Pricing Derivatives with Counterparty Risk and Collateralization: A Fixed Point Approach

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Abstract

This paper studies a valuation framework for financial contracts subject to reference and counterparty default risks with collateralization requirement. We propose a fixed point approach to analyze the mark-to-market contract value with counterparty risk provision, and show that it is a unique bounded and continuous fixed point via contraction mapping. This leads us to develop an accurate iterative numerical scheme for valuation. Specifically, we solve a sequence of linear inhomogeneous PDEs, whose solutions converge to the fixed point price function. We apply our methodology to compute the bid and ask prices for both defaultable equity and fixed-income derivatives, and illustrate the non-trivial effects of counterparty risk, collateralization ratio and liquidation convention on the bid-ask spreads.

Keywords: bilateral counterparty risk, collateralization, credit valuation adjustment, fixed point method, contraction mapping

JEL Classification: G12, G13, G23, C63

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1 Introduction

Counterparty risk has played an important role during the 2008 financial crisis. According to the Bank for International Settlements (BIS\(^1\)) two-thirds of counterparty risk losses during the crisis were from counterparty risk adjustments in MtM valuation whereas the rest were due to actual defaults. In order to account for the counterparty risk, recent regulatory changes, such as Basel III, incorporate the counterparty risk adjustments in the calculation of capital requirement. On the other hand, the use of collateral in the derivative market has increased dramatically. According to the survey conducted by the International Swaps and Derivatives Association (ISDA) in 2013\(^2\), the percentage of all trades subject to collateral agreements in the over-the-counter (OTC) market increases from 30% in 2003 to 73.7% in 2013. OTC market participants continue to adapt collateralization and counterparty risk adjustments in their MtM valuation methodologies for various contracts, including forwards, total return swaps, interest rate swaps and credit default swaps.

When an OTC market participant trades a financial claim with a counterparty, the participant is exposed not only to the price change and default risk of the underlying asset but also to the default risk of the counterparty. To reflect the counterparty default risk in MtM valuation, three adjustments are calculated in addition to the counterparty-risk free value of the claim. While credit valuation adjustment (CVA) accounts for the possibility of the counterparty’s default, debt valuation adjustment (DVA) is calculated to adjust for the participant’s own default risk. In addition, collateral interest payments and the cost of borrowing generate funding valuation adjustment (FVA). In the industry, the valuation adjustment incorporating CVA, DVA, FVA and collateralization, is called total valuation adjustment (XVA)\(^3\).

In this paper, we study a valuation framework for financial contracts accounting for XVA. We consider two current market conventions for price computation. The main difference in the two conventions rises in the assumption of the liquidation value – either counterparty risk-free value or MtM value with counterparty risk provision – upon default. Brigo et al. (2012) and Brigo and Morini (2011) show that the values under the two conventions have significant differences and large impacts on net debtors and creditors.

With counterparty risk provision, the MtM value is defined implicitly via a risk-neutral expectation. This gives rise to major challenges in analyzing and computing the contract value. We propose a novel fixed point approach to analyze the MtM value. Our methodology involves solving a sequence of inhomogeneous linear PDEs, whose classical solutions are shown via contraction mapping mapping

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\(^1\)See BIS press release at http://www.bis.org/press/p110601.pdf
\(^2\)Survey available at http://www2.isda.org/functional-areas/research/surveys/margin-surveys/
\(^3\)The terminology can be found in Carver (2013) and Capponi (2013), among others.
arguments to converge to the unique fixed point price function. This approach also motivates us to
develop an iterative numerical scheme to compute the values of a variety of financial claims under
different market conventions.

In related studies, Fujii and Takahashi (2013) incorporate BCVA and under/over collateralization,
and calculate the MtM value by simulation. Henry-Labordère (2012) approximates the MtM
value by numerically solving a related nonlinear PDE through simulation of a marked branching
diffusion, and provides conditions to avoid a “blow-up” of the simulated solution. Burgard and Kjaer
(2011) also consider a similar nonlinear PDE in their study of hedging strategies inclusive of funding
costs, and taking into account closeout payments exchanged at the time when either party of the
contract defaults. In contrast, our fixed point methodology works directly with the price definition
in terms of a recursive expectation, rather than heuristically stating and solving a nonlinear PDE.
Our contraction mapping result allows us to solve a series of linear PDE problems with bounded
classical solutions, and obtain a unique bounded continuous MtM value.

Our model also provides insight on the bid-ask prices of various financial contracts. The XVA
is asymmetric for the buyer and the seller. As such, the incorporation of adjustment to unilateral
or bilateral counterparty risk leads to a non-zero bid-ask spread. In other words, counterparty risk
reveals itself as a market friction, resulting in a transaction cost for OTC trades. In addition, we
examine the impact of various parameters such as default rate, recovery rate, collateralization ratio
and effective collateral interest rate. We find that a higher counterparty default rate and funding
cost reduce the MtM value, whereas the market participant’s own default rate and collateralization
ratio have positive price effects. For claims with a positive payoff, such as calls and puts, we establish
a number of price dominance relationships. In particular, when collateral rates are low, the bid-ask
prices are dominated by the counterparty risk-free value. Moreover, the bid-ask prices decrease
when we use the MtM value rather than counterparty risk-free value for the liquidation value upon
default.

The recent regulatory changes and post-crisis perception of counterparty risk have motivated
research on the analysis of XVA. Brigo et al. (2014) consider an arbitrage-free valuation framework
with bilateral counterparty risk and collateral with possible rehypothecation. Capponi (2013) studies
the arbitrage-free valuation of counterparty risk, and analyzes the impact of default correlation,
collateral migration frequency and collateral re-hypothecation on the collateralized CVA. Thompson
(2010) analyzes the effect of counterparty risk on insurance contracts and examines the moral hazard
of the insurers. Brigo and Chourdakis (2009) focus on the valuation of CDS with counterparty risk
that is correlated with the reference default. Hull and White (2012) investigate the wrong way risk –
the additional risk generated by the correlation between the portfolio return and counterparty
default risk.

Let us give an outline of the rest of the paper. In Sect. 2, we formulate the MtM valuation
of a generic financial claim with default risk and counterparty default risks under collateralization. In
Sect. 3 we provide a fixed point theorem and a recursive algorithm for valuation. In Sect. 4, we
compute the MtM values of various defaultable equity claims and derive their bid-ask prices. In
Sect. 5 we apply our model to price a number of defaultable fixed-income claims. Sect. 6 concludes
the paper, and the Appendix contains a number of longer proofs.
2 Model Formulation

In the background, we fix a probability space \((\Omega, \mathcal{F}, \mathbb{Q})\), where \(\mathbb{Q}\) is the risk-neutral pricing measure. In our model, there are three defaultable parties: a reference entity, a market participant, and a counterparty dealer. We denote them respectively as parties 0, 1, and 2. The default time \(\tau_i\) of party \(i \in \{0, 1, 2\}\) is modeled by the first jump time of an exogenous doubly stochastic Poisson process. Precisely, we define

\[
\tau_i = \inf \left\{ t \geq 0 : \int_0^t \lambda^{(i)}_u \, du > E_i \right\},
\]

where \(\{E_i\}_{i=0,1,2}\) are unit exponential random variables that are independent of the intensity processes \((\lambda^{(i)}_t)_{t \geq 0}, i \in \{0, 1, 2\}\). Throughout, each intensity process is assumed to be of Markovian form \(\lambda^{(i)}_t \equiv \lambda^{(i)}(t, S_t, X_t)\) for some bounded positive function \(\lambda^{(i)}(t, s, x)\), and is driven by the pre-default stock price \(S\) and the stochastic factor \(X\) satisfying the SDEs

\[
dS_t = (r(t, X_t) + \lambda^{(0)}(t, S_t, X_t)) S_t \, dt + \sigma(t, S_t) S_t \, dW_t, \tag{2.2}
\]

\[
dX_t = b(t, X_t) \, dt + \eta(t, X_t) \, d\tilde{W}_t. \tag{2.3}
\]

Here, \((W_t)_{t \geq 0}\) and \((\tilde{W}_t)_{t \geq 0}\) are standard Brownian motions under \(\mathbb{Q}\) with an instantaneous correlation parameter \(\rho \in (-1, 1)\). The risk-free interest rate is denoted by \(r_t \equiv r(t, X_t)\) for some bounded positive function. At the default time \(\tau_0\), the stock price will jump to value zero and remain worthless afterwards. This “jump-to-default model” for \(S\) is a variation of those by Merton (1976), Carr and Linetsky (2006), and Mendoza-Arriaga and Linetsky (2011).

2.1 Mark-to-Market Value with Counterparty Risk Provision

A defaultable claim is described by the triplet \((g, h, l)\), where \(g(S_T, X_T)\) is the payoff at maturity \(T\), \((h(S_t, X_t))_{0 \leq t \leq T}\) is the dividend process, and \(l(\tau_0, X_{\tau_0})\) is the payoff at the default time \(\tau_0\) of the reference entity. We assume continuous collateralization which is a reasonable proxy for the current market where daily or intraday margin calls are common (see Fujii and Takahashi 2013). For party \(i \in \{1, 2\}\), we denote by \(\delta_i\) the collateral coverage ratio of the claim’s MtM value. We use the range \(0 \leq \delta_i \leq 120\%\) since dealers usually require over-collateralization up to 120% for credit or equity linked notes (see Ramaswamy 2011, Table 1).

We first consider pricing of a defaultable claim without bilateral counterparty risk. We call this value counterparty-risk free (CRF) value. Precisely, the ex-dividend pre-default time \(t\) CRF value of the defaultable claim with \((g, h, l)\) is given by

\[
\Pi(t, s, x) := \mathbb{E}_{t,s,x} \left[ e^{-\int_t^T (r_u + \lambda^{(0)}_u) \, du} g(S_T, X_T) + \int_t^T e^{-\int_t^v (r_u + \lambda^{(0)}_u) \, du} \left( h(S_u, X_u) + \lambda^{(0)}_u l(u, X_u) \right) \, du \right]. \tag{2.4}
\]

The shorthand notation \(\mathbb{E}_{t,s,x}[\cdot] := \mathbb{E}[\cdot | S_t = s, X_t = x]\) denotes the conditional expectation under \(\mathbb{Q}\) given \(S_t = s, X_t = x\).
Incorporating counterparty risk, we let $\tau = \min\{\tau_0, \tau_1, \tau_2\}$, which is the first default time among the three parties with the intensity function $\lambda(t, s, x) = \sum_{k=0}^{2} \lambda^{(k)}(t, s, x)$. The corresponding three default events $\{\tau = \tau_0\}, \{\tau = \tau_1\}$ and $\{\tau = \tau_2\}$ are mutually exclusive. When the reference entity defaults ahead of parties 1 and 2, i.e. $\tau = \tau_0$, the contract is terminated and party 1 receives $l(\tau_0, X_{\tau_0})$ from party 2 at time $\tau_0$. When either the market participant or the counterparty defaults first, i.e. $\tau < \tau_0$, the amount that the remaining party gets depends on unwinding mechanism at the default time. We adopt the market convention where the MtM value with counterparty risk provision, denoted by $P$, is used to compute the value upon the participant’s defaults (see Fujii and Takahashi 2013, Henry-Labordère 2012).

Throughout, we use the notations $x^+ = x \mathbf{1}_{\{x \geq 0\}}$ and $x^- = -x \mathbf{1}_{\{x < 0\}}$. Suppose that party 2 defaults first, i.e. $\tau = \tau_2$. If the MtM value at default is positive ($P_{\tau_2} \geq 0$), then party 1 incurs a loss only if the contract is under-collateralized by party 2 ($\delta_2 < 1$) since the amount $\delta_2 P_{\tau_2}^+$ is secured as a collateral. As a result, with the loss rate $L_2$ (i.e. 1 - recovery rate) for party 2, the total loss of party 1 at $\tau_2$ is $L_2 (1 - \delta_2)^+ P_{\tau_2}^+$. On the other hand, suppose that the MtM value is negative ($P_{\tau_2} < 0$). Party 1 has a loss only if party 1 puts collateral more than the MtM value $P_{\tau_2}$, i.e. the contract is over-collateralized ($\delta_1 \geq 1$). In this case, party 1’s total loss is the product of the party 2’s loss rate and the exposure, i.e. $L_2 (\delta_1 - 1)^+ P_{\tau_2}^-$. Therefore, the remaining value of the party 1’s position at the default time $\tau_2$ is

$$P_{\tau_2} - L_2 (1 - \delta_2)^+ P_{\tau_2}^+ - L_2 (\delta_1 - 1)^+ P_{\tau_2}^-.$$  \hspace{1cm} (2.5)

Next, we consider the case when party 1 defaults first, i.e. $\tau = \tau_1$. We denote by $L_1$ the loss rate of party 1. If the MtM value of party 1’s position at the default is negative ($P_{\tau_1} < 0$) and the contract is under-collateralized ($\delta_1 < 1$), party 2’s loss is $L_1 (1 - \delta_1)^+ P_{\tau_1}^-$. Similarly, when the MtM value is positive ($P_{\tau_1} \geq 0$) and the contract is over-collateralized ($\delta_1 \geq 1$), party 2 incurs a loss of the amount $L_1 (\delta_2 - 1)^+ P_{\tau_1}^+$. Because of the bilateral nature of the contract, party 2’s loss is party 1’s gain. Therefore, at the default time $\tau_1$, the value of party 1’s position is

$$P_{\tau_1} + L_1 (1 - \delta_1)^+ P_{\tau_1}^- + L_1 (\delta_2 - 1)^+ P_{\tau_1}^+. \hspace{1cm} (2.6)$$

Moreover, the market participant is exposed to funding cost associated with collateralization over the period since the collateral rate and funding rate do not coincide with the risk-free rate. When the liquidation value of the contract $P_t$ is positive to party 1 at time $t$, party 2 posts collateral $\delta_2 P_t^+$ to party 1. To keep the collateral, party 1 continuously pays collateral interest at rate $c_2$ to party 2 until any default time or expiry. On the other hand, when $P_t$ is negative to party 1, party 1 borrows $\delta_1 P_t^-$ to post collateral to party 2. As a result, party 1 receives interest payments at rate $c_1$ proportional to collateral amount. We call $c_i$ the effective collateral rate of party $i$ ($i = 1, 2$), which is the nominal collateral rate minus the funding cost rate of party $i$. The rates $c_1$ and $c_2$ can be both negative in practice if the funding costs are high. Therefore, party 1 has the following cash
flow generated by the collateral and effective collateral rates:

\[ 1(t_0 \leq t \leq T) \left( c_1 \delta_1 P^+_t - c_2 \delta_2 P^+_t \right), \quad 0 \leq t \leq T. \quad (2.7) \]

The aforementioned cash flow analysis implies that the pre-default MtM value with counterparty risk (CR) provision is given by

\[
P(t, s, x) = \mathbb{E}_{t,s,x} \left[ e^{-\int_t^T (r_u + \lambda_u) du} g(S_T, X_T) + \int_t^T e^{-\int_t^u (r_v + \lambda_v) dv} \left( h(S_u, X_u) + \lambda_u(0) l(t, X_u) \right) du \right. \\
+ \left. \int_t^T \lambda_u^{(2)} e^{-\int_t^u (r_v + \lambda_v) dv} \left( (1 - L_2 (1 - \delta_2^+) + L_2 (\delta_1 - 1)^+) P^+_u - (1 - L_1 (1 - \delta_1)^+) P^-_u \right) du \right. \\
+ \left. \int_t^T \lambda_u^{(1)} e^{-\int_t^u (r_v + \lambda_v) dv} \left( (1 - L_1 (\delta_2 - 1)^+) P^+_u - (1 - L_1 (1 - \delta_1)^+) P^-_u \right) du \right. \\
+ \left. \int_t^T e^{-\int_t^u (r_v + \lambda_v) dv} \left( c_1 \delta_1 P^-_u - c_2 \delta_2 P^+_u \right) du \right]. \quad (2.8)\]

The first line accounts for the terminal cash flow, the dividend, and the payoff at the reference asset’s default \((\tau = \tau_0)\). The second line and the third line are the cash flows at party 2’s default \((\tau = \tau_2)\) in (2.5) and party 1’s default \((\tau = \tau_1)\) in (2.6), respectively. The last line results from the collateral and effective collateral rates in (2.7). To simplify, we introduce the following notations

\[
\tilde{r}(t, s, x) = r(t, x) + \lambda(t, s, x), \\
\alpha(t, s, x) = L_2 \lambda^{(2)}(t, s, x)(1 - \delta_2^+) - L_1 \lambda^{(1)}(t, s, x)(\delta_2 - 1)^+ + c_2 \delta_2, \\
\beta(t, s, x) = L_1 \lambda^{(1)}(t, s, x)(1 - \delta_1^+) - L_2 \lambda^{(2)}(t, s, x)(\delta_1 - 1)^+ + c_1 \delta_1, \\
f(t, s, x, y) = h(s, x) + \lambda(0)(t, s, x) l(t, x) + (\lambda^{(1)} + \lambda^{(2)} - \beta)(t, s, x)y + (\beta - \alpha)(t, s, x)y^+. \quad (2.12)\]

This allows to express (2.8) in the equivalent but simplified form:

\[
P(t, s, x) = \mathbb{E}_{t,s,x} \left[ e^{-\int_t^T \tilde{r}_u du} g(S_T, X_T) + \int_t^T e^{-\int_t^u \tilde{r}_v dv} f(u, S_u, X_u, P_u) du \right], \quad (2.13)\]

where \(\tilde{r}_t \equiv \tilde{r}(t, S_t, X_t)\) as defined in (2.9).

**Remark 2.1.** As an alternative of MtM value with CR provision, the liquidation value at the time of default can be evaluated as the CRF value of the claim. In other words, at the default time \(\tau < \tau_0\), the liquidation value is evaluated as \(\Pi_\tau\) rather than \(P_\tau\). Replacing \(P_u\) in (2.13) with \(\Pi_u\) for \(t \leq u \leq T\) gives the MtM value without CR provision (see Henry-Labordère (2012)):

\[
\tilde{P}(t, s, x) = \mathbb{E}_{t,s,x} \left[ e^{-\int_t^T \tilde{r}_u du} g(S_T, X_T) + \int_t^T e^{-\int_t^u \tilde{r}_v dv} f(u, S_u, X_u, \Pi_u) du \right]. \quad (2.14)\]

To conclude this section, we summarize the symbols and their financial meanings in Table 1 which we will use frequently throughout this paper.
2.2 Bid-Ask Prices

In OTC trading, market participants, such as dealers, may take a long position as a buyer or a short position as a seller. Without counterparty risk, the buyer’s CRF bid price $\Pi^b(t, s, x)$ for a claim with payoff $(g, h, l)$ is given by (2.4). The MtM value of the seller’s position satisfies (2.4) by replacing $(g, h, l)$ with $(-g, -h, -l)$, the negative of which gives the seller’s CRF ask price $\Pi^s(t, s, x)$. In fact, the bid-ask prices are identical, i.e. $\Pi^b(t, s, x) = \Pi^s(t, s, x)$.

Similarly for the case with counterparty risk provision, the buyer’s bid price is $P^b(t, s, x) = P(t, s, x)$ as in (2.13). The seller’s ask price is given by

$$P^s(t, s, x) = \mathbb{E}_{t, s, x} \left[ e^{-\int_t^T \tilde{r}_u \, du} g(S_T, X_T) + \int_t^T e^{-\int_t^{u} \tilde{r}_v \, dv} \tilde{f}(u, S_u, X_u, P^s_u) \, du \right],$$

(2.15)

where

$$\tilde{f}(t, s, x, y) = h(s, x) + \lambda^{(0)}(t, s, x) l(t, x) + (\lambda^{(1)} + \lambda^{(2)} - \beta)(t, s, x) y - (\beta - \alpha)(t, s, x) y^{-}.$$  

(2.16)

Since $\tilde{f}(t, s, x, y)$ is different from $f(t, s, x, y)$ in (2.12), the symmetry observed in the CRF prices generally no longer holds in the presence of bilateral counterparty risk. Most importantly, such an asymmetry generates bid-ask spreads for defaultable claims. For any contract with counterparty risk provision, the participant can quote two prices: $P^b(t, s, x)$ as a buyer or $P^s(t, s, x)$ as a seller. In addition, since the payoff components $(g, h, l)$ can be negative, the bid and/or ask prices also can be negative (see Figure 4).

The total valuation adjustment (XVA) is defined as a deviation of the MtM value from the CRF value, namely, $\Pi - P^b$ for a long position and $P^s - \Pi$ for a short position. The bid-ask spread accounting for the XVA with CR provision is defined as $S(t, s, x) = P^s(t, s, x) - P^b(t, s, x)$.

The two factors $\alpha$ and $\beta$ in (2.10) and (2.11) that appear in $f$ and $\tilde{f}$ summarize the effects of counterparty risk and collateralization on the bid-ask prices. Specifically, $\alpha$ explains the effect of positive counterparty exposure of the MtM value $P^+_u$ while $\beta$ explains the effect of negative exposure $P^-_u$. When the two parameters have the same value ($\alpha = \beta$), the two functions $f$ and $\tilde{f}$ in (2.12) and (2.16) are identical. Therefore, the bid-ask prices $P^b$ and $P^s$ are equal. Such a price symmetry also arises in a number of other scenarios: (i) when both parties have perfect collateralization ratio ($\delta_1 = \delta_2 = 1$) and the same effective collateral rate ($c_1 = c_2$); (ii) when both parties have zero collateralization ratio ($\delta_1 = \delta_2 = 0$) with the same effective default rate ($L_1 \lambda^{(1)} = L_2 \lambda^{(2)}$), and (iii) when both parties have the same effective collateral rate ($c_1 = c_2$) with the same effective default

| Symbol | Definition | Symbol | Definition for party $i \in \{1, 2\}$ |
|--------|------------|--------|-------------------------------------|
| $P$    | MtM value with CR provision | $R_i$ | Recovery rate |
| $\tilde{P}$ | MtM value without CR provision | $c_i$ | Effective collateral rate |
| $\Pi$  | CRF value | $\delta_i$ | Collateralization ratio |
| $\tau_0$ | Default time of reference asset | $\tau_i$ | Default time |
| $\lambda^{(0)}$ | Default intensity of reference asset | $\lambda^{(i)}$ | Default intensity |

Table 1: Summary of notations.
rate and collateralization ratio \((L_1 \lambda^{(1)} = L_2 \lambda^{(2)}, \delta_1 = \delta_2)\).

**Remark 2.2.** When the counterparty risk-free value \(\Pi\) is used to estimate the liquidation value upon default, the seller’s bid price is given by

\[
\hat{P}^s(t, s, x) = \mathbb{E}_{t, s, x} \left[ e^{-\int_t^T \tilde{r}_u du} g(S_T, X_T) + \int_t^T e^{-\int_t^u \tilde{r}_v dv} \tilde{f}(u, S_u, X_u, \Pi_u) du \right],
\]

(2.17)

where \(\tilde{f}\) is defined in (2.16). In contrast to (2.13), the price function on the LHS does not appear on the RHS.

### 3 Fixed Point Method

The defining equation (2.13) has a recursive form whereby the price function \(P\) appears on both sides. Denote the spacial domain by \(D := \mathbb{R}^+ \times \mathbb{R}\). For any function \(w \in C_b([0, T] \times D, \mathbb{R})\), we define the operator \(M\) by

\[
(Mw)(t, s, x) = \mathbb{E}_{t, s, x} \left[ e^{-\int_t^T \tilde{r}_u du} g(S_T, X_T) + \int_t^T e^{-\int_t^u \tilde{r}_v dv} f(u, S_u, X_u, w(u, S_u, X_u)) du \right].
\]

(3.1)

Then, we recognize from (2.13) that the MtM value with counterparty risk provision satisfies \(P = MP\). This motivates us to show that the operator \(M\) has a unique fixed point, and therefore, guarantees the existence and uniqueness of the MtM value \(P\).

We discuss our fixed point approach by first showing that the operator \(M\) defined in (3.1) preserves boundedness and continuity. To this end, we outline a number of conditions according to Heath and Schweizer (2000).

1. **(C1)** We define

\[
\Gamma(t, s, x) = \begin{bmatrix} r(t, x) + \lambda^{(0)}(t, s, x) \\ \gamma(t, x) \end{bmatrix} \quad \text{and} \quad \Sigma(t, s, x) = \begin{bmatrix} \sigma(t, s) s & 0 \\ \rho \eta(t, x) & \sqrt{1 - \rho^2} \eta(t, x) \end{bmatrix}.
\]

The coefficients \(\Gamma\) and \(\Sigma\) are locally Lipschitz-continuous in \(s\) and \(x\), uniformly in \(t\). That is, for each compact subset \(F\) of \(D\), there is a constant \(K_F < \infty\) such that for \(\psi \in \{\Gamma, \Sigma\},\)

\[
|\psi(t, s_1, x_1) - \psi(t, s_2, x_2)| \leq K_F \| (s_1, x_1) - (s_2, x_2) \| \quad \forall t \in [0, T], (s_1, x_1), (s_2, x_2) \in F,
\]

where \(\| \cdot \|\) is the Euclidean norm in \(\mathbb{R}^2\).

2. **(C2)** For all \((t, s, x) \in [0, T] \times D\), the solution \((S, X)\) neither explodes nor leaves \(D\) before \(T\), i.e.

\[
\mathbb{Q} \left( \sup_{t \leq u \leq T} \| (S_u, X_u) \| < \infty \right) = 1 \quad \text{and} \quad \mathbb{Q} \left( (S_u, X_u) \in D, \forall u \in [t, T] \right) = 1.
\]

3. **(C3)** The functions \(h\) and \(g\) are bounded and continuous, and \(r, l\) and \(\lambda^{(i)}, i \in \{0, 1, 2\}\), are positive, continuous and bounded.
Lemma 3.1. Given any function \( w \in C_b([0,T] \times \mathcal{D}, \mathbb{R}) \), it follows that \( v := \mathcal{M}w \in C_b([0,T] \times \mathcal{D}, \mathbb{R}) \).

Proof. The boundedness of \( v \) follows directly from that of \( w, h, g, r, \) and \( \lambda^{(i)} \) (see condition (C3)). To prove the continuity of \( v \), we first observe that

\[
(t, s, x) \mapsto e^{-\int_t^T \tilde{r}_u \, du} g(S_T, X_T) + \int_t^T e^{-\int_t^u \tilde{r}_v \, dv} f(u, S_u, X_u, w(u, S_u, X_u)) \, du \tag{3.2}
\]

is continuous \( \mathbb{Q} \)-a.s. Indeed, the continuity of \((S, X)\) implies that the mapping \((t, s, x) \mapsto g(S_T, X_T)\) is continuous \( \mathbb{Q} \)-a.s. Also, \((t, s, x, u) \mapsto \tilde{r}(u, S_u, X_u)\) and \((t, s, x, u) \mapsto f(u, S_u, X_u, w(u, S_u, X_u))\) are uniformly continuous and bounded \( \mathbb{Q} \)-a.s. on compact subsets of \([0,T] \times \mathcal{D} \times [t, T]\). Hence, the mapping in (3.2) is continuous \( \mathbb{Q} \)-a.s. Recall that \( v = \mathcal{M}w \) is the expectation of the RHS in (3.2), which is bounded continuous, so \( v \) is also continuous for \((t, s, x) \in [0,T] \times \mathcal{D}\) by Dominated Convergence Theorem.

3.1 Contraction Mapping

Next, we show that the mapping \( \mathcal{M} \) is a contraction. By the boundedness of \( \alpha(t, s, x), \beta(t, s, x) \) and \( \lambda^{(i)}(t, s, x) \) for \( i \in \{0, 1, 2\} \), we can define a finite positive constant by

\[
L = \sup_{(t, s, x) \in [0,T] \times \mathcal{D}} \left\{ |\lambda^{(1)}(t, s, x) + \lambda^{(2)}(t, s, x) - \beta(t, s, x)| + |\beta(t, s, x) - \alpha(t, s, x)| \right\}.
\]

Proposition 3.2. The mapping \( \mathcal{M} \) defined in (3.1) is a contraction on the space \( C_b([0,T] \times \mathcal{D}, \mathbb{R}) \) with respect to the norm

\[
\|w\|_\gamma := \sup_{(t, s, x) \in [0,T] \times \mathcal{D}} e^{-\gamma(T-t)}|w(t, s, x)|, \tag{3.3}
\]

for \( L \leq \gamma < \infty \). In particular, \( \mathcal{M} \) has a unique fixed point \( w^* \in C_b([0,T] \times \mathcal{D}, \mathbb{R}) \).

Proof. From (2.12), we observe that \( |f(t, s, x, y_1) - f(t, s, x, y_2)| \leq L |y_1 - y_2|, \) for \((t, s, x) \in [0,T] \times \mathcal{D}\). This implies \( f \) is Lipschitz-continuous in \( y \), uniformly over \((t, s, x)\). By Lemma 3.1, the operator \( \mathcal{M} \) maps \( C_b([0,T] \times \mathcal{D}, \mathbb{R}) \) into itself. For \((t, s, x) \in [0,T] \times \mathcal{D}, \ w_1, w_2 \in C_b([0,T] \times \mathcal{D}, \mathbb{R}), \) and \( \gamma > 0, \)
we have
\[
e^{-\gamma(T-t)}|\mathcal{M}w_1(t, s, x) - (\mathcal{M}w_2)(t, s, x)|
= e^{-\gamma(T-t)} \mathbb{E}_{t,s,x} \left[ \int_t^T e^{-\gamma u} \tilde{\tau}_v du (f(u, S_u, X_u, w_1(u, S_u, X_u)) - f(u, S_u, X_u, w_2(u, S_u, X_u))) du \right] \\
\leq e^{-\gamma(T-t)} \mathbb{E}_{t,s,x} \left[ \int_t^T e^{-\gamma(T-u)} L |w_1(u, S_u, X_u) - w_2(u, S_u, X_u)| e^{\gamma(T-u)} du \right]
\leq e^{-\gamma(T-t)} L \|w_1 - w_2\|_\gamma \int_t^T e^{\gamma(T-u)} du
\leq \frac{L}{\gamma} \|w_1 - w_2\|_\gamma.
\]

We have used the condition that \(\tilde{\tau}_v \geq 0\) for \((i)\), and the fact that \(f\) is Lipschitz in \(y\) for \((ii)\). The inequality \((iii)\) is implied by the norm in \((3.3)\). As a result, for any \(\gamma > L \geq 0\), \(\mathcal{M}\) is a contraction.

The norm \(\|\cdot\|_\gamma\) is equivalent to the supremum norm \(\|\cdot\|_\infty\) on the space \(C_0([0, T] \times \mathcal{D}, \mathbb{R})\). A similar norm is used in [Becherer and Schweizer, 2005] and [Leung and Sirca, 2009] in their studies of reaction diffusion PDEs arising from indifference pricing.

Using the fact that \(\mathcal{M}\) is a contraction proved in Proposition 3.2, there exists a sequence of functions \((P^{(n)})_{n \geq 0}\) that satisfy \(P^{(n+1)} = \mathcal{M} P^{(n)}\), \(\forall n \geq 0\), and the sequence converges to the fixed point \(P\). The convergence does not rely on the choice of the initial function. Indeed, one can simply pick any bounded continuous function as a starting point, e.g. \(P^{(0)} = 0\ \forall (t, s, x)\), and iterate to have a sequence \((P^{(n)})_{n \geq 0}\) that resides in \(C_0([0, T] \times \mathcal{D}, \mathbb{R})\).

Furthermore, we can show that for each \(n \geq 1\), \(P^{(n)} \equiv P^{(n)}(t, s, x)\) is a classical solution of the following inhomogeneous PDE problem:

\[
\frac{\partial P^{(n)}}{\partial t} + \mathcal{L} P^{(n)} - \tilde{\tau}(t, s, x) P^{(n)} + f(t, s, x, P^{(n-1)}) = 0,
\]

\[
P^{(n)}(T, s, x) = g(s, x), \quad (3.4)
\]

where the operator \(\mathcal{L}\) is defined by

\[
\mathcal{L} := \frac{1}{2} \sigma(t, s)^2 s^2 \frac{\partial^2}{\partial s^2} + \frac{1}{2} \eta(t, x)^2 \frac{\partial^2}{\partial x^2} + \rho \eta(t, x) \sigma(t, s) s \frac{\partial^2}{\partial s \partial x} \\
\quad + \hat{\tau}(t, s, x) s \frac{\partial}{\partial s} + b(t, x) \frac{\partial}{\partial x}. \quad (3.5)
\]

In order to prove the result, we need the following additional conditions, adapted in our notation from \((A3') - (A3d')\) of [Heath and Schweizer, 2000].

(C4) There exists a sequence \((D_n)_{n \in \mathbb{N}}\) of bounded domains with closure \(\bar{D}_n \subset D\) such that \(\bigcup_{n=1}^\infty D_n = D\) and each \(D_n\) has a \(C^2\)-boundary.
As in Heath and Schweizer (2000), one can take $D_n = \left[\frac{1}{n}, n\right] \times \left[-n, n\right] \subset \mathbb{R}_+ \times \mathbb{R}$. For each $n$, we require that

\[(C5) \quad b(t, x), \ a(t, s, x) := \Sigma(t, s, x) \Sigma'(t, s, x), \text{ and } \tilde{r}(t, s, x) \text{ be uniformly Lipschitz-continuous on } [0, T] \times \bar{D}_n, \text{ where } \Sigma' \text{ denotes the transpose matrix of } \Sigma,\]

\[(C6) \quad a(t, s, x) \text{ be uniformly elliptic on } \mathbb{R}^2 \text{ for } (t, s, x) \in [0, T) \times D_n, \text{ i.e. there is } \delta_n > 0 \text{ such that } y^T a(t, s, x) y \geq \delta_n \|y\|^2 \text{ for all } y \in \mathbb{R}^2,\]

\[(C7) \quad f(t, s, x, y) \text{ be uniformly Hölder-continuous on } [0, T] \times \bar{D}_n \times \mathbb{R}.\]

The conditions (C1) – (C7) are quite general, and they allow for various models, including the Heston, CEV, and thus, geometric Brownian motion models for equity, and the Ornstein-Uhlenbeck and Cox-Ingersoll-Ross models for the stochastic factor $X$ (Heath and Schweizer, 2000, Sect. 2).

The triplet $(g, h, l)$, default intensities $\lambda^{(i)}$ and interest rate $r$ can be easily chosen to satisfy the boundedness and continuity conditions in (C3), as we will do in our examples in Sections 4 and 5.

**Theorem 3.3.** Under conditions $\text{(C1)} - \text{(C7)}$, there exists a sequence of bounded classical solutions $(P^{(n)}) \subset C^{1,2}_b([0, T] \times \mathcal{D}, \mathbb{R})$ of the PDE problem (3.4) that converges to the fixed point $P \in C^b([0, T] \times \mathcal{D}, \mathbb{R})$ of the operator $M$.

We provide the proof in Appendix A.1. The insight of Proposition 3.2 and Theorem 3.3 is that we can construct and solve a series of inhomogeneous but linear PDEs whose classical solutions converge to a unique fixed point price function $P$ as in (2.13). Recent studies by Burgard and Kjaer (2011) and Henry-Labordère (2012) evaluate the MtM value $P(t, s, x)$ by working with the associated nonlinear PDE of the form:

$$\frac{\partial P}{\partial t} + LP - \tilde{r}(t, s, x) P + f(t, s, x, P) = 0,$$

for $(t, s, x) \in [0, T) \times \mathcal{D}$, with terminal condition $P(T, s, x) = g(s, x)$, for $(s, x) \in \mathcal{D}$. The nonlinearity of (3.6) poses major challenges on analyzing and numerically solving for $P$. Henry-Labordère (2012) provides a method to approximate the solution that involves replacing the nonlinear term $f$ with a polynomial and simulating a marked branching diffusion. This method, however, does not guarantee that the solution from simulation will resemble the solution of the nonlinear PDE, and does not ensure any regularity, such as continuity or boundedness of, either solution. Henry-Labordère (2012) provides conditions on the chosen polynomial to avoid a “blow-up” of the simulation algorithm. In contrast, our fixed point methodology circumvents this issue by establishing that the pricing definition in (2.13) is a contraction mapping, as opposed to working with the nonlinear PDE. As a result, we solve a series of linear PDE problems with bounded classical solutions. In the limit, a unique bounded continuous MtM value $P$ is obtained.

### 3.2 Numerical Implementation

Our contraction mapping methodology lends itself to a recursive numerical algorithm. As mentioned in the previous section, we iteratively solve a sequence of linear inhomogeneous PDEs (3.4). At
each iteration, the error is measured in terms of the maximum difference between two consecutive solutions \( P^{(n)} \) and \( P^{(n-1)} \) over the entire domain \([0, T] \times \mathcal{D}\). We continue the iteration procedure until the error is less than the pre-defined tolerance level \( \bar{\epsilon} \).

For implementation, we use the standard Crank-Nicolson finite difference method (FDM) to obtain the values (see, among others, Wilmott et al. (1995) and Strikwerda (2007)). We restrict the domain \([0, T] \times \mathcal{D}\) to a finite domain \( \bar{\mathcal{D}} = \{(t, s, x) : 0 \leq t \leq T, X \leq x \leq \bar{X}, 0 \leq s \leq \bar{S}\} \). The parameters \( \bar{S}, X \) and \( \bar{X} \) are sufficiently large enough to preserve the accuracy of the numerical solutions. We discretize the function \( P^{(n)}(t, s, x) \) as \( P^{(n)}(t_i, s_j, x_k) \) where \( i \in \{0, \ldots, N\} \), \( j \in \{0, \ldots, M\} \) and \( k \in \{0, \ldots, L\} \) with \( \Delta t = T/N \), \( \Delta s = \bar{S}/M \), \( \Delta x = (\bar{X} - X)/L \) and \( t_i = i \Delta t \), \( s_j = j \Delta s \), \( x_k = k \Delta x \). Our numerical procedure is summarized in Algorithm 1.

**Algorithm 1** Fixed Point Algorithm for Evaluating the MtM Value

```
set \( n = 1 \), \( P^+ = P^{(0)} \)

solve for \( P^{(1)} \) from PDE \((3.4)\)

set \( \epsilon = \|P^{(1)} - P^{(0)}\|_\infty \)

while \( \epsilon > \bar{\epsilon} \) do
    set \( n = n + 1 \), \( P^+ = P^{(n-1)} \)
    solve for \( P^{(n)} \) from PDE \((3.4)\)
    set \( \epsilon = \|P^{(n)} - P^{(n-1)}\|_\infty \)
end while

return \( P^{(n)} \)
```

For the CRF value, we solve the linear PDE associated with \( \Pi = \Pi(t, s, x) \) in \((2.4)\), namely,

\[
\frac{\partial \Pi}{\partial t} + \mathcal{L} \Pi - \tilde{r}(t, s, x) \Pi(t, s, x) + h(s, x) + \lambda^{(0)}(t, s, x) l(t, x) = 0,
\]

for \((t, s, x) \in [0, T] \times \mathcal{D}\), with terminal condition \( \Pi(T, s, x) = g(s, x) \), for \((s, x) \in \mathcal{D}\). The CRF value becomes an input to the PDE problem for the MtM value without provision, given by

\[
\frac{\partial \hat{P}}{\partial t} + \mathcal{L} \hat{P} - \tilde{r}(t, s, x) \hat{P} + f(t, s, x, \Pi(t, s, x)) = 0,
\]

for \((t, s, x) \in [0, T] \times \mathcal{D}\), and \( \hat{P}(T, s, x) = g(s, x) \), for \((s, x) \in \mathcal{D}\). Again, we apply the Crank-Nicolson FDM method to compute their values.

**Remark 3.4.** The assumption on the boundedness of the default intensities does not encapsulate the local intensity model, where the reference default intensity function is of the form: \( \lambda(t, s) = cs^{-p} \), for some \( p, c > 0 \). See Carr and Linetsky (2006); Carr and Madan (2010); Linetsky (2006); Madan and Unal (1998), among others. As a close alternative, one can cap the exploding intensity and set \( \lambda(t, s) = cs^{-p} \wedge M \) for some arbitrarily chosen large constant \( M \). With this modification, our model and contract mapping results still apply. Since the default intensity function is finite except at \( s = 0 \), in numerical implementation using finite difference, one can set the default intensity at the layer \( s = 0 \) to be the large value \( M \) instead of an infinite value.
4 Defaultable Equity Derivatives with Counterparty Risk

We now apply our valuation methodology to value a number of defaultable equity claims. Specifically, we will derive and compare the MtM values with and without counterparty risk provisions as well as the CRF value. Moreover, we will analyze and illustrate the bid-ask prices.

As a special case of (2.2), we model the pre-default stock price process by

$$dS_t = \left( r + \lambda(0) \right) S_t \, dt + \sigma S_t \, dW_t,$$

(4.1)

where we assume constant interest rate $r$ and default rates $\lambda(i), i \in \{0, 1, 2\}$. In addition, we let

$$\lambda = \sum_{k=0}^{2} \lambda^{(k)},$$

and set

$$\alpha = L_2 \lambda^{(2)} (1 - \delta_2)^+ - L_1 \lambda^{(1)} (\delta_2 - 1)^+ + c_2 \delta_2,$$

(4.2)

$$\beta = L_1 \lambda^{(1)} (1 - \delta_1)^+ - L_2 \lambda^{(2)} (\delta_1 - 1)^+ + c_1 \delta_1.$$  

(4.3)

4.1 Call Spreads

Let us consider a generic call spread with the terminal payoff:

$$g(S_T) = \begin{cases} 
  m_2 & \text{if } S_T > K + \epsilon_2, \\
  \frac{(m_1 + m_2)}{\epsilon_1 + \epsilon_2} (S_T - K) & \text{if } K - \epsilon_1 \leq S_T \leq K + \epsilon_2, \\
  -m_1 & \text{if } S_T < K - \epsilon_1,
\end{cases}$$

(4.4)

with $m_1, m_2, \epsilon_1, \epsilon_2 > 0$, where $m_1/\epsilon_1 = m_2/\epsilon_2 =: M$. The payoff resembles that of a long position of $M$ call options with strike $K - \epsilon_1$, a short position of $M$ call options with strike $K + \epsilon_2$ and short $m_1$ notional of zero coupon bond with the same maturity. Similar positions can be achieved when two OTC traders buy and sell call options with different strikes, plus/minus some cash. As $\epsilon_1$ and $\epsilon_2$ in (4.4) go to zero, the payoff converges to that of a digital call position covered in Henry-Labordère (2012).

With the terminal payoff $g$ in (4.4), dividend $h = 0$, and value at reference default $l(\tau_0) = -m_1 e^{-r(T-\tau_0)}$, the CRF value of the spread contract admits the formula

$$\Pi(t,s) = M \left( C^{BS}(t,s;T,K - \epsilon_1,r + \lambda^{(0)},\sigma) - C^{BS}(t,s;T,K + \epsilon_2,r + \lambda^{(0)},\sigma) \right) - e^{-r(T-t)} m_1,$$

where $C^{BS}(t,s;T,K,r,\sigma)$ is the Black-Scholes call option price at time $t$ with spot price $s$, maturity $T$, strike price $K$, risk-free rate $r$ and volatility $\sigma$. From (2.13), the MtM value with counterparty risk provision is given by

$$P(t,s) = \mathbb{E}_{t,s} \left[ e^{-(r+\lambda)(T-t)} g(S_T) + \int_t^T e^{-(r+\lambda)(u-t)} f(u,S_u,P_u) \, du \right],$$

(4.5)

where $f(t,s,y) := \lambda^{(0)} l(t) + (\lambda^{(1)} + \lambda^{(2)} - \beta) y + (\beta - \alpha) y^+$. The MtM value without counterparty risk provision $\hat{P}(t,s)$ is similarly obtained replacing $P_u$ in (4.5) with $\Pi_u$. 

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The model for $S$ in (4.4), the triple $(g, h, l)$, and other (constant) coefficients satisfy the conditions (C1)-(C7) with domain $D = \mathbb{R}_+$ (see also [Heath and Schweizer 2000, Sect. 2]). We numerically compute the MtM value $P(t, s)$ by Algorithm 1 from Section 3.2. For the iterative PDE (3.4), we adopt the coefficients in this section and the terminal payoff $g(S_T)$ given in (4.4). In Table 2 we show the convergence of the MtM values with provision for three different contracts where $\epsilon_1 = \epsilon_2 = \{2, 1, 0.01\}$. The first column of each contract shows the value of the MtM value of the contract at spot $s = 10$ for each step $0 \leq n \leq 5$. The second column of each contract shows the supremum norm $\epsilon = \|P^{(n)} - P^{(n-1)}\|_\infty$ for each step $0 \leq n \leq 5$. The algorithm stops at $n = 5$ for all three contracts.

| $n$ | $\epsilon_1 = \epsilon_2 = 2$ | $\epsilon_1 = \epsilon_2 = 1$ | $\epsilon_1 = \epsilon_2 = 0.01$ |
|-----|-----------------|-----------------|-----------------|
| 0   | $P^{(0)}(0, 10)$ | $P^{(0)}(0, 10)$ | $P^{(0)}(0, 10)$ |
| 1   | 0.1197           | 0.1293           | 0.1326           |
| 2   | 0.1387           | 0.1490           | 0.1526           |
| 3   | 0.1377           | 0.1479           | 0.1515           |
| 4   | 0.1377           | 0.1480           | 0.1516           |
| 5   | 0.1377 $< 10^{-5}$ | 0.1480 $< 10^{-5}$ | 0.1516 $< 10^{-5}$ |

Table 2: Convergence of the MtM values with provision $P(0, s)$ of call spread contract at spot price $s = 10$ (at-the-money) and $m_1 = m_2 = 1$. Parameters: $K = 10$, $T = 2$, $t = 0$, $r = 2\%$, $\sigma = 25\%$, $\lambda^{(1)} = 3\%$, $\lambda^{(1)} = 5\%$, $\lambda^{(2)} = 15\%$, $R_1 = 40\%$, $R_2 = 40\%$, $\delta_1 = \delta_2 = 0$, $\bar{\epsilon} = 10^{-5}$, $\bar{S} = 40$, $\Delta S = 0.01$, $\Delta t = 1/1000$.

Let us visualize the convergence of the MtM value with CR provision $P^{(n)}(0, s)$ in Figure 1 (left). Using the tolerance level $\bar{\epsilon} = 10^{-5}$ for the maximum difference over each iteration, the algorithm stops after 4 iterations. As we can see, the price functions $P^{(3)}(0, s)$ and $P^{(4)}(0, s)$ over $0 \leq s \leq \bar{S} = 40$ are not visibly distinguishable.

![Figure 1](image1.png)

Figure 1: (Left) Convergence of the MtM values with provision of a call spread $P(0, s)$ for $s \in [10, 18]$. (Right) Comparison of the three MtM values of a call spread $\{\Pi(0, s), \bar{P}(0, s), P(0, s)\}$ over the spot price. Parameters are given in Table 2.

In Figure 1 (right), we plot three different values $\Pi(0, s)$, $\bar{P}(0, s)$ and $P(0, s)$ for $0 \leq s \leq \bar{S}$. As we can see, the ordering of these three values can change completely depending on the spot price. For example, for large spot prices, we observe that the CRF value dominates the other two MtM values, but it is lowest when the spot price is small. Furthermore, the value without provision...
dominates the value with provision for high spot prices, and the opposite holds true for low spot prices.

Next, we look at the sensitivity of the MtM values with respect to the counterparty’s or own default risk, collateralization ratio and effective collateral rate. In Figure 2, the MtM values are decreasing in the counterparty default rate (left) and increasing in the participant’s own default rate (right), as is intuitive. Note that the MtM value with provision moves more rapidly with respect to the counterparty default rate, but the MtM value without provision is more sensitive in the participant’s own default rate.

In Figure 3 (left), an increase in $\delta_2$ reduces counterparty-risk exposure, and therefore, increases the MtM values with and without counterparty risk provision. The rate of increase in contract value slows down when the collateralization ratio exceed 1. In the over-collateralized range $[1, 1.2]$, party 1 is no longer exposed to the counterparty’s default risk. The increase in the contract value (from party 1’s perspective) results from the possibility of collecting the excess collateral upon party 1’s own default.

In practice, if the participant’s funding cost rate is high, the effective collateral rate can be negative (see Burgard and Kjaer (2011)). This implies a net interest payment by the participant for the long position due to collateralization. As the effective collateral rate becomes more negative, the contract values with and without provision decrease as we observe on the right panel of Figure 3.

We illustrate the bid-ask prices $P^b$ and $P^s$ of a call spread in Figure 4. On the left panel where the participant is assumed to be default-free, we observe the dominance of the three prices: $P^s \geq \Pi \geq P^b$. However, in the bilateral counterparty-risk case, the ordering of prices is different in in-the-money (ITM) and out-of-money (OTM) ranges. We see that $\Pi \geq P^s \geq P^b$ in the ITM range, but $P^s \geq P^b \geq \Pi$ in the OTM range.
4.2 Equity Forwards

Equity forward contracts are commonly traded in the OTC market. With stock $S$ as the underlying asset, we consider a forward with maturity $T$. The initial forward price $F_0$ is set so that the contract has zero value at inception. When the underlying stock defaults, the stock price goes to zero, and the buyer has to pay the discounted value $e^{-r(T-\tau_0)}F_0$ at the default time. The contract cash flow is described by the triplet $(g,h,l) = (S_T - F_0,0,-e^{-r(T-\tau_0)}F_0)$. As the underlying stock price fluctuates over time, the MtM value also varies.

The MtM value of a long forward contract $P(t,s)$ with provision (see (4.5)) is computed using Algorithm 1. In Table 3 we show the convergence result of the MtM value at time $t = 1$ when the stock price $S_1 = 20$, with initial forward price $F_0 = 10$. The first column of each case shows the value of the forward contract for each step $n \in \{1,\ldots,6\}$. The second column of each case shows...
the error in terms of the supremum norm \( \| P^{(n)} - P^{(n-1)} \|_{\infty} \) over the whole domain \([0, T] \times \mathcal{D}\), and with tolerance \( \bar{\epsilon} = 10^{-5} \) the algorithm stops at \( n = 6 \) in both cases. The number of iterations may depend on the initial value \( P^{(0)} \), threshold \( \bar{\epsilon} \) and upper bound \( \bar{S} \). As we observe, for sufficiently large upper bounds \( \bar{S} \in \{30, 40\} \), the convergent prices are the same.

Table 3: Convergence of the values of a forward contract when spot price \( S_1 = 20 \) at \( t = 1 \), with maximum stock price \( \bar{S} \in \{30 \) (left column),40 (right column)\}. Other common parameters: \( F_0 = 10, T = 3, r = 2\% \), \( \sigma = 25\% \), \( \lambda^{(0)} = 3\% \), \( \lambda^{(1)} = 5\% \), \( \lambda^{(2)} = 15\% \), \( R_1 = 40\% \), \( R_2 = 40\% \), \( \delta_1 = \delta_2 = 0 \), \( \bar{\epsilon} = 10^{-5} \), \( \Delta S = 0.05 \), \( \Delta t = 1/500 \).

| \( S_1 = 20 \) | \( S = 30 \) | \( S = 40 \) |
|---|---|---|
| \( n = 0 \) | Value | \( \epsilon \) | Value | \( \epsilon \) | Value | \( \epsilon \) |
| 0 | 0 | - | 0 | - |
| 1 | 8.6900 | 17.5592 | 8.6900 | 26.4284 |
| 2 | 7.6124 | 2.1391 | 7.6124 | 3.2034 |
| 3 | 7.6777 | 0.1289 | 7.6777 | 0.1927 |
| 4 | 7.6751 | 0.0052 | 7.6751 | 0.0077 |
| 5 | 7.6752 | 0.0002 | 7.6752 | 0.0002 |
| 6 | 7.6752 | < 10\(^{-5}\) | 7.6752 | < 10\(^{-5}\) |

At time \( t \), when the stock price is \( s \), the CRF value of a long forward is given by

\[
\Pi(t, s) = (s - e^{-r(T-t)}F_0).
\]

(4.6)

In order to compute the MTM value of a long forward contract \( \hat{P}^b \) without counterparty risk provision, we apply (4.6) to (2.14) and obtain

\[
\hat{P}^b(t, s) = \Pi(t, s) + \mathbb{E}_{t,s} \left[ \int_t^T e^{-(r+\lambda)(u-t)} \left( (\beta - \alpha) \Pi^+(u, S_u) - \beta \Pi(u, S_u) \right) du \right]
\]

\[
= \Pi(t, s) + \mathbb{E}_{t,s} \left[ \int_t^T e^{-(r+\lambda)(u-t)} \left( (\beta - \alpha)(S_u - e^{-r(T-u)}F_0) + - \beta(S_u - e^{-r(T-u)}F_0) \right) du \right]
\]

\[
= \Pi(t, s) + \int_t^T e^{-\lambda(u-t)} \left( (\beta - \alpha) \mathbb{E}_{t,s} [e^{-r(u-t)}(S_u - e^{-r(T-u)}F_0)] - \beta(s - e^{-r(T-u)}F_0) \right) du.
\]

To simplify the above equation, we notice that

\[
\mathbb{E}_{t,s} [e^{-r(u-t)}(S_u - e^{-r(T-u)}F_0)] = C^{BS}(t, s; T, e^{-r(T-u)}F_0, r + \lambda^{(0)}, \sigma).
\]

We apply similar arguments to the seller’s MtM value, and summarize as follows.

**Proposition 4.1.** The bid-ask prices without counterparty risk provision, \( \hat{P}^b(t, s) \) and \( \hat{P}^s(t, s) \), of a stock forward without counterparty risk provision are given by

\[
\hat{P}^b(t, s) = \Pi(t, s) + \int_t^T e^{-\lambda(u-t)} \left( (\beta - \alpha) C^{BS}(u, s; T, e^{-r(T-u)}F_0, r + \lambda^{(0)}, \sigma) - \beta(s - e^{-r(T-u)}F_0) \right) du,
\]

\[
\hat{P}^s(t, s) = \Pi(t, s) + \int_t^T e^{-\lambda(u-t)} \left( (\alpha - \beta) C^{BS}(u, s; T, e^{-r(T-u)}F_0, r + \lambda^{(0)}, \sigma) - \alpha(s - e^{-r(T-u)}F_0) \right) du,
\]

with \( \Pi(t, s) \) satisfying (4.6).
The fair forward price makes the MtM value of the contract equal to zero at the start of the contract. The CRF value in (4.6) implies that \( F_0 = e^{rT}S_0 \). However, the fair forward price in presence of bilateral counterparty risk is found implicitly. Precisely, the fair forward price \( F^*_0 \) of a long (resp. short) position makes the MtM value with counterparty risk provision satisfies \( P^b(0, S_0; F^b_0) = 0 \) (resp. \( P^s(0, S_0; F^s_0) = 0 \)).

In Figure 5, we plot the bid-ask prices \( P^b \) and \( P^s \) of the forward contract with counterparty risk provision together with the CRF value. On the left panel, all three values increase as the underlying stock price increases. Similar to the call spread case, the price ordering changes from \( \Pi \geq P^s \geq P^b \) in the ITM range to \( P^s \geq P^b \geq \Pi \) in the OTM range. On the right panel, both MtM values decrease significantly as the counterparty default rate increases. However, the MtM with provision moves more rapidly, similar to Figure 2.

![Graph](http://example.com/graph.png)

Figure 5: (Left) The value of a stock forward at time 0 is increasing in the spot price. The fair forward price is the spot price at which the contract value is zero. (Right) The MtM forward contract values of a stock forward are decreasing in the counterparty’s default rate \( \lambda^{(2)} \), with \( F_0 = 10, t = 1 \) and \( T = 3 \). Parameters: \( r = 2\%, \sigma = 25\%, \lambda^{(0)} = 3\%, \lambda^{(1)} = 5\%, R_1 = 40\%, \lambda^{(2)} = 15\%, \lambda^{(2)} = 40\%, \delta_1 = \delta_2 = 0, \bar{\epsilon} = 10^{-5}, \Delta S = 0.05, \Delta t = 1/500. \)

Remark 4.2 (Total Return Swap). Total return swaps (TRS) are also traded over the counter as an alternative to equity stock forwards. The swap buyer will receive the increase in equity value at swap expiration date \( T, S_T - S_0 \). On the other hand, the buyer continuously pays a premium rate \( p \geq r \) until the expiration date and also pays the decrease in the equity value at the expiration date to the swap seller. The TRS is represented as the triplet \( (S_T - S_0, -p, -S_0 e^{-r(T-T_0)}) \).

4.3 Claims with Positive Payoffs

Some derivatives have positive payoffs, i.e. the triplet \( g, h, l \geq 0 \), including calls, puts and digital options. For both conventions, the nonlinear property of the price function disappears since we can substitute the nonlinear functions \( P^+(t, s) \) and \( \Pi^+(t, s) \) in (3.6) and (3.8) by the linear functions \( P(t, s) \) and \( \Pi(t, s) \). From this observation, we derive the formulae for the bid-ask prices of derivatives with positive payoffs.
Proposition 4.3. For any claim with \( g, h, l \geq 0 \), the bid-ask prices of with counterparty risk provision satisfy

\[
P^b(t, s) = \mathbb{E}_{t,s} \left[ e^{-(r + \alpha + \lambda(0))(T-t)} g(S_T) + \int_t^T e^{-(r + \alpha + \lambda(0))(u-t)} (h(S_u) + \lambda(0) l(u, S_u)) \, du \right], \tag{4.7}
\]

\[
P^s(t, s) = \mathbb{E}_{t,s} \left[ e^{-(r + \beta + \lambda(0))(T-t)} g(S_T) + \int_t^T e^{-(r + \beta + \lambda(0))(u-t)} (h(S_u) + \lambda(0) l(u, S_u)) \, du \right]. \tag{4.8}
\]

Without counterparty risk provision (see Remark 2.1), the bid-ask prices satisfy

\[
\hat{P}^b(t, s) = \Pi(t, s) - \mathbb{E}_{t,s} \left[ \int_t^T \alpha e^{-(r + \lambda)(u-t)} \Pi(u, S_u) \, du \right], \tag{4.9}
\]

\[
\hat{P}^s(t, s) = \Pi(t, s) - \mathbb{E}_{t,s} \left[ \int_t^T \beta e^{-(r + \lambda)(u-t)} \Pi(u, S_u) \, du \right]. \tag{4.10}
\]

We apply these results to call options on a defaultable stock.

Example 4.4. The bid price of a European call option with provision is given by

\[
P^b(t, s) = e^{-\alpha(T-t)} C^{BS}(t, s; T, K, r + \lambda(0), \sigma). \tag{4.11}
\]

Moreover, the bid price of a European call option without provision is given by

\[
\hat{P}^b(t, s) = \left( 1 - \frac{\alpha}{\lambda(1) + \lambda(2)} \right) \left( 1 - e^{-(\lambda(1) + \lambda(2))(T-t)} \right) C^{BS}(t, s; T, K, r + \lambda(0), \sigma). \tag{4.12}
\]

In both (4.11) and (4.12), the bid prices have two components: the CRF value of the call option, i.e. Black-Scholes price, and a multiplier that depends on the parameter \( \alpha \) (see (4.2)). First, suppose that \( \alpha = 0 \). This happens, for example, when the contract is perfectly collateralized and effective collateral rate is zero, i.e. \( \delta_2 = c_2 = 0 \). In this case, both bid prices are identical to the CRF value \( \Pi(t, s) = C^{BS}(t, s; T, K, r + \lambda(0), \sigma) \). The parameter \( \alpha \) is positive when the contract is under-collateralized by party 2, i.e. \( \delta_2 < 1 \). In this case, the call option buyer is exposed to counterparty default risk, and consequently, the bid prices are less than the CRF value \( \Pi(t, s) \). On the other hand, \( \alpha \) becomes negative when the contract is over-collateralized by party 2 and the collateral rate is negligible, i.e. \( \delta_2 > 1 \) and \( c_2 = 0 \). The call buyer receives additional financial benefit from excess collateral since the buyer only returns a fraction \( 1 - L_1 \) of the over-collateralized amount at the buyer’s own default. As a result, the bid prices are greater than the CRF value.

Figure 6 illustrates the two bid prices (with and without provision) for various maturities and spot prices. We observe the dominance of the price without provision over the price with provision. Moreover, the difference of two prices increase as maturity or spot price increases. This wider level of difference is attributed to the fact that the difference of counterparty risk exposure increases as the maturity increases.

Next, we derive a number of price dominance relationships for the bid-ask prices and the CRF value of claims with positive payoffs.
Figure 6: The bid prices of a call option with and without counterparty risk provision are increasing in stock price \(s \in [5, 15]\) (left) and in maturity \(T \in [0, 5]\) (right). Parameters: \(\lambda^{(0)} = 5\%, \lambda^{(2)} = 10\%, S_0 = 10, T = 1, t = 0, r = 2\%, \sigma = 25\%, R_1 = R_2 = 40\%, K = S_0 = 10\) and \(\delta_1 = \delta_2 = 0\).

**Proposition 4.5.** Consider any claim with the triplet \(g, h, l \geq 0\). If \(\alpha, \beta \geq 0\), then the following price dominance relationships hold:

\[
\hat{P}^b(t, s), \hat{P}^s(t, s) \leq \Pi(t, s) \quad \text{and} \quad P^b(t, s), P^s(t, s) \leq \Pi(t, s). \tag{4.13}
\]

Furthermore, if \(\lambda^{(1)} + \lambda^{(2)} \geq \alpha, \beta \geq 0\), then we have

\[
P^b(t, s) \leq \hat{P}^b(t, s) \leq \Pi(t, s) \quad \text{and} \quad P^s(t, s) \leq \hat{P}^s(t, s) \leq \Pi(t, s). \tag{4.14}
\]

We provide a proof in Appendix A.3. In market conditions where the effective collateral rates \(c_1, c_2\) are much lower than the counterparty/own default rates \(\lambda^{(1)}, \lambda^{(2)}\), the conditions for (4.13) and (4.14) are satisfied. In turn, since the buyer’s (resp. seller’s) MtM value with provision assumes \(P^b\) (resp. \(P^s\)) as the liquidation value, which is lower than the liquidation value \(\Pi\) in the case without provision according to (4.13), this implies the price dominance relationships in (4.14), as is presented in Figure 6.

5 Defaultable Fixed-Income Derivatives with Counterparty Risk

We now consider defaultable fixed-income claims under bilateral counterparty risk setting. We model the stochastic factor \(X\) by two affine diffusions: Ornstein-Uhlenbeck (OU) and Cox-Ingersoll-Ross (CIR) processes. Under the OU model, the factor process \(X\) follows

\[
dX_t = \kappa(\theta - X_t)\, dt + \sigma\, d\tilde{W}_t,
\]
with positive constant parameters $\kappa, \theta, \sigma > 0$ that represent the speed of mean reversion, long-term mean, and volatility, respectively. In the CIR model, the factor process $X$ follows

$$dX_t = \kappa(\theta - X_t) dt + \sigma \sqrt{X_t} dW_t,$$

where the positive parameters $\{\kappa, \theta, \sigma\}$ satisfy $\kappa, \theta, \sigma > 0$ and the Feller condition $2\kappa \theta > \sigma^2$ so that $X$ stays positive at all times a.s. Both models for $X$ satisfy the corresponding conditions among (C1)-(C7) (Heath and Schweizer, 2000, Sections 2.1-2.2), and we will consider here swaps whose payoff are bounded continuous.

A generic fixed-income contract is described by the triplet $(g(x), h(x), l(t, x))$, with default times $\{\tau_i\}_{i=0,1,2}$ as in (2.1) and Markovian intensities of the form $\lambda_i^{(j)} = \lambda^{(j)}(t, X_t)$. From (2.13), the MtM value with counterparty risk provision is given by

$$P(t, x) = \mathbb{E}_{t,x} \left[ e^{-\int_t^T (r(u)+\lambda(u, X_u)) du} g(X_T) + \int_t^T e^{-\int_u^t (r(v)+\lambda(v, X_v)) dv} f(u, X_u, P_u) du \right], \quad (5.1)$$

where $f(t, x, y) = h(x) + \lambda^{(0)}(t, x) l(t, x) + (\lambda^{(1)}(t, x) + \lambda^{(2)}(t, x) - \beta(t, x)) y + (\beta(t, x) - \alpha(t, x)) y^+$. The parameters $\alpha(t, x)$ and $\beta(t, x)$ are defined similarly to the definitions (2.10) and (2.11). Noticing that $P(t, x)$ in (5.1) is a special case of $P(t, s, x)$ in (2.13), we use Algorithm 1 to find the MtM value by solving iteratively the following PDE:

$$\frac{\partial P^{(n)}}{\partial t} + \mathcal{L}_X P^{(n)} - (r(t) + \lambda(t, x)) P^{(n)} + f(t, x, P^{(n-1)}) = 0, \quad (5.2)$$

for $(t, x) \in [0, T) \times \mathbb{R}$, with $P^{(n)}(T, x) = g(x)$ for $x \in \mathbb{R}$ ($x \in \mathbb{R}_+$ for the CIR case). The MtM value without counterparty risk provision $\tilde{P}(t, x)$ is similarly obtained by replacing $P(u, X_u)$ with $\Pi(u, X_u)$ in the right hand side of (5.1).

For numerical examples, we assume the constant interest rate $r$ and constant default rates $\lambda^{(1)}$ and $\lambda^{(2)}$. The reference default intensity is modeled by $\lambda^{(0)}_t = (X_t)^+ \wedge \bar{X}$ under the OU or CIR model for $X$. In other words, we impose an upper bound $\bar{X}$ and a lower bound $\underline{X} = 0$ on the reference default intensity, and this allows us to apply our contraction mapping result in Theorem 3.3. For implementation, we apply the FDM and Algorithm 1 in Section 3.2 to (5.2) to obtain the MtM value with counterparty risk provision.

### 5.1 Credit Default Swaps (CDS)

After the 2008 financial crisis, CDS contracts are traded in a new standardized way whereby the protection buyer pays at a fixed premium rate, along with a non-zero (positive/negative) upfront payment at the start of the contract. A long position of a CDS contract written on the default event payoff are bounded continuous.

In Table 4, we show the convergence of the CDS MtM values. The first and second columns for each model show the values of the CDS at $x = 2\%$ and $x = 8\%$, respectively. In addition, the third column shows the error in terms of the supremum norm $\epsilon = \|P^{(n)} - P^{(n-1)}\|_\infty$ over the entire grid for each step $n = 1, 2, \ldots, 7$. The algorithm stops at $n = 5$ under the OU model, and at $n = 7$ under
the CIR model with the common tolerance level $\bar{\epsilon} = 10^{-5}$.

| $n$ | $P^{(n)}(0, 2\%)$ | $P^{(n)}(0, 10\%)$ | $\bar{\epsilon}$ | $P^{(n)}(0, 2\%)$ | $P^{(n)}(0, 10\%)$ | $\bar{\epsilon}$ |
|-----|------------------|-------------------|-----------------|------------------|-------------------|-----------------|
| 0   | 0                | 0                 | -               | 0                | -                 | 0               |
| 1   | -0.1326          | 0.05218           | 0.9048          | -0.1130          | 0.1569            | 0.3950          |
| 2   | -0.1526          | 0.04518           | 0.0992          | -0.1131          | 0.1176            | 0.0911          |
| 3   | -0.1515          | 0.04562           | 0.0060          | -0.1130          | 0.1248            | 0.0159          |
| 4   | -0.1516          | 0.04560           | 0.0002          | -0.1130          | 0.1238            | 0.0022          |
| 5   | -0.1516          | 0.04560           | $< 10^{-5}$     | -0.1130          | 0.1239            | $< 10^{-4}$     |
| 6   | -                | -                 | -               | -0.1130          | 0.1239            | $< 10^{-5}$     |
| 7   | -                | -                 | -               | -0.1130          | 0.1239            | $< 10^{-5}$     |

Table 4: Convergence of MtM values of a CDS in the OU/CIR model with reference default rate $x = 2\%$ and $x = 10\%$. For the OU model: $\theta = 3\%, \sigma = 3.5\%, \kappa = 5\%$. For the CIR model: $\theta = 3\%, \sigma = 5\%, \kappa = 5\%$. Other common Parameters: $T = 5$, $t = 0$, $p = 100bps$, $r = 2\%$, $\psi_0 = 0$, $\psi_1 = 5\%$, $\psi_2 = 25\%$, $w_0 = 1$, $w_1 = w_2 = 0$, $\lambda^{(1)} = 5\%$, $\lambda^{(2)} = 25\%$, $R_1 = R_2 = 40\%$, $\delta_1 = \delta_2 = 0$, $\bar{\epsilon} = 10^{-5}$, $X = 20\%$, $\bar{X} = 0$, $\Delta X = 0.001$, $\Delta t = 1/500$.

Figure 7: The CDS upfront prices under the CIR Model are increasing in the counterparty recovery rate $R_2$ (left) and decreasing in the counterparty default rate $\lambda^{(2)}$ (right). Other parameters are same as in Table 4 along with $x = 8\%$.

In Figure 7, we compare the three MtM values in terms of the counterparty recovery rate $R_2 = 1 - L_2$ and the counterparty default rate $\lambda^{(2)}$. The CRF value is given by (A.9) and is independent of $R_2$ and $\lambda^{(2)}$, so it is flat across the $x$-axes in Figure 7. An increase in counterparty recovery rate or a decrease in counterparty default rate improves both MtM values, whereas the CRF value does not change. In both cases, the MtM value without provision dominates the MtM value with provision.

5.2 Total Return Swaps (TRS)

Total return swaps (TRS) on defaultable bonds are OTC traded, and their MtM values are subject to counterparty risk. A TRS is also referred to as a bond forward (see Schönbucher 2003, Chap. 2.5). Fix a maturity of $T$ years, we consider a TRS on a zero-recovery defaultable bond with maturity $T' \geq T$. The value of the defaultable bond, denoted by $C$, is given in (A.5) for the OU model and in
for the CIR model. Given no default up to the swap maturity $T$, the swap buyer receives the difference between the bond price $C(T, X_T; T')$ and the pre-specified strike $K = C(0, X_0; T')$ from the swap seller. If the bond defaults before time $T$, the swap buyer pays the strike $K$ to the seller. Until the first default time $\tau$ or expiration date $T$, the buyer continues to pay at the risk-free rate plus a spread $r + p$. For the MtM upfront value $P(t, x)$ with counterparty risk provision, we follow (5.1) with the triplet

$$
g(x) = C(T, x; T') - K, \quad h(t, x) = -K(r + p), \quad l(t, x) = -K.
$$

Figure 8: (Left) CDS bid-ask upfront prices under the CIR model. (Right) TRS bid-ask upfront prices under the CIR model. Other common parameters: $x = 8\%$, $T = 3$, $T' = 10$, $t = 0$, $p = 100bps$, $r = 2\%$, $\theta = 3\%$, $\kappa = 5\%$, $\psi_0 = 0$, $\psi_1 = 5\%$, $\psi_2 = 25\%$, $w_0 = 1$, $w_1 = w_2 = 0$, $\lambda^{(1)} = 5\%$, $R_1 = 40\%$, $\lambda^{(2)} = 25\%$, $R_2 = 40\%$, $\delta_1 = \delta_2 = 0$, $\bar{\epsilon} = 10^{-8}$, $\bar{X} = 20\%$, $\bar{X} = 0$, $\Delta X = 0.001$, $\Delta t = 1/500$.

In Figure 8, we obtain the bid-ask upfront prices with provision for a CDS and a TRS, along with the counterparty risk-free upfront values, under the CIR model. Since both CDS and TRS are swaps, the MtM values can be positive or negative, depending on the reference default intensity. As buying a CDS is similar to longing default risk, the CDS upfront value is increasing in the reference default intensity (left). On the other hand, since the TRS buyer is shorting default risk, the TRS upfront value decreases as the reference asset’s default intensity $\lambda^{(0)}$ increases (right).

6 Conclusion

In summary, we have discussed a valuation framework to analytically study and numerically compute financial contracts subject to reference and counterparty default risks with bilateral collateralization. In addition to the underlying price dynamics used in this paper, our fixed point approach and the corresponding iterative numerical algorithm can potentially be adapted to price derivatives with counterparty risk in other models. The challenge lies in efficiently and accurately solving a sequence of inhomogeneous PDE problems. Our model also sheds light on the role played by counterparty risk and collateralization in the formation of bid-ask spreads.

Several interesting and practically important problems remain for future investigation. For
example, what are the price and risk impacts of utilizing multiple counterparties for OTC trading? A recent study by Bo and Capponi (2013) derives the XVA of a CDS portfolio with a large number of entities. Apart from pricing, counterparty risk should also be incorporated into static or dynamic strategies for derivatives trading (see e.g. Jiao and Pham (2011)), as well as mark-to-market timing Leung and Liu (2012). In view of the financial crisis, counterparty risk has also become a key component in the design of clearing houses. Answers to these questions will be useful not only for individual or institutional investors, but also for regulators.

A Appendix

A.1 Proof of Theorem 3.3
For \( w \in C_b^{1,2}([0, T] \times D, \mathbb{R}) \) and the operator \( M \) in (3.1), we define \( v \equiv v(t, s, x) \) by

\[
v := Mw = \mathbb{E}_{t,s,x} \left[ e^{-\int_s^T \tilde{r}_u \, du} g(S_T, X_T) + \int_t^T e^{-\int_t^s \tilde{r}_v \, dv} f(u, S_u, X_u, w(u, S_u, X_u)) \, du \right]. \tag{A.1}
\]

Equation (A.1) admits the same form as (1.2) of Heath and Schweizer (2000). By Theorem 1 of Heath and Schweizer (2000), \( v \) is a classical solution \( (C^{1,2}([0, T] \times D, \mathbb{R})) \) of the PDE:

\[
\frac{\partial v(t, s, x)}{\partial t} + Lv(t, s, x) - \bar{r}(t, s, x) v(t, s, x) + f(t, s, x, w(t, s, x)) = 0, \tag{A.2}
\]

for \( (t, s, x) \in [0, T) \times D \) under certain conditions. To apply their result, we verify the sufficient conditions (A1),(A2),(A3') and (A3d')–(A3e') in their Theorem 1. The conditions (A1), (A2), (A3') and (A3d')–(A3e') are identical to our conditions (C1), (C2), (C4) and (C5) – (C6). Since \( w \in C_b^{1,2}([0, T] \times D, \mathbb{R}) \), it is Lipschitz-continuous on \([0, T] \times D_n\), \( \forall n \). Combined with condition (C7), it implies that the composition \( (t, s, x) \to f(t, s, x, w(t, s, x)) \) is uniformly Hölder-continuous on \([0, T] \times D_n \times \mathbb{R} \), thus satisfying (A3d'). Lastly, the boundedness of \( v \) from Lemma 3.1 corresponds to (A3e'). Therefore, we conclude that \( v \) is a bounded classical solution (i.e. \( C_b^{1,2}([0, T] \times D, \mathbb{R}) \)) of the PDE (A.2) for \( (t, s, x) \in [0, T) \times D \).

Now, let’s select the initial function \( P(0) \in C_b^{1,2}([0, T] \times D, \mathbb{R}) \), e.g. \( P(0) = 0 \). Then, the subsequent functions \( P(n) = MP^{(n-1)} \), \( n = 1, 2, \ldots \), are also \( C_b^{1,2}([0, T] \times D, \mathbb{R}) \) and satisfy the linear inhomogeneous PDE (3.4). By Proposition 3.2 the contraction mapping \( M \) ensures the sequence \( (P(n)) \) to converge to a unique fixed point \( P \in C_b([0, T] \times D, \mathbb{R}) \).

A.2 Proof of Proposition 4.3
First, we denote \( \bar{\lambda} :=\lambda^{(1)} + \lambda^{(2)} \) and \( \bar{h}(t, s) := h(t, s) + \lambda^{(0)} l(t, s) \). Applying the positive payoff to the definition of \( P^b = P \) in (2.13), the MtM value with CR provision is given by

\[
P^b(t, s) = \mathbb{E}_{t,s} \left[ e^{-(r+\lambda)(T-t)} g(S_T) + \int_t^T e^{-(r+\lambda)(u-t)} \bar{h}(u, S_u) \, du + \int_t^T (\bar{\lambda} - \alpha) e^{-(r+\lambda)(u-t)} P^b(u, S_u) \, du \right]. \tag{A.3}
\]

To prove (4.7), we substitute it into the RHS of (A.3) and verify that it indeed reduces to (4.7).
To this end, we get

\[
P^b(t, s) = \mathbb{E}_{t,s} \left[ e^{-(r+\lambda)(T-t)} g(S_T) + \int_t^T e^{-(r+\lambda)(u-t)} \tilde{h}(u, S_u) \, du \\
+ \int_t^T (\tilde{\lambda} - \alpha) e^{-(r+\lambda)(u-t)} \left\{ e^{-(r+\alpha+\lambda(0))(T-u)} g(S_T) + \int_u^T e^{-(r+\alpha+\lambda(0))(v-u)} \tilde{h}(v, S_v) \, dv \right\} \, du \right]
\]

\[
= \mathbb{E}_{t,s} \left[ e^{-(r+\alpha+\lambda(0))(T-t)} g(S_T) + \int_t^T e^{-(r+\lambda)(u-t)} \tilde{h}(u, S_u) \, du + \int_t^T \int_t^v (\tilde{\lambda} - \alpha) e^{-(\tilde{\lambda}-\alpha) (v-u)} e^{-(\alpha-\lambda(0)) u} + \lambda^T \tilde{h}(v, S_v) \, du \, dv \right]
\]

\[
= \mathbb{E}_{t,s} \left[ e^{-(r+\alpha+\lambda(0))(T-t)} g(S_T) + \int_t^T e^{-(r+\lambda)(u-t)} \tilde{h}(u, S_u) \, du + \int_t^T (e^{-(r+\alpha+\lambda(0))(u-t)} - e^{-(r+\lambda)(u-t)}) \tilde{h}(u, S_u) \, du \right]
\]

\[
= \mathbb{E}_{t,s} \left[ e^{-(r+\alpha+\lambda(0))(T-t)} g(S_T) + \int_t^T e^{-(r+\alpha+\lambda(0))(u-t)} \tilde{h}(u, S_u) \, du \right].
\]

Since the last equality resembles (4.7), we conclude. The same steps will yield the proof for expression (4.8).

To verify (4.9), we use the expressions of \( \Pi \) in (2.4) and \( \hat{P}^b = \hat{P} \) in (2.14) to get

\[
\hat{P}^b(t, s) = \mathbb{E}_{t,s} \left[ e^{-(r+\lambda)(T-t)} g(S_T) + \int_t^T e^{-(r+\lambda)(u-t)} \tilde{h}(u, S_u) \, du - \int_t^T \alpha e^{-(r+\lambda)(u-t)} \Pi(u, S_u) \, du \right.

\[
+ \int_t^T \tilde{\lambda} e^{-(r+\lambda)(u-t)} \Pi(u, S_u) \, du \right]
\]

\[
= \mathbb{E}_{t,s} \left[ e^{-(r+\lambda)(T-t)} g(S_T) + \int_t^T e^{-(r+\lambda)(u-t)} \tilde{h}(u, S_u) \, du - \int_t^T \alpha e^{-(r+\lambda)(u-t)} \Pi(u, S_u) \, du \right.

\[
+ \int_t^T \tilde{\lambda} e^{-(r+\lambda)(u-t)} \left\{ \int_u^T e^{-(\alpha+\lambda(0))(v-u)} \tilde{h}(v, S_v) \, dv + e^{-(r+\lambda)(0)}(T-u) g(S_T) \right\} \, du \right]
\]

\[
= \mathbb{E}_{t,s} \left[ e^{-(r+\lambda)(T-t)} g(S_T) - \int_t^T \alpha e^{-(r+\lambda)(u-t)} \Pi(u, S_u) \, du \right.

\[
+ \int_t^T e^{-(r+\lambda)(u-t)} \tilde{h}(u, S_u) \, du + \int_t^T (1 - e^{-\tilde{\lambda}(v-t)}) e^{-(r+\lambda(0))(v-t)} \tilde{h}(v, S_v) \, dv \right]
\]

\[
= \mathbb{E}_{t,s} \left[ e^{-(r+\lambda)(T-t)} g(S_T) - \int_t^T \alpha e^{-(r+\lambda)(u-t)} \Pi(u, S_u) \, du + \int_t^T e^{-(r+\lambda(0))(u-t)} \tilde{h}(u, S_u) \, du \right]
\]

\[
= \Pi(t, s) - \mathbb{E}_{t,s} \left[ \int_t^T \alpha e^{-(r+\lambda)(u-t)} \Pi(u, S_u) \, du \right].
\]

Applying the same steps to the definition of \( \hat{P}^s \) in (2.17), we obtain the equation (4.10).
A.3 Proof of Proposition 4.5

From (4.9) and (4.10) and the condition \( \alpha, \beta \geq 0 \), we obtain the inequalities \( \hat{P}^b(t, s) \leq \Pi(t, s) \) and \( \hat{P}^s(t, s) \leq \Pi(t, s) \). The price expressions (4.7) and (4.8) imply that

\[
P^b(t, s) = \mathbb{E}_{t,s} \left[ e^{-\int_t^T (r + \alpha + \lambda(0)) \, du} g(S_T) + \int_t^T e^{-\int_t^r (r + \alpha + \lambda(0)) \, dv} \tilde{h}(u, S_u) \, du \right]
\]

\[
\leq \mathbb{E}_{t,s} \left[ e^{-\int_t^T (r + \lambda(0)) \, du} g(S_T) + \int_t^T e^{-\int_t^r (r + \lambda(0)) \, dv} \tilde{h}(u, S_u) \, du \right] = \Pi(t, s).
\]

(A.4)

Similar arguments give \( \hat{P}^s(t, s) \leq \Pi(t, s) \). Hence, we conclude (4.13).

Next, applying the definition of \( \hat{P}^b \equiv \hat{P} \) in (2.14) along with the inequality (A.4), we get

\[
\hat{P}^b(t, s) = \mathbb{E}_{t,s} \left[ e^{-(T-t)(r + \lambda)} g(S_T) + \int_t^T e^{-(u-t)(r + \lambda)} \tilde{h}(u, S_u) \, du + \int_t^T (\lambda - \alpha) e^{-(u-t)(r + \lambda)} \Pi(u, S_u) \, du \right]
\]

\[
\geq \mathbb{E}_{t,s} \left[ e^{-(T-t)(r + \lambda)} g(S_T) + \int_t^T e^{-(u-t)(r + \lambda)} \tilde{h}(u, S_u) \, du + \int_t^T (\lambda - \alpha) e^{-(u-t)(r + \lambda)} P^b(u, S_u) \, du \right]
\]

\[
= P^b(t, s).
\]

The last equality follows from the definition of \( P^b \) in (2.13). From the definition of \( \hat{P}^s \) in (2.17) and the inequality \( 0 \leq P^s \leq \Pi \) in (4.13), we obtain

\[
\hat{P}^s(t, s) = \mathbb{E}_{t,s} \left[ e^{-(T-t)(r + \lambda)} g(S_T) + \int_t^T e^{-(u-t)(r + \lambda)} \tilde{h}(u, S_u) \, du + \int_t^T (\lambda - \beta) e^{-(u-t)(r + \lambda)} \Pi(u, S_u) \, du \right]
\]

\[
\geq \mathbb{E}_{t,s} \left[ e^{-(T-t)(r + \lambda)} g(S_T) + \int_t^T e^{-(u-t)(r + \lambda)} \tilde{h}(u, S_u) \, du + \int_t^T (\lambda - \beta) e^{-(u-t)(r + \lambda)} P^s(u, S_u) \, du \right]
\]

\[
= P^s(t, s).
\]

The last equality holds from the definition of \( P^s \) in (2.15). Hence, we conclude (4.14).

A.4 CRF CDS and TRS Prices with OU and CIR Reference Default Intensities

Here we summarize the computation of the CRF values of CDS and TRS. Let us assume that the risk free rate is a time-deterministic function \( r(t) \), and the default intensities \( \lambda(i), i \in \{0, 1, 2\} \), are of the form \( \lambda(i)(t, X_t) = \psi_i(t) + w_i X_t \), where \( X \) follows the OU or CIR model. This specification is more general than the one used in the numerical examples in Sections 5.1, 5.2 and it yields analytic expressions for the pre-default prices (without counterparty risk provision) of the credit default swaps and total return swaps considered therein. These prices are used for comparison analysis (see Figures 7 and 8).

In the OU model, the pre-default zero-coupon bond price with maturity \( T \) and zero recovery meets the formula [Schönbucher 2003 Chap. 7.1.1]

\[
C_1(t, x; T) = e^{-\int_t^T r(u) \, du} e^{-\int_t^T \psi_0(u) \, du} \mathbb{E}_{t,x} \left[ e^{-\int_t^T w_0 X_s \, ds} \right]
\]

\[
= e^{-\int_t^T (r(u) + \psi_0(u)) \, du} e^{A_1(T-t) - B_1(T-t) w_0 x},
\]

(A.5)

where

\[
A_1(u) = \int_0^u \left( \frac{1}{2} \sigma^2 B_1(v)^2 - \kappa \theta B_1(v) \right) \, dv, \quad B_1(u) = \frac{1 - e^{\kappa u}}{\kappa}, \quad 0 \leq u \leq T.
\]
In the CIR model, the bond price is given by (Schönbucher, 2003, Chap. 7.2)

\[ C_2(t, x; T) = e^{-\int_t^T r(u) du} e^{-\int_t^T \psi_0(u) du} \pi_{t,x} \left[ e^{-\int_t^T w_0 X_u du} \right] \]

\[ = e^{-\int_t^T (r(u)+\psi_0(u)) du} A_2(T-t) e^{-B_2(T-t)x}, \tag{A.6} \]

where

\[ A_2(u) = \left[ \frac{2\Xi e^{\frac{\kappa}{2}(\Xi+\kappa)}}{(\Xi+\kappa)(e^{\Xi u}-1)+2\Xi} \right]^{\frac{2\Xi \theta}{\sigma^2}}, \quad B_2(u) = \left[ \frac{2(e^{\Xi u}-1) w_0}{(\Xi+\kappa)(e^{\Xi u}-1)+2\Xi} \right], \tag{A.7} \]

for \( 0 \leq u \leq T \) with constant \( \Xi = \sqrt{\kappa^2 + 2\sigma^2 w_0} \).

In turn, we can summarize the prices for the CDS discussed in Section 5.1. Under the OU model, the buyer pays the pre-default \textit{upfront price} for the CDS:

\[ \Pi(t, x) = \int_t^T C_1(t, x; u) \left( w_0 x e^{-\kappa (u-t)} + (\kappa \theta - \frac{\sigma^2}{\kappa})(u-t) + \frac{\sigma^2}{\kappa}(1-e^{-\kappa (u-t)}) - p \right) du, \tag{A.8} \]

where \( C_1(t, x; u) \) is the pre-default zero coupon bond price with maturity \( T \) and zero recovery given in (A.5). Under the CIR model, the pre-default upfront price of a CDS with maturity \( T \) and premium rate \( p \) is given by

\[ \Pi(t, x) = \int_t^T C_2(t, x; u) \left[ w_0 (\kappa \theta B_2(u-t) + B_2(u-t)x) - p \right] du, \tag{A.9} \]

with \( C_2(t, x; u) \) in (A.6) and \( B_2(u) \) in (A.7). See Chap. 7 of Schönbucher (2003).

As for the total return swap described in Section 5.2, its CRF upfront value at time \( t \leq T \), is given by

\[ \Pi(t, x) = \mathbb{E}_{t,x} \left[ e^{-\int_t^T (r(u)+\lambda^{(0)}(u,X_u)) du} (C(T, X_T; T') - K) \right. \]

\[ - \int_t^T \lambda^{(0)}_u e^{-\int_t^u (r(v)+\lambda^{(0)}(v,X_v)) dv} K du - \int_t^T e^{-\int_t^u (r(v)+\lambda^{(0)}(v,X_v)) dv} K (r(u) + p) du \]

\[ = C(t, x; T') - K \left( 1 + p \int_t^T C(t, x; u) du \right). \tag{A.10} \]

For the CRF prices in Figures 7 and 8, we set \( \psi_0 = 0 \), and assume constant interest rate \( r \). The counterparty default rates \( \lambda^{(1)} \) and \( \lambda^{(2)} \) are also constant, and the reference default rate is set to be \( \lambda^{(0)}(t, X_t) = X_t \). This amounts to setting \( w_1 = w_2 = 0 \), and \( w_0 = 1 \) in (A.5)-(A.10) to obtain the CRF values.

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