The puzzle of the $\pi \to \gamma \gamma^*$ transition form factor

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We study the $P \to \gamma \gamma^*$ ($P = \pi^0, \eta, \eta'$) transition form factors by means of the local-duality (LD) version of QCD sum rules. For the case of $\eta$ and $\eta'$, the conventional LD model provides a good description of the existing data. However, for the $\pi$ form factor we find disagreement with recent BABAR results for high $Q^2$ even though the accuracy of the LD approximation is expected to increase with $Q^2$. It remains mysterious why the $\eta$ and $\eta'$ form factors to virtual photons, on the one hand, and the $\pi$ form factor, on the other hand, show a qualitatively different behaviour corresponding to a rising with $Q^2$ violation of local duality in the pion case. In a quantum mechanical example we show that, for a bound-state size of about 1 fm, the LD sum rule provides an accurate prediction for the form factor for $Q^2 \geq$ a few GeV$^2$.

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1. INTRODUCTION

The form factor describing the two-photon transition of a light pseudoscalar meson $P$ is one of the simplest hadronic form factors in QCD. The corresponding amplitude

$$\langle \gamma(q_1)\gamma(q_2)|P(p)\rangle = i\epsilon_{\varepsilon_1 \varepsilon_2 q_1 q_2} F_{P \to \gamma \gamma}(q_1^2, q_2^2) \quad (1.1)$$

contains only one invariant form factor, $F_{P \to \gamma \gamma}(q_1^2, q_2^2)$. We shall be interested in the case of one real and one virtual photons, $q_1^2 = 0$ and $-q_2^2 = Q^2 \geq 0$, and define $F_{P \gamma}(Q^2) \equiv F_{P \to \gamma \gamma}(q_1^2 = 0, q_2^2 = -Q^2)$. For the pion case, the form factor $F_{\pi \gamma}(Q^2)$ has the following properties: (i) In the chiral limit of massless quarks and a massless pseudoscalar $\pi$, the form factor at $Q^2 = 0$ is given by the axial anomaly $[1]$, $F_{\pi \gamma}(Q^2 = 0) = 1/(2\sqrt{2}\pi f_\pi)$, $f_\pi = 130$ MeV. (ii) At large $Q^2$, perturbative QCD (pQCD) predicts the asymptotic behaviour $[2]$, $F_{\pi \gamma}(Q^2 \to \infty) \to \sqrt{2}f_\pi/Q^2$. Brodsky and Lepage proposed a simple interpolating formula between these two values $[2]$

$$F_{\pi \gamma}(Q^2) = \frac{1}{2\sqrt{2}\pi^2 f_\pi} \left(1 + \frac{Q^2}{4\pi^2 f_\pi^2}\right)^{-1}. \quad (1.2)$$

This formula may not provide an accurate description of the form factor at small nonzero $Q$ but one may expect its accuracy to increase rather fast at $Q^2$ values larger than a few GeV$^2$. In contrast to these expectations, the BABAR collaboration recently reported a surprising result for the behaviour of the $F_{\pi \gamma}$ form factor $[3]$: the product $Q^2 F_{\pi \gamma}(Q^2)$ does not saturate at large $Q^2$ but increases further (Fig. 1).

![Fig. 1: Form factor $F_{\pi \gamma}$ as function of $Q^2$: experimental data from $[3]$ (red) and $[4, 5]$ (black); the solid line represents the interpolation $[1, 2]$](image)
This result has already attracted a lot of attention in the literature (see, e.g. [6] and references therein). The aim of our analysis is to study the $P\gamma$ transition form factor for $P = \pi^0, \eta, \eta'$ by making use of the local-duality version of QCD sum rules [7]. This approach allows one to study hadron form factors without knowing subtle details of their structure and to consider on equal footing form factors of different hadrons.

We find that for both $\eta$ and $\eta'$ transitions, the LD sum rule provides a satisfactory description of the form factors in the region $Q^2 = 5 - 100 \text{ GeV}^2$ [1, 5, 8, 6]. The result from the LD approximation for the $\pi$ case, on the other hand, does not agree with the BABAR data.

In order to test the accuracy of the LD sum rule for the $P\gamma$ transition form factor, we consider, in parallel to QCD, a quantum-mechanical potential model. In the latter case, the form factor can be obtained both by the LD sum rule and by an exact calculation. Comparing these results with each other provides a probe of the LD approximation [10]. Here we find that, independently of the details of the confining potential, the LD sum rule reaches the accuracy of a few percent already at relatively low values of the momentum transfer.

It remains mysterious why, in contrast to the success of the LD approximation for the $\eta$ and $\eta'$ transitions and to the experience from quantum mechanics, the results from the LD approximation strongly contradict the BABAR result.

This Letter is organized as follows: In the next section, we recall the structure of the $\langle VAV \rangle$ amplitude and the relation of $F_{\pi\gamma}$ to the axial anomaly. Section 3 presents the dispersion representation for $\langle VAV \rangle$ and touches the issue of multiloop radiative corrections to this quantity. Section 4 reviews the LD approximation for the $P\gamma$ form factor and studies the expected accuracy of this approximation making use of a quantum-mechanical testing ground. Section 5 applies the LD sum rule to the $F_{P\gamma}$, $P = \pi^0, \eta, \eta'$ form factors. Section 6 summarizes our conclusions.

2. THE THREE-POINT FUNCTION $\langle VAV \rangle$ AND THE AXIAL ANOMALY

Let us start with the amplitude of two-photon production from the vacuum induced by the axial-vector current of nearly massless quarks of one flavour, $j_5^\mu = \bar{q}\gamma_\mu\gamma_5 q$, with $\varepsilon_{1,2}$ denoting the photon polarization vectors:

$$\langle \gamma(q_1)\gamma(q_2) \rangle_0^\alpha(x = 0)|0\rangle = T_{\mu\nu\beta}(p|q_1, q_2)\varepsilon^\beta_{1,2}. \quad p = q_1 + q_2. \quad (2.1)$$

The amplitude $T_{\mu\nu\beta}$ is obtained from the vacuum expectation value of the $T$-product of two vector and one axial-vector currents and will be referred to as the $\langle VAV \rangle$ amplitude. Vector-current conservation yields the following relations:

$$T_{\mu\nu\beta}(p|q_1, q_2)\varepsilon^\alpha_1 = 0, \quad T_{\mu\nu\beta}(p|q_1, q_2)\varepsilon^\beta_2 = 0. \quad (2.2)$$

The general decomposition of the amplitude contains four invariant form factors and may be written as

$$T_{\mu\nu\beta}(p|q_1, q_2) = -\frac{p_\mu}{p^2 - m_P^2} \epsilon_{\alpha\beta q_1 q_2} i F_A + (q_1^2 \epsilon_{\mu\alpha\beta q_2} - q_1 \epsilon_{\alpha q_1 q_2} - \frac{p_\mu}{p^2 - m_P^2} q_2^2 \epsilon_{\alpha q_1 q_2}) i F_1$$

$$+ (q_2^2 \epsilon_{\mu q_2 q_1} - q_2 \epsilon_{\mu q_2 q_1} - \frac{p_\mu}{p^2 - m_P^2} q_1^2 \epsilon_{q_1 \alpha q_2}) i F_2 + (q_1 q_2) \epsilon_{\alpha q_1 q_2} i F_3. \quad (2.3)$$

The explicit calculation of these form factors at one and two loops in QCD leads to $F_3 = 0$ [11]. Therefore, we shall omit the corresponding term. The absence of any contact terms in $T_{\mu\nu\beta}$ can be verified by reducing out one of the photons and using the conservation of the electromagnetic current [12].

The parameterization [2, 3] takes into account the pole at $p^2 = m_P^2$, related to the contribution of the lightest pseudoscalar state. Apart from the pole at $p^2 = m_P^2$, the amplitude has no singularities at small $p^2$. In the chiral limit, $m_P = 0$, both the second and third Lorentz structures in (2.3) are transverse with respect to $p_\mu$, and the form factor $F_A$ represents the axial anomaly [4]. According to the Adler–Bardeen theorem [13], the axial anomaly is saturated by the one-loop expression, $F_A = 1/(2\pi^2)$, and remains non-renormalized by higher-order corrections.

Separating the longitudinal and the transverse structures for the case $m_P \neq 0$, we obtain

$$T_{\mu\nu\beta}(p|q_1, q_2) = -\frac{p_\mu}{p^2}(\alpha q_1 q_2) i F_A + \frac{p_\mu}{p^2} m_P^2 \epsilon_{\alpha q_1 q_2} i (F_A + q_1^2 F_1 + q_2^2 F_2)$$

$$+(q_1^2 \epsilon_{\mu q_1 q_2} - q_1 \epsilon_{q_1 \alpha q_2} - \frac{p_\mu}{p^2} q_1^2 \epsilon_{q_1 \alpha q_2}) i F_1$$

$$+ (q_2^2 \epsilon_{q_2 q_1} - q_2 \epsilon_{q_2 q_1} - \frac{p_\mu}{p^2} q_2^2 \epsilon_{q_2 q_1}) i F_2. \quad (2.4)$$

Beyond the chiral limit, the first two terms contain two poles: [14]: a “kinematical” pole at $p^2 = 0$, which cancels the corresponding singularities in the transverse Lorentz structures, and the “dynamical” pole at $p^2 = m_P^2$, corresponding
to the $\pi$ meson. The full amplitude is regular at $p^2 = 0$. That is, the pole at $p^2 = 0$ in the transverse Lorentz structures is of purely kinematic origin and does not correspond to a massless particle. By forming the divergence, one obtains

$$i p^\mu T_{\mu\alpha\beta} = \epsilon_{\alpha\beta q_1 q_2} \left[ F_A - \frac{m^2}{m^2 - p^2} (F_A + q_1^2 F_1 + q_2^2 F_2) \right]. \quad (2.5)$$

As is clear from (2.5), the transition form factor of interest is given by the linear combination of $F_A$, $F_1$, and $F_2$

$$F_{\gamma\gamma \rightarrow \gamma\gamma} = -\frac{1}{f_p} (F_A + q_1^2 F_1 + q_2^2 F_2) |_{p^2 = m^2}. \quad (2.6)$$

Thus, the $P \rightarrow \gamma\gamma$ form factor is proportional to the axial anomaly only at one kinematical point, $q_1^2 = q_2^2 = 0$. At this point, the pion form factor in the chiral limit is protected from radiative corrections by the Adler–Bardeen theorem; the one-loop result represents the exact result. Hence, in the chiral limit one expects the pion pole at $p^2 = 0$ to emerge in one-loop diagrams for the $\langle VAV \rangle$ amplitude. Indeed, in this unique situation a pole dual to a single hadron state emerges from the single one-loop diagram of perturbation theory [13].

However, already if one of the photons is virtual, the transition form factor $F_{\gamma\gamma \rightarrow \gamma\gamma}$ and the axial anomaly are not proportional to each other. It is the topic of the next two sections to analyze the form factor $F_{\gamma\gamma}(Q^2)$ by means of dispersive sum rules [10].

3. DISPERSION REPRESENTATIONS FOR $\langle VAV \rangle$ AND THE AXIAL ANOMALY

We now discuss the one-loop expression for the amplitude. To this end, a slightly different parameterization, obtained by setting $(F_A + q_1^2 F_1 + q_2^2 F_2)/(p^2 - m^2) = F_0$, proves to be convenient:

$$T_{\mu\alpha\beta}(p|q_1, q_2) = -p_\mu \epsilon_{\alpha\beta q_1 q_2} | F_0 + (q_1^2 \epsilon_{\alpha\beta q_1 q_2} - q_1 \epsilon_{\mu q_1 \beta q_2}) | F_1 + (q_2^2 \epsilon_{\mu\beta q_2 q_1} - q_2 \epsilon_{\mu q_2 \alpha q_1}) | F_2. \quad (3.1)$$

As follows from (2.6), $F_0$ contains the contribution of the pseudoscalar meson of our interest.

We also consider the transition amplitude of the pseudoscalar current operator $\bar{q} \gamma_5 q$:

$$\langle \gamma(q_1) \gamma(q_2) | \bar{q} \gamma_5 q | 0 \rangle = \epsilon_{\alpha\beta q_1 q_2} \epsilon^\alpha_1 \epsilon^\beta_2 F_5(q_1^2, q_2^2, p^2). \quad (3.2)$$

The two-photon amplitude of the divergence of the axial current takes the form

$$\langle \gamma(q_1) \gamma(q_2) | \partial^\mu j^\mu_\beta | 0 \rangle = \epsilon_{\alpha\beta q_1 q_2} \epsilon^\alpha_1 \epsilon^\beta_2 (p^2 F_0 - q_1^2 F_1 - q_2^2 F_2). \quad (3.3)$$

The case of our interest is $q_1^2 = 0$, then the form factor $F_1$ does not contribute to the divergence. In perturbation theory, the form factors $F_0$, $F_2$, and $F_5$ may be written in terms of their spectral representations in $p^2$ (with $q_2^2 = -Q^2$):

$$F_i(p^2, q^2) = \frac{1}{\pi} \int_{4m^2}^{\infty} \frac{ds}{s - p^2} \Delta_i(s, q^2). \quad (3.4)$$

To one-loop order, the spectral densities read [12, 17, 10]

$$\Delta_0(s, q^2) = -\frac{1}{2\pi} \frac{1}{(s - q^2)^2} \left[ -q^2 w + 2m^2 \log \left( \frac{1 + w}{1 - w} \right) \right],$$

$$\Delta_2(s, q^2) = -\frac{1}{2\pi} \frac{1}{(s - q^2)^2} \left[ -s w + 2m^2 \log \left( \frac{1 + w}{1 - w} \right) \right],$$

$$\Delta_5(s, q^2) = -\frac{1}{2\pi} \frac{m}{s - q^2} \log \left( \frac{1 + w}{1 - w} \right), \quad w = \sqrt{1 - 4m^2/s}. \quad (3.5)$$

Obviously, the absorptive parts $\Delta_i$ obey the classical equation of motion for the divergence of the axial current

$$s \Delta_0(s, q^2) - q^2 \Delta_2(s, q^2) = 2m \Delta_5(s, q^2). \quad (3.6)$$

The form factors then satisfy

$$p^2 F_0(p^2, q^2) - q^2 F_2(p^2, q^2) = 2m F_5(p^2, q^2) - \frac{1}{\pi} \int_{4m^2}^{\infty} ds \Delta_0(s, q^2). \quad (3.7)$$
The last integral is equal to \(-1/2\pi\), independently of the values of \(m\) and \(q^2\), and represents the axial anomaly 1:

\[
p^2 F_0(p^2, q^2) - q^2 F_2(p^2, q^2) = 2m F_5(p^2, q^2) + \frac{1}{2\pi^2}.
\] (3.8)

In the chiral limit \(m = 0\) and for \(q^2 = 0\), the form factor \(F_0\) develops a pole related to a massless pseudoscalar meson 15. The residue of this pole is again the axial-anomaly 1/2\(\pi^2\).

As is clear from (3.7), the anomaly represents the integral of \(\Delta_0\), the spectral density of the form factor \(F_0\). Adler and Bardeen tell us that the anomaly is non-renormalized by multiloop corrections. The easiest realization of this requirement is obtained by setting \(s, q^2 \neq 0\). Then one may ask oneself how it may happen that the anomaly nevertheless remains non-renormalized by multiloop corrections? The only possible answer we see is that the non-renormalization of the anomaly is reached due to some conspiral property of multiloop contributions to \(\Delta_0(s, q^2)\). An argument in favour of this possibility comes from explicit two-loop calculations 11, 18 which report the non-renormalizability of the full \(\langle V AV \rangle\) vertex to the two-loop accuracy. However, if so, the full form factor \(F_0\) is given by its one-loop expression. Then, this expression should develop the pion pole, known to be present in the full amplitude for any value of \(q^2\). But, obviously, this pole does not emerge in the one-loop expression for \(F_0\) if \(q^2 \neq 0\)!

This requires that multiloop corrections to the form factor \(F_0\) do not vanish. Then one may ask oneself how it may happen that the anomaly nevertheless remains non-renormalized by multiloop corrections? The only possible answer we see is that the non-renormalization of the anomaly is reached due to some conspiral property of multiloop contributions to \(\Delta_0(s, q^2)\). An argument in favour of this possibility comes from explicit two-loop calculations 11, 18 which report the non-renormalizability of the full \(\langle V AV \rangle\) vertex to the two-loop accuracy. However, if so, the full form factor \(F_0\) is given by its one-loop expression. Then, this expression should develop the pion pole, known to be present in the full amplitude for any value of \(q^2\). But, obviously, this pole does not emerge in the one-loop expression for \(F_0\) if \(q^2 \neq 0\)!

So we conclude that, in spite of the fact that explicit calculations yield a non-renormalizability of the full \(\langle V AV \rangle\) vertex to two-loop accuracy 11, 18, the conjecture of 11 that this result might hold to all orders of the perturbative expansion may not be valid.

4. \(F_{P,\gamma}\) FROM A LOCAL-DUALITY SUM RULE FOR \(F_0\)

As is obvious from (2.3), the contribution of the light pseudoscalar constitutes a part of the form factor \(F_0\). The Borel sum rule for the corresponding Lorentz structure reads \((Q^2 = -q^2 > 0)\)

\[
\int ds \exp(-s\tau)\Delta_0(s, Q^2) = -f_P F_{P,\gamma}(Q^2) \exp(-m^2_P \tau) + \text{contributions of excited states}.
\] (4.1)

Exploiting the concept of duality, the contribution of the excited states is assumed to be dual to the high-energy region of the diagrams of perturbation theory above an effective threshold \(s_{\text{eff}}\). After that, setting the Borel parameter \(\tau = 0\) (which yields the so-called local-duality limit), we arrive at the LD sum rule for a pseudoscalar \(\bar{q}q\)-meson

\[
s_{\text{eff}}(Q^2) \int_{4m^2} ds \Delta_0(s, Q^2) = -f_P F_{P,\gamma}(Q^2), \quad \text{for large } Q^2. \quad \text{(4.2)}
\]

The spectral density \(\Delta_0\) to one-loop order is given by (3.5); two-loop corrections were found to be absent 11, 18. As discussed above, higher-loop corrections to \(\Delta_0\) cannot vanish; so the l.h.s. of (4.2) is known to be \(O(\alpha_s^2)\) accurate. Recall that all details of the nonperturbative dynamics are encoded in a single quantity, the effective threshold \(s_{\text{eff}}(Q^2)\). The effective threshold is an essential parameter of the method of dispersive sum rules; it is not identical to the physical threshold and is not universal (i.e., it is specific for the correlator under consideration); moreover, in general it depends on \(Q^2\).

In the chiral limit, the LD expression for the form factor for the one-flavour case is particularly simple:

\[
F_{P,\gamma}(Q^2) = \frac{1}{2\pi^2 f_P} \frac{s_{\text{eff}}(Q^2)}{s_{\text{eff}}(Q^2) + Q^2}.
\] (4.3)

Apart from neglecting \(\alpha_s^2\) and higher-order corrections to the spectral density \(\Delta_0\), no approximations have been done up to now: we have just considered the LD limit \(\tau = 0\); for an appropriate choice of \(s_{\text{eff}}(Q^2)\) the form factor may still be calculated exactly. Approximations come into the game when we consider a model for \(s_{\text{eff}}(Q^2)\).

Irrespective of the behaviour of \(s_{\text{eff}}(Q^2)\), at \(Q^2 = 0\) the form factor is related to the axial anomaly: \(F_{P,\gamma}(0) = 1/(2\pi^2 f_P)\). QCD factorization requires \(s_{\text{eff}}(Q^2) \rightarrow 4\pi^2 f_P^2\) for large \(Q^2\). The simplest model compatible with this requirement is obtained by setting

\[
s_{\text{eff}}(Q^2) = 4\pi^2 f_P^2 \quad \text{(4.4)}
\]
for all values of $Q^2$. We shall refer to the choice yielding for the neutral pion case the Brodsky–Lepage result \cite{1,2}, as the “conventional LD model”. The relevance and the expected accuracy of the LD model may be tested in those cases where the form factor $F_{\pi\gamma}(Q^2)$ is known, i.e., may be calculated by other theoretical approaches or measured experimentally. Then, the exact effective threshold may be reconstructed from \cite{1,3}, in this way probing the accuracy of the LD model. Let us play this game making use of a quantum-mechanical model, where the exact form factor may be calculated from the solution of the Schrödinger equation, and then compared with the result of the sum rule for a three-point function of nonrelativistic field theory.

A. LD sum rule in quantum mechanics

The analogue of the $\pi\gamma$ form factor in quantum mechanics is given by \cite{23,24}

$$F_{\text{NR}}(q) = \int_0^\infty dT \langle \Psi | J(q) \exp(-HT) | r = 0 \rangle,$$  

(4.5)

where the current operator $J(q)$ is introduced by the kernel $\langle r'|J(q)|r \rangle = \exp(iq \cdot r) \delta^3(r - r')$, the Hamiltonian $H$ governing the nonrelativistic potential model reads

$$H = \frac{k^2}{2m} + V(r),$$

(4.6)

and $\Psi$ is the corresponding ground state. Technically, the LD model for the form factor \cite{1,4,5} is constructed from the quantum-mechanical analogue of the three-point function in the same way as for the case of the elastic form factor (for details, consult \cite{10}).

Recall, however, an essential conceptual difference between the $P\gamma$ form factor and the elastic form factor with respect to the factorization of these quantities at large momentum transfers: The factorization of the elastic form factor requires the presence of both Coulomb and confining terms in the interaction. The factorization of the $P\gamma$ form factor does not require the presence of a Coulombic term and emerges also for a purely confining interaction. The LD model for a given form factor is tightly related to its factorization properties; specifically, the LD model for the analogue of the $P\gamma$ form factor may be formulated in quantum mechanics for the case of a purely confining potential.

Figure 2 depicts the exact effective threshold $k_{\text{eff}}(Q)$ — the quantum-mechanical counterpart of the effective threshold $s_{\text{eff}}(Q^2)$ — for the example of the harmonic-oscillator potential $V(r) = m\omega^2 r^2/2$, for parameter values relevant for hadron physics: a reduced mass of the light quark of $m = 0.175$ GeV and an interaction strength of $\omega = 0.5$ GeV, which lead to a size of the ground state around 1 fm, a typical size of a ground-state hadron in QCD. We have

![Graph showing $k_{\text{eff}}(Q)$ vs. $Q$](image)

Fig. 2: Effective threshold $k_{\text{eff}}(Q)$ for the three-point function in quantum mechanics, recalculated from the exact form factor $F_{\text{NR}}(Q)$, Eq. (4.5), in the harmonic-oscillator potential model. $R_g \equiv |\Psi(r = 0)|^2$.

\footnote{In an alternative approach to the $P\gamma$ form factor \cite{22}, the pseudoscalar meson is described by a set of distribution amplitudes of increasing twist which are treated as nonperturbative inputs. In our analysis, the deviation of the effective threshold $s_{\text{eff}}(Q^2)$ from its asymptotic value $4\pi^2 f_p^2$ corresponds to some extent to the contribution of higher-twist distribution amplitudes in the approach of \cite{22}.}
checked that a similar picture for the effective threshold emerges for other confining potentials; moreover, adding the Coulomb potential changes this picture only slightly.

From the behaviour of \( k_{\text{eff}}(Q) \), we conclude that the LD model may be expected to work increasingly well already for \( Q^2 \) above a few GeV\(^2\). Inspired by this result, we now look what the LD model predicts for \((\pi, \eta, \eta') \to \gamma \gamma^*\) transitions.

5. THE \((\pi^0, \eta, \eta') \to \gamma \gamma^*\) FORM FACTOR

A. The \(\pi^0 \to \gamma \gamma^*\) form factor

Taking into account the \(\pi^0\) flavour structure and choosing the relevant interpolating current \( j_{\mu}^5 = (\bar{u}\gamma_{\mu}\gamma_5 u - \bar{d}\gamma_{\mu}\gamma_5 d) / \sqrt{2} \), the LD sum rule reads

\[
F_{\pi^0\gamma}(Q^2) = \frac{N_c}{\sqrt{2}} \left( \frac{4}{9} - \frac{1}{9} \right) \frac{1}{2\pi^2 f_\rho^2} s_{\text{eff}}(Q^2) + Q^2, \quad N_c = 3,
\]

which by setting \( s_{\text{eff}}(Q^2) = 4\pi^2 f_\rho^2 \), \( f_\rho = 130\ \text{MeV} \), leads to the Brodsky-Lepage formula (1.2). Figure 1 shows the corresponding plot. Figure 3 represents the “experimental” effective threshold recalculated from the form factor data via Eq. (5.1). This “experimental” effective threshold may be well approximated by a linearly rising function of \( Q^2 \). Surprisingly, the BABAR data show very strong — and growing with \( Q^2 \) — violations of local duality! This observation is in absolute contradiction to our experience from quantum mechanics. Let us investigate next what happens in the case of the \(\eta\) and \(\eta'\) mesons.

B. The \((\eta, \eta') \to \gamma \gamma^*\) form factor

The simple expression (1.2) is sometimes erroneously assumed also for the \(\eta\) and \(\eta'\) cases [5, 25]. However, the naïve replacement \( f_\pi \to f_{\eta,\eta'} \) in (1.2) yields wrong expressions for \( F_{(\eta,\eta')\gamma} \). The correct way to proceed is to take into account the presence of two — nonstrange and strange — components in the \(\eta\) and \(\eta'\) mesons and their mixing. Making use of the \(\eta-\eta'\) mixing scheme from [26, 27] (see also [28]), the flavour structure of \(\eta\) and \(\eta'\) may be described as follows

\[
|\eta\rangle = \frac{\bar{u}u + \bar{d}d}{\sqrt{2}} \cos \phi - |\bar{s}s\rangle \sin \phi,
\]
\[
|\eta'\rangle = \frac{\bar{u}u + \bar{d}d}{\sqrt{2}} \sin \phi + |\bar{s}s\rangle \sin \phi, \quad \phi \simeq 39.3^0,
\]

The corresponding expression for the form factors take the form

\[
F_{\eta\gamma}(Q^2) = \frac{N_c}{\sqrt{2}} \left( \frac{4}{9} + \frac{1}{9} \right) F_n(Q^2) \cos \phi - \frac{N_c}{9} F_s(Q^2) \sin \phi,
\]
\[
F_{\eta'\gamma}(Q^2) = \frac{N_c}{\sqrt{2}} \left( \frac{4}{9} + \frac{1}{9} \right) F_n(Q^2) \sin \phi + \frac{N_c}{9} F_s(Q^2) \cos \phi.
\]

![Fig. 3: Effective threshold \(s_{\text{eff}}(Q^2)\) recalculated from the data [] (red) and [] (black) by means of the LD relation (5.1) for the form factor \(F_{\pi^0\gamma}(Q^2)\).](image)
Here \( F_n(Q^2) \) and \( F_s(Q^2) \) are the form factors describing the transition of the nonstrange and \( \bar{s}s \)-components, respectively. The corresponding LD sum-rule for these form factors have a simple form

\[
F_{n\gamma}(Q^2) = \frac{1}{f_n} \int_{0}^{\Delta_0(s, Q^2)} ds \Delta_n(s, Q^2),
\]

\[
F_{s\gamma}(Q^2) = \frac{1}{f_s} \int_{0}^{\Delta_s(s, Q^2)} ds \Delta_s(s, Q^2).
\]

(5.4)

\( \Delta_n \) and \( \Delta_s \) correspond to \( \Delta_0 \) with different quark masses in the loop. In numerical calculations we set \( m_u = m_d = 0 \) and \( m_s = 100 \) MeV. Accordingly, the LD model involves two separate effective thresholds for the nonstrange and the strange components \( [27] \):

\[
s^{(n)}_{\text{eff}} = 4\pi^2 f_n^2, \quad f_n \approx 1.07 f_\pi, \quad s^{(s)}_{\text{eff}} = 4\pi^2 f_s^2, \quad f_s \approx 1.36 f_\pi.
\]

(5.5)

The LD model may not perform well for small values of \( Q^2 \), where the true effective threshold is smaller than the LD threshold. However, for larger \( Q^2 \) the LD model gives reasonable predictions for the form factors, as illustrated by Fig. 4.

![Fig. 4: LD form factors \( F_{n\gamma} \) and \( F_{s\gamma} \) vs. \( Q^2 \), compared with the experimental data presented in [4, 5] (coloured dots) and \( \bigcirc \) (black dots). The data points “borrowed” from the timelike momentum transfer \( q^2 = -Q^2 = 112 \) GeV\(^2 \) \( [8] \). \( q^2 F_{n\gamma}(q^2) = 0.229 \pm 0.03 \pm 0.008 \) GeV and \( q^2 F_{s\gamma}(q^2) = 0.251 \pm 0.019 \pm 0.008 \) GeV are not shown in this plot.](image)

6. CONCLUSIONS

In this paper, we analyzed the \( P \to \gamma \gamma^* \) transitions for \( P = \pi^0, \eta, \eta' \).

I. We emphasized that the \( P \to \gamma \gamma^* \) form factor is proportional to the axial anomaly only if both photons are on-shell; if at least one of these photons is virtual, this proportionality is lost. As a result, the \( P \to \gamma \gamma^* \) form factor in the chiral limit is not protected from receiving higher-order radiative corrections by the Adler–Bardeen theorem. Moreover, for virtual photons, the one-loop expression for the \( \langle V AV \rangle \) amplitude does not develop a pole at \( p^2 = 0 \), related to a massless pseudoscalar. Therefore, the one-loop result for the form factor \( F_0 \) cannot represent the full result for this quantity. \( F_0 \) should receive radiative corrections at higher orders in the loop expansion, in spite of the absence of two-loop radiative corrections to the \( \langle V AV \rangle \) amplitude reported in \( [11, 18] \).

Interestingly, the form factor \( F_0 \) and the axial anomaly are given by dispersive integrals involving the same function, \( \Delta_0(s, q^2) \). By the argument given above, radiative corrections (coming from three and more loops) to the spectral density \( \Delta_0(s, q^2) \) cannot vanish. Then, the Adler–Bardeen theorem requires some specific consprial properties of multiloop corrections to \( \Delta_0(s, q^2) \) enforcing the vanishing of their integrals over \( s \).

II. We applied the local-duality version of QCD sum rules to the \( (\pi^0, \eta, \eta') \to \gamma \gamma^* \) transition form factors. An attractive feature of this approach is the possibility to study form factors of bound states without knowing subtle...
details of their structure. Moreover, it allows one to consider on equal footing form factors of different bound states. Our findings may be summarized as follows:

1. We tested the accuracy of the LD model for the $P\gamma$ transition form factor in quantum mechanics. We calculated this form factor from the solution of the Schrödinger equation and compared with the result of the quantum-mechanical LD sum rule. This comparison reveals that for a usual bound state, with a typical hadronic extension of about 1 fm, the LD sum rule is expected to yield accurate predictions for the form factor for $Q^2$ larger than a few GeV$^2$; this accuracy increases with $Q^2$. At small but nonzero $Q^2$, deviations from the LD model depend on subtle details of the confining interaction.

2. Surprisingly, the BABAR data for the pion form factor exhibit an extreme violation of local duality in the $\pi\gamma$ form factor even at $Q^2 = 40$ GeV$^2$. Moreover, the violation of local duality increases with $Q^2$ in the range $Q^2 = 10 \sim 40$ GeV$^2$.

3. Even more surprisingly — taking into account the strong disagreement in the pion case — the LD predictions agree with the experimental data for both the $\eta$ and $\eta'$ mesons in the rather broad range $Q^2 = 4 \sim 100$ GeV$^2$.

The question why the nonstrange component in $\eta$ and $\eta'$, on the one hand, and the pion, on the other hand, should lead to a qualitatively different behaviour of the $P \to \gamma\gamma^*$ form factor remains mysterious. So far no compelling theoretical explanation for this strange phenomenon has been found (see also [22, 23]).

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