Matching Conditions on Capillary Ripples: Polarization

Estudio de las condiciones de empalme para las oscilaciones de intercara entre dos fluidos. Polarización.

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The matching conditions at the interface between two non-mixed fluids at rest are obtained directly using the equation of movement of the whole media. This is a non-usual point of view in hydrodynamics courses and our aim is to fix ideas about the intrinsic information contained in the matching conditions, on fluids in this case. Afterward, it is analyzed the polarization of the normal modes at the interface and it is shown that this information can be achieved through a physical analysis and reinforced later by the matching conditions. A detailed analysis of the matching conditions is given to understand the role that plays the continuity of the stress tensor through the interface on the physics of the surface particle movement. The main importance of the viscosity of each medium is deduced.

I. INTRODUCTION

The boundary conditions constitute the key feature in any theory of surface waves. It is through them that one introduces in the analysis the physical consequences of specific surface effects, such as local changes of mass or stresses, thus going beyond the simple case in which one merely matches two semi-infinite bulk media at an interface (which, in particular, can be a free surface).

In the last years, increased attention has been paid to the properties of capillary waves by physicists and chemists. Ripples represent one of the few cases in which the relation between the dynamical properties of a surface and liquid flows can be predicted completely. The study of capillary ripples has clarified which properties of a liquid surface determine the surface’s resistance against deformation.

The boundary conditions for the stress at the interface are derived from the principle that the forces acting upon such an “interfacial” element result not only from viscous stresses in the liquid but also from stresses existing in the deformed interface. The cause of the difference is that an interface, unlike a three dimensional liquid, can not enjoy the property of incompressibility. Work done on an element of liquid is partially degrade into heat by viscous friction and partially transformed into kinetic energy which is transmitted to adjoining elements. Work done on an interfacial element leads, at least partially, to an increment of the surface potential energy. It is this potential energy of the deformed interface which enables the whole system, including the interface, to carry out an oscillatory motion.

In this article, first, it will be shown a non-traditional way to obtain the matching conditions at the interface between two non-mixed fluid at rest, considering the equation of motion of the whole media directly. Usually, university courses do not use this approach and state the boundary conditions from outside the constitutive equations governing the studied problem. It is important from a pedagogical point of view to evidence that in fact, the matching conditions at the interfaces are contained, almost in all cases, in the equation of movement for the whole system taken as the composition of all media. The

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other cases arise when the fluid interfaces have intrinsic properties not included on the equations of the media in that cases mentioned before.

This point of view was already used in [9–13] to elaborate a formalism, based on Surface Green Functions, which establish an isomorphism between solids and fluids related to the interface normal modes. The aim of this paper is to emphasize the fact that this way to establish the matching conditions allows us to introduce the physical characteristics related with the boundary conditions in a firmer floating.

II. EQUATIONS FOR THE BULK IN A VISCOUS INCOMPRESSIBLE FLUID.

To achieve a system of equations which describes the oscillatory motion of a viscous incompressible fluid, the starting point is the linearized Navier-Stokes equation for a viscous fluid [15], which reads:

\[
\rho \frac{\partial V_i}{\partial t} = \frac{\partial \tau_{ij}}{\partial x_j} + \rho F_{i,\text{ext}}
\]  

where \( \rho \) is the density of equilibrium, \( V_i \) the components of the fluid particle, \( F_{i,\text{ext}} \) are the external forces and \( \tau_{ij} \) is the stress tensor which, for a viscous incompressible fluid, has the following form [15],

\[
\tau_{ij} = -p\delta_{ij} + \eta \left( \frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i} \right)
\]  

(2)

Here \( p \) are the small variations of pressure, \( \eta \) is the viscosity parameter and \( \delta_{ij} \) is the unit matrix elements. The subscripts \( i \) and \( j \) take the values \( y, z \) identically. In (1) the sum over repeated subscripts is understood. It had been assumed that the system is symmetric with respect to the \( x \) direction.

Putting (2) in (1) it is obtained

\[
-\rho \frac{\partial V_y}{\partial t} + \frac{\partial}{\partial y} \left( -p + 2\eta \frac{\partial V_y}{\partial y} \right) + \frac{\partial}{\partial z} \left( \eta \left( \frac{\partial V_y}{\partial z} + \frac{\partial V_z}{\partial y} \right) \right) = 0
\]  

(3)

\[
-\rho \frac{\partial V_z}{\partial t} + \frac{\partial}{\partial y} \left( \eta \left( \frac{\partial V_z}{\partial y} + \frac{\partial V_y}{\partial z} \right) \right) + \frac{\partial}{\partial z} \left( -p + 2\eta \frac{\partial V_z}{\partial z} \right) = 0
\]  

(4)

where we have neglected the external forces.

Also the continuity equation is needed, which expresses that the volume of an element of the incompressible fluid does not change during the motion. It has the form [14]

\[
\frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} = 0
\]  

(5)

The first step is to obtain the equation of motion and its solution of the each medium taken as infinite. So, the parameters as density and viscosity are considered constants in the whole medium and it leads to transform equations (3) and (4) to:

\[
-\rho \frac{\partial V_y}{\partial t} - \frac{\partial p}{\partial y} + \eta \nabla^2 V_y = 0
\]  

(6)

\[
-\rho \frac{\partial V_z}{\partial t} - \frac{\partial p}{\partial z} + \eta \nabla^2 V_z = 0
\]  

(7)

where \( \nabla^2 = (\partial^2 / \partial y^2 + \partial^2 / \partial z^2) \).

The solution of the system (6), (7) and (12) written as a vector field velocity, can be putted as the sum of an irrotational field (related with the longitudinal mode) and a divergence free field (related with the transverse modes) [1], i.e.

\[
V = V_1 + V_2
\]  

(8)

which satisfy:

\[
\nabla \times V_1 = 0
\]  

(9)

\[
\nabla \cdot V_2 = 0
\]  

(10)

Any irrotational field is characterized by a scalar function, the “potential” function \( \varphi(y, z, t) \), such that

\[
V_1 = -\nabla \varphi
\]  

(11)

and the divergence free field can be described by a vector function [1], the “stream” or vorticity function \( \psi(y, z, t) \), such that

\[
V_2 = \left( -\frac{\partial \psi}{\partial z}, \frac{\partial \psi}{\partial y} \right)
\]  

(12)

The velocity components can thus be written in terms of the potential and the stream functions, as:

\[
V_y = \frac{\partial \varphi}{\partial y} - \frac{\partial \psi}{\partial z}
\]  

(13)

\[
V_z = -\frac{\partial \varphi}{\partial z} + \frac{\partial \psi}{\partial y}
\]  

(14)

Substitution of eqns. (13) and (14) in the continuity condition (5) gives

\[
\nabla^2 \varphi = 0
\]  

(15)

and substitution into eqns. (6) and (7) leads to

\[
\frac{\partial}{\partial y} \left( \rho \frac{\partial \varphi}{\partial t} - p \right) + \frac{\partial}{\partial z} \left( \rho \frac{\partial \psi}{\partial t} - \eta \nabla^2 \psi \right) = 0
\]  

(16)

\[
\frac{\partial}{\partial z} \left( \rho \frac{\partial \varphi}{\partial t} - p \right) - \frac{\partial}{\partial y} \left( \rho \frac{\partial \psi}{\partial t} - \eta \nabla^2 \psi \right) = 0
\]  

(17)

Equations (16) and (17) are simultaneously satisfied if one considers:
\[ \rho \frac{\partial \varphi}{\partial t} - p = C_1 \]  
\[ \rho \frac{\partial \psi}{\partial t} - \eta \nabla^2 \psi = C_2 \] (18) (19)

The constants \( C_1 \) and \( C_2 \) are obtained from the condition at zero flow, and it gives raise to \( C_1 = \rho_0 \) and \( C_2 = 0 \) where \( \rho_0 \) is the reference atmospheric pressure.

The solution of equations (18), (18) and (19) are looking for as the following

\[ \varphi(y, z, t) = \Phi(z) e^{i(\kappa y - \omega t)} \] (20)
\[ \psi(y, z, t) = \Psi(z) e^{i(\kappa y - \omega t)} \] (21)

where the parameters \( \kappa \) and \( \omega \) are the wavevector and frequency of the wave respectively.

Substituting eqns. (20) and (21) in (18) and (19) gives the \( z \)-dependence of the functions \( \varphi \) and \( \psi \), which satisfy:

\[ \frac{d^2 \Phi(z)}{dz^2} - \kappa^2 \Phi(z) = 0 \] (22)
\[ \frac{d^2 \Psi(z)}{dz^2} - \kappa^2 \Psi(z) = 0 \] (23)

with \( \kappa_1^2 = \kappa^2 - i \frac{\rho \omega}{\eta} \)

Equations (22) and (23) lead to the solutions:

\[ \Phi(z) = C_1 e^{\kappa z} + C_2 e^{-\kappa z} \] (24)
\[ \Psi(z) = C_3 e^{i\kappa z} + C_4 e^{-i\kappa z} \] (25)

and in combination with eqns. (20) and (21) they give a solution of the form:

\[ \varphi(y, z, t) = \left( C_1 e^{\kappa z} + C_2 e^{-\kappa z} \right) e^{i(\kappa y - \omega t)} \] (26)
\[ \psi(y, z, t) = \left( C_3 e^{i\kappa z} + C_4 e^{-i\kappa z} \right) e^{i(\kappa y - \omega t)} \] (27)

where \( C_1, C_2, C_3 \) and \( C_4 \) are constants to be determined by boundary and matching conditions.

Then, the expressions for the varying velocity components and the pressure are finally obtained by substitution of eqns. (20) and (27) into (18), (19) and (18), respectively.

### III. INTERFACE PROBLEM: MATCHING CONDITIONS.

Now we will match the two media. We consider a surface which at rest coincides with the plane \( z = 0 \) and it separates medium \( M_1 \) at \( z < 0 \) from medium \( M_2 \) at \( z > 0 \). Each one is viscous and incompressible.

First of all, any solution has to fulfill the continuity of the velocity field across the surface according to

\[ V_y^{(1)} = V_y^{(2)} \quad \text{at } z = 0 \] (28)
\[ V_z^{(1)} = V_z^{(2)} \quad \text{at } z = 0 \] (29)

Superscript 1 denotes medium \( M_1 \) and superscript 2 denotes medium \( M_2 \). The rest of the matching conditions at the interface are derived from the system of equations which govern the whole system. This is no usually done in the normal program courses at the Universities and we consider that it is important to state that the matching conditions are, almost in all of the cases, contained in the equation of movement for the whole system taken as the composition of each one. These are eqns. (6) and (7) where the parameters \( \eta \) and \( \rho \) are constant, but taking different values on each medium. Integrating these equations through the surface about \( z = 0 \) from \( -\epsilon \) to \( +\epsilon \) and later taking \( \epsilon \to 0 \), it is obtained from eq. (3)

\[ \left[ \eta_1 \left( \frac{\partial V_y^{(1)}}{\partial z} + \frac{\partial V_y^{(1)}}{\partial y} \right) \right]_{z = 0^-} = \left[ \eta_2 \left( \frac{\partial V_y^{(2)}}{\partial z} + \frac{\partial V_y^{(2)}}{\partial y} \right) \right]_{z = 0^+} \] (30)

and from eq. (4)

\[ \left[ -p_1 + 2\eta_1 \frac{\partial V_z^{(1)}}{\partial z} \right]_{z = 0^-} = \left[ -p_2 + 2\eta_2 \frac{\partial V_z^{(2)}}{\partial z} \right]_{z = 0^+} - p_\gamma \] (31)

where \( p_\gamma \) is the jump due to the surface tension according with the Laplace Law [14]. The other terms in eqns. (3) and (4) vanish when \( \epsilon \to 0 \) because they are continuous or have a finite jump at \( z = 0 \).

It is easily seen according to eq. (2) that eqns. (30) and (31) are the conditions of the continuity of the stress tensor components through the interface, as expected.

On the other hand, the boundary conditions of the problem are regularity at \( z \to \pm \infty \), which leads to eliminate 4 of the 8 constants appearing in (26) and (27) for the two media. Using the resulting functions \( \varphi \) and \( \psi \) for each medium in conditions (28)- (29) the following system is met:

\[
\begin{bmatrix}
-ik & ik & -q_{t1} & -q_{t2} \\
-1 & -1 & i & -i \\
-2ik^2\eta_1 & -2ik^2\eta_2 & -\eta_1Q_{t1} & \eta_2Q_{t1} \\
d_{41} & \eta_2Q_{t2} & d_{43} & 2ik\eta_2q_{t2}
\end{bmatrix}
\begin{bmatrix}
C_1 \\
C_2 \\
C_3 \\
C_4
\end{bmatrix}
= 0
\]

(32)

to determine the remaining constants, with the definitions \( d_{41} = -\eta_1Q_{t1} - i\Upsilon \), \( d_{43} = 2ik\eta_2q_{t1} - \Upsilon \), \( \Upsilon = \gamma\kappa_i^2/\omega \), \( Q_{t1} = 2\kappa_i^2 - i\rho_1/\eta_1 \) and \( Q_{t2} = 2\kappa_i^2 - i\rho_2/\eta_2 \).

In deducing the expression in system (32) which comes from (34) it was considered that

\[ p_\gamma = -\frac{\partial^2 z}{\partial y^2} \] (33)
on the surface, but as we are dealing with the velocity field, then it is obtained that:

\[ \frac{\partial p_\gamma}{\partial t} = -\gamma \frac{\partial^2 V_z}{\partial y^2} \]  

(34)

which finally leads to:

\[ p_\gamma = \frac{\gamma \kappa^2}{i\omega} (-\kappa C_1 + i\kappa C_3) e^{i(\kappa y - \omega t)} \]  

(35)

according to the substitution \( \partial/\partial t \rightarrow -i\omega \) and \( \partial/\partial y \rightarrow -\kappa^2 \).

The vanishing of the determinant of \([2]\) leads to the equation for the dispersion relation of the existing surface modes. It can be written as:

\[
\omega^2 \left[ (\rho_1 + \rho_2)(\rho_1 q_{12} + \rho_2 q_{11}) - \kappa(\rho_1 - \rho_2)^2 \right] + \\
+ \gamma \kappa^3 \left[ \rho_1 (\kappa - q_{12}) + \rho_2 (\kappa - q_{11}) \right] + \\
4 \kappa^3 (\eta_2 - \eta_1)^2 (\kappa - q_{12}) + \\
+ 4i \kappa^2 \omega (\eta_2 - \eta_1)(\rho_1 \kappa - \rho_2 \kappa - \rho_1 q_{12} + \rho_2 q_{11}) = 0
\]

(36)

This is the dispersion relation for capillary waves for the interface of two viscous incompressible fluids.

In order to obtain the constants \( C_1, C_2, C_3 \) and \( C_4 \), an initial stimulus is needed according to an initial value problem \([1]\) but this method, simple at the beginning, becomes rather complicated later and it is not good for a quite general study.

As was said on the introduction, the aim of this paper is not to get inside the dispersion relation of the normal modes at the interface of two viscous fluids at rest. For a better study of this subject we recommend paper \([3]\). We get here to show a way of solution also different from the usually taken as only the velocity vector as a function of an "stream function". From now on, we will put our attention on the matching conditions and we will show that more that a mathematical information of the matching can be found on it, but also the physics of the polarization can be deduced and how the interface moves in its oscillation.

From eqns. \([30]\) and \([31]\) it can be obtained more information about the polarization of the modes on the interface. This will be done in the next section.

IV. MATCHING CONDITIONS AND POLARIZATION.

There are two possible modes on the fluid: one in which the fluid particle moves in the direction of the wave propagation called longitudinal with notation \( L(V_y,0) \) and another transverse to the direction of the wave propagation and normal to the interface with notation \( TN(0,V_z) \).

The longitudinal mode is related to the fluid compressibility because this motion of the fluid particles is only possible when its volume changes \([4]\). This analysis also holds when the mode is on the interface, but this does not mean that there are no longitudinal modes of oscillation on the interface when the fluids involved are incompressible. It had been shown in \([1]\) that the interface, when oscillating, must be considered as a compressible one, because precisely its change in area is responsible for the increasing of its potential energy and therefore, for its oscillation. It is important to state that this is a fundamental argument to understand the movement of any interface in hydrodynamics.

Nevertheless, now it can be shown that in spite of the compressibility of the interface, there does not exist pure longitudinal mode if we are dealing with incompressible media. Let us demonstrate this.

If a point \( y_0 \) on the interface is considered moving with velocity, say \( V^S_{y0} \) in the \( y \)-axis direction, then, according to the continuity of velocity, the point \((y_0, -\epsilon)\) in \( M_1 \) and the point \((y_0, +\epsilon)\) in \( M_2 \) must have the same velocity \( V^S_{y0} \) if \( \epsilon \) is small enough. As the interface is compressible, at the point \( y_1 \) near enough \( y_0 \) the velocity can be, for instance, \( V^S_y \) different in general from \( V^S_{y0} \) but as the media are incompressible, at the point \((y_1, -\epsilon)\) and \((y_1, +\epsilon)\) the velocity must be \( V^S_{y0} \). See Fig. 1. This is not in conformity with the continuity of the velocity through the surface and hence the pure longitudinal mode is not possible and only the TN mode seems to be valid when the media are incompressible.

After these considerations during the above demonstration, the student can keep the idea that the interface oscillations can only occur in the \( z \)-axis. This is the accurate moment to show to the student that things not always are as they apparently seem to be, because that assumption does not take into account the different properties of each medium, whose response depends on its fundamental parameters, as density and viscosity, which are different for each medium. Hence, it is evident that it must be analyzed, precisely, the interface matching conditions.

It is useful to compare and to support the previous qualitative analysis with a quantitative and more profound one regarding the interface matching conditions.

Recalling carefully eqns. \([30]\) and \([31]\) and supposing that such a wave propagates in \( y \) direction with movement only in \( z \) direction (TN mode), then \( V^{(1)}_y = V^{(2)}_y = 0 \) and eq. \([30]\) becomes

\[
\eta_1 \frac{\partial V^{(1)}_z}{\partial y} \bigg|_{z=0^-} = \eta_2 \frac{\partial V^{(2)}_z}{\partial y} \bigg|_{z=0^+}
\]

(37)

It is known that \( V_z \) is continuous along the interface for all points. Then, the derivative with respect to \( y \) is also the same in both hands of \([37]\) and this expression only holds if \( \eta_1 = \eta_2 \), i.e., if the media have the same viscosity. It does not mean for the interface to disappear because the density of each medium can be different, as they apparently seem to be, but as the media are incompressible, at the point \((y_1, -\epsilon)\) and \((y_1, +\epsilon)\) the velocity must be \( V^S_{y0} \). See Fig. 1. This is not in conformity with the continuity of the velocity through the surface and hence the pure longitudinal mode is not possible and only the TN mode seems to be valid when the media are incompressible.
fulfill the continuity of the stress tensor in the $y$ direction given by eq. (30) yielding to a component of movement along $y$ direction. This mode will be called Sagittal mode or $S(V_y,V_z)$.

The above analysis was done for the general case. Now we are able to take the particular case in which one of the media is vacuum, for instance, $M_2$, with $\eta_2 = 0$. Then, condition (30) becomes

$$\left[ \eta_1 \left( \frac{\partial V_y^{(1)}}{\partial z} + \frac{\partial V_z^{(1)}}{\partial y} \right) \right]_z = 0$$

and it can be seen that also $V_y$ must be non zero on the surface to hold eq. (38) with the corresponding Sagittal polarization movement.

With this analysis on the conditions of stress component continuity in $y$ direction along the interface, it can be seen that if the two media are viscous (at least one of them), the fluid particle of the interface moves in a Sagittal mode which combines movement in both directions: along the wave propagation in $y$ direction, and normal to the interface in $z$ direction.

Then, it is qualitatively clear that the viscosity of each media plays a fundamental role in the coupling of modes even for incompressible fluids. In spite of that, it could be a mistake to say that the modes decouple if the viscosities are equal. It should not be forgotten that eq. (31) is also important in the characterization of the interface particle behaviour and it includes the pressure on each side of the surface. According to eq. (31), the pressure is associated with the longitudinal mode and the inertial effect of the fluid particle according with the density of the media. Then, it contains the information of each components of the velocity and also of the density and according to eq. (31) the pressure has a jump through the interface. This result, in combination with the analysis of eq. (31) make difficult to understand the role played by the densities of each media on the surface polarization movement, and it can not be reached from this only analysis. This point is still a matter of investigation.

V. CONCLUSIONS

The present work is an attempt to give an example, using the hydrodynamics, of how the study of the interface matching conditions allows us to make a plentiful and rich in details discussion. Moreover, of how the interface matching conditions contain a sufficient information to conclude that the interface oscillation must be with a Sagittal mode and not with neither a pure longitudinal, nor a pure transversal one. This movement has been shown to be close related to the physical properties of the media such as viscosities and that fact allows us to establish rigorously that those are the parameter which characterize the interface movement and the response of each medium to an stimulus coming from the other one.

It was seen that viscosity is the main parameter in the coupling of the two modes to achieve a Sagittal one, nevertheless within the framework of this formalism it is difficult to determine the role of viscosity and of the density ratios in the coupling of modes. This aspect needs further investigation.

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FIG. 1. Relation of velocities on the interface between two viscous fluids.
\[ v_{yo} \]

\[ v_{y1}^{(2)} \]

\[ y_0 \]

\[ y_1 \]

\[ +\varepsilon \]

\[ -\varepsilon \]

\[ v_{yo}^{s} \]

\[ v_{y1}^{s} \]