Non-Elitist Genetic Algorithm as a Local Search Method

Anton V. Eremeev

Omsk Branch of Sobolev Institute of Mathematics SB RAS
13, Pevstov str., 644099, Omsk, Russia
e-mail: eremeev@ofim.oscsbras.ru

Abstract. Sufficient conditions are found under which the iterated non-elitist genetic algorithm with tournament selection first visits a local optimum in polynomially bounded time on average. It is shown that these conditions are satisfied on a class of problems with guaranteed local optima (GLO) if appropriate parameters of the algorithm are chosen.

Key words: genetic algorithm, local search, approximation solution.

Introduction

The genetic algorithm (GA) proposed by J. Holland [10] is a randomized heuristic search method, based on analogy with the genetic mechanisms observed in nature and employing a population of tentative solutions. Different modifications of GA are widely used in the areas of operations research pattern recognition, artificial intelligence etc. (see e.g. [13, 16]). Despite of numerous experimental investigations of these algorithms, their theoretical analysis is still at an early stage [5].

In this paper, the genetic algorithms are studied from the prospective of local search for combinatorial optimization problems, and the NP optimization problems in particular [2]. The major attention is payed to identification of the situations where the GA finds a local optimum in polynomially bounded time on average. Here and below we assume that the randomness is generated only by the randomized operators of selection, crossover and mutation within the GA. In what follows, we call a value polynomially bounded, if there exists a polynomial in the length of the problem input, which bounds the value from above. Throughout the paper we use the terms efficient algorithm or polynomial-time algorithm for an algorithm with polynomially bonded running time. A problem which is solved by such an algorithm is polynomially solvable.

This study is motivated by the fact that the GAs are often considered to be the local search methods (see e.g. [1] [11] [14]). Therefore a topical question is: In what circumstances the GA efficiency is due to its similarity with the local search?

1 Standard definitions and algorithm description

NP Optimization Problems. In what follows, the standard definition of an NP optimization problem is used (see e.g. [3]). By \(\{0,1\}^*\) we denote the set of all strings with symbols from \(\{0,1\}\) and arbitrary string length. For a string \(S \in \{0,1\}^*\), the symbol \(|S|\) will denote its length. In what follows \(\mathbb{N}\) denotes the set of positive integers, and given a string \(S \in \{0,1\}^*\), the symbol \(|S|\) denotes the length of the string \(S\).
Definition 1 An $NP$ optimization problem $\Pi$ is a triple $\Pi = (\text{Inst}, \text{Sol}(I), f_I)$, where $	ext{Inst} \subseteq \{0, 1\}^*$ is the set of instances of $\Pi$ and:

1. The relation $I \in \text{Inst}$ is computable in polynomial time.

2. Given an instance $I \in \text{Inst}$, $\text{Sol}(I) \subseteq \{0, 1\}^{n(I)}$ is the set of feasible solutions of $I$, where $n(I)$ stands for the dimension of the search space $\{0, 1\}^{n(I)}$. Given $I \in \text{Inst}$ and $x \in \{0, 1\}^{n(I)}$, the decision whether $x \in \text{Sol}(I)$ may be done in polynomial time, and $n(I) \leq \text{poly}(|I|)$ for some polynomial poly.

3. Given an instance $I \in \text{Inst}$, $f_I : \text{Sol}(I) \rightarrow \mathbb{N}$ is the objective function (computable in polynomial time) to be maximized if $\Pi$ is an $NP$ maximization problem or to be minimized if $\Pi$ is an $NP$ minimization problem.

Without loss of generality we will consider in our analysis only the maximization problems. The results will hold for the minimization problems as well. The symbol of problem instance $I$ may often be skipped in the notation, when it is clear what instance is meant from the context.

Definition 2 A combinatorial optimization problem is polynomially bounded, if there exists a polynomial in $|I|$, which bounds the objective values $f_I(x)$, $x \in \text{Sol}(I)$ from above.

An algorithm for an $NP$ maximization problem $\Pi = (\text{Inst}, \text{Sol}(I), f_I)$ has a guaranteed approximation ratio $\rho$, $\rho \geq 1$, if for any instance $I \in \text{Inst}$, $\text{Sol}(I) \neq \emptyset$, it delivers a feasible solution $x$, such that $f_I(x) \geq \max\{f_I(x) | x \in \text{Sol}(I)\}/\rho$.

Neighborhoods and local optima. Let a neighborhood $\mathcal{N}_I(y) \subseteq \text{Sol}(I)$ be defined for every $y \in \text{Sol}(I)$. The mapping $\mathcal{N}_I : \text{Sol}(I) \rightarrow 2^{\text{Sol}(I)}$ is called the neighborhood mapping. This mapping is supposed to be efficiently computable (see e.g. [2]).

Definition 3 If the inequality $f_I(y) \leq f_I(x)$ holds for all neighbors $y \in \mathcal{N}_I(x)$ of a solution $x \in \text{Sol}(I)$, then $x$ is called a local optimum w.r.t. the neighborhood mapping $\mathcal{N}_I$.

Suppose $D(\cdot, \cdot)$ is a metric on $\text{Sol}(I)$. The neighborhood mapping

$$\mathcal{N}_I(x) = \{y : D(x, y) \leq R\}, \quad x \in \text{Sol}(I),$$

is called a neighborhood mapping of radius $R$ defined by metric $D(\cdot, \cdot)$.

A local search method starts from some feasible solution $y_0$. Each iteration of the algorithm consists in moving from the current solution to a new solution in its neighborhood, such that the value of objective function is increased. The way to choose an improving neighbor, if there are several of them, will not matter in this paper. The algorithm continues until it will reach a local optimum.

Genetic Algorithms. The simple GA proposed in [10] has been intensively studied and exploited over four decades. A plenty of variants of GA have been developed since publication of the simple GA, sharing the basic ideas, but using different population management strategies, selection, crossover and mutation operators [14].
The GA operates with populations $X^t = (x^1, \ldots, x^\lambda)$, $t = 1, 2, \ldots$, which consist of $\lambda$ genotypes. In terms of the present paper the genotypes are strings from $B = \{0, 1\}^n$. For convenience we assume that the number of genotypes $\lambda$ is even.

In a selection operator $\text{Sel} : B^\lambda \rightarrow \{1, \ldots, \lambda\}$, each parent is independently drawn from the previous population $X^t$ where each individual in $X^t$ is assigned a selection probability depending on its fitness $\Phi(x)$. Below we assume the following natural form of the fitness function:

- if $x \in \text{Sol}$ then $\Phi(x) = f(x)$;
- if $x \not\in \text{Sol}$ then its fitness is defined by some penalty function, such that $\Phi(x) < \min_{y \in \text{Sol}} f(y)$.

In this paper we consider the tournament selection operator [9]: draw $k$ individuals uniformly at random from $X^t$ (with replacement) and choose the best of them as a parent.

A pair of offspring genotypes is created using the randomized operators of crossover $\text{Cross} : B \times B \rightarrow B \times B$ and mutation $\text{Mut} : B \rightarrow B$. In general, we assume that operators $\text{Cross}(x, y)$ and $\text{Mut}(x)$ are efficiently computable randomized routines. We also assume that there exists a positive constant $\varepsilon$ which does not depend on $I$, such that the fitness of at least one of the genotypes resulting from crossover $(x', y') = \text{Cross}(x, y)$ is not less than the fitness of the parents $x, y \in B$ with probability at least $\varepsilon$, i.e.

$$P\{\max\{\Phi(x'), \Phi(y')\} \geq \max\{\Phi(x), \Phi(y)\}\} \geq \varepsilon \quad (1)$$

for any $x, y \in B$.

When a population $X^{t+1}$ of $\lambda$ offspring is constructed, the GA proceeds to the next iteration $t + 1$. An initial population $X^0$ is generated randomly. One of the ways of initialization consists in independent choice of all bits in genotypes.

To simplify the notation below, $GA$ will always denote the non-elitist genetic algorithm with the following outline.

**Algorithm $GA$**

Generate the initial population $X^0$, assign $t := 0$.

**While** termination condition is not met **do**:

**Iteration** $t + 1$:

**For** $j$ from 1 to $\lambda/2$ **do**:

- Tournament selection: $x := x^{\text{Sel}(x^t),t}$, $y := x^{\text{Sel}(x^t),t}$.
- Mutation: $x' := \text{Mut}(x)$, $y' := \text{Mut}(y)$.
- Crossover: $(x^{2j-1,t+1}, x^{2j,t+1}) := \text{Cross}(x', y')$.

**End for**.

$t := t + 1$.

**End while.**

The population size $\lambda$ and tournament size $k$, in general may depend on problem instance $I$. The termination condition may be required to stop a genetic algorithm when a solution of sufficient quality is obtained or the computing time is limited, or because the population is
"trapped" in some unpromising area and it is preferable to restart the search. In theoretical analysis of the \(\mathcal{G}A\) it is often assumed that the termination condition is never met. In order to incorporate the possibility of restarting the search, we will also consider the iterated \(\mathcal{G}A\), which has the following outline.

Algorithm iterated \(\mathcal{G}A\)

Repeat:
  Generate the initial population \(X^0\), assign \(t := 0\).
  While termination condition \(t > t_{\text{max}}\) is not met do:
    Iteration \(t + 1\):
      For \(j\) from 1 to \(\lambda/2\) do:
        Tournament selection: \(x := x^{\text{Sel}(X^t),t}\), \(y := x^{\text{Sel}(X^t),t}\).
        Mutation: \(x' := \text{Mut}(x)\), \(y' := \text{Mut}(y)\).
        Crossover: \((x^{2j-1,t+1}, x^{2j,t+1}) := \text{Cross}(x', y')\).
      End for.
    \(t := t + 1\).
  End while.
Until false.

Examples of mutation and crossover operators. As examples of crossover and mutation we can consider the well-known operators of bitwise mutation \(\text{Mut}^*\) and single-point crossover \(\text{Cross}^*\) from the simple \(\mathcal{G}A\) [10].

The crossover operator computes \((x', y') = \text{Cross}^*(x, y)\), given \(x = (x_1, ..., x_n)\), \(y = (y_1, ..., y_n)\), such that with probability \(P_c\),

\[
x' = (x_1, ..., x_\chi, y_{\chi+1}, ..., y_n), \quad y' = (y_1, ..., y_\chi, x_{\chi+1}, ..., x_n),
\]

where the random number \(\chi\) is chosen uniformly from 1 to \(n-1\). With probability \(1 - P_c\) both parent individuals are copied without any changes, i.e. \(x' = x\), \(y' = y\).

Condition (I) is fulfilled for the single-point crossover with \(\varepsilon = 1 - P_c\), if \(P_c < 1\) is a constant. Condition (I) would also be satisfied with \(\varepsilon = 1\), if an optimized crossover operator was used (see e.g., [4, 7]).

The bitwise mutation operator \(\text{Mut}^*\) computes a genotype \(x' = \text{Mut}^*(x)\), where independently of other bits, each bit \(x'_i\), \(i = 1, ..., n\), is assigned a value \(1 - x_i\) with probability \(P_m\) and with probability \(1 - P_m\) it keeps the value \(x_i\).

2 Expected Hitting Time of a Local Optimum

Suppose an NP maximization problem \(\Pi = (\text{Inst}, \text{Sol}(I), f_I)\) is given and a neighborhood mapping \(\mathcal{N}_I\) is defined. Let \(m\) denote the number of all non-optimal values of objective function \(f\), i.e. \(m = |\{f \mid f = f(x), \ x \in \text{Sol}\}| - 1\). Then starting from any feasible solution the local search method finds a local optimum within at most \(m\) steps. Let us compare this process to the computation of a \(\mathcal{G}A\).
Let $s$ be a lower bound on the probability that the mutation operator transforms a given solution $x$ into a specific neighbor $y \in \mathcal{N}(x)$, i.e.

$$s \leq \min_{x \in \text{Sol}, \ y \in \mathcal{N}(x)} \mathbb{P}\{\text{Mut}(x) = y\}.$$  

The greater the value $s$, the more consistent is the mutation with the neighborhood mapping $\mathcal{N}$.

Let the size of population $\lambda$, the tournament size $k$ and the bound $s$ be considered as functions of the input data $I$. The symbol $e$ denotes the base of the natural logarithm.

**Lemma 1** If $X^0$ contains a feasible solution, $k \geq r\lambda$, $r > 0$, $m > 1$, $s > 0$ and

$$\lambda \geq \frac{2(1 + \ln m)}{s\varepsilon(1 - 1/e^{2r})},$$  

then the GA visits a local optimum until iteration $m$ with probability at least $1/e$.

**Proof.** Note that in the initial population, the individual of greatest fitness is a feasible solution. Let an event $E_{j+1}^{t+1}$, $j = 1, \ldots, \lambda/2$, consist in fulfilment of the following three conditions:

1. An individual $x^t_*$ of greatest fitness in population $X^t$ is selected at least once when the $j$-th pair of offspring is computed;

2. Mutation operator applied to $x^t_*$ performs the best improving move within the neighborhood $\mathcal{N}(x^t_*)$, i.e. $\Phi(\text{Mut}(x^t_*)) = \max_{y \in \mathcal{N}(x^t_*)} \Phi(y)$.

3. When the crossover operator is applied for computing the $j$-th pair of offsprings, at least one of its outputs has the fitness not less than $\max_{y \in \mathcal{N}(x^t_*)} \Phi(y)$;

Let $p$ denote the probability of union of the events $E_{j+1}^{t+1}$, $j = 1, \ldots, \lambda/2$. In what follows we construct a lower bound $\ell \leq p$, which holds for any population $X^t$ containing a feasible solution. According to the outline of the GA, $\mathbb{P}\{E_1^{t+1}\} = \ldots = \mathbb{P}\{E_{\lambda/2}^{t+1}\}$. Let us denote this probability by $q$. Given a population $X^t$, the events $E_{j+1}^{t+1}$, $j = 1, \ldots, \lambda/2$, are independent, so $p \geq 1 - (1 - q)^{\lambda/2} \geq 1 - e^{-q^{\lambda/2}}$. Now $q$ may be bounded from below:

$$q \geq s\varepsilon \left(1 - \left(1 - \frac{1}{\lambda}\right)^{2k}\right).$$  

Note that $(1 - 1/\lambda)^{2k} \leq (1 - 1/\lambda)^{2r\lambda} \leq 1/e^{2r}$. Therefore

$$q \geq s\varepsilon \left(1 - \frac{1}{e^{2r}}\right) = sc,$$  

where $c = \varepsilon \left(1 - \frac{1}{e^{2r}}\right)$. In what follows we shall use the fact that conditions [2] and [3] imply

$$\lambda \geq \frac{2}{s\varepsilon (1 - 1/e^{2r})} \geq 2/q.$$  

(4)

To bound probability $p$ from below, we first note that for any $z \in [0, 1]$ holds

$$1 - \frac{z}{e} \geq e^{-z}.$$  

(5)
Assume $z = e^{-q\lambda/2 + 1}$. Then in view of inequality (4), $z \leq 1$, and consequently,

$$p \geq \exp\left\{ -e^{1-q\lambda/2} \right\} \geq \exp\left\{ -e^{1-sc\lambda/2} \right\}. \quad (6)$$

Now the right-hand side expression from (6) may be used as a lower bound $\ell = \exp\left\{ -e^{1-sc\lambda/2} \right\}$.

Let us now consider a sequence of populations $X_0, X_1, \ldots$. Note that $\ell_m$ is a lower bound for the probability to reach a local optimum in a series of at most $m$ iterations, where that at each iteration the best found solution is improved, until a local optimum is found. Indeed, suppose $A_t = E_1^t + \ldots + E_{\lambda/2}^t$, $t = 1, 2, \ldots$. Then

$$P\{A_1 \& \ldots \& A_m\} = P\{A_1\} \prod_{t=1}^{m-1} P\{A_{t+1}|A_1 \& \ldots \& A_t\} \geq \ell_m. \quad (7)$$

In view of condition (2), we find a lower bound for the probability to reach a local optimum in a sequence of at most $m$ iterations where the best found solution is improved in each iteration:

$$\ell_m = \exp\left\{ -me^{1-sc\lambda/2} \right\} \geq \exp\left\{ -me^{1-\ln m} \right\} = 1/e. \quad \square$$

Many well-known NP optimization problems, such as the Maximum Satisfiability Problem and the Maximum Cut Problem have a set of feasible solutions equal to the whole search space $\{0, 1\}^n(I)$. The following proposition applies to the problems with such property.

**Proposition 1** If $\text{Sol}(I) = \{0, 1\}^n(I)$ for all $I \in \text{Inst}$ and the conditions of Lemma 1 hold, then a local optimum is reached in at most $em$ iterations of the $GA$ on average.

**Proof.** Consider a sequence of series of the $GA$ iterations, where the length of each series is $m$ iterations. Suppose, $D_i$, $i = 1, 2, \ldots$, denotes an event of absence of local optima in the population throughout the $i$-th series. The probability of each event $D_i$, $i = 1, 2, \ldots$, is at most $\mu = 1 - 1/e$ according to Lemma 1. Analogously to the bound (7) we obtain the inequality $P\{D_1 \& \ldots \& D_i\} \leq \mu^i$.

Let $\eta$ denote the random variable, equal to the number of the first run where a local optimum was obtained. By the properties of expectation (see e.g. [8]),

$$E[\eta] = \sum_{i=0}^{\infty} P\{\eta > i\} = 1 + \sum_{i=1}^{\infty} P\{D_1 \& \ldots \& D_i\} \leq 1 + \sum_{i=1}^{\infty} \mu^i = e.$$

Consequently, the average number of iterations until a local optimum is first obtained is at most $em$. $\square$

Suppose the termination condition in the iterated $GA$ is $t > t_{\text{max}} = m$. Then execution of this algorithm may be viewed as a sequence of independent runs of the $GA$, where the length of each run is $m$ iterations.

Let $\lceil \cdot \rceil$ denote rounding up. In conditions of Lemma 1, given the parameters

$$\lambda = 2 \left[ \frac{1 + \ln m}{sc(1 - 1/e^{2r})} \right], \quad k = \lceil r\lambda \rceil, \quad (8)$$

$$\frac{1 + \ln m}{sc(1 - 1/e^{2r})}$$

...
the probability that GA finds a local optimum during the first \( m \) iterations is \( \frac{1}{e} \). So the total number of populations computed in the iterated GA until it first visits a local optimum is at most \( em \).

The operators Mut and Cross are supposed to be efficiently computable and the tournament selection requires \( O(\lambda) \) time. Therefore the time complexity of computing a pair of offspring in the GA is polynomially bounded and the following theorem holds.

**Theorem 1** If problem \( \Pi = (\text{Inst}, \text{Sol}(I), f_I) \) and the function \( s^{-1}(I) \) are polynomially bounded and population \( X^0 \) contains a feasible solution at every run, then the iterated GA with suitable choice of parameters first visits a local optimum on average in polynomially bounded time.

Note that a slight modification of the proof of Theorem 4 from [12] yields the result of the above theorem in the case when the crossover operator is not used. The proof in [12], however, is based on a more complex method of drift analysis.

Often the neighborhood mappings for NP optimization problems are polynomially bounded, i.e. the cardinality \( |N(x)| \), \( x \in \text{Sol} \) is a polynomially bounded value [15]. In such cases there exists a mutation operator \( \text{Mut}(x) \) that generates a uniform distribution over the set \( N(x) \), and the condition on function \( s^{-1}(I) \) in Theorem 1 is satisfied.

Let \( \delta(x,y) \) denote the Hamming distance between \( x \) and \( y \).

**Definition 4** [2] Suppose \( \Pi \) is an NP optimization problem. A neighborhood mapping \( N \) is called \( K \)-bounded, if for any \( x \in \text{Sol} \) and \( y \in N(x) \) holds \( \delta(x,y) \leq K \), where \( K \) is a constant.

The bitwise mutation operator \( \text{Mut}^* \) with probability \( P_m^\delta(x,y)(1 - P_m)^{n-\delta(x,y)} \) outputs a string \( y \), given a string \( x \). Note that probability \( P_m^j(1 - P_m)^{n-j} \), as a function of \( P_m \), \( P_m \in [0,1] \), attains its minimum at \( P_m = j/n \). The following proposition gives a lower bound for the probability \( \mathbb{P}\{ \text{Mut}^*(x) = y \} \), which is valid for any \( y \in N(x) \), assuming that \( P_m = K/n \).

**Proposition 2** Suppose the neighborhood mapping \( N \) is \( K \)-bounded, \( K \leq n/2 \) and \( P_m = K/n \). Then for any \( x \in \text{Sol} \) and any \( y \in N(x) \) holds

\[
\mathbb{P}\{ \text{Mut}^*(x) = y \} \geq \left( \frac{K}{en} \right)^K.
\]

**Proof.** For any \( x \in \text{Sol} \) and \( y \in N(x) \) we have

\[
\mathbb{P}\{ \text{Mut}^*(x) = y \} = \left( \frac{K}{n} \right)^{\delta(x,y)} \left( 1 - \frac{K}{n} \right)^{n-\delta(x,y)} \geq \left( \frac{K}{n} \right)^K \left( 1 - \frac{K}{n} \right)^{n-K},
\]

since \( P_m = K/n \leq 1/2 \). Now \( \frac{\partial}{\partial n}(1 - K/n)^{n-K} < 0 \) for \( n > K \), and besides that, \( (1 - K/n)^{n-K} \to 1/e^K \) as \( n \to \infty \). Therefore \( (1 - K/n)^{n-K} \geq 1/e^K \), which implies the required inequality. \( \square \)
3 Analysis of Guaranteed Local Optima Problems

In this section Theorem 1 is used to estimate the GA capacity of finding the solutions with guaranteed approximation ratio.

Definition 5 [2] A polynomially bounded NP optimization problem Π belongs to the class GLO of Guaranteed Local Optima problems, if the following two conditions hold:

1) At least one feasible solution \( y_I \in \text{Sol} \) is efficiently computable for every instance \( I \in \text{Inst} \);
2) A \( K \)-bounded neighborhood mapping \( N_I \) exists, such that for every instance \( I \), any local optimum of \( I \) with respect to \( N_I \) has a constant guaranteed approximation ratio.

The class GLO contains such well-known NP optimization problems as Maximum Satisfiability, Maximum Cut and the following problems on graphs with bounded vertex degree: Independent Set Problem, Dominating Set Problem and Vertex Cover [2].

If a problem Π belongs to GLO and \( n/2 \geq K \) then in view of Proposition 2 for any \( x \in \text{Sol} \) and \( y \in N(x) \), the bitwise mutation operator with \( P_m = K/n \) satisfies the condition \( \mathbb{P}\{\text{Mut}^*(x) = y\} \geq 1/\text{poly}(\vert I \vert) \), where poly is some polynomial. If \( n/2 < K < n \), then probability \( \mathbb{P}\{\text{Mut}^*(x) = y\} \) is bounded from below by a positive constant. Therefore, Theorem 1 implies the following

Corollary 1 If Π ∈ GLO and population \( X^0 \) at every run contains a feasible solution, then given suitable values of parameters, the iterated GA with bitwise mutation first visits a solution with a constant guaranteed approximation ratio in polynomially bounded time on average.

Conclusion

The obtained results indicate that if a local optimum is efficiently computable by the local search method, it is also computable in expected polynomial time by the iterated GA with tournament selection. The same applies to the GA without restarts, if the set of feasible solutions is the whole search space. Besides that, given suitable parameters, the iterated GA with tournament selection and bitwise mutation approximates any problem from GLO class within a constant ratio in polynomial time on average.

4 Acknowledgements

Supported by Russian Foundation for Basic Research grants 12-01-00122 and 13-01-00862.

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