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Chain conditions in amalgamated algebras along an ideal

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Abstract Let $A$ and $B$ be two rings, let $J$ be an ideal of $B$ and let $f : A \to B$ be a ring homomorphism. In this paper, we study when the amalgamation of $A$ with $B$ along $J$ with respect to $f$ is a $\phi$-ring. Hence, we study two different chain conditions over this structure. Namely, the nonnil-Noetherian condition and the Noetherian spectrum condition.

Mathematics Subject Classification 13E05 · 13E99 · 13B25 · 13J05

1 Introduction

Throughout this paper, all rings are commutative with unity. We denote by $\text{Nilp}(R)$ the set of nilpotent elements of the ring $R$. By $(a)$ we denote the ideal of $R$ generated by $a \in R$.

Let $A$ and $B$ be two rings, let $J$ be an ideal of $B$ and let $f : A \to B$ be a ring homomorphism. In this setting, we can consider the following subring of $A \times B$:

$$A \triangleleft f J := \{(a, f(a) + j) \mid a \in A, j \in J\}$$

called the amalgamation of $A$ with $B$ along $J$ with respect to $f$ (introduced and studied by D’Anna, Finocchiaro, and Fontana in [11] and [12]). This construction is a generalization of the amalgamated duplication of a ring along an ideal (introduced and studied by D’Anna and Fontana in [8], [9] and [10]). Moreover, other classical
constructions (such as the $A+XB[X]$, $A+XB[[X]]$, and the $D+M$ constructions) can be studied as particular cases of the amalgamation [11, Examples 2.5 and 2.6] and other classical constructions, such as the Nagata’s idealization, cf. [17, page 2], and the CPI extensions (in the sense of Boisen and Sheldon [7]) are strictly related to it [11, Example 2.7 and Remark 2.8].

On the other hand, the amalgamation $A \bowtie J$ is related to a construction proposed by Anderson in [1] and motivated by a classical construction due to Dorroh [14], concerning the embedding of a ring without identity in a ring with identity. An ample introduction on the genesis of the notion of amalgamation is given in [11, Section 2]. Also, the authors consider the iteration of the amalgamation process, giving some geometrical applications of it.

One of the key tools for studying $A \bowtie J$ is based on the fact that the amalgamation can be studied in the frame of pullback constructions [11, Section 4]. This point of view allows the authors in [11] and [12] to provide an ample description of various properties of $A \bowtie J$, in connection with the properties of $A$, $J$ and $f$. Namely, in [11], the authors studied the basic properties of this construction (e.g., characterizations for $A \bowtie J$ to be a Noetherian ring, an integral domain, a reduced ring) and they characterized those distinguished pullbacks that can be expressed as an amalgamation. Moreover, in [12], they pursue the investigation on the structure of the rings of the form $A \bowtie J$, with particular attention to the prime spectrum, to the chain properties and to the Krull dimension.

Recall from [3] and [13] that a prime ideal of $R$ is called a divided prime ideal if $P \subseteq (x)$ for every $x \in R \setminus P$; thus a divided prime ideal is comparable to every ideal of $R$. In [2], [4] and [5], the author paid attention to the class of rings

$$\mathcal{H} = \{ R \mid R \text{ is a commutative ring and } \text{Nilp}(R) \text{is a divided prime ideal of } R \}$$

Observe that if $R$ is an integral domain, then $R \in \mathcal{H}$. If $R \in \mathcal{H}$, then $R$ is called a $\phi$-ring.

Let $A$ and $B$ be two rings, let $J$ be an ideal of $B$ and let $f : A \to B$ be a ring homomorphism. In this paper, we study when the amalgamation of $A$ with $B$ along $J$ with respect to $f$ is a $\phi$-ring.

Recall that a ring $R$ is said to be nonnil-Noetherian if each ideal of $R$ which is not contained in the nilradical of $R$ is finitely generated. The treatment of this notion in the context of the class of rings of the form $A \bowtie J$, with particular attention to the prime spectrum, to the chain properties and to the Krull dimension.

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Let $A$ and $B$ be two rings, let $J$ be an ideal of $B$ and let $f : A \to B$ be a ring homomorphism. In this paper, we study when the amalgamation of $A$ with $B$ along $J$ with respect to $f$ is a $\phi$-ring.

Recall that a ring $R$ is said to be nonnil-Noetherian if each ideal of $R$ which is not contained in the nilradical of $R$ is finitely generated. The treatment of this notion in the context of the class of $\phi$-rings was established in [6], where the author proved that many of the properties of Noetherian rings are true for non-nil-Noetherian rings. Trivially, Noetherian rings are non-nil-Noetherian but the converse is not true in general, cf. [6, Theorem 3.4]. In Sect. 2, we characterize when $A \bowtie J$ is non-nil-Noetherian provided it is $\phi$-ring. Recall that a ring $R$ has Noetherian spectrum if it satisfies the ascending chain condition for radical ideals. Every non-nil-Noetherian ring has Noetherian spectrum and the converse is false; cf. [16, Proposition 1.8 and Remark 1.9]. In Sect. 2, we characterize when $A \bowtie J$ is of Noetherian spectrum.

### 2 Main results

We begin with the following result in which we study when $A \bowtie J$ is a $\phi$-ring.

**Theorem 2.1** Let $A$ and $B$ be two rings, $J$ be an ideal of $B$ and let $f : A \to B$ be a ring homomorphism. If $A \bowtie J$ is a $\phi$-ring then the following properties hold:

1. $A$ is a $\phi$-ring.
2. If $J$ is a prime ideal of $f(A) + J$ or $f^{-1}(J) \subseteq \text{Nilp}(A)$ then $f(A) + J$ is a $\phi$-ring.

Conversely, assume that $J = \text{Nilp}(B)$ and that $f^{-1}(J) \subseteq \text{Nilp}(A)$ then if $f(A) + J$ and $A$ are $\phi$-rings then so is $A \bowtie J$.

**Proof** Clearly, $\text{Nilp}(A \bowtie J) = \{(a, f(a) + j) \mid a \in \text{Nilp}(A), j \in \text{Nilp}(B) \cap J\}$.

Assume that $A \bowtie J$ is a $\phi$-ring. Then, $\text{Nilp}(A \bowtie J)$ is a prime ideal of $A \bowtie J$. From [12, Proposition 2.6], there are two possible cases:

**Case 1** There exists a prime ideal $P$ of $A$ such that

$$\text{Nilp}(A \bowtie J) = P \bowtie J := \{(p, f(p) + j) \mid p \in P, j \in J\}$$

Then, $\text{Nilp}(A) = P$, and so it is prime. Consider $a \notin \text{Nilp}(A)$, then, $(a, f(a)) \notin \text{Nilp}(A \bowtie J)$. Hence, $\text{Nilp}(A \bowtie J) \subseteq (a, f(a)) = A \bowtie J$ since $\text{Nilp}(A \bowtie I)$ is a divided prime ideal of $A \bowtie J$. Thus, for each $x \in \text{Nilp}(A)$, there exists $(b, f(b) + j) \in A \bowtie J$ such that $(x, f(x)) = (b, f(b) + j)(a, f(a))$. Hence,
\[ x = ba, \text{ and so } \text{Nilp}(A) \subseteq aA. \text{ Thus, } \text{Nilp}(A) \text{ is a divided prime ideal of } A. \text{ Consequently, } A \text{ is a } \phi\text{-ring. Moreover, } \{0\} \times J \subseteq P \Rightarrow J = \text{Nilp}(A \Rightarrow J). \text{ Hence, } J \subseteq \text{Nilp}(f(A) + J). \]

Case 2: There exists a prime ideal \( Q \) of \( B \) with \( J \not\subseteq Q \) such that

\[
\text{Nilp}(A \Rightarrow J) = \overline{Q} := \{(a, f(a) + j) \mid a \in A, j \in J, f(a) + j \in Q\}
\]

Let \( j \in J \setminus Q \). Then, \( (0, j) \not\in \overline{Q} \). Thus, \( \text{Nilp}(A \Rightarrow J) = \overline{Q} \) and \( A \Rightarrow J \subseteq \{0\} \times J \) since \( \text{Nilp}(A \Rightarrow J) \) is a divided prime ideal of \( A \Rightarrow J \). Hence, \( \text{Nilp}(A) = \{0\} \). Let \( x, y \in A \) such that \( xy = 0 \). Then, \( (x, f(x))(y, f(y)) = (0, 0) \in \overline{Q} \). Hence, \( (x, f(x)) \in \overline{Q} \) or \( (y, f(y)) \in \overline{Q} \). Thus, \( x = 0 \) or \( y = 0 \).

Consequently, \( A \) is an integral domain. So, \( A \) is a \( \phi\)-ring.

Assume that \( J \) is a prime ideal of \( f(A) + J \). Let \((f(a) + j)(f(b) + j') \in \text{Nilp}(f(A) + J)\). There exists a positive integer \( k \) such that \( |(f(a) + j)(f(b) + j')|^k = 0 \). Hence, \( (f(a)b)^k \in J \). Thus, \( f(a) \in J \) or \( f(b) \in J \). Suppose that \( f(a) \in J \). Then, \((0, f(a) + j)(f(b) + j') \in A \Rightarrow J \) and we have \( (0, f(a) + j)(f(b) + j') = (0, f(a) + j)(f(b) + j) \in \text{Nilp}(A \Rightarrow J) \). Then, \((0, f(a) + j) \in \text{Nilp}(A \Rightarrow J)\) or \((f(b) + j') \in \text{Nilp}(A \Rightarrow J) \). Hence, \( f(a) + j \in \text{Nilp}(f(A) + J) \) or \( f(b) + j' \in \text{Nilp}(f(A) + J) \). Consequently, \( \text{Nilp}(f(A) + J) \) is a prime ideal of \( f(A) + J \). Let \((x, f(x))(y, f(y)) = (0, 0) \in \overline{Q} \). Hence, \( (x, f(x)) \in \overline{Q} \) or \( (y, f(y)) \in \overline{Q} \). Thus, \( x = 0 \) or \( y = 0 \).

Remark 2.2 The assumption “\( J \) is a prime ideal of \( f(A) + J \) or \( f^{-1}(J) \subseteq \text{Nilp} (A) \)” is necessary to show that \( f(A) + J \) is a \( \phi\)-ring. Consider the homomorphism of rings \( f : \mathbb{Z} \rightarrow \mathbb{Z}/6\mathbb{Z} ; n \mapsto \overline{n} \) and set \( J = \{0\} \) the zero ideal of \( \mathbb{Z}/6\mathbb{Z} \). Clearly, \( \mathbb{Z} \) is a \( \phi\)-ring (since it is an integral domain). Moreover, \( A \Rightarrow J \cong \mathbb{Z} \). Then, \( A \Rightarrow J \) is also a \( \phi\)-ring. But, \( f(\mathbb{Z}) + \mathbb{Z}/6\mathbb{Z} = \mathbb{Z}/6\mathbb{Z} \) is not a \( \phi\)-ring. Indeed, \( \text{Nilp}(\mathbb{Z}/6\mathbb{Z}) = \{0\} = J \) which is not a prime ideal of \( \mathbb{Z}/6\mathbb{Z} \) since \( 2 \times 3 = 6 = 0 \in \text{Nilp}(\mathbb{Z}/6\mathbb{Z}) \). We can see also that \( f^{-1}(J) = 6\mathbb{Z} \not\subseteq \text{Nilp}(\mathbb{Z}) = \{0\} \).

Recall that if \( A = B, f = id_A \) and \( J \) is an ideal of \( A \), the ring \( A \Rightarrow \mathbb{B} \) coincides with the amalgamated duplication of \( A \) along the ideal \( J \) defined in [10], as follows:

\[
A \Rightarrow J = \{(a, a + j) \mid a \in A, j \in J\}
\]
Corollary 2.3  Let $A$ and $B$ be two rings and let $f : A \to B$ be a ring homomorphism and assume that $f^{-1}(\text{Nilp}(B)) \subseteq \text{Nilp}(A)$. Then, $A \bowtie \text{Nilp}(B)$ is a $\phi$-ring if and only if $A$ and $f(A) + \text{Nilp}(B)$ are $\phi$-rings.

In particular, for each ring $A$, $A \bowtie \text{Nilp}(A)$ is a $\phi$-ring if and only if $A$ is a $\phi$-ring.

Proof  The general case follows immediately from Theorem 2.1, while the particular case follows from the general one when $A = B$ and $f = \text{id}_A$.

Using Corollary 2.3, we can construct a new family of $\phi$-rings.

Example 2.4  Let $A$ be a $\phi$-ring which is not integral domain. Set $A_1 = A \bowtie \text{Nilp}(A)$ and for each $i \geq 1$ set $A_{i+1} = A_i \bowtie \text{Nilp}(A_i)$. Then, $(A_i)_{i \geq 1}$ is a family of a $\phi$-rings which are not integral domains.

Proof  The fact that $A_i$ is a $\phi$-ring for each $i \geq 1$ follows from Corollary 2.3. If $A_i$ is an integral domain for some $i \geq 1$ then by induction and by [11, Remark 5.3], $A$ is an integral domain, a contradiction.

Proposition 2.5  Let $A$ and $B$ be two rings, $J$ an ideal of $B$ and let $f : A \to B$ be a ring homomorphism. If $A \bowtie J$ is a non-nil-Noetherian ring then so are $A$ and $f(A) + J$.

Proof  By [16, Proposition 1.3], every homomorphic image of a non-nil-Noetherian ring is non-nil-Noetherian. Thus, if $A \bowtie J$ is non-nil-Noetherian, then so are $A$ and $f(A) + J$ (by [11, Proposition 5.1(3)]).

Remark 2.6  Let $A$ and $B$ be two rings, $J$ an ideal of $B$ and let $f : A \to B$ be a ring homomorphism. Set $\bar{A} = A/\text{Nilp}(A), \bar{B} = B/\text{Nilp}(B), \pi : B \to \bar{B}$ the canonical projection, and $\bar{J} = \pi(J)$. Consider the ring homomorphism $\bar{f} : \bar{A} \to \bar{B}$ defined by setting $\bar{f}(\bar{a}) = f(a)$. It is easy to see that $\bar{f}$ is well defined and it is clearly a ring homomorphism. The kernel of the restriction to $A \bowtie J$ of the canonical projection $A \times B \to \bar{A} \times \bar{B}$ is obviously $\text{Nilp}(A \bowtie J)$ and the image is $\bar{A} \bowtie \bar{J}$. Hence, we have the following isomorphism of rings:

$$
\psi : (A \bowtie J)/\text{Nilp}(A \bowtie J) \to \bar{A} \bowtie \bar{J} \\
(a, f(a) + j) \mapsto (\bar{a}, \bar{f}(a) + \bar{j})
$$

Moreover, the rings $\bar{f}(\bar{A}) + \bar{J}$ and $(f(A) + J)/\text{Nilp}(f(A) + J)$ are always isomorphic. Indeed, if $\lambda$ is the restriction to $f(A) + J$ of the projection $B \to \bar{B}$, then clearly $\text{Im}(\lambda) = \bar{f}(\bar{A}) + \bar{J}$ and $\ker(\lambda) = \text{Nilp}(f(A) + J)$.

In what follows we characterize $A \bowtie J$ to be non-nil-Noetherian under the assumption that it is a $\phi$-ring.

Theorem 2.7  Let $A$ and $B$ be two rings, $J \neq [0]$ an ideal of $B$ and let $f : A \to B$ be a ring homomorphism. If $A \bowtie J$ is a $\phi$-ring, the following are equivalent:

1. $A \bowtie J$ is a non-nil-Noetherian ring.
2. $A$ and $f(A) + J$ are non-nil-Noetherian rings and $f^{-1}(J) \subseteq \text{Nilp}(A)$.

Proof  (1) $\Rightarrow$ (2) By Proposition 2.5, we have only to prove that $f^{-1}(J) \subseteq \text{Nilp}(A)$. By [6, Theorem 2.2], $(A \bowtie J)/\text{Nilp}(A \bowtie J)$ is a Noetherian domain. Thus, by Remark 2.6, $\bar{A} \bowtie \bar{J}$ is a Noetherian domain.

If $\bar{J} = [0]$, then, $J \subseteq \text{Nilp}(B)$. Thus, $\text{Nilp}(A \bowtie J) = \{(a, f(a) + j) \mid a \in \text{Nilp}(A), j \in J\} = \text{Nilp}(A) \bowtie J$.

Let $x \in f^{-1}(J)$. If $\text{Nilp}(A \bowtie J) \subseteq (x, 0)A \bowtie J$, then $J = [0]$, which is impossible. Then, $(x, 0)A \bowtie J \subseteq \text{Nilp}(A \bowtie J)$. Thus, $x \in \text{Nilp}(A)$.

If $\bar{J} \neq [0]$ then, by [11, Proposition 5.2], $\bar{f}^{-1}(\bar{J}) = 0$. Consequently, $f^{-1}(J) \subseteq \text{Nilp}(A)$.

(2) $\Rightarrow$ (1) Let $\bar{x} \in \bar{f}^{-1}(\bar{J})$. Then, $\bar{f}(\bar{x}) = f(\bar{x}) \in \bar{J}$. So, there exists an element $j \in J$ such that $(f(x) - j) \in \text{Nilp}(B)$. Hence, there is an integer $k$ such that $(f(x) - j)^k = 0$. Thus, $(x^k) \in J$. Consequently, $x^k \in \text{Nilp}(A)$. Thus, $x \in \text{Nilp}(A)$, and $\bar{x} = 0$. Hence, $\bar{f}^{-1}(\bar{J}) = [0]$. On the other hand, by Theorem 2.1, $A$ and $f(A) + J$ are $\phi$-rings. Thus, [6, Theorem 2.2], $\bar{A}$ and $(f(A) + J)/\text{Nilp}(f(A) + J) \cong \bar{f}(\bar{A}) + \bar{J}$ are Noetherian domains. Hence, by [11, Proposition 5.6], $(\bar{A} \bowtie \bar{J})$ is a Noetherian ring. Moreover, $\bar{A} \bowtie \bar{J} \cong (A \bowtie J)/\text{Nilp}(A \bowtie J)$ which is an integral domain as $A \bowtie J$ is a $\phi$-ring.


Corollary 2.8 Let $A$ be a $\phi$-ring. Then, $A \triangleleft \lhd \text{Nilp}(A)$ is nonnil-Noetherian if and only if $A$ is nonnil-Noetherian.

Proof Follows from Theorem 2.7 and Corollary 2.3.

If $J$ is a finitely generated $A$-module with the structure naturally induced by $f$, and $J$ is a nonnil ideal of $B$, then the Noetherian and nonnil-Noetherian conditions coincide over the amalgamation of $A$ with $B$ along $J$ with respect to $f$.

Proposition 2.9 Assume that $J \not\subseteq \text{Nilp}(B)$ and at least one of the following conditions holds:

1. $J$ is a finitely generated $A$-module (with the structure naturally induced by $f$).
2. $f$ is a finite homomorphism.

Then, $A \triangleright \triangleleft J$ is nonnil-Noetherian if and only if $A \triangleright \triangleleft J$ is Noetherian.

Proof From [11, Proposition 5.7], under one of the conditions made above, $A \triangleright \triangleleft J$ is Noetherian if and only if $A$ is Noetherian. If $A \triangleright \triangleleft J$ is Noetherian then clearly it is nonnil-Noetherian. Conversely, assume that $A \triangleright \triangleleft J$ is nonnil-Noetherian. Let $P$ be a prime ideal of $A$. Then, $P \triangleright \triangleleft J$ is a prime ideal of $A \triangleright \triangleleft J$. Moreover, $P \triangleright \triangleleft J \not\subseteq \text{Nilp}(A \triangleright \triangleleft J)$ since $J \not\subseteq \text{Nilp}(B)$. Thus, $P \triangleright \triangleleft J$ is a finitely generated ideal of $A \triangleright \triangleleft J$. Hence, $P$ is a finitely generated ideal of $A$. Thus, every prime ideal of $A$ is finitely generated. So, $A$ is Noetherian. Consequently, $A \triangleright \triangleleft J$ is Noetherian.

In what follows, we give an example of a ring homomorphism $f : A \to B$ and an ideal $J$ of $B$ such that $A$ and $f(A) + J$ are nonnil-Noetherian rings and $A \triangleright \triangleleft J$ is not nonnil-Noetherian.

We recall this construction. For a ring $R$, let $B$ be an $R$-module. Consider

$$R(+)B = \{(r, b) \mid r \in R \text{ and } b \in B\}$$

and let $(r, b)$ and $(s, c)$ two elements of $R(+)B$. Define:

1. $(r, b) = (s, c)$ if $r = s$ and $b = c$.
2. $(r, b) + (s, c) = (r + s, b + c)$.
3. $(r, b)(s, c) = (rs, rc + sb)$.

Under these definitions $R(+)B$ becomes a commutative ring with identity called the Nagata’s idealization of $B$ in $R$.

Example 2.10 Set $A = \mathbb{Z}(+)\mathbb{Q}$ and consider the surjective ring homomorphism $f : A \to \mathbb{Z}/6\mathbb{Z}$; $f((n, q)) = n$. Consider $J = 3\mathbb{Z}/6\mathbb{Z} = \{0, 3\}$ the ideal of $\mathbb{Z}/6\mathbb{Z}$. Then, $A$ and $f(A) + J$ are nonnil-Noetherian rings. However, $A \triangleright \triangleleft J$ is not.

Proof By [6, Theorem 3.4], $A$ is a nonnil-Noetherian ring which is not a Noetherian ring. On the other hand, $f(A) + J = \mathbb{Z}/6\mathbb{Z}$ is a Noetherian ring, and so a nonnil-Noetherian ring with $\text{Nilp}(\mathbb{Z}/6\mathbb{Z}) = \{0\}$. Moreover, $J \not\subseteq \text{Nilp}(\mathbb{Z}/6\mathbb{Z})$ and $J$ is a finitely generated $A$-module (with the structure naturally induced by $f$) since $J = 3\mathbb{Z}/6\mathbb{Z} = 3\mathbb{Z}/6\mathbb{Z}$ is $3\mathbb{Z}(A) = 3\mathbb{Z}$. If we suppose that $A \triangleright \triangleleft J$ is nonnil-Noetherian, then by Proposition 2.9, $A \triangleright \triangleleft J$ is Noetherian, and so is $A$, which is impossible.

We end this paper with a characterization of $A \triangleright \triangleleft J$ to be of Noetherian spectrum.

Proposition 2.11 The ring $A \triangleright \triangleleft J$ has Noetherian spectrum if and only if $A$ and $f(A) + J$ have Noetherian spectrum.

In particular, if $B$ has Noetherian spectrum, then $A \triangleright \triangleleft J$ has Noetherian spectrum if and only if $A$ has Noetherian spectrum.

Proof The result follows immediately by applying [15, Corollary 1.6], keeping in mind the fiber product structure of $A \triangleright \triangleleft J = \pi \circ f \times \pi$ where $\pi$ is the canonical surjection $f(A) + J \to (f(A) + J)/J$, the fact that $(f(A) + J)/J$ is isomorphic to $A \triangleright \triangleleft J/(f(A) + J)/J$, and the fact that every subspace of a Noetherian topological space is still Noetherian.

In the particular case, $A \triangleright \triangleleft J = \pi_1 \circ f \times \pi_1$ where $\pi_1$ is the canonical surjection, $\pi_1 : B \to B/J$. □

We have the following consequence of the previous proposition.
Corollary 2.12 Let $A$ be a ring and $I$ an ideal of $A$. Then, $A \ni I$ has Noetherian spectrum if and only if $A$ has Noetherian spectrum.

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