Z$_2$-Graded Cocycles in Higher Dimensions

C. Ekstrand
Department of Theoretical Physics,
Royal Institute of Technology,
S-100 44 Stockholm, Sweden

Abstract

Current superalgebras and corresponding Schwinger terms in 1 and 3 space dimensions are studied. This is done by generalizing the quantization of chiral fermions in an external Yang-Mills potential to the case of a Z$_2$-graded potential coupled to bosons and fermions.

1 Introduction

Since the suggestion of supersymmetry as a possible symmetry in nature there has been an increased interest in theories allowing for a Z$_2$-graded extension. In this paper we will find Z$_2$-graded generalizations of the Lundberg[1] and Mickelsson-Faddeev[2,3] cocycle, and the underlying formalism. The cocycles under consideration appear in the representation theory of the infinite dimensional Lie algebras Map(M; g) of smooth maps from a d-dimensional compact manifold M to a compact, semi-simple Lie algebra g (e.g. g = su(N)). These maps can be interpreted as gauge transformations in some gauge theory.

When M is a C$^\infty$ manifold with a Riemannian- and a spin structure there is a natural embedding of Map(M; g) and A(M), the set of Yang-Mills configurations, into some sets g$_p$ and Gr$_p$, p = (d + 1)/2, respectively. How this is made will be described later. The Lie algebras g$_p$ and Grassmannians Gr$_p$ are used in the algebraic investigation of the representations of Map(M; g). To define them one considers a separable Hilbert space h and a grading operator F$_0$ (i.e. F$_0$ = F$_0^*$ = F$_0^{-1}$; the star denotes the Hilbert space adjoint) such that the spaces h$_k$ = (1 ± F$_0$)h are infinite dimensional. Then g$_p$ is defined as the Lie algebra of all bounded operators X on h such that ([F$_0$, X]$^*$[F$_0$, X])$^p$ is trace class (we recall, see ref. 4, that an operator a on h is trace class if $\sum_n |(f_n, af_n)|$ is finite for an arbitrary orthonormal basis $\{f_n\}$ in h, and in that case its Hilbert space trace $tr(a) = \sum_n (f_n, af_n)$ exists, i.e. it is finite and basis independent). Similarly, Gr$_p$ is defined as the set of grading operators F such that ((F - F$_0$)$^*$ (F - F$_0$))$^p$ is trace class. The embedding described above

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is then possible for the case $h = L^2(M; V)$, where $V$ is a representation space of $\mathfrak{g}$.

The representation theory for the case $p = 1$ ($d = 1$) is well-understood (see for example ref. [5]). The reason is that elements in $\mathfrak{g}_p$ can only be implemented (or second quantized) in the physical relevant representation of the field algebra if $p = 1$. It is also known how to make a $\mathbb{Z}_2$ extension of the $p = 1$ case and a graded Lundberg cocycle has been obtained[3].

In higher dimensions the situation is much more difficult. A straightforward attempt to second quantize elements in $\mathfrak{g}_p$, $p > 1$ will fail. One will have to deal with infinities corresponding to the divergencies arising in certain Feynman diagrams in gauge theory models. There exists several different renormalization methods to handle such divergencies. The one we will use is close to the one used in ref. [7]. Here the implementations is by forms obtained by an appropriate multiplicative regularization. For this, the Grassmannians $Gr_p$ needs to be introduced. The case $d = 3$ and chiral fermions leads to the Mickelsson-Rajeev cocycle[8]. A corresponding boson cocycle has been obtained in ref. [9].

In this paper we will generalize the results described above to a $\mathbb{Z}_2$-graded case. A local form of the cocycles will be calculated, and some applications motivated by gauge theory models will be considered.

2 Preliminaries

To fix notation we summarize some basic facts about $\mathbb{Z}_2$-graded vector spaces and algebras. An element $v$ in a $\mathbb{Z}_2$-graded vector space $V = V_0 \oplus V_1$ is said to be homogeneous of degree $\alpha$, $\deg(v) = \alpha$, if $v \in V_\alpha$, $\alpha \in \mathbb{Z}_2 = \{0, 1\}$. If $V$ is also an algebra with grading preserving multiplication, i.e. $v \in V_\alpha, w \in V_\beta \Rightarrow vw \in V_{\alpha+\beta}$, then it is called a $\mathbb{Z}_2$-graded algebra. We define the supercommutator $[\cdot, \cdot]: V \times V \to V$, to be the bilinear map

$$[v, w] = vw - (-1)^{\deg(v)\deg(w)}wv.$$  (1)

This formula, as well as all other formulas in the paper, is by linearity also defined for non-homogeneous elements. Equipped with the supercommutator, $V$ becomes a Lie superalgebra (this subject is described in ref. [10]). Every linear operator $X$ on $V$ can be written in matrix form

$$X = \begin{pmatrix} X_{00} & X_{\bar{0}\bar{1}} \\ X_{\bar{1}0} & X_{\bar{1}\bar{1}} \end{pmatrix}$$  (2)

corresponding to the decomposition $V = V_0 \oplus V_1$. Then $\deg(X_{\alpha\beta}) = \alpha + \beta$ defines a grading which provides every algebra of linear operators on $V$ with a $\mathbb{Z}_2$-structure.

We will now consider a $\mathbb{Z}_2$-graded Hilbert space and define certain operators acting thereon. Let $h = h_0 \oplus h_1$ be an infinite dimensional separable $\mathbb{Z}_2$-graded
Hilbert space with the subspaces $h_0$ and $h_1$ both infinite dimensional. Let $P_\alpha$ denote the orthogonal projection onto $h_\alpha$: $h_\alpha = P_\alpha h$, and introduce the Klein operator $\gamma = P_0 - P_1$. We denote by $\mathcal{B}$ the algebra of bounded operators on $h$ and by $\mathcal{B}_{2p}$ the Schatten ideal classes (see ref. [4] consisting of operators $X \in \mathcal{B}$ such that $(X^* X)^p$ has a converging Hilbert space trace (or supertrace, which will be defined later). Especially, $\mathcal{B}_1$ and $\mathcal{B}_2$ are the ideal of trace class and Hilbert-Schmidt operators, respectively. We define

$$Gr_p (F_0) = \{ F \in \mathcal{B}; F^2 = 1 \text{ and } F - F_0 \in \mathcal{B}_{2p} \}$$

for some fixed operator $F_0 \in \mathcal{B}$ obeying

$$[F_0, \gamma] = 0$$
$$F_0 = F_0^* = F_0^{-1}.$$  \hspace{1cm} (4)

The operator $F_0$ will be thought as being the sign of the free Hamiltonian. It is also useful to introduce the classes

$$g_p (F) = \{ X \in \mathcal{B}; [F, X]_{s} \in \mathcal{B}_{2p} \}.$$  \hspace{1cm} (5)

We regard the subspaces $h_0$ and $h_1$ as the one-particle spaces of charged bosons and fermions, respectively. Using the bosonic and fermionic Fock spaces $\mathcal{F}_B (h_0)$ and $\mathcal{F}_F (h_1)$ (defined in ref. [5] for example) with vacua $\Omega_B$ and $\Omega_F$ we construct the Fock space $\mathcal{F}_\gamma (h) = \mathcal{F}_B (h_0) \otimes \mathcal{F}_F (h_1)$ with vacuum $\Omega = \Omega_B \otimes \Omega_F$. The bosonic and fermionic creation and annihilation operators can be combined to the operators

$$a(f) = a_B (P_0 f) \otimes 1_1 + 1_0 \otimes a_F (P_1 f)$$

and $a^\dagger (f) = a(f)^*$, $\forall f \in h$, the Fock space adjoint, which act on $\mathcal{F}_\gamma (h)$. They obey $a(f) \Omega = 0$, and have a common, dense and invariant domain in $\mathcal{F}_\gamma (h)$. Introducing the grading

$$\deg (a^\dagger (P_\alpha f)) = \alpha = \deg (a(P_\alpha f)), \quad \alpha \in \mathbb{Z}_2, f \in h$$

we can extend the canonical commutation relations (CCR) and anticommutation relations (CAR) of the bosonic and fermionic creation and annihilation operators to the canonical supercommutator relations (CSR)

$$[a(f), a^\dagger (g)]_s = (f, g) 1, \quad [a(f), a(g)]_s = 0 \quad \forall f, g \in h$$

where $(\cdot, \cdot)$ is the scalar product in $h$. We now define the CSR algebra over $h$ to be the quotient of the free * algebra with complex coefficients, generated by $a(f), a^\dagger (f), f \in h$, and the identity $1$, by (the two sided * algebra generated by) $a^\dagger (f) = a(f)^*$, $a^\dagger (c_1 f_1 + c_2 f_2) = c_1 a^\dagger (f_1) + c_2 a^\dagger (f_2)$ and the relations in (8). A representation of this algebra was constructed above, usually referred...
to as the free (Fock-Cook) representation of the CSR algebra. From a physical point of view this representation is not satisfactory since it provides us with a Hamiltonian operator that is unbounded from below. We will instead consider a representation where this is avoided (a so called highest weight representation – see ref. [11]). Using the elements in the free representation the so called quasi-free representation of the CSR algebra can be constructed:

\[
\begin{align*}
\tilde{a}(f; F_0) &= \tilde{a}(P_+ F_0 f) + \tilde{a} (J P_{\pm} F_0 f) \\
\tilde{a}^\dagger(f; F_0) &= \tilde{a}^\dagger (P_+ F_0 f) - \tilde{a} (\gamma J P_{\pm} F_0 f)
\end{align*}
\]

where \( J \) is a conjugation in \( \mathfrak{h} \) (an antilinear norm-preserving operator obeying \( J^2 = 1 \)) commuting with \( \gamma \) and with \( P_{\pm} = \frac{(1 \pm F_0)}{2} \). The procedure of going from the free to the quasi-free representation is a generalization of the well known process of ‘filling the Dirac sea’.

### 3 Second Quantization and \( \mathbb{Z}_2 \)-graded cocycles

For certain bounded operators \( X \) on \( \mathfrak{h} \) the corresponding Fock space operator \( d\Gamma(X; F_0) \) can be defined. It is by definition the operator on \( \mathcal{F}_\gamma(h) \) that has vanishing vacuum expectation value

\[
< \Omega, d\Gamma(X; F_0)\Omega >= 0
\]

and satisfies

\[
[d\Gamma(X; F_0), a^\dagger(f; F_0)]_a = a^\dagger(X f; F_0), \quad f \in \mathfrak{h}.
\]

where \( < \cdot, \cdot > \) denotes the scalar product in \( \mathcal{F}_\gamma(h) \). The relation \( [d\Gamma(X; F_0), a^\dagger(f; F_0)]_a = a^\dagger(X f; F_0), \quad f \in \mathfrak{h} \) defines \( d\Gamma(X; F_0) \) up to an additive c-number possibly depending on \( X \). One way to fix the value of the c-number is by using eq. \( (10) \). Let \( \{f_n\}_{n=-\infty}^{\infty} \) be a complete system of orthonormal homogeneous vectors in \( \mathfrak{h} \) with \( F_0 f_n = \lambda_n f_n \), where the eigenvalues are indexed such that \( \lambda_{n<0} = -1 \) and \( \lambda_{n\geq0} = 1 \). Then it is easy to see that if the operator \( d\Gamma(X; F_0) \) exists, it must be of form

\[
d\Gamma(X; F_0) = \sum_{n,m=-\infty}^{\infty} (f_n, X f_m) : a^\dagger(f_n; F_0) a(f_m; F_0) :
\]

where the normal ordering is defined by

\[
: a^\dagger(f_n; F_0) a(f_m; F_0) : = \begin{cases}
(-1)^{\deg(f_n)} a(f_m; F_0) a^\dagger(f_n; F_0) & \text{if } n = m < 0 \\
a^\dagger(f_n; F_0) a(f_m; F_0) & \text{otherwise}.
\end{cases}
\]

The necessary and sufficient condition for existence of \( d\Gamma(X; F_0) \) is that \( X \in \mathfrak{g}_1(F_0) \).
The operator $X$ will throughout the paper be thought of as the generator of an infinitesimal gauge transformation. The wave functions on which it acts are assumed to be dependent on some $F \in \mathcal{G}_1(F_0)$ and second quantized according to the polarization in $h$ given by $F_0$. To take into account the action a gauge transformation has on $F$ (corresponding to its action on the Hamiltonian) we introduce the operator $G(X;F_0) = d\Gamma(X;F_0) + \mathcal{L}_X$, with the Lie derivative $\mathcal{L}_X, X \in \mathfrak{g}_1(F_0)$ acting on functionals $m(F), F \in \mathcal{G}_1(F_0)$, as

$$\mathcal{L}_X m(F) = \frac{d}{dt} m \left( F - t [F,X]_{(s)} + O(t^2) \right) \big|_{t=0}. \tag{14}$$

Taking the supercommutator of two $G$’s gives the relation

$$[G(X;F_0),G(Y;F_0)]_s = G([X,Y]_s;F_0) - c(X,Y;F_0), \quad X,Y \in \mathfrak{g}_1(F). \tag{15}$$

The Schwinger term

$$c(X,Y;F_0) = d\Gamma([X,Y]_s;F_0) - [d\Gamma(X;F_0) d\Gamma(Y;F_0)]_s. \tag{16}$$

can be calculated by taking the vacuum expectation value of (16) and performing some algebraic manipulations. The result is

$$c_1(X,Y) = c(X,Y;F_0) = -\frac{1}{4} \text{str} \left( F_0 [F_0,X]_{(s)} [F_0,Y]_{(s)} \right) = -\frac{1}{2} \text{str}_c \left( X [F_0,Y]_{(s)} \right). \tag{17}$$

Since $\deg(F_0) = \bar{0}$ our notation $(s)$ is to illustrate that the commutator as well as the supercommutator may be used in the equations. The conditional trace, conditional supertrace and ordinary supertrace can be defined in terms of the ordinary trace according to

$$\text{str}_c(a) = \text{tr}_c(\gamma a) = \frac{1}{2} \text{tr} \left( \gamma (a + F_0 a F_0) \right), \quad a + F_0 a F_0 \in \mathcal{B}_1$$

$$\text{str}(a) = \text{tr}(\gamma a) \quad a \in \mathcal{B}_1. \tag{18}$$

The term $c$ is a graded generalization of the Lundberg cocycle[4]. The graded Jacobi identity is equivalent with the 2-cocycle relation

$$c([X,Y]_s,Z;F_0) + \text{graded cyclic perm.} = 0, \quad X,Y,Z \in \mathfrak{g}_1(F_0). \tag{19}$$

If a different condition than (10) had been used to determine the $c$-number ambiguity in (11), then $G(X;F_0)$ would have been changed to $G(X;F_0) + \zeta(X)$ for some complex valued function $\zeta$ of $\mathfrak{g}_1(F_0)$. This would in turn change the cocycle $c$ to $c + \delta \zeta$, with

$$\delta \zeta(X,Y) = \zeta([X,Y]_s). \tag{20}$$
The new cocycle is in the same cohomology class as the old one since they differ only by a function of type $\delta \zeta$, a coboundary.

When considering the corresponding situation for some $F \in Gr_1(F_0)$ we define $\tilde{G}(X, F), X \in g_1(F)$ (or equivalently $X \in g_1(F_0)$) by

$$\tilde{G}(X, F) = U^*(F)G(X, F_0)U(F)$$

(21)

for some unitary operator $U(F)$ on $F_+(h)$. Obviously it obeys a relation as the one in (15) and with the same Schwinger term. Changing $\tilde{G}(X, F)$ by a $c$-number $\zeta(X; F)$ gives a new kind of coboundary:

$$\delta \zeta(X, Y; F) = \zeta([[X, Y], F] - \mathcal{L}_X \zeta(Y; F) + (-1)^{\deg(X)\deg(Y)}\mathcal{L}_Y \zeta(X; F).$$

(22)

Since the Schwinger term then can be $F$-dependent, the 2-cocycle relation will now be

$$c([[X, Y], Z; F] - \mathcal{L}_X c(Y, Z; F) + \text{graded cyclic perm.} = 0, \quad X, Y, Z \in g_1(F).$$

(23)

$d\Gamma(X; F)$ does no longer exist as an operator when $X \in g_2(F)$ and $F \in Gr_2(F_0)$. Therefore some additional regularization is needed. A useful observation is that $d\Gamma(X; F)$ can still be defined as a sesquilinear form $[X, Y; F]$. Proceeding as in the ungraded case we make use of this by finding a coboundary $(\delta b)(X, Y; F)$ for $X, Y \in g_1(F)$, $F \in Gr_1(F_0)$ such that $(\delta b)(X, Y; F)$ is divergent when we extend to $g_2(F)$ respective $Gr_2(F_0)$ and the divergency is such that $c_1(X, Y) + (\delta b)(X, Y; F)$ will be finite.

We will now try to find such a $b$. First we state:

**Lemma 1** Let $b(X; F) = \frac{1}{8} \str((F - F_0)F_0([X, F]_s))$ for $X \in g_1(F), F \in Gr_1(F_0)$. Then

$$\langle \delta b \rangle(X, Y; F) = \frac{1}{8} \str(F[[F_0, X]_s], [F_0, Y]_s).$$

(24)

**Proof** Using the definition of coboundary (22) we get

$$\langle \delta b \rangle(X, Y; F) = \frac{1}{8} \str((F - F_0)F_0([X, F]_s))$$

$$- \frac{1}{8} \str([F, X]_s F_0 [F_0, Y]_s)$$

$$+ \frac{1}{8}(-1)^{\deg(X)\deg(Y)}\str([F, Y]_s F_0 [F_0, X]_s).$$

(25)

Using the graded Jacobi identity for the first term while adding and subtracting the whole expression by the term

$$\frac{1}{8} \str([F_0, X]_s F_0 [F_0, Y]_s) - (-1)^{\deg(X)\deg(Y)} (X \leftrightarrow Y)$$

(26)
Theorem 1

The right hand side of (25) can be written as

\[
\frac{1}{8} \text{str} \left((F - F_0)F_0 \left[ [F_0, X]_{(s)}, Y \right]_{s}\right)
\]
\[
+ \frac{1}{8} (-1)^{\deg(X)\deg(Y)\text{str}} \left([F - F_0, Y]_{s} F_0 [F_0, X]_{(s)}\right)
\]
\[
+ \frac{1}{8} (-1)^{\deg(X)\deg(Y)\text{str}} \left([F_0, Y]_{(s)} F_0 [F_0, X]_{(s)}\right)
\]
\[
- (-1)^{\deg(X)\deg(Y)} (X \leftrightarrow Y).
\]

(27)

With the formula for the supercommutator of a product: \([AB, C]_s = A[B, C]_s + (-1)^{\deg(B)\deg(C)}[A, C]_s B\) we get

\[
(\delta b)(X, Y; F) = \frac{1}{8} \text{str} \left(\left((F - F_0)F_0 [F_0, X]_{(s)}, Y \right]_{s}\right)
\]
\[
- \frac{1}{8} (-1)^{\deg(X)\deg(Y)\text{str}} \left((F - F_0) [F_0, Y]_{(s)} [F_0, X]_{(s)}\right)
\]
\[
+ \frac{1}{8} (-1)^{\deg(X)\deg(Y)\text{str}} \left([F_0, Y]_{(s)} F_0 [F_0, X]_{(s)}\right)
\]
\[
- (-1)^{\deg(X)\deg(Y)} (X \leftrightarrow Y).
\]

(28)

The first term vanish since it is the supertrace of a supercommutator. Since \(F_0[F_0, Y]_{(s)} = -[F_0, Y]_{(s)}F_0\) the third term cancels the \(F\)-independent part of the second term. We are then left with (24) which was to be proven.

**Theorem 1** For \(X, Y \in \mathfrak{g}_1(F)\), \(F \in \text{Gr}_1(F_0)\) and \(c_2(X, Y; F) = c_1(X, Y) + \delta b(X, Y; F)\) it holds that

\[
c_2(X, Y; F) = \frac{1}{8} \text{str}_c \left(\left((F - F_0) \left[ [F_0, X]_{(s)}, [F_0, Y]_{(s)}\right]\right)\right)
\]

(29)

which exists even for \(X, Y \in \mathfrak{g}_2(F)\), \(F \in \text{Gr}_2(F_0)\)

The form of \(c_2\) is obtained by adding the expressions (17) and (24). That it is convergent for the given classes is easily seen by the following calculation:

\[
c_2(X, Y; F) = \frac{1}{16} \text{str} \left(\left(\left((F - F_0) + F_0(F - F_0)F_0\right) \left[ [F_0, X]_{(s)}, [F_0, Y]_{(s)}\right]\right)\right)
\]
\[
= \frac{1}{16} \text{str} \left(\left((F - F_0)^2F_0 \left[ [F_0, X]_{(s)}, [F_0, Y]_{(s)}\right]\right)\right).
\]

(30)

Thus it can be written as a supertrace of a trace class operator and is therefore well defined. The function \(c_2\) is a graded generalization of the Mickelsson-Rajeev cocycle [8]. When \(X \in \mathfrak{g}_p(F), F \in \text{Gr}_p(F_0), p > 2\) the fundamental ideas are the same as for \(p = 2\), and for this reason we omit to study these cases.
4 A local form of the cocycles

We will now determine the form of the cocycles for the case $h = L^2(\mathbb{R}^d) \otimes V$, where the finite dimensional $\mathbb{Z}_2$-graded space $V$ introduces a natural grading in $h$ by: $f : \mathbb{R}^d \to V$, $f \in h_{\alpha}$. Several of the calculations in this section are based on rules for PSDO’s, summarized in the appendix. From now on we restrict to the case of $X, Y$ and $F$ being PSDO’s of order 0. The operators $X$ and $Y$ are also assumed to be compactly supported (in configuration space).

We add the condition that the (super)commutator with $F_0$ for $X$ and $Y$ should to be a PSDO of order $-1$, while the corresponding condition for $F$ is that $F - F_0$ should be of order $-1$ (the motivation for these conditions will become clear later). It is equivalent with the fact that the operators are in $\mathfrak{g}_p(F_0)$ respective $Gr_p(F_0)$ for $2p = d + 1$. This is a consequence of the fact that a PSDO on $\mathbb{R}^d$ is traceclass if and only if its symbol is of order $-d - 1$ (see appendix). It means that when $d = 1$ no regularization has to be done and the cocycle of interest is $c_1$. When $d = 3$ the operators under consideration are in $\mathfrak{g}_2$ and $Gr_2$, respectively. A renormalization procedure has to be performed, leading to the cocycle $c_2$.

For a compactly supported PSDO $a$ of form (74) and of order $k$ the cut-off supertrace can be expanded as

$$
\text{str}_\Lambda(a) = \int d^dx \int d^dq (2\pi)^d P_\Lambda(q) \text{str}_V \sigma(a)(q, x) = c_{k+d}(a)\Lambda^{k+d} + \ldots + c_1(a)\Lambda + c_{\text{log}}(a) \log\left(\frac{\Lambda}{\Lambda_0}\right) + c_0(a)\Lambda^0 + c_{-1}(a)\Lambda^{-1} + \ldots
$$

(31)

with $\Lambda_0$ some scale parameter, $P_\Lambda(q) = \Theta(1 - \frac{|q|}{\Lambda})$, and $\Theta$ is the Heaviside function. The $\Lambda$-independent term $c_0(a)$ is of particular interest to us. We will refer to it as the regularized supertrace $\text{STR}(a)$. It is just the supertrace when the argument is a trace class operator. The operators that will be considered here are such that $c_{\text{log}}(a)$ is zero and thus the regularized supertrace is well defined. A PSDO that is compactly supported, homogeneous of order $-d$ and whose symbol can be written as a total derivative in momentum space has the property that the cut-off supertrace is $\Lambda$-independent and equal to the regularized supertrace of the operator. This is a nice property so we now try to find coboundaries that can be added to the cocycles (17) and (29) in a way that the resulting new cocycles can be written as a cut-off supertrace of such an operator. An example of a PSDO whose symbol to highest order can be written as a total derivative with respect to some momentum space variable is the supercommutator of two PSDO’s

$$
\text{str}_V (\sigma ([a, b]_\alpha))(q, x)
$$
\[ = -i \sum_{k=1}^{d} \text{str}_V \left( \frac{\partial}{\partial q_k} \left( \sigma(a) \frac{\partial}{\partial x_k} \sigma(b) \right) \right) \]

\[-(-1)^{\text{deg}(a)\text{deg}(b)} \frac{\partial}{\partial x_k} \left( \left( \frac{\partial}{\partial q_k} \sigma(b) \right) \sigma(a) \right) \right) (q, x) \]

+ lower order terms. \hspace{1cm} (32)

This is shown by using the formula (77) for calculating the symbol of the product of two PSDO’s. The second term in the expression is uninteresting to us since it vanishes when integrating if one of the operators under consideration is compactly supported. The lower order terms in the expression can also be written as a sum of total derivatives of momentum and configuration variables. Motivated by the discussion above we try to find some functions \( b_1 \) and \( b_2 \) such that the new cocycles

\[ \tilde{c}_1 = c_1 - \delta b_1 \]

\[ \tilde{c}_2 = c_2 - \delta b_2 - \delta b_1 \]

(33)

can be written as a regularized supertrace of a supercommutator.

The choice

\[ b_1 = \frac{1}{2} \text{STR} (F_0 X) \]

\[ b_2 = \frac{1}{8} \text{STR} (FF_0 [F_0, X]_{(s)}) \]

(34)

will do if the restriction \( \sigma(F_0)(q, x) = \sigma(F_0)(q) \) is put on \( F_0 \). From a physical point of view this is reasonable if thinking of \( F_0 \) as the sign of the free Hamiltonian. It implies that

\[ \text{STR}(F_0 a F_0) = \text{STR}(a) \]

(35)

is true for \( a \) trace class. The relation

\[ \text{str}_C (a) = \text{STR}(a) \]

(36)

holds therefore if \( a \) is conditional trace class since it obviously holds when \( a \) is trace class. Now, using (17), (32)–(36) and the linearity of the regularized supertrace, we obtain

\[ \tilde{c}_1 (X, Y) = - \frac{1}{2} \text{STR} ([X, F_0 Y]_{s}) \]

\[ = \frac{i}{2} \int dx \int_{|q| \leq \Lambda} \frac{dq}{2\pi} \text{str}_V \frac{\partial}{\partial q} \left( \sigma(X)_0 (q, x) \right) \]

\[ - \left( \sigma(F_0)_0 (q) \sigma(Y)_0 (q, x) \right) \]

(37)
where $\sigma(\cdot)_k$ stands for the piece of the symbol that is homogeneous of order $k$. Similarly, using

$$\text{str}_\Lambda \left( \left[ F_0, [X,Y]_{(s)} \right] \right) = 0$$

and the $F$-dependent part of the calculation in the proof of Lemma 1 with the supertrace replaced by the regularized supertrace (remember that the regularized supertrace of a supercommutator is not necessarily zero and the expression in (27) is therefore useful) together with (29) and (33)–(36) give

$$\tilde{c}_2(X,Y; F) = \frac{i}{8} \sum_{k=1}^{3} \int d^3x \int_{|q| \leq \Lambda} \frac{d^3q}{(2\pi)^3} \text{str}_\Lambda \left[ \left( F - F_0 \right) \left[ F_0, X \right]_{(s)} \left[ F_0, Y \right]_{(s)} \right]$$

Assuming that the first term vanishes (as it will in the special cases below) we use (32) to obtain

$$\tilde{c}_2(X,Y; F) = \frac{i}{8} \sum_{k=1}^{3} \int d^3x \int_{|q| \leq \Lambda} \frac{d^3q}{(2\pi)^3} \frac{\partial}{\partial q_k} \left( \sigma(F - F_0)_{-1}(q,x) \sigma(F_0)_{0}(q) \sigma(F_0, X)_{(s)}_{-1}(q,x) \frac{\partial}{\partial x_k} \sigma(Y)_{0}(q,x) \right)$$

$$\tilde{c}_2(X,Y; F) = \frac{i}{8} \sum_{k=1}^{3} \int d^3x \int_{|q| \leq \Lambda} \frac{d^3q}{(2\pi)^3} \frac{\partial}{\partial q_k} \left( \sigma(F - F_0)_{-1}(q,x) \sigma(F_0)_{0}(q) \sigma(F_0, X)_{(s)}_{-1}(q,x) \frac{\partial}{\partial x_k} \sigma(Y)_{0}(q,x) \right)$$

$$\tilde{c}_2(X,Y; F) = \frac{i}{8} \sum_{k=1}^{3} \int d^3x \int_{|q| \leq \Lambda} \frac{d^3q}{(2\pi)^3} \frac{\partial}{\partial q_k} \left( \sigma(F - F_0)_{-1}(q,x) \sigma(F_0)_{0}(q) \sigma(F_0, X)_{(s)}_{-1}(q,x) \frac{\partial}{\partial x_k} \sigma(Y)_{0}(q,x) \right)$$

5 Two examples

Some concrete examples will now be considered. The operators of interest are the Hamiltonian $H = H_0 + A$ and compactly supported gauge transformations $e^X$. The free Hamiltonian $H_0$, $A$ and $X$ are assumed to be PSDO’s such that $\sigma(H_0^2)(q,x) = |q|^2$, while $A$ and $X$ are assumed to be of order 0. Let $F(0) = \text{sgn}(H(0))$, using the spectral theorem with $\text{sgn}(x) = 1 (-1)$ for $x \geq 0$ ($x < 0$). Then

$$\sigma(H^2)(q,x) = |q|^2 \left( 1 + \frac{\sigma(H_0)\sigma(A) + \sigma(A)\sigma(H_0)}{|q|^2} \right) (q,x) + O(|q|^0)$$

$$\sigma(H^2)(q,x) = |q|^2 \left( 1 + \frac{\sigma(H_0)\sigma(A) + \sigma(A)\sigma(H_0)}{|q|^2} \right) (q,x) + O(|q|^0)$$
and using (77),

$$\sigma(1/\sqrt{H^2})(q, x) = \frac{1}{|q|} \left( 1 - \frac{\sigma(H_0)\sigma(A) + \sigma(A)\sigma(H_0)}{2|q|^2} \right) (q, x) + O(|q|^{-3})$$

(42)

implying

$$\sigma(F)(q, x) = \frac{1}{|q|} \left( \sigma(H_0) + \frac{\sigma(A)}{2} - \frac{\sigma(H_0)\sigma(A)\sigma(H_0)}{2|q|^2} \right) (q, x) + O(|q|^{-2}).$$

(43)

Thus we get that $F - F_0$ is a PSDO of order $-1$.

Performing an infinitesimal gauge transformation gives

$$\sigma(H_0 + A) \rightarrow \sigma(H_0 + A_X) = \sigma(H_0 + A - [A, X], - [H_0, X]_{(s)}) + O(|q|^{-1}).$$

(44)

Since both $A$ and $A_X$ are PSDO’s of order 0 we see that a gauge transformation must obey the condition that $[H_0, X]_{(s)}$ is of order 0. This in turn implies that $[F_0, X]_{(s)}$ is of order $-1$.

The way the operators have been introduced here they fulfill all restrictions put in the former section. The results obtained there can therefore be used. Two different choices of $H_0, A$ and $X$ will be considered in the next section. For each choice the cocycles (37) and (40) will be calculated. Note that we only have to specify $H_0, A$ and $X$ to highest order (as PSDO’s) since it is only this part of them that appears in the formulas for the cocycles. In both cases $V = V_{\text{spin}} \otimes V_{\text{color}} \otimes V_{\mathbb{Z}_2}$, that is, the boson and fermion wavefunctions will be assumed to have an equal number of spin and color (internal degrees) components. It is useful to define some additional (super)traces:

$$\text{str}_V = \text{tr}_{\text{spin}} \text{str}_{\text{color}} = \text{tr}_{\text{spin}} \text{tr}_{\text{color}} \text{str}_{\mathbb{Z}_2}.$$

The equations (37) and (40) for the cocycles has a momentum integration over a domain that contains the origin. Now, since there are factors $\frac{1}{|q|}$ in the integrand, coming from $\sigma(F(0))$, the integral is not well defined. To avoid this problem the factors $\frac{1}{|q|}$ will be regularize close to the origin in momentum space. It turns out (see below) that the formulas for the cocycles are independent of how such an infrared regularization is made. It is so since Stokes’ theorem may be used and thus the cocycles depend only on the behavior of the integrand in a neighborhood of the boundary of the integration domain in momentum space. We denote our regularized function by $\frac{1}{|q|} r$ and define it to be equal to some smooth function when $|q| \leq \delta$.

5.1 Case 1

The operators will here be written in a matrix form according to the decomposition $h = h_0 \oplus h_1$, just as was done in (2). To get a rather simple form of the
coclces \([\mathbb{M}]\) and \([\mathbb{N}]\) we mimic the chiral fermion case and assume that also the energy spectrum in the boson space is determined by the fermion Hamiltonian. We choose to consider vector potentials that are of the same form in all four sectors and it should be a generalization of the ‘ordinary’ potential in the fermion to fermion sector. Thus:

\[
\sigma(H_0)(q, x) = \begin{pmatrix} \frac{q}{0} & 0 \\ 0 & \frac{q}{0} \end{pmatrix}, \quad \sigma(A)(q, x) = A(x) = \begin{pmatrix} A_{00}(x) & A_{01}(x) \\ A_{10}(x) & A_{11}(x) \end{pmatrix}
\]

where \(S = \left\{ \sum_{i=1}^{3} S_i \sigma_j \right\} \quad d = 1 \quad d = 3 \quad S = q, A, A_{\alpha\beta}(x), \quad \alpha, \beta \in \mathbb{Z}_2\) (45)

where \(\sigma_j, j = 1, 2, 3\) are the usual Pauli spin matrices and the \(A_{\alpha\beta}\)’s are assumed to commute with these. Motivated by the ungraded case (the ‘usual’ gauge transformations leading to the conservation of the electric charge) we restrict ourselves to the case of \(X\) being a multiplication operator; \((Xf)(x) = Xf(x)\), commuting with the Pauli matrices.

When \(d = 1\) a simple calculation leads to

\[
\tilde{c}_1(X; Y) = \frac{i}{2} \int dx \int \frac{dq}{2\pi} \text{str}_{V} \left( X(x) \frac{\partial}{\partial x} \left( \frac{q}{|q|} Y(x) \right) \right)
\]

This is recognized as a graded generalization of the affine Kac-Moody cocycle (see ref. [11]).

Let us now consider the case \(d = 3\). Since \(\text{tr}(\sigma_j) = 0, j = 1, 2, 3\), the first term in (39) vanishes and thus we may use equation (40). Since

\[
\sigma(F_0, X)_{(a)}(q, x) = \sum_{j=1}^{3} \left( -i \frac{\partial}{\partial q_j} \frac{\hat{q}_j}{|q|} \right) \frac{\partial}{\partial x_j} X(x) + \mathcal{O}(|q|^{-2})
\]

and

\[
\sigma(F - F_0)(q) = \frac{1}{|q|} \sum_{i=1}^{3} A_i(x) \left( \sigma_i - \frac{\hat{q}_i}{|q|} \right) + \mathcal{O}(|q|^{-2})
\]

we obtain

\[
\tilde{c}_2(X, Y; A) = -\frac{1}{8} \sum_{i, j, k=1}^{3} \text{str}_{\text{color}} J^\Lambda_{ijk} \int d^3 x \left( A_i(x) \frac{\partial X}{\partial x_j}(x) \frac{\partial Y}{\partial x_k}(x) \right) - (-1)^{\text{deg}(X)\text{deg}(Y)} (X \leftrightarrow Y)
\]

where

\[
J^\Lambda_{ijk} = -\int_{|q| \leq \Lambda} \frac{d^3 q}{(2\pi)^3} \text{tr}_{\text{spin}} \frac{\partial}{\partial q_k} \left( \left( \sigma_i - \frac{\hat{q}_i}{|q|^2} \right) \frac{\hat{q}_j}{|q|} \right) \left( \frac{\partial}{\partial q_j} \frac{\hat{q}_k}{|q|^2} \right).
\]
Since $J^\Lambda_{ijk}$ is of the form of an integration over a derivation we may replace the expression $\frac{\partial |q_i|}{\partial |q_i|}$ with 1 whenever it occurs in the integrand as long as $\Lambda$ is greater than the small regularization parameter $\delta$. Therefore the relation

$$\frac{\partial}{\partial q_i} \left| \frac{q_i}{|q_i|} \right| = 1$$

holds under the integration. Using this together with $\text{tr}_{\text{spin}}(\sigma_i \sigma_j \sigma_k) = 2i\epsilon_{ijk}$ and Stokes’ theorem gives

$$J^\Lambda_{ijk} = -\int_{|q| \leq \Lambda} \frac{d^3q}{(2\pi)^3} \frac{\partial}{\partial q_k} \left( 2i\epsilon_{ijl} \frac{q_l}{|q|^3} \right) = -2i\epsilon_{ijl} \frac{\delta_{kl}}{6\pi^2} = \frac{i}{3\pi^2} \epsilon_{ijk}$$

which combined with (49) results in

$$\tilde{c}_2(X, Y; A) = -\frac{i}{24\pi^2} \sum_{i,j,k=1}^3 \int d^3x \text{tr}_{\text{color}} \left( \epsilon_{ijk} A_i(x) \frac{\partial X}{\partial x_j}(x) \frac{\partial Y}{\partial x_k}(x) \right) -(-1)^{\text{deg}(X)\text{deg}(Y)} (X \leftrightarrow Y). \quad (52)$$

This is recognized as a graded generalization of the Mickelsson-Faddeev cocycle $[2,3]$. The calculation performed above is close to a corresponding calculation $[15]$ of the Schwinger term for chiral fermions.

### 5.2 Case 2

Massless bosons can be described by a vector-valued (needed for the spin and the internal degrees of freedom) function $\varphi$ obeying the Klein-Gordon equation

$$\Box \varphi = 0. \quad (53)$$

Introduce the conjugate momentum, $\pi^* = \frac{\partial}{\partial \varphi} \varphi$, in order to get an equivalent description containing two-component functions with the Hamiltonian

$$\tilde{H}_0 = \begin{pmatrix} 0 & i \\ -i|D|^2 & 0 \end{pmatrix} \quad (54)$$

with $|D|^2 = -\Delta$, the Laplacian, so that:

$$i \frac{\partial}{\partial t} \left( \begin{array}{c} \varphi \\ \pi^* \end{array} \right) = \tilde{H}_0 \left( \begin{array}{c} \varphi \\ \pi^* \end{array} \right). \quad (55)$$

Combining bosons with chiral fermions leads to the study of the Hamiltonian

$$H_0 = \begin{pmatrix} 0 & i & 0 \\ -i|D|^2 & 0 & 0 \\ 0 & 0 & D \end{pmatrix} \quad (56)$$
with \( \sigma(D)(q,x) = q \) as in the former example, acting on column vectors of form

\[
\begin{pmatrix}
\varphi \\
\pi^* \\
\psi
\end{pmatrix}
\] (57)

where \( \psi \) denotes the fermionic wavefunction. The case when the fields \( \varphi, \pi^* \) and \( \psi \) have an equal number of spin components will be considered. We choose to consider gauge transformations \( X \) of the form

\[
X(x) = \begin{pmatrix}
X_{\varphi}(x) & iX_{\varphi \pi^*}(x) \frac{D}{D^2} & X_{\varphi \psi}(x) \\
-iDX_{\varphi \pi^*}(x) & DX_{\varphi}(x) \frac{D}{D^2} & -iDX_{\varphi \psi}(x) \\
X_{\psi \pi^*}(x) & iX_{\psi \varphi}(x) \frac{D}{D^2} & X_{\psi}(x)
\end{pmatrix}
\] (58)

where the \( X_{\varphi_1 \varphi_2}(x) \)’s are multiplication operators commuting with the Pauli matrices. The motivation for this choice is the following. We would like to have multiplication operators commuting with the Pauli matrices that mixes the \( \varphi \) and \( \psi \) among themselves. This motivates the choice of the four elements in the corners of the matrix. The form of the remaining elements (to highest order) follow directly from this and the condition that the (super)commutator with \( H_0 \) should be of order 0 together with the claim that the form of the gauge transformation should be preserved under the supercommutator. Taking the (super)commutator with \( H_0 \) gives

\[
[H_0, X(x)](s) = \begin{pmatrix}
[D, X_{\varphi \pi^*}(x)] & i[D, X_{\varphi \varphi}(x)] \frac{D}{D^2} & [D, X_{\varphi \psi}(x)] \\
-iD[D, X_{\varphi \pi^*}(x)] & D[D, X_{\varphi \pi^*}(x)] \frac{D}{D^2} & -iD[D, X_{\varphi \psi}(x)] \\
[D, X_{\psi \pi^*}(x)] & i[D, X_{\psi \varphi}(x)] \frac{D}{D^2} & [D, X_{\psi \psi}(x)]
\end{pmatrix}
\] (59)

which motivates us to consider vector potentials of form

\[
A(x) = \begin{pmatrix}
A_{\varphi}(x) & iA_{\varphi \pi^*}(x) \frac{D}{D^2} & A_{\varphi \psi}(x) \\
-iDA_{\varphi \pi^*}(x) & DA_{\varphi}(x) \frac{D}{D^2} & -iDA_{\varphi \psi}(x) \\
A_{\psi \pi^*}(x) & iA_{\psi \varphi}(x) \frac{D}{D^2} & A_{\psi}(x)
\end{pmatrix}
\] (60)

where the \( A_{\varphi_1 \varphi_2}(x) \)’s are as in the former section (in the case \( d = 3 \) they are a sum of Pauli matrices multiplied with multiplication operators that commutes with the Pauli matrices).

To easier see the structure of the operators, they will be conjugated by

\[
W = \frac{1}{\sqrt{2}} \begin{pmatrix}
|D|^{-1/2} & |D|^{-1/2} & 0 \\
-i|D|^{-1/2} & i|D|^{-1/2} & 0 \\
0 & 0 & \sqrt{2}
\end{pmatrix}
\] (61)

according to

\[
\Phi \rightarrow \Phi^W = W^{-1} \Phi
\]
\[ H_0 \to H_0^W = W^{-1}H_0W \]
\[ X \to X^W = W^{-1}XW \]
\[ A \to A^W = W^{-1}AW. \] (62)

Then we get
\[ H_0^W = \begin{pmatrix}
    D & 0 & 0 \\
    0 & -D & 0 \\
    0 & 0 & D
\end{pmatrix} \] (63)

\[ X^W(x) = \begin{pmatrix}
    |D|^{1/2} (X_{\varphi}(x) + X_{\varphi^*}(x)) |D|^{-1/2} \\
    0 \\
    |D|^{1/2} (X_{\varphi}(x) - X_{\varphi^*}(x)) |D|^{-1/2}
\end{pmatrix} \]
\[ \begin{pmatrix}
    \sqrt{2}X_{\psi}(x)|D|^{-1/2} \\
    0 \\
    0 \end{pmatrix} \]
\[ (x) \begin{pmatrix}
    \sqrt{2}|D|^{1/2}X_{\varphi}(x) \\
    0 \\
    X_{\psi}(x)
\end{pmatrix} \]
\[ A^W(x) = \begin{pmatrix}
    |D|^{1/2} (A_{\varphi}(x) + A_{\varphi^*}(x)) |D|^{-1/2} \\
    0 \\
    |D|^{1/2} (A_{\varphi}(x) - A_{\varphi^*}(x)) |D|^{-1/2}
\end{pmatrix} \]
\[ \begin{pmatrix}
    \sqrt{2}|D|^{1/2}A_{\varphi}(x) \\
    0 \\
    0 \end{pmatrix} \]
\[ (x) \begin{pmatrix}
    \sqrt{2}|D|^{1/2}A_{\varphi}(x) \\
    0 \\
    A_{\psi}(x)
\end{pmatrix}. \]

It is easy to check that the formulas for the cocycles (67) and (60) are independent of such a conjugation (use (43) and identities as \[ |H_0^W| = |H_0| \]). Thus, \[ W \]-conjugated operators can be used in the calculations.

For \( d = 1 \) we use (63) and (64) in (67) to get
\[ \tilde{c}_1(X,Y) = \frac{i}{2} \int dx \int_{|q| \leq \Lambda} \frac{dq}{2\pi} \text{str} \frac{\partial}{\partial q} T(q, x) \] (64)

where \( T \) is a matrix \( \{T_{ij}\}_{ij} \) with the diagonal entries
\[ T_{11}(q, x) = |q|^{-1/2} (X_{\varphi}(x) + X_{\varphi^*}(x)) \frac{q}{|q|} \frac{\partial}{\partial x} (Y_{\varphi} + Y_{\varphi^*}) \] (65)
The factor $\frac{\partial}{\partial q} \frac{2}{q}$ might be replaced by $2\delta(q)$ under the momentum integration leading to a trivial $q$-integration. Evaluating the supertrace and collecting terms gives

$$\tilde{c}_1(X, Y) = \frac{i}{2\pi} \int dx \text{tr}_{\text{color}} \left( 2X_{\varphi\pi} \frac{\partial}{\partial x} Y_{\varphi\pi} + 2X_{\varphi\pi} \frac{\partial}{\partial x} Y_{\varphi\pi} - 2X_{\psi\psi} \frac{\partial}{\partial x} Y_{\varphi\psi} - 2X_{\psi\psi} \frac{\partial}{\partial x} Y_{\varphi\psi} \right)$$

where $\text{tr}_{\text{color}}$ is defined by $\text{tr}_{\text{color}} Z^2 = \text{str}_{\text{color}}$, and $\text{str}_{\text{color}}$ is defined on matrices $\{a_{ij}\}_{ij}$ of form (64) by

$$\text{str}_{\text{color}}(a) = a_{11} + a_{22} - a_{33}. (67)$$

When $d = 3$, the first term in (39) will vanish by the same reason as for the former case.

We now get

$$\sigma \left( [F^W_0, X^W]_{(1)} \right)(q, x) = \frac{1}{|q|} \sum_{i=1}^{3} \sigma(A^W_i)_0(q, x) \left( \sigma_i - \frac{q_i |q|}{q} \right) + O(|q|^{-2}) (68)$$

and

$$\sigma (F^W - F^W_0)(q) = \frac{1}{|q|} \sum_{i=1}^{3} \sigma(A^W_i)_0(q, x) \left( \sigma_i - \frac{q_i |q|}{q} \right) + O(|q|^{-2}) (69)$$

leading to

$$\tilde{c}_2(X, Y; A) = -\frac{1}{8} \sum_{i,j,k=1}^{3} J^A_{ijk} \int d^3x \text{str}_{\text{color}} \left( \sigma(A^W_i)_0(q, x) \frac{\partial \sigma(X^W)_0(q, x)}{\partial x_j} \frac{\partial \sigma(Y^W)_0(q, x)}{\partial x_k} + O(|q|^{-2}) \right) (70)$$

where the expression under the $x$-integration actually is $q$-independent. Introducing the operators

$$\tilde{X}(x) = \begin{pmatrix} X_{\varphi\varphi}(x) + X_{\varphi\pi}(x) & 0 & \sqrt{2}X_{\varphi\psi}(x) \\ 0 & X_{\varphi\pi}(x) - X_{\varphi\pi}(x) & 0 \\ \sqrt{2}X_{\psi\varphi}(x) & 0 & X_{\psi\psi}(x) \end{pmatrix}$$

and

$$\tilde{A}(x) = \begin{pmatrix} A_{\varphi\varphi}(x) + A_{\varphi\pi}(x) & 0 & \sqrt{2}A_{\varphi\psi}(x) \\ 0 & A_{\varphi\pi}(x) - A_{\varphi\pi}(x) & 0 \\ \sqrt{2}A_{\psi\varphi}(x) & 0 & A_{\psi\psi}(x) \end{pmatrix}$$
and using (51) the equation above can be rewritten as

\[
\tilde{c}_2(X,Y;A) = -\frac{i}{24\pi^2} \sum_{i,j,k=1}^{3} \int d^3x \operatorname{strcolor} \left( \epsilon_{ijk} \tilde{A}_i(x) \frac{\partial \tilde{X}}{\partial x_j}(x) \frac{\partial \tilde{Y}}{\partial x_k}(x) \right) \\
- (-1)^{\deg(X)\deg(Y)} (X \leftrightarrow Y) \tag{71}
\]

which is similar to the result (52) in the former example.

**Acknowledgments:**

I am grateful to J. Mickelsson for interesting me in the field under consideration. I would also like to thank him and E. Langmann for many useful discussions.

**Pseudodifferential operators (PSDO)**

Here some facts about PSDO will be summarized (see ref. [16] and ref. [17]). A PSDO \( a \) on the Hilbert space \( L^2(\mathbb{R}^d) \otimes V \), \( V \) a finite dimensional vector space and \( N = \dim(V) \), is determined by its symbol \( \sigma(a) \) which is a smooth \( N \times N \) matrix valued function, such that

\[
(af) (x) = \frac{1}{(2\pi)^{d/2}} \int d^dq \sigma(a)(q,x) \hat{f}(q)e^{iqx} \tag{72}
\]

where

\[
\hat{f}(q) = \frac{1}{(2\pi)^{d/2}} \int d^dx f(x)e^{-iqx} \quad f \in L^2(\mathbb{R}^d) \otimes V. \tag{73}
\]

The PSDO we are concerned with admits asymptotic expansion of their symbols as

\[
\sigma(a)(q,x) \sim \sigma(a)_k(q,x) + \sigma(a)_{k-1}(q,x) + \sigma(a)_{k-2}(q,x) + \ldots \tag{74}
\]

where \( k \) is an integer and

\[
\sigma(a)_j(\mu q, x) = \mu^j \sigma(a)_j(q, x), \quad \text{for } \mu > 1, |q| \gg 1. \tag{75}
\]

Such a PSDO is said to be of order \( k \). A PSDO \( a \) that has only one term in its expansion

\[
\sigma(a)(q,x) \sim \sigma(a)_k(q,x) \tag{76}
\]

is said to be homogeneous of order \( k \).

The symbol for the product of two PSDO’s is given asymptotically by

\[
\sigma(ab)(q,x) \sim \sum m! \frac{(-i)^{|m|}}{m!} (\partial_q^m \sigma(a)(q,x) \partial_x^m \sigma(b)(q,x)) \tag{77}
\]
where the sum is over all sets of nonnegative integers \( m = (m_1, ..., m_d), \quad |m| = m_1 + ... + m_d, \quad \partial^m_x = \frac{\partial^{m_1}}{\partial x_1} ... \frac{\partial^{m_d}}{\partial x_d}, \) etc., and \( m! = m_1! ... m_d! \).

Finally, the trace of a trace class operator in \( \mathbb{R}^d \) is given by

\[
\text{tr} a = \frac{1}{(2\pi)^d} \int d^d x d^d q \text{tr} V \sigma(a)(q, x). \tag{78}
\]

Thus, an operator with compact support in configuration space is Hilbert-Schmidt if and only if it has a symbol of order \(-d/2\) or less.

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