Structural modification based analysis and design of tuned mass dampers

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Abstract. This article discusses the design of tuned vibration absorber for structures. A central notion in the identification of the parameters of a tuned mass damper (TMD) absorber for MDOF systems is that of modal mass of the original structure. The modal mass, as it is well known, is associated with the mode of vibration of the original structure whose frequency is required to be absorbed. The estimation of the modal mass can be achieved by either using a mass, stiffness, and damping model of the structure or, for an existing in service structure, from experimental modal analysis. In both of the above cases inaccuracies are introduced due to modeling errors and experimental noise. This article sets the theory of TMD in the context of structural modification. This enables the design of TMD by using only the frequency response function of the primary system without taking into consideration the original mass or modal mass. The theory developed herein is for one degree of freedom undamped primary vibrating system. The proposed structural modification based methodology of designing TMD is demonstrated on a specific single degree of freedom vibrating system.

1. Introduction
Vibration absorbers are one degree of freedom mass, spring, damper (MSD) systems that are attached on structures that vibrate excessively when subjected to external dynamic loads. The effective mitigation of excessive vibrations is normally achieved by tuning the vibration absorber to the frequency of these vibrations. The basic theory that underlies the vibration absorption of undamped one degree of freedom systems using MSD absorbers is well developed and reported in many textbooks of vibration theory [1].

The MSD absorber theory was subsequently extended to take into consideration (a) the effect of damping in the primary structure on the design of MSD absorbers [2, 3] and (b) the difficulties of using them to more complicated structures such as beams [4], plates, shells [5] and cylinders [6]. An outcome of the theory extension to more complicated structures is that the modal (and effective) mass are essential in designing vibration absorbers [7, 8]. However, estimation of the modal and effective mass for structures in service requires experimental modal analysis [9] which, especially for large and complicated structures, could be expensive and cumbersome [10].

This paper, through the theory of structural modification, extends the original theory of designing MSD absorber systems in order to overcome the difficulties involved when they are used on more complicated structures. The theory herein is developed for the undamped single degree of freedom system. The only requirement in designing the optimal MSD absorber is the frequency response function of the primary system. The actual values of the primary mass and
spring constant are not required. It is expected that the extension of this theory to multi-degree of freedom systems will require only the measured frequency response function at the point of the attachment of the MSD absorber. Section 2 outlines the relevant structure modification theory whereas in section 3 this theory is exploited to find the optimal MSD. Finally section 4 summarises the conclusions of the work presented in this paper.

2. Structural modification theory in the context of spring mass damper absorber
This section develops a methodology of designing MSD absorbers that is based on the theory of structural modification. The advantage of the proposed methodology is that it requires only measured FRFs on the primary structure. The objective in mind is to design a MSD absorber to eliminate unwanted vibrations of a lightly damped structure excited at a single mode and, possibly, the frequency of excitation varies within a range around the resonance corresponding to the mode. Hence the primary structure can be thought as an undamped single degree of freedom system as shown in figure 1 of mass \( m \) and stiffness \( k \). The MSD absorber of mass, \( m_a \), spring, \( k_a \) and damper \( c_a \) is attached as shown in figure 2. In subsection 2.1 the relevant theory of structural modification in the context of MSD is presented and in subsection 2.2 the theory needed to develop the methodology is discussed.

![Figure 1. The single degree of freedom primary system consisting of spring of stiffness \( k \) and mass \( m \).](image1)

![Figure 2. Mass, \( m_a \), spring \( k_a \), damper, \( c \), absorber attached to the primary system of figure 1.](image2)

2.1. Theory of structural modification
The dynamics of a structure change due to modifications of its mass, stiffness and damping properties. The theory of structural modification deals with all aspects involved when such changes occur. In general, structural modification is separated to direct and inverse. In the direct modification, the structural modifications are known and the aim is to find the changes in the dynamic behaviour whereas in the inverse modification where the aim is to design the appropriate passive mass, damping, and stiffness modifications which bring about the desired dynamic behavior [11, 12, 13].

The vibration absorption problem can be viewed as a structural modification of the primary structure, shown in figure 1, by the the MSD absorber of figure 2. This modification involves the addition of a mass, \( m_a \), at a new degree of freedom, and a spring of stiffness \( k_a \) and damper of damping coefficient \( c \) connecting the mass at the new degree of freedom with the original structure. The theory developed in [11, 13], by introducing the effect of damping \( c \), can be readily extended to obtain the modified point frequency response function, \( h^{\text{mod}}(\omega) \), at \( m \) from the original point FRF, \( h^{\text{or}}(\omega) \), as follows

\[
h^{\text{mod}}(\omega) = \frac{h^{\text{or}}(\omega) \left(-\omega^2 m_a + k_a + j\omega c\right)}{-\omega^2 m_a + k_a + j\omega c - (\omega^2 m_a k_a + j\omega^3 m_a c) h^{\text{or}}(\omega)}
\] (1)
where \( m_a, k_a \) and \( c \) are the mass, spring constant and the damping of the MSD absorber, \( \omega \) the forcing frequency and \( j \) the imaginary unit. This equation demonstrates how the original frequency response function is affected by modifying the original structure by a mass, spring and damper absorber and the subsequent development of a design methodology for finding the optimal vibration absorber is totally relied on it.

2.2. Vibration absorber theory based on structural modification

The vibration absorber theory states that the optimal \( m_a \) and \( k_a \) parameters of an undamped absorber, to be attached on a structure excited at a frequency \( \omega \) close to one of its natural frequencies, are those that render the natural frequency of the vibration absorber \( \omega_{na} = \sqrt{\frac{k_a}{m_a}} \) equal to the forcing frequency. This can be also seen by first setting the damping, \( c \), in equation 1, to 0 and then by rendering the numerator zero. In practice when the mass (or the effective mass) of the original structure is known the mass of the absorber \( m_a \) is chosen to be a fraction of the original (effective) mass and then the stiffness \( k_a \) is selected to tune the absorber’s natural frequency to the desired frequency.

In order to facilitate the development the mass and spring constant of the primary system, figure 1, are taken to be \( m = 30 \) Kg and \( k = 300 \) kN/m and hence its natural frequency is \( 100 \) rad/s. The magnitude of the actual frequency response function of this system is shown in figure 3. According to the classical theory of vibration absorbers the natural frequency of the undamped vibration absorber \( \omega_{na} \) should be \( 100 \) rad/s. Choosing the absorber mass \( m_a \) to be 10 Kg the spring constant of the absorber becomes 100 kN/m. Setting \( c = 0 \) in equation 1 the modified FRF is computed and its magnitude is plotted in figure 4.

The two new resonance frequencies observed in figure 4 can be obtained from the original FRF \( h_{or} (\omega) \) by setting the damping and the denominator of equation 1 to zero. By doing so, it is found that the new resonance frequencies should satisfy the following equalities,

\[
\| h_{or} \| (\omega) = \pm \frac{\omega^2 m + k}{\omega^2 m k}
\]

where \( \| \cdot \| \) denotes the absolute if the argument is real or the magnitude if the argument is complex. A graphical solution is given in figure 3 in where the dashed and the dash-dotted curves correspond to the positive and the negative function respectively given in equation 2. The newly introduced natural frequencies are the abscissas of the intersections of these two curves with the magnitude, \( \| h_{or} \| (\omega) \), of the original frequency response function. It can be seen from figure 3 that these values are \( \Omega_1 = 75.21 \) rad/s and \( \Omega_2 = 135.46 \) rad/s.

Damping is introduced in the absorber in order to reduce the amplitude vibrations that could occur when the frequency band of the excitation contains frequencies that are close to the newly introduced resonances. Figure 4 depicts the effect of various values of damping, namely \( c = 250, 500 \) and \( 750 \) \( \text{Nm/s} \) on the magnitude of the modified frequency response function. It can be concluded that, even though damping reduces the resonance, high vibration amplitudes that extend over a frequency band might occur. As a consequence an optimum value of damping should be sought. In the classical literature this is done by observing that all curves of the amplitudes of the modified frequency response functions, figure 4, intersect at two specific frequency values, say \( \omega_k \) and \( \omega_l \). Hence the FRF magnitudes at \( \omega_k \) and \( \omega_l \) are independent of the damping.

The values of these frequencies can be obtained from the original frequency response function \( h_{or} (\omega) \) utilizing the structural modification theory. Since the FRF values at \( \omega_k \) and \( \omega_l \) are independent of the damping \( c \) the magnitudes of the modified frequency response functions for \( c = 0 \) given by,
The optimal MSD absorber parameters are obtained by first finding the absorber mass, \( m_a \), and stiffness, \( k_a \) that render the FRF magnitudes at \( \omega_k \) and \( \omega_l \) equal. Then the optimal damping is obtained by ensuring that the equal FRF magnitudes at \( \omega_l \) and \( \omega_k \) are also the maximum values over the excitation frequency range.

### 3.1. Optimal absorber mass and stiffness

It is assumed that absorber mass \( m_a \) is kept at 10Kg and thereby the absorber spring constant \( k_a \) that would render the FRF magnitudes at \( \omega_k \) and \( \omega_l \) equal must be found. This can be achieved by setting \( h^\text{mod}_{\omega_k=0} (\omega_1) = h^\text{mod}_{\omega_k=0} (\omega_2) \) which is equivalent to \( h^\text{mod}_{\omega_k=0} (\omega_1) = -h^\text{mod}_{\omega_k=0} (\omega_2) \).

Setting \( 
\begin{align*}
  a &= h^\text{or} (\omega_k) - h^\text{or} (\omega_l) + \left( \omega_k^2 - \omega_l^2 \right) m_a h^\text{or} (\omega_k) h^\text{or} (\omega_l)
\end{align*}
\end{align*}


Substituting the values $\omega_k = 78.85 \text{rad/s}$, $\omega_l = 117.37 \text{rad/s}$ and $m_a = 10 \text{ Kg}$ in 6, 7 and 8 and solving the quadratic polynomial the new absorber value is computed to be $k_a = 3600/7 \text{ N/m}$. The magnitude of the modified frequency response functions of the absorber with the optimal mass, $m_a$ and spring constant $k_a$ values for various damping values, namely, $c = 250$, 500 and 750 $\text{Nm/s}$ are shown in figure 7.

Since the optimal absorber mass and spring constant changed the values of the resonance frequencies, $\omega_1$ and $\omega_2$ and of the frequencies, $\omega_k$ and $\omega_l$ at which the FRF magnitude is independent of the damping changed. Their values can be computed from the original FRF using the techniques described in section 2.2. The values obtained by doing this are $\omega_1 = 55.42$, $\omega_2 = 108.32$, $\omega_k = 57.24$ and $\omega_l = 96.95 \text{rad/s}$. These values are also seen in figure 7.

### 3.2. Optimal absorber damping

The optimal damping value, as it is stated at the beginning of section 2.2 is the value that maximizes the magnitude of the modified FRF, $h_{\text{mod}}(\omega)$ given by equation 1 at the frequencies $\omega_k$ and $\omega_l$. The magnitude of $\|h_{\text{mod}}\| (\omega)$ is given by,

$$
\|h_{\text{mod}}\| (\omega) = \sqrt{N_R^2 + N_I^2 / D_R^2 + D_I^2} \tag{9}
$$

where $N_R$, $D_R$, $N_I$ and $D_I$ are the real and imaginary parts of the numerator and denominator of $h_{\text{mod}}(\omega)$, equation 1, and are given by,

$$
N_R (\omega) = (-\omega^2 m_a + k_a) h^{or} (\omega) \tag{10}
$$

$$
N_I (\omega) = \omega c h^{or} (\omega) \tag{11}
$$

$$
D_R (\omega) = -\omega^2 m_a + k_a - \omega^2 m_a k_a h^{or} (\omega) \tag{12}
$$
\[ D_I(\omega) = c \left( \omega^3 m_a h^{\text{or}}(\omega) + \omega \right) \] (13)

Therefore the optimal damping renders the derivative \( \frac{d}{d\omega} \| h^{\text{mod}}(\omega) \|^2 \) zero at \( \omega_k \) and \( \omega_l \).

The derivative \( \frac{d}{d\omega} \| h^{\text{mod}}(\omega) \|^2 \) is zero when the following quantity becomes zero,

\[
2 \left( D_R^2(\omega) + D_I^2(\omega) \right) \left( N_R(\omega) \frac{dN_R(\omega)}{d\omega} + N_I(\omega) \frac{dN_I(\omega)}{d\omega} \right) \\
-2 \left( N_R^2(\omega) + N_I^2(\omega) \right) \left( D_R(\omega) \frac{dD_R(\omega)}{d\omega} + D_I(\omega) \frac{dD_I(\omega)}{d\omega} \right)
\] (14)

where

\[
\frac{dN_R(\omega)}{d\omega} = -2\omega m_a h^{\text{or}}(\omega) + \left(-\omega^2 m_a + k_a\right) \frac{dh^{\text{or}}(\omega)}{d\omega} \] (15)

\[
\frac{dN_I(\omega)}{d\omega} = c \left( h^{\text{or}}(\omega) + \omega \frac{dh^{\text{or}}(\omega)}{d\omega} \right) \] (16)

\[
\frac{dD_R(\omega)}{d\omega} = -2\omega m_a - 2\omega m_a k_a h^{\text{or}}(\omega) - \omega^2 m_a k_a \frac{dh^{\text{or}}(\omega)}{d\omega} \] (17)

\[
\frac{dD_I(\omega)}{d\omega} = c \left( 3\omega^2 m_a h^{\text{or}}(\omega) + \omega^3 m_a \frac{dh^{\text{or}}(\omega)}{d\omega} + 1 \right) \] (18)

Evaluating expression 14 at \( \omega_l \) and \( \omega_k \) two quartic polynomials in \( c \) are obtained whose one of their factors yields the optimal damping value. The value obtained for the parameters of this example is \( 588.281 \text{ Ns/m} \) and occurs for the \( \omega_l = 96.95 \text{ rad/s} \) value. This value along with \( m_a = 20 \text{ Kg} \) and \( k_a = 36007 \text{ N/m} \) constitute the optimal values of the MSD absorber. The curve of the modified frequency response function corresponding to the optimum damping value is shown in figure 8.
4. Conclusions
In this article the structural modification theory was employed to extend the theory of TMDs. It was shown that using this theory the design of damped dynamic absorbers can be achieved by using only the measured frequency response function on the original structure. Structural modification provided the means (a) to compute the newly created resonance frequencies and the invariant frequencies at once, (b) to retune the MSD absorber when damping in the absorber is introduced and (c) compute the optimal damping of the retuned MSD. The only information required to achieve this was the frequency response function of the primary structure. This work will be extended to lightly damped primary structures, multi degree of freedom systems and continuum systems.

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