Distributed Continuous-Time Algorithm for Constrained Convex Optimizations via Nonsmooth Analysis Approach

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Abstract

This technical note studies the distributed optimization problem of a sum of nonsmooth convex cost functions with local constraints. At first, we propose a novel distributed continuous-time projected algorithm, in which each agent knows its local cost function and local constraint set, for the constrained optimization problem. Then we prove that all the agents of the algorithm can find the same optimal solution, and meanwhile, keep the states bounded while seeking the optimal solutions. We conduct a complete convergence analysis by employing nonsmooth Lyapunov functions for the stability analysis of differential inclusions. Finally, we provide a numerical example for illustration.

Index Terms

Constrained distributed optimization, continuous-time algorithms, multi-agent systems, nonsmooth analysis, projected dynamical systems.

I. INTRODUCTION

The distributed optimization of a sum of convex functions is an important class of decision and data processing problems over network systems, and has been intensively studied in recent years
In addition to the discrete-time distributed optimization algorithms (e.g., [1], [2]), continuous-time multi-agent solvers have recently been applied to distributed optimization problems as a promising and useful technique [3]–[8], thanks to the well-developed continuous-time stability theory.

Constrained distributed optimization, in which the feasible solutions are limited to a certain region or range, is significant in a number of network decision applications, including multirobot motion planning, resource allocation in communication networks, and economic dispatch in power grids. In practice, local constraints in the distributed optimization design are often necessary due to the performance limitations of the agents in computation and communication capacities as well as task requirements of privacy and security. For example, in large-scale optimization problems, the computation/communication capacity of a single agent may not be enough to handle all the constraints of the agents; in alignment or resource allocation problems, each agent’s feasible choice is limited to a certain range, while the agents may not want to share their private information with others; and in strategic social networks, the agents keep their own limit constraints or budget constraints confidential for security concerns. However, due to the consideration of local constraints, the design of such algorithms, to minimize the global cost functions within the feasible set while allowing the agents operate with only local cost functions and local constraints, is a very difficult task. Conventionally, the projection method has been widely adopted in the algorithm design for constrained optimization [9], [10] and related problems [11]. [6] constructed a primal-dual type continuous-time projected algorithm to solve a distributed optimization problem, where each agent has its own private constraint function, while [8] proposed a continuous-time distributed projected dynamics for constrained optimization, where the agents share the same constraint set. Moreover, [12] presented a primal-dual continuous-time projected algorithm for distributed nonsmooth optimization, where each agent has its own local bounded constraint set, though its auxiliary variables may be asymptotically unbounded.

The purpose of this technical note is to propose a novel continuous-time projected algorithm for distributed nonsmooth convex optimization problems where each agent has its own general local constraint set. The main contributions of the note are four folds. Firstly, a distributed continuous-time algorithm is proposed for the agents to find the same optimal solution based only on local cost functions and local constraint sets, by combining primal-dual methods for saddle
point seeking and projection methods for set constraints. The proposed algorithm is consistent with those in [3]–[5] when there were no constraints in the optimization problem. Secondly, nonsmooth cost functions are considered here, while smooth cost functions were discussed in most continuous-time distributed optimization designs [6], [7]. To solve the complicated problem, nonsmooth Lyapunov functions are employed along with the stability theory of differential inclusions (resulting from the nonsmooth cost functions) to conduct a complete and original convergence analysis. Thirdly, our proposed algorithm is proved to solve the optimization problem and have bounded states while seeking the optimal solutions, and therefore, further improves the recent interesting result in [12], whose algorithm may have asymptotically unbounded states. Finally, different from the strict/strong convexity in existing results [6], [7], general convexity is investigated. In fact, our nonsmooth analysis techniques also guarantee the convergence of the algorithm even when the problem has a continuum of optimal solutions due to the convexity. Therefore, the convergence analysis provides additional insights and understandings for continuous-time distributed optimization algorithms compared with [3], [5]–[7].

The remainder of this note is organized as follows. In Section II, notations and definitions are presented and reviewed. In Section III, a constrained convex (nonsmooth) optimization problem is formulated and a distributed continuous-time projected algorithm is proposed. In Section IV, a complete proof is presented to show that the algorithm state is bounded and the agents’ estimates are convergent to the same optimal solution, and simulation studies are carried out for illustration. Finally, in Section V, concluding remarks are given.

II. MATHEMATICAL PRELIMINARIES

In this section, we introduce necessary notations, definitions and preliminaries about graph theory and projection operators.

A. Notations

Let \( \mathbb{R} \) denote the set of real numbers; let \( \mathbb{R}^n \) and \( \mathbb{R}^{n \times m} \) denote the set of \( n \)-dimensional real column vectors and the set of \( n \)-by-\( m \) real matrices, respectively; \( \mathcal{B}(\mathbb{R}^q) \) denotes the collection of all subsets of \( \mathbb{R}^q \); \( I_n \) denotes the \( n \times n \) identity matrix and (\( \cdot \))\(^T\) denotes the transpose. Furthermore, \( \| \cdot \| \) denotes the Euclidean norm. Write \( \text{rank}(A) \) for the rank of a matrix \( A \), \( \text{range}(A) \) for the range of \( A \), \( \ker(A) \) for the kernel of \( A \), \( \lambda_{\text{max}}(A) \) for the largest eigenvalue of \( A \), \( 1_n \) for the \( n \times 1 \)
ones vector, \( \mathbf{0}_n \) for the \( n \times 1 \) zeros vector, and \( A \otimes B \) for the Kronecker product of matrices \( A \) and \( B \). Denote \( A > 0 \) (or \( A \geq 0 \)) when matrix \( A \in \mathbb{R}^{n \times n} \) is positive definite (or positive semi-definite). Also, denote \( \overline{S} \) as the closure of a subset \( S \subset \mathbb{R}^n \), \( \text{int}(S) \) as the interior of \( S \), \( \mathcal{N}_S(x) \) as the normal cone of \( S \) at an element \( x \in S \), \( \mathcal{T}_S(x) \) as the tangent cone of \( S \) at an element \( x \in S \), and \( \mathcal{B}_\epsilon(p), p \in \mathbb{R}^n \) as the open ball centered at \( p \) with radius \( \epsilon > 0 \). Denote \( \text{dist}(x, M) \) as the distance from a point \( x \) to a set \( M \) (that is, \( \text{dist}(x, M) \triangleq \inf_{p \in M} \|p - x\| \)), and \( x(t) \) approaches \( M \) if \( x(t) \to M \) as \( t \to \infty \) (that is, for each \( \epsilon > 0 \), there is \( T > 0 \) such that \( \text{dist}(x(t), M) < \epsilon \) for all \( t > T \)).

B. Graph Theory

A weighted undirected graph \( G \) is denoted by \( G(V, E, A) \), where \( V = \{1, \ldots, n\} \) is a set of nodes, \( E \subset V \times V \) is a set of edges, \( A = [a_{i,j}] \in \mathbb{R}^{n \times n} \) is a weighted adjacency matrix such that \( a_{i,j}, a_{j,i} > 0 \) if \( \{i, j\} \in E, j \neq i \), and \( a_{i,j} = 0 \) otherwise. The weighted Laplacian matrix is \( L_n = D - A \), where \( D \in \mathbb{R}^{n \times n} \) is diagonal with \( D_{i,i} = \sum_{j=1,j\neq i}^{n} a_{i,j}, i \in \{1, \ldots, n\} \). In this note, we call \( L_n \) the Laplacian matrix and \( A \) the adjacency matrix of \( G \) for convenience when there is no confusion. Specifically, if the weighted undirected graph \( G \) is connected, then \( L_n \geq 0 \), \( \text{rank}(L_n) = n - 1 \), and \( \ker(L_n) = \{k\mathbf{1}_n : k \in \mathbb{R}\} \).

C. Projection Operator

Define \( P_K(\cdot) \) as a projection operator given by \( P_K(u) = \arg \min_{v \in K} \|u - v\|, \) where \( K \subset \mathbb{R}^n \).

Lemma 2.1: \([20]\) If \( K \subset \mathbb{R}^n \) is a closed convex set, then

\[
(u - P_K(u))^T(v - P_K(u)) \leq 0, \quad \forall u \in \mathbb{R}^n, \quad \forall v \in K.
\]  \hspace{1cm} (1)

III. PROBLEM DESCRIPTION AND OPTIMIZATION ALGORITHM

A. Problem Description

Consider a network of \( n \) agents interacting over a graph \( G \). There is a local cost function \( f^i : \mathbb{R}^q \to \mathbb{R} \) and a local feasible constraint set \( \Omega_i \subset \mathbb{R}^q \) for all \( i \in \{1, \ldots, n\} \). The global cost function of the network is \( f(x) = \sum_{i=1}^{n} f^i(x) \), and the feasible set is the intersection of local
constraint sets, that is, \( x \in \Omega_0 \triangleq \bigcap_{i=1}^n \Omega_i \subset \mathbb{R}^q \). Then we will provide a distributed algorithm to solve

\[
\min_{x \in \Omega_0} f(x), \quad f(x) = \sum_{i=1}^n f^i(x), \quad x \in \Omega_0 \subset \mathbb{R}^q,
\]

where each agent only uses its own local cost function, its local constraint, and the shared information of its neighbors through constant local communications.

To ensure the wellposedness of problem (2), the following assumption is needed.

**Assumption 3.1:**
1) The weighted graph \( G \) is connected and undirected with symmetric weighted Laplacian matrix \( L_n \).
2) For all \( i \in \{1, \ldots, n\} \), \( f^i \) is continuous and convex on an open set containing \( \Omega_i \), and \( \Omega_i \subset \mathbb{R}^q \) is closed and convex with \( \bigcap_{i=1}^n \text{int}(\Omega_i) \neq \emptyset \).
3) There exists at least one optimal solution to problem (2).

**Remark 3.1:** Problem (2) covers many problems in recent distributed optimization studies. For example, it introduces the constraints compared with the unconstrained optimization model in [4]. Moreover, it generalizes the model in [8] by allowing heterogeneous constraints, and extends the models in [6] and [12], which considered function constraints and hyper box (sphere) constraints, respectively.

Let \( x^i(t) \in \Omega_i \subset \mathbb{R}^q \) be the estimate of agent \( i \) at time instant \( t \geq 0 \) for the optimal solution. Let \( L \triangleq L_n \otimes I_q \in \mathbb{R}^{nq \times nq} \), where \( L_n \in \mathbb{R}^{n \times n} \) is the Laplacian matrix of \( G \). Denote \( x \triangleq [x_1^T, \ldots, x_n^T]^T \in \Omega \subset \mathbb{R}^{nq} \) and \( f(x) \triangleq \sum_{i=1}^n f^i(x) \) with \( x \in \Omega \), where \( \Omega \triangleq \prod_{i=1}^n \Omega_i \) is the Cartesian product of \( \Omega_i, i \in \{1, \ldots, n\} \). Then we arrive at the following lemma by directly analyzing the optimality condition.

**Lemma 3.1:** Suppose Assumption 3.1 holds and \( \alpha > 0 \). \( x^* \in \Omega_0 \subset \mathbb{R}^q \) is an optimal solution to problem (2) if and only if there exist \( x^* = 1_n \otimes x^* \in \Omega \subset \mathbb{R}^{nq} \) and \( \lambda^* \in \mathbb{R}^{nq} \) such that

\[
0_{nq} \in \left\{ P_{T_{\Omega}(x^*)}(-g(x^*) - \alpha L \lambda^*) : g(x^*) \in \partial f(x^*) \right\},
\]

\[
L x^* = 0_{nq},
\]

where \( T_{\Omega}(x^*) \) is the tangent cone of \( \Omega \) at an element \( x^* \in \Omega \) and \( P_{T_{\Omega}(x^*)}(\cdot) \) is the projection operator to \( T_{\Omega}(x^*) \).
Proof: According to Theorem 3.33 in [10], \( x^* \) is an optimal solution to problem (2) if and only if
\[
0_q \in \partial f(x^*) + \mathcal{N}_{\Omega_0}(x^*),
\]
where \( \mathcal{N}_{\Omega_0}(x^*) \) is the normal cone of \( \Omega_0 \) at \( x^* \in \Omega_0 = \bigcap_{i=1}^n \Omega_i \). Note that \( f^i(\cdot), i = 1, \ldots, n, \) is convex and \( \bigcap_{i=1}^n \text{Int}(\Omega_i) \neq \emptyset \) by Assumption 3.1. It follows from Theorem 2.85 and Lemma 2.40 in [10] that \( \partial f(x^*) = \sum_{i=1}^n \partial f^i(x^*) \) and \( \mathcal{N}_{\Omega_0}(x^*) = \sum_{i=1}^n \mathcal{N}_{\Omega_i}(x^*) \). To prove this lemma, one only needs to show (4) holds if and only if (3) is satisfied.

Suppose (3) holds. Since graph \( G \) is connected, there exists \( x^* \in \mathbb{R}^q \) such that \( x^* = 1_n \otimes x^* \in \mathbb{R}^{nq} \) because of (3a). Note that \( 0_{nq} = P_{\Omega_0(x^*)}(-g(x^*) - \alpha L\lambda^*) \) if and only if \( -g(x^*) - \alpha L\lambda^* \in \mathcal{N}_{\Omega}(x^*) \). Let \( a_{i,j} \) be the \((i, j)\)th entry of the adjacency matrix of \( G \) and \( \lambda^* = [(\lambda_{1*}^*)^T, \ldots, (\lambda_{n*}^*)^T]^T \in \mathbb{R}^{nq} \) with \( \lambda_{i*}^* \in \mathbb{R}^q, i \in \{1, \ldots, n\} \). Then (3a) holds if and only if there exists \( g_i(x^*) \in \partial f^i(x^*) \) such that \( -g_i(x^*) - \alpha \sum_{j=1}^n a_{i,j}(\lambda_{i*}^* - \lambda_{j*}^*) \in \mathcal{N}_{\Omega_i}(x^*), i = 1, \ldots, n \). Because \( L_n = L_n^T \) by Assumption 3.1, \( \sum_{i=1}^n \sum_{j=1}^n a_{i,j}(\lambda_{i*}^* - \lambda_{j*}^*) = 1/2 \sum_{i=1}^n \sum_{j=1}^n (a_{i,j} - a_{j,i})(\lambda_{i*}^* - \lambda_{j*}^*) = 0_q \) and \( -\sum_{i=1}^n g_i(x^*) \in \sum_{i=1}^n \mathcal{N}_{\Omega_i}(x^*) = \mathcal{N}_{\Omega_0}(x^*) \). Since \( \sum_{i=1}^n g_i(x^*) \in \sum_{i=1}^n \partial f^i(x^*) = \partial f(x^*), (4) \) is thus proved.

Conversely, suppose (4) holds. Let \( x^* = 1_n \otimes x^* \). (3b) is clearly true. It follows from (4) that there exists \( g_i(x^*) \in \partial f^i(x^*) \) such that \( -\sum_{i=1}^n g_i(x^*) \in \sum_{i=1}^n \mathcal{N}_{\Omega_i}(x^*) \). Choose \( z_i(x^*) \in \mathcal{N}_{\Omega_i}(x^*), i = 1, \ldots, n \), such that \( -\sum_{i=1}^n g_i(x^*) = \sum_{i=1}^n z_i(x^*) \). Next, define vectors \( l_i(x^*) \triangleq z_i(x^*) + g_i(x^*), i = 1, \ldots, n \). It is clear that \( \sum_{i=1}^n l_i(x^*) = 0_q \). Note that \( L \) is symmetric by Assumption 3.1. By the fundamental theorem of linear algebra, the sets \( \ker(L) \) and \( \text{range}(L) \) form an orthogonal decomposition of \( \mathbb{R}^{nq} \). Define \( l(x^*) \triangleq [l_1(x^*)^T, \ldots, l_n(x^*)^T]^T \in \mathbb{R}^{nq} \). For all \( x = 1_n \otimes x \in \ker(L), l(x^*)^T x = \sum_{i=1}^n l_i(x^*)^T x = 0 \) and hence, \( l(x^*) \in \text{range}(L) \) and there exists \( \lambda^* \in \mathbb{R}^{nq} \) such that \( l(x^*) = -\alpha L\lambda^* \). Thus, there exists \( \lambda^* = [(\lambda_{1*}^*)^T, \ldots, (\lambda_{n*}^*)^T]^T \in \mathbb{R}^{nq} \) with \( \lambda_{i*}^* \in \mathbb{R}^q \) such that \( -g_i(x^*) - \alpha \sum_{j=1}^n a_{i,j}(\lambda_{i*}^* - \lambda_{j*}^*) = -g_i(x^*) + l_i(x^*) = z_i(x^*) \in \mathcal{N}_{\Omega_i}(x^*), i = 1, \ldots, n \), where \( a_{i,j} \) is the \((i, j)\)th entry of the adjacency matrix of \( G \). Hence, there exist \( g(x^*) \in \partial f(x^*) \) and \( \lambda^* \in \mathbb{R}^{nq} \) such that \( -g(x^*) - \alpha L\lambda^* \in \mathcal{N}_{\Omega}(x^*), \) equivalently, \( 0_{nq} = P_{\Omega_0(x^*)}(-g(x^*) - \alpha L\lambda^*) \). (3a) is proved. \( \blacksquare \)
B. Distributed Continuous-Time Projected Algorithm

For the optimization problem (2), we propose a distributed optimization algorithm as follows:

\[
\dot{x}_i(t) = P_{T_{\Omega_i}(x_i(t))} \left[ -g_i(x_i(t)) - \alpha \sum_{j=1}^{n} a_{i,j}(x_i(t) - x_j(t)) \right. \\
\left. - \alpha \sum_{j=1}^{n} a_{i,j}(\lambda_i(t) - \lambda_j(t)) \right], \\
\dot{\lambda}_i(t) = \alpha \sum_{j=1}^{n} a_{i,j}(x_i(t) - x_j(t)),
\]

where \( t \geq 0, i \in \{1, \ldots, n\} \), \( x_i(0) = x_{i0} \in \Omega_i \subseteq \mathbb{R}^q \), \( \lambda_i(0) = \lambda_{i0} \in \mathbb{R}^q \), \( \alpha > 0 \), and \( a_{i,j} \) is the \((i, j)\)th element of the adjacency matrix of graph \( G \), \( T_{\Omega_i}(x_i(t)) \) is the tangent cone of \( \Omega_i \) at an element \( x_i(t) \in \Omega_i \) and \( P_{T_{\Omega_i}(x_i(t))}(\cdot) \) is the projection operator to \( T_{\Omega_i}(x_i(t)) \).

**Remark 3.2:** Algorithm (5) is motivated by the primal-dual type continuous-time algorithms, which was firstly proposed in [3] and later on extended in [4], [6], [7], [12]. If the state constraints are relaxed to \( \Omega_i = \mathbb{R}^q, i \in \{1, \ldots, n\} \), then algorithm (5) is consistent with the algorithm proposed in Section IV of [4]. Algorithm (5) also incorporates projection operation to handle constraints, which had also been adopted in [8] and [12]. However, [8] only handled homogeneous constraints, and [12] may produce unbounded states, which may be hard to implement in practice. Here our proposed algorithm (5) handles the problems with local constraints and can guarantee the boundedness of states.

IV. MAIN RESULTS

In this section, we first introduce additional preliminaries for nonsmooth analysis, and then give the convergence analysis of the algorithm with an illustrative simulation.

A. Nonsmooth Analysis

To study our algorithm, we need concepts related to nonsmooth analysis. Consider a differential inclusion [15] in the form of

\[
\dot{x}(t) \in \mathcal{H}(x(t)), \quad x(0) = x_0, \quad t \geq 0,
\]
where $\mathcal{H} : \mathbb{R}^q \to \mathfrak{B}(\mathbb{R}^q)$ is a set-valued map with nonempty compact values. Let $\tau > 0$. A solution of (6) defined on $[0, \tau] \subset [0, \infty)$ is an absolutely continuous function $x : [0, \tau] \to \mathbb{R}^q$ such that (6) holds for almost all $t \in [0, \tau]$ (in the sense of Lebesgue measure). Recall that the solution $t \mapsto x(t)$ to (6) is a right maximal solution if it cannot be extended forward in time. We assume that all right maximal solutions to (6) exist on $[0, \infty)$. A set $\mathcal{M}$ is said to be weakly invariant (resp., strongly invariant) with respect to (6) if $\mathcal{M}$ contains a maximal solution (resp., all maximal solutions) of (6) for every $x_0 \in \mathcal{M}$. A point $x_*$ is an almost cluster point (see [15, p. 311]) of a measurable function $\phi(\cdot)$ when $t \to \infty$ if $\mu\{t \geq 0 : \|\phi(t) - x_*\| \leq \varepsilon\} = \infty$ for all $\varepsilon > 0$, where $\mu(\cdot)$ is the Lebesgue measure.

Let $\mathcal{D}$ be a compact, strongly positive invariant set with respect to (6). Let $W$ be a nonnegative lower semicontinuous (see [15, p. 22]) function defined on $\mathbb{R}^q \times \mathbb{R}^q$ and $V$ be a nonnegative lower semicontinuous and inf-compact (see [15, p. 292]) function defined on $\mathbb{R}^q$. Assume there exists an upper semicontinuous (see [15, p. 41]) map $\tilde{\mathcal{H}}(x)$ with closed values such that $\mathcal{H}(x) \subset \tilde{\mathcal{H}}(x)$ for all $x \in \mathcal{D}$ and $0_q \in \tilde{\mathcal{H}}(x)$ if and only if $0_q \in \mathcal{H}(x)$, we introduce a result for the existence of an almost cluster point.

**Lemma 4.1:** If $\phi(\cdot) \in \mathbb{R}^q$ is a solution of (6) with $\phi(0) = x_0 \in \mathcal{D}$ such that

$$V(\phi(t)) - V(\phi(s)) + \int_s^t W(\phi(\tau), \dot{\phi}(\tau))d\tau \leq 0, \quad t \geq s \geq 0,$$

then $\phi(\cdot)$ and $\dot{\phi}(\cdot)$ have almost cluster points $x_*$ and $v_*$, which satisfy $W(x_*, v_*) = 0$. If, in addition, $W(x, v) > 0$ for all $x \in \mathbb{R}^q$ and all $v \neq 0_q$, then $x_*$ is an equilibrium of the differential inclusion (6).

**Proof:** By [15, Proposition 5, p. 311], $\phi(\cdot)$ and $\dot{\phi}(\cdot)$ have almost cluster points $x_*$ and $v_*$ which satisfy $W(x_*, v_*) = 0$.

If, in addition, $W(x, v) > 0$ for all $x \in \mathbb{R}^q$ and all $v \neq 0_q$, then $v_* = 0_q$. Let $\{t_i\}_{i=1}^{\infty}$ be an increasing nonnegative sequence such that $t_i \to \infty$ and $\{\phi(t_i), \dot{\phi}(t_i)\} \to (x_*, 0_q)$. Clearly, $\dot{\phi}(t_i) \in \mathcal{H}(\phi(t_i)) \subset \tilde{\mathcal{H}}(\phi(t_i))$ for all $i \in \{1, 2, \ldots, \infty\}$. Because $\tilde{\mathcal{H}}(\cdot)$ is upper semicontinuous, $0_q \in \tilde{\mathcal{H}}(x_*)$ by definition. Recall that $0_q \in \mathcal{H}(x_*)$ is equivalent to $0_q \in \mathcal{H}(x_*)$, $x_*$ is an equilibrium of the differential inclusion (6).

Furthermore, we introduce a lemma, which is inspired by [18, Proposition 3.1] and is used in the convergence analysis.

**Lemma 4.2:** Let $\mathcal{D}$ be a compact, strongly positive invariant set with respect to (6), and
\(\phi(\cdot) \in \mathbb{R}^q\) be a solution of (6) with \(\phi(0) = x_0 \in D\). If \(z\) is an almost cluster point of \(\phi(\cdot)\) and a Lyapunov stable equilibrium of (6), then \(z = \lim_{t \to \infty} \phi(t)\).

**Proof:** Suppose \(z\) is an almost cluster point of \(\phi(\cdot)\) and \(z\) is Lyapunov stable. Let \(\varepsilon > 0\). Since \(z\) is Lyapunov stable, there exists \(\delta = \delta(\varepsilon, z) > 0\) such that the solution \(\tilde{\phi}(t)\) of system (6) with \(\tilde{\phi}(0) = y \in B_c(z)\) satisfies that \(\tilde{\phi}(t) \in B_c(z)\) for all \(t \geq 0\). Since \(z\) is an almost cluster point of \(\phi(\cdot)\), there exists \(h = h(\delta, x_0) > 0\) such that \(\phi(h) \in B_c(z)\). It follows from our construction of \(\delta\) that \(\phi(t) \in B_c(z)\) for all \(t \geq h\). Because \(\varepsilon > 0\) is arbitrary, \(z = \lim_{t \to \infty} \phi(t)\).

**B. Convergence Analysis**

Let \(x \triangleq [x_1^T, \ldots, x_n^T]^T \in \Omega \subset \mathbb{R}^{nq}\) and \(\lambda \triangleq [\lambda_1^T, \ldots, \lambda_n^T]^T \in \mathbb{R}^{nq}\) with \(\Omega \triangleq \prod_{i=1}^n \Omega_i\). Algorithm (5) can be written in a compact form

\[
\begin{bmatrix}
x(t)
\lambda(t)
\end{bmatrix} \in \mathcal{F}(x(t), \lambda(t)), \ x(0) = x_0 \in \Omega, \ \lambda(0) = \lambda_0 \in \mathbb{R}^{nq},
\]

where \(\mathcal{F}(x, \lambda) \triangleq \left\{ \begin{bmatrix} P_{\mathcal{T}_\Omega(x)}[-\alpha Lx - \alpha \lambda g(x)] \\
\alpha Lx
\end{bmatrix} : g(x) \in \partial f(x) \right\}\) and \(L = L_n \otimes I_q \in \mathbb{R}^{nq \times nq}\).

**Remark 4.1:** The optimization algorithm (7) is of the form \(\dot{x}(t) \in P_{\mathcal{T}_K(x(t))}[\mathcal{H}(x(t))]\), where \(x(0) = x_0 \in K\), \(K\) is a closed convex subset of \(\mathbb{R}^q\), and \(\mathcal{H}\) is an upper semicontinuous map with nonempty compact convex values. It follows from Proposition 2 of [15, p. 266] and Theorem 1 of [15, p. 267] that algorithm (7) has right maximal solutions on \([0, \infty)\). Since \(P_{\mathcal{T}_K(x(t))}[\mathcal{H}(x(t))] \subset \mathcal{T}_K(x(t))\), \(K\) is a strongly invariant set to \(\dot{x}(t) \in P_{\mathcal{T}_K(x(t))}[\mathcal{H}(x(t))]\). In addition, \(P_{\mathcal{T}_K(x(t))}[\mathcal{H}(x(t))] \subset \mathcal{H}(x(t)) - N_K(x(t)), \ 0_q \in P_{\mathcal{T}_K(x(t))}[\mathcal{H}(x(t))]\) if and only if \(0_q \in \mathcal{H}(x(t)) - N_K(x(t))\), and \(\mathcal{H}(x(t)) - N_K(x(t))\) is upper semicontinuous because both \(\mathcal{H}(x(t))\) and \(N_K(x(t))\) are upper semicontinuous. Hence, Lemma 4.1 can be applied to the convergence analysis of algorithm (7).

Because \(L_n\) is symmetric by Assumption 3.1, \(L_n\) can be factored as \(L_n = QAQ^T\) by the symmetric eigenvalue decomposition, where \(Q\) is an orthogonal matrix and \(\Lambda\) is a diagonal matrix whose diagonal entries are the eigenvalues of \(L_n\). Define a diagonal matrix \(\Lambda \in \mathbb{R}^{n \times n}\) such that \(\Lambda_{i,i} = 1/\Lambda_{i,i}\) if \(\Lambda_{i,i} > 0\) and \(\Lambda_{i,i} = 2k\alpha\) if \(\Lambda_{i,i} = 0\) for \(i \in \{1, \ldots, n\}\). The following lemma provides a result when \(\alpha > 0\) and \(0 < k < \frac{1}{\alpha \lambda_{\max}(L_n)}\).
Lemma 4.3: Consider algorithm (7) under Assumption 3.1 with $0 < k < \frac{1}{\alpha \lambda_{\max} (L_n)}$. Then $Q_n = k\alpha^2 Q \left( \frac{1}{k\alpha} \overline{X} - I_n \right) Q^T > 0$ and $\alpha L_n - k\alpha^2 L_n^2 = L_n Q_n L_n$.

Proof: With $0 < k < \frac{1}{\alpha \lambda_{\max} (L_n)}$, it is easy to prove $Q_n > 0$. Because $L_n = Q\Lambda Q^T$ and $\Lambda \Lambda = \Lambda$ by the definition of $\Lambda$,

$$L_n Q_n L_n = k\alpha^2 L_n Q \left( \frac{1}{k\alpha} \overline{X} - I_n \right) Q^T L_n$$

$$= k\alpha^2 Q\Lambda Q^T [Q \left( \frac{1}{k\alpha} \overline{X} - I_n \right) Q^T] Q\Lambda Q^T$$

$$= \alpha Q\Lambda \overline{X} \Lambda Q^T - k\alpha^2 (Q\Lambda Q^T)^2$$

$$= \alpha Q\Lambda Q^T - k\alpha^2 (Q\Lambda Q^T)^2$$

$$= \alpha L_n - k\alpha^2 L_n^2$$

which implies the conclusion.

If 3) of Assumption 3.1 holds, there exists $(x^*, \lambda^*) \in \Omega \times \mathbb{R}^{nq}$ satisfying (3) by Lemma 3.1. Let $x^* \in \Omega$ and $\lambda^* \in \mathbb{R}^{nq}$ be the vectors such that (3) is satisfied. Define

$$V_1^*(x, \lambda) \triangleq \frac{1}{2} \|x - x^*\|^2 + \frac{1}{2} \|\lambda - \lambda^*\|^2,$$

$$V_2^*(x, \lambda) \triangleq f(x) - f(x^*) + \alpha \frac{1}{2} x^T L x + \alpha x^T \Lambda \lambda.$$  

Remark 4.2: Functions $V_1^*(x, \lambda)$ and $V_2^*(x, \lambda)$ are constructed to form the candidates of Lyapunov functions in the theoretical analysis. Function $V_1^*(x, \lambda)$ is also used as a Lyapunov function in [4] to prove algorithm convergence of unconstrained distributed optimization, which is a very good result. In the analysis of [4], the cost function was assumed to have a finite number of critical points and the quadratic Lyapunov functions were used. However, in this note, the cost functions are assumed to be convex, which means that the cost function may have infinitely many solutions (or infinitely many critical points). Function $V_2^*(x, \lambda)$ uses the convexity property to tackle convex cost functions (see part (iii) and (iv) of proof to Lemma 4.4). ♦

Recall that if $\phi(\cdot)$ is a solution of (6) and $V : \mathbb{R}^q \rightarrow \mathbb{R}$ is locally Lipschitz and regular (see [17] p. 39), then $\dot{\phi}(t)$ and $\dot{V}(\phi(t))$ exist almost everywhere. Next, we give the following result, whose proof is given in Appendix.

Lemma 4.4: Suppose Assumption 3.1 holds. Let $V_1^*(x, \lambda)$ and $V_2^*(x, \lambda)$ be as defined in (8) and (9), and let $(x(t), \lambda(t))$ be a trajectory to algorithm (5) or (7).

(i) $\dot{V}_1^*(x(t), \lambda(t)) \leq -\alpha x^T(t) L x(t) \leq 0$ for almost all $t \geq 0$. 

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(iii) \( \dot{V}_2^*(x(t), \lambda(t)) \leq -\|\dot{x}(t)\|^2 + \alpha^2 x^T(t) L^2 x(t) \) for almost all \( t \geq 0 \).

(iv) With \( V^*(x, \lambda) \) defined in part (iii) for \( 0 < k < \frac{1}{\alpha \lambda_{\text{max}}(L_n)} \), \( \dot{V}^*(x(t), \lambda(t)) \leq -k\|\dot{x}(t)\|^2 - \dot{\lambda}^T(t) Q \dot{\lambda}(t) \leq 0 \) for almost all \( t \geq 0 \), where \( Q \in \mathbb{R}^{nq \times nq} \) is positive definite.

Based on Lemmas 4.2 and 4.4, we obtain our main result for state boundedness and convergence of the proposed algorithm.

**Theorem 4.1:** Suppose Assumption 3.1 holds and let \((x(t), \lambda(t))\) be a trajectory to algorithm (5) or (7). Then

(i) \((x(t), \lambda(t))\) is bounded;

(ii) \((x(t), \lambda(t))\) converges to a point \((\bar{x}, \bar{\lambda})\) such that \(\bar{x} = 1_n \otimes \bar{x}\) and \(\bar{x}\) is an optimal solution to problem (2).

**Proof:** In this theorem, part (i) claims that an equilibrium point of algorithm (7) is Lyapunov stable and any trajectory of algorithm (7) is bounded; part (ii) further claims that any trajectory of algorithm (7) converges to one of the equilibria of algorithm (7).

(i) Let \( V_1^*(x, \lambda) \) be as defined in (8). It is clear that \( V_1^*(x, \lambda) \) is positive definite, \( V_1^*(x, \lambda) = 0 \) if and only if \((x, \lambda) = (x^*, \lambda^*)\), and \( V_1^*(x, \lambda) \to \infty \) as \((x, \lambda) \to \infty \).

By (i) of Lemma 4.4 \( \dot{V}_1^*(x(t), \lambda(t)) \leq 0 \) for almost all \( t \geq 0 \). Hence, \( D \triangleq \left\{ (x, \lambda) \in \Omega \times \mathbb{R}^{nq} : V_1^*(x, \lambda) \leq M \right\} \), where \( M > 0 \), is strongly positive invariant. Note that \( V_1^*(x, \lambda) \) is positive definite and \( V_1^*(x, \lambda) \to \infty \) as \((x, \lambda) \to \infty \). Set \( D \) is bounded and the solution \((x(t), \lambda(t))\) is also bounded. Part (i) is thus proved.

(ii) Let \( V^*(x, \lambda) \) be as defined in (iii) of Lemma 4.4. Due to (iv) of Lemma 4.4 \( \dot{V}^*(x(t), \lambda(t)) \leq -k\|\dot{x}(t)\|^2 - \hat{\lambda}^T(t) Q \hat{\lambda}(t) \leq 0 \) for almost all \( t \geq 0 \), where \( Q \in \mathbb{R}^{nq \times nq} \) is positive definite. Define \( W(x, \lambda) = k\|\dot{x}\|^2 + \hat{\lambda}^T Q \hat{\lambda} \). It is clear that \( W(x, \lambda) = 0 \) if and only if \( \dot{x} = 0_{nq} \) and \( \dot{\lambda} = 0_{nq} \).

Recall that \((x(t), \lambda(t))\) is bounded by (i) and \( V^*(x, \lambda) \) is inf-compact and nonnegative with all \((x, \lambda) \in \Omega \times \mathbb{R}^{nq}\) by (iii) of Lemma 4.4. Note that

\[
V^*(x(t), \lambda(t)) - V^*(x(s), \lambda(s)) = \int_s^t \dot{V}^*(x(\tau), \lambda(\tau)) d\tau \\
\leq - \int_s^t W(\dot{x}(\tau), \dot{\lambda}(\tau)) d\tau.
\]

By Lemma 4.1 \((x(t), \lambda(t))\) has an almost cluster point \((\bar{x}, \bar{\lambda}) \in \Omega \times \mathbb{R}^{nq}\) and \((\bar{x}, \bar{\lambda})\) is an equilibrium point of (7).
Define a function $\bar{V}(x, \lambda) \triangleq \frac{1}{2}\|x - \bar{x}\|^2 + \frac{1}{2}\|\lambda - \bar{\lambda}\|^2$. It is clear that $\bar{V}(x, \lambda)$ is positive definite, $\bar{V}(x, \lambda) = 0$ if and only if $(x, \lambda) = (\bar{x}, \bar{\lambda})$, and $\bar{V}(x, \lambda) \to \infty$ if $(x, \lambda) \to \infty$. Because $(\bar{x}, \bar{\lambda})$ is an equilibrium point of (7), $(\bar{x}, \bar{\lambda})$ satisfies (3). Moreover, it follows from $(i)$ of Lemma 4.4 that $\hat{V}(x(t), \lambda(t))$ along the trajectories of (5) satisfies $\dot{\bar{V}}(x(t), \lambda(t)) \leq 0$ for almost all $t \geq 0$. Hence, $(\bar{x}, \bar{\lambda})$ is a Lyapunov stable equilibrium point to the system (5).

Clearly, $(\bar{x}, \bar{\lambda})$ is an almost cluster point of $(x(t), \lambda(t))$ and $(\bar{x}, \bar{\lambda})$ is a Lyapunov stable equilibrium. According to Lemma 4.2, $(x(t), \lambda(t))$ converges to $(\bar{x}, \bar{\lambda})$ as $t \to \infty$. Because $(\bar{x}, \bar{\lambda})$ is an equilibrium point of (7), there exists $\bar{x} \in \Omega_0 \subset \mathbb{R}^q$ such that $\bar{x} = 1_n \otimes \bar{x}$ and $\bar{x}$ is an optimal solution to problem (2) by Lemma 3.1. Part $(ii)$ is thus proved. ♦

**Remark 4.3:** Theorem 4.1 shows the convergence of the proposed algorithm. The convergence analysis, in fact, can also be conducted following the method in [14]. ♦

**Remark 4.4:** The convergence analysis in this note is based on nonsmooth Lyapunov functions, which can be regarded as an extension of the analysis on basis of smooth Lyapunov functions used in [3], [4], [7]. Moreover, the novel technique proves that algorithm (5) is able to solve optimization problems with a continuum of optimal solutions, and therefore, improves some previous ones in [3], [7], which only handle problems with only one optimal point. ♦

### C. Numerical Simulation

The following is a numerical example for illustration.

Consider the optimization problem (2) with $x \in \mathbb{R}$, where $\Omega_i = \{x \in \mathbb{R} : i - 12 \leq x \leq i - 2\}$ and nonsmooth cost functions are

$$f^i(x) = \begin{cases} 
-x + i - 5, & \text{if } x < i - 5, \\
0, & \text{if } i - 5 \leq x \leq i + 5, \quad i = 1, \ldots, 5. \\
x - i - 5, & \text{if } x \geq i + 5,
\end{cases}$$

The adjacency matrix of the information sharing graph $G$ of algorithm (5) is given by

$$A = \begin{bmatrix}
0 & 1 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 0
\end{bmatrix}.$$ 

It can be easily verified that $\Omega_0 = \cap_{i=1}^5 \Omega_i = [-7, -1]$ and the optimal solution is $x = -1$, which is on the boundary of the constraint set $\Omega_0$. If there are no set constraints ($\Omega_i = \mathbb{R}$), every point in the set $[0, 6]$ is an optimal solution.
The trajectories of estimates for $x$ versus time are shown in Fig. 1. It can be seen that all the agents converge to the same optimal solution which satisfies all the local constraints and minimizes the sum of local cost functions, without knowing other agents’ constraints or feasible sets. Fig. 2 shows the trajectories of the auxiliary variable $\lambda_i$’s and verifies the boundedness of the algorithm trajectories. Fig. 3 shows the trajectories of functions $V_1^*(x, \lambda)$ and $V_2^*(x, \lambda)$ versus time.
V. CONCLUSIONS

In this note, a novel distributed projected continuous-time algorithm has been proposed for a distributed nonsmooth optimization under local set constraints. By virtue of projected differential inclusions and nonsmooth analysis, the proposed algorithm has been proved to be convergent while keeping the states bounded. Furthermore, based on the stability theory and convergence results for nonsmooth Lyapunov functions, the algorithm has been shown to solve the convex optimization problem with a continuum of optimal solutions. Finally, the algorithm performance has also been illustrated via a numerical simulation.

APPENDIX

PROOF OF LEMMA 4.4

(i) Let \((\mathbf{x}(t), \lambda(t))\) be a trajectory to algorithm (5) or (7). Recall that \(\hat{V}_1^*(\mathbf{x}(t), \lambda(t))\) and \((\dot{\mathbf{x}}(t), \dot{\lambda}(t))\) exist for almost all \(t \geq 0\). Suppose \(\hat{V}_1^*(\mathbf{x}(t), \lambda(t))\) and \((\dot{\mathbf{x}}(t), \dot{\lambda}(t))\) exist at a positive time instant \(t\). By (7), there exists \(g(\mathbf{x}(t)) \in \partial f(\mathbf{x}(t))\) such that \(\dot{\mathbf{x}}(t) = P_{T_{\Omega}(x(t))}[-\alpha L\mathbf{x}(t) - \alpha L\lambda(t) - g(\mathbf{x}(t))]\) and \(\dot{\lambda}(t) = \alpha L\mathbf{x}(t)\).

Clearly, \(\dot{\mathbf{x}}(t) = P_{T_{\Omega}(x(t))}[-\alpha L\mathbf{x}(t) - \alpha L\lambda(t) - g(\mathbf{x}(t))]\) implies
\[-\alpha L\mathbf{x}(t) - \alpha L\lambda(t) - g(\mathbf{x}(t)) - \dot{\mathbf{x}}(t) \in N_{\Omega}(\mathbf{x}(t)),\]
where \(N_{\Omega}(\mathbf{x}(t)) \triangleq \{d \in \mathbb{R}^{nq} : d^T(\bar{x} - \mathbf{x}(t)) \leq 0, \forall \bar{x} \in \Omega\}\) is the normal cone of \(\Omega\) at an element
\( x(t) \in \Omega \). Hence,

\[
(\alpha Lx(t) + \alpha L\lambda(t) + g(x(t)) + \dot{x}(t))^T(x(t) - \dot{x}) \leq 0,
\]

for all \( \dot{x} \in \Omega \).

By choosing \( \dot{x} = x^* \),

\[
(\alpha Lx(t) + \alpha L\lambda(t) + g(x(t)) + \dot{x}(t))^T(x(t) - x^*) \leq 0. \tag{10}
\]

By Assumption 3.1 and (3b), we have \( L = L^T \) and \( Lx^* = 0_{nq} \), therefore,

\[
x^T(t)(x(t) - x^*) \leq -\alpha x^T(t)Lx(t) - \alpha x^T(t)L\lambda(t) - g(x(t))^T(x(t) - x^*). \tag{11}
\]

Furthermore, it follows from \( \dot{\lambda}(t) = \alpha Lx(t) \) that

\[
\frac{1}{2} \frac{d}{dt} \|\lambda(t) - \lambda^*\|^2 = \alpha (\lambda(t) - \lambda^*)^T Lx(t). \tag{12}
\]

In view of (11) and (12),

\[
\frac{d}{dt} V^*_1(x(t), \lambda(t)) \leq -\alpha x^T(t)Lx(t) - g(x(t))^T(x(t) - x^*)
\]

\[
-\alpha \lambda^*^T Lx(t)
\]

\[
= -(g(x(t)) - g(x^*))^T(x(t) - x^*)
\]

\[
-(g(x^*) + \alpha L\lambda^*)^T(x(t) - x^*)
\]

\[
-\alpha x^T(t)Lx(t), \tag{13}
\]

where \( g(x^*) \in \partial f(x^*) \) is chosen such that \( P_{\Omega}(x^*)(-g(x^*) - \alpha L\lambda^*) = 0_{nq} \).

Note that \( P_{\Omega(x^*)}(-g(x^*) - \alpha L\lambda^*) = 0_{nq} \) implies \( -g(x^*) - \alpha L\lambda^* \in \mathcal{N}_\Omega(x^*) \), where \( \mathcal{N}_\Omega(x^*) \)

is the normal cone of \( \Omega \) at an element \( x^* \in \Omega \). Hence,

\[
(-g(x^*) - \alpha L\lambda^*)^T(p - x^*) \leq 0
\]

for all \( p \in \Omega \). Since \( x(t) \in \Omega \), we have

\[
(-g(x^*) - \alpha L\lambda^*)^T(x(t) - x^*) \leq 0. \tag{14}
\]

Because \( f(x) \) is convex, \( (g(x(t)) - g(x^*))^T(x(t) - x^*) \geq 0 \) with \( g(x(t)) \in \partial f(x(t)) \) and \( g(x^*) \in \partial f(x^*) \). It follows from (13) that

\[
\frac{d}{dt} V^*_1(x(t), \lambda(t)) \leq -\alpha x^T(t)Lx(t) \leq 0. \tag{15}
\]
(ii) Let \((x(t), \lambda(t))\) be a trajectory to algorithm \([5]\) or \([7]\). Recall that \(\dot{V}_2^*(x(t), \lambda(t))\) and \((\dot{x}(t), \dot{\lambda}(t))\) exist for almost all \(t \geq 0\). Suppose \(\dot{V}_2^*(x(t), \lambda(t))\) and \((\dot{x}(t), \dot{\lambda}(t))\) exist at a positive time instant \(t\). Since \(f(x)\) is convex in \(x\),

\[
\begin{align*}
    f(x(t)) - f(x(t - h)) & \leq \langle p, x(t) - x(t - h) \rangle, \\
    f(x(t + h)) - f(x(t)) & \geq \langle p, x(t + h) - x(t) \rangle.
\end{align*}
\]

for all \(p \in \partial f(x(t))\) and \(h \in (0, t]\).

Dividing both sides of the inequalities by \(h \in (0, t]\) and letting \(h \to 0\), we obtain

\[
\frac{d}{dt} f(x(t)) = \langle p, \dot{x(t)} \rangle, \quad \forall p \in \partial f(x(t)).
\]

By \([7]\), there exists \(g(x(t)) \in \partial f(x(t))\) such that \(\dot{x}(t) = P_{T_{\Omega}(x(t))}[\alpha Lx(t) - \alpha L\lambda(t) - g(x(t))]\) and \(\dot{\lambda}(t) = \alpha Lx(t)\). Choose \(p = g(x(t))\). Then \(\frac{d}{dt} f(x(t)) = g(x(t))^T \dot{x}(t)\). Hence,

\[
\frac{d}{dt} V_2^*(x(t), \lambda(t)) = (\alpha Lx(t) + \alpha L\lambda(t) + g(x(t)))^T \dot{x}(t)
\]

\[
+ \alpha^2 x^T(t) L^2 x(t).
\]

Set \(K = T_{\Omega}(x(t)), v = 0_{nq} \in K, u = -[\alpha Lx(t) + \alpha L\lambda(t) + g(x(t))] \in \mathbb{R}^{nq}\), and \(P_K(u) = \dot{x}(t)\) in \([1]\). It follows from \([1]\) that

\[
[\alpha Lx(t) + \alpha L\lambda(t) + g(x(t))]^T \dot{x}(t) \leq -\|\dot{x}(t)\|^2.
\]

Hence, \(\frac{d}{dt} V_2^*(x(t), \lambda(t)) \leq -\|\dot{x}(t)\|^2 + \alpha^2 x^T(t) L^2 x(t)\), which follows from \([17]\).

(iii) Let \(0 < k < \frac{1}{\alpha \lambda_{\text{max}}(L_n)}\) and note that \(Lx^* = L^T x^* = 0_{nq}\). It can be easily verified that

\[
\begin{align*}
    V^*(x, \lambda) &= V_1^*(x, \lambda) + k V_2^*(x, \lambda) \\
    &= J_1(x, \lambda) + J_2(x) + J_3(x),
\end{align*}
\]

where \(J_1(x, \lambda) = \frac{1}{2} \|x - x^*\|^2 + \frac{1}{2} \|\lambda - \lambda^*\|^2 + k \alpha (x - x^*)^T L (\lambda - \lambda^*)\), \(J_2(x) = k \alpha \frac{1}{2} x^T L x\), and \(J_3(x) = k [f(x) - f(x^*) + \alpha (x - x^*)^T \lambda^*]\). To prove \(V^*(x, \lambda)\) is nonnegative for all \((x, \lambda) \in \Omega \times \mathbb{R}^{nq}\), we show \(J_1(x, \lambda) \geq 0, J_2(x) \geq 0, \) and \(J_3(x) \geq 0\) for all \((x, \lambda) \in \Omega \times \mathbb{R}^{nq}\).

Since \(L\) is positive semi-definite, 

\[
J_2(x) = k \alpha \frac{1}{2} x^T L x \geq 0,
\]

and \((x - x^*)^T L ((x - x^*) + (\lambda - \lambda^*)) \geq 0\) for all \((x, \lambda) \in \Omega \times \mathbb{R}^{nq}\). Hence, 

\[
(x - x^*)^T L (x - x^*) + (\lambda - \lambda^*)^T L (\lambda - \lambda^*) \geq 0.
\]
Let \( \mu_i, i = 1, \ldots, n, \) be the eigenvalues of \( L \in \mathbb{R}^{n \times n}. \) Since the eigenvalues of \( I_q \) are 1, it follows from the properties of Kronecker product that the eigenvalues of \( L = L_n \otimes I_q \) are \( \mu_i \times 1, i = 1, \ldots, n. \) Thus, \( \lambda_{\text{max}}(L_n) = \lambda_{\text{max}}(L). \)

Because of Assumption 3.1, \( L = L^T. \) By (19),

\[
\frac{k\alpha}{2} (x - x^*)^T L (\lambda - \lambda^*) \geq -\frac{k\alpha}{2} (x - x^*)^T L (x - x^*) - \frac{k\alpha}{2} \lambda_{\text{max}}(L_n) \| x - x^* \|^2 - \frac{k\alpha}{2} \lambda_{\text{max}}(L_n) \| \lambda - \lambda^* \|^2.
\]

Due to \( 1 - k\alpha \lambda_{\text{max}}(L_n) > 0, \)

\[
J_1(x, \lambda) \geq \frac{1}{2}(1 - k\alpha \lambda_{\text{max}}(L_n)) \| x - x^* \|^2 + \frac{1}{2}(1 - k\alpha \lambda_{\text{max}}(L_n)) \| \lambda - \lambda^* \|^2 \geq 0.
\]  

(20)

Since \( f(x) \) is convex in \( x \in \Omega, \)

\[
J_3(x) = k[f(x) - f(x^*) + \alpha(x - x^*)^T L \lambda^*] \geq k[(p + \alpha L \lambda^*)^T (x - x^*)], \quad \forall p \in \partial f(x^*).
\]

Note that there exists \( g(x^*) \in \partial f(x^*) \) such that \( \left| P_{\Omega_0}(x^*) - g(x^*) - \alpha L \lambda^* \right| = 0, \) which follows from (3a). Choose \( p = g(x^*). \) In light of (14) and similar arguments above (14),

\[
(p + \alpha L \lambda^*)^T (x - x^*) \geq 0
\]

for all \( x \in \Omega \) with \( p = g(x^*). \) Hence,

\[
J_3(x) \geq 0, \quad \forall x \in \Omega.
\]

(21)

In view of (18), (20), and (21), \( V^*(x, \lambda) = V^*_1(x, \lambda) + kV^*_2(x, \lambda) \) is nonnegative with all \( (x, \lambda) \in \Omega \times \mathbb{R}^{nq}. \)

(iv) It follows from part (i) and (ii) that \( \dot{V}^*(x(t), \lambda(t)) \leq -x^T(t)[\alpha L - k\alpha^2 L^2]x(t) - k\| \dot{x}(t) \|^2 \)

for almost all \( t \geq 0. \)
With $Q_n > 0$ as defined in Lemma 4.3, we have
\[ L_n Q_n L_n = \alpha L_n - k \alpha^2 L_n^2 \]
by Lemma 4.3. Define $Q = Q_n \otimes I_q > 0$. Recalling $\dot{\lambda}(t) = \alpha L x(t)$, it can be easily proved that
\[ x^T(t)(\alpha L - k \alpha^2 L^2)x(t) = x^T(t)LQLx(t) = \dot{\lambda}^T(t)Q \dot{\lambda}(t). \]
Hence, $\dot{V}^*(x(t), \lambda(t)) \leq -k\|\dot{x}(t)\|^2 - \dot{\lambda}^T(t)Q \dot{\lambda}(t) \leq 0$ for almost all $t \geq 0$.

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