Some results of exact solutions for the two-spin-flip number problem of the spin-1 Heisenberg chain

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Abstract. The eigenvalues and eigenvectors of the spin-1 Heisenberg chain are not exactly solvable by using of the Bethe Ansatz method. As an attempt, we propose a new format of the basic states and derive the exact solutions for the two-spin-flip number problem of the spin-1 Heisenberg chain. We discuss qualitatively some basic features of the above model for different types of wave number values.

1. Introduction

An exactly solved quantum system model means all the eigenvalues and eigenvectors of the model can be analytically solved from the Hamiltonian. It is related to a very important mathematical concept, namely the notion of integrability. If a model is integrable, then this model can be exactly diagonalised by means of some special skills and techniques (e.g. the Bethe Ansatz). The motivation to quest for exact solutions of quantum system model should be enumerated as follows, (i) obtaining the correct information of many-body interacting system and getting a clearer understanding; (ii) clarifying the special features of the low-dimension quantum systems; (iii) combining with numerical methods (numerical diagonalizations, Monte Carlo calculations, renormalization group approaches, etc.) and expending to general cases.

The quantum magnet, also being called a spin chain in one dimension (1D), can be investigated theoretically using a model introduced by Heisenberg in 1928 [1]. The spin-1/2 Heisenberg model was solved exactly by Bethe in 1931, who developed a method now known as the Bethe Ansatz [2]. This method is based on the assumption of plane wave solutions for the wave functions of the systems. The Bethe ansatz is an exact method for the calculation of eigenvalues and eigenvectors of a limited but select class of quantum many-body model systems. Although the eigenvalues and eigenvectors of a finite system may be obtained with less effort from a brute force numerical diagonalization, the Bethe ansatz offers two important advantages: (i) all eigenstates are characterized by a set of quantum numbers which can be used to distinguish them according to specific physical properties; (ii) in many cases the eigenvalues and the physical properties derived from them can be evaluated in the thermodynamic limit [3-6].

There are two higher spin chain models which are integrable, so that are exactly solvable by the Bethe Ansatz. The first integrable model is the Uimin-Lai-Sutherland Hamiltonian, which exhibits SU(3) symmetry [7]. The second integrable model is the Babujan-Takhtajan Hamiltonian, which exhibits SU(2) symmetry. In 1982, Babujan and Takhtajan extended the integrable spin-1/2 model to
higher spin chains by means of a method called Algebraic Bethe Ansatz, and obtain their corresponding integrable Hamiltonians \[8, 9\].

The spin-1 Heisenberg chain is the simplest typical higher spin chain model, unfortunately it is not soluble by using of the Bethe Ansatz method \[10\]. This means the basic states, which construct the wave functions of the model, cannot be expressed by the Bethe Ansatz format for all spin flip cases. However, it is also possible that the other general formats of the basic states exist, making use of which we can acquire the exact solutions for the spin-1 Heisenberg model. In this paper, we propose a new format of the basic states (like the Bethe Ansatz format, but are different) and derive the exact solutions for the two-spin-flip (2-spin-flip) number problem of the spin-1 Heisenberg chain.

2. Exact solutions for the two-spin-flip number problem of the spin-1 chain

2.1. Models
The quantum spins obey SU(2) symmetry. In the spin-1/2 case, the spin operators arising in the Heisenberg Hamiltonian are simply given by the Pauli matrices, \( \hat{S} = (1/2) \hat{\sigma} \). However, in the spin-1 case, the spins could be in three states, up (1 or \( \uparrow \)), zero (0 or \( \rightarrow \)) and down (-1 or \( \downarrow \)), respectively. In order to obtain a description in terms of spin operators, a three dimensional representation of \( SU(2) \) is needed. The components of the spin-1 operators \( \hat{S} \) can be written down as

\[
\hat{S}_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \hat{S}_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \hat{S}_z = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},
\] (1)

obeying the SU(2) canonical commutation relations \[ [\hat{S}^i_n, \hat{S}^j_n] = i\hat{S}^k_n \epsilon^{ijk} \delta_{mn} \]. We define the spin flip operators \( \hat{S}^+ \) and \( \hat{S}^- \) as \( \hat{S}^\pm = \hat{S}^x \pm i\hat{S}^y \). The spin commutation relations (with \( \eta = 1 \)) are

\[
[S^+, S^j] = \pm S^j, \quad [S^+, S^-] = 2S^z.
\] (2)

The Hamiltonian of the Heisenberg model composing of the spins \( S_n = (S_{nx}, S_{ny}, S_{nz}) \) with the quantum number \( s = 1 \) on a \( N \) sites 1D lattice satisfying periodic boundary conditions \( S_{N+1} = S_1 \) is given by

\[
\hat{H} = \sum_{n=1}^{N} S_n \cdot S_{n+1} = \sum_{n=1}^{N} \left(S^+_n S^-_{n+1} + S^-_n S^+_{n+1} + S^z_n S^z_{n+1} \right) = \sum_{n=1}^{N} \left[1/2 \left(S^+_n S^-_{n+1} + S^-_n S^+_{n+1} \right) + S^z_n S^z_{n+1} \right],
\] (3a)

with \( \left[S^+_n, S^z_{n+1}\right] = \pm S^z_{n+1} \delta_{mn}, \quad \left[S^+_n, S^-_{n+1}\right] = 2S^z_{n+1} \delta_{mn} \).

\( \hat{H} \) acts on a Hilbert space with the dimension of \( 2^N \) spanned by the orthogonal basis vectors \( |\sigma_i, K, \sigma_n\rangle \), where \( \sigma_n = \uparrow \) represents an up spin and \( \sigma_n = \downarrow \) a down spin at site \( n \).

The application of the operators \( S^+_n, S^-_n \) on the basis vector \( |\sigma_i, K, \sigma_n\rangle \) yields the results summarized in table 1.

**Table 1.** Rules governing the application of the spin operators on the basis vectors \( |\sigma_i, K, \sigma_n\rangle \) with \( \sigma_n = \uparrow, \rightarrow, \downarrow \).

|          | \( |\uparrow, K\rangle \) | \( |\rightarrow, K\rangle \) | \( |\downarrow, K\rangle \) |
|----------|------------------|------------------|------------------|
| \( \hat{S}^+_n \) | 0                | \( |\uparrow, K\rangle \) | \( |\rightarrow, K\rangle \) |
| \( \hat{S}^-_n \) | \( |\rightarrow, K\rangle \) | 0                | \( |\downarrow, K\rangle \) |
| \( \hat{S}^z_n \) | \( |\rightarrow, K\rangle \) | 0                | \( |\downarrow, K\rangle \) |
2.2. Derivation of exact solutions
Supposing the initial state \( |\text{ini}\rangle = |1\rangle \), we have \( \hat{a}_n^\dagger |n\rangle = |0\rangle_n \), \( \hat{a}_n^\dagger \hat{a}_n |1\rangle_n = |1\rangle_n - |0\rangle_n \), \( \hat{a}_n^\dagger \hat{a}_n |1\rangle_n = 0 \), where \( \hat{a}_n^\dagger \) is a fermionic annihilation operator.

In the case of 2-spin-flip number, the wave functions (eigenvectors) can be expressed by the superposition of the basis states
\[
\Psi_2 = \sum_{i=1}^{N} \sum_{j=1}^{N} \phi(n_1, n_2) \hat{a}_{n_1}^\dagger \hat{a}_{n_2}^\dagger |\text{ini}\rangle^N .
\]

where \( \phi(n_1, n_2) \) are the coefficients. The Schrodinger's equation is
\[
\hat{H} \Psi_2 = E_2 \Psi_2 = E_2 \sum_{i=1}^{N} \sum_{j=1}^{N} \phi(n_1, n_2) \hat{a}_{n_1}^\dagger \hat{a}_{n_2}^\dagger |\text{ini}\rangle^N ,
\]
and the periodic boundary (P.B.) conditions are given by
\[
\phi(n_1, n_2) = \phi(n_2, n_1 + N) .
\]

Substituting the Hamiltonian of equation (3a) for the Schrodinger's equation (5), we obtain the expansion equations in the different cases shown below:

(i) When \( n_2 > n_1 + 1 \), on the P.B. conditions \( \phi(n_1, n_2) = \phi(n_1, n_1 + N) \), we have
\[
\hat{H} \Psi_2 = \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \phi(n_1, n_2) \hat{H}_{i,j} \hat{a}_{n_1}^\dagger \hat{a}_{n_2}^\dagger |\text{ini}\rangle^N ; \quad (7)
\]
\[
\begin{align*}
\hat{H} \Psi_2 &= (J/2) \sum_{i=1}^{N} \sum_{j=1}^{N} \phi(n_1, n_2) \hat{a}_{n_2}^\dagger \hat{a}_{n_1}^\dagger \hat{a}_{n_1}^\dagger \hat{a}_{n_2}^\dagger |\text{ini}\rangle^N + (J/2) \sum_{i=1}^{N} \sum_{j=1}^{N} \phi(n_1, n_2) (N-6) \hat{a}_{n_2}^\dagger \hat{a}_{n_1}^\dagger |\text{ini}\rangle^N .
\end{align*}
\]

(ii) When \( n_2 = n_1 + 1 \), we have
\[
\hat{H} \Psi_2 = (J/2) \sum_{i=1}^{N} \sum_{j=1}^{N} \phi(n_1, n_2) \hat{a}_{n_2}^\dagger \hat{a}_{n_1}^\dagger \hat{a}_{n_1}^\dagger \hat{a}_{n_2}^\dagger |\text{ini}\rangle^N ; \quad (8)
\]

(iii) When \( n_2 = n_1 \), on the P.B. conditions \( \phi(n_1, n_1) = \phi(n_1 + N, n_1 + N) \), we have
\[
\hat{H} \Psi_2 = (J/2) \sum_{i=1}^{N} \sum_{j=1}^{N} \phi(n_1, n_2) \hat{a}_{n_2}^\dagger \hat{a}_{n_1}^\dagger \hat{a}_{n_1}^\dagger \hat{a}_{n_2}^\dagger |\text{ini}\rangle^N + (J/2) \sum_{i=1}^{N} \sum_{j=1}^{N} \phi(n_1, n_2) (N-6) \hat{a}_{n_2}^\dagger \hat{a}_{n_1}^\dagger |\text{ini}\rangle^N .
\]

According to the results above, we derive the corresponding equations describing the relations between the coefficients in the different cases as follows.

If \( n_2 > n_1 + 1 \), then
\[
\phi(n_1 + 1, n_2) + \phi(n_1 + 1, n_1) + \phi(n_1, n_2 + 1) + \phi(n_1, n_2 - 1) = \phi(n_1, n_2) (2E/J + 8); \quad (10)
\]

If \( n_2 = n_1 + 1 \), then
\[
\phi(n_1 + 1, n_2) + \phi(n_1 - 1, n_2) + \phi(n_1, n_2 + 1) + \phi(n_1, n_2 - 1) = \phi(n_1, n_2) (2E/J + 6); \quad (11a)
\]
\[
\text{or } \phi(n - 1, n + 1) + \phi(n, n + 2) + \phi(n + 1, n + 1) + \phi(n, n) = \phi(n, n + 1) (2E/J + 6); \quad (11b)
\]

If \( n_2 = n_1 \), then
\[
\phi(n_1 - 1, n_2) + \phi(n_1, n_2 + 1) = \phi(n_1, n_2) (2E/J + 8); \quad (12a)
\]
\[
\phi(n - 1, n + 1) + \phi(n, n + 1) = \phi(n, n) (2E/J + 8). \quad (12b)
\]

where the energy value \( E = E_2 - E_0 \). \( E_2 \) is the eigenvalue, the constant \( E_0 = NJ \).

We can simplify the above equations as follows:

1) (2): \( \phi_\lambda(n_1 + 1, n_2) + \phi_\lambda(n_1, n_2 - 1) - \phi_\lambda(n_1, n_2 + 1) = 2\phi_\lambda(n_1, n_2) \)

or \( \phi_\lambda(n_1 + 1, n + 1) + \phi_\lambda(n, n) - \phi_\lambda(n + 1, n + 1) - \phi_\lambda(n, n) = 2\phi_\lambda(n, n + 1) \);

2) (3): \( \phi_\lambda(n_1 + 1, n_2) + \phi_\lambda(n_1, n_2 - 1) = 8\phi_\lambda(n_1, n_2) - 8\phi_\lambda(n_1, n_2) \)

or \( \phi_\lambda(n_1 + 1, n) + \phi_\lambda(n, n - 1) = 8\phi_\lambda(n, n) - 8\phi_\lambda(n, n) \).
\[ \phi(n_1, n_2) = (1 - \delta_{n_1 n_2}) \phi_A(n_1, n_2) + \delta_{n_1 n_2} \phi_B(n_1, n_2) \]

with \( \phi_A(n_1, n_2) = A_{12} e^{i \delta_{n_1 n_2} \delta_{n_1 n_2}} + A_{21} e^{i \delta_{n_1 n_2} \delta_{n_1 n_2}} \), \( \phi_B(n_1, n_2) = B e^{i \delta_{n_1 n_2}} \).

The first term \((1 - \delta_{n_1 n_2}) \phi_A(n_1, n_2)\) gives the coefficients for two-body basis states when \( n_1 \neq n_2 \), \( \phi_A(n_1, n_2) \) is a Bethe Ansatz format expression, the second term \( \delta_{n_1 n_2} \phi_B(n_1, n_2) \) gives the coefficients for one-body basis states when \( n_1 = n_2 \), \( \phi_B(n_1, n_2) \) is an exponential function.

2.3. Some results of exact solutions

In this subsection, we give the derivation of the exact solutions for the two-spin-flip number problem of the spin-1 chain making use of the coefficients format in equation (16) and the relations between the coefficients in equations (10) - (14).

Substituting equation (15) for equation (10), we have
\[ 2(\cos k_1 + \cos k_2) \left( A_{12} e^{i \delta_{n_1 n_2} \delta_{n_1 n_2}} + A_{21} e^{i \delta_{n_1 n_2} \delta_{n_1 n_2}} \right) = (A_{12} e^{i \delta_{n_1 n_2} \delta_{n_1 n_2}} + A_{21} e^{i \delta_{n_1 n_2} \delta_{n_1 n_2}})(2E/J + 8). \]

Form above equation, we solve out \( E \) as
\[
E = J(\cos k_1 + \cos k_2) - 4J.
\]

The eigenvalue of the spin-1 chain can be calculated by
\[
E_2 = E + E_0 = J(\cos k_1 + \cos k_2) - 4J + NJ.
\]

Substituting equation (15) for equation (13), we have
\[
2 \left( A_{12} e^{i \delta_{n_1 n_2} \delta_{n_1 n_2}} + A_{21} e^{i \delta_{n_1 n_2} \delta_{n_1 n_2}} \right) = \left( A_{12} e^{i \delta_{n_1 n_2} \delta_{n_1 n_2}} + A_{21} e^{i \delta_{n_1 n_2} \delta_{n_1 n_2}} \right) - Be^{ik_2} - Be^{ik_1},
\]
\[
= 2 \left( A_{12} e^{i \delta_{n_1 n_2} \delta_{n_1 n_2}} + A_{21} e^{i \delta_{n_1 n_2} \delta_{n_1 n_2}} \right)
\]
\[
= \left[ A_{12} e^{i \delta_{n_1 n_2} + 1 - 2e^{ik_2}} + A_{21} e^{i \delta_{n_1 n_2} + 1 - 2e^{ik_1}} \right] e^{i(k_1 + k_2) n} = B(e^{ik_1} + 1),
\]
\[
= \left[ A_{12} e^{i \delta_{n_1 n_2} + 1 - 2e^{ik_2}} + A_{21} e^{i \delta_{n_1 n_2} + 1 - 2e^{ik_1}} \right] e^{i(k_1 + k_2) n} = B(e^{ik_2} + 1).
\]

Form above equation, we obtain
\[
A_{12} e^{i \delta_{n_1 n_2} + 1 - 2e^{ik_1}} + A_{21} e^{i \delta_{n_1 n_2} + 1 - 2e^{ik_2}} = B(e^{ik_1} + 1),
\]

with \( k_1 + k_2 - k_3 = 0 \) for all \( n \).

Similarly, substituting equation (15) for equation (14), we have
\[
\left( A_{12} e^{i \delta_{n_1 n_2} \delta_{n_1 n_2}} + A_{21} e^{i \delta_{n_1 n_2} \delta_{n_1 n_2}} \right) + \left( A_{12} e^{i \delta_{n_1 n_2} \delta_{n_1 n_2}} + A_{21} e^{i \delta_{n_1 n_2} \delta_{n_1 n_2}} \right) = 8 \left( A_{12} e^{i \delta_{n_1 n_2} \delta_{n_1 n_2}} + A_{21} e^{i \delta_{n_1 n_2} \delta_{n_1 n_2}} \right) - 8Be^{ik_2},
\]
\[
\left[ A_{12} e^{i \delta_{n_1 n_2} + e^{ik_2} - 8} + A_{21} e^{i \delta_{n_1 n_2} + e^{ik_2} - 8} \right] e^{i(k_1 + k_2 - k_3) n} = -8Be^{ik_2},
\]
\[
\left[ A_{12} e^{i \delta_{n_1 n_2} + e^{ik_2} - 8} + A_{21} e^{i \delta_{n_1 n_2} + e^{ik_2} - 8} \right] e^{i(k_1 + k_2 - k_3) n} = -8B.
\]

Form above equation, we obtain
\[
A_{12} e^{i \delta_{n_1 n_2} + e^{ik_2} - 8} + A_{21} e^{i \delta_{n_1 n_2} + e^{ik_2} - 8} = -8B,
\]

with \( k_1 + k_2 - k_3 = 0 \) for all \( n \).

Combining equation (18) and (19), we have
\[
8A_{12} e^{i \delta_{n_1 n_2} + 1 - 2e^{ik_1}} + 8A_{21} e^{i \delta_{n_1 n_2} + 1 - 2e^{ik_2}} = -\left( A_{12} e^{i \delta_{n_1 n_2} + e^{ik_2} - 8} + A_{21} e^{i \delta_{n_1 n_2} + e^{ik_2} - 8} \right) e^{i(k_1 + k_2 - k_3 + 1)}.
\]

Form above equation, we obtain
\[
A_{12} = \frac{-8e^{ik_1 + 1 - 2e^{ik_1}} + e^{ik_2} + e^{ik_2} - 8e^{ik_1 + 1}}{8e^{ik_1 + 1 - 2e^{ik_1}} + e^{ik_2} + e^{ik_2} - 8e^{ik_1 + 1}} = \frac{e^{ik_2} + e^{ik_2}}{e^{ik_1} + e^{ik_2}} - 16e^{ik_1}.
\]

Form above equation, we obtain
\[
A_{21} = \frac{-8e^{ik_1 + 1 - 2e^{ik_1}} + e^{ik_2} + e^{ik_2} - 8e^{ik_1 + 1}}{8e^{ik_1 + 1 - 2e^{ik_1}} + e^{ik_2} + e^{ik_2} - 8e^{ik_1 + 1}} = \frac{e^{ik_2} + e^{ik_2}}{e^{ik_1} + e^{ik_2}} - 16e^{ik_2}.
\]
Equation (19) can be rewritten as
\[ A_{21}\left[ A_{12}/A_{21}\right]\left(e^{i\lambda_i} + e^{i\lambda_j} - 8\right) + \left(e^{i\lambda_i} + e^{i\lambda_j} - 8\right) = -8B \] or
\[ A_{21}\left[ A_{12}/A_{21}\right]\left(\lambda_i + \lambda_j - 8\right) + \left(\lambda_i + \lambda_j - 8\right) = -8B. \]

We obtain
\[ A_{21} = -\frac{8B}{\left( A_{12}/A_{21}\right)\left(e^{i\lambda_i} + e^{i\lambda_j} - 8\right) + \left(e^{i\lambda_i} + e^{i\lambda_j} - 8\right)}, \] (21a)

or
\[ A_{21} = -\frac{8B}{\left( A_{12}/A_{21}\right)\left(e^{i\lambda_i} + e^{i\lambda_j} - 8\right) + \left(e^{i\lambda_i} + e^{i\lambda_j} - 8\right)}, \] (21b)

Finally, the basic equations for the coefficients can be summarized as follows,
\[ e^{i\theta} = -\frac{e^{i\lambda_i} \left(e^{ik_{12}} + 1\right)^2 - 16e^{i\lambda_i}}{e^{i\lambda_j} \left(e^{ik_{12}} + 1\right)^2 - 16e^{i\lambda_j}} = \frac{A_{12}}{A_{21}}, \] (22a)

\[ B = \text{constant}, \quad A_{21} = -\frac{8B}{e^{i\theta} \left(e^{i\lambda_j} + e^{i\lambda_i} - 8\right) + \left(e^{i\lambda_j} + e^{i\lambda_i} - 8\right)}, \] (22b)

We also consider the P.B. conditions in the different cases mentioned above, and derive some necessary equations shown below.

When \( n_2 \geq n_1 + 1 \), the P.B. condition is
\[ \phi_A(n_1, n_2) = \phi_A(n_2, n_1 + N), \] (23)

We have \( A_{12} = A_{21} \) if \( n_1, n_2 \) are allowed, \( n_1, n_2 \) pairs are restricted, \( \lambda_1 \in \{1, 2, \Lambda, N\} \) are called Bethe quantum numbers.

Adding two equations in equation (22a), we get
\[ \lambda_i + \lambda_j = \lambda_k, \] (28a)

Because of the relation \( 
\lambda_i + \lambda_j = \lambda_k, \)

Our task is to find all \( (\lambda_i, \lambda_j) \) pairs and corresponding \( \theta \) which lead to solutions of equations (22a) and (27a). Then we acquire the coefficients of all eigenvectors. The allowed \( (\lambda_i, \lambda_j) \) pairs are restricted
to $0 \leq \lambda_1 \leq \lambda_2 \leq N - 1$. Switching $\lambda_1$ and $\lambda_2$ simply interchanges $k_1$ and $k_2$ and produces the same solution. There are $N (N + 1)/2$ pairs that meet this restriction, this is equal to the total number of the eigenvectors of the two-spin-flip number $N$-sites spin-1 chain. The solutions of the wave numbers $k_1$, $k_2$ and $\theta$, can be determined analytically or computationally. Some of them have real values $k_1$ and $k_2$, and others yield complex conjugate momenta.

In the following part, we discuss qualitatively some basic features of the two-spin-flip number spin-1 chain for different cases of the wave numbers $k_1$, $k_2$ and $k_3$. The numerical results about the exact solutions and their interpretations, as well as all eigenvalues and eigenvectors that span the entire Hilbert space will be provided elsewhere.

(i) The total wave number $k = k_1 + k_2$ is a real number, can be regarded as the presentation of the two superimposed magnons for the two-body states $(n_1 \neq n_2)$. $k$ is the quantum number associated with the translational symmetry of Hamiltonian and exists independently of the S1 Ansatz. On the other hand, because $k_3 = k_1 + k_2 = k$, $k$ is also the one-magnon ($s = 1$) wave number of the one-body states $(n_1 = n_2)$;

(ii) In case of real solutions $k_1$ and $k_2$ for the two-body states, we should see that the two magnons $(s_1 = s_2 = 0)$ scatter off each other. It can be characterized as two-magnon scattering states. The magnon interaction is reflected in the phase shift $\theta$ and in the deviation of the momenta $k_1$ and $k_2$ from the values of the one-magnon wave numbers;

(iii) In case of complex solutions of $k_2 = k'_1$ for the two-body states, we should see that the two magnons $(s_1 = s_2 = 0)$ form the bound states. It can be characterized as two-magnon bound states. The bound state manifests itself in the enhanced probability that the two flipped spins are on neighboring sites of the lattice. This property of the wave function is best captured by the calculations of the weight distribution $\langle \phi(n_1, n_2) \rangle$ (the absolute values of the coefficients of the basic states);

(iv) As a verification check, the comparison between the results above and the numerical calculation data (numerical diagonalizations or Monte Carlo calculations) will be carried out in the next step;

(v) The method described in this paper is different from that in reference [11] at the following two points. (a) The wave function of the present work (equation (15)) consists of the two-body and one-body terms (basic states) with nonequivalent coefficients $A$ (a set of coefficients $A_{12}$ and $A_{21}$) and $B$, respectively. While the wave function in reference [11] (equation (2.8)) only contains a two-body term with a set of coefficients $A$ and $B$. (b) Therefore, the S matrixes derived from the wave function in this paper (equation (20)) and reference [11] (equation (4.4)) are different. Additionally, our work gives the relation (equation (21)) between the coefficients $A$ ($A_{12}$, $A_{21}$) and $B$, which is not necessary in reference [11];

(vi) We will consider the expanding of our method to the 3 flip states or larger flip number cases. On the other hand, whether a many-body S matrix can be factorized into two-body S matrixes in this wave function format is still a problem, the possibility of which should be investigated by some rigorous mathematical methods (like those for the Yang-Baxter equation).

4. Summary
We quests for the exact solutions of the spin-1 Heisenberg chain using a new format of the coefficients of the basic states (like Bethe Ansatz format but different) for the two-spin-flip number problem. We derived the relations between the internal coefficients of the different three terms and the basic equations for solving out the three wave numbers included in the coefficients. We also discussed qualitatively some basic features of the two-spin-flip number spin-1 chain for the different types of the wave number values.

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