Near Heisenberg limited parameter estimation precision by a dipolar Bose gas reservoir engineering

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(Dated: October 11, 2018)

We propose a scheme to obtain the Heisenberg limited parameter estimation precision by immersing atoms in a thermally equilibrated quasi-one-dimensional dipolar Bose-Einstein condensate reservoir. We show that the collisions between the dipolar atoms and the immersed atoms can result in a controllable nonlinear interaction through tuning the relative strength and the sign of the dipolar and contact interaction. We find that the repulsive dipolar interaction reservoir is preferential for the spin squeezing and the appearance of an entangled non-Gaussian state. As an useful resource for quantum metrology, we also show that the non-Gaussian state results in the phase estimation precision in the Heisenberg scaling, outperforming that of the spin-squeezed state.

PACS numbers:

I. INTRODUCTION

One of main goals of quantum metrology is to achieve parameter (or phase) estimation precision beyond the shot-noise limit. Atomic spin squeezed states (SSSs) play an important role in quantum phase estimation and have been widely studied in the past few decades ever since the pioneer work of Kitagawa and Ueda, who showed that the SSSs can be dynamically generated from the so-called one-axis twisting interaction (OAT) among spin-1/2 particles. As useful quantum resource, the SSSs have been proposed to achieve such a sub-shot-noise limited phase sensitivity experimentally. Recently, Stroble et al. have demonstrated that the OAT can also generate entangled non-Gaussian states (ENGSs), which can outperform the spin-squeezed state. The OAT interaction has been proposed and demonstrated using ion traps. Rydeberg atoms, nitrogen-vacancy centers and atomic Bose-Einstein condensates (BECs).

BECs due to their unique coherence properties and the controllable nonlinearity, have attracted much attention for quantum metrology. Experimental realizations of the OAT model has been proposed and demonstrated through Feshbach resonances or spatially separating the components of BECs. Besides, the atomic BECs also often as the reservoirs suitable for engineering is considered widely. For instance, one can drastically enhance the OAT interaction by placing a two-state condensate in a completely different special BEC reservoir.

So far, the studies of bosonic atoms for metrology are mainly focused on s-wave contact interaction. However, for ultracold atoms there also exists long-rang magnetic dipole-dipole interaction (MDDI). In experiments, dipolar BECs have been realized for atoms with large magnetic dipole moments. Furthermore, both the sign and the strength of the effective dipolar interaction can be tuned via a fast rotating orienting field. Very recently, Yuan et al. have used the quasi-2D dipolar BEC as reservoir engineering to study the non-Markovian dynamics of an impurity atom. Therefore, the effects of the MDDI should be considered in the realizations of the OAT model based on dipolar BEC.

In this paper, we realize the OAT model induced by the reservoir dephasing noise, which has been widely viewed as one of the main obstacles for quantum metrology. We consider the dynamics of two-mode BEC consisting of N atoms coupled to a one-dimensional (1D) dipolar Bose gas reservoir. It show that, the collisions interaction between the dipolar BEC reservoir and the immersed atoms can be described by a spin-boson model. Through calculating the SS, ENGSs can last for a very long time under the dephasing noise. It can monotonically increase in the regimes without SS ($\xi_R > 1$), next successively undergoes metastable entangled states and entanglement suddenly increase, corresponding to $F_Q \approx N^2/2$ and $F_Q \sim N^2$, respectively. According to Cramér-Rao theorem, $\Delta \theta \geq 1/\sqrt{F_Q}$, we know that $F_Q > N$ means that the states are entangled and useful for sub-shot-noise-limited phase-estimation precision; and $F_Q = N^2$ is the maximal entangled states, corresponding to the Heisenberg limit. This confirms that the phase estimation sensitivity can approach to Heisenberg limit, when using ENGSs for metrology.

The paper is organized as follows. In Sec. II, we give the model of immersed atoms interacting with the thermally equilibrated quasi-1D dipolar BEC reservoir. In Sec. III, we study the dynamics evolution of the atoms due to the dephasing...
noise. The SS and entanglement dynamical behaviors are discussed in Sec. IV and Sec. V. Finally, we draw our conclusion in Sec. VI.

FIG. 1: Schematic diagrams of (a) $N$ two-level atoms (red) immersed in a quasi-1D dipolar gas (green) and (b) the tuning of the dipole-dipole interaction via a fast rotating orienting field. (d)-(e) show the Wigner function of the initial coherent spin state, the spin squeezed state, and entangled non-Gaussian state (similar to the spin cat state) for the two-level atom system. The negative values of Wigner function correspond to the quantum states.

II. FORMULATION

We consider a system of $N$ two states $^{87}\text{Rb}$ atoms with up $|\uparrow\rangle$ and down states $|\downarrow\rangle$ immersed in a quasi-1D dipolar gas reservoir. And the system atoms are confined in a harmonic trap that is independent of the internal states [see Fig 1(a)]. In general, the interaction between the system atoms and the reservoir is described by the Hamiltonian

$$H = H_A + H_B + H_{AB},$$

(1)

where $H_A$ is the two-state atomic Hamiltonian, $H_B$ is the dipolar gas reservoir Hamiltonian, and $H_{AB}$ describes their interaction.

A. Two level atom system

The spin Hamiltonian provides the most intuitive description in the internal case of two-mode trapped atomic BEC

$$H_A = \lambda J_z + \chi J_z^2.$$

(2)

Here, we define the pseudo-spin operator $\boldsymbol{J} \equiv (J_x, J_y, J_z)$ based on space orbitals as $\boldsymbol{J} = (a_1^*, a_1)\sigma(\alpha_1, a_1)^T/2$, where $\sigma$ is the Pauli matrices, and $\tilde{a}_1$ denotes the annihilation operators of the atom states. The energy difference of the two states $\lambda$ and nonlinearity $\chi$ depend on the mean filed wave-function of the two modes. We assume that the two modes have the same spatial orbital

$$\Phi_M = (\pi\ell_A^2)^{-3/4}e^{-\left((x^2+y^2+z^2)/(2\ell_A^2)\right)}.$$

(3)

where $\ell_A = \sqrt{n/(m_A\omega_A)}$ with $\omega_A$ being the trap frequency, and $m_A$ being the mass of atom. Therefore, $\chi = (g_{11} + g_{22} - 2g_{12})/[2(2\pi)^{1/2}\ell_A^2]$ with coupling constants $g_{ij} = 4\pi\hbar^2\alpha_{ij}/m_A$ and $\alpha_{ij}$ being $s$-wave scattering length. For $^{87}\text{Rb}$ and the chosen hyperfine states, $\alpha_{ij}$ are almost equal: $a_{11} = 100.44 a_0, a_{22} = 95.47 a_0$ and $a_{12} = 97.7 a_0$, with $a_0$ being the Bohr radius. Then, the nonlinearity $\chi$ is close to zero. We point out that $\chi$ is tunable via Feshbach resonance, but the price of these methods is significantly increased atom losses [59]. Below, we choose $\chi = 0$, and apply a dipolar gas reservoir to induce a stronger nonlinearity interaction.

B. Bogoliubov modes of quasi-1D dipolar gas reservoir

In second-quantized form, the many-body Hamiltonian of the 1D dipolar BEC is

$$H_B = \int dx \hat{\Psi}_B^\dagger(x)\hat{H}\Psi_B(x) + \frac{1}{2} \int dx dx' \hat{\Psi}_B^\dagger(x')\hat{V}(x-x')\hat{\Psi}_B(x')\hat{\Psi}_B(x).$$

(4)

where $\hat{\Psi}_B(x)$ is the field operator and $\hat{h} = -\frac{\hbar^2}{2m_B}\frac{\partial}{\partial x}$ is the single-particle Hamiltonian with $m_B$ being the mass of the atom. Here, we have assumed that the dipolar BEC to be confined in a cylindrically symmetric trap with a transverse trapping frequency $\omega_L$ and negligible longitudinal confinement $\omega_Z$, along the $x$ direction, i.e., $\omega_L/\omega_Z \gg 1$. In three dimensions, the two-body interaction is

$$V^{3D}(r) = g_B\delta(r) + \frac{3c_{dd}d_1 - 3(\mu_m \cdot \hat{r})^2}{4\pi r^3},$$

(5)

where the contact interaction strength is $g_B = 4\pi\hbar^2 a_B/m_B$ with $a_B$ being the $s$-wave scattering length; the dipolar interaction strength is $c_{dd} = 4\pi\hbar^2 a_{dd}/m_B$, where $a_{dd} = \mu_0\mu_m^2 m_B/(12\pi\hbar^2)$ is a length scale characterizing the MDDI with $\mu_0$ the vacuum permeability, $\mu_m$ the magnetic dipole moment; here $\hat{r} = r/r$ is a unit vector.

To obtain the effective 1D interaction potential, $V(x-x')$, in Hamiltonian [4]. We assume that the transverse wave function of all the reservoir atoms is

$$\Psi_L(y, z) = (\pi\ell_B^2)^{-1/2}e^{-\left((y^2+z^2)/(2\ell_B^2)\right)},$$

(6)

with $\ell_B = \sqrt{n/(m_B\omega_B)}$ being the width of the Gaussian function. By integrating out the $y$ and $z$ variables, we can obtain the Fourier transform of the 1D interaction potential (as shown in Appendix A)

$$\tilde{V}_{1D}(k) = \frac{g_B}{2\pi\ell_B} \left[ 1 - \epsilon_{dd}\tilde{v}(k) \right]$$

(7)
with $\epsilon_{dd} \equiv \epsilon_d/g_B = a_{dd}/a_B$, where

$$\bar{\epsilon}(k) = 1 - \frac{3}{2} \frac{k^2}{\ell_B^2} \exp \left[ \frac{k^2}{2} \frac{\ell_B^2}{\ell} \right] \Gamma \left( 0, \frac{k^2 \ell_B^2}{2} \right),$$

(8)

with $\Gamma(0, x)$ being the incomplete Gamma function.

To proceed, in the degenerate regime, the bosonic field can be decomposed as

$$\hat{\Psi}_B(r) = \Psi_\perp(y, z) \left[ \sqrt{n_0} + \frac{1}{\sqrt{L}} \sum_k \left( u_k \hat{b}_k e^{i\delta s_k} - v_k \hat{b}_k^* e^{-i\delta s_k} \right) \right],$$

(9)

with $n_0$ being the condensate linear density, $L$ the length of the reservoir, $\hat{b}_k (\hat{b}_k^*)$ the annihilation (creation) operators of the Bogoliubov modes with momentum $k$. And its Bogoliubov modes are

$$u_k = 1/2 \left( \sqrt{\epsilon_k/E_k} + \sqrt{E_k/\epsilon_k} \right),$$

$$v_k = 1/2 \left( \sqrt{\epsilon_k/E_k} - \sqrt{E_k/\epsilon_k} \right),$$

(10)

with $E_k = \hbar^2 k^2/(2m_B)$ being the free-particle energy. Where the excitation energy is $\epsilon_k = \sqrt{E_k^2 + 2n_0 E_k \bar{\epsilon}(k)}$.

$$\epsilon_k = \frac{1}{2} \lambda_0 \omega \sqrt{(k \ell_B)^4 + \eta (k \ell_B)^2 \left[ 1 - \epsilon_{dd} \bar{\epsilon}(k) \right]},$$

(11)

with dimensionless parameters $\eta = 8n_0 a_B$. Hence, the Hamiltonian for the collective excitations is

$$H_B = \sum_{k \neq 0} \epsilon_k \hat{b}_k^* \hat{b}_k.$$

(12)

The sum over Bogoliubov modes exclude the zero mode and will act as the reservoir under our model.

C. Interaction Hamiltonian

We assume that the reservoir atoms are coupled with the up state $|\uparrow\rangle$ of the system atoms via a Raman transition [31,51]

$$H_{AB} = g_{AB} \hat{\alpha}_+ \hat{\alpha}_\uparrow \int dr |\Phi_A(r)|^2 \Psi_B^\dagger(r) \Psi_B(r),$$

(13)

where $g_{AB} = 2\pi \hbar^2 a_{AB}/m_B$ with the atoms and reservoir scattering length $a_{AB}$ and reduced mass $m_{AB} = m_A m_B/(m_A + m_B)$. By substituting Eqs. (3) and (9) into the above interaction Hamiltonian and omitting the square terms about $\hat{b}_k$ and $\hat{b}_k^*$, we have

$$H_{AB} \approx \delta_{\perp} \hat{\alpha}_+ \hat{\alpha}_\uparrow + \alpha_{\perp}^2 \sum_k g_k (\hat{b}_k + \hat{b}_k^*),$$

(14)

where

$$\delta_{\perp} = g_{AB} n_0 \int dy dz |\Psi_B(y, z)|^2 |\Phi_A(y, z)|^2 \int dx |\Phi_A(x)|^2$$

$$= \frac{2\pi \hbar^2 a_{AB} n_0}{m_{AB} (\ell_A^2 + \ell_B^2)}.$$  

(15)

and

$$g_k = \frac{2\pi \hbar^2 a_{AB}}{m_{AB} (\ell_A^2 + \ell_B^2)} \sqrt{n_0 E_k / L_k} \exp \left( -\frac{k^2 \ell_B^2}{4} \right).$$

(16)

III. SYSTEM DYNAMICAL EVOLUTION

In the interaction picture with respect to $H_B$, the total Hamiltonian is

$$H_I(t) = (\lambda + \delta) J_z + N_f \sum_k g_k (\hat{b}_k \hat{b}_k^* e^{i\omega t} - \hat{b}_k^* \hat{b}_k e^{-i\omega t}) - i\Gamma_{\text{loss}} N_f,$$

(17)

which is a non-Hermitian dephasing spin-boson model with $N_f = (J_z + N/2)$ being the up state number operator. Here the non-Hermitian term $\Gamma_{\text{loss}}$ is phenomenologically introduced to describe the one-body particle loss rate, owing to inelastic collisions between the system atoms and the noncondensed thermal atoms. It results in the particles be kicked out from the system. Such a kind of loss is a typical dissipation effect and has been widely studied [24,52,55].

By using of Magnus expansion [39], the time evolution operator can be read as $U(t) = e^{-iH_{\text{tot}} t}$, where the effective Hamiltonian is (see Appendix B for details)

$$H_{\text{eff}} = \lambda' J_z + \Delta(t) J_z^\perp + iJ_z \sum_k (\alpha_k \hat{b}_k - \alpha_k^* \hat{b}_k) - i\Gamma_{\text{loss}} N_f,$$

(18)

with $\lambda' = \lambda + \Delta(t)$ and $\alpha_k = g_k (1 - e^{i\omega t})/(\omega t \omega)$. From the above equation, we can find that the collisional interaction between the atoms and reservoir induce a nonlinear term $\propto J_z^\perp$, corresponding to the OAT Hamiltonian, and the noise induced nonlinear strength is [57]

$$\Delta(t) = \frac{1}{t} \int_0^t d\omega \omega \omega - \sin(\omega t) \frac{\omega}{\omega^2}.$$  

(19)

Here, the reservoir spectral density $J(\omega)$ defined as $J(\omega) = \sum_{k \neq 0} |g_k|^2 \delta(\omega - \epsilon_k)/\hbar$. In the continuum limit, $L^{-1} \sum_k \rightarrow (2\pi)^{-1} \int dk$, we have

$$J(\omega) = \Theta \hbar \omega \ell_B^3 \sum_{k \neq 0} f(k) \delta \left( \omega - \epsilon(k) \right)$$

$$= \Theta \hbar \omega \ell_B^3 \sum_{k \neq k(\omega)} \frac{f(k_\omega)}{k_\omega} \left| \epsilon(k) \right|^{-1},$$

(20)

where $f(k) \equiv k^2 e^{-k^2 \ell_B^2/2}$ with $k_\omega$ being the roots of the equation $\epsilon(k) = \hbar \omega$, and the dimensionless parameter is

$$\Theta = \frac{n_0 \ell_B^3 g_{AB}^2 (m_A + m_B)^2}{\pi m_A^2 (\ell_A^2 + \ell_B^2)^2}. $$

(21)

Assuming that the initial state of the total system is given by

$$\rho_T(0) = |\Phi(0)\rangle \langle\Phi(0)| \otimes \rho_B,$$

(22)

where $|\Phi(0)\rangle \equiv 1/\sqrt{2} |\uparrow\rangle + |\downarrow\rangle$ is CSS, with the probability amplitudes $c_m = 2^{-J} (c_{-J}^m)^{1/2}$ and total spin $J = N/2$ for a system consisting of $N$ condensed atoms. And the density matrix of reservoir read as
$$\rho_B = \Pi_i[1 - \exp(-\beta \omega_i)] \exp(-\beta \omega_i b_i^\dagger b_i),$$  
with $\beta$ the inverse temperature. With the help of Eq. (18), the
time-evolution reduced matrix elements of the atom system at
any later time $t$ is found by tracing over the reservoir degrees of
freedom
$$\rho_{jm,jn}(t) = e^{-it'_{j,m-n}} e^{it\Delta(t)\sigma^2} \times e^{-\Gamma_{int}(m+n)\sigma^2} e^{-it\sigma^2 \gamma(t)} \rho_{jm,jn}(0), \tag{24}$$
with decoherence function
$$\gamma(t) = \frac{1}{t} \int_0^\infty d\omega J(\omega) \coth \left( \frac{\hbar \omega}{2 k_B T} \right) \frac{1 - \cos(\omega t)}{\omega^2}, \tag{25}$$
which is a function of temperature $T$.

In Eq. (24), $\Delta(t)$ and $\gamma(t)$, respectively, correspond to
the unitary and non-unitary evolution due to the effects of the
reservoir. Equations (19) and (25) show that both $\Delta(t)$ and $\gamma(t)$
depend on the reservoir spectral density $J(\omega)$, which can
be controlled by tuning the MDDI of the dipolar Bose gas
reservoir $\sigma^2$. This is the main difference from Fesh-
bach resonances method whose control is on the system atoms
directly and will strongly enhance three-body loss near reso-
nances regime beside the one-body loss. Therefore, in our
model the one-body particle loss due to the inelastic collisions
of the noncondensed atoms maybe the main loss mechanism
that need considering. When temperature is low enough the
values of one-body loss rate $\Gamma_{loss}$ will be very small, since
there is only a small number of thermal atoms with sufficient
energy to knock atoms out of condensate $\sigma^2$. For simplicity,
hereafter we only consider the case of $T \rightarrow 0$, and choose $\Gamma_{loss}$
as a free parameter.

IV. SPIN SQUEEZING AND ENTANGLED
NON-GAUSSIAN SPIN STATES

A. Spin squeezing parameter

Now, a state is regarded as squeezed if the variance of
one spin component normal to the mean spin vector $\langle J \rangle =
Tr \frac{1}{N} \rho_B$, is lower than the Heisenberg limited value. The
SS parameter defined by Wineland is $\sigma^2$
$$\xi_R^2 = N \langle (\Delta J_R)_{\min}^2 \rangle^2 / \langle |\langle J \rangle| \rangle^2,$$
where $(\Delta J_R)_{\min}$ represents the minimal variance of the
spin component perpendicular to the mean spin direction $\hat{R}_0 \equiv \langle J \rangle / |\langle J \rangle|$. Where the mean spin is $|\langle J \rangle| =
\sqrt{\langle J_x^2 \rangle + \langle J_y^2 \rangle + \langle J_z^2 \rangle}$. A state is spin squeezed if $\xi_R^2 < 1$. In
addition, the smaller $\xi_R^2$ is, the stronger the squeezing is. If
$\Gamma_{loss} = 0$, the squeezing parameter can be evaluated explicitly
$$\xi_R^2 = \frac{4 + (N - 1) \left( \tilde{A} - \sqrt{\tilde{A}^2 + \tilde{B}^2} \right)}{4 e^{-2\gamma(t)} [\cos(t\Delta(t))]^{2N-2}}, \tag{26}$$
which does not depend on $\lambda'$. And the optimally squeezed
direction is $\phi_{opt} = [\pi + \tan^{-1}(\tilde{B}/\tilde{A})]$, with
$$\tilde{A} = 1 - \cos^N \gamma(t), \tag{27}$$
$$\tilde{B} = -4 \sin(t\Delta(t)) \cos^{N-2} \gamma(t) \exp[-t\gamma(t)].$$

When considering the one-body losses, $\Gamma_{loss} \neq 0$, the form of
SS has given in Appendix A.

Equation (26) indicates that the dephasing noise plays two
different roles: On one hand, it can generate the SS by creating
the nonlinear interaction $\Delta(t)$; on the other hand, it degrades
the degree of SS via the decoherence function $\gamma(t)$.

FIG. 2: Time dependence of $\Delta(t)$ and $\gamma(t)$ for different $\epsilon_{dd}$. Here we
choose $\Theta = 1.5 \times 10^{-2}$ and $\eta = 5$. Note that for repulsive MDDI the
values of noise-induced nonlinear interaction $\Delta(t)$ approach to their
steady values $\Delta(\infty)$, while decoherence function $\gamma(t)$ is decreasing with
time.

FIG. 3: Spin squeezing dynamics of two-mode BEC consisting of
$N = 100$ atoms coupled to a 1D dipolar Bose gas reservoir. (a) Squeezing parameter for different values of $\epsilon_{dd}$: from top to bottom $\epsilon_{dd} = 1, 0, -1$. Here $\Gamma_{loss} = 0$. (b) Squeezing parameter for different values of loss parameters with $\epsilon_{dd} = -1$, from top to bottom $\Gamma_{loss} = 0.01 \Delta(\infty), 0.002 \Delta(\infty)$, and 0.
B. QFI and entanglement of non-Gaussian spin states

To investigate the entanglement created by the dipolar BEC reservoir, we can also consider the QFI. In general, the states that are entangled and useful for sub-shot-noise-limited parameter-estimation precision is identified by the QFI criterion $F_Q > N$. The QFI $F_Q$ with respect to $\theta$, acquired by an SU(2) rotation, can be described as

$$F_Q[\rho(\theta, t), \hat{J}_\theta] = i\hbar C_i t^2,$$

(28)

where $\rho(\theta, t) = \exp(-i\theta\hat{J}_\theta)\rho(t)\exp(i\theta\hat{J}_\theta)$ with \( \hat{I} \) being the optimal rotation direction, and the matrix element for the symmetry matrix $C$ is

$$C_{ij} = \sum_{k, l} \frac{(p_i - p_j)^2}{p_i + p_j} [\langle \langle J_i | J_l \rangle \langle J_0 | J_l \rangle + \langle \langle J_i | J_l \rangle \langle J_0 | J_l \rangle ] ,$$

(29)

with $p_i(|i\rangle)$ being the eigenvalues (eigenvectors) of $\rho(\theta, t)$.

For simplicity, we first consider the case of small $N$, e.g., $N = 2$. When setting $\lambda' = 0$ and neglecting the particle loss, e.g., $\Gamma_{loss} = 0$, the QFI can be calculated analytically

$$F_Q[\rho(\theta, t), \hat{J}_\theta] = \text{max}[C_{xx}, C_{zz}, C_{yy}],$$

(30)

with

$$C_{xx} = \frac{4 \sin^2[2\gamma(t)t] + 16 e^{2\gamma(t)t}}{1 + 3 e^{4\gamma(t)t}} \left[ 1 - \frac{16 \cos^2[\Delta(t)t]}{e^{-6\gamma(t)t} [1 - e^{-4\gamma(t)t}]^2 + 16} \right]$$

(31)

in $x$-axis direction, and

$$C_{yy} + C_{zz} = \sqrt{(C_{yy} + C_{zz})^2 + 4C_{yz}^2}$$

(32)

in $y_z$ plane, which can also be obtained by using Eq. (29) (see Appendix B).

The maximal QFI can be found in $x$ axis direction

$$F_Q^{\max} = \frac{4 \sin^2[2\gamma(t)t_{opt}] + 16 e^{2\gamma(t)t_{opt}}}{1 + 3 e^{4\gamma(t)t_{opt}}}$$

(33)

when choosing optimal interrogation time $t_{opt} = \pi/[2\Delta(t)]$. Equation (33) reveals that the values of $\gamma(t)$ is directly related to the QFI. One finds $F_Q^{\max} \rightarrow N^2$ (the Heisenberg limit) if $\gamma(t) \rightarrow 0$. Fortunately, Fig. 2(b) indicates that the nearly neglectable $\gamma(t)$ can be obtained when the dipolar Bose gas reservoir with repulsive MDDI. Therefore, the main limitation of Heisenberg scaling is one-body loss mechanism in our scheme. Next, we consider large $N$ and the case of $\Gamma_{loss} \neq 0$ numerically.

V. RESULTS AND DISCUSSION

As a concrete example, we consider a BEC reservoir of $^{162}\text{Dy}$ atoms, for which we have $\mu_m = 9.9 \mu_B$ and $a_B = 112 a_0$ with $\mu_B$ the Bohr magneton. This means that $a_{dd} \approx 131 a_0$ and dipolar interaction is mainly attractive since $\epsilon_{dd} > 1$, then the attraction is stronger than the short-range repulsion. However, not only the contact interaction strength can be tunable via Feshbach resonance, but also both the sign and the strength of the effective dipolar interaction can be tuned by the use of a fast rotating orienting field. Later, we will consider the values of $\epsilon_{dd} \in [-1, 1]$, which is repulsive (attractive) MDDI for $\epsilon_{dd} < 0$ ($\epsilon_{dd} > 0$).

![FIG. 4:](image)

(a) Time dependence of QFI amplification rate with repulsive interaction $\epsilon_{dd} = -1$. We show three cases with $N = 10, 30$ and 50. (b) The maximal QFI amplification rate $F_Q^{\max}/N$ as a function of atom number $N$ with $\epsilon_{dd} = -1$. (c) QFI amplification rate and (d) the corresponding optimal evolution time $\omega_s t_{opt}$ with respect to $\epsilon_{dd}$ when $N = 100$. Here $\Gamma_{loss} = 0$.

Numerically, it is convenient to introduce the dimensionless units: $\hbar \omega_s$ for energy, $\omega_s^{-1}$ for time, and $\ell_B = [\hbar/(m \omega_s)]^{1/2}$ for length. To obtain the values of dimensionless parameter $\eta$ and $Q$, we assume a quasi-1D trap with $\omega_s = 2\pi \times 20 \text{Hz}$ and $\omega_x = 2\pi \times 10^3 \text{Hz}$; and the corresponding harmonic oscillator width is $\ell_B = \ell_A \approx 2.5 \times 10^{-7} \text{m}$. We assume that the linear density of the quasi-1D condensate is $n_0 = 10^8 \text{m}^{-1}$, and the $s$-wave scattering length between Rb and Dy atoms is $a_{AB} \sim 5 \text{nm}$. Then, we shall take $\eta = 5$ and $\Theta = 1.5 \times 10^{-2}$ in the results presented below.

Since both the SS and QFI depend on $\gamma(t)$ and $\Delta(t)$, let us first investigate the time dependence of dephasing factor. From Fig. 2, we can clearly see that the squeezing rate $\Delta(t)$ is nearly constant, $\Delta(\infty)$. Whereas $\gamma(t)$ is decreasing with time, what is more we can get very small $\gamma(t)$ (e.g. $10^{-3 \sim 10^{-4}}$) when $\epsilon_{dd} < 0$. Comparing $\Delta(t)$ and $\gamma(t)$, we can also find that the values of $\Delta(t)$ (squeezing rate) is larger than $\gamma(t)$ (dispersive rate) for $\epsilon_{dd} < 0$, which means that we can obtain strong squeezing and large QFI by the reservoir’s engineering with repulsive MDDI.

In Fig. 3, we plot the SS $\xi^2_B$ dynamics of two-mode BEC consisting $N$ atoms coupled to a 1D dipolar Bose gas reservoir for various $\epsilon_{dd}$ values. As is shown in Fig. 3(a), the optimal squeezing can be achieved within short time scale, and after a transient time, it is lost ($\xi^2_B > 1$) and then ENSGs are produced. However, for repulsive-dipolar-interaction reservoir, we can obtain stronger SS, due to their smaller dispersive rate $\gamma(t)$. In Fig. 3(b), we further plot the time evolution of $\xi^2_B$ for various atom loss rates. It indicates that the squeezing degree degraded with the increasing of the atom loss rate.
Figure 4 illustrates the QFI amplification rates with respect to the initial state (CSS) $F_Q/N$. In Fig. 4(a), we present the time dependence of QFI amplification rates with repulsive interaction. We see that differ from the case of SS, the amplified QFI can last for a very long time, which means that the ENGSs can be produced and achieve the maximal even in the regimes without squeezing. The optimal QFI first monotonically increasing and then reach a metastable $\sim N/2$ in the $yz$ plane. Subsequently, the QFI suddenly increasing in the $x$-axis direction at the optimal interrogation time $t_{opt}$. As shown in Fig. 4(b), the maximal amplification rate $F_Q^{\text{max}}/N$ is proportional to the atom number $N$, and the scale factor is $\sim N$, which is the Heisenberg scale. What is more, compared with the attractive MDDI reservoir the repulsive interaction can induce larger QFI, this result is presented in Fig. 4(c).

Comparison of Figs. 4(a) and 5(a), we can see that for not too large atom loss rates $\Gamma_{\text{loss}}$ the optimal evolution time is $t_{opt} = \pi/[2\Delta(t)]$, which is the same as the case of $N = 2$.

To demonstrate the near Heisenberg-limited sensitivity with the ENGSs realized by our model, we calculate the fidelity between the optimal ENGSs and spin cat states [14], which are the maximal entangled states and have the Heisenberg-limited sensitivity for metrology. It is given by

$$|\Psi\rangle_{\text{cat}} = \frac{1}{\sqrt{2}} \left( | \frac{\pi}{2}, 0 \rangle - e^{-i\theta} | \frac{\pi}{2}, \pi \rangle \right),$$

where $| \theta, \phi_{0} \rangle \equiv e^{i\theta(J_{x} \sin \phi_{0} - J_{y} \cos \phi_{0})}|j,j\rangle$ is the CSS. With the definition of fidelity, we have

$$F_{Q} = \text{tr} \sqrt{Q_{\text{cat}}^{1/2} \rho(t_{opt}) Q_{\text{cat}}^{1/2}},$$

where $Q_{\text{cat}} = |\Psi\rangle_{\text{cat}} \langle \Psi|$.

Form Fig. 6, we can find that the fidelity depends on both the $\epsilon_{dd}$ and particle number $N$. The maximal values occurs at $\epsilon_{dd} = -1$. Because of the dissipative rate $\gamma(t)$, the fidelity decrease with the increase of $N$. As shown in Fig. 6(a), when $\Gamma_{\text{loss}} = 0$ we get $F_{Q} \approx 0.94, 0.85$, and $0.80$ for $N = 10, 30$, and $50$. And the effect of $\Gamma_{\text{loss}}$ on fidelity also is shown in Fig. 6(b).

VI. CONCLUSION

In summary, we have realized the SSSs and ENGSs by immersing atoms in a thermally equilibrated quasi-1D dipolar BEC reservoir. It has demonstrated that the repulsive-dipolar-interaction reservoir can induce better SS and entanglement. We have shown that owing to the dephasing noise, even in the regimes without SS the ENGSs can successively undergo highly metastable entangled states and entanglement suddenly increase. To explain the highly sensitivity for metrology, we calculated the fidelity between the optimal ENGSs and spin cat states, and found that the optimal ENGS is similar to the spin cat state. It has confirmed that by the use of ENGSs for metrology, the phase estimation sensitivity can surpass that by SS and even approach to Heisenberg limit for neglectable parameters.

FIG. 5: (a) QFI vs time $\omega_{J}t$ for various $\Gamma_{\text{loss}}$ with $N = 2$. The solid line (black) corresponds to the analytic solution given in Eqs. (30-33). (b) The maximal QFI as a function of atom number $N$ for various $\Gamma_{\text{loss}}$. The shaded area indicates the regime between shot-noise limit and Heisenberg limit. (c) The rates of the Heisenberg limit $F_{Q}^{\text{max}}/N^{2}$ with respect to atom number $N$ for different values of loss parameters $\Gamma_{\text{loss}}$. Other parameters are $\epsilon_{dd} = -1$, from top to bottom $\Gamma_{\text{loss}} = 0$, $0.001\Delta(\infty)$, $0.002\Delta(\infty)$, and $0.005\Delta(\infty)$.

FIG. 6: (a) The fidelity between our optimal ENGSs and spin cat state with respect to $\epsilon_{dd}$ for different values of atom number $N$. Here we choose $\Gamma_{\text{loss}} = 0$. (b) The fidelity as a function of atom number $N$ for various $\Gamma_{\text{loss}}$ with $\epsilon_{dd} = -1$. 

\[ |\Psi\rangle_{\text{cat}} = \frac{1}{\sqrt{2}} \left( | \frac{\pi}{2}, 0 \rangle - e^{-i\theta} | \frac{\pi}{2}, \pi \rangle \right), \]

\[ F_{Q} = \text{tr} \sqrt{Q_{\text{cat}}^{1/2} \rho(t_{opt}) Q_{\text{cat}}^{1/2}}, \]
atom loss rates. The effect of the atom loss rate as a free parameter has also been considered.

Finally, we give two remarks on the above obtained results: First, these results we have obtained in this paper are based on the negligible spatial evolution of the immersed condensate wave functions. Usually this assumption is enough to capture the basic processes and physics, and detailed consideration of the impact of spatial dynamics can be investigated by adopting the multi-configurational time-dependent Hartree for bosons method [58]. Second, the scheme we proposed in this work can also suit the case that the system atoms are weak or no interaction two-level impurity atoms which are not condensate.

Acknowledgments

This work was supported by the NSFC under Grant No. 11547159. G.R.J. acknowledges support from the Major Research Plan of the NSFC (Grant No. 91636108).

Appendix A: Derivation of Eq. (7)

Here, we present a detailed derivation of the Fourier transform of the effective 1D interaction potential. It can be obtained by integrating out the y and z variables as

\[
\tilde{V}_{1D}(k) = \frac{1}{2\pi} \int \int dydz |\Psi_\perp(y, z)|^2 F_{1c}^{-1} \left[ F_{2c} \left[ \Psi_\perp(y, z) \right] \right] \tilde{V}(k) \\
= \frac{1}{2\pi} \int \int dydz |\Psi_\perp(y, z)|^2 F_{1c}^{-1} \left[ \frac{1}{2\pi} e^{-i(k_B^2 z^2 + k_B^2 y^2)/4} \left[ g_B - c_d \left( 1 - 3(\hat{\mu}_m \cdot \hat{e}_k)^2 \right) \right] \right] ,
\]

(A1)

When assuming the dipole moments lie on the xz plane forming an angle \( \varphi \) to x axis, i.e.,

\[
\hat{\mu}_m = (\cos \varphi, 0, \sin \varphi).
\]

(A2)

Then, we have

\[
\tilde{V}_{1D}(k) = \frac{1}{(2\pi)^2} \int d\phi \int dk_\perp k_\perp e^{-i(k_B^2 \varphi)/2} \left[ g_B - c_d \left( 1 - 3\left( \frac{k \cos \varphi + k_\perp \cos \varphi \sin \varphi}{k^2 + k_\perp^2} \right) \right) \right] \\
= \frac{g_B}{2\pi l_B^2} - \frac{c_d}{2\pi l_B^2} \left[ 1 - \frac{3}{2} \sin^2 \varphi \right] \left[ 1 - \frac{3}{2} k_B^2 \exp \left( k_B^2 l_B^2 \right) \left( \frac{k_B^2 l_B^2}{2} \right) \Gamma \left( 0, \frac{k_B^2 l_B^2}{2} \right) \right] \\
= \frac{g_B}{2\pi l_B^2} - \frac{c_d}{2\pi l_B^2} \left[ 1 - \bar{c}_{dd} \left[ 1 - \frac{3}{2} k_B^2 \exp \left( k_B^2 l_B^2 \right) \left( \frac{k_B^2 l_B^2}{2} \right) \Gamma \left( 0, \frac{k_B^2 l_B^2}{2} \right) \right] \right] ,
\]

(A3)

where \( \bar{c}_{dd} = \bar{c}_d/g_B \) with \( k_\perp = \sqrt{k_B^2 + k_\perp^2} \) and \( \bar{c}_d = c_d \left( 1 - \frac{3}{2} \sin^2 \varphi \right) \). Clearly, the effective 1D dipolar interaction vanishes at the magical angle \( \alpha_n = 54.74^\circ \), and it is attractive (repulsive) for \( \alpha < \alpha_n (\alpha > \alpha_n) \). In the main text, we have dropped the tilde on \( \bar{c}_d \) and \( \bar{c}_{dd} \), and we will only consider the values of \( \epsilon_{dd} \in [-1, 1] \).

Appendix B: Time evolution operator \( U(t) \)

The time evolution operator can be obtained by using Magnus expansion

\[
U(t) \equiv T_+ \exp \left[ -i \int_0^t H_I(t')dt' \right] = \exp \left[ \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} F_n(t) \right] .
\]

(B1)
\[ F_1(t) = \int_0^\infty H_1(t')dt' = i\hbar J_z + \left( J_z + \frac{N}{2} \right) i \sum_k \left( g_k b_k e^{i\omega_k t} + g_k^* b_k^* e^{-i\omega_k t} \right) \]
\[ = i\hbar J_z + \left( J_z + \frac{N}{2} \right) \sum_k (\alpha_k b_k^\dagger - \alpha_k^* b_k) - i\Gamma_{\text{loss}}, \quad (B2) \]
\[ F_2(t) = \int_0^\infty ds \int_0^\infty ds' [H_2(s), H_2(s')] \]
\[ = -2iN^2 \sum_k |g_k|^2 \int_0^\infty ds \int_0^\infty ds' \sin \omega_k (s-s') = -2iN^2 \Delta(t), \quad (B3) \]

with the amplitudes \( \alpha_k = -ig_k \int_0^\infty e^{i\omega_k s} ds/t = g_k (1 - e^{i\omega_k s})/\omega_k t, \) since \([H_2(s), H_2(s')] = -2iN^2 \sum_k |g_k|^2 \sin \omega_k (s-s'), \) which commutes with the high order terms. It is worth to point out that the commutator of the interaction Hamiltonian at two different times is an operator but not a \( C \) number as considering in the single bit case, which can induce the nonlinear interaction; the noise-induced nonlinear interaction strengthen \( \Delta(t) \) can recast as

\[ \Delta(t) = \frac{1}{i} \sum_k |g_k|^2 \int_0^\infty ds \int_0^\infty ds' \sin \omega_k (s-s') = \frac{1}{i} \int_0^\infty d\omega J(\omega) \int_0^\infty ds \int_0^\infty ds' \sin \omega (s-s') \]
\[ = \frac{1}{i} \int_0^\infty d\omega J(\omega) \frac{\omega t - \sin(\omega t)}{\omega^2}, \quad (B4) \]

where we have used the relation \( \sum_k |g_k|^2 \to \int_0^\infty d\omega J(\omega). \) Then,

\[ U(t) = \exp \left[ -iF_1(t) - \frac{1}{2}F_2(t) \right] = \exp(-i\lambda J_z) \exp \left[ \left( J_z + \frac{N}{2} \right) \sum_k (\alpha_k b_k^\dagger - \alpha_k^* b_k) \right] \exp[i\Gamma_{\text{loss}}] \]
\[ = \exp(-i\lambda' J_z) \exp \left[ i\Delta(t) J_z \right] \exp(-i\Gamma_{\text{loss}}) \exp[i\phi_0(t)] \exp \left[ \sum_k (\alpha_k b_k^\dagger - \alpha_k^* b_k) \right], \quad (B5) \]

where \( \lambda' = \lambda - N\Delta(t) \) and \( \phi_0(t) \) is the global phase and will be dropped.

**Appendix C: Spin squeezing with \( \Gamma_{\text{loss}} \neq 0 \)**

In this Appendix, we present a detailed of SS with \( \Gamma_{\text{loss}} \neq 0. \) To this end, we assume, without loss of generality, that \( \hat{n}_0 = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \phi) \), where \( \theta = \tan^{-1} \left( \sqrt{\langle J_1^2 \rangle + \langle J_2^2 \rangle} \right) \) and \( \phi = \tan^{-1} \left( \langle J_2 \rangle / \langle J_z \rangle \right) \) are polar and azimuthal angles, respectively. We then define two mutually perpendicular unit vectors \( \hat{n}_1 = (-\sin \phi, \cos \phi, 0) \) and \( \hat{n}_2 = (\cos \phi \cos \theta, \cos \phi \sin \theta, -\sin \phi) \). Clearly, both \( \hat{n}_1 \) and \( \hat{n}_2 \) are perpendicular to \( \hat{n}_0 \) such that \( (\hat{n}_1, \hat{n}_2, \hat{n}_0) \) form a right-hand frame. Now, the minimal fluctuation of a spin component perpendicular to the mean spin is

\[ (\Delta J_{n_k})^2_{\text{min}} = \frac{1}{2} \left( C - \sqrt{A^2 + B^2} \right), \quad (C1) \]

and the mean spin is

\[ \langle J \rangle = \sqrt{\langle J_z \rangle^2 + \langle J_1 \rangle^2 + \langle J_2 \rangle^2} = \sqrt{\langle J_z \rangle^2 + \langle J_1 \rangle^2}, \quad (C2) \]

where

\[ A = \frac{\sin^2 \phi}{2} \left[ j(j+1) - 3\langle J_z^2 \rangle \right] \frac{1 + \cos^2 \phi}{2} \text{Re} \langle J_z^2 e^{-2i\theta} \rangle + \sin \phi \cos \theta \text{Re} \langle J_z (2J_z + 1) e^{-i\theta} \rangle, \]
\[ B = \cos \theta \text{Im} \langle J_z^2 e^{-2i\theta} \rangle + \sin \theta \text{Im} \langle J_z (2J_z + 1) e^{-i\theta} \rangle, \]
\[ C = j(j+1) - \langle J_z^2 \rangle - \text{Re} \langle J_z^2 e^{-2i\theta} \rangle - \frac{\sin^2 \theta}{2} \left[ j(j+1) - 3\langle J_z^2 \rangle \right] + \frac{1 + \cos^2 \phi}{2} \text{Re} \langle J_z^2 e^{-2i\theta} \rangle - \frac{\sin(2\phi)}{2} \text{Re} \langle J_z (2J_z + 1) e^{-i\theta} \rangle, \quad (C3) \]
with

\[ \langle J_x \rangle = \int e^{ixt} e^{-Nt_\text{tot}t} e^{-\gamma t} [\cos[t\Delta(t)] \cosh(\Gamma_{\text{loss}} t) + i \sin[t\Delta(t)] \sinh(\Gamma_{\text{loss}} t)]^2^{-1} \]

\[ \langle J_0^2 \rangle = \int \left( j - \frac{1}{2} \right) e^{-Nt_\text{tot}t} e^{-\gamma t} [\cos(2t\Delta(t))] \cosh(\Gamma_{\text{loss}} t) + i \sin(2t\Delta(t))] \sinh(\Gamma_{\text{loss}} t)]^2^{-2} \]

\[ \langle J_z \rangle = \frac{-j}{2} e^{-Nt_\text{tot}t} \sinh(2\Gamma_{\text{loss}} t) [1 + \cosh(2\Gamma_{\text{loss}} t)]^{-1} \]

\[ \langle J_\perp^2 \rangle = \frac{j e^{-Nt_\text{tot}t}}{2(1 + e^{2\Gamma_{\text{loss}} t})^2} [1 + \cosh(2\Gamma_{\text{loss}} t)]^2 [2e^{2\Gamma_{\text{loss}} t} + j(1 - e^{2\Gamma_{\text{loss}} t})^2]. \]

\[ \langle J_\perp (2J_\perp + 1) \rangle = 2j \left[ j - \frac{1}{2} \right] e^{i xt} e^{-Nt_\text{tot}t} e^{-\gamma t} \cos[t\Delta(t)] \cosh(\Gamma_{\text{loss}} t) + i \sin[t\Delta(t)] \sinh(\Gamma_{\text{loss}} t)]^2^{-2} \times [\cos[t\Delta(t)] \sinh(\Gamma_{\text{loss}} t) + i \sin[t\Delta(t)] \cosh(\Gamma_{\text{loss}} t)]. \]

**Appendix D: Matrix elements of** $C_\perp$ **for** $N = 2$

The matrix elements for the symmetric matrix $C$ in $yz$ plane are

\[ C_{yy} = 4 \left[ \frac{\beta_+}{p + p_-} (p - p_-)^2 + \frac{\beta_-}{p + p_+} (p - p_+)^2 \right], \]

\[ C_{zz} = 4 \left[ \frac{\alpha_+}{p + p_-} (p - p_-)^2 + \frac{\alpha_-}{p + p_+} (p - p_+)^2 \right], \]

\[ C_{yz} = 4 \sqrt{2} \left[ \frac{(p - p_-)^2 \alpha_+ \alpha_- \text{Im} \beta_+}{p + p_+} + \frac{(p - p_+)^2 \alpha_- \alpha_- \text{Im} \beta_-}{p + p_-} \right], \] (D1)

with

\[ p = \frac{1}{4} (1 - e^{-4\gamma t}), \quad p_\pm = \frac{1}{8} e^{-4\gamma t} \left[ 1 + 3 e^{4\gamma t} \pm \Xi \right], \]

\[ \Xi = \sqrt{(1 - e^{4\gamma t})^2 + 16 e^{4\gamma t}}, \] (D2)

\[ \alpha_\pm = \frac{2 \sqrt{2}}{\sqrt{16 + e^{-4\gamma t}} [1 - e^{4\gamma t} \pm \Xi]^2}, \]

\[ \beta_\pm = \frac{-e^{i\Delta(t)} e^{-3\gamma t} [1 - e^{4\gamma t} \pm \Xi]}{\sqrt{16 + e^{-4\gamma t}} [1 - e^{4\gamma t} \pm \Xi]^2}. \] (D3)
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