On the asymptotics of the solution of a wave operator with holomorphic coefficients and its application in mechanics

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Abstract. We study the problem of wave propagation in the medium whose velocity characteristics change under an external impact. The aim of our study is constructing the asymptotics of the solution of a wave operator with a variable coefficient for the Laplacian at infinity. This study allows us to study the elements of transport infrastructure for the presence of forced oscillation modes leading to the destruction of the object.

1. Introduction

In this study, we will consider the problem of wave propagation in the medium whose velocity characteristics change under an external impact. We will study a three-dimensional case. We will consider the problem for the D'Alembert wave operator with a variable time-dependent coefficient of the Laplacian. G A Askaryan was one of the first physicists who obtained an equation of such type in 1961. In his study [5], he considered the self-focusing phenomenon in light beams, which is one of the effects of self-action of light and manifests itself in the concentration of the light-beam energy in the nonlinear medium with a refractive index that increases with increasing light intensity. Askaryan has noted in his paper that the ionizing, thermal, and separating effect of the beam of intensive radiation on the medium can be so strong that it leads to a drastic difference between the medium properties in the beam and outside it, which results in the waveguide propagation of the beam and eliminates its geometrical and diffraction divergence; this interesting phenomenon can be called the self-focusing of the electromagnetic beam. Later, the foundations of a mathematically rigorous theoretical description of this phenomenon were laid by V I Talanov in [6]. In mechanical vibrational data systems, the effect occurs when an external action on the medium, changing the speed characteristics. For example, when compacting the soil at the support of a bridge or building. This study allows us to examine the elements of transport infrastructure for the presence of forced oscillation modes leading to the destruction of the object.

The main operator that describes these effects is the D'Alembert wave operator with a variable time-dependent coefficient of the Laplacian.

\[
\left( \frac{d}{dt} \right)^2 u(x,t) - a(t)\Delta u(x,t) = 0
\]  

(1)

The aim of our study is constructing the asymptotics of a solution to equation (1) at \( t \rightarrow \infty \).

Using the variable separation method, we obtain
\[ u(x,t) = Y(x)v(t). \]

We obtain two equations for determining the functions \( Y(x) \) and \( v(t) \)
\[
\Delta Y(x) + \lambda Y(x) = 0
\]
(2.1)
\[
\left( \frac{d}{dt} \right)^2 v(t) + a(t) \lambda v(t) = 0
\]
(2.2)

Equation (2.1) is the Sturm--Liouville problem for the Laplace operator. This problem is already well studied in [7].

In this case, problem (2.1) is reduced to solving a homogeneous Fredholm equation of the second kind with a symmetric kernel. In the general case, the solution \( Y(x) \) can be represented by using the Green's function
\[
Y(x) = \lambda \int_{D} G(x,x_0)Y(x_0)dV_{x_0},
\]
where \( G(x,x_0) \) --- Green function of the internal Dirichlet problem for the Laplace operator.

Now let us proceed to studying equation (2.2).

It is assumed here that the function \( a(t) > 0 \) is regular at infinity. This means that there is an exterior of the circle \( |x| > R \) such that the function \( a(t) \) can be expanded in it into the convergent power series
\[
a(t) = \sum_{j=0}^{\infty} b_j t^j.
\]

Let us make the substitution of variables \( t = \frac{1}{r} \). We get
\[
\frac{d}{dt} v(t) = \frac{dv}{dr} \frac{dr}{dt} = -\frac{1}{t^2} \frac{dv}{dr} = -r^2 \frac{dv}{dr}.
\]

The equation takes the form
\[
\left( -r^2 \frac{d}{dr} \right)^2 v + a_i(r) \lambda v = 0 \tag{3}
\]

Here \( a_i(r) = \sum_{j=0}^{\infty} b_j r^j \). We have obtained an equation with an irregular singular point \( r = 0 \).

Infinity, generally speaking, is an irregular singular point of equation (3). In the particular case when it is a regular singular point, the problem of constructing the asymptotics of solutions is solved.

As is known, the asymptotics in the neighborhood of regular singular points are conormal, see for example [4], namely, they have the form
\[
\sum_i t^i \sum_{j=0}^{k} a_{i} j \ln \frac{1}{t},
\]
where \( a_{i}, s_{j} \) are some complex numbers.

It was shown in [8] that any homogeneous ordinary differential equation with holomorphic coefficients of order \( n \) can be represented in the form
\[ \hat{H} u = H \left( r, -r^k \frac{d}{dr} \right) u = 0, \]

where \( \hat{H} \) is the differential operator with holomorphic coefficients:

\[ H(r, p) = \sum_{i=0}^{\infty} a_i(r)p^i. \]

Here, \( a_i(r) \) are holomorphic functions; \( a_n(0) \neq 0 \); a formula for calculating the minimum integer non-negative value of \( k \) is obtained. Depending on this value of \( k \), we can divide the equations into three types; each of them corresponds to its own type of asymptotic behavior. The equations for which \( k = 0 \) are attributed to the equations of the first type. In this case, \( r = 0 \) is a non-singular point and the solutions are holomorphic functions. The equations of the second type are those for which \( k = 1 \); in this case, \( r = 0 \) is a regular singular point and, as mentioned above, the asymptotic behavior of the solution is conormal. The equations of the third type are those for which \( k > 1 \); the singular point \( r = 0 \) is irregular. In this case, equation (4) is an equation of non-Fuchsian type. Let us divide the equations of non-Fuchsian type into two classes as follows. We assign to the first class the equations such that the polynomial \( H_0(p) = H(0, p) \), which is called the principal symbol, has only simple roots.

We call such equations the non-Fuchsian equations of the first type. The second class includes the remaining equations, i.e., the non-Fuchsian equations such that the principal symbol has not only simple, but also multiple roots. We call them the non-Fuchsian equations of the second type. Earlier, the case of non-Fuchsian equations of the first type for linear equations and their systems was considered, for example, in Sternberg’s paper [9] and in many other studies. In these papers, the asymptotic expansions of solutions to the non-Fuchsian equations of the first type were constructed; they were obtained in the form of products of the corresponding exponentials by divergent power series as follows:

\[ \sum_{j=1}^{n} e^{j/2} \sum_{r=1}^{\infty} a_{rj} r^r \sum_{j=0}^{\infty} b_j r^j. \]

In the particular case, this asymptotics has the form

\[ u = \sum_{j=1}^{n} e^{\alpha_i/r} \sum_{k=0}^{\infty} a_{ik} r^k, \]

where \( \alpha_i, i = 1, \ldots, n \) are the roots of polynomial \( H_0(p) \), \( \sigma_j \) and \( a_i^k \) are some complex numbers.

However, if the asymptotic expansion

\[ u \approx u_1 + u_2 + \ldots + u_n = e^{\lambda_1/r} \sum_{k=0}^{\infty} a_{1k} r^k + e^{\lambda_2/r} \sum_{k=0}^{\infty} a_{2k} r^k + \ldots + e^{\lambda_n/r} \sum_{k=0}^{\infty} a_{nk} r^k \]  

has at least two terms corresponding to values \( \lambda_1 \) and \( \lambda_2 \) with different real parts (for definiteness, let us assume that \( \text{Re} \lambda_1 > \text{Re} \lambda_2 \)), there is a significant difficulty in interpreting the obtained expansion.

The fact is that all terms in the first element corresponding to the value \( \lambda_1 \) (the dominant element) are of greater order at \( r \to 0 \) than any of the terms in the second (recessive) element. Therefore, to interpret expansion (6), it is necessary to summarize the series (generally speaking, divergent) corresponding to the dominant element. Consideration of the recessive components of the expansion of the solution \( u \) of equation (4) is important, in particular, for constructing uniform asymptotics of the solutions in the complex case, where the point \( r \) moves on the complex plane and the dominant and
recessive elements of the expansion can exchange their roles. In other words, the plane is conditionally divided into sectors in which one of the elements is dominant, another is recessive, and when passing from one sector to another, their leadership changes (the recessive element becomes the dominant one and vice versa). However, in the vicinity of the boundaries of these sectors, several elements are of equal order and none of them can be neglected. This phenomenon occurs, for example, when considering the Euler example, as well as when constructing the asymptotics of the solution for problem (1) and, in general, for all non-Fuchsian asymptotics (5). As a result, to study the asymptotic expansions of solutions to equation (1), it is necessary to introduce a regular method for summing up divergent series to construct uniform asymptotics of solutions with respect to the variable \( r \), i.e., representing asymptotic expansions not only in certain sectors, but also in the entire vicinity of the considered point.

In the late 1980s, a mathematical apparatus suitable for summing up such series was first introduced by J. Ecalle [7] based on the Laplace-Borel transform and the concept of a resurgent function.

The apparatus for interpreting and constructing asymptotic expansions of the form (5), based on the Laplace-Borel transform, is called resurgent analysis. The main idea of resurgent analysis is that the formal Borel transform \( \tilde{u}_j(p) \) are the power series with respect to the dual variable \( p \) that converge in the neighborhood of the points \( p = \lambda_j \). The inverse Borel transform gives a regular way to sum up these series. However, it is necessary to prove an infinite extendibility of functions \( \tilde{u}_j(p) \) i.e., their extendibility along any path on the Riemann surface \( \tilde{u}_j(p) \) not passing through some discrete set depending on the function (an exact definition of the concept of infinite extendibility is given below).

The proof of this fact, as a rule, was very complicated when applying resurgent analysis to constructing asymptotic forms of solutions to differential equations. For equations with degenerations, the proof of infinite extendibility was obtained in the works of V. Shatalov and M. Korovina [1, 3]. This result allows applying the resurgent analysis methods to constructing uniform asymptotics for solutions of linear differential equations with holomorphic coefficients in the spaces of functions of exponential growth.

Owing to this result, in [1, 11], uniform asymptotics of solutions were constructed for the case where the roots of the principle symbol \( H_0(p) = H(0, p) \) are simple. Thus, the question of constructing uniform asymptotics of solutions for this case was closed by directly applying the Laplace-Borel transform to the corresponding equation. We will apply the results of these works in this study.

2. Definitions and auxiliary statements
In this section we give the definitions of some notions of the resurgent analysis we will use below.

Let \( S_{r,e} \) denote the sector \( S_{r,e} = \{ \rho | -\varepsilon < \arg \rho < \varepsilon, \rho < R \} \). We will say that the function \( f \) is analytical on and is of exponential growth no more than \( k \), if there are nonnegative constants \( C \) and \( \alpha \) such that in the sector \( S_{r,e} \) the following inequality is valid:

\[
|f| < Ce^{-\frac{\alpha}{|\rho|}}
\]

Let us denote by \( E_k(S_{r,e}) \) the space of functions holomorphic in the domain \( S_{r,e} \) and of \( k \)-exponential growth in zero.

Let us denote by \( E(C) \) the space of integer functions of exponential growth.
The \( k \)-Laplace-Borel transform of the function \( f(r) \in E_k(S_{R,\varepsilon}) \) is the mapping \( B_k : E_k(S_{R,\varepsilon}) \rightarrow \mathbb{E}(\tilde{\Omega}_{R,\varepsilon})/\mathbb{E}(C) \) defined by the formula

\[
B_k f = \int_0^{r_0} e^{-p/r^k} f(r) \frac{dr}{r^{k+1}}.
\]

Here, \( r_0 \) denotes an arbitrary point of the sector. The inverse \( k \)-Laplace-Borel transform is defined by the formula:

\[
B_k^{-1} \tilde{f} = \frac{k}{2\pi i} \int_{\gamma} e^{p/r^k} \tilde{f}(p) dp.
\]

The contour \( \gamma \) is shown in Figure 1.

**Figure 1.** The contour \( \gamma \).

It should be noted that for the \( k \)-Laplace–Borel transform, the following formulae are true:

\[
B_k \circ \left(-\frac{1}{k} r^{k+1} \frac{\partial}{\partial r}\right)f(r) = p B_k f,
\]

\[
\frac{\partial}{\partial p} \circ B_k f = -B_k \left(\frac{1}{r^k} f(r)\right).
\]

(7)

Now, we can formulate the definition of the \( k \)-resurgent function

**Definition 1.** The function \( \tilde{f} \) is called infinitely extendable, if for any number \( R \), there is a discrete set of points \( Z_R \) in \( C \) such that the function \( \tilde{f} \) is analytically extended from the initial domain of definition along any path with a length smaller than \( R \), which does not pass through \( Z_R \).

**Definition 2.** The element \( f \) of the space \( E_k(S_{R,\varepsilon}) \) is called the \( k \)-resurgent function, if its \( k \)-Laplace–Borel transform \( \tilde{f} = B_k f \) is infinitely extendable.
3. Main results

**Theorem.** If \( b_0 \neq 0, \lambda \neq 0 \) then the asymptotics of the solution to equation (1) in the space of exponentially growing functions in the neighborhood of infinity by \( t \) has the form

\[
u(x, t) \approx \left( e^{ct} + \sum_{i=0}^{\infty} A_i t^{-i} \right) Y(x) \]

and the numbers \( c_i, i = 1, 2 \) are the roots of the polynomial \( p^2 + b_0 \lambda \).

If \( b_0 = 0, b_1 \neq 0, \lambda \neq 0 \) then the asymptotics of the solution has the form

\[
u(x, t) \approx t^{\lambda} \left( e^{c_1 \sqrt{t}} \sum_{i=0}^{\infty} A_i t^{-i/2} + e^{c_2 \sqrt{t}} \sum_{i=0}^{\infty} A_i t^{i/2} \right) Y(X) \]

Here, we denote by \( c_i, i = 1, 2 \) are the roots of the polynomial \( p^2 + 4b_1 \lambda \); and by \( \sum_{i=0}^{\infty} A_i r^i, j = 1, 2 \), the asymptotic series.

If \( b_0 = 0, b_1 = 0, \lambda = 0 \) then the asymptotics of the solution to equation (1) can be represented as the product of the function \( Y(X) \) and the corresponding conormal asymptotic behavior.

**Proof**

Let us consider that for the function \( a_i(r) = \sum_{j=0}^{\infty} b_j r^j \), the condition \( b_0 \neq 0, \lambda \neq 0 \) is satisfied. In this case, equation (3) can be rewritten as follows:

\[
\left( -r^2 \frac{d}{dr} \right)^2 v + b_0 \lambda v + b_1 \lambda rv + \lambda r^2 g(r)v = 0
\]

(8)

Here, \( g(r) = \sum_{j=2}^{\infty} p_j r^{j-2} \) is the corresponding holomorphic function.

The principal symbol of the differential operator of equation (8) has the form \( p^2 + b_0 \lambda \). Let us denote by \( c_i, i = 1, 2 \) the roots of this polynomial. Let us find the asymptotics at the nonzero simple root \( c_1 \). It follows from the results obtained in [1] that the solution to equation (8) is a resurgent function and its solution has a WKB asymptotics. Let us construct it.

We shift the root \( c_1 \) to zero using the substitution \( v = e^{c_1} v_1 \). Since the equality

\[
\left( -r^2 \frac{d}{dr} \right)^2 v + b_0 \lambda v = e^{c_1} \left( \left( r^2 \frac{d}{dr} \right)^2 + 2c_1 \left( r^2 \frac{d}{dr} \right) + c_1^2 + b_0 \lambda \right) v_1
\]

is satisfied, we can rewrite equation (7) in the form

\[
\left( -r^2 \frac{d}{dr} \right)^2 v_1 - 2c_1 \left( r^2 \frac{d}{dr} \right) + b_1 \lambda rv_1 + \lambda r^2 g(r)v = 0
\]

Let us perform one more substitution \( v_1 = r^{c_1} v_2 \).
\[
\left(r^2 \frac{d}{dr}\right)^2 v_1 = r^{\sigma_1} \left(\sigma_1 + 1\right) r^2 + 2\sigma_1 \left(r^2 \frac{d}{dr}\right) + \left(r^2 \frac{d}{dr}\right)^2 v_2
\]

We get the equation
\[
\left(-r^2 \frac{d}{dr}\right)^2 v_2 - 2c_1 \left(r^2 \frac{d}{dr}\right) v_2 + 2c_1 \sigma_1 r + b_1 \lambda r v_2 + \lambda r^2 g(r) v_2 + \left(\sigma_1 \left(\sigma_1 + 1\right) r^2 + 2\sigma_1 \left(r^2 \frac{d}{dr}\right) v_2 = 0
\]

Let us set
\[
\sigma_1 = -\frac{b_1 \lambda}{2c_1}.
\]

Equation (7) can be rewritten in the final form as follows:
\[
\left(-r^2 \frac{d}{dr}\right)^2 v_2 - 2c_1 \left(-r^2 \frac{d}{dr}\right) v_2 - 2\sigma_1 \left(-r^2 \frac{d}{dr}\right) v_2 + r^2 g_1(r) v_2 = 0
\]

Here, \( g_1(r) = \lambda g(r) + \sigma_1 \left(\sigma_1 + 1\right) \). Let us perform the Laplace-Borel transform. Using the designation \( \hat{v} = Bv \), we get
\[
p^2 \hat{v}_2 + 2c_1 p \hat{v}_2 - 2\sigma_1 \int p \hat{v}_2 dp + \int \int g_1(0) \hat{v}_2(p_1) dp_1 dp_2 + \int \int (Bg_1(r) \hat{v}_2(r))(p_1) dp_1 dp_2 dp_3 = f.
\]

Here, \( g_2(r) = g_1(r) - g_1(0) \), \( f \) denotes an arbitrary holomorphic function. The principal symbol of the differential operator in equation (10) has the form \( p \left(p + 2c_1\right) \). It was shown in [1] that the asymptotics of the solution in the neighborhood of the simple root \( p = 0 \) may be represented in the form
\[
\frac{A}{p} + \ln p \sum_{i=0}^{\infty} a_i p^i
\]

Here, \( A \) is some constant; the series \( \sum_{i=0}^{p} a_i p^i \) converges in some neighborhood of the point \( p = 0 \).

Let us find the inverse Laplace-Borel transform of the function (11):
\[
B^{-1} \left(\frac{A}{p} + \ln p \sum_{i=0}^{p} a_i p^i\right) = \sum_{i=0}^{\infty} A^i r^i.
\]

Here, \( \sum_{i=0}^{\infty} A^i r^i \) denotes the corresponding asymptotic series. It follows from (12) that the asymptotic term for the function \( v_1(r) \), which corresponds to the root of the principal symbol \( c_1 \) in the neighborhood of the point \( r = 0 \) has the form \( r^{\sigma_1} \sum_{i=0}^{\infty} A^i r^i \); the corresponding asymptotic term for the function \( v(r) \) has the form

\[ \text{...} \]
The asymptotic term corresponding to the root \( c_1 \) can be found in a similar way.

**Lemma 1.** Let \( b_0 \neq 0, \lambda \neq 0 \), then, the asymptotics of the solution to equation (3) in the neighborhood of the point \( r = 0 \) has the form

\[
\nu \approx e^{r^2} r^{c_1} \sum_{i=0}^{\infty} A_i^1 r^i
\]

Here, the numbers \( c_i, i = 1, 2 \) are the roots of the polynomial \( p^2 + b_0 \lambda \); the numbers \( \sigma_i, i = 1, 2 \) are determined according to formula (5); \( \sum A_i^j r^j, j = 1, 2 \) are the corresponding asymptotic series.

Let the condition \( b_0 = 0, b_1 \neq 0, \lambda \neq 0 \) be satisfied. Then, we can rewrite equation (3) as follows:

\[
\left(-r^2 \frac{d}{dr}\right)^2 v + b_1 \lambda rv + \lambda r^2 g(r)v = 0
\]

Here, unlike the previous case, \( H_0(p) = p^2 \), that is, the principal symbol has a multiple root at zero, so the method that we used earlier in this case is not applicable. In order to apply it, we transform equation (13).

Since the equality

\[
\left(-r^2 \frac{d}{dr}\right)^2 = r \left( \frac{3}{2} \frac{d}{dr} \right)^2 + \frac{1}{2} \left( \frac{3}{2} \frac{d}{dr} \right)
\]

holds, equation (10) can be rewritten in the form:

\[
\left(-r^2 \frac{d}{dr}\right)^2 v + \frac{1}{2} r^2 \left( \frac{3}{2} \frac{d}{dr} \right) v + b_1 \lambda v + \lambda r g(r)v = 0
\]

In the same way as it was done above, we make the replacement \( v = r^{\sigma} v_1 \). It is easy to show that equation (14) can be rewritten in the form

\[
\left(-r^3 \frac{d}{dr}\right)^2 v_1 + \left( \frac{1}{2} + 2 \sigma \right) r^2 \left( \frac{3}{2} \frac{d}{dr} \right) v_1 + b_1 \lambda v_1 + r (\lambda g(r) + \sigma (\sigma + 1)) v_1 = 0
\]

Let \( \sigma = -\frac{1}{4} \),

\[
\left(-2r^3 \frac{d}{dr}\right)^2 v_1 + 4b_1 \lambda v_1 + 4r \left( \frac{\lambda g(r)}{16} - \frac{3}{16} \right) v_1 = 0
\]

The principal symbol of the differential operator in equation (13) has the form \( p^2 + 4b_1 \lambda \). Let us denote by \( c_i, i = 1, 2 \) the simple roots of this polynomial. It follows from the results obtained in [1]
that the asymptotics of the solution to equation (14) will be the WKB asymptotics. We construct the asymptotics of this equation similarly to the previous case. It will have the following form:

**Lemma 2.** Let \( b_0 = 0, b_1 \neq 0, \lambda \neq 0 \), then, the asymptotics of the solution to equation (3) in the neighborhood of the point \( r = 0 \) has the form

\[
v \approx e^{\varepsilon r} \sum_{i=0}^{\infty} \lambda^i A_i r^i + e^{-\varepsilon r} \sum_{i=0}^{\infty} \lambda^{-i} A_i r^{-i}
\]

Here \( \varepsilon, i = 1,2 \) are the roots of polynomial \( p^2 + 4b_1 \lambda \).

Now, let \( b_0 = 0, b_1 = 0 \), then, equation (3) takes the form

\[
\left(-r^2 \frac{d}{dr}\right)^2 + \lambda r^2 g(r) v = 0
\]

In this case, the principal symbol has the form \( H_0(p) = p^2 \) and has a root of the second order at zero; therefore, the method we used before is not applicable in this case. Since the equality

\[
r^2 \frac{d}{dr} r^2 \frac{d}{dr} = r^3 \left( r \frac{d}{dr} \left( r \frac{d}{dr} \right) + r^3 \frac{d}{dr} \right)
\]

is valid, we can rewrite equation (16) as follows:

\[
\left( r \frac{d}{dr} \right)^2 + r \frac{d}{dr} + \lambda g(r) v = 0
\]

This is an equation of Fuchsian type. The point \( r = 0 \) is a regular singular point of equation (17). The problem of constructing such solutions is well-studied (see, for example, [4]). It is reasonable to search for a solution to this equation in the Sobolev weight spaces and the asymptotics of its solution is conormal. It is obvious that if \( \lambda = 0 \), equation (4) it is also an equation of Fuchsian type. The theorem is proved.

**Remark.** If equation (1) has the form

\[
\left( \frac{d}{dt} \right)^2 + a^2 \lambda v = 0
\]

where \( a \neq 0 \) is a real constant, then all asymptotics of solutions to this equation in the space of the functions of exponential growth in the neighborhood of infinity with respect to variable \( \$t\$ \) can be represented in the form

\[
u(x,t) = (A_1 e^{ct} + A_2 e^{ct}) Y(x)
\]

Here, \( c_i, i = 1,2 \), are the roots of the polynomial \( p^2 + a^2 \lambda \); \( A_i \) are arbitrary constants.

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