Countable Tightness, Elementary Submodels and Homogeneity

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Abstract

We show (in $\mathsf{ZFC}$) that the cardinality of a compact homogeneous space of countable tightness is no more than $2^{\aleph_0}$.

1 Introduction

A space $X$ is said to have countable tightness ($t(X) = \aleph_0$) if whenever $A \subseteq X$ and $x \in \overline{A}$, there is a countable $B \subseteq A$ such that $x \in \overline{B}$. A space $X$ is homogeneous if for every $x, y \in X$ there is a homeomorphism $f$ of $X$ onto $X$ with $f(x) = y$. It is known (see [3]) that any compact space of countable tightness contains a point with character at most $2^{\aleph_0}$; if the space is also homogeneous then it follows that $|X| \leq 2^{2^{\aleph_0}}$. In [1], Arhangel’skii conjectured that in fact $|X| \leq 2^{\aleph_0}$ for any such space. The main goal of this paper is to give a proof (in $\mathsf{ZFC}$) to Arhangel’skii’s conjecture. This is achieved in Theorem 3.2. As a corollary of this we also confirm a conjecture of I. Juhász, P. Nyikos and Z. Szentmiklóssy (see [4]), stating that it is consistent that every homogeneous $T_5$ compactum is first countable.

Our main tool will be the “Elementary Submodels technique”: Given a topological space $(X, \tau)$, let $M$ be an elementary submodel of $H(\theta)$ (i.e. the set of all sets of hereditary cardinality less than $\theta$) for a “large enough” regular cardinal $\theta$. Then one uses properties of $X \cap M$, $\tau \cap M$ and $M$ itself to get results about $(X, \tau)$. A model $M$ is countably closed if any countable sequence of elements of $M$ is in $M$ (i.e. $M^\omega \subseteq M$). For more details and a good introduction to the technique see [2]. Let us just say that in each
specific application, one takes $\theta$ large enough for $H(\theta)$ to contain all sets of interest in the context under discussion. In this sense we will just say $M \prec V$. In Section 2, we prove some basic facts in the context of elementary submodels of countably tight compact spaces; we also give answer (Theorem 2.2) to a question of L.R. Junqueira and F. Tall.

2 Elementary submodels

Fix a compact Hausdorff space $(X, \tau)$ with $t(X) = \aleph_0$ and fix a countably closed $M \prec V$ with $X, \tau \in M$. Let $Z = X \cap M \subseteq X$ with the subspace topology.

One of the main goals of this section is to show that $Z$ is a retract of $X$. The following result suggests what the retraction will be.

Lemma 2.1 For every $x \in X$ there is a $q_x \in Z$ such that for all $U \in \tau \cap M$ either $q_x \notin U$ or $x \in U$.

Proof. Fix $x \in X$ and assume there is not such a $q_x$. Then for each $q \in Z$ we can fix a $U_q \in \tau \cap M$ such that $q \in U_q$ and $x \notin U_q$. Since $Z$ is compact we get that $Z \subseteq \bigcup_{q \in Q} U_q$ for some finite $Q \subseteq Z$. On the other hand $x \notin \bigcup_{q \in Q} U_q \in M$ so by elementarity there is an $x' \in (X \cap M) \setminus \bigcup_{q \in Q} U_q$ which is impossible. ♠

Just by elementarity and the fact that $X$ is Hausdorff it is immediate that $\tau \cap M$ separates points in $X \cap M$. We prove now that in fact $\tau \cap M$ separates points in $Z$.

Lemma 2.2 If $p_0, p_1 \in Z$ and $p_0 \neq p_1$ then there are $U_0, U_1 \in \tau \cap M$ such that $p_0 \in U_0$, $p_1 \in U_1$ and $U_0 \cap U_1 = \emptyset$.

Proof. Given $p_0, p_1 \in Z$ with $p_0 \neq p_1$ we use regularity of $X$ to fix $U'_i, V_i \in \tau$ for $i \in 2$ such that $p_i \in V_i$, $\overline{V_i} \subseteq U'_i$ and $U'_0 \cap U'_1 = \emptyset$. Since $t(X) = \aleph_0$, there are countable $A_0, A_1 \subseteq M$ such that $A_i \subseteq V_i$ and $p_i \in \overline{A_i}$ for $i \in 2$. Since $M$ is countably closed we have that $A_0, A_1 \in M$ and hence by elementarity $M \models \exists U_0, U_1 \in \tau [ \overline{A_0} \subseteq U_0, \overline{A_1} \subseteq U_1, U_0 \cap U_1 = \emptyset ]$. ♠

Corollary 2.1 For every $x \in X$ there is a unique $q_x \in Z$ such that for all $U \in \tau \cap M$ either $q_x \notin U$ or $x \in U$. 
Definition 2.1 In view of the last result we define the function $r_M : X \to Z$ by $r_M(x) = q_x$ for $x \in X$.

Lemma 2.3 $r_M$ is continuous.

Proof. Fix $W \in \tau$ (not necessarily in $M$) with $W \cap Z \neq \emptyset$, fix $x \in r_M^{-1}(W)$ and let $q = r_M(x)$. We need to show that there is a $V \in \tau$ such that $x \in V \subseteq r_M^{-1}(W)$.

For each $p \in Z \setminus W$ we use Lemma 2.2 to get $U_p, V_p \in \tau \cap M$ such that $p \in U_p, q \in V_p$ and $U_p \cap V_p = \emptyset$. Since $Z \setminus W$ is closed (and hence compact) we get that $Z \setminus W \subseteq U := \bigcup_{p \in P} U_p$ for some finite $P \subseteq Z \setminus W$. Clearly $q \in V := \bigcap_{p \in P} V_p$ and $U \cap V = \emptyset$. Also by elementarity we have that $U, V \in M$.

Since $q \in V \in M$ we get (from the definition of $r_M$) that $x \in V$. To see that $V \subseteq r_M^{-1}(W)$, fix $y \in V$ and note that if $r_M(y)$ was in $Z \setminus W$ then it would be in $U$ and since $U \in M$ we would get that $y \in U$; but this is impossible since $U \cap V = \emptyset$ and hence $r_M(y) \in W$.

It is clear that $r_M(x) = x$ for all $x \in Z$. So we actually have that $r_M$ is a retraction. Looking closer at the last proof, we see that $q \in V \cap Z \subseteq W \cap Z$, so we also showed the following

Corollary 2.2 The set $\{U \cap Z : U \in \tau \cap M\}$ is a base for the topology of $Z$.

For reference, we summarize our results in the following

Theorem 2.1 Let $(X, \tau)$ be a compact space of countable tightness and let $M \prec V$ be countably closed with $X, \tau \in M$. Then $\tau_M : X \to X \cap M$ is a retraction and $\{U \cap X \cap M : U \in \tau \cap M\}$ is a base for $X \cap M$.

In [5], Junqueira and Tall define the space $X_M$ as the set $X \cap M$ with the topology $\tau_M$ generated by $\{U \cap M : U \in \tau \cap M\}$. They ask (Problem 7.22) if there is a consistent example of a compat $T_2$ space $X$ with countable tightness and a countably closed $M$ such that $X_M$ is not normal. A consequence of Theorem 2.1 is that $X_M$ is a subspace of $X$; this was already proved in [5] (Theorem 2.11). In the next theorem we make use of this fact to give a negative answer to their question.
Theorem 2.2 If \((X, \tau)\) is a compact Hausdorff space of countable tightness and \(M\) is countably closed then \(X \cap M\) \((= X_M)\) is normal.

**Proof.** Fix two disjoint closed \(E, F \subseteq X \cap M\). We claim that \(\overline{E}\) and \(\overline{F}\) are still disjoint. Therefore, since \(X \cap M\) is compact (and hence normal), \(\overline{E}\) and \(\overline{F}\) (and hence \(E\) and \(F\)) can be separated.

Now suppose (seeking a contradiction) that \(p \in \overline{E} \cap \overline{F}\). Since \(X\) is countably tight, there are countable \(C_E \subseteq E\) and \(C_F \subseteq F\) such that \(p \in \overline{C_E} \cap \overline{C_F}\). Since \(C_E\) and \(C_F\) are countable, they are in \(M\) and thus by elementarity there is a \(q \in X \cap M\) with \(q \in \overline{C_E} \cap \overline{C_F}\). But this is impossible since \(E\) and \(F\) are closed and disjoint in \(X \cap M\). Hence \(E \cap F = \emptyset\).

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### 3 Homogeneity

The following lemma was proved in \cite{3}.

**Lemma 3.1** If \(X\) is a compact \(T_2\) space of countable tightness then there are a countable set \(S \subseteq X\) and a non-empty closed \(G_\delta\) set \(H\) in \(X\) with \(H \subseteq S\).

In general, given a point \(p \in X\) one cannot expect to get \(p \in H\) in the previous lemma. For example if \(X = \kappa \cup \{\infty\}\) is the one point compactification of an uncountable discrete \(\kappa\) then \(p = \infty\) is a counter example. In this space the closure of any countable set is countable, but on the other hand any \(G_\delta\) subset containing \(p\) must be uncountable. However the following is obvious.

**Corollary 3.1** If \(X\) is a compact homogeneous \(T_2\) space of countable tightness and \(p \in X\) then there are a countable set \(S \subseteq X\) and a closed \(G_\delta\) set \(H\) in \(X\) with \(p \in H\) and \(H \subseteq S\).

Now we are ready to prove our first important result.

**Theorem 3.1** Suppose \((X, \tau)\) is a compact homogeneous Hausdorff space of countable tightness. Then \(w(X) \leq 2^{\aleph_0}\).

**Proof.** By Corollary 3.1 and regularity of \(X\) we can fix functions \(\psi : X \times \omega \to \tau\) and \(S : X \to [X]^{\aleph_0}\) such that for all \(x \in X\) and \(n \in \omega\):

\[\]
1. \( x \in \psi(x, n) \) and \( \psi(x, n + 1) \subseteq \psi(x, n) \).
2. \( H(x) := \bigcap_{m \in \omega} \psi(x, m) = \bigcap_{m \in \omega} \psi(x, m) \subseteq S(x) \).

Now fix a countably closed \( M \prec V \) with \( X, \tau, \psi, S \in M \) and \( |M| \leq 2^{\aleph_0} \).

Let \( Z = X \cap M \) and \( r_M : X \rightarrow Z \) as in Section 2. We know (by Theorem 2.1) that \( w(Z) \leq |\tau \cap M| \leq 2^{\aleph_0} \). We will show that in fact \( X = Z \) and hence \( w(X) \leq 2^{\aleph_0} \).

Fix now \( x \in X \) and let \( p = r_M(x) \in Z \). By Theorem 2.1 for each \( n \in \omega \) we can find \( U_n \in \tau \cap M \) such that \( \psi(p, n + 1) \cap Z \subseteq U_n \cap Z \subseteq \psi(p, n) \cap Z \).

Also by countable tightness of \( X \) we can fix \( A \in [X \cap M]^{\aleph_0} \) such that \( p \in \overline{A} \).

Now since \( A, \psi, H \) and \( < U_n : n \in \omega > \) are all in \( M \) (since \( M \) is countably closed), we get by elementarity that there must be a \( q \in X \cap M \) such that

1. \( q \in \bigcap_{n \in \omega} U_n \).
2. \( q \in \overline{A} \).
3. \( \overline{A} \cap \bigcap_{n \in \omega} U_n = \overline{A} \cap H(q) \).

Since \( p \) satisfies conditions 1 and 2, we get that \( p \in H(q) \) and hence \( x \in H(q) \) by definition of \( r_M \). But on the other hand \( S(q) \subseteq X \cap M \) and \( H(q) \subseteq \overline{S(q)} \) and therefore \( x \in Z \). This shows that \( X = Z \) which is what we wanted.

\[ \blacksquare \]

**Remark 3.1** With more work one can show that in fact \( X \subseteq M \), showing then that \( |X| \leq 2^{\aleph_0} \). However we shall use a different (perhaps simpler) strategy to get this result.

The following is a well known fact about compact homogeneous spaces. In fact it was shown in [6] that it also holds for any compact power homogeneous space.

**Lemma 3.2** If \( X \) is compact and homogeneous, then \( |X| \leq w(X)^{\pi \chi(X)} \).

Putting together the two previous results and the fact that \( \pi \chi(X) \leq t(X) \) for any compact space, we immediately get our main result.

**Theorem 3.2** If \( X \) is compact, homogeneous and \( t(X) = \aleph_0 \) then \( |X| \leq 2^{\aleph_0} \).

Using the fact that \( |X| = 2^{\chi(X)} \) for compact homogeneous spaces, we get:
Corollary 3.2 \((2^{\aleph_0} < 2^{\aleph_1})\) If \(X\) is compact, homogeneous and \(t(X) = \aleph_0\) then \(X\) is first countable.

In [6], J. van Mill asked whether every \(T_5\) (i.e. hereditarily normal) homogeneous compact space has cardinality \(2^{\aleph_0}\). In [4], I. Juhász, P. Nyikos and Z. Szentmiklóssy proved that the answer is yes in forcing extension resulting by adding \((2^{\aleph_1})^V\) Cohen reals. They also showed that after adding \(\aleph_2\) Cohen reals, every \(T_5\) homogeneous compact space has countable tightness. Putting this together with Theorem 3.2 and assuming for example that \(2^{\aleph_0} = \aleph_2\) and \(2^{\aleph_1} = \aleph_3\) in the ground model, we get a confirmation of a conjecture proposed by them in the same paper.

Theorem 3.3 It is consistent that every homogeneous \(T_5\) compact space is first countable.

References

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