Fresh perspective on gauging the conformal group

M.P. Hobson\textsuperscript{1,} and A.N. Lasenby\textsuperscript{1,2}

\textsuperscript{1}Astrophysics Group, Cavendish Laboratory, JJ Thomson Avenue, Cambridge CB3 0HE, UK
\textsuperscript{2}Kavli Institute for Cosmology, Madingley Road, Cambridge CB3 0HA, UK

(Dated: received 27 January 2021; accepted 4 April 2021)

We consider the construction of gauge theories of gravity that are invariant under local conformal transformations. We first clarify the geometric nature of global conformal transformations, in both their infinitesimal and finite forms, and the consequences of global conformal invariance for field theories, before reconsidering existing approaches for gauging the conformal group, namely auxiliary conformal gauge theory and biconformal gauge theory, neither of which is generally accepted as a complete solution. We then demonstrate that, provided any matter fields belong to an irreducible representation of the Lorentz group, the recently proposed extended Weyl gauge theory (eWGT) may be considered as an alternative method for gauging the conformal group, since eWGT is invariant under the full set of local conformal transformations, including inversions, as well as possessing conservation laws that provide a natural local generalisation of those satisfied by field theories with global conformal invariance, and also having an ‘ungauged’ limit that corresponds to global conformal transformations. By contrast, although standard Weyl gauge theory also enjoys the first of these properties, it does not share the other two, and so cannot be considered a valid gauge theory of the conformal group.

I. INTRODUCTION

In the classical description of a physical system, any property has meaning only relative to the same property of some other reference system, and not in any absolute sense\textsuperscript{1}. Thus, any measurement corresponds to calculating the ratio of two quantities with the same units. By using ‘natural units’, all physical quantities can be expressed in terms of length, and so the description of physical systems should be invariant under the group of transformations that leave the ratios of lengths unchanged, namely the global conformal transformations. These include Poincaré transformations, which preserve length, together with global scale changes and special conformal transformations (SCTs), all of which are connected to the identity, and so may be considered in their infinitesimal forms. In addition, conformal transformations also include inversions, which are both finite and discrete, and hence excluded from the infinitesimal transformations.

The freedom to make an arbitrary choice of units at any point in space and time further suggests that the description of physical systems should, in fact, be invariant under local conformal transformations, which therefore motivates the study of gauging the conformal group. This is usually performed by considering only the infinitesimal transformations, hence excluding inversions, and allowing the constant group parameters to become arbitrary functions of position. Field theories constructed to be invariant under these local transformations are known as conformal gauge theories, and have been widely studied since the 1970s as potential modified gravity theories.

One finds, however, that this standard approach to gauging the conformal group and the resulting class of auxiliary conformal gauge theories (ACGTs)\textsuperscript{1,2} suffer from serious theoretical difficulties. Most notably, SCTs are not represented in the final structure of AGCTs, since the corresponding gauge field can be algebraically eliminated from the theory. More precisely, one may show that for any self-consistent ACGT action, the resulting field equation for the SCT gauge field can be solved and substituted back into the action to obtain an effective action that is independent of this gauge field, which is thus termed an auxiliary field (hence the name for this class of theories). Thus, in this approach, it appears that the symmetry reduces back to the local Weyl group.

These difficulties have motivated an alternative approach, known as biconformal gauging\textsuperscript{3–14}, which is built on the observation that the reduction to the local Weyl group that occurs in ACGTs is associated with the breaking of the symmetry that exists between the generators of translations and SCTs in the conformal algebra. Biconformal gauging preserves this symmetry by construction, although again considers only transformations that are connected to the identity. The resulting biconformal gauge theories (BCGT) are successful in circumventing many of the problems encountered in the standard approach and have some very interesting and promising features. Nonetheless, the resulting requirement of an eight-dimensional base manifold complicates their physical interpretation.

Neither of these approaches is thus currently generally accepted, and so the role of the conformal group in the construction of gauge theories of gravity remains uncertain. In this paper, we therefore consider an alternative approach to gauging the conformal group, which is moti-
vated in part by consideration of finite conformal transformations, which are therefore not necessarily connected to the identity and so include inversions.

Our reasons for including inversions explicitly are twofold. First, from a physical perspective, it is well known that both the Faraday action for the electromagnetic field and the Dirac action for a massless spinor field are invariant not only under the elements of the conformal group that are connected to the identity, but also under inversions [15–18]. Second, from a mathematical viewpoint, if one considers finite conformal transformations, rather than infinitesimal ones, then the inversion operation effectively replaces the SCT as the fourth distinct element of the conformal group, since the SCT is merely the composition of an inversion, a finite translation and a second inversion. Indeed, this correspondence extends to the action of the elements of the finite conformal group on fields, provided the latter belong to an irreducible representation of the Lorentz group. Moreover, the inversion is itself the composition of a scaling and reflection, both of which are position dependent in prescribed ways. Thus, when one gauges the finite conformal group, the only transformations to consider beyond those of the local Weyl group are gauged reflections, which have not been addressed previously, to our knowledge. Since reflections are merely improper Lorentz transformations, however, they may be localised straightforwardly by gauging the full Lorentz group, rather than only the restricted Lorentz group that is usually considered. Once again, this approach extends to the action of finite conformal transformations on fields that belong to an irreducible representation of the Lorentz group.

These considerations suggest an alternative means of circumventing the difficulties associated with the gauging of SCTs discussed above. In particular, it follows that both Weyl gauge theory (WGT) and the recently proposed extended Weyl gauge theory (eWGT) [19] already accommodate all the gauged symmetries of the full finite conformal group, without the need to introduce any more gauge fields, provided that each occurrence of a restricted Lorentz transformation in the finite transformation laws for the covariant derivative and gauge fields, respectively, instead denotes an element of the full Lorentz group. In this way, both WGT and eWGT actions constructed in the usual way are invariant under (finite) local conformal transformations. As we will show, however, only eWGT also possesses conservation laws that provide a natural local generalisation of those satisfied by field theories with global conformal invariance, and has an ‘ungauged’ limit that corresponds to global conformal transformations. We conclude in Section V. In addition, in Appendix A we include a simple derivation of the finite forms for the action on the coordinates of every element of the conformal group (not just those connected to the identity) directly from their defining requirement, without integrating infinitesimal forms. Finally, in Appendix B we present a brief outline of the consequences of general global and local symmetries for field theories, focussing in particular on Noether’s first and second theorems, the latter being discussed surprisingly rarely in the literature.

II. GLOBAL CONFORMAL INVARIANCE

In Minkowski spacetime, conformal coordinate transformations \( x^\mu \rightarrow x'^\mu \) from some Cartesian inertial coordinate system \( x^\mu \) are those that leave the light cone (and hence causal structure) invariant, such that

\[
    ds^2 = \eta_{\mu\nu} \, dx^\mu \, dx^\nu = \Omega^2(x) \eta_{\mu\nu} \, dx'^\mu \, dx'^\nu, \tag{1}
\]

for some (real) function \( \Omega(x) \). Indeed, more generally, conformal transformations preserve the ‘angle’ or inner product between any two vectors, which is equivalent to the invariance of the ratio of their lengths.
A. Infinitesimal global conformal transformations

We first briefly discuss infinitesimal global conformal transformations, since these are the more commonly considered form in the literature. This allows us both to establish our notation and to clarify the geometric meaning of the transformations in a manner that enables a transparent extension to their finite forms presented in Section II B below.

For an infinitesimal coordinate transformation \( x'^\mu = x^\mu + \xi^\mu(x) \) to satisfy (1), it is readily established that one requires

\[
\partial_\alpha \xi_\beta = \frac{1}{n} (\partial_\mu \xi^\mu) \eta_{\alpha \beta},
\]

which is the conformal Killing equation in \( n \)-dimensional Minkowski spacetime, although hereinafter we will concentrate exclusively on the case \( n = 4 \). One may show in the usual (albeit rather intricate) manner that the most general solution for \( \xi^\mu(x) \) has the form

\[
\xi^\mu(x) = a^\mu + \omega^{\mu \nu} x^\nu + \rho x^2 + c^\mu x^2 - 2c \cdot x x^\mu,
\]

where the 15 infinitesimal parameters \( a^\mu, \omega^{\mu \nu}, \rho \) and \( c^\mu \) are constants, i.e. not functions of spacetime position, and we use the shorthand notation \( x^2 \equiv \eta_{\mu \nu} x^\mu x^\nu \) and \( c \cdot x \equiv \eta_{\mu \nu} c^\mu x^\nu \), in which \( \eta_{\mu \nu} = \text{diag}(1, -1, -1, -1) \) is the Minkowski metric in Cartesian inertial coordinates. If the four parameters \( c^\mu \) defining the so-called special conformal transformation (SCT) vanish, then (3) reduces to an infinitesimal global Weyl transformation. Moreover, if the parameter \( \rho \) defining the dilation (or scale transformation) also vanishes, then (3) further reduces to an infinitesimal global Poincaré transformation, consisting of a restricted Lorentz rotation defined by the six parameters \( \omega^{\mu \nu} \) and a spacetime translation defined by the four parameters \( a^\mu \). The group of global Poincaré transformations is the isometry group of Minkowski spacetime, such that \( \Omega^2(x) = 1 \).

A more intuitive geometric interpretation of an infinitesimal global conformal transformation may be arrived at directly using (2), from which one may show that

\[
\partial_\beta \omega^\alpha_\beta = \omega^\alpha_\sigma + g \delta^\alpha_\beta,
\]

where \( \omega^\alpha_\beta = -\omega^\beta_\alpha \) and \( g \) represent (in general) a position-dependent infinitesimal rotation and dilation, respectively, which must satisfy the conditions (assuming dimensionality \( n \geq 3 \))

\[
\partial_\alpha \omega^\alpha_\beta - 2 \delta^\alpha_\beta \partial_\alpha g = 0,
\]

\[
\partial_\alpha \omega^\alpha_\beta g = 0.
\]

Successive integration of equations (2) and (4), from the last to the first, then yields \( g(x) = \rho - 2c \cdot x, \omega^\alpha_\beta(x) = \omega^\alpha_\beta + 4c^\alpha_\beta \) and the expression (3) for \( \xi^\alpha(x) \), as before, where \( \rho, \omega^\alpha_\beta \) and \( c^\mu \) are again constants.

The action of an infinitesimal conformal transformation on some field \( \varphi(x) \) defined on the spacetime may be determined by first considering the 11-parameter (little) subgroup, say \( H(1, 3) \), of the conformal group \( C(1, 3) \), obtained by setting \( a^\mu = 0 \), which leaves the origin \( x^\mu = 0 \) invariant. Its generator matrices \( \Sigma_{\mu \nu}, \Delta \) and \( \kappa_\mu \), corresponding to Lorentz rotations, dilations and SCTs, respectively, satisfy the commutation relations

\[
[\Sigma_{\mu \nu}, \Sigma_{\rho \sigma}] = \eta_{\mu \rho} \Sigma_{\nu \sigma} - \eta_{\nu \rho} \Sigma_{\mu \sigma} + \eta_{\mu \sigma} \Sigma_{\nu \rho} - \eta_{\nu \sigma} \Sigma_{\mu \rho},
\]

\[
[\Sigma_{\mu \nu}, \kappa_\rho] = \eta_{\mu \rho} \kappa_\nu - \eta_{\nu \rho} \kappa_\mu,
\]

\[
[\Sigma_{\mu \nu}, \Delta] = 0,
\]

\[
\Delta, \Delta = 0, \quad [\kappa_\mu, \kappa_\nu] = 0, \quad [\kappa_\mu, \Delta] = \kappa_\mu.
\]

Using the method of induced representations, one may then show that the action of a full infinitesimal conformal transformation on some field \( \varphi(x) \) leads to a ‘form’ variation \( \delta_0 \varphi(x) \equiv \varphi'(x) - \varphi(x) \) given by

\[
\delta_0 \varphi(x) = (a^\mu P_\mu + \frac{1}{2} \omega^{\mu \nu} M_{\mu \nu} + \rho D + c^\mu K_\mu) \varphi(x),
\]

where the 15 generators of the conformal group \( C(1, 3) \) have the forms

\[
P_\mu = -\partial_\mu,
\]

\[
M_{\mu \nu} = x_\mu \partial_\nu - x_\nu \partial_\mu + \Sigma_{\mu \nu},
\]

\[
D = -x \cdot \partial + \Delta,
\]

\[
K_\mu = (2x_\mu x \cdot \partial - x^2 \partial_\mu) + 2(x' \cdot \Sigma_{\mu \nu} - x_\mu \Delta) + \kappa_\mu,
\]

which correspond to translations, Lorentz rotations, dilations and SCTs, respectively. Note that the generator of the SCT can be expressed in terms of (parts of) the other generators as \( K_\mu = x^2 P_\mu + 2x_\mu\Sigma_{\mu \nu} - 2x_\mu D + \kappa_\mu \).

The generators \( \Sigma, M, P, D \) satisfy the commutation relations

\[
[M_{\mu \nu}, M_{\rho \sigma}] = \eta_{\mu \rho} M_{\nu \sigma} - \eta_{\nu \rho} M_{\mu \sigma} + \eta_{\mu \sigma} M_{\nu \rho} - \eta_{\nu \sigma} M_{\mu \rho},
\]

\[
[M_{\mu \nu}, P_\rho] = \eta_{\rho \mu} P_\nu - \eta_{\rho \nu} P_\mu,
\]

\[
[M_{\mu \nu}, K_\rho] = \eta_{\rho \mu} K_\nu - \eta_{\rho \nu} K_\mu,
\]

\[
[M_{\mu \nu}, D] = 0
\]

\[
[P_\mu, K_\nu] = 2(M_{\mu \nu} + \eta_{\mu \nu} D),
\]

\[
[D, D] = 0,
\]

\[
[P_\mu, D] = -P_\mu,
\]

\[
[K_\mu, D] = -K_\mu,
\]

\[
[K_\mu, K_\nu] = 0,
\]

which define the Lie algebra of the conformal group. Note that, as expected, one recovers the Lie algebra of the Weyl group \( W(1, 3) \) by ignoring commutators containing \( K_\mu \), and if one also ignores commutators containing \( D \) one recovers the Lie algebra of the Poincaré group \( P(1, 3) \).

From (5) and (6), it is straightforward to show that the form variation can also be written as

\[
\delta_0 \varphi(x) = [\xi^\mu(x) P_\mu + \frac{1}{2} \omega^{\mu \nu}(x) \Sigma_{\mu \nu} + g(x) \Delta + c^\mu K_\mu] \varphi(x),
\]

where \( \xi^\mu(x) \) is given by (3), \( \omega^{\mu \nu}(x) = \omega^{\mu \nu} + 4c^{\mu \nu} x^2 \) and \( g(x) = \rho - 2c \cdot x \), which are clearly all functions of

\footnote{For an infinitesimal transformation \( x'^\mu = x^\mu + \xi^\mu(x) \), the ‘form’ variation \( \delta_0 \varphi(x) \equiv \varphi'(x) - \varphi(x) \) is related to the ‘total’ variation \( \delta \varphi(x) \equiv \varphi'(x) - \varphi(x) \) by \( \delta_0 \varphi(x) = \delta \varphi(x) - \xi^\mu \partial_\mu \varphi(x) = \delta \varphi(x) + \xi^\mu P_\mu \varphi(x) \).}
spacetime position. The form variation (10) is of particular
interest when the field \( \varphi(x) \) belongs to an irreducible
representation of the Lorentz group, since the action of
the conformal group is considerably simplified because
the matrix generators \( \Delta \) and \( \kappa_I \) have particularly sim-
ples forms. First, according to Schur’s lemma, any matrix
that commutes with the generators \( \Sigma_{\mu\nu} \) must be a mul-
tiple of the identity. Indeed, one has \( \Delta = wI \), where \( I \)
is the identity and \( w \) is the Weyl weight (or scaling di-
mension) of the field \( \varphi(x) \). Then, from \( [\kappa_\mu, \Delta] = \kappa_\mu \), one
finds that \( \kappa_\mu = 0 \). In this case, with \( \Delta = wI \) and \( \kappa_\mu = 0 \),
it is worth noting that the form variation (10) may be
considered as a particular example of an infinitesimal lo-
cal Weyl transformation, consisting of the combination
of particular forms of position-dependent translation, rota-
tion and dilation; this is consistent with the geometric
interpretation of an infinitesimal global conformal trans-
formation expressed in (4).

Fields that transform according to (10) under a confor-
mal transformation are called primary fields. There also
exist non-primary fields, the most important example of
which is the derivative \( \partial_\nu \varphi(x) \) of a primary field. It is
straightforward to show that

\[
\delta_0(\partial_\nu \varphi) = \Theta \partial_\mu \varphi - (\omega^\nu \delta_\mu^\nu + \rho \delta_\mu^\nu) \partial_\nu \varphi + 2(c^\nu \Sigma_{\nu\mu} - c_\mu \Delta) \varphi,
\]

where \( \Theta \equiv \xi^\alpha P_\alpha + \frac{1}{2} \omega^\alpha \Sigma_{\alpha\beta} + \rho \Delta + c^\alpha \kappa_\alpha \) is the quantity
in square brackets on the RHS of (10), with generators
appropriate to the nature of \( \varphi(x) \), and the final term on
the RHS of (10) shows that \( \partial_\nu \varphi(x) \) is non-primary. In-
deed, although the transformation law is linear, this final
term also means that it is inhomogeneous. It is clear that
this behaviour results solely from the SCT; if \( c^\mu = 0 \), and
hence \( \omega^{\mu\nu} = \omega^{\nu\mu} \) and \( \rho = \rho \), one recovers a global Weyl
transformation and (11) becomes homogeneous and of
an analogous form to the transformation law (10) of the
original field \( \varphi(x) \) with \( c^\mu = 0 \), once the generators
have been augmented to accommodate the additional vector
index on the partial derivative \( \partial_\mu \).

B. Finite global conformal transformations

We now discuss the finite form of global conformal trans-
formations, which are central to our later consider-
ations, but are not often described in the gauge theory
literature. We again seek to clarify the geometric nature
of the transformations, deriving novel (to our knowledge)
finite conditions that reduce to equations (11) and (12)
in the infinitesimal limit, and focus in particular on the role
played by inversions.

The finite forms for the action on the coordinates of the
elements of the conformal group \( C(1,3) \) correspon-
ting to translations, restricted Lorentz rotations (jointly
Poincaré transformations) and dilations (jointly Weyl
transformations) are easily found by obtaining the inter-
gal curves of the corresponding infinitesimal expressions.
Again denoting the 15 now finite constant parameters of
the group by \( a, \omega, \rho \) and \( c \), these finite forms are, respec-
tively,

\[
x^\mu = x^\mu + a^\mu, \quad x^\nu = \Lambda^\nu_\nu(x^\nu), \quad x^\mu = c^\mu x^\mu,
\]

where \( \Lambda^\nu_\mu(x) \) is a restricted Lorentz transformation ma-
rix satisfying \( \eta_{\mu\nu} \Lambda^\nu_\rho \Lambda^\rho_\mu = \eta_{\mu\nu} \) and \( \det \Lambda^\nu_\mu = 1 \).

The same procedure can be used to find the finite form
for the action of a SCT, but more geometrical insight is
obtained by first introducing the inversion transforma-
tion, which may be taken to have the form

\[
x^\mu = \frac{x^\mu}{x^2},
\]

where \( x^2 = \eta_{\mu\nu} x^\mu x^\nu \neq 0 \). This discrete transforma-
tion is clearly also an element of the full conformal group
(although not one connected to the identity), since the
new (Minkowski spacetime) metric is given by \( \eta'_\mu_\nu(x) = \eta_{\mu\nu}/(x^2)^2 \). If one then considers the composite trans-
formation consisting of an inversion, followed by a finite
translation through \( c^\mu \), followed by a second inversion,
one finds

\[
x^\mu = \frac{x^\mu + c^\mu x^2}{1 + 2c \cdot x + c^2 x^2},
\]

which reduces to the infinitesimal SCT in (13) for small
\( c^\mu \). Since every smooth conformal transformation of a
pseudo-Euclidean (Euclidean) space of dimension \( n \geq 3 \)
can be represented as a composition of isometry, dilata-
tion and inversion (22), the expression (14) must rep-
resent a finite SCT. It is worth mentioning that, although
both the numerator and denominator of (14) vanish for
\( x^2 = -c^2 / x^2 \), this point is mapped to infinity and hence
the finite SCT is not defined globally. Indeed, in order to
define finite SCTs globally, one must consider a confor-
marly compactified Minkowski spacetime, which includes
an additional special point at infinity and its null cone,
but we will not consider this subtlety any further here.

Although not usually presented in the literature, one
can in fact derive the finite forms for the action on the
coordinates of every element of the conformal group (not
just those connected to the identity) directly from the
defining requirement (11), without having to integrate in-
finitesimal forms. As just mentioned, one may consider
every smooth conformal transformation as a composition
of isometry, dilation and inversion. The coordinate trans-
formation matrix for the inversion (13) is given by

\[
X^\mu_\nu \equiv \frac{\partial x^\mu}{\partial x^\nu} = \frac{1}{x^2} \left( \delta^\mu_\nu - 2x^\mu x^\nu/x^2 \right) = \frac{1}{x^2} P^\mu_\nu(\hat{x}),
\]

which one may identify (23) as the composition of a
(position-dependent) dilation \( 1/x^2 \) and reflection \( P^\mu_\nu(\hat{x}) \)
in the hyperplane perpendicular to the unit vector \( \hat{x} \equiv
x^\mu/\sqrt{|x^2|} \). It is straightforward to show that \( P^\mu_\nu(\hat{x}) \)
is an improper Lorentz transformation matrix, satisfying
\( \eta_{\mu\nu} P^\nu_\rho(\hat{x})P^\rho_\mu(\hat{x}) = \eta_{\mu\nu} \) and \( \det P^\mu_\nu(\hat{x}) = -1 \). Thus, with
no loss of generality, one may write the transformation
matrix of any smooth finite conformal transformation in the form
\[ X^\mu = \Omega(x) \Lambda^\mu(x), \] (16)
where, in general, \( \Lambda^\mu(x) \) represents a position-dependent finite Lorentz transformation (either proper or improper) and \( \Omega(x) \) represents a position-dependent finite dilation; indeed, one sees immediately that \( \Omega \) satisfies the defining requirement [1]. As shown in Appendix A, one may further use (1) to derive conditions on \( \Lambda^\mu(x) \) and \( \Omega(x) \), which may be written as (for dimensionality \( n \geq 3 \))
\[ \Lambda, \partial_\mu \Lambda \gamma^\beta - 2\delta_\mu^\alpha \partial_\beta \ln \Omega = 0, \tag{17a} \]
\[ 2\Omega \partial_\alpha \partial_\beta \Omega + \eta_{\alpha \beta} (\partial_\Omega) (\partial^\gamma \Omega) - 4 (\partial_\alpha \Omega) (\partial_\beta \Omega) = 0. \tag{17b} \]

It is straightforward to check that, on writing \( x'^\mu = x^\mu + \xi^\mu(x) \), \( \Lambda^\mu(x) \equiv \delta^\mu_\alpha + \Omega^\mu_\nu(x) \) and \( \Omega(x) = e^{\nu(x)} \approx 1 + \delta(x) \), the expressions (16) reduce correctly in the infinitesimal limit to those given in [4,5]. Moreover, as discussed further in Appendix A, the conditions (17) may be used to determine directly the action on coordinates of the four distinct finite elements of the conformal group, namely position-independent translations, rotations and scalings, together with inversions (and hence also SCTs).

In the space of fields, the action of the finite elements of the conformal group that are connected to the identity (thus excluding inversions) is formally given by the exponential of the operator appearing on the RHS of (7), such that for some field \( \psi(x) \) one has
\[ \psi'(x) = \exp(a^\mu P_\mu + \frac{1}{2} \omega^{\mu\nu} M_{\mu\nu} + \rho D + c^\mu K_\mu) \psi(x), \tag{18} \]
where the group parameters \( a, \omega, \rho \) and \( c \) are now finite constants.

It is again of particular interest to consider the case of some physical field \( \phi(x) \) belonging to an irreducible representation of the Lorentz group. Setting \( c_\mu = 0 \) for the moment, thereby neglecting the finite SCTs and considering just the Weyl group, one may describe the action of a finite transformation as
\[ \phi'(x') = e^{\omega^\mu S(\omega)} \phi(x), \tag{19} \]
where \( S(\omega) \) is the matrix corresponding to the element \( \omega \) of the restricted Lorentz group (or SL(2,C) group) in the representation to which \( \phi(x) \) belongs (we have suppressed Lorentz indices on these objects for notational simplicity), and \( \omega \) is the Weyl (or conformal) weight of the field \( \phi(x) \). Indeed, we have adopted the form (19) for the action of finite global Weyl transformations on physical fields in our previous work [13].

To determine the explicit form for the action of a finite SCT on some such physical field \( \phi(x) \), it is again convenient to consider first the action of an inversion, for which the transformation matrix is given by \( \Omega(x) \). Thus, the action of an inversion on, for example, a vector field \( V^\mu(x) \) of Weyl (scaling) weight \( w \), is given by
\[ V^\mu(x') = (x^2)^{-w} V^\mu(\hat{x}) V^{\nu(\hat{x})}. \tag{20} \]

Analogous transformation laws hold for higher-rank tensor fields. One may also show that the action of an inversion on a spinor field \( \psi(x) \) of Weyl weight \( w \) is given by
\[ \psi'(x') = (x^2)^{-w} (\gamma \cdot \hat{x}) \psi(x), \tag{21} \]
where \( \gamma = \{ \gamma^\mu \} \) denotes the set of Dirac matrices. It is worth noting that the quantities \( I^\mu_{\nu}(\hat{x}) \) and \( \gamma \cdot \hat{x} \) have previously been identified [22] as matrices that decouple the Lorentz indices on tensor and spinor fields, respectively, from SCTs, but surprisingly without giving their geometric interpretation as a reflection in the hyperplane perpendicular to the unit vector \( \hat{x} \). It is straightforward to show that these two decoupling matrices are related by the useful formula
\[ (\gamma \cdot \hat{x})\gamma^\mu(\gamma \cdot \hat{x}) = -I^\mu_{\nu}(\hat{x}) \gamma^\nu. \tag{22} \]

Recalling that a SCT is the composition of an inversion, followed by a translation \( c^\mu \), followed by a second inversion, it is now straightforward to show that the action of a finite SCT on, for example, a vector field or spinor field of Weyl weight \( w \) is given by, respectively,
\[ V^{\mu}(x') = [\sigma(x,c)]^{-w} I^\mu_{\alpha}(\hat{x}') I^\alpha_{\nu}(\hat{x}) V^{\nu}(x), \tag{23a} \]
\[ \psi'(x') = [\sigma(x,c)]^{-w} (\gamma \cdot \hat{x}') (\gamma \cdot \hat{x}) \psi(x), \tag{23b} \]
where we have defined the (inverse) scaling \( \sigma(x,c) \equiv 1 + 2c \cdot x + c^2 x^2 \) that appears in the denominator of the coordinate transformation \( \sigma(x,c) \) resulting from a finite SCT. The transformations (23) are thus the composition of a reflection in the hyperplane perpendicular to \( \hat{x} \), a reflection in the hyperplane perpendicular to \( \hat{x}' \) and a scaling. Combining the two reflections, the action of a finite SCT on the fields thus consists of a rotation in the hyperplane defined by \( \hat{x} \) and \( \hat{x}' \) (through twice the angle between the two directions) and a scaling. It is worth noting that, since the resulting rotation is composed of two reflections, it is of a special (or constrained) form; in four dimensions not all (proper) Lorentz rotations can be constructed in this way.

The above geometrical interpretation of a finite SCT is consistent with the action of an infinitesimal SCT on a (primary) field that belongs to an irreducible representation of the Lorentz group, which is given by \( \Delta = w I, \kappa_\mu = 0, \omega_{\mu\nu} = 0 \) and \( \rho = 0 \), and describes the combination of an infinitesimal scaling and rotation (both position dependent). Indeed, it is straightforward to show that in the limit of small \( \sigma^\mu \), the transformations (23) yield precisely the forms \( \sigma^{\mu\nu}(x) = 4c^{[\mu} x^{\nu]} \) and \( \rho(x) = -2c \cdot x \) for the infinitesimal parameters appearing in (20). Moreover, having introduced the reflection operator \( I^\mu_{\nu}(\hat{x}) \), it is worth noting that the generator \( K_\mu \) for SCTs in (12) can be written as \( K_\mu = -x^{2} I^\mu_{\nu}(\hat{x}) \partial_\nu + 2(x' \Sigma_{\mu\nu} - x_\mu \Delta) + \kappa_\mu \).

Finally, we note that since the action both of inversions and SCTs on fields that belong to an irreducible representation of the Lorentz group consists of a scaling and a Lorentz rotation, albeit an improper rotation for
inversions, then the action of any element of the full finite conformal group on such a field may be written in the form \(12\), provided one extends the definition of \(S(\omega)\) to include matrices corresponding to elements of the full Lorentz group, and allows \(\rho\) and \(\omega\) to become particular functions of spacetime position.

### C. Global conformal invariant field theory

As our final topic in the discussion of global conformal invariance, we describe its consequences for field theories, since this is sometimes unclear in the literature and it is key for our later considerations to identify the appropriate conservation laws to compare with those obtained once the conformal group is gauged. The conservation laws discussed below are inevitably determined by considering only the infinitesimal forms of the transformations, and hence include only those simply connected to the identity. Nonetheless, we also briefly consider the consequences of invariance under inversions.

Consider a Minkowski spacetime \(\mathcal{M}\), labelled using Cartesian inertial coordinates, in which the dynamics of some set of fields \(\varphi_i(x)\) \((i = 1, 2, \ldots)\) is described by the action

\[
S = \int L(\varphi_i, \partial_t \varphi_i) \, dt^4,
\]

such as that considered in Appendix 13. The index \(i\) here again merely labels different matter fields, rather than denoting the tensor or spinor components of individual fields (which are suppressed throughout). It is also worth noting that these fields may include a scalar compensator field (often denoted also by \(\phi\)) with Weyl weight \(w = -1\), which may be used to replace mass parameters in the standard forms of Lagrangians for massive matter fields to achieve global conformal invariance (for example by making the substitution \(m \rightarrow m/\phi\)) in the action for a massive Dirac field \(\psi\), where \(\mu\) is a dimensionless parameter but \(\mu\) is dimensionless parameter but \(\mu\) has dimensions of mass in natural units) 19, 24.

The consequences of invariance of the action 24 under an infinitesimal global conformal transformation (connected to the identity) may be determined by substituting the forms 8 and 7 into the general expressions 12 and 13. Recall that the operators \(\delta_\theta\) and \(\partial_\mu\) commute, and equating to zero the coefficients multiplying the constant parameters \(a^\mu\), \(\omega^{\mu\nu}\), \(\rho\) and \(c^\alpha\), respectively, leads to the conditions

\[
\partial_\mu L - \frac{\partial L}{\partial \dot{\varphi}_i} \partial_\mu \varphi_i - \frac{\partial L}{\partial (\partial_\alpha \varphi_i)} \partial_\mu \partial_\alpha \varphi_i = 0,
\]

\[
\partial L \sum_{\mu \nu} \varphi_i + \frac{\partial L}{\partial (\partial_\alpha \varphi_i)} \sum_{\mu \nu} \partial_\alpha \varphi_i + (\eta_{\alpha \beta} \partial_\rho - \eta_{\alpha \rho} \partial_\beta) \partial_\mu \varphi_i = 0,
\]

\[
\frac{\partial L}{\partial L \Delta \varphi_i} + \frac{\partial L}{\partial (\partial_\alpha \varphi_i)} (\Delta - 1) \partial_\alpha \varphi_i + 4L = 0,
\]

\[
\frac{\partial L}{\partial L k_\mu \varphi_i} + \frac{\partial L}{\partial (\partial_\alpha \varphi_i)} [k_\mu \partial_\alpha \varphi_i + 2(\sum_{\mu \nu} \eta_{\mu \nu} \Delta) \varphi_i] = 0,
\]

which hold up to a total divergence of any quantity that vanishes on the boundary of the integration region in 23. The first condition is equivalent to requiring that \(L\) has no explicit dependence on spacetime position \(x\), and this condition has been used to derive the second and third conditions. Moreover, the first three conditions have all been used to derive the final condition. In particular, this means that for the action to be invariant under SCTs (which is necessary for conformal invariance), it must be Poincaré and scale invariant, in addition to satisfying the condition 25a. Conversely, an action that is invariant under Poincaré transformations and SCTs is necessarily scale invariant.

Again adopting the forms 8 and 17, one finds that the general expression 18 for the Noether current becomes

\[
J^\mu = -a^\alpha t^\mu_{\alpha} + \frac{1}{2} \omega^{\alpha\beta} M^\mu_{\alpha\beta} + \rho D^\mu + c^\alpha K^\mu_{\alpha},
\]

where the coefficients of the parameters of the conformal transformation are defined by

\[
t^\mu_{\alpha} \equiv \frac{\partial L}{\partial (\partial_\mu \varphi_i)} \partial_\alpha \varphi_i - \delta^\mu_\alpha L,
\]

\[
M^\mu_{\alpha\beta} \equiv \frac{\partial L}{\partial (\partial_\alpha \varphi_i)} \partial_\beta \varphi_i - \frac{\partial L}{\partial (\partial_\beta \varphi_i)} \partial_\alpha \varphi_i + 2\eta_{\alpha\beta} \Delta L,
\]

\[
D^\mu \equiv -x^\alpha t^\mu_{\alpha} + j^\mu,
\]

\[
K^\mu_{\alpha} \equiv (2x^\alpha x^\beta - \delta^\alpha_\beta x^2) t^\mu_{\beta} + 2x^\beta (s^\mu_{\alpha\beta} - \eta_{\alpha\beta} j^\mu) + k^\mu_{\alpha},
\]

which are the (total) canonical energy-momentum, angular momentum, dilation current and special conformal current, respectively, of the fields \(\varphi_i\), and we have also defined the quantities

\[
s^\mu_{\alpha\beta} \equiv \frac{\partial L}{\partial (\partial_\mu \varphi_i)} \sum_{\alpha\beta} \delta^\mu_{\alpha\beta} \varphi_i,
\]

\[
j^\mu \equiv \frac{\partial L}{\partial (\partial_\mu \varphi_i)} \Delta \varphi_i,
\]

\[
k^\mu_{\alpha} \equiv \frac{\partial L}{\partial (\partial_\mu \varphi_i)} k_{\alpha} \varphi_i,
\]

which are the (total) canonical spin angular momentum, intrinsic dilation current and intrinsic special conformal current of the fields. In 27d, it is worth noting that the first term on the RHS can be written as \(-x^2 I^\alpha_{\beta}(\hat{x}) \theta^\mu_{\beta}\).
If the field equations $\delta L / \delta \phi_i = 0$ are satisfied, then invariance of the action (24) reduces to the conservation law $\partial_\mu J^\mu = 0$. Since the parameters of a global conformal transformation in (25) are constants, one thus obtains separate conservation laws of the form (29), given by

\[
\begin{align*}
\partial_\mu t^\mu_\alpha &= 0, \\
\partial_\mu s^\mu_{\alpha\beta} + 2t_{(\alpha|\beta)} &= 0, \\
\partial_\mu j^\mu &= 0, \\
\partial_\mu k^\mu_\alpha + 2(s^\mu_{\alpha\mu} - j_\alpha) &= 0,
\end{align*}
\]

which again hold up to a total divergence, and may be considered as the “on-shell” specialisation of the conditions (25). As previously, the first condition has been used to derive the second and third conditions, and the first three conditions have all been used to derive the final condition. It is worth noting that the conditions (25) and (29) in fact hold for any subset of terms in the Lagrangian $\mathcal{L}$ for which the resulting action is conformally invariant.

As mentioned earlier, if the fields $\phi_i(x)$ belong to irreducible representations of the Lorentz group, which is usually the case for physical fields, then $\Delta = wI$ and $\kappa_\mu = 0$, and hence $k^\mu_\alpha$ also vanishes. In this case, it is usual to define the “field virial” (30)

\[
\mathfrak{V}_\alpha \equiv \frac{\partial L}{\partial (\partial_\beta \phi_i)}(\Sigma_{\alpha\beta} - \eta_{\alpha\beta}w)\phi_i = s^\beta_{\alpha\beta} - j_\alpha.
\]

From (29d), one sees that for (any subset of) an action that is Poincaré and scale invariant also to be conformally invariant, the field virial $\mathfrak{V}_\alpha$ must vanish up to a total divergence. Remarkably, this last condition is found to hold for all renormalizable field theories involving particles with spin 0, 1/2 and 1, even though scale invariance (and hence conformal invariance) is, in general, broken in such theories.

As is well known, the conservation laws (29a) and (29d), resulting from translational and Lorentz invariance, respectively, can both be expressed in terms of the single Belinfante energy-momentum tensor (28)

\[
t^\mu_\nu = t^\mu_\nu + \frac{1}{2} \partial_\lambda (s^\mu_{\nu\lambda} + s^\nu_{\mu\lambda} - s^{\lambda\mu\nu}),
\]

as the two properties $\partial_\mu t^\mu_\nu = 0$ and $t^\mu_\nu = t^\nu_\mu$. Moreover, for theories describing fields $\phi_i(x)$ that belong to irreducible representations of the Lorentz group and where the field virial (30) vanishes up to a total divergence, scale invariance is thus equivalent to conformal invariance, and one may further define the improved energy-momentum tensor

\[
\theta^\mu_\nu = t^\mu_\nu - \frac{1}{6} (\partial^\mu \partial^\nu - \eta^\mu_\nu \Box^2) \sum_i \phi_i^2,
\]

such that the remaining conditions in (29) can be expressed in terms of this single quantity as $\partial_\mu \theta^\mu_\nu = 0$, $\theta^\mu_\nu = \theta^\nu_\mu$ and $\theta^\mu_\mu = 0$.

Since the discussion thus far relies on infinitesimal transformations, it applies only to elements of the conformal group that are continuously connected to the identity, and hence neglects invariance of the action (24) under inversions, which are intrinsically both finite and discrete. As mentioned in the Introduction, this issue is of particular interest since it is straightforward to show that both the Faraday action for the electromagnetic field and the Dirac action for a massless spinor field are invariant only not under the continuous elements of the conformal group considered above, but also under inversions.

Additional conserved quantities can be generated by discrete symmetries. For example, theories invariant under spatial inversion $x^i = -x^i$ ($i = 1, 2, 3$) conserve parity. It is far less straightforward, however, to determine directly the consequences of invariance of a general action (24) under the (conformal) inversion (39). Nonetheless, one may gain some insight by considering the large-parameter limit of the finite SCT (14), which is given by (25)

\[
x'^\mu = \frac{c^\mu}{c^2} + \frac{1}{c^2} T^\mu_\nu (c) x^\nu + O \left( \frac{1}{c^3} \right).
\]

Thus, a large-$c$ SCT consists of the composition of an inversion (39), a reflection in the hyperplane perpendicular to $\hat{c}$, a scale transformation by $1/c^2$ and a translation by $c^\mu/c^2$. An action that is invariant under translations, scale transformations and SCTs, as we considered above, must therefore also be invariant under the combination of just an inversion and a reflection in the hyperplane perpendicular to $\hat{c}$. Hence, if the action is invariant under an inversion alone, it must also be invariant under reflection in an arbitrary hyperplane, which provides a covariant generalisation of the (one-dimensional) parity and time-reversal transformations.

### III. Previous Approaches to Conformal Gauging

As noted in the Introduction, there are strong theoretical reasons to consider gauging the conformal group with a view to constructing theories of the gravitational interaction that are invariant under local conformal transformations. Previous approaches have focussed on infinitesimal transformations and hence considered gauging only the elements of the conformal group that are connected to the identity (namely translations, Lorentz rotations, dilations and SCTs). This is achieved, in principle, by allowing the 15 parameters of the group $\alpha, \omega, \rho$ and $c$ to become independent arbitrary functions of position.

#### A. Gauging a spacetime symmetry group

Some early approaches to constructing a gauge theory of a spacetime symmetry group $\mathcal{G}$ (in our case...
the conformal group) encountered complications arising from attempting to draw too close an analogy with the gauge theories of internal symmetries \([33, 34]\). In modern terminology, the corresponding gauge fields (or Yang–Mills potentials) were introduced as the components of a connection on a principal fiber bundle with spacetime as base space and \(G\) as fiber. If the whole of a spacetime group \(G\) is gauged in the Yang–Mills sense, however, the gauged ‘internal translations’ prevent the identification of the translational gauge fields with a vierbein in the geometric interpretation of the gauge theory.

It was the gauging of the Poincaré group by Kibble \([35]\) that first revealed how to achieve a meaningful gauging of groups that act on the points of spacetime as well as on the components of physical fields. The essence of Kibble’s approach was to note that when the parameters of the Poincaré group become independent arbitrary functions of position, this leads to a complete decoupling of the translational parts from the rest of the group, and the former are then interpreted as arising from a general coordinate transformation (GCT; or spacetime diffeomorphisms, if interpreted actively). Thus the action of the gauged Poincaré group was considered as a GCT \(x^\mu \to x'^\mu\), together with the local action of its Lorentz subgroup \(H\) on the orthonormal tetrad basis vectors \(e_i(x)\) that define local Lorentz reference frames, where we adopt the common convention that Latin indices (from the start of the alphabet) refer to anholonomic local tetrad frames, while Greek indices refer to holonomic coordinate frames. This approach to gauging can be straightforwardly extended to more general spacetime symmetry groups \([14, 27, 31, 32]\).

The physical model envisaged in Kibble’s approach is an underlying Minkowski spacetime in which a set of matter fields \(\varphi_i\) is distributed continuously. The field dynamics are described by a matter action \(S_M = \int L_M(\varphi_i, \partial_\mu \varphi_i) dx\) that is invariant under the global action of \(G\). One then gauges the group \(G\) by demanding that the matter action be invariant with respect to (infinitesimal, passively interpreted) GCT and the local action of the subgroup \(H\), obtained by setting the translation parameters of \(G\) to zero (which leaves the origin \(x^\mu = 0\) invariant), and allowing the remaining group parameters to become independent arbitrary functions of position. One is thus led to the introduction of new field variables, which are interpreted as gravitational gauge fields. These are used to assemble a covariant derivative \(\mathcal{D}_\mu \varphi\) that transforms in the same way under the action of the gauged group \(G\) as \(\partial_\mu \varphi\) does under the global action of \(G\). The matter action in the presence of gravity is then typically obtained by the minimal coupling procedure of replacing partial derivatives in the special-relativistic matter Lagrangian by covariant ones, to obtain \(S_M = \int h^{-1} L_M(\varphi_i, \mathcal{D}_\mu \varphi_i) dx\) , where the factor containing \(h \equiv \det(h_{\mu}^{\nu})\) (here \(h_{\mu}^{\nu}\) is the translational gauge field) is required to make the integrand a scalar density rather than a scalar.

In addition to the matter action, the total action must also contain terms describing the dynamics of the free gravitational gauge fields. Following the normal procedure used in gauging internal symmetries, Kibble first constructed covariant field-strength tensors for the gauge fields by commuting covariant derivatives, i.e. by considering \([\mathcal{D}_a, \mathcal{D}_b] \varphi\). The free gravitational action then takes the form \(S_G = \int h^{-1} L_G d^4x\), where \(L_G\) is some Lagrangian that depends on the field strengths and is such that \(S_G\) is invariant under the action of the gauged group \(G\). The total action is taken as the sum of the matter and gravitational actions, and variation of the total action with respect to the gauge fields leads to coupled gravitational field equations.

Following Kibble’s work, several other approaches to gauging a spacetime symmetry group have been proposed, in which, for example, the transformations are interpreted actively, or one considers finite rather than infinitesimal transformations \([37, 40]\), but in terms of the final locally valid field equations that these formulations reach, given an initial total action, they are equivalent to Kibble’s original method.

Finally, it is worth noting that Kibble’s gauge approach to gravitation is most naturally interpreted as a field theory in Minkowski spacetime \([38, 40]\), in the same way as the gauge field theories describing the other fundamental interactions, and this is the viewpoint that we shall adopt in this paper. It is more common, however, to reinterpret the mathematical structure of such gauge theories geometrically, where in particular the translational gauge field \(h_{\mu}^{\nu}\) is considered as the components of a vierbein system in a more general spacetime \([37]\). These issues are discussed in more detail elsewhere \([19, 27]\).

More recent approaches to gauging a spacetime symmetry group adopt the geometric interpretation wholeheartedly, and are usually expressed in the language of fiber bundles. In this view, it is clear from the discussion above that only the subgroup \(H\) should act on the fibers, not the whole of \(G\) (i.e. no ‘internal translation’). The simplest and most natural translation of the scheme into fiber bundle language consists of expressing the gauge theory of a spacetime symmetry group \(G\) in terms of the group manifold \(\mathcal{G}\); specifically, in terms of the principal fiber bundle \(\mathcal{G}(\mathcal{G}/H, H)\), where the coset space \(\mathcal{G}/H\) is spacetime \([31, 41]\).

Indeed, this viewpoint is embodied in the so-called quotient manifold method \([42, 43]\), which may be considered as an inversion of Kibble’s approach, and is usually expressed in the language of differential forms as follows. Consider some Lie group \(G\) possessing a Lie subgroup \(H\). The Maurer–Cartan structure equations for \(\mathcal{G}\) read

\[
\mathbf{d}\omega^A - f_{BC}^A \omega^B \wedge \omega^C = 0, \tag{34}
\]

where \(f_{BC}^A\) are the structure constants of the algebra of the group \(G\). These equations constitute an integrability condition that give a 1-form \(\omega^A\) on \(\mathcal{G}\) that carries the basic infinitesimal information about the group’s structure. One may thus define the exponential map of the corresponding Lie algebra and hence a local group ac-
tion. One then takes the quotient $\mathcal{G}/\mathcal{H}$, which is necessarily a manifold $\mathcal{M}$ (usually interpreted as spacetime), and the 1-forms provide its connection. The result is a principal fiber bundle with local $\mathcal{H}$ symmetry and base manifold $\mathcal{M}$. This structure is then modified by generalizing the manifold, and by changing the connection. Changing the manifold has no effect on the local structure, but changing the connection modifies the Maurer–Cartan equations (to yield the Cartan equations), resulting in curvature 2-forms

$$\mathcal{R}^A = d\omega^A - f_{BC}^A \omega^B \wedge \omega^C.$$  \hspace{1cm} (35)

Two restrictions are placed on these curvatures. First, the curvatures must characterize the manifold only, which requires them to be ‘horizontal’, i.e. bilinear in the connections. Second, one requires integrability of the Cartan equations, which leads to the Bianchi identities satisfied by the curvatures.

Thus, once one has made the choice of $\mathcal{G}$ and $\mathcal{H}$, this approach determines: the physical arena $\mathcal{M}$, the local symmetry group $\mathcal{H}$, the relevant field-strength tensors $\mathcal{R}^A$, and any structures inherited from $\mathcal{G}$. While other structures may be imposed, such as additional (compensator) scalar fields, it is usual to consider only those arising directly from properties of the gauge group. To complete a gravity theory, one finally constructs a local action of $\mathcal{H}$, by analogy with (10), the form variation of a (primary) field is given by

$$\delta_0 \varphi(x) = -\xi^\mu(x) \partial_\mu \varphi(x) + \varepsilon(x) \varphi(x),$$  \hspace{1cm} (36)

where $\varepsilon(x) \equiv \frac{1}{2} \varepsilon^{ab}(x) \Sigma_{ab} + \rho(x) \Delta + c^a(x) \kappa_a$ is an element of the localised (little) subgroup $\mathcal{H}$, and $\varepsilon^{ab}(x)$, $\rho(x)$, $c^a(x)$ and $\xi^\mu(x)$ are now independent arbitrary functions of position. Consequently, the transformation law of the derivative $\partial_\mu \varphi(x)$ is no longer given by (11).

The construction of a covariant derivative that transforms like (11) under the gauged conformal group is typically achieved in two steps. First, one defines the $\mathcal{H}$-covariant derivative

$$D_\mu \varphi(x) \equiv \partial_\mu + \tilde{\Gamma}_\mu(x) \varphi(x),$$  \hspace{1cm} (37)

where $\tilde{\Gamma}_\mu(x)$ is a linear combination of the generators of $\mathcal{H}$ that depends on the gauge fields corresponding to local Lorentz rotations, local dilations, and local special conformal transformations, respectively (the bars appearing in these definitions are to distinguish the corresponding quantities from others to be defined later). In the second step, we define a `generalised $\mathcal{H}$-covariant' derivative, linearly related to $D_\mu \varphi$, by

$$\tilde{D}_\mu \varphi(x) \equiv h_{\mu}(x) D_\mu \varphi(x),$$  \hspace{1cm} (38)

where we have introduced the translational gauge field $h_{\mu}(x)$. It is assumed that $h_{\mu}(x)$ has an inverse, usually denoted by $b^\mu (x)$, such that $h_{\mu}(x) b^\mu (x) = \delta^\mu_\nu$ and $h_{\mu}(x) b^\nu (x) = \delta^\mu_\nu$ (where, for brevity, we henceforth typically drop the explicit $x$-dependence).

Under the action of an infinitesimal GCT $x^\mu \rightarrow x^\mu + \xi^\mu(x)$ and the infinitesimal local action of $\mathcal{H}$, we require $\tilde{D}_\mu \varphi$ to have an analogous transformation law to (11), namely

$$\delta_0 (\tilde{D}_\mu \varphi) = -\xi^\mu \partial_\mu \tilde{D}_\mu \varphi + \varepsilon \tilde{D}_\mu \varphi - (\bar{\omega}^\mu a + \rho \delta^\mu_a) \tilde{D}_b \varphi + 2(\varepsilon \Sigma_{ab} - c_a \Delta) \varphi,$$  \hspace{1cm} (39)
but where \( \varpi^{ab}, g, c^a \) and \( \xi^\mu \) are independent arbitrary functions of position. This requirement leads uniquely to the transformation laws

\[
\delta_0 h_a^\mu = -\xi^\nu \partial_\nu h_a^\mu + h_a^\nu \partial_\nu \xi^\mu - (\varpi_b^a + g \delta^a_b) h_b^\mu, \quad (40a)
\]

\[
\delta_0 \bar{\Gamma}^\mu = -\xi^\nu \partial_\nu \bar{\Gamma}^\mu - \bar{\Gamma}_\nu \partial_\nu \xi^\mu - \partial_\mu \xi^\nu + [\bar{\Gamma}^\mu, \xi^\nu].
\]

One sees that \( h_a^\mu \) transforms as a GCT vector, a local Lorentz 4-vector, and has Weyl weight \( w \) laws of all the gauge fields \( A^\mu \). In the general case, \( \bar{\Gamma}^\mu \) transforms as a covariant GCT vector, but has \( \bar{\Gamma}^\mu \) in the transformation laws of \( \xi^\mu \) in (40a) shows that \( \bar{\Gamma}^\mu \) is not the connection for the gauge group \( \mathcal{H} \). Indeed, it was already apparent from the corresponding final term in (39) that \( \bar{\Gamma}_a^\nu \) is not an \( \mathcal{H} \)-covariant derivative in the usual sense; its transformation law is linear but inhomogeneous. As mentioned earlier, this behaviour originates in the final term of the transformation law (11) for \( \partial_\mu \varphi \) under the action of a global conformal transformation, and can be traced to the fact that translations do not form an invariant sub-

derivative, one finds

\[
[\bar{D}_\mu, \bar{D}_\nu] \varphi = \left( \frac{1}{2} R^{ab \mu \nu \rho \sigma} \bar{\Gamma}_{ab} + H_{\mu \nu \rho \sigma} + S^{a \mu \nu \rho} \kappa_a \right) \varphi, \quad (43)
\]

where \( A^{ab \mu}(x), B_{\mu}(x) \) and \( f^a_{\mu}(x) \) are the gauge fields corresponding to local Lorentz rotations, local dilations, and local special conformal transformations, respectively. It is worth pointing out that this assumed form for \( \bar{\Gamma}_a^\nu \) constitutes a choice of how to include the gauge fields, and leads directly to their required transformation laws

de derivative, one finds

\[
[\bar{D}_\mu, \bar{D}_\nu] \varphi = \left( \frac{1}{2} R^{ab \mu \nu \rho \sigma} \bar{\Gamma}_{ab} + H_{\mu \nu \rho \sigma} + S^{a \mu \nu \rho} \kappa_a \right) \varphi, \quad (43)
\]

where we have defined the ‘\( \mathcal{H} \)-rotational’, ‘\( \mathcal{H} \)-dilational’ and the ‘\( \mathcal{H} \)-special conformal’ field-strength tensors, respectively (the reason for notating the last of these with an asterisk will become clear shortly). In terms of the gauge fields \( A^{ab \mu}, B_{\mu} \) and \( f^{a \mu} \), the field strengths have the forms

\[
R^{ab \mu \nu} \equiv 2(\partial_\nu A^{ab \mu} + \eta_{cd} A^{ac \mu} A^{bd \nu}), \quad (44a)
\]

\[
H_{\mu \nu \rho \sigma} \equiv 2(\partial_\rho B_{\mu \nu} - \partial_\sigma B_{\mu \nu} + \partial_\mu B_{\rho \sigma}), \quad (44b)
\]

\[
S^{a \mu \nu \rho} \equiv 2(\partial_\nu f^{a \mu} + A^a_{\rho \mu} f^{a \nu} - B_{\rho \mu} f^{a \nu}) = 2D^a_{\mu \nu \rho} f^{a \nu}, \quad (44c)
\]

For the sake of brevity, in the final expression we have introduced the derivative operator \( D^a_{\mu \nu \rho} \equiv \partial_\mu + \frac{1}{2} A^{ab \mu \nu} \bar{\Gamma}_{ab} + w b_{\mu \nu} \), familiar from WGT 19, 27, where \( w \) is the Weyl weight of the field on which it acts. All three field strengths in (44) transform covariantly under GCT and local Lorentz rotations in accordance with their respective index structures, and also under local dilations with the Weyl weights \( w(R^{ab \mu \nu}) = 0, w(H_{\rho \sigma \mu \nu}) = 0 \) and \( w(S^{a \mu \nu \rho}) = -1 \), respectively, but none of them transforms covariantly under local SCTs, as we discuss further below.

Before doing so, however, we next consider the commutator of two ‘generalised \( \mathcal{H} \)-covariant’ derivatives. Since
where the action of $D_\mu \varphi$ differs from $\bar\Delta_\mu \varphi$, this commutator differs from $[\bar\Delta_\mu \varphi]$ by an additional term containing the derivatives of $h_a^{\mu}$, and reads

$$[\bar\Delta_\mu, D_\nu] \varphi = (\frac{1}{2} R_{\nu \mu}^{abcd} \Sigma_{ab} + H_{\mu \nu} \Delta + S^{a \mu} \kappa_a - T^{a \mu} \varphi) \varphi,$$

where $R_{\nu \mu}^{abcd} \equiv h_{\nu \mu}^{abcd} R_{\nu \mu}^{abcd}$, $H_{\mu \nu} = h_{\nu \mu}^{abcd} H_{\mu \nu}$ and $S^{a \mu} \equiv h_{\nu \mu}^{abcd} S^{a \mu}_{\nu \mu}$, and the ‘$H$-translational’ field strength of the gauge field $h_a^{\mu}$ is given by

$$T^{a \mu} \equiv h_{\nu \mu}^{abcd} T^{a \mu}_{\nu \mu},$$

which clearly has the same form as $S^{a \mu}$, but with $f_a^{\mu}$ replaced by $h_a^{\mu}$. It is worth noting that $R_{\nu \mu}^{abcd}$, $H_{\mu \nu}$ and $T^{a \mu}$ have the same functional forms of the gauge fields as the rotational, dilational and translational gauge field strengths, respectively, in WGT [10].

It is straightforward to show that $R_{\nu \mu}^{abcd}$, $H_{\mu \nu}$, $S^{a \mu}$ and $T^{a \mu}$ are GCT scalars and transform covariantly under local Lorentz rotations and under local dilations, with weights $w(R_{\nu \mu}^{abcd}) = w(H_{\mu \nu}) = -2$, $w(S^{a \mu}) = -3$ and $w(T^{a \mu}) = -1$, respectively. As one might expect, however, the transformation laws under local SCTs are more complicated, and are given by

$$\delta_\alpha R_{\nu \mu}^{abcd} = 4 c^{[a} T^{b]}_{\nu \mu} + 8 \delta_{[a}^{[i} D_{d]}^{[j]} b),$$

$$\delta_\alpha H_{\mu \nu} = -2 c_a T^{a \mu \nu} + 4 \delta_{[a}^{[i} D_{d]}^{[j]} c_a,$$

$$\delta_\alpha S^{a \mu} = -2 c_b (R_{\nu \mu}^{abcd} + 8 \delta_{[a}^{[i} h_{d]}^{[j]} f_{b]}^{[j]}),$$

$$+ 2 c^\mu (H_{\mu \nu} + 4 f_{[a}^{[i} |h_d|^{[j]}),$$

$$\delta_\alpha T^{a \mu} = 0,$$

where the action of $D_\alpha$ assumes that $w(\alpha^a) = -1$. Thus, there is a ‘mixing’ of the transformation laws of the field strengths, which arises from mixing of the transformation laws of the gauge fields themselves, as described above. Moreover, one sees that the transformation laws also depend on the gauge fields directly, rather than just through the field strengths. Indeed, it is only the $H$-translational field strength $T^{a \mu}$ that transforms covariantly (indeed, invariantly) under local SCTs.

The transformation laws mean that the only combination of terms containing field strengths that may be included in the total Lagrangian to obtain an action that is invariant under local conformal transformations is $\phi^2 (\beta_1 T_{abc} T^{abc} + \beta_2 T_{abc} T^{abc} + \beta_3 T_{a} T^{a})$, where the $\beta_i$ are dimensionless parameters and $\phi$ is some (compensator) scalar field with Weyl weight $w(\phi) = -1$. This behaviour differs markedly from that encountered in WGT or PGT, in which all the field strengths transform covariantly under all the localised transformations. It is therefore necessary to adapt Kibble’s approach slightly, as we now outline, to reduce the complications arising from the gauging of SCTs.

The above complications arise, in part, from the fact that the quantity $\bar\Gamma_\mu$ is not the connection for the gauge group $\mathcal{H}$, as is apparent from its transformation law [10]. The nature of $\bar\Gamma_\mu$ can be better understood by considering a purely internal $SO(2,4)$ symmetry, to which the conformal group $C(1,3)$ is isomorphic, with generators $\pi_a$, $\Sigma_{ab} \Delta$ and $\kappa_a$ satisfying a set of commutation rules analogous to [10], where $\pi_a$ is the translational generator. One then defines the quantity

$$\bar\Gamma_\mu \equiv b^\mu \pi_a + \bar\Gamma_\mu,$$

which transforms under the simultaneous local action of $\mathcal{H}$ and a GCT as

$$\delta_\alpha \bar\Gamma_\mu = -\xi^\mu \partial_\mu \bar\Gamma_\mu - \bar\Gamma_\nu \partial_\mu \xi^\nu - \partial_\mu \varepsilon - [\bar\Gamma_\mu, \varepsilon],$$

and hence is a connection for the group $SO(2,4)$. Moreover, the transformation laws for the two parts of $\bar\Gamma_\mu$ correspond precisely those found in [10] (where we recall that $b^\mu_a$ is the inverse of $h_a^{\mu}$). Thus, $b^\mu_a$ and $\bar\Gamma_\mu$ together constitute a connection for the group $SO(2,4)$.

If one then defines the new ‘$G$-covariant’ derivative operator $\bar\Delta_\mu \varphi = \partial_\mu + \bar\Gamma_\mu$, the resulting commutator reads

$$[\bar\Delta_\mu, D_\nu] \varphi = (\frac{1}{2} \bar R_{\nu \mu}^{abcd} \Sigma_{ab} + \bar H_{\mu \nu} \Delta + S^{a \mu} \kappa_a - T^{a \mu} \bar\varphi) \varphi,$$

where the new rotational and dilational ‘$G$-covariant’ field strengths are given in terms of those defined in [14] as

$$\bar R_{\nu \mu}^{abcd} = R_{\nu \mu}^{abcd} + 8 f_{[a}^{[i} |h_d|^{[j]}),$$

$$\bar H_{\mu \nu} = H_{\mu \nu} + 4 f_{[a}^{[i} |h_d|^{[j]}),$$

One may define the corresponding GCT scalar field strengths $\bar R_{\nu \mu}^{abcd}$ and $\bar H_{\mu \nu}$, in an analogous manner to that used above. The resulting set of field strengths again transform covariantly under local Lorentz rotations and local dilations (with the same weights as given previously), but now transform under local SCTs as

$$\delta_\alpha \bar R_{\nu \mu}^{abcd} = 4 c^{[a} T^{b]}_{\nu \mu} + 8 \delta_{[a}^{[i} D_{d]}^{[j]} b),$$

$$\delta_\alpha \bar H_{\mu \nu} = -2 c_a T^{a \mu \nu} + 4 \delta_{[a}^{[i} D_{d]}^{[j]} c_a,$$

$$\delta_\alpha S^{a \mu} = -2 c_b (\bar R_{\nu \mu}^{abcd} + 8 \delta_{[a}^{[i} h_{d]}^{[j]} f_{b]}^{[j]}),$$

$$\delta_\alpha T^{a \mu} = 0,$$

Once again, there is ‘mixing’ between these transformation laws, although now they depend only on the field strengths (and on the parameters of the local SCT, as expected).

1. Auxiliary conformal gauge theory with non-zero torsion

Given the transformation laws [52], the most general, parity-even free-gravitational action (containing no compensator scalar fields) that is invariant under local conformal transformations is uniquely determined (up to an overall multiple), and given by [13]

$$S_G = \alpha \int h_{-1} (\bar R_{abcd} \bar R^{abcd} + 4 T_{abc} S^{abc} + 2 \bar H_{abcd} \bar H^{abcd}) d^4 x.$$
The phenomenology of the resulting gravity theory remains to be fully explored, but one can show [6] that the gauge field $f^a_\mu$ corresponding to local SCTs acts as an auxiliary field (hence the name for this approach), since its field equation may be used to eliminate it from the action [63]. Hence it appears that the symmetry reduces back to the local Weyl group. In principle, the total Lagrangian could again also include the combination $\phi^2(\beta_1 T^{abc}_{\ aba} T^{abc}_{\ aba} + \beta_2 T^{abc}_{\ aba} T^{abc}_{\ aba} + \beta_3 T^{abc}_{\ aba} T^{abc}_{\ aba})$, where the $\beta_i$ are dimensionless parameters and $\phi$ is some (compensator) scalar field with Weyl weight $w(\phi) = -1$, together possibly with an additional kinetic term for $\phi$, but these additional terms appear not to have been considered previously.

2. Auxiliary conformal gauge theory with vanishing torsion

One sees from [52] that, since $T^{*a}_{\ cd}$ transforms covariantly under the full gauged conformal group, one can consistently set it to zero, if desired. In this case, $\tilde{\mathcal{R}}^{ab}_{\ cd}$ and $\tilde{\mathcal{H}}_{cd}$ then also become fully covariant, and so the number of terms that may be included in a total action that remains invariant under local conformal transformations is considerably increased. Moreover, the condition $T^{*a}_{\ cd} = 0$ can be used to eliminate the rotational gauge field $A^{ab}_{\ \mu}$ by writing it in terms of the translational and dilational gauge fields $h_a^\mu$ and $B_\mu$. This torsionless special case of auxiliary conformal gauge theory has been studied more extensively [1, 3]. The only admissible Lagrangian term that is linear in the gauge field strengths $\phi^2 \tilde{\mathcal{R}}$, but it may be shown that variation of the resulting action with respect to the gauge field $f^a_\mu$ leads to inconsistencies [12]. Attention has therefore focussed on Lagrangians consisting of (arbitrary) linear combinations of terms quadratic in the field strengths $\tilde{\mathcal{R}}^{ab}_{\ cd}$ and $\tilde{\mathcal{H}}_{cd}$ and their contractions (and without compensator scalar fields). It may be shown, however, that in every such case, the gauge field $f^a_\mu$ may again be eliminated from the original action by its own field equation [24, 4], such that the resulting action depends only on $h_a^\mu$ and $B_\mu$. Indeed, every such action is found to be equivalent to [6]

$$S_G = \int h^{-1}(0^{a}_{abcd} 0^{abcd} + \beta \mathcal{H}_{ab}\mathcal{H}^{ab}) d^4 x,$$  (54)

where $0^{abcd}$ is the conformal tensor, defined by

$$0^{abcd} = 0^{0}_{abcd} - \eta_{[a}\eta_{[b} 0^{0}_{c]d]} + \eta_{[a} \eta_{b]} 0^{0}_{c]d] + \frac{1}{3} \eta_{[a} \eta_{b]} \eta_{c]} 0^{0}_{d]} 0^{0}_{d]} R,$$  (55)

in which $0^{0}_{abcd} = 2 h^{\mu}_{ab} h_x (\delta_{\mu[a} A^{ab}_{\ \nu]v} + A^{a}_{c[a} A^{ab}_{\ \nu]} )$ is the gauge theory equivalent of the Riemann tensor, obeying the usual symmetries and identities, and the quantities $0^{0}_{abcd}$ are the Ricci rotation coefficients

$$0^{0}_{abcd} = h^{a}_{b} \phi^{(a}_{[b|l]} [h_{c]}_{d]} - h^{a}_{d} \phi^{(a}_{[b|l]} [h_{c]}_{b]}_{d]} - h^{a}_{d} \phi^{(a}_{[b|l]} [h_{c]}_{b]} _{d]} - h^{a}_{d} \phi^{(a}_{[b|l]} [h_{c]}_{b]} _{d]} - h^{a}_{d} \phi^{(a}_{[b|l]} [h_{c]}_{b]} _{d]}$$  (56)

which depend entirely on the translational gauge field $h^\mu_a$ (and its inverse) [19]. Thus, with the elimination of $f^a_\mu$, the symmetry appears once again to have reduced back to the local Weyl group.

We conclude this section by noting that the auxiliary conformal gauge theories that we have constructed using (a slight generalisation of) Kibble’s approach [6] are identical to those obtained using the quotient manifold method of gauging, in which the Lie group $G$ is the conformal group (more precisely, the elements of it that are connected to the identity) with the 15 generators $\{ A^{a}_{\mu}, M_{ab}, D, K_a \}$ given in [8], $H$ is the inhomogeneous Weyl group with 11 generators $\{ M_{ab}, D, K_a \}$, and the quotient $G/H$ is thus a homogeneous four-dimensional manifold $M$ (interpreted as spacetime). In particular, this approach leads to field-strength tensors that agree precisely with those used to construct the action [63].

C. Ungauging the conformal group

In our discussion in Section [14, A] below of the relative merits of eWGT and WGT as putative gauge theories of the conformal group, one of our key considerations will be the ‘ungauged’ limit of each theory. We therefore give an account here of the process typically used for determining this limit and propose some modifications of it, before applying our adapted version to ACGT as an exemplar.

According to Lord [21], in order to justify that the local action of the (little) subgroup $H$, together with general diffeomorphisms (or GCT) on $M$, does indeed constitute a true gauge theory of a spacetime group $G$, one must show that the limiting case of ‘ungauged’ transformations does in fact correspond to the correct global action of $G$ on $M$ and on fields in $M$. Lord demonstrates that this holds for the auxiliary conformal gauge theory (ACGT) described above, and also for analogous gauge theories based on the de Sitter case (by Wigner–Inonii contraction of the de Sitter case) the Poincaré group; the ‘ungauged’ limit of Poincaré gauge theory is also considered by Hehl [20].

The ‘ungauged’ limit corresponds to vanishing gauge field strengths. For ACGT, one thus requires $\tilde{\mathcal{R}}^{ab}_{\ cd}, \tilde{\mathcal{H}}_{cd}, S^{*a}_{\ cd}$ and $T^{*a}_{\ cd}$ to vanish. In this limit, the coordinate system and $H$-gauge can be chosen such that

$$h^\mu_a (x) = \delta^\mu_a, \quad A^{ab}_{\ \mu} (x) = 0, \quad B_\mu (x) = 0, \quad f^a_\mu (x) = 0.$$  (57)

In this reference system, the first condition means that the distinction between Latin and Greek indices is lost. It is important, however, to retain this distinction (and that between calligraphic and non-calligraphic quantities) when considering behaviour under any subsequent GCT and $H$-gauge transformation.

From the transformation laws [40] and [42] of the gauge fields, Lord notes simply that in order for any subsequent GCT and $H$-gauge transformation to preserve
the relations (57), one requires
\[
\begin{align*}
\partial_\beta \xi^\alpha &= \omega^{\alpha\beta} + \phi \delta^\alpha_\beta, \\
\partial_\mu \omega^{\alpha\beta} &= 4c^{[\alpha A^{0]}}_\beta, \\
\partial_\mu \phi &= -2c\mu, \\
\partial_\mu e^\alpha &= 0.
\end{align*}
\]
(58a)
(58b)
(58c)
(58d)
Successive integration of these equations (from the last to the first) is straightforward and yields an expression for \(\xi^\alpha(x)\) of the form (3) for an infinitesimal global conformal transformation. Similarly, the transformation law (30) reduces to that given in (11) for the action of an infinitesimal global conformal transformation on (a primary) physical field. Hence, Lord concludes that ACGT has the correct ‘ungauged’ limit.

It is not clear from Lord’s discussion, however, why the ‘ungauged’ limit should be derived by requiring that the relations (57) be preserved. Although preserving these relations ensures that the covariant derivative remains equal to the simple partial derivative, it is certainly unnecessary for the field strength tensors to remain zero, to which the ‘ungauged’ limit corresponds. Indeed, since these tensors are GCT scalars and transform covariantly under global Lorentz rotations and local dilations, and according to (52) under local SCTs, they will remain zero under any subsequent GCT and \(H\)-gauge transformation, which in general will not preserve the relations (57).

Moreover, as we now show, the final three relations in (57) are superfluous for identifying global conformal transformations as the ‘ungauged’ limit of ACGT. Requiring only that the first relation in (57) be preserved (which ensures the equivalence of Latin and Greek indices before and after the transformation) leads immediately to the first equation in (58), which is simply a consequence of demanding that \(\delta_0 h_\mu^\alpha = 0\). It is straightforward to show, however, that the first relation in (58) is both a necessary and sufficient condition for \(\xi^\alpha(x)\) to satisfy the four-dimensional conformal Killing equation in Minkowski spacetime given in (2), from which it follows that the most general solution for \(\xi^\alpha(x)\) has the form (3) of an infinitesimal global conformal transformation. The remaining three conditions in (58) then follow automatically, which in turn means that the final three relations in (57) are also preserved. Thus, for ACGT, the requirement that these further relations be preserved is superfluous, and the correct ‘ungauged’ limit can be identified by requiring only that the first relation in (57) is preserved.

It is clear, however, that imposing this reduced requirement on other gauge theories will not in general isolate the correct ‘ungauged’ limit. Consider WGT, for example, for which \(G\) is the inhomogeneous Weyl group and \(H\) is the homogeneous Weyl group. The structure of WGT is easily obtained from that of ACGT by setting \(c^a \equiv 0\) and \(f^a\mu \equiv 0\) throughout Section III.13. The transformation law (10) for the translational gauge field therefore holds unchanged in WGT. Thus, requiring only the first relation in (57) to be preserved and following the same argument as above leads to the erroneous conclusion that the ‘ungauged’ limit of WGT also corresponds to global conformal transformations, rather than global Weyl transformations, for which \(c^a = 0\). This further condition can be obtained only by requiring the relations \(A^{ab}_{\mu} = 0\) and \(B_{\mu} = 0\) in (57), which are also preserved (recall that \(f^a_{\mu} \equiv 0\) in WGT). That this does indeed lead to the correct ‘ungauged’ limit of global Weyl transformations can be seen immediately from the transformation laws for \(A^{ab}_{\mu}\) and \(B_{\mu}\) in WGT, which are given by (72) with \(c^a = 0\).

Given the lack of a clear rationale for identifying the ‘ungauged’ limit of a gravitational gauge theory by imposing Lord’s condition that (the appropriate subset of) the relations (57) should be preserved, it is of interest to investigate an alternative prescription. This may be motivated most naturally by considering more closely the identification of the ‘ungauged’ limit with the requirement that field-strength tensors should vanish.

To this end, let us consider the ACGT covariant derivative of some matter field \(\varphi(x)\), which from (47), (48) and (49) is given by
\[
\delta_0 \varphi = h_\mu c^\mu (\partial_\mu + \frac{1}{2} A_{\mu}^{ab} \Sigma_{ab} + B_\mu \Delta + f^a_\mu \kappa_a) \varphi. 
\]
(59)

It is clear that the dynamics of the matter field will be sensitive to the translational gauge field \(h_\mu^a\), irrespective of the nature of \(\varphi\). This is not the case, however, for the other gauge fields \(A^{ab}_{\mu}\), \(B_{\mu}\) and \(f^a_\mu\). Depending on the nature of \(\varphi\), the dynamics of the matter field may be insensitive to one or more of these gauge fields. To establish the ‘ungauged’ limit, one should therefore consider the ‘subsidiary’ field-strength tensors obtained from \(\tilde{R}_{\mu}^{ab}, \tilde{H}_{\mu}, S^{a}_{cd}\) and \(T^{a}_{cd}\) by including only those terms that depend on \(A^{ab}_{\mu}, B_{\mu}\) or \(f^a_\mu\), respectively (or, equivalently, by setting the other gauge fields in each case to be identically zero).

Thus, starting from the relations (57), one should demand that under subsequent GCT and \(H\)-gauge transformations that preserve the first relation (such that \(\delta_0 h_\mu^a = 0\), all ‘subsidiary’ field-strength tensors remain zero. Such tensors are still covariant under GCTs, but (typically) not so under general \(H\)-gauge transformations, and hence will not automatically remain zero, even if one starts from the set of relations (57). Thus, demanding that they do so can impose further constraints on the allowed nature of the subsequent GCT and \(H\)-gauge transformations, beyond the requirement imposed by \(\delta_0 h_\mu^a = 0\) that the most general solution for \(\xi^\alpha(x)\) is the infinitesimal global conformal transformation (3).

A straightforward way of imposing this requirement is to demand that, under subsequent \(H\)-gauge transformations satisfying \(\delta_0 h_\mu^a = 0\), the change in each ‘full’ field strength \(\tilde{R}_{\mu}^{ab}, \tilde{H}_{\mu}, S^{a}_{cd}\) and \(T^{a}_{cd}\) arising from the change in each gauge field \(A^{ab}_{\mu}\), \(B_{\mu}\) or \(f^a_\mu\) should vanish separately. It is clear that the condition \(\delta_0 h_\mu^a = 0\) guarantees that the variation in the field-strength tensors vanishes if the variation in their non-calligraphic counterparts does. The transformation laws
of the latter (assuming $\delta_0 h_a^\mu = 0$) are given in terms of the transformations of the other gauge fields by

$$
\delta_0 \tilde{R}^{ab}_{\mu \nu} = 2 \partial_\mu \delta_0 A^{ab}_\nu + 8 \delta_0 \tilde{R}^{[a}_{\mu} \delta_0 f^{b]}_{\nu},
$$

(60a)

$$
\delta_0 \tilde{H}_{\mu \nu} = 2 \partial_\mu \delta_0 B_{\nu} + 4 \eta_{\mu \nu} \delta_0 f^a_\tau \delta_0 Y^{a}_{\tau},
$$

(60b)

$$
\delta_0 S^{*a}_{\mu \nu} = 2 \partial_\mu \delta_0 f^a_\nu,
$$

(60c)

$$
\delta_0 T^{*a}_{\mu \nu} = 2 \delta_0 A^a_{b [\mu} \delta_0 Y^{b}_{\nu]} + 2 \delta_0 B_{[\mu} \delta_0 Y^{a}_{\nu]}.
$$

(60d)

We thus require that each term on the RHS of these equations should vanish separately. From the transformation laws of these gauge fields, this requirement is satisfied only if $\delta_0 A^{ab}_{\mu}, \delta_0 B_{\mu}$ and $\delta_0 f^a_\mu$ vanish. This is, however, equivalent merely to the final three relations in (57) being preserved, which follows automatically from our initial requirement that the first relation in (57) is preserved. Thus, no further conditions apply and one correctly deduces that the most general solution for $\xi^a(x)$ has the form of an infinitesimal global conformal transformation.

Let us now repeat the above process for WGT. In this case, the field strengths are $R^{ab}_{\mu \nu}, H_{\mu \nu}$ and $T^{*a}_{\mu \nu}$, where the first two may be obtained from their counterparts in ACGT by setting $f^a_\mu = 0$, as is clear from (61). Thus, if one again starts from the conditions (67) (again recalling that $f^a_\mu = 0$ in WGT) and demands only that the first condition is preserved under subsequent GCT and H-gauge transformations, the transformation laws of the WGT field strengths are given in terms of the transformations of the WGT gauge fields $A^{ab}_{\mu}$ and $B_{\mu}$ by the corresponding expressions in (60) with $\delta_0 f^a_\mu \equiv 0$. From the transformation laws of the WGT gauge fields $A^{ab}_{\mu}$ and $B_{\mu}$, which may be obtained from (62) by setting $c^a \equiv 0$, one may show that our additional requirement is satisfied only if $\delta_0 A^{ab}_{\mu}$ and $\delta_0 B_{\mu}$ vanish. These further conditions correspond to the second and third relations in (67) being preserved, which in turn requires $c^a = 0$. Thus, one correctly deduces that the most general solution for $\xi^a(x)$ has the form of a global Weyl transformation.

Finally, it is a simple matter to verify that an analogous procedure applied to PGT leads to the correct identification of the corresponding ‘ungauged’ limit as global Poincaré transformations. Indeed, for PGT, WGT and ACGT, this procedure is found to be equivalent to requiring that (the appropriate subset of) the relations (57) are preserved, as Lord originally suggested. As we will see in Section XV, however, these two approaches are not always equivalent.

### D. Biconformal gauging

Before moving on to discuss our new approach for gauging the conformal group in Section XV, we conclude this section with a brief discussion of an existing alternative scheme, known as biconformal gauging, which leads to a very different conformal gauge theory to ACGT, with several interesting properties.

As discussed in Section XI in the standard approach to gauging the conformal group, one may eliminate the gauge field $f^a_\mu$ corresponding to local SCTs, which implies that the symmetry has reduced back to the local Weyl group. As pointed out by Wheeler [3], however, this reduction is rather curious, since in addition to the elimination of $f^a_\mu$, the symmetry between the generators $P_a$ and $K_a$ in the Lie algebra of the conformal group has also been lost. Indeed, the elimination of $f^a_\mu$ occurs because one chooses to identify the translational gauge field $h_a^\mu$ with the vierbein (at least in the geometrical interpretation), as is done in PGT. In gauging the conformal group, however, one can make the alternative choice of identifying the SCT gauge field $f^a_\mu$ with the vierbein, in which case the translational gauge field $h_a^\mu$ may be eliminated instead. Thus, in the case of the conformal group, there is an additional symmetry between the two generators $P_a$ and $K_a$, which is broken by one’s (arbitrary) choice for identifying the vierbein.

An alternative approach to gauging the conformal group, which preserves the symmetry between $P_a$ and $K_a$ by construction, is the so-called biconformal gauging. Expressed in terms of the quotient manifold method, in biconformal gauging the Lie group $G$ is again the conformal group (only the elements connected to the identity) with the 15 generators $\{P_a, M_{ab}, D, K_a\}$, but the Lie subgroup $H$ is now the homogeneous Weyl group with the seven generators $\{M_{ab}, D\}$. The resulting quotient $G/H$ is thus an eight-dimensional manifold, called biconformal space, which has a number of very interesting properties, as we now briefly describe.

Biconformal space is spanned by the basis 1-forms $h_a$ and $f^a$, derived from the translations and special conformal transformations, respectively. If the dilational curvature vanishes, then the non-degenerate 2-form $h_a \wedge f^a$ is also closed (i.e. an exact differential), and hence symplectic. Moreover, the Killing metric is non-degenerate when restricted to the base manifold, so that the group structure determines a metric, rather than imposing one by hand. If the basis 1-forms $h_a$ and $f^a$ are separately in involution and orthogonal, then biconformal space can be considered as a form of relativistic phase space, consisting of separate configuration and momentum metric submanifolds. Moreover, the signatures of these submanifolds are severely limited and, in particular, the notion of time emerges naturally, since the configuration space must be Lorentzian and is therefore interpreted as spacetime. It is usually argued that the full biconformal space should be interpreted as representing ‘the world’, since both classical and quantum mechanics take their most elegant forms in phase space and, moreover, a phase space is required to formulate the uncertainty principle.

To define a dynamical theory, one writes down an action on the full biconformal space. Its symplectic structure means that the volume element is dimensionless, and so an action linear in the curvatures can be conformally
invariant, without introducing any additional (compensating) fields. If one assumes the torsion to vanish and the momentum subspace is flat, the theory reduces to general relativity on the spacetime tangent bundle.

It is clear that biconformal gauge theory (BCGT) has some very interesting properties and is worthy of continued investigation, but we will not pursue it further here.

IV. NEW APPROACH TO CONFORMAL GAUGING

In an earlier paper [19], we introduced a novel alternative to standard Weyl gauge theory, in which we proposed an ‘extended’ form for the transformation law of the rotational gauge field under finite local dilations, given by

\[ A_{\mu}^{cd} = A_{\mu}^{cd} + \theta(b_{\mu}^{a} Q^{d} - b^{d}_{\mu} Q^{a}), \quad (61) \]

Thus, for general values of \( \theta \), neither \( R_{cd}^{ab} \) nor \( T_{bc}^{a} \) transforms covariantly. For \( \theta = 0 \), however, one recovers the ‘normal’ transformation law for the A-field, such that \( R_{cd}^{ab} \) transforms covariantly under local dilations, but \( T_{bc}^{a} \) transforms inhomogeneously. By contrast, for \( \theta = 1 \) one obtains a covariant transformation law for \( T_{bc}^{a} \), but an inhomogeneous one for \( R_{cd}^{ab} \). The extended A-field transformation law (61) accommodates these extreme cases in a balanced manner.

We therefore developed our so-called ‘extended’ Weyl gauge theory (eWGT), which is based on the construction of a new form of covariant derivative \( D_{\mu} \phi \) that transforms in the same way under local Weyl transformations as \( \partial_{\mu} \phi \) does under the global Weyl transformations, but where the rotational gauge field introduced is assumed to transform under local dilations as \( (61) \). The resulting theory has a number of interesting features, which we summarise briefly below.

A. Extended Weyl gauge theory (eWGT)

In eWGT, the spacetime group \( G \) under consideration is the inhomogeneous Weyl group and its subgroup \( \mathcal{H} \) is the homogeneous Weyl group, as in standard WGT. It is also assumed that any physical (matter) fields \( \phi(x) \) belong to an irreducible representation of the Lorentz group (as is usually the case in physical theories). Consequently, as discussed in Sections 11A and 11B, the generator of dilations takes the simple form \( \Delta = w I \), where \( w \) is the Weyl weight of \( \phi \). Adopting Kibble’s general methodology, the gauged action of \( \mathcal{G} \) on such a field is considered as a GCT \( x^{\mu} \rightarrow x'^{\mu} \), together with the local action of \( \mathcal{H} \), such that (in finite form) one obtains a local version of (19), namely

\[ \phi'(x') = e^{w \phi(x)} S(\omega(x)) \phi(x). \quad (63) \]

1. Covariant derivative

Following the usual approach, the construction of the covariant derivative in eWGT is achieved in two steps. First, one defines the ‘\( \mathcal{H} \)-covariant’ derivative

\[ D_{\mu}^{\mathcal{H}} \phi(x) \equiv \left[ \partial_{\mu} + \Gamma_{\mu}^{\mathcal{H}}(x) \right] \phi(x), \quad (64) \]

where \( \Gamma_{\mu}^{\mathcal{H}}(x) \) is a linear combination of the generators of \( \mathcal{H} \) that depends on the gauge fields. Second, one constructs the ‘generalised \( \mathcal{H} \)-covariant’ derivative, linearly related to \( D_{\mu}^{\mathcal{H}} \phi \) by

\[ D_{\mu}^{\mathcal{H}} \phi(x) \equiv h_{\mu} \phi(x) D_{\mu}^{\mathcal{H}} \phi(x), \quad (65) \]

where \( h_{\mu}(x) \) is the translational gauge field, again assumed to have the inverse \( b^{a}_{\mu}(x) \).
In eWGT, however, one does not adopt the standard approach of introducing each gauge field in $\Gamma_{\mu}^a(x)$ as the linear coefficient of the corresponding generator, such as in (41), since this would lead directly to the standard WGT transformation laws for the rotational and dilatational gauge fields (given in infinitesimal form by the first two relations in (42) with $c^a \equiv 0$). Rather, in order to accommodate our proposed extended transformation law under local dilations, one is led to introduce the ‘rotational’ gauge field $A_{\mu}^{ab}(x)$ and the ‘dilational’ gauge field $B_{\mu}(x)$ in a very different way, so that:

$$\Gamma_{\mu}^a = A_{\mu}^{ab} \Sigma_{ab} + (B_{\mu} - \frac{1}{3} T_{\mu}) \Delta, \quad (66)$$

in which $T_{\mu} = b^a T_a$, where $T_a \equiv T_{\mu}^{ab}$ is the trace of the PGT torsion, and we have introduced the modified

$$\begin{align*}
h_{\mu}^{ab}(x') &= X_{\mu}^{ab} e^{-\phi(x)} \Lambda_{\mu}^{ab}(x) h_{\mu}^{ab}(x), \\
A_{\mu}^{ab}(x') &= X_{\mu}^{ab} [A_{\mu}^{ab}(x) \Lambda_{\mu}^{ab}(x) + \Lambda_{\mu}^{ab}(x) \partial_{\mu} \Lambda_{\mu}^{ab}(x) + 2 \theta_{\mu}^{ab}(x) \partial_{\mu} \Lambda_{\mu}^{ab}(x)], \\
B_{\mu}(x') &= X_{\mu}^{ab} [B_{\mu}(x) - \theta Q_{\mu}(x)],
\end{align*} \quad (69)$$

where $X_{\mu}^{ab} \equiv \partial x_{\mu}^{ab} / \partial x'^{ab}$ are the elements of the GCT transformation matrix and $X_{\mu}^{ab} \equiv \partial x'^{ab} / \partial x_{\mu}^{ab}$ are the elements of its inverse. Hence, we have achieved our goal of accommodating the A-field transformation (61) under local dilations, while recovering the full transformation law in WGT for the special case $\theta = 0$. By contrast, the transformation law for $B_{\mu}$ reduces to that in WGT for the special case $\theta = 1$. Unlike the transformation

A-field

$$A_{\mu}^{ab} \equiv A_{\mu}^{ab} + (b^a_{\mu} B^b - b^b_{\mu} B^a), \quad (67)$$

where $B_{\mu} = h_{\mu}^{ab} B_a$. It is worth noting that $A_{\mu}^{ab}$ is not considered to be a fundamental field, but merely a shorthand for the above combination of the gauge fields $h_{\mu}^{ab}$ (or its inverse), $A_{\mu}^{ab}$ and $B_{\mu}$. Similarly, $T_{\mu}$ is merely a shorthand for the corresponding function of the gauge fields $h_{\mu}^{ab}$ (or its inverse) and $A_{\mu}^{ab}$.

It is straightforward to show that, if $\varphi$ has Weyl weight $w$, then (65) does indeed transform covariantly with Weyl weight $w-1$, as required, under the gauged (finite) action of $G$, such that:

$$D_{\mu}^{\nu} \varphi'(x') = e^{(w-1)\varphi(x)} \Lambda_{\mu}^{a}(x) S(\omega(x)) D_{\mu}^{\nu} \varphi(x), \quad (68)$$

given the gauge fields transform according to (69).

3 We denote the ‘dilational’ gauge field here by $B_{\mu}$ and also use a different sign convention in order to harmonise our notation with that of WGT. To recover the notation of our original paper (16), one should make the replacements $B_{\mu} \rightarrow -V_{\mu}$, $H_{\mu} \rightarrow -H_{\mu}$, $\zeta_{\mu} \rightarrow -\zeta_{\mu}$, and similarly for their counterparts carrying only Latin indices and/or daggers.

4 The derivative (65) does in fact transform covariantly as in (68) under the much wider class of gauge field transformations in which $\theta Q_{\mu}(x)$ is replaced in (69) by an arbitrary vector field $Y_{\mu}(x)$. Indeed, if one also makes this replacement in (61), one finds that the WGT (and PGT) matter actions for the massless Dirac field and the electromagnetic field are still invariant under local dilations, although the discussion regarding the transformation properties of $\mathcal{R}_{ab}^{cd}$ and $T_{ab}^{cd}$ following (62) requires appropriate modification. The covariance of $D_{\mu}^{\nu} \varphi$ under this wider class of transformations allows one to identify a further gauge symmetry of eWGT, namely under the simultaneous transformations $A_{\mu}^{ab} \rightarrow A_{\mu}^{ab} + b^a_{\mu} Y_{\mu}^{b} - b^b_{\mu} Y_{\mu}^{a}$ and $B_{\mu} \rightarrow B_{\mu} + Y_{\mu}$, where $Y_{\mu} = h_{\mu}^{ab} Y_{\mu}^{a}$ and $Y_{\mu}$ is an arbitrary vector field. Under this symmetry, both $A_{\mu}^{ab}$ and $B_{\mu} - \frac{1}{3} T_{\mu}$ remain unchanged and thus $D_{\mu}^{\nu} \varphi$ is invariant, as too are the eWGT field strengths and action. One may make use of this symmetry of eWGT to self-consistently choose a gauge in which either $B_{\mu}$ or $T_{\mu}$ is set to zero, which can considerably simplify subsequent calculations.

2. Field strengths

The eWGT gauge field strengths are defined in the usual way in terms of the commutator of the covariant derivatives. Considering first the eWGT $H$-covariant derivative, one finds that:

$$[D_{\mu}, D_{\nu}] \varphi = (\frac{1}{2} R_{\mu \nu}^{ab} \Sigma_{ab} + H_{\mu \nu}^{\mu} \Delta) \varphi, \quad (70)$$

which is of an analogous form to the corresponding result in WGT (which may be obtained from (43) by setting $f^a_{\mu} = 0$ and hence $S^{a \mu \nu} = 0$), but the eWGT field strengths have very different dependencies on the gauge
fields. In particular, one finds
\[
R^{ab}_{\mu\nu} \equiv 2(\partial_{\mu} A^{ab}_{\nu} + \eta_{\mu\nu} A^{ac}_{\mu} A^{db}_{\nu}),
\]
\[
H^1_{\mu\nu} \equiv 2\delta_{\mu}(B_{\nu} - \frac{1}{2}T_{\nu}),
\]  
(71)
both of which transform covariantly under GCT and local Lorentz rotations in accordance with their respective index structures, and are invariant under local dilations.

Considering next the commutator of two ‘generalised \(\mathcal{H}\)-covariant’ derivatives, one finds
\[
[D_a, D_d] \varphi = \left(\frac{1}{2} R^{ab}_{cd} \Sigma_{ab} + \mathcal{H}_{cd} \Delta - \mathcal{T}^{a}_{cd} D_d \varphi\right),
\]  
(72)
where \(R^{ab}_{cd} = h_c^{\mu} h_d^{\nu} R^{ab}_{\mu\nu}\) and \(\mathcal{H}_{cd} = h_c^{\mu} h_d^{\nu} H^1_{\mu\nu}\), and the translational field strength is given by
\[
\mathcal{T}^{a}_{bc} \equiv h_b^{\mu} h_c^{\nu} \mathcal{T}^{a}_{\mu\nu} \equiv 2h_b^{\mu} h_c^{\nu} D^1_{\mu\nu}. \]
(73)
We note that \(R^{ab}_{cd}\) and \(\mathcal{T}^{a}_{bc}\) are given in terms of their counterparts \(R^{ab}_{cd}\) and \(\mathcal{T}^{a}_{bc}\) in PGT by
\[
R^{ab}_{cd} = R^{ab}_{cd} + 4\delta^{[a}_{[c} [D^1_{d]} B^a] - 2B^2 \delta^{[a}_{[c} B^b] - 2B^2 \mathcal{T}^{b}_{cd},
\]
\[
\mathcal{T}^{a}_{bc} = T^{a}_{bc} + 4\delta^{[a}_{[c} T^1_{b]} d], \]
(74)
where \(B^2 = B^a B_a\) and for brevity we have introduced the derivative operator \(D_{a} \equiv h_a^{\mu} D_{\mu} \equiv h_a^{\mu}(\partial_{\mu} + \frac{1}{2} A^{ab}_{\mu} \Sigma_{ab})\) familiar from PGT. It is particularly important to note that the trace of the eWGT torsion vanishes identically, namely \(T^1_{a} \equiv T^1_{ba} = 0\), so that \(\mathcal{T}^{a}_{bc}\) is completely trace free (contraction on any pair of indices yields zero). \(R^{ab}_{cd}\), \(\mathcal{H}_{cd}\) and \(\mathcal{T}^{a}_{cd}\) are GCT scalars and transform covariantly under local Lorentz transformations and under local dilations with weights \(w(R^{ab}_{cd}) = w(\mathcal{H}_{cd}) = -2\) and \(w(\mathcal{T}^{a}_{cd}) = -1\) respectively.

3. Action

As in other gravitational gauge theories, the total action in eWGT consists typically of kinetic terms for any matter field(\(s\) \(\varphi\)), terms describing the coupling of the matter field(\(s\) to the gravitational gauge fields (and possibly to each other), and (kinetic) terms describing the dynamics of the free gravitational gauge fields.

Since \(D^1_{a} \varphi\) is constructed to have an analogous transformation law under extended local Weyl transformations to that of \(\partial_\mu \varphi\) under global Weyl transformations, one may immediately construct a matter action that is fully invariant under the extended gauged Weyl group from one that is invariant under global Weyl transformations by employing the usual minimal coupling procedure of replacing partial derivatives by covariant ones to obtain
\[
S_M = \int h^{-1} L_M(\varphi_i, D^1_{a} \varphi_i) d^4x.
\]
(75)

As mentioned previously, the set of fields \(\varphi_i\) may already include a scalar compensator field (denoted also by \(\phi\)) with Weyl weight \(w = -1\), for example in a Yukawa coupling term of the form \(\mu \varphi \bar{\psi} \psi\) with a massless Dirac field \(\psi\) (since this allows for the Dirac field to acquire a mass dynamically upon adopting the Einstein gauge \(\phi = \phi_0\) \([19, 24]\), together perhaps with kinetic and quartic potential terms for \(\phi\) of the form \(\nu D^1_{\alpha} \phi D^1_{\alpha} \phi - \lambda \phi^4\) (where \(\mu, \nu, \lambda\) are dimensionless parameters).

The terms in the total action that describe the dynamics of the free gravitational gauge fields are constructed from the gauge field strengths. In contrast to ACGT, the eWGT field strengths all transform covariantly under the full group of localised transformations, and so may be used straightforwardly to construct the free-gravitational action. The requirement of local scale invariance requires the free-gravitational Lagrangian \(L_G\) to be a relative scalar with Weyl weight \(w(L_G) = -4\), which may therefore contain an arbitrary linear combination \(L_R = \sum_{i} \lambda_i L_i\), of the six distinct terms quadratic in \(R^1_{abcd}\) and its contractions, and a term \(L_M = \frac{1}{2} \mathcal{H}_{ab} \mathcal{H}^{ab}\). In principle, one could also include quartic terms in \(\mathcal{T}^{1}_{abc}\) (which has no non-trivial contractions, unlike its counterparts in PGT, WGT and ACGT), or cross terms such as \(R^1_{[ab]} \mathcal{H}^{ab}\) — but these are not usually considered. Thus, one typically has
\[
S_G = \int h^{-1} (L_M + L_R) d^4x,
\]
(76)
where any parameters in the action are dimensionless.

In particular, \(L_M\) cannot contain the linear Einstein–Hilbert analogue term \(L_R \equiv -\frac{1}{2} R^1\) (where \(R^1 \equiv R^1_{abcd}\) and the factor of \(-1/2\) is conventional) or \(L_T \equiv \beta_1 T^1_{abc} T^{1abc} + \beta_2 T^1_{abc} T^{1bac}\). Nonetheless, such terms can be included in the total Lagrangian if they are multiplied by a compensator scalar field term \(\phi^2\) \([44]\). Such combinations are therefore usually considered not to belong to the free gravitational Lagrangian and are instead added to the matter Lagrangian \(L_M\) \([15]\). Thus, the matter Lagrangian may have an extended form, including all interactions of the matter fields with the gravitational gauge fields, which is given by \(L_M^+ \equiv L_M + \phi^2 (L_R + L_T)\) (in which the parameters \(\alpha, \beta\) are again dimensionless), such that the corresponding action has the functional dependences
\[
S_M = \int h^{-1} L_M^+(\varphi_i, D^1_{a} \varphi_i, R^1, \mathcal{T}^{1}_{abc}) d^4x,
\]
(77)
an action \(S_M\) that no longer satisfies other (required) invariance properties of the original one; indeed, this occurs for the Faraday action of the electromagnetic field, for which the minimal coupling procedure destroys electromagnetic gauge invariance. In such cases, the gauge action must be modified to restore the original (required) invariances, so that \(S_M\) does not have the form given above.

\(^5\) It should be noted, however, that this minimal coupling procedure can (as in other gravitational gauge theories \([15]\)) result in
where the set of fields $\varphi_i$ includes the scalar compensator. In any case, it is only the form of the total Lagrangian $L_T = L_M + L_G$ that is relevant for the field equations.

Finally, it is worth noting that terms containing covariant derivatives of (contracted) field strengths, such as $D^a_i D^b_i R^{ab}$ or $D^a_i D^b_i R$, are of Weyl weight $w = -4$ and so can, in principle, be included in $L_G$. In eWGT, however, such terms contribute only surface terms to the action, as a consequence of the trace $T^a_a$ of the eWGT torsion vanishing identically. Thus, such terms have no effect on the resulting field equations, and so may be omitted (at least classically); this is not true in general for other gauge theories, such as PGT, WGT and ACGT.

4. Field equations

The eWGT field equations are obtained by varying the total action $S_T$ with respect to the gravitational gauge fields $h^a_{\mu}$, $A^a_{\mu}$ and $B_\mu$, together with the matter fields $\varphi_i$ (which may include a scalar compensator field $\phi$). Defining $\tau^a_{\mu} \equiv \delta L_T/\delta h^a_{\mu}$, $\sigma_{ab}^\mu \equiv 2\delta L_T/\delta A^a_{\mu}$ and $\zeta^a \equiv \delta L_T/\delta B_\mu$, where $L_T \equiv h^{-1} L_T$, the set of gravitational field equations are most naturally expressed in terms of their counterparts carrying only Latin indices $\tau^a_{\mu} h^a_{\mu}$, $\sigma_{ab} \equiv \sigma_{ab}^\mu b_\mu$, and $\zeta^a \equiv \zeta^\mu b_\mu$, as

\[ \tau^a_{\mu} = 0, \]  
\[ \sigma_{ab} = 0, \]  
\[ \zeta^a = 0. \]  

(78a, 78b, 78c)

The quantities $\tau^a_{\mu}$, $\sigma_{ab}$ and $\zeta^a$ are clearly scalars under GCT, and it is straightforward to show that each of them also transforms covariantly under local Lorentz rotations and local dilations, as expected, with Weyl weights $w = 0$, $w = 1$ and $w = 1$ respectively. Moreover, with one exception, these transformation properties also hold for the corresponding quantities obtained from any subset of the terms in $L_T$ that transforms covariantly with weight $w = -4$ under local Lorentz rotations and local dilations (for example, $L_M$, $L_M + L_G$ or $L_G$ separately). The exception relates to quantities corresponding to $\tau^a_{\mu}$, which transform covariantly under local dilations only if one considers all the terms in $L_T$, and then only by virtue of the $A^a_{\mu}$ field equation $\tau^a_{\mu}$.

This unusual feature is a result of the extended transformation law $\delta A^a_{\mu} = D^a_i (\partial \varphi_i)_{ab} b_\mu$, which leads one to introduce the related quantities

\[ \tau^{ia}_{b} \equiv \tau^a_{\mu} - \sigma_{cb} D^c_{\mu} - \sigma^{ca} c B_\mu. \]  

(79)

These do transform covariantly when one considers only some subset of the terms in $L_T$ that themselves transform covariantly and with weight $w = -4$ under local Lorentz rotations and local dilations. It is therefore more convenient to replace $\tau^{ia}_{b}$ with the alternative field equation

\[ \tau^{ia}_{b} = 0. \]  

(80)

Indeed, this field equation emerges naturally if one adopts an alternative variational principle, in which $A^a_{\mu}$ is replaced by $A^{ia}_{b \mu}$ in the set of field variables; this approach also considerably shortens the calculations involved in deriving all the gravitational field equations.

Another unusual feature of the eWGT field equations, which also emerges most naturally from the alternative variational principle, is that for any total Lagrangian $L_T$ in which the gravitational gauge fields appear only through eWGT covariant derivatives or field strengths (which is usually the case), one may show that

\[ \zeta^a \equiv \sigma_{ab}^\mu b_\mu. \]  

(81)

Consequently, in this generic case, the $B$-field equation $\tau^{ia}_{b}$ is no longer independent, but merely the relevant contraction of the $A$-field equation $\tau^{ia}_{b}$. Moreover, the relation $\delta L_T/\delta \varphi_i \equiv \partial L_T/\partial (D^a_i)_{\mu} \delta \varphi_i$ also holds for the corresponding quantities obtained from any subset of the terms in $L_T$ that transforms covariantly with weight $w = -4$ under local Lorentz rotations and local dilations.

Finally, the remaining (matter) field equations are obtained by varying $S_T$ with respect to the fields $\varphi_i$ which may include a scalar compensator field $\phi$. In the (usual) case in which $L_T$ is a function of the matter fields only through $\varphi_i$ and $D^a_i \varphi_i$, these may be shown to have the simple (and manifestly covariant) forms

\[ \partial L_T/\partial \varphi_i \equiv \partial L_T/\partial (D^a_i \varphi_i) = 0, \]  

(82)

where $\partial L_T/\partial \varphi_i \equiv [\partial L_T(\varphi_i, D^a_i)_{\mu}/\partial \varphi_i]_{\mu} = 0$, so that $\varphi_i$ and $D^a_i \varphi_i$ are treated as independent variables.

5. Conservation laws

Invariance of $S_T$ under (infinitesimal) GCTs, local Lorentz rotations and extended local dilations, respectively, lead to conservation laws of the general form $\partial L_T/\partial \varphi_i$, as discussed in Appendix B. These can be written in the following manifestly covariant form:

\[ \partial L_T/\partial \varphi_i \equiv \partial L_T/\partial (D^a_i \varphi_i) = 0, \]  

(82)

The definition of $\sigma_{ab}^\mu$ used here differs by a factor of 2 from that used in our original paper [12]. In order to allow for a more straightforward comparison to be made with the canonical spin-

angular-momentum tensor. To recover the notation of our original paper [12], one should make the replacement $\sigma_{ab}^\mu \rightarrow 2 \sigma_{ab}^\mu$, and similarly for its counterpart carrying only Latin indices.
where we have defined the quantities $\zeta^{1a} \equiv \xi^{a} - \sigma^{ka} b_{k}$ and $H_{ab} = 2h_{a}^{\mu} h_{b}^{\nu} \partial_{[\mu} B_{\nu]}$, which are both easily verified to be GCT scalars and to transform covariantly under local Lorentz rotations and local dilations.

These conservation equations have a very different form to those in WGT. In particular, invariance of Lorentz rotations and local dilations. The third conservation law (83c) is unusual in being an algebraic condition on the trace $\delta L_{T}/\delta \psi_{i}$ and invariance laws. The third conservation law (83c) is unusual in being an algebraic condition on the trace $\delta L_{T}/\delta \psi_{i}$ and invariance laws.

Finally, it is worth noting that the conservation laws also hold for corresponding quantities obtained from any subset of terms in $L_{T}$ that is covariant under local Lorentz transformations and under local dilations with weight $w = -4$ (e.g. $L_{M}$, $L_{M^{+}}$ or $L_{G}$ separately).

B. Finite local conformal invariance

As we noted in Sections 11A and 11B for physical fields $\phi_{1}(x)$ that belong to irreducible representations of the Lorentz group, as is assumed in eWGT, the action (both infinitesimal and finite) of a general element of the conformal group that is connected to the identity corresponds to a combination of a translation, (proper) Lorentz rotation and dilation; in particular, a SCT corresponds merely to a Lorentz rotation and dilation that depend on spacetime position $x$ in a prescribed way. Since translations, (restricted) Lorentz rotations and dilations are already gauged in eWGT and WGT, then so too are SCTs and hence any element of the conformal group that is connected to the identity.

As discussed in Section 11B however, the full conformal group also includes the inversion operation (13), which is finite and discrete, and hence not connected to the identity. Moreover, it is worth recalling that a SCT is merely the composition of an inversion, a translation and a second inversion. In Section 11B we demonstrated that an inversion, together with its action on physical fields that belong to an irreducible representation of the Lorentz group, consists of the composition of a dilation $1/x^{2}$ and a reflection $I^{\mu}_{\nu}(\hat{x})$ in the hyperplane perpendicular to $\hat{x}$, both of which are clearly position dependent in a prescribed way. In particular, under an inversion, physical fields are acted upon by $I^{\mu}_{\nu}(\hat{x})$ for each tensor index and by $\gamma \cdot \hat{x}$ for each 4-spinor index.

Since dilations are already gauged in eWGT and WGT, the only new operation to consider is the reflection. To our knowledge, the gauging of reflections has not been addressed previously, but the most natural approach is to generalise the reflection in the hyperplane perpendicular to $\hat{x}$ at each point to a reflection in the hyperplane perpendicular to some unit vector $n(x)$ that can vary arbitrarily with spacetime position $x$. As usual, this gauged transformation should be completely decoupled from GCTs, and so we denote the reflection matrix at each spacetime point by $I^{a}_{b}(n(x))$, which operates on each Latin tensor index carried by a field (or, equivalently, $\gamma \cdot n(x)$ for each spinor index).

From the discussion in Section 11B however, $I^{a}_{b}(n(x))$ corresponds to a finite improper Lorentz transformation matrix at each spacetime point. Thus eWGT already accommodates gauged reflections, without the need to introduce any more gauge fields, provided that each occurrence of the restricted Lorentz transformation matrix $A^{a}_{b}(x)$ in the finite transformation laws (83) and (99) for the covariant derivative and the existing gauge fields, respectively, is extended to denote a general transformation matrix of the full Lorentz group (which consists of proper Lorentz rotations and spacetime reflections) and, in particular, is given by $I^{a}_{b}(n(x))$ under gauged reflections. Indeed, the same holds true for WGT, for which the finite transformation laws of the gauge fields are given by (69), with $\theta = 0$ in (69b) and $\theta = 1$ in (69c).

Thus, provided all matter fields $\phi_{1}(x)$ are assumed to belong to irreducible representations of the Lorentz group and with the above modest extension to the transformation laws of the gauge fields, both WGT and eWGT accommodate all the gauged symmetries of the full conformal group, such that actions constructed in the usual way in each theory are invariant under (finite) local conformal transformations. As we now demonstrate below, however, WGT cannot be considered as a true gauge theory of the conformal group in the usual sense, whereas eWGT can be interpreted as such.
C. Local conformal conservation laws

As discussed in Section II C, if one considers a field theory in Minkowski spacetime that describes the dynamics of a set of fields \( \phi_i(x) \) that belong to irreducible representations of the Lorentz group, then for the action (24) to be invariant under global conformal transformations (that are connected to the identity), one requires the first three conservation laws in (29) to hold ‘on-shell’ (which together ensure Poincaré and scale invariance) and the field virial (30) to vanish (which ensures the additional invariance under SCTs), up to a total divergence.

We now consider in more detail the forms of the conservation laws in WGT and eWGT, both of which we have just demonstrated have actions that are invariant under local conformal transformations. As we will see, eWGT has very different conservation laws to WGT,

\[
(D^*_c + T^*_c)(h\tau^c_{\alpha}) + h(\tau^c_{\beta}T^b_{\alpha\beta} - \frac{1}{2}\sigma_{ab}R_{cd} - \xi^c H_{cd}) = 0, \quad (84a)
\]

\[
(D^*_c + T^*_c)(h\sigma_{ab}^c) + 2h\tau_{ab} = 0, \quad (84b)
\]

\[
(D^*_c + T^*_c)(h\zeta^c) - h\tau^c = 0, \quad (84c)
\]

where \( \tau^a_{\mu} \equiv \delta L_{M^\mu}/\delta a^\mu \), \( \sigma_{ab}^c \equiv 2\delta L_{M^c}/\delta A^{ab} \) and \( \zeta^c \equiv \delta L_{M^c}/\delta B_{\mu} \) (in which \( L_{M^c} \equiv h^{-1}L_{M^\mu} \)), and their counterparts carrying only Latin indices \( \tau^a_{\mu} \equiv \tau^a_{\mu}h^\mu_{\beta} \), \( \sigma_{ab}^c \equiv \sigma_{ab}^c h^\mu_{\beta} \) and \( \zeta^c \equiv \zeta^c h_{\beta}^\mu \) are most naturally considered as the (total) dynamical energy-momentum, spin-angular-momentum and dilaton current, respectively, of the matter fields. The above conservation laws are clearly invariant under GCTs and transform covariantly under the local action of the subgroup \( H \) of homogeneous Weyl transformations, as expected.

The conservation laws (34) provide a natural generalisation for localised Weyl transformations of the first three conservation laws in (29). There is not, however, any further conservation law corresponding to the generalisation of the condition that the field virial (30) should vanish up to a total divergence, which was necessary to ensure that the original action (24) be invariant under SCTs, in addition to global Weyl transformations, and hence invariant under global conformal transformations (connected to the identity). The absence of such a further conservation law in WGT demonstrates that it does not constitute a gauge theory of the conformal group in the usual sense.

One should note, however, that the quantities in (34) are dynamical currents, whereas those in (29) are canonical. It is therefore of interest to compare the forms of these two types of current. This comparison is facilitated by first separating the contributions to the dynamical matter currents resulting from each of the terms in \( L_{M^\mu} = L_M + \phi^2 R + \phi^2 T_{\mu\nu} \), which we denote by

\[
\tau^a_{\beta} = (\tau_M)^a_{\beta} + (\tau_R)^a_{\beta} + (\tau_T)^a_{\beta}, \quad \text{and similarly for } \sigma_{ab}^c \text{ and } \zeta^c.
\]

We then introduce the following covariant canonical currents of the matter fields (25)

\[
t^{\alpha}_{\beta} \equiv \frac{\partial L_M}{\partial (D^*_a \phi^i)} D^*_b \phi^i - \delta^a_{b} L_M, \quad (85a)
\]

\[
\sigma^{\alpha}_{\beta} \equiv \frac{\partial L_M}{\partial (D^*_a \phi^i)} \sigma_{ab}^c \phi^i, \quad (85b)
\]

\[
\zeta^{\alpha} \equiv \frac{\partial L_M}{\partial (D^*_a \phi^i)} \zeta^c \phi^i, \quad (85c)
\]

which provide a natural generalisation of the standard canonical currents \( t^{a}_{\beta}, \sigma^{a}_{\beta} \), and \( j^\mu \) in (27) and (28). By considering the form of the WGT covariant derivative \( D^*_a \phi^i \), one may show directly that the covariant canonical currents and the dynamical currents derived from \( L_M \) alone are essentially equivalent in WGT, since \( h(\tau_M)^a_{\beta} \equiv t^{a}_{\beta}, h(\sigma_M)^a_{\beta} \equiv \sigma^{a}_{\beta} \), and \( h(\zeta_M)^a \equiv j^a \). These equivalences may also be derived by demanding the coincidence of the currents \( J^\mu \) and \( S^\mu \) derived from \( L_M \), which are discussed in Appendix B 2.

2. eWGT

Let us now repeat the above analysis for a matter action of the form \( S_M = \int h^{-1}L_{M^\mu}(\phi, D^*_a \phi^i, R^i, T^a_{abc}) \, d^4x \) in eWGT, as given in (77). In this case, from the combination of (81) and (83) (applied only to \( S_M \) and assuming all the matter equations of motion to hold), one instead
obtains the conditions

\[ D_c^i (h \tau^{ic} \phi) + h (\tau^{ic} \xi^{b} D_{bc}^i - \frac{1}{2} \sigma_{ab}^{\phi} R^{ic \phi}_{ab}) = 0, \quad (86a) \]
\[ D_c^i (h \sigma^{ic} \phi) + 2 h \tau_{ab}^{\phi} = 0, \quad (86b) \]
\[ h \tau^{ic} \phi = 0, \quad (86c) \]
\[ h (\zeta^a - \sigma^{ab} b) = 0, \quad (86d) \]

where \( \tau^{ic} \phi \) is defined in (79). The conditions (86) have a somewhat different form from their WGT counterparts in (34). In particular, (86) shows that the trace of the modified dynamical energy-momentum tensor vanishes. This is reminiscent of the vanishing trace of the improved energy-momentum tensor (22), which encodes the invariance of theories under global scale transformations. In (86), however, one has not used the Belinfante procedure to combine the translational and rotational currents, but instead retained the distinction between them. Thus, \( \tau^{ic} \phi \) remains non-symmetric, which is appropriate when working in terms of the tetrad rather than the metric, and also allows one straightforwardly to accommodate torsion. Most important in eWGT, however, is the additional final condition (86d), which is analogous to a covariant generalisation of the condition that the field virial should vanish. Thus the eWGT conservation laws provide a natural local generalisation of all of the usual conservation laws (29) for theories that are invariant under global conformal transformations (and contain only fields that belong to irreducible representations of the Lorentz group).

As was the case in our consideration of the WGT conservation laws, however, it is also of interest to consider the relationship between the dynamical currents in (86) and their canonical counterparts. This comparison is again facilitated by first separating the contributions to the dynamical matter currents resulting from each of the terms in \( L_M^+ = L_M + \delta^2 L_{\mathcal{R}_1} + \delta^2 L_{\mathcal{T}^{12}}, \) in a similar manner to that used for WGT. We also define a set of eWGT covariant canonical currents \( t^{ia} \), \( s^{ic} \), and \( j^{ia} \) in an analogous manner to their WGT counterparts in (86), but with each occurrence of the WGT covariant derivative \( D_a^i \varphi_i \) replaced by the eWGT covariant derivative \( D_a^i \varphi_i \). By considering the form of the latter, one may again directly relate the dynamical and covariant canonical currents, but in eWGT these relationships are somewhat more complicated than those in WGT. In particular, one finds (after a lengthy calculation) that

\[ h (\tau_M)^{i a} = t^{ia} \phi + \frac{1}{1} (D_{ab}^i \varphi_i - \delta_b^a D_{ab}^i \varphi_i), \quad (87a) \]
\[ h (\sigma_M)_a^c = s^{ic} \phi + \frac{1}{3} (\delta_b^a \varphi_i - \delta_b^i \varphi_i), \quad (87b) \]
\[ h (\zeta_M)^a = j^{ia} \phi - \delta^{ia} \phi, \quad (87c) \]

Once again, these equivalences may also be derived by demanding the coincidence of the currents \( J^a \) and \( S^a \) as derived from \( L_M \), as discussed in Appendix [22]. Substituting the above expressions into (86), for the restricted case in which \( L_M \) is the full matter Lagrangian density, yields the covariant canonical conservation laws

\[ D_c^i (t^{ic} \phi) + t^{ic} \phi D_{bc}^i - \frac{1}{2} t^{ic} \phi R^{ic \phi}_{ab} = 0, \quad (88a) \]
\[ D_c^i (s^{ic} \phi) + 2 t^{ic} \phi = 0, \quad (88b) \]
\[ D_c^i j^{ic} - t^{ic} \phi = 0, \quad (88c) \]

and the final condition (86d) is satisfied identically.

The expressions (86) clearly represent a natural local generalisation of the first three conservation laws in (29). Moreover, one sees from (86d) that, provided \( (\zeta_M)^a \) vanishes up to a total divergence, then so too should \( s^{ic} \phi - j^{ia} \phi \), which provides a replacement additional condition that is a natural generalisation of the analogous requirement on the field virial (30) for globally conformal invariant theories. This requirement is indeed satisfied, not only by \( (\zeta_M)^a \) but also by \( \zeta^a \) evaluated from the extended matter Lagrangian density \( L_{M^+} \), which is given by

\[ h \zeta^a = h (\zeta_M)^a - \frac{1}{2} (\nu + 3 a) D^{ia} \phi^2, \quad (89) \]

provided the terms in \( L_{M^+} \) corresponding to the non-compensator matter fields \( \varphi_i \) do not contain the dilaton gauge field \( B_\mu \). This occurs naturally if the \( \varphi_i \) correspond to the Dirac field and/or the electromagnetic field [19]. Thus, in terms of the covariant canonical currents, the eWGT conservation laws once again provide a natural local generalisation of all of the usual conservation laws (29) for theories that are invariant under global conformal transformations.

### D. Ungauging eWGT

In Section III C we considered the process of ‘ungauging’ ACGT, and obtained the correct limit of global conformal transformations. We also considered ‘ungauging’ WGT and found the limit to correspond to global Weyl transformations, which again shows that WGT cannot be considered as a true gauge theory of the conformal group. In this section, we consider the ‘ungauged’ limit of eWGT.

One begins by requiring the field-strength tensors \( R^{ab \phi}_{cd}, \quad H^a_{\mu} \) and \( T^{ia \phi}_{cd} \) to vanish in the ‘ungauged’ limit. Similarly to WGT, in this limit, the coordinate system and the H-gauge can be chosen such that

\[ h_a^\mu (x) = \delta_a^\mu, \quad A_{\mu} (x) = 0, \quad B_\mu (x) = 0. \quad (90) \]

It is worth noting that, in this reference system, the eWGT covariant derivative reduces to a partial derivative. Thus, the eWGT covariant canonical currents \( t^{ia} \phi, \quad s^{ic} \phi \), and \( j^{ia} \phi \) reduce to the standard ones in (27) and (28), and the conditions (86) and (87) reduce, respectively, to the first three conservation laws in (29) and the vanishing of the field virial (30) up to a total divergence. Indeed, we note further that all these reductions also occur under the less restrictive set of conditions \( h_a^\mu (x) = \delta_a^\mu \) and
As previously, the condition $\delta_0 h_{a}^{\mu} = 0$ guarantees that the variation in the field-strength tensors vanishes if the variation in the non-calligraphic counterparts does so. The transformation laws of the latter (assuming $\delta_0 h_{a}^{\mu} = 0$) are given very simply in terms of the transformations of the quantities $A_{ab}^{\mu}$ by

$$
\begin{align}
\delta_0 R_{\mu \nu}^{ab} &= 2 \partial_{[\mu} \delta_0 A_{\nu]}^{ab}, \\
\delta_0 H_{\mu \nu}^{\gamma} &= \frac{2}{3} \delta_0 \partial_{[\mu} A_{\nu]}^{\gamma}, \\
\delta_0 T_{\mu \nu}^{a} &= 2 \delta_0 A_{[\mu}^{a} \partial_{\nu]}^{b} - \frac{2}{3} \delta_0 \partial_{[\mu} A_{\nu]}^{b}.
\end{align}
$$

Since these expressions depend solely on $\delta_0 A_{ab}^{\mu}$, our procedure is equivalent to demanding only that the variation in each field-strength tensor vanishes, but this is satisfied by construction. Alternatively, one may show this directly by making use of the transformation law $\delta_0 A_{ab}^{\mu} = -2\partial_{[\mu} A_{\nu]}^{a b}$, which may be derived by taking the infinitesimal limits of (69) and assuming $\delta_0 h_{a}^{\mu} = 0$. Substituting this form for $\delta_0 A_{ab}^{\mu}$ into (92), one finds that (92a) and (92c) vanish identically, and (92b) vanishes by virtue of the condition (53). Hence no further conditions apply to this solution, which is sufficient to show that the ‘ungauged’ limit of eWGT corresponds to global conformal transformations; this differs markedly from WGT, for which we showed in Section III.C that the ‘ungauged’ limit corresponds to global Weyl transformations.

It is worth noting that under the global conformal transformation (4), the second and third conditions in (90) are not preserved. Indeed, one finds

$$
\begin{align}
\delta_0 A^{ab} &\equiv 4(1 - \theta)\delta_0 A^{ab} &= 4 \theta c_{\mu},
\end{align}
$$

which in turn lead to $\delta_0 A_{ab}^{\mu} = 4 \delta_0 \partial_{[\mu} A_{\nu]}^{a b}$. Thus, as anticipated in Section III.C although applying our ‘ungauging’ approach to WGT is equivalent to requiring that all the conditions in (90) are preserved, this equivalence does not hold when it is applied to eWGT. Indeed, as is clear from (93), the latter requirement in eWGT would lead to the condition $c_{\mu} = 0$, which corresponds to a global Weyl transformation.

2. Finite transformations

We may extend our discussion to finite transformations, which also include inversions. Starting again from the relations (90), one should demand that under subsequent finite GCT and $\mathcal{H}$-gauge transformations that preserve the first relation (such that $\delta_0 h_{a}^{\mu} = 0$ and so ensuring the equivalence of Latin and Greek indices before and after the transformation), all ‘subsidary’ field-strength tensors remain zero. The infinitesimal form of the eWGT transformation law for $h_{a}^{\mu}$ is easily obtained from its finite form in (69), and is identical to that given in (39) for ACGT (and WGT). Thus, following the argument given in Section III.C, the most general solution for $\xi^{a}(x)$ has the form (4) of an infinitesimal global conformal transformation. One now has to check, however, if any further conditions apply to this solution by our requirement on the behaviour of the ‘subsidiary’ field-strength tensors.

By analogy with the discussion in Section III.C, a straightforward way of imposing this requirement is to demand that, under subsequent GCT and $\mathcal{H}$-gauge transformations satisfying $\delta_0 h_{a}^{\mu} = 0$, the change in each ‘full’ field strength $\mathcal{R}^{ab}_{\mu \nu}, H_{\mu \nu}^{a},$ and $T^{a}_{\mu \nu}$ arising from the change in either combination of gauge fields $A_{ab}^{\mu}$ or $B_{ab}^{\mu} - \frac{1}{3} T_{\mu}$ should vanish separately. Unlike the cases considered in Section III.C, however, the quantities $A_{ab}^{\mu}$ and $B_{ab}^{\mu} - \frac{1}{3} T_{\mu}$ are not independent. Indeed, starting from (69) and assuming $\delta_0 h_{a}^{\mu} = 0$, it is easily shown that $
abla_{\mu}(B_{ab}^{\mu} - \frac{1}{3} T_{\mu}) = \frac{1}{3} \delta_0 A_{ab}^{\mu}$. Thus, one need consider only the changes in $\mathcal{R}^{ab}_{\mu \nu}, H_{\mu \nu}^{a},$ and $T^{a}_{\mu \nu}$ arising from the change in the combination $A_{ab}^{\mu}$ of the gauge fields.

As previously, the condition $\delta_0 h_{a}^{\mu} = 0$ guarantees that the variation in the field-strength tensors vanishes if the variation in their non-calligraphic counterparts does so. The transformation laws of the latter (assuming $\delta_0 h_{a}^{\mu} = 0$) are given very simply in terms of the transformations of the quantities $A_{ab}^{\mu}$ by

$$
\begin{align}
\delta_0 R_{\mu \nu}^{ab} &= 2 \partial_{[\mu} \delta_0 A_{\nu]}^{ab}, \\
\delta_0 H_{\mu \nu}^{\gamma} &= \frac{2}{3} \delta_0 \partial_{[\mu} A_{\nu]}^{\gamma}, \\
\delta_0 T_{\mu \nu}^{a} &= 2 \delta_0 A_{[\mu}^{a} \partial_{\nu]}^{b} - \frac{2}{3} \delta_0 \partial_{[\mu} A_{\nu]}^{b}.
\end{align}
$$

Since these expressions depend solely on $\delta_0 A_{ab}^{\mu}$, our procedure is equivalent to demanding only that the variation in each field-strength tensor vanishes, but this is satisfied by construction. Alternatively, one may show this directly by making use of the transformation law $\delta_0 A_{ab}^{\mu} = -2\partial_{[\mu} A_{\nu]}^{a b}$, which may be derived by taking the infinitesimal limits of (69) and assuming $\delta_0 h_{a}^{\mu} = 0$. Substituting this form for $\delta_0 A_{ab}^{\mu}$ into (92), one finds that (92a) and (92c) vanish identically, and (92b) vanishes by virtue of the condition (53). Hence no further conditions apply to this solution, which is sufficient to show that the ‘ungauged’ limit of eWGT corresponds to global conformal transformations; this differs markedly from WGT, for which we showed in Section III.C that the ‘ungauged’ limit corresponds to global Weyl transformations.

It is worth noting that under the global conformal transformation (4), the second and third conditions in (90) are not preserved. Indeed, one finds

$$
\begin{align}
\delta_0 A^{ab} &\equiv 4(1 - \theta)\delta_0 A^{ab} &= 4 \theta c_{\mu},
\end{align}
$$

which in turn lead to $\delta_0 A_{ab}^{\mu} = 4 \delta_0 \partial_{[\mu} A_{\nu]}^{a b}$. Thus, as anticipated in Section III.C although applying our ‘ungauging’ approach to WGT is equivalent to requiring that all the conditions in (90) are preserved, this equivalence does not hold when it is applied to eWGT. Indeed, as is clear from (93), the latter requirement in eWGT would lead to the condition $c_{\mu} = 0$, which corresponds to a global Weyl transformation.
conditions apply arising from our requirement on the behaviour of the ‘subsidiary’ field-strength tensors.

By analogy with the infinitesimal case, we demand that, under subsequent GCT and $H$-gauge transformations satisfying $h^\nu_{\alpha}(x') = \delta^\nu_\alpha$, the change in each ‘full’ field strength $R^{l_{ab}}_{\mu\nu}$, $H^l_{\mu\nu}$ and $T^{l_{ab}}_{\mu\nu}$ arising from the change in either combination of gauge fields $A^{l_{ab}}_{\mu\nu}$ or $B_\mu - \frac{1}{4} T_\mu$ should vanish separately. Starting from (67) and assuming $h^\nu_{\alpha}(x') = \delta^\nu_\alpha$, one may show that $B'_\mu(x') - \frac{1}{4} T'_\mu(x') = \frac{1}{2} A^{l_{ab}}_{\mu\nu}(x')$, and so again one need only consider changes in $R^{l_{ab}}_{\mu\nu}$, $H^l_{\mu\nu}$ and $T^{l_{ab}}_{\mu\nu}$ arising from the change in the combination $A^{l_{ab}}_{\mu\nu}$ of the gauge fields.

The condition $h^\nu_{\alpha}(x') = \delta^\nu_\alpha$ guarantees that the transformed field-strength tensors vanish if their non-colligraphic counterparts do so. The transformation laws of the latter (assuming $h^\nu_{\alpha}(x') = \delta^\nu_\alpha$) are given in terms of $A^{l_{ab}}_{\mu\nu}(x')$ by

\begin{align}
R^{l_{ab}}_{\mu\nu} &= 2 \partial^\nu [A^{l_{ab}}_{\mu\nu}] + 2 A^{l_{ab}}_{\epsilon}[\epsilon_{\mu\nu}] A^{l_{ab}}_{\epsilon\nu} \tag{94a}, \\
H^l_{\mu\nu} &= \frac{2}{3} \partial^\nu A^{l_{ab}}_{\epsilon\nu} b_{\epsilon\mu} \tag{94b}, \\
T^{l_{ab}}_{\mu\nu} &= 2 A^{l_{ab}}_{\epsilon[b\delta\epsilon\mu]} - \frac{2}{3} \delta^\nu_{\epsilon} A^{l_{ab}}_{\epsilon\nu} b_{\epsilon\mu} \tag{94c}.
\end{align}

As in the infinitesimal case, since these expressions depend solely on $A^{l_{ab}}_{\mu\nu}$, our procedure is equivalent to demanding only that each transformed field-strength tensor vanishes, but this is again satisfied by construction. Alternatively, one may show this directly by making use of the transformation law $A^{l_{ab}}_{\mu\nu} = -2 \delta^a_{\mu} \delta^b_{\nu}$, which may be derived from (69) with the assumption $h^\nu_{\alpha}(x') = \delta^\nu_\alpha$. Substituting this form for $A^{l_{ab}}_{\mu\nu}$ into (21) one finds that (24) and (25) vanish identically, and (21a) vanishes by virtue of the condition (71b). Hence no further conditions apply to the solution, so that the ‘ungauged’ limit of eWGT corresponds to finite global conformal transformations, including inversions.

V. CONCLUSIONS

We have reconsidered the process of gauging of the conformal group and the resulting construction of gravitational gauge theories that are invariant under local conformal transformations. The standard approach leads to auxiliary conformal gauge theories (ACGT), so called because they suffer from the problem that the gauge field corresponding to special conformal transformations can be eliminated from the theory using its own equation of motion, so that the symmetry appears to reduce back to the local Weyl group. Such theoretical difficulties with AGCT have led to the development of an alternative bi-conformal gauging and the construction of its associated biconformal gauge field theories (BCGT). Although these theories possess some very interesting and promising properties, their physical interpretation is complicated by their requirement of an eight-dimensional base manifold. Thus, the role played by local conformal invariance in gravitational gauge theories remains uncertain.

We have therefore revisited the recently proposed extended Weyl gauge theory (eWGT), which was previously noted to implement Weyl scaling in a novel way that may be related to gauging of the full conformal group. We demonstrated this relationship here by first showing that, provided any physical matter fields belong to an irreducible representation of the Lorentz group, eWGT is indeed invariant under the full set of (finite) local conformal transformations, including inversions. This property is, however, also shared by standard WGT, as might be expected from the theoretical shortcomings of ACGT. Nonetheless, we also showed that eWGT has two further properties not shared by WGT. First, the conservation laws of eWGT provide a natural local generalisation of those satisfied by field theories with global conformal invariance, in particular that the field virial should vanish; this is the key criterion for an action to be invariant under SCTs, in addition to the remainder of the global conformal group (connected to the identity). Second, we showed that the ‘ungauged’ limit of eWGT corresponds to global conformal transformations, rather than global Weyl transformations. These findings suggest that eWGT can be regarded as a valid alternative gauge theory of the conformal group, despite not having been derived by direct consideration of the localisation of its group parameters. Therefore, eWGT might be considered as a ‘concealed’ conformal gauge theory (CCGT).

ACKNOWLEDGMENTS

The authors thank Will Barker for useful comments.

Appendix A: Direct derivation of finite global conformal transformations

For a finite coordinate transformation $x'^\mu = f^\mu(x)$ in $n$-dimensional Minkowski spacetime to satisfy the defining condition (1) to be conformal, one immediately requires

\begin{equation}
(\partial_\alpha f_\gamma)(\partial_\beta f_\gamma) = \frac{1}{4}(\partial_\gamma f_\mu)(\partial^\mu f_\gamma)\eta_{\alpha\beta},
\end{equation}

where $(\partial_\gamma f_\mu)(\partial^\mu f_\gamma) = n\Omega^2$ and $\partial_\gamma f_\mu = \partial x'^\mu/\partial x^\gamma$ is the coordinate transformation matrix. Equation (A1) is the finite version of the conformal Killing equation (2), to which it reduces in the infinitesimal limit $x'^\mu \approx x^\mu + \xi^\mu(x)$.

Acting on (A1) with $\partial_\lambda$, cyclically permuting the indices $\lambda$, $\alpha$ and $\beta$ to obtain two further equivalent equations and subtracting the first equation from the sum of the other two, one obtains

\begin{equation}
2(\partial_\alpha \partial_\beta f_\gamma)(\partial_\lambda f_\gamma) = \frac{1}{4}(\eta_{\lambda\alpha}\partial_\beta + \eta_{\beta\lambda}\partial_\alpha - \eta_{\alpha\beta}\partial_\lambda)\Omega^2,
\end{equation}

of which we will make use shortly.
Another useful equation may be obtained by first acting on \( \text{eq. (A1)} \) with \( \partial^\beta \), then acting on the resulting equation with \( \partial_\beta \) and finally using \( \partial_\lambda \) again. This yields

\[
[n_\alpha \beta \Box^2 + (n-2)\partial_\alpha \partial_\beta] \Omega^2 + 2(\partial_\alpha \partial_\beta f^\gamma) \Box^2 f_\gamma - 2(\partial_\alpha \partial_\lambda f^\gamma)(\partial_\beta \partial_\lambda f_\gamma) = 0, \tag{A3}
\]

\[
(n-1)(n-2)\partial_\alpha \partial_\beta \Omega^2 + 2(n-1)[(\partial_\alpha \partial_\beta f^\gamma) \Box^2 f_\gamma - (\partial_\alpha \partial_\lambda f^\gamma)(\partial_\beta \partial_\lambda f_\gamma)] - \eta_{\alpha \beta} [(\Box^2 f^\gamma)(\Box^2 f_\gamma) - (\partial_\alpha \partial_\lambda f^\gamma)(\partial_\beta \partial_\lambda f_\gamma)] = 0, \tag{A4}
\]

As discussed in Section 1113, one may write the transformation matrix of a smooth conformal transformation in the form 116, such that

\[
\partial_\nu f^\mu = \Omega(x) \Lambda^\mu_{\nu}(x), \tag{A6}
\]

where \( \Lambda^\mu_{\nu}(x) \) is, in general, a position-dependent Lorentz rotation matrix (either proper or improper). First, substituting \( \text{eq. (A6)} \) into \( \text{eq. (A2)} \), one finds that \( \Omega(x) \) and \( \Lambda^\mu_{\nu}(x) \) from which one may derive two further useful equations. First, contracting \( \text{eq. (A3)} \) with \( \eta^{\alpha \beta} \), one obtains

\[
(n-1)\Box^2 \Omega^2 + (\Box^2 f^\gamma)(\Box^2 f_\gamma) - (\partial_\alpha \partial_\beta f^\gamma)(\partial_\lambda \partial_\beta f_\lambda) = 0. \tag{A5}
\]

must satisfy the relation

\[
\partial_\mu \Lambda^\gamma_{\mu} = (\Lambda^{\gamma}_{\mu} \partial_\beta - \eta_{\mu \beta} \Lambda^{\gamma}_{\lambda} \partial_\lambda) \ln \Omega, \tag{A7}
\]

from which one may straightforwardly obtain the result \( \text{eq. (17a)} \), namely

\[
\Lambda^{\gamma}_{\alpha} \partial_\mu \Lambda^{\gamma}_{\beta} - 2\eta_{\mu \beta} \partial_\beta \ln \Omega = 0. \tag{A8}
\]

Then, substituting \( \text{eq. (A6)} \) into \( \text{eq. (A1)} \) and \( \text{eq. (A5)} \), respectively, and using the result \( \text{eq. (A7)} \), one finds

\[
(n-1)2\Omega \Box^2 \Omega + (n-4)(\partial_\gamma \Omega)(\partial^\gamma \Omega) = 0, \tag{A9}
\]

\[
(n-1)(n-2)[2\Omega \partial_\alpha \partial_\beta \Omega + \eta_{\alpha \beta} (\partial_\gamma \Omega)(\partial^\gamma \Omega) - 4(\partial_\alpha \Omega)(\partial_\beta \Omega)] = 0, \tag{A10}
\]

where the second result matches \( \text{eq. (17b)} \) for \( n \geq 3 \).

In order to solve \( \text{eqs. (A8), (A10)} \) for \( \Omega \) and \( \Lambda^\mu_{\nu} \), it is in fact more convenient to work in terms of the reciprocal dilation \( \sigma \equiv 1/\Omega \), for which \( \text{eqs. (A8), (A10)} \) become

\[
\Lambda^{\gamma}_{\alpha} \partial_\mu \Lambda^{\gamma}_{\beta} + 2\delta^{\gamma}_{\mu} \delta^{\beta}_{\beta} \ln \sigma = 0, \tag{A11}
\]

\[
(n-1)(n-2)(\partial_\alpha \partial_\beta \sigma)(\partial^\gamma \sigma) - 2\sigma \Box^2 \sigma = 0, \tag{A12}
\]

Assuming \( n \geq 3 \), \( \text{eq. (A13)} \) immediately implies that \( \partial_\alpha \partial_\beta \sigma = 0 \) for \( \alpha \neq \beta \). One thus requires \( \sigma(x) = q_\mu(x^\mu) \), i.e. the sum of \( n \) functions, each of which is a function only of the corresponding coordinate (and possibly a constant). Substituting this form back into \( \text{eq. (A13)} \) with \( \alpha = \beta \) and adopting the signature \( \eta_{\alpha \beta} = \text{diag}(1, -1, \ldots, -1) \), one finds that

\[
q''_{\mu} = -q''_{\mu} = \frac{1}{2} \sigma(\partial_\gamma \ln \sigma)(\partial^\gamma \ln \sigma), \tag{A14}
\]

where \( \text{prime denotes differentiation with respect to the function argument and the index} \ i \text{ runs from} 1 \text{ to} n-1 \). Thus, up to a sign, each \( q''_{\mu} \) must be equal to the same constant. Consequently, \( \sigma(x) \) must have the form

\[
\sigma(x) = a + 2c_\mu x^\mu + bx^2, \tag{A15}
\]

where \( a, b \) and \( c_\mu \) are constants. Finally, substituting this form back into \( \text{eq. (A14)} \) or \( \text{eq. (A12)} \), yields the condition that \( ab = c^2 \).

Let us first assume that the vector \( c_\mu \) is non-null. In this case, there are three non-trivial possibilities:

(i) \( a \neq 0 \) and \( b = 0 = c_\mu \), so that \( \sigma = a \);  

(ii) \( b \neq 0 \) and \( a = 0 = c_\mu \), so that \( \sigma = bx^2 \);  

(iii) \( a, b \) and at least one component of \( c_\mu \) are non-zero, so that \( \sigma = a(1 + 2c_\mu x^\mu + x^2) \), where \( c_\mu \equiv c_\mu/a \).

Turning then to the case where the vector \( c_\mu \) is non-zero but null, one requires at least one of \( a \) and \( b \) to be zero. Hence, there are three further possibilities:

(iv) \( a \neq 0 \) and \( b = 0 \), so that \( \sigma = a + 2c_\mu x^\mu \);  

(v) \( a = 0 \) and \( b \neq 0 \), so that \( \sigma = 2c_\mu x^\mu + bx^2 \);  

(vi) \( a = 0 = b \), so that \( \sigma = 2c_\mu x^\mu \).

For each of the above possible forms for the reciprocal scale factor \( \sigma(x) \), one may now use \( \text{eq. (A11)} \) to determine the form of the corresponding (proper or improper) Lorentz transformation \( \Lambda^\mu_{\nu}(x) \), and hence the full transformation matrix \( X^\mu_{\nu}(x) = [1/\sigma(x)] \Lambda^\mu_{\nu}(x) \) in each case. Before proceeding, however, it should be noted that \( \text{eq. (A11)} \) is insufficient to determine \( \Lambda^\mu_{\nu}(x) \) fully, since if \( \Lambda^\mu_{\nu}(x) \) satisfies \( \text{eq. (A11)} \), then so too does \( \Lambda^{\lambda}_{\mu}(x) \Lambda^\mu_{\nu} \), where \( \Lambda^\lambda_{\nu} \) may be any position-independent Lorentz transformation matrix.
transformation matrix. With this caveat in mind, we now consider each of the possible forms for $\sigma(x)$ listed above.

(i) For $\sigma = a$, one requires $\Lambda_{\mu \nu} = \text{constant}$. Thus $X_{\mu \nu}$ corresponds to a combination of a position-independent scaling $1/a$, (proper or improper) Lorentz transformation and translation (namely a global Weyl transformation).

(ii) For $\sigma = bx^2$, (A11) is solved by $\Lambda_{\mu \nu}(x) = I_{\mu \nu}(x)$, which corresponds to a reflection in the hyperplane perpendicular to the unit vector $\hat{x}$. Thus $X_{\mu \nu}$ corresponds an inversion followed by a position-independent scaling $1/b$.

(iii) For $\sigma = a(1 + 2\tilde{c}_\mu x^\mu + \tilde{c}^2 x^2)$, (A11) is solved by $\Lambda_{\mu \nu}(x) = I_{\mu \nu}(x)$, where $x^\mu$ is given by (A14) with $c_\mu$ replaced by $\tilde{c}_\mu$. Thus $X_{\mu \nu}$ corresponds to a SCT, in which the intermediate translation is through the null vector $\tilde{c}$, followed by a position-independent scaling $1/a$.

(iv) For $\sigma = a + 2c_\mu x^\mu$, (A11) is solved by $\Lambda_{\mu \nu}(x) = I_{\mu \nu}(x)$, where $x^\mu$ is given by (A14) with $c_\mu$ replaced by $\tilde{c}_\mu$, and for which $\tilde{c}^2 = 0$. Thus $X_{\mu \nu}$ corresponds to a SCT, in which the intermediate translation is through the null vector $\tilde{c}$, followed by a position-independent scaling $1/a$.

(v) For $\sigma = 2c_\mu x^\mu + bx^2 = b(2\tilde{c}_\mu x^\mu + x^2)$, where $\tilde{c}_\mu \equiv c_\mu / b$, (A11) is solved by $\Lambda_{\mu \nu}(x) = I_{\mu \nu}(\tilde{n}(x))$, where the unit vector $\tilde{n}(x)$ has components

$$\tilde{n}^\mu(x) = \frac{x^\mu + \tilde{c}^\mu}{\sqrt{x^2 + 2\tilde{c} \cdot x}}. \quad (A16)$$

It is straightforward to show that the resulting $X_{\mu \nu}$ corresponds to a translation through $\tilde{c}^\mu$, followed by an inversion, followed by a position-independent scaling $1/b$.

(vi) For $\sigma = 2c_\mu x^\mu$, (A11) is solved by $\Lambda_{\mu \nu}(x) = I_{\mu \nu}(\tilde{c})$. In a similar way to case (v), it is straightforward to show that the resulting $X_{\mu \nu}$ corresponds to a translation through $c^\mu$ in the limit $c^\mu \to \infty$, followed by an inversion.

In deriving the above solutions, we have made use of the following results. First, if $\Lambda_{\mu \nu}(x) = I_{\mu \nu}(\tilde{n}(x))$, which corresponds to a reflection in the hyperplane perpendicular to a position-dependent unit vector $\tilde{n}(x)$, the first term in (A11) may be written as

$$\Lambda_\gamma^\alpha \partial_\mu \Lambda_\gamma^\beta = 4\tilde{n}^\alpha \partial_\mu \tilde{n}^\beta. \quad (A17)$$

By then considering the identity $\tilde{n}^\gamma \partial_\alpha \tilde{n}^\beta = 0$, one quickly finds that (A11) implies that $\tilde{n}_\mu \propto \partial_\mu \sigma$. Second, if $\Lambda_{\mu \nu}(x) = I_{\mu \nu}(\tilde{n}(x))$, which corresponds to a reflection in the hyperplane perpendicular to a position-dependent unit vector $\tilde{m}(x)$, followed by a reflection in the hyperplane perpendicular to $\tilde{n}(x)$ (which together constitute a local rotation in the hyperplane defined by $\tilde{m}(x)$ and $\tilde{n}(x)$, through twice the angle between them), then the first term in (A11) may be written as

$$\Lambda_\gamma^\alpha \partial_\mu \Lambda_\gamma^\beta = 4(\tilde{m}^\alpha \partial_\mu \tilde{m}^\beta + \tilde{n}^\alpha \partial_\mu \tilde{n}^\beta + 2\tilde{m}^\alpha \tilde{n}^\beta \tilde{m}_\gamma \partial_\mu \tilde{n}^\gamma - 2\tilde{m} \cdot \tilde{n} \tilde{m}^\alpha \partial_\mu \tilde{n}^\beta). \quad (A18)$$

As expected, cases (i)-(vi) contain only the four distinct finite elements of the conformal group, namely position-independent translations, rotations and scalings, together with inversions.

Appendix B: Global and local symmetries in field theory

Consider a Minkowski spacetime $\mathcal{M}$, labelled using Cartesian inertial coordinates $x^\mu$, in which the dynamics of some set of fields $\chi(x) = \{\chi_i(x)\}$ (i = 1, 2, ...) is described by the action

$$S = \int L(\chi, \partial_\mu \chi) \, d^4x. \quad (B1)$$

It should be understood here that the index $i$ merely labels different matter fields, rather than denoting the tensor or spinor components of individual fields (which are suppressed throughout). It is worth noting that, in general, the fields $\chi_i(x)$ may include matter fields $\varphi_i(x)$ and gauge fields $g_i(x)$.

Invariance of the action (B1) under an infinitesimal coordinate transformation $x'^\mu = x^\mu + \xi^\mu(x)$ and form variations $\delta_0 \chi_i(x)$ in the fields (where the latter do not necessarily result solely from the coordinate transformation), implies that, up to a total divergence of any quantity that vanishes on the boundary of the integration region, one has

$$\delta_0 L + \partial_\mu (\xi^\mu L) = 0, \quad (B2)$$

where the form variation of the Lagrangian is given by

$$\delta_0 L = \frac{\partial L}{\partial \chi_i} \delta_0 \chi_i + \frac{\partial L}{\partial (\partial_\mu \chi_i)} \delta_0 (\partial_\mu \chi_i), \quad (B3)$$

and, according to the usual summation convention, there is an implied sum on the index $i$.

The invariance condition (B2) can alternatively be rewritten as

$$\frac{\partial L}{\partial \chi_i} \delta_0 \chi_i + \partial_\mu J^\mu = 0, \quad (B4)$$

where $\delta L / \delta \chi_i$ denotes the standard variational derivative and the Noether current $J^\mu$ is given by

$$J^\mu = \frac{\partial L}{\partial (\partial_\mu \chi_i)} \delta_0 \chi_i + \xi^\mu L. \quad (B5)$$

If the field equations $\delta L / \delta \chi_i = 0$ are satisfied, then (B4) reduces to the (on-shell) conservation law $\partial_\mu J^\mu = 0$, which is the content of Noether’s first theorem and applies both to global and local symmetries.
1. Global symmetries

Let us first consider an action invariant under a global symmetry. In the context of constructing gauge theories, it is usual first to consider an action of the form

\[ S = \int L(\varphi, \partial \varphi) \, d^4x, \]  

(B6)

where the Lagrangian density \( L \) and the Lagrangian \( L \) coincide and depend only on a set of matter fields \( \varphi(x) = \{ \varphi_i(x) \} \) and their first derivatives. Moreover, we will consider only the case for which the action of the global symmetry on the coordinates and fields can be realised linearly.

In this case, the coordinate transformation and the resulting form variations of the fields that leave the action invariant can be written as, respectively,

\[ \xi^\mu(x) = \lambda^j \xi_j^\mu(x), \quad \delta_0 \varphi_i(x) = \lambda^j G_{ij} \varphi_i(x) \]  

(B7)

where \( \lambda^j \) are a set of constant parameters, \( \xi_j^\mu(x) \) are given functions and \( G_{ij} \) are the generators of the global symmetry corresponding to the representation to which \( \varphi_i \) belongs. Note that, for each value of \( j \), the parameter \( \lambda^j \) typically represents a set of infinitesimal constants carrying one or more coordinate indices; for example, if one considers global conformal invariance, then \( \{ \lambda^1(x), \lambda^2(x), \lambda^3(x), \lambda^4(x) \} = \{a(x), \omega(x), \rho(x), c(x)\} \), where \( a(x) \) is interpreted as an infinitesimal general coordinate transformation (GCT) and is usually denoted instead by \( \xi^a(x) \).

The Noether current (B5) then takes the form

\[ J^\mu = \lambda^j \left( \frac{\partial L}{\partial (\partial_\mu \varphi_i)} G_{ij} \varphi_i + \epsilon_j^\mu L \right). \]  

(B8)

Since the parameters \( \lambda^j \) are constants, the (on-shell) conservation law \( \partial_\mu J^\mu = 0 \) hence leads to a separate condition for each value of \( j \), given by

\[ \partial_\mu \left( \frac{\partial L}{\partial (\partial_\mu \varphi_i)} G_{ij} \varphi_i + \epsilon_j^\mu L \right) = 0, \]  

(B9)

which again hold up to a total divergence of any quantity that vanishes on the boundary of the integration region of the action (B6).

\[ \delta S = \int \lambda^j \left[f_{ij}(\chi, \partial \chi) \frac{\delta L}{\delta \chi_i} - \partial_\mu \left( f_{ij}^\mu \frac{\delta L}{\delta \chi_i} \right) \right] + \partial_\mu (J^\mu - S^\mu) \, d^4x, \]  

(B11)

where we define the new current \( S^\mu \equiv -\lambda^j f_{ij}^\mu \delta L/\delta \chi_i \).

Since the \( \lambda^j \) are arbitrary functions, for the action to be invariant one requires the separate conditions

\[ f_{ij} \frac{\delta L}{\delta \chi_i} - \partial_\mu \left( f_{ij}^\mu \frac{\delta L}{\delta \chi_i} \right) = 0, \]  

(B12)

\[ \partial_\mu (J^\mu - S^\mu) = 0, \]  

(B13)

where the former hold for each value of \( j \) separately and the latter holds up to a total divergence of a quantity than vanishes on the boundary of the integration region.

The first set of conditions (B12) are usually interpreted as conservation laws, which are covariance under the local symmetry, although not manifestly so in the form given above. The condition (B13) implies that \( J^\mu = S^\mu + \partial_\nu Q^\nu \), where \( Q^\nu = -Q^{\mu \nu} \), so the two currents coincide up to a total divergence. By contrast with the case of a global symmetry, if the field equations \( \delta L/\delta \chi_i = 0 \) are satisfied, then the conservation laws (B12) hold identically and \( S^\mu \) vanishes. Thus, the conditions (B12) (B13) effectively contain no information on-shell, which is essentially the content of Noether’s second theorem [46].

Nonetheless, the on-shell condition that all the field equations \( \delta L/\delta \chi_i = 0 \) are satisfied can only be imposed if \( L \) is the total Lagrangian density, and not if \( L \) corresponds only to some subset thereof (albeit one for which the corresponding action should still be invariant under the local symmetry). In particular, suppose one is considering a field theory for which the total Lagrangian density \( L_T = L_M + L_G \), where \( L_G \) contains every term that depends only on the gauge fields \( g_i \) and/or their derivatives, and \( L_M \) contains all the remaining terms. Thus, if \( L = L_M \), then only the matter field equations \( \delta L/\delta \chi_i = 0 \) can be imposed, whereas if \( L = L_G \), none of the field equations can be imposed. In either case, the surviving terms in (B12) (B13) do contain information [47].
[1] J. Crispim-Romao, A. Ferber, P.G.O. Freund, Nucl. Phys. B 126, 429 (1977)
[2] J. Crispim-Romao, Nuc. Phys. B 145, 535 (1978)
[3] M. Kaku, P.K. Townsend, P. van Nieuwenhuizen, Phys. Lett. B 69, 304 (1977)
[4] M. Kaku, P.K. Townsend, P. van Nieuwenhuizen, Phys. Rev. D 17, 3179 (1978)
[5] E.A. Lord, P. Goswami, Pramana J. Phys. 25, 635 (1985)
[6] J.T. Wheeler, Phys. Rev. D 44, 1769 (1991)
[7] E.A. Ivanov, J. Niederle, Phys. Rev. D 25, 976 (1982)
[8] E.A. Ivanov, J. Niederle, Phys. Rev. D 25, 988 (1982)
[9] J.T. Wheeler, J. Math. Phys. 39, 299 (1998)
[10] A. Wehner, J.T. Wheeler, Nuc. Phys. B 557, 380 (1999)
[11] J.S. Hazboun, J.T. Wheeler, J. Phys. Conf. Ser. 360, 012013 (2012)
[12] J.T. Wheeler, J. Phys. Conf. Ser. 462, 012059 (2013)
[13] J.T. Wheeler, Phys. Rev. D 90, 025027 (2014)
[14] J.T. Wheeler, Nucl. Phys. B 943, 114624 (2019)
[15] E. Cunningham, Proc. London Math. Soc. 8, 77 (1910)
[16] H. Bateman, Proc. London Math. Soc. 8, 223 (1910)
[17] H. Bateman, Proc. London Math. Soc. 8, 469 (1910)
[18] J.A. Schouten, J. Haantjes, Kon. Ned. Akad. Wet. Proc. 39, 1059 (1936)
[19] A.N. Lasenby, M.P. Hobson, J. Math. Phys. 57, 092505 (2016)
[20] F.W. Hehl, in Proceedings of the 6th course of the International School of Cosmology and Gravitation, edited by P.G. Bergmann and V. de Sabbata (Plenum, New York, 1978)
[21] E.A. Lord, P. Goswami, J. Math. Phys. 27, 3051 (1986)
[22] B. Dubrovin, S. Novikov, A. Fomenko, Sovremennaya Geometriya (Nauka, Moscow, 1979)
[23] B.S. deWitt, The Global Approach to Quantum Field Theory, (Clarendon Press, Oxford, 2003)
[24] P.A.M. Dirac, Proc. R. Soc. A 333, 403 (1973)
[25] S.-H. Ho, R. Jackiw, S.-Y. Pi, J. Phys. A: Math. Theor. 44, 225401 (2011)
[26] S. Coleman, R. Jackiw, Ann. Phys. 67, 552 (1971)
[27] M. Blagojevic, Gravitation and Gauge Symmetries, (IOP Publishing, Bristol, 2002)
[28] F.J. Belinfante, Physica 7, 449 (1940)
[29] C. Callan, S. Coleman, R. Jackiw, Ann. Phys. 59, 42 (1970)
[30] J.T. Miller, G.N. Fleming, Phys. Rev. 174, 1625 (1968)
[31] R. Utiyama, Phys. Rev. 101, 1597 (1956)
[32] D. Sciama, Rev. Mod. Phys. 36, 463 (1964)
[33] D. Ivanenko, G. Sardanashvily, Phys. Rep. 94, 1 (1983)
[34] E.A. Lord, P. Goswami, J. Math. Phys. 27, 2415 (1986)
[35] T.W.B. Kibble, J. Math. Phys. 2, 212 (1961)
[36] J.P. Harnad, R.B. Pettitt, J. Math. Phys. 17, 1827 (1976)
[37] F.W. Hehl, P. von der Heyde, G.D. Kerlick and J.M. Nester, Rev. Mod. Phys. 48, 393 (1976)
[38] C. Wiesendanger, Class. Quant. Grav. 13, 681 (1996)
[39] N. Mukunda, in Gravitation, Gauge Theories and the Early Universe, edited by B.R. Iyer, N. Mukunda and C.V. Vishveshwara (Kluwer, Dordrecht, 1989), p. 467
[40] A.N. Lasenby, C. Doran and S.F. Gull, Phil. Trans. R. Soc. Lond. A 356, 487 (1998)
[41] S. MacDowell, R. Mansouri, Phys. Rev. Lett. 38, 739 (1977)
[42] Y. Ne'eman, T. Regge, La Rivista del Nuovo Cimento (1978-1999) 1(5), 1 (1978)
[43] Y. Ne'eman, T. Regge, Phys. Lett. B 74, 54 (1978)
[44] T. Padmanabhan, Class. Quant. Grav. 2, L105 (1985)
[45] P.D. Mannheim, Prog. Part. Nucl. Phys. 56, 340 (2006)
[46] S.G. Avery, B.U.W. Schwab, JHEP, 02, 031 (2016)
[47] M. Forger, H. Romer, Annals Phys. 309, 306 (2004)