Partial Domination and Irredundance Numbers in Graphs

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Abstract

A dominating set of a graph $G = (V, E)$ is a vertex set $D$ such that every vertex in $V(G) \setminus D$ is adjacent to a vertex in $D$. The cardinality of a smallest dominating set of $D$ is called the domination number of $G$ and is denoted by $\gamma(G)$. A vertex set $D$ is a $k$-isolating set of $G$ if $G - N_G[D]$ contains no $k$-cliques. The minimum cardinality of a $k$-isolating set of $G$ is called the $k$-isolation number of $G$ and is denoted by $\iota_k(G)$. Clearly, $\gamma(G) = \iota_1(G)$. A vertex set $I$ is irredundant if, for every non-isolated vertex $v$ of $G[I]$, there exists a vertex $u$ in $V \setminus I$ such that $N_G(u) \cap I = \{v\}$. An irredundant set $I$ is maximal if the set $I \cup \{u\}$ is no longer irredundant for any $u \in V(G) \setminus I$. The minimum cardinality of a maximal irredundant set is called the irredundance number of $G$ and is denoted by $ir(G)$. Allan and Laskar [1] and Bollobás and Cockayne [2] independently proved that $\gamma(G) < 2ir(G)$, which can be written $\iota_1(G) < 2ir(G)$, for any graph $G$. In this paper, for a graph $G$ with maximum degree $\Delta$, we establish sharp upper bounds on $\iota_k(G)$ in terms of $ir(G)$ for $\Delta - 2 \leq k \leq \Delta + 1$.

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1 Notation

Let $G$ be a simple graph with vertex set $V(G)$ and edge set $E(G)$ of order $n(G) = |V(G)|$ and size $m(G) = |E(G)|$. We denote the degree of $v$ in $G$ by $deg_G(v)$ and denote the maximum degree of $G$ by $\Delta(G)$. A neighbor of a vertex $v$ in $G$ is a vertex $u$ which is adjacent to $v$. The open neighbor set $N_G(v)$ of a vertex $v$ in $G$ is the set of neighbors of $v$. That is $N_G(v) = \{u \in V(G) | uv \in E(G)\}$. The closed neighbor set of $v$ is $N_G[v] = N_G(v) \cup \{v\}$. For a subset $S \subseteq V(G)$, we use $N_S(v)$ to denote $N_G(v) \cap S$ and $deg_S(v) = |N_G(v) \cap S|$. If $v \in S$ and $deg_S(v) = 0$, then $v$ is an isolated vertex in $S$. The neighbor set of a vertex subset $S$ of $G$ is the set $N_G(S) = \cup_{v \in S} N_G(v)$. The closed neighbor set of $S$ in $G$ is the set $N_G[S] = N_G(S) \cup S$. For a vertex $x \in S$, the private neighbor set of $x$ with respect to $S$ is $N_G[x] \setminus N_G[S \setminus \{x\}]$ and is denoted by $PN(x, S)$. In particular, a vertex $y \in V(G) \setminus S$ is
A private neighbor of \( x \) with respect to \( S \) if \( N_S(y) = \{x\} \). It is worth noting that if \( x \) an isolated vertex in \( S \), then \( x \in PN(x, S) \). The subgraph of \( G \) induced by \( S \) is denoted by \( G[S] \). The subgraph obtained from \( G \) by deleting all vertices in \( S \) and all edges incident with vertices in \( S \) is denoted by \( G - S \). The distance between two vertices \( u \) and \( v \) in a connected graph \( G \) is the length of a shortest \((u, v)\)-path in \( G \) and is denoted by \( d_G(u, v) \). A complete graph on \( n \) vertices is denoted by \( K_n \). For a graph \( G \), if \( H \) is a subgraph of \( G \) isomorphic to \( K_k \), then \( H \) is called a \( k \)-clique. A \( k \)-clique \( H \) is adjacent to a vertex \( v \) (vice versa) if \( v \in V(H) \) or \( v \) is adjacent to a vertex of \( H \). A block \( B \) of a graph \( G \) is a maximal subgraph of \( G \) such that \( B \) itself does not contain a cut vertex. A graph \( G \) is a block graph if every block of \( G \) is a complete graph while \( G \) is a cactus if each edge of \( G \) belongs to at most one cycle. The cyclomatic number of \( G \) is \( \mu(G) = m(G) - n(G) + k(G) \) where \( k(G) \) is the number of components of \( G \).

A subset \( I \subseteq V(G) \) is called irredundant if \( PN(x, I) \neq \emptyset \) for all \( x \in I \). Thus, for a vertex \( x \in I \), there exists a vertex \( y \in V(G) \setminus I \) such that \( N_I(y) = \{x\} \) or \( x \) is an isolated vertex in \( I \). An irredundant set \( I \) is said to be maximal if \( I \) is not a proper subset of any irredundant set of \( G \). The cardinality of a smallest maximal irredundant set of \( G \) is called the irredundance number of \( G \) and is denoted by \( ir(G) \).

For vertex subsets \( S \) and \( R \) of \( G \), we say that \( S \) dominates \( R \) if every vertex in \( R \setminus S \) is adjacent to a vertex in \( S \). We write \( S \succ R \) if \( S \) dominates \( R \). If \( S = \{s\} \), we write \( s \succ R \). Moreover, \( S \) is a dominating set of \( G \) if \( S \succ V(G) \). The cardinality of a smallest dominating set of \( G \) is called the domination number of \( G \) and is denoted by \( \gamma(G) \).

Let \( \mathcal{F} \) be a family of graphs. A vertex subset \( S \subseteq V(G) \) is said to be \( \mathcal{F} \)-isolating if \( G - N_G[S] \) does not contain \( H \) as an induced subgraph for all \( H \in \mathcal{F} \). The \( \mathcal{F} \)-isolation number of \( G \), denoted by \( \iota_{\mathcal{F}}(G) \), is the minimum cardinality of an \( \mathcal{F} \)-isolating set of \( G \). This notion was introduced by Caro and Hansberg \cite{CH}. If \( \mathcal{F} = \{K_k\} \), we say that a \( \{K_k\} \)-isolating set is \( K_k \)-isolating and we denote \( \iota_{\{K_k\}}(G) \) by \( \iota_k(G) \). A smallest \( K_k \)-isolating set is called a \( \iota_k \)-set. Clearly, the \( K_1 \)-isolating sets are the dominating sets and \( \iota_1(G) = \gamma(G) \). Moreover every \( K_k \)-isolating set is \( K_{k+1} \)-isolating and \( \iota_{k+1}(G) \leq \iota_k(G) \).

Every minimal dominating set is irredundant and thus \( ir(G) \leq \gamma(G) \) for every graph \( G \). An upper bound on \( \gamma(G) \) in terms of \( ir(G) \) was independently established by Allan and Laskar \cite{AL} and by Bollobás and Cockayne \cite{BC}.

**Theorem 1** \cite{AL} \cite{BC} Let \( G \) be a graph. Then \( \gamma(G) < 2ir(G) \).

Several authors lowered this bound for particular classes of graphs. In 1991, Damaschke \cite{D} reduced the upper bound of Theorem 1 for trees.

**Theorem 2** \cite{D} Let \( T \) be a tree. Then \( \gamma(T) < 3ir(T)/2 \).

The upper bound of Theorem 2 was proved to be true by Volkman \cite{V}, in 1998, for some classes of tree-like graphs as block graphs, cacti and the graphs whose cyclomatic number is at most two. Volkman further conjectured that:
Conjecture 1. If $G$ is a cactus, then $5\gamma(G) < 8ir(G)$.

Volkman’s conjecture was proved by Zverovich [14]. Favaron, Kabanov and Puech [7] showed that $\gamma(G) \leq 3ir(G)/2$ if $G$ is claw-free and that the bound is sharp in different classes of claw-free graphs.

Theorem 1 can be written $\iota_1(G) \leq 2ir(G) - 1$. Our aim is to generalize this result and to find a sharp upper bound on $\iota_k(G)$ in terms of $ir(G)$ for some larger values of $k$.

2 Main results

If a graph $G$ with maximum degree $\Delta(G) = \Delta$ contains a $k$-clique then $k \leq \Delta + 1$ since every vertex in a $k$-clique is adjacent to at least $k - 1 \leq \Delta$ vertices. When $\Delta \leq 1$, the graph $G$ is the union of $p_1$ paths of length one and $p_0$ isolated vertices yielding that $p_1 = \iota_2(G) \leq \iota_1(G) = p_0 + p_1 = ir(G)$. We assume throughout that $\Delta \geq 2$ and we consider the four cases $\Delta - 2 \leq k \leq \Delta + 1$. In this section, we give the bounds on $\iota_k(G)$ in terms of $ir(G)$ without proofs and show that the bounds are attained. The proofs are given in the following sections.

Theorem 3 Let $G$ be a graph with maximum degree $\Delta \geq 2$. If $\Delta \leq k \leq \Delta + 1$, then $\iota_k(G) \leq ir(G)$.

To show the sharpness of the bound when $k = \Delta + 1$, we let $G^1_t$ be the disjoint union of $t$ copies $K^1_k, K^2_k, \ldots, K^t_k$ of a clique of size $k = \Delta + 1$ vertices. Let $x_i \in V(K^i_k)$. We see that $deg_{G^1_t}(x_i) = \Delta$. Clearly, $\{x_1, x_2, \ldots, x_t\}$ is an $\iota_k$-set and also a smallest maximal irredundant set. Therefore, $ir(G^1_t) = t = \iota_k(G^1_t)$.

When $k = \Delta$, we let $K^1_k, K^2_k, \ldots, K^t_k$ be $2t$ copies of a clique of size $k = \Delta$ vertices. Let $x_i \in V(K^i_k)$ and let $P^i_2 = x_1^i x_2^i$ be $t$ copies of $P_2$ vertices. The graph $G^2_{2t}$ is obtained by joining $x_1^i$ to $x_{2i-1}^i$ and $x_2^i$ to $x_{2i}^i$. So, $deg_{G^2_{2t}}(x_i) = \Delta$ for all $1 \leq i \leq 2t$. Clearly, $\{x_1, x_2, \ldots, x_{2t}\}$ is an $\iota_k$-set, implying that $\iota_k(G^2_{2t}) = 2t$. Now, we let $I$ be a smallest maximal irredundant set of $G^2_{2t}$. If $I \cap \{x_1^i \cup V(K^i_k)\} = \emptyset$ for some $1 \leq i \leq t$, then $x_{2i-1}^i$ has a private neighbor with respect to $I \cup \{x_{2i}^i\}$ where $x_{2i-1}^i \in V(K^{2i-1}_k) \setminus \{x_{2i-1}^i\}$. Thus, $I \cup \{x_{2i-1}^i\}$ is an irredundant set of $G^2_{2t}$ contradicting the maximality of $I$. Hence, $I \cap \{x_1^i \cup V(K^i_k)\} \neq \emptyset$ for all $1 \leq i \leq t$. Similarly, $I \cap \{x_2^i \cup V(K^i_k)\} \neq \emptyset$. This yields $ir(G^2_{2t}) = |I| \geq 2t$. On the other hand, we see that $\{x_1, x_2, \ldots, x_{2t}\}$ is a maximal irredundant set of $G^2_{2t}$. By the minimality of $ir(G^2_{2t})$, we have $ir(G^2_{2t}) \leq 2t$. These implies that $ir(G^2_{2t}) = 2t = \iota_k(G^2_{2t})$.

Theorem 4 Let $G$ be a graph with maximum degree $\Delta \geq 2$. If $k = \Delta - 1$, then $\iota_k(G) \leq \frac{(\Delta - 4)ir(G)}{2\Delta - 2}$.

When $\Delta = k + 1 \geq 7$, we show the sharpness of the bound of Theorem 4 by constructing connected graphs $G(k, l)$ that satisfy $\iota_k(G(k, l)) = \left\lfloor \frac{(\Delta - 4)ir(G(k, l))}{2\Delta - 2} \right\rfloor$. We first construct the
graph \( L(k) \) as follows. Let \( x_1x_2x_3 \) be a path of length two, \( y_1, y_2, y_3 \) be three distinct vertices and \( K^0_k, K^1_k, K^2_k \) and \( K^3_k \) be four disjoint \( k \)-cliques. The graph \( L(k) \) is constructed as follows:

- for \( 1 \leq i \leq 3 \), join \( y_i \) to all vertices in \( \{x_i\} \cup V(K^i_k) \),
- join \( x_2 \) to \( \Delta - 3 \) vertices in \( K^0_k \) and, for \( i \in \{1, 3\} \), join \( x_i \) to \( t_i \) vertices in \( K^0_k \) in such a way that \( (i) t_1 + t_3 = \Delta + 1 \) and \( 1 \leq t_1, t_3 \leq \Delta - 3 \) and \( (ii) \) every vertex in \( K^0_k \) is adjacent to exactly 2 vertices in \( \{x_1, x_2, x_3\} \).

The above construction is possible because there are 2\( k \) edges from \( K^0_k \) to \( \{x_1, x_2, x_3\} \) and there are \( \Delta - 3 + \Delta + 1 = 2\Delta - 2 = 2k \) edges from \( \{x_1, x_2, x_3\} \) to \( K^0_k \). It can be observed that the vertex \( y_i \) has degree \( k + 1 = \Delta \) and the vertex \( x_2 \) is adjacent to \( \Delta - 3 \) vertices in \( K^0_k \). Further, \( \deg_{L(k)}(x_1) = \deg_{L(k)}(x_3) \leq \Delta - 1 \) while \( \deg_{L(k)}(x_2) = \Delta \). Now, we let \( l \) be a positive integer such that

\[
4l = \left\lceil \frac{(3\Delta-4)3l}{2\Delta-2} \right\rceil.
\]

Note that \( \ell \) is at most 3 for \( \Delta = 7 \), at most 2 until \( \Delta = 10 \) and \( \ell = 1 \) for bigger \( \Delta \).

The graph \( G(k, l) \) that satisfies \( \iota_k(G(k, l)) = \left\lceil \frac{(3\Delta-4)\iota(G(k, l))}{2\Delta-2} \right\rceil \) is obtained from \( l \) copies \( L_1, \ldots, L_\ell \) of the graph \( L(k) \) by joining \( x_3 \) of \( L_i \) to \( x_1 \) of \( L_{i+1} \) for all \( 1 \leq i \leq l - 1 \). For the sake of convenient, we may relabel \( x_1, x_2, x_3, y_1, y_2, y_3, K^0_k, K^1_k, K^2_k \) and \( K^3_k \) in \( L(k) \) to be \( x^i_1, x^i_2, x^i_3, y^i_1, y^i_2, y^i_3, K^{i,0}_k, K^{i,1}_k, K^{i,2}_k \) and \( K^{i,3}_k \) respectively in \( L_i \). We let \( z_i \) be a vertex in \( K^{i,0}_k \).

Clearly, \( \bigcup_{i=1}^{\ell} \{y^i_1, y^i_2, y^i_3, z_i\} \) is a \( K_k \)-isolating set of \( G(k, l) \). By the minimality of \( \iota_k(G(k, l)) \), we have \( \iota_k(G(k, l)) \leq 4l \). Let \( S \) be an \( \iota_k \)-set of \( G(k, l) \). To be adjacent to \( K^{i,0}_k, K^{i,1}_k, K^{i,2}_k \) and \( K^{i,3}_k \) we have that \( S \cap (y^i_j \cup V(K^{i,j}_k)) \neq \emptyset \) and \( S \cap (V(K^{i,0}_k) \cup \{x^i_1, x^i_2, x^i_3\}) \neq \emptyset \) for all \( 1 \leq i \leq \ell \) and \( 1 \leq j \leq 3 \). Therefore, \( \iota_k(G(k, l)) \geq 4l \) implying that \( \iota_k(G(k, l)) = 4l \).

Next, we let \( I \) be a smallest maximal irredundant set of \( G(k, l) \). By maximality of \( I \), we have that \( I \cap (\{x^i_1, y^i_1\} \cup V(K^{i,j}_k)) \neq \emptyset \). This yields \( \iota(G(k, l)) \geq 3l \). Clearly, \( \bigcup_{i=1}^{\ell} \{x^i_1, x^i_2, x^i_3\} \) is a maximal irredundant set of \( G(k, l) \). Hence,

\[
\iota_k(G(k, l)) = 4l = \left\lceil \frac{(3\Delta-4)3l}{2\Delta-2} \right\rceil = \left\lceil \frac{(3\Delta-4)\iota(G(k, l))}{2\Delta-2} \right\rceil.
\]

An example of the graph \( G(k, l) \) when \( k = 6, \Delta = k + 1 = 7 \) and \( \ell = 3 \) is illustrated by Figure [I]. In the figure, we let a circle or an oval to denote a clique of order 6. We see that \( \omega_6(G(6, 3)) = 12 \) while \( \iota(G(6, 3)) = 9 \).
Theorem 5 Let $G$ be a graph with maximum degree $\Delta \geq 3$. If $k = \Delta - 2$, then $\iota_k(G) \leq \frac{3\ir(G)}{2}$.

For connected graphs satisfying the upper bound of Theorem 5, we let $x_1x_2x_3x_4$ be a path of length 3, $K^0_k, K^1_k$ and $K^2_k$ be 3 $k$-cliques. The graph $D(k)$ is constructed as follows:

- join $x_1$ to all vertices in $V(K^1_k)$,
- join $x_4$ to all vertices in $V(K^2_k)$ and
- join both $x_2$ and $x_3$ to all vertices in $V(K^0_k)$.

Then, for $t \geq 2$, the graph $D(k,t)$ is obtained by the disjoint union of $t$ copies $D_1, ..., D_t$ of $D(k)$. We relabel $x_i$ and $K^j_k$ of $D(k)$ to be $x_i^j$ and $K^j_k$ in $D_t$ respectively for all $1 \leq i \leq 4$, $0 \leq j \leq 2$ and $1 \leq \ell \leq t$. Then, add the edges $x_4^\ell x_1^{\ell+1}$ for $1 \leq \ell \leq t - 1$. It can be easily checked that $\cup_{\ell=1}^t \{x_1^\ell, x_2^\ell, x_4^\ell\}$ is a smallest $\iota_k$-set and $\cup_{\ell=1}^t \{x_2^\ell, x_3^\ell\}$ is a smallest maximal irredundant set of $D(k,t)$. Thus $\iota_k(D(k,t)) = 3t$ and $\ir(D(k,t)) = 2t$ implying that $\iota_k(D(k,t)) = \frac{3\ir(D(k,t))}{2}$.

We saw that for $k = 1$, $\iota_1(G) = \gamma(G)$. Hence, Theorem 4 confirms the fact that paths and cycles have equal irredundance and domination numbers and Theorem 5 admits the following Corollary.

Corollary 1 Let $G$ be a graph with maximum degree $\Delta = 3$. Then $\gamma(G) \leq \frac{3\ir(G)}{2}$ and the bound is sharp.
Note that for cubic graphs, the slightly larger bound $\gamma(G) \leq \frac{27}{10}ir(G)$ can be deduced from the properties $\gamma(G) \leq \frac{3n}{4}\gamma_G \delta_2$ and $ir(G) \geq \frac{2n}{3\Delta} \delta_3$.

The following graph $H$ is an example of subcubic graph for which the equality in Corollary 1 is attained. $H$ consists of $k$ paths $x_i^1, x_i^2, \ldots, x_i^k$ with $1 \leq i \leq k$ and the $2k$ edges $x_i^j x_i^j$ and $x_i^j x_i^{j+1 \mod(k)}$. It can be checked that $\bigcup_{i=1}^k \{x_i^1, x_i^j, x_i^{j+1}\}$ is a $\gamma(H)$-set and $\bigcup_{i=1}^k \{x_i^1, x_i^j\}$ is an $ir(H)$. Hence $\gamma(H) = \frac{3ir(H)}{2}$.

Let $c_\Delta(k)$ be the least upper bound of $\frac{\gamma_G}{ir(G)}$ for all graphs $G$ of maximum degree $\Delta$. From our results in Theorems 3, 4 and 5, we proved that the graphs $G$ of given maximum degree $\Delta$ satisfy $\gamma_G \leq c_\Delta(k)ir(G)$ for $\Delta - 2 \leq k \leq \Delta + 1$ with $c_\Delta(\Delta - 2) = \frac{3}{2}$, $c_\Delta(\Delta - 1) = \frac{3\Delta - 4}{2\Delta - 2}$ and $c_\Delta(\Delta + 1) = c_\Delta(\Delta) = 1$. For the smallest possible value of $k$ which is 1, it follows by Theorem 1 that $c_\Delta(1) < 2$. For an arbitrary large $\Delta$, we conjecture that:

**Conjecture 2** Let $G$ be a graph with maximum degree $\Delta$. Then $\gamma_G \leq c_\Delta(k)ir(G)$ where $c_\Delta(1), c_\Delta(2), \ldots, c_\Delta(\Delta)$ is a strictly decreasing sequence such that $1 \leq c_\Delta(k) < 2$ for $1 \leq k \leq \Delta$.

Now, we fix $k = 1$ and let $c_\Delta(1) = c_\Delta$. By Corollary 1, $c_3 = \frac{3}{2}$. We think that $c_\Delta$ converges to 2 when $\Delta \to \infty$ and we give a second conjecture generalizing Corollary 1.

**Conjecture 3** Let $G$ be a graph with maximum degree $\Delta \geq 2$. Then $\gamma_G \leq c_\Delta ir(G)$ where $c_\Delta$ is a strictly increasing sequence such that $c_2 = 1$ and $c_\Delta \to 2$ when $\Delta \to \infty$.

The following example shows that if the previous conjecture is true, then $c_4 \geq \frac{5}{3} = \frac{2\Delta - 3}{4\Delta - 1}$.

The graph $G$ is constructed from 3 disjoint paths $P_6$, $c_1b_1a_1a_2b_2c_2$, $1 \leq i \leq 3$ and six new vertices $u, v, w, x, y, z$ by adding the 12 edges $ua_11, ua_12, va_21, va_22, wa_11, wa_12, xa_11, xa_12, ya_21, ya_23, za_21, za_23$. Then $\bigcup_{1 \leq i \leq 3} \{a_i, a_2i\}$ is a smallest maximal irredundant set and $\bigcup_{1 \leq i \leq 3} \{b_i, b_2i\} \cup \{a_21, a_23, u, z\}$ is a minimum dominating set. Thus $ir(G) = 6$, $\gamma(G) = 10$ and $\frac{\gamma(G)}{ir(G)} = \frac{5}{3} = \frac{2\Delta - 3}{4\Delta - 1}$.

## 3 Preliminary and preparation

We need the following properties of a maximal irredundant set which were established by Bollobás and Cockayne 2).

**Theorem 6** Let $I$ be a maximal irredundant set of a graph $G$. If there exists a vertex $u$ which is not dominated by any vertex in $I$, then there exists a vertex $x \in I$ such that

(a) $PN(x, I) \subseteq N(u)$ and
(b) if there exists a pair of non-adjacent vertices $x_1$ and $x_2$ in $PN(x, I)$, then, for each $1 \leq i \leq 2$, there exists $y_i \in I \setminus \{x\}$ such that $x_i > PN(y_i, I)$. 


Observations:
1. In part (a) of Theorem 6, the vertices $x$ of $I$ such that $PN(x, I) \subseteq N(u)$ are not isolated in $I$ since in this case, $x$ would belong to $PN(x, I)$ yielding that $PN(x, I)$ could not be dominated by $u$.

2. Similarly in (b) the vertex $y_i$ is not isolated in $I$ for otherwise $y_i \in PN(y_i, I)$ and the private neighbor $x_i$ of $x$ would be adjacent to another vertex $y_i$ of $X$, a contradiction.

We give now the notation that we will use in the proofs of our main results. In the following, let $G$ be a graph. We let

$I = \text{a smallest maximal irredundant set of } G,$

$U = \text{the set of vertices which are not dominated by } I,$

$P = \text{the set of vertices } y \in PN(x, I) \setminus I \text{ for some } x \in I \text{ and}$

$S = N(I) \setminus (P \cup I).$

Therefore, every vertex in $S$ is adjacent to at least two vertices in $I$. Clearly, the sets $I, U, P$ and $S$ form a partition of $V(G)$. By Theorem 6(a), for every vertex $u \in U$, there exists a non-isolated vertex $x \in I$ such that $PN(x, I) \subseteq N(u)$. Further, we partition the set $I$ as follows.

$Z = \text{the set of all isolated vertices of } I,$

$A = I \setminus Z,$ the set of non isolated vertices of $I,$

$Q = \text{the set of vertices } q \in A \text{ such that } PN(q, I) \not\subseteq N(u) \text{ for all } u \in U,$

$B = A \setminus Q,$ the set of non isolated vertices $q$ of $I$ whose $I$-private neighborhood is dominated by some $u \in U,$

$N = \text{a maximal set of vertices } x \in B \text{ such that } PN(x, I) \text{ is a clique or, for each } x' \in PN(x, I), x' \succ PN(y, I) \text{ for some } y \in N,$

and

$M = B \setminus N.$

We note that if $U \neq \emptyset$, then $B \neq \emptyset$ by Theorem 6(a) and the partition of $B$ refers to Theorem 6(b). From the definition, if $x \in M$ then $PN(x, I)$ is not a clique and thus $|PN(x, I)| \geq 2,$ and there exists a vertex $x' \in PN(x, I)$ such that $x'$ does not dominate $PN(y, I)$ for any $y \in N$ but $x' \succ PN(w, I)$ for some $w \in I \setminus N$ (by Theorem 6(b)).

Note also that for no vertex $z$ of $Z$, $PN(z, I) \subseteq N(u)$ since every isolated vertex $z$ of $I$ belongs to $PN(z, I)$. 
Figure 2: The partition $I \cup P \cup S \cup U$ of the graph and the partition $N \cup M \cup Q \cup Z$ of $I$.

4 Proofs

Throughout this section, $I$ is a minimum maximal irredundant set as in Section 3. Let $N = \{a_1, a_2, ..., a_{|N|}\}$ and choose $a'_i \in PN(a_i, I)$ for all $1 \leq i \leq |N|$. The following lemma is a property of $\hat{N} = \{a'_1, a'_2, ..., a'_{|N|}\}$.

Lemma 1 $\hat{N} \succ \cup_{i=1}^{|N|} PN(a_i, I)$.

Proof. If $PN(a_i, I)$ is a clique, then $a'_i \succ PN(a_i, I)$. If $PN(a_i, I)$ is not a clique, let $a \in PN(a_i, I)$. From the definition of $N$, $a \succ PN(a_j, I)$ for some $a_j \in N$, which implies $a'_j a \in E(G)$. Therefore $\hat{N} \succ PN(a_i, I)$ for every vertex $a_i$ of $N$. \qed

Now, we are ready to prove our three theorems.
4.1 Proof of Theorem 3

We first consider the case when \( k = \Delta + 1 \). The cliques \( K_k \) are isolated in \( G \). Let \( G_i, 1 \leq i \leq h, \) be the components of \( G \) isomorphic to \( K_k \) and \( G_i, h + 1 \leq i \leq \ell, \) the other ones if any. Then \( \mu_k(G) = h \) and \( \mu(G) \geq \ell \geq h = \mu_k \). Moreover \( \mu_k(G) = \mu(G) \) if and only if every component of \( G \) is a clique \( K_k \).

We now consider the case when \( k = \Delta \). Let \( H \) be a \( k \)-clique of \( G \).

Claim 1 If \( V(H) \cap I = \emptyset \), then \( V(H) \cap S = \emptyset \)

Proof. We may suppose to the contrary that \( V(H) \cap I = \emptyset \) but \( V(H) \cap S \neq \emptyset \). Let \( v \in V(H) \cap S \). Thus \( \deg_H(v) = \Delta - 1 \) because \( |V(H)| = k = \Delta \). Since \( v \in S, v \) is adjacent to at least two vertices in \( I \) which are not in \( H \). Therefore, \( \deg_G(v) \geq \deg_H(v) + 2 = \Delta + 1 \), a contradiction. \( \Box \)

Claim 2 If \( V(H) \subseteq U \) and \( v \in V(H) \), then there exists \( x \in N \) such that \( v \succ PN(x, I) \).

Proof. By Observation 1 after Theorem 6 \( v \succ PN(x, I) \) for some \( x \in B \). Since \( d_U(v) \geq d_H(v) = \Delta - 1 \), \( |PN(x, I)| \leq 1 \). Thus \( x \in N \) since \( PN(x, I) \) is a clique. \( \Box \)

Claim 3 If \( V(H) \cap (P \cup I) = \emptyset \), then, for all \( v \in V(H) \), there exists \( a_i' \in \hat{N} \), such that \( va_i' \in E(G) \).

Proof. By Claim 1 \( V(H) \cap S = \emptyset \) and thus \( V(H) \subseteq U \). By Claim 2 \( v \succ PN(a_i, I) \) for some \( a_i \in N \). Thus \( va_i' \in E(G) \). \( \Box \)

Let now \( X = \hat{N} \cup M \cup Q \cup Z \) and let \( H \) be a \( k \)-clique of \( G \). If \( V(H) \cap I \neq \emptyset \), then \( H \) contains or is adjacent to a vertex of \( X \) by the definition of \( \hat{N} \). If \( V(H) \cap P \neq \emptyset \), then \( H \) is adjacent to a vertex of \( X \) by Lemma 4. If \( V(H) \cap (P \cup I) = \emptyset \), then \( H \) is adjacent to a vertex of \( \hat{N} \) by Claim 3. Therefore \( X \) is a \( K_k \)-isolating set of \( G \) and \( \mu_k(G) \leq |X| = |\hat{N} \cup M \cup Q \cup Z| = \mu(G) \). This completes the proof. \( \Box \)

4.2 Proof of Theorem 4

Now \( k = \Delta - 1 \). Recall that if \( x \in M \), then \( |PN(x, I)| \geq 2 \). We let

\[
M' = \text{the vertex subset of } M \text{ such that } |PN(x, I)| = 2 \text{ for all } x \in M',
\]

\[
\bar{M} = M \setminus M',
\]

\[
S' = \text{the subset of } S \text{ such that each vertex in } S' \text{ is not adjacent to any vertex in } \bar{M} \cup Q \cup Z.
\]

Therefore, every vertex in \( S' \) is adjacent to at least two vertices in \( N \cup M' \). We let further
\( \mathcal{H} \) = the set of all \( k \)-cliques \( H \) such that \( V(H) \subseteq S' \).

To prove Theorem \[4\] we prove in Lemma \[2\] a stronger result

**Lemma 2** Let \( G \) be a graph with maximum degree \( \Delta = k + 1 \). Let \( s \) be the maximum number of \( k \)-cliques contained in \( S' \) to which a vertex in \( N \cup M' \) can be adjacent. When \( S' = \emptyset \) or does not contain any \( k \)-clique, let \( s = 0 \). Then

\[
\iota_k(G) \leq \frac{(3\Delta - 4)ir(G)}{2\Delta - 2} - s + 1 \quad \text{if } 1 \leq s \leq \Delta - 2, \\
\iota_k(G) \leq ir(G) \quad \text{if } s = 0.
\]

**Proof.** We need the following claims.

**Claim 4** The \( k \)-cliques of \( G[S] \) are components of \( S \). In particular, the \( k \)-cliques in \( \mathcal{H} \) are pairwise disjoint.

**Proof.** Let \( u \) be a vertex of a \( k \)-clique \( H \) contained in \( S \). Then \( \Delta \geq d_G(u) \geq d_S(u) + d_I(u) \geq (k - 1) + 2 \). Since \( k = \Delta - 1 \), \( d_S(u) = k - 1 \), which shows that \( H \) forms a component of \( G[S] \).

\[(\ast)\]

**Claim 5** Let \( H \) be a \( k \)-clique such that \( V(H) \cap U \neq \emptyset \) and \( V(H) \cap P = \emptyset \). If \( u \in V(H) \cap U \), then \( PN(x, I) \subseteq N_G(u) \) for some \( x \in N \cup M' \).

**Proof.** Since \( u \in U \), there exists \( x \in B \) such that \( PN(x, I) \subseteq N_G(u) \). If \( x \in \tilde{M} \), then \( |PN(x, I)| \geq 3 \) which implies that \( deg_P(u) \geq 3 \). Since \( V(H) \cap P = \emptyset \), it follows that \( deg_G(u) \geq deg_H(u) + deg_P(u) \geq (\Delta - 2) + 3 = \Delta + 1 \), a contradiction. Therefore, \( x \in N \cup M' \).

\[(\ast)\]

**Claim 6** \( |\mathcal{H}| \leq \frac{|N \cup M'|(\Delta - 2)}{2k} \).

**Proof.** By Claim \[4\] there are exactly \( 2k|\mathcal{H}| \) edges from the \( |\mathcal{H}| \) \( k \)-cliques of \( S' \) to vertices in \( N \cup M' \). For a vertex \( x \in N \cup M' \), \( deg_I(x) \geq 1 \) because \( x \) is non-isolated in \( I \) and \( deg_P(x) \geq 1 \) because \( x \) has at least one private neighbor with respect to \( I \). Thus, \( deg_S(x) \leq \Delta - 2 \). By a double counting, we have

\[
2k|\mathcal{H}| \leq (\Delta - 2)|N \cup M'|
\]

implying that \( |\mathcal{H}| \leq \frac{|N \cup M'|(\Delta - 2)}{2k} \).

Recall that \( M' \) is the vertex subset of \( M \) such that \( |PN(x, I)| = 2 \) for all \( x \in M' \). In the following, we let
Moreover, we let $x \in G - \{b\}$, $\tilde{P} \subseteq B$ and by the definition of $P$, $x \notin H \cap H$ adjacent to $H \cap H$ contradicting $x \notin H \cap H$. Therefore, $\tilde{P} = \emptyset$. Now, we will show that the set of all private neighbors of vertices in $V$ contains no $k$-clique and thus $G[S \cap N[X \cup \tilde{M} \cup Q \cup Z]]$ contains no $k$-clique.

Now, we will show that the set

$$T = \tilde{N} \cup Y \cup X \cup \tilde{M} \cup Q \cup Z$$

is a $K_k$-isolating set of $G$. We assume to the contrary that there exists a $k$-clique $H$ in $G - N_G[T]$. By the choice of $T$ and Lemma 1, we have that

$$V(H) \cap (T \cup N \cup PN \cup MP \cup \tilde{M} \cup P \cup Q \cup PZ \cup (S \setminus S')) = \emptyset \tag{1}$$

and $H$ is not entirely contained in $S'$. Hence $V(H) \subseteq W \cup U \cup S'$ and $V(H) \cap (W \cup U) \neq \emptyset$.

Suppose that $V(H) \cap U \neq \emptyset$. Let $w_1 \in V(H) \cap U$. So, $w_1$ is a private neighbor of a vertex $x_i \in M'$ whose second private neighbor $y_i$ is in $Y$. Since $Y \subseteq T$, $y_i w_1 \notin E(G)$. By Theorem 1(b), $w_1 \succ PN(x, I)$ for some $x \in I \setminus Z$. We first consider the case when $x \in N \cup M'$. So $PN(x, I) \cap (\tilde{N} \cup Y) \neq \emptyset$ implying that $w_i$ is adjacent to a vertex in $\tilde{N} \cup Y$. Thus, $w_i \in N_G[T]$, contradicting $H$ is in $G - N_G[T]$. Hence $x \in \tilde{M} \cup Q$ and $w_i$ has at least one neighbor $x'$ in $PM \cup PQ$. Since $V(H) \cap (PM \cup PQ) = \emptyset$, $x' \notin V(H)$. On the other hand, $x_i \in M' \subseteq B$ and by the definition of $B$, there exists a vertex $u$ in $U$ dominating $PN(x_i, I)$. Hence $u$ is adjacent to $y_i$ and to $w_i$, and since $y_i \in T$, $u \notin V(H)$. Now $w_i$ has at least three neighbors $u, x', x_i$ not in $V(H)$ and $d_G(w_i) \geq 3 + d_H(w_i) = 3 + (\Delta - 2) = \Delta + 1$, a contradiction. Hence $V(H) \cap W = \emptyset$ and thus $V(H) \subseteq U \cup S'$ with $V(H) \cap U \neq \emptyset$. Let $u \in V(H) \cap U$. By Claim 5 $u$ is adjacent to a vertex in $\tilde{N} \cup Y$. Therefore, $V(H) \cap N_G[T] \neq \emptyset$ contradicting $H$ is in $G - N_G[T]$. So, $T$ is a $K_k$-isolating set of $G$. 

\[ M' = \{x_1, x_2, ..., x_t\} \text{ where } |M'| = t, \]
\[ PN(x, I) = \{y_i, w_i\}, \]
\[ Y = \{y_i : 1 \leq i \leq t\}, \]
\[ W = \{w_i : 1 \leq i \leq t\}. \]
Hence,

\[
\iota_k(G) \leq |T| \leq |\hat{N} \cup Y \cup \tilde{M} \cup Q \cup Z| + |X| \\
\leq ir(G) + |X| \\
= ir(G) + \frac{|N \cup M'|(\Delta - 2)}{2k} - s + 1 \\
\leq \frac{(3\Delta - 4)ir(G)}{2\Delta - 2} - s + 1 \quad \text{when } s \neq 0
\]

and \(\iota_k(G) \leq ir(G)\) when \(s = 0\) since in this case, \(|X| = 0\).

From the previous lemma, \(\iota_k(G) \leq \max\{\frac{(3\Delta - 4)ir(G)}{2\Delta - 2}, ir(G)\} = \frac{(3\Delta - 4)ir(G)}{2\Delta - 2}\), which proves Theorem 4. We see that Equality in Theorem 4 requires \(s = 1\) as in the example given in Section 2. In the appendix, we construct a graph satisfying the equality in Lemma 2 when \(s \geq 2\).

### 4.3 Proof of Theorem 5

We recall that:

\(M' = \text{the subset of } M \text{ such that } |PN(x, I)| = 2 \text{ for all } x \in M'\).

Then we define further that:

\(M'' = \text{the subset of } M \text{ such that } |PN(x, I)| = 3 \text{ for all } x \in M''\),

\(\overline{M} = M \setminus (M' \cup M'')\),

\(Q' = \text{the subset of } Q \text{ such that } |PN(x, I)| = 1 \text{ for all } x \in Q'\),

\(\check{Q} = \text{the subset of } Q \text{ such that } |PN(x, I)| \geq 2 \text{ for all } x \in \check{Q}, \text{ that is } \check{Q} = Q \setminus Q'\),

\(S'' = \text{the subset of } S \text{ such that every vertex in } S'' \text{ is adjacent in } I \text{ to only vertices in } N \cup M' \cup M'' \cup Q'\)

and

\(\mathcal{H}' = \text{the family of } k\text{-cliques } H \text{ such that } V(H) \subseteq S''\).

The following claim is a direct consequence of Theorem 6(b).

**Claim 7** Let \(x \in M' \cup M''\) and \(x' \in PN(x, I)\) such that \(x'\) is not adjacent to any vertex in \(PN(x, I)\). Then \(x' \succ PN(y, I)\) for some \(y \in A\).

When \(k = \Delta - 2\) and \(\Delta > 3\), it is possible that two \(k\text{-cliques in } \mathcal{H}' \text{ intersect. Let } H \text{ and }
be two different \( k \)-cliques in \( H' \) such that \( V(H) \cap V(H') \neq \emptyset \). Let \( s = |V(H) \cap V(H')| \) and \( a \in V(H) \cap V(H') \). Then

\[
d_{V(H) \cup V(H')}(a) = d_{V(H)}(a) + d_{V(H')} (a) - d_{V(H) \cap V(H')}(a) = 2(k - 1) - (s - 1) = 2k - s - 1.
\]

Remind that \( \text{deg}_I(a) \geq 2 \) because \( a \in V(H) \subseteq S \). Thus,

\[
k + 2 = \Delta \geq d_G(a) \geq d_I(a) + d_{V(H) \cup V(H')}(a) \geq 2k - s + 1
\]

implying that \( s \geq k - 1 \).

Since \( H \) and \( H' \) are different, it follows that \( s \leq k - 1 \) which yields \( s = k - 1 \). Hence two intersecting \( k \)-cliques of \( S'' \) share exactly \( k - 1 \) vertices and each of these vertices, say \( x \), has no other neighbor in \( S'' \) since \( d_{V(H \cup H')} (x) = k + 2 = \Delta \). We call isolated clique a \( k \)-clique of \( S'' \) intersecting no other one and twin a set of two intersecting \( k \)-cliques of \( S'' \). Note that a twin is either a \((k+1)\)-clique or a \((k+1)\)-clique minus one edge. Clearly, an isolated clique has \( k \) vertices while a twin has \( k + 1 \) vertices. We let \( t_1 \) and \( t_2 \) be the number of isolated cliques and twins in \( H' \) respectively. The following claim establishes an upper bound of \( t_1 + t_2 \).

**Claim 8** \( t_1 + t_2 \leq \frac{|N \cup M' \cup M'' \cup Q'|}{2} \)

**Proof.** Let \( e(H', N \cup M' \cup M'' \cup Q') \) be the number of edges between the vertices of the cliques in \( H' \) and the vertices in \( N \cup M' \cup M'' \cup Q' \). By the definition of \( H' \), there are at least two edges from each vertex of cliques in \( H' \) to \( N \cup M' \cup M'' \cup Q' \). Because \( H' \) only consists of isolated cliques and twins, it follows that

\[
2k(t_1 + t_2) \leq 2kt_1 + 2(k + 1)t_2 \leq e(H', N \cup M' \cup M'' \cup Q').
\]

On the other hand, since every vertex \( v \in N \cup M' \cup M'' \cup Q' \) is not isolated in \( I \) and has at least one private neighbor in \( P \), we have that \( \text{deg}_{S''}(v) \leq \Delta - 2 \). Hence

\[
e(H', N \cup M' \cup M'' \cup Q') \leq (\Delta - 2)|N \cup M' \cup M'' \cup Q'| = k|N \cup M' \cup M'' \cup Q'|.
\]

By Equations (2) and (3), \( t_1 + t_2 \leq \frac{|N \cup M' \cup M'' \cup Q'|}{2} \) which proves Claim 8. Note that equality in Claim 8 implies, from the first inequality in (2), that \( t_2 = 0 \), that is all the \( k \)-cliques of \( S'' \) are isolated. (3)

In the following, for an isolated clique or a twin \( H \), we let \( x_H \) be a vertex in \( H \). We let \( X = \{x_H : H \text{ is an isolated clique or a twin of } H' \} \). So, \( X \) is a \( K_k \)-isolating set of \( S'' \) and

\[
|X| \leq \frac{|N \cup M' \cup M'' \cup Q'|}{2}
\]

by Claim 8. Recall that \( \tilde{N} = \{a'_1, a'_2, ..., a'_{|X|}\} \). Similarly, for a vertex \( b \in M' \cup M'' \), we choose one vertex \( b' \) from \( PN(b, I) \) and, for a vertex \( c \in Q' \), we call \( c' \) the unique vertex of \( PN(c, I) \). Then, we let

\[
\tilde{M} = \{b' : b \in M' \cup M''\} \text{ and }
\]
\[ \hat{Q} = \{ c' : c \in Q' \}. \]

For the rest of this paper, we aim to prove that

\[ T = \hat{N} \cup \hat{M} \cup \bar{M} \cup \hat{Q} \cup \tilde{Q} \cup Z \cup X \]

is a \( K_k \)-isolating set of \( G \). We assume to the contrary that there exists a \( k \)-clique \( H \) in \( G - N_G[T] \). By the choice of \( T \), we have

\[ V(H) \cap I = \emptyset. \] (4)

Claim 9 \( V(H) \cap U = \emptyset \).

**Proof.** We assume to the contrary that there exists \( u \in V(H) \cap U \). By Theorem 8(b) and the definition of \( B \), \( PN(x,I) \subseteq N_G(u) \) for some \( x \in B \). If \( x \in N \cup M' \cup M'' \), then \( u \) is adjacent to \( x' \) for some \( x' \in \hat{N} \cup \hat{M} \subseteq T \) contradicting \( H \) is in \( G - N_G[T] \). Thus, \( x \in \bar{M} \). If \( V(H) \cap PN(x,I) \neq \emptyset \), then \( H \) is adjacent to a vertex \( x \in \bar{M} \subseteq T \), again contradicting \( H \) is in \( G - N_G[T] \). Thus, \( V(H) \cap PN(x,I) = \emptyset \). Because \( x \in \bar{M} \), we have \( |PN(x,I)| \geq 4 \). Thus,

\[ deg_{PN(x,I)}(u) \geq 4 \]

a contradiction. So, \( V(H) \cap U = \emptyset \) and this proves Claim 9.

Claim 10 \( V(H) \cap PN(b,I) = \emptyset \) for all \( b \in M' \cup M'' \).

**Proof.** Assume to the contrary that there exists \( b \in M' \cup M'' \) such that \( V(H) \cap PN(b,I) \neq \emptyset \). Let \( u \in V(H) \cap PN(b,I) \). Thus, \( u \in PN(b,I) \). By the definition of \( M \), \( u \) is adjacent to a vertex in \( U \). Thus, \( deg_U(u) \geq 1 \). Clearly, \( deg_I(u) = 1 \) because \( u \) is a private neighbor of \( b \).

By the assumption that \( H \) is in \( G - N_G[T] \), \( u \) is not adjacent to \( b' \) which is selected in \( \hat{M} \) from \( PN(b,I) \). Hence, \( u \) does not dominate \( PN(b,I) \). By Claim 7, \( u \) dominates \( PN(y,I) \) for some \( y \in A \setminus \{ b \} \).

If \( y \in (N \cup M' \cup M'') \setminus \{ b \} \), then \( u \) is adjacent to a vertex \( y' \in \hat{N} \cup \hat{M} \) where \( y' \) is selected from \( PN(y,I) \), contradicting \( H \) is in \( G - N_G[T] \).

If \( y \in \bar{M} \), then \( V(H) \cap PN(y,I) = \emptyset \) because \( \bar{M} \subseteq T \). Since \( |PN(y,I)| \geq 4 \), we have

\[ deg_{G}(u) \geq deg_{PN(y,I)}(u) + deg_{H}(u) \geq 4 + (k - 1) = \Delta + 1, \]

a contradiction.

Finally, we consider the case when \( y \notin Q \). Because \( \hat{Q} \subseteq T \), it follows that \( y \in \tilde{Q} \). In this case \( |PN(y,I)| \geq 2 \). Similarly, \( V(H) \cap PN(y,I) = \emptyset \) because \( \tilde{Q} \subseteq T \). By Claim 9 and Equation 4,
\[ \text{deg}_G(u) \geq \text{deg}_{P_N(y,l)}(u) + \text{deg}_H(u) + \text{deg}_U(u) + \text{deg}_I(u) \geq 2 + (k - 1) + 1 + 1 = \Delta + 1, \]

a last contradiction proving Claim 10 (c).

Now, for a \( k \)-clique \( H \) which is in \( G - N_G[T] \), we have by Equation (4), Claim 10 and Lemma 10 that

\[ V(H) \cap (I \cup P) = \emptyset. \]

Moreover, by Claim 9 \( V(H) \subseteq S \). Because \( \mathcal{M} \cup \hat{Q} \cup \mathcal{Z} \subseteq T \), it follows that every vertex in \( H \) is adjacent to only vertices in \( N \cup M' \cup M'' \cup Q' \). Therefore, \( V(H) \subseteq S' \), namely, \( H \in \mathcal{H}' \). It follows that \( H \) contains or is adjacent to a vertex in \( X \), a contradiction since \( X \subseteq T \). Therefore, \( T \) is an \( K_k \)-isolating set of \( G \).

By the minimality of \( \iota_k(G) \) and Claim 8 we have

\[ \iota_k(G) \leq |T| = |\hat{N} \cup \hat{M} \cup \hat{M} \cup \hat{Q} \cup \hat{Q} \cup Z \cup X| \]
\[ = |\hat{N} \cup \hat{M} \cup \hat{M} \cup \hat{Q} \cup \hat{Q} \cup Z| + |X| \]
\[ \leq \text{ir}(G) + \frac{|N \cup M' \cup M'' \cup Q'|}{2} \]
\[ \leq \text{ir}(G) + \frac{3 \text{ir}(G)}{2} = \frac{3 \text{ir}(G)}{2}. \]

This completes the proof. \( \square \)

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**Appendix**

We give a construction of graphs satisfying the bound of Lemma 2 when \( s \geq 2 \) since an example with \( s = 1 \) was already shown in Section 2. First, we construct the graph \( S(k) \) which is obtained from two paths \( x_1x_2x_3x_4 \) and \( x_5x_6x_7x_8 \) of length 3 and five \( k \)-cliques \( K_k^0, K_k^1, ..., K_k^4 \) by joining edges as follows:

- Join \( x_1, x_4, x_5 \) and \( x_8 \) to every vertex of \( K_k^1, K_k^2, K_k^3 \) and \( K_k^4 \), respectively.

Further, we let \( V(K_k^0) = \{ z_1, ..., z_k \} \). Then,

- Join \( x_2 \) to \( z_1, ..., z_{k-1} \), \( x_3 \) to \( z_2, ..., z_k \) and join \( x_6 \) to \( z_1 \) and \( z_k \).

Note that \( deg_{S(k)}(x_2) = deg_{S(k)}(x_3) = k + 1 = \Delta \) while \( d_{S(k)}(x_6) = 4 \) and \( d_{S(k)}(x_7) = 2 \). An example of the graph \( S(k) \) when \( k = 8 \) is illustrated by Figure 3.
Further, we construct the graph $F(k, s)$ which is obtained from a path $a_1a_2a_3a_4$ of length 3 and $s + 2$ of $k$-cliques $K'_k$, $K''_k$ and $\tilde{K}_1^k, ..., \tilde{K}_s^k$ by putting edges as follows:

- join $a_1$ and $a_4$ to every vertex of $K'_k$ and $K''_k$, respectively,
- join $a_2$ to two vertices of $\tilde{K}_k^1$ and to one vertex of each $\tilde{K}_k^2, ..., \tilde{K}_k^s$ and
- join $a_3$ to $\Delta - 2$ vertices in $\tilde{K}_k^1$ to share exactly one common neighbor with $a_2$.

It is worth noting that exactly one vertex in $\tilde{K}_k^1$ has degree $\Delta$ and the other $k - 1$ vertices have degree $\Delta - 1$. An example the graph $F(k, s)$ when $k = 8$ and $s = 6$ is illustrated by Figure 4.
Figure 4: The graph $F(8, 6)$.

Now, we are ready to construct the graph $H(k, s, l)$ from one copy of $F(k, s)$ and $l$ copies $S_1, ..., S_l$ of $S(k)$ where $l$ is given by the equation

$$(s - 1)(2k - 1) = (2\Delta - 6)(l - 1) + r$$

for some $0 \leq r < 2\Delta - 6$. We may relabel $x_j$ in $S(k)$ to be $x^i_j$ in $S_i$ for all $1 \leq j \leq 8$ and $1 \leq i \leq l$. Further, we let

$L_1 = \bigcup_{i=2}^s \tilde{K}^i_k$ in the copy of $F(k, s)$ and

$L_2 = \bigcup_{i=1}^{l-1} \{x^i_6, x^i_7\} \cup \{x^i_0\}$ in the $l$ copies of $S(k)$.

Then we add edges as follows, in such a way that the maximum degree of $H(k, s, l)$ remains equal to $\Delta = k + 1$.

- join $x^i_7$ to all the $k - 1$ vertices of degree $\Delta - 1$ in $V(\tilde{K}^i_k)$,
- join $x^i_0$ to $r' = \min\{r, \Delta - 4\}$ vertices in $L_1$,
- for $1 \leq i \leq l - 1$, join $x^i_6$ to $\Delta - 4$ vertices of $L_1$ and $x^i_7$ to $\Delta - 2$ vertices of $L_1$ in such a way that every vertex in $L_1$ is adjacent to at most two vertices in $L_2 \cup \{a_2, a_3\}$ to respect the value of the maximum degree.

The adjacency condition above is possible since $s - 1 + (2\Delta - 6)(l - 1) + r' \leq 2(s - 1)k$ from Equation (5).
We show that the graph \( H(k, s, l) \) when \( k = 8, s = 6 \) and \( l = 7 \) satisfies the upper bound of Lemma 2. For the sake of convenience, we let \( H = H(8, 6, 7) \). Clearly, \( \{a_2, x_2, \ldots, x_7\} \cup \{a_1, a_4\} \cup (\cup_{i=1}^{7}\{x_1^i, x_4^i, x_5^i, x_8^i\}) \) is a \( K_8 \)-isolating set of \( H \). Thus \( v_8(H) \leq 38 \). Let \( S \) be a smallest \( v_8 \)-set of \( G \). To be adjacent to all the 8-cliques, \( S \) intersects each disjoint subset as follows: (i) \( S \cap (\{a_1\} \cup V(K_8')) \neq \emptyset \), (ii) \( S \cap (\{a_4\} \cup V(K_8'')) \neq \emptyset \), (iii) \( S \cap (\{x_1^i\} \cup V(K_8^{i, 2})) \) for all \( 1 \leq i \leq 7 \) and \( j \in \{1, 4, 5, 8\} \), (iii) \( S \cap (\{x_2, x_3, x_6\} \cup V(K_8')) \neq \emptyset \) and (iv) \( S \cap (\{x_2, x_3, x_6\} \cup V(K_8^{i, 0})) \neq \emptyset \) for all \( 1 \leq i \leq 7 \). By (i) - (iv), \( S \) has at least \( 2 + 28 + 1 + 7 = 38 \) vertices implying that \( v_8(H) = 38 \).

We next let \( I \) be a smallest maximal irredundant set of \( H \). Hence, every vertex in \( I \) has a private neighbor with respect to \( I \). If \( I \cap (\{a_1, a_2\} \cup V(K_8')) = \emptyset \), then \( a_1 \) is not a private neighbor of any vertex in \( I \). Thus, for a vertex \( h \in V(K_8') \), \( a_1 \) is a private neighbor of \( h \) with respect to \( I \cup \{h\} \). Therefore, \( I \cup \{h\} \) is an irredundant set of \( H \) containing \( I \) contradicting maximality of \( I \). So, \( I \cap (\{a_1, a_2\} \cup V(K_8')) \neq \emptyset \). Similarly, \( I \cap (\{a_3, a_4\} \cup V(K_8'')) \neq \emptyset \) and \( I \cap (\{x_2, x_3, x_6\} \cup V(K_8^{i, 0})) \neq \emptyset \) for all \( j \in \{1, 5\}, j' \in \{4, 8\} \) and \( 1 \leq i \leq 7 \). This implies that \( ir(H) = |I| \geq 1 + 1 + 28 = 30 \). On the other hand, we observe that \( \{a_2, a_3\} \cup (\cup_{i=1}^{7}\{x_1^i, x_4^i, x_5^i, x_8^i\}) \) is an irredundant set of \( G \). By minimality of \( ir(H) \), we have \( ir(H) \leq 30 \) implying that \( ir(H) = 30 \). We see that, in the graph \( H = H(8, 6, 7) \), we have \( k = 8, \Delta = 9, s = 6 \) and \( l = 7 \). Thus, \( v_8(H) = 38 = 30 + \left\lceil \frac{30(9-2)}{2(8)} \right\rceil - 6 + 1 = ir(H) + \left\lceil \frac{ir(H)(\Delta-2)}{2k} \right\rceil - s + 1 \) satisfying the bound of Lemma 2 when \( 1 \leq s \leq \Delta - 2 \).