A Spacetime Calculation of 
the Calabrese-Cardy Entanglement Entropy

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Abstract

We calculate Sorkin’s spacetime entanglement entropy of a Gaussian scalar field for complementary regions in the 2d cylinder spacetime and show that it has the Calabrese-Cardy form. We find that the cut-off dependent term is universal when we use a covariant UV cut-off as in [1]. In addition, we show that the relative size-dependent term exhibits complementarity. Its coefficient is however not universal and depends on the choice of pure state. It asymptotes to the universal form within a natural class of pure states.

The Calabrese-Cardy formula for the entanglement entropy (EE) of a CFT for an interval $I_s$ of length $s$ in a circle $C_\ell$ of circumference $\ell$ is given by

$$S = \frac{c}{3} \ln \left( \frac{\ell}{\pi \epsilon} \right) + \frac{c}{3} \ln(\sin(\alpha \pi)) + c_1$$

where $\alpha = s/\ell$, $c$ is the CFT central charge, $\epsilon$ is a UV cut-off and $c_1$ is a non-universal constant [2]. This formula has been shown to apply to a diverse range of two dimensional systems which fall within the same universality class, including a geometric realisation by Ryu and Takayanagi [3] and others [4]. Entanglement entropy (EE) was first proposed in [5] as a possible contributor to black hole entropy. Hence understanding Eqn. (1) from a spacetime perspective is of broad interest.

As a follow up to their earlier work, Calabrese and Cardy studied the unitary time evolution of the EE for an interval $I_s$ inside a larger interval $I \supset I_s$. Starting with a pure state, which is an eigenstate of a ”pre-quench” Hamiltonian, and then quenching the system at $t = 0$, they used path integral techniques to show that the EE increases with time. It then saturates after the “light-crossing” time, in keeping with causality [6]. This corresponds to the “time” required for the domain of dependence of $I_s$ to be fully defined. Seeking out a covariant formulation of EE is therefore of interest both to understanding the results of [6]

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in a spacetime language as well as more generally in QFT and quantum gravity. Such a formulation is moreover in keeping with the broader framework of AQFT, where observables are associated with spacetime regions rather than spatial hypersurfaces [7].

In [8] Sorkin proposed a spacetime formula for the EE of a Gaussian scalar field $\Phi$ in a globally hyperbolic subregion $\mathcal{O}$ of a globally hyperbolic spacetime $(M, g)$, with respect to its causal complement $\mathcal{O}^c$. It uses the restriction of the Wightmann function $W(x, x')$ in $M$ to $\mathcal{O}$, and the Pauli-Jordan function $i\Delta(x, x')$ which appears in the Peierl’s spacetime commutation relation $[\hat{\Phi}(x), \hat{\Phi}(x')] = i\Delta(x, x')$.

Sorkin’s spacetime EE (SSEE) of $\mathcal{O}$ with respect to $\mathcal{O}^c$ is

$$S = \sum_{\mu} \mu \ln(|\mu|), \quad \hat{W}|_{\mathcal{O}} \circ \chi = \mu(i\Delta) \circ \chi,$$

where $\chi \not\in \text{Ker}(\hat{\Delta})$ and where

$$A \circ v(x) \equiv \int_{\mathcal{O}} dV x'A(x, x')v(x').$$

It is motivated by the finite system Wightmann function for a Gaussian state which is a direct sum of identical systems with two degrees of freedom [8]. The SSEE formula generalises the calculation of EE for a state at a given time to that associated with a spacetime region.

In [11] the SSEE for nested causal diamonds $\mathcal{D}_s \subset \mathcal{D}_S$ was shown to yield the first, cut-off dependent term of Eqn. (1) with $c = 1$ when $s << S$. Since $\mathcal{D}_s$ is the domain of dependence of $I_s$, this is the natural spacetime analogue of $\mathcal{I}_s \subset \mathcal{I}_S$. In this work we calculate the SSEE for the spacetime analogue of $\mathcal{I}_s \subset \mathcal{C}_\ell$ for finite $\ell$ and additionally, find the same $\alpha$-dependence as Eqn. (1), thus explicitly demonstrating complementarity. A natural spacetime analogue of $\mathcal{C}_\ell$ is its (zero momentum) Cauchy completion, which is the $d = 2$ cylindrical spacetime $(M, g)$ with $ds^2 = -dt^2 + dx^2$, $x + \ell \sim x$. The domains of dependence of $\mathcal{I}_s$ and its complement $\mathcal{I}_{s-} \subset \mathcal{C}_\ell$ are the causal diamonds $\mathcal{D}_s$ and $\mathcal{D}_{s-}$ respectively, as shown in Fig 1.

In what follows we use a mixture of analytical and numerical methods to solve the SSEE eigenvalue problem.

Figure 1: The spacetime analogues of $\mathcal{I}_s, \mathcal{I}_{s-} \subset \mathcal{C}_\ell$ are their domains of dependence $\mathcal{D}_s$ and $\mathcal{D}_{s-}$ in $(M, g)$ shown in green and red respectively.
We will find it convenient to work with the Sorkin-Johnston (SJ) formulation \[9, 10, 11, 12, 13\], where the SJ spectrum provides the required (covariant) UV cut-off with which to calculate \(S\), as was done in \[1\]. For a compact globally hyperbolic region \((M, g)\) of a spacetime it follows from \(\text{Ker}(\hat{\square}) = \text{Im}(i\hat{\Delta})\) \[14\] that the eigenmodes of the integral Hermitian operator \(i\hat{\Delta}\) provide a covariant orthonormal basis (the SJ modes) with respect to the \(L^2\) norm on \((M, g)\) \[7\]. The SJ vacuum or Wightmann function is given by the positive part of \(i\hat{\Delta}\). Since the SJ spectrum is covariant so is a UV cut-off in this basis.

For our calculation of \(S\) we will use the SJ vacuum \(W_\tau\) for a free massless scalar field in a slab \((M_\tau, g)\) of height \(2\tau\) in the cylinder spacetime \[10\], and its restriction to \(D_s \subset M_\tau\),

\[W_\tau(x, t; x', t') = \sum_{m \in \mathbb{Z}} q_m \psi_m(x, t)\psi^*_m(x', t'),\]  

(4)

where \(\{\psi_m, q_m\}\) are the \(L^2\) normalised positive frequency SJ eigenmodes and eigenvalues in \(M_\tau\) \[10\]:

\[\psi_m(x, t) = \left(\frac{(1 - \zeta_m)}{2\sqrt{2}lc_m} e^{2\pi|m|lt} + \frac{(1 + \zeta_m)}{2\sqrt{2}lc_m} e^{-2\pi|m|lt}\right)e^{2\pi mx},\]

\[q_m = \ell s_m c_m, \quad \zeta_m = \frac{c_m}{s_m}, \quad \gamma = \frac{2\tau}{l}, \quad m \in \mathbb{Z},\]

\[c_m^2 = \tau (1 + \text{sinc}(2|m|\pi\gamma)), \quad s_m^2 = \tau (1 - \text{sinc}(2|m|\pi\gamma)).\]

(5)

The \(m = 0\) “zero mode” in particular takes the form

\[\psi_0(t) = \frac{1}{2\sqrt{\tau l}} \left(1 - i\frac{\sqrt{3}}{\tau} t\right), \quad q_0 = \frac{2}{\sqrt{3}} \tau^2.\]

(6)

Unlike the standard vacuum on the cylinder, \(W_\tau\) is \(\tau\)-dependent. Each \(W_\tau\) can however be viewed as a pure (non-vacuum) state in \(M_\bar{\tau}\) for any \(\bar{\tau} > \tau\), as we will later show. To accommodate both \(D_s\) and \(D_{\ell-s}\) in our calculations, we require \(2\tau \geq s, \ell - s\).

The SJ modes in \(D_s\) are naturally expressed in terms of the light cone coordinates \(u = \frac{1}{\sqrt{2}}(t - x), v = \frac{1}{\sqrt{2}}(t + x)\) and come in the two mutually orthogonal series \[15\]

\[f_k = e^{-iku} - e^{-ikv}, \quad k = 2\sqrt{2}n\pi/s\]

\[g_\kappa = e^{-inu} + e^{-inkv} - 2\cos\left(\frac{\kappa s}{2\sqrt{2}}\right), \quad \tan\left(\frac{\kappa s}{2\sqrt{2}}\right) = \frac{\kappa s}{\sqrt{2}}\]

(7)

with eigenvalues \(\lambda_k = \frac{s}{2\sqrt{2}k}\) and \(\lambda_\kappa = \frac{s}{2\sqrt{2}\kappa}\), respectively, and with \(L^2\) norm in \(D_s\)

\[||f_k||^2 = s^2, \quad ||g_\kappa||^2 = s^2 \left(1 - 2\cos^2\left(\frac{\kappa s}{2\sqrt{2}}\right)\right).\]

(8)
Since $i\tilde{\Delta}$ is diagonal in this basis we will use it to transform Eqn. (2) to the matrix form

$$\hat{W}_r|_{D_s} X = \mu \Lambda X,$$

(9)

where $\Lambda$ is the diagonal matrix $\{\lambda_k, \lambda_\nu\}$. For $X \not\in \text{Ker}(i\tilde{\Delta})$, we can invert this to suggestively write

$$\hat{\rho} X = \Lambda^{-1}\hat{W}_r|_{D_s} X = \mu X,$$

(10)

so that $\mathcal{S}$ can be viewed as the von-Neumann entropy of $\hat{\rho}$. The spectrum of $\hat{\rho}$ is unbounded and hence needs a UV cut-off. As in [1] we use the covariant UV-cut off with respect to the SJ spectrum $\{\lambda_k, \lambda_\nu\}$. For large $\kappa$ the condition $\tan(\kappa s/2\sqrt{2}) = \kappa s/\sqrt{2}$ can be approximated by $\kappa \approx \sqrt{2}(2n + 1)\pi/s$, so that a consistent choice of cut-off for both sets of eigenvalues is $\epsilon = k_{\max}^{-1} = s/(2\sqrt{2\max n}\pi)$. We also need to ensure that this same cut-off is used in the causal complement, i.e., $k_{\max} = 2\sqrt{2n'_{\max}}\pi/(\ell - s)$, where $n'$ denotes the quantum number for the SJ spectrum in $D_{\ell-s}$, so that $\epsilon = \frac{\ell\alpha}{2\sqrt{2\pi n_{\max}}} = \frac{\ell(1-\alpha)}{2\sqrt{2\pi n'_{\max}}}$. We expand the SJ modes in $M_r$ in terms of those in $D_s$ to obtain the non-zero matrix elements for $\hat{W}_r|_{D_s}$ for general $\alpha, \gamma$. Suppressing the $\tau, D_s$ labels, these are

$$\hat{W}_{kk'} = \frac{s^4}{32\pi} \sum_{m>0} \frac{1}{|m|\zeta_m} \left( \eta_m^- \text{sinc}(x_m^-) - \eta_m^+ \text{sinc}(x_m^+) \right) \times \left( \eta_m^- \text{sinc}(x_m'^-) - \eta_m^+ \text{sinc}(x_m'^+) \right),$$

$$\hat{W}_{k\kappa} = \frac{s^4}{32\pi} \sum_{m>0} \frac{1}{|m|\zeta_m} \left( \eta_m^- \text{sinc}(z_m^-) + \eta_m^+ \text{sinc}(z_m^+) \right) \times \left( \eta_m^- \text{sinc}(z_m'^-) + \eta_m^+ \text{sinc}(z_m'^+) \right)$$

$$+ \hat{W}_{k\kappa'}^{(0)}$$

(11)

where $x_m = (n \pm \alpha m)\pi$, $x_m' = (n' \pm \alpha m)\pi$, $z_m = \kappa s/2\sqrt{2} \pm \alpha m\pi$, $z_m' = \kappa' s/2\sqrt{2} \pm \alpha m\pi$, and the contribution from the zero mode is

$$\hat{W}_{k\kappa'}^{(0)} = \frac{s^4}{2\sqrt{3}} \frac{\tau}{\ell} \cos(\kappa s/(2\sqrt{2})) \cos(\kappa' s/(2\sqrt{2}))$$

$$\times \left( 1 + \sqrt{\frac{3}{2}} \frac{1}{\kappa\tau} \right) \left( 1 + \sqrt{\frac{3}{2}} \frac{1}{\kappa'\tau} \right).$$

(12)

Our strategy is to construct $\hat{\rho}$ from these matrix elements and to solve for its eigenvalues using a numerical matrix solver. However, each matrix elements in Eqn. (11) is an infinite sum over the quantum number $m$ and hence not amenable to explicit calculation. We therefore need to find a closed form expression for the above matrix elements.

We notice that when $\gamma$ takes half-integer values (for which the SJ vacuum Hadamard [10]), $\zeta_m = 1$ for $m \neq 0$, which leads to a considerable simplification. Further, let $\alpha$ be rational, so that we can write $\alpha = p/q$, with $p, q \in \mathbb{Z}$, and $p, q > 0$ being relatively prime. For these choices of $\alpha$ and $\gamma$, the infinite sums of Eqn. (11) reduce to the following finite sums over Polygamma functions $\Psi(x)$ and $\Psi^{(1)}(x)$.
\[
\begin{align*}
\tilde{W}_{kk'} &= \frac{s^4}{8\pi n} \left[ \delta_{n,n'} (\alpha \Theta(n) \sum_m \delta_{n,m} + \frac{1}{\pi^2 \alpha q^2 n} \sum_{r=1}^{q-1} \sin^2(r\alpha \pi) \left[ -\alpha q \Psi\left(\frac{r}{q}\right) + \alpha q \Psi\left(\frac{\alpha r - n}{\alpha q}\right) \right]
+ n \Psi(\alpha r - n) - n' \Psi(\alpha r - n') \right]
+ \frac{1}{\alpha q^2 s^2 \kappa^2 \pi^2} \left( \sum_{r=1}^{q-1} \Omega(\kappa, \kappa', \alpha, r) \left[ \alpha q \pi \left( \Psi\left(\frac{r}{q} - \frac{\kappa s}{\eta}\right) - \Psi\left(\frac{r}{q}\right) \right) + \frac{\kappa s}{2\sqrt{2}} \Psi(\frac{r}{q} - \frac{\kappa s}{\eta}) \right]
+ \frac{\kappa s}{2\sqrt{2}} \Psi(\frac{r}{q} - \frac{\kappa s}{\eta}) \right]
+ \frac{s^2 \kappa \kappa'}{2} \left[ \gamma_e \left( 1 - \frac{\kappa s}{\eta} \right) - \kappa \Psi\left( \frac{r}{q} - \frac{k s}{\eta} \right) \right]
+ \frac{s^2 \kappa \kappa'}{2} \left[ \gamma_e \left( 1 - \frac{k s}{\eta} \right) - \kappa \Psi\left( \frac{r}{q} - \frac{k s}{\eta} \right) \right], \quad \eta = 2\sqrt{2} \alpha q \pi \quad (13)
\end{align*}
\]

where \(\gamma_e\) represents the Euler-Mascheroni constant and

\[
\Omega(\kappa, \kappa', \alpha, r) = \kappa \kappa' \frac{s^2}{2} \cos^2(\alpha r \pi) + \sin^2(\alpha r \pi) - (\kappa + \kappa') \frac{s}{2\sqrt{2}} \sin(2\alpha r \pi). \quad (14)
\]

We are now in a position to solve for the eigenvalues of \(\hat{\rho}\) using Mathematica’s numerical eigenvalue solver. We consider a range of values of \(\alpha, \gamma\) and the cut-off \(n_{\text{max}}/\alpha\) given in the table below.

| \(\alpha\) | \(\frac{1}{10}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{1}{3}, \frac{3}{4}, \frac{3}{5}, \frac{2}{3}, \frac{1}{2}, 1, 5, 10\) |
|----------|--------------------------------------------------|
| \(\gamma\) | 1, 2, 4, 6, 8, 16, 21.5, 32, 40.3, 100, 200, 1000, 2000 |
| \(n_{\text{max}}/\alpha\) | 1000, 1200, 1400, 1600, 1800, 2000, 2200, 2400, 2600 |

In the list of \(\gamma\) values, we have also included the specific non-half-integer value of \(\gamma = 40.3\) for which \(\zeta_m \sim 1\) even for \(m = 1\). In general, we note that \(\zeta_m \sim 1\) for \(m >> \gamma^{-1}\). The
error coming from small \( m \) terms has been explicitly calculated in this case as a function of \( m \) and seen to be small. For the special case \( \alpha = 0 \), \( S \) is trivially zero, while for \( \alpha = 1 \), the domain of dependence of \( C_\ell \) is no longer a causal diamond, but all of \( M_\tau \). Since \( \hat{W}_\tau \) is the SJ vacuum and therefore pure, \( S = 0 \).

Fig. 2 shows the results of simulations for these various \( \alpha \) and \( \gamma \) values, for a fixed choice of cut-off \( n_{\text{max}}/\alpha = 2600 \). It is already clear that \( S \) satisfies complementarity. This is much more explicit in Fig. 3 where we vary over the cut-off. Our numerical results suggest that

\[
S = \frac{c(\gamma)}{3} \ln \left( \frac{\ell}{\pi \epsilon} \right) + f(\gamma) \ln(\sin(\alpha \pi)) + c_1(\gamma). \tag{15}
\]

Using the best-fit curves in Figs. 3-5 and the associated data in the appendix, we find that

\[
c(\gamma) \sim 1 \quad \text{and} \quad f(\gamma) \sim 0.33 + a/\gamma + b/\gamma^2
\]

\[
c_1(\gamma) \sim a' \log \gamma + b'. \tag{16}
\]

Thus, the first term of Eqn. 1 is reproduced for any choice of \( \alpha, \gamma \). This generalises the results of [1] where this was shown in the limit of \( \alpha \ll 1 \). The dependence on \( \alpha \), i.e.,
Figure 3: Log-linear plot of $S$ vs $\frac{n_{\text{max}}}{\alpha}$ for different $\alpha$ fitted to $S = a \log (n_{\text{max}}/\alpha) + b$ for $\gamma = 1000$.

Figure 4: A plot of $f(\gamma)$ vs. $\gamma$ for different values of $n_{\text{max}}/\alpha$, fitted to $0.33 + a/\gamma + b/\gamma^2$. The inset figure shows the smaller $\gamma$ values.

Figure 5: A log-linear plot of $c_1(\gamma)$ vs $\gamma$ for different values of $n_{\text{max}}/\alpha$ fitted to $a \log \gamma + b$. 
the second term of Eqn. (1) is also reproduced and hence exhibits complementarity for any \( \alpha \) (see Fig. 2, 3). Its coefficient however is not universal and depends on \( \gamma \) as shown in Fig. 4. However, as \( \gamma >> 1 \), \( f(\gamma) \) does asymptote to the universal value \( 1/3 \). Finally, the non-universal constant \( c_1(\gamma) \) diverges logarithmically with \( \gamma \) as shown in Fig. 5. This can be traced to the IR divergence in the zero modes of the massless theory.

The behaviour of \( f(\gamma) \) can be viewed as a dependence on the choice of the pure state \( W_\tau \) in \( M_\tau \), for \( M_\tau \supset M_\sigma \). From Eqn. (4) we see that \( \hat{W}_\tau \) is a state in \( M_\sigma \), i.e., \( \hat{W}_\tau = \hat{R}_\tau + i\Delta/2 \), where \( \hat{R}_\tau \) is real and symmetric.

Expanding \( i\hat{\Delta} \) in the SJ modes \( \{\psi_m^{(\tau)}\} \) of \( M_\sigma \) and \( \hat{W}_\tau \) in \( \{\psi_m^{(\tau)}\} \), and inserting in Eqn. (2) we see that term by term

\[
g_m^\tau \psi_m^{(\tau)}(x, t) A_m = \mu g_m^\tau [\psi_m^{(\tau)}(t, x) \hat{A}_m - \psi_m^{*(\tau)}(t, x) \hat{B}_m],
\]

where \( A_m = (\psi_m^{(\tau)}, \chi)_\tau, \hat{A}_m = (\psi_m^{(\tau)}, \chi)_\tau, \hat{B}_m = (\psi_m^{*(\tau)}, \chi)_\tau \) and \( (., .)_\tau \) is the \( L^2 \) inner product in \( M_\sigma \). Expanding \( \psi_m^{(\tau)} = a_m \psi_m^{(\tau)} + b_m \psi_m^{*(\tau)} \), \( a_m = \frac{s_m^\tau}{2c_m} (\zeta_m^{(\tau)} + \zeta_m^{(\tau)}), b_m = \frac{s_m^\tau}{2c_m} (\zeta_m^{(\tau)} - \zeta_m^{(\tau)}) \) this simplifies to

\[
g_m^\tau a_m (a_m \hat{A}_m + b_m \hat{B}_m) = \mu g_m^\tau \hat{A}_m
\]
\[
g_m^\tau b_m (a_m \hat{A}_m + b_m \hat{B}_m) = -\mu g_m^\tau \hat{B}_m. \tag{17}
\]

The solutions for this are either \( a_m \hat{A}_m + b_m \hat{B}_m = 0 \Rightarrow \mu = 0 \), or \( a_m \hat{B}_m + b_m \hat{A}_m = 0 \Rightarrow \mu = \frac{g_m^\tau}{g_m^\tau} (a_m^2 - b_m^2) = 1 \), which means that \( \hat{W}_\tau \) is a pure state in \( M_\sigma \). Thus \( f(\gamma) \) can be viewed as the dependence on the choice of pure state in \( M_\sigma \) for \( M_\tau \subset M_\sigma \).

We end with some remarks. While we have demonstrated complementarity for certain rational values of \( \alpha \), an analytic demonstration using Eqn. (13) seems non-trivial, in part because the UV regulated matrices \( \hat{\rho}_\alpha \) and \( \hat{\rho}_{1-\alpha} \) are of different dimensions. Conversely, complementarity implies that if \( n_{\text{max}} > n'_{\text{max}} \), \( \rho_\alpha = \hat{\rho}_{1-\alpha} \oplus 1_N \oplus 0_N \), where \( 0 \) is the zero matrix and \( N = (n_{\text{max}} - n'_{\text{max}})/2 \).

In our computations we find that the eigenvalues of \( \hat{\rho} \) (which always come in pairs \( (\mu, 1-\mu) \)) exhibit the surprising feature that all but one pair hovers around the values 0 and 1, thus contributing most significantly to \( \mathcal{S} \). Indeed, the \( \mathcal{S} \) calculated using the largest few pairs of eigenvalues accounts for most of the entropy (see appendix).

Finally, it would be interesting to calculate the non-zero mass case which is IR divergence free. While the small mass approximation of the SJ modes in \( \mathcal{D}_a \) is known \cite{16}, the challenge will be to obtain closed form expressions for the matrix elements of \( \hat{W} \) as we have done.

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Appendix: Supporting Data

In this appendix we present some plots with additional data which were used to compute the coefficients $c(\gamma)$, $f(\gamma)$ and $c_1(\gamma)$ in the Entanglement Entropy.

Fig. 6 shows the dependence of $S$ on $\alpha$ for different values of $\gamma$ and with three different values of $n_{\text{max}}/\alpha$ (1200, 2000 and 2600). The SSEE can be fitted to the form $S = a_1 \log(\sin(\alpha \pi)) + b_1$ where the coefficient $a_1$ corresponds to $f(\gamma)$ in Eqn. (15). The values of $a_1$ and $b_1$ along with their errors are given in the tables in Fig. 6. $a_1$ and therefore $f(\gamma)$ can be seen to be independent of $n_{\text{max}}/\alpha$. It is however dependent on $\gamma$ and asymptotes to the universal value of $1/3$ in the Calaberse-Cardy formula for $\gamma >> 1$. We fit $f(\gamma)$ values to the form
\[ f(\gamma) = 0.33 + a_2/\gamma + b_2/\gamma^2 \] (18)
and find the $a_2 \approx -0.48$ and $b_2 \approx 0.23$ with the error given in the tables of Fig. 6.

Fig. 7 shows the dependence of $S$ on $n_{\text{max}}/\alpha$ for different $\alpha$ and with three different values of $\gamma$ (16, 200 and 1000). Here SSEE can be fitted to the form, $S = a_3 \log(n_{\text{max}}/\alpha) + b_3$. As is clear from the tables in this figure $a_3 \approx 0.33 \approx 1/3$ for all $\alpha$ and $\gamma$ with the order of error given in the table. $b_3$ however depends on $\alpha$ and $\gamma$. This suggests that $c(\gamma) \approx 1$ in Eqn. (15).

In order to extract $c_1(\gamma)$ we subtract the first term in Eqn. (15) (which depends on $n_{\text{max}}/\alpha$) using $c(\gamma)/3$ given by the values of $a_3$ in the table of Fig. 7 from the values of $b_1$ in the table of Fig. 6 for $n_{\text{max}}/\alpha = 1200, 2000$ and 2600. We find that the difference (or $c_1(\gamma)$) is independent of the choice of $n_{\text{max}}/\alpha$ which is as expected. We fit the dependence on $\gamma$ by
\[ c_1(\gamma) = a_4 \log(\gamma) + b_4 \] (19)
and the values for the coefficients are given in the table in the Fig. 5.

We also find that the eigenvalues (which always come in pairs $(\mu, 1 - \mu)$) exhibit the surprising feature that all but one pair hovers around the values 0 and 1 and hence contributes significantly to $S$. In Fig. 8 we also show the comparison of the eigenvalues obtained in the two complementary regions, we find that they differ only in the numbers of (0, 1) pairs. Further, if we calculate $S$ for the largest pairs of eigenvalues, we find that the error is small, as shown in Fig 9.
Figure 6: $S$ vs $\alpha$ for different $\gamma$ with $n_{max}/\alpha = 1200, 2000$ and 2600 fitted to $S = a \log(\sin(\pi \alpha)) + b$. The fit parameters are shown in the table.
Figure 7: A log-linear plot of $S$ vs $n_{\text{max}}/\alpha$ for different $\alpha$ with $\gamma = 16$, 200 and 1000 fitted to $S = a \log (n_{\text{max}}/\alpha) + b$. The fit parameters are shown in the table which show $a \sim 1/3$. This is also true for other values of $\gamma$. The curves for complementary values of $\alpha$ are indistinguishable.
Figure 8: A plot comparing the eigenvalues $\mu$ of the entropy equation for one choice of complementary regions with $\alpha = 0.2, 0.8$ for $n_{\text{max}}/\alpha = 2600$ and different choices of $\gamma$. On the left are the eigenvalues associated with the $f_k$ matrix elements which are independent of $\gamma$ and on the right are those associated with the $g_\kappa$, which are $\gamma$ dependent. We note that the number of eigenvalues differ in both regions but only in the number of $(1, 0)$ pairs which leads to the equality of the SSEE in these complementary regions. Further, the significant contribution comes from the $g_\kappa$ matrix elements of which there are precisely two which are substantially different from $(1, 0)$. These increase with $\gamma$ and are the main contributors to $c_1(\gamma)$.

Figure 9: In order to estimate the contribution of the pairs $(\mu, 1 - \mu)$, we plot the percentage error in the SSEE when only the largest pairs (one, two and three represented in blue, orange and green respectively) of eigenvalues are considered, as a function of the different parameters $\gamma, n_{\text{max}}/\alpha$ and $\alpha$. In each case, we see that the error goes down to < 1% even when only the 3 largest eigenvalues are retained.

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