CALABI'S INHOMOGENEOUS EINSTEIN MANIFOLD IS GLOBALLY SYMPLECTOMORPHIC TO $\mathbb{R}^{2n}$

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Abstract. We construct explicit global symplectic coordinates for the Calabi's inhomogeneous Kähler–Einstein metric on tubular domains.

1. Introduction

Let $\omega$ be a Kähler form on a $n$-dimensional complex manifold $M$ diffeomorphic to $\mathbb{R}^{2n} \simeq \mathbb{C}^n$. One basic and fundamental question from the symplectic point of view is to understand when $(M, \omega)$ admits global symplectic coordinates, i.e. when there exists a global diffeomorphism $\Psi : M \to \mathbb{R}^{2n}$ such that $\Psi^* \omega_0 = \omega$, where $\omega_0 = \sum_{j=1}^n dx_j \wedge dy_j$ is the standard symplectic form on $\mathbb{R}^{2n}$ (the existence of a local symplectic diffeomorphism is guaranteed by the celebrated Darboux Theorem). In general the previous question has a negative answer after Gromov’s discovery [6] of the existence of exotic symplectic structures on $\mathbb{R}^{2n}$ (see also [1] for an explicit construction of a 4-dimensional symplectic manifold diffeomorphic to $\mathbb{R}^4$ which cannot be symplectically embedded in $(\mathbb{R}^4, \omega_0)$). Therefore it is natural to look for sufficient conditions, related to the Riemannian or to the complex structure of the manifold involved, which assure the existence of global symplectic coordinates. D. McDuff [5] proved a global version of Darboux Theorem for complete and simply-connected Kähler manifolds with non positive sectional curvature and in [5] the first author and A. Di Scala provide a construction of a global symplectomorphism from bounded symmetric domains equipped with the Bergman metric to $(\mathbb{R}^{2n}, \omega_0)$, using the theory of Jordan triple systems. Constructions of explicit global symplectic coordinates on some complex domains (e.g. Reinhardt domains or Lebrun’s Ricci–flat metric on $\mathbb{C}^2$) is given in [4] and [7].

In this paper we construct explicit global symplectic coordinates for the Calabi’s inhomogeneous Kähler–Einstein form $\omega$ on the complex tubular domains $M = \frac{1}{2} D_a \oplus i \mathbb{R}^n \subset \mathbb{C}^n$, $n \geq 2$, where $D_a \subset \mathbb{R}^n$ is the open ball of...
$\mathbb{R}^n$ centered at the origin and of radius $a$. Our main result is the following result (see next section for details):

**Theorem 1.** For all $n \geq 2$, the Kähler manifold $(M, \omega)$ is globally symplectomorphic to $(\mathbb{R}^{2n}, \omega_0)$ via the map:

$$\Phi: M \rightarrow \mathbb{R}^n \oplus i\mathbb{R}^n \simeq \mathbb{R}^{2n}, (x, y) \mapsto (\text{grad } f, y),$$

(1)

where $f: D_a \rightarrow \mathbb{R}, x = (x_1, \ldots, x_n) \mapsto f(x)$ is a Kähler potential for $\omega$, i.e. $\omega = \frac{i}{2} \partial \bar{\partial} f$, and $\text{grad } f = (\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n})$.

Notice that in [3, p. 23] Calabi provides an explicit formula for the curvature tensor of $(M, g)$ (he needs this formula to show that the metric $g$ associated to the Kähler form $\omega$ is not locally homogeneous). On the other hand it seems a difficult task to compute the sign of the sectional curvature of $g$ using Calabi’s formula. Consequently, it is not clear if $g$ satisfies or not the assumptions of McDuff’s theorem, namely if its sectional curvature is nonpositive. Nevertheless, it is worth pointing out that the proof of McDuff’s result is telling us that there exist global symplectic coordinates, but it is not giving any criterion to compute explicitly them as we did in Theorem 1.

We finally remark that our result should be used to give an explicit description of all Lagrangian submanifolds of $(M, \omega)$ which have classically played an important role in symplectic geometry.

## 2. CALABI’S METRIC AND THE PROOF OF THEOREM 1

Consider the complex tubular domain $M = \frac{1}{2} D_a \oplus i\mathbb{R}^n \subset \mathbb{C}^n$, as in the introduction. Let $g$ be the metric on $M \subset \mathbb{C}^n$ whose associated Kähler form is given by:

$$\omega = \frac{i}{2} \partial \bar{\partial} f(z_1 + \bar{z}_1, \ldots, z_n + \bar{z}_n),$$

(2)

where $f: D_a \rightarrow \mathbb{R}$ is a radial function $f(x_1, \ldots, x_n) = Y(r)$, for $r = (\sum_{j=1}^n x_j^2)^{1/2}$, for $x_j = (z_j + \bar{z}_j)/2$, $y_j = (z_j - \bar{z}_j)/2i$, that satisfies the differential equation:

$$(Y'/r)^{n-1}Y'' = e^Y,$$

(3)

with initial conditions:

$$Y'(0) = 0, \quad Y''(0) = e^{Y(0)/n}.$$  

(4)

In [3], Calabi proved that the Kähler metric $g$ so defined is smooth, Einstein, complete and not locally homogeneous. This was indeed the first example of such a metric. The reader is also referred to [3] for an alternative and easier proof of the fact that this metric is complete but not locally homogeneous.
Proof of Theorem 1. Let us prove first that the map $\Phi$ given by (1) satisfies $\Phi^*\omega_0 = \omega$. In order to simplify the notation we write $\partial f / \partial x_j = f_j$ and $\partial^2 f / \partial x_j \partial x_k = f_{jk}$. The pull-back of $\omega_0$ through $\Phi$ reads:

$$\Phi^*\omega_0 = \sum_{j=1}^{n} df_j \wedge dy_j = \sum_{j,k=1}^{n} f_{jk} \, dx_k \wedge dy_j = \frac{i}{2} \sum_{j,k=1}^{n} f_{jk} \, dz_j \wedge d\bar{z}_k,$$

thus the desired identity follows by:

$$\omega = \frac{i}{2} \partial\bar{\partial} f(z_1 + \bar{z}_1, \ldots, z_n + \bar{z}_n) = \frac{i}{2} \sum_{j,k=1}^{n} f_{jk} \, dz_j \wedge d\bar{z}_k.$$

Observe now that since $\omega$ and $\omega_0$ are non-degenerate it follows by the inverse function theorem that $\Phi$ is a local diffeomorphism. In order to conclude the proof it is then enough to verify that $\Phi$ is a proper map, from which it follows it is a covering map and hence a global diffeomorphism. In our situation this is equivalent to:

$$\lim_{(x,y) \to \partial M} \Phi(x,y) = \infty$$

or equivalently:

$$\lim_{x \to \partial D_a} ||\text{grad } f(x)|| = \infty.$$

This readily follows by $f_j(x) = \frac{z_j}{r} Y'(r)$ and the fact that $Y'(r)$ tends to infinity as $r \to \alpha$ (see [3, p. 21]).

\[\square\]

References

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