Stationary or static space-times and Young tableaux

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Abstract. Algebraic curvature tensors possess generators which can be formed from symmetric or alternating tensors $S$, $A$ or tensors $\theta$ with an irreducible $(21)$-symmetry. In differential geometry examples of curvature formulas are known which contain generators on the basis of $S$ or $A$ realized by differentiable tensor fields in a natural way. We show that certain curvature formulas for stationary or static space-times contain such differentiable realizations of generators based on $\theta$. The tensor $\theta$ is connected with the timelike Killing vector field of the space-time. $\theta$ lies in a special symmetry class from the infinite family of irreducible $(21)$-symmetry classes. We determine characteristics of this class. In particular, this class allows a maximal reduction of the length of the curvature formulas. We use a projection formalism by Vladimirov, Young symmetrizers and Littlewood-Richardson products. Computer calculations were carried out by means of the packages Ricci and PERMS.

1. Introduction
The subject of the present paper is the search for examples of Riemann tensors which contain terms with a structure of certain generators of algebraic curvature tensors.

Let $V$ be a finite-dimensional $\mathbb{K}$-vector space, where $\mathbb{K}$ is the field of real or complex numbers. We denote by $T_r V$ the $\mathbb{K}$-vector space of covariant tensors of order $r$ over $V$.

**Definition 1.1** The $\mathbb{K}$-vector space $A(V) \subset T_4 V$ of all algebraic curvature tensors is the set of all tensors $\mathcal{R} \in T_4 V$ which satisfy for all $w, x, y, z \in V$

\begin{align*}
\mathcal{R}(w, x, y, z) &= \mathcal{R}(w, x, z, y) = -\mathcal{R}(y, z, w, x) \\
\mathcal{R}(w, x, y, z) + \mathcal{R}(w, y, z, x) + \mathcal{R}(w, z, x, y) &= 0.
\end{align*}

For algebraic curvature tensors several types of generators are known. For instance, algebraic curvature tensors can be generated by the following tensors:

\begin{align*}
\gamma(S)_{\kappa\lambda\mu\nu} &:= S_{\kappa\nu} S_{\lambda\mu} - S_{\kappa\mu} S_{\lambda\nu}, & S &\in S^2(V), \\
\alpha(A)_{\kappa\lambda\mu\nu} &:= 2A_{\kappa\lambda} A_{\mu\nu} + A_{\kappa\mu} A_{\lambda\nu} - A_{\kappa\nu} A_{\lambda\mu}, & A &\in A^2(V),
\end{align*}

where $S^p(V)$, $A^p(V)$ denotes the spaces of totally symmetric/alternating $p$-forms over $V$. P. Gilkey [1, pp.41-44, P.236] and B. Fiedler [2] gave different proofs for

**Theorem 1.2** $\mathcal{A}(V) = \text{Span}_{S \in S^2(V)} \{\gamma(S)\} = \text{Span}_{A \in A^2(V)} \{\alpha(A)\}$.

But $\mathcal{A}(V)$ possesses also generators on the basis of products $U \otimes w$ or $w \otimes U$, $U \in T_3 V$, $w \in T_1 V$, where $U$ has a so-called irreducible $(21)$-symmetry. B. Fiedler proved in [3].
Theorem 1.3  Let $r \in \mathbb{K}[S_3]$ be a minimal right ideal belonging to the partition $(21) + 3$ and let $T_r$ be the symmetry class of tensors $U \in T_3 V$ that is defined by $r$. Further, let $y_0$ be the Young symmetrizer of the Young tableau

$$
t := \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}.
$$

(5)

Then the following statements are equivalent

(i) $A(V) = \text{Span}_{U \in T_r, \omega \in T_3 V} \{y_0^r(U \otimes \omega)\} = \text{Span}_{U \in T_r, \omega \in T_3 V} \{y_0^r(\omega \otimes U)\}$.

(ii) $r$ is different from the right ideal $v_0 := f_0 \cdot \mathbb{K}[S_3]$ which is generated by the idempotent

$$
f_0 := \frac{1}{2} (\text{id} - (13)) - \frac{1}{6} \sum_{p \in S_3} \text{sign}(p) p.
$$

In differential geometry or general relativity theory (GR), many examples of Riemann tensors are known, in which expressions of the type (3) or (4) occur (see e.g. [4, Thm.2.1],[5, Thm.2],[6, Sec.3]). In the present paper we want to show that there exist also curvature formulas in which generators from Theorem 1.3 appear realized by differentiable tensor fields in a natural way. We found such generators in curvature formulas of stationary and static space-times of GR.

Let $(M,g)$ be a 4-dimensional pseudo-Riemannian manifold of class $C^\infty$ whose fundamental tensor $g$ has signature $(+ -- -)$. Let $\nabla$ be the Levi-Civita connection of $g$.

Definition 1.4  (i) $(M,g)$ is called a stationary space-time if there exists a timelike Killing field $\xi$ on $M$ characterized by the conditions

$$
\xi_{\mu,\nu} + \xi_{\nu,\mu} = 0, \quad g_{\mu\nu} \xi^\mu \xi^\nu > 0.
$$

(6)

(ii) A stationary $(M,g)$ is called a static space-time if the Killing field $\xi$ is hypersurface-orthogonal, i.e. $\xi$ satisfies the additional condition $\xi_{[\mu,\nu,\lambda]} = 0$.

If $(M,g)$ is stationary then one can construct local coordinates $t, x^1, x^2, x^3$ around every point $p \in M$ such that $g_{\mu\nu} = g_{\mu\nu}(x^1, x^2, x^3)$ and $\xi = \partial_t$. If $(M,g)$ is static, then we can choose these local coordinates $t, x^1, x^2, x^3$ in such a way that

$$
ds^2 = g_{\mu\nu} dx^\mu dx^\nu = f(x^1, x^2, x^3) dt^2 - h_{ab}(x^1, x^2, x^3) dx^a dx^b
$$

(7)

$$
\mu, \nu = 0, 1, 2, 3 \quad a, b = 1, 2, 3.
$$

Here $ds^2 := h_{ab} dx^a dx^b$ is a positive definite, 3-dimensional metric.

Now we formulate the main results of our paper.

Proposition 1.5  Let $\tau^\mu := (\xi_0 \xi^\alpha)^{-1/2} \xi^\mu$ be the timelike unit vector field which is proportional to the above Killing field $\xi^\mu$. Further, let $Z_{\kappa,\lambda,\mu}$ be the (covariant) spatial projection (35) of the covariant Riemann tensor $R_{\kappa,\lambda,\mu}$, let $P_{\kappa,\lambda,\mu}$ be the (covariant) 3-dimensional curvature tensor from (34) and let $F_{\mu}$ be the field (26). Then the following formula holds:

$$
-Z_{\kappa,\lambda,\mu} = P_{\kappa,\lambda,\mu} - \frac{3}{4} F_{\nu} F_{\nu} \tau_{\kappa} \tau_{\mu} + \frac{3}{4} F_{\nu} F_{\nu} \tau_{\lambda} \tau_{\mu} + \frac{3}{4} F_{\nu} F_{\nu} \tau_{\kappa} \tau_{\nu} - \frac{3}{4} F_{\nu} F_{\mu} \tau_{\lambda} \tau_{\nu} +
$$

$$
F_{\nu} \tau_{\mu} [\kappa, \lambda] \tau_{\nu} \tau_{[\kappa, \lambda]} + 1/2 F_{\nu} \tau_{\nu} \tau_{[\kappa, \lambda]} + 1/2 F_{\nu} \tau_{\nu} \tau_{[\kappa, \lambda]} +
$$

$$
1/2 F_{\nu} \tau_{\nu} \tau_{\kappa} \tau_{\lambda} - 1/2 F_{\nu} \tau_{\nu} \tau_{\kappa} \tau_{\lambda} + 1/2 F_{\nu} \tau_{\nu} \tau_{\kappa} \tau_{\lambda} -
$$

$$
1/2 F_{\nu} \tau_{\nu} \tau_{\kappa} \tau_{\lambda} - 1/2 F_{\nu} \tau_{\nu} \tau_{\kappa} \tau_{\lambda} + 1/2 F_{\nu} \tau_{\nu} \tau_{\kappa} \tau_{\lambda} -
$$

$$
1/2 F_{\nu} \tau_{\nu} \tau_{\kappa} \tau_{\lambda} - 1/2 F_{\nu} \tau_{\nu} \tau_{\kappa} \tau_{\lambda} + 1/2 F_{\nu} \tau_{\nu} \tau_{\kappa} \tau_{\lambda} -
$$

$$
2 \tau_{[\kappa, \lambda]} \tau_{[\mu, \nu]} + \tau_{[\kappa, \lambda]} \tau_{[\mu, \nu]} - \tau_{[\kappa, \lambda]} \tau_{[\mu, \nu]}
$$

(8)

1 See section 4 for details about symmetry classes of tensors, Young tableaux and Young symmetrizers.
Theorem 1.6 Let $M_p$ be the tangent space of $M$ at a point $p \in M$.

(i) At every $p \in M$ the tensor $(\tau_\lambda \tau_{[\mu;\nu]})|_p$ belongs to a symmetry class $T_p \subseteq T_3 M_p$ whose defining right ideal $\tau \subset \mathbb{K}[S_3]$ has a decomposition $\tau = \tau_1 \oplus \tau_2$ into 2 minimal right ideals $\tau_i$ which is described by the Littlewood-Richardson product

$$[1^2][1] \sim [2 1] + [1^3].$$

(ii) At every $p \in M$ the tensor $(F_\lambda \tau_{[\mu;\nu]})|_p$ lies in a symmetry class $T_p \subseteq T_4 M_p$ whose defining right ideal $\tau \subset \mathbb{K}[S_3]$ has a decomposition $\tau = \tau_1 \oplus \ldots \oplus \tau_5$ into 5 minimal right ideals $\tau_i$ which is described by the Littlewood-Richardson products

$$[2 1][1] \sim [3 1] + [2^2] + [2 1^2], \quad [1^3][1] \sim [2 1^2] + [1^4].$$

At most the product $[2 1][1]$ yields a contribution to the symmetry class $A(V)$ of algebraic curvature tensors which belongs to $[2^2]$.

Lemma 1.7 For the above fields $\xi^\mu$ and $\tau^\mu$ the conditions $\xi_{[\lambda} \xi_{\mu;\nu]} = 0$ and $\tau_{[\lambda} \tau_{\mu;\nu]} = 0$ are equivalent.

Theorem 1.8 A stationary space-time is static iff $\tau_{[\lambda} \tau_{\mu;\nu]} = 0$ or, equivalently, iff the part $[1^3]$ of (9) vanishes. In a static space-time the tensor $\tau_{\lambda} \tau_{[\mu;\nu]}$ belongs to an irreducible $(2 1)$-symmetry class.

Because of (9) the tensor $\tau_{\lambda} \tau_{[\mu;\nu]}$ possesses a decomposition

$$\tau_{\lambda} \tau_{[\mu;\nu]} = \theta_{\lambda \mu \nu} + \tau_{[\lambda} \tau_{\mu;\nu]}, \tag{11}$$

where $\theta_{\lambda \mu \nu} := \tau_{\lambda} \tau_{[\mu;\nu]} - \tau_{[\lambda} \tau_{\mu;\nu]}$ is the unique part of $\tau_{\lambda} \tau_{[\mu;\nu]}$ which has an irreducible $(2 1)$-symmetry. (10) leads to the remarkable consequence that $\tau_{[\lambda} \tau_{\mu;\nu]}$ does not yield a contribution to (8) even in the case of a stationary space-time.

Theorem 1.9 The substitution (11) transforms (8) into

$$-Z_{\kappa \lambda \mu \nu} = P_{\kappa \lambda \mu \nu} - \frac{3}{4} F_\lambda F_{\nu \kappa} \tau_{\mu \tau} + \frac{3}{4} F_\kappa F_{\nu \lambda} \tau_{\mu \tau} + \frac{3}{4} F_\nu F_{\lambda \kappa} \tau_{\mu \tau} - \frac{3}{4} F_\mu F_{\lambda \kappa} \tau_{\nu \tau} - F_\nu \theta_{\mu \kappa \lambda} + F_\nu \theta_{\mu \kappa \lambda} - \frac{1}{2} F_\nu \theta_{\lambda \kappa \mu} + \frac{1}{2} F_\lambda \theta_{\kappa \nu \mu} + \frac{1}{2} F_\lambda \theta_{\kappa \nu \mu} - \frac{1}{2} F_\nu \theta_{\kappa \mu \lambda} + \frac{1}{2} F_\nu \theta_{\kappa \mu \lambda} - \frac{1}{2} F_\lambda \theta_{\nu \kappa \mu} - \frac{1}{2} F_\nu \theta_{\kappa \nu \lambda} - F_\nu \theta_{\nu \mu \lambda} + F_\nu \theta_{\lambda \mu \nu} -$$

$$2 \tau_{[\kappa;\lambda]} \tau_{[\mu;\nu]} + \tau_{[\nu;\mu]} \tau_{[\kappa;\lambda]} - \tau_{[\kappa;\lambda]} \tau_{[\mu;\nu]} \tag{12}$$

In (12) the tensor $\tau_{[\lambda} \tau_{\mu;\nu]}$ does not appear.

Theorem 1.10 Let $\theta_{\lambda \mu \nu} := \tau_{\lambda} \tau_{[\mu;\nu]} - \tau_{[\lambda} \tau_{\mu;\nu]}$ be the tensor field from (11).

(i) The symmetry properties of $\theta$ are described by the relations

$$0 = \theta_{\lambda \mu \nu} + \theta_{\mu \nu \lambda} + \theta_{\lambda \nu \mu}$$

$$0 = -\theta_{\lambda \mu \nu} + \theta_{\nu \mu \lambda} + \theta_{\mu \lambda \nu}$$

$$0 = \theta_{\lambda \mu \nu} - \theta_{\mu \lambda \nu} + \theta_{\mu \nu \lambda}$$

$$0 = \theta_{\nu \mu \lambda} + \theta_{\nu \lambda \mu} \tag{13}$$
\[ -Z_{\kappa\lambda\mu\nu} = P_{\kappa\lambda\mu\nu} - \frac{3}{4} F_\lambda F_\nu \tau_\kappa \tau_\mu + \frac{3}{4} F_\kappa F_\nu \tau_\lambda \tau_\mu + \frac{3}{4} F_\lambda F_\mu \tau_\kappa \tau_\nu - \frac{3}{4} F_\kappa F_\mu \tau_\lambda \tau_\nu \]
\[ + \tau_\kappa [\tau_\lambda \nu] - \tau_\kappa [-\tau_\lambda \nu] - 2\tau_\kappa \tau_\lambda \tau_\mu \nu - 2\tau_\kappa \tau_\mu \tau_\lambda \nu - \frac{3}{2} F_\kappa \theta_\mu \lambda \nu + \frac{3}{2} F_\kappa \theta_\mu \lambda \nu - \frac{3}{2} F_\mu \theta_\kappa \lambda \nu \]
\[ = \frac{1}{2} F_\kappa (\theta \otimes F)_{\mu\kappa\lambda\nu} \text{ according to Theorem 1.3} \]

2. A projection formalism

In investigations of stationary or static space-times one can use a projection formalism which is described for instance in [7, pp. 180] or [8, pp. 49]. We apply the formalism from the book [8, 2.2, A projection formalism] in our paper.

The formalism of Vladimirov starts with the assumption that a timelike unit vector field \( \tau^\mu \) is given on \( M \) which describes the 4-speed of a "continuum of observers". If we define
\[ h_{\mu\nu} := \tau_\mu \tau_\nu - g_{\mu\nu} \quad \text{and} \quad g_{\mu\nu} := \eta_{\mu\alpha} h_{\alpha\nu}, \]
then we obtain the following decompositions of the metric tensors:
\[ g_{\mu\nu} :\tau_{\mu\nu} = 0 \quad \text{and} \quad h_{\mu\nu} = 0, \quad h_{\mu\nu} = 0. \]
A simple consequence of \( \tau^\mu \tau_\mu = g_{\mu\nu} \tau^\mu \tau_\nu = 1 \) is
\[ h_{\mu\nu} \tau_\nu = 0 \quad \text{and} \quad h_{\mu\nu} \tau_\nu = 0. \]
If \( B_{\mu_1 \ldots \mu_r} \nu_1 \ldots \nu_s \) are the coordinates of a \( r \)-times covariant and \( s \)-times contravariant tensor \( B \) then we can define the time component of \( B \)
\[ \tilde{B} := B_{\mu_1 \ldots \mu_r} \nu_1 \ldots \nu_s \tau^{\mu_1} \ldots \tau^{\mu_r} \nu_1 \ldots \nu_s. \]
and the spatial projection of \( B \)
\[ \tilde{B}_{\alpha_1 \ldots \alpha_r} \beta_1 \ldots \beta_s := (-1)^{r+s-1} h_{\alpha_1}^{\mu_1} \ldots h_{\alpha_r}^{\mu_r} h_{\nu_1}^{\beta_1} \ldots h_{\nu_s}^{\beta_s} B_{\mu_1 \ldots \mu_r} \nu_1 \ldots \nu_s. \]
Furthermore we can form mixed projections, for instance
\[ \tilde{B}_{\alpha_1 \ldots \alpha_r \beta_1 \ldots \beta_s} := (-1)^{r+s-1} h_{\alpha_1}^{\mu_1} \ldots h^{\mu_r \nu_r} h_{\nu_1}^{\beta_1} \ldots h_{\nu_s}^{\beta_s} B_{\mu_1 \ldots \mu_r} \nu_1 \ldots \nu_s. \]
Because of (17) the projections (19) and (20) fulfill
\[ \tau^{\alpha_1} \tilde{B}_{\alpha_2 \ldots \alpha_r \ldots} = 0 \quad \text{and} \quad \tau^{\beta_1} \tilde{B}_{\ldots \beta_2 \ldots \beta_s} = 0. \]
If we use the time component \( d\tau := \tau^\mu dx^\mu \) and the spatial projection \( d\tilde{x}^\nu := -h^\nu_\mu dx^\mu \) of \( dx^\mu \) and set \( d\tilde{t} := h_{\mu\nu} d\tilde{x}^\mu d\tilde{x}^\nu \) then we obtain
\[ ds^2 = (\tau_\mu \tau_\nu - h_{\mu\nu}) dx^\mu dx^\nu = d\tilde{t}^2 - h_{\mu\nu} d\tilde{x}^\mu d\tilde{x}^\nu = d\tilde{r}^2 - d\tilde{t}^2. \]
Finally, (15), (17) and 4 = \( g_{\mu\nu} g^{\mu\nu} \) lead to
\[ h_{\alpha}^\mu h_{\mu\beta} = -h_{\alpha\beta} \quad \text{and} \quad h_{\alpha\mu} h^{\alpha\mu} = -h^{\mu\nu} \quad \text{and} \quad h_{\mu\nu} h^{\mu\nu} = 3. \]
Now we consider the decomposition
\[ \tau_{\mu\nu} = \frac{1}{2} (\tau_{\mu\nu} - \tau_{\nu\mu}) + \frac{1}{2} (\tau_{\mu\nu} + \tau_{\nu\mu}) \]
of the covariant derivative \( \tau_{\mu\nu} \). If we covariantly differentiate \( 1 = \tau_{\mu} \tau^\mu \) we obtain immediately
\[ \tau_{\mu\nu} \tau^\mu = 0, \quad (\tau_{\mu\nu} - \tau_{\nu\mu}) \tau^\mu \tau^\nu = 0, \quad (\tau_{\mu\nu} + \tau_{\nu\mu}) \tau^\mu \tau^\nu = 0. \]
**Definition 2.1** The following quantities $F_\alpha, A_{\alpha\beta}, D_{\alpha\beta}$ are useful in formulas for the Riemann tensor:

\[
F_\alpha := -(\tau_{\mu\nu} - \tau_{\nu\mu}) \tau^\nu h^\mu_\alpha = -(\tau_{\mu\nu} + \tau_{\nu\mu}) \tau^\nu h^\mu_\alpha = (\tau_{\alpha\nu} - \tau_{\nu\alpha}) \tau^\nu,
\]

\[
A_{\alpha\beta} := \frac{1}{2} (\tau_{\mu\nu} - \tau_{\nu\mu}) h^\mu_\alpha h^\nu_\beta, \quad D_{\alpha\beta} := -\frac{1}{2} (\tau_{\mu\nu} + \tau_{\nu\mu}) h^\mu_\alpha h^\nu_\beta.
\]

Obviously, $D$ is a symmetric tensor, $D_{\alpha\beta} = D_{\beta\alpha}$, whereas $A$ is skew-symmetric, $A_{\alpha\beta} = -A_{\beta\alpha}$.

**Lemma 2.2** The tensor $A_{\alpha\beta}$ satisfies (see [8, p.51])

\[
A_{\alpha\beta} = \frac{1}{2} (\tau_{\alpha\beta} - \tau_{\beta\alpha}) + \frac{1}{2} (\tau_\alpha F_\beta - \tau_\beta F_\alpha).
\]

**Proof.** Taking into account (15) and (27) we can write

\[
A_{\alpha\beta} = \frac{1}{2} (\tau_{\mu\nu} - \tau_{\nu\mu}) \tau^\nu \tau^\alpha \tau^\beta - \frac{1}{2} (\tau_{\mu\nu} - \tau_{\nu\mu}) \tau^\nu \tau^\alpha \delta^\beta_\gamma
\]

\[
- \frac{1}{2} (\tau_{\mu\nu} - \tau_{\nu\mu}) \tau^\nu \tau^\beta \delta^\alpha_\mu + \frac{1}{2} (\tau_{\mu\nu} - \tau_{\nu\mu}) \delta^\mu_\nu,
\]

where $\delta^\mu_\nu$ denotes the Kronecker symbol. The first summand of (29) vanishes because of (25). But then (26) leads to (28). \(\square\)

We use the following definition of the Christoffel symbols and the Riemann tensor of $\nabla$:

**Definition 2.3** Let $\nabla$ be the Levi-Civita connection of $g$. Then we define the Christoffel symbols $\Gamma^\lambda_{\mu\nu}$ and the Riemann tensor $R^\lambda_{\mu\nu\alpha}$ of $\nabla$ by

\[
\Gamma^\lambda_{\mu\nu} := \frac{1}{2} g^{\lambda\gamma} (\partial_\mu g_{\nu\gamma} + \partial_\nu g_{\mu\gamma} - \partial_\gamma g_{\mu\nu})
\]

\[
R^\lambda_{\mu\nu\alpha} := \partial_\nu \Gamma^\lambda_{\mu\alpha} - \partial_\mu \Gamma^\lambda_{\nu\alpha} + \Gamma^\kappa_{\nu\mu} \Gamma^\lambda_{\kappa\alpha} - \Gamma^\kappa_{\mu\nu} \Gamma^\lambda_{\kappa\alpha}.
\]

The book [8] by Vladimirov uses (30), too, but defines the Riemannian curvature tensor by

\[
\check{R}^\lambda_{\alpha\mu\nu} := \partial_\nu \Gamma^\lambda_{\alpha\mu} - \partial_\mu \Gamma^\lambda_{\alpha\nu} + \Gamma^\kappa_{\alpha\mu} \Gamma^\lambda_{\kappa\nu} - \Gamma^\kappa_{\alpha\nu} \Gamma^\lambda_{\kappa\mu}.
\]

The transformation between (31) and (32) reads

\[
R^\lambda_{\mu\nu\alpha} = -\check{R}^\lambda_{\alpha\mu\nu}.
\]

**Lemma 2.4** The Christoffel symbols $\Gamma^\lambda_{\mu\nu}$ can be expressed by $\tau^\mu, h_{\mu\nu}, F_\alpha, A_{\alpha\beta}$ and $D_{\alpha\beta}$ in the following way

\[
\Gamma^\mu_{\alpha\beta} = \left( L^\mu_{\alpha\beta} + h^\mu_{\alpha\beta} \right) - \tau^\mu (A_{\alpha\beta} - D_{\alpha\beta} + F_\alpha \tau_\beta) + F^\mu \tau_\alpha \tau_\beta + \left( \tau_\alpha A^\mu_\beta + \tau_\beta A^\mu_\alpha \right) - \tau_\alpha \tau^\mu_\beta,
\]

where $L^\mu_{\alpha\beta} := \frac{1}{2} h_{\mu\epsilon} (\partial_\beta h_{\alpha\epsilon} + \partial_\epsilon h_{\alpha\beta} - \partial_\alpha h_{\beta\epsilon})$ denotes the ”Christoffel symbols” of the ”3-dimensional metric” $h$. (See [8, p.53].)

**Definition 2.5** We denote by $P$ the tensor

\[
P^\lambda_{\alpha\beta\mu} := -h^\mu_{\alpha\beta} h^\nu_{\mu\beta} h^\kappa_{\mu\beta} \left( \partial_\mu L^\nu_{\alpha\nu\mu} + \partial_\nu L^\kappa_{\alpha\nu\mu} - \partial_\kappa L^\nu_{\alpha\nu\mu} - L^\nu_{\alpha\nu\mu} \right), \quad L^\nu_{\alpha\nu\mu} := L^\nu_{\alpha\nu} + h^\nu_{\alpha\nu},
\]

which can be considered the curvature tensor assigned to $h$. (See [8, p.55]).
The right-hand side of (34) is equal to the right-hand side of formula (3.30) in [8, p.55]. We adapted only the left-hand side of (34) to our definition (31) of the curvature tensor by means of (33). Note that $L_{\sigma\nu}$ is not symmetric with respect to $\sigma, \nu$ in general.

The Riemann tensor $R$ possesses three spatial projections.

**Definition 2.6** We denote by $Z_{\mu\nu\kappa}^{\lambda}, Y_{\mu\nu\kappa}, X_{\nu\kappa}$ the following three spatial projections of the Riemann tensor $R$:

$$Z_{\mu\nu\kappa}^{\lambda} := h^\alpha_\mu h^\beta_\nu h^\gamma_\kappa R^{\lambda}_{\alpha\beta\gamma\delta}, \quad Y_{\mu\nu\kappa} := h^\alpha_\mu h^\beta_\nu h^\gamma_\kappa \tau^{\lambda}_{\alpha\beta\gamma\delta}, \quad X_{\nu\kappa} := -h^\beta_\nu h^\alpha_\kappa \tau^{\lambda}_{\alpha\beta\gamma\delta}. \quad (35)$$

In the present paper we consider only $Z_{\mu\nu\kappa}^{\lambda}$.

**Proposition 2.7** The spatial projection $Z$ of the Riemann tensor $R$ satisfies

$$-Z_{c\gamma\kappa}^{\lambda} = P_{c\gamma\kappa}^{\lambda} + 2 A^{\lambda}_{\kappa} A_{\gamma} + (D^c_{\gamma} + A_{\gamma}^{\lambda}) (D_{\gamma\kappa} + A_{\gamma\kappa}) - (D_{\gamma}^{\lambda} + A_{\gamma}^{\lambda}) (D_{c\kappa} + A_{c\kappa}) \quad (36)$$

$$-Z_{c\gamma\kappa\lambda} = P_{c\gamma\kappa\lambda} + 2 A_{\kappa\lambda} A_{\gamma} + (D_{\gamma\kappa} + A_{\gamma\kappa}) (D_{\gamma\lambda} + A_{\gamma\lambda}) - (D_{\gamma\lambda} + A_{\gamma\lambda}) (D_{c\kappa} + A_{c\kappa}) \quad (37)$$

where $P_{c\gamma\kappa\sigma} = g_{\lambda\sigma} P_{c\gamma\kappa}^{\sigma} = -h_{\lambda\sigma} P_{c\gamma\kappa}^{\sigma}$.

**Proof.** Relation (36) is equal to the relation (3.39) in [8, p.56], in which a transformation (33) of the left-hand side was carried out. From (36) we obtain (37) by lowering of $\lambda$ by means of $g_{\lambda\sigma}$. The relation $P_{c\gamma\kappa\lambda} = -h_{\lambda\sigma} P_{c\gamma\kappa}^{\sigma}$ is a consequence of (16) and (17). $\Box$

3. **Stationary and static space-times**

Now we apply the projection formalism of Section 2 to a stationary space-time. Let $\xi^\mu$ be the timelike Killing field of such a space-time and $\tau^\mu$ be the timelike unit vector field which is proportional to $\xi^\mu$, i.e. we have $\xi^\mu = \phi \tau^\mu$.

3.1. **Proof of Lemma 1.7**

From $\xi^\mu = \phi \tau^\mu$ we obtain

$$\xi_{\mu;\nu} = (\partial_\nu \phi) \tau_{\mu} + \phi \tau_{\mu;\nu}. \quad (38)$$

and $\xi_{[\lambda} \xi_{\mu;\nu]} = \phi \tau_{[\lambda} \tau_{\mu]} (\partial_\nu \phi) + \phi^2 \tau_{[\lambda} \tau_{\mu;\nu]} = \phi^2 \tau_{[\lambda} \tau_{\mu;\nu]}$, since $\tau_{[\lambda} \tau_{\mu]} = 0$. Consequently, $\xi_{[\lambda} \xi_{\mu;\nu]} = 0$ and $\tau_{[\lambda} \tau_{\mu;\nu]} = 0$ are equivalent. $\Box$

3.2. **Proof of Proposition 1.5**

First we show

**Lemma 3.1** If $\tau^\mu$ is proportional to a Killing field $\xi^\mu$, i.e. $\xi^\mu = \phi \tau^\mu$ is fulfilled, then $D_{\mu\nu} = 0$.

**Proof.** The Killing equation $\xi_{(\mu;\nu)} = 0$ and (38) lead to $0 = \tau_{(\mu} \partial_{\nu)} \phi + \phi \tau_{(\mu;\nu)}$. But then (27) and (17) yield $D_{\alpha\beta} = -\tau_{(\mu;\nu)} h^{\alpha}_{\mu} h^{\beta}_{\nu} = -\phi^{-1} \tau_{(\mu} \partial_{\nu)} \phi h^{\alpha}_{\mu} h^{\beta}_{\nu} = 0$. $\Box$

Now Proposition 1.5 can be proved in the following way. Because of $D_{\mu\nu} = 0$ we can transform (37) into

$$-Z_{c\gamma\kappa\lambda} = P_{c\gamma\kappa\lambda} + 2 A_{\kappa\lambda} A_{\gamma} + A_{c\lambda} A_{\gamma\kappa} - A_{\gamma\lambda} A_{c\kappa}. \quad (39)$$

If we substitute $A_{\alpha\beta}$ by means of (28) in (39) and use the notation $\tau_{(\mu;\nu)} = \frac{1}{2} (\tau_{\mu;\nu} - \tau_{\nu;\mu})$, then we obtain (8). We determined the long formula (8) by means of the Mathematica package Ricci [9]. The Mathematica notebook of this calculation can be downloaded from [10]. $\Box$
4. Symmetry classes of tensors

We denote by $\mathbb{K}[S_r]$ the group ring of the symmetric group $S_r$.

**Definition 4.1** If $T \in T_r V$ and $a = \sum_{p \in S_r} a(p) p \in \mathbb{K}[S_r]$, then we denote by $aT$ the $r$-times covariant tensor

$$(aT)(v_1, \ldots, v_r) := \sum_{p \in S_r} a(p) T(v_{p(1)}, \ldots, v_{p(r)}) , \quad (aT)_{i_1 \ldots i_r} = \sum_{p \in S_r} a(p) T_{i_{p(1)} \ldots i_{p(r)}} .$$

Because of (40), the group ring elements $a \in \mathbb{K}[S_r]$ are called symmetry operators for the tensors $T \in T_r V$. Further we denote by $*: \mathbb{K}[S_r] \to \mathbb{K}[S_r]$ the operator

$$* : a = \sum_{p \in S_r} a(p) p \mapsto a^* := \sum_{p \in S_r} a(p)p^{-1} .$$

**Definition 4.2** Let $\mathfrak{r} \subseteq \mathbb{K}[S_r]$ be a right ideal of $\mathbb{K}[S_r]$. Then the tensor set

$$T_{\mathfrak{r}} := \{ aT \mid a \in \mathfrak{r}, T \in T_r V \}$$

is called the symmetry class of $r$-times covariant tensors defined by $\mathfrak{r}$. $T_{\mathfrak{r}}$ is called irreducible if $\mathfrak{r}$ is minimal.

**Proposition 4.3** Let $e \in \mathbb{K}[S_r]$ be a generating idempotent of a right ideal $\mathfrak{r} \subseteq \mathbb{K}[S_r]$. Then a tensor $T \in T_r V$ lies in the symmetry class $T_{\mathfrak{r}}$ of $\mathfrak{r}$ iff $eT = T$.

Important special symmetry operators are Young symmetrizers, which are defined by means of Young tableaux. A Young tableau $t$ of $r \in \mathbb{N}$ is an arrangement of $r$ boxes such that

(i) the numbers $\lambda_i$ of boxes in the rows $i = 1, \ldots, l$ form a decreasing sequence $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_l > 0$ with $\lambda_1 + \ldots + \lambda_l = r$,

(ii) the boxes are filled with the numbers $1, 2, \ldots, r$ in any order.

For instance, the following graphic shows a Young tableau of $r = 15$.

$$\begin{align*}
\lambda_1 &= 5 \\
\lambda_2 &= 4 \\
\lambda_3 &= 4 \\
\lambda_4 &= 2 \\
&\{11 \quad 2 \quad 5 \quad 4 \quad 12 \quad 9 \quad 6 \quad 13 \quad 3 \quad 8 \quad 16 \quad 1 \quad 7 \quad 10 \quad 14 \}\end{align*}$$

Obviously, the unfilled arrangement of boxes, the Young frame, is characterized by a partition $\lambda = (\lambda_1, \ldots, \lambda_l) \vdash r$ of $r$.

If a Young tableau $t$ of a partition $\lambda \vdash r$ is given, then the Young symmetrizer $y_t$ of $t$ is defined by$^3$

$$y_t := \sum_{p \in \mathcal{H}_t} \sum_{q \in \mathcal{V}_t} \text{sign}(q) p \circ q$$

where $\mathcal{H}_t$, $\mathcal{V}_t$ are the groups of the horizontal or vertical permutations of $t$ which only permute numbers within rows or columns of $t$, respectively. The Young symmetrizers of $\mathbb{K}[S_r]$ are essentially idempotent and define decompositions

$$\mathbb{K}[S_r] = \bigoplus_{\lambda \vdash r} \bigoplus_{t \in \text{ST}_\lambda} \mathbb{K}[S_r] \cdot y_t , \quad \mathbb{K}[S_r] = \bigoplus_{\lambda \vdash r} \bigoplus_{t \in \text{ST}_\lambda} y_t \cdot \mathbb{K}[S_r]$$

$^2$ See [11, pp.127], [12, Lemma III.2.2], [13].

$^3$ We use the convention $(p \circ q)(i) := p(q(i))$ for the product of two permutations $p, q$.
of $\mathbb{K}[S_r]$ into minimal left or right ideals. In (44), the symbol $ST_\lambda$ denotes the set of all standard tableaux of the partition $\lambda$. Standard tableaux are Young tableaux in which the entries of every row and every column form an increasing number sequence.\(^4\)

The inner sums of (44) are minimal two-sided ideals

$$a_\lambda := \bigoplus_{t \in ST_\lambda} \mathbb{K}[S_r] \cdot y_t = \bigoplus_{t \in ST_\lambda} y_t \cdot \mathbb{K}[S_r] \quad (45)$$

of $\mathbb{K}[S_r]$. The set of all Young symmetrizers $y_t$ which lie in $a_\lambda$ is equal to the set of all $y_t$ whose tableau $t$ has the frame $\lambda \vdash r$. Furthermore two minimal left ideals $l_1, l_2 \subseteq \mathbb{K}[S_r]$ or two minimal right ideals $r_1, r_2 \subseteq \mathbb{K}[S_r]$ are equivalent iff they lie in the same ideal $a_\lambda$. Now we say that a symmetry class $T_r$ belongs to $\lambda \vdash r$ iff $r \subseteq a_\lambda$.

S.A. Fulling, R.C. King, B.G. Wybourne and C.J. Cummins showed in [14]

**Theorem 4.4 (Fulling, King, Wybourne, Cummins)**

Let $y_t$ be the Young symmetrizer of the standard tableau (5). Then a tensor $T \in T_s V$ lies in $A(V)$ if and only if $\frac{1}{r_t} y_t^* T = T$.

The group ring element $\frac{1}{r_t} y_t^*$ is a primitive idempotent. Since $y_t \in a_{(2^2)}$, the minimal ideals $l := \mathbb{K}[S_i] \cdot y_t, r = r_t = y_t^* \cdot \mathbb{K}[S_i]$ satisfy $l, r \subseteq a_{(2^2)}$, i.e. the symmetry class $A(V)$ of algebraic curvature tensors belongs to the partition $\lambda = (2^2)^+ 4$.

The following proposition guarantees that we can use Littlewood-Richardson products to determine information about symmetry classes which contain product tensors such as $\tau \in T_{\lambda \tau}[\mu \nu]$ and $F_{\nu \tau}[\mu \nu]$.

**Proposition 4.5**

Let $r_i \subseteq \mathbb{K}[S_r]$ ($i = 1, \ldots, m$) be right ideals and $T^{(i)} \in T_{r_i} T_s V$ be $r_i$-times covariant tensors from the symmetry classes characterized by the $r_i$. Consider the product

$$T := T^{(1)} \otimes \cdots \otimes T^{(m)} \in T_s V, \quad r := r_1 + \cdots + r_m. \quad (46)$$

For every $i$ we define an embedding

$$i_i : S_{r_i} \to S_r, \quad (i_i s)(k) := \begin{cases} \Delta_i + s(k - \Delta_i) & \text{if } r_{i-1} < k \leq r_i \\ k & \text{else} \end{cases} \quad (47)$$

where $\Delta_i := r_0 + \cdots + r_{i-1}$ and $r_0 := 0$. Then all product tensors (46) belong to the symmetry class $T_r$ of the right ideal

$$r := \left(\mathbb{K}[S_r] \cdot \mathcal{L}(\mathbb{I}_1 \cdots \mathbb{I}_m)\right)^* = \left(\mathbb{K}[S_r] \cdot (\mathbb{I}_1 \otimes \cdots \otimes \mathbb{I}_m)\right)^* \quad (48)$$

where $\mathbb{I}_i := i_i(l_i)$ are the embeddings of the left ideals $l_i = r_i^*$ into $\mathbb{K}[S_r]$ induced by the $i_i$. If $\dim V \geq r$, then the right ideal $r$ does not contain a proper right subideal $\tilde{r} \subset r$ such that all tensors (46) lie also in the symmetry class $T_{\tilde{r}}$ of $\tilde{r}$.

Let $\tilde{\omega}_G : G \to GL(\mathbb{K}[G])$ denote the regular representation of a finite group $G$ defined by $\tilde{\omega}_g(f) := g \cdot f, g \in G, f \in \mathbb{K}[G]$. If we use the left ideals $l_i = r_i^*$ to define subrepresentations $\alpha_i := \tilde{\omega}_{S_{r_i}} | l_i, \beta := \tilde{\omega}_{S_r} | r$, then the representation $\beta$ is equivalent to the Littlewood-Richardson product of the $\alpha_i$ (see B. Fiedler [12, Sec.III.3.2.2]):

$$\beta \sim \alpha_1 \alpha_2 \cdots \alpha_m := \alpha_1 \# \cdots \# \alpha_m \uparrow S_r \quad (49)$$

(‘$\#$’ denotes the outer tensor product of the above representations.) This result corresponds to statements of S.A. Fulling et al. [14]. Relation (49) allows us to determine information about the structure of the right ideal (48) by means of the Littlewood-Richardson rule.\(^7\)

\(^4\) About Young symmetrizers and Young tableaux see for instance the references given in [3, Footnote 6].

\(^5\) Theorem 4.4 is a special case of a more general theorem about symmetry classes of $R_{ijkl}(a_1 \ldots a_n)$ in [14].

\(^6\) See B. Fiedler [12, Sec.III.3.2] and B. Fiedler [15].

\(^7\) About the Littlewood-Richardson rule see for instance the references in [3, Footnote 13].
5. On symmetry classes for $\tau_\lambda \tau_{[\mu;\nu]}$ and $F_\kappa \tau_\lambda \tau_{[\mu;\nu]}$

5.1. Proof of Theorem 1.6
Because of Proposition 4.5 we can determine information about a symmetry class containing $\tau_\lambda \tau_{[\mu;\nu]}$ from the corresponding Littlewood-Richardson product. Since the symmetries of $\tau_\lambda$ and $\tau_{[\mu;\nu]}$ are generated by the Young symmetrizers of the tableaux $\begin{ytableau} 1 \\
2 
\end{ytableau}$ and $\begin{ytableau} 2 \\
3 
\end{ytableau}$ we have to calculate the Littlewood-Richardson product $[1\,2][1]$. The use of the Littlewood-Richardson rule yields the graphic

\[ \begin{ytableau} 1 \\
2 
\end{ytableau} \times \begin{ytableau} 2 \\
3 
\end{ytableau} \sim \begin{ytableau} 1 & 2 \\
3 & 4 
\end{ytableau} + \begin{ytableau} 1 & 2 \\
4 & 3 
\end{ytableau} \]

which can be translated into (9). The relation (9) means that the right ideal $r$ of the symmetry class containing $\tau_\lambda \tau_{[\mu;\nu]}$ possesses a decomposition $r = r_1 \oplus r_2$ into 2 minimal right ideals which belong to the partitions $(2\,1), (1\,3) \vdash 3$.

Now we see that information about the right ideal $r$ of a symmetry class for $F_\kappa \tau_\lambda \tau_{[\mu;\nu]}$ can be gained from the Littlewood-Richardson products $[2\,1][1]$ and $[1\,3][1]$. For these products the Littlewood-Richardson rule yields (10) since it leads to the graphics

\[ \begin{ytableau} 1 & 2 \\
3 & 4 
\end{ytableau} \times \begin{ytableau} 1 \\
2 
\end{ytableau} \sim \begin{ytableau} 1 & 2 \\
3 & 4 
\end{ytableau} + \begin{ytableau} 1 & 2 \\
4 & 3 
\end{ytableau} , \quad \begin{ytableau} 1 & 2 \\
3 & 4 
\end{ytableau} \times \begin{ytableau} 1 \\
2 
\end{ytableau} \sim \begin{ytableau} 1 & 2 \\
3 & 4 
\end{ytableau} + \begin{ytableau} 1 & 2 \\
4 & 3 
\end{ytableau} \]

Theorem 4.4 tells us that the symmetry class $A(V)$ is defined by the right ideal $r = y^*_t \cdot K[S_4]$ generated by the Young symmetrizer of the tableau $\begin{ytableau} 1 \\
2 
\end{ytableau}$. This right ideal $r$ is minimal and belongs to $(2\,2) \vdash 4$ because $y_t \in a(2\,2)$. But only $[2\,1][1]$ possesses a part $[2\,2]$ which belongs to a minimal right ideal of $(2\,2)$. This proves the last assertion of Theorem 1.6. □

5.2. Proof of Theorem 1.8
One and only one symmetry class $T_\varepsilon$ of $T_3V$ belongs to the partition $(1\,3) \vdash 3$. It is defined by the right ideal $\tilde{r} := y^*_\varepsilon \cdot K[S_3]$ where $y_\varepsilon$ is the Young symmetrizer of the Young tableau

\[ \begin{ytableau} 1 \\
2 
\end{ytableau} \]

$\frac{1}{6}y^*_\varepsilon$ yields the alternation of a tensor from $T_3V$, i.e. it holds $\frac{1}{6}(y_\varepsilon T)_{\lambda\mu\nu} = \tau_{[\lambda;\mu;\nu]}$ if we abbreviate $T_{\lambda\mu\nu} := \tau_\lambda \tau_{[\mu;\nu]}$. Thus, $\tau_{[\lambda;\mu;\nu]} = 0$ means that the part of $\tau_\lambda \tau_{[\mu;\nu]}$ which lies in $T_\varepsilon$ vanishes. □

5.3. Proof of Theorem 1.9
Because of (10) the Littlewood-Richardson product $[1\,3][1]$ has no part that belongs to the partition $(2\,2) \vdash 4$. This leads us to the assumption that the terms $\tau_{[\lambda;\mu;\nu]}$ will possibly fall out of (8) if we carry out the substitution (11) in (8). We verified this by a Mathematica calculation using the tensor package Ricci [9]. In this calculation we showed that the lines 3, 4, 5 of (8) vanish if we replace all expressions $\tau_{[\lambda;\mu;\nu]}$ by the coordinates of an arbitrary alternating tensor $a_{\lambda\mu\nu}$ of order 3. A record of this calculation can be found in [10, curvterms.nb]. □

5.4. Proof of Theorem 1.10
The proof uses results from [16, 3]. In [16] we showed by means of discrete Fourier transforms for symmetric groups that every minimal right ideal $r \subset a(2\,1) \subset K[S_3]$ is generated by exactly
one element of the following set of (primitive) idempotents$^8$

$$\zeta_\nu := \frac{1}{3} \{ [1, 2, 3] + \nu [1, 3, 2] + (1 - \nu) [2, 1, 3] - \nu [2, 3, 1] + (-1 + \nu) [3, 1, 2] - [3, 2, 1] \}, \ \nu \in \mathbb{K}$$

$$\eta := \frac{1}{3} \{ [1, 2, 3] - [2, 1, 3] - [2, 3, 1] + [3, 2, 1] \}.$$ 

On the other hand $\theta_{\lambda\mu\nu} = \tau_{\lambda\tau[\mu\nu]} - \tau[\lambda\tau_{\mu\nu}]$ is generated by the idempotent symmetry operator $\rho := \frac{1}{3} \{ [1, 2, 3] - [1, 3, 2] \} - \frac{1}{3} \sum_{p \in S_3} \text{sign}(p) \rho$ from $\tau_{\lambda\mu\nu}$. We showed by means of the Mathematica package PERMS [17] in the notebook [10, curvterms.nb] that $\eta \cdot \rho \neq \rho, \rho \cdot \eta \neq \eta$ and

$$\zeta_\nu \cdot \rho = \rho \text{ and } \rho \cdot \zeta_\nu = \zeta_\nu \Leftrightarrow \nu = -1.$$ 

Consequently, the tensor $\theta_{\lambda\mu\nu}$ belongs to the symmetry class $T_3$ which is defined by the right ideal $\tau := \zeta_{-1} \cdot \mathbb{K}[S_3]$ with generating idempotent $\zeta_{-1}$.

This right ideal $\tau$ is different from the right ideal $\tau_0$ in Theorem 1.3 since a result of [3] says that $\tau_0 = \zeta_{1/2} \cdot \mathbb{K}[S_3]$. In the notebook [10, part16a.nb] for the paper [3] we proved that all tensors of the symmetry class of $\zeta_{-1}$ satisfy the identities (13). Using (13) we transformed (12) into (14) by means of Ricci [9] in [10, curvterms.nb]. Finally, we see by a comparison with formulas in [3, Sec.4.3] that the last line of (14) is equal to $\frac{1}{2} y_t (\theta \otimes F)_{\kappa \lambda \mu \nu}$, where $t$ is the tableau (5).

Remark : In [3] we showed that the symmetry operator $\zeta_{-1}$ produces tensors $U_{\lambda\mu\nu}$ of a (2 1)-symmetry class which admits the index commutation symmetry $U_{\lambda\nu\mu} = -U_{\lambda\mu\nu}$. Obviously the above tensor $\theta_{\lambda\mu\nu}$ possesses this index commutation symmetry. A further result of [3] says that then the coordinates of algebraic curvature tensors $y_t (w \otimes U) (t$ given by (5)) can be reduced to a sum with a minimal length of 4 summands. This effect can be observed in the above reduction of the number of terms $F_n \theta_{\lambda\mu\nu}$ in formula (12), too.

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