Control Contraction Metrics on Finsler Manifolds

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**Abstract**—Control Contraction Metrics (CCMs) provide a nonlinear controller design involving an offline search for a Riemannian metric and an online search for a shortest path between the current and desired trajectories. In this paper, we generalize CCMs to Finsler geometry, allowing the use of non-Riemannian metrics. We provide open loop and sampled data controllers. The sampled data control construction presented here does not require real time computation of globally shortest paths, simplifying computation.

I. INTRODUCTION

Control synthesis for general nonlinear systems remains a challenging problem, and no one technique is recognised as universally applicable [1], [2], [3], [4]. Two popular classes of solution are explicit control constructions based on classical Lyapunov theory and model predictive techniques involving real time optimisation.

A Lyapunov function characterizes the stability of a system and is related to the intuitive idea of energy decaying in stable systems. A control Lyapunov function (CLF) is necessary and sufficient for controllability of a system [5], and for large classes of systems (those affine in controls), the construction of a controller given a CLF is simple [7]. However, control Lyapunov functions are in general difficult to find [8].

The Control Contraction Metric (CCM) method of control synthesis, introduced by Manchester and Slotine [9], simplifies the search for a Lyapunov function. Rather than explicitly search for a Lyapunov function, a convex search is performed for a CCM which measures distance between trajectories. The CCM may be thought of as inducing an infinite family of local Lyapunov functions. Online computation involves a search for a minimal path and integration of a local, differential control law along this minimal path. CCM controllers have more efficient online computation than nonlinear Model Predictive Control [10], and have been applied in several application areas, including mechanical systems [11], decentralized and distributed control [12], [13], [14] and collision-free motion planning [15].

CCMs are based closely on contraction analysis, introduced by Lohmiller and Slotine [16]. The central idea is that if all nearby trajectories converge to each other, then all trajectories converge to one nominal trajectory and the system is stable. The idea that global properties can be inferred from the local behaviour of trajectories does away with the need to construct global functions.

CCMs are inherently Riemannian - the Lyapunov function locally induced by a CCM is a Riemannian metric, similar to a traditional global quadratic Lyapunov function. The restriction to Riemannian metrics precludes the use of certain desirable non-Riemannian Lyapunov functions, for example \( p \)-norms with \( p \neq 2 \) and consensus algorithms based upon the Hilbert projection metric [17].

Contracting systems were first considered in terms of non-Riemannian Finsler metrics by Lewis [18]. Recent work on contraction analysis has also made use of non-Riemannian metrics, to identify system properties [19], [20], analyse a wider class of systems [21], [22] and generalize contraction ideas to allow different behaviours [23].

Forni and Sepulchre have recently provided a framework for contraction analysis that encompasses approaches using Riemannian metrics and approaches using other metrics through a generalization to Finsler geometry [24]. Furthermore, they have taken the first steps towards unifying contraction analysis and traditional Lyapunov theory, in an effort to make the powerful tools of Lyapunov theory available in contraction analysis.

The primary contribution of this paper is a generalization of CCMs to Finsler manifolds, unifying the frameworks of Manchester and Slotine [9] and Forni and Sepulchre [24]. This removes the restriction that metrics are Riemannian, allowing CCM controllers to be applied to a larger class of systems. Further, we provide a sampled data control construction that does not require the computation of minimal paths between trajectories. This controller allows the CCM method to be applied in cases where either a minimal path does not exist, or it is too costly to compute a minimal path online.

The remainder of this paper is structured as follows. In Section II we present the notation used throughout and review some fundamental definitions and results. In Section III we provide a differential characterization of stable closed loop systems. In Section IV we state and prove the fundamental result of this paper: a generalization of the open loop controller given by Manchester and Slotine [9] to Finsler manifolds. We use this open loop controller to construct several sampled data, closed loop controllers in Section V. Concluding remarks are made in Section VI.

II. NOTATION AND PRELIMINARIES

In this section, we introduce the notation used throughout the paper and review several important results and definitions that underlie the work presented in the following sections.

We adopt the notation of [24] and [9]. A manifold \( \mathcal{M} \) is a couple \( (\mathcal{M}, \mathcal{A}) \) where \( \mathcal{M} \) is a set and \( \mathcal{A} \) is a maximal atlas...
of \( \mathcal{M} \) that induces a Hausdorff, second countable topology. The tangent space at \( x \) and tangent bundle for \( \mathcal{M} \) are denoted by \( T_x \mathcal{M} \) and \( T \mathcal{M} \) respectively. We denote by \( \mathbb{R}^+ \) the set \( \{x \in \mathbb{R} : x \geq 0 \} \).

This paper considers control-affine nonlinear dynamical systems defined over a manifold \( \mathcal{M} \) of dimension \( n \). These take the form
\[
\dot{x} = f(x) + B(x)u
\]
where \( f \) is a \( C^1 \) vector field that maps \( x, t \in \mathcal{M} \times \mathbb{R} \) to vectors in \( T_x \mathcal{M} \), \( B \) is a smooth function and \( u \in \mathbb{R}^n \). A trajectory is a couple \((x, u), x : \mathbb{R}^+ \rightarrow \mathcal{M}, u : \mathbb{R}^+ \rightarrow \mathbb{R}^n\), such that \( x \) and \( u \) satisfy (1) for all \( t \in \mathbb{R}^+ \). Analysis of the differential dynamics, which characterize the linearization of the system (1) along trajectories, yield the results of this paper. The differential dynamics are given by
\[
\dot{\delta}_x = A(x, u) \delta_x + B(x) \delta_u, \tag{2}
\]
where \( A = \frac{\partial f}{\partial x} + \sum_{i=1}^n \frac{\partial f}{\partial x_i} u_i \), \( b_i \) represents the \( i^{th} \) column of \( B \) and \( u_i \) represents the \( i^{th} \) element of \( u \).

Several classes of real functions are referred to in the definitions that follow. A class \( K \) function \( \alpha \) is a locally Lipschitz function \( \alpha : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) which is strictly increasing with \( \alpha(0) = 0 \). A function \( \beta : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) belongs to class \( KL \) if, for all \( t \geq 0 \), \( \beta(\cdot, t) \) is a class \( K \) function and for all \( s \geq 0 \), \( \beta(s, \cdot) \) is nonincreasing and \( \lim_{t \rightarrow \infty} \beta(s, t) = 0 \).

Forni and Sepulchre [24] characterize stability of systems on manifold in the following definition.

**Definition 1 (Incremental stability [24]):** Consider the system \( \dot{x} = f(x, t) \) on a manifold \( \mathcal{M} \). Let \( d : \mathcal{M} \times \mathcal{M} \rightarrow \mathbb{R}^+ \) be a continuous distance on \( \mathcal{M} \). The system is
- **incrementally stable** on \( \mathcal{M} \) with respect to \( d \) if there exists a class \( K \) function \( \alpha \) such that, for any two trajectories \( x_1(t), x_2(t) : \mathcal{M} \rightarrow \mathcal{M} \), for all \( t \geq t_i \),
  \[
  d(x_1(t), x_2(t)) \leq \alpha(d(x_1(0), x_2(0))).
  \]
- **incrementally asymptotically stable** on \( \mathcal{M} \) if it is incrementally stable and, for any two trajectories \( x_1(t), x_2(t) : \mathcal{M} \rightarrow \mathcal{M} \),
  \[
  \lim_{t \rightarrow \infty} d(x_1(t), x_2(t)) = 0.
  \]
- **incrementally exponentially stable** on \( \mathcal{M} \) if there exists \( K \geq 1 \) and \( \lambda > 0 \) such that, for any two trajectories \( x_1(t), x_2(t) : \mathcal{M} \rightarrow \mathcal{M} \), for all \( t \geq t_i \),
  \[
  d(x_1(t), x_2(t)) \leq Ke^{-\lambda(t-t_i)}d(x_1(t_i), x_2(t_i)).
  \]

We extend these definitions to systems with control input, considering whether or not the system can be made to converge to a particular trajectory.

**Definition 2 (Stabilizability/controllability):** Consider the system (1) on a manifold \( \mathcal{M} \). Let \( d : \mathcal{M} \times \mathcal{M} \rightarrow \mathbb{R}^+ \) be a continuous distance on \( \mathcal{M} \). A trajectory \( x^* : \mathbb{R} \rightarrow \mathcal{M} \) of the system (1) is
- **controllable (stabilizable)** on \( \mathcal{M} \) with respect to \( d \) if there exists an open loop (resp. closed loop) control signal and a class \( K \) function \( \alpha \) such that, for any trajectory \( x(t) : \mathbb{R} \rightarrow \mathcal{M} \), for all \( t \geq t_i \),
  \[
  d(x^*(t), x(t)) \leq \alpha(d(x^*, x)).
  \]
- **asymptotically controllable (stabilizable)** on \( \mathcal{M} \) if there exists an open loop (resp. closed loop) control signal such that it is incrementally stable and, for any trajectory \( x(t) : \mathbb{R} \rightarrow \mathcal{M} \),
  \[
  \lim_{t \rightarrow \infty} d(x^*(t), x(t)) = 0.
  \]
- **exponentially controllable (stabilizable)** on \( \mathcal{M} \) if there exists an open loop (resp. closed loop) control signal, \( K \geq 1 \) and \( \lambda > 0 \) such that, for any trajectory \( x^*(t) : \mathbb{R} \rightarrow \mathcal{M} \), for all \( t \geq t_i \),
  \[
  d(x^*(t), x(t)) \leq Ke^{-\lambda(t-t_i)}d(x^*(t_i), x(t_i)).
  \]

All three types of controllability (stabilizability) are termed universal if any choice of trajectory \( x^* \) is controllable (stabilizable).

Classical Lyapunov theory and its recent extension to contraction analysis [24] tells us that the existence of a (Finsler-) Lyapunov function is equivalent to stability of a system. A candidate Finsler-Lyapunov function is defined as follows.

**Definition 3 (Finsler-Lyapunov function [24]):** Consider a manifold \( \mathcal{M} \) and a \( C^1 \) function \( V : T \mathcal{M} \rightarrow \mathbb{R}^+ \) such that, for all \( (x, \delta_x) \in T \mathcal{M} \),
\[
\gamma \in C \quad F(x, \delta_x)^p \leq V(x, \delta_x) \leq c_2 F(x, \delta_x)^p. \tag{3}
\]

\( F \) satisfies:
1. \( F \) is a \( C^1 \) function for every \( (x, \delta_x) \in T \mathcal{M} \) such that \( \delta_x \neq 0 \);
2. \( F(x, \delta_x) > 0 \) for each \( (x, \delta_x) \in T \mathcal{M} \) such that \( \delta_x \neq 0 \);
3. \( F(x, \lambda \delta_x) = \lambda F(x, \delta_x) \) for every \( \lambda \geq 0 \) and every \( (x, \delta_x) \in T \mathcal{M} \);
4. \( F(x, \delta_{x_1} + \delta_{x_2}) < F(x, \delta_{x_1}) + F(x, \delta_{x_2}) \) for every \( (x, \delta_{x_1}), (x, \delta_{x_2}) \in T \mathcal{M} \) such that \( \delta_{x_1} \neq \lambda \delta_{x_2} \) for any \( \lambda \in \mathbb{R} \).

We refer to a manifold \( \mathcal{M} \) endowed with a Finsler structure \( F \) as a Finsler manifold.

A key result of Forni and Sepulchre [24] is that, for a system with no control input, if a candidate Finsler-Lyapunov function \( V \) can be found such that
\[
V = \frac{\partial V}{\partial x} \dot{x} + \frac{\partial V}{\partial \delta_x} \dot{\delta}_x \leq -\alpha(V),
\]
then the system is incrementally stable, with the type of stability depending on the form of \( \alpha \). In this case, \( V \) is called a contraction measure.

The Finsler structure \( F \) endows the manifold \( \mathcal{M} \) with a global measure of distance. Before defining this distance, we introduce some necessary notation. A curve \( \gamma \) on a manifold \( \mathcal{M} \) is a function \( \gamma : I \subset \mathbb{R} \rightarrow \mathcal{M} \). We denote \( \partial \gamma / \partial s \) by \( \gamma_s \). A curve is regular if \( \gamma_s \neq 0 \) for all \( s \). The space \( \Gamma(x_1, x_2) \)
is defined as the set of all curves γ : [0, 1] → M such that γ(0) = x₁ and γ(1) = x₂.

**Definition 4 (Finsler distance):** Given a candidate Finsler-Lyapunov function V with Finsler structure F and defining I = [0, 1], the distance d : M × M → ℝ⁺ is given by

\[ d(x₁, x₂) = \inf_{Γ(x₁, x₂)} \int_0^1 F(γ(s), γₙ(s))ds \] (4)

Note that in general, d is not symmetric, that is, d(x₁, x₂) ≠ d(x₂, x₁). d is however positive definite and satisfies the triangle inequality. If \( \int_0^1 F(γ(s), γₙ(s))ds = d(x₁, x₂) \), γ is a minimizing geodesic. The Finsler manifold M is said to be forward geodesically complete if every geodesic γ(s) defined on s ∈ [a, b] can be extended to a geodesic defined on s ∈ [a, ∞). The Hopf-Rinow Theorem states that any two points x₁, x₂ ∈ M can be connected by a minimizing geodesic if M is forward geodesically complete. We refer the reader to Bao et al. [25] for a full treatment of the Hopf-Rinow theorem and geodesics on Finsler manifolds.

For notational convenience, this paper only considers time invariant f, B and V. However, the results extend in a straightforward manner to the time varying case.

**III. CONTRACTION OF SYSTEMS WITH CONTROL INPUTS**

We begin in a similar manner to [9, Prop. 1], by examining conditions on a contraction measure V for a system with control inputs, when a controller has been found that makes the system contract while allowing all solutions of the dynamics (1) to remain feasible.

**Proposition 1:** Suppose that, for the system (1) on a smooth manifold M, there exists a smooth feedback control of the form \( u = k(x, t) + v \) such that there exists a candidate Finsler-Lyapunov function V (with Finsler structure F) that gives

\[ \dot{V} = \frac{∂V}{∂x} \dot{x} + \frac{∂V}{∂δ_x} δ_x ≤ -α(V) \] (5)

for every \( t ∈ ℝ, x ∈ M, δ_x ∈ T_xM \) and \( v ∈ ℝ^n \) and some \( α : ℝ⁺ → ℝ⁺ \). Then, for all \( δ_x ≠ 0 \),

\[ \frac{∂V}{∂δ_x} B = 0 \implies \frac{∂V}{∂x} (f + Bu) + \frac{∂V}{∂δ_x} Aδ_x ≤ -α(V). \] (6)

**Proof:** We first note that \( δ_u = \frac{∂u}{∂δ_x} δ_x = Kδ_x \). Substituting the dynamics (1) and differential dynamics (2) in (5) gives

\[ \frac{∂V}{∂x} f + \frac{∂V}{∂δ_x} B(k + v) + \frac{∂V}{∂δ_x} \left( \frac{∂f}{∂x} + \sum_{i=1}^m \frac{∂b_i}{∂x} (k_i + v_i) \right) δ_x + \frac{∂V}{∂δ_x} B K δ_x ≤ -α(V). \]

As this is affine in v, for the right hand side to remain bounded for unbounded v, we require

\[ \frac{∂V}{∂x} b_i + \frac{∂V}{∂δ_x} \frac{∂b_i}{∂δ_x} = 0 \] (7)

For all i, where \( b_i \) is the i-th column of B. This gives

\[ \frac{∂V}{∂x} f + \frac{∂V}{∂δ_x} \frac{∂f}{∂x} δ_x + \frac{∂V}{∂δ_x} B K δ_x ≤ -α(V). \]

The result follows from letting \( (∂V/∂δ_x) B = 0 \) and adding (7).

**IV. OPEN LOOP CONTROL SYNTHESIS**

The result of Proposition 1 leads to the question of whether Condition (6) implies the existence of a stabilizing control law. In this section, we show that if a contraction measure V can be found such that (6) is true, the system can be universally stabilized by an open loop control signal. This provides a generalization of the open loop results of [9, Th. 1].

**Theorem 1:** Consider the system (1), (2) on a smooth manifold M with \( f ∈ C² \). Suppose there exists a candidate Finsler-Lyapunov function \( V ∈ C∞ \) with Finsler structure F such that

\[ \frac{∂V}{∂δ_x} B = 0 \implies \frac{∂V}{∂x} (f + Bu) + \frac{∂V}{∂δ_x} Aδ_x < -α(V) \] (8)

for every \( t ∈ ℝ⁺, x ∈ M, u ∈ ℝ^n \) and \( δ_x ∈ T_xM, δ_x ≠ 0 \). Furthermore, suppose that for all compact subsets \( X ⊂ ℝ^n \), for all \( x ∈ X \), and for all compact subsets \( Y ⊂ ℝ^n \) not containing 0, for all \( δ_x ∈ Y, \) the ratio

\[ \frac{∂V}{∂δ_x} B B^T \frac{∂V}{∂δ_x} \] (9)

is bounded, where

\[ \frac{∂V}{∂u} = \frac{∂V}{∂x} B + \frac{∂V}{∂δ_x} δ_x. \]

Then there exists an open loop control law such that the system is

- universally controllable on M if \( α(s) = 0 \) for all \( s ≥ 0 \);
- universally asymptotically controllable on M if \( α \) is a class K function;
- universally exponentially controllable on M if \( α(s) = λ s > 0 \) for each \( s > 0 \).

We refer to the function V as a (Finsler) control contraction metric (CCM).

The condition that (9) is bounded is true for any system meeting the conditions of Proposition 1 or the strong conditions given in [9, Sec. III. A.]. We now give the control law construction and then prove stabilizability.

**A. Open Loop Control Construction**

We construct a local control law that stabilizes the differential dynamics (2), following the construction of [9, Lemma 2]. We then integrate this along a path connecting an arbitrary target trajectory with the current trajectory and apply the control signal corresponding to the integral evaluated at the current trajectory.
Define
\[ a(x, \delta_x, u) = \frac{\partial V}{\partial x} (f + Bu) + \frac{\partial V}{\partial \delta_x} A\delta_x + \alpha(V) \]
and
\[ b(x, \delta_x) = \frac{\partial V}{\partial \delta_x} B^T \frac{\partial V}{\partial x}. \]

Let
\[ \rho(x, \delta_x, u) = \begin{cases} 0 & \text{if } a < 0 \\ \frac{a + \sqrt{a^2 + b^2}}{b} & \text{otherwise.} \end{cases} \]

The differential feedback control is given by
\[ k_\delta(x, \delta_x, u) = -\rho(x, \delta_x, u) B(x)^T \frac{\partial V(x, \delta_x)}{\partial \delta_x}. \tag{10} \]

The open loop control signal to stabilize the system to a target trajectory \((x^*(t), u^*(t)) \in C \times \mathbb{R}^n\) in a time interval \(T\) of the form \([t_i, t_e]\), \([t_i, t_e]\), or \([t_i, \infty)\) is then calculated as follows:

1. Measure \(x(t_i)\) and construct a smooth path \(c(t_i, s) \in \Gamma(x^*(t_i), x(t_i))\).
2. Solve the following equation for \(k_p\):
\[ k_p(c, u^*, t, s) = u^* + \int_0^s k_\delta(c(t, s), c_s(t, s), k_p(c, u^*, t, s), t) \, ds, \]
where \(c_s = \partial c/\partial s\).
3. For each \(t \in T\), apply the control signal \(u(t) = k_p(c(t, s), u^*(t), t, 1)\), where \(c(t, s)\) is the forward image of \((t, s)\) with the path of controls defined in Equation (2). That is, for all \(s \in [0, 1]\) and \(t \in T\), \(c(t, s)\) is a solution to
\[ \frac{d}{dt} c(t, s) = f(c(t, s), t) + B(c(t, s), t) k_p(c(t, u^*(t), t, s)). \]

**Proof:** [Proof of Theorem 1] It follows from [9, Lemma 2] that the differential control (10) makes the extended system dissipative with respect to the storage function \(V\) and supply rate \(\alpha(V)\), that is,
\[ \dot{V} = \frac{\partial V}{\partial x} (f + Bu) + \frac{\partial V}{\partial \delta_x} A\delta_x + Bk_\delta < -\alpha(V). \tag{11} \]
Indeed, substituting (10) into the left hand side of (11) gives
\[ \dot{V} = a - \rho b - \alpha(V) < -\alpha(V) - \sqrt{a^2 + b^2} < -\alpha(V). \]

We now show that the differential control signal (10) is integrable along regular curves in \(M\). That is, for any regular curve \(c : [0, 1] \to M\) and any \(v_0 \in \mathbb{R}^n, t \in \mathbb{R}^+\), a unique solution of the following integral equation exists on \(s \in [0, 1]\):
\[ v(s) = v_0 + \int_0^s k_\delta(c(s), c_s(s), v(s)) \, ds. \tag{12} \]

The condition (3) implies either \(b > 0\) or \(a < 0\). It follows from [7, Th. 1] that \(\rho\) is smooth for all \(x, u\) and \(\delta_x \neq 0\), and the apparent discontinuity at \(b = 0\) is removed by setting \(\rho = 0\) when \(b = 0\). Smoothness of \(\rho\) implies smoothness of \(k_\delta\) when \(\delta_x \neq 0\), which is the case in Equation (12) where \(\delta_x\) is set to \(c_s(s)\), which is non-zero by regularity of \((c(s), \delta(s))\).

It follows from [26, Th. 3.2] that a unique solution to (12) exists if \(k_\delta\) is a globally Lipschitz function with respect to its third argument for \(s \in [0, 1]\). As \(B\) and \(V\) are continuously differentiable and smooth respectively, the product \(B(\partial V/\partial \delta_x)\) is bounded on closed intervals. Hence if \(\rho\) is Lipschitz with respect to \(u\) on \(s \in [0, 1]\), too, then \(k_\delta\). As \(\rho\) is smooth, it is globally Lipschitz if its derivative with respect to its third argument is bounded. This is clear for \(b \leq 0\). For \(b > 0\), noting that the only dependence \(\rho\) has on \(u\) is via \(a\), we have
\[ \frac{\partial \rho}{\partial u} = \frac{\partial}{\partial u} \left( \frac{\alpha(u)}{b} \right) = \frac{1}{b} \frac{a(u)}{\sqrt{a^2(u) + b^2}}. \]
Since \(a = 0 \Longrightarrow b > 0\) and \(a\) is affine in \(u\), the only term that can be unbounded is \((1/b)(\partial a/\partial u)\). However, this is precisely the term which is bounded by Condition (9). Hence \(\rho\) is Lipschitz with respect to \(u\) for \(s \in [0, 1]\) and a solution to (12) exists.

We now show that applying the control law of Section IV-A makes the initial trajectory \(x\) converge to the (arbitrary) chosen trajectory \(x^*\). Universal exponential stabilizability follows.

Consider a regular curve \(c(t, s) \in \Gamma(x^*(t), x(t))\). Then, for all \(t \geq t_i\), we have \(c(t, 0) = x^*(t)\) and \(c(t, 1) = x(t)\). Furthermore, for all \(t \geq t_i\) and all \(s \in [0, 1]\), \(c_s = \partial c/\partial s\) satisfies the differential dynamics (2):
\[ \frac{d}{dt} c_s(t, s) = A c_s + B k_\delta. \]
It follows from (11) that
\[ \frac{d}{dt} V(c(t), c_s(t)) < -\alpha(V(c(t), c_s(t))). \tag{13} \]

We now consider three cases, corresponding to the three forms of \(\alpha\) given in the statement of Theorem 1. If \(\alpha(s) = 0\), (13) gives \(\frac{d}{dt} V(c(t), c_s(t)) < 0\), so \(V(c(t), c_s(t)) < V(c(t_i), c_s(t_i))\) for all \(t \geq t_i\). As \(V\) is non-negative, this gives \(V(c(t), c_s(t))^{1/p} < V(c(t_i), c_s(t_i))^{1/p}\).
It follows that
\[ d(x^*(t), x(t)) \leq \int_I F(c(t), c_s(t)) \, ds \leq c_1 \int_I V(c(t), c_s(t))^{1/p} \, ds \leq c_1 \int_I V(c(t_i), c_s(t_i))^{1/p} \, ds \leq \left( \frac{c_2}{c_1} \right)^{1/p} \int_I F(c(t_i), c_s(t_i)) \, ds. \]
As the choice of \(x^*\) is arbitrary, this implies that the system is universally controllable.
If $\alpha(V)$ is a class $K$ function, \cite{13} again gives $\frac{d}{dt} V(c(t), c_s(t)) < 0$, so the system is universally controllable. Furthermore, as shown in the proof of \cite{24, Th. 1}, there exists a $KL$ function $\beta$ such that

$$V(c(t), c_s(t)) \leq \beta(C(c(t_i), c_s(t_i), t - t_i)).$$

Integrating with respect to $s$,

$$\int_I F(c(t), c_s(t)) ds \leq c_1^{-1} \int_I \beta(V(c(t_i), c_s(t_i)), t - t_i) ds$$

$$d(x^*(t), x(t)) \leq c_1^{-1} \int_I \beta(V(c(t_i), c_s(t_i)), t - t_i) ds$$

$$\lim_{t \to \infty} d(x^*(t), x(t)) \leq c_1^{-1} \int_I \beta(V(c(t_i), c_s(t_i)), t - t_i) ds$$

where the final equality follows from the definition of a $KL$ function and Lebesgue’s dominated convergence theorem. As the choice of $x^*$ is arbitrary, universal asymptotic controllability follows.

If $\alpha(V) = \lambda V$, \cite{13} and \cite{27, Th. 6.1} give

$$V(c(t), c_s(t)) < e^{-\lambda(t-t_i)} V(c(t_i), c_s(t_i))$$

$$\int_I F(c(t), c_s(t)) ds \leq c_1^{-1} \int_I \beta(V(c(t_i), c_s(t_i)), t - t_i) ds$$

$$d(x^*(t), x(t)) \leq c_1^{-1} \int_I \beta(V(c(t_i), c_s(t_i)), t - t_i) ds$$

That is, the trajectories $x$ and $x^*$ converge exponentially with rate $\lambda/p$. Furthermore, if $c(t_i)$ is a minimising geodesic,

$$d(x^*(t), x(t)) < \left(\frac{c_2}{c_1}\right)^{\frac{1}{p}} e^{-\frac{1}{p}(t-t_i)}$$

$$d(x^*(t_i), x(t_i)).$$

This implies that, if $c$ is a minimising geodesic, the overshoot is bounded above by $(c_2/c_1)^{1/p}$. As the choice of $x^*$ is arbitrary, this proves Theorem \cite{1} for the case $\alpha(V) = \lambda V$.

Remark 1: The control scheme proposed in Section \cite{V-A} appears difficult to compute. Computation of this scheme is dealt with (under the restricted class of Riemannian metrics) in several other papers. Manchester and Slotine \cite{9} define a continuous feedback control which removes the need for Step 3 of the open loop construction. Leung and Manchester \cite{10} present a pseudospectral approach for the computation of the path in Step 1, and show that this approach is more efficient than nonlinear Model Predictive Control. While we do not detail any computational methods, the following two examples illustrate the construction of open loop controllers for simple systems.

Example 1: Let $M = \mathbb{R}^2$ and consider the system

$$\dot{x} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u.$$  

Let $V = \delta^2_1 + \delta^2_2$, with one possible Finsler structure given by $V^{1/4}$. Then

$$\frac{\partial V}{\partial \delta_x} B = 4 \delta^3_1,$$

which is zero at $\delta_1 = 0$, and

$$\frac{\partial V}{\partial \delta_x} \left( \frac{\partial f}{\partial x} + \frac{\partial B}{\partial x} u \right) \delta_x = 4 \left( \delta^4_1 - \delta^2_2 \right),$$

which is strictly less than $(-4 + \varepsilon) \delta^2_2$ when $\delta_1 = 0$. Furthermore, the numerator of \cite{9} is always zero as $\partial B/\partial x = 0$, so the ratio \cite{9} is bounded. Hence it follows from Theorem \cite{1} that this system is universally controllable.

Treating $\varepsilon$ as zero, we have

$$\rho = \frac{-1 - \sqrt{1 + 16\delta_1}}{4\delta^2_1}$$

$$k_3 = -\delta_1 - \delta_1 \sqrt{1 + 16\delta^2_1}.$$

Now suppose that our initial position is $(1, 1)$ and our desired trajectory is $x^*(t) = (0, 0)$, $u^* = 0$. A path connecting our initial and desired position is given by $c(t, s) = (s, s)$. We then have the following equation for $k_p$:

$$k_p = \int_0^s \frac{\partial c}{\partial s} - \frac{\partial c}{\partial s} \sqrt{1 + 16 \left( \frac{\partial c}{\partial s} \right)^4} \, ds.$$  

This is solved approximately by discretising both in time and with respect to $s$ along the curve $c(t, s)$. At each time step, Equation \cite{15} is solved approximately by quadrature. The forward image of each discretised point on $c(t, s)$ is then calculated by numerical integration of the system dynamics with the newly computed control signal.

Figure \cite{1} illustrates the time response of the unstable state given this control scheme. The response of the same system with control calculated using the Riemannian metric $V = \delta^2_1 + \delta^2_2$ is also illustrated.
Example 2: Consider the one dimensional system $\dot{\vartheta} = -\sin \vartheta + u$, which approximates an overdamped pendulum. Let $\mathcal{M} = [0, \pi]$. Note that the system has a stable equilibrium at $\vartheta = 0$ (the downright equilibrium) and an unstable equilibrium at $\vartheta = \pi$ (the upright equilibrium). We compare control computed with a Riemannian (and hence symmetric) metric $V_1 = 4\delta_x^2$, and an asymmetric Finsler metric $V_2 = (2\sqrt{\delta_x} - \delta_0)^2$ (which is the square of a Randers metric [25, Sec. 1.3C]). Both metrics satisfy the conditions of Theorem 1 with a Finsler structure given by $F_i = \sqrt{V_i}$, except that $V_2$ is only $C^1$ and not smooth. This means the proof of integrability does not apply. However, we find in this case that a controller can be computed.

The induced Finsler distance $d(x_1, x_2)$ may be thought of as the cost to go from state $x_1$ to state $x_2$. If $x_1 > x_2$, use of the asymmetric $V_2$ and $F_2$ gives $d(x_1, x_2) > d(x_2, x_1)$.

In real terms, it is more costly to rotate the pendulum downwards than upwards. Intuitively, there is no reason for a “cost to go” to be symmetric - in this case, rotating the pendulum downwards represents a loss of potential energy and can be deemed as more expensive than the corresponding gain in potential energy. The controller computed with this asymmetric metric uses larger control input to move the pendulum from $\vartheta = 0$ (the downright equilibrium) and an unstable equilibrium at $\vartheta = \pi$ (the upright equilibrium). We compare control computed with a Riemannian (and hence symmetric) metric $V_1 = 4\delta_x^2$, and an asymmetric Finsler metric $V_2 = (2\sqrt{\delta_x} - \delta_0)^2$ (which is the square of a Randers metric [25, Sec. 1.3C]). Both metrics satisfy the conditions of Theorem 1 with a Finsler structure given by $F_i = \sqrt{V_i}$, except that $V_2$ is only $C^1$ and not smooth. This means the proof of integrability does not apply. However, we find in this case that a controller can be computed.

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Example 3: Let $\mathcal{M} = \mathbb{R}$ and consider the system $\dot{x} = -x + x^2u$.

This example illustrates the importance of the ratio (9) remaining bounded. With a Lyapunov function of $V = \delta_x^2$ and Finsler structure $F = \sqrt{V}$, we have

$$\frac{\partial V}{\partial \delta_x} B = 2\delta_x x^2,$$

and

$$\frac{\partial V}{\partial \delta_x} \left( \frac{\partial f}{\partial x} + \frac{\partial B}{\partial x} u \right) \delta_x = -2\delta_x^2 + 4\delta_x^2 u x.$$

Hence the system and Lyapunov function meet the requirement (9). However, computing the ratio (9) gives $1/x^3$, which is unbounded as $x \to 0$. This means the control signal cannot be integrated. Intuitively, if we begin with an initial condition of zero, we are at a stable equilibrium with no control input and can never leave.

V. CLOSED LOOP CONTROL SYNTHESIS

A natural question is whether the open loop results of the previous section can be adapted to a closed loop controller. In this section, we develop sampled data feedback controllers that guarantee universal stabilizability. We first give a general sampled data control construction, then prove its properties.
A. Closed Loop Control Construction

Given a continuous time period \( T = [t_i, t_e] \), we construct a closed loop control calculated at discrete times in \( T \) as follows.

1) At the initial time \( t_i \), measure the present state \( x(t_i) \) and construct a path \( c_i \in F(x(t_i), x(t_i)) \).

2) Run the open loop control constructed in Section V-A on the interval \( [t_i, t_{i+1}] \) for some \( t_{i+1} > t_i \).

3) At time \( t_{i+1} \), compute a new path \( c_{i+1} \) such that

\[
\int I V(c_{i+1}(t_{i+1}), c_{s,i+1}(t_{i+1})) \frac{d}{dt} ds \\
\leq \int I V(c_{i}(t_{i+1}), c_{s,i}(t_{i+1})) \frac{d}{dt} ds \tag{16}
\]

for all \( s \in I \) and return to step 2.

Note that the sample times \( t_i, t_{i+1}, \ldots \) may be chosen arbitrarily.

Proposition 2: Consider the system (1, 2) on a smooth manifold \( M \) with \( f \in C^2 \). Suppose there exists a Finsler-Lyapunov function \( V \in C^\infty \) with Finsler structure \( F \) that satisfies the conditions of Theorem 1. Then the system is

- universally stabilizable on \( M \) via the control construction of Section V-A if \( \alpha = 0 \).
- universally exponentially stabilizable on \( M \) via the control construction of Section V-A if \( \alpha(V) = -\lambda V \) for fixed \( \lambda > 0 \).

Remark 2: In the open loop case, if \( \alpha \) is a class \( K \) function, the system exhibits asymptotic stabilizability. However, as we have no information about the rate of convergence, for the sampled data controller we can only guarantee regular stabilizability (the distance between the current and target trajectories remains bounded for all time).

Proof: [Proof of Proposition 2] First consider the case \( \alpha = 0 \). Then, for the first time period, (13) gives

\[
\int I V(c(t), c_s(t)) \frac{d}{dt} ds \leq \int I V(c_i(t_i), c_{s,i}(t_i)) \frac{d}{dt} ds.
\]

On the second time period, we have

\[
\int I V(c_{i+1}(t), c_{s,i+1}(t)) \frac{d}{dt} ds \\
\leq \int I V(c_{i+1}(t_{i+1}), c_{s,i+1}(t_{i+1})) \frac{d}{dt} ds \\
\leq \int I V(c_i(t_{i+1}), c_{s,i}(t_{i+1})) \frac{d}{dt} ds \\
\leq \int I V(c_i(t_{i+1}), c_{s,i}(t_{i+1})) \frac{d}{dt} ds.
\]

By induction, it follows that, on any time period,

\[
\int I V(c_{i+j}(t), c_{s,i+j}(t)) \frac{d}{dt} ds < e^{-\alpha(t-t_i)} \int I V(c_i(t_i), c_{s,i}(t_i)) \frac{d}{dt} ds.
\]

This gives, for all \( t \),

\[
d(x^*(t), x(t)) \leq \int I F(c_{i+j}(t), c_{s,i+j}(t)) ds \\
\leq c_1^{-\frac{1}{p}} \int I V(c_{i+j}(t), c_{s,i+j}(t)) \frac{d}{dt} ds \\
< c_1^{-\frac{1}{p}} \int I V(c_i(t_i), c_{s,i}(t_i)) \frac{d}{dt} ds \\
< \left( \frac{c_2}{c_1} \right)^{t-t_i} \int I F(c_i(t_i), c_{s,i}(t_i)) ds.
\]

This proves universal exponential stabilizability for the case \( \alpha = 0 \).

Now consider the case \( \alpha(V) = -\lambda V \). On the first time interval, (14) gives

\[
\int I V(c(t), c_s(t)) \frac{d}{dt} ds < e^{-\frac{\lambda}{\alpha}(t-t_i)} \int I V(c_i(t_i), c_{s,i}(t_i)) \frac{d}{dt} ds.
\]

On the second time period, we have

\[
\int I V(c_{i+1}(t), c_{s,i+1}(t)) \frac{d}{dt} ds \\
< e^{-\frac{\lambda}{\alpha}(t-t_i+1)} \int I V(c_{i+1}(t_{i+1}), c_{s,i+1}(t_{i+1})) \frac{d}{dt} ds \\
\leq e^{-\frac{\lambda}{\alpha}(t-t_i+1)} \int I V(c_i(t_{i+1}), c_{s,i}(t_{i+1})) \frac{d}{dt} ds \\
\leq e^{-\frac{\lambda}{\alpha}(t-t_i)} \int I V(c_i(t_i), c_{s,i}(t_i)) \frac{d}{dt} ds.
\]

By induction, it follows that, on any time period,

\[
\int I V(c_{i+j}(t), c_{s,i+j}(t)) \frac{d}{dt} ds < e^{-\frac{\lambda}{\alpha}(t-t_i)} \int I V(c_i(t_i), c_{s,i}(t_i)) \frac{d}{dt} ds.
\]

This gives, for all \( t \),

\[
d(x^*(t), x(t)) \leq \int I F(c_{i+j}(t), c_{s,i+j}(t)) ds \\
\leq c_1^{-\frac{1}{p}} \int I V(c_{i+j}(t), c_{s,i+j}(t)) \frac{d}{dt} ds \\
< c_1^{-\frac{1}{p}} e^{-\frac{\lambda}{\alpha}(t-t_i+1)} \int I V(c_i(t_{i+1}), c_{s,i}(t_i)) \frac{d}{dt} ds \\
\leq \left( \frac{c_2}{c_1} \right)^{t-t_i} e^{-\frac{\lambda}{\alpha}(t-t_i)} \int I F(c_i(t_i), c_{s,i}(t_i)) ds.
\]

This proves universal exponential stabilizability for \( \alpha(V) = -\lambda V \).

Example 4: Consider the case \( M = \mathbb{R}^n \) with a Finsler-Lyapunov function \( V(x, \delta x) = \delta x^T M \delta x \) for some matrix \( M \). Suppose there exist bounds \( c_1, c_2 \) such that, for the Finsler structure \( F = \sqrt{\delta x^T M \delta x} \), \( c_1 F(x, \delta x)^2 \leq V(x, \delta x) \leq c_2 F(x, \delta x)^2 \). \( F \) gives a Riemannian structure on \( \mathbb{R}^n \). If \( V \) satisfies (8) for the system (1, 2), \( M \) is a control contraction metric in the sense of [9, Th. 1]. In this setting, the construction of Section V-A is a generalization.
of the sampled data controller given in [9] in two senses. Firstly, it provides universal stabilizability under the weaker dissipation condition \( \alpha = 0 \). Secondly, it does not require computation of minimising geodesics - any initial path \( c_i \) may be used, and the only condition on subsequent paths is that they are no longer than the forward image of the previous path (at the same time \( t \)). This allows, for example, the path to be refined at sample points via a local search for a shorter path, without requiring the solution of a global shortest path.

VI. CONCLUSIONS

This work generalizes Control Contraction Metrics to Finsler manifolds. This allows a larger class of metrics to be used to measure distances between trajectories, increasing the class of problems to which CCM methods can be applied.

The sampled data controllers constructed in Section V do no require computation of shortest paths between points (minimising geodesics). This allows the controllers to be applied in cases where minimising geodesics either do not exist or are too computationally expensive to compute.

Further work remains to be done and will be the subject of future papers. The open loop controller constructed in Section IV requires the Finsler-Lyapunov function to be smooth. This precludes the use of certain desirable metrics. This precludes the use of certain desirable metrics.

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