Learning and Selfconfirming Equilibria in Network Games*

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Abstract

Consider a set of agents who play a network game repeatedly. Agents may not know the network. They may even be unaware that they are interacting with other agents in a network. Possibly, they just understand that their payoffs depend on an unknown state that in reality is an aggregate of the actions of their neighbors. Each time, every agent chooses an action that maximizes her subjective expected payoff and then updates her beliefs according to what she observes. In particular, we assume that each agent only observes her realized payoff. A steady state of such dynamic is a selfconfirming equilibrium given the assumed feedback. We characterize the structure of the set of selfconfirming equilibria in network games and we relate selfconfirming and Nash equilibria. Thus, we provide conditions on the network under which the Nash equilibrium concept has a learning foundation, despite the fact that agents may have incomplete information. In particular, we show that the choice of being active or inactive in a network is crucial to determine whether agents can make correct inferences about the payoff state and hence play the best reply to the truth in a selfconfirming equilibrium. We also study learning dynamics and show how agents can get stuck in non–Nash selfconfirming equilibria. In such dynamics, the set of inactive agents can only increase in time, because once an agent finds it optimal to be inactive, she gets no feedback about the payoff state, hence she does not change her beliefs and remains inactive.

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1 Introduction

Imagine an online social network, like Twitter, with many users. Let us consider a simultaneous-moves game, in which each user $i$ decides her level of activity $a_i \geq 0$ in the social network. The payoff that agents get from their activity depends on the social interaction. In particular, active user $i$ receives idiosyncratic externalities, that can be positive and negative, from the other users with whom she is in contact in the social network. The externality from user $i$ to user $j$ is proportional to the time that they both spend on the social network, $a_i$ and $a_j$. Sticking to a quadratic specification, that allows for linear best replies, let us assume that the payoff of $i$ from this game is

$$u_i(a_i, a_{-i}) = \alpha_i a_i - \frac{1}{2} a_i^2 + \sum_{j \in I \setminus \{i\}} z_{ij} a_i a_j.$$  

(1)

In eq. (1), $I$ is the set of agents in the social network and $a_i$ is the level of activity of $i \in I$, while $\alpha_i$ represents the individual pleasure of $i$ from being active on the social network in isolation, which results in the bliss point of activity in autarchy. Parameter $\alpha_i$ can also be negative, and in this case $i$ would not be active in isolation. For each $j \in I \setminus \{i\}$, there is some exogenous level of externality from $j$ to $i$ denoted by $z_{ij}$. We say that $j$ affects $i$, or that $j$ is a peer of $i$, if $z_{ij} \neq 0$.

Later on, in this paper, we will also consider an extra global term in the payoff function

$$u_i(a_i, a_{-i}) = \alpha a_i - \frac{1}{2} a_i^2 + \sum_{j \in I \setminus \{i\}} z_{ij} a_i a_j + \beta \sum_{k \in j \in I \setminus \{i\}} a_k.$$  

(2)

We can interpret this extra term as an additional pleasure that $i$ gets from being member (even if not active) of an online social network that is overall popular.

In this paper, the network described by the matrix $Z$ of all the $z_{ij}$’s is exogenous. As a first approximation, this fits a directed online social network like Twitter or Instagram, where users cannot decide who follows them. Under this interpretation, $i$ receives positive or negative externalities from those who follow her, that are proportional to her activity. $i$ acquires popularity from being active or not in the social network. Payoff represents what $i$ can indirectly observe about her own popularity (i.e. likes that she receives, people congratulating with her in real world conversations, and so on...). We imagine that $i$ cannot choose the style of what she writes, since she just follows her exogenous nature. In this interpretation, $a_i$ represents the amount of tweets that $i$ writes, and this can make her more or less popular for those who follow her, according to how her style combines with the (typically unobserved) tastes of each of her followers.

Since we are going to analyze learning dynamics and their steady states, we also have to specify what agents observe after their choices, because this affects how they update their beliefs. Twitter

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1 This is the class of games originally analyzed by Ballester et al. (2006). Bramoullé et al. (2014) is one of the more recent papers providing results for such linear-qadratic network games, and they discuss also how to generalize to games that have the same best-reply functions. Zenou (2016) surveys many applications.
user $i$ typically observes perfectly her own activity level $a_i$, but she may not observe the sign of the externalities and the activity of others. However, she gets indirect measures of her level of popularity that come from her conversations and experiences in the real world, where her popularity from Twitter affects her social and professional real life. Players of this game may have wrong beliefs about the details of the game they are playing (e.g. the structure of the network, or the value of the parameters) and about the actions of other players. With this, they update their beliefs in response to the feedback they receive, which will be their (possibly indirectly measured) payoff. This updating process may lead to a learning dynamic that does not converge to a Nash equilibrium of the game.

In this paper we address the following question: Assuming simple updating rules, under what circumstances do learning dynamics converge to a Nash equilibrium of the game and when, instead, do they just converge to a selfconfirming equilibrium where agents best reply to confirmed but possibly wrong beliefs? This question is per se interesting, and with our answers we provide novel theoretical tools for the analysis of network games. However, the application of the model to online social networks that we just anticipated can also help in understanding why we may easily observe apparently non–optimal best responses by economic agents in such an environment, such as agents who get stuck into “inactivity traps.”

Section 2 presents our baseline model. For this setting, we characterize the set of selfconfirming equilibria in Section 3, and we study the learning process in Section 4. In Section 5 we analyze a more general model that accounts for global externalities. Section 6 concludes. We devote appendices to proofs and technical results. Appendix A analyzes properties of feedback and selfconfirming equilibria in a class of games including as special cases the network games that we consider. Appendix B reports existing results in linear algebra, that we use to find sufficient conditions for reaching interior Nash equilibria in network games. Appendix C contains the proofs of our propositions.

## 2 The framework

Consider a set $I$ of agents, with cardinality $n = |I|$ and generic element $i$, located in a network. Let the network be characterized by an adjacency matrix $Z \in \mathbb{R}^{I \times I}$, where entry $z_{ij}$ specifies whether agent $i$ is linked to agent $j \neq i$ and the weight of this link, and we let $z_{ii} = 0$ by convention. In what follows we consider the case of directed networks, so that, given $i, j \in I$, we allow $z_{ij} > 0$, and $z_{ji} = 0$. Externality weights are an unknown parameter of the model. We assume that there are commonly known upper and lower bounds $\bar{w}$ and $w$ in the weighted externalities, that can be positive or negative, between players. We let $\Theta \subseteq [w, \bar{w}]^{I \times I}$ denote the compact set of possible weighted networks $Z$. The network game is parametrized by $Z \in \Theta$.

Throughout the paper we will play with different properties and specifications of matrix $Z$. To simplify the notation we will often decompose it in a way that distinguishes between the actual
links, that specify if there is an externality between two players, and the magnitude and the sign of this externality. We call \( Z_0 \in \{0,1\}^{I \times I} \) the basic underlying representation of the network, the adjacency matrix whose \( ij \) element specifies whether the action of \( j \) has an externality on \( i \). We think of it as a link from \( i \) to \( j \) because \( j \) is one of \( i \)'s peers. \( Z_0 \) is a directed network.

On top of that we build \( Z \) adding weights on the links of \( Z_0 \). This can be done in several ways, depending on how much heterogeneity we want to allow for. We will write \( Z = \gamma Z_0 \) when all links bear the same level of externality \( \gamma \in [w, \bar{w}] \). We will write \( Z = \Gamma Z_0 \), where \( \Gamma \) is a diagonal matrix, when we want to specify that each player \( i \) is affected by the same weight \( \gamma_i \in [w, \bar{w}] \) from all her peers, but these \( \gamma_i \)'s are heterogeneous. We will also consider the case in which the existing links may have weights of different signs but the same intensity. That is, we write \( Z = S \odot Z_0 \) (in which the operator \( \odot \) is the Hadamard product), for \( \gamma \in [w, \bar{w}] \), and \( S \in \{-\gamma, \gamma\}^{I \times I} \). Finally, when we write simply \( Z \) we consider the case of a directed weighted network \( Z \in \Theta \). Many of our results will hold for this most general case.

Each agent \( i \in I \) chooses an action \( a_i \) from interval \( A_i = [0, \bar{a}_i] \), where the upper bound \( \bar{a}_i \) is “sufficiently large”.\(^2\) For each \( i \in I \), \( A_{-i} := \times_{j \neq i} A_j \) denotes the set of feasible action profiles \( a_{-i} = (a_j)_{j \in I \setminus \{i\}} \) for players different from \( i \). Similarly, defining \( N_i := \{ j \in I : z_{ij} \neq 0 \} \) as the set of the neighbors of a given agent \( i \), \( A_{N_i} := \times_{j \in N_i} A_j \) denotes the set of feasible action profiles \( a_{N_i} := (a_j)_{j \in N_i} \) of \( i \)'s neighbors.

For each \( i \in I \), we posit a set (interval) \( X_i = [\underline{x}_i, \bar{x}_i] \) of payoff states for \( i \), with the interpretation that \( i \)'s payoff is determined by her action \( a_i \) and by her payoff state \( x_i \) according to a continuous utility function \( v_i : A_i \times X_i \rightarrow \mathbb{R} \). The payoff state \( x_i \) is in turn determined by the actions of \( i \)'s neighbors and is unknown to \( i \) at the time of his choice. For each agent \( i \in I \) and matrix \( Z \), we consider a parametrized aggregator of the coplayer’s actions \( \ell_i : A_{-i} \times \Theta \rightarrow X_i \) of the following form: \( \ell_i \) is continuous, its range \( \ell_i (A_{-i} \times \Theta) \) is connected, and for each \( Z \in \Theta \), the section of \( \ell_i \) at \( Z \) is\(^3\)

\[
\ell_{i,Z} : A_{-i} \rightarrow X_i, \quad a_{-i} \mapsto \sum_{j \neq i} z_{ij} a_j.
\]

Note, since \( X_i \) is the codomain of \( \ell_i \), we are effectively assuming that, for every \( Z \in \Theta \),

\[
\underline{x}_i \leq \sum_{j \in N_i^-} z_{ij} \bar{a}_j, \quad \bar{x}_i \geq \sum_{j \in N_i^+} z_{ij} \bar{a}_j,
\]

where \( N_i^- := \{ j \in I : z_{ij} < 0 \} \) denote the set of neighbors of player \( i \) that have a negative effect.

\(^2\)Note that in the network literature it is common to assume \( A_i = \mathbb{R}_+ \). However, for the games we consider, we can always find an upper bound \( \bar{a} \) on actions such that the problem is unchanged when actions are bounded above by \( \bar{a} \).

\(^3\)In principle we can allow for non-linear aggregators, as in Feri and Pin (2017). However, in this paper, we focus on the linear case.
on the payoff state of $i$. Similarly, $N_i^+ := \{ j \in I : z_{ij} > 0 \}$ denotes the set of neighbors of player $i$ that have a positive effect on the payoff state of $i$.

The overall payoff function that associates each action profile $(a_i, a_{-i})$ with a payoff for agent $i$ is thus parametrized by the adjacency matrix $Z$:

$$u_i : A_i \times A_{-i} \times \Theta \to \mathbb{R},$$

$$(a_i, a_{-i}, Z) \mapsto v_i(a_i, \ell_i(a_{-i}, Z)).$$

We assume that each agent $i$ knows how her payoff depends on her action and her payoff state, that is, we assume that $i$ knows function $v_i$, but we do not assume that $i$ knows $Z$. Actually, from the perspective of our analysis, agent $i$ might even ignore that the payoff state $x_i$ aggregates her neighbors’ activities according to some weighted network structure, because we are not modeling how $i$ reasons strategically.\footnote{If the parametrized payoff functions and the parameter space $\Theta$ are common knowledge, strategic reasoning according to the epistemic assumptions of rationality and common belief in rationality can be captured by a simple incomplete-information version of the rationalizability concept. See, e.g., Chapter 7 of Battigalli (2018) and the references therein.}

If $v_{i,x_i} : A_i \to \mathbb{R}$ is strictly quasi-concave for each $x_i$, there is a unique best reply $r_i(x_i)$ to each payoff state $x_i$. Although the aggregator is linear, if this “proximate” best reply function $r_i : X_i \to A_i$ is non-linear,\footnote{More precisely, not affine.} then also the best reply $r_i(\ell_i(a_{-i}, Z))$ is non-linear in $a_{-i}$. Linearity obtains if and only if $v_i$ is quadratic in $a_i$ and linear in $x_i$. Without substantial loss of generality, among such utility functions we consider the following form, generalizing equation (1) that we discussed earlier:

$$v_i : A_i \times X_i \to \mathbb{R},$$

$$(a_i, x_i) \mapsto \alpha_i a_i - \frac{1}{2}a_i^2 + a_i x_i.$$ \hspace{1cm} (4)

Note that $v_i$ in eq. (4) is continuous and strictly concave in $a_i$. Thus, $G = \langle I, \Theta, (A_i, u_i)_{i \in I} \rangle$, with $u_i$ defined by eqs. (3)-(4), is a parametrized nice game (see Moulin 1984 for a definition of nice game, and Appendix A for a generalization, with results for non-linear-quadratic network games).

We assume that the game is repeatedly played by agents maximizing their instantaneous payoff. After each play agents get some feedback. Let $M$ be an abstract set of “messages” (e.g., monetary outcomes). The information obtained by agent $i \in I$ at the end of each period is described by a feedback function $f_i : A_i \times X_i \to M$. Assuming that $i$ knows how her feedback is determined by the payoff state given her action, if she receives message $m$ after action $a_i$ she infers that the state $x_i$ belongs to the “ex post information set”

$$f_{i,a_i}^{-1}(m) := \{ x_i' \in X_i : f_i(a_i, x_i') = m \}.$$ 

This completes the description of the object of our analysis. The structure

$$NG = \langle I, \Theta, (A_i, X_i, v_i, \ell_i, f_i)_{i \in I} \rangle$$
is a (parameterized) network game with feedback, or simply network game. Our analysis depends on assumptions about the payoff functions and the feedback functions. Here we present the strongest assumptions, the Appendix contains a more general analysis.

**Definition 1.** A network game with feedback NG is **linear-quadratic** if the utility function of each player has the linear-quadratic form (4).

In this case, the proximate best-reply function is

\[
    r_i(x_i) = \begin{cases} 
        0, & \text{if } x_i \leq -\alpha_i, \\
        \alpha_i + x_i, & \text{if } -\alpha_i < x_i < \bar{a}_i - \alpha_i, \\
        \bar{a}_i, & \text{if } x_i \geq \bar{a}_i - \alpha_i. 
    \end{cases}
\]

(5)

Even if agent \(i\) may play a best reply to the aggregate \(x_i\), it is possible to write the derived best reply to the actions of others as

\[
    r_i(\ell_i(a_{-i}, Z)) = \begin{cases} 
        0, & \text{if } \sum_{j \neq i} z_{ij}a_j \leq -\alpha_i, \\
        \alpha_i + \sum_{j \neq i} z_{ij}a_j, & \text{if } -\alpha_i < \sum_{j \neq i} z_{ij}a_j < \bar{a}_i - \alpha_i, \\
        \bar{a}_i, & \text{if } \sum_{j \neq i} z_{ij}a_j \geq \bar{a}_i - \alpha_i. 
    \end{cases}
\]

(6)

**Definition 2.** Feedback \(f_i\) satisfies **observability if and only if player \(i\) is active (OiffA)** if section \(f_i,a_i\) is injective for each \(a_i \in (0,\bar{a}_i]\) and constant for \(a_i = 0\); \(f_i\) satisfies **just observable payoffs (JOP)** relative to \(v_i\) if there is a function \(\bar{v}_i : A_i \times M \rightarrow \mathbb{R}\) such that

\[
    \forall (a_i, x_i) \in A_i \times X_i, \quad v_i(a_i, x_i) = \bar{v}_i(a_i, f_i(a_i, x_i))
\]

and the section \(\bar{v}_i,a_i : M \rightarrow \mathbb{R}\) is injective for each \(a_i \in A_i\). A network game with feedback NG satisfies **observability by active players** if feedback \(f_i\) satisfies OiffA, for each player \(i \in I\), and it satisfies **just observable payoffs** if \(f_i\) satisfies JOP for each player \(i \in I\).

In a game with just observable payoffs, because of injectivity of the feedback function, agents infer their realized payoff from the message they get, but no more than that, that is, inferences about the payoff state can be obtained by looking at the preimages of the payoff function. For example, the feedback could be a total benefit, or revenue function

\[
    f_i : A_i \times X_i \rightarrow \mathbb{R}, \\
    (a_i, x_i) \mapsto \alpha_i a_i + a_i x_i,
\]

with the payoff given by the difference between benefit and activity cost \(C_i(a_i)\):

\[
    v_i : A_i \times X_i \rightarrow \mathbb{R}, \\
    (a_i, x_i) \mapsto f_i(a_i, x_i) - C_i(a_i).
\]

Under the reasonable assumption that agent \(i\) knows her cost function, when she chooses \(a_i\) and then gets message \(m\), she infers that her payoff is \(\bar{v}_i(a_i, m) = m - C_i(a_i)\). Thus, each section \(\bar{v}_i,a_{-i}\) \((a_i \in A_i)\) is indeed injective. If the feedback/benefit function is \(f_i(a_i, x_i) = \alpha_i a_i + a_i x_i\), then it satisfies observability if and only if \(i\) is active.
Remark 1. If NG is linear-quadratic and satisfies just observable payoffs, then it satisfies observability by active players. If NG satisfies observability by active players, then

\[ f_{i,a_i}^{-1}(f_i(a_i, x_i)) = \begin{cases} X_i, & \text{if } a_i = 0, \\ \{x_i\}, & \text{if } a_i > 0 \end{cases} \]  

for every agent \( i \in I \) and action-state pair \((a_i, x_i) \in A_i \times X_i\).

Most of our analysis focuses on linear-quadratic network games with just observable payoffs. This implies that agents who are active get as feedback a message enabling them to perfectly determine the state. Conversely, inactive agents get a completely uninformative message.

To choose an action, subjectively rational agents must have some deterministic or probabilistic conjecture about the payoff state \( x_i \). We refer to conjectures about the state as shallow conjectures, as opposed to deep conjectures, which concern the specific network topology and the actions of other players \((a_{-i})\). In linear-quadratic network games (more generally, in nice games with feedback), it is sufficient to focus on deterministic shallow conjectures. Indeed, for every probabilistic conjecture \( \mu_i \in \Delta(X_i) \), there exists a deterministic conjecture \( \hat{x}_i \in X_i \) that justifies the same action \( a^*_i \) as the unique best reply (see the discussion in A.1).

2.1 Selfconfirming equilibrium

We analyze a notion of equilibrium which is broader than Nash equilibrium. Recall that our approach allows for the possibility of agents who are unaware of the full game around them. In equilibrium, agents best respond to conjectures consistent with the feedback that they receive, which is not necessarily fully revealing. We believe that this approach fits well to a networked environment where agents’ knowledge and the information they receive are only local.\(^6\)

Definition 3. A profile \((a^*_i, \hat{x}_i)_{i \in I} \in \times_{i \in I} (A_i \times X_i)\) of actions and (shallow) deterministic conjectures is a selfconfirming equilibrium (SCE) at \( Z \) if, for each \( i \in I \),

1. (subjective rationality) \( a^*_i = r_i(\hat{x}_i) \),
2. (confirmed conjecture) \( f_i(a^*_i, \hat{x}_i) = f_i(a^*_i, \ell_i(a^*_i, Z)) \).

The two conditions require that 1) each agent best responds to her own conjectures; 2) the conjectures in equilibrium must belong to the ex-post information set so that the expected feedback coincides with the actual feedback at \( \ell_i(a^*_i, Z) \). We say that \( a^* = (a^*_i)_{i \in I} \) is a selfconfirming equilibrium.

\(^6\)In a context of endogenous strategic network formation, McBride (2006) applies the conjectural equilibrium concept, which is essentially the same as selfconfirming equilibrium for games with feedback (see Battigalli et al. (1992) and the discussions in Battigalli et al. 2015). More recently, also Lipnowski and Sadler (2017) and Frick et al. (2018) have adopted self-confirming equilibrium notions to describe network games. Their assumptions and their results are different and independent from ours.
**Action profile** at $Z$ if there exists a corresponding profile of conjectures $(\hat{x}_i)_{i \in I}$ such that $(a_i^*, \hat{x}_i)_{i \in I}$ is a selfconfirming equilibrium at $Z$, and we let $A^{SCE}_Z$ denote the set of such profiles. Also, for any adjacency matrix $Z \in \Theta$, we denote by $A^{NE}_Z$ the set of (pure) Nash equilibria of the (nice) game determined by $Z$, that is,

$$A^{NE}_Z := \{ a^* \in \times_{i \in I} A_i : \forall i \in I, a^*_i = r_i (\ell_i (a_{-i}^*, Z)) \}.$$  

Nice games satisfy all the standard assumptions for the existence of Nash equilibria.$^7$ Hence, we obtain the existence of selfconfirming equilibria for each $Z \in \Theta$. Indeed a Nash equilibrium is a selfconfirming equilibrium with correct conjectures. To summarize:

**Remark 2.** For every $Z$, there is at least one Nash equilibrium, and every Nash equilibrium is a selfconfirming profile of actions:

$$\forall Z \in \Theta, \emptyset \neq A^{NE}_Z \subseteq A^{SCE}_Z.$$  

### 3 A characterization of SCE

In this section we characterize the set $A^{SCE}_Z$ of selfconfirming equilibrium profiles of actions in linear-quadratic network games with just observable payoffs. All our proofs are derived from the results in Appendix A and Appendix B, which refer to the case of generic network games without the restriction to linear best replies, and are stated in Appendix C. We start with the simplest case in which every agent necessarily finds it subjectively optimal to be active (that is, being inactive is dominated – see Lemma A in Appendix A).

**Proposition 1.** Consider a network game $NG$ satisfying observability by active players. Assume that, for every $i \in I$ and for every $\hat{x}_i \in X_i$, $r_i (\hat{x}_i) > 0$. Then, for each $Z \in \Theta$, $A^{SCE}_Z = A^{NE}_Z$.

Assume that $\alpha_i$ (from eqs (4) and (5)) is such that $\alpha_i > 0$. Assume further that $Z = \gamma Z_0$, with $\gamma > 0$ and that $Z_0 \in \{0, 1\}^{I \times I}$. This represents the standard case of local complementarities studied by Ballester et al. (2006). If $\gamma (n - 1) < 1$ there is a unique Nash equilibrium which is also interior. Our proposition states that, in this case, if being inactive is not justifiable as a best reply to any shallow conjecture, then there is only one selfconfirming equilibrium action profile, which necessarily coincides with the unique Nash equilibrium.

We now consider a more general case in which agents may be inactive. Let $I_0$ denote the set of players for whom being inactive is justifiable. Note that, by Lemma A in Appendix A,

$$I_0 = \{ i \in I : \min r_i (X_i) = 0 \}.$$  

$^7$Since the self-map $a \mapsto (r_i (a_{-i}, Z))_{i \in I}$ is continuous on the convex and compact set $A = \times_{i \in I} [0, \bar{a}_i]$, by Brouwer’s Theorem it has a fixed point.
Also, for each $\mathbf{Z} \in \Theta$ and non-empty subset of players $J \subseteq I$, let $A^{NE}_{J, \mathbf{Z}}$ denote the set of Nash equilibria of the auxiliary game with player set $J$ obtained by imposing $a_i = 0$ for each $i \in I \setminus J$, that is,

$$A^{NE}_{J, \mathbf{Z}} = \left\{ \mathbf{a}^*_J \in \times_{j \in J} A_j : \forall j \in J, a^*_j = r_j \left( \ell_j \left( \mathbf{a}^*_{J \setminus \{j\}} \right), \mathbf{0}_{I \setminus J}, \mathbf{Z} \right) \right\},$$

where $\mathbf{0}_{I \setminus J} \in \mathbb{R}^{I \setminus J}$ is the profile that assigns 0 to each $i \in I \setminus J$. If $J = \emptyset$, let $A^{NE}_{J, \mathbf{Z}} = \{ \emptyset \}$ by convention, where $\emptyset$ is the pseudo-action profile such that $(\emptyset, \mathbf{0}_I) = \mathbf{0}_I$.

We relate the set of selfconfirming equilibria to the sets of Nash equilibria of such auxiliary games.

**Proposition 2.** Suppose that network game with feedback $\text{NG}$ is linear-quadratic and satisfies just observable payoffs. Then, for each $\mathbf{Z} \in \Theta$, the set of selfconfirming action profiles is

$$A^{SCE}_{\mathbf{Z}} = \bigcup_{I \setminus J \subseteq I_0} A^{NE}_{J, \mathbf{Z}} \times \{ \mathbf{0}_{I \setminus J} \},$$

that is, in each SCE profile $\mathbf{a}^*$, a subset $I \setminus J$ of players for whom being inactive is justifiable choose 0, and every other player chooses the best reply to the actions of her coplayers. Therefore, in each SCE profile $\mathbf{a}^*$ and for each player $i \in I$,

$$a^*_i = 0 \Rightarrow x_i \leq -\alpha_i,$$

$$a^*_i > 0 \Rightarrow \left( \alpha_i + \sum_{j \in I} z_{ij}a^*_j > 0 \land a^*_i = \min \left\{ \bar{a}_i, \alpha_i + \sum_{j \in I} z_{ij}a^*_j \right\} \right). \quad (8)$$

In every SCE we can partition the set of agents in two subsets. Agents in $J \subseteq I$ are active, i.e., they choose a strictly positive action, agents in $I \setminus J$ instead choose the null action. Start considering the latter. Since they play $a^*_i = 0$, they get null payoff independently of others' actions. But, since every conjecture $\hat{x}^i \in (-\infty, -\alpha_i]$ is consistent with this payoff, their conjecture is (trivially) consistent with their feedback. As for agents in $J$, since they choose a strictly positive action $a^*_i > 0$, they receive a message that enables them to infer the true payoff state $x^i$; with this, they necessarily choose the objective best reply to their neighbours actions, whether or not they are aware of them. Note that, if being inactive is justifiable for every agent ($I_0 = I$), then $\mathbf{0}_I \in A^{SCE}_{\mathbf{Z}}$ for every $\mathbf{Z} \in \Theta$.

This implies that the set of selfconfirming equilibria can be characterized by means of the sets of Nash equilibria of the auxiliary games in which only active agents are considered. If, for example, there is a unique interior Nash equilibrium for the auxiliary game corresponding to every subset of active players, then $|A^{SCE}_{\mathbf{Z}}| = 2^{|J|}$, that is, there are exactly $2^n$ SCE action profiles. A.3 discusses the equilibrium characterization for the generalized case of non linear-quadratic network games.

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*As we do in set theory with the empty set, when we consider functions whose domain is a subset of some index set $I$, it is convenient to have a symbol for the pseudo-function with empty domain. For example, if $I = \mathbb{N}$, such functions are (finite and countably infinite) sequences, or subsequences, and $\emptyset$ is the empty sequence.*
**Example 1.** Consider Figure 1, representing a network between 4 nodes. We set $\alpha_i = 0.1$ for each player $i$. Let us first assume that each arrow represents a positive externality of 0.2 (and arrows point to the source of the externality). In this case we have one NEs, but 16 possible SCEs, one for each subset of the players that we allow to be active. Table 1 reports the action of players in each case (we omit redundant pairs and singletons). Note that player 3, when active, always plays the same action $a_3 = 0.1$, because she is not affected by any externality. Other players, instead, play differently when active, according to who else is active.

![Figure 1: A network between 4 nodes. Every arrow is for an externality of equal magnitude and sign.](image)

|        | All   | {1, 2, 3} | {1, 2, 4} | {1, 3, 4} | {2, 3, 4} | {1, 2}  | {1, 3}  | {1, 4}  | ... | $\emptyset$ |
|--------|-------|-----------|-----------|-----------|-----------|---------|---------|---------|-----|-----------|
| $a_1$  | 0.1292| 0.1       | 0.125     | 0.1292    | 0         | 0.1     | 0.1     | 0.125   | 0   | 0         |
| $a_2$  | 0.1750| 0.14      | 0.15      | 0         | 0.144     | 0.12    | 0       | 0       | 0   | 0         |
| $a_3$  | 0.1   | 0.1       | 0         | 0.1       | 0.1       | 0.1     | 0.1     | 0       | 0   | 0         |
| $a_4$  | 0.1458| 0         | 0.125     | 0.1458    | 0.12      | 0       | 0       | 0.125   | 0   | 0         |

Table 1: Self confirming equilibria of the network from Figure 1, with all positive externalities of 0.2. The unique Nash Equilibrium is in bold.

Consider now the same network, but assume that each arrow represents a negative externality of 0.6. In this case we have more NEs (there is not a NE where all players are active, but there are 3 NEs), but less than 16 SCEs (there are 13), because for some subset $J$ of players (such as $J = I = \{1, 2, 3, 4\}$) there is no SCE in which all its elements are active. Table 2 reports the actions of players in each case (we omit redundant pairs and singletons).
Table 2: Self confirming equilibria of the network from Figure 1, with all negative externalities of −0.6. Nash Equilibria are in bold.

This simple example shows that moving from a case of full complementarity to a case of full substitutability, we may increase the number of Nash Equilibria and decrease the number of SCEs. However, even in the limiting case where substitution effects are extremely strong, the two sets of equilibria will not coincide, because the strategy profile in which everyone is inactive will be a an SCE but not an NE.

3.1 Assumptions about the network

Next, we focus on the network $Z$. We list below some properties of matrix $Z$ that are not maintained assumptions. In different parts of the paper we will use some of these assumptions to have sufficient conditions for the existence and stability of selfconfirming equilibria. We refer to Appendix B for a deeper discussion on these assumptions and their implications.

**Assumption 1.** Matrix $Z$ of size $n$ has bounded values, i.e. $|z_{ij}| < \frac{1}{n}$ for all $i$ and $j$.

**Assumption 2.** Matrix $Z$ has the same sign property i.e., for every $i, j$, $\text{sign}(z_{ij}) = \text{sign}(z_{ji})$, where the sign function can have values $-1$, $0$ or $1$.

**Assumption 3.** Matrix $Z$ is negative, i.e. $z_{ij} < 0$ for all $i$ and $j$,

We recall here that the spectral radius $\rho(Z)$ of $Z$ is the largest absolute value of its eigenvalues.

**Assumption 4.** Matrix $Z$ is limited, i.e. $\rho(Z) < 1$.

In Section 2 we discussed how, in some cases, we can write $Z$ as $Z = \Gamma Z_0$, where $\Gamma$ is a diagonal matrix, and $Z_0$ is the basic underlying representation of the network. When this is possible, matrix $Z$ represents a basic network combined with an additional idiosyncratic effect by which every agent $i$ weights the effects of the others on her. This effect is modeled by the parameter $\gamma_i$.

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9The sign condition is the one used in Bervoets et al. (2016) to prove convergence to Nash equilibria in network games, under a particular form of learning.

10Then the payoff of $i \in I$ at a given profile $a$ of the original game is

$$u_i(a) = \alpha a_i - \frac{1}{2} a_i^2 + a_i \gamma_i \sum_{j \in I} z_{0,ij} a_j = \alpha a_i - \frac{1}{2} a_i^2 + a_i \sum_{j \in I} z_{ij} a_j.$$
assumption adds an additional condition on $Z_0$.

**Assumption 5.** Matrix $Z$ is symmetrizable, i.e. it can be written as $Z = \Gamma Z_0$, with $\Gamma$ diagonal and $Z_0$ symmetric. Moreover, $\Gamma$ has all positive entries in the diagonal.

Note that if $Z$ is symmetrizable then all its eigenvalues are real. Moreover, since $\Gamma$ has all positive entries, Assumption 5 implies the sign condition from Assumption 2. Our final assumption is discussed in Bramoullé et al. (2014) and combines Assumptions 4 and 5 above.

**Assumption 6.** $Z = \Gamma Z_0$ is symmetrizable-limited, i.e. $Z$ is symmetrizable and, for every $i, j$, $z_{ij} = z_{0,ij} \sqrt{\gamma_i \gamma_j}$, is limited.

Our previous results from Section 3, about the characterization of selfconfirming equilibria, state that we can choose any subset of agents and have them inactive in a SCE. However we cannot ensure that the other agents are active, because their best response in the reduced game could be null. The next result goes in the direction of specifying under which sufficient conditions this does not happen. Given the matrix $Z$, and given $J \subseteq I$, we call $Z_J$ the submatrix who has only rows and columns corresponding to the elements of $J$.

**Proposition 3.** Consider a set $J \subseteq I$. Let us assume that $Z_J$ satisfies at least one of the three conditions below:

1. it has bounded values (Assumption 1),
2. it is negative and limited (Assumptions 3 and 4),
3. or it is symmetrizable–limited (Assumption 6).

Then, we have the two following results:

1. $A_{NE}^J = \{a_{NE}^J\}$, such that $a_{NE}^J > 0$;
2. There exists $a^* \in A_{SCE}^Z$ such that $a^* = \{a_{NE}^J\} \times \{0_{I \setminus J}\}$.

Proposition 3 provides sufficient conditions to have an arbitrary set of active and inactive players in a selfconfirming equilibrium. In this case the set of selfconfirming equilibria has cardinality equal to the cardinality of the power set $2^I$, that is $2^n$.

We provide here below two examples, one with all positive externalities, the other with mixed externalities.

**Example 2.** Consider $n$ players, and a randomly generated network between them, of the type $Z = \Gamma Z_0$, generated by the following generating process. $Z_0$ is undirected, generated by an Erdős
and Rényi (1960) process for which each link is i.i.d., and such that its expected number of overall 
links (i.e., counted in both directions) is $k \cdot n$, for some $k \in \mathbb{R}_+$. This means that the expected 
number of links for each player is $k$. It is well known that this model predicts, as $n$ goes to infinity, 
that $Z_0$ will have no clustering and, when $k \geq 2$, a connected giant component.

$\Gamma$ is a diagonal matrix, such that each element $\gamma_i$ in the diagonal is positive and is generated 
by some i.i.d. random process with mean $\mu$ and variance $\sigma^2$. 

In this case, Füredi and Komlós (1981) prove that the expected highest eigenvalue of $Z$, as $n$ grows, is 

$$E(\lambda_i) = k\mu + \frac{\sigma^2}{\mu} + O\left(\frac{1}{\sqrt{n}}\right).$$

From Proposition 3, under Assumption 6, as $n$ tends to infinity, $Z$ is symmetrizable–limited if $E(\lambda_i) < 1$, which implies that 

$$\frac{\mu - \sigma^2}{\mu^2} > k.$$ 

Clearly, a necessary condition for previous inequality to hold is that $\mu > \sigma^2$.

When this happens, as $n$ grows to infinity, we will always have a unique NE of the game where all 
players are active.

Note that this limiting result excludes the possibility (because the expected clustering of $Z_0$ goes 
to 0) that there is a subset $J$ of players, that have a dense sub–network between them, and a high 
realization of $\gamma_i$’s, such that there does not exist $a^* \in A^{SCE}_Z$, for which 

$$a^* = \{a^{NE}_J\} \times \{0_{I \setminus J}\}.$$ 

In fact, if this was the case, because of only positive externalities, we would not even have an all 
active equilibrium for the whole population of $n$ agents.

**Example 3.** Proposition 3 provides alternative conditions, that are only sufficient, for interior NE 
in an auxiliary game in which only agents in $J$ are considered. Figure 2 provides an example of 
game that do not satisfy any of them, but still has a unique interior NE. We set $\alpha_i = 0.1$ for each 
player $i$. Every blue arrow stands for a positive externality of 0.2 (so, the blue arrows represent 
just the first case from Example 1). The two red arrows stand for a negative externality of 0.2. 
This network game has a unique NE, and 16 SCE. Table 3 shows them all (redundant couples and 
singleton are omitted).

| All   | {1, 2, 3} | {1, 2, 4} | {1, 3, 4} | {2, 3, 4} | {1, 2}  | {1, 3}  | {1, 4}  | {2, 3}  | ∅      |
|-------|-----------|-----------|-----------|-----------|---------|---------|---------|---------|--------|
| $a_1$ | 0.1257    | 0.1       | 0.125     | 0.128     | 0       | 0.1     | 0.125   | 0       | 0      |
| $a_2$ | 0.1603    | 0.1346    | 0.15      | 0         | 0.144   | 0.12    | 0       | 0       | 0.1154 |
| $a_3$ | 0.0412    | 0.731     | 0         | 0.720     | 0.1     | 0       | 0.1     | 0       | 0.0729 |
| $a_4$ | 0.1336    | 0         | 0.125     | 0.14      | 0.12    | 0       | 0       | 0.125   | 0      |

Table 3: Self confirming equilibria of the network from Figure 2, with positive externalities of 0.2 
and negative externalities of $-0.2$. The unique Nash Equilibrium is in bold.
4 Learning process

We have not considered any dynamics yet. Definition 3 of selfconfirming equilibrium, characterized also by the conditions stated in Proposition 2, identifies steady states: If agents happen to have selfconfirming conjectures and play accordingly, then they have no reason to move away from it. However we may wonder how agents get to play SCE action profiles, and if these profiles are stable.

We first notice that SCE has solid learning foundations.\(^{11}\) The following result is specifically relevant for this paper (see Gilli (1999) and Chapter 6 of Battigalli (2018)). Consider a sequence in time of action profiles, given by \((a_t)^\infty_{t=0}\). Then, if \((a_t)^\infty_{t=0}\) is consistent with adaptive learning\(^{12}\) and \(a_t \to a^*\), it follows that \(a^*\) must be a selfconfirming equilibrium action profile.

Of course, the limit of the trajectory may or may not be a Nash equilibrium. Let us now consider a best response dynamics. This generates trajectories that—by construction—are consistent with adaptive learning. With this, we prove convergence (under reasonable assumptions), hence convergence to an SCE.

To ease the analysis we consider best reply dynamics for shallow conjectures. For each period \(t \in \mathbb{N}\) and each agent \(i \in I\), \(a_{i,t} = r(\hat{x}_{i,t})\) is the best reply to \(\hat{x}_{i,t}\). After actions are chosen, given the feedback received, agents update their conjectures. If conjectures are confirmed then an agent keeps past conjecture, otherwise she updates using as new conjecture the conjecture that would

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\(^{11}\)See, for example, Battigalli et al. (1992), Battigalli and Marinacci (2016), Fudenberg and Kreps (1995), and the references therein.

\(^{12}\)In a finite game, a trajectory \((a_t)^\infty_{t=0}\) is consistent with adaptive learning if for every \(\hat{t}\), there exists some \(T\) such that, for every \(t > \hat{t} + T\) and \(i \in I\), \(a_{i,t}\) is a best reply to some deep conjecture \(\mu_i\) that assigns probability 1 to the set of action profiles \(a_{-i}\) consistent with the feedback received from \(\hat{t}\) through \(t - 1\). The definition for compact, continuous games is a bit more complex (cf Milgrom and Roberts (1991)), who assume perfect feedback).
have been correct in the past period. In details,

$$\hat{x}_{i,t+1} = \begin{cases} 
\hat{x}_{i,t} & \text{if } a_{i,t} = 0, \\
\ell_i(a_{-i,t}, Z) & \text{if } a^*_{i,t} > 0,
\end{cases} \quad (9)$$

and, from (5) (considering that the upper bound $\bar{a}_i$ is set so that it is never reached) we have simply

$$a_{i,t+1} = r_i(\hat{x}_{i,t+1}) = \begin{cases} 
0, & \text{if } \hat{x}_{i,t} \leq -\alpha_i, \\
\alpha_i + \hat{x}_{i,t+1}, & \text{if } \hat{x}_{i,t} > -\alpha_i.
\end{cases}$$

Coherently with the previous analysis, this update rule states that if an agent $i$ at time $t$ is inactive ($a_{i,t} = 0$), past conjectures are confirmed and thus kept. If instead the agent is active ($a_{i,t} > 0$), feedback is such that agents can perfectly infer the payoff state $x_{i,t} = \ell_i(a_{-i,t}, Z)$, and so they update conjectures according to (9). This is one possible adaptive learning dynamics. The result cited above implies that if the dynamics described above converges, then it must converge to a selfconfirming equilibrium, i.e., a rest point where players keep repeating their choices.

In this section we analyze the stability of such rest points in the simplest possible case of robustness to small perturbations, as in Bramoullé and Kranton (2007). However, we will not consider perturbations to the strategy profile, but perturbations on the profile of conjectures.

**Definition 4** (Learning process). Each player $i \in I$ starts at time 0 with a belief, and beliefs are represented by a vector of shallow deterministic conjectures $\hat{x}_0 = (\hat{x}_i, 0)_{i \in I}$. In each period $t$ players best reply to their conjectures: for each $i \in I$, $a_{i,t} = \max\{\alpha_i + \hat{x}_{i,t}, 0\}$.

At the beginning of each period $t+1$ each player $i$ keeps his $t$-period shallow conjecture if he was inactive, and updates his conjecture to period-$t$ revealed payoff state if he was active, that is,

$$\hat{x}_{i,t+1} = \frac{u_i(a_i)}{\alpha_i} - \alpha_i + \frac{1}{2}a_{i,t}.$$

Even if we consider the case of linear best replies, from equations (8) and (9), the system is not linear because

$$\hat{x}_{i,t+1} = \begin{cases} 
\hat{x}_{i,t} & \text{if } \hat{x}_{i,t} \leq -\alpha_i, \\
\sum_{j \in I} z_{ij}a_{j,t} & \text{if } \hat{x}_{i,t} > -\alpha_i,
\end{cases}$$

and for every other player $j$, we have that $a_{j,t} = \max\{\alpha_j + \hat{x}_{j,t}, 0\}$.

Clearly an SCE of the game, as defined in the beginning of Section 3, is always a rest point of this learning dynamic. We now consider the stability of such rest points $a^*$. Say that a profile of conjectures $\hat{x}$ is consistent with $a^*$ if $a^*_i = r_i(\hat{x}_i)$ for every $i \in I$.

**Definition 5.** A selfconfirming action profile $a^* \in A^{SCE}_Z$ is locally stable if there are a profile of conjectures $\hat{x}$ and $\epsilon > 0$ consistent with $a^*$ such that the learning dynamics starting from any $\hat{x}'$ with $\|\hat{x}' - \hat{x}\| < \epsilon$ converges back to $\hat{x}$. 

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4.1 Results

Each SCE is characterized by a set of active agents. So, given a strategy profile \( a = (a_i)_{i \in I} \), let \( I_a = \{ i \in I : a_i > 0 \} \) denote the set of active players. With this, for each action profile \( a \), \( Z_{I_a} \) denotes the submatrix with rows and columns corresponding to players who are active in \( a \). This allows us to characterize locally stable selfconfirming equilibria.

**Proposition 4.** Consider \( a^* \in A_{\mathbb{Z}}^{SCE} \). \( a^* \) is locally stable if

- Assumption 4 holds for matrix \( Z_{I_{a^*}} \);
- for some \( \tilde{x} \) consistent with \( a^* \) and every \( i \in I \setminus I_{a^*} \), \( \alpha_i + \tilde{x}_i < 0 \).

Intuitively, consider a sufficiently small perturbation of players’ conjectures. The first condition ensures that active players keep being active and their actions converge back to the Nash equilibrium of the auxiliary game with player set \( I_{a^*} \). The second condition ensures that inactive players keep being inactive. Next, we provide alternative sufficient conditions that allow to characterize the subsets of active agents associated to SCEs.

**Proposition 5.** Consider a selfconfirming strategy profile \( a^* \in A_{\mathbb{Z}}^{SCE} \). If \( Z_{I_{a^*}} \) satisfies at least one of the three conditions below:

1. it has bounded values (Assumption 1),
2. it is negative and limited (Assumptions 3 and 4),
3. or it is symmetrizable–limited (Assumption 6),

then \( a^* \) is locally stable and, for every \( J \subseteq I_{a^*} \), there exists a locally stable selfconfirming equilibrium \( a^{**} \in A_{\mathbb{Z}}^{SCE} \) such that

1. \( A_{J,\mathbb{Z}}^{NE} = \{ a_J^{NE} \} \), with \( a_J^{NE} > 0_J \);
2. \( \{ a^{**} \} = \{ a_J^{NE} \} \times \{ 0_{I \setminus J} \} \).

The proof is based on results from linear algebra. In fact, if an adjacency matrix satisfies one of the conditions from Proposition 5, then also every submatrix of that matrix satisfies that property.

We know that there may be SCEs that are not Nash equilibria, because some agents are inactive even if this is not a best response to the actions of the others. Proposition 5 tells us two additional things. Under the stated conditions, for any given SCE \( a^* \) with set of active agents \( I_{a^*} \), any subset \( J \subseteq I_{a^*} \) of those agents is associated to a stable SCE where all agents in \( J \) are active, and the other agents are inactive. Second, since the empty subset of agents is trivially associated to the stable SCE where every agent is inactive, for every network game there is always a subset \( J \) of agents associated to a stable SCE where all and only the agents in \( J \) are active.
4.2 Examples and discussion

The following example shows that we can reach SCEs that are not NE also if the initial beliefs induce all positive actions at the beginning of the learning dynamic.

Example 4. Consider the case of 4 players, with the network matrix $\mathbf{Z} \in \{-0.2, 0, 0.2\}^{I \times I}$ shown in Figure 2, and, for every $i$, $\alpha_i = 0.1$. This is a case of general externalities, that can be positive or negative. Figure 3 shows the learning dynamics of actions and beliefs that start from different initial conditions. In one case (left panels) we converge to the unique Nash equilibrium of this game (the dotted lines), in the other (right panels) the learning dynamics put, after 2 rounds, one player out from the active agents, and the remaining 3 converge to a selfconfirming equilibrium which is not Nash. ▲

![Figure 3: General strategic externalities. Starting from different beliefs on the same network (from Figure 2), the learning process may converge to the unique Nash equilibrium (left panels) or to a SCE which is not a Nash equilibrium (right panels). Note that actions are just an upward translation of beliefs, by the quantity $\alpha$.](image)

The next example (which does also not satisfy the local stability conditions of Proposition 5) shows that convergence may not occur even in a simple case of positive externalities.

Example 5. Now consider again the network from Example 1 (Figure 1), with 4 nodes. Even if there are only positive externalities, the magnitude of $\gamma$ may imply convergence or not. If $\gamma < 1$, there is convergence. If instead $\gamma \geq 1$ there is divergence. Figure 4 shows two cases, with $\gamma = 0.9$.
and $\gamma = 1$ respectively, starting from the same initial beliefs. Note that, nodes/players 1 and 4 reinforce each other, and this gives rise to an oscillating behavior of their beliefs.

![Beliefs]

Figure 4: Only positive externalities. Starting from the same beliefs on the same network structure (from Figure 1), the learning process may converge or not depending on the size of $\gamma$: $\gamma = 0.9$ in the left panel; $\gamma = 1$ in the right panel. We report only beliefs because, as in Figure 4, actions are just an upward translation of beliefs, of amount $\alpha$..

Our notion of stability with respect to conjectures relates to the standard notion of stability with respect to actions in the following way. First of all, since played actions are justified by some conjectures, the only reason for these actions to change is a perturbation of the surrounding conjectures, but this is not a sufficient condition. If all agents are active, the two definitions have the same the consequences in terms of stability, since a perturbation with respect to actions happens if and only if every agent’s conjecture is perturbed. However, if a selfconfirming equilibrium has inactive agents, then those inactive agents who play a corner solution do not show perturbation in actions when their conjectures are perturbed. This implies that if an action profile is stable with respect to actions perturbations, then it is also stable under conjectures perturbations, but the converse does not hold.

5 Local and Global externalities

As anticipated when discussing Eq. (2), we consider now an extension to the case of equation (4), in which we add a global externality term with no strategic effects. For each $i \in I$, we posit an interval $Y_i = [\underline{y}_i, \bar{y}_i]$, a coefficient $\beta \in \mathbb{R}$, and we consider the following aggregator:

\[ g = \sum_{j \neq i} \left[ \min\left(\frac{y_{ij}}{\beta}, 1\right) - \min\left(\frac{y_{ji}}{\beta}, 1\right) \right] \]

This aggregator $g$ sums up the actions of all the agents in the network except agent $i$. We could have considered agent $i$ as well, but we opted for this specification so as not to change the first order condition with respect to the case with just local externalities.
We assume that every agent $i \in I$ knows $Y_i$. Then, we let $y_i = g_i (a_{-i}, \beta)$ and we maintain the assumption that $x_i = \ell_i (a_{-i}, Z)$. The new parametrized utility function is

$$v_i : A_i \times X_i \times Y_i \rightarrow \mathbb{R}$$

$$v_i (a_i, x_i, y_i) \mapsto \alpha_i a_i - \frac{1}{2} a_i^2 + a_i x_i + y_i,$$

where both $x_i$ and $y_i$ are unknown. The general form of the feedback function is

$$f_i : A_i \times X_i \times Y_i \rightarrow M.$$

Deterministic shallow conjectures for each $i \in I$ are now determined by the pair $(\hat{x}_i, \hat{y}_i) \in X_i \times Y_i$.

**Definition 6.** A profile $(a^*_i, \hat{x}_i, \hat{y}_i)_{i \in I} \in \times_{i \in I} (A_i \times X_i \times Y_i)$ of actions and (shallow) deterministic conjectures is a **selfconfirming equilibrium** at $Z$ and $\beta$ of a linear quadratic network game with feedback and global externalities if, for each $i \in I$,

1. (subjective rationality) $a^*_i = r_i (\hat{x}_i)$,
2. (confirmed conjecture) $f_i (a^*_i, \hat{x}_i, \hat{y}_i) = f_i (a^*_i, \ell_i (a^*_{-i}, Z), g_i (a^*_{-i}, \beta))$.

Notice that the rationality condition is unchanged with respect to the case of only local externalities since best-reply conditions are not affected by the global externality term. To compare this game with the linear-quadratic network game with only local externalities, we consider the case of just observable payoffs. Then, without loss of generality we can assume that $f_i = v_i$ for every $i \in I$.

With this, we can characterize the SCE set as follows:

**Proposition 6.** Fix $Z \in \Theta$ and $\beta$. Every selfconfirming equilibrium profile $(a^*_i, \hat{x}_i, \hat{y}_i)_{i \in I} \in \times_{i \in I} (A_i \times X_i \times Y_i)$ of a linear-quadratic network game with global externalities and just observable payoffs is such that, for every $i \in I$,

1. if $a^*_i = 0$, then $\hat{x}_i \in (-\infty, -\alpha_i]$, $\hat{y}_i = y_i$;
2. if $a^*_i > 0$, then $a^*_i = \alpha_i + \hat{x}_i$, $\hat{y}_i = y_i + a^*_i (x_i - \hat{x}_i)$.

We discuss how the presence of the global externality term in the utility function changes radically the characterization of selfconfirming equilibria. As before, we assume that players observe their own realized payoffs. Yet, when global externalities are present, observability by active players does not hold anymore. Inactive players have correct conjectures about the global externality, but may have correct or incorrect conjectures about the local externality. Active players, on the
other hand, are not able to determine precisely the magnitude of the local effects with respect to the global effects. Given any strictly positive action \( a_i^* \), the confirmed conjectures condition yields \( (\hat{y}_i - y_i) = a_i^* (x_i - \hat{x}_i) \). Then, in equilibrium, if agent \( i \) overestimates (underestimates) the local externality, she must compensate this error by underestimating (overestimating) the global externality. Then, compared to the case of only local externalities, we have that: (i) active agents choose a best response to a (typically) wrong conjecture about \( x \); thus, (ii) it is not possible to characterize SCE by means of Nash equilibria of the auxiliary games restricted to the active players.

We present now a simple example showing how wrong conjectures about local and global externalities may have a big effect on equilibrium actions.

**Example 6.** Consider three agents in a line network. Let agent 2 be at the center of the line. Then, for every \((a^*, Z, \beta)\), \( \ell_2 (a^*_{-2}, Z) \) is proportional to \( g_2 (a^*_{-2}, \beta) \), always with the same ratio, while this is not true for agents 1 and 3. We assume that each agent thinks to be playing in a complete network, so every \( i \in I \) thinks that \( \ell_i (a^*_i, Z) \) is always proportional to \( g_i (a^*_i, \beta) \), with the same ratio. In this case agents 1 and 3 think to be more central than what they actually are.

Table 4 provides the Nash equilibria for the actual network and for the complete network, and the selfconfirming equilibrium actions for the case described above.

|       | Line NE | Complete NE | SCE    |
|-------|---------|-------------|--------|
| \( a_1 \) | 0.130   | 0.167       | 1.569  |
| \( a_2 \) | 0.152   | 0.167       | 1.679  |
| \( a_3 \) | 0.130   | 0.167       | 1.569  |

Table 4: Simulations for the case of \( \alpha = 0.1, \gamma = 0.2, \) and \( \beta = 1 \). Columns refer to 1) Nash Equilibrium of the line network; 2) Nash equilibrium of complete network; 3) SCE in the line network in which each \( i \in I \) believes that \( \ell_i (a^*_i, Z) = \frac{\gamma}{\beta} g_i (a^*_i, \beta) \).

Simulations show that if agents overestimate the impact of local externalities this generates a **multiplier** effect that makes equilibrium actions increase at a level even larger that what would be predicted in a complete network by Nash equilibrium. This is the result of how agents misinterpret their feedbacks. In details, thinking to be in a complete network makes agents 1 and 2 overestimate local externalities. Take for instance agent 1. Given any \( a_{-1} \), she chooses a best reply higher than the Nash equilibrium one since she overestimates the local externality. This high action has the effect of increasing the global externality term for agent 3. Agent 3, by overestimating local externality, partly attributes this higher global externality to the local externality term, and chooses an action larger than predicted by Nash equilibrium. The choice of agent 3 increases in turns the global externality perceived by agent 1, and so on. At the same time agent 2, as neighbors choose higher actions, increases her own action level. This effect goes on and a multiplier effect seems to be at place. In the limit, selfconfirming equilibrium actions are almost ten times larger than the
complete network Nash equilibrium.

5.1 Learning with Global Externalities

We now consider the learning process that originates from an adaptive updating of conjectures, as we did for the case of only local externalities. For an easy reference, we rewrite here Eq. (2) as a payoff function that depends on players’ actions, with the time index and specifying \( x_{i,t} \) and \( y_{i,t} \) as functions of co-players’ actions:

\[
\begin{align*}
    u_{i,t}(a_{i,t}, a_{-i,t}) &= \alpha a_{i,t} - \frac{1}{2} a_{i,t}^2 + a_{i,t} \sum_{j \in I \setminus \{i\}} z_{ij} a_{j,t} + \beta \sum_{k \in I \setminus \{i\}} a_{k,t}.
\end{align*}
\]

To ease the analysis, we assume the same parameter \( \alpha \) for each player and we focus on the case of strictly positive justifiable actions. We obtain this by assuming that \( \alpha > 0 \) and that all the elements of \( Z \) are nonnegative. This case, however, is a bit more complex since, at each time, there are infinitely many collections of feasible pairs \((\hat{x}_{i,t}, \hat{y}_{i,t})_{i \in I}\) compatible with adaptive learning. For every \( i \in I \), and each time \( t \), let \( m_{i,t} = f_i(a_{i,t}, x_{i,t}, y_{i,t}) = u_i(a_{i,t}, a_{-i,t}) \) be the message agents receive. Then, given \( \hat{x}_{i,t}, \hat{y}_{i,t} \) is uniquely determined. In details, at each time period, agent \( i \)’s conjecture is a pair \((\hat{x}_{i,t}, \hat{y}_{i,t})\) consistent with the message received at the previous period. We obtain

\[
\dot{y}_{i,t+1} = m_{i,t} - \alpha a_{i,t} + \frac{1}{2} (a_{i,t})^2 - a_{i,t} \hat{x}_{i,t+1}.
\]

Given message \( m_{i,t-1} \), and considering that agents perfectly recall their past actions, \( \hat{y}_{i,t} \) is uniquely determined as a function of \( \hat{x}_{i,t} \). We can just focus on the dynamics of \( \hat{x}_{i,t} \). The dynamics of \( \hat{x}_{i,t} \) is given by the following equation

\[
\dot{x}_{i,t+1} = \frac{m_{i,t} - \dot{y}_{i,t+1}}{a_{i,t}} - \alpha + \frac{1}{2} a_{i,t}
\]

To avoid bifurcations at each time period, we need to use simplifying assumptions. We define

\[
c_{i,t} := \frac{\hat{x}_{i,t}^14}{\hat{y}_{i,t}}
\]

Then,

Assumption 7. For each \( i \in I \) and for each \( t \in \mathbb{N} \), \( c_{i,t} = c_{i,t+1} = c_i \).

We call \( c_i \) the perceived centrality of player \( i \). For each player, this parameter describes what she thinks to be the share of the activity in her neighborhood with respect to the sum of all externalities. In doing so, we implicitly assume that players think that not all the other players play the null action \( a_{k,t} = 0 \). This is actually a reasonable assumption, because under positive externalities any best response \( a_{k,t} \) should be at least \( \alpha \).
the actions of the population. This perceived share has a strong relationship with the Bonacich centrality. In the unique Nash equilibrium $a^*$ of the game, where all actions are positive, we have
\[ a^*_i = \alpha + x_i = \alpha + \sum_{j \in I \setminus \{i\}} z_{ij} a^*_j. \]

The profile of Bonacich centrality measures $b$ is the unique solution of the linear system\(^\text{15}\)
\[ b_i = \alpha + \sum_{j \in I \setminus \{i\}} z_{ij} b_j. \]

So, when beliefs are correct, as in the Nash equilibrium, we have $b_i = a_i$ and $c_i = \frac{b_i - \alpha}{y_i}$. Now, in the Nash equilibrium we have also $\frac{1}{y_i} - \frac{1}{y_j} = \beta \frac{a_j - a_i}{y_i y_j}$. If the number of players is large, we have $y_i \gg a_i$ and $y_j \gg a_j$, which implies $\frac{1}{y_i} \simeq \frac{1}{y_j}$, and so every $c_i$ is roughly the same linear rescaling of $b_i$.

From equation (11), and expressing the message as the observed payoff, we get that the following learning dynamic
\[ \hat{x}_{i,t+1} = x_{i,t} + \frac{y_{i,t}}{a_{i,t}} - \frac{\hat{y}_{i,t+1}}{a_{i,t}}. \]

Plugging in $c_{i,t} = \frac{\hat{x}_{i,t}}{y_{i,t}}$ we get
\[ \hat{x}_{i,t+1} = \frac{c_{i,t}}{1 + c_{i,t} a_{i,t}} (a_{i,t} x_{i,t} + y_{i,t}) . \]

We define the true centrality of player $i$ at time $t$ as
\[ c'_i,t = \frac{x_{i,t}}{y_{i,t}} . \]

Note that $c'_i,t \in [0, \frac{\sum_{j \neq i} z_{ij}}{\beta}]$. For this reason, we also assume that the perceived centrality of each player $i$ is such that $c_i \in \left(0, \frac{\sum_{j \neq i} z_{ij}}{\beta}\right]$, and this specifies the set of all admissible perceived centralities. The dynamic, then, can be written as
\[ \hat{x}_{i,t+1} = c_i y_{i,t} a^*_i c'_i,t + 1 \]
\[ a^*_i c'_i,t + 1 \]
which implies that the conjecture is correct only when $c_i = c'_i,t$.

We look at best responses $a_{i,t+1} = \alpha + \hat{x}_{i,t+1}$, and study existence and characterization of the steady state of this learning process. Recall that $y_{i,t} = \beta \sum_{j \neq i} a_{j,t}$. To find a fixed point we look at the system of $n$ equations
\[ H_i(a^*, c, \beta, Z) := \alpha + c_i \left( \beta \sum_{j \neq i} a^*_j \right) \frac{a^*_i c'_i,t + 1}{a^*_i c_i + 1} - a^*_i = 0 . \]

\text{15}In general, independently of any game defined on the network, Bonacich centrality is a network centrality measure that depends on a parameter $\alpha > 0$. It is defined exactly as the solution of that same linear system. For a detailed discussion on this see Dequiet and Zenou 2017
For comparison, we also study the system of equations that provide the Nash Equilibrium of this network game, namely:

\[
F_i(a^*, \beta, Z) := \alpha + \sum_{j \in I} z_{ij} a_j^* - a_i^* = 0
\]

(16)

Let \( A \subset [\alpha, \infty)^I \) denote the set of the solutions of the system (15). We have the following result.

**Proposition 7.** If the system defined by (16) admits a solution, then for each vector \( c \) of perceived centralities also the system defined by (15) admits a solution. Moreover, the system implies a homeomorphism \( \Phi \) between all profiles \( c \) and \( A \). Homeomorphism \( \Phi \) is monotone with respect to the lattice order of the two sets.

The previous result provides information only on the steady states of our dynamical system. Note however that the homeomorphism is implied by the particular learning dynamic that we are assuming, which is based on constant belief centralities. Here below we show a result that provides sufficient conditions for convergence of the learning dynamic. We impose as a sufficient condition that local and global externalities are not too large.

**Proposition 8.** If, for each player \( i \in I \),

\[
0 < c_i \beta (n - 1) < \sum_{j \neq i} z_{ij} < 2
\]

then the dynamic defined by the learning process (15) always converges to its unique solution, which is stable.

It should be noted that we are not requiring that \( |\sum_{j \neq i} z_{ij}| < 1 \), which would imply that Assumption 4 hold.

**Example 7.** Under the conditions of Proposition 8, we use equation (14) to run dynamical systems converging to the SCE implicitly defined by (15). This allows us to provide a graphical illustration of Proposition 7, for the case of three nodes. As in Example 6, we do this for the case of a line network (where each of the two links is bidirectional), and for the case of a complete network. Figure 5 shows the results. We can start from any pattern of perceived centralities for the three nodes. The left panel shows the profile of perceived centralities when at least one node has maximal perceived centrality (the three faces of the cube have different colors, according to which node has the maximal centrality). The central panel shows the corresponding SCE conjectures \( \hat{x} \) when the network is a line (the node that has perceived centrality 1 in the red dots is the central node). The right panel shows the corresponding SCE beliefs \( \hat{x} \) when the network is a complete triangle. The figure suggests that homeomorphism \( \Phi \) (from Proposition 7) is highly non linear, because of the self reinforcement process in beliefs that we discussed in Example 6. The figure also shows that, as stated by Proposition 7, homeomorphism \( \Phi \) respects the lattice order on the two sets.

Proposition 7 tells us that a non-negative shift in each perceived centrality will always result in a non-negative shift in each agent’s action in the resulting SCE. However, Proposition 8 gives an implicit warning. Too high perceived centralities may imply that the sufficient conditions for
Figure 5: Simulations showing the homeomorphism of Proposition 8 for the case of 3 nodes. The left panel shows vectors of preceived centralities. The central panel shows the corresponding SCE beliefs $\hat{x}$ when the network is a line (the node that has perceived centrality 1 in the red dots is the central node). The right panel shows the corresponding SCE beliefs $\hat{x}$ when the network is a complete triangle.

stability are lost, and convergence to the corresponding SCE may be lost. Note also that, summing up equation (2) for all the players, the aggregate welfare is maximized if the vector of actions satisfies the linear system

$$a_i^* = \alpha + (n - 1)\beta + \sum_{j \in I \setminus \{i\}} (z_{ij} + z_{ij})a_j^* .$$

Social platforms like Facebook and Twitter often provide information to users about the activity of their peers. A rationale for this marketing strategy can be that these companies want to change the beliefs of players, making them feel more important (i.e. central) in the social network. Even a benevolent social planner may want to set the perceived centralities to the level for which the social optimum is achieved. However, according to our model, if perceived centralities are too high, the
system may become unstable.

6 Conclusion

In this paper we lay the basis for a novel approach to network games. Many of the applications of those games mimic large societies with million of nodes and non regular distribution of connections. It is natural to assume that players are not aware of the complete structure of the network; thus, they do not perform sophisticated strategic reasoning possibly leading to a Nash equilibrium, but just best–respond to some to subjective beliefs affected by information feedback they receive. We analyze simple adaptive dynamics and show that in some cases they converge to stable Nash equilibria. However, we characterize also those situations in which feasible stable outcomes are not Nash equilibria, but rather selfconfirming equilibria in which some (if not all) players have wrong beliefs and yet the feedback they receive is consistent with such beliefs. We also show that simple biases in the perception of own centrality in the network may lead players to play action profiles that are very far from the unique Nash equilibrium of the game.

The natural application of this approach is to online social platforms like Facebook and Twitter. Using a linear quadratic structure for the payoff function we have also laid the ground for a tractable welfare analysis of the model. However, policy implications are not straightforward if we want to consider the long–run benefits of connections and not only about the instantaneous payoffs of the users of those platforms.

Our analysis does not account for the strategic reasoning that agents can perform given some commonly know features of the network. For example, known results about rationalizability imply that, if the (nice) network game has strategic complementarities and is common knowledge, then sophisticated strategic reasoning leads to Nash equilibrium. If only some aspects of the network game are commonly known, then both strategic reasoning and learning affect the long-run outcome, which is a kind of rationalizable self-confirming equilibrium. This is a topic we are working on.

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16On nice games with strategic complementaries see, e.g., Chapter 5 of Battigalli (2018) and the references therein.
Appendix A  Selfconfirming equilibria in parametrized nice games with aggregators

In this section we develop a more general analysis of selfconfirming equilibria in a class of games that contains the linear-quadratic network games with feedback. To ease reading, we make this section self-contained repeating some definitions from the main text.

A parametrized nice game with aggregators and feedback is a structure

\[ G = \langle I, \Theta, (A_i, \ell_i, v_i, f_i)_{i \in I} \rangle \]

where

- \( I \) is the finite players set, with cardinality \( n = \vert I \vert \) and generic element \( i \).
- \( \Theta \subseteq \mathbb{R}^m \) is a compact parameter space.
- \( A_i = [0, \bar{a}_i] \subseteq \mathbb{R}_+ \), a closed interval, is the action space of player \( i \) with generic element \( a_i \in A_i \).
- \( X_i = [\underline{x}_i, \bar{x}_i] \subseteq \mathbb{R} \), a closed interval, is the a space of payoff states for \( i \).
- \( \ell_i : A_{-i} \times \Theta \rightarrow X_i \) (where \( A_{-i} = \times_{j \in I \setminus \{i\}} A_j \)) is a continuous parametrized aggregator of the actions of \( i \)'s coplayers such that its range \( \ell_i (A_{-i} \times \Theta) \) is connected.\(^{17}\)
- \( v_i : A_i \times X_i \rightarrow \mathbb{R} \) is the payoff (utility) function of player \( i \), which is strictly quasi-concave in \( a_i \) and continuous,\(^{18}\) and from which we derive the parameterized payoff function

\[ u_i : A_i \times A_{-i} \times \Theta \rightarrow \mathbb{R}, \quad (a_i, a_{-i}, \theta) \mapsto v_i (a_i, \ell_i (a_{-i}, \theta)). \]

Thus, \( x_i = \ell_i (a_{-i}, \theta) \) is the payoff relevant state that \( i \) has to guess in order to choose a subjectively optimal action. With this, for each \( \theta \in \Theta \), \( \langle I, (A_i, u_i, \theta)_{i \in I} \rangle \) is a nice game (Moulin, 1979), and \( \langle I, \Theta, (A_i, u_i)_{i \in I} \rangle \) is a parametrized nice game. We let

\[ r_i : X_i \rightarrow A_i, \quad x_i \mapsto \arg \max_{a_i \in A_i} v_i (a_i, x_i) \]

denote the best reply function of player \( i \).

\(^{17}\)Since the range of each section \( \ell_{i, \theta} \) must be a closed interval, we require that the union of the closed intervals \( \ell_{i, \theta} (A_{-i}) \ (\theta \in \Theta) \) is also an interval, which must be closed because \( \Theta \) is compact and \( \ell_i \) continuous.

\(^{18}\)That is, \( v_i \) is jointly continuous in \( (a_i, x_i) \) and, for each \( x_i \in [\underline{x}_i, \bar{x}_i] \), the section \( v_i, x_i : [0, \bar{a}_i] \rightarrow \mathbb{R} \) has a unique maximizer \( a_i^* \) (that typically depends on \( x_i \)), it is strictly increasing on \([0, a_i^*] \), and it is strictly decreasing on \([a_i^*, \bar{a}_i] \). Of course, the monotonicity requirement holds vacuously when the relevant subinterval is a singleton.
• Let \( M \subseteq \mathbb{R} \) be a set of “messages,” \( f_i : A_i \times X_i \to M \) is a feedback function that describes what \( i \) observes (a “message,” e.g., a monetary outcome) after taking any action \( a_i \) given any payoff state \( x_i \).\(^{19}\)

On top of the formal assumptions stated above, we maintain the following informal assumption about players’ knowledge of the game:

• Each player \( i \) knows \( v_i \) and \( f_i \).

Unless we explicitly say otherwise, we instead do not assume that \( i \) knows \( \theta \), or function \( \ell_i \), or even that \( i \) understands that his payoff is affected by the actions of other players. However, since \( i \) knows the feedback function \( f_i : A_i \times X_i \to M \) and the action he takes, what \( i \) infers about the payoff state \( x_i \) after he has taken action \( a_i \) and observed message \( m \) is that

\[
x_i \in f_{i,a_i}^{-1}(m) := \{ x_i' : f_i(a_i,x_i') = m \}.
\]

A.1 Conjectures

**Definition A.** A shallow conjecture for \( i \) is a probability measure \( \mu_i \in \Delta(X_i) \). A (deep) conjecture for \( i \) is a probability measure \( \bar{\mu}_i \in \Delta(A_{-i} \times \Theta) \). An action \( a_i^* \) is justifiable if there exists a shallow conjecture \( \mu_i \) such that

\[
a_i^* \in \arg\max_{a_i \in A_i} \int_{X_i} v_i(a_i,x_i) \mu_i(dx_i);
\]

in this case we say that \( \mu_i \) justifies \( a_i^* \). Similarly, we say that (deep) conjecture \( \bar{\mu}_i \in \Delta(A_{-i} \times \Theta) \) justifies \( a_i^* \) if the shallow conjecture induced by \( \bar{\mu}_i \) (\( \mu_i = \bar{\mu}_i \circ \ell_i^{-1} \in \Delta(X_i) \)) justifies \( a_i^* \).

**Remark 3.** If \( a_i \mapsto v_i(a_i,x_i) \) is strictly concave for each \( x_i \), then also \( a_i \mapsto \int_{X_i} v_i(a_i,x_i) \mu_i(dx_i) \) is strictly concave and the map

\[
\mu_i \mapsto \arg\max_{a_i \in A_i} \int_{X_i} v_i(a_i,x_i) \mu_i(dx_i)
\]

is a continuous function.\(^{20}\)

The following lemma summarizes well known results about nice games (see, e.g., Battigalli 2018) and some straightforward consequences for the more structured class of nice games with aggregators considered here:

**Lemma A.** The best reply function \( r_i : X_i \to A_i \) is continuous, hence its range \( r_i(X_i) \) is a closed interval, just like \( X_i \). Furthermore, for each given \( a_i^* \in A_i \), the following are equivalent:

\[^{19}\]Here the assumption that \( M \) is a set of real numbers is without loss of generality, because the same holds for the set of payoff states \( X_i \).

\[^{20}\]When \( \Delta(X_i) \) is endowed with the topology of weak convergence of measures.
• \( a^*_i \) is justifiable,

• \( a^*_i \in r_i (X_i) \) (that is, \( a^*_i \) is justified by a deterministic shallow conjecture),

• there is no \( a_i \) such that \( v_i (a^*_i, x_i) < v_i (a_i, x_i) \) for all \( x_i \in X_i \) (that is, \( a^*_i \) is not dominated by any other pure action).

**Corollary A.** Suppose that the aggregator \( \ell_i \) is onto. Then, an action of player \( i \) is justifiable if an only if it is justified by a deep conjecture.

**Proof.** The “if” part is trivial. For the “only if” part, fix a justifiable action \( a^*_i \) arbitrarily. By Lemma A, there is some \( x_i \in X_i \) such that \( a^*_i = r_i (x_i) \). Since the aggregator \( \ell_i \) is onto, there is some \( \ell_i^{-1} (x_i) \) such that

\[
a^*_i = \arg \max_{a_i \in A_i} u_i (a_i, a_{-i}, \theta).
\]

Hence \( a^*_i \) is justified the deep conjecture \( \delta_{(a_{-i}, \theta)} \), that is, the Dirac measure supported by \( (a_{-i}, \theta) \).

\[\blacksquare\]

With this, from now on we restrict our attention to (shallow, or deep) deterministic conjectures.

**A.2 Feedback properties**

**Definition B.** Feedback \( f_i \) satisfies observable payoffs (OP) relative to \( v_i \) if there is a function \( \tilde{v}_i : A_i \times M \to \mathbb{R} \) such that

\[
v_i (a_i, x_i) = \tilde{v}_i (a_i, f_i (a_i, x_i))
\]

for all \( (a_i, x_i) \in A_i \times X_i \); if the section \( \tilde{v}_{i,a_i} \) is injective for each \( a_i \in A_i \), then we say that \( f_i \) satisfies just observable payoffs (JOP) relative to \( v_i \). Game \( G \) satisfies (just) observable payoffs if, for each player \( i \in I \), feedback \( f_i \) satisfies (J)OP relative to \( v_i \).

If \( f_i \) satisfies JOP, we may assume without loss of generality that \( f_i = v_i \), because, for each action \( a_i \), the partitions of \( X_i \) induced by the preimages of \( v_{i,a_i} \) and \( f_{i,a_i} \) coincide:

**Remark 4.** Feedback \( f_i \) satisfies JOP relative to \( v_i \) if and only if

\[
\forall a_i \in A_i, \quad \left\{ v_{i,a_i}^{-1} (u) \right\}_{u \in v_{i,a_i} (X_i)} = \left\{ f_{i,a_i}^{-1} (m) \right\}_{m \in f_{i,a_i} (X_i)}.
\]

**Proof.** (Only if) Fix \( a_i \in A_i \). Since \( f_i \) satisfies JOP relative to \( v_i \), \( v_{i,a_i} (X_i) = (\tilde{v}_{i,a_i} \circ f_{i,a_i}) (X_i) \) (by OP), for each \( u \in v_{i,a_i} (X_i) \) there is a unique message \( m_{a_i,u} = \tilde{v}_{i,a_i}^{-1} (u) \) (by injectivity of \( \tilde{v}_{i,a_i} \)), and

\[
v_{i,a_i}^{-1} (u) = \left\{ x_i \in X_i : v_i (a_i, x_i) = u \right\} = \left\{ x_i \in X_i : \tilde{v}_i (a_i, f_i (a_i, x_i)) = u \right\} = \left\{ x_i \in X_i : f_i (a_i, x_i) = m_{a_i,u} \right\} = f_{i,a_i}^{-1} (m_{a_i,u}),
\]

28
which implies eq. (a).

(If) Suppose that eq. (a) holds. For every $a_i \in A_i$ and $m \in f_{i,a_i}(X_i)$ select some $\xi_i(a_i, m) \in f_{i,a_i}^{-1}(m)$. Let

$$D := \bigcup_{a_i \in A_i} \{a_i\} \times f_{i,a_i}(X_i)$$

With this,

$$\xi_i : D \to X_i$$

is a well defined function. Domain $D$ is the set of action-message pairs for which the definition of $\bar{v}_i$ matters. Define $\bar{v}_i$ as follows:

$$\bar{v}_i(a_i, m) = \begin{cases} v_i(a_i, \xi_i(a_i, m)) & \text{if } (a_i, m) \in D, \\ 0 & \text{otherwise.} \end{cases}$$

By construction, eq. (a) implies that

$$\forall (a_i, x_i) \in A_i \times X_i, \bar{v}_i(a_i, f_i(a_i, x_i)) = v_i(a_i, x_i).$$

Hence, OP holds. Furthermore, for all $a_i \in A_i, m', m'' \in f_{ai}(X_i)$,

$$m' \neq m'' \Rightarrow \xi_i(a_i, m') \neq \xi_i(a_i, m'') \Rightarrow v_i(a_i, \xi_i(a_i, m')) \neq v_i(a_i, \xi_i(a_i, m'')) \Rightarrow \bar{v}_i(a_i, m') \neq \bar{v}_i(a_i, m'')$$

where the first and the second implications follow from eq. (a) ($\xi_i(a_i, m')$ and $\xi_i(a_i, m'')$ belong to different cells of the coincident partitions, hence yield different utilities), and the third holds by construction. Therefore, $\bar{v}_{i,a_i}$ is injective for every $a_i$, which means the JOP holds.

**Definition C.** Feedback $f_i$ satisfies **observability if and only if $i$ is active (OiffA)** if section $f_{i,a_i}$ is injective for each $a_i > 0$ and constant for $a_i = 0$. Game $G$ satisfies **observability by active players** if OiffA holds for each $i$.

**Remark 5.** If $NG$ is linear-quadratic and satisfies just observable payoffs, then it satisfies observability by active players.

**Proof.** By Remark 4 JOP implies that, for each $a_i \in A_i$,

$$\left\{ v_{i,a_i}^{-1}(u) \right\}_{u \in v_{i,a_i}(X_i)} = \left\{ f_{i,a_i}^{-1}(m) \right\}_{m \in f_{i,a_i}(X_i)}.$$

The linear-quadratic form of $v_i$ implies that, for every $x_i \in X_i$,

$$v_{i,0}^{-1}(v_{i,0}(x_i)) = X_i$$
These equalities imply that \( f_{i,0} \) is constant and \( f_{i,a_i} \) is injective for \( a_i > 0 \), that is, \( NG \) satisfies observability by active players.

**Definition D.** Feedback \( f_i \) satisfies **own-action independence** (OAI) of feedback about the state if, for all justifiable actions \( a_i^*, a_i^0 \) and all payoff states \( \hat{x}, x \in X_i \),

\[
f_i (a_i^*, \hat{x}) = f_i (a_i^*, x) \Rightarrow f_i (a_i^0, \hat{x}) = f_i (a_i^0, x).
\]

Game \( G \) satisfies own-action independence of feedback about the state if, for each player \( i \in I \), feedback \( f_i \) satisfies OAI.

In other words, OAI says that if player \( i \) cannot distinguish between two payoff states \( \hat{x} \) and \( x \) when he chooses some given justifiable action \( a_i^* \), then he cannot distinguish between these two states when he chooses any other justifiable action \( a_i^0 \). This is equivalent to requiring that the partitions of \( X_i \) of the form \( \{f_{i,a_i}^{-1} (m)\}_{m \in f_{i,a_i} (X_i)} \) coincide across justifiable actions, i.e., across actions \( a_i \in r_i (X_i) \) (see Lemma A).

The following lemma—which holds for any game, not just nice games—states that, under payoff observability and own-action independence, an action is justified by a confirmed conjecture if and only if it is a best reply to the actual payoff state:

**Lemma B.** If \( f_i \) satisfies payoff observability relative to \( v_i \) and own-action independence of feedback about the state, then for all \( (a_i^*, x_i) \in A_i \times X_i \) the following are equivalent:

1. there is some \( \hat{x} \in X_i \) such that \( a_i^* \in \arg \max_{a \in A_i} v_i (a, \hat{x}) \) and \( f_i (a_i^*, \hat{x}) = f_i (a_i^*, x) \),
2. \( a_i^* \in \arg \max_{a \in A_i} v_i (a, x_i) \).

**Proof.** (Cf. Battigalli et al. 2015, Battigalli 2018) It is obvious that (2) implies (1) independently of the properties of \( f_i \). To prove that (1) implies (2), suppose that \( f_i \) satisfies OP-OAI and let \( \hat{x} \) be such that (1) holds. Let \( a_i^0 \) be a best reply to the actual state \( x \). We must show that also \( a_i^* \) is a best reply to \( x \). Note that both \( a_i^* \) and \( a_i^0 \) are justifiable; hence, by OAI, \( f_i (a_i^*, \hat{x}) = f_i (a_i^*, x) \) implies \( f_i (a_i^0, \hat{x}) = f_i (a_i^0, x) \). Using OP, condition (1), and OAI as shown in the following chain of equalities and inequalities, we obtain

\[
\begin{align*}
v_i (a_i^*, x) & \overset{\text{(OP)}}{=} \tilde{v}_i (a_i^*, f_i (a_i^*, x)) \overset{(1)}{=} \tilde{v}_i (a_i^*, f_i (a_i^*, \hat{x})) \overset{\text{(OP)}}{=} v_i (a_i^*, \hat{x}) \overset{(1)}{\geq} v_i (a_i^*, x), \\
v_i (a_i^0, x) & \overset{\text{(OP)}}{=} \tilde{v}_i (a_i^0, f_i (a_i^0, x)) \overset{\text{(1, OAI)}}{=} \tilde{v}_i (a_i^0, f_i (a_i^0, \hat{x})) \overset{\text{(OP)}}{=} v_i (a_i^0, \hat{x}) \overset{\text{(OP)}}{=} v_i (a_i^0, x).
\end{align*}
\]

Since \( a_i^0 \) is a best reply to \( x \) and \( v_i (a_i^*, x) \geq v_i (a_i^0, x) \), it must be the case that also \( a_i^* \) is a best reply to \( x \). □
Corollary B. Suppose that $G$ satisfies payoff observability and own-action independence of feedback about the state, then the sets of selfconfirming action profiles and Nash equilibrium action profiles coincide for each $\theta$:

$$\forall \theta \in \Theta, \quad A_{\theta}^{SCE} = A_{\theta}^{NE}.$$  

Proof By Remark 2, we only have to show that $A_{\theta}^{SCE} \subseteq A_{\theta}^{NE}$. Fix any $a^* = (a_i^*)_{i \in I} \in A_{\theta}^{SCE}$ and any player $i$. By definition of SCE, there is some $\hat{x}_i \in X_i$ such that $a_i^* \in r_i(\hat{x}_i)$ and $f_i(a_i^*, \hat{x}_i) = f_i(a_i^*, \ell_i(a_{-i}, \theta))$. By Lemma B $a_i^* \in r_i(\ell_i(a_{-i}, \theta))$. This holds for each $i$, hence $a^* \in A_{\theta}^{NE}$. 

Corollary B provides sufficient conditions for the equivalence between SCE and NE. Next, we give sufficient conditions that allow a characterization of $A_{\theta}^{SCE}$ by means of Nash equilibria of auxiliary games.

A.3 Equilibrium Characterization

If $a_i \in [0, \bar{a}_i]$ is interpreted as an activity level (e.g., effort) by player $i$, then it makes sense to say that $i$ is active if $a_i > 0$ and inactive otherwise. Let $I_0$ denote the set of players for whom being inactive is justifiable. Note that, by Lemma A,

$$I_0 = \{ i \in I : \min r_i(X_i) = 0 \}.$$  

Also, for each $\theta \in \Theta$ and nonempty subset of players $J \subseteq I$, let $A_{J, \theta}^{NE}$ denote the set of Nash equilibria of the auxiliary game with players set $J$ obtained by letting $a_i = 0$ for each $i \in I\setminus J$, that is,

$$A_{J, \theta}^{NE} = \left\{ a_j \in \times_{j \in J} A_j : \forall j \in J, a_j^* = r_j(\ell_j(a_{-j}, 0)) \right\},$$  

where $0_{I \setminus J} \in \mathbb{R}^{I\setminus J}$ is the profile that assigns 0 to each $i \in I\setminus J$. If $J = \emptyset$, let $A_{\emptyset, \theta}^{NE} = \emptyset$ by convention.

Lemma C. Suppose that the parametrized nice game with aggregators and feedback $G$ satisfies observability by active players. Then, for each $\theta$, the set of selfconfirming action profiles is

$$A_{\theta}^{SCE} = \bigcup_{I \cap J \subseteq I_0} A_{J, \theta}^{NE} \times \{0_{I \setminus J}\}.$$  

Proof Let $J$ be the set of players $i$ such that $a_i^* > 0$. Fix $\theta \in \Theta$ arbitrarily. Let $a^* \in A_{\theta}^{SCE}$ and fix any $i \in I$. If $a_i^* = 0$, then 0 is justifiable for $i$, that is, $i \in I_0$. If $a_i^* > 0$, OiifA implies that $f_i(a_i^*)$ is injective, that is, action $a_i^*$ reveals the payoff state, hence the (shallow) conjecture justifying $a_i^*$ is correct: $a_i^* = r_i(\ell_i(a_{-i}, \theta))$. Thus, $a^* = (a_j^*, a_{-j}^*)$ so that $a_i^* = 0$ for each $i \in I\setminus J \subseteq I_0$, and $a_j^* = r_j(\ell_j(a_{-j}^*, 0)) > 0$ for each $j \in J$. Hence,

$$a^* = (a_j^*, a_{-j}^*) \in A_{J, \theta}^{NE} \times \{0_{I \setminus J}\}$$  

with $I \setminus J \subseteq I_0$.  

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Let \( I \setminus J \subseteq I_0 \) and \((a^*_j, a^*_{I \setminus J}) \in A^{NE}_\Theta \times \{0_{I \setminus J}\} \). Since \( G \) satisfies \( OiffA \), for each \( i \in I \setminus J \), any conjecture justifying \( a^*_i = 0 \) (any \( \hat{x}_i \in r_i^{-1}(0) \)) is trivially confirmed. For each \( j \in J \), \( a^*_j > 0 \) is by assumption the best reply to the correct, hence confirmed, conjecture \( x^*_j = \ell_i \left(a^*_{I \setminus \{j\}}, 0_{I \setminus J}, \theta \right) \). Hence, \((a^*_j, a^*_{I \setminus J}) = (a^*_j, 0_{I \setminus J}) \in A^{SCE}_\Theta \). ■

Appendix B Interior Nash equilibria

Propositions 1 and 2 in Section 3 show that in our framework there exists an equivalence between any selfconfirming equilibrium and the Nash equilibrium of a reduced game in which only active agents are considered and there is also \( OiffA \). Moreover, we can set any subset of agents to be inactive. We now provide some results about existence of these selfconfirming equilibria, that will be useful in proving Proposition 3 in Section 3. We first present sufficient conditions that are present in the literature for the existence of interior Nash equilibria, then we provide some original results.

In this appendix we formulate the problem as a linear algebra problem. We consider a square matrix \( Z \in \mathbb{R}^{n \times n} \) such that \( z_{ii} = 0 \) for all \( i \in \{1, \ldots, n\} \). We call \( I \) the identity matrix, \( \lambda_{\text{max}}(Z) \) the maximal eigenvalue of \( Z \), \( \rho(Z) \) the spectral radius of \( Z \) (i.e. the largest absolute value of its eigenvalues), \( 1 \) is the vector of all 1’s, \( 0 \) is the vector of all 0’s, and \( \gg \) is the strict partial ordering between vectors (meaning that all the elements in the first vector are pairwise strictly greater than the elements in the second vector).

**Proposition C.** If for all \( i \), \( z_{ii} = 0 \), for all \( j \neq i \), \( z_{ij} \leq 0 \), and if \( \rho(Z) < 1 \), then \((I - Z)^{-1} 1 \gg 0\). 21

There are also results when the sign of the externalities are mixed. We recall that the matrix \( Z \) is symmetrizable if there exists a diagonal matrix \( \Gamma \) and a symmetric matrix \( Z_0 \) such that \( Z = \Gamma Z_0 \). Note that if \( Z \) is symmetrizable then all its eigenvalues are real. If for all \( i \), \( z_{ii} = 0 \), and \( Z \) is symmetrizable, we define the symmetric matrix \( \tilde{Z} \) to be such that \( \tilde{z}_{ij} = z_{ij} \sqrt{\gamma_i \gamma_j} \).

**Proposition D.** If for all \( i \), \( z_{ii} = 0 \), \( Z \) is symmetrizable, and if \( |\lambda_{\text{max}}(\tilde{Z})| < 1 \), then \((I - Z)^{-1} 1 \gg 0\). 22

We provide here below an alternative condition, which does also guarantee all positive solutions.

**Proposition E.** Consider a square matrix \( Z \in \mathbb{R}^{n \times n} \) such that:

- \( z_{ii} = 0 \) for all \( i \in \{1, \ldots, n\} \);

---

21This is Theorem 1 in Ballester et al. (2006). The same result is in Appendix A in Stańczak et al. (2006).

22See Section VI of Bramoullé et al. (2014), generalizing Proposition 2 therein. Note that in their payoff specification externalities have a minus sign, while in (4) we have a plus sign: this is why we have a condition on the maximal eigenvalue and not on the minimal eigenvalue.
\[ |z_{ij}| < \frac{1}{n} \text{ for all } i, j \in \{1, \ldots, n\}. \]

Then \((I - Z)^{-1} 1 \gg 0.\)

**Proof:** Call \(B = (I - Z).\) First of all, by Gershgorin circle theorem, \(^{23}\) \(B\) has all eigenvalues strictly between 0 and 2, so \(\det(B) \neq 0.\)

Consider the \(n\) vectors \(b^1, \ldots, b^n\) given by the \(n\) rows of \(B\), and take the hyperplane in \(\mathbb{R}^n\) passing by those \(n\) points:

\[ H = \{ h \in \mathbb{R}^n : \exists \alpha \in \mathbb{R}^n \text{ with } \alpha' \cdot 1 = 1 \text{ and } h = B'\alpha \}. \]

Now, consider the following vector

\[ v = B^{-1} 1. \]

Component \(v_i\) of \(v\) is exactly the sum of the elements in \(i^{th}\) row of \(B^{-1}\). However, \(v\) is also a vector perpendicular to \(H\). That is because for any \(h \in H\) we have

\[ h \cdot v = (B'\alpha)' \cdot B^{-1} 1 = \alpha' 1 = \sum_{i=1}^{n} \alpha_i = 1, \]

which is a constant.

Now, we want to show that \(H\) does not pass through the convex region of vectors with all non-positive elements: \(H \cap (-\infty, 0]^n = \emptyset.\)

In fact, it is impossible to find \(\alpha \in \mathbb{R}^n\), such that \(\alpha' \cdot 1 = 1\) and \(B'\alpha \ll 0.\)

If it was the case, by absurdum, we could take \(k = \arg \max_{i \in \{1, \ldots, n\}} \{\alpha_i\} \) \((\alpha^k > 0 \text{ because } \sum_{i=1}^{n} \alpha_i = 1), \) and write

\[ \alpha b^k = \alpha_k + \sum_{j \neq k} \alpha_j b_{jk} > \alpha_k - \sum_{j \neq k} |\alpha_j||z_{jk}| > \alpha_k \left(1 - \sum_{j \neq k} |z_{jk}|\right) > 0, \]

which would be a contradiction.

Finally, we show that if an hyperplane \(H\) satisfies \(H \cap (-\infty, 0]^n = \emptyset,\) then its perpendicular vector from the origin has all positive elements, and this would close the proof.

We do so by induction on \(n.\)

1. \(n = 2: \) This is easy to show graphically;

2. **Induction hypothesis:** Suppose it is true for \(n = m - 1;\)

^{23}https://en.wikipedia.org/wiki/Gershgorin_circle_theorem
3. **Induction step:** In $\mathbb{R}^m$, a vector $v$ from the origin which is perpendicular to an hyperplane $H$ not passing through the origin can be obtained in the following way. For each dimension $i \in \{1, \ldots, m\}$ take $V^{-i} = \{ v \in \mathbb{R}^m : v_i = 0 \}$. Call $H^{-i}$ the intersection of $H$ with $V^{-i}$, and take a vector $v_{-i} \in V^{-i}$ from the origin that is perpendicular to $H^{-i}$. By the induction hypothesis $v_{-i}$ has all positive elements. We can obtain the vector $v$ from the origin that is perpendicular to $H$ by rescaling each $v_{-i}$, such that $v_{-i}$ is the projection of $v$ on $H^{-i}$. By construction, $v$ will have all positive elements.

Notice that, if $Z$ satisfies the conditions of Proposition E, then it must also hold that $|\lambda_{max}(Z)| < 1$, because of the **Gershgorin circle theorem**. However, the condition that $|\lambda_{max}(Z)| < 1$ is in general not sufficient to guarantee that $(I - Z)^{-1} 1 \gg 0$.

**Appendix C  Proofs**

**Proof of Proposition 1**

**Proof.** Since every agent is active, **state observability by active players** implies **own action independence of the feedback about the state**. Then, the result derives from Corollary B in Appendix A.

**Proof of Proposition 2**

**Proof.** By Remark 5, $NG$ satisfies observability by active players. Hence, Lemma C in Appendix A and the best reply equation (6) yield the result.

**Proof of Proposition 3**

**Proof.** Condition (i), (ii) and (iii) correspond, respectively, to the conditions in Propositions E, C and D from Appendix B.

**Proof of Proposition 4**

**Proof.** If for every $i \in I \backslash I_a^*$ we have that $\alpha + \hat{x}_i < 0$, then changing their $\hat{x}_i$ such that the inequality is still strict, will not make them become active. So, let us focus on the subset $I_a^*$ of active agents. If we perturb locally the beliefs, we will perturb locally also their actions. Assumption 4 garantees that the discrete dynamical system defined for actions by (8) and (9) is stable. So, the variation to beliefs can always be small enough such that: all their actions remain strictly positive; we are in a neighborhood of $a^*$ in the actions’ space, such that the discrete dynamical system defined for actions by (8) and (9) converges back to $a^*$.  

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Proof of Proposition 5

Proof. When we remove elements from $J_a$ and set them to 0, it is as if we delete corresponding rows and columns in the $Z_{J_a}$ matrix. By the Cauchy interlace theorem applied to symmetrizable matrices (see Kouachi 2016) we know that the eigenvalues of the new matrix are between the minimal and the maximal eigenvalues of the old matrix. ■

Proof of Proposition 6

A selfconfirming equilibrium is such that, for all $i \in I$, rationality implies

$$a_i^* = \max\{0, \alpha_i + \hat{x}_i\} .$$

Each agent then thinks that

$$m^* = \alpha_i a_i^* - \frac{1}{2} (a_i^*)^2 + a_i^* \hat{x}_i + \hat{y}_i ,$$

so that

$$\hat{y}_i = m^* - \alpha_i a_i^* + \frac{1}{2} (a_i^*)^2 - a_i^* \hat{x}_i .$$

Substituting the expression of the true payoff function

$$m^* = \alpha_i a_i^* - \frac{1}{2} (a_i^*)^2 + a_i^* x_i + y_i$$

into it, we get the dependence between $\hat{y}_i$ and $\hat{x}_i$:

$$\hat{y}_i = y_i + a_i^* (x_i - \hat{x}_i) .$$

The first and second items in the proposition are derived, respectively, if $a_i^* = 0$ or $a_i^* > 0$. ■

Proof of Proposition 7

Proof. First, we derive some properties. Each equation in the system given by (15) can be written also as a parabola $b_1 a_i^2 + b_2 a_i + b_3 = 0$, in the following way

$$H_i(a, c, Z) = \begin{cases} c_i a_i^2 + \left(1 - \alpha c_i - c_i \left(\sum_{j \in I} z_{ij} a_{j,t}\right)\right) a_i \\ \equiv b_1 \end{cases}$$

$$- \left(1 + c_i \left(\beta \sum_{j \neq i} a_{j,t}\right)\right) = 0 .$$

(b)
So, the solution \( a_i^\ast \) to \( \ell_i(a, c, Z) = 0 \) lies in the right–arm of an upward parabola, where \( \left. \frac{d\ell_i}{da_i} \right|_{a_i=a_i^\ast} > 0 \). With respect to \( c_i \), each \( \ell_i(a, c, Z) \) is a linear equation.

Note also that each \( a_i \) is bounded in the interval
\[
\alpha < a_i < \alpha + \left( \sum_{j \in N_i} z_{ij} a_j \right) + \frac{\beta \sum_{k \neq i} a_k}{a_i}.
\]

Considering that \( a_i^\ast \) is increasing in \( b_3 \) and decreasing in \( b_2 \), it is easy to see that each \( a_i^\ast \) increases in each \( a_j \), with \( j \neq i \).

Second, we show that there is a homeomorphism. There is a continuous function defined from each \( c \in [0, 1]^n \) to an element \( a \in A \), that is because

- either \( c_i = 0 \) and then \( a_i^\ast = \alpha \);
- or \( c_i > 0 \) and then each \( a_i^\ast \) is continuously increasing in each \( x_j \) with \( j \neq i \).

\[ \lim_{c_i \to 0} a_i^\ast = \alpha. \]

\( a_i^\ast \) is bounded above by
\[
\alpha + \left( \sum_{j \in N_i} z_{ij} a_j \right) + \frac{\beta \sum_{j \neq i} a_j}{a_i^\ast}.
\]

Since the system defined by (16) admits a solution, also this system has a finite solution.

This function is one–to–one and invertible, because for each \( a \in A \), we obtain a unique vector \( c \in [0, 1]^n \), and since we obtain it from a linear system of equations, also the inverse function from \( A \) to \( [0, 1]^n \) is continuous.

To analyze the relation between \( a^\ast \) and \( c \), we can apply the implicit function theorem to \( F_i(a, c, Z) \).

We can compute
\[
\frac{dF_i}{dc_i} = \frac{\beta \sum_{j \neq i} a_{j,t}}{(a_i c_i + 1)^2}
\]

Now, since
\[
\ell_i(a, c, Z) = -(a_i c_i + 1) F_i(x, c) \ ,
\]
we have that \( \ell_i(a, c, Z) \), with respect to \( a_i \), has the same zeros as \( F_i(a, c, Z) \), and that, for each \( a_i \), \( \ell_i(a, c, Z) \) is negative if and only if \( F_i(a, c, Z) \) is positive. As they are both continuous functions, this means that since \( \left. \frac{d\ell_i}{da_i} \right|_{a_i=a_i^\ast} > 0 \), we have \( \left. \frac{dF_i}{da_i} \right|_{a_i=a_i^\ast} < 0 \). So, we obtain that
\[
\left. \frac{da_i}{dv_i} \right|_{a_i=a_i^\ast} = - \left. \frac{\partial F_i}{\partial c_i} \right|_{a_i=a_i^\ast} \left. \frac{\partial F_i}{\partial a_i} \right|_{a_i=a_i^\ast} > 0 \ .
\]

This shows that \( a_i^\ast \) is increasing with \( v_i \), and the other way round.

Proof of Proposition 8

Proof. We consider the system (15)

\[ F_i(a, v, Z) = \alpha + c_i \left( \beta \sum_{j \neq i} a_{j,t} \right) \left( \frac{a_i c_i' + 1}{a_i c_i + 1} \right) - a_i = 0, \]

with \( c_i' = \frac{\sum_{j \neq i} z_{i,j} a_{j,t}}{\beta \sum_{j \neq i} a_{j,t}} \). We can compute its Jacobian, with respect to \( a \), and check that each row of the Jacobian sum to less than 1, so that the process is always a contraction. The Jacobian \( J \) is such that:

\[
\begin{align*}
J_{ij} &= \frac{v_i}{a_i c_i + 1} \left( \beta + a_i z_{i,j} \right) \\
J_{ii} &= c_i \left( \beta \sum_{j \neq i} a_j \right) \left( \frac{c_i' - c_i a_i c_i' + 1}{a_i c_i + 1} \right) - 1
\end{align*}
\]

The sum of each row of the Jacobian is

\[ \sum_j J_{ij} = \frac{c_i}{a_i c_i + 1} \left( \beta \left( \sum_{j \neq i} a_j \right) \left( c_i' - c_i a_i c_i' + 1 \right) + a_i \left( \sum_{j \neq i} z_{i,j} \right) + \beta(n - 1) \right) - 1 \quad (d) \]

Let us analyze expression (d) with respect to \( a_i \), for any \( a_i \geq 0 \). As \( a_i \to \infty \), we have that expression (d) is equal to

\[ \sum_{j \neq i} z_{i,j} - 1, \quad (e) \]

whose absolute value is less than one by assumption.

If \( a_i \to 0 \), expression (d) becomes

\[ c_i \beta \left( \sum_{j \neq i} a_j \right) \left( c_i' - c_i \right) + (n - 1) - 1. \quad (f) \]

An interior maximum or minimum of the numerical expression (d), with respect to \( a_i \), must satisfy first order condition

\[
- \left( \frac{c_i}{a_i c_i + 1} \right)^2 \left( \beta \left( \sum_{j \neq i} a_j \right) \left( c_i' - c_i a_i c_i' + 1 \right) + a_i \left( \sum_{j \neq i} z_{i,j} \right) + \beta(n - 1) \right) \\
+ \frac{c_i}{a_i c_i + 1} \left( \beta \left( \sum_{j \neq i} a_j \right) \left( \frac{c_i}{a_i c_i + 1} \right) \left( c_i' - c_i a_i c_i' + 1 \right) + \left( \sum_{j \neq i} z_{i,j} \right) \right) = 0
\]

Last expression can be simplified and results in

\[ v_i \beta(n - 1) = \sum_{j \neq i} z_{i,j}, \]

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which is independent on \( a_i \). So, the only candidates for being minima or maxima for expression (d) are its value in the extrema, namely (e) and (f).

Also, the sign of the first derivative of (d) with respect to \( a_i \) is equal to the sign of \( \sum_{j \neq i} z_{i,j} - c_i \beta (n - 1) \). So, if \( c_i \beta (n - 1) < \sum_{j \neq i} z_{i,j} \) we have that (d) is strictly increasing in \( a_i \), and then (e) is strictly greater than (f).

The value of (e) is between \(-1\) and \(1\), by assumption, because \( 0 < \sum_{j \neq i} z_{i,j} < 2 \).

The quantity in (f) is minimized by \( v_i \to 0 \); and \( c'_i \to 0 \). In this case (f) goes to \(-1\) from the right, and for any \( c_i > 0 \) it will be greater than \(-1\). This complete the proof, because we have shown that any row of the Jacobian \( J \) sums to a number between \(-1\) an \(1\).

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