STATISTICAL NATURE OF SKYRME–FADDEEV MODELS IN 2+1 DIMENSIONS AND NORMALIZABLE FERMIONS

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The Skyrme–Faddeev model has planar soliton solutions with the target space $\mathbb{C}P^N$. An Abelian Chern–Simons term (the Hopf term) in the Lagrangian of the model plays a crucial role for the statistical properties of the solutions. Because $\Pi_3(\mathbb{C}P^1) = \mathbb{Z}$, the term becomes an integer for $N = 1$. On the other hand, for $N > 1$, it becomes perturbative because $\Pi_3(\mathbb{C}P^N)$ is trivial. The prefactor $\Theta$ of the Hopf term is not quantized, and its value depends on the physical system. We study the spectral flow of normalizable fermions coupled with the baby-Skyrme model ($\mathbb{C}P^N$ Skyrme–Faddeev model). We discuss whether the statistical nature of solitons can be explained using their constituents, i.e., quarks.

Keywords: topological soliton, skyrmion, spin statistics, spectral flow

DOI: 10.1134/S0040577919090010

1. Introduction

The Skyrme–Faddeev model is an example of a field theory that admits finite-energy knotted solitons. The classical soliton solutions of the Skyrme–Faddeev model can be useful for describing the strong-coupling sector of the Yang–Mills theory. It was shown in [1] that in the case of the complex projective target space $\mathbb{C}P^N$, a Skyrme–Faddeev-type model has an infinite number of exact soliton solutions in the integrable sector if the coupling constants satisfy a special relation. The existence of vortex solutions of the model outside the integrable sector was confirmed numerically for an appropriate choice of potentials [2]. We note that the model is essentially equivalent to the so-called baby-Skyrme model [3], which is a (2+1)-dimensional analogue of the Skyrme model.

It is well known that from the quantum standpoint, soliton solutions have a special property (have “fractional” spin statistics) if the Hopf term (theta term) is included in the action of the model [4]. Because $\Pi_3(\mathbb{C}P^1) = \mathbb{Z}$, this term becomes the Hopf invariant and can therefore be represented as a total derivative, which does not influence the classical equations of motion. On the other hand, because $\Pi_4(\mathbb{C}P^1)$ is trivial, the coupling constant $\Theta$ (prefactor of the Hopf term) is not quantized. As shown in [4], in the model with the Hopf Lagrangian, solitons with a unit topological charge have the spin $\Theta/2\pi$, which can be fractional. For a fermionic model coupled with a $\mathbb{C}P^1$-valued field, $\Theta$ can be found, for example, by using perturbation theory [5], [6].

For $N > 1$, the algebra $\Pi_3(\mathbb{C}P^N)$ is already trivial, and the Hopf term is then perturbative, i.e., it is not a homotopy invariant, and this in the general case means that the contribution from the term can

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always be fractional even if we choose an integer $n$ in the anyon angle $\Theta = n\pi$. It was pointed out in [7] that an analogue of the Wess–Zumino–Witten term appears in a $\mathbb{CP}^N$-valued field and plays a role similar to the Hopf term for $N = 1$ [8]. As a result, the soliton can be quantized as an anyon with the statistics angle $\Theta$ and such a Hopf-like term.

Here, we solve the fermionic model coupled with the baby-Skyrme model or the $\mathbb{CP}^N$ Skyrme–Faddeev model. The basic property of the localizing mode of fermions on a topological soliton is understood in terms of a basis from the Atiyah–Singer index theorem [9]. The index for the Dirac operator $D$ can be defined as $\dim \ker D - \dim \ker D^\dagger$, which is related to the Casimir energy of the fermions. The spectral flow analysis in the chiral-invariant model (the Skyrme model) is a simple realization of the theorem [10]. When the number $B$ of the one-particle spectra passes from the positive to the negative continuum, the size or strength of the background skyrmions changes. As a result, the Casimir energy has $B$ states, and this corresponds to solitons. In the Skyrme model, the Wess–Zumino–Witten term is topological and is the origin of the topological nature of solitons (skyrmions) and fermions. We therefore believe that the statistical nature of the soliton is related to the localizing fermions. Because the Hopf term of the $\mathbb{CP}^N$ model is not topological, we expect that the consistency between the statistical nature of the soliton and the nature of the localizing fermions is broken. Analysis of the spectral flow argument yields new information about this consistency (or inconsistency).

We consider only the case $N = 2$, but the generalization to large $N$ is straightforward.

2. The baby-Skyrme model

2.1. The model and the topological charge. We introduce the $\mathbb{CP}^1$ model, the so-called baby-Skyrme model. The full canonical quantization of this model was already studied in [11] but without the Hopf term in the action. There are many studies of fermions coupled with the nonlinear sigma model [5], [6], [8], including analysis of the spectral flow (see, e.g., [12] for a recent study). The Lagrangian of this model is written as

$$L_{\text{baby}} = M^2 \partial_\mu \vec{n} \cdot \partial^\mu \vec{n} + \frac{1}{e^2} (\partial_\mu \vec{n} \times \partial_\nu \vec{n}) \cdot (\partial_\sigma \vec{n} \times \partial^\sigma \vec{n}) - \mu^2 V, \quad (1)$$

where $\vec{n} = (n_1, n_2, n_3)$ is constrained to the surface of a sphere of unit radius: $\vec{n} \cdot \vec{n} = 1$. The positive parameter $M^2$ is a coupling constant with the dimension of mass, and the coupling constant $e^{-2}$ has the dimension of inverse mass ($e^2$ must be negative for the Hamiltonian to be positive). The potential term containing no derivative is denoted by $\mu^2 V$, and $\mu^2$ is the coupling strength. The boundary condition $\vec{n}_\infty = (0, 0, 1)$ allows a one-point compactification of the space $\mathbb{R}^2 \to S^2$. Therefore, skyrmions arise for a map $\vec{n}: S^2 \to S^2$. This map belongs to an equivalence class characterized by the homotopy group $\pi_2(S^2) = \mathbb{Z}$. The solutions called baby skyrmions are obtained by solving the Euler–Lagrange equations for (1) by introducing an appropriate ansatz or by simplifying the Hamiltonian with numerical algorithms, for example, simulated annealing. For our numerical analysis, we use the standard hedgehog ansatz

$$\vec{n} = (\sin f(r) \cos n\varphi, \sin f(r) \sin n\varphi, \cos f(r)) \quad (2)$$

with the boundary condition

$$f(0) = \pi, \quad f(\infty) = 0, \quad n \in \mathbb{Z}. \quad (3)$$

The topological invariant is

$$Q_{\text{top}} = -\frac{1}{4\pi} \int d^2 x \vec{n} \cdot (\partial_1 \vec{n} \times \partial_2 \vec{n}). \quad (4)$$

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In terms of ansatz (2) with boundary condition (3), we easily obtain \( Q_{\text{top}} \equiv n \). To discuss quantization, it is more convenient to use the \( SU(2) \)-valued field \( U := \vec{\tau} \cdot \vec{n} \). We can rewrite the Lagrangian as

\[
\mathcal{L}_{\text{baby}} = \frac{M^2}{2} \text{Tr}(\partial_\mu U \partial^\mu U) - \frac{1}{8e^2} \text{Tr}([\partial_\mu U, \partial_\nu U]^2) - \mu^2 V.
\]

The topological current is defined in terms of \( U \) as

\[
j^\mu(U) = \frac{i}{16\pi} \epsilon^{\mu\nu\lambda} \text{Tr}(U \partial_\nu U \partial_\lambda U),
\]

and topological charge (4) is expressed by

\[
Q_{\text{top}} = \frac{i}{16\pi} \int d^2 x j^0(U) = \frac{i}{16\pi} \int d^2 x \epsilon_{ij} \text{Tr}(U \partial_i U \partial_j U).
\]

We again mention that the analogue of the Wess–Zumino–Witten term for baby skyrmions was already discussed in the literature [7], [8]. It is given by

\[
S^{(3)}_{\text{WZW}} = \frac{\Theta}{128\pi^2} \int_{M_4} d^4 x \epsilon^{\mu\nu\alpha\beta} \text{Tr}[U \partial_\mu U \partial_\nu U \partial_\alpha U \partial_\beta U]
\]

or, in a more convenient form easily obtained from (8),

\[
S^{(3)}_{\text{WZW}} = \Theta \int_{\partial M_4} d^3 x \mathcal{L}_{\text{Hopf}}, \quad \mathcal{L}_{\text{Hopf}} := \frac{1}{4\pi^2} \epsilon^{\mu\nu\alpha} a_\mu \partial_\nu a_\alpha,
\]

where \( a_\mu := -iZ^1 \partial_\mu Z \). These complex coordinates and the field \( U \) are related by \( U := 1 - 2Z \otimes Z^1 \). Because \( \Pi_4(\mathbb{C}P^1) \) is trivial, the prefactor \( \Theta \) does not require quantization. This coefficient is sometimes called the anyon angle because it is related to the fractional angular momentum of the baby skyrmions.

For the subsequent analysis, we introduce dimensionless coordinates \((t, \rho, \varphi)\) defined as

\[
x^0 = r_0 t, \quad x^1 = r_0 \rho \cos \varphi, \quad x^2 = r_0 \rho \sin \varphi,
\]

where the length scale \( r_0 \) is defined in terms of the coupling constants, \( r_0^2 = 4/(M^2|e|^2) \), and the speed of light is \( c = 1 \) in the natural units. The line element \( ds^2 \) is

\[
ds^2 = r_0^2(dt^2 - d\rho^2 - \rho^2 d\varphi^2).
\]

### 2.2. Normalizable fermions.

The fermion–vortex system was first studied by Jackiw and Rossi [13]. It is well known that a fermionic effective model coupled to the baby skyrmion with a gap \( mU \) by integrating over the fermionic field leads to an effective Lagrangian containing a baby-Skyrme-like model and some topological terms including the Hopf term [5], [6].

We consider a gauged model

\[
\mathcal{L}_{\text{term}} = i\bar{\psi}\gamma^\alpha(\partial_\alpha - iA_\alpha)\psi - m\bar{\psi}U\psi \equiv i\bar{\psi}D_\alpha\psi, \quad \alpha = 1, 2, 3,
\]

where \( m \) is the coupling constant of baby skyrmions to fermions. The matrices \( \gamma^\alpha \) are defined standardly: \( \gamma^1 = -i\sigma_1, \gamma^2 = -i\sigma_2, \) and \( \gamma^3 = \sigma_3 \), where \( \sigma_\alpha \) are standard Pauli matrices. Under an appropriate rescaling
of the Lagrangian, i.e., \( \psi \to r_{\theta}^{-1} \psi, A_\alpha \to r_{\theta}^{-1} A_\alpha, \) and \( m \to r_{\theta}^{-1} m, \) the system becomes dimensionless. The Euclidean partition function is

\[
Z = e^{\omega(U)} = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp(i \bar{\psi} \mathcal{D}_A \psi) = (\det \mathcal{D}_A)^{n_c},
\]

where \( n_c \) is the fermion degeneracy. We separate the effective action \( \omega(U) \) into real and imaginary parts:

\[
\omega(U) = n_c \text{Sp} \log(\mathcal{D}_A) = \text{Re} \omega(U) + i \text{Im} \omega(U),
\]

\[
\text{Re} \omega(U) = \frac{n_c}{2} \text{Sp} \log \mathcal{D}_A \mathcal{D}_A, \quad \text{Im} \omega(U) = \frac{n_c}{2i} \text{Sp} \log \left( \frac{i \mathcal{D}_A}{i \mathcal{D}_A} \right),
\]

where \( \text{Sp} \) denotes a full trace containing a functional and also a matrix trace involving the flavor and the spinor indices and \( \text{Tr} \) denotes the usual matrix trace.

There are many papers devoted to calculating the effective action \( \omega(U) = \log Z \) in terms of a derivative expansion. The result contains both the action of the model (in the real part) and the topological terms (in the imaginary part). After a lengthy calculation (see the appendix), we obtain the effective action in the Minkowski space–time:

\[
\text{Re} \omega(U) \bigg|_{A=0} = n_c \frac{|m|}{16\pi} \int d^3x \text{Tr}(\partial_\mu U \partial^\mu U) + \text{(higher derivative terms)},
\]

\[
\text{Im} \omega(U) = -n_c \int d^3x \left( j^\mu(U) A_\mu + \pi \text{sgn}(m) \mathcal{L}_{\text{Hopf}}(U) \right).
\]

If we regard the effective action as the action of a baby-Skyrme-type model, then we can define the anyon angle \( \Theta \) as \( \Theta \equiv n_c \pi \text{sgn}(m), \) and it is then quantized as usual spin statistics. Therefore, similarly to the case of the Skyrme model, the fermion number of the baby skyrmion coincides with its topological charge. Indeed, the invariance of \( \omega(U) \) under an isosinglet transformation leads to a conserved fermion current \[14\]. The analysis is perturbative, i.e., the expansion is only justified for small momenta compared with a physical cutoff.

There is another definition of the fermion number in the skyrmion background related to the distorted Dirac vacuum \[15\], \[16\],

\[
N_{\text{Cas}} := \frac{1}{2} \sum_\mu \left[ \text{sgn}(\epsilon_\mu) - \text{sgn}(\epsilon_\mu^0) \right],
\]

where \( \epsilon_\mu \) are the eigenvalues of the Hamiltonian

\[
i \mathcal{D} \bigg|_{A=0} := \gamma_3(i \partial_3 - \mathcal{H}_{\text{ferm}}), \quad \mathcal{H}_{\text{ferm}} = -i \gamma_3 \gamma_k \partial_k + \gamma_3 m U,
\]

and \( \epsilon_\mu^0 \) is the similar eigenvalue at \( U = 1. \) The number \( N_{\text{Cas}} \) is the Casimir energy, which counts the number of the negative energy levels minus those of the vacuum background. Expression (17) is directly obtained from the effective action \( \omega(U) \). Therefore, at least within the perturbative regime, both results should coincide: \( N_{\text{Cas}} = Q_{\text{top}}. \) In what follows, we numerically confirm this coincidence using the spectral flow analysis.

### 2.3. Numerical analysis

We present typical baby skyrmions with the topological charges \( Q_{\text{top}} = 1, 3 \) in Fig. 1. The explicit form of the Hamiltonian is

\[
\mathcal{H}_{\text{ferm}} = \begin{pmatrix}
mU & -e^{-i\varphi} \left( \partial_\rho - \frac{i \partial_\varphi}{\rho} \right) \\
e^{i\varphi} \left( \partial_\rho + \frac{i \partial_\varphi}{\rho} \right) & -mU
\end{pmatrix}.
\]


It can be shown that the Hamiltonian $H$-definition conditions. We can construct the plane wave basis as a cylinder of large radius $\mathcal{K}$-component of the isospin Pauli matrices.

The plane wave basis is defined as:

$$
\phi^{(p)}_{\mathcal{K}}(k^{(p)}, x) = \sum_{\epsilon} \phi^{(p)}_{\mathcal{K}}(k^{(p)}, x) \theta_{\epsilon}(k^{(p)}, x),
$$

where $\theta_{\epsilon}(k^{(p)}, x)$ is the third component of the orbital angular momentum and $\tau_3$ is the third component of the isospin Pauli matrices.

We briefly explain the numerical method for the spectrum of these fermions. According to the Rayleigh–Ritz variational principle, we can obtain the upper bound of the spectrum for each $\mathcal{K}$ from the secular equation:

$$
\det(A - iB) = 0,
$$

where

$$
A_{k^{(p)}k^{(q)}} = \int d^3 x \phi^{(p)}(k^{(p)}, x)^\dagger \mathcal{H}_{\text{ferm}} \phi^{(q)}(k^{(q)}, x),
$$

$$
B_{k^{(p)}k^{(q)}} = \int d^3 x \phi^{(p)}(k^{(p)}, x)^\dagger \phi^{(q)}(k^{(q)}, x).
$$

The plane wave basis is defined as

$$
\phi^{(u)}_{\mathcal{K}}(k^{(u)}, x) := \langle x | (u) \mathcal{K}; i \rangle = N^{(u)}_{i} \left(\epsilon_{i, \epsilon}^{(u)} + J_{\mathcal{K} - \frac{1}{2} + \frac{\sigma_3}{2}}(k^{(u)}, \rho) e^{i(K - \frac{1}{2} - \frac{\sigma_3}{2}) \phi}\right) \otimes \left(\begin{array}{c} 1 \\ 0 \end{array} \right),
$$

$$
\phi^{(d)}_{\mathcal{K}}(k^{(d)}, x) := \langle x | (d) \mathcal{K}; i \rangle = N^{(d)}_{i} \left(\epsilon_{i, \epsilon}^{(d)} + J_{\mathcal{K} + \frac{1}{2} + \frac{\sigma_3}{2}}(k^{(d)}, \rho) e^{i(K + \frac{1}{2} + \frac{\sigma_3}{2}) \phi}\right) \otimes \left(\begin{array}{c} 0 \\ 1 \end{array} \right),
$$

where

$$
\omega_{i, \epsilon}^{(p)+} = \omega_{i, \epsilon}^{(p)-} = 1, \quad \omega_{i, -\epsilon}^{(p)+} = \omega_{i, -\epsilon}^{(p)-} = \frac{-\text{sgn}(\epsilon) k_i^{(p)}}{|\epsilon| + m}, \quad \epsilon > 0, \quad p = u, d.
$$

We can construct the plane wave basis as a cylinder of large radius $D$. As a result of imposing the boundary conditions

$$
J_{\mathcal{K} + \frac{1}{2} - \frac{\sigma_3}{2}}(k^{(u)} D) = 0, \quad J_{\mathcal{K} - \frac{1}{2} + \frac{\sigma_3}{2}}(k^{(d)} D) = 0,
$$

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Fig. 2. The spectral flow corresponding to the solutions shown in Fig. 1 for (a) \( n = 1 \) and (b) \( n = 3 \): the background field is the vacuum field \( U_\infty \) for \( \lambda = 0 \) and the soliton field \( U \) for \( \lambda = 1 \). We plot the first 18 levels (9 positive and 9 negative energy levels at the vacuum \( \lambda = 0 \)). The coupling constant is chosen as \( m = 1.0 \), and the potential parameter is \( \tilde{\mu}^2 = 0.005 \).

we obtain a discrete set of wave numbers \( k^{(u)} \) and \( k^{(d)} \). We then have the orthogonality conditions
\[
\int_0^D d\rho \rho J_\nu(k_i^{(p)} \rho)J_\nu(k_j^{(p)} \rho) = \int_0^D d\rho \rho J_{\nu \pm 1}(k_i^{(p)} \rho)J_{\nu \pm}(k_j^{(p)} \rho) = \delta_{ij} \frac{D^2}{2} |J_{\nu \pm 1}(k_i^{(p)} D)|^2, \tag{26}
\]
where \( \nu = \frac{K}{2} \pm \frac{n}{2} \). We can solve Eq. (21) numerically. For the entire infinite set of wave numbers (which means an infinite size of the matrices), the eigenvalue \( \epsilon \) becomes exact. The normalization constants of the basis vectors are
\[
\mathcal{N}^{(u)}_i = \left[ \frac{2\pi D^2 |\epsilon|}{|\epsilon| + m} (J_K - \frac{\nu}{2} (k_i^{(u)} D))^2 \right]^{-1/2},
\]
\[
\mathcal{N}^{(d)}_i = \left[ \frac{2\pi D^2 |\epsilon|}{|\epsilon| + m} (J_K + \frac{\nu}{2} (k_i^{(d)} D))^2 \right]^{-1/2}. \tag{27}
\]

To obtain the spectral flow, we use the linearly interpolated field
\[
U(x, \lambda)_{\text{intp}} = (1 - \lambda)U_\infty + \lambda U(x), \quad 0 \leq \lambda \leq 1,
\]
where \( U(x) \) is the field with a nontrivial topology and \( U_\infty \) is the asymptotic field. For a given \( \lambda \), we smoothly connect the vacuum and the solitonic states. We substitute \( U_{\text{intp}}(x, \lambda) \) in Hamiltonian (19) and solve eigenproblem (21) by the standard matrix diagonalization algorithm of LAPACK.

In Fig. 2, we present some typical spectral flow results. Some special energy levels pass from the positive to the negative continuum. The number of fermionic levels crossing zero is always equal to the topological charges of \( U(x) \). As is seen, these levels are normalizable modes, and the behavior is then described by the index theorem.

The eigenfunction \( \psi(x) \) of Hamiltonian (19) in terms of the eigenstates of (21) has the form
\[
\psi_{K, \mu}(x) := \langle x | K; \mu \rangle = \sum_i [\langle x | (u) K; i \rangle \langle (u) K; i | \mu \rangle + \langle x | (d) K; i \rangle \langle (d) K; i | \mu \rangle]. \tag{28}
\]
We note that the grand spin \( K \) is a good quantum number; matrices (22) are block diagonal. Therefore, the summation for each \( K \) in (28) is only over \( i \). Hence, we can compute the normalized density of the fermion mode
\[
d_K(\rho) = |\tilde{\varphi} \psi_{K,0}(x) \gamma_3 \psi_{K,0}(x)|, \tag{29}
\]
Fig. 3. Fermion density (29): the eigenfunction is computed with the soliton background at $\lambda = 1$. The curves correspond to the bold lines in Fig. 2 (two lines almost merge in Fig. 2b). The potential parameter is $\tilde{\mu}^2 = 0.1$.

where $\psi_{K,0}$ indicates the mode of the spectral flow. Plots of the density are shown in Fig. 3. For numerical calculations, we chose the cylinder radius $D = 50.0$ and 512 points for discretizing the momenta. These spectral flow modes localize on the soliton and then become normalizable modes. The results show that the anyon angle $\Theta \equiv n_c \pi \text{sgn}(m)$ is determined by the number of the normalizable fermionic modes. The nature of the spin statistics of the baby skyrmion is thus consistent with the nature of the localized fermions.

3. The CP$^N$ model

3.1. The model and the field equations. We briefly sketch the Skyrme–Faddeev model on the target space $\mathbb{C}P^N$. We introduce the Lagrangian of the form

$$\mathcal{L}_{SF} = \frac{M^2}{2} \text{Tr} (\partial_\mu \Phi \partial^\nu \Phi) + \frac{1}{e^2} \text{Tr} ([\partial_\mu \Phi, \partial_\nu \Phi]^2) + \frac{\beta}{2} (\text{Tr} ([\partial_\nu \Phi \partial^\mu \Phi]^2) + \gamma (\text{Tr} (\partial_\mu \Phi \partial_\nu \Phi))^2 - \mu^2 V(\Phi),$$

(30)

where $M^2$ is a coupling constant with the dimension of mass, the coupling constants $e^{-2}$, $\beta$, and $\gamma$ have the dimension of inverse mass, and $V$ denotes a potential term that contains no derivative term and does not break local symmetries of the model. For the field $\Phi$, we use the parameterization with $N$ complex fields $u := \{u_i\}, i = 1, \ldots, N,$

$$\Phi = I_{(N+1) \times (N+1)} + \frac{2}{\vartheta^2} \begin{pmatrix} -u \otimes u^\dagger & iu \\ -iu^\dagger & -1 \end{pmatrix}, \quad \vartheta := \sqrt{1 + u^\dagger \cdot u}. $$

(31)

In terms of these fields, Lagrangian (30) is written as

$$\mathcal{L}_{SF} = -\frac{1}{2}[M^2 \eta_{\mu\nu} + C_{\mu\nu}] \tau^{\mu\nu} - \mu^2 V,$$

(32)

where

$$\tau_{\mu\nu} := -\frac{4}{\vartheta^2} [\vartheta^2 \partial_\nu u^\dagger \cdot \partial_\mu u - (\partial_\nu u^\dagger \cdot u)(u^\dagger \cdot \partial_\mu u)],$$

$$C_{\mu\nu} := M^2 \eta_{\mu\nu} - \frac{4}{e^2} \left[(\beta e^2 - 1) \tau^{\mu}_\rho \eta_{\nu\rho} + (\gamma e^2 - 1) \tau_{\mu\nu} + (\gamma e^2 + 2) \tau_{\rho\rho} \right].$$

(33)
Variation with respect to \( u_i^* \) leads to equations that after multiplication by the function inverse to \( \Delta_{ij}^2 \), i.e., by

\[
\Delta_{ij}^2 = \frac{1}{1 + u^\dagger \cdot u} (\delta_{ij} + u_i u_j^*),
\]

can be written in the form

\[
(1 + u^\dagger \cdot u)\partial^\mu (C_{\mu\nu} \partial^\nu u_i) - C_{\mu\nu}[(u^\dagger \cdot \partial^\mu u)\partial^\nu u_i + (u^\dagger \cdot \partial^\nu u)\partial^\mu u_i] +
\]

\[
+ \frac{\mu^2}{4} (1 + u^\dagger \cdot u) \sum_{k=1}^N \left[ (\delta_{ik} + u_i u_k^*) \frac{\delta V}{\delta u_k^*} \right] = 0.
\]

(34)

In what follows, we consider some example potentials. In the simplest case, where the potential is a function of absolute values of the fields, \( V(|u|^2, \ldots, |u_N|^2) \), the contribution related to this potential becomes

\[
\sum_{k=1}^N \left[ (\delta_{ik} + u_i u_k^*) \frac{\delta V}{\delta u_k^*} \right] = u_i \left[ \frac{\delta V}{\delta |u_i|^2} + \sum_{k=1}^N |u_k|^2 \frac{\delta V}{\delta |u_k|^2} \right].
\]

We consider the ansatz

\[
u_j = f_j(\rho)e^{i\eta_j \varphi}.
\]

(35)

The constants \( n_i \) form a set of integers. We define the diagonal matrix \( \lambda \equiv \text{diag}(n_1, \ldots, n_N) \) to simplify the form of some formulas below. The expressions \( \tau_{\mu\nu} \) have the forms

\[
\tau_{\rho\rho} \equiv \theta(\rho) = -\frac{4}{\rho^2} [\vartheta^2 f'T.f' - (f'T.f)(f'T.f')],
\]

\[
\tau_{\varphi\varphi} \equiv \omega(\rho) = -\frac{4}{\rho^2} [\vartheta^2 f'T.\lambda^2.f - (f'T.\lambda.f)^2],
\]

\[
\tau_{\rho\varphi} = -\tau_{\varphi\rho} \equiv \zeta(\rho) = -i\frac{4}{\rho^2} [\vartheta^2 f'T.\lambda.f - (f'T.\lambda.f)(f'T.f)],
\]

(36)

where the prime denotes the derivative with respect to \( \rho \) and \( T \) denotes transposition. The equations of motion in dimensionless coordinates become

\[
(1 + f'T.f) \left[ \frac{1}{\rho}(\rho \tilde{C}_{\rho\rho} f_k')' + \frac{i}{\rho} \left( \frac{\tilde{C}_{\varphi\varphi}}{\rho} \right)'(\lambda.f)_k - \frac{1}{\rho^2} \tilde{C}_{\varphi\varphi}(\lambda^2.f)_k \right] -
\]

\[
- 2 \left[ \tilde{C}_{\rho\rho}(f'T.f)' f_k' - \frac{1}{\rho^2} \tilde{C}_{\varphi\varphi}(f'T.\lambda.f)(\lambda.f)_k \right] +
\]

\[
+ \tilde{\mu}^2 \left( 1 + f'T.f \right)^2 \left[ \frac{\delta V}{\delta f_k^2} + \sum_{i=1}^N f_i^2 \frac{\delta V}{\delta f_i^2} \right] = 0
\]

(37)

for each \( k = 1, \ldots, N \), where we set \( \tilde{C}_{\mu\nu} := C_{\mu\nu}/M^2 \) and \( \tilde{\mu}^2 := (\vartheta^2/M^2)\mu^2 \). The components \( \tilde{C}_{\mu\nu} \) in the equations of motion are

\[
\tilde{C}_{\rho\rho} = -1 + (\beta e^2 - 1) \left( \theta + \frac{\omega}{\rho^2} \right) + (2\gamma e^2 + 1)\theta,
\]

\[
\tilde{C}_{\varphi\varphi} = -\rho^2 + \rho^2(\beta e^2 - 1) \left( \theta + \frac{\omega}{\rho^2} \right) + (2\gamma e^2 + 1)\omega,
\]

\[
\tilde{C}_{\rho\varphi} = -\tilde{C}_{\varphi\rho} = -3i\zeta.
\]

(38)
In the numerical computation, it is useful to introduce the scaled coordinate \( y \) and the variables \( g_j \):

\[
\rho = \sqrt{\frac{1 - y}{y}}, \quad f_j = \frac{1}{\sqrt{N}} \sqrt{\frac{1 - g_j}{g_j}}, \quad y \in (0, 1], \quad g_j \in (0, 1].
\]  

(39)

Taking the results in [17] and also the discussion in [1] into account, we can determine the topological charge in the present model. The field \( u \) yields a map from the plane \((x^1, x^2)\) to \( \mathbb{C}P^N \). But for the energy to be finite, the field must tend to a constant at spatial infinity. The plane \((x^1, x^2)\) is then topologically compactified into \( S^2 \), and the finite energy field configuration defines a map \( S^2 \rightarrow \mathbb{C}P^N \), which is classified into the homotopy classes of the group \( \pi_2(\mathbb{C}P^N) \). There is a theorem [17] according to which \( \pi_2(G/H) = \pi_1(H)_G \), where \( \pi_1(H)_G \) is the subset of \( \pi_1(H) \) formed by closed paths in \( H \) that can be contracted to a point in \( G \). In our case, the homotopy group is thus given by

\[
\pi_2(\mathbb{C}P^N) = \pi_1(SU(N) \otimes U(1))_{SU(N+1)} = \mathbb{Z}.
\]  

(40)

The topological charge, an element of the homotopy group, is given by the integral of the topological current defined in terms of the field \( \Phi \) as

\[
j^\mu(\Phi) = \frac{i}{16\pi} e^{\rho \lambda} \text{Tr}(\Phi \partial_\nu \Phi \partial_\lambda \Phi).
\]  

(41)

As noted in [1], [18], the topological charge is in fact equal to the number of poles of \( u_i \) including poles at infinity. Because the solutions behave as a holomorphic function near the boundaries, i.e., \( u_i \sim \rho^{n_i} e^{i n_i \phi} \) (where \( n_i \in \mathbb{Z} \)) near the origin and at spatial infinity, the topological charge is given by \( Q_{\text{top}} = n_{\text{max}} - |n_{\text{min}}| \), where \( n_{\text{max}} \) and \( n_{\text{min}} \) are the greatest positive integer and the least negative integer in the set \( n_1, \ldots, n_N \).

We now give the potential term in explicit form. In the general case, potential terms are a function of fields, which vanish at spatial infinity and preserve the local symmetries of the model. In this model, the simplest choice is the “old baby”-type potential \( \text{Tr}(1 - \Phi^{-1}_\infty \Phi) \), where \( \Phi_\infty \) is the value of the field at spatial infinity, i.e., \( \Phi_\infty := \lim_{\rho \rightarrow \infty} \Phi(\rho) \). Assuming that the solution and its holomorphic counterpart have the same asymptotic behavior at spatial infinity, we find that inverse of the principal variable \( \Phi \) goes to \( \Phi^{-1}_\infty := \text{diag}(-1, 1, 1) \) as \( \rho \rightarrow \infty \) for \( n_1 > n_2 > 0 \). We note that the inverse of the principal variable goes to \( \Phi^{-1}_0 := \text{diag}(1, 1, -1) \) as \( \rho \rightarrow 0 \). The expression \( \text{Tr}(1 - \Phi^{-1}_0 \Phi) \) can then be included as the “new baby” potential with two vacuums [19]. Finally, we consider the general form of the potential

\[
V = [(\text{Tr}(1 - \Phi^{-1}_0 \Phi))^a (\text{Tr}(1 - \Phi^{-1}_\infty \Phi))^b = \]

\[
= \frac{(g_1 + g_2 - 2g_1 g_2) (1 + g_2) a g_1 g_2}{(g_1 + g_2)^{a+b}},
\]  

(42)

where the integers \( a \geq 0 \) and \( b > 0 \).

Assuming that for \( n_2 < 0 \), the field \( u_2 \) behaves at zero as its holomorphic counterpart, i.e., \( \sim \rho^{n_2} \), we find that it tends to diverge as \( \rho \rightarrow 0 \). The inverse of the principal variable \( \Phi \) then goes to \( \Phi^{-1}_0 := \text{diag}(1, -1, 1) \) as \( \rho \rightarrow 0 \). The general form of the potential becomes

\[
V = \frac{(1 + |u_1|^2)^a (1 + |u_2|^b)}{(1 + |u_1|^2 + |u_2|^2)^{a+b}} = \frac{g_1 g_2^b (1 + g_1)^a (1 + g_2)^b}{(g_1 + g_2)^{a+b}},
\]  

(43)

where the integers \( a \geq 0 \) and \( b > 0 \).
3.2. Normalizable fermion modes. For the target space $\mathbb{C}P^N$, fermions with chiral symmetry coupled to the soliton were first discussed in [17]. It was confirmed that the normalizable zero mode of the fermion appears by virtue of the index theorem.

We consider a gauged model corresponding to Lagrangian (30),

$$L_{\text{ferm}} = i\bar{\psi} \gamma^\alpha (\partial_\alpha - ia_\alpha) \psi - m\bar{\psi} \Phi \equiv i\bar{\psi} D A \psi, \quad \alpha = 1, 2, 3. \quad (44)$$

The Euclidean partition function is defined as

$$Z = e^{\omega(\Phi)} = \int D\psi D\bar{\psi} \exp(i\bar{\psi} D A \psi) = (\det i D A)^{n_c}. \quad (45)$$

We obtain the real and the imaginary part of the effective action:

$$\text{Re} \omega(\Phi) \big|_{A=0} = n_c \frac{|m|}{16\pi} \int d^3 x \, \text{Tr}(\partial_\mu \Phi \partial^\mu \Phi) + (\text{higher derivative terms}), \quad (46)$$

$$\text{Im} \omega(\Phi) = -n_c \int d^3 x \left( j^\mu(\Phi) A_\mu + \pi \text{sgn}(m) L_{\text{Hopf}}(\Phi) \right). \quad (47)$$

The explicit form of the current $j^\mu$ coincides with (41). Consequently, as noted in [5], [6], [8], the anyon angle $\Theta$ is determinable in this fermionic context as $\Theta \equiv n_c \pi \text{sgn}(m)$ if the vortices are coupled to the fermionic field. But because $\Pi_3(\mathbb{C}P^N)$ is trivial, the Hopf term $L_{\text{Hopf}}$ itself is perturbative, and the value of the integral depends on the background classical solutions. Consequently, we cannot expect that this value becomes an integer. As a result, the solitons are always anyons even if $\Theta = n\pi, n \in \mathbb{Z}$ [20].

3.3. Numerical analysis. In Fig. 4, we show plots of typical soliton solutions for the topological charges $Q_{\text{top}} = 3, 4$. We chose the potential with $a = 0$ and $b = 2$ in (42) or (43).

The Hamiltonian has the form

$$H_{\text{ferm}} = -i\gamma_3 \gamma_k \partial_k + \gamma_3 m \Phi = \begin{pmatrix} m \Phi & -e^{-i\varphi} \left( \partial_\rho - \frac{i \partial_\varphi}{\rho} \right) \\ e^{i\varphi} \left( \partial_\rho + \frac{i \partial_\varphi}{\rho} \right) & -m \Phi \end{pmatrix}, \quad (48)$$

where $i D A \big|_{A=0} := \gamma_3 (i \partial_3 - H_{\text{ferm}})$. It can be shown that $H_{\text{ferm}}$ commutes with the operator of angular momentum, which we call a grand spin as before,

$$\mathcal{K} = \ell_3 + \frac{\sigma_3}{2} - \frac{n_1}{2} \left( \lambda_3 + \frac{1}{\sqrt{3}} \lambda_8 \right) + \frac{n_2}{2} \left( \lambda_3 - \frac{1}{\sqrt{3}} \lambda_8 \right), \quad (49)$$

Fig. 4. The solutions $g_1$ and $g_2$ with the potential with $(a, b) = (0, 2)$: the other parameters are $(\beta e^2, \gamma e^2, \tilde{\mu}^2) = (5.0, 5.0, 1.0)$. 

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where $\ell_3 = (\mathbf{r} \times \mathbf{p})_3 = -i \partial / \partial \varphi$ and $\lambda_3$ and $\lambda_s$ are Gell-Mann matrices.

The plane wave basis is

$$
\phi_{\mathcal{K}}^{(u)}(k_{i}^{(u)}, x) = N_i^{(u)} \left( \omega_{i,e}^{(u)} J_{\mathcal{K} - \frac{1}{2} - \frac{2n_1 - n_2}{3}}(k_{i}^{(u)} \rho)e^{i(\mathcal{K} - \frac{1}{2} + \frac{2n_1 - n_2}{3}) \varphi} \right) \otimes \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},
$$

$$
\phi_{\mathcal{K}}^{(d)}(k_{i}^{(d)}, x) = N_i^{(d)} \left( \omega_{i,e}^{(d)} J_{\mathcal{K} - \frac{1}{2} + \frac{2n_1 - n_2}{3}}(k_{i}^{(d)} \rho)e^{i(\mathcal{K} - \frac{1}{2} - \frac{2n_1 - n_2}{3}) \varphi} \right) \otimes \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},
$$

$$
\phi_{\mathcal{K}}^{(s)}(k_{i}^{(s)}, x) = N_i^{(s)} \left( \omega_{i,e}^{(s)} J_{\mathcal{K} - \frac{1}{2} + \frac{2n_1 - n_2}{3}}(k_{i}^{(s)} \rho)e^{i(\mathcal{K} - \frac{1}{2} - \frac{2n_1 - n_2}{3}) \varphi} \right) \otimes \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},
$$

where

$$
\omega_{i,e}^{(p)+} = \omega_{i,e}^{(p)-} = 1, \quad \omega_{i,e}^{(p)+} = \omega_{i,e}^{(p)-} = -\frac{\text{sgn}(\epsilon) k_{i}^{(p)}}{|\epsilon| + m}, \quad \epsilon > 0, \quad p = u, d, s. \quad (50)
$$

We construct the plane wave basis in a cylinder of large radius $\rho = D$. As a result of imposing the boundary conditions

$$
J_{\mathcal{K} - \frac{1}{2} + \frac{2n_1 - n_2}{3}}(k_{i}^{(u)} D) = 0, \quad J_{\mathcal{K} - \frac{1}{2} - \frac{2n_1 - n_2}{3}}(k_{i}^{(d)} D) = 0, \quad J_{\mathcal{K} - \frac{1}{2} + \frac{2n_1 - n_2}{3}}(k_{i}^{(s)} D) = 0,
$$

we obtain the discrete set of wave numbers $k_{i}^{(u)}$, $k_{i}^{(d)}$, and $k_{i}^{(s)}$. The orthogonality conditions are given by the relations

$$
\int_0^D d\rho \, \rho J_{\nu}(k_{i}^{(p)} \rho) J_{\nu}(k_{i}^{(q)} \rho) = \int_0^D d\rho \, \rho J_{\nu+1}(k_{i}^{(p)} \rho) J_{\nu+1}(k_{i}^{(q)} \rho) = \delta_{ij} \frac{D^2}{2} [J_{\nu+1}(k_{i}^{(p)} D)]^2, \quad (51)
$$

where

$$
\nu = \mathcal{K} - \frac{1}{2} + \frac{2n_1 - n_2}{3}, \quad \mathcal{K} = \frac{1}{2} + \frac{n_1 - n_2}{3}, \quad \mathcal{K} = \frac{1}{2} - \frac{n_1 + n_2}{3}.
$$

The eigenproblem can be solved numerically. If we take the entire infinite set of wave numbers (which means an infinite size of the matrices), then the spectrum $\epsilon$ becomes exact. The normalization constants for the basis vectors are

$$
N_{i}^{(u)} = \sqrt{\frac{2\pi D^2 |\epsilon|}{|\epsilon| + m} \left( J_{\mathcal{K} + \frac{1}{2} + \frac{2n_1 - n_2}{3}}(k_{i}^{(u)} D) \right)^2}^{-1/2},
$$

$$
N_{i}^{(d)} = \sqrt{\frac{2\pi D^2 |\epsilon|}{|\epsilon| + m} \left( J_{\mathcal{K} + \frac{1}{2} - \frac{2n_1 - n_2}{3}}(k_{i}^{(d)} D) \right)^2}^{-1/2}, \quad (52)
$$

$$
N_{i}^{(s)} = \sqrt{\frac{2\pi D^2 |\epsilon|}{|\epsilon| + m} \left( J_{\mathcal{K} + \frac{1}{2} - \frac{2n_1 - n_2}{3}}(k_{i}^{(s)} D) \right)^2}^{-1/2}.
$$

To obtain the spectral flow, we use the linearly interpolated field

$$
\Phi(x, \lambda)_{\text{int}} = (1 - \lambda) \Phi_\infty + \lambda \Phi(x), \quad 0 \leq \lambda \leq 1,
$$

where $\Phi(x)$ is the field with a nontrivial topology and $\Phi_\infty(x)$ is the asymptotic field. In Fig. 5, we present some typical plots of spectral flows. Some special energy levels pass from the negative to the positive...
The spectral flow corresponding to the solutions in Fig. 4: the first 34 levels (17 positive and 17 negative for the vacuum background $\lambda = 0$) are shown. The coupling constant is chosen as $m = 2.0$.

The number of fermionic levels crossing zero is always equal to the topological charge of the field $\Phi(x)$. As is seen, these levels are normalizable modes, and their behavior is then described by the index theorem.

The normalizable fermion density is given by (29). Plots of it are shown in Fig. 6. To save computer time in the numerical calculation, we chose the cylinder radius $D = 20.0$ and 192 points for discretizing the
Fig. 6. Fermion density (29) for the model with potential (42) with \((a, b) = (0, 2)\), and the other parameters \((\beta e^2, \gamma e^2, \bar{\mu}^2) = (1.1, 1.1, 1.0)\): the eigenfunctions are computed with the solitonic background at \(\lambda = 1\). The curve types correspond to the curves in the insets in Fig. 5.

4. Generalization to larger \(N\)

We have considered the cases \(N = 1, 2\), but our analysis can be directly generalized to larger \(N\). In fact, soliton equations of motion (37) are written for general \(N\): moreover, some results can be found in [2]. We therefore concentrate on discussing how the fermionic part should be treated for larger \(N\). It is easy to show that for general \(N\), the conserved quantum number commuting with Hamiltonian (48) is

\[
\mathcal{K} = \ell_3 + \frac{\sigma_3}{2} + \sum_{k=1}^{N} n_k \left[ \frac{1}{3} I_{(N+1) \times (N+1)} - \hat{I}_k \right],
\]

where \(\hat{I}_k = \text{diag}(0, \ldots, 1, 0, \ldots, 0)\). For \(N = 2\), (53) of course coincides with (49).

The plane wave basis is written as

\[
\phi^{(1)}_{\mathcal{K}}(k_i^{(1)}, \mathbf{x}) = N^{(1)}_i \left( \omega_{i,\epsilon}^{(1)} f_{P}(k_i^{(1)} \rho) e^{iP\phi} \left( \begin{array}{c} 1 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{array} \right) \right),
\]

where \(N^{(1)}_i = \sqrt{\frac{1}{2\pi}} \frac{1}{\sqrt{2i\epsilon}} \frac{1}{\sqrt{k_i^{(1)} \rho}} \frac{1}{\sqrt{\omega_{i,\epsilon}^{(1)}}} \).
and we introduce the notation

\[ \phi^{(k)}_k(k^{(k)}, x) = N^{(k)}_k \left( \begin{array}{c} \omega^{(k)+}_i J_{P-n_1+n_k}(k^{(k)}_i) e^{i(P-n_1+n_k)\varphi} \\ \omega^{(k)-}_i J_{P+1-n_1+n_k}(k^{(k)}_i) e^{i(P+1-n_1+n_k)\varphi} \end{array} \right) \otimes \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \]

(\text{where } k = 2, \ldots, N - 1 \text{ and } 1 \text{ in the column matrix is in the } k\text{th position}),

\[ \phi^{(N)}_k(k^{(N)}, x) = N^{(N)}_k \left( \begin{array}{c} \omega^{(N)+}_i J_{P-n_1}(k^{(N)}_i) e^{i(P-n_1)\varphi} \\ \omega^{(N)-}_i J_{P+1-n_1}(k^{(N)}_i) e^{i(P+1-n_1)\varphi} \end{array} \right) \otimes \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, \]

where

\[ \omega^{(p)+}_{i,\epsilon} = \omega^{(p)-}_{i,\epsilon} = 1, \quad \omega^{(p)+}_{i,\epsilon} = \omega^{(p)-}_{i,\epsilon} = -\text{sgn}(\epsilon) k^{(p)}_i \frac{1}{|\epsilon| + m}, \quad \epsilon > 0, \quad p = 1, \ldots, N, \]

and we introduce the notation

\[ P := K - \frac{1}{2} + \frac{2n_1 - \Sigma_{k=2}^{N} n_k}{3}. \]

The wave numbers \( k^{(k)} \) are discretized by the boundary conditions

\[ J_{P-n_1+n_k}(k^{(k)}_i) D = 0, \quad k = 1, \ldots, N - 1, \quad J_{P-n_1}(k^{(N)}_i) D = 0. \]

The orthogonality conditions are

\[ \int_0^D d\rho J_\nu(k^{(p)}_i) J_\nu(k^{(p)}_j) = \int_0^D d\rho J_\nu\pm 1(k^{(p)}_i) J_\nu\pm 1(k^{(p)}_j) = \delta_{ij} \frac{D^2}{2} |J_\nu\pm 1(k^{(p)}_i) D|, \]

where \( \nu = P - n_1 + n_k, \quad P - n_1 (k = 1, \ldots, N - 1) \). The normalization constants for the basis vectors are

\[ N^{(k)}_k = \left[ \frac{2\pi D^2 |\epsilon|}{|\epsilon| + m} (J_{P+1-n_1+n_k}(k^{(k)}_i) D)^2 \right]^{-1/2}, \quad k = 1, \ldots, N - 1, \]

\[ N^{(N)}_k = \left[ \frac{2\pi D^2 |\epsilon|}{|\epsilon| + m} (J_{P+1-n_1}(k^{(N)}_i) D)^2 \right]^{-1/2}. \]

In terms of this basis, Eq. (21) can be solved numerically.

5. Summary

We have studied the spectrum of the fermions coupled with the baby-Skyrme model or the \( CP^N \) Skyrme–Faddeev-type model. We computed the spectral flow of the fermionic one-particle state giving the level-crossing picture. The baby-skyrmions are assumed to be anyons because \( \Pi_4(\mathbb{C} P^1) \) is trivial. But the anyon angle \( \Theta = n_\pi \text{sgn}(m) \) should be an integer corresponding to the number \( Q_{\text{top}} \) of normalizable modes of fermions. Solutions of the \( CP^N \) model are anyons because the Hopf term is perturbative. On the other hand, \( Q_{\text{top}} \) normalizable states are localized on the soliton. Hence, in the case of the target space \( CP^N \), there is an inconsistency between the statistical natures of fermions and solitons. Perhaps, this inconsistency can be overcome; we will devote our next paper to this.
Appendix

In this appendix, we use the notation in [15]. The partition function is given by the integral

\[ Z = \int \mathcal{D} \psi \mathcal{D} \bar{\psi} \exp \left\{ \int d^3 x i \bar{\psi} \mathcal{D}_A \psi \right\} = (\det i \mathcal{D}_A)^{n_c}, \]  

(59)

where \( \psi \) and \( \bar{\psi} \) are Dirac fields and \( i \mathcal{D}_A = i \gamma_\alpha (\partial_\alpha - i A_\alpha) - m U \), where \( A_\alpha \) is a \( U(1) \) gauge field. The matrices \( \gamma_\alpha \) are defined as \( \gamma_\alpha = -i \sigma_\alpha \). The effective action \( \omega = n_c \log \det i \mathcal{D}_A \) is split into the real and imaginary parts

\[ \text{Re} \omega = \frac{n_c}{2} \text{Sp} \log \left( 1 + \frac{\text{Im} \gamma_\alpha \partial_\alpha U}{-\partial^2 + m^2} \right) \]

\[ = \frac{n_c}{2} \int d^3 x \int \frac{dk}{(2\pi)^3} \text{Tr} \log \left( 1 + \frac{\text{Im} \gamma_\alpha \partial_\alpha U}{-\partial^2 + m^2} \right) e^{ikx} = \]

\[ = \frac{n_c}{2} \int d^3 x \int \frac{dk}{(2\pi)^3} \text{Tr} \log \left( 1 + \frac{\text{Im} \gamma_\alpha \partial_\alpha U}{k^2 + m^2 - 2ik\partial - \partial^2} \right). \]

Expanding this expression in powers of \( 2ik\partial + \partial^2 \) and \( \gamma_\alpha \partial_\alpha U \), we obtain the first nonzero term

\[ \text{Re} \omega^{(2)} = -n_c \frac{|m|}{16\pi} \int d^3 x \text{Tr} (\partial_\mu U \partial_\mu U). \]

(62)

Taking the spinor trace and switching to the Minkowski metric, we obtain action (15).

Taking the variation \( \mathcal{D} \to \mathcal{D} + \delta \mathcal{D} \), \( \mathcal{D}_A \to \mathcal{D}_A + \delta \mathcal{D}_A \) as \( A_\mu \to 0 \), we obtain the variation of the imaginary part of the action,

\[ \delta \text{Im} \omega = \frac{n_c}{2i} \text{Tr} \left( \frac{1}{\mathcal{D} \mathcal{D}^\dagger} \delta \mathcal{D} - \frac{1}{\mathcal{D}^\dagger \mathcal{D}} \delta \mathcal{D} \right). \]

(63)

It contains the product of three derivatives:

\[ \delta \text{Im} \omega^{(3)} = -n_c \frac{\text{sgn}(m)}{32\pi} \int d^3 x e^{\mu\nu\lambda} \text{Tr} (\partial_\mu U \partial_\nu U \partial_\lambda U U) \delta U). \]

(64)

In terms of the new variable \( a_\mu := -i Z^\dagger \partial_\mu Z \), we can write the last formula as

\[ \text{Im} \delta \omega^{(3)} = n_c \frac{\text{sgn}(m)}{2\pi} \int d^3 x e^{\mu\nu\lambda} \delta a_\mu \partial_\nu a_\lambda. \]

(65)

It can be shown that this expression coincides with the variation of action (9). For \( A_\mu \neq 0 \), we also have the two-derivative component

\[ \text{Im} \delta \omega^{(2)} |_{A_\mu \neq 0} = -\frac{n_c}{16\pi i} \delta A_\mu \int d^3 x e^{\mu\nu\lambda} \text{Tr} (\partial_\nu U \partial_\lambda U U). \]

(66)

These two components contribute to the final expression (16).
Acknowledgments. One of the authors (N. S.) thanks the conference organizers of MQFT-2018 for the kind accommodation and hospitality.

Conflicts of interest. The authors declare no conflicts of interest.

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