From a simple Lagrangian the equations of motion for a particle with spin are derived. The spin is shown to be conserved on the particle’s world-line. In the absence of a spin the equation coincides with that of a geodesic. The equations of motion are valid for massless particles as well, since mass does not enter the equations explicitly.

1. Introduction

According to the equivalence principle (EP) of the general theory of relativity (GR), motion of structureless test particles in a gravitational background is determined only by the spacetime geometry: particle worldlines are the geodesics of the spacetime. Things become more complicated for test particles which are not structureless and carry, for example, a non-vanishing charge or spin. In such cases, the worldline of a particle, in general, is no longer a geodesic, but is modified by electromagnetic and/or spin-gravity forces (for, e.g., [1, 2] and references therein). The problem of the motion of classical spinning particle in external fields has occupied scientists for nearly all of the last century. Covariant equations of motion for a relativistic particle in an electromagnetic field were first written more than 70 years ago by Frenkel [3]. For the case of a gravitational field, Mathisson [4] and Papapetrou [5] found the energy-momentum and angular momentum propagation equations for a rotating test body (“pole-dipole particle”) according to the Einstein’s GR. Tulczyjew [6], Beiglboeck [7] and Madore [8] developed these into laws of motion by adding a definition of a centre-of-mass world line. Later, Dixon [9] generalized these treatments and made them more rigorous. All theses works dealt with
the motion of extended rigid bodies or tops. It is far from obvious whether one can observe in practice the spin corrections to the equations of motion of elementary particles. However, the problem of influence of the spin on the trajectory of a particle in an external field is not of only theoretical interest. Spin-dependent corrections certainly exist in differential cross sections of scattering processes. It was proposed long ago to separate charged particles of different polarizations through the spin interaction with external fields in a storage ring of accelerator 10. Though this proposal is being discussed rather actively now (see a review in Ref. [11]) it is not yet clear whether it is feasible technically. The EP can be put to test in an astrophysical setting, a recent proposal being based on the analysis of the differential time delay between the arrival of left and right-handed circularly polarized (LCP and RCP) signals from the millisecond pulsar PSR 1937+214 12. However, far few papers exist on the theoretical foundations of possible deviations of the photon motion from geodesics (see, e.g., Ref. [12, 13]). Here we report yet another approach to this long-standing problem. If an appropriate Lagrangian density is taken into account, then the photon equation of motion (modified geodesic equation) can be found from the Euler-Lagrange equations.

2. Formalism

The generalized concept exists that classical particles follow the path of the least spacetime distance between the endpoints, even when space is curved by gravity. Thus, in (pseudo-) Riemannian spaces the geodesic equation is found from the variation of an action \( S \), identified with the parameter \( s \) of a curve interpreted as its length. The same method has been used to find the geodesic equation of light where, however, one technical problem arises: photon’s worldline has no length. One way to avoid this difficulty is to consider the motion of a massive vector particle and, if the obtained equation does not contain mass explicitly, simply put the mass to zero; with one further condition that the four velocity of a particle be a null vector. This method has been widely used for scalar particles and is known to give the equation of geodesic regardless of mass. In this work we apply this approach, which makes it possible to use the length parameter of the action principle.

Usually in order to describe the behavior of a field in a given gravitational background, one solves the corresponding field equations for a given metric. If the goal is to describe waves, one can take the corresponding wave solution. In a spherically symmetric spacetime the solutions contain factors expressed in terms of spherical harmonics. This method works well when the wavelength is comparable to the scales under consideration. It is not so in the case of light propagating in the vicinity of a massive object and, to describe the propagation of light as an electromagnetic field, one would have to employ the spherical harmonics of a very high order. The corresponding solution would look too complicated and tell little about the behavior of light.

Many efforts have been spent in the last decades to work out a simple approach
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to this problem that would give a satisfactory approximation to the wave as some curves that could be called “rays” and, at the same time, would take into account the polarization of light. In the present work we try to work out a simple approximation of this type. Our idea consists of the following: we consider a massive vector field obeying the Proca equation, describing the propagation of this field, in some restricted domain of the spacetime. The shape of the domain can be chosen as that of some world-tube transverse to the wave, with the cross-section comparable to the wavelength, or, say, not greater than two orders of magnitude. As this tube is timelike there must exist a timelike curve Λ in its interior, which specifies a local coordinate system as the time axis. And if the tube is not too wide, this coordinate system would cover the entire interior. If $s$ is the proper time on the curve Λ, the curve can be chosen in such a way that the field equation reduces to

$$\frac{D}{Ds} \frac{DA_i}{Ds} + m^2 A_i = 0,$$

the same way as it happens in standard Cartesian coordinates for Minkowskian spacetime. Correspondingly, the field Lagrangian is $- \dot{A}^2 + m^2 A^2$, where dot stands for the covariant derivative on $s$, if and only if the curve Λ is chosen properly. This Lagrangian should contain one more term responsible for the shape of Λ, yielding the geodesic equation when we switch off the field. The form of this term is well known: $1/2m \dot{x}^2$, thus our final Lagrangian is

$$2L = m\dot{x}^2 - \dot{A}^2 + m^2 A^2,$$

and coupling between the field and the shape of the curve Λ is incorporated in the form of a covariant derivative $\dot{A}$, which contains the product of connection, velocity $\dot{x}$ and the field.

The derivation of the conservation laws is more convenient in orthonormal frames. In what follows, $e_i^a$ will denote the components of an orthonormal 1-form frame field,

$$\theta^a = e_i^a(x)dx^i,$$

and $e^i_a$ the components of its dual vector frame field,

$$e_a = e_i^a(x)\frac{\partial}{\partial x^i}.$$  

Here frame indices are always $a, b, c, ...$; coordinate indices are $i, j, k, ...$. The metric tensor can be expressed as

$$g = g_{ij}dx^i \otimes dx^j = \delta_{ab}\theta^a \otimes \theta^b.$$  

The connection 1-form for these frames may be introduced through the first structure equation:

$$d\theta^a = \omega^a_b \wedge \theta^b, \quad \omega_{ab} + \omega_{ba} = 0$$  

(6)
and the connection coefficients $\gamma_{abc}$ are that of the expansion of this 1-form in the local frames $\{\theta^a\}$:

$$\omega^e_b = \gamma^e_{ab} \theta^a .$$

Thus,

$$\dot{A}^a = \frac{dA^a}{ds} + \gamma^a_{bc} A^c \frac{dx^b}{ds}$$

(8)
or $\dot{A}^a$ is a covariant derivative in orthonormal frame on a curve $x^i(s)$ with $\gamma_{bc}^a$ being a spin connection.

3. Equations of Motion

Each generalized coordinate has its conjugate generalized momentum:

$$p_a \equiv \frac{\partial L}{\partial \dot{x}^a} = m\eta_{ab}\dot{x}^b - \eta_{db} \dot{A}^b A^c \gamma^d_{ac} .$$

(9)

and

$$E_a \equiv \frac{\partial L}{\partial A^a} = -\dot{A}^b \eta_{ab} ,$$

(10)

Corresponding to them the Euler-Lagrange equations are

$$\frac{d}{ds} p_a = \frac{\partial L}{\partial x^a}$$

(11)

and

$$\frac{d}{ds} E_a = \frac{\partial L}{\partial A^a} .$$

(12)

Let us first consider equations for the $E_a$ (12). Right hand side of this equation is:

$$\frac{\partial L}{\partial A^a} = -\dot{A}^c \eta_{bc} \gamma^b_{ca} \dot{x}^c + m^2 A^c \eta_{ac} .$$

(13)

And Euler-Lagrange equations for $E_a$ will be:

$$\left( \frac{d}{ds} E_a - E_a \gamma^b_{ca} \dot{x}^c \right) - m^2 A^c \eta_{ac} = 0 ,$$

(14)

where we used Eq. 10. Expression in the brackets is a covariant derivative for $E_a$, thus, we obtain:

$$\dot{E}_a - m^2 A^c \eta_{ac} = 0 .$$

(15)

Again using Eq. 10, we finally obtain:

$$\ddot{A}^b \eta_{ab} + m^2 A^b \eta_{ab} = 0 ,$$

(16)

which, as we can see, reduces to Proca equation for the four-vector field $A_\mu(x)$. 

$$\left( \Box + m^2 \right) A_\mu = 0 .$$

(17)
We describe spin of the particle in this model directly by a tensor of spin $S_{ab}$:

$$mS_{ab} = \frac{1}{2} \left( \dot{A}_b A_a - \dot{A}_a A_b \right) .$$  \hspace{1cm} (18)

To obtain the equation of motion for spin, we write

$$m \frac{dS_{ab}}{ds} = \frac{1}{2} \frac{d}{ds} \left[ \dot{A}_a A_b - \dot{A}_b A_a \right] .$$ \hspace{1cm} (19)

Due to (17) all partial derivatives of $A$ vanish and we are left with:

$$m \frac{dS_{ab}}{ds} = \frac{1}{2} \dot{x}^c \left( \gamma_{cd} S_{ab} + \gamma_{cb} S_{ad} \right) ,$$ \hspace{1cm} (20)

Here $e^a_i$ is the matrix introduced in the equation (4) and derivatives $(e^b_i)_a$ are obtained from the first structure equation (6).

Using our definition of a spin tensor (Eq. 18), we obtain:

$$\frac{dS_{ab}}{ds} = \dot{x}^c \left( \gamma_{cd} S_{ab} + \gamma_{cb} S_{ad} \right) ,$$ \hspace{1cm} (21)

or

$$DS_{ab} = 0 .$$ \hspace{1cm} (22)

Thus, spin is transported parallel to itself along the worldline.

To derive Euler-Lagrange equations for generalized momentum, we return to the coordinate basis. In this case Lagrangian (Eq. 2) will take a form

$$2L = mg_{ij} \dot{x}^j \dot{x}^j - \dot{g}_{ij} \left( \frac{dA^i}{ds} + \Gamma^i_{kl} \dot{x}^k A^l \right) \left( \frac{dA^j}{ds} + \Gamma^j_{kl} \dot{x}^k A^l \right) + m^2 g_{ij} A^i A^j ,$$ \hspace{1cm} (23)

where $\Gamma^i_{kl}$ now are Christoffel symbols. Generalized momentum from here is:

$$p_i = \frac{\partial L}{\partial \dot{x}^i} = mg_{ij} \dot{x}^j - \Gamma^m_{il} A^l \dot{A}_m .$$ \hspace{1cm} (24)

In coordinate basis Eq. 11 becomes:

$$\frac{d}{ds} p_i = \frac{\partial L}{\partial x^i} .$$ \hspace{1cm} (25)

LHS of this equation is:

$$\frac{dp_i}{ds} = mg_{ij} \frac{d\dot{x}^j}{ds} + m \dot{x}^j \dot{x}^k \partial_h g_{ij} - \dot{x}^j A^l \dot{A}_m \partial_j \Gamma^m_{il} - \Gamma^m_{il} \left( \frac{dA^l}{ds} \dot{A}_m + A^l \frac{d\dot{A}_m}{ds} \right) ,$$ \hspace{1cm} (26)

It must be noted that while Eq. 2 represents a non-relativistic Lagrangian of the free mass point when we remove the field part, here we use the relativistic form; in this case both forms are actually equivalent. When we are considering relativistic kinematics (free motion of a mass point, for example), the trajectory will be a geodesic regardless of what form a Lagrangian we take.
where derivatives $\partial_i$ are defined as $\partial_i = \frac{\partial}{\partial x^i}$. RHS is:

$$\frac{\partial L}{\partial x^i} = \frac{1}{2} m \partial_i g_{jk} \dot{x}^j \dot{x}^k + \partial_i g_{mn} \left(-\dot{A}^m \ddot{A}^n + m^2 A^m A^n \right) - g_{mn} \dot{x}^k A^l \dot{A}^n \partial_i \Gamma_{kl}^m.$$  \tag{27}$$

Subtracting (27) from (26), we obtain:

$$mg_{ij} \ddot{x}^j + m \dot{x}^k \dot{x}^j \partial_k g_{ij} - \frac{1}{2} m \partial_i g_{mn} m^2 A^m A^n - \Gamma_{il}^m \dot{A}^l A^m A^n - g_{mn} \dot{x}^k A^l \dot{A}^n \partial_i \Gamma_{kl}^m = 0.$$  \tag{28}$$

Using

$$\partial_i g_{mn} = \frac{1}{2} \left(g_{kn} \Gamma_{k}^{mi} + g_{km} \Gamma_{m}^{ki} \right),$$  \tag{29}$$

and some algebra, we obtain from (28)

$$mg_{ij} \ddot{x}^j + m \dot{x}^k \dot{x}^j \partial_k g_{ij} - \frac{1}{2} m \partial_i g_{mn} m^2 A^m A^n - \Gamma_{il}^m \dot{A}^l A^m A^n - \Gamma_{il}^m \dot{A}^l A^m A^n - \ddot{x}^j A^l \dot{A}^m \partial_j \Gamma_{il}^m + \dot{x}^k A^l \dot{A}^n \partial_i \Gamma_{kl}^m = 0.$$  \tag{30}$$

We may notice that terms in the brackets are zero due to the Proca equation (17) and terms containing $\dot{A} \dot{A}$ cancel; the remaining terms

$$g_{ij} \frac{D \dot{x}^j}{D s} = \dot{x}^j A^l \dot{A}^m \left(\partial_j \Gamma_{il}^k - \partial_i \Gamma_{jl}^k + \Gamma_{im}^k \Gamma_{jl}^m - \Gamma_{mj}^k \Gamma_{il}^m \right).$$  \tag{31}$$

With the definitions of the curvature tensor and spin tensor (18), the equation takes the evidently covariant form:

$$g_{ij} \frac{D \dot{x}^j}{D s} = R_{jil}^k \dot{x}^j S_{kl}^i.$$  \tag{32}$$

This equation coincides with the Papapetrou equation 5. It must be pointed out that in case of a zero spin this equation becomes a geodesic. If we reparametrise the curve with some new parameter $\lambda$ in such a way that

$$g_{ij} \frac{d x^i}{d \lambda} \frac{d x^j}{d \lambda} = 0,$$  \tag{33}$$

we can rewrite Eq. 32 as

$$g_{ij} \frac{D \dot{x}^j}{D \lambda} = R_{jil}^k \dot{x}^j S_{kl}^i.$$  \tag{34}$$

This equation is valid for the massless particles with spin as well, since mass does not enter the equation explicitly.

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