Factorization of the Non-Stationary Schrödinger Operator

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Abstract

We consider a factorization of the non-stationary Schrödinger operator based on the parabolic Dirac operator introduced by Cerejeiras/Kähler/Sommen. Based on the fundamental solution for the parabolic Dirac operators, we shall construct appropriated Teodorescu and Cauchy-Bitsadze operators. Afterwards we will describe how to solve the nonlinear Schrödinger equation using Banach fixed point theorem.

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1 Introduction

Time evolution problems are of extreme importance in mathematical physics. However, there is still a need for special techniques to deal with these problems, specially when non-linearities are involved.

For stationary problems, the theory developed by K. Gürlebeck and W. Sprößig [10], based on an orthogonal decomposition of the underlying function space in terms of the subspace of null-solutions of the corresponding Dirac operator, has been successfully applied to a wide range of equations, for instance Lamé, Navier-Stokes, Maxwell or Schrödinger equations [6], [10], [11], [2] or [13]. Unfortunately, there is no easy way to extend this theory directly to non-stationary problems.

In [7] the authors proposed an alternative approach in terms of a Witt basis. This approach allowed a successful application of the already existent techniques of elliptic function theory (see [10], [6]) to non-stationary problems in time-varying domains. Namely, a suitable orthogonal decomposition for the
underlying function space was obtained in terms of the kernel of the parabolic Dirac operator and its range after application to a Sobolev space with zero boundary-values.

In this paper we wish to apply this approach to study the existence and uniqueness of solutions of the non-stationary nonlinear Schrödinger equation.

Initially, in section two, we will present some basic notions about complexified Clifford algebras and Witt basis. In section three we will present a factorization for the operators $(±i\partial_t - \Delta)$ using an extension of the parabolic Dirac operator introduced in [1]. For the particular case of the non-stationary Schrödinger operator we will present the corresponding Teodorescu and Cauchy-Bitsadze operators in analogy to [10]. Moreover, we will obtain some direct results about the decomposition of $L_p$-spaces and the resolution of the linear Schrödinger problem.

In the last section we will present an algorithm to solve numerically the non-linear Schrödinger and we prove its convergence in $L_2$-sense using Banach’s fixed point theorem.

2 Preliminaries

We consider the $m$-dimensional vector space $\mathbb{R}^m$ endowed with an orthonormal basis $\{e_1, \ldots, e_m\}$.

We define the universal Clifford algebra $\mathcal{C}\ell_{0,m}$ as the $2^m$-dimensional associative algebra which preserves the multiplication rules $e_ie_j + e_je_i = -2\delta_{i,j}$. A basis for $\mathcal{C}\ell_{0,m}$ is given by $e_0 = 1$ and $e_A = e_{h_1} \cdots e_{h_k}$, where $A = \{h_1, \ldots, h_k\} \subset M = \{1, \ldots, m\}$, for $1 \leq h_1 < \cdots < h_k \leq m$. Each element $x \in \mathcal{C}\ell_{0,m}$ will be represented by $x = \sum x_A e_A, x_A \in \mathbb{R}$, and each non-zero vector $x = \sum_{j=1}^m x_j e_j \in \mathbb{R}^m$ has a multiplicative inverse given by $\frac{1}{2|x|}$. We denote by $\overline{x}\mathcal{C}\ell_{0,m}$ the (Clifford) conjugate of the element $x \in \mathcal{C}\ell_{0,m}$, where

$$\overline{x}\mathcal{C}\ell_{0,m} = 1, \overline{e_j}\mathcal{C}\ell_{0,m} = -e_j, \overline{ab}\mathcal{C}\ell_{0,m} = \overline{b}\mathcal{C}\ell_{0,m}\overline{a}\mathcal{C}\ell_{0,m}.$$

We introduce the complexified Clifford algebra $\mathcal{C}\ell_m$ as the tensorial product

$$\mathbb{C} \otimes \mathcal{C}\ell_{0,m} = \left\{ w = \sum_A z_A e_A, \ z_A \in \mathbb{C}, \ A \subset M \right\}$$

where the imaginary unit interacts with the basis elements via $ie_j = e_ji, j = 1, \ldots, m$. The conjugation in $\mathcal{C}\ell_m = \mathbb{C} \otimes \mathcal{C}\ell_{0,m}$ will be defined as $\overline{w} = \sum_A \overline{z_A} \mathcal{C}\ell_{0,m}$. Let us remark that for $a, b \in \mathcal{C}\ell_m$ we have $|ab| \leq 2^m|a||b|$.

We introduce the Dirac operator $D = \sum_{j=1}^m e_j\partial_{x_j}$. It factorizes the $m$-dimensional Laplacian, that is, $D^2 = -\Delta$. A $\mathcal{C}\ell_m$-valued function defined on an open domain $\Omega$, $u : \Omega \subset \mathbb{R}^m \mapsto \mathcal{C}\ell_m$, is said to be left-monogenic if it satisfies $Du = 0$ on $\Omega$ (resp. right-monogenic if it satisfies $uD = 0$ on $\Omega$).

A function $u : \Omega \mapsto \mathcal{C}\ell_m$ has a representation $u = \sum_A u_A e_A$ with $\mathbb{C}$-valued components $u_A$. Properties such as continuity will be understood component-wisely. In the following we will use the short notation $L_p(\Omega), C^k(\Omega)$, etc.,
instead of $L_p(\Omega, C\ell_m)$, $C^k(\Omega, C\ell_m)$. For more details on Clifford analysis, see [5], [12], [4] or [9].

Taking into account [7] we will imbed $\mathbb{R}^m$ into $\mathbb{R}^{m+2}$. For that purpose we add two new basis elements $f$ and $f^\dagger$ satisfying

$$f^2 = f^\dagger 2 = 0, \quad ff^\dagger + f^\dagger f = 1, \quad fe_j + e_j f = f^\dagger e_j + e_j f^\dagger = 0, \quad j = 1, \ldots, m.$$  

This construction will allows us to use a suitable factorization of the time evolution operators where only partial derivatives are used.

# 3 Factorization of time-evolution operators

In this section we will study the forward/backward Schrödinger equations,

$$(\pm i\partial_t - \Delta)u(x, t) = 0, \quad (x, t) \in \Omega, \quad (1)$$

where $\Omega \subset \mathbb{R}^m \times \mathbb{R}^+$, $m \geq 3$, stands for an open domain in $\mathbb{R}^m \times \mathbb{R}^+$. We remark at this point that $\Omega$ is a time-varying domain and, therefore, not necessarily a cylindric domain.

Taking account the ideas presented in [1] and [7] we introduce the following definition

**Definition 3.1.** For a function $u \in W^1_p(\Omega)$, $1 < p < +\infty$, we define the forward (resp. backward) parabolic Dirac operator

$$D_{x, \pm it}u = (D + f\partial_t \pm if^\dagger)u, \quad (2)$$

where $D$ stands for the (spatial) Dirac operator.

It is obvious that $D_{x, \pm it} : W^1_p(\Omega) \rightarrow L_p(\Omega)$.

These operators factorize the correspondent time-evolution operator $[1]$, that is

$$(D_{x, \pm it})^2 u = (\pm i\partial_t - \Delta)u. \quad (3)$$

Moreover, we consider the generic Stokes’ Theorem

**Theorem 3.2.** For each $u, v \in W^1_p(\Omega)$, $1 < p < +\infty$, it holds

$$\int_{\Omega} vd\sigma_{x, t} u = \int_{\partial\Omega} [(vD_{x, -it})u + v(D_{x, +it})]d\Omega$$

where the surface element is $d\sigma_{x, t} = (D_x + f\partial_t)|dxdt$, the contraction of the homogeneous operator associated to $D_{x, -it}$ with the volume element.

We now construct the fundamental solution for the time-evolution operator $-\Delta - i\partial_t$. For that purpose, we consider the fundamental solution of the heat operator

$$e(x, t) = \frac{H(t)}{(4\pi t)^{m/2}} \exp \left( -\frac{|x|^2}{4t} \right), \quad (4)$$
where \( H(t) \) denotes the Heaviside-function. Let us remark that the previous fundamental solution verifies
\[
(-\Delta + \partial_t)e(x,t) = \delta(x)\delta(t).
\]

We apply to (4) the rotation \( t \rightarrow it \). There we obtain
\[
(-\Delta - i\partial_t)e(x, it) = -\Delta e(x, it) + \partial_t e(x, it) = \delta(x)\delta(it) = -i\delta(x)\delta(t),
\]
i.e., the fundamental solution for the Schrödinger operator \(-\Delta - i\partial_t\) is
\[
e_-(x, t) = ie(x, it) = i H(t)(4\pi it/m^2 \exp\left(i \frac{|x|^2}{4t}\right)). \tag{5}
\]

Then we have

**Definition 3.3.** Given the fundamental solution \( e_- = e_-(x, t) \) we have as fundamental solution \( E_- = E_-(x, t) \) for the parabolic Dirac operator \( D_{x, -it} \) the function
\[
E_-(x, t) = e_-(x, t)D_{x, -it} = \frac{H(t)}{(4\pi it)^{\frac{3}{2}}} \exp\left(i \frac{|x|^2}{4t}\right) \left(\frac{-x}{2t} + \frac{f}{2t} - \frac{im}{2t} + f^*\right) \tag{6}
\]

If we replace the function \( v \) by the fundamental solution \( E_- \) in the generic Stoke’s formula presented before, we have, for a function \( u \in W^1_p(\Omega) \) and a point \((x_0, t_0) \notin \partial\Omega\), the Borel-Pompeiu formula,
\[
\int_{\partial\Omega} E_-(x - x_0, t - t_0)d\sigma_{x,t} u(x, t) = u(x_0, t_0) + \int_{\Omega} E_-(x - x_0, t - t_0)(D_{x, +it}u)dxdt. \tag{7}
\]

Moreover, if \( u \in \ker(D_{x, +it}) \) we obtain the Cauchy’s integral formula
\[
\int_{\partial\Omega} E_-(x - x_0, t - t_0)d\sigma_{x,t} u(x, t) = u(x_0, t_0).
\]

Based on expression (7) we define the Teodorescu and Cauchy-Bitsadze operators.

**Definition 3.4.** For a function \( u \in L_p(\Omega) \) we have
(a) the Teodorescu operator
\[
T_- u(x_0, t_0) = \int_{\Omega} E_-(x - x_0, t - t_0)u(x, t)dxdt \tag{8}
\]
(b) the Cauchy-Bitsadze operator
\[
F_- u(x_0, t_0) = \int_{\partial\Omega} E_-(x - x_0, t - t_0)d\sigma_{x,t} u(x, t), \tag{9}
\]
for \((x_0, t_0) \notin \partial\Omega\).
Using the previous operators, (7) can be rewritten as
\[ F_- u = u + T_- D_{x,-it} u, \]
whenever \( v \in W^1_p(\Omega), \ 1 < p < \infty. \)

Moreover, the Teodorescu operator is the right inverse of the parabolic Dirac operator \( D_{x,-it} \), that is,
\[
D_{x,-it} T u = \int_\Omega D_{x,-it} E_-(x - x_0, t - t_0) u(x, t) dx dt \\
= \int_\Omega \delta(x - x_0, t - t_0) u(x, t) dx dt \\
= u(x_0, t_0),
\]
for all \((x_0, t_0) \in \Omega.\)

In view of the previous definitions and relations, we obtain the following results, in an analogous way as in [7].

**Theorem 3.5.** If \( v \in W^2_p(\partial\Omega) \) then the trace of the operator \( F_- \) is
\[
\text{tr}(F_- v) = \frac{1}{2} v - \frac{1}{2} S_- v, \tag{10}
\]
where
\[
S_- v(x_0, t_0) = \int_{\partial\Omega} E_-(x - x_0, t - t_0) d\sigma_{x,t} v(x, t)
\]
is a generalization of the Hilbert transform.

Also, the operator \( S_- \) satisfies \( S_-^2 = I \) and, therefore, the operators
\[
P = \frac{1}{2} I + \frac{1}{2} S_- , \quad Q = \frac{1}{2} I - \frac{1}{2} S_- 
\]
are projections into the Hardy spaces.

Taking account the ideas presented in [7] an immediate application is given by the decomposition of the \( L_p \)-space.

**Theorem 3.6.** The space \( L_p(\Omega) \), for \( 1 < p \leq 2 \), allows the following decomposition
\[
L_p(\Omega) = L_p(\Omega) \cap \ker(D_{x,-it}) \oplus D_{x,it} \left( W^1_p(\Omega) \right) ,
\]
and we can define the following projectors
\[
P_- : L_p(\Omega) \to L_p(\Omega) \cap \ker(D_{x,-it}) \\
Q_- : L_p(\Omega) \to D_{x,-it} \left( W^1_p(\Omega) \right) .
\]
Proof. Let us denote by \((-\Delta - i\partial_t)_0^{-1}\) the solution operator of the problem
\[
\begin{align*}
(-\Delta - i\partial_t)u &= f \text{ in } \Omega \\
u &= 0 \text{ on } \partial\Omega
\end{align*}
\]

As a first step we take a look at the intersection of the two subspaces \(D_{x,-it} \left( \overset{\circ}{W}^1_p(\Omega) \right)\) and \(L^p(\Omega) \cap \ker (D_{x,-it})\).

Consider \(u \in L^p(\Omega) \cap \ker (D_{x,-it}) \cap D_{x,-it} \left( \overset{\circ}{W}^1_p(\Omega) \right)\). It is immediate that \(D_{x,-it}u = 0\) and also, because \(u \in D_{x,-it} \left( \overset{\circ}{W}^1_p(\Omega) \right)\), there exist a function \(v \in \overset{\circ}{W}^1_p(\Omega)\) with \(D_{x,-it}v = u\) and \((-\Delta - i\partial_t)v = 0\).

Since \((-\Delta - i\partial_t)_0^{-1}f\) is unique (see [14]) we get \(v = 0\) and, consequently, \(u = 0\), i.e., the intersection of this subspaces contains only the zero function. Therefore, our sum is a direct sum.

Now let us \(u \in L^p(\Omega)\). Then we have
\[
u_2 = D_{x,-it}(-\Delta - i\partial_t)_0^{-1}D_{x,-it}u \in D_{x,-it} \left( \overset{\circ}{W}^1_p(\Omega) \right).
\]

Let us now apply \(D_{x,-it}\) to the function \(u_1 = u - u_2\). This results in
\[
D_{x,-it}u_1 = D_{x,-it}u - D_{x,-it}u_2 = D_{x,-it}u - D_{x,-it}D_{x,-it}(-\Delta - i\partial_t)_0^{-1}D_{x,-it}u = D_{x,-it}u - D_{x,-it}(-\Delta - i\partial_t)(-\Delta - i\partial_t)_0^{-1}D_{x,-it}u = 0
\]
i.e., \(D_{x,-it}u_1 \in \ker (D_{x,-it})\). Because \(u \in L^p(\Omega)\) was arbitrary chosen our decomposition is a decomposition of the space \(L^p(\Omega)\).

In a similar way we can obtain a decomposition of the \(L^p(\Omega)\) space in terms of the parabolic Dirac operator \(D_{x,+it}\). Moreover, let us remark that the above decompositions are orthogonal in the case of \(p = 2\).

Using the previous definitions we can also present an immediate application in the resolution of the linear Schrödinger problem with homogeneous boundary data.

**Theorem 3.7.** Let \(f \in L^p(\Omega), 1 < p \leq 2\). The solution of the problem
\[
\begin{align*}
(-\Delta - i\partial_t)u &= f \text{ in } \Omega \\
u &= 0 \text{ on } \partial\Omega
\end{align*}
\]
is given by \(u = T_-Q_-T_-f\).
Proof. The proof of this theorem is based on the properties of the operator $T_-$ and of the projector $Q_-$. Because $T_-$ is the right inverse of $D_{x, -it}$, we get

$$D_{x, -it}^2 u = D_{x, -it}(Q_- T_- f) = D_{x, -it}(T_- f) = f.$$

4 The Non-Linear Schrödinger Problem

In this section we will construct an iterative method for the non-linear Schrödinger equation and we study its convergence. As usual, we consider the $L_2$-norm

$$||f||^2 = \int_\Omega |f|^2 dx dt,$$

where $|\cdot|$ denotes the scalar part.

Moreover, we also need the mixed Sobolev spaces $W^{\alpha, \beta}_p(\Omega)$. For this we introduce the convention

$$\Omega^t = \{x : (x, t) \in \Omega\} \subset \mathbb{R}^m$$

$$\Omega^x = \{t : (x, t) \in \Omega\} \subset \mathbb{R}^+.$$

Then, we say that

$$u \in W^{\alpha, \beta}_p(\Omega) \iff \begin{cases} u(\cdot, t) \in W^{\alpha}_p(\Omega^t), \forall t \\ u(x, \cdot) \in W^{\beta}_p(\Omega^x), \forall x \end{cases}$$

Under this condition we will study the (generalized) non-linear Schrödinger problem:

$$-\Delta_x u - i\partial_t u + |u|^2 u = f \quad \text{in} \quad \Omega$$
$$u = 0 \quad \text{on} \quad \partial\Omega,$$

where $|u|^2 = \sum_A |u_A|^2$. We can rewrite (11) as

$$D_{x, -it}^2 u + M(u) = 0,$$

where $M(u) = |u|^2 u - f$. It is easy to see that

$$u = -T_- Q_- T_- (M(u))$$

is a solution of (12) by means of direct application of $D_{x, -it}^2$ to both sides of the equation.

We remark that for $u \in W^{2,1}_2(\Omega)$, we get

$$||D_{x, -it} u|| = ||Q_- T_- M(u)|| = ||T_- M(u)||.$$
We now prove that (13) can be solved by the convergent iterative method

\[ u_n = -T_\cdot Q_\cdot T_\cdot (M(u_{n-1})) \]  

(14)

For that purpose we need to establish some norm estimations. Initially, we have that

\[ \|u_n - u_{n-1}\| = \|T_\cdot Q_\cdot T_\cdot [M(u_{n-1}) - M(u_{n-2})]\| \leq C_1\|M(u_{n-1}) - M(u_{n-2})\|, \]

(15)

where \( C_1 = \|T_\cdot Q_\cdot T_\cdot \| = \|T\|_2 \).

We now estimate the factor \( \|M(u_{n-1}) - M(u_{n-2})\| \). We get

\[ \|M(u_{n-1}) - M(u_{n-2})\| = \|u_{n-1}^2 - u_{n-1} - u_{n-2}^2 u_{n-2}\| \leq \|u_{n-1}^2 (u_{n-1} - u_{n-2})\| + \|u_{n-1} - u_{n-2}^2 u_{n-2}\| \leq 2^{m+1}\|u_{n-1} - u_{n-2}\| (\|u_{n-1}\|^2 + \|u_{n-2}\|\|u_{n-1} - u_{n-2}\|), \]

We assume \( K_n := 2^{m+1} (\|u_{n-1}\|^2 + \|u_{n-2}\|\|u_{n-1} - u_{n-2}\|) \) so that

\[ \|u_n - u_{n-1}\| \leq C_1 K_n \|u_{n-1} - u_{n-2}\|. \]

Moreover, we have additionally that

\[ \|u_n\| = \|T_\cdot Q_\cdot T_\cdot M(u_{n-1})\| \leq 2^{m+1} C_1 \|u_{n-1}\|^3 + C_1 \|f\| \]  

(16)

holds.

In order to prove that indeed we have a contraction we need to study the auxiliary inequality

\[ 2^{m+1} C_1 \|u_{n-1}\|^3 + C_1 \|f\| \leq \|u_{n-1}\|, \]

that is,

\[ \|u_{n-1}\|^3 - \frac{\|u_{n-1}\|}{2^{m+1} C_1} + \frac{\|f\|}{2^{m+1}} \leq 0. \]

(17)

The analysis of (17) will be made considering two cases

**Case I:** When \( \|u_{n-1}\| \geq 1 \), we can establish the following inequality in relation to (17)

\[ \|u_{n-1}\|^2 - \frac{\|u_{n-1}\|}{3 \cdot 2^{m+1}} + \frac{\|f\|}{2^{m+1}} \leq \|u_{n-1}\|^3 - \frac{\|u_{n-1}\|}{2^{m+1} C_1} + \frac{\|f\|}{2^{m+1}}. \]
Then, from (17), we have
\[
\|u_{n-1}\|^2 - 2\|f\| + \|f\|^2 \leq 0
\]
\[
\|u_{n-1}\|^2 - \frac{1}{6} \cdot 2^{m+1} + \frac{1}{36} \cdot 2^{2m+2} + \frac{1}{2} \cdot 2^{m+1} - \frac{1}{36} \cdot 2^{2m+2} \leq 0
\]
\[
\left(\|u_{n-1}\| - \frac{1}{6} \cdot 2^{m+1}\right)^2 \leq \frac{1}{36} \cdot 2^{2m+2} - \|f\| \leq \frac{1}{2^{m+1}} \left(\frac{1}{36} \cdot 2^{2m+2} - \|f\|\right).
\] (18)

If \(\|f\| \leq \frac{1}{36} \cdot 2^{m+1}\), then
\[
\|u_{n-1}\| \leq \|u_{n-1}\| - \frac{1}{6} \cdot 2^{m+1}
\]
where \(W = \sqrt{\frac{1}{36} \cdot 2^{2m+2}}\).

In consequence, if
\[
\frac{1}{6} \cdot 2^{m+1} - W \leq \|u_{n-1}\| \leq \frac{1}{6} \cdot 2^{m+1} + W
\]
then we have from (16) the desired inequality
\[
\|u_n\| \leq \|u_{n-1}\|.
\]

Furthermore, we have now to study the remaining case. Assuming now that \(\|u_{n-1}\| \leq \frac{1}{6} \cdot 2^{m+1} - W\), we have
\[
\|u_n\| \leq 2^{m+1} C_1 \left(\frac{1}{6} \cdot 2^{m+1} - W\right)^3 + C_1 \|f\| \leq \frac{1}{6} \cdot 2^{m+1} - W
\]
and \(\|u_{n-1}\| \leq \frac{1}{6} \cdot 2^{m+1} - W, \|u_{n-2}\| \leq \frac{1}{6} \cdot 2^{m+1} - W\) so that it holds
\[
\|u_{n-1} - u_{n-2}\| \leq 2 \left(\frac{1}{6} \cdot 2^{m+1} - W\right).
\]

With the previous relations we can estimate the value of \(K_n\)
\[
K_n = 2^{m+1} \left(\|u_{n-1}\|^2 + \|u_{n-2}\|\|u_{n-1} - u_{n-2}\|\right)
\leq 2^{m+1} \left[\left(\frac{1}{6} \cdot 2^{m+1} - W\right)^2 + 2 \left(\frac{1}{6} \cdot 2^{m+1} - W\right)^2\right]
\leq 3 \cdot 2^{m+1} \left(\frac{1}{6} \cdot 2^{m+1} - W\right)
= \frac{1}{2} - 3 \cdot 2^{m+1} W < \frac{1}{2},
\] (19)
which implies that

\[ ||u_{n-2}|| \leq R := \frac{1}{3 \cdot 2m+1}. \]

Finally, we have that

\[ ||u_n - u_{n-1}|| \leq \mathcal{K}_n ||u_{n-1} - u_{n-2}||, \]

with \( \mathcal{K}_n < \frac{1}{2} \).

**Case II:** When \( ||u_{n-1}|| < 1 \), we can establish the following inequality

\[ ||u_{n-1}||^4 - \frac{||u_{n-1}||^2}{3 \cdot 2m+1} + \frac{||f||}{2m+1} \leq ||u_{n-1}||^3 - \frac{||u_{n-1}||}{2m+1} C_1 + \frac{||f||}{2m+1}. \]

Then, from (17), we have

\[ ||u_{n-1}||^4 - \frac{||u_{n-1}||^2}{3 \cdot 2m+1} + \frac{||f||}{2m+1} \leq 0 \]

\[ \Leftrightarrow \left( ||u_{n-1}||^2 - \frac{1}{6 \cdot 2m+1} \right)^2 \leq \frac{1}{36 \cdot 2^2m+2} - \frac{||f||}{2m+1}. \] (20)

Again, if \( ||f|| \leq \frac{1}{36 \cdot 2^2m+2} \) then

\[ \left| ||u_{n-1}||^2 - \frac{1}{6 \cdot 2m+1} \right| \leq W, \]

where \( W = \sqrt{\frac{1}{36 \cdot 2^2m+2} - \frac{||f||}{2m+1}}. \)

As a consequence,

\[ \frac{1}{6 \cdot 2m+1} - W \leq ||u_{n-1}||^2 \leq \frac{1}{6 \cdot 2m+1} + W \]

\[ \Leftrightarrow \sqrt{\frac{1}{6 \cdot 2m+1} - W} \leq ||u_{n-1}|| \leq \sqrt{\frac{1}{6 \cdot 2m+1} + W} \]

leads to \( ||u_n|| \leq ||u_{n-1}||. \)

Again, considering now the case of \( ||u_{n-1}|| \leq \sqrt{\frac{1}{6 \cdot 2m+1} - W} \), we obtain

\[ ||u_n|| \leq 2^{m+1} C_1 \left( \sqrt{\frac{1}{6 \cdot 2m+1} - W} \right)^3 \leq \sqrt{\frac{1}{6 \cdot 2m+1} - W} \]

and

\[ ||u_{n-1}|| \leq \sqrt{\frac{1}{6 \cdot 2m+1} - W} \]

\[ ||u_{n-2}|| \leq \sqrt{\frac{1}{6 \cdot 2m+1} - W} - W \]

\[ ||u_{n-1} - u_{n-2}|| \leq 2 \sqrt{\frac{1}{6 \cdot 2m+1} - W}. \]
With the previous relations we can estimate the value of $K_n$

\[ K_n = 2^{m+1} \left( \|u_{n-1}\|^2 + \|u_{n-2}\| \|u_{n-1} - u_{n-2}\| \right) \]

\[ \leq 2^{m+1} \left[ \left( \frac{1}{6 \cdot 2^{m+1}} - W \right) + 2 \left( \frac{1}{6 \cdot 2^{m+1}} - W \right) \right] \]

\[ = 3 \cdot 2^{m+1} \left( \frac{1}{6 \cdot 2^{m+1}} - W \right) \]

\[ = \frac{1}{2} - 3 \cdot 2^{m+1} W < \frac{1}{2} \] \hspace{1cm} (21)

which implies that

\[ \|u_{n-2}\| \leq R := \frac{1}{3 \cdot 2^{m+1}}. \]

Finally, we have that

\[ \|u_n - u_{n-1}\| \leq K_n \|u_{n-1} - u_{n-2}\|, \]

with $K_n < \frac{1}{2}$.

The application of Banach’s fixed point, to the previous conclusions, results in the following theorem

**Theorem 4.1.** The problem (11) has a unique solution $u \in W^{2,1}_2(\Omega)$ if $f \in L^2(\Omega)$ satisfies the condition

\[ \|f\| \leq \frac{1}{36 \cdot 2^{m+1}}. \]

Moreover, our iteration method (14) converges for each starting point $u_0 \in W^{1,1}_2(\Omega)$ such that

\[ \|u_0\| \leq \frac{1}{6 \cdot 2^{m+1}} + W, \]

with $W = \sqrt{\frac{1}{36 \cdot 2^{2(m+1)}}} - \|f\| \cdot \frac{2^{m+1}}{2^{m+1}}$.

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