COHEN LENSTRA PARTITIONS AND MUTUALLY ANNIHILATING MATRICES OVER A FINITE FIELD

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Abstract. Motivated by questions in algebraic geometry, Yifeng Huang recently derived generating functions for counting mutually annihilating matrices and mutually annihilating nilpotent matrices over a finite field. We give a different derivation of his results using statistical properties of random partitions chosen from the Cohen-Lenstra measure.

1. Introduction

Motivated by questions in algebraic geometry, Yifeng Huang [8] derived the following two generating functions.

\[
\sum_{n \geq 0} \frac{|\{A, B \in \text{Mat}_n(F_q) : AB = BA = 0\}|}{|GL(n, q)|} u^n = \frac{1}{1 - u} \sum_{a \geq 0} \frac{u^a}{(1/q)a(u/q)a}.
\]

\[
\sum_{n \geq 0} \frac{|\{A, B \in \text{Nil}_n(F_q) : AB = BA = 0\}|}{|GL(n, q)|} u^n = \frac{1}{(u/q)^\infty} \sum_{c \geq 0} \frac{u^{2c}}{q^{c^2}(1/q)c(u/q)c}.
\]

Here \(\text{Mat}_n(F_q)\) is the set of \(n \times n\) matrices over the finite field \(F_q\), and \(\text{Nil}_n(F_q)\) is the set of \(n \times n\) nilpotent matrices over the finite field \(F_q\). Also, we have used the (standard) notation

\[(x)_i = (1 - x)(1 - x/q)(1 - x/q^2) \cdots (1 - x/q^{i-1}).\]

Our main purpose here is to show that these two generating functions can be rederived using statistical properties of a Cohen-Lenstra measure on partitions. Huang’s lovely paper used analytic ideas related to Cohen-Lenstra heuristics, but our approach is probabilistic and different.

2. Cohen-Lenstra random partitions

To begin we give some notation. We let \(\lambda\) be a partition of some non-negative integer \(|\lambda|\) into parts \(\lambda_1 \geq \lambda_2 \geq \cdots\). We let \(m_i(\lambda)\) denote the number of parts of size \(i\), and we define

\[\lambda'_i = m_i(\lambda) + m_{i+1}(\lambda) + \cdots.\]

So \(\lambda'_i\) is the number of parts of \(\lambda\), and for convenience we also denote this by \(l(\lambda)\). Moreover, if one represents \(\lambda\) by a diagram with row lengths \(\lambda_1, \lambda_2, \cdots\), then \(\lambda'_i\) is the size of the \(i\)th column of the diagram of \(\lambda\).

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In a very influential paper [1], the number theorists Cohen and Lenstra defined a probability measure $P$ on the set of all partitions $\lambda$ of all natural numbers. The definition of the measure $P$ is given by the formula

$$P(\lambda) = (1/q)^\infty \frac{1}{|\text{Aut}(\lambda)|},$$

where $\text{Aut}(\lambda)$ is the automorphism group of a finite abelian group of type $\lambda$. Although we won’t use it, we mention the explicit formula

$$|\text{Aut}(\lambda)| = q^{\sum_i (\lambda'_i)^2} \prod_i (1/q)_{m_i(\lambda)}.$$

In later work, Fulman [3] studied a more general probability measure

$$P_u(\lambda) = \left(\frac{u}{q}\right)^\infty \frac{u^{[\lambda]}}{|\text{Aut}(\lambda)|}.$$

We will use the following result from [4], which gives a way to generate random partitions from the measure $P_u$.

**Theorem 2.1.** Starting with $\lambda'_0 = \infty$, define in succession $\lambda'_1 \geq \lambda'_2 \geq \cdots$ according to the rule that if $\lambda'_i = a$, then $\lambda'_{i+1} = b$ with probability

$$K(a, b) = \frac{u^b(1/q)_a(u/q)_a}{q^{2a}(1/q)_{a-b}(u/q)_b}.$$

This gives the following corollary, the first part of which will be used in proving equation (1) and the second part of which will be used in proving equation (2).

**Corollary 2.2.**

1. For a non-negative integer $a$,

$$\sum_{\lambda : \lambda'_i = a} P_u(\lambda) = \frac{(u/q)^\infty u^a}{(u/q)_a q^{2a}(1/q)_a}.$$

2. For non-negative integers $a \geq b$,

$$\sum_{\lambda : \lambda'_i = a, m_1(\lambda) = b} P_u(\lambda) = \frac{u^{2a-b}(u/q)^\infty}{q^{2a+(a-b)^2}(1/q)_{a-b}(1/q)_a(u/q)_a}.$$ 

**Proof.** For the first result, Theorem 2.1 implies that

$$\sum_{\lambda : \lambda'_i = a} P_u(\lambda) = K(\infty, a).$$

For the second result, note that $m_1(\lambda) = \lambda'_1 - \lambda'_2$. So Theorem 2.1 implies that

$$\sum_{\lambda : \lambda'_i = a, m_1(\lambda) = b} P_u(\lambda) = K(\infty, a)K(a, a - b)$$

and the result follows. □
3. Mutually annihilating matrices

The purpose of this section is to rederive (1) and (2).

To begin, we recall the Jordan form of an element of $\text{Mat}_n(F_q)$. This associates to each monic, irreducible polynomial $\phi$ over $F_q$ a partition $\lambda_\phi$ such that

$$\sum \phi d(\phi)|\lambda_\phi| = n,$$

where $d(\phi)$ is the degree of $\phi$. For further background on Jordan forms over finite fields, one can consult Chapter 6 of [7].

Lemma 3.1 is proved in Stong [9] and calculates the number of elements of $\text{Mat}_n(F_q)$ with given Jordan form.

**Lemma 3.1.** Suppose that

$$\sum \phi d(\phi)|\lambda_\phi| = n,$$

so that $\{\lambda_\phi\}$ is a possible Jordan form of an element of $\text{Mat}_n(F_q)$. Then the number of elements of $\text{Mat}_n(F_q)$ with Jordan form $\{\lambda_\phi\}$ is equal to

$$\frac{|GL(n, q)|}{\prod \phi |\text{Aut}(\lambda_\phi)|_{q \rightarrow q^d(\phi)}}.$$

Here the notation $|\text{Aut}(\lambda_\phi)|_{q \rightarrow q^d(\phi)}$ means that we place $q$ by $q$ in the formula for $|\text{Aut}(\lambda_\phi)|$.

3.1. **Proof of (1).** The following lemma is crucial to deriving (1).

**Lemma 3.2.** Let $A$ be an element of $\text{Mat}_n(F_q)$. Then the number of $B$ such that $AB = BA = 0$ is equal to $q^{m^2}$, where $m$ is the number of Jordan blocks of $A$ with eigenvalue 0.

**Proof.** Clearly the number we are computing is a similarity invariant: if $A$ is replaced by $UAU^{-1}$, the set of $B$ such that $AB = BA = 0$ is replaced by $UBU^{-1}$. Now write $A = \text{diag}(C, N)$ where $C$ is invertible and $N$ is nilpotent (in Jordan form). If $AB = BA = 0$, then $B$ commutes with $A$ and so $B = \text{diag}(0, L)$ with $NL = LN = 0$.

So it reduces to looking at the nilpotent part of $A$. So assume that $A$ is nilpotent with Jordan blocks corresponding to a given partition and consider its centralizer.

If $B$ is in the centralizer, then we can write $B = (B_{ij})$ blocking it up with respect to the Jordan blocks. Then $AB = 0$ if and only if the image of $B$ is in the kernel of $A$. Note that if we write $A = \text{diag}(J_1, ..., J_m)$ with the $J_i$ Jordan blocks then $AB = (J_iB_{ij})$ and there is a one dimensional space of possible $B_{ij}$ with this property (with the $B$ in the centralizer). Thus the set of $B$ with $AB = BA = 0$ has dimension $m^2$ where $m$ is the number of Jordan blocks.

Now we proceed to the main result of this subsection.
Proof. (Of (1)) It follows from Lemmas 3.1 and 3.2 that the number of pairs \( A, B \) in \( \text{Mat}_n(F_q) \) such that \( AB = BA = 0 \) is equal to \( |GL(n,q)| \) multiplied by

\[
\sum_{\{\lambda_0\}} \prod_{\phi} \frac{q^{l(\lambda_z)^2}}{|Aut(\lambda_\phi)|_{q \to q^{d(\phi)}}},
\]

where the sum is over all Jordan forms of \( \text{Mat}_n(F_q) \) and \( l(\lambda_z) \) is the number of parts of \( \lambda_z \).

Separating out the contribution from the polynomial \( \phi(z) = z \), we obtain

\[
|GL(n,q)| \text{ multiplied by the coefficient of } u^n \text{ in }
\sum_{\lambda} q^{(\lambda_z)^2} \frac{u^{|\lambda|}}{|Aut(\lambda)|} \prod_{\phi \neq z} \frac{u^{d(\phi)|\lambda|}}{|Aut(\lambda)|_{q \to q^{d(\phi)}}}.
\]

Now for any finite group, the sum over conjugacy classes of the reciprocal of centralizers sizes is equal to 1. Applying this to \( GL(n,q) \) gives that

\[
\prod_{\phi \neq z} \frac{u^{d(\phi)|\lambda|}}{|Aut(\lambda)|_{q \to q^{d(\phi)}}} = \frac{1}{1-u}.
\]

We conclude that

\[
\sum_{n \geq 0} \frac{|\{A,B \in \text{Mat}_n(F_q) : AB = BA = 0\}|}{|GL(n,q)|} u^n
\]

is equal to

\[
\frac{1}{1-u} \sum_{\lambda} q^{(\lambda_z)^2} \frac{u^{|\lambda|}}{|Aut(\lambda)|}.
\]

This is exactly

\[
\frac{1}{1-u} \frac{1}{(u/q)_\infty} \sum_{a \geq 0} q^{a^2} \sum_{\lambda: \lambda_z = a} P_u(\lambda).
\]

By part 1 of Corollary 2.2 this is equal to

\[
\frac{1}{1-u} \frac{1}{(1/q)_a (u/q)_a} \sum_{a \geq 0} u^a,
\]

as claimed. \( \square \)

3.2. Proof of (2). The following lemma is crucial to deriving (2).

**Lemma 3.3.** Let \( A \) be a nilpotent element of \( \text{Mat}_n(F_q) \). Then the number of nilpotent \( B \) such that \( AB = BA = 0 \) is equal to \( q^{m_d^2 - d} \), where \( m \) is the number of Jordan blocks of \( A \) and \( d \) is the number of Jordan blocks of \( A \) of size 1.

**Proof.** We argue as in the proof of Lemma 3.2. So let \( B = (B_{ij}) \) commute with and annihilate \( A \). We saw there is a one dimensional choice for each \( B_{ij} \) independently. It is well known that the centralizer of \( A \) modulo the Jacobson radical is a direct product \( M_{d_1}(q) \times \ldots \times M_{d_r}(q) \) where \( d_i \) is the number of Jordan blocks of size \( i \). Thus, our computation reduces to the case where all Jordan blocks have the same size \( d \). If \( i > 1 \), we see that \( BA = 0 \)
implies that the image of $B$ is contained in the kernel of $A$ and so $B$ is in the Jacobson radical. In this case any $B$ with $AB = BA = 0$ is nilpotent (and indeed $B^2 = 0$). If $i = 1$ and $d = d_1$, then the centralizer is the full matrix ring $M_d(q)$ and it is well known (see [2] or [6]) that the number of nilpotent matrices of size $d$ is $q^{d^2-d}$. This completes the proof. □

Now we proceed to the main result of this subsection.

Proof. (Of (2)) From Lemma 3.1, the number $n \times n$ of nilpotent matrices over $F_q$ with Jordan type $\lambda$ is equal to

$$|GL(n, q)| / |Aut(\lambda)|.$$

So Lemma 3.3 implies that

$$\sum_{n \geq 0} \frac{|\{A, B \in Nil_n(F_q) : AB = BA = 0\}| u^n}{|GL(n, q)|}$$

is equal to

$$\sum_{\lambda} \frac{u^{\lambda}}{|Aut(\lambda)|} q^{(\lambda')^2 - m_1(\lambda)},$$

where the sum is over all partitions of all natural numbers.

This is equal to

$$\frac{1}{(u/q)^{\infty}} \sum_{\lambda} P_u(\lambda) q^{(\lambda')^2 - m_1(\lambda)}.$$

By part 2 of Corollary 2.2, this is equal to

$$\sum_{a \geq b \geq 0} \frac{u^{2a-b}}{q^{(a-b)^2+b}(1/q)_b(1/q)_{a-b}(u/q)_{a-b}}.$$

Letting $c = a - b$, this becomes

$$\sum_{b,c \geq 0} \frac{u^{2c+b}}{q^{c+b}(1/q)_b(1/q)_c(u/q)_c} = \sum_{b,c \geq 0} \frac{u^b}{q^b(1/q)_b} \sum_{c \geq 0} \frac{u^{2c}}{q^{2c}(1/q)_c(u/q)_c} = \frac{1}{(u/q)^{\infty}} \sum_{c \geq 0} \frac{u^{2c}}{q^{2c}(1/q)_c(u/q)_c},$$

where the last step used the well known identity

$$\sum_{b \geq 0} \frac{u^b}{q^b(1/q)_b} = \frac{1}{(u/q)^{\infty}}.$$

□

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