POINCARÉ DUALITY OF WONDERFUL COMPACTIFICATIONS AND TAUTOLOGICAL RINGS

DAN PETERSEN

Abstract. Let \( g \geq 2 \). Let \( M_{g,n}^{tr} \) be the moduli space of \( n \)-pointed genus \( g \) curves with rational tails. Let \( C^n_g \) be the \( n \)-fold fibered power of the universal curve over \( M_g \). We prove that the tautological ring of \( M_{g,n}^{tr} \) has Poincaré duality if and only if the same holds for the tautological ring of \( C^n_g \). We also obtain a presentation of the tautological ring of \( M_{g,n}^{tr} \) as an algebra over the tautological ring of \( C^n_g \). This proves a conjecture of Tavakol. Our results are valid in the more general setting of wonderful compactifications.

1. Introduction

Let \( g \geq 2 \), and let \( M_g \) be the moduli space of smooth curves of genus \( g \). Let \( C_g \to M_g \) be the universal curve, and let \( C^n_g \) be its \( n \)-fold fibered power. We denote by \( R^\bullet(C^n_g) \) the tautological ring of \( C^n_g \), which is a subalgebra of its rational Chow ring \( A^\bullet(C^n_g) \) generated by certain geometrically natural classes. The tautological ring was introduced by Mumford [Mumford 1983] and has been intensely studied since then, in particular because of the series of conjectures known as the Faber conjectures [Faber 1999; Pandharipande 2002], and because of the role it plays in Gromov–Witten theory. By [Looijenga 1995; Faber 1997], it is known that \( R^{g-2+n}(C^n_g) \cong \mathbb{Q} \) and \( R^k(C^n_g) = 0 \) for \( k > g-2+n \). One part of the Faber conjectures asserts that \( R^\bullet(C^n_g) \) is a Poincaré duality algebra with socle in degree \( g-2+n \); that is, that the cup-product pairing into the top degree is perfect.

One can also consider the space of \( n \)-pointed curves of genus \( g \) with rational tails, \( M_{g,n}^{tr} \). The space \( M_{g,n}^{tr} \) is defined as the preimage of \( M_g \) under the forgetful map \( M_{g,n} \to M_g \) between Deligne–Mumford compactifications. It, too, has a tautological ring, and according to the Faber conjectures \( R^\bullet(M_{g,n}^{tr}) \) is also a Poincaré duality algebra with socle in degree \( g-2+n \). It is believed among experts that these two conjectures ‘should’ be equivalent. For example, [Pixton 2013, Appendix A] writes:

“Also, instead of doing computations in \( M_{g,n}^{tr} \) we work with \( C^n_g \), the \( n \)th power of the universal curve over \( M_g \). The tautological rings of these two spaces are very closely related, and it seems likely that the Gorenstein discrepancies are always equal in these two cases.”

However, I am not aware of any precise result along these lines in the literature. The goal of this note is to prove an explicit relationship between the two tautological rings, which in particular implies that \( R^\bullet(M_{g,n}^{tr}) \) will have Poincaré duality if and only the same is true

The author is supported by the Danish National Research Foundation through the Centre for Symmetry and Deformation (DNRF92).
for $R^\bullet(C^n_g)$ (Theorem 2.7). In fact, our results give an expression for the Gorenstein discrepancies of $R^\bullet(M^n_{g,n})$ in terms of those of $R^\bullet(C^n_g)$ for $1 \leq m \leq n$. We also deduce from results of Fulton–MacPherson and Li a presentation of $R^\bullet(M^n_{g,n})$ as an algebra over $R^\bullet(C^n_g)$ (Proposition 2.11). This proves a conjecture of Tavakol.

We remark that it is likely that $R^\bullet(M^n_{g,n})$ does not have Poincaré duality in general. Counterexamples to the analogous conjectures for the spaces $M^n_{g,n}$ and $M^n_{ct_{g,n}}$ have been constructed in [Petersen and Tommasi 2014; Petersen 2013]. The conjecture that the Pixton’s extension of the Faber–Zagier relations give rise to all relations in the tautological rings would imply that $R^\bullet(M^n_{g,n})$ fails to have Poincaré duality in general [Pixton 2013; Pandharipande, Pixton, and Zvonkine 2015; Janda 2013]; as would Yin’s conjecture that all relations on the symmetric power $C^n_g$ should arise from motivic relations on the universal jacobian [Yin 2014]. See also the discussion in [Faber 2013].

What we prove is a general result about intersection rings of wonderful compactifications [Li 2009b]. Let $Y$ be a smooth variety, and $G$ a collection of subvarieties of $Y$ which form a building set (see Subsection 2.1). The wonderful compactification $Y_G$ is obtained from $Y$ by a sequence of blow-ups in smooth centers, given by intersections of the elements of $G$. Our motivation for considering these is that if we take $Y = C^n_g$ and $G$ the collection of all diagonal loci, then $Y_G \cong M^n_{g,n}$. It is perhaps worth pointing out that in this particular case — where $Y$ is given by an $n$-fold cartesian power, and $G$ consists of all diagonals — the wonderful compactification reduces to the compactification introduced in [Fulton and MacPherson 1994].

The Chow ring of $Y_G$ can be expressed [Li 2009a] in a combinatorial fashion in terms of the Chow rings of $Y$ and the Chow rings of certain intersections of elements of $G$, which we call burrows, see Theorem 2.13. An identical formula works also for the respective cohomology rings. Suppose now that $Y$ is also compact. Then the cohomology rings of $Y$ and all burrows will have Poincaré duality, and so will the cohomology ring of $Y_G$ (as all these spaces are smooth and compact). One might therefore guess that Poincaré duality of $H^\bullet(Y_G)$ can be deduced purely combinatorially from Poincaré duality for $Y$ and for the burrows. This turns out to be true, and quite easy to prove (independently of Li’s result mentioned in the first sentence of this paragraph): the inductive structure of the wonderful compactification implies that one only needs to check that Poincaré duality is preserved under two ‘basic’ operations, where we either blow up in a single smooth center or form a projective bundle. In particular, the same phenomenon — that Poincaré duality for $Y$ and for all burrows implies Poincaré duality of $Y_G$ — will work equally well on the level of Chow rings, and for not necessarily compact $Y$. Moreover, as will be crucial for us, the argument works identically for the tautological rings.

2. POINCARÉ DUALITY AND WONDERFUL COMPACTIFICATIONS

2.1. Wonderful compactifications. We refer to the papers [Li 2009; Li 2009a] for precise definitions of wonderful compactification, building set and nest, as well as for proofs of the below assertions. Instead we state only as much as is needed for the logic of the proof.

Let $Y$ be a smooth variety. Let $G$ be a building set in $Y$. This means that $G$ is a collection of closed subvarieties of $Y$ satisfying certain conditions regarding the combinatorics of how the varieties in $G$ may intersect each other. These conditions state in particular that
any nonempty intersection of elements of $G$ is smooth. The \textit{wonderful compactification} $Y_G$ is obtained from $Y$ by an iterative procedure as follows: let $X$ be an element of minimal dimension\footnote{As explained by Li, there are other possible orders in which one can perform the blow-ups, but this one will suffice for our purposes.} in $G$, and let $Y^{(1)} = \text{Bl}_X Y$. There is an induced building set $G^{(1)}$ in $Y^{(1)}$ which consists of the dominant transforms of all elements of $G$. Recall that the dominant transform coincides with the strict transform except for varieties contained in the center of the blow-up, in which case the strict transform is empty and the dominant transform is the preimage of the subvariety. We define (by induction) $Y_G = Y^{(1)}_{G^{(1)}}$. The reason this makes sense as an inductive definition is that eventually we obtain a variety $Y^{(n)}$ with a building set $G^{(n)}$ all of whose elements are Cartier divisors, after which all further blow-ups are canonically isomorphisms and $Y_G = Y^{(n)}$.

The wonderful compactification $Y_G$ is again smooth, containing $Y^o = Y \setminus \bigcup_{X \in G} X$ as the complement of a strict normal crossing divisor. If $Y$ is compact, then $Y_G$ is a compactification of $Y^o$, which explains the awkward terminology (specifically, that a ‘wonderful compactification’ is not necessarily compact). In practice one often starts with the space $Y^o$ and one wishes to compactify it so that its complement is a normal crossing divisor, which is useful e.g. in mixed Hodge theory \cite{Deligne1971}. By Hironaka’s theorem such a compactification always exists, but the advantage of $Y_G$ is its explicit description and combinatorial structure. The irreducible components of the normal crossing divisor correspond bijectively to the elements of $G$, and we write $E_X$ for the divisor corresponding to $X$.

If $G$ is a building set, then there are certain distinguished subsets $N \subseteq G$ called \textit{nests}. They can be defined by a combinatorial condition, but they have the following geometric interpretation: a set $\{X_1, \ldots, X_k\} \subseteq G$ is a nest if and only if $E_{X_1} \cap \cdots \cap E_{X_k} \neq \emptyset$. We say that a subvariety $Z \subset Y$ is a \textit{burrow} if it has the form $\bigcap_{X \in N} X$ for some nest $N$ (this terminology does not appear in Li’s work). Every burrow for the building set $G^{(i+1)}$ is obtained from a burrow for $G^{(i)}$ either as a blow-up in a smaller burrow, or as a projective bundle of some rank.

\textbf{Remark 2.1.} Here are some examples of wonderful compactifications.

1. Let $X$ be a smooth variety, and $Y = X^n$ its $n$-fold cartesian power. For every subset $I \subseteq [n]$ we have a diagonal

$D_I = \{(x_1, \ldots, x_n) \in Y : x_i = x_j \text{ for } i, j \in I\}$.

The collection of all diagonals $D_I$ with $|I| \geq 2$ form a building set, and the corresponding wonderful compactification is the Fulton–MacPherson compactification \cite{Fulton1994} of $Y^o = F(X, n)$, the configuration space of $n$ distinct ordered points on $X$.

2. As remarked in Fulton and MacPherson’s original paper, their construction makes sense just as well for a smooth family of algebraic varieties $X \to S$ over a smooth base, and the collection of diagonals in the $n$-fold fibered power of $X$ over $S$. In particular, we can consider the universal family $C_g \to \mathcal{M}_g$ over the moduli space of curves of genus $g$ and its $n$-fold fibered power $C^g_n$. For $I \subseteq [n]$ we have the diagonals $D_I = \{(C; x_1, \ldots, x_n) \in C^g_n : x_i = x_j \text{ for } i, j \in I\}$ which form a building set (this time we can even take $|I| \geq 3$ if we wish, since the diagonals with $|I| = 2$ are already Cartier divisors). The resulting wonderful compactification is exactly $\mathcal{M}^t_{g,n}$.
(3) In the previous examples, we can instead consider polydiagonals, i.e. arbitrary intersections of diagonals. These also form a building set, and the corresponding wonderful compactification was first introduced by [Ulyanov 2002].

(4) Let \( Y \) be a complex vector space, and suppose that the building set \( G \) is a collection of subspaces. In this case, \( Y_G \) is the wonderful model of the subspace arrangement, introduced by [De Concini and Procesi 1995].

(5) Let \( Y = \mathbb{P}^n \), and choose \( n + 2 \) points in general position on \( Y \). Let \( G \) be the collection of all projective subspaces spanned by these points. This is a building set, and \( Y_G \cong \mathcal{M}_{0,n+3} \), by a construction of [Kapranov 1993].

(6) Let \( Y = (\mathbb{P}^1)^n \), and take \( G \) to be the set of all diagonals \( D_I \), as well as all subsets of the form \( D_I,p = \{(x_1, \ldots, x_n) \in Y : x_i = p \text{ for } i \in I\} \) for \( p \in \{0, 1, \infty\} \). In this example, too, \( Y_G \cong \mathcal{M}_{0,n+3} \).

2.2. Preservation of Poincaré duality. In this section we shall consider operations on algebraic varieties which preserve the property of having a Chow ring with Poincaré duality. This property may seem a bit unnatural, except in the very special case of a smooth compact variety whose Chow ring maps isomorphically onto the cohomology ring (e.g. one with an algebraic cell decomposition). Nevertheless we can of course study it. Later we will observe that all propositions below remain valid if the Chow rings are replaced with tautological rings, in the cases we are interested in.

**Proposition 2.2.** Let \( Y \) be a smooth variety and \( i: Z \hookrightarrow Y \) a smooth closed subvariety of codimension \( c \). Suppose that \( A^d(Y) \cong A^{d-c}(Z) \cong \mathbb{Q} \) and that both intersection rings vanish above these degrees. The following are equivalent:

1. \( A^\bullet(\text{Bl}_Z Y) \) has Poincaré duality.
2. \( A^\bullet(Y) \) and \( A^\bullet(Z) \) both have Poincaré duality, and \( 0 \neq [Z] \in A^\bullet(Y) \).

**Proof.** Under the hypotheses, we have

\[
(*) \quad A^i(\text{Bl}_Z Y) \cong A^i(Y) \oplus \bigoplus_{k=1}^{c-1} A^{i-k}(Z) \cdot E^k,
\]

where \( E \) stands for the class of the exceptional divisor. The ring structure is given by the rules

\[
\alpha \cdot E = i^*(\alpha) \cdot E
\]

for \( \alpha \in A^\bullet(Y) \), and

\[
\alpha \cdot E^c = (-1)^c i_* (\alpha) + \sum_{i=1}^{c-1} (-1)^i \alpha c_i \cdot E^{c-i}
\]

for \( \alpha \in A^\bullet(Z) \), where \( c_i \in A^i(Z) \) is the \( i \)-th Chern class of the normal bundle \( N_{Z \subset Y} \). It follows easily that the intersection matrix describing the pairing

\[
A^i(\text{Bl}_Z Y) \otimes A^{d-i}(\text{Bl}_Z Y) \to A^d(\text{Bl}_Z Y) \cong \mathbb{Q}
\]

becomes block upper triangular when the summands in (*) are ordered as

\[
1, E, E^2, \ldots, E^{c-1}
\]

in degree \( i \), and ordered as

\[
1, E^{c-1}, E^{c-2}, \ldots, E
\]
Proof. Let \( s \) be a smooth variety. Let \( E \) be a rank \( r \) vector bundle. Then \( A^*(Y) \) has Poincaré duality (with socle in degree \( d \)) if and only if \( A^*(P(E)) \) has Poincaré duality (with socle in degree \( d + r - 1 \)).

Proof. The proof is analogous to the previous one, but simpler, using the projective bundle formula. \( \square \)

An analogous statement can be proven for a wonderful compactification by iterating the previous two propositions.

Proposition 2.4. Let \( Y \) be a smooth variety. Let \( G \) be a building set in \( Y \). Suppose that there exists an integer \( d \) such that for every burrow \( Z \subseteq Y \) (including \( Z = Y \), corresponding to the empty nest), \( A^{d - \text{codim} Z}(Z) \cong \mathbb{Q} \) and that \( A^*(Z) \) vanishes above this degree. The following are equivalent:

1. \( A^*(Y_G) \) has Poincaré duality.
2. For every burrow \( Z \), \( A^*(Z) \) has Poincaré duality, and \( |Z| \neq 0 \) in \( A^*(Y) \).

Proof. Let \( Y^{(1)} \) be the variety obtained by blowing up an element \( Z \in G \) of minimal dimension, as described above. Since \( Z \) is a burrow, \( A^*(Y^{(1)}) \) is a Poincaré duality algebra by Proposition 2.2. Every burrow in \( Y^{(1)} \) is either a projective bundle over a burrow in \( Y \) or a blow-up of one in a smaller burrow; either way, the hypotheses in the theorem hold also for \( Y^{(1)} \) and \( G^{(1)} \) by Propositions 2.2 and 2.3, so we are done by induction. \( \square \)

Propositions 2.2, 2.3 and 2.4 remain valid with identical proof also for cohomology rings. We could also consider certain subalgebras of the Chow or cohomology rings, which is necessary for the applications to tautological classes that we will consider. Let us state this as a separate proposition:
Proposition 2.5. Let $Y$ be a smooth variety. Let $G$ be a building set in $Y$. Suppose that for every burrow $Z \subseteq Y$ we have a subalgebra $R^*(Z) \subseteq A^*(Z)$ containing the Chern classes of all normal bundles, such that the collection $\{R^*(Z)\}$ is closed under pullback and pushforwards. Define $R^*(Y_G)$ to be the span of all elements of the form
\[
\alpha \cdot E_{x_1} \cdots E_{x_k}
\]
with $\alpha \in R^*(X_1 \cap \cdots \cap X_k)$. It is not hard to see that this is well defined and that $R^*(Y_G)$ is an algebra. Suppose that there exists an integer $d$ such that for every burrow $Z$, $R^{d-\text{codim} Z}(Z) \cong \mathbb{Q}$, and that $R^*(Z)$ vanishes above this degree. The following are equivalent:

1. $R^*(Y_G)$ has Poincaré duality.
2. For every burrow $Z$, $R^*(Z)$ has Poincaré duality, and $[Z] \neq 0$ in $R^*(Y)$.

Proof. The proof is identical. 

Remark 2.6. Suppose, in the situation of the previous proposition, that all restriction maps $R^*(Y) \to R^*(Z)$ are surjective. (This is true in the case of $C^n_g$. Then one could equivalently define $R^*(Y_G)$ more simply as the algebra over $R^*(Y)$ generated by all divisor classes $E_X$.

Theorem 2.7. Fix $g \geq 2$ and $n \geq 1$. The following are equivalent:

1. The tautological ring $R^*(\mathcal{M}_{g,n}^\text{rt})$ has Poincaré duality.
2. The tautological rings $R^*(C_g^m)$ have Poincaré duality for all $m = 1, \ldots, n$.
3. The tautological ring $R^*(C_g^m)$ has Poincaré duality.

Proof. Let us first discuss the equivalence of (1) and (2). Apply Proposition 2.5 with $Y = C^n_g$ and $G$ the set of diagonals. Then $Y_G \cong \mathcal{M}_{g,n}^\text{rt}$. Each burrow $Z$ is an intersection of diagonals, so it is isomorphic to some $C_g^m$ and we can let $R^*(Z)$ be its usual tautological ring. Then $R^*(Y_G)$ (defined as in Proposition 2.5) coincides with the usual tautological ring $R^*(\mathcal{M}_{g,n}^\text{rt})$. By [Looijenga 1992, Faber 1997], $R^{g-2+m}(C_g^m) \cong \mathbb{Q}$, and the tautological ring vanishes above this degree. For each burrow $Z$, the restriction of the class $[Z]$ to any fiber of the map $C_g^m \to \mathcal{M}_g$ is nonzero. Thus the previous proposition applies, and we conclude that (1) and (2) are equivalent.

We need to argue that (3) implies (2). So assume that $R^*(C_g^m)$ fails to have Poincaré duality for some $1 \leq m < n$. Then there exists $0 \neq \alpha \in R^*(C_g^m)$ which pairs to zero with everything in complementary degree. Let $\pi : C_g^n \to C_g^m$ be the forgetful map. By the projection formula, $\pi^*(\alpha)$ pairs to zero with everything in complementary degree, and the map $\pi^*$ is injective on Chow groups. Thus $R^*(C_g^m)$ fails to have Poincaré duality, too.

Remark 2.8. Special cases of Proposition 2.5 have been proven by more involved arguments in [Tavakol 2011, Section 5], [Tavakol 2011a, Section 4], [Tavakol 2014], Section 8 and [Tavakol 2014a]. Roughly, Tavakol has in all cases proved statements of the following form: for certain algebraic curves $X$, the intersection matrix describing the pairing into the top degree in a tautological ring of a Fulton–MacPherson compactification $X[n]$ (or a similar space) is block triangular, with diagonal blocks expressed in terms of pairings in the tautological ring of $X^m$, $m \leq n$. Tavakol’s proofs have used certain filtrations of the tautological rings, and explicit bases for the tautological rings given by ‘standard monomials’.
2.3. $R^\bullet(M_{g,n}t)$ as an algebra over $R^\bullet(C_g^n)$. Suppose that $X \rightarrow S$ is a family of smooth varieties of relative dimension $m$ over a smooth base. Let $Y = X^n$ be the $n$th fibered power over $S$, and let $G$ be the building set given by the diagonal loci. In this case, the wonderful compactification $Y_G$ coincides with the Fulton–MacPherson compactification $X[n]$ introduced in [Fulton and MacPherson 1994].

Let $i: Z \hookrightarrow Y$ be a closed smooth subvariety of a smooth variety. We define a Chern polynomial for $Z \subset Y$ to be a polynomial $P_{Z \subset Y}(t) \in A^\bullet(Y)[t]$ of the form

$$t^d + c_1t^{d-1} + \ldots + c_d$$

where $d = \text{codim} Z$, $c_i$ is a class in $A^i(Y)$ whose restriction to $A^i(Z)$ is the $i$th Chern class of the normal bundle $N_{Z \subset Y}$, and $c_d = [Z]$. If the restriction map $A^\bullet(Y) \rightarrow A^\bullet(Z)$ is surjective then a Chern polynomial always exists, and in this case one can moreover simplify the blow-up formula in Equation \[\Box\]: we have

$$A^\bullet(\text{Bl}_Z Y) = A^\bullet(Y)[E]/(P_{Z \subset Y}(-E), J_{Z \subset Y} \cdot E),$$

where $J_{Z \subset Y}$ denotes the kernel of $A^\bullet(Y) \rightarrow A^\bullet(Z)$.

Fulton and MacPherson have determined $A^\bullet(X[n])$ as an algebra over $A^\bullet(X^n)$. Their result is stated and proved under the simplifying assumption that $A^\bullet(X^n) \rightarrow A^\bullet(X^m)$ is surjective for all $m < n$. For every $S \subseteq \{1, \ldots, n\}$, define

$$D_S = \{(x_1, \ldots, x_n) \in X^n : x_i = x_j \text{ if } i, j \in S\}$$

and let $J_S = \ker(A^\bullet(X^n) \rightarrow A^\bullet(D_S))$. For any $i, j \in \{1, \ldots, n\}$ let $P_{i,j}(t) \in A^\bullet(X^n)[t]$ be the Chern polynomial of $D_{\{i,j\}} \hookrightarrow X^n$.

**Theorem 2.9** (Fulton–MacPherson). Suppose that $A^\bullet(X^n) \rightarrow A^\bullet(X^m)$ is surjective for $m < n$. There is an isomorphism

$$A^\bullet(X[n]) \cong A^\bullet(X^n)[\{E_S\}]/\text{relations},$$

in which there is a generator $E_S$ for every $S \subseteq \{1, \ldots, n\}$ with $|S| \geq 2$, and the relations are given by

1. $E_S \cdot E_T = 0$ unless $S \cap T = \emptyset, S, T$,
2. $J_S \cdot E_S = 0$,
3. For any distinct $i, j \in \{1, \ldots, n\}$, $P_{i,j}(-\sum_{i,j} E_S) = 0$.

**Remark 2.10.** There exists an alternative presentation which is less economical but sometimes more practical. Suppose that $\{S_1, \ldots, S_k\}$ are disjoint subsets of $\{1, \ldots, n\}$ with $|S_i| \geq 2$, and that $S_i \subseteq T$ for $i = 1, \ldots, k$. Let $W = \bigcap_{i=1}^k D_{S_i}$. If $k = 0$ then we set $W = X^n$. Then there is a relation

$$P_{D_T \subset W}(-E_T) \cdot \prod_{i=1}^k E_{S_i} = 0.$$ 

This follows from the fact that $P_{D_T \subset W}(-E_T)$ vanishes in $A^\bullet(\text{Bl}_T W)$. We can replace the third class of relations in Theorem 2.9 with the collection of all these relations, as $S_1, \ldots, S_k$ and $T$ varies.

Theorem 2.9 also gives a presentation of $R^\bullet(M_{g,n}t)$ as an algebra over $R^\bullet(C_g^n)$. To fix notation, let $D_S$ (as above) denote the diagonal loci in $C_g^n$, and denote by the same symbol their classes in $A^\bullet(C_g^n)$. Let $K \in A^1(C_g)$ be the first Chern class of the relative dualizing sheaf of
π : C_g → M_g, and let K_i ∈ A^1(C^n_g) be the pullback of K from the i-th factor. Thus \( R^* (C^n_g) \) is generated by the classes D_{ij}, K_i and the \( \kappa \)-classes. (The \( \kappa \)-classes are pulled back from \( M_g \), where they are defined as \( \kappa_d = \pi_* K^{d+1} \).)

**Proposition 2.11.** There is an isomorphism

\[
R^* (M^r_{g,n}) \cong R^* (C^n_g) / \{ E_S \} / \text{relations},
\]

in which there is a generator \( E_S \) for every \( S \subseteq \{ 1, \ldots, n \} \) with \( |S| \geq 3 \), and the relations are given by

1. \( E_S \cdot E_T = 0 \) unless \( S \cap T \in \{ \emptyset, S, T \} \),
2. \( D_{ij} \cdot E_S = 0 \) for \( i \in S, j \notin S \),
3. \( (D_{ij} + K_1) \cdot E_S = 0 \) for \( i, j \in S \),
4. \( (K_i - K_j) \cdot E_S = 0 \) for \( i, j \in S \),
5. For any distinct \( i, j \in \{ 1, \ldots, n \} \), \( \sum_{(i,j) \subseteq S} E_S = 0 \).

**Proof.** Theorem 2.9 gives a presentation of \( A^* (M^r_{g,n}) \) over \( A^* (C^n_g) \). The subalgebra generated over \( R^* (C^n_g) \) by the exceptional divisors \( E_S \) is exactly \( R^* (M^r_{g,n}) \), and so we can read off a presentation for the tautological ring of \( M^r_{g,n} \) from the theorem.

Observe first that when \( S = \{ i, j \} \), we have \( E_S = D_{ij} \). Thus we can take the \( E_S \) with \( |S| \geq 3 \) as generators. From the first relation in Theorem 2.9, we see that we then need to impose the additional relation \( D_{ij} \cdot E_S = 0 \) for \( i \in S, j \notin S \).

To determine the ideals \( J_S \), observe that the restriction map

\[
R^* (C^n_g) \to R^* (D_S)
\]

has a section. Modulo the ideal \( J_S \) we have relations \( K_i - K_j = 0, \Delta_{ij} + K_j = 0 \) (which follows from excess intersection) and \( D_{ik} - D_{jk} = 0 \), where \( i, j \in S \) and \( k \notin S \), and modulo these relations every element of \( R^* (C^n_g) \) is in the image of the section. Thus these relations generate \( J_S \) and we can replace the relation \( J_S \cdot E_S = 0 \) with the third and fourth of the relations in our list, viz. \( (D_{ij} + K_j) \cdot E_S = (K_i - K_j) \cdot E_S = 0 \) for \( i, j \in S \). (The relation \( (D_{ik} - D_{jk}) \cdot E_S = 0 \) for \( i, j \in S \) and \( k \notin S \) can be omitted since it follows from the second relation in our list.)

Finally, the Chern polynomial is given by \( P_{ij}(t) = t + D_{ij} \). Keeping in mind that \( E_{ij} = D_{ij} \), the final relation follows. \( \square \)

**Remark 2.12.** Again a slightly different presentation can be obtained as in Remark 2.10. In this form, Proposition 2.11 was conjectured in [Tavakol 2014].

One can also give an additive description of \( R^* (M^r_{g,n}) \) in terms of \( R^* (C^n_g) \), \( m \leq n \). More generally, for any wonderful compactification, the Chow ring of \( Y_G \) was calculated in [Li 2009].

Let \( G \) be a building set, and suppose that \( N \subseteq G \) is a nest. A function \( \mu : N \to \mathbb{Z}_{\geq 0} \) is called **standard** if for all \( X \in N \) we have

\[
\mu(X) < \text{codim}(X) - \text{codim} \left( \bigcap_{Z \subseteq N, Z \subseteq X} Z \right).
\]
For such a function, we denote $\|\mu\| = \sum_{X \in N} \mu(X)$.

The following theorem specializes in particular to give a direct sum decomposition of $R^\bullet(M_{g,n}^t)$ whose summands correspond to tautological rings of $C_g^m$, $m \leq n$, with a degree shift. One can for instance express the Gorenstein discrepancies of $M_{g,n}^t$ in terms of those for $C_g^m$ for $m \leq n$. We omit the details, as the procedure should be clear by now.

**Theorem 2.13 (Li).** Let $Y$ be a smooth variety, $G$ be a building set on $Y$. Then

$$A^\bullet(Y_G) = \bigoplus_N \bigoplus_\mu A^{-\|\mu\|}(\bigcap_{X \in N} X).$$

Here the first summation runs over all nests $N \subseteq G$, and the second over all standard functions $\mu: N \to \mathbb{Z}_{>0}$.

Note that the summation includes in particular the empty nest $N = \emptyset$, corresponding to the single summand $A^\bullet(Y)$.

A map from the right hand side to the left hand side is defined as follows: given a nest $N$, a standard function $\mu$ and an element $\alpha \in A^{-\|\mu\|}(\bigcap_{X \in N} X)$, the element

$$\alpha \cdot \prod_{X \in N} E_X^{\mu(X)}$$

is well defined in $A^\bullet(Y_G)$. To see that the map is surjective one uses relations analogous to those in Remark 2.10. Specifically, suppose that $N$ is a nest and $X \in N$. Let $Z_1, \ldots, Z_k$ be the minimal elements of $\{Z \in N : X \subsetneq Z\}$, and let $W = \bigcap_{i=1}^k Z_i$. Then

$$P_{X \subset W}(-E_X) \cdot \prod_{i=1}^k Z_i = 0.$$

In general a monomial $\prod_X E_X^{\mu(X)}$ can only be nonzero if the set $\{X : \mu(X) > 0\}$ is a nest, and successive applications of the relation (***) will reduce any such monomial to a linear combination of ones in which the exponents $\mu$ is a standard function. To see this, note that the degree of $P_{X \subset W}$ is

$$\text{codim}(X) - \text{codim}\left(\bigcap_{Z \subseteq W, X \subseteq Z} Z\right),$$

which is precisely the upper bound appearing in the definition of a standard function.

Using this, we can also state a ‘multiplicative’ version of Li’s Theorem 2.13.

**Proposition 2.14.** Let $Y$ be a smooth variety, $G$ a building set. Assume that $A^\bullet(Y) \to A^\bullet(Z)$ is surjective for every burrow $Z$. Then

$$A^\bullet(Y_G) = A^\bullet(Y)[[E_X]]/\text{relations},$$

where there is a generator for all $X \in G$, and the relations have the form

1. $E_X \cdot E_{X'} = 0$ if the divisors $E_X$ and $E_{X'}$ are disjoint (i.e. $\{X, X'\}$ is not a nest),
2. $J_X \cdot E_X = 0$, where $J_X = \ker(A^\bullet(Y) \to A^\bullet(X))$,
3. all relations as in Equation (***).
Remark 2.15. Suppose as in Proposition 2.4 that there exists an integer $d$ such that for every burrow $Z \subseteq Y$ (including $Z = Y$, corresponding to the empty nest), $A^{d - \text{codim}} Z(Z) \cong \mathbb{Q}$ and $A^\bullet(Z)$ vanishes above this degree. Then it is not hard to prove that two summands $(N, \mu)$ and $(N', \mu')$ in the decomposition have a nontrivial pairing into the top degree if and only if $N = N'$ and

$$\mu(X) + \mu'(X) \geq \text{codim}(X) - \text{codim} \left( \bigcap_{Z \in N, X \subset Z} Z \right)$$

for all $X \in N$. It follows from this that the intersection matrices describing the pairing into the top degree are block-triangular. This gives another approach to Proposition 2.4.

References

De Concini, Corrado and Procesi, Claudio (1995). “Wonderful models of subspace arrangements”. Selecta Math. (N.S.) 1 (3), 459–494.

Deligne, Pierre (1971). “Théorie de Hodge. II”. Inst. Hautes Études Sci. Publ. Math. (40), 5–57.

Faber, Carel (1997). “A non-vanishing result for the tautological ring of $\mathcal{M}_g$”. Unpublished. arXiv:math/9711219.

Faber, Carel (1999). “A conjectural description of the tautological ring of the moduli space of curves”. Moduli of curves and abelian varieties. Aspects Math., E33. Braunschweig: Vieweg, 109–129.

Faber, Carel (2013). “Tautological algebras of moduli spaces of curves.” Moduli spaces of Riemann surfaces. Providence, RI: American Mathematical Society (AMS); Princeton, NJ: Institute for Advanced Study (IAS), 197–219.

Fulton, William and MacPherson, Robert (1994). “A compactification of configuration spaces”. Ann. of Math. (2) 139 (1), 183–225.

Janda, Felix (2013). “Tautological relations in moduli spaces of weighted pointed curves”. Preprint. arXiv:1306.6580.

Kapranov, Mikhail M. (1993). “Chow quotients of Grassmannians. I". I. M. Gel’fand Seminar. Vol. 16. Adv. Soviet Math. Amer. Math. Soc., Providence, RI, 29–110.

Li, Li (2009a). “Chow motive of Fulton-MacPherson configuration spaces and wonderful compactifications”. Michigan Math. J. 58 (2), 565–598.

Li, Li (2009b). “Wonderful compactification of an arrangement of subvarieties”. Michigan Math. J. 58 (2), 535–563.

Looijenga, Eduard (1995). “On the tautological ring of $\mathcal{M}_g$”. Invent. Math. 121 (2), 411–419.

Mumford, David (1983). “Towards an enumerative geometry of the moduli space of curves”. Arithmetic and geometry, Vol. II. Vol. 36. Progr. Math. Boston, MA: Birkhäuser Boston, 271–328.

Pandharipande, Rahul (2002). “Three questions in Gromov-Witten theory”. Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002). Beijing: Higher Ed. Press, 503–512.

Pandharipande, Rahul, Pixton, Aaron, and Zvonkine, Dimitri (2015). “Relations on $\overline{\mathcal{M}}_{g,n}$ via 3-spin structures”. J. Amer. Math. Soc. 28 (1), 279–309.

Petersen, Dan (2013). “The tautological ring of the space of pointed genus two curves of compact type”. Preprint. arXiv:1310.7369.

Petersen, Dan and Tommasi, Orsola (2014). “The Gorenstein conjecture fails for the tautological ring of $\overline{\mathcal{M}}_{2,n}$”. Invent. Math. 196 (1), 139–161.
Pixton, Aaron (2013). “The tautological ring of the moduli space of curves”. PhD thesis. Princeton University.

Tavakol, Mehdi (2011). “The tautological ring of $M^{ct}_{1,n}$.” English. *Ann. Inst. Fourier* **61** (7), 2751–2779.

Tavakol, Mehdi (2014a). “A conjectural connection between $R^*(C^n_g)$ and $R^*(M^{ct}_{g,n})$”. Preprint. arXiv:1412.5036.

Tavakol, Mehdi (2014b). “Tautological classes on the moduli space of hyperelliptic curves with rational tails”. Preprint. arXiv:1406.7403.

Tavakol, Mehdi (2014c). “The tautological ring of the moduli space $M^{ct}_{2,n}$.” *International Mathematics Research Notices* **2014** (24), 6661–6683.

Ulyanov, Alexander P. (2002). “Polydiagonal compactification of configuration spaces”. *J. Algebraic Geom.* **11** (1), 129–159.

Yin, Qizheng (2014). “Cycles on curves and Jacobians: a tale of two tautological rings”. arXiv:1407.2216.

E-mail address: danpete@math.ku.dk

Institut for Matematiske Fag, Københavns Universitet, Universitetsparken 5, 2100 København Ø