Mean-Field Delayed BSDEs with Jumps

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Abstract. We establish sufficient conditions for the existence and uniqueness of mean-field backward stochastic differential equations with time delayed generator in the sense that at \( t \), the generator may depend on previous values up to a delay constant \( \delta \) not on the whole past as in Delong and Imkeller [10], [13]. For sufficiently small delay constant \( \delta \) and for any finite time horizon, we get a unique solution.

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1 Introduction

Backward stochastic differential equations (BSDEs) appear in their linear form as an adjoint equation when Bismut [7] was dealing with stochastic optimal control. After that, this theory has been developed also for the nonlinear case, we refer to namely to the seminal work by Pardoux and Peng [19] and also to Pardoux [20], El Karoui et al [14]. The first work applying in finance was made by El Karoui et al [14].

Given a driven Brownian motion \( B \), a generator \( f : \Omega \times [0, T] \times \mathbb{R}^2 \to \mathbb{R} \) and a terminal condition \( \xi \). Solving a BSDE consists in finding a couple of processes \((Y(t), Z(t))_{t \geq 0}\) adapted to the considered filtration (the Brownian one), such that, at time \( t \), \((Y(t), Z(t))_{t \geq 0}\) satisfies the equation

\[
Y(t) = \xi + \int_t^T f(s, Y(s), Z(s))ds - \int_t^T Z(s)dB(s), \ 0 \leq t \leq T. \quad (1.1)
\]

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The crucial question now is which conditions should be satisfied by the generator $f$ and the terminal value $\xi$ in order to get the existence and the uniqueness of such a solution to (1.1).

In this paper we are interested in a generalisation of this above BSDE, where at time $s$ the coefficient $f$ depends on past information and the law of the solution process and we consider also the discontinuous case. More precisely, we are interested in the Mean-Field Delayed BSDE (MF-DBSDE) with jumps of the form

$$
\begin{align*}
Y(t) &= \xi + \int_t^T f(s, Y_s, Z_s, K_s(\cdot), P(Y_s, Z_s, K_s(\cdot))) ds - \int_t^T Z(s) dB(s) \\
&\quad - \int_t^T \int_{\mathbb{R}} K(s, \zeta) \tilde{N}(ds, d\zeta), t \in [0, T], \\
Y(t) &= Y(0), Z(t) = 0, K(t, \cdot) = 0, t < 0.
\end{align*}
$$

(1.2)

Here, for a given delay constant $\delta > 0$,

$$
\begin{align*}
Y_s &= (Y(s + r))_{r \in [-\delta, 0]}, \\
Z_s &= (Z(s + r))_{r \in [-\delta, 0]}, \\
K_s(\cdot) &= (K(s + r, \cdot))_{r \in [-\delta, 0]},
\end{align*}
$$

where information on the past of the solution process $(Y, Z, K)$ is considered. We remark that in this case, i.e., when the dependence of the coefficient $f$ on past information is studied, the process $f(s, Y_s, Z_s, K_s(\cdot), P(Y_s, Z_s, K_s(\cdot)))$ is already $\mathbb{F}$-adapted, and coincides $ds dP$-a.e. with its optional projection.

The case of DBSDEs in both continuous and discontinuous case have been studied by Delong and Imkeller $[10]$ and $[13]$. The authors consider the coefficients depending on the full past of the path of the solution. Following the same approach of Delong and Imkeller $[10]$, Agram and Røse $[5]$ obtained existence and uniqueness of MF-DBSDE where the mean-field term considered is the expectation of the state and also we refer to the paper by Ma and Liu $[15]$, in this framework.

The case when the mean-field is represented by the law of the state has been studied by Carmona and Delarue $[9]$ and with discrete delay and implicit terminal condition has been studied by Agram $[3]$. The last two papers, the probability measures are defined in the Wasserstein metric space $\mathcal{P}_2$ and the Wasserstein distance $W_2$ is used.

In this note we study MF-BSDEs of type (1.2) driven by a Brownian motion and a jumps and the filtration is that generated by both the Brownian motion and the independent Poisson random measure. We consider the Hilbert space of measures $\mathcal{M}$ introduced by Agram and Øksendal $[2]$, $[1]$. The delay of the state processes is taking up to a delay constant $\delta$ not of the hole past as in Delong and Imkeller $[10]$ and this will help to get existence and uniqueness of the MF-BSDE (1.2) for any finite time horizon $T$ and any Lipschitz constant $C$ but for sufficiently small delay constant $\delta$.

BSDEs with jumps have been studied by many authors, for example, we refer to Tang and Li $[22]$, Barles et al $[3]$, Royer $[21]$ Sulem and Quenez $[18]$ and Øksendal and Sulem $[17]$. These type of MF-DBSDE generalises the classical BSDE and have turned out to be useful in various applications, namely in finance and in stochastic control. While in finance the delay imposes in the modelling by the fact that agents have often only time-delayed information,
for example: If one wants to find an investment strategy and an investment portfolio which replicate a liability or meet a target which depends on the applied strategy or the portfolio depends on its past values, then the DBSDEs are the best tool to solve this financial problem. BSDEs with delay can also arise in portfolio management problems, variable annuities, unit-linked products and participating contract. For more details about applications of such equations, we refer to Delong [11].

Next sections are devoted to the study of the MF-DBSDEs as follows:

- in section 2, we introduce the adequate spaces of processes and we fixe suitable assumptions on the driver and the terminal value.
- in section 3, we give a theorem on the existence and the uniqueness of the solutions.

2 Background

Let $B(t)$, $t \geq 0$ be a 1-dimentional Brownian motion, and $\tilde{N}(dt, d\zeta)$, $t \geq 0$ be an independent compensated Poisson random measure, with compensator $\nu(d\zeta)dt$, on a probability space $(\Omega, \mathcal{F}, P)$. Let $\mathcal{F}=(\mathcal{F}_t)_{t\geq0}$ denote the natural filtration satisfying the usual conditions of right continuity and completeness, associated with $B$ and $N$. Let $\delta > 0$, and extend the filtration by letting $\mathcal{F}_t = \mathcal{F}_0$ for $t \in [-\delta, 0]$. We consider the MF-DBSDE, for $r \in [-\delta, 0]$ :

$$
\begin{cases}
Y(t) = \xi + \int_t^T f(s, Y_s, Z_s, K_s(\cdot), P(Y_s, Z_s, K_s(\cdot)))ds \\
\quad - \int_t^T Z(s)dB(s) - \int_t^T \int_E K(s, \zeta)\tilde{N}(ds, d\zeta), t \in [0, T], \\
Y(t) = Y(0), Z(t) = 0, K(t, \cdot) = 0, t < 0,
\end{cases}
$$

(2.1)

where

$$
Y_s(r, \omega) := \begin{cases}
Y(s + r, \omega), & r \in [-\delta, 0], s + r \geq 0, \omega \in \Omega, \\
Y(0, \omega), & s + r < 0,
\end{cases}
$$

$$
Z_s(r, \omega) := \begin{cases}
Z(s + r, \omega), & r \in [-\delta, 0], s + r \geq 0, \omega \in \Omega, \\
0, & s + r < 0,
\end{cases}
$$

$$
K_s(r, \omega)(\zeta) := \begin{cases}
K(s + r, \omega, \zeta), & r \in [-\delta, 0], s + r \geq 0, \omega \in \Omega, \zeta \in E, \\
0, & s + r < 0,
\end{cases}
$$

where $E = \mathbb{R}_0 := \mathbb{R} - \{0\}$ and $\xi \in L^2(\Omega, \mathcal{F}_T)$.

Here, for each $t$, $(Y_t, Z_t, K_t(\cdot))$ is assumed to belong to the space

$$
\mathcal{S}_\infty^2 \times L^2 \times \mathbb{H}^2,
$$

of functionals defined below.
• $S^2_\infty = L^\infty(\Omega, \mathcal{D}[-\delta, 0], \mathbb{R})$ consists of the càdlàg functions

$$\alpha : [-\delta, 0] \to \mathbb{R}.$$ 

Let $S^2_\infty$ be equipped with the norm

$$\| \alpha \|^2_{S^2_\infty} := \mathbb{E}\left[ \sup_{r \in [-\delta, 0]} |\alpha(r)|^2 \right] < \infty.$$ 

We refer to [16] for more on this space in connection with stochastic functional differential equations.

• $L^2 = L^2_{-\delta}(\mathbb{R})$ is the space of all functions

$$\sigma : [-\delta, 0] \to \mathbb{R},$$

Borel measurable, such that

$$\| \sigma \|^2_{L^2} := \int_{-\delta}^0 |\sigma(r)|^2 dr < \infty.$$ 

• $L^2(\nu)$ consists of Borelian functions $K : E \to \mathbb{R}$, such that

$$\| K \|^2_{L^2(\nu)} := \int_E K(\zeta)^2 \nu(d\zeta) < \infty.$$ 

• $H^2 = L^2_{-\delta \times \nu}(\mathbb{R})$ is the space of all functions

$$\theta : [-\delta, 0] \times E \to \mathbb{R},$$

Borel measurable, such that

$$\| \theta \|^2_{H^2} := \int_{-\delta}^0 \int_E |\theta(r, \zeta)|^2 \nu(d\zeta) dr < \infty.$$ 

• $L^2(\Omega, \mathcal{F}_T)$ is the set of square integrable random variables which are $\mathcal{F}_T$-measurable. We also define the following spaces:

• $S^2_T$ consists of the $\mathcal{F}$-adapted càdlàg processes

$$Y : \Omega \times [0, T] \to \mathbb{R},$$

such that $\mathbb{E}\left[ \sup_{t \in [0, T]} |Y(t)|^2 \right] < \infty$. We equip $S^2_T$ with the norm

$$\| Y \|^2_{S^2_T} := \mathbb{E}\left[ \sup_{t \in [0, T]} e^{\beta t} |Y(t)|^2 \right], \ \beta > 0.$$
\textbf{L}^2_T consists of the \( \mathbb{F} \)-predictable processes

\[ Z : \Omega \times [0, T] \to \mathbb{R}, \]

such that \( \mathbb{E}[\int_0^T |Z(t)|^2 \, dt] < \infty \). We equip \( \text{L}^2_T \) with the norm

\[ \| Z \|_{L^2_T}^2 := \mathbb{E}[\int_0^T e^{\beta t} |Z(t)|^2 \, dt], \quad \beta > 0. \]

\textbf{H}_T^2 consists of the \( \mathbb{F} \)-predictable processes

\[ K : \Omega \times [0, T] \times E \to \mathbb{R}, \]

such that \( \mathbb{E}[\int_0^T \int_E K(t, \zeta)^2 \nu(d\zeta) \, dt] < \infty \). We equip \( \text{H}_T^2 \) with the norm

\[ \| K \|_{H^2_T}^2 := \mathbb{E}[\int_0^T \int_E e^{\beta t} K(t, \zeta)^2 \nu(d\zeta) \, dt], \quad \beta > 0. \]

Notice that if \((Y, Z, K) \in \text{S}^2_T, \beta \times \text{L}^2_T, \beta \times \text{H}^2_T, \beta\), then for a.e. \( t \in [-\delta, T] \), the segment process \((Y_t, Z_t, K_t(\cdot))\) belongs to \( \text{L}^2 \times \text{L}^2 \times \text{H}^2 \) for a.e. \( t \), \( P \)-a.s.

In this section, we, as in Agram and Øksendal \[1\], \[2\], construct an Hilbert space \( \mathcal{M}(\cdot) \) of (random) measures.

\textbf{Definition 2.1} Let \( \tilde{\mathcal{M}}(\mathbb{R}) \) denote the set of random measures \( \mu \) on \( \mathbb{R} \) such that

\[ \mathbb{E}[\int_\mathbb{R} |\hat{\mu}(y)|^2 e^{-y^2} \, dy] < \infty, \quad (2.2) \]

where

\[ \hat{\mu}(y) = \int_\mathbb{R} e^{ixy} \, d\mu(x) \]

is the Fourier transform of the measure \( \mu \).

If \( \mu, \eta \in \tilde{\mathcal{M}}(\mathbb{R}) \) we define the inner product \( \langle \mu, \eta \rangle_{\tilde{\mathcal{M}}(\mathbb{R})} \), by

\[ \langle \mu, \eta \rangle_{\tilde{\mathcal{M}}(\mathbb{R})} = \mathbb{E}[\int_\mathbb{R} \text{Re}(\hat{\mu}(y)\hat{\eta}(y)) e^{-y^2} \, dy], \]

where, in general, \( \text{Re}(z) \) denotes the real part of the complex number \( z \), and \( \bar{z} \) denotes the complex conjugate of \( z \).

The norm \( \| \cdot \|_{\tilde{\mathcal{M}}(\mathbb{R})} \) associated to this inner product is given by

\[ \| \mu \|^2_{\tilde{\mathcal{M}}(\mathbb{R})} = \langle \mu, \mu \rangle_{\tilde{\mathcal{M}}(\mathbb{R})} = \mathbb{E}[\int_\mathbb{R} |\hat{\mu}(y)|^2 e^{-y^2} \, dy]. \]

The space \( \tilde{\mathcal{M}}(\mathbb{R}) \) equipped with the inner product \( \langle \mu, \eta \rangle_{\tilde{\mathcal{M}}(\mathbb{R})} \) is a pre-Hilbert space.

Let \( \mathcal{M}(\mathbb{R}^m) \) denote the set of random measures \( \mu = \mu(\omega) \) on \( \mathbb{R}^m \) such that

\begin{itemize}
  \item We denote by \( \mathcal{M}(\cdot) \) the completion of \( \tilde{\mathcal{M}}(\cdot) \).
  \item We denote by \( \mathcal{M}_0(\cdot) \) the set of all deterministic elements of \( \mathcal{M}(\cdot) \).
\end{itemize}
Definition 2.2 \( \bullet \) \( \mathcal{M}(\mathbb{R}^2 \times L^2(\nu)) \) is the space of random measures \( \mu \) on \( \mathbb{R}^2 \times L^2(\nu) \), such that

\[
\|\mu\|_{\mathcal{M}(\mathbb{R}^2 \times L^2(\nu))} := \mathbb{E}\left[\int_{\mathbb{R}^2} \int_{\mathbb{R}^3} |\hat{\mu}(y_1, y_2, y_3)|^2 \exp(-\sum_{j=1}^{3} y_j^2) \nu(dy_1, y_2, y_3)(d\zeta) \right] < \infty,
\]

where \( \hat{\mu}(y_1, y_2, y_3, \zeta) = \hat{\mu}(y_1, y_2, y_3) \) is the Fourier transform of the measure \( \mu \) parametrized at \( \zeta \), i.e.,

\[
\hat{\mu}(y_1, y_2, y_3) := \int_{\mathbb{R}^3} \exp(-2\pi i \sum_{j=1}^{3} x_j y_j) d\mu(x_1, x_2, x_3); \quad y_1, y_2, y_3 \in \mathbb{R}.
\]

Lemma 2.3 Let \( X^{(1)}, X^{(2)}, \tilde{X}^{(1)}, \tilde{X}^{(2)} \) and \( X^{(3)}, \tilde{X}^{(3)} \) be random variables in \( L^2(P) \) and in \( L^2(\nu) \) respectively. Then

\[
\|\mathcal{L}(X^{(1)}, X^{(2)}, X^{(3)}) - \mathcal{L}(\tilde{X}^{(1)}, \tilde{X}^{(2)}, \tilde{X}^{(3)})\|^2_{\mathcal{M}_0(\mathbb{R}^2 \times L^2(\nu))} \\
\leq C \mathbb{E}[\{(X^{(1)} - \tilde{X}^{(1)})^2 + (X^{(2)} - \tilde{X}^{(2)})^2 + \int_{\mathbb{R}^2} (X^{(3)}(\zeta) - \tilde{X}^{(3)}(\zeta))^2 \nu(\zeta)\}],
\]

where \( \mathcal{L}(X) = P_X \).

Proof Let \( X = (X^{(1)}, X^{(2)}, X^{(3)}), \tilde{X} = (\tilde{X}^{(1)}, \tilde{X}^{(2)}, \tilde{X}^{(3)}) \) and \( y = (y_1, y_2, y_3) : \)

\[
\|\mathcal{L}(X) - \mathcal{L}(\tilde{X})\|^2_{\mathcal{M}_0(\mathbb{R}^2 \times L^2(\nu))} \\
= \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} |\tilde{\mathcal{L}}(X)(y, \zeta) - \tilde{\mathcal{L}}(\tilde{X})(y, \zeta)|^2 e^{-y^2} dy \min(1, \zeta^2) \nu(\zeta) \\
= \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} e^{-ixy} d\mathcal{L}(X)(x) - \int_{\mathbb{R}^3} e^{-ixy} d\mathcal{L}(\tilde{X})(x)|^2 e^{-y^2} dy \min(1, \zeta^2) \nu(\zeta) \\
\leq \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \mathbb{E}[e^{-ixy} - e^{-i\tilde{X}y}]|^2 e^{-y^2} dy \min(1, \zeta^2) \nu(\zeta) \\
\leq \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \mathbb{E}[(X - \tilde{X})y]|^2 e^{-y^2} dy \min(1, \zeta^2) \nu(\zeta) \\
\leq \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} y^2 \mathbb{E}[(X - \tilde{X})^2] e^{-y^2} dy \min(1, \zeta^2) \nu(\zeta) \\
= \mathbb{E}[(X^{(1)} - \tilde{X}^{(1)})^2] \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} y^2 e^{-y^2} dy \min(1, \zeta^2) \nu(\zeta) \\
+ \mathbb{E}[(X^{(2)} - \tilde{X}^{(2)})^2] \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} y^2 e^{-y^2} dy \min(1, \zeta^2) \nu(\zeta) \\
+ \int_{\mathbb{R}^2} \mathbb{E}[(X^{(3)} - \tilde{X}^{(3)})^2(\zeta)] \min(1, \zeta^2) \nu(\zeta) \int_{\mathbb{R}^3} y^2 e^{-y^2} dy.
\]

Note that \( \int_{\mathbb{R}^2} \min(1, \zeta^2) \nu(\zeta) < \infty \) for all Lévy measure \( \nu \). \( \square \)

Definition 2.4 \( \bullet \) Define \( \mathcal{M}^3(\mathbb{R}^2 \times L^2(\nu)) \) to be the Hilbert space of all path segments \( \overline{\mu} = \{\mu(s)\} \) of measure-valued processes \( \mu(\cdot) \) with \( \mu(s) \in \mathcal{M}(\mathbb{R}^2 \times L^2(\nu)) \) for each \( s \in [-\delta, 0] \), equipped with the norm

\[
\|\overline{\mu}\|_{\mathcal{M}^3(\mathbb{R}^2 \times L^2(\nu))} := \int_{-\delta}^{0} \|\mu(s)\|_{\mathcal{M}(\mathbb{R}^2 \times L^2(\nu))} ds.
\]
• $\mathcal{M}_0(\mathbb{R}^2 \times L^2(\nu))$ and $\mathcal{M}_0^\delta(\mathbb{R}^2 \times L^2(\nu))$ denote the set of deterministic elements of $\mathcal{M}(\mathbb{R}^2 \times L^2(\nu))$ and $\mathcal{M}^\delta(\mathbb{R}^2 \times L^2(\nu))$, respectively.

There are several advantages with working with this Hilbert space $\mathcal{M}$, compared to the Wasserstein metric space:

• A Hilbert space has a useful stronger structure than a metric space.
• The distance is not continuous but the norm is.
• The Wasserstein metric space $\mathcal{P}_2$ deals only with probability measures with finite second moment, while the Hilbert space deals with any (random) measure satisfying (2.2).

Let us give some examples of measures:

**Example 2.5 (Measures)** We consider here a 1-dimensional case:

1. Suppose that $\mu = \delta_{x_0}$, the unit point mass at $x_0 \in \mathbb{R}$. Then $\delta_{x_0} \in \mathcal{M}_0(\mathbb{R})$ and
   \[ \int_{\mathbb{R}} e^{ixy} d\mu(x) = e^{ix_0y}, \]
   and hence
   \[ \|\mu\|_{\mathcal{M}_0(\mathbb{R})}^2 = \int_{\mathbb{R}} |e^{ix_0y}|^2 e^{-y^2} dy < \infty. \]

2. Suppose $d\mu(x) = f(x) dx$, where $f \in L^1(\mathbb{R})$. Then $\mu \in \mathcal{M}_0(\mathbb{R})$ and by Riemann-Lebesgue lemma, $\hat{\mu}(y) \in C_0(\mathbb{R})$, i.e. $\hat{\mu}$ is continuous and $\hat{\mu}(y) \to 0$ when $|y| \to \infty$. In particular, $|\hat{\mu}|$ is bounded on $\mathbb{R}$ and hence
   \[ \|\mu\|_{\mathcal{M}_0(\mathbb{R})}^2 = \int_{\mathbb{R}} |\hat{\mu}(y)|^2 e^{-y^2} dy < \infty. \]

3. Suppose that $\mu$ is any finite positive measure on $\mathbb{R}$. Then $\mu \in \mathcal{M}_0(\mathbb{R})$ and
   \[ |\hat{\mu}(y)| \leq \int_{\mathbb{R}} d\mu(y) = \mu(\mathbb{R}) < \infty, \text{ for all } y, \]
   and hence
   \[ \|\mu\|_{\mathcal{M}_0(\mathbb{R})}^2 = \int_{\mathbb{R}} |\hat{\mu}(y)|^2 e^{-y^2} dy < \infty. \]

4. Next, suppose $x_0 = x_0(\omega)$ is random. Then $\delta_{x_0(\omega)}$ is a random measure in $\mathcal{M}(\mathbb{R})$. Similarly, if $f(x) = f(x, \omega)$ is random, then $d\mu(x, \omega) = f(x, \omega) dx$ is a random measure in $\mathcal{M}(\mathbb{R})$. 

7
3 Existence and uniqueness

The objective of this section is to give a theorem on the existence and the uniqueness of the solution of equation (2.1). For the proof we use the fixed point argument which is a classical tool to prove the existence and the uniqueness of BSDEs.

Definition 3.1

- A functional
  \[ f : \Omega \times [0, T] \times \mathbb{L}^2 \times \mathbb{H}^2 \times \mathcal{M}^\delta(\mathbb{R}^2 \times L^2(\nu)) \to \mathbb{R}, \]
  is progressively measurable.

- A process
  \[(Y, Z, K) \in S_T^2 \times L_T^2 \times H_T^2\]
  is said to be a solution to (2.1) if
  \[ \int_0^T |f(s, Y_s, Z_s, K_s(\cdot), P_{(Y_s, Z_s, K_s(\cdot))})| ds < +\infty \text{ P-a.s.,} \]
  and
  \[
  \begin{aligned}
  Y(t) &= \xi + \int_t^T f(s, Y_s, Z_s, K_s(\cdot), P_{(Y_s, Z_s, K_s(\cdot))})ds - \int_t^T Z(s)dB(s) \\
  &\quad - \int_t^T \int F(K(s, \zeta)N(ds, d\zeta), t \in [0, T], \\
  Y(t) &= Y(0), Z(0) = K(t, \cdot) = 0, t < 0.
  \end{aligned}
  \]

We impose the following set of assumptions which will guarantee the existence and the uniqueness of the solution of the MF-DBSDE (2.1).

Assumptions

Let \( f \) be a functional generator and \( \xi \) the terminal condition. Suppose that:

(i) \( \xi \in L^2(\Omega, \mathcal{F}_T). \)

(ii) For all \( t \in [0, T] \), we have
  \[ |f(t, 0, 0, 0, P_0)| < c, \]
  where \( P_0 \) is the Dirac measure with mass at zero and \( c \) is a given constant.

(iii) For all \( t \in [0, T] \) and for all \( y_i \in \mathbb{L}^2, z_i \in \mathbb{L}^2, k_i(\cdot) \in \mathbb{H}^2 \) and \( \eta_i \in \mathcal{M}_0(\mathbb{L}^2 \times \mathbb{L}^2 \times \mathbb{H}^2), i = 1, 2 \), we have for a constant \( C > 0 \) and for some probability measure \( \mu \) on \( [-\delta, 0] \times \mathcal{B}[-\delta, 0] \) where \( \mathcal{B} \) stands for the Borel sets of \( [-\delta, 0] \), such that
  \[
  |f(t, y_1, z_1, k_1(\cdot), \eta_1) - f(t, y_2, z_2, k_2(\cdot), \eta_2)|^2 \\
  \leq C \int_{-\delta}^0 (|Y_1(t + r) - Y_2(t + r)|^2 + |Z_1(t + r) - Z_2(t + r)|^2 \\
  + \int_F |K_1(t + r, \zeta) - K_2(t + r, \zeta)|^2 \nu(d\zeta) \\
  + ||\eta_1(r) - \eta_2(r)||^2_{\mathcal{M}(\mathbb{R}^2 \times L^2(\nu))} \mu(dr), \ P-a.s.
  \]
The following theorem gives the existence and the uniqueness of the solution of a MF-DBSDE with jumps under assumptions (i)-(iii).

**Theorem 3.2** Let us suppose the above assumptions (i)-(iii), with \( \rho > \mu(\{0\}) \). Then for sufficiently small \( \delta_\rho > 0 \), it holds for all \( \delta \in (0, \delta_\rho) \), the MF-DBSDE (2.1) admits a unique solution \( (Y, Z, K) \in S_T^2 \times L_T^2 \times H_T^2 \).

**Remark 3.3** In general, the Lipschitz condition with \( \|y_1 - y_2\|_{S_T^2} \) instead of \( \|y_1 - y_2\|_{L_T^2} \) is considered; see, for instance Delong and Imkeller [10], [13]. Recall that:

\[
\|y_1 - y_2\|_{L_T^2} \leq C \|y_1 - y_2\|_{S_T^2},
\]

they obtain the solution of the DBSDE for a sufficiently small time horizon \( T > 0 \) or a sufficiently small Lipschitz constants. We take here a condition which is more restrictive but it allows us to obtain the existence and uniqueness of our MF-DBSDE with jumps (2.1) for any finite time horizon \( T > 0 \) and for any Lipschitz constant but for a sufficiently small delay constant \( \delta > 0 \).

**Proof** Let us define the mapping

\[
\Phi : (L^2(F_0) \times L^2_T) \times L_T^2 \times H_T^2 \rightarrow (L^2(F_0) \times L^2_T) \times L_T^2 \times H_T^2
\]

by setting

\[
\Phi ((U(0), U), V, Q) := ((Y(0), Y), Z, K),
\]

where for \( U(0) \in L^2(F_0), U \in L_T^2, U(t) = U(0), t < 0, V \in L_T^2, V(t) = 0, t < 0, Q \in H_T^2, Q(t) = 0, t < 0, (Y, Z, K) \in S_T^2 \times L_T^2 \times H_T^2 \) is the unique solution of the MF-DBSDE with jumps

\[
\begin{cases}
Y(t) = \xi + \int_t^T f(s, U_s, V_s, Q_s(\cdot), P_{U,s,V,s,Q_s(\cdot)}(\cdot)) ds - \int_t^T Z(s) dB(s) \\
\quad - \int_t^T \int_E K(s, \zeta) \tilde{N}(ds, d\zeta), t \in [0, T], \\
Y(0) = Y(0), Z(t) = 0, K(t, \cdot) = 0, t < 0.
\end{cases}
\]

For \( \beta > 0 \) and \( ((U(0), U), V, Q) \in (L^2(F_0) \times L_T^2) \times L_T^2 \times H_T^2 \) we introduce the norm

\[
\|(U(0), U), V, Q\|_\beta := \|(U(0), U), V, Q\|_\beta := (E[|U(0)|^2])
\]

\[
+ E\left[ \int_0^T e^{\beta s} (|U(s)|^2 + |V(s)|^2 + \int_E |Q(s, \zeta)|^2 \nu(d\zeta)) ds \right]\frac{1}{2}.
\]

Note that \( (L^2(F_0) \times L_T^2) \times L_T^2 \times H_T^2 \) endowed with this norm is a Banach space. We will show that for suitably chosen \( \beta > 0, \delta \in (0, \delta_0) \), the mapping

\[
\Phi : (L^2(F_0) \times L_T^2) \times L_T^2 \times H_T^2, \|\cdot\|_\beta) \rightarrow (L^2(F_0) \times L_T^2) \times L_T^2 \times H_T^2, \|\cdot\|_\beta)
\]

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is contracting, i.e., there is a unique fixed point \(((Y(0), Y), Z, K) \in (L^2(F_0) \times L_T^2) \times L_T^2 \times H_T^2)\) of \(\Phi\). Consequently,

\[
\begin{align*}
Y(t) &= \xi + \int_t^T f(s, Y_s, Z_s, K_s, P_{Y,Y_s, Z_s, K_s}) \, ds - \int_t^T Z(s) \, dB(s) \\
&\quad - \int_t^T \int_E K(s, \zeta) \tilde{N}(ds, d\zeta), t \in [0, T], \\
Y(t) &= Y(0), \ Z(t) = 0, \ K(t, \cdot) = 0, t < 0.
\end{align*}
\]

In particular \(Y\) has a continuous version and by standard estimations, there exists a constant \(C \in \mathbb{R}\), such that

\[
\mathbb{E}[\sup_{t \in [0,T]} |Y(t)|^2] \leq C(\mathbb{E}[|\xi|^2] + \mathbb{E}[\int_t^T |f(s, U_s, V_s, Q_s, \cdot, P_{U_s, V_s, Q_s})|^2 \, ds] \\
+ \mathbb{E}[\int_t^T |Z(s)|^2 \, ds] + \mathbb{E}[\int_t^T \int_E |K(s, \zeta)|^2 \nu(d\zeta)ds] < \infty.
\]

Consequently, \(Y \in S_T^p\). Let us consider \(((U(0), U), V, Q), ((U'(0), U'), V', Q') \in (L^2(F_0) \times L_T^2) \times L_T^2 \times H_T^2\) and let us use the simplified notations:

\[
\begin{align*}
\Phi((U(0), U), V, Q) &:= ((Y(0), Y), Z, K), \\
\Phi((U'(0), U'), V', Q') &:= ((Y'(0), Y'), Z', K'), \\
((U(0), U), V, Q) &:= ((U(0), U), V, Q) - ((U'(0), U'), V', Q'), \\
((Y(0), Y), Z, K) &:= ((Y(0), Y), Z, K) - ((Y'(0), Y'), Z', K').
\end{align*}
\]

Applying Itô’s formula to \((e^{\beta t} |\tilde{Y}(t)|^2)|_{t \geq 0}\) and using the Lipschitz condition \((ii)\), we get

\[
e^{\beta t} |\tilde{Y}(t)|^2 + \int_t^T e^{\beta s} |\tilde{Y}(s)|^2 + |\tilde{Z}(s)|^2 + \int_E |\tilde{K}(s, \zeta)|^2 \nu(d\zeta)ds \\
\leq 2 \int_t^T e^{\beta s} |\tilde{Y}(s)| \times \\
\times |f(s, U_s, V_s, Q_s, \cdot, P_{U_s, V_s, Q_s})| - f(s, U'_s, V'_s, Q'_s, \cdot, P_{U'_s, V'_s, Q'_s})|ds \\
- 2 \int_t^T e^{\beta s} \tilde{Y}(s) \cdot \tilde{Z}(s) \, dB(s) - 2 \int_t^T \int_E e^{\beta s} \tilde{Y}(s') \cdot \tilde{K}(s, \zeta) \tilde{N}(ds, d\zeta) \\
\leq 2 \int_t^T e^{\beta s} |\tilde{Y}(s)| \times \\
\times \mathbb{E}[\int_0^T \left( |\tilde{U}(s + r)|^2 \nu(d \zeta) \right) \mu(dr) \\
+ \sqrt{\pi} \mathbb{E}[\int_0^T \left( |\tilde{V}(s + r)|^2 \nu(d \zeta) \right) \mu(dr)]ds \\
- 2 \int_t^T e^{\beta s} \tilde{Y}(s) \cdot \tilde{Z}(s) \, dB(s) - 2 \int_t^T \int_E e^{\beta s} \tilde{Y}(s') \cdot \tilde{K}(s, \zeta) \tilde{N}(ds, d\zeta). \tag{3.1}
\]

Using for every term in the integrand of the Lebesgue integral at the right hand side of the above equality, the estimate \(2C(1 + \int_{\mathbb{R}^d} y^2 e^{-y^2} dy)ab \leq 2\rho C'^2 a^2 + \frac{1}{\rho} b^2\), we obtain

\[
e^{\beta t} |\tilde{Y}(t)|^2 + \int_t^T e^{\beta s} |\tilde{Y}(s)|^2 + |\tilde{Z}(s)|^2 + \int_E |\tilde{K}(s, \zeta)|^2 \nu(d\zeta)ds \\
\leq 12 \rho C'^2 \int_t^T e^{\beta s} |\tilde{Y}(s)|^2 \, ds \\
+ \frac{1}{\rho} \int_t^T e^{\beta s} \int_0^T \left( |\tilde{U}(s + r)|^2 \nu(d \zeta) \right) \mu(dr)ds \\
- 2 \int_t^T e^{\beta s} \tilde{Y}(s) \cdot \tilde{Z}(s) \, dB(s) - 2 \int_t^T \int_E e^{\beta s} \tilde{Y}(s') \cdot \tilde{K}(s, \zeta) \tilde{N}(ds, d\zeta),
\]

where \(C'^2 = C^2(1 + (\int_{\mathbb{R}^d} y^2 e^{-y^2} dy)^2 + 2\int_{\mathbb{R}^d} y^2 e^{-y^2} dy).\)
Choose $\beta = 1 + 12\rho C^2$, the last line in (3.1) constitutes a sum of martingale differences (with the necessary integrability properties) then, taking expectation, we get

$$
\mathbb{E}[e^{\beta t} \int_0^T e^{\beta s} \left( |\bar{U}(s+r)|^2 + |\bar{V}(s+r)|^2 + \int_F |\bar{Q}(s+r, \zeta)|^2 \nu(d\zeta)\right) ds |\mathcal{F}_t] \\
\leq \frac{1}{\rho} \mathbb{E}[\int_0^T e^{\beta s} \left( |\bar{U}(s)|^2 + |\bar{V}(s)|^2 + \int_F |\bar{Q}(s, \zeta)|^2 \nu(d\zeta)\right) ds |\mathcal{F}_t]. 
$$

(3.2)

By changing the variables $v = s + r$, we have

$$
\int_0^T e^{\beta s} \int_0^s \left( |\bar{U}(s+r)|^2 + |\bar{V}(s+r)|^2 + \int_F |\bar{Q}(s+r, \zeta)|^2 \nu(d\zeta)\right) ds \mu(dr) \\
= \int_0^T e^{-\beta r} \int_0^r e^{\beta(s+r)} \left( |\bar{U}(s+r)|^2 + |\bar{V}(s+r)|^2 + \int_F |\bar{Q}(s+r, \zeta)|^2 \nu(d\zeta)\right) ds \mu(dr) \\
= \int_0^r e^{-\beta r} \int_0^T e^{\beta v} \left( |\bar{U}(v)|^2 + |\bar{V}(v)|^2 + \int_F |\bar{Q}(v, \zeta)|^2 \nu(d\zeta)\right) dv \mu(dr) \\
\leq \int_0^r e^{-\beta r} \mu(dr) \left( |\bar{U}(0)|^2 + \int_0^T e^{\beta s} \left( |\bar{U}(s)|^2 + |\bar{V}(s)|^2 + \int_F |\bar{Q}(s, \zeta)|^2 \nu(d\zeta)\right) ds\right).
$$

(3.3)

Combining (3.3) with (3.2) and taking $t = 0$ and taking conditional expectation, we obtain

$$
\mathbb{E}[|\bar{Y}(0)|^2] + \mathbb{E}[\int_0^T e^{\beta s} \left( |\bar{Y}(s)|^2 + |\bar{Z}(s)|^2 + \int_F |\bar{K}(s, \zeta)|^2 \nu(d\zeta)\right) ds] \\
\leq \frac{1}{\rho} \int_0^0 e^{-\beta r} \mu(dr) \mathbb{E}[|\bar{U}(0)|^2 + \int_0^T e^{\beta s} \left( |\bar{U}(s)|^2 + |\bar{V}(s)|^2 + \int_F |\bar{Q}(s, \zeta)|^2 \nu(d\zeta)\right) ds].
$$

for $\delta \in (0, 1)$. As

$$
\frac{1}{\rho} \int_0^0 e^{-\beta r} \mu(dr) \to \frac{1}{\rho} \mu(\{0\}) < 1, \text{ as } \delta \to 0,
$$

there is some $\delta_0 > 0$ such that for all $\delta \in (0, \delta_0)$: $\frac{1}{\rho} \int_0^0 e^{-\beta r} \mu(dr) < 1$, i.e., $\Phi : ((L^2(\mathcal{F}_0) \times L^2_T) \times L^2_T \times H^2_T, ||| \cdot |||_\beta) \to ((L^2(\mathcal{F}_0) \times L^2_T) \times L^2_T \times H^2_T, ||| \cdot |||_\beta)$ has a unique fixed point: $(Y(0), Y, Z, K) \in ((L^2(\mathcal{F}_0) \times L^2_T) \times L^2_T \times H^2_T, ||| \cdot |||_\beta)$. \[\Box\]

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