\( \mathcal{N} = 4 \) SUSY Yang–Mills: three loops made simple(r)

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September 25, 2018

Abstract

We construct universal parton evolution equation that produces space- and time-like anomalous dimensions for the maximally super-symmetric \( \mathcal{N} = 4 \) Yang–Mills field theory model, and find that its kernel satisfies the Gribov–Lipatov reciprocity relation in three loops. Given a simple structure of the evolution kernel, this should help to generate the major part of multi-loop contributions to QCD anomalous dimensions, due to classical soft gluon radiation effects.

1 Introduction

Super-symmetric Yang–Mills theories, which neighbour QCD in their physical content but are easier to handle, may play an important rôle in shedding light upon multi-loop QCD results. In this work we employ the \( \mathcal{N} = 4 \) super-symmetric Yang–Mills (\( \mathcal{N} = 4 \) SYM) field theory in order to better understand some important features of the QCD parton evolution.

Soft gluons and the “inheritance” idea. Permanent fight for increasing precision of QCD predictions necessitates higher order perturbative calculations. Such calculations, and markedly the recently completed three loop analysis of DIS anomalous dimensions \([1]\), yield lengthy expressions reach in hidden physical content that one has to grasp. A higher loop expression necessarily contains certain structures inherited from the preceding orders of the PT expansion. This is the case, for instance, for the renormalization effects that build up the physical coupling \( \alpha_{\text{ph}} \) determining in all orders the intensity of soft gluon emission given by the Born level expression \([2,3]\).

We expect this “inheritance” feature to go deeper and to include, in particular, real and virtual radiative effects due to multiple soft gluons. The basis for such expectation is the celebrated, virtually unknown, Low–Burnett–Kroll theorem \([4]\) which fully applies to QCD as well as to SYM quantum field theories. It tells us that soft gluon radiation has in fact classical nature. It is independent of the quantum state of the radiating parton system, and does not change it. Therefore it should be possible to find a procedure for iteratively “dressing” an underlying lower order hard parton splitting process by soft gluons. The realisation of this programme would allow one to generate, by simple means, the corrections that the anomalous dimensions...
dimension receives from soft gluons in all orders of the perturbative expansion. There is a good reason to believe that soft gluon radiation effects are responsible for the major part of high order contributions to the QCD anomalous dimension. At the two loop level this has been explicitly demonstrated in a particular example of the time-like heavy quark fragmentation function in [2].

To verify this expectation at the three loop level we use as a testing ground the $\mathcal{N}=4$ SYM model. Since soft radiation is universal, the results of the present study are directly applicable to QCD (modulo adjustment of colour factors). Indeed, the diagonal one loop anomalous dimensions ($q \to q$, $g \to g$) contain the universal classical part of the gluon emission probability,

$$\frac{dw}{dx} = C_p \frac{\alpha_{ph}}{\pi} \left[ \frac{x}{1-x} + (1-x)g_p(x) \right].$$  \hspace{1cm} (1.1)

Being proportional to the “colour charge squared” $C_p$, it is otherwise insensitive to the nature of the radiating particle $p$. Quantum effects encoded in $g_p$ obviously depend on $p$ ($g_{1/2} = \frac{1}{4}$ for a fermion, $g_1 = x + x^{-1}$ for a vector emitter, etc.). Importantly, as a consequence of the LBK theorem, the genuine quantum contribution to (1.1) is down, relative to the universal classical piece, by two powers in the $(1-x)$ counting. In the standard approach, in higher orders classical and quantum effects get mixed by the $Q^2$ evolution, resulting in cumbersome expressions for anomalous dimensions.

In the $\mathcal{N}=4$ SYM theory, $g_p = 0$ in (1.1) at one loop: quantum effects due to boson and fermion fields compensate each other, leaving us with a pure “classical physics” at the Born level, and thus provide a perfect ground for testing the “inheritance” idea. The very idea of extraction of the third loop contribution to the $\mathcal{N}=4$ SYM anomalous dimension [5] from the corresponding QCD result [1] was based on the observation that the two-loop expression for the anomalous dimension $\gamma_{\text{SYM}}$ is built of the functions characterised by the same level of transcendentality [6]. In fact, such functions (with transcendentality $\tau = 2k - 1$ in the $k$th order of perturbation theory) are nothing but finite “leftovers” of the real–virtual cancellation of infrared gluon divergences. Bearing this in mind, the study of higher orders in the “universal anomalous dimension” of the $\mathcal{N}=4$ super-multiplet of twist-two operators [5] should help us to understand the structure of the iteration of pure classical gluon emissions in QCD.

Analytic link between parton distribution and fragmentation functions. It is important to stress that apart from being large, soft gluon effects are also responsible for generating enhanced $\ln^n(1-x)$ corrections in the quasi-elastic kinematics, $1-x \ll 1$, both for parton distribution functions in space-like process, $x_B \to 1$, and fragmentation functions in time-like parton cascades, $x_F \to 1$. Such corrections translate into $\ln N$ terms in the asymptotics of anomalous dimensions at large $N$. In one loop [7–9], where $\gamma(N) \simeq C_p \frac{\alpha_s}{\pi}\psi(N) \propto \ln N$, the task of analytic continuation was solved in [10] where it was proposed to treat certain logarithms in arithmetic sense, $\ln(1-x) \Rightarrow \ln|1-x|$; in the QCD context this idea had been implemented in [9]. However, in higher orders the power $n$ of the $(\ln N)^n$ enhancement in the anomalous dimension increases, thus impairing the analytic link between DIS and $e^+e^-$ annihilation channels. For a thorough discussion of the problem at the two–loop level see [11] and references therein.

The $\mathcal{N}=4$ SYM framework is perfectly suited for shedding light on the problem of finding a way to analytically relate these cross-channel processes. For the benefit of the reader, let us
recall this problem in more detail.

In the DIS case the large virtual momentum \( q \) transferred from the incident lepton to the target nucleon with momentum \( P \) is space-like, \( q^2 < 0 \). Inelasticity of the process is conveniently characterised by the Bjorken variable \( x_B = -q^2/(2Pq) \). Inclusive fragmentation of an \( e^+e^- \) pair with total momentum \( q \) (large positive invariant mass squared \( q^2 \)) into a final state hadron with momentum \( P \) is characterised by the Feynman variable \( x_F = 2(Pq)/q^2 \) (hadron energy fraction in the \( e^+e^- \) cms.). The fact that Bjorken and Feynman variables are indicated by the same letter is certainly not accidental. In both channels \( 0 \leq x \leq 1 \) though these variables are actually mutually reciprocal, \( x_F \Leftrightarrow 1/x_B \), rather than identical. One \( x \) becomes the inverse of the other after the crossing operation \( P_\mu \rightarrow -P_\mu \):

\[
x_B = \frac{-q^2}{2(Pq)}, \quad x_F = \frac{2(Pq)}{q^2}.
\] (1.2)

Apart from the difference in the hadron momentum \( P \) belonging to the initial state in the DIS and final state in the \( e^+e^- \) case, the Feynman diagrams for the two processes are basically the same. “Mass singularities” that emerge when additional momenta in the Feynman diagrams become collinear to \( P \) are therefore also the same. That is why in the two processes a similar parton interpretation emerges in terms of QCD evolution equations.

The common structure of Feynman diagrams and the reciprocity (1.2) are the origin of the close relation between deep inelastic structure functions and \( e^+e^- \) annihilation inclusive fragmentation functions. In particular, the space- and time-like evolution anomalous dimensions turn out to be related.

**Drell-Levy-Yan relation.** Back in 1969, Drell, Levy and Yan have proposed that the two channels can be linked by analytical continuation [12]. If one takes the space-like splitting function \( \tilde{\gamma}^{(S)}(x) \) — the Mellin transform of the anomalous dimension \( \gamma^{(S)}(N) \) — and **analytically continues** it from the physical region \( x \equiv x_B \leq 1 \) to \( x > 1 \), such a procedure would yield the time-like splitting function \( \tilde{\gamma}^{(T)}(x_F) \) with \( x_F = x^{-1} < 1 \):

\[
\tilde{\gamma}^{(T)}(x) = -x^{-1} \tilde{\gamma}^{(S)}(x^{-1}).
\] (1.3)

**Gribov-Lipatov reciprocity.** Gribov and Lipatov in their first systematic study of DIS and \( e^+e^- \) annihilation processes in the QFT framework [7], suggested that the common dynamics in the two channels results simply in the identity

\[
\tilde{\gamma}^{(T)}(x) = \tilde{\gamma}^{(S)}(x).
\] (1.4a)

Taken together with the DLY relation (1.3) this leads to the relation known as the GL reciprocity:

\[
\tilde{\gamma}(x) = -x \tilde{\gamma}(x^{-1}).
\] (1.4b)

In the Mellin moment space,

\[
\gamma(N) = M[\tilde{\gamma}(x)] \equiv \int_0^1 \frac{dx}{x} x^N \tilde{\gamma}(x),
\] (1.5)

(1.4) translates into the symmetry \( N \rightarrow -(N+1) \); in other words, in order to satisfy the GL reciprocity, the anomalous dimension \( \gamma(N) \) has to be a function of a single variable \( N(N+1) \) —
the Casimir operator of the collinear $SO(1,2)$ subgroup of the $SO(2,4)$ conformal group [13]. A group symmetry nature of the GL reciprocity was advocated already in the early 70’s when the issue was first raised [14]. Indeed, since $x$ is a light-cone momentum fraction, $y = \ln x$ represents parton (pseudo)rapidity: $k_0 = m_\perp \cosh y$, $k_z = m_\perp \sinh y$. Therefore in the Mellin transformation, 

$$\int \frac{dx}{x} x^N = \int dy e^{yN} \implies \int d\phi e^{i\phi N}, \quad \phi = iy,$$

an analogy between the Lorenz boost parameter $y$ and an imaginary rotation angle allows one to look upon the conjugate variable $N$ as “angular momentum”, with $J^2 = N(N + 1)$ the corresponding Casimir operator.

True in one loop (leading logarithmic approximation), see e.g. (1.1), the relations (1.4) are known to break already in the second loop [15, 16].

**Reciprocity respecting evolution equation.** This letter presents a development of the project that aims to reorganise the QFT parton evolution picture [17] in such a way as to preserve a close relation between space- and time-like parton dynamics. In [2,18] a “reciprocity respecting” equation (RRE) for QCD anomalous dimensions has been suggested, based on a notion of the universal “evolution kernel” $P(x, \alpha)$, identical for two channels, that satisfies the Gribov–Lipatov relation in all orders:

$$\partial_\tau D^{(T/S)}(T/S)(N, Q^2) \equiv \gamma_\sigma(N, \alpha) D^{(T/S)}(N, Q^2) = \int_0^1 \frac{dz}{z} z^N P(z, \alpha) D^{(T/S)}(N, z^\sigma Q^2). \quad (1.6)$$

Here $\tau = \ln Q^2$, while $\sigma = +1$ for $e^+e^-$ annihilation parton fragmentation functions $D^{(T)}$ (time-like evolution), and $\sigma = -1$ for DIS parton distributions $D^{(S)}$ (space-like). Using the Taylor expansion trick we obtain the formal solution of (1.6):

$$\partial_\tau D^{(T/S)}(T/S)(N, Q^2) = \int_0^1 \frac{dz}{z} z^N P(z, \alpha) z^\sigma \partial_\tau D^{(T/S)}(N, Q^2). \quad (1.7)$$

At the three-loop level, this equation has been used to analyse the $x \rightarrow 1$ behaviour of the splitting functions $\tilde{\gamma}^{(T/S)}(x)$ in [18]. There the all-order relation $C = -\sigma A^2$ between the magnitudes of the $A(1-x)^{-1}$ and $C \ln(1-x)$ singularities was established, observed by Moch, Vermaseren and Vogt for quark and gluon space-like anomalous dimensions [1]. The prediction of (1.7) for the time-like case was verified at the third loop by explicit calculation of the non-singlet anomalous dimension in [19].

The analysis of next subleading contributions in the $x \rightarrow 1$ limit attempted in [18] was complicated by the presence of scheme dependent contributions proportional to the $\beta$ function, starting from the constant term $\tilde{\gamma}(x) = O(1)$, which contribution also had to be under full “inheritance” control, according to the LBK wisdom. It is indeed in the $\mathcal{N}=4$ SYM model, having $\beta(\alpha) \equiv 0$.

Here the RRE (1.7) readily reduces to a pair of complementary relations which hold in all orders and for arbitrary $N$:

$$\gamma^{(T/S)}(N) = P(N + \sigma \cdot \gamma^{(T/S)}(N)), \quad (1.8a)$$
$$P(N) = \gamma^{(T/S)}(N - \sigma \cdot P(N)). \quad (1.8b)$$

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1 We thank G. Korchemsky for communicating us an elegant solution of the problem encountered in [18].
Expanding (1.8a) results in

\[ \gamma \equiv \gamma^{(T/S)}[\alpha] = \mathcal{P} + \sigma \gamma \dot{\mathcal{P}} + \frac{1}{2} \gamma^2 \ddot{\mathcal{P}} + \sigma \frac{1}{3!} \gamma^3 \dddot{\mathcal{P}} + \mathcal{O}(\alpha^5), \]

where dots indicate derivatives with respect to \( N \) and \( \mathcal{P} \equiv \mathcal{P}(N) \). Since \( \gamma = \mathcal{O}(\alpha) \), this equation gives iteratively a tower of inherited higher order contributions to the anomalous dimensions in both channels. Solving (1.9) perturbatively gives

\[ \frac{1}{2}(\gamma^{(T)} + \gamma^{(S)}) \equiv \mathcal{P} + R, \quad R = \left[ \mathcal{P} \dot{\mathcal{P}}^2 + \frac{1}{2} \mathcal{P}^2 \dddot{\mathcal{P}} \right] + \mathcal{O}(\alpha^5), \] (1.10a)

\[ \frac{1}{2}(\gamma^{(T)} - \gamma^{(S)}) \equiv V = \mathcal{P} \dot{\mathcal{P}} + \left[ \mathcal{P} \dot{\mathcal{P}}^3 + \frac{3}{2} \mathcal{P}^2 \dddot{\mathcal{P}} \ddot{\mathcal{P}} + \frac{1}{6} \mathcal{P}^3 \dddot{\mathcal{P}} \dot{\mathcal{P}} \right] + \mathcal{O}(\alpha^6). \] (1.10b)

The difference of space- and time-like anomalous dimensions \( V \) in (1.10b) does not have a linear term in the evolution kernel \( \mathcal{P} \); therefore, knowing \( \mathcal{P} \) at order \( k \), one predicts \( V \) at the order \( k + 1 \). Moreover, since the induced terms in (1.10) contain powers of \( \mathcal{P} \), the lowest order evolution kernel \( \mathcal{P}_1 \propto \ln N \) generates a tree of logarithmically enhanced terms, up to \( \alpha^k \ln^k N/N^{k+1} \) in order \( k \) of the perturbative expansion.

If the large-\( N \) singularity of the evolution kernel \( \mathcal{P} \) turns out to be stable under perturbation, given \( \mathcal{P} \) at order \( k \), (1.9) would predict all logarithmically enhanced contributions \( f_m(N) \cdot \ln^m N, \ 2 \leq m \leq k + 1 \) at the next order \( k + 1 \), where \( f_m(N) = \mathcal{O}(N^{-k-1}) \) are known functions that are regular at \( N = \infty \).

We translate the space-like \( \mathcal{N} = 4 \) SYM anomalous dimension recently calculated by Kotikov, Lipatov, Onishchenko and Velizhanin in three loops [5] into the corresponding evolution kernel \( \mathcal{P} \) of (1.8). We find that \( \mathcal{P}_3 \) is given by a rather transparent expression, much more compact than \( \gamma_3 \), exhibits only a single power of the \( \ln N \) enhancement at large \( N \) and satisfies the GL reciprocity (1.4). Grasping internal logic of the second and third order contributions to \( \mathcal{P} \) should help to gain better understanding of the structure of multi-loop QCD anomalous dimensions.

The paper is organised as follows. In section 2 we present the evolution kernel \( \mathcal{P} \) in three loops. In section 3 we expose in detail essential properties of the answer, including important kinematical limits. In section 4 we present our conclusions and discuss perspectives for future studies.

## 2 Answer

We construct the RR evolution kernel,

\[ \mathcal{P}(\alpha, N) = \sum_{k=1} a^k \mathcal{P}_k(N), \quad \gamma^{(T/S)}(\alpha, N) = \sum_{k=1} a^k \gamma^{(T/S)}_k(N), \quad a = \frac{C_A}{\pi}, \] (2.11)

employing the space-like anomalous dimension, \( \gamma = \gamma^{(S)} \), for the \( \mathcal{N} = 4 \) SYM model. From the perturbative expansion of (1.10),

\[ \gamma_1 = \mathcal{P}_1, \quad \gamma_2 = \mathcal{P}_2 - \mathcal{P}_1 \dot{\mathcal{P}}_1, \]

\[ \gamma_3 = \mathcal{P}_3 - \left[ \dot{\mathcal{P}}_1 \mathcal{P}_2 + \mathcal{P}_1 \ddot{\mathcal{P}}_1 \right] + \left[ \mathcal{P}_1 \dot{\mathcal{P}}_1^2 + \frac{1}{2} \mathcal{P}_1^2 \dddot{\mathcal{P}}_1 \right]. \] (2.12)
Within our normalisation convention \( (2.11) \), one has \( \gamma_k = \gamma_{\text{uni}}^{(k-1)}/4^k \), with \( \gamma_{\text{uni}} \) the anomalous dimension in the original notation of [5]. The latter is given in terms of harmonic sum functions \( S_{\pm a}(N) \) and \( S_{-p,a,...}(N) \), see [20, 21] and Appendix \[A\]. It reads

\[
\begin{align*}
\gamma_1 &= -S_1; \quad (2.13a) \\
\gamma_2 &= \frac{1}{2} S_3 + S_1 S_2 + \left( \frac{1}{2} S_{-3} + S_1 S_{-2} - S_{-2},1 \right); \quad (2.13b) \\
\gamma_3 &= -\frac{1}{2} S_5 - \left[ \left( S_1^2 S_3 + \frac{1}{2} S_2 S_3 + S_1 S_2^2 + \frac{3}{2} S_1 S_4 \right) - S_1 \left[ 4 S_{-4} + \frac{1}{2} S_{-2}^2 + 2 S_2 S_{-2} - 6 S_{-3,1} - 5 S_{-2,2} + 8 S_{-2,1,1} \right] \\
&\quad - \left( \frac{1}{2} S_2 + 3 S_{1}^2 \right) S_{-3} - S_3 S_{-2} + (S_2 + 2 S_1^2) S_{-2,1} + 12 S_{-2,1,1,1} \\
&\quad - 6 (S_{-3,1,1} + S_{-2,1,2} + S_{-2,2,1}) + 3 (S_{-4,1} + S_{-3,2} + S_{-2,3}) - \frac{3}{2} S_{-5}. \quad (2.13c)
\end{align*}
\]

In order to arrive at a compact expression for the third loop evolution kernel \( P_3 \) we introduce the characteristic function \( \varphi \),

\[
d\varphi = \frac{dz}{z} \ln \frac{(1+z)^2}{4z} = \ln \cosh^2 \left( \frac{1}{2} \ln z \right), \quad (2.14a)
\]

which is invariant under the transformation \( z \to z^{-1} \), construct integrals

\[
\Phi_\tau(x) = \frac{1}{\Gamma(\tau)} \int_x^1 \frac{dz}{z} (\varphi(x) - \varphi(z))^{\tau-1}, \quad (2.14b)
\]

(where subscript \( \tau \) marks transcendentality of the function), and define Mellin transforms

\[
\hat{Y}_{-m}(N) = (-1)^N M \left[ \frac{x}{1 + x} \Phi_{m-1}(x) \right]. \quad (2.14c)
\]

Functions \( \hat{Y}_{-m}(N) \) are given by linear combinations

\[
\begin{align*}
\hat{Y}_{-3} &= \hat{S}_{-3} - 2 \hat{S}_{1,-2}; \\
\hat{Y}_{-4} &= \hat{S}_{-4} - 2 (\hat{S}_{1,-3} + \hat{S}_{2,-2}) + 4 \hat{S}_{1,1,-2}; \\
\hat{Y}_{-5} &= \hat{S}_{-5} - 2 (\hat{S}_{3,-2} + \hat{S}_{2,-3} + \hat{S}_{1,-4}) + 4 (\hat{S}_{2,1,-2} + \hat{S}_{1,2,-2} + \hat{S}_{1,1,-3}) - 8 \hat{S}_{1,1,1,-2}, \quad (2.14d)
\end{align*}
\]

where hat crowns a sum with subtracted \( N = \infty \) value:

\[
\hat{S}_\theta(N) \equiv S_\theta(N) - S_\theta(\infty). \quad (2.15)
\]

Here \( \hat{S}_{m,-p}(N) \) are complementary harmonic sums,

\[
\hat{S}_{a_1,a_2,...,a_n}(N) = (-1)^n \sum_{k_1 = N+1}^\infty \sum_{k_2 = k_1 + 1}^\infty \cdots \sum_{k_n = k_{n-1} + 1}^\infty \frac{\text{sgn} a_1 k_1}{k_1^{a_1}} \frac{\text{sgn} a_2 k_1}{k_1^{a_2}} \ldots \frac{\text{sgn} a_n k_1}{k_1^{a_n}}, \quad (2.16)
\]

whose relation to standard multi-index harmonic sums is displayed in Appendix \[A\]. The functions \( (2.14d) \) are built of harmonic sums with one, and the rightmost, negative index (therefore the notation \( Y_{-m} \)). Internal logic of the constructs \( (2.14d) \) is transparent and makes generalisation straightforward.

In these terms, the \( \mathcal{N} = 4 \) SYM evolution kernel, in first three orders of the perturbative expansion in the physical coupling,

\[
a_{ph} = a \left( 1 - \frac{1}{2} \zeta_2 a + \frac{1}{20} \zeta_2^2 a^2 + \ldots \right), \quad (2.17)
\]
takes a compact form
\begin{align}
P_1 &= -S_1; \quad (2.18a) \\
P_2 &= \frac{1}{2}S_3 - \frac{1}{2}Y_3 + B_2; \quad (2.18b) \\
P_3 &= -\frac{1}{2}S_5 + \frac{3}{2}Y_5 + B_3 + \zeta_2 \cdot \frac{1}{2}S_3 \\
&\quad + S_1 \cdot \left[ Y_4 - \frac{1}{2}(S_4 + S_2^2) + \zeta_2 \cdot \frac{1}{2}S_2 \right]. \quad (2.18c)
\end{align}

Here \( B_2 = \frac{3}{4}\zeta_3 \), \( B_3 = -\frac{1}{8}\zeta_2\zeta_3 - \frac{5}{4}\zeta_5 \).

### 3 Discussion

Let us list some characteristic features of the RR evolution kernel (2.18).

The evolution kernel (2.18) (as well as anomalous dimensions) contains harmonic sums with positive and negative indices. These contributions have different origin and play markedly different roles in extreme kinematical limits. Harmonic sums bearing a negative index (indices) oscillate with \( N \), see (A.1c) and (2.14c):
\[
P(N) = P^\text{pos}(N) + (-1)^N \cdot P^\text{neg}(N).
\]

The presence of the oscillating factor \((-1)^N\) calls for introducing two separate functions, analytically continued from even and odd moments \( N \). Indeed, in QCD one has two independent non-singlet anomalous dimensions, replacing (3.19) by
\[
\gamma_{ns}^{(\pm)} = pqq \pm p\bar{q}.
\]

The two anomalous dimensions have different signature with respect to \( s \leftrightarrow u \ (x \to -x) \) crossing. In \( \mathcal{N} = 4 \) SYM “quarks” are Majorana fermions, making \( q \) and \( \bar{q} \) physically indistinguishable. In the light-cone formulation of \( \mathcal{N} = 4 \) SYM [22, 23], in the study of anomalous dimensions [24] the negative signature does not appear [25], since operators built of the unique superfield of the theory have vanishing odd moments. Therefore, in the \( \mathcal{N} = 4 \) SYM context the only possible continuation to non-integer moments consists of dropping the factor \((-1)^N\), as it has been done in [5], resulting in \( \gamma_{\text{uni}} \equiv \gamma^{(+)} \). We should keep in mind, however, that both signatures are present in the QCD case, where the maximal transcendentality structures of (2.18) are also present and are responsible for a significant part of higher order contributions.

### 3.1 Inheritance

Compared to the standard anomalous dimension (2.13), the presence of harmonic sums with positive indices in the RR evolution kernel (2.18) is minimal:
\[
\gamma_2 = \frac{1}{2}S_3 + S_1S_2 + \cdots, \quad P_2 = \frac{1}{2}S_3 + \cdots;
\]
\[
\gamma_3 = -\frac{1}{2}S_5 - [S_1^2S_3 + \frac{1}{2}S_2S_3 + S_1S_2^2 + \frac{3}{2}S_1S_4] + \cdots, \quad P_3 = -\frac{1}{2}S_5 + \cdots,
\]
where \( \cdots \) mark contributions of negative index sums. Thus, all non-linear combinations of \( S_i \) in the anomalous dimension has been generated by the RRE and are therefore physically “inherited” from the first loop.
3.2 Negative index sums: relation to Feynman graph discontinuities

Sums with one negative index contribute to the anomalous dimension starting from the second loop. Their appearance can be traced to non-planar diagrams that can be cut both in $s$ and $u$, and thus have non-zero Mandelstam double spectral function, $\rho_{su} \neq 0$.

The diagrams for non-singlet QCD quark evolution shown in Fig. 1 can be related by the crossing transformation $s \leftrightarrow u \simeq -s$ which, in the language of the Bjorken $x$ variable, formally corresponds to the reflection $x \rightarrow -x$. It is therefore only natural that the second

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{diagram.png}
\caption{On the origin of negative index harmonic sums}
\end{figure}

loop non-singlet $q_i \rightarrow \bar{q}_i$ anomalous dimension described by the graph (b), is proportional, in $x$ representation, to the first loop $q_i \rightarrow q_i$ splitting function evaluated at the reflected point,

$$p_{q\bar{q}}(x) = a_p^2 \left( C_A - C_F \right) p_{q\bar{q}}(-x) \cdot \Phi_2(x), \quad p_{q\bar{q}}(x) = \frac{1 + x^2}{2(1 - x)}.$$ (3.22)

The proportionality factor $\Phi_2(x)$ is a function of transcendentality 2, defined in (2.14b). It is important to stress that this factor appears, invariably, in second loop anomalous dimensions, multiplying the full first loop splitting functions $p_{BA}(-x)$ in all parton transitions, $B \rightarrow A$, including non-diagonal ones, see [16]. Using three-loop results of [1], it is straightforward to verify that in the non-singlet QCD anomalous dimensions $\gamma_{ns}^{(\pm)}$ every harmonic sum (maximal transcendentality or not) bearing a negative index is proportional to $2C_F - C_A$ (colour suppressed), that is has non-planar origin.

Let us compare contributions of positive and negative index sums. For the second loop evolution kernel we have

$$\begin{align*}
M^{-1} \left[ \frac{1}{2} \hat{S}_3 \right] &= \frac{1}{2} \hat{P}_1(x) \cdot \left( -\frac{1}{2} \ln^2 x \right) \equiv \hat{P}_2^{\text{pos}}(x), \\
M^{-1} \left[ -\frac{1}{2} \hat{Y}_{-3} \cdot (-1)^N \right] &= -\frac{1}{2} \hat{P}_1(-x) \cdot \left( -\Phi_2(x) \right) \equiv \hat{P}_2^{\text{neg}}(x),
\end{align*}$$ (3.23a,b)

with $\hat{P}_1(x) = x/(1 - x)$ the one loop splitting function, the inverse Mellin image of $P_1(N) = -S_1(N)$. The fact that these two terms originate from different cuts of the same diagram can

\footnote{called $S_2(x)$ in the original papers [15], [16].}
be seen from internal structure of corresponding contributions:

\[ \tilde{P}^{\text{pos}}_2(x) = \frac{1}{2} \frac{x}{1-x} \cdot \int_x^1 \frac{dz}{z} \frac{1+z}{1+z} \ln z, \]  
(3.24a)

\[ \tilde{P}^{\text{neg}}_2(x) = \frac{1}{2} \frac{x}{1+x} \cdot \int_x^1 \frac{dz}{z} \frac{1-z}{1+z} \ln z. \]  
(3.24b)

where we have used (2.14a) to transform the integral

\[ \Phi_2(x) \equiv \int_x^1 \frac{dz}{z} \ln \left( \frac{(1+x)^2}{(1+z)^2} \right) = \int_x^1 \frac{dz}{2z} \frac{z-1}{z+1} \ln z. \]

Higher order positive index contributions, \( \hat{S}_{2k-1} \) in (2.18) with \( k \) the order of the perturbative expansion, have a simple origin and a simple structure:

\[ \hat{S}_3(N) = \frac{1}{2!} \frac{d^2}{dN^2} S_1(N), \quad \text{M}^{-1}[\hat{S}_3] = \frac{1}{2!} x \frac{\ln^2 x}{x-1}; \]  
(3.25a)

\[ \hat{S}_5(N) = \frac{1}{4!} \frac{d^4}{dN^4} S_1(N), \quad \text{M}^{-1}[\hat{S}_5] = \frac{1}{4!} x \frac{\ln^4 x}{x-1}. \]  
(3.25b)

Given that the negative and positive index contributions are related by the \( s \leftrightarrow u \) crossing, and guided by the second loop example, a possibility to generate the negative index piece from its positive counterpart does not look implausible.

### 3.3 Gribov–Lipatov reciprocity

The Gribov–Lipatov reciprocity is respected by each piece entering the evolution kernel. Indeed, it is easy to see from (A.1) that the inverse Mellin images of harmonic sums \( \hat{S}_a \) with odd \( a \), as well as negative index sums \( \hat{S}_{-b} \) with \( b \) even, satisfy the relation

\[ F(x) = -x F(x^{-1}) \]  
(3.26)

(For the distribution \( \text{M}^{-1}[S_1] \) this relation holds for \( x \neq 1 \).) The same is true for functions \( \hat{Y}_{-m} \), see (2.14),

\[ \text{M}^{-1}[-(-1)^N \hat{Y}_{-m}] = \frac{x}{1+x} \cdot \Phi_{m-1}(x), \]  
(3.27a)

which transform according to (3.26), due to

\[ \Phi_r(x^{-1}) = -\Phi_r(x). \]  
(3.27b)

Finally, an observation that (3.26) is stable under convolution of two RR functions, concludes the proof that every item in (2.18) satisfies the GL reciprocity relation (1.4).

In the moment space, as we have discussed above, one expects symmetry with respect to the substitution

\[ \mathcal{P}(N) = \mathcal{P}(-N-1), \]  
(3.28)

and thus the dependence on the combination \( N(N+1) \). Indeed, in QCD, for example, the second loop non-singlet positive signature evolution kernel reads [2]

\[ \tilde{\mathcal{P}}^{\text{ns,+}}_2(x) = \left( \frac{1}{2} C_A - C_F \right) \left[ p_{qq}(x) \cdot \frac{1}{2} \ln^2 x + p_{qq}(-x) \cdot \Phi_2(x) \right] \]
\[ - \frac{1}{4} C_F \left( \frac{1}{2} (1+x) \ln x + (1-x) \right) + \frac{1}{4} C_A (1-x). \]  
(3.29)
It contains “algebraic” pieces whose dependence on the “Casimir” is transparent,
\[
M_1 [1 - x] = \frac{1}{N(N + 1)}, \quad M_1 [(1 + x) \ln x] = \frac{2N(N + 1) + 1}{N^2(N + 1)^2}. \tag{3.30}
\]

For maximal transcendentality contributions this property also holds, in certain sense. It cannot
be applied to integer \(N\) since the reflected image of, say, the leading harmonic sum \(S_1(N)\) has
poles at each positive integer point:
\[
S_1(-N - 1) = S_1(N) - \frac{\pi \cos \pi N}{\sin \pi N}. \tag{3.31}
\]

However, the symmetry holds for half-integer values of \(N\), from where the relation \(\text{(3.28)}\) can
be continued onto the entire complex \(N\)-plane.

### 3.4 Large \(x\)

Quasi-elastic scattering, \((1 - x) \ll 1\), is determined by the behaviour at large \(N\), where
\[
\begin{align*}
\hat{S}_a(N) & \simeq N^{-a+1}, \quad \hat{S}_{-a}(N) \simeq \frac{1}{2} N^{-a}, \quad \tag{3.32a} \\
\hat{Y}_{-a}(N) & = \mathcal{O}(N^{-2a+2}) \quad (a \geq 2). \quad \tag{3.32b}
\end{align*}
\]

The second loop evolution kernel \(\text{(2.18a)}\) is “regular” at \(N = +\infty\) (that is, does not contain
\(\ln N\) terms) and is dominated by the positive index term,
\[
\hat{S}_3(N) \simeq N^{-2} \implies \delta \hat{P}_2(x) \sim (1 - x). \tag{3.33}
\]

It is down by two powers in \((1 - x)\) counting as compared to the Born term \(a_{ph}(1 - x)^{-1}\), while
the negative index contribution is suppressed ever further. Power counting looks natural: the
graph Fig\(\text{(11)b}\) contains in the intermediate state two hard partons, each producing the phase
space suppression factor \(d^3k/k \propto (1 - x)^2\), resulting in
\[
\hat{Y}_{-3}(N) \propto N^{-4} \implies \delta \hat{P}_2(x) \sim (1 - x)^3 \propto (1 - x)^{-1} \cdot [(1 - x)^2]^2.
\]

Unlike the second loop, the third loop contribution to the kernel \(\mathcal{P}_3\) exhibits a single power
of logarithmic enhancement, \(S_1(N) \propto \ln N\). Now the positive index term is negligible, \(\hat{S}_5 \propto
N^{-4} \Rightarrow a_{ph}^3(1 - x)^3\), and it is the last singular term in \(\text{(2.18a)}\) that turns out to be leading in
this limit:
\[
\begin{align*}
\hat{S}_2 \cdot S_1(N) & \hat{S}_{-2}(N) \simeq \frac{\zeta_2 \ln N}{4 N^2} \implies \delta \hat{P}_3(x) \sim (1 - x) \ln(1 - x). \quad \tag{3.34}
\end{align*}
\]

It is worth noticing that the origin of the contribution \(\text{(3.31)}\) is rather specific. Namely, it
originates from harmonic sums with two negative indices. In QCD, maximal transcendentality
sums of this nature in the third loop are \([1]\)
\[
\begin{align*}
\frac{1}{6\gamma} & \Rightarrow \frac{1}{4} \left[ 2C_F (2C_F - C_A) \left[ -2(C_F + C_A)S_{-3,2} + 2(3C_F - C_A)S_{-2,3} \\
& + 4C_A S_{-2,-2,1} + 4(3C_F - 2C_A)S_{-2,1,-2} + 2(3C_A - 2C_F)S_{1,-2,-2} \right].
\end{align*}
\]

Equating the colour factors, \(C_F = C_A\), one obtains a logarithmically enhanced contribution to
the universal \(N=4\) SYM anomalous dimension,
\[
\frac{1}{6\gamma} \Rightarrow S_{-3,-2} + S_{-2,-3} - S_{1,-2,-2} - S_{-2,1,-2} - S_{-2,-2,1} = -\frac{1}{2} S_1(S_{-2}^2 + S_4). \tag{3.35}
\]
The term built of positive index sums, $S_1S_4$, is taken care of by the RRE, while the squared negative index sum, $S_2^{-2}$, propagates to the evolution kernel (2.18c):

$$-\frac{1}{2}S_1S_2^{-2} = -\frac{1}{2}S_1(\hat{S}_{-2} - \frac{1}{2}\zeta_2)^2 = -\frac{1}{2}\zeta_2^2 \cdot S_1 - \frac{1}{2}S_1\hat{S}_{-2}^2 + \frac{1}{2}\zeta_2S_1\hat{S}_{-2}. \quad (3.36)$$

The first term on the r.h.s. participates in forming the $\alpha^3$ correction to the physical coupling, the second is $\mathcal{O}(\ln N/N^3)$. The last term is a cross-product of the subtracted sum, $\hat{S}_{-2} \propto N^{-2}$, and $\hat{S}_{-2}(\infty) = -\frac{1}{2}\zeta_2$, and gives rise to a bizarre correction (3.34).

The third loop contribution (3.34) looks as a radiative correction to the previous order kernel (3.33). Taken at face value, the presence of this logarithmically enhanced contribution, though subleading in the overall $(1-x)$ counting, negates our attempt to construct a perturbative scheme such that relatively large higher order corrections in all orders would be generated automatically. The fact that this proposal stumbled at the level of the third loop, calls for a deeper insight into the physical origin of contributions described by harmonic sums with two negative indices, in order to incorporate this eventuality into the “inheritance” framework by further elaborating the RR equation and its evolution kernel.

### 3.5 Small $x$

Small-$x$ behaviour of the anomalous dimension, and of the evolution kernel, is determined by the rightmost singularity in the $N$ plane. In reality, the BFKL limit in $N=4$ SYM is described by the anomalous dimension $\gamma_{\text{uni}}(N-2)$ having a pole at $N=1$. For the sake of simplicity, we will ignore the shift of the argument and discuss the behaviour of $\mathcal{P}(N)$ at $N = 1 + \omega$ for $\omega \to 0$. Participating harmonic sums have the following expansion at this point:

$$S_1(\omega) = -\omega^{-1} + \zeta_2 \cdot \omega + \sum_{k=3} (-1)^k \zeta_k \omega^{k-1}, \quad (3.37a)$$

$$\hat{S}_a(\omega) = -\omega^{-a} - \zeta_a + \mathcal{O}(\omega), \quad a \geq 2; \quad (3.37b)$$

$$\hat{S}_{-m}(\omega) = -\hat{S}_m(\omega) + \mathcal{O}(\omega^0), \quad \hat{S}_{-2}(\omega) = \omega^{-2} - \frac{1}{2}\zeta_2 + \mathcal{O}(\omega); \quad (3.37c)$$

$$\hat{Y}_{-m}(\omega) = \hat{S}_{-m}(\omega) - \zeta_2 \hat{S}_{-m+2} + \ldots; \quad \omega = N + 1 \to 0, \quad (3.37d)$$

where the factor $(-1)^N$ has been suppressed in $\hat{S}_{-m}$ and $\hat{Y}_{-m}$. At small $x$, positive and negative index sums are equally important.

According to (3.37), on the second line of (2.18c) the leading and first subleading singularities cancel,

$$\hat{Y}_{-4} - \frac{1}{2}(\hat{S}_{-4} + \hat{S}_{-2}^2) + \frac{1}{2}\zeta_2 \hat{S}_{-2} \simeq (\omega^{-4} - \zeta_2\omega^{-2}) - \frac{1}{2}(\omega^{-4} + [\omega^{-2} - \frac{1}{2}\zeta_2^2]) + \frac{1}{2}\zeta_2\omega^{-2} = \mathcal{O}(\omega^{-1}), \quad (3.38)$$

and the evolution kernel can be approximated as

$$\mathcal{P}_1 = -S_1 \simeq \omega^{-1} - \zeta_2\omega + \ldots, \quad (3.39a)$$

$$\mathcal{P}_2 \simeq \frac{1}{2}\hat{S}_3 - \frac{1}{2}\hat{Y}_{-3} \simeq -\omega^{-3} + \frac{1}{2}\zeta_2\omega^{-1} + \ldots, \quad (3.39b)$$

$$\mathcal{P}_3 \Rightarrow -\frac{1}{2}\hat{S}_5 + \frac{1}{2}\hat{Y}_{-5} + \frac{1}{2}\hat{S}_3 \simeq 2\omega^{-5} - 2\zeta_2\omega^{-3} + \ldots. \quad (3.39c)$$
To obtain the anomalous dimension one has to add to (3.39) the generated pieces (1.10). From (3.39) one easily derives perturbative expansion coefficients for the generated reciprocity respecting (R) and violating terms (V):

\[
R_2 = 0, \quad (3.40a)
\]

\[
V_2 = \mathcal{P}_1 \dot{\mathcal{P}}_1 = -\omega^{-3} + \mathcal{O}(1); \quad (3.40b)
\]

\[
R_3 = \mathcal{P}_1 \dot{\mathcal{P}}_1^2 + \frac{1}{2} \mathcal{P}_1^2 \dot{\mathcal{P}}_1 = 2\omega^{-5} - 3\zeta_2\omega^{-3} + \mathcal{O}(1), \quad (3.40c)
\]

\[
V_3 = \mathcal{P}_1 \dot{\mathcal{P}}_2 + \dot{\mathcal{P}}_1 \mathcal{P}_2 = 4\omega^{-5} - 3\zeta_2\omega^{-3} + \mathcal{O}(\omega^{-2}). \quad (3.40d)
\]

The inherited contributions,

\[
R_2 + \sigma V_2 = -\sigma\omega^{-3} + \mathcal{O}(1), \quad (3.41a)
\]

\[
R_3 + \sigma V_3 = 2(1 + 2\sigma)\omega^{-5} - (1 + 3\sigma)\zeta_2\omega^{-3} + \ldots, \quad (3.41b)
\]

also exhibit double logarithmic series and contain subleading singularities. Adding the genuine third loop evolution kernel (3.39) and the generated pieces (3.41), for the space-like channel, \(\sigma = -1\), we observe cancellation of double logarithmic terms, together with single logarithmic singularities:

\[
\gamma^{(S)} \simeq \frac{a_{ph}}{\omega} + 0 \cdot \frac{a_{ph}^2}{\omega^2} + 0 \cdot \frac{a_{ph}^3}{\omega^3} + \zeta_2 \frac{a_{ph}^2}{2\omega} + \ldots, \quad \omega = N + 1 \ll 1, \quad (3.42)
\]

the latter corresponding to two famous “zeroes” in the BFKL anomalous dimension [26].

For the time-like evolution, \(\sigma = +1\), combining (3.39) and (3.41) produces

\[
\gamma^{(T)} \simeq \frac{a_{ph}}{\omega} - 2 \cdot \frac{a_{ph}^2}{\omega^3} + 8 \cdot \frac{a_{ph}^3}{\omega^5} + \zeta_2 \cdot \left(\frac{a_{ph}^2}{2\omega} - \frac{6 a_{ph}^3}{\omega^3}\right) + \ldots. \quad (3.43)
\]

### 3.6 RRE and coherence in small-\(x\) physics

Thus, in the small-\(x\) region the evolution kernel \(\mathcal{P}_k\) contains double logarithmic enhancement factors, \((a_{ph}/\omega)^{k-1}\), multiplying the one loop anomalous dimension, \(a_{ph}/\omega\). In fact, the leading singular contributions given by the first terms in (3.39) represent the start of the all-order series,

\[
\mathcal{P}_{DL} = \frac{1}{2} \left(\sqrt{\omega^2 + 4a_{ph}^2} - \omega\right) \simeq \frac{a_{ph}}{\omega} - \frac{a_{ph}^2}{\omega^3} + \frac{2 a_{ph}^3}{\omega^5} - 5 \frac{a_{ph}^4}{\omega^7} + 14 \frac{a_{ph}^5}{\omega^9} + \ldots, \quad (3.44)
\]

that are characteristic for the evolution kernel. Its origin lies simply in the choice of the logarithmic ordering variable that separates successive parton splittings, known as “parton evolution time”. As has been shown in [18], in order to unify the space- and time-like parton dynamics one has to choose parton fluctuation lifetime, \(k+/k^2\), as a common evolution variable for both channels. It is this choice that leads to the non-local evolution equation (1.6) whose universal kernel \(\mathcal{P}\) no longer coincides with the anomalous dimension. The latter acquires an additional term, \(\sigma V\), which is opposite in sign in the two channels. This specific contribution, which is solely responsible for the breaking of the GL reciprocity, is of pure kinematical origin.
and is easy to derive since, in a given order of perturbative expansion, it is inherited from lower orders.

Inherited $R$ and $V$ contributions (1.10) also possess DL enhanced terms, see (3.41). They combine with those of the kernel proper (3.41) and cancel in the space-like case to produce single logarithmic BFKL series (3.42). The cancellation of DL enhanced terms in $\gamma^{(S)}$ holds in all orders, the reason being a general physical observation, due to V.N. Gribov, that inelastic diffraction vanishes in the forward direction.

Indeed, consider a space-like scattering process in which partron $p$ emits a soft gluon, $p \to p' + k_1$ with $k_{1+} \ll p_+$, which gluon, in turn, radiates a still softer gluon $k_2$ ($k_{2+} \ll k_{1+}$). The logarithmic condition, according to the fluctuation lifetime ordering, reads

$$\frac{k_{1t}^2}{k_{1+}} < \frac{k_{2t}^2}{k_{2+}}.$$  

This region includes, apart from $k_{1t}^2 < k_{2t}^2$, a $k_{1t}$ integration over the interval

$$k_{2t}^2 < k_{1t}^2 < \frac{k_{1+}}{k_{2+}} \cdot k_{2t}^2,$$

which is rather large in the soft kinematics and contributes as $\alpha \ln x \cdot \alpha \ln^2 x$ to the evolution kernel $P_2$. However, in the anomalous dimension this contribution cancels when one takes into account emission of $k_2$ off the partons $p$ and $p'$. Physically, the process can be viewed upon as break-up (inelastic diffraction) of the incident particle $p$ in the external gluon field of transverse size $\lambda_\perp \sim 1/k_{2t}$. However, in the kinematical region (3.45b) the transverse size $1/k_{1t}$ of the parton fluctuation $p \to p' + k_1$ is smaller than the resolution power of the probe, $\lambda_\perp$. In these circumstances the destructive interference between $k_2$ interacting with the initial (p) and final state ($p' + k_1$) comes onto the stage and eliminates (3.45b) reducing lifetime ordering to the transverse momentum one,

$$(P + R) - V : \quad k_{2t}^2/k_+ \implies k_{1t}^2.$$ 

In the language of the RRE, it is the role of generated DL terms to take care of this transformation due to coherent suppression of the part of the phase space (3.45b) kinematically allowed to soft gluons by the dynamically unaware, unsuspecting lifetime ordering.

For the time-like evolution, the very same generated double logs add to those in $P$, resulting in the characteristic DL series for the time-like anomalous dimension,

$$\gamma_{\text{DL}}^{(T)} = \frac{1}{4} \left( \sqrt{\omega^2 + 8a_{\text{ph}} - \omega} \right) = \frac{a_{\text{ph}}}{\omega} - 2 \frac{a_{\text{ph}}^2}{\omega^3} + 8 \frac{a_{\text{ph}}^3}{\omega^5} - 40 \frac{a_{\text{ph}}^4}{\omega^7} + 224 \frac{a_{\text{ph}}^5}{\omega^9} + \ldots ,$$  

whose first three terms are present in (3.43). This structure of the time-like anomalous dimension is equivalent to ordering soft gluon cascades in angles:

$$(P + R) + V : \quad k_2^2/k_+ \to k_1^2/k_+.$$ 

The angular ordering following, once again, from destructive soft gluon interference in the angle disordered kinematics [27], can be considered to be an image of the transverse momentum ordering in the space-like case, via the RRE.
In spite of the fact that the evolution kernel $\mathcal{P}$ does not correspond to a clever choice of the evolution variable in either T- or S-channel, its universality can be exploited to relate DIS and $e^+e^-$ anomalous dimensions. In particular, the RRE links together two puzzling results that were never thought to be of a common origin [28]. They are: the absence of the $\alpha^2$ and $\alpha^3$ terms in the BFKL anomalous dimension (3.42) in the DIS problem and, on the other hand, phenomenon of \textit{exact} angular ordering [29, 30] that seems to hold, unexpectedly, down to the next-to-next-to-leading order in $e^+e^-$ anomalous dimension [31].

4 Conclusions

Maximal transcedentality Euler–Zagier harmonic sums that describe anomalous dimensions of twist-two operators in $\mathcal{N}=4$ super-symmetric Yang–Mills theory are genetically related, from QFT point of view, to multiple radiation of soft gluons. The word “soft” here stands for lack of better terminology. In fact, these gluons may be perfectly “hard”, carrying a large (light-cone) momentum fraction $x = O(1)$. One would rather call them \textit{classical}. The point is as follows. Mellin image of the Born level $\mathcal{N}=4$ SYM anomalous dimension,

$$\gamma_1(N) = -S_1(N) = \psi(N+1) - \psi(1) \implies \tilde{\gamma}_1(x) = \frac{x}{1-x} \ (x < 1), \quad (4.47)$$

is nothing but the universal part of the gluon radiation probability which, according to the celebrated Low–Burnett–Kroll theorem [4] has \textit{classical nature}. Such radiation does not depend on quantum properties of the emitter, but its (colour) charge, and does not affect quantum state of the radiating system. \textit{Classical} (or LBK) gluons (4.47) constitute a significant part of diagonal QCD splitting functions,

$$\tilde{\gamma}_{g\to q}(x) = \frac{C_F\alpha}{\pi} \left[ \frac{x}{1-x} + \frac{(1-x)}{2} \right], \quad (4.48a)$$

$$\tilde{\gamma}_{g\to g}(x) = \frac{C_A\alpha}{\pi} \left[ \frac{x}{1-x} + (1-x) \cdot (x + x^{-1}) \right]. \quad (4.48b)$$

The \textit{quantum} parts of emission probabilities, depending on the details of the emitter, are down by \textit{two powers} of the gluon momentum fraction $(1-x)$, in perfect accord with the LBK wisdom.

“Classical” does not necessarily mean “simple”. However, amazing simplifications do arise, now and then, in the QFT context where classical radiation is concerned. Thus, \textit{exact} Parke–Taylor QCD amplitudes [32] which describe radiation of arbitrary number of gluons with “classical” helicities, $2 \to n$, and generalise production probabilities [33] of soft gluons with $x_i \ll 1$ to arbitrary momentum fractions $x \sim 1$. Hence, a breathtaking possibility of an \textit{all-order} prediction [34] of the magnitude of the “cusp anomalous dimension” [35], which is just another name for intensity of soft (classical) gluon radiation — the “physical” (or “bremsstrahlung”) QCD coupling [2, 3].

From this perspective, dynamics of the $\mathcal{N}=4$ SYM model has a good chance to be “classical”, to extent, since, thanks to super-symmetry, quantum effects due to fermions and bosons have cancelled both in the anomalous dimension (4.47) and in the $\beta$-function. This idea finds a strong support in the conjectured gauge–string duality [36] — AdS/CFT correspondence [37]. A powerful property of integrability, both in the high energy Regge asymptotics of scattering
amplitudes [38, 39] and in the scale dependence of quasi-partonic operators [40] manifests itself most profoundly in the $\mathcal{N} = 4$ SYM case [41]. Descending from SYM back to QCD by reducing the number of super-symmetries, one finds integrability property to survive in certain sub-sectors of QCD parton dynamics, such as baryon wave function and maximal helicity multi-gluon operators [42].

All that makes this model a perfect playing ground for unveiling internal structure of multi-gluon radiation in QCD, whose effects spoil perturbative expansions and bear major responsibility for complexity of higher order calculations, and of QCD anomalous dimensions in particular. In the present work we chose $\mathcal{N} = 4$ SYM model in order to learn how to single out higher loop effects, and structures, that are inherited from the lower orders of the perturbative expansion. To this end we used bookkeeping based on the concept of the Gribov–Lipatov reciprocity respecting evolution equation, RRE, [2, 18] which treats space- and time-like parton multiplication in one go.

Power of the LBK wisdom about the classical nature of radiation described by (4.47) revealed itself most remarkably in this model. The RRE applied to the space-like anomalous dimension $\gamma_{uni}$, known in three loops [5], has generated the major part of higher order contributions in terms of a compact “evolution kernel” $P$ linked to $\gamma_{uni}$ by a non-linear relation (1.8). Simplicity of the evolution kernel is suggestive and invites deeper studies.

Note added. When this project was under completion, we learned about the work by B. Basso and G. Korchemsky in which the reciprocity respecting equation (1.8) has naturally emerged as a consequence of the conformal symmetry.

Acknowledgements

We are grateful to Gregory Korchemsky, Anatoly Kotikov, Lev Lipatov and Gavin Salam for illuminating discussions.

A Harmonic sum representation

The answer contains harmonic sums,

$$S_a = \sum_{k=1}^{N} k^{-a}, \quad S_{-a} = \sum_{k=1}^{N} (-k)^{-a}; \quad \hat{S}_p(N) \equiv S_p(N) - S_p(\infty),$$

having the following Mellin representations:

$$S_1(N) = \mathcal{M} \left[ \frac{x}{(x-1)_+} \right] = \psi(N+1) - \psi(1), \quad (A.1a)$$

and ($a \geq 2$)

$$S_a(N) - \zeta_a \equiv \hat{S}_a(N) = \frac{(-1)^{a-1}}{\Gamma(a)} \mathcal{M} \left[ \frac{x \ln^{a-1} x}{x-1} \right]; \quad (A.1b)$$

$$S_{-a}(N) + \zeta_a(1-2^{-a}) \equiv \hat{S}_{-a}(N) = \frac{(-1)^{a-1}}{\Gamma(a)} \mathcal{M} \left[ \frac{x \ln^{a-1} x}{x+1} \right] \cdot (-1)^N. \quad (A.1c)$$
As for generalized multi-index Euler–Zagier harmonic sums, a representation alternative to that in (2.13) is better suited for analysis of the large-$N$ asymptotics. It employs the basis of sums with the negative index moved from the head to the tail of the index vector, $S_{\vec{m},-p}$ instead of $S_{-p,\vec{m}}$. The latter may contain a logarithmic enhancement factor, $S_{-p,\vec{m}} \propto S_1^F(N)$, with $\ell$ the number of the rightmost +1 indices, e.g., $S_{-2,1,1,1} \propto S_3^2 S^{-2} = O(\ln^3 N/N^2)$.

Crucial simplification of the answer is achieved in terms of \textit{complementary harmonic sums} $S_{\vec{m},-p}$ which we define by a recursive relation

$$S_{a_1,\ldots,a_n}(N) = S_{a_1,\ldots,a_n}(N) - \sum_{k=1}^{n-1} S_{a_1,\ldots,a_k}(N) \cdot S_{a_{k+1},\ldots,a_n}(\infty), \quad n \geq 2.$$  \hfill (A.2)

In particular,

$$S_{a,-p}(N) = S_{a,-b}(N) - S_{a}(N)S_{-p}(\infty),$$

$$S_{a,b,-p}(N) = S_{a,b,-p}(N) - S_{a}(N)S_{b,-p}(\infty) - S_{a,b}(N)S_{-p}(\infty),$$

$$S_{a,b,c,-p}(N) = S_{a,b,c,-p}(N) - S_{a}(N)S_{b,c,-p}(\infty) - S_{a,b}(N)S_{c,-p}(\infty) - S_{a,b,c}(N)S_{-p}(\infty).$$

These functions are finite at $N = \infty$. Subtracting the value at infinity, one arrives at (2.16). The case of all positive indices $a_i$ but the very last one, $a_n = -p$, is special in that such sums fall fast with $N$ and contain no $\ln N$ factors accompanying any subleading power $N^{-k}$.

Mellin transforms of participating multi-index sums are as follows:

$$\Gamma S_{a,-p} = \text{M} \left[ \frac{x}{1+x} \int_x^1 \frac{dz}{z+1} \ln^{a-1} \frac{z}{1} \ln^{p-1} \frac{1}{z} \right],$$

$$\Gamma S_{a,b,-p} = \text{M} \left[ \frac{x}{1+x} \int_x^1 \frac{dz}{z+1} \ln^{p-1} \frac{1}{z} \int_x^1 \frac{dt}{1+t} \ln^{b-1} \frac{z}{t} \ln^{a-1} \frac{1}{x} \right], \hfill (A.3a)$$

$$\Gamma S_{a,b,c,-p} = \text{M} \left[ \frac{x}{1+x} \int_x^1 \frac{dy}{1+y} \ln^{a-1} \frac{y}{x} \int_y^1 \frac{dt}{1+t} \ln^{b-1} \frac{t}{y} \int_t^1 \frac{dz}{1+z} \ln^{c-1} \frac{z}{t} \ln^{p-1} \frac{1}{z} \right],$$

where

$$\Gamma = (-1)^N \cdot \left( \prod_{g \in (a,\ldots,p)} \Gamma(g) \right)^{-1}. \hfill (A.3b)$$

Constructing linear combinations (2.14d) using (A.3a), one arrives at (2.14e).

\section*{B Shifting negative indices to the right}

Here we list relations that help to transform the original representation (2.13) in terms of harmonic sums $S_{-p,\vec{m}}$ of [5] into the negative-on-the-right form, $S_{\vec{m},-p}$:

Two indices

$$S_{-2,1} = -S_{1,-2} + S_1 S_{-2} + S_{-3}, \quad S_{-2,2} = -S_{2,-2} + S_2 S_{-2} + S_{-4},$$

$$S_{-2,3} = -S_{3,-2} + S_3 S_{-2} + S_{-5}, \quad S_{-3,1} = -S_{1,-3} + S_1 S_{-3} + S_{-4},$$

$$S_{-3,2} = -S_{2,-3} + S_2 S_{-3} + S_{-5}, \quad S_{-4,1} = -S_{1,-4} + S_1 S_{-4} + S_{-5}.$$
Three indices

\[
S_{-2,1,1} = S_{1,1,-2} + S_1(S_{-3} - S_{1,-2}) + S_{-2}S_{1,1} - S_{2,-2} - S_{1,-3} + S_{-4},
\]

\[
S_{-3,1,1} = S_{1,1,-3} + S_1(S_{-4} - S_{1,-3}) + S_{-3}S_{1,1} - S_{2,-3} - S_{1,-4} + S_{-5};
\]

\[
S_{-2,2,1} = S_{-2,1,2} = (S_{2,1,-2} + S_{1,2,-2}) + S_{1}(S_{2}S_{-2} - S_{2,-2} + S_{-4})
\]
\[+ S_2(S_{-3} - S_{1,-2}) + S_3S_{-2} - 2S_{3,-2} - S_{2,-3} - S_{1,-4} + 2S_{-5}.
\]

Four indices

\[
3S_{-2,1,1,1} = -3S_{1,1,1,-2} + S_{1,1}(2S_{-3} - 3S_{1,-2}) + S_{1,1}S_{-3} + S_{3}S_{-2}
\]
\[+ S_1(3S_{1,1,-2} + (S_{1,1} + S_{2})S_{-2} - 3S_{2,-2} - 3S_{1,-3} + 3S_{-4})
\]
\[+ 3(S_{1,2,-2} + S_{2,1,-2} + S_{1,1,-3}) - 3(S_{1,-4} + S_{2,-3} + S_{3,-2} - S_{-5}).
\]

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