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PSEUDO-PERIODIC HOMEOMORPHISMS AND DEGENERATION OF RIEMANN SURFACES

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Abstract. We will announce two theorems. The first theorem will classify all topological types of degenerate fibers appearing in one-parameter families of Riemann surfaces, in terms of “pseudoperiodic” surface homeomorphisms. The second theorem will give a complete set of conjugacy invariants for the mapping classes of such homeomorphisms. This latter result implies that Nielsen’s set of invariants [Surface transformation classes of algebraically finite type, Collected Papers 2, Birkhäuser (1986)] is not complete.

Let \( \{F_\xi\} \) be a family of Riemann surfaces parametrized by complex numbers \( \xi \). As \( \xi \) approaches a special value, say, 0, \( F_\xi \) changes its “shape” and finally gets singularities becoming a singular surface \( F_0 \). This degeneration phenomenon has long been studied. Here we study it from the topological point of view. We will show that the topological types of the degenerate fibers can be completely classified in terms of certain surface mapping classes introduced by Nielsen [Ni2] some fifty years ago. We will also give a complete set of conjugacy invariants for such mapping classes.

Throughout this paper all manifolds will be oriented, and all homeomorphisms between them will be orientation-preserving. \( \Sigma_g \) will denote a closed surface of genus \( g \). Details will appear in [MM2].

1. Pseudoperiodic homeomorphisms

A homeomorphism \( f : \Sigma_g \rightarrow \Sigma_g \) and its mapping class \([f]\) are called in this paper pseudoperiodic if \([f]\) is either of finite order or reducible and in the latter case all component mapping classes are of finite order. (Cf. [Th, G].) It was Nielsen [Ni2] who first studied these mapping classes under the name of surface...
transformation classes of algebraically finite type. Let us recall some conjugacy invariants introduced by Nielsen. (See [Ni2, G].)

Suppose \( f \) is reduced by a system of simple closed curves \( C = C_1 \cup C_2 \cup \cdots \cup C_r \). \( C \) is called admissible if each connected component of \( \Sigma_g \setminus C \) has negative Euler characteristic. If \( g \geq 2 \), such a system always exists. With each curve \( C_j \) of an admissible system \( C \) is associated a rational number \( s(C_j) \) called the \textit{screw number}. This measures the amount of Dehn twist performed by \( f^\alpha \) about \( C_j \), where \( \alpha = \alpha(C_j) \) is the smallest positive integer such that \( f^\alpha(C_j) = C_j \). An admissible system \( C \) is precise if \( s(C_j) \neq 0 \) for each \( C_j \). A precise system always exists and is unique up to isotopy.

A curve \( C_j \) is amphidrome if \( \alpha \) is even and \( f^{\alpha/2}(C_j) = -C_j \).

We say that a pseudoperiodic homeomorphism \( f \) is of negative twist if either \( [f] \) is of finite order, or, when \( [f] \) is reducible, \( s(C_j) < 0 \) for each curve \( C_j \) in a precise system \( C \).

2. Degenerating Family

By a \textit{degenerating family} (of Riemann surfaces) of genus \( g \) we mean a triple \((M, D, \varphi)\) consisting of a noncompact complex surface \( M \); an open unit disk \( D = \{ \xi \in \mathbb{C} \mid |\xi| < 1 \} \); and a surjective, proper, and holomorphic map \( \varphi : M \to D \). All fibers \( F_\xi = \varphi^{-1}(\xi) \) are assumed to be connected, and outside the origin \( \varphi|_{\varphi^{-1}(D^*)} : \varphi^{-1}(D^*) \to D^* \) is assumed to be a smooth fiber bundle with fiber \( \Sigma_g \), where \( D^* = D - \{0\} \). The family is \textit{minimal} if it is free from \((-1)\)-curves.

Two families, \((M_i, D_i, \varphi_i), i = 1, 2, \) are \textit{topologically equivalent} \((\sim^{\text{TOP}})\) if there exist homeomorphisms \( H : M_1 \to M_2 \) and \( h : D_1 \to D_2 \) satisfying \( h(0) = 0 \) and \( h \varphi_1 = \varphi_2 H \). We are interested in the following set:

\[
\mathcal{S}_g = \{ \text{minimal degenerating families of genus } g \}/\sim^{\text{TOP}}.
\]

Given a degenerating family of genus \( g \), the \textit{monodromy homeomorphism} \( f : \Sigma_g \to \Sigma_g \) around the central fiber \( F_0 \) is determined as usual (up to isotopy and conjugation). By the results of Imayoshi [I], Shiga and Tanigawa [ST], and Earle and Sipe [ES], \( f \) is a pseudoperiodic homeomorphism of negative twist. (There is an alternative topological proof, [MM2]s.) Let \( \mathcal{M}_g \) be the mapping class group of \( \Sigma_g \) and \( \hat{\mathcal{M}}_g \) the set of conjugacy classes in \( \mathcal{M}_g \). Let \( \mathcal{P}_g^- \) denote the subset of \( \hat{\mathcal{M}}_g \) represented by pseudoperiodic mapping classes of negative twist. Then we have a well-defined map

\[
\rho : \mathcal{S}_g \to \mathcal{P}_g^-.
\]

**Theorem 1.** For \( g \geq 2 \), \( \rho : \mathcal{S}_g \to \mathcal{P}_g^- \) is bijective.

The corresponding map for \( g = 1 \) is surjective, and the “kernel” consists of multiple fibers. (Cf. [K].) Using Theorem 1 and the construction in §3, one can topologically recover Namikawa and Ueno’s classification of singular fibers of genus 2 [NU]. Since \( \mathcal{M}_g \cong \text{Aut}(\pi_1 \Sigma_g)/\text{Inn}(\pi_1 \Sigma_g) \), we have

**Corollary 1.1.** If \( g \geq 2 \), the action of the monodromy on \( \pi_1 \Sigma_g \) determines the topological equivalence class of \((M, D, \varphi)\). In particular, if the action is trivial, \( F_0 \) is nonsingular.
Note that the action on $H_1(\Sigma;\mathbb{Z})$ is not sufficient [NU]. For an explicit algebraic calculation of nonabelian monodromy see [O].

**Corollary 1.2.** Given a pseudoperiodic homeomorphism of negative twist $f : \Sigma_g \to \Sigma_g$, there exists a degenerating family $(M, D, \varphi)$ whose monodromy homeomorphism coincides with $f$ up to isotopy and conjugation.

A closely related existence theorem has been independently announced by Earle and Sipe [ES, §7]. See [MM1] for a short abstract of our result, where we adopted a sign convention opposite to the one here.

3. **Generalized quotient**

The idea in proving Theorem 1 is to construct the inverse map of $\rho : \mathcal{S}_g \to \mathcal{P}_g^-$. A Riemann surface with nodes $S$ was introduced by Bers [B]. We will call the underlying topological space of $S$ a chorizo space (chorizo = Spanish sausage), which we allow to have boundaries and not to be connected. A chorizo space below will be numerical in the sense that to each irreducible component is attached a positive integer called the multiplicity.

For a pseudoperiodic homeomorphism of negative twist $f : \Sigma_g \to \Sigma_g$ we can construct a numerical chorizo space called the *generalized quotient* $S_f$ of $f$ as follows:

Decompose $\Sigma_g$ as $\Sigma_g = A \cup B$, where $A$ is the union of annular neighborhoods of the curves in the precise system $C$ such that $f(A) = A$. We assume that $f|B : B \to B$ is periodic. The quotient space $B/(f|B)$ is an orbifold. Let $p$ be a cone point, $(m, \lambda, \sigma)$ the valency of $p$ [Ni1]; that is, if $x \in B$ is a point over $p$, $m$ is the smallest positive integer such that $f^m(x) = x$, $f^m$ is the rotation around $x$ through the angle $2\pi\delta/\lambda$ ($0 < \delta < \lambda, \gcd(\lambda, \delta) = 1$) and $\sigma$ is the integer determined by $\delta \sigma \equiv 1 \pmod{\lambda}$, $0 < \sigma < \lambda$.

By the Euclidean algorithm we obtain a sequence of integers $n_0 > n_1 > \cdots > n_l = 1$ such that $n_0 = \lambda, n_1 = \lambda - \sigma, n_{i-1} + n_{i+1} \equiv 0 \pmod{n_i}, i = 1, \ldots, l - 1$. Set $m_i = mn_i$ ($i = 0, 1, \ldots, l$).

Let $\text{Ch}(B)$ be the chorizo space constructed from $B/(f|B)$ by replacing a neighborhood of each cone point with the numerical chorizo space shown in Figure 1, which consists of a disk and $l$ spheres.

Let $A_j$ be an annular neighborhood of $C_j$. The boundary curves $S_1$ and $S_2$ of $A_j$ have their valencies $(m^{(1)}, \lambda^{(1)}, \sigma^{(1)})$ and $(m^{(2)}, \lambda^{(2)}, \sigma^{(2)})$, when regarded as boundary curves of the periodic part $B$ [Ni1].

Suppose $C_j$ is not amphidrome. Then $m^{(1)} = m^{(2)} = \alpha(C_j)$. Let $m$ be this common value.

**Lemma.** There exists uniquely a sequence of positive integers $n_0, n_1, \ldots, n_l$ ($l \geq 1$) satisfying the following conditions:

(i) $n_0 = \lambda^{(1)}, n_l = \lambda^{(2)}$;
(ii) $n_1 \equiv \sigma^{(1)} \pmod{\lambda^{(1)}}, n_{l-1} \equiv \sigma^{(2)} \pmod{\lambda^{(2)}}$;
(iii) $n_{i-1} + n_{i+1} \equiv 0 \pmod{n_i}, i = 1, 2, \ldots, l - 1$;
(iv) $(n_{i-1} + n_{i+1})/n_i \geq 2, i = 1, 2, \ldots, l - 1$; and
(v) $\sum_{i=0}^{l-1} 1/n_i = \alpha(C_j)$.

Let $\text{Ch}(A_j)$ be the chorizo space shown in Figure 2, which consists of two disks and $l - 1$ spheres, $m_i$ being defined to be $mn_i$. 


We consider the spaces \( \text{Ch}(f^iA_j) \) \((i = 0, 1, \ldots, m - 1)\) identical : \( \text{Ch}(A_j) = \text{Ch}(fA_j) = \cdots = \text{Ch}(f^{m-1}A_j) \).

Finally, suppose \( A_j \) is amphidrome. Then \( S_1 \) and \( S_2 \) have the same valency \((2m, \lambda, \sigma)\), where \( 2m = \alpha(C_j) \). Let \( n_0, n_1, \ldots, n_l \) be a sequence of integers satisfying
\[
n_0 \geq n_1 \geq \cdots \geq n_l = 1, \quad n_0 = \lambda, n_1 = \sigma, \quad n_i - 1 + n_{i+1} \equiv 0 \pmod{n_i}, \quad \text{and} \quad \sum_{i=0}^{l-1} 1/n_i n_{i+1} = (1/2)|s(C_j)|.
\]

Let \( \text{Ch}(A_j) \) be the chorizo space shown in Figure 3, which consists of a disk and \( l + 2 \) spheres. Again we consider the spaces \( \text{Ch}(f^iA_j) \) \((i = 0, 1, \ldots, (m/2) - 1)\) identical.

Now the generalized quotient \( S_f \) is defined to be the union of \( \text{Ch}(B) \) and \( \text{Ch}(A_j) \)'s, \( A_j \) running over all the annuli in \( A \). A natural projection \( \pi : \Sigma_g \to S_g \) can be defined.

Let \( C_\pi \) be the mapping cylinder of \( \pi \). We construct an “open book” \( \overline{M} \) with a “page” \( C_\pi \). (See [Ta, W].) Then \( M = \text{int}(\overline{M}) \) has a complex structure, and we obtain a degenerating family \((M, D, \varphi)\) whose monodromy coincides with \( f \). Blow down \((-1)\)-curves, if any, in \( M \). All of the process is topologically canonical, and we get the inverse map \( \sigma : \mathcal{P}_g \to \mathcal{S}_g \) of \( \rho : \mathcal{S}_g \to \mathcal{P}_g \), proving its bijectivity.

4. Conjugacy invariants

We define the partition graph \( X_f \) associated with a pseudoperiodic homeomorphism of negative twist \( f : \Sigma_g \to \Sigma_g \) as follows: Let \( C \) be a precise system. The vertices (resp. the edges) of \( X_f \) are in one-to-one correspondence to the connected components \( b \) of \( \Sigma_g - C \) (resp. the curves \( \{C_i\} \) in \( C \)). An edge \( e(C_i) \) joins vertices \( v(b) \) and \( v(b') \) if and only if \( C_i \) is in the adherence of \( b \) and also of \( b' \). The refined partition graph \( \overline{X}_f \) is obtained from \( X_f \) by subdividing those edges \( e(C_i) \) that correspond to amphidrome curves by their middle points.

A periodic map \( \psi_f : \overline{X}_f \to \overline{X}_f \) is induced from \( f \). The quotient graph \( Y_f = \overline{X}_f/\psi_f \) is a weighted graph in the sense that each vertex (and each edge) carries a positive integer called the weight, which is the number of the vertices (resp. the edges) of \( \overline{X}_f \) over the vertex (resp. the edge) of \( Y_f \).

The conjugacy class of the periodic action \( \psi_f : \overline{X}_f \to \overline{X}_f \) can be interpreted as a cohomology class \( c_f \) in a suitably defined weighted cohomology group \( H^*_W(Y_f) \). The weighted graph \( Y_f \) also serves as the decomposition diagram of \( S_f \), and there is a natural collapsing map \( \eta_f : S_f \to Y_f \).
Theorem 2. The triple \((S_f, Y_f, c_f)\) determines the conjugacy class of the mapping class \([f]\).

Nielsen’s set of invariants introduced in [Ni2] has exactly the same amount of information as \((S_f, Y_f)\) but lacks \(c_f\). Thus his assertion [Ni2, §15; G, Theorem 13.4] that his invariants are complete is incorrect.

Here is an example. Let \(\Sigma_6\) be the surface shown in Figure 4, which has a system of curves \(C = \{C_1, \ldots, C_5\}\). Let \(f_k : \Sigma_6 \to \Sigma_6 \ (k = 1, 2)\) be a homeomorphism such that \(f_k^*(\Sigma_6 - C) \simeq \text{id}\), \(f_k(b_i) = b_{i+k}\), \(f_k(C_j) = C_{j+k}\), and \(s(C_j) = -1\). (Indices are taken modulo 5.) Although Nielsen’s invariants are the same for \([f_1]\) and \([f_2]\), these mapping classes are not conjugate, because the actions on the partition graphs are not conjugate.

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