Quantum Properties of the Electron Field in Kerr–Newman Black Hole Manifolds

F. Belgiorno

Dipartimento di Fisica, Università di Milano, 20133 Milano, Italy, and
I.N.F.N., Sezione di Milano, Italy

M. Martellini

Dipartimento di Fisica, Università di Milano, 20133 Milano, Italy
I.N.F.N., Sezione di Milano, Italy, and
Landau Network at “Centro Volta”, Como, Italy

(March 24, 2022)

Abstract

We study some spectral features of the one–particle electron Hamiltonian obtained by separating the Dirac equation in a Kerr–Newman black hole background. We find that the essential spectrum includes the whole real line. As a consequence, there is no gap in the spectrum and discrete eigenvalues are not allowed for any value of the black hole charge \( Q \) and angular momentum \( J \). Our spectral analysis will be also related to the dissipation of the black hole angular momentum and charge.

PACS: 04.62.+v, 02.30.Tb, 04.70.Dy
Keywords: Dirac equation, Quantum field theory in Curved Spacetime

*E-mail: belgiorno@mi.infn.it
†E-mail: maurizio.martellini@mi.infn.it


I. INTRODUCTION

In this letter we show that in a Kerr–Newman black hole background the reduced one-particle Hamiltonian for the electron field, obtained from the Dirac equation by separation of variables [1–4], is characterized by a peculiar spectral feature: Its essential spectrum is given by $\mathbb{R}$. As a consequence, the absence of gap in the spectrum and of discrete eigenvalues is deduced. The above results hold for any value of $Q, J$. We will study the general Kerr–Newman case. The Reissner–Nordström case $J = 0, Q \neq 0$ has been studied in [5] and here we extend the results of [5] to the Schwarzschild case $J = 0, Q = 0$.

Some qualitative considerations about the black hole loss of charge and/or angular momentum are also allowed by our spectral analysis. As it is well known, the presence of the Klein paradox [7,8] has been related to the quantum dissipation of the black hole charge and angular momentum [9–12]. The conditions for the existence of the Klein paradox are studied. A further discussion is found in the conclusions.

II. ONE–PARTICLE ELECTRON HAMILTONIAN IN KERR–NEWMAN BLACK HOLE MANIFOLDS

The metric of a Kerr–Newman black hole of mass $M$, angular momentum $J$ and charge $Q$ in Boyer–Lindquist coordinates $(t, r, \theta, \phi)$ is [13,14]

$$ds^2 = -\left(\frac{\Delta - a^2 \sin^2 \theta}{\Sigma}\right) dt^2 - \frac{2a \sin^2 \theta (r^2 + a^2 - \Delta)}{\Sigma} dtd\phi + \left[\frac{(r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta}{\Sigma}\right] \sin^2 \theta d\phi^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2$$

where $a = J/M$, $\Sigma \equiv r^2 + a^2 \cos^2 \theta$ and $\Delta \equiv r^2 + a^2 + Q^2 - 2Mr = (r - r_+)(r - r_-)$; $r_+ \geq r_- > 0$ correspond to the event horizon radius and to the Cauchy horizon radius respectively. We will consider only the external region $r \in (r_+, \infty)$. The electromagnetic vector potential is $A_\mu = (Qr)/\Sigma(-1, 0, 0, a \sin^2 \theta)$. The radial part of the separated Dirac equation for a charged field [1–4] in the case of the electron field (mass $m_e$, charge $-e$) is given by

$$\left(D_0 + \frac{i(-eQr)}{\Delta}\right)R_-(r) = (\lambda + im_e r)R_+(r)$$

$$(\Delta D^\dagger_{\frac{3}{2}} + i(-eQr))R_+(r) = (\lambda - im_e r)R_-(r)$$

where $D_0 = \partial_r + iK/\Delta$; $D^\dagger_{\frac{3}{2}} = \partial_r - iK/\Delta + (r - M)/\Delta$; $K = (r^2 + a^2)\omega + am$. $\omega$ appearing in $K$ represents the one–particle energy of the solution of the separated Dirac equation, $\lambda$ is the eigenvalue for the $\theta$-dependent part of the spinor and $m$ is the semi–integer azimuthal quantum number. A rather straightforward algebraic manipulation of the above equations,

\footnote{Some of our results for Reissner–Nordström represent a rigorous implementation of the ones found in [6], which are based on a WKB approximation.}
obtained by posing \( R_- = F + iG, \) \( R_+ = (F - iG)/\sqrt{\Delta} \) (cf. 3), allows to get the following system:

\[
\begin{align*}
\frac{dF}{dr} &= +\frac{\lambda}{\sqrt{\Delta}} F + \left( \frac{K}{\Delta} + \frac{m_e r}{\sqrt{\Delta}} + \frac{-eQr}{\Delta} \right) G \\
\frac{dG}{dr} &= -\frac{\lambda}{\sqrt{\Delta}} G + \left( -\frac{K}{\Delta} + \frac{m_e r}{\sqrt{\Delta}} - \frac{eQr}{\Delta} \right) F.
\end{align*}
\]

We define, by means of the change of variable \( dx/dr = \delta/\Delta, \) where \( \delta \equiv r^2 + a^2, \) the tortoise–like coordinate \( x; \) in the non extremal case one gets \( x = r + (r_+^2 + a^2)/(r_+ - r_-) \cdot \log((r - r_+)/r_+) - (r_-^2 + a^2)/(r_+ - r_-) \cdot \log((r - r_-)/r_-) \) and in the extremal one \( x = r + 2r_+ \cdot \log((r - r_+)/r_+) - (r_+^2 + a^2)/(r - r_+). \) In both cases, \( x \in (-\infty, +\infty). \) Then, it is easy to show that the above system is equivalent to the eigenvalue problem

\[
H \vec{W} = \omega \vec{W}
\]

where \( \vec{W} \equiv (F, G)^T; \) \( H \) is the one–particle (reduced) Hamiltonian in matrix form

\[
H = H_0 + V(r(x))
\]

with

\[
H_0 = \begin{bmatrix}
0 & -\partial_x \\
\partial_x & 0
\end{bmatrix}
\]

and

\[
V(r(x)) = \begin{bmatrix}
-am + m_e r \sqrt{\Delta} - eQr \\
-\lambda \sqrt{\Delta} & -am - m_e r \sqrt{\Delta} - eQr 
\end{bmatrix}.
\]

The Hamiltonian \( (2), \) which is the one–particle energy operator calculated by separation of variables, in the following will be also called partial wave operator. We define \( \Omega \equiv a/\delta; \) \( \Phi \equiv (Qr/\delta); \) these quantities, valued at the event horizon, become the black hole angular velocity \( \Omega_h \) and the black hole electric potential \( \Phi_h \) respectively. Then the potential can be rewritten as

\[
V(r(x)) = \begin{bmatrix}
-m\Omega - e\Phi + m_e r \sqrt{\Delta} \\
-\lambda \sqrt{\Delta} & -m\Omega - e\Phi - m_e r \sqrt{\Delta}
\end{bmatrix}.
\]

The properties of the operator \( H \) can be analyzed in the framework of the so called Dirac systems of ordinary differential equations [15]. It is easy to show that \( H \) is formally self-adjoint in the Hilbert space \( L^2(\mathbb{R}, dx)^2 \) and that it is also essentially self-adjoint in \( C_0^\infty(\mathbb{R})^2. \) Indeed, the so called limit point case holds at \( -\infty \) and at \( +\infty \) (cf. corollary to the theorem 6.8 in [15]) and so its deficiency indices are \( (0, 0) \) (cf. theorem 5.7 of [15]). This means that there is no need to select boundary conditions for the electron one–particle Hamiltonian on the given manifold.
III. ESSENTIAL SPECTRUM

We show that the discrete spectrum of the reduced one–particle Hamiltonian is empty and that there is no gap in the spectrum, as a consequence of the fact that the essential spectrum \( \sigma_e \) coincides with \( \mathbb{R} \). The decomposition method \[15\] and theorems 16.5 and 16.6 of \[15\] allow us to find out the essential spectrum of the reduced Hamiltonian \( H \). An analogous study was carried out in the Reissner–Nordström case in \[5\], where the absence of gap and of the discrete spectrum was implicitly shown for Reissner-Nordström black holes. Here the same reasoning is applied to the general Kerr–Newman case. Physical consequences of our spectral analysis will be also considered and a second quantization formalism will be understood in the following section. Moreover, it is necessary to recall that the correct physical state for quantum field theory in a non extremal black hole background has to be selected according to suitable analyticity requirements for the metric and the fields on the extended manifold \[17\]. For the (eternal) Kerr–Newman non extremal case the one–particle Hamiltonian vacuum is not the correct physical Hartle–Hawking state; nevertheless, it is still related to the physical state: On the extended Schwarzschild background its “heating up” at the black hole temperature in the scalar field case gives rise to the Hartle–Hawking state \[18\]. Cf. also \[19,20\] for the scalar field case in the Kerr background. Extremal black holes cannot a priori be treated on the same foot as the non extremal ones, because of the lack of a geometric temperature. Cf. also \[21\]. The correct physical state is not known. We choose conservatively the Boulware (one–particle Hamiltonian) vacuum.

Before applying the mathematical tools cited above, it is useful to introduce the following constant shift in \( \omega \): \( \omega \rightarrow \omega - e\Phi_h - m\Omega_h \) in such a way that the eigenvalue problem \( (1) \) becomes an eigenvalue problem (with eigenvalue \( \omega \)) for the Hamiltonian \( (2) \) with the shifts \( e\Phi \rightarrow e(\Phi - \Phi_h); m\Omega \rightarrow m(\Omega - \Omega_h) \) in the potential. We will call the shifted potential again \( V \). Our choice is purely conventional. See also the discussion in the following section. According to the decomposition method, in order to locate the essential spectrum \( \sigma_e \) of \( H \), it is sufficient to locate the essential spectrum of the restrictions of \( H \) to suitable subintervals of \( \mathbb{R} \). Particularly, let \( H_- \) and \( H_+ \) be self–adjoint extensions of the reduced Hamiltonian restricted to the intervals \((-\infty,0]\) and \([0,\infty)\) respectively. Then it can be proved that \( \sigma_e(H) = \sigma_e(H_-) \cup \sigma_e(H_+) \) (cf. theorem 11.5 of \[13\]). In order to apply theorem 16.5 of \[15\], we determine the limits of the potential \( V \) for \( x \rightarrow +\infty \) and \( x \rightarrow -\infty \):

\[
\lim_{x \to +\infty} V(r(x)) \equiv V_+ = \begin{bmatrix} m_e + m\Omega_h + e\Phi_h & 0 \\ 0 & -m_e + m\Omega_h + e\Phi_h \end{bmatrix}
\]

and

\[
\lim_{x \to -\infty} V(r(x)) \equiv V_- = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.
\]

Indeed, theorem 16.5 of \[15\] states that, if the \( w_- \leq w_+ \) are the eigenvalues of \( V_+ \) (analogously for \( V_- \)), there is no essential spectrum contribution from the open interval \( (w_-, w_+) \).

\[2\] Cf. also \[16\], the theorems above correspond to Korollar 6.9 and Satz 6.10 respectively.
Then in the case of $H_+$ one gets $\sigma_e(H_+) \cap (-m_e + e\Phi_h + m\Omega_h, m_e + e\Phi_h + m\Omega_h) = \emptyset$. (For $H_-$ the following trivial result holds: $\sigma_e(H_-) \cap \emptyset = \emptyset$). The above result holds in the non extremal case as well as in the extremal one. Moreover, one can apply theorem 16.6 of [15] that will enable us to locate exactly the essential spectrum contribution arising from near $+\infty$ and from near the horizon. We first study $H_+$ near $x = +\infty$. According to theorem 16.6 of [15], if $|\cdot|$ stays for a norm in $C^2[0,\infty)$ and if for some $d \in (0, +\infty)$ the limit $\lim_{x \to +\infty} \frac{1}{x} \int_{a}^{y(x)} ds |V(r(s)) - V_+| = 0$ then the essential spectrum of $H_+$ contains the complement of the open interval $(w_-, w_+)$ (of course, the above limit is trivially zero if the integral is finite for $x \to +\infty$). It is easy to show that the above integrand near $+\infty$ behaves as $\frac{1}{r}$ and so $\lim_{x \to +\infty} \frac{1}{x} \int_{a}^{y(x)} ds |V(r(s)) - V_+| = 0$, as can be also verified by using L’Hospital’s rule. Then one can conclude that $\sigma_e(H_+) \supset (-\infty, -m_e + e\Phi_h + m\Omega_h] \cup [m_e + e\Phi_h + m\Omega_h, +\infty)$.

In the case of $H_-$, the study is analogous but it is necessary to distinguish the non extremal case and the extremal one. By defining the variable $y = -x$ one can apply theorem 16.6 of [15] by evaluating $\lim_{y \to +\infty} \frac{1}{y} \int_{a}^{y} dy |V(r(y)) - V_-| = \lim_{r \to r_+} \frac{1}{y(r)} \int_{r}^{y(r)} ds |V(r) - V_-|$. The integrand is singular as $r \to r_+$ and so there is an integrable singularity. Then the above limit is trivially 0.

In the extremal case the integrand behaves as $(r - r_+)^{-\frac{1}{2}}$ and the most singular contribution of the integral behaves as $\log(r - r_+)$. The factor $\frac{1}{y}$ behaves as $r - r_+$ in the limit $r \to r_+$ and so one gets $\lim_{r \to r_+} \frac{1}{y(r)} \int_{r}^{y(r)} ds |V(r) - V_-| = 0$. Then in both cases $\sigma_e(H_-) \supset \mathbb{R}$. All the above results allow us to conclude that

$$\sigma_e(H_+) = (-\infty, -m_e + e\Phi_h + m\Omega_h] \cup [m_e + e\Phi_h + m\Omega_h, +\infty)$$
$$\sigma_e(H_-) = \mathbb{R}.$$  

Then it is evident that $\sigma_e(H) = \mathbb{R} = \sigma(H)$ and so one has to conclude that the discrete spectrum is void and that there is no gap in the spectrum even if the Dirac field is massive.

### IV. QUALITATIVE PHYSICAL CONSEQUENCES

We will limit ourselves to qualitative considerations. For an explicit calculation of the effective action see

---

3The Klein condition in the scalar field case identifies the so called superradiant modes. But we recall that the superradiance phenomenon is not possible for Dirac fields.

4Our shift implies that the positive states $\omega > m_e + e\Phi_h + m\Omega_h$ and the negative states $\omega < -m_e + e\Phi_h + m\Omega_h$ at $+\infty$ have to be compared with the positive states $\omega > 0$ and the negative
We will start our discussion by summarizing the Reissner–Nordström case; then a study of the Kerr case will precede the analysis of the Kerr–Newman one.

1. \( J=0 \) cases

The case \( J=0, Q \neq 0 \) has been analyzed in \[5\]. Therein it is shown that for all the partial wave operators the same Klein condition \( m_e < e \Phi_h \) (for \( Q > 0 \)) holds. Then a positive charge flow towards \( +\infty \), signalling a discharge of the black hole, is possible. We stress that the Schwarzschild case \( J=0, Q=0 \) is a particular case of the above result. There is no possibility to find a Klein region but also in the Schwarzschild case one gets \( \sigma_e = R = \sigma \). The disappearance of the mass gap in the massive Klein–Gordon case in a Schwarzschild background is discussed in \[24\]. Cf. also \[25\].

2. Kerr case

For the sake of definiteness, we will assume \( \Omega_h > 0 \). We deduce that, if \( |m| \Omega_h < m_e \), there is no overlap of the asymptotic negative energy states at \( +\infty \) and the positive energy states at \( -\infty \); if \( |m| \Omega_h > m_e \), then a Klein region exists: for \( m > 0 \) one gets an overlap of the asymptotic negative energy states at \( +\infty \) and the positive energy states at \( -\infty \); if \( m < 0 \) then asymptotic positive states at \( +\infty \) overlap asymptotic negative states at \( -\infty \). It follows that the loss of angular momentum of an uncharged Kerr black hole in line of principle can take place also by mean of charged currents. Indeed, when \( m \Omega_h < -m_e \), there is a negative charge flow \( j_- \) towards \( +\infty \). In the case \( m \Omega_h > m_e \), there is a positive charge flow \( j_+ \) towards \( +\infty \). Globally \( j_+ + j_- = 0 \) so no net black hole charge can arise. Qualitatively one expects that to a lower value of \( |m| \) corresponds a bigger contribution to the effective action for the angular momentum loss of the black hole (cf. e.g. \[22\] for the massless scalar field case). For an estimate of \( |m| \) see the appendix.

We note that there are remarkable differences with respect to the Reissner–Nordström case. Indeed, in the Kerr case the Klein paradox exists whichever physical parameters one has, because the Klein condition depends on \( m \) and definitively there are partial wave operators implementing it and all but for a finite number of partial wave operators (that are symmetrically placed in \( m \) with respect to 0) are affected by a Klein overlap region. Moreover the Klein region size increases with \( |m| \), whereas in the Reissner–Nordström case it is the same for all the partial wave operators.

\[ \text{states } \omega < 0 \text{ at the horizon. If the shift is not implemented, then the positive states } \omega > m_e \text{ and the negative states } \omega < -m_e \text{ at } +\infty \text{ have to be compared with the positive states } \omega > -e \Phi_h - m \Omega_h \text{ and the negative states } \omega < -e \Phi_h - m \Omega_h \text{ at the horizon. In other words, the extremes of the gap in the spectrum at } +\infty \text{ “coalesce” into the single point } 0 \text{ at the horizon in the shifted potential case and into the point } -e \Phi_h - m \Omega_h \text{ at the horizon in the non shifted one. The Klein condition is the same in both cases.} \]
Here we choose $Q > 0$, $\Omega_h > 0$. A Klein region exists if $m < -(m_e + e\Phi_h)/\Omega_h \equiv n_-$; $m > +(m_e - e\Phi_h)/\Omega_h \equiv n_+$. The former condition means to get $m < n_- < 0$ and a negative charge flow $j_-$ towards $+\infty$; the latter implies $n_+ > 0$ if $m_e > e\Phi_h$ and $n_+ < 0$ if $m_e < e\Phi_h$ and a positive charge flow $j_+$ towards infinity. We note that $|n_+| > |n_-|$, so that more partial waves contribute to the positive charge flow leaving the black hole: The Klein condition is asymmetric in such a way as to couple the loss of angular momentum with a loss of charge and a net current leaving the black hole takes place. The asymmetry is stronger when $m_e < e\Phi_h$ (note that this is the discharge condition for a Reissner–Nordström black hole) because in this case $j_+$ gets contributions from the lowest values of $|m|$. For an explicit evaluation of $n_+, n_-$ see the appendix. Summarizing, the introduction of a positive charge of the black hole originates a net charge flux leaving the black hole, associated with a contextual loss of angular momentum: $j_+ + j_- > 0$.

V. CONCLUSIONS

The one–particle Hamiltonian for charged Dirac particles around a black hole of the Kerr–Newman family has been shown to be characterized by the absence of discrete eigenvalues and of gap in its spectrum. The peculiar properties of the essential spectrum contribution from near the event horizon represent the reason for such a behavior. As a consequence, we can note that, even in the case the Hawking effect and/or the Klein effect are strongly suppressed, no stationary quantum mechanical orbits around a black hole can exist. In particular, a charged black hole cannot form a sort of “exotic” atomic system having the charged black hole as its nucleus and around an electronic cloud, at least as far as the external field approximation holds. Such a possibility is instead left open e.g. in the case of a naked Reissner–Nordström singularity [26].

The conditions allowing the presence of the Klein paradox, which is related with the angular momentum and charge dissipation, have been also determined.

APPENDIX

It is interesting, for an evaluation of $n_+, n_-$, to restore the physical dimensions:

$$n_+ = \frac{m_e c^2}{\hbar c} \left( \frac{c}{\Omega_h r_+} \right) r_+(1 - \gamma \Phi_h^*)$$
$$n_- = -\frac{m_e c^2}{\hbar c} \left( \frac{c}{\Omega_h r_+} \right) r_+(1 + \gamma \Phi_h^*)$$

where we have defined the adimensional factors $\gamma \equiv e/(m_e \sqrt{G})$ and $\Phi_h^* \equiv (\sqrt{c} \Phi_h)/c^2$. In the case of the electron field one gets $\gamma \sim 10^{21}$ and $(m_e c^2)/\hbar c \simeq 2.59 \cdot 10^{12}$ (meters)$^{-1}$. The following inequalities can be useful: $\Phi_h^* < 1$; $\Omega_h r_+ < c/2$. Then $c/(\Omega_h r_+) \in (2, +\infty)$. Moreover, the introduction of the Planck length $l_{pl} = 1.6 \cdot 10^{-35}$ (meters) allows us to write $n_+ = +4.18 \cdot 10^{-23} \left( c/(\Omega_h r_+) \right) \cdot (r_+/l_{pl}) \cdot (1 - \gamma \Phi_h^*)$; $n_- = -4.18 \cdot 10^{-23} \left( c/(\Omega_h r_+) \right) \cdot (r_+/l_{pl}) \cdot (1 + \gamma \Phi_h^*)$. In the case of a Kerr black hole ($Q = 0$) one gets $n_+ \geq +8.36 \cdot 10^{-23} r_+/l_{pl}$;
$n_- \leq -8.36 \cdot 10^{-23} \frac{r_+}{l_{pt}}$ and it results that, even in the case of a very fast rotation, only for microscopic black holes with $r_+ \leq r_0 \sim 10^{-12}$ meters a low $|m|$ contribution to the loss of angular momentum by means of the massive electron field is possible. In the astrophysical case of an event horizon radius order of the Sun Schwarzschild radius one finds $|m| \geq |m_0| \sim 10^{16}$. For a slow rotation $|m|$ becomes obviously bigger. As far as the general Kerr–Newman case is concerned, we note that for $1 \gg \gamma \Phi^*_h$ the behavior is nearly the same as for a Kerr black hole so that an astrophysical black hole will involve in the dissipation of its charge and angular momentum only big values of $|m|$. If instead $1 \ll \gamma \Phi^*_h$, then all the small values of $|m|$ are involved in the discharge process however slow the rotation of the hole may be.
REFERENCES

[1] D. Page, Phys. Rev. D 14 (1976), 1509.
[2] B. Carter and R.G. McLenagahan, Generalized master equations for wave equation separation in a Kerr or Kerr-Newman black hole background, in: Proc. second Marcel Grossmann Meeting on General Relativity (Trieste, 1979), ed. R. Ruffini (North–Holland, Amsterdam, 1982).
[3] C.L. Pekeris, Phys. Rev. A 35 (1987), 14.
[4] E.G. Kalnins and W. Miller, Jr., J. Math. Phys. 33 (1991), 286.
[5] F. Belgiorno, Phys. Rev. D 58 (1998), 084017.
[6] M. Soffel, B. Müller and W. Greiner, J. Phys. A 10 (1977), 551.
   M. Soffel, B. Müller and W. Greiner, Phys. Rep. 85 (1982), 51.
[7] B. Thaller, The Dirac Equation; (Springer-Verlag, Berlin, 1992).
[8] W. Greiner, Müller and J. Rafelski, Quantum Electrodynamics of Strong Fields, with an introduction into modern relativistic quantum mechanics. (Springer-Verlag, Berlin, 1985).
[9] G.W. Gibbons, Commun. Math. Phys. 44 (1975), 245.
[10] T. Damour, Klein paradox and vacuum polarization, in: Proc. first Marcel Grossmann Meeting on General Relativity (Trieste, 1975), ed. R. Ruffini (North–Holland, Amsterdam, 1977) p. 459.
[11] N. Deruelle, Classical and quantum states in black hole physics, in: Proc. first Marcel Grossmann Meeting on General Relativity (Trieste, 1975), ed. R. Ruffini (North–Holland, Amsterdam, 1977) p. 483.
[12] T. Damour and R. Ruffini, Phys. Rev. Lett. 35 (1975), 463.
[13] R. Wald, General Relativity (Chicago University Press, Chicago, 1984).
[14] B. Carter, Phys. Rev. 174 (1968), 1559.
[15] J. Weidmann, Spectral Theory of Ordinary Differential Operators. Lecture Notes in Mathematics 1258 (Springer-Verlag, Berlin, 1987).
[16] J. Weidmann, Math. Z 119 (1971), 349.
[17] J.B. Hartle and S.W. Hawking, Phys. Rev. D 13 (1976), 2188.
   G.W. Gibbons and M.J. Perry, Proc. R. Soc. Lond. A 358 (1978), 467.
[18] B.S. Kay and R.M. Wald, Phys. Reports 207 (1991), 49.
[19] W. Israel, Phys. Lett. A 57 (1976), 107.
[20] V.P. Frolov and K.S. Thorne, Phys. Rev. D 39 (1989), 2125.
[21] L. Vanzo, Phys. Rev. D 55, (1997) 2192.
[22] B.S. DeWitt, Physics Reports 19 (1975), 295.
[23] W.A. Hiscock and L.D. Weems, Phys. Rev. D 41 (1990), 1142.
[24] B.S. Kay, Commun. Math. Phys. 100 (1985), 57.
[25] B.S. Kay, Mathematical aspects of quantum field theory in curved space–time, in: Mathematical Problems in Theoretical Physics, Proc. international conference on mathematical physics (Lausanne 1979). Lecture Notes in Physics 116 (Springer–Verlag, Berlin, 1980).
[26] F. Belgiorno and M. Martellini-in progress.