system. In the simplest case of a reflectionless system, due to the presence of \( p \) in the structure of the Lax-Novikov integral \( \mathcal{P} \), the latter can be factorized into the product of two non-singular operators, \( \mathcal{P} = \mathcal{A}^{-}(p\mathcal{A}^{+}) \). In correspondence with this, the given reflectionless system \( H_{n} \), can be generated from and intertwined with the free particle \( H_{0} \) not only by the order \( n \) differential operators \( \mathcal{A}^{-} \) and \( \mathcal{A}^{+} \), but also by the operators \( \mathcal{A}^{-}p \) and \( p\mathcal{A}^{+} \) of differential order \( n + 1 \). As a consequence, the extended system \( \mathcal{H} \) composed from \( H_{0} \) and \( H_{n} \) has not only a pair of supercharges \( Q_{a} \), \( a = 1, 2 \), constructed from the operators \( \mathcal{A}^{-} \) and \( \mathcal{A}^{+} \), but also possesses a pair of supercharges \( S_{a} \) of differential order \( n + 1 \) constructed from the intertwining operators \( \mathcal{A}^{-}p \) and \( p\mathcal{A}^{+} \). The anti-commutators of the supercharges \( Q_{a} \) and \( Q_{b} \) produce a polynomial of order \( n \) in \( \mathcal{H} \), while \( S_{a} \) and \( S_{b} \) anti-commute for a polynomial of order \( n + 1 \) in \( \mathcal{H} \). The anti-commutator of \( Q_{a} \) and \( S_{b} \) gives rise to an additional even generator \( \mathcal{L} \) of the superalgebra composed from \( pH_{0}^{n} \) and Lax-Novikov integral \( \mathcal{P} \). As a result, instead of the \( \mathcal{N} = 2 \) (nonlinear in the case of \( n > 1 \)) Poincaré supersymmetry generated by two supercharges and Hamiltonian \( \mathcal{H} \), we obtain an exotic nonlinear \( \mathcal{N} = 4 \) Poincaré supersymmetry which includes an additional bosonic integral \( \mathcal{L} \). Analogous exotic nonlinear \( \mathcal{N} = 4 \) Poincaré supersymmetric structure describes extended systems \( \mathcal{H} \) composed from isospectral, or almost isospectral pairs of reflectionless systems with multi-soliton potentials \( u_{n}(x, \ldots) \) and \( u_{n'}(x, \ldots) \), where the ellipsis corresponds to the sets of \( 2n \) and \( 2n' \) parameters characterizing the amplitudes and phases of the \( n \)- and \( n' \)-soliton solutions of the KdV equation. The concrete form of the superalgebra depends on the choice of those parameters, and its supercharges undergo some restructuring associated with lowering their differential
orders each time when some sets of the amplitude parameters in \( H_n \) coincide with those in super-partner reflectionless system \( H_{n'} \) \([5]\). Additional restructuring in supercharges and exotic nonlinear superalgebra generated by them can also happen for special values of the phase differences associated with the coinciding pairs of the soliton amplitudes in potentials \( u_n(x, \ldots) \) and \( u_{n'}(x, \ldots) \) \([5]\). In all the cases, however, a pair of supercharges are matrix operators of some even differential order, while another pair of supercharges has an odd differential order. This is related to the nature of the Lax-Novikov integrals of the subsystems \( H_n \) and \( H_{n'} \), which have odd differential orders \( 2n + 1 \) and \( 2n' + 1 \), and that supercharges from different pairs effectively provide the factorization of Lax-Novikov integrals into two non-singular differential operators. For such extended quantum systems \( \mathcal{H} \), the phenomenon of transmutation between the exact and partially broken exotic nonlinear supersymmetries was observed and interpreted in terms of the soliton scattering in \([8]\). In the case of the unbroken supersymmetry, the unique ground state of the system \( \mathcal{H} \) is a zero mode of all the odd generators \( Q_a \) and \( S_a \) and of the even generators \( \mathcal{H} \) and \( \mathcal{L} \) of the superalgebra. Coherently with this, the operator \( \mathcal{L} \) is a central element of the exotic nonlinear \( \mathcal{N} = 4 \) Poincaré superalgebra. In the phase of the partially broken exotic nonlinear supersymmetry, the even generator \( \mathcal{L} \) mutually transforms the pairs of the supercharges \( Q_a \) and \( S_a \) by means of the commutator, and the system \( \mathcal{H} \) has a doubly degenerate lowest energy level, whose corresponding states are annihilated by a part of the supercharges \([5, 8]\).

A similar exotic nonlinear \( \mathcal{N} = 4 \) Poincaré supersymmetric structure also describes extended systems \( \mathcal{H} \) composed from isospectral pairs of the finite-gap quantum systems, and finite-gap systems with soliton defects. In the latter case, the fine structure of the exotic supersymmetry controls the nature and propagation of soliton defects with energies which can be introduced into different prohibited zones of the finite-gap systems \([6]\).

The KdV equation has also rational solutions, in which the dynamics of the moving poles is governed by the Calogero-Moser systems. Such solutions can be obtained via an appropriate limit procedure from multi-soliton solutions by exploiting the Galilean symmetry of the KdV equation. They also can be obtained directly from the free particle by applying to it singular generalized Darboux transformations based on zero-energy eigenstates \( \psi_0 = 1 \) and \( \psi_0(x) = x \) of the free particle and Jordan states corresponding to the same zero energy \([9]\). The simplest case of the Darboux-generated in this way system corresponds to the two-particle Calogero model with the omitted center of mass coordinate. Due to a singular nature of the Darboux transformation, the Schrödinger symmetry of a free particle reduces and transforms into conformal \( \mathfrak{sl}(2, \mathbb{R}) \) symmetry of the generated Calogero system with a non-degenerate continuous spectrum \((0, \infty)\). Though the Darboux-dressed momentum operator \( \mathcal{P} = \mathcal{A}^- p \mathcal{A}^+ \) in this case commutes with the generated Hamiltonian \( H_n \), it is a formal, non-physical integral of motion since acting on non-degenerate eigenstates of \( H_n \) it transforms them into non-physical, formal eigenstates of \( H_n \) which do not satisfy the Dirichlet boundary condition at \( x = 0 \) \([10]\). Coherently with a non-physical nature of the formal Lax-Novikov integral, the corresponding extended system \( \mathcal{H} \) is described by (a non-linear in general case) \( \mathcal{N} = 2 \) Poincaré supersymmetry generated by supercharges \( Q_a \) constructed from the operators \( \mathcal{A}^- \) and \( \mathcal{A}^+ \), but the exotic \( \mathcal{N} = 4 \) Poincaré supersymmetry is lost.

Then the natural question arises whether it is possible to somehow restore the exotic nonlinear \( \mathcal{N} = 4 \) Poincaré supersymmetry in the quantum systems associated with rational solutions of the KdV equation.

In \([9]\) it was recently shown that this indeed can be achieved via the \( \mathcal{PT} \)-regularization \( x \to x + i \alpha, \alpha \in \mathbb{R}, \alpha \neq 0 \), of Darboux transformations. The key point of such a com-
plex shift is that it allows to recuperate the Schrödinger symmetry in a Darboux-generated system, where, however, a higher derivative Lax-Novikov integral expands its algebra and transforms into a non-linear one. The obtained in such a way systems possess several interesting properties. They are not only refectionless, but are perfectly invisible since in them the transmission amplitude itself, and not only its modulus, is equal to one. Another peculiarity is that each of them contains a unique bound state of zero energy at the very edge of the continuous part of the spectrum, which is described by a quadratically integrable wave function, and in this sense they are zero-gap quantum systems. The paired perfectly invisible systems are described by different forms of the nonlinearly extended generalized super-Schrödinger symmetry, which can include or not include the superconformal \( \mathfrak{osp}(2,2) \) symmetry in the form of a sub-superalgebra depending on the unbroken or partially broken phase of the exotic nonlinear \( N = 4 \) Poincaré supersymmetry in them. The potentials of such perfectly invisible \( \mathcal{PT} \)-invariant quantum systems can be promoted to the solutions of the complexified KdV equation (or higher equations of the hierarchy), which exhibit, particularly, a behaviour typical for extreme (rogue) waves.

The simplest system with the unbroken exotic \( N = 4 \) nonlinear supersymmetry is described by the Hamiltonian and supercharges

\[
\mathcal{H} = \begin{pmatrix} H_1^0 & 0 \\ 0 & H_0 \end{pmatrix}, \quad Q_1 = \begin{pmatrix} 0 & D_1 \\ D_1^\# & 0 \end{pmatrix}, \quad S_1 = \begin{pmatrix} 0 & -iD_1P_0 \\ iP_0D_1^\# & 0 \end{pmatrix},
\]

and \( Q_2 = i\sigma_3Q_1, \ S_2 = i\sigma_3S_1. \) Here \( P_0 = p = -i\frac{\partial}{\partial x} \) is the momentum operator of the free particle \( H_0 = -\frac{\partial^2}{\partial x^2}, \ D_1 = \xi \frac{\partial}{\partial x} - \xi^{-1} = \frac{\partial}{\partial x} - \xi^{-1}, \) and \( D_1^\# = -\xi^{-1}\frac{\partial}{\partial x} = -\frac{\partial}{\partial x} - \xi^{-1} \) are constructed on the base of \( \xi = x + i\alpha \) which is a non-physical zero-energy eigenstate of \( H_0. \) Operators \( D_1 \) and \( D_1^\# \) are the Darboux generators \( \mathcal{A}^- \) and \( \mathcal{A}^+ \) for the super-partners \( H_0 \) and \( H_1^\alpha = -\frac{\partial^2}{\partial x^2} + 2\xi^{-2} \). The \( \mathcal{H}, Q_a \) and \( S_a \) generate the non-linear superalgebra \( [\mathcal{H}, Q_a] = [\mathcal{H}, S_a] = 0, \)

\[
\{Q_a, Q_b\} = 2\delta_{ab}\mathcal{H}, \quad \{S_a, S_b\} = 2\delta_{ab}\mathcal{H}^2, \quad \{Q_a, S_b\} = 2\varepsilon_{ab}\mathcal{L}, \]

where

\[
\mathcal{L} = \begin{pmatrix} \mathcal{P}_1^\alpha & 0 \\ 0 & H_0\mathcal{P}_0 \end{pmatrix}
\]

is the bosonic integral of motion being a central charge of this superalgebra. The kernel of the Lax-Novikov integral \( \mathcal{P}_1^\alpha = D_1\mathcal{P}_0D_1^\# \) of the \( \mathcal{PT} \)-regularized two-particle Calogero subsystem \( H_1^\alpha = \ker \mathcal{P}_1^\alpha = \text{span} \{\xi^{-1}, \xi, \xi^3\}. \) Here \( \xi^{-1} \) is the zero-energy bound state of \( H_1^\alpha, \) while \( \xi \) and \( \xi^3 \) are its Jordan states, \( H_1^\alpha\xi = 2\xi^{-1}, \) \( H_1^\alpha\xi^3 = -4\xi. \) The unique ground state of the system \( \mathcal{H} \) of zero-energy \( \Psi_0 = (D_11, 0)^t = (\xi^{-1}, 0)^t \) is annihilated by all the supercharges \( Q_a \) and \( S_a \) as well as by the even generator \( \mathcal{L}. \)

The set of the even operators \( \mathcal{H}, (I - \frac{1}{2}\Sigma), \ K = \text{diag} (K_1^\alpha, K_0^\alpha), \ D = \text{diag} (D_1^\alpha, D_0^\alpha), \) and odd operators \( Q_a \) and \( \lambda_1 = -\xi\sigma_2 - 2tQ_1, \ \lambda_2 = i\sigma_3\lambda_1 \) generate the \( \mathfrak{osp}(2|2) \) superalgebra of the matrix system \( \mathcal{H}. \) Here \( I = \text{diag} (1, 1) \) and \( \Sigma = \sigma_3; \ D_0^\alpha = \frac{1}{4}\{G_0^\alpha, \mathcal{P}_0\} \) and \( K_0^\alpha = (G_0^\alpha)^2 \) are the generators of conformal \( \mathfrak{sl}(2, \mathbb{R}) \) symmetry of \( H_0 \) being its time-dependent, dynamical integrals of motion constructed on the base of its generator of Galileo transformations \( G_0^\alpha = \xi - 2t\mathcal{P}_0, \) while \( D_1^\alpha = \frac{1}{4}\{\xi, \mathcal{P}_0\} - tH_1^\alpha \) and \( K_1^\alpha = \xi^2 - 8tD_1^\alpha - 4t^2H_1^\alpha \) are the analogous \( \mathfrak{sl}(2, \mathbb{R}) \) generators for \( H_1^\alpha. \) Extension of the set of the generators of superconformal \( \mathfrak{osp}(2|2) \) symmetry of the system \( \mathcal{H} \) by the even integral \( \mathcal{L} \) gives rise to the expansion of the set of
the integrals of motion by the set of the even integrals
\[ \Sigma, \quad \mathcal{P}_- = \frac{1}{2} (1 - \sigma_3) \mathcal{P}_0, \quad \mathcal{G}_- = \frac{1}{2} (1 - \sigma_3) \mathcal{G}_0, \quad \mathcal{G} = \operatorname{diag} \left( \mathcal{G}_1, \frac{1}{2} \{ \mathcal{G}_0, H_0 \} \right), \]
and by the second order supercharges \( S_a \) and the odd integrals \( \mu_1 = \frac{1}{2} \{ \xi, \mathcal{P}_0 \} \sigma_1 - \frac{i}{2} \{ \xi, \mathcal{P}_0 \} \sigma_2 - 2 t S_1, \mu_2 = i \sigma_3 \mu_1, \kappa_1 = \xi^2 \sigma_1 - 4 t \mu_1 - 4 t^2 S_1, \) and \( \kappa_2 = i \sigma_3 \kappa_1, \) where \( \mathcal{G}_1 = D_1 \mathcal{G}_0 D_1^\# \) is the Darboux-dressed free particle integral \( \mathcal{G}_0. \) The resulting nonlinear (quadratic) superalgebra is generated by ten even and ten odd integrals of the system \( \mathcal{H} \) including a trivial even central charge \( \mathcal{L}, \) and represents a nonlinearly super-extended Schrödinger algebra with the \( \mathfrak{osp}(2|2) \) sub-superalgebra. The nontrivial bosonic generators \( (\mathcal{L}, \mathcal{H}, \mathcal{G}, \mathcal{P}_-, \Sigma = \sigma_3, \mathcal{D}, \mathcal{V}, \mathcal{G}_-, \mathcal{K}, \mathcal{R}) \) are eigenstates of the dilatation generator \( \mathcal{D}, [\mathcal{D}, \mathcal{O}] = i s_\mathcal{O} \mathcal{O}, \) with the eigenvalues given by \( s_\mathcal{O} = (3/2, 1, 1/2, 1/2, 0.0, -1/2, -1/2, -1, -3/2). \) Analogously, for the fermionic generators \( (S_a, \mathcal{Q}_a, \mu_a, \lambda_a, \kappa_a), \) \( s_\mathcal{O} = (1, 1/2, 0, -1/2, -1). \) The peculiarity of the nonlinear superalgebra, whose explicit form is described in [11], is that the (anti-)commutators of the generators of the \( \mathfrak{osp}(2|2) \) sub-superalgebra with any other generator is linear in generators.

A simple example of the system in the phase of the partially broken phase of the exotic nonlinear \( \mathcal{N} = 4 \) Poincaré supersymmetry is given by the Hamiltonian \( \mathcal{H} = \operatorname{diag} (H_1^{a_1}, H_1^{a_2}) \) composed from two Calogero systems regularized by different complex shifts \( a_1 > a_2. \) The subsystems \( H_1^{a_1} \) and \( H_1^{a_2} \) can be intertwined by the second order differential operators \( D_{a_1} D_{a_2} \) and \( D_{a_2} D_{a_1} \) via the ‘virtual’ free particle system, \( (D_{a_1} D_{a_2}) H_1^{a_2} = H_1^{a_1} (D_{a_2} D_{a_1}) \), \( (D_{a_2} D_{a_1}) H_1^{a_1} = H_1^{a_2} (D_{a_1} D_{a_2}) \). However, there also exists the first order intertwiners, \( D = \frac{d}{dx} + \mathcal{W}, D^# = -\frac{d}{dx} + \mathcal{W}, \) where \( \mathcal{W} = \frac{1}{\xi_1 - \xi_2} - \frac{1}{\xi_1 - \xi_2}, \xi_j = x + i a_j, D H_1^{a_1} = H_1^{a_2} D, \]

\[ D^# H_1^{a_2} = H_1^{a_1} D^#. \]

The supercharges and Lax-Novikov integral of this extended system are
\[ Q_1 = \begin{pmatrix} 0 & D^# \\ D & 0 \end{pmatrix}, \quad S_1 = \begin{pmatrix} 0 & D_{a_1} D_{a_2}^# \\ D_{a_2} D_{a_1}^# & 0 \end{pmatrix}, \]
\[ \mathcal{L}_1 = \begin{pmatrix} \mathcal{P}_{a_2} & 0 \\ 0 & \mathcal{P}_{a_1} \end{pmatrix}, \]
\[ Q_2 = \sigma_3 Q_1, S_2 = \sigma_3 S_1. \] They satisfy nontrivial superalgebraic relations
\[ \{Q_a, Q_b\} = 2 \delta_{ab} (\mathcal{H} - \Delta^2), \quad \{S_a, S_b\} = 2 \delta_{ab} \mathcal{H}^2, \quad \{Q_a, S_b\} = 2 (\epsilon_{ab} \mathcal{L}_1 + i \delta_{ab} \Delta \mathcal{H}). \]

The exotic nonlinear supersymmetry here is in the spontaneously partially broken phase: the doublet of the bound states \( \Psi_0^\pm = (D_{a_2} 1, \pm D_{a_1} 1)^t = (\xi_1^{-1}, \mp \xi_1^{-1})^t \) of zero energy at the very edge of the fourfold degenerate continuous spectrum are not annihilated by the first order supercharges, \( Q_1 \Psi_0^\pm = \pm i \Delta \Psi_0^\pm. \)

In the case of the system \( \mathcal{H} = \operatorname{diag}(H_1^{a_2}, H_1^{a_1}), \) its nonlinear superconformal algebra is more complicated [11]. The numbers of the even and odd generators are the same as in the previous example, but no odd fermionic generator has a definite scaling dimension, i.e. is not an eigenstate of the dilatation operator \( \mathcal{D}. \) As a consequence, the \( \mathfrak{osp}(2|2) \) superalgebra is not contained as a sub-superalgebra in this case.

It is interesting to note that the two simplest \( PT \)-regularized Calogero models \( H_1^\alpha = -\frac{d^2}{dx^2} + \ell (\ell + 1) \xi^{-2} \) with \( \ell = 1, 2 \) control stability properties of the \( PT \)-regularized kinks in the field-theoretical Liouville and \( SU(3) \) conformal Toda systems [9].
Consider now the state $\psi_{a,\gamma}^{(1)} = \gamma \xi^{-1} + \xi^2$, $\xi = x + i\alpha$, $\alpha \in \mathbb{R}$, $\gamma = 12\tau + i\nu\alpha^3$, $\nu \in (1, \infty)$, $\tau \in (-\infty, \infty)$, which is a linear combination of the bound state $\xi^{-1}$ of the system $H_{1}^a = -\frac{d^2}{dx^2} + \frac{2}{\xi^2}$ of zero eigenvalue and of its non-physical partner $\xi^2$ of the same zero energy. Taking it as a seed state for the generalized Darboux transformation, we obtain a superpotential $W_{a,\gamma}^{(1)} = \frac{d}{dx} \left( \ln \psi_{a,\gamma}^{(1)} \right) = -\xi^{-1} + 3\xi^2(\xi^3 + \gamma)^{-1}$, and generate the super-partner systems $H_{\pm} = -\frac{d^2}{dx^2} + V_{\pm}$ given in a usual way by the potentials $V_{\pm} = (W_{a,\gamma}^{(1)})^2 \pm (W_{a,\gamma}^{(1)})'$. This yields $V_+ = 2\xi^{-2}$, i.e. $H_+ = H_{1}^a$, and $H_- = H_{2}^{a,\gamma} = -\frac{d^2}{dx^2} + V_-$, where
\[
V_- = -2 \left( \ln W(\xi, -\gamma + \xi^3) \right)^{''} = \frac{6}{\xi^2} - 6\gamma \frac{4\xi^3 + \gamma}{\xi^2(\xi^3 + \gamma)^2} := V(x; \alpha, \gamma(\tau, \nu)).
\]

The first equality with the Wronskian $W$ means here that the system $H_-$ can also be produced directly from the free particle system by taking as the set of the seed states for the generalized Darboux transformation the non-physical zero-energy eigenstate $\xi$ of the free particle and a linear combination $-\gamma + \xi^3$ of its zero-energy eigenstate $-\gamma$ and its Jordan state $\xi^3$, $H_0\xi^3 = -6\xi$. As a function of $x$ and $\tau$, the potential $V(x; \alpha, \gamma(\tau, \nu))$ obeys the system of the coupled nonlinear equations $u_{\tau} - 6uu_x + u_{xxx} = 0$ being regular function for all values of $x$ and $\tau$. In the case $\alpha = 0$, potential $V$ takes the form of the well known singular rational solution $u(x, \tau) = 6x\frac{x^3 - 24\tau}{(x^3 + 12\tau)^3}$ of the KdV equation. Note also that the potential $V(x; \alpha, \gamma(\tau, \nu))$ as a function of $x$ satisfies simultaneously the higher stationary equation of the KdV hierarchy, $30u^2u_x - 20u_xu_{xxx} - 10wu_{xxx} + u_{xxxxx} = 0$. The real and imaginary parts of the potential $u(x, \tau) = v(x, \tau) + iw(x, \tau)$ obey the system of the coupled nonlinear equations $v_{\tau} - 3(v^2 - w^2)x + v_{xxx} = 0$ and $w_{\tau} - 6iw_x + w_{xxx} = 0$, and represent some two-soliton waves. For appropriately chosen parameters $\alpha$ and $\nu$, they reveal the behaviour typical for extreme (rouge) waves [9], see Figure 1.

![Figure 1: Evolution of real $v(x, \tau)$ (on the left) and imaginary $w(x, \tau)$ (on the right) parts of the potential $V_+(x; \alpha, \gamma(\tau, \nu))$ as a complex $PT$-symmetric solution of the KdV equation at $\alpha = 100$, $\nu = 5$; dashed lines: $\tau = -10^7$, continuous lines: $\tau = 0$, dotted lines: $\tau = 10^7$.](image-url)

In conclusion we note that the rational extensions of the harmonic oscillator, or of the conformal Alfaro, Fubini, Furlan model (AFF) [12, 13, 14] can be constructed from the indicated systems by applying to them dual Darboux transformations with intertwining operators to be differential operators of the even and odd orders [15, 16, 4]. The produced in such a way systems reveal a “finite-gap” structure in their discrete spectra, but the dual Darboux schemes generate from the harmonic oscillator or the AFF model the pairs of the systems described by the Hamiltonians mutually shifted for a nonzero constant. As a consequence, instead of the Lax-Novikov type integrals, in this case nontrivial ladder operators
are generated, which allow to connect finite “valence bands” with equidistant infinite part of the spectrum. Using them, one can construct three pairs of the ladder operators which encode the spectral peculiarities of the system and form a complete spectrum-generating set of the ladder operators. Such rationally extended systems are characterized by nonlinearly deformed extended conformal (Newton-Hooke) symmetry. They also can be related to the free particle via the singular Darboux transformations and by the conformal bridge construction described in a recent paper [17]. Both Darboux and conformal bridge transformations substantially use zero-energy eigenstates and Jordan states corresponding to zero-energy.

Identifying a spatial reflection $R$ as a $\mathbb{Z}_2$-grading operator, the nonlinear $\mathcal{N} = 2$ Poincaré supersymmetry can be revealed in many purely bosonic non-extended quantum systems in the form of the bosonized supersymmetry [18, 19, 20]. In such systems, the Lax-Novikov integrals $\mathcal{P}$ play the role of the local supercharge, while the second supercharge $i\mathcal{R}\mathcal{P}$ is nonlocal due to the nonlocal nature of the reflection operator. Similarly, a hidden superconformal symmetry is identified in the quantum harmonic oscillator system [21].

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