On A Sturm-Liouville Type Problem with Retarded Argument which Contains a Spectral Parameter in the Boundary Condition

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Abstract
The aim of this study is to investigate a discontinuous problem of the Sturm-Liouville type with a retarded argument containing a spectral parameter in the boundary condition and two additional conditions of transmission at the two discontinuity points. Also, eigenparameter dependent boundary conditions and obtains asymptotic formulas for the eigenvalues and eigenfunctions. And the spectral parameter is real. And the real valued function is continuous on that interval and it has a finite limit. The goal of this article is to obtain asymptotic formulas for eigenvalues of eigenfunctions for problem of the form:

\[ z'' + \rho(x)z(x - \Delta(x)) + \lambda z(x) = 0 \]  

on \( \mathbb{R}, \) with boundary conditions

\[ \alpha_1 z(0) + \alpha_2 \dot{z}(0) = 0 \]
\[ \lambda z(\pi) + \dot{z}(\pi) = 0 \]

And transmission conditions

\[ z(a - 0) - \beta_1 z(a + 0) = 0 \]
\[ \dot{z}(a - 0) - \beta_2 \dot{z}(a + 0) = 0 \]

The real valued function \( \rho(x) \) is continuous in \( [0,a) \cup (a,\pi], \) and has a finite limit, \( \rho(a \mp 0) = \lim_{x \to a \pm 0} \rho(x), \) the real valued function \( \Delta(x) \geq 0 \) is continuous in \( [0,a) \cup (a,\pi] \) and has a finite limit, \( \Delta(a \mp 0) = \lim_{x \to a \pm 0} \Delta(x), x - \Delta(x) \geq 0, \) if \( x \in [0,a); x - \Delta(x) \geq a, \) if \( x \in (a,\pi]; \lambda \) is a real spectral parameter; \( \alpha_1, \alpha_2, \beta_1 \) and \( \beta_2 \) are arbitrary real numbers, and not zero.

This article is devoted to studying the properties of the eigenvalues and eigenfunctions of the boundary value problem (1-5). Boundary value problems for differential equations of the second order with retarded argument are studied earlier by other researchers [2-4], and various physical applications of such problems also addressed by another group [1]. Similarly, another group obtained the asymptotic formulas for the eigenvalues and eigenfunctions of a discontinuous boundary value problem with retarded argument which contains a spectral parameter in the boundary conditions change.

Keyword
Differential Equation with Retarded Argument, Transmission Conditions, Asymptotic of Eigenvalues and Eigenfunctions

1. Introduction
We consider the boundary value problem for the differential equation:

\[ z'' + \rho(x)z(x - \Delta(x)) + \lambda z(x) = 0 \]  

On \( [0,a) \cup (a,\pi], \) with boundary conditions

\[ \alpha_1 z(0) + \alpha_2 \dot{z}(0) = 0 \]  

\[ \lambda z(\pi) + \dot{z}(\pi) = 0 \]  

And transmission conditions

\[ z(a - 0) - \beta_1 z(a + 0) = 0 \]  

\[ \dot{z}(a - 0) - \beta_2 \dot{z}(a + 0) = 0 \]
appears in both differential equation and boundary conditions.

Let \( m_1(x, \lambda) \) be a solution of equation (1) on \([0, a]\), satisfying the initial conditions
\[
\begin{align*}
m_1(0, \lambda) &= \alpha_2, \quad m_1(0, \lambda) = -\alpha_1 \quad (6)
\end{align*}
\]

The condition (6) defines a unique solution of equation (1) on \([0, a]\) ([1] p.12).

After defining the above solution, we shall define the solution \( m_2(x, \lambda) \) of equation (1) on \([a, \pi]\), by means of the solution \( m_1(x, \lambda) \) by the initial conditions
\[
\begin{align*}
m_2(a, \lambda) &= \beta_1^{-1} m_1(a, \lambda) \\
m_2'(a, \lambda) &= \beta_1^{-2} m_1'(a, \lambda) \quad (7)
\end{align*}
\]

The conditions (7) are defined as a unique solution of equation (1) on \([a, \pi]\).

As a result, the function \( m(x, \lambda) \) is defined on \([0, a] \cup (a, \pi]\) by the function
\[
m(x, \lambda) = \begin{cases} m_1(x, \lambda) & x \in [0, a) \\
m_2(x, \lambda) & x \in (a, \pi]\end{cases}
\]
is such a solution of equation (1) on \([0, a] \cup (a, \pi]\), which satisfies one of the boundary conditions and both transmission conditions.

**Lemma 1.** Let \( m(x, \lambda) \) be a solution of equation (1) and \( \lambda > 0 \), then the following integral equations hold
\[
\begin{align*}
m_1(x, \lambda) &= \alpha_2 \cos tx - \frac{\alpha_1}{t} \sin tx - \frac{1}{\pi} \int_0^\pi \frac{\rho(t) \sin (t(x - \tau))}{t - \sqrt{\lambda}} m_1(r - \Delta(t, \lambda)) d\tau, \quad (t = \sqrt{\lambda}, \lambda > 0) \quad (8)
\end{align*}
\]
\[
\begin{align*}
m_2(x, \lambda) &= \frac{1}{\beta_1} m_1(a, \lambda) \cos(t(x - a)) + \frac{m_1(a, \lambda)}{\beta_1} \sin(t(x - a)) - \frac{1}{\pi} \int_a^\pi \rho(t) \sin(t(x - \tau)) m_2(r - \Delta(t, \lambda)) d\tau, \quad (t = \sqrt{\lambda}, \lambda > 0) \quad (9)
\end{align*}
\]

**Proof.** To prove this, it is enough to substitute

\(-t^2 m_1(r, \lambda) - m_1(r, \lambda)\) and \(-t^2 m_2(r, \lambda) - m_2(r, \lambda)\) instead of \(-\rho(t)m_1(r - \Delta(t, \lambda))\) and \(-\rho(t)m_2(r - \Delta(t, \lambda))\) in the integrals in (8) and (9) respectively by parts twice.

**Theorem 1.** We have only simple eigenvalues of the problem (1)-(5).

**Proof.** Let \( \lambda \) be an eigenvalue of the problem (1) - (5) and
\[
\tilde{\psi}(x, \lambda) = \begin{cases} \tilde{\psi}_1(x, \lambda) & x \in [0, a) \\
\tilde{\psi}_2(x, \lambda) & x \in (a, \pi]\end{cases}
\]
be a corresponding eigenfunction. Then from (2) and (6), it follows that the determinant
\[
W[\tilde{\psi}_1(0, \lambda), m_1(0, \lambda)] = \begin{vmatrix} \tilde{\psi}_1(0, \lambda) & \alpha_2 \\ \tilde{\psi}_1'(0, \lambda) & -\alpha_1 \end{vmatrix} = 0
\]

And by Theorem 2.2.2 in [1], the functions \( \tilde{\psi}_1(x, \lambda) \), and \( m_1(x, \lambda) \) are linearly dependent on \([0, a]\).

We can also prove that the functions \( \tilde{\psi}_2(x, \lambda) \), and \( m_2(x, \lambda) \) are linearly dependent on \([a, \pi]\).

Hence
\[
v_1(x, \lambda) = k_1 m_1(x, \lambda) \quad (i=1, 2) \quad (10)
\]

For some \( k_1 \neq 0 \) and \( k_2 \neq 0 \), we must show that \( k_1 = k_2 \).

Suppose that \( k_1 \neq k_2 \) from the equality (4) and (10) we have
\[
\begin{align*}
v_1(a - 0, \lambda) - \beta_1 v_1(a + 0, \lambda) &= \nu_1(a, \lambda) - \beta_1 \nu_2(a, \lambda) \\
k_1 m_1(a, \lambda) - \beta_1 k_2 m_2(a, \lambda) &= k_1 m_1(a, \lambda) - \beta_1 k_2 m_2(a, \lambda)
\end{align*}
\]

Since \( \beta_1 (k_1 - k_2) \neq 0 \) it follows that
\[
m_2(\lambda) = 0 \quad (11)
\]

By the same technique from equality (5), we can show that
\[
m_2(a, \lambda) = 0 \quad (12)
\]

From the fact that \( m_2(x, \lambda) \) is a solution of the differential equation (1) on \([a, \pi]\) and satisfies the initial conditions (11) and (12) it follows that \( m_1(x, \lambda) = 0 \) identically on \([a, \pi]\). ([1, p.12 Theorem 1.2.1]).

By using this we may also find
\[
m_1(a, \lambda) = m_1(a, \lambda) = 0
\]

From the latter discussions of \( m_2(x, \lambda) \) it follows that \( m_1(x, \lambda) = 0 \) identically on \([0, a] \cup (a, \pi]\). But this contradicts (6).

**2. An Existence Theorem**

The function \( m(x, \lambda) \) defined in section 1 is a non-trivial solution of equation 1 satisfying conditions (2), (4) and (5) putting \( m(x, \lambda) \) to (3) we get the characteristic equation.

\[
G(\lambda) \equiv m^*(\pi, \lambda) + \lambda m(\pi, \lambda) = 0 \quad (13)
\]

By theorem (1) the set of eigenvalues of the boundary value problem (1) - (5) coincides with the set of real roots of equation (13).

Let \( \rho_1 = \int_0^\pi |\rho(t)| d\tau \) and \( \rho_2 = \int_\pi^a |\rho(t)| d\tau \).

**Lemma 2.**

1) Let \( \lambda \geq 4\rho_1^2 \), then for the solution \( m_1(x, \lambda) \) of equation (8), the following inequality holds
\[
|m_1(x, \lambda)| \leq \frac{1}{\rho_1} \sqrt{4\rho_1^2 + \alpha_1^2} \quad x \in [0, a] \quad (14)
\]

2) Let \( \lambda \geq \max \{4\rho_1^2, 4\rho_2^2\} \), then for the solution \( m_1(x, \lambda) \) of equation (9) the following inequality holds
\[
|m_2(x, \lambda)| \leq \frac{4}{\rho_1\rho_2} \sqrt{4\rho_1^2 + \alpha_1^2} \quad x \in [a, \pi] \quad (15)
\]

**Proof.** Let \( A_{12} = \max \{|m_1(x, \lambda)|\} \). Then from (8), it follows that for every \( \lambda > 0 \), the following inequality holds
\[
A_{12} \leq \frac{\sqrt{\alpha_2^2 + \alpha_1^2}}{\rho_1^2 + \frac{1}{\rho_1^2}} + A_{12} \rho_1
\]

If \( \tau \geq 2\rho_1 \) we get (14). Differentiating (8) with respect to
x we have

\[ m_1'(x, \lambda) = - \alpha_2 \sin tx - \alpha_1 \cos tx - \int_0^x \rho(t) \cos t(x - t) \, dt \]

From (16) and (14), it follows that for \( t \geq 2 \rho_1 \), the following inequality holds.

\[ |m_1'(x, \lambda)| \leq \sqrt{t^2 \alpha_2^2 + \alpha_1^2} + \sqrt{4 \rho_1^2 \alpha_2^2 + \alpha_1^2} \]

Hence

\[ |m_1'(x, \lambda)| \leq \frac{1}{\rho_1} \sqrt{4 \rho_1^2 \alpha_2^2 + \alpha_1^2} \]  

Let \( A_{21} = \max \{|m_2(x, \lambda)|\} \), Then from (9), (14) and (17) it follows that for \( t \geq 2 \rho_1 \), the following inequality holds

\[ A_{21} \leq \frac{1}{\rho_1} \sqrt{4 \rho_1^2 \alpha_2^2 + \alpha_1^2} + \frac{1}{\rho_2} A_{22} \]

Hence if \( \lambda \geq \max \{4 \rho_1^2, 4 \rho_2^2\} \) we get (15).

**Theorem 2.** We have an infinite set of positive eigenvalues of the problem (1) – (5).

**Proof.** Differentiating (9) with respect to \( x \), we get

\[ m_2'(x, \lambda) = \frac{1}{\rho_1} m_1(a, \lambda) \sin t(x - a) + \frac{1}{\rho_2} m_2(a, \lambda) \cos t(x - a) - \int_a^x \rho(t) \cos t(x - t) m_2(t - \Delta \tau, \lambda) \, dt \]  

From (8), (9), (13), (16) and (18) we get

\[ \frac{1}{\rho_1} \left( \alpha_2 \cos ta - \alpha_1 \sin ta - \int_a^x \rho(t) \sin t(a - t) m_1(t - \Delta \tau, \lambda) \, dt \right) \sin (\pi - a) + \frac{1}{\rho_2} (- \alpha_2 \sin ta - \alpha_1 \cos ta - \int_a^x \rho(t) \cos t(a - t) m_1(t - \Delta \tau, \lambda) \, dt) \sin t(\pi - a) - \int_a^x \rho(t) \cos t(\pi - t) m_2(t - \Delta \tau, \lambda) \, dt + \lambda \frac{1}{\rho_1} [\alpha_2 \cos ta - \alpha_1 \sin ta - \int_a^x \rho(t) \sin t(a - t) m_1(t - \Delta \tau, \lambda) \, dt] \sin t(\pi - a) - \int_a^x \rho(t) \cos t(\pi - t) m_2(t - \Delta \tau, \lambda) \, dt] = 0 \]  

There are two possible cases:

1) \( \alpha_2 \neq 0 \)

2) \( \alpha_2 = 0 \)

Without delay, we need to consider case 1 only while other cases may be considered analogically. Let \( \lambda \) be sufficiently large, then by (14) and (15) equation (19) may be written in the form of

\[ t \sin \tau + O(1) = 0 \]  

Obviously, for large \( s \) equation (20) has an infinite set of roots, then the theorem is proved.

### 3. Asymptotic Formulas for Eigenvalues and Eigenfunctions

In this section, asymptotic properties of eigenvalues and eigenfunctions are displayed in which we assumed that \( s \) is sufficiently large.

From (8) and (14), we get:

\[ m_1(x, \lambda) = O(1) \quad \text{on} \quad [0,a] \]  

From (9) and (15), we get

\[ m_2(x, \lambda) = O(1) \quad \text{on} \quad [a,\pi] \]  

The existence and continuity of the derivatives \( m_1'(x, \lambda) \) for \( 0 \leq x \leq a \), \( |\lambda| < \infty \) and \( m_2'(x, \lambda) \) for \( a \leq x \leq \pi \), \( |\lambda| < \infty \), follow from Theorem 1.4.1 in [1].

\[ m_1'(x, \lambda) = O(1), \quad x \in [0,a], \quad \text{and} \quad m_2'(x, \lambda) = O(1), \quad x \in [a,\pi] \]  

**Theorem 3.** For each sufficiently large \( N \), there is exactly a unique eigenvalue of the problem (1) – (5) near \( N^2 \).

**Proof.** We consider the expression which is denoted by \( O(1) \) in equation (20).

In case, if formulas (21– 23) are concerned, it can be shown by differentiation with respect to \( t \) that for large \( t \), this expression has bounded derivative. It is obvious that for large \( t \) the roots of equation (20) are situated close to entire numbers. Therefore, we need to indicate that for large \( n \), only one root (20) lies near to each \( N \).

We consider the function \( \theta(t) = t \sin \tau + O(1) \).

It’s derivative which has the form \( \frac{d}{dt} \theta(t) = \sin \tau + t \cos \tau \sin \tau + O(1) \), does not vanish for \( t \) close to \( N \) for sufficiently large \( N \).

Thus our assertion follows by Rolle’s Theorem.

Let \( N \) be sufficiently large. What follows we shall denote by \( \lambda_n = \frac{t_n^2}{N} \) the eigenvalue of the problem (1) – (5) situated near \( N^2 \).

We set \( t_N = N + \delta_N \). Then from (20), it follows that \( \delta_N = O\left(\frac{1}{N}\right) \). Consequently

\[ t_N = N + O\left(\frac{1}{N}\right) \]  

The formula (24) makes it possible to obtain asymptotic expressions for the eigenfunction of the problem (1 - 5).

From (8), (16) and (21) we get

\[ m_1(x, \lambda) = \alpha_2 \cos tx + O\left(\frac{1}{t}\right) \]  

\[ m_1'(x, \lambda) = - \tau \alpha_2 \sin tx + O(1) \]  

From (9), (22) and (26) we get

\[ m_2(x, \lambda) = \frac{\alpha_2 \cos tx + O\left(\frac{1}{t}\right)}{\rho_1} \]  

By putting (24) in (25) and (27) we derive that

\[ v_{N} = m_1(x, \lambda_n) = \alpha_2 \cos (N\tau) + O\left(\frac{1}{N}\right) \]
\[ v_{2N} = m_2(x, \lambda_n) = \frac{\alpha_2}{\beta_1} \cos (N x) + O \left( \frac{1}{N} \right) \]

Hence the eigenfunctions \( v_N(x) \) have the following asymptotic representation:

\[
\begin{align*}
    v_N(x) &= \begin{cases} 
        \frac{\alpha_2}{\beta_1} \cos (N x) + O \left( \frac{1}{N} \right) & x \in (0, a] \\
        \frac{\alpha_2}{\beta_1} \cos (N x) + O \left( \frac{1}{N} \right) & x \in (a, \pi]
    \end{cases}
\end{align*}
\]

In some circumstances, more exact asymptotic formulas that depend on the retardation may be considered.

Let us assume that the following conditions are fulfilled:

\( a \) The derivatives \( \rho'(x) \) and \( \Delta'(x) \) exist and are bounded in \((0, a] \cup (a, \pi]\) and have finite limits.

\[ \rho(a \mp 0) = \lim_{x \to a \mp 0} \rho(x) \quad \text{and} \quad \Delta(a \mp 0) = \lim_{x \to a \mp 0} \Delta(x) \]

\( b \) \( \Delta(x) < 1 \) in \((0, a] \cup (a, \pi]\), \( \Delta(0) = 0 \) and \( \lim_{x \to a \mp 0} \Delta(x) = 0 \).

By using \( b \), we have

\[ x - \Delta(x) \geq 0 \] for \( x \in (0, a] \) and \( x - \Delta(x) \geq a \) \( x \in (a, \pi] \)

\[ (28) \]

From \((25), (27)\) and \((28)\) we have

\[ m_1(\tau - \Delta(\tau), \lambda) = \frac{\alpha_2}{\beta_1} \cos (\tau - \Delta(\tau)) + \frac{\alpha_2}{\beta_1} \cos \left( \frac{\alpha_2}{\beta_1} \right) + O \left( \frac{1}{\tau} \right) \quad (29) \]

\[ m_2(\tau - \Delta(\tau), \lambda) = \frac{\alpha_2}{\beta_1} \cos (\tau - \Delta(\tau)) + \frac{\alpha_2}{\beta_1} \cos \left( \frac{\alpha_2}{\beta_1} \right) + O \left( \frac{1}{\tau} \right) \quad (30) \]

Under conditions \( a \) and \( b \) the following two formulas can be proved by the same technique in \([1] \) Lemma 3.3.3

\[ \int_0^\pi \rho(\tau) \cos (2\tau - \Delta(\tau)) d\tau = O \left( \frac{1}{\tau} \right) \quad (31) \]

\[ \int_0^\pi \rho(\tau) \sin (2\tau - \Delta(\tau)) d\tau = O \left( \frac{1}{\tau} \right) \]

From \((29), (30)\) and \((31)\) after long operations we have

\[ \frac{\alpha_1}{\beta_1} \cos \left( \frac{\alpha_2}{\beta_1} \right) - \frac{\alpha_2}{\beta_1} \cos \left( \frac{\alpha_2}{\beta_1} \right) \int_0^\pi \rho(\tau) \cos (\Delta(\tau)) d\tau + \frac{\alpha_2}{\beta_1} \cos \left( \frac{\alpha_2}{\beta_1} \right) \int_0^\pi \rho(\tau) \sin (\Delta(\tau)) d\tau + O \left( \frac{1}{\tau} \right) = 0 \]

Again, if we take \( t_N = N + \delta_N \), for sufficiently large \( N \), we obtain

\[ \delta_N = \frac{1}{N \pi} \left( \frac{\alpha_1}{\alpha_2} - 1 - \frac{1}{\pi} \int_0^\pi \rho(\tau) \cos (N\Delta(\tau)) d\tau \right) + O \left( \frac{1}{N^2} \right) \]

And finally

\[ t_N = N + \frac{1}{N \pi} \left( \frac{\alpha_1}{\alpha_2} - 1 - \frac{1}{\pi} \int_0^\pi \rho(\tau) \cos (N\Delta(\tau)) d\tau \right) + O \left( \frac{1}{N^2} \right) \]

\[ (32) \]

Thus we have proved the following theorem.

\textbf{Theorem 4}. If conditions \( a \) and \( b \) are satisfied, then the eigenvalues \( \lambda_n = t_N^2 \) of the problem \((1)-(5)\) have the equation \((32)\) asymptotic formula for \( N \rightarrow \infty \).

\textbf{Proof}. Now we obtain a sharper asymptotic formula for the eigenfunctions.

From \((8)\) and \((29)\)

\[ m_1(x, \lambda) = \frac{\alpha_2}{\beta_1} \cos \left( \frac{\alpha_2}{\beta_1} x \right) \cos (x - \tau) \cos (\tau - \Delta(\tau)) d\tau + O \left( \frac{1}{\tau^2} \right) \quad (33) \]

Thus replacing \( \beta \) by \( \beta_N \) we have

\[ v_{1N}(x) = \frac{\alpha_2}{\beta_1} \left\{ \cos (x - \tau) \cos (\tau - \Delta(\tau)) d\tau + O \left( \frac{1}{\tau^2} \right) \right\} \quad (34) \]

From \((9), (30), (31)\) and replacing \( \beta \) by \( \beta_N \) we have

\[ v_{2N}(x) = \frac{\alpha_2}{\beta_1} \left\{ \cos (x - \tau) \cos (\tau - \Delta(\tau)) d\tau + O \left( \frac{1}{\tau^2} \right) \right\} \]

\[ (35) \]

\[ \sin N \pi \left( \frac{\alpha_1}{\alpha_2} - 1 - \frac{1}{\tau} \int_0^\pi \rho(\tau) \cos (N\Delta(\tau)) d\tau \right) \]

\[ + \frac{1}{2} \int_0^\pi \rho(\tau) \cos (N\Delta(\tau)) d\tau \] + \[ O \left( \frac{1}{\tau^2} \right) \]

\[ (36) \]

\[ 4. \textbf{Conclusions} \]

In this current study, initially we obtained asymptotic formulas for eigenvalues and eigenfunctions for discontinuous Sturm-Liouville type problem with retarded argument which contains a spectral parameter in the boundary condition. Later on, under additional conditions \( a \) and \( b \) obtained more exact asymptotic formulas, which depend upon the retardation.

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