Coherence and decoherence in the Harper-Hofstadter model

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We quantum-simulated the 2D Harper-Hofstadter (HH) lattice model in a highly elongated tube geometry—three sites in circumference—using an atomic Bose-Einstein condensate. In addition to the usual transverse (out-of-plane) magnetic flux, piercing the surface of the tube, we threaded a longitudinal flux $\Phi_L$, down the axis of the tube. This geometry evokes an Aharonov-Bohm interferometer, where noise in $\Phi_L$ would readily decohere the interference present in trajectories encircling the tube. We observe this behavior only when transverse flux is a rational fraction of the flux-quantum, and remarkably find that for irrational fractions the decoherence is absent. Furthermore, at rational values of transverse flux, we show that the time evolution averaged over the noisy longitudinal flux matches the time evolution at nearby irrational fluxes. Thus, the appealing intuitive picture of an Aharonov-Bohm interferometer is insufficient. Instead, we quantitatively explain our observations by transforming the HH model into a collection of momentum-space Aubry-André models.

Understanding how and when closed quantum systems lose or retain coherence is a central intellectual and practical question for quantum technologies. For example, modern optical atomic clocks operate in highly optimized decoherence-free subspaces created by using “clock” states that are insensitive to the environment, as well as using lasers at magic wavelengths and polarizations that give only common mode energy shifts. In rare cases, such as collisional narrowing or environment assisted tunneling, random processes can enhance coherence. Here we add to this list the quasi-periodic lattice model of the Harper-Hofstadter (HH) model in a highly-elongated tube geometry—a 1D quasi-crystal—by showing that the dynamics can be made immune to environmental noise.

Ultracold atomic gases in optical lattices can mimic Aharonov-Bohm (AB) phase factors using the optical phase of interfering laser beams. Even in units of the magnetic flux quantum $\Phi_0 = \hbar/q$ (for Planck’s constant $h$ and charged $q$), these systems realize large tunable magnetic fluxes $\Phi = a^2 B/\Phi_0$ per lattice platelette (lattice constant $a$). Planar geometries, narrow Hall ribbons, and even tubes have been realized in experiment. In the tube geometry, the longitudinal flux $\Phi_L$ threading the tube has significant physical consequences. For example, adiabatically ramping $\Phi_L$ by one flux quantum would drive one cycle of Laughlin’s topological charge pump which can probe both non-interacting and many-body topological systems.

The HH model, initially formulated to describe electrons moving in a 2D crystalline lattice subject to a transverse magnetic field; in terms of the AB phase $\Phi$ the HH model takes the form

$$\hat{H} = -J S_m \sum_{n=0}^{\infty} e^{i(2\pi \Phi_n + \phi)} |m + 1, n\rangle \langle m, n| + h.c.$$  

For isotropic tunneling $J_s = J_x$, the resulting “Hofstadter-butterfly” energy spectrum was one of the first quantum fractals ever predicted. In planar geometries the additional uniform Peierls phase $\phi$ has no physical consequence, however, for a tube $M$ sites in circumference $[M = 3$ depicted in Fig. 1(a)], the uniform Peierls phase contributes $M \phi/(2\pi)$ to the longitudinal flux $\Phi_L$. Using the synthetic dimension approach, we assembled our 2D lattice by combining the sites of a 1D optical lattice with three internal atomic states to respectively define the longitudinal ($e_x$) and azimuthal ($e_s$) directions of our tube. We perform AB-like interference experiments [Fig. 1(a)] in which particles prepared in a wavepacket at site $m = 1$ along $e_s$, but extended along $e_x$, are released and potentially interfere as they rapidly encircle the tube.

In this Letter, we report three key observations summarized in Fig. 1(b), where $\langle \cdot \rangle$ denotes the average over $\phi$ uniformly sampled from $[0, 2\pi)$, and $|\cdot|$ marks the time-average. (i) For rational transverse flux $\Phi = P/Q$ (expressed in reduced form), the time-evolving population in each $m$-site depends strongly on $\phi$ and therefore exhibits large uncertainties, as one would expect for an AB interferometer. (ii) This dependence decreases with increasing $Q$ and vanishes for irrational $\Phi$. (iii) The $\phi$-averaged dynamics at rational $\Phi$ is equal to the that at nearby irrational $\Phi$. In all cases, our numerical simulations are in excellent agreement with both mean evolution and $\phi$-sensitivity. While our experimentally probed $\Phi$ near 2/3, these observations are generalized by the numerical simulation shown in Fig. 1(c) that plots the time-averaged sensitivity to $\phi$; this curve is approximated by an everywhere discontinuous Thomae-like function. The spatial extent $w$ limits the degree to which $\Phi$ can be distinguished to $\approx a/w$, broadening the otherwise singular peaks.

Implementation We performed these experiments using $^{87}$Rb BECs in the $|S_1/2, F = 1\rangle$ electronic ground state manifold in a crossed optical dipole trap. The longitudinal Thomas-Fermi radius $R_{TF} = 11.5(5)$ $\mu$m was
We adiabatically loaded BECs into the ground state basis were (\(B\) field resulting energy differences \(\delta\epsilon\)).

A pair of “Raman” laser beams counter-propagating along \(e_x\) [Fig. 1(d)] (wavelength \(\lambda_R\), and recove wavevector \(k_R = 2\pi/\lambda_R\) resonantly Raman coupled the \(m\) states with strength \(\Omega_R = 0.296(6)\)\(E_L\), providing hopping \(J_a = 0.111(3)E_L\) along \(e_x\). Because \(|m| = 2\) was Raman-coupled to \(|m = 0\), we adopt periodic labels, i.e., \(|m\rangle \equiv |\text{mod}(m,3)\rangle\). Since \(\Phi = k_R/E_L\) [24], we tuned \(\Phi = 2/3 + \Delta\Phi\) by varying the Raman wavelength from 770.94(1) nm \((\Delta\Phi = 2/87)\) to 806.46(1) nm \((\Delta\Phi = -1/141)\), with range limited by the increasing power requirement as the detuning from the excited states increased. While \(\phi\) could be tuned by changing the phase of any of the Raman lasers or displacing the optical lattice, we instead phase-shifted the CDD field \(B_m\), giving the same effect in the CDD basis [25]. Each experimental run randomly sampled a Peierls phase \(\phi\) uniformly distributed from 0 to \(2\pi\).

In the data presented here, we began with BECs in \(|m = 1\) and initiated dynamics by abruptly introducing \(J_a\) for a time \(t\) up to 1.8 ms, at which time the lattice, Raman, and dipole trap lasers were simultaneously extinguished. During the subsequent 21 ms time-of-flight (TOF) we applied a magnetic field gradient to spatially separate atoms in the three \(|m\rangle\) states. We then used absorption imaging to detect the resulting density distribution, yielding the longitudinal momentum distributions of each \(|m\rangle\) state.

Figure 2(b) shows representative TOF data after time evolution for \(\Delta\Phi = 1/84\) and initial state \(|m = 1\) (middle row). Each crystal momentum state of the longitudinal lattice consists of momentum states that are imaged as horizontally spaced diffraction orders with spacing \(2\hbar k_L\). The synthetic lattice sites are resolved vertically; the diffraction orders in these sites are shifted by \(\Delta\Phi = 1/84\). While \(\phi\) could be tuned by changing the phase of any of the Raman lasers or displacing the optical lattice, we instead phase-shifted the CDD field \(B_m\), giving the same effect in the CDD basis [25]. Each experimental run randomly sampled a Peierls phase \(\phi\) uniformly distributed from 0 to \(2\pi\).

Physical picture We present an intuitive picture explaining when the dynamics depends on \(\phi\), focusing on the case relevant to our experiment, with \(M = 3\) and initial state \(m = 1\). For each \(m\) state, Fig 2(a) plots the sinusoidal band structure of the longitudinal lattice potential as a function of crystal momentum \(q\). Given the BEC’s RMS spatial extent \(w = 0.46R_{TF} = 20(1)a\) (see SM), the corresponding momentum-space density distribution had RMS width \(w_q \approx h/(2w) = 4.0(2)\times 10^{-3} \times 2\hbar k_L\), schematically indicated by the grey Gaussian in the left panel. The azimuthal tunneling from Raman coupling induces transitions (tan arrows) that change the crystal momentum by \(2\hbar k_L\Phi\), as well as imparting the phase \(\phi\). After evolving under this coupling a period of time, some amplitude will remain in the ini-
giving a Gaussian wavepacket of width 0
|\text{est} band of the longitudinal lattice. The filled grey curve FIG. 2. Momentum space AA model. (a) The lowest band of the longitudinal lattice. The filled grey curve represents a Gaussian wavepacket of width $0.064 \times 2\hbar k_L$. The hollow and solid green rectangles, spaced by $2\hbar k_L$ in crystal momentum, are equivalent in the energy spectrum. The arrows mark Raman-induced coupling starting in state $|j = 0\rangle = |m_0 = 1, q_0 = 0\rangle$ with $\Delta \Phi = 1/30$. (b) A TOF data taken at $t = 1.506$ ms and $\Delta \Phi = 1/84$ show the momentum distribution associated with each $|m\rangle$ site. (c) AA ring corresponding to (a). The bottom panel zooms into the section with $j$ in the range of $\pm 5$. The horizontal lines marking the on-site energy resulting sampled from the dispersion in (a), are colored in accordance to their $|m\rangle$ state, and the grey bars indicate the discretely sampled Gaussian wavepacket.

initial state, while some will undergo three transitions, returning to the initial $|m\rangle$ state (right panel), with crystal momentum shifted $3\Delta \Phi \times 2\hbar k_L$. This closes a $\phi$-sensitive momentum-space interferometer, provided the final wavepacket has non-negligible overlap with the initial state, i.e., $3\Delta \Phi \times 2\hbar k_L \gtrsim w_q$.

Model We make the intuitive picture quantitative by Fourier transforming the 2D HH Hamiltonian along $e_x$ giving $\hat{H} = \sum_{q} H_{AA}(q_0)$ over Hamiltonians

\[
\hat{H}_{AA}(q_0) = -2J_x \sum_j \cos \left(\frac{2\pi (j\Phi + \frac{q_0}{2\hbar k_L})}{\sqrt{N}} \right) |j\rangle \langle j| - J_s \left( e^{i\phi} \sum_j |j + 1\rangle \langle j| \text{h.c.} \right) . \tag{2}
\]

each labeled by crystal momentum $q_0$. Each $\hat{H}_{AA}(q_0)$ is a realization of the 1D Aubry-Andrey (AA) lattice with nearest-neighbor hopping strength $J_s$ and sinusoidal potential with depth $4 J_s$, and phase set by $q_0$. The sites of this AA lattice $|j\rangle \equiv |m_0 + j, q_0 + j\Phi \times 2\hbar k_L\rangle$ are labeled by azimuthal site-index $m$ along with longitudinal crystal momentum $q$. As shown in Fig. 2(a), the sinusoidal potential originates from Raman transitions changing the crystal momentum by $2\hbar k_L\Phi$ as $m$ is incremented, in effect sampling the lowest band of the longitudinal lattice. For rational $\Phi$, each $H_{AA}(q_0)$ describes a ring [Fig. 2(c)] of size $N_{AA} = \text{LCM}(M, Q)$ (LCM denotes the least common multiple) since $|m_0 + N_{AA}, q_0 + N_{AA}\Phi \times 2\hbar k_L\rangle$ coincides with the initial state $|m_0, q_0\rangle$. For irrational $\Phi$ (incommensurate potential) the AA model is 1D quasicrystal with a prototypical metal ($J_s > J_x$) to insulator ($J_s < J_x$) transition [27, 28], and at criticality $J_s = J_x$ it is a quantum fractal showing features of quantum chaos [29].

Together with $\Phi$, the momentum space width $w_q$ specifies the initial occupation of AA lattice sites: the mapping in Fig. 2(a) shows that every third AA site potentially samples the initial wavepacket until $|3j\Delta \Phi \times 2\hbar k_L \gtrsim w_q$ when the site’s crystal momentum falls outside the initial wavepacket. A representative AA lattice is shown in Fig. 2(c) where the grey bars result from sampling the filled grey curve in Fig. 2(a). In the AA model’s metallic phase ($J_s > J_x$, where our experiments took place) amplitude ballistically expands from each initially occupied site, while in the insulating phase amplitude remains exponentially localized near each initial sites [28]. When $3|\Delta \Phi | \times 2\hbar k_L \ll w_q$, the AA ring contains multiple initially occupied sites; as the system evolves in the metallic phase, amplitude originating in different sites can overlap and interfere. In contrast, for $3|\Delta \Phi | \times 2\hbar k_L \gg w_q$ only isolated sites are initially occupied and the interference is absent. Any interference depends strongly on $\phi$, which appears in the tunneling term of Eq. (2).

We quantify the degree of interference by the normalized variance $\text{var}[P_m(t)]/(P_m(t))^2$ in each HH $m$-site. The analytic treatment in the SM shows that for $3|\Delta \Phi | \times 2\hbar k_L \gtrsim w_q$ this normalized variance is proportional to $f(3|\Delta \Phi | \times 2\hbar k_L)^2$, where $f(q) \equiv \int dq \psi^* (q + \delta q) \psi(q)$ is the form factor expressing the overlap integral anticipated by our intuitive description, and $\psi(q)$ is the momentum-space wavefunction.

Discussion Figure 3 summarizes our data for three $\Delta \Phi$ and compares the experimental data with our numerical modeling. Figure 3(a) depicts a control case where we converted our tube geometry into a ribbon by removing the Raman coupling between $|m = 0\rangle$ and $|m = 2\rangle$. The top row depicts the AA lattice along with the initial state (grey bars), showing the transition from multiple sources for $\Delta \Phi \approx 0$, to a single isolated source for $\Delta \Phi = 1/84$. In the ribbon case, multiple sources are present, but the disconnected links in the lattice prevent any potential interference, and the $\phi$-sensitivity vanishes for all transverse flux $\Phi$. The accompanying numerical simulations used parameters obtained from fits to the data in Fig. 1(b), lie within the calibration uncertainties (see SM). To obtain quantitative agreement with our data, we include a phenomenological dephasing parameter obtained from fitting the decaying sinusoid of Fig. 3(a).

The second row of Fig. 3 plots the three diffraction peaks in the range $-1/2 < q/(2\hbar k_L) < 1/2$ observed after an evolution time $t = 1.506$ ms. When $\Delta \Phi$ was small enough to allow interference between atoms originating
from different AA lattice sites [Fig. 3(b)], the corresponding crystal momentum difference between different paths to the same order is too small to be resolved. When $3|\Delta\Phi| \times 2\hbar k_L \gg w_q$, the momentum difference associated with the different AA became resolvable, causing each order in the TOF image to fragment into multiple overlapping sub-orders. In the ribbon case, only three orders are present irrespective of $\Delta\Phi$.

The third row of Fig. 3 compares the observed time dependence (top) with the prediction of our model (bottom). Each symbol marks the observed fractional population in the initial state $P_{m=1}$ for a single measurement (no averaging), and the bottom panel is a histogram of the predicted trajectories for all $\phi$. The experimental data is highly variable only for small $\Delta\Phi$ with a manifestly non-Gaussian distribution. The noisy dynamics begin at $t \approx \pi \hbar / (3 J_a) = 0.185$ ms when population was significantly transferred out of the initial state, and interference became possible [31]. The numerical model fully captures the observed spread of data, its time-dependence and even the non-Gaussian distribution, including singular features, i.e., caustics [32]. The ribbon case lacks interference and exhibits only technical noise.

Figure 3(a) plots the time-averaged normalized variance for fully connected rings (black data) as well as ribbons (grey data). A sharp peak is present at $\Phi = 2/3$ for the tube geometry, while the signal is featureless for the ribbon geometry. The measured time dependence of the normalized variance, shown in Fig. 4(b), is peaked at $\Delta\Phi = 0$ for all times, but with variable amplitude. In both cases, the numerical calculations [Fig. 4(c), and solid curve in Fig. 4(a)] are nearly indistinguishable from the data.

Lastly, Fig. 4(d,e) plots the $\phi$-averaged time evolution both for experiment (d) and numerics (e); neither show any feature at $\Delta\Phi = 0$ where the noise feature is maximal in Fig. 4(b,c). This can be directly understood in the Hall-tube picture as follows: the HH describing the system in the neighborhood of site $n$ can be transformed to that at $n = 0$ by a suitable change of $\phi$. For irrational transverse fluxes, the system uniformly samples $\phi$, leading to spatially self-averaged time evolution. Although individual systems at rational flux lack the spatial self-averaging effect, averaging over $\phi$ recovers the uniform sampling in the irrational case, resulting in similar mean time evolution as nearby irrational fluxes.

**Outlook** The realization of the HH model in the highly-asymmetric geometry proved an ideal testbed of the pheno-
FIG. 4. **Transverse flux ∆Φ dependance.** (a) Phase sensitivity characterized by the time averaged normalized variance is plotted for the fully connected (black) and fragmented (grey) AA rings. The solid line results from the numerical simulation of the full model with the experimentally measured Thomas-Fermi radius of 11.5 µm. The baseline of the numerical simulation is shifted by the averaged value of the experimental data away from the peak. Normalized variance (b) and mean time evolution (d) data were linearly interpolated from unequally spaced transverse fluxes probed experimentally. (c,e) are their corresponding numerical simulations, respectively.

nomena in this study, and the choice Φ ≈ 2/3 minimized the distance between the adjacent initially occupied AA lattice sites for M = 3. The onset time for noisy dynamics should increase with M, and if M were comparable to the longitudinal extent, we would not have observed any phase sensitivity within the experimental time scale.

Our AA model makes additional predictions that go beyond the experimental observations presented here. At longer evolution times than in our experiments, AA rings in the metallic phase with just one initially occupied site can exhibit long-time interference as amplitude fully encircles the ring. While this work did not address the question of giving φ explicit time-dependence, a simple argument suggests that sufficiently slowly varying φ(t) will still leave the dynamics unchanged for irrational ∆Φ near Φ = 2/3 and in the AA insulating phase. In the momentum space AA picture, φ(t) is equivalent to the potential $j d\phi/dt$ (i.e., a uniform electric field). We might expect that the dynamics will be unchanged, provided that both $J_s$ and $J_x$ are large compared to the energy shift from the electric field $\xi d\phi/dt$ over the extent of the the AA localization length $\xi$. If true, this would be a new type of quasidisordered-stabilised decoherence-free subspace.

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[17] Summing the phase factors around the tube gives $\Phi_L = M[\Phi_n + \phi/(2\pi)]$; $\Phi_L$ depends on longitudinal position n owing to the contribution $2\pi\Phi_n$ from the transverse flux.
The Thomae function is zero for irrational $\Phi$ and $1/Q$ for rational $\Phi$. Our analysis in the SM shows that the expected structure is proportional to a modified Thomae function that becomes $1/$LCM($Q,M$) for rational $\Phi$.

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The position-space Raman matrix elements $\propto e^{i(2kR_x+\Phi)}|m+1,x\rangle\langle m,x|$, sampled on a lattice with positions $x=na$, yield the phase factor in Eq. (1), with $\Phi = kR/kL$.

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For a 1D chain of three sites, population is maximally transferred out of the initial state at $t = \pi\hbar/(3J_s)$ with periodic boundaries and $t = \pi\hbar/(2\sqrt{2}J_s)$ with hard-wall boundaries.

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Supplementary Information

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S1. EXPERIMENTAL DETAILS

A. Experimental sequence

Each experiment began with a Bose-Einstein condensate (BEC) of $^{87}$Rb in a far-detuned crossed dipole trap, whose trapping frequencies were $(f_x, f_y, f_z) \approx (44, 53, 159)$ Hz in $|F = 1, m_F = -1\rangle$ sublevel, where $F$ and $m_F$ are the total atomic and magnetic angular momentum, respectively. Owing to the central role of the system size in our discussion, we independently determined the longitudinal Thomas-Fermi radius $R_{TF}$ of the BEC using both TOF and in-situ images. We used the Castin-Dum equations [S1] to obtain $R_{TF} = 11.6(5)$ µm from TOF images. We used digitally refocused in-situ images to give $R_{TF} = 11.4(6)$ µm. Because these two measurements share the same magnification, their uncertainties are correlated, and take the average $R_{TF} = 11.5(5)$ µm.

The rf field for the dynamical decoupling $B_{rf}$ [Fig. 1(d) in the main text] was assigned a randomized phase right before the atoms were adiabatically loaded into the $|m = 0\rangle$ pseudo-spin state. This rapid adiabatic passage mapped the magnetic sublevels $|m_F = -1, +1, 0\rangle$ to $|m = 0, 1, 2\rangle$ pseudo-spin states [S2], which are insensitive to the magnetic field and facilitate cyclic Raman coupling. The magnetic field $B_0$ and rf strength $\Omega_{rf}$ were stabilized using the procedure detailed in Sec. [S1C]. After that, we applied an rf π-pulse, along the direction orthogonal to both $B_{rf}$ and $B_0$, to prepare the atoms in the $|m = 1\rangle$ state. We removed any remaining atoms in $|m = 0\rangle$ by transferring them to the $F=2$ hyperfine manifold and blowing them away with a resonant light pulse. Meanwhile, the atoms were adiabatically loaded into the 1D optical lattice with a ramp duration of 200 ms. We then switched on the Raman coupling for various time $t$, followed by simultaneous sudden extinction of the Raman, optical lattice and dipole trap beams, to study the dynamics.

To achieve the momentum and pseudo-spin resolved imaging, we reversed the process of loading the atoms into the rf dressed states. Namely, during a time-of-flight of 21.3 ms, we adiabatically ramped down the magnetic field, followed by ramping down $\Omega_{rf}$ to zero. As a result, states $|m = 0, 1, 2\rangle$ mapped back to $|m_F = -1, +1, 0\rangle$, respectively, which were then separated using the Stern-Gerlach effect with a bias field perpendicular to the lattice beam propagation direction. The absorption image in Fig. 2(b) in the main text does not show the $|m\rangle$ states in the same order as observed in the experiments ($|m = 1\rangle$ and $|m = 2\rangle$ states were switched). Additionally, due to the high magnetic field gradient applied, a harmonic potential perpendicular to the bias field stretches $|m_F = +1\rangle$ and compresses $|m_F = -1\rangle$ states by $3 \sim 4\%$. The optical density profiles of $|m_F = \pm 1\rangle$ sublevels were rescaled with linear interpolation such that the distances between the neighboring orders, which should all be the two-lattice-photon recoil momenta, is the same for all the sublevels.
B. Raman setup

The Raman beams were almost colinear with the counter-propagating optical lattice beams, with the lattice beam bisecting the angle $\beta = 0.34(6)$° between the two Raman beams. In this configuration, the transverse flux per lattice plaquette is $\Phi = k_R \cos(\beta/2)/k_L \approx k_R/k_L$. The tiny angle $\beta$ was introduced to avoid retro-reflected beams. The Raman beams were carefully aligned such that there was no momentum transfer along the transverse direction so the dynamics of our system are essentially one-dimensional.

For phase coherence between Raman, optical lattice beams and $B_{\text{rf}}$, we detected the beatnote of the two Raman beams in the vicinity of the retro mirror of the optical lattice. There were two or three frequency components in the beatnote corresponding to the two or three Raman transitions. The $|m = 0\rangle \leftrightarrow |m = 1\rangle$ transition has the largest intensity and is relatively far away from the other peaks, and was therefore chosen to be the locking frequency. The local oscillator was taken from the same Direct Digital Synthesized (DDS) signal generator as $\Omega_{\text{ext}}$ for the dynamical decoupling and the Raman beatnote can be in phase. The feedback was applied to the Raman beam that only contained one frequency component [Raman 1 in Fig. 1(d) in the main text]. In the rotating frame, the frequency differences between the two Raman beams are $\delta \omega_{m,m+1}$, where $m = 0, 1, 2$. In the lab frame, they are $\delta \omega_{m,m+1} - \omega_{\text{rf}}$, so the frequencies of the beatnotes are $\sim 22$ MHz.

C. Magnetic field and rf locks

We then applied two microwave pulses of duration $T_{\mu w} = 100$ µs, separated by 1/60 s to partially transfer the atoms to $|F = 2, m_F = -1\rangle$ sublevel. At the locking point, the two pulses are blue and red detuned by $1/(2T_{\mu w})$ from the resonance, and each pulse transfers $\sim 5\%$ of the atoms. The lock point $B_{\text{lock}} \approx 31.37$ G was chosen such that the rf frequency $\omega_{\text{rf}}/(2\pi) = 22.1$ MHz is resonant with the $|F = 1, m_F = -1\rangle \leftrightarrow |F = 1, m_F = 0\rangle$ transition.

The error signal was the imbalance of the two transfers, $\left(\text{OD}_{\text{int}}^{(1)} - \text{OD}_{\text{int}}^{(2)}\right) / \left(\text{OD}_{\text{int}}^{(1)} + \text{OD}_{\text{int}}^{(2)}\right)$, where $\text{OD}_{\text{int}}^{(i)}$, $i = 1, 2$, is the integrated optical density of the two images, and was fed to the next experimental run, setting its magnetic fields. Our procedure extends earlier work [S3], and for further specifics see Ref. [S5].

After the atoms were transferred to $|m = 0\rangle$ state, we used a similar technique to stabilize the strength of the rf field $\Omega_{\text{rf}}$. The microwave pulses drove the transition that has the largest sensitivity to $\Omega_{\text{rf}}$ and almost least sensitive to the magnetic field, namely, from $|m = 0\rangle$ to the state that asymptotically goes to $|F = 2, m_F = -2\rangle$ with decreasing magnetic field.

S2. Simulations and Reduction to HH Model

We start with the full Hamiltonian

$$
\hat{H}_{\text{full}} = \left[\frac{\hbar^2 \hat{k}^2}{2 M_a} + \frac{V_L}{2} \cos(2k_L \hat{x}) + V_{\text{ext}}(\hat{x})\right] \otimes \hat{1} + \left[\sum_m \delta_m |m\rangle \langle m| - \sum_m \frac{\Omega_{R,m}}{2} e^{i(2k_R \hat{x} + \phi)} |m + 1\rangle \langle m| + \text{H.c.}\right],
$$

(S1)

describing the light-matter interaction of our three-state atoms of mass $M_a$ including a state independent confining potential $V_{\text{ext}}(\hat{x})$. In our experiment, we measured $V = 5.0(1) E_L$ lattice depth by suddenly applying the lattice potential and fitting the resulting Kapitza-Dirac time evolution [S6]. We obtained the Raman coupling strength $\Omega_{R,m} = \Omega_{R} = 0.296(6) E_L$ by separately measuring the Rabi frequencies of each transition and adjusting them to be equal within our uncertainties. This process was repeated each time we changed the wavelength of the titanium sapphire laser producing the Raman laser beams. The parameters $\delta_m = 0.00(2) E_L$ describe detuning from Raman resonance. The stated uncertainties include the variation across the measured transverse fluxes.

The ground-band behavior of $\hat{H}_{\text{full}}$ can be approximated by the tight binding HH Hamiltonian

$$
\hat{H}(\phi) = -J_x \sum_{m,n} |m, n + 1\rangle \langle m, n| - J_y \sum_{m,n} e^{i2\pi(\Phi_n + \phi)} |m + 1, n\rangle \langle m, n| + \text{h.c.},
$$

(S2)

where in analogy to the relation between the Planck constant $\hbar$ and $\hbar \equiv \hbar/(2\pi)$, we define $\phi = \phi/(2\pi)$. We connected these two models by numerically solving the first term in Eq. (S1) to obtain the ground-band Wannier orbitals $w_0(x - na)$ for each lattice site $n$, from which we obtained the longitudinal tunneling strength

$$
J_x = -\int dx w_0^*(a) \left[ -\frac{\hbar^2}{2 M_a} \frac{d^2}{dx^2} + \frac{V_L}{2} \cos(2k_L \hat{x}) \right] w_0(0) = 0.066(2) E_L.
$$

(S3)
FIG. S1. **Full and reduced model comparison** (a) Simulation using the full light matter Hamiltonian, and (b) Simulation using the reduced HH Hamiltonian. Both cases modeled a 1.8 ms evolution time. The first row plots the normalized variance averaged over the evolution time as a function of $\Phi$. The second row plots the occupation probability $P_{m=1}(t)$ of the initial state, as a function of both $\Phi$ and $t$. The third row shows the normalized variance, as a function of both $\Phi$ and $t$.

We then projected second term in Eq. (S1) to the lowest band subspace giving the synthetic dimension tunneling strength

$$J_s = \frac{\Omega R}{2} \int dx |w_0(x)|^2 e^{2ikRx} = 0.111(3)E_L.$$  \hspace{1cm} (S4)

where the integral defines a Lamb-Dicke suppression factor, equal to 0.75 for our lattice depth.

**A. Displacement property of HH Hamiltonian**

The Hamiltonian Eq. (S2) at longitudinal site $n$ is identical to the Hamiltonian at site $n = 0$ with the Peierls phase factor changed to $\phi' = \Phi n + \delta$. In terms of the displacement operator that shifts the longitudinal sites as $|n + \delta n\rangle = \hat{D}(\delta n)|n\rangle$, the displacement property of the Hamiltonian is

$$\hat{D}(\delta n)\hat{H}\hat{D}^\dagger(\delta n) = \hat{H}(\phi - \Phi \delta n)$$  \hspace{1cm} (S5)

This property was used in the main text to explain the similarity between the time evolution at rational transverse flux $\Phi$ and nearby irrational fluxes, as well as the suppression of uncertainty in the time evolution at irrational fluxes.

**B. Simulations**

All of the numerical simulations presented in the main manuscript were complete real space 1D discrete variable representation (DVR) simulations of the full light-matter Hamiltonian [S7]. Here we compare full-system simulations to those of the reduced Hofstadter model, and show: (1) their qualitative time-evolution is the same, but after long times their behavior differs quantitatively; and (2) the predicted peaks in normalized variance as a function of $\Phi$ are indistinguishable. We used a real space DVR method rather than momentum space band structure approach for two
reasons: (1) band structure simulations are not possible for irrational flux, since there is no periodic potential; and (2) our observed noise signature depended critically on the spatial extent of our system.

Our simulations of non-interacting atoms included a harmonic potential $V_{\text{ext}}$ that served to define the spatial extent of the ground state wavefunction

$$\psi_{\text{HO}}(x) \propto \exp \left[ -\frac{1}{2} \left( \frac{x}{\ell_{\text{HO}}} \right)^2 \right],$$

with harmonic oscillator length $\ell_{\text{HO}}$. Because the squared form factor $|f(q)|^2$ governs the coherence peak width, we selected the frequency of the harmonic potential frequency so that the RMS width of $|f(q)|^2$ from the numerical simulation was equal to that resulting from $n(x) \propto \left( 1 - (x/\ell_{\text{TF}})^2 \right)^2$, the 1D profile resulting from integrating a 3D Thomas-Fermi profile along both transverse directions. This yields the harmonic oscillator length $\ell_{\text{HO}} = 20\sqrt{2\pi}R_{\text{TF}}/77 \approx 0.651R_{\text{TF}}$ giving a spatial density profile with RMS width $w = \ell_{\text{HO}}/\sqrt{2} \approx 0.460R_{\text{TF}}$. The associated widths in momentum space are $\kappa_{\text{HO}} = 1/\ell_{\text{HO}}$ with a momentum-density RMS width $w_q = 1/(2\kappa_{\text{HO}})$. With the definition of the potential, every term in both the light matter Hamiltonian and the HH Hamiltonian are fully defined.

We validated our calibrations by performing least squares fits to the data in the middle panel of Fig. 1(b), including $P_0(t)$ and $P_2(t)$, for a ring-coupling geometry at $\Delta \Phi = -1/141$, as well as the data in Fig. 3(d) for the ribbon geometry. The resulting best fit coefficients $(\Omega_{R,0}, \Omega_{R,1}, \Omega_{R,2}) = (0.94, 0.98, 0.98)\Omega_{R}$ and $(\delta_0, \delta_1, \delta_2) = (0, 0.02)E_L$ are consistent with the uncertainties of our independent calibrations. All of our simulations include a single phenomenological fitting parameter $\tau = 1.5(1)$ ms, obtained only from the fit to the ribbon geometry data, to capture the slow decay of the observed coherent evolution.

We conclude with a side-by-side comparison of the dynamics of these two descriptions, as depicted in Fig. S1, with the full light matter simulation in (a) and the HH description in (b). Firstly, the time-averaged noise variance, plotted in the first row as a function of $\Phi$, is indistinguishable confirming that our key observation is a property both of our true physical system as well as the reduced HH model. The second and third rows display the time evolution of the initial state probability $P_{m=1}(t)$ and the noise variance, respectively. These data show that the exact time evolution of these simulations share the same dynamical time scale and qualitative features, but differ markedly in the quantitative evolution. Still in both cases, the probability $P_{m=1}(t)$ shows no feature associated with the peak in the noise variance.

To quantitatively understand the variance of the time evolution in our experiments, we reduce the 2D HH model to 1D AA model. We begin by Fourier transforming the real dimension

$$|m,n\rangle = \int_{-1/2}^{1/2} dq \, e^{-i2\pi qn} |m,q\rangle,$$

(S6)

to realize a series of AA lattices in momentum space:

$$\hat{H}_{\text{AA}}(q_i) = -2J_x \sum_j \cos[2\pi(q_i + j\Phi)] \langle j; q_i | j; q_i \rangle - J_x \sum_j e^{i\phi} \sum_{|j+1|; q_i} \langle j; q_i | j; q_i \rangle + \text{h.c.}$$

(S7)

In the main text, we approximated irrational transverse fluxes by nearby rational numbers within the finite resolution of the system. Here, we instead approximate rational numbers by near-irrational numbers. In this case, each AA Hamiltonian $\hat{H}_{\text{AA}}(q_i)$ describes an infinite chain, and the total Hamiltonian $\hat{H}$ sums over an infinite set of such AA chains, where the crystal momentum $q_i$ is written in units of two-photon recoil momenta $2\hbar k_L$. Here, both the pseudo-spin and crystal momentum are labeled periodically, namely $|m + M\rangle = |m\rangle$ and $|q + 1\rangle = |q\rangle$. Each chain contains a countably infinite set of sites labeled by $|j; q_i\rangle \equiv |m, q\rangle = |m_i + j, q_i + j\Phi\rangle$, where $j \in \mathbb{Z}$. In what follows, we take $m_i = 0$ without any loss of generality.

Figure S2 shows an example of an AA chain with $\Phi = P_2/Q_2 + \Delta \Phi$, and $\Delta \Phi = 1/84 + (\sqrt{5} - 1)/15000$, close to the simple rational fraction $P_2/Q_2 = 2/3$. This chain is nearly indistinguishable from the corresponding AA ring, with $N_{\text{AA}} = \text{LCM}(M, Q) = 84$ sites, where the sites around $j = 84$ are near-replicas of those around $j = 0$. Thus encircling a rational-flux-ring of size $N_{\text{AA}}$ corresponds to traveling between sites of an irrational-flux-chain spaced by $N_{\text{AA}}$ sites. We note that even AA rings can contain these near-replicas and they are spaced by $N_{\text{rep}} = M/|\text{LCM}(M, Q)|$ sites.

Provided $|\Delta \Phi| \ll \text{LCM}(M, Q)|\Delta \Phi| \lesssim w_q$. At long times that are not probed in our experiments, we expect to see the growth of the variance as the atoms travel to the nearest replica, for the $\phi$-sensitivity suppressed cases at short time scales.
FIG. S2. Irrational flux AA chains An AA chain with \( q = 0, \Phi = P_I/Q_r + \Delta \Phi \) where \( \Delta \Phi = 1/84 + (\sqrt{5} - 1)/15000 \) and \( P_I/Q_r = 2/3 \). The grey bars show the unraveling of a Gaussian wavepacket with width \( w_q = 1/(2\pi w) \), where \( w = 23 \).

C. Mean time evolution

The initial state is a wavepacket of width \( w_q \) in momentum space with initial pseudo-spin state \( |m_i = 0\rangle \):

\[
|\tilde{\psi}\rangle = \int dq_0 \tilde{\psi}_0(q_0) |0, q_0\rangle
\]

which is unraveled in the AA lattices as

\[
|\tilde{\psi}\rangle = \int dq \tilde{D}(q_0) \sum_j \tilde{\psi}_{0,M,j}(q_0) |M,j; q_0\rangle,
\]

where \( \tilde{\psi}_{0,j}(q_0) = \tilde{\psi}_0(q_0 + j\Phi) \) denotes the wavefunction in the chain containing site \( |j = 0; q_0\rangle = |0, q_0\rangle \). The sum over \( j \) includes all sites within a chain. \( \tilde{D}(q_0) \) is an everywhere discontinuous function and takes the value either 1 or 0, reminiscent of the Dirichlet function, such that the integral over \( q_0 \) (summing over infinite chains) includes all the momenta within \( [0,1) \) without repeating the momenta already connected within a chain.

At time \( t \), an initial site \( |j; q_0\rangle \) evolves to

\[
|j(t); q_0\rangle = \sum_{j'} \alpha_{j,j'}(t; q_0) e^{i\phi(j'-j)} |j'; q_0\rangle
\]

where \( \alpha_{j,j'}(t; q_0) = \langle j'; q_0 | \tilde{U}_{AA}(t; q_0) | j; q_0 \rangle \) describes the time evolution from site \( j \) to \( j' \) with \( \phi = 0 \). We can therefore interpret \( \alpha_{j,j'}(t; q_0) \) as the wavefunction at time \( t \) for a particle starting in \( |j'; q_0\rangle \) expressed in the \( |j; q_0\rangle \) basis, with \( \phi = 0 \).

When \( \phi \neq 0 \), the time evolution only contributes a phase factor \( e^{i\phi(j'-j)} \), as can be seen from the expansion of the evolution operator:

\[
\tilde{U}_{AA} = e^{-i\hat{H}_{AA}t} = I - \frac{it\hat{H}_{AA}}{\hbar} + \frac{1}{2!} \left( -\frac{it\hat{H}_{AA}}{\hbar} \right)^2 + \cdots
\]

The initial state has pseudo-spin \( M j_1, j_1 \in \mathbb{Z} \). Consequently, the total wavefunction of chain \( q_1 \) at time \( t \) evolves to

\[
|\tilde{\psi}(t); q_1\rangle = \sum_{j_1,j_2} \tilde{\psi}_{0,M,j_1}(q_1) \alpha_{M,j_1,j_2}(t; q_1) e^{i\phi(j_2-Mj_1)} |j_2; q_1\rangle
\]

with the probability to arrive in the final site \( |j; q_0\rangle \) equal to

\[
\tilde{P}_j(t; q_0) = \left| \langle j; q_0 | \tilde{\psi}(t); q_1 \rangle \right|^2 = \sum_{j_1,j_2} \tilde{\psi}_{0,M,j_1}(q_1) \tilde{\psi}_{0,M,j_2}(q_1) \alpha_{M,j_1,j_2}^*(t; q_1) \alpha_{M,j_2,j}(t; q_1) e^{i\phi(Mj_2-Mj_1)}
\]

Inserting \( \langle \exp[i\phi M(j_1-j_2)] \rangle_{\phi} = \delta_{j_1,j_2} \) into the above equation, leads to the averaged probabilities:

\[
\langle \tilde{P}_j(t; q_0) \rangle_{\phi} = \sum_{j_1} \left| \tilde{\psi}_{0,M,j_1}(q_1) \right|^2 |\alpha_{M,j_1,j}(t; q_1)|^2
\]
This is an incoherent sum over the probabilities of the atoms starting from each initial site weighted by the probability of traveling from that site to the final site.

Assuming the initial wavepacket is very narrow, the Hamiltonian in the vicinity of an initially occupied site $|j_0; q_i\rangle$ is close to that at $|j_0 = 0; q_i\rangle$ and independent of $q_i$. This allows us to replace the evolution from site $j_0$ to site $j$ with that from site 0 to $j - j_0$, i.e., $\alpha_{j_0,j}(t; q_i) \rightarrow \alpha_{0,j-j_0}(t)$, leading to probability depending only on the distance traveled.

Our experiments measured the probability to arrive at the final state with pseudo-spin $m_i + M\ell, \ell \in \mathbb{Z}$:

$$\langle P_{m_i}(t) \rangle_\phi = \int dq_i D(q_i) \sum_\ell \langle \tilde{P}_{m_i + M\ell}(t; q_i) \rangle_\phi$$

\begin{align*}
&\approx \int dq_i D(q_i) \sum_{\ell_1, \ell_2} |\tilde{\psi}_{0,0,M\ell_1}(q_i)|^2 |\alpha_{0,m_i + M(\ell_2 - \ell_1)}(t)|^2 \quad \text{We used } \alpha_{j_0,j}(t; q_i) \rightarrow \alpha_{0,j-j_0}(t). \\
&= \int dq_i D(q_i) \sum_{\ell_1} |\tilde{\psi}_{0,0,M\ell}(q_i)|^2 \sum_{\ell_2} |\alpha_{0,m_i + M\ell_2}(t)|^2 \quad \text{We changed the variable as } \ell_2 - \ell_1 \rightarrow \ell_2.
\end{align*}

$$= \sum_\ell |\alpha_{0,m_i + M\ell}(t)|^2 \quad (\tilde{\psi} | \tilde{\phi} \rangle = \int dq_i D(q_i) \sum_{\ell_1} |\tilde{\psi}_{0,0,M\ell_1}(q_i)|^2 = 1)$$

See Eq. [S9]

This equation depends only on the evolution of a particle starting at AA lattice site $|j = 0\rangle$. We consider a small range of transverse fluxes in the vicinity of a rational fraction $\Phi_r$ of the flux quantum. Within the range of propagation, the small deviation of the transverse flux is not resolved, i.e., $\alpha_{0,j-j_0}(t) \approx \alpha_{0,j-j_0}(t)$. We hereby conclude that the mean time evolution in the vicinity of $\Phi_r$ is a smooth function of $\Phi_r$.

### D. Variance of the time evolution

The variance of the time evolution for uniformly sampled Peierls phase $\phi$ is defined as

$$\text{var}_{m_i}(t) \equiv \langle P_{m_i}(t)^2 \rangle_\phi - \langle P_{m_i}(t) \rangle_\phi^2$$

We calculate $\text{var}_{m_i}(t)$ beginning

$$\langle P_{m_i}(t)^2 \rangle_\phi = \int dq_1 D(q_1) \int dq_2 D(q_2) \sum_{\ell_1, \ell_2} \langle \tilde{P}_{m_i + M\ell_1}(t; q_1) \tilde{P}_{m_i + M\ell_2}(t; q_2) \rangle_\phi$$

We will insert Eq. [S11]

$$\approx \int dq_1 D(q_1) \int dq_2 D(q_2) \sum_{j_1, j_1'} \sum_{j_2, j_2'} \left< e^{iM\phi(j_1+j_2-j_1'-j_2')} \right> \left< \tilde{\psi}^{AA*}_{0,j_1'}(q_1) \tilde{\psi}^{AA}_{0,j_1}(q_1) \tilde{\psi}^{AA}_{0,j_2}(q_2) \tilde{\psi}^{AA}_{0,j_2'}(q_2) \right>$$

$$\times \left< \alpha^{*}_{0,m_i + M(\ell_1-j_1)}(t) \alpha_{0,m_i + M(\ell_1-j_1)}(t) \alpha^{*}_{0,m_i + M(\ell_2-j_2)}(t) \alpha_{0,m_i + M(\ell_2-j_2)}(t) \right>$$

We will change the variables as $j_1 \rightarrow j_1', j_2 \rightarrow j_2, i = 1, 2$

$$= \int dq_1 D(q_1) \int dq_2 D(q_2) \sum_{j_1, j_1'} \sum_{j_2, j_2'} \left< e^{iM\phi(j_1+j_2)} \right> \left< \tilde{\psi}^{AA*}_{0,j_1'+j_1}(q_1) \tilde{\psi}^{AA}_{0,j_1}(q_1) \tilde{\psi}^{AA}_{0,j_2}(q_2) \tilde{\psi}^{AA}_{0,j_2'}(q_2) \right>$$

$$\times \left< \alpha^{*}_{0,m_i + M(\ell_1-j_1)}(t) \alpha_{0,m_i + M(\ell_1-j_1)}(t) \alpha^{*}_{0,m_i + M(\ell_2-j_2)}(t) \alpha_{0,m_i + M(\ell_2-j_2)}(t) \right>$$

We will use $\left< e^{iM\phi(j_1+j_2)} \right> = \delta_{j_1,-j_2}$

$$= \sum_j \left< \int dq_1 D(q_1) \int dq_2 D(q_2) \sum_{j_1, j_1'} \tilde{\psi}^{AA}_{0,j_1'+j_1}(q_1) \tilde{\psi}^{AA}_{0,j_1}(q_1) \tilde{\psi}^{AA}_{0,j_2}(q_2) \tilde{\psi}^{AA}_{0,j_2'}(q_2) \right>$$

$$\times \left< \alpha^{*}_{0,m_i + M(\ell_1-j_1)}(t) \alpha_{0,m_i + M(\ell_1-j_1)}(t) \alpha^{*}_{0,m_i + M(\ell_2-j_2)}(t) \alpha_{0,m_i + M(\ell_2-j_2)}(t) \right>$$
The expression in the first square bracket of the last step in the above equation is

$$[\cdot]$$

As noted below Eq. (S9), the wavefunction was unraveled to the AA chain according to the momentum distribution.

$$\int dq_1 D(q_1) \int dq_2 D(q_2) \sum_{j_1,j_2} \tilde{\psi}_0^* [q_1 + M(j + j_1^\ell) \Phi] \tilde{\psi}_0 [q_1 + Mj_1' \Phi] \tilde{\psi}_0^* [q_2 + M(-j + j_2^\ell) \Phi] \tilde{\psi}_0 [q_2 + Mj_2' \Phi]$$

We will reverse the unraveling, i.e. \( \int dq_i D(q_i) \sum_{j_i} \to \int dq_i, i = 1, 2 \)

$$\int dq_1 dq_2 \tilde{\psi}_0^* (q_1 + M \Phi j) \tilde{\psi}_0(q_1) \tilde{\psi}_0^* (q_2 - M \Phi j) \tilde{\psi}_0(q_2) = \left[ \int dq_1 \tilde{\psi}_0^* (q_1 + M \Phi j) \tilde{\psi}_0(q_1) \right]^2 = |f(M \Phi j)|^2 \equiv S(M \Phi j),$$

(S18)

where \( f \) is the atomic form factor, and \( S \) is the static structure factor \( S_0. \) From the normalization of the wavefunction, we have \( f(0) = 1 \) and \( S(0) = 1. \)

Combining Eqs. (S17, S18), we arrive at

$$\langle P_{q_i}^2(t) \rangle_{\phi} \approx \sum_j |f(M \Phi j)|^2 \left| \sum_\ell \alpha_0, m_i + M(\ell - j) (t) \alpha_0, m_i + M \ell t (t) \right|^2.$$  \hspace{1cm} (S19)

Importantly, if we also perform a time average we de-phase the cross-terms in the double sum

$$\langle \cdots \rangle = \sum_{\ell_1, \ell_2} \left( \alpha_{0, m_i + M(\ell_1 - j) (t)} \alpha_{0, m_i + M \ell_1 t (t)} \alpha_{0, m_i + M(\ell_2 - j) (t)} \alpha_{0, m_i + M \ell_2 t (t)} \right) \delta_{\ell_1, \ell_2}$$

(S20)

$$= \delta_{j,0} \left( \left[ \sum_\ell \left| \alpha_{0, m_i + M \ell t} \right|^2 \right] \langle \rho \rangle_\ell + (1 - \delta_{j,0}) \sum_\ell \langle |\alpha_{0, m_i + M(\ell - j) (t)}|^2 \alpha_{0, m_i + M \ell t} \rangle_\ell \right)$$

(S21)

The Kronecker delta functions locate those cases that match conjugated terms with non-conjugated terms, and the last term avoids a double counting error.

We now shift the first term in the last line to the LHS to get the time-averaged variance

$$\langle \text{var}_{m_i} \rangle_t = \sum_{j \neq 0} |f(M \Phi j)|^2 \sum_\ell \langle |\alpha_{0, m_i + M(\ell - j) (t)}|^2 \alpha_{0, m_i + M \ell t} \rangle_\ell$$

$$= \sum_{j \neq 0} |f(M \Phi j)|^2 \sum_\ell \langle |\alpha_{0, i - M j} (t)|^2 \alpha_{0, i} (t) \rangle_\ell,$$

(S23)

having used \( f(0) = 1. \) In the second line, we defined the total variance by summing over final states.

Due to the narrow wavepacket assumption, the sites that could possibly contribute to the form factor \( f \) are separated by \( \text{LCM}(M, Q_i), \) rather than \( M \) (Fig. S3), where the nearby rational transverse flux \( \Phi_i = P_i/Q_i \) with \( P_i \) and \( Q_i \) being co-prime. In the main text and the experiments, \( P_i/Q_i = 2/3, \) while here we derive a general case. Therefore, \( f(M \Phi j)^2 = |f[M \Phi j]|^2 \delta_{Mj, \text{LCM}(M, Q_i)} \)

$$= \left| f \left[ \text{LCM}(M, Q_i) \Delta \Phi j + \text{LCM}(M, Q_i) \frac{P_i}{Q_i} j \right] \right|^2 \delta_{Mj, \text{LCM}(M, Q_i)}$$

(S24)

Only the non-integer part of the variable matters because the crystal momentum \( q \) is periodically labeled. If \( \text{LCM}(M, Q_i)|\Delta \Phi| \gg w_q, \) then the form factor and hence the variance approach zero for \( j \neq 0. \) When \( \text{LCM}(M, Q_i)|\Delta \Phi| \lesssim w_q, \) the closest distance between initially occupied sites on the chain is \( \text{LCM}(M, Q_i) \) (Fig. S3). This leads to the final form of the total variance

$$\langle \text{var} \rangle_t = \sum_{j \neq 0} |f[\text{LCM}(M, Q_i) \Delta \Phi j]|^2 \sum_\ell \langle |\alpha_{0, i - \text{LCM}(M, Q_i) j} (t)|^2 \alpha_{0, i} (t) \rangle_\ell,$$

(S25)
FIG. S3. Irrational flux AA chains with \( M \neq \text{LCM}(M, Q_r) \). An AA chain with \( q_i = 0, \Phi = P_r/Q_r + \Delta \Phi \) where \( \Delta \Phi = 1/615 + (\sqrt{5} - 1)/500000 \) and \( P_r/Q_r = 1/2 \). The grey bars show the unraveling of a Gaussian wavepacket with width \( w_q = 1/(2\pi w) \), where \( w = 23 \). These parameters were not probed in our experiments.

We now consider some limiting behavior:

1. for \( \text{LCM}(M, Q_r)|\Delta \Phi| \gg w_q \), then only \( f(0) \) is practically nonzero and we get
   \[
   \langle \text{var} \rangle_t = 0 \tag{S26}
   \]

2. for \( \text{LCM}(M, Q_r)|\Delta \Phi| \approx w_q \), we take only the first terms in the \( j \)-sum
   \[
   \langle \text{var} \rangle_t = 2|f| \text{LCM}(M, Q_r)|\Delta \Phi|^2 \sum_i \langle |\alpha_{0,i-LCM(M,Q_r)}(t)|^2 |\alpha_{0,i}(t)|^2 \rangle_t \tag{S27}
   \]
   we now assume that for long time \( |\alpha_{0,i-LCM(M,Q_r)}(t)|^2 \approx |\alpha_{0,i}(t)|^2 \) to give
   \[
   \langle \text{var} \rangle_t = 2|f| \text{LCM}(M, Q_r)|\Delta \Phi|^2 \left[ \sum_i \langle |\alpha_{0,i}(t)|^4 \rangle_t \right] . \tag{S28}
   \]

3. We extend (2) by assuming from the beginning that enough time has passed so the wavepackets have spread over a range wide compared to the initial distribution, giving
   \[
   \langle \text{var} \rangle_t = \sum_{j \neq 0} |f| \text{LCM}(M, Q_r)|\Delta \Phi|^2 \sum_i \langle |\alpha_{0,i-LCM(M,Q_r)}(t)|^2 |\alpha_{0,i}(t)|^2 \rangle_t \\
   \approx \sum_{j \neq 0} |f| \text{LCM}(M, Q_r)|\Delta \Phi|^2 \left[ \sum_i \langle |\alpha_{0,i}(t)|^4 \rangle_t \right] \tag{S29}
   \]
   where we made the far-field replacement \( |\alpha_{0,i-LCM(M,Q_r)}|^2 \approx |\alpha_{0,i}|^2 \). Note that \( \sum_i \langle |\alpha_{0,i}(t)|^4 \rangle_t \) is the inverse participation ratio in the AA lattice.

4. for \( \Delta \Phi \to 0 \), we get
   \[
   \langle \text{var} \rangle_t = \sum_{j,i} \langle |\alpha_{0,i-LCM(M,Q_r)}(t)|^2 |\alpha_{0,i}(t)|^2 \rangle_t - \sum_i \langle |\alpha_{0,i}(t)|^4 \rangle_t \tag{S30}
   \]
   \[
   \approx \frac{1}{\text{LCM}(M, Q_r)} - \sum_i \langle |\alpha_{0,i}(t)|^4 \rangle_t \rightarrow \frac{1}{\text{LCM}(M, Q_r)} \tag{S31}
   \]
   in the second line we assumed that the wavefunction is uniform on the scale of \( \text{LCM}(M, Q_r) \) sites (a poor assumption in the insulating phase, and even in our data there is a \( \approx 2 \times \) difference in typical averaged populations) and broke
the first term into LCM\((M, Q_t)\) individual terms \(|\alpha_{0,i-LCM(M,Q_t)}m(t)|^2|\alpha_{0,t}|^2\) with \(m\) from 0 to LCM\((M, Q_t) - 1\). With that we had a new sum over \(m\), that we used to complete the \(j\) sum to be on every lattice site. At which point we used the normalization condition to do this double sum, and the \(\rightarrow\) is assuming a spreading wavepacket for which the \(\sum_i\) will fall to zero.

### S3. AA LATTICES IN REAL SPACE

In accordance with our TOF data, we explained our observations in a momentum space picture. In this section, we will derive an analogous description in real space that gives the dual AA model, where the noise-suppression for irrational transverse flux results from spatial self-averaging. Inserting the Fourier expansion

\[
|m, n\rangle = \frac{1}{\sqrt{M}} \sum_{q_m} e^{-i2\pi q_m m} |q_m, n\rangle
\]  

(S32)

where \(q_m \in \{0, \frac{1}{M}, \ldots, \frac{M-1}{M}\}\), into Hamiltonian Eq. (S2), leads to

\[
\hat{H}_{AA}(\bar{\Phi} - q_m) = -J_x \sum_n (|q_m, n + 1\rangle \langle q_m, n| + \text{h.c.}) - 2J_s \sum_n \cos[2\pi(\Phi n + \bar{\Phi} - q_m)] |q_m, n\rangle \langle q_m, n| 
\]

(S33)

where \(\hat{H} = \sum q_m \hat{H}_{AA}(\bar{\Phi} - q_m)\). The original HH model is transformed into \(M\) decoupled AA chains in real space (Fig. S4). The quasi-momentum \(q_m\) shifts the phase of the corresponding chain.

**FIG. S4.** An example of the HH model Fourier transformed into decoupled AA chains, with \(M = 3\), \(\Delta \Phi = \Phi - 2/3 = 1/147\) and \(\bar{\Phi} = 0\).

#### A. Mean time evolution

The initial state, with pseudo-spin \(m_i = 0\), is a superposition of \(q_m\) state and a wavepacket of width \(w\) in real space:

\[
|\psi\rangle = \left[ \sum_n \psi_0(n) |n\rangle \right] \otimes \frac{1}{\sqrt{M}} \sum_{q_m} |q_m\rangle
\]

(S34)

The atoms evolve along all \(M\) chains, and the wavefunction at time \(t\) is

\[
|\psi(t)\rangle = \frac{1}{\sqrt{M}} \sum_{q_m} \hat{U}_{AA}(\bar{\Phi} - q_m) \sum_n \psi_0(n) |q_m, n\rangle
\]

(S35)
where $\hat{U}_{AA}(\phi)$ is the evolution operator governed by the Hamiltonian $\hat{H}_{AA}(\phi)$. The probability of the final state to have pseudo-spin $m_t$ at time $t$ is

$$P_{m_t}(t) = \sum_n \langle \langle m_t, n | \psi(t) \rangle \rangle^2$$

We will insert Eqs. (S32) and (S35). Also note $\hat{U}_{AA}$ does not change $|q_m\rangle$ and $\sum_n |n\rangle \langle n| = 1$

$$= \frac{1}{M^2} \sum_{q_m, q_m'} e^{i2\pi m_t(q_m - q_m')} \psi_0^* (n') \psi_0(n) \langle n' | \hat{U}_{AA}^\dagger (\phi - q_m') \hat{U}_{AA}(\phi - q_m) |n\rangle$$

The spatial AA model and therefore its evolution operator have the same displacement property [Eq. (S30)] as the HH model. We will shift site $n$ to 0.

$$= \frac{1}{M^2} \sum_{n, \delta n} \psi_0^*(n + \delta n) \psi_0(n) \sum_{q_m, q_m'} e^{i2\pi m_t(q_m - q_m')} \langle \delta n | \hat{U}_{AA}^\dagger (\phi + \Phi n - q_m') \hat{U}_{AA}(\phi + \Phi n - q_m) |0\rangle$$

The mean time evolution is

$$\langle P_{m_t}(t) \rangle \phi$$

$$= \frac{1}{M} \sum_{\delta n} \sum_{\delta q_m} e^{-i2\pi m_t \delta q_m} \left\langle \langle \delta n | \hat{U}_{AA}^\dagger (\phi) \hat{U}_{AA}(\phi + \delta q_m) |0\rangle \right\rangle \phi \left( \sum_{n, \delta n} \psi_0^*(n + \delta n) \psi_0(n) \right)$$

The quantity in the bracket describes the interference of atoms initially located in sites $n = 0$ and $\delta n$. The small wavefunction spread approximation echoes the narrow momentum distribution approximation made in the momentum-space picture. Next, we will quantitatively analyze how the deviation of the transverse flux $\Delta \Phi = \Phi - \Phi_0$ affects the mean time evolution. If we denote the farthest site from $n = 0$ that has nonzero contribution to Eq. (S37) as $\delta n_{\text{max}}$, then we require the atoms not to travel too far to resolve the difference in the transverse flux yet $\Delta \Phi \delta n_{\text{max}} \ll 1$. The cosine term that modulates the on-site energy in the AA Hamiltonian $\hat{H}_{AA}(\phi)$ [Eq. (S33)] can then be approximated as

$$\cos \left( 2\pi \left[ (\Phi_t + \Delta \Phi) n + \phi \right] \right) = \cos(2\pi \Delta \Phi n) \cos[2\pi(\Phi_t n + \phi)] - \sin(2\pi \Delta \Phi n) \sin[2\pi(\Phi_t n + \phi)]$$

$$\approx \cos[2\pi(\Phi_t n + \phi)] - 2\pi \Delta \Phi n \sin[2\pi(\Phi_t n + \phi)]$$

(S38)

This allows us to separate the contribution of $\Phi_t$ to the Hamiltonian from $\hat{H}_{AA}(\Phi_t, \phi)$ (writing out the transverse phase dependence for clarity). Given $e^{(A + B)} = e^A e^B + O([B, A])$, the evolution operator can be approximated as

$$\hat{U}_{AA}^{(1)}(\Phi, \phi) \approx \hat{U}_{AA}(\Phi_t, \phi) e^{-2\pi i 2J_1 \Delta \Phi \phi n/[2\pi(\Phi_t n + \phi)] \times t/\hbar}$$

(S39)

Plugging this into Eq. (S37), we have

$$\langle P_{m_t}(t) \rangle_\phi \approx \frac{1}{M} \left( \sum_{\delta q_n} e^{-i2\pi m_t \delta q_m} \sum_{\delta n} \left\langle \left\langle \delta n | \hat{U}_{AA}^\dagger(\Phi_t, \phi) \hat{U}_{AA}(\Phi_t, \phi + \delta q_m) |0\right\rangle \right\rangle \phi \right)$$

The approximation in Eq. (S39) requires $2J_s t/\hbar \ll 1$ in order for the higher order terms to be small. Together with the earlier approximation $\Delta \Phi \delta n_{\text{max}} \ll 1$, which validates Eq. (S38), the phase factor within the bracket, linear with both $t$ and $\Delta \Phi$ in Eq. (S40), remains small and smooth.

To summarize, we obtained the mean time evolution by shifting the sites to the vicinity of $n = 0$. This is possible because the dynamics on different sites of the AA chains is equivalent to that on the same site under different $\phi$. The global transverse phase dependence ($\Delta \Phi n$) is averaged out by uniformly sampling $\phi$. Consequently, the system can only resolve the transverse phase within the range of the propagation ($\Delta \Phi \delta n_{\text{max}}$), leading to similar mean time evolution observed across the full range of experimentally probed transverse fluxes.
B. Variance of the time evolution

Without averaging over $\bar{\phi}$, when shifting from site $n$ to 0, $\phi$ is changed to $\Phi n + \bar{\phi}$. At irrational fluxes $\Phi$, for a large system, summing over $n$ randomly samples the phases, equivalent to averaging over $\bar{\phi}$. Therefore, each individual time evolution is expected to evolve as the averaged evolution—a spatial self-averaging effect. On the other hand, at rational fluxes, averaging over $n$ only allows sampling $Q_r$ different phases in each chain and a total of $LCM(M, Q_r)$ phases including all chains. For example, at $P_r/Q_r = 2/3$ and $\bar{\phi} = 0$, only three phases $0, 2\pi/3$ and $4\pi/3$ are sampled, inequivalent to averaging over $\bar{\phi}$. Thus, each time evolution is expected to have its own unique trajectory.

To calculate the variance of the time evolution, we calculate the two terms $\langle P_m(t) \rangle_\phi$ and $\langle P_m(t)^2 \rangle_\phi$ separately. The former follows straightforwardly from Eq. (S37):

$$\langle P_m(t) \rangle_\phi \approx \frac{1}{M^2} \sum_{\delta q_{m,1}, \delta q_{m,2}} e^{-i2\pi m_1(\delta q_{m,1} + \delta q_{m,2})} \times \sum_{\delta n_1, \delta n_2} \left\langle \langle \delta n_1 | \tilde{U}_{AA}(\bar{\phi}) \tilde{U}_{AA}(\bar{\phi} + \delta q_{m,1}) | 0 \rangle \langle \delta n_2 | \tilde{U}_{AA}^*(\bar{\phi}) \tilde{U}_{AA}^*(\bar{\phi} + \delta q_{m,2}) | 0 \rangle \right\rangle_\phi$$

(S41)

For the latter, we have

$$\langle P_m(t)^2 \rangle_\phi = \frac{1}{M^2} \sum_{n_1, \delta n_1, n_2, \delta n_2} \psi_0^*(n_1 + \delta n_1) \psi_0(n_1) \psi_0^*(n_2 + \delta n_2) \psi_0(n_2) \times \langle \langle \delta n_1 | \tilde{U}_{AA}^*(\bar{\phi} + \Phi n_1 - q_{m,1}') \tilde{U}_{AA}(\bar{\phi} + q_{m,1} - q_{m,2} - q_{m,1}') | 0 \rangle \langle \delta n_2 | \tilde{U}_{AA}^*(\bar{\phi} + \Phi n_2 - q_{m,2}') \tilde{U}_{AA}(\bar{\phi} + q_{m,2} = 0) | 0 \rangle \rangle_\phi$$

(S42)

from Eq. (S36), where we performed an analogous procedure as in Eq. (S37). Under the assumption of small spread of the wavepacket, we made the following approximation:

$$\sum_n \psi_0^*(n + \delta n_1) \psi_0(n) \psi_0^*(n + \Delta n + \delta n_2) \psi_0(n + \Delta n) \approx \sum_n |\psi_0(n)|^2 |\psi_0(n + \Delta n)|^2 \equiv \tilde{S}(\Delta n)$$

(S43)

where $\tilde{S}(\Delta n)$ is the autocorrelation function, which is the Fourier transform of the static structure factor introduced in Eq. (S18). This can be explicitly seen as follows:

$$\int dq \psi^*(q + \delta q) \psi(q) = \int dx_1 \int dx_2 \psi^*(x_1) \psi(x_2)e^{-i2\pi q x_1} \int \frac{dq}{2\pi} e^{-i2\pi q(x_1 - x_2)} = \int dx |\psi(x)|^2 e^{-i2\pi \delta q x}$$

(S44)

$$\mathcal{F}[S(\delta q)] = \int d\delta q S(\delta q)e^{-i2\pi \delta q \Delta n} = \int d\delta q \left| \int dx |\psi(x)|^2 e^{-i2\pi \delta q x} \right|^2 e^{-i2\pi \delta q \Delta n} = \int d\delta q \int dx_1 \int dx_2 |\psi(x_2)|^2 |\psi(x_1)|^2 e^{i2\delta q(x_2 - x_1)} e^{-i2\pi \delta q \Delta n} = \int dx_1 \int dx_2 |\psi(x_2)|^2 |\psi(x_1)|^2 \delta(x_2 - x_1 - \Delta n)$$

(S45)

As phases wrap around $2\pi$, only the non-integer part of the variable in $\tilde{U}_{AA}$ matters. To take out the integer part of the variable and highlight the role of $\Delta \Phi$, We will change the variable $\Delta n \rightarrow a Q_r + b$, where $a, b \in \mathbb{Z}$ and $b = 0, 1, \cdots, Q_r - 1$. Therefore, $\Phi \Delta n = P_r a + b P_r = \Phi a Q_r + \Delta \Phi b \approx a Q_r + \Phi a Q_r$.
term because it is small under our assumption of small $\Delta \Phi$. Ignoring the contribution of $b$ to $\tilde{S}$, as $Q_r \ll w$, leads to

$$\langle P_{m_1}(t)^2 \rangle_{\tilde{\phi}} \approx \frac{1}{M^2} \sum_{\delta q_m} \tilde{S}(aQ_r) e^{-i2\pi m_1(\delta q_m, \delta q_m, 2)/M} \int_{-\infty}^{\infty} d\Theta \tilde{S} \left( \frac{\Theta - b}{\text{LCM}(M, Q_r)} \right) / \Delta \Phi$$

We will swap the range of $b/\text{LCM}(M, Q_r)$ and $\Theta$ to remove some redundant integer contribution to the variable in $\tilde{U}_{AA}$, while the range of the variable in $\tilde{S}$ remains the same.

Let us consider the regime $1/\Delta \Phi \ll w$, where $\tilde{S}$ is almost independent of $\Theta$, and thereby $\Theta$ only showing up in the bracket. The relative phases in the two brackets average out by the integral over $\Theta$, allowing averaging over the Peierls phase to act on each bracket individually rather than their product. As a result, we have

$$\langle P_{m_1}(t)^2 \rangle_{\tilde{\phi}} \approx \frac{1}{M^2 \Delta \Phi \text{LCM}(M, Q_r)} \sum_{\delta q_m} \tilde{S}(aQ_r) e^{-i2\pi m_1(\delta q_m, \delta q_m, 2)/M} \int_{-\infty}^{\infty} d\Theta \tilde{S} \left( \frac{\Theta - b}{\text{LCM}(M, Q_r)} \right) / \Delta \Phi$$

where $\Delta \Phi_{M, Q_r} \equiv \Delta \Phi \text{LCM}(M, Q_r)$. The function $\sigma(x) \equiv \left[ \sum_n \tilde{S}(n/x) \right] / x - 1$ is defined so that $\text{var}_{m_1}(t)/\langle P_{m_1}(t)^2 \rangle_{\tilde{\phi}} = \sigma(\Delta \Phi_{M, Q_r})$. When $1/x \to 0$, $\sigma(x) \to \int dx' \tilde{S}(x') - 1 = 0$, which can be seen from the definition of $\tilde{S}$ [Eq. (S43)] and the normalization of the wavefunction. As a result, when $1/\Delta \Phi_{M, Q_r} \ll w$, the uncertainty of the time evolution becomes vanishingly small. This is the same condition as what we obtained in the momentum-space picture.
FIG. S5. Time evolution uncertainty plotted in (a) linear and (b) log scale for a Gaussian wavefunction. In both cases, the thick red curve comes from exactly evaluating Eq. (S51), while the black curve results from the small $w \Delta \Phi_{M,Q,r}$ expansion and the black curve results from the large $w \Delta \Phi_{M,Q,r}$ expansion. The vertical grey line marks $2 \pi w \Delta \Phi_{M,Q,r} = \pi/2$ were we expect the validity of these expansions to crossover. The pink lines mark the maximum possible noise for quantum states drawn completely at random: the top line at 2 is for $M \to \infty$, and the bottom line at 0.8 is for $M = 3$.

C. The system size

Assuming Gaussian wavepackets, we are able to derive the analytical expression of the relative variance $\sigma(\Delta \Phi_{M,Q,r})$.

$$\psi_0(n) = \frac{1}{\pi^{1/4} w^{1/2}} \exp \left[-\frac{1}{2} \left(\frac{n}{w}\right)^2\right],$$  

(S49)

we have

$$\tilde{S}(\Delta n) = \sum_n |\psi_0(n)|^2 |\psi_0(n + \Delta n)|^2 = \frac{1}{(2\pi)^{1/2} w} \exp \left[-\frac{\Delta n^2}{2w^2}\right]$$  

(S50)

Then,

$$\sigma(\Delta \Phi_{M,Q,r}) = \frac{1}{(2\pi)^{1/2} w \Delta \Phi_{M,Q,r}} \vartheta_3 \left[0, \frac{i}{2\pi \Delta \Phi_{M,Q,r}^2 w^2}\right] - 1$$  

(S51)

where $\vartheta_3(z, \tau) = \sum_{n=-\infty}^{\infty} q^{n^2} \eta^{n}$ is the third elliptic theta function, where $q = e^{i\pi \tau}$ and $\eta = e^{2\pi i z}$.

For $w \Delta \Phi_{M,Q,r} \gtrsim 1/4$, using one of the Jacobi identities $\vartheta_3(z/\tau, -1/\tau) = \alpha \vartheta_3(z, \tau)$, where $\alpha = \sqrt{-i\tau} \exp \left[\pi i z^2 / \tau\right]$, and keeping only the $n = 0, \pm 1$ terms, the function is very well described by

$$\sigma(\Delta \Phi_{M,Q,r}) \to 2 \exp \left[-\frac{1}{2} \left(2\pi \Delta \Phi_{M,Q,r} w\right)^2\right]$$  

(S52)

For $w \Delta \Phi_{M,Q,r} \lesssim 1/4$, keeping only the $n = 0$ term, the function is very well described by

$$\sigma(\Delta \Phi_{M,Q,r}) \to \frac{\sqrt{2\pi}}{2\pi \Delta \Phi_{M,Q,r} w} - 1$$  

(S53)

However, it is worth noting that this limit is inconsistent with the assumption made in Eq. (S48), which has led to the definition of $\sigma(\Delta \Phi_{M,Q,r})$.

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