A PRIORI ESTIMATES FOR MULTIDIMENSIONAL BSDES WITH INTEGRABLE DATA

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Abstract. We study Backward Stochastic Differential Equations on a probability space equipped with a Brownian filtration. We assume that the terminal value and the generator at zero are merely integrable. Moreover, the generator is assumed to be non-increasing with respect to the value variable (with no restrictions on the growth) and Lipschitz continuous, with sublinear growth, with respect to the control variable. We provide a priori estimate and stability result for solutions to the aforementioned BSDEs.

1. Introduction

Let us fix a complete probability space \((\Omega, \mathcal{F}, P)\) and a \(d\)-dimensional Brownian motion \(B\) on \((\Omega, \mathcal{F}, P)\). Let \(\mathcal{F}\) be the standard augmentation of the filtration generated by \(B\). We say that a pair \((Y, Z)\), consisting of \(\mathcal{F}\)-progressively measurable processes, is a solution to a Backward Stochastic Differential Equation with data \((\tau, \xi, f)\) (notation: BSDE\(_{\tau}(\xi, f)\)), where \(\tau\) is a bounded \(\mathcal{F}\)-stopping time (terminal time), \(\xi\) is an \(\mathcal{F}_\tau\)-measurable random vector (terminal condition), and \(f: \Omega \times [0, \tau] \times \mathbb{R}^k \times \mathbb{R}^{d \times k} \to \mathbb{R}^k\) is an \(\mathcal{F}\)-progressively measurable process with respect to the first two variables (generator), if

\[
Y_t = \xi + \int_0^\tau f(r, Y_r, Z_r) \, dr - \int_0^\tau Z_r \, dB_r, \quad t \in [0, \tau].
\]

In [1] Briand, Delyon, Hu, Pardoux and Stoica have proven that under the following weak assumptions on the data

\begin{enumerate}[(H1)]
\item there is \(\lambda \geq 0\) such that \(|f(t, y, z) - f(t, y, z')| \leq \lambda |z - z'|\) for \(t \in [0, \tau]\), \(y, z, z' \in \mathbb{R}^{dk}\),
\item there is \(\mu \in \mathbb{R}\) such that \((y-y', f(t, y, z) - f(t, y', z)) \leq \mu |y - y'|^2\) for \(t \in [0, \tau]\), \(y, y' \in \mathbb{R}^k, z \in \mathbb{R}^{d \times k}\),
\item for every \((t, z) \in [0, \tau] \times \mathbb{R}^{dk}\) the mapping \(\mathbb{R}^k \ni y \to f(t, y, z)\) is continuous,
\item either \(k \geq 2\) and \(\mathbb{E}\left(\int_0^\tau \sup_{|y| \leq M} \left|f(r, y, 0)\right| \, dr\right) < \infty\) for any \(M > 0\) or \(k = 1\) and \(\int_0^\tau |f(r, y, 0)| \, dr < \infty\) for any \(y \in \mathbb{R}\),
\item \(\xi \in L^p(\mathcal{F}_\tau), f(\cdot, 0, 0) \in L^p_p(0, \tau),\)
\end{enumerate}

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(Z) there exists an $\mathbb{F}$-progressively measurable process $g \in L^1_B(0, \tau)$ and $\gamma \geq 0$, $\kappa \in [0,1)$ such that
\[ |f(t, y, z) - f(t, y, 0)| \leq \gamma (g_t + |y| + |z|)^\kappa, \quad t \in [0, \tau], \ y \in \mathbb{R}^k, \ z \in \mathbb{R}^{d \times k}; \]
with $p = 1$ in (H5), there exists a solution $(Y, Z)$ to (1.1) such that $Y$ is of class (D) and $Z \in \mathcal{H}^s(0, \tau)$, $s \in (0, 1)$. Moreover, under these assumptions there exists at most one solution $(Y, Z)$ to (1.1) such that $Y$ is of class (D). It is interesting that, although almost 20 years have passed since the publication of the above theorem, surprisingly such fundamental result as a priori estimate for solutions to (1.1) is still missing from the literature (besides the special case when $f$ is independent of $\cdot$-variable). The aim of the present paper is to fill this gap.

We shall prove (see Theorems 3.4, 3.5) that for any $a, b \in (0, 1)$ there exists a continuous strictly increasing function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$, with $\varphi(0) = 0$, which depends only on $\varepsilon_{\sup \tau, \lambda, \mu, \gamma, \kappa, a, b}$, such that
\begin{equation}
\sup_{\sigma \leq \tau} \mathbb{E}|Y_{\sigma} - \bar{Y}_{\sigma}| + \mathbb{E} \sup_{t \leq \tau} |Y_t - \bar{Y}_t|^a + \mathbb{E} \left( \int_0^\tau |Z_{\tau} - \bar{Z}_\tau|^2 \, dr \right)^{b/2} \\
\leq \varphi(\mathbb{E}|\xi - \bar{\xi}| + \mathbb{E} \int_0^\tau |f - \bar{f}(r, Y_r, \bar{Y}_r)| \, dr)
\end{equation}
for any solutions $(Y, Z), (\bar{Y}, \bar{Z})$ of BSDE$^r(\xi, f)$, BSDE$^r(\bar{\xi}, \bar{f})$, respectively, with $Y, \bar{Y}$ being of class (D) (here $\xi, \bar{\xi}, f, \bar{f}$ satisfy the same assumptions as $\xi, f$). As a corollary to the above a priori estimate, we get a stability result for BSDEs with $L^1$-data.

As far as we know the only paper concerned with stability results for BSDEs with $L^1$-data (we omit in this comment papers with generators independent of $\cdot$-variable) is the paper by S.J. Fan [2], where the author has proven the following convergence
\begin{equation}
\sup_{t \leq T} \mathbb{E}|Y_t - Y^n_t| + \mathbb{E} \left( \int_0^\tau |Z_{\tau} - Z^n_{\tau}|^2 \, dr \right)^{b/2} \to 0,
\end{equation}
provided $\mathbb{E}|\xi - \bar{\xi}| \to 0$ and $|f - \bar{f}| \leq \varepsilon_n \searrow 0$. Here $(Y^n, Z^n)$ is a solution to BSDE$^T(\xi_n, f_n)$, with $(\xi_n, f_n)$ satisfying the same assumptions as $(\xi, f)$, and $(\varepsilon_n)$ is a decreasing sequence of positive numbers. We see that (1.2) readily implies (1.3), and even stronger convergence, without assuming boundedness of $|f - \bar{f}|$ (note that S.J. Fan considered even weaker than (H2) one-sided Osgood condition). In [2] the author conjectured that in general a priori estimates for BSDEs with $L^1$-data cannot hold (see the comments in the first paragraph on page 1863 in [2]). Our main result disproves this conjecture.

From the theoretical and practical point of view a priori estimates and stability results describe one of the most fundamental features of any type of equations. Here, we would like to mention just about one crucial application of our results. The result by Briand, Delyon, Hu, Pardoux and Stoica allow one, among others, to define, for any fixed bounded stopping times $\alpha \leq \beta$, the following operator (so-called nonlinear expectation)
\[ \mathbb{E}^\beta_{\alpha, \beta} : L^1(\mathcal{F}_\beta) \to L^1(\mathcal{F}_\alpha), \quad \mathbb{E}^\beta_{\alpha, \beta} (\eta) := Y^\beta_{\alpha, \beta}, \]
where $Y^\beta_{\alpha, \beta}$ is a process of class (D) being the first component of a solution to BSDE$^\beta(\eta, f)$, with $k = 1$. The concept of nonlinear expectation has been introduced by Peng [4], originally for $p = 2$ and Lipschitz continuous $f$, and appeared to be a crucial notion in models of mathematical finance and control theory. Thanks to the results in [1] this notion is also
well defined on $L^1$. However, to apply this operator in practice, some basic properties of it are indispensable. One of them is stability, which is still missing in the literature, i.e. we ask what one can say about the difference

$$E|E^f_{a,b}(\eta_1) - E^f_{a,b}(\eta_2)|$$

for $\eta_1, \eta_2 \in L^1(\mathcal{F}_\beta)$. This leads us to stability results for BSDEs with $L^1$-data. As a corollary to our main results we have the following inequality

$$E|E^f_{a,b}(\eta_1) - E^f_{a,b}(\eta_2)| \leq \varphi(E|\eta_1 - \eta_2|),$$

for any $\eta_1, \eta_2 \in L^1(\mathcal{F}_\beta)$, where $\varphi$ is described in (1.2).

2. Backward SDEs with $L^p$-data: the case $p > 1$

For any $n \geq 1$ and $x \in \mathbb{R}^n$ by $|x|$ we denote the euclidean norm of the vector $x$. Let $\beta$ be a bounded stopping time and $p > 0$. By $\mathcal{S}^p_{\mathcal{F}}(0, \beta)$ we denote the set of all $\mathcal{F}$-progressively measurable $\mathbb{R}^k$-valued processes $Y$ such that $E\sup_{0 \leq t \leq \beta} |Y_t|^p < \infty$. We set

$$\|Y\|_{\mathcal{S}^p_{\mathcal{F}}(0, \beta)} := (E\sup_{0 \leq t \leq \beta} |Y_t|^p)^{\frac{1}{p}}, \quad p > 1, \quad |Y|_{\mathcal{S}^p_{\mathcal{F}}(0, \beta)} := E\sup_{0 \leq t \leq \beta} |Y_t|^p, \quad p \in (0, 1).$$

Let $r, q \geq 1$. By $L^p_{\mathcal{F}}(0, \beta)$ we denote the set of all $\mathcal{F}$-progressively measurable, $\mathbb{R}^k$-valued processes $X$ such that

$$\|X\|_{L^p_{\mathcal{F}}(0, \beta)} := \left( E\left( \int_0^\beta |X_r|^p \, dr \right)^\frac{1}{p} \right) < \infty.$$

$L^p_{\mathcal{F}}(0, \beta)$ is the shorthand for $L^{r,q}_{\mathcal{F}}(0, \beta)$.

Let $\mathcal{G} \subset \mathcal{F}$ be a $\sigma$-field. $L^r(\mathcal{G})$ denotes the set of all $\mathcal{G}$-measurable random vectors $X$ such that $E|X|^r < \infty$. By $\mathcal{H}_{\mathcal{F}}(0, \beta)$ we denote the space of all $\mathcal{F}$-progressively measurable $\mathbb{R}^{d \times k}$-valued processes $Z$ such that $P\left( \int_0^\beta |Z_r|^2 \, dr < \infty \right) = 1$. $\mathcal{H}^s_{\mathcal{F}}(0, \beta), s > 0$, is a subspace of $\mathcal{H}_{\mathcal{F}}(0, \beta)$ consisting of $Z$ satisfying $E\left( \int_0^\beta |Z_r|^2 \, dr \right)^s < \infty$. We set

$$\|Z\|_{\mathcal{H}^s_{\mathcal{F}}(0, \beta)} := \left( E\left( \int_0^\beta |Z_r|^2 \, dr \right)^s \right)^{\frac{1}{s}}, \quad s > 1, \quad |Z|_{\mathcal{H}^s_{\mathcal{F}}(0, \beta)} := E\left( \int_0^\beta |Z_r|^2 \, dr \right)^s, \quad s \in (0, 1).$$

We say that $\mathcal{F}$-progressively measurable process $X$ is of class (D) on $[[0, \beta]]$ if the family $\{|X_r|, \tau \text{ is a stopping time}, \tau \leq \beta \}$ is uniformly integrable. For $p \geq 1$, by $D^p_{\mathcal{F}}(0, \beta)$ we denote the set of all $\mathcal{F}$-progressively measurable, $\mathbb{R}^k$-valued processes $Y$ such that $|Y|^p$ is of class (D) on $[[0, \beta]]$. We equip $D^p_{\mathcal{F}}(0, \beta)$ with the norm

$$\|Y\|_{D^p_{\mathcal{F}}(0, \beta)} := \left( \sup_{\sigma \leq \beta} E|Y_\sigma|^p \right)^{\frac{1}{p}},$$

where the supremum is taken over the set of stopping times $\sigma$. Throughout the paper, we adopt the convention that any $l$-dimensional random vector $X$, with $l < k$, is considered as a member of the space of $k$-dimensional random vectors by the inclusion operator

$$X = (X^1, \ldots, X^l) \mapsto (X^1, \ldots, X^l, 0, \ldots, 0) \in \mathbb{R}^k.$$

Let $\hat{X}$ be a $k$-dimensional $\mathcal{F}_\beta$-adapted random vector.
Definition 2.1. We say that a pair $(Y, Z)$ of $\mathbb{F}$-adapted processes is a solution to backward stochastic differential equation on the interval $[[0, \beta]]$ with right-hand side $f$ and terminal value $\xi$ ($\text{BSDE}^\beta(\xi, f)$ for short) if

(a) $Y$ is a continuous process and $Z \in \mathcal{H}_\mathbb{F}(0, \beta)$,
(b) $f(\cdot, Y, Z)$ is $\mathbb{F}$-progressively measurable and $\int_0^\beta |f(r, Y_r, Z_r)| dr < \infty$,
(c) $Y_t = \xi + \int_t^\beta f(r, Y_r, Z_r) dr - \int_t^\beta Z_r dB_r$, $t \in [0, \beta]$.

Remark 2.2. Let $\beta$ be a bounded stopping time and $T := \text{esssup} \beta$. Let $\hat{\xi} \in L^1(\mathcal{F}_\beta)$. Observe that if $(Y^\beta, Z^\beta)$ is a solution to $\text{BSDE}^\beta(\xi, f)$, then $(Y_{-\lambda \beta}, 1_{[0, \beta]} Z^\beta)$ is a solution to $\text{BSDE}^T(\hat{\xi}, 1_{[0, \beta]} f)$. Conversely, if $(Y^T, Z^T)$ is a solution to $\text{BSDE}^T(\hat{\xi}, 1_{[0, \beta]} f)$, then $(Y^T, Z^T)$ is a solution to $\text{BSDE}^\beta(\xi, f)$.

In light of the above remark, we may focus on BSDEs with deterministic terminal time. In the remainder of the paper, we fix a number $T > 0$. Let us adopt the shorthands $\|\cdot\|_T := \|\cdot\|_{\mathcal{S}_T^p(0, T)}$, $\|\cdot\| := \|\cdot\|_{\mathcal{S}_T^p(0, T)}$, $\|\cdot\|_{L^p_{\mathcal{F}}(0, T)} := \|\cdot\|_{L^p_{\mathcal{F}}(0, T)}$, $\|\cdot\|_{L^p} := \|\cdot\|_{L^p_{\mathcal{F}}(0, T)}$, $\|\cdot\|_{H^p} := \|\cdot\|_{H^p_{\mathcal{F}}(0, T)}$, $\|\cdot\|_{\mathcal{D}} := \|\cdot\|_{\mathcal{D}^p(0, T)}$.

The results presented below have been proven in [1].

Theorem 2.3. Let $p > 1$.

(i) Assume that (H1)–(H5) are in force. Then there exists a solution $(Y, Z) \in \mathcal{S}_T^p(0, T) \times \mathcal{H}_{\mathbb{F}}^p(0, T)$ to $\text{BSDE}^T(\xi, f)$.

(ii) Assume that (H1),(H2) are satisfied. Then there exists at most one solution $(Y, Z)$ to $\text{BSDE}^T(\xi, f)$ such that $Y \in \mathcal{S}_T^p(0, T)$.

Proposition 2.4. Let $p > 1$. Assume that (H1), (H2), (H5) are satisfied. Let $(Y, Z)$ be a solution to $\text{BSDE}(\xi, f)$ such that $Y \in \mathcal{S}_T^p(0, T)$. Then there exists $c_p > 0$, depending only on $p$, such that

\[
E\left[ \sup_{0 \leq t \leq T} e^{\alpha t} |Y_t|^p + \left( \int_0^T e^{2\alpha r} |Z_r|^2 dr \right)^{\frac{p}{2}} \right] \leq c_p E\left[ e^{\alpha T} |\xi|^p + \left( \int_0^T e^{\alpha r} |f(r, 0, 0)| dr \right)^p \right],
\]

for any $\alpha \geq \mu + \frac{\lambda^2}{10(p-1)}$.

3. Backward SDEs with $L^p$-data: the case $p = 1$

3.1. Preliminary results. The following result has been proven in [1].

Theorem 3.1. Let $p = 1$. Assume that (H1)–(H5), (Z) are in force. Then the following assertions hold.

(i) There exists a solution $(Y, Z)$ of $\text{BSDE}^T(\xi, f)$ such that $Y$ is of class (D) and $Z \in \mathcal{H}_{\mathbb{F}}^p(0, T)$, $s \in (0, 1)$.

(ii) There exists at most one solution $(Y, Z)$ to $\text{BSDE}^T(\xi, f)$ such that $Y$ is of class (D).

Proof. The assertion (i) follows from [1, Theorem 6.3]. As to (ii), by [1, Theorem 6.2], there exists at most one solution $(Y, Z)$ to $\text{BSDE}^T(\xi, f)$ such that $Y$ is of class (D) and $Z \in \mathcal{H}_{\mathbb{F}}^p(0, T)$, $s \in (0, 1)$. So, it is enough to show that if $(Y, Z)$ is a solution to $\text{BSDE}^T(\xi, f)$
such that $Y$ is of class (D), then $Z \in \mathcal{H}^s_T(0,T)$, $s \in (0,1)$. This follows at once from [3, Remark 2.1] and [1, Lemma 3.1].

**Remark 3.2.** Let $a \in \mathbb{R}$. Observe that if $(Y, Z)$ is a solution to BSDE$^T(\xi, f)$, then $(\bar{Y}, \bar{Z})$ is a solution to BSDE$^T(\bar{\xi}, \bar{f})$, where

\[
(\bar{Y}_t, \bar{Z}_t) := (e^{at}Y_t, e^{at}Z_t), \quad \bar{\xi} := e^{aT}\xi, \quad \bar{f}(t, y, z) := e^{at}f(t, e^{-at}y, e^{-at}z) - ay
\]

Clearly, if $(\xi, f)$ satisfies any of conditions (H1), (H3)–(H5), then $(\bar{\xi}, \bar{f})$ satisfies it too. If $f$ satisfies (H2), then $\bar{f}$ satisfies (H2) but with $\mu$ replaced by $\mu - a$, and if $f$ satisfies (Z), then $\bar{f}$ satisfies (Z) with $(\gamma, g_\mu)$ replaced by $(\gamma e^{aT}, ge^{-\frac{a}{\gamma}})$.

We let $\text{sgn}(x) := \frac{x}{|x|}$, $x \in \mathbb{R}^k$, $x \neq 0$, and $\text{sgn}(x) = 0$, $x = 0$.

**Proposition 3.3.** Let $p = 1$. Assume that $f$ does not depend on $z$ and that (H2) is in force. Let $(Y, Z)$ be a solution to BSDE$^T(\xi, f)$ such that $Y$ is of class (D). Then for any $\alpha \geq \mu$,

\[
\|e^{aY}\|_{D^1} \leq \mathbb{E}\left(e^{aT}\xi + \int_0^T e^{ar}|f(r, 0)|dr\right).
\]  

Moreover, $Y \in \mathcal{S}^q_T(0,T)$, $q \in (0,1)$ and

\[
|e^{aY}|_{S^q} \leq \frac{1}{1-q}\left[\mathbb{E}\left(e^{aTq}\xi + \int_0^T e^{arq}|f(r, 0)|dr\right)^q\right].
\]

Furthermore, if $k = 1$, then

\[
\mathbb{E}\int_0^T e^{ar}|f(r, Y_r)|dr \leq 2\mathbb{E}\left(e^{aT}|\xi| + \int_0^T e^{ar}|f(r, 0)|dr\right).
\]

**Proof.** In light of Remark 3.2, we may assume that $\mu \leq 0$ in condition (H2) and $a = 0$. Let us define

\[
\tau_n = \inf\{t \geq 0 : \int_0^t |Z_r|^2dr \geq n\} \wedge T.
\]

By the Itô-Tanaka formula (see [1, Corollary 2.3]), for any stopping time $\sigma \leq \tau_k$,

\[
|Y_\sigma| \leq |Y_{\tau_n}| + \int^T_{\sigma} \langle \text{sgn}(Y_r), f(r, Y_r) \rangle dr - \int^T_{\sigma} \langle \text{sgn}(Y_r)Z_r, dB_r \rangle.
\]

By (H2)

\[
\langle \text{sgn}(Y_r), f(r, Y_r) - f(r, 0) \rangle \leq 0.
\]

From the above and (3.4) we deduce that

\[
|Y_\sigma| + \int^T_{\tau_n} \langle \text{sgn}(Y_r), f(r, Y_r) - f(r, 0) \rangle dr \leq |Y_{\tau_n}| + \int_0^T |f(r, 0)|dr - \int^T_{\tau_n} \langle \text{sgn}(Y_r)Z_r, dB_r \rangle.
\]

By the definition of $(\tau_n)$, we have that $\int_0^{\wedge \tau_n} \langle \text{sgn}(Y_r)Z_r, dB_r \rangle$ is a martingale. Therefore

\[
\mathbb{E}|Y_\sigma| + \mathbb{E}\int^T_{\tau_n} \langle \text{sgn}(Y_r), f(r, Y_r) - f(r, 0) \rangle dr \leq \mathbb{E}|Y_{\tau_n}| + \int_0^T |f(r, 0)|dr,
\]

Passing to the limit with $n \to \infty$ we get (3.1). By [3, Remark 2.1] we have

\[
\mathbb{E}\sup_{t \leq T}|Y_t|^{q} \leq \frac{1}{1-q}\|Y\|_{D^1}^{q}.
\]
for any \( q \in (0, 1) \). This combined with (3.1) yields (3.2). In case \( k = 1 \)
\[
\mathbb{E} \int_{\sigma}^{T_n} \left| (\text{sgn}(Y_r), f(r, Y_r) - f(r, 0)) \right| dr = \mathbb{E} \int_{\sigma}^{T_n} |f(r, Y_r) - f(r, 0)| dr.
\]
Therefore, letting \( n \to \infty \) in (3.6) gives
\[
\mathbb{E} \int_{0}^{T} |f(r, Y_r) - f(r, 0)| dr \leq \mathbb{E} \left( |\xi| + \int_{0}^{T} |f(r, 0)| dr \right). \tag{3.8}
\]
From this we easily conclude (3.3).
\[\square\]

3.2. Main results. We shall adopt the following notation: \( e_a : \mathbb{R} \to \mathbb{R} \), with \( e_a(x) := e^{ax}, x \in \mathbb{R} \). We will use frequently the following function
\[
C(x, y, z) := 2(1 + y)x[(z + \lambda) \vee 1](1 \vee T)^3, \quad x, y, z \geq 0.
\]
Observe that for any \( x, y_1, y_2, z \geq 0 \)
\[
C(x, y_1 + y_2, z) \leq C(x, y_1, z) + C(x, y_2, z). \tag{3.9}
\]

**Theorem 3.4.** Let \( p = 1 \). Assume (H1)–(H5), (Z).

(i) For any \( q \in (\kappa, 1) \) there exists \( c_{\kappa, q} > 0 \) depending only on \( q, \kappa \) such that for any solution \((Y, Z)\) to BSDE\((\xi, f)\) such that \( Y \) is of class (D) and any \( a \geq \mu + \frac{\lambda^2}{1/(q - 1)} \), we have
\[
\|e_a Y\|_{D^1} + \mathbb{E} \left( \int_{0}^{T} e^{2ar}|Z_r|^2 dr \right)^{\frac{1}{2}} \leq C(c_{\kappa, q}, |e_{-\frac{q}{2}} g|_{L_T^2}, e^{aT}\gamma) \psi_1 \left( e^{aT}\mathbb{E} |\xi| + \mathbb{E} \int_{0}^{T} e^{ar}|f(r, 0, 0)| dr \right), \tag{3.10}
\]
where \( \psi_1(x) = x + x^2(1-q), x \geq 0 \).

(ii) Let \( q \in (\kappa, 1) \) and \( k = 1 \). Then for any solution \((Y, Z)\) to BSDE\((\xi, f)\), such that \( Y \) is of class (D), and any \( a \geq \mu + \frac{\lambda^2}{1/(q - 1)} \), we have
\[
\mathbb{E} \int_{0}^{T} e^{ar}|f(r, Y_r, Z_r)| dr \leq C^2(c_{\kappa, q}, |e_{-\frac{q}{2}} g|_{L_T^2}, e^{aT}\gamma) \psi_2 \left( e^{aT}\mathbb{E} |\xi| + \mathbb{E} \int_{0}^{T} e^{ar}|f(r, 0, 0)| dr \right), \tag{3.11}
\]
with \( c_{\kappa, q} \) as in (i), and \( \psi_2(x) = x + x^2(1-q)^2, x \geq 0 \).

**Proof.** By Remark 3.2, we may assume that \( 0 = a \geq \mu + \frac{\lambda^2}{1/(q - 1)} \). Throughout the proof \( c_{\gamma_1, \ldots, \gamma_k} \) denote a constant, which may vary from line to line, but it depends only on parameters \( \gamma_1, \gamma_2, \ldots, \gamma_k \). Throughout the proof, we frequently use the following elementary inequality
\[
x^b \leq x^a, \quad x \geq 0, \quad 0 \leq a \leq b \leq 1. \tag{3.12}
\]
Fix \( q \in (\kappa, 1) \). Set \( p_0 := \frac{q}{q - 1} > 1 \). We let \( f^0(t, y) := f(t, y, 0) \). Let \((Y^0, Z^0)\) be a solution to BSDE\(^F(\xi, f^0)\), such that \( Y^0 \) is of class (D) and \( Z^0 \in \mathcal{H}_T^2(0, T), s \in (0, 1) \) (see Theorem 3.1).

Observe that the pair \((\bar{Y}, \bar{Z}) \) := \((Y - Y^0, Z - Z^0)\) is a solution to BSDE\(^F(0, F)\) with
\[
F(t, y, z) := f(t, y + Y^0_t, z + Z^0_t) - f(t, Y^0_t, 0).
\]
It is an elementary check that $F$ satisfies conditions (H1)–(H3), (Z) and (H5) for any $p \geq 1$. By [1, Corollary 2.3]

$$|\bar{Y}_t| \leq \mathbb{E}\left( \int_0^T |F(r,0,Z_r)| \, dr \right) = \mathbb{E}\left( \int_0^T |f(r,Y^0_r,Z_r) - f(r,Y^0_0,0)| \, dr \right)$$

$$\leq \gamma \mathbb{E}\left( \int_0^T (g_r + |Y^0_r| + |Z_r|)^\kappa \, dr \right).$$

Therefore, by Doob’s inequality,

$$\mathbb{E} \sup_{t \leq T} |\bar{Y}_t|^{p_0} \leq c_{p_0} T^{p_0-1} \mathbb{E} \int_0^T (g_r + |Y^0_r| + |Z_r|)^q \, dr < \infty$$

Consequently, by [1, Lemma 3.1] $\bar{Z} \in \mathcal{H}^{p_0}_{T_p}(0,T)$. Observe that conditions (H1) and (Z) (for $f$) together imply that

$$|F(t,0,0)| \leq (\lambda + \gamma)(g_t \mathbb{1}_{\{|Z|^2 \geq 1\}} + |Y^0_t| + |Z^0_t|)^{\kappa}, \quad t \in [0,T].$$

Therefore, by Proposition 2.4 and [1, Lemma 3.1],

$$\mathbb{E} \sup_{t \leq T} |\bar{Y}_t|^{p_0} + \mathbb{E}\left( \int_0^T \bar{Z}_r^2 \, dr \right)^{p_0/2} \leq c_{p_0} \mathbb{E}\left( \int_0^T |F(r,0,0)| \, dr \right)^{p_0} \leq c_{p_0} (\gamma + \lambda) \mathbb{E}\left( \int_0^T (|g_r| \mathbb{1}_{\{|Z|^2 \geq 1\}} + |Y^0_r| + |Z^0_r|)^\kappa \, dr \right)^{p_0}$$

$$\leq 3^{p_0} c_{p_0} (\gamma + \lambda) \left[ \mathbb{E}\left( \int_0^T g_r^\kappa \mathbb{1}_{\{|Z|^2 \geq 1\}} \, dr \right)^{p_0} + T^{p_0} \mathbb{E} \sup_{t \leq T} |Y^0_t|^q + T^{p_0(2-q)} \mathbb{E}\left( \int_0^T |Z^0_t|^2 \, dr \right)^{\frac{q}{2}} \right]$$

$$\leq 3^{p_0} (1 \vee T)^{p_0} c_{p_0} (\gamma + \lambda) \left[ \mathbb{E}\left( \int_0^T g_r^\kappa \mathbb{1}_{\{|Z|^2 \geq 1\}} \, dr \right)^{p_0} + \mathbb{E} \sup_{t \leq T} |Y^0_t|^q + \mathbb{E}\left( \int_0^T |Z^0_t|^2 \, dr \right)^{\frac{q}{2}} \right]$$

$$\leq 3^{p_0} (1 \vee T)^{p_0} c_{p_0} (\gamma + \lambda) \left[ \mathbb{E}\left( \int_0^T g_r^\kappa \mathbb{1}_{\{|Z|^2 \geq 1\}} \, dr \right)^{p_0} + c_q \left( \mathbb{E} \int_0^T |f(r,0,0)| \, dr + \mathbb{E} |\xi| \right)^q \right].$$

(3.13)

For brevity, we let $T_1 := 1 \vee T$, $\gamma_\lambda := (\lambda + \gamma) \vee 1$ and $A := 3^{p_0} T_1^{p_0} c_{p_0} \gamma_\lambda$. Now, we shall estimate the first term on the right-hand side of (3.13). By using Hölder’s inequality, we compute that

$$\mathbb{E}\left( \int_0^T g_r^\kappa \mathbb{1}_{\{|Z|^2 \geq 1\}} \, dr \right)^{p_0} \leq T^{p_0-1} \mathbb{E}\left( \int_0^T g_r^\kappa \mathbb{1}_{\{|Z|^2 \geq 1\}} \, dr \right)$$

$$\leq T^{p_0-1} \left( \mathbb{E} \int_0^T g_r \, dr \right)^q \mathbb{E}\left( \int_0^T \mathbb{1}_{\{|Z|^2 \geq 1\}} \, dr \right)^{1-q}.$$ 

(3.14)

By Hölder’s inequality again

$$\left[ \mathbb{E}\left( \int_0^T \mathbb{1}_{\{|Z|^2 \geq 1\}} \, dr \right) \right]^{1-q} = \left[ \mathbb{E}\left( \int_0^T \mathbb{1}_{\{|Z|^2 \geq 1\}} \, dr \right) \right]^{1-q}$$

$$\leq \left[ \mathbb{E} \int_0^T |Z_r|^q \, dr \right]^{1-q} \leq T^{(2-q)(1-\gamma)} \left[ \left( \mathbb{E} \int_0^T |Z_r|^2 \, dr \right)^{\gamma} \right]^{1-q}.$$ 

(3.15)
Applying, respectively, [1, Lemma 3.1], Proposition 3.3 and Jensen’s inequality, we find that
\[
\left[ \mathbb{E} \left( \int_0^T |Z_t^0| \, dt \right)^{1-q} \right]^{1-q} \leq c_q \left[ \mathbb{E} \sup_{t \in [0,T]} |Y_t^0|^{q} + \mathbb{E} \left( \int_0^T |f(r,0,0)| \, dr \right)^q \right]^{1-q}
\leq c_q \left( \frac{1}{1-q} \left( \mathbb{E} |\xi| + \mathbb{E} \int_0^T |f(r,0,0)| \, dr \right)^q + \mathbb{E} \left( \int_0^T |f(r,0,0)| \, dr \right)^q \right)^{1-q} \tag{3.16}
\leq c_q \left( \mathbb{E} \int_0^T |f(r,0,0)| \, dr + \mathbb{E} |\xi| \right)^{q(1-q)}
\]
Set \( K := \mathbb{E} \int_0^T |f(r,0,0)| \, dr + \mathbb{E} |\xi| \). Combining (3.13)–(3.16) implies that
\[\|\tilde{Y}\|_{L^q_{S^0}} + \|\tilde{Z}\|_{F^0} \leq c_q A \left( T^1 \|g\|_{L^q_T} K^{q(1-q)} + K^q \right). \tag{3.17}\]
Hence
\[\|\tilde{Y}\|_{D^1} \leq c_q A \left( T^1 \|g\|_{L^q_T} K^{q(1-q)} + K^q \right), \quad |\tilde{Z}|_{F^0} \leq c_q A \left( T^1 \|g\|_{L^q_T} K^{q(1-q)} + K^q \right). \]
Consequently,
\[\|Y\|_{D^1} \leq (c_q A)^{1/p_0} \left( T^1 \|g\|_{L^q_T} K^{q(1-q)} + K^q \right)^{1/p_0} + \|Y^0\|_{D^1}, \tag{3.18}\]
\[\|Z\|_{F^0} \leq (c_q A)^{q/p_0} \left( T^1 \|g\|_{L^q_T} K^{q(1-q)} + K^q \right)^{q/p_0} + |Z^0|_{F^0}. \tag{3.19}\]
By Proposition 3.3, [1, Lemma 3.1] and (3.12),
\[\|Y\|_{D^1} \leq (c_q A)^{1/p_0} \left( T^1 \|g\|_{L^q_T} K^{q(1-q)} + K^q \right)^{1/p_0} + K \]
\[\leq (1 + \|g\|_{L^q_T}) c_{\kappa,q} \gamma T^1 (K^{\kappa^2(1-q)} + K) = \frac{1}{2} C(c_{\kappa,q}, \|g\|_{L^q_T}, \gamma) (K^{\kappa^2(1-q)} + K), \]
and
\[|Z|_{F^0} \leq (c_q A)^{q/p_0} \left( T^1 \|g\|_{L^q_T} K^{q(1-q)} + K^q \right)^{q/p_0} + c_q K^q \]
\[\leq (1 + \|g\|_{L^q_T}) c_{\kappa,q} \gamma T^1 (K^{\kappa^2(1-q)} + K) = \frac{1}{2} C(c_{\kappa,q}, \|g\|_{L^q_T}, \gamma) (K^{\kappa^2(1-q)} + K). \]
We see that (3.18), (3.19) imply (3.10). In order to obtain (3.11), we look at \((Y,Z)\) as a solution to BSDE\(^T\) \(\xi, f_Z\), where \(f_Z(t,y) := f(t,y,Z_t)\), in other words, we freeze \(Z\) in the driver \(f\). From this perspective \((Y,Z)\) is a solution to BSDE with the driver independent of \(z\) variable. Therefore, by (3.3) applied to \(f_Z\), we have
\[\mathbb{E} \int_0^T |f(r,Y_r,Z_r)| \, dr \leq 2 \left( \mathbb{E} |\xi| + \mathbb{E} \int_0^T |f(r,0,Z_r)| \, dr \right). \tag{3.20}\]
By (Z) and (H1)
\[\mathbb{E} \int_0^T |f(r,0,Z_r)| \, dr \leq \gamma \lambda \mathbb{E} \int_0^T g^r \mathbb{1}_{\{Z_r \geq 1\}} + |Z_r|^\kappa \, dr + \mathbb{E} \int_0^T |f(r,0,0)| \, dr. \tag{3.21}\]
By Hölder’s inequality
\[\mathbb{E} \int_0^T g^r \mathbb{1}_{\{Z_r \geq 1\}} \, dr \leq \|g\|_{L^q_T} \left( \mathbb{E} \int_0^T \mathbb{1}_{\{Z_r \geq 1\}} \, dr \right)^{1-\kappa}, \tag{3.22}\]
\[\left( \mathbb{E} \int_0^T \mathbb{1}_{\{Z_r \geq 1\}} \, dr \right)^{1-\kappa} \leq \left( \mathbb{E} \int_0^T |Z_r|^\kappa \, dr \right)^{1-\kappa} \leq \left[ T^{-1/q} \mathbb{E} \left( \int_0^T |Z_r|^q \, dr \right)^{q/2} \right]^{1-\kappa}, \tag{3.23}\]
Assume that (H1)-(H5), (Z) hold. Let
\[ \square \]
where
\[ (f) \]
hand side of (f)
\[ (3.20) \]
\[ \text{Proof.} \]
Set
\[ \text{By virtue of (3.20)-(3.24), we conclude that} \]
\[ \mathbb{E} \int_0^T |f(r,Y_r,Z_r)| dr \leq 2 \left[ K + \gamma \lambda |g|_{L^2} \right] \mathbb{E} \left[ \int_0^T |Z_r|^2 dr \right]^{q/2} \]
\[ \leq 2 \left[ K + \gamma \lambda |g|_{L^2} T^2 |Z|^1_{\mathcal{H}^q} + \gamma \lambda T |Z|^2_{\mathcal{H}^q} \right]. \]  
This combined with (3.19) gives, with the shorthand
\[ C = C(c_{\kappa,q}, \|g\|_{L^2}, \gamma), \]
\[ \mathbb{E} \int_0^T |f(r,Y_r,Z_r)| dr \leq 2 \left[ K + T^2 \gamma \lambda C^{1-\kappa}(K^{\kappa q(1-q)(1-\kappa)} + K^{q(1-\kappa)}) \right. \]
\[ \left. + \gamma \lambda T \gamma |C|^q (K^{\kappa^2(1-q)} + K^{\kappa}) \right]. \]
Observe that \( \kappa^2(1-q)^2 \) is smaller than any exponent of a power with base \( K \) on the right-hand side of (3.26). Therefore, from (3.12) and (3.26), we conclude that
\[ \mathbb{E} \int_0^T |f(r,Y_r,Z_r)| dr \leq C^2 \left( K + K^{\kappa^2(1-q)^2} \right). \] 
This completes the proof. \( \square \)

**Theorem 3.5.** Let \( p = 1 \). Consider a function \( \tilde{f} : \Omega \times [0,T] \times \mathbb{R}^k \times \mathbb{R}^d \rightarrow \mathbb{R}^k \) and \( \tilde{\xi} \in L^1(\mathcal{F}_T) \). Assume that (H1)-(H5), (Z) hold. Let \( (Y,Z), (\tilde{Y},\tilde{Z}) \) be solutions to BSDE\( ^T (\xi, f), \)
BSDE\( ^T (\tilde{\xi}, \tilde{f}) \), respectively, such that \( Y, \tilde{Y} \) are of class (D). Suppose that \( \mathbb{E} \int_0^T |f(r,Y_r,Z_r) - \tilde{f}(r,\tilde{Y}_r,\tilde{Z}_r)| dr < \infty \).

(i) For any \( q \in (\kappa,1) \) there exists \( c_{\kappa,q} > 0 \) depending only on \( \kappa, q \) such that for any \( a \geq \mu + \frac{\lambda^2}{18(\sqrt{\kappa})^2} \),
\[ \|e_a(Y - \tilde{Y})\|_{D^1} + \|e_a(Z - \tilde{Z})\|_{\mathcal{H}^q} \leq C(c_{\kappa,q}, \|e_a - \hat{g}\|_{L^2}, 2e^{\alpha T} \gamma) \psi_3(|\xi - \tilde{\xi}|_{L^1} + \|f - \tilde{f}(\cdot, Y, Z)\|_{L^1}) \] (3.28)
where \( \hat{g}_t := g_t + 3|\hat{Y}|_{k} + |\hat{Z}|_{k}, \hat{\kappa} := \sqrt{\kappa q}, \) and \( \psi_3(x) = x + x^{\kappa^2(1-q)} \), \( x \geq 0 \).

(ii) Let \( q \in (\kappa,1) \) and \( k = 1 \). Then for any \( a \geq \mu + \frac{\lambda^2}{18(\sqrt{\kappa})^2} \),
\[ \mathbb{E} \int_0^T e^{a r} |f(r,Y_r,Z_r) - \tilde{f}(r,\tilde{Y}_r,\tilde{Z}_r)| dr \leq C^2(c_{\kappa,q}, \|e_a - \hat{g}\|_{L^2}, 2e^{\alpha T} \gamma) \psi_4(|\xi - \tilde{\xi}|_{L^1} + \|f - \tilde{f}(\cdot, Y, Z)\|_{L^1}) \] (3.29)
with \( c_{\kappa,q}, \hat{\kappa}, \hat{\gamma} \) as in (i), and \( \psi_4(x) := x + x^{\kappa^2(1-q)^2} \).

**Proof.** Set
\[ C := C(c_{\kappa,q}, \|e_a - \hat{g}\|_{L^2}, 2e^{\alpha T} \gamma). \] Let \( (Y, Z), (\tilde{Y}, \tilde{Z}) \) be as in the assertion of the theorem. By Theorem 3.1 \( Z \in L^2(0,T), s \in (0,T). \) Observe that
\[ Y_t - \tilde{Y}_t = \xi - \tilde{\xi} + \int_0^T F(r,Y_r - \tilde{Y}_r,Z_r - \tilde{Z}_r) dr - \int_t^T (Z_r - \tilde{Z}_r) dB_r, \ t \in [0,T], \]
where
\[ F(t,y,z) = f(t,y + \tilde{Y}_t,z + \tilde{Z}_t) - \tilde{f}(t,\tilde{Y}_t,\tilde{Z}_t). \] In other words \( Y - \tilde{X}, Z - \tilde{\xi} \) is a solution to BSDE\( ^T (\xi - \tilde{\xi}, F) \). By the assumptions made on \( \tilde{f} \), process \( \tilde{f}(\cdot, \tilde{Y}, \tilde{Z}) \) is \( \mathbb{F} \)-progressively...
measurable. Obviously, $F$ satisfies (H1), (H2), with the same constants, and (H3), (H5). By
Theorem 3.1 $Z - \bar{Z} \in H^p_{\mathbb{F}}(0, T)$, $s \in (0, 1)$. As a result, $\bar{Z} \in H^p_{\mathbb{F}}(0, T)$, $s \in (0, 1)$. Now, by (Z), we have
$$|F(t, y, z) - F(t, y, 0)| = |f(t, y + \bar{Y}_t, z + \bar{Z}_t) - f(t, y + \bar{Y}_t, \bar{Z}_t)| \leq 2\gamma(g_t + |\bar{Y}_t| + |\bar{Z}_t| + |y| + |z|)^\kappa.$$  
Let us take $\beta \in (\kappa, q)$. Then
$$(g_t + |\bar{Y}_t| + |\bar{Z}_t| + |y| + |z|)^\kappa = \left((g_t + |\bar{Y}_t| + |\bar{Z}_t| + |y| + |z|)^\beta\right)^\kappa$$
$$\leq \left((g_t \vee 1) + |\bar{Y}_t|^{\beta} + |\bar{Z}_t|^{\beta} + (|y| \vee 1) + (|z| \vee 1)\right)^\beta \leq (g_t + 3 + |\bar{Y}_t|^{\beta} + |\bar{Z}_t|^{\beta} + |y| + |z|)^\beta.$$  
Let us define $\hat{g}_t := g_t + 3 + |\bar{Y}_t|^{\beta} + |\bar{Z}_t|^{\beta}$. By the fact that $\bar{Y}$ is of class (D) and $\bar{Z} \in H^p_{\mathbb{F}}(0, T)$, $s \in (0, 1)$, we have $\hat{g} \in L^p_F(0, T)$. Thus, $F$ satisfies (Z), with $g$ replaced by $\hat{g}$, $\gamma$ replaced by $2\gamma$ and $\kappa$ replaced by $\beta$.

**Step 1.** Suppose that $F$ satisfies (H4). Letting $\beta = \sqrt{\alpha q}$ and applying Theorem 3.1 give the desired inequalities. **Step 2.** The general case. By the very definition of a solution to BSDE$^T(\xi, f)$,
$$\int_0^T |\tilde{f}(r, \bar{Y}_r, \bar{Z}_r)| dr < \infty.$$  
(3.30)
Thus, the sequence
$$\sigma_n := \inf\{t > 0 : \int_0^t |\tilde{f}(r, \bar{Y}_r, \bar{Z}_r)| dr \geq n\} \wedge T, \quad n \geq 1,$$  

satisfies $P(\exists_{n \geq 1} \sigma_n = T) = 1$. Let $\alpha_n := \inf\{t > 0 : |\bar{Y}_t - \bar{Y}_0| \geq n\} \wedge T$. Since $\bar{Y}$ is càdlàg, we also have $P(\exists_{n \geq 1} \alpha_n = T) = 1$. Let $\tau_n := \sigma_n \wedge \alpha_n$, and $a_0 := \mathbb{E}|Y_0|$. Then, by (Z),
$$\int_0^{\tau_n} \sup_{|y| \leq M} |F(t, y, 0)| dt$$
$$\leq \int_0^{\tau_n} \sup_{|y| \leq M} \left(|f(t, y + \bar{Y}_t, \bar{Z}_t) - f(t, y + \bar{Y}_t, 0)| + |f(t, y + \bar{Y}_t, 0)| + |\tilde{f}(t, \bar{Y}_t, \bar{Z}_t)|\right) dt$$
$$\leq \gamma \int_0^{\tau_n} (g_t + |\bar{Y}_t| + |\bar{Z}_t| + M)^\kappa dt + \int_0^{\tau_n} \sup_{|y| \leq M+n+a_0} |f(t, y, 0)| dt + n.$$  
In consequence, $F$ satisfies (H4) with $f$ replaced by $F$ and $\tau$ replaced by $\tau_n$ (recall that $\bar{Z} \in H^p_{\mathbb{F}}(0, T)$, $s \in (0, 1)$). Clearly, $(\bar{Y} - \bar{Y}, \bar{Z} - \bar{Z})$ is a solution to BSDE$^{\tau_n}(Y_{\tau_n} - Y_{\tau_n}, F)$. By Step 2,
$$\|e_a(Y - \bar{Y})\|_{H^p(0, \tau_n)} + \|e_a(Z - \bar{Z})\|_{H^p(0, \tau_n)} \leq C\psi_3(\|Y_{\tau_n} - \bar{Y}_{\tau_n}\|_{L^1} + \|f - \tilde{f}(\cdot, \bar{Y}, \bar{Z})\|_{L^p_f}),$$  
and
$$\mathbb{E} \int_0^{\tau_n} e^{\alpha x} |f(r, Y_r, Z_r) - \tilde{f}(r, \bar{Y}_r, \bar{Z}_r)| dr \leq C^2\psi_4(\|Y_{\tau_n} - \bar{Y}_{\tau_n}\|_{L^1} + \|f - \tilde{f}(\cdot, \bar{Y}, \bar{Z})\|_{L^p_f}).$$  
By sending $n \to \infty$ and using the fact that $Y, \bar{Y}$ are of class (D) and $P(\exists_{n \geq 1} \tau_n = T) = 1$, we conclude the result.

**Corollary 3.6.** Let $p = 1$. Let $\tilde{f} : \Omega \times [0, T] \times \mathbb{R}^k \times \mathbb{R}^{d \times k} \to \mathbb{R}^k$ and $\bar{\xi} \in L^1(\mathcal{F}_T)$. Assume that $(\xi, f)$ and $(\bar{\xi}, \tilde{f})$ satisfy (H1)-(H5), (Z). Let $(Y, Z), (\bar{Y}, \bar{Z})$ be solutions to BSDE$^T(\xi, f)$,
BSDE$^T(\xi, f)$, respectively, such that $Y, \bar{Y}$ are of class (D). Suppose that $\mathbb{E}\int_0^T |f(r, \bar{Y}_r, Z_r) - \bar{f}(r, \bar{Y}_r, \bar{Z}_r)|dr < \infty$. Let

$$K_a := e^{aT} \mathbb{E}[\xi] + \mathbb{E}\int_0^T e^{ar}|\bar{f}(r, 0, 0)|dr, \quad \delta f := f - \bar{f}, \quad \delta \xi := \xi - \bar{\xi},$$

and

$$L_a(x) := C(x, \|e_{-\frac{a}{\kappa}}g\|_{L^1}, e^{a^T}\gamma)\psi_1(K_a), \quad x \geq 0,$$

with $\psi_1$ as in Theorem 3.4(i).

(i) For any $q \in (\kappa, 1)$ there exists $c_{\kappa,q} > 0$ - depending only on $\kappa, q$ - such that for any $a \geq \mu + \frac{\lambda^2}{1\wedge (\frac{\kappa}{\kappa-1})}$,

$$\|e_{-\kappa}(Y - \bar{Y})\|_{\mathcal{D}^1} + \|e_{\kappa}(Z - \bar{Z})\|_{\mathcal{H}^0} \leq T_1\bar{L}^3(c_{\kappa,q})\psi_3(\|\delta \xi\|_{L^1} + \|\delta f(\cdot, \bar{Y}, \bar{Z})\|_{L^1}),$$

(3.31)

where $\hat{\kappa} := \sqrt{\kappa^2}$, and $\psi_3(x) = x + x^{2\kappa(1-q)}$, $x \geq 0$.

(ii) Let $q \in (\kappa, 1)$ and $k = 1$. Then for any $a \geq \mu + \frac{\lambda^2}{1\wedge (\frac{\kappa}{\kappa-1})}$,

$$\mathbb{E}\int_0^T e^{ar}|f(r, Y_r, Z_r) - \bar{f}(r, \bar{Y}_r, \bar{Z}_r)|dr \leq T_1\bar{L}^3(c_{\kappa,q})\psi_4(\|\delta \xi\|_{L^1} + \|\delta f(\cdot, \bar{Y}, \bar{Z})\|_{L^1}),$$

(3.32)

with $c_{\kappa,q}, k$ as in (i), and $\psi_4(x) := x + x^{k(1-q)^2}$.

Proof. Clearly, $a \geq \mu + \frac{\lambda^2}{1\wedge (\frac{\kappa}{\kappa-1})}$. By using Theorem 3.4, we have

$$\|e_{-\kappa}(g + 3 + |\bar{Y}|^{\hat{\kappa}} + |\bar{Z}|^{\hat{\kappa}})\|_{L^p} \leq \|e_{-\kappa}g\|_{L^p_L} + 4\|e_{-a}\|_{L^p_L} + T\|e_\kappa\bar{Y}\|_{\mathcal{D}^1} + T_1\|e_\kappa\bar{Z}\|_{\mathcal{H}^{\sqrt{\kappa}}}$$

$$\leq \|e_{-\kappa}g\|_{L^p_L} + 4\left(\frac{1}{\kappa}\wedge T\right) + T_1C(c_{\kappa,q}, \|e_{-\kappa}g\|_{L^p_L}, e^{a^T}\gamma)\psi_1(K_a)$$

$$\leq 5T_1C(c_{\kappa,q}, \|e_{-\kappa}g\|_{L^p_L}, e^{a^T}\gamma)\psi_1(K_a).$$

Therefore, by (3.9),

$$C(c_{\kappa,q}, \|e_{-\kappa}(g + 3 + |\bar{Y}|^{\hat{\kappa}} + |\bar{Z}|^{\hat{\kappa}})\|_{L^p_L}, 2e^{a^T}\gamma) \leq 5T_1C^2(c_{\kappa,q}, \|e_{-\kappa}g\|_{L^p_L}, e^{a^T}\gamma)\psi_1(K_a).$$

(3.33)

From this and Theorem 3.5 one easily concludes the desired inequalities. \hfill \Box

Corollary 3.7. Let $p = 1$, $\beta$ be a bounded stopping time, $\eta, \bar{\eta} \in \mathcal{F}_\beta$ and $f$ satisfy (H1)--(H5), (Z). Then there exists $C > 0$ - depending only on $\lambda, \mu, \sup \beta, \gamma, \kappa$ - such that

$$\mathbb{E}[f^\beta_{\alpha}E^\eta_{\alpha, \beta} - E^\beta_{\alpha, \beta}(\bar{\eta})] \leq C(1 + \|g\|_{L^1_L})^2(1 + \|\bar{\eta}\|_{L^1(\mathcal{F}_\beta)} + \|f(\cdot, 0, 0)\|_{L^1_L(\mathcal{F}_\beta)})^2\psi(\|\eta - \bar{\eta}\|_{L^1(\mathcal{F}_\beta)})$$

for any stopping time $\alpha \leq \beta$, where $\psi(x) = x + x^{\frac{1}{4}(1-\kappa^2)}$, $x \geq 0$.

Proof. It follows directly from Corollary 3.6 applied with $q = \frac{1}{2}(1 + \kappa)$. \hfill \Box

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