Namaste Franklin, from the magic squares of Narayana Pandita

Maya Mohsin Ahmed
maya.ahmed@gmail.com

Abstract

Narayana Pandita constructed magic squares as a superimposition of two squares, folded together like palms in the Indian greeting, Namaste. In this article, we show how to construct Franklin squares of every order, as a superimposition of two squares. We also explore the myriad of similarities in construction and properties of Franklin and Narayana squares.

1 Introduction.

A magic square is a square matrix whose entries are non-negative integers, such that the sum of the numbers in every row, in every column, and in each diagonal is the same number called the magic sum. Narayana Pandita, in the thirteenth century, showed how to construct the magic squares in Table 1 ([6], [7]). The well-known squares F1, F2, and F3, that appear in Table 2, were constructed by Benjamin Franklin, in the eighteenth century ([1], [5], [8]). We start with looking at the mind blowing structure and properties of these squares.

Table 1: Narayana squares N1 and N2.

| 60 | 53 | 44 | 37 | 13 | 20 | 29 |
|----|----|----|----|----|----|----|
| 3 | 14 | 19 | 30 | 59 | 54 | 43 |
| 58 | 55 | 42 | 39 | 2 | 5 | 18 |
| 1 | 16 | 17 | 32 | 57 | 56 | 41 |
| 61 | 52 | 45 | 36 | 5 | 12 | 21 |
| 6 | 11 | 22 | 21 | 02 | 51 | 46 |
| 65 | 60 | 47 | 34 | 10 | 25 | 26 |
| 8 | 9 | 24 | 23 | 64 | 49 | 48 |
| 248 | 233 | 216 | 201 | 184 | 169 | 152 |
| 7 | 26 | 39 | 58 | 71 | 90 | 103 |
| 246 | 235 | 214 | 203 | 182 | 171 | 150 |
| 5 | 28 | 37 | 60 | 69 | 92 | 101 |
| 244 | 237 | 212 | 205 | 180 | 173 | 148 |
| 4 | 29 | 36 | 61 | 68 | 93 | 100 |
| 224 | 233 | 216 | 201 | 184 | 169 | 152 |
| 8 | 25 | 40 | 57 | 72 | 89 | 104 |
| 24 | 25 | 40 | 57 | 72 | 89 | 104 |
| 121 | 121 | 121 | 121 | 121 | 121 | 121 |

Table 2: Franklin squares F1, F2, and F3.

| 248 | 233 | 216 | 201 | 184 | 169 | 152 |
| 7 | 26 | 39 | 58 | 71 | 90 | 103 |
| 246 | 235 | 214 | 203 | 182 | 171 | 150 |
| 5 | 28 | 37 | 60 | 69 | 92 | 101 |
| 244 | 237 | 212 | 205 | 180 | 173 | 148 |
| 4 | 29 | 36 | 61 | 68 | 93 | 100 |
| 224 | 233 | 216 | 201 | 184 | 169 | 152 |
| 8 | 25 | 40 | 57 | 72 | 89 | 104 |
| 24 | 25 | 40 | 57 | 72 | 89 | 104 |
| 121 | 121 | 121 | 121 | 121 | 121 | 121 |
Table 2: Franklin squares.

|   | F1 |   | F2 |   |   |   |
|---|----|---|----|---|---|---|
| 52 | 61 | 4 | 13 | 20 | 29 | 36 | 45 |
| 14 | 3  | 62 | 51 | 46 | 40 | 45 | 38 |
| 53 | 60 | 5 | 12 | 21 | 28 | 37 | 44 |
| 11 | 6  | 59 | 43 | 58 | 27 | 22 | 38 |
| 55 | 58 | 7 | 10 | 23 | 26 | 39 | 42 |
| 9  | 8  | 57 | 56 | 41 | 40 | 25 | 24 |
| 50 | 63 | 2 | 15 | 18 | 31 | 34 | 47 |
| 16 | 1  | 64 | 49 | 48 | 32 | 32 | 17 |

|   | F3 |   |   |
|---|----|---|---|
| 200 | 217 | 232 | 249 |
| 198 | 219 | 230 | 248 |
| 60 | 37 | 28 | 5 |
| 201 | 216 | 233 | 248 |
| 55 | 42 | 23 | 10 |
| 203 | 214 | 235 | 246 |
| 53 | 44 | 21 | 12 |
| 205 | 212 | 237 | 241 |
| 51 | 46 | 19 | 13 |
| 207 | 210 | 239 | 242 |
| 49 | 48 | 17 | 16 |
| 196 | 221 | 228 | 253 |
| 92 | 38 | 30 | 2 |
| 194 | 224 | 228 | 255 |
| 64 | 33 | 32 | 1 |

By a **continuous property**, we mean that if we imagine the square as the surface of a torus (i.e., if we glue opposite sides of the square together), then the property can be translated without effect on the corresponding sums. See Figure 2 for examples. From now on, row sum, column sum, or bend diagonal sum, etc. mean that we are adding the entries of those elements. **Franklin squares** were defined in [3] as follows.

**Definition 1.1** (Franklin Square). Consider an integer, $n = 2^r$ such that $r \geq 3$. Let the magic sum be denoted by $M$ and $N = n^2 + 1$. We define an $n \times n$ Franklin square to be a $n \times n$ matrix with the following properties:

1. Every integer from the set $\{1, 2, \ldots, n^2\}$ occurs exactly once in the square. Consequently,
   \[M = \frac{n}{2}N.\]

2. All the the half rows, half columns add to one-half the magic sum. Consequently, all the rows and columns add to the magic sum.

3. All the bend diagonals add to the magic sum, continuously (see Figures 1 and 2).

4. All the $2 \times 2$ sub-squares add to $2N$, continuously.

Observe that Franklin squares are not traditional magic squares because the main diagonals do not add to the magic sum. In this article, we restrict our discussion to Narayana squares of order $2^r$, where $r \geq 3$. This is for comparison with Franklin squares. For more general Narayana squares, see [6] and [7].
Definition 1.2 (Narayana Square). Consider an integer, \( n = 2^r \) such that \( r \geq 3 \). Let the magic sum be denoted by \( M \) and \( N = n^2 + 1 \). We define an \( n \times n \) Narayana square to be a \( n \times n \) matrix with the following properties:

1. Every integer from the set \( \{1, 2, \ldots, n^2\} \) occurs exactly once in the square. Consequently,
   \[
   M = \frac{n}{2} N.
   \]
2. All the rows and columns add to the magic sum.
3. All the pandiagonals add to the magic sum (see Figure 2).<br>
4. All the \( 2 \times 2 \) sub-squares add to \( 2N \), continuously.

Thus, Narayana squares are magic squares with additional properties. Though the squares were constructed by two different people, across different centuries and continents, the similarities in properties and construction are remarkable, and we address some of the properties in the next proposition.

Proposition 1.1. Consider an integer, \( n = 2^r \) such that \( r \geq 3 \). Let the magic sum be denoted by \( M \), \( m = n^2/2 + 1 \), and \( N = n^2 + 1 \). The similarities and differences in the defining properties of Franklin and Narayana squares are given below.
Figure 3: Right and left Pandiagonals.

| Franklin Square | Narayana Square |
|-----------------|-----------------|
| \( M = \frac{n}{2}N \) | \( M = \frac{n}{2}N \) |
| **2** All the rows and columns add to \( M \) | All the rows and columns add to \( M \). |
| **3** All the \( 2 \times 2 \) sub-squares add to \( 2N \), continuously. | All the \( 2 \times 2 \) sub-squares add to \( 2N \), continuously. |
| **4** All the the half row sums add to \( M/2 \). | Half row sums add either to \( (n/4)m \) or \( M - (n/4)m \). |
| **5** Half column sums add to \( M/2 \). | Half column sums add to \( M/2 \pm n^2/8 \). |
| **6** All Bend diagonal sums add to \( M \). | Left and right bend diagonal sums add to \( M \pm n/2 \). Top and bottom bend diagonal sums add to \( M \pm n^2/2 \). |
| **7** Pandiagonal sums add to \( M \pm n^2/2 \). | All pandiagonals add to \( M \). |

Proof of Proposition 1.1 is covered in Section 2 and Section 3. Many other properties of these squares are also revealed in these sections.

We describe Narayana’s method of constructing magic squares. For constructing an \( n \times n \) Narayana square, we start with two sequences, namely,

\[ \text{Mulapankti sequence: } 1, 2, \ldots, n \]

and

\[ \text{Ganapankti sequence: } 0, n, 2n, 3n, \ldots, (n-1)n. \]

We construct arrays consisting of two rows, from these sequences, as shown below (also see Examples 1.1 and 1.2).

\[ \text{Mulapankti array} \]

\[
\begin{array}{cccc}
1 & 2 & \ldots & \frac{n}{2} - 1 & \frac{n}{2} \\
n & n - 1 & \ldots & \frac{n}{2} + 2 & \frac{n}{2} + 1
\end{array}
\]
Ganapankti array

\[
\begin{array}{cccccc}
0 & n & 2n & \ldots & n(\frac{n}{2} - 2) & n(\frac{n}{2} - 1) \\
n(n-1) & n(n-2) & n(n-3) & \ldots & n(\frac{n}{2} + 1) & n^2
\end{array}
\]

Two squares, a covering one called Chadaka, and a square to be covered called Chadya are formed as follows. We start with the last column of the Mulapankti array, working backwards. Each column of the Mulapankti array is written horizontally, repeatedly, to form the first \(n/2\) rows of the Chadya. The two entries in each column are inverted, and again written horizontally, to form the last \(n/2\) rows of the Chadya. The Chadaka is constructed using the Ganapankti array. This construction is similar to Chadya, except that, the columns of the Ganapankti array are written vertically, repeatedly, to form the first \(n/2\) columns of the Chadaka. Again, the entries of the Ganapankti columns are reversed to construct the last \(n/2\) columns. The Chadya and the Chadaka are then superimposed like the folding of palms in a Namaste to form the Narayana magic square. In other words, the Chadaka is flipped about a vertical edge and added to the Chadya to get the magic square. The square we get by flipping the Chadaka about a vertical edge is called flipped Chadaka from now onwards. Example 1.1 and Example 1.2 demonstrates the Narayana construction for the \(8 \times 8\) and \(16 \times 16\) Narayana square, respectively.

**Example 1.1.** Narayana’s method of constructing \(8 \times 8\) magic squares.

We demonstrate how to construct the Narayana square N1 in Table 1.

| Mulapankti sequence: 1, 2, 3, 4, 5, 6, 7, 8. |
|---------------------------------------------|
| Ganapankti sequence: 0, 8, 16, 24, 32, 48, 56. |

| Mulapankti array | Ganapankti array |
|------------------|------------------|
| 1 2 3 4          | 0 8 16 24        |
| 8 7 6 5          | 56 48 40 32      |

Each column of the Mulapankti array is written horizontally, repeatedly, to form the first 4 rows of the Chadya.

The entries of the Mulapankti array columns are flipped, and again written horizontally, repeatedly, to form the last four rows of the Chadya.
The columns of the Ganapankti array are written vertically, repeatedly, to form the first four columns of the Chadaka.

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8 \\
9 & 10 & 11 & 12 \\
13 & 14 & 15 & 16 \\
\end{array}
\]

The entries of the Ganapankti array columns are flipped to construct the last four columns.

Finally, the Chadya and Chadaka are superimposed, like the folding of palms in a Namaste to form the Narayana magic square N1. That is, the Chadaka is flipped along a vertical edge and added to the Chadya to get N1.

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8 \\
9 & 10 & 11 & 12 \\
13 & 14 & 15 & 16 \\
\end{array}
\]

\[
\begin{array}{cccc}
16 & 15 & 14 & 13 \\
12 & 11 & 10 & 9 \\
8 & 7 & 6 & 5 \\
4 & 3 & 2 & 1 \\
\end{array}
\]

\[
\begin{array}{cccc}
60 & 56 & 52 & 48 \\
51 & 47 & 43 & 39 \\
42 & 38 & 34 & 30 \\
31 & 27 & 23 & 19 \\
\end{array}
\]\n
Example 1.2. Narayana’s method of constructing 16 \times 16 magic squares.

In this example, we construct the 16 \times 16 Narayana square N2 in Table II.

Mulapankti sequence: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16.

Ganapankti sequence: 16, 32, 48, 64, 80, 96, 112, 128, 144, 160, 176, 192, 208, 224, 240.

The Chadya and the Chadaka of N2 is given below. The Chadya and the flipped Chadaka are added together to get the 16 \times 16 Narayana square N2.
Table 3: Chadya and Chadaka of Franklin square F1.

| Chadya of N2 | Chadaka of N2 |
|-------------|--------------|
| 8 9 8 9 8 9 9 8 8 9 8 9 8 9 8 9 | 112 96 80 64 48 32 16 0 |
| 7 10 7 10 7 10 7 10 7 10 7 10 7 10 7 10 | 128 144 160 176 192 208 224 240 |
| 6 11 6 11 6 11 6 11 6 11 6 11 6 11 6 11 | 112 96 80 64 48 32 16 0 |
| 5 12 5 12 5 12 5 12 5 12 5 12 5 12 5 12 | 128 144 160 176 192 208 224 240 |
| 4 13 4 13 4 13 4 13 4 13 4 13 4 13 4 13 | 112 96 80 64 48 32 16 0 |
| 3 14 3 14 3 14 3 14 3 14 3 14 3 14 3 14 | 128 144 160 176 192 208 224 240 |
| 2 15 2 15 2 15 2 15 2 15 2 15 2 15 2 15 | 112 96 80 64 48 32 16 0 |
| 1 16 1 16 1 16 1 16 1 16 1 16 1 16 1 16 | 128 144 160 176 192 208 224 240 |
| 9 8 9 8 9 8 9 8 9 8 9 8 9 8 9 8 | 112 96 80 64 48 32 16 0 |
| 10 7 10 7 10 7 10 7 10 7 10 7 10 7 10 | 128 144 160 176 192 208 224 240 |
| 11 6 11 6 11 6 11 6 11 6 11 6 11 6 11 | 112 96 80 64 48 32 16 0 |
| 12 5 12 5 12 5 12 5 12 5 12 5 12 5 12 5 12 | 128 144 160 176 192 208 224 240 |
| 13 4 13 4 13 4 13 4 13 4 13 4 13 4 13 4 | 112 96 80 64 48 32 16 0 |
| 14 3 14 3 14 3 14 3 14 3 14 3 14 3 14 | 128 144 160 176 192 208 224 240 |
| 15 2 15 2 15 2 15 2 15 2 15 2 15 2 15 2 | 112 96 80 64 48 32 16 0 |
| 16 1 16 1 16 1 16 1 16 1 16 1 16 1 16 | 128 144 160 176 192 208 224 240 |

Franklin never revealed his method of constructing his squares. In [1], a method to construct Franklin squares using Hilbert basis was developed. Later in [4], a method to construct Franklin squares of every order, in particular, F1 and F3, was derived.

In Section 2 we describe and prove the method developed in [4]. Let \( N = n^2 + 1 \). The strategy is to first place the numbers \( i \), where \( i = 1, 2, \ldots, n^2/2 \), and then place the numbers \( N - i \), such that all the defining properties of the square are satisfied. In this article, we call this method the \( N - i \) method. Then we show how the \( N - i \) method produces two squares Chadya and Chadaka which can be superimposed, like folded palms in a Namaste, to construct Franklin squares. In other words, the method of constructing Franklin squares in [4] is only a slight modification of Narayana’s construction in [7]. See Tables 3 and 4 for examples. The \( N - i \) method cannot be directly used to create F2. See [2] for a method to construct Franklin square F2.

In Section 3 we develop an \( N - i \) method to construct Narayana squares of every order. We also use the \( N - i \) method to create new Narayana squares. Moreover, we show that the \( N - i \) method to construct Narayana squares, is the same as the original Chadya-Chadaka method given by Narayana Pandita.
In this section, we describe the \( N \rightarrow i \) method to construct Franklin squares. We also show that the \( N \rightarrow i \) method can also be rewritten as a Chadya-Chadaka method.

Let \( N = n^2 + 1 \). The strategy is to first place the numbers \( i \), where \( i = 1, 2, \ldots, n^2/2 \), and then place the numbers \( N-i \), such that, all the defining properties of the square are satisfied. We start by dividing the Franklin square in to two sides: the left side consisting of the first \( n/2 \) columns and the right side consisting of the last \( n/2 \) columns. The construction of the right and left sides are largely independent of each other. Each side is further divided in to three parts: the Top part consisting of the first \( n/4 \) rows, the Middle part consisting of the middle \( n/2 \) rows, and the Bottom part consisting of the last \( n/4 \) rows.

**Distance of a column** in a given side is defined as the number of columns between the given column and the center of the side. For example, consider the left side. Here, the distance of the \( n/4 \) th and the \( n/4 + 1 \) th column of the square (the middle columns of the side) is zero whereas the distance of the first and last column of the left side is \( n/4 \).

Each side is build, partially, two equidistant columns at a time. For the left side of the Franklin square we start from the middle two columns of the side, and navigate outwards, two columns, at a time. For the right side, we start with the first and last columns, and navigate inward towards the center.

Given a side and a pair of equidistant columns, we denote the column on the left of the center as \( C_l \) and the column on the right of the center as \( C_r \). Consider a given part with \( r \) rows and a starting number \( A \). There are only two operations involved for such a part. An \( Up \) operation where consecutive numbers from \( A \) to \( A+r \) are filled, in consecutive rows, starting from the bottom row of the part, upwards, alternating between the columns \( C_l \) and \( C_r \). The only other operation is the \( Down \) operation where consecutive numbers from \( A \) to \( A+r \) are filled, in consecutive rows, alternating between the columns \( C_l \) and \( C_r \).
starting from the top row of the part in a downward direction. The starting columns of the parts are different for the two sides and is given below.

| Starting column |
|-----------------|
| **Operation**   | **Part** | **Left side** | **Right side** |
|                 | Bottom   | $C_l$          | $C_r$          |
|                 | Middle   | $C_l$          | $C_r$          |
| Top             | $C_l$    | $C_r$          |
|                 | Bottom   | $C_r$          | $C_l$          |
| Down            | Middle   | $C_l$          | $C_r$          |
| Top             | $C_r$    | $C_l$          |

For a given pair of equidistant columns, the sequence of operations depends on the parity of the distance, and is as described below.

| Even distance | Odd distance |
|---------------|--------------|
| Part          | Bottom       | Top         | Middle      | Operation | Up | Down | Down | |
| Operation     | $A$          | $A + n/4$   | $A + n/2$   | Starting Number | $A$ | $A + n/2$ | $A + 3n/4$ | |

This sequence is same for the both the sides and will place $n$ consecutive numbers in the chosen two columns. But the starting number for the entire sequence of operations, depends on the distance and the side. For the left side, if the distance is $d$ then the starting number is $nd + 1$. Whereas for the right side, for columns at a distance of $n/4 - d$, the starting number is $nd + 1 + n^2/4$.

Finally, we complete the square as follows. For each side, the empty cells in a row are filled with $N - i$, where $i$ is the entry in the same row in the equidistant column.

**Example 2.1.** In Table 5, the steps of partially filling the left side of the the $16 \times 16$ Franklin square F3 are illustrated. We start with the middle pair of columns. The distance of this pair from the center is zero, hence the starting number is $nd + 1 = 16 \times 0 + 1 = 1$. The sequence of operation is

| Part | Bottom | Top | Middle |
|------|--------|-----|--------|
| Operation | Up   | Up  | Down  |
| Starting Number | 1    | 5   | 9     |

That is, we enter the numbers from 1 to 4 in the bottom part, starting from $C_l$, using the Up operation. Next we enter the numbers from 5 to 8 in the top part, starting from $C_l$, using the Up operation. Finally, we enter the numbers from 8 to 16 in the middle part starting from column $C_r$, using the Down operation.

In Step 2 of Table 5, we consider equidistant columns of distance 1 from the center. The starting number is $nd + 1 = 16 \times 1 + 1 = 17$. Since the distance is odd, the sequence
of operation is

| Part | Middle | Top | Bottom |
|------|--------|-----|--------|
| Operation | Up     | Down | Down   |
| Starting Number | 17    | 25  | 29     |

Thus, the numbers 17 to 32 are placed in the two columns using the above sequence of operations. Steps 3 and 4 demonstrate the placement of numbers from 33 to 64 in the rest of the columns of the left side of F3. See Table 7 for the placement of the numbers from 65 to 256 in the right side of the $16 \times 16$ Franklin square. See Table 9, for the placement of numbers from 1 to 32 for the $8 \times 8$ Franklin Square F1.

The filling of the empty cells with $N - i$, where $i$ is the entry in the same row in the equidistant column, for the left side of the square F3 is given in Table 8. The completion of the right side of F3 is shown in Table 8. The final step of filling empty cells for the square F1 is given in Table 17.

The numbers $n^2/4 + 1$ to $n^2/2$ are entered in the right side of Franklin square, starting from the last and first column, working inwards. Consequently, we derive the right side from the left side as follows. We swap the first $n/4$ columns with the last $n/4$ columns of the partially filled left side and add $n^2/4$ to each entry. For example, we swap the first four columns with the last four columns and add 64 to each element of the square in Step 4 of Table 5. This gives us the square in Step 4 of Table 6. See Table 11 for an illustration. The last step of filling the empty squares with $N - i$, involve subtractions by $n^2/4$. See Table 12 for the example of F3. This means that, once the left side is build, we swap the first half columns with last half columns and then add and subtract $n^2/4$, appropriately, to get the right side. For example, the first eight columns of the square F3 is swapped with the last eight columns of the left side of F3 and 64 is added and subtracted as shown in Table 13.

Summarizing, we derive the following algorithm for constructing Franklin squares.

**Algorithm 2.1.** (Constructing $n \times n$ Franklin squares.)

1. **Partial filling of the left side.**
   
   Start with the two middle columns of left side, and then work outwards two equidistant columns at a time. Fill $n$ numbers at every step. We follow the sequence of operations, as described above, according to the parity of the distance of the columns.

2. **Subtractions from N to complete the left side.**
   
   Each empty cell of the left side is filled with the number $N - i$ where $i$ is the entry in the same row in the equidistant column. This gives us the left side of the Franklin square.

3. **Constructing the right side of the Franklin square from the left side.**
   
   Swap the first $n/4$ columns with the last $n/4$ columns of the left side to build the right side of the Franklin square. For odd rows of this modified square, $n^2/4$ is added to every entry in the first half of the row, and $n^2/4$ is subtracted from every entry.
Table 5: Constructing the Left side of the Franklin square F3.

| Step 1 | Step 2 |
|--------|--------|
| 8      | 8      |
| 25     | 25     |
| 40     | 40     |
| 7      | 7      |
| 6      | 27     |
| 38     | 38     |
| 5      | 24     |
| 10     | 10     |
| 11     | 11     |
| 13     | 13     |
| 14     | 14     |
| 15     | 15     |
| 16     | 16     |
| 3      | 3      |
| 4      | 4      |
| 5      | 5      |
| 9      | 9      |
| 10     | 11     |
| 14     | 15     |
| 16     | 18     |
| 29     | 30     |
| 31     | 31     |
| 34     | 34     |
| 29     | 1      |

| Step 3 | Step 4 |
|--------|--------|
| 8      | 8      |
| 25     | 25     |
| 40     | 40     |
| 7      | 7      |
| 6      | 27     |
| 38     | 38     |
| 5      | 24     |
| 10     | 10     |
| 11     | 11     |
| 22     | 24     |
| 12     | 12     |
| 21     | 21     |
| 13     | 13     |
| 20     | 20     |
| 4      | 4      |
| 15     | 15     |
| 23     | 24     |
| 19     | 20     |
| 14     | 14     |
| 16     | 18     |
| 17     | 18     |
| 29     | 30     |
| 31     | 31     |
| 34     | 34     |
| 1      | 2      |
| 32     | 32     |

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Table 6: Constructing the Right side of the Franklin square F3.

| Step 1 | Step 2 |
|--------|--------|
| 72   | 72 89 104 |
| 70   | 91 102 |
| 73   | 88 105 |
| 75   | 80 107 |
| 77   | 84 109 |
| 79   | 82 111 |
| 68   | 93 100 |
| 66   | 95 98 |

| Step 3 | Step 4 |
|--------|--------|
| 72 89 104 | 72 89 104 121 |
| 70 91 102 | 70 91 102 123 |
| 73 88 105 | 73 88 105 120 |
| 75 80 107 | 75 80 107 118 |
| 77 84 109 | 77 84 109 116 |
| 79 82 111 | 79 82 111 114 |
| 68 93 100 | 68 93 100 126 |
| 66 95 98 | 66 95 98 127 |

Table 7: The left side of the Franklin square F3.

| N - 57 | N - 40 | N - 25 | N - 8 | 8 | 25 | 40 | 57 |
|--------|--------|--------|--------|---|----|----|----|
| 58  | 39   | 26   | 7   | N - 7 | N - 26 | N - 39 | N - 58 |
| N - 59 | N - 38 | N - 27 | N - 6 | 6 | 27 | 38 | 59 |
| 60 | 37 | 28 | 5 | N - 5 | N - 28 | N - 37 | N - 60 |
| N - 56 | N - 41 | N - 24 | N - 9 | 9 | 24 | 41 | 56 |
| 55 | 42 | 23 | 10 | N - 10 | N - 23 | N - 42 | N - 55 |
| N - 54 | N - 43 | N - 22 | N - 11 | 11 | 22 | 43 | 54 |
| 53 | 44 | 21 | 12 | N - 12 | N - 21 | N - 44 | N - 53 |
| N - 52 | N - 45 | N - 20 | N - 13 | 13 | 20 | 45 | 52 |
| 51 | 46 | 19 | 14 | N - 14 | N - 19 | N - 46 | N - 51 |
| N - 50 | N - 47 | N - 18 | N - 15 | 15 | 18 | 47 | 50 |
| 49 | 48 | 17 | 16 | N - 16 | N - 17 | N - 48 | N - 49 |
| N - 49 | N - 36 | N - 29 | N - 4 | 4 | 29 | 36 | 61 |
| 62 | 35 | 30 | 3 | N - 3 | N - 30 | N - 35 | N - 62 |
| N - 63 | N - 34 | N - 31 | N - 2 | 2 | 31 | 34 | 63 |
| 64 | 33 | 32 | 1 | N - 1 | N - 32 | N - 33 | N - 64 |

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Table 8: The right side of the Franklin square F3.

| 72 | 89 | 104 | 121 | N − 121 | N − 104 | N − 89 | N − 72 |
| 70 | 91 | 102 | 123 | N − 123 | N − 102 | N − 91 | N − 70 |
| 73 | 88 | 106 | 120 | N − 120 | N − 106 | N − 93 | N − 73 |
| 75 | 86 | 107 | 118 | N − 118 | N − 107 | N − 86 | N − 75 |
| 77 | 84 | 109 | 116 | N − 116 | N − 109 | N − 84 | N − 77 |
| 79 | 82 | 111 | 114 | N − 114 | N − 111 | N − 82 | N − 79 |
| 68 | 93 | 100 | 125 | N − 125 | N − 100 | N − 93 | N − 68 |
| 66 | 95 | 98 | 127 | N − 127 | N − 98 | N − 95 | N − 66 |
| 65 | 90 | N − 97 | N − 128 | 128 | 97 | 96 | 65 |

Table 9: Construction of $8 \times 8$ Franklin Square F1.

### Step 1

```
4 3
5 0
6 8
2 1
```

### Step 2

```
4 13
14 3
5 12
11 6
7 10
9 8
2 15
16 1
```

### Step 3

```
4 13
14 3
9 12
21
9
24
8
18
16
```

### Step 4

```
4 13
14 3
9 12
21
9
24
8
18
16
```

Table 10: Final step in construction of F1

| N − 13 | N − 4 | 4 | 13 | 20 | 29 | N − 29 | N − 20 |
| N − 14 | 3 | N − 3 | N − 14 | N − 19 | N − 30 | 30 | 19 |
| N − 12 | N − 5 | 5 | 12 | 21 | 28 | N − 28 | N − 21 |
| 11 | 6 | N − 6 | N − 11 | N − 22 | N − 27 | 27 | 22 |
| N − 10 | N − 7 | 7 | 10 | 23 | 26 | N − 26 | N − 23 |
| 9 | 8 | N − 8 | N − 9 | N − 24 | N − 25 | 25 | 24 |
| N − 15 | N − 2 | 2 | 15 | 18 | 31 | N − 31 | N − 18 |
| 16 | 1 | N − 1 | N − 16 | N − 17 | N − 32 | 32 | 17 |
Table 11: Step 4 of Table 6 in construction of right side of F3 derived, alternately, by swapping columns of partially built left side, and adding 64.

|    | 8  | 25 | 40 | 57 | N – 57 | N – 40 | N – 25 | N – 8 |
|----|----|----|----|----|--------|--------|--------|------|
| N + 1 | 6 | 27 | 38 | 59 | N – 59 | N – 38 | N – 27 | N – 6 |
| N + 2 | 9 | 24 | 41 | 56 | N – 56 | N – 41 | N – 24 | N – 9 |
| N + 3 | 11 | 22 | 43 | 54 | N – 54 | N – 43 | N – 22 | N – 11 |
| N + 4 | 15 | 29 | 47 | 61 | N – 61 | N – 47 | N – 29 | N – 14 |
| N + 5 | 2 | 31 | 54 | 63 | N – 63 | N – 54 | N – 31 | N – 2 |
| N + 6 | 3 | 35 | 62 | 64 | N – 64 | N – 62 | N – 35 | N – 3 |
| N + 7 | 13 | 20 | 45 | 52 | N – 52 | N – 45 | N – 20 | N – 13 |
| N + 8 | 15 | 18 | 47 | 50 | N – 50 | N – 47 | N – 18 | N – 15 |
| N + 9 | 16 | 17 | 48 | 49 | N – 49 | N – 48 | N – 17 | N – 16 |
| N + 10 | 29 | 36 | 61 | 64 | N – 64 | N – 36 | N – 29 | N – 4 |
| N + 11 | 31 | 34 | 63 | 64 | N – 64 | N – 34 | N – 31 | N – 2 |
| N + 12 | 33 | 37 | 64 | 64 | N – 64 | N – 37 | N – 33 | N – 1 |

Table 12: Constructing the right side of F3.

|    | 8  | 25 | 40 | 57 | N – 57 | N – 40 | N – 25 | N – 8 |
|----|----|----|----|----|--------|--------|--------|------|
| N + 1 | 6 | 27 | 38 | 59 | N – 59 | N – 38 | N – 27 | N – 6 |
| N + 2 | 9 | 24 | 41 | 56 | N – 56 | N – 41 | N – 24 | N – 9 |
| N + 3 | 11 | 22 | 43 | 54 | N – 54 | N – 43 | N – 22 | N – 11 |
| N + 4 | 15 | 29 | 47 | 61 | N – 61 | N – 47 | N – 29 | N – 14 |
| N + 5 | 2 | 31 | 54 | 63 | N – 63 | N – 54 | N – 31 | N – 2 |
| N + 6 | 3 | 35 | 62 | 64 | N – 64 | N – 62 | N – 35 | N – 3 |
| N + 7 | 13 | 20 | 45 | 52 | N – 52 | N – 45 | N – 20 | N – 13 |
| N + 8 | 15 | 18 | 47 | 50 | N – 50 | N – 47 | N – 18 | N – 15 |
| N + 9 | 16 | 17 | 48 | 49 | N – 49 | N – 48 | N – 17 | N – 16 |
| N + 10 | 29 | 36 | 61 | 64 | N – 64 | N – 36 | N – 29 | N – 4 |
| N + 11 | 31 | 34 | 63 | 64 | N – 64 | N – 34 | N – 31 | N – 2 |
| N + 12 | 33 | 37 | 64 | 64 | N – 64 | N – 37 | N – 33 | N – 1 |

Table 13: Deriving the right side of the Franklin square F3 from the left side.

|    | 8  | 25 | 40 | 57 | N – 57 | N – 40 | N – 25 | N – 8 |
|----|----|----|----|----|--------|--------|--------|------|
| N + 1 | 6 | 27 | 38 | 59 | N – 59 | N – 38 | N – 27 | N – 6 |
| N + 2 | 9 | 24 | 41 | 56 | N – 56 | N – 41 | N – 24 | N – 9 |
| N + 3 | 11 | 22 | 43 | 54 | N – 54 | N – 43 | N – 22 | N – 11 |
| N + 4 | 15 | 29 | 47 | 61 | N – 61 | N – 47 | N – 29 | N – 14 |
| N + 5 | 2 | 31 | 54 | 63 | N – 63 | N – 54 | N – 31 | N – 2 |
| N + 6 | 3 | 35 | 62 | 64 | N – 64 | N – 62 | N – 35 | N – 3 |
| N + 7 | 13 | 20 | 45 | 52 | N – 52 | N – 45 | N – 20 | N – 13 |
| N + 8 | 15 | 18 | 47 | 50 | N – 50 | N – 47 | N – 18 | N – 15 |
| N + 9 | 16 | 17 | 48 | 49 | N – 49 | N – 48 | N – 17 | N – 16 |
| N + 10 | 29 | 36 | 61 | 64 | N – 64 | N – 36 | N – 29 | N – 4 |
| N + 11 | 31 | 34 | 63 | 64 | N – 64 | N – 34 | N – 31 | N – 2 |
| N + 12 | 33 | 37 | 64 | 64 | N – 64 | N – 37 | N – 33 | N – 1 |
of the second half. For even rows, \( n^2/4 \) is subtracted from every entry in the first half of the row, and \( n^2/4 \) is added to every entry of the second half. This gives us the right side of the Franklin square.

We proceed to show that Algorithm 2.1 produces a Franklin square. We start with the following lemma which describes many properties of the Franklin square, constructed using Algorithm 2.1.

**Lemma 2.1.** Let \( a_{i,j} \) denote the entries of a \( n \times n \) Franklin square, where \( i = 1, 2, \ldots, n \) and \( j = 1, 2, \ldots, n \).

1. Pair of entries of adjacent rows in a column add to \( N \pm 1 \) except for the rows \( n/4 \) and \( 3n/4 \), as follows.

   Consider a row \( i \in \{1, 2, \ldots, n\} \setminus \{n/4, 3n/4, n\} \). In the top and bottom parts of the square, that is, when \( i \leq n/4 \) or \( n/2 < i < n \), we have

   \[
   a_{i,j} + a_{i+1,j} = \begin{cases}
   N + 1, & \text{if } j \text{ is odd, and } i \text{ is odd,} \\
   N - 1, & \text{if } j \text{ is odd, and } i \text{ is even,} \\
   N - 1, & \text{if } j \text{ is even, and } i \text{ is odd,} \\
   N + 1, & \text{if } j \text{ is even, and } i \text{ is even.}
   \end{cases}
   \]

   For the middle part, the situation is exactly opposite. That is, when \( n/4 < i \leq n/2 \), we have

   \[
   a_{i,j} + a_{i+1,j} = \begin{cases}
   N - 1, & \text{if } j \text{ is odd, and } i \text{ is odd,} \\
   N + 1, & \text{if } j \text{ is odd, and } i \text{ is even,} \\
   N + 1, & \text{if } j \text{ is even, and } i \text{ is odd,} \\
   N - 1, & \text{if } j \text{ is even, and } i \text{ is even.}
   \end{cases}
   \]

   Finally, we consider rows \( n/4 \) and \( 3n/4 \).

   \[
   a_{n/4,j} + a_{n/4+1,j} = \begin{cases}
   N + n/4, & \text{if } j \text{ is odd,} \\
   N - n/4, & \text{if } j \text{ is even.}
   \end{cases}
   \]

   \[
   a_{3n/4,j} + a_{3n/4+1,j} = \begin{cases}
   N - 3n/4, & \text{if } j \text{ is odd,} \\
   N + 3n/4, & \text{if } j \text{ is even.}
   \end{cases}
   \]

2. Consider a row \( i \leq n/2 \) and let \( m_i = n/2 - 1 - 2(i - 1) \). Equidistant entries across the Horizontal axis add to either \( N + m_i \) or \( N - m_i \) as follows.

   \[
   a_{i,j} + a_{n-i+1,j} = \begin{cases}
   N + m_i, & \text{if } i \text{ is odd and } j \text{ is odd,} \\
   N - m_i, & \text{if } i \text{ is odd and } j \text{ is even,} \\
   N - m_i, & \text{if } i \text{ is even and } j \text{ is odd,} \\
   N + m_i, & \text{if } i \text{ is even and } j \text{ is even.}
   \end{cases}
   \]

3. Let \( m = n^2/2 + 1 \). Equidistant entries across the vertical axis add to either \( m \) or \( 2N - m \) as follows.
If $i$ is odd

$$a_{i,j} + a_{i,n-j+1} = \begin{cases} 2N - m, & \text{if } j \leq n/4, \\ m, & \text{if } n/4 < j < n/2. \end{cases}$$

If $i$ is even, the exact opposite is true. That is,

$$a_{i,j} + a_{i,n-j+1} = \begin{cases} m, & \text{if } j \leq n/4, \\ 2N - m, & \text{if } n/4 < j < n/2. \end{cases}$$

Proof. The square inherits these properties by construction. 

**Proposition 2.1.** Algorithm 2.1 produces a Franklin square.

Proof. Let $a_{i,j}$ denote the entries of an $n \times n$ square constructed by Algorithm 2.1. Let $N = n^2 + 1$ and $M$ denote the magic sum.

1. $2 \times 2$ sub-square sums add to $2N$ continuously.

   Consider a row $i \in \{1, 2, \ldots, n\} \setminus \{n/4, 3n/4, n\}$. By Part 1 in Lemma 2.1

   if $a_{i,j} + a_{i+1,j} = N + 1$, then $a_{i,j+1} + a_{i+1,j+1} = N - 1$.

   On the other hand,

   if $a_{i,j} + a_{i+1,j} = N - 1$, then $a_{i,j+1} + a_{i+1,j+1} = N + 1$.

   Consequently, for all $i \in \{1, 2, \ldots, n\} \setminus \{n/4, 3n/4, n\}$ and all $j$,

   $$a_{i,j} + a_{i+1,j} + a_{i,j+1} + a_{i+1,j+1} = 2N.$$ 

Now we consider the rows $n/4$ and $3n/4$. By Part 1 in Lemma 2.1

if $a_{n/4,j} + a_{n/4+1,j} = N + n/4$, then $a_{n/4,j+1} + a_{n/4+1,j+1} = N - n/4$, and

if $a_{n/4,j} + a_{n/4+1,j} = N - n/4$, then $a_{n/4,j+1} + a_{n/4+1,j+1} = N + n/4$.

Also,

if $a_{3n/4,j} + a_{3n/4+1,j} = N + 3n/4$, then $a_{3n/4,j+1} + a_{3n/4+1,j+1} = N - 3n/4$, and

if $a_{3n/4,j} + a_{3n/4+1,j} = N - 3n/4$, then $a_{3n/4,j+1} + a_{3n/4+1,j+1} = N + 3n/4$.

Consequently, all the $2 \times 2$ sub-squares, within the Franklin square, add to $2N$. Next, we verify the continuity of this property.

Part 2 of Lemma 2.1 implies

$$a_{1,j} + a_{n,j} = \begin{cases} N + m_1, & \text{if } j \text{ is odd}, \\ N - m_1, & \text{if } j \text{ is even}. \end{cases}$$
Consequently, the $2 \times 2$ sub-squares formed by rows 1 and $n$ add to $2N$. Part 3 of Lemma 2.1 implies

$$a_{i,1} + a_{i,n} = \begin{cases} 2N - m, & \text{if } i \text{ is odd,} \\ m, & \text{if } i \text{ is even.} \end{cases}$$

Thus, the $2 \times 2$ sub-squares formed by columns 1 and $n$ add to $2N$. This proves that the continuity property of $2 \times 2$ sub-squares hold for squares constructed by Algorithm 2.1.

2. Half row and half column sums add to $M/2$. Row and column sums add to $M$.

By Part 1 of Lemma 2.1, quarter column sums are as follows.

$$a_{1,j} + a_{2,j} + \cdots + a_{n/4,j} = \begin{cases} \frac{n}{8}(N + 1), & \text{when } j \text{ is odd,} \\ \frac{n}{8}(N - 1), & \text{when } j \text{ is even.} \end{cases}$$

$$a_{n/4+1,j} + a_{n/4+2,j} + \cdots + a_{n/2,j} = \begin{cases} \frac{n}{8}(N - 1), & \text{when } j \text{ is odd,} \\ \frac{n}{8}(N + 1), & \text{when } j \text{ is even.} \end{cases}$$

$$a_{n/2+1,j} + a_{n/2+2,j} + \cdots + a_{3n/4,j} = \begin{cases} \frac{n}{8}(N - 1), & \text{when } j \text{ is odd,} \\ \frac{n}{8}(N + 1), & \text{when } j \text{ is even.} \end{cases}$$

$$a_{3n/4+1,j} + a_{3n/4+2,j} + \cdots + a_{n,j} = \begin{cases} \frac{n}{8}(N + 1), & \text{when } j \text{ is odd,} \\ \frac{n}{8}(N - 1), & \text{when } j \text{ is even.} \end{cases}$$

Consequently, for all $j$,

$$a_{1,j} + a_{2,j} + \cdots + a_{n/2,j} = \frac{n}{4}N,$$

$$a_{n/2+1,j} + a_{n/2+1,j} + \cdots + a_{n,j} = \frac{n}{4}N.$$

That is, all the half columns add to $(n/4)N$, which is half the magic sum. Therefore, all the columns add to the magic sum. By construction, the way subtractions were done from $N$, (see Part 2 in Algorithm 2.1), half rows add to $(n/4)N$. Hence full rows add to $(n/2)N$ which is the magic sum.

3. Bend diagonals add to $M$.

To prove that the left bend diagonals add to the magic sum, we add the entries, pairwise, where each pair is equidistant from the horizontal axis. Let $1 \leq j < n$, then by Part 2 of Lemma 2.1, if $1 \leq i < n/4$, and if,

$$a_{i,j} + a_{n-i+1,j} = N + m_i, \text{ then, } a_{i+1,j+1} + a_{n-i,j+1} = N + m_i, \text{ and if }$$

$$a_{i,j} + a_{n-i+1,j} = N - m_i, \text{ then, } a_{i+1,j+1} + a_{n-i,j+1} = N - m_i.$$
Observe that, if \( n/4 + 1 \leq i < n/2 \), then the signs for \( m_i \) in the above sums are exactly opposite, by Part 2 of Lemma 2.1. Let \( 1 \leq j \leq n/2 + 1 \), then, if \( j \) is odd, the left bend diagonal sum starting with row 1 and column \( j \) adds as follows.

\[
\left[(a_{1,j} + a_{n,j}) + (a_{2,j+1} + a_{n-1,j+1}) + \cdots + (a_{n/4,j+n/4-1} + a_{n/4+1,j+n/4-1})\right]
+ \left[(a_{n/4+1,j+n/4} + a_{n/4+1,j+n/4}) + \cdots + (a_{n/2,j+n/2-1} + a_{n/2+1,j+n/2-1})\right]
= \left[N + (\frac{n}{2} - 1) + N - (\frac{n}{2} - 3) - N + (\frac{n}{2} - 5) + \cdots + N - 1\right]
+ \left[N - (\frac{n}{2} - 1) + N + (\frac{n}{2} - 3) - N - (\frac{n}{2} - 5) + \cdots + N + 1\right]
= \frac{n}{2}N = M.
\]

A similar argument gives us that the even bend diagonals also add to \( M \). Consequently, the left bend diagonals, when \( j = 1, 2, \ldots, n/2 + 1 \), add to the magic sum.

Let \( n/2 + 1 < j \leq n \), the left bend diagonal sums are

\[
(a_{1,j} + a_{n,j}) + (a_{2,j} + a_{n-1,j+1}) + \cdots + (a_{n-j+1,n} + a_{j,n})
+(a_{n-j+2,1} + a_{j-1,1}) + (a_{n-j+3,2} + a_{j-2,2}) + \cdots + (a_{n/4,j-n/4+1} + a_{n/4+1,j-n/4-1}).
\]

By Part 2 of Lemma 2.1 since the sums depend only on the parity of \( j \), we get that the necessary cancellations happen, and these sums, also, add to \( M \).

For example, in the case of of F1 (see Table 10), the seventh bend diagonal sum is

\[
(a_{1,7} + a_{8,7}) + (a_{2,8} + a_{7,8}) + (a_{3,1} + a_{6,1}) + (a_{4,2} + a_{5,2})
= (N + 3) + (N + 1) + (N - 3) + (N - 1) = 4N = M.
\]

Thus, the left bend diagonals add to the magic sum, continuously. The proof that all right bend diagonals add to the magic sum, is similar to the case of left bend diagonals. The proof depends, mainly, on the fact that equidistant entries across the horizontal axis add to either \( N + m_i \) or \( N - m_i \), and all \( m_i \) cancel in the final sum.

Similar argument is used to prove that the top and bottom bend diagonals add to magic sum. By Part 3 of Lemma 2.1 pairs of equidistant entries across the vertical axis add to \( m \) or \( 2N - m \).

For \( 1 \leq i \leq n/2 + 1 \), let \( i \) be odd, then the \( i \)-th top bend diagonal sum is given
\[
(a_{i,1} + a_{i,n}) + (a_{i+1,2} + a_{i+1,n-1}) + \cdots + (a_{i+n-1,1} + a_{i+n-1,n})
\]
\[
+ \left[ (a_{i+\frac{n}{4}+1} + a_{i+\frac{n}{4},n}) + \cdots + (a_{i+\frac{n}{2}+1} + a_{i+\frac{n}{2},n}) \right]
\]
\[
= [(2N - m) + m + \cdots + m] + [(2N - m) + m + \cdots + m]
\]
\[
= \frac{n}{2}N = M.
\]

When \(i\) is even, \(m\) and \(2N - m\) are replaced with each other, wherever they appear in the above sum. Thus, the top bend diagonal sums add to \(M\) when \(1 \leq i \leq n/2 + 1\).

For \(n/2 + 1 < i \leq n\), the top bed diagonal sums are
\[
(a_{i,1} + a_{i,n}) + (a_{i+1,2} + a_{i+1,n-1}) + \cdots (a_{n,n-i+1} + a_{n,i})
\]
\[
+ (a_{1,n-i+2} + a_{1,i-1}) + (a_{2,n-i+3} + a_{2,i-2}) + \cdots + (a_{i-n,i-1} + a_{i-n,i+1})
\]

Again, by Part 3 of Lemma 2.1, it can be checked that the top bed diagonal sums add to \(M\).

For example, in the case of F1 (see Table 10), the seventh top bend diagonal sum is
\[
(a_{7,1} + a_{7,8}) + (a_{8,2} + a_{8,7}) + (a_{1,3} + a_{1,6}) + (a_{2,4} + a_{2,5})
\]
\[
= (2N - m) + m + m + (2N - m) = 4N = M.
\]

Consequently, the top bend diagonals add to the magic sum, continuously. A similar proof, applying Part 3 of Lemma 2.1, shows that the bottom bend diagonals add to the magic sum, continuously.

Thus, a square constructed by Algorithm 2.1 is a Franklin square.

\[\square\]

We now show that the Algorithm 2.1 is very similar to Narayana’s method. That is, we show how a Franklin square can be constructed as a superimposition of Chadya and Chadaka. We illustrate the derivation of a Chadya and a Chadaka square using the example of F3, before formulating an algorithm. Since \(n\) numbers are placed at every step, the entries in a partially completed left side of a Franklin square, before the final step of subtraction from \(N\), can be rewritten as multiples on \(n\). For example, for F3, since 16 numbers are placed at every step, the entries in the partially completed left side of Franklin square F3, in Step 4 of Table 5, can be rewritten as multiples of 16, as follows.
Since \( N - rn - i = (n^2 + 1) - rn - i = (n^2 - (r + 1)n) + (n + 1 - i) \), the subtractions from \( N \) for \( F3 \), in Table \([7]\), can be rewritten as follows.

|     | 8 + 192 | 9 + 208 | 8 + 224 | 9 + 240 | 8 | 9 + 16 | 8 + 32 | 9 + 48 |
|-----|---------|---------|---------|---------|---|--------|--------|--------|
| 10 + 48 | 7 + 32  | 10 + 16 | 7      | 10 + 240 | 7 + 224 | 10 + 208 | 7 + 192 |
| 6 + 192 | 11 + 208 | 6 + 224 | 11 + 240 | 6 | 11 + 16 | 6 + 32 | 11 + 48 |
| 12 + 48 | 5 + 32  | 12 + 16 | 5      | 12 + 240 | 5 + 224 | 12 + 208 | 5 + 192 |
| 9 + 192 | 8 + 208 | 9 + 224 | 8 + 240 | 9 | 8 + 16 | 9 + 32 | 8 + 48 |
| 4 + 192 | 10 + 32 | 4 + 16 | 10 | 7 + 240 | 10 + 224 | 7 + 208 | 10 + 192 |
| 11 + 192 | 6 + 208 | 11 + 224 | 6 + 240 | 11 | 6 + 16 | 11 + 32 | 6 + 48 |
| 5 + 192 | 12 + 32 | 5 + 16 | 12 | 5 + 240 | 12 + 224 | 5 + 208 | 12 + 192 |
| 13 + 192 | 4 + 208 | 13 + 224 | 4 + 240 | 13 | 4 + 16 | 13 + 32 | 4 + 48 |
| 3 + 192 | 14 + 32 | 3 + 16 | 14 | 3 + 240 | 14 + 224 | 3 + 208 | 14 + 192 |
| 15 + 192 | 2 + 208 | 15 + 224 | 2 + 240 | 15 | 2 + 16 | 15 + 32 | 2 + 48 |
| 1 + 192 | 16 + 32 | 1 + 16 | 16 | 1 + 240 | 16 + 224 | 1 + 208 | 16 + 192 |
| 13 + 192 | 4 + 208 | 13 + 224 | 4 + 240 | 4 | 13 + 16 | 4 + 32 | 13 + 48 |
| 15 + 192 | 16 + 32 | 16 + 16 | 16 | 16 + 240 | 16 + 224 | 16 + 208 | 16 + 192 |

Consequently, the left side of the Franklin square can then be split into two squares. We call these squares Chadaya and flipped Chadaka of the left side of a Franklin square. For example, the left side of \( F3 \) is the sum of the Chadaya and flipped Chadaka of the left side of \( F3 \), as shown below.

\[
\text{Left side of } F3 = \text{Chadaya of left side of } F3 + \text{Flipped Chadaka of left side of } F3
\]
We now apply Step 3 of Algorithm 2.1. To get the right side from the left side, we swap the first \( n/4 \) columns with the last \( n/4 \) columns of both the Chadya and Chadaka of the left side of the Franklin square, and then add and subtract \( n^2/4 \). In case of F3, we get

| 8 9 8 9 8 9 8 9 | 0 16 32 48 192 208 224 240 | 64 64 64 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 |
| 10 7 10 7 7 7 10 7 | 240 224 208 192 48 32 16 0 | $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 |
| 6 11 6 11 6 11 6 11 | 0 16 32 48 192 208 224 240 | $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 |
| 12 5 12 5 12 5 12 5 | 240 224 208 192 48 32 16 0 | $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 |
| 9 8 9 8 9 8 9 8 | 0 16 32 48 192 208 224 240 | $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 |
| 7 10 7 10 7 10 7 10 | 240 224 208 192 48 32 16 0 | $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 |
| 11 6 11 6 11 6 11 6 | 0 16 32 48 192 208 224 240 | $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 |
| 5 12 5 12 5 12 5 12 | 240 224 208 192 48 32 16 0 | $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 |
| 13 4 13 1 13 4 13 4 | 0 16 32 48 192 208 224 240 | $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 |
| 3 14 3 14 3 14 3 14 | 240 224 208 192 48 32 16 0 | $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 |
| 15 2 15 2 15 2 15 2 | 0 16 32 48 192 208 224 240 | $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 |
| 1 16 1 16 1 16 1 16 | 240 224 208 192 48 32 16 0 | $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 |
| 4 13 4 13 4 13 4 13 | 0 16 32 48 192 208 224 240 | $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 |
| 14 3 14 3 14 3 14 3 | 240 224 208 192 48 32 16 0 | $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 |
| 2 15 2 15 2 15 2 15 | 0 16 32 48 192 208 224 240 | $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 |
| 16 1 16 1 16 1 16 1 | 240 224 208 192 48 32 16 0 | $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 $-$ 64 |

Observe that the Chadaya, is unchanged by the column swaps. Thus the Chadaya of the left side and right side of the Franklin square are the same. The Flipped Chadaka of the right side is obtained by adding the last two squares in the equation above.

\[
\text{Right side of F3} = \quad \text{Chadya of right side of F3} \quad + \quad \text{Flipped Chadaka of right side of F3}
\]

| 8 9 8 9 8 9 8 9 | 64 80 96 112 128 144 160 176 |
| 10 7 10 7 7 7 10 7 | 176 160 144 128 112 96 80 64 |
| 6 11 6 11 6 11 6 11 | 64 80 96 112 128 144 160 176 |
| 12 5 12 5 12 5 12 5 | 176 160 144 128 112 96 80 64 |
| 9 8 9 8 9 8 9 8 | 64 80 96 112 128 144 160 176 |
| 7 10 7 10 7 10 7 10 | 176 160 144 128 112 96 80 64 |
| 11 6 11 6 11 6 11 6 | 64 80 96 112 128 144 160 176 |
| 5 12 5 12 5 12 5 12 | 176 160 144 128 112 96 80 64 |
| 13 4 13 4 13 4 13 4 | 64 80 96 112 128 144 160 176 |
| 3 14 3 14 3 14 3 14 | 176 160 144 128 112 96 80 64 |
| 15 2 15 2 15 2 15 2 | 64 80 96 112 128 144 160 176 |
| 16 1 16 1 16 1 16 1 | 176 160 144 128 112 96 80 64 |
| 4 13 4 13 4 13 4 13 | 64 80 96 112 128 144 160 176 |
| 2 15 2 15 2 15 2 15 | 64 80 96 112 128 144 160 176 |
| 16 1 16 1 16 1 16 1 | 176 160 144 128 112 96 80 64 |

Putting the left and right sides together, we get the Chadya and the Flipped Chadaka of the Franklin square. The flipped Chadaka is flipped again to get the Chadaka of the Franklin square. See Table 4 for the Chadya and Chadaka of F3.

We, now, describe a new Algorithm to construct Franklin squares as superimposition of Chadya and Chadaka squares. This algorithm, as we have seen, is just a rewriting of Algorithm 2.1.

**Algorithm 2.2.** 1. We start with placing the numbers 1 to \( n \) in the \( n/4 \) and \( n/4 + 1 \)th columns, using the following operation sequence.
For example, in the case of the Franklin square F3, we get

| Part | Bottom | Top | Middle |
|------|--------|-----|--------|
| Operation | Up | Up | Down |
| Starting Number | 1 | 1 + n/4 | 1 + n/2 |

Observe that this is Step 1 of Table 5.

2. Let a be the entry in a row in Step [1]. Then the row is filled with a and n + 1 − a in alternate columns. This gives us the Chadya of the square. In the case of F3, we get

3. To construct the Chadaka, we start with the left side of the Franklin square. We place zeroes in the two middle rows, in the same sequence, as in Step [2] of Algorithm [2,7]. This places zeroes, alternating between the middle two columns, starting with the n/4 + 1 th column, in a downward direction.

In the case of F3, this step will produce
4. Next, we fill the empty cells in the middle two columns with $n^2 - n$.

Filling the empty cells with $256 - 16 = 240$ for F3, we get

```
- - - - 0 - - -
- - 240 0 - - -
- - - - - - - -
- - - - - - - -
- - - - - - - -
- - - - - - - -
- - - - - - - -
- - - - - - - -
```

5. We place $n \times i$ where $i = 1, \ldots, n/4 - 1$, to the right or left of zero depending on which side of zero is empty. Finally, we place $n^2 - (i+1)n$, where $i = 1, \ldots, n/4 - 1$ to the right or left of $n^2 - n$, depending on which side of $n^2 - n$ is empty. This gives us the flipped Chadaka of the left side of the Franklin square.

In case of F3, we get

```
192 208 224 240 0 16 32 48
48 32 16 0 240 224 208 192
192 208 224 240 0 16 32 48
48 32 16 0 240 224 208 192
192 208 224 240 0 16 32 48
48 32 16 0 240 224 208 192
192 208 224 240 0 16 32 48
48 32 16 0 240 224 208 192
192 208 224 240 0 16 32 48
48 32 16 0 240 224 208 192
192 208 224 240 0 16 32 48
48 32 16 0 240 224 208 192
```
6. The flipped Chadaka of the right side of the Franklin square from the right side is constructed by applying Step 3 of Algorithm 2.1. Swap the first \( n/4 \) columns with the last \( n/4 \) columns of the Chadaka of the left side to build the Chadaka of the right side of the Franklin square. For odd rows of this modified square, \( n^2/4 \) is added to every entry in the first half of the row, and \( n^2/4 \) is subtracted from every entry of the second half. For even rows, \( n^2/4 \) is subtracted from every entry in the first half of the row, and \( n^2/4 \) is added to every entry of the second half. This gives us the flipped Chadaka of the right side of the Franklin square.

Thus, we get the flipped Chadaka of the right side of the Franklin square F3, as shown below.

![Table showing the calculation process]

We put the flipped Chadakas of the two sides to get the flipped Chadaka of F3.

![Table showing the final Chadaka]

Because the Chadya of the left side and the right side are the same, we can also construct the two sides separately. For example, the left side of Franklin square F1

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is constructed as follows.

| Chadya | Flipped Chadaka |
|--------|-----------------|
| 4 5 1 5 | 48 56 0 8 |
| 6 3 6 5 | 8 0 56 48 |
| 5 4 5 4 | 48 56 0 8 |
| 3 6 3 6 | 8 0 56 48 |
| 7 2 7 2 | 48 56 0 8 |
| 1 8 1 8 | 8 0 56 48 |
| 2 / 2 / 2 / 2 | 48 56 0 8 |
| 8 1 8 1 | 8 0 56 48 |

The right side of Franklin square F1 is derived below.

| Chadya | Flipped Chadaka |
|--------|-----------------|
| 4 5 1 5 | 0 8 48 56 |
| 6 3 6 5 | 56 48 8 0 |
| 5 4 5 4 | 0 8 48 56 |
| 3 6 3 6 | 48 56 0 8 |
| 7 2 7 2 | 0 8 48 56 |
| 1 8 1 8 | 56 48 8 0 |
| 2 / 2 / 2 / 2 | 0 8 48 56 |
| 8 1 8 1 | 56 48 8 0 |

See Tables 3 and 4 for the Chadya and Chadaka of F1 and F3, respectively.

**Lemma 2.2.** Consider an \( n \times n \) Franklin square. Let the magic sum be denoted by \( M \), then pandiagonals add to \( M \pm n^2/2 \).

**Proof.** We first look at left pandiagonals (see Figure 3). Since every pandiagonal starts from the first row, let \( P_{1,c} \) denote a pandiagonal that starts from column \( c \). Let \( a_{i,j} \) denote the entries of a \( n \times n \) Franklin square.

Let \( a_{i,j} \) belong to a pandiagonal and let \( 1 \leq i \leq n/2 \). If \( 1 \leq j \leq n/2 \), then \( a_{n/2+i,n/2+j} \) also belong to the pandiagonal. On the other hand if \( j > n/2 \), then \( a_{n/2+i,n/2+j} \) belongs to the pandiagonal. Thus, every pandiagonal \( P_{1,c} \) is made up of \( n/2 \) paired entries.

Let \( 1 \leq i \leq n/2 \), define \( s_{i,j} \) to be

\[
    s_{i,j} = \begin{cases} 
    a_{i,j} + a_{n/2+i,n/2+j} & \text{if } 1 \leq j \leq n/2 \\
    a_{i,j} + a_{i,j-n/2} & \text{if } n/2 < j \leq n.
    \end{cases}
\]

Observe that \( s_{i,j} = s_{i,j+n/2} \). Therefore, by the continuity property of pandiagonals, it is sufficient to consider \( s_{i,j} \), where \( 1 \leq i,j \leq n/2 \), to derive pandiagonal sums. Each pandiagonal sum contains \( n/2 \) such paired sums. Let \( ch_{i,j} \) and \( cd_{i,j} \) denote the entries of the Chadya and Chadaka, respectively. Let \( y_{i,j} = ch_{i,j} + ch_{n/2+i,n/2+j} \) and \( d_{i,j} = cd_{i,j} + cd_{n/2+i,n/2+j} \). Then \( s_{i,j} = y_{i,j} + d_{i,j} \).

By construction, for \( 1 \leq i \leq n/4 \) and \( 1 \leq j \leq n/2 \),
\[ y_{i,j} = \begin{cases} 
\frac{5n}{4} + 1 & \text{if } i \text{ is odd and } j \text{ is odd.} \\
\frac{3n}{4} + 1 & \text{if } i \text{ is odd and } j \text{ is even.} \\
\frac{3n}{4} + 1 & \text{if } i \text{ is even and } j \text{ is odd.} \\
\frac{5n}{4} + 1 & \text{if } i \text{ is even and } j \text{ is even.} 
\end{cases} \]

Let \( n/4 < i \leq n/2 \) and \( 1 \leq j \leq n/2 \). Then

\[ y_{i,j} = \begin{cases} 
\frac{3n}{4} + 1 & \text{if } i \text{ is odd and } j \text{ is odd.} \\
\frac{5n}{4} + 1 & \text{if } i \text{ is odd and } j \text{ is even.} \\
\frac{5n}{4} + 1 & \text{if } i \text{ is even and } j \text{ is odd.} \\
\frac{3n}{4} + 1 & \text{if } i \text{ is even and } j \text{ is even.} 
\end{cases} \]

For example, in the case of \( F_1 \),

\[
[y_{i,j}] = \begin{bmatrix}
4 & 5 & 6 \\
3 & 4 & 5 \\
2 & 3 & 4
\end{bmatrix}
+ \begin{bmatrix}
1 & 1 & 2 \\
2 & 2 & 2 \\
8 & 8 & 8
\end{bmatrix}
= \begin{bmatrix}
11 & 7 & 11 \\
7 & 11 & 7 \\
11 & 11 & 11
\end{bmatrix}
= \begin{bmatrix}
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2}
\end{bmatrix}
\]

Check that, in the case of the Franklin square \( F_3 \),

\[
[y_{i,j}] = \begin{bmatrix}
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2}
\end{bmatrix}
\]

Consequently, when we add all the \( y_{i,j} \) along any left pandiagonal, we always get the sum to be

\[
\frac{n}{2} \left( \frac{5n}{4} + 1 + \frac{3n}{4} + 1 \right) = \frac{n^2}{2} + \frac{n}{2}.
\]

Let \( 1 \leq j \leq \frac{n}{4} \). When \( i \) is odd, we get

\[
d_{i,j} = n^2 + 2(j - 1)n,
d_{\frac{n}{4} + j} = n^2 - \left( \frac{n}{2} - 2(j - 1) \right) n.
\]

When \( i \) is even, we get

\[
d_{i,j} = n^2 - 2jn,
d_{\frac{n}{4} + j} = n^2 + \left( \frac{n}{2} - 2(j - 1) \right) n.
\]
For example, in the case of F3, we get

$$[d_{i,j}] =
\begin{array}{cccccccc}
192 & 208 & 224 & 240 & 0 & 16 & 32 & 48 \\
48 & 32 & 16 & 0 & 240 & 224 & 208 & 192 \\
192 & 208 & 224 & 240 & 0 & 16 & 32 & 48 \\
48 & 32 & 16 & 0 & 240 & 224 & 208 & 192 \\
192 & 208 & 224 & 240 & 0 & 16 & 32 & 48 \\
48 & 32 & 16 & 0 & 240 & 224 & 208 & 192 \\
192 & 208 & 224 & 240 & 0 & 16 & 32 & 48 \\
48 & 32 & 16 & 0 & 240 & 224 & 208 & 192 \\
\end{array}
+ \begin{array}{cccccccc}
0 & 16 & 32 & 48 & 192 & 208 & 224 & 240 \\
0 & 16 & 32 & 48 & 192 & 208 & 224 & 240 \\
0 & 16 & 32 & 48 & 192 & 208 & 224 & 240 \\
0 & 16 & 32 & 48 & 192 & 208 & 224 & 240 \\
0 & 16 & 32 & 48 & 192 & 208 & 224 & 240 \\
0 & 16 & 32 & 48 & 192 & 208 & 224 & 240 \\
0 & 16 & 32 & 48 & 192 & 208 & 224 & 240 \\
0 & 16 & 32 & 48 & 192 & 208 & 224 & 240 \\
\end{array}
+ \begin{array}{cccccccc}
64 & 64 & 64 & 64 & 64 & 64 & 64 & 64 \\
64 & 64 & 64 & 64 & 64 & 64 & 64 & 64 \\
64 & 64 & 64 & 64 & 64 & 64 & 64 & 64 \\
64 & 64 & 64 & 64 & 64 & 64 & 64 & 64 \\
64 & 64 & 64 & 64 & 64 & 64 & 64 & 64 \\
64 & 64 & 64 & 64 & 64 & 64 & 64 & 64 \\
64 & 64 & 64 & 64 & 64 & 64 & 64 & 64 \\
64 & 64 & 64 & 64 & 64 & 64 & 64 & 64 \\
\end{array}

= \begin{array}{cccccccc}
224 & 208 & 224 & 240 & 0 & 16 & 32 & 48 \\
256 & 208 & 224 & 240 & 0 & 16 & 32 & 48 \\
256 & 208 & 224 & 240 & 0 & 16 & 32 & 48 \\
256 & 208 & 224 & 240 & 0 & 16 & 32 & 48 \\
256 & 208 & 224 & 240 & 0 & 16 & 32 & 48 \\
256 & 208 & 224 & 240 & 0 & 16 & 32 & 48 \\
256 & 208 & 224 & 240 & 0 & 16 & 32 & 48 \\
256 & 208 & 224 & 240 & 0 & 16 & 32 & 48 \\
\end{array}
+ \begin{array}{cccccccc}
\frac{n^2}{2} & n^2 & 4n & n^2 + 2n & n^2 + 4n & n^2 + 6n & n^2 + 8n & n^2 + 10n \\
\frac{n^2}{2} & n^2 + 2n & n^2 + 4n & n^2 + 6n & n^2 + 8n & n^2 + 10n & n^2 + 12n & n^2 + 14n \\
\frac{n^2}{2} & n^2 + 4n & n^2 + 6n & n^2 + 8n & n^2 + 10n & n^2 + 12n & n^2 + 14n & n^2 + 16n \\
\frac{n^2}{2} & n^2 + 6n & n^2 + 8n & n^2 + 10n & n^2 + 12n & n^2 + 14n & n^2 + 16n & n^2 + 18n \\
\frac{n^2}{2} & n^2 + 8n & n^2 + 10n & n^2 + 12n & n^2 + 14n & n^2 + 16n & n^2 + 18n & n^2 + 20n \\
\frac{n^2}{2} & n^2 + 10n & n^2 + 12n & n^2 + 14n & n^2 + 16n & n^2 + 18n & n^2 + 20n & n^2 + 22n \\
\frac{n^2}{2} & n^2 + 12n & n^2 + 14n & n^2 + 16n & n^2 + 18n & n^2 + 20n & n^2 + 22n & n^2 + 24n \\
\frac{n^2}{2} & n^2 + 14n & n^2 + 16n & n^2 + 18n & n^2 + 20n & n^2 + 22n & n^2 + 24n & n^2 + 26n \\
\end{array}

Check that in the case of F1, we get

$$[d_{i,j}] =
\begin{array}{cccccccc}
\frac{n^2}{2} & \frac{n^2}{2} + 2n & \frac{n^2}{2} - 4n & \frac{n^2}{2} - 2n \\
\frac{n^2}{2} - 3n & \frac{n^2}{2} - 4n + 2n & \frac{n^2}{2} - 6n & \frac{n^2}{2} - 8n \\
\frac{n^2}{2} - 4n & \frac{n^2}{2} - 4n + 2n & \frac{n^2}{2} - 6n & \frac{n^2}{2} - 8n \\
\frac{n^2}{2} - 5n & \frac{n^2}{2} - 4n + 2n & \frac{n^2}{2} - 6n & \frac{n^2}{2} - 8n \\
\frac{n^2}{2} - 6n & \frac{n^2}{2} - 4n + 2n & \frac{n^2}{2} - 6n & \frac{n^2}{2} - 8n \\
\frac{n^2}{2} - 7n & \frac{n^2}{2} - 4n + 2n & \frac{n^2}{2} - 6n & \frac{n^2}{2} - 8n \\
\frac{n^2}{2} - 8n & \frac{n^2}{2} - 4n + 2n & \frac{n^2}{2} - 6n & \frac{n^2}{2} - 8n \\
\frac{n^2}{2} - 9n & \frac{n^2}{2} - 4n + 2n & \frac{n^2}{2} - 6n & \frac{n^2}{2} - 8n \\
\end{array}
$$

Consequently, when we add all the $d_{i,j}$ along a left pandiagonal, $P_{1,c}$, we get the sum to be

$$\frac{n^2}{2}n^2 - n^2, \text{ if } c \text{ is odd, and }$$

$$\frac{n^2}{2}n^2, \text{ if } c \text{ is even.}$$

Recall that the magic sum $M = \frac{n}{2}(n^2 + 1)$. The pandiagonal sum is the sum of all $s_{i,j}$ along a pandiagonal. Since $s_{i,j} = y_{i,j} + d_{i,j}$, for a left pandiagonal $P_{1,c}$ when $c$ is odd, the pandiagonal sum add to

$$\frac{n^2}{2} + \frac{n^2}{2} + \frac{n^2}{2} - n^2 = \frac{n^2}{2}(n^2 + 1) - \frac{n^2}{2} = M - \frac{n^2}{2}.$$ 

On the other hand, for a left pandiagonal $P_{1,c}$, when $c$ is even, the pandiagonal sum add to

$$\frac{n^2}{2} + \frac{n^2}{2} + \frac{n^2}{2} = \frac{n^2}{2}(n^2 + 1) + \frac{n^2}{2} = M + \frac{n^2}{2}.$$ 

For example, in the case of F1, the entries in the pandiagonal sum of $P_{1,1}$ is shown in bold below. Observe that the pandiagonal sum is $M - \frac{n^2}{2}$. 

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The entries of the pandiagonal sum of $P_{1,2}$ is shown in bold below. Check that the pandiagonal sum is $M + \frac{n^2}{2}$.

![Table](image)

The proof is similar for right pandiagonals. Thus, all the pandiagonals add to $M \pm \frac{n^2}{2}$.

Even though the Franklin square F2 is not constructed using the $N-i$ method, observe that all the pandiagonals of F2, also, add to $M \pm \frac{n^2}{2}$.

## 3 New method to construct Narayana square.

In this section, we develop the $N-i$ method to construct Narayana squares. We then show that the $N-i$ method is the same as the Chadya and Chadaka method of Section 1. Finally, we modify the $N-i$ method to create a new Narayana square.

We start by dividing the Narayana square in to two sides: the *left side* consisting of the first $n/2$ columns and the *right side* consisting of the last $n/2$ columns. Each side is further divided in to two parts: the *Top part* consisting of the first $n/2$ rows, and the *Bottom part* consisting of the last $n/2$ rows.

Let $N = n^2 + 1$. The strategy is to first place the numbers $i$, where $i = 1, 2, \ldots, n^2/2$, and then place the numbers $N-i$, such that all the defining properties of the square are satisfied.

*Distance of a column* in a given side is defined as the number of columns between the given column and the left edge of the side. Consequently, the distance of the $j$ th column of a side is $j - 1$.

The square is build, partially, two equidistant columns, one from each side of the square, at a time. Given a pair of equidistant columns, we denote the column in left side as $C_l$ and the column in the right side as $C_r$. 
Consider a given part with \( r \) rows and a starting number \( A \). There are only two operations involved for such a part. An Up operation where consecutive numbers from \( A \) to \( A + r \) are filled, in consecutive rows, starting from the bottom row of the part, and column \( C_l \), upwards, alternating between the columns \( C_l \) and \( C_r \). The only other operation is the Down operation where consecutive numbers from \( A \) to \( A + r \) are filled, in consecutive rows, alternating between the columns \( C_l \) and \( C_r \), starting from the top row of the part, and column \( C_r \), in a downward direction.

For a given pair of equidistant columns, the sequence of operations depends on the parity of the distance, and is as described below.

| Even distance | Odd distance |
|---------------|--------------|
| \( \begin{array}{c|c|c} \text{Part} & \text{Top} & \text{Bottom} \\ \hline \text{Operation} & Up & Down \\ \hline \text{Starting Number} & A & A + n/2 \\ \end{array} \) | \( \begin{array}{c|c|c} \text{Part} & \text{Bottom} & \text{Top} \\ \hline \text{Operation} & Up & Down \\ \hline \text{Starting Number} & A & A + n/2 \\ \end{array} \) |

This sequence will place \( n \) consecutive numbers in the chosen two columns. If the distance is \( d \), then the starting number for the entire sequence of operations, is \( nd + 1 \).

Finally, we complete the square by placing the numbers \( N - i, i = 1, 2, \ldots n^2/2 \) as follows. At this stage, the odd rows of the left part and the even rows of the right part are empty. Subtractions from \( N \) occur across diagonal parts. The odd rows of the top left part are obtained by subtracting the entries from the corresponding cells of the bottom right part, from \( N \). The odd rows of the bottom left part are obtained by subtracting the entries in the corresponding cells in the top right part, from \( N \). Similarly, the even rows of the top right part is obtained by subtracting the entries in the corresponding cells in the bottom left part, from \( N \). The even rows of the bottom right part is obtained by subtracting the entries in the corresponding cells in the top left part, from \( N \).

**Example 3.1.** In Table 14, the steps of filling the numbers 1 to 32 in the \( 8 \times 8 \) Narayana square \( N1 \) are shown. We start with the first columns of the left and right side. The distance of this pair from their respective left edges, is zero. Hence the starting number is \( nd + 1 = 8 \times 0 + 1 = 1 \). Here \( C_l \) is column 1, and \( C_r \) is column 5 of the square. The sequence of operation is

\( \begin{array}{c|c|c} \text{Part} & \text{Top} & \text{Bottom} \\ \hline \text{Operation} & Up & Down \\ \hline \text{Starting Number} & 1 & 5 \\ \end{array} \)  

That is, we enter the numbers from 1 to 4 in the top part, starting from \( C_l \), using the Up operation. Next we enter the numbers from 5 to 8 in the bottom part, starting from \( C_r \), using the Down operation. See Step 1 of Table 14.

In Step 2 of Table 14, we consider equidistant columns of distance 1. The starting number is \( nd + 1 = 8 \times 1 + 1 = 9 \). Since the distance is odd, the sequence of operation is

\( \begin{array}{c|c|c} \text{Part} & \text{Bottom} & \text{Top} \\ \hline \text{Operation} & Up & Down \\ \hline \text{Starting Number} & 9 & 13 \\ \end{array} \)
Table 14: Construction of $8 \times 8$ Narayana square N1: filling the numbers from 1 to 32.

| Step 1 | Step 2 |
|--------|--------|
| 3 14 1 16 | 3 14 1 16 |
| 1 16 4 2 | 1 16 4 2 |
| 5 15 2 10 | 5 15 2 10 |
| 8 9 6 11 | 8 9 6 11 |

Step 3

| Step 3 | Step 4 |
|--------|--------|
| 3 14 19 17 15 18 | 3 14 19 17 15 18 |
| 1 16 11 12 21 20 13 20 | 1 16 11 12 21 20 13 20 |
| 6 11 12 21 8 24 | 6 11 12 21 8 24 |
| 1 16 11 12 21 8 24 | 1 16 11 12 21 8 24 |

Table 15: Construction of Narayana square N1.

| N - 8 | N - 12 | N - 21 | N - 28 |
|-------|--------|--------|--------|
| 3     | 14     | 19     | 30     |
| N - 7 | N - 10 | N - 23 | N - 26 |
| 1     | 16     | 17     | 32     |
| N - 4 | N - 13 | N - 20 | N - 29 |
| 6     | 11     | 22     | 27     |
| N - 2 | N - 15 | N - 18 | N - 31 |
| 8     | 9      | 24     | 25     |

| N - 8 | N - 12 | N - 21 | N - 28 |
|-------|--------|--------|--------|
| 4     | 14     | 20     | 29     |
| N - 6 | N - 11 | N - 22 | N - 27 |
| 2     | 15     | 18     | 31     |
| N - 8 | N - 9  | N - 24 | N - 25 |
| N - 3 | N - 14 | N - 19 | N - 30 |
| 7     | 10     | 23     | 26     |
| N - 1 | N - 16 | N - 17 | N - 32 |

Thus, the numbers 9 to 16 are placed in the two columns using the above sequence of operations. Steps 3 and 4 demonstrate the placement of numbers from 17 to 32 in the rest of the columns of the square N1. See Table 16 for the placement of the numbers from 1 to 128 in the $16 \times 16$ Narayana square N2.

The filling of the empty cells in square N1 with $N - i$, where $i$ is the entry in the corresponding cell in a diagonal part, is given in Table 16. The final step of filling empty cells for the square N2 is given in Table 17.

Summarizing, we derive the following algorithm for constructing Narayana squares.

**Algorithm 3.1.** (Constructing $n \times n$ Narayana squares.)

1. Partial filling of the left side.

Start with the first two columns of each side, and then work outwards two equidistant columns at a time. Fill $n$ numbers at every step. We follow the sequence of operations according to the parity of the distance of the columns, as explained above.
Table 16: Partially filled rows of the $16 \times 16$ Narayana square N2.

| $N - 9$ | $N - 24$ | $N - 41$ | $N - 56$ | $N - 73$ | $N - 88$ | $N - 105$ | $N - 120$ |
|---------|----------|----------|----------|----------|----------|-----------|-----------|
| 7       | 26       | 39       | 58       | 71       | 90       | 103       | 122       |
| $N - 11$ | $N - 22$ | $N - 43$ | $N - 54$ | $N - 75$ | $N - 86$ | $N - 107$ | $N - 118$ |
| 5       | 28       | 37       | 60       | 69       | 92       | 101       | 124       |
| $N - 13$ | $N - 20$ | $N - 45$ | $N - 52$ | $N - 77$ | $N - 84$ | $N - 109$ | $N - 116$ |
| 3       | 30       | 35       | 62       | 67       | 94       | 99        | 126       |
| $N - 15$ | $N - 18$ | $N - 47$ | $N - 50$ | $N - 79$ | $N - 82$ | $N - 111$ | $N - 114$ |
| 1       | 32       | 33       | 64       | 65       | 96       | 97        | 128       |

| $N - 13$ | $N - 20$ | $N - 45$ | $N - 52$ | $N - 77$ | $N - 84$ | $N - 109$ | $N - 116$ |
|----------|----------|----------|----------|----------|----------|-----------|-----------|
| 9       | 24       | 41       | 56       | 73       | 88       | 105       | 120       |

Table 17: Final step in the construction of the $16 \times 16$ Narayana square N2.

$N - 13$ | $N - 20$ | $N - 45$ | $N - 52$ | $N - 77$ | $N - 84$ | $N - 109$ | $N - 116$ | $N - 118$ | $N - 119$ | $N - 120$ | $N - 121$ | $N - 122$ | $N - 123$ | $N - 124$ | $N - 125$ | $N - 126$ | $N - 127$ | $N - 128$

| 9       | 24       | 41       | 56       | 73       | 88       | 105       | 120       |

| $N - 13$ | $N - 20$ | $N - 45$ | $N - 52$ | $N - 77$ | $N - 84$ | $N - 109$ | $N - 116$ | $N - 118$ | $N - 119$ | $N - 120$ | $N - 121$ | $N - 122$ | $N - 123$ | $N - 124$ | $N - 125$ | $N - 126$ | $N - 127$ | $N - 128$

| 1       | 32       | 33       | 64       | 65       | 96       | 97        | 128       | 129       | 130       | 131       | 132       | 133       | 134       | 135       | 136       | 137       | 138       | 139       |

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2. Subtractions from $N$ to complete the square.

Let $a_{i,j}$ denote the entries of the square. Let $1 \leq i, j \leq n/2$.

The empty cells in the top part and left side of the square, that is, when $i$ is odd, are given by

$$a_{i,j} = N - a_{n/2+i, n/2+j}.$$ 

The empty cells in the bottom part and left side of the square, that is, when $i$ is odd, are given by

$$a_{n/2+i,j} = N - a_{i, n/2+j}.$$ 

The empty cells in the top part and right side of the square, that is, when $i$ is even, are given by

$$a_{i,n/2+j} = N - a_{n/2+i,j}.$$ 

The empty cells in the bottom part and right side of the square, that is, when $i$ is even, are given by

$$a_{n/2+i, n/2+j} = N - a_{i,j}.$$ 

It will be established, soon, that Algorithm 3.1 is the same as Narayana’s original Chadya-Chadaka method of construction. Since this is an ancient, well known method, Algorithm 3.1 needs no proof. However, proving that the Algorithm works, gives us an opportunity to explore many interesting properties of the square.

**Lemma 3.1.** Let $a_{i,j}$ denote the entries of a $n \times n$ Narayana square. If the number of rows or columns of two entries, from a given axis, is the same, the entries are called equidistant.

1. Pair of entries of adjacent rows in a column add to $N \pm 2i$ except for the rows $i = n/2$ and $i = n$, as follows.

   Consider a row $i \in \{1, 2, \ldots, n\} \setminus \{n/2, n\}$. In the top part of the square, that is, when $1 \leq i < n/2$, we have

   $$a_{i,j} + a_{i+1,j} = \begin{cases} N - 2i, & \text{if } j \text{ is odd,} \\ N + 2i, & \text{if } j \text{ is even.} \end{cases}$$

   For the bottom part, the situation is exactly opposite. That is, when $n/2 < i < n$, we have

   $$a_{i,j} + a_{i+1,j} = \begin{cases} N - 2i, & \text{if } j \text{ is even,} \\ N + 2i, & \text{if } j \text{ is odd.} \end{cases}$$

   Finally, we consider rows $n/2$ and $n$.

   $$a_{n/2,j} + a_{n/2+1,j} = \begin{cases} N - (n/2 - 1), & \text{if } j \text{ is odd,} \\ N + (n/2 - 1), & \text{if } j \text{ is even.} \end{cases}$$

   $$a_{n,j} + a_{1,j} = \begin{cases} N - (n/2 - 1), & \text{if } j \text{ is even,} \\ N + (n/2 - 1), & \text{if } j \text{ is odd.} \end{cases}$$
2. Let \( m = n^2/2 + 1 \), then equidistant entries from center in half rows add to \( m \) or \( 2N - m \) as described below. Consider the left side, that is \( 1 \leq j \leq n/2 \),

\[
a_{i,j} + a_{i,n/2+1-j} = \begin{cases} 
m, & \text{if } i \text{ is even}, \\
2N - m, & \text{if } i \text{ is odd}.
\end{cases}
\]

For the right side, where \( n/2 < j \leq n \),

\[
a_{i,j} + a_{i,n+1-j} = \begin{cases} 
m, & \text{if } i \text{ is odd}, \\
2N - m, & \text{if } i \text{ is even}.
\end{cases}
\]

3. Equidistant entries from center in half columns add to \( N \pm n/2 \) as follows. For top part, that is, \( 1 \leq i \leq n/2 \),

\[
a_{i,j} + a_{\frac{n}{2}+1-i,j} = \begin{cases} 
N - \frac{n}{2}, & \text{if } j \text{ is odd}, \\
N + \frac{n}{2}, & \text{if } j \text{ is even}.
\end{cases}
\]

For bottom part, that is, \( n/2 < i \leq n \),

\[
a_{i,j} + a_{\frac{n}{2}+1-i,j} = \begin{cases} 
N - \frac{n}{2}, & \text{if } j \text{ is even}, \\
N + \frac{n}{2}, & \text{if } j \text{ is odd}.
\end{cases}
\]

4. Let \( m_i = n/2 - 1 - 2(i - 1) \), for \( 1 \leq i \leq n/2 \). Equidistant entries across the horizontal axis add to \( N \pm m_i \) as follows.

When \( 1 \leq i \leq n/4 \),

\[
a_{i,j} + a_{n+1-i,j} = \begin{cases} 
N + m_i, & \text{for } j \text{ odd}, \\
N - m_i, & \text{for } j \text{ even}.
\end{cases}
\]

When \( n/4 + 1 \leq i \leq n/2 \),

\[
a_{i,j} + a_{n+1-i,j} = \begin{cases} 
N - m_i, & \text{for } j \text{ odd}, \\
N + m_i, & \text{for } j \text{ even}.
\end{cases}
\]

5. Let \( m_j = n/2 - 1 - 2(j - 1) \), where \( 1 \leq j \leq n/2 \). Equidistant entries across the vertical axis add to \( N \pm m_j n \), as shown below.

When \( 1 \leq j \leq n/4 \),

\[
a_{i,j} + a_{i,n+1-j} = \begin{cases} 
N + m_j n, & \text{for } i \text{ odd}, \\
N - m_j n, & \text{for } i \text{ even}.
\end{cases}
\]
When \( n/4 + 1 \leq j \leq n/2 \),

\[
a_{i,j} + a_{i,n+1-j} = \begin{cases} 
N - m_j n, & \text{for } i \text{ odd}, \\
N + m_j n, & \text{for } i \text{ even}.
\end{cases}
\]

**Proof.** The square inherits these properties by construction. \( \square \)

**Corollary 3.1.** Consider an \( n \times n \) Narayana square. Let \( M \) denote the magic sum and let \( m = n^2/2 + 1 \). Then,

1. Half row sums add either to \((n/4)m\) or \(M - (n/4)m\).
2. Half column sums add to \(M/2 \pm n^2/8\).

**Proof.**

1. By Part 2 of Lemma 3.1, adding the \( n/4 \) equidistant pairs in half rows, we get

\[
a_{i,1} + a_{i,2} + \cdots + a_{i,\frac{n}{2}} = \begin{cases} 
M - (\frac{n}{4})m, & \text{if } i \text{ is odd}, \\
(\frac{n}{4})m, & \text{if } i \text{ is even}.
\end{cases}
\]

\[
a_{i,\frac{n}{2}+1} + a_{i,\frac{n}{2}+2} + \cdots + a_{i,n} = \begin{cases} 
M - (\frac{n}{4})m, & \text{if } i \text{ is even}, \\
(\frac{n}{4})m, & \text{if } i \text{ is odd}.
\end{cases}
\]

Consequently, for odd rows, left half row sums add to \( M - (n/4)m \), and right half row sums add to \((n/4)m\). On the other hand, for even rows, left half row sums add to \((n/4)m\), and right half row sums add to \( M - (n/4)m \).

2. By Part 3 of Lemma 3.1, when \( j \) is odd, and \( 1 \leq i \leq n/2 \),

\[
a_{i,j} + a_{\frac{n}{2}+1-i,j} = \begin{cases} 
N - \frac{n}{2}, & 1 \leq i \leq \frac{n}{2} \\
N + \frac{n}{2}, & \frac{n}{2} < i \leq n.
\end{cases}
\]

Therefore, for odd \( j \), the top half columns add to

\[
\frac{n}{4} \left( N - \frac{n}{2} \right) = \frac{M}{2} - \frac{n^2}{8},
\]

and the bottom half columns add to \( M/2 + n^2/8 \).

Similarly, by Part 3 of Lemma 3.1, we get, when \( j \) is even, the top half columns add to \( M/2 + n^2/8 \), and the bottom half columns add to \( M/2 - n^2/8 \).

Thus, half column sums add to \( M/2 \pm n^2/8 \). \( \square \)
Proposition 3.1. Algorithm 3.1 produces a Narayana square.

Proof. Let $a_{i,j}$ denote the entries of a $n \times n$ square constructed by Algorithm 3.1.

1. $2 \times 2$ sub-square sums.

Consider a row $i \in \{1, 2, \ldots, n\} \setminus \{n/2, n\}$.

By Part 1 of Lemma 3.1,

if $a_{i,j} + a_{i+1,j} = N + 2i$, then $a_{i,j+1} + a_{i+1,j+1} = N - 2i$.

On the other hand,

if $a_{i,j} + a_{i+1,j} = N - 2i$, then $a_{i,j+1} + a_{i+1,j+1} = N + 2i$.

Consequently, for all $i \in \{1, 2, \ldots, n\} \setminus \{n/2, n\}$ and all $j$,

$a_{i,j} + a_{i+1,j} + a_{i+1,j} + a_{i+1,j+1} = 2N$.

Now we consider the row $n/2$. By Part 1 of Lemma 3.1 we get

if $a_{n/2,j} + a_{n/2+1,j} = N + (n/2 - 1)$, then $a_{n,j} + a_{n,j+1} = N - (n/2 - 1)$,

if $a_{n/2,j} + a_{n/2+1,j} = N - (n/2 - 1)$, then $a_{n,j} + a_{n,j+1} = N + (n/2 - 1)$.

Consequently, all the $2 \times 2$ sub-squares, within the Narayana square, add to $2N$.

Next, we verify the continuity of this property.

By Part 4 of Lemma 3.1,

if $a_{n,j} + a_{1,j} = N + (n/2 - 1)$ then $a_{n,j+1} + a_{n,j+1} = N - (n/2 - 1)$ and

if $a_{n,j} + a_{1,j} = N - (n/2 - 1)$ then $a_{n,j+1} + a_{n,j+1} = N + (n/2 - 1)$.

Consequently, the $2 \times 2$ sub-squares formed by rows 1 and $n$ add to $2N$. Part 5 of Lemma 3.1 implies

$$a_{i,1} + a_{i,n} = \begin{cases} N + (n/2 - 1)n, & \text{if } i \text{ is even,} \\ N - (n/2 - 1)n, & \text{if } i \text{ is odd.} \end{cases}$$

Thus, the $2 \times 2$ sub-squares formed by columns 1 and $n$ add to $2N$. This proves that the continuity property of $2 \times 2$ sub-squares hold for squares constructed by Algorithm 3.1.
2. Row and column sums.

By Corollary 3.1 when \( i \) is odd, left half row sums add to \( M - (n/4)m \), and right half row sums add to \((n/4)m\). Consequently, when \( i \) is odd the \( i \)-th row sum is \( M \).

Similarly, applying Corollary 3.1 we see that necessary cancellations happen, when we add row half sums and column half sums, to form row sums and column sums, respectively. Consequently, row and column sums add to \( M \).

3. Pandiagonal sums.

Let \( a_{i,j} \) belong to a pandiagonal. Then, as we saw in Lemma 2.2, if \( 1 \leq j \leq n/2 \), then \( a_{n/2+i,n/2+j} \) also belongs to the pandiagonal. On the other hand if \( j > n/2 \), then \( a_{n/2+i,j-n/2} \) belongs to the pandiagonal. That is, every pandiagonal is made up of \( n/2 \) paired entries.

Consider \( 1 \leq i, j \leq n/2 \). By Step 2 of Algorithm 3.1, we get
\[
a_{i,j} + a_{n/2+i,n/2+j} = N, \\
a_{i,n/2+j} + a_{n/2+i,j} = N.
\]

Consequently, every pandiagonal sum adds to \((n/2)N = M\).

Thus, Algorithm 3.1 produces a Narayana square.

Lemma 3.2. Left and right bend diagonal sums add to \( M \pm n/2 \). Top and bottom bend diagonal sums add to \( M \pm n^2/2 \).

Proof. Let \( 1 \leq j \leq n/2 + 1 \), then, if \( j \) is odd, then by Part 4 of Lemma 3.1 the left bend diagonal sum starting with row 1 and column \( j \) adds as follows.

\[
\left[ (a_{1,j} + a_{n,j}) + (a_{2,j+1} + a_{n-1,j+1}) + \cdots + (a_{2n/4,j+n/4} - 1 + a_{2n/4+1,j+n/4} - 1) \right] \\
+ \left[ (a_{2n/4+1,j+n/4} + a_{3n/4,j+n/4}) + \cdots + (a_{n/2,j+n/2-1} + a_{n/2+1,j+n/2-1}) \right] \\
= \left[ (N + (n/2 - 1)) + (N - (n/2 - 3)) + (N + (n/2 - 5)) + \cdots + (N + 3) + (N - 1) \right] \\
+ \left[ (N - 1) + (N + 3) + \cdots + (N - (n/2 - 3)) + (N + (n/2 - 1)) \right] \\
= 2 \left[ (N + (n/2 - 1)) + (N - (n/2 - 3)) + (N + (n/2 - 5)) + \cdots + (N + 3) + (N - 1) \right] \\
= 2 \left[ \frac{n}{4}N + \frac{n}{2} - 1 - \frac{n}{2} + 3 + \frac{n}{2} - 5 + \cdots + 3 - 1 \right] \\
= 2 \left[ \frac{n}{2}N + \frac{n}{8} \times 2 \right] \\
= \frac{n}{2}N + \frac{n}{2} = M + \frac{n}{2}.
\]
A similar argument gives us that when \( j \) is even, the left bend diagonals add to \( M - n/2 \).

Now, let \( n/2 + 1 < j \leq n \), the left bend diagonal sums are

\[
(a_{i,j} + a_{n,j}) + (a_{2,j} + a_{n-1,j+1}) + \cdots + (a_{n-j+1,n} + a_{j,n})
\]

\[
+ (a_{n-j+2,1} + a_{j-1,1}) + (a_{n-j+3,2} + a_{j-2,2}) + \cdots + (a_{\frac{n-j}{2},1 - \frac{n-j}{2}} + a_{\frac{n}{2} + 1, j - \frac{n-j}{2} - 1}).
\]

Check that, by Part 4 of Lemma 3.1, we get that these sums also add to \( M \pm n/2 \).

For example, in the case of \( N1 \) (see Table 15), the seventh bend diagonal sum is

\[
(a_{1,7} + a_{8,7}) + (a_{2,8} + a_{7,8}) + (a_{3,1} + a_{6,1}) + (a_{4,2} + a_{5,2})
\]

\[
= (N + 3) + (N - 1) + (N - 1) + (N + 3) = 4N + 4 = M + \frac{n}{2}.
\]

Thus, the left bend diagonals add to \( M \pm n/2 \), continuously. The proof that all right bend diagonals add to \( M \pm n/2 \), is similar to the case of left bend diagonals. The proof depends, mainly, on the fact that equidistant entries across the horizontal axis add to either \( N + m_i \) or \( N - m_i \), and all \( m_i \) cancel in the final sum.

Equidistant entries across the vertical axis is used to prove that the top and bottom bend diagonals add to magic sum. By Part 5 of Lemma 3.1, pairs of equidistant entries across the vertical axis add to \( N - m_i n \) or \( N + m_i n \).

For \( 1 \leq i \leq n/2 + 1 \), let \( i \) be odd, then the \( i \)-th top bend diagonal sum is given below.

\[
\left[ (a_{i,1} + a_{i,n}) + (a_{i+1,2} + a_{i+1,n-1}) + \cdots + (a_{i + \frac{n}{4} - 1, \frac{n}{4}} + a_{i + \frac{3n}{4} - 1, \frac{3n}{4} + 1}) \right]
\]

\[
+ \left[ (a_{i + \frac{n}{4}, \frac{n}{4} + 1} + a_{i + \frac{3n}{4}, \frac{3n}{4} + 1}) + \cdots + (a_{i + \frac{n}{2} - 1, \frac{n}{2}} + a_{i + \frac{n}{2}, \frac{n}{2}}) \right]
\]

\[
= \left[ (N + (\frac{n}{2} - 1)n) + (N - (\frac{n}{2} - 3)n) + \cdots (N + 3n) + (N - n) \right]
\]

\[
+ \left[ (N - n) + (N + 3n) + (N - (\frac{n}{2} - 3)n) + (\frac{n}{2} - 1)n) + (N + (\frac{n}{2} - 1)n) \right]
\]

\[
= 2 \left[ \frac{n}{4} N + \frac{n}{8}(2n) \right]
\]

\[
= \frac{n}{2} N + \frac{n^2}{2} = M + \frac{n^2}{2}.
\]

When \( i \) is even, it can be checked that the the \( i \)-th top bend diagonal sum is \( M - n^2/2 \). Thus, the top bend diagonal sums add to \( M \pm n^2/2 \) when \( 1 \leq i \leq n/2 + 1 \).

For \( n/2 + 1 < i \leq n \), the top bed diagonal sums are

\[
(a_{i,1} + a_{i,n}) + (a_{i+1,2} + a_{i+1,n-1}) + \cdots (a_{n,n-i+1} + a_{n,i})
\]

\[
+ (a_{1,n-i+2} + a_{1,i-1}) + (a_{2,n-i+3} + a_{2,i-2}) + \cdots + (a_{i - \frac{n}{2} - 1, \frac{n}{2}} + a_{i - \frac{n}{2}, \frac{n}{2} + 1})
\]
Again, by Part 5 of Lemma 3.1 it can be checked that the top bed diagonal sums add to \( M + n^2/2 \).

For example, in the case of N1 (see Table 15), the seventh top bend diagonal sum is 

\[
(a_{7,1} + a_{7,8}) + (a_{8,2} + a_{8,7}) + (a_{1,3} + a_{1,6}) + (a_{2,4} + a_{2,5})
\]

\[
= (N + 3n) + (N - n) + (N - n) + (N + 3n) = 4N + 4n = M + n^2/2.
\]

Consequently, the top bend diagonals add to \( M + n^2/2 \), continuously. A similar proof, applying Part 5 of Lemma 3.1 shows that the bottom bend diagonals add to \( M + n^2/2 \), continuously.

We proceed to show that the \( N - i \) method is the same as Narayana Pandit’s Chadya-Chadaka method described in Section 1. We use the example of the Narayana square N2 to demonstrate our derivation. This process is very similar to the derivation of the Chadya and Chadaka squares of Franklin squares in Section 2. Since 16 numbers are filled at every step, Table 16 can be rewritten in terms of multiples of 16 as shown below.

\[
\begin{array}{cccccccccccccccc}
7 & 10 & 16 & 7 & 13 & 10 & 48 & 7 & 64 & 10 & 80 & 7 & 96 & 10 & 112
\
10 & 16 & 5 & 32 & 12 & 48 & 5 & 64 & 12 & 80 & 5 & 96 & 12 & 112
\
14 & 32 & 14 & 48 & 3 & 64 & 14 & 80 & 3 & 96 & 14 & 112
\
1 & 16 & 1 & 32 & 16 & 48 & 1 & 64 & 16 & 80 & 1 & 96 & 16 & 112
\
8 & 16 & 1 & 96 & 16 & 32 & 9 & 48 & 8 & 64 & 9 & 80 & 8 & 96 & 9 & 112
\
6 & 1 & 96 & 6 & 32 & 11 & 48 & 6 & 64 & 11 & 80 & 6 & 96 & 11 & 112
\
4 & 13 & 4 & 32 & 13 & 48 & 4 & 64 & 13 & 80 & 4 & 96 & 13 & 112
\
2 & 15 & 16 & 2 & 32 & 13 & 48 & 2 & 64 & 15 & 80 & 2 & 96 & 15 & 112
\end{array}
\]

Since \( N - i - rN = (n^2 + 1) - r(n + 1 - i) \), Table 17 becomes

\[
\begin{array}{cccccccccccccccc}
8 & 9 & 16 & 8 & 32 & 9 & 48 & 8 & 64 & 9 & 80 & 8 & 96 & 9 & 112
\
6 & 1 & 96 & 6 & 32 & 11 & 48 & 6 & 64 & 11 & 80 & 6 & 96 & 11 & 112
\
4 & 13 & 4 & 32 & 13 & 48 & 4 & 64 & 13 & 80 & 4 & 96 & 13 & 112
\
2 & 15 & 16 & 2 & 32 & 13 & 48 & 2 & 64 & 15 & 80 & 2 & 96 & 15 & 112
\end{array}
\]

\[
\begin{array}{cccccccccccccccc}
9 & 16 & 9 & 32 & 8 & 48 & 9 & 64 & 8 & 80 & 9 & 96 & 8 & 112
\
11 & 6 & 16 & 11 & 32 & 6 & 48 & 11 & 64 & 6 & 80 & 11 & 96 & 6 & 112
\
13 & 4 & 16 & 13 & 32 & 4 & 48 & 13 & 64 & 4 & 80 & 13 & 96 & 4 & 112
\
15 & 2 & 16 & 15 & 32 & 2 & 48 & 15 & 64 & 2 & 80 & 15 & 96 & 2 & 112
\end{array}
\]

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Consequently, the square can be split as the Chadya and flipped Chadaka of $N_2$.

| Chadya | Flipped Chadaka |
|--------|-----------------|
| 8 9 8 5 8 9 8 9 8 9 8 9 8 9 8 9 | 230 224 208 192 176 160 144 128 0 16 32 48 64 80 96 112 |
| 7 10 7 10 7 10 7 10 7 10 7 10 7 10 7 10 | 0 16 32 48 64 80 96 112 240 224 208 192 176 160 144 128 |
| 6 11 6 11 6 11 6 11 6 11 6 11 6 11 6 11 | 0 16 32 48 64 80 96 112 240 224 208 192 176 160 144 128 |
| 5 12 5 12 5 12 5 12 5 12 5 12 5 12 5 12 | 0 16 32 48 64 80 96 112 240 224 208 192 176 160 144 128 |
| 4 13 4 13 4 13 4 13 4 13 4 13 4 13 4 13 | 0 16 32 48 64 80 96 112 240 224 208 192 176 160 144 128 |
| 3 14 3 14 3 14 3 14 3 14 3 14 3 14 3 14 | 0 16 32 48 64 80 96 112 240 224 208 192 176 160 144 128 |
| 2 15 2 15 2 15 2 15 2 15 2 15 2 15 2 15 | 0 16 32 48 64 80 96 112 240 224 208 192 176 160 144 128 |
| 1 16 1 16 1 16 1 16 1 16 1 16 1 16 1 16 | 0 16 32 48 64 80 96 112 240 224 208 192 176 160 144 128 |
| 9 8 9 8 9 8 9 8 9 8 9 8 9 8 9 8 | 0 16 32 48 64 80 96 112 240 224 208 192 176 160 144 128 |
| 10 7 10 7 10 7 10 7 10 7 10 7 10 7 10 7 | 0 16 32 48 64 80 96 112 240 224 208 192 176 160 144 128 |
| 11 6 11 6 11 6 11 6 11 6 11 6 11 6 11 6 | 0 16 32 48 64 80 96 112 240 224 208 192 176 160 144 128 |
| 12 9 12 9 12 9 12 9 12 9 12 9 12 9 12 9 | 0 16 32 48 64 80 96 112 240 224 208 192 176 160 144 128 |
| 13 14 13 14 13 14 13 14 13 14 13 14 13 14 13 14 | 0 16 32 48 64 80 96 112 240 224 208 192 176 160 144 128 |
| 14 15 14 15 14 15 14 15 14 15 14 15 14 15 14 15 | 0 16 32 48 64 80 96 112 240 224 208 192 176 160 144 128 |
| 15 16 15 16 15 16 15 16 15 16 15 16 15 16 15 16 | 0 16 32 48 64 80 96 112 240 224 208 192 176 160 144 128 |
| 16 1 16 1 16 1 16 1 16 1 16 1 16 1 16 1 | 0 16 32 48 64 80 96 112 240 224 208 192 176 160 144 128 |

Thus, the $N - i$ method is the same as the original Chadya-Chadaka method of Narayana (see Example 1.2). However, the $N - i$ method can be easily modified to create new Narayana squares.

**Example 3.2. Constructing New Narayana Square.**

We first enter the numbers 1 to 32 as shown below. We begin in the second column of the last row of the square. This step is slightly different from Step 1 in Algorithm 3.1.

```
| - | - | - | 28 | 5 | 12 | 21 |
| 27 | 0 | 11 | 22 | - | - | - |
| 26 | 14 | - | - | 32 | 10 | 23 |
| - | - | - | 29 | 4 | 13 | 20 |
| 30 | 3 | 14 | 19 | - | - | - |
| 31 | 2 | 15 | 18 | - | - | - |
```

Next we do the necessary subtractions from $N$. This step is the same as in Algorithm 3.1.
Thus, we get a new square which can be checked to be a new Narayana square.

\[
\begin{array}{cccccccc}
36 & 61 & 52 & 45 & 28 & 5 & 12 & 21 \\
27 & 6 & 11 & 22 & 35 & 62 & 61 & 46 \\
34 & 63 & 50 & 47 & 26 & 7 & 10 & 23 \\
25 & 8 & 9 & 24 & 33 & 64 & 49 & 48 \\
37 & 60 & 55 & 44 & 29 & 4 & 13 & 20 \\
30 & 3 & 14 & 19 & 38 & 59 & 54 & 43 \\
39 & 58 & 56 & 42 & 31 & 2 & 15 & 18 \\
32 & 1 & 16 & 17 & 40 & 57 & 56 & 41
\end{array}
\]

Also verify that the new Narayana square has the following additional properties: Half row sums add either to \((n/4)m\) or \(M - (n/4)m\); Half column sums add to \(M/2 \pm n^2/8\); Left and right bend diagonal sums add to \(M \pm n^2/2\); Top and bottom bend diagonal sums add to \(M \pm n^2/2\).

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