Inversion Theorem for Bilinear Hilbert Transform

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Abstract

An approximation result for the bilinear Hilbert transform is proved and used for the inversion of the bilinear Hilbert transform. Also, p-Lebesgue points \((p \geq 1)\) are analyzed.

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1 Introduction

The aim of this paper is to give an inversion theorem for the bilinear Hilbert transform (BHT) defined in appropriate classes of functions and distributions. More precisely, for functions, the product \(f(x)g(x)\) is obtained as the inversion of the BHT at Lebesgue points (Theorem 1.1).

In several papers Lacey and Thiele ([4]-[6]) had studied the continuity of \(H_{\alpha}(f,a)(x) = \lim_{\varepsilon \to 0} \int_{|t| < \varepsilon \leq |t|} f(x-y)g(x+\alpha y)\frac{dy}{y}, \quad \alpha \in \mathbb{R} \setminus \{0, -1\}\)
where \(f \in L^2(\mathbb{R})\) and \(g \in L^\infty(\mathbb{R})\), respectively \(f \in L^{p_1}(R)\) and \(g \in L^{p_2}(\mathbb{R})\), \(1 < p_1, p_2 < \infty\). Their main result is the affirmative answer on the Calderon conjecture, first for \(p_1 = 2, p_2 = \infty\) ([5]), then for \(p_1, p_2 \in (1, \infty)\). Let \(2/3 < p = \frac{p_1 p_2}{p_1 + p_2}\) or \(p_1 = 2, p_2 = \infty\) and \(p = 2\). Then their main result is \(||H_{\alpha}(f,a)||_{L^p} \leq C||f||_{L^{p_1}}||a||_{L^{p_2}}, \quad f \in L^{p_1}, \quad a \in L^{p_2}\), where \(C > 0\) depends on \(\alpha, p_1, p_2\). We refer to [7] and the references therein for further reading on multi-linear operators given by singular multipliers.

The bilinear Hilbert transform \(H_{\alpha} : L^2 \times L^\infty \to L^2\) respectively, \(H_{\alpha} : L^{p_1} \times L^{p_2} \to L^p\), was extended in [1] to \(\mathcal{D}'_{L^2} \times \mathcal{D}_{L^\infty} \to \mathcal{D}'_{L^2}\), respectively, \(\mathcal{D}'_{L^2} \times \mathcal{D}_{L^{p_2}} \to \mathcal{D}'_{L^1}\) (with suitable parameters) as a hypocontinuous, respectively,
continuous mapping. The trilinear Hilbert transform on $\mathcal{D}_{L^p} \times \mathcal{D}_{L^q} \times \mathcal{D}_A \to \mathcal{D}_{L^r}$ is studied in [3]. This analysis is based on [7]. Moreover the BHT of ultradistributions is analyzed in [2].

Recall [9]: Let $f \in L^p_{\text{loc}}, p \geq 1$. Then $x \in \mathbb{R}$ is a $p$-Lebesgue point of $f$, $x \in A^p_{f}$, if
\[
\frac{1}{r} \int_{|t|<r} |f(x-t) - f(x)|^p dt \to 0, \text{as } r \to 0.
\]

If $p = 1$, we will use notation $A^1_f = A_f$.

Our aim is to prove the following theorem.

**Theorem 1.1** Let $f \in L^2(\mathbb{R})$, $g \in L^\infty(\mathbb{R})$, respectively $f \in L^{p_1}(\mathbb{R})$, $g \in L^{p_2}(\mathbb{R})$, $\frac{1}{p_1} + \frac{1}{p_2} \leq 1$. Let $x \in A^2_{f} \cap A^\infty_{g}$, respectively, $x \in A^{p_1}_{f} \cap A^{p_2}_{g}$. Then
\[
f(x)g(x) = i\pi (\lim_{\varepsilon \to 0} H_{\alpha,\varepsilon}(f,g)(x) - H_{\alpha}(f,g)(x)),
\]
where
\[
H_{\alpha,\varepsilon}(f,g)(x) = \int_{\mathbb{R}} f(x-t)g(x+\alpha t) \frac{dt}{t + i\varepsilon}, \quad x \in \mathbb{R}, \quad \varepsilon \in (0,1).
\]

This theorem is stated in [1].

**Outline of the proof.**

Since
\[
\lim_{\varepsilon \to 0} H_{\alpha,\varepsilon}(f,g)(x) - H_{\alpha}(f,g)(x) =
\]
\[
\lim_{\varepsilon \to 0} \left( \int_{-\infty}^{\infty} f(x-t)g(x+\alpha t) \frac{dt}{t^2 + \varepsilon^2} - \int_{0<|t|<\varepsilon} f(x-y)g(x+\alpha y) \frac{dy}{y} \right) \tag{1}
\]
\[
+ \lim_{\varepsilon \to 0} \int_{0<|t|<\varepsilon} f(x-y)g(x+\alpha y) \frac{dy}{y} - H_{\alpha}(f,g)(x)
\]
\[
- i \lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} [f(x-t)g(x+\alpha t) - f(x)g(x)] \frac{\varepsilon dt}{t^2 + \varepsilon^2} \tag{2}
\]
we have to prove that (1) and (2) tend to zero as $\varepsilon \to 0$, if $x \in A^2_{f} \cap A^\infty_{g}$, respectively, $x \in A^{p_1}_{f} \cap A^{p_2}_{g}$.
For this proof we need an appropriate analysis of p-Lebesgue points. Section 2 is devoted to p-Lebesgue points, Section 3 contains a preparation of Theorem 1.25, in [8], Chapter I, in the context of BHT and finally, at the end of Section 3 Theorem 1.1 is proved. This theorem is used in the appendix for the the inversion theorem of BHT in the spaces of distributions and ultradistributions.

2 On p-Lebesgue points

Lemma 1 Let $1 \leq p_2 \leq p_1$ and $x \in A_{f^{p_1}}$. Then $x \in A_{f^{p_2}}$.

Proof: Let $p = \frac{p_1}{p_2}$ and $q = \frac{p_1}{p_1 - p_2}$. Since $\frac{1}{p} + \frac{1}{q} = 1$ we have

$$\frac{1}{r} \int_{|t|<r} |f(x-t) - f(x)|^{p_2} dt \leq \frac{1}{r^{\frac{1}{p} + \frac{1}{q}}} \left( \int_{|t|<r} |f(x-t) - f(x)|^{p_1} dt \right)^{\frac{1}{p}} \cdot \left( \int_{|t|<r} dt \right)^{\frac{1}{q}} = \left( \frac{1}{r} \int_{|t|<r} |f(x-t) - f(x)|^{p_1} dt \right)^{\frac{1}{p}}$$

and this implies the assertion.

Lemma 2 If $f \in L^{p_1}_{loc}$, $g \in L^{p_2}_{loc}$, where $\frac{1}{p_1} + \frac{1}{p_2} = 1$ and $x \in A_{f^{p_1}} \cap A_{g^{p_2}}$, then $x \in A_{fg}$.

Proof: We have

$$K = \frac{1}{r} \int_{|t|<r} |f(x-t)g(x-t) - f(x)g(x)| dt$$

$$\leq \frac{1}{r} \int_{|t|<r} |f(x-t) - f(x)| \cdot |g(x-t)| dt$$

$$+ \frac{1}{r} \int_{|t|<r} |g(x-t) - g(x)| \cdot |f(x)| dt$$

$$= I + J.$$

3
For the first integral $I$ we have:

$$I \leq \frac{1}{r} \left( \int_{|t|<r} |f(x-t) - f(x)|^{p_1} dt \right)^{\frac{1}{p_1}} \left( \int_{|t|<r} |g(x-t) - g(x) + g(x)|^{p_2} dt \right)^{\frac{1}{p_2}}$$

$$\leq \frac{1}{r^{\frac{1}{p_1}} + \frac{1}{p_2}} \left( \int_{|t|<r} |f(x-t) - f(x)|^{p_1} dt \right)^{\frac{1}{p_1}} \left[ \left( \int_{|t|<r} |g(x-t) - g(x)|^{p_2} dt \right)^{\frac{1}{p_2}} + \left( \int_{|t|<r} |g(x)|^{p_2} dt \right)^{\frac{1}{p_2}} \right]$$

$$= \left( \frac{1}{r} \int_{|t|<r} |f(x-t) - f(x)|^{p_1} dt \right)^{\frac{1}{p_1}} \left[ \left( \frac{1}{r} \int_{|t|<r} |g(x-t) - g(x)|^{p_2} dt \right)^{\frac{1}{p_2}} + |g(x)|^{\frac{2}{r^{p_2}}} \right].$$

Because $x \in A^p_f \bigcap A^p_g$, it follows that

$$\frac{1}{r} \int_{|t|<r} |f(x-t) - f(x)|^{p_1} dt \to 0 \text{ as } r \to 0, \quad i = 1, 2.$$

So integral $I \to 0$ as $r \to 0$. Integral $J$ also tends to zero as $r \to 0$, according to Lemma 1 and we obtain that $K$ tends to zero and that $x \in A_{fg}$.

**Lemma 3** Let $f \in L^{p_1}_{loc}$, $g \in L^{p_2}_{loc}$, where $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} < 1$, $x \in A^p_f \bigcap A^p_g$. Then $x \in A^p_{fg}$.

**Proof:** First we note that $fg \in L^{p_3}_{loc}$. We have

$$K = \frac{1}{r} \int_{|t|<r} |f(x-t)g(x-t) - f(x)g(x)|^{p_3} dt$$

$$\leq \frac{1}{r} \int_{|t|<r} |g(x-t) - g(x)|^{p_3} \cdot |f(x-t)|^{p_3} dt$$

$$+ \frac{1}{r} \int_{|t|<r} |f(x-t) - f(x)|^{p_3} \cdot |g(x)|^{p_3} dt$$

$$= I + J.$$
Since \( \frac{p_3}{p_1} + \frac{p_3}{p_2} = 1 \), for the first integral \( I \) we have:

\[
I \leq \frac{1}{r} \left( \int_{|t|<r} |f(x-t)|^{p_1} dt \right)^{\frac{p_3}{p_1}} \cdot \left( \int_{|t|<r} |g(x-t) - g(x)|^{p_2} dt \right)^{\frac{p_3}{p_2}} \\
\leq \frac{1}{r} \left( \int_{|t|<r} |f(x-t) - f(x)|^{p_1} dt + \int_{|t|<r} |f(x)|^{p_1} dt \right)^{\frac{p_3}{p_1}} \cdot \left( \int_{|t|<r} |g(x-t) - g(x)|^{p_2} dt \right)^{\frac{p_3}{p_2}} \\
\leq \frac{1}{r^{\frac{p_3}{p_1}} + \frac{p_3}{p_2}} \left( \int_{|t|<r} |f(x-t) - f(x)|^{p_1} dt \right)^{\frac{p_3}{p_1}} \left( \int_{|t|<r} |g(x-t) - g(x)|^{p_2} dt \right)^{\frac{p_3}{p_2}} \\
+ \frac{1}{r} \left( \int_{|t|<r} |f(x)|^{p_1} dt \right)^{\frac{p_3}{p_1}} \left( \int_{|t|<r} |g(x-t) - g(x)|^{p_2} dt \right)^{\frac{p_3}{p_2}} \\
= \frac{1}{r^{\frac{p_3}{p_1}}} \left( \int_{|t|<r} |f(x-t) - f(x)|^{p_1} dt \right)^{\frac{p_3}{p_1}} \frac{1}{r^{\frac{p_3}{p_2}}} \left( \int_{|t|<r} |g(x-t) - g(x)|^{p_2} dt \right)^{\frac{p_3}{p_2}} \\
+ \frac{1}{r} \left[ |f(x)|^{p_1} \left( \int_{|t|<r} |g(x-t) - g(x)|^{p_2} dt \right)^{\frac{p_3}{p_2}} \right] \\
= \frac{1}{r^{\frac{p_3}{p_1}}} \left( \int_{|t|<r} |f(x-t) - f(x)|^{p_1} dt \right)^{\frac{p_3}{p_1}} \frac{1}{r^{\frac{p_3}{p_2}}} \left( \int_{|t|<r} |g(x-t) - g(x)|^{p_2} dt \right)^{\frac{p_3}{p_2}} \\
+ 2\frac{p_3}{p_1} |f(x)|^{p_1} \frac{1}{r^{\frac{p_3}{p_2}}} \left( \int_{|t|<r} |g(x-t) - g(x)|^{p_2} dt \right)^{\frac{p_3}{p_2}} \\
= \left[ \left( \frac{1}{r} \int_{|t|<r} |f(x-t) - f(x)|^{p_1} dt \right)^{\frac{p_3}{p_1}} + 2\frac{p_3}{p_1} |f(x)|^{p_3} \right] \\
\cdot \left( \frac{1}{r} \int_{|t|<r} |g(x-t) - g(x)|^{p_2} dt \right)^{\frac{p_3}{p_2}} \to 0 \text{ as } r \to 0.
\]

Now we consider

\[
J = \frac{|g(x)|^{p_3}}{r} \int_{|t|<r} |f(x-t) - f(x)|^{p_3} dt.
\]
Since $p_3 \leq p_1$, it follows that $f \in L^p_{\text{loc}}$. By Lemma 1, $J \to 0$ as $r \to 0$ and the Lemma is proved because $K \to 0$ as $r \to 0$.

**Lemma 4** Let $f_i \in L^p_{\text{loc}}, i = 1, \ldots, n$, \( \sum_{i=1}^{n} \frac{1}{p_i} = \frac{1}{q_{n-1}} < 1, F = \prod_{i=1}^{n} f_i \) and $x \in \bigcap_{i=1}^{n} A_{f_i}^{p_i}$. Then $x \in A_{F}^{q_{n-1}}$.

**Proof:** Note that $F \in L^{q_{n-1}}_{\text{loc}}$. Put $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q_1}$. Lemma 3 implies that $x \in A_{f_1 f_2}^{q_1}$. Now we have $x \in A_{f_i f_j}^{q_i}$ for $i \geq 3$ and $\frac{1}{q_{n-2}} + \frac{1}{p_n} = \frac{1}{q_{n-1}}$ and again, using Lemma 3 it follows that $x \in A_{f}^{q_{n-1}}$.

**Lemma 5** Let $f_i \in L^p_{\text{loc}}, i = 1, \ldots, n$, \( \sum_{i=1}^{n} \frac{1}{p_i} = 1, F = \prod_{i=1}^{n} f_i \) and $x \in \bigcap_{i=1}^{n} A_{f_i}^{p_i}$ implies that $x \in A_{F}$.

**Proof:** The proof follows by Lemmas 4 and 2.

### 3 An Approximation Lemma for BHT

Our aim is to prove the following version of Lemma 1.2 Ch.VI in [8].

**Lemma 6** Let $f \in L^p_1(\mathbb{R}), g \in L^p_2(\mathbb{R}), p_1, p_2 \in [1, \infty), \frac{1}{p_1} + \frac{1}{p_2} = 1$. Let $x \in A_f^{p_1} \cap A_g^{p_2}$. Then $x \in A_{fg}$ and

$$
\lim_{\varepsilon \to 0} \left\{ \int_{-\infty}^{\infty} f(x-t)g(x+\alpha t) \frac{t}{t^2 + \varepsilon^2} dt - \int_{0<\varepsilon \leq |t|} f(x-t)g(x+\alpha t) \frac{dt}{t} \right\} = 0.
$$

For the proof of Lemma 6 we need a version of Theorem 1.25, Ch. I in [8]. This is Lemma 7:

**Lemma 7** Let $\varphi \in L^1(\mathbb{R}), \psi(x) = \text{ess.sup} |t| \geq |x| |\varphi(t)|, x \in \mathbb{R}$. Assume that $\psi \in L^1(\mathbb{R})$. Let $\varphi_\varepsilon = \frac{1}{\varepsilon^p}(\frac{x}{\varepsilon})$, $\varepsilon > 0$. Assume $f \in L^{p_1}(\mathbb{R}), g \in L^{p_2}(\mathbb{R}), p_1, p_2 \in [1, \infty), \frac{1}{p_1} + \frac{1}{p_2} = 1$. Then $x \in A_{fg}$ and

$$
\lim_{\varepsilon \to 0} \int_{\mathbb{R}} f(x-t)g(x+\alpha t)\varphi_\varepsilon(t) dt = f(x)g(x) \int_{\mathbb{R}} \varphi(t) dt.
$$
Proof of Lemma 7: First we prove the following assertion:

Let \( \delta > 0 \). Then there exists \( \eta > 0 \) such that

\[
\frac{1}{r} \int_{|t|<r} |f(x-t)g(x+\alpha t) - f(x)g(x)| dt < \delta \text{ if } r \leq \eta.
\] (3)

We have

\[
\frac{1}{r} \int_{|t|<r} |f(x-t)g(x+\alpha t) - f(x)g(x+\alpha t) + f(x)g(x+\alpha t) - f(x)g(x)| dt
\]

\[
\leq \frac{1}{r} \int_{|t|<r} |f(x-t) - f(x)| \cdot |g(x+\alpha t)| dt + \frac{1}{r} \int_{|t|<r} |f(x)| \cdot |g(x+\alpha t) - g(x)| dt = I_1 + I_2.
\]

We estimate \( I_1 \) as follows:

\[
I_1 \leq \frac{1}{r} \left( \int_{|t|<r} |f(x-t) - f(x)| \, dt \right)^{\frac{1}{p_1}} \left( \int_{|t|<r} |g(x+\alpha t) - g(x)| \, dt \right)^{\frac{1}{p_2}}
\]

\[
\leq \left( \frac{1}{r} \int_{|t|<r} |f(x-t) - f(x)| \, dt \right)^{\frac{1}{p_1}} \left( \frac{1}{r} \int_{|t|<r} |g(x+\alpha t) - g(x)| \, dt + \frac{1}{r} \int_{|t|<r} |g(x)| \, dt \right)^{\frac{1}{p_2}}.
\]

Substituting \( \alpha t = u \) we obtain

\[
I_1 \leq \left( \frac{1}{r} \int_{|t|<r} |f(x-t) - f(x)| \, dt \right)^{\frac{1}{p_1}} \left( \frac{\alpha}{\alpha r} \int_{|t|<\alpha r} |g(x-t) - g(x)| \, dt + C \right)^{\frac{1}{p_2}}.
\]

Assumptions on \( f \) and \( g \) imply that the right hand side tends to zero as \( r \to 0 \). Similarly for \( I_2 \) we have the same inequality because

\[
I_2 \leq \frac{1}{r} \left( \int_{|t|<r} |f(x)| \, dt \right)^{\frac{1}{p_1}} \left( \int_{|t|<r} |g(x+\alpha t) - g(x)| \, dt \right)^{\frac{1}{p_2}}
\]

\[
\leq C \left( \frac{\alpha}{\alpha r} \int_{|t|<\alpha r} |g(x-t) - g(x)| \, dt \right)^{\frac{1}{p_2}}.
\]

The last expression tends to zero as \( r \to 0 \). Thus, (3) is proved.
Let $a = \int_{\mathbb{R}} \varphi_\varepsilon(t) dt = \int_{\mathbb{R}} \varphi(t) dt$. We have

$$\left| \int_{\mathbb{R}} f(x - t)g(x + \alpha t)\varphi_\varepsilon(t) dt - af(x)g(x) \right| \leq$$

$$\leq \left| \int_{|t|<\eta} [f(x - t)g(x + \alpha t) - f(x)g(x)]\varphi_\varepsilon(t) dt \right| + \left| \int_{|t|\geq\eta} [f(x - t)g(x + \alpha t) - f(x)g(x)]\varphi_\varepsilon(t) dt \right| = I_1 + I_2.$$

As in [8], p. 14, we will use the properties of the function $\psi$. Function $\psi$ is radial ($\psi(x_1) = \psi(x_2)$ if $|x_1| = |x_2|$). Put $\psi_0(r) = \psi(x)$, $|x| = r$ then $\psi_0$ is a decreasing function of $|r|$. We will use the fact that

$$\lim_{t\to 0} t\psi_0(t) \to 0 \text{ as } t \to 0 \text{ or } t \to \infty. \quad (4)$$

Put

$$d_+(t) = |f(x - t)g(x + \alpha t) - f(x)g(x)|, \quad t > 0,$$

$$d_-(t) = |f(x - t)g(x + \alpha t) - f(x)g(x)|, \quad t < 0.$$

We have

$$d_+ \leq d_{1,+} + d_{2,+}, \quad d_- \leq d_{1,-} + d_{1,-},$$

where

$$d_{1,+} = |f(x-t)g(x-t) - f(x)g(x)|, \quad d_{2,+} = |f(x-t)g(x+\alpha t) - f(x-t)g(x-t)|, \quad t > 0,$$

and $d_{1,-}$ and $d_{2,-}$ are defined in the same way, for $t < 0$.

Put

$$D_{i,+}(t) = \int_{0}^{t} d_{i,+}(s) ds, \quad t > 0, \quad \text{and} \quad D_{i,-}(t) = \int_{0}^{t} d_{i,-}(s) ds, \quad t < 0, \quad i = 1, 2.$$

We will estimate $D_{1,+}(t)$ and in the same way one can estimate $D_{1,-}(t)$. For the estimates of $D_{2,+}(t)$ and $D_{2,-}(t)$ one has to make the decomposition

$$d_{2,+}(t) \leq |f(x-t)||g(x+\alpha t) - g(x)|, \quad t > 0,$$
similarly for $d_{2,-}(t), t < 0$, and to repeat the proof as for $D_{1,+}(t)$, with a simple change of variable $at = -s$ in the integrals related to $d_{2,+}$ and $d_{2,+}$. We have

\[
I_1 \leq \int_0^{\eta} d_{1,+}(t)\varphi_\varepsilon(t)dt + \int_0^{\eta} d_{2,+}(t)\varphi_\varepsilon(t)dt \\
+ \int_{-\eta}^{0} d_{1,-}(t)\varphi_\varepsilon(t)dt + \int_{-\eta}^{0} d_{2,-}(t)\varphi_\varepsilon(t)dt = I_{11} + I_{12} + I_{13} + I_{14}.
\]

We will estimate $I_{11}$; the similar estimate holds for $I_{12}, I_{13}$ and $I_{14}$. For every $\varepsilon < 1$,$ \]
\[
I_{11} = \left| \frac{D_{1,+}(\eta)}{\varepsilon} \right| - \int_0^{\eta} D_{1,+}(t)d\left(\frac{1}{\varepsilon}\psi_0\left(\frac{t}{\varepsilon}\right)\right) dt \\
\leq \left| \frac{D_{1,+}(\eta)}{\eta} \right| \sup_{\varepsilon < 1} \frac{\eta}{\varepsilon}\psi_0\left(\frac{\eta}{\varepsilon}\right) - \int_0^{\eta/\varepsilon} D_{1,+}(\varepsilon t)d\left(\psi_0(\varepsilon t)\right) dt.
\]

Note (cf. (4)),

\[
\frac{D_{1,+}(\eta)}{\eta} \rightarrow 0 \text{ as } \eta \rightarrow 0 \text{ and } \sup_{\varepsilon < 1} \frac{\eta}{\varepsilon}\psi_0\left(\frac{\eta}{\varepsilon}\right) < \infty.
\]

Now, we estimate the second integral. By the assumption,

\[
D_{1,+}(\varepsilon t) \leq \varepsilon\eta, t < \eta.
\]

With this we have:

\[
\left| \int_0^{\eta} D_{1,+}(t)d\left(\frac{1}{\varepsilon}\psi_0\left(\frac{t}{\varepsilon}\right)\right) dt \right| \leq C\eta \int_0^{\infty} |d\psi_0(s)|.
\]

Thus, for every $\delta > 0$, there exists $\eta_0$ such that

\[
I_{11} < \delta, \eta < \eta_0.
\]
The same holds for $I_{12}, I_{13}, I_{14}$. Let us estimate $I_2 = I_{21} + I_{22} + I_{23} + I_{24}$. Actually, we will estimate only $I_{21}$ since the other parts can be estimated in the same way.

$$I_{21} \leq \sup_{\varepsilon < 1, t > \eta} \frac{1}{t} \left| \varepsilon^{-1} \varphi \left( \frac{t}{\varepsilon} \right) \right| \int_{|t| > \eta} \left| f(x-t)g(x-t) - f(x)g(x) \right| dt$$

$$\leq \frac{1}{\eta} \sup_{\varepsilon < 1, t > \eta} \frac{t}{|t|} \left| \varepsilon^{-1} \varphi \left( \frac{t}{\varepsilon} \right) \right| \int_{|t| > \eta} \left| f(x-t)g(x-t) - f(x)g(x) \right| dt.$$ 

Now the assertion follows by (3).

Thus we have that for every $\delta > 0$ there exists $\varepsilon_0$ such that

$$\left| \int \limits_\mathbb{R} f(x-t)g(x+\alpha t)\varphi_\varepsilon(t)dt - af(x)g(x) \right| < C\delta.$$ 

This completes the proof of Lemma 7.

**Proof of Lemma 6:** The proof follows by the use of the same idea as in Lemma 1.2 Ch.VI in [8]. Put

$$\varphi(t) = \begin{cases} \frac{t}{t^2 + 1} - \frac{1}{t}, & |t| \geq 1 \\ \frac{t}{t^2 + 1}, & |t| < 1, \end{cases}$$ 

$\varphi_\varepsilon(t) = \varepsilon^{-1} \varphi \left( \frac{t}{\varepsilon} \right)$, $\varepsilon > 0$, $t \in \mathbb{R}$ and $\psi(x) = \sup_{|t| \geq |x|} |\varphi(t)|$, $x \in \mathbb{R}$. Note $\psi \in L^1(\mathbb{R})$ and

$$\int \limits_\mathbb{R} f(x-t)g(x+\alpha t)\varphi_\varepsilon(t)dt \leq \int \limits_{0<|\varepsilon|<|t|} \frac{f(x-t)g(x+\alpha t)}{t} dt \leq \int \limits_{0<|\varepsilon|<|t|} \frac{f(x-t)g(x+\alpha t)}{t} dt$$

$$= \int \limits_\mathbb{R} f(x-t)g(x+\alpha t)\varphi_\varepsilon(t)dt.$$ 

Now the proof follows by Lemma 7 and the observation (as in [8], p.218) that

$$\lim_{\varepsilon \to 0} \int \limits_\mathbb{R} f(x-t)g(x+\alpha t)\varphi_\varepsilon(t)dt = f(x)g(x) \int \limits_\mathbb{R} \varphi(t)dt = 0,$$

since $\int \varphi(t)dt = 0$.

**Remark.** With the same arguments as for Lemma 6 we have
Let $f \in L^{p_1}$, $g \in L^{p_2}$, where $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} < 1$, $x \in A^p_f \cap A^p_g$. Then it follows that $x \in A^p_{fg}$ and

$$
\lim_{\varepsilon \to 0} \left\{ \int_{\mathbb{R}} f(x-t)g(x+\alpha t) \frac{t}{t^2 + \varepsilon^2} dt - \int_{0<|t|} f(x-t)g(x+\alpha t) \frac{dt}{t} \right\} = 0.
$$

**Proof of Theorem 1.1.** Let us return to the proof of Theorem 1.1. in Section 1.

By Lemma 6, it follows that (1) tends to zero as $\varepsilon \to 0$. Since

$$
\lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} |f(x-t)g(x+\alpha t) - f(x)g(x)| \frac{\varepsilon dt}{t^2 + \varepsilon^2}
$$

$$
= \lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} |f(x-\varepsilon t)g(x+\alpha \varepsilon t) - f(x)g(x)| \frac{dt}{t^2 + 1}
$$

by the Lebesgue theorem, (2) tends to zero as $\varepsilon \to 0$.

This completes the proof of Theorem 1.1.

### 4 Appendix

As a consequence of Theorem 1.1, we recall and reprove parts (iii) and (iv) of Theorem 5 in [1].

**Theorem 4.1** Let $f \in \mathcal{D}'_{L^2}(\mathbb{R})$, $g \in \mathcal{D}_{L^\infty}(\mathbb{R})$, respectively $f \in \mathcal{D}'_{L^q}(\mathbb{R})$, $g \in \mathcal{D}_{L^{p_2}}(\mathbb{R})$, $1 < p_1, p_2$, $p = \frac{p_1 p_2}{p_1 + p_2}$, $q = \frac{p}{p_1 - 1}$ and $q_1 = \frac{p_1}{p_1 - 1}$. Then in the sense of convergence in $\mathcal{D}'_{L^2}$, respectively $\mathcal{D}'_{L^{q_1}}$ holds:

$$
f(x)g(x) = \frac{i}{\pi} \left( \lim_{\varepsilon \to 0} H^*_\alpha,\varepsilon(f,g)(x) - H^*_\alpha(f,g)(x) \right),
$$

where

$$
<H^*_\alpha,\varepsilon(f,g), \psi> = <f, H^*_\alpha,\varepsilon(\psi,g)>, \quad \psi \in \mathcal{D}_{L^2}, \text{respectively} \quad \psi \in \mathcal{D}_{L^{p_1}}
$$

and

$$
<H^*_\alpha(f,g), \psi> = <f, H^*_\alpha(\psi,g)>, \quad \psi \in \mathcal{D}_{L^2}, \text{respectively} \quad \psi \in \mathcal{D}_{L^{p_1}}
$$

are defined in [1], Definition 6 and 1.
**Proof:** We use the simple property
\[
\left( \frac{d}{d\varepsilon} \right)^m H_{\alpha,\varepsilon}(f,g)(x) = \sum_{k=0}^{m} \binom{m}{k} (H_{\alpha,\varepsilon}(f^{(k)},g^{(m-k)})(x))
\]
and
\[
\lim_{\varepsilon \to 0^+} < H_{\alpha,\varepsilon}^*(f,g), \psi > = \lim_{\varepsilon \to 0^+} < f, H_{\alpha,\varepsilon}(g,\psi) >
\]
\[
= < f, H_\alpha(g,\psi) - i\pi g \psi >
\]
\[
= < f, H_\alpha(g,\psi) > - < f, i\pi g \psi >
\]
\[
= < H_\alpha^*(f,g), \psi > - < i\pi f g, \psi >
\]
\[
= < H_\alpha^*(f,g) - i\pi f g, \psi > .
\]
which is the consequence of Lemma 6 for \(g\) and \(\psi\). This implies the above assertions.

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