THE HOMOTOPY TYPE OF THE COMPLEMENT OF A COORDINATE
SUBSPACE ARRANGEMENT

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Abstract. The homotopy type of the complement of a complex coordinate subspace arrangement
is studied by fathoming out the connection between its topological and combinatorial structures.
A family of arrangements for which the complement is homotopy equivalent to a wedge of spheres
is described. One consequence is an application in commutative algebra: certain local rings are
proved to be Golod, that is, all Massey products in their homology vanish.

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lemma.
1. Introduction

In this paper we study connections between the topology of the complements of certain complex arrangements, and algebraic and combinatorial objects associated to them.

Let
\[ \mathcal{A} = \{ L_1, \ldots, L_r \} \]
be a complex subspace arrangement in \( \mathbb{C}^n \), that is, a finite set of complex linear subspaces in \( \mathbb{C}^n \).

For such an arrangement \( \mathcal{A} \), define its support \( |\mathcal{A}| \) as \( |\mathcal{A}| = \bigcup_{i=1}^r L_i \subset \mathbb{C}^n \) and its complement \( U(\mathcal{A}) \) as
\[ U(\mathcal{A}) = \mathbb{C}^n \setminus |\mathcal{A}|. \]

Arrangements and their complements play a pivotal role in many constructions of combinatorics, algebraic and symplectic geometry, etc.; they also arise as configuration spaces for different classical mechanical systems. Special problems connected with arrangements and their complements arise in different areas of mathematics and mathematical physics. The multidisciplinary nature of the subject results in ongoing theoretical improvements, a constant source of new applications and the penetration of new ideas and techniques in each of the component research areas. It is the interplay of methods from seemingly disparate areas that makes the theory of subspace arrangements a vivid and appealing field of research.

In the study of arrangements it is important to get a detailed description of the topology of their complements, including properties such as homology groups, cohomology rings, homotopy type, and so on. In this paper we are concerned with the homotopy type of the complement of a complex coordinate subspace arrangement. A complex coordinate subspace of \( \mathbb{C}^n \) is given by
\[ L_\sigma = \{ (z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_{i_1} = \cdots = z_{i_k} = 0 \} \]
where \( \sigma = \{ i_1, \ldots, i_k \} \) is a subset of \([n] = \{ 1, \ldots, n \}\), allowing us to define a complex coordinate subspace arrangement \( \mathcal{C}A \) in \( \mathbb{C}^n \) as a family of coordinate subspaces \( L_\sigma \) for \( \sigma \subset [n] \). The main topological space we study, naturally associated to the complex coordinate subspace arrangement \( \mathcal{C}A \), is the complement \( U(\mathcal{C}A) \) in \( \mathbb{C}^n \). Our results are obtained by studying the topological and combinatorial structures of \( U(\mathcal{C}A) \) with the help of commutative and homological algebra, combinatorics and homotopy theory.

It has been known for some time that hyperplane arrangements have a torsion free cohomology ring. Recently it was proved \[ \text{[S]} \] that after suspending the complement of a hyperplane arrangement it becomes homotopy equivalent to a wedge of spheres. The case of complex coordinate subspace arrangements is much more complicated. Already at the cohomology level, there is a more intricate structure. The Buchstaber-Panov formula for \( H^*(U(\mathcal{C}A)) \) \[ \text{[BP]} \] detects torsion in special cases, implying that even stably \( U(\mathcal{C}A) \) cannot always be homotopy equivalent to a wedge of spheres. That makes the question of when the complement of a coordinate subspace arrangement is homotopy
equivalent to a wedge of spheres more difficult and therefore more interesting. The main goal of this paper is to describe a family of coordinate subspace arrangements for which the complement is homotopy equivalent to a wedge of spheres.

The basic connections between the topology, combinatorics and commutative algebra of coordinate subspace arrangements are established as follows.

Let $K$ be a simplicial complex on the vertex set $[n]$. We shall consider only complexes that are finite, abstract simplicial complexes represented by their collection of faces. Every simplicial complex $K$ on the vertex set $[n]$ defines a complex arrangement of coordinate subspaces in $\mathbb{C}^n$ via the correspondence

$$K \ni \sigma \mapsto \text{span}\{e_i : i \notin \sigma\}$$

where $\{e_i\}_{i=1}^n$ is the standard basis for $\mathbb{C}^n$. Equivalently, for each simplicial complex $K$ on the set $[n]$, we associate the complex coordinate subspace arrangement

$$\mathcal{CA}(K) = \{L_\sigma | \sigma \notin K\}$$

and its complement

$$U(K) = \mathbb{C}^n \setminus \bigcup_{\sigma \notin K} L_\sigma.$$ 

On the other hand, to $K$ and a commutative ring $R$ with unit there is an associated algebraic object, the Stanley-Reisner ring $R[K]$, also known in the literature as the face ring of $K$. Denote by $R[v_1, \ldots, v_n]$ the graded polynomial algebra on $n$ variables where $\deg(v_i) = 2$ for each $i$ over $R$. The Stanley-Reisner ring of a simplicial complex $K$ on the vertex set $[n]$ is the quotient ring

$$R[K] = R[v_1, \ldots, v_n]/I_K$$

where $I_K$ is the homogeneous ideal generated by all square free monomials $v^\sigma = v_{i_1} \cdots v_{i_s}$ such that $\sigma = \{v_{i_1}, \ldots, v_{i_s}\} \notin K$.

Coming back to topology and following the Buchstaber-Panov approach to toric topology, there are another two topological spaces associated to a simplicial complex $K$ and its Stanley-Reisner ring $R[K]$. The first space arises as a topological realisation of the Stanley-Reisner ring. It is the Davis-Januszkiewicz space $DJ(K)$, whose cohomology ring is isomorphic to the Stanley-Reisner ring $R[K]$. The Davis-Januszkiewicz space maps by an inclusion into the classifying space of the $n$-dimensional torus. The homotopy fibre of this inclusion can be identified with another torus space, the moment-angle complex $Z_K$, which has as a deformation retract the complement $U(K)$ of the complex coordinate subspace arrangement $B[8.0]$. Different models of $DJ(K)$ and $Z_K$ as well as their additional properties will be addressed later on in Section 2. These homotopic identifications show that the problem of determining the homotopy type of the complement of complex coordinate
subspace arrangements is equivalent to determining the homotopy type of the moment-angle complex $\mathcal{Z}_K$. To do this we need to closely examine the homotopy fibration sequence

$$\mathcal{Z}_K \to DJ(K) \xrightarrow{\text{incl}} BT^n.$$  

The main technique employed for understanding of this filtration is Mather’s Cube Lemma [M], which relates homotopy pullbacks and homotopy pushouts in a cubical diagram. This is applied iteratively as $K$ is built up one face at a time, in a prescribed order. An analysis of the component homotopy fibration and cofibration sequences produces our main result, Theorem 1.1 (see below).

To find a suitable simplicial complex $K$ whose $U(K)$ will be homotopy equivalent to a wedge of spheres, we first look at its cohomology ring. As $U(K)$ is homotopy equivalent to $\mathcal{Z}_K$, this is the same as looking at the cohomology ring of $\mathcal{Z}_K$. The integral cohomology of $\mathcal{Z}_K$ has been calculated in [BP, 7.6 and 7.7]. If $\mathcal{Z}_K$ is to be homotopy equivalent to a wedge of spheres then we need to consider simplicial complexes $K$ for which all Massey products in $H^*(\mathcal{Z}_K)$ vanish. That will not imply that $\mathcal{Z}_K$ is itself homotopic to a wedge of spheres but at least on the cohomological level there will be no obstructions to that claim. Combinatorists, from their point of view, have studied simplicial complexes and associated to them certain Tor algebras that correspond to the cohomology of $\mathcal{Z}_K$ as in our case. They have determined several classes of complexes for which it can be shown that all Massey products in associated Tor algebras vanish.

One such class is of shifted complexes. A simplicial complex $K$ is *shifted* if there is an ordering on the vertex set such that whenever $\sigma$ is a simplex of $K$ and $v'<v$, then $(\sigma-v)\cup v'$ is a simplex of $K$. Gasharov, Peeva and Welker [GPW] showed that when $K$ is a shifted complex, then all Massey products in $H^*(\mathcal{Z}_K)$ are trivial. In this case we obtain much stronger result by determining the homotopy type of $\mathcal{Z}_K$.

**Theorem 1.1.** *Let $K$ be a shifted complex. Then $U(K)$ is homotopy equivalent to a wedge of spheres.*

Previously, the only known cases of simplicial complexes $K$ for which the complement $U(K)$ has the homotopy type of a wedge of spheres occurred when $K$ was a disjoint union of $n$ vertices. When $n = 2$ or $n = 3$, these are classical results of low dimensional topology, while the general case was proved by the authors [GT]. The result in Theorem 1.1 is much more general. Notice for example that any full $k$-dimensional skeleton of the standard simplicial complex on $n$ vertices $\Delta^n$ is a shifted complex.

Let $\mathcal{F}_t$ be the family of simplicial complexes $K$ for which the moment-angle complex $\mathcal{Z}_K$ has the property that $\Sigma^t \mathcal{Z}_K$ is homotopy equivalent to a wedge of spheres. Our next theorem describes the influence that combinatorial operations on simplicial complexes have with respect to $\mathcal{F}_t$. 
Theorem 1.2. Let $K_1 \in \mathcal{F}_t$ and $K_2 \in \mathcal{F}_s$ for some non-negative integers $t$ and $s$. The effect on family membership of the simplicial complex $K$ resulting from the following operations on $K_1$ and $K_2$ is:

1. the disjoint union of simplicial complexes:
   if $K = K_1 \coprod K_2$, then $K \in \mathcal{F}_m$ where $m = \max\{t, s\}$;
   
2. gluing along a common face:
   if $K = K_1 \bigcup_{\sigma} K_2$, then $K \in \mathcal{F}_m$ where $\sigma$ is a common face of $K_1$ and $K_2$ and $m = \max\{t, s\}$;
   
3. the join of simplicial complexes:
   if $K = K_1 \ast K_2$, then $K \in \mathcal{F}_m$ where $m = \max\{t, s\} + 1$.

As a corollary we specify the operations on simplicial complexes for which $\mathcal{F}_0$ is closed.

Corollary 1.3. Let $K_1$ and $K_2$ be simplicial complexes in $\mathcal{F}_0$. Then $\mathcal{F}_0$ is closed for the following operations on simplicial complexes:

1. the disjoint union of simplicial complexes,
   $K = K_1 \coprod K_2 \in \mathcal{F}_0$;

2. gluing along a common face,
   $K = K_1 \bigcup_{\sigma} K_2 \in \mathcal{F}_0$, where $\sigma$ is a common face of $K_1$ and $K_2$.

The information we have obtained on complex subspace arrangements has an application in commutative algebra. Let $R$ be a local ring. One of the fundamental aims of commutative algebra is to describe the homology ring of $R$, that is $\text{Tor}_R(k, k)$, where $k$ is a ground field. The first step in understanding $\text{Tor}_R(k, k)$ is to obtain information about its Poincaré series $P(R)$, more specifically, whether $P(R)$ is a rational function. A far reaching contribution to this problem was made by Golod. A local ring $R$ is Golod if all Massey products in $\text{Tor}_{k[v_1, \ldots, v_n]}(R, k)$ vanish. Golod proved that if a local ring is Golod, then its Poincaré series represents a rational function and it is determined by $P(\text{Tor}_{k[v_1, \ldots, v_n]}(R, k))$. Although being Golod is an important property, not many Golod rings are known. Using our results on the homotopy type of the complement of a coordinate subspace arrangement, we are able to use homotopy theory to gain some insight into these difficult homological-algebraic questions. The main results are as follows.

Theorem 1.4. For a simplicial complex $K$,

$$P(k[K]) \leq \frac{t(1 + t)^n}{t - P(H^*(U(K)))}.$$ 

Equality is obtained when $k[K]$ is Golod.

Theorem 1.5. If $K \in \mathcal{F}_0$, then $k[K]$ is a Golod ring.

Combining Theorems 1.4 and 1.5 we obtain the following result.
Corollary 1.6. For a simplicial complex $K \in \mathcal{F}_0$,

$$P(k[K]) = \frac{t(1 + t)^n}{t - P(H^\ast(U(K)))}.$$ 

To close, let us remark that all the techniques used in this paper can be also applied to real and quaternionic coordinate subspace arrangements by changing the ground ring from complex numbers to real, quaternionic numbers respectively. In those cases Theorem 1.1 describes the homotopy type of the complement of real, quaternionic coordinate subspace arrangements. For real arrangements instead of torus spaces and $\mathbb{C}P^\infty$, we look at spaces with an action of $\mathbb{Z}/2$ (also considered as $S^0$) and $\mathbb{R}P^\infty$, respectively; while in the case of quaternionic arrangements we deal with $S^3$ spaces and $\mathbb{H}P^\infty$.

The disposition of the paper is as follows. Section 2 catalogues the main objects of study and states various properties they satisfy. Sections 3 through 9 build up to and deal with the primary focus of the paper, Theorem 1.1. Sections 8 through 6 establish the preliminary homotopy theory. Included are identifications of the homotopy types of various pushouts, a review of homotopy actions, the general statement of Mather’s Cube Lemma and a finer analysis of a special case involving homotopy actions, and several properties of the fat wedge. Section 7 considers a particular pattern of successive inclusions of one coordinate subspace into another which we term a regular sequence. Such a sequence need not always exist, but when it does we show there is a measure of control over the homotopy types of the successive homotopy fibres obtained from including the coordinate subspaces into the full coordinate space $X_1 \times \cdots \times X_n$. Section 8 gives conditions guaranteeing the existence of regular sequences, which are based on the properties of a shifted complex. Section 9 puts together all the material in Sections 3 through 8 to prove Theorem 1.1. At this point, the class of simplicial complexes for which $Z_K$ is homotopy equivalent to a wedge of spheres includes the shifted complexes. Section 10 shows that there are other simplicial complexes $K$ which have $Z_K$ homotopy equivalent (or stably homotopy equivalent) to a wedge of spheres by proving Theorem 1.2 and Corollary 1.3. Finally, Section 11 turns to commutative algebra considering Golod’s rings and their properties, and proves Theorems 1.4 and 1.5.

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2. The main objects: their definitions and properties

As mentioned in the introduction the main objective of this paper is the study of arrangements and their complements from topological point of view. To pass from the combinatorial concept of arrangements to a topological one, we use different topological models associated to simplicial
complexes $K$ and their algebraic counterparts, the Stanley-Reisner rings $\mathbb{Z}[K]$ (or the face rings) of $K$.

The purpose of this section is to present the main objects which we are going to use and to set the notation. We rely heavily on constructions in toric topology introduced and studied by Buchstaber and Panov [BP].

2.1. The Davis-Januszkiewicz space. The topological realisation of the Stanley-Reisner ring $\mathbb{Z}$ is called the Davis-Januszkiewicz space $DJ(K)$. The first model of $DJ(K)$ is a Borel-type construction due to Davis and Januszkiewicz [DJ]. For our purposes we use another model of $DJ(K)$ given by Buchstaber-Panov [BP]. In what follows, we identify the classifying space of the circle $S^1$ with the infinite-dimensional projective space $\mathbb{C}P^\infty$, and therefore the classifying space $BT^n$ of the $n$-torus with the $n$-fold product of $\mathbb{C}P^\infty$. For an arbitrary subset $\sigma \subset [n]$, define the $\sigma$-power of $BT^n$ as

$$BT^\sigma = \{(x_1, \ldots, x_n) \in BT^n \mid x_i = * \text{ if } i \notin \sigma\}.$$

**Definition 2.1.** Let $K$ be a simplicial complex on the index set $[n]$. The Davis-Januszkiewicz space is given as the cellular subcomplex

$$DJ(K) = \bigcup_{\sigma \in K} BT^\sigma \subset BT^n.$$

Buchstaber and Panov justified the name of this topological model by proving the following.

**Proposition 2.2** (Buchstaber-Panov [BP]). The cohomology of $DJ(K)$ is isomorphic to the Stanley-Reisner ring $\mathbb{Z}[K]$. Moreover, the inclusion of cellular complexes $i: DJ(K) \rightarrow BT^n$ induces the quotient epimorphism

$$i^*: \mathbb{Z}[v_1, \ldots, v_n] \rightarrow \mathbb{Z}[K]$$

in cohomology.

Recently, Notbohm-Ray [NR] showed that the Davis-Januszkiewicz spaces are uniquely determined, up to homotopy equivalence, by their cohomology ring. This implies that all models of Davis-Januszkiewicz spaces are mutually homotopy equivalent.

2.2. The moment-angle complex. Realise the torus $T^n$ as a subspace of $\mathbb{C}^n$

$$T^n = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid |z_i| = 1, \text{ for } i = 1, \ldots, n\}$$

contained in the unit polydisc

$$(D^2)^n = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid |z_i| \leq 1, \text{ for } i = 1, \ldots, n\}.$$

For an arbitrary subset $\sigma \subset [n]$, define

$$B_\sigma = \{(z_1, \ldots, z_n) \in (D^2)^n \mid |z_i| = 1 \text{ if } i \notin \sigma\}.$$
Definition 2.3. Let $K$ be a simplicial complex on the index set $[n]$. Define the moment-angle complex $Z_K$ by

$$Z_K = \bigcup_{\sigma \in K} B_\sigma \subset (D^2)^n.$$ 

Observe that since each $B_\sigma$ is invariant under the action of $T^n$, the moment-angle complex $Z_K$ is a $T^n$-space. Buchstaber and Panov showed that the moment-angle complex is another topological model of the Stanley-Reisner ring $\mathbb{Z}[K]$ by proving that the $T^n$-equivariant cohomology $H^*_T(Z_K)$ is isomorphic to $\mathbb{Z}[K]$.

The following description of the moment-angle complex $Z_K$ together with its relation to the complement of an arrangement plays the pivotal role in our approach to determine the homotopy type of the complement of a complex coordinate subspace arrangement.

Proposition 2.4 (Buchstaber-Panov [BP]). The moment-angle complex $Z_K$ is the homotopy fibre of the embedding

$$i: DJ(K) \to BT^n.$$ 

Recall from [1] that $U(K)$ denotes the complement of the complex coordinate subspace arrangement associated to a simplicial complex $K$.

Theorem 2.5 (Buchstaber-Panov [BP]). There is an equivariant deformation retraction

$$U(K) \to Z_K.$$ 

2.3. The cohomology of moment-angle complexes and shifted complexes. Theorem 2.5 insures that the homotopy type of the complement $U(K)$ of a complex coordinate subspace arrangement can be obtained by finding the homotopy type of the moment-angle complex $Z_K$.

In our study of the homotopy type of $U(K)$ we specialised by asking for which simplicial complexes $K$ the complement $U(K)$ is homotopy equivalent to a wedge of spheres. To begin we look at the cohomology ring of $Z_K$ finding those simplicial complexes for which there is no cohomological obstruction for $Z_K$ to be homotopic to a wedge of spheres. Buchstaber and Panov [BP] described the cohomology algebra of $Z_K$ by proving that there is an isomorphism

$$H^*(Z_K; k) \cong \text{Tor}_{k[v_1, \ldots, v_n]}(k[K], k).$$ 

as graded algebras.

Definition 2.6. The Stanley-Reisner ring $k[K]$ is Golod if all Massey products in $\text{Tor}_{k[v_1, \ldots, v_n]}(k[K], k)$ vanish.

This definition provides a class of rings $k[K]$ for which $Z_K$ might be homotopic to a wedge of spheres. Although being Golod is an interesting property of a ring, there are not many examples of Golod rings. The one that is going to be of use for us comes from combinatorics.
**Definition 2.7.** A simplicial complex $K$ is *shifted* if there is an ordering on its set of vertices such that whenever $\sigma \in K$ and $v' < v$, then $(\sigma - v) \cup v' \in K$.

Notice that any full $i$-th skeleton $\Delta^i(n-1)$ of the standard simplicial complex $\Delta^{n-1}$ (also denoted by $\Delta(n)$) on $n$ vertices is shifted.

**Proposition 2.8** (Gasharov, Peeva and Welker [GPW]). If $K$ is shifted, then its face ring $k[K]$ is Golod.

### 3. Preliminary homotopy decompositions

The purpose of this section is to identify the homotopy type of several pushouts. We begin by stating Mather’s Cube Lemma [M], which relates homotopy pullbacks and homotopy pushouts in a cubical diagram.

**Lemma 3.1.** Suppose there is a homotopy commutative diagram

```
  E ↓  F
   ↓  ↓  ↓
A   G   H
   ↓  ↓  ↓
C → B → D.
```

Suppose the bottom face $A - B - C - D$ is a homotopy pushout and the sides $E - G - A - C$ and $E - F - A - B$ are homotopy pullbacks.

(a) If the top face $E - F - G - H$ is also a homotopy pushout then the sides $G - H - C - D$ and $F - H - B - D$ are homotopy pullbacks.

(b) If the sides $G - H - C - D$ and $F - H - B - D$ are also homotopy pullbacks then the top face $E - F - G - H$ is a homotopy pushout.

□

We next set some notation. Let $X_1$ and $X_2$ be spaces, and let $1 \leq j \leq 2$. Let $\pi_j : X_1 \times X_2 \to X_j$ be the projection onto the $j^{th}$ factor and let $i_j : X_j \to X_1 \times X_2$ be the inclusion into the $j^{th}$ factor. Let $q_j : X_1 \vee X_2 \to X_j$ be the pinch map onto the $j^{th}$ wedge summand. As well, unless otherwise specified, we adopt the Milnor-Moore notation of denoting the identity map on a space $X$ by $X$. 
Lemma 3.2. Let $A$, $B$ and $C$ be spaces. Define $Q$ as the homotopy pushout

\[
\begin{array}{ccc}
A \times B & \xrightarrow{\ast \times B} & C \times B \\
\downarrow \pi_1 & & \downarrow \\
A & \to & Q.
\end{array}
\]

Then $Q \simeq (A \ast B) \vee (C \times B)$.

Proof. Consider the diagram of iterated homotopy pushouts

\[
\begin{array}{ccc}
A \times B & \xrightarrow{\pi_2} & B \\
\downarrow \pi_1 & & \downarrow \\
A & \xrightarrow{\ast} & A \ast B \\
\end{array}
\quad
\begin{array}{ccc}
i_2 & \to & C \times B \\
\downarrow & & \downarrow \\
i & \to & A \ast B \\
\end{array}
\quad
\begin{array}{ccc}
t & \to & Q.
\end{array}
\]

Here, it is well known that the left square is a homotopy pushout, and the right homotopy pushout defines $Q$. Note that $i_2 \circ \pi_2 \simeq \ast \times B$. The outer rectangle in an iterated homotopy pushout diagram is itself a homotopy pushout, so $\overline{Q} \simeq Q$. The right pushout then shows that the homotopy cofibre of $C \times B \to Q$ is $\Sigma B \vee (A \ast B)$. Thus $t$ has a left homotopy inverse. Further, $s \circ i_2 \simeq \ast$ so pinching out $B$ in the right pushout gives a homotopy cofibration $C \times B \to Q \xrightarrow{r} A \ast B$ with $r \circ t$ homotopic to the identity map. \hfill $\square$

Lemma 3.3. Let $A$, $B$, $C$ and $D$ be spaces. Define $Q$ as the homotopy pushout

\[
\begin{array}{ccc}
A \times B & \xrightarrow{\ast \times B} & C \times B \\
\downarrow A \times \ast & & \downarrow \\
A \times D & \to & Q.
\end{array}
\]

Then $Q \simeq (A \ast B) \vee (C \times B) \vee (A \times D)$.

Proof. Let $Q_1$ be the homotopy pushout of the maps $A \times D \to Q$ and $A \times D \xrightarrow{\pi_1} A$. Then there is a diagram of iterated homotopy pushouts

\[
\begin{array}{ccc}
A \times B & \xrightarrow{\ast \times B} & C \times B \\
\downarrow A \times \ast & & \downarrow \\
A \times D & \to & Q \\
\downarrow \pi_1 & & \downarrow \\
A & \to & Q_1.
\end{array}
\]

Observe that the outer rectangle is also a homotopy pushout, so by Lemma 3.2 we have $Q_1 \simeq (A \ast B) \vee (C \times B)$. Further, the outer rectangle shows that the map $A \to Q_1$ is null homotopic.
Since $A \times B \xrightarrow{A \times \ast} A \times D$ is homotopic to the composite $A \times B \xrightarrow{\pi_1} A \xrightarrow{i_1} A \times D$, there is an iterated homotopy pushout diagram

$$
\begin{array}{cccccc}
A \times B & \xrightarrow{\ast \times B} & C \times B \\
\downarrow{\pi_1} & & \downarrow{} \\
A & \xrightarrow{} & Q_1 \\
\downarrow{i_1} & & \downarrow{} \\
A \times D & \xrightarrow{} & Q.
\end{array}
$$

Since $A \rightarrow Q_1$ is null homotopic, we can pinch out $A$ in the lower pushout to obtain a homotopy pushout

$$
\begin{array}{cccccc}
\ast & \xrightarrow{} & Q_1 \\
\downarrow & & \downarrow{} \\
A \times D & \xrightarrow{} & Q.
\end{array}
$$

Hence $Q \simeq Q_1 \lor (A \times D) \simeq (A \ast B) \lor (C \times D) \lor (A \times D)$. \hfill \Box

**Lemma 3.4.** Let $A$, $B$ and $C$ be spaces. Define $Q$ as the homotopy pushout

$$
\begin{array}{cccccc}
A \times (B \lor C) & \xrightarrow{\pi_2} & B \lor C \\
\downarrow{A \times q_2} & & \downarrow{} \\
A \times C & \xrightarrow{} & Q.
\end{array}
$$

Then $Q \simeq C \lor (A \ast B)$.

**Proof.** First consider the homotopy pushout

$$
\begin{array}{cccccc}
B & \xrightarrow{} & B \lor C \\
\downarrow{q_2} & & \downarrow{} \\
\ast & \xrightarrow{} & C.
\end{array}
$$

In general, if $M$ is the homotopy pushout of maps $X \xrightarrow{f} Y$ and $X \xrightarrow{g} Z$ then an easy application of the Cube Lemma (Lemma 3.1) shows that $N \times M$ is the homotopy pushout of $N \times X \xrightarrow{N \times f} N \times Y$ and $N \times X \xrightarrow{N \times g} N \times Z$. In our case, taking the product with $A$ gives a homotopy pushout

$$
\begin{array}{cccccc}
A \times B & \xrightarrow{} & A \times (B \lor C) \\
\downarrow{\pi_1} & & \downarrow{A \times q_2} \\
A & \xrightarrow{i_1} & A \times C.
\end{array}
$$
Now consider the diagram of iterated homotopy pushouts

\[
\begin{array}{ccccccccc}
A \times B & \xrightarrow{\pi_2} & A \times (B \vee C) & \xrightarrow{\pi_1} & B \vee C & \xrightarrow{q_1} & B \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
A & \xrightarrow{i_1} & A \times C & \xrightarrow{A \times q_2} & Q & \xrightarrow{q_2} & Q'
\end{array}
\]

where the right pushout defines \( Q' \). Because the squares are all homotopy pushouts so is the outermost rectangle. Thus, as the top row is homotopic to the projection \( \pi_2 \), we see that \( Q' \simeq A \ast B \).

The right pushout then implies there is a homotopy cofibration \( C \rightarrow Q \rightarrow Q' \simeq A \ast B \).

On the other hand, the composite \( A \times B \rightarrow A \times (B \vee C) \xrightarrow{\pi_2} B \vee C \) is homotopic to the composite \( A \times B \xrightarrow{\pi_2} B \xrightarrow{j_1} B \vee C \), where \( j_1 \) is the inclusion. Thus there is an iterated homotopy pushout diagram

\[
\begin{array}{ccccccccc}
A \times B & \xrightarrow{\pi_2} & B & \xrightarrow{j_1} & B \vee C \\
\downarrow & & \downarrow & & \downarrow \\
A & \xrightarrow{A \ast B} & Q.
\end{array}
\]

As \( q_1 \circ j_1 \) is homotopic to the identity map on \( B \), the composite \( A \ast B \rightarrow Q \rightarrow Q' \simeq A \ast B \) is homotopic to the identity map. Hence the homotopy cofibration \( C \rightarrow Q \rightarrow A \ast B \) splits as \( Q \simeq C \vee (A \ast B) \).

\[\square\]

**Lemma 3.5.** Suppose there is a homotopy pushout

\[
\begin{array}{cccccc}
A \times B & \xrightarrow{f} & D \\
\downarrow \ast \times B & & \downarrow \\
C \times B & \xrightarrow{g} & E
\end{array}
\]

where the restriction of \( f \) to \( B \) is null homotopic. Then \( g \) factors through a map \( g' : C \times B \rightarrow E \) and \( g' \) has a left homotopy inverse.

**Proof.** As the restriction of \( f \) to \( B \) is null homotopic, the homotopy commutativity of the diagram in the statement of the Lemma implies that the restriction of \( g \) to \( B \) is also null homotopic. Pinching \( B \) out on the left side results in a homotopy pushout

\[
\begin{array}{cccccc}
A \times B & \xrightarrow{f'} & D \\
\downarrow \ast \times B & & \downarrow \\
C \times B & \xrightarrow{g'} & E \\
\downarrow & & \downarrow \\
Y & & Y
\end{array}
\]

for maps \( f' \) and \( g' \). Since \( \ast \times B \) is null homotopic, we have \( Y \simeq (C \times B) \vee \Sigma (A \times B) \), implying that \( g' \) has a left homotopy inverse. \[\square\]
4. A review of homotopy actions

This section is a brief reminder of some properties of homotopy actions. Suppose there is a homotopy fibration

$$F \to E \to B.$$  

Let $\partial : \Omega B \to F$ be the connecting map in the homotopy fibration sequence. Then there is a canonical homotopy action $\theta : F \times \Omega B \to F$ such that:

(a) $\theta$ restricted to $F$ is homotopic to the identity map,
(b) $\theta$ restricted to $\Omega B$ is homotopic to $\partial$, and
(c) there is a homotopy commutative diagram

$$
\begin{array}{ccc}
\Omega B \times \Omega B & \xrightarrow{\mu} & \Omega B \\
\downarrow{\partial \times \Omega B} & & \downarrow{\partial} \\
F \times \Omega B & \xrightarrow{\theta} & F.
\end{array}
$$

A special case is given by the path-loop fibration $\Omega B \to \mathcal{P}B \to B$. Here, the homotopy action $\theta : \Omega B \times \Omega B \to \Omega B$ is homotopic to the loop multiplication.

Next, the homotopy action is natural for maps of homotopy fibration sequences. If there is a homotopy fibration diagram

$$
\begin{array}{ccc}
F & \xrightarrow{f} & E \xrightarrow{g} \to B \\
\downarrow{h} & & \downarrow{h} \\
F' & \xrightarrow{f'} & E' \xrightarrow{g'} \to B',
\end{array}
$$

then there is a homotopy commutative diagram of actions

$$
\begin{array}{ccc}
F \times \Omega B & \xrightarrow{\theta} & F \\
\downarrow{f \times \Omega h} & & \downarrow{f} \\
F' \times \Omega B' & \xrightarrow{\theta'} & F'.
\end{array}
$$

One example of this, we will make use of, is the following.

**Lemma 4.1.** Suppose $F \to E \xrightarrow{f} B$ is a homotopy fibration with homotopy action $\theta : F \times \Omega B \to F$. Then the homotopy fibration $F \to E \times X \xrightarrow{f \times X} B \times X$ has a homotopy action $\theta' : F \times (\Omega B \times \Omega X) \to F$ which factors as

$$
\begin{array}{ccc}
F \times (\Omega B \times \Omega X) & \xrightarrow{\theta'} & F \\
\downarrow{F \times \pi_1} & & \downarrow{F \times \pi_1} \\
F \times \Omega B & \xrightarrow{\theta} & F
\end{array}
$$

where $\pi_1$ is the projection.
Proof. Projecting, we obtain a homotopy pullback

\[
\begin{array}{ccc}
F & \rightarrow & E \times X \\
\downarrow & & \downarrow f \times X \\
F & \rightarrow & E \\
\end{array}
\begin{array}{ccc}
\downarrow & & \downarrow \pi_1 \\
\pi_1 & & \pi_1 \\
\end{array}
\begin{array}{ccc}
E \times X & \rightarrow & B \times X \\
\downarrow & & \downarrow \\
E & \rightarrow & B.
\end{array}
\]

The asserted homotopy commutative diagram now follows from the naturality of the homotopy action. \qed

5. A special case of the Cube Lemma

This section describes a particular case of the Cube Lemma which involves a homotopy action in the homotopy pushout of fibres. Suppose there is a homotopy pushout

\[
\begin{array}{ccc}
A & \rightarrow & B \\
\downarrow & & \downarrow \\
C & \rightarrow & D.
\end{array}
\]

Suppose there is a space \( Z \) and a map \( D \rightarrow Z \). Map each of \( A, B, C \) and \( D \) into \( Z \) and take homotopy fibres; name these \( E, F, G \) and \( H \) respectively. Then there is a homotopy commutative cube

\[
\begin{array}{ccc}
E & \rightarrow & F \\
\downarrow & & \downarrow \\
G & \rightarrow & H \\
\downarrow & & \downarrow \\
A & \rightarrow & B \\
\downarrow & & \downarrow \\
C & \rightarrow & D
\end{array}
\]

in which the bottom face is a homotopy pushout and all four sides are homotopy pullbacks. Lemma 3.1 then says that the top face is also a homotopy pushout. In practice, we will have \( Z = C \times Y \) for some space \( Y \), together with two additional conditions, described in the following proposition.

**Proposition 5.1.** Suppose there is a decomposition \( Z = C \times Y \) such that:

(i) the composite \( C \rightarrow D \rightarrow C \times Y \) is homotopic to the inclusion of the first factor;

(ii) the composite \( B \rightarrow D \rightarrow C \times Y \) has a right homotopy inverse when looped.

Let \( M \) be the homotopy fibre of the map \( A \rightarrow C \). Then:

(a) \( E \simeq M \times \Omega Y \) and \( G \simeq \Omega Y \);
(b) the homotopy pushout of fibres becomes

\[
\begin{array}{ccc}
M \times \Omega Y & \xrightarrow{g} & F \\
\downarrow \pi & & \downarrow \pi \\
\Omega Y & \xrightarrow{} & H
\end{array}
\]

where \(\pi\) is the projection and the restriction of \(g\) to \(\Omega Y\) is null homotopic;

(c) the map \(g\) is homotopic to the composite

\[
M \times \Omega Y \xrightarrow{g|M \times \Omega Y} F \times \Omega Y \xrightarrow{F \times i} F \times (\Omega C \times \Omega Y) \xrightarrow{\theta} F
\]

where \(g|M\) is the restriction of \(g\) to \(M\), \(i\) is the inclusion into the second factor, and \(\theta\) is the homotopy action of \(\Omega C \times \Omega Y\) on \(F\).

Proof. First consider the effect of condition (i) on the cube, in particular, on the face \(E - G - A - C\). Let \(PY\) be the path space of \(Y\). The inclusion \(C \rightarrow C \times Y\) can be replaced up to homotopy equivalence by the product \(C \times PY \rightarrow C \times Y\). The map \(A \rightarrow C\) is then replaced by the product map \(A \times * \rightarrow C \times PY\). Composing into \(C \times Y\) then gives a homotopy pullback

\[
\begin{array}{ccc}
N & \rightarrow & A \times * \\
\downarrow & & \downarrow \\
* \times \Omega Y & \rightarrow & C \times PY
\end{array}
\]

which defines the space \(N\). Since the maps defining the homotopy pullback are all product maps, \(N\) is homotopy equivalent to the product \(N_1 \times N_2\), where \(N_1\) is the homotopy pullback of the maps \(A \rightarrow C\) and \(* \rightarrow C\), and \(N_2\) is the homotopy pullback of the maps \(* \rightarrow PY\) and \(\Omega Y \rightarrow PY\). That is, \(N_1 \simeq M\) and \(N_2 \simeq \Omega Y\). Further, the map \(N \rightarrow \Omega Y\) is homotopic to the projection \(M \times \Omega Y \rightarrow \Omega Y\). This proves part (a), that \(E \simeq M \times \Omega Y\) and \(G \simeq \Omega Y\), and also shows in part (b) that the map \(E \rightarrow G\) corresponds to the projection.

Next, consider the cube face \(E - F - A - B\). Observe that the connecting map \(\Omega C \times \Omega Y \rightarrow N\) for the fibration along the top row of the pullback defining \(N\) corresponds to the product map \(\Omega C \times \Omega Y \xrightarrow{\delta \times \Omega Y} M \times \Omega Y\), where \(\delta\) is the connecting map in the homotopy fibration sequence \(\Omega C \rightarrow M \rightarrow A \rightarrow C\). Using the homotopy equivalence \(E \simeq M \times \Omega Y\) we are considering the homotopy pullback diagram

\[
\begin{array}{ccc}
\Omega C \times \Omega Y & \xrightarrow{\delta \times \Omega Y} & M \times \Omega Y \\
\downarrow \gamma & & \downarrow g \\
\Omega C \times \Omega Y & \xrightarrow{} & F
\end{array}
\]

where \(\gamma\) is the connecting map. Condition (ii) implies that \(\gamma\) is null homotopic. The homotopy commutativity of the left square then immediately implies that the restriction of \(g\) to \(\Omega Y\) is null homotopic. This completes the proof of part (b).
The naturality of the homotopy action applied to the homotopy pullback in the previous paragraph gives a homotopy commutative diagram

$$
\begin{array}{ccc}
(M \times \Omega Y) \times (\Omega C \times \Omega Y) & \overset{\theta'}{\longrightarrow} & M \times \Omega Y \\
\downarrow g \times (\Omega C \times \Omega Y) & & \downarrow g \\
F \times (\Omega C \times \Omega Y) & \overset{\theta}{\longrightarrow} & F,
\end{array}
$$

where $\theta'$ and $\theta$ are the respective actions. Since the homotopy fibration sequence $\Omega C \times \Omega Y \overset{\delta}{\longrightarrow} M \times \Omega Y \longrightarrow \Omega Y$ is a product of fibration sequences, $\theta'$ is homotopic to the product of the actions of the individual fibrations. That is, the homotopy fibration sequence $\Omega C \overset{\delta}{\longrightarrow} M \longrightarrow A$ has a homotopy action $\theta'' : M \times \Omega C \longrightarrow M$, while the homotopy fibration sequence $\Omega Y \longrightarrow \Omega Y \longrightarrow \mathcal{P}Y \longrightarrow Y$ has a homotopy action $\mu : \Omega Y \times \Omega Y \longrightarrow \Omega Y$ given by the loop multiplication. The map $\theta'$ is then the composite

$$
\theta' : (M \times \Omega Y) \times (\Omega C \times \Omega Y) \overset{M \times T \times \Omega Y}{\longrightarrow} (M \times \Omega C) \times (\Omega Y \times \Omega Y) \overset{\theta'' \times \mu}{\longrightarrow} M \times \Omega Y
$$

where $T$ is the map which interchanges factors. Precomposing with the inclusion of factors 1 and 4, $M \times \Omega Y \overset{j \times i}{\longrightarrow} (M \times \Omega Y) \times (\Omega C \times \Omega Y)$, we have $\theta' \circ (j \times i)$ homotopic to the identity map. The homotopy commutative diagram of actions above then results in a string of homotopies

$$
g \simeq g \circ \theta' \circ (j \times i) \simeq \theta \circ (g \times 1_{\Omega Y \times \Omega Y}) \circ (j \times i) \simeq \theta \circ (g|_M \times i)
$$

which proves part (c).

**Corollary 5.2.** There is a homotopy cofibration

$$
M \times \Omega Y \overset{g'}{\longrightarrow} F \longrightarrow H
$$

where $g'$ is an extension of $g$ to $M \times \Omega Y$.

**Proof.** Consider the homotopy pushout of fibres in Proposition 5.1. We know that the restriction of $g$ to $\Omega Y$ is null homotopic. Since the projection $\pi$ has a right inverse, the map $\Omega Y \longrightarrow H$ is also null homotopic. Thus the factor $\Omega Y$ in the left column of the homotopy pushout can be pinched out, resulting in a new homotopy pushout

$$
\begin{array}{ccc}
M \times Y & \overset{g'}{\longrightarrow} & F \\
\downarrow & & \downarrow \\
* & \longrightarrow & H
\end{array}
$$

which is exactly the asserted homotopy cofibration. \qed
6. Proper coordinate subspaces of the fat wedge

Let \( X_1, \ldots, X_n \) be path-connected spaces. In this section we investigate properties of the homotopy fibre of the inclusion of the fat wedge \( FW(1, \ldots, n) \) into the product \( X_1 \times \cdots \times X_n \). Here,

\[
FW(1, \ldots, n) = \{(x_1, \ldots, x_n) \mid \text{at least one } x_i \text{ is } \ast\}.
\]

Including the fat wedge into the product gives a homotopy fibration

\[
F^n \rightarrow FW(1, \ldots, n) \rightarrow X_1 \times \cdots \times X_n
\]

which defines the space \( F^n \). Porter [P] showed that \( F^n \) is homotopy equivalent to \( \Omega X_1 \ast \cdots \ast \Omega X_n \) by examining certain subspaces of contractible spaces. Doeraene [D] reproduced this result in a more general setting by using the Cube Lemma. We include a proof using the Cube Lemma for the sake of completeness.

**Lemma 6.1.** There is a homotopy equivalence \( F^n \simeq \Omega X_1 \ast \cdots \ast \Omega X_n \).

**Proof.** We induct on \( n \). When \( n = 1 \), we have \( FW(1) = \ast \) and so \( F^1 = \Omega X_1 \). Assume \( F^{n-1} \simeq \Omega X_1 \ast \cdots \ast \Omega X_{n-1} \). Observe that there is a topological pushout

\[
\begin{array}{ccc}
FW(1, \ldots, n-1) & \xrightarrow{i} & FW(1, \ldots, n-1) \times X_n \\
\downarrow & & \downarrow \\
X_1 \times \cdots \times X_{n-1} & \rightarrow & FW(1, \ldots, n)
\end{array}
\]

where \( i \) is the inclusion into the first factor. Mapping all four corners into \( X_1 \times \cdots \times X_n \) and taking homotopy fibres gives homotopy fibrations

\[
(2) \quad F^n \rightarrow FW(1, \ldots, n) \rightarrow X_1 \times \cdots \times X_n
\]

\[
(3) \quad F^{n-1} \rightarrow FW(1, \ldots, n-1) \times X_n \rightarrow X_1 \times \cdots \times X_n
\]

\[
(4) \quad \Omega X_n \rightarrow X_1 \times \cdots \times X_{n-1} \rightarrow X_1 \times \cdots \times X_n
\]

\[
(5) \quad F^{n-1} \times \Omega X_n \rightarrow FW(1, \ldots, n-1) \rightarrow X_1 \times \cdots \times X_n.
\]

Note that homotopy fibration (4) is the product of the identity fibration * \( \rightarrow X_1 \times \cdots \times X_{n-1} \rightarrow X_1 \times \cdots \times X_{n-1} \) and the path-loop fibration \( \Omega X_n \rightarrow * \rightarrow X_n \). This relates to both homotopy fibrations (2) and (3). Homotopy fibration (5) is the product of the fibration \( F^{n-1} \rightarrow FW(1, \ldots, n-1) \rightarrow X_1 \times \cdots \times X_{n-1} \) and the identity fibration above. Hence the inclusion \( i \) induces a map of fibres \( F^{n-1} \times \Omega X_n \rightarrow F^{n-1} \) which is the projection onto the first factor. Homotopy
fibration is the product of the fibration \( F^{n-1} \rightarrow FW(1, \ldots, n-1) \rightarrow X_1 \times \cdots \times X_{n-1} \) and the path-loop fibration. Hence the inclusion \( FW(1, \ldots, n-1) \rightarrow X_1 \times \cdots \times X_{n-1} \) induces a map of fibrations \( F^{n-1} \times \Omega X_n \rightarrow \Omega X_n \) which is the projection onto the second factor. Collecting all this information on the homotopy fibres, Lemma 3.1 says that there is a homotopy pushout of fibres

\[
\begin{array}{ccc}
F^{n-1} \times \Omega X_n & \xrightarrow{\pi_1} & F^{n-1} \\
\downarrow \pi_2 & & \downarrow \\
\Omega X_n & \rightarrow & F^n.
\end{array}
\]

It is well known that in general the homotopy pushout of the projections \( A \times B \rightarrow A \) and \( A \times B \rightarrow B \) is homotopy equivalent to \( A \ast B \). Thus, in our case, \( F^n \simeq F^{n-1} \ast \Omega X_n \). The inductive hypothesis on \( F^{n-1} \) then implies that \( F^n \simeq \Omega X_1 \ast \cdots \ast \Omega X_n \). □

For \( 1 \leq i \leq n \), let \( X_1 \times \cdots \times \hat{X}_i \times \cdots \times X_n \) be the subspace of \( X_1 \times \cdots \times X_n \) in which the \( i^{th} \)-coordinate is fixed as \( * \). Let \( FW(1, \ldots, \hat{i}, \ldots, n) \) be the fat wedge in \( X_1 \times \cdots \times \hat{X}_i \times \cdots \times X_n \). Let \( B_i = X_i \times FW(1, \ldots, \hat{i}, \ldots, n) \). Observe that each \( B_i \) is a subspace of \( FW(1, \ldots, n) \) and there is a topological pushout

\[
\begin{array}{ccc}
FW(1, \ldots, \hat{i}, \ldots, n) & \rightarrow & B_i \\
\downarrow & & \downarrow \\
X_1 \times \cdots \times \hat{X}_i \times \cdots \times X_n & \rightarrow & FW(1, \ldots, n).
\end{array}
\]

Consider the sequence of inclusions \( B_i \rightarrow FW(1, \ldots, n) \rightarrow X_1 \times \cdots \times X_n \). Using Lemma 6.1 we obtain a homotopy pullback

\[
\begin{array}{ccc}
F_i & \xrightarrow{h_i} & \Omega X_1 \ast \cdots \ast \Omega X_n \\
\downarrow & & \downarrow \\
B_i & \rightarrow & FW(1, \ldots, n) \\
\downarrow & & \downarrow \\
X_1 \times \cdots \times X_n & \rightarrow & X_1 \times \cdots \times X_n
\end{array}
\]

which defines the map \( h_i \).

**Lemma 6.2.** The map \( h_i \) is null homotopic.

**Proof.** Consider the homotopy pushout in diagram (6). We wish to apply Proposition 5.1 with \( A = FW(1, \ldots, \hat{i}, \ldots, n) \), \( B = B_i \), \( C = X_1 \times \cdots \times \hat{X}_i \times \cdots \times X_n \), \( D = FW(1, \ldots, n) \), and \( Z = X_1 \times \cdots \times X_n \). We need to check that the two conditions in Proposition 5.1 hold. Observe that \( Z = C \times X_i \) and \( C \rightarrow Z \) is the inclusion of the first factor so condition (i) is satisfied. Since \( B_i = X_i \times FW(1, \ldots, \hat{i}, \ldots, n) \) and the map \( FW(1, \ldots, \hat{i}, \ldots, n) \rightarrow X_1 \times \cdots \hat{X}_i \times \cdots \times X_n \) has a right homotopy inverse when looped, the map \( B_i \rightarrow X_1 \times \cdots \times X_n \) also has a right homotopy inverse.
when looped, and so condition (ii) is satisfied. Proposition 5.1 then says that when the four corners of the pushout in diagram (6) are mapped into \(X_1 \times \cdots \times X_n\) and homotopy fibres are taken, there is a homotopy pushout of fibres

\[
\begin{array}{ccc}
(\Omega X_1 \times \cdots \times \Omega X_n) \times \Omega X_i & \xrightarrow{g} & F_i \\
\pi & \downarrow & \downarrow h_i \\
\Omega X_i & \xrightarrow{} & \Omega X_1 \times \cdots \times \Omega X_n
\end{array}
\]  

(7)

where \(\pi\) is the projection, the restriction of \(g\) to \(\Omega X_i\) is null homotopic, and \(g\) is determined by the action of \(\Omega X_1 \times \cdots \times \Omega X_n\) on \(F_i\).

We next examine how \(g\) is determined by this action. Since \(B_i = X_i \times FW(1, \ldots, \hat{i}, \ldots, n)\), we can project to obtain a homotopy pullback

\[
\begin{array}{ccc}
F_i & \xrightarrow{} & F_i \\
\downarrow & & \downarrow \\
B_i & \xrightarrow{} & FW(1, \ldots, \hat{i}, \ldots, n) \\
\downarrow & & \downarrow \\
X_1 \times \cdots \times X_n & \xrightarrow{\pi} & \hat{X}_i \times \cdots \times X_n.
\end{array}
\]

Lemma 4.1 says that \(g\) factors through a projection,

\[
\begin{array}{ccc}
(\Omega X_1 \times \cdots \times \Omega X_n) \times \Omega X_i & \xrightarrow{\pi} & \Omega X_1 \times \cdots \times \Omega X_i \\
\xrightarrow{} & \xrightarrow{} & \xrightarrow{} \\
(\Omega X_1 \times \cdots \times \Omega X_i) & \xrightarrow{g'} & F_i.
\end{array}
\]

The projection of \(g\) lets us define a composite

\[
g' : (\Omega X_1 \times \cdots \times \Omega X_i \times \cdots \times \Omega X_n) \times \Omega X_i \xrightarrow{\pi} \Omega X_1 \times \cdots \times \Omega X_i \rightarrow F_i.
\]

We can use \(g'\) to pinch out the factor of \(\Omega X_i\) in diagram (7) in order to obtain a homotopy cofibration

\[
(\Omega X_1 \times \cdots \times \Omega X_i \times \cdots \times \Omega X_n) \times \Omega X_i \xrightarrow{g'} F_i \xrightarrow{h_i} \Omega X_1 \times \cdots \times \Omega X_n.
\]

To simplify notation, let \(Y = \Omega X_1 \times \cdots \times \Omega X_i \times \cdots \times \Omega X_n\). Since \(Y\) is a suspension, \(Y \times \Omega X_i \simeq Y \lor (Y \land \Omega X_i)\). So \(g'\) can alternatively be described by the composite \(Y \times \Omega X_i \xrightarrow{\pi} \Omega X_1 \times \cdots \times \Omega X_i \rightarrow Y \rightarrow F_i\), where \(\pi\) is the pinch map. Thus \(Y \land \Omega X_i\) is sent trivially into \(F_i\) by \(g'\) and so \(\Sigma Y \land \Omega X_i\) retracts off the homotopy cofibre \(\Omega X_1 \times \cdots \times \Omega X_n\) of \(g'\). But \(\Sigma Y \land \Omega X_i \simeq \Omega X_1 \times \cdots \times \Omega X_n\). Thus in the homotopy cofibration sequence \(Y \times \Omega X_i \xrightarrow{g'} F_i \xrightarrow{h_i} \Omega X_1 \times \cdots \times \Omega X_n \xrightarrow{\delta} \Sigma(Y \times \Omega X_i)\), the map \(\delta\) has a left homotopy inverse and hence \(h_i\) is null homotopic. □
In what follows a coordinate subspace denotes an arbitrary union of $X_{i_1} \times \ldots \times X_{i_j}$ for some $1 \leq i_1 < \ldots < i_j \leq n$. We now use the spaces $B_i$ and Lemma 6.2 to generalise to the case of any proper coordinate subspace of $FW(1, \ldots, n)$.

**Proposition 6.3.** Suppose $A$ is a proper coordinate subspace of $FW(1, \ldots, n)$. Include $A$ into $FW$ and then include into $X_1 \times \cdots \times X_n$ to obtain a homotopy pullback

\[
\begin{array}{ccc}
F & \xrightarrow{h} & \Omega X_1 \ast \cdots \ast \Omega X_n \\
\downarrow & & \downarrow \\
A & \rightarrow & FW(1, \ldots, n) \\
\downarrow & & \downarrow \\
X_1 \times \cdots \times X_n & \xrightarrow{h} & X_1 \times \cdots \times X_n \\
\end{array}
\]

which defines the map $h$. Then $h$ is null homotopic.

**Proof.** First observe that the inclusion of $A$ into $FW(1, \ldots, n)$ factors through $B_i$ for some $i$. This statement really just follows from the definitions. In terms of coordinates,

$$B_i = \{(x_1, \ldots, x_m) \mid \text{at least one of } x_1, \ldots, \hat{x}_i, \ldots, x_m \text{ is } *\}.$$ 

If the inclusion of $A$ into $FW(1, \ldots, n)$ does not factor through $B_i$, then $A$ must contain a sequence of the form $(x_1, \ldots, x_n)$ in which each of $x_1, \ldots, \hat{x}_i, \ldots, x_n$ is not $. Since $A$ is a coordinate subspace, every sequence of this form must be in $A$. (Note that $A$ is a subspace of $FW(1, \ldots, n)$ so this forces $x_i$ to be $*$ in each such sequence.) If this is true for $1 \leq i \leq n$, then all of $FW(1, \ldots, n)$ is contained in $A$, contradicting the hypothesis that $A$ is a proper coordinate subspace of $FW(1, \ldots, n)$.

The factorization of $A \rightarrow FW(1, \ldots, n)$ through $B_i$ results in a diagram of iterated homotopy pullbacks

\[
\begin{array}{ccc}
F & \xrightarrow{h_i} & \Omega X_1 \ast \cdots \ast \Omega X_n \\
\downarrow & & \downarrow \\
A & \rightarrow & FW(1, \ldots, n) \\
\downarrow & & \downarrow \\
X_1 \times \cdots \times X_n & \xrightarrow{h_i} & X_1 \times \cdots \times X_n \\
\end{array}
\]

The outer rectangle is the homotopy pullback defining $h$, so $h$ factors through $h_i$. But $h_i$ is null homotopic by Lemma 6.2 and so $h$ is null homotopic. \qed

7. **Homotopy fibres associated to regular sequences**

Let $X_1, \ldots, X_n$ be path-connected spaces. Let $A$ and $B$ be two coordinate subspaces of $X_1 \times \cdots \times X_n$, where $B \subseteq A$. Let $F_A$ and $F_B$ be the homotopy fibres of the inclusions of $A$ and $B$ respectively into $X_1 \times \cdots \times X_n$. Observe that there is a map of fibres $F_B \rightarrow F_A$. The purpose of this section
is to consider the homotopy types of $F_A$ and $F_B$ and how these are related by the map of fibres. In general, not much could be expected to be said. We show that if $A$ is built up from $B$ by what we call a regular sequence, and if the homotopy type of $F_B$ is of a certain description, then the homotopy type of $F_A$ is of the same description and there is control over the map of fibres. All this is made concrete in Theorem 7.2 and Proposition 7.5.

We begin by defining what is meant by a regular sequence. Let \{1, \ldots, n\} be a subset of \{1, \ldots, n\}, where \(i_1 < \cdots < i_m\). Let \{j_1, \ldots, j_{n-m}\} be the complement of \{1, \ldots, n\}, where \(j_1 < \cdots < j_{n-m}\). Let $FW(i_1, \ldots, i_m)$ be the fat wedge in $X_{i_1} \times \cdots \times X_{i_m}$. Let $A_0$ and $A$ be coordinate subspaces of $X_1 \times \cdots \times X_n$ such that $X_1 \vee \cdots \vee X_n \subseteq A_0$ and $A_0 \subseteq A$. Then $A$ can be built up iteratively from $A_0$ by a sequence of topological pushouts

\[
FW(i_1, \ldots, i_m) \xrightarrow{\quad} A_{k-1} \\
\downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\
X_{i_1} \times \cdots \times X_{i_m} \xrightarrow{\quad} A_k
\]

where $1 \leq k \leq l$, and $A_l = A$. There may be many choices of sequences of pushouts which realise $A$ in this way. A particular type of sequence, if it exists, is well suited to identifying the homotopy fibre of the inclusion $A \hookrightarrow X_1 \times \cdots \times X_n$.

**Definition 7.1.** Let $X_1, \ldots, X_n$ be path-connected spaces. A coordinate subspace $A$ of $X_1 \times \cdots \times X_n$ is **regular** if the sequence

\[
A_0 \subseteq A_1 \subseteq \cdots \subseteq A_l = A
\]

has the following property for each $1 \leq k \leq l$. Let \{\(s_1, \ldots, s_r\}\} be the largest subset of \{\(j_1, \ldots, j_{n-m}\)\} for which $A_{k-1}$ can be written as a product $A_{k-1} = A'_{k-1} \times X_{s_1} \times \cdots \times X_{s_r}$ (permuting the coordinates if necessary). Then there is a topological pushout

\[
M_{k-1} \xrightarrow{\quad} N_{k-1} \\
\downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\
FW(i_1, \ldots, i_m) \xrightarrow{\quad} A_{k-1}
\]

where $M_{k-1}$ is a proper coordinate subspace of $FW(i_1, \ldots, i_m)$.

The definition of a regular sequence may seem on first reading to be a bit mystifying, but it arises naturally when considering coordinate subspaces associated to shifted complexes. It might be useful at this point to briefly skip ahead to Examples 8.2 and 8.3 in order to see the connection.

To go along with the definition, we establish some notation. Let \{\(t_1, \ldots, t_{n-m-r}\)\} be the complement of \{\(s_1, \ldots, s_r\)\} in \{\(j_1, \ldots, j_{n-m}\)\}. Let $S = X_{s_1} \times \cdots \times X_{s_r}$ and $T = X_{t_1} \times \cdots \times X_{t_{n-m-r}}$, so $S \times T = X_{j_1} \times \cdots \times X_{j_{n-m}}$ and $A_{k-1} = A'_{k-1} \times S$. 

For $0 \leq k \leq l$, let $F_k$ be the homotopy fibre of the inclusion $A_k \to X_1 \times \cdots \times X_n$. Observe that if $A_{k-1} = A'_{k-1} \times S$ then there is a diagram of iterated homotopy pullbacks

\[
\begin{array}{c}
F_{k-1} \\
\downarrow \\
A'_{k-1} \\
\downarrow \\
X_i \times \cdots \times X_{i_m} \times T
\end{array}
\quad
\begin{array}{c}
F_{k-1} \\
\downarrow \\
A_{k-1} \\
\downarrow \\
X_i \times \cdots \times X_{i_m} \times S \times T
\end{array}
\quad
\begin{array}{c}
\pi \\
\downarrow \\
X_i \times \cdots \times X_{i_m} \times T
\end{array}
\]

where $i$ and $\pi$ are the inclusion and projection respectively.

In Theorem 7.2 we make the seemingly odd assumption that the fibre $F_0$ is a co-$H$ space. However, in the context of coordinate subspace arrangements, this condition arises naturally, as we are trying to show that certain homotopy fibres (labelled $Z_K$) are homotopy equivalent to wedges of spheres for appropriate simplicial complexes $K$, in which case the fibres $Z_K$ are co-$H$ spaces.

**Theorem 7.2.** Suppose there is a regular sequence of coordinate subspaces

\[ A_0 \subseteq A_1 \subseteq \cdots \subseteq A_l = A. \]

Assume that $F_0$ is a co-$H$ space. Then the following hold:

(a) for $1 \leq k \leq l$, there is a homotopy cofibration

\[
(\Omega X_{i_1} \times \cdots \times \Omega X_{i_m}) \times (\Omega X_{j_1} \times \cdots \times \Omega X_{j_{n-m}}) \to F_{k-1} \to F_k
\]

and a homotopy decomposition

\[
(\Omega X_{i_1} \times \cdots \times \Omega X_{i_m}) \times (\Omega X_{j_1} \times \cdots \times \Omega X_{j_{n-m}}) \simeq C_{k-1} \vee D_{k-1}
\]

where $C_{k-1}$ maps trivially into $F_{k-1}$ and $D_{k-1}$ retracts off $F_{k-1}$;

(b) there is a homotopy decomposition $F_{k-1} \simeq D_{k-1} \vee E_{k-1}$ for some space $E_{k-1}$;

(c) $F_k$ is a co-$H$ space and there is a homotopy decomposition $F_k \simeq \Sigma C_{k-1} \vee E_{k-1}$.

**Proof.** As the proof of part (a) is lengthy, we begin by assuming that part (a) has been proved and show that parts (b) and (c) hold. With $F_0$ as the base case, we inductively assume that $F_{k-1}$ is a co-$H$ space. Let $E_{k-1}$ be the homotopy cofibre of $D_{k-1} \to F_{k-1}$. By part (a), this map has a left homotopy inverse $F_{k-1} \to D_{k-1}$. Since $F_{k-1}$ is a co-$H$ space, we can add to obtain a composite $F_{k-1} \to F_{k-1} \vee F_{k-1} \to D_{k-1} \vee E_{k-1}$ which is a homotopy equivalence. This proves part (b).
Next, including $D_{k-1}$ into $C_{k-1} \vee D_{k-1}$ we obtain a homotopy pushout

\[
\begin{array}{ccc}
D_{k-1} & \longrightarrow & C_{k-1} \vee D_{k-1} \\
\downarrow & & \downarrow \\
D_{k-1} & \longrightarrow & C_{k-1} \\
\downarrow & & \downarrow \\
F_{k-1} & \longrightarrow & E_{k-1} \\
\downarrow & & \downarrow \\
F_k & \longrightarrow & F_k.
\end{array}
\]

By part (a) the map $C_{k-1} \longrightarrow F_{k-1}$ is null homotopic, so in the pushout the map $C_{k-1} \longrightarrow E_{k-1}$ is also null homotopic. Hence $F_k \simeq \Sigma C_{k-1} \vee E_{k-1}$. Finally, $E_{k-1}$ is a retract of $F_{k-1}$ which has been inductively assumed to be a co-$H$ space, so $E_{k-1}$ is also a co-$H$ space. Thus $F_k$ is a wedge of two co-$H$ spaces and so is itself a co-$H$ space.

We now prove part (a).

**Step 1. Setting up:** Consider the pushout in diagram (8). We apply Proposition 5.1 with $A = FW(i_1, \ldots, i_m)$, $B = A_{k-1}$, $C = X_{i_1} \times \cdots \times X_{i_m}$, $D = A_k$ and $Z = X_1 \times \cdots \times X_n$. We need to check that conditions (i) and (ii) of Proposition 5.1 hold. Observe that $Z = C \times (X_{j_1} \times \cdots \times X_{j_{n-m}})$ and $C \longrightarrow Z$ is the inclusion of the first factor, so condition (i) is satisfied. Since $X_1 \vee \cdots \vee X_n$ is a subspace of $A_{k-1}$ and the inclusion $X_1 \vee \cdots \vee X_n \longrightarrow X_1 \times \cdots \times X_n$ has a right homotopy inverse when looped, the inclusion $A_{k-1} \longrightarrow X_1 \times \cdots \times X_n$ also has a right homotopy inverse when looped, and so condition (ii) is also satisfied. Proposition 5.1 insures that when the four corners of the pushout in diagram (8) are mapped into $X_1 \times \cdots \times X_n$ and homotopy fibres are taken, there is a homotopy pushout of fibres

\[
(\Omega X_{i_1} \times \cdots \times \Omega X_{i_m}) \times (\Omega X_{j_1} \times \cdots \times \Omega X_{j_{n-m}}) \xrightarrow{g} F_{k-1}
\]

\[
\begin{array}{ccc}
\Omega X_{j_1} \times \cdots \Omega X_{j_{n-m}} & \longrightarrow & F_k \\
\downarrow & & \downarrow \\
(\Omega X_{i_1} \times \cdots \times \Omega X_{i_m}) \times (\Omega X_{j_1} \times \cdots \times \Omega X_{j_{n-m}}) & \longrightarrow & F_k.
\end{array}
\]

where $\pi$ is the projection, the restriction of $g$ to $\Omega X_{j_1} \times \cdots \times \Omega X_{j_{n-m}}$ is null homotopic, and $g$ is determined by the action of $\Omega X_1 \times \cdots \times \Omega X_n$ on $F_{k-1}$. As the restriction of $g$ to $\Omega X_{j_1} \times \cdots \times \Omega X_{j_{n-m}}$ is null homotopic, we can pinch out this factor in diagram (9) and, as in Corollary 5.2, obtain a homotopy cofibration

\[
(\Omega X_{i_1} \times \cdots \times \Omega X_{i_m}) \times (\Omega X_{j_1} \times \cdots \times \Omega X_{j_{n-m}}) \xrightarrow{g'} F_{k-1} \longrightarrow F_k.
\]

where $g'$ is an extension of $g$ to the half-smash.
Step 2. The summand $C_{k-1}$: The decomposition $A_{k-1} = A'_{k-1} \times S$ implies that there is a homotopy pullback

$$
\begin{array}{ccc}
F_{k-1} & \longrightarrow & F_{k-1} \\
\downarrow & & \downarrow \\
A_{k-1} & \longrightarrow & A'_{k-1} \\
\downarrow & & \downarrow \\
X_1 \times \cdots \times X_n & \longrightarrow & (X_{i_1} \times \cdots \times X_{i_m}) \times T
\end{array}
$$

where $\pi$ is the projection. Lemma 4.1 says that the map $g$ in diagram (9) factors through a projection,

$$
(\Omega X_{i_1} \ast \cdots \ast \Omega X_{i_m}) \times (\Omega S \times \Omega T) \xrightarrow{g} F_{k-1}
$$

where $\tilde{g}$ is the restriction of $g$ to $(\Omega X_{i_1} \ast \cdots \ast \Omega X_{i_m}) \times \Omega T$.

Let $Y = \Omega X_{i_1} \ast \cdots \ast \Omega X_{i_m}$. Since the restriction of $g$ to $(\Omega S \times \Omega T)$ is null homotopic, the restriction of $\tilde{g}$ to $\Omega T$ is null homotopic. Thus $\tilde{g}$ factors through $Y \times \Omega T$. It was only necessary to choose some extension $g'$ of $g$ to the half-smash in (9) in order to obtain the homotopy cofibration, so we could have taken $g'$ to be the composite $Y \times (\Omega S \times \Omega T) \xrightarrow{1 \times \pi} Y \times \Omega T \xrightarrow{\tilde{g}} F_k$.

Step 3. The summand $D_{k-1}$: It remains to show that $D_{k-1} = Y \vee (Y \wedge \Omega S \wedge \Omega T)$ is a retract of $F_{k-1}$.

Again, we consider $A_{k-1} = A'_{k-1} \times S$ where $S = X_{s_1} \times \cdots \times X_{s_r}$. Observe that $\{i_1, \ldots, i_m\}$ and $\{s_1, \ldots, s_r\}$ are disjoint sets in $\{1, \ldots, n\}$ so the inclusion $FW(i_1, \ldots, i_m) \hookrightarrow A_{k-1}$ of diagram (8) factors as a composite $FW(i_1, \ldots, i_m) \xrightarrow{q} A'_{k-1} \xrightarrow{\pi} A_{k-1}$. Define the space $A''_k$ as the topological pushout

$$
\begin{array}{ccc}
FW(i_1, \ldots, i_m) & \longrightarrow & A'_{k-1} \\
\downarrow & & \downarrow \\
X_{i_1} \times \cdots \times X_{i_m} & \longrightarrow & A''_k.
\end{array}
$$

Since $A$ is regular, there is a topological pushout

$$
\begin{array}{ccc}
M_{k-1} & \longrightarrow & N_{k-1} \\
\downarrow & & \downarrow \\
FW(i_1, \ldots, i_m) & \longrightarrow & A'_{k-1}
\end{array}
$$

(11)
where $M_{k-1}$ is a proper coordinate subspace of $FW(i_1, \ldots, i_m)$. Note that all the spaces in diagram $(11)$ are coordinate subspaces of $X_{i_1} \times \cdots \times X_{i_m} \times T$. We intend to map the four corners of the pushout into $X_{i_1} \times \cdots \times X_{i_m} \times T$, take homotopy fibres, and apply the Cube Lemma. Before doing so we identify the homotopy fibres. Let $F_M$ be the homotopy fibre of the inclusion $M_{k-1} \rightarrow X_{i_1} \times \cdots \times X_{i_m}$. By Lemma 6.1, the homotopy fibre of the inclusion $FW(i_1, \ldots, i_m) \rightarrow X_{i_1} \times \cdots \times X_{i_m}$ is homotopy equivalent to $\Omega X_{i_1} \ast \cdots \ast \Omega X_{i_m}$. Including $X_{i_1} \times \cdots \times X_{i_m}$ into $X_{i_1} \times \cdots \times X_{i_m} \times T$ we obtain a homotopy pullback

$$
\begin{array}{ccc}
F_M \times \Omega T & \longrightarrow & M_{k-1} \\
\downarrow & & \downarrow \\
\Omega X_{i_1} \ast \cdots \ast \Omega X_{i_m} \times \Omega T & \longrightarrow & FW(i_1, \ldots, i_m) \times \Omega T \\
\downarrow & & \downarrow \\
X_{i_1} \times \cdots \times X_{i_m} \times T & \longrightarrow & X_{i_1} \times \cdots \times X_{i_m} \times T
\end{array}
$$

for some map $h$. Let $F_N$ be the homotopy fibre of the inclusion $N_{k-1} \rightarrow X_{i_1} \times \cdots \times X_{i_m} \times T$. The definition of a regular sequence includes the hypothesis that $X_1 \vee \cdots \vee X_n \subseteq A_0$, and so $X_1 \vee \cdots \vee X_n \subseteq A_{k-1}$. Having projected away from coordinates $s_1, \ldots, s_i$, we have $X_{i_1} \vee \cdots \vee X_{i_m} \vee X_{j_1} \vee \cdots \vee X_{j_t} \subseteq A'_{k-1}$. As diagram $(11)$ is a homotopy pushout and $FW(i_1, \ldots, i_m)$ intersects $X_{j_1} \vee \cdots \vee X_{j_t}$ at a point, we must have $X_{j_1} \vee \cdots \vee X_{j_t} \subseteq N_{k-1}$. Thus $\Omega T = \Omega X_{j_1} \ast \cdots \ast \Omega X_{j_t}$ retracts off $\Omega N_{k-1}$. Therefore, in the homotopy pullback

$$
\begin{array}{ccc}
F_M \times \Omega T & \longrightarrow & M_{k-1} \\
\downarrow & & \downarrow \\
F_N & \longrightarrow & N_{k-1} \\
\downarrow & & \downarrow \\
X_{i_1} \times \cdots \times X_{i_m} \times T & \longrightarrow & X_{i_1} \times \cdots \times X_{i_m} \times T
\end{array}
$$

(the pullback defines the map $\tilde{f}$) the restriction of $\tilde{f}$ to $\Omega T$ is null homotopic. Now recall from Step 2 that the homotopy fibre of the inclusion $A'_{k-1} \rightarrow X_{i_1} \times \cdots \times X_{i_m} \times T$ is homotopy equivalent to $F_{k-1}$. Thus, when the four corners of the pushout in diagram $(11)$ are mapped into $X_{i_1} \times \cdots \times X_{i_m} \times T$ and homotopy fibres are taken, Lemma 3.1 implies that there is a homotopy pushout of fibres

$$
\begin{array}{ccc}
F_M \times \Omega T & \longrightarrow & F_N \\
\downarrow & & \downarrow \\
(\Omega X_{i_1} \ast \cdots \ast \Omega X_{i_m}) \times \Omega T & \longrightarrow & F_{k-1}
\end{array}
$$

for some map $\tilde{g}$. We can identify $\tilde{g}$: it is the restriction of the map $g$ in diagram $9$ to $(\Omega X_{i_1} \ast \cdots \ast \Omega X_{i_m}) \times \Omega T$. This is because, as in the proof of Proposition 6.1 (c), the map $\tilde{g}$ is determined by the action of $\Omega X_{i_1} \times \cdots \times \Omega X_{i_m} \times \Omega T$ on $F_{k-1}$. But the pullback

$$
\begin{array}{ccc}
F_{k-1} & \longrightarrow & A'_{k-1} \\
\downarrow & & \downarrow \\
F_{k-1} & \longrightarrow & A_k \\
\downarrow & & \downarrow \\
X_{i_1} \times \cdots \times X_{i_m} \times T & \longrightarrow & X_{i_1} \times \cdots \times X_{i_m} \times S \times T
\end{array}
$$
obtained from including $A_{k-1}'$ into $A_{k-1} = A_{k-1}' \times S$ implies that the action of $\Omega X_{i_1} \times \cdots \times \Omega X_{i_m} \times \Omega T$ on $F_{k-1}$ is the restriction of the action of $\Omega X_{i_1} \times \cdots \times \Omega X_{i_m} \times \Omega T \times \Omega S$ on $F_{k-1}$, that is, the action of $\Omega X_1 \times \cdots \times \Omega X_n$ on $F_{k-1}$, and the latter action determines $g$. Consequently, the factorization of $g$ through $g'$ implies that the restriction $g$ factors as a composite (using the notation from Step 2)

\[ \gamma : Y \times \Omega T \to Y \times (\Omega S \times \Omega T) \to F_{k-1}. \]

Note that $D_{k-1}$ was defined as $Y \vee (Y \wedge \Omega T) \simeq Y \times \Omega T$, and we are trying to prove precisely that $\gamma$ has a left homotopy inverse.

Consider diagram (12). Since $M_{k-1}$ is a proper coordinate subspace of $FW(i_1, \ldots, i_m)$, Proposition 6.3 implies that $h$ is null homotopic. Thus $h \times \Omega T$ is homotopic to $* \times \Omega T$. We have seen that the restriction of $\bar{f}$ to $\Omega T$ is null homotopic. Lemma 8.3 now applies, and shows that $\gamma$ has a left homotopy inverse. □

We now condense some of the information coming out of Theorem 7.2 by concentrating on how the fibre $F_0$ of the starting point $A_0$ of the regular sequence relates to the fibre $F_l$ of the ending point $A_l$ of the sequence. Let $\theta$ be the composite

\[ \theta : F_0 \to F_1 \to \cdots \to F_l. \]

In particular, we want to know how the homotopy type of $F_0$ influences that of $F_l$. This requires a suitable hypothesis on the homotopy type of $F_0$ to get going. We now define a class of spaces which will do the job.

**Definition 7.3.** Let $G^n_1$ be the collection of spaces $F$ which are homotopy equivalent to a wedge of summands of the form $\Sigma \Omega X_{i_1} \ast \cdots \ast \Omega X_{i_m}$, where $1 \leq i_1 < \cdots < i_m \leq n$.

Consider Definition 7.3 in the case of primary interest, when $X_i = \mathbb{C}P^\infty$ for $1 \leq i \leq n$. Then $\Omega X_i \simeq S^1$ and so $\Sigma \Omega X_{i_1} \ast \cdots \ast \Omega X_{i_m} \simeq S^{m+1}$, in which case $F$ is homotopy equivalent to a wedge of spheres. As spaces which are homotopy equivalent to wedges of spheres will appear repeatedly, it will be convenient to introduce an abbreviated way of saying this.

**Definition 7.4.** Let $W$ be the collection of spaces $F$ which are homotopy equivalent to a wedge of spheres.

**Proposition 7.5.** Assume the hypotheses of Theorem 7.2. Suppose in addition that (for all path-connected spaces $X_1, \ldots, X_n$) we have $F_0 \in G^n_1$. Consider the map of fibres $\theta : F_0 \to F_l$. The following hold:

(a) $F_l \in G^n_1$, and

(b) there is a homotopy decomposition $F_0 \simeq F_0^1 \vee F_0^2$ where $F_0^1, F_0^2 \in G^n_1$, the restriction of $\theta$ to $F_0^1$ is null homotopic, and the restriction of $\theta$ to $F_0^2$ has a right homotopy inverse.
Theorem 7.2 gives that for $1 \leq k \leq l$, there is a homotopy cofibration

$$(\Omega X_{i_1} \times \cdots \times \Omega X_{i_m}) \times (\Omega X_{j_1} \times \cdots \times \Omega X_{j_{n-m}}) \xrightarrow{f_k} F_{k-1} \xrightarrow{g_k} F_k$$

and there are homotopy decompositions

$$(\Omega X_{i_1} \times \cdots \times \Omega X_{i_m}) \times (\Omega X_{j_1} \times \cdots \times \Omega X_{j_{n-m}}) \simeq C_{k-1} \vee D_{k-1},$$

$$F_{k-1} \simeq D_{k-1} \vee E_{k-1},$$

$$F_k \simeq \Sigma C_{k-1} \vee E_{k-1}$$

where the restriction of $f_k$ to $C_{k-1}$ is null homotopic and the restriction of $f_k$ to $D_{k-1}$ has a right homotopy inverse. This implies that the restriction of $g_k$ to $D_{k-1}$ is null homotopic and the restriction of $g_k$ to $E_{k-1}$ has a right homotopy inverse.

First observe that, in general, there are homotopy decompositions $(\Sigma A) \times B \simeq \Sigma A \vee (\Sigma A \wedge B)$ and $\Sigma(A \times B) \simeq \Sigma A \vee \Sigma B \vee (\Sigma A \wedge B)$. Using the first decomposition and iterating on the second, we see that

$$(\Omega X_{i_1} \times \cdots \times \Omega X_{i_m}) \times (\Omega X_{j_1} \times \cdots \times \Omega X_{j_{n-m}}) \in W.$$ 

The fact that Theorem 7.2 holds for all path-connected spaces $X_1, \ldots, X_n$ means that the decompositions are independent of the particular choices of those spaces. This lets us make an advantageous choice of $X_1, \ldots, X_n$, observe how the decompositions behave in this special case, and then infer the general decompositions.

The advantageous choice is to take $X_i = \mathbb{C}P^\infty$ for $1 \leq i \leq n$. Then $\Omega X_i \simeq S^1$ and

$$(\Omega X_{i_1} \times \cdots \times \Omega X_{i_m}) \times (\Omega X_{j_1} \times \cdots \times \Omega X_{j_{n-m}}) \in W.$$ 

Thus $C_{k-1}, D_{k-1} \in W$. The hypothesis on $F_0$ implies that in this case $F_0 \in W$. Thus the homotopy equivalence $F_0 \simeq D_0 \vee E_0$ implies that $E_0 \in W$. Hence $F_1 \in W$. Inductively, we see that $C_{k-1}, D_{k-1}, E_{k-1}, F_k \in W$ for all $1 \leq k \leq l$. In particular, $F_l \in W$. We next describe the decomposition $F_0 \simeq F_0^1 \vee F_0^2$. The decomposition $F_k \simeq \Sigma C_{k-1} \vee E_{k-1}$ gives a retraction of $E_{k-1}$ off $F_k$. Consider how this relates to the decomposition $F_k \simeq D_k \vee E_k$. Since $D_k, E_k \in W$, we can choose subwedges of spheres $E_{D_k}, E_{E_k}$ of $D_k, E_k$ respectively such that $E_{k-1} \simeq E_{D_k} \vee E_{E_k}$. Let $F_0^1 = D_1 \vee E_{D_1} \vee \cdots \vee E_{D_{l-1}}$, and let $F_0^2 = E_{E_{l-1}}$. Then $F_0^1 \vee F_0^2 \simeq F_0$. The condition that $g_k$ is null homotopic when restricted to $D_k$ then implies that it is null homotopic when restricted to $E_{D_k}$, and so collectively we see that the restriction of $\theta$ to $F_0^1$ is null homotopic. The condition that $g_{l-1}$ has a right homotopy inverse when restricted to $E_{l-1}$ implies that the restriction of $\theta$ to $F_0^2 = E_{E_{l-1}}$ has a right homotopy inverse.

Now consider the general case. Observe that by keeping track of the indices $i_s$ and $j_t$ on each copy of $\Omega X_{i_s} \simeq S^1$ and $\Omega X_{j_t} \simeq S^1$ in the special case, we can discern which wedge summands of $(\Omega X_{i_1} \times \cdots \times \Omega X_{i_m}) \times (\Omega X_{j_1} \times \cdots \times \Omega X_{j_{n-m}})$ are in $C_{k-1}$ and which are in $D_{k-1}$. In particular,
we see that \( C_{k-1}, D_{k-1} \in G^n_1 \) for each \( 1 \leq k \leq l \). The same index bookkeeping on the successive decompositions in the special case then implies that \( E_{k-1}, F_k \in G^n_1 \) for \( 1 \leq k \leq l \) in particular, \( F_1 \in G^n_1 \), proving part (a) – and there is a decomposition \( F_0 \simeq F_0^1 \cup F_0^2 \) such that \( F_0^1, F_0^2 \in G^n_1 \), the restriction of \( \theta \) to \( F_0^1 \) is null homotopic and the restriction of \( \theta \) to \( F_0^2 \) has a right homotopy inverse, which proves part (b).

\[ \Box \]

8. The existence of regular sequences

In this section we give a general set of conditions which guarantees the existence of regular sequences. In Examples 8.2 and 8.3 we then give particular instances which will be used later in Section 9. The set of conditions is phrased in terms of shifted complexes. Recall that a simplicial sequences. In Examples 8.2 and 8.3 we then give particular instances which will be used later in

\[ \text{Proposition 8.1. Let } L \text{ and } K \text{ be two shifted complexes on the index set } [n], \text{ where } L \text{ is contained within star}(1) \text{ of } K \text{ and } K \text{ has no disjoint points. Fix path-connected spaces } X_1, \ldots, X_n. \text{ Let } B \]
and $A$ be the coordinate subspaces of $X_1 \times \cdots \times X_n$ which correspond to $L$ and $K$ respectively. Let $\text{Star}(1) \subseteq A$ be the coordinate subspace which corresponds to $\text{star}(1) \subseteq K$. Then there is a sequence of coordinate subspaces

$$B = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_l = \text{Star}(1)$$

which is regular.

Before beginning with the proof of Proposition 8.1 we give two examples which will be used subsequently. Observe that since all $n$ vertices are in $L$, the coordinate subspace $X_1 \vee \cdots \vee X_n$ is contained in $B$.

**Example 8.2.** Let $K$ be a connected shifted complex. Let $L$ be the disjoint union of the $n$ vertices of $K$. Consider $\text{star}(1)$ in $K$. Then $B = X_1 \vee \cdots \vee X_n$, $A = \text{Star}(1)$, and Proposition 8.1 says that there exists a sequence of coordinate subspaces

$$X_1 \vee \cdots \vee X_n = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_l = \text{Star}(1)$$

which is regular.

**Example 8.3.** Let $K$ be a connected shifted complex. Let $L$ be link(1). Let $\text{star}_R(2)$ be star(2) in $\text{rest}\{2, \ldots, n\}$. Then $B = \text{Link}(1)$ and $A = \text{Star}_R(2)$. To apply Proposition 8.1 we need to check that (within $\text{rest}\{2, \ldots, n\}$) link(1) is contained in star$_R(2)$. Let $(i_1, \ldots, i_m)$ be a simplex of link(1). If $i_1 = 2$ then $(i_1, \ldots, i_m)$ is clearly in star$_R(2)$. If $i_1 \neq 2$, then as link(1) $\subseteq$ star(1) (in $K$), the definition of link(1) says there exists a simplex $(1, 2, \ldots, i_1 - 1, i_1, \ldots, i_m)$ in star(1). The restricted simplex $(2, \ldots, i_1 - 1, i_1, \ldots, i_m)$ is therefore in star$_R(2)$. Thus star$_R(2)$ contains all the simplices in link(1). Proposition 8.1 then implies that there is a sequence of coordinate subspaces

$$\text{Link}(1) = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_l = \text{Star}_R(1)$$

which is regular.

**Proof of Proposition 8.1** We adjoin subspaces to $B$ in two separate iterations. These adjunctions correspond to gluing simplices to $L$ one at a time until star(1) in $K$ is obtained.

**Iteration 1:** Since $K$ is connected and shifted, every vertex in $K$ is connected by an edge to the vertex 1, that is, the simplex $(1,j)$ is in $K$ for every $2 \leq j \leq n$. Now $L$ may contain disjoint points. If so, since $L$ is shifted, the simplices $(1,j)$ will not be in $L$ for $j \geq j_0$, where $j_0$ is the first vertex not connected to 1. In terms of coordinate subspaces, each $X_j$ is a wedge summand of $B$, and $B$ contains the coordinate subspaces $X_1 \times X_j$ for $j < j_0$. The point of this first iteration is to adjoin the coordinate subspaces $X_1 \times X_j$ for $j \geq j_0$. They will be adjoined in left lexicographical order.
The adjunction is realised by a homotopy pushout

\[
\begin{array}{c}
X_1 \vee X_j \\ \downarrow \\ X_1 \times X_j \\
\end{array} \longrightarrow A_{k-1} \\
\begin{array}{c}
\downarrow \\ \downarrow \\ A_k \\
\end{array}
\]

which defines the space \( A_k \). Here, we begin with the \( j_0 \) case, where \( A_0 = B \), so \( k = j - 1 - j_0 \). To show that this sequence is regular, we need to show that there is a homotopy pushout

\[
\begin{array}{c}
M_{k-1} \\ \downarrow \\ X_1 \vee X_j \\
\end{array} \longrightarrow A_{k-1} \\
\begin{array}{c}
\downarrow \\ \downarrow \\ N_{k-1} \\
\end{array}
\]

Take \( M_{k-1} = X_1 \). Observe that by the iteration to this point, \( A_{k-1} \) is the wedge \( X_1 \vee \cdots \vee X_n \) with the coordinate subspaces \( X_1 \times X_1 \) adjoined for \( 2 \leq i \leq j - 1 \). In particular, \( X_j \) is a wedge summand of \( A_{k-1} \). Let \( N_{k-1} \) be the complementary wedge summand of \( A_{k-1} \), so \( A_{k-1} \simeq X_j \vee N_{k-1} \). Then it is clear that \( M_{k-1} = X_1 \) includes into \( N_{k-1} \), the diagram above homotopy commutes, and it is in fact a homotopy pushout.

**Iteration 2:** First observe that at the end of Iteration 1, all the coordinate subspaces \( X_1 \times X_j \) for \( 2 \leq j \leq n \) have been adjoined to \( \text{Star}(1) \). So \( A_{n-j_0} = X_1 \times (X_2 \vee \cdots \vee X_n) \).

We now adjoin the remaining coordinate subspaces of \( \text{Star}(1) \) in a two-step process. The idea is to adjoin all the remaining coordinate subspaces corresponding to the two-dimensional simplices of \( \text{star}(1) \) in lexicographic order, then the coordinate subspaces corresponding to the three-dimensional simplices of \( \text{star}(1) \), and so on. Suppose all the coordinate subspaces corresponding to the \( (m-2) \)-dimensional simplices in \( \text{star}(1) \) have been adjoined. Suppose \( (1, i_2, \ldots, i_m) \) is the simplex of dimension \( m - 1 \) of least lexicographic order whose corresponding coordinate subspace has not already been adjoined. To perform the adjunction it is necessary that the coordinate subspaces corresponding to the boundary of \( (1, i_2, \ldots, i_m) \) have already been adjoined. The boundary is composed of the simplices

\[
(1, i_2, \ldots, i_{m-1}), (1, i_3, \ldots, i_m), \ldots, (1, i_2, \ldots, i_{m-2}, i_m), \text{ and } (i_2, \ldots, i_m).
\]

All the coordinate subspaces corresponding to boundary simplices starting with the vertex 1 have already been adjoined by inductive hypothesis: all the simplices are of dimension \( m - 2 \) and are all clearly in \( \text{star}(1) \). The lexicographical ordering implies that the coordinate subspace corresponding to the simplex \( (i_2, \ldots, i_m) \) has not yet been adjoined. So we first need to adjoin the coordinate subspace corresponding to \( (i_2, \ldots, i_m) \) and then adjoin the coordinate subspace corresponding to \( (1, i_2, \ldots, i_m) \). Note that the coordinate subspaces corresponding to the boundary simplices of \( (i_2, \ldots, i_m) \) have already been adjoined because \( \text{star}(1) \) being shifted means that if \( \tau \) is a simplex in
the boundary of \((i_2, \ldots, i_m)\) then \((1, \tau)\) is also a simplex of \(\text{star}(1)\), and as its dimension is \(m - 2\), the corresponding coordinate subspace has already been adjoined by inductive hypothesis.

The two-step gluing process is realised by the homotopy pushouts

\[
\begin{align*}
F W(i_2, \ldots, i_m) & \longrightarrow A_{k-1} \\
X_{i_1} \times \cdots \times X_{i_m} & \longrightarrow A_k
\end{align*}
\]

where the pushouts define the spaces \(A_k\) and \(A_{k+1}\). Observe that if we assume \(A_{k-1} \simeq X_1 \times A'_{k-1}\) – this is true for the base case \(A_{n-j_0}\) as mentioned at the beginning of this iteration – then the two-step process in adjoining the coordinate subspace corresponding to the simplex \((1, i_2, \ldots, i_m)\) implies that \(A_{k+1} \simeq X_1 \times A'_{k+1}\). Thus if we show that the two-step process is itself a regular sequence, then the entire iteration is a string of two-step regular sequences and so is a regular sequence, completing the proof.

For the \(k - 1\) case, as \(A_{k-1} \simeq X_1 \times A'_{k-1}\), the definition of a regular sequence enforces us to project onto \(A'_{k-1}\) and look for a homotopy pushout

\[
\begin{align*}
M_{k-1} & \longrightarrow N_{k-1} \\
F W(i_2, \ldots, i_m) & \longrightarrow A_{k-1}
\end{align*}
\]

where \(M_{k-1}\) is a proper coordinate subspace of \(F W(i_2, \ldots, i_m)\). Having projected away from variable 1, this homotopy pushout is really a lower dimensional case which builds up \(\text{Star}(2)\) within \(\text{Rest}\{2, \ldots, n\}\). The inductive hypothesis on dimension means that we can assume that this homotopy pushout exists. For the \(k\) case, we need to show that there is a homotopy pushout

\[
\begin{align*}
M_k & \longrightarrow N_k \\
F W(1, i_2, \ldots, i_m) & \longrightarrow A_k
\end{align*}
\]

where \(M_k\) is a proper coordinate subspace of \(F W(1, i_2, \ldots, i_m)\). Let \(M_{k-1} = X_{i_2} \times F W(i_3, \ldots, i_m)\). (Note that \(M_{k-1}\) equals \(\text{Star}(1)\) in \(F W(i_2, \ldots, i_m)\).) Observe that if such a homotopy pushout exists, then \(N_k\) needs to contain all the coordinate subspaces of \(A_k\) except \(X_{i_2} \times \cdots \times X_{i_m}\). But this is exactly the description of \(A_{k-1}\), so by taking \(N_k = A_{k-1}\) we obtain the desired homotopy pushout.
9. The homotopy type of $Z_K$ for shifted complexes

Recall that if $K$ is a simplicial complex on the index set $[n]$, then there is a corresponding Davis-Januszkiewicz space $DJ(K)$ and a homotopy fibration

$$Z_K \to DJ(K) \to \prod_{i=1}^{n} BT.$$  

One of the main goals of the paper is to prove Theorem 1.1 which we restate as:

**Theorem 9.1.** If $K$ is a shifted complex, then $Z_K$ is homotopy equivalent to a wedge of spheres. That is, $K \in F_0$.

It is well known (and easy to prove) that if $K$ is shifted then each of link(1), star(1), and rest{2, . . . , n} is shifted, star(1) = (1) * link(1), and there is a topological pushout

$$\begin{array}{ccc}
\text{link(1)} & \longrightarrow & \text{rest\{2, . . . , n\}} \\
\downarrow & & \downarrow \\
\text{star(1)} & \longrightarrow & K.
\end{array}$$

This results in a corresponding homotopy pushout of Davis-Januszkiewicz spaces

$$\begin{array}{ccc}
DJ(\text{link(1)}) & \longrightarrow & DJ(\text{rest\{2, . . . , n\}}) \\
\downarrow & & \downarrow \\
DJ(\text{star(1)}) & \longrightarrow & DJ(K)
\end{array}$$

where $DJ(\text{star(1)}) = BT \times DJ(\text{link(1)})$. Mapping the four corners into $\prod_{i=1}^{n} BT$ and taking homotopy fibres gives a cube as in Lemma 3.1 and in particular a homotopy pushout of fibres

$$\begin{array}{ccc}
Z_{\text{link(1)}} & \longrightarrow & Z_{\text{rest\{2, . . . , n\}}} \\
\downarrow & & \downarrow \\
Z_{\text{star(1)}} & \longrightarrow & Z_K.
\end{array}$$

We wish to show that each of $Z_{\text{link(1)}}$, $Z_{\text{rest\{2, . . . , n\}}}$, and $Z_{\text{star(1)}}$ is homotopy equivalent to a wedge of spheres, and then identify the maps in the homotopy pushout in order to show that $Z_K$ is also homotopy equivalent to a wedge of spheres.

This topological problem can be reformulated more generally for coordinate subspaces. We still assume that $K$ is a shifted complex on the index set $[n]$. Let $X_1, \ldots, X_n$ be path-connected spaces. Let $A$ be the coordinate subspace of $X_1 \times \cdots \times X_n$ associated to $K$. Then there is a homotopy pushout

$$\begin{array}{ccc}
\text{Link(1)} & \longrightarrow & \text{Rest\{2, . . . , n\}} \\
\downarrow & & \downarrow \\
\text{Star(1)} & \longrightarrow & A
\end{array}$$

(13)
where \( \text{Star}(1) \simeq X_1 \times \text{Link}(1) \). Now compose each of the four corners with the inclusion \( A \rightarrow X_1 \times \cdots \times X_n \) and take homotopy fibres. Let \( F_L, F_S, F_R, \) and \( F_A \) be the homotopy fibres of the respective inclusions of \( \text{Link}(1), \text{Star}(1), \text{Rest}\{2, \ldots, n\}, \) and \( A \) into \( X_1 \times \cdots \times X_n \). Then Lemma 3.1 says there is a homotopy pushout of fibres

\[
\begin{array}{ccc}
F_L & \longrightarrow & F_R \\
\downarrow & & \downarrow \\
F_S & \longrightarrow & F_A
\end{array}
\]

The homotopy pushout in (14) can be refined. First, consider the map \( F_L \rightarrow F_S \). As \( \text{link}(1) \) is a simplicial complex on the vertices \( \{2, \ldots, n\} \), the space \( \text{Link}(1) \) is a coordinate subspace of \( X_2 \times \cdots \times X_n \). Thus \( F_L \simeq \Omega X_1 \times F_L \) where \( F_L \) is the homotopy fibre of the inclusion \( F_L \rightarrow X_2 \times \cdots \times X_n \). Continuing, as \( \text{Star}(1) \simeq X_1 \times \text{Link}(1) \), there is a homotopy pullback

\[
\begin{array}{ccc}
\Omega X_1 \times F_L & \longrightarrow & \text{Link}(1) \\
\downarrow & & \downarrow \\
F_S & \longrightarrow & X_1 \times \text{Link}(1)
\end{array}
\]

As the map \( \text{Link}(1) \rightarrow X_1 \times \text{Link}(1) \) is the inclusion of the second factor, the previous homotopy pullback shows that \( F_L \simeq F_S \) and the map \( \Omega X_1 \times F_L \rightarrow F_S \) is the projection. Next, consider the map \( F_L \rightarrow F_R \). As \( \text{Rest}\{2, \ldots, n\} \) is a coordinate subspace of \( X_2 \times \cdots \times X_n \), we have \( F_R \simeq \Omega X_1 \times F_R \) where \( F_R \) is the homotopy fibre of \( \text{Rest}\{2, \ldots, n\} \rightarrow X_2 \times \cdots \times X_n \). As \( \text{Link}(1) \) is a subspace of \( \text{Rest}\{2, \ldots, n\} \), the map \( F_L \rightarrow F_R \) becomes \( \Omega X_1 \times F_S \overset{\Omega X_1 \times \gamma}{\longrightarrow} \Omega X_1 \times F_R \) for some map \( \gamma \). Collecting all this information on the homotopy fibres, the homotopy pushout in diagram (14) becomes a homotopy pushout

\[
\begin{array}{ccc}
\Omega X_1 \times F_S & \longrightarrow & \Omega X_1 \times F_R \\
\downarrow & & \downarrow \\
F_S & \longrightarrow & F_A
\end{array}
\]

The goal is to identify the homotopy type of \( F_A \). We do this in Proposition 9.2. It may be useful to recall the definition of \( G_1^n \) in 7.3.

**Proposition 9.2.** Let \( K \) be a shifted complex on the index set \([n]\). Let \( X_1, \ldots, X_n \) be path-connected spaces and let \( A \) be the coordinate subspace of \( X_1 \times \cdots \times X_n \) which corresponds to \( K \). Use the notation and setup established in diagrams (13) and (15). Then the following hold:

(a) \( F_S \in G_1^n \) and \( F_R \in G_2^n \);

(b) there is a homotopy decomposition \( F_S \simeq F^1_S \vee F^2_S \) such that \( F^1_S, F^2_S \in G_1^n \), the restriction of \( \gamma \) to \( F^1_S \) is null homotopic, and the restriction of \( \gamma \) to \( F^2_S \) has a right homotopy inverse;
(c) \( F_A \in \mathcal{G}_n^1 \).

**Proof.** We induct on \( n \), the number of vertices. When \( n = 1 \), we have \( A = X_1 \), \( \text{Star}(1) = X_1 \), \( \text{Rest}\{2, \ldots, n\} = * \), and \( \text{Link}(1) = * \). Composing into (the product space) \( X_1 \) and taking homotopy fibres, we immediately see that \( F_S \simeq * \), \( \overline{F_R} \simeq * \), \( \gamma \) is homotopic to the map from the basepoint to itself so part (b) trivially holds, and \( F_A \simeq * \).

Assume the Proposition holds for \( n-1 \) vertices. First, applying Proposition 7.5 (a) to the regular sequence from \( X_1 \vee \cdots \vee X_n \) to \( \text{Star}(1) \) in Example 8.2 shows that \( F_S \in \mathcal{G}_n^1 \). Next, since \( \text{Rest}\{2, \ldots, n\} \) is a shifted complex on the vertices \( \{2, \ldots, n\} \), the inductive hypothesis implies that \( F_R \in \mathcal{G}_n^2 \). This proves part (a).

Assume part (b) for the moment. Lemma 3.4 then applies to show there is a homotopy equivalence

\[
F_A \simeq F_S^2 \vee (\Omega X_1 \ast F_S^1).
\]

Thus \( A \in \mathcal{G}_n^1 \), proving part (c).

To prove part (b), we need to closely examine the map \( F_S \xrightarrow{\gamma} F_R \). This was defined in the setup for diagram (15) by a homotopy pullback

\[
\begin{array}{ccc}
F_S & \longrightarrow & \text{Link}(1) \longrightarrow X_2 \times \cdots \times X_n \\
\downarrow{\gamma} & & \downarrow \\
\overline{F_R} & \longrightarrow & \text{Rest\{2, \ldots, n\} \longrightarrow X_2 \times \cdots \times X_n}.
\end{array}
\]

By definition, \( \text{Star}_R(2) \) is a coordinate subspace of \( \text{Rest}\{2, \ldots, n\} \). In Example 8.3 we showed that \( \text{Link}(1) \) is a coordinate subspace of \( \text{Star}_R(2) \). Thus there is a diagram of iterated homotopy pullbacks

\[
\begin{array}{ccc}
F_S & \longrightarrow & \text{Link}(1) \longrightarrow X_2 \times \cdots \times X_n \\
\downarrow{\delta} & & \downarrow \\
F_S & \longrightarrow & \text{Star}_R(2) \longrightarrow X_2 \times \cdots \times X_n \\
\downarrow{\epsilon} & & \downarrow \\
\overline{F_R} & \longrightarrow & \text{Rest\{2, \ldots, n\} \longrightarrow X_2 \times \cdots \times X_n}.
\end{array}
\]

where the pullbacks define the space \( \overline{F_S} \) and the maps \( \delta \) and \( \epsilon \). Hence \( \gamma \simeq \epsilon \circ \delta \). We deal with each of \( \delta \) and \( \epsilon \) one at a time.

Applying Proposition 7.6 (b) to the regular sequence from \( \text{Link}(1) \) to \( \text{Star}_R(2) \) in Example 8.3 shows that \( F_S \simeq E_1 \vee E_2 \) where \( E_1, E_2 \in \mathcal{G}_1^1 \), the restriction of \( \delta \) to \( E_1 \) is null homotopic, and the restriction of \( \delta \) to \( E_2 \) has a right homotopy inverse.

For \( \epsilon \), we appeal to the inductive hypothesis. Let \( \text{link}_R(2) \) be \( \text{link}(2) \) within \( \text{rest}\{2, \ldots, n\} \). Since \( \text{rest}\{2, \ldots, n\} \) is a shifted complex, it is the pushout of \( \text{star}_R(2) \) and \( \text{rest}\{3, \ldots, n\} \) over \( \text{link}_R(2) \).
This results in a homotopy pushout of the corresponding coordinate subspaces (in $X_2 \times \cdots \times X_n$)

$$\text{Link}_R(2) \longrightarrow \text{Rest}\{3, \ldots, n\} \quad \text{Star}_R(2) \longrightarrow \text{Rest}\{2, \ldots, n\}.$$ 

Let $F_L$ and $F_R$ respectively be the homotopy fibres of the inclusions of $\text{Link}_R(2)$ and $\text{Rest}\{3, \ldots, n\}$ into $X_2 \times \cdots \times X_n$. Recall that $F_S$ and $F_R$ respectively have been defined as the homotopy fibres of the inclusions of $\text{Star}_R(2)$ and $\text{Rest}\{2, \ldots, n\}$ into $X_2 \times \cdots \times X_n$. As in diagram (14), when all four corners of the pushout above are mapped into $X_2 \times \cdots \times X_n$, we obtain a homotopy pushout of fibres

$$F_L \longrightarrow F_R \quad F_S \longrightarrow \bar{F}_R.$$ 

Arguing as for diagram (15), this homotopy pushout of fibres refines to a homotopy pushout

$$\Omega X_2 \times F_S \overset{1 \times \bar{\Sigma}}{\longrightarrow} \Omega X_2 \times \bar{F}_R$$

where $\bar{F}_R$ is the homotopy fibre of the inclusion $\text{Rest}\{3, \ldots, n\} \longrightarrow X_3 \times \cdots \times X_n$. Let $\varphi : F_S \longrightarrow \bar{F}_R$ be the map along the bottom row. Since the underlying shifted complex $\text{rest}\{2, \ldots, n\}$ is on $n - 1$ vertices, by inductive hypothesis we can assume that there is a homotopy decomposition $F_S \simeq D_1 \vee D_2$ where $D_1, D_2 \in G^2_n$, the restriction of $\bar{\Sigma}$ to $D_1$ is null homotopic while the restriction of $\bar{\Sigma}$ to $D_2$ has a right homotopy inverse. Lemma 3.3 then implies that there is a homotopy equivalence $\bar{F}_R \simeq D_2 \vee (\Omega X_2 \ast D_1)$, the restriction of $\varphi$ to $D_1$ is null homotopic while the restriction of $\varphi$ to $D_2$ has a right homotopy inverse.

Now consider the composite $\gamma : F_S \overset{\delta}{\longrightarrow} F_S \overset{\epsilon}{\longrightarrow} \bar{F}_R$. Since the restriction of $\delta$ to $E_2$ has a right homotopy inverse, the decomposition $F_S \simeq D_1 \vee D_2$ results in a decomposition $E_2 \simeq E^1_2 \vee E^2_2$ where $E^1_2$ retracts off $D_1$. Let $F_S^1 = E^1_2 \vee E^1_2$, and let $F_S^2 = E^2_2$. Then the conditions on $\delta$ and $\epsilon$ imply that the restriction of $\gamma$ to $F_S^1$ is null homotopic while the restriction of $\gamma$ to $F_S^2$ has a right homotopy inverse, proving part (b). 

With Proposition 9.2 in hand, we can prove Theorem 9.1 as a special case.

**Proof of Theorem 9.1.** In this case, each space $X_i$ equals $BT$, the classifying space of the torus, the coordinate subspace $A$ equals $DJ(K)$, and the homotopy fibre $F_A$ equals $Z_K$. Proposition 9.2 (c) says that $Z_K \in G^1_n$, meaning that $Z_K$ is homotopy equivalent to a wedge of summands of the form...
\[ \Omega BT_{i_1} \ast \cdots \ast \Omega BT_{i_m}. \] Such a summand is homotopy equivalent to \( S^{m+1} \) since \( \Omega BT \simeq S^1 \). Thus \( Z_K \) is homotopy equivalent to a wedge of spheres, and so \( K \in \mathcal{F}_0 \).

A special case of a shifted complex is the full \( i \)-skeleton \( \Delta^i(n) \) of the standard simplex \( \Delta^{n-1} \) on \( n \) vertices. For path-connected spaces \( X_1, \ldots, X_n \), let \( T^n_k \) be the coordinate subspace associated to \( \Delta^{n-k}(n) \). Specifically,

\[
T^n_k = \{(x_1, \ldots, x_n) \in X_1 \times \cdots \times X_n \mid \text{at least } k \text{ of the } x_i's \text{ are } \ast \}.
\]

In particular, \( T^n_0 = X_1 \times \cdots \times X_n \), \( T^n_1 \) is the fat wedge, \( T^n_{n-1} = X_1 \vee \cdots \vee X_n \), and \( T^n_n = \ast \). Let \( F^n_k \) be the homotopy fibre of the inclusion \( T^n_k \to T^n_0 \). By Theorem 9.1, \( F^n_k \) is homotopy equivalent to a wedge of spheres. This wedge can be calculated explicitly using the iteration in Proposition 8.1 to reproduce a result first obtained in a different context by Porter [P]. For a space \( X \) and a positive integer \( j \), let \( j \cdot X \) be the wedge sum of \( j \) copies of \( X \). Let \( X^{(j)} \) be the \( j \)-fold smash of \( X \) with itself.

**Theorem 9.3** (Porter). For \( n \geq 1 \), let \( X_1, \ldots, X_n \) be path-connected spaces. Let \( k \) be such that \( 1 \leq k \leq n - 1 \). Then there is a homotopy equivalence

\[
F^n_k \simeq \bigvee_{j=n-k+1}^n \left( \bigvee_{1 \leq i_1 < \cdots < i_j \leq n} \binom{j-1}{n-k} \Sigma^{n-k} \Omega X_{i_1} \wedge \cdots \wedge \Omega X_{i_j} \right).
\]

**Corollary 9.4.** As in Theorem 9.3, if \( X_i = X \) for each \( 1 \leq i \leq n \) then there is a homotopy equivalence

\[
F^n_k \simeq \bigvee_{j=n-k+1}^n \left( \binom{n}{j} \binom{j-1}{n-k} \Sigma^{n-k} (\Omega X)^{(j)} \right).
\]

The case of relevance to us is Corollary 9.4 applied when \( X = \mathbb{C}P^\infty \). Then \( T^n_0 \) corresponds to \( DJ(K) \) for \( K = \Delta^{n-k}(n) \). The homotopy fibre \( F^n_k \) corresponds to \( Z_K \). Since \( \Omega X \simeq S^1 \), we obtain:

**Corollary 9.5.** If \( K = (\Delta^{n-k}(n)) \), then

\[
Z_K \simeq \bigvee_{j=n-k+1}^n \binom{n}{j} \binom{j-1}{n-k} S^{n-k+j}.
\]

10. **Topological extensions**

At this point, we have shown that if a simplicial complex \( K \) is shifted then its moment-angle complex \( Z_K \) is homotopy equivalent to a wedge of spheres. Next, we want to consider other simplicial complexes \( K \) for which \( Z_K \) is homotopy equivalent to a wedge of spheres, or for which \( Z_K \) is stably equivalent to a wedge of spheres. We phrase this as follows.
**Definition 10.1.** Let $\mathcal{F}_t$ be the family of simplicial complexes $K$ for which the moment-angle complex $Z_K$ has the property that $\Sigma' Z_K$ is homotopy equivalent to a wedge of spheres.

The definition of $\mathcal{F}_t$ gives rise to a filtration

$$\mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_t \subset \ldots \subset \mathcal{F}_\infty.$$  

This filtration does not account for all simplicial complexes $K$. As mentioned in the introduction, torsion can occur in the cohomology ring of $Z_K$ for certain simplicial complexes $K$, making it impossible for $Z_K$ to be even stably homotopic to a wedge of spheres.

The problem we want to consider next is how the filtration level is affected when combinatorial operations are applied to two simplicial complexes from possibly different filtration levels. The combinatorial operations we look at are: the disjoint union of simplicial complexes, gluing along a common face and the join of simplicial complexes. Recall that for given simplicial complexes $K_1$ and $K_2$ on sets $S_1$ and $S_2$ respectively the join $K_1 \ast K_2$ is the simplicial complex

$$K_1 \ast K_2 := \{ \sigma \subseteq S_1 \cup S_2 \mid \sigma = \sigma_1 \cup \sigma_2, \sigma_1 \in K_1, \sigma_2 \in K_2 \}$$

on the set $S_1 \cup S_2$.

**Theorem 10.2.** Let $K_1 \in \mathcal{F}_t$ and $K_2 \in \mathcal{F}_s$ for some non-negative integers $t$ and $s$. The effect on family membership of the simplicial complex $K$ resulting from the following operations on $K_1$ and $K_2$ is:

1. gluing along a common face:
   - if $K = K_1 \cup_{\sigma} K_2$, then $K \in \mathcal{F}_m$ where $\sigma$ is a common face of $K_1$ and $K_2$ and $m = \max\{t, s\}$;
2. the disjoint union of simplicial complexes:
   - if $K = K_1 \coprod K_2$, then $K \in \mathcal{F}_m$ where $m = \max\{t, s\}$;
3. the join of simplicial complexes:
   - if $K = K_1 \ast K_2$, then $K \in \mathcal{F}_m$ where $m = \max\{t, s\} + 1$.

**Proof.** (1) Let $DJ(K_i)$, $i = 1, 2$, $DJ(\sigma)$ and $DJ(K)$ be the corresponding Davis-Januszkiewicz spaces. Each vertex in $K_i$, $\sigma$ or $K$ corresponds to a coordinate in $DJ(K_i)$, $DJ(\sigma)$ or $DJ(K)$ respectively. List the vertices of $K_1$ as $\{1, \ldots, l, \ldots, m\}$, where the vertices of $\sigma$ are $\{l + 1, \ldots, m\}$. List the vertices of $K_2$ as $\{l + 1, \ldots, m, \ldots, n\}$. Regard $DJ(K_1)$ as a subspace of $\prod_{i=1}^{m} \mathbb{C} P^{\infty}$. Let $D_1$ be the image of $DJ(K_1)$ under the map $\prod_{i=1}^{m} \mathbb{C} P^{\infty} \to \prod_{i=1}^{l} \mathbb{C} P^{\infty}$ given by the inclusion of the first $m$ coordinates. Similarly, regard $DJ(K_2)$ as a subspace of $\prod_{i=l+1}^{n} \mathbb{C} P^{\infty}$, and let $D_2$ be its image under the map $\prod_{i=l+1}^{n} \mathbb{C} P^{\infty} \to \prod_{i=1}^{n} \mathbb{C} P^{\infty}$ given by the inclusion of the last $n - l$ coordinates. Since $\sigma$ is a simplex, $DJ(\sigma)$ is a product of $m - l$ copies of $\mathbb{C} P^{\infty}$. Let $D_3$ be the image of $DJ(\sigma)$ in $\prod_{i=1}^{m} \mathbb{C} P^{\infty}$ under the map $\prod_{i=l+1}^{m} \mathbb{C} P^{\infty} \to \prod_{i=1}^{n} \mathbb{C} P^{\infty}$ given by the inclusion of the middle $m - l$ coordinates.
coordinates. Let $D$ be the topological pushout

\begin{align*}
D_1 & \longrightarrow D_1 \\
\downarrow & \downarrow \\
D_2 & \longrightarrow D.
\end{align*}

Then $D = DJ(K)$ and is a subspace of $\prod_{i=1}^{m} \mathbb{C}P^\infty$.

For notational convenience, let $BT^n = \prod_{i=1}^{n} \mathbb{C}P^\infty$. Map each of the four corners of pushout (16) into $BT^n$ and take homotopy fibres. This gives homotopy fibrations

$$F \longrightarrow D \longrightarrow BT^n$$

$$F_1 \times N \longrightarrow D_1 \longrightarrow BT^n$$

$$M \times F_2 \longrightarrow D_2 \longrightarrow BT^n$$

$$M \times N \longrightarrow D_3 \longrightarrow BT^n$$

where the first homotopy fibration defines $F$, $F_1$ is the homotopy fibre of $D_1 \longrightarrow \prod_{i=1}^{m} \mathbb{C}P^\infty$, $F_2$ is the homotopy fibre of $D_2 \longrightarrow \prod_{i=1}^{n} \mathbb{C}P^\infty$, $M = \prod_{i=1}^{l} S^1$, and $N = \prod_{i=m+1}^{n} S^1$. Including $D_3$ into $D_1$ gives a homotopy pullback diagram

$$\begin{array}{ccc}
\Omega BT^n & \longrightarrow & M \times N \\
\downarrow \downarrow & & \downarrow \downarrow \\
\Omega BT^n & \longrightarrow & D_3 \longrightarrow BT^n \\
\end{array}$$

for some map $\theta$ of fibres. We now identify $\theta$. With $BT^m = \prod_{i=1}^{m} \mathbb{C}P^\infty$, the pullback just described is the product of the homotopy pullback

$$\begin{array}{ccc}
\Omega BT^m & \longrightarrow & M \times N \\
\downarrow \downarrow & & \downarrow \downarrow \\
\Omega BT^m & \longrightarrow & D_3 \longrightarrow BT^m \\
\end{array}$$

and the path-loop fibration $N \longrightarrow * \longrightarrow \prod_{i=m+1}^{n} \mathbb{C}P^\infty$. So $\theta = \theta' \times N$. Further, $M = \prod_{i=1}^{l} S^1$ is a retract of $\Omega BT^m \simeq \prod_{i=1}^{l} S^1$ and $\Omega BT^m \longrightarrow F_1$ is null homotopic since $\Omega BT^m$ is a retract of $\Omega D_1 = \Omega DJ(K_1)$. Hence $\theta' \simeq *$ and so $\theta \simeq * \times N$. A similar argument for the inclusion of $D_3$ into $D_2$ shows that the map of fibres $M \times N \longrightarrow M \times F_2$ is homotopic to $M \times *$.

Collecting all this information about homotopy fibres, Lemma 3.1 shows that there is a homotopy pushout

$$\begin{array}{ccc}
M \times N & \longrightarrow & F_1 \times N \\
\downarrow \downarrow & & \downarrow \downarrow \\
M \times F_2 & \longrightarrow & F.
\end{array}$$
Lemma 3.3 then gives a homotopy decomposition

\[ F \simeq (M \ast N) \vee (M \times F_2) \vee (F_1 \times N). \]

We want to show that \( \Sigma^m F \) is homotopy equivalent to a wedge of spheres, where \( m = \max\{t, s\} \). If so, then as \( D = DJ(K) \), we have \( F = Z_K \) and hence \( K \in \mathcal{F}_{\max\{t, s\}} \), proving (1).

To show \( \Sigma^m F \) is homotopy equivalent to a wedge of spheres, we show that each of \( \Sigma^m (M \ast N) \), \( \Sigma^m (M \times F_2) \), and \( \Sigma^m (F_1 \times N) \) are homotopy equivalent to wedges of spheres. First, observe that the suspension of a product of spheres is homotopy equivalent to a wedge of spheres. Since \( M \) and \( N \) are products of copies of \( S^1 \), \( M \ast N \) is therefore homotopy equivalent to a wedge of spheres, and hence \( \Sigma^m (M \ast N) \) is as well. Second, if \( m \geq 1 \) then \( \Sigma^m (M \times F_2) \simeq \Sigma^m (M \wedge F_2) \vee \Sigma^m F_2 \). By hypothesis, \( \Sigma^m F_2 \) is homotopy equivalent to a wedge of spheres. As \( m \geq 1 \), \( \Sigma^m (M \wedge F_2) \) is the \((m - 1)\)-fold suspension of \((\Sigma M) \wedge F_2 \). But \( \Sigma M \) is homotopy equivalent to a wedge of spheres, so \((\Sigma M) \wedge F_2 \) is homotopy equivalent to a wedge of suspensions of \( F_2 \). Therefore \( \Sigma^m (M \wedge F_2) \) is homotopy equivalent to a wedge of \( m \)-fold suspensions of \( F_2 \), implying that it is homotopy equivalent to a wedge of spheres. If \( m = 0 \), then \( F_2 \) is still homotopy equivalent to a wedge of spheres, so we can write \( F_2 \simeq \Sigma^2 F_2 \), where \( \Sigma^2 F_2 \) is a wedge of spheres. We then have \( M \times F_2 \simeq M \times (\Sigma^2 F_2) \simeq \Sigma^2 (M \times F_2) \) and the decomposition into a wedge of spheres now follows as in the \( m \geq 1 \) case. The decomposition of the summand \( F_1 \times N \) into a wedge of spheres is exactly as for \( M \times F_2 \).

(2) Let \( K = K_1 \coprod K_2 \) be the disjoint union of two simplicial complexes \( K_1 \) and \( K_2 \) on the index sets \([m]\) and \([n]\) respectively. Then their disjoint union \( K = K_1 \coprod K_2 \) is a simplicial complex on the index set \([m + n]\) obtained as the result of gluing \( K_1 \) to \( K_2 \) along the empty face. Applying part (1) then shows that \( K \in \mathcal{F}_m \), where \( m = \max\{t, s\} \). Moreover, the homotopy type of \( Z_K \) is given by

\[ Z_K \simeq \left( \prod_{i=1}^{m} S^1 \ast \prod_{j=1}^{n} S^1 \right) \vee \left( Z_{K_1} \times \prod_{i=1}^{n} S^1 \right) \vee \left( \prod_{i=1}^{m} S^1 \times Z_{K_2} \right). \]

(3) The Stanley-Reisner ring of the join \( K_1 \ast K_2 \) of two simplicial complexes \( K_1 \) and \( K_2 \) on the index sets \([m]\) and \([n]\) has the following form:

\[ \mathbb{Z}[K_1 \ast K_2] = \mathbb{Z}[K_1] \otimes \mathbb{Z}[K_2]. \]

Therefore the fibration

\[ DJ(K_1 \ast K_2) \longrightarrow BT^{m+n} \]

associated to the join of \( K_1 \) and \( K_2 \) is the product fibration

\[ DJ(K_1) \times DJ(K_2) \longrightarrow BT^m \times BT^n. \]

Hence \( Z_{K_1 \ast K_2} \simeq Z_{K_1} \times Z_{K_2} \). This proves part (3) and finishes the proof of the Theorem. \( \square \)

As a corollary we specify the operations on simplicial complexes for which \( \mathcal{F}_0 \) is closed.
Corollary 10.3. Let $K_1$ and $K_2$ be simplicial complexes in $\mathcal{F}_0$. Then $\mathcal{F}_0$ is closed for the following operations on simplicial complexes:

1. gluing along a common face,
   \[ K = K_1 \cup_{\sigma} K_2 \in \mathcal{F}_0, \text{ where } \sigma \text{ is a common face of } K_1 \text{ and } K_2. \]

2. the disjoint union of simplicial complexes,
   \[ K = K_1 \bigsqcup K_2 \in \mathcal{F}_0; \]

11. Algebra

Let $A$ be a polynomial ring on $n$ variables $k[x_1, \ldots, x_n]$ over a field $k$ and let $R = A/I$, where $I$ is homogeneous ideal. In this section we shall be interested in the nature of $\text{Tor}_R(k, k)$; specifically, in identifying a class of rings $R$ for which all Massey products in $\text{Tor}_A(R, k)$ vanish and how this impacts upon the Poincaré series of $R$. Recall that the Poincaré series of $R$ is the formal power series

\[ P(R) = \sum_{i=0}^{\infty} b_i t^i \]

where $b_i = \dim_k \text{Tor}_i^R(k, k)$ are the Betti numbers of $R$. It has been conjectured by Kaplansky and Serre that $P(R)$ always represents a rational function. The regular local rings were the first rings for which $P(R)$ was explicitly computed. In this case Serre [Se] showed that $P(R) = (1 + t)^n$. Tate [T] showed that if $R$ is a complete intersection, then there exist non-negative integers $m, n$ such that

\[ P(R) = \frac{(1 + t)^n}{(1 - t^2)^m}. \]

Golod [G] made a far reaching contribution to the problem by showing that if certain homology operations on the Koszul complex vanish, then there exist non-negative integers $n, c_1, \ldots, c_n$ such that

\[ P(R) = \frac{(1 + t)^n}{1 - \sum_{i=1}^{n} c_i t^{i+1}}. \]

In general not much is known about the rationality of $P(R)$; although there is an inequality due to Golod [G] showing that $P(R)$ is always bounded (coefficient-wise) by a rational function.

In the past, describing various properties of $\text{Tor}_R(k, k)$ has been largely an algebraic problem. Further on, we translate the problem of rationality of the Poincaré series into topology by using recent results of toric topology. Then by using our results on the homotopy type of the complement of a coordinate subspace arrangement, we find a class of rings $R$ for which $P(R)$ is a rational function determined by $P(\text{Tor}_A(R, k))$.

In what follows $R$ will be the Stanley-Reisner ring $k[K]$ of an arbitrary simplicial complex $K$ on $n$ vertices. Recall from Definition 2.6 that the Stanley-Reisner ring $k[K]$ is Golod if all Massey products in $\text{Tor}_k(k[K], k)$ vanish. Buchstaber and Panov [BP] proved that

\[ \text{Tor}_k^*(k[K], k) \cong H^*(\Omega DJ(K); k). \]
This isomorphism now lets us exploit the topological properties of the loop space $\Omega DJ(K)$ to obtain further information about $\text{Tor}_R(k, k)$. Looking at the split fibration

$$\Omega Z_K \to \Omega DJ(K) \to T^n$$

we have

$$\text{Tor}_R^n(k, k) \cong H^*(\Omega DJ(K)) = H^*(T^n) \otimes H^*(\Omega Z_K).$$

A calculation using the bar resolution shows that

$$P(H^*(\Omega Z_K)) \leq P(T(\Sigma^{-1} H^*(Z_K)))$$

where $\Sigma^{-1} H^*(Z_K)$ is the desuspension of the module $H^*(Z_K)$. Therefore

$$P(R) \leq (1 + t)^n P(T(\Sigma^{-1} H^*(Z_K))) = \frac{t(1 + t)^n}{t - P(H^*(Z_K))}.$$

Looking at the Eilenberg-Moore spectral sequence (the bar resolution) that computes the cohomology of the fibre in the path-loop fibration $\Omega Z_K \to * \to Z_K$, we conclude that the above equality is reached when the differentials are trivial. According to May, the differentials are determined by the Massey products and therefore they are trivial when all the Massey products in $H^*(Z_K)$ vanish. As $H^*(Z_K) \cong \text{Tor}_{k[v_1, \ldots, v_n]}(k[K], k)$ \cite{BP}, an equality for $P(R)$ is obtained when the Stanley-Reisner ring $k[K]$ is Golod. This proves the following theorem.

**Theorem 11.1.** For a simplicial complex $K$,

$$(17) \quad P(k[K]) \leq \frac{t(1 + t)^n}{t - P(H^*(Z_K))}.$$  

Equality is obtained when $k[K]$ is Golod.

We proceed by describing a new class of Golod rings using topological methods.

**Theorem 11.2.** If $K \in F_0$, then $k[K]$ is a Golod ring.

**Proof.** By definition of the family $F_0$, when $K \in F_0$ then $Z_K$ is homotopy equivalent to a wedge of spheres. Therefore in the cohomology of $Z_K$ all cup products and higher Massey products are trivial. On the other hand, recall that Buchstaber and Panov \cite{BP} proved that

$$H^*(Z_K) \cong \text{Tor}_{k[v_1, \ldots, v_n]}(k[K], k).$$

Therefore in $\text{Tor}_{k[v_1, \ldots, v_n]}(k[K], k)$ all Massey products are trivial. Now by definition, the ring $k[K]$ is Golod. \hfill \Box

We finish by proving that the Poincaré series of a ring belonging to the class defined in Theorem 11.2 represents a rational function.
Corollary 11.3. If $K \in \mathcal{F}_0$, then the Poincaré series of the ring $k[K]$ has the following form

$$P(k[K]) = \frac{t(1 + t)^n}{t - P(H^*(\mathcal{Z}_K))}.$$

Proof. As $k[K]$ is a Golod ring, in (17) equality holds. \qed

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