Non-singular “Gauss” black hole from non-locality: a simple model with a de Sitter core, mass gap, and no inner horizon

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Abstract

Cutting out an infinite tube around \( r = 0 \) formally removes the Schwarzschild singularity, but without a physical mechanism this procedure seems ad hoc and artificial. In this Essay we provide such a mechanism by means of non-locality. Motivated by the Gauss law we define a suitable radius variable as the inverse of a regular non-local potential, and use this variable to model a non-singular black hole. The resulting geometry has a de Sitter core, but for generic values of the regulator there is no inner horizon, saving this model from potential issues via mass inflation. An outer horizon only exists for masses above a critical threshold, thereby reproducing the conjectured “mass gap” for black holes in non-local theories. The geometry’s density and pressure terms decrease exponentially, thereby rendering it an almost-exact vacuum solution of the Einstein equations outside of astrophysical black holes.

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1 Introduction

The presence of singularities inside black holes is a robust prediction of General Relativity. However, it is commonly believed that a suitable UV completion of gravity ameliorates this behavior and renders all physical quantities finite in proximity to the center of the black hole. While there are indications that putative theories of quantum gravity feature regular black holes in their semiclassical limits, an explicit derivation of such objects proves cumbersome.

For this reason, Bardeen [1] considered a simple modification of the Schwarzschild metric that is manifestly finite at $r = 0$ but reproduces the large-distance behavior known from General Relativity. Others have followed similar approaches and have developed a rich family of non-singular black hole geometries [2–10] (and references therein). In this Essay we focus on static regular black holes and postpone a discussion of time-dependent formation (and evaporation) to later studies. Static non-singular black hole geometries typically have several properties:

1. They do not solve the vacuum Einstein equations exactly, but their Einstein tensor decreases polynomially with distance away from the center at $r = 0$.
2. In addition to the outer event horizon at $r \approx 2GM$ there exists an inner horizon at $r \sim \ell$ as well, where $\ell$ is the regularization scale.
3. Close to $r = 0$ the geometry approaches a de Sitter form.
4. The curvature upper bound is given by $1/\ell^2$ and is independent of the black hole mass, which is also called the “limiting curvature condition.” [4,11–13]

Moreover, in the spherically symmetric and static case, the regularity is achieved by replacing the mass parameter $M$ by a mass function $M(r)$ that scales in a suitable fashion to remove the singularity at $r = 0$. A well-known model is that of Hayward [2],

$$\text{d}s^2 = -f_H(r)\text{d}t^2 + \frac{\text{d}r^2}{f_H(r)} + r^2\text{d}\Omega^2, \quad f_H(r) = 1 - \frac{2Mr^2}{r^3 + 2M\ell^2},$$

(1)

where $\ell > 0$ is the regularization length scale. The complicated appearance of the black hole’s mass parameter $M$ in the denominator of the function $f_H(r)$ guarantees the limiting curvature condition. Typically, the function $f_H(r)$ has two zeroes, corresponding to the inner horizon and the outer horizon, respectively.

In this Essay we propose a different route. Starting from a modified radius variable we construct a non-singular metric that has no such inner horizon but still features a de Sitter core. Moreover, the form of the modified radius variable is motivated by recent results in non-local gravity, thereby removing a layer of ambiguities.
2 Modified radius variable

In a local field theory in four spacetime dimensions, the potential of a point particle decreases monotonically with the inverse spatial distance (in suitable units),

\[ \phi_{\text{loc}} = -\frac{1}{r}. \]  

(2)

Similarly, the field strength decreases with the inverse area, due to Gauss’ law. Now, reversing this logic, one could measure the field strength and thereby deduce the radial distance away from the source. As the field strength diverges, one reaches \( r = 0 \). For the sake of simplicity, but without loss of generality, in what follows we shall consider the potential as the fundamental variable, for which similar mathematical properties hold true. Hence, one may be inclined to define a radius to be the inverse of the potential,

\[ r \equiv -\frac{1}{\phi_{\text{loc}}}. \]  

(3)

However, the singularity of the local potential is deemed unphysical since it gives rise to infinite forces and accelerations. It is possible to modify the equations of motion for scalar potentials, and at the linear level a class of non-local theories has proven particularly successful in removing the divergence at \( r = 0 \) \([14–20]\); for earlier work in non-commutative geometry and regular black holes see Refs. \([21–24]\). Within a quantum-mechanical approach to the singularity problem one also encounters non-local terms \([25,26]\). For these reasons we consider the non-local equation

\[ e^{-\ell^2 \nabla^2} \nabla^2 \phi_{\text{nl}} = 4\pi G M \delta^{(3)}(\mathbf{x}), \]  

(4)

and one may show that the potential of such an infinite-derivative theory is given by

\[ \phi_{\text{nl}} = -\frac{\text{erf} \left( \frac{r}{2\ell} \right)}{r}, \]  

(5)

in suitable units. At large distances \( r \gg \ell \) one recovers \( \phi_{\text{loc}} \), but the potential is finite and regular at \( r = 0 \). Using this non-locally regularized potential we may now define a modified radial distance

\[ \tilde{r} \equiv -\frac{1}{\phi_{\text{nl}}} = \frac{r}{\text{erf} \left( \frac{r}{2\ell} \right)}. \]  

(6)

In Fig. 1 we plot the local potential with its regularized, non-local counterpart, as well as the two corresponding radius variables. As becomes apparent, the modified radius variable \( \tilde{r} \) has a minimal
value proportional to the regulator scale $\ell$:

$$\tilde{r}(r \to 0) = \sqrt{\pi} \ell + \mathcal{O}(r^2).$$

At large distances, however, the two radial coordinates approach each other exponentially fast [27],

$$\tilde{r}(r \to \infty) = r + \frac{2\ell}{\sqrt{\pi}} e^{-r^2/(4\ell^2)}.$$  

(8)

Hence, taking this non-locally modified radius variable $\tilde{r}$ as the physical radius variable effectively cuts out the region $r \in [0, \sqrt{\pi} \ell]$ from the manifold, while rapidly approaching the standard radius definition for distances larger than $\ell$.

Figure 1: Newtonian and regularized potential (left), and corresponding radius functions (right).

3 Non-singular black hole model

Let us now explore the ramifications for a static and spherically symmetric black hole spacetime subjected to the formal substitution $r \mapsto \tilde{r}(r)$,

$$\text{d}s^2 = -f_{\text{nl}}(r) \text{d}t^2 + \frac{\text{d}r^2}{f_{\text{nl}}(r)} + \tilde{r}^2 \text{d}\Omega^2, \quad f_{\text{nl}}(r) = 1 - \frac{2M}{\tilde{r}(r)}, \quad \tilde{r} = \frac{r}{\text{erf}(\sqrt{\frac{r}{2\ell}})}.$$  

(9)

Due to its motivation via the non-local Gauss law (and the appearance of the error function $\text{erf}(x)$ as well as Gaussian factors $e^{-r^2/(4\ell^2)}$ in the radius and curvature) we shall refer to it as the “Gauss” model. Note that this is not a coordinate transformation since we explicitly keep $r$ as the coordinate radius variable. However, it is clear that circles of $r = \text{const}$ now have the proper circumference $2\pi \tilde{r}(r)$. Unlike usually assumed in non-singular black hole models, we here explicitly rescale the spherical geometry as well, which is a necessary step to render this black hole model
finite at \( r = 0 \); see, however, Simpson and Visser for a similar model \([7]\).

### 3.1 Horizons

Let us briefly compare the metric function \( f_{nl}(r) \) to that of General Relativity and Hayward, see Fig. 2. For generic values of \( \ell \) and \( M \), where we assume that \( M/\ell > 1 \), it is clear that the behavior at \( r = 0 \) is rather different. In the General Relativity case one has the standard spacelike singularity, whereas the Hayward model is de Sitter-like. At \( r = 0 \) the Gauss model behaves as

\[
f(r \ll \ell) \approx 1 - \frac{2M}{\sqrt{\pi \ell}} + \frac{Mr^2}{6\sqrt{\pi \ell^3}},
\]

which shows that for large masses \( 2M > \sqrt{\pi \ell} \) the geometry is indeed de Sitter-like at the origin.

![Figure 2: Metric functions for the Schwarzschild, Hayward, and Gauss black hole.](image)

The striking difference between the Gauss and Hayward model lies in the absence of an inner horizon for the latter. An apparent horizon is located wherever the following condition is satisfied:

\[
(\nabla r)^2 = g^{rr} = 0,
\]

such that the locations of apparent horizons correspond to the zeros of the metric function \( f(r) \), or, equivalently, wherever the vector field \( \partial^\mu r = \delta^\mu_r \) becomes null. While the outer horizons are roughly located around \( r \sim 2M \), modulo small corrections due to \( \ell \), there is an inner horizon for the Hayward model, but none for the Gauss and Schwarzschild black hole.
Since inner horizons make black holes susceptible to mass inflation \cite{28,29}, the generic absence of such a structure in this model is an interesting feature of the non-local regulator. While more work is needed to understand the precise origins, it is likely due to the fact that our model is intrinsically non-polynomial. In this way, the absence of the inner horizon would be directly inherited from the “ghost-free property” of non-local gravity which in turn heavily relies upon entire non-polynomial functions for the gravitational propagator \cite{20,30}, such as $e^{-\ell^2 \square}$ as employed in Eq. (4).

In fact, one may check that substituting the complicated function $\text{erf}(x)$ by a rational approximation $x^2/(1 + x^2)$ gives rise to an inner horizon. (The substitution $x/(1 + x)$ is not allowed since it induces a conical singularity around $r = 0$.) For this reason we believe that the absence of the inner horizon is indeed due to the non-rational form of our modification.

### 3.2 Mass gap

It is well known that in higher-derivative as well as non-local infinite-derivative theories of gravity there exists a mass gap for the dynamical formation of black holes via a spherically symmetric collapse of null dust \cite{31,32}, and this mass gap is proportional to the regularization scale. In other words, small black holes do not form unless their mass parameter exceeds a critical value.

In the present context, note that the modified radius variable $\tilde{r}$ is always larger than the minimal distance $\sqrt{\pi \ell}$. For this reason the apparent horizon condition (11) can only be satisfied if

$$M > M_0 = \frac{\sqrt{\pi \ell}}{2},$$

that is, the mass parameter exceeds a critical value. As expected, in the limiting case of $\ell \to 0$ this mass gap vanishes as one recovers the Schwarzschild case. While the considerations presented in this Essay are focused on the time-independent scenario, it is still interesting that they qualitatively reproduce the mass gap found in dynamical situations.

If the mass is less than the critical value, $m < m_0$, the resulting geometry is horizonless but regular at $r = 0$. Specifically, $r = 0$ then corresponds to a wormhole throat moving forward in time, just as in the Simpson–Visser case \cite{7}; for comments on possible analytic continuations see below.

### 3.3 Regularity, curvature invariants, and limiting curvature condition

To show the regularity of this metric one may calculate several scalar curvature invariants. We focus here on the Ricci scalar $R$, the square of the traceless Ricci tensor $S^2 = (S_{\mu\nu})^2$, as well as
the square of the Weyl tensor $C^2 = (C_{\mu\nu\rho\sigma})^2$. They take the following form at $r = 0$ [33]:

$$R = \frac{3\sqrt{\pi}M + (6 - 2\pi)\ell}{3\pi\ell^3} - \frac{7\sqrt{\pi}M + 2(10 + \pi)\ell}{60\pi\ell^5} r^2 + \mathcal{O}(r^4),$$  \hfill (13)

$$S^2 = \frac{9\pi M^2 + 4(3 - 2\pi)\sqrt{\pi}M\ell + 2(18 + \pi^2)\ell^2}{36\pi^2\ell^6} - \frac{63\pi M^2 + 2(51 - 13\pi)\sqrt{\pi}M\ell + 4[90 + (15 + \pi)\pi]\ell^2}{1080\pi^2\ell^8} r^2 + \mathcal{O}(r^4),$$  \hfill (14)

$$C^2 = \frac{[3\sqrt{\pi}M - (6 + \pi)\ell]^2}{27\pi^2\ell^6} - \frac{81\pi M^2 - 3\sqrt{\pi}(74 + 13\pi)M\ell + 4(5 + \pi)(6 + \pi)\ell^2}{270\pi^2\ell^8} r^2 + \mathcal{O}(r^4)$$  \hfill (15)

Somewhat cumbersome expressions aside, it is clear that the scalar curvature at $r = 0$ is positive for large masses $M > (2\pi - 6)\ell/(3\sqrt{\pi})$, consistent with our previous estimate $2M > \sqrt{\pi}\ell$. Moreover, the invariants are all manifestly finite as well as regular at $r = 0$, since no linear terms in $r$ appear.

However, the invariants’ behavior at $r = 0$ is not bounded by a universal constant. Demanding that the curvature scales at most Planckian for typical astrophysical black holes,

$$R \sim \frac{M_\odot}{\ell^3} \lesssim \frac{1}{\ell_p^2},$$  \hfill (16)

gives the constraint that $\ell \gtrsim 10^{-22}$ m, which is thirteen orders of magnitude larger than the Planck scale. Using this as a reference value, we can now estimate the order of magnitude of deviations from the Schwarzschild black hole outside the horizon of an astrophysical black hole, given by

$$e^{-M_\odot^2/\ell^2} \approx e^{-10^{50}} \approx 0.$$  \hfill (17)

This is to be compared to the case of polynomial non-singular black holes, where deviations are equal to simple powers of $(\ell/M_\odot) \sim 10^{-25}$.

### 3.4 Properties of $r = 0$ and geodesic (in)completeness

The location $r = 0$ corresponds to $\tilde{r} = \sqrt{\pi}\ell$ and hence the metric is

$$ds^2|_{r=0} = \left(\frac{2M}{\sqrt{\pi}\ell} - 1\right) dt^2 + \pi\ell^2 d\Omega^2,$$  \hfill (18)

which is nothing but a sphere of surface area $4\pi^2\ell^2$ factored with another spatial direction $t$, provided the mass parameter $M$ is large enough. It would be interesting to study the response of this “sphere” to infalling matter.

It is easy to show that a radial geodesic in a static metric with $-g_{tt} = g^{rr} = f(r)$ has a conserved quantity $E = f(r)\dot{t}$, where the dot denotes differentiation with respect to the affine parameter $\lambda$.
Then, the equation of motion simply becomes

\[ r^2 = E^2, \]  

implying that any radial geodesic can reach the surface of that sphere \((r = 0)\) at finite affine parameter, which in turn implies geodesic incompleteness [34]; for an application to non-singular black holes see Ref. [35]. However, this might not be a serious drawback since many regular black hole models are geodesically incomplete. For a recent review on this topic we refer to [36]. In this particular case it seems that continuing the variable \(r\) to the entire range of \(\mathbb{R}\) would solve that issue and potentially give rise to a wormhole-type geometry; see also Simpson and Visser [7].

4 Conclusions and outlook

In this essay we have constructed a non-singular “Gauss” black hole from the principle of a non-local regulator that “cuts out” a piece of spacetime with radii less than the non-local regularization scale. The presented geometry has several interesting features: first and foremost it has no inner horizon despite having a de Sitter core. In that sense it is a concrete realization of a loop hole in Frolov’s theorem that non-singular rational de Sitter black holes must have an inner horizon [4], since the Gauss black hole does not have a rational \(r\)-dependence. Second, its deviation from the Einstein vacuum decreases exponentially fast. And third, perhaps most interestingly, it seemingly relates the absence of an inner horizon to the non-polynomial form of the graviton propagator.

However, this model does not satisfy the limiting curvature condition, thereby placing a constraint of \(\ell \gtrsim 10^{-22} \text{ m}\) on the scale of non-locality, when applied to astrophysical black holes. It remains to be seen if and how this condition can be studied further.

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