The quantum dynamics of two qubits inside two distant microcavities connected via a single-mode optical fiber

Van Hieu Nguyen, Bich Ha Nguyen and Hai Trieu Duong

Institute of Materials Science, Vietnam Academy of Science and Technology, 18 Hoang Quoc Viet, Hanoi, Vietnam
E-mail: nhieu@iop.vast.ac.vn

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Abstract
For application to studying the transmission of quantum information, also called quantum communication, between two identical qubits placed inside two identical single-mode microcavities connected via a single-mode optical fiber, the time evolution of this system is investigated. In the Markovian approximation, the von Neumann equation for its reduced density matrix contains a completely positive linear operator called the Liouvillian operator describing the decoherence of this system due to its interaction with the environment. By using the Linblad formula for the Liouvillian operator, a system of rate equations can be derived. In the special case of resonance between the energy difference of two states in each qubit and the energy of the fiber mode, the rate equations for the system excited up to the first level are solved in first order approximation with respect to the decoherence constants. It is shown that when there is no decoherence, the perfect quantum state transmission between two qubits can take place if the physical parameters of the system satisfy definite conditions. A possible extension to studying the system excited to high energy states is also discussed.

Keywords: quantum communication, microcavity, decoherence, perfect quantum state transmission

Classification numbers: 3.00, 3.01

1. Introduction
Theoretical and experimental studies of quantum information (QI) transfer or transmission between two qubits are primary topics in the physics of quantum information and quantum computation [1, 2]. While the physical origin of the QI transfer between two nearby spin-qubits may be their direct magnetic spin–spin coupling (see recent review [3] and references therein), that of the QI transmission between two distant qubits is their interactions through the intermediary of some quantum system, for example the electromagnetic radiation propagating through an optical fiber (OF). Each qubit may be a trapped two-level atom or ion placed inside a microcavity (MC), and for performing the QI transmission between two distant qubits Cirac et al [4] proposed using an experimental set-up consisting of two trapped two-level atoms or ions placed inside two single-mode MCs connected via an OF. Subsequently, similar models of quantum state transmission between two qubits placed inside two single-mode MCs connected via an OF were studied by Pelizzari [5] and Sarafini et al [6]. The dynamical theory of a two-level atomic system interacting with photons in a single-mode MC through the allowed electrical dipole transition of an electron between two atomic levels—cavity quantum electrodynamics (CQED)—has been a topic of intensive theoretical research in the last decade [7]. The polariton-like strongly coupled system of a two-level quantum dot (QD) and a photon in a single-mode MC has also been investigated in many experimental works [8–10]. In view of successful experimental realization of the coupling between an OF and a MC [11], fabrication of the above-mentioned QI processing systems is feasible.

The purpose of this work is to continue the theoretical study of the time evolution of the density matrix of a
QI processing system of the form proposed in [4] and subsequently investigated in [5, 6]. With the attempt to derive explicit analytical expressions of elements of the density matrix in terms of physical parameters of the QI processing systems in some particular cases, we limit this work to study of a system with two single-mode MCs and a single-mode OF. In [6] it was noted that for transmission through a not too long OF, this limitation is reasonable.

In section 2 rate equations for elements of the reduced density matrix between the ground and single excitation states are derived. The solution of this system of rate equations in the lowest order approximation with respect to decoherence constants is studied in section 3. In the special resonance case, explicit analytical expressions of reduced density matrix elements are derived. Similar reasoning can be applied to numerical calculations of these matrix elements in the general non-resonance case. Section 4 is devoted to a discussion on the reduced density matrix of the subsystem of two qubits. From the general form of expressions of their 16 matrix elements it follows that a closed system of 16 linear differential equations (rate equations) does not exist for them. Section 5 is the conclusion.

2. Rate equations for ground and single excitation states

Following [4–6] we study a QI processing system consisting of two identical single-electron-two-level atoms ‘a’ and ‘b’ placed inside two corresponding identical single-mode MCs ‘a’ and ‘b’ connected via a single-mode OF. Suppose that the photon–electron interaction inside each MC is the allowed electrical dipole transition between two energy levels with some effective coupling constant $f$. We choose the ground state energy of atoms to be zero and denote by $\varepsilon$ the energy of their excited states, by $\omega_0$ and $\omega$ the energies of cavity modes and the fiber mode, respectively, and by $\lambda$ and $e^{\pm\phi}\lambda$ the coupling constants of mixing the fiber mode with cavity modes in MCs ‘a’ and ‘b’, respectively. The presence of phase factors $e^{\pm\phi}$ is due to the electromagnetic radiation propagation from one MC to other. In the rotating wave approximation the electron–photon system without the interaction of the environment has the following total Hamiltonian:

$$H = \varepsilon (a_1^+ a_1 + b_1^+ b_1) + \omega_0 (\gamma_0^a a_0 + \gamma_0^b b_0) + \omega (\gamma_0^a a_0 + \gamma_0^b b_0) + f (a_1^+ a_2 + b_1^+ b_2 + b_2^+ b_1),$$

where $a_1^+$ or $b_1^+$ (or $a_2^+$ or $b_2^+$) and $a_1$ or $b_1$ (or $a_2$ or $b_2$) are electron creation and destruction operators at the upper (lower) energy level in the MC ‘a’ and ‘b’, respectively, $\gamma_0^a$ or $\gamma_0^b$ and $\gamma_a$ or $\gamma_b$ are photon creation and destruction operators in the MC ‘a’ and ‘b’, respectively, and $\gamma$ and $\gamma^*$ are those in the OF.

Interactions between the system and the environment may cause two phenomena: energy renormalization (Lamb shift) and decoherence of the system. In this work we shall neglect energy renormalization. Then in the Markovian approximation, the time evolution of reduced density matrix $\rho$ of the system is governed by the von Neumann equation

$$\frac{d\rho}{dt} = -i [H, \rho] + L \rho,$$

where $L$ is a linear completely positive operator called the Liouvillian operator. Its general form was derived by Gorini, Kossakowski and Sudarshan (GKS) [12].

In the study of the decoherence of open quantum systems the Lindblad formula [13] is often used, a special case of that derived by GKS [12]. The concrete form of the Lindblad formula for each system depends on the physical mechanisms of its decoherence. In the case of the quantum system with Hamiltonian (1), there are three main decoherence mechanisms:

1. the electron relaxation from excited levels;
2. the photon leakage from MCs;
3. the photon leakage from the OF.

Liouvillian operator $L$ is determined by the following Lindblad formula:

$$L \rho = \frac{1}{\alpha_i} \left( [\sigma_{i}^{-}, \rho \sigma_{i}^{(+)}] + [\sigma_{i}^{-}, \rho \sigma_{i}^{(+)}] \right)$$

where $\alpha_i$ and $\alpha_i$ are positive constants, and $\sigma_{i}^{-}$, $\sigma_{i}^{(+)}$ and $\sigma_{i}^{(+)}$, $\sigma_{i}^{-}$ are lowering and raising operators for electrons of atoms in the corresponding MCs. From the von Neumann equation (2) with Hamiltonian (1) and Liouvillian operator (3), the system of rate equations for elements of reduced density matrix $\rho$ follows. The vacuum state vector is denoted by $|0\rangle$. The Hilbert space of state vectors of the electron–photon system with one electron in each MC is an (infinite) direct sum of linear subspaces invariant with respect to the action of the Hamiltonian $H$:

- One-dimensional subspace $V_0$ with the ground state as its basis vector

$$|\psi_0\rangle = a_1^0 b_1^0 |0\rangle.$$

- Five-dimensional subspace $V_1$ with five single excitation states as its basis vectors

$$|\psi_1\rangle = a_1^0 b_1^1 |0\rangle,$$

$$|\psi_2\rangle = a_1^1 b_1^0 |0\rangle,$$

$$|\psi_3\rangle = a_2^0 b_2^0 |0\rangle,$$

$$|\psi_4\rangle = a_2^0 b_1^1 |0\rangle.$$

- Thirteen-dimensional subspace $V_2$ with 13 double excitation states as its basis vectors

$$|\psi_5\rangle = a_1^0 a_2^0 |0\rangle,$$

$$|\psi_6\rangle = a_1^0 b_2^0 |0\rangle,$$

$$|\psi_7\rangle = a_1^1 b_1^0 |0\rangle,$$

$$|\psi_8\rangle = a_2^0 b_1^1 |0\rangle,$$

$$|\psi_{10}\rangle = a_1^0 a_2^0 |0\rangle,$$

$$|\psi_{11}\rangle = a_1^0 b_2^0 |0\rangle,$$

$$|\psi_{12}\rangle = a_1^1 b_1^0 |0\rangle,$$

$$|\psi_{13}\rangle = \frac{1}{\sqrt{2}} (a_2^1 + b_1^0) |0\rangle,$$

$$|\psi_{14}\rangle = \frac{1}{\sqrt{2}} (a_2^1 b_1^0 |0\rangle,$$

$$|\psi_{15}\rangle = \frac{1}{\sqrt{2}} (a_2^1 + b_1^0) |0\rangle,$$

$$|\psi_{16}\rangle = a_1^0 a_2^0 |0\rangle,$$

$$|\psi_{17}\rangle = a_1^0 a_2^0 |0\rangle.$$
and other linear invariant (with respect to $H$) subspaces of multiple excitation states.

Consider the simple case when only ground and single excitation states of the system are involved in the QI transmission process. In this case the reduced density matrix may have non-vanishing elements $\rho_{ij}$ only between two basis vectors in six-dimensional subspace $V_0 \oplus V_1$, $i, j = 0, 1, 2, \ldots, 5$. They satisfy the following rate equations:

$$
\frac{d\rho_{00}}{dt} = \alpha_1 (\rho_{11} + \rho_{22}) + \alpha_2 (\rho_{33} + \rho_{44}) + \alpha_3 \rho_{55},
$$

(7.1)

$$
\frac{d\rho_{01}}{dt} = i\epsilon \rho_{01} + i f \rho_{03} - \frac{1}{2} \alpha_3 \rho_{01},
$$

(7.2)

$$
\frac{d\rho_{02}}{dt} = i\epsilon \rho_{02} + i f \rho_{04} - \frac{1}{2} \alpha_3 \rho_{02},
$$

(7.3)

$$
\frac{d\rho_{03}}{dt} = i \omega_0 \rho_{03} + i f \rho_{01} + i \lambda \rho_{05} - \frac{1}{2} \alpha_3 \rho_{03},
$$

(7.4)

$$
\frac{d\rho_{04}}{dt} = i \omega_0 \rho_{04} + i f \rho_{02} + i e^{i\psi} \lambda \rho_{05} - \frac{1}{2} \alpha_3 \rho_{04},
$$

(7.5)

$$
\frac{d\rho_{05}}{dt} = i \omega_0 \rho_{05} + i f \rho_{03} + i e^{i\psi} \lambda \rho_{04} - \frac{1}{2} \alpha_3 \rho_{05},
$$

(7.6)

$$
\frac{d\rho_{10}}{dt} = -i\epsilon \rho_{10} - i f \rho_{30} - \frac{1}{2} \alpha_3 \rho_{10},
$$

(7.7)

$$
\frac{d\rho_{11}}{dt} = i f (\rho_{13} - \rho_{31}) - \alpha_3 \rho_{11},
$$

(7.8)

$$
\frac{d\rho_{12}}{dt} = i f (\rho_{14} - \rho_{32}) - \alpha_3 \rho_{12},
$$

(7.9)

$$
\frac{d\rho_{13}}{dt} = i (\omega_0 - \epsilon) \rho_{13} + i f (\rho_{11} - \rho_{33}) + i \lambda \rho_{15} + \frac{1}{2} (\alpha_3 + \alpha_3) \rho_{13},
$$

(7.10)

$$
\frac{d\rho_{14}}{dt} = i (\omega_0 - \epsilon) \rho_{14} + i f (\rho_{12} - \rho_{34}) + i e^{i\psi} \lambda \rho_{15} + \frac{1}{2} (\alpha_3 + \alpha_3) \rho_{14},
$$

(7.11)

$$
\frac{d\rho_{15}}{dt} = i (\omega - \epsilon) \rho_{15} - i f \rho_{35} + i \lambda \rho_{13} + i e^{i\psi} \lambda \rho_{14} - \frac{1}{2} (\alpha_3 + \alpha_3) \rho_{15},
$$

(7.12)

$$
\frac{d\rho_{20}}{dt} = -i\epsilon \rho_{20} - i f \rho_{40} - \frac{1}{2} \alpha_3 \rho_{20},
$$

(7.13)

$$
\frac{d\rho_{21}}{dt} = i f (\rho_{23} - \rho_{41}) - \alpha_3 \rho_{21},
$$

(7.14)

$$
\frac{d\rho_{22}}{dt} = i f (\rho_{24} - \rho_{42}) - \alpha_3 \rho_{22},
$$

(7.15)

$$
\frac{d\rho_{23}}{dt} = i (\omega_0 - \epsilon) \rho_{23} + i f (\rho_{21} - \rho_{43}) + i \lambda \rho_{25} - \frac{1}{2} (\alpha_3 + \alpha_3) \rho_{23},
$$

(7.16)
the unitary transformation between the eigenstate basis with

\[ \frac{d\rho_{31}}{dt} = i (\varepsilon - \omega) \rho_{31} + \alpha \rho_{31} - i \lambda \rho_{31}, \]
\[ \frac{d\rho_{32}}{dt} = i (\varepsilon - \omega) \rho_{32} + i \lambda \rho_{32} - i e^{i \omega} \rho_{32}, \]
\[ \frac{d\rho_{33}}{dt} = i \rho_{33} + i \lambda \rho_{33} - i e^{i \omega} \rho_{33}, \]
\[ \frac{d\rho_{34}}{dt} = i (\varepsilon - \omega) \rho_{34} + i \lambda \rho_{34} - i e^{i \omega} \rho_{34}, \]
\[ \frac{d\rho_{35}}{dt} = i \lambda (\rho_{35} - \rho_{33}) + i e^{i \omega} \rho_{35} - \alpha \rho_{35}. \]

3. Solving the rate equations for ground and single excitation states

In order to study the time evolution of the QI transmission system using two two-level atoms placed inside two single-mode MCs connected by a single-mode OF, it is necessary to solve rate equations with given initial conditions. In the lowest order approximation with respect to small decoherence constants \( \alpha_s, \alpha_c, \) and \( \alpha, \) 36 elements \( \rho_{ij}(t) \) of the reduced density matrix between ground and single excitation states are determined by equations (7.1)–(7.36) and have the following expressions:

\[ \rho_{ij}(t) = \sum_{k,l=1}^{5} U_{ik}(t) \rho_{kl}(0) U_{jk}^{*}(t), \]
\[ \rho_{00}(t) = \sum_{k=1}^{5} U_{ik}(t) \rho_{k0}(0), \]
\[ \rho_{0j}(t) = \sum_{i=1}^{5} \rho_{i0}(0) U_{ij}^{*}(t), \]
\[ \rho_{00}(0) = \sum_{i=1}^{5} \rho_{ij}(t), \]
\[ U_{ik}(t) = \sum_{\mu=1}^{36} C_{\mu} e^{-i E_{\mu} t} e^{-\gamma_{\mu} t}, \]

where \( E_{\mu} \) are eigenvalues of the total Hamiltonian, \( \gamma_{\mu} \) are corresponding damping constants and \( C_{\mu} \) are coefficients of the unitary transformation between the eigenstate basis with unit vectors \( |\psi_{\mu}\rangle, \)

\[ H |\psi_{\mu}\rangle = E_{\mu} |\psi_{\mu}\rangle, \]

and the natural basis with unit vectors \( |\psi_i\rangle \) given by equations (5), \( \mu = 1, 2, 3, 4, 5, i = 1, 2, 3, 4, 5, \)

\[ C_{\mu i} = \langle \psi_{\mu} | \psi_i \rangle. \]

Two eigenvalues \( E_1 \) and \( E_2 \) as well as two damping constants \( \gamma_1 \) and \( \gamma_2 \) and corresponding coefficients \( C_{1i} \) and \( C_{2i} \) with \( i = 1, 2, 3, 4, 5, \) have simple expressions:

\[ E_1 = \frac{1}{2} (\varepsilon + \omega_0 + \Delta), \quad E_2 = \frac{1}{2} (\varepsilon + \omega_0 - \Delta), \]

\[ \Delta = \sqrt{(\varepsilon - \omega_0)^2 + 4 f^2}, \]

\[ \gamma_1 = \frac{f^2}{\Delta (\Delta + \omega_0 - \varepsilon)} \alpha_s + \frac{1}{4} \frac{\Delta + \omega_0 - \varepsilon}{\Delta} \alpha_c, \]

\[ \gamma_2 = \frac{f^2}{\Delta (\Delta - \omega_0 + \varepsilon)} \alpha_s + \frac{1}{4} \frac{\Delta - \omega_0 + \varepsilon}{\Delta} \alpha_c, \]

\[ C_{1i} = \frac{\Delta + \omega_0 - \varepsilon}{\Delta}, \quad C_{12} = - e^{i \omega} C_{11}, \]

\[ C_{13} = \frac{E_1 - \varepsilon}{f} C_{11}, \quad C_{14} = - e^{i \omega} C_{13}, \quad C_{15} = 0, \]

\[ C_{21} = - \frac{1}{2} \sqrt{1 + \frac{\omega_0 - \varepsilon}{\Delta}}, \quad C_{22} = - e^{i \omega} C_{21}, \]

\[ C_{23} = \frac{E_2 - \varepsilon}{f} C_{21}, \quad C_{24} = - e^{i \omega} C_{23}, \quad C_{25} = 0. \]

In the special resonance case \( \omega = \varepsilon, \) three eigenvalues \( E_3, E_4 \) and \( E_5 \) as well as three damping constants \( \gamma_3, \gamma_4 \) and \( \gamma_5 \) and corresponding coefficients \( C_{3i}, C_{4i} \) and \( C_{5i} \) with \( i = 1, 2, 3, 4, 5 \) also have simple expressions:

\[ E_3 = \frac{1}{2} (\varepsilon + \omega_0 + \Delta'), \quad E_4 = \frac{1}{2} (\varepsilon + \omega_0 - \Delta'), \quad E_5 = \varepsilon, \]

\[ \Delta' = \sqrt{(\varepsilon - \omega_0)^2 + 4 f^2 + 8 \lambda^2}, \]

\[ \gamma_3 = \frac{f^2}{\Delta' (\Delta' + \omega_0 - \varepsilon)} \alpha_s + \frac{1}{4} \frac{\Delta' + \omega_0 - \varepsilon}{\Delta'} \alpha_c + \frac{2 \lambda^2}{\Delta' (\Delta' + \omega_0 - \varepsilon)} \alpha, \]

\[ \gamma_4 = \frac{f^2}{\Delta' (\Delta' - \omega_0 + \varepsilon)} \alpha_s + \frac{1}{4} \frac{\Delta' - \omega_0 + \varepsilon}{\Delta'} \alpha_c + \frac{2 \lambda^2}{\Delta' (\Delta' - \omega_0 + \varepsilon)} \alpha, \]

\[ \gamma_5 = \frac{\lambda^2}{f^2 + 2 \lambda^2} \alpha_s + \frac{1}{2} \frac{f^2}{f^2 + 2 \lambda^2} \alpha, \]

\[ C_{31} = \frac{f}{\sqrt{\Delta' (\Delta' + \omega_0 - \varepsilon)}}, \quad C_{32} = e^{i \omega} C_{31}, \]

\[ C_{33} = \frac{1}{2} \sqrt{\frac{\Delta' + \omega_0 - \varepsilon}{\Delta'}}, \quad C_{34} = e^{i \omega} C_{33}, \]

\[ C_{35} = \frac{2 \lambda}{\sqrt{\Delta' (\Delta' + \omega_0 - \varepsilon)}}. \]
\[
C_{41} = -\frac{\gamma}{\sqrt{\Lambda (\Lambda - \omega_0 + \nu)}} \quad C_{42} = e^{i\phi} C_{41},
\]
\[
C_{43} = \frac{\gamma}{\sqrt{\Lambda (\Lambda - \omega_0 + \nu)}}, \quad C_{44} = e^{i\phi} C_{43},
\]
\[
C_{45} = -\frac{2\lambda}{\sqrt{\Lambda (\Lambda - \omega_0 + \nu)}},
\]
\[
C_{51} = -\frac{\lambda}{\sqrt{f^2 + 2\lambda^2}}, \quad C_{52} = e^{i\phi} C_{51},
\]
\[
C_{53} = C_{54} = 0, \quad C_{55} = -\frac{f}{\sqrt{f^2 + 2\lambda^2}}.
\]

In the general non-resonance case, \(\alpha \neq \omega\), three eigenvalues \(E_5, E_4, E_3\) are three roots of the algebraic equation
\[
(E - \omega_0) (E - \gamma) (E - \omega) - f^2 (E - \omega) - 2\lambda^2 (E - \gamma) = 0,
\]
and three damping constants \(\gamma_\mu\) and corresponding coefficients \(C_{\mu i}, \mu = 3, 4, 5, i = 1, 2, 3, 4, 5\) have the following expressions:
\[
\gamma_\mu = \Lambda^2 \left[ \alpha_0 + \frac{(E_\mu - \gamma)}{f} \alpha_0 + \frac{2\lambda^2}{f^2} \left( \frac{E_\mu - \gamma}{E_\mu - \omega} \right)^2 \right],
\]
\[
C_{\mu 1} = A_\mu, \quad C_{\mu 2} = e^{i\phi} A_\mu, \quad C_{\mu 3} = \frac{E_\mu - \gamma}{f} A_\mu,
\]
\[
C_{\mu 4} = e^{i\phi} C_{\mu 3}, \quad C_{\mu 5} = \frac{2\lambda}{f} \frac{E_\mu - \gamma}{E_\mu - \omega} A_\mu,
\]
where
\[
A_\mu = \frac{1}{\sqrt{2}} \frac{f}{E_\mu - \gamma} \frac{1}{\sqrt{1 + \frac{f^2}{(E_\mu - \gamma)} + \frac{2\lambda^2}{(E_\mu - \omega)}}},
\]
\[
e^{-i\phi} U_{12}(t) = e^{i\phi} U_{21}(t)
\]
\[
e^{-i(\omega_0 + \nu/2)t} \left[ \cos \frac{\Delta}{2} \left( \frac{e^{-i\gamma t} + e^{-\gamma t}}{2} + \frac{\gamma}{\Delta} \right) - i \sin \frac{\Delta}{2} \left( \frac{e^{-i\gamma t} - e^{-\gamma t}}{2} + \frac{\gamma}{\Delta} \right) \right]
\]
\[
e^{-i(\omega_0 + \nu/2)t} \left[ \frac{f^2}{f^2 + 2\lambda^2} \left( \cos \frac{\Delta}{2} \left( \frac{e^{-i\gamma t} + e^{-\gamma t}}{2} + \frac{\gamma}{\Delta} \right) - i \sin \frac{\Delta}{2} \left( \frac{e^{-i\gamma t} - e^{-\gamma t}}{2} + \frac{\gamma}{\Delta} \right) \right) \right]
\]
\[
e^{-i(\omega_0 + \nu/2)t} e^{i(\omega_0 - \nu/2)t} e^{-\gamma t}
\]
\[
U_{13}(t) = U_{31}(t) = U_{24}(t) = U_{42}(t)
\]
\[
e^{-i(\omega_0 + \nu/2)t} \left[ \frac{f_0}{\Delta} \left( \cos \frac{\Delta}{2} \left( \frac{e^{-i\gamma t} + e^{-\gamma t}}{2} - i \sin \frac{\Delta}{2} \left( \frac{e^{-i\gamma t} - e^{-\gamma t}}{2} \right) \right) \right]
\]
\[
e^{-i(\omega_0 + \nu/2)t} \left[ \frac{f_0}{\Delta} \left( \cos \frac{\Delta}{2} \left( \frac{e^{-i\gamma t} + e^{-\gamma t}}{2} + \frac{\gamma}{\Delta} \right) - \frac{\gamma}{\Delta} \left( \frac{e^{-i\gamma t} - e^{-\gamma t}}{2} + \frac{\gamma}{\Delta} \right) \right) \right]
\]
\[
e^{-i(\omega_0 + \nu/2)t} \left( \frac{f_0}{\Delta} \cos \frac{\Delta}{2} \left( \frac{e^{-i\gamma t} + e^{-\gamma t}}{2} \right) - \frac{\gamma}{\Delta} \left( \frac{e^{-i\gamma t} - e^{-\gamma t}}{2} \right) \right]
\]
\[
e^{-i(\omega_0 + \nu/2)t} \left( \frac{f_0}{\Delta} \cos \frac{\Delta}{2} \left( \frac{e^{-i\gamma t} + e^{-\gamma t}}{2} \right) \right)
\]
\[
e^{-i(\omega_0 + \nu/2)t} \left( \frac{f_0}{\Delta} \cos \frac{\Delta}{2} \left( \frac{e^{-i\gamma t} + e^{-\gamma t}}{2} \right) \right)
\]
\[
e^{-i(\omega_0 + \nu/2)t} \left( \frac{f_0}{\Delta} \cos \frac{\Delta}{2} \left( \frac{e^{-i\gamma t} + e^{-\gamma t}}{2} \right) \right)
\]
\[
e^{-i(\omega_0 + \nu/2)t} \left( \frac{f_0}{\Delta} \cos \frac{\Delta}{2} \left( \frac{e^{-i\gamma t} + e^{-\gamma t}}{2} \right) \right)
\]
\[
e^{-i(\omega_0 + \nu/2)t} \left( \frac{f_0}{\Delta} \cos \frac{\Delta}{2} \left( \frac{e^{-i\gamma t} + e^{-\gamma t}}{2} \right) \right)
\]
\[ e^{-i\omega t}\mathcal{H}_1(t) = e^{i\phi} U_{\mathcal{H}_1}(t) = \\
\frac{1}{2} \left\{ \cos \frac{\Delta}{2} t \left( \frac{e^{-\gamma_1 t} + e^{-\gamma_2 t}}{2} - \frac{\epsilon - \omega_0 e^{-\gamma_1 t} - e^{-\gamma_2 t}}{\Delta} \right) \right. \\
+ i \sin \frac{\Delta}{2} t \left( \frac{e^{-\gamma_1 t} - e^{-\gamma_2 t}}{2} - \frac{\epsilon - \omega_0 e^{-\gamma_1 t} + e^{-\gamma_2 t}}{\Delta} \right) \left. \right\} \]
The curves $|U_i(t)|^2$, $i = 1, 2, 3, 4, 5$ for the systems satisfying conditions (28.1) and (28.2) at $n = 1$, $p = 1$, $\epsilon - \omega_0 = 5$, $m \geq 2$ and $n = 2$, $p = 1$, $\epsilon - \omega_0 = 5$, $m \geq 3$ are plotted in figures 3 and 4, while those satisfying conditions (29.1) and (29.2) are plotted in figures 5 and 6.

In the first case equation (26) hold at

$$T_n = 2p \frac{2\pi}{|\epsilon - \omega_0|} = (2n + 1) \frac{2\pi}{\sqrt{(\epsilon - \omega_0)^2 + 4f^2}} = 2m \frac{2\pi}{\sqrt{(\epsilon - \omega_0)^2 + 4f^2 + 8\lambda^2}},$$

(30)

while in the second case equation (26) hold at

$$T_n = (2p + 1) \frac{2\pi}{|\epsilon - \omega_0|} = 2n \frac{2\pi}{\sqrt{(\epsilon - \omega_0)^2 + 4f^2}} = 2(m + 1) \frac{2\pi}{\sqrt{(\epsilon - \omega_0)^2 + 4f^2 + 8\lambda^2}}.$$

(31)

If positive integers $2p$, $2n+1$ and $2m$ cannot be decomposed as $2p = (2v + 1)2p_0$, $2n + 1 = (2v + 1)(2n_0 + 1)$ and $2m = (2v + 1)2m_0$ with some positive integers $m_0 > n_0 \geq p_0$ and $v$, then $T_n$ determined by equation (30) is the earliest time moment at which the perfect quantum state transmission between two qubits takes place. Similarly, if the non-negative integers $2p + 1$, $2n$ and $2m + 1$ cannot be decomposed as $2p + 1 = (2v + 1)(2p_0 + 1)$, $2n = (2v + 1)(2n_0)$ and $2m + 1 = (2v + 1)(2m_0 + 1)$ with some non-negative integers

Figure 2. $|U_i|^2$ versus time $t$ in the double resonance case $\epsilon = \omega = \omega_0$ with $n = 1$ and $m = 2$ (a) or $m = 3$ (b).

Figure 3. $|U_i(t)|^2$ versus time $t$ in the single resonance case $\epsilon = \omega \neq \omega_0$ with $n = 1$, $p = 1$, $\epsilon - \omega_0 = 5$ and (a) $m = 2$ or (b) $m = 10$.

Figure 4. $|U_i(t)|^2$ versus time $t$ in the single resonance case $\epsilon = \omega \neq \omega_0$ with $n = 2$, $p = 1$, $\epsilon - \omega_0 = 5$ and (a) $m = 3$ or (b) $m = 10$.

and

$$\left( \frac{f}{\epsilon - \omega_0} \right)^2 = \frac{4n^2 - (2p + 1)^2}{4(2p + 1)^2}. \quad (29.2)$$
between different subsystems with state vectors the efficiency of the quantum state transmission or transfer $T$ transmission between two qubits at some time moment place.

perfect quantum state transmission between two qubits takes by equation (31) is the earliest time moment at which the perfect quantum state transmission between two qubits takes place.

In special cases with the perfect quantum state transmission between two qubits at some time moment $T_n$, the efficiency of the quantum state transmission or transfer between different subsystems with state vectors $|\psi_i\rangle$, $i = 1, 2, 3, 4, 5$ can be measured by the physical quantity

$$W_{ij} = \frac{1}{T_n} \int_0^{T_n} |U_{ij}(t)|^2 dt$$

(32)
called the average rate of the transmission or transfer between two related quantum states $|\psi_i\rangle$ and $|\psi_j\rangle$, $i \neq j$, $W_{ij}$ being the average rate of the presence of the system in the initial state $|\psi_i\rangle$. In the double resonance case ($\omega_0 = \varepsilon = \omega$) with coupling constants $f$ and $\lambda$ satisfying condition (25), we have

$$W_{11} = W_{12} = W_{21} = W_{22} = \frac{1}{8} \left[ 3 - \frac{(2n + 1)^2}{m^2} + 3 \frac{(2n + 1)^4}{16m^4} \right].$$

(33.1)

$$W_{13} = W_{14} = W_{23} = W_{24} = W_{31} = W_{32} = W_{42} = \frac{1}{8} \left[ 1 + \frac{(2n + 1)^2}{4m^2} \right],$$

(33.2)

$$W_{15} = W_{25} = W_{35} = W_{52} = \frac{3}{4} \left[ 1 - \frac{(2n + 1)^2}{4m^2} - \frac{(2n + 1)^4}{16m^4} \right].$$

(33.3)

$$W_{33} = W_{34} = W_{43} = W_{44} = \frac{1}{4}.$$ (33.4)

$$W_{55} = \frac{1}{2} \left[ 1 - \frac{(2n + 1)^2}{2m^2} + 3 \frac{(2n + 1)^4}{16m^4} \right].$$ (33.5)

In the single resonance case $\omega_0 \neq \varepsilon = \omega$ the average rates are

$$W_{11} = W_{12} = W_{21} = W_{22} = \frac{1}{8} \left[ 3 - \frac{(2n + 1)^2}{m^2} + \frac{4p^2}{(2n + 1)^2} \right] + \frac{1}{16} \left[ 3 + \frac{p^2}{m^2} \right] \left[ \frac{(2n + 1)^2 - 4p^2}{(m^2 - p^2)^2} \right].$$ (34.1)

$$W_{13} = W_{14} = W_{23} = W_{24} = W_{31} = W_{32} = W_{42} = \frac{1}{8} \left[ (2n + 1)^2 - 4p^2 \right] \left[ \frac{1}{(2n + 1)^2} + \frac{1}{4m^2} \right].$$ (34.2)

$$W_{15} = W_{25} = W_{35} = W_{52} = \frac{1}{64} \left[ 3 + \frac{p^2}{m^2} \right] \left[ (2n + 1)^2 - 4p^2 \right] \left[ \frac{4m^2 - (2n + 1)^2}{(m^2 - p^2)^2} \right].$$ (34.3)
\( W_{33} = W_{34} = W_{43} = W_{44} = \frac{1}{4} \left[ \frac{1}{(2n+1)^2} + \frac{1}{4m^2} \right] \),

\( W_{35} = W_{45} = W_{53} = W_{54} = \frac{4m^2 - (2n+1)^2}{16m^2} \),

\( W_{55} = \frac{1}{32} \left[ \frac{4m^2 - (2n+1)^2}{(m^2 - p^2)^2} \right] + \frac{1}{16} \left[ (2n+1)^2 - 4p^2 \right] \),

if coupling constants \( f, \lambda \) and energy difference \( \varepsilon - \omega_0 \) satisfy conditions (28.1) and (28.2), or

\[
W_{11} = W_{12} = W_{21} = W_{22} = \frac{3}{8} \left[ 3 - 4 \frac{4n^2 - (2p+1)^2}{(2n+1)^2 - (2p+1)^2} + \frac{(2p+1)^2}{4n} \right] + \frac{1}{4} \left[ \frac{(2p+1)^2}{(2m+1)^2} \right] \],

\[
W_{13} = W_{14} = W_{23} = W_{24} = W_{31} = W_{32} = W_{35} = W_{42} = \frac{1}{8} \left[ 4n^2 - (2p+1)^2 \right] \left[ \frac{1}{4n^2} + \frac{4}{(m+1)^2} \right] \],

\[
W_{15} = W_{25} = W_{51} = W_{52} = \frac{1}{4} \left[ 3 + \frac{(2p+1)^2}{(2m+1)^2} \right] \times \frac{4n^2 - (2p+1)^2}{(2n+1)^2 - (2p+1)^2} \frac{1}{(2n+1)^2 - 4n^2},

\[
W_{33} = W_{34} = W_{43} = W_{44} = \frac{1}{4} + \frac{1}{8} \left( 2n+1 \right)^2 \frac{1}{4n^2} + \frac{1}{2m+1} \left( 2n+1 \right)^2 \],

\[
W_{35} = W_{45} = W_{53} = W_{54} = \frac{1}{4} \left( 2n+1 \right)^2 - \frac{4n^2}{(m+1)^2},

\[
W_{55} = \frac{1}{2} \left[ 1 + \frac{(2p+1)^2}{(2m+1)^2} \right] \frac{4n^2 - (2p+1)^2}{(2m+1)^2 - (2p+1)^2} \],

if \( f, \lambda \) and \( \varepsilon - \omega_0 \) satisfy conditions (29.1) and (29.2). The values of average rates \( W_{ij} \) at different values of integers \( m, n \) or \( m, n, p \) are represented in figures 7 and 8.

4. The reduced density matrix of the two-qubit subsystem

Reduced density matrix \( \rho \) is a linear operator in the Hilbert space of the state vector of the system consisting of two qubits
and photons in the OF and two MCs. The trace of ρ with respect to indices labeling states of photons is the reduced density matrix ρ^{qq} of the subsystem of two qubits. In the Hilbert space of state vectors of the system with Hamiltonian (1) let us choose a basis consisting of state vectors of states with given numbers of photons in the OF and in each MC:

\[ |i, j; n_a \gamma_a, n_b \gamma_b, n_y \rangle = \frac{1}{\sqrt{n_a!}} (\gamma_a^{n_a}) \frac{1}{\sqrt{n_b!}} (\gamma_b^{n_b}) \frac{1}{\sqrt{n_y!}} (\gamma_y^{n_y}) a_i^* b_j^* |0 \rangle. \]

(36)

The elements of reduced density matrix ρ^{qq} of the two-qubit subsystem are

\[ \rho_{(i)k(l)}^{qq}(t) = \rho_{(i)k(l)}^{qq}(t) |k, l \rangle = \sum_{n_a, n_b, n=0} \rho_{(i)k(l)j(m)}^{qq}(t) |k, l \rangle \rho_{j(m)}^{qq}(t) |k, l \rangle. \]

(37)

From von Neumann equation (1) with Liouvillian operator (3) it follows that differential equations determining matrix elements of ρ(t) between two states excited at nth and mth levels contain only contributions from those between two states excited at nth, (n + 1)th and mth, (m + 1)th levels. Therefore, if at the initial time moment t = 0 the system of qubits and photon is excited only up to the nth level, then at any subsequent time moment t > 0 only matrix elements of ρ(t) between states excited up to the nth level may be non-vanishing. In particular, if at the initial time moment t = 0 the system of qubits and photons is excited only up to the first level, then seven elements ρ_{(11)(k)}^{qq}(t) and ρ_{(k)(11)}^{qq}(t), k, l = 1, 2, of reduced density matrix ρ^{qq}(t) of the two-qubit subsystem always vanish:

\[ \rho_{(11)(k)}^{qq}(t) = \rho_{(k)(11)}^{qq}(t) = 0, \quad k, l = 1, 2, \]

(38.1)

and nine other elements are

\[ \rho_{(12)(12)}^{qq}(t) = \rho_{11}(t) = \sum_{k=1}^{5} U_{1k}(t) U_{k1}(t)^* \rho_{k1}(0), \]

(38.2)

\[ \rho_{(12)(21)}^{qq}(t) = \rho_{12}(t) = \sum_{k=1}^{5} U_{1k}(t) U_{2k}(t)^* \rho_{k2}(0), \]

(38.3)

\[ \rho_{(12)(22)}^{qq}(t) = \rho_{10}(t) = \sum_{k=1}^{5} U_{1k}(t) \rho_{k0}(0), \]

(38.4)

\[ \rho_{(21)(12)}^{qq}(t) = \rho_{21}(t) = \sum_{k=1}^{5} U_{2k}(t) U_{1k}(t)^* \rho_{k1}(0), \]

(38.5)

\[ \rho_{(21)(21)}^{qq}(t) = \rho_{22}(t) = \sum_{k=1}^{5} U_{2k}(t) U_{2k}(t)^* \rho_{k2}(0), \]

(38.6)

\[ \rho_{(22)(12)}^{qq}(t) = \rho_{20}(t) = \sum_{k=1}^{5} U_{2k}(t) \rho_{k0}(0), \]

(38.7)

\[ \rho_{(22)(22)}^{qq}(t) = \rho_{01}(t) = \sum_{k=1}^{5} U_{1k}(t)^* \rho_{k0}(0), \]

(38.8)

It is worth noting that elements (38.2)–(38.10) of reduced density matrix ρ^{qq}(t) of the two-qubit subsystem depend not only on their 9 initial values ρ_{ij}(0), i, j = 0, 1, 2, but also on 36 other quantities ρ_{il}(0), 0 ≤ i ≤ 5, 3 ≤ l ≤ 5 and ρ_{kl}(0), 0 ≤ j ≤ 5, 3 ≤ k ≤ 5. This means that a closed system of linear differential equations for 16 elements ρ_{(i)(j)}^{qq}(t) of reduced density matrix ρ^{qq}(t) of the two-qubit system does not exist.

For application to quantum computation it is necessary to generate quantum gates of the two-qubit subsystem. Because 7 elements (38.1) of reduced density matrix ρ^{qq}(t) of this subsystem vanish if there are only ground and single excitation states of the system of qubits and photons, in order to generate the CNOT quantum gate it is necessary to use at least double excitation states of the system of qubits and photons. At this level of excitations the reduced density matrix ρ(t) is determined by a closed system of 18 × 18 = 324 differential equations for its 324 matrix elements. It is necessary to elaborate a suitable computational technique to solve (numerically) such a large system of rate equations in order to determine 16 matrix elements of reduced density matrix ρ^{qq}(t) of the two-qubit subsystem.

5. Conclusion

The time evolution of a quantum communication system consisting of two identical qubits placed inside two identical single-mode microcavities connected via a single-mode optical fiber was investigated. The interaction of this quantum communication system with the environment causes its decoherence. Rate equations for elements of its reduced density matrix were established in the Markovian approximation with the use of the Linblad formula for the Liouvillian operator. In general they might be solved by means of numerical methods. However, in the special case of resonance between the energy difference ε of two states in each qubit and energy ω of the fiber mode, ε = ω, rate equations for the system excited up to the first level can be solved analytically in the first order approximation with respect to decoherence constants. Their analytical expressions were derived. They give a complete picture of the excitation transmission or transfer between different states of the quantum communication system. The conditions necessary for the perfect quantum state transfer between two qubits in the system without decoherence were presented, and formulae for the average rates of different transmission and transfer processes were established. Possible extension to the case of highly excited systems was also discussed.
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