THE REPRESENTATION TYPE OF JACOBIAN ALGEBRAS

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Abstract. We show that the representation type of the Jacobian algebra \( \mathcal{P}(Q, S) \) associated to a 2-acyclic quiver \( Q \) with non-degenerate potential \( S \) is invariant under QP-mutations. We prove that, apart from very few exceptions, \( \mathcal{P}(Q, S) \) is of tame representation type if and only if \( Q \) is of finite mutation type. We also show that most quivers \( Q \) of finite mutation type admit only one non-degenerate potential up to weak right equivalence. In this case, the isomorphism class of \( \mathcal{P}(Q, S) \) depends only on \( Q \) and not on \( S \).

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1. Introduction

1.1. Cluster algebras and Jacobian algebras. Cluster algebras were invented by Fomin and Zelevinsky [FZ1] in an attempt to obtain a combinatorial approach to the dual of Lusztig’s canonical basis of quantum groups. Another motivation for cluster algebras was the concept of total positivity, which dates back about 80 years, and was generalized and connected to Lie theory by Lusztig in 1994, see [Lu1, Lu2]. By now numerous connections between cluster algebras and other branches of mathematics have been discovered, e.g. representation theory of quivers and algebras and Donaldson-Thomas invariants of 3-Calabi-Yau categories.

By definition, the cluster algebra \( \mathcal{A}_B \) associated to a skew-symmetrizable integer matrix \( B \) is the subalgebra of a field of rational functions generated by an inductively constructed set of so-called cluster variables. (Starting with \( B \) and a set of initial cluster variables the other cluster variables are obtained via iterated seed mutations.) In this paper, we assume that the matrix \( B \) is skew-symmetric (for arbitrary skew-symmetrizable matrices \( B \) the theory of cluster algebras is much less developed). The set of skew-symmetric integer matrices corresponds bijectively to the
set of 2-acyclic quivers. We denote $\mathcal{A}_Q := \mathcal{A}_B$ if the quiver $Q$ corresponds to the skew-symmetric matrix $B$. The set of cluster algebras $\mathcal{A}_Q$ can be divided naturally into two classes. Namely, the quiver $Q$ is either of finite or of infinite mutation type.

One of the main links between the theory of cluster algebras and the representation theory of algebras is given by the work of Derksen, Weyman and Zelevinsky [DWZ1, DWZ2] on quivers with potentials $(Q, S)$ and the representations of the associated Jacobian algebras $\mathcal{P}(Q, S)$. Here $S$ is a non-degenerate potential on a 2-acyclic quiver $Q$, and $\mathcal{P}(Q, S)$ is by definition the completed path algebra of $Q$ modulo the closure of the ideal generated by the cyclic derivatives of $S$. Our first main result (Theorem 1.2) shows that the mutation type of $Q$ is closely related to the representation type of $\mathcal{P}(Q, S)$.

Let $\Gamma := \Gamma(Q, S)$ be the Ginzburg dg-algebra associated to $(Q, S)$, see [Gi] and also the survey article [Ke]. The derived category $\mathcal{D}_{f.d.}(\Gamma)$ of dg-modules over $\Gamma$ with finite-dimensional total homology is a 3-Calabi-Yau category. It has a natural $t$-structure whose heart is canonically equivalent to the category $\text{mod}(\mathcal{P}(Q, S))$ of finite-dimensional modules over the Jacobian algebra $\mathcal{P}(Q, S)$. The triangle quotient $\mathcal{D}_{\text{perf}}(\Gamma)/\mathcal{D}_{f.d.}(\Gamma)$ is the cluster category associated to $(Q, S)$ defined by Amiot [A1, A2]. Here $\mathcal{D}_{\text{perf}}(\Gamma)$ is the perfect derived category of $\Gamma$. Our second main result (Theorem 1.4) implies that for most quivers $Q$ of finite mutation type, there is essentially just one way to associate a 3-Calabi-Yau category and a cluster category to $Q$. (If one drops the assumption of the non-degeneracy of $S$, then these categories no longer reflect the mutations of seeds of the cluster algebra $\mathcal{A}_Q$ associated to $Q$.)

Jacobian algebras are not only important for the study of cluster algebras, they also appear in other contexts such as the study of BPS spectra of certain classes of quantum field theories. Quivers with potentials arising from triangulations of marked oriented surfaces often are of particular importance, see for example [ACCERV, BS, C, CV, Sm].

1.2. Representation type of Jacobian algebras. For definitions related to quivers with potentials and Jacobian algebras we refer to Section 2. By Drozd’s [D] celebrated Tame-Wild Theorem, any finite-dimensional algebra over an algebraically closed field is either of tame or of wild representation type. This is called the tame-wild dichotomy or representation type dichotomy for finite-dimensional algebras. One of our aims is to determine the representation type of Jacobian algebras. The following statement may not come as a surprise, but the proof is not so straightforward.

**Theorem 1.1.** For any non-degenerate potential $S$ on a 2-acyclic quiver $Q$, the representation type of the Jacobian algebra $\mathcal{P}(Q, S)$ is preserved under $Q\!P$-mutation.

The quiver $Q$ is of finite cluster type if there are only finitely many cluster variables in the cluster algebra $\mathcal{A}_Q$ associated to $Q$. We say that $Q$ is of finite mutation type if there are only finitely many quivers mutation equivalent to $Q$. Otherwise, $Q$ is of infinite mutation type.

Fomin and Zelevinsky [FZ2] proved that a quiver $Q$ is of finite cluster type if and only if it is mutation equivalent to a Dynkin quiver. In particular, $Q$ is of finite mutation type in this case. Using this, it is not difficult to show that for any non-degenerate potential $S$ on $Q$ the Jacobian algebra $\mathcal{P}(Q, S)$ is representation-finite if and only if $Q$ is of finite cluster type.

Based on work by Fomin, Shapiro and Thurston [FST], the quivers of finite mutation type were classified by Felikson, Shapiro and Tumarkin [FeST]. We recall their classification in Section 5.

We call a quiver $Q$ Jacobi-tame or Jacobi-wild if for all non-degenerate potentials $S$ the Jacobian algebra $\mathcal{P}(Q, S)$ is of tame or wild representation type, respectively. Otherwise, we call $Q$ Jacobi-irregular.
The following theorem is our first main result. (The quivers $T_1$, $T_2$, $X_6$, $X_7$ and $K_m$ appearing in the statement can be found in Sections 5.1 and 7.13. They are of finite mutation type. Note that the mutation equivalence classes of the quivers $T_1$, $T_2$ and $K_m$ contain only one quiver up to isomorphism.)

**Theorem 1.2.** Let $Q$ be a 2-acyclic quiver. If $Q$ is not mutation equivalent to one of the quivers $T_1$, $T_2$, $X_6$, $X_7$ or $K_m$ with $m \geq 3$, then the following hold:

(i) $Q$ is Jacobi-tame if and only if $Q$ is of finite mutation type.

(ii) $Q$ is Jacobi-wild if and only if $Q$ is of infinite mutation type.

For the exceptional cases the following hold:

(iii) If $Q$ is mutation equivalent to one of the quivers $X_6$, $X_7$ or $K_m$ with $m \geq 3$, then $Q$ is Jacobi-wild.

(iv) If $Q$ is mutation equivalent to one of the quivers $T_1$ or $T_2$, then $Q$ is Jacobi-irregular.

The following result is a direct consequence of Theorems 1.1 and 1.2.

**Corollary 1.3.** Let $S$ be a non-degenerate potential on $Q$, and assume that $Q$ is not equal to $T_1$ or $T_2$. Then the representation type of $\mathcal{P}(Q, S)$ depends only on $Q$, i.e. it is independent from $S$, and is preserved under QP-mutation.

Using Theorem 1.1, Drozd’s Tame-Wild Theorem and Felikson, Shapiro and Tumarkin’s [FeST] classification of quivers of finite mutation type, the proof of Theorem 1.2 is reduced to the problem of showing that the Jacobian algebras $\mathcal{P}(Q, S)$ arising from triangulations of marked surfaces and also a small number of exceptional Jacobian algebras are tame.

### 1.3. Classification of potentials

The existence of non-degenerate potentials (over uncountable fields) was proved in [DWZ1]. However, it can be rather difficult to construct a non-degenerate potential explicitly. Potentials are usually studied up to right equivalence. One of several reasons is that two Jacobian algebras $\mathcal{P}(Q, S_1)$ and $\mathcal{P}(Q, S_2)$ are isomorphic provided $S_1$ and $S_2$ are right equivalent. In this paper we introduce a slightly more general notion, namely that of weak right equivalence, that still implies isomorphism of Jacobian algebras, see Section 2.5 for the definition.

It turns out that QP-mutations of weakly right equivalent QPs are again weakly right equivalent, and that QPs weakly right equivalent to non-degenerate ones are non-degenerate as well.

If $Q$ is a quiver which is mutation equivalent to some acyclic quiver, then there is only one non-degenerate potential on $Q$, up to right equivalence. (This follows from Derksen, Weyman and Zelevinsky’s result that QP-mutation induces a bijection between right equivalence classes of potentials, and by the trivial fact that an acyclic quiver has only one potential, namely the zero-potential $S = 0$.) In general it seems to be hopeless to classify all non-degenerate potentials on an arbitrary 2-acyclic quiver. However, the following theorem, which is our second main result, says that most quivers of finite mutation type admit only one non-degenerate potentials up to right equivalence or weak right equivalence. Fomin, Shapiro and Thurston [FeST] associate to each triangulation $\tau$ of a marked surface $(\Sigma, M)$ a quiver $Q(\tau)$ of finite mutation type, see Section 4 for more details. By Felikson, Shapiro and Tumarkin [FeST], most quivers of finite mutation type arise in this way.

**Theorem 1.4.** Assume that $Q = Q(\tau)$ for a triangulation $\tau$ of a marked surface $(\Sigma, M)$. Then the following hold:

(i) If the boundary of $\Sigma$ is non-empty, and $(\Sigma, M)$ is not a torus with $|M| = 1$, then there exists only one non-degenerate potential $S$ on $Q$ up to right equivalence.
(ii) If the boundary of $\Sigma$ is empty, and
\[ |M| \geq \begin{cases} 
5 & \text{if } \Sigma \text{ is a sphere,} \\
3 & \text{otherwise,} 
\end{cases} \]
then there exists only one non-degenerate potential $S$ on $Q$ up to weak right equivalence.

There is a short list of mutation classes of quivers of finite mutation type that do not arise from triangulations of surfaces. We show that for most of these exceptional quivers there exists again only one non-degenerate potential up to right equivalence.

Assume that $Q = Q(\tau)$ for a triangulation $\tau$ of a marked surface $(\Sigma, M)$ with empty boundary, and $(\Sigma, M)$ is not a sphere with $|M| = 4$. If $|M| = 1$, then we show that there exist at least two non-degenerate potentials on $Q$ up to weak right equivalence. For the case $|M| = 2$, it remains an open problem to classify the non-degenerate potentials.

For a triangulation $\tau$ of a marked surface, Labardini $\text{[LF1, LF2]}$ defined and studied a potential $S(\tau)$ on $Q(\tau)$. (The definition of $S(\tau)$ is recalled in Section 8.1.) Our proof of Theorem 1.4 does not rely on Labardini’s previous results. However, using his results we get that the potential $S$ appearing in Theorem 1.4 is equal to $S(\tau)$ up to weak right equivalence.

1.4. Which tame algebras do occur? In many cases, tame Jacobian algebras are deformations of skewed-gentle algebras. But it remains a challenge to understand their representation theory as most representation theoretic knowledge gets lost under deformations.

Let $Q$ be a quiver of finite mutation type, and let $S$ be a non-degenerate potential on $Q$. As mentioned above, in most cases we have $Q = Q(\tau)$ for some triangulation $\tau$ of some marked surface $(\Sigma, M)$, and $P(Q, S)$ is isomorphic to $P(Q(\tau), S(\tau))$, where $S(\tau)$ is the Labardini potential on $Q(\tau)$.

1. (i) If $(\Sigma, M)$ is unpunctured (i.e. all marked points are in the boundary of $\Sigma$), then the Jacobian algebra $P(Q(\tau), S(\tau))$ is a gentle algebra for all triangulations $\tau$ of $(\Sigma, M)$, see $\text{[ABCP]}$.
2. (ii) If the boundary of $\Sigma$ is empty, and $(\Sigma, M)$ is not a sphere with $|M| = 4$, then there exists a triangulation $\sigma$ of $(\Sigma, M)$ such that $P(Q(\sigma), S(\sigma))$ is a skewed-gentle algebra, and for all $\tau$ the algebras $P(Q(\tau), S(\tau))$ and $P(Q(\sigma), S(\sigma))$ are related by a composition of finitely many nearly Morita equivalences, see Section 3.3 for the definition of a nearly Morita equivalence. Skewed-gentle algebras belong to the class of clannish algebras. Therefore, one can classify the indecomposable $P(Q(\sigma), S(\sigma))$-modules combinatorially $\text{[CB2]}$.
3. If the boundary of $\Sigma$ is empty, and $(\Sigma, M)$ is not a sphere with $|M| = 4$, then the algebra $P(Q(\tau), S(\tau))$ is symmetric for all $\tau$, see $\text{[L2]}$. These algebras show a similar behaviour as algebras of quaternion type, compare $\text{[E] V}$.
4. If $(\Sigma, M)$ is a sphere with $|M| = 4$, or if $Q$ is mutation equivalent to one of the exceptional quivers $E_{m}^{(1, 1)}$ (see Section 5.1), then for all $\tau$ the algebra $P(Q(\tau), S(\tau))$ is closely related to the class of tubular algebras, see $\text{[GG]}$ for further details.

1.5. This article is organized as follows. Some definitions and basic results on quivers with potentials and Jacobian algebras are recalled in Section 2. In Section 3 we show that the representation type of a Jacobian algebra is invariant under QP-mutations. Section 4 consists of a reminder on the construction of quivers associated to triangulations of marked surfaces, and on the relation between flips of a triangulation and mutations of the corresponding quiver. This is taken from Fomin, Shapiro and Thurston’s $\text{[FST]}$ work on cluster algebras arising from surfaces. In Section 5
we recall the classification of mutation finite quivers due to Felikson, Shapiro and Tumarkin \cite{FeliksonShapiroTumarkin}. In Section \ref{section:tameness} we give a tameness criterion for Jacobian algebras based on a general result by Geiß \cite{Geiss}, who showed that deformations of tame algebras are tame. Apart from some exceptional cases, the representation type of Jacobian algebras defined by quivers with non-degenerate potential is determined in Section \ref{section:representation-type}. Here we rely heavily on the tameness criterion from Section \ref{section:tameness} and we construct triangulations of marked surfaces satisfying some favourable properties. In Section \ref{section:non-degenerate-potentials} we classify all non-degenerate potentials for most quivers of finite mutation type, and we provide a uniqueness criterion for non-degenerate potentials. Finally, Section \ref{section:exceptional-cases} deals with the exceptional cases.

2. Preliminaries

2.1. Let $\mathbb{C}$ be the field of complex numbers. For a $\mathbb{C}$-algebra $A$ let $\text{mod}(A)$ be the category of finite-dimensional left $A$-modules. If not mentioned otherwise, by an $A$-module we mean a module in $\text{mod}(A)$.

2.2. Basic algebras. A quiver is a quadruple $Q = (Q_0, Q_1, s, t)$ with $Q_0$ and $Q_1$ finite sets and two maps $s, t : Q_1 \to Q_0$. The elements in $Q_0$ and $Q_1$ are called vertices and arrows of $Q$, respectively. We say that an arrow $\alpha$ in $Q_1$ starts in $s(\alpha)$ and ends in $t(\alpha)$.

A path of length $m \geq 1$ in $Q$ is a tuple $(\alpha_1, \ldots, \alpha_m)$ of arrows of $Q$ such that $s(\alpha_i) = t(\alpha_{i+1})$ for all $1 \leq i \leq m - 1$. We also just write $\alpha_1 \cdots \alpha_m$ instead of $(\alpha_1, \ldots, \alpha_m)$, and we set $s(\alpha_1 \cdots \alpha_m) := s(\alpha_m)$ and $t(\alpha_1 \cdots \alpha_m) := t(\alpha_1)$. Additionally, for each vertex $i$ of $Q$ there is a path $e_i$ of length $0$ with $s(e_i) = t(e_i) = i$.

The path algebra of $Q$ is denote by $\mathbb{C} \langle Q \rangle$. The paths in $Q$ form a $\mathbb{C}$-basis of $\mathbb{C} \langle Q \rangle$. Let $\mathbb{C} \langle \langle Q \rangle \rangle$ be the completed path algebra of $Q$. As a $\mathbb{C}$-vector space we have

$$\mathbb{C} \langle \langle Q \rangle \rangle = \prod_{m \geq 0} \mathbb{C} Q_m$$

where $\mathbb{C} Q_m$ is a $\mathbb{C}$-vector space with a basis labeled by the paths of length $m$ in $Q$. The multiplication of $\mathbb{C} \langle Q \rangle$ and $\mathbb{C} \langle \langle Q \rangle \rangle$ is induced by the concatenation of paths. Both algebras are naturally graded by the length of paths.

Let

$$\mathfrak{m} := \prod_{m \geq 1} \mathbb{C} Q_m$$

be the arrow ideal of $\mathbb{C} \langle \langle Q \rangle \rangle$. For any subset $U \subseteq \mathbb{C} \langle \langle Q \rangle \rangle$ let

$$\overline{U} := \bigcap_{p \geq 0} (U + \mathfrak{m}^p)$$

be the $\mathfrak{m}$-adic closure of $U$.

An algebra $\Lambda$ is called basic provided it is isomorphic to an algebra of the form $\mathbb{C} \langle \langle Q \rangle \rangle / I$, where $I$ is an ideal with $I \subseteq \mathfrak{m}^2$. For a basic algebra $\Lambda = \mathbb{C} \langle \langle Q \rangle \rangle / I$ define $\overline{\Lambda} := \mathbb{C} \langle \langle Q \rangle \rangle / \overline{I}$, and for $p \geq 2$ let $\Lambda_p := \mathbb{C} \langle \langle Q \rangle \rangle / (I + \mathfrak{m}^p)$ be the $p$-truncation of $\Lambda$. The algebras $\Lambda_p$ are obviously finite-dimensional, and we have $\overline{\Lambda} = \varprojlim (\Lambda_p)$, i.e. $\overline{\Lambda}$ is the inverse limit of the algebras $\Lambda_p$. It is straightforward to show that

$$\text{mod}(\Lambda) = \text{mod}(\overline{\Lambda}) = \bigcup_{p \geq 2} \text{mod}(\Lambda_p).$$
2.3. The representation type of a basic algebra. Let
\[ \Lambda = \mathbb{C} \langle \langle Q \rangle \rangle / I \]
be a basic algebra. Then \(\Lambda\) is called representation-finite if there are only finitely many indecomposable \(\Lambda\)-modules in \(\text{mod}(\Lambda)\) up to isomorphism. We say that \(\Lambda\) is tame provided for each dimension vector \(d\) there are finitely many \(\Lambda\)-\(\mathbb{C}[X]\)-bimodules \(M_1, \ldots, M_t\), which are free of finite rank as \(\mathbb{C}[X]\)-modules, such that all indecomposable \(\Lambda\)-modules \(X\) with \(\text{dim}(X) = d\) are isomorphic to a module of the form
\[ M_i \otimes_{\mathbb{C}[X]} \mathbb{C}[X]/(X - \lambda) \]
for some \(1 \leq i \leq t\) and some \(\lambda \in \mathbb{C}\). Clearly, every representation-finite algebra is tame. The algebra \(\Lambda\) is called wild if there is a \(\Lambda\)-\(\mathbb{C}(X,Y)\)-bimodule \(M\), which is free of finite rank as a \(\mathbb{C}(X,Y)\)-module, such that the exact functor
\[ M \otimes_{\mathbb{C}(X,Y)} - : \text{mod}(\mathbb{C}(X,Y)) \rightarrow \text{mod}(\Lambda) \]
maps indecomposable to indecomposable and non-isomorphic to non-isomorphic modules. An exact functor with these properties is called a representation embedding. It follows that for every finitely generated \(\mathbb{C}\)-algebra \(B\) there is a representation embedding \(\text{mod}(B) \rightarrow \text{mod}(\Lambda)\). The following theorem is due to Drozd \([D]\). A more detailed proof can be found in \([CB1]\). Since
\[ \text{mod}(\Lambda) = \bigcup_{p \geq 2} \text{mod}(\Lambda_p), \]
Drozd’s Theorem also holds for arbitrary basic algebras, and not just for finite-dimensional ones.

**Theorem 2.1** (Drozd). Every finite-dimensional algebra is either tame or wild, but not both.

We say that \(\Lambda\) is of finite, tame or wild representation type provided \(\Lambda\) is representation-finite, tame or wild, respectively.

2.4. Quivers with potentials and Jacobian algebras. Let \(Q\) be a quiver. A path \(\alpha_1 \cdots \alpha_m\) of length \(m \geq 1\) in \(Q\) is a cycle or more precisely an \(m\)-cycle if \(s(\alpha_m) = t(\alpha_1)\). Quivers without cycles are called acyclic. A quiver is 2-acyclic if it does not contain any 2-cycles. Note that there is a bijection between the set of 2-acyclic quivers and the set of skew-symmetric integer matrices. More precisely, let \(Q = (Q_0, Q_1, s, t)\) be a quiver with \(Q_0 = \{1, \ldots, n\}\). Define a matrix \(B_Q = (b_{ij}) \in M_{n}(\mathbb{Z})\) by
\[ b_{ij} = |\{\alpha \in Q_1 \mid s(\alpha) = j, t(\alpha) = i\}| - |\{\alpha \in Q_1 \mid s(\alpha) = i, t(\alpha) = j\}|. \]
Then \(Q \mapsto B_Q\) gives a bijection
\[ \{\text{2-acyclic quivers with vertices } 1, \ldots, n\} \rightarrow \{\text{skew-symmetric matrices in } M_{n}(\mathbb{Z})\}. \]

An element \(S \in \mathbb{C} \langle \langle Q \rangle \rangle\) is a potential on \(Q\) if \(S\) is a (possibly infinite) linear combination of cycles in \(Q\). The pair \((Q,S)\) is called a quiver with potential or just a \(QP\). A \(QP\) \((Q,S)\) is 2-acyclic if \(Q\) is 2-acyclic.

We recall Derksen, Weyman and Zelevinsky’s \([DWZ1]\) definition of the Jacobian algebra \(P(Q,S)\) associated to a quiver with potential \((Q,S)\). For a cycle \(\alpha_1 \cdots \alpha_m\) in \(Q\) and an arrow \(\alpha \in Q_1\) define
\[ \partial_\alpha(\alpha_1 \cdots \alpha_m) := \sum_{p: \alpha_p = \alpha} \alpha_{p+1} \cdots \alpha_m \alpha_1 \cdots \alpha_{p-1}. \]
Then we extend by linearity and continuity to define the cyclic derivative \(\partial_\alpha(S)\) of a potential \(S\) on \(Q\). Let \(\partial(S) := \{\partial_\alpha(S) \mid \alpha \in Q_1\}\). (Note that our usage of \(\partial(S)\) differs from the one in \([DWZ1]\).)
Define
\[ \mathcal{P}(Q, S) := \mathbb{C} \langle \langle Q \rangle \rangle / J(S) \]
where \( J(S) \) is the \( \mathfrak{m} \)-adic closure of the ideal \( I(S) \) generated by the set \( \partial(S) \) of cyclic derivatives of \( S \).

2.5. Right equivalences. For a quiver \( Q \) let \( R := R_Q \) be the semisimple subalgebra of \( \mathbb{C} \langle \langle Q \rangle \rangle \) which is generated by the idempotents \( e_1, \ldots, e_n \) of \( \mathbb{C} \langle \langle Q \rangle \rangle \). Using the canonical inclusion \( R \to \mathbb{C} \langle \langle Q \rangle \rangle \), we can see \( \mathbb{C} \langle \langle Q \rangle \rangle \) as an \( R \)-algebra.

**Proposition 2.2** ([DWZ1 Proposition 2.4]). Let \( Q \) and \( Q' \) be quivers on the same vertex set, and let their respective arrow spans be \( A \) and \( A' \). Every pair \((\varphi^{(1)}, \varphi^{(2)})\) of \( R \)-bimodule homomorphisms \( \varphi^{(1)}: A \to A', \varphi^{(2)}: A \to \mathfrak{m}(Q')^2 \), extends uniquely to a continuous \( R \)-algebra homomorphism \( \varphi: \mathbb{C} \langle \langle Q \rangle \rangle \to \mathbb{C} \langle \langle Q' \rangle \rangle \) such that \( \varphi|_A = \varphi^{(1)} + \varphi^{(2)} \). Furthermore, \( \varphi \) is an \( R \)-algebra isomorphism if and only if \( \varphi^{(1)} \) is an \( R \)-bimodule isomorphism.

Let us recall some definitions from [DWZ1 Section 2]. An \( R \)-algebra automorphism \( \varphi: \mathbb{C} \langle \langle Q \rangle \rangle \to \mathbb{C} \langle \langle Q \rangle \rangle \) is

- **unitriangular** if \( \varphi^{(1)} \) is the identity of \( A \);
- **of depth** \( \ell < \infty \) if \( \varphi^{(2)}(A) \subseteq \mathfrak{m}^{\ell+1} \), but \( \varphi^{(2)}(A) \nsubseteq \mathfrak{m}^{\ell+2} \);
- **of infinite depth** if \( \varphi^{(2)}(A) = 0 \).

The depth of \( \varphi \) will be denoted by \( \text{depth}(\varphi) \).

Thus for example, the identity is the only unitriangular \( R \)-algebra automorphism of \( \mathbb{C} \langle \langle Q \rangle \rangle \) of infinite depth. Also, the composition of unitriangular automorphisms is unitriangular.

Let \( V := V_Q \) be the \( \mathfrak{m} \)-adic closure of the \( \mathbb{C} \)-vector subspace of \( \mathbb{C} \langle \langle Q \rangle \rangle \) generated by all elements of the form \( \alpha_1 \alpha_2 \ldots \alpha_d - \alpha_2 \ldots \alpha_d \alpha_1 \) with \( \alpha_1 \ldots \alpha_d \) a cycle in \( Q \). Then \( V \) is also a \( \mathbb{C} \)-vector subspace of \( \mathbb{C} \langle \langle Q \rangle \rangle \), and by definition two potentials \( S \) and \( S' \) are **cyclically equivalent** if their difference belongs to \( V \). In this case, we write \( S \sim_{\text{cyc}} S' \).

Let \((Q, S)\) and \((Q', S')\) be QPs such that \( Q \) and \( Q' \) have the same set of vertices. Thus we have \( R := R_Q = R_{Q'} \). An \( R \)-algebra isomorphism \( \psi: \mathbb{C} \langle \langle Q \rangle \rangle \to \mathbb{C} \langle \langle Q' \rangle \rangle \) is called a **right equivalence** if the potentials \( \psi(S) \) and \( S' \) are cyclically equivalent. In this case, we say that \((Q, S)\) and \((Q', S')\) are **right equivalent**, and we write \( \psi: (Q, S) \to (Q', S') \).

Two potentials \( S \) and \( S' \) are called **weakly right equivalent** if the potentials \( S \) and \( tS' \) are right equivalent for some \( t \in \mathbb{C}^* \). In this case, the Jacobian algebras \( \mathcal{P}(Q, S) \) and \( \mathcal{P}(Q, S') \) are isomorphic.

Two cycles \( v \) and \( w \) in \( Q \) are **rotationally equivalent** if \( v = w_2 w_1 \) for some paths \( w_1 \) and \( w_2 \) in \( Q \) such that \( w = w_1 w_2 \). In this case, we write \( v \sim_{\text{rot}} w \). Note that \( v \sim_{\text{rot}} w \) if and only if \( v \sim_{\text{cyc}} w \).

A cycle \( v \) **appears** in a potential \( S = \sum_{w} \lambda_w w \) if there is some \( w \) with \( \lambda_w \neq 0 \) and \( w \sim_{\text{rot}} v \). In this case, we also say that \( S \) **contains** \( v \). Two potentials \( S \) and \( S' \) on a quiver \( Q \) are **rotationally disjoint** if there is no cycle \( v \) in \( Q \) appearing in both potentials \( S \) and \( S' \).

Given a non-zero element \( u \in \mathbb{C} \langle \langle Q \rangle \rangle \), we denote by \( \text{short}(u) \) the unique integer such that \( u \in \mathfrak{m}^{\text{short}(u)} \) but \( u \notin \mathfrak{m}^{\text{short}(u)+1} \). We also set \( \text{short}(0) = \infty \). If \( S \) is a **finite potential**, i.e. if \( S \) is a linear combination of finitely many cycles, then we denote by \( \text{long}(S) \) the length of the longest cycle appearing in \( S \).
2.6. Mutations of quivers. Let \( Q = (Q_0, Q_1, s, t) \) be a 2-acyclic quiver with set of vertices \( Q_0 = \{1, \ldots, n\} \). The \textit{mutation} \( \mu_k(Q) \) of \( Q \) at \( k \) is another 2-acyclic quiver obtained from \( Q \) in three steps:

1. For each pair \((\beta, \alpha)\) of arrows in \( \{(\beta, \alpha) \in Q_1^2 \mid s(\beta) = k = t(\alpha)\} \) add a new arrow \([\beta\alpha] \) with \( s([\beta\alpha]) = s(\alpha) \) and \( t([\beta\alpha]) = t(\beta) \).
2. Each arrow \( \alpha \) with \( k \in \{s(\alpha), t(\alpha)\} \) is replaced by an arrow \( \alpha^* \) in the opposite direction.
3. Remove a maximal collection of pairwise disjoint 2-cycles.

Mutations of quivers or equivalently of skew-symmetric matrices were introduced and studied by Fomin and Zelevinsky [FZ1] as part of their definition of a cluster algebra.

2.7. Mutations of quivers with potential. Let \( (Q, S) \) be a quiver with potential, and let \( P(Q, S) \) be the corresponding Jacobian algebra. Let \( k \) be a vertex of \( Q \) such that \( k \) does not lie on a 2-cycle. In this situation, Derksen, Weyman and Zelevinsky [DWZ1, Section 5] define the \textit{premutation} \( \tilde{\mu}_k(Q, S) := (\tilde{Q}, \tilde{S}) \) in three steps:

1. For every pair of arrows \((\beta, \alpha)\) in \( Q_{2,k} := \{(\beta, \alpha) \in Q_1^2 \mid s(\beta) = k = t(\alpha)\} \) add a new arrow \([\beta\alpha] \) with \( s([\beta\alpha]) = s(\alpha) \) and \( t([\beta\alpha]) = t(\beta) \).
2. Each arrow \( \alpha \) with \( k \in \{s(\alpha), t(\alpha)\} \) is replaced by an arrow \( \alpha^* \) in opposite direction.
3. Set
\[
\tilde{S} := [S] + \Delta_k(Q)
\]
where \([S]\) is obtained from \( S \) by replacing (after possibly some rotation of cycles) each occurrence of a sequence of arrows \( \beta\alpha \) with \((\beta, \alpha) \in Q_{2,k}\) by \([\beta\alpha] \), and
\[
\Delta_k(Q) := \sum_{(\beta, \alpha) \in Q_{2,k}} [\beta\alpha]\alpha^* \beta^*.
\]

Following [DWZ1, Definition 5.5] let \( \mu_k(Q, S) \) be the reduced part of the QP \( \tilde{\mu}_k(Q, S) \). Note that the QP \( \mu_k(Q, S) \) is only defined up to right equivalence. The QP \( \mu_k(Q, S) \) is the QP-\textit{mutation} of \((Q, S)\) at \( k \).

2.8. Non-degenerate and rigid potentials. For cluster algebra purposes one wishes that a QP \((Q, S)\) and also all possible iterations of QP-mutations of \((Q, S)\) are 2-acyclic. In this case, \( S \) and also \((Q, S)\) is called \textit{non-degenerate}. Derksen, Weyman and Zelevinsky [DWZ1] have shown that (for uncountable ground fields) any given 2-acyclic quiver admits at least one non-degenerate potential.

To decide whether a given potential \( S \) is non-degenerate turns out to be an extremely difficult problem. At least in principle, one needs to apply all possible sequences of QP-mutations and check that one always obtains 2-acyclic QPs. There is, however, a condition on QPs that guarantees non-degeneracy without the need of applying QP-mutations. This condition is called \textit{rigidity}. A potential \( S \) on \( Q \) and also \((Q, S)\) is called \textit{rigid} if every oriented cycle in \( Q \) is cyclically equivalent to an element of the Jacobian ideal \( J(S) \).

2.9. Restriction of potentials. For a QP \((Q, S)\) and a subset \( I \) of the vertex set \( Q_0 \) let \((Q|_I, S|_I)\) be the \textit{restriction} of \((Q, S)\) to \( I \). By definition (see [DWZ1, Definition 8.8]), \( Q|_I \) is the full subquiver of \( Q \) with vertex set \( I \), and \( S|_I \) is obtained from \( S \) by deleting all summands of the form \( \lambda_w w \) with \( w \) not a cycle in the subquiver \( Q|_I \) of \( Q \). The following statement follows from [DWZ1, Proposition 8.9] and [LF2, Corollary 22].

\textbf{Proposition 2.3.} Let \((Q, S)\) be a QP, and let \( I \) be a subset of \( Q_0 \). If \((Q, S)\) is non-degenerate (resp. rigid), then \((Q|_I, S|_I)\) is non-degenerate (resp. rigid).
2.10. The appearance of cycles in non-degenerate potentials.

**Proposition 2.4.** Let \((Q,S)\) be a QP, and let \(I\) be a subset of \(Q_0\) such that the following hold:

(i) \(Q|_I\) contains exactly \(t\) arrows \(\alpha_1, \ldots, \alpha_t\);
(ii) \(c := \alpha_1 \cdots \alpha_t\) is a cycle in \(Q\);
(iii) The vertices \(s(\alpha_1), \ldots, s(\alpha_t)\) are pairwise different;
(iv) \(S\) is non-degenerate.

Then the cycle \(c\) appears in \(S\).

**Proof.** The restriction \(S|_I\) of \(S\) is a non-degenerate potential on \(Q|_I\) by Proposition 2.3. By our assumptions the quiver \(Q|_I\) is a cyclically oriented quiver of Euclidian type \(A_{t-1}\). This implies that the cycle \(c\) appears in \(S|_I\). (It is fairly easy to see that a potential \(W\) on \(Q|_I\) is non-degenerate if and only if \(c\) appears in \(W\) up to rotation.) Therefore \(c\) appears also in \(S\). \(\Box\)

**Corollary 2.5.** Let \((Q,S)\) be a QP with \(S\) non-degenerate. Suppose that \(\alpha\beta\gamma\) is a 3-cycle in \(Q\) such that there are no multiple arrows between the three vertices \(s(\alpha), s(\beta)\) and \(s(\gamma)\). Then \(\alpha\beta\gamma\) appears in \(S\).

3. Mutation invariance of representation type

In this section we prove Theorem 1.1. After some preparations in Section 3.2 where we prove some elementary properties of split morphisms between free modules over commutative \(\mathbb{C}\)-algebras, we show in Section 3.3 that the mutation of certain bimodules commutes with taking tensor products. This is the main ingredient for the proof of Theorem 1.1 which is given in Subsection 3.4.

3.1. Mutations of representations. Let \((Q,W)\) be a QP, and assume that \(k\) is a vertex of \(Q\) which does not lie on a 2-cycle. Define \((\bar{Q}, \bar{W}) := \tilde{\mu}_k(Q,W)\). If \(M\) is a representation of \(\mathcal{P}(Q,W)\), Derksen, Weyman and Zelevinsky [DWZ1] defined a representation \(\bar{\mu}_k(M) := \bar{M}\) of \(\mathcal{P}(Q,W)\). To describe it properly, let

\[
M(k_{\text{in}}) := \bigoplus_{\alpha \in Q_1: t(\alpha) = k} M(s(\alpha)) \quad \text{and} \quad M(k_{\text{out}}) := \bigoplus_{\beta \in Q_1: s(\beta) = k} M(t(\beta)).
\]

Furthermore, we need the three linear maps

\[
M(\alpha_k) : M(k_{\text{in}}) \to M(k), \quad (m_\alpha)_{\alpha \in Q_1: t(\alpha) = k} \mapsto \sum_{\alpha \in Q_1: t(\alpha) = k} M(\alpha)(m_\alpha),
\]

\[
M(\beta_k) : M(k) \to M(k_{\text{out}}), \quad m \mapsto (M(\beta)(m))_{\beta \in Q_1: s(\beta) = k},
\]

\[
M(\gamma_k) : M(k_{\text{out}}) \to M(k_{\text{in}}), \quad (m_\beta)_{\beta \in Q_1: s(\beta) = k} \mapsto \left(\sum_{\beta \in Q_1: s(\beta) = k} M(\partial_{\beta,a}(W))(m_\beta)\right)_{\alpha \in Q_1: t(\alpha) = k}.
\]

The definition of \(\partial_{\beta,a}(W)\) can be found in [DWZ2] Section 4). Furthermore, we have to choose retractions \(p_\alpha : M(k_{\text{in}}) \to \text{Ker}(M(\alpha_k))\) to the inclusion \(i_\alpha\), and \(p_\gamma : M(k_{\text{out}}) \to \text{Ker}(M(\gamma_k))\) to the inclusion \(i_\gamma\).

Now, we define \(\bar{M} := \bar{\mu}_k(M)\) as follows:

\[
\bar{M}(i) = \begin{cases} M(i) & \text{if } i \neq k, \\ \text{Ker}(M(\gamma_k))/\text{Im}(M(\beta_k)) \oplus \text{Ker}(M(\alpha_k)) & \text{if } i = k, \end{cases}
\]

and on arrows
\( M((\beta \alpha)) = M(\beta)M(\alpha) \) for all \((\beta, \alpha) \in \mathbb{Q}_{2,k}\),

\( \tilde{M}(\gamma) = M(\gamma) \) for all \( \gamma \in Q_1 \cap \mathbb{Q}_1\),

\( \tilde{M}(\alpha_k) = (p_{\alpha_k}^{-1})_1 : M(k_{in}) \to M(k) \), with \( \tilde{M}(k_{in}) = M(k_{out}) \),

and \( p : \text{Ker}(M(\gamma_k)) \to \text{Ker}(M(\gamma_k))/\text{Im}(M(\alpha_k)) \) the canonical projection,

\( \tilde{M}(\beta_k) = (0, i_{\alpha_k}) : M(k) \to M(k_{out}) = M(k_{in}) \).

Recall, that \( \tilde{M}(\alpha_k) \equiv ((\tilde{M}(\beta^*))_{\beta \in Q_1 : s(\beta) = k})^T \) and \( \tilde{M}(\beta_k) \equiv ((\tilde{M}(\alpha^*))_{\alpha \in Q_1 : t(\beta) = k}) \).

Note that our description for mutations of representations is slightly simplified with respect to the original definition by Derksen, Weyman and Zelevinsky, as for example in [DWZ2, Section 4]. First of all, we only consider \textit{undecorated} representations. Moreover, we use for \( M(k) \) a less symmetric, though isomorphic definition which is more convenient for our purpose. (The original definition makes it more clear that \( \tilde{M}(k) \) should be viewed as a glueing of \( \text{Ker}(M(\alpha_k)) \) and \( \text{Coker}(M(\beta_k)) \) along \( \text{Im}(M(\gamma_k)) \approx M(k_{out})/\text{Ker}(M(\gamma_k)) \).

### 3.2. Generic Images and Kernels

Let \( R \) be a commutative \( \mathbb{C} \)-algebra. In this subsection all tensor products are over \( R \) so that we write \( \otimes \) instead of \( \otimes_R \) for typographical reasons.

We say that a homomorphism \( g \) in \( \text{Hom}_R(R^m, R^n) \) splits if there exist submodules \( K' \leq R^m \) and \( I' \leq R^n \) such that \( R^m = \text{Ker}(g) \oplus K' \) and \( R^n = \text{Im}(g) \oplus I' \).

**Lemma 3.1.** For \( g \in \text{Hom}_R(R^m, R^n) \) consider the following diagram consisting of the obvious inclusions and projections:

\[
\begin{array}{ccccc}
0 & \longrightarrow & \text{Ker}(g) & \longrightarrow & R^m \\
i_0 & & \downarrow g & & \downarrow \text{Im}(g) & \longrightarrow & R^n \\
& & \downarrow i_1 & & \downarrow p \text{Coker}(g) & \longrightarrow & 0 \\
0 & \longrightarrow & 0 & \longrightarrow & 0
\end{array}
\]

Assume that \( g \) splits. Then for each \( R \)-module \( S \) the following hold:

(a) The maps \( i_1 \otimes S \) and \( i_0 \otimes S \) induce isomorphisms of vector spaces

\[
(3.1) \quad \overline{i_1 \otimes S} : \text{Im}(g) \otimes S \xrightarrow{\sim} \text{Im}(g \otimes S) \quad \text{and}
\]

\[
(3.2) \quad \overline{i_0 \otimes S} : \text{Ker}(g) \otimes S \xrightarrow{\sim} \text{Ker}(g \otimes S), \quad \text{respectively.}
\]

In particular, with \( \overline{i_1^S} : \text{Im}(g \otimes S) \to R^n \otimes S \) and \( \overline{i_0^S} : \text{Ker}(g \otimes S) \to R^m \otimes_R S \) we have \( \overline{i_1^S} \circ (\overline{i_1 \otimes S}) = i_1 \otimes S \) and \( \overline{i_0^S} \circ (\overline{i_0 \otimes S}) = i_0 \otimes S \).

(b) Let \( p_0 \) be a retraction for \( i_0 \), i.e. \( p_0 \circ i_0 = 1_{\text{Ker}(g)} \), then \( \overline{(i_0 \otimes S)} \circ (p_0 \otimes S) \) is a retraction for \( \overline{i_0^S} \).

**Proof.** In the above diagram the two short exact sequences split by hypothesis. In particular, they remain exact under the functor \(- \otimes S\).

(a) Since \(- \otimes S\) is right exact we have

\[
\text{Im}(g \otimes S) = \text{Ker}(p \otimes S) = \text{Im}(i_1 \otimes S).
\]

By our hypothesis \( i_1 \otimes S \) is injective which implies \((3.1)\).
By the same token

\[ \text{Ker}(g \otimes S) = \text{Ker}(\bar{g} \otimes S) = \text{Im}(i_0 \otimes S). \]

Again, by hypothesis \( i_0 \otimes S \) is injective which implies (3.2).

(b) We have

\[ ((i_0 \otimes S) \circ (p_0 \otimes S) \circ i_0^S) \circ (i_0 \otimes S) = (i_0 \otimes S) \circ (p_0 \otimes S) \circ (i_0 \otimes S) = (i_0 \otimes S) \circ \text{Id}_{\text{Ker}(g \otimes S)}, \]

which implies our claim, since \( i_0 \otimes S \) is an isomorphism.

\[ \square \]

**Lemma 3.2.** Consider split morphisms \( f \in \text{Hom}_R(R^l, R^m) \) and \( g \in \text{Hom}_R(R^m, R^n) \) with \( g \circ f = 0 \). We have then inclusions \( i_1 : \text{Im}(f) \hookrightarrow \text{Ker}(g) \), \( i_0 : \text{Ker}(g) \hookrightarrow R^m \) and \( i_1 = i_0 \circ i : \text{Im}(f) \hookrightarrow R^m \). Moreover, we have the projection \( p : \text{Ker}(g) \to \text{Ker}(g) / \text{Im}(f) \). With this notation we obtain for any \( R \)-module \( S \) the following commutative diagram with exact rows and all vertical arrows isomorphisms:

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & \text{Im}(f) \otimes S & \xrightarrow{\bar{i} \otimes S} & \text{Ker}(g) \otimes S & \xrightarrow{p \otimes S} & \text{Ker}(g) / \text{Im}(f) & \xrightarrow{\bar{s}} & 0 \\
& & \downarrow{i_1 \otimes S} & & \downarrow{i_0 \otimes S} & & \downarrow{s} & & \\
0 & \longrightarrow & \text{Im}(f \otimes S) & \xrightarrow{i \otimes S} & \text{Ker}(g \otimes S) & \xrightarrow{p} & \text{Ker}(g \otimes S) / \text{Im}(f \otimes S) & \xrightarrow{\bar{s}} & 0 \\
\end{array}
\]

The morphisms in the bottom row are the canonical inclusion and projection.

**Proof.** Since the morphisms \( f \) and \( g \) split, we know that \( i_0 \otimes S \) and \( i_1 \otimes S \) are injective, and that \( i_0 \otimes S \) and \( i_1 \otimes S \) are isomorphisms by Lemma 3.1. Since \( (i_0 \otimes S) \circ (i \otimes S) = i_1 \otimes S \), also \( i \otimes S \) is injective. It remains to show that in the above diagram the left square commutes. To this end let \( i_0^S : \text{Ker}(g \otimes S) \hookrightarrow R^m \otimes S \) and \( i_1^S : \text{Im}(f \otimes S) \hookrightarrow R^m \otimes S \) be the canonical inclusions. Then \( i_0^S \circ i_1^S = i_1^S \). Thus, by Lemma 3.1 (a) we have

\[ i_1^S \circ (i_0 \circ (i_1 \otimes S)) = i_1 \otimes S \circ (i \otimes S) = i_0 \circ (i_0 \otimes S) \circ (i \otimes S) = i_0 \circ (i_0 \otimes S) \circ (i \otimes S). \]

Since \( i_0^S \) is injective the required commutativity follows.

\[ \square \]

**Lemma 3.3.** Let \( R \) be an integral domain and \( g \in \text{Hom}_R(R^m, R^n) \), then there exists some \( d \in R \) such that the localization \( g_d : R^m_d \to R^n_d \) splits.

**Proof.** Let \( F \) be the fraction field of \( R \) and \( g_F : F^m \to F^n \) the localization of \( g \). If \( r \) is the rank of \( g_F \) we can find \( F \)-bases \( \underline{b} = (b_1, \ldots, b_m) \) resp. \( \underline{b}' = (b_1', \ldots, b_n') \) of \( F^m \) resp. \( F^n \) such that \( (b_1, \ldots, b_{m-r}) \) is a basis of \( \text{Ker}(g_F) \) and \( g(b_{m-r+i}) = b'_i \) for \( 1 \leq i \leq r \).

Denote by \( (v_1, \ldots, v_m) \) resp. \( (v_1', \ldots, v_n') \) the standard basis of \( F^m \) resp. \( F^n \). Thus,

\[ b_i = \sum_{j=1}^m x_{ij} v_j \quad \text{and} \quad b'_i = \sum_{j=1}^n x'_{ij} v'_j \]

for certain \( x_{ij}, x'_{ij} \in R \) and \( y_{ij}, y'_{ij} \in R \setminus \{0\} \). With

\[ d := \prod_{i,j=1}^m y_{ij} \cdot \prod_{i,j=1}^n y'_{ij} \cdot \det((x_{ij})_{i,j=1,\ldots,m}) \cdot \det((x'_{ij})_{i,j=1,\ldots,n}) \]

we can view \( \underline{b} \) as a \( R_d \)-basis of \( R_d^m \) and \( \underline{b}' \) as a \( R_d \)-basis of \( R_d^n \). Moreover, \( (b_1, \ldots, b_{m-r}) \) is a \( R_d \)-basis of \( \text{Ker}(g_d) \) and \( (b'_1, \ldots, b'_r) \) is a \( R_d \)-basis of \( R_d^n \). Thus, \( g_d \) splits.

\[ \square \]

**Remark 3.4.** It follows from the proof that in Lemma 3.3 above we may assume that \( \text{Ker}(g_d) \) and \( \text{Im}(g_d) \) are free \( R_d \)-modules.
3.3. Mutation of Bimodules. In this section \((Q,W)\) is a QP over the field \(\mathbb{C}\), \(\mathcal{P}(Q,W)\) is the corresponding Jacobian algebra, and \(R\) denotes a commutative \(\mathbb{C}\)-algebra.

We view an \(\mathcal{P}(Q,W)\)-\(R\)-bimodule \(M\) (which is free as an \(R\)-module) as a nilpotent covariant representation of \(Q\) into the category of (free) \(R\)-modules which respects the relations \(\partial_a(W)\) for all arrows \(a\) of \(Q\).

The mutation operation for modules as described in Section 3.1 also makes sense for any \(\mathcal{P}(Q,W)\)-\(R\)-bimodule \(M\) which is finitely generated and free as an \(R\)-module, if we assume that the corresponding \(R\)-module homomorphisms \(M(\alpha_k)\) and \(M(\gamma_k)\) between free \(R\)-modules split. In fact, this allows us to choose the required retractions \(p_\alpha\) and \(p_\gamma\).

If \(R\) is a domain, and \(M\) is just finitely generated and free as an \(R\)-module, we can find \(d \in R\) such that the localized \(\mathcal{P}(Q,W)-R_d\)-bimodule \(M_d\) has the property that \(M_d(\alpha_k), M_d(\beta_k)\) and \(M_d(\gamma_k)\) split. Moreover, we may assume then that the \(\mathcal{P}(Q,W)-R_d\)-bimodule \(\tilde{\mu}_k(M_d)\) is free (and finitely generated) as an \(R_d\)-module.

**Proposition 3.5.** Let \((Q,W)\) be a QP, and let \(k\) be a vertex of \(Q\) which does not lie on a 2-cycle, and let \(R\) be a \(\mathbb{C}\)-algebra which is a domain. Then consider the premutation \((\tilde{Q},\tilde{W}) := \tilde{\mu}_k(Q,W)\). Let \(M\) be a \(\mathcal{P}(Q,W)\)-\(R\)-bimodule which is free and finitely generated as an \(R\)-module, and suppose that the maps \(M(\alpha_k), M(\beta_k)\) and \(M(\gamma_k)\) split. Then we have for each \(R\)-modules \(S\) a canonical \(\mathcal{P}(Q,W)\)-module isomorphism

\[
\tilde{\mu}_k(M) \otimes_R S \cong \tilde{\mu}_k(M \otimes_R S).
\]

**Proof.** Recall that

\[
(\tilde{\mu}_k(M) \otimes_R S)(k) = \left(\frac{\text{Ker}(M(\gamma_k))}{\text{Im}(M(\beta_k))} \oplus \text{Ker}(M(\alpha_k))\right) \otimes_R S,
\]

and

\[
(\tilde{\mu}_k(M \otimes_R S))(k) = \left(\frac{\text{Ker}(M(\gamma_k) \otimes_R S)}{\text{Im}(M(\beta_k) \otimes_R S)} \oplus \text{Ker}(M(\alpha_k) \otimes_R S)\right).
\]

For the rest of the proof, all tensor products are over \(R\), so that we will write \(\otimes\) instead of \(\otimes_R\) for typographical reasons. Given our hypotheses, we may use by Lemma 3.1 (b) the map \(p^S_\alpha := (\tilde{i}_\alpha \otimes S) \circ (p_\alpha \otimes S)\) as retraction for the natural inclusion \(i^S_\alpha: \text{Ker}(M(\alpha_k) \otimes S) \hookrightarrow M(\text{im}) \otimes S\), and the map \(p^S_\gamma := (\tilde{i}_\gamma \otimes S) \circ (p_\gamma \otimes S)\) as retraction for the natural inclusion \(i^S_\gamma: \text{Ker}(M(\gamma_k) \otimes S) \hookrightarrow M(\text{out}) \otimes S\). We will take these maps for the definition of the structure maps of \(\tilde{\mu}_k(M \otimes S)\).

Moreover, our hypotheses imply that we have by Lemma 3.1 (a) and Lemma 3.2 a natural isomorphism

\[
\varphi_k := \left(\begin{array}{cc}
\tilde{c}^S_{\gamma,\beta} & 0 \\
0 & \tilde{c}^S_{\alpha,\alpha \otimes S}
\end{array}\right): \left(\frac{\text{Ker}(M(\gamma_k))}{\text{Im}(M(\beta_k))} \oplus \text{Ker}(M(\alpha_k))\right) \otimes S \cong \left(\frac{\text{Ker}(M(\gamma_k) \otimes S)}{\text{Im}(M(\beta_k) \otimes S)} \oplus \text{Ker}(M(\alpha_k) \otimes S)\right).
\]

We set \(\varphi_i = \text{Id}_{M(i) \otimes S}\) for all \(i \in Q_0 \setminus \{k\}\) and claim the \(\varphi = (\varphi_j)_{j \in Q_0}\) is the required isomorphism of \(\mathcal{P}(Q,W)\)-modules. By the construction of \(\varphi\) it is sufficient to show that it is in fact a homomorphism of representations of \(\mathcal{P}(Q,W)\). From the definition of mutation of representations it follows that to this end it is sufficient to verify the two equalities

\[
\varphi_k \circ ((\tilde{\mu}_k(M))(\alpha_k)) \otimes S = ((\tilde{\mu}_k(M \otimes S))(\alpha_k)) \text{ and}
\]

\[
(\tilde{\mu}_k(M))(\beta_k) = ((\tilde{\mu}_k(M \otimes S))(\beta_k)) \circ \varphi_k.
\]

Now, after expanding, Equation (3.3) reads as follows:

\[
c^S_{\gamma,\beta} \circ ((p \otimes S) \circ (p_\gamma \otimes S)) = p^S \circ ((\tilde{i}_\gamma \otimes S) \circ (p_\gamma \otimes S)) \text{ and}
\]

\[
(\tilde{i}_\alpha \otimes S) \circ ((p_\alpha \otimes S) \circ (M(\gamma_k) \otimes S)) = ((\tilde{i}_\alpha \otimes S) \circ (p_\alpha \otimes S)) \circ (M(\gamma_k) \otimes S).
\]
The first of these equations holds by Lemma 3.2, whilst the second one is trivial. Finally, after expanding, Equation 3.4 becomes
\[ i_\alpha \otimes S = i_\alpha^S \circ (i_\alpha \otimes S), \]
which holds by Lemma 3.1 (a).

3.4. Mutation invariance of tameness. Recall that all simple \( \mathbb{C}[X] \)-modules are of the form
\[ S_\lambda := \mathbb{C}[X]/(X - \lambda) \]
with \( \lambda \in \mathbb{C} \), and that \( S_\lambda \cong S_\mu \) if and only if \( \lambda = \mu \). Let \( d \in \mathbb{C}[X] \), and let \( \mathbb{C}[X]_d \) be the localization of \( \mathbb{C}[X] \) at \( p \). By a slight abuse of notation, for \( \lambda \in \mathbb{C} \) with \( p(\lambda) \neq 0 \), we also write \( S_\lambda \) for the simple \( \mathbb{C}[X]_d \)-module \( \mathbb{C}[X]_d/(X - \lambda) \).

In the above definition, some of the bimodules might be trivial in the sense that \( M_i \otimes \mathbb{C}[X]_d S_\lambda \cong M_i \otimes \mathbb{C}[X]_d S_\mu \) for all \( \lambda, \mu \in \mathbb{C} \) with \( d(\lambda) \neq 0 \neq d(\mu) \).

A Jacobian algebra \( \mathcal{P}(Q,W) \) is tame, if for each dimension vector \( v \in \mathbb{N}^{Q_0} \) there exists a polynomial \( d \in \mathbb{C}[X] \), and a finite number of \( \mathcal{P}(Q,W) \)-\( \mathbb{C}[X]_d \)-bimodules \( M_1, \ldots, M_m \), which are finitely generated and free as \( \mathbb{C}[X]_d \)-modules, with the following property: Each indecomposable \( \mathcal{P}(Q,W) \)-module \( N \) with \( \dim(N) = v \) is isomorphic to \( M_i \otimes \mathbb{C}[X]_d S_\lambda \) for some \( 1 \leq i \leq m \) and some \( \lambda \in \mathbb{C} \) with \( d(\lambda) \neq 0 \). It is not difficult to show that this definition of tameness is equivalent to the one from Section 2.3.

Using Drozd’s Tame-Wild Theorem 2.1, the invariance of the representation type of Jacobian algebras under mutation, as stated in Theorem 1.1, follows from the following result.

**Theorem 3.6.** Let \( (Q,W) \) be a QP and suppose that \( k \in Q_0 \) does not lie on a 2-cycle, so that \( \tilde{\mu}_k(Q,W) \) is defined. Then the following are equivalent:

(i) \( \mathcal{P}(Q,W) \) is tame;  
(ii) \( \mathcal{P}(\tilde{\mu}_k(Q,W)) \) is tame;  
(iii) \( \mathcal{P}(\mu_k(Q,W)) \) is tame.

**Proof.** Define \( (\tilde{Q},\tilde{W}) := \tilde{\mu}_k(Q,W) \). Assume that \( \mathcal{P}(Q,W) \) is tame. Let \( v \in \mathbb{N}^{Q_0} \) be a dimension vector. We may suppose that \( v \) is not the dimension vector of the simple \( \mathcal{P}(\tilde{Q},\tilde{W}) \)-module \( S_k \). It follows from (the proof of) [DWZ1, Theorem 10.13] that each indecomposable \( \mathcal{P}(\tilde{Q},\tilde{W}) \)-module \( N \) with \( \dim(N) = v \) is isomorphic to a module of the form \( \tilde{\mu}_k(M) \) for some \( \mathcal{P}(Q,W) \)-module \( M \) with \( \dim(M(i)) = \dim(N(i)) \) for all \( i \in Q_0 \setminus \{k\} \) and \( \dim(M'(k)) \leq \dim(N(k_{\text{in}})) + \dim(N(k_{\text{out}})). \)

Thus, since \( \mathcal{P}(Q,W) \) is tame, there exist some \( d \in \mathbb{C}[X] \) and some \( \mathcal{P}(Q,W) \)-\( \mathbb{C}[X]_d \)-bimodules \( M_1, \ldots, M_m \), which are finitely generated and free as \( \mathbb{C}[X]_d \)-modules, such that the following holds: Each indecomposable \( \mathcal{P}(\tilde{Q},\tilde{W}) \)-module \( N \) with \( \dim(N) = v \) is isomorphic to \( \tilde{\mu}_k(M_i \otimes \mathbb{C}[X]_d S_\lambda) \) for some \( 1 \leq i \leq m \) and some \( \lambda \in \mathbb{C} \) with \( d(\lambda) \neq 0 \). By Lemma 3.3 we may assume without loss of generality that the maps \( M_i(\alpha_k), M_i(\beta_k) \) and \( M_i(\gamma_k) \) (defined in 3.1) split for all \( 1 \leq i \leq m \). In fact, if necessary we just localize and add a finite number of trivial bimodules. Then, by Proposition 3.5 we have for the \( \mathcal{P}(\tilde{Q},\tilde{W}) \)-\( \mathbb{C}[X]_d \)-bimodules \( \tilde{\mu}_k(M_i) \) that \( \tilde{\mu}_k(M_i \otimes \mathbb{C}[X]_d S_\lambda) \cong \tilde{\mu}_k(M_i) \otimes \mathbb{C}[X]_d S_\lambda \) for all \( i \) and all \( \lambda \in \mathbb{C} \) with \( d(\lambda) \neq 0 \). Thus \( \mathcal{P}(\tilde{Q},\tilde{W}) \) is tame. The opposite direction is proved in the same way since mutation of QPs is involutive on isomorphism classes of Jacobian algebras. Note that the QP-mutation at a vertex \( k \) is defined as long as \( k \) does not lie on a 2-cycle, so reducedness is not required. Thus (i) and (ii) are equivalent.

The equivalence of (ii) and (iii) is straightforward. \( \square \)
3.5. Nearly Morita equivalence for Jacobian algebras. In this section, we discuss an alternative strategy for proving Theorem 1.1. For a $K$-algebra $\Lambda$ and an $A$-module $M \in \text{mod}(\Lambda)$ let $\text{add}(M)$ be the additive subcategory generated by $M$, i.e. $\text{add}(M)$ consists of the $A$-modules that are isomorphic to finite direct sums of direct summands of $M$. The $M$-stable category

\[ \text{mod}_M(\Lambda) := \text{mod}(\Lambda)/\text{add}(M) \]

has by definition the same objects as $\text{mod}(\Lambda)$, and the morphism spaces are the morphism spaces from $\text{mod}(\Lambda)$ modulo the subspaces of morphism factoring through some object in $\text{add}(M)$.

Let $S$ be a potential on $Q$, and let $(Q', S') = \mu_k(Q, S)$ be a QP-mutation of $(Q, S)$. Set $\Lambda := P(Q, S)$ and $\Lambda' := P(Q', S')$. Buan, Iyama, Reiten and Smith [BIRS] prove the following result showing that the mutation of Jacobian algebras almost induces a Morita equivalence of the corresponding module categories.

**Theorem 3.7** ([BIRS, Theorem 7.1]). There is an equivalence of additive categories

\[ f: \text{mod}_{S_k}(\Lambda) \to \text{mod}_{S_k}(\Lambda') \]

Using Crawley-Boevey [CB2, Theorem 4.4], Krause [K Corollary 3.4] proved that stable equivalences of dualizing algebras preserve the representation type. For details and definitions we refer to [K]. Combining Krause’s result with Theorem 3.7 we get the following statement.

**Theorem 3.8.** Suppose that $\Lambda$ and $\Lambda'$ are dualizing algebras. Then the representation types of $\Lambda$ and $\Lambda'$ coincide.

If $\Lambda$ is finite-dimensional, then $\Lambda$ and $\Lambda'$ are both dualizing algebras. Thus we got a proof for Theorem 1.1 for all finite-dimensional Jacobian algebras. (Recall from [DWZ] that mutations of finite-dimensional Jacobian algebras are again finite-dimensional.) We expect that a slight modification of Krause’s proof of [K Corollary 3.4] yields a proof of Theorem 3.8 also for infinite dimensional Jacobian algebras $\Lambda$ and $\Lambda'$.

4. Triangulations of marked surfaces and quiver mutations

4.1. This section is aimed at recalling some definitions and facts on quivers arising from triangulations of oriented surfaces with marked points. Our main reference for this section is [FST].

4.2. Triangulations of marked surfaces. A marked surface, or simply a surface, is a pair $(\Sigma, M)$, where $\Sigma$ is a compact connected oriented surface with (possibly empty) boundary, and $M$ is a finite set of points on $\Sigma$, called marked points, such that $M$ is non-empty and has at least one point from each connected component of the boundary of $\Sigma$. The marked points that lie in the interior of $\Sigma$ are called punctures, and the set of punctures of $(\Sigma, M)$ is denoted by $P$. A marked surface $(\Sigma, M)$ is unpunctured if $P = \emptyset$.

Throughout the paper we will always assume that $(\Sigma, M)$ is none of the following:

- an unpunctured monogon, digon or triangle;
- a once-punctured monogon or digon;
- a sphere with less than four punctures.

By a monogon (resp. digon, triangle) we mean a disk with exactly one (resp. two, three) marked point(s) on the boundary. A disk (resp. annulus) is by definition a surface of genus 0 with exactly one (resp. two) boundary component(s). A sphere is a surface with empty boundary and genus 0, and by a torus we mean a surface with possibly non-empty boundary and genus 1.

Let $(\Sigma, M)$ be a marked surface.
(1) An arc in \((\Sigma, \mathbb{M})\) is a curve \(i\) in \(\Sigma\) such that the following hold:
- the endpoints of \(i\) belong to \(\mathbb{M}\);
- \(i\) does not intersect itself, except that its endpoints may coincide;
- the relative interior of \(i\) is disjoint from \(\mathbb{M}\) and from the boundary of \(\Sigma\);
- \(i\) does not cut out an unpunctured monogon nor an unpunctured digon.

(2) An arc whose endpoints coincide will be called a loop.

(3) Two arcs \(i_1\) and \(i_2\) are isotopic if there exists an isotopy \(H : I \times \Sigma \to \Sigma\) such that \(H(0, x) = x\) for all \(x \in \Sigma\), \(H(1, i_1) = i_2\), and \(H(t, m) = m\) for all \(t \in I := [0, 1]\) and all \(m \in \mathbb{M}\). Arcs will be considered up to isotopy and orientation.

(4) Two arcs are compatible if there are arcs in their respective isotopy classes whose relative interiors do not intersect.

(5) An ideal triangulation, or simply a triangulation, of \((\Sigma, \mathbb{M})\) is any maximal collection of pairwise compatible arcs whose relative interiors do not intersect each other.

The pairwise compatibility of any collection of arcs can be realized simultaneously. That is, given any collection of pairwise compatible arcs, it is always possible to find representatives in their isotopy classes whose relative interiors do not intersect each other.

For an ideal triangulation \(\tau\), the valency \(\text{val}_\tau(p)\) of a puncture \(p \in \mathbb{P}\) is the number of arcs in \(\tau\) incident to \(p\), where each loop at \(p\) is counted twice.

Let \(\tau\) be an ideal triangulation of a surface \((\Sigma, \mathbb{M})\).

(1) For each connected component of the complement in \(\Sigma\) of the union of the arcs in \(\tau\), its topological closure \(\triangle\) will be called an ideal triangle, or simply a triangle, of \(\tau\).

(2) An ideal triangle \(\triangle\) is called interior if its intersection with the boundary of \(\Sigma\) consists only of (possibly none) marked points. Otherwise it will be called non-interior.

(3) An interior ideal triangle \(\triangle\) is self-folded if it contains exactly two arcs \(a\) and \(b\) of \(\tau\), see Figure 1. The arc \(b\) is called the folded side of the ideal triangle.

\begin{figure}[h]
\centering
\includegraphics[width=0.2\textwidth]{self_folded_triangle.png}
\caption{Self-folded triangle.}
\end{figure}

The number \(n\) of arcs in any ideal triangulation of \((\Sigma, \mathbb{M})\) is determined by the following numerical data:
- the genus \(g = g(\Sigma)\) of \(\Sigma\);
- the number \(b = b(\Sigma)\) of boundary components of \(\Sigma\);
- the number \(p = p(\Sigma, \mathbb{M})\) of punctures;
- the number \(c = c(\Sigma, \mathbb{M})\) of marked points that lie on the boundary of \(\Sigma\).

Indeed, we have
\[ n = 6g + 3b + 3p + c - 6, \]
a formula that can be proved using the definition and basic properties of the Euler characteristic. Hence \(n\) is an invariant of \((\Sigma, \mathbb{M})\), called the rank of \((\Sigma, \mathbb{M})\).
4.3. **Flips of tagged triangulations and mutations of quivers.** Let $\tau$ be an ideal triangulation of $(\Sigma, M)$ and let $i \in \tau$ be an arc. If $i$ is not the folded side of a self-folded triangle, then there exists exactly one arc $i'$, different from $i$, such that $\sigma := (\tau \setminus \{i\}) \cup \{i'\}$ is an ideal triangulation of $(\Sigma, M)$. We say that $\sigma$ is obtained from $\tau$ by applying a flip at $i$, or by flipping the arc $i$, and we write $\sigma = f_i(\tau)$. In order to be able to flip the folded sides of self-folded triangles, Fomin, Shapiro and Thurston [FST] introduce the notion of a tagged arc. A tagged triangulation of $(\Sigma, M)$ is then defined to be any maximal collection of pairwise compatible tagged arcs.

All tagged triangulations of $(\Sigma, M)$ have the same cardinality (equal to the rank $n$ of $(\Sigma, M)$) and every collection of $n - 1$ pairwise compatible tagged arcs is contained in precisely two tagged triangulations. This means that every tagged arc in a tagged triangulation can be replaced by a uniquely defined, different tagged arc that together with the remaining $n - 1$ arcs forms a tagged triangulation. In analogy with the ordinary case, this combinatorial replacement is called a **flip**.

To every tagged triangulation $\tau$ Fomin-Shapiro-Thurston associate a skew-symmetric $n \times n$ integer matrix $B(\tau)$. When $\tau$ is actually an ideal triangulation the matrix $B(\tau)$ is defined as follows. The rows and columns of $B(\tau)$ correspond to the arcs of $\tau$. Let $\pi_+ : \tau \to \tau$ be the function that is the identity on the set of arcs that are not folded sides of self-folded triangles of $\tau$, and sends the folded side of a self-folded triangle to the unique loop of $\tau$ enclosing it. For each non-self-folded ideal triangle $\bigtriangleup$ of $\tau$, let $B_{ij}^{\bigtriangleup}$ be the $n \times n$ integer matrix defined by

$$b_{ij}^{\bigtriangleup} = \begin{cases} 1 & \text{if } \bigtriangleup \text{ has sides } \pi_+(i) \text{ and } \pi_+(j), \text{ with } \pi_+(j) \text{ preceding } \pi_+(i) \\ -1 & \text{if the same holds, but in the counter-clockwise order;} \\ 0 & \text{otherwise.} \end{cases}$$

The **signed adjacency matrix** $B(\tau)$ is then defined as

$$B(\tau) = \sum_{\bigtriangleup} B^{\bigtriangleup},$$

where the sum runs over all non-self-folded triangles of $\tau$.

In the general situation, where the tagged triangulation $\tau$ is not necessarily an ideal triangulation, the signed adjacency matrix $B(\tau)$ is defined as the signed adjacency matrix of an ideal triangulation $\tau^0$ obtained from $\tau$ by deletion of notches, see [FST] for details.

Note that all entries of $B(\tau)$ have absolute value at most 2. Moreover, $B(\tau) = (b_{ij})$ is skew-symmetric, hence gives rise to a 2-acyclic quiver, the **adjacency quiver** $Q(\tau)$, whose vertices $1, \ldots, n$ correspond to the tagged arcs in $\tau$, with $b_{ij}$ arrows from $j$ to $i$ whenever $b_{ij} > 0$. Numerous examples of triangulations $\tau$ and quivers $Q(\tau)$ can be found in Section [4]. For a tagged triangulation $\tau$ and the associated ideal triangulation $\tau^0$, the adjacency quivers $Q(\tau)$ and $Q(\tau^0)$ are isomorphic.

The next result due to Fomin, Shapiro and Thurston [FST] shows that flipping tagged arcs is compatible with quiver mutation.

**Proposition 4.1** ([FST] Proposition 4.8 and Lemma 9.7). Let $\tau$ and $\sigma$ be tagged triangulations. If $\sigma$ is obtained from $\tau$ by flipping the tagged arc $k$ of $\tau$, then $Q(\sigma) = \mu_k(Q(\tau))$.

## 5. Quivers of finite mutation type

### 5.1. Classification of quivers of finite mutation type.** A quiver of finite mutation type is called **exceptional** if it is not of the form $Q(\tau)$ for some triangulation $\tau$ of some marked surface $(\Sigma, M)$. The acyclic quivers of finite mutation type were classified by Buan and Reiten. They...
show that these are exactly the quivers of type $A_n, D_n, E_m, \tilde{A}_n, \tilde{D}_n$ and $\tilde{E}_m$ for $m = 6, 7, 8$ and the $m$-Kronecker quiver $K_m$ with two vertices and $m \geq 3$ arrows. Representatives of mutation equivalence classes of the exceptional acyclic quivers of finite mutation type are displayed in Figure 2. Up to mutation equivalence, there are only five exceptional non-acyclic quivers. One representative from each mutation equivalence class is displayed in Figure 3.

$K_m$

$E_6$

$E_7$

$E_8$

$\tilde{E}_6$

$\tilde{E}_7$

$\tilde{E}_8$

**Figure 2.** Acyclic exceptional quivers of finite mutation type

**Figure 3.** Non-acyclic exceptional quivers of finite mutation type

Based on the work of Fomin, Shapiro and Thurston [FST], the quivers of finite mutation type were classified by Felikson, Shapiro and Tumarkin [FeST].
Theorem 5.1 ([FeST, Main Theorem]). A 2-acyclic quiver $Q$ is of finite mutation type if and only if one of the following hold:

(i) $Q = Q(\tau)$ for some triangulation $\tau$ of some marked surface $(\Sigma, \mathcal{M})$;
(ii) $Q$ is mutation equivalent to one of the quivers $X_6, X_7, K_m$ for $m \geq 3$, $E_m, \widetilde{E}_m, E_m^{(1,1)}$ for $m = 6, 7, 8$.

5.2. Block decomposition of quivers arising from triangulations. We recall some notation and results from [FeST, Section 13]. Each non-self-folded triangle of a triangulation $\tau$ of a marked surface gives rise to a certain quiver called a block. The quiver $Q(\tau)$ associated to $\tau$ is then obtained by glueing these blocks in the obvious way (and by deleting possible 2-cycles). For details we refer to [FeST]. In Figure 4 we display the six different kinds of blocks arising from the six different kinds of non-self-folded triangles, see [FeST] for details. Note that some of the vertices of the blocks are white ($\circ$) and others are black ($\bullet$). (Actually, there is yet another kind of non-self-folded triangle. However, it occurs only in one exceptional case. Namely, take the sphere with 4 punctures and its triangulation consisting of three self-folded triangles and the non-self-folded triangle whose sides are the three non-folded sides of the three self-folded triangles.)

One can also start with a list of block quivers, and then glue them (without referring to a triangulation) as follows: Let $B_1, \ldots, B_t$ be a list of copies of blocks, and for $1 \leq k \leq t$ let $W_k$ be the set of white vertices of the quiver $B_k$. Let $W = W_1 \cup \cdots \cup W_t$ be the disjoint union of the sets $W_k$. A glueing map for $B_1, \ldots, B_t$ is a map $g : W \rightarrow W$ such that the following hold:

(g1) $g^2 = \text{id}_W$;
(g2) For $1 \leq k \leq t$, the restriction of $g$ to $g(W_k) \cap W_k$ is the identity.

Let $\text{pre.glue}(B_1, \ldots, B_t; g)$ be the quiver obtained from the disjoint union of the quivers $B_1, \ldots, B_t$ by identifying the vertices $i$ and $g(i)$, where $i$ runs through $W$. Next, let $\text{glue}(B_1, \ldots, B_t; g)$ be the quiver obtained from $\text{pre.glue}(B_1, \ldots, B_t; g)$ by removing 2-cycles in the usual sense. Following Fomin, Shapiro and Thurston [FeST], we say that a connected quiver $Q$ is block decomposable if $Q = \text{glue}(B_1, \ldots, B_t; g)$ for some $B_1, \ldots, B_t$ and $g$ as above. The trivial quiver with just one vertex and no arrows is also called block decomposable.

Theorem 5.2 ([FeST, Theorem 13.3]). For a quiver $Q$ the following are equivalent:

(i) $Q$ is block decomposable;
(ii) $Q = Q(\tau)$ for some triangulation $\tau$ of some marked surface.

6. Tame algebras and deformations of Jacobian algebras

6.1. One of our aims is to determine which Jacobian algebras $\mathcal{P}(Q, S)$ with $(Q, S)$ a non-degenerate QP are of tame representation type. We will show that for most quivers $Q$ of finite mutation type, $\mathcal{P}(Q, S)$ is a skewed-gentle algebra or a deformation of a skewed-gentle algebra.
6.2. Deformations of algebras. The following theorem due to Crawley-Boevey \cite{CB4} is a slightly more general version of a theorem by Geiß \cite{G2}. It is a crucial tool for the proof of our first main result Theorem 1.2.

**Theorem 6.1** (\cite[Theorem B]{CB4}). Let $A$ be a finite-dimensional $\mathbb{C}$-algebra, and let $X$ be an irreducible affine variety over $\mathbb{C}$, and let $f_1, \ldots, f_r : X \to A$ be morphisms of affine varieties (where $A$ has its natural structure as an affine space). For $x \in X$ set 
\[
A(x) := A/(f_1(x), \ldots, f_r(x)).
\]
Let $x_0, x_1 \in X$ such that the following hold:

(i) $A(x_0)$ is tame;
(ii) $A(x) \cong A(x_1)$ for all $x$ in some dense open subset of $X$.

Then $A(x_1)$ is tame.

In most applications, the variety $X$ appearing in Theorem 6.1 will be just the affine line $\mathbb{C}$. There are several notions of deformations of algebras. In this article, we say that an algebra $B$ is a deformation of an algebra $A$ if in the situation of Theorem 6.1 we have $A \cong A(x_0)$ and $B \cong A(x_1)$.

6.3. Deformations of Jacobian algebras. As before, let $Q$ be a 2-acyclic quiver. Let 
\[
S = \sum_{t \geq 3} \sum_{w_t} \mu_{w_t} w_t
\]
be a potential on $Q$, where $\mu_{w_t} \in \mathbb{C}$ and the $w_t$ run over all cycles of length $t$ in $Q$. Recall that 
\[
\text{short}(S) = \min\{t \geq 3 \mid \mu_{w_t} \neq 0 \text{ for some } w_t\},
\]
and define 
\[
S_{\text{min}} := \sum_{t = \text{short}(S)} \sum_{w_t} \mu_{w_t} w_t.
\]

Let $I(S)$ be the ideal of $\mathbb{C} \langle\langle Q \rangle\rangle$ generated by the cyclic derivatives $\partial_\alpha(S)$ of $S$. Furthermore, for $p \geq 3$ let 
\[
S_p := \sum_{3 \leq t \leq p} \sum_{w_t} \mu_{w_t} w_t
\]
be the $p$-truncation of $S$. We also set $S_2 := 0$. The following lemma is straightforward.

**Lemma 6.2.** For $p \geq 2$ we have $I(S_p) + m^p = I(S) + m^p$.

For $\lambda \in \mathbb{C}^*$ define 
\[
S(\lambda) := \sum_{t \geq 3} \sum_{w_t} \lambda^{t - \text{short}(S)} \mu_{w_t} w_t,
\]
and set $S(0) := S_{\text{min}}$. Let $S(\lambda)_p$ denote the $p$-truncation of the potential $S(\lambda)$.

We define an algebra homomorphism 
\[
f_\lambda : \mathbb{C} \langle\langle Q \rangle\rangle \to \mathbb{C} \langle\langle Q \rangle\rangle
\]
by $\alpha \mapsto \lambda \alpha$ for all $\alpha \in Q_1$. Clearly, $f_\lambda$ is an isomorphism for all $\lambda \neq 0$.

**Lemma 6.3.** For all $\alpha \in Q_1$ and $\lambda \neq 0$ we have 
\[
f_\lambda(\partial_\alpha(S)) = \sum_{t \geq 3} \sum_{w_t} \lambda^{t-1} \mu_{w_t} \partial_\alpha(w_t) \quad \text{and} \quad \partial_\alpha(S(\lambda)) = \lambda^{\text{short}(S)} f_\lambda(\partial_\alpha(S)).
\]
Proof. The first equality follows from the definition of \( f_\lambda \) and the fact that \( \partial_\alpha(w_t) \) is a linear combination of paths of length \( t - 1 \). We have
\[
\chi^{\text{short}(S)} f_\lambda(\partial_\alpha(S)) = \chi^{\text{short}(S)} \sum_{t \geq 3} \sum_{w_t} \chi^{t-1} \mu_{w_t} \partial_\alpha(w_t)
\]
\[
= \sum_{t \geq 3} \sum_{w_t} \chi^{t-\text{short}(S)} \mu_{w_t} \partial_\alpha(w_t)
\]
\[
= \partial_\alpha(S(\lambda)).
\]
Thus the second equality also holds. \( \square \)

**Corollary 6.4.** For \( \lambda \neq 0 \) we have \( f_\lambda(I(S)) = I(S(\lambda)) \).

Define \( \Lambda := \mathcal{P}(Q, S) \) and \( \Lambda(\lambda) := \mathcal{P}(Q, S(\lambda)) \). Note that \( \Lambda = \Lambda(1) \). For \( p \geq 2 \) we have
\[
\Lambda(\lambda)_p = \mathbb{C}\langle\langle Q\rangle\rangle/(I(S(\lambda)) + m^p) = \mathbb{C}\langle\langle Q\rangle\rangle/(I(S(\lambda)_p) + m^p)
\]
and \( \mathbb{C}\langle\langle Q\rangle\rangle_p = \mathbb{C}\langle\langle Q\rangle\rangle/m^p \).

**Lemma 6.5.** For \( \lambda \neq 0 \) we have \( \Lambda \cong \Lambda(\lambda) \) and \( \Lambda_p \cong \Lambda(\lambda)_p \).

**Proof.** For \( \lambda \neq 0 \) we get \( f_\lambda(m^p) = m^p \). Now the lemma follows from Corollary 6.4. \( \square \)

**Proposition 6.6.** If \( \Lambda(0)_p \) is tame, then \( \Lambda_p \) is also tame.

**Proof.** For each \( \alpha \in Q_1 \) and \( p \geq 2 \) define a map
\[
d_{\alpha,p} : \mathbb{C} \to \mathbb{C}\langle\langle Q\rangle\rangle_p
\]
by \( \lambda \mapsto \partial_\alpha(S(\lambda)_p) + m^p \). The vector spaces \( \mathbb{C} \) and \( \mathbb{C}\langle\langle Q\rangle\rangle_p \) can be seen as affine spaces, and \( d_{\alpha,p} \) is then obviously a morphism of affine varieties.

For each \( \lambda \in \mathbb{C} \) there is a canonical isomorphism
\[
\Lambda(\lambda)_p \cong \mathbb{C}\langle\langle Q\rangle\rangle_p/I(\lambda)_p,
\]
where \( I(\lambda)_p \) is the ideal of \( \mathbb{C}\langle\langle Q\rangle\rangle_p \) generated by the elements \( d_{\alpha,p}(\lambda) \), where \( \alpha \) runs through \( Q_1 \).

Using Lemma 6.5, the result follows now from Theorem 6.1. \( \square \)

A basic algebra \( \Lambda \) is tame if and only if all \( p \)-truncations \( \Lambda_p \) are tame. This follows from the fact that
\[
\text{mod}(\Lambda) = \bigcup_{p \geq 2} \text{mod}(\Lambda_p).
\]
Thus the following tameness criterion for Jacobian algebras follows directly from Proposition 6.6.

**Corollary 6.7.** Let \( Q \) be a 2-acyclic quiver, and let \( S \) be a potential on \( Q \). If \( \mathcal{P}(Q, S_{\text{min}}) \) is tame, then \( \mathcal{P}(Q, S) \) is also tame.

6.4. **Gentle and skewed-gentle algebras.** We repeat the definition of a gentle and a skewed-gentle algebra. These are typical classes of tame algebras.

Let \( Q = (Q_0, Q_1, s, t) \) be a quiver, and let \( \Lambda \) be an algebra isomorphic to an algebra of the form \( \mathbb{C}\langle\langle Q\rangle\rangle/I \) or \( \mathbb{C}\langle\langle Q\rangle\rangle/\mathbb{C}\langle Q\rangle \). Following \[AS\] we call \( \Lambda \) a gentle algebra if the following hold:

\( \text{(g1)} \) For each \( k \in Q_0 \) we have \( |\{\alpha \in Q_1 \mid s(\alpha) = k\}| \leq 2 \), and \( |\{\alpha \in Q_1 \mid t(\alpha) = k\}| \leq 2 \);
\( \text{(g2)} \) The ideal \( I \) is generated by a set of paths of length 2;
\( \text{(g3)} \) If for some arrow \( \beta \in Q_1 \) there exist two paths \( \alpha_1 \beta \neq \alpha_2 \beta \) of length 2, then precisely one of these paths belongs to \( I \).
(g4) If for some arrow $\beta \in Q_1$ there exist two paths $\beta \gamma_1 \neq \beta \gamma_2$ of length 2, then precisely one of these paths belongs to $I$.

Note that our definition of a gentle algebra is slightly more general than the original definition from [AS].

Let $Q = (Q_0, Q_1, s, t)$ be a quiver, and let $\Lambda$ be an algebra isomorphic to an algebra of the form $\mathbb{C}\langle\langle Q \rangle\rangle/I$ or $\mathbb{C}\langle Q \rangle/I$. Following [GP, Definition 4.2] we call $\Lambda$ a skewed-gentle algebra if there is a set

$$L \subseteq \{ \alpha \in Q_1 | s(\alpha) = t(\alpha) \}$$

of loops of $Q$ such that the following hold:

1. (sg1) For each $k \in Q_0$ we have $| \{ \alpha \in Q_1 | s(\alpha) = k \} | \leq 2$, and $| \{ \alpha \in Q_1 | t(\alpha) = k \} | \leq 2$;
2. (sg2) The ideal $I$ is generated by a set of paths of length 2, and by all elements of the form $\alpha^2 - \alpha$ with $\alpha \in L$;
3. (sg3) If for some arrow $\beta \in Q_1 \setminus L$ there exist two paths $\alpha_1 \beta \neq \alpha_2 \beta$ of length 2, then precisely one of these paths belongs to $I$;
4. (sg4) If for some arrow $\beta \in Q_1 \setminus L$ there exist two paths $\beta \gamma_1 \neq \beta \gamma_2$ of length 2, then precisely one of these paths belongs to $I$.

On the first glance, skewed-gentle algebras do not seem to be basic algebras, since the elements $\alpha^2 - \alpha$ with $\alpha \in L$ are not contained in $m_2$. However, it is not difficult to show that skewed-gentle algebras are indeed basic algebras. Note also that every gentle algebra is a skewed-gentle algebra.

**Theorem 6.8.** Skewed-gentle algebras are tame.

**Proof.** Skewed-gentle algebras belong to the class of clannish algebras introduced by Crawley-Boevey [CB2]. The tameness of clannish algebras follows from [CB2, Theorem 3.8].

6.5. **Examples.**

6.5.1. Let $H$ be the quiver

\[
\begin{array}{c}
\circ 1' \\
\downarrow \delta' \\
2 \quad 3' \\
\downarrow \gamma \\
\circ 2 \\
\downarrow \delta'' \\
\circ 1'' \\
\end{array}
\Rightarrow
\begin{array}{c}
\circ 5' \\
\downarrow \alpha' \\
4 \quad 3 \\
\downarrow \beta \\
\circ 3 \\
\downarrow \alpha'' \\
\circ 5'' \\
\end{array}
\]

of affine type $\widetilde{D}_6$. Its path algebra $\mathbb{C}(H)$ is isomorphic to $\mathbb{C}\langle Q \rangle/I$, where $Q$ is the quiver

$$\varepsilon_1 \circ 1 \overset{\delta}{\longrightarrow} 2 \overset{\gamma}{\longrightarrow} 3 \overset{\beta}{\longrightarrow} 4 \overset{\alpha}{\longrightarrow} 5 \overset{\varepsilon_5}{\circlearrowleft}$$

and the ideal $I$ is generated by $\varepsilon_1^2 - \varepsilon_1$ and $\varepsilon_5^2 - \varepsilon_5$. It follows that $\mathbb{C}(H)$ is a skewed-gentle algebra.
6.5.2. Let $H$ be the quiver

$$
\begin{array}{ccc}
1 & \overset{\beta_1}{\rightarrow} & 2' \\
\downarrow{\gamma} \quad & & \uparrow{\alpha_1} \\
3 & \overset{\beta_2}{\leftarrow} & 2'' \\
\end{array}
$$

and let $S := \alpha_1 \beta_1 \gamma + \alpha_2 \beta_2 \gamma$. Thus the ideal $I(S)$ of $\mathbb{C}\langle H \rangle$ is generated by the elements $\beta_i \gamma$, $\gamma \alpha_i$ for $i = 1, 2$ and $\alpha_1 \beta_1 + \alpha_2 \beta_2$. One easily checks that $\mathcal{P}(H, S) = \mathbb{C}\langle H \rangle / J(S) = \mathbb{C}(H) / I(S)$. (The first equality holds by the definition of a Jacobian algebra.) The Jacobian algebra $\mathcal{P}(H, S)$ is isomorphic to the algebra $\mathbb{C}\langle Q \rangle / I$, where $Q$ is the quiver

$$
\begin{array}{ccc}
1 & \overset{\alpha_1}{\rightarrow} & 2' \\
\downarrow{\gamma} \quad & & \uparrow{\alpha} \\
3 & \overset{\beta_2}{\leftarrow} & 2'' \\
\end{array}
$$

and $I$ is generated by the elements $\alpha \beta$, $\beta \gamma$, $\gamma \alpha$ and $\varepsilon^2 - \varepsilon$. Thus $\mathcal{P}(H, S)$ is a skewed-gentle algebra.

6.5.3. Let $Q$ be the quiver

$$
\begin{array}{ccc}
1 & \overset{\beta_2}{\rightarrow} & 2 \\
\downarrow{\gamma} \quad & & \uparrow{\alpha_1} \\
3 & \overset{\beta_1}{\leftarrow} & 2' \\
\end{array}
$$

and let

$$S := \alpha_1 \beta_1 \gamma_1 + \alpha_2 \beta_2 \gamma_2 - \alpha_1 \beta_2 \gamma_1 \alpha_2 \beta_1 \gamma_2.$$

Using the notation from Section 6.3 we get

$$S(\lambda) := \alpha_1 \beta_1 \gamma_1 + \alpha_2 \beta_2 \gamma_2 - \lambda^2 \alpha_1 \beta_2 \gamma_1 \alpha_2 \beta_1 \gamma_2$$

for $\lambda \in \mathbb{C}$. By Lemma 6.5, we have $\mathcal{P}(Q, S) \cong \mathcal{P}(Q, S(\lambda))$ for all $\lambda \neq 0$. The algebra $\mathcal{P}(Q, S(0))_p$ is isomorphic to the path algebra $\mathbb{C}\langle Q \rangle$ modulo the ideal generated by $\alpha_1 \beta_1$, $\beta_1 \gamma_1$, $\gamma_1 \alpha_1$, $\alpha_2 \beta_2$, $\beta_2 \gamma_2$, $\gamma_2 \alpha_2$ and all paths of length $p$. Thus $\mathcal{P}(Q, S(0))_p$ is a factor algebra of a gentle algebra. It follows that $\mathcal{P}(Q, S(0))_p$ is tame for all $p$. Thus $\mathcal{P}(Q, S(0)) = \mathcal{P}(Q, S_{\min})$ is tame. Now Corollary 6.7 implies that $\mathcal{P}(Q, S)$ is tame as well.

6.6. Jacobian algebras as deformations of skewed-gentle algebras. We say that a quiver $Q$ has no double arrows provided the number of arrows from $i$ to $j$ is at most one for all vertices $i$ and $j$ of $Q$.

**Proposition 6.9.** Let $Q = \text{glue}(B_1, \ldots, B_t; g)$ be a block decomposable quiver. Assume that the following hold:

1. For all $i \neq j$ we have $|g(W_i) \cap W_j| \leq 1$;
2. Each 3-cycle in $Q$ is contained in one of the blocks $B_1, \ldots, B_t$;
3. None of the blocks $B_1, \ldots, B_t$ is of type V.
Let \( S \) be any non-degenerate potential on \( Q \). Then the Jacobian algebra \( \mathcal{P}(Q,S) \) is a deformation of a skewed-gentle algebra. In particular, \( \mathcal{P}(Q,S) \) is tame. Assume additionally that \( Q = \text{glue}(B_1, \ldots, B_t; g) \) satisfies the following:

\[ \text{(gl6) Each of the blocks } B_1, \ldots, B_t \text{ is of type I or II.} \]

Then \( \mathcal{P}(Q,S) \) is a deformation of a gentle algebra.

**Proof.** Assume that (gl3), (gl4) and (gl5) are satisfied. Note that condition (gl3) in Proposition 6.9 implies that \( Q \) does not have any double arrows, and that pre\( \text{glue}(B_1, \ldots, B_t; g) \) does not have any 2-cycles. Now let \( S \) be a non-degenerate potential on \( Q \), and let \( S_{\text{min}} \) be defined as in Section 6.3. By Corollary 2.5 we know that \( S_{\text{min}} \) contains all 3-cycles of \( Q \) up to rotation. Now one easily checks that the algebra \( \mathcal{P}(Q,S_{\text{min}}) \) is a skewed-gentle algebra. (Blocks of type IIIa, IIIb or IV are handled in the same way as in Examples 6.5.1 and 6.5.2.) In particular, \( \mathcal{P}(Q,S_{\text{min}}) \) is tame. By Corollary 6.7 this implies that \( \mathcal{P}(Q,S) \) is tame. Similarly, if we assume additionally (gl6), it follows easily that \( \mathcal{P}(Q,S_{\text{min}}) \) is a gentle algebra. \( \square \)

6.7. **Definition of gentle and skewed-gentle triangulations.** We will show in Section 7 that for most marked surfaces \((\Sigma, \mathcal{M})\) there is a triangulation \( \tau \) such that \( Q = Q(\tau) = \text{glue}(B_1, \ldots, B_t; g) \) satisfies the assumptions (gl3), (gl4), (gl5) and (gl6) of Proposition 6.9. In this case, we call \( \tau \) a gentle triangulation. If \( Q(\tau) \) satisfies (gl3), (gl4) and (gl5), then \( \tau \) is a skewed-gentle triangulation. If \( Q(\tau) \) is the trivial quiver (this only happens if \( \tau \) is one of the two triangulations of an unpunctured 4-gon), then \( \tau \) is also called a gentle or skewed-gentle triangulation.

Note that our definition of a gentle or skewed-gentle triangulation is quite restrictive. For example, condition (gl3) implies that the quiver \( Q(\tau) \) of a skewed-gentle triangulation \( \tau \) has no double arrows. Let \( \tau \) be a triangulation, set \( Q := Q(\tau) \), and let \( S \) be a non-degenerate potential on \( Q \). If \( \tau \) is a gentle triangulation, we get that \( \mathcal{P}(Q,S_{\text{min}}) \) is a gentle algebra, but \( \mathcal{P}(Q,S) \) is in general not gentle. It can also happen that both \( \mathcal{P}(Q,S) \) and \( \mathcal{P}(Q,S_{\text{min}}) \) are gentle algebras, even though \( \tau \) is not a gentle triangulation. For skewed-gentle triangulations the situation is analogous.

7. **The representation type of Jacobian algebras: Regular cases**

7.1. **Representation type and mutation type.** Let \( Q \) be a 2-acyclic quiver. Recall that \( Q \) is of finite cluster type if the associated cluster algebra \( \mathcal{A}_Q \) has only finitely many cluster variables. Apart from a few exceptional cases, which are treated separately in Section 9 the following theorem will be proved in this section. The proof relies heavily on the construction of suitable triangulations of marked surfaces, which can be found in Sections 7.4 and 7.8.

**Theorem 7.1.** Assume that \( Q \) is not mutation equivalent to one of the quivers \( T_1 \), \( T_2 \), \( X_6 \), \( X_7 \) or \( K_m \) with \( m \geq 3 \). Then for any non-degenerate potential \( S \) on \( Q \) the following hold:

- (f) \( \mathcal{P}(Q,S) \) is representation-finite if and only if \( Q \) is of finite cluster type.
- (t) \( \mathcal{P}(Q,S) \) is tame if and only if \( Q \) is of finite mutation type.
- (w) \( \mathcal{P}(Q,S) \) is wild if and only if \( Q \) is of infinite mutation type.

The quivers \( T_1 \), \( T_2 \), \( X_6 \), \( X_7 \) and \( K_m \) will also be studied in Section 9.

7.2. **Reduction to quivers of finite mutation type.**

**Lemma 7.2.** Part (f) of Theorem 7.1 is true.
Lemma 7.3. Assume that $Q$ is of infinite mutation type. Then $\mathcal{P}(Q,S)$ is wild for all non-degenerate potentials $S$ on $Q$.

Proof. Combine Lemma 7.3 and Drozd’s Theorem 2.1

Lemma 7.5. Suppose that part (t) of Theorem 7.1 is true. Then (w) is also true.

Proof. This follows directly from Drozd’s Theorem 2.1

To prove Theorem 7.1 it remains to study $\mathcal{P}(Q,S)$ for $Q$ of finite mutation type and decide when $\mathcal{P}(Q,S)$ is tame.

7.3. Proof of Theorem 7.1

The following lemma takes care of a large part of the proof of Theorem 7.1

Lemma 7.6. Let $(\Sigma, M)$ be a marked surface which is not equal to a torus with $|M| = 1$, or to a sphere with $|M| = 4$. Let $Q = Q(\tau)$ for some triangulation $\tau$ of $(\Sigma, M)$. Then $\mathcal{P}(Q,S)$ is tame for all non-degenerate potentials $S$ on $Q$.

Proof. If $(\Sigma, M)$ is a monogon with $|P| = 2$, then there is a triangulation $\sigma$ of $(\Sigma, M)$ such that $Q(\sigma)$ is an acyclic quiver of type $A_3$. There is only one potential $S$ on $Q(\sigma)$, namely $S = 0$, and $\mathcal{P}(Q(\sigma),S)$ is tame. In case $(\Sigma, M)$ is a sphere with $|M| = 5$, a monogon with $|P| \geq 3$, a digon, a triangle or an annulus with $|M \setminus P| = 2$, there is a skewed-gentle triangulation $\sigma$ of $(\Sigma, M)$ by Theorem 7.1 and Sections 7.7 and 7.8. In all remaining cases, there is a gentle triangulation $\sigma$ of $(\Sigma, M)$ by Theorem 7.8. Thus by Proposition 6.9 in each of the above cases, the Jacobian algebra $\mathcal{P}(Q(\sigma),W)$ is a tame for any non-degenerate potential $W$ on $Q(\sigma)$.

By [FST], Proposition 4.8, $Q(\tau)$ and $Q(\sigma)$ are mutation equivalent quivers. Since the potential $S$ is non-degenerate, we see that $(Q(\tau),S)$ is QP-mutation equivalent to $(Q(\sigma),W)$ for some non-degenerate potential $W$ on $Q(\sigma)$. Thus $\mathcal{P}(Q(\sigma),W)$ is tame. Now Theorem 7.1 implies that $\mathcal{P}(Q(\tau),S)$ is tame.
To finish the proof of Theorem 7.1, we need the following lemma, which will be proved in Section 9.

**Lemma 7.7.** For the following quivers $Q$ and all non-degenerate potentials $S$ on $Q$, the Jacobian algebra $P(Q,S)$ is tame:

1. $Q$ is mutation equivalent to one of the quivers $E_m, \tilde{E}_m$ or $E_m^{(1,1)}$ for $m = 6, 7, 8$;
2. $Q = Q(\tau)$ for some triangulation $\tau$ of a sphere $(\Sigma, M)$ with empty boundary and $|M| = 4$.

### 7.4. Gentle triangulations.

**Theorem 7.8.** Let $(\Sigma, M)$ be a marked surface which is not equal to one of the following:

- a mongon, a digon, or a triangle;
- an annulus with $|M \setminus \mathbb{P}| = 2$;
- a sphere with $|M| = 4, 5$;
- a torus with $|M| = 1$.

Then there exists a gentle triangulation $\tau$ of $(\Sigma, M)$.

We prove Theorem 7.8 by induction. The rather lengthy induction base is dealt with in Section 7.5 and the induction step is performed in Section 7.6. In the course of the proof, we draw several triangulations $\tau$ of marked surfaces and their adjacency quivers $Q(\tau)$. The shaded regions in the quivers $Q(\tau)$ correspond to the interior non-self-folded triangles of $\tau$. Vertices with the same label have to be identified.

### 7.5. Proof of Theorem 7.8: Induction base.

#### 7.5.1. Unpunctured 4-gon.
An unpunctured 4-gon has only two triangulations, and their adjacency quiver is the trivial quiver with one vertex and no arrows. By definition these triangulations are gentle.

#### 7.5.2. Unpunctured annulus $(\Sigma, M)$ with $|M| = 3$.
In Figure 5 we show a gentle triangulation $\tau$ and its quiver $Q(\tau)$ for an unpunctured annulus $(\Sigma, M)$ with $|M| = 3$.

![Figure 5](image)

**Figure 5.** A gentle triangulation for an unpunctured annulus $(\Sigma, M)$ with $|M| = 3$.

#### 7.5.3. Unpunctured surface $(\Sigma, M)$ with genus 0, three boundary components and $|M| = 3$.
As shown in Figure 6, an unpunctured surface $(\Sigma, M)$ with genus 0, three boundary components and $|M| = 3$ has a gentle triangulation.
7.5.4. **Sphere** $(\Sigma, M)$ with $|M| = 6$. Each sphere $(\Sigma, M)$ with $|M| = 6$ has a gentle triangulation $\tau$. Figure 7 illustrates the construction of such a $\tau$ and its quiver $Q(\tau)$. (The punctures are denoted by $p_1, \ldots, p_6$.)

7.5.5. **Torus** $(\Sigma, M)$ with empty boundary and $|M| = 2$. Each torus $(\Sigma, M)$ with empty boundary and $|M| = 2$ has a gentle triangulation $\tau$. Figure 8 illustrates the construction of such a $\tau$ and its quiver $Q(\tau)$. (In the picture, in order to obtain a torus one has to identify the two sides labeled by 1 and also the two sides labeled by 2 along the indicated orientations. The two punctures of the torus are denoted by $p$ and $q$.)

7.5.6. **Torus** $(\Sigma, M)$ with $|M| = 2$ and $|M \setminus \mathbb{P}| = 1$. In the case of a torus with one boundary component, one marked point $p$ on this component, and one puncture $q$, the triangulation shown in Figure 9 is a gentle triangulation.

7.5.7. **Torus** $(\Sigma, M)$ with two boundary components and $|M| = 2$. In the case of a torus with two boundary components, and one marked point on each of these components, the triangulation shown in Figure 10 is gentle.

7.5.8. **Unpunctured torus** $(\Sigma, M)$ with one boundary component and $|M| = 2$. In the case of an unpunctured torus with one boundary component and two marked points $p$ and $q$ on this component, the triangulation displayed in Figure 11 is gentle.
Figure 8. A gentle triangulation for a torus $(\Sigma, M)$ with empty boundary and $|M| = 2$.

Figure 9. A gentle triangulation for a torus with one boundary component, one marked point on it, and one puncture.

Figure 10. A gentle triangulation for a torus with two boundary components and one marked point on each of these.

7.5.9. **Surfaces** $(\Sigma, M)$ with empty boundary, $|M| = 1$ and genus at least two. Let $g \geq 2$. Let $P_g$ be a $4g$-gon with vertices $q_1, \ldots, q_{4g}$ ordered clockwise along the boundary of $P_g$ as shown in
Figure 11. A gentle triangulation for an unpunctured torus with one boundary component and two marked points on it.

Figure 12. For each $1 \leq i \leq 4g$ we glue the two sides labeled by $i$ along the orientation shown in Figure 12. This yields a surface $\Sigma$ of genus $g$. We denote the corresponding identification map by $\pi: P_g \to \Sigma$. For each $1 \leq t \leq 2g$ let $t'$ be the diagonal of $P_g$ that connects $q_{2t-2}$ with $q_{2t}$. (We set $q_{4g+1} := q_1$.) The set $\{t' \mid 1 \leq t \leq 2g\}$ forms a $2g$-gon inscribed in $P_g$. By inserting arcs $t''$ from $q_{2t}$ to $q_{2t+2}$ for $2 \leq t \leq 2g-2$, as illustrated in Figure 13 for $g = 3$, we obtain a triangulation $T$ of this $2g$-gon. Let $s_1, \ldots, s_{4g}$ be the sides of $P_g$, where $s_k$ connects $q_k$ with $q_{k+1}$. (We set $q_{4g+1} := q_1$.) Then the set

$$\tau := \{\pi(s_l) \mid 1 \leq l \leq 4g\} \cup \{\pi(t') \mid 1 \leq t \leq 2g\} \cup \{\pi(t'') \mid 2 \leq t \leq 2g-2\}$$

is a triangulation of the once-punctured surface $(\Sigma, M)$, where $M = \{\pi(q_1)\}$. (Note that $\pi(q_1) = \pi(q_2) = \cdots = \pi(q_{4g})$, and $\pi(s_{4k+\ell}) = \pi(s_{4k+\ell+2})$ for $0 \leq k \leq g-1$ and $\ell = 1, 2$.) For $g \geq 2$ the quiver $Q(\tau)$ looks as indicated in Figure 13. It is easy to see that $\tau$ is a gentle triangulation. Thus we obtained a gentle triangulation for surfaces $(\Sigma, M)$ with empty boundary, $|M| = 1$ and genus $g \geq 2$. 

Figure 12. A $4g$-gon $P_g$ with a side pairing that yields a once-punctured surface $\Sigma$ with empty boundary and genus $g$. 

$q_0 := q_{4g}$.)
7.5.10. **Surfaces** \((\Sigma, \mathcal{M})\) with non-empty boundary, \(|\mathcal{M}| = 1\) and genus \(g \geq 2\). Let \((\Sigma', \mathcal{M}')\) be a once-punctured surface with empty boundary and genus \(g \geq 2\). Let \(\sigma\) be a gentle triangulation of \((\Sigma', \mathcal{M}')\) as constructed in Section 7.5.9. Let \((\Sigma, \mathcal{M})\) be the surface obtained from \((\Sigma', \mathcal{M}')\) by cutting out a disc whose boundary contains the unique marked point in \(\mathcal{M}'\) (hence \(\mathcal{M} = \mathcal{M}'\)). We cut such a disc as shown in Figure 14 on the left. In particular, the elements of \(\sigma\) (which are arcs in \((\Sigma', \mathcal{M}')\)) remain arcs in \((\Sigma, \mathcal{M})\). We complete \(\sigma\) to a triangulation \(\tau\) of \((\Sigma, \mathcal{M})\) by adding an arc \(k\) as shown in Figure 14. The quiver \(Q(\tau)\) is shown in Figure 14 on the right. Now a straightforward check shows that \(\tau\) is a gentle triangulation.
7.6. Proof of Theorem 7.8: Induction step.

Lemma 7.9. Suppose that a marked surface \((\Sigma, M)\) has a gentle triangulation \(\sigma\), and assume that a marked surface \((\Sigma', M')\) can be obtained from \((\Sigma, M)\) by one of the following operations:

(a) Adding a puncture;
(b) Adding a boundary component with exactly one marked point on it;
(c) Adding a marked point to a boundary component.

Then \((\Sigma', M')\) has a gentle triangulation.

Proof. (a) Let \(k\) be an arc of \(\sigma\). The arc \(k\) is part of exactly two triangles \(\triangle_1\) and \(\triangle_2\). (These triangles are not self-folded, since \(\sigma\) is a gentle triangulation.) Pick any point \(q\) lying on the arc \(k\) such that \(q\) is not one of the end points of \(k\). Declare \(q\) to be a new puncture, so that \((\Sigma', M') := (\Sigma, M \cup \{q\})\) is obtained from \((\Sigma, M)\) by adding a puncture. As illustrated in Figure 15, we draw two new arcs \(k_1\) and \(k_2\) contained in \(\triangle_1\) and \(\triangle_2\), respectively. The result is a triangulation \(\tau\) of \((\Sigma', M')\). Using the assumption that \(\sigma\) is a gentle triangulation, it is easy to verify that \(\tau\) is also a gentle triangulation. (The lower part of Figure 15 shows how \(Q(\tau)\) is obtained from \(Q(\sigma)\) by replacing the full subquiver of \(Q(\sigma)\) shown on the left by the quiver shown on the right.)

(b) Pick a triangle \(\triangle\) of \(\sigma\), and a topological open disc \(D \subseteq \triangle\) whose closure does not intersect any of the arcs in \(\sigma\). In this way we obtain a surface \(\Sigma' := \Sigma \setminus D\) that has one more boundary component than \(\Sigma\). Let \(q\) be a point on the new boundary component, and draw four arcs in \((\Sigma', M \cup \{q\})\) as shown in Figure 16. Using the fact that \(\sigma\) is a gentle triangulation, we get that \(\tau\) is also gentle.

(c) Let \(C\) be any boundary component of \(\Sigma\). Pick any point \(q\) on \(C \setminus M\). Then \((\Sigma', M') := (\Sigma, M \cup \{q\})\) is obtained from \((\Sigma, M)\) by adding a marked point to a boundary component of \(\Sigma\). Draw the unique arc in \((\Sigma', M')\) that joins \(q\) to a point in \(M\). This completes \(\sigma\) to a triangulation \(\tau\) of \((\Sigma', M')\). This is illustrated in Figure 17. (Note that some of the marked points \(q_1, q_2, q_3\) in Figure 17 might be identical.) Again, since \(\sigma\) is gentle, we see that \(\tau\) must be gentle as well. \(\square\)
To finish the proof of Theorem 7.8, observe that any marked surface \((\Sigma, M)\) different from the surfaces excluded in the assumptions of Theorem 7.8 can be obtained from one of the surfaces treated in Section 7.5 by repeatedly adding punctures, boundary components, and marked points to boundary components. Now Theorem 7.8 follows from Lemma 7.9.

7.7. A skewed-gentle triangulation for a sphere \((\Sigma, M)\) with \(|M| = 5\). Each sphere \((\Sigma, M)\) with \(|M| = 5\) has a skewed-gentle triangulation \(\tau\). Figure 18 illustrates the construction of such a \(\tau\) and its quiver \(Q(\tau)\).

7.8. Skewed-gentle triangulations for monogons, digons and triangles.

7.8.1. Monogons. Let \((\Sigma, M)\) be a monogon with \(t \geq 3\) punctures. For \(t \geq 4\), Figure 19 shows a skewed-gentle triangulation \(\tau\) of \((\Sigma, M)\) together with the quiver \(Q(\tau)\). (The marked point on the boundary of \(\Sigma\) is labeled by \(p\).) The case \(t = 3\) is dealt with in Figure 20.

7.8.2. Digons. Let \((\Sigma, M)\) be a digon with \(t + 1 \geq 2\) punctures. Figure 21 shows a skewed-gentle triangulation \(\tau\) of \((\Sigma, M)\) together with the quiver \(Q(\tau)\).

7.8.3. Triangles. Let \((\Sigma, M)\) be a triangle with \(t \geq 1\) punctures. Figure 22 shows a skewed-gentle triangulation \(\tau\) of \((\Sigma, M)\) together with the quiver \(Q(\tau)\).
Figure 18. A skewed-gentle triangulation for a sphere \( (\Sigma, M) \) with \( |M| = 5 \)

Figure 19. A skewed-gentle triangulation for a monogon \( (\Sigma, M) \) with \( t \geq 4 \) punctures.
7.9. **Skewed-gentle triangulations for punctured annuli** $(\Sigma, \mathcal{M})$ with $|\mathcal{M} \setminus \mathcal{P}| = 2$. Let $(\Sigma, \mathcal{M})$ be an annulus with $|\mathcal{M} \setminus \mathcal{P}| = 2$ and $|\mathcal{P}| = t - 1 \geq 1$ punctures. Figure 23 shows a skewed-gentle triangulation $\tau$ of $(\Sigma, \mathcal{M})$ together with the quiver $Q(\tau)$.

7.10. **Skewed-gentle triangulations for a punctured torus** $(\Sigma, \mathcal{M})$ with $|\mathcal{M} \setminus \mathcal{P}| = 1$. Let $(\Sigma, \mathcal{M})$ be a torus with $|\mathcal{M} \setminus \mathcal{P}| = 1$ and $|\mathcal{P}| = t \geq 1$ punctures. Figure 24 shows a skewed-gentle triangulation $\tau$ of $(\Sigma, \mathcal{M})$ together with the quiver $Q(\tau)$. 
7.11. **Triangulations of unpunctured surfaces.** Let $Q$ be a 2-acyclic quiver. A potential $S$ on $Q$ is called a 3-cycle potential if $S = w_1 + \cdots + w_t$, where $w_1, \ldots, w_t$ is a complete set of representatives of rotation equivalence classes of 3-cycles in $Q$. Note that for $Q = Q(\tau)$ and $\tau$ a gentle triangulation of some marked surface $(\Sigma, M)$, the rotation equivalence classes of 3-cycles in $Q$ correspond bijectively to the interior triangles of $\tau$.

Combining Proposition 7.13 with [ABCP, Theorem 2.7] yields the following.

**Proposition 7.10.** Let $(\Sigma, M)$ be an unpunctured marked surface, and let $\tau$ be a triangulation of $(\Sigma, M)$ such that $Q(\tau)$ has no double arrows. Let $S$ be a 3-cycle potential on $Q(\tau)$. Then the following hold:

1. $\tau$ is a gentle triangulation;
2. Each cycle $\nu$ of length at least 4 in $Q(\tau)$ is rotation equivalent to a cycle of the form $c \partial_\alpha(S)$ for some arrow $\alpha$ appearing in exactly one summand of $S$ and some path $c$ of length at least 2 in $Q(\tau)$.

The condition that an arrow $\alpha$ appears in exactly one summand of a potential $S$ on a quiver $Q$ can be rephrased by saying that the cyclic derivative $\partial_\alpha(S)$ is a scalar multiple of a single path in $Q$.

7.12. **Skewed-gentle triangulations for surfaces with non-empty boundary.** The following result will be needed in Section 8 for the classification of non-degenerate potentials for surfaces with non-empty boundary.
Theorem 7.11. Let $(\Sigma, M)$ be a marked surface with non-empty boundary. Assume that $(\Sigma, M)$ is not a monogon, and not a torus with $|M| = 1$. Then there exist a triangulation $\tau$ of $(\Sigma, M)$, and a 3-cycle potential $S \in \mathbb{C}\langle \langle Q(\tau) \rangle \rangle$, such that the following hold:

1. $\tau$ is a skewed-gentle triangulation;
2. Each cycle $w$ of length at least 4 in $Q(\tau)$ is rotation equivalent to a cycle of the form $c\partial_{\alpha}(S)$ for some arrow $\alpha$ appearing in exactly one summand of $S$ and some path $c$ of length at least 2 in $Q(\tau)$.

Proof. For $(\Sigma, M)$ a digon or a triangle, the triangulations shown in Figures 21 and 22, respectively, satisfy the properties (sg) and (u). If $(\Sigma, M)$ is an annulus with $|M \setminus P| = 2$, then the triangulation in Figure 23 satisfies (sg) and (u).

Now assume that $(\Sigma, M)$ is not a digon, triangle or annulus with $|M \setminus P| = 2$. Let $(\Sigma, M_0)$ be the marked surface obtained from $(\Sigma, M)$ by deleting the punctures, i.e. $M_0 := M \setminus P$. Let $\tau_0$ be any triangulation of $(\Sigma, M_0)$ such that $Q(\tau_0)$ has no double arrows. (Such a triangulation exists by
Figure 24. A skewed-gentle triangulation for a punctured torus \((\Sigma, M)\) with \(|M \setminus \mathcal{P}| = 1\).

Theorem 7.8. By Proposition 7.10 we know that \(\tau_0\) is a gentle triangulation satisfying condition (u). In particular, \(\tau_0\) satisfies (sg).

Since \(\partial \Sigma \neq \emptyset\), the triangulation \(\tau_0\) must have a non-interior triangle, say \(\triangle_0\). At least one side of the triangle \(\triangle_0\) belongs to the boundary. Let \(p_1\) and \(p_2\) be its (possibly identical) end points. Put \(t\) punctures \(q_1, \ldots, q_t\) inside \(\triangle_0\). As illustrated in Figure 24, we draw \(t\) arcs \(b_1, \ldots, b_t\) from \(p_1\) to \(p_2\), and for each \(1 \leq i \leq t\) an arc \(c_i\) from \(p_1\) to \(q_i\), and \(t\) arcs \(d_i\) from \(p_1\) to itself, such that \(c_i\) and \(d_i\) form a self-folded triangle for each \(i\). Together with the arcs form the triangulation \(\tau_0\) of \((\Sigma, M_0)\), we obtain a triangulation \(\tau_t\) of the marked surface \((\Sigma, M_t)\) where \(M_t := M_0 \cup \{q_1, \ldots, q_t\}\).

Let \(C_t\) be the quiver shown in the lower part of Figure 25. The side \(a_3\) of \(\triangle_0\) belongs to the boundary of \((\Sigma, M_t)\), and at most one of the sides \(a_1\) or \(a_2\) might also be part of the boundary. We obtain \(Q(\tau_t)\) from the disjoint union of the quivers \(Q(\tau_0)\) and \(C_t\) by adding an arrow \(a_1 \to b_1\), provided \(a_1\) does not belong to the boundary, and an arrow \(b_1 \to a_2\), provided \(a_2\) is not part of the boundary. Now, using the fact that \(\tau_0\) is a gentle triangulation satisfying (u), one easily checks that \(\tau_t\) is a skewed-gentle triangulation also satisfying (u). To be more precise, given a cycle \(w\) of length at least 4 in \(Q(\tau_t)\), one can always find a summand \(\alpha \beta \gamma\) of any given 3-cycles potential \(S\) on \(Q(\tau_t)\) such that one arrow \(\eta\) of \(\alpha \beta \gamma\) appears only once in \(S\) and \(w' = c \partial_\eta(S)\) for some path \(c\) and some rotation \(w'\) of \(w\). This finishes the proof. \(\square\)
It is easy to see that for the triangulation $\tau_t$ constructed in the proof of Theorem 7.11 the Jacobian algebra $P(Q(\tau_t), S(\tau_t))$ is a skewed-gentle algebra. (This follows directly by combining [ABCP, Theorem 2.7] and the example in Section 6.5.2, compare also the recent preprint [QZ].)

7.13. Triangulations without double arrows. As a straightforward corollary of Theorem 7.8 we get the following result.

**Proposition 7.12.** Let $Q$ be a 2-acyclic quiver of finite mutation type which is not equal to one of the quivers $T_1$, $T_2$ or $K_m$ with $m \geq 2$ (see Figure 26). Then $Q$ is mutation equivalent to a quiver without double arrows.

We were informed by Sefi Ladkani that he obtained a proof of Proposition 7.12 independently. The quivers $T_1$, $T_2$ and $K_m$ in the statement of Proposition 7.12 are displayed in Figure 26. Recall that the mutation equivalence class of each of these three quivers contains just one quiver up to isomorphism.
Figure 26. Quivers of finite mutation type with non-removable double arrows.

The next result is just a re-statement of Proposition 7.12

Proposition 7.13. Let \((\Sigma, M)\) be a marked surface which is not equal to a torus \((\Sigma, M)\) with \(|M| = 1\), or to an annulus with \(|M| = 2\). Then \((\Sigma, M)\) has a triangulation \(\tau\) such that \(Q(\tau)\) has no double arrows.

8. Classification of non-degenerate potentials: Regular cases

8.1. Construction of a non-degenerate potentials for quivers associated to triangulations of surfaces. Let \((\Sigma, M)\) be a marked surface. As before, let \(\mathbb{P}\) be its set of punctures.

For technical reasons, we introduce some quivers that are obtained from signed adjacency quivers by adding some 2-cycles in specific situations. Let \(\tau\) be a triangulation of \((\Sigma, M)\). For each puncture \(p\) incident to exactly two arcs of \(\tau\), we add to \(Q(\tau)\) a 2-cycle that connects those arcs and call the resulting quiver the unreduced signed adjacency quiver \(\hat{Q}(\tau)\). It is clear that \(Q(\tau)\) can always be obtained from \(\hat{Q}(\tau)\) by deleting 2-cycles.

Now let \(x = (x_p)_{p \in \mathbb{P}}\) be a choice of non-zero scalars \(x_p \in \mathbb{C}\). Following \([LF2]\) we recall the definition of a potential \(S(\tau, x)\) on \(Q(\tau)\).

(i) Each interior non-self-folded triangle \(\triangle\) of \(\tau\) gives rise to a unique (up to rotation equivalence) oriented 3-cycle \(\alpha^\triangle \beta^\triangle \gamma^\triangle\) in \(\hat{Q}(\tau)\). Define 
\[
\hat{S}^\triangle(\tau, x) := \alpha^\triangle \beta^\triangle \gamma^\triangle.
\]
(ii) If the interior non-self-folded triangle \(\triangle\) with sides \(j, k\) and \(l\), is adjacent to two self-folded triangles as displayed in Figure 27, define (up to rotation equivalence) 
\[
\hat{U}^\triangle(\tau, x) := x_p^{-1} x_q^{-1} \alpha \beta \gamma,
\]
where \(p\) and \(q\) are the punctures enclosed in the self-folded triangles adjacent to \(\triangle\). Otherwise, if \(\triangle\) is adjacent to less than two self-folded triangles, define \(\hat{U}^\triangle(\tau, x) := 0\).

Figure 27. Definition of \(\hat{U}^\triangle(\tau, x)\).
(iii) If a puncture $p$ is adjacent to exactly one arc $i$ of $\tau$, then $i$ is the folded side of a self-folded triangle of $\tau$ and around $i$ we have the configuration shown in Figure 28. In case both $k$ and $l$ are indeed arcs of $\tau$ (and not part of the boundary of $\Sigma$), we define (up to rotation equivalence)

$$\hat{S}^p(\tau, x) := -x_p^{-1} \alpha \beta \gamma.$$ 

Otherwise, if either $k$ or $l$ is a boundary segment, we define $\hat{S}^p(\tau, x) := 0.$

(iv) If a puncture $p$ is adjacent to more than one arc, delete all the loops incident to $p$ that enclose self-folded triangles. The arrows between the remaining arcs adjacent to $p$ give rise to a unique (up to rotation equivalence) cycle $\alpha^p_1 \ldots \alpha^p_{d_p}$ (without repeated arrows) around $p$ in the counterclockwise orientation (defined by the orientation of $\Sigma$). Note that some of the remaining arcs might still be loops. We define (up to rotation equivalence)

$$\hat{S}^p(\tau, x) := x_p \alpha^p_1 \ldots \alpha^p_{d_p}.$$ 

The definition of $\alpha^p_1 \ldots \alpha^p_{d_p}$ is illustrated in Figure 29.

(v) The un reduced potential $\hat{S}(\tau, x) \in \mathbb{C}(\langle \hat{Q}(\tau) \rangle)$ associated to $\tau$ and $x$ is defined as

$$\hat{S}(\tau, x) := \sum_\triangle \left( \hat{S}^\triangle(\tau) + \hat{U}^\triangle(\tau, x) \right) + \sum_{p \in P} \left( \hat{S}^p(\tau, x) \right),$$

where the first sum runs over all interior non-self-folded triangles of $\tau$.

(vi) We define $(Q(\tau), \hat{S}(\tau, x))$ to be the reduced part of $(\hat{Q}(\tau), \hat{S}(\tau, x)).$

(vii) Define

$$S(\tau) := S(\tau, x)$$

provided $x_p = 1$ for all $p \in \mathbb{P}$.

If $\tau$ is a triangulation such that every puncture of $(\Sigma, M)$ is incident to at least three arcs of $\tau$, then $\hat{Q}(\tau)$ is already 2-acyclic, hence equal to $Q(\tau)$, and therefore $\hat{S}(\tau, x) = S(\tau, x)$. Moreover,
the triangulation $\tau$ does not have self-folded triangles. This implies that $S(\tau, x)$ takes a simpler form, namely we have

$$S(\tau, x) = \sum_\Delta (\alpha_\Delta \beta_\Delta \gamma_\Delta) + \sum_{p \in \mathbb{P}} (x_p \alpha_1^p \cdots \alpha_d^p).$$

The only situation where one needs to apply reduction to $(\hat{Q}(\tau), \hat{S}(\tau, x))$ in order to obtain $S(\tau, x)$ is when there is some puncture incident to exactly two arcs of $\tau$. The reduction is done explicitly in [LF2, Section 3].

For a tagged triangulation $\tau$, the definition of $S(\tau, x)$ can be found in [LF3].

**Theorem 8.1 ([LF3]).** Let $(\Sigma, \mathbb{M})$ be a marked surface which is not equal to a sphere with $|\mathbb{M}| = 4, 5$. If $\tau$ and $\sigma$ are tagged triangulations of $(\Sigma, \mathbb{M})$ related by the flip of a tagged arc $k$, then $\mu_k((Q(\tau), S(\tau, x)))$ is right equivalent to $(Q(\sigma), S(\sigma, x))$.

Theorem 8.1 has the following immediate consequence.

**Corollary 8.2 ([LF3]).** Let $(\Sigma, \mathbb{M})$ be a marked surface which is not a sphere with $|\mathbb{M}| = 4, 5$. Then for any tagged triangulation $\tau$ of $(\Sigma, \mathbb{M})$, the potentials $S(\tau, x)$ are non-degenerate.

In the case of a sphere with five punctures, only a weaker version of Theorem 8.1 has been proved. Namely, ideal triangulations related by a flip have QPs related by the corresponding QP-mutation, see [LF2], but the tagged version of this statement has not yet been proved for this case.

**Proposition 8.3 ([LF3]).** Suppose $(\Sigma, \mathbb{M})$ is a marked surface with non-empty boundary. Then for any two choices $x = (x_p)_{p \in \mathbb{P}}$ and $y = (y_p)_{p \in \mathbb{P}}$ of non-zero scalars $x_p, y_p \in \mathbb{C}$, the QPs $(Q(\tau), S(\tau, x))$ and $(Q(\tau), S(\tau, y))$ are right equivalent.

### 8.2. Potentials for surfaces with empty boundary

In this section we prove the following classification theorem.

**Theorem 8.4.** Let $(\Sigma, \mathbb{M})$ be a marked surface with empty boundary. Assume that

$$|\mathbb{M}| \geq \begin{cases} 6 & \text{if } \Sigma \text{ is a sphere}, \\ 3 & \text{otherwise}. \end{cases}$$

Then for any tagged triangulation $\tau$ of $(\Sigma, \mathbb{M})$ the quiver $Q(\tau)$ admits only one non-degenerate potential up to weak right equivalence.

We conjecture that Theorem 8.4 holds as well for all marked surfaces $(\Sigma, \mathbb{M})$ with genus $g(\Sigma) \geq 1$ and $|\mathbb{M}| = 2$. We will see in Section 9 that for surfaces $(\Sigma, \mathbb{M})$ with empty boundary and $|\mathbb{M}| = 1$ there are at least two non-degenerate potentials up to weak right equivalence.

We now prove a series of lemmas leading to a proof of Theorem 8.4.

**Lemma 8.5.** Suppose $(\Sigma, \mathbb{M})$ is a marked surface with empty boundary different from a sphere with $|\mathbb{M}| = 4, 5$. Let $x = (x_p)_{p \in \mathbb{P}}$ and $y = (y_p)_{p \in \mathbb{P}}$ be arbitrary choices of non-zero scalars. Then for every tagged triangulation $\sigma$ of $(\Sigma, \mathbb{M})$, the QPs $(Q(\sigma), S(\sigma, x))$ and $(Q(\sigma), S(\sigma, y))$ are weakly right equivalent.

**Proof.** We start with a couple of general observations. Let $Q$ be any 2-acyclic quiver (not necessarily arising from a triangulation). Then the following hold:

(1) If $S_1, S_2, S_3$ are potentials on $Q$ such that $S_1$ is weakly right equivalent to $S_2$, and $S_2$ is weakly right equivalent to $S_3$, then $S_1$ is weakly right equivalent to $S_3$;
(2) If \(S_1\) and \(S_2\) are weakly right equivalent potentials on \(Q\), then \(\mu_k(Q, S_1)\) and \(\mu_k(Q, S_2)\) are weakly right equivalent for any vertex \(k \in Q_0\). This follows from the following obvious facts:

- for any non-zero scalar \(\lambda\), the potential \([\lambda S_2] + \Delta_k(Q) \in \mathbb{C}[\langle \tilde{\mu}_k(Q) \rangle]\) is right equivalent to \(\lambda([S_2] + \Delta_k(Q)) = \lambda \tilde{\mu}_k(S_2)\), and hence, \([S_1] + \Delta_k(Q) = \mu_k(S_1)\) is right equivalent to \(\lambda \tilde{\mu}_k(S_2)\);
- if \(\varphi: \mathbb{C}[\langle \tilde{\mu}_k(Q) \rangle] \to \mathbb{C}[\langle \tilde{\mu}_k(Q) \rangle]\) is a right equivalence between \((\tilde{\mu}_k(Q), \tilde{\mu}_k(S_2))\) and \((\tilde{\mu}_k(Q)_{\text{red}}, \tilde{\mu}_k(S_2)_{\text{red}}) \oplus (\tilde{\mu}_k(Q)_{\text{triv}}, \tilde{\mu}_k(S_2)_{\text{triv}})\), then \(\varphi\) is also a right equivalence between \((\tilde{\mu}_k(Q), \lambda \tilde{\mu}_k(S_2))\) and \((\tilde{\mu}_k(Q)_{\text{red}}, \lambda \tilde{\mu}_k(S_2)_{\text{red}}) \oplus (\tilde{\mu}_k(Q)_{\text{triv}}, \lambda \tilde{\mu}_k(S_2)_{\text{triv}})\);
- for any non-zero scalar \(\lambda\), the QP \((\tilde{\mu}_k(Q)_{\text{red}}, \lambda \tilde{\mu}_k(S_2)_{\text{red}})\) is reduced and the QP \((\tilde{\mu}_k(Q)_{\text{triv}}, \lambda \tilde{\mu}_k(S_2)_{\text{triv}})\) is trivial.

These facts, together with Theorem \ref{thm:main}, imply that, in order to prove Lemma \ref{lem:main} it suffices to show the mere existence of a triangulation \(\tau\) such that \((Q(\tau), S(\tau, 1))\) is weakly right equivalent to a scalar multiple of \((Q(\tau), S(\tau, x))\).

Suppose \(\tau\) is a triangulation such that every puncture \(p \in P\) has valency \(\text{val}_\tau(p) \geq 3\). It follows that there is a puncture \(p\) with \(\text{val}_\tau(p) \geq 4\). For the existence of such a triangulation \(\tau\) we refer to \cite{Ladkan}. Now choose any puncture \(q\). A minor modification of the proof of \cite[Proposition 11.2]{Ladkan} proves that \(S(\tau, x)\) is right equivalent to \(S(\tau, w)\), where \(w = (w_p)_{p \in P}\) is defined by \(w_q = \prod_{p \in P} w_p\), and \(w_p = 1\) for \(p \neq q\).

Let \(a\) be the number of arrows of \(Q(\tau)\), and \(r\) be the number of punctures of \((\Sigma, M)\). Note that the integer \(a - 3r\) is positive (this follows from the fact that all punctures have valency at least 3, and at least one puncture has valency at least 4). Let \(\xi \in \mathbb{C}\) be an \((a - 3r)\)-root of \(w_q\). For each puncture \(p\), let \(z_p = \xi^{a_p - 3}\), where \(a_p\) is the number of arrows in the cycle that surrounds \(p\). Note that \(\prod_{p \in P} z_p = w_q\). As in the previous paragraph, this implies that \(S(\tau, z)\) is right equivalent to \(S(\tau, w)\).

We see that, in order to prove that \(S(\tau, 1)\) is right equivalent to a scalar multiple of \(S(\tau, x)\), it is enough to show that \(S(\tau, 1)\) is right equivalent to a scalar multiple of \(S(\tau, z)\). But this is easy, let \(\varphi\) be the \(R\)-algebra automorphism of \(R(\langle Q(\tau) \rangle)\) given by \(\varphi: \beta \mapsto \xi \beta\) for every arrow \(\beta\) of \(Q\). Then \(\varphi\) is a right equivalence between \((S(\tau, 1), \xi^3 S(\tau, z))\).

Let \((\Sigma, M)\) be a surface with empty boundary, and let \(\tau\) be a triangulation of \((\Sigma, M)\) such that every puncture has valency at least 3. Let \(Q = Q(\tau)\) be the quiver of \(\tau\). Following Ladkan \cite{Ladkan} we define two maps \(f, g: Q_1 \to Q_1\) as follows. Each triangle \(\triangle\) of \(\tau\) gives rise to an oriented 3-cycle \(\alpha^\triangle \beta^\triangle \gamma^\triangle\) on \(Q(\tau)\). We set \(f\alpha^\triangle = \gamma^\triangle\), \(f\beta^\triangle = \alpha^\triangle\) and \(f\gamma^\triangle = \beta\). Now, given any arrow \(\alpha\) of \(Q(\tau)\), the quiver \(Q(\tau)\) has exactly two arrows starting at the terminal vertex of \(\alpha\). One of these two arrows is \(f\alpha\). We define \(g\alpha\) to be the other arrow.

Note that the map \(f\) (resp. \(g\)) splits the arrow set of \(Q(\tau)\) into \(f\)-orbits (resp. \(g\)-orbits). The set of \(f\)-orbits is in one-to-one correspondence with the set of triangles of \(\tau\). All \(f\)-orbits have exactly three elements. The set of \(g\)-orbits is in one-to-one correspondence with the set of punctures of \((\Sigma, M)\). For every arrow \(\alpha\) of \(Q(\tau)\), we denote by \(m_\alpha\) the size of the \(g\)-orbit of \(\alpha\). Note that \((g^{m_\alpha - 1}\alpha)(g^{m_\alpha - 2}\alpha) \ldots (g\alpha)\alpha\) is a cycle surrounding the puncture corresponding to the \(g\)-orbit of \(\alpha\).

For Lemmas \ref{lem:orbit1} \ref{lem:orbit2} \ref{lem:orbit3} and \ref{lem:orbit4} let \(\{\alpha_1, \ldots, \alpha_T\}\) (resp. \(\{\beta_1, \ldots, \beta_{|P|}\}\)) be a system of representatives for the action of the map \(f\) (resp. \(g\)) on the set of arrows of \(Q(\tau)\). Thus, each \(f\)-orbit (resp. each \(g\)-orbit) has exactly one representative in the set \(\{\alpha_1, \ldots, \alpha_T\}\) (resp. \(\{\beta_1, \ldots, \beta_{|P|}\}\)).
We also use the following notations
\[ A := \sum_{k=1}^{\ell} (f^2 \alpha_k)(f \alpha_k)(\alpha_k) \quad \text{and} \quad B := \sum_{t=1}^{\|\|} x_t \left( (g^{m_{\beta_t}}^{-1} \beta_t) (g^{m_{\beta_t}}^{-2} \beta_t) \ldots (g \beta_t)(\beta_t) \right). \]

Note that, with these notations, we have \( S(\tau, x) = A + B \).

**Lemma 8.6.** Let \((\Sigma, M)\) be a marked surface with empty boundary. Suppose \( \tau \) is a tagged triangulation of \((\Sigma, M)\) such that the following hold:

\[
\begin{align*}
(8.1) & \quad \text{Every puncture has valency at least four;} \\
(8.2) & \quad \text{None of the arcs in} \ \tau \ \text{is a loop;} \\
(8.3) & \quad Q(\tau) \text{ has no double arrows.}
\end{align*}
\]

Then every cycle in \( Q(\tau) \) that is rotationally disjoint from \( S(\tau, x) \), is rotationally equivalent to a cycle of one of the following types:

1. \((f^2 \alpha) (f \alpha) \alpha) \) for some \( n > 1 \);
2. \((g^{m_{\beta}}^{-1}) (g^{m_{\beta}}^{-2}) \ldots (g \beta)(\beta) \) for some \( n > 1 \);
3. \((f^2 \beta)(f \beta) \lambda(f^2 \alpha)(f \alpha) \rho \) for some arrows \( \alpha, \beta, \lambda, \rho \), such that \( \lambda = (g^{-1} f \beta) \lambda' \) or \( \rho = \rho'(g f^2 \beta) \).

**Proof.** Let \( \xi = \alpha_1 \ldots \alpha_r \) be any cycle on \( Q(\tau) \). Denote \( \alpha_{r+1} = \alpha_1 \), and notice that for every \( \ell = 1, \ldots, r \), we have either \( \alpha_{\ell} = f \alpha_{\ell+1} \) or \( \alpha_{\ell} = g \alpha_{\ell+1} \). Let \( f g \xi \) be the length-\( r \) sequence of \( f \) and \( g \) paths that has an \( f \) at the \( \ell \)-th place if \( \alpha_{\ell} = f \alpha_{\ell+1} \) and a \( g \) otherwise.

If \( f g \xi \) consists only of \( f \)s, then \( \xi \) is rotationally equivalent to \((f^2 \alpha)(f \alpha) \alpha) \) for some arrow \( \alpha \) and some \( n \geq 1 \). If \( f g \xi \) consists only of \( g \)s, then \( \xi \) is rotationally equivalent to \((g^{m_{\beta}}^{-1}) (g^{m_{\beta}}^{-2}) \ldots (g \beta)(\beta) \) for some arrow \( \beta \) and some \( n \geq 1 \). Therefore, if \( \xi \) is rotationally disjoint from \( S(\tau, x) \) and \( f g \xi \) involves only \( f \)s or only \( g \)s, then \( \xi \) is rotationally equivalent to a cycle of type (I) or (II).

Suppose that at least one \( f \) and at least one \( g \) appear in \( f g \xi \). Rotating \( \xi \) if necessary, we can assume that \( f g \xi \) starts with an \( f \) followed by a \( g \), i.e., \( f g \xi = (f, g, \ldots) \). This means that \( \alpha_1 = f \alpha_2 \) and \( \alpha_2 = g \alpha_3 \). In particular, the arrows \( \alpha_1 \) and \( \alpha_2 \) are contained in a common triangle \( \Delta \) of \( \tau \). Since no arc in \( \tau \) is a loop, the vertices of \( \Delta \) are three different punctures. Hence the puncture associated to \( \alpha_2 \) is not incident to the arc in \( \tau \) which is opposite to \( \alpha_2 \) in \( \Delta \).

Consider the path \( \alpha_3 \ldots \alpha_r \). It starts precisely at the arc in \( \tau \) which is opposite to \( \alpha_2 \) in \( \Delta \), and it ends at an arc incident to the puncture associated to \( \alpha_2 \). Since all elements of \( \{g^n \alpha_3 \mid n \in \mathbb{Z}\} \) are arrows connecting arcs that are incident to the puncture associated to \( \alpha_2 \), we deduce that there is some \( \ell \in \{3, \ldots, r-1\} \) such that \( \alpha_\ell = f \alpha_{\ell+1} \). This means that \( \xi \) is rotationally equivalent to a cycle of type (III).

Lemma 8.6 is proved.

It is easy to check that a triangulation \( \tau \) satisfying the assumptions \( (8.1), (8.2) \) and \( (8.3) \) in Lemma 8.6 is a gentle triangulation, and that \( Q(\tau) = \text{glue}(B_1, \ldots, B_t; g) \) with \( B_1, \ldots, B_t \) blocks of type II.

**Lemma 8.7.** Let \((\Sigma, M)\) be a marked surface with empty boundary, and let \( \tau \) be a tagged triangulation of \((\Sigma, M)\) satisfying \( (8.1), (8.2) \) and \( (8.3) \). If \( S' \in \mathbb{C} \langle \langle Q(\tau) \rangle \rangle \) is a potential rotationally disjoint from \( S(\tau, x) \), then \( Q(\tau), S(\tau, x) + S' \) is right equivalent to \( (Q(\tau), S(\tau, x) + W) \) for some potential \( W \in \mathbb{C} \langle \langle Q(\tau) \rangle \rangle \) involving only cycles of types I and III.
Proof. By Lemma 8.6 up to cyclical equivalence we can write \( S' = S'_I + S'_{II} + S'_{III} \), with
\[
S'_I = \sum_{k=1}^{\ell} \left( (f^2 \alpha_k)(f \alpha_k) \left( \sum_{n=1}^{\infty} y_{k,n} \alpha_k ((f^2 \alpha_k)(f \alpha_k) \alpha_k)^n \right) \right),
\]
\[
S'_{II} = \sum_{t=1}^{\mid \mathbb{P} \mid} \left( (g^m \beta_t - 1 \beta_t)(g^m \beta_t - 2 \beta_t) \ldots (g \beta_t) \left( \sum_{n \geq 1} z_{t,n} \beta_t ((g^m \beta_t - 1 \beta_t)(g^m \beta_t - 2 \beta_t) \ldots (g \beta_t) \beta_t)^n \right) \right),
\]
\[
S'_{III} = \sum_{\alpha \in Q_1(\tau)} (f^2 \alpha)(f \alpha) \nu_\alpha,
\]
such that for each \( \alpha \in Q_1(\tau) \), \( \nu_\alpha \) is a possibly infinite linear combination of paths of the form \( \lambda(f^2 \beta)(f \beta) \rho \) as in the above description of cycles of type III. Define an \( R \)-algebra automorphism \( \psi \) of \( \mathbb{C} \langle\langle Q(\tau) \rangle\rangle \) according to the rule
\[
\beta_t \mapsto \beta_t - \sum_{n \geq 1} x_t^{-1} z_{t,n} \beta_t ((g^m \beta_t - 1 \beta_t)(g^m \beta_t - 2 \beta_t) \ldots (g \beta_t) \beta_t)^n
\]
for \( 1 \leq t \leq |\mathbb{P}| \). It is clear that \( \psi \) is unitriangular of positive depth. Moreover, we have \( \text{depth}(\psi) \geq \text{short}(S'_{II}) - r \), where \( r := \max\{\text{val}_r(p) \mid p \in \mathbb{P}\} \).

Direct computation yields
\[
\psi(S(\tau, x) + S') \sim_{\text{cyc}} S(\tau, x) + \psi(A) - A + \psi(S'_I) + \psi(S'_{II}) + S'_{III}.
\]
Note that
- \( \psi(A) - A \) involves only cycles of type III;
- \( \psi(S'_I) \) involves only cycles of type I or III;
- \( \psi(S'_{II}) - S'_{III} \) involves only cycles of type II, and \( \text{short}(\psi(S'_{II}) - S'_{III}) > \text{short}(S'_{III}) \) since \( \psi \) has positive depth;
- \( \psi(S'_{III}) \) involves only cycles of type III.

Summarizing, given \( S' = S'_I + S'_{II} + S'_{III} \), we can guarantee the existence of an \( R \)-algebra automorphism \( \psi \) of \( \mathbb{C} \langle\langle Q(\tau) \rangle\rangle \) such that
- \( \psi \) is unitriangular and \( \text{depth}(\psi) \geq \text{short}(S'_{III}) - r \);
- \( \psi(S(\tau, x) + S') \) is cyclically equivalent to \( S(\tau, x) + S'' \) for some potential \( S'' \) with the property that, when written in the form \( S'' = S''_I + S''_{II} + S''_{III} \), it satisfies \( \text{short}(S''_{III}) > \text{short}(S'_{III}) \).

Therefore, the lemma follows from a limit process. \( \square \)

**Lemma 8.8.** Let \( (\Sigma, M) \) be a marked surface with empty boundary, and let \( \tau \) be a tagged triangulation of \( (\Sigma, M) \) satisfying (S.1), (S.2) and (S.3). If \( W \in \mathbb{C} \langle\langle Q(\tau) \rangle\rangle \) is a potential involving only cycles of types I and III, then \( (Q(\tau), S(\tau, x) + W) \) is right equivalent to \( (Q(\tau), S(\tau, x) + U) \) for some potential \( U \) involving only cycles of type III.

Proof. By Lemma 8.6 up to cyclical equivalence we can write \( W = W_I + W_{III} \) with
\[
W_I = \sum_{k=1}^{\ell} \left( (f^2 \alpha_k)(f \alpha_k) \left( \sum_{n=1}^{\infty} y_{k,n} \alpha_k ((f^2 \alpha_k)(f \alpha_k) \alpha_k)^n \right) \right),
\]
\[
W_{III} = \sum_{\alpha \in Q_1(\tau)} (f^2 \alpha)(f \alpha) \nu_\alpha,
\]
such that for each \( \alpha \in Q_1(\tau) \), \( \nu_\alpha \) is a possibly infinite linear combination of paths of the form \( \lambda(f^2\beta)(f\beta)\rho \) as in the above description of cycles of type III. Define an \( R \)-algebra automorphism \( \varphi \) of \( \mathbb{C}\langle\langle Q(\tau)\rangle\rangle \) according to the rule

\[
\alpha_k \mapsto \alpha_k - \sum_{n=1}^{\infty} y_{k,n} \alpha_k \left((f^2\alpha_k)(f\alpha_k)\alpha_k\right)^n
\]

for \( 1 \leq k \leq \ell \). It is clear that \( \varphi \) is unitriangular with \( \text{depth}(\varphi) = \text{short}(W_1) - 3 > 0 \).

Direct computation yields

\[
\varphi(S(\tau, x) + W) \sim_{\text{cyc}} S(\tau, x) + \varphi(B) - B + \varphi(W_1) - W_1 + \varphi(W_{III}).
\]

Note that

- \( \varphi(B) - B \) involves only cycles of type III;
- \( \varphi(W_1) - W_1 \) involves only cycles of type I, and \( \text{short}(\varphi(W_1) - W_1) > \text{short}(W_1) \) since \( \varphi \) has positive depth;
- \( \varphi(W_{III}) \) involves only cycles of type III.

Summarizing, given \( W = W_I + W_{III} \), we can guarantee the existence of an \( R \)-algebra automorphism \( \varphi \) of \( \mathbb{C}\langle\langle Q(\tau)\rangle\rangle \) such that the following hold:

- \( \varphi \) is unital and \( \text{depth}(\varphi) = \text{short}(W_1) - 3 \);
- \( \varphi(S(\tau, x) + W) \) is cyclically equivalent to \( S(\tau, x) + W' \) for some potential \( W' \) with the property that, when written in the form \( W' = W_I' + W_{II}' + W_{III}' \), it satisfies \( W_{II}' = 0 \) and \( \text{short}(W_1') > \text{short}(W_1) \).

Therefore, the lemma follows from a limit process. \( \square \)

**Lemma 8.9.** Let \((\Sigma, \mathbb{M})\) be a surface with empty boundary, and let \( \tau \) be a (tagged) triangulation of \((\Sigma, \mathbb{M})\) satisfying \((\mathbf{S.1})\), \((\mathbf{S.2})\) and \((\mathbf{S.3})\). If \( U \in \mathbb{C}\langle\langle Q(\tau)\rangle\rangle \) is a potential involving only cycles of type III, then \((Q(\tau), S(\tau, x) + U)\) is right equivalent to \((Q(\tau), S(\tau, x))\).

**Proof.** By Lemma \((\mathbf{S.6})\) the following holds:

\( (\ast) \) Every cycle appearing in \( U \) is rotationally equivalent to a cycle of the form

\[
(f^2\beta)(f\beta)\lambda(f^2\alpha)(f\alpha)\rho
\]

for some arrows \( \alpha, \beta \), and some paths \( \lambda, \rho \) such that \( \lambda = (g^{-1}f\beta)\lambda' \) or \( \rho = \rho'(gf^2\beta) \).

For each cycle appearing in \( U \) we choose exactly one such expression \( (f^2\beta)(f\beta)\lambda(f^2\alpha)(f\alpha)\rho \). Once this choice has been made, we see that

\[
U \sim_{\text{cyc}} \sum_{\alpha \in Q_1(\tau)} (f^2\alpha)(f\alpha)\nu_\alpha,
\]

where, for each \( \alpha \in Q_1(\tau) \), \( \nu_\alpha \) is a possibly infinite linear combination of paths of the form \( \rho(f^2\beta)(f\beta)\lambda \) as above. Define an \( R \)-algebra automorphism \( \eta \) of \( \mathbb{C}\langle\langle Q(\tau)\rangle\rangle \) according to the rule

\[
\alpha \mapsto \alpha - \nu_\alpha, \quad \alpha \in Q_1(\tau).
\]

It is clear that \( \eta \) is unital with \( \text{depth}(\eta) = \text{short}(U) - 3 > 0 \).

The key observation here is that \( \eta(A) \sim_{\text{cyc}} A - U + U' \), with \( U' \) a potential involving only cycles of type III, and such that \( \text{short}(U') > \text{short}(U) \). To see this, note that

\[
\sum_{\alpha \in Q_1(\tau)} (f^2\alpha)(f\alpha)\nu_\alpha \sim_{\text{cyc}} \sum_{k=1}^{\ell} ((f^2\alpha_k)(f\alpha_k)(\nu_{\alpha_k}) + (f^2\alpha_k)(\nu_{f\alpha_k})(\alpha_k) + (\nu_{f^2\alpha_k})(f\alpha_k)(\alpha_k))
\]
and
\[
\eta(A) = \sum_{k=1}^{\ell} (f^2\alpha_k - \nu f^2\alpha_k)(f\alpha_k - \nu f\alpha_k)(\alpha_k - \nu \alpha_k)
\]
\[
= \sum_{k=1}^{\ell} ((f^2\alpha_k)(f\alpha_k)(\alpha_k))
\]
\[
- \sum_{k=1}^{\ell} ((f^2\alpha_k)(\nu \alpha_k) + (f^2\alpha_k)(\nu f\alpha_k)(\alpha_k) + (\nu f^2\alpha_k)(f\alpha_k)(\alpha_k))
\]
\[
+ \sum_{k=1}^{\ell} ((f^2\alpha_k)(\nu f\alpha_k)(\nu \alpha_k) + (\nu f^2\alpha_k)(f\alpha_k)(\nu \alpha_k) + (\nu f^2\alpha_k)(\nu f\alpha_k)(\alpha_k) - (\nu f^2\alpha_k)(\nu f\alpha_k)(\nu \alpha_k))
\]
By (⋆) and Lemma 8.6 the potential
\[
\sum_{k=1}^{\ell} ((f^2\alpha_k)(\nu f\alpha_k)(\nu \alpha_k) + (\nu f^2\alpha_k)(f\alpha_k)(\nu \alpha_k) + (\nu f^2\alpha_k)(\nu f\alpha_k)(\alpha_k) - (\nu f^2\alpha_k)(\nu f\alpha_k)(\nu \alpha_k))
\]
involves only cycles that are rotationally equivalent to cycles of type III.

Direct computation yields
\[
\eta(S(\tau, x) + U) \sim_{\text{cyc}} S(\tau, x) + \eta(B) - B + \eta(U) - U + U'.
\]

Note that
- \(\eta(B) - B\) involves only cycles of type III (this follows from (⋆) and Lemma 8.6 when we expand \(\eta(B)\) in a way similar to the way we expanded \(\eta(A)\) above), and \(\text{short}(\eta(B) - B) \geq \text{short}(B) + \text{depth}(\eta) = \text{short}(B) + \text{short}(U) - 3 > \text{short}(U)\) since \(\tau\) satisfies (8.1);
- \(\eta(U) - U\) involves only cycles of type III (this follows from (⋆) and Lemma 8.6 when we expand \(\eta(U)\) in a way similar to the way we expanded \(\eta(A)\) above), and \(\text{short}(\eta(U) - U) > \text{short}(U)\) since the depth of \(\eta\) is positive.

Summarizing, given \(U = U_3\), we can guarantee the existence of an \(R\)-algebra automorphism \(\eta\) of \(C\langle\langle Q(\tau)\rangle\rangle\) such that
- \(\eta\) is uniterangular and \(\text{depth}(\eta) = \text{short}(U) - 3\);
- \(\varphi(S(\tau, x) + U)\) is cyclically equivalent to \(S(\tau, x) + U'\) for some potential \(U'\) with the property that, when written in the form \(U' = U'_1 + U'_2 + U'_3\), it satisfies \(U'_1 = 0 = U'_2\) and \(\text{short}(U') > \text{short}(U)\).

Therefore, the lemma follows from a limit process. \(\square\)

The following lemma is a direct consequence of Lemmas 8.7, 8.8, and 8.9.

**Proposition 8.10.** Let \((\Sigma, \mathbb{M})\) be a marked surface with empty boundary, and let \(\tau\) be a tagged triangulation of \((\Sigma, \mathbb{M})\) satisfying (8.1), (8.2) and (8.3). Then, for any potential \(S' \in C\langle\langle Q(\tau)\rangle\rangle\) rotationally disjoint from \(S(\tau, x)\), the QP \((Q(\tau), S(\tau, x) + S')\) is right equivalent to \((Q(\tau), S(\tau, x))\).

We now give a sufficient condition on \((\Sigma, \mathbb{M})\) for the existence of a triangulation \(\tau\) satisfying (8.1), (8.2) and (8.3).
Lemma 8.11. Let \((\Sigma, M)\) be a marked surface with empty boundary. Assume that
\[ |M| \geq \begin{cases} 
6 & \text{if } \Sigma \text{ is a sphere,} \\
3 & \text{otherwise.}
\end{cases} \]
Then there exists a triangulation \(\tau\) of \((\Sigma, M)\) satisfying the conditions \((8.1)\), \((8.2)\) and \((8.3)\).

Proof. If \((\Sigma, M)\) is a sphere with \(|M| \geq 6\), the triangulation obtained from Figure 7 in combination with the procedure of adding punctures as described in Figure 15 satisfies \((8.1)\), \((8.2)\) and \((8.3)\).

In the case of positive-genus surfaces, we shall prove the lemma by induction on the genus of \(\Sigma\). In Figure 30 we see a triangulation \(\sigma\) of the torus with exactly three punctures \((T, \{p_1, p_2, p_3\})\). Straightforward inspection shows that every puncture is incident to at least 4 arcs in \(\sigma\), i.e. that \(\sigma\) satisfies \((8.1)\). It is also clear that every arc in \(\sigma\) connects punctures \(p_i, p_j\), with \(p_i \neq p_j\), i.e. that \(\sigma\) satisfies \((8.2)\). Clearly, the quiver \(Q(\sigma)\), drawn in Figure 30 as well, does not have double arrows, i.e. \(\sigma\) satisfies \((8.3)\). Taking this \(\sigma\) as starting point, if we now begin adding punctures and arcs as indicated in Figure 15, we obtain a triangulation satisfying \((8.1)\), \((8.2)\) and \((8.3)\) for the closed torus with \(n \geq 3\) punctures. The assertion of the lemma is thus proved when the genus of \(\Sigma\) is 1.

Now, let \((\Sigma', M')\) be a positive-genus surface with empty boundary and at least three punctures, and suppose that \((\Sigma', M')\) has a triangulation \(\tau'\) satisfying \((8.1)\), \((8.2)\) and \((8.3)\). Let \(\Delta_1\) be any ideal triangle of \(\tau'\) (recall that, by definition, \(\Delta_1\) is the topological closure in \(\Sigma'\) of a connected component of the complement in \(\Sigma'\) of the union of all arcs in \(\tau'\)). Since no arc in \(\tau'\) is a loop, \(\Delta_1\) is homeomorphic to a closed disc. Similarly, if we take a triangle \(\Delta_2\) of the triangulation \(\sigma\) defined in the first paragraph of the ongoing proof, then \(\Delta_2\) is homeomorphic to a closed disc.
The sides of $\triangle_1$ (resp. $\triangle_2$) are three different arcs of $\tau'$ (resp. $\sigma$) that are not loops. Let $j_1$, $j_2$ and $j_3$ (resp. $k_1$, $k_2$ and $k_3$) be the arcs in $\tau'$ (resp. $\sigma$) that are the sides of $\triangle_1$ (resp. $\triangle_2$), ordered along the clockwise orientation of $\triangle_1$ (resp. the counterclockwise orientation of $\triangle_2$). Let $\triangle_1^\circ$ (resp. $\triangle_2^\circ$) be the interior of $\triangle_1$ in $\Sigma'$ (resp. of $\triangle_2$ in the torus), and glue $\Sigma' \setminus (\triangle_1^\circ)$ to $T \setminus (\triangle_2^\circ)$ by gluing $j_1$ with $k_1$, $j_2$ with $k_2$, and $j_3$ with $k_3$. The result of this gluing is the well-known connected sum $\Sigma' \# T$, which is a surface whose genus exceeds by one the genus of $\Sigma'$.

Since $M' \subseteq \Sigma' \setminus (\triangle_1^\circ)$ we can view $M'$ as a set of marked points in $\Sigma := \Sigma' \# T$ via the canonical inclusion $(\Sigma' \setminus (\triangle_1^\circ)) \hookrightarrow \Sigma$. Note that under the inclusion $(T \setminus (\triangle_2^\circ)) \hookrightarrow \Sigma$, the marked points $p_1, p_2, p_3$ of $(T, \{p_1, p_2, p_3\})$ are mapped to points that lie already in the image of $M'$ under the inclusion $\Sigma' \hookrightarrow \Sigma$.

Under the inclusions of $\Sigma' \setminus (\triangle_1^\circ)$ and $T \setminus (\triangle_2^\circ)$ in $\Sigma$, we can view both $\tau'$ and $\sigma$ as sets of arcs on $\Sigma$. Their union $\tau := \tau' \cup \sigma$ is then a triangulation of $\Sigma$ and satisfies (8.1), (8.2) and (8.3), as the reader can easily verify. This finishes the inductive step of our proof. \hfill $\Box$

**Lemma 8.12.** Let $(\Sigma, M)$ be a marked surface with empty boundary, and let $\tau$ be a tagged triangulation of $(\Sigma, M)$ satisfying (8.1), (8.2) and (8.3). Each non-degenerate potential $S$ on $Q(\tau)$ is of the form $S = S(\tau, x) + S'$, where $S'$ is rotationally disjoint from $S(\tau, x)$.

**Proof.** For $p \in \mathbb{P}$ let $Q_p(\tau)$ be the full subquiver of $Q(\tau)$ whose vertices correspond to the arcs in $\tau$ incident to $p$. It is clear that $Q_p(\tau)$ contains the canonical cycle $S_p := \alpha_1^p \alpha_2^p \cdots \alpha_d^p$ around $p$. Since $\tau$ does not contain any loops, we have $d_p = \text{val}_\tau(p)$. Now suppose that there is another arrow $\alpha$ in $Q_p(\tau)$. This implies that two of the arcs incident to $p$, say $k$ and $l$, are both incident to a puncture $q$ different from $p$. Each triangle of $\tau$ is an interior triangle, since $\Sigma$ has an empty boundary. Thus $k, l$ and a third arc form the sides of a triangle. But now $m$ has to be a loop, a contradiction. It follows that $Q_p(\tau)$ is isomorphic to a cyclically oriented quiver of type $A_{d_p-1}$. Now let $S$ be a non-degenerate potential on $Q(\tau)$. Since $\tau$ is a gentle triangulation, we know that $S$ contains all 3-cycles in $Q(\tau)$. Furthermore, by Proposition 2.4, the cycles $S_p$ with $p \in \mathbb{P}$ also appear in $S$. Now the result follows from the definition of $S(\tau, x)$.

Combining Proposition 8.10 and Lemmas 8.11 and 8.12 yields Theorem 8.4.

### 8.3. The sphere with five punctures.

**Proposition 8.13.** Let $\sigma$ be a triangulation of a sphere $(\Sigma, M)$ with $|M| = 5$. Then $Q(\sigma)$ admits only one non-degenerate potential up to weak right equivalence.

Since QP-mutation maps weakly right equivalent QPs to QPs that are again weakly right equivalent, to prove Proposition 8.13 it suffices to show the existence of a triangulation $\tau$ whose quiver $Q(\tau)$ admits only one non-degenerate potential up to weak right equivalence. We choose $\tau$ to be the triangulation depicted in Figure 31. Let $x = (x_1, x_2, x_3, x_4, x_5)$ be a tuple of non-zero scalars. Then we have

$$S(\tau, x) = -x_1^{-1}\alpha_1^1 \beta_1^1 \gamma_1^1 \delta_1 - x_2^{-1}\alpha_2^2 \beta_2^2 \gamma_2^2 \delta_2 - x_3^{-1}\alpha_3^3 \beta_3^3 \gamma_3^3 \delta_3 + x_4 \alpha_3^3 \beta_3^3 \alpha_2^2 \beta_2^2 \alpha_1^1 \beta_1 + x_5 \gamma_3^3 \delta_3 \gamma_2^2 \delta_2 \gamma_1^1 \delta_1.$$

The fact that $Q(\tau)$ admits only one non-degenerate potential up to weak right equivalence is a consequence of the following lemma, whose proof can be achieved in steps that are similar to the proofs of Lemmas 8.6, 8.7, 8.8, 8.9 and 8.12.

**Lemma 8.14.**

1. Let $\lambda = -x_1^{-1}\alpha_1^1 \beta_1^1 \gamma_1^1 \delta_1 - x_2^{-1}\alpha_2^2 \beta_2^2 \gamma_2^2 \delta_2 - x_3^{-1}\alpha_3^3 \beta_3^3 \gamma_3^3 \delta_3$. Every cycle on $Q(\tau)$ rotationally disjoint from $S(\tau, x)$ is rotation equivalent to a cycle of one of the following types:

   (I) $(\alpha_i \beta_i \gamma_i \delta_i)^n$ for some $n > 1$ and some $i = 1, 2, 3$.
related by a flip have QPs related by QP-mutation, and since ideal triangulations connected by a sequence of flips involving only ideal triangulations. Since ideal triangulations

Proof. Let $Q$ be the degenerate potential on $\tau$ weakly right equivalent, where punctures of $(Q, \tau)$ are related by the flip of a tagged arc $S$. Let $T, T'$ be tagged triangulations of a sphere $\Sigma$ with $5$ punctures. Consequently, for any tagged triangulation $\tau$ of the five-punctured sphere, the QP $(Q(\tau), S(\tau, x))$ is non-degenerate.

Corollary 8.15. Let $\tau$ and $\sigma$ be tagged triangulations of a sphere $(\Sigma, M)$ with $|M| = 5$. If $\tau$ and $\sigma$ are related by the flip of a tagged arc $i \in \tau$, then the QPs $\mu_i(Q(\tau), S(\tau, x))$ and $(Q(\sigma), S(\sigma, x))$ are weakly right equivalent, where $x$ is an a priori fixed tuple of non-zero scalars attached to the five punctures of $(\Sigma, M)$. Consequently, for any tagged triangulation $\tau$ of the five-punctured sphere, the QP $(Q(\tau), S(\tau, x))$ is non-degenerate.

Proof. Let $T$ be the ideal triangulation depicted in Figure 31. By Proposition 8.13, any non-degenerate potential on $Q(T)$ is weakly right equivalent to $S(T, x)$. Since $Q(T)$ admits a non-degenerate potential over $\mathbb{C}$ (cf. [DWZ1]), and since potentials weakly right equivalent to non-degenerate ones are themselves non-degenerate, this implies that $S(T, x)$ is non-degenerate.

Suppose $T'$ is an ideal triangulation of the sphere with five punctures. Then $T'$ and $T$ are connected by a sequence of flips involving only ideal triangulations. Since ideal triangulations related by a flip have QPs related by QP-mutation, and since $S(T, x)$ is non-degenerate, we deduce

Figure 31. A triangulation of a sphere with 5 punctures.
that $S(T', x)$ is non-degenerate. Given the definition of the QPs associated to tagged triangulations (cf. [LF3, Definition 3.2]), this implies that both $S(\tau, x)$ and $S(\sigma, x)$ are non-degenerate (recall that $\tau$ and $\sigma$ have been assumed to be tagged triangulations).

Let us denote the underlying potential of the QP $\mu_0(Q(\tau), S(\tau, x))$ by $\mu_i(S(\tau, x))$. Since $S(\tau, x)$ is non-degenerate, $\mu_i(S(\tau, x))$ is a non-degenerate potential on the quiver $\mu_i(Q(\tau)) = Q(\sigma)$. By Proposition 8.13 $\mu_i(S(\tau, x))$ and $S(\sigma, x)$ are weakly right equivalent. Corollary 8.15 is proved. \[\square\]

In the situation of Corollary 8.15 we conjecture that the QPs $\mu_i(Q(\tau), S(\tau, x))$ and $(Q(\sigma), S(\sigma, x))$ are right equivalent and not only weakly right equivalent.

It was proven in [LF2] that for ideal triangulations $\tau$ and $\sigma$ of the 5-punctured sphere that are related by the flip of an arc $i \in \tau$, the QPs $\mu_i(Q(\tau), S(\tau, x))$ and $(Q(\sigma), S(\sigma, x))$ are right equivalent (hence weakly right equivalent). However, [LF2] does not deal with the case of flips involving arbitrary tagged triangulations of the sphere with five punctures. This case is not dealt with either in the more recent paper [LF3].

8.4. A uniqueness criterion for non-degenerate potentials. The main result of this subsection is Theorem 8.19, which gives a criterion for a potential to be the only non-degenerate potential, up to right equivalence, on a given quiver.

For the following lemma, we use the convention that $m^\infty = 0$.

**Lemma 8.16** ([DWZ1, Equation (2.4)]). If $\varphi$ is a unitriangular $R$-algebra automorphism of $\mathbb{C}[\langle Q \rangle]$, then for every $n \geq 0$ and every $u \in m^n$ we have $\varphi(u) - u \in m^{n + \text{depth}(\varphi)}$.

The following lemma strengthens [LF3, Lemma 2.4], which in turn appears implicitly in [DWZ1].

**Lemma 8.17.** Let $Q$ be a quiver, and let $(\psi_n)_{n > 0}$ be a sequence of unitriangular $R$-algebra automorphisms of $\mathbb{C}[\langle Q \rangle]$. Suppose that $\lim_{n \to \infty} \text{depth}(\psi_n) = \infty$. Then the following hold:

(i) The limit
$$\psi := \lim_{n \to \infty} \psi_n \psi_{n-1} \ldots \psi_1$$
is a well-defined unitriangular $R$-algebra automorphism of $\mathbb{C}[\langle Q \rangle]$.

(ii) If $S$ and $(S_n)_{n > 0}$ are a potential and a sequence of potentials, respectively, on $Q$ such that $\lim_{n \to \infty} \text{short}(S_n) = \infty$, and $\psi_n$ is a right equivalence $(Q, S + S_n) \to (Q, S + S_{n+1})$ for all $n > 0$, then $\psi$ is a right equivalence $(Q, S + S_1) \to (Q, S)$.

**Proof.** We can suppose, without loss of generality, that $\text{depth}(\psi_n) < \infty$ for all $n > 0$ (this is because the only unitriangular automorphism of $\mathbb{C}[\langle Q \rangle]$ that has infinite depth is the identity). Since $\lim_{n \to \infty} \text{depth}(\psi_n) = \infty$, there exists $m_2 > 0$ such that $\text{depth}(\psi_k) > \text{depth}(\psi_1)$ for all $k \geq m_2$. And once we have a positive integer $m_n$, we can find $m_{n+1} > m_n$ such that $\text{depth}(\psi_k) > \text{depth}(\psi_{m_n})$ for all $k \geq m_{n+1}$. Set $\varphi_n := \psi_n \psi_{m_n - 1} \ldots \psi_{m_{n-1}+1}$, with the convention that $m_1 = 1$ and $m_0 = 0$. Then $(\text{depth}(\varphi_n))_{n > 0}$ is a strictly increasing sequence of positive integers.

Note that the limit $\lim_{n \to \infty} \varphi_n \varphi_{n-1} \ldots \varphi_1$ is equal to the limit $\lim_{n \to \infty} \psi_n \psi_{n-1} \ldots \psi_1$. In order to show that $\varphi := \lim_{n \to \infty} \varphi_n \varphi_{n-1} \ldots \varphi_1$ is well-defined, it suffices to show that for every $u \in m$ and every $d > 0$ the sequence $(u_d^n)_{n > 0}$ formed by the components of degree $d$ of the elements $u_n := \varphi_n \ldots \varphi_1(u)$ eventually stabilizes as $n \to \infty$. Note that $\text{depth}(\varphi_n) \geq n - 1$ for all $n > 0$. From this and Lemma 8.19 we deduce that, for a given $u \in m$, there exists a sequence $(v_n)_{n > 0}$ such that $v_n \in m^n$ and $u_n = u + \sum_{j=1}^n v_j$ for all $n > 0$. From this we see that $u_{n+1}^{(d)} = u_{n+1}^{(d)}$ for $n > d$, so the sequence $(u_n^{(d)})_{n > 0}$ stabilizes. This proves (i).
As before, let $V := V_Q$ be the $m$-adic closure of the $\mathbb{C}$-vector subspace of $\mathbb{C} \langle \langle Q \rangle \rangle$ generated by all elements of the form $\alpha_1 \alpha_2 \cdots \alpha_d - \alpha_2 \cdots \alpha_d \alpha_1$ with $\alpha_1 \cdots \alpha_d$ a cycle in $Q$.

Since compositions of right equivalences are again right equivalences, $\varphi_n$ is a right equivalence $(Q, S + S_{m_n}) \to (Q, S + S_{m_{n+1}})$ for all $n > 0$. We deduce that
\[
\varphi_n \varphi_{n-1} \cdots \varphi_2 \varphi_1 (S + S_1) - (S + S_{m_{n+1}})
\]

is contained in $V$. The fact that both sequences $(\varphi_n \varphi_{n-1} \cdots \varphi_2 \varphi_1 (S + S_1))_{n>0}$ and $(-(S + S_{m_{n+1}}))_{n>0}$ are convergent in $\mathbb{C} \langle \langle Q \rangle \rangle$ implies that
\[
(\varphi_n \varphi_{n-1} \cdots \varphi_2 \varphi_1 (S + S_1) - (S + S_{m_{n+1}}))_{n>0}
\]

converges as well. Since $V$ is closed, we get that
\[
\varphi(S + S_1) - S = \lim_{n \to \infty} (\varphi_n \varphi_{n-1} \cdots \varphi_2 \varphi_1 (S + S_1) - (S + S_{m_{n+1}}))
\]
is an element of $V$. Thus (ii) follows. \hfill \Box

**Lemma 8.18.** Let $S$ be a finite potential and $m \geq \text{long}(S)$. If every cycle $\xi$ in $Q$ of length greater than $m$ is cyclically equivalent to an element of the form $\sum_{\eta \in Q_1} u_\eta \partial_\eta(S)$ with $\text{short}(u_\eta) + \text{short}(\partial_\eta(S)) \geq \text{length}(\xi)$ for all $\eta \in Q_1$, then the following hold:

1. For every non-zero potential $S'$ such that $\text{short}(S') > m$ there exist a potential $S''$ with $\text{short}(S'') > \text{short}(S')$ and a right equivalence $\psi: (Q, S + S') \to (Q, S + S'')$ which is unitriangular and satisfies $\text{depth}(\psi) \geq \text{short}(S') - \text{long}(S)$;
2. For every non-zero potential $S'$ such that $\text{short}(S') > m$, the QP $(Q, S + S')$ is right equivalent to $(Q, S)$.

**Proof.** Let $S'$ be a non-zero potential such that $\text{short}(S') > m$. Up to cyclic equivalence, we can assume that
\[
S' = \sum_{\eta \in Q_1} u_\eta \partial_\eta(S)
\]
for some elements $u_\eta \in e_{t(\eta)} \mathbb{C} \langle \langle Q \rangle \rangle e_{s(\eta)}$, with
\[
\text{short}(u_\eta) + \text{short}(\partial_\eta(S)) \geq \text{short}(S') > m \geq \text{long}(S).
\]

Note that $\text{short}(u_\eta) > 1$ for every $\eta \in Q_1$. Define a unitriangular $R$-algebra automorphism $\psi$ of $\mathbb{C} \langle \langle Q \rangle \rangle$ by the rule $\eta \mapsto \eta - u_\eta$ (that $\psi$ is indeed an automorphism and indeed unitriangular follows from the fact that $\text{short}(u_\eta) > 1$ for every $\eta \in Q_1$). Then we have
\[
\text{depth}(\psi) = \min \{ \text{short}(u_\eta) - 1 \mid \eta \in Q_1 \}.
\]

Let $\alpha \in Q_1$ be an arrow such that $\text{short}(u_\alpha) - 1 = \text{depth}(\psi)$. Then
\[
\text{short}(S') - \text{long}(S) \leq \text{short}(S') - \text{short}(\partial_\alpha(S)) - 1 \leq \text{short}(u_\alpha) - 1 = \text{depth}(\psi).
\]

Write
\[
S = \sum_{k=1}^{t} x_k \eta_{k,1} \eta_{k,2} \cdots \eta_{k,m_k}
\]
for some non-zero scalars $x_1, \ldots, x_t \in \mathbb{C}$ and arrows $\eta_{k,i} \in Q_1$ for $1 \leq i \leq m_k$ and $1 \leq k \leq t$. Then we have
\[
\psi(S) = \sum_{k=1}^{t} x_k (\eta_{k,1} - u_{\eta_{k,1}})(\eta_{k,2} - u_{\eta_{k,2}}) \cdots (\eta_{k,m_{k}} - u_{\eta_{k,m_{k}}}).
\]

Expanding each product $(\eta_{k,1} - u_{\eta_{k,1}})(\eta_{k,2} - u_{\eta_{k,2}}) \cdots (\eta_{k,m_{k}} - u_{\eta_{k,m_{k}}})$, we see that it is possible to write it as $T_{k,1} + T_{k,2} + T_{k,3}$, where

1. $T_{k,1} = \eta_{k,1} \eta_{k,2} \cdots \eta_{k,m_{k}}$,
(2) \( T_{k,2} = -\sum_{i=1}^{m_k} \eta_{k,1} \cdots \eta_{k,i-1}u_{\eta_{k,i}}\eta_{k,i+1} \cdots \eta_{k,m_k} \).

(3) \( T_{k,3} \) consists of all the summands that involve at least two factors of the form \( u_{\eta_{k,1}} \).

Hence, we get

\[
\psi(S) = \sum_{k=1}^t x_k T_{k,1} + \sum_{k=1}^t x_k T_{k,2} + \sum_{k=1}^t x_k T_{k,3}.
\]

We obviously have \( S = \sum_{k=1}^t x_k T_{k,1} \). Furthermore, \( \sum_{k=1}^t x_k T_{k,2} \) is cyclically equivalent to \(-S'\). Therefore, \( \psi(S + S') \) is cyclically equivalent to

\[
S + \sum_{k=1}^t x_k T_{k,3} + (\psi(S) - S').
\]

Since we clearly have \( \text{short}(\sum_{k=1}^t x_k T_{k,3}) > \text{short}(S') \), and since \( \text{short}(\psi(S') - S') \geq \text{short}(S') + \text{depth}(\psi) > \text{short}(S') \), setting \( S'' = \sum_{k=1}^t x_k T_{k,3} + (\psi(S') - S') \) we have \( \text{short}(S'') \geq \text{short}(S') \). This finishes the proof of the first statement.

An inductive use of the first statement yields a sequence \((S_n)_{n>0}\) of potentials and a sequence \(\psi_n\) of unitriangular \(R\)-algebra automorphisms of \(\mathbb{C}\langle\langle Q\rangle\rangle\), with the following properties:

- \( S_1 = S' \);
- \( \text{short}(S_{n+1}) > \text{short}(S_n) \);
- \( \text{depth}(\psi_n) \geq \text{short}(S_n) - \text{long}(S) \);
- \( \psi_n \) is a right equivalence \((Q, S_n) \to (Q, S_{n+1})\).

The second property implies \( \lim_{n \to \infty} S_n = 0 \). The second and third properties imply that \( \lim_{n \to \infty} \text{depth}(\psi_n) = \infty \). We can thus apply Lemma 8.17 and conclude that the limit \( \psi = \lim_{n \to \infty} \psi_n \psi_{n-1} \cdots \psi_1 \) is a well-defined right equivalence \((Q, W) \to (Q, S)\). This proves the second statement and hence finishes the proof of Lemma 8.18.

**Theorem 8.19.** Suppose \((Q, S)\) is a QP that satisfies the following three properties:

(i) \( S \) is a finite potential;

(ii) Every cycle \( \xi \) in \( Q \) of length greater than \( \text{long}(S) \) is cyclically equivalent to an element of the form

\[
\sum_{\eta \in Q_1} u_\eta \partial_\eta(S)
\]

with

\[
\text{short}(u_\eta) + \text{short}(\partial_\eta(S)) \geq \text{length}(\xi)
\]

for all \( \eta \in Q_1 \);

(iii) Every non-degenerate potential on \( Q \) is right equivalent to \( S + S' \) for some potential \( S' \) with \( \text{short}(S') \geq \text{long}(S) \).

Then \( S \) is non-degenerate and every non-degenerate potential on \( Q \) is right equivalent to \( S \).

**Proof.** Let \( W \) be any non-degenerate potential on \( Q \). By condition (iii), we can assume that \( W = S + S' \) for some potential \( S' \) with \( \text{short}(S') \geq \text{long}(S) \). By conditions (i) and (ii) we can apply Lemma 8.18 to deduce that \((Q, W)\) is right equivalent to \((Q, S)\).

Since \( Q \) admits at least one non-degenerate potential (see [DWZ1]), and since, as we just showed, any non-degenerate potential on \( Q \) is right equivalent to \( S \), we see that \( S \) must be non-degenerate. This finishes the proof. \(\square\)
8.5. **Potentials for surfaces with non-empty boundary.** The main aim of this subsection is to prove the following theorem. Its proof relies on Theorem 7.11.

**Theorem 8.20.** Suppose \((\Sigma, \mathcal{M})\) is a marked surface with non-empty boundary, and \((\Sigma, \mathcal{M})\) is not a torus with \(|\mathcal{M}| = 1\). For any tagged triangulation \(\tau\) of \((\Sigma, \mathcal{M})\) the quiver \(Q(\tau)\) admits only one non-degenerate potential up to right equivalence.

*Proof.* Let \(\tau\) be a tagged triangulation of some arbitrary marked surface \((\Sigma, \mathcal{M})\), and suppose that \(\sigma\) is another tagged triangulation related to \(\tau\) by a flip at \(k\). By [FST] we know that \(\mu_k(Q(\tau)) = Q(\sigma)\). We know from [DWZ1] that a QP-mutation at \(k\) induces a bijection between the sets of right equivalence classes of non-degenerate potentials on \(Q(\tau)\) and on \(Q(\sigma)\). Thus, to prove a uniqueness result as stated in the theorem, it is enough to exhibit a single triangulation of \((\Sigma, \mathcal{M})\) whose quiver admits only one non-degenerate potential up to right equivalence.

Now assume that \((\Sigma, \mathcal{M})\) is a marked surface with non-empty boundary, and assume that \((\Sigma, \mathcal{M})\) is not a torus with \(|\mathcal{M}| = 1\). First, we assume additionally that \((\Sigma, \mathcal{M})\) is not a monogon. For \((\Sigma, \mathcal{M})\) a digon, a triangle, or an annulus with \(|\mathcal{M} \setminus \mathcal{P}| = 2\), let \(\tau\) be the triangulation displayed in Figure 21, 22, or 23 respectively. Otherwise, let \(\tau := \tau_i\) be the triangulation constructed in the proof of Theorem 7.11. Then \(\tau\) is a skewed-gentle triangulation.

Now let \(S\) be a 3-cycle potential on \(Q(\tau)\). (Note that \(S = S(\tau, 1)\).) We claim that \(S\) is, up to right equivalence, the unique non-degenerate potential on \(Q(\tau)\). To prove this claim, we show that the potential \(S\) satisfies the conditions (i), (ii) and (iii) in Theorem 8.19. It clearly satisfies (i), and by Theorem 7.11 (u) it also satisfies condition (ii). To show (iii), let \(W\) be any non-degenerate potential on \(Q(\tau)\). Without loss of generality, we can assume that the cycles appearing in \(W\) are pairwise different up to rotation equivalence. Since \(Q(\tau)\) does not have double arrows, every 3-cycle in \(Q(\tau)\) appears in \(W\), up to rotation equivalence, see Corollary 2.5. For each 3-cycle \(w\) appearing in \(W\) there is an arrow \(\eta\) that only occurs in \(w\) and in none of the other 3-cycles appearing in \(W\), see Theorem 7.11 (u). Thus, applying a suitable rescaling of arrows, we can assume without loss of generality, that each of these 3-cycles occurs with a coefficient equal to 1. In other words, we can assume that \(W = S + S'\), where \(S\) is our chosen 3-cycle potential, and all cycles appearing in \(S'\) have length at least 4. Thus we have \(\text{short}(S') > \text{long}(S) = 3\). So \(S\) satisfies condition (iii) in Theorem 8.19. It follows that \(S\) is the unique non-degenerate potential on \(Q(\tau)\), up to right equivalence.

Finally, let \((\Sigma, \mathcal{M})\) be a monogon with \(t \geq 2\) punctures. For \(t = 2\) one easily constructs a triangulation \(\tau\) of \((\Sigma, \mathcal{M})\) such that \(Q(\tau)\) is acyclic. Thus there is just one (non-degenerate) potential on \(Q(\tau)\). For \(t \geq 4\) (respectively \(t = 3\)) let \(\tau\) be the triangulation of \((\Sigma, \mathcal{M})\) displayed in Figure 19 (resp. Figure 20). Using the same techniques as the ones explained in Sections 8.2 and 8.3 one shows that \(Q(\tau)\) admits only one non-degenerate potential up to right equivalence. \(\square\)

9. **Exceptional cases**

9.1. In this section we classify the non-degenerate potentials on a list of exceptional quivers of finite mutation type, and we determine the representation type of the corresponding Jacobian algebras. Summarizing, we will prove the following statement.

**Theorem 9.1.** Let \(Q\) be a 2-acyclic quiver.

1. If \(Q\) is mutation equivalent to one of the quivers \(E_m, \overline{E}_m\) or \(E_m^{(1,1)}\) with \(m \geq 3\), then \(Q\) is Jacobi-tame. Furthermore, there exists only one non-degenerate potential on \(Q\) up to right equivalence.
(ii) If $Q$ is mutation equivalent to one of the quivers $X_6$, $X_7$ or $K_m$ with $m \geq 3$, then $Q$ is Jacobi-wild. Furthermore, for the cases $X_6$ and $K_m$ there exists only one non-degenerate potential on $Q$ up to right equivalence.

(iii) If $Q$ is (mutation equivalent to) one of the quivers $T_1$ or $T_2$, then $Q$ is Jacobi-irregular. In particular, there are at least two non-degenerate potentials on $Q$ up to weak right equivalence.

Furthermore, we classify all non-degenerate potentials for the sphere with 4 punctures, and we show that for surfaces $(\Sigma, M)$ with empty boundary and $|M| = 1$ there are at least two non-degenerate potentials up to weak right equivalence.

For some exceptional quivers $Q$ of finite mutation type there are non-degenerate potentials $S$ on $Q$ such that the Jacobian algebra $\mathcal{P}(Q, S)$ is wild. This is proved with the help of Galois coverings. For convenience we recall the relevant definitions and results in Section 9.10.

9.2. Acyclic quivers.

9.2.1. Classification of non-degenerate potentials.

Lemma 9.2. Assume that $Q$ is mutation equivalent to an acyclic quiver. Then there exists only one non-degenerate potential on $Q$.

Proof. An acyclic quiver $Q$ has only one potential, namely the zero potential $S = 0$, and $\mathcal{P}(Q, S)$ is isomorphic to the path algebra $\mathbb{C}\langle Q \rangle$. The potential $S$ has to be non-degenerate by [DWZ1, Corollary 7.4]. Since $\text{QP}$-mutation induces a bijection on right equivalence classes of non-degenerate potentials [DWZ1, Theorem 5.7], the result follows. □

9.2.2. Representation type. The following result is well known, see for example [R].

Theorem 9.3. For an acyclic quiver $Q$ the following hold:

(i) $\mathbb{C}\langle Q \rangle$ is representation-finite if and only if $Q$ is a Dynkin quiver.

(ii) $\mathbb{C}\langle Q \rangle$ is tame if and only if $Q$ is a Euclidean quiver.

(iii) In all other cases, $\mathbb{C}\langle Q \rangle$ is wild.

As a consequence of Theorems 1.1 and Lemma 9.2 we get the following result.

Proposition 9.4. Assume that $Q$ is mutation equivalent to an acyclic quiver $Q'$. Then $Q$ is Jacobi-wild if and only if $\mathbb{C}\langle Q' \rangle$ is wild.

In particular, the quivers $K_m$ with $m \geq 3$ are Jacobi-wild, and the quivers mutation equivalent to one of the quivers $E_m$ or $\tilde{E}_m$ are Jacobi-tame.

9.3. The quivers $E_m^{(1,1)}$.

9.3.1. Classification of non-degenerate potentials.

Lemma 9.5. Let $Q$ be mutation equivalent to one of the quivers $E_m^{(1,1)}$. Then there exists only one non-degenerate potential $S$ on $Q$ up to right equivalence.
Proof. Let $E$ be the quiver

![Quiver Diagram]

with arrows $\alpha_i : 1 \to i + 1$ and $\beta_i : i + 1 \to 5$ for $i = 1, 2, 3$ and $\gamma_j : 5 \to 1$ for $j = 1, 2$. The quiver $\mu_5\mu_4\mu_3\mu_2(E)$ is acyclic. Thus up to right equivalence there is only one non-degenerate potential $S$ on $E$. It is straightforward to check that up to right equivalence we have

$$S = \gamma_1(\beta_1 \alpha_1 + \beta_2 \alpha_2) + \gamma_2(\beta_2 \alpha_2 + \beta_3 \alpha_3).$$

Potentials of this type can already be found in [L1, Section 1.2]. Now let $Q$ be one of the quivers $E_m^{(1,1)}$ for $m = 6, 7, 8$. Clearly, $E$ is a full subquiver of $Q$, and every cycle in $E$ is also a cycle in $Q$. Thus, if $W$ is a non-degenerate potential on $Q$, then $W$ itself is a non-degenerate potential on $E$ by Proposition 2.3, and hence there exists a right-equivalence $\varphi : (E, W) \to (E, S)$. This right-equivalence $\varphi$ can be extended to a right-equivalence between $(Q, W)$ and $(Q, S)$ by sending every arrow $\delta \in Q \setminus E_1$ to itself. We see that $S$ is, up to right equivalence, the only non-degenerate potential on $Q$. □

9.3.2. Representation type.

**Proposition 9.6.** Let $Q$ be mutation equivalent to one of the quivers $E_m^{(1,1)}$. Then $Q$ is Jacobi-tame.

**Proof.** As shown above, there is only one non-degenerate potential $S$ on $Q$ up to right equivalence. The tameness of $\mathcal{P}(Q, S)$ is proved by relating $\mathcal{P}(Q, S)$ to Ringel’s [R] tubular algebras. A detailed proof can be found in [CG]. □

9.4. The quiver $X_6$.

9.4.1. Classification of non-degenerate potentials. Let $X_6$ be the quiver displayed below

![Quiver Diagram]

and set

$$W := \gamma_1 \beta_1 \alpha_1 + \gamma_2 \beta_2 \alpha_2 + \gamma_1 \beta_2 \alpha_2' \gamma_2 \beta_1 \alpha_1'.$$

Up to right equivalence $W$ is the only non-degenerate potential on $X_6$. Indeed, let $Q$ be the quiver obtained from $X_6$ by deleting the vertex 5 and the arrow $\delta$. Then $(Q, W)$ is the QP associated to a triangulation of an annulus with two marked points on the boundary and one puncture, hence it is a non-degenerate QP. Moreover, $Q$ admits only one non-degenerate potential up to right equivalence by Theorem 8.20. If $S$ is a non-degenerate potential on $X_6$, then $(Q, S)$ is non-degenerate by Proposition 2.3, hence there exists a right-equivalence $\varphi(Q, S) \to (Q, W)$. This right equivalence can be extended to a right equivalence between $(X_6, S)$ and $(X_6, W)$ by sending the arrow $\delta$ to itself.
9.4.2. Representation type.

**Lemma 9.7.** Assume that $Q$ is mutation equivalent to the quiver $X_6$. Then $Q$ is Jacobi-wild.

**Proof.** The Jacobian algebra $\Lambda_2 := \mathcal{P}(X_6, W)$ with $X_6$ and $W$ as defined above is obviously finite-dimensional. Note, that with the $\mathbb{Z} \times \mathbb{Z}$-grading on $\mathbb{C}\langle X_6 \rangle$ defined by $|\alpha'_1| = (1, 0), |\alpha_1| = (1, 1) = |\alpha_2|, |\alpha'_2| = (0, 1)$ and $|\beta_i| = |\gamma_i| = |\delta| = 0$ the potential $W \in \mathbb{C}\langle X_6 \rangle$ becomes homogeneous (of degree $(1, 1)$). This yields a $\mathbb{Z} \times \mathbb{Z}$-grading on $\Lambda_2 = \mathcal{P}(X_6, W)$. We highlight in the corresponding Galois-covering $\tilde{\Lambda}_2$ a convex hypercritical subcategory which belongs to the frame $\tilde{E}_7$. Thus $\Lambda_2$ is wild.

Notice, that $\tilde{\Lambda}$ is defined by a $\mathbb{Z} \times \mathbb{Z}$-periodic quiver with all possible commutativities and the only zero-relations $\gamma_i \beta_i$ for $i \in \{1, 2\}$ wherever applicable.

9.5. The quiver $X_7$.

9.5.1. Classification of non-degenerate potentials. For quivers mutation equivalent to $X_7$, the classification of non-degenerate potentials up to right equivalence is still an open problem. We conjecture that there are at least two potentials up to right equivalence.

9.5.2. Representation type.

**Lemma 9.8.** Assume that $Q$ is mutation equivalent to the quiver $X_7$. Then $Q$ is Jacobi-wild.

**Proof.** Suppose that $S$ is a non-degenerate potential on $X_7$. An obvious QP-restriction yields a non-degenerate potential $W$ on $X_6$ and a surjective algebra homomorphism $f: \mathcal{P}(X_7, S) \to \mathcal{P}(X_6, W)$. This yields an exact embedding $\text{mod}(\mathcal{P}(X_6, W)) \to \text{mod}(\mathcal{P}(X_7, S))$. Thus, if $\mathcal{P}(X_6, W)$ is wild, then $\mathcal{P}(X_7, S)$ is also wild. We proved already above that $\mathcal{P}(X_6, W)$ is wild. This finishes the proof.
9.6. **Torus** $(\Sigma, \mathbb{M})$ **with empty boundary and** $|\mathbb{M}| = 1$.

9.6.1. **Classification of non-degenerate potentials.** Let $Q = Q(\tau)$ for some triangulation $\tau$ of a torus $(\Sigma, \mathbb{M})$ with empty boundary and $|\mathbb{M}| = 1$. Then $Q$ is isomorphic to the quiver $T_1$ displayed below.

$$
T_1 := \begin{array}{c}
1 \\
\gamma_1 \\
\beta_1 \\
\alpha_2 \\
\beta_2 \\
\alpha_1 \\
2 \\
\end{array}
$$

The non-degenerate potentials on $T_1$ have been classified by Geuenich [Geu] in the following sense. He shows that a potential $S$ on $T_1$ is non-degenerate if and only if $S_{\min}$ is non-degenerate. Furthermore, Geuenich proves that the potential $S_{\min}$ of a non-degenerate potential $S$ is either right equivalent to the potential $S_{\text{tame}} := \alpha_1 \beta_1 \gamma_1 + \alpha_2 \beta_2 \gamma_2$, or to the potential $S_{\text{wild}} := \gamma_2 \beta_2 \alpha_1 + \gamma_1 \beta_2 \alpha_2$.

9.6.2. **A tame non-degenerate potential on** $T_1$. For

$$
S := S_{\text{tame}} = \alpha_1 \beta_1 \gamma_1 + \alpha_2 \beta_2 \gamma_2
$$

the Jacobian algebra $\mathcal{P}(T_1, S)$ is a gentle algebra and therefore tame.

9.6.3. **A wild non-degenerate potential on** $T_1$. Let

$$
W := S_{\text{wild}} = \gamma_2 \beta_2 \alpha_1 + \gamma_2 \beta_1 \alpha_2 + \gamma_1 \beta_2 \alpha_2.
$$

The Jacobian algebra $\mathcal{P}(T_1, W)$ is infinite-dimensional. However, we will see that the finite-dimensional quotient $\Lambda_3 := \mathcal{P}(T_1, W)/(\gamma_1 \beta_1 \alpha_1 \gamma_1 \beta_1)$ is wild. Note that $W \in \mathbb{C}\langle T_1 \rangle$ is homogeneous with respect to the $\mathbb{Z} \times \mathbb{Z}$-grading of $\mathbb{C}\langle T_1 \rangle$ given by $|\alpha_1| = |\beta_1| = |\gamma_1| = (1,0)$ and $|\alpha_2| = |\beta_2| = |\gamma_2| = (0,1)$. This yields on $\mathcal{P}(T_2, W)$ and $\Lambda_3$ a $\mathbb{Z} \times \mathbb{Z}$-grading.

9.7. **Torus** $(\Sigma, \mathbb{M})$ **with non-empty boundary and** $|\mathbb{M}| = 1$. 

Notice, that $\tilde{\Lambda}$ is defined by a $\mathbb{Z} \times \mathbb{Z}$-periodic quiver with all possible commutativities and the zero-relations $\beta_2 \alpha_2$, $\gamma_2 \beta_2$, $\alpha_2 \gamma_2$ and $\gamma_1 \beta_1 \alpha_1 \gamma_1 \beta_1$ wherever applicable.
9.7.1. Classification of non-degenerate potentials. Let $Q = Q(\tau)$ for some triangulation $\tau$ of a torus $(\Sigma, M)$ with non-empty boundary and $|M| = 1$. Then $Q$ is isomorphic to the quiver $T_2$ displayed below.

![Diagram of quiver $T_2$]

**Proposition 9.9.** The quiver $Q := T_2$ admits exactly two non-degenerate potentials up to right equivalence. More specifically, the potentials

$$S = \alpha_1\beta_1\gamma_1 + \alpha_2\beta_2\gamma_2 \quad \text{and} \quad W = \alpha_1\beta_1\gamma_1 + \alpha_1\beta_2\gamma_2 + \alpha_2\beta_2\delta\gamma_1$$

are both non-degenerate and not right equivalent to each other, and any non-degenerate potential on $Q$ is right equivalent to $S$ or to $W$.

**Proof.** Note that $e_2m^3e_2 \subseteq J(S)$. Were $(Q,S)$ and $(Q,W)$ right equivalent, there would exist an $R$-algebra automorphism $\varphi$ of $\mathbb{C}[(Q)]$ such that $\varphi(J(S)) = J(W)$. We would therefore have $e_2m^3e_2 = \varphi(e_2me_2)^3 \subseteq \varphi(J(S)) = J(W)$. Hence, in order to show that $(Q,S)$ and $(Q,W)$ are not right equivalent it is enough to show that $e_2m^3e_2$ is not contained in $J(W)$. We claim that the element $\alpha_2\beta_2\gamma_2$ of $e_2m^3e_2$ does not belong to $J(W)$. To prove this claim it suffices to exhibit a nilpotent representation $\hat{M}$ of $Q$ which is annihilated by the cyclic derivatives of $W$ and such that $\alpha_2\beta_2\gamma_2$ does not act as the zero map on $\hat{M}$. A straightforward check shows that the representation

![Diagram of nilpotent representation $\hat{M}$]

satisfies all these requirements. We conclude that $(Q,S)$ and $(Q,W)$ are not right equivalent.

To show that, up to right equivalence, there are no other non-degenerate potentials on $Q$ besides $S$ and $W$, we first show:

**Lemma 9.10.**

(1) For any potential $S'$ on $Q$ such that $\text{short}(S') > 3$, the QP $(Q, S + S')$ is right equivalent to $(Q,S)$.

(2) For any potential $W'$ on $Q$ such that $\text{short}(W') > 4$, the QP $(Q, W + W')$ is right equivalent to $(Q,W)$.

**Proof.** The potential $S$ obviously satisfies the hypothesis of Lemma 8.18 with $m = 3$. To see that $W$ satisfies such hypothesis with $m = 4$, first notice that any cycle in $Q$ passing through the arrow $\delta$ is rotationally equivalent to a cycle that has the path $\beta_2\delta\gamma_1 = \partial_{\alpha_1}(W)$ as a factor. Furthermore, any cycle on $Q$ of length greater than 4 not passing through $\delta$ is rotationally equivalent to a cycle that has one of the following paths as a factor:

- $\alpha_1\beta_1\gamma_1\alpha_1\beta_1\gamma_1 = \alpha_1\beta_1\partial_{\beta_1}(W)\beta_1\gamma_1$
\[
\begin{align*}
\alpha_1 \beta_1 \gamma_1 \alpha_1 \beta_2 \gamma_2 &= \alpha_1 \beta_1 \partial_{\delta_1}(W) \beta_2 \gamma_2 \\
\alpha_1 \beta_1 \gamma_1 \alpha_2 \beta_1 \gamma_1 &= \alpha_1 \beta_1 \alpha_2 \beta_2 \gamma_1 - \alpha_1 \beta_1 \alpha_2 \beta_2 \gamma_2 = \alpha_1 \beta_1 \partial_{\alpha_1}(W) - \alpha_1 \beta_1 \partial_{W}(W) \gamma_2 \\
\alpha_1 \beta_1 \gamma_1 \alpha_2 \beta_2 \gamma_2 &= \alpha_1 \beta_1 \partial_{W}(W) \gamma_2 \\
\alpha_1 \beta_2 \gamma_2 \alpha_2 \beta_1 \gamma_1 &= \partial_{\gamma_2}(W) \gamma_2 \alpha_1 \beta_1 \gamma_1 \\
\alpha_1 \beta_2 \gamma_2 \alpha_2 \beta_2 \gamma_2 &= \partial_{\gamma_2}(W) \gamma_2 \alpha_2 \beta_2 \gamma_2 \\
\alpha_2 \beta_1 \gamma_1 \alpha_2 \gamma_1 \beta_1 \gamma_1 + \alpha_2 \beta_1 \gamma_1 \alpha_2 \beta_2 \gamma_1 - \alpha_2 \beta_1 \gamma_1 \alpha_2 \beta_2 \gamma_2 &= \alpha_2 \beta_1 \gamma_1 \alpha_2 \partial_{\alpha_1}(W) - \alpha_2 \beta_1 \partial_{W}(W) \gamma_2 \\
\alpha_2 \beta_1 \gamma_1 \alpha_2 \beta_2 \gamma_2 &= \alpha_2 \beta_1 \partial_{W}(W) \gamma_2 \\
\alpha_2 \beta_2 \gamma_2 \alpha_2 \beta_1 \gamma_1 &= \alpha_2 \beta_2 \gamma_2 \alpha_1 \beta_1 \gamma_1 + \alpha_2 \beta_2 \gamma_2 \alpha_2 \beta_2 \gamma_1 - \alpha_2 \beta_2 \gamma_2 \alpha_2 \beta_2 \gamma_2 = \alpha_2 \beta_2 \partial_{\alpha_1}(W) \alpha_2 \beta_1 \gamma_1 - \alpha_2 \beta_2 \partial_{\alpha_2}(W) \alpha_2 \beta_1 \gamma_1 \\
\alpha_2 \beta_2 \gamma_2 \alpha_2 \beta_2 \gamma_2 &= \alpha_2 \beta_2 \gamma_2 \alpha_2 \beta_2 \gamma_2 + \alpha_2 \beta_1 \gamma_1 \alpha_2 \beta_2 \gamma_2 - \alpha_2 \beta_1 \gamma_1 \alpha_2 \beta_2 \gamma_2 = \alpha_2 \beta_1 \partial_{\alpha_1}(W) \alpha_2 \beta_2 \gamma_2 - \alpha_2 \beta_1 \partial_{W}(W) \gamma_2.
\end{align*}
\]

This readily implies that \(W\) satisfies the hypothesis of Lemma \ref{lem:8.18} with \(m = 4\). \(\square\)

Let \(T\) be any non-degenerate potential on \(Q\). Up to cyclic equivalence, we can write it as

\[
T = x_1 \alpha_1 \beta_1 \gamma_1 + x_2 \alpha_2 \beta_2 \gamma_2 + x_3 \alpha_1 \beta_2 \gamma_1 + y_1 \alpha_1 \beta_1 \gamma_1 + y_2 \alpha_2 \beta_2 \gamma_2 + y_3 \alpha_2 \beta_2 \gamma_1 + T',
\]

where \(x_1, x_2, y_1, y_2, z_1, z_2\) are scalars and \(T' \in m^5\).

A straightforward computation shows that the degree-2 component of \(\bar{\mu}_4 \bar{\mu}_3(T)\) is

\[
x_1 \alpha_1 \beta_1 \gamma_1 + x_2 \alpha_2 \beta_2 \gamma_2 + x_3 \alpha_1 \beta_2 \gamma_1 + y_1 \alpha_2 \beta_1 \gamma_1 + y_2 \alpha_2 \beta_2 \gamma_2 + y_3 \alpha_2 \beta_2 \gamma_1.
\]

Since \(\mu_4 \mu_3(Q, T)\) is (right equivalent to) the reduced part of \(\mu_4 \mu_3(Q, T) = (\bar{\mu}_4 \bar{\mu}_3(Q), \bar{\mu}_4 \bar{\mu}_3(T))\) and since \(T\) is non-degenerate, this implies that the matrix

\[
X = \begin{bmatrix}
  x_1 & x_2 & x_3 \\
  y_1 & y_2 & y_3
\end{bmatrix}
\]

has rank 2. If the first two columns of this matrix are linearly independent, then the rule \(\psi : \alpha_1 \mapsto x_1 \alpha_1 + y_1 \alpha_2, \alpha_2 \mapsto x_2 \alpha_1 + y_2 \alpha_2\), gives rise to an automorphism \(\psi\) of \(C\langle Q \rangle\). Its inverse \(\psi^{-1}\) maps \(T\) to \(\psi^{-1}(T) = S + \psi^{-1}(x_3 \alpha_1 \beta_2 \delta_1 + y_3 \alpha_2 \beta_2 \delta_1 + T')\). Since \(\text{short} (\psi^{-1}(x_3 \alpha_1 \beta_2 \delta_1 + y_3 \alpha_1 \beta_2 \delta_1 + T')) > 3\), we can apply Lemma \ref{lem:9.10} to conclude that \(\psi^{-1}(T)\), and hence \(T\), is right equivalent to \(S\).

If the first two columns of \(X\) are linearly dependent, then

\[
\begin{bmatrix}
  u_1 & u_3 \\
  u_2 & u_4
\end{bmatrix}
\begin{bmatrix}
  x_1 & x_2 & x_3 \\
  y_1 & y_2 & y_3
\end{bmatrix}
= \begin{bmatrix}
  1 & u_1 & u_2 \\
  0 & 0 & 1
\end{bmatrix}
\]

for some invertible matrix

\[
D = \begin{bmatrix}
  u_1 & u_3 \\
  u_2 & u_4
\end{bmatrix}
\]
The rule \( \psi : \alpha_1 \mapsto u_1 \alpha_1 + u_2 \alpha_2, \alpha_2 \mapsto u_3 \alpha_1 + u_4 \alpha_2 \) gives rise to an \( R \)-algebra automorphism of \( \mathbb{C}[[Q]] \) and sends \( T \) to
\[
\psi(T) = \alpha_1 \delta_1 \gamma_1 + w_1 \alpha_1 \beta_2 \gamma_2 + w_2 \alpha_1 \beta_3 \delta_1 + \alpha_2 \beta_2 \delta_1 + \psi(T'),
\]
Since the degree-2 component of \( \tilde{\mu}_4(\psi(T)) \) is \( w_1 \alpha_1 \beta_2 \gamma_2 \) and since \( T \) is non-degenerate, we deduce that \( w_1 \neq 0 \). The rule \( \varphi : \gamma_2 \mapsto w_1^{-1} \gamma_2 - w_1^{-1} w_2 \delta \gamma_1 \) sends \( \psi(T) \) to
\[
\varphi \psi(T) = W + \varphi \psi(T')
\]
Noticing that \( \text{short}(\varphi \psi(T')) \geq 5 \), we can apply Lemma \( \ref{lemma:9.10} \) to conclude that \( \varphi \psi(T) \), and hence \( T \), is right equivalent to \( W \).

We already know that the potential \( S \) is Jacobi-finite and rigid. Since \( W \) is non-degenerate, it makes sense to ask whether \( W \) is also Jacobi-finite and/or rigid. Here is the answer:

**Proposition 9.11.** The potential \( W \) is Jacobi-finite and non-rigid.

**Proof.** The finite-dimensionality of the Jacobian algebra \( \mathcal{P}(Q,W) \) is already evident in the proof of Lemma \( \ref{lemma:9.10} \). To show it is not rigid, we first establish a result concerning cyclic equivalence for general quivers.

**Lemma 9.12.** Let \( Q \) be any quiver and let \( C \) be a system of the representatives for the equivalence relation of rotation on the set of all cycles on \( Q \) (thus, two cycles are equivalent if they can be obtained from each other by rotation, and \( C \) contains exactly one element from each equivalence class). If \( \sum_{\xi \in C} x_{\xi} \xi \) is a (possibly infinite) linear combination of elements of \( C \) that is cyclically equivalent to zero, then \( x_{\xi} = 0 \) for all \( \xi \in C \).

**Proof.** That \( u = \sum_{\xi \in C} x_{\xi} \xi \) is cyclically equivalent to zero means that it is the limit of a sequence \( (u_n)_{n>0} \) of elements of \( \mathbb{C}[[Q]] \) that can be written as finite \( \mathbb{C} \)-linear combinations of elements of the form \( \alpha_1 \alpha_2 \ldots \alpha_d - \alpha_2 \ldots \alpha_d \alpha_1 \) with \( \alpha_1 \ldots \alpha_d \) a cycle on \( Q \). Let \( d > 0 \), then there exists \( N > 0 \) such that \( u_n^{(d)} = u_{n+1}^{(d)} = u^{(d)} \) for all \( n \geq N \).

Given a cycle \( \alpha_1 \alpha_2 \ldots \alpha_d \) on \( Q \), and scalars \( x_1, \ldots, x_d \), we have
\[
(9.1) \quad \sum_{k=1}^{d} x_k (\alpha_k \alpha_{k+1} \ldots \alpha_d \alpha_1 \ldots \alpha_{k-1} - \alpha_{k+1} \ldots \alpha_d \alpha_1 \ldots \alpha_{k-1} \alpha_k) = \sum_{k=1}^{d} (x_k - x_{k-1}) \alpha_k \alpha_{k+1} \ldots \alpha_d \alpha_1 \ldots \alpha_{k-1},
\]
(with the convention that \( \alpha_{d+1} = \alpha_1 \) and \( x_0 = x_d \)) which means that \( (9.1) \) is non-zero if and only if \( x_1, \ldots, x_d \) are not all the same scalar, in which case, in the right hand side of \( (9.1) \) will have at least two terms appearing with non-zero scalar. Since such two terms are rotationally equivalent, we see that we cannot obtain \( u_n^{(d)} = u^{(d)} \) as a finite sum of elements of the form \( (9.1) \) unless \( u^{(d)} = 0 \). Therefore \( u = 0 \).

To show that \( W \) is not rigid, it is enough to exhibit a cycle which is not cyclically equivalent to any element of the Jacobian ideal \( J(W) \). We claim that \( \alpha_2 \beta_2 \gamma_2 \) is such cycle. Indeed, suppose that \( \alpha_2 \beta_2 \gamma_2 \) is cyclically equivalent to an element \( u \) of \( J(W) \). Then we can take \( u \) to have the form
\[
(9.2) \quad u = \sum_{\eta \in Q_1} u_{\eta} \partial_{\eta}(W).
\]
A direct check shows that the only summand in this expression that contributes with a rotation of \( \alpha_2 \beta_1 \gamma_1 \) or \( \alpha_2 \beta_2 \gamma_2 \) to the expression of \( u \) as a (possibly infinite) linear combination of cycles, is the summand with \( \eta = \alpha_1 \). Moreover, the rotations \( \beta_1 \gamma_1 \alpha_2, \gamma_1 \alpha_2 \beta_1, \beta_2 \gamma_2 \alpha_2 \) and \( \gamma_2 \alpha_2 \beta_2 \), are not contributed at all by any summand in \( \mathcal{M}_2 \). Even more, in the expression of \( u \) as a (possibly infinite) \( \mathbb{C} \)-linear combination of cycles, the cycles \( \alpha_2 \beta_1 \gamma_1 \) and \( \alpha_2 \beta_2 \gamma_2 \) must appear accompanied with the same scalar \( x \). Since \( \alpha_2 \beta_2 \gamma_2 - u \) is cyclically equivalent to zero, we see that \( x = 1 \) and, at the same time, \( x = 0 \), contradiction that proves that \( \alpha_2 \beta_2 \gamma_2 \) is not cyclically equivalent to any element of the Jacobian ideal \( J(W) \). We conclude that \( W \) is not rigid.

9.7.2. A tame non-degenerate potential on \( T_2 \). It is not hard to check that 
\[
S := \alpha_1 \beta_1 \gamma_1 + \alpha_2 \beta_2 \gamma_2
\]
is a non-degenerate potential on \( T_2 \), and that \( \mathcal{P}(T_2, S) \) is a gentle algebra and therefore tame.

9.7.3. A wild non-degenerate potential on \( T_2 \). For 
\[
W := \alpha_1 \beta_1 \gamma_1 + \alpha_1 \beta_2 \gamma_2 + \alpha_2 \beta_2 \delta \gamma_1
\]
the Jacobian algebra \( \Lambda_1 := \mathcal{P}(T_2, W) \) is finite-dimensional. Note that \( W \in \mathbb{C}[\langle T_2 \rangle] \) is homogenous with respect to the \( \mathbb{Z} \times \mathbb{Z} \)-grading on \( \mathbb{C}[\langle T_2 \rangle] \) given by \( |\alpha_1| = (1, 1), |\alpha_2| = (1, 0), |\delta| = (0, 1), |\beta_1| = 0, |\gamma_i| = 0 \). This yields a \( \mathbb{Z} \times \mathbb{Z} \)-grading on \( \Lambda_1 \).

We highlight in the corresponding Galois covering \( \tilde{\Lambda}_2 \) a convex hypercritical subcategory. It belongs to one of the frames which are concealed of type \( \tilde{\mathcal{E}}_7 \), see [Un] p. 150. Thus \( \Lambda_1 = \mathcal{P}(T_2, W) \) is wild.

\[
\tilde{\Lambda}_1 := 
\begin{array}{c}
\begin{array}{ccc}
4 & \gamma_2 & \alpha_2 \\
\beta_2 & \gamma_1 & \alpha_1 & 3 \\
2 & \beta_1 & 1 & \delta \\
\end{array}
& & \\
\begin{array}{ccc}
\delta & \beta_2 & \gamma_1 \\
\beta_1 & 1 & \delta \\
3 & \alpha_1 & 4 & \gamma_2 \\
\end{array}
\end{array}
\]

Note that the Galois covering \( \tilde{\Lambda}_1 \) is (strongly) simply connected. It is given by a \( \mathbb{Z} \times \mathbb{Z} \)-periodic quiver together with all possible commutativities and the zero relations \( \gamma_1 \alpha_1, \alpha_1 \beta_2, \beta_2 \delta \gamma_1 \) and \( \gamma_1 \alpha_2 \beta_2 \) wherever applicable.
9.8. Surfaces \((\Sigma, \mathcal{M})\) with empty boundary and \(|\mathcal{M}| = 1\).

**Proposition 9.13.** Let \(\tau\) be a triangulation of a marked surface \((\Sigma, \mathcal{M})\) with empty boundary and \(|\mathcal{M}| = 1\). Then there exist at least two non-degenerate potentials on \(Q(\tau)\) that are not weakly right equivalent.

**Proof.** Consider the definition of \(S(\tau, x)\) for \(x = x\) a non-zero scalar (attached to the unique puncture of \((\Sigma, \mathcal{M})\)). Since for any triangulation of \((\Sigma, \mathcal{M})\) the unique puncture always has valency greater than two, one can extend the definition of \(S(\tau, x)\) to include the situation where \(x = x = 0\). One obtains \(S(\tau, 0) = S(\tau, x)_{\text{min}}\) for any scalar \(x = x\).

In [LF2, Theorem 30] it is proved that, for \(x = x \neq 0\), the QPs \(\mu_t(Q(\sigma_1), S(\sigma_1, x))\) and \((Q(\sigma_2), S(\sigma_2, x))\) are right equivalent for any two triangulations \(\sigma_1\) and \(\sigma_2\) related by the flip of an arc \(i \in \sigma_1\). As pointed out by Ladkani in [L0, Section 4.3], the same proof from [LF2, Theorem 30] is still valid when one allows \(x = x = 0\). This readily implies the non-degeneracy of \(S(\tau, 0)\). On the other hand, it is easily seen that the Jacobian algebra \(\mathcal{P}(Q(\tau), S(\tau, 0))\) is infinite-dimensional, while \(\mathcal{P}(Q(\tau), S(\tau, x))\) is finite-dimensional for \(x = x \neq 0\) as shown by Ladkani in [L2]. The proposition follows.

**Remark 9.14.** It has been proved by Geuenich [Gen], that in the case of the once-punctured torus, the quiver \(Q(\tau)\) has infinitely many non-degenerate potentials which are pairwise not weakly right equivalent.

9.9. The sphere with four punctures. Let \((Q^{(1)}, W^{(1)}_t)\) be the QP defined below. We have \(Q^{(1)} = Q(\tau)\), where \(\tau\) is the usual tetrahedron triangulation of a sphere \((\Sigma, \mathcal{M})\) with \(|\mathcal{M}| = 4\). Let \(\Lambda_t := \mathcal{P}(Q^{(1)}, W^{(1)}_t)\) be the corresponding Jacobian algebra.

It is easy to see that each quiver which is mutation equivalent to \(Q^{(1)}\) is isomorphic to one of the quivers \(Q^{(i)}\) for \(i = 1, 2, 3, 4\) below. The aim of this section is to prove the following result:

**Proposition 9.15.** For the quiver \(Q := Q^{(1)}\) the following hold:

(i) Each non-degenerate potential on \(Q^{(1)}\) is right equivalent to one of the potentials \(W^{(1)}_t\) with \(t \in \mathbb{C} \setminus \{0, 1\}\);

(ii) The Jacobian algebras \(\Lambda_t\) and \(\Lambda_{t'}\) are isomorphic if and only if \(t' \in \{t, t^{-1}\}\).

**Corollary 9.16.** Let \(Q = Q(\tau)\) for a triangulation \(\tau\) of a sphere \((\Sigma, \mathcal{M})\) with \(|\mathcal{M}| = 4\). Then there are infinitely many non-degenerate potentials on \(Q\) up to weak right equivalence.

The Jacobian algebras \(\Lambda_t\) are finite-dimensional and (weakly) symmetric [L2] and tame of “tubular type” [GG]. The algebra \(\Lambda_1\) is an infinite dimensional clannish algebra, and the corresponding potential \(W^{(1)}_t\) is degenerate.

We have to consider the following four quivers with potential where we assume that the coefficients \(t, t_{ij}, t_{ijk}\) belong to \(\mathbb{C}^+\). For typographical reasons we identify each of the quivers along the dotted line. See also [GKO] p. 117] for essentially the same list.

\[
\begin{array}{ccccccc}
1 & \gamma_{11} & 5 & \beta_{11} & 3 & \alpha_{11} & 2 \\
2 & \gamma_{21} & 6 & \beta_{21} & 4 & \alpha_{21} & 2 \\
\end{array}
\]

\[
W^{(1)}_t := (t - 1)\gamma_{21}\beta_{11}\alpha_{12} + \sum_{i,j,k=1}^2 \gamma_{ik}\beta_{kj}\alpha_{ji}
\]

\[
W^{(1)}_{\text{gen}} := \sum_{i,j,k=1}^2 t_{kji}\gamma_{ik}\beta_{kj}\alpha_{ji},
\]
We start by collecting some elementary facts about the above family of quivers with potential.

**Lemma 9.17.**  
(a) The potential $W_{\text{gen}}^{(1)}$ on the quiver $Q^{(1)}$ above is right equivalent to the potential $W_t^{(1)}$ if and only if

$$t = \frac{t_{112}t_{121}t_{211}t_{222}}{t_{111}t_{122}t_{212}t_{221}}.$$

Moreover, $(Q^{(1)}, W_t^{(1)}) = \mu_1(Q^{(2)}, W_t^{(2)}).$

(b) The potential $W_{\text{gen}}^{(2)}$ on $Q^{(2)}$ is right equivalent to $W_t^{(2)}$ if and only if

$$t = \frac{t_{111}t_{122}}{t_{12}t_{21}}.$$

(c) The potentials $W_{t/(t-1)}^{(3,2)}$ and $W_{(1-t)^{-1}}^{(3,3,3)}$ on $Q^{(3)}$ are right equivalent to $W_t^{(3,1)}$ if $t \neq 1$. Moreover, $(Q^{(3)}, W_t^{(3,1)}) = \mu_6(Q^{(2)}, W_t^{(2)}).$

(d) The potentials $W_{t/(t-1)}^{(4,2)}$ and $W_{(1-t)^{-1}}^{(4,3)}$ on $Q^{(4)}$ are right equivalent to $W_t^{(4,1)}$ if $t \neq 1$. Moreover, $(Q^{(4)}, W_t^{(4,1)}) = \mu_1(Q^{(3)}, W_t^{(3,1)}).$
We have that \( \nu \) is a bijection and \( \nu \) for all example from Lemma 9.17(c) that \( \nu \) also by Lemma 9.19. For Lemma 9.18. Let \( \mathbb{C} := \mathbb{C} \setminus \{0,1\} \) and consider the involutions \( f_1 \) and \( f_2 \) of \( \mathbb{C} \) which act as \( f_1(t) = t^{-1} \) and \( f_2(t) = 1 - t \). Then the following hold:

(i) The set \( F := \{ \text{Id}_\mathbb{C}, f_1, f_2, f_2 \circ f_1, f_1 \circ f_2, f_1 \circ f_2 \circ f_1 \} \) is with the composition of functions a Coxeter group of type \( A_2 \). In particular, \( f_1 \circ f_2 \circ f_1 = f_2 \circ f_1 \circ f_2 \).

(ii) If \( t \in \mathbb{C} \), then the orbit \( F(t) := \{ f(t) \mid f \in F \} \) is the set \( \{ t, t^{-1}, 1 - t, (t - 1)/t, 1/(t - 1), t/(1 - t) \} \).

We leave the easy proof to the reader. Putting together these facts we get the following:

**Lemma 9.19.** For \( t \in \mathbb{C} \setminus \{0,1\} \) let

\[
S_4(t) := \{(Q^{(i)}_s,W^{(i)}_s) \mid i \in \{1,2,3,4\}, s \in F(t)\}
\]

where we abbreviate \( W^{(i)}_s := W^{(i,1)}_s \) for \( i = 3,4 \). Then, if \( j \in \{1,2,\ldots,6\} \) and \( (Q,W) \in S_4(t) \) we have \( \mu_j(Q,W) \) isomorphic to an element of \( S_4(t) \). In particular, the elements of \( S_4(t) \) are non-degenerate QPs.

A slightly less precise result can be obtained by using tubular cluster categories of type \((2,2,2,2)\) see [GG1, 2.5.1].

Note that up to cyclic equivalence we may assume that any cycle in \( Q^{(2)} \) starts (and ends) in the vertex 1. The length of any cycle is a multiple of 4.

**Lemma 9.20.** We have for the above described quiver \( Q^{(2)} \) the following:

(a) Consider for \( t \in \mathbb{C} \) the ideal \( I_t := \langle \partial x W^{(2)}_t \mid x \in Q_1^{(1)} \rangle \) in the path algebra \( \mathbb{C} Q^{(2)}_t \). If \( t \neq 1 \), then any path \( p \) of length 4 with \( s(p) \neq t(p) \) belongs to \( I_t \) and any path of length 5 belongs to \( I_t \). In particular, \( \mathbb{C} Q^{(1)}_t/I_t \) is isomorphic to the Jacobian algebra \( \mathcal{P}(Q^{(2)}_t,W^{(2)}_t) \).

(b) Let \( W = W^{(2)}_t + W' \) be a potential on \( Q^{(2)}_t \), where \( W' \) is a possibly infinite linear combination of cycles of length at least 8 and \( t \neq 1 \), then \( W \) is right equivalent to \( W^{(2)}_t \).
Proof. (a) Because of Lemma 9.17(b) it is sufficient to analyze paths which start in the vertex 1 or 5. Modulo \(I_t\) we have for example
\[
\delta_2 \alpha_1 \beta_1 \gamma_1 \delta_{i_4} W_4^{(2)} \delta_2 \alpha_2 \beta_2 \gamma_1 = t \delta_2 \alpha_1 \beta_1 \gamma_1,
\]
which implies \(\delta_2 \alpha_1 \beta_1 \gamma_1 \in I_t\) if \(t \neq 1\). This proves (a). Part (b) is a direct consequence of Lemma 8.18, since \(W_t^{(2)}\) is homogeneous of degree 3 with respect to the path length grading. \(\square\)

**Lemma 9.21.** Let \(W_t := W_t^{(2)} + W'\) be a potential on \(Q^{(2)}\) where \(W'\) is a possibly infinite linear combination of cycles of length greater than 4. Then \(W_t\) is degenerate for \(t = 1\).

Proof. The QP \(\tilde{Q}, \tilde{W}_t := \tilde{\mu}_6 \tilde{\mu}_5 \mu_4 \mu_3 (Q^{(2)}, W_t^{(2)})\) looks as follows:

\[
\tilde{Q} : \begin{array}{c}
\alpha_1^2 \\
\alpha_2^2 \\
\beta_2^2 \\
\beta_1^2 \\
\gamma_1^2 \\
\gamma_2^2 \\
\delta_1^2 \\
\delta_2^2 \\
\end{array}
\] \[
\tilde{W}_t = \begin{array}{c}
\sum_{i=1}^{2} (\beta_i \alpha_i \gamma_i + \delta_i \gamma_i) \\
(t - 1) [\alpha_1 \beta_1 \gamma_2 \alpha_2] + \sum_{i,j=1}^{2} [\alpha_i \beta_i \gamma_j \delta_j]\end{array}
\]

Here \(\tilde{W}'\) is (possibly infinite) sum of cycles of length strictly greater than 2 built from the “composed” arrows \([\alpha_i \beta_i]\) and \([\gamma_i \delta_i]\), since up to cyclic equivalence we may assume that all cycles in \(W'\) start in the vertex 1.

Next, note that \(\mu_4 \mu_3 (Q^{(2)}, W_t^{(2)}) = \tilde{\mu}_6 \tilde{\mu}_5 \mu_4 \mu_3 (Q^{(2)}, W_t^{(2)})\) is 2-acyclic, and the vertices 5 and 6 are not joined by an arrow in \(\mu_4 \mu_3 (Q^{(2)})\). Thus,
\[
\tilde{\mu}_5 (\tilde{\mu}_4 \mu_4 \mu_3 (Q^{(2)}, W_t^{(2)}))_{\text{red}} = \tilde{\mu}_5 \tilde{\mu}_4 \mu_4 \mu_3 (Q^{(2)}, W_t^{(2)})_{\text{red}}.
\]
By [DWZ1] Proposition 4.15 the reduced part of \(\tilde{Q}, \tilde{W}_1\) is 2-acyclic if and only if \(\text{det}(111) \neq 0\). This is the case if and only \(t \neq 1\). This necessary condition for \(W_t\) being non-degenerate does clearly not depend on the choice of \(W'\). \(\square\)

**Proof of Proposition 9.17.** Note, that the quiver \(Q^{(1)}\) has 8 oriented cycles of length 3. These cycles are up to cyclic equivalence of the form \(C_{kji} := \gamma_{ik} \beta_{kj} \alpha_{ji}\) for \(i,j,k \in \{1, 2\}\). Each \(C_{kji}\) has to appear in a non-degenerate potential with a coefficient \(t_{kji} \in \mathbb{C}^*\).

By Lemma 9.17(a) it is sufficient to prove the claim of Proposition 9.15 by proving the analogous claim for \(Q^{(2)}\) in place of \(Q^{(1)}\). Let \(W = W_4 + W_{\geq 8}\) a non-degenerate potential on \(Q^{(2)}\), where \(W_4\) comprises all cycles of length 4 which appear in \(W\) with a non-zero coefficient. Now, we have
\[
\tilde{\mu}_1 (Q^{(2)}, W) = (Q^{(1)}, [W_4] + [W_{\geq 8}] + C_{111} + C_{121} + C_{211} + C_{221})\]

Note that \([W_4] = \sum_{i,j=1}^{2} t_{ij} C_{ij}\) and \([W_{\geq 8}]\) consists of cycles of length greater or equal to 6. By the above remark we conclude that \(W_4\) is of the form \(W_4^{(2)}\), and after rescaling we may assume that \(W_4 = W_t^{(2)}\) with \(t \in \mathbb{C}^*\). Now part (i) of Proposition 9.15 follows directly from Lemmas 9.20 9.21 and 9.19 Part (ii) can be checked easily by a direct calculation. \(\square\)

9.10. **Reminder on Galois coverings.** Let \(\Lambda\) be a finite dimensional basic \(\mathbb{C}\)-algebra. We say that a group grading \(\Lambda = \oplus_{g \in G} \Lambda_g\) for a group \(G\) is **compatible** if

- \(\Lambda_g \cdot \Lambda_h \subset \Lambda_{gh}\) for all \(g, h \in G\),
- \(\text{rad} \Lambda = \oplus_{g \in G} (\Lambda_g \cap \text{rad} \Lambda)\), where \(\text{rad} \Lambda\) is the Jacobson radical of \(\Lambda\),
- \(G\) is generated by \(\{g \in G \mid \Lambda_g \neq 0\}\).
If $1_\Lambda = e_1 + \cdots + e_n$ is a decomposition of the unit into primitive orthogonal idempotents, we can construct in this situation a Galois covering $\tilde{\Lambda}$ which is a locally bounded $\mathbb{C}$-category with objects $G \times \{1, \ldots, n\}$, morphism spaces $\tilde{\Lambda}((g, i), (h, j)) = (e_j \Lambda g^{-1} e_i, g)$ for all $((g, i), (h, j)) \in G \times \{1, \ldots, n\}$ and the obvious composition coming from the multiplication in $\Lambda$. Obviously, $G$ acts freely on $\tilde{\Lambda}$ and the orbit category can be identified with the subcategory $\Lambda'$ of the $\Lambda$-right modules which has the following objects: $\{e_1 \Lambda, \ldots, e_n \Lambda\}$. It is a standard task to realise that for a locally bounded category $\tilde{\Lambda}$ the category of finitely presented $\mathbb{C}$-linear functors $\tilde{\Lambda} \rightarrow \mathbb{C}$-mod behaves pretty much like the module category of a finite dimensional algebra and that in this context an appropriate version of the tame-wild theorem holds, see [DS2] for more details. In particular, a locally bounded category is wild if and only if it contains a finite subcategory which is wild. An important idea behind this construction is that the category of $\tilde{\Lambda}$-modules is equivalent to the category of $G$-graded $\Lambda$-modules, see for example [Gr].

If $\Lambda$ has a Galois covering which is wild, then $\Lambda$ is also wild. This follows from [DS1, Proposition 2]. Thus, if we can identify in a Galois covering $\tilde{\Lambda}$ of $\Lambda$ a convex subcategory which belongs to Unger’s list of concealed minimal wild algebras [Un], we can conclude that $\Lambda$ is wild.

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