Uncertainty relations and approximate quantum error correction

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The uncertainty principle can be understood as constraining the probability of winning a game in which Alice measures one of two conjugate observables, such as position or momentum, on a system provided by Bob, and he is to guess the outcome. Two variants are possible: either Alice tells Bob which observable she measured, or he has to furnish guesses for both cases. Here I derive new uncertainty relations for both, formulated directly in terms of Bob’s guessing probabilities. For the former these relate to the entanglement that can be recovered by action on Bob’s system alone. This gives a condition for approximate quantum error correction in terms of the recoverability of “amplitude” and “phase” information, implicitly used in the recent construction of efficient quantum polar codes. I also find a new relation on the guessing probabilities for the latter game, which has application to wave-particle duality relations.

Beyond their foundational appeal, uncertainty relations have become an important tool in quantum information theory, particularly entropic formulations (for a review, see [1]). One way to frame the recent statements is in terms of a guessing game [2]. In the game Bob prepares a quantum system and gives to Alice, who then measures one of two conjugate observables, such as position or momentum. She then asks Bob to guess the outcome of her measurement, and he wins the game if he can guess correctly. There are two versions of the game, depending on whether Alice tells Bob which observable she measured. If she does, then Bob need only furnish a guess for one observable, but if not he has no choice but to guess what the outcome would be in either case.

According to the uncertainty principle, the latter variant must be impossible to win with any reliability, but the former game can be won if Bob supplies Alice with one half of a maximally entangled state. For instance, if the observables are orthogonal angular momentum components of a spin-1/2 particle, then Bob can win the game by supplying a spin-singlet state. Upon learning which observable Alice measured, he performs the same measurement on his spin and reports the opposite result. Quantitative bounds on the game formulated using entropy can be found in [3, 4] for conjugate observables, and subsequent work has generalized the statements to arbitrary observables and entropy measures.

In this article I derive uncertainty relations for both variants of the game, formulated directly in terms of the guessing probabilities, rather than entropies. Guessing probability is a more directly operational quantity than entropy, which yields more straightforward quantitative constraints on the guessing game and an even simpler interpretation of the resulting uncertainty relations. For the latter game, call it version 2, I find a bound which constrains the allowed combinations of guessing probabilities. In accordance with intuition from the uncertainty principle, it implies that if Bob can reliably guess one of the observables, then he can do little better than to randomly guess the other. The bound builds on a closely related uncertainty relation for min- and max-entropies [5], and turns out to be related to wave-particle duality relations [6].

For the former game, version 1, the new uncertainty relation provides a converse to the sufficiency of using entanglement to win the game. It implies that if Bob can reliably guess either observable, then it must necessarily be possible to recover a high-fidelity entangled state by acting on his system alone. This can be viewed as a sufficient condition for approximate quantum error correction, and the decomposition into guessing two observables is reminiscent of the focus on “amplitude” and “phase” errors in constructions of exact quantum error correcting codes. Here, however, the focus is shifted away from errors and onto classical amplitude and phase information, i.e. the information about the two observables. Thus, whenever a channel can reliably transmit both kinds of classical information, it can reliably transmit entanglement. The appeal of this condition is that it reduces the quantum task to two simpler, more classical tasks, but does not require the picture of amplitude and phase errors.

A similar observation was made in [3], quantifying reliability in terms of entropy. Here the link is more direct, however, as the proof proceeds by building an entanglement recovery map from Bob’s guessing strategies, which can be viewed as measurements. This ensures that properties of the classical decoding measurements can be transferred to the entanglement recovery map. Indeed, this approximate error-correcting condition was implicitly used in the construction of quantum polar codes by the author and others [7], and the constructive link is crucial in showing that the quantum codes are efficiently decodable for suitable channels.

1 Uncertainty guessing games

Let us restrict our attention to finite-dimensional quantum systems, fix a dimension $d$, and suppose that Alice measures one of the generalized Pauli operators $X = \sum_{s=0}^{d-1} |s + 1\rangle \langle s|$ or $Z = \sum_{s=0}^{d-1} \omega^s |s\rangle \langle s|$. Here $\{|s\rangle\}$ is a
fixed basis and ω is a primitive dth root of unity. Denote the eigenvectors of X by |x⟩ = 1/\sqrt{d} \sum_{2}^{d-1} \omega^{ixz} |z⟩. These observables are conjugate in the sense that any eigenstate of one has uniform overlap with any eigenstate of the other, namely 1/\sqrt{d}.

Bob is free to prepare any conceivable quantum state of many systems, call it ψ, and give a d-dimensional subsystem denoted A to Alice for measurement. We can describe his procedure for guessing Alice’s outcome as performing a measurement on his remaining systems, as was done in the spin-singlet example above. Generally, both guessing measurements are POVMs, and let us denote by \( \Lambda_z \) (\( \Gamma_x \)) his POVM for guessing the \( Z \) (\( X \)) outcome.

The distinction between the two versions of the game is whether the two POVMs must be performed simultaneously or not, i.e. if they commute. They need not in version 1, when Alice tells Bob which observable she measured. But in version 2 Alice demands both guesses, so Bob must perform both measurements. In the latter case it is convenient to regard the commutation of the POVMs as arising from the fact that they are measurements on different subsystems, call them \( B \) and \( E \).

In either case we are chiefly interested in the optimal probability that Bob guesses correctly, which for the \( Z \) observable is given by

\[
P(Z^A|B)_\psi := \max_{\Lambda_z} \text{Tr} \left[ \sum_z (|z\rangle \langle z| \otimes \Lambda_z^B) \psi^{AB} \right].
\]

(1)

Here the optimization is over all valid POVMs, i.e. d positive semidefinite operators \( \Lambda_z \) with the property that \( \sum_z \Lambda_z = \mathbb{I} \). The optimal guessing probability for the \( X \) observable is entirely analogous. Also important is the maximum entanglement fidelity that can be obtained from \( \psi \) by acting on Bob’s systems alone,

\[
F(A|B)_\psi := \max_\sigma F (\Phi^{AB}, \sigma^{AB}(\psi^{AB})),
\]

(2)

where \( F(\rho, \sigma) := \text{Tr} \sqrt{\sqrt{\rho} \sigma \sqrt{\rho}} \) is the fidelity. Here the maximum is over quantum channels \( \Phi^{AB} \) taking \( B \) to \( A' \cong A \) and \( \Phi^{AB} \) is any maximally-entangled state.

2 Version 1: Noncommuting guesses

Bob can win the bipartite game for any dimension \( d \) by preparing an entangled state \( |\Phi\rangle^{AB} = 1/\sqrt{d} \sum_{2}^{d-1} |z\rangle^A |z\rangle^B \). No matter which observable Alice measures on \( A \), Bob performs the same measurement on \( B \). Clearly, for \( Z \), Alice and Bob’s measurement outcomes always match, and thus \( P(Z^A|B)_\psi = 1 \). The same conclusion holds for \( P(X^A|B)_\psi \), since a simple calculation shows that in this case their outcomes always sum to zero modulo \( d \).

In fact, entanglement in \( \psi \) is necessary to win the game. This conclusion also holds approximately and is quantitatively captured by the relation

\[
\text{acos} F(A|B)_\psi \leq \text{acos} P(Z^A|B)_\psi + \text{acos} P(X^A|B)_\psi,
\]

(3)

where acos is the inverse of the cosine function.

Let us now turn to the proof of (3). We can actually show a more general statement, somewhat outside the scope of the game, but useful in the context of quantum error correction. It turns out that it is not strictly necessary for both guessing probabilities to be close to unity to conclude that entanglement can be recovered from \( \psi \). We only need to show that \( Z^A \) is recoverable from Bob’s system and, separately, that \( X^A \) is recoverable from Bob’s system under the additional assumption that \( Z \) is perfectly recoverable. More concretely, let \( \psi^{X^A}_Z = U_{Z^A}^{X^A,\psi}(U_{Z^A}^{X^A,\psi})^\dagger \), where \( U_{Z^A}^{X^A,\psi} = \sum_{z=0}^{d-1} |z\rangle \langle z| \otimes |z\rangle^A \). Essentially, \( U_{Z^A} \) copies the \( Z \)-value of \( A \) to \( A' \), and therefore \( P(Z^A|B)_\psi \approx 1 \). Then, we can show

Theorem 1. For any bipartite state \( \psi^{AB} \),

\[
\text{acos} F(A|B)_\psi \leq \text{acos} P(Z^A|B)_\psi + \text{acos} P(X^A|B')_\psi.
\]

(4)

Before proceeding to the proof, first note that (4) implies (3) by monotonicity of acos and the following.

Lemma 1. For any bipartite state \( \psi^{AB} \),

\[
P(X^A|B')_\psi \geq P(X^A|B)_\psi.
\]

(5)

This statement is certainly plausible under the intuition that it is easier to guess \( X \) when given \( A' \) as well as \( B \). However, the process of generating \( \psi_Z \) from \( \psi \) alters the \( X \) observable, and so this reasoning does not immediately apply. Nevertheless, the statement holds due to conjugacy of \( X^A \) and \( U_Z \).

Proof. Start by noting that \( U_{Z^A}^{X^A,\psi} = 1/\sqrt{d} \sum_{x=0}^{d-1} |x\rangle^A \otimes (Z^{-x})^A \). Let \( |\psi\rangle^{AB} \) be a purification of \( \psi^{AB} \) and \( |\psi_Z\rangle^{AB} = U_{Z^A}^{X^A,\psi} |\psi\rangle^{AB} \). Since \( A \) and \( A' \) are interchangable in \( |\psi_Z\rangle \), it follows that

\[
|\psi_Z\rangle_A^{AB} = 1/\sqrt{d} \sum_{x=0}^{d-1} |x\rangle^A \otimes (Z^{-x})^A |\psi\rangle_A^{AB}.
\]

(6)

The action of \( Z^{-x} \) will be to shift the \( X \) value of \( A' \) by \(-x\). But if the \( X \) value of \( A' \) in \( |\psi_Z\rangle_A^{AB} \) is recoverable from \( B \), then by comparing the value on \( A' \) and \( B \), we can accurately determine the value of the shift. To this end, let \( \Gamma_x^B \) be the optimal measurement in \( P(X^A|B')_\psi \). Define the new measurement with elements \( \Gamma_x^B = \sum_{x'=0}^{d-1} \tilde{\Gamma}_{x'}^B \otimes \Gamma_x^Z \), where \( \tilde{\Gamma}_{x'}^B \) is the projector onto \( |x'\rangle^B \). This measurement yields the difference between the outcome of the guessing measurement \( \Gamma \) on \( B \) and the \( X \) measurement on \( A' \). Notice that \( (Z^{-y})^{B} \sum_{x'=0}^{d-1} \tilde{\Gamma}_{x'}^B \otimes \Gamma_x^Z = \mathbb{I}_B^Z \). Using the form of \( \psi_Z \) in (6) to compute \( P(X^A|B')_\psi \) gives

\[
P(X^A|B')_\psi \geq \text{Tr} \left[ \sum_{x=0}^{d-1} \tilde{\Gamma}_{x'}^B \otimes \Gamma_x^Z \right] |\psi_Z\rangle_A^{AB} \]

(7a)

\[
= \langle \psi \rangle_A \sum_{x=0}^{d-1} \langle \psi \rangle_A^{AB} \]

(7b)

\[
= P(X^A|B')_\psi.
\]

(7c)

establishing the claim.

Proof of Theorem 1. First consider the properties of the coherent implementations of Alice’s measurements. The
coherent Z measurement is given by $U_Z$, and the analogous $U_X$ is simply $U_X = \sum_{x=0}^{d-1} |x\rangle \langle x| \otimes |x\rangle^A$. Performing one after the other yields

$$U_X^{A\Lambda_0}U_Z^{A\Lambda_0} = \frac{1}{\sqrt{d}} \sum_{x=0}^{d-1} \omega^{-xz} |x\rangle^A \otimes |z\rangle^A \langle x| \langle z|^A. \quad (8)$$

The phase $\omega^{-xz}$ can be removed by a controlled-phase operation $V^{A\Lambda'} = \sum_{x=0}^{d-1} |x\rangle \langle x|^{A'} \otimes (Z^x)^A$. Defining $W^{A\Lambda'\Lambda}_x := V^{A\Lambda'} U^{A\Lambda_0} U^{A\Lambda_0}$, we find

$$W^{A\Lambda'\Lambda}_x = \frac{1}{\sqrt{d}} \sum_{x=0}^{d-1} |x\rangle^{A'} \otimes |x\rangle^A \otimes 1^{|A\Lambda|}. \quad (9)$$

This operator transfers $A$ to $A'$ and then creates a maximally-entangled state in $A''$.

Thus, if Bob can simulate the action of $U_X$ and $U_Z$ by coherent measurements on his system, he should be able to create a high-fidelity entangled state in $A''$. Suppose that the optimal measurements for guessing $Z$ and $X$ are $A'$ and $I_{A''}$, respectively. Define the coherent implementations of his two measurements, $V_{Z}^{A''}|A\Lambda| := \sum_{x} |z\rangle^A \otimes \sqrt{A_{x}}$, and $V_{X}^{A''}|A\Lambda| := \sum_{x} |x\rangle^A \otimes \sqrt{F_{Z}}$, and consider the fidelity between $V_{X}^{A''}|A\Lambda| \otimes V_{Z}^{A''}|A\Lambda|$ and $W^{A''}|A\Lambda|$. Since $V_{X}^{A''}|A\Lambda| \otimes V_{Z}^{A''}|A\Lambda|$ is an operation solely on Bob’s systems, we have

$$F(A\Lambda|B)_\psi \geq F(W|\psi), V_{X}^{A''}|A\Lambda| \otimes V_{Z}^{A''}|A\Lambda|). \quad (10)$$

The operation of $V_{X}^{A''}|A\Lambda| \otimes V_{Z}^{A''}|A\Lambda|$ is shown as a quantum circuit in Figure 1. Using the triangle inequality and unitary invariance of the fidelity [8, Ch. 10], (3) and (4) allow us to break the problem of quantum transmission down into two classical pieces. This gives additional structure to the problem of designing encoding and decoding mechanisms and allows a large flexibility in adapting each to the particular channel at hand. This can help guide our search for reliable codes and encoders. And since we can now efficiently construct classical transmission to quantum, this gives us some structure with which to construct efficient and practical decoders. As mentioned above, this link is crucial in constructing efficient quantum polar codes [7].

Moreover, shifting the focus away from errors to information allows yet more flexibility in adapting an error-correction scheme to a particular channel. This can be illustrated in the original example of an approximate error-correcting code by Leung et al. [9], where just four physical qubits suffice to protect one encoded logical qubit from the action of the amplitude damping channel to first order in the damping probability. (Exact correction requires five qubits.) Even though amplitude damping is not a combination of amplitude and phase errors, we can understand the operation of the approximate code as enabling reliable transmission of amplitude and phase information to first order. Recently we have applied this approach to find structured decoders for approximate codes designed for the amplitude damping channel based on nonlinear classical codes [10].

![Figure 1: The quantum circuit recovering entanglement from a bipartite state $\psi_{AB}$, when measurement of $Z$ or $X$ on $A$ can be reliably predicted by measurement of $B$ or $BA'$, respectively. The associated measurements $A_Z$ and $I_X$ are performed coherently in sequence, the latter taking results of the former into account, followed by a controlled-phase gate applied to the ancilla systems. The procedure also leaves the input state in systems $A'$ and $B$.](image)

3 Approximate quantum error correction

Both (3) and (4) can be regarded as conditions for approximate quantum error correction. Suppose we are interested in transmitting entanglement through a given quantum channel by inputting one half of some fixed bipartite state. This results in an output state $\psi_{AB}^\Lambda$, and measuring either $X$ or $Z$ of system $A$ results in an output that corresponds to input of an $X$ or $Z$ eigenstate to the channel. With either set of $X$ or $Z$ inputs we could hope to send classical information through the channel, and (3) or (4) imply that if both of these classical tasks are reliable on average, then it is also possible to transmit quantum information. Here the average is taken over the choice of $X$ or $Z$ inputs, the probabilities of which are determined by the associated measurement results.

$$Q(Z^A|B)_\psi := F(Z^A, |z\rangle^B \otimes |\psi\rangle^B), \quad (13)$$

which quantifies how close the $Z$ outcome of $A$ is to being uniformly distributed and independent of the conditional state in $B$. 

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| Image Reference | Caption |
|-----------------|---------|
| ![Figure 1](image) | The quantum circuit recovering entanglement from a bipartite state $\psi_{AB}$, when measurement of $Z$ or $X$ on $A$ can be reliably predicted by measurement of $B$ or $BA'$, respectively. The associated measurements $A_Z$ and $I_X$ are performed coherently in sequence, the latter taking results of the former into account, followed by a controlled-phase gate applied to the ancilla systems. The procedure also leaves the input state in systems $A'$ and $B$. |
Theorem 2. For $\psi^{AB E}$ a pure state,
\[ \cos F(A|B)_\psi \leq \cos P(Z^A|B|)_\psi + \cos Q(Z^A|E)_\psi. \] (14)
\[ \cos F(A|B)_\psi \leq \cos Q(X^A|A'E)_\psi + \cos Q(Z^A|E)_\psi. \] (15)

The first bound says that if the pure state $\psi^{AB E}$ can be used to create a secret key between Alice and Bob, a uniformly-distributed classical random variable independent of $E$, then the same state can be transformed into a maximally entangled state. A similar relation was used by Devetak in the achievability part of the quantum noisy channel coding theorem [11]. The second states that entanglement is recoverable from $B$ when system $E$ cannot predict $Z$ or $X$, the latter case even aided by knowledge of $Z$. This is broadly similar to the “decoupling” statement of Schumacher and Westmoreland [12], but formulated as decoupling of $X$ and $Z$ information, not of the quantum state itself. The proof below makes clear that $Q(Z^A|E)_\psi \geq P(X^A|A'B)_\psi$, and likewise $Q(X^A|A'E)_\psi \geq P(X|A'B)_\psi$. Thus, the latter condition is the strongest as it implies the former, and both imply (4).

Proof. The proof proceeds by replacing each of the two terms in the bound of (11) by fidélities.

To establish (14) we use $Q(Z^A|E)_\psi$ to construct an appropriate $V_X$ in $F(U_X|\psi_Z), V_X|\psi_Z).$. Start with $F(V_{X}_{\psi}, \pi_x \otimes \psi^E)$ and observe that $W|\psi = VU_X|\psi_Z)$ is a purification of $\pi_x \otimes \psi^E$, as is $U_X|\psi_Z)$. Of course, $|\psi_Z)$ is a purification of $\psi_{XE}$, so
\[ Q(Z^A|E)_\psi = \max_{V} \langle \psi_Z) | (U_X|A'^A|) V^{A'B} \psi_{XE}(Z^A) \rangle, \] (16)
where $V$ is an isometry from $A'B$ to $A'^A'B$. Calling the optimizer $V_X$ and applying it to (11) gives (14).

For (15) we use $Q(X^A|A'E)_\psi$, to construct an appropriate $V_X$ in $F(|\psi_Z), V_X|\psi_Z)$ for $Q(X^A|A'E)_\psi$, the relevant state is the $X$-measured version of $\psi_Z$:
\[ \psi_{XE,\psi}^{AB E} = \text{Tr} _A[U_X|A'^A|\psi_Z^{AB E}(U_X|A'^A|)] \]
\[ = \frac{1}{2} \sum _x |\tilde{x}\rangle \langle \tilde{x}| \otimes (Z^x \psi^E), \] (17a)

where in the final step we have interchanged the $A$ and $A'$ labels. Calling the optimizer $V_X$ and applying it to (11) with $V_X$ from the previous argument gives (15). \hfill \square

4 Version 2: Commuting guesses

The uncertainty principle implies that Bob cannot always win version 2 of the uncertainty game, for to do so would require preparing a state that is simultaneously an eigenstate of $X$ and $Z$. In fact, we can obtain a quantitative approximate statement in this direction from the above results. Using Bob’s optimal measurement $V_X$ in (16) we obtain $Q(Z^A|E)_\psi \geq P(X^A|BA')_\psi$, and by Lemma 1 this implies $Q(Z|E)_\psi \geq P(X|A')_\psi$. Thus, the larger Bob’s probability of guessing $X$ using $B$, the more the distribution of $Z$ looks uniform and independent of the system $E$. A tighter relation comes from an entropic uncertainty relation for min- and max-entropies [5], which in the present notation reads
\[ \max _\sigma F(\psi^E, \pi^A \otimes \sigma^E) \geq P(X^A|B)_\psi. \] (19)

Figure 2: Feasible and achievable guessing probabilities for version 2 of the uncertainty game, for the case $d = 64$.

But what are the possible combinations of guessing probabilities? The fidelity quantity $Q$ does have an operational meaning, but is not so immediately related to the guessing game. It turns out that (19) can be transformed into a constraint on the set of possible guessing probabilities. In particular, we have the following bounds.

Theorem 3. For any tripartite state $\psi^{AB E}$,
\[ P(X^A|E)_\psi + (P(X^A|B)_\psi - \frac{1}{2})^2 \leq 1, \] (20)
\[ P(X^A|B)_\psi + (P(Z^A|E)_\psi - \frac{1}{2})^2 \leq 1. \] (21)

The proof is based on a bound relating the trace distance and fidelity. Suppose $A_E$ is the optimal measurement in $P(Z^A|E)_\psi$ and define $T^{AE} = \sum _{\epsilon|\epsilon} |\epsilon\rangle \langle \epsilon| \otimes F^{E}$. Then $\text{Tr} (T^{AE}(\psi_{XE}^E - \pi_x \otimes \sigma^E)) = P(Z^A|E)_\psi - \frac{1}{2}$. Maximizing this expression over all possible POVM elements gives the trace distance: $\delta(\rho, \sigma) := \max _{\rho, \sigma} \text{Tr} (T(\rho - \sigma))$, and so we can appeal to the bound $\delta(\rho, \sigma)^2 + F(\rho, \sigma) \leq 1$ [8, §9.2.3] to infer
\[ P(Z^A|E)_\psi - \frac{1}{2})^2 + F(\psi^E, \pi^A \otimes \sigma^E) \leq 1. \] (22)

Using (19) in (22), choosing $\sigma^E$ to be the fidelity optimizer, immediately gives (21). The other inequality follows by interchanging observables and $B$ and $E$ systems.
Theorem 3 tells us more precisely how the probability of guessing the outcome of one observable tends to its minimum as the probability of guessing the other goes to unity, as illustrated in Fig. 2 for \( d = 64 \). The bounds are nearly tight when one guessing probability is large: Bob can simply interpolate between the X and Z bases by preparing a state in the family \( |\theta\rangle = \frac{1}{\sqrt{2}} (\cos \theta |0\rangle + \sin \theta |1\rangle) \), for \( \theta \in [0, \frac{\pi}{2}] \) and \( \mathcal{N} \) the appropriate normalization, and always guess the outcomes will be \( Z = 0 \) and \( X = 0 \).

The bounds are loose for \( P(Z^i|B) \approx P(X^i|B) \), but here we can appeal to bounds for a related uncertainty game. Instead of one party guessing Alice’s outcome, [13] supposes there are two, and each is told which observable was measured. This is more information than Bob receives in version 2 of the present game, so the guessing probabilities here must be smaller. Nevertheless, by having two parties, there exist commuting guessing measurements for the two observables, meaning constraints derived in [13] also apply here. Figure 2 shows that their Theorem 8 together with Theorem 3 give a nearly-tight characterization of allowed guessing probabilities.

For qubits, the situation is even better, as we can appeal to a bound from [6] which links the min-max uncertainty relation (19) to wave-particle duality relations. For \( d = 2 \) using their Eq. (6) instead of (22) in (19) gives \((2P(Z^A|E)_{\psi} - 1)^2 + (2P(X^A|B)_{\psi} - 1)^2 \leq 1\), which precisely matches the achievable strategy given above. Moreover, since (22) is tighter than Eq. (6) for \( d > 2 \), Theorem 3 leads to a tightened version of the wave-particle duality relation for symmetric interferometers in [6, Theorem 1] by using the definitions therein of the particle distinguishability \( \mathcal{D} = (dP(Z^A|E)_\psi - 1)/(d - 1) \) and the visibility \( \mathcal{V} = \max_{\psi} (dP(X^A|B)_\psi - 1)/(d - 1) \), where the maximization is over all observables conjugate to Z.

5 Conclusions and open questions
I have given uncertainty relations in the form of bounds on the guessing probabilities in the two variants of the uncertainty game. The uncertainty relation for the first version yields a sufficient condition for approximate quantum error correction, and a simple modification of the proof yields two stronger but nonconstructive sufficient conditions. In combination with [13, Theorem 8], the bounds on the second version were found to be essentially tight, but tightness of the first is an open question. Furthermore, following the approach of [6], the relation for the second version yields a new wave-particle duality relation for multi-path interferometers.

Finally, it is interesting to consider if channel versions of the uncertainty relations could hold. For instance, we may ask if reliable transmission of classical X and Z information even in the worst-case implies that the given channel is close to the identity channel. However, a counterexample constructed in [14] shows that this is not the case. The particular channel is such that Bob’s probabilities of guessing X and Z in the worst case are \((d + \sqrt{2} - 2)/d^2\) and exactly 1, respectively, and yet two particular channel inputs lead to completely distinct outputs to the channel environment. However, the information-disturbance tradeoff of [15] requires the environment output of channels close to identity to be essentially independent of the input. Hence, no channel analog of Theorem 1 or of (19) can hold, though neither statement of Theorem 2 is apparently ruled out. Moreover, the worst-case probability of guessing Z from the output to the environment is \(1/(d - 1)\), and therefore (21) (the tighter of the two bounds in this case) does hold. It would be interesting to determine if worst-case versions of Theorems 2 and 3 hold for channels generally; doubly so for the latter since it is derived from (19), which we just observed does not hold in this setting.

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