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THE $SO_q(N, \mathbb{R})$-SYMMETRIC HARMONIC OSCILLATOR ON THE QUANTUM EUCLIDEAN SPACE $R_q^N$ AND ITS HILBERT SPACE STRUCTURE

ABSTRACT

We show that the isotropic harmonic oscillator in the ordinary euclidean space $\mathbb{R}^N$ ($N \geq 3$) admits a natural q-deformation into a new quantum mechanical model having a q-deformed symmetry (in the sense of quantum groups), $SO_q(N, \mathbb{R})$. The q-deformation is the consequence of replacing $\mathbb{R}^N$ by $R_q^N$ (the corresponding quantum space). This provides an example of quantum mechanics on a noncommutative geometrical space. To reach the goal, we also have to deal with a sensible definition of integration over $R_q^N$, which we use for the definition of the scalar product of states.
1. Introduction

The development of Physics has been often characterized by the introduction of some more general and accurate theory as sort of deformations of already known and accepted ones. A well-known example is special relativity, which can be viewed as a deformation of Galileo’s relativity; the velocity of light plays the role of deformation parameter. Another example is quantum mechanics, which can be seen as a deformation of classical mechanics, Planck constant being the deformation parameter.

Within the recent increasing interest for quantum groups the question has been raised \[1\] whether these fascinating mathematical objects can replace (or generalize) Lie groups in the description of the fundamental symmetries of physics, since they can be considered as continuous deformations of Lie groups themselves \[2\],\[3\]. One may ask whether the axioms of quantum mechanics are compatible with a more general description of continuous symmetries than the usual one, i.e. the one provided by the theory of Lie groups and of their unitary representations over the Hilbert spaces of physical states; many new possibilities in this direction seem to be open \[4\].

In particular it looks tempting to consider deformations of the symmetries of space(time) \[5\],\[6\],\[7\],\[8\],\[9\]; in such a case quantum groups and/or the underlying quantum spaces \[3\] replace classical space(time) and represent examples of noncommutative geometries \[10\]. Such geometries look promising for describing the microscopic structure of spacetime.

In this work we consider a specific finite-dimensional quantum mechanical model with a symmetry Lie group \(G\) and ask whether it admits a \(q\)-deformation (\(q\) \(\equiv\) parameter of deformation) such that the symmetry of the \(q\)-deformed model be described by the corresponding quantum group \(G_q\). Actually, in the deformation not only we should replace the Lie group symmetry of the system (i.e. of its hamiltonian) by the quantum group symmetry, but first we should replace the concept of covariance of physical laws w.r.t. a Lie group by the concept of their covariance w.r.t. a quantum group.

In fact we are going to build the (time independent) harmonic oscillator on the \(N\)-dimensional real quantum euclidean space (we will call it \(R^N_q\), and \(N \geq 3\)) as the deformation of the classical isotropic harmonic oscillator on \(R^N\). Correspondingly, the symmetry group \(SO(N,\mathbb{R})\) of rotations is deformed into the quantum group \(SO_q(N,\mathbb{R})\). The construction of the \(q\)-deformed quantum model is performed in “coordinate representation”.

As known, there are two dually related ways to look at Lie groups, at the spaces of their representation, and to \(q\)-deform them (see for instance \[3\]). In the first case one considers a Lie group \(G\) (or its Lie algebra \(\mathcal{G}\)) and its action on a representation space \(V\); in this
case one deforms the universal enveloping algebra $\mathcal{U}(G)$ of $G$ and its representations. In the second case, one considers the Hopf algebra $Fun(G)$ of functions on the group $G$ and its corepresentations; then deformation involves the latter objects. The present work is based on the second approach.

It is convenient to recall how one can formulate covariance of the physical description using the language of coaction and corepresentations; we do this job in the specific case in which we consider a point particle in ordinary 3-dimensional space $\mathbf{R}^3$ and we take the group of rotations $SO(3, \mathbf{R})$ of $\mathbf{R}^3$ as the symmetry group $G$.

$Fun(G)$ is the commutative algebra of functions on the group $G := SO(3, \mathbf{R})$. The functions can be expressed as power series in the basic variables $T_{ij} \in Fun(G)$, $i, j = 1, 2, 3$, which are defined by $T_{ij}(g) = g^i_j \ (g \in G$ and $\|g^i_j\| \in \text{adj}(G)$, where $\text{adj}(G)$ denotes the adjoint representation of $G$). Any vector $\vec{V} \equiv (V^i)$ provides a fundamental corepresentation of the (left) coaction $\phi_L$ of $Fun(G)$:

$$\phi_L(V^i) := T_{ij} \bigotimes V^j$$  \hspace{1cm} (1.1)

For instance if $\vec{V} = \vec{X}$ ($\vec{X} \equiv$the position operator of the quantum particle in a reference frame $S$), then the position operator $\vec{X}'$ of the particle in the frame $S'$ obtained from $S$ by a rotation $g$ will be given by

$$[\phi_L(X^i)](g, \cdot) := [T_{ij} \bigotimes X^j](g, \cdot) = T_{ij}(g)X^i = g^i_jX^j = X'^i. \hspace{1cm} (1.2)$$

$\phi_L$ is extended as an algebra homomorphism to higher rank corepresentations of $Fun(G)$. For instance:

$$\phi_L(X^iX^j) := \phi_L(X^i)\phi_L(X^j). \hspace{1cm} (1.3)$$

This implies for the commutator $[X^i, X^j]$

$$\phi_L([X^i, X^j]) = T_{ij}^h T_{k}^j \bigotimes X^h X^k - T_{ij}^h T_{k}^i \bigotimes X^k X^h. \hspace{1cm} (1.4)$$

This formula is consistent with the commutation relations

$$[X^i, X^j] = 0, \hspace{1cm} [T_{ij}^h, T_{k}^j] = 0 \hspace{1cm} (1.5)$$

It is easy to see from formula (1.4) that deformations of the commutation relations (1.5)$_a$ and (1.5)$_b$ are strictly coupled; the use of the quantum space $\mathbf{R}^3_q$ and of the quantum group $SO_q(3, \mathbf{R})$ lets one perform a consistent $q$-deformation of both.

To understand the line of development of the present work, let us briefly review the basic mathematical tools which allow the formulation of classical (i.e. undeformed) quantum
mechanics in the coordinate representation \( \Pi \) over the \( N \)-dimensional space \( \mathbb{R}^N \). We can summarize its main ingredients in the following list (with self-evident notation):

- **1)** There exists a vector calculus on \( \mathbb{R}^N \) which is covariant w.r.t. the group \( SO(N, \mathbb{R}) \) of rotations of \( \mathbb{R}^N \). \( x \equiv (x^i) \in \mathbb{R}^N \) is the coordinate vector.

- **2)** There exists a \( (SO(N, \mathbb{R}) \text{-covariant}) \) differential calculus \( D \) on \( \mathbb{R}^N \) (derivatives \( \equiv \partial^i \in D \)).

- **3)** There exists an antilinear involutive antihomomorphism defined on the algebra of functions of \( x, \partial \), the so-called complex conjugation \( \ast \).

- **4)** The vectors belonging to the Hilbert space \( \mathcal{H} \) are represented by

\[
\Pi : |u> \in \mathcal{H} \to \psi_u(x) \in L^2(\mathbb{R}^N)
\]

- **5)** Relevant operators (observables etc.) are represented in terms of functions of \( x, \partial \). Eigenvalue equations are represented by differential equations (at least in a domain dense in \( L^2(\mathbb{R}^N) \)).

- **6)** Scalar products are evaluated by means of Riemann integration,

\[
<u|v> = \int d^N x \, \psi_u^\ast \psi_v,
\]

which satisfies Stoke’s theorem and therefore automatically makes the momentum operators \( \frac{i}{\hbar} \partial^i \) hermitean.

- **7)** The Schroedinger equation for the harmonic oscillator on \( \mathbb{R}^N \) admits an algebraic solution by means of the creation and destruction operators (which are also represented using \( x, \partial \))

To construct the q-deformed model we use a q-deformed version of each of these ingredients. The analogs of points 1), 2), 3) are thoroughly developed in Ref. [3],[11],[12],[13]; the analogs of the remaining points are essentially new. They are constructed here using some partial results given in Ref. [14]. Using these q-deformed tools, we show that a sensible q-deformed harmonic oscillator on \( \mathbb{R}_q^N \) (with symmetry \( SO_q(N, \mathbb{R}) \)) can be constructed. In other words we will show that such a model satisfies the fundamental axioms of quantum mechanics.

The plan of the work is as follows.

Section 2. is a summary (based essentially on Ref. [3],[11],[14]) of already presented results concerning the quantum group \( SO_q(N, \mathbb{R}) \), the quantum space \( \mathbb{R}_q^N \), the (two) \( SO_q(N, \mathbb{R}) \text{-covariant} \) differential calculi \( D, \bar{D} \) on \( \mathbb{R}_q^N \), the (time-independent) Schroedinger equation of the harmonic oscillator on \( \mathbb{R}_q^N \) and its algebraic solution. The Schroedinger equation is formulated here in terms of the q-deformed laplacians of \( D, \bar{D} \) and it is solved
using a suitable generalization of the classical creation/destruction operators. The spectrum is bounded from below, as physics requires. The eigenfunctions are the “q-deformed” Hermite functions.

Sections 3.,4. deal with the definition of integration over $\mathbb{R}^N_q$. Integration is thoroughly defined using Stoke’s theorem only on some functions of the type $\text{polynomial} \cdot \text{gaussian}$; the latter will be involved in the definition of the scalar products of states of the harmonic oscillator. To define integration in full generality one should carefully delimit the domain of functions on which commutation of integration and infinite sums makes sense; this is out of the scope of this work. In Sect. 3 we analyse the desired requirements that an honest definition of integration should satisfy; among them Stoke’s theorem plays a special role. In Sect. 4 and appendix B we carry out the construction of the integral for the abovementioned relevant functions; at the end of that section we comment on a surprising feature regarding the behaviour of integration under dilatation of the integration variables, a sort of “quantized” scaling invariance.

In the remaining sections we construct the Hilbert space of the harmonic oscillator. First a pre-Hilbert space $\mathcal{H}$ is introduced by representing the states in two different ways inside the space $\text{Fun}(\mathbb{R}^N_q)$ (Sect. 5), so as to construct in the simplest possible way hermitean operators as functions of coordinates and derivatives. The two representations ($\Pi, \bar{\Pi}$) correspond respectively to $D, \bar{D}$. It is shown that the position/momentum operators and the hamiltonian of the harmonic oscillator are observables, i.e. hermitean operators. In Sect. 6 we find some observables commuting with the hamiltonian, namely the angular momentum components and the square angular momentum, and we determine the spectrum and eigenfunctions of the latter. With the help of these results we prove (Sect. 7) the positivity of the scalar product introduced in Sect. 5. This allows the completion of $\mathcal{H}$ into a Hilbert space $[\mathcal{H}]$. Section 8 contains the conclusions of the present work.

In most cases any relation admitting a barred and an unbarred version will be explicitly written only in the unbarred representation, the barred version being immediately available after some simple replacements (see Section 2.).

q-deformed harmonic oscillators have already been treated by other authors [15] starting from a purely algebraic approach, in the sense that creation/destruction operators with some prescribed commutations relation are postulated from the very beginning without any reference to a geometrical framework. The deformation considered there concerned the well-known hidden $su(n)$ symmetries of the harmonic oscillator hamiltonians. Here and in [16], on the contrary, the deformation concerns the rotation symmetry of the space itself: in other words, a geometrical framework is the starting point and creation/destruction
operators are constructed out of the deformed “coordinates” and “derivatives”.

2. Notation and Preliminary results

In Ref. [11] the differential calculus on the $N$-dimensional ($N \geq 3$) real quantum euclidean space (we will call it $\mathbb{R}^N_q$ in the sequel) was developed along the successful line already followed in Ref. [7], the guiding principles being essentially Leibniz rule together with nilpotency for the exterior derivative, and the requirement of covariance with respect to the quantum group $SO_q(N, \mathbb{R})$ for the whole calculus. As noticed by its authors, the differential calculus explicitly developed there was one of the two possible (linearly independent) versions; the second one can be obtained from the first by very simple replacements. Under complex conjugation each of the two calculi is mapped into the other. In a recent work [13] it is shown that these two calculi generate the same ring, which we will call $\text{Diff}(\mathbb{R}^N_q)$ in the sequel, namely barred derivatives can be expressed as some nonlinear (and quite complicated) functions of coordinates and unbarred ones, and so on. For the line of development of this work we don’t need this result, and we will essentially treat the two calculi as independent objects.

In this section we first recollect some basic definitions and relations characterizing the quantum group $SO_q(N, \mathbb{R})$, the algebra $\text{Fun}(\mathbb{R}^N_q)$ of functions on $\mathbb{R}^N_q$ (which is generated by the noncommuting coordinates $x (= \{x^i\}, i = 1, ..., N)$), its two differential calculi (which will be denoted by $D, \bar{D}$) and the action of the complex conjugation $\ast$ on all of them. For further details we refer the reader to [3], [11],[13]. Then we summarize the results of [14] concerning the Schroedinger equation of the harmonic oscillator on $\mathbb{R}^N_q$ and its solution.

2.1. The Real Quantum Euclidean Space $\mathbb{R}^N_q$ and its Two Differential Calculi

The real section $SO_q(N, \mathbb{R})$ ($q \in \mathbb{R}^+$) of the quantum group $SO_q(N)$ [3] is taken as the symmetry quantum group of the whole construction. As known, elements of the Hopf algebra $\text{Fun}(SO_q(N))$ (the algebra of “functions” on the quantum group $SO_q(N)$) are formal ordered power series in the generating elements $\{T^i_j\}, i, j = 1, 2, ..., N$. The latter satisfy the relations

\[ TCT^i = 1_{SO_q(N)}CT = T^iCT \quad (2.1) \]

\[ \hat{R}(T \otimes T) = (T \otimes T)\hat{R}. \quad (2.2) \]
Here $1_{SO_q(N)}$ is the unit element of $Fun(SO_q(N))$, $C = \|C_{ij}\|$ is the metric matrix (which is its own inverse, $C^{-1} = C$), and $\hat{R} = \|\hat{R}^{ij}_{hk}\|$ is the braid matrix, defined on $C^N \otimes C^N$; $\hat{R}$ is symmetric: $\hat{R}^T = \hat{R}$. Both $C$ and $\hat{R}$ depend on $q$ and are real for $q \in \mathbb{R}$. $\hat{R}$ satisfies the Yang-Baxter equation (in the “braid” version)

$$\hat{R}_{12}\hat{R}_{23}\hat{R}_{12} = \hat{R}_{23}\hat{R}_{12}\hat{R}_{23},$$

(2.3)

and admits the very useful decomposition

$$\hat{R}_q = qP_S - q^{-1}P_A + q^{1-N}P_1 \quad \hat{R}_q^{-1} = q^{-1}P_S - qP_A + q^{N-1}P_1.$$  

(2.4)

$P_S, P_A, P_1$ are the projection operators onto the three eigenspaces of $\hat{R}$ (the latter have respectively dimensions $\frac{N(N+1)}{2} - 1, \frac{N(N-1)}{2}, 1$): they project the tensor product $x \otimes x$ of the fundamental corepresentation $x$ of $SO_q(N)$ into the corresponding irreducible corepresentations (the symmetric, antisymmetric and singlet, namely the q-deformed versions of the corresponding ones of $SO(N)$). The projector $P_1$ is related to the metric matrix $C$ by $P_1^{ij} = \frac{C^{ij}C_{hk}}{Q_N}$ (the factor $Q_N$ is defined by $Q_N := C^{ij}C_{ij}$). $\hat{R}^{\mp 1}, C$ satisfy the relations

$$C_{mi}\hat{R}^{\pm 1}_{hk} = \hat{R}^{\mp 1}_{mj}C_{nk}$$

(2.5)

As direct consequences of (2.2),(2.3),(2.5), for any polynomial $f(t) \in C(t)$ we find

$$f(\hat{R})(T \otimes T) = (T \otimes T)f(\hat{R}),$$

(2.6)

$$f(\hat{R}_{12})\hat{R}_{23}\hat{R}_{12} = \hat{R}_{23}\hat{R}_{12}f(\hat{R}_{23}),$$

(2.7)

$$[f(\hat{R}), P \cdot (C \otimes C)] = 0$$

(2.8)

($P$ is the permutator: $P^{ij}_{hk} := \delta^i_h\delta^j_k$); in particular this holds for $f(\hat{R}) = \hat{R}^{\mp 1}, P_A, P_S, P_1$.

The algebra $O_q^N$ (in the notation of Ref. [3]) is defined as the space of formal series in the ordered powers of the $\{x^i\}$ variables, modulo the relations

$$P_A^{ij}_{hk}x^hx^k = 0.$$  

(2.9)

For instance, for $N = 3$ eq.’s (2.9) amount to the three independent relations

$$x^1x^2 - qx^2x^1 = 0, \quad x^2x^3 - qx^3x^2 = 0, \quad x^1x^3 - x^3x^1 + (q^\frac{1}{2} - q^{-\frac{1}{2}})(x^2)^2 = 0.$$  

(2.10)

For $q = 1$ and any $N$ $P_A^{ij}_{hk} = \frac{1}{2}(\delta^i_h\delta^j_k - \delta^i_k\delta^j_h)$ so that the $x^i$ coordinates become commuting variables and their order in each monomial doesn’t matter any more (classical geometry).

The exterior derivative $d$ of the differential calculus $D$ maps the space $\Lambda^p_q(O^N)$ of $p$-forms into the one $\Lambda^{p+1}_q(O^N)$ of $(p+1)$-forms, is nilpotent and satisfies Leibniz rule:

$$d^2 = 0, \quad \quad d\alpha_p := (d\alpha_p - (-1)^p\alpha_p d) \in \Lambda^{p+1}_q, \quad \alpha_p \in \Lambda^p_q.$$  

(2.11)
In particular if \( f \in O_q^N = \Lambda_q^0 \), \( df \) is a 1-form. We denote by \( \xi^i := dx^i \) the exterior derivatives of the basic coordinates \( x^i \); \( \{\xi^i\} \) is a basis of \( \Lambda_q^1 \). The decomposition \( d = \xi^i \partial_i \) defines the derivatives \( \partial_i \) corresponding to each coordinate \( x^i \). Indices are raised and lowered through the metric matrix \( C \), for instance

\[
\partial_i = C_{ij} \partial^j, \quad \partial^i = C^{ij} \partial_j, \quad C^{ij} := (C^{-1})_{ij} = C_{ij}
\]

(2.12)

Among the "commutation" relations between \( x^i \)'s, \( \xi^i \)'s, \( \partial_i \)'s we mention the following:

\[
x^i \xi^j = q \hat{R}^{ij}_{hk} x^h x^k
\]

(2.13)

\[
\partial^i x^j = C^{ij} + q \hat{R}^{-1}_{hk} x^h \partial^k
\]

(2.14)

\[
P_A^{ij}_{hk} \partial^h \partial^k = 0
\]

(2.15)

Higher degree forms can be defined as wedge products of 1-forms: the wedge products of the basic 1-forms \( \xi^i \)'s are defined as their tensor products modulo the relations

\[
P_S(\xi \otimes \xi) = 0 \quad P_1(\xi \otimes \xi) = 0.
\]

(2.16)

The wedge product is denoted by \( \wedge \) as usual. Therefore \( P_S(\xi \wedge \xi) = 0 \), \( P_1(\xi \wedge \xi) = 0 \).

The (left) coaction \( \phi_L : O_q^N \rightarrow Fun(SO_q(N)) \otimes O_q^N \) of \( SO_q(N) \) is defined on the basic variables \( x^i, \partial^i, \xi^i \) by

\[
\phi_L \circ d = (1_{SO_q(N)} \otimes d) \circ \phi_L
\]

\[
\phi_L : x^i \rightarrow T^i_j \otimes x^j, \quad \phi_L : \partial^i \rightarrow T^i_j \otimes \partial^j, \quad \phi_L : \xi^i \rightarrow T^i_j \otimes \xi^j
\]

(2.17)

and is extended as an homomorphism. The square length \( xCx := x^i C_{ij} x^j \) and the laplacian \( \Delta := \partial^i \partial_i = \partial^i C_{ij} \partial^j \) are central elements respectively in \( O_q^N \) and in the algebra of the \( \partial \) derivatives; they are scalars under the coaction of the quantum group:

\[
\phi_L(xCx) = 1_{SO_q(N)} \otimes (xCx), \quad \phi_L(\Delta) = 1_{SO_q(N)} \otimes \Delta.
\]

(2.18)

The above relations define the differential calculus \( D = \{d, \xi^i, \partial^i\} \). The barred calculus \( \overline{D} = \{\overline{d}, \overline{\xi^i}, \overline{\partial^i}\} \) can be obtained from the unbarred one replacing \( d, \xi^i, \partial^i, \Delta, \Lambda_q, q, \hat{R}_q \) by \( \overline{d}, \overline{\xi^i}, \overline{\partial^i}, \overline{\Delta}, \overline{\Lambda_q}, q^{-1}, \hat{R}_q^{-1} \). For instance:

\[
\overline{\partial^i x^j} = C^{ij} + q^{-1} \hat{R}^{ij}_{hk} x^h \overline{\partial^k}
\]

(2.19)

**Note:** in the sequel we will usually omit relations and definitions concerning the barred calculus/representation. Once and for all, we inform the reader that they can be obtained
from the ones concerning the unbarred calculus/representation through the above replace-
ments and, more generally, through the replacement \( \mathcal{O} \to \bar{\mathcal{O}} \) for all new objects \( \mathcal{O} \) that we are going to introduce in the work.

In Ref. [13] it is shown that all objects of \( \bar{D} \) lie in the ring \( Diff(\mathbb{R}^N_q) \) generated by \( D \), namely can be expressed as some (nonlinear) functions of the unbarred objects (and viceversa). In the sequel we will only use the relation

\[
\bar{\Delta} = q^N G_q \Delta, \tag{2.20}
\]

where \( G_q \) denotes the dilatation operator belonging to this ring with action

\[
G_q f(x, \partial, \bar{\partial}) := f(q^{-\frac{1}{2}} x, q^{\frac{1}{2}} \partial, q^{\frac{1}{2}} \bar{\partial}), \tag{2.21}
\]

whose explicit expression in terms of \( x, \partial \) is not needed here.

If \( q \in \mathbb{R} \) one can introduce an antilinear involutive antihomomorphism \( * \):

\[
*^2 = id \quad (AB)^* = B^* A^* \tag{2.22}
\]

on \( Fun(SO_q(N)), Diff(\mathbb{R}^N_q) \). Since the point \( q = 0 \) is singular for the \( \hat{R} \) and \( C \) matrices, in the sequel we will specialize the discussion to the case \( q \in \mathbb{R}^+ \). On the basic variables \( T^i_j \) * is defined by

\[
(T^i_j)^* = C^{li} T^l_m C_{jm}, \tag{2.23}
\]

whereas it maps \( x^i, \xi^i, \partial^i, d \) into a combination of \( x^i, \bar{\xi}^i, \bar{\partial}^i, \bar{d} \) respectively, in the following way

\[
(x^i)^* = x^j C_{ji}, \quad (\xi^i)^* = \bar{\xi}^j C_{ji}, \quad (\partial^i)^* = -q^{-N} \bar{\partial}^j C_{ji}, \quad d^* = -\bar{d}. \tag{2.24}
\]

The square length turns out to be real, whereas the two laplacians are mapped one into the other:

\[
(x C x)^* = x C x \quad (\Delta)^* = \bar{\Delta} q^{-2N}. \tag{2.25}
\]

**Note:** The algebra \( Fun(SO_q(N)) \) (resp. \( O^N_q \)), defined by (2.1),(2.2) (resp. (2.9)), \( q \in \mathbb{R} \), endowed with \( * \) is denoted by \( Fun(SO_q(N,\mathbb{R})) \) (resp. \( Fun(\mathbb{R}^N_q) \)) and will be called the “algebra of functions on the quantum group \( SO_q(N,\mathbb{R}) \)” (resp. the “algebra of functions on the quantum space \( \mathbb{R}^N_q \)”).

We list now some formulae and definition which will useful in the sequel. For any function \( f(x) \in O^N_q \), \( \partial^i f \) can be expressed in the form

\[
\partial^i f = \check{f}^i + \check{f}^j_i \partial^i \quad \check{f}^i, \check{f}^j_i \in Fun(\mathbb{R}^N_q) \tag{2.26}
\]
upon using (2.14) to move step by step the derivatives to the right of each \(x^i\) variable of each term of the power expansion of \(f\), as far as the extreme right. Similarly to what has been done in formula (2.11), we denote \(\hat{f}^i\) by \(\partial^i f\):

\[
\partial^i f := \partial^i f - \hat{f}^j \partial^j (= \hat{f}^i).
\]  

(2.27)

In an analogous way we can define \(\bar{\partial}^i f\).

The q-exponential function is introduced by

\[
\exp_q[Z] := \sum_{n=0}^{\infty} \frac{Z^n}{(n)_q!}; \quad (n)_q := \frac{q^n - 1}{q - 1};
\]  

(2.28)

for our scopes its usefulness lies essentially in the relation

\[
\partial^i \{\exp_q[\frac{\alpha(xCx)}{\mu}]\} = \alpha^i \exp_q[\frac{\alpha(xCx)}{\mu}] + \exp_q[\frac{q^2 \alpha(xCx)}{\mu}] \partial^i
\]  

(2.29)

which implies \(\partial^i \exp_q[\frac{\alpha(xCx)}{\mu}] \propto x^i \exp_q[\frac{\alpha(xCx)}{\mu}]\). From the definition (2.28) it is easy to check the following q-derivative property for the exponentials

\[
\frac{\exp_q[qZ] - \exp_q[Z]}{q - 1} = Z \exp_q[Z]
\]  

(2.30)

Finally, from (2.14), (2.19) it is easy to derive:

\[
\Delta x^i = \mu \partial^i + q^2 x^i \Delta \quad \partial^i (xCx) = \mu x^i + q^2 (xCx) \partial^i
\]  

(2.31)

where \(\mu := 1 + q^{2-N}\). From (2.31) it is easy to derive:

\[
\Delta (xCx)^h = \frac{\mu^3 q^{N+2h-2}}{q^2 - 1} h_{q^2} (xCx)^{h-1} B - \frac{\mu^2 (q^2 + 1)}{q^2 - 1} h_{q^4} (xCx)^{h-1} + q^{4h} (xCx)^h \Delta; \quad \Delta
\]  

(2.32)

the operator \(B\) is defined by

\[
B := 1 + \frac{q^2 - 1}{\mu} x^i \partial_i
\]  

(2.33)

and satisfies the properties \(B(xCx) = q^2 (xCx)B, \ B\Delta = q^{-2}\Delta B\).

2.2. The Schroedinger equation for the harmonic oscillator.

In Ref. [14] we introduced the ” hamiltonians ”

\[
h_\omega := \frac{1}{2} (-q^N \Delta + \omega^2 xCx) \quad \tilde{h}_\omega := \frac{1}{2} (-q^{-N} \tilde{\Delta} + \omega^2 (xCx))
\]  

(2.34)
corresponding to the calculi \( D, \bar{D} \). If \( q = 1 \) both coincide with the Hamiltonian of the classical harmonic isotropic oscillator on \( \mathbb{R}^N \). The eigenvalues of \( h_\omega, \bar{h}_\omega \) coincide and

\[
h_\omega^* = \bar{h}_\omega. \tag{2.35}
\]

The eigenvalues \( \{E_n\}_{n=1,2...} \) are given by

\[
E_n = \omega(q^{\frac{N}{2}} - 1 + q^{1-\frac{N}{2}})[\frac{N}{2} + n]_q \quad n \geq 0; \tag{2.36}
\]

the corresponding \( n^{th} \) level eigenfunctions are combinations of the “q-deformed Hermite functions” \( \psi_n^{i_1i_2...i_n}, \bar{\psi}_n^{i_1i_2...i_n} \)

\[
h_\omega \psi_n^{i_1i_2...i_n} = E_n \psi_n^{i_1i_2...i_n} \quad \psi_n^{i_1i_2...i_n} := a_n^{i_n} + a_{n-1}^{i_{n-1}} + ... a_1^{i_1} \psi_0
\]

\[
\bar{h}_\omega \bar{\psi}_n^{i_1i_2...i_n} = E_n \bar{\psi}_n^{i_1i_2...i_n} \quad \bar{\psi}_n^{i_1i_2...i_n} := \bar{a}_n^{i_n} + \bar{a}_{n-1}^{i_{n-1}} + ... \bar{a}_1^{i_1} \bar{\psi}_0, \tag{2.37}
\]

where the indices \( i, j \), \( j = 1, ..., n \), belong to \( \{1, 2, ..., N\} \). Here \( \psi_0, \bar{\psi}_0 \) denote the ground state eigenfunctions

\[
\psi_0 := \exp_q\left[-\frac{q^{-N}\omega xC_x}{\mu}\right] \quad \bar{\psi}_0 := \exp_{q^{-2}}\left[-\frac{q^N\omega xC_x}{\bar{\mu}}\right], \tag{2.38}
\]

and

\[
a_h^{i^+} := b_h(q)(x^i - \frac{q^{2-h}}{\omega} \bar{\partial}^i)G_q \quad \bar{a}_h^{i^+} := b_h(q^{-1})(x^i - \frac{q^{-2}}{\omega} \bar{\partial}^i)G_{q^{-1}} \quad i = 1, 2, ..., N \tag{2.39}
\]

are the ”creation” operators at level \( h \) in the unbarred and barred scheme respectively. The operator \( G_q \) was defined in (2.21) and

\[
[n]_q := \frac{q^n - q^{-n}}{q - q^{-1}} \tag{2.40}
\]

are the q-deformed integers: \( [n]_q \xrightarrow{q \rightarrow 1} n \). At this stage we are free to fix the coefficients \( b_h(q) \) as we wish.

The operators

\[
a_h^i := d_h(q)(x^i + \frac{q^{h+N}}{\omega} \bar{\partial}^i)G_q \quad \bar{a}_h^i := d_h(q^{-1})(x^i + \frac{q^{-h-N}}{\omega} \bar{\partial}^i)G_{q^{-1}} \tag{2.41}
\]

are destruction operators (at level \( h - 2 \)), since \( a_h^i \psi_{h-1}, \bar{a}_h^i \psi_{h-1} \) are eigenvectors of level \( (h - 2) \). Again, at this stage we are free to fix the coefficients \( d_n \) as we wish. In section
5 we will fix \( b_n, d_n \) so as to build in the simplest way well-defined position/momentum observables.

As noticed in Ref. [14], the spectrum is bounded from below and increasing with \( n \) for any \( q \in \mathbb{R}^+ \). Energy levels are invariant under the replacement \( q \to q^{-1} \) and have the same degeneracy as in the classical case. They are not equidistant as in the classical case \((q = 1)\) and the difference between neighbouring energy levels diverges with \( n \).

In the classical case any function of the type \( P(x) \exp\left[ -\frac{\omega(xCx)}{\mu} \right] \) (\( P(x) \) being a polynomial) can be expressed as a combination of particular functions of this type, the well-known Hermite functions with characteristic constant \( \omega \). Moreover, any eigenfunction of the corresponding harmonic oscillator hamiltonian is a combination of Hermite functions of one (and the same) level. Quite similar results hold also in the q-deformed case. The formulation of these results is however slightly technical; the reader who is not interested in details can skip the rest of this section without relevant consequences for the general understanding.

According to the notation introduced in [14], let

\[
P_n(x) := a \text{ polynomial in } x \text{ containing only powers of degree } p = n(\text{mod } 2). \quad (2.42)
\]

Using the q-derivative property (2.30) one checks that a function of the type polynomial \( \cdot \) gaussian can be expressed in infinitely many equivalent ways:

\[
P_n(x) \exp_q^2\left[ -\frac{\omega q^{-N-m}(xCx)}{\mu} \right] = \equiv P_{n+2h}(x) \exp_q^2\left[ -\frac{\omega q^{-N-m-2h}(xCx)}{\mu} \right], \quad h \geq 0. \quad (2.43)
\]

Having defined the spaces \( \Psi_n, V_n, V \) and \( S \) by

\[
\Psi_n := \text{Span}_\mathbb{C}\{\psi_n \text{'s of formula (2.37)}\} \quad (2.44)
\]

\[
V_n := \text{Span}_\mathbb{C}\{P_n(x) \exp_q^2\left[ -\frac{\omega q^{-N}(xCx)}{\mu} \right] \}, \quad (2.45)
\]

\[
V := \sum_{n=0}^{\infty} V_n \quad (2.46)
\]

and

\[
S := \text{Span}_\mathbb{C}\{\text{eigenfunctions } \psi \text{ of } h_{\omega} \text{ of the form } \psi = P(x) \exp_q^2\left[ -\frac{\alpha(xCx)}{\mu} \right] \}, \quad (2.47)
\]
then one can prove

**Proposition 2.1:**

\[ V = S = \bigoplus_{n=0}^{\infty} \Psi_n. \] (2.48)

This equality has the following

**Corollary 2.2:** No eigenfunctions of \( h_\omega \) other than the ones belonging to the \( \Psi_n \)'s can be found in \( V = S \). Correspondingly, no eigenvalues other than \( E_n \)'s \( n = 1, 2, \ldots \).

For the proof [14] of statement (2.48) one makes essential use of relation (2.43) and of the property

\[ P_{A h k} a_{n}^{h+} a_{n-1}^{k+} = 0 \] (2.49)

and proves the

**Lemma 2.3**

\[ V_n = \bigoplus_{0 \leq h \leq \frac{n}{2}} \Psi_{n-2h}; \quad \text{dim}(\Psi_n) = \text{dim}(M_n) = \binom{N + n - 1}{N - 1}, \quad n \in \mathbb{N}. \] (2.50)

Here

\[ M_n := \text{Span}_\mathbb{C}\{x^{i_1}x^{i_2} \ldots x^{i_n}, i_h = 1, 2, \ldots N\}. \] (2.51)

Note that \( V = \bigoplus_{n=0}^{\infty} \Psi_n \). \( V \) can be split into subspaces \( V^+, V^- \) of opposite parity:

\[ V^+ := \bigoplus_{h=0}^{\infty} \Psi_{2h} \quad V^- := \bigoplus_{h=0}^{\infty} \Psi_{2h+1}, \quad \Rightarrow \quad V = V^+ \oplus V^- \] (2.52)

3. **Integration: formal requirements**

In Ref. [7] the authors propose a definition of integration over the quantum hyperplane essentially based on the requirements of linearity and of validity of Stoke’s theorem (of course in such an approach the latter is no more a “theorem”). Denoting by \( < f > \), \( \int \omega_n \) respectively the integral of a function \( f \) and of an n-form \( \omega_n \) over the n-dimensional hyperplane (as usual they are related by definition by the identity \( < f > = \int dV f \), where \( dV \) denotes the volume form), Stoke’s theorem takes respectively the forms

\[ < \partial^i f > = 0 \quad i = 1, 2, \ldots, n, \quad \int d\omega_{n-1} = 0; \] (3.1)
\(\partial^i f\) denotes the (total derivative) function which was introduced in formula (2.27). In the classical case, if \(f = P_n(x)\exp[-a|x|^2]\) \((P_n \text{ denotes a polynomial of degree } n \text{ in } x \text{ and } |x|^2 \text{ the square lenght})\), then

\[
\partial^i P_n(x)\exp[-a|x|^2] = P_{n-1}(x)\exp[-a|x|^2] + P_{n+1}\exp[-a|x|^2];
\]

relations (3.1), (3.2) imply

\[
< P_{n-1}(x)\exp[-a|x|^2] > + < P_{n+1}\exp[-a|x|^2] > = 0.
\]

Relation (3.3) allows to recursively define the integral \(< f >\) (for any function \(f\) of the same kind) in terms of \(< \exp[-a|x|^2] >\) (which fixes the normalization of the integration). The same holds in the q-deformed case, provided one has defined the generalization of the exponential (the so-called q-exponential).

The integration over the hyperplane defined according to (3.1), (3.2) has the following properties: a) it is continuous in \(q\); b) it is covariant w.r.t. \(GL_q(n)\) (in the sense that will defined below); c) it coincides with the classical Riemann integral for \(q=1\) (by a suitable choice of the normalization factor); d) it satisfies the reality condition

\[
< f >^* = < f^* >
\]

for any \(q \in \mathbb{R}^+\); therefore: d) the positivity condition,

\[
< f^* f > \geq 0, \quad < f^* f > = 0 \Leftrightarrow f = 0,
\]

holds, at least in a \((f\text{-dependent})\) neighbourhood of \(q=1\), since it holds for \(q=1\). If there exists a neighbourhood \(U \subset \mathbb{R}^+\) of \(q=1\) such that positivity holds \(\forall q \in U\), in this neighbourhood a scalar product can be introduced through the definition

\[
(f, g) := < f^* g >,
\]

and one can convert into a Hilbert space a suitable subspace of the algebra of functions on the quantum hyperplane.

In the case of the real quantum euclidean space the situation seems complicated by the fact that there exist two sets of linearly independent derivatives belonging respectively to the differential calculi \(D, \bar{D}\), hence potentially two kinds of integrations \(< >, \ll \gg\) and two versions of Stoke’s theorem:

\[
< \partial^i f | = 0 \quad i = 1, 2, ..., N; \quad \int d\omega_{n-1} | = 0
\]
\[ \ll \bar{\partial}^i f \gg = 0 \quad i = 1, 2, \ldots, N; \quad \int d\omega_n = 0. \quad (3.7) \]

At first sight the reality condition (3.4) for each of the two integrations \(\ll\gg, \ll\gg\) seems no more guaranteed by Stoke’s theorems (3.7) because \(\ast\) maps derivatives \(\partial \in D\) into derivatives \(\bar{\partial} \in \bar{D}\) (and vice versa). Quite surprisingly, in next section we will see that the two integrations coincide; therefore relation (3.4) holds.

For the moment we keep \(\ll\gg, \ll\gg\) distinct. We list the requirements that these integrations should satisfy and show that they are compatible with each other. In next section one of these requirements, Stoke’s theorem, will be used to recursively define the integrations. Through the relation

\[ \ll f \gg = \int dV f \quad (3.8) \]

statements regarding integral of functions can be translated into ones regarding integrals of \(N\)-forms, and vice versa, so usually they will be written only in one of the two versions.

We would like an integration \(\ll\gg\) to be defined on a not too poor subspace \(V\) of \(\text{Fun}(\mathbf{R}^N_q)\) and to satisfy:

1) linearity;
2) covariance;
3) continuity in \(q\) and correspondence principle for \(q \to 1\);
4) reality;
5) positivity.

Of course linearity means

\[ \ll \alpha f + \beta g \gg = \alpha \ll f \gg + \beta \ll g \gg, \quad \alpha, \beta \in \mathbb{C} \quad f, g \in V \quad (3.9) \]

and one has to check that if \(f\) vanishes because of relations (2.9), then so does \(\ll f \gg\), in other terms

\[ f(x) = A_{ij} P^{ij}_{hk} x^h x^k \cdot g(x) \quad \Rightarrow \quad \ll f \gg = 0. \quad (3.10) \]

By covariance we mean

\[ 1_{SO_q(N)} \ll f \gg = (id_{SO_q(N)} \otimes \ll \gg) \circ \phi_L(f), \quad (3.11) \]

where \(1_{SO_q(N)}\) and \(id_{SO_q(N)}\) denote respectively the unit element and the identity operator on \(\text{Fun}(SO_q(N, \mathbf{R}))\), and \(\phi_L\) is the left coaction of \(SO_q(N, \mathbf{R})\) on \(\text{Fun}(\mathbf{R}^N_q)\). More explicitly, if \(f^{i_1 i_2 \ldots i_k} := x^{i_1} x^{i_2} \ldots x^{i_k} g(x C x)\), then covariance means

\[ 1_{SO_q(N)} \ll f^{i_1 i_2 \ldots i_k} \gg = T^{i_1}_{j_1} T^{i_2}_{j_2} \ldots T^{i_k}_{j_k} \ll f^{j_1 j_2 \ldots j_k} \gg, \quad (3.12) \]

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in other words the numbers \(< f_{i_1i_2...i_k} >\), \(i_j = 1, 2, ..., N\), are the components of an "isotropic" tensor; in the classical case relation (3.12) corresponds to the well-known property of tensors such as

\[
\int d^N x \ g(|x|^2) x^i = 0, \quad \int d^N x \ g(|x|^2) x^i x^j \propto \delta^{ij},
\]

\[
\int d^N x \ g(|x|^2) x^i x^j x^k x^l \propto (\delta^{ij}\delta^{kl} + \delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}), ...
\]

(3.13)

namely the property that the latter are invariant under an orthogonal transformation of the coordinates \(x^i \rightarrow x'^i := g^i_j x^j\). The simplest nontrivial example of a tensor satisfying (3.12) is for \(k = 2\), \(< f^{ij} >\propto C^{ij}\). In general tensors satisfying (3.12) involve matrix products among \(\hat{R}\)-matrices (or, equivalently, \(\hat{R}^{-1}\)-matrices) and contractions with metric matrices \(C\): the former reorder indices by means of the RTT relations (2.2), whereas the latter transform a couple of neighbouring \(T\)-matrices into a commuting number (see relation (2.1)). Therefore an integral \(< x^{i_1}...x^{i_k} g(xC x) >\) should be factorizable as a product

\[
< x^{i_1}...x^{i_k} g(xC x) > = S^{i_1...i_k} \alpha_g,
\]

(3.14)

and \(S^{i_1...i_k} = 0\) for \(k\) odd; the \(g\)-dependence of the RHS of (3.14) is concentrated in the constant \(\alpha_g\), which essentially is a (yet unspecified) integral along the “radial” direction. Explicit solutions \(S^{i_1...i_k}, \bar{S}^{i_1...i_k}\) satisfying (3.12) will be found in section 4.

Point 3) means that we require a q-deformed integral to reduce to a classical one (with some integration measure \(\rho(x) d^N x\)) when \(q=1\). Maybe it is timely to recall the fact that the \(x^i\) coordinates are not real (even for \(q=1\)), but are complex combinations of the usual real cartesian coordinates; the latter can be used to perform the integral when \(q=1\).

The reality and positivity conditions 4), 5) in the form (3.4), (3.5) or in some other form should guarantee that the definition (3.6) or what takes its place introduces an honest scalar product \((\ , \ , \)\) in a suitable subspace \(\mathcal{V}\) of \(Fun(\mathbb{R}_q^N)\)

\[
(f, g)^* = (g, f); \quad (f, f) \geq 0, \quad (f, f) = 0 \Leftrightarrow f = 0 \quad f, g \in \mathcal{V},
\]

(3.15)

to convert this subspace into a Hilbert space.

To the five previous points we add a requirement characterizing the specific problem we are dealing with here, namely that the hamiltonian (or the position/momentum operators) of the harmonic oscillator be hermitean operators w.r.t. \((\ , \ , \)\). As it will be clear in the sequel, we are led to ask for the validity of
6) Stoke’s theorem

in the form (3.7). We will see that, as in the classical case, point 6) involves a definite choice of the ” radial ” part of the integration (whereas the latter is left unspecified by 2) alone). As already noticed, Stoke’s theorem is a formidable tool to define (up to a normalization factor) the corresponding integration.

Now we briefly discuss compatibility of requirements 1) - 6).

It is straightforward to check that linearity is compatible with covariance because of property (2.6) (where we take \( f(\hat{R}) = \mathcal{P}_A \)). Requirement 3) is obviously compatible with 1),2) since classical integration is linear and there exist \( SO(N,\mathbb{R}) \)-invariant integration measures. It is straightforward to prove that reality (3.4) is compatible with linearity (because of relation (2.6), where again we take \( f(\hat{R}) = \mathcal{P}_A \)), with covariance (apply \( \ast \) to eq. (3.12) and use definitions (2.23),(2.24)) and with the correspondence principle (classical real integrations satisfy the reality condition). Positivity in the form (3.5) is clearly compatible with requirements 1),3) and with reality (3.4). At this stage is not easy to understand if it is compatible with covariance. Using the results which will be presented in Sect.’s 6.,7. one could prove that this is the case. The question is not strictly relevant for the solution of problem (3.15) in the specific case of the harmonic oscillator, since the scalar product \( (\, , \, ) \) that we are going to introduce in Sect. 5 is not of the form (3.6). In fact, we will prove that the scalar product introduced there is positive definite.

Let us analyze now the compatibility of Stoke’s theorems (3.7) with points 1) - 5). It is straightforward to check that both (3.7)\(_a\) and (3.7)\(_b\) are compatible with linearity: in fact this compatibility is reduced to the consistency of both differential calculi \( D, \bar{D} \) with the q-space relations (2.9) and the commutation relations (2.13). Similarly, the compatibility with covariance is guaranteed by the fact that the exterior derivative commutes with the coaction (see formula (2.17)\(_a\)). As for the correspondence principle, compatibility is ensured by the fact that in the limit \( q \to 1 \) both \( D \) and \( \bar{D} \) go to the classical differential calculus, and Riemann integration (on smooth functions) satisfies Stoke’s theorem. In order to show that Stoke’s theorems are compatible with reality in the form (3.4), we first show that

**Proposition 3.1:** the integrations \( < \, >, \ll \, \gg \) satisfying Stoke’s theorems (3.7)\(_a\) and (3.7)\(_b\) are compatible with reality in the form

\[
< f >^\ast = \ll f^\ast \gg,
\]

Then compatibility with reality in the form (3.4) will follow from the equality \( < \, > = \ll \, \gg \),

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which we will prove in next section.

Proof: Let us consider the spaces of formal relations

\[ \mathcal{F} = \text{Span}_C \{ \partial^i f - \partial^j f | - f_j^i \partial^j = 0, \quad i, j = 1, \ldots, N \ f \in \mathcal{V} \} \]  

(3.17)

\[ \bar{\mathcal{F}} := \text{Span}_C \{ \bar{\partial}^i f - \bar{\partial}^j f | - \bar{f}_j^i \bar{\partial}^j = 0, \quad i, j = 1, \ldots, N \ f \in \mathcal{V} \}, \]  

(3.18)

where: 1) \( \partial^i f |, f_j^i, \bar{\partial}^i f |, \bar{f}_j^i \) are the functions introduced in formula (2.27); 2) \( \mathcal{V} \) is some subspace of \( \text{Fun}(\mathbb{R}^N_q) \) closed under complex conjugation and containing also the functions \( \partial^i f |, f_j^i, \bar{\partial}^i f |, \bar{f}_j^i \) for any \( f \in \mathcal{V} \). In the classical case the space \( V_{cl} \) of functions of the type \( P(x) \exp[-a|x|^2] \) (\( P \) being a polynomial) is an example of such a subspace \( \mathcal{V} \), and we will see that an analogous space will be available in the q-deformed case, too. Under these assumptions it is immediate to recognize that the two sets (3.17),(3.18) are mapped into each other by \( * \), since \( * : \mathcal{D} \to \bar{\mathcal{D}} \) and \( * : \bar{\mathcal{D}} \to \mathcal{D} \). In other terms \( \mathcal{F}^* = \bar{\mathcal{F}} \). If we define subspaces \( A, \bar{A} \subset \mathcal{V} \) as the linear spans of functions \( \partial^i f | \) and \( \bar{\partial}^i f | \) respectively, the previous remark implies

\[ A^* = \bar{A} \]  

(3.19)

For each \( a \in A \) let \( \bar{a} \in \bar{A} \) be the function such that \( a^* = \bar{a} \). Stoke’s theorems respectively imply

\[ (3.7)_a \quad \Rightarrow \quad < a > = 0 = < a >^* \quad \forall a \in A \]  

(3.20)

\[ (3.7)_b \quad \Rightarrow \quad \ll \bar{a} \gg = 0 = \ll \bar{a} \gg^* \quad \forall \bar{a} \in \bar{A}, \]  

(3.21)

hence reality in both forms (3.4) and (3.16) is trivially satisfied for the integrals \( < a >, \ll \bar{a} \gg \). If \( q=1 \) and we take \( \mathcal{V} = V_{cl} \) one easily realizes that any \( f \in \mathcal{V} \) can be expressed in the form

\[ f = a + c_f f_0, \quad a \in A, \ c_f \in \mathbb{C} \]  

(3.22)

(as anticipated at the beginning of this section), where \( f_0 \) is defined by \( f_0 := \exp[-a|x|^2] \). Consequently

\[ < f > = c_f < f_0 >. \]  

(3.23)

For self-evident reasons we call \( f_0 \) the reference function of the integral. In next sections we will see that a similar situation occurs also in the q-deformed case, for instance by taking \( \mathcal{V} = V \) (\( V \) was defined in Sect. 2.) and \( f_0 := \exp_q[-a\frac{(x \bar{C} x)}{\mu}] \). In any case \( f_0 \) should be a real function not belonging to \( A \) and should go to a smooth rapidly decreasing classical function in the limit \( q \to 1 \). Taking the complex conjugate of eq. (3.22) we get

\[ f^* = \bar{a} + c_f^* f_0, \quad \bar{a} \in \bar{A}, \ c_f \in \mathbb{C}, \]  

(3.24)
which implies

$$\langle f^* \rangle = c^* \langle f_0 \rangle$$

(3.25)

We are still free to fix \( \langle f_0 \rangle, \langle f_0 \rangle \) as we like. If we impose the reality condition in the form (3.16) on the reference function we see that it is transferred to all functions belonging to \( \mathcal{V} \), as claimed ♦.

Since \( f^*_0 = f_0 \), the reality condition (3.16) on the reference function reads \( \langle f_0 \rangle = \langle f_0 \rangle^\ast \). In the sequel we will take \( \langle f_0 \rangle \in \mathbb{R}^+ \).

Finally the compatibility of Stoke’s theorem with the positivity condition in the form (3.5) is left as an open question, but again is not relevant for our specific problem; whereas we will see in Sect. 7 that the scalar product in the Hilbert space of the harmonic oscillator, defined using the integral \( \langle \rangle \), is positive defined.

4. Integration: construction

In this section we use Stoke’s theorem (in its two versions (3.7)) as a tool for constructing the integrations. The systematic enforcement of Stoke’s theorems generates a set of formal relations between integrals of different functions. We determine these relations in two steps. First, we find out the isotropic tensors \( S^{i_1 \ldots i_k}, \tilde{S}^{i_1 \ldots i_k} \); hence, according to (3.14), the integrals \( \langle f \rangle, \langle f \rangle \) of a non scalar function \( f \) will be expressed in terms of integrals of a scalar one. Due to the fact that \( S^{i_1 \ldots i_k}, \tilde{S}^{i_1 \ldots i_k} \) turn out to be proportional, the two integrations coincide. Second, we determine the equations relating integrals of different scalar functions; in this way we will be able to express integrals of scalar functions in terms of the integral \( \langle f_0 \rangle \) of a particular one, what we call the reference function \( f_0 \). \( \langle f_0 \rangle \) is a normalization constant and can be fixed quite arbitrarily (see the end of the preceding section). So to say, the second step amounts to integration over the radial coordinate. As an example we will explicitly consider in this section the reference function \( f_0 = \exp[\frac{-\alpha x^2 C x}{\mu}] \); in section 5. we will take another reference function which is conceived for defining the scalar products of states of the harmonic oscillator. In this way one can define the integrals for infinitely many independent functions \( \{f_i\}_{i \in \mathbb{N}} \) and therefore for finite combinations of them. This is enough for the scopes of this work, since it will enable us to define a positive definite scalar product inside the span of states of the harmonic oscillator (see section 5.); then the completion of this pre-Hilbert space will be done w.r.t. the corresponding norm. Nevertheless, to further enlarge the domain of definition of the
integrals one could consider functions admitting series expansions in the \( \{f_i\} \), and we will briefly address this problem at the end of this section.

The preliminary discussion of the previous section has shown that the two basic integrations \(< >, \ll \gg \) are linear, covariant and coincide with the classical Riemann integration for \( q=1 \). Therefore the explicit recursive application of the two Stoke’s theorems will determine (up to a factor) isotropic tensors \( S^{i_1...i_k}, \bar{S}^{i_1...i_k} \) (see (3.12)). As we are going to see, up to a factor these tensors coincide and do not depend on the choice of the function \( g(xCx) \) in formula (3.14). The relevant results of this section are summarized in Propositions 4.1, 4.2, 4.3.

The choice \( g = \exp_q[-\frac{\alpha xCx}{\mu}] \) (or, alternatively, we could take \( g = \exp_{q-2}[-\frac{\alpha xCx}{\mu}] \)) is particularly convenient for this goal. Using relation (2.29) we find

\[
\partial^{i_1} x^{i_2}...x^{i_k} \exp_q[-\frac{\alpha xCx}{\mu}] = \\
= -\alpha x^{i_1} x^{i_2}...x^{i_k} \exp_q[-\frac{\alpha xCx}{\mu}] + \exp_q[-\frac{q^2 \alpha xCx}{\mu}] \partial^{i_1} x^{i_2}...x^{i_k} = \\
= -\alpha x^{i_1} x^{i_2}...x^{i_k} \exp_q[-\frac{\alpha xCx}{\mu}] + \exp_q[-\frac{q^2 \alpha xCx}{\mu}] M_{k,j_3...j_k} x^{j_3}...x^{j_k} 
\]

(4.1)

where the tensors \( M_{i_1...i_k}^{j_1...j_k}, N_{i_1...i_k}^{j_1...j_k} \) are introduced by the defining relation

\[
\partial^{i_1} x^{i_2}...x^{i_k} =: M_{k,j_3...j_k}^{i_1...i_k} x^{j_3}...x^{j_k} + N_{k,j_1...j_k}^{i_1...i_k} x^{j_1}...x^{j_{k-1}} \partial^{j_k} 
\]

(4.2)

Taking the integral \(< > \) of (4.1) and applying Stoke’s theorem we find

\[
< x^{i_1}...x^{i_k} \exp_q[-\frac{\alpha xCx}{\mu}] > = \frac{1}{\alpha} M_{k,j_3...j_k}^{i_1...i_k} < x^{j_3}...x^{j_k} \exp_q[-\frac{q^2 \alpha xCx}{\mu}] > 
\]

(4.3)

Starting from \( k = 0, 1 \) and noting that Stoke’s theorem (or, equivalently, covariance) imply \(< x^i \exp_q[-\frac{\alpha xCx}{\mu}] > = 0 \), we see that the recursive application of relation (4.3) determines tensors \( S_k^{i_1...i_k} \). The result is summarized in the

**Proposition 4.1:** the tensors

\[
S_k^{i_1...i_k} := 0 \text{ if } k \text{ is odd} \quad (4.4)
\]

\[
S_{2n} := M_{2n} \cdot M_{2(n-1)} \cdot ... M_2 
\]

(4.5)

satisfy the covariance condition (3.12).
Here we have used the shorthand notation
\[
(M_{2k} \cdot M_{2(k-1)})_j^{i_1i_2...i_{2k}} = M_{2k,j^{l_1l_2...l_{2k}}} M_{2(k-1),j^{l_3l_4...l_{2k}}}.
\]
(4.6)
As a direct consequence of the proposition and of relation (4.3), the integral (4.3) will vanish when \(k\) is odd and
\[
<x^{i_1}x^{i_2}...x^{i_{2n}}exp_{q^2}[\frac{q^2αCx}{μ}] >_x S^{i_1i_2...i_{2n}}_{2n}.
\]
(4.7)
Similarly one can determine tensors \(\bar{S}^{i_1...i_k}_{2n}\) satisfying the analog of relation (3.12) for the integration \(\ll\gg\); to this end we only need to replace \(\partial, M, N, S, \hat{R}^{-}\) with \(\bar{\partial}, \bar{M}, \bar{N}, \bar{S}, \bar{R}^{-}\) in the preceding formulae.

**Proposition 4.2:**
\[
\bar{S}^{i_1...i_k}_{2n} \propto S^{i_1...i_k}_{2n}.
\]
(4.8)
**Proof:** this immediately follows from the very useful formulae
\[
\Delta^n x^{i_1}x^{i_2}...x^{i_{2n}}| = (μ)^{n}n_q^{2!}S^{i_1i_2...i_{2n}}_{2n}
\]
(4.9)
\[
\bar{\Delta}^n x^{i_1}x^{i_2}...x^{i_{2n}}| = (\bar{μ})^{n}n_{\bar{q}}^{-2!}\bar{S}^{i_1i_2...i_{2n}}_{2n}
\]
(4.10)
and from equations (2.20),(2.21). The proof of relations (4.9),(4.10) is by induction and will be given in appendix A ♦.

Next, it is easy to realize that equation (4.7) can be generalized to
\[
<x^{i_1}x^{i_2}...x^{i_{2n}}g(xCx) >_x S^{i_1i_2...i_{2n}}_{2n}.
\]
(4.11)
with any function \(g(xCx)\). In fact, looking at the power series defining \(g\) one immediately finds that \(\partial^i g(xCx) = \tilde{g}(xCx)x^i\), with some functions \(\tilde{g}, \in Fun(R^N_q)\). Then, applying both sides of (4.2) to \(g\) and taking the integral \(< >\) we find
\[
0 = <\partial^i x^{i_2}...x^{i_{2n}}g > = M_{2n,j^{i_1i_2...i_{2n}}} <x^{j_3}...x^{i_{2n}}g > + N_{2n,j^{i_1...i_{2n}}} <x^{j_3}...x^{j_{2n}}\tilde{g} >.
\]
(4.12)
This result holds for any function \(g\), in particular for the previous choice \(g = exp_{q^2}[\frac{αCx}{μ}]\); by comparison with (4.3), (4.5), we infer the invertibility of the matrices \(N_{2n}\), the relations
\[
N^{-1}_{2n} \cdot S_{2n} \propto S_{2n}
\]
(4.13)
and hence the relations
\[
<x^{i_1}...x^{i_{2n}}\tilde{g} > = c_n,g S^{i_1...i_{2n}}_{2n},
\]
(4.14)
for any function $\tilde{g}(xCx)$. By contracting the free indices $i_1, i_2, \ldots, i_{2n}$ with $C_{i_1 i_2} \ldots C_{i_{2n-1} i_{2n}}$ we reduce the determination of the constant $c_n, \tilde{g}$ to the evaluation of the integral of a purely scalar function. The same arguments can be applied to the integration $\ll \gg$. Thus we are led to the

**Proposition 4.3:**

\[
\langle x^{i_1} \ldots x^{i_{2n}} \tilde{g} \rangle = \frac{S_{2n}^{i_1 \ldots i_{2n}}}{S_{2n}} \langle (xCx)^n \tilde{g} \rangle,
\]

\[
\ll x^{i_1} \ldots x^{i_{2n}} \tilde{g} \gg = \frac{S_{2n}^{i_1 \ldots i_{2n}}}{S_{2n}} \ll (xCx)^n \tilde{g} \gg;
\]

here

\[
S_{2n} := C_{i_1 i_2} \ldots C_{i_{2n-1} i_{2n}} S_{2n}^{i_1, i_2, \ldots, i_{2n}}.
\]

Using formulae (2.32), (4.9) it is easy to show that the constant $S_{2n}$ is positive for any $q \in \mathbf{R}^+$. Let us analyze the \textit{radial} dependence of the two integrals $\langle \rangle, \ll \gg$. We introduce the operators

\[
B := 1 + \frac{q^2 - 1}{\mu} x^i \partial_i = q^{-N}(1 + \frac{q^2 - 1}{\mu} \partial^i x_i)
\]

\[
\bar{B} := 1 + \frac{q^{-2} - 1}{\bar{\mu}} x^i \bar{\partial}_i = q^N(1 + \frac{q^{-2} - 1}{\bar{\mu}} \bar{\partial}^i x_i);
\]

it is straightforward to check that $B(xCx) = q^2(xCx)B$, $\bar{B}(xCx) = q^{-2}(xCx)\bar{B}$ and therefore

\[
Bf(xCx) = f(q^2xCx)B, \quad \bar{B}f(xCx) = f(q^{-2}xCx)\bar{B},
\]

for any $f \in O_q(N)$ depending only on $(xCx)$; hence

\[
q^{-N}(f + \frac{q^2 - 1}{\mu} \partial^i x_i f) = f(q^2xCx), \quad q^N(f + \frac{q^{-2} - 1}{\bar{\mu}} \bar{\partial}^i x_i f) = f(q^{-2}xCx).
\]

By taking the integrals $\langle \rangle, \ll \gg$ respectively of (4.20)$_a$, (4.20)$_b$ and by applying Stoke’s theorems (3.7) we find the formal relations

\[
\langle f(q^2xCx) \rangle q^N = \langle f(xCx) \rangle, \quad \ll f(q^2xCx) \gg q^N = \ll f(xCx) \gg
\]

for both integrations. As we will see in a moment, equation (4.21) is sufficient to determine the integral of any scalar function in terms of that of the reference function $f_0$; if we set $\langle f_0 \rangle = \ll f_0 \gg (\in \mathbf{R}^+)$, this implies the formal relation

\[
\langle f \rangle = \ll f \gg,
\]

(4.22)
at least for $f = f(xCx)$. But looking back at relations (4.16),(4.8) we realize that previous
equation holds for any $f$. This concludes the proof of the

**Proposition 4.4:** the two integrations $\langle \cdot \rangle$, $\ll \gg$ (formally) coincide.

Since the integral $\langle f \rangle$ of any $f \in Fun(R^N_q)$, if it exists, is reduced to a combinations of
integrals of radial functions by means of relation (4.15), then property (4.21) is generalized
by the

**Proposition 4.5:**

$$\langle f(qx) \rangle q^N = \langle f(x) \rangle.$$  \hfill (4.23)

This fundamental relation characterizes the integration defined by means of Stoke’s theo-
rem and will be called ” scaling property ” for reasons which will become clear at the end
of this section.

So far we have not specified the domain of functions $f \in Fun(R^N_q)$ for which the integral
$\langle f \rangle$ can be defined. Therefore all the previous relations were purely formal. Now we
pick up a particular reference function $f_0(xCx)$. We ask what are the functions $f$ such that
the corresponding integral $\langle f \rangle$ can be reduced to the one $\langle f_0 \rangle$ by means of iterated
application of Stoke’s theorem and of linearity, and turn out to be finite. Of course we
wish to include in this space of ” integrable ” functions as many $f \in Fun(R^N_q)$ as possible.

As an example we take $f_0 = exp_{q^2}[-\frac{\alpha x C x}{\mu}]$, $\alpha > 0$, which for q=1 reduces to a
well known smooth rapidly decreasing classical function, the gaussian. First we consider
functions $f$ of the type $f(xCx) = P(xCx) exp_{q^2}[-\frac{\alpha x C x}{\mu}]$, $P$ being an arbitrary polynomial.
Using property (4.23) and the q-derivative property (2.30) of the exponential we show the

**Proposition 4.6:**

$$\langle exp_{q^2}[-\frac{\alpha x C x}{\mu}] (xCx)^h \rangle = \left( \frac{\mu}{\alpha} \right)^h \left( h - 1 + \frac{N}{2} \right) q^2 \cdots \left( \frac{N}{2} \right) q^2 \cdot q^{-h(N+h-1)c},$$  \hfill (4.24)

where $c := \langle exp_{q^2}[-\frac{\alpha x C x}{\mu}] \rangle \in R^+$ plays the role of normalization factor.

**Proof:**

$$\langle exp_{q^2}[-\frac{\alpha x C x}{\mu}] (xCx)^{k-1} \rangle = q^{N+2(k-1)} \langle exp_{q^2}[-\frac{q^2 \alpha x C x}{\mu}] (xCx)^{k-1} \rangle =$$

$$= q^{N+2(k-1)} \langle exp_{q^2}[-\frac{\alpha x C x}{\mu}] (xCx)^{k-1} \rangle +$$

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\[- \frac{\alpha}{\mu} (q^2 - 1)q^{N+2(k-1)} < \exp_q^2 [- \frac{\alpha xC x}{\mu}] (x C x)^k >, \quad (4.25)\]

whence

\[< \exp_q^2 [- \frac{\alpha xC x}{\mu}] (x C x)^k > = (\frac{\mu}{\alpha})(k - 1 + \frac{N}{2})_q q^{N-2(k-1)} < \exp_q^2 [- \frac{\alpha xC x}{\mu}] (x C x)^{k-1} >; \quad (4.26)\]

applying \(h\) times formula (4.26) for \(k = h, h - 1, \ldots, 1\) we find (4.24) \(\diamondsuit\).

Relations (4.15), (4.24) allow to define the integration \(< >\), on all functions of the type \(f = P(x)f_0\), where \(P(x)\) is an arbitrary polynomial in \(x\) and \(f_0 := \exp_q^2 [- \frac{\alpha xC x}{\mu}]\). We could enlarge the domain of definition of the integrations by admitting functions \(P(x)\) in the form of power series \(P(x) = \sum_{n=0}^{\infty} \sum_{i_1, i_2, \ldots, i_n} A_{i_1 i_2 \ldots i_n} x^{i_1} x^{i_2} \ldots x^{i_n}\) such that the series

\[\sum_{n=0}^{\infty} \sum_{i_1, i_2, \ldots, i_n} A_{i_1 i_2 \ldots i_n} < x^{i_1} x^{i_2} \ldots x^{i_n} f_0 > \quad (4.27)\]

converges; the integral \(< f >\) would then be defined as the limit (4.27). A further step towards the enlargement of the domain of definition of the integrations could be done along the following lines. In the previous formula we could take \(P = P(xC x) = \sum_{n=0}^{\infty} a_n (xC x)^n\) with coefficients \(a_n \in \mathbb{R}\) and such that \(< P(xC x) \exp_q^2 [- \frac{\alpha xC x}{\mu}] > := c'\) is finite. Then we could define a new reference function by letting \(f_0 := P(xC x) \exp_q^2 [- \frac{\alpha xC x}{\mu}]\): by means of formulae (4.15), (4.16), (4.21) we should be able to evaluate \(< \tilde{P}(x)f_0 >\) in terms of \(c'\) for all polynomials \(\tilde{P}(x)\). Thus one could include in the domain of integrable functions also functions \(f\) susceptible of a decomposition \(f = \tilde{P}(x)f_0\), \(\tilde{P}(x) = \sum_{n=0}^{\infty} \sum_{i_1, i_2, \ldots, i_n} A_{i_1 i_2 \ldots i_n} x^{i_1} x^{i_2} \ldots x^{i_n}\) such that the series (4.27) with this new \(f_0\) converges. It is natural to figure that to the new choice of the reference function there should correspond an actual enlargement of the domain of integrable functions. This operation could be iterated in a sort of continuation of the functional \(< >\) so as to enlarge to the maximum possible size the space of integrable functions. It is out of the scope of this work to face this problem by analyzing which conditions the coefficients \(\{A_{i_1 i_2 \ldots i_n}\}\) of an expansion of the type \(f = f_0 \sum_{n=0}^{\infty} \sum_{i_1, i_2, \ldots, i_n} A_{i_1 i_2 \ldots i_n} x^{i_1} x^{i_2} \ldots x^{i_n}\) should satisfy in order that \(f\) be integrable\(^{(1)}\).

We just briefly note that, having defined the integrations \(< >\) using Stoke’s theorems (3.7), one could define a new integration \(< >_\rho\) satisfying (at least) requirements 1) - 3) of the preceding section, by setting

\[< f >_\rho := < f \cdot \rho >; \quad (4.28)\]
the "weight" $\rho$ should be a real scalar function.

Now let us come back to property (4.23). By its iterative application we find

$$< f(q^n x) > q^{nN} = < f(x) > , \quad n \in \mathbb{Z},$$

(4.29)

or, equivalently, in differential form notation

$$\int dV \; q^{nN} f(q^n x) = \int dV \; f(x) , \quad n \in \mathbb{Z}.$$  

(4.30)

Relation (4.30) states that under the change of integration variables $x \rightarrow ax$ with $a = q^n$ the integral $\int$ is invariant if we let $dV$ transform according to $dV \rightarrow a^N dV$, namely according to the law of transformation of $d^N x$. This explains the name "scaling property" for relations (4.21),(4.23),(4.29), (4.30). In both the classical and the q-deformed case this property characterizes the integrals satisfying Stoke's theorem; for $q=1$ the latter reduce to the usual Riemann integral, which has a "homogeneous" (i.e. translation invariant) measure.

One can now ask if the scaling property holds even if the dilatation parameter $a \notin Q := \{ q^n , \; n \in \mathbb{Z} \}$. One can easily check that this is not the case (let for instance $f_0 = \exp_q [ -b (xCx) ]$ and let $f = \exp_q [ -ab (xCx) ]$ be the function which we want to integrate by choosing $f_0$ as reference function). In other terms, the function

$$F(a) := < f(ax) > a^N$$

(4.31)

is periodic in the variable $b = \ln(a)$ with period $\ln(q)$, but is not constant. To be specific, assume for instance $q > 1$. If $a \in \mathbb{R}^+$ is fixed and $q^n < a < q^{n+1}$ the function $F$ fluctuates around the value $F(1) = < f(x) >$, the width of the fluctuation being the same $\forall n \in \mathbb{Z}$, therefore also around large $a$. But if we take the deformation parameter $q$ very close to 1, then $Q$ is, so to say, "almost dense" in any interval contained in $\mathbb{R}^+$, i.e. $a$ can be approximated quite well by an element of $Q$. In other terms, at a macroscopic scale (i.e. for $a \in \mathbb{R}^+$ such that $|\ln(a)_{ln(q)}| \gg 1$) deviations from the classical scaling property would not be detectable, even though they would be relevant at microscopic ones (i.e. for $a \in \mathbb{R}^+$, $|\ln(a)_{ln(q)}| \sim 1$). This surprising feature might be considered as a very interesting indication of the occurrence of a dishomogeneity of the observable properties of space when the usual euclidean commutative space is replaced by the corresponding quantum space.

Let us consider now eq. (4.29) from the dimensional viewpoint. The series expansion of the function $f$ makes sense only if $f(x)$ is of the form $f(x) = g(c_0 x)$, where $c_0$ is some constant with dimension of inverse length, $[c_0] = L^{-1}$ (in the case of the harmonic oscillator...
we can take $c_0 = \sqrt{\omega}$). For the sake of brevity assume that $[g] = 1$. Since the volume form has dimension $L^N$, then $[< f(x) >] = [< g(c_0 x) >] = L^N$; this implies

$$< f(x) > = < g(c_0 x) > \propto c_0^{-N}$$  \hspace{1cm} (4.32)

We cannot require property (4.32) to be valid for a scaling $c_0 \to c_0' := \alpha c_0$ of the fundamental constant with an arbitrary $\alpha \in \mathbb{R}^+$, otherwise the scaling property of the integral would hold for an arbitrary scaling parameter as well. The scaling is not incompatible with the “quantized” scaling property (4.29) only if $c_0' \in \mathcal{I}_{c_0} := \{c = q^m c_0 : \ m \in \mathbb{Z}\}$. In other words choosing an integration implies a restriction on the admitted values for the fundamental constants characterizing the system.

5. The Hilbert space of the harmonic oscillator and the observables $R^i$, $P_j$, $H_\omega$

In this Section we define the pre-Hilbert space of states of the harmonic oscillator on $\mathbb{R}^N_q$ and define the observables Hamiltonian, position and momentum. We first generate the space through the application of creation operators to the ground state, then endow it with a scalar product which mixes the barred and unbarred representation; the latter is designed to make differential operators (such as the hamiltonian) hermitean in a straightforward way. The second part is technical and may be skipped by the reader without serious consequences for the general understanding. It starts just after formula (5.20) and deals with the determination of some coefficients which appear in the definition of the creation/destruction operators.

We introduce the pre-Hilbert space $\mathcal{H}$ of the $SO_q(N)$-symmetric (isotropic) harmonic oscillator with characteristic constant $\omega$ in the following way. Let $|0\rangle$ be the ground state with the energy $E_0$ given in formula (2.36). We introduce a direct ($\Pi, V$) and a barred ($\bar{\Pi}, \bar{V}$) representation ($V, \bar{V}$ were defined in Sect. 2) by first assuming

$$\exp_{q^2}[-\frac{\omega q^{-N}(x^C x)}{\mu}] \in V$$

$|0\rangle \in \mathcal{H}$

$$\exp_{q^{-2}}[-\frac{\omega q^N(x^C x)}{\mu}] \in \bar{V}.$$  \hspace{1cm} (5.1)
Creation and destruction operators $A^+, A^i$ are to be represented respectively by

$$b_n(q)(x^i - \frac{q^{-n-2}}{\omega} \partial^i)G_q \equiv a_n^{+i} \tag{5.2}$$

when acting on states of level $(n - 1)$ (to give states of level $n$), and by

$$d_n(q)(x^i + \frac{q^{n+N}}{\omega} \partial^i)G_q \equiv a_n^i \tag{5.3}$$

when acting on states of level $(n - 1)$ (to give states of level $(n - 2)$); the operator $G_q$ was defined in formula (2.21), and the coefficients $b_n, d_n$ will be fixed below. The space $\mathcal{H}_n$ of states of level $n$ will be introduced as linear span of the vectors

$$|i_n, i_{n-1}, ... i_1 > := A^{+i_n} A^{+i_{n-1}} ... A^{+i_1} |0 > \tag{5.4}$$

The vector $|i_n, ... i_1 >$ can be assigned the $SO_q(N, \mathbb{R})$ transformation law

$$\phi_l(|i_n, ... i_1 >) = T^n_{j_n} ... T^1_{j_1} \otimes |j_n, ... j_1 > \tag{5.5}$$

since both $\psi_n^{i_n ... i_1}$ and $\bar{\psi}_n^{i_n ... i_1}$ have transformation laws of this kind. Any $|u_n > \in \mathcal{H}_n$ is an eigenvector with eigenvalue

$$E_n = \omega \frac{1}{2} (q^{\frac{N}{2}} - 1 + q^{1-\frac{N}{2}})[\frac{N}{2} + n]_q, \quad n \geq 0 \tag{5.6}$$

of the hamiltonian $H_\omega$, which is represented by

$$h_\omega = \frac{1}{2} (-q^N \Delta + \omega^2 (xCx)) \tag{5.7}$$

$$h_\omega = \frac{1}{2} (-q^{-N} \bar{\Delta} + \omega^2 (xCx));$$

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\( \mathcal{H} \) itself is defined as

\[
\mathcal{H} := \bigoplus_{n=1}^{\infty} \mathcal{H}_n. \tag{5.8}
\]

By the above construction any vector \( |u> \in \mathcal{H} \) will be represented both by a vector \( \psi_u \in V \) and by a vector \( \bar{\psi}_u \in \bar{V} \):

\[
|u> \quad \text{and} \quad \bar{\psi}_u.
\]

With reference to the notation of Sect. 2., we know that any function of the type \( \psi_u = P_n(x) \exp q \left[ -\frac{\omega q^{-n-N-2m} x C x}{\mu} \right] \) belongs to \( V \). From the above construction the corresponding \( \bar{\psi}_u := \bar{\Pi} \psi_u \in \bar{V} \) will be of the form \( \bar{\psi}_u = \bar{P}_n(x) \exp q \left[ -\frac{\omega q^{+n+N+2m} x C x}{\mu} \right] \) where the polynomial \( \bar{P}_n(x) \) is obtained from \( P_n(x) \) by the following steps: 1) writing \( P_n(x) \exp q \left[ -\frac{\omega q^{-n-N-2m} x C x}{\mu} \right] \) as a combinations of the \( \psi_m \)'s of formula (5.4); 2) replacing \( \psi_m \)'s by \( \bar{\psi}_m \)'s. If we consider the explicit form of \( \psi_m, \bar{\psi}_m \) involving only the coordinates (without derivatives) the second step amounts to the substitutions \( q \leftrightarrow q^{-1}, \bar{R} \leftrightarrow \bar{R}^{-1} \); in particular if the \( \bar{R}, \bar{R}^{-1} \) matrices are written in terms of the projectors \( \mathcal{P}_S, \mathcal{P}_A, \mathcal{P}_1 \) alone, then we only need to interchange \( q \) with \( q^{-1} \).

We define the scalar product of two vectors \( |v>, |u> \in \mathcal{H} \) by the sum of two “conjugate” terms:

\[
(u, v) := <\bar{\psi}_u^* \psi_v> + <\bar{\psi}_u \psi_v^*>. \tag{5.10}
\]

Indeed \((u, v)\) is manifestly sesquilinear and (using property (3.4))

\[
(v, u)^* = <\bar{\psi}_v^* \psi_u> = <\bar{\psi}_u^* \psi_v> = <\psi_u \bar{\psi}_v> = (u, v) \tag{5.11}
\]

as required (see relation (3.15)). Relation (5.11) implies that \((u, u) \in \mathbb{R} \); its positivity (i.e. \((u, u) \geq 0 \) and \((u, u) = 0 \iff u = 0) \forall q \in \mathbb{R}^+ \) will be proved in Sect. 7. Here we just note that it must hold at least in a \(|u>-dependent) neighbourhood of \( q=1, \) as it holds for \( q=1 \) and \((u, u) \) is a continuous function of \( q \).

The abstract definition of the hermitean conjugate \( T^\dagger \) of an operator \( T \) is the usual one

\[
(u, Tv) = (T^\dagger u, v). \tag{5.12}
\]

We have chosen for the scalar product the (apparently cumbersome) form (5.11) to make the operator \( H_\omega \) hermitean. Let us check that this is the case. Using the notation introduced in formula (2.27)

\[
\Delta f = f'(x) + f_j(x, \partial) \partial^j =: \Delta f| + f_j(x, \partial) \partial^j, \quad f \in \text{Fun}(\mathbb{R}_q^N) \tag{5.13}
\]
and the relation \( \bar{\Delta} = q^{2N} \Delta^* \) it is straightforward to show that

\[
(\Delta f|g := f^* \cdot (q^{-2N} \bar{\Delta}) - \partial^j f_j^* g |. \quad (5.14)
\]

Hence

\[
< \psi_u^* \bar{h} \psi_v > = < \psi_u^* (\omega^2 x C x - q^{-N} \bar{\Delta}) \psi_v > \quad (5.15)
\]

and

\[
< (h \psi_u)^* \bar{\psi}_v > = -q^N < (\Delta \psi_u)^* \bar{\psi}_v > + < \omega^2 x C x \psi_u^* \bar{\psi}_v >
\]

\[
= -q^{-N} < \psi_u^* \bar{\Delta} \bar{\psi}_v > + < \omega^2 x C x \psi_u^* \bar{\psi}_v > + q^N < \partial^j \psi_u^* \bar{j} \bar{\psi}_v >; \quad (5.16)
\]

The last term vanishes because of Stoke’s theorem (3.7) (in fact \( \partial^j \) are derivatives of \( \bar{\partial} \) type), therefore \( < \psi_u^* \bar{h} \psi_v > = < (h \psi_u)^* \bar{\psi}_v > \). Similarly one proves that \( < \bar{\psi}_u^* h \psi_v > = < (\bar{h} \psi_u)^* \bar{\psi}_v > \). Hence we find the

**Proposition 5.1:** the Hamiltonian \( H \) is hermitean:

\[
(u, H \psi v) = (H \psi u, v). \quad (5.17)
\]

As an immediate consequence of the hermiticity of the hamiltonian, if \( |u|, |v| \) are two eigenvectors of \( H \) with different eigenvalues, then

\[
(u, v) = 0. \quad (5.18)
\]

Looking back at the previous proof we see that in fact a stronger property holds:

\[
n \neq m \quad \Rightarrow \quad < \bar{\psi}_n^* \psi_m > = 0, \quad < \psi_n^* \bar{\psi}_m > = 0 \quad \psi_p \in \Psi_p, \quad \bar{\psi}_p \in \bar{\Psi}_p \quad (5.19)
\]

(\( \Psi_p, \bar{\Psi}_p \) were defined in formula (2.44))

For the evaluation of the scalar products \( (< , >) \) it is only necessary to find out integrals of the type \( < (xC)x^k f (xC) > \) with

\[
f = \exp_q^{-2} \left[ -\frac{\omega q^{N+k+2m} x C x}{\mu} \right] \exp_q^{-2} \left[ -\frac{\omega q^{-N-k-2m} x C x}{\mu} \right], \quad (5.20)
\]

since their tensor structure is already determined by the general knowledge of the tensors \( S^{i_1 \cdots i_n} \) of Section 4.; these integrals will be determined in Appendix B.

We still have to fix the coefficients \( b_n, d_n \) to complete the definitions (5.2),(5.3) of \( A^i, A^{+i} \). We determine them imposing two requirements: 1) that creation/destruction operators are hermitean conjugate of each-other, equation (5.23); 2) that the position
operators do not contain derivatives, equation (5.34). The final result is shown in equations (5.36), (5.38). The reader not interested in these computations might simply give a glance to these formulae. Before starting, we mention two relations which we will use in doing this job:

$$< (\bar{a}_{n+1}^i a_n^{i+j} \psi)^* \psi >= q^{1-N} \frac{b_{n+1}(q^{-1})b_n(q^{-1})}{d_n(q)d_{n-1}(q)} < \bar{\psi}^* a_{n-1}^i a_n^{i'} \psi > C_{j'j} C_{i'i} \tag{5.21}$$

$$< (a_{n+1}^i a_n^{i+j} \psi)^* \bar{\psi} >= q^{1+N} \frac{b_{n+1}(q)b_n(q)}{d_n(q^{-1})d_{n-1}(q^{-1})} < \psi^* a_{n-1}^i a_n^{i'} \bar{\psi} > C_{j'j} C_{i'i}; \tag{5.22}$$

they can be derived using Stoke’s theorem (3.7) and the scaling property (4.31).

First we require that

$$(A^{+i})^\dagger = A^i C_{li} \quad (A^i)^\dagger = A^{+i} C_{li}; \tag{5.23}$$

using the orthogonality relations (5.18) this requirement reduces to

$$(A^{+i} u_n, u_{n+1}) = (u_n, A^i u_{n+1}) C_{li} \quad A^{+i} u_n, u_{n+1}) = (u_{n+1}, A^{+i} u_n) C_{li}, \quad \forall n \geq 0, \forall u_n \in \mathcal{H}_n, \quad u_{n+1} \in \mathcal{H}_{n+1}. \tag{5.24}$$

Condition (5.24) is equivalent to

$$\begin{cases} (A^{+i} A^{+j} u_m, u_{m+2}) = (u_m, A^{k} A^{+i} u_{m+2}) C_{kj} C_{li}, \quad \forall m \geq 0, \\ (A^i A^j u_{m+2}, u_m) = (u_{m+2}, A^{+k} A^{+i} u_m) C_{kj} C_{li}, \quad \forall u_m \in \mathcal{H}_m, u_{m+2} \in \mathcal{H}_{m+2} \end{cases}$$

$$(A^{+i} u_0, u_1) = (u_0, A^i u_1) C_{li} \quad (A^i u_0, u_1) = (u_1, A^{+i} u_0) C_{li} \tag{5.25}$$

The implication (5.24) ⇒ (5.25) is trivial. To prove the converse one we just need to express the vectors $|u_n>$ as combinations of the vectors (5.4). Writing down conditions (5.25) explicitly in terms of $b_n, d_n$ and using relations (5.21), (5.22), we can check that these conditions are satisfied if

$$\frac{b_{m+2}(q^{-1})b_{m+1}(q^{-1})}{q^{N+1}d_{m+1}(q)d_m(q)} < \bar{\psi}^* u_m a_{m+1}^j a_{m+2}^i \psi_{u_m+2} > < \bar{\psi}^* u_m a_{m+2}^j a_{m+3}^i \psi_{u_m+2} > \forall q \in \mathbb{R}^+, \tag{5.26}$$

and similarly for the conjugate term. According to formula (5.19) we can replace in the RHS of (5.26) the function $a_{m+1}^j a_{m+2}^i \psi_{u_m+2}$ by the one $P_{\psi_m} [a_{m+1}^j a_{m+2}^i \psi_{u_m+2}]$ ($P_{\psi_m}$ denotes the projector onto the space $\Psi_m$). Using formula (B.9) we can check that

$$P_{\psi_m} [a_{m+1}^j a_{m+2}^i \psi_{u_m+2}] =$$
\[
\begin{align*}
q^{m+1+N} + q^{-m-1} \frac{q^{m+N} + q^{-m}}{q^{m+3+N} + q^{-1-m} q^{m+2+N} + q^{-m} d_{m+3}(q)d_{m+2}(q)} a_{m+2}^i a_{m+3}^j u_{m+2}^i
\end{align*}
\] (5.27)

Collecting this information we find that (5.26) and hence (5.25) are satisfied if
\[
\frac{b_{m+2}(q^{-1})}{d_{m+3}(q)} = q^{N+1} \frac{1 + q^{2m+4+N}}{1 + q^{2m+N}} \frac{d_{m+2}(q)}{b_{m+1}(q^{-1})} \quad \forall m \geq 0, \quad \forall q \in \mathbb{R}^+.
\] (5.28)

As for conditions (5.25), an explicit computation shows that they are satisfied if
\[
d_2(q) = b_1(q^{-1}) q^{-N-1} \frac{1 + q^N}{1 + q^2} \phi(q) \quad \forall q \in \mathbb{R}^+,
\] (5.29)

where the constant \(\phi(q)\) is defined in formula (B.8). Solving the recursive equation (5.28) by taking relation (5.29) as the initial condition, we find
\[
\frac{d_{m+2}(q)}{b_{m+1}(q^{-1})} = \left[ \frac{(1 + q^2)q^N}{\phi(1 + q^{2+N})} \right]^{(-1)^{m+1}} q^{-N-1} \frac{1 + q^{2m+N}}{1 + q^{2m+2+N}} \quad \forall q \in \mathbb{R}^+, \quad m \geq 0.
\] (5.30)

Summing up, relation (5.30) guarantees that equation (5.23) is satisfied.

As a direct consequence of equation (5.23), if \(f_i \in \mathbb{C}\) are numbers such that
\[
C_{ml} f_i^* = f_m,
\] (5.31)

then the operators \(f_i(A^l + A^{l+1}), \frac{d}{dx}(A^l - A^{l+1})\) are hermitean operators. There exist \(N\) independent solutions \(f_i^i, i = 1, 2, \ldots, N\) of equations (5.31). For instance if \(N = 3\) we can take
\[
\|f_i^i\| = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & q^{\frac{N}{2}} \\ 0 & 1 & 0 \\ 0 & i & q^{\frac{N}{2}} \end{pmatrix}.
\] (5.32)

In general, given \(N\) solutions \(f_i^i\) of (5.31) we define
\[
R^i := \frac{1}{\sqrt{\omega}} f_i^i(A^l + A^{l+1}) \quad \quad P^i := \frac{1}{i\sqrt{\omega}} f_i^i(A^l - A^{l+1}).
\] (5.33)

Additional conditions on the coefficients \(b_n, d_n\) arise if we impose the requirement that the operators \(\Pi(R^i), \tilde{\Pi}(R^i)\) (acting respectively in \(V, \tilde{V}\)) contain no derivative:
\[
g_n(q)f_i^i x^j G_q \text{ on } \Psi_n
\] (5.34)

\[
g_n(q^{-1})f_i^i x^j G_q \text{ on } \tilde{\Psi}_n
\]
Equation (5.34) and the definitions (5.2), (5.3) of $A_l, A^\dagger_l$ imply

$$d_{m+1}(q) = q^{-2m-N}b_{m+1}(q), \quad g_m(q) = \frac{1+q^{-2m-N}}{\sqrt{\omega}}b_{m+1}(q). \tag{5.35}$$

Relations (5.30), (5.35) determine all the coefficients $b_n, d_n, n \geq 1,$ in terms of $b_1$. The solutions are

$$d_n(q) = q^{2-2n-N}b_n(q), \quad b_n(q) = \frac{(1+q^N)q^{2n-2}}{1+q^{2n-2+N}} \cdot \begin{cases} b_1(q) & \text{if } n \text{ is odd} \\ \phi \frac{1+q^{N+2}}{1+q^2}q^{-1}b_1(q^{-1}) & \text{if } n \text{ is even}. \end{cases} \tag{5.36}$$

Consequently

$$g_n = \begin{cases} \frac{1+q^{-N}}{\sqrt{\omega}}b_1(q) & \text{if } n \text{ is even} \\ \frac{(1+q^N)(1+q^{-2-N})}{(1+q^2)\sqrt{\omega}}q^{2}b_1(q^{-1}) & \text{if } n \text{ is odd} \end{cases} \tag{5.37}$$

We are still free to fix $b_1$. In the sequel we will take

$$b_1(q) = \frac{\sqrt{2}\omega}{1+q^{-N}} \implies g_+ = \sqrt{2}, \quad g_- = \phi q^3 \frac{1+q^{-2-N}}{} \frac{1+q^2}{\sqrt{2}}. \tag{5.38}$$

The above choice is such that $b_n(1) = \sqrt{\frac{2}{q^N}} = d_n(1)$, so that in the classical limit $q = 1$ $A^\dagger, A^\dagger$ give the classical creation/annihilation operators. Let

$$\mathcal{H}^+ := \sum_{h=0}^{\infty} \mathcal{H}_{2h} \quad \mathcal{H}^- := \sum_{h=0}^{\infty} \mathcal{H}_{2h+1} \tag{5.39}$$

We can summarize formulae (5.34), (5.37) by

$$g_\pm(q)f^i x^l G_q \psi_u \quad \Pi \quad \bar{\psi}_u = \Pi(\Pi(\Pi(|u>) \in V_+, \quad \bar{\psi}_u = \Pi(\Pi(|u>) \in V_+).$$

$$R^i |u> \begin{cases} \Pi & \text{if } |u> \in \mathcal{H}_\pm , \end{cases} \tag{5.40}$$

where $\psi_u = \Pi(|u>) \in V_\pm, \quad \bar{\psi}_u = \Pi(\Pi(|u>) \in V_\pm$. It is natural to call the observables $R^i$ the position operators of the system, since they reduce to the ordinary position operators when $q = 1$. Because of our requirement (5.34) $\Pi(R^i)G_{q^{-1}}, \Pi(R^i)G_q$ act as pure multiplication by a combination of coordinates $x^i$. The classical commutation relations $[R^i, R^j] = 0$ are replaced by the new ones

$$\check{\mathcal{P}}^{ij}_{A h k} R^h R^k = 0, \quad \check{\mathcal{P}}_A := (f \otimes f) \mathcal{P}_A(f^{-1} \otimes f^{-1}) \tag{5.41}$$
Up to a factor there exists only one quadratic function of the $R^i$’s which is a scalar and an observable, $R^2 := \frac{1}{2} R^i (f^{-1} T C f^{-1})_{ij} R^j$ (notice that the matrix $(f^{-1} T C f^{-1})$ is hermitean), therefore we will call it the square length. Since the action of $R^i$ flips the parity of a vector, $R^2$ is represented in the same way on all of $\mathcal{H}$:

\[ R^2 \Rightarrow \begin{pmatrix} \phi^{2^{1 + \frac{N}{2} + q - q - 1 - \frac{N}{2}}} q^{1 - \frac{N}{2}} x C x G q^2 \\ \phi^{2^{1 + \frac{N}{2} + q - q - 1 - \frac{N}{2}}} q^{1 + \frac{N}{2}} x C x G q^{-2} \end{pmatrix}. \quad (5.42) \]

According to the definition (5.33) the observable $P^i$ will be represented by

\[ P^i |u_n > \Rightarrow \begin{pmatrix} \Pi \frac{1}{i \sqrt{\omega}} b_{n+1}(q) f^i [(q^{2 - n - N} - 1) x^l + 2 q^{1 - n} \partial^j] G q \psi u_n \\ \Pi \frac{1}{i \sqrt{\omega}} b_{n+1}(q^{-1}) f^i [(q^{2n + N} - 1) x^l + 2 q^{n - 1} \partial^j] G q^{-1} \bar{\psi} u_n. \end{pmatrix} \quad if \ |u_n > \in \mathcal{H}_n \quad (5.43) \]

They will be called momentum operators, since they reduce to the ordinary momentum operators in $\mathbb{R}^N$ when $q = 1$. Contrary to the classical case, from formula (5.43) we recognize that $\Pi(P^i) G q^{-1}, \bar{\Pi}(P^i) G q$ are not pure derivatives. A straightforward computation shows that the classical commutation relations $[P^i, P^j] = 0$ are replaced by the new ones

\[ \bar{P}^j_{A, h k} P^h P^k = 0. \quad (5.44) \]

The reader might ask why we have not defined the position/momentum operators so that the square length and the square momentum be represented in $V$ (resp. in $\bar{V}$) by $xCx, q^N \Delta$ (resp. $xCx, q^{-N} \bar{\Delta}$). The reason is that the operators $xCx, q^N \Delta$ (resp. $xCx, q^{-N} \bar{\Delta}$) do not map all of $V$ (resp. $\bar{V}$) into itself (for instance, if $\psi_0$ denotes the ground state in $V$, then it is easy to check that $(xCx)\psi_0 \notin V$). Therefore the hamiltonian (5.7) cannot be written as a combination of the square length and square momentum. This difficulty seems difficult to overcome by modifying the hamiltonian and therefore its eigenfunctions and $V$ itself. For instance, looking for the eigenfunctions of a hamiltonian of the type $q^{1 + \frac{N}{2}} (\omega^2 x C x - q^{-2 - N} \Delta) G q^2$, which is formally hermitean, one can find that they are functions belonging to the space $F := \{ f = P(x) \exp_q^2 [-\frac{\omega q^{1 - \frac{N}{2}} x C x}{\mu}] \}$; here, contrary to the eigenfunctions $\psi_n$ of $\hbar_\omega$, the exponent in the gaussian is independent of the degree $n$ of the polynomial $P(x)$. One immediately realizes that the operator $q^{1 + \frac{N}{2}} x C x G q^2$ maps $F$
out of itself, i.e. it is not a well defined operator in $F$; $xCx$, instead, would be. Therefore it would be natural to consider the latter as the representative in $F$ of the square length. Similarly, $q^{-1-\frac{N}{2}}\Delta G_{q^2}$ is not a well defined operator in $F$, whereas $q^{-N}\Delta$ is. Thus again we would see that this new hamiltonian cannot be written as a combination of the square length and of the square momentum.

6. The observables $L^2, L_m$

We look for some other hermitean operators such that they commute with the hamiltonian $H_\omega$ and with each other. To this end in this section we search the analog of the angular momentum. We will mainly give explicit definitions and relations in the unbarred representation; the barred ones can be found performing the usual substitutions.

As a primary requirement, the components of the angular momentum should commute with any scalar function of the coordinates and of the momenta. In the classical case they are antisymmetrized products of coordinates and derivatives of the type $\frac{1}{i}(y^i \partial^j - y^j \partial^i)$ or their combinations. Therefore we first look at the commutation relations of the operators $L_{ij} := P_{A \; kj} x^h \partial^k = -q^{-2}P_{A \; hjk} \partial^h x^k$ with $xCx, \Delta$. Using formula (2.31) we find

$$L_{ij} xCx = q^2 xCx L_{ij}, \quad L_{ij} \Delta = q^{-2} \Delta L_{ij}. \quad (6.1)$$

It immediately follows that

$$[G_{q^2} L_{ij}, xCx] = 0 = [G_{q^2} L_{ij}, \Delta] \quad (6.2)$$

Next, it is easy to show that $G_{q^2} L_{ij}$ commutes with any scalar polynomial $I$ (in particular $h_\omega$) obtained combining $x^i$’s and $\partial^i$’s:

$$[I(x, \partial), G_{q^2} L_{ij}] = 0. \quad (6.3)$$

Actually any such polynomial can be written as a polynomial in $xCx, \Delta$ alone (see Appendix C). For this reason, we propose the operators $G_{q^2} L_{ij}$’s (and their barred partners) as candidates to the role of angular momentum components.

By squaring the $L_{ij}$ we obtain the scalar operator $L^2$:

$$L^2 := L_{ij} L_{ji} = x^h \partial^k P_{A \; hjk} x^j \partial_i \quad (6.4)$$
To obtain the last expressions in (6.4) we have used the property (2.8) (where we take \( f(\hat{R}) = p_A \)), and \( P_A^2 = P_A \). Of course \( G_{q^4}L^2 \) commutes with any scalar function \( I(x, \partial) \) and in particular with \( \hbar \omega \).

We want to find out eigenvalues and eigenfunctions of \( G_{q^4}L^2 \). From the above property it is clear that if \( P(x) \) is an eigenvector of \( G_{q^4}L^2 \), then for any function \( p = p(xCx) \) \( g := P(x)p(xCx) \) is an eigenvector of \( G_{q^4}L^2 \) with the same eigenvalue. A little thinking will convince the reader that, just as in the classical case, after factorizing a possible function \( p(xCx) \) the eigenvectors \( P(x) \) of \( G_{q^4}L^2 \)

\[
G_{q^4}L^2 P(x) = cP(x)
\]

(6.5)
can be written as homogeneous polynomials:

\[
P(x) = p(xCx)A_{i_1i_2...i_n}x^{i_1}x^{i_2}...x^{i_n}
\]

(6.6)

Actually \( G_{q^4}L^2 \) is homogeneous in both \( x \), \( \partial \) with the same degree (two), hence it transforms any homogeneous polynomial of degree \( n \) into another one of the same degree. Now we specify the form of the \( A_{i_1i_2...i_n} \) coefficients. We are going to prove that just as in the classical case and up to factors \( p(xCx) \) the set of homogeneous polynomials

\[
P_{i_1i_2...i_n} := P_{n,S}A_{i_1i_2...i_n}x^{i_1}x^{i_2}...x^{i_n}, \quad l_i = 1, ..., N
\]

(6.7)
is a complete set of eigenvectors of degree \( n \) of both \( G_{q^4}L^2 \) and \( G_{q^4}L^2 \). Here \( P_{n,S} \) is the \( q \)-deformed symmetric projector acting on \( \otimes^n \mathbb{C} \) (in particular \( P_{2,S} = P_S \)), whose properties will be briefly discussed in Appendix D. The main property of these projectors is that

\[
P_{n,S}P_{A \ i, (i+1)} = 0 = P_{n,S}P_{1 \ i, (i+1)}, \quad 1 \leq i \leq n - 1
\]

(6.8)

where, for any matrix \( F \) defined on \( C \otimes C \) we denote by \( F_{i,i+1} \) \( (1 \leq i \leq n - 1) \) the matrix defined by \( F_{i,i+1} := 1 \otimes ... 1 \otimes F \otimes 1 \otimes ... \otimes 1 \) (\( F \) at the \( i^{th} \) and \( (i + 1)^{th} \) place). Since all the projectors are symmetric, the above properties hold also if we multiply \( P_{n,S} \) by \( P_{A \ i, (i+1)}, P_{1 \ i, (i+1)} \) from the left. Relations (6.8) imply

\[
P_{n,S}P_{S \ i, (i+1)} = P_{n,S} \quad 1 \leq i \leq n - 1.
\]

(6.9)

Let us consider the space \( M_n \) of homogeneous polynomials of degree \( n \) (see formula 2.51) and its two projections

\[
M_n^S = P_{n,S}M_n \quad M_n^1 = (P_1 \otimes 1_{n-2})M_n.
\]

(6.10)
Proposition 6.1: $M_n$ can be decomposed into the direct sum

$$M_n = \bigoplus_{0 \leq m \leq \frac{n}{2}} M_{n,n-2m} \quad M_{n,n-2m} := (xCx)^m M_{n-2m}.$$  \hspace{1cm} (6.11)

In other words

$$1_{M_n} := \bigoplus_{0 \leq m \leq \frac{n}{2}} (P_1 \otimes \ldots \otimes P_1 \otimes P_{n-2m}, S)$$  \hspace{1cm} (6.12)

is the identity operator on $M_n$.

Proof: Let us consider $M_k$. Because of formula (2.9), (6.8)

$$M_k = M_k^S \oplus M_k^1 = [P_{k,S} \oplus (P_1 \otimes 1_{k-2})]M_k.$$  \hspace{1cm} (6.13)

By repeated application of formula (6.13) for $k = n, n-2, \ldots$ we arrive at the claim $\diamondsuit$.

We can evaluate the dimensions of the spaces $M_k^S$ in a straightforward way, since we know the dimension of $M_l$ as a function of $l$ (see formula (2.50)), and $M_k^1$ is generated by \{\(x^{i_1} \ldots x^{i_{k-2}} x^{j_{k-1}} x^{j_k} P_{1_{j_{k-1},j_k}}\), i.e., by $\{x^{i_1} \ldots x^{i_{k-2}} (xCx)\}$:

$$\dim(M_k^1) = \dim(M_{k-2}) = \binom{N + k - 3}{N - 1}$$

$$\dim(P_k^S) := \dim(M_k^S) = \dim(M_k) - \dim(M_k^1) = \dim(M_k) - \dim(M_{k-2})$$  \hspace{1cm} (6.14)

Proposition 6.2: $M_n^S$ are eigenspaces of $2q^2 G_q^4 \mathcal{L}^2$, $2q^{-2} G_q^{-4} \bar{\mathcal{L}}^2$ with the same eigenvalue

$$l_n^2 = \frac{2}{q + q^{-1}} \frac{q^{N - 2} + q^{-2 - N}}{q^{N - 2} + q^{1-N}} [n]_q [N + n - 2]_q.$$  \hspace{1cm} (6.15)

Note: we have included in the definition of these operator a factor 2 so that for $N = 3$ and $q=1$ the eigenvalues reduce to the classical ones $n(n + 1)$ of the classical square angular momentum in three dimensions. Moreover, as the energies $E_n$, the eigenvalues (6.15) are invariant under the transformation $q \rightarrow q^{-1}$.

Proof: To reach the goal we first transform $\mathcal{L}^2$ into a more suitable form, which explicitly shows its scalar character. By quite a lengthy calculation one can show that

$$\mathcal{L}^2 = \alpha_N(q)x^i \partial_i + \beta_N(q)x^i x^j \partial_j \partial_i + \gamma_N(q)xCx$$  \hspace{1cm} (6.16)

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where
\[ \alpha_N(q) := \frac{(q^{2-N} + q^{N-2})(q^{1-N} - q^{N-1})}{(q^{1-N} + q^{N-1})(q^{-2} - q^2)}, \]
\[ \beta_N(q) := \frac{q^3 + q^{N-1}}{\mu(q + q^{-1})}, \]
\[ \gamma_N(q) := -\frac{(q^{5-N} + q)(1 + q^{-N})}{\mu^2(q + q^{-1})}. \] (6.17)

Now notice that property (6.9) together with the relations
\[ \mathcal{P}^i_{j h k} x^h \partial^k = \mathcal{P}^j_{i h k} \partial^h x^k, \] (6.18)
implies
\[ \mathcal{P}_{n, S b_1 b_2 \ldots b_n} a_1 a_2 \ldots a_n x^{b_1} \ldots x^{b_{i-1}} \partial^{b_i} x^{b_{i+1}} \ldots x^{b_n} = 0. \] (6.19)

Similarly, upon use of formula (2.31)
\[ \mathcal{P}_{n, S b_1 b_2 \ldots b_n} a_1 a_2 \ldots a_n x^{b_1} \ldots x^{b_{i-1}} \Delta x^{b_i} \ldots x^{b_n} = 0 \] (6.20)

This means that when applying \( \mathcal{L}^2 \) to \( \mathcal{P}_{n, S l_1 l_2 \ldots l_n} x^{i_1} x^{i_2} \ldots x^{i_n} \) we can forget all the terms (which we will denote by dots) containing powers of \( \Delta \) or where the index \( b_i \) of a derivative \( \partial^{b_i} \) is contracted with an index of \( \mathcal{P}_{n, S} \). The term with coefficient \( \gamma_N \) in the RHS of (6.16) can be ignored, whereas
\[ (x^a \partial_a)x^{b_i} = x^{b_i} + q^2 x^{b_i}(x^a \partial_a) + \ldots \]
\[ (x^a x^b \partial_b \partial_a)x^{b_i} = (1 + q^2)x^{b_i}(x^a \partial_a) + q^4 x^{b_i}(x^a x^b \partial_b \partial_a) + \ldots \] (6.21)

By a somewhat lengthy calculation we find
\[ G_q \mathcal{L}^2 \mathcal{P}_{S l_1 l_2 \ldots l_n} = c_n \mathcal{P}_{S l_1 l_2 \ldots l_n} \] (6.22)
where
\[ c_n := \frac{q^{-2} q^{\frac{N}{2} - 2} + q^{2 - \frac{N}{2}}}{q + q^{-1} q^{\frac{N}{2} - 1} + q^{1 - \frac{N}{2}} [n]_q [N + n - 2]_q}; \] (6.23)

\( \mathcal{P}_{S l_1 \ldots l_n}(x) \) are therefore eigenvectors of \( G_q \mathcal{L}^2 \), with eigenvalues depending only on \( n \). In the derivation we have used the formula (2.28)_b and the relation
\[ \sum_{k=1}^{n} q^{-2k} k q^2 = q^2 \frac{n q^2 (n + 1) q^2}{2 q^2}, \] (6.24)
(the latter can be easily proved iteratively). In the same way we can show that the eigenvalues \( \bar{c}_n \) of \( G_{q^{-4}} \mathcal{L}^2 \) are given by \( \bar{c}_n = q^4 c_n \). We see that \( M^S_n \) is eigenspace of both \( 2q^2 G_{q^4} \mathcal{L}^2 \) and \( 2q^{-2} G_{q^{-4}} \mathcal{L}^2 \) with the same eigenvalue \( l^2_n \) given in formula (6.15) \( \diamond \).

As a direct consequence of propositions 6.1, 6.2 we find the

**Corollary 6.3:** relation (6.11) provides the decomposition of \( M_n \) into the direct sum of eigenspaces \( M_{n,n-2m}^S \) of \( 2q^2 G_{q^4} \mathcal{L}^2 \) and \( 2q^{-2} G_{q^{-4}} \mathcal{L}^2 \) with eigenvalues \( l^2_{n-2m} \), \( m = 0, 1, ... \lceil \frac{n}{2} \rceil \).

Since \([h_\omega, G_{q^4} \mathcal{L}^2] = 0\), it is possible to find eigenvectors of \( h_\omega, G_{q^4} \mathcal{L}^2 \) at the same time. Using again property (6.9) it is quite easy to realize that

\[
[(\mathcal{P}_1 \otimes ... \otimes \mathcal{P}_1 \otimes \mathcal{P}_{n-2m,S}) \psi_n]^{l_1,...l_n} \propto \mathcal{P}_S^{l_{2m+1}...l_n} p_{n,m}(xC)\exp[q^2\omega q^{-n-N}x Cx / \mu], \tag{6.25}
\]

(with suitable polynomials \( p_{n,m}, 0 \leq m \leq \lceil \frac{n}{2} \rceil \)), hence these functions are eigenvectors of \( 2q^2 G_{q^4} \mathcal{L}^2 \) with eigenvalue \( l^2_{n-2m} \). The same holds for the analogous combinations of \( \bar{\psi} \)'s. Using the property (2.49) \((\mathcal{P}_A i,i+1 \psi_n = 0)\) we conclude that \( 1_{M_n} \) is the identity operator in \( \Psi_n, \bar{\Psi}_n \) as well. Therefore

\[
\Psi_n = \bigoplus_{0 \leq m \leq \lceil \frac{n}{2} \rceil} \Psi_{n,n-2m} \quad (\text{resp. } \bar{\Psi}_n = \bigoplus_{0 \leq m \leq \lceil \frac{n}{2} \rceil} \bar{\Psi}_{n,n-2m}), \tag{6.26}
\]

where

\[
\Psi_{n,n-2m} := (\mathcal{P}_1 \otimes ... \otimes \mathcal{P}_1 \otimes \mathcal{P}_{n-2m,S}) \psi_n \quad (\text{resp. } \bar{\Psi}_{n,n-2m} := (\mathcal{P}_1 \otimes ... \otimes \mathcal{P}_1 \otimes \mathcal{P}_{n-2m,S}) \bar{\psi}_n), \tag{6.27}
\]

is the eigenspace of \( h_\omega, 2q^2 G_{q^4} \mathcal{L}^2 \) (resp. of \( \bar{h}_\omega, 2q^{-2} G_{q^{-4}} \mathcal{L}^2 \)) with eigenvalues \( E_n, l^2_{n-2m} \).

The above discussion shows that we are in the right condition to define a square angular momentum operator \( \mathcal{L}^2 \) in \( \mathcal{H} \). We set

\[
\begin{align*}
2q^2 G_{q^4} \mathcal{L}^2 \quad 
\end{align*}
\]

\[
\begin{align*}
2q^{-2} G_{q^{-4}} \mathcal{L}^2 \quad &.\tag{6.28}
\end{align*}
\]
We introduce the subspaces \( \mathcal{H}_{n,n-2m} \subset \mathcal{H} \) by

\[
\begin{align*}
\mathcal{H}_{n,n-2m} & \xrightarrow{\Pi} \Psi_{n,n-2m} \\
\mathcal{H}_{n,n-2m} & \xrightarrow{\bar{\Pi}} \bar{\Psi}_{n,n-2m}.
\end{align*}
\] (6.29)

We summarize the preceding results in the

**Proposition 6.4:** \( \mathcal{H}_{n,n-2m} \) \((n \geq 0, 0 \leq m \leq \frac{n}{2})\) is an eigenspace of the operators \( H_\omega, L^2 \) defined by (5.7),(6.28) with eigenvalues \( E_n, l^2_{n-2m} \) (see (5.6),(6.15)) respectively. Moreover

\[
\mathcal{H} = \bigoplus_{n=0}^{\infty} \bigoplus_{0 \leq m \leq \frac{n}{2}} \mathcal{H}_{n,n-2m}.
\] (6.30)

It remains to show that \( L^2 \) is hermitean. The proof is similar to the one we gave for \( H_\omega \); one uses Stoke’s theorem (3.7), the scaling property (4.31) of \( < > \) and the relation

\[
L^{2*} = q^{-2N-4} \bar{L}^2.
\] (6.31)

The latter can be drawn using relation (2.24), and formula (2.8). Explicitly:

\[
(L^2 u, v) = (u, L^2 v).
\] (6.32)

The direct consequence of formulae (5.23),(6.32) is that \( \mathcal{H}_{n,m} \) are orthogonal subspaces of \( \mathcal{H} \), i.e.

\[
u \in \mathcal{H}_{n,k}, \ v \in \mathcal{H}_{n',k'} \ and \ (n,k) \neq (n',k') \Rightarrow (u,v) = 0.
\] (6.33)

The previous proof actually implies a stronger property:

\[
< \bar{\psi}^*_{n,k} \psi_{n',k'} > = 0 \Rightarrow < \psi^*_{n,k} \bar{\psi}_{n',k'} > \quad if \quad (n,k) \neq (n',k'),
\] (6.34)

where \( \psi_{p,h} \in \Psi_{p,h}, \ \bar{\psi}_{p,h} \in \bar{\Psi}_{p,h} \).

Now we show how to construct the observables “angular momentum components”. As already noted at the beginning of this section, \( G_{q^2 L^{ij}} \) (respectively \( G_{q^{-2} \bar{L}^{ij}} \)) commutes with any scalar function \( I(x,\partial) \) (respectively \( I(x,\bar{\partial}) \)), in particular with \( h_\omega, G_{q^2 L^2} \) (respectively with \( h_\omega, G_{q^{-2} \bar{L}^2} \)). We therefore look for combinations \( G_{q^2 L_m} := m_{ij} G_{q^2 L^{ij}}, \ G_{q^{-2} \bar{L}_m} := \bar{m}_{ij} G_{q^{-2} \bar{L}^{ij}} \) which can be considered as the representatives in \( V, \bar{V} \) of one and
the same operator in $\mathcal{H}$. To this end they should have the same eigenvalues and there should exist an isomorphism between the corresponding eigenfunctions.

Consider a function $\chi(x)$ which is an eigenvector of $G_q^2 L_m$:

$$G_q^2 L_m \chi| = \lambda \chi.$$  \hspace{1cm} (6.35)

Up to a factor $f(xCx)$ $\chi$ must be a homogeneous polynomial of the type

$$\chi_D := D_{l_1 \ldots l_k} P_{\mathbb{S}}^{l_1 l_2 \ldots l_k}, \quad k \geq 1, \quad D_{l_1 \ldots l_k} \in \mathbb{C},$$  \hspace{1cm} (6.36)

since $[g_q^2 L^2, G_q^2 L_m] = 0$. Hence equation (6.35) explicitly reads

$$G_q^2 m_{ij} D_{l_1 \ldots l_k} \mathcal{P}_{\mathbb{S}}^{l_1 l_2 \ldots l_k} \mathcal{P}^A P^B_{hh} x^h \partial^k x^{i_1 i_2 \ldots i_k} \chi| = \lambda \chi_D.$$  \hspace{1cm} (6.37)

Using relations (6.8),(6.9) one can easily show that the RHS can be rewritten in the following way:

$$G_q^2 m_{ij} D_{l_1 \ldots l_k} \mathcal{P}_{\mathbb{S}}^{l_1 l_2 \ldots l_k} \mathcal{P}^A P^B_{hh} x^h q^2 C_{k j_1} x^{i_2 \ldots i_k} \chi| = \lambda \chi_D.$$  \hspace{1cm} (6.38)

On the other hand, using exactly the same relations and arguments one can show that $G_{q^{-2}} \bar{\mathcal{L}}_m \chi_D| = q^2 G_q^2 L_m \chi_D|$, hence we conclude that $\chi_D$ is an eigenvector of both $q^{-1} G_{q^{-2}} \bar{\mathcal{L}}_m$ and $q G_q^2 L_m$ with the same eigenvalue. We are led to guess that we can introduce a well defined operator $L_m$ on $\mathcal{H}$ by the definition

$$L_m \begin{array}{c} \Pi \rightarrow \\ \Pi \end{array} qG_q^2 \bar{\mathcal{L}}_m.$$  \hspace{1cm} (6.39)

Using the same arguments employed when proving the hermiticity of $L^2$ one can formally show that $L^m$ is hermitean provided the coefficients $m_{ij}$’s satisfy the condition

$$[(C \otimes C) \cdot m^*]_{ij} = m_{ji}.$$  \hspace{1cm} (6.40)

Under this assumption it follows that inside each subspace $\mathcal{H}_{n, n-2h}$ there exists a complete set of eigenfunctions of $L_m$, that its eigenvalues $\lambda$ are real, and that the eigenfunctions are of the form

$$D_{l_2 h+1 \ldots l_n} (\mathcal{P}_1 \otimes \ldots \otimes \mathcal{P}_1 \otimes \mathcal{P}_{n-2h,i_1 \ldots i_k})^{l_1 l_2 \ldots l_k} |i_1 i_2 \ldots i_n > \in \mathcal{H}_{n, n-2h},$$  \hspace{1cm} (6.41)
where the coefficients $D$’s satisfy condition (6.37) with $k = n - 2h$. Summing up, for any set of coefficients $\{m_{ij}\}$ satisfying condition (6.40) $L_m$ is a well-defined observable on $\mathcal{H}$ commuting with the Hamiltonian and the square angular momentum:

$$[H_\omega, L_m] = 0, \quad [L^2, L_m] = 0. \quad (6.42)$$

For $q = 1$ and a suitable normalization it coincides with a particular component of the angular momentum. When $N = 3$ $H_\omega, L^2, L_m$ make up a complete set of commuting observables, at least in a neighbourhood of $q = 1$, since this is the case when $q = 1$. In a forthcoming paper [18] we will study the spectrum and eigenfunctions of the operators $L_m$.

7. Positivity of the scalar product

In this section we prove the positivity of the scalar product $(\ , \ )$; in this way the proof that $\mathcal{H}$ is a pre-Hilbert space is finished. Then completion of $\mathcal{H}$ can be performed in the standard way. The section is not essential for a conceptual understanding of the work and may be skipped by the reader if he/she is not interested in computations.

Proposition 7.1: $\forall q \in \mathbb{R}^+$ the scalar product introduced in section 5 is positive definite:

$$(u, u) \geq 0 \quad (u, u) = 0 \iff u = 0, \quad u \in \mathcal{H}. \quad (7.1)$$

Proof: The results of the preceding section imply that it is sufficient to prove positivity inside each subspace $\mathcal{H}_{n,n-2m}$. The most general $\mid u > \in \mathcal{H}_{n,k}$, $k = n - 2m$, $0 \leq m \leq \frac{n}{2}$ is of the form

$$D_{l_1l_2...l_k}^\dagger \bar{\psi}^{l_1l_2...l_k}_{n,(k,S)}$$

$$u \quad \rightarrow \quad D_{l_1l_2...l_k} \psi^{l_1l_2...l_k}_{n,(k,S)} \quad (7.2)$$

(see (6.25)) where

$$\psi^{l_1l_2...l_k}_{n,(k,S)} := (a^+_n Ca^+_{n-1}...a^+_k Ca^+_{k+1})\psi^{l_1l_2...l_k}_{k,S}$$

$$\bar{\psi}^{l_1l_2...l_k}_{n,(k,S)} := (\bar{a}^+_n \bar{C}a^+_{n-1}...\bar{a}^+_k \bar{C}a^+_{k+1})\bar{\psi}^{l_1l_2...l_k}_{k,S} \quad (7.3)$$
and

\[ \psi_{k,S}^{l_1 l_2 \ldots l_k} := \mathcal{P}_{k,S} l_1 \ldots l_k \psi_{k}^{l_1 \ldots l_k} = t_k(q) \mathcal{P}_{k,S} l_1 \ldots l_k x^{i_1} \ldots x^{i_k} \exp q^2 - \frac{\omega q^{-k-N} x C x}{\mu} \]

\[ \tilde{\psi}_{k,S}^{l_1 l_2 \ldots l_k} := \mathcal{P}_{k,S} l_1 \ldots l_k \tilde{\psi}_{k}^{l_1 \ldots l_k} = t_k(q^{-1}) \mathcal{P}_{k,S} l_1 \ldots l_k x^{i_1} \ldots x^{i_k} \exp q^{-2} - \frac{\omega q^{k+N} x C x}{\bar{\mu}}. \] (7.4)

Here \( a_m C a_{m+1} := a_i^m C_{ij} a_{m+1}^j \) and \( a_{m+i}, a_m^i \) are the creation/destruction operators introduced in (5.2), (5.3). A glance at formula (5.36) and an easy calculation show that the coefficients \( t_k(q) \) and \( t_k(q^{-1}) \) are positive for \( \forall q \in \mathbb{R}^+ \). In the rest of this section \( a \propto b \) will mean \( a = \sigma b, \sigma > 0 \). The square norm of \( u \) is given by

\[ (u, u) = D_{p_1 \ldots p_k}^{*} \mathcal{D}_{l_1 \ldots l_k} \left< (\psi_{n,(k,S)}^{p_1 \ldots p_k}) \psi_{n,(k,S)}^{l_1 \ldots l_k} \right> + \left< (\psi_{n,(k,S)}^{p_1 \ldots p_k}) \tilde{\psi}_{n,(k,S)}^{l_1 \ldots l_k} \right> \] (7.5)

We are going to show our claim by proving that each one of the conjugate terms in the RHS of relation (7.5) is positive.

**Lemma 7.2:**

\[ \left< (\tilde{\psi}_{n,(k,S)}^{p_1 \ldots p_k}) \psi_{n,(k,S)}^{l_1 \ldots l_k} \right>_\propto \left< (\psi_{n,(k,S)}^{p_1 \ldots p_k}) \tilde{\psi}_{n,(k,S)}^{l_1 \ldots l_k} \right>_\propto \]

\[ \mathcal{P}_{k,S} l_1 \ldots l_k \mathcal{P}_{k,S} p_1 \ldots p_k C^{h_1 j_1} \ldots C^{h_k j_k} \frac{S^{h_k \ldots h_1 i_1 \ldots i_k}}{S_{2k}} \rho_k \] (7.6)

where

\[ \rho_k := \exp q^{-2} - \frac{\omega q^{k+N} x C x}{\bar{\mu}} \left< (\psi_{n,(k,S)}^{p_1 \ldots p_k}) \tilde{\psi}_{n,(k,S)}^{l_1 \ldots l_k} \right> \] (7.7)

**Proof of Lemma 7.2:** From the definition (7.3) and formulae (5.21), (5.22), (5.36) it follows

\[ \left< (\tilde{\psi}_{n,(k,S)}^{p_1 \ldots p_k}) \psi_{n,(k,S)}^{l_1 \ldots l_k} \right>_\propto \left< (\psi_{n,(k,S)}^{p_1 \ldots p_k}) \tilde{\psi}_{n,(k,S)}^{l_1 \ldots l_k} \right>_\propto \] (7.8)

where

\[ f_{k,S}^{l_1 \ldots l_k} := (ak C a_{k+1}) \ldots (a_{n-4} C a_{n-3}) (a_{n-2} C a_{n-1}) \psi_{n,(k,S)}^{l_1 l_2 \ldots l_k}. \] (7.9)

Because of formula (5.19), only the component of \( f_{k,S}^{l_1 \ldots l_k} \) belonging to \( \Psi_k \) contributes to the integral (7.8). Looking at formula (B.10), we can decompose the operator \( (a_{n-2} C a_{n-1}) \) in the following way

\[ (a_{n-2} C a_{n-1}) = \alpha_{n-1,2}(a_{n+2} C a_{n+1}) + \beta_{n-1,2}(a_{n} C a_{n+1}) + \gamma_{n-1,2}(a_{n} C a_{n+1}) + \] (7.10)

\[ + \delta_{n-1,2}(a_{n+2} C a_{n+1}), \]
which is appropriate to clearly display the result of its action on $\Psi_n$: we see that it maps $\psi_{n,(k,S)}'$ into a combination of functions $\psi_{n+2}'$, $\psi_n'$, $\psi_{n-2}'$ belonging respectively to $\Psi_{n+2}$, $\Psi_n$, $\Psi_{n-2}$. Next, the operator $(a_{n-4}C_{n-3})$ acts on $\psi_{n+2}'$, $\psi_n'$, $\psi_{n-2}'$. For each of these three functions we choose the appropriate decomposition of $(a_{n-4}C_{n-3})$. Doing the same job again and again, we end up with a combination of functions belonging to $\Psi_{2n-k}$, $\Psi_{2n-2-k}$, $\ldots$ $\Psi_k$. It is not difficult to realize that

$$P \Psi_k(f^{l_1\ldots l_k}) = \prod_{h=1}^m \beta_{n-2h+1,2}(a_{k+2}C_{k+3})\ldots(a_{n}C_{n+1})\psi_{n,(k,S)}^{l_1l_2\ldots l_k}, \quad (7.11)$$

where $P \Psi_k$ denotes the projector on $\Psi_k$. Since all coefficients $\beta_{l,m}$ are positive for $q \in \mathbb{R}^+$, by picking the explicit definition (7.3) of $\psi_{n,(k,S)}'$ we find

$$P \Psi_k(f^{l_1\ldots l_k}) \propto (a_{k+2}C_{k+3})\ldots(a_{n}C_{n+1})(a^+_{n}C^+_{n-1})\psi_{n-2,(k,S)}^{l_1l_2\ldots l_k}, \quad (7.12)$$

In the appendix B it is proved that

$$(a_{n}C_{n+1})(a^+_{n}C^+_{n-1})\psi_{n-2,(k,S)}^{l_1l_2\ldots l_k} \propto \psi_{n-2,(k,S)}^{l_1l_2\ldots l_k} \quad (7.13)$$

(see formula (C.12), (C.23)); hence

$$P \Psi_k(f^{l_1\ldots l_k}) \propto (a_{k+2}C_{k+3})\ldots(a_{n-2}C_{n-1})\psi_{n-2,(k,S)}^{l_1l_2\ldots l_k} \quad (7.14)$$

using $m = \frac{n-k}{2}$ times the same kind of argument we conclude that

$$P \Psi_k(f^{l_1\ldots l_k}) \propto \psi_{k,S}^{l_1l_2\ldots l_k}. \quad (7.15)$$

From eq.'s (7.8), (5.19), (7.15), (7.4), (4.15) it follows that

$$< (\psi_{n,(k,S)}^{p_1\ldots p_k})^*\psi_{n,(k,S)}^{l_1\ldots l_k} > \propto < (\psi_{k,S}^{p_1\ldots p_k})^*\psi_{k,S}^{l_1\ldots l_k} > \propto$$

$$\propto P_{k,S}^{l_1\ldots l_k} P_{k,S}^{p_1\ldots p_k} C_{h_1j_1} \ldots C_{h_kj_k}.$$  

$$< x^{h_k}\ldots x^{h_1} x^{i_1} \ldots x^{i_k} e^{xq\frac{\omega q k - N}{\mu xC}} e^{xq\frac{-\omega q k + N}{\mu xC}} > = RHS(7.6)$$

Similarly one can show the claim for $< (\psi_{n,(k,S)}^{p_1\ldots p_k})^*\psi_{n,(k,S)}^{l_1\ldots l_k} >$. \diamond

**Lemma 7.3:**

$$D_{p_1p_2\ldots p_k}^* P_{k,S}^{p_1p_2\ldots p_k} D_{l_1l_2\ldots l_k} P_{k,S}^{l_1l_2\ldots l_k} C_{h_1j_1} C_{h_2j_2} \ldots C_{h_kj_k} S_{h_k\ldots h_{i_1}\ldots i_k} > 0. \quad (7.16)$$
Proof of Lemma 7.3: First, using property (E.6), (E.5) of the symmetric projectors we can rewrite LHS(7.16) in the following way:

\[
[(\otimes^k C) \cdot D_{j_1\ldots j_k}^T P_{k,S} ]_{j_1\ldots j_k} \cdot P_{k,S} ]_{l_1\ldots l_k} S_{h_1\ldots h_k} D_{l_1\ldots l_k}.
\]  (7.17)

In appendix E we prove the following relation:

\[
P_{j_1\ldots j_k} P_{l_1\ldots l_k} = \sigma_k [P_{k,S} \cdot (\otimes^k C) P_{k,S} ]_{j_1\ldots j_k} \cdot l_1\ldots l_k;
\]  (7.18)

a similar result could be proved in the barred case. Using property (E.6) (E.5) again, we can rewrite the RHS of (7.17) as a sum of positive terms

\[
\sum_{m_1, m_2, \ldots, m_k} \|[(\otimes^k C) \cdot P_{k,S} \cdot D]_{m_1\ldots m_k} \|^2;
\]  (7.19)

this expression is always \( \geq 0 \) and is zero if and only if \( P_{k,S} \cdot D = 0 \iff u = 0 \) (in fact \( (\otimes^k C) \) is a nondegenerate matrix) \( \Box \).

Now we can complete the proof of Proposition 7.1: since \( \rho_k > 0 \) (see formula (B.24)), \( S_{2k} > 0 \), from the two preceding lemmata we immediately derive the thesis \( \Box \).

Now we can introduce a norm \( \| \cdot \| \) in \( \mathcal{H} \) by setting

\[
\|u\|^2 = (u, u).
\]  (7.20)

The completion \( [\mathcal{H}] \) of \( \mathcal{H} \) w.r.t. this norm can be performed in the standard way. It induces completions \([V], [\bar{V}] \subset Fun(\mathbb{R}^N_q)\) of \( V, \bar{V} \). It would be interesting to investigate if the latter can be characterized in an intrinsic way, e.g. by characterizing their (formal) power expansion in \( x_i \)'s. This is left as a possible subject for some future work.

8. Conclusions

We have shown that the quantum harmonic oscillator on \( \mathbb{R}^N \) with symmetry \( SO(N, \mathbb{R}) \) admits a \( q \)-deformation into the harmonic oscillator on the quantum space \( \mathbb{R}^N_q \) with symmetry \( SO_q(N, \mathbb{R}) \), for any \( q \in \mathbb{R}^+ \).

In fact this \( q \)-deformed harmonic oscillator has a lower bounded energy spectrum; generalizing the classical algebraic construction, the Hilbert space of physical states is
built applying construction operators to the (unique) ground state. The scalar product is strictly positive for any \( q \in \mathbb{R}^+ \). Observables are defined as hermitean operators, as usual. In particular we have constructed the observables hamiltonian, square angular momentum, angular momentum components, position operators, momentum operators. As in the classical case, the first two and any angular momentum component commute with each other; when \( N = 3 \) they make up a complete set of observables.

Both spectra of the hamiltonian and of the square angular momentum are discrete, and the eigenvalues have the same degeneracy as in the non-deformed case. The \( q \)-deformed eigenvalues are invariant under the replacement \( q \to q^{-1} \) and can be obtained from the classical ones essentially by the replacement \( n \to [n]_q \), where \([n]_q := \frac{q^n - q^{-n}}{q - q^{-1}} \). Energy levels are no more equidistant; their difference increases with \( n \).

Guiding ideas for the construction were \( SO_q(N, \mathbb{R}) \)-covariance and correspondence principle in the classical limit \( q \to 1 \). Essential tools were the two differential calculi on \( \mathbb{R}^N_q \), the corresponding (and coinciding) two integrations on this quantum space and the corresponding two representations of the Hilbert space into the space of functions on \( \mathbb{R}^N_q \). A sort of quantized scaling property of the integral under dilatation of the integration variables and correspondingly a quantization of the dimensional constant \( \omega \) (the characteristic constant of the harmonic oscillator) has been singled out.

Appendix A

We prove formulae (4.9), (4.10) by induction. For \( n = 1 \) (4.9), (4.10) are true, since \( \Delta x^{i_1} x^{i_2} = \mu \partial^{i_1} x^{i_2} = \mu C^{i_1 i_2} \Delta x^{i_1} x^{i_2} = \tilde{\mu} \partial^{i_1} x^{i_2} = \tilde{\mu} C^{i_1 i_2} \). Now assume that they are true for \( n = m - 1 \). Then

\[
\Delta^m x^{i_1} x^{i_2} \ldots x^{i_{2m}} = \mu \Delta^{m-1} \partial^{i_1} x^{i_2} \ldots x^{i_{2m}} + q^2 \Delta^{m-1} x^{i_1} \Delta x^{i_2} \ldots x^{i_{2m}} = \\
= \mu(1 + q^2) \Delta^{m-1} \partial^{i_1} x^{i_2} \ldots x^{i_{2m}} + q^2 \Delta^{m-2} x^{i_1} \Delta^2 x^{i_2} \ldots x^{i_{2m}} = \\
= \ldots \ldots = \\
= \mu m_q^2 \Delta^{m-1} \partial^{i_1} x^{i_2} \ldots x^{i_{2m}} + q^{2m} x^{i_1} \Delta^m x^{i_2} \ldots x^{i_{2m}}. \tag{A.1}
\]

The second term in the last expression is zero, since the \( 2m \) derivatives contained in \( \Delta^m \) act on \((2m - 1)\) coordinates \( x \); using the definition (4.2) of the tensor \( M_{2m} \), the induction hypothesis and the definition (4.5) of \( S_{2m} \) we are able to rewrite the first term as

\[
\mu m_q^2 M_{2m, j_3 \ldots j_{2m}}^{i_1 i_2 \ldots i_{2m}} \Delta^{m-1} x^{j_3} \ldots j_{2m} = (\mu)^m m_q^2 !M_{2m, j_3 \ldots j_{2m}}^{i_1 i_2 \ldots i_{2m}} S_{2(m-1)}^{j_3 \ldots j_{2m}} =
\]

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which shows that (4.9) is true also for \( n = m \). In a similar way one proves (4.10).

\[ (A.2) \]

**Appendix B**

In this appendix we first show how to evaluate integrals of the type

\[ < (xC)^m \exp_{-2} \left[ -\frac{\omega q^{N+k} xC}{\mu} \right] \exp_{-2} \left[ -\frac{\omega q^{-k} xC}{\mu} \right] > (B.1) \]

taking \( f_0 := \exp_{-2} \left[ -\frac{\omega q^{N} xC}{\mu} \right] \exp_{-2} \left[ -\frac{\omega q^{-N} xC}{\mu} \right] \) as reference function. The outcomes results, together with formulae (4.15),(4.16), will allow the determination of all integrals involved in the scalar products of vectors of \( \mathcal{H} \). Second, we give some results concerning the action of creation/destruction operators on functions \( \psi \in V \).

We start from

\[ < (xC)^m \exp_{-2} \left[ -\frac{\omega q^{k-N} xC}{\mu} \right] \exp_{-2} \left[ -\frac{\omega q^{-k-N} xC}{\mu} \right] > = q^{N+2m} < (xC)^m \exp_{-2} \left[ -\frac{\omega q^{k-N} xC}{\mu} \right] \exp_{-2} \left[ -\frac{\omega q^{-k-N} xC}{\mu} \right] > (B.2) \]

which is a direct consequence of the scaling property (4.23) of the integrals. Using the q-derivative properties (2.30) of the exponentials to expand the functions \( \exp_{-2} \left[ -\frac{\omega q^{k-N} xC}{\mu} \right] \exp_{-2} \left[ -\frac{\omega q^{-k-N} xC}{\mu} \right] \) we find

\[ < (xC)^{m+1} \exp_{-2} \left[ -\frac{\omega q^{k-N} xC}{\mu} \right] \exp_{-2} \left[ -\frac{\omega q^{-k-N} xC}{\mu} \right] > = \left( \frac{q^{1-N} + q^N}{\omega} \right). \]

\[ \left[ \frac{N}{2} + m \right]_q \left[ \frac{m-k}{2} \right]_q \cdot \left[ \frac{N}{2} + m - 2 \right]_q \cdots \left[ \frac{N}{2} \right]_q \left[ \frac{m-k}{2} ! \right]_q \cdot \right. \]

\[ < \exp_{-2} \left[ -\frac{\omega q^{k-N} xC}{\mu} \right] \exp_{-2} \left[ -\frac{\omega q^{-k-N} xC}{\mu} \right] > = \left( \frac{q^{1-N} + q^N}{\omega} \right)^m \left[ \frac{N}{2} + m - 1 \right]_q \left[ \frac{N}{2} + m - 2 \right]_q \cdots \left[ \frac{N}{2} \right]_q \left[ \frac{m-k}{2} ! \right]_q \cdot \]

\[ \left( B.3 \right) \]

i.e.

\[ (B.4) \]

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Now consider the integral $< f_0 >$. If $k = 2l$, upon use of the q-derivative properties (2.30) one finds

$$
<f_0> = <\exp_q^{-2} \left[ -\frac{\omega q^{2l+N} xC_x}{\mu} \right] \exp_q[ -\frac{\omega q^{-2l-N} xC_x}{\mu}] > \prod_{h=0}^{l-1} \left( 1 - q^{2(h-l)-N} \omega \frac{q^2-1}{\mu} xC_x \right) \prod_{h=0}^{l-1} \left( 1 - q^{2(l-h)+N} \omega \frac{q^{-2}-1}{\mu} xC_x \right).
$$

(B.5)

Expanding the products contained in the square brackets and using formula (B.4) to evaluate all the integrals one finds

$$
<f_0> = z_k <\exp_q^{-2} \left[ -\frac{\omega q^{k+N} xC_x}{\mu} \right] \exp_q[ -\frac{\omega q^{-k-N} xC_x}{\mu}] >
$$

(B.6)

with a suitable constant $z_k$. If $k$ is even, this formula, together with (B.4), allows to evaluate any integral (B.1) in terms of $< f_0 >$ (which is taken as the normalization factor of the integral). If $k$ is odd, by repeating the previous steps we obtain

$$
<f_0'> = z'_k <\exp_q^{-2} \left[ -\frac{\omega q^{k+N} xC_x}{\mu} \right] \exp_q[ -\frac{\omega q^{-k-N} xC_x}{\mu}] >
$$

(B.7)

where $f_0' := \exp_q^{-2} \left[ -\frac{\omega q^{N+1} xC_x}{\mu} \right] \exp_q[ -\frac{\omega q^{-N-1} xC_x}{\mu}]$. Following the line suggested at the end of sect. 4, it is possible to find the constant $\phi(q)$ such that

$$
<\exp_q^{-2} \left[ -\frac{\omega q^{1+N} xC_x}{\mu} \right] \exp_q[ -\frac{\omega q^{-1-N} xC_x}{\mu}] > = \phi(q)
$$

and

$$
\phi(q) <\exp_q^{-2} \left[ -\frac{\omega q^N xC_x}{\mu} \right] \exp_q[ -\frac{\omega q^{-N} xC_x}{\mu}] >
$$

(B.8)

and to show that it is positive $\forall q \in \mathbb{R^+}$. We don’t perform here this computation, but just notice that by continuity the positivity of $\phi$ must hold at least in a neighbourhood of $q = 1$, since $\phi(1) = 1$. In this way all the integrals (B.1) are evaluated in terms of the normalization constant $< f_0 >$. Note that from the definition (B.8) it follows $\phi(q^{-1}) = \phi(q)$.

**********

From the definition (5.2), (5.3) of the creation/destruction operators it immediately follows

$$
\frac{a_i^n}{d_n(q)} = \frac{q^{n+N} + q^{2-n-m} a_i^{n+m}}{q^{n+N+m} + q^{2-n-m} d_{n+m}(q)} + \frac{q^{n+N+m} - q^{n+N} a_{n+m}^{+i}}{q^{n+N+m} + q^{2-n-m} b_{n+m}(q)} m \in \mathbb{Z},
$$

(B.9)
whence

\[ a_{n-1}C_{an} := a_{n-1}^i C_{ij} a_n^j = \alpha_{n,m}(q)(a_{n+m+1}^+ C_{an+m}) + \beta_{n,m}(q)(a_{n+m-1} C_{an+m}) + \]

\[ + \gamma_{n,m}(q)(a_{n+m-1}^+ C_{an+m}) + \delta_{n,m}(q)(a_{n+m+1} C_{an+m}), \quad (B.10) \]

where \( \beta_{n,m}(q) \) is positive \( \forall q \in \mathbb{R}^+ \).

**********

We know that

\[ (a_n C_{an+1})(a_n^+ C_{an-1}^+) \psi_{n-2,(k,S)}^{l_1 \ldots l_k} = v_{n-2,k} \psi_{n-2,(k,S)}^{l_1 \ldots l_k} \quad (B.11) \]

(the function \( \psi_{n-2,(k,S)} \) \( \left( k = n - 2m \right) \) was defined in (7.3)), since both sides are eigentunctions of \( h_{\omega}, G_q^4 \mathcal{L}_2 \) with the same eigenvalues and have the same transformation properties under the action of the quantum group \( SO_q(N, \mathbb{R}) \). We now show that the constant \( v_{n-2,k} \) is positive \( \forall q \in \mathbb{R}^+ \). In the sequel by \( a \propto b \) we will mean \( a = \sigma b \) with \( \sigma > 0 \). Note that \( \psi_{l_1 \ldots l_k}^{n-2,(k,S)} \) can be written in the form

\[ \psi_{n-2,(k,S)}^{l_1 \ldots l_k} = [c(x C x)^{m-1} + \ldots] \exp_{q^2} \left[ -\frac{q^{-N+2} \omega C x}{\mu} \right] P_{k,S}^{l_1 \ldots l_k} x^{i_1 \ldots i_k}, \quad (B.12) \]

where (as in the sequel) the dots in the square bracket denote lower degree powers of \( (x C x) \). The strategy will be to find out \( v_{n-2,k} \) by only looking at the term of highest degree in \( x C x \) at each step of the derivation. From the definition (5.2),(5.3) of the creation/destruction operators and the definition (4.18) of the \( B \) operator we get

\[ (a_n C_{an+1})(a_n^+ C_{an-1}^+) = q^{-3}[x C x + \frac{q^{2(n+1+N)}}{\omega^2} \Delta + \frac{q^{n+2N} \mu^2}{\omega(q^2 - 1)} B + \text{const.}] \cdot \]

\[ [x C x + \frac{q^{10 - 2n}}{\omega^2} \Delta - \frac{q^{2-n+N} \mu^2}{\omega(q^2 - 1)} B + \text{const.}] G_q^4 d_{n}(q) d_{n+1}(q) b_{n}(q) b_{n-1}(q) \quad (B.13) \]

The \( \Delta \)'s in the first and second square bracket have to act respectively on functions belonging to \( G_q^2 \Psi_n \) and to \( G_q^4 \Psi_{n-2} \), therefore they can be respectively replaced by \( (q^{-N-4} \omega^2 x C x - q^{-N-2} E_n) \) and \( (q^{-N-8} \omega^2 x C x - q^{-N-4} E_{n-2}) \). Hence

\[ (a_n C_{an+1})(a_n^+ C_{an-1}^+) \propto E \cdot F, \quad (B.14) \]

where

\[ E := [x C x (1 + q^{N+2(n-1)}) + \frac{q^{n+2N} \mu^2}{\omega(q^2 - 1)} B + \text{const.}] \]
\[ F := \left[ xC(1 + q^2(1 - n) - N) - \frac{q^{2-n+N}\mu^2}{\omega(q^2 - 1)}B + \text{const.} \right]q^4. \] (B.15)

From formulae (6.19) and (6.21), one easily derives the identity

\[ B\mathcal{P}_{k,S}^l_{i_1,...i_k} x^{i_1}...x^{i_k} = \frac{q^{2k} + q^{2-N}}{\mu} \mathcal{P}_{k,S}^l_{i_1,...i_k} x^{i_1}...x^{i_k} \] (B.16)

Using the fundamental property (4.19) of \( B \), formulae (B.12) and (B.16) we find

\[ F\psi_{n-2,(k,S)}^{l_1,...l_k} = \left[ c(xCx)^m q^{4-4m} (1 + q^2(1-n) - N) \exp_q^2 \left[ -\frac{q^{2-N-2}\omega x C x}{\mu} \right] + \right. \]
\[ -c\mu q^{4-n-2m+N}(q^{2k} + q^{2-N})(xCx)^m-1 \exp_q^2 \left[ -\frac{q^{2-N}\omega x C x}{\mu} \right] + ...]. \]
\[ \mathcal{P}_{k,S}^l_{i_1,...i_k} x^{i_1}...x^{i_k}, \] (B.17)

applying the q-derivative property (2.30) to the exponential \( \exp_q^2 \left[ -\frac{q^{2-N}\omega x C x}{\mu} \right] \) we find

\[ F\psi_{n-2,(k,S)}^{l_1,...l_k} \propto [c(xCx)^m + ...] \exp_q^2 \left[ -\frac{q^{2-N-2}\omega x C x}{\mu} \right] \mathcal{P}_{k,S}^l_{i_1,...i_k} x^{i_1}...x^{i_k}. \] (B.18)

After similar steps one can see that the result of the action of \( E \) on \( \psi_{n-2,(k,S)} \) is

\[ E \cdot F\psi_{n-2,(k,S)}^{l_1,...l_k} \propto e[c(xCx)^m+1+...] \exp_q^2 \left[ -\frac{q^{2-N-2}\omega x C x}{\mu} \right] \mathcal{P}_{k,S}^l_{i_1,...i_k} x^{i_1}...x^{i_k} \] (B.19)

where

\[ e := (1 - q^{2m})(1 - q^{2(n-m-1)+N}). \] (B.20)

Using again the q-derivative property (2.30), we increase by 4 the degree of the q-power in the exponent and we lower by 2 the degree of the polynomial in \( (xCx) \) contained in the square bracket, with the result that eq. (B.11) holds with \( v_{n-2,k} \) given by

\[ v_{n-2,k} \propto e \left( \frac{-\mu}{\omega(q^2 - 1)q^{n-N-2}} \right) \left( \frac{-\mu}{\omega(q^2 - 1)q^{n-N}} \right). \] (B.21)

We see that \( v_{n-2,k} > 0 \quad \forall q \in \mathbb{R}^+ \).

*************

Using the explicit definition of the creation/destruction operators and the Schroedinger equation for \( \psi_n \in \Psi_n \) it is straightforward to show that

\[ a^+_n C a_{n+1} \psi_n = \sigma B G q^2 \psi_n \quad a_{n+2} C a^+_n \psi_n = \sigma' B G q^2 \psi_n, \quad \sigma, \sigma' > 0; \] (B.22)
if \( \psi_n \) is the scalar eigenfunction of level \( n = 2m \), then \( BG_{q^2} \) acts as the identity operator and therefore

\[
a_n^+ C_{n+1} \psi_n = \sigma \psi_n \quad a_{n+2} C_{n+1}^+ \psi_n = \sigma' \psi_n, \quad \sigma, \sigma' > 0. \tag{B.23}
\]

We show that

\[
< \exp_{q^2} \left[ - \frac{\omega q^{k+N} xC x}{\mu} \right] (xC x)^k \exp_{q^2} \left[ - \frac{\omega q^{-k-N} xC x}{\mu} \right] > > 0 \quad \forall q \in \mathbb{R}^+ \tag{B.24}
\]

First we consider the case \( k = 2h \). Using the scaling property (4.21) of the integral we find

\[
LHS(B.24) = q^{-k(k+1)} < \psi_0 (xC xG_{q^2})^k \psi_0 >. \tag{B.25}
\]

Looking back at formulae (5.33)\(_a\), (5.42) it is easy to prove that \( (xC xG_{q^2})^k \psi_0 \) can be decomposed in the following way

\[
(xC x)G_{q^2} = \frac{q+q^{-1}}{\omega (1+q^{-2-N})} [(a_n C_{n+1}) + (a_n^+ C_{n+1}) + (a_{n+2} C_{n+1}) + (a_{n+2}^+ C_{n+1})] \forall n \geq 0. \tag{B.26}
\]

Only the component \( \mathcal{P}\psi_0 ((xC xG_{q^2})^k \psi_0) \) belonging to \( \Psi_0 \) of the function \( (xC xG_{q^2})^k \psi_0 \) gives a nonvanishing contribution to the integral (B.25), because of property (5.19). Using the decomposition (B.26) with \( n = 0, 2, ..., 2(k-1) \), properties (B.11),(B.21),(B.23) we see that

\[
\mathcal{P}\psi_0 ((xC xG_{q^2})^k \psi_0) = \tau_k \psi_0, \tag{B.27}
\]

where \( \tau_k \) is given by a sum of positive constants \( \forall q \in \mathbb{R}^+ \). This proves (B.24) in the case \( k = 2l \).

If \( k = 2l + 1 \) an analogous reduction shows that

\[
< \exp_{q^2} \left[ - \frac{\omega q^{1+N} xC x}{\mu} \right] (xC x)^{k+1} \exp_{q^2} \left[ - \frac{\omega q^{-1-N} xC x}{\mu} \right] > =
\]

\[
= \tau'_k < \exp_{q^2} \left[ - \frac{\omega q^{1+N} xC x}{\mu} \right] (xC x) \exp_{q^2} \left[ - \frac{\omega q^{-1-N} xC x}{\mu} \right] >, \tag{B.28}
\]

where \( \tau'_k > 0 \forall q \in \mathbb{R}^+ \). Formulae (B.3), (B.8) imply

\[
< \exp_{q^2} \left[ - \frac{\omega q^{1+N} xC x}{\mu} \right] (xC x) \exp_{q^2} \left[ - \frac{\omega q^{-1-N} xC x}{\mu} \right] > =
\]

\[
= \frac{2-N}{\omega} + q^{N-2} xC x \left[ \frac{N}{2} q \phi(q) \right] < \exp_{q^2} \left[ - \frac{\omega q^{N} xC x}{\mu} \right] \exp_{q^2} \left[ - \frac{\omega q^{-N} xC x}{\mu} \right] >. \tag{B.29}
\]
Since $\phi(q)$ is positive $\forall q \in \mathbb{R}^+$, (B.24) is proved for any $k$.

**Appendix C**

In this appendix we show that any scalar polynomial $I(x, \partial)$ (resp. $I(x, \bar{\partial})$) in $x^i, \partial^j$ (resp. $x^i, \bar{\partial}^j$) can be expressed as a polynomial in the (ordered) variables $xCx, \Delta$ (resp. $xCx, \bar{\Delta}$) alone. We limit ourselves to the unbarred case; the proof for the barred case is a word by word repetition of the proof of the former, after obvious replacements.

To be a scalar $I$ must be a polynomial in scalar variables of the type

$$I^2 = \eta^{i_1}_e (\eta^{i_2}_{e_2}) ... (\eta^{i_n}_{e_n})_{i_n} ... (\eta^{i_2}_{e_2})_{i_2} \eta^{i_1}_{e_1}$$

where $\varepsilon_i, \varepsilon'_j = +, -, \eta_+ := x$ and $\eta_- := \partial$. From here we see that $I$ can only contain terms of even degree in $\eta^i$; we denote by $I_{2m}$ a scalar polynomial of degree $2m$ and containing only even powers of $\eta^i$. The only four independent $I_2$ are 1, $xCx, \Delta, x^i \partial_i$, and they all can be expressed as polynomials in $xCx, \Delta$ because of formula (2.32).

Our claim amounts to showing that for any $I_{2m} (m \geq 0)$ there exist an ordered polynomial $P_1(xCx, \Delta)$ in $xCx, \Delta$ such that

$$I_{2m} = P_1(xCx, \Delta)$$

The claim is obviously true for $m = 0$. The general proof is by induction: assume that it is true for $m = k$. Since any $I_{2(k+1)}$ can be written as a polynomial in $\bar{I}_{2n}$ variables with $n \leq k + 1$, it is sufficient to prove the claim for a $\bar{I}_{2(k+1)}$ whatsoever. By the induction hypothesis and the very definition (C.1) of the $\bar{I}$ variables $\bar{I}_{2(k+1)}$ can be written in the form

$$\bar{I}_{2(k+1)} = (\eta^i_\varepsilon) \bar{P}(xCx, \Delta)(\eta^i_{\varepsilon'})$$

with some polynomial $\bar{P}$. Decomposing the latter in a sum of monomials and using formulae

$$\partial^i(xCx) = \mu x^i + q^2(xCx)\partial^i \quad x^i \Delta = q^{-2}\Delta x^i - \mu q^{-2}\partial^i$$

to move the $\eta^i$'s step by step through all the factors $xCx, \Delta$ as far as the extreme right we will be able to write the RHS of (C.3) as a combination of terms of the type $\bar{P}'(xCx, \Delta) \cdot (\eta^i_{\varepsilon''}) (\eta^i_{\varepsilon'})_i$; but $(\eta^i_{\varepsilon''}) (\eta^i_{\varepsilon'})_i$ is a polynomial of the type $I_2$ for which the claim (C.2) holds, hence it holds also for $\bar{I}_{2(k+1)}$ and the statement (C.2) is completely proved.
Appendix D

In this appendix we list a few properties of the projectors $P_k^S$ defined in formula (6.8). In a forthcoming paper we will show that

$$P_{k,S} = \pi_k(\hat{R}_{i,i+1}, P_{i,i+1}^1), \quad i = 1, 2, \ldots, k-1 \quad (D.1)$$

$$P_{k,S}^T = P_{k,S} \quad (D.2)$$

where $\pi_k$ is a polynomial in $\hat{R}_{i,i+1}, P_{i,i+1}^1$ such that

$$\pi_k(\hat{R}_{i,i+1}, P_{i,i+1}^1) = \pi_k(\hat{R}_{k-i,k-i+1}, P_{k-i,k-i+1}^1). \quad (D.3)$$

Let us denote by $P_k$ the permutator on $\otimes^k \mathbb{C}^N$ defined by

$$P_k(v_1 \otimes v_2 \otimes \ldots \otimes v_k) = v_k \otimes \ldots \otimes v_2 \otimes v_1, \quad v_i \in \mathbb{C}^N; \quad (D.4)$$

Using the relation (2.8) it is easy to check that

$$P_k \cdot (\otimes^k C) \hat{R}_{i,i+1} = \hat{R}_{k-i,k-i+1} P_k \cdot (\otimes^k C) \quad P_k \cdot (\otimes^k C) P_{i,i+1}^1 = P_{k-i,k-i+1}^1 P_k \cdot (\otimes^k C) \quad (D.5)$$

Relations (D.3), (D.5) imply

$$[P_k \cdot (\otimes^k C), P_{k,S}] = 0 \quad (D.6)$$

Here we give, as an example, the explicit form of $P_{3,S}$ in terms of $\hat{R}_{i,i+1}, (P_1)_{i,i+1}$:

$$P_{3,S} = \frac{1}{3q^2} \{ 1 + q(\hat{R}_{12} + \hat{R}_{23}) + q^2(\hat{R}_{12} \hat{R}_{23} + \hat{R}_{23} \hat{R}_{12}) + q^3 \hat{R}_{12} \hat{R}_{23} \hat{R}_{12} +$$

$$- \frac{(q^N - 1)^2}{2(q^{N+2} - 1)} [(2q^2 + 1 - q^4)((P_1)_{12} + (P_1)_{23}) + 2q^4 Q_N((P_1)_{12}(P_1)_{23} + (P_1)_{23}(P_1)_{12})$$

$$q^2(q + q^{-1})(\hat{R}_{12}(P_1)_{23} + (P_1)_{23} \hat{R}_{12} + \hat{R}_{23}(P_1)_{12} + (P_1)_{12} \hat{R}_{23}) +$$

$$+ (\hat{R}_{12}(P_1)_{23} \hat{R}_{12} + \hat{R}_{23}(P_1)_{12} \hat{R}_{23})] \} \quad (D.7)$$

Appendix E

We give a brief proof of relation (7.18). From relation (4.9) we infer that

$$P_{k,S} \frac{j_k \ldots j_1}{h_k \ldots h_1} P_{k,S} \frac{l_1 l_2 \ldots l_k}{i_1 i_2 \ldots i_k} s_{h_k \ldots h_1 i_1 \ldots i_k} =$$

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\[= \sigma'_k \mathcal{P}_{k,S} j_k \ldots j_1 \mathcal{P}_{k,S} l_1 \ldots l_k \Delta^k x^{h_k \ldots h_1 x^{i_1} \ldots x^{i_k}}, \quad \sigma'_k > 0. \quad (E.1)\]

Using relations (6.18),(6.19),(6.20) we can rewrite the RHS in the following way:

\[
\sigma'_k \mathcal{P}_{k,S} j_k \ldots j_1 \mathcal{P}_{k,S} l_1 \ldots l_k \Delta^{k-1} (\mu \partial^{h_k} + q^2 x^{h_k} \Delta) x^{h_{k-1} \ldots h_1 x^{i_1} \ldots x^{i_k}} = \\
= \sigma'_k \mathcal{P}_{k,S} j_k \ldots j_1 \mathcal{P}_{k,S} l_1 \ldots l_k \Delta^{k-1} [\mu k q^2 x^{h_k} \ldots x^{h_2} \partial^{h_1} x^{i_1} \ldots x^{i_k}] + \\
q^2 x^{h_k} \ldots x^{h_1} \Delta x^{i_1} \ldots x^{i_k}. \quad (E.2)\]

The second term in the square brackets will yield a vanishing contribution. In fact, the operator \(\Delta^{k-1}\) can transform at most \((k-1)\) of the \(k\) \(x^{h_i}\) into \(\partial^{h_i}\), and the remaining \(x^{h_i}\)'s can be moved to the left of all derivatives using property (6.18); such an expression is zero, since it contains a number \(l > k\) of derivatives acting on \(x^{i_1} \ldots x^{i_k}\) (i.e. on their left). Using \(k\) times the same kind of argument we end up with

\[
\mathcal{P}_{k,S} j_k \ldots j_1 \mathcal{P}_{k,S} l_1 \ldots l_k S^{h_k \ldots h_1 i_1 \ldots i_k} = \\
\sigma''_k \mathcal{P}_{k,S} j_k \ldots j_1 \mathcal{P}_{k,S} l_1 \ldots l_k [\partial^{h_k} \ldots \partial^{h_2} \partial^{h_1} x^{i_1} \ldots x^{i_k}], \quad \sigma''_k > 0. \quad (E.3)\]

Now let us perform the remaining derivations in the RHS of (E.3). Using relation (2.5) it becomes

\[
\sigma''_k \mathcal{P}_{k,S} j_k \ldots j_1 \mathcal{P}_{k,S} l_1 \ldots l_k [\partial^{h_k} \ldots \partial^{h_2} (C^{h_2 i_1} x^{i_2} \ldots x^{i_k} + q C^{h_1 q} \hat{R}^{i_1 i_2 x^{i_3} \ldots x^{i_k} + \ldots})], \quad (E.4)\]

and using relations (6.9) it can be written in the form

\[
\sigma''_k \mathcal{P}_{k,S} j_k \ldots j_1 \mathcal{P}_{k,S} l_1 \ldots l_k [\partial^{h_k} \ldots \partial^{h_2} k q^2 C^{h_1 i_1} x^{i_2} \ldots x^{i_k}] = \\
= \ldots \ldots = \\
= \sigma_k \mathcal{P}_{k,S} j_k \ldots j_1 \mathcal{P}_{k,S} l_1 \ldots l_k C^{h_1 i_1} C^{h_2 i_2} \ldots C^{h_k i_k} = \\
= \sigma_k [\mathcal{P}_{k,S} (\otimes^k C) \mathcal{P}_{k,S}]^{j_k \ldots j_1}, \quad \sigma_k > 0 \quad (E.5)\]

where for the last equality we have used properties (D.2),(D.6). Relation (7.18) is thus proved.

Notes

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To do this job one has to manage q-series. We hope to report useful results in this direction elsewhere [17].

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