Localization in Coupled Finite Vibro-Impact Chains: Discrete Breathers and Multibreathers

I. Grinberg\textsuperscript{a,*}, O.V. Gendelman\textsuperscript{a}

\textsuperscript{a}Faculty of Mechanical Engineering, Technion – Israel Institute of Technology, Haifa, Israel

Abstract

We examine the dynamics of strongly localized periodic solutions (discrete breathers) in two-dimensional array of coupled finite one-dimensional chains of oscillators. Localization patterns with both single and multiple localization sites (multibreathers) are considered. The model is scalar, i.e. each particle can move only parallel to the axis of the chain it belongs to. The model involves symmetric parabolic on-site potential with rigid constraints (the displacement domain of each particle is finite) and a linear nearest-neighbor coupling in the chain, and also between the neighbors in adjacent chains. When the particle approaches the constraint, it undergoes an elastic Newtonian impact. The rigid impact constraints are the only source of nonlinearity in the system. The model allows easy computation of highly accurate approximate solutions for the breathers and multibreathers with an arbitrary set of localization sites in conservative setting. The vibro-impact nonlinearity permits explicit derivation of a monodromy matrix for the breather and multibreather solutions. Consequently, the stability of the derived breather and multibreather solutions can be studied in the framework of simple methods of linear algebra, without additional approximations. It is shown that due to the coupling of the chains, the breather solutions can undergo the symmetry breaking.

Keywords: discrete breather, localization, vibroimpact

PACS: 05.45.Yv, 63.20.Pw, 63.20.Ry

Localization in discrete dynamical systems with nonlinearity has been of growing interest in recent years\cite{8, 9, 30, 5, 29, 22, 2, 22, 3, 9}. Nonlinearity enables localization even in a purely homogeneous system, without the need for a disorder as is the case of ordinary linear systems. The primary phenomenon discussed in this work is the Discrete Breather (DB), often referred to as intrinsic localized mode (ILM) or discrete soliton. The DB is a strongly localized periodic regime of a lattice-based system. The strong localization of the DB is often exponential, however, in nonlinear systems it may even be hyper-exponential\cite{2}. Simply put, the DB can be described as an oscillating envelope localized around a single lattice site, or several sites in the case of a Multibreather (MB). The DBs have been theoretically studied, and even experimentally observed, in many fields of physics. Among the physical systems where DBs have been encountered, are superconducting Josephson junctions\cite{27}, nonlinear magnetic metamaterials\cite{19}, electrical lattices\cite{7}, micro-mechanical cantilever arrays\cite{16, 18, 17, 25, 24}, Bose-Einstein condensates\cite{28}, graphene sheets\cite{10}, and chains of mechanical oscillators\cite{13, 6, 12, 14, 21}.

Either nonlinearity or disorder are essential features in order to encounter the DBs, hence, exact analytic solutions are scarce. Theoretical work on this phenomenon is primarily restricted to approximate techniques and numerical study\cite{8, 9, 4, 23}. Some known exceptions are the completely integrable Ablowitz-Ladik model\cite{1}, chains with homogeneous interaction\cite{20} and vibro-impact chains\cite{13, 21, 12, 14}. The latter has recently been expanded to asymmetric DBs in symmetric and asymmetric settings\cite{15}. Two former types of models – Ablowitz-Ladik and homogeneous interactions – do not allow extension to higher dimensions. This Letter shows that the vibro-impact models can efficiently describe the DBs in a two-dimensional (2D) lattice.

We consider the DBs in the 2D lattice that comprises several identical finite vibro-impact chains. Each chain’s masses are linearly coupled to their counterparts in the adjacent chains. The particles are allowed to move only in the directions of the chains they belong to, so this system can be classified as the 2D scalar model. Due to the complexity of the system, the treated model is restricted to a conservative setup. The approach is based on the refs. \cite{13, 21, 12, 14, 15}, however the chain coupling does not allow exact analytical solution in this case. The approach is based on generalized Fourier series representation of the solution. An approximate solution is obtained by truncation of the Fourier series and then solving the resulting set of linear algebraic equations. Numerical study shows a very rapid convergence of the series and therefore a very small error. Furthermore, the nature of the model allows...
writing the monodromy matrix explicitly, and therefore the linear stability of the DB and MB solutions can be investigated without further approximation\cite{28}. Stability analysis of the solutions reveals that pitchfork bifurcation is possible despite the conservative symmetric model due to the coupling of the chains.

As mentioned above, each particle is coupled by linear springs to its neighbors in the chain, as well as to its counterparts in the adjacent chains. Besides, each mass is subject to an identical on-site coupling – a linear spring with a symmetric pair of impact barriers located at distances $u_{n,m} = ± 1$ from the trivial equilibrium position. This unit scaling does not restrict the generality. The Hamiltonian of the systems that includes $(M + 1)$ chains of $(N + 1)$ masses is written as follows:

$$H = \sum_{m=0}^{M} \sum_{n=0}^{N} \left( \frac{1}{2} p_{n,m}^2 + V(u_{n,m}) \right) + \sum_{m=0}^{M-1} \sum_{n=0}^{N} W_1(u_{n,m} - u_{n+1,m}) + \sum_{m=0}^{M-1} \sum_{n=0}^{N} W_2(u_{n,m} - u_{n,m+1}) + \sum_{m=0}^{M} W_1(u_{N,n,m} - u_{0,m}) + \sum_{n=0}^{N} W_2(u_{n,M} - u_{n,0}),$$

(1)

$$V(x) = \begin{cases} \frac{\gamma_1}{2} x^2, & |x| < 1, \\ \text{infinity}, & |x| = 1 \end{cases},$$

(2)

$$W_1(x) = \frac{\gamma_2}{2} x^2,$$

(3)

$$W_2(x) = \frac{\gamma_3}{2} x^2,$$

(4)

where $p_{n,m}$ is the momentum of each particle, $\gamma_1$, $\gamma_2$ and $\gamma_3$ are the on-site coupling and chain coupling stiffness coefficients respectively and $V(x)$, $W_1(x)$ and $W_2(x)$ are the on-site, coupling and chain coupling potentials, respectively. In principle, the rigid on-site barriers without the linear anchoring spring are sufficient to provide the existence of the DB solutions\cite{12}; in this work we consider more general case of $\gamma_1 \geq 0$.

We adopt here the traditional Newtonian model of the elastic impacts. Namely, when at a certain time instance $t = t_b$ some particle achieves the impact barrier ($u_n(t_b) = ± 1$), its velocity is instantaneously modified according to the following law:

$$u_n(t_b^+) = -u_n(t_b^-).$$

(5)

We examine the conservative model where there is no external force applied to the masses and no energy loss in the process, i.e. all impacts are elastic. The periodicity of the MB allows us to predict the time instances of the collisions, and to express them as periodic external excitation in the following manner:

$$\ddot{u}_{n,m} + \gamma_1 u_{n,m} + \gamma_2 (2u_{n,m} - u_{n+1,m} - u_{n-1,m}) + \gamma_3 (2u_{n,m} - u_{n,m+1} - u_{n,m}) = 2p_{n,m} \delta_{nk} \delta_{mr} \alpha(t),$$

(6)

where $\alpha(t) = \sum_{j=-\infty}^{\infty} \left( \delta \left( t - \frac{\pi (2j+1)}{\omega} \right) - \delta \left( t - \frac{2\pi j}{\omega} \right) \right)$ describes the periodic impacts, $\delta(t)$ is the dirak delta function, $\delta_{nm}$ is the Kronecker delta and, along with $k$ and $r$, determines the localization sites, and $2p_{n,m}$ is the magnitude of the change of momentum during the impact. Since the impacts are elastic $p_{n,m}$ determines the magnitude of the velocity of the colliding mass just before, and right after the collision. Additionally, finite chains with periodic boundary conditions are considered, i.e. $u_{N+1,n}(t) = u_{0,n}(t)$ and $u_{n,M+1}(t) = u_{n,0}(t)$ where $N + 1$ is the number masses in each chain and $M + 1$ is the number of chains. Without loss of generality, we let the mass denoted by $m = 0$ and $n = 0$ to be a localization site. Note, however, that at least one site cannot be a localization site in order to excludes a trivial case, when all sites in the lattice are excited.

The terms for the periodic impacts can also be written in the form of generalized Fourier series:

$$\alpha(t) = -\frac{2\omega}{\pi} \sum_{j=0}^{\infty} \cos ((2j + 1) \omega t).$$

(7)

The solution can now also be written as a generalized Fourier series in a similar form:

$$u_{n,m} = \sum_{j=0}^{\infty} u_{n,m,j} \cos ((2j + 1) \omega t).$$

(8)

Plugging into the equations of motion and separating to different harmonics yields:

$$- (2j + 1)^2 \omega^2 u_{n,m,j} + \gamma_1 u_{n,m,j} + \gamma_2 (2u_{n,m,j} - u_{n+1,m,j} - u_{n-1,m,j}) + \gamma_3 (2u_{n,m,j} - u_{n,m+1,j} - u_{n,m,j}) = -\frac{4\omega^2 p_{n,m}}{\pi} \delta_{nk} \delta_{mr}.$$

(9)

Additionally to this infinite set of equation, there is also the impact condition for each impacting mass:

$$u_{k,r}(0) = \sum_{j=0}^{\infty} (-1)^{k+r} u_{k,r,j} = ± 1,$$

(10)

where the ± sign determines whether the oscillations of the specific site is in or out-of phase with respect to the other localization sites.

In order to obtain an approximate solution, we limit the number of harmonics considered to the first $L + 1$. Thus, we obtain a set of $(L + 1) (N + 1) (M + 1)$ linear equations which can be solved in order to find an approximate solution. It is important to note that if a solutions exists, convergence is certain and $L$ determines the accuracy of the obtained approximation.

We investigate the stability of the MB solutions with the help of Floquet theory\cite{29}. Since the explored model allows explicit construction of the monodromy matrix, it is easy to find its eigenvalues for every set of parameters.
Thus, broad regions of the parameter space can be explored for various structures of the breathers with minimal numerical efforts. Moreover, the eigenvectors corresponding to the unstable Floquet multipliers can be easily computed and examined to give a qualitative insight into physical mechanisms of the loss of stability.

The governing equations of motion for can be re-written in the following equivalent form:

$$
\ddot{u} = A\ddot{u},
$$

where,

$$
\ddot{u} = \begin{bmatrix} \ddot{x} \\ \ddot{\bar{x}} \end{bmatrix},
$$

$$
\ddot{x} = \begin{bmatrix} u_{00} & \cdots & u_{(N-1)M} & u_{NM} \end{bmatrix}^T
$$

and:

$$
A = \begin{bmatrix}
0 & \bar{A} & 0 \\
A_2 & A_1 & A_2 \\
0 & A_2 & 0 \\
A_2 & 0 & \cdots & 0 & A_2 \\
0 & A_2 & \cdots & A_2 & 0 \\
& & & & \\
\end{bmatrix} 2(N+1)(M+1) \times 2(N+1)(M+1),
$$

$$
\tilde{A} = \begin{bmatrix} A_1 & A_2 & \cdots & 0 & 0 & A_2 \\
A_2 & A_1 & A_2 & 0 & \cdots & 0 \\
\vvdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & A_2 & A_1 \\
A_2 & 0 & \cdots & 0 & A_2 & A_1 \\
\end{bmatrix} (N+1)(M+1) \times N(M+1),
$$

$$
A_1 = \begin{bmatrix} \sigma & -\gamma_2 & 0 & \cdots & 0 & -\gamma_2 \\
-\gamma_2 & \sigma & -\gamma_2 & 0 & \cdots & 0 \\
\vvdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & \sigma & -\gamma_2 & 0 \\
-\gamma_2 & 0 & \cdots & 0 & -\gamma_2 & \sigma \\
-\gamma_2 & 0 & \cdots & 0 & -\gamma_2 & \sigma \\
\end{bmatrix} (N+1)(N+1),
$$

$$
A_2 = -\gamma_3 I_{(N+1)(N+1)},
$$

where $\sigma = \gamma_1 + 2\gamma_2 + 2\gamma_3$.

Here $\tilde{A}$ is the Laplace adjacency matrix of the system. For the forced-damped model, minor modification is required:

$$
\tilde{\ddot{u}} = \tilde{A} \tilde{\ddot{u}} + \tilde{F},
$$

where $\tilde{F} = F(t) \begin{bmatrix} 0 & \cdots & 0 & 1 & \cdots & 1 \end{bmatrix}^T$.

All considered solutions are symmetric in a sense that the successive impacts for each particle are divided by half-period intervals, and absolute amounts of momentum transferred to given particle in the course of given impact is the same. From the above equation, it is easy to derive the matrix, that describes the evolution of the perturbed phase trajectory between two successive impacts:

$$
L = \exp \left( \frac{\pi}{\omega} \tilde{A} \right).
$$

To describe the evolution of the perturbed phase trajectory in the course of impacts, we apply a formalism of the salutation matrix. Since the impacts are instantaneous independent events, they can be treated separately and then combined to result in the following salutation matrix:

$$
S = \begin{bmatrix} \hat{S} & 0 \\
\hat{S} & \hat{S} \end{bmatrix},
$$

where,

$$
\hat{S} = \begin{bmatrix}
\epsilon_{00} & 0 & \cdots & 0 \\
0 & \epsilon_{10} & 0 & \cdots \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \epsilon_{(N-1)M} \\
0 & 0 & \cdots & 0 & 1 \\
\end{bmatrix} (N+1)(M+1) \times (N+1)(M+1),
$$

$$
s_{ij} = 1 - (1 + \epsilon) \delta_{ik} \delta_{jr},
$$

$$
\hat{s}_{ij} = \frac{(1 + \epsilon) \delta_{ik} \delta_{jr} \tilde{u}_{ij}(\phi-)}{\tilde{p}_{ij}},
$$

Due to the symmetry of the even functions composing the Fourier series, the monodromy matrix can be written compactly as follows:

$$
M = (LS)^2.
$$

As mentioned above, the eigenvalues of this monodromy matrix are computed numerically for the given parameter values. The resulting stability pattern in the space of parameters are exemplified in the next section.

An example of the DB shape i.e. the displacements of the masses at the time instance of the impact, is presented in fig. 1, where one can appreciate already the strong localization of the solution. For qualitative understanding of the properties of the analytically obtained DB solution, and in order to verify the accuracy of the numerical algorithms, we compare the results of the analysis to numerical simulations. The simulations were performed in MatLab; the vibro-impact was modeled according to the impact law using built-in event-driven algorithms with Runge-Kutta (RK) solver. The results of the numerical simulations were in good agreement with the analytic solution despite the approximation as illustrated in fig. 2. It is important to note that the number of masses in these simulations is relatively large; hence, the number of leading harmonics chosen here is not very large to maintain a reasonable compu-
Numerical examination shows great improvement in accuracy when further increasing the number of harmonics at the cost of a much longer simulation.

Unless stated otherwise, the parameters chosen for all simulations are:

\[ N = 10, \quad M = 4, \quad \omega = 1.2, \quad \gamma_1 = 0.1, \quad \gamma_2 = 0.01, \quad \gamma_3 = 0.02. \] (26)

Figure 1: The frame of a DB for \( \gamma_2 = 0.2 \), i.e. the displacements of the masses at the instance of the impact.

Figure 2: Displacement of one of the impacting masses for a multibreather with two localization sites at (0,0) and (3,2), i.e. \( n = 3 \) and \( m = 2 \). Dashed gray line corresponds to the analytical result and solid black line corresponds to numerical result.

Figure 3 shows a typical existence-stability map for the MB. Similarly to previous work, the mechanism for loss of stability discovered in this example of a conservative MB is the Neimark Sacker Bifurcation. In fig. 4 we demonstrate weak correlation between the stability threshold and the number of chains except for a very small number of chains. This weak correlation is explained by the strong localization of the solution. Note that this may also be interpreted as the correlation between the stability threshold and the number of particles in the chain as the equations of motion shows these are interchangeable. Another interesting result is in the case of \( M = 1 \), i.e. two coupled chains, where we can see the threshold differs in the lower frequencies. At lower frequencies the dashed marking in fig. 4 corresponds to pitchfork bifurcation, that is previously not encountered for a conservative DB. For clarification, the difference for \( M = 1 \) is seen in fig. 5.

Figure 6 shows the eigenvectors corresponding to the pitchfork bifurcation. While it is difficult to show numerically, we can learn from the strongly localized form of the eigenvectors that it is likely to lead to breaking of symmetry as often associated with the pitchfork bifurcation. This is an interesting and somewhat surprising finding as the calculations in ref. [15] clearly shows that an asymmetric DB is not possible for the single conservative chain.

To conclude, in this work, we derive the approximate DB and MB solutions for coupled vibro-impact chains.
algebraic equations that has to be solved and may cost solution, when external forcing and damping are included. The second challenge is a derivation of the approximate solution, when external forcing and damping are included. The difficulty lies with the addition of nonlinearity to the algebraic equations that has to be solved and may cost in a less accurate approximation and larger computational effort.

The authors are grateful to Israel Science Foundation (grant 838/13) for financial support.

References

[1] M. J. Ablowitz and J. F. Ladik. Nonlinear differential-difference equations and fourier analysis. Journal of Mathematical Physics, 17(6):1011–1018, 1976.

[2] P. W. Anderson. Absence of diffusion in certain random lattices. Phys. Rev., 109:1492–1505, Mar 1958.

[3] O.O. Bendiksen. Localization phenomena in structural dynamics. Chaos, Solitons & Fractals, 11(10):1621 – 1660, 2000.

[4] David Cai, A.R. Bishop, Niels Grunbech-Jensen, and Boris A. Malomed. Moving solitons in the damped ablowitz-ladik model driven by a standing wave. Phys. Rev. E, 50:R694–R697, Aug 1994.

[5] David K Campbell. Nonlinear physics: Fresh breather. Nature, 432(7016):455–456, 2004.

[6] J. Cuevas, L. Q. English, P. G. Kevrekidis, and M. Anderson. Discrete breathers in a forced-damped array of coupled pendula: Modeling, computation, and experiment. Phys. Rev. Lett., 102:224101, Jun 2009.

[7] L. Q. English, F. Palmero, P. Candiani, J. Cuevas, R. Carretero-González, P. G. Kevrekidis, and A. J. Sievers. Generation of localized modes in an electrical lattice using subharmonic driving. Phys. Rev. Lett., 108:084101, Feb 2012.

[8] S. Flach and C.R. Willis. Discrete breathers. Physics Reports, 295(5):181 – 264, 1998.

[9] Sergej Flach and Andrey V. Gorbach. Discrete breathers – advances in theory and applications. Physics Reports, 467(1-3):116, 2008.

[10] Alberto Fraile, Emmanuel N. Koukaras, Konstantinos Papageorgiou, Nikos Lazarides, and G.P. Tsironis. Long-lived discrete breathers in free-standing graphene. Chaos, Solitons & Fractals, 87:262 – 267, 2016.

[11] Mats H. Fredriksson and Arne B. Nordmark. On normal form calculations in impact oscillators. Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences, 456(1994):315–329, 2000.

[12] O. V. Gendelman. Exact solutions for discrete breathers in a forced-damped chain. Phys. Rev. E, 87:062911, Jun 2013.

[13] O. V. Gendelman and L. I. Manevitch. Discrete breathers in vibroimpact chains: Analytic solutions. Phys. Rev. E, 78:026609, Aug 2008.

[14] Itay Grinberg and Oleg V. Gendelman. Localization in finite vibroimpact chains: Discrete breathers and multibreathers. Phys. Rev. E, 94:032204, Sep 2016.

[15] Itay Grinberg and Oleg V. Gendelman. Localization in infinite vibroimpact chains: Discrete breathers and multibreathers. Phys. Rev. E, 102:224101, Jun 2019.

[16] G. Gottlieb. Bifurcations and loss of orbital stability in nonlinear viscoelastic beam arrays subject to parametric actuation. Journal of Sound and Vibration, 329(18):3835 – 3855, 2010.

[17] Eyal Krenig, Boris A. Malomed, M. C. Cross, and Ron Lifshitz. Intrinsic localized modes in parametrically driven arrays of nonlinear resonators. Phys. Rev. E, 80:046202, Oct 2009.

[18] Masayuki Kimura and Takashi Hikihara. Coupled cantilever array with tunable on-site nonlinearity and observation of localized oscillations. Physics Letters A, 373(14):1257 – 1260, 2009.

[19] N. Lazarides, M. Eleftheriou, and G. P. Tsironis. Discrete breathers in nonlinear magnetic metamaterials. Phys. Rev. Lett., 97:157406, Oct 2006.

[20] A. A. Ovchinnikov and S. Flach. Discrete breathers in systems with homogeneous potentials: Analytic solutions. Phys. Rev. Lett., 83:248–251, Jul 1999.
[21] Nathan Perchikov and O.V. Gendelman. Dynamics and stability of a discrete breather in a harmonically excited chain with vibro-impact on-site potential. *Physica D: Nonlinear Phenomena*, 292-293:8 – 28, 2015.
[22] C. Pierre and E.H. Dowell. Localization of vibrations by structural irregularity. *Journal of Sound and Vibration*, 114(3):549 – 564, 1987.
[23] F. Romeo and O.V. Gendelman. Discrete breathers in forced chains of oscillators with cubic nonlinearities. *Procedia IUTAM*, 19:236–243, 2016.
[24] M. Sato, B. E. Hubbard, and A. J. Sievers. Colloquium : Nonlinear energy localization and its manipulation in micromechanical oscillator arrays. *Rev. Mod. Phys.*, 78:137–157, Jan 2006.
[25] M. Sato, S. Imai, N. Fujita, S. Nishimura, Y. Takao, Y. Sada, B. E. Hubbard, B. Ilic, and A. J. Sievers. Experimental observation of the bifurcation dynamics of an intrinsic localized mode in a driven 1d nonlinear lattice. *Phys. Rev. Lett.*, 107:234101, Nov 2011.
[26] S.H. Strogatz. *Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry, and Engineering*. Advanced book program. Westview Press, 1994.
[27] E. Trías, J. J. Mazo, and T. P. Orlando. Discrete breathers in nonlinear lattices: Experimental detection in a josephson array. *Phys. Rev. Lett.*, 84:741–744, Jan 2000.
[28] Andrea Trombettoni and Augusto Smerzi. Discrete solitons and breathers with dilute bose-einstein condensates. *Phys. Rev. Lett.*, 86:2353–2356, Mar 2001.
[29] Alexander F Vakakis, Oleg V Gendelman, Lawrence A Bergman, D Michael McFarland, Gaëtan Kerschen, and Young Sup Lee. *Nonlinear targeted energy transfer in mechanical and structural systems*, volume 156. Springer Science & Business Media, 2008.
[30] Alexander F Vakakis, Leonid I Manevitch, Yuri V Mikhlin, Valery N Pilipchuk, and Alexandr A Zevin. *Normal modes and localization in nonlinear systems*. Springer, 1996.