Convergence analysis of a finite difference method for stochastic Cahn–Hilliard equation

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Abstract. This paper presents the convergence analysis of the spatial finite difference method (FDM) for the stochastic Cahn–Hilliard equation with Lipschitz nonlinearity and multiplicative noise. Based on fine estimates of the discrete Green function, we prove that both the spatial semi-discrete numerical solution and its Malliavin derivative have strong convergence order 1. Further, by showing the negative moment estimates of the exact solution, we obtain that the density of the spatial semi-discrete numerical solution converges in $L^1(\mathbb{R})$ to the exact one. Finally, we apply an exponential Euler method to discretize the spatial semi-discrete numerical solution in time and show that the temporal strong convergence order is nearly $3/8$, where a difficulty we overcome is to derive the optimal Hölder continuity of the spatial semi-discrete numerical solution.

1. Introduction

Consider the following stochastic Cahn–Hilliard equation

$$\partial_t u + \Delta^2 u = \Delta f(u) + \sigma(u) \dot{W}, \quad \text{in } [0,T] \times \mathcal{O}$$

(1.1)

with initial condition $u(0,\cdot) = u_0$ and homogeneous Dirichlet boundary conditions (DBCs) $u = \Delta u = 0$ on $\partial\mathcal{O}$. Here, $\mathcal{O} := (0, \pi)$, $T > 0$, and $\{W(t,x), (t,x) \in [0,T] \times \mathcal{O}\}$ is a Brownian sheet defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Assume that $u_0 : \mathcal{O} \to \mathbb{R}$ is a deterministic continuous function, and $\sigma$ is bounded and globally Lipschitz continuous. Eq. (1.1) is a well-known phenomenological model to describe the complicated phase separation. In the original form, $f$ is the derivative of the homogeneous free energy $F$ which contains a logarithmic term and in some cases can be approximated by an even-degree polynomial with a positive dominant coefficient [4], for example, $F(x) = \frac{1}{4}x^4 - \frac{1}{2}x^2$. In this case, the truncation technique is usually used to localize (1.1) such that the sequence $\{u_R\}_{R \geq 1}$ given by

$$\partial_t u_R + \Delta^2 u_R = \Delta f_R(u_R) + \sigma(u) \dot{W}, \quad \text{in } [0,T] \times \mathcal{O}$$

(1.2)

can approximate $u$ in some sense (see e.g., [4, 8]), where $\{f_R\}_{R \geq 1}$ satisfies the global Lipschitz condition. Hence, an effective numerical method applied to Eq. (1.2) is expected to approximate Eq. (1.1) well. With this consideration, the present work investigates numerical
methods for stochastic Cahn–Hilliard equations with Lipschitz nonlinearity (i.e., \( f \) is globally Lipschitz continuous), which includes the linearized Cahn–Hilliard equation \((f = 0)\).

The existing results on the numerical methods for stochastic Cahn–Hilliard equations mainly focus on the strong convergence analysis. Without being too exhaustive, we mention \([5, 21]\) on the finite element approximation for the case of \( f = 0 \). For the case of polynomial nonlinearity and additive noise, \([19]\) and \([14]\) respectively obtain the strong convergence of a spatial semi-discretization and a full discretization; \([10, 24]\) establish the strong convergence rates of full discretizations based on the finite element method and spectral Galerkin method in space, respectively. Concerning the case of multiplicative noise, \([8]\) presents the sharp strong convergence rate for a full discretization by using the spectral Galerkin method in space. In addition to the strong convergence analysis, the convergence analysis of densities of numerical solutions is also meaningful, which provides a theoretical foundation to approximate the density of the exact solution of the original system by means of numerical methods. There have been plenty results on density convergence of numerical solutions for various stochastic systems (see e.g., \([3, 6, 9, 16, 18, 22]\)), of which we have not yet found relevant results for stochastic Cahn-Hilliard equations. The present paper aims to approximate the density of the exact solution of Eq. (1.1) via a spatial finite difference method (FDM) and present the strong convergence rate and density convergence of the associated numerical solution.

The spatial FDM has been employed to numerically solve, for instance, stochastic heat equations \([15, 11, 1]\) and stochastic wave equations \([7]\). First, we give subtle error estimates between the discrete Green function of the spatial FDM and the exact one. Then under the globally Lipschitz condition on \( f \), we obtain the strong convergence order 1 for the spatial semi-discrete numerical solution \( u^n(t, x) \) based on the FDM, with \( \pi_n \) being the spatial stepsize. Further, it is shown that the Malliavin derivative of \( u^n(t, x) \) has the strong convergence order 1 as well. Combining the above results with the negative moment estimates of the exact solution, we deduce that the spatial semi-discrete numerical solution admits a density, which converges in \( L^1(\mathbb{R}) \) to the density of the exact solution.

For more effective computation, we further discretize \( u^n \) via an exponential Euler method in time and obtain the full discretization \( u^{m,n}(t, x) = \{u^{m,n}(t, x), (t, x) \in [0, T] \times \Omega\} \), where \( T_m \) denotes the temporal stepsize. As an explicit method, the exponential Euler method is more computationally efficient than the implicit method and does not suffer from the CFL condition. By investigating the temporal Hölder continuity of the spatial semi-discrete numerical solution, we attain the strong convergence rate of the proposed method for Eq. (1.1) with Lipschitz nonlinearity, namely

\[
\|u^{m,n}(t, x) - u(t, x)\|_{L^p(\Omega)} \leq C(\epsilon)(n^{-1} + m^{-\frac{3}{8}} + \epsilon),
\]  

where \( 0 < \epsilon \ll 1 \). The spatial convergence order 1 and temporal convergence order nearly \( \frac{3}{8} \) in (1.3) are optimal in the sense that they coincide with the mean-square spatial and temporal Hölder continuity exponents of the exact solution, respectively. On the basis of (1.3), a localized argument leads to an \( L^p(\Omega; \mathbb{R}) \) convergence order localized on a set of arbitrarily large probability for Eq. (1.1) with polynomial nonlinearity. With the independent interest, when \( f \) is a polynomial of degree 3 with a positive dominant coefficient, we also establish the Hölder continuity and the uniform moment estimate of the exact solution. These are prepared for the density convergence analysis of the numerical solution of the spatial FDM for Eq. (1.1) with polynomial nonlinearity in our future work.

The rest of this paper is organized as follows. Section 2 gives the Hölder continuity and the uniform moment estimate of the exact solution. Then we introduce the spatial FDM and study its strong convergence order in Section 3. Section 4 presents the density convergence
of the spatial semi-discrete numerical solution. In Section 5 we further apply an exponential Euler method to obtain a full discretization and obtain its strong convergence order.

2. Preliminaries

Let $\mathcal{C}^\alpha(O)$ be the space of $\alpha$-Hölder continuous functions on $O$ if $\alpha \in (0, 1)$, and be the space of $\alpha$ times continuously differentiable functions on $O$ if $\alpha \in \mathbb{N}$. For $d \geq 1$, we denote by $\| \cdot \|$ and $\langle \cdot, \cdot \rangle$ the Euclidean norm and inner product of $\mathbb{R}^d$, respectively. For $1 \leq q \leq \infty$, we denote by $\| \cdot \|_{L^q}$ the usual norm of the space $L^q(O) := L^q(O; \mathbb{R})$. For $1 \leq p < \infty$ and a Banach space $(H, \| \cdot \|_H)$, let $L^p(\Omega; H)$ be the space of $H$-valued random variables with bounded $p$th moment, endowed with the norm $\| \cdot \|_{L^p(\Omega; H)} := \left( \mathbb{E}[\| \cdot \|_H^p] \right)^{\frac{1}{p}}$. Especially, we write $\| \cdot \|_p := \| \cdot \|_{L^p(\Omega, \mathbb{R})}$ for short. Hereafter, we use $C$ to denote a generic positive constant that may change from one place to another and depend on several parameters but never on the space and time stepsizes. Without illustrated, the supremum with respect to $t \in [0, T]$ (respectively, $x \in \mathcal{O}$ and $(t, x) \in [0, T] \times \mathcal{O}$) is denoted by $\sup_t$ (respectively, $\sup_x$ and $\sup_{t, x}$).

In this section, we present the regularity estimate of the exact solution of Eq. (1.1) under Assumption 1 or 2.

Assumption 1. $f$ satisfies the globally Lipschitz condition, i.e., there is $K > 0$ such that

$$\| f(y) - f(z) \| \leq K \| y - z \|, \quad \forall \ y, z \in \mathbb{R}.$$  

Assumption 2. $f$ is a polynomial of degree 3 with a positive dominant coefficient, i.e.,

$$f(x) = a_0x^3 + a_1x^2 + a_2x + a_3 \text{ with } a_0 > 0.$$  

The physical importance of the Dirichlet problem is pointed out to us by M. E. Gurtin: it governs the propagation of a solidification front into an ambient medium which is at rest relative to the front [12]; see [13, 8] and references therein for the study of Cahn–Hilliard equation with DBCs. In this case, the Green function associated to $\partial_t + \Delta^2$ is given by $G_t(x, y) = \sum_{j=1}^{\infty} e^{-\lambda_j^2 t} \phi_j(x) \phi_j(y)$, $t \in [0, T]$, $x, y \in \mathcal{O}$, where $\lambda_j = -j^2$, $\phi_j(x) = \sqrt{2/\pi} \sin(jx)$, $j \geq 1$. It is known that $\{\phi_j, j \geq 1\}$ forms an orthonormal basis of $L^2(O)$. Denote $G_t(x, y) := \int_{\mathcal{O}} G_t(x, y) v(y) dy$, $v \in C(\mathcal{O})$. Similar to [4] Lemma 1.2, there exist $C, c > 0$ such that

$$|G_t(x, y)| \leq \frac{C}{t^{1/4}} \exp \left(-c \frac{|x - y|^{4/3}}{|t|^{1/3}} \right), \quad (2.1)$$

$$|\Delta G_t(x, y)| \leq \frac{C}{t^{3/4}} \exp \left(-c \frac{|x - y|^{4/3}}{|t|^{1/3}} \right). \quad (2.2)$$

The well-posedness of the stochastic Cahn–Hilliard equation under Assumption 2 with NBCs has been established in [4]. Since the Green function with DBCs and NBCs share similar properties, the existence and uniqueness of the solution to Eq. (1.1) under Assumption 2 can be obtained in an almost same way, and we present an outline of the idea here. For $R \geq 1$, let $K_R : \mathbb{R} \to \mathbb{R}$ be an even smooth cut-off function satisfying

$$K_R(x) = 1, \quad \text{if } |x| < R; \quad K_R(x) = 0, \quad \text{if } |x| \geq R + 1,$$  

and $|K_R| \leq 1$, $|K_R'| \leq 2$. Consider a sequence of SPDEs

$$\partial_t \bar{u}_R(t, x) + \Delta^2 \bar{u}_R(t, x) = \Delta \left( K_R(\|\bar{u}_R(t, \cdot)\|_{L^q}) f(\bar{u}_R(t, x)) \right) + \sigma(\bar{u}_R(t, x)) \dot{W}(t, x)$$  

(2.4)

with DBCs and $\bar{u}_R(0, \cdot) = u_0 \in L^q(O)$ for some $q \geq 4$. Define the stopping times

$$\tau_R := \inf \{t \geq 0 : \|\bar{u}_R(t, \cdot)\|_{L^q} \geq R \}, \quad R \geq 1.$$
Using the uniqueness of the solution of Eq. (2.4), it is concluded from the local property of stochastic integrals that for $R' > R$, $\bar{u}_R(t, \cdot) = \bar{u}_{R'}(t, \cdot)$ for $t \leq \tau_R$, so that a process $u$ can be defined by $u(t, \cdot) = \bar{u}_R(t, \cdot)$ for $t \leq \tau_R$. Set $\tau_\infty = \lim_{R \to \infty} \tau_R$. Then $u$ is the unique solution of (1.1) on the interval $[0, \tau_\infty)$. Further, $\{u_R\}_{R \geq 1}$ are $\{\mathcal{F}_t\}_{t \in [0,T]}$ adapted stochastic processes such that for $\rho \in [q, \infty)$,

$$
\sup_{R \geq 1} \mathbb{E} \left[ \sup_t \|\bar{u}_R(t, \cdot)\|_{L^\rho}^\rho \right] \leq C(T, \rho, q) \tag{2.5}
$$

(see the second inequality in P794 of [4]). Based on (2.5), $\tau_\infty = +\infty$ a.s. (see [4] (2.36)), and thus under Assumption 2, Eq. (1.1) admits a global solution, i.e.,

$$
u(t, x) = \mathbb{G}_t u_0(x) + \int_0^t \int_\mathcal{O} \Delta G_{t-s}(x, y) f(u(s, y)) dy ds + \int_0^t \int_\mathcal{O} G_{t-s}(x, y) \sigma(u(s, y)) W(ds, dy), \quad (t, x) \in [0, T] \times \mathcal{O}.
$$

It follows from Fatou’s lemma and (2.5) that for $\rho \in [q, \infty)$,

$$
\mathbb{E} \left[ \sup_t \|u(t, \cdot)\|_{L^\rho}^\rho \right] \leq \liminf_{\rho \to \infty} \mathbb{E} \left[ 1_{\{T \leq \tau_R\}} \sup_t \|\bar{u}_R(t, \cdot)\|_{L^\rho}^\rho \right] \leq C(T, \rho, q), \tag{2.6}
$$

which also holds for any $\rho \geq 1$ and $q \geq 1$ in view of the Hölder inequality and the continuity of $u_0$. Besides, under Assumption 1, a standard Picard approximation argument shows that Eq. (1.1) admits a unique solution satisfying (2.6).

Similar to [4] Lemma 1.8, we have the following regularity of $G$.

**Lemma 2.1.** For $\alpha \in (0, 1)$, there exists $C = C_\alpha$ such that for $x, y \in \mathcal{O}$ and $t > s$,

$$
\int_0^t \int_\mathcal{O} |G_{t-r}(x, z) - G_{t-r}(y, z)|^2 dz dr \leq C |x - y|^2,
$$

$$
\int_0^s \int_\mathcal{O} |G_{t-r}(x, z) - G_{s-r}(x, z)|^2 dz dr + \int_s^t \int_\mathcal{O} |G_{t-r}(x, z)|^2 dz dr \leq C |t - s|^{\frac{3}{2} \alpha}.
$$

Based on (2.6) and Lemma 2.1 we present the Hölder continuity of the exact solution. Under either Assumption 1 or Assumption 2 there exists some constant $K_0$ such that

$$
|f(x)| \leq K_0 (1 + |x|^3). \tag{2.7}
$$

**Lemma 2.2.** Let Assumption 1 or 2 hold, $u_0 \in C^2(\mathcal{O})$, and $\alpha \in (0, 1)$. Then for $p \geq 1$, there exists some constant $C = C(\alpha, p, T, K_0)$ such that

$$
\|u(t, x) - u(s, y)\|_p \leq C(|t - s|^\frac{3}{2} + |x - y|), \quad \forall (t, x), (s, y) \in [0, T] \times \mathcal{O}. \tag{2.8}
$$

**Proof.** We first prove

$$
\sup_{t, x} \|u(t, x)\|_p \leq C(p, T). \tag{2.9}
$$

To this end, we write $u(t, x) = \sum_{i=1}^3 u_i(t, x)$ with

$$
u_1(t, x) := \mathbb{G}_t u_0(x), \tag{2.10}
$$

$$
u_2(t, x) := \int_0^t \int_\mathcal{O} \Delta G_{t-r}(x, z) f(u(r, z)) dz dr, \tag{2.11}
$$

$$
u_3(t, x) := \int_0^t \int_\mathcal{O} G_{t-r}(x, z) \sigma(u(r, z)) W(dr, dz). \tag{2.12}
$$
Let us state a useful property in [4] Lemma 1.6. For any $\rho \in [1, \infty]$, $q \in [\rho, +\infty]$, and $1/\gamma = 1/q - 1/\rho + 1 \in [0, 1]$, the linear operator $\mathcal{L}_t$ defined by

$$
\mathcal{L}_t(v)(t, x) = \int_{t_0}^{t} \int_{\Omega} \Delta G_{t-s}(x, y)v(s, y)dyds, \quad 0 \leq t_0 < t \leq T, \quad x \in \Omega, \quad v \in L^1(t_0, T; L^p(\Omega))
$$

is a mapping from $L^1(t_0, T; L^p(\Omega))$ to $L^{\infty}(t_0, T; L^q(\Omega))$ with

$$
\|\mathcal{L}_t(v)(t, \cdot)\|_{L^q} \leq C \int_{t_0}^{t} (t - s)^{-\frac{3}{4} + \frac{1}{4\gamma}} \|v(s, \cdot)\|_{L^p} ds. \quad (2.13)
$$

It follows from (2.6) and (2.7) that

$$
\mathbb{E}[\|f(u(r, \cdot))\|^p_{L^p}] \leq C(1 + \mathbb{E}[\|u(r, \cdot)\|^p_{L^p}]) \leq C(p, T, K_0), \quad \forall \ r \in [0, T]. \quad (2.14)
$$

Applying (2.13) with $t_0 = 0$, $q = \gamma = \infty$ and $\rho = 1$ leads to

$$
\|u_2(t, \cdot)\|_{L^\infty} \leq C \int_{0}^{t} (t - r)^{-\frac{3}{4}} \|f(u(r, \cdot))\|_{L^1} dr,
$$

which combined with the Minkowski inequality and (2.14) implies

$$
\sup_x \|u_2(t, x)\|_p \leq C \int_{0}^{t} (t - r)^{-\frac{3}{4}} \left(\mathbb{E}[\|f(u(r, \cdot))\|^p_{L^p}]\right)^{\frac{1}{p}} dr \leq C(p, T, K_0)t^{\frac{3}{4}}. \quad (2.15)
$$

Since $\sigma$ is bounded, the Burkholder inequality [4] Theorem B.1 and (2.1) yield

$$
\|u_3(t, x)\|^2 \leq C(p) \int_{0}^{t} \int_{\Omega} \Delta G_{t-r}(x, z)dzdr \leq C(p)t^{\frac{3}{4}}, \quad (2.16)
$$

for $(t, x) \in [0, T] \times \Omega$. In addition, (2.14) implies $|u_1(t, x)| \leq C \|u_0\|_{C(\Omega)}$ for $x \in \Omega$, which together with (2.15) and (2.16) completes the proof of (2.3).

Without loss of generality, assume that $s < t$. Notice that $u(t, x) = u(t, y) = G_tu_0(x) - G_tu_0(y) + I_f + I_\sigma$ with

$$
I_f : = \int_{0}^{t} \int_{\Omega} [\Delta G_{t-r}(x, z) - \Delta G_{t-r}(y, z)] f(u(r, z))dzdr,
$$

$$
I_\sigma : = \int_{0}^{t} \int_{\Omega} [G_{t-r}(x, z) - G_{t-r}(y, z)] \sigma(u(r, z))W(dr, dz),
$$

and $u(t, x) - u(s, x) = G_tu_0(x) - G_su_0(x) + J^-_f + J^-_\sigma + J^-_f + J^-_\sigma + J^-_f + J^-_\sigma$ with

$$
J^-_f : = \int_{0}^{s} \int_{\Omega} [\Delta G_{t-r}(x, z) - \Delta G_{s-r}(x, z)] f(u(r, z))dzdr,
$$

$$
J^-_\sigma : = \int_{0}^{s} \int_{\Omega} [G_{t-r}(x, z) - G_{s-r}(x, z)] \sigma(u(r, z))W(dr, dz),
$$

$$
J^-_f : = \int_{s}^{t} \int_{\Omega} \Delta G_{t-r}(x, z)f(u(r, z))dzdr,
$$

$$
J^-_\sigma : = \int_{s}^{t} \int_{\Omega} G_{t-r}(x, z)\sigma(u(r, z))W(dr, dz),
$$

where the explicit dependence of $I_f, I_\sigma, J^-_f, J^-_\sigma, J^-_f, J^-_\sigma$ on $t, s, x, y$ is dropped for simplicity. Using [4] Lemma 2.3 and the assumption $u_0 \in C^2(\Omega)$, we get $|G_tu_0(x) - G_tu_0(y)| + |G_tu_0(x) -
\(G_s u_0(x) \leq C(|t-s|^{1/2} + |x-y|).\) By the Burkholder inequality, the boundedness of \(\sigma,\) and Lemma 2.1 we obtain \(\|I_{|\sigma|}\|^2 + \|J_{|\sigma|}\|^2 \leq C(|t-s|^{1/2} + |x-y|)^2).\) For any \(\alpha_1 > 0,\)
\[e^{-x} \leq C_{\alpha_1} x^{-\alpha_1}, \quad \forall \, x > 0.\] (2.17)

By the orthogonality of \(\{\phi_j\}_{j=1}^\infty \) in \(L^2(\mathcal{O}),\) (2.17), and \(|\phi_j(x) - \phi_j(y)| \leq j|x-y|\) for \(j \geq 1,
\[
\int_0^1 |\Delta G_s(x, z) - \Delta G_s(y, z)|^2 \, dz = \sum_{j=1}^\infty j^4 e^{-2js} |\phi_j(x) - \phi_j(y)|^2 \leq C|x-y|^2 \sum_{j=1}^\infty j^{6-4\rho} s^{-\rho},
\]
with \(\rho > 0.\) Choosing \(\rho \in (\frac{3}{4}, 2)\) and using the Cauchy–Schwarz inequality,
\[
\int_0^1 \int_0^1 |\Delta G_s(x, z) - \Delta G_s(y, z)| \, dz \, ds \leq \sqrt{\pi} \int_0^1 \left( \int_0^1 |\Delta G_s(x, z) - \Delta G_s(y, z)|^2 \, dz \right)^{\frac{1}{2}} \, ds \leq C|x-y| \left( \sum_{j=1}^\infty j^{6-4\rho} \right)^{\frac{1}{2}} \int_0^1 s^{-\rho} \, ds \leq C|x-y|.
\] (2.18)

Taking advantage of (2.18) and (2.29), we obtain
\[
|I_f|_p \leq C|x-y| \sup_{t,x} |f(u(t, x))|_p \leq C|x-y|(1 + \sup_{t,x} |u(t, x)|^{3}_{3p}) \leq C|x-y|.
\]

Similarly, the orthogonality of \(\{\phi_j\}_{j=1}^\infty \) in \(L^2(\mathcal{O}),\) the Cauchy–Schwarz inequality, and (2.17) yield that for \(\rho > \frac{3}{4},\)
\[
\int_s^t \int_0^1 |\Delta G_{t-r}(x, z)| \, dz \, dr \leq \sqrt{\pi} \int_s^t \left( \int_0^1 |\Delta G_{t-r}(x, z)|^2 \, dz \right)^{\frac{1}{2}} \, dr \leq C \int_s^t \left( \sum_{j=1}^\infty j^{4e^{-2j^2}} (t-r)^{-\rho} \right)^{\frac{1}{2}} \, dr \leq C(t-s)^{1-\frac{3}{4}}.
\] (2.19)

which together with (2.29) implies that for \(\alpha \in (0, 1),\)
\[
|J_f|_p \leq \int_s^t \int_0^1 |\Delta G_{t-r}(x, z)|(1 + |u(r, z)|^3_{3p}) \, dz \, dr \leq C(t-s)^{\frac{3}{2}}\alpha.
\]

Observe that for any \(\alpha_2 \in (0, 1),\)
\[
1 - e^{-x} \leq C_{\alpha_2} x^{\alpha_2}, \quad x \geq 0.
\] (2.20)

Then the orthogonality of \(\{\phi_j\}_{j=1}^\infty \) in \(L^2(\mathcal{O}),\) the Cauchy–Schwarz inequality, (2.20) and (2.17) imply that for \(\alpha_3 \in (0, \frac{3}{8})\) and \(\rho = \frac{13}{8} + \alpha_3 < 2\) with \(4 - 4\rho + 8\alpha_3 < -1,\) it holds that
\[
\int_0^1 \int_0^1 |\Delta G_{t-r}(x, z) - \Delta G_{s-r}(x, z)| \, dz \, dr \leq \sqrt{\pi} \int_0^1 \left( \int_0^1 |\Delta G_{t-r}(x, z) - \Delta G_{s-r}(x, z)|^2 \, dz \right)^{\frac{1}{2}} \, dr \leq C \int_0^1 \left( \sum_{j=1}^\infty j^{4e^{-2j^2}} |1 - e^{-j^4(t-s)|^2} \right)^{\frac{1}{2}} \, dr \leq C \left( \sum_{j=1}^\infty j^{4e^{-4\rho+8\alpha_3}} \right)^{\frac{1}{2}} \int_0^1 |s-r|^{-\frac{3}{2}} \, dr (t-s)^{\alpha_3} \leq C(t-s)^{\alpha_3}.
\] (2.21)
Hence, it follows from (2.8) that $\|J_f(t)\|_p \leq C(t - s)^{\frac{3}{2} \alpha}$ with $\alpha \in (0, 1)$. Finally, combining the above estimates, we obtain (2.8), which completes the proof. □

**Lemma 2.3.** Under the same conditions of Lemma [2.2] for any $p \geq 1$,

$$
\mathbb{E} \left[ \sup_{t,x} |u(t, x)|^p \right] \leq C(p, T, K_0).
$$

**Proof.** Based on (2.8) with $\alpha = \frac{2}{3}$, we apply [17] Theorem C.6] with $H_1 = \frac{1}{\delta}, H_2 = 1, H = 5$, $k = p > 6$, $q = 1 - \frac{6}{p}$, and $\delta = \frac{1}{2}(q + 1 - H/p)$ to obtain that there exists $C_1$ such that

$$
\mathbb{E} \left[ \sup_{(t,x) \neq (s, y)} \frac{|u(t, x) - u(s, y)|^p}{(|t - s|^\frac{1}{p} + |x - y|)^{p-6}} \right] \leq C_1.
$$

Hence, for $p > 6$,

$$
\mathbb{E} \left[ \sup_{t,x} |u(t, x)|^p \right] = \mathbb{E} \left[ \sup_{t,x} |u(t, x) - u(0, 0)|^p \right] \leq C_1 (T^{\frac{1}{4}} + \pi)^{p-6}.
$$

(2.22)

For $1 \leq p \leq 6$, the desired result follows from (2.22) and the Hölder inequality. The proof is completed. □

### 3. Finite difference method

In this section, we introduce the spatial FDM method for Eq. (1.1) and derive its strong convergence rate. Given a function $w$ on the mesh $\mathcal{O}^h := \{0, h, 2h, \ldots, \pi\}$, we define the difference operator

$$
\delta_h w_i := \frac{w_{i-1} - 2w_i + w_{i+1}}{h^2}, \quad \delta_h^2 w_i := \frac{w_{i-2} - 4w_{i-1} + 6w_i - 4w_{i+1} + w_{i+2}}{h^4},
$$

for $i \in \mathbb{Z}_n := \{1, 2, \ldots, n - 1\}$, where $w_i := w(ih)$. The compatibility conditions $u_0(0) = u_0(\pi) = 0$ and $u_0''(0) = u_0''(\pi) = 0$ are direct results of DBCs and the initial condition. One can approximate $u(t, kh)$ via $\{u^n(t, kh)\}_{n \geq 2}$, where $u^n(0, kh) = u_0(kh)$ and

$$
\begin{align*}
\frac{du^n(t, kh)}{dt} + \delta_h^2 u^n(t, kh)dt &= \delta_h f(u^n(t, kh))dt + n\pi^{-1}\sigma(u^n(t, kh))d(W(t, (k+1)h) - W(t, kh)), \\
&= u^n(t, 0) = u^n(t, \pi) = 0, \quad u^n(t, -h) + u^n(t, h) = u^n(t, (n - 1)h) + u^n(t, (n + 1)h) = 0,
\end{align*}
$$

(3.1)

for $t \in [0, T]$ and $k \in \mathbb{Z}_n$, under the boundary conditions

$$
u^n(t, 0) = u^n(t, \pi) = 0, \quad u^n(t, -h) + u^n(t, h) = u^n(t, (n - 1)h) + u^n(t, (n + 1)h) = 0,
$$

for $t \in (0, T]$. For $k \in \mathbb{Z}_n \cup \{0\}$, we use the polygonal interpolation

$$
u^n(t, x) := u^n(t, kh) + (n\pi^{-1}x - k)(u^n(t, (k + 1)h) - u^n(t, kh)), \quad \forall x \in [kh, (k + 1)h].$$

To solve (3.1), we introduce

$$
U(t) = (U_1(t), \ldots, U_{n-1}(t))^\top, \quad \beta_t = (\beta_t^1, \ldots, \beta_t^{n-1})^\top
$$

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with $U_k(t) := w^n(t, kh)$ and $\beta_t^k := \sqrt{\frac{\pi}{2n}}(W(t, (k+1)h) - W(t, kh))$ for $k \in \mathbb{Z}_n$, where the explicit dependence of $U(t)$ and $\beta_t$ on $n$ is omitted. Let

$$A_n := \frac{n^2}{\pi^2} \begin{bmatrix} -2 & 1 & 0 & \cdots & \cdots & 0 \\
1 & -2 & 1 & \ddots & \ddots & \vdots \\
0 & 1 & -2 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 1 & -2 & 1 \\
0 & \cdots & \cdots & 0 & 1 & -2 \end{bmatrix}.$$

Then (3.1) can be rewritten into an $(n-1)$-dimensional SDE

$$\begin{align*}
dU(t) + A_n U(t) dt &= A_n F_n(U(t)) dt + \sqrt{n/\pi} \Sigma_n(U(t)) dB_t \\
&= \sqrt{n/\pi} \Sigma_n(U(t)) dB_t, \quad t \in [0, T].
\end{align*}
$$

(3.2)

with the initial condition $U(0) = (u_0(h), \ldots, u_0((n-1)h))^\top$ and the coefficients

$$F_n(U(t)) = (f(U_1(t)), \ldots, f(U_{n-1}(t)))^\top, \quad \Sigma_n(U(t)) = \text{diag}(\sigma(U_1(t)), \ldots, \sigma(U_{n-1}(t))).$$

Under Assumption 1, (3.2) admits a unique strong solution which satisfies

$$U(t) = \exp(-A_n^2 t) U(0) + \int_0^t A_n \exp(-A_n^2 (t-s)) F_n(U(s)) ds
\begin{align*}
&+ \sqrt{n/\pi} \int_0^t \exp(-A_n^2 (t-s)) \Sigma_n(U(s)) dB_s, \quad t \in [0, T].
\end{align*}
$$

(3.3)

For $j \in \mathbb{Z}_n$, $e_j = (e_j(1), \ldots, e_j(n-1))^\top$ given by

$$e_j(k) = \sqrt{\frac{\pi}{n}} \phi_j(kh) = \sqrt{2/\pi} \sin(jkh), \quad k \in \mathbb{Z}_n,$n

(3.4)

is an eigenvector of $A_n$ associated with the eigenvalue $\lambda_{j,n} = -j^2 c_{j,n}$, where

$$c_{j,n} := \sin^2 \left( \frac{j}{2n} \pi \right) / \left( \frac{j}{2n} \pi \right)^2$$

satisfies $\frac{1}{\pi^2} \leq c_{j,n} \leq 1$. The vectors $\{e_i\}_{i=1}^{n-1}$ form an orthonormal basis of $\mathbb{R}^{n-1}$. In particular,

$$\langle e_i, e_j \rangle = \frac{\pi}{n} \sum_{k=1}^{n-1} \phi_i(kh) \phi_j(kh) = \int_0^\pi \phi_i(\kappa_n(y)) \phi_j(\kappa_n(y)) dy \delta_{ij},$$

(3.5)

where $\kappa_n(y) = h[y/h]$ with $[\cdot]$ being the floor function (see e.g., [15]). It is verified that $1 - \frac{\sin a}{a} \leq \frac{1}{6} a^2$ for all $a \in \left[0, \frac{\pi}{2} \right]$, which indicates that for $j \in \mathbb{Z}_n$,

$$0 \leq 1 - c_{j,n} = \left( 1 + \sin \left( \frac{j}{2n} \pi \right) / \left( \frac{j}{2n} \pi \right) \right) \left( 1 - \sin \left( \frac{j}{2n} \pi \right) / \left( \frac{j}{2n} \pi \right) \right) \leq \frac{\pi^2 j^2}{12n^2}.$$n

(3.6)

Introduce the discrete kernel

$$G^n_l(x, y) = \sum_{j=1}^{n-1} \exp(-\lambda_{j,n}^2 t) \phi_{j,n}(x) \phi_{j,n}(y),$$

where $\phi_{j,n}(x) = \phi_j(kh) + (n\pi^{-1} x - k)(\phi_j((k+1)h) - \phi_j(kh))$ for $x \in [kh, (k+1)h]$, $k \in \mathbb{Z}_n \cup \{0\}$. Define the discrete Dirichlet Laplacian $\Delta_n$ by $\Delta_n w(y) = 0$ for $y \in [0, h)$,

$$\Delta_n w(y) = \frac{n^2}{\pi^2} \left( w(\kappa_n(y) + \frac{\pi}{n}) - 2w(\kappa_n(y)) + w(\kappa_n(y) - \frac{\pi}{n}) \right), \quad y \in [h, \pi),$$

(3.7)
where \( w : \mathcal{O} \to \mathbb{R} \) with \( w(0) = w(\pi) = 0 \). Since \( \Delta_n \phi_j(\kappa_n(y)) = \lambda_{j,n} \phi_j(\kappa_n(y)) \),
\[
\Delta_n G^n_t(x, y) = \sum_{j=1}^{n-1} \lambda_{j,n} \exp(-\lambda_{j,n}^2 t) \phi_{j,n}(x) \phi_j(\kappa_n(y)).
\]

Similar to [15 Section 2], based on [3.3], the diagonalization of the matrix \( A_n \), [5.4] and \( u^n(t, kh) = \sum_{k=1}^{n-1} (U(t, e_j)e_j(k)) \), one has
\[
u^n(t, x) = \int_0^t G^n_t(x, y) u_0(\kappa_n(y)) dy + \int_0^t \int_0^t G^n_{t-s}(x, y) f(u^n(s, \kappa_n(y))) dy ds
\]
\[+ \int_0^t \int_0^t G^n_{t-s}(x, y) \sigma(u^n(s, \kappa_n(y))) W(ds, dy), \quad (t, x) \in [0, T] \times \mathcal{O}. \quad (3.8)
\]

The follow lemma characterizes the error between \( G \) and \( G^n \).

**Lemma 3.1.** There exists some constant \( C = C(T) \) such that for any \( x \in \mathcal{O} \) and \( t \in (0, T) \),
\[
\int_0^t \int_\mathcal{O} |\Delta_n G^n_s(x, y) - \Delta G_s(x, y)| dy ds \leq C n^{-1}, \quad (3.9)
\]
\[
\int_0^t \int_\mathcal{O} |G^n_s(x, y) - G_s(x, y)|^2 dy ds \leq C n^{-2}. \quad (3.10)
\]

**Proof.** For \( s \in [0, T] \) and \( x, y \in \mathcal{O} \), denote \( M_1^{s,x,y} = \sum_{j=1}^{n-1} \lambda_{j,n} e^{-\lambda_{j,n}^2 s} \phi_{j,n}(x) (\phi_j(\kappa_n(y)) - \phi_j(y)) \),
\( M_2^{s,x,y} = \sum_{j=1}^{n-1} (\lambda_{j,n} e^{-\lambda_{j,n}^2 s} \phi_{j,n}(x) - \lambda_j e^{-\lambda_j^2 s} \phi_j(x)) \phi_j(y) \) and \( M_3^{s,x,y} = \sum_{j=1}^{n} - \lambda_j e^{-\lambda_j^2 s} \phi_j(x) \phi_j(y) \). Since \( \{ \phi_j \}_{j \geq 1} \) is an orthonormal basis of \( L^2(\mathcal{O}) \), it holds that
\[
\int_\mathcal{O} |\Delta_n G^n_s(x, y) - \Delta G_s(x, y)|^2 dy = \int_\mathcal{O} |\sum_{i=1}^3 M_i^{s,x,y}|^2 dy 
\]
\[\leq 3 \int_\mathcal{O} |M_1^{s,x,y}|^2 dy + 3 \sum_{j=1}^{n-1} |\lambda_{j,n} e^{-\lambda_{j,n}^2 s} \phi_{j,n}(x) - \lambda_j e^{-\lambda_j^2 s} \phi_j(x)|^2 + 3 \sum_{j=1}^{\infty} e^{-2\lambda_j^2 s} \quad (3.11)
\]

It follows from the boundedness of \( \{ \phi_j \}_{j \geq 1} \) and [2.17] that for \( \alpha_1 \in (\frac{5}{4}, 2) \),
\[
\int_0^t \left( \sum_{j=n}^{\infty} j^{4-2\alpha_1} \right)^{\frac{1}{2}} ds \leq C \int_0^t \left( \sum_{j=n}^{\infty} j^{4-4\alpha_1 s-\alpha_1} \right)^{\frac{1}{2}} ds \leq C(\alpha_1)n^{\frac{5-4\alpha_1}{2}}. \quad (3.12)
\]

By [3.6], we have \( |\lambda_j - \lambda_j| \leq C j^4/n^2 \), and thus \( \lambda_j^2 - \lambda_j^2 \leq |\lambda_j - \lambda_j| ||\lambda_j + \lambda_j|| \leq C j^6/n^2 \), which along with [2.20] yields \( |e^{-\lambda_j^2 s} - e^{-\lambda_j^2 s}| = e^{-\lambda_j^2 s} |1 - e^{-(\lambda_j^2 - \lambda_j^2 s)s} | \leq e^{-\lambda_j^2 s} \frac{26}{7} \). Besides, it can be verified that \( |\phi_{j,n}(x) - \phi_j(x)| \leq C j/n \). Therefore, for \( \rho, \rho_1 > 0 \)
\[
\sum_{j=1}^{n-1} |\lambda_{j,n} e^{-\lambda_{j,n}^2 s} \phi_{j,n}(x) - \lambda_j e^{-\lambda_j^2 s} \phi_j(x)|^2 
\]
\[\leq \sum_{j=1}^{n-1} |\lambda_{j,n} - \lambda_j|^2 e^{-2\lambda_j^2 s} + \sum_{j=1}^{n-1} \lambda_j^2 e^{-\lambda_j^2 s} - e^{-\lambda_j^2 s} |^2 + \sum_{j=1}^{n-1} \lambda_j^2 e^{-2\lambda_j^2 s} |\phi_{j,n}(x) - \phi_j(x)|^2 
\]
\[\leq C n^{-4} \sum_{j=1}^{n-1} j^{8-4\rho} s^{-\rho} + C n^{-4} \sum_{j=1}^{n-1} j^{16-4\rho_1} s^{2-\rho_1} + C n^{-2} \sum_{j=1}^{n-1} j^{6-4\rho} s^{-\rho}. \]
Choosing \( \rho = \rho_1 - 2 \in (0, 2) \), we obtain
\[
\int_0^t \left( \sum_{j=1}^{n-1} \left| \lambda_{j,n} e^{-\lambda_j^2 t, n_s^2} \phi_{j,n}(x) - \lambda_j e^{-\lambda_j^2 s} \phi_j(x) \right|^2 \right)^{\frac{1}{2}} ds \leq C \int_0^t (n^{5-4\rho} s^{-\rho})^{\frac{1}{2}} ds \leq C n^{\frac{5-4\rho}{2}}. 
\]  
(3.13)

We recall the following inequality in \([15]\) Lemma 3.2:
\[
\int_{\Omega} |w(y) - w(\kappa_n(y))|^2 dy \leq C n^{-2} \int_{\Omega} \left| \frac{d}{dy} w(y) \right|^2 dy, \quad \text{for } w \in C^1(\Omega). 
\]  
(3.14)

Introducing \( B_n(s, x, y) = \sum_{j=1}^{n-1} \lambda_{j,n} e^{-\lambda_j^2 t, n_s^2} \phi_{j,n}(x) \phi_j(y) \) and making use of (3.14), we obtain
\[
\int_{\Omega} |M_1^{s,x,y}|^2 dy = \int_{\Omega} |B_n(s, x, y) - B_n(s, x, \kappa_n(y))|^2 dy \leq C n^{-2} \sum_{j=1}^{n-1} j^6 e^{-2\lambda_j^2 s} \phi_{j,n}(x)^2.
\]

Hence, it follows from (2.17) that for \( \rho \in \left( \frac{4}{3}, 2 \right) \),
\[
\int_0^t \left( \int_{\Omega} |M_1^{s,x,y}|^2 dy \right)^{\frac{1}{2}} ds \leq C \int_0^t \left( \frac{1}{n^2} \sum_{j=1}^{n-1} j^{6-4\rho} s^{-\rho} \right)^{\frac{1}{2}} ds \leq C n^{-1}.
\]  
(3.15)

Combining (3.11)-(3.13) with (3.15) allows us to deduce
\[
\int_0^t \int_{\Omega} |\Delta_n G_n^s(x, y) - \Delta G_s(x, y)| dy ds \leq \sqrt{\pi} \int_0^t \left( \int_{\Omega} |\Delta_n G_n^s(x, y) - \Delta G_s(x, y)|^2 dy \right)^{\frac{1}{2}} ds \leq C n^{-1},
\]
which proves (3.9). It remains to prove (3.10). Set \( H_n(t, x, y) := \sum_{j=1}^{n-1} \exp(-\lambda_j^2 t) \phi_j(x) \phi_j(y). \)

Then \( \int_{\Omega} |G_n^s(x, y) - G_s(x, y)|^2 dy \leq 4 \sum_{k=1}^{\beta} J_k(s, x) \), where
\[
J_1(s, x) := \sum_{j=n}^{\infty} \exp(-2\lambda_j^2 s), \quad J_2(s, x) := \int_{\Omega} |H_n(s, x, y) - H_n(s, x, \kappa_n(y))|^2 dy,
\]
\[
J_3(s, x) := \sum_{j=1}^{n-1} \left| \exp(-\lambda_j^2 s) - \exp(-\lambda_{j,n}^2 s) \right|^2, \quad J_4(s, x) := \sum_{j=1}^{n-1} e^{-2\lambda_j^2 n_s^2} |\phi_{j,n}(x) - \phi_j(x)|^2.
\]

For the first term, we have \( \int_0^t J_1(s, x) ds \leq C \sum_{j=n}^{\infty} j^{-4} \leq C n^{-3} \). For \( 2 < \alpha < 3 \), \( J_3(s, x) \leq \sum_{j=1}^{n-1} e^{-2\lambda_j^2 n_s^2} (1 - e^{-(\lambda_j^2 - \lambda_{j,n}^2) s^2})^2 \leq C \sum_{j=1}^{n-1} j^{12} n_s^{4 \alpha} s^{-2-\alpha} \leq C n^{9 - 4\alpha} s^{-2-\alpha} \), which implies that \( \int_0^t J_3(s, x) ds \leq C n^{-3+\epsilon} \) with arbitrarily small \( \epsilon > 0 \). Since \( |\phi_{j,n}(x) - \phi_j(x)| \leq C j/n \), it holds that \( J_4(s, x) \leq C \sum_{j=1}^{n-1} j^{-4\alpha} s^{-\alpha} j^2 \leq C n^{-2-\alpha} \) for \( \frac{3}{4} < \alpha < 1 \), and thus \( \int_0^t J_4(s, x) ds \leq C n^{-2} \). Using (3.14), we arrive at
\[
\int_0^t J_2(s, x) ds \leq C n^{-2} \int_0^t \int_{\Omega} |\frac{d}{dy} H_n(s, x, y)|^2 dy ds \leq C n^{-2} \sum_{j=1}^{n-1} \int_0^t j^2 \exp(-2j^4 s) ds \leq C n^{-2}.
\]

Combining the above estimates completes the proof of (3.10). \( \Box \)

For \( n \geq 2 \), denote by \( \mathbb{U}(t) := (u(t, h), \ldots, u(t, (n-1)h))^\top \) the exact solution of Eq. (1.1) on spatial grid points, where the explicit dependence of \( \mathbb{U}(t) \) on \( n \) is omitted. We introduce the following auxiliary process \( \{ \mathbb{U}(t), t \in [0, T] \} \) by
\[
d\mathbb{U}(t) + A_n^2 \mathbb{U}(t) dt = A_n F_n(\mathbb{U}(t)) dt + \sqrt{n/\pi} \Sigma_n(\mathbb{U}(t)) d\beta_t, \quad t \in [0, T]
\]
with initial value $\tilde{U}(0) = U(0)$. Let $\tilde{u}^n = \{\tilde{u}^n(t, x), (t, x) \in [0, T] \times \mathcal{O}\}$ satisfy
\[
\tilde{u}^n(t, x) = \int_\mathcal{O} G^n_t(x, y)u_0(\kappa_n(y)) dy + \int_0^t \int_\mathcal{O} \Delta_n G^n_{t-s}(x, y)f(u(s, \kappa_n(y))) dy ds
+ \int_0^t \int_\mathcal{O} G^n_{t-s}(x, y)\sigma(u(s, \kappa_n(y))) W(ds, dy).
\] (3.16)

Then $\tilde{U}_k(t) = \tilde{u}^n(t, kh)$ for $k \in \mathbb{Z}_n$ and $t \in [0, T]$. In order to estimate $\|u^n(t, x) - u(t, x)\|_p$, it suffices to estimate $\|\tilde{u}^n(t, x) - u(t, x)\|_p$ and $\|\tilde{u}^n(t, x) - u^n(t, x)\|_p$, where the first term is tackled as follows.

**Lemma 3.2.** Suppose that Assumption 1 or 2 holds and $u_0 \in C^3(\mathcal{O})$. Then for any $p \geq 1$, there exists some constant $C = C(p, T, K_0)$ such that for any $(t, x) \in [0, T] \times \mathcal{O}$,
\[
\|\tilde{u}^n(t, x) - u(t, x)\|_p \leq C n^{-1}.
\]

**Proof.** Recall that $u = u_1 + u_2 + u_3$, where $u_i$, $i = 1, 2, 3$, are defined in $[2.10]-[2.12]$, respectively. Similarly, for $(t, x) \in [0, T] \times \mathcal{O}$, we introduce $\tilde{u}^n_i(t, x) := \int_\mathcal{O} G^n_t(x, y)u_0(\kappa_n(y)) dy$,
\[
\tilde{u}^n_2(t, x) := \int_0^t \int_\mathcal{O} \Delta_n G^n_{t-s}(x, y)f(u(s, \kappa_n(y))) dy ds,
\]
\[
\tilde{u}^n_3(t, x) := \int_0^t \int_\mathcal{O} G^n_{t-s}(x, y)\sigma(u(s, \kappa_n(y))) W(ds, dy),
\]
and divide the proof into three parts.

**Part 1:** Following the proof of [3] Lemma 2.3, we use the PDE satisfied by $G$ to write $u_1(t, x) = u_0(x) - \int_0^t \int_\mathcal{O} \Delta G_r(x, z)u_0'(z) dz dr$. As a numerical counterpart,
\[
\tilde{u}^n_1(t, x) - \tilde{u}^n(0, x) = \int_\mathcal{O} \int_0^t \frac{\partial}{\partial r} G^n_r(x, z)u_0(\kappa_n(z)) dz dr
= - \int_0^t \int_\mathcal{O} \Delta^2_n G^n_r(x, z)u_0(\kappa_n(z)) dz dr = - \int_0^t \int_\mathcal{O} \Delta_n G^n_r(x, z)\Delta_n u_0(z) dz dr,
\] (3.17)
where in the last step we have used the fact that
\[
\int_\mathcal{O} \Delta_n v(z) w(\kappa_n(z)) dz = \int_\mathcal{O} v(\kappa_n(z)) \Delta_n w(z) dz,
\]
for $v, w : \mathcal{O} \to \mathbb{R}$ with $v = w = 0$ on $\partial \mathcal{O}$. Here, $\tilde{u}^n(0, kh) = u_0(kh)$ for $k \in \mathbb{Z}_n \cup \{0\}$, and $\tilde{u}^n(0, x) = u_0(kh) + (n\pi^{-1}x - k)u_0((k+1)h) - u_0(kh))$ for $x \in [kh, (k+1)h]$, $k \in \mathbb{Z}_n \cup \{0\}$. In particular, when $u_0 \in C^1(\mathcal{O})$, it holds that
\[
|\tilde{u}^n(0, x) - \tilde{u}^n(0, y)| \leq C|x - y|, \quad x, y \in \mathcal{O}.
\] (3.18)

By $u_0 \in C^3(\mathcal{O})$ and (3.7), there exist $\theta_1, \theta_2 \in (0, 1)$ such that for $z \in [h, \pi)$,
\[
|u_0''(z) - \Delta_n u_0(z)| = |u_0''(z) - \frac{1}{2} u_0''(\kappa_n(z) + \frac{\pi}{n}) - \frac{1}{2} u_0''(\kappa_n(z) - \theta_2 \frac{\pi}{n})| \leq C n^{-1},
\]
and for $z \in [0, h)$, $|u_0''(z) - \Delta_n u_0(z)| = |u_0''(z)| = |u_0''(z) - u_0''(0)| \leq C n^{-1}$. Therefore, using (2.2) and (3.9), a direct calculation gives
\[
|\tilde{u}^n_i(t, x) - u_1(t, x)| \leq \frac{C}{n} + \int_0^t \int_\mathcal{O} |\Delta_n G^n_r(x, z) - \Delta G_r(x, z)| dz dr
+ \int_0^t \int_\mathcal{O} |\Delta G_r(x, z)||u_0''(z) - \Delta_n u_0(z)| dz dr \leq C n^{-1}.
\] (3.19)
**Part 2:** The error \( \tilde{u}_3^N(t, x) - u_3(t, x) \) is divided into

\[
\tilde{u}_3^N(t, x) - u_3(t, x) = \int_0^t \int_\Omega [G_{t-s}^n(x, y) - G_{t-s}(x, y)] \sigma(u(s, \kappa_n(y))) W(ds, dy) \\
+ \int_0^t \int_\Omega G_{t-s}(x, y) [\sigma(u(s, \kappa_n(y))) - \sigma(u(s, y))] W(ds, dy).
\]

The Burkholder inequality, the boundedness and Lipschitz continuity of \( \sigma \), (3.10), (2.11), and Lemma 2.2 imply

\[
\|\tilde{u}_3^N(t, x) - u_3(t, x)\|_p^2 \\
\leq \int_0^t \int_\Omega |G_{t-s}^n(x, y) - G_{t-s}(x, y)|^2 dyds + C \int_0^t \int_\Omega G_{t-s}^2(x, y) \|u(s, \kappa_n(y)) - u(s, y)\|_p^2 dyds \\
\leq Cn^{-2} + C \int_0^t (t-s)^{-\frac{1}{2}} \sup_{y \in \Omega} \|u(s, \kappa_n(y)) - u(s, y)\|_p^2 ds \leq Cn^{-2}. \tag{3.20}
\]

**Part 3:** Notice that \( \|\tilde{u}_2^N(t, x) - u_2(t, x)\|_p \leq I_1 + I_2 \), where

\[
I_1 := \int_0^t \int_\Omega |\Delta_n G_{t-s}^n(x, y) - \Delta G_{t-s}(x, y)| \|f(u(s, \kappa_n(y)))\|_p dyds,
\]
\[
I_2 := \int_0^t \int_\Omega |\Delta G_{t-s}(x, y)| \|f(u(s, \kappa_n(y))) - f(u(s, y))\|_p dyds.
\]

It follows from (3.9), (2.7) and Lemma 2.3 that

\[
I_1 \leq C \int_0^t \int_\Omega |\Delta_n G_{t-s}^n(x, y) - \Delta G_{t-s}(x, y)| dyds \left(1 + \sup_{t, x} \|u(t, x)\|_p^3\right) \leq Cn^{-1}.
\]

Under Assumption 1 or 2 \( \|f(a_1) - f(a_2)\| \leq c_0(1 + a_1^2 + a_2^2)(a_2 - a_1) \). Hence, the Hölder inequality together with Lemmas 2.2 and 2.3 yields that for any \( (s, x) \in [0, T] \times \Omega \),

\[
\|f(u(s, \kappa_n(x))) - f(u(s, x))\|_p \\
\leq C \|u(s, \kappa_n(x)) - u(s, x)\|_3p (1 + \|u(s, \kappa_n(x))\|_3p^2 + \|u(s, x)\|_3p^2) \leq Cn^{-1}.
\]

Therefore, \( I_2 \leq Cn^{-1} \int_0^t \int_\Omega |\Delta G_{t-s}(x, y)| dyds \leq Cn^{-1} \), in view of (2.11). In conclusion, \( \|\tilde{u}_2^N(t, x) - u_2(t, x)\|_p \leq Cn^{-1} \), which along with (3.19) and (3.20) finishes the proof. \( \square \)

By Lemma 3.2 we present the strong convergence rate of the spatial FDM for Eq. (1.1). We would like to mention that Theorem 3.3 also holds for stochastic Cahn–Hilliard equations with NBCs.

**Theorem 3.3.** Suppose that Assumption 1 holds and \( u_0 \in C^3(\Omega) \). Then for every \( p \geq 1 \), there exists some constant \( C = C(p, T, K) \) such that for any \( (t, x) \in [0, T] \times \Omega \),

\[
\|u^n(t, x) - u(t, x)\|_p \leq Cn^{-1}.
\]

**Proof.** Denote \( e^n(t, x) := u^n(t, x) - u(t, x) \). In view of (3.8) and (3.16),

\[
u^n(t, x) - \tilde{u}^n(t, x) = \int_0^t \int_\Omega \Delta_n G_{t-s}^n(x, y) \left[ f(u^n(s, \kappa_n(y))) - f(u(s, \kappa_n(y))) \right] dyds \\
+ \int_0^t \int_\Omega G_{t-s}^n(x, y) \left( \sigma(u^n(s, \kappa_n(y))) - \sigma(u(s, \kappa_n(y))) \right) W(ds, dy).
\]
By the expressions of $G^n_t$ and $\Delta_n G^n_t$ and (2.17), we arrive at that for $0 < \epsilon \ll 1$,
\[
|G^n_t(x, y)| \leq C_\epsilon t^{-\frac{1}{2} - \epsilon}, \quad |\Delta_n G^n_t(x, y)| \leq C_\epsilon t^{-\frac{3}{2} - \epsilon}, \quad \forall \ t \in (0, T], \ x, y \in \mathcal{O}.
\] (3.21)

Hence, the Cauchy–Schwarz inequality with respect to the measure $|\Delta_n G^n_{t-s}(x, y)|dy ds$, Lemma 3.2 and the Minkowski and Burkholder inequalities yield that for $0 < \epsilon \ll 1$,
\[
\|e^n(t, x)\|_p \leq C n^{-2} + C \int_0^t \int_{\mathcal{O}} |\Delta_n G^n_{t-s}(x, y)||u^n(s, \kappa_n(y)) - u(s, \kappa_n(y))|^2 dy ds
\]
\[
\quad + C \int_0^t \int_{\mathcal{O}} |G^n_{t-s}(x, y)||u^n(s, \kappa_n(y)) - u(s, \kappa_n(y))|^2 dy ds
\]
\[
\leq C n^{-2} + C_\epsilon \int_0^t \int_{\mathcal{O}} (t-s)^{-\frac{3}{2} - \epsilon}|u^n(s, \kappa_n(y)) - u(s, \kappa_n(y))|^2 dy ds,
\]
in which the second step used (3.21). Taking the supremum over $x$ produces
\[
\sup_x \|e^n(t, x)\|_p \leq C n^{-2} + C_\epsilon \int_0^t (t-s)^{-\frac{3}{2} - \epsilon}\sup_x \|e^n(s, x)\|_p ds,
\] (3.22)
which along with the Gronwall lemma with weak singularities (see e.g., [15] Lemma 3.4) completes the proof. □

4. Convergence of density

For real-valued random variables $X, Y$, we write $d_{TV}(X, Y)$ to indicate the total variation distance between $X$ and $Y$, i.e.,
\[
d_{TV}(X, Y) = 2 \sup_{A \in \mathcal{B}(\mathbb{R})} \{|\mathbb{P}(X \in A) - \mathbb{P}(Y \in A)|\} = \sup_{\phi \in \Phi} |\mathbb{E}[\phi(X)] - \mathbb{E}[\phi(Y)]|,
\]
where $\Phi$ is the set of continuous functions $\phi : \mathbb{R} \to \mathbb{R}$ which are bounded by 1, and $\mathcal{B}(\mathbb{R})$ is the Borel $\sigma$-algebra of $\mathbb{R}$. Furthermore, if $\{X_n\}_{n \geq 1}$ and $X_\infty$ have the densities $p_{X_n}$ and $p_{X_\infty}$ respectively, then
\[
d_{TV}(X_n, X_\infty) = \|p_{X_n} - p_{X_\infty}\|_{L^1(\mathbb{R})}.
\] (4.1)

In this section, we show that for $k \in \mathbb{Z}_n$, the spatial semi-discrete numerical solution $u^n(T, kh)$ admits a density, which converges in $L^1(\mathbb{R})$ to the density of the exact solution $u(T, kh)$.

4.1. Malliavin calculus. We start with introducing some notations in the context of the Malliavin calculus with respect to the space-time white noise (see e.g., [23]). The isonormal Gaussian family $\{W(h), h \in \mathcal{H}\}$ corresponding to $\mathcal{H} := L^2([0, T] \times \mathcal{O})$ is given by the Wiener integral $W(h) = \int_0^T \int_{\mathcal{O}} h(s, y)W(ds, dy)$. Denote by $\mathcal{S}$ the class of smooth real-valued random variables of the form
\[
X = \varphi(W(h_1), \ldots, W(h_n)),
\] (4.2)
where $\varphi \in C^\infty_p(\mathbb{R}^n)$, $h_i \in \mathcal{H}$, $i = 1, \ldots, n$, $n \geq 1$. Here $C^\infty_p(\mathbb{R}^n)$ is the space of all $\mathbb{R}$-valued smooth functions on $\mathbb{R}^n$ whose partial derivatives have at most polynomial growth. The Malliavin derivative of $X \in \mathcal{S}$ of the form (4.2) is an $\mathcal{H}$-valued random variable given by $DX = \sum_{i=1}^n \partial_i \varphi(W(h_1), \ldots, W(h_n))h_i$, which is also a random field $DX = \{D_{\theta, \xi}X, (\theta, \xi) \in [0, T] \times \mathcal{O}\}$ with $D_{\theta, \xi}X = \sum_{i=1}^n \partial_i \varphi(W(h_1), \ldots, W(h_n))h_i(\theta, \xi)$ for almost everywhere $(\theta, \xi, \omega) \in [0, T] \times \mathcal{O} \times \Omega$. For any $p \geq 1$, we denote the domain of $D$ in $L^p(\Omega; \mathbb{R})$ by $\mathbb{D}^{1,p}$, meaning that $\mathbb{D}^{1,p}$ is the closure of $\mathcal{S}$ with respect to the norm
\[
\|X\|_{1,p} = \left(\mathbb{E} \left[\|X\|^p + \|DX\|^p_{\mathcal{H}}\right]\right)^{\frac{1}{p}}.
\]
We define the iteration of the operator \( D \) in such a way that for \( X \in \mathcal{S} \), the iterated derivative \( D^kX \) is an \( \mathcal{S} \)-valued random variable. More precisely, for \( k \in \mathbb{N}_+ \), \( D^kX = \{D_{r_1 \theta_1} \cdots D_{r_k \theta_k} X, (r_i, \theta_i) \in [0, T] \times \mathcal{O}\} \) is a measurable function on the product space \([0, T] \times \mathcal{O}\). Then for \( p \geq 1 \), \( k \in \mathbb{N} \), denote by \( \mathbb{D}^{k,p} \) the completion of \( \mathcal{S} \) with respect to the norm \( \|X\|_{k,p} = \left( \mathbb{E}\left[|X|^p + \sum_{j=1}^k \|D^jX\|_{\mathbb{S}^j}^p \right]\right)^{\frac{1}{p}} \). Define \( \mathbb{D}^{k,\infty} := \bigcap_{p \geq 1} \mathbb{D}^{k,p} \) and \( \mathbb{D}^\infty := \bigcap_{k \geq 1} \mathbb{D}^{k,\infty} \) to be topological projective limits.

We close this part by the following proposition, which allows us to obtain the convergence of density of a sequence of random variables from the convergence in \( \mathbb{D}^{1,2} \).

**Proposition 4.1.** \[22\] Theorem 4.2 Let \( \{X_n\}_{n \geq 1} \) be a sequence in \( \mathbb{D}^{1,2} \) such that each \( X_n \) admits a density. Let \( X_\infty \in \mathbb{D}^{2,4} \) and let \( 0 < \alpha \leq 2 \) be such that \( \mathbb{E}[\|DX_\infty\|_H^\alpha] < \infty \). If \( X_n \to X_\infty \) in \( \mathbb{D}^{1,2} \), then there exists a constant \( c > 0 \) depending only on \( X_\infty \) such that for any \( n \geq 1 \),

\[
\text{d}_{TV}(X_n, X_\infty) \leq c\|X_n - X_\infty\|_{1/2}^{\alpha/2}.
\]

**4.2. Convergence in \( \mathbb{D}^{1,2} \).** In this part, we extend the strong convergence of the spatial FDM to the convergence in \( \mathbb{D}^{1,2} \). It is shown in \[8\] Proposition 3.1 or \[4\] Lemma 3.2] that if \( f(x) = (x^3 - x)K_R(x) \), then for any \((t, x) \in [0, T] \times \mathcal{O}\), \( u(t, x) \in \mathbb{D}^{1,2} \) and satisfies

\[
D_{r,z}u(t,x) = G_{t-r}(x,z)\sigma(u(r,z)) + \int_r^t \int_\mathcal{O} \Delta G_{t-s}(x,y)f'(u(s,y))D_{r,z}u(s,y)dyds
+ \int_r^t \int_\mathcal{O} G_{t-s}(x,y)\sigma'(u(s,y))D_{r,z}u(s,y)W(ds, dy),
\]

if \( r \leq t \), and \( D_{r,z}u(t,x) = 0 \), if \( r > t \). Their proofs rely on the global Lipschitz continuity of \( f_R \), and thus (4.3) holds naturally whenever \( f \) satisfies Assumption \[1\]. Further, we impose Assumption \[3\] to study the regularity of the exact solution in the Malliavin Sobolev space.

**Assumption 3.** For some integer \( k \geq 1 \), \( f \) and \( \sigma \) have bounded derivatives up to order \( k \).

**Lemma 4.2.** Under Assumptions \[4\] and \[3\], \( u(t, x) \in \mathbb{D}^{k,\infty} \) for any \((t, x) \in [0, T] \times \mathcal{O}\). Moreover, for any \( p \geq 1 \), there exists \( C = C(k,p,T) \) such that

\[
\sup_{t, x} \|u(t, x)\|_{k,p} \leq C.
\]

**Proof.** Define the Picard approximation by \( w^0(t,x) = u_0(x), (t,x) \in [0,T] \times \mathcal{O} \), and for \( i \in \mathbb{N} \),

\[
w^{i+1}(t,x) = G_{t}u_0(x) + \int_0^t \int_\mathcal{O} \Delta G_{t-s}(x,y)f(w^i(s,y))dyds
+ \int_0^t \int_\mathcal{O} G_{t-s}(x,y)\sigma(w^i(s,y))W(ds, dy), \quad (t, x) \in [0, T] \times \mathcal{O}.
\]

Fix \((t,x) \in [0,T] \times \mathcal{O}\). In view of \[23\] Lemma 1.5.3], the proof of \( u(t, x) \in \mathbb{D}^{k,\infty} \) boils down to proving that

(i) \( \{w^i(t, x)\}_{i \geq 1} \) converges to \( u(t, x) \) in \( L^p(\mathcal{O}; \mathbb{R}) \) for every \( p \geq 1 \).

(ii) for any \( p \geq 1 \), \( \sup_{t, x} \|w^i(t, x)\|_{k,p} < \infty \).

Property (i) and property (ii) with \( k = 1 \) and \( p = 2 \) can be obtained in the same way as in \[4\] Lemma 3.2] (the sequence \( \{w^i(t, x)\}_{i \geq 1} \) corresponds to \( \{u_{n,k}(t, x)\}_{k \geq 1} \) in \[4\]). The proof of property (ii) with general \( k, p \geq 1 \) is omitted since it is standard and similar to those for other kinds of SPDEs with Lipschitz continuous coefficients; see \[2\] Proposition 4.3] for the case of stochastic heat equations, \[25\] Theorem 1] for the case of stochastic wave equations. \( \square \)
Similar to properties (i) and (ii), the standard Picard approximation also shows that for any \((t, x) \in [0, T] \times \mathcal{O}, u^n(t, x) \in \mathbb{D}^{1,2}.

**Proposition 4.3.** Suppose that \(u_0 \in C^3(\mathcal{O})\) and Assumptions \(^1\) and \(^2\) hold for \(k = 2\). Then there exists some constant \(C\) such that for any \((t, x) \in [0, T] \times \mathcal{O},
\[
\mathbb{E} \left[ \| Du^n(t, x) - Du(t, x) \|_b^2 \right] \leq C n^{-2}.
\]

**Proof.** By the chain rule and (3.8), we obtain
\[
D_{r,z} u^n(t, x) = G_{t-r}^n(x, z) \sigma(u^n(r, \kappa_n(z)))
\]

\[
+ \int_r^t \int_{\mathcal{O}} \Delta_n G_{t-s}^n(x, y) f'(u^n(s, \kappa_n(y))) D_{r,z} u^n(s, \kappa_n(y)) dy ds
\]

\[
+ \int_r^t \int_{\mathcal{O}} G_{t-s}^n(x, y) \sigma'(u^n(s, \kappa_n(y))) D_{r,z} u^n(s, \kappa_n(y)) W(ds, dy),
\]

(4.4)

if \(r \leq t\), and \(D_{r,z} u^n(t, x) = 0\), if \(r > t\). Combining (4.4) and (4.3), we write
\[
D_{r,z} u^n(t, x) - D_{r,z} u(t, x) := I_{t,x}^n(r, z) + J_{t,x}^n(r, z) + K_{t,x}^n(r, z),
\]

(4.5)

where for \(r > t\), \(I_{t,x}^n(r, z) = J_{t,x}^n(r, z) = K_{t,x}^n(r, z) = 0\), and for \(r \leq t\),
\[
I_{t,x}^n(r, z) = \int_r^t \int_{\mathcal{O}} \Delta_n G_{t-s}^n(x, y) f'(u^n(s, \kappa_n(y))) [D_{r,z} u^n(s, \kappa_n(y)) - D_{r,z} u(s, \kappa_n(y))] dy ds
\]

\[
+ \int_r^t \int_{\mathcal{O}} \Delta_n G_{t-s}^n(x, y) f'(u^n(s, \kappa_n(y))) [D_{r,z} u(s, \kappa_n(y)) - D_{r,z} u(s, y)] dy ds
\]

\[
+ \int_r^t \int_{\mathcal{O}} \Delta_n G_{t-s}^n(x, y) \sigma'(u^n(s, \kappa_n(y))) D_{r,z} u(s, \kappa_n(y)) W(ds, dy)
\]

\[
=: J_{t,x}^{n,1}(r, z) + J_{t,x}^{n,2}(r, z) + J_{t,x}^{n,3}(r, z) + J_{t,x}^{n,4}(r, z),
\]

\[
K_{t,x}^n(r, z) = \int_r^t \int_{\mathcal{O}} G_{t-s}^n(x, y) \sigma'(u^n(s, \kappa_n(y))) [D_{r,z} u^n(s, \kappa_n(y)) - D_{r,z} u(s, \kappa_n(y))] W(ds, dy)
\]

\[
+ \int_r^t \int_{\mathcal{O}} G_{t-s}^n(x, y) \sigma'(u^n(s, \kappa_n(y))) [D_{r,z} u(s, \kappa_n(y)) - D_{r,z} u(s, y)] W(ds, dy)
\]

\[
+ \int_r^t \int_{\mathcal{O}} \Delta_n G_{t-s}^n(x, y) \sigma'(u^n(s, \kappa_n(y))) - \Delta G_{t-s}^n(x, y) \sigma'(u(s, y)) D_{r,z} u(s, y) W(ds, dy)
\]

\[
=: K_{t,x}^{n,1}(r, z) + K_{t,x}^{n,2}(r, z) + K_{t,x}^{n,3}(r, z).
\]

When \(r > t\), we always set \(J_{t,x}^{n,i}(r, z) = K_{t,x}^{n,j}(r, z) = 0\) for \(i = 1, 2, 3, 4\) and \(j = 1, 2, 3, 4\). Hereafter, let \(\epsilon < 1\) be an arbitrarily fixed positive number. A combination of Lemma \(^2\) and Theorem \(^3\) reveals that for any \(p \geq 2\),
\[
\| u^n(s, \kappa_n(y)) - u(s, y) \|_p \leq \| u^n(s, \kappa_n(y)) - u(s, \kappa_n(y)) \|_p + \| u(s, \kappa_n(y)) - u(s, y) \|_p
\]

\[
\leq C n^{-1},
\]

(4.6)
for all \((s, y) \in [0, T] \times \mathcal{O}\). Then the Lipschitz continuity of \(f'\), the Minkowski and Cauchy-Schwarz inequalities and (3.21) produce

\[
\|J_{t,x}^{n,3}\|_{L^2(\Omega; \mathcal{H})} \leq C_n \int_0^t \left( t - s \right)^{-\frac{3}{4}} \|u^n(s, \kappa_n(y)) - u(s, y)\|_4 \|Du(s, y)\|_{L^1(\Omega; \mathcal{H})} dy ds \\
\leq Cn^{-1} \sup_{t,x} \|Du(t, x)\|_{L^1(\Omega; \mathcal{H})} \leq Cn^{-1},
\]

where (4.6) and Lemma 4.2 were used in the second line. Similarly, by the boundedness of \(f'\), Lemma 2.3, Lemma 4.2, and (3.9),

\[
\|J_{t,x}^{n,4}\|_{L^2(\Omega; \mathcal{H})} \leq \int_0^t \int_\Omega |\Delta_n G_{t-s}^n(x, y) - \Delta G_{t-s}(x, y)||f'(u(s, y))Du(s, y)||_{L^2(\Omega; \mathcal{H})} dy ds \\
\leq C \int_0^t \int_\Omega |\Delta_n G_{t-s}^n(x, y) - \Delta G_{t-s}(x, y)| dy ds \leq Cn^{-1}.
\]

Since \(\sigma\) is bounded and Lipschitz continuous, it follows from the Minkowski inequality, (4.6), (3.10), and (2.1) that for \(p \geq 2\),

\[
\|I_{t,x}^n\|_{L^p(\Omega; \mathcal{H})} \leq \int_0^t \int_\Omega \|G_{t-r}^n(x, z)\|_{p}^2 dz dr \\
\leq 2 \int_0^t \int_\Omega \|G_{t-r}^n(x, z) - G_{t-r}(x, z)\|_{p}^2 \|\sigma(u^n(r, \kappa_n(z)))\|_{p}^2 dz dr \\
+ 2 \int_0^t \int_\Omega \|G_{t-r}(x, z)\|_{p}^2 \|\sigma(u^n(r, \kappa_n(z))) - \sigma(u(r, z))\|_{p}^2 dz dr \leq Cn^{-2}.
\]

Replacing \(\sigma\) by \(\sigma'\) in the above inequality, we also have

\[
\int_0^t \int_\Omega \|G_{t-s}^n(x, y)\|_{p}^2 \|\sigma'(u^n(s, \kappa_n(y))) - G_{t-s}(x, y)\|_{p}^2 dy ds \leq Cn^{-2},
\]

which along with the Burkholder inequality for Hilbert space valued martingales (see e.g. (4.18)), the Hölder inequality and Lemma 4.2 indicates

\[
\|K_{t,x}^{n,3}\|_{L^2(\Omega; \mathcal{H})} \leq C \int_0^t \int_\Omega \|G_{t-s}^n(x, y)\|_{p}^2 \|\sigma'(u^n(s, \kappa_n(y))) - G_{t-s}(x, y)\|_{p}^2 dy ds \\
\leq Cn^{-2} \sup_{t,x} \|Du(t, x)\|_{L^2(\Omega; \mathcal{H})} \leq Cn^{-2}.
\]

In order to estimate \(I_{t,x}^{n,2}\) and \(K_{t,x}^{n,2}\), we claim that for \(p \geq 2\), there exists some constant \(C = C(p, T)\) such that for any \(x_1, x_2 \in \mathcal{O}\) and \(t \in (0, T)\),

\[
\|Du(t, x_1) - Du(t, x_2)\|_{L^p(\Omega; \mathcal{H})} \leq C|x_1 - x_2|.
\]

(4.7)

Indeed, from (4.3), we have that for \(r \leq t\),

\[
D_{r,z}u(t, x_1) - D_{r,z}u(t, x_2) = [G_{t-r}(x_1, z) - G_{t-r}(x_2, z)] \sigma(u(r, z)) \\
+ \int_0^t \int_\Omega [\Delta G_{t-s}(x_1, y) - \Delta G_{t-s}(x_2, y)] f'(u(s, y)) D_{r,z}u(s, y) dy ds \\
+ \int_0^t \int_\Omega [G_{t-s}(x_1, y) - G_{t-s}(x_2, y)] \sigma'(u(s, y)) D_{r,z}u(s, y) W(ds, dy) \\
=: L_1(r, z) + L_2(r, z) + L_3(r, z).
\]
For \( r > t \), let \( L_i(r, z) = 0, \ i = 1, 2, 3 \). The boundedness of \( \sigma \) and Lemma 2.1 indicate
\[
\|L_1\|_{L^p(\Omega; \mathcal{B})} \leq C \iint_0^t \int_{\mathcal{O}} |G_{t-r}(x_1, z) - G_{t-r}(x_2, z)|^2 \, dz \, dr \leq C|x_1 - x_2|^2.
\]
Since \( f' \) is bounded, it follows from (2.18) and Lemma 4.2 that
\[
\|L_2\|_{L^p(\Omega; \mathcal{B})} \leq C \int_0^t \int_{\mathcal{O}} |\Delta G_{t-s}(x_1, y) - \Delta G_{t-s}(x_2, y)| \|Du(s, y)\|_{L^p(\Omega; \mathcal{B})} \, dy \, ds \leq C|x_1 - x_2|.
\]
Similarly, it follows from the Burkholder inequality, Lemmas 2.1 and 4.2 that
\[
\|L_3\|_{L^p(\Omega; \mathcal{B})} \leq C|x_1 - x_2|.
\]
Gathering the above estimates of \( L_1, L_2 \) and \( L_3 \), we obtain (4.7). By means of (3.21), it can be verified that
\[
\|J_{t,x}^{n,2}\|_{L^2(\Omega; \mathcal{B})} + \|K_{t,x}^{n,2}\|_{L^2(\Omega; \mathcal{B})} \leq C n^{-1}.
\]
Substituting the above estimates of \( \|P_{t,x}^{n}\|_{L^2(\Omega; \mathcal{B})}, \|J_{t,x}^{n,2}\|_{L^2(\Omega; \mathcal{B})}, \ i = 2, 3, 4, \) and \( \|K_{t,x}^{n,2}\|_{L^2(\Omega; \mathcal{B})}, j = 2, 3, \) into (4.8), we deduce that for \( 0 < \epsilon < 1 \),
\[
\|Du^n(t, x) - Du(t, x)\|_{L^2(\Omega; \mathcal{B})}^2 \leq C n^{-2} + C \int_0^t \int_{\mathcal{O}} |\Delta_n G_{t-s}(x, y)| \|Du^n(s, \kappa_n(y)) - Du(s, \kappa_n(y))\|_{L^2(\Omega; \mathcal{B})} \, dy \, ds
\]
\[
+ C \int_0^t \int_{\mathcal{O}} |G_{t-s}(x, y)|^2 \|Du^n(s, \kappa_n(y)) - Du(s, \kappa_n(y))\|_{L^2(\Omega; \mathcal{B})}^2 \, dy \, ds
\]
\[
\leq C n^{-2} + C \int_0^t \int_{\mathcal{O}} \int_{0}^{t-s} (t-s)^{-\delta - \epsilon} \|Du^n(s, \kappa_n(y)) - Du(s, \kappa_n(y))\|_{L^2(\Omega; \mathcal{B})}^2 \, dy \, ds \, ds,
\]
where the last step used the Hölder inequality and (3.21). Similar to (3.22), by taking the supremum over \( x \in \mathcal{O} \) on both sides of (4.8) and applying the Gronwall lemma with weak singularities (see e.g., [15] Lemma 3.4), we complete the proof.

### 4.3. Convergence of density

In this part, we present the convergence of density of the numerical solution \( \{u^n(T, kh)\}_{n \geq 2} \) for \( k \in \mathbb{Z}_n \). In order to apply Proposition 4.1 with \( X_\infty = u(T, x) \), we impose Assumption 4 and investigate the negative moment estimate of \( Du(t, x) \).

**Assumption 4.** There exists some \( \sigma_0 > 0 \) such that \( |\sigma(x)| > \sigma_0 \), for any \( x \in \mathbb{R} \).

**Lemma 4.4.** Let \( x \in \mathcal{O} \) and Assumptions 1 and 4 hold. Then there is \( \rho \in (0, 1) \) such that
\[
\mathbb{E} \left[ \|Du(T, x)\|_{\mathcal{B}}^{2\rho} \right] \leq C(\rho, T).
\]

**Proof.** To prove (4.9), we need to use [8] Proposition 3.2, which is summarized as follows: under Assumption 1 if \( x_i \in \mathcal{O}, \ i = 1, \ldots, d \), are distinct points, then for some \( p_0 > 0 \), there exists \( \varepsilon_0 = \varepsilon_0(p_0) \) such that for all \( \varepsilon \in (0, \varepsilon_0) \),
\[
\sup_{\xi \in \mathbb{R}^d, \|\xi\| = 1} \mathbb{P} \left( \xi^\top C(t) \xi \leq \varepsilon \right) \leq \varepsilon^{p_0},
\]
where \( C(t) := \langle Du(t, x_i), Du(t, x_j) \rangle_{\mathcal{B}} \) denotes the Malliavin covariance matrix of \( \{u(t, x_1), \ldots, u(t, x_d)\} \) (the notation \( u \) corresponds to \( X_R \) in [8]).

As a consequence of (4.10) with \( d = 1 \) and \( t = T \), we have that for all \( 0 < \varepsilon \leq \varepsilon_0 \),
\[
\mathbb{P} (\|Du(T, x)\|_{\mathcal{B}}^2 \leq \varepsilon) \leq \varepsilon^{p_0},
\]
which implies that for any \( \rho < p_0 \),
\[
\sum_{n=1}^{\infty} n^{\rho-1} \mathbb{P} (\|Du(T, x)\|_{\mathcal{B}}^{-2} \geq n) \leq \sum_{n=1}^{\lfloor \varepsilon_0^{-1} \rfloor} n^{\rho-1} \sum_{n=\lfloor \varepsilon_0^{-1} \rfloor + 1}^{\infty} n^{\rho-1} n^{-p_0} \leq C(\rho, \varepsilon_0).
Then we have that for \(0 < \rho < \min\{p_0, 1\}\) and \(Z := \|Du(T, x)\|_{H^{-2}}^2\),
\[
E[Z^\rho] \leq 1 + \sum_{n=1}^{\infty} (n+1)^\rho P(n \leq Z < n+1) \leq 2 + \sum_{n=1}^{\infty} ((n+1)^\rho - n^\rho) P(Z \geq n)
\leq 2 + \rho \sum_{n=1}^{\infty} n^{\rho-1} P(Z \geq n) \leq C(\rho, \varepsilon_0),
\]
which implies (4.9). The proof is completed. \(\square\)

We are ready to give the main result of this section, which states that for \(k \in \mathbb{Z}_n\), the density of the numerical solution \(u^n(T, kh)\) exists and converges in \(L^1(\mathbb{R})\) to the density of the exact solution. The readers are referred to [41, 8] for the existence of the density \(p_{u(t,x)}\) of the exact solution \(u(t,x)\) for any \((t,x) \in [0,T] \times \mathcal{O}\).

**Theorem 4.5.** Suppose that Assumptions [1, 3] and [4] hold for \(k = 2\), and \(u_0 \in C^3(\mathcal{O})\). Then for any \(k \in \mathbb{Z}_n\), \(u^n(T, kh)\) admits a density \(p_{u^n(T, kh)}\), and moreover
\[
\lim_{n \to \infty} \|p_{u(T, kh)} - p_{u^n(T, kh)}\|_{L^1(\mathbb{R})} = 0.
\]

**Proof.** By [23, Theorem 2.3.3] and [3, 2], we obtain that under Assumption [1] for any \(t \in (0, T]\), the law of \(U(t)\) is absolutely continuous with respect to the Lebesgue measure on \(\mathbb{R}^{n-1}\). Thus, \(\{u^n(T, kh)\}_{k \in \mathbb{Z}_n}\) admits a density. Theorem 3.3, Lemma 4.2, Proposition 4.3, and Lemma 4.4 indicate that the conditions of Proposition 4.1 are fulfilled for \(\alpha = 2\rho\), \(X_n = u^n(T, kh)\) and \(X_\infty = u(T, kh)\). As a result,
\[
\lim_{n \to \infty} d_{TV}(u(T, kh), u^n(T, kh)) = 0,
\]
which together to (4.1) completes the proof. \(\square\)

## 5. Full discretization

For the purpose of effective computation, we combine the spatial FDM with a temporal exponential Euler method to obtain the full discretization of Eq. (1.1), and give the strong convergence rate of the fully discrete numerical solution in this section.

Let \(\{t_i = i\tau, i = 0, 1, \ldots, m\} \) (with \(m \geq 2\)) be a uniform partition of \([0, T]\), where \(\tau := T/m\) is the uniform time stepsize. Denote by \(\eta_m(s) = \tau \lfloor s/\tau \rfloor\) the largest time grid point smaller than \(s\). By replacing \(s\) in (3.8) by \(\eta_m(s)\), we obtain the full discretization \(u^{m,n} = \{u^{m,n}(t,x); (t,x) \in [0,T] \times \mathcal{O}\}\) given by
\[
u^{m,n}(t,x) = \int_{\mathcal{O}} G^n_{t,x}(y) u_0(\kappa_n(y)) dy + \int_0^t \int_{\mathcal{O}} \Delta_n G^n_{t-s\eta_m}(y) f(u^{m,n}(\eta_m(s), \kappa_n(y))) dy ds + \int_0^t \int_{\mathcal{O}} G^n_{t-s\eta_m}(y) \sigma(u^{m,n}(\eta_m(s), \kappa_n(y))) W(ds, dy).
\]

The discrete Green function \(G^n\) satisfies the following estimates.

**Lemma 5.1.** Let \(\gamma \in (0, \frac{3}{8})\). Then for any \(x, y \in \mathcal{O}\) and \(s, t \in [0,T]\) with \(s < t\),
\[
\left| \int_0^t \int_{\mathcal{O}} |G^n_{t-r}(x,z) - G^n_{t-r}(y,z)|^2 dz dr \right| \leq C|x - y|^2,
\]
\[
\int_0^s \int_\Omega |G^n_{t-r}(x, z) - G^n_{s-r}(x, z)|^2 \, dz \, dr + \int_t^s \int_\Omega |G^n_{t-r}(x, z)|^2 \, dz \, dr \leq C_s|t-s|^{2\gamma}.
\]

**Proof.** The proof is similar to that of [4, Lemma 1.8]. It can be verified that
\[
|\phi_{j,n}(x) - \phi_{j,n}(y)| \leq \sqrt{2\pi^{-1}} j |x-y|.
\]
A combination of (3.5) and (5.2) implies
\[
\int_0^t \int_\Omega |G^n_{t-r}(x, z) - G^n_{s-r}(y, z)|^2 \, dz \, dr \leq \sum_{j=1}^{n-1} \frac{1}{2\lambda_{j,n}^2} |\phi_{j,n}(x) - \phi_{j,n}(y)|^2 \leq C|x-y|^2.
\]
By the uniform boundedness of \(\phi_{j,n}\) and (3.3),
\[
\int_\Omega |G^n(x, z)|^2 \, dz = \sum_{j=1}^{n-1} \exp(-2\lambda_{j,n}^2 t)|\phi_{j,n}(x)|^2 \leq C t^{-\frac{1}{4}} \int_0^\infty e^{-s^4} \, dz \leq C t^{-\frac{1}{4}},
\]
which indicates \(\int_t^s \int_\Omega |G^n_{t-r}(x, z)|^2 \, dz \, dr \leq C|t-s|^\frac{3}{4}\). Using (3.5) and (2.20), we obtain
\[
\int_0^s \int_\Omega |G^n_{t-r}(x, z) - G^n_{s-r}(x, z)|^2 \, dz \, dr \leq \sum_{j=1}^{n-1} \left(1 - \exp(-\lambda_{j,n}(t-s))^2 \right) \frac{1 - \exp(-2\lambda_{j,n}^2 s)}{2\lambda_{j,n}^2} |\phi_{j,n}(x)|^2
\]
\[
\leq C \sum_{j=1}^{n-1} \lambda_{j,n}^2 (t-s)^{2\gamma} \leq C(t-s)^{2\gamma},
\]
where \(\gamma < \frac{3}{8}\). The proof is completed. \(\square\)

In a similar way, one can prove the following lemma.

**Lemma 5.2.** Let \(\alpha \in (0,1)\). Then for any \(x, y \in \Omega\) and \(s, t \in [0,T]\) with \(s < t\),
\[
\int_s^t \int_\Omega |\Delta_n G^n_{t-r}(x, z)| \, dz \, dr \leq C|t-s|^{\alpha},
\]
\[
\int_0^s \int_\Omega |\Delta_n G^n_{t-r}(x, z) - \Delta_n G^n_{s-r}(y, z)| \, dz \, dr \leq C(\alpha)(|x-y| + |t-s|)^{\alpha}.
\]

**Proof.** By using the orthogonality of \(\{\phi_j \circ \kappa_n\}_{j \in \mathbb{Z}_n}\) and the Cauchy–Schwarz inequality, (5.3) and (5.5) with \(x = y\) and (5.3) with \(t = s\) can be obtained similarly as in (2.19), (2.21) and (2.18), respectively. \(\square\)

**Proposition 5.3.** Let Assumption 7 hold and \(u_0 \in C^2(\Omega)\). Then for any \(\alpha \in (0,1)\) and \(p \geq 1\), there exists \(C = C(p, T, \alpha)\) such that for any \((t, x) \in [0, T] \times \Omega\),
\[
\|u^n(t, x) - u^n(s, y)\|_p \leq C(|t-s|^{\frac{3\alpha}{4}} + C|x-y|).
\]

**Proof.** As a result of Theorem 3.3 and Lemma 2.3 for \(n \geq 2\) and \((t, x) \in [0, T] \times \Omega\),
\[
\|u^n(t, x)\|_p \leq \|u^n(t, x) - u^n(t, x)\|_p + \|u(t, x)\|_p \leq C(p, T, K).
\]
Recall that \(\tilde{u}^n_1(t, x) := \int_\Omega G^n(x, y) u_0(\kappa_n(y)) \, dy\). It follows from (3.17) and (3.18) that
\[
|\tilde{u}^n_1(t, x) - \tilde{u}^n_1(t, y)| \leq C|x-y| + \int_0^t \int_\Omega |\Delta_n G^n(x, z) - \Delta_n G^n(y, z)| |\Delta_n u_0(z)| \, dz \, dr.
\]
Since \(u_0 \in C^2(\Omega), |\Delta_n u_0(z)| \leq C \text{ for } z \in \Omega, \text{ and hence (5.5) implies } |\tilde{u}^n_1(t, x) - \tilde{u}^n_1(t, y)| \leq C|x-y|\). Similarly, based on (3.17), (5.4), and (5.5), one also has that for any \(\alpha \in (0,1)\),
\[ |\tilde{u}^n_t(t, x) - \tilde{u}^n_t(s, x)| \leq C|t - s|^{\frac{3\alpha}{8}}. \]

Based on a standard argument as in the proof of Lemma 4.2, it follows from (5.6) and Lemmas 5.1 and 5.2 that \[ \|u^n(t, x) - \tilde{u}^n_t(t, x) - u^n(s, y) + \tilde{u}^n_t(s, y)\|_p \leq C(t - s)^{\frac{3\alpha}{8}} + C|x - y| \]. The proof is completed.

**Theorem 5.4.** Suppose that Assumption 1 holds and \( u_0 \in C^3(\mathcal{O}) \). Then for every \( p \geq 1 \) and \( 0 < \epsilon \ll 1 \), there exists some constant \( C = C(p, T, K, \epsilon) \) such that for any \((t, x) \in [0, T] \times \mathcal{O},\]

\[ \|u^{m,n}(t, x) - u(t, x)\|_p \leq C(n^{-1} + m^{-\frac{3-\epsilon}{8}}). \]

**Proof.** Let \((t, x) \in [0, T] \times \mathcal{O}\) and \( 0 < \epsilon \ll 1 \). By virtue of Theorem 3.3, it remains to show

\[ \|u^{m,n}(t, x) - u^n(t, x)\|_p \leq Cm^{-\frac{3-\epsilon}{8}}. \]

By (3.8), (5.1), the Minkowski inequality, the Burkholder inequality, and the Lipschitz continuity of \( \sigma \) and \( b \), we obtain that for any \( p \geq 2, \]

\[
\begin{align*}
\|u^{m,n}(t, x) - u^n(t, x)\|_p^2 & \leq CH_{1,m,n} + CH_{2,m,n} + CQ_{1,m,n} + CQ_{2,m,n} \\
& + C \int_0^t \int_\mathcal{O} |\Delta_n G^n_{t-\eta_m(s)}(x, y)||u^{m,n}(\eta_m(s), \kappa_n(y)) - u^n(\eta_m(s), \kappa_n(y))\|_p^2 dy ds \\
& + C \int_0^t \int_\mathcal{O} |G^n_{t-\eta_m(s)}(x, y)||\sigma(u^n(\eta_m(s), \kappa_n(y))) - \sigma(u^n(s, \kappa_n(y)))\|_p^2 dy ds,
\end{align*}
\]

where

\[
H_{1,m,n} := \left( \int_0^t \int_\mathcal{O} |\Delta_n G^n_{t-\eta_m(s)}(x, y) - \Delta_n G^n_{t-s}(x, y)||f(u^n(\eta_m(s), \kappa_n(y)))\|_p^2 dy ds \right)^{\frac{1}{2}},
\]

\[
H_{2,m,n} := \left( \int_0^t \int_\mathcal{O} |\Delta_n G^n_{t-s}(x, y)||f(u^n(\eta_m(s), \kappa_n(y))) - f(u^n(s, \kappa_n(y)))\|_p^2 dy ds \right)^{\frac{1}{2}},
\]

\[
Q_{1,m,n} := \left( \int_0^t \int_\mathcal{O} |G^n_{t-\eta_m(s)}(x, y) - G^n_{t-s}(x, y)||\sigma(u^n(\eta_m(s), \kappa_n(y)))\|_p^2 dy ds \right)^{\frac{1}{2}},
\]

\[
Q_{2,m,n} := \left( \int_0^t \int_\mathcal{O} |G^n_{t-s}(x, y)||\sigma(u^n(\eta_m(s), \kappa_n(y))) - \sigma(u^n(s, \kappa_n(y)))\|_p^2 dy ds \right)^{\frac{1}{2}}.
\]

Taking advantage of Corollary 5.3 and (3.21), we obtain

\[
H_{2,m,n} + Q_{2,m,n} \leq C\epsilon \sup_t |\eta_m(t) - t|^{\frac{3-\epsilon}{8}} \leq C\epsilon m^{-\frac{3-\epsilon}{4}}.
\]

Similar to the proof of (5.3), one has that for \( \alpha < \frac{3}{8}, \)

\[
\int_0^t \int_\mathcal{O} |G^n_{t-\eta_m(s)}(x, y) - G^n_{t-s}(x, y)|^2 dy ds \leq C\epsilon \sup_t |\eta_m(t) - t|^{2\alpha},
\]

which along with the boundedness of \( \sigma \) shows that \( Q_{1,m,n} \leq Cm^{-\frac{3(1-\alpha)}{4}} \). Similar to (5.5) with \( x = y \), we also have that for \( \alpha < \frac{3}{8}, \)

\[
\int_0^t \int_\mathcal{O} |\Delta_n G^n_{t-\eta_m(s)}(x, y) - \Delta_n G^n_{t-s}(x, y)| dy ds \leq C\epsilon \sup_t |\eta_m(t) - t|^{\alpha},
\]

which along with (5.6) reveals that \( H_{1,m,n} \leq C\epsilon m^{-\frac{3-\epsilon}{4}} \). Gathering the above estimates together yields that for any \( t \in [0, T], \)

\[
\sup_x \|u^{m,n}(t, x) - u^n(t, x)\|_p \leq C\epsilon m^{-\frac{3-\epsilon}{4}}.
\]
\[ \leq Cm^{-\frac{3-\epsilon}{4}} + C \int_0^t \int_\Omega |\Delta_n G_{t-n\eta_m}^n(x,y)| \sup_x \|u^{m,n}(\eta_m(s),x) - u^n(\eta_m(s),x)\|_p^2 \,dy\,ds \]
\[ + C \int_0^t \int_\Omega \|G_{t-n\eta_m}^n(x,y)\| \sup_x \|u^{m,n}(\eta_m(s),x) - u^n(\eta_m(s),x)\|_p^2 \,dy\,ds. \]

Letting \( a(t) := \sup_x \|u^{m,n}(t,x) - u^n(t,x)\|_p^2, \quad t \in [0,T] \) and using (3.21), we obtain
\[ a(t) \leq Cm^{-\frac{3-\epsilon}{4}} + C \int_0^t (t - \eta_m(s))^{-\frac{3-\epsilon}{4}} a(\eta_m(s)) \,ds, \quad t \in [0,T]. \tag{5.7} \]

Hence, for any \( k = 1, 2, \ldots, m \),
\[ a(t_k) \leq Cm^{-\frac{3-\epsilon}{4}} + C \int_0^{t_k} (t_k - \eta_m(s))^{-\frac{3-\epsilon}{4}} a(\eta_m(s)) \,ds = Cm^{-\frac{3-\epsilon}{4}} + C \sum_{i=0}^{k-1} t_k^{-\frac{3-\epsilon}{4}} a(t_i), \]
which together with the discrete Gronwall lemma (see e.g. [20] Lemma A.4) implies that \( \sup_{0 \leq k \leq m} a(t_k) \leq Cm^{-\frac{3-\epsilon}{4}} \). Finally, taking (5.7) into account gives
\[ a(t) \leq Cm^{-\frac{3-\epsilon}{4}} + C \int_0^t (t - \eta_m(s))^{-\frac{3-\epsilon}{4}} \,ds \leq Cm^{-\frac{3-\epsilon}{4}}, \quad t \in [0,T]. \]

Thus the proof is complete. \( \square \)

**Remark 5.5.** The application of the orthogonality of \( \{\phi_j \circ \kappa_n\}_{j \in \mathbb{Z}_n} \) plays a key role to obtain the temporal convergence order nearly \( \frac{3}{4} \) of the full discretization in Theorem 5.4. For example, if the left hand of (5.4) is estimated in the following way
\[ \int_0^s \int_\Omega |\Delta_n G_{t-r}^n(x,z) - \Delta_n G_{s-r}^n(x,z)| \,dz \,dr \]
\[ \leq C \int_0^s \sum_{j=0}^{n-1} |\lambda_{j,n}| \exp(-\lambda_{j,n}^2(s-r))[1 - \exp(-\lambda_{j,n}^2(t-s))] \,dr \]
\[ \leq C \sum_{j=1}^{n-1} \lambda_{j,n}^2(t-s) \alpha \leq C \sum_{j=1}^\infty j4^{\beta} (t-s) \alpha \leq C_\alpha(t-s) \alpha, \]
with \( \alpha \in (0, \frac{1}{4}) \), then we can only obtain \( \int_0^s \int_\Omega |\Delta_n G_{t-r}^n(x,z) - \Delta_n G_{s-r}^n(x,z)| \,dz \,dr \leq C(\alpha)|t - s|^{\frac{1}{2} - \epsilon} \) with \( 0 < \epsilon \ll 1 \). As a result, the temporal Hölder continuity exponent of \( u^n \) is only nearly \( \frac{1}{4} \), which leads to that the temporal convergence order of the exponential Euler method is only nearly \( \frac{1}{4} \).

Combining Theorem 5.4 with a localized argument, we show an \( L^p(\Omega; \mathbb{R}) \) convergence order localized on a set of arbitrarily large probability for Eq. (1.1) with polynomial nonlinearity.

**Corollary 5.6.** Suppose that Assumption 2 hold and \( u_0 \in C^3(\Omega) \). Then for any \( R \geq 1, \ 0 < \epsilon \ll 1 \) and \( p \geq 1 \), there exists \( C = C(R,T,p,\epsilon) \) such that
\[ \mathbb{E} \{ \Omega_{Rn} \Omega_{Rn}^m |u^{m,n}(t,x) - u(t,x)|^p \} \leq C(n^{-1} + m^{-\frac{3-\epsilon}{4}}). \]

**Proof.** For \( R \geq 1 \), denote \( \Omega_R := \{ \omega \in \Omega : \sup_{t,x} |u(t,x,\omega)| \leq R \} \). Set \( f_R = K_R f \) with \( K_R \) defined by (2.3). Consider the localized Cahn-Hilliard equation
\[ \partial_t u_R + \Delta^2 u_R = \Delta f_R(u_R) + \sigma(u_R)\dot{W}, \quad R \geq 1 \]
\[ \tag{5.8} \]
with $u_R(0, \cdot) = u_0$ and DBCs. Then the local property of stochastic integrals shows $u = u_R$ (i.e., for any $(t, x) \in [0, T] \times \mathcal{O}$, $u(t, x) = u_R(t, x)$) on $\Omega_R$ a.s. Consider the numerical solution $u^{m,n}_R$ of (5.8) based on the FDM in space and the exponential Euler method in time, i.e.,

$$u^{m,n}_R(t, x) = \int_\mathcal{O} G^n_t(x, y)u_0(\kappa_n(y))dy + \int_0^t \int_\mathcal{O} \Delta_n G^n_{t-\eta_m(s)}(x, y)f_R(u^{m,n}_R(\eta_m(s), \kappa_n(y)))dyds$$

$$+ \int_0^t \int_\mathcal{O} C^n_{t-\eta_m(s)}(x, y)\sigma(u^{m,n}_R(\eta_m(s), \kappa_n(y)))W(ds, dy),$$

(5.9)

for $n, m \geq 2$ and $(t, x) \in [0, T] \times \mathcal{O}$. Setting $\Omega^{m,n}_R := \{\omega \in \Omega : \sup_{t,x} |u^{m,n}(t, x, \omega)| \leq R\}$, and comparing (5.9) with (5.1), it follows from the local property of stochastic integrals that $u^{m,n}_R = u^{m,n}$ on $\Omega^{m,n}_R$ a.s. For fixed $R \geq 1$, since $f_R$ satisfies Assumption [1], Theorem 5.4 indicates that there exists some constant $C = C(R, T, p, \epsilon)$ such that for $0 < \epsilon \ll 1$ and $p \geq 1$,

$$\mathbb{E} [\|u^{m,n}_R(t, x) - u_R(t, x)\|^p] \leq C(n^{-1} + m^{-3-\epsilon/p}).$$

Since $u^{m,n}(t, x)$ and $u(t, x)$ have almost surely continuous trajectories, we have $\lim_{R \to \infty} \mathbb{P}(\Omega_R) = \lim_{R \to \infty} \mathbb{P}(\Omega^{m,n}_R) = 1$, which implies $\lim_{R \to \infty} \mathbb{P}(\Omega_R \cap \Omega^{m,n}_R) = 1$. By $u = u_R$ on $\Omega_R$ a.s., and $u^{m,n}_R = u^{m,n}$ on $\Omega^{m,n}_R$ a.s., we obtain

$$\mathbb{E} \left[1_{\Omega_R \cap \Omega^{m,n}_R} |u^{m,n}_R(t, x) - u(t, x)|^p \right] \leq \mathbb{E} \left[|u^{m,n}_R(t, x) - u_R(t, x)|^p \right] \leq C(n^{-1} + m^{-3-\epsilon/p}).$$

The proof is completed. $\square$

Remark 5.7. Theorems 5.5 and 4.3 indicate that when applying the spatial FDM to the localized Cahn–Hilliard equation (1.2), the associated numerical solution is strongly convergent and the density of the numerical solution converges in $L^1(\mathbb{R})$. In addition, Section 2 gives the uniform moment estimate and Hölder continuity of the exact solution for Eq. (1.1) with $f$ being a polynomial of degree 3 with a positive dominant coefficient. We expect to combine the above results with the localization technique to study the strong convergence of the spatial FDM and the density convergence of the associated numerical solution for the stochastic Cahn–Hilliard equation with polynomial nonlinearity and multiplicative noise in the future.

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