Some Examples of $m$-Isometries

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Abstract. We obtain the admissible sets on the unit circle to be the spectrum of a strict $m$-isometry on an $n$-finite dimensional Hilbert space. This property gives a better picture of the spectrum of an $m$-isometry. We determine that the only $m$-isometries on $\mathbb{R}^2$ are 3-isometries and isometries giving by $I + Q$, where $Q$ is a nilpotent operator. Moreover, on real Hilbert space, we obtain that $m$-isometries preserve volumes. Also, we present a way to construct a strict $(m+1)$-isometry with a given $m$-isometry, using ideas of Aleman and Suciu (Integr Equ Oper Theory 85:259–287, 2016, Proposition 5.2) on infinite dimensional Hilbert space.

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1. Introduction

Let $H$ be a Hilbert space. Denote by $L(H)$ the algebra of bounded linear operators on $H$. For $T \in L(H)$ we consider the adjoint operator $T^* \in L(H)$, which is the unique map that satisfies

$$\langle x, Ty \rangle = \langle T^* x, y \rangle,$$

for every $x, y \in H$. Given $T \in L(H)$, denote by $Ker(T)$ and $R(T)$, the kernel and range of $T$, respectively. For a positive integer $m$, an $m$-isometry is an operator $T \in L(H)$ which satisfies the condition

$$(yx - 1)^m(T) := \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} T^k T^* T^k = 0; \quad (1.1)$$

equivalently

$$\sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \|T^k x\|^2 = 0, \quad (1.2)$$
for every \( x \in H \). A strict \( m \)-isometry, with \( m > 1 \), is an \( m \)-isometry which is not an \((m - 1)\)-isometry. This class of operators was introduced by Agler in [2] and was studied by Agler and Stankus in [4–6].

Let \( n \) be a positive integer. Recall that \( Q \in L(H) \) is an \( n \)-nilpotent if \( Q^n = 0 \) and \( Q^{n-1} \neq 0 \).

A notion related with \( m \)-isometries is the following: An operator \( T \in L(H) \) is an isometric \( n \)-Jordan if there exist an isometry \( A \in L(H) \) and an \( n \)-nilpotent \( Q \in L(H) \) such that \( T = A + Q \) with \( AQ = QA \).

**Theorem 1.1** [13, Theorem 2.2]. Any isometric \( n \)-Jordan operator is a strict \((2n - 1)\)-isometry.

Actually, a much stronger result is true. Indeed in [15, Theorem 3], it is obtained a generalization of Theorem 1.1 for \( m \)-isometries: if \( T \) is an \( m \)-isometry, \( Q \) is an \( n \)-nilpotent operator and they commute, then \( T + Q \) is a \((2n + m - 2)\)-isometry. See also [25,28]. Moreover, in [15], it was proved that this property, in general, is false in Banach space context.

Another generalization of isometry was obtained in [13, Proposition 2.6] for sub-isometry \( n \)-Jordan operator. Recall that \( T \) is a sub-isometry \( n \)-Jordan operator if \( T \) is the restriction of an isometry \( n \)-Jordan operator \( J \) to an invariant subspace of \( J \).

Notice that Theorem 1.1 gives an easy way to construct examples of \( m \)-isometries, for an odd \( m \). It is sufficient to choose the identity operator as the isometry and any \( n \)-nilpotent operator with \( n = \frac{m+1}{2} \).

At a first glance, we could think that all the \( m \)-isometries come from isometric \( n \)-Jordan operators. However, this is not true, since there are strict \( m \)-isometries for even \( m \), see [8, Proposition 9]. What can we say about \( m \)-isometries with odd \( m \)? Recently, Yarmahmoodi and Hedayatian have proven that the only isometric \( n \)-Jordan weighted shift operators on \( \ell^2(\mathbb{N}) \) are isometries [30, Theorem 1]. However, there are \( m \)-isometries that are not isometric \( n \)-Jordan, since Athavale in [8] gave examples of strict \( m \)-isometries with the weighted shift operator for all integers \( m \).

Some authors have given examples of \( m \)-isometries, for example with the unilateral or bilateral weighted shift [1,12,14,18] and with the composition operator [14,16,23]. Other ways to construct examples of \( m \)-isometries is developing different tools like tensor product [19], functional calculus [24], on Hilbert-Schmidt class [17] and with \( C_0 \)-semigroups [10,21,29].

The purpose of this paper is to make a clear picture of \( m \)-isometries on finite dimensional Hilbert spaces. In Sect. 2, we begin with the study of \( m \)-isometries on \( \mathbb{R}^2 \) and on \( \mathbb{R}^n \), with \( n \geq 3 \). We give all the 3-isometries on \( \mathbb{R}^2 \). Also, we obtain the expression of \( m \)-isometries and study how this class of operators change volumes on \( \mathbb{R}^n \). Moreover, we study the case of complex Hilbert space, where we show the admissible sets on the unit circle to be the spectrum of an \( m \)-isometry. In Sect. 3, we reproduce similar ideas of Aleman and Suciu [7, Proposition 5.2] to define a 3-isometry using a given 2-isometry. In fact, we obtain a way to construct a strict \((m + 1)\)-isometry using a weaker condition than strict \( m \)-isometry.

In particular, we will answer the following problems:
Problem 1.2. Let $T \in L(H)$ with finite dimensional Hilbert space $H$ and $m$ be an odd integer. Are all strict $m$-isometries of the form $\lambda I + Q$, where $Q$ is a nilpotent operator and $\lambda$ is a complex number with modulus 1?

Problem 1.3. Let $T \in L(\mathbb{R}^n)$. How does an $m$-isometry $T$ change volumes?

Problem 1.4. Let $H$ be any finite dimensional Hilbert space and let $T$ be an $m$-isometry with odd $m$. What can we say about the spectrum?

2. $m$-Isometries on Finite Dimensional Hilbert Space

Recall some important properties of the spectrum of an $m$-isometry.

Denote $\overline{D}$ and $\partial D$ the closed unit disk and the unit circle, respectively.

**Lemma 2.1.** Let $m$ be a positive integer, $H$ be a Hilbert space and $T \in L(H)$ be an $m$-isometry. Then

1. [4, Lemma 1.21] $\sigma(T) = \overline{D}$ or $\sigma(T) \subseteq \partial D$.
2. [3, Lemma 19] The eigenvectors of $T$ corresponding to distinct eigenvalues are orthogonal.

**Remark 2.2.** (1) Notice that any $m$-isometry on a finite dimensional space is bijective.
(2) It is well known that if $Q$ is $k$-nilpotent on an $n$-dimensional vector space, then $k \leq n$.

Denote $I_m(H) := \{ T \in L(H) : T \text{ is an } m\text{-isometry} \}$.

The following theorem gives a nice picture of $m$-isometries on finite dimensional Hilbert space.

**Theorem 2.3.** ([13, Theorem 2.7], [3, page 134]) Let $H$ be an $n$-finite dimensional Hilbert space and $T \in L(H)$. Then

1. $T$ is a strict $m$-isometry if and only if $T$ is an isometric $k$-Jordan operator, where $m = 2k - 1$ with $k \leq n$.
2. $I_1(H) = I_2(H) \subsetneq I_3(H) = I_4(H) \subsetneq \cdots \subsetneq I_{2n-1}(H) = I_j(H)$ for all $j \geq 2n - 1$.

**Proof.** We include the proofs for completeness.

(1) Assume that $T$ is a strict $m$-isometry on $H$. Then the spectrum of $T$, $\sigma(T) = \{ \lambda_1, \lambda_2, \ldots, \lambda_s \}$, where $\lambda_i$ are eigenvalues of modulus 1, since the spectrum of $T$ must be in the unit circle and $m$ is odd [4, Lemma 1.21 & Proposition 1.23]. By part (2) of Lemma 2.1, the spectral subspaces of $T$, $H_i := \text{Ker}(T - \lambda_i)^{n_i}$ are mutually orthogonal and

$$T \cong T_{|H_1} \oplus \cdots \oplus T_{|H_s},$$

where $n_1, \ldots, n_s$ are positive integers such that $\text{Ker}(T - \lambda_i)^{n_i} = \text{Ker}(T - \lambda_i)^N$ for all $N \geq n_i$. Moreover, for all $j \in \{1, \ldots, s\}$, we have that $\sigma(T_{|H_j}) = \{ \lambda_j \}$ and $T_{|H_j}$ is of the form $\lambda_j + Q_j$ for some nilpotent operator $Q_j$. So,
\( T = A + Q \) for some isometry, in fact unitary diagonal operator \( A \) and some nilpotent operator \( Q \) such that \( AQ = QA \).

The converse is consequence of Theorem 1.1.

(2) Let us prove that \( I_{2\ell-1}(H) = I_{2\ell}(H) \) for all \( \ell \in \mathbb{N} \). Recall that if \( T \) is \((2\ell)\)-isometry, then \( T \) is bijective and so \( T \) is \((2\ell - 1)\)-isometry \([4, \text{Proposition 1.23}]\). Moreover, the highest degree of nilpotent operator on \( n \)-dimensional Hilbert space is \( n \). The result is a consequence of Theorem 1.1. \( \Box \)

2.1. \( m \)-Isometries on Real Hilbert Spaces

Next, we study the \( m \)-isometries on \( \mathbb{R}^n \).

Based on the above results, we obtain all \( m \)-isometries on \( \mathbb{R}^2 \).

**Theorem 2.4.** If \( T \in L(\mathbb{R}^2) \) is a strict \( m \)-isometry, then \( m = 1 \) or \( m = 3 \) and \( T = A + Q \), where \( A \) is an isometry and \( Q \) is a nilpotent operator of order 2 that commutes.

Recall that the isometries on \( \mathbb{R}^2 \) are given by

\[
R_\theta := \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \quad \text{and} \quad S_\theta := \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix},
\]

where

1. \( R_\theta \) is a rotation (about 0) and its determinant \( \det(R_\theta) \) is 1 and
2. \( S_\theta \) is a symmetry with respect to the straight line of equation \( x_2 = \tan(\theta/2)x_1 \) and \( \det(S_\theta) = -1 \).

And the non-zero nilpotent operators on \( \mathbb{R}^2 \) are \( \lambda M, \lambda N \) and \( \lambda Q_k \) where

\[
M := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad N := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad Q_k := \begin{pmatrix} 1 & k \\ 0 & -1 \end{pmatrix},
\]

(2.1)

with \( k \neq 0 \) and \( \lambda \in \mathbb{C}\setminus\{0\} \).

We are interested in studying isometries that commute with nilpotent operators on \( \mathbb{R}^2 \).

**Lemma 2.5.** The unique isometries in \( L(\mathbb{R}^2) \) that commute with a non-zero nilpotent operator are the trivial cases, that is, \( \pm I \).

**Proof.** Simple calculations prove that

\[
R_\theta M = MR_\theta \iff R_\theta N = NR_\theta \iff R_\theta Q_k = Q_k R_\theta \iff \sin \theta = 0 \iff \theta = 0 \text{ or } \theta = \pi.
\]

That is, the unique isometries of type \( R_\theta \) which commute with some non-zero nilpotent (hence with all nilpotent operator) are \( R_0 = I \) and \( R_\pi = -I \).

Analogously, we have that

\[
S_\theta M = MS_\theta \iff S_\theta N = NS_\theta \iff S_\theta Q_k = Q_k S_\theta \iff \sin \theta = \cos \theta = 0,
\]

which it is impossible. Hence there are not isometries \( S_\theta \) which commute with some non-zero nilpotent operator. \( \Box \)

Taking into account Theorem 2.3 we give the unique strict 3-isometries on \( \mathbb{R}^2 \). Indeed, we answer Problem 1.2 for \( n = 2 \) in the following result:
Theorem 2.6. The strict 3-isometries on $\mathbb{R}^2$ are of the form $\pm I + Q$, where $Q$ is a non-zero nilpotent operator given in (2.1).

Proof. It is immediate by Theorem 2.4 and Lemma 2.5. \hfill \Box

Let $T \in L(\mathbb{R}^n)$ with $n \geq 3$ and let us consider the following $n$ conditions:

$(M_k)$ \hspace{1cm} $S_k(Tx_1, Tx_2, \ldots, Tx_k) = S_k(x_1, x_2, \ldots, x_k)$

for all $x_1, x_2, \ldots, x_k \in \mathbb{R}^n$ and $k = 1, 2, \ldots, n$, where $S_k(x_1, x_2, \ldots, x_k)$ denotes the $k$-dimensional measure of the set

$$\left\{ \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_k x_k \ : \ 0 \leq \lambda_i \leq 1, \ \text{for} \ i = 1, 2, \ldots, k \right\}.$$

Lemma 2.7. Let $T \in L(\mathbb{R}^n)$. Then

(1) [26, Teorema II] $T$ satisfies the conditions $(M_1)$, $(M_2)$, $\ldots$, $(M_{n-1})$ if and only if $T$ is an isometry.

(2) [20] The condition $(M_n)$ is equivalent to $\det(T) = \pm 1$.

An easy application of Theorem 1.1 gives that, for example in $\mathbb{R}^3$, we have strict 3-isometries giving by $\pm I + Q$, where $Q$ is a 2-nilpotent operator, and strict 5-isometries giving by $\pm I + Q$, where $Q$ is a 3-nilpotent operator.

The next result gives answer to Problems 1.2 and 1.3, for $n \geq 3$, where $n$ is the dimension of the Hilbert space.

Theorem 2.8. Let $n \geq 3$. Then the following properties follow:

(1) There are non-trivial strict $m$-isometries on $L(\mathbb{R}^n)$ for any odd $m$ less than $2n - 1$, that is, there exists an isometry different from $\pm I$ such that commutes with a non-zero $k$-nilpotent operator with $k \in \{1, 2, \ldots, n-1\}$.

(2) The $m$-isometries preserve volumes.

Proof. (1) Define

$$A(x_1, x_2, \ldots, x_n) := (-x_1, x_2, \ldots, x_n)$$

$$Q_j(x_1, x_2, \ldots, x_n) := (0, x_3, x_4, \ldots, x_{j+1}, 0, \ldots, 0).$$

Then $A$ is an isometry and $Q_j$ is a $j$-nilpotent operator such that

$$AQ_j(x_1, x_2, \ldots, x_n) = Q_j A(x_1, x_2, \ldots, x_n) = (0, x_3, x_4, \ldots, x_{j+1}, 0, \ldots, 0),$$

for all $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. By Theorem 1.1, we get that $A + Q_j$ is a non-trivial strict $(2j - 1)$-isometry for $j = 1, \ldots, n - 1$.

(2) By Lemma 2.7, it will be enough to prove that $\det(A + Q) = \pm 1$ for all isometries $A$ that commute with a nilpotent operator $Q$. Since $AQ = QA$, then $\sigma(A + Q) = \sigma(A)$ by [31, Proposition 1.1]. According to the spectrum of an isometry on a finite dimensional space, we have that the spectrum of $A$ is a closed subset of the unit circle. By [9, page 150], the determinant of $T$ is the product of the eigenvalues of $T$, counting multiplicity. Hence $\det(T) = \pm 1$. \hfill \Box

The converse of part (2) of Theorem 2.8 is not true, as proves the following example:
Example 2.9. Let \( T := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} \). Then \( \det(T) = 1 \) and \( T \) is not a 3-isometry, since

\[
\| T^3 x \|^2 - 3 \| T^2 x \|^2 + 3 \| T x \|^2 - \| x \|^2 \neq 0,
\]

for \( x := (1, 1, 0) \).

2.2. \( m \)-Isometries on Complex Hilbert Spaces

We recall the following results about the spectrum of \( m \)-isometries:

Lemma 2.10 [13, Theorem 4.4]. Let \( H \) be an infinite dimensional Hilbert space.

1. If \( K \) is any compact subset of \( \partial \mathbb{D} \), then there exists a strict \( m \)-isometry for any odd number \( m \) such that \( \sigma(T) = K \).
2. If \( K \) is the closed unit disk, then there exists a strict \( m \)-isometry for any integer number \( m \) such that \( \sigma(T) = K \).

The main aim of this section is to solve Problem 1.4.

Let \( T \in L(\mathbb{C}^n) \) be an \( m \)-isometry. It is clear that \( \sigma(T) \subseteq \partial \mathbb{D} \) by part (1) of Lemma 2.1 and \( \sigma(T) \) has at most \( n \) different eigenvalues. Indeed if \( K := \{ \lambda_1, \ldots, \lambda_n \} \) with \( \lambda_i \) different complex numbers on the unit circle, then it is possible to define an isometry \( T \) such that \( \sigma(T) = K \). In particular, the following operator:

\[
T(x_1, \ldots, x_n) := (\lambda_1 x_1, \ldots, \lambda_n x_n)
\]

is an isometry on \( \mathbb{C}^n \) with \( \sigma(T) = \{ \lambda_1, \ldots, \lambda_n \} \).

In the following theorem we prove that any \( m \)-isometry with \( m \geq 3 \) on \( \mathbb{C}^n \) cannot have \( n \) different eigenvalues.

Theorem 2.11. Any strict \((2k - 1)\)-isometry on \( \mathbb{C}^n \) with \( 2 \leq k \leq n \) has at most \( n - 1 \) distinct eigenvalues.

Proof. Assume that \( T \in L(\mathbb{C}^n) \) is a strict \((2k - 1)\)-isometry with \( \sigma(T) = \{ \lambda_1, \ldots, \lambda_n \} \) where \( \lambda_1, \ldots, \lambda_n \) are different eigenvalues of \( T \). Then \( T \) could be written as \( T = PSP^{-1} \), for some \( P \in L(\mathbb{C}^n) \) where

\[
S := \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{pmatrix}
\]

and \( |\lambda_i| = 1 \) for \( i \in \{1, \ldots, n\} \), by part (1) of Lemma 2.1. Since \( T \) is a strict \((2k - 1)\)-isometry, by part (2) of Lemma 2.1, the operator \( P \) is a unitary operator. This means that \( T \) is unitarily equivalent to \( S \); therefore, \( T \) is a unitary operator, which is a contradiction. \( \square \)

Theorem 2.12. The strict \((2k - 1)\)-isometries on \( \mathbb{C}^n \), with \( 2 \leq k \leq n \) are of the form \((\lambda_1 I_{n_1} \oplus \cdots \oplus \lambda_\ell I_{n_\ell}) + Q\), with \( \ell \in \{1, \ldots, n - k + 1\} \), where \( Q \) is a \( k \)-nilpotent, \( |\lambda_j| = 1 \) for all \( j \in \{1, \ldots, \ell\} \) and \( n_1 + \cdots + n_\ell = n \).
Proof. Suppose that $T$ is a strict $(2k - 1)$-isometry. By Theorem 2.3, we have that $T = U + Q$, where $U$ is a unitary operator and $Q$ is a $k$-nilpotent operator such that $UQ = QU$.

Assume, by contradiction, that $T$ has at least $n - k + 2$ distinct eigenvalues. That means

$$\sigma(T) = \{\lambda_1, \ldots, \lambda_r\}, \text{ with } r \geq n - k + 2.$$  

Then $\mathbb{C}^n = H_{\lambda_1} \oplus \cdots \oplus H_{\lambda_r}$, where $H_{\lambda_i} := Ker(T - \lambda_i I)^{n_i}$ and $n_i$ is the order of multiplicity of the eigenvalue $\lambda_i$. Denote $T_{|H_i}$ the restriction operator of $T$ to $H_i$, for $1 \leq i \leq r$. Then $T_{|H_i} = \lambda_i I_{n_i} + Q_i$, where $Q_i$ is a $h_i$-nilpotent with $1 \leq h_i \leq n_i$. By part (2) of Lemma 2.1, we conclude that $T$ could be written as

$$T = (\lambda_1 I_{n_1} \oplus \cdots \oplus \lambda_r I_{n_r}) + (Q_1 \oplus \cdots \oplus Q_r),$$

where $Q_1 \oplus \cdots \oplus Q_r$ is a $k_{0}$-nilpotent, with $k_{0} := \max_{i=1,\ldots,r}\{h_i\}$ and $k_{0} < k$. Then we get a contradiction. □

Corollary 2.13. If $T \in L(\mathbb{C}^n)$ is a strict $(2k - 1)$-isometry, with $2 \leq k \leq n$, then $\sigma(T) \subseteq \{\lambda_1, \ldots, \lambda_{n-k+1}\} \subseteq \partial \mathbb{D}$.

Corollary 2.14. Any $(2n - 1)$-isometry on $\mathbb{C}^n$ is of the form $\lambda I + Q$, where $Q$ is an $n$-nilpotent operator and $\lambda \in \partial \mathbb{D}$. In particular the spectrum is a single point on the unit circle.

3. Construction of an $(m + 1)$-Isometry from an $m$-Isometry

In this section we present a method to construct a Hilbert space $H_k$ and an $(m + 1)$-isometry on $H_k$ from an $m$-isometry $T^k$ on a Hilbert space $H$ for some integer $k$. Our result is based on the construction given by Aleman and Suciu in [7, Proposition 5.2] for $m = 2$ and $k = 1$.

Henceforth $H$ will denote an infinite dimensional Hilbert space.

Given $S \in L(H)$, $x \in H$ and an integer $\ell \geq 1$, it is defined as follows:

$$\beta_\ell(S, x) := \frac{1}{\ell!} \sum_{j=0}^{\ell} (-1)^{\ell-j} \binom{\ell}{j} \|S^j x\|^2.$$  

Note that $S$ is an $m$-isometry if and only if $\beta_m(S, x) = 0$ for all vector $x \in H$.

Consider $\mathbb{C}[z]$ the space of all complex polynomials. Given $p \in \mathbb{C}[z]$, we write

$$p(z) = \sum_{n \geq 0} p_n z^n$$

and define $Lp \in \mathbb{C}[z]$ in the following way:

$$Lp(z) := \sum_{n \geq 1} p_n z^{n-1} = \frac{p(z) - p(0)}{z}.$$  

We have that $\mathbb{C}[z]$ is an inner product space with the norm $\|\cdot\|_2$ given by

$$\|p\|^2_2 := \sum_{n \geq 0} |p_n|^2.$$
Also if we consider a new norm on $\mathbb{C}[z]$ defined by
\[ \|p\|_k^2 := \|p\|_2^2 + \sum_{n \geq 0} \|(L^{nk}p)(T)x_0\|^2 , \]
it is obtained that $\mathbb{C}[z]$ is an inner product space with $\| \cdot \|_k$. Denote $H_k$ its completion with the new norm.

The following combinatorial result will be useful:

**Lemma 3.1** [22, Eq. 0.151 (4)]. If $m$ is any positive integer, then
\[ \sum_{k=0}^{m} (-1)^k \binom{n}{k} = (-1)^m \binom{n-1}{m} , \]
for any integer $n \geq m + 1$.

Recall that the class of $m$-isometries is stable under powers. However, the converse is not true. See [11,27].

**Theorem 3.2.** Let $T \in L(H)$ such that $T^k$ is a strict $m$-isometry on $R(T^k)$, for some $k$, and $x_0 \in H \setminus \{0\}$ such that $\beta_{m-1}(T^k, T^k x_0) \neq 0$.

(1) For every $p \in \mathbb{C}[z]$ and $j \in \mathbb{N}$,
\[ \| M^{kj}_z p \|_k^2 = \|p\|_k^2 + \sum_{i=1}^{j} \| T^{ki} p(T)x_0 \|^2 , \]
where $M_z$ denotes the multiplication operator defined by $M_z p := z p$.

(2) For every $p \in \mathbb{C}[z]$ and $\ell \geq 1$,
\[ \beta_{\ell+1}(M_z^k, p) = \frac{\ell!}{(\ell + 1)!} \beta_{\ell}(T^k, T^k p(T)x_0) . \] (3.1)

(3) The extension of $M_z^k$ to $H_k$ is an $(m + 1)$-isometry.

**Proof.** (1) Let $p$ be any polynomial in $\mathbb{C}[z]$ and $j \in \mathbb{N}$. Then we will prove that
\[ \| M^{kj}_z p \|_k^2 = \|p\|_k^2 + \sum_{i=1}^{j} \| T^{ki} p(T)x_0 \|^2 , \] (3.2)
by induction. For $j = 1$ we need to prove that
\[ \| M_z^k p \|_k^2 = \|p\|_k^2 + \| T^k p(T)x_0 \|^2 , \] (3.3)
for any polynomial $p$.

Let $p(z) := \sum_{n \geq 0} p_n z^n$. Then
\[ \| M_z^k p \|_k^2 = \|z^k p\|_k^2 = \|z^k p\|_2^2 + \sum_{n \geq 0} \|(L^{nk}z^k p)(T)x_0\|^2 \]
\[ = \|p\|_2^2 + \|(z^k p)(T)x_0\|^2 + \sum_{n \geq 1} \|(L^{nk}z^k p)(T)x_0\|^2 \]
\[ = \|p\|_2^2 + \| T^k p(T)x_0 \|^2 + \sum_{n \geq 0} \|(L^{nk}p)(T)x_0\|^2 \]
\[ = \|p\|_k^2 + \| T^k p(T)x_0 \|^2 . \]
Then (3.3) holds.

Suppose that (3.2) is true for \( j \). Let us prove it for \( j + 1 \). Then

\[
||M_z^{(j+1)}p||_k^2 = ||M_z^j(M_z^k p)||_k^2 = ||M_z^k p||_k^2 + \sum_{i=1}^{j} ||T^{ki}(M_z^k p)(T)x_0||^2
\]

\[
= ||z^k p||_k^2 + \sum_{i=1}^{j} ||T^{ki}T^k p(T)x_0||^2
\]

\[
= ||p||_2^2 + \sum_{n \geq 0} ||(L^{nk} z^k p)(T)x_0||^2 + \sum_{i=1}^{j} ||T^{ki+1}(p(T)x_0)||^2
\]

\[
= ||p||_2^2 + ||T^k p(T)x_0||^2 + \sum_{n \geq 0} ||(L^{nk} p)(T)x_0||^2 + \sum_{i=2}^{j+1} ||T^{ki} p(T)x_0||^2
\]

\[
= ||p||_2^2 + \sum_{i=1}^{j+1} ||T^{ki} p(T)x_0||^2.
\]

So we prove (3.2).

(2) For \( \ell \in \mathbb{N} \), we have

\[
\beta_{\ell+1}(M_z^k, p) = \frac{1}{(\ell+1)!} \sum_{j=0}^{\ell+1} (-1)^{\ell+1-j} \binom{\ell+1}{j} ||M_z^j p||_k^2
\]

\[
= \frac{1}{(\ell+1)!} \left( (-1)^{\ell+1} ||p||_k^2 + \sum_{j=1}^{\ell+1} (-1)^{\ell+1-j} \binom{\ell+1}{j} ||M_z^j p||_k^2 \right)
\]

\[
= \frac{1}{(\ell+1)!} \left( (-1)^{\ell+1} ||p||_k^2 + \sum_{j=1}^{\ell+1} (-1)^{\ell+1-j} \binom{\ell+1}{j} \left( ||p||_k^2 + \sum_{i=1}^{j} ||T^{ki} p(T)x_0||^2 \right) \right)
\]

\[
= \frac{1}{(\ell+1)!} \sum_{j=1}^{\ell+1} (-1)^{\ell+1-j} \binom{\ell+1}{j} \sum_{i=1}^{j} ||T^{ki} p(T)x_0||^2
\]

\[
= \frac{1}{(\ell+1)!} \sum_{j=1}^{\ell+1} ||T^{k+1} p(T)x_0||^2 \sum_{i=j}^{\ell+1} (-1)^{\ell+1-i} \binom{\ell+1}{i},
\]

where \( p \) is any polynomial in \( \mathbb{C}[z] \).

Using Lemma 3.1, in the last sum, we have that

\[
\sum_{i=j}^{\ell+1} (-1)^{\ell+1-i} \binom{\ell+1}{i} = -\sum_{i=0}^{j-1} (-1)^{\ell+1-j} \binom{\ell+1}{j} = (-1)^{\ell+j-1} \binom{\ell}{j-1}.
\]

So,

\[
\beta_{\ell+1}(M_z^k, p) = \frac{1}{(\ell+1)!} \sum_{j=1}^{\ell+1} ||T^{k+1} p(T)x_0||^2 (-1)^{\ell+j-1} \binom{\ell}{j-1}
\]

\[
= \frac{1}{(\ell+1)!} \sum_{j=0}^{\ell} (-1)^{\ell-j} \binom{\ell}{j} ||T^{k+1} p(T)T^k x_0||^2
\]

\[
= \frac{\ell!}{(\ell+1)!} \beta_\ell(T^k, T^k p(T)x_0).
\]

So, (3.1) is proved.
It is enough to prove that $\beta_{m+1}(M^k_z, p) = 0$ for all $p \in \mathbb{C}[z]$. This is a consequence of (3.1), since $T^k$ is an $m$-isometry on $R(T^k)$. □

**Corollary 3.3** [7, Proposition 5.2]. Let $T$ be a 2-isometry on a Hilbert space $H$. Fix $x_0 \in H\setminus\{0\}$ and let $H_1$ be the completion of the space of analytic polynomials with respect to the norm

$$\|p\|^2_1 := \|p\|^2_2 + \sum_{n \geq 0} \|(L^n p)(T)x_0\|^2.$$ 

Then the multiplication operator by the independent variable $M_z p := zp$ extends to a 3-isometry on $H_1$.

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**References**

[1] Abdullah, B., Le, T.: The structure of $m$-isometric weighted shift operators. Oper. Matrices 10(2), 319–334 (2016)
[2] Agler, J.: A disconjugacy theorem for Toeplitz operators. Am. J. Math. 112, 1–14 (1990)
[3] Agler, J., Helton, W., Stankus, M.: Classification of hereditary matrices. Linear Algebra Appl. 274, 125–160 (1998)
[4] Agler, J., Stankus, M.: $m$-isometric transformations of Hilbert space. I. Integr. Equ. Oper. Theory 21(4), 383–429 (1995)
[5] Agler, J., Stankus, M.: $m$-isometric transformations of Hilbert space. II. Integr. Equ. Oper. Theory 23(1), 1–48 (1995)
[6] Agler, J., Stankus, M.: $m$-isometric transformations of Hilbert space. III. Integr. Equ. Oper. Theory 24(4), 379–421 (1996)
[7] Aleman, A., Suciu, L.: On ergodic operator means in Banach spaces. Integr. Equ. Oper. Theory 85, 259–287 (2016)
[8] Athavale, A.: Some operator theoretic calculus for positive definite kernels. Proc. Am. Math. Soc. 112(3), 701–708 (1991)
[9] Axler, S.: Down with determinants!. Am. Math. Mon. 102(2), 139–154 (1995)
[10] Bermúdez, T., Bonilla, A., Zaway, H.: $C_0$-semigroups of $m$-isometries on Hilbert spaces. J. Math. Anal. Appl. 472(2), 879–893 (2019)
[11] Bermúdez, T., Mendoza, C Díaz, Martinón, A.: Powers of $m$-isometries. Stud. Math. 208(3), 249–255 (2012)
[12] Bermúdez, T., Martinón, A., Negrín, E.: Weighted shift operators which are $m$-isometries. Integr. Equ. Oper. Theory 68(3), 301–312 (2010)
[13] Bermúdez, T., Martinón, A., Noda, J.: An isometry plus a nilpotent operator is an $m$-isometry. Applications. J. Math. Anal. Appl. 407(2), 505–512 (2013)

[14] Bermúdez, T., Martinón, A., Noda, J.: Weighted shift and composition operators on $\ell_p$ which are $(m, q)$-isometries. Linear Algebra Appl. 505, 152–173 (2016)

[15] Bermúdez, T., Martinón, A., Müller, V., Noda, J.: Perturbation of $m$-isometries by nilpotent operators. Abstr. Appl. Anal. 745479, 1–6 (2014)

[16] Botelho, F.: On the existence of $n$-isometries on $\ell_p$ spaces. Acta Sci. Math. (Szeged) 76, 183–192 (2010)

[17] Botelho, F., Jamison, J.E., Zheng, B.: Strict isometries of arbitrary orders. Linear Algebra Appl. 436(9), 3303–3314 (2012)

[18] Chô, M., Öta, S., Tanahashi, K.: Invertible weighted shift operators which are $m$-isometries. Proc. Am. Math. Soc 141(12), 4241–4247 (2013)

[19] Duggal, B.P.: Tensor product of $n$-isometries II. (English summary). Funct. Approx. Comput 4(1), 27–32 (2012)

[20] Fleming, W.H.: Undergraduate Texts in Mathematics. Functions of several variables, 2nd edn. Springer, New York (1977)

[21] Gallardo-Gutiérrez, E.A., Partington, J.R.: $C_0$-semigroups of 2-isometries and Dirichlet spaces. Rev. Mate. Iberoam. 34(3), 1415–1425 (2018)

[22] Gradshteyn, I.S., Ryzhik, I.M.: Table of Integrals, Series, and Products. Academic Press, New York (1980)

[23] Gu, C.: High order isometric composition operators on $\ell_p$ spaces and infinite graphs with polynomial growth (2019) (preprint)

[24] Gu, C.: Functional calculus for $m$-isometries and related operators on Hilbert spaces and Banach spaces. Acta Sci. Math. (Szeged) 81(3–4), 605–641 (2015)

[25] Gu, C., Stankus, M.: Some results on higher order isometries and symmetries: products and sums with a nilpotent operator. Linear Algebra Appl. 469, 500–509 (2015)

[26] Guivernau, A.: Transformaciones que conservan el área. Gaceta Mate. 5–6, 63–67 (1980)

[27] Jablonski, Z.J.: Complete hyperexpansivity, subnormality and inverted boundedness conditions. Integr. Equ. Oper. Theory 44(3), 316–336 (2002)

[28] Le, T.: Algebraic properties of operator roots of polynomials. J. Math. Anal. Appl. 421(2), 1238–1246 (2015)

[29] Rydhe, E.: An Agler-type model theorem for $C_0$-semigroups of Hilbert space contractions. J. Lond. Math. Soc. 93(2), 420–438 (2016)

[30] Yarmahmoodi, S., Hedayatian, K.: Isometric $N$-Jordan weighted shift operators. Turk. J. Math. https://doi.org/10.3906/mat-1507-54

[31] Yarmahmoodi, S., Hedayatian, K., Yousefi, B.: Supercyclicity and hypercyclicity of an isometry plus a nilpotent (English summary). Abstr. Appl. Anal. 686832, 1–11 (2011)
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