Comments on the radial distribution of charged particles in a magnetic field

H. Backe

Johannes Gutenberg-University Mainz, Institute for Nuclear Physics, D-55099 Mainz, Germany

Abstract
Magnetic guiding fields in combination with energy dispersive semiconductor detectors have been employed already more than 50 years ago for in-beam internal conversion electron spectroscopy. Even then it was recognized that efficiency modulations may appear as function of the electron energy, arising when electrons hit a baffle or miss the sensitive area of the detector. Current high precision beta decay experiments of polarized neutrons with conceptional similar experimental devices resulted in a detailed study of the point spread function (PSF). The latter describes the radial probability distribution of mono-energetic electrons at the detector plane. Singularities occur as function of the radial detector coordinate which have been investigated and discussed by Sjue at al. (Rev. Scient. Instr. 86, 023102 (2015)), and Dubbers (arXiv:1501.05131v1 [physics.ins-det]). In this comment a rather precise numerical representation of the PSF is presented and compared with approximations in the mentioned papers.

Keywords: Charged particles in magnetic fields

1. Introduction
In two recent papers [1, 2] the radial spread of charged particles moving in a solenoidal magnetic guiding field has been investigated. The physical back-

1corresponding author; E-Mail: backe@kph.uni-mainz.de
ground behind this attempt is based on the fact that the distribution of the
particles at a detector with finite radius is a potential source of systematic er-
rors in high precision experiments. To the latter belongs the measurement of
the beta asymmetry in the decay of polarized neutrons, see, e.g., Ref. [3]. In
principle, this fact is known since long. In the late 60th of the last century
nuclear spectroscopists developed solenoidal transport systems, equipped with
Si(Li) detectors as energy dispersive elements, for in-beam internal conversion
electron spectroscopy. It was pointed out already in one of the first publications
in detail [4] that the phase relation between polar emission angle and the radial
coordinate at a circular Si(Li) detector, or a baffle between target and detector,
results in unwanted fluctuations of the transmission probability as function of
the electron energy. Efficiency modulations on the 10 % level were reported,
e.g., in Ref. [5]. In order to overcome this problem, the unwanted wiggles were
averaged out by wobbling the magnetic field strength, see, e.g., also Ref. [6, 7].

From the work of Sjue et al. [1] and Dubbers [2] it is now well known
that such efficiency modulations originate from singularities which appear in
the so-called mono-energetic point spread function (PSF) as function of the
radius coordinate. These singularities have been mathematically treated in the
mentioned papers by means of different approaches. The probability density at
the detector plane is presented by Sjue et al. with the aid of an integral equation
while explicitly by Dubbers, cf. Eq. (14) [1] and Eq. (12) [2], respectively. The
ways to find solutions are quite different. Sjue et al. solve the integral equation
numerically, while Dubbers presents and discusses analytical approximations.

This contribution describes an alternative approach with which numerical
solutions of arbitrary accuracy for the PSF can be obtained. It is based on the
mathematically correct parameter representation of both, the radius coordinate
$R(\cos \theta)$ and the probability density $dP(\cos \theta)/dR$ at the detector plane. Para-
meter is $\cos \theta$ which intrinsically is a function of the polar emission angle $\theta$ at
the source. Notice that for rotational symmetry $d(\cos \theta)$ is just proportional to
the solid angle element $d\Omega$. In the next section first some mathematical details
are described. Results will be compared in section 3 with those presented in
Figure 1: (a) Impact loci curve at the detector plane for polar emission angle variations in the interval $1 \geq \cos \theta \geq 0.2$ with decrements $\Delta \cos \theta = 0.001$. It is called in the text trajectory. Azimuthal emission angle $\varphi = 0$. (b) Scatter plot for $\cos \theta$ and $\varphi$ randomly distributed. Parameters of Ref. \cite{1} taken: $B = 0.326$ Tesla, $pc = 0.976$ MeV, $r_0 = p/(eB) = 0.01$ m, $z_0 = 0.10$ m, and $z_0/r_0 = 10$. Ref. \cite{1, 2}. The paper closes with conclusions in section 4.

2. Radial distribution at the detector plane

For the sake of convenience the nomenclature of Sjue et al., \cite{1} will be adapted in the following. A right-handed coordinate system is defined with the magnetic field $\vec{B}$ coinciding with the $\hat{z}$ direction. The detector is placed in the (x,y) plane at a distance $z_0$ from the origin of the coordinate system in which the point source is located. The charged particle starts with polar angle $\theta$ and azimuthal angle $\varphi$, the latter defined with respect to the y axis. The point of impact at the detector plane is given by \cite{1} Eq. (4)

$$\frac{\vec{r}}{r_0} = \sqrt{1 - \cos^2 \theta} \left\{ \hat{x} \left[ \left( 1 - \cos \left( \frac{z_0}{r_0 \cos \theta} \right) \right) \cos \varphi + \sin \left( \frac{z_0}{r_0 \cos \theta} \right) \sin \varphi \right] + \hat{y} \left[ - \left( 1 - \cos \left( \frac{z_0}{r_0 \cos \theta} \right) \right) \sin \varphi + \sin \left( \frac{z_0}{r_0 \cos \theta} \right) \cos \varphi \right] \right\}$$

(1)

with $r_0 = p/(q \cdot B)$ the maximum projected orbital radius, $p$ the momentum, and $q$ the charge of the particle. Fig. \cite{1} (a) depicts the loci the radius vector
\( \vec{r}/r_0 \) traverses according to Eq. (1) if the cosine of the polar emission angle \( \theta \) is decremented in steps \( \Delta \cos \theta = 0.001 \). This impact loci curve must be distinguished from the projected orbit of the electron on its way from the target to the detector and will be called in the following trajectory.

The radius coordinate in the \((x,y)\) plane reads [1, Eq. (12)]

\[
\frac{R(\cos \theta)}{r_0} = \sqrt{\left(\frac{x}{r_0}\right)^2 + \left(\frac{y}{r_0}\right)^2} = \sqrt{2(1 - \cos^2 \theta) \left( 1 - \cos \left( \frac{z_0}{r_0 \cos \theta} \right) \right)}.
\]

The differential probability \( dP_c \) per normalized differential radius interval \( d\left(\frac{R}{r_0}\right) \) is

\[
\frac{dP_c(\cos \theta)}{d(\frac{R}{r_0})} = \frac{dP_c(\cos \theta)}{d\cos \theta} \cdot \frac{d\cos \theta}{d(\frac{R}{r_0})} = \frac{dP_c}{d(\frac{R}{r_0})/d\cos \theta}.
\]

For the example of an isotropically emitting source with \( dP(\cos \theta)/d\cos \theta = 1 \) and \( d(\frac{R}{r_0})/d\cos \theta \) calculated from Eq. (2) one obtains after some algebraic manipulations

\[
\frac{dP_c(\cos \theta)}{dR/r_0} = \frac{r_0 \cos^2 \theta \sqrt{1 - \cos^2 \theta}}{(1 - \cos^2 \theta) z_0 \cos \left( \frac{z_0}{2r_0 \cos \theta} \right) + 2r_0 \cos^3 \theta \sin \left( \frac{z_0}{2r_0 \cos \theta} \right)},
\]

or normalized to the unit area

\[
\frac{dP_c(\cos \theta)}{dA} = \frac{1}{2\pi R r_0} \frac{dP_c(\cos \theta)}{d(\frac{R}{r_0})}.
\]

It can be shown that Eq. (5) agrees with Eq. (14) of Ref. [2].

Treating \( \cos \theta \) as a free parameter, both the radial detector coordinate \( R(\cos \theta) \) and \( dP_c(\cos \theta)/dR \) can be calculated with Eq. (2) and (4), respectively. If the parameter \( \cos \theta \) is varied within the interval \([1,0]\) one gets an impression how \( dP_c/d(\frac{R}{r_0}) \) evolves as function of \( \frac{R}{r_0} \). A corresponding parametric plot is depicted in Fig. 2 (a). Shown are branches which start at \( R/r_0 = 0 \) and end again at \( R/r_0 = 0 \) for each closed trajectory at the detector plane depicted in Fig. 1 (a). However, what is wanted, is the sum of all individual contribution for a chosen normalized radius \( \frac{R}{r_0} \). This is the sum

\[
\frac{dP(\frac{R}{r_0})}{d(\frac{R}{r_0})} = \sum_{n=m}^{\infty} \left( \frac{dP_c(\cos \theta|_n)}{d(\frac{R}{r_0})} + \frac{dP_c(\cos \theta|_{\leq n})}{d(\frac{R}{r_0})} \right).
\]
Figure 2: (a) Parametric representation of the differential probability \( dP_c/(dR/r_0) \) as function of the normalized radius coordinate \( R/r_0 \). The parameter \( \cos \theta \) has been varied in the interval \([1, 0.27]\), i.e. 73% of the emitted intensity has been exhausted. For \( \cos \theta = 1 \) the curve starts at the origin, evolves with decreasing \( \cos \theta \) to the first spike which corresponds to the largest radius of the inner trajectory in Fig. 1(a), returns to \( R/r_0 = 0 \) and evolves from there to the next spike. (b) Sum of 500 trajectories as function of the the normalized radius coordinate \( R/r_0 \). Magnetic field and geometrical parameters the same as quoted in Fig. 1.

with \( \cos \theta_{n>}^\ast \) and \( \cos \theta_{n<}^\ast \) the two solutions of the equation

\[
\sqrt{2(1 - \cos^2 \theta)} \left( 1 - \cos \left( \frac{z_0}{r_0 \cos \theta} \right) \right) = \frac{R}{r_0}
\]

for the \( n^{th} \) trajectory in the interval

\[
\frac{z_0}{r_0 2\pi (n + n_f)} < \cos \theta \leq \min \left[ \frac{z_0}{r_0 2\pi (n - 1 + n_f)}, 1 \right],
\]

and \( n_f = \text{floor}(z_0/(r_0 2\pi)) \). The lower limit of the summation \( m \) is the smallest integer for which \( R/R_n > 1 \) holds with \( R_n \) the maximum radius of the \( n^{th} \) trajectory in the interval defined by the inequality (8).

3. Results and Discussion

In Fig 2(b) numerical results on the basis of Eq. (6) are depicted. Totally 500 trajectories at the detector plane of Fig. 1(a) have been taken into account. The accuracy of the calculation at \( R/r_0 = 1.92 \) and \( R/r_0 = 0.6 \) has been estimated to be in the order of 1.8 %, and 0.6 %, respectively. The accuracy can be improved by extending the summation in Eq. (6) over more trajectories.
Figure 3: (a) Random distribution as shown in Fig. 1 (b) analyzed at the detector plane with a radial resolution of 0.8 % or for the treated example of 80 µm. (b) Comparison of the numerical true PSF of Dubbers [2] with the approximations presented in chapter 3 in red with the exact solution in black.

However, it should be mentioned that the series expansion Eq. (6) converges rather slowly.

In the interval $1.92 < R/r_0 \leq 2$ infinitely many spikes appear. It is not necessary to treat them in a mathematically exact manner. This fact is demonstrated in Fig. 3(a). A number of $10^6$ impact points at the detector plane were generated by randomly distributing $\cos \theta$ and $\phi$ in Eq. (1). A small sample is shown in Fig. 1(b). The generated distribution has been analyzed with a virtual detector of 0.8 % spatial resolution in radial direction, corresponding for the chosen example with $r_0 = 1$ cm to 80 µm. It can clearly be seen that all the spikes in the mentioned interval result in a mean converging against 0.5. It should be mentioned that the same argument holds more or less for a finite beam spot size of the same order of magnitude. Fig. 3(a) is fully in accord with Fig. 4 of Sjue et al. [1]. Probably the procedure applied by the authors to generate their Fig. 4 was nothing else than what has been just described here.

In Fig. 3(b) the exact results are compared with the approximations elaborated by Dubbers [2]. In due distance from the spikes and for outer trajectories in Fig. 1(a) with many revolutions, his approximation apparently seems to be rather good. For the important inner trajectory the approximation is rather poor, and even normalization is not preserved.
Finally the question should be addressed whether the considerations on the PSF of Ref. [1] can be applied for field configurations other than homogeneous ones. It has been pointed out by Dubbers [2, ch. 5] that in axially symmetric, continuously descending magnetic fields the general formulas remain valid after replacement of $R$ by $R \cdot \sqrt{B(z_0)/B(0)}$ with $B(0)$ and $B(z_0)$ the magnetic fields on-axis at the source and detector position, respectively. It is certainly true that the underlying adiabatic invariance considerations makes some valuable statements on the movement of charged particles in inhomogeneous magnetic fields [8, p. 592f], see also Ref. [5]. However, whether in the fundamental Eq. (2) the phase $z_0/(r_0 \cos \theta)$ of the cosine function can simply be replaced by $\sqrt{B(0)B(z_0)} z_0/[(B_\rho)_p \cos \theta]$, with $(B_\rho)_p = p/q$, maintaining otherwise the functional dependence is questionable and needs to be discussed in terms of quantitative accuracy considerations. For more general magnetic field configurations it seems unlikely to find analytical solutions equivalent to Eq. (2) and (4) and experimentalists would be well advised to investigate their instruments from the beginning by Monte-Carlo simulations performed with exact orbit calculations.

4. Conclusions

A method has been described with which mono-energetic point spread functions can be calculated with arbitrary accuracy for a homogeneous magnetic guiding field. It has been shown by Monte-Carlo simulations that a finite detector resolution or a finite target spot size smear out the singularities for trajectories in the detector plane originating from polar emission angles approaching $\theta = \pi/2$. The results of Sjue et al. [1] are fully in accord with the results obtained in this paper. Although Dubbers [2] presents the correct parameter representation for the probability density function, which is not explicitly quoted by Sjue et al. [1], his analytical approximations for the singularities appear for the innermost trajectories to be rather inaccurate with even a significant violation of the normalization.
In any case, the subject addressed in Ref. [1, 2] is appealing and certainly beneficial for intuitional and educational purposes.

Acknowledgements

Calculations have been performed with the Wolfram Mathematica 8.0 package. Pictures were prepared with the LevelScheme scientific figure preparation system by M. A. Caprio, Department of Physics, University of Notre Dame, Version 3.53 (January 10, 2013) [9].

References

[1] S. K. L. Sjue, L. J. Broussard, M. Makela, L. McGaughey, A. R. Young, B. A. Zeck, Radial distribution of charged particles in a magnetic field, Review of Scientific Instruments 86 (2015) 023102 (6 pp).

[2] D. Dubbers, Magnetic guidance of charged particles, arXiv:1501.05131v1 [physics.ins-det] (2015) 16 pp.

[3] D. Dubbers, L. Raffelt, B. Märkisch, F. Friedl, H. Abele, The point spread function of electrons in a magnetic field, and the decay of the free neutron, Nuclear Instruments and Methods A 763 (2014) 112–119.

[4] B. Klank, R. A. Ristinen, Design and Performance of a Transport Solenoid - Si(Li) Detector Conversion Electron Spectrometer for On-Line Use with Accelerators, in: J. H. Hamilton, J. C. Manthuruthil (Eds.), Radioactivity in Nuclear Spectroscopy, Modern Techniques and Applications, Vol. I, Proceedings of the International Conference on Radioactivity in Nuclear Spectroscopy (Vanderbilt University) (1969).

[5] K. Kotajima, R. Beringer, A Magnetic Solenoid Electron Transporter, The Review of Scientific Instruments 41 (1970) 632–635.
[6] Th. Lindblad, C. G. Lindén, An On-line Multichannel Electron Spectrometer with High Transmission, Nuclear Instruments and Methods 126 (1975) 397–406.

[7] H. Backe, L. Richter, R. Willwater, E. Kankeleit, E. Kuphal, Y. Nakayama, B. Martin, In-Beam Spectroscopy of Low Energy Conversion Electrons with a Recoil Shadow Method - A New Possibility for Subnanosecond Lifetime Measurements, Zeitschrift für Physik A 285 (1978) 159–169.

[8] J. D. Jackson, Classical Electrodynamics, 3rd Edition, John Wiley & Sons, Inc., New York, Chichester, Weinheim, Brisbane, Singapore, Toronto.

[9] M. A. Caprio, LevelScheme: A level scheme drawing and scientific figure preparation system for Mathematica, Computer Physics Communications 171 (2005) 107–118.