On the intrinsic geometry of a unit vector field

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Abstract

We study the geometrical properties of a unit vector field on a Riemannian 2-manifold, considering the field as a local imbedding of the manifold into its tangent sphere bundle with the Sasaki metric. For the case of constant curvature $K$, we give a description of the totally geodesic unit vector fields for $K = 0$ and $K = 1$ and prove a non-existence result for $K \neq 0, 1$. We also found a family $\xi_\omega$ of vector fields on the hyperbolic 2-plane $L^2$ of curvature $-c^2$ which generate foliations on $T_1L^2$ with leaves of constant intrinsic curvature $-c^2$ and of constant extrinsic curvature $-\frac{c^2}{4}$.

Keywords: Sasaki metric, vector field, sectional curvature, totally geodesic submanifolds.

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Introduction

A unit vector field $\xi$ on a Riemannian manifold $M$ is called holonomic if $\xi$ is a field of normals of some family of regular hypersurfaces in $M$ and non-holonomic otherwise. The geometry of non-holonomic unit vector fields has been developed by A. Voss at the end of the 19-th century. The foundations of this theory can be found in [1]. Recently, the geometry of a unit vector field has been considered from another point of view. Namely, let $T_1M$ be the unit tangent sphere bundle of $M$ endowed with the Sasaki metric [9]. If $\xi$ is a unit vector field on $M$, then one may consider $\xi$ as a mapping $\xi: M \rightarrow T_1M$ so that the image $\xi(M)$ is a submanifold in $T_1M$ with the metric induced from $T_1M$. So, one may apply the methods from the study of the geometry of submanifolds to determine geometrical characteristics of a unit vector field. For example, the unit vector field $\xi$ is said to be minimal if $\xi(M)$ is of minimal volume with respect to the induced metric [6]. A number of examples of locally minimal vector unit fields has been found (see [2, 3, 7]). On the other hand, using the geometry of submanifolds, we may find the Riemannian, Ricci or scalar curvature of a unit vector field using the second fundamental form of the submanifold $\xi(M) \in T_1M$ found in [11]. In this paper we apply this approach to the simplest case when the base space is 2-dimensional and hence the submanifold $\xi(M) \in T_1M$ is a hypersurface.

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1 The results

Let $\xi$ be a given unit vector field. Denote by $e_0$ a unit vector field such that $\nabla e_0 \xi = 0$. Denote by $e_1$ a unit vector field, orthogonal to $e_0$, such that $\nabla e_1 \xi = \lambda \eta$, where $\eta$ is a unit vector field, orthogonal to $\xi$. The function $\lambda$ is a signed singular value of a linear operator $\nabla : T^1 M \to \xi^\perp$ (acting as $(\nabla X)\xi = \nabla_X \xi$). Set $\nabla \xi \xi = k \eta$, $\nabla \eta \eta = \kappa \xi$. The functions $k$ and $\kappa$ are the signed geodesic curvatures of the integral curves of the fields $\xi$ and $\eta$ respectively. We prove that $\lambda^2 = k^2 + \kappa^2$.

Denote the signed geodesic curvatures of the integral curves of the fields $e_0$ and $e_1$ as $\mu$ and $\sigma$ respectively. Then $\nabla e_0 e_0 = \mu e_1$, $\nabla e_1 e_1 = \sigma e_0$.

The rientations of the frames $(\xi, \eta)$ and $(e_0, e_1)$ are independent. Set $s = 1$ if the orientations are coherent and $s = 0$ otherwise.

The following result (Lemma 3.2) is a basic tool for the study.

Let $M$ be a 2-dimensional Riemannian manifold of Gaussian curvature $K$. The second fundamental form $\Omega$ of the submanifold $\xi \subset T^1 M$ is given by

$$\Omega = \begin{bmatrix} -\mu \frac{\lambda}{\sqrt{1 + \lambda^2}} & (-1)^{s+1} \frac{K}{2} + \frac{\epsilon_0(\lambda)}{1 + \lambda^2} \\ (-1)^{s+1} \frac{K}{2} + \frac{\epsilon_0(\lambda)}{1 + \lambda^2} & e_1 \left( \frac{\lambda}{\sqrt{1 + \lambda^2}} \right) \end{bmatrix}.$$

Using the formula for the sectional curvature of $T^1 M^n$, we find an expression for the Gaussian curvature of $\xi(M^2)$ (Lemma 3.3).

The Gaussian curvature $K_\xi$ of a hypersurface $\xi(M) \subset T^1 M$ is given by

$$K_\xi = \frac{K^2}{4} + \frac{K(1 - K)}{1 + \lambda^2} + \frac{(-1)^{s+1}}{1 + \lambda^2} e_0(K) + \frac{1}{2} \mu e_1 \left( \frac{1}{1 + \lambda^2} \right) - \left( (-1)^{s+1} \frac{K}{2} + \frac{\epsilon_0(\lambda)}{1 + \lambda^2} \right)^2,$$

where $K$ is the Gaussian curvature of $M$.

As applications of these Lemmas, we prove the following theorems.

**Theorem 3.1** Let $M^2$ be a Riemannian manifold of constant Gaussian curvature $K$. A unit vector field $\xi$ generating a totally geodesic submanifold in $T^1 M^2$ exists if and only if $K = 0$ or $K = 1$. Moreover,
if $K = 0$, then $\xi$ is either a parallel vector field or moving along a family of parallel geodesics with constant angle speed. Geometrically, $\xi(M^2)$ is either $M^2$ imbedded isometrically into $M^2 \times S^1$ as a factor or a (helical) flat submanifold in $M^2 \times S^1$;

(b) if $K = 1$, then $\xi$ is a vector field on a standard sphere $S^2$ which is parallel along the meridians and moving along the parallels with a unit angle speed. Geometrically, $\xi(M^2)$ is a part of totally geodesic $\mathbb{R}P^2$ locally isometric to sphere $S^2$ of radius 2 in $T_1S^2 \overset{\text{isom}}{\approx} \mathbb{R}P^3$.

Theorem 3.3 Let $M^2$ be a space of constant Gaussian curvature $K$. Suppose that $\xi$ is a unit geodesic vector field on $M^2$. Then $\xi(M^2)$ has constant Gaussian curvature in one of the following cases:

(a) $K = -c^2 < 0$ and $\xi$ is a normal vector field for the family of horocycles on the hyperbolic 2-plane $L^2$ of curvature $-c^2$. In this case, $K_\xi = -c^2$ and therefore $\xi(M^2)$ is locally isometric the base space;

(b) $K = 0$ and $\xi$ is a parallel vector field on $M^2$. In this case $K_\xi = 0$ and $\xi(M^2)$ is also locally isometric to the base space;

(c) $K = 1$ and $\xi$ is any (local) geodesic vector field on the standard sphere $S^2$. In this case, $K_\xi = 0$.

Theorem 3.4 Let $L^2$ be a hyperbolic 2-plane of constant curvature $-c^2$. Then $T_1L^2$ admits a hyperfoliation with leaves of constant intrinsic curvature $-c^2$ and of constant extrinsic curvature $-\frac{c^2}{4}$. The leaves are generated by unit vector fields making a constant angle with a pencil of parallel geodesics on $L^2$.

2 Basic definitions and preliminary results

Let $(M, g)$ be an $(n + 1)$ – dimensional Riemannian manifold with metric $g$. Let $\nabla$ denote the Levi-Civita connection on $M$. Then $\nabla_X \xi$ is always orthogonal to $\xi$ and hence, $(\nabla_X \xi) \overset{\text{def}}{=} \nabla_X \xi : T_pM \rightarrow \xi_p^\bot$ is a linear operator at each $p \in M$. We define an adjoint operator $(\nabla \xi)^* X : \xi_p^\bot \rightarrow T_pM$ by

$$\langle (\nabla \xi)^* X, Y \rangle_g = \langle X, \nabla Y \xi \rangle_g.$$ 

Then there is an orthonormal frame $e_0, e_1, \ldots, e_n$ in $T_pM$ and an orthonormal frame $f_1, \ldots, f_n$ in $\xi_p^\bot$ such that

$$(\nabla \xi)e_0 = 0, \quad (\nabla \xi)e_\alpha = \lambda_\alpha f_\alpha, \quad (\nabla \xi)^* f_\alpha = \lambda_\alpha e_\alpha, \quad \alpha = 1, \ldots, n, \quad (1)$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are real-valued functions.

Definition 2.1 The orthonormal frames satisfying (1) are called singular frames for the linear operator $(\nabla \xi)$ and the real valued functions $\lambda_1, \lambda_2, \ldots, \lambda_n$ are called the (signed) singular values of the operator $\nabla \xi$ with respect to the singular frame.
Remark that the sign of the singular value is defined up to the directions of the vectors of the singular frame.

For each \( \tilde{X} \in T(p, \xi)TM \) there is a decomposition
\[
\tilde{X} = X^h + X^v
\]
where \((\cdot)^h\) and \((\cdot)^v\) are the horizontal and vertical lifts of vectors \(X_1\) and \(X_2\) from \(T_pM\) to \(T(p, \xi)TM\). The Sasaki metric is defined by the scalar product of the form
\[
\langle\langle \tilde{X}, \tilde{Y} \rangle \rangle = \langle X_1, Y_1 \rangle + \langle X_2, Y_2 \rangle,
\]
where \(\langle\cdot, \cdot\rangle\) means the scalar product with respect to metric \(g\).

The following lemma has been proved in [11].

**Lemma 2.1** At each point \((p, \xi) \in \xi(M) \subset TM\) the vectors
\[
\begin{cases}
\tilde{e}_0 = e_0^h, \\
\tilde{e}_\alpha = \frac{1}{\sqrt{1 + \lambda_\alpha^2}} (e_\alpha^h + \lambda_\alpha f_\alpha^v), \quad \alpha = 1, \ldots, n,
\end{cases}
\]
form an orthonormal frame in the tangent space of \(\xi(M)\) and the vectors
\[
\tilde{n}_\sigma = \frac{1}{\sqrt{1 + \lambda_\sigma^2}} (-\lambda_\sigma e_\sigma^h + f_\sigma^v), \quad \sigma = 1, \ldots, n,
\]
form an orthonormal frame in the normal space of \(\xi(M)\).

Let \(R(X, Y)\xi = [\nabla_X, \nabla_Y] \xi - \nabla_{[X,Y]} \xi\) be the curvature tensor of \(M\). Introduce the following notation
\[
r(X, Y)\xi = \nabla_X \nabla_Y \xi - \nabla_{\nabla_X Y} \xi.
\]
Then, evidently,
\[
R(X, Y)\xi = r(X, Y)\xi - r(Y, X)\xi.
\]

The following Lemma has also been proved in [11].

**Lemma 2.2** The components of second fundamental form of \(\xi(M) \subset T_1M\) with respect to the frame \([\xi]\) are given by
\[
\tilde{\Omega}_{\sigma00} = \frac{1}{\sqrt{1 + \lambda_\sigma^2}} \langle r(e_0, e_\sigma)\xi, f_\sigma \rangle,
\]
\[
\tilde{\Omega}_{\sigma0\alpha} = \frac{1}{\sqrt{1 + \lambda_\sigma^2}(1 + \lambda_\alpha^2)} \left[ \langle r(e_\alpha, e_0)\xi + r(e_0, e_\alpha)\xi, f_\sigma \rangle + \lambda_\sigma \lambda_\alpha \langle R(e_\sigma, e_\alpha)\xi, f_\alpha \rangle \right],
\]
\[
\tilde{\Omega}_{\sigma\alpha\beta} = \frac{1}{\sqrt{1 + \lambda_\sigma^2}(1 + \lambda_\alpha^2)(1 + \lambda_\beta^2)} \left[ \langle r(e_\alpha, e_\beta)\xi + r(e_\beta, e_\alpha)\xi, f_\sigma \rangle + \lambda_\alpha \lambda_\sigma \langle R(e_\sigma, e_\alpha)\xi, f_\beta \rangle + \lambda_\beta \lambda_\sigma \langle R(e_\sigma, e_\beta)\xi, f_\alpha \rangle \right],
\]
where \(\{e_0, e_1, \ldots, e_n; f_1, \ldots, f_n\}\) is a singular frame of \((\nabla \xi)\) and \(\lambda_1, \ldots, \lambda_n\) are the corresponding singular values.
Let $\tilde{\nabla}$ and $\nabla$ be the Levi-Civita connections of the Sasaki metric of $TM$ and the metric of $M$ respectively. The Kowalski formulas [8] give the covariant derivatives of combinations of lifts of vector fields.

**Lemma 2.3 (O.Kowalski)** Let $X$ and $Y$ be vector fields on $M$. Then at each point $(p, \xi) \in TM$ we have

$$\tilde{\nabla}_X Y = (\nabla_X Y)^h - \frac{1}{2} (R(X, Y)\xi)^v,$$

$$\tilde{\nabla}_X Y^v = \frac{1}{2} (R(\xi, Y)X)^h + (\nabla_X Y)^v,$$

$$\tilde{\nabla}_X Y^h = \frac{1}{2} (R(\xi, X)Y)^h,$$

$$\tilde{\nabla}_X Y^v = 0,$$

where $R$ is the Riemannian curvature tensor of $(M, g)$.

This basic result allows to find the curvature tensor of $TM$ (see [8]) and the curvature tensor of $T_1M$ (see [4]). As a corollary, it is not too hard to find an expression for the sectional curvature of $\xi(M)$. It is well-known that $\xi^v$ is a unit normal for $T_1M$ as a hypersurface in $TM$. Thus, $\tilde{X} = X^h + X^v_1$ is tangent to $T_1M$ if and only if $\langle X^v_2, \xi \rangle = 0$.

Let $\tilde{X} = X^h_1 + X^v_2$ and $\tilde{Y} = Y^h_1 + Y^v_2$, where $X_2, Y_2 \in \xi^\perp$, form an orthonormal base of a 2-plane $\tilde{\pi} \subset T_{(p, \xi)}T_1M$. Then we have [5]:

$$K(\tilde{\pi}) = \langle R(X_1, Y_1)Y_1, X_1 \rangle - \frac{3}{4} \|R(X_1, Y_1)\xi\|^2 + \frac{1}{4} \|R(\xi, Y_2)X_1 + R(\xi, X_2)Y_1\|^2 + \|X_2\|^2 \|Y_2\|^2 - \langle X_2, Y_2 \rangle^2 + 3\langle R(X_1, Y_1)Y_2, X_2 \rangle - \langle R(\xi, X_2)X_1, R(\xi, Y_2)Y_1 \rangle + \langle (\nabla_X R)(\xi, Y_2)Y_1, X_1 \rangle + \langle (\nabla_Y R)(\xi, X_2)X_1, Y_1 \rangle. \quad (5)$$

Combining the results of Lemma 2.1, Lemma 2.2 and (5), we can write an expression for the sectional curvature of $\xi(M)$.

**Lemma 2.4** Let $\tilde{X}$ and $\tilde{Y}$ be orthonormal vectors which span a 2-plane $\tilde{\pi}$ tangent to $\xi(M) \subset T_1M$. Denote by $K_\xi(\tilde{\pi})$ the sectional curvature $\xi(M)$ with respect to metric, induced by Sasaki metric of $T_1M$. Then

$$K_\xi(\tilde{\pi}) = \tilde{K}(\tilde{\pi}) + \sum_{\sigma} \left( \Omega_\sigma(\tilde{X}, \tilde{X})\Omega_\sigma(\tilde{Y}, \tilde{Y}) - \Omega^2_\sigma(\tilde{X}, \tilde{Y}) \right), \quad (6)$$

where $\tilde{K}(\tilde{\pi})$ is the sectional curvature of $T_1M$ given by [6], $\Omega_\sigma$ are the components of the second fundamental form of $\xi(M)$ given by Lemma 2.2 and the vectors are given with respect to the frame [5].
3 The 2-dimensional case

Let $M$ be a 2-dimensional Riemannian manifold. The following proposition gives useful information about the relation between the singular values of the $(\nabla \xi)$-operator, geometric characteristics of the integral curves of singular frame and the Gaussian curvature of the manifold.

Lemma 3.1 Let $\xi$ be a given smooth unit vector field on $M^2$. Denote by $e_0$ a unit vector field on $M^2$ such that $\nabla_{e_0} \xi = 0$. Let $\eta$ and $e_1$ be the unit vector fields on $M^2$ such that $(\xi, \eta)$ and $(e_0, e_1)$ form two orthonormal frames on $M^2$. Denote by $\lambda$ a signed singular value of the operator $(\nabla \xi)$. Then we have

$$\nabla_{e_1} \xi = \lambda \eta,$$

and the following relations hold:

(a) if $k = \langle \nabla_\xi \xi, \eta \rangle$ is a signed geodesic curvature of a $\xi$-curve and $\kappa = \langle \nabla_\eta \eta, \xi \rangle$ is a signed geodesic curvature of a $\eta$-curve, then

$$\lambda^2 = k^2 + \kappa^2;$$

(b) if $K$ is the Gaussian curvature of $M^2$, then

$$(-1)^s K = e_0(\lambda) - \lambda \sigma,$$

where $\sigma = \langle \nabla_{e_1} e_1, e_0 \rangle$ is a signed geodesic curvature of a $e_1$-curve and

$$s = \begin{cases} 1 & \text{if the frames } (\xi, \eta) \text{ and } (e_0, e_1) \text{ have the same orientation,} \\ 0 & \text{if the frames } (\xi, \eta) \text{ and } (e_0, e_1) \text{ have an opposite orientation} \end{cases}.$$

Proof. (a) If $(\xi, \eta)$ is an orthonormal frame on $M^2$, then

$$\nabla_\xi \xi = k \eta, \quad \nabla_\xi \eta = -k \xi,$$
$$\nabla_\eta \xi = -\kappa \eta, \quad \nabla_\eta \eta = \kappa \xi.$$ (7)

Geometrically, the functions $k$ and $\kappa$ are the signed geodesic curvatures of $\xi$- and $\eta$-curves respectively.

In a similar way we get

$$\nabla_{e_0} e_0 = \mu e_1, \quad \nabla_{e_0} e_1 = -\mu e_0,$$
$$\nabla_{e_1} e_0 = -\sigma e_1, \quad \nabla_{e_1} e_1 = \sigma e_0.$$ (8)

where $\mu$ and $\sigma$ are the signed geodesic curvatures of the $e_0$- and $e_1$-curves respectively.

Let $\omega$ be an angle function between $\xi$ and $e_0$. Then we have two possible decompositions:

$$\text{Or}(+) \begin{cases} e_0 = \cos \omega \xi + \sin \omega \eta, \\ e_1 = -\sin \omega \xi + \cos \omega \eta, \end{cases} \quad \text{Or}(-) \begin{cases} e_0 = \cos \omega \xi + \sin \omega \eta, \\ e_1 = \sin \omega \xi - \cos \omega \eta. \end{cases}$$
In the case $O(r(+)$ we have
\[
\nabla_{e_0} \xi = (k \cos \omega - \kappa \sin \omega) \eta,
\]
\[
\nabla_{e_1} \xi = -(k \sin \omega + \kappa \cos \omega) \eta,
\]
and due to the choice of $e_0$ and $e_1$ we see that
\[
\begin{align*}
    k \cos \omega - \kappa \sin \omega &= 0, \\
    k \sin \omega + \kappa \cos \omega &= -\lambda.
\end{align*}
\]
So, for the case of $O(r(+) \quad k = -\lambda \sin \omega, \quad \kappa = -\lambda \cos \omega.
In a similar way, for the case of $O(r(-) \quad k = \lambda \sin \omega, \quad \kappa = \lambda \cos \omega. In both cases
\[
\lambda^2 = k^2 + \kappa^2.
\]
(b) Due to the choice of the frames,
\[
\langle R(e_0, e_1)\xi, \eta \rangle = \langle \nabla_{e_0} \nabla_{e_1} \xi - \nabla_{e_1} \nabla_{e_0} \xi - \nabla_{e_0} e_1 - \nabla e_0 e_1, \eta \rangle = \langle \nabla_{e_0} (\lambda \eta) - \nabla_{-\mu e_0 + \sigma e_1} \xi, \eta \rangle = e_0(\lambda) - \lambda \sigma.
\]
On the other hand,
\[
\langle R(e_0, e_1)\xi, \eta \rangle = \begin{cases}
-K & \text{for the case of } O(r(+), \\
+K & \text{for the case of } O(r(-).
\end{cases}
\]
Set $s = 1$ for the case $O(r(+)$ and $s = 0$ for the case $O(r(-).$ Combining the results, we get $(-1)^s K = e_0(\lambda) - \lambda \sigma$, which completes the proof.

The result of Lemma 2.2 can also be simplified in the following way.

**Lemma 3.2** Let $M$ be a 2-dimensional Riemannian manifold of Gaussian curvature $K$. In terms of Lemma 3.1 the second fundamental form of the submanifold $\xi(M) \subset T_1 M$ can be presented in two equivalent forms:

(i) 
\[
\Omega = \begin{bmatrix}
-\mu \frac{\lambda}{\sqrt{1 + \lambda^2}} & \frac{(-1)^{s+1} K}{2} + \frac{e_0(\lambda)}{1 + \lambda^2} \\
(1 + 1)^{s+1} K \frac{e_0(\lambda)}{1 + \lambda^2} & e_1 \left( \frac{\lambda}{\sqrt{1 + \lambda^2}} \right)
\end{bmatrix},
\]

(ii) 
\[
\Omega = \begin{bmatrix}
-\mu \frac{\lambda}{\sqrt{1 + \lambda^2}} & \frac{1}{2} \left( \sigma \lambda + \frac{1 - \lambda^2}{1 + \lambda^2} e_0(\lambda) \right) \\
\frac{1}{2} \left( \sigma \lambda + \frac{1 - \lambda^2}{1 + \lambda^2} e_0(\lambda) \right) & e_1 \left( \frac{\lambda}{\sqrt{1 + \lambda^2}} \right)
\end{bmatrix}.
\]
Proof.

At each point \((p, \xi) \in \xi(M)\) the vectors
\[
\begin{align*}
\tilde{e}_0 &= e_0^h, \\
\tilde{e}_1 &= \frac{1}{\sqrt{1 + \lambda^2}}(e_1^h + \lambda \eta^v)
\end{align*}
\]
form an orthonormal frame in the tangent space of \(\xi(M)\) and
\[
\tilde{n} = \frac{1}{\sqrt{1 + \lambda^2}}(-\lambda e_1^h + \eta^v),
\]
is a unit normal for \(\xi(M) \subset T_1M\).

Thus we see that in a 2-dimensional case the components of \(\Omega\) take the form
\[
\begin{align*}
\Omega_{00} &= \frac{1}{\sqrt{1 + \lambda^2}} \langle r(e_0, e_0) \xi, \eta \rangle, \\
\Omega_{11} &= \frac{1}{(1 + \lambda^2)^{3/2}} \langle r(e_1, e_1) \xi, \eta \rangle, \\
\Omega_{01} &= \frac{1}{2(1 + \lambda^2)} \left[ \langle r(e_1, e_0) \xi + r(e_0, e_1) \xi, \eta \rangle + \lambda^2 \langle R(e_1, e_0) \xi, \eta \rangle \right].
\end{align*}
\]

Keeping in mind (4), (8) and (9), we see that
\[
\begin{align*}
\langle r(e_0, e_0) \xi, \eta \rangle &= -\mu \lambda, \\
\langle r(e_0, e_1) \xi, \eta \rangle &= e_0(\lambda), \\
\langle r(e_1, e_0) \xi, \eta \rangle &= \sigma \lambda, \\
\langle r(e_1, e_1) \xi, \eta \rangle &= e_1(\lambda), \\
\langle R(e_0, e_1) \xi, \eta \rangle &= (-1)^s K.
\end{align*}
\]

So we have
\[
\begin{align*}
\Omega_{00} &= -\mu \frac{\lambda}{\sqrt{1 + \lambda^2}}, \\
\Omega_{11} &= \frac{e_1(\lambda)}{(1 + \lambda^2)^{3/2}} = e_1 \left( \frac{\lambda}{\sqrt{1 + \lambda^2}} \right), \\
\Omega_{01} &= \frac{1}{2(1 + \lambda^2)}(e_0(\lambda) + \lambda \sigma - \lambda^2 (-1)^s K) = \begin{cases} (-1)^{s+1} \frac{K}{2} + \frac{e_0(\lambda)}{1 + \lambda^2} \\ \frac{1}{2} \left( \sigma \lambda + \frac{1 - \lambda^2}{1 + \lambda^2} e_0(\lambda) \right) \end{cases}
\end{align*}
\]
where Lemma 3.1 (b) has been applied in two ways.

\[
\boxdot
\]

3.1 Totally geodesic vector fields

The main goal of this section is to prove the following theorem.

**Theorem 3.1** Let \(M^2\) be a Riemannian manifold of constant Gaussian curvature \(K\). A unit vector field \(\xi\) generating a totally geodesic submanifold in \(T_1M^2\) exists if and only if \(K = 0\) or \(K = 1\). Moreover,
a) If \( K = 0 \), then \( \xi \) is either a parallel vector field or is moving along a family of parallel geodesics with constant angle speed. Geometrically, \( \xi(M^2) \) is either \( M^2 \) imbedded isometrically into \( M^2 \times S^1 \) as a factor or a (helical) flat submanifold in \( M^2 \times S^1 \);

b) If \( K = 1 \), then \( \xi \) is a vector field on a sphere \( S^2 \) which is parallel along the meridians and moving along the parallels with a unit angle speed. Geometrically, \( \xi(M^2) \) is a part of totally geodesic \( RP^2 \) locally isometric to sphere \( S^2 \) of radius 2 in \( T_1S^2 \approx RP^3 \).

The proof will be divided into a series of separate propositions.

**Proposition 3.1** Let \( M^2 \) be a Riemannian manifold. Let \( D \) be a domain in \( M^2 \) endowed with a semi-geodesic coordinate system such that \( ds^2 = du^2 + f^2 dv^2 \), where \( f(u,v) \) is some non-vanishing function. Denote by \( (e_0, e_1) \) an orthonormal frame in \( D \) and specify \( e_0 = \partial_u, e_1 = \frac{1}{f} \partial_v \). If \( \xi \) is a unit vector field in \( D \) parallel along \( u \)-geodesics, then \( \xi \) can be written given as

\[
\xi = \cos \omega e_0 + \sin \omega e_1,
\]

where \( \omega = \omega(v) \) is an angle function and

(a) a singular frame for \( \xi \) may be chosen as \( \{e_0, e_1, \eta = -\sin \omega e_0 + \cos \omega e_1\} \);

(b) a singular value for \( \xi \) in this case is \( \lambda = e_1(\omega) - \sigma \), where \( \sigma \) is a signed geodesic curvature of the \( e_1 \)-curves.

**Proof.** Indeed, if \( \xi \) is parallel along \( u \)-geodesics, then evidently the angle function \( \omega \) between \( \xi \) and the \( u \)-curves does not depend on \( u \). So this function has the form \( \omega = \omega(v) \) and \( \xi = \cos \omega e_0 + \sin \omega e_1 \). Moreover, since

\[
\nabla_{e_0} e_0 = 0, \quad \nabla_{e_0} e_1 = 0, \\
\nabla_{e_1} e_0 = \frac{\partial f}{\partial v} e_1, \quad \nabla_{e_1} e_1 = -\frac{\partial f}{\partial u} e_0,
\]

we see that \( \sigma = -\frac{\partial f}{\partial v} \) and \( \nabla_{e_1} \xi = (e_1(\omega) - \sigma) \eta \), where \( \eta = -\sin \omega e_0 + \cos \omega e_1 \).

Therefore, \( \lambda = e_1(\omega) - \sigma \) and the proof is complete.

**Proposition 3.2** Let \( M^2 \) be a Riemannian manifold of constant negative curvature \( K = -\frac{1}{\rho^2} < 0 \). Then there is no totally geodesic unit vector field on \( M^2 \).

**Proof.** Suppose \( \xi \) is totally geodesic unit vector field on \( M^2 \). Set \( \Omega \equiv 0 \) in Lemma 3.2. Then \( \lambda \mu \equiv 0 \). If \( \lambda \equiv 0 \) in some domain \( D \subset M^2 \), then \( \xi \) is parallel in this domain and hence \( M^2 \) is flat in \( D \), which contradicts the hypothesis. Suppose that \( \mu \equiv 0 \) at least in some domain \( D \subset M^2 \). This means that \( e_0 \)-curves are geodesics in \( D \) and the field \( \xi \) is parallel along them. Choose a family of \( e_0 \)-curves and the orthogonal trajectories as a local coordinate net in \( D \). Then the first fundamental form of \( M^2 \) takes the form

\[
ds^2 = du^2 + f^2 dv^2,
\]
where \( f(u,v) \) is some function. Since \( M^2 \) is of constant curvature \( K = -\frac{1}{r^2} \), the function \( f \) satisfies the equation

\[
 f_{uu} - \frac{1}{r^2} f = 0.
\]

The general solution of this equation is

\[
 f(u,v) = A(v) \cosh(u/r) + B(v) \sinh(u/r).
\]

There are two possible cases:

(i) \( A^2(v) \equiv B^2(v) \) over the whole domain \( D \);
(ii) \( A^2(v) \not\equiv B^2(v) \) in some subdomain \( D' \subset D \).

Case (i). In this case, in dependence of the signs of \( A(v) \) and \( B(v) \),

\[
 f(u,v) = A(v)e^{u/r} \quad \text{or} \quad f(u,v) = A(v)e^{-u/r}.
\]

Consider the first case (the second case can be reduced to the first one after the parameter change \( u \to -u \)). Making an evident \( v \)-parameter change, we reduce the metric to the form

\[
 ds^2 = du^2 + r^2 e^{2u/r} dv^2.
\]

Applying Proposition 3.1 for \( f = re^{u/r} \), we get \( \lambda = \frac{1}{r}(\omega' e^{-u/r} + 1) \). Setting \( \Omega_{11} \equiv 0 \), we see that \( e_1(\lambda) \equiv 0 \). Hence \( \omega'' = 0 \), i.e., \( \omega = av + b \). Therefore,

\[
 \lambda = \frac{1}{r} \left( a e^{-u/r} + 1 \right).
\]

Considering \( \Omega_{01} \equiv 0 \) (with \( s = 1 \) because of \( \text{Or}(+) \)-case), we get

\[
 -\frac{1}{2r^2} + \frac{1}{r^2}(ae^{-u/r} + 1)^2 = -\frac{(\frac{1}{r^2} + 1)(ae^{-u/r} + 1)^2 - a^2 e^{-2u/r}}{2r^2[1 + \frac{1}{r^2}(ae^{-u/r} + 1)^2]} \neq 0,
\]

and hence, this case is not possible.

Case (ii). Choose a subdomain \( D' \subset D \) such that \( A^2(v) < B^2(v) \) or \( A^2(v) > B^2(v) \) over \( D' \). Then the function \( f \) may be presented respectively in two forms:

(a) \( f(u,v) = \sqrt{B^2 - A^2} \sinh(u/r + \theta) \) \quad or \quad
(b) \( f(u,v) = \sqrt{A^2 - B^2} \cosh(u/r + \theta) \),

where \( \theta(v) \) is some function.

Consider the case (a). After a \( v \)-parameter change, the metric in \( D' \) takes the form

\[
 ds^2 = du^2 + r^2 \sinh^2(u/r + \theta) dv^2.
\]
Applying Proposition 3.1 for \( f = r \sinh(u/r + \theta) \), we get
\[
\lambda = \frac{\omega'}{r \sinh(u/r + \theta)} + \frac{1}{r} \coth(u/r + \theta).
\]
Considering \( \Omega_{11} \equiv 0 \), we have \( e_1(\lambda) \equiv 0 \) which implies the identity
\[
\omega'' \sinh(u/r + \theta) - \omega' \theta' \cosh(u/r + \theta) - \theta' \equiv 0.
\]
From this we get \( \omega'' = 0 \), \( \theta' = 0 \) and hence
\[
\theta = \text{const}, \quad \omega = a v + b \quad (a, b = \text{const}).
\]
After a parameter change we reduce the metric to the form
\[
ds^2 = du^2 + r^2 \sinh^2(u/r) dv^2.
\]
Applying Proposition 3.1 for \( f = r \sinh(u/r) \), we get \( \lambda = \frac{a + \cosh(u/r)}{r \sinh(u/r)} \). The substitution into \( \Omega_{01} \) gives
\[
-\frac{1}{2} \left( \frac{1}{r^2} + 1 \right) [a + \cosh(u/r)]^2 - a^2 + 1 
\]
\[
-\frac{1}{2} \frac{r^2 \sinh^2(u/r) + [a + \cosh(u/r)]^2}{r^2 \cosh^2(u/r) + [a + \cosh(u/r)]^2} \neq 0,
\]
which completes the proof for the polar case.

The Cartesian case consideration gives \( \omega = a v + b \), \( \lambda = \frac{a + \sinh(u/r)}{r \cosh(u/r)} \) and
\[
\Omega_{01} = -\frac{1}{2} \left( \frac{1}{r^2} + 1 \right) [a + \sinh(u/r)]^2 - a^2 - 1
\]
\[
-\frac{1}{2} \frac{r^2 \cosh^2(u/r) + [a + \sinh(u/r)]^2}{r^2 \sinh^2(u/r) + [a + \sinh(u/r)]^2} \neq 0,
\]
which completes the proof.

\begin{proof}
Suppose \( \xi \) is totally geodesic unit vector field on \( M^2 \). The same arguments as in Proposition 3.2 lead to the case \( \mu = 0 \) at least in some domain \( D \subset M^2 \). So, choose again a family of \( e_0 \)-curves and the orthogonal trajectories as a local coordinate net in \( D \). Then the first fundamental form of \( M^2 \) can be expressed as \( ds^2 = du^2 + f^2 dv^2 \), where \( f(u, v) \) is some function. Since \( M^2 \) is of constant curvature \( K = r^{-2} \), the function \( f \) satisfies the equation
\[
f_{uu} + \frac{1}{r^2} f = 0.
\]
The general solution of this equation \( f(u, v) = A(v) \cos(u/r) + B(v) \sin(u/r) \) may be presented in two forms:

(a) \( f(u, v) = \sqrt{A^2 + B^2} \sin(u/r + \theta) \) or

(b) \( f(u, v) = \sqrt{A^2 + B^2} \cos(u/r + \theta) \),
\end{proof}
where \( \theta(v) \) is some function.

Consider first, the case (a). After \( v \)-parameter change, the metric in \( D \) takes the form
\[
ds^2 = du^2 + r^2 \sin^2(u/r + \theta) \, dv^2.
\]
Applying Proposition 3.1 for \( f = r \sin(u/r + \theta) \), we get
\[
\lambda = \frac{\omega'}{r \sin(u/r + \theta)} + \frac{1}{r} \cot(u/r + \theta).
\]
Setting \( \Omega_{11} \equiv 0 \), we find \( e_1(\lambda) \equiv 0 \) which implies the identity
\[
\omega'' \sin(u/r + \theta) - \omega' \theta' \cos(u/r + \theta) + \theta' \equiv 0.
\]
From this \( \omega'' = 0 \), \( \theta' = 0 \) and we have again \( \left\{ \begin{align*}
\theta &= \text{const}, \\
\omega &= av + b \quad a, b = \text{const}.
\end{align*} \right. \)

After a suitable \( u \)-parameter change, we reduce the metric to the form
\[
ds^2 = du^2 + r^2 \sin^2(u/r) \, dv^2
\]
Applying Proposition 3.1 for \( f = r \sin(u/r) \), we get \( \lambda = \frac{a + \cos(u/r)}{r \sin(u/r)} \). Substitution into \( \Omega_{01} \) gives
\[
1 \left( \frac{1}{r^2} - 1 \right) [a + \cos(u/r)]^2 + a^2 - 1 = 0,
\]
which is possible only if \( r = 1 \) and \( |a| = 1 \). So, we obtain to the standard sphere metric
\[
ds^2 = du^2 + \sin^2 u \, dv^2
\]
and ( after the \( \pm v + b \to v \) parameter change ) the unit vector field
\[
\xi = \left\{ \cos v, \frac{\sin v}{\sin u} \right\}.
\]
This vector field is parallel along the meridians of \( S^2 \) and moves helically along the parallels of \( S^2 \) with unit angle speed.

For the case (b) one can find \( \omega = av + b \), \( \lambda = \frac{a - \sin(u/r)}{r \cos(u/r)} \) and
\[
\Omega_{01} = \frac{1}{2} \left( \frac{1}{r^2} - 1 \right) [a - \sin(u/r)]^2 + a^2 - 1 = 0.
\]
which gives \( r = 1 \) and \( |a| = 1 \) as a result. Thus, we have a metric
\[
ds^2 = du^2 + \cos^2 u \, dv^2
\]
and a vector field \( \xi = \left\{ \cos v, \frac{\sin v}{\cos u} \right\} \). It is easy to see that the results of cases (a) and (b) are geometrically equivalent.
Introduce the local coordinates \((u, v, \omega)\) on \(T_{1}S^{2}\), where \(\omega\) is the angle between arbitrary unit vector \(\xi\) and the coordinate vector field \(X_{1} = \{1, 0\}\). The first fundamental form of \(T_{1}S^{2}\) with respect to these coordinates is

\[d\tilde{s}^{2} = du^{2} + dv^{2} + 2 \cos u \, dv \, d\omega + d\omega^{2}.\]

The local parameterization of the submanifold \(\xi(S^{2})\), generated by the given field, is \(\omega = v\) and the induced metric on \(\xi(S^{2})\) is

\[d\tilde{s}^{2} = du^{2} + 2(1 + \cos u) \, dv^{2} = du^{2} + 4 \cos^{2} u/2 \, dv^{2}.\]

Thus, \(\xi(S^{2})\) is locally isometric to sphere \(S^{2}\) of radius 2. Since \(T_{1}S^{2} \cong \mathbb{R}P^{3}\) and there are no other totally geodesic submanifolds in \(\mathbb{R}P^{3}\) except \(\mathbb{R}P^{2}\), we see that \(\xi(S^{2})\) is a part of \(\mathbb{R}P^{2}\). So the proof is complete.

**Proposition 3.4** Let \(M^{2}\) be a Riemannian manifold of constant zero curvature \(K = 0\). Then a totally geodesic unit vector field \(\xi\) on \(M^{2}\) is either parallel or moves along the family of parallel geodesics with constant angle speed. Geometrically, \(\xi(M^{2})\) is either \(E^{2}\) imbedded isometrically into \(E^{2} \times S^{1}\) as a factor or a helical flat submanifold in \(E^{2} \times S^{1}\).

**Proof.** Suppose \(\xi\) is totally geodesic unit vector field on \(M^{2}\). Set \(\Omega \equiv 0\) in Lemma 3.2. Then \(\lambda \mu \equiv 0\). If \(\lambda \equiv 0\) over some domain \(D \subset M^{2}\), then \(\xi\) is parallel in this domain.

Suppose \(\lambda \neq 0\) in a domain \(D \subset M^{2}\). Then \(\mu \equiv 0\) on at least a subdomain \(D' \subset D\). This means that the \(e_{0}\)-curves are geodesics in \(D'\) and the field \(\xi\) is parallel along them. Choose a family of \(e_{0}\)-curves and the orthogonal trajectories as a local coordinate net in \(D'\). Then the first fundamental form of \(M^{2}\) takes the form \(ds^{2} = du^{2} + f^{2} \, dv^{2}\) and since \(M^{2}\) is of zero curvature, \(f\) satisfies the equation

\[f_{uu} = 0.\]

A general solution of this equation is \(f(u, v) = A(v)u + B(v)\). There are two possible cases:

(a) \(A(v) \neq 0\) in some subdomain \(D'' \subset D'\);
(b) \(A(v) \equiv 0\) over the whole domain \(D'\).

Case (a). The function \(f\) may be presented over \(D''\) in the form

\[f(u, v) = A(v)(u + \theta),\]

where \(\theta(v) = B(v)/A(v)\). After a \(v\)-parameter change, the metric in \(D''\) takes the form \(ds^{2} = du^{2} + (u + \theta)^{2} \, dv^{2}\). Applying Proposition 3.1 for \(f = u + \theta\), we get \(\lambda = \frac{\omega' + 1}{u + \theta}\). Setting \(\Omega_{11} \equiv 0\), we obtain the identity

\[\omega''(u + \theta) - (\omega' + 1)\theta' \equiv 0.\]
From this we get \( \omega'' = 0 \) or \( \omega' = -1 \). In the first case, \( \lambda = 0 \) and the field \( \xi \) is parallel again. In the second case \( \theta = \text{const}, \omega = av + b, a, b = \text{const} \).

Making a parameter change, we reduce the metric to the form
\[
ds^2 = du^2 + u^2 dv^2
\]
Applying Proposition 3.1 with \( f(u, v) = u \), we get \( \lambda = a + 1 \). The substitution into \( \Omega_{01} \) gives the condition
\[
-\frac{a + 1}{u^2 + (a + 1)^2} = 0
\]
which is possible only if \( a = -1 \). But this means that again \( \lambda = 0 \) and hence \( \xi \) is a parallel vector field.

Case (b). After a \( v \)-parameter change, the metric takes the form
\[
ds^2 = du^2 + dv^2.
\]
Applying Proposition 3.1 for \( f \equiv 1 \), we get \( \lambda = \omega' \). Setting \( \Omega_{11} \equiv 0 \), we find \( \omega'' = 0 \). This means that \( \omega = av + b \) and \( \xi \) is either parallel along the \( u \)-lines \( (a = 0) \) or moves along the \( u \)-lines helically with constant angle speed.

Let \( (u, v, \omega) \) be standard coordinates in \( E^2 \times S^1 \). Then the first fundamental form of \( E^2 \times S^1 \) is
\[
d\tilde{s}^2 = du^2 + dv^2 + d\omega^2.
\]
If \( a = 0 \), then with respect to these coordinates the local parameterization of \( \xi(E^2) \) is \( \omega = \text{const} \) and \( \xi(E^2) \) is nothing else but \( E^2 \) isometrically imbedded into \( E^2 \times S^1 \). If \( a \neq 0 \), then the local parameterization of \( \xi(E^2) \) is \( \omega = av + b \) and the induced metric is
\[
d\tilde{s}^2 = du^2 + (1 + a^2) dv^2
\]
which is flat. The imbedding is helical in the sense that this submanifold meets each flat element of the cylinder \( p : E^2 \times S^1 \to S^1 \) under constant angle \( \varphi = \arccos \frac{1}{\sqrt{1 + a^2}} \). So the proof is complete.

\section{The curvature}

The main goal of this section is to obtain an explicit formula for the Gaussian curvature of \( \xi(M^2) \) and apply it to some specific cases. The first step is the following lemma.

\begin{lemma}
Let \( \xi \) be a unit vector field on a 2-dimensional Riemannian manifold of Gaussian curvature \( K \). In terms of Lemma 3.1, the sectional curvature \( K_{T_1M}(\xi) \) of \( T_1M \) along 2-planes tangent to \( \xi(M) \) is given by
\[
K_{T_1M}(\xi) = \frac{K^2}{4} + \frac{K(1-K)}{1+\lambda^2} + (-1)^{s+1} \frac{\lambda}{1+\lambda^2} \epsilon_0(K).
\]
\end{lemma}
Proof. Let \( \hat{\pi} \) be a 2-plane tangent to \( \xi(M) \). Then \( \hat{X} = e_0^h \) and \( \hat{Y} = \frac{1}{\sqrt{1 + \lambda^2}}(e_1^h + \lambda \eta^e) \) form an orthonormal basis of \( \hat{\pi} \). So we may apply Lemma 3.2 setting \( X_1 = e_0, X_2 = 0, Y_1 = \frac{1}{\sqrt{1 + \lambda^2}} e_1, Y_2 = \frac{\lambda}{\sqrt{1 + \lambda^2}} \eta \).

We get

\[
\langle R(X_1, Y_1)X_1, X_1 \rangle = \frac{1}{1 + \lambda^2} \langle R(e_0, e_1)e_1, e_0 \rangle = \frac{1}{1 + \lambda^2} K,
\]

\[
\|R(X_1, Y_1)e_1\|^2 = \frac{1}{1 + \lambda^2} \|R(e_0, e_1)e_1\|^2 = \frac{1}{1 + \lambda^2} K^2,
\]

\[
\|R(\xi, Y_2)X_1\|^2 = \frac{\lambda^2}{1 + \lambda^2} \|R(\xi, \eta)e_0\|^2 = \frac{\lambda^2}{1 + \lambda^2} K^2,
\]

\[
\langle (\nabla_{X_1} R)(\xi, Y_2)X_1, X_1 \rangle = \frac{\lambda}{1 + \lambda^2} \langle (\nabla_{e_0} R)(\xi, \eta)e_1, e_0 \rangle = \frac{-(-1)^s \lambda}{1 + \lambda^2} \epsilon_0(K),
\]

where \( K \) is the Gaussian curvature of \( M \). Applying directly Lemma 3.2 we obtain

\[
K_{T_1 M}(\xi) = \frac{1}{1 + \lambda^2} \left( K - \frac{3}{4} K^2 + \frac{\lambda^2 K^2}{4} + (-1)^s \lambda \epsilon_0(K) \right)
\]

\[
= \frac{1}{1 + \lambda^2} \left( K(1 - K) + \frac{(1 + \lambda^2) K^2}{4} + (-1)^s \lambda \epsilon_0(K) \right)
\]

\[
= \frac{K^2}{4} + \frac{K(1 - K)}{1 + \lambda^2} + (-1)^s \frac{\lambda}{1 + \lambda^2} \epsilon_0(K).
\]

Now we have the following.

**Lemma 3.4** Let \( \xi \) be a unit vector field on a 2-dimensional Riemannian manifold \( M \). In terms of Lemma 3.2, the Gaussian curvature \( K_\xi \) of the hypersurface \( \xi(M) \in T_1 M \) is given by

\[
K_\xi = \frac{K^2}{4} + \frac{K(1 - K)}{1 + \lambda^2} + (-1)^s \lambda \frac{\epsilon_0(K)}{1 + \lambda^2} + \frac{1}{2} \mu e_1 \left( \frac{1}{1 + \lambda^2} \right) - \left( (-1)^s \frac{K}{2} + \frac{\epsilon_0(\lambda)}{1 + \lambda^2} \right)^2,
\]

where \( K \) is the Gaussian curvature of \( M \).

**Proof.** In our case, one can easily reduce the formula (5) to the form

\[
K_\xi = K_{T_1 M}(\xi) + \det \Omega.
\]

Applying Lemma 3.2 we see that

\[
\det \Omega = -\mu \frac{\lambda}{\sqrt{1 + \lambda^2}} e_1 \left( \frac{\lambda}{\sqrt{1 + \lambda^2}} \right) - \left( (-1)^s \frac{K}{2} + \frac{\epsilon_0(\lambda)}{1 + \lambda^2} \right)^2 =
\]

\[
-\frac{1}{2} \mu e_1 \left( \frac{\lambda^2}{1 + \lambda^2} \right) - \left( (-1)^s \frac{K}{2} + \frac{\epsilon_0(\lambda)}{1 + \lambda^2} \right)^2 =
\]

\[
\frac{1}{2} \mu e_1 \left( \frac{1}{1 + \lambda^2} \right) - \left( (-1)^s \frac{K}{2} + \frac{\epsilon_0(\lambda)}{1 + \lambda^2} \right)^2.
\]

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Combining this result with Lemma 3.3, we get what was claimed.

As an application of Lemma 3.4 we prove the following theorems.

**Theorem 3.2** Let $M^2$ be a 2-dimensional Riemannian manifold of Gaussian curvature $K$. Suppose that $\xi$ is a unit geodesic vector field on $M^2$. Then the submanifold $\xi(M^2) \subset T_1 M^2$ has non-positive extrinsic curvature.

**Proof.** By definition, the extrinsic curvature of a submanifold is the difference between the sectional curvature of the submanifold and the sectional curvature of ambient space along the planes, tangent to the submanifold. In our case, this is $\det \Omega$. If $\xi$ is a geodesic vector field, then we may choose $e_0 = \xi$ and then $\mu = k = 0$. Therefore, for the extrinsic curvature we get

$$-\left( (-1)^{s+1} \frac{K}{2} + \frac{e_0(\lambda)}{1 + \lambda^2} \right)^2 \leq 0.$$ 

**Theorem 3.3** Let $M^2$ be a space of constant Gaussian curvature $K$. Suppose that $\xi$ is a unit geodesic vector field on $M^2$. Then $\xi(M^2)$ has constant Gaussian curvature in one of the following cases:

(a) $K = -c^2 < 0$ and $\xi$ is a normal vector field for the family of horocycles on the hyperbolic 2-plane $L^2$. In this case $K_\xi = -c^2$ and therefore $\xi(L^2)$ is locally isometric to $L^2$;

(b) $K = 0$ and $\xi$ is a parallel vector field on $M^2$. In this case $K_\xi = 0$ and $\xi(M^2)$ is also locally isometric to $M^2$;

(c) $K = 1$ and $\xi$ is any (local) geodesic vector field on the standard sphere $S^2$. In this case $K_\xi = 0$.

**Proof.** Since $\xi$ is geodesic, we may set $e_0 = \xi$, $e_1 = \eta$, $s = 1$. Taking into account (7) and (8), we see that $\lambda = -\kappa = -\sigma$. Lemma 3.1 (b) gives $-K = -e_0(\sigma) + \sigma^2$. So the result of Lemma 3.3 takes the form

$$K_\xi = \frac{K^2}{4} + \frac{K(1-K)}{1+\sigma^2} - \left( \frac{K}{2} - \frac{e_0(\sigma)}{1+\sigma^2} \right)^2 =$$

$$\frac{K^2}{4} + \frac{K(1-K)}{1+\sigma^2} - \left( \frac{K}{2} - \frac{K + \sigma^2}{1+\sigma^2} \right)^2 =$$

$$\frac{K(1-K)}{1+\sigma^2} + \frac{K(1+\sigma^2)}{1+\sigma^2} - \left( \frac{K + \sigma^2}{1+\sigma^2} \right)^2 =$$

$$K - \left( \frac{K + \sigma^2}{1+\sigma^2} \right)^2.$$ 

Suppose that $K_\xi$ is constant. Then the following cases should be considered:
(a) $\sigma = \text{const} \neq 0$. This means that the orthogonal trajectories of the field $\xi$ consist of curves of constant curvature. With respect to this natural coordinate system, the metric of $M^2$ takes the form $ds^2 = du^2 + f^2 dv^2$. Set $\sigma = -c$. Then the function $f$ should satisfy the equation

$$\frac{f_u}{f} = c$$

the general solution of which is $f(u, v) = A(v)e^{cu}$. After $v$-parameter change we obtain metric of the form

$$ds^2 = du^2 + e^{2cu} dv^2.$$

So, the manifold $M^2$ is locally isometric to the hyperbolic 2-plane $L^2$ of curvature $-c^2$ and the field $\xi$ is a geodesic field of (internal or external) normals to the family of horocycles.

(b) $\sigma = 0$. Then evidently $\xi$ is a parallel vector field and therefore the manifold $M^2$ is locally Euclidean which implies $K_\xi = 0$.

(c) $\sigma$ is not constant. Then $K_\xi$ is constant if $K = 1$ only. So, $M^2$ is contained in a standard sphere $S^2$ and the curvature of $\xi(S^2)$ does not depend on $\sigma$. Thus, the field $\xi$ is any (local) geodesic vector field. Evidently, $K_\xi = 0$ for this case.

The case (a) of the Theorem 3.3 has an interesting generalization of the following kind.

**Theorem 3.4** Let $L^2$ be a hyperbolic 2-plane of curvature $-c^2$. Then $T_1L^2$ admits a hyperfoliation with leaves of constant intrinsic curvature $-c^2$ and of constant extrinsic curvature $-\frac{c^2}{4}$. The leaves are generated by unit vector fields making a constant angle with a pencil of parallel geodesics on $L^2$.

**Proof.** Consider $L^2$ with metric $ds^2 = du^2 + e^{2cu} dv^2$ and a family of vector fields

$$\xi_\omega = \cos \omega X_1 + \sin \omega X_2 \quad (\omega = \text{const}),$$

where $X_1 = \{1, 0\}$, $X_2 = \{0, e^{-cu}\}$ are the unit vector fields.

Since $\nabla X_1 \xi_\omega = 0$, we may set $e_0 = X_1$, $e_1 = X_2$ and therefore we have $\sigma = -c$, $\lambda = c$. Then, setting $K = -c^2$ and $\lambda = c$ in Lemma 3.3 we get

$$K_\xi = -c^2.$$

The extrinsic curvature of $\xi(L^2)$ is also constant since

$$\det \Omega = -\frac{1}{4}c^2.$$

Now fix a point $\tilde{P}_\infty$ at infinity boundary of $L^2$ and draw a pencil of parallel geodesics from $\tilde{P}_\infty$ through each point of $L^2$. Define a family of submanifolds $\xi_\omega(L^2)$ for this pencil. Evidently, through each point $(p, \zeta) \in T_1L^2$ there passes only one submanifold of this family. Thus, a family of submanifolds $\xi_\omega$ form a
hyperfoliation on $T_1L^2$ of constant intrinsic curvature $-c^2$ and constant extrinsic curvature $-\frac{c^2}{r^2}$.

Geometrically, $\xi_\omega(L^2)$ is a family of coordinate hypersurfaces $\omega = \text{const}$ in $T_1L^2$. Indeed, let $(u,v,\omega)$ form a natural local coordinate system on $T_1L^2$. Then the metric of $T_1L^2$ has the form

$$ds^2 = du^2 + 2e^{2u}dv^2 + 2dvd\omega + d\omega^2.$$  

With respect to these coordinates, the coordinate hypersurface $\omega = \text{const}$ is nothing else but $\xi_\omega(L^2)$ and the induced metric is

$$ds^2 = du^2 + 2e^{2u}dv^2.$$  

Evidently, its Gaussian curvature is constant and equal to $-c^2$.  

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