Using Trust Region Method with BFGS Technique for Solving Nonlinear Systems of Equations

Hasan H. Dwallet and Mushtak A. K. Shiker

1 Department of Mathematics, College of Education for Pure Sciences, University of Babylon, Hilla-Iraq.

E-mail: 1 mathhasan74@gmail.com, 2 mmttmhh@yahoo.com

Abstract. The major aim of this paper is to find a solution for systems of non-linear equations, a new method is suggested by combining the trust region method (TRM) with BFGS technique, which will help us to find the solution of the non-linear system in less time and less effort, by using BFGS in a new equation based on [Xiangrong Li, Bopeng Wang & Wujie Hu]. Then the convergence of the new algorithm was provided and also calculate the numerical results. By comparing the suggested method with three other methods in terms of time, functions evaluation and number of iterations, the numerical result will tell us that our method is the best.

Keywords: BFGS Update. Global convergence. Nonlinear equations. Trust Region Radius.

1. Introduction

Assume the systems of nonlinear equations:

\[ F(x) = 0, \ x \in \mathbb{R}^n, \]

where \( F: \mathbb{R}^n \rightarrow \mathbb{R}^n \) is continuously differentiable where,

\[ F(x) := (F_1(x), F_2(x) \ldots F_n(x))^T \]

Problem (1) often arises in a wide field as applied sciences, medical applications, economics, finance [1]. Traditional iterative methods for solving (1) the line search method (LSM) and TRM [2]. LSM is considered one of the important iterative methods for solving systems of nonlinear equations, and its working mechanism depends on searching in a new iteration along the descent direction in each iteration.

TRM works to find an area around the current iteration so that the step length is determined first and then the direction of the search, or the direction and length of the step is determined simultaneously, in this work we will combine the trust region method with the line search method to find a new method based on BFGS technique, LSM reduces the number of step rejections and thus reduces the time consumed. Either the BFGS technique avoids the use of the Jacobian matrix or the second derivative as it works with the quasi-Newton method. [3].

For symmetric nonlinear equations whose Jacobian matrix \( f(x) \) of \( f(x) \) has asymmetry of all
\( x \in n \), then Quasi-Newton depended on BFGS approaches that can be used in their solution [4]. Suppose \( \varphi \) is the norm function definite by

\[
\varphi(x) = \frac{1}{2} \|f(x)\|^2
\]

Assume that \( f(x) \) equal to zero, so the nonlinear problem (1) is equivalent to the next global optimization problem

\[
\min \varphi(x), x \in \mathbb{R}^n, \tag{2}
\]

In common TRM, at each iterative point \( x_k \). In TRM to obtain each step, we seek a solution of the subproblem

\[
\min d_k(p), \|d\| \leq \Delta, \tag{3}
\]

where

\[
d_k(p) = \frac{1}{2} \|f(x_k) + \nabla f(x_k)d\|^2.
\]

So, Zhang and Wang present a new TRM:

\[
\min \varphi_k(p) \text{ s.t } c^p \|f(x_k)\|^\gamma, \tag{4}
\]

Where \( p \) is a positive integer and \( 0.5 < \gamma < 1, 0 < c < 1 \). We can get, superlinear convergence under local bound, the weakest case is the non-degeneracy. Global convergence needs non-degeneracy of \( \nabla f(x^*) \). we do not have the algorithm that has a property that includes the iterative sequence found by the algorithm \( f(x_k) \to 0 \) without assuming that \( \nabla f(x^*) \) is not generated [5].

So TRM works well when the Jacobian or Hessian computation is inexpensive. It is easy to find that one common disadvantage of two former methods in the process of finding or calculating the Jacobian matrix \( J(x) \) in each iteration, which the consumption of effort and time, especially for widespread problems. According to the above observations, we can now present a new method that has global convergence, as well as you do not need to calculate the Jacobian matrix \( J(x) \) in each iteration. This enables us to use Quasi-Newton's update matrix [2, 3]. In our work, the experimental step can be obtained at each iterative point \( x_k \) by solving the following subproblem

\[
\min q_k(d) \text{ such that } \|d\| \leq \Delta_k, \tag{5}
\]

Where

\[
q_k(d) = \frac{1}{2} \|f(x_k) + B_k d\|^2.
\]

the TRR \( \Delta_k \) is defined by \( \Delta_k = \|f(x_k)\|^2 \) and \( B_k \) is generated by the following BFGS formula,

\[
B_{k+1} = B_k - \frac{B_k s_k y_k^T B_k}{s_k^T B_k s_k} + \frac{y_k y_k^T}{s_k^T y_k} \tag{6}
\]

where \( s_k = x_{k+1} - x_k, \ y_k = f(x_{k+1}) - f(x_k), \ x_{k+1} \) is the following iteration, and \( B_0 \) is a primary symmetric positive definite matrix. By \( y_k = f(x_{k+1}) - f(x_k) \), so, we get the approximate relatives,

\[
y_k = f(x_{k+1}) - f(x_k) \approx \nabla f(x_{k+1}) s_k \tag{7}
\]
Since $B_{k+1}$ achieves the secant equation $B_{k+1}S_k = y_k$, we have approximately
\[ B_{k+1}S_k \approx \nabla f(x_{k+1})S_k \]
(8)
This illustrates that $B_{k+1}$ approximates $\nabla f_{k+1}$ lengthways direction $s_k[6, 7]$. We will use the following abbreviations during the search where:

\[ f_k = f(x_k), B_k \in \mathbb{R}^{n \times n} \text{ is a symmetric matrix}, ||\cdot|| \text{ denotes Euclidean norm}[8]. \]

We will define the ratio $r_k^p$ (11). The $\rho_k$ ratio represents the actual reduction (9) to the predicate reduction (10).

\[ Aredk(d_k^p) = \varphi(x_k + d_k^p) - \varphi(x_k), \]
(9)
and
\[ Predk(d_k^p) = qk(d_k^p) - qk(0) \]
(10)
So,
\[ r_k^p = \frac{Aredk(d_k^p)}{Predk(d_k^p)} \]
(11)
We will use the ratio (11) to determine the acceptable step or the rejected as follows: There are three cases. (i) If the value of the $r_k$ is equal to 1 then, the model has a good agreement and the trust region can be expanded and thus we will get a new radius where $x_{k+1} = x_k + d_k$. (ii) If the $r_k$ value is negative or zero then the step is rejected and the trust region is shrinking and the subproblem will be resolved to catch a new value for i.e $x_{k+1} = x_k + \alpha_k d_k$. (iii) If the $r_k$ value is positive but not equal to (1) then the step is accepted and the radius is not altered. The ratio $r_k$, it is very important to choose whether the trial step is rejected or not and how to correct the TRR [9, 10]. The authors had many papers in various scientific fields see [11-24], but in this paper, depending on [6]. We will adopt a new approach to solve the non-linear system of unconstrained equations by integrating TRM method with BFGS technique. the remainder of this work will be divided as follows. In the second section, the algorithm is presented. In the third section, we demonstrate the global convergence of the proposed technology. We show some preliminary numerical results for the proposed algorithm in the fourth section. Finally, in the fifth section, we present some concluding observations.

2. The new method:

Algorithm 1.

Step0: Given constants $\rho, c \in (0, 1), \ p = 0, \ \epsilon > 0, \ x_0 \in \mathbb{R}^n, B_0 \in \mathbb{R}^n \times \mathbb{R}^n$ is symmetric and positive definite. Let $k := 0$;

Step 1: If $\|g_k\| < \epsilon$, stop. Otherwise, go to step 2;

Step 2: Solve the subproblem (5) with $\Delta = \Delta_k$ to get $d_k$;

Step 3: Calculate $Aredk(d_k^p), Predk(d_k^p)$, also
\[ r_k^p = \frac{Aredk(d_k^p)}{Predk(d_k^p)} \]
If $r_k^p < p$, so we assume $p = p + 1$, go to step 2. Otherwise, go to step 4;

Step 4: Let $x_{k+1} = x_k + d_k^p, y_k = f_{k+1} - f_k$, if $y_k^t d_k^p > 0$, update $B_{k+1}$ by (6), otherwise, let $B_{k+1} = B_k$;
Lemma 2.1

If \( d^p_k \) is the solution of (5), then,

\[
-predk(d^p_k) = \frac{1}{2} \| B_k g_k \| \min \left[ \frac{\| B_k g_k \|}{\| B_k \|^2} \right],
\]

(12)

Proof:

Since \( d^p_k \) is the solution of (5), for any \( \alpha \in [0,1] \), we have:

\[
-Predk(d^p_k) \geq -\frac{\Delta_k}{\| B_k g_k \|} B_k g_k
\]

\[
= \alpha \Delta_k \| B_k g_k \| - \frac{1}{2} \alpha^2 \Delta^2 (B_k B_k g_k)^T / \| B_k g_k \|^2
\]

\[
\geq \alpha \Delta_k \| B_k g_k \| - \frac{1}{2} \alpha^2 \Delta^2 \| B_k B_k \|
\]

Hence, we get,

\[
Predk(d^p_k) \geq \max_{0 \leq \alpha \leq 1} \left[ \alpha \Delta_k \| B_k g_k \| - \frac{1}{2} \alpha^2 \Delta^2 \| B_k \|^2 \right]
\]

The proof is completed.

3. Convergence analysis

To structure the global convergence for the trust region algorithm, we must introduce the following results:

Assumptions:

(i) Assume the level set \( \Omega \),

\[
\Omega = \{ x | \varphi(x) \leq \varphi(x_0) \},
\]

(13)

be limited.

(ii) \( f(x) \) is twice continuously differentiable on an open convex set \( \Omega_1 \) containing \( \Omega \).

(iii) The following relation,

\[
\| [\nabla f(x_k) - B_k f(x_k)] \| = o(d^p_k),
\]

(14)

holds.

(iv) The matrices \( \{ B_k \} \) are uniformly restricted on 1, This leads to a presence positive constants

\[ 0 < M_0 \leq M \] such that,

\[
M_0 \leq \| B_k \| \leq M, \forall k,
\]

(15)

Assumption (ii) indicates that \( \exists M_1 > 0 \) s. t

\[
\| \nabla f(x_k)^T \nabla f(x_k) \| \leq M_1, \forall k,
\]

(16)

Lemma 3.1
Let Assumption i holds and \( \{x_k\} \) be generated by Algorithm 1. If \( d_k^p \) is the solution of (5).

Then,

\[
|\text{Are}d_k(d_k^p) - \text{Predk}(d_k^p)| = O(d_k^p)^2.
\]

**Proof:**

By (9), (10), and Assumption, we have,

\[
|\text{Are}d_k(d_k^p) - \text{Predk}(d_k^p)| = \frac{1}{2} \| f_k + \nabla f(x_k)d_k^p + o \left( \|d_k^p\|^2 \|f_k + B_kd_k^p\| \right) - \|f_k + B_kd_k^p\| \|d_k^p\| - o \left( \|d_k^p\|^4 \right) \]
\[
\leq \| \nabla f(x_k) - B_kf_k \| \|d_k^p\| + o \left( \|d_k^p\|^2 \right) + o \left( \|d_k^p\|^4 \right)
\]
\[
= o \left( \|d_k^p\|^2 \right)
\]

The proof is completed.

**Lemma 3.2**

Suppose the assumption that i hold and \( \{x_k\} \) is obtained from algorithm 1. Then algorithm 1 does not rotate indefinitely in the internal cycle.

**Proof:**

If Algorithm 1 loops in the internal cycle at \( x_k \) infinitely, i.e., \( p \to \infty, \rho_k \to 0 \) and \( c^p \to 0 \). Obviously, \( \|g_k\| \geq \epsilon \), otherwise the algorithm stops. Then we get

\[
\|d_k^p\| \leq \Delta_k = \|g_k\| \to 0.
\]

By lemma 2.1 and 3.1, we have,

\[
|\rho_k - 1| = \frac{|\text{Are}d_k(d_k^p) - \text{Predk}(d_k^p)|}{\|\text{Predk}(d_k^p)\|} \leq \frac{2o \left( \|d_k^p\|^2 \right)}{\Delta_k \|B_kg_k\|} \to 0.
\]

Therefore, for \( p \) sufficiently large

\[
\rho_k^p \geq \rho, \quad (17)
\]

this contradiction the fact \( \rho_k^p < \rho \). Then finished the proof.

**Lemma 3.3**

Suppose the assumption that i hold and \( \{x_k\} \) is obtained from algorithm 1. Then \( \{x_k\} \subset \Omega \).

Moreover, \( \{\varphi(x_k)\} \) converges.

**Proof**

By the algorithm definition, we get,

\[
\rho_k^p \geq \rho > 0 \quad (18)
\]

Then, Lemma 2.1 indicates,

\[
\varphi(x_{k+1}) \leq \varphi(x_k) \leq \cdots \leq \varphi(x_0).
\]

This means \( \{x_k\} \subset \Omega \). If \( \varphi(x_k) \geq 0 \), then \( \varphi(x_k) \) converges. This finished the proof.

**Theorem 3.1**

Suppose the assumption that i hold and \( \{x_k\} \) is obtained from algorithm 1. Then the algorithm either stops finitely or generates an infinite sequence \( \{x_k\} \) such that

\[
\lim_{k \to \infty} \|g_k\| = 0 \quad (19)
\]
Proof:

We use the contradiction principle to prove, assuming that the algorithm continues to operate for several steps. Now we are let's assume the following relationship

$$\lim_{k \to \infty} \|B_k g_k\| = 0,$$

(20)

is true. By (15), we may obtain (19). Thus, it also shows (20).

Suppose, that $$\exists \varepsilon > 0$$, where $$\varepsilon$$ a constant and a subsequence $$\{k_j\}$$ achieve

$$\|B_k g_{k_j}\| \geq \varepsilon,$$

(21)

Assume $$K = \{k \mid \|B_k g_k\| \geq \varepsilon \}$$, then by Assumption 1 and $$\|B_k g_k\| \geq \varepsilon, \forall k \in K$$, is bounded away from 0. We can adopt $$\|B_k\| \geq \varepsilon$$, $$\forall k \in K$$.

From the definition of Algorithm 1 and Lemma 2.1, we get,

$$\sum_{k \in K} [\varphi(x_k) - \varphi(x_{k+1})] \geq - \sum_{k \in K} \text{Pred}\,(d_k^p) \geq \sum_{k \in K} \rho \cdot \frac{1}{2} \min [c^{p_k x}, \frac{\varepsilon}{M^2}] \cdot \varepsilon$$

where $$p^k$$ is the biggest $$p$$ value attained in the internal circle at the iterative point $$x_k$$. By Lemma 3.3, we show that $$\{\varphi(x_k)\}$$ is convergent, then we get

$$\sum_{k \in K} \rho \cdot \frac{1}{2} \min [c^{p_k x}, \frac{\varepsilon}{M^2}] \cdot \varepsilon < +\infty$$

Thus $$p_k \to +\infty$$ as, $$k \to +\infty$$ and $$k \in K$$. So we can let $$p_k \geq 1$$ for all $$k \in K$$. Giving to the compute of $$p_k$$ ($$k \in K$$) in the inner circle, the solution $$d_k$$ matching to the next subproblem,

$$\min q_k (d) = \frac{1}{2} \|g(x_k) + B_k d\|_2^2.$$

s.t

$$\|d\| \leq c^{p_k-1}$$

(22)

Let $$x_{k+1} = x_k + d_k^T$$, we get

$$\varphi(x_k) - \varphi(x_{k+1}) < -\text{Pred}(d_k^p) < \rho,$$

(23)

then, from Lemma 2.1,

$$-\text{Pred}(d_k^p) \geq \frac{1}{2} \min [c^{p_k x}, \frac{\varepsilon}{M^2}] \cdot \varepsilon$$

From lemma 3.1, we find,

$$\varphi(x_k) - \varphi(x_{k+1}) - \text{Pred}(d_k^p) = o \left(\|d_k^p\|^2\right) = o \left(c^{2(p_k-1)}\right).$$

Hence,

$$\frac{\varphi(x_k) - \varphi(x_{k+1})}{\text{Pred}(d_k^p)} - 1 \leq \frac{o c^{2(p_k-1)}}{0.5 \min [c^{p_k x}, \frac{\varepsilon}{M^2}] \cdot \varepsilon}.$$ 

So, $$p_k \to +\infty$$ as, $$k \to +\infty$$ and $$k \in K$$.

Hence, it contradicts (23). This proof is finished.

4. Numerical Results.
The numerical tests to compare the results of a new algorithm TTRf 66 with three algorithms was reported.

TTRHHf1: Which produced by Shiker, M. A. K. and Sahib, Z. [23], it used the descent method.

TTRHHf5: This algorithm created by Dwail, H. H. and Shiker, M. A. K. [10].

TTRzjg: which produced by Shi, Z. J. and Guo, J. [3], it used the steepest descent method.

We will use a computer with CPU–time 1.70 GHZ and 8.00 GB Ram, each algorithm codes are written in MATLAB R2014a, this work checks the information for problems in algorithms convergence.

We use \( \mu_1 = 0.6; \mu_2 = 0.9; c = 0.5; \) epsilon =\( 10^{-5} \); \( p = 0.6 \) and number of entire of iteration override 20000, we test the problems

\[
P_1: f = 100 \cdot (x_2 - x_1^2) + (1 - x_1)^2,
\]

\[
P_2: f = (x_2 - x_1^2)^2 + (1 - x_1)^2.
\]

\[
P_3: f = (x_2 - x_1^2)^2 + (1 - x_1)^2.
\]

\[
P_4: f = (x_2 - x_1^2)^2 - (1 - x_1).
\]

\[
P_5: f = \sin(x_2)^3 + \cos(x_1)^3.
\]

\[
P_6: f = \sin(x_2)^{1/3} + \cos(x_1)^{1/3},
\]

\[
P_7: f = \sin(x_2)^2 - \cos(x_1)^2.
\]

The mathematical marks of every algorithm are reported in tables 4.1, 4.2, and 4.3, where table 4.1 include the number of iterations, table 4.2 include the number of functions, and table 4.3 has the CPU times.

**Table 4.1: Number of Iterations**

|       | TTRf66 | TTRzjg | TTRHHf1 | TTRHHf5 |
|-------|--------|--------|---------|---------|
| \( P_1 \) | 2      | 4      | 4       | 3       |
| \( P_2 \) | 2      | 4      | 4       | 3       |
| \( P_3 \) | 2      | 4      | 4       | 3       |
| \( P_4 \) | 2      | 4      | 4       | 3       |
| \( P_5 \) | 2      | 4      | 4       | 3       |
| \( P_6 \) | 2      | 4      | 4       | 3       |
| \( P_7 \) | 2      | 4      | 4       | 3       |

**Table 4.2: Number of Functions Evaluation**

|       | TTRf66 | TTRzjg | TTRHHf1 | TTRHHf5 |
|-------|--------|--------|---------|---------|
| \( P_1 \) | 2      | 4      | 3       | 4       |
| \( P_2 \) | 2      | 4      | 3       | 4       |
| \( P_3 \) | 2      | 4      | 3       | 4       |
| \( P_4 \) | 2      | 4      | 3       | 4       |
| \( P_5 \) | 2      | 4      | 3       | 4       |
| \( P_6 \) | 2      | 4      | 3       | 4       |
| \( P_7 \) | 2      | 4      | 3       | 4       |

**Table 4.3: CPU times**

|       | TTRf66 | TTRzjg | TTRHHf1 | TTRHHf5 |
|-------|--------|--------|---------|---------|
| \( P_1 \) | 5.123  | 0.056  | 8.003   | 7.004   |
| \( P_2 \) | 2.345  | 6.654  | 6.783   | 5.524   |
| \( P_3 \) | 2.274  | 8.538  | 5.419   | 5.155   |
In the former tables, we can see that a new algorithm (TTRf66) is the better among all the tested algorithms since it requires a number of iterations and functions evaluation fewer than other algorithms. Also, the CPU time that new lines want to solve the problems is less than the CPU time that other methods won't. So, we will express the algorithm (TTRf66) is more exact and robustness than other algorithms and it is so efficient for solving the nonlinear systems of equations.

5. Conclusion.

We introduced a new approach by combining TRM with BFGS technique, which contributed to solving the system in less time and less effort, and we proved that the new method is better than other famous algorithms through numerical results, we used a new radius \( \Delta_k = \|f(x_k)\|^2 \), and natural direction search \( d_k = -p \ast g_k \)

6. References

[1] Yuan G, Wei Z and Lu X 2011 A BFGS trust-region method for nonlinear equations, Computing, 92: 317–333.

[2] Zeng M and Zhou G 2016 A trust region spectral method for large-scale systems of nonlinear equations. Journal of Inequalities and Applications 8: 4, p. 174-186

[3] Shi ZJ and Guo J 2008 Anew trust region method with adaptive radius. Compute Optima Appl.41, p. 225 - 242.

[4] Dwail H H, Mahdi M M, Wasi H A, Hashim K H, Dreeb N K, Hussein A H and Shiker M A K 2020 A new modified TR algorithm with adaptive radius to solve a nonlinear systems of equations, “in press”, accepted paper for publication in Journal of Physics: Conference Series, International Conference of Modern Applications on Information and Communication Technology (ICMAICT) - Iraq.

[5] Zhang J & Wang Y 2003 A new trust region method for nonlinear equations, Math Methods Oper. Res. 58, p. 283–298.

[6] Xiangrong Li, Bopeng W and Wujie H 2017 A modified nonmonotone BFGS algorithm for unconstrained optimization, Journal of Ineq. and Appl, 1, p. 183-195.

[7] Hashim K H and Shiker M A K 2020 Using a new line search method with gradient direction to solve nonlinear systems of equations, “in press”, accepted paper for publication in Journal of Physics: Conference Series, International Conference of Modern Applications on Information and Communication Technology (ICMAICT) - Iraq.

[8] Zhou Q and Zhang C 2014 A new nonmonotone adaptive trust region method based on simple quadratic models, J Appl Math Compute, 272. P. 107–115.

[9] Dwail H H and Shiker M A K 2020 Reducing the time that TRM requires to solve systems of nonlinear equations, IOP Conf. Ser.: Mater. Sci. Eng. 928 042043

[10] Dwail H H and Shiker M A K 2020 Using a trust region method with nonmonotone technique to solve unrestricted optimization problem, J. Phys.: Conf. Ser. 1664 012128.
[11] Hassan Z A H H and Shiker M A K 2018 Using of generalized baye’s theorem to evaluate the reliability of aircraft systems, Journal of Engineering and Applied Sciences, (Special Issue 13), p. 10797-10801.

[12] Shiker M A K and Amini K 2018 A new projection-based algorithm for solving a large scale nonlinear system of monotone equations, Croatian operational research review, 9: 1, p. 63-73.

[13] Wasi H A and Shiker M A K 2020 A modified of FR method to solve unconstrained optimization, “in press”, accepted paper for publication in Journal of Physics: Conference Series, International Conference of Modern Applications on Information and Communication Technology (ICMAICT) - Iraq.

[14] Wasi H A and Shiker M A K 2020 A new hybrid CGM for unconstrained optimization problems, J. Phys.: Conf. Ser. 1664 012077.

[15] Mahdi M M and Shiker M A K 2020 Three-Term of New Conjugate Gradient Projection Approach under Wolfe Condition to Solve Unconstrained Optimization Problems, Journal of Advanced Research in Dynamical and Control Systems, 12: 7, p. 788-795.

[16] Mahdi M M and Shiker M A K 2020 A new projection technique for developing a Liou-Storey method to solve nonlinear systems of monotone equations , J. Phys.: Conf. Ser. 1591 012030.

[17] Mahdi M M and Shiker M A K 2020 Three terms of derivative free projection technique for solving nonlinear monotone equations, J. Phys.: Conf. Ser. 1591 012031.

[18] Mahdi M M and Shiker M A K 2020 Solving systems of nonlinear monotone equations by using a new projection approach, “in press”, accepted paper for publication in Journal of Physics: Conference Series, International Conference of Modern Applications on Information and Communication Technology (ICMAICT) - Iraq.

[19] Mahdi M M and Shiker M A K 2020 A New Class of Three-Term Double Projection Approach for Solving Nonlinear Monotone Equations J. Phys.: Conf. Ser. 1664 012147.

[20] Hussein H A and Shiker M A K 2020 A modification to Vogel’s approximation method to solve transportation problems, J. Phys.: Conf. Ser. 1591 012029.

[21] Hussein H A, Shiker M A K and Zabiba M S M 2020 A new revised efficient of VAM to find the initial solution for the transportation problem, J. Phys.:Conf. Ser. 1591 012032.

[22] Hussein H A and Shiker M A K 2020 Two New Effective Methods to Find the Optimal Solution for the Assignment Problems, Journal of Advanced Research in Dynamical and Control Systems, 12: 7, p 49-54.

[23] Shiker M A K and Sahib Z 2018 A modified trust-region method for solving unconstrained optimization, Journal of Engineering and Applied Sciences, 13: 22, p. 9667–9671.

[24] Wasi H A and Shiker M A K 2020 Proposed CG method to solve unconstrained optimization problems, “in press”, accepted paper for publication in Journal of Physics: Conference Series, International Conference of Modern Applications on Information and Communication Technology (ICMAICT) - Iraq.