INTERPOLATION OF EXPONENTIAL-TYPE FUNCTIONS ON A
UNIFORM GRID BY SHIFTS OF A BASIS FUNCTION

JEREMY LEVESLEY
Department of Mathematics, University of Leicester
Leicester, UK

XINPING SUN
Department of Mathematics
Missouri State University, USA

FAHD JARAD AND ALEXANDER KUSHPEL
Department of Mathematics, Çankaya University
Ankara, Turkey

Dedicated to the memory of Ward Cheney

Abstract. In this paper, we present a new approach to solving the problem of interpolating a continuous function at \((n+1)\) equally-spaced points in the interval \([0,1]\), using shifts of a kernel on the \((1/n)\)-spaced infinite grid. The archetypal example here is approximation using shifts of a Gaussian kernel. We present new results concerning interpolation of functions of exponential type, in particular, polynomials on the integer grid as a step en route to solve the general interpolation problem. For the Gaussian kernel we introduce a new class of polynomials, closely related to the probabilistic Hermite polynomials and show that evaluations of the polynomials at the integer points provide the coefficients of the interpolants. Finally we give a closed formula for the Gaussian interpolant of a continuous function on a uniform grid in the unit interval (assuming knowledge of the discrete moments of the Gaussian).

1. Introduction. In the mathematical literature pertaining to radial basis functions, there have been mainly two approaches to constructing interpolants with Gaussian kernels. The first involves interpolating a function \(f \in C(\mathbb{R})\) on the \(h\)-spaced grid \(h\mathbb{Z}\) by an interpolant of the form

\[
\sum_{z \in \mathbb{Z}} \alpha_z \psi(z - x/h),
\]

where

\[
\psi(x) = \frac{\exp(-\|x\|^2/2)}{\sqrt{2\pi}}.
\]

Analysis of this so-called cardinal approximation has been done in a series of papers of Cheney, Baxter, Riemenschneider, Sivakumar and others [8, 38, 25, 7, 3, 30, 31].
In the context of $sk$-splines this subject has extensively been studied by Kushpel, Levesley and others in [19, 20, 21, 22, 23, 24, 13]. The second concentrates on interpolating a continuous function on a finite subset $Y$ of a compact interval (e.g $[0, 1]$). Under this circumstance, the interpolant one seeks is of the form

$$\sum_{y \in Y} \alpha_y \psi(y - x).$$

There are multidimensional set-ups for both approaches. Modern mathematical literature abounds in developing error estimates for approximation schemes in this context. We refer readers to [15, 16, 13, 24, 11, 26] and the references therein.

Approximation methods involving sparse-grid algorithms have been recently proven effective and efficient; see [14]. Some sophisticated multi-level sparse grid kernel interpolation schemes have been constructed by authors of [12]. We are currently motivated to develop sparse-grid algorithms for high-dimensional approximation with the Gaussian kernel and derive error estimates for $C^k$-functions with polynomial growth. However, there are several obstacles en route to achieving these goals. The main purpose of the current paper is to clear a few obstacles out of the way. First and foremost, we face the problem of interpolating a function at the $(n + 1)$ equally-spaced points $ih, i = 0, 1, \ldots, n$ with $h = 1/n$, where $n \in \mathbb{N}$. The approach we take here differs from those discussed in the above references. We first interpolate the given $(n + 1)$ data by a degree $n$ polynomial, and then interpolate the polynomial by a radial basis function interpolant on $h\mathbb{Z}$.

Functions of exponential type\footnote{Some variants of functions of exponential type are also referred to as “band limited functions” in the literature.} are often utilized as a half-way house in deriving error estimates for Sobolev space functions; see [27, 28]. As such, it is worthwhile to study the effect of the interpolation scheme when the target functions are of exponential type, and in particular, polynomials, which we anticipate to play a significant role in our future effort to obtain more nuanced error estimates for $C^k$-functions with polynomial growth. Interestingly enough, we observe in the analytic number theory literature that interpolation goes the opposite way in the sense that functions of exponential type are employed to approximate the Gaussian and other useful radial basis functions; see [5, 6]. We hope that interactions of the two seemingly inverse research tracks will create synergistic results.

The layout of the paper is as follows. In Section 2, we will consider a general kernel $\psi$ and study the operator $T$ induced by the Toeplitz matrix $\psi(j - k), j, k \in \mathbb{Z}$. The action of $T$ on an $f \in C(\mathbb{R})$ takes the form:

$$\sum_{j \in \mathbb{Z}} \psi(j - k)f(j), \quad k \in \mathbb{Z}.$$
the coefficients of a Gaussian interpolant in terms of these polynomials. In Section 4, we show how to interpolate general functions on a uniform grid in the unit interval via interpolation by polynomials. This is not a convergent approximation scheme. However, numerical experiments suggest that a residual correction scheme, in which the error at the previous level is interpolated using Gaussian interpolation on a grid with half the point separation, converges with rate greater than any polynomial. This convergence behavior is also observed in the numerical experiments in the multilevel sparse grid paper [12]. In [17] such a convergence rate is observed for periodic multilevel approximation by continuous convolution. While the discrete version of this is quasi-interpolation and not interpolation, the quasi-interpolation result is indicative of the rate of convergence that can be achieved using translates of the Gaussian at different scales.

2. Interpolation with general kernels. Let us assume we have a positive function \( \psi \) defined on \( \mathbb{R} \) such that all the “discrete moments” \( M_k \):

\[
M_k = \sum_{j \in \mathbb{Z}} j^k \psi(j), \quad k = 0, 1, \ldots, N,
\]

are finite. Let \( S_N \) be the vector space over \( \mathbb{R} \) consisting of sequences \( s = \{ s(j) \}_{j \in \mathbb{Z}} \) for which there exists a constant \( C \) independent of \( j \in \mathbb{Z} \) such that \( |s(j)| \leq C|j|^N \). Each \( \psi \) specified above induces a linear operator \( T_\psi \) from \( S_N \) to itself represented in the following format:

\[
\forall s \in S_N, \quad T_\psi(s) = t, \quad t(\ell) = \sum_{j \in \mathbb{Z}} s(j)\psi(\ell - j), \quad \ell \in \mathbb{Z}.
\]

In some contexts, it is mathematically facilitating to consider \( T_\psi(s) \) as a binary operation between \( \psi \) and \( s \), denoted by \( \psi \star s \). This operation is commutative, i.e., \( \psi \star s = s \star \psi \).

Literature abounds in studying operators of the form:

\[
\forall f \in V, \quad T_\phi(f)(x) = \sum_{j \in \mathbb{Z}} f(j)\phi(x - j), \quad x \in \mathbb{R}.
\]

Here \( \phi \) is a prescribed piecewise smooth function, and \( V \) is a certain subspace of \( C(\mathbb{R}) \). Depending on the function \( \phi \) employed therein and the scientific disciplines in which such operators are utilized, the action of the operator \( T_\phi \) is called “sampling operation” [40], “cardinal interpolation” or “quasi-cardinal interpolation” [35]. The common goal is to reconstruct or approximate a target function \( f \) by one of the form as given in the right hand side of (3) utilizing sampled values \( f(j) \) of \( f \) at integers. Various applications of compactly supported functions (e.g. splines) propelled the study of these operators and the related approximation schemes to a crescendo during the 80’s of the last century; see [9, 10, 33, 32], and the references therein.

We are interested in investigating the action of the operators (2) on \( \text{span}\{p|_{\mathbb{Z}} : p \in \Pi_N\} \), where \( \Pi_N \) denotes the vector space of polynomials of degree \( N \) or less. Let \( p_k(x) = x^k \), for \( 0 \leq k \leq N \). We seek coefficients \( a_{k,j}, \quad j \in \mathbb{Z} \), such that

\[
I[p_k](x) := \sum_{j \in \mathbb{Z}} a_{k,j}\psi(x - j),
\]

interpolates \( p_k \) at all the integers. In other words

\[
I[p_k](\ell) = \ell^k, \quad 0 \leq k \leq N, \quad \ell \in \mathbb{Z}.
\]
We will show that if coefficients $a_{k,j}$ (as polynomials in $j$ of degree $k$) exist, then they are unique. We then construct such coefficients (of polynomial form). Thus $I[p_k]$ is the unique interpolant with coefficients of polynomial form. Furthermore, these coefficients are constructible in a recursive fashion.

An operator of the form as in (2) is the restriction to $\mathbb{Z}$ of the corresponding operator in (3), in which the well-known Strang-Fix condition ([10, 9]) gives a full characterization of the capacity of polynomial reproduction in terms of the Fourier transform of $\phi$ provided that $\phi$ is compactly supported. de Boor [4] provided a rich family of polynomials contained in the linear span of integer translates of a compactly supported function. Ron ([33, 32]) established a necessary and sufficient condition for the global linear independence of integer translates of a compactly supported distribution. The pioneering work has lent us useful tools, and furthermore enable us to develop the the results on interpolating polynomials on $\mathbb{Z}$ in the current section in a parallel way to what has been done for compactly supported functions. We will further acknowledge this connection in pertinent contexts.

In the current section, our main purpose is to use minimal assumptions on $\psi$ to show that an operator of the form as in (2) is one-to-one and onto on span\{ $p \mid \mathbb{Z}$ : $p \in \Pi_N$ \}. We will achieve this goal via interpolation using elementary and easily-accessible proofs. In the Appendix, however, we extend results of existence and uniqueness of interpolating polynomial on $\mathbb{Z}$ to include a large class of functions of exponential type. In particular, the proof of Proposition 5 (in the Appendix) has essentially provided a new way of proving the global linear independence of integer translates of a function that has suitable decay, and satisfies the conditions of Wiener’s lemma [18] and Carlson’s theorem [2]. We will devote efforts to the enhancement of this approach in the future.

**Lemma 2.1.** Let $r_k$ be a polynomial of degree $k$ for some $0 \leq k \leq N$, with $N$ as defined in (1). Suppose that

$$f(\ell) := \sum_{j \in \mathbb{Z}} r_k(j) \psi(j - \ell) = 0, \quad \ell \in \mathbb{Z}.$$ 

Then $r_k \equiv 0$.

**Proof.** We proceed by induction. If $k = 0$ then $r_k = c$ for some constant $c$. Then,

$$c \sum_{j \in \mathbb{Z}} \psi(j - \ell) = 0,$$

and since $\psi$ is positive, $c = 0$.

Assume that the result holds true for all polynomials of degree $< k$. Let $\Delta f(x) = f(x + 1) - f(x)$, $x \in \mathbb{R}$, be the forward difference operator. Then

$$\Delta f(\ell) = f(\ell + 1) - f(\ell)$$

$$= \sum_{j \in \mathbb{Z}} r_k(j) \psi(j - \ell - 1) - \sum_{j \in \mathbb{Z}} r_k(j) \psi(j - \ell)$$

$$= \sum_{j \in \mathbb{Z}} r_k(j + 1) \psi(j - \ell) - \sum_{j \in \mathbb{Z}} r_k(j) \psi(j - \ell)$$

$$= \sum_{j \in \mathbb{Z}} (r_k(j + 1) - r_k(j)) \psi(j - \ell)$$

$$= \sum_{j \in \mathbb{Z}} \Delta r_k(j) \psi(j - \ell) = 0.$$
Note that $\Delta r_k$ is a polynomial of degree $k - 1$, which satisfies $\Delta r_k(j) = 0, j \in \mathbb{Z}$. By the induction hypothesis, we have that $\Delta r_k = 0$. It follows that $r_k$ is a constant. We use the induction hypothesis once more to conclude that $r_k$ is identically zero.

Remark that the main technique in the proof above is analogous to that of Proposition 1.1 of [4] that asserts the following: If $M_0 = \sum_{j \in \mathbb{Z}} \psi(j) = 1$, then $\forall p \in \Pi_N, (p|_{\mathbb{Z}} - \psi \ast p) \in \Pi_{N-1}$. The back-substitution technique we use in the current section has also been fully developed and applied in [4] for compactly-supported functions. With a minor modification, it is readily applicable in the non-compact setting herein.

The result of Lemma 2.1 can be equivalently stated as follows. There is no nontrivial polynomial $p$, $\deg(p) \leq N$, such that

$$\sum_{j \in \mathbb{Z}} p(j) \psi(j - \ell) = 0, \ \ell \in \mathbb{Z}.$$ 

Upon a slight modification, this fits into the theory of shift invariant spaces [33] and the general framework of global linear independence of integer translates of a compactly supported distribution established by Ron [32].

In the sequel, we will use the above fact without further declaration.

**Lemma 2.2.** For $k = 0, 1, \cdots, N$,

$$\sum_{j \in \mathbb{Z}} j^k \psi(j - \ell) = \sum_{i=0}^{k} \binom{k}{i} M_{k-i} \ell^i, \ \ell \in \mathbb{Z}.$$ 

**Proof.** Substituting the expression for $M_{k-m}$ from (1) we have

$$\sum_{i=0}^{k} \binom{k}{i} M_{k-i} \ell^i$$

$$= \sum_{i=0}^{k} \binom{k}{i} \left\{ \sum_{j \in \mathbb{Z}} j^{k-i} \psi(j) \right\} \ell^i$$

$$= \sum_{j \in \mathbb{Z}} \psi(j) \left\{ \sum_{i=0}^{k} \binom{k}{i} j^{k-i} \ell^i \right\}$$

$$= \sum_{j \in \mathbb{Z}} \psi(j)(j+\ell)^k$$

$$= \sum_{j \in \mathbb{Z}} j^k \psi(j - \ell),$$

by simply renumbering the sum. \hfill \Box

Let

$$B_{k,i} = M_{k-i} \binom{k-1}{i-1}, k = 1 \cdots, N + 1, i = 1, \cdots, k.$$ 

Since

$$\sum_{j \in \mathbb{Z}} j^{k-1} \psi(j - \ell) = \sum_{i=1}^{k} B_{k,i} \ell^{i-1}, \ \ell = 1, \cdots, N + 1,$$

2Proposition 1.1 of [4] is a multi-dimensional result. Here we only cite the result for the special case $d = 1$. 

we can use backward elimination to see that
\[ \ell^{k-1} = \sum_{j \in \mathbb{Z}} \left( \sum_{i=1}^{k} A_{k,i} j^{i-1} \right) \psi(j - \ell), \; k = 1, \ldots, N + 1, \] (5)
for some numbers \( A_{k,i}, \; k = 1, \ldots, N + 1, \; i = 1, \ldots, k \). Define the polynomial
\[ a_k(j) = \sum_{i=1}^{k} A_{k+1,i+1} j^{i-1}. \]
Then the coefficients \( a_{k,j} \) defined in (4) satisfy \( a_{k,j} = a_k(j) \).

An immediate consequence of Lemma 2.1 and (5) is

**Theorem 2.3.** The coefficients \( a_{k,j} \) of the interpolant \( I[p_k] \) given in (4) are unique. Moreover, they are polynomials (in \( j \)) of degree \( k \).

Let
\[ q_k(x) = \sum_{m=0}^{k} M_{k-m} \binom{k}{m} x^m. \] (6)
In the next theorem we prove a generating function style relationship for the polynomials \( q_k \) and the interpolating functions \( I[p_k] \). As a corollary of this theorem we give a recursive formula for the errors between \( I[p_k] \) and \( p_k \).

**Theorem 2.4.** For \( k = 0, 1, \ldots, N \),
\[ \sum_{j \in \mathbb{Z}} (j - x)^k \psi(j - x) = \sum_{i=0}^{k} \binom{k}{i} q_i(-x) I[p_{k-i}](x). \] (7)

**Proof.** If we substitute (5) into Lemma 2.2 we have
\[ \sum_{j \in \mathbb{Z}} j^k \psi(j - \ell) = \sum_{i=0}^{k} \binom{k}{i} M_{k-i} \sum_{j \in \mathbb{Z}} a_i(j) \psi(j - \ell), \; \ell \in \mathbb{Z}, \]
and by the linear independence of \( \psi(\cdot - \ell), \ell \in \mathbb{Z} \),
\[ j^k = \sum_{i=0}^{k} \binom{k}{i} M_{k-i} a_i(j), \; j \in \mathbb{Z}. \] (8)
We now expand the left hand side of (7) to get
\[ \sum_{j \in \mathbb{Z}} (j - x)^k \psi(j - x) = \sum_{j \in \mathbb{Z}} \sum_{i=0}^{k} \binom{k}{i} (-x)^i j^{k-i} \psi(j - x) \]
\[ = \sum_{i=0}^{k} \binom{k}{i} (-x)^i \sum_{m=0}^{k-i} \binom{k-i}{m} M_{k-i-m} a_m(j) \psi(j - x) \]
\[ = \sum_{i=0}^{k} \binom{k}{i} (-x)^i \sum_{m=0}^{k-i} \binom{k-i}{m} M_{k-i-m} \left( \sum_{j \in \mathbb{Z}} a_m(j) \psi(j - x) \right) \]
\[ = \sum_{i=0}^{k} \binom{k}{i} (-x)^i \sum_{m=0}^{k-i} \binom{k-i}{m} M_{k-i-m} I[p_m](x). \]
In deriving the above equations, we have used (4) and (8). Making use of the formula
\[
\binom{k}{i} \binom{k - i}{m} = \binom{k}{m} \binom{k - m}{i},
\]
we reorder the final sum above as follows.
\[
\sum_{j \in \mathbb{Z}} (j - x)^k \psi(j - x)
\]
\[
= \sum_{m=0}^{k} I[p_m](x) \sum_{i=0}^{k-m} \binom{k}{i} \binom{k - i}{m} M_{k-m-i}(-x)^i
\]
\[
= \sum_{m=0}^{k} \binom{k}{m} I[p_m](x) \sum_{i=0}^{k-m} \binom{k - m}{i} M_{k-m-i}(-x)^i
\]
\[
= \sum_{m=0}^{k} \binom{k}{m} I[p_m](x) q_{k-m}(-x).
\]
In deriving the last equation, we have used (6).

We define the error of interpolation
\[
E_k(x) := I[p_k](x) - p_k(x),
\]
and
\[
\chi_k(x) := \sum_{j \in \mathbb{Z}} (j - x)^k \psi(j - x) - M_k, \quad k = 0, 1, \ldots, N.
\]
Next, we obtain a recursive formula which, upon an appropriate inversion, allows us to write the errors in terms of the functions \(\chi_k\) and \(q_k\).

**Corollary 1.** For \(k = 0, 1, \ldots, N\),
\[
\chi_k(x) = \sum_{i=0}^{k} \binom{k}{i} q_i(-x) E_{k-i}(x).
\]

**Proof.** Since
\[
I[p_{k-i}](\ell) = p_{k-i}(\ell), i = 0, 1, \ldots, k, \ell \in \mathbb{Z},
\]
by Theorem 2.4, we have for \(k = 0, 1, \ldots, N\),
\[
\sum_{i=0}^{k} \binom{k}{i} q_i(\ell) \ell^k = \sum_{j \in \mathbb{Z}} (j - \ell)^k \psi(j - \ell) = M_k.
\]
Since a polynomial is uniquely determined by its values on the integers we have
\[
\sum_{i=0}^{k} \binom{k}{i} q_i(-x)x^k = M_k.
\]
Subtracting this equation from (7) we see that
\[
\chi_k(x) = \sum_{i=0}^{k} \binom{k}{i} q_i(-x) I[p_{k-i}](x) - M_k
\]
\[
= \sum_{i=0}^{k} \binom{k}{i} q_i(-x) I[p_{k-i}](x) - \sum_{i=0}^{k} \binom{k}{i} q_i(-x) p_{k-i}(x)
\[ \sum_{i=0}^{k} \binom{k}{i} q_i(-x) (I[p_{k-i}] - p_{k-i}(x)) = \sum_{i=0}^{k} \binom{k}{i} q_i(-x) E_{k-i}(x). \]

**Remark 1.** Theorem 2.4 gives a recursive formula for computing the interpolant of any polynomial, as long as one knows the interpolants for polynomials of lower degree. Likewise, the result of the corollary expresses the error between a polynomial and its interpolant in the same fashion. If the Gaussian kernel is employed to do interpolation, then we will have more interesting information to offer. We will study this topic in the next section.

### 3. The Gaussian kernel

In this section we study exclusively the case of

\[ \psi(x) = \frac{\exp(-x^2/2)}{\sqrt{2\pi}}. \]

Pertinent to the contents of this paper will be the probabilistic Hermite polynomials \( H_{\epsilon_k}, k = 0, 1, \cdots \). These may be defined in a number of ways, but for us perhaps the most appropriate one is via Rodrigues formula:

\[ H_{\epsilon_k}(x) := \exp\left(\frac{x^2}{2}\right) \frac{d^k}{dx^k} \exp(-x^2/2), k = 0, 1, \cdots. \]

We have the following explicit representation of these polynomials (see e.g. \cite{2}):

\[ H_{\epsilon_k}(x) = \sum_{i=0}^{\lfloor k/2 \rfloor} (-1)^{k-i} \frac{k!}{i!(k-2i)!} x^{k-2i}, k = 0, 1, \cdots, \]

where \( \lfloor x \rfloor \) is the greatest integer less than or equal to \( x \). The polynomial \( H_{\epsilon_k} \) has a close cousin that is often referred to as the probabilistic polynomial of negative variance:

\[ nH_{\epsilon_k}(x) = \sum_{i=0}^{\lfloor k/2 \rfloor} \frac{k!}{i!(k-2i)!} x^{k-2i}, k = 0, 1, \cdots, \]

which has the same coefficients in absolute value, but the coefficients are all positive.

The probabilistic polynomials of negative variance arise very naturally in this study as they are the result of the continuous convolution of the Gaussian with the polynomials of appropriate degree:

**Lemma 3.1.** Let \( \psi(x) = \exp(-x^2/2) \). Then

\[ nH_{\epsilon_k}(x) = \int_{-\infty}^{\infty} y^k \psi(y-x) dy = \sum_{i=0}^{\lfloor k/2 \rfloor} \binom{k}{2i} C_{2i} x^{k-2i}, \]

where

\[ C_k = \int_{-\infty}^{\infty} y^k \psi(y) dy = \begin{cases} (k-1)!!, & k \text{ even}, \\ 0, & k \text{ odd}. \end{cases} \]
Proof. It is well-known (see e.g. [29]) that
\[ \int_{-\infty}^{\infty} y^k \psi(y) dy = \begin{cases} (k - 1)!!, & k \text{ even}, \\ 0, & k \text{ odd}. \end{cases} \]

To prove the first equation, we make a simple change of variable \( w = y - x \):
\[
\int_{-\infty}^{\infty} y^k \psi(x - y) dy = \int_{-\infty}^{\infty} (w + x)^k \psi(w) dw
= \sum_{i=0}^{k} \binom{k}{i} x^i \int_{-\infty}^{\infty} w^{k-i} \psi(w) dw
= \sum_{i=0}^{k} \binom{k}{i} C_k x^i
= \sum_{i=0}^{\lfloor k/2 \rfloor} \binom{k}{i} C_{2i} x^{k-2i}.
\]

We remind readers that all the odd degree terms have zero coefficients. If we substitute the value for \( C_{2i} \) we see that \( nHe_k \) is the probabilistic Hermite polynomial of negative variance:
\[
nHe_k(x) = \sum_{i=0}^{\lfloor k/2 \rfloor} \frac{k!}{i!(k-2i)!} x^{k-2i},
\]

A fascinating relationship between \( He_k \) and \( nHe_k \) is the so-called umbral composition (see [2]):
\[
\sum_{i=0}^{\lfloor k/2 \rfloor} \frac{k!}{i!(k-2i)!} \frac{He_{k-2i}(x)}{2^i} = x^k.
\]

Using Lemma 3.1 and the second equation above we have
\[
x^k = \sum_{i=0}^{\lfloor k/2 \rfloor} (-1)^i \frac{k!}{2^i i!(k-2i)!} \int_{-\infty}^{\infty} y^{k-2i} \psi(x - y) dy
= \int_{-\infty}^{\infty} He_k(y) \psi(y - x) dy, \tag{9}
\]
so that we can recover the monomials by integrating against the probabilistic Hermite polynomials. Of course, this gives us an idea of what will happen in the discrete case.

To do this, we need an analogue of the probabilistic Hermite polynomial for the discrete case. We define
\[
dHe_k(x) = \sum_{i=0}^{\lfloor k/2 \rfloor} (-1)^i \binom{k}{2i} M_{2i} x^{k-2i}, \quad k = 0, 1, \cdots, \tag{10}
\]
where $M_{2k}$ is the discrete moment as defined in (1). We also let
\[ dnHe_k = q_k, \quad k = 0, 1, \ldots, \]
where $q_k$ is as defined in (6). In other words
\[ dnHe_k(x) = \sum_{i=0}^{\lfloor k/2 \rfloor} \binom{k}{2i} M_{2i} x^{k-2i}. \quad (11) \]

Equation (9) suggests that a closed formula for the interpolant $I[p_k](x)$ resembles
\[ \sum_{j \in \mathbb{Z}} dHe_k(j) \psi(j - x), \quad x \in \mathbb{R}. \]

In the next result, we will show that this is indeed the case (up to a constant very close to 1) for $k = 0, 1, 2, 3$. For $k \geq 5$ we need to make some modifications to $\tilde{H}_k$ for a closed form.

**Lemma 3.2.** For $k = 0, 1, \cdots$, and $\ell \in \mathbb{Z}$,
\[ \sum_{j \in \mathbb{Z}} dHe_k(j) \psi(j - \ell) \]
\[ = \sum_{i=0}^{\lfloor k/4 \rfloor} \left\{ \sum_{m=0}^{2i} (-1)^m \binom{k}{2m} \binom{k-2m}{4i-2m} M_{2m} M_{4i-2m-2m} \right\} \ell^{k-4i}. \]

**Proof.** By Theorem 2.2, Equations (10) and (11), we have, for $\ell \in \mathbb{Z}$,
\[ \sum_{j \in \mathbb{Z}} dHe_k(j) \psi(j - \ell) \]
\[ = \sum_{j \in \mathbb{Z}} \left\{ \sum_{i=0}^{\lfloor k/2 \rfloor} (-1)^i \binom{k}{2i} M_{2i} j^{k-2i} \right\} \psi(j - \ell) \]
\[ = \sum_{i=0}^{\lfloor k/2 \rfloor} (-1)^i \binom{k}{2i} M_{2i} \left\{ \sum_{j \in \mathbb{Z}} j^{k-2i} \psi(j - \ell) \right\} \]
\[ = \sum_{i=0}^{\lfloor k/2 \rfloor} (-1)^i \binom{k}{2i} M_{2i} \left\{ \sum_{m=0}^{(k-2i)/2} (-1)^m \binom{k-2i}{2m} M_{2m} \ell^{k-2i-2m} \right\}. \]

Rearranging we obtain
\[ \sum_{j \in \mathbb{Z}} dHe_k(j) \psi(j - \ell) \]
\[ = \sum_{i=0}^{\lfloor k/2 \rfloor} \left\{ \sum_{m=0}^{i} (-1)^m \binom{k}{2m} \binom{k-2m}{2i-2m} M_{2m} M_{2i-2m} \right\} \ell^{k-2i} \]
\[ = \sum_{i=0}^{\lfloor k/4 \rfloor} \left\{ \sum_{m=0}^{2i} (-1)^m \binom{k}{2m} \binom{k-2m}{4i-2m} M_{2m} M_{4i-2m} \right\} \ell^{k-4i}, \]

where we have used the fact that, if $i$ is odd, then
\[ \sum_{m=0}^{i} (-1)^m \binom{k}{2m} \binom{k-2m}{2i-2m} M_{2m} M_{2i-2m} = 0, \]
which is true because
\[
\binom{k}{2m} \binom{k - 2m}{2i - 2m} = \frac{k!}{(k - 2m)! (2m)!} \frac{(k - 2m)!}{(k - 2i)!(2i - 2m)!} \\
= \frac{k!}{(k - 2i + 2m)! (2i - 2m)!} \frac{(k - 2i)!(2m)!}{(k - 2i)!} \\
= \binom{k}{2(i - m)} \binom{k - 2(i - m)}{2m}. 
\]

As we can see, for \( k = 0, 1, 2, 3 \), the above lemma gives an exact formula. Interestingly, suppose that we replace \( \mathcal{M}_k \) by \( \mathcal{C}_k \), \( k = 0, 1, \cdots \), then the correction terms above are all zero. This is why we get the umbral composition formula for the probabilistic Hermite polynomials. For higher degrees we need to modify the polynomial in the summation for interpolation. To this end we introduce the polynomials \( Q_k(x) \), which we define by
\[
Q_k = \frac{1}{\mathcal{M}_0} n H_{\mathcal{C}_k}, \quad k = 0, 1, 2, 3, 
\]
and for \( k = 4, 5, \cdots \),
\[
Q_k = \frac{1}{\mathcal{M}_0} n H_{\mathcal{C}_k} - \sum_{i=1}^{\lfloor k/4 \rfloor} \left\{ \sum_{m=0}^{2i} (-1)^j \binom{k}{2m} \binom{k - 2m}{4i - 2m} M_{2m, M_{4i-2m}} \right\} Q_{k-4i}. 
\]

Using Lemma 3.2, we immediately get the main result of this section

**Theorem 3.3.** For \( k = 0, 1, \cdots, \) and \( \psi \) the Gaussian, we have
\[
I[p_k](x) = \sum_{j \in \mathbb{Z}} Q_k(j) \psi(j - x).
\]

4. **Gaussian interpolant on a \( h \)-spaced points in an interval.** In this section, we give a recipe for computation of a new Gaussian kernel interpolant to a function defined at equally spaced points \( X = \{0, \frac{1}{n}, \frac{2}{n}, \cdots, 1\} \).

The construction of this interpolant utilizes the full \( (1/n) \)-spaced infinite grid. As such, it is different from most of Gaussian kernel interpolants constructed with conventional procedures. However, for all the practical computational purposes, only a small number of centres outside of the interval of interpolation are required. The rapid decay of the Gaussian kernel offsets the error incurred by dropping terms (shifts) of the interpolant far from the interpolation interval.

In this case we seek an interpolant of the form
\[
I_n[f](x) = \sum_{j \in \mathbb{Z}} a_n^j (f)(j - nx), \quad x \in [0, 1],
\]
for \( f \in C([0, 1]) \). To do this we follow the following recipe:
1. Interpolate \( f \) on \( X \) with a degree \( n \) polynomial

\[
P_n[f](x) = \sum_{i=0}^{n} c_i(f)t_i(x),
\]

where \( t_i, i = 0, \ldots, n \) is a basis for the degree \( n \) polynomials.

2. Interpolate the scaled polynomial \( S_{1/n}P_n \):

\[
x \mapsto P_n \left( \frac{1}{n} x \right),
\]

at the integers \( 0, 1, \ldots, n \), and evaluate the result at \( nx \):

\[
I_n[f](x) = I[S_{1/n}P_n](nx)
\]

\[
= \sum_{i=0}^{n} c_i(S_{1/n}f)I[t_i](nx).
\]

With the basis of monomials this becomes

\[
I_n[f](x) = \sum_{j \in \mathbb{Z}} \sum_{i=0}^{n} c_i(S_{1/n}f)Q_i(j)\psi(j-nx),
\]

where \( Q_i, i = 0, \ldots, n \) are as given in (12) and (13). The coefficients in the expression must be interpreted as the appropriate ones for the monomial basis.

Of course, the monomial basis is not a good basis to use numerically, as the Vandermonde matrix which arises by direct interpolation is very ill-conditioned. Instead one could use the Newton basis for polynomials, and the divided difference approach. This will give an alternative and more complicated algorithm for the computation of appropriate coefficients, involving Stirling numbers of the first kind (see e.g. [1]). The main purpose of this paper is to elucidate a process with which one computes Gaussian interpolants via polynomial interpolation.

We appreciate that the coefficients \( Q_k \) in (12) are somewhat complicated. However, one may compute these numbers in an off-line procedure much like the computation of zeros of orthogonal polynomials. These coefficients arise as a result of using the monomial basis for interpolation. As indicated above, if a more stable basis (e.g., the Newton basis) is adopted for this computation environment, then the computation of the corresponding coefficients may simplify. We leave the door open for future research in seeking more stable algorithms for computing Gaussian or other related kernel interpolants.

5. Conclusion. As main results of this paper, we have shown that the interpolant to a polynomial using a suitable kernel has polynomial coefficients. More importantly, a kernel interpolant to a polynomial is constructible recursively, as is the way in which we express the error between the polynomial and its kernel interpolant. For the Gaussian kernel, we provide closed formulas for the coefficients of the kernel interpolant to a polynomial. These are given in terms of a new class of polynomials that closely resemble the classical probabilistic Hermite polynomials. Via interpolating polynomials, we find a way to construct a kernel interpolant to a function defined on an equally-spaced grid of a compact interval. In theory, this interpolant uses shifts of the kernel on a full infinite grid. In numerical implementation, however, since the coefficients in the expansion are polynomial, and the Gaussian decays more rapidly, only a small number of shifts of the kernel centered outside of the interpolation interval is needed. These have cleared the way for our
future work in which we will investigate numerical aspects of this process. Our
goals are to obtain stable and efficient algorithms for the computation of the inter-
polant, and to develop error estimates for $C^k$–functions having polynomial growth.
With the stationary interpolation scheme, the error will not go to zero as the grid
spacing contracts, but the errors estimate will be useful for analysing the residual
approximation algorithm that is detailed in [12].

Appendix: Functions of exponential type

In this appendix we show that we can extend the results of Section 2 beyond the
polynomial case to that of functions of exponential type. For $f \in L(\mathbb{R})$, we use the
following Fourier transform pair:

$$
\hat{f}(\xi) = \int_{\mathbb{R}} e^{-2\pi i \xi x} f(x) dx, \quad \check{f}(x) = \int_{\mathbb{R}} e^{2\pi i \xi x} f(\xi) d\xi.
$$ (14)

We assume that both Fourier transform and inverse Fourier transform have been
properly extended to the Schwartz class of tempered distributions. Let $W$ denote
the collection of all functions $\psi \in C(\mathbb{R})$ that decays rapidly. That is, there exists a
constant $C > 0$, such that for any $N \in \mathbb{N}$, the following inequality holds true:

$$
|\psi(x)| \leq \frac{C}{1 + |x|^N}, \quad x \in \mathbb{R}.
$$ (15)

Each $\psi \in W$ induces a periodic function $\tilde{\psi}$ on $\mathbb{R}$:

$$
\tilde{\psi}(x) := \sum_{z \in \mathbb{Z}} \psi(z) e^{2\pi i x z}, \quad x \in \mathbb{R}.
$$ (16)

The period of the above function is 1. We will use $[0,1]$ as the fundamental interval.
We are particularly interested in the subset $W^*$ of $W$ defined by

$$
W^* := \{ \psi \in W : \tilde{\psi}(x) \neq 0, \quad x \in [0,1] \}.
$$ (17)

**Lemma 5.1.** For each $\psi \in W^*$, there exists a sequence of complex numbers $a_z$, $z \in \mathbb{Z}$, such that

$$
\sum_{z \in \mathbb{Z}} |a_z z^k| < \infty
$$

for any $k \in \mathbb{Z}_+$, and

$$
\frac{1}{\psi(x)} = \sum_{z \in \mathbb{Z}} a_z e^{2\pi i x z}, \quad x \in \mathbb{R}.
$$

**Proof.** First off, Wiener’s lemma [18, p. 228] asserts that there exists a sequence of
complex numbers $a_z$, $z \in \mathbb{Z}$, such that

$$
\sum_{z \in \mathbb{Z}} |a_z| < \infty,
$$

and

$$
\frac{1}{\psi(x)} = \sum_{z \in \mathbb{Z}} a_z e^{2\pi i z x}, \quad x \in \mathbb{R}.
$$

The rapid decay of the function $\psi$ ensures that the periodic function $\tilde{\psi}$ (see (16)) is
infinitely differentiable on $\mathbb{R}$. Since $\tilde{\psi}(x) \neq 0$ for all $x \in \mathbb{R}$, this property of smooth-
ness of the function $\tilde{\psi}$ passes on to the function $(\tilde{\psi})^{-1}$, whose Fourier coefficients
$a_z$, $z \in \mathbb{Z}$, therefore enjoy the desired decay condition. \qed
We will refer to the inequality as displayed in (17) Wiener’s condition. Let $\mathcal{S}$ and $\mathcal{S}'$ denote, respectively, the Schwartz classes of functions and tempered distributions. For each given $0 \leq \sigma < \infty$, let $E_\sigma$ denote the class of analytic functions of exponential type $\sigma$. We will focus on a subclass $E_\sigma^*$ of $E_\sigma$ defined by:

$$E_\sigma^* := \{ f \in E_\sigma : \exists C > 0 \text{ and } N \in \mathbb{N} \text{ such that } |f(x)| \leq C(1 + |x|^N), \; x \in \mathbb{R} \}.$$  

**Proposition 1.** Let $\psi \in W^*$. For each $\sigma \geq 0$, and every $f \in E_\sigma^*$ there exists a $g \in E_\sigma^*$, such that

$$f(j) = \sum_{z \in \mathbb{Z}} g(z) \psi(j - z), \quad j \in \mathbb{Z}.$$  

**Proof.** By the Paley-Wiener theorem [39, p. 162], we may write

$$f = \hat{T}, \; T \in \mathcal{S}' , \; \text{supp}(T) \subset [-\sigma/(2\pi), \sigma/(2\pi)].$$

The factor $(2\pi)^{-1}$ is the result of the particular format of the Fourier transform pair we use in the present paper; see Equation (14). Since $\psi \in W^*$, we have

$$\frac{1}{\psi(x)} = \sum_{z \in \mathbb{Z}} a_z e^{2\pi i z x}, \; x \in \mathbb{R},$$

in which the Fourier coefficients $a_z , \; z \in \mathbb{Z}$, decay rapidly thanks to Lemma 5.1. That is, for any $k \in \mathbb{Z}_+$, we have

$$\sum_{z \in \mathbb{Z}} |a_z z^k| < \infty.$$  

Thus, the following equation defines $T_\psi$ as a Schwartz class distribution:

$$\left\langle \frac{T}{\psi}, \phi \right\rangle := \sum_{z \in \mathbb{Z}} a_z (T, e_z \cdot \phi), \; \phi \in \mathcal{S},$$

where the angle brackets denote the action of the distribution on a test function. Here $e_z$ denotes the function $x \mapsto e^{2\pi i z x}$. We also have

$$\text{supp} \left( \frac{T}{\psi} \right) \subset [-\sigma/(2\pi), \sigma/(2\pi)].$$

We use the same Paley-Wiener Theorem mentioned above to conclude that

$$g := \left( \frac{T}{\psi} \right)$$

is an element of $E_\sigma^*$. Thus, there exists a constant $C > 0$ and an $N \in \mathbb{N}$ such that

$$g(x) \leq C(1 + |x|^N), \; x \in \mathbb{R}.$$  

Now consider the function

$$f^*(x) := \sum_{z \in \mathbb{Z}} g(x - z) \psi(z).$$
Fix each fixed \( M > 0 \) and every \( x \in [-M, M] \), we have
\[
\left| \sum_{z \in \mathbb{Z}} g(x - z) \psi(z) \right| \leq C \sum_{z \in \mathbb{Z}} \left( 1 + |x - z|^N \right) \left( 1 + |z|^{(N+2)} \right)^{-1} \leq C_N M^N \sum_{z \in \mathbb{Z}} (1 + |z|)^{-2}.
\]

Here \( C_N \) is a constant depending only on \( N \). Thus the series converges uniformly on every compact subset of \( \mathbb{R} \). Therefore the function \( f^* \) is continuous on \( \mathbb{R} \) and has at most polynomial growth. We calculate its (distributional) inverse Fourier transform:
\[
(f^*)' = (g)' \cdot \hat{\psi} = \frac{T}{\psi} \cdot \hat{\psi} = T.
\]
This shows that both \( f \) and \( f^* \) are the Fourier transform of the distribution \( T \), meaning that they are the same function. In particular, we have
\[
\sum_{z \in \mathbb{Z}} g(j - z) \psi(z) = \sum_{z \in \mathbb{Z}} g(z) \psi(j - z) = f(j), \quad j \in \mathbb{Z}.
\]
This completes the proof. \( \square \)

For the uniqueness of the coefficients, we have the following result.

**Proposition 2.** Assume that \( 0 < \epsilon < \pi \). Let \( g \in E^*_{\pi - \epsilon} \), \( \psi \in W^* \), and let \( f \) be defined by
\[
f(x) := \sum_{z \in \mathbb{Z}} g(z) \psi(x - z),
\]
Then, in order that \( f(j) = 0 \), \( j \in \mathbb{Z} \), it is necessary and sufficient that \( g(\zeta) = 0 \), \( \zeta \in C \).

**Proof.** Of course, only the necessity part needs a proof. Assume that \( f(j) = 0 \), \( j \in \mathbb{Z} \). Write
\[
f(j) = \sum_{z \in \mathbb{Z}} g(z) \psi(j - z) = \sum_{z \in \mathbb{Z}} g(j - z) \psi(z), \quad j \in \mathbb{Z},
\]
and consider the function \( f^* \) defined by,
\[
f^*(x) = \sum_{z \in \mathbb{Z}} g(x - z) \psi(z).
\]
We remind readers that \( f(j) = f^*(j), \quad j \in \mathbb{Z} \). From the proof of Proposition 1, we observe that \( f^* \) is continuous on \( \mathbb{R} \) and has at most polynomial growth. The (distributional) Fourier transform of \( f^* \) can be easily calculated to be
\[
\hat{f}^*(\xi) := \hat{g}(\xi) \sum_{z \in \mathbb{Z}} \psi(z) e^{2\pi i z x} = \hat{g}(\xi) \hat{\psi}(\xi).
\]
Since \( g \in E^{\ast}_{\pi - \epsilon} \), \( \hat{g} \) is supported in \([-1/2 + \epsilon, 1/2 - \epsilon]\). Thus \( f^{\ast} \in E^{\ast}_{\pi - \epsilon} \). Resorting to Carlson’s theorem,\(^3\) we conclude that \( f^{\ast}(\zeta) = 0 \), \( \zeta \in \mathcal{C} \). The Fourier transform of \( f^{\ast} \) is therefore also zero. Since 
\[
\hat{f}^{\ast}(\zeta) = \hat{g}(\zeta)\hat{\psi}(\zeta),
\]
and \( \hat{\psi}(\zeta) \neq 0 \), we have \( \hat{g}(\zeta) = 0 \), \( \zeta \in \mathbb{R} \). That is, \( \hat{g} \) is the zero distribution. Hence \( g \) is identically zero.

Propositions 1 and 2 imply the following result.

**Corollary 2.** Let \( 0 < \epsilon < \pi \) be given, and let \( \psi \in W^{\ast} \). For each \( g \in E^{\ast}_{\pi - \epsilon} \), there exists a unique \( f \in E^{\ast}_{\pi - \epsilon} \), such that
\[
\sum_{z \in \mathbb{Z}} \psi(j - z)f(z) = g(j), \quad j \in \mathbb{Z}.
\]

Suppose that \( f \) is radial (even), and that for some \( \delta > 0 \) we have 
\[
|f(x)| \leq A(1 + x^2)^{-1/2 + \delta},
\]
and
\[
|\hat{f}(\xi)| \leq A(1 + \xi^2)^{-1/2 + \delta},
\]
where \( A > 0 \) is a constant. Then the following Poisson summation formula holds true; see [37, p.252].
\[
\sum_{z \in \mathbb{Z}} f(z)e^{2\pi i z x} = \sum_{z \in \mathbb{Z}} \hat{f}(x + z).
\]
Thus, Wiener’s condition is satisfied if both \( \psi \) and \( \hat{\psi} \) have the decay rate shown in (15), and \( \hat{\psi} \) is positive. Specifically, the Gaussian kernel satisfies this condition.

We remind readers that for any \( 0 < \epsilon < \pi \), and any polynomial \( p \), we have \( p \in E^{\ast}_{\pi - \epsilon} \). To interpolate a polynomial, we do not need any extra decay condition other than what has been imposed on functions from \( W^{\ast} \). Moreover, any \( \psi \in \mathcal{C} \) satisfying Wiener’s condition suffices.

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\(^3\)Carlson’s theorem asserts that a function in \( E_{\pi - \epsilon}^{\ast} \) that vanishes on all the positive integers is identically zero; see [34].
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E-mail address: j1101e.ac.uk
E-mail address: XSunMissouriState.edu
E-mail address: fahd@cancaya.edu.tr
E-mail address: kushpel@cankaya.edu.tr