ON GEOMETRY OF NUMBERS AND UNIFORM APPROXIMATION TO THE VERONESE CURVE

JOHANNES SCHLEISCHITZ

Abstract. Consider the classical problem of rational simultaneous approximation to a point in $\mathbb{R}^n$. The optimal lower bound on the gap between the induced ordinary and uniform approximation exponents has been established by Marnat and Moschevitin in 2018. Recently Nguyen, Poels and Roy provided information on the best approximating rational vectors to points where the gap is close to this minimal value. Combining the latter result with parametric geometry of numbers, we effectively bound the dual linear form exponents in the described situation. As an application, we slightly improve the upper bound for the classical exponent of uniform Diophantine approximation $\hat{\lambda}_n(\xi)$, for even $n \geq 4$. Unfortunately our improvements are small, for $n = 4$ only in the fifth decimal digit. However, the underlying method in principle can be improved with more effort to provide better bounds. We indeed establish reasonably stronger results for numbers which almost satisfy equality in the estimate by Marnat and Moschevitin. We conclude with consequences on the classical problem of approximation to real numbers by algebraic numbers/integers of absolutely bounded degree.

Keywords: exponents of Diophantine approximation, regular graph, parametric geometry of numbers
Math Subject Classification 2010: 11J13, 11J82

1. Consequences of a recent Theorem in geometry of numbers

Classical topics in Diophantine approximation are to study simultaneous rational approximation to points $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ and to find small values of linear forms in $\xi$ with integer coefficients. This naturally leads to investigation of classical exponents of Diophantine approximation. Denote by $\omega(\xi)$ the (possibly infinite) supremum of $\omega$ such that

$$Y_{\xi} := \max_{1 \leq i \leq n} |x\xi_i - y_i| \leq x^{-\omega}$$

has infinitely many integer vector solutions $\underline{y} = (x, y_1, \ldots, y_n) \in \mathbb{Z}^{n+1}$. This is usually referred to as the ordinary exponent of simultaneous approximation. Similarly, let the uniform exponent of simultaneous approximation denoted by $\hat{\omega}(\xi)$ be the supremum of $\omega$ such that

$$0 < x \leq X, \quad Y_{\xi} \leq x^{-\omega}$$

Middle East Technical University, Northern Cyprus Campus, Kalkanli, Güzelyurt
johannes.schleischitz@univie.ac.at.
has a solution \( x \in \mathbb{Z}^{n+1} \) for all large \( X \). For linear forms, let \( \omega^*(\xi) \) and \( \tilde{\omega}^*(\xi) \) respectively be the (possibly infinite) supremum of \( \omega^* \) such that

\[
\max_{1 \leq j \leq n} |a_j| \leq X, \quad 0 < |a_0 + a_1 \xi_1 + \cdots + a_n \xi_n| \leq X^{-\omega^*}
\]

has a solution in integers \( a_j \) for certain arbitrarily large \( X \) and all large \( X \), respectively. Variants of Dirichlet’s Box principle yield the lower bounds

\[
\omega(\xi) \geq \tilde{\omega}(\xi) \geq \frac{1}{n}, \quad \omega^*(\xi) \geq \tilde{\omega}^*(\xi) \geq n,
\]

for any \( \xi \in \mathbb{R}^n \). Finding refined relations between these exponents is an important topic in Diophantine approximation. Assume in the sequel that \( \xi \) is linearly independent over \( \mathbb{Q} \) together with \( \{1\} \). Marnat and Moshchevitin [8] recently proved a remarkable (sharp ) improvement of the trivial inequalities \( \omega(\xi) \geq \tilde{\omega}(\xi) \) and \( \omega^*(\xi) \geq \tilde{\omega}^*(\xi) \) that had been conjectured by Schmidt and Summerer [20]. The first can be stated as

\[
(2) \quad \tilde{\omega}(\xi) + \frac{\tilde{\omega}(\xi)^2}{\omega(\xi)} + \cdots + \frac{\tilde{\omega}(\xi)^n}{\omega(\xi)^{n-1}} \leq 1,
\]

where we mean that \( \tilde{\omega}(\xi) = 1 \) implies \( \omega(\xi) = \infty \). We do not require the dual linear form estimate in this paper. See also [19] for a previous weaker inequality, and the PhD thesis of Rivard-Cooke [11] for a simplified proof of (2) and a conjectured generalization. Equality in (2) is obtained in a case that Schmidt and Summerer in [20] refer to as the regular graph. We omit the geometrical motivation behind it, but point out that in this scenario the logarithms of the numbers \( x \) realizing the exponent \( \omega(\xi) \) in (1) (for appropriate \( y_i \)) form almost a geometric sequence with ratio \( \omega(\xi)/\tilde{\omega}(\xi) \). The recent preprint by Nguyen, Poels and Roy [9] provides a clearer picture by essentially showing that if \( \omega(\xi), \tilde{\omega}(\xi) \) almost satisfy identity in the estimate (2), then similar properties must still be satisfied. Indeed this information is provided in a compact form in Theorem 1.4 from [9], which we rephrase below upon omitting some subtle additional information not required here. With \( Y_{\underline{x}} \) defined in (1), for a parameter \( X > 1 \) we derive the quantity

\[
L_{\underline{x}}(X) = \min \{Y_{\underline{x}} : 1 \leq x \leq X \},
\]

where the minimum is taken over all integer vectors \( \underline{x} = (x, y_1, \ldots, y_n) \) with first coordinate positive and not exceeding \( X \). The induced sequence of vectors \( \underline{x} \) realizing the minimum for some \( X \) are sometimes referred to as best approximations or minimal points.

**Theorem 1.1** (Nguyen, Poels, Roy). Let \( \xi \in \mathbb{R}^n \) with \( \mathbb{Q} \)-linearly independent coordinates together with \( \{1\} \). Suppose that there exist positive real numbers \( a, b, \alpha, \beta \) such that for all large enough \( X \) we have

\[
(3) \quad bX^{-\beta} \leq L_{\underline{x}}(X) \leq aX^{-\alpha}.
\]

Then \( \alpha \leq \beta \) and

\[
(4) \quad \epsilon = \epsilon_{\alpha, \beta} := 1 - \left( \frac{\alpha^2}{\beta} + \cdots + \frac{\alpha^n}{\beta^{n-1}} \right) \geq 0.
\]
If
\[ \epsilon \leq \frac{1}{4n} \left( \frac{\alpha}{\beta} \right)^n \min \{ \alpha, \beta - \alpha \}, \]
then there exists an unbounded sequence of integer vectors \( \bar{x}_j = (x_j, y_{j,1}, \ldots, y_{j,n}) \) with the properties
- \[ |\alpha \log x_{j+1} - \beta \log x_j| \leq C + 4\epsilon (\beta/\alpha)^n \log x_{j+1} \]
- \[ |\log Y_{\bar{x}_j} + \beta \log x_{j+1}| \leq C + 4\epsilon (\beta/\alpha)^2 \log x_{j} \]
- the vectors \( \bar{x}_j, \bar{x}_{j+1}, \ldots, \bar{x}_{j+n} \) are linearly independent
- There is no vector \( \bar{x} = (x, y_1, \ldots, y_n) \) so that \( 1 \leq x \leq x_j \) and \( Y_{\bar{x}} < Y_{\bar{x}_j} \)

Clearly assumption (5) implies
\[ \alpha \leq \hat{\omega}(\xi) \leq \omega(\xi) \leq \beta, \]
and conversely any \( \xi \) satisfies (5) for any \( \alpha < \hat{\omega}(\xi) \) and \( \beta > \omega(\xi) \) with suitable \( a, b \). We will thus occasionally identify \( \alpha \) and \( \beta \) with \( \hat{\omega}(\xi) \) and \( \omega(\xi) \) respectively. Then \( \epsilon = 0 \) corresponds to equality in (2). In the situation of Theorem 1.1, we use the description of best approximating vectors to provide estimates on the dual exponents \( \omega^*(\xi), \hat{\omega}^*(\xi) \).

**Theorem 1.2.** Assume \( \xi = (\xi_1, \ldots, \xi_n) \) satisfies condition (3) of Theorem 1.1 with given \( a, b, \alpha, \beta \). With \( \epsilon \) as in (4) satisfying (5), let
\[ \phi := \frac{4\epsilon \beta^{n-1}}{\alpha^n}, \quad \rho := \frac{4\epsilon \beta^2}{\alpha^2}. \]

Then we have
\[ \hat{\omega}^*(\xi) \geq \frac{(\beta - \rho)S}{(\frac{\alpha}{\beta} - \phi)^{-n} + (\beta - \rho)(1 - S)}, \quad S := \sum_{j=1}^{n} \left( \frac{\alpha}{\beta} + \phi \right)^{1-j}. \]

Moreover
\[ \omega^*(\xi) \geq \frac{\rho^2 - \beta^2 - (\beta + \rho)^2 T}{\rho - \beta + (\beta + \rho)^2 T}, \quad T := \sum_{j=1}^{n-1} (\frac{\alpha}{\beta} + \phi)^j. \]

Conversely we have the upper bounds
(8) \[ \hat{\omega}^*(\xi) \leq (\beta - \rho)^{-1} \left( \frac{\alpha}{\beta} - \phi \right)^{-n}, \]
and
(9) \[ \omega^*(\xi) \leq (\beta - \rho)^{-1} \left( \frac{\alpha}{\beta} - \phi \right)^{-n-1}. \]

All bounds in Theorem 1.2 can in principle be improved as will be indicated in its proof, thereby implying better bounds in Theorem 2.2 below as well. However we do not attempt to optimize the method as it would lead to a significantly more cumbersome proof. A consequence of Theorem 1.2 is that equality in (2) is sufficient for all classical exponents to attain the values as in the corresponding regular graph, and by continuity reasons they cannot differ much from them if the error in (2) is sufficiently small.
Corollary 1.3. Assume $\xi = (\xi_1, \ldots, \xi_n)$ satisfies equality in (2). Then

$$\hat{\omega}^\ast(\xi) = \frac{\omega(\xi)^{n-1}}{\omega(\xi)^n}, \quad \omega^\ast(\xi) = \frac{\omega(\xi)^n}{\omega(\xi)^{n+1}}.$$ 

Moreover, for any $\varepsilon > 0$ and $c < 1$, if we identify $\alpha = \hat{\omega}(\xi)$ and $\beta = \omega(\xi)$, upon $\alpha \leq c$ there exists $\delta = \delta(n, c) > 0$ such that the estimate $\varepsilon = \varepsilon_{\alpha, \beta} < \delta$ implies

$$\frac{\omega(\xi)^{n-1}}{\omega(\xi)^n} - \varepsilon \leq \hat{\omega}^\ast(\xi) \leq \frac{\omega(\xi)^{n-1}}{\omega(\xi)^n} + \varepsilon, \quad \frac{\omega(\xi)^n}{\omega(\xi)^{n+1}} - \varepsilon \leq \omega^\ast(\xi) \leq \frac{\omega(\xi)^n}{\omega(\xi)^{n+1}} + \varepsilon.$$

Proof. We identify $\alpha = \hat{\omega}(\xi)$ and $\beta = \omega(\xi)$. Let $\varepsilon > 0$. First fix $\alpha \in [1/n, 1)$. Let $\beta_0$ be the solution to equality in (2). Then by continuity of the bounds in Theorem 1.2 in $\alpha, \beta$, there is some $\delta > 0$ that depends on $n, \varepsilon, \alpha$ such that $\beta \in (\beta_0 - \delta, \beta_0 + \delta)$ implies that the bound expressions differ from the respective values $\omega(\xi)^{n-1}/\omega(\xi)^n$ and $\omega(\xi)^n/\omega(\xi)^{n+1}$ by less than $\varepsilon$. By implicit function theorem, for given $\varepsilon$ we have that the solution $\beta$ to (5) depends continuously on $\alpha$, and for $\varepsilon = 0$ it becomes $\beta_0$. Hence the same claim holds if $\varepsilon = \varepsilon_{\alpha, \beta}$ in (4) is smaller than some modified $\delta > 0$. Since we restrict $\alpha$ to the compact interval $[1, c]$ and (a lower bound for) $\delta$ depends continuously on $\alpha$, we may choose $\delta > 0$ independent of $\alpha$. \hfill \qed

Unfortunately $\delta$ depends in a sensitive way on $\varepsilon$ (that is, on $\alpha, \beta$), consequently the numerical improvements in Theorem 2.2 below are small.

2. An application to the Veronese curve

We consider $\xi = (\xi, \xi^2, \ldots, \xi^n)$ on the Veronese curve. As customary we write

$$\lambda_n(\xi) = \omega(\xi), \quad \hat{\lambda}_n(\xi) = \hat{\omega}(\xi), \quad w_n(\xi) = \omega^\ast(\xi), \quad \hat{w}_n(\xi) = \hat{\omega}^\ast(\xi),$$

for the intensely studied classical exponents of Diophantine approximation on the Veronese curve. The exponents $w_n(\xi)$ date back to Mahler [7], others have been defined in [11]. For $n = 1$ we have $\hat{\lambda}_1(\xi) = 1$ for any irrational $\xi$, see Khintchine [5]. The consequence $\hat{\omega}(\xi) \leq 1$ for any $\xi \notin \mathbb{Q}^n$ agrees with (2). While without restriction to the Veronese curve the uniform exponent $\hat{\omega}(\xi)$ attains the maximum value 1 for certain $\xi \in \mathbb{R}^n$ with $\mathbb{Q}$-linearly independent coordinates with $\{1\}$ no matter how large $n$ is (see Poels [10] for general classes of manifolds containing points with this property), on the Veronese curve the exponent is always significantly smaller. Only for $n = 2$ the optimal bound is known, given as

$$\hat{\lambda}_2(\xi) \leq \frac{\sqrt{5} - 1}{2} = 0.6180 \ldots$$

The inequality was found by Davenport and Schmidt [4], the optimality is due to Roy [13]. For even $n$ the following estimates are from [17].
Theorem 2.1 (Schleischitz). Let \( n \geq 2 \) be an even integer. For any transcendental real number \( \xi \) we have \( \hat{\lambda}_n(\xi) \leq \tau_n \) where \( \tau_n \) is the solution of
\[
\left(\frac{n}{2}\right)^n t^{n+1} - \left(\frac{n}{2} + 1\right) t + 1 = 0
\]
in the interval \( \left(\frac{2}{n+2}, \frac{2}{n}\right) \).

The bound \( \tau_n \) is obtained as the solution for identity in (2) for \( \hat{\lambda}_n(\xi) \) when \( \hat{\lambda}_n(\xi) = 2/n \).

For \( n = 2 \) we confirm the optimal estimate (11). Other numerical bounds are
\[
\hat{\lambda}_4(\xi) \leq 0.370635 \ldots, \quad \hat{\lambda}_6(\xi) \leq 0.268186 \ldots, \quad \hat{\lambda}_{20}(\xi) \leq 0.092803 \ldots.
\]

For large \( n \), the bound is of order
\[
\tau_n = \frac{2}{n} - \frac{\chi}{n^2} + o\left(n^{-2}\right), \quad n \to \infty,
\]
where \( \chi = 3.18 \ldots \) can be expressed as a zero of some power series, see [17]. For sake of completeness, we remark that for the case of odd \( n \) the bound \( \hat{\lambda}_n(\xi) \leq 2/(n + 1) \) was established by Laurent [6], an improvement for \( n = 3 \) is due to Roy [12]. Application of (6) from our new Theorem 1.2 leads to a small improvement that can be stated in the following way.

Theorem 2.2. Let \( n \geq 4 \) be an even integer and \( \xi \) be any transcendental real number. For \( \alpha \in [1/n, 1] \) define
\[
\epsilon_\alpha = 1 - \alpha - \frac{n}{2} \alpha^2 - \cdots - \left(\frac{n}{2}\right)^{n-1} \alpha^n, \quad S_\alpha = \sum_{j=1}^{n} \left(\frac{n \alpha}{2} + \frac{2^{n+1} \epsilon_\alpha}{n^{n-1} \alpha^n}\right)^{1-j}.
\]

Then \( \hat{\lambda}_n(\xi) \leq \sigma_n \) with \( \sigma_n \) the solution of the implicit equation
\[
\left(\frac{2}{n} - \frac{16 \epsilon_\alpha}{n^{4/2}}\right) S_\alpha \left(\frac{n \alpha}{2} - \frac{2^{n+1} \epsilon_\alpha}{n^{n-1} \alpha^n}\right)^{-n} + \left(\frac{2}{n} - \frac{16 \epsilon_\alpha}{n^{4/2}}\right)(1 - S_\alpha) = \mu_n := \max\{2n - 2, w(n)\},
\]
for \( \alpha \) in the interval \( (0, 1) \), where \( w(n) \) is the solution of
\[
\frac{(n-1)w}{w-n} - w + 1 = \left(\frac{n-1}{w-n}\right)^n
\]
in the interval \( [n, 2n-1) \). For \( n \geq 10 \) the maximum expression in (14) becomes \( 2n - 2 \).

The left hand side in (14) is (10) with \( \beta = 2/n \). It can be checked and follows from the proof that \( \sigma_n < \tau_n \) for \( n \geq 4 \), however the quantities differ only by a very small amount. For example we obtain the numerical bounds
\[
\hat{\lambda}_4(\xi) \leq 0.370629 \ldots, \quad \hat{\lambda}_6(\xi) \leq 0.268183 \ldots,
\]
that may be compared with (12). The asymptotics (13) remain unaffected and the improvement occurs in a lower order term. Improvements can be made via improving the estimate (6) in Theorem 1.2. Another source of improvement would be better bounds for the exponent \( \hat{w}_n(\xi) \), related to \( \mu_n \) in the theorem. On the other hand, increasing the
bound for \( \epsilon \) in (5) would not lead to better bounds when combined with Theorem 1.2 in its present form.

The sensitive dependence of (6), (7), (8), (9) on \( \epsilon = \epsilon_{a,\beta} \) is the key problem why the improvement compared to Theorem 2.1 is small. We wonder about estimates in the optimal case \( \epsilon = 0 \). This is partly motivated by the fact that for \( n = 2 \) identity in (11) is attained (only) for numbers \( \xi \) with \( (\xi, \xi^2) \) with \( \epsilon = 0 \) inducing a regular graph, see [13]. We also include a result for the case where the difference from the regular graph is very small, derived by continuity reasoning.

**Theorem 2.3.** Let \( n \geq 4 \) be even and let \( \xi \in \mathbb{R} \). First assume \( \xi = (\xi, \xi^2, \ldots, \xi^n) \) satisfies equality in (2). If \( n \in \{4, 6, 8\} \), then

\[
\hat{\lambda}_4(\xi) < 0.3588, \quad \hat{\lambda}_6(\xi) < 0.2540, \quad \hat{\lambda}_8(\xi) < 0.1968,
\]

and as \( n \to \infty \) we have the asymptotical bound

\[
\hat{\lambda}_n(\xi) \leq \frac{\Theta + o(1)}{n}, \quad n \to \infty,
\]

where \( \Theta = 1.7564 \ldots \) is the solution to \( e^t/t = 2 \sqrt{e} \) with \( t > 1 \).

When we drop equality assumption in (2), we still have the following continuity result. For \( n \geq n_0 \) and every \( \epsilon > 0 \), there exists \( \delta_n > 0 \) such that if \( \alpha = \hat{\lambda}_n(\xi) \in [1/n, 1] \) and \( \beta = \lambda_n(\xi) \in [1/n, \infty] \) are linked by \( \epsilon = \epsilon_{a,\beta} < \delta_n \), then we have

\[
\hat{\lambda}_n(\xi) \leq \frac{\Theta + \epsilon}{n}.
\]

Unfortunately, similar to Corollary 1.3 the bound \( \delta_n \) for the conclusion (17) is very small when \( \epsilon \) is small. On the other hand, we remark that for \( n = 8 \) the bound in (15) is already smaller than \( 2/(n + 2) = 0.2 \) from Theorem 2.1.

We derive a corollary on approximation to real numbers by algebraic integers. For an algebraic integer \( \alpha \) denote by \( H(\alpha) \) its height, i.e. the naive height (maximum modulus among its coefficients) of its irreducible minimal polynomial over \( \mathbb{Z}[T] \). By a well-known argument of Davenport and Schmidt [4], from Theorem 2.2 and Theorem 2.3 we infer

**Corollary 2.4.** Let \( n \geq 4 \) be an even integer, \( \xi \) a real transcendental number and \( \epsilon > 0 \). Then for \( \sigma_n \) as in Theorem 2.2 the inequality

\[
|\xi - \alpha| < H(\alpha)^{-\frac{1+\epsilon}{\sigma_n}}
\]

has infinitely many solutions in real algebraic integers \( \alpha \) of degree at most \( n + 1 \) (or real algebraic numbers of degree precisely \( n \)). If \( \xi = (\xi, \xi^2, \ldots, \xi^n) \) satisfies equality in (2), then for \( n \geq n_0 \) and \( \Theta \) as in Theorem 2.3 the estimate

\[
|\xi - \alpha| < H(\alpha)^{-\frac{1+\epsilon}{\Theta}}
\]

has infinitely many solutions \( \alpha \) as above.

For the implication on algebraic numbers of precise degree \( n \) see Bugeaud and Teulié [3]. It would be very desirable to make (19) unconditional, the bounds for the exponent in (18) are currently best known and only of order \(-n/2 - O(1)\).
3. Proof of Theorem 1.2

We first introduce parametric geometry of numbers that our proof is based on. We follow the introductory paper of Schmidt and Summerer [18]. Let \( n \geq 1 \) an integer and \( \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n \) be fixed. For a parameter \( Q > 1 \) and every \( 1 \leq j \leq n + 1 \), let \( \psi_j(Q) \) be the supremum of \( \nu \) for which the system

\[
|x| \leq Q^{1+\nu}, \quad Y_\xi \leq Q^{-1/n+\nu}
\]

has \( j \) linearly independent integral solution vectors \( x = (x_1, \ldots, x_n) \), with \( Y_\xi \) as in (1).

Let \( q = \log Q \) and define the functions \( L_j(q) \) from \( \psi_j(Q) \) via

\[
L_j(q) = \frac{\psi_j(Q)}{q}, \quad 1 \leq j \leq n + 1.
\]

It can be checked that \( L_j(q) \) are piecewise linear with slopes among \( \{-1, 1/n\} \). In fact, any \( L_j(q) \) is locally induced by a function

\[
L_j(q) = \max \{ \log x - q, \log Y_\xi + q \},
\]

more precisely we may write

\[
L_j(q) = \min \max_{1 \leq i \leq j} L_{x_i}(q),
\]

with minimum taken over all sets of \( j \) linearly independent vectors \( x_1, \ldots, x_j \in \mathbb{Z}^{n+1} \). We infer

\[
-1 \leq \psi_j \leq \frac{1}{n}, \quad 1 \leq j \leq n + 1,
\]

where we have put

\[
\psi_j = \liminf_{Q \to \infty} \psi_j(Q) = \liminf_{q \to \infty} \frac{L_j(q)}{q}, \quad \overline{\psi}_j = \limsup_{Q \to \infty} \psi_j(Q) = \limsup_{q \to \infty} \frac{L_j(q)}{q}.
\]

Another important property of the functions highlighted in [18] is

\[
\sum_{j=1}^{n+1} L_j(q) = O(q^{-1}). \tag{22}
\]

This is equivalent to Minkowski’s Second Convex Body Theorem. In particular, on long intervals on average one function \( L_j(q) \) will decay with slope \(-1\) whereas the remaining functions rise with slope \( 1/n \). The next result linking the quantities \( \overline{\psi}_{n+1}, \overline{\psi}_{n+1} \) with classical exponents is already more or less contained in work of Schmidt and Summerer [18].

**Proposition 3.1.** We have

\[
(1 + \omega^*(\xi)^{-1})(1 + \overline{\psi}_{n+1}) = \frac{n + 1}{n}, \tag{23}
\]

and

\[
(1 + \omega^*(\xi)^{-1})(1 + \overline{\psi}_{n+1}) = \frac{n + 1}{n}. \tag{24}
\]
Proof. Define $\omega_{n+1}(\xi)$ and $\hat{\omega}_{n+1}(\xi)$ as the supremum of $\omega$ such that (1) has $n + 1$ linearly independent solution vectors $(x, y_1, \ldots, y_{n})$ for arbitrarily large $X$ and all large $X$, respectively. Mahler’s Theorem on Polar Convex Bodies implies $\hat{\omega}^\ast(\xi)^{-1} = \omega_{n+1}(\xi)$ and similarly $\omega^\ast(\xi)^{-1} = \hat{\omega}_{n+1}(\xi)$, as essentially established in [18]. With these identifications, the identities from [18, Theorem 1.4] in the case of the $(n + 1)$-st successive minimum instead the first (see also [14]) turn into the claims. □

The following direct consequence of Theorem 1.1 is just formulated for convenience.

**Proposition 3.2.** Let $\xi$ satisfy the hypothesis of Theorem 1.1 and consider the induced sequence $(x_j)_{j \geq 1}$ with $x_j = (x_j, y_{j,1}, \ldots, y_{j,n})$ from its claim. Then for $\ell \geq 0$ any fixed integer as $j \to \infty$ we have

$$
\left(\frac{\alpha}{\beta} - \phi + o(1)\right) \log x_{j+\ell} \leq \log x_j \leq \left(\frac{\alpha}{\beta} + \phi + o(1)\right) \log x_{j+\ell}, \quad j \geq 1,
$$

where

$$
\phi := \frac{4\epsilon \beta^{n-1}}{\alpha^n}.
$$

Moreover, again as $j \to \infty$ we have

$$
(-\beta - \rho + o(1)) \log x_j \leq \log Y_j \leq (-\beta + \rho + o(1)) \log x_j, \quad j \geq 1,
$$

where

$$
\rho := \frac{4\epsilon \beta^2}{\alpha^2}.
$$

**Proof.** The first property of Theorem 1.1 can be formulated

$$
\log x_j = \left(\frac{\alpha}{\beta} + \phi_j + o(1)\right) \log x_{j+1}, \quad j \to \infty,
$$

with some $|\phi_j| \leq \phi$. The first claim (25) follows. The second claim (26) follows directly from the second property of Theorem 1.1. □

In the proof, in accordance with notation in (1) we write

$$
Y_j = \max_{1 \leq i \leq n} |x_j \xi_i - y_{j,i}|, \quad j \geq 1,
$$

when $x_j = (x_j, y_{j,1}, \ldots, y_{j,n})$ are the vectors from Theorem 1.1.

**Proof of Theorem 1.2.** We start with (6), which turns out to be the most tedious calculation. In view of (23), we have to provide lower bounds for $\psi_{n+1}(1)$. We look at the first successive minimum. Let $(q_k)_{k \geq 1}$ denote the sequence of local minima of $L_1(q)$ and $(r_k)_{k \geq 1}$ the sequence of its local maxima, labeled $q_k < r_k < q_{k+1}$. We estimate the first $n$ functions $L_1(q), L_2(q), \ldots, L_n(q)$ from above in intervals $q \in I_k := [q_k, q_{k+1})$. Since $[q_m, \infty)$ is the disjoint union of these intervals $I_k$ over $k \geq m$ and by (22)

$$
\psi_{n+1}(e^q) = \frac{L_{n+1}(q)}{q} + \sum_{j=1}^{n} \frac{L_j(q)}{q} - O(q^{-1}), \quad q > 0,
$$

We have

$$
\hat{\psi}_{n+1}(e^q) = \frac{L_{n+1}(q)}{q} + \sum_{j=1}^{n} \frac{L_j(q)}{q} - O(q^{-1}), \quad q > 0,
$$

where

$$
\hat{\psi}_{n+1}(e^q) = \frac{L_{n+1}(q)}{q} + \sum_{j=1}^{n} \frac{L_j(q)}{q} - O(q^{-1}), \quad q > 0.
$$
this will lead to the desired lower bound for $\psi_{n+1}$.

Let $x_k \in \mathbb{Z}^{n+1}$ be the vector inducing $q_k$, so that $L_{x_k}(q)$ has a local minimum at $q = q_k$. Note that these are precisely the best approximation vectors from Theorem 1.1. Also note that $L_{x_k}(r_k) = L_{x_{k+1}}(r_k)$ since every maximum of $L_1(q)$ is a minimum of $L_2(q)$. Write $x_k = (x_k, y_{k,1}, \ldots, y_{k,n})$. Notice now that the sets of vectors
\[
\{x_{k-n+1}, x_{k-n+2}, \ldots, x_k\}, \quad \{x_{k-n+2}, x_{k-n+3}, \ldots, x_{k+1}\}
\]
are both linearly independent by the third claim of Theorem 1.1. By (21) we infer that $L_{x_k}(q)$ decreases with slope $1$, by (22) the remaining functions $L_2(q), \ldots, L_{n+1}(q)$ must increase with slope $1/n$, up to an error of $O(1)$. Hence the minimum of $\psi_{n+1}(Q) = L_{n+1}(q)/q$ with $q \in [q_k, q_{k+1}]$ (or equivalently $Q \in [e^{q_k}, e^{q_{k+1}}]$) must essentially be taken within $q \in [q_k, r_k]$ up to an error of $o(1)$, more precisely
\[
\min_{q \in [q_k, q_{k+1}]} \frac{L_{n+1}(q)}{q} \geq \min_{q \in [q_k, r_k]} \frac{L_{n+1}(q)}{q} - O(q^{-1}).
\]
Since the error term does not affect $\psi_{n+1}$, the claim is shown.

Next we further split the interval $[q_k, r_k]$ into two intervals $[q_k, s_k]$ and $[s_k, r_k]$ where $s_k$ is the first coordinate of the point where $L_{z_{q_k-n+1}}(q)$ meets $L_{z_{k+1}}(q)$, by (20) that is the solution of
\[
\log x_{k+1} - q = \log Y_{k-n+1} + \frac{q}{n},
\]
which yields
\[
s_k = \frac{n}{n+1}(\log x_{k+1} - \log Y_{k-n+1}).
\]
In the interval $q \in [q_k, s_k]$ we estimate the left expression in (28), that is
\[
\sum_{j=1}^{n} \frac{L_{z_{q_k-n+1}}(q)}{q}.
\]
Since every $L_{z_q}(q)$ reaches its minimum at $q_i$ and the sequence $(q_i)_{i \geq 1}$ is clearly increasing, in this interval all involved functions increase with slope $1/n$. Thus
\[
L_{z_q}(q_k) = \log Y_i + \frac{1}{n}, \quad 1 \leq i \leq k,
\]
and the sum (30) reaches its maximum at the right end point \( q = s_k \) and we conclude
\[
\max_{q \in [q_k, s_k]} \sum_{j=1}^{n} L_{k-n+j}(q) \leq \frac{\sum_{j=1}^{n} L_{k-n+j}(s_k)}{s_k} \leq \frac{\sum_{j=1}^{n} (\log Y_{k-n+j} + \frac{s_k}{n})}{s_k} = 1 + \sum_{j=1}^{n} \frac{\log Y_{k-n+j}}{s_k}.
\]
Thus by (29) we infer
\[
\max_{q \in [q_k, s_k]} \sum_{j=1}^{n} L_{k-n+j}(q) \leq 1 + \frac{n + 1}{n} \cdot \sum_{j=1}^{n} \frac{\log Y_{k-n+j}}{\log x_{k+1} - \log Y_{k-n+1}}.
\]
Finally, in the interval \( q \in [s_k, r_k] \) we estimate the right expression in (28), that is
\[
\sum_{j=2}^{n+1} \frac{L_{k-n+j}(q)}{q}.
\]
Since in this interval \( L_1(q) \) is induced by \( L_{s_k}(q) \) and increases with slope \( 1/n \), the graphs of \( L_{s_k}(q) \) and \( L_{s_k+1}(q) \) do not intersect in its interior but at a value \( q \geq r_k \) (otherwise \( L_{s_k+1}(q) \) would induce falling \( L_1(q) \) in some partial interval). This means \( L_{s_k+1}(q) \) decays with slope \(-1\) in the entire interval \([s_k, r_k] \). Thus the numerator sum in (32) has slope \(-1 + (n-1)q / n = -1/n \) in it. It follows from the general bound \( L_{s_k}(q) \leq q/n \) for every \( q \) (by (20)) that the expression (32) decays in the interval \([s_k, r_k] \) and therefore attains its maximum at the left end point \( s_k \), which leads to the same bound as in (31) again.

This bound (31) remains to be estimated, which we perform via Theorem 1.1. We first readily verify that by \( x_{k+1} > 1 \) and \( Y_{n-k+1} < 1 \) the expression is increasing in all involved variables \( x_{k+1}, Y_{k-n+1}, \ldots, Y_k \). Thus we have to find upper bounds for each variable.

By (26) applied to \( j = k - n + 1 \) up to \( j = k \), we obtain
\[
\max_{q \in [q_k, q_{k+1}]} \sum_{j=1}^{n} L_j(q) \leq 1 + \frac{n + 1}{n} \cdot \frac{\sum_{j=1}^{n}(-\beta + \rho + o(1)) \log x_{k-n+j}}{\log x_{k+1} - (-\beta + \rho + o(1)) \log x_{k-n+1}} \quad k \to \infty.
\]
Observe that \( \rho < \beta \) as this is equivalent to \( \alpha^2/\beta > 4\epsilon \), but since \( n \geq 2 \) and \( 0 < \alpha < \beta \) we compute
\[
\frac{\alpha^2}{\beta} = \beta \left( \frac{\alpha}{\beta} \right)^2 \geq \beta \left( \frac{\alpha}{\beta} \right)^n \geq (\beta - \alpha) \left( \frac{\alpha}{\beta} \right)^n > \frac{\left( \frac{\beta}{\alpha} \right)^n (\beta - \alpha)}{n} \geq 4\epsilon,
\]
where the most right inequality follows from (5). Hence we verify that this time the expression (33) is decreasing in \( x_{k-n+2}, \ldots, x_k \) but increasing in \( x_{k+1} \). For \( x_{k-n+1} \) the situation is unclear, depending on the sign of \( \log x_{k+1} - (\beta - \rho) \sum_{j=2}^{n} \log x_{k-n+j} \). Thus we want to find lower bounds for \( x_{k-n+2}, \ldots, x_k \) and upper bounds for \( x_{k+1} \) in terms of \( x_{k-n+1} \).
It can be checked similar to (34) that $\alpha / \beta - \phi > 0$. By (25) we can estimate
\[
\log x_{k-n+j} \geq \frac{\log x_{k-n+1}}{((\frac{\alpha}{\beta} + \phi)^{j-1} + o(1))}, \quad 1 \leq j \leq n, \quad \log x_{k+1} \leq \frac{\log x_{k-n+1}}{((\frac{\alpha}{\beta} - \phi)^{n} + o(1))}.
\]
We remark that these are crude estimates that can be improved by a refined analysis of the interplay of the possible quotients of the occurring $\log x_j$, however things become rather messy. Similar situations will occur more often below when we apply Proposition 3.1. Inserting these bounds in (33) we divide numerator and denominator by $x_{n-k+1}$ to conclude
\[
\max_{q \in [q_k, q_{k+1}]} \sum_{j=1}^{n} \frac{L_j(q)}{q} \leq 1 + \frac{n+1}{n} \cdot \frac{\sum_{j=1}^{n} (\frac{1}{(\frac{\alpha}{\beta} + \phi)^{j-1} + o(1)})}{\frac{1}{(\frac{\alpha}{\beta} - \phi)^{n} + o(1)} + \beta - \rho - o(1)}
= 1 + \frac{n+1}{n} (-\beta + \rho) \cdot \frac{\sum_{j=1}^{n} (\frac{1}{(\frac{\alpha}{\beta} + \phi)^{j-1}})}{\frac{1}{(\frac{\alpha}{\beta} - \phi)^{n}} + \beta - \rho} + o(1), \quad k \to \infty.
\]
Now we finally use (27) to conclude
\[
\psi_{n+1} = \limsup_{q \to \infty} \frac{L_{n+1}(q)}{q} = \limsup_{q \to \infty} \max_{k \in [q_k, q_{k+1}]} \frac{L_{n+1}(q)}{q} \geq -1 + \frac{n+1}{n} (-\beta + \rho) \cdot \frac{\sum_{j=1}^{n} (\frac{1}{(\frac{\alpha}{\beta} + \phi)^{j-1}})}{\frac{1}{(\frac{\alpha}{\beta} - \phi)^{n}} + \beta - \rho}.
\]
Using (23) leads to the stated bound (6) after some rearrangements.

We turn to the lower bound (7) for $\omega^v(\xi)$. Here we look at the values $q = q_k$ where the functions $L_{x_k}(q)$ attain their minimum values. Again since $(q_k)_{k \geq 1}$ increases, all
\[
L_{x_k-n+1}(q_k), L_{x_k-n+2}(q), \ldots, L_{x_k-1}(q_k)
\]
increase with slope $1/n$ at $q = q_k$. Hence
\[
L_{x_k-n+j}(q_k) = \log Y_{k-n+j} + \frac{q_k}{n}, \quad 1 \leq j \leq n,
\]
since clearly also for $j = n$ we can use that representation as there is equality in the expressions of (20) for $L_{x_k}(q)$ at $q = q_k$. From the linear dependence of $L_{x_k-n+1}, L_{x_k-n+2}, \ldots, L_{x_k-1}$ by Theorem 1.1 and (27), if we let $Q_k = e^{q_k}$ we infer
\[
\psi_{n+1}(Q_k) = \frac{L_{n+1}(q_k)}{q_k} \geq -\sum_{j=1}^{n} \frac{L_{x_k-n+j}(q_k)}{q_k} - O(q_k^{-1}) = -\sum_{j=1}^{n} \frac{\log Y_{k-n+j}}{q_k} - 1 - O(q_k^{-1}).
\]
Now $q_k$ is by (20) given as the solution of
\[
\log x_k - q_k = \log Y_k + \frac{q_k}{n}
\]
hence
\[
q_k = \frac{n}{n+1} \cdot (\log x_k - \log Y_k).
\]
Plugging in yields
\[
\psi_{n+1}(Q_k) \geq \frac{n+1}{n} \cdot \frac{\sum_{j=1}^{n} \log Y_{k-n+j}}{\log Y_k - \log x_k} - 1 - O(q_k^{-1}).
\]
We see that the right hand side expression is decreasing in $x_k$ and in $Y_k, \ldots, Y_{k-1}$, the situation is unclear for $Y_k$ depending on the sign of $\log x_k + \sum_{j=1}^{n-1} \log Y_j$. Thus we look for lower bounds for the other variables in terms of $Y_k$. By (26) we have

$$\psi_{n+1}(Q_k) \geq \frac{n+1}{n} \cdot \frac{\log Y_k - (\beta + \rho + o(1)) \sum_{j=1}^{n-1} \log x_{k-j}}{\log Y_k(1 + \frac{1}{\beta+\rho}) + o(1)} - 1, \quad k \to \infty.$$  

Now this expression is increasing in $Y_k$ so again by (26) we can estimate

$$\psi_{n+1}(Q_k) \geq \frac{n+1}{n} \cdot \frac{(\rho - \beta + o(1)) \log x_k - (\beta + \rho + o(1)) \sum_{j=1}^{n-1} \log x_{k+j}}{(\rho - \beta + o(1))(1 + \frac{1}{\beta+\rho}) \log x_k} - 1.$$  

Since $\rho - \beta < 0$, the right hand side is increasing in $x_k-1, \ldots, x_k$. Thus by (25) we can replace the sum in the numerator expression by $T + o(1)$ with

$$T := \sum_{j=1}^{n-1} \frac{\alpha}{\beta + \phi}^j,$$

without making the expression larger. Dividing denominator and numerator by $\log x_k$ and dropping the negligible lower order terms yields

$$\psi_{n+1} \geq \limsup_{k \to \infty} \psi_{n+1}(Q_k) \geq \frac{n+1}{n} \cdot \frac{(\rho - \beta) - (\beta + \rho)T}{(\rho - \beta)(1 + \frac{1}{\beta+\rho})} - 1.$$  

Inserting in (24) gives (7).

We turn to the upper bounds. Here we have to find upper bounds for $\overline{\psi}_{n+1}$ and $\underline{\psi}_{n+1}$, respectively. For the uniform exponent we choose $q = u_k$ as the points where the graphs of $L_{x-k} (q)$ and $L_{x_k} (q)$ meet, that is the solution of

$$L_{x-k} (q) = L_{x_k} (q).$$  

Since at this position clearly $L_{x-k} (q)$ increases whereas $L_{x_k} (q)$ decreases, from (20) we obtain

$$u_k = \frac{n}{n+1} (\log x_k - \log Y_k).$$  

Again since $x_k, x_k-1, \ldots, x_k$ are linearly independent, by (21) we have to bound each of the corresponding functions $L_{x_k}, k - n \leq j \leq k$, at position $u_k$. It is easy to see that $L_{x_k} (u_k) = L_{x_k-n} (u_k)$ is the maximum among the values, since the rising part of the graph of $L_{x_k} (q)$ will intersect the falling part of each of $L_{x_k-n} (q), \ldots, L_{x_k-1} (q)$ before (i.e. at smaller $q$ values) it eventually meets $L_{x_k} (q)$ at $q = u_k$. Moreover by (20) the function value is given as $L_{x_k} (u_k) = \log x_k - u_k$. Hence we estimate

$$\underline{\psi}_{n+1} \leq \liminf_{k \to \infty} \frac{L_{x_k} (u_k)}{u_k} \leq \liminf_{k \to \infty} \frac{\log x_k - u_k}{u_k} = \liminf_{k \to \infty} \frac{\log x_k}{u_k} - 1.$$  

Using the representation (35) of $u_k$ yields

$$\underline{\psi}_{n+1} \leq \liminf_{k \to \infty} \frac{n+1}{n} \cdot \frac{\log x_k}{\log x_k - \log Y_{k-n}} - 1.$$
We check that the expression increases in \( Y_{k-n} \). Hence we can use (26) to estimate
\[
\psi_{n+1} \leq \liminf_{k \to \infty} \frac{n+1}{n} \cdot \frac{\log x_k}{\log x_k - (-\beta + \rho) \log x_{k-n}} - 1
\]
Again since \( \beta > \rho \) we see the right hand side decays in \( x_{k-n} \), application of (25) with \( \ell = n \) yields
\[
\psi_{n+1} \leq \frac{n+1}{n} \cdot \frac{1}{1 + (\beta - \rho)(\frac{\alpha}{\beta} - \phi)^n} - 1.
\]
Plugging this into (23) we get (9).

Finally we show (9). For this we estimate \( \overline{\psi}_{n+1} \) from above to apply (24). Recall we defined \( u_k \) as the value where \( L_{\xi_k-n} \) meets \( L_{\xi_k} \). Consider the interval \( [u_k, u_{k+1}) \). We have seen above that the last successive minimum function at \( u_k \) is at most
\[
L_{n+1}(u_k) \leq L_{\xi_k}(u_k) = L_{\xi_k-n}(u_k) = \frac{u_k}{n} + \log Y_{k-n}.
\]
Now let \( p_k \) be the point where \( L_{\xi_k-n} \) meets \( L_{\xi_k+1} \). Clearly \( p_k > u_k \). Similarly as in (35) above we get
\[
p_k = \frac{n}{n+1}(\log x_{k+1} - \log Y_{k-n}).
\]
comparison with (35) for index \( k + 1 \) gives \( p_k < u_{k+1} \) since \( (Y_j)_{j \geq 1} \) decreases. We split \( [u_k, u_{k+1}) \) into \( [u_k, p_k) \) and \( [p_k, u_{k+1}) \). It is readily verified that for \( q \in [u_k, p_k) \) we have
\[
L_{\xi_k-n}(q) > L_{\xi_k-n+1}(q) > \cdots > L_{\xi_k}(q)
\]
and these vectors are linearly independent, we have
\[
L_{n+1}(q) \leq L_{\xi_k-n}(q), \quad q \in [q_k, p_k).
\]
In the other partial interval \( [p_k, u_{k+1}) \) we similarly see that
\[
L_{\xi_k+1} > L_{\xi_k-n+2}(q) > \cdots > L_{\xi_k}
\]
and again by their linear independence and the definition of \( u_{k+1} \) we conclude
\[
L_{n+1}(q) \leq L_{\xi_k+1}(q), \quad q \in [p_k, u_{k+1}).
\]
Since \( L_{\xi_k+1} \) which decays with slope \(-1\) in the latter interval \( [p_k, u_{k+1}) \), the corresponding values \( L_{\xi_k+1}(q)/q \) decrease in this interval. Thus we only need to take into account \( [u_k, p_k] \) when looking for upper bounds for \( \overline{\psi}_{n+1} \). Moreover, by (38) and since \( L_{\xi_k-n}(q) \) increases in this interval with slope \( 1/n \), the quantity \( L_{\xi_k-n}(q)/q \) increases in \( [u_k, p_k] \), it suffices to consider the right end point \( q = p_k \). From (37) and since \( [u_k, \infty) \) is the disjoint union of the intervals \( [u_k, u_{k+1}) \) over \( k \geq m \), a very similar argument as for (36) above yields
\[
\overline{\psi}_{n+1} \leq \frac{n+1}{n} \cdot \frac{1}{1 + (\beta - \rho)(\frac{\alpha}{\beta} - \phi)^{n+1}} - 1.
\]
Again using (24) we obtain the bound (9). \( \square \)
4. Proof of Theorems 2.2, 2.3

The proof of Theorem 2.2 is based on a case distinction $\lambda_n(\xi) > \frac{2}{n}$ and $\lambda_n(\xi) \leq \frac{2}{n}$. The first case is dealt with by the following result from [17].

**Theorem 4.1** (Schleischitz). Let $n \geq 2$ be an even integer. Assume $\xi$ is transcendental and satisfies $\lambda_n(\xi) > \frac{2}{n}$. Then

$$\hat{\lambda}_n(\xi) \leq \frac{2}{n+2}.$$  

It can be verified that the bound $2/(n+2)$ is smaller than $\sigma_n$ in Theorem 2.2 (thus clearly also smaller than $\tau_n$ in Theorem 2.1).

In the latter case $\lambda_n(\xi) \leq \frac{2}{n}$ we use Theorem 1.2. We recall that the slightly weaker bounds in Theorem 2.1 in [17] followed from Theorem 4.1 combined with (2), corresponding to $\epsilon = 0$ in (4). The key observation for the improvement is that when $b = 2/n$ and $\epsilon = 0$, then the bound for the exponent $\hat{w}_n(\xi)$ we obtain from (6) is larger than known upper bounds for this exponent rephrased in Theorem 4.2 below. Thus by continuity we expect that if $\epsilon$ is sufficiently small, we still get a contradiction by (6).

We turn to the upper bound indicated above. Indeed, in contrast to general points in $\mathbb{R}^n$ where $\hat{\omega}^*(\xi) = \infty$ may occur, for points on the Veronese curve the exponent is bounded in terms of $n$. The following currently best known bound is a consequence of Bugeaud and Schleischitz [2] when incorporating the results from [8], already observed in [16]. The paper [2] in turn improved on a previous bound by Davenport, Schmidt [4].

**Theorem 4.2** (Bugeaud, Schleischitz). For any $n \geq 1$ and transcendental real $\xi$ we have $\hat{w}_n(\xi) \leq \mu_n$ where $\mu_n$ is as defined in Theorem 2.2.

Recall (10) for the following proof.

**Proof of Theorem 2.2.** We may assume $\lambda_n(\xi) \leq 2/n$ by Theorem 4.1. Observe that by (2) this implies an upper bound for $\hat{\lambda}_n(\xi)$ that is just slightly larger than $\sigma_n$. Denote this inferred bound by $\Psi_n$ and write $I_n := (\sigma_n, \Psi_n)$.

Now assume on the contrary that $\hat{\lambda}_n(\xi) > \sigma_n$ for some $\xi$. Then $\hat{\lambda}_n(\xi) \in I_n$, and again from (2) we obtain a lower bound for $\lambda_n(\xi)$ just slightly smaller than $2/n$. Denote the bound by $\Phi_n$ and the resulting range for $\lambda_n(\xi)$ by $J_n := (\Phi_n, 2/n)$. For given parameters $\alpha, \beta$ write

$$W_{\alpha, \beta} := \frac{(\beta - 4\beta^2\alpha^2)S}{(\beta - 4\beta^2\alpha^2)^{-n} + (\beta - 4\beta^2\alpha^2)(1 - S)}$$

where $\epsilon = \epsilon_{\alpha, \beta}$ is defined in (4), and $S = S_{\alpha, \beta}$ in (6). Then in particular $W_{\alpha} := W_{\alpha, 2/n}$ denotes the left hand side in (14). By construction $\epsilon_{\alpha} = \epsilon_{\alpha, 2/n}$ and $S_{\alpha} = S_{\alpha, 2/n}$ and $\sigma_n$ is the solution for $\alpha$ to equality $W_{\alpha} = \mu_n$, thus

$$W_{\sigma_n, 2/n} = \mu_n.$$
A brief calculation verifies that \( \epsilon = \epsilon_{\alpha,\beta} \) from (11) satisfies (15) when
\[
(40) \quad \alpha \in I_n, \quad \beta \in J_n \cup \left( \frac{2}{n}, \frac{2}{n} + \epsilon \right) =: K_n
\]
for some small \( \epsilon = \epsilon(n) > 0 \) (independent of \( \alpha \)).

Next observe that by the strict inequality \( \hat{\lambda}_n(\xi) > \sigma_n \), we have that the hypothesis (3) of Theorem 1.1 holds for every pair \((\alpha, \beta)\) with \( \alpha \in (\sigma_n, \hat{\lambda}_n(\xi)) \subseteq I_n \) and \( \beta > 2/n \), and suitable \( a, b \). Thus, hypothesis (3) holds in particular for \( \alpha \in I_n \) and \( \beta \in (2/n, 2/n + \epsilon) \subseteq K_n \). Hence we can apply Theorem 1.2 for any
\[
(41) \quad \alpha \in (\sigma_n, \hat{\lambda}_n(\xi)) \subseteq I_n, \quad \beta \in \left( \frac{2}{n}, \frac{2}{n} + \epsilon \right) \subseteq K_n,
\]
and (6) yields for any pair \( \alpha, \beta \) as in (41) the inequality
\[
\hat{w}_n(\xi) = \hat{\omega}^*(\xi) \geq W_{\alpha,\beta}.
\]
A short calculation further shows that when \( \beta \in K_n \) is fixed, the expression \( W_{\alpha,\beta} \) increases as \( \alpha \) increases in \( I_n \). Thus
\[
W_{\alpha,\beta} > W_{\sigma_n,\beta},
\]
with strict inequality because \( \alpha > \sigma_n \) strictly. By continuity of \( W_{\alpha,\beta} \) in the second argument, for any fixed \( \alpha > \sigma_n \) we still have
\[
(43) \quad W_{\alpha,\beta} > W_{\sigma_n,2/n}
\]
if \( \beta \) is sufficiently close to \( 2/n \) (alternatively one can start with \( \mu_n + \epsilon \) for arbitrarily small \( \epsilon > 0 \) in the right hand side of (14), and use continuity to derive the contradiction below). Thus for any pair \( \alpha, \beta \) as in (11), combining (39), (42), (43) upon making \( \epsilon \) smaller if necessary we conclude
\[
\hat{w}_n(\xi) = \hat{\omega}^*(\xi) \geq W_{\alpha,\beta} > W_{\sigma_n,2/n} = \mu_n.
\]
This contradicts Theorem 4.2. Thus we cannot have \( \hat{\lambda}_n(\xi) > \sigma_n \). \( \square \)

Finally we prove Theorem 2.3 with a similar method.

**Proof of Theorem 2.3.** We proceed as in the proof of Theorem 2.2. Let us first assume \( \epsilon = 0 \) for some \( \alpha, \beta \), i.e. there is equality in (2) and we are in the situation of the regular graph. Then \( \alpha = \hat{\lambda}_n(\xi) \) and \( \beta = \lambda_n(\xi) \) and by Corollary 1.3 we have identity
\[
(44) \quad \hat{w}_n(\xi) = \hat{\omega}^*(\xi) = \frac{\omega(\xi)^{n-1}}{\hat{\omega}(\xi)^n} = \frac{\lambda_n(\xi)^{n-1}}{\hat{\lambda}_n(\xi)^n}.
\]
It is easily checked that upon equality in (2) the expressions are increasing as functions in \( \alpha = \hat{\lambda}_n(\xi) \). By Theorem 4.2, the exponent \( \hat{\lambda}_n(\xi) \) is thus bounded by the solution to
\[
(45) \quad \mu_n = \frac{\lambda_n(\xi)^{n-1}}{\hat{\lambda}_n(\xi)^n},
\]
with the exponents \( \lambda_n(\xi) \) and \( \hat{\lambda}_n(\xi) \) linked by an identity in (2). For \( n \in \{4, 6, 8\} \) we derive the stated numerical bounds (14) with some computation. For large \( n \), we have \( \mu_n = 2n - 2 < 2n \) and with some analysis of the regular graph the claimed asymptotics.
can be derived. We give some more details. We use identity [15, (31)] which, upon identifying 
\[ \hat{\omega}_{n+1}(\xi) - 1 = \omega(\xi) \] 
(see the proof of Proposition 3.1), can be written 
\[ (1 + \omega^*(\xi)) \cdot \left( 1 + \frac{1}{\omega^*(\xi)} \right)^n = \left( 1 + \frac{1}{\omega(\xi)} \right) \cdot (1 + \omega(\xi))^n. \] 
Moreover by (44) and (45) we have 
\[ \hat{\omega}^*(\xi)/n = \mu_n/n = 2 - o(1) \] 
and then also \( n\omega^*(\xi)/n = 2 + o(1) \) as can be derived from identity [15, (33)], we see that the left hand side in (46) is of order 
\( (2\sqrt{e} + o(1))^n \) as \( n \to \infty \). Thus so is the right hand side and we readily conclude 
\( n\omega(\xi) = \Theta + o(1) \). Finally the smaller quantity \( n\hat{\omega}(\xi) = n\hat{\lambda}_n(\xi) \) will be asymptotically of the same order as \( n \to \infty \) (see (30) in [15]), thus (16) follows.

Finally (17) follows by a similar continuity argument as in Corollary 1.3. First assume \( \alpha \in [1/n, 1) \) is fixed. By Corollary 1.3 and its proof, the expression \( \hat{\omega}_n(\xi) = \hat{\omega}^*(\xi) \) depends continuously on \( \epsilon \) if \( \beta = \beta(\epsilon) \) is such that there is identity in (5). Moreover for \( \epsilon = 0 \) and \( \alpha = \hat{\lambda}_n(\xi) \) larger than claimed, we get a contradiction \( \hat{\omega}_n(\xi) > \mu_n \) as we have proved above. Thus for \( \epsilon \in [0, \delta_n] \) with \( \delta_n > 0 \) small enough in dependence of \( n, \alpha, \epsilon \), in case of larger \( \alpha = \hat{\lambda}_n(\xi) \) we still obtain the same contradiction \( \hat{\omega}_n(\xi) > \mu_n \). Finally again as we can restrict \( \alpha = \hat{\lambda}_n(\xi) \) to a compact interval like \( [1/n, 1/2] \), we can choose \( \delta_n \) uniformly in \( \alpha \), depending only on \( n \) and \( \epsilon \). □

References

[1] Y. Bugeaud, M. Laurent. Exponents of Diophantine Approximation and Sturmian Continued Fractions. *Ann. Inst. Fourier (Grenoble)* 55 (2005), 773–804.
[2] Y. Bugeaud, J. Schleischitz. On uniform approximation to real numbers. *Acta Arith.* 175 (2016), 255–268.
[3] Y. Bugeaud, O. Teulié. Approximation d’un nombre réel par des nombres algébriques de degré donné. (French) [Approximation of a real number by algebraic numbers of a given degree]. *Acta Arith.* 93 (2000), no. 1, 77–86.
[4] H. Davenport, W. M. Schmidt. Approximation to real numbers by algebraic integers. *Acta Arith.* 15 (1969), 393–416.
[5] Y. A. Khintchine, Über eine Klasse linearer Diophantischer Approximationen. *Rendiconti Palermo* 50 (1926), 170–195.
[6] M. Laurent. Simultaneous rational approximation to the successive powers of a real number. *Indag. Math. (N.S.)* 14 (2003), no. 1, 45–53.
[7] K. Mahler, Zur Approximation der Exponentialfunktionen und des Logarithmus I.II. *J. reine angew. Math.* 166 (1932), 118–150.
[8] A. Marnat, N. Moshchevitin. An optimal bound for the ratio between ordinary and uniform exponents of Diophantine approximation. *arXiv: 1802.03081*.
[9] N. A. V. Nguyen, A. Poels, D. Roy. A transference principle for simultaneous Diophantine approximation. *arXiv: 1908.11777*.
[10] A. Poels. A class of maximally singular sets for rational approximation. *arXiv: 1909.12159*
[11] M. Rivard-Cooke. Parametric Geometry of Numbers. PhD thesis, University of Ottawa, 2019; https://ruor.uottawa.ca/handle/10393/38871.
[12] D. Roy. On simultaneous rational approximations to a real number, its square, and its cube. *Acta Arith.* 133 (2008), no. 2, 185–197.
[13] D. Roy. Approximation to real numbers by cubic algebraic integers I. *Proc. London Math. Soc.* (3) 88 (2004), no. 1, 42–62.
[14] J. Schleischitz. Two estimates concerning classical diophantine approximation constants. *Publ. Math. Debrecen* 84/3-4 (2014), 415–437.

[15] J. Schleischitz. Some notes on the regular graph defined by Schmidt and Summerer and uniform approximation. *J. J. Algebra, Number Theory Appl.* 39 (2017), no. 2, 115–150.

[16] J. Schleischitz. Uniform approximation and best approximation polynomials. *Acta Arith.* 185 (2018), no. 3, 249–274.

[17] J. Schleischitz. An equivalence principle between polynomial and simultaneous Diophantine approximation. *to appear in Ann. Scuola Normale Superiore di Pisa*. [arXiv:1704.00055](https://arxiv.org/abs/1704.00055).

[18] W.M. Schmidt, L. Summerer. Parametric geometry of numbers and applications. *Acta Arith.* 140 (2009), no. 1, 67–91.

[19] W.M. Schmidt, L. Summerer. Diophantine approximation and parametric geometry of numbers. *Monatsh. Math.* 169 (2013), 51–104.

[20] W.M. Schmidt, L. Summerer. Simultaneous approximation to three numbers. *Mosc. J. Comb. Number Theory* 3 (2013), no. 1, 84–107.